# Linearization Continued

## Section 8.4 Follow-Up

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# Review



- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:
  *For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where $c_1$ denotes the first Chern class of the bundle $TW$.*

- We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

# Review

- We used the chosen frame $\{Z_i\}$ to define a chart centered at $u$ of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ Y = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

- We regard $Y(s, t)$ as a tangent vector to $W$ in some Euclidean embedding.

# Review

- We seek to compute the composite map in charts:

$$\begin{tikzcd}
W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \arrow[r, "\iota"] \arrow[rrr, bend left, dashed, "\mathcal{F}_u"] & \mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, bend left, dotted, "\bar{\mathcal{F}}"] & L^p\left(\mathbb{R}\times S^1;TW\right) \arrow[r] & L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)
\end{tikzcd}$$

$$u \xrightarrow{\quad\bar{\mathcal{F}}\quad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

$$(y_1,\ldots,y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)$$

# Add a Tangent



$$\begin{aligned}
\mathcal{F}(u) &= \quad \tfrac{\partial u}{\partial s} \quad + J(u)\tfrac{\partial u}{\partial t} \quad - J(u) X_t(u) \\
\mathcal{F}(u+Y) &= \quad \tfrac{\partial (u+Y)}{\partial s} \quad + J(u+Y)\tfrac{\partial (u+Y)}{\partial t} \quad - J(u+Y) X_t(u+Y)
\end{aligned}$$

Extract the part that is linear in $Y$ and collect terms:

$$\begin{aligned}
&(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\qquad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)
\end{aligned}$$

.

# Review



Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}
(d\mathcal{F})_u(Y) &= \left( \frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right) \\
&\qquad + \left( (dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \right) \\
&= \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) \\
&\quad + \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right. \\
&\qquad\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right).
\end{aligned}$$

Use the fact that this is $O_1 + O_0$ in $Y$.

# Review



Study $O_1$ first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where $J_0$ is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^n))$ is a linear operator of order 0.

# Order 0 Symmetry in the Limit

**Theorem (8.4.4, CR + Symmetric in the Limit)**

*If $u$ solves Floer's equation, then*

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

*where*

1. *$S$ is linear*
2. *$S$ tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s, t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \textit{uniformly in } t$$

# Proof



Collect terms in the order zero part:

$$\begin{aligned}
O_0 = S(y_1, \cdots, y_{2n}) &= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\
&\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right] \\
&= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i) X_t \right) \right. \\
&\qquad \left. + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right].
\end{aligned}$$

- Claim: the terms in blue and orange vanish in the limit $s \longrightarrow \pm\infty$, so it suffices to prove that the red term limits to a symmetric operator.

# Proof: Blue Term Vanishes

$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since $u$ is a solution and

$$\frac{\partial u}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

*This seems to be the full argument for the blue term.*

# Proof: Orange Term Vanishes (1 and 3)



$$\frac{\partial Z_i}{\partial s} \xrightarrow{s \longrightarrow \pm\infty} 0$$

Follows since the frame $Z_i$ was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \, \partial t} \quad \curvearrowright Z_i \xrightarrow{s \longrightarrow \pm\infty} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \xrightarrow{s \longrightarrow \pm\infty} 0.$$

*This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in t?*

# Proof: Symmetry



Write the remaining red term as

$$A := A(y_1, \ldots, y_{2n}) = \sum y_i \left( J(u)\frac{\partial Z_i}{\partial t} - J(u)\left(dX_t\right)_u(Z_i) \right).$$

Extract the $j$th component:

$$A_j = \sum y_i \left\langle J(u)\frac{\partial Z_i}{\partial t} - J(u)\left(dX_t\right)_u(Z_i), \quad Z_j \right\rangle.$$

We'll show that

$$\lim_{s \to \pm\infty} \left\langle J(u)\frac{\partial Z_i}{\partial t} - J(u)\left(dX_t\right)_u(Z_i), Z_j \right\rangle - \left\langle J(u)\frac{\partial Z_j}{\partial t} - J(u)\left(dX_t\right)_u Z_j, \quad Z_i \right\rangle = 0.$$

# Proof: Symmetry

Use the fact that the frame $\{Z_i\}$ is unitary:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \langle J(u) Z_i, Z_j \rangle \\
&= \left\langle (dJ)_u \left( \frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle + \left\langle J(u) Z_i, \frac{\partial Z_j}{\partial t} \right\rangle \\
&= \left\langle (dJ)_u \left( \frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle - \left\langle Z_i, J(u) \frac{\partial Z_j}{\partial t} \right\rangle .
\end{aligned}$$

# Proof: Symmetry



Therefore it suffices to show

$$\begin{aligned} & -\left\langle J(u)\left(dX_t\right)_u\left(Z_i\right), \quad Z_j\right\rangle \\ & +\left\langle J(u)\left(dX_t\right)_u\left(Z_j\right), \quad Z_i\right\rangle \\ & -\left\langle (dJ)_u\left(\frac{\partial u}{\partial t}\right) Z_i, \quad Z_j\right\rangle \end{aligned}$$

$$\overset{s \longrightarrow \pm\infty}{\longrightarrow} 0.$$

# Proof: Symmetry

Using the fact that

$$\left( \frac{\partial u}{\partial t} - X_t(u) \right) \xrightarrow{s \longrightarrow \pm\infty} 0$$

we can equivalently show

$$\begin{aligned} &- \left\langle J(u) \left( dX_t \right)_u (Z_i) , \quad Z_j \right\rangle \\ &+ \left\langle J(u) \left( dX_t \right)_u (Z_j) , \quad Z_i \right\rangle \\ &- \langle (dJ)_u \left( X_t \right) Z_i , \qquad Z_j \rangle \end{aligned}$$

$$\xrightarrow{s \longrightarrow \pm\infty} 0$$

*For a fixed $(s, t)$, this expression only depends on $Z_i$ at the point $u(s, t)$.*

# Lemma



**Lemma**: For $p \in W$, $\{Z_i\}$ a unitary basis of $T_pW$,

$$\begin{aligned}
&- \left\langle J(p)\left(dX_t\right)_p\left(Z_i\right), \quad Z_j\right\rangle \\
&+ \left\langle J(p)\left(dX_t\right)_p\left(Z_j\right), \quad Z_i\right\rangle \\
&- \langle (dJ)_p\left(X_t\right) Z_i, \qquad Z_j\rangle \\
&= 0.
\end{aligned}$$

**Claim**: This lemma immediately concludes the previous proof?

# Proof of Lemma



Extend $\{Z_i\}$ to a chart containing $p$ and use the Leibniz rule to rewrite

$$-\left\langle J(p)\left(dX_t\right)_p\left(Z_i\right), Z_j\right\rangle + \left\langle J(p)\left(dX_t\right)_p\left(Z_j\right), Z_i\right\rangle - \langle (dJ)_p\left(X_t\right) Z_i, Z_j\rangle = 0$$

as

$$-\left\langle J\left(dX_t\right)\left(Z_i\right), Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle + \left\langle J\left(dZ_i\right)\left(X_t\right), Z_j\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle$$

$$= \left\langle J\left[X_t, Z_i\right], Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle.$$

where we'll rewrite the red terms.

# Proof of Lemma



Now use

$$X_t\langle JZ_i,\ Z_j\rangle = 0 \implies \langle d\left(JZ_i\right)\left(X_t\right), Z_j\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle = 0.$$

We now rewrite the RHS from before:

$$\langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right), Z_i\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle$$

$$\begin{aligned}
&= \langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right) - J\left(dZ_j\right)\left(X_t\right), Z_i\rangle \\
&= \langle J\left[X_t, Z_i\right], Z_j\rangle - \langle J\left[X_t, Z_j\right], Z_i\rangle \\
&= \omega\left(\left[X_t, Z_i\right], Z_j\right) - \omega\left(\left[X_t, Z_j\right], Z_i\right).
\end{aligned}$$

The symmetry follows from $\omega$ being closed and

$$\begin{aligned}
0 &= d\omega\left(X_t, Z_i, Z_j\right) \\
&= X_t \cdot \omega\left(Z_i, Z_j\right) - Z_i \cdot \omega\left(X_t, Z_j\right) + Z_j \cdot \omega\left(X_t, Z_i\right) \\
&\quad - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) - \omega\left(\left[Z_i, Z_j\right], X_t\right) \\
&= -X_t \cdot \langle Z_i, JZ_j\rangle + Z_i \cdot \left(dH_t\right)\left(Z_j\right) - Z_j \cdot \left(dH_t\right)\left(Z_i\right) \\
&\quad - \left(dH_t\right)\left(\left[Z_i, Z_j\right]\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\
&= d\left(dH_t\right)\left(Z_i, Z_j\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\
&= -\omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right). \qquad \blacksquare
\end{aligned}$$

# Linearization of Hamilton's Equation

Recall

$$(d\mathcal{F})_u = \bar{\partial} Y + SY = \left(\bar{\partial} + S\right) Y$$

Now think of $S$ as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm \coloneqq \lim_{s \longrightarrow \pm\infty} S(s, \cdot\,) \quad \text{respectively}$$

## Theorem

*The equation*

$$\partial_t Y = J_0 S^\pm Y$$

*is a linearization of Hamilton's equation*

$$\frac{\partial z}{\partial t} = X_t(z) \quad at \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & for\ S^- \\ y = \lim_{s \longrightarrow \infty} u & for\ S^+ \end{cases} \quad respectively.$$

# Proof



We first linearize Hamilton's equation at $x$:

$$\frac{\partial z}{\partial t} = X_t(z) \quad \overset{\text{linearized}}{\Longrightarrow} \quad \frac{\partial Y}{\partial t} = (dX_t)_x Y.$$

So write $Y = \sum y_i Z_i$ to obtain

$$\begin{aligned}
\sum_i \frac{\partial y_i}{\partial t} Z_i &= \sum_i y_i \left( -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i) \right) \\
&= \sum_i \sum_j y_i \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i), Z_j \right\rangle Z_j \\
&= \sum_i \sum_j y_j \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle Z_i
\end{aligned}$$

$$\implies \frac{\partial y_i}{\partial t} = \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle y_j.$$