Homological Algebra Problem Sets

Problem Set 3

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Table of Contents

Contents

Tab	ole of Contents	2
1	Wednesday, February 17	3

Table of Contents

1 Wednesday, February 17

Problem 1.0.1 (Prove Corollary 2.3.2)

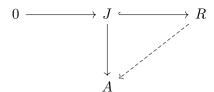
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have a = br for some $b \in A$.

Solution:

Note: we'll assume R is commutative.

 \Leftarrow : Suppose $A \in R$ -Mod is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \subseteq R$:



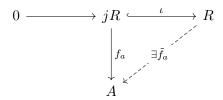
Link to Diagram

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that A = jA, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write x = jr for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$f_a: J = jR \to R$$

 $x = jr \mapsto r\mathbf{a}.$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:



Link to Diagram

We can now use the fact that

$$r\mathbf{a} = f_a(jr)$$

$$= \tilde{f}_a(\iota(jr))$$

$$= \tilde{f}_a(jr)$$

$$= jr\tilde{f}_a(1_R)$$

$$= rj\tilde{f}_a(1_R)$$

$$\Rightarrow \mathbf{a} = j\tilde{f}_a(1_R) \in jA,$$

using R-linearity and $j, r \in R$ since R is commutative where in the last step we have canceled an r on the left, using the fact a PID is a domain and thus has no nonzero zero divisors, and in particular r is not a zero divisor and is thus left-cancelable. So in the definition of divisibility, we can take

$$\mathbf{a}' \coloneqq \tilde{f}_a(1_R),$$

and letting a range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)

Calculate $\operatorname{Ext}_{\mathbb{Z}}^{i}(\mathbb{Z}/p,\mathbb{Z}/q)$ for distinct primes p,q.

Problem 1.0.3 (Weibel 2.3.2)

For $A \in \mathbf{Ab}$, define $I(A) := \bigoplus_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \to I(A)$. Show that e_A is

injective.

Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)

If $U: \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_iF) \cong L_i(UF).$$

Problem 1.0.5 (Weibel 2.4.3)

If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_iF(A) \cong L_{i-1}FM$$

Show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$.

i > 2.

Problem 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

- a. A is a projective R-module.
- b. $\operatorname{Hom}_R(\cdot, A)$ is an exact functor.
- c. $\operatorname{Ext}_R^{i\neq 0}(A,B)=0$ and for all B, i.e. A is $\operatorname{Hom}_R(\,\cdot\,,B)$ -acyclic for all B.
- d. $\operatorname{Ext}_{R}^{1}(A,B)$ vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)

Show that colim is left adjoint to Δ , and conclude that colim is right-exact when when \mathcal{A} is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on \mathbf{Ab} .

Wednesday, February 17