

# Problem Set 1

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## 1 Problem 5

### 1.1 Part 1

Let  $A \in \text{Mat}(n, n)$  be a positive definite  $n \times n$  matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and  $B \in \text{Mat}(n, n)$  be positive semi-definite, so

$$\langle v, Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v, (A + B)v \rangle &= \langle v, Av \rangle + \langle v, Bv \rangle \\ &> \langle v, Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

## 1.2 Part 2

## 2 Problem 6

### 2.1 Part 1

Let  $M = S^2$  as a smooth manifold, and consider a vector field on  $M$ ,

$$X : M \rightarrow TM$$

We want to show that there is a point  $p \in M$  such that  $X(p) = 0$ .

Every vector field on a compact manifold without boundary is complete, and since  $S^2$  is compact with  $\partial S^2 = \emptyset$ ,  $X$  is necessarily a complete vector field.

Thus every integral curve of  $X$  exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point  $p \in M$ .

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular,  $\phi_0 = \text{id}_M$ , and  $\phi_1 \in \text{Diff}(M, M)$ . Moreover  $\phi_0$  is homotopic to  $\phi_1$  via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to  $\phi_0$  and  $\phi_1$ . For an arbitrary map  $f : M \rightarrow M$ , we have

$$\Lambda(f) = \sum_k \text{Tr} \left( f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where  $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned}\Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M),\end{aligned}$$

the Euler characteristic of  $M$ .

Since homotopic maps induce equal maps on homology, we also have  $\Lambda(\phi_1) = \chi(M)$ .

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have  $\chi(S^2) = 2 \neq 0$ , and thus  $\phi_1$  has a fixed point  $p_0$ , thus

$\frac{\partial}{\partial t} \phi_t(p_0) \Big|_{t=1}$  so

$$\begin{aligned}\phi_t(p) &= p \\ \implies \frac{\partial}{\partial t} \phi_t(p) &= \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} &= 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies X(\phi_1(p_0)) &:= \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 && \text{by definition of } \phi_1\end{aligned}$$

so  $X(\phi_1(p_0)) = 0$ , which shows that  $p_0$  is a zero of  $X$ . So  $X$  has at least one zero, as desired.  $\square$

## 2.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned}s : S^2 &\rightarrow S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\rightarrow (\mathbf{x}, [1, 1])\end{aligned}$$

which is the identity on the  $S^2$  component and assigns the constant vector  $[1, 1]$  to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc}
 \mathbb{R}^2 & \longrightarrow & TS^2 \\
 & & \downarrow s \\
 & & S^2
 \end{array}$$

can *not* have a nowhere vanishing section.  $\square$