Mathematics Subject GRE Workshop

Agenda

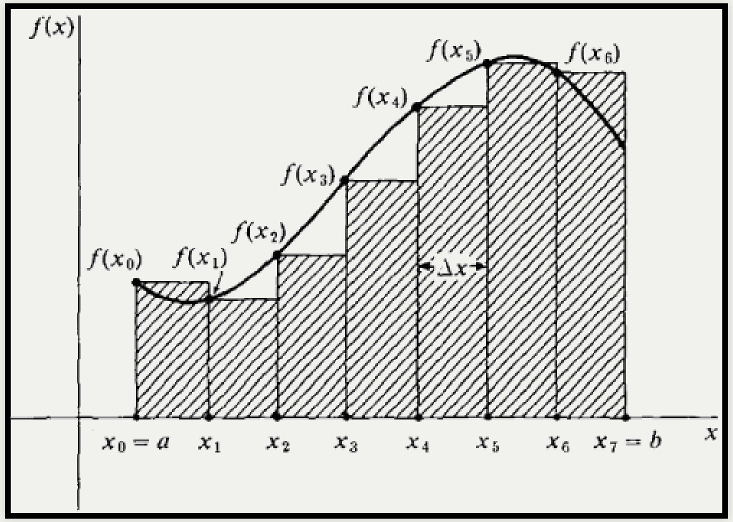
- Description of Mathematics Subject GRE
- Topics it covers
- Exam logistics
- Recommended resources
- Study techniques/tips
- Review of topics + sample problems

What is the Mathematics Subject GRE?

- Different from the Math section of the *General* GRE
- Required of graduate student applicants to many Math Ph.D. programs
- Tests a breadth of undergraduate topics

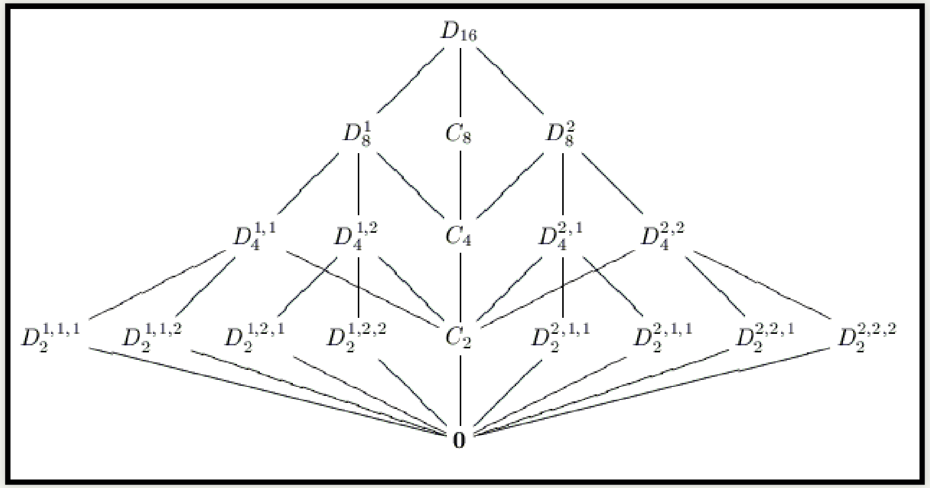
Topics

- Calculus (50%)
  - Single Variable
  - Multivariable
  - Differential Equations



“Algebra” (25%)

- Linear Algebra
- Abstract Algebra
- Number Theory



Mixed Topics (25%)

- Real Analysis
- Logic / Set Theory
- Discrete Mathematics
- Point-Set Topology
- Complex Analysis
- Combinatorics
- Probability

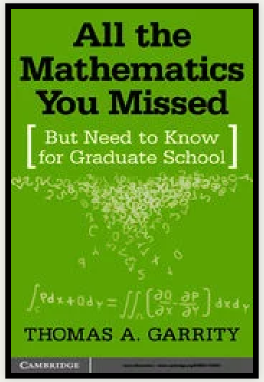
$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x \text{ is rational, } x = \frac{p}{q} \text{ in lowest terms and } q > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Logistics

- Multiple choice, 5 choices
- 66 questions, 170 minutes
- No downside to guessing
- Only offered 3x/year
- Need to register ~2 months in advance

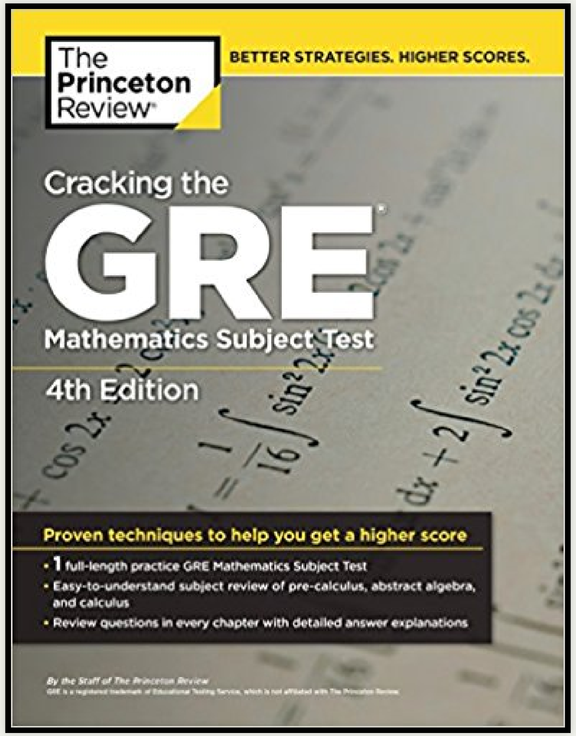
References

Garrity, All the Mathematics You Missed (But Need to Know for Graduate School)



Good high-level overview of undergrad topics.

The Princeton Review, Cracking the Math GRE Subject Test



“Calculus: The Greatest Hits”, good breadth.

Shallow treatment of Algebra, Real Analysis, Topology, Number Theory.

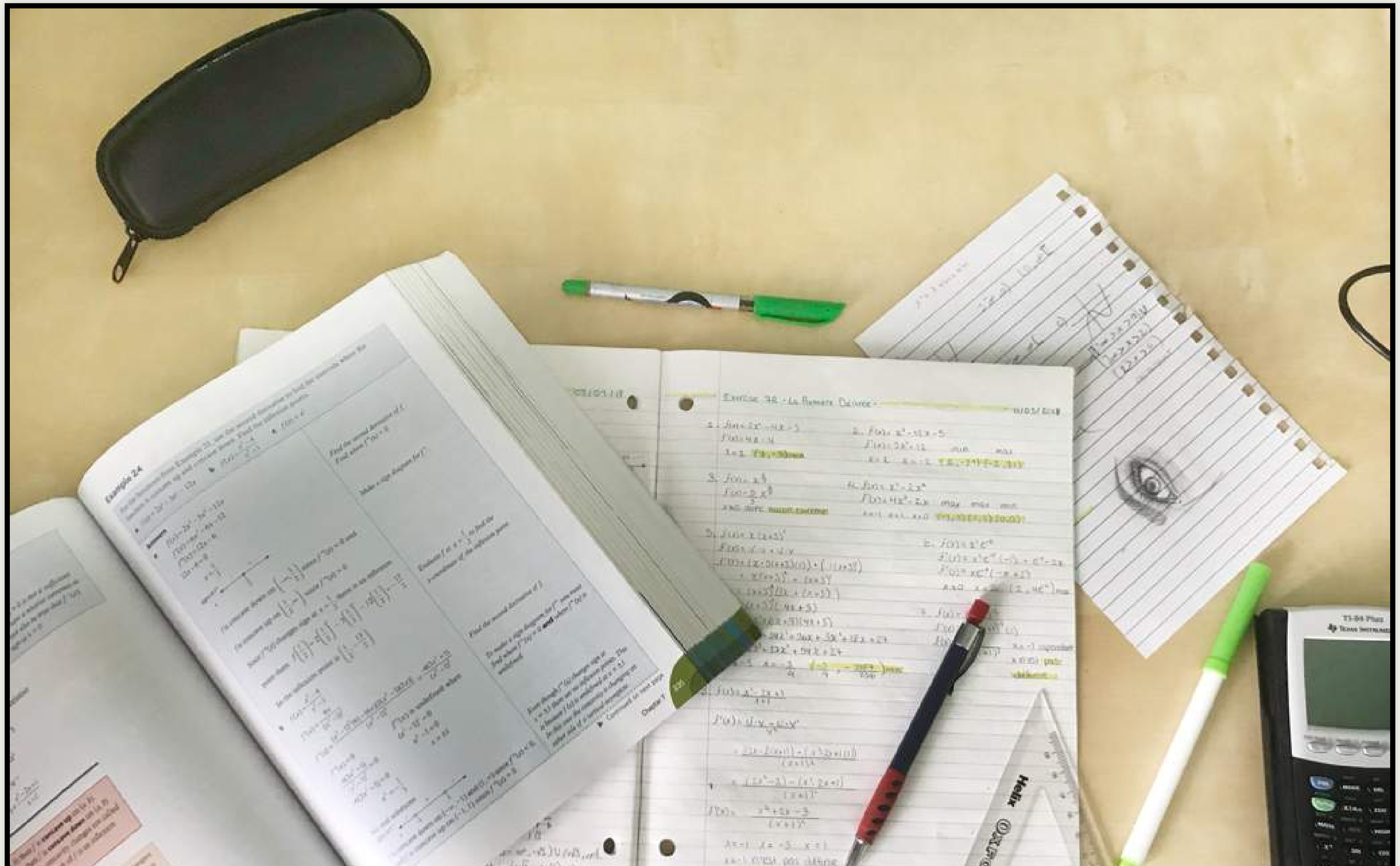
Five Official Practice Exams (with Solutions)

- GR 1268
- GR 0568
- GR 9367
- GR 8767
- GR 9768

All old and *significantly* easier than exams in recent years.

Aim for 90th percentile in < 2 hours.

General Tips



Math-Specific Tips

- Focus on lower div
- For Calculus, focus on speed: median ≤ 1 minute
- Drill *a lot* of problems
  - Seriously, a lot.
  - *Seriously.*
- Should memorize formulas and definitions
  - No time to rederive!
- Save actual exams as diagnostic tools

Study Tips

- Start early
  - Steady practice paced over 3-9 months is 100x more effective than 1 month of cramming
- Speed is important
- Spaced repetition, e.g. Anki
- Replicate exam conditions
- Build mental stamina
  - i.e. 2-3 hours of uninterrupted problem solving
- Self care!!
  - Sleep
  - Eat right

Single Variable Calculus

Differential

- Computing limits
- Showing continuity
- Computing derivatives
- Rolle’s Theorem
- Mean Value Theorem
- Extreme Value Theorem
- Implicit Differentiation
- Related Rates
- Optimization
- Computing Taylor expansions
- Computing linear approximations

Integral

- Riemann sum definition of the integral
- The fundamental theorem of Calculus (both forms)
- Computing antiderivatives
  - u-substitutions
  - Partial fraction decomposition
  - Trigonometric Substitution
  - Integration by parts
  - Specific integrands
- Computing definite integrals
- Solids of revolution
- Series (see real analysis section)

Computing Limits

- Tools for finding $\lim_{x \to a} f(x)$, in order of difficulty:
  - Plug in: equal to $f(a)$ if $f \in C^0(N_\varepsilon(a))$
  - Algebraic Manipulation
  - L'Hopital's Rule (only for indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$)
    - For $\lim f(x)^{g(x)} = 1^\infty, \infty^0, 0^0$, let $L = \lim f^g \implies \ln L = \lim g \ln f$
  - Squeeze theorem
  - Take Taylor expansion at a
  - Monotonic + bounded (for sequences)

Use Simple Techniques

When possible, of course.

$$\frac{a}{b+\sqrt{c}} = \frac{a}{b+\sqrt{c}}\left(\frac{b-\sqrt{c}}{b-\sqrt{c}}\right) = \frac{a(b-\sqrt{c})}{b^2-c}$$

$$\frac{1}{ax^2+bx+c} = \frac{1}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2}$$

The Fundamental Theorems of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b \frac{\partial}{\partial x} f(x)\,dx = f(b) - f(a)$$

First form is usually skimmed over, but very important!

FTC Alternative Forms

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} g(t)dt = g(b(x))b'(x) - g(a(x))a'(x)$$

Commuting D and I

Commuting a derivative with an integral

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

$$+ f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x)$$

(Derived from chain rule)

Set

$$a(x) = a, b(x) = b, f(x,t) = f(t) \implies \frac{\partial}{\partial x} f(t) = 0,$$

then commute to derive the FTC.

Applications of Integrals

- Solids of Revolution
  - Disks: $A = \int \pi r(t)^2 \, dt$
  - Cylinders: $A = \int 2\pi r(t)h(t) \, dt$
- Arc Lengths
  - $ds = \sqrt{dx^2 + dy^2}, \qquad L = \int \, ds$

Series

There are 6 major tests at our disposal:

- **Comparison Test**
  - $a_n < b_n$ and $\sum b_n < \infty \implies \sum a_n < \infty$
  - $b_n < a_n$ and $\sum b_n = \infty \implies \sum a_n = \infty$
  - You should know some examples of series that converge and diverge to compare to.
- **Ratio Test**

$$R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

  - $R < 1$: absolutely convergent
  - $R > 1$: divergent
  - $R = 1$: inconclusive

More Series

- **Root Test**

$$R = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

  - $R < 1$: convergent
  - $R > 1$: divergent
  - $R = 1$: inconclusive
- **Integral Test**

$$f(n) = a_n \implies \sum a_n < \infty \iff \int_1^\infty f(x)dx < \infty$$

More Series

- **Limit Test**
$$\lim_{n\to\infty}\frac{a_n}{b_n} = L < \infty \implies \sum a_n < \infty \iff \sum b_n < \infty$$
- **Alternating Series Test**
$$a_n \downarrow 0 \implies \sum (-1)^n a_n < \infty$$

Advanced Series

- **Cauchy Criteria**:
  - Let $s_k = \sum_{i=1}^{k} a_i$ be the k-th partial sum, then

$$\sum a_i \text{ converges } \iff \{s_k\} \text{ is a Cauchy sequence,}$$

- **Weierstrass M Test**:

$$\sum_{n=1}^{\infty} |\|f_n\|_\infty| < \infty \implies$$

$$\exists f \in C^0 \ \ni \ \sum_{n=1}^{\infty} f_n \rightrightarrows f$$

  - i.e. define $M_k = \sup\{f_k(x)\}$ and require that $\sum |M_k| < \infty$
  - "Absolute convergence in the sup norms implies uniform convergence"

Multivariable Calculus

General Concepts

- Vectors, div, grad, curl
- Equations of lines, planes, parameterized curves
  - And finding intersections
- Multivariable Taylor series
  - Computing linear approximations
- Multivariable optimization
  - Lagrange Multipliers
- Arc lengths of curves
- Line/surface/flux integrals
- Green’s Theorem
- The divergence theorem
- Stoke’s Theorem

Geometry in $\mathbb{R}^3$

Lines

$$Ax + By + C = 0,\ \mathbf{x} = \mathbf{p} + t\mathbf{v},$$

$$\mathbf{x} \in L \iff \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

Planes

$$Ax + By + Cz + D = 0,\ \mathbf{x}(t, s) = \mathbf{p} + t\mathbf{v}_1 + s\mathbf{v}_2$$

$$\mathbf{x} \in P \iff \langle \mathbf{x} - \mathbf{p}, \mathbf{n} \rangle = 0$$

Distances to lines/planes: project onto orthogonal complement.

Tangent Planes/Linear Approximations

Let $S \subseteq \mathbb{R}^3$ be a surface. Generally need a point $\mathbf{p} \in S$ and a normal $\mathbf{n}$.

Key Insight: The gradient of a function is normal to its level sets.

Case 1: $S = \{[x, y, z] \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$

i.e. it is the zero set of some function $f : \mathbb{R}^3 \to \mathbb{R}$

- ∇f is a vector that is normal to the zero level set.
- So just write the equation for a tangent plane $\langle \mathbf{n}, \mathbf{x} - \mathbf{p}_0 \rangle$.

Tangent Planes/Linear Approximations

Case 2: S is given by $z = g(x, y)$

- Let $f(x, y, z) = g(x, y) - z$, then

$$\mathbf{p} \in S \iff \mathbf{p} \in \{[x, y, z] \in \mathbb{R}^3 \mid f(x, y, z) = 0\}.$$

- Then ∇f is normal to level sets, compute $\nabla f = [\frac{\partial}{\partial x} g, \frac{\partial}{\partial y} g, -1]$
- Proceed as in previous case.

Optimization

Single variable: solve $\frac{\partial}{\partial x} f(x) = 0$ to find critical points c_i then check min/max by computing $\frac{\partial^2}{\partial x^2} f(c_i)$.

Multivariable: solve $\nabla f(\mathbf{x}) = 0$ for critical points $\mathbf{c}_i$, then check min/max by computing the determinant of the Hessian:

$$H_f(\mathbf{a}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{bmatrix}.$$

Optimization

Lagrange Multipliers:

$$\text{Optimize } f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) = c$$
$$\implies \nabla \mathbf{f} = \lambda \nabla \mathbf{g}$$

- Generally a system of nonlinear equations
  - But there are a few common tricks to help solve.

Multivariable Chain Rule

$$\begin{tikzcd}
& u \arrow[r] \arrow[dd] \arrow[ddr] & x \\
z \arrow[ur] \arrow[urr] \arrow[dr] \arrow[drr] & & \\
& v \arrow[r] \arrow[uur] & y
\end{tikzcd}$$

Multivariable Chain Rule

To get any one derivative, sum over all possible paths to it:

$$\begin{aligned}\left(\frac{\partial z}{\partial x}\right)_y = &\left(\frac{\partial z}{\partial x}\right)_{u,y,v} \\ &+ \left(\frac{\partial z}{\partial v}\right)_{x,y,u}\left(\frac{\partial v}{\partial x}\right)_y \\ &+ \left(\frac{\partial z}{\partial u}\right)_{x,y,v}\left(\frac{\partial u}{\partial x}\right)_{v,y} \\ &+ \left(\frac{\partial z}{\partial u}\right)_{x,y,v}\left(\frac{\partial u}{\partial v}\right)_{x,y}\left(\frac{\partial v}{\partial x}\right)_y\end{aligned}$$

Subscripts denote variables held constant while differentiating.

Linear Approximation

Just use Taylor expansions.

Single variable case:

$$f(x) = f(p) + f'(p)(x - p)$$
$$+ f''(p)(x - a)^2 + O(x^3)$$

Multivariable case:

$$f(\mathbf{x}) = f(\mathbf{p}) + \nabla f(\mathbf{p})(\mathbf{x} - \mathbf{a})$$
$$+ (\mathbf{x} - \mathbf{p})^T H_f(p)(\mathbf{x} - \mathbf{p}) + O(\|\mathbf{x} - \mathbf{p}\|_2^3)$$

Linear Algebra

Big Theorems

- Rank Nullity:

$$|\ker(A)| + |\mathrm{im}\,(A)| = |\mathrm{domain}(A)|$$

- Fundamental Subspace Theorems

$$\mathrm{im}\,(A) \perp \ker(A^T), \qquad \ker(A) \perp \mathrm{im}\,(A^T)$$

- Compute
    - Determinant, trace, inverse, subspaces, eigenvalues, etc
    - Know properties too!
- Definitions
    - Vector space, subspace, singular, consistent system, etc

Fundamental Spaces

- Finding bases for various spaces of A:
  - $\text{rowspace} A / \text{im } A^T \subseteq \mathbb{R}^n$
    - Reduce to RREF, and take nonzero rows of $\text{RREF}(A)$.
  - $\text{colspace} A / \text{im } A \subseteq \mathbb{R}^m$:
    - Reduce to RREF, and take columns with pivots from original A.

Fundamental Spaces

- $\operatorname{nullspace}(A)/\ker A$:
  - Reduce to RREF, zero rows are free variables, convert back to equations and pull free variables out as scalar multipliers.
- Eigenspace:
  - Recall the equation:
$$\lambda \in \operatorname{Spec}(A) \iff \exists \mathbf{v}_\lambda \ni A\mathbf{v}_\lambda = \boldsymbol{\lambda}\mathbf{v}_\lambda$$
  - For each $\lambda \in \operatorname{Spec}(A)$, compute $\ker(\lambda I - A)$

Big List of Equivalent Properties

Let A be an $n \times n$ matrix representing a linear map $L : V \to W$

TFAE:

- A is invertible and has a unique inverse A^{-1}
- A^T is invertible
- $\det(A) \neq 0$
- The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $b \in \mathbb{R}^m$
- The homogeneous system $A\mathbf{x} = 0$ has only the trivial solution $\mathbf{x} = 0$
- $\operatorname{rank}(A) = \dim(W) = n$
  - i.e. A is full rank
- $\operatorname{nullity}(A) := \dim(\operatorname{nullspace}(A)) = \dim(\ker L) = 0$

Big List of Equivalent Properties

- $A = \prod_{i=1}^{k} E_i$ for some finite k, where each E_i is an elementary matrix.
- A is row-equivalent to the identity matrix I_n
- A has exactly n pivots
- The columns of A are a basis for $W \cong \mathbb{R}^n$
  - i.e. $\text{colspace}(A) = \mathbb{R}^n$
- The rows of A are a basis for $V \cong \mathbb{R}^n$
  - i.e. $\text{rowspace}\,(A) = \mathbb{R}^n$
- $(\text{colspace}\,(A))^{\perp} = \left(\text{rowspace}\,(A^T)\right)^{\perp} = \{\mathbf{0}\}$
- Zero is not an eigenvalue of A.
- A has n linearly independent eigenvectors

Various Other Topics

- Quadratic forms
- Projection operators
- Least Squares
- Diagonalizability, similarity
- Canonical forms
- Decompositions ($QR, VDV^{-1}, SVD,$ etc)

Ordinary Differential Equations

Easy IVPs

- Should be able to immediately write solutions to any initial value problem of the form
$$\sum_{i=0}^{n} \alpha_i y^{(i)}(x) = f(x)$$
  - Just write the characteristic polynomial.

Easy IVPs

- Example: A second order homogeneous equation

$$ay'' + by' + cy = 0 \mapsto ax^2 + bx + c = 0$$

  - Two distinct roots:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

  - One real root:

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

  - Complex conjugates $\alpha \pm \beta i$:

$$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

More Easy IVPs

- The Logistic Equation

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P \implies P(t) = \frac{P_0}{\frac{P_0}{C} + e^{-rt}(1 - \frac{P_0}{C})}$$

- Separable

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

More Easy IVPs

- Systems of ODEs

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t) \implies \mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\ \mathbf{v}_i$$

for each eigenvalue/eigenvector pair $(\lambda_i, \mathbf{v}_i)$.

Less Common Topics

- Integrating factors
- Change of Variables
- Inhomogeneous ODEs (need a *particular solution*)
  - Variation of parameters
  - Annihilators
  - Undetermined coefficients
  - Reduction of Order
  - Laplace Transforms
  - Series solutions
- Special ODEs
  - Exact
  - Bernoulli

Topics: Number Theory

Definitions

- The fundamental theorem of arithmetic:

$$n \in \mathbb{Z} \implies n = \prod_{i=1}^{n} p_i^{k_i}, \quad p_i \text{ prime}$$

- Divisibility and modular congruence:

$$x \mid y \iff y = 0 \mod x \iff \exists c \ni y = xc$$

- Useful fact:

$$x = 0 \mod n \iff x = 0 \mod p_i^{k_i} \; \forall i$$

(Follows from the Chinese remainder theorem since all of the $p_i^{k_i}$ are coprime)

Definitions

- GCD, LCM

$$xy = \gcd(x, y)\, \mathrm{lcm}(x, y)$$

$$d \mid x \text{ and } d \mid y \implies d \mid \gcd(x, y)$$

$$\text{and } \gcd(x, y) = d \gcd\left(\frac{x}{d}, \frac{y}{d}\right)$$

  - Also works for $\mathrm{lcm}(x, y)$
  - Computing $\gcd(x, y)$:
    - Take prime factorization of x and y,
    - Take only the distinct primes they have in common,
    - Take the minimum exponent appearing

The Euclidean Algorithm

Computes GCD, can also be used to find modular inverses:

$$\begin{aligned} a &= q_0 b + r_0 \\ b &= q_1 r_0 + r_1 \\ r_0 &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_k &= q_{k+2} r_{k+1} + \mathbf{r_{k+2}} \\ r_{k+1} &= q_{k+3} r_{k+2} + 0 \end{aligned}$$

Back-substitute to write $ax + by = \mathbf{r_{k+2}} = \gcd(a, b)$.

(Also works for polynomials!)

Definitions

- Coprime

$$a \text{ is coprime to } b \iff \gcd(a,b) = 1$$

- Euler's Totient Funtion

$$\phi(a) = |\{x \in \mathbb{N} \ni x \leq a \text{ and } \gcd(x,a) = 1\}|$$

  - Computing ϕ:

$$\gcd(a,b) = 1 \implies \phi(ab) = \phi(a)\phi(b)$$

$$\phi(p^k) = p^k - p^{k-1}$$

  - Just take the prime factorization and apply these.

Definitions

Know some group and ring theoretic properties of $\mathbb{Z}/n\mathbb{Z}$

- $\mathbb{Z}/n\mathbb{Z}$ is a field $\iff$ n is prime.
  - So we can solve equations with inverses:
    $ax = b \mod n \iff x = a^{-1}b \mod n$
- But there will always be *some* units; in general,

$$|(\mathbb{Z}/n\mathbb{Z})^\times| = \phi(n)$$

  and is cyclic when $n = 1, 2, 4, p^k, 2p^k$

Chinese Remainder Theorem

The system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_r \pmod{m_r} \end{aligned}$$

has a unique solution $x \mod \prod m_i$ iff $\gcd(m_i, m_j) = 1$ for each pair i, j.

Chinese Remainder Theorem

The solution is given by

$$x = \sum_{j=1}^{r} a_j \frac{\prod_i m_i}{m_j} \left(\left[\frac{\prod_i m_i}{m_j} \right]^{-1}_{\bmod m_j} \right)$$

Seems symbolically complex, but actually an easy algorithm to carry out by hand.

Chinese Remainder Theorem

Ring-theoretic interpretation: let $N = \prod n_i$, then

$$\gcd(i, j) = 1 \ \ \forall (i, j) \implies \mathbb{Z}_N \cong \bigoplus \mathbb{Z}_{n_i}$$

Theorems

- Fermat's Little Theorem and Euler's Theorem

$$a^p = a \mod p$$

$$p \nmid a \implies a^{p-1} = 1 \mod p$$

$$\text{and in general,}$$

$$a^{\phi(p)} = 1 \mod p$$

- Wilson's Theorem

$$n \text{ is prime} \iff (n-1)! = -1 \mod n$$

Advanced Topics

- Mobius Inversion
- Quadratic residues
- The Legendre/Jacobi Symbols
- Quadratic Reciprocity

Topics: Abstract Algebra

Definitions

- Group, ring, subgroup, ideal, homomorphism, etc
- Order, Center, Centralizer, orbits, stabilizers
- Common groups: $S_n, A_n, C_n, D_{2n}, \mathbb{Z}_n$, etc

Structure

- Structure of S_n
  - e.g. Every element is a product of disjoint cycles, and the order is the lcm of the order of the cycles.
  - Generated by (e.g.) transpositions
  - Cycle types
  - Inversions
  - Conjugacy classes
  - Sign of a permutation
- Structure of $\mathbb{Z}_n$

$$\mathbb{Z}_{pq} = \mathbb{Z}_p \oplus \mathbb{Z}_q \iff (p,q) = 1$$

Basics

Group Axioms

- Closure: $a, b \in G \implies ab \in G$
- Identity: $\exists e \in G \mid a \in G \implies ae = ea = a$
- Associativity: $a, b, c \in G \implies (ab)c = a(bc)$
- Inverses: $a \in G \implies \exists b \in G \mid ab = ba = e$

One step subgroup test:

$$H \leq G \iff a, b \in H \implies ab^{-1} \in H$$

Useful Theorems

Cauchy's Theorem

- If $|G| = n = \prod p_i^{k_i}$, then for each i there exists a subgroup H of order p_i.

The Sylow Theorems

- If $|G| = n = \prod p_i^{k_i}$, for each i and each $1 \leq k_j \leq k_i$ then there exists a subgroup $H_{i,j}$ for all orders $p_i^{k_j}$.
  - Note: partial converse to Cauchy's theorem.

Classification of Abelian Groups

Suppose $|G| = n = \prod_{i=1}^{m} p_i^{k_i}$

$$G \cong \bigoplus_{i=1}^{n} G_i \text{ with } |G_i| = p_i^{k_i} \text{ and}$$

$$G_i \cong \bigoplus_{j=1}^{k} \mathbb{Z}_{p_i^{\alpha_j}} \text{ where } \sum_{j=1}^{k} \alpha_j = k_i$$

G decomposes into a direct sum of groups corresponding to its prime factorization. For each component, you take the corresponding prime, write an integer partition of its exponent, and each unique partition yields a unique group.

Ring Theory

- Definition: $(R, +, \times)$ where $(R, +)$ is abelian and $(R, times)$ is a monoid.
- Ideals: $(I, +) \leq (R, +)$ and $r \in R, x \in I \implies rx \in I$
- Noetherian: $I_1 \subseteq I_2 \subseteq \cdots \implies \exists N \ni I_N = I_{N+1} = \cdots$
  - (Ascending chain condition)
- Differences between prime and irreducible elements
  - Prime: $p \mid ab \implies \mid a$ or $p \mid b$
  - Irreducible: x irreducible $\iff \nexists a, b \in R^\times \ni p = ab$
- Various types of rings and their relations:

Topics: Real Analysis

- Properties of Metric Spaces
- The Cauchy-Schwarz Inequality
- Definitions of Sequences and Series
- Testing Convergence of sequences and series
- Cauchy sequences and completeness
- Commuting limiting operations:
  - $[\frac{\partial}{\partial x}, \int dx]$
- Uniform and point-wise continuity
- Lipschitz Continuity

Big Theorems

- **Completeness**: Every Cauchy sequence in $\mathbb{R}^n$ converges.
- **Generalized Mean Value Theorem**

$$f, g \text{ differentiable on } [a, b] \implies$$

$$\exists c \in [a, b] : [f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$$

  - Take $g(x) = x$ to recover the usual MVT
- **Bolzano-Weierstrass**: every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- **Heine-Borel**: in $\mathbb{R}^n$, X is compact $\iff$ X is closed and bounded.

Topics: Point-Set Topology

General Concepts

- Open/closed sets
- Connected, disconnected, totally disconnected, etc
- Mostly topics related to metric spaces

Useful Facts

- Topologies are closed under
  - Arbitrary unions:

$$U_j \in \mathcal{T} \implies \bigcup_{j \in J} U_i \in \mathcal{T}$$

  - Finite intersections:

$$U_i \in \mathcal{T} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$$

- In $\mathbb{R}^n$, singletons are closed, and thus so are finite sets of points
  - Useful for constructing counterexamples to statements

Topics: Complex Analysis

General Concepts

- n-th roots:

$$e^{\frac{ki}{2\pi n}}, \qquad k = 1, 2, \cdots n - 1$$

- The Residue theorem:

$$\oint_C f(z)\, dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$$

  - Exams often include one complex integral
  - Need a number of other theorems for actually computing residues

Topics: Discrete Mathematics + Combinatorics

General Concepts

- Graphs, trees
- Recurrence relations
- Counting problems
  - e.g. number of nonisomorphic structures
- Inclusion-exclusion, etc

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example Problems

Example Problem 1

8. Which of the following is NOT a group?

   (A) The integers under addition

   (B) The nonzero integers under multiplication

   (C) The nonzero real numbers under multiplication

   (D) The complex numbers under addition

   (E) The nonzero complex numbers under multiplication

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C, because $\mathbb{Z} - \{0\}$ lacks inverses
(Would need to extend to $\mathbb{Q}$)

Example Problem 2

19. If z is a complex variable and $\overline{z}$ denotes the complex conjugate of z, what is $\lim_{z \to 0} \frac{(\overline{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Example Problem 2

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$$L = \lim_{(a,b)\to 0} \frac{(a-bi)^2}{(a+bi)^2} = \lim_{(a,b)\to 0} \frac{a^2 - b^2 - 2abi}{a^2 - b^2 + 2abi}$$

$$a = 0 \implies L = 1$$

$$a = b \implies L = -1$$

So E, because the limit needs to be path-independent.

Example Problem 3

24. Consider the system of linear equations

$$\begin{aligned} w + 3x + 2y + 2z &= 0 \\ w + 4x + y \quad\quad &= 0 \\ 3w + 5x + 10y + 14z &= 0 \\ 2w + 5x + 5y + 6z &= 0 \end{aligned}$$

with solutions of the form (w, x, y, z), where w, x, y, and z are real. Which of the following statements is FALSE?

(A) The system is consistent.

(B) The system has infinitely many solutions.

(C) The sum of any two solutions is a solution.

(D) $(-5, 1, 1, 0)$ is a solution.

(E) Every solution is a scalar multiple of $(-5, 1, 1, 0)$.

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Don’t row-reduce or invert! Just one computation

Example Problem 3

So D, A are true. C is true because it's a homogeneous system. B is true because $A\mathbf{x} = 0 \implies A(t\mathbf{x}) = tA\mathbf{x} = 0$which means $t\mathbf{x}$ is a solution for every t. By process of elimination, E must be false.

Example Problem 4

42. Let $\mathbb{Z}^+$ be the set of positive integers and let d be the metric on $\mathbb{Z}^+$ defined by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$$

for all $m, n \in \mathbb{Z}^+$. Which of the following statements are true about the metric space $(\mathbb{Z}^+, d)$?

I. If $n \in \mathbb{Z}^+$, then $\{n\}$ is an open subset of $\mathbb{Z}^+$.

II. Every subset of $\mathbb{Z}^+$ is closed.

III. Every real-valued function defined on $\mathbb{Z}^+$ is continuous.

(A) None (B) I only (C) III only (D) I and II only (E) I, II, and III

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Note $N_{\frac{1}{2}}(x) = \{x\}$, so every singleton is open. Any subset of $\mathbb{Z}$ is a countable union of its singletons, so every subset of $\mathbb{Z}$ is open. The complement any set is one such subset, so every subset is clopen. The inverse image of any subset of $\mathbb{R}$ under any $f : \mathbb{Z} \to \mathbb{R}$ is a subset of $\mathbb{Z}$,