Homological Algebra Problem Sets

Problem Set 2

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Problem 1.0.1 (Weibel 1.3.3)
Prove the 5-lemma. Suppose the following rows are exact:

$$
\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}
$$

Link to Diagram

a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
b. Show that if f_2, f_4 are epi and f_5 is monic, then f_3 is an epi.
c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):
Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with "$\hookrightarrow$" and the epis with "$\twoheadrightarrow$":

$$
\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[d, hook, "f_4"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4
\end{tikzcd}
$$

Link to Diagram

It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2.
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \to A_2 \to A_3$, so $x = 0$.

$$
\begin{tikzcd}
y_5 \in A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] \arrow[dr, phantom, "3"] & y_2 \in A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] \arrow[dr, phantom, "2"] & x \in A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "1"] & y_1 \in A_4 \arrow[d, hook, "f_4"] \\
y_4 \in B_1 \arrow[r, "\partial_1^B"] & y_3 \in B_2 \arrow[r, "\partial_2^B"] & 0 \in B_3 \arrow[r, "\partial_3^B"] & 0 \in B_4
\end{tikzcd}
$$

Link to Diagram

Solution (Part (b)):
Similarly, it suffices to consider the following portion of the diagram:

$$
\begin{tikzcd}
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}
$$

Link to Diagram

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4, pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \to A_5 \to B_5$.
- By injectivity of f_5, the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z - y \mapsto y_1 - y_1 = 0$ under $B_3 \to B_4$, using that $d(z-y) = d(z) - d(y)$ since these are R-module morphisms.

- By exactness of the bottom row, pull $z - y$ back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2, pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z - y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z - y$ under $A_3 \to B_3$.
- But then $y_3 - \zeta_3 \mapsto z - (z - y) = y$ under $A_3 \to B_3$ as desired.

$$
\begin{tikzcd}
 & \zeta_3 \arrow[dddddd, dashed] & & \\
\zeta_2 \arrow[ur, dotted] \arrow[ddd, dotted, bend right] & y_3 \arrow[r, dotted] \arrow[dddd, dashed, bend left] & y_2 \arrow[r, dotted] \arrow[ddd, dotted, bend left] & 0 \arrow[ddd, dotted, bend left] \\
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] \arrow[dr, phantom, "3"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "2"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] \arrow[dr, phantom, "1"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5 \\
\zeta_1 \arrow[ddr, dotted] & y \arrow[r, dotted] & y_1 \arrow[r, dotted] & 0 \\
 & z & & \\
 & z - y & &
\end{tikzcd}
$$

Link to Diagram

Solution (Part (c)):
Given the previous two result, if the outer maps are isomorphisms then f_3 is both monic and epi. Using a technical fact that monic epis are isomorphisms in a category $\mathcal{C}$ if and only if $\mathcal{C}$ is *balanced* and that all abelian categories are balanced, f_3 is isomorphism.

Problem 1.0.2 (Weibel 1.4.2)
Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$
\begin{aligned}
C_n &\cong Z_n \oplus B_n' \\
Z_n &= B_n \oplus H_n'.
\end{aligned}
$$

Show that C is split exact if and only if $H_n' = 0$.

Solution:
For this problem, we'll use the fact that if d is an epimorphism, it satisfies the right-cancellation property: if $f \circ d = g \circ d$, then $f = g$. In particular, if $d_n = d_n s_{n-1} d_n$ with $d_n : C_n \to C_{n-1}$

surjective and $s_{n-1} : C_{n-1} \to C_n$, we can conclude $\mathbb{1}_{C_n} = d_n s_{n-1}$. We'll also use the fact that if we have a SES in any abelian category $\mathcal{A}$, then the following are equivalent:

- The sequence is split on the left.
- The sequence is split on the right.
- The middle term is isomorphic to the direct sum of the outer terms.

Fixing notation, we'll write $C \coloneqq (\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots)$, and we'll use concatenation fg to denote function composition $f \circ g$.
$\implies$:
Suppose C is split, so we have maps $\{s_n\}$ such that $\partial_n = \partial_n s_{n-1} \partial_n$.

Claim: The short exact sequence

$$0 \to Z_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

admits a right-splitting $f : B_{n-1} \to C_n$, and thus there is an isomorphism

$$C_n \cong Z_n \oplus B_n' \coloneqq Z_n \oplus B_{n-1}.$$

Proof (?).
That this sequence is exact follows from the fact that it can be written as

$$0 \to \ker \partial_n \hookrightarrow C_n \overset{\partial_n}{\twoheadrightarrow} \operatorname{im} \partial_n \to 0.$$

We proceed by constructing the splitting f. Noting that $s_{n-1} : C_{n-1} \to C_n$ and $B_{n-1} \leq C_{n-1}$, the claim is that its restriction $f \coloneqq s_{n-1}|_{B_{n-1}}$ works. It suffices to show that $(C_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{f} C_n)$ composes to the identity map $\mathbb{1}_{C_n}$. This follows from the splitting assumption, along with right-cancellability since ∂_n is surjective onto its image:

$$\partial_n = \partial_n s_{n-1} \partial_n \xRightarrow{\text{right-cancel } \partial_n} \mathbb{1}_{C_n} = \partial_n s_{n-1} \coloneqq \partial_n f.$$

■

Claim: The SES

$$0 \to B_n \overset{\iota_{BZ}}{\hookrightarrow} Z_n \overset{\pi}{\twoheadrightarrow} \frac{Z_n}{B_n} \to 0$$

admits a left-splitting $f : Z_n \to B_n$, and thus there is an isomorphism

$$Z_n \cong B_n \oplus H_n' \coloneqq B_n \oplus H_n(C) \coloneqq B_n \oplus \frac{Z_n}{B_n}.$$

Proof (?).
We proceed by again constructing the splitting $f : Z_n \to B_n$. The situation is summarized in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[ddr, two heads, "\partial_{n+1}"'] & C_n \arrow[l, bend right, dashed, "s_n"'] \arrow[r] & \cdots \\
& & Z_n \arrow[u, hook, "\iota_Z"'] & \\
& & B_n \arrow[u, hook, "\iota_{BZ}"'] &
\end{tikzcd}$$

Link to Diagram

So a natural candidate for the map f is the composition

$$Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} B_n,$$

so $f \coloneqq \partial_{n+1} s_n \iota_Z$. We can simplify this slightly by regarding $Z_n \leq C_n$ as a submodule to suppress ι_Z, and identify s_n with its restriction to Z_n to write $f \coloneqq \partial_{n+1} s_n$. The claim is then that $f \iota_{BZ} = \mathbb{1}_{B_n}$. Anticipating using the fact that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$, we post-compose with ∂_{n+1} and compute:

$$\begin{aligned}
f \iota_{BZ} \partial_{n+1} &= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_n) \\
&= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{s_n|_{B_n}} C_{n+1} \xrightarrow{\partial_{n+1}}) \\
&= \partial_{n+1} s_n \partial_{n+1} \\
&= \partial_{n+1},
\end{aligned}$$

where in the last step we've used the splitting hypothesis and the fact that it remains true when everything is restricted to the submodule $B_n \leq C_n$. Using surjectivity of ∂_{n+1} onto B_n, we can now conclude as before:

$$f \iota_{BZ} \partial_{n+1} = \partial_{n+1} \xRightarrow{\text{right-cancel } \partial_n} f \iota_{BZ} = \mathbb{1}_{B_n}.$$

■

Problem 1.0.3 (Weibel 1.4.3)
Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

Solution:
$\Longleftarrow$:

C **is split**: Suppose $\mathbb{1}_C$ is nullhomotopic, so that there exist maps

$$s_n : C_n \to C_{n+1} \qquad\qquad \mathbb{1}_{C_n} = \partial_{n+1} s_n + s_{n-1}\partial_n.$$

We then have the following situation:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathbb{1}_{C_{n+1}}"'] & C_n \arrow[r, "\partial_n"] \arrow[l, dashed, bend right, "s_n"'] \arrow[d, "\mathbb{1}_{C_n}"] \arrow[dl, "s_n"] & C_{n-1} \arrow[r] \arrow[l, dashed, bend right, "s_{n-1}"'] \arrow[d, "\mathbb{1}_{C_{n-1}}"] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here the nullhomotopy is outlined in red, and the map relevant to the splitting in green. Note that $s_n : C_n \to C_{n+1}$ is a candidate for a splitting, we just need to show that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$. We can proceed by post-composing the LHS with the identity $\mathbb{1}_C$, which allows us to substitute in the nullhomotopy:

$$\begin{aligned}
\partial_{n+1} &= \mathbb{1}_{C_n}\partial_{n+1} \\
&= (\partial_{n+1}s_n + s_{n-1}\partial_n)\,\partial_{n+1} \\
&= \partial_{n+1}s_n\partial_{n+1} + s_{n-1}\partial_n\partial_{n+1} \\
&= \partial_{n+1}s_n\partial_{n+1} + s_{n-1}0 && \text{since } \partial^2 = 0 \\
&= \partial_{n+1}s_n\partial_{n+1}.
\end{aligned}$$

C **is exact**: This follows from the fact that since $\mathbb{1}_C = \partial s + s\partial$ are equal as maps of chain complexes, the images $D_1 \coloneqq \mathbb{1}_C(C)$ and $D_2 \coloneqq (\partial s + s\partial)(C)$ are equal as chain complexes and have equal homology. Evidently $D_1 = C$, and on the other hand, each graded piece $(D_2)_n$ *only* consists of boundaries coming from various pieces of C, since $\partial s + s\partial$ necessarily lands in the images of the maps ∂_n. Thus $C_n(D_2) \subseteq B_n(D_2) = \varnothing$, i.e. there are no chains (or cycles) in D_2 which are *not* boundaries, and thus $H_n(D_2) \coloneqq Z_n(D_2)/B_n(D_2) = 0$ for all n. We can thus conclude that $C = D_2 \implies H(C) = H(D_2) = 0$, so C must be exact.

$\implies$: Suppose C is split. Then by exercise 1.4.2, we have R-module decompositions

$$\begin{aligned}
C_n &\cong Z_n \oplus B_{n-1} \\
Z_n &\cong B_n \oplus H_n
\end{aligned}$$

$$\implies C_n \cong B_n \oplus B_{n-1} \oplus H_n.$$

Supposing further that C is exact, we have $H_n = 0$, and thus $C_n \cong B_n \oplus B_{n-1}$. We first note that in this case, we can explicitly write the differential ∂_n. Letting $\mathbb{1}_n$ denote the identity on C_n, where by abuse of notation we also write this for its restriction to any submodules, we have:

$$\begin{tikzcd}
C_n \arrow[r, "\partial_n"] & C_{n-1} \\
\rotatebox{90}{=} & \rotatebox{90}{=} \\
B_n \arrow[r, dashed, "0"] \arrow[ddr, dashed, "0"] & B_{n-1} \\
\oplus & \oplus \\
B_{n-1} \arrow[uur, "\mathbb{1}_n"] \arrow[r, dashed, "0"] & B_{n-2}
\end{tikzcd}$$

Link to Diagram

We can thus write ∂_n as the matrix

$$\partial_n = \begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

Similarly using this decomposition, we can construct a map $s_n : C_n \to C_{n+1}$:

$$\begin{tikzcd}
C_n \arrow[r, "s_n"] & C_{n+1} \\
\rotatebox{90}{=} & \rotatebox{90}{=} \\
B_n \arrow[r, dashed, "0"] \arrow[ddr, "\mathbb{1}_n"] & B_{n+1} \\
\oplus & \oplus \\
B_{n-1} \arrow[uur, dashed, "0"] \arrow[r, dashed, "0"] & B_n
\end{tikzcd}$$

Link to Diagram

We can write this as the following matrix:

$$s_n = \begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix}.$$

We can now verify that s_n is a nullhomotopy from a direct computation:

$$\begin{aligned}
\partial_{n+1}s_n + s_{n-1}\partial_n &= \begin{bmatrix} 0 & \mathbb{1}_{n+1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbb{1}_{n-1} & 0 \end{bmatrix}\begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_{B_n} & 0 \\ 0 & \mathbb{1}_{B_{n-1}} \end{bmatrix} \\
&= \mathbb{1}_{C_n},
\end{aligned}$$

expressed as a map $B_n \oplus B_{n-1} \to B_n \oplus B_{n-1}$.

Problem 1.0.4 (Weibel 1.4.5)
Show that chain homotopy classes of maps form a quotient category K of $\mathrm{Ch}(R\text{-mod})$ and that the functors H_n factor through the quotient functor $\mathrm{Ch}(R\text{-mod}) \to K$ using the following steps:

1. Show that chain homotopy equivalence is an equivalence relation on $\left\{ f : C \to D \;\middle|\; f \text{ is a chain map} \right\}$. Define $\mathrm{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
2. Let $f \simeq g : C \to D$ be two chain homotopic maps. If $u : B \to C, v : D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $\mathrm{Ch}(R\text{-mod}) \to K$ is an additive functor.
4. Is K an abelian category? Explain.

Try at least two parts.

Solution (Part 1):

Claim 1: Chain homotopy equivalence defines an equivalence relation on $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$.

Proof (of claim 1).
We recall that for morphisms $f, g \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$, we have $f \simeq g \iff f - g \simeq 0 \iff \exists s : A \to B[1]$ such that $\partial^B s + s\partial^A = f - g$.

- **Reflexive:** We want to show that $f \simeq f$, i.e. $f - f = 0 \simeq 0$. Producing the map $s = 0$ works, since $\partial s + s\partial = \partial 0 + 0\partial = 0$.
- **Symmetric**: Suppose $f \simeq g$, so there exists an s such that $\partial s + s\partial = f - g$. Then taking $s' \coloneqq -s$ produces a chain homotopy $g - f \simeq 0$, since we can write

$$\begin{aligned}
\partial s' + s'\partial &= \partial(-s) + (-s)\partial \\
&= -\partial s - s\partial \\
&= -(\partial s + s\partial) \\
&= -(f - g) \\
&= g - f.
\end{aligned}$$

- **Transitive**: Suppose $f \simeq g$ and $g \simeq h$, we want to show $f \simeq h$. By assumption we have maps s, s' such that $\partial s + s\partial = f - g$ and $\partial s' + s'\partial = g - h$, so set $s'' \coloneqq s + s'$. We can then check that this is a chain homotopy from f to h:

$$\begin{aligned}
\partial s'' + s''\partial &= \partial(s + s') + (s + s')\partial \\
&= (\partial s + s\partial) + (\partial s' + s'\partial) \\
&= (f - g) + (g - h) \\
&= f - h,
\end{aligned}$$

where we've used that abelian categories are enriched over abelian groups, so we have a commutative and associative addition on homs.

■

Claim 2: $(\mathrm{Hom}_K(A, B), \oplus) \in \mathrm{Ab}$, where we define an addition on equivalence classes by

$$[f] \oplus [g] \coloneqq [f + g].$$

Proof (of claim 2).
We check the group axioms directly:

- **Closure under operation**: We can check that $[f] \oplus [g] \coloneqq [f+g] \coloneqq [g']$ makes sense, since $g' \coloneqq f + g \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)$, making $[g']$ a well-defined equivalence class of maps in $\mathrm{Hom}_K(A, B)$.

- **Two-sided Identities**: The equivalence class $[0] \coloneqq \left\{ f \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B) \;\middle|\; f \simeq 0 \right\}$ serves as an identity, where we take the zero map as a representative. This follows from the fact that
$$[0] \oplus [f] \coloneqq [0 + f] = [f] = [f] \oplus [0].$$

- **Associativity**: This again follows from the abelian group structure on the original hom:
$$[f] \oplus ([g] \oplus [h]) = [f + (g + h)] = [(f + g) + h] = ([f] \oplus [g]) \oplus [h].$$

- **Two-sided inverses**: Let $\ominus[f] \coloneqq [-f]$ be a candidate for the inverse with respect to $\oplus$. To see that this works, we have
$$[f] \oplus (\ominus[f]) \coloneqq [f] \oplus [-f] = [f - f] = [0],$$
and a similar calculation goes through to show it's also a left-sided inverse.

- **Well-definedness**: We need to show that if $[f] = [f']$ and $[g] = [g']$, the sums agree, so $[f] \oplus [g] = [f'] \oplus [g']$. We'll use the fact that $[f] = [f'] \iff f - f' \simeq 0$ (and similarly for g), so we can compute:
$$\begin{aligned}
([f] \oplus [g]) \ominus \big([f'] \oplus [g']\big) &\coloneqq [f + g] \ominus [f' - g'] \\
&\coloneqq [(f + g) - (f' - g')] \\
&= [(f - f') + (g - g')] \\
&= [f - f'] \oplus [g - g'] \\
&= [0] \oplus [0] && \text{using } f \simeq f',\ g \simeq g' \\
&\coloneqq [0].
\end{aligned}$$

■

Solution (Part 2):
We have the following situation, where the double-arrows denote chain homotopies:

$$
\begin{tikzcd}
C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & D \arrow[d, "v"] \\
B \arrow[u, "u"] \arrow[r, bend left, "vfu"] \arrow[r, bend right, "vgu"'] \arrow[r, phantom, "\exists\Downarrow"] & E
\end{tikzcd}
$$

Link to Diagram

We want to show that $vgu \simeq vfu$, or equivalently that $vgu - vfu \simeq 0$. This is immediate:

$$vgu - vfu = v(gu - fu) = v(f - g)u \simeq v0u = 0.$$

Alternatively, as an explicit computation, if we assume $f \simeq g$ then there is a nullhomotopy s for $f - g$, in which case the map $s' \coloneqq vsu$ works as a nullhomotopy for $vfu - vgu$:

$$
\begin{aligned}
vfu - vgu = v(f-g)u &= v(\partial s + s\partial)u \\
&= v\partial su + vs\partial u \\
&= \partial vsu + vsu\partial && \text{since } u, v \text{ are chain maps} \\
&\coloneqq \partial s' + s'\partial.
\end{aligned}
$$

We can now define a composition map on $\mathcal{K}$:

$$
\begin{aligned}
\circ : \operatorname{Hom}_K(A_1, A_2) \times \operatorname{Hom}_K(A_2, A_3) &\to \operatorname{Hom}_K(A_1, A_3) \\
([f], [g]) &\mapsto [f \circ g].
\end{aligned}
$$

The previous argument then precisely says:

1. If $u : B \to C$ and $f, gC \to D$ with $[f] = [g]$, the composition $[u] \circ [f] = [u] \circ [g] : B \to D$ is well-defined, and
2. If $v : D \to E$ and $f, g : C \to D$ with $[f] = [g]$, the composition $[f] \circ [v] = [g] \circ [v]$ is well-defined.

So this composition on $\mathcal{K}$ is well-defined. Moreover,

- There is an identity morphism $[\mathbb{1}_A] \in \operatorname{Hom}_K(A, A)$ coming from the class of $\mathbb{1}_C \in \operatorname{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, A)$,
- It is associative, since $[h] \circ [gf] \coloneqq [hgf] \coloneqq [hg] \circ [f]$, and
- It is unital, in the sense that $[\mathbb{1}_B] \circ [f] \coloneqq [\mathbb{1}_A \circ f] = [f] = [f \circ \mathbb{1}_A] \coloneqq [f] \circ [\mathbb{1}_B]$ for any $f : A \to B$.

So this data satisfies all of the axioms of a category (Weibel A.1.1).

Solution (Part 3):

To see that $f_0 + f_1 \simeq g_0 + g_1$, we have

$$\begin{aligned}(f_0 + f_1) - (g_0 + g_1) &= (f_0 - g_0) + (f_1 - g_1) \\ &\simeq 0 + 0 = 0.\end{aligned}$$

To be explicit, if s_i are chain homotopies for f_i, g_i, we can take $s \coloneqq s_0 + s_1$:

$$\begin{aligned}(f_0 + f_1) - (g_0 + g_1) &= (f_0 - g_0) + (f_1 - g_1) \\ &= (\partial s_0 + s_0 \partial) + (\partial s_1 + s_1 \partial) \\ &= (\partial s_0 + \partial s_1) + (s_0 \partial + s_1 \partial) \\ &= \partial (s_0 + s_1) + (s_0 + s_1)\partial \\ &\coloneqq \partial s + s \partial.\end{aligned}$$

That K forms an additive category is a consequence of the following facts:

- K has products, since $\mathcal{A}$ had products and $\mathrm{Ob}(K) \coloneqq \mathrm{Ob}(\mathcal{A})$.
- K has a zero object, for the same reason.
- Composition distributes over addition, i.e.

$$\begin{aligned}&[f]([g] \oplus [h]) \\ &\quad \coloneqq [f][g+h] \\ &\quad \coloneqq [f(g+h)] \\ &\quad \coloneqq [fg + fh] \\ &\quad = [fg] \oplus [fh].\end{aligned}$$

Moreover, the quotient functor $\mathrm{Ch}(R\text{-mod}) \to K$ is an additive functor, since the maps $Q : \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)/\sim$ are morphisms of abelian groups, using the fact that Q commutes with both additions:

$$Q(f+g) = [f+g] \coloneqq [f] \oplus [g] = Q(f) + Q(g).$$

Alternatively, we can note that the set of all $f \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A}}(A, B)$ which are nullhomotopic form a subgroup H, and since everything is abelian we can form the quotient $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B)/H$ observe that this is isomorphic as a group to $\mathrm{Hom}_K(A, B)$.

Solution (Part 4):
We can first note that K is an additive category, since we still have a zero object and products inherited from $\mathrm{Ch}(\mathcal{A})$.

Note: I don't see a great way to prove that any particular category is abelian or not! Checking the axioms listed in Appendix A.4 seems quite difficult.

Problem 1.0.5 (Weibel 1.5.1)

Let $\mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C)$, so

$$\mathrm{cone}(C)_n = C_{n-1} \oplus C_n.$$

Show that $\mathrm{cone}(C)$ is split exact, with splitting map given by $s(b,c) \coloneqq (-c, 0)$.

Solution:
Fixing notation, let

- ∂_n be the nth differential on C,
- $\mathbb{1}_C$ be the identity chain map on C,
- $\mathbb{1}_n : C_n \to C_n$ be the nth graded component of $\mathbb{1}_C$,
- $\widehat{C} \coloneqq \mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C)$,
- $\widehat{\partial}_n$ be the nth differential on $\widehat{C}$, and
- $\widehat{\mathbb{1}}$ be the identity on $\widehat{C}$,
- $\widehat{\mathbb{1}}_n : \widehat{C}_n \to \widehat{C}_n$ be the nth component of $\widehat{\mathbb{1}}$.

From exercise 1.4.2, it suffices to show that $\widehat{\mathbb{1}}$ is nullhomotopic. Since we have a direct sum decomposition $\mathrm{cone}(C)_n \coloneqq C_{n-1} \oplus C_n$, we can write $\widehat{\mathbb{1}}$ as a block matrix

$$\begin{bmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & \mathbb{1}_n \end{bmatrix}.$$

We can similarly write down the differential on $\mathrm{cone}(C)$ in block form:

$$\begin{tikzcd}
\mathrm{cone}(C)_{n+1} \arrow[r, "\widehat{\partial}_{n+1}"] & \mathrm{cone}(C)_n
\end{tikzcd}$$

$$\begin{tikzcd}
C_n \arrow[r, "-\partial_n"] \arrow[dr, "-\mathbb{1}_n" near start] & C_{n-1} \\
\oplus & \oplus \\
C_{n+1} \arrow[r, "\partial_{n+1}"'] \arrow[ur, dashed, "0" near start] & C_n
\end{tikzcd}$$

Link to Diagram

This yields

$$\widehat{\partial}_n \coloneqq \begin{bmatrix} -\partial_{n-1} & 0 \\ -\mathbb{1}_{n-1} & \partial_n \end{bmatrix}.$$

Similarly, the map $s_n(b,c) = (-c,0)$ can be written as

$$s_n = \begin{bmatrix} 0 & -\mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

We can thus proceed by a direct computation:

$$\begin{aligned}
\partial_{n+1}s_n + s_{n-1}\partial_n &= \begin{bmatrix} -\partial_n & 0 \\ -\mathbb{1}_n & \partial_{n+1} \end{bmatrix}\begin{bmatrix} 0 & -\mathbb{1}_n \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathbb{1}_{n-1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\partial_{n-1} & 0 \\ -\mathbb{1}_{n-1} & \partial_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & \partial_n \mathbb{1}_n \\ 0 & \mathbb{1}_n \mathbb{1}_n \end{bmatrix} + \begin{bmatrix} \mathbb{1}_{n-1}\mathbb{1}_{n-1} & -\mathbb{1}_{n-1}\partial_n \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_{n-1}\mathbb{1}_{n-1} & \partial_n \mathbb{1}_n - \mathbb{1}_{n-1}\partial_n \\ 0 & \mathbb{1}_n \mathbb{1}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_{n-1}\mathbb{1}_{n-1} & 0 \\ 0 & \mathbb{1}_n \mathbb{1}_n \end{bmatrix} \\
&= \widehat{\mathbb{1}}_n.
\end{aligned}$$

Problem 1.0.6 (Weibel 1.5.2)
Let $f : C \to D \in \mathrm{Mor}(\mathrm{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(-s, f) : \mathrm{cone}(C) \to D.$$

Solution:
Remark 1.0.1: As a notational convention for this problem, I'll take vectors v to be column vectors, and v^t will denote a row vector. I've also written $f \simeq 0$ to denote that f is nullhomotopic.

$\implies$: Suppose that $f \simeq 0$, so there are maps s_n such that the following diagrams commute for every n:

$$\begin{tikzcd}
 & \cdots \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[dl, dashed, "s_n"'] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_{n+1}^D"] & D_n \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

Write $\widehat{C} \coloneqq \mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C) \coloneqq C[1] \oplus C$, we then want to construct a lift $\widehat{f}$ of f such that the following diagram commutes:

$$\begin{tikzcd} \widehat{C} \arrow[dr, dashed, "\widehat{f}"] & \\ C \arrow[u, "\iota_{\widehat{C}}"] \arrow[r, "f"'] & D \end{tikzcd}$$

Link to Diagram

where $\iota_{\widehat{C}}$ is the following inclusion of C into its cone:

$$\begin{tikzcd} C_n \arrow[r, "\iota_{\widehat{C}}"] & \widehat{C}_n \\ C_n \arrow[r, dashed, "0"] \arrow[dr, "\mathbb{1}_{C_n}"] & C_{n-1} \\ & \oplus \\ & C_n \end{tikzcd}$$

Link to Diagram

We define the map $\widehat{f}$ in the following way:

$$\begin{tikzcd} C_n \arrow[r, "\iota_{\widehat{C}}"] & \widehat{C}_n \arrow[r, "\widehat{f}_n"] & D_n \\ C_n \arrow[r, dashed, "0"] \arrow[dr, "\mathbb{1}_{C_n}"] & C_{n-1} \arrow[ddr, "-s_{n-1}"] & \\ & \oplus & \\ & C_n \arrow[r, "f_n"] & D_n \end{tikzcd}$$

Link to Diagram

That this is a lift follows from computing the composition, which can be done in block matrices:

$$\widehat{f}_n \circ \iota_{\widehat{C},n} = \begin{bmatrix} -s_{n-1} \\ f_n \end{bmatrix}^t \begin{bmatrix} 0 \\ \mathbb{1}_{C_n} \end{bmatrix} = [f_n \mathbb{1}_{C_n}] = f_n,$$

where the first matrix acts as a row vector. It only remains to check that $\widehat{f}$ defines a chain map, which follows from the following computation:

$$
\begin{aligned}
\partial_n^D \widehat{f}_n - \widehat{f}_{n-1}\widehat{\partial}_n &= [\partial_n^D] \begin{bmatrix} -s_{n-1} \\ f_n \end{bmatrix}^t - \begin{bmatrix} -s_{n-2} \\ f_{n-1} \end{bmatrix} \begin{bmatrix} -\partial_n^C & 0 \\ -\mathbb{1}_{C_n} & \partial_{n+1}^C \end{bmatrix} \\
&= \begin{bmatrix} \partial_n^D(-s_{n-1}) - s_{n-2}\partial_n^C + f_{n-1}\mathbb{1}_{C_n} \\ \partial_n^D f_n - f_{n-1}\partial_n^C \end{bmatrix}^t \\
&= \begin{bmatrix} f_{n-1} - (\partial_n^D s_{n-1} + s_{n-2}\partial_n^C) \\ 0 \end{bmatrix}^t && \text{since } f \text{ a chain map } \implies \partial f = f\partial \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{since } f \simeq 0 \implies \partial s + s\partial = f \\
&= 0.
\end{aligned}
$$

$\Longleftarrow$: Suppose we have a lift $\widehat{f} : \widehat{C} \to D$; then define the following maps as the proposed splittings:

$$
\begin{aligned}
s_n : C_{n-1} &\to D_n \\
c &\mapsto \widehat{f}_n(-c, 0).
\end{aligned}
$$

There are two relevant facts to observe:

1. We have $f = \tilde{f}\iota_{\widehat{C}}$ where $\iota_{\widehat{C}}(c) \coloneqq (0, c) \in \widehat{C}$ is inclusion into the second direct summand, and in particular

$$
f_n(c) = \widehat{f}_n \iota_{\widehat{C},n}(c) = \widehat{f}_n(0, c).
$$

$$
\begin{tikzcd}
(0,c) \in C_{n-1} \oplus C_n \arrow[rrr, dashed, mapsto, bend left] & & & \\
& \widehat{C_n} \arrow[dr, "\widehat{f}_n"] & & \\
c \arrow[uu, dashed, mapsto] \arrow[rrr, dashed, mapsto, bend right] & C_n \arrow[u, "\iota_{\widehat{C},n}"] \arrow[r, "f_n"'] & D_n & \widehat{f}_n(0,c) = f_n(c)
\end{tikzcd}
$$

Link to Diagram

2. Since $\widehat{f}$ is a chain map, we have for each n

$$\partial_n^D \widehat{f}_n(x,y) = \widehat{f}_{n+1}\widehat{\partial}_n(x,y) \qquad \text{as maps } \widehat{C}_n \to D_n.$$

We now proceed to compute at the level of elements that s defines a splitting:

$$\begin{aligned}
\partial_{n+1}^D s_n(c) + s_{n-1}\partial_n^D(c) &\coloneqq \partial_{n+1}^D \widehat{f}_{n+1}(c,0) + \widehat{f}_n \partial_n^C(c,0) \\
&= \partial_{n+1}^D \widehat{f}_{n+1}(c,0) + \widehat{f}_n(\partial_n^C(c), 0) \\
&= \widehat{f}_n \widehat{\partial}_{n+1}(c,0) + \widehat{f}_n(\partial_n^C(c),0) && \text{by (2)} \\
&= \widehat{f}_n\left(\begin{bmatrix} -\partial_n^C & 0 \\ -\mathbb{1}_n^C & \partial_{n+1}^C \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix} \right) + \widehat{f}_n(\partial_n^C(c),0) \\
&= \widehat{f}_n\left(\begin{bmatrix} -\partial_n^C(c) \\ c \end{bmatrix} \right) + \widehat{f}_n(\partial_n^C(c),0) \\
&= \widehat{f}_n(-\partial_n^C(c), c) + \widehat{f}_n(\partial_n^C(c),0) \\
&= \widehat{f}_n(-\partial_n^C(c) + \partial_n^C(c), c) && \text{since } f_n \text{ is an } R\text{-mod morphism} \\
&= \widehat{f}_n(0,c) \\
&= f_n(c) && \text{by (1).}
\end{aligned}$$

Problem 1.0.7 (Extra)

a. Show that free implies projective.

b. Show that $\mathrm{Hom}_R(M, \cdot)$ is left-exact.

c. Show that P is projective if and only if $\mathrm{Hom}_R(P, \cdot)$ is exact.

Solution:

Proof (of (a)).
Suppose M is free, so we have a set S and an injection $S \hookrightarrow M$ such that every map in $\hom_{\mathsf{Set}}(S, Y)$ for $Y \in R\text{-mod}$ lifts to a unique map in $\hom_{R\text{-mod}}(M, Y)$. Suppose further that we have the following situation; we seek to construct a lift $\tilde{h} : M \to A$:

$$
\begin{tikzcd}
& s_i \arrow[dddddd, bend left=60, dotted, mapsto] \\
& \rotatebox[origin=c]{-90}{$\in$} \\
& S \arrow[d, hook, "\iota"] \arrow[ddl, bend right, "h"'] \\
& M \arrow[d, "g"] \arrow[dl, dashed, bend right, "\exists!\tilde{h}"'] \\
A \arrow[r, two heads, "f"'] & B \arrow[r] & 0 \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
\alpha_i & \beta_i \arrow[l, dotted, mapsto]
\end{tikzcd}
$$

Link to Diagram

This lift exists by first considering $s_i \in S$ and noting that since $\beta_i \coloneqq \iota g(s_i) \in B$ and f is surjective, there exist *some* elements α_i in A such that $f(\alpha_i) = \beta_i$ for each i. So define the map

$$
\begin{aligned}
h : S &\to A \\
s_i &\mapsto \alpha_i.
\end{aligned}
$$

By the universal property of free modules, this lifts to a map $\tilde{h} : M \to A$, so M is projective.

∎

Proof (of (b)).
Suppose we have a SES

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

The claim is that $\mathrm{Hom}_R(M, \cdot)$ yields an exact sequence

$$0 \to \mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C) \to \cdots,$$

where $f_*(\alpha) = f \circ \alpha$ for $\alpha : M \to A$ and similarly $g_*(\beta) = g \circ \beta$ for $\beta : M \to B$. To show that this is exact, it suffices to show three things:

1. $\ker f_* = 0$,
2. $\operatorname{im} f_* \subseteq \ker g_*$, and
3. $\ker g_* \subseteq \operatorname{im} f_*$.

Proceeding with each part:

1. By definition, if $\beta : M \to B$ satisfies $\beta \in \ker f_*$, and thus $f\beta = 0$. Since the original sequence was exact, f is injective, thus a monomorphism, thus satisfies the left-cancellation property. So we can immediately conclude that $\beta = 0$.
2. Let $\beta : M \to B$ be in $\operatorname{im} f_*$, so there exists some $\alpha : M \to A$ with $\beta = f\alpha$. We want to show that $\beta \in \ker g_*$, so we can apply g_* to obtain $g_*(\beta) = g_*(f\alpha) \coloneqq gf\alpha$. But by exactness of the first sequence, $gf = 0$, so $gfa = 0$.
3. Let $\beta' \in \ker g_*$, so $g\beta' = 0$. In order to show $\beta' \in \operatorname{im} f_*$, we want to construct some $\alpha : M \to A$ such that $\beta' = f_*(\alpha) \coloneqq f\alpha$. Considering $m \in M$, we know $g\beta'(m) = 0$ and thus $\beta'(m) \in \ker g = \operatorname{im} f$ by exactness of the first sequence. So there exists some $a_m \in A$ with $f(a_m) = \beta'(m)$, and we can define a map

$$\begin{aligned} \alpha : M &\to A \\ m &\mapsto a_m. \end{aligned}$$

   By construction, we then have $f\alpha(m) = f(a_m) \coloneqq \beta'(m)$ for every $m \in M$, so $f\alpha = \beta'$.

■

Proof (of (c)).
Assume the same setup as (b).
$\implies$: Suppose P is projective, so it satisfies the following universal property:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists ! \tilde{\gamma}"'] \arrow[d, "\gamma"] & \\
B \arrow[r, two heads, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Using the results of (b), it suffices to check exactness at $\operatorname{Hom}_R(P, C)$ in the following sequence:

$$0 \to \operatorname{Hom}_R(P, A) \xrightarrow{f_*} \operatorname{Hom}_R(P, B) \xrightarrow{g_*} \operatorname{Hom}_R(P, C) \to 0,$$

or equivalently that g_* is surjective. Using that $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective, the universal property above means that every $\gamma \in \operatorname{Hom}_R(P, C)$ lifts to a map $\tilde{\gamma} \in \operatorname{Hom}_R(P, B)$ where $g\tilde{\gamma} = \gamma$. Since $g_*(\tilde{\gamma}) \coloneqq g\tilde{\gamma}$, this precisely means that $\gamma \in \operatorname{im} g_*$.

$\impliedby$: Reversing the above argument, if $\operatorname{Hom}_R(P, \cdot)$ is exact, then every $P \xrightarrow{\gamma} C$ has a preimage under g_*, which is precisely a lift $P \xrightarrow{\tilde{\gamma}} B$. So P satisfies the universal property of projective modules.

■