# Title

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## Contents



## 1 Friday January 17th

For a $R \subset T$ a subring of a ring, the set of intermediate rings is a large/interesting class of rings. Recall: uncountably many rings between $\mathbb{Z}$ and $\mathbb{Q}$! Taking $R$ a PID and $T$ its fraction field, a similar result will hold.

Define $I \trianglelefteq R$ as the kernel of a ring morphism. This implies that $I \subset (R, +)$ with the absorption property $RI \subset I$. Conversely, any $I$ satisfying these two properties is the kernel of a ring morphism: namely $R \longrightarrow R/I$. This makes sense as a group morphism.

**Exercise** Define $xy + I = (x + I)(y + I)$, need to check well-definedness. Write out

$$(x + i_1)(y + i_2) = \cdots$$

Need to check that

$$i_1 y + i_2 x + i_1 i_2 \in I,$$

but the absorption property does precisely this.

Note that if we were in a non-commutative setting, this would define a left ideal. These don't have to coincide with right ideals – there are rings where the former satisfy properties that the latter does not.

*Example:* The subrings of $R = \mathbb{Z}$ are of the form $n\mathbb{Z}$ for $n \geq 0$, with the usual quotient.

**Definition 1.0.1** (Proper Ideals).
An ideal $I \trianglelefteq R$ is *proper* iff $I \subsetneq R$.

**Exercise** An ideal $I$ is not proper iff $I$ contains a unit.

**Exercise** $R$ is a field iff the only ideals are $0, R$.

**Definition 1.0.2** (Lattice Structure of Ideals).
Let $\mathcal{I}(R)$ be the set of all ideals in $R$. What structure does it have? It is partially ordered under

inclusion. It is a complete lattice, i.e. every element has an infimum (GLB) and a supremum (LUB).

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Namely, for a family of ideals $\theset{I_j}$, the **infimum** is the intersection and **s
$$
\generators{y} = \theset{ \sum^n r_i y_i \suchthat n\in \NN_{> 0},~ r_i \in R,~ y_i\in y}
.$$
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**Exercise** For $I_1, I_2 \trianglelefteq R$, it is the case that $I_1 + I_2 \coloneqq \{i_1 + i_2\} = \langle I_1, I_2 \rangle$.

**Theorem 1.1** ***(Lattice Isomorphism Theorem for Rings).***
Let $I \trianglelefteq R$ and $\phi : R \longrightarrow R/I$, and define $\ell(I) = \{I \subset J \trianglelefteq R\}$. Then we can define maps

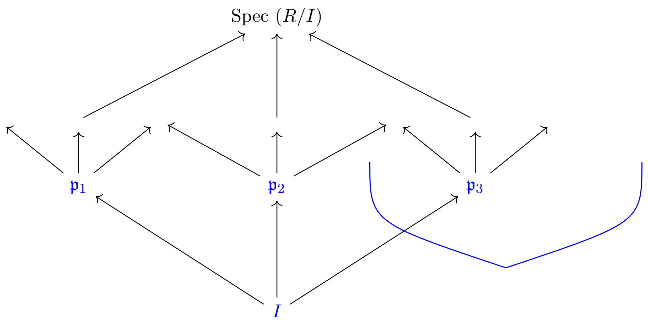
$$
\begin{aligned}
\Phi : \ell(R) &\longrightarrow \ell(R/I) \\
J &\mapsto \frac{I+J}{J},
\end{aligned}
$$

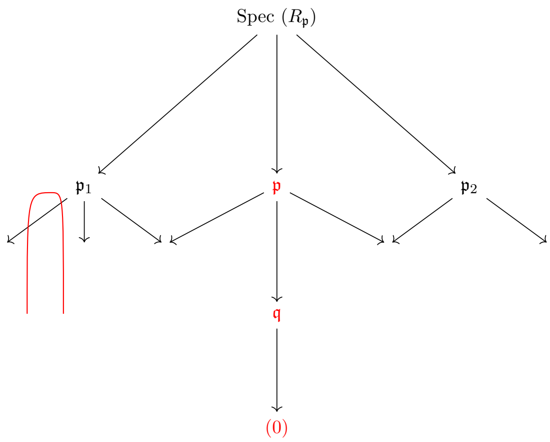
and

$$
\begin{aligned}
\Psi : \ell(R/I) &\longrightarrow \ell(R) \\
J \trianglelefteq R/I &\mapsto \phi^{-1}(J).
\end{aligned}
$$

We can check that $\Psi \circ \Phi)(J) = I + J$, and $\Phi \circ \Psi(J) = J$ $(= J/I?)$ So $\Psi$ has a left inverse and is thus injective. Its image is the collection of ideals that contain $J$, and $\Psi : \ell(R/I) \longrightarrow \ell_I(R)$ **is a bijection** and is in fact a lattice isomorphism with $\ell_I(R) \subset \ell(R)$.

Note that this gives us everything above (?) an ideal in the ideal lattice; the dual notion will come from localization.




**Remarks** The ideal lattice $\ell(R)$ is

- A complete lattice under subset inclusion,
- A commutative monoid under addition
- A commutative monoid under *multiplication*, which we'll define.

**Definition 1.1.1** (Product of Ideals)**.**
For $I, J \trianglelefteq R$, we define

$$IJ = \left\langle ij \;\middle|\; i \in I,\ j \in J \right\rangle .$$

Note that we have to take the ideal generated by products here.

For $\langle x \rangle = (x)$ a principal ideal and $\langle y \rangle$ principal, we do have $(x)(y) = (xy)$. Note that $IJ \subset I \bigcap J$, whereas the sum was larger than $I, J$.

**Exercise** Note that $(\ell(R), \cdot)$ has an absorbing element, namely $(0)I = (0)$. For $(M, +)$ a commutative monoid and $M \hookrightarrow G$ a group, then multiplication by $x$ is injective and so for all $y \in M$, $xz = yz \implies x = y$, so $M$ is cancellative.

Question: what if we consider $\mathcal{I}^\bullet(R)$ the set of nonzero ideals of $R$. Does this help? We will see next time.