Moduli Spaces

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1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, ... Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

1.1 Representability

Last time: Fix an S-scheme, i.e. a scheme over S.

Then there is a map

$$\begin{aligned}\mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{\mathrm{Sch}/S}(T, x).\end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned}h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form}\end{aligned}$$

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd} .$$

Theorem 1.1 ***(Yoneda).***

$$\hom_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 1.2.

$$\hom_{\mathrm{Sch}/S}(x,y) \cong \hom_{\mathrm{Fun}}(h_x,h_y).$$

Definition 1.2.1 (Moduli Functor)**.**
A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 1.2.2 (Moduli Space)**.**
A **moduli space** for that "something" appearing above is an $M \in \mathrm{Obj}(\mathrm{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \mathrm{Spec}\ (k)$.

h_m is the functor of points over M.

Remark (1) $h_m(\mathrm{Spec}\ (k)) = M(\mathrm{Spec}\ (k)) \cong$ "families over Spec k" $= F(\mathrm{Spec}\ k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M, and $\mathrm{id}_M \in \mathrm{Mor}_{\mathrm{Sch}/S}(M,M) = \xi_{Univ}$ is the universal family.

Every family is uniquely the pullback of ξ_{Univ}. This makes it much like a classifying space.

For $T \in \mathrm{Sch}/S$,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\mathrm{Univ}}). \end{aligned}$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\begin{array}{ccc} \xi_M & & \xi_{M'} \\ M & \xrightarrow{f} & M' \\ & & \\ M' & \xrightarrow{g} & M \\ \xi_{M'} & & \xi_M \end{array}$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.
A length 2 subscheme of $\mathbb{A}^1_k$ (??) then

$$F(S) = \left\{ V(x^2 + bx + c) \right\} \subset \mathbb{A}'_5$$

where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

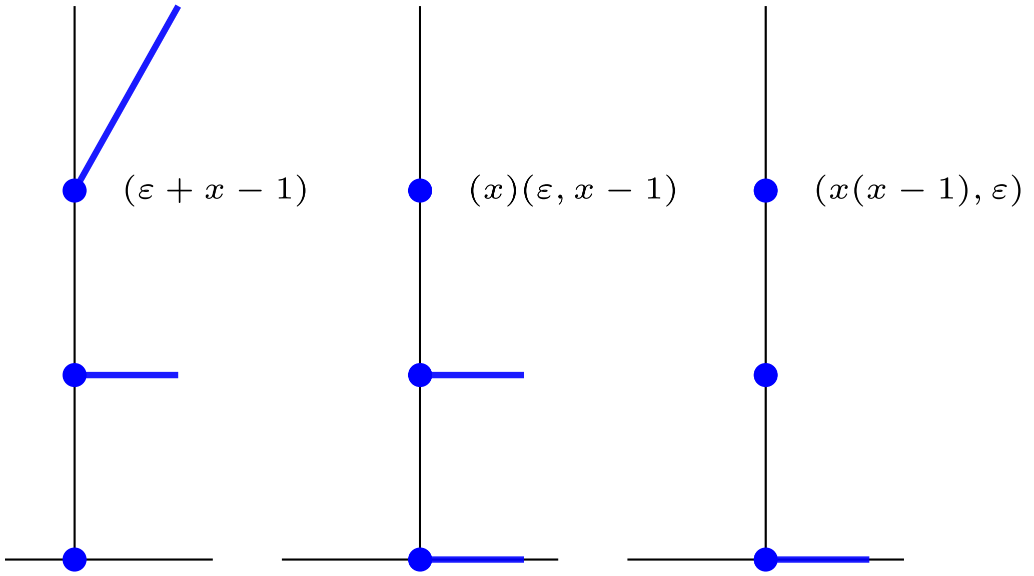
Then F is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider Spec $k[\varepsilon]$.



F	✓	x	x
F'	✓	x	x
F''	✓	✓	✓

$$\begin{tikzcd}
\text{Spec } k \arrow[r, hook, "i"] & \text{Spec } k[\varepsilon] & & = F'(\text{Spec } k) \arrow[dr] & \\
F(\text{Spec } k[\varepsilon]) \arrow[r, "F(i)"] & F(\text{Spec } k) \arrow[urr] & & & = F''(\text{Spec } k) \\
T_pF^{','} \arrow[u, hook, "\subset"] & P = V(x(x-1)) \arrow[u, "\in"] & & &
\end{tikzcd}$$

We think of $T_pF^{',''}$ as the tangent space at p.

If F is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\text{Spec } k[\varepsilon]) \arrow[r] & M(\text{Spec } k) \\
T_pM \arrow[u, hook, "\subset"] \arrow[r] & p \arrow[u, hook, "\subset"']
\end{tikzcd}$$

$$\begin{tikzcd}
& \text{Spec } k \arrow[dl, hook'] \arrow[dr, "?"] & \\
\text{Spec } k[\varepsilon] \arrow[rr] & & \text{Spec } \mathcal{O}_{M,p} \subset M
\end{tikzcd}$$

$$\begin{tikzcd}
& k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
\mathfrak{m}_p \arrow[u, hook] & (\varepsilon) \arrow[u, hook] \\
\mathfrak{m}_p^2 \arrow[u, hook] & 0 \arrow[u, hook]
\end{tikzcd}$$

Moreover, $T_pM = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda\varepsilon
\end{aligned}$$

$$\lambda^* : \text{Spec } (k[\varepsilon]) \longrightarrow \text{Spec } (k[\varepsilon])$$

$$\begin{aligned}
\lambda : M(\text{Spec } (k[\varepsilon])) &\longrightarrow M(\text{Spec } (k[\varepsilon])) \\
\cup \qquad &\qquad\quad \cup \\
T_pM &\longrightarrow T_pM.
\end{aligned}$$

Conclusion: If F is representable, for each $p \in F(\text{Spec } k)$ there exists a unique point of T_pF that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$\begin{tikzcd} F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}/T) &\longrightarrow \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

Proposition 1.3.
$\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(S) = \mathcal{O}_s^{n+1} &\longrightarrow L \longrightarrow 0/\sim. \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $F(f)$ is given by pullbacks.

Remark $\mathbb{P}^n_S$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ T \mapsto F_S(T) &= \left\{ \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim . \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of Proposition).
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

Claim: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{tikzcd} F_i \arrow[r] & F \\ F_{ij} \arrow[u] \arrow[r] & F_j \arrow[u] \end{tikzcd}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example .
For $n = 1$, we can glue along open subschemes

$$\begin{tikzcd} & F_0 \\ F_{01} \arrow[ur] \arrow[dr] & \\ & F_1 \end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd}
& & & F_0 \arrow[ddr] \\
& & F_{01} \arrow[ur] \arrow[dr] \\
F_{012} \arrow[urr] \arrow[r] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\
& & F_{12} \arrow[ur] \arrow[dr] \\
& & & F_2 \arrow[uur]
\end{tikzcd}$$

This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).
Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S.
RHS to LHS comes from gluing.

■

Proof (of Claim).

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.
We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec $k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.
What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I}{}_{F} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

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2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space $\mathbb{P}^n_{\mathbb{Z}}$ constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}^n_{\mathbb{Z}}} : \mathbb{P}^n_{\mathbb{Z}}\mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ s \mapsto \mathbb{P}^n_{\mathbb{Z}}(s) &= \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for L a line bundle, up to isomorphisms of diagrams:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, Rightarrow, no head] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}^n_{\mathbb{Z}}$ where

$$F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \;\middle|\; s_i \text{ is invertible} \right\}$$

are representable and can be glued together, and projective space represents this functor.

Remark Because projective space represents this functor, there is a universal object:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) & \end{tikzcd}$$

and other functors are pullbacks of the universal one. (Moduli Space)

Exercise Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over Spec $\mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

Definition 2.0.1 (Equalizer)**.**
For a category C, we say a diagram $X \longrightarrow Y \rightrightarrows Z$ is an *equalizer* iff it is universal with respect to the property:

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, shift left] \arrow[r, shift right] & Z \\ & S \arrow[ul, dashed, "\exists!"] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Note that X is the universal object here.

Example 2.1.
For sets, $X = \left\{ y \;\middle|\; f(y) = g(y) \right\}$ for $Y \xrightarrow{f,g} Z$.

Definition 2.0.2 (Coequalizer)**.**
A **coequalizer** is the dual notion,

$$\begin{tikzcd} & S & \\ Z \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[u] \arrow[r] & X \arrow[ul, dashed, "\exists!"'] \end{tikzcd}$$

Example 2.2.
Take $C = \mathrm{Sch}/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$\begin{tikzcd} X_\alpha \arrow[r] & X \\ X_{\alpha\beta} \arrow[u] \arrow[r] & X_\beta \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} X_\beta \arrow[r] & X \\ X_{\beta\alpha} \arrow[u] \arrow[r] & X_\alpha \arrow[u] \end{tikzcd}$$

These are canonically isomorphic.

In Sch/S, we have

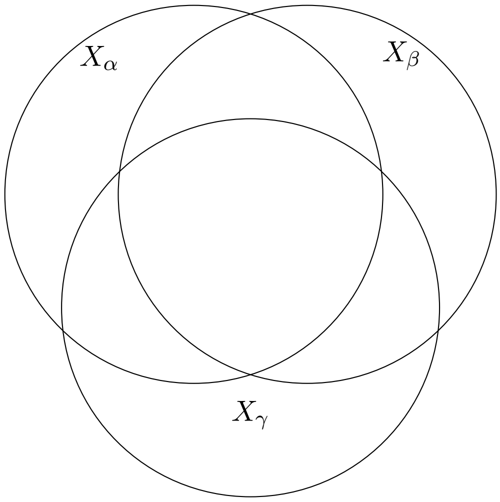
$$\coprod_{\alpha\beta} X_{\alpha\beta} \overset{f_{\alpha\beta}}{\underset{g_{\alpha\beta}}{\rightrightarrows}} \coprod_\alpha X_\alpha \longrightarrow X$$

where

$$\begin{aligned} f_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\alpha \\ g_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\beta; \end{aligned}$$

this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the **cocycle condition**:

1. $$\varphi_{\alpha\beta}^{-1}\left(X_{\beta\alpha} \bigcap X_{\beta\gamma}\right) = X_{\alpha\beta} \bigcap X_{\alpha\gamma},$$

   noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

$$\begin{tikzcd}
X_{\alpha\beta} \bigcap X_{\alpha\gamma} \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\alpha\beta}"'] & & X_{\gamma\alpha} \bigcap X_{\gamma\beta} \\
& X_{\beta\alpha} \bigcap X_{\beta\gamma} \arrow[ur, "\varphi_{\beta\gamma}"'] &
\end{tikzcd}$$

Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \longrightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 2.0.3 (Zariski Sheaf)**.**
A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_α, the following is an equalizer:

$$F(T) \longrightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 2.3.
Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus

if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_{\alpha} T_\alpha \longrightarrow T,$$

since giving a local map to X that agrees on intersections if enough to specify a map from $T \longrightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 2.0.4 (Subfunctors, Open/Closed Functors)**.**
Suppose we have a morphism $F' \longrightarrow F$ in the category $\mathrm{Fun}(\mathrm{Sch}/S, \mathrm{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all T/S.
- ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over T, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U.

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F', which maps to F (noting a section is the same as a map to the functor of points), and since $T \longrightarrow F$ and $F' \longrightarrow F$, we can form the fiber product $F' \times_F T$:

$$\begin{tikzcd}
F' \arrow[r] & F \\
F' \times_F T \arrow[u] \arrow[r, "g"] & T \arrow[u, "\xi"']
\end{tikzcd}$$

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \longrightarrow F$ is open/closed/locally closed iff $F' \times_F T$ is representable and g is open/closed/locally closed.

I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \longrightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

Definition 2.0.5 (Open Covers)**.**
A collection of open subfunctors $F_\alpha \subset F$ is an **open cover** iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \longrightarrow F$, the T_α in the following diagram are an open cover of T:

$$\begin{tikzcd} F_\alpha \arrow[r] & F \\ T_\alpha \arrow[u] \arrow[r] & T \arrow[u, "\xi"'] \end{tikzcd}$$

Example 2.4.
Given

$$F(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}$$

and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \longrightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

Proposition 2.1.
A Zariski sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ with a representable open cover is representable.

Proof .
Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \longrightarrow F_\alpha$ have the correct gluing maps.

This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \longrightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x; we'll show that x represents F.
First produce a map $x \longrightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

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Lemma 2.2.
If $E \longrightarrow F$ is a map of functors and E, F are zariski sheaves, where there are open covers $E_\alpha \longrightarrow E, F_\alpha \longrightarrow F$ with commutative diagrams

$$\begin{tikzcd} E \arrow[r] & F \\ E_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$\begin{tikzcd} X \arrow[r] & F \\ X_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

Example 2.5.
For S and E a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} / \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E.

Then the restriction $F_i \longrightarrow \mathbb{P}E$ were $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i, $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 2.6.
For E locally free over S of rank n, take $r < n$ and consider the functor $\mathrm{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} / \sim$ (a Grassmannian) where Q is locally free of rank k.

Exercise

a. Show that this is representable
b. For the Plucker embedding

   $$\mathrm{Gr}(k, E) \longrightarrow \mathbb{P} \wedge^k E,$$

   a section over T is given by $f^*E \longrightarrow Q \longrightarrow 0$ corresponding to

   $$\wedge^k f^*E \longrightarrow \wedge^k Q \longrightarrow 0,$$

   noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \mathrm{Spec}\ k$, then E is a k-vector space V, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n.

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \longrightarrow Q \longrightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for $\mathbb{P}^n$ of all flat families of subschemes.

3 Thursday January 16th

3.1 Subfunctors

A functor $F' \subset F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is **open** iff for all $T \xrightarrow{\xi} F$ where $T = h_T$ and $\xi \in F(T)$.

We can take fiber products:

$$\begin{tikzcd} F' \arrow[rr] & & F \\ \underset{\text{Representable}}{F' \times_F T} \arrow[u] \arrow[rr, "\text{Open}"] & & T \arrow[u] \end{tikzcd}$$

So we can think of "inclusion in F" as being an *open condition*: for all T/S and $\xi \in F(T)$, there exists an open $U \subset T$ such that for all covers $f : T' \longrightarrow T$, we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff f factors through U.

Suppose $U \subset T$ in Sch/T, we then have

$$h_{U/T}(T') = \begin{cases} \emptyset & T' \longrightarrow T \text{ doesn't factor} \\ \{\mathrm{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is $h_{U/T}(T') = \hom_T(T', U)$.

By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{ (a, b) \in F'(T) \times T(T) \;\middle|\; \xi(b) = \iota(a) \text{ in } F(T) \right\},$$

where $F' \xrightarrow{\iota} F$ and $T \xrightarrow{\xi} F$.

So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of F/T' as sections of F over T/T' (?).

$$\begin{tikzcd}
F' \arrow[r, "\iota"] & F & \\
F' \times_F T \arrow[u] \arrow[r] & T \arrow[u, "\xi"] & \\
& & T' \arrow[ul, "f"] \arrow[uul, bend right, "f\circ\xi"']
\end{tikzcd}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U/S}(T'),$$

and so for $U \subset T$ in Sch/S we have $h_{U/S} \subset h_{T/S}$ is the functor of maps that factor through U. We just identify $h_{U/S}(T') = \hom_S(T', U)$ and $h_{T/S}(T') = \hom_S(T', T)$.

Example 3.1.
$\mathbb{G}_m, \mathbb{G}_a$. $\mathbb{G}_a$ represents giving a global function, $\mathbb{G}_m$ represents giving an invertible function.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{G}_a \\
T' \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & T \arrow[u, "f\in\mathcal{O}_T(T)"']
\end{tikzcd}$$

where $T' = \{f \neq 0\}$ and $\mathcal{O}_T(T)$ are global functions.

3.2 Actual Geometry: Hilbert Schemes

The best moduli space!

Want to parameterize families of subschemes over a fixed object. Fix k a field, X/k a scheme; we'll parameterize subschemes of X.

Definition 3.0.1 (Hilbert Functor)**.**
The hilbert functor is given by

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

which sends T to closed subschemes $Z \subset X \times_S T \longrightarrow T$ which are flat over T.

Here flatness replaces the Cartier condition.

Definition 3.0.2 (Flatness)**.**
For $X \xrightarrow{f} Y$ and $\mathbb{F}$ a coherent sheaf on X, f is flat over Y iff for all $x \in X$ the stalk F_x is a flat

$\mathcal{O}_{y,f(x)}$-module.

Note that f is flat if $\mathcal{O}_x$ is.

Flatness corresponds to varying continuously.

Warning: Unless otherwise stated, assume schemes are Noetherian.

Note that everything works out if we only path with finite covers.

Remark If X/k is projective, so $X \subset \mathbb{P}_k^n$, we have line bundles $\mathcal{O}_x(1) = \mathcal{O}(1)$. For any sheaf F over X, there is a hilbert polynomial $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$. (i.e. we twist by $\mathcal{O}(1)$ n times.)

The cohomology of F isn't changed by the pushforward into $\mathbb{P}_n$ since it's a closed embedding, i.e.

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

Fact (First) For $n \gg 0$, $\dim_k H^0 = \dim M_n$, the nth graded piece of M, which is a graded module over the homogeneous coordinate ring whose $i_* F = \tilde{M}$.

In general, for L ample of X and F coherent on X, we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

Proposition 3.1.
For $f : X \longrightarrow S$ projective, i.e. there is a factorization:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^n \times S \ni \mathcal{O}(1) \arrow[d] \\
& S
\end{tikzcd}$$

If S is reduced, irreducible, locally Noetherian, then f is flat $\iff$ $P_{\mathcal{O}_{x_s}}$ is constant for all $s \in S$.

To be more precise, look the base change to X_1, and the pullback of the fiber? $\mathcal{O}\Big|_{x_i}$?

Note: not using the word "integral" here! S is flat $\iff$ the hilbert polynomial over the fibers are constant.

Example 3.2.
The zero-dimensional subschemes $Z \in \mathbb{P}_k^n$, then P_Z is the length of Z, i.e. $\dim_k(\mathcal{O}_Z)$, and

$$P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z).$$

For two closed points in $\mathbb{P}^2$, $P_Z = 2$.

Consider the affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$, which is given by

$$\text{Spec } k[x,y]/(y,x^2) \cong k[x]/(x^2)$$

and $P_Z = 2$. I.e. in flat families, it has to record how the tangent directions come together.

Example 3.3.
Consider the flat family $xy = 1$ (flat because it's an open embedding) over $k[x]$, here we have points running off to infinity.

Proposition 3.2 ***(Modified Characterization of Flatness for Sheaves).***
A sheaf F is flat iff P_{F_S} is constant.

3.2.1 Proof

Assume $S = \text{Spec } A$ for A a local Noetherian domain.

Lemma 3.3.
For F a coherent sheaf on X/A is flat, we can take the cohomology via global sections $H^0(X;F(n))$. This is an A-module, and is a free A-module for $n \gg 0$.

Proof (of Lemma).
Assumed X was projective, so just take $X = \mathbb{P}^n_A$ and let F be the pushforward. There is a correspondence sending F to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module $\sum_{n \gg 0} H^0(\Pi^m_A; F(n))$ This is equal to $\oplus_{n \gg 0} H^0(\mathbb{P}^m_A; F(n))$ and taking the associated sheaf ($Y \mapsto \tilde{Y}$, as per Hartshorne's notation) which is free, and thus F is free.

See tilde construction in Hartshorne, essentially amounts to localizing free tings.

Conversely, take an affine cover U_i of X. We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume $H^i(\mathbb{P}^m; F(n)) = 0$ for $n \gg 0$, and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^m; F(n)) \longrightarrow \mathcal{C}^0(\underline{U}; F(n)) \longrightarrow \cdots \longrightarrow C^m(\underline{U}; F(n)) \longrightarrow 0.$$

Assuming F is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right.
Finally, local Noetherian + finitely generated flat implies free.

■

By the lemma, we want to show $H^0(\mathbb{P}^m; F(n))$ is free for $n \gg 0$ iff the hilbert polynomials on the fibers P_{F_S} are all constant.

Claim 1 (1)**.**
It suffices to show that for each point $s \in \text{Spec } A$, we have

$$H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$$

for $k(S)$ the residue field, for $n \gg 0$.

Note that P_{F_s} measures the rank of the LHS.

$\Longrightarrow$: The dimension of RHS is constant, whereas the LHS equals $P_{F_S}(n)$.

$\Longleftarrow$: If the dimension of the RHS is constant, so the LHS is free.

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank.

For M a finitely generated module over A, find $0 \longrightarrow A^n \longrightarrow M \longrightarrow Q$ is surjective after tensoring with $\text{Frac}(A)$, and tensoring with $k(S)$ for a closed point, if $\dim A^n = \dim M$ then $Q = 0$.

Proof (of Claim 1).
By localizing, we can assume s is a closed point. Since A is Noetherian, its ideal is f.g. and we have

$$A^m \longrightarrow A \longrightarrow k(S) \longrightarrow 0.$$

We can tensor with F (viewed as restricting to fiber) to obtain

$$F(n)^m \longrightarrow F(n) \longrightarrow F_S(n) \longrightarrow 0.$$

Because F is flat, this is still exact.

We can take $H^*(x, \cdot)$, and for $n \gg 0$ only H^0 survives. This is the same as tensoring with $H^0(x, F(n))$.

■

Definition 3.3.1 (Hilbert Polynomial Subfunctor)**.**
Given a polynomial $P \in \mathbb{Z}[n]$ for X/S projective, we define a subfunctor by picking only those with Hilbert polynomial p fiberwise as $\text{Hilb}^P_{X/S} \subset \text{Hilb}_{X/S}$. This is given by $Z \subset X \times_S T$ with $P_Z = P$.

Theorem 3.4 ***(Grothendieck).***
If S is Noetherian and X/S projective, then $\text{Hilb}^P_{X/S}$ is representable by a projective S-scheme.

See cycle spaces in analytic geometry.

4 Thursday January 23

Some facts about the Hilbert polynomial:

1. For a subscheme $Z \subset \mathbb{P}^n_k$ with $\deg P_z = \dim Z = n$, then

$$p_z(t) = \deg z t^n/(n!) + O(t^{n-1}).$$

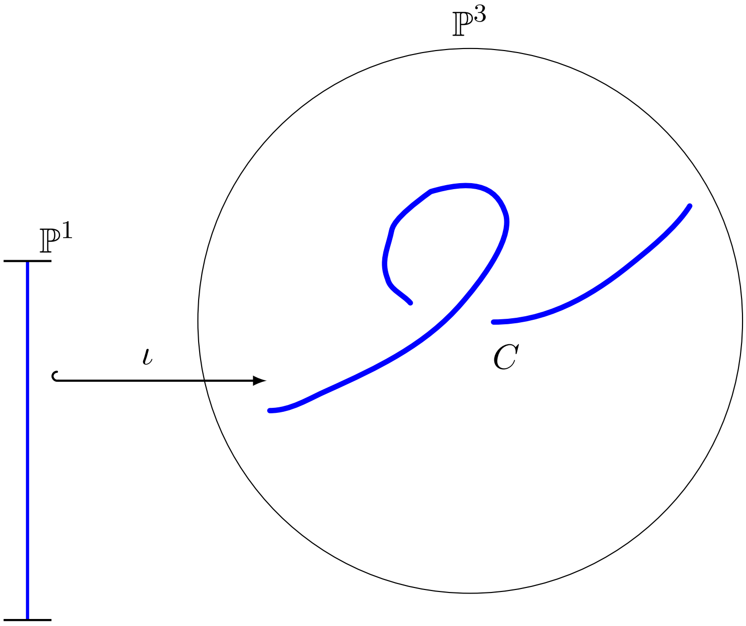
2. We have $p_z(t) = \chi(\mathcal{O}_z(t))$, consider the sequence

$$0 \longrightarrow I_z(t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{(t)} \longrightarrow \mathcal{O}_z^{(t)} \longrightarrow 0,$$

then $\chi(I_z(t)) = \dim H^0(\mathbb{P}^n, J_z(t))$ for $t \gg 0$, and $p_z(0)$ is the Euler characteristic of $\mathcal{O}_Z$.

Serre vanishing, Riemann-Roch, ideal sheaf.

Example 4.1 (Good to keep in mind)**.**
The twisted cubic:



Then

$$p_C(t) = (\deg C)t + \chi(\mathcal{O}_{\mathbb{P}^1}) = 3t + 1.$$

4.0.1 Hypersurfaces

Recall that length 2 subschemes of $\mathbb{P}^1$ are the same as specifying quadratics that cut them out, each such $Z \subset \mathbb{P}^1$ satisfies $Z = V(f)$ where $\deg f = d$ and f is homogeneous. So we'll be looking at $\mathbb{P}H^0(\mathbb{P}^n_k, \mathcal{O}(d))^\vee$, and the guess would be that this is $\mathrm{Hilb}_{\mathbb{P}^n_k}$

Resolve the structure sheaf

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

so we can twist to obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

Then

$$\chi(\mathcal{O}_D(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(t-d)),$$

which is

$$\binom{n+t}{n} - \binom{n+t-d}{n} = \frac{dt^{n-1}}{(n-1)!} + O(t^{n-2}).$$

Lemma 4.1.
Anything with the Hilbert polynomial of a degree d hypersurface is in fact a degree d hypersurface.

We want to write a morphism of functors

$$\mathrm{Hilb}_{\mathbb{P}^n_k}^{P_{n,d}} \longrightarrow \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee.$$

which sends flat families to families of equations cutting them out.

Want

$$Z \subset \mathbb{P}^n \times S \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee \longrightarrow L \longrightarrow 0.$$

This happens iff

$$0 \longrightarrow L^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$$

with torsion-free quotient.

Note that we use $L^\vee$ instead of $\mathcal{O}_s$ because of scaling.

We have

$$\begin{aligned} &0 \longrightarrow I_z \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S} \longrightarrow \mathcal{O}_z \longrightarrow 0 \\ &0 \longrightarrow I_z(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \mathcal{O}_z(d) \longrightarrow 0 \quad \text{by twisting.} \end{aligned}$$

We then consider $\pi_s : \mathbb{P}^n \times S \longrightarrow S$, and apply the pushforward to the above sequence noting that it is not right-exact.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \pi_{s*}I_z(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_{\mathbb{P}^n\times S}(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_z(d) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L^\vee = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) \arrow[r] & \text{locally free} \arrow[r] & 0 &
\end{tikzcd}
$$

Note: above diagram may be off horizontally? Todo: check.

This equality follows from flatness, cohomology, and base change. In particular, we need the following facts

The scheme-theoretic fibers, given by $H^0(\mathbb{P}^n, I_z(d))$ and $H^0(\mathbb{P}^n, \mathcal{O}_z(d))$, are all the same dimension.

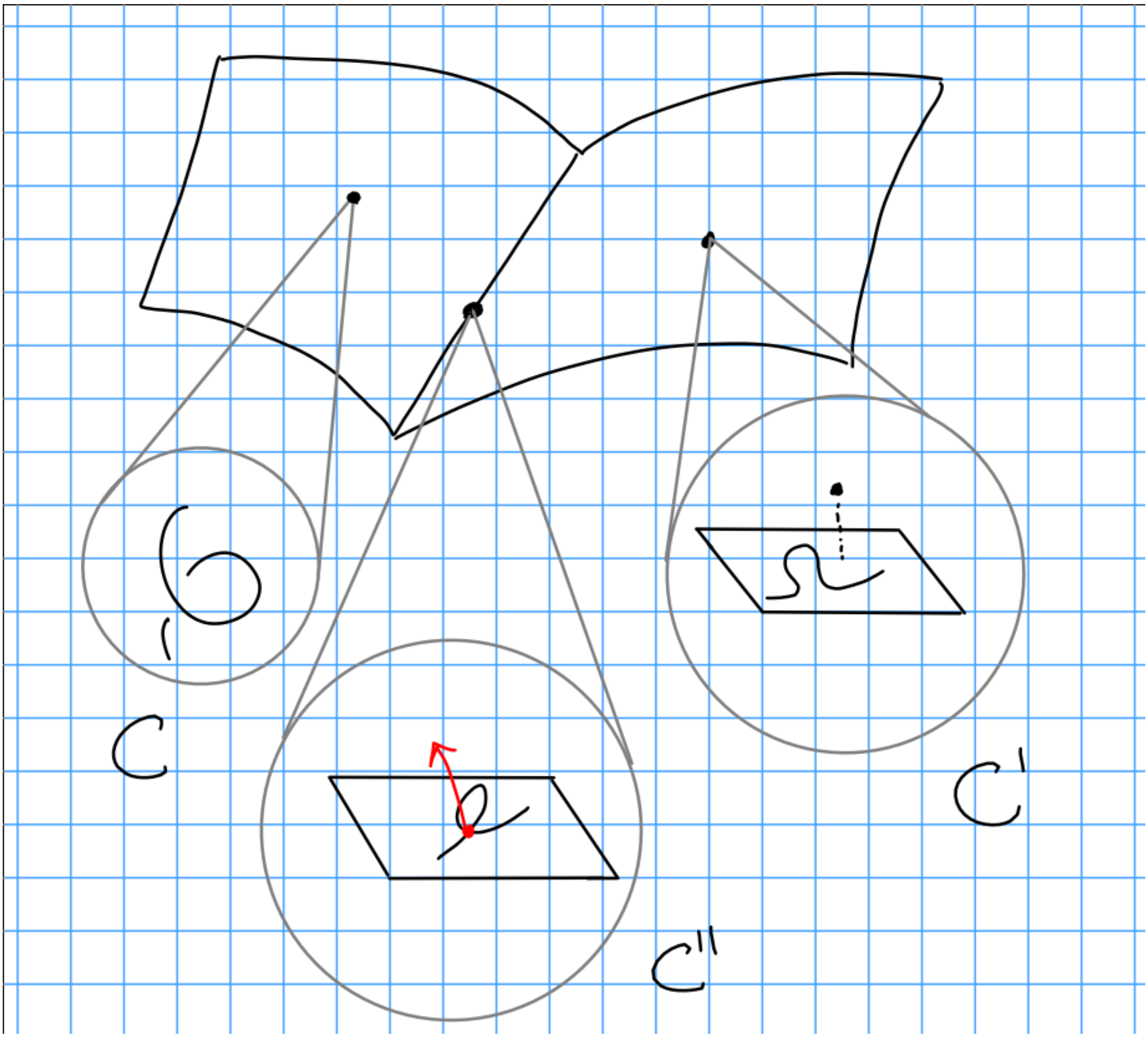
Using

1. Cohomology and base change, i.e. for $X \xrightarrow{f} Y$ a map of Noetherian schemes (or just finite-type) and F a sheaf on X which is flat over Y, there is a natural maop (not usually an isomorphism) $R^i f_* f \otimes k(y) \longrightarrow H^i(x_y, F|_{x_y})$, but is an isomorphism if $\dim H^i(x_y, F|_{x_y})$ is constant, in which case $R^i f_* f$ is locally free.
2. If $Z \subset \mathbb{P}^n_k$ is a degree d hypersurface, then independently we know $\dim H^0(\mathbb{P}^n, I_z(d)) = 1$ and $\dim H^0(\mathbb{P}^n, \mathcal{O}_z(d)) = \binom{d+n}{n} - 1$.

To get a map going backwards, we take the universal degree 2 polynomial and form $V(a_{00}x_0^2 + a_{11}x_1^2 + a_{12}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2) \subset \mathbb{P}^2 \times \mathbb{P}^5$.

Next example: twisted cubics.

Consider a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ obtained by taking a basis of a homogeneous cubic polynomial. The canonical example is $(x, y) \longrightarrow (x^3, x^2y, xy^2, y^3)$. Then $P_C(t) = 3t+1$, and $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_k}$ has a component with generic point a twisted cubic, and another component with points a curve disjoint union a point, and the overlap are nodal curves with a "fat" 3-dimensional point:



Then $P_{C'} = 1 + \tilde{P}$, the hilbert polynomial of just the base without the disjoint point, so this equals $1+P_{2,3} = 1+(3t+0) = 3t+1$. For $P_{C''}$, we take the sequence $0 \longrightarrow k \longrightarrow \mathcal{O}_{C''} \longrightarrow \mathcal{O}_{C''\text{reduced}} \longrightarrow 0$, so $P_{C''} = 1 + P_{C''\text{red}} = 3t+1$.

Note: flat families have to have the same constant Hilbert polynomial.

Note that we can get paths in this space from $C \longrightarrow C''$ and $C' \longrightarrow C''$ by collapsing a twisted cubic onto a plane, and sending a disjoint point crashing into the node on a nodal cubic.

We're mapping $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, and there is a natural action of $\mathbb{P}\mathrm{GL}(4) \curvearrowright \mathbb{P}^3$, so we get a map

$$\mathbb{P}\mathrm{GL}(4) \times \mathbb{P}^3 \longrightarrow \mathbb{P}^3.$$

Let $c \in \mathbb{P}^3$ and let $\mathcal{C}$ be the preimage. This induces (?) a map $\mathbb{P}\mathrm{GL}(4) \longrightarrow \mathrm{Hilb}_{\mathbb{P}^3}^{3t+1}$ where the fiber over $[C]$ in the latter is $\mathbb{P}\mathrm{GL}(2) = \mathrm{Aut}(\mathbb{P}^1)$. By dimension counting, we find that the dimension of the twisted cubic component is $15 - 3 = 12$.

The 15 in the other component comes from 3-dim choices of plane, 3-dim choices of a disjoint point, and $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee \cong \mathbb{P}^9$, yielding 15 dimensions.

To show that these are actually different components, we use Zariski tangent spaces. Let T_1 be the tangent space of the twisted cubic component, then $\dim T_1 \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_k} = 12$, and similarly the dimension of the tangent space over the C' component is 15.

Algebra fact: Let A be Noetherian and local, then the dimension of the Zariski tangent space, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$, the Krull dimension. If this is an equality, then A is regular.

Thus dimensions of the tangent spaces give an upper bound.

Proposition: If X/k is projective and P is a Hilbert polynomial, then $[Z] \in \mathrm{Hilb}^P_{X/k}$, i.e. a closed subscheme of X with hilbert polynomial p (note there's an ample bundle floating around) then the tangent space is $\hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.