

# 8.8 Part 2, Computing the Index of $L$

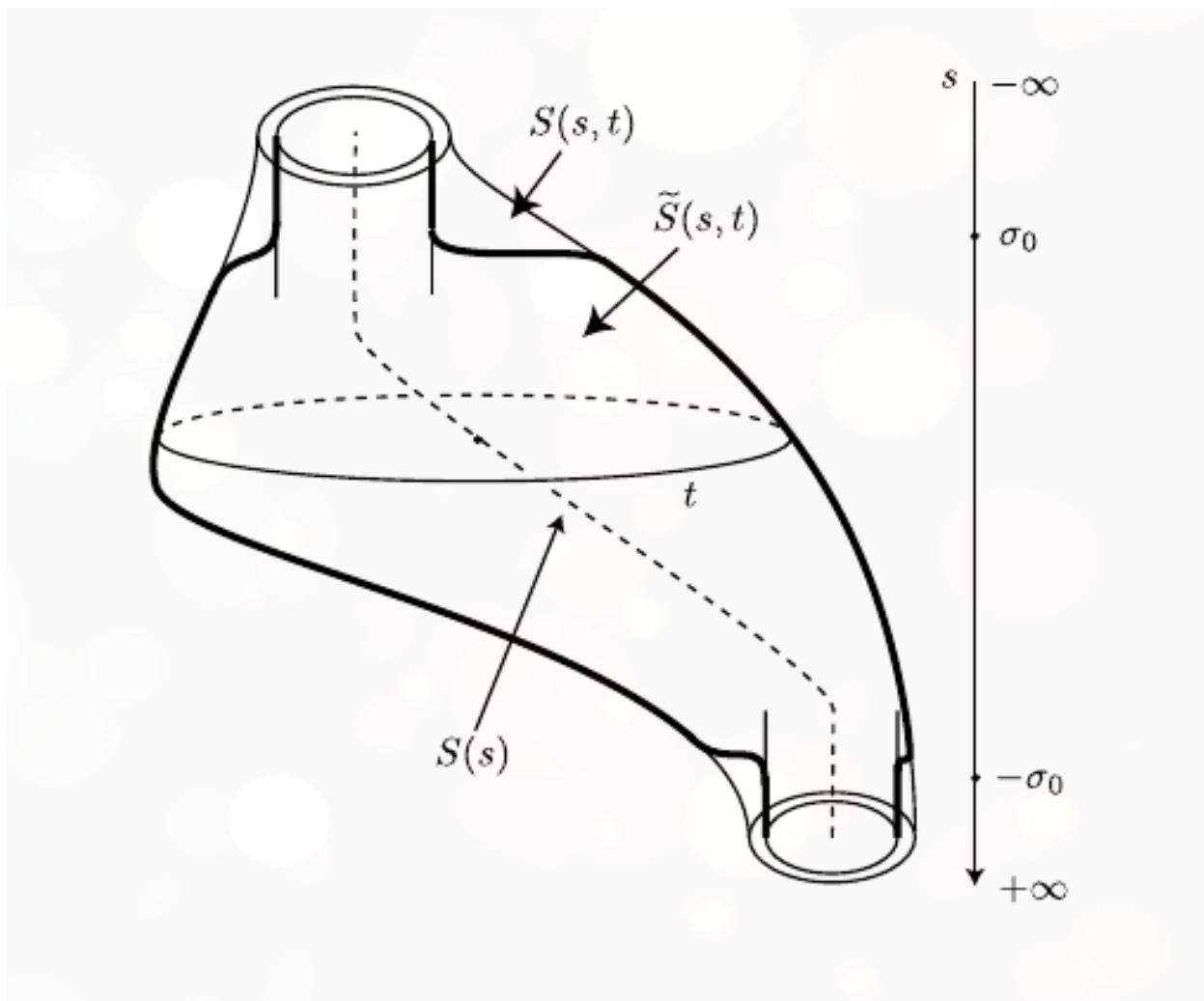
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What we're trying to prove:

- 8.1.5:  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .
- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$

$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

- 8.7: Shows  $L$  is Fredholm
- By the end of 8.8: replace  $L$  by  $L_1$  with the same *index*

- (not the same kernel/cokernel)
- Compute  $\text{Ind } L_1$ : explicitly describe  $\ker L_1, \text{coker } L_1$ .
- Replace in two steps:
  - $L \rightsquigarrow L_0$ , modified outside  $B_{\sigma_0}(0)$  in  $s$ .
  - \* Replace  $S(s, t)$  by a matrix

$$\tilde{S}(s, t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

- \* Idea: approximate by cylinders at infinity.
- \* Use invariance of index under small perturbations.
- $L_0 \rightsquigarrow L_1$  by a homotopy, where  $S_\lambda : S \rightsquigarrow S(s)$  a diagonal matrix that is a constant matrix *outside*  $B_\varepsilon(0)$ .
- \* Use invariance of index under homotopy.

## 0.1 Main Results

- Theorem 8.8.1:

$$\text{Ind}(L) = \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing  $L$  to  $L_1$  Construct an operator

$$\begin{aligned} L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

where  $S : \mathbb{R} \longrightarrow \text{Mat}(2n; \mathbb{R})$  is a path of diagonal matrices depending on  $\text{Ind}(R^\pm(t))$ ; then

$$\text{Ind}(L) = \text{Ind}(L_1) = \text{Ind}(R^-(t)) - \text{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing  $L_1$  to  $R^\pm$ . Let  $k^\pm := \text{Ind}(R^\pm)$ ; then  $\text{Ind}(L_1) = k^- - k^+$ .
- Lemma 8.8.4:  $\text{Ind}(L_0) = \text{Ind}(L)$ .
- Han's Talk:
  - Prop 8.8.3, using Lemma 8.8.5
- Me
  - Proof of 8.8.5

## 0.2 8.8.5:

Used in the proof of 8.8.3,  $\text{Ind}(L_1) = k^- - k^+$ .

Setup:

We have

$$J_0^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & \\ & J_0^1 & \\ & & \ddots \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \text{Mat}(2; \mathbb{R}).$$

We had a path of diagonal matrices:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$

Statement: let  $p > 2$  and define

$$\begin{aligned} F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

This looks like  $L_1$  for  $n = 1$ ?

1. Suppose  $a_1(s) = a_2(s)$  and define  $a^\pm := a_1^\pm = a_2^\pm$ . Then

$$\begin{aligned} \dim \text{Ker } F &= 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\} \\ \dim \text{Ker } F^* &= 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^+ < 2\pi\ell < a^- \right\}. \end{aligned}$$

2. Suppose  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ , then

$$\begin{aligned} \dim \text{Ker } F &= \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\} \\ \dim \text{Ker } F^* &= \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}. \end{aligned}$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

## 0.3 Proof

### 0.3.1 Assertion 1

**Assertion 1:** Suppose  $a_1(s) = a_2(s)$  and define  $a^\pm := a_1^\pm = a_2^\pm$ . Then

$$\begin{aligned} \dim \text{Ker } F &= 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\} \\ \dim \text{Ker } F^* &= 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^+ < 2\pi\ell < a^- \right\}. \end{aligned}$$

Step 1: Transform to Cauchy-Riemann Equations

- Write  $a(s) = a_1(s) = a_2(s)$ .
- Start with equation on  $\mathbb{R}^2$ ,

$$Y(s, t) = (Y_1(s, t), Y_2(s, t))$$

- Replace with equation on  $\mathbb{C}$ :

$$Y(s, y) = Y_1(s, t) + iY_2(s, t)$$

- Rewrite the PDE  $F(Y) = 0$  as  $\bar{\partial}Y + S(s)Y = 0$ , i.e.

$$\frac{\partial}{\partial s} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let  $Y = B\tilde{Y}$  where  $B \in \text{GL}(1, \mathbb{C})$  satisfies  $(\bar{\partial} + S)B = 0$  to obtain  $\bar{\partial}\tilde{Y} = 0$ .

- Can choose  $B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix}$  where  $\frac{\partial b}{\partial s} = -a(s)b(s)$ .
- Explicitly, we can take the integral  $b(s) = e^{\int_0^s -a(t) dt} = e^{-A(s)}$

- Remark: for some constants  $C_i$ , we have

$$A(s) = \begin{cases} C_1 + a^- s & s \leq -\sigma_0 \\ C_2 + a^+ s & s \geq \sigma_0 \end{cases}.$$

- Remark: the new  $\tilde{Y}$  satisfies CR. It is continuous and  $L^1_{\text{loc}}$  and thus by elliptic regularity  $C^\infty$ . Its real/imaginary parts are  $C^\infty$  and harmonic.

Step 2: ?

- Identify  $s + it \in \mathbb{R} \times S^1$  with  $u = e^{2\pi z}$
- Apply Laurent's theorem to  $\tilde{Y}(u)$  on  $\mathbb{C} \setminus \{0\}$  to obtain an expansion of  $\tilde{Y}$  in  $z$ .
- Deduce that the solutions of the system are given by

$$\tilde{Y}(s + it) = \sum_{\ell \in \mathbb{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where  $c_\ell \in \mathbb{C}$  and this sequence converges for all  $s, t$ .

- Write in real coordinates as

$$\tilde{Y}(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s\ell} \left( \alpha_\ell \begin{pmatrix} \cos 2\pi\ell t \\ \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -\sin 2\pi\ell t \\ \cos 2\pi\ell t \end{pmatrix} \right).$$

- Return to  $Y = B\tilde{Y}$ :

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s\ell} \left( \alpha_\ell \begin{pmatrix} e^{-A(s)} \cos 2\pi\ell t \\ e^{-A(s)} \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -e^{-A(s)} \sin 2\pi\ell t \\ e^{-A(s)} \cos 2\pi\ell t \end{pmatrix} \right).$$

- For  $s \geq s_0$ , for some constants  $K_i$  we can write

$$Y(s, t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^-)s + K} (\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t) \\ e^{(2\pi\ell - a^-)s + K'} (\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t) \end{pmatrix}.$$

and for  $s \geq s_0$

$$Y(s, t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^+)s + C} (\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t) \\ e^{(2\pi\ell - a^+)s + C'} (\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t) \end{pmatrix}.$$

- Then  $Y \in L^p \iff$  the exponential terms die at infinity. Forces the conditions:
  - $\ell \neq 0 \implies \alpha_\ell = \beta_\ell = 0$  or  $2\pi\ell < a^+$ .
  - $\ell = 0 \implies (a_0 = 0 \text{ or } a^+ > 0)$  and  $(\beta_0 = 0 \text{ or } a^+ > .0)$ .

This further forces

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } a^- < 2\pi\ell < a^+ & \ell \neq 0 \\ \left( \alpha_0 = 0 \text{ or } a^- < 0 < a^+ \right) \text{ and } \left( \beta_0 = 0 \text{ or } a^- < 0 < a^+ \right) & \ell = 0 \end{cases}.$$

- Finitely many such  $\ell$  that satisfy these conditions
- Sufficient conditions for  $Y(s, t) \in W^{1,p}$ .

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

I.e.  $F = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s).$

- Compute dimension of space of solutions:

$$\dim \text{Ker } F = 2\# \left\{ \ell \in \mathbf{Z}^* \mid a^- < 2\pi\ell < a^+ \right\} = 2\# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\}.$$

Test:

$$\mathbb{1}[\{x\}].$$

Use this to deduce  $\dim \ker F^*$ :

- $Y \in \ker F^* \iff Z(s, t) := Y(-s, t)$  is in the kernel of the operator

$$\tilde{F} : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Z \mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y.$$

- Obtain  $\ker F^* \cong \ker \tilde{F}$ .

■<sub>1</sub>

### 0.3.2 Assertion 2

**Assertion 2:** Suppose  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ , then

$$\begin{aligned} \dim \operatorname{Ker} F &= \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\} \\ \dim \operatorname{Ker} F^* &= \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}. \end{aligned}$$

We use the following lemma

- Lemma 8.8.7:

$$\sup_{s \in \mathbb{R}} \|S(s)\| < 1 \implies \text{the elements in } \ker F, \ker F^* \text{ are independent of } t.$$

- Proof: see Proposition 10.1.7, in subsection 10.4.a.
- We know (?)

$$\mathbf{a}(s) := \begin{bmatrix} a_1(s) \\ a_2(s) \end{bmatrix}, \quad \mathbf{Y} := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \ker F \implies \frac{\partial \mathbf{Y}}{\partial s} = -\mathbf{a}(s)\mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c} \exp(-\mathbf{A}(s)), \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma) d\sigma.$$

- Explicitly,

$$\begin{cases} \frac{\partial Y_1}{\partial s} = -a_1(s)Y_1 \\ \frac{\partial Y_2}{\partial s} = -a_2(s)Y_2 \end{cases} \implies Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma) d\sigma.$$

- As before,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

- Thus  $Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+)$ , establishing the formula for  $\dim \ker F$ .
- Get formula for  $\dim \ker F^*$  using  $\tilde{F}$  as before.

■<sub>2</sub>

(End of Chapter)

### 1 8.8.3:

- Previously: we replaced  $S$  with a path of diagonal matrices only depending on  $s$ .
- Obtained an operator  $L_1$ .
- Statement: Prop 8.8.3: Reducing  $L_1$  to  $R^\pm$ . Let  $k^\pm := \operatorname{Ind}(R^\pm)$ ; then  $\operatorname{Ind}(L_1) = k^- - k^+$ .

**1.1 Proof**

- From section 8.7:  $\text{coker } L_1 \cong \ker L_1^*$ :
- Recall definition of  $L_1$ :

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

- We can write the adjoint:

$$L_1^* : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^q(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S^T(s)Z$$

where  $\frac{1}{p} + \frac{1}{q} = 1$  are conjugate exponents.

- Now apply Lemma 8.8.3, which computes the kernels explicitly
- Consider four cases, depending on parity of  $k^\pm - n$ , show all 4 lead to  $\text{Ind}(L_1) = k^- - k^+$ .  
 $S_{k^\pm}$  are certain diagonal matrices, and
  1.  $k^- \equiv k^+ \equiv n \pmod{2}$ .
  2.  $k^- \equiv n, k^+ \equiv n-1 \pmod{2}$
  3.  $k^- \equiv n-1, k^+ \equiv n \pmod{2}$ .
  4.  $k^- \equiv k^+ \equiv n-1 \pmod{2}$

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