

Mapping Class Groups

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1 | Setup

- All manifolds:
 - Connected
 - Oriented
 - 2nd countable (countable basis)
 - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
 - With boundary (possibly empty)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Curves: simple, closed, oriented
- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \left\{ f : X \rightarrow Y \mid f \text{ is continuous} \right\}.$$

1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
 - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology \mathcal{O}_{CO} :

$$U \in \mathcal{O}_{\text{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

1.1.1 Mapping Spaces

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{\text{CO}}) \quad \text{where } \mathcal{O}_{\text{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
 - $\text{Homeo}(X, Y)$ the subspace of homeomorphisms
 - $\text{Imm}(X, Y)$, the subspace of immersions (injective map on tangent spaces)
 - $\text{Emb}(X, Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
 - $C^k(X, Y)$, the subspace of $k \times$ differentiable maps
 - $C^\infty(X, Y)$ the subspace of smooth maps
 - $\text{Diffeo}(X, Y)$ the subspace of diffeomorphisms
 - $C^\omega(X, Y)$ the subspace of analytic maps
 - $\text{Isom}(X, Y)$ the subspace of isometric maps (for Riemannian metrics)
 - $[X, Y]$ homotopy classes of maps

1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact.}$$

– In words: $f_n \rightarrow f$ uniformly on every compact set.

- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{f_\alpha\} \subset \text{hom}_{\text{Top}}(X, Y)$$

form a compact subset of $\text{Map}(X, Y)$?

- Essentially answered by:

Theorem 1.1 (Ascoli).

If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \text{hom}_{\text{Top}}(X, Y)$ has compact closure $\iff \mathcal{F}$ is equicontinuous and $F_x := \{f(x) \mid f \in \mathcal{F}\} \subset Y$ has compact closure.

Corollary 1.2 (Arzela).

If $\{f_n\} \subset \text{hom}_{\text{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every x , it contains a uniformly convergent subsequence.

1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
 - Can take G to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} := \text{hom}_{\text{TopGrp}}(G, S^1)$$

where we consider $S^1 \subset \mathbb{C}$.

- Can integrate with respect to the Haar measure μ , define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\text{Map}(\cdot, Y)$:

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\ X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\ X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}. \end{aligned}$$

- Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, Y^X) \\ H : X \times Z &\rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\ (x, z) &\mapsto H(x, z) \iff z \mapsto H(\cdot, z). \end{aligned}$$

- Categorically, $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top .
- A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space PY , the based loop space ΩY .
- Importance in homotopy theory: the path space fibration

$$\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y$$

- Plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences.
- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps f, g are homotopic \iff they are connected by a path in $\text{Map}(X, Y)$.

Picture!

2.1.1 Proof

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, Y^X) \cong \text{Map}(X \times I, Y),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \rightarrow Y$, the LHS are paths in the space of maps.

2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\text{Map}(I, \cdot)$:

$$\begin{aligned}
\mathcal{P}\text{Map}(X, Y) &:= \text{Map}(I, Y^X) \\
\mathcal{P}^2\text{Map}(X, Y) &:= \mathcal{P}\text{Map}(I, Y^X) = \text{Map}(I, (Y^X)^I) = \text{Map}(I, Y^{XI}) \\
&\vdots \\
\mathcal{P}^n\text{Map}(X, Y) &:= \mathcal{P}^{n-1}\text{Map}(I, Y^{XI}) = \text{Map}(X, Y^{XI^n}).
\end{aligned}$$

- Can interpret

$$\mathcal{P}^2\text{Map}(X, Y) = \mathcal{P}\text{Map}(X \times I, Y).$$

as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

3 | Defining the Mapping Class Group

3.1 Isotopy

- Define a homotopy between $f, g : X \rightarrow Y$ as a map $F : X \times I \rightarrow Y$ restricting to f, g on the ends.
 - Equivalently: a *path*, an element of $\text{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \rightarrow Y$ to be homeomorphisms for every t .
 - Equivalently: a path in the subspace of homeomorphisms, an element of $\text{Map}(I, \text{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\text{Map}(X, Y)$, subspace of homeomorphisms.

3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
 - In a general category \mathcal{C} , we can always define the group $\text{Aut}_{\mathcal{C}}(X)$.
 - * If the group has a topology, we can consider $\pi_0\text{Aut}_{\mathcal{C}}(X)$, the set of path components.
 - * Since groups have identities, we can consider $\text{Aut}_{\mathcal{C}}^0(X)$, the path component containing the identity.
 - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X)/\text{Aut}_{\mathcal{C}}^0(X).$$

- Here the \pm indicates that we take both orientation preserving and non-preserving automorphisms.
- Has an index 2 subgroup of orientation-preserving automorphisms, $\text{MCG}^+(X)$.
- Can define $\text{MCG}_{\partial}(X)$ as those that restrict to the identity on ∂X .

Picture: quotienting out by identity component

3.3 Definitions in Several Categories

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with \mathcal{O}_{CO} .

- Taking $\text{MCG}_{\text{Top}}^{\pm}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking $\text{MCG}_{C^{\infty}}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f] \in [X, Y]\}.$$

- Taking $\text{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences*.

Image

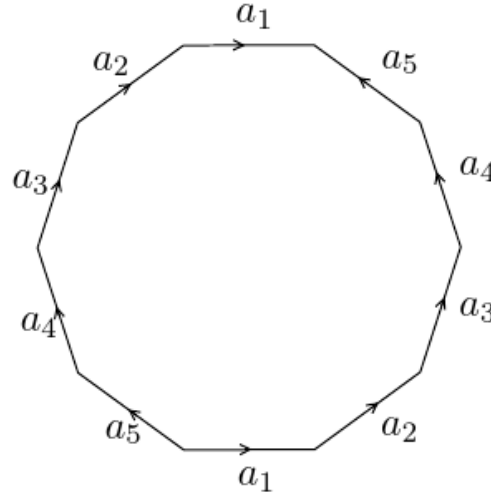


Figure 2.2 Rotation by $2\pi/10$ gives an order 10 element of $\text{Mod}(S_2)$.

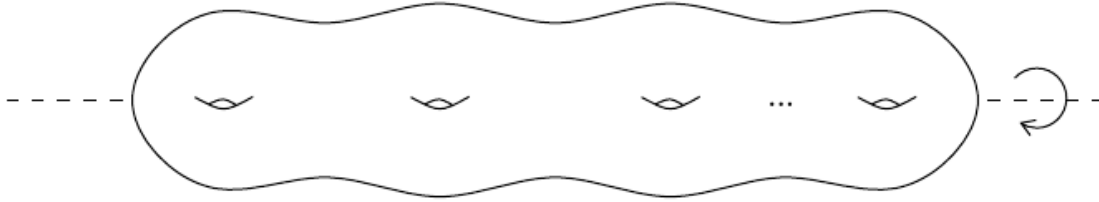


Figure 2.3 The rotation by π about the indicated axis is a hyperelliptic involution.

Figure 1: Image

3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
 - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi\text{Aut}(X)$.
- For surfaces: For Σ a genus g surface, $\text{MCG}(S)$ acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \rightarrow \text{MCG}(\Sigma) \rightarrow T(\Sigma) \rightarrow \mathcal{M}_g \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to X .

- $T(S)$ is the moduli space of complex structures on S , up to the action of homeomorphisms that are isotopic to the identity:
 - Points are isomorphism classes of marked Riemann surfaces
 - Equivalently the space of hyperbolic metrics
- Used in the Nielsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
 - Periodic,
 - Reducible (preserves some simple closed curves), or
 - Pseudo-Anosov (has directions of expansion/contraction)

Picture: \mathcal{M}_g .

4 | Examples of MCG

4.1 The Plane: Straight Lines

- $\text{MCG}_{\text{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, take the straight-line homotopy:

$$F : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$$
$$F(x, t) = tf(x) + (1 - t)x.$$

Picture: parameterize line between x and $f(x)$ and flow along it over time.

4.2 The Closed Disc: The Alexander Trick

- $\text{MCG}_{\text{Top}}(\mathbb{D}^2) = 1$: for any $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that $f|_{\partial\mathbb{D}^2} = \text{id}$, take

$$F : \mathbb{D}^2 \times I \rightarrow \mathbb{D}^2$$

$$F(x, t) := \begin{cases} tf\left(\frac{x}{t}\right) & \|x\| \in [0, t) \\ x & \|x\| \in [1-t, 1] \end{cases}.$$

- This is an isotopy from f to the identity.
- Interpretation: “cone off” your homeomorphism over time:

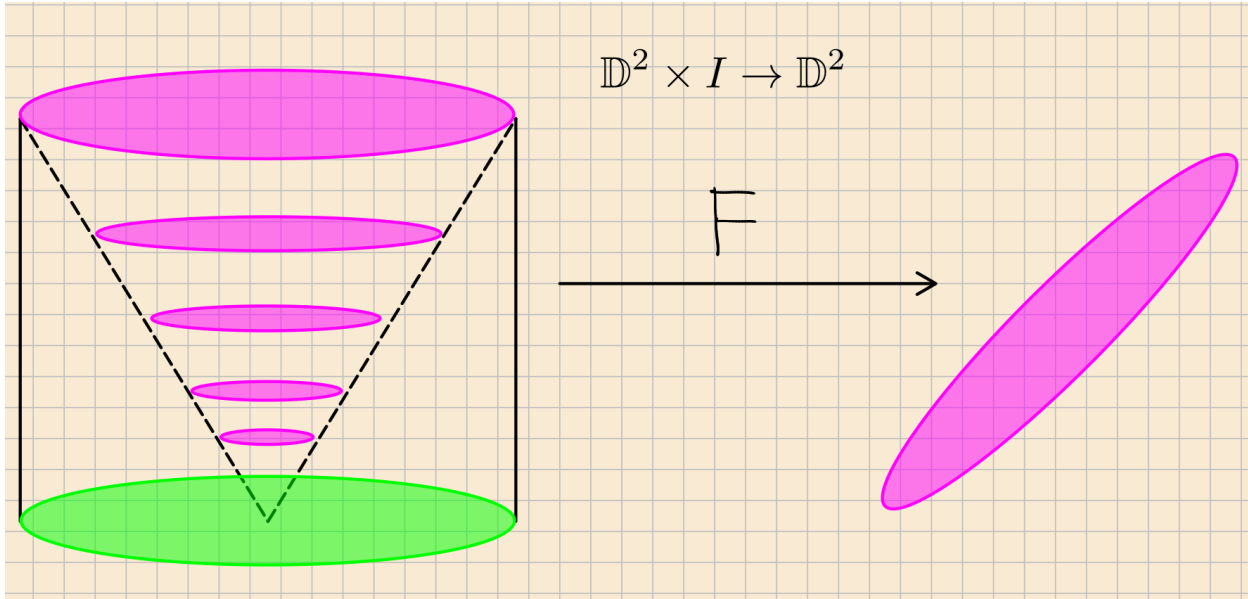


Figure 2: Image

- Note that this won't work in the smooth category: singularity at origin

4.3 Overview of Big Results

- The word problem in $\text{MCG}(\Sigma_g)$ is solvable
- Any finite group is $\text{MCG}(X)$ for some compact hyperbolic 3-manifold X .
- For $g \geq 3$, the center of $\text{MCG}(\Sigma_g)$ is trivial and $H_1(\text{MCG}(\Sigma_g); \mathbb{Z}) = 1$
 - Why care: same as abelianization of the group.

Theorem 4.1 (Dehn-Neilsen-Baer).

Let Σ_g be compact and oriented with $\chi(\Sigma_g) < 0$. Then

$$\text{MCG}_{\partial}^+(\Sigma_g) \cong \text{Out}_{\partial}(\pi_1(\Sigma_g)) \cong_{\text{Grp}} \pi_0 \text{ho}_{\partial}(\Sigma_g).$$

- For $g \geq 4$, $H_2(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$

- Why care: used to understand surface bundles

$$\begin{array}{ccc} \Sigma_g & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

- Find the classifying space $B\mathrm{Diffeo}$
- Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\mathrm{Diffeo}(\Sigma_g)] \cong [S^{n-1}, \mathrm{Diffeo}(\Sigma_g)]$$

- Theorem (Earle-Ells): For $g \geq 2$, $\mathrm{Diffeo}_0(\Sigma_g)$ is contractible. As a consequence, $\mathrm{Diffeo}(\Sigma_g) \twoheadrightarrow \mathrm{Mod}(\Sigma_g)$ is a homotopy equivalence, and there is a correspondence:

$$\left\{ \begin{array}{c} \text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B \end{array} \right\} / \text{Bundle isomorphism} \iff \left\{ \begin{array}{c} \text{Monodromy Representations} \\ \rho: \pi_1(B) \rightarrow \mathrm{MCG}(\Sigma_g) \end{array} \right\} / \text{Conjugacy}.$$

5 | Dehn Twists

- $\text{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation

Claim: Let $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, then $\text{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto \exp(2\pi i|z|)z. \end{aligned}$$

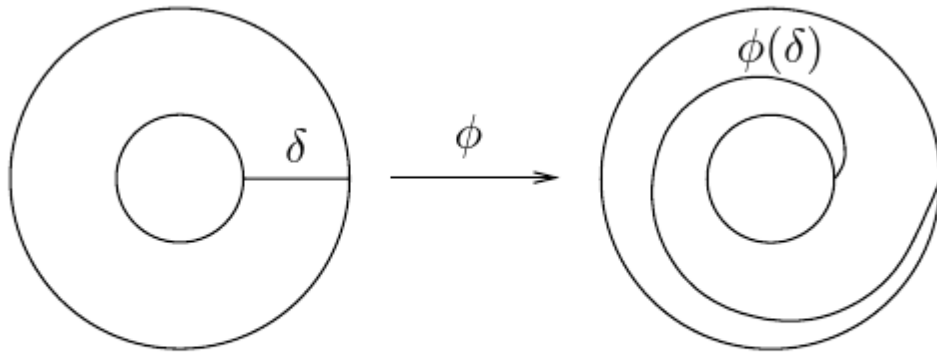


Figure 2.5 A generator for $\text{Mod}(A)$.

Figure 3: Image

6 | MCG of the Torus

6.1 Setup

Definition 6.0.1 (Special Linear Group).

$$\mathrm{SL}(n, \mathbb{k}) = \left\{ M \in \mathrm{GL}(n, \mathbb{k}) \mid \det M = 1 \right\} = \ker \det_{\mathbb{G}_m}.$$

Definition 6.0.2 (Symplectic Group).

$$\mathrm{Sp}(2n, \mathbb{k}) = \left\{ M \in \mathrm{GL}(2n, \mathbb{k}) \mid M^t \Omega M = \Omega \right\} \leq \mathrm{SL}(2n, \mathbb{k})$$

where Ω is a nondegenerate skew-symmetric bilinear form on \mathbb{k} .

Example:

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Definition 6.0.3 (Algebraic Intersection).

A bilinear antisymmetric form on middle homology:

$$\hat{\iota} : H_1(\Sigma_g; \mathbb{Z}) \otimes H_1(\Sigma_g; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

Note that this is a symplectic pairing.

- There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned} \rho : \mathrm{MCG}(\Sigma) &\rightarrow \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_*. \end{aligned}$$

- For a surface of finite genus $g \geq 1$, elements in $\mathrm{im} \rho$ preserve the *algebraic intersection form*
- Thus there is an interesting surjective representation:

$$0 \rightarrow \mathrm{Tor}(\Sigma_g) \hookrightarrow \mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}(2g; \mathbb{Z}) \rightarrow 0.$$

- Kernel is the *Torelli group*, interesting because the symplectic group is well understood, so questions about MCG reduce to questions about Tor.

Theorem 6.1 (*Mapping Class Group of the Torus*).

The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

6.2 Proof

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\text{Homeo}(X, X), \circ) &\rightarrow (\text{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*. \end{aligned}$$

- This will descend to the quotient $\text{MCG}(X)$ iff

$$\text{Homeo}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\text{id})$$

- This is true here, since any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned} \tilde{\sigma} : \text{MCG}(X) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

Claim: $\text{im}(\tilde{\sigma}) \subseteq \text{SL}(2, \mathbb{Z})$.

Proof .

- Algebraic intersection numbers in Σ_2 correspond to determinants
- $f \in \text{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2 of Farb and Margalit

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- We can thus freely restrict the codomain to define the map

$$\begin{aligned} \sigma : \text{MCG}(X) &\rightarrow \text{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

6.2.1 Surjectivity

Claim: σ is surjective.

- \mathbb{R}^2 is the universal cover of Σ_2 , with deck transformation group \mathbb{Z}^2 .
- Any $A \in \text{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \text{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

Claim: \tilde{A} is equivariant wrt \mathbb{Z}^2

Proof .

$$\text{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that

$$S^2 = 1, \quad T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

So if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $\tilde{A} \in \mathrm{SL}(2, \mathbb{Z})$, we have $\tilde{A}\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. A preserves any integer lattice

$$\Lambda = \{p\mathbf{v}_1 + q\mathbf{v}_2 \mid p, q \in \mathbb{Z}\}.$$

■

- So \tilde{A} descends to a well-defined map

$$\psi_{\tilde{A}} : \Sigma_2 \circlearrowleft = \mathbb{R}^2 / \mathbb{Z}^2 \circlearrowleft$$

which is still a linear self-homeomorphism.

- There is a correspondence

$$\left\{ \begin{array}{c} \text{Primitive curves in} \\ \pi_1(\Sigma_2) \cong \mathbb{Z}^2 \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Primitive vectors in } \mathbb{Z}^2 \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{array} \right\} /_{\text{homotopy}},$$

where an element x is *primitive* iff it is not a multiple of another element.

- By changing basis, you can associate a unique primitive vector to M (all components coprime)
- By the correspondence, changing a map by a homotopy corresponds to the same primitive vector
- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

6.2.2 Injectivity

Claim: σ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.

Proposition 6.2 (Hatcher 1B.9).

Let X be a connected CW complex and Y a $K(G, 1)$. Then there is a map

$$\mathrm{hom}_{\mathrm{Grp}}(\pi_1(X; x_0), \pi_1(Y; y_0)) \rightarrow \mathrm{hom}_{\mathrm{Top}}((X; x_0), (Y; y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map. Moreover, the map is unique up to homotopies fixing x_0 .

- Thus there is a correspondence

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } \Sigma_2 \circlearrowleft \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Group morphisms } \mathbb{Z}^2 \circlearrowleft \end{array} \right\}.$$

- Claim: any element $f \in \mathrm{MCG}(\Sigma_2)$ has a representative φ which fixes any given basepoint
- So if $f \in \ker \sigma$, then $f \simeq \varphi \simeq \mathrm{id}$ are homotopic, so $\ker \sigma = 1$.