Title

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Contents

1 Lecture 15: The L-Polynomial

Recall that we had $Z(t) + F(t) + G(t)$:

$$(q-1)F(t) = \sum_{0 \le \deg C \le 2g-2} q^{\ell(C)} t^{\deg(C)}$$
$$(q-1)G(t) = h\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t}\right).$$

Note that $F(t)$ is a polynomial of degree at most $2g-2$, and clearing denominators in $G(t)$ yields a polynomial of degree at most $2g$

Definition 1.0.1 (The L-polynomial)
The L-polynomial is defined as

$$L(t) := (1-t)(1-qt)Z(t) = (1-t)(1-qt)\sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[t].$$

It turns out that the degree bound of $2g$ is sharp, and the coefficients closer to the middle are most interesting:

Theorem 1.0.2 *(?).*
Let $K/\mathbb{F}_q$ be a function field of genus $g \ge 1$, then

a. $\deg L = 2g$.
b. $L(1) = h$
c. $L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right)$.
d. Writing $L(t) = \sum_{j=1}^{2g} a_j t^j$,

- $a_0 = 1$ and $a_{2g} = q^g$.
- For all $0 \le j \le g$, we have $a_{2g-j} = q^{g-j} a_j$.
- $a_1 = |\Sigma(K/\mathbb{F}_q)| - (q+1)$, which notably does not depend on g.
- Write $L(t) = \prod_{j=1}^{2g}(1-\alpha_j t) \in \mathbb{C}[t]$ [a]

e. The $\alpha_j \in \overline{\mathbb{Z}}$ [b] (which were *a priori* in $\mathbb{C}$) and can be ordered such that for all $1 \le j \le g$, we have $a_j a_{g+j} = q$. [c]

f. If $L_r(t) = (1-t)(1-q^r t)Z_r(t)$ then $L_r(t) = \prod_{j=1}^{2g}(1-\alpha_j^r t)$, where K_r is the constant extension $K\mathbb{F}_{q^r}/\mathbb{F}_{q^r}$

[a] The polynomial isn't monic, but rather has a constant coefficient, so this expansion is somewhat more natural than (say) $\prod(t-\alpha)$.
[b] $\overline{\mathbb{Z}}$ denotes the algebraic integers.
[c] This is the first hint at the Riemann hypothesis: if for example they all had the same complex modulus, this would force $|a_j| = \sqrt{q}$. Thus proving that they all have the same absolute value is 99% of the content!

Note that the α_j are reciprocal roots.

Proof (of a).
We saw from $Z(t) = F(t) + G(t)$ that $\deg L \leq 2g$. Equality will follow from the proof of (d) part 1, since this would imply that $a_{2g} = q^g \neq 0$.

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Proof (of b).
Our formula $Z(t) = F(t) + G(t)$ and Schmidt's theorem (showing $\delta = 1$) gives

$$L(t) = (1-t)(1-qt)F(t) + \frac{h}{q-1}\left(q^g t^{2g-2}(1-t) - (1-qt)\right),$$

where we've expanded G but not F because it involves various $\ell(D)$ which are difficult to compute. It is some polynomial though, and we can evaluate L at 1 to get $L(1) = h$. Thus the class number is the sum of the coefficients!

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Proof (of c).
This follows easily from the functional equation for $Z(t)$, which we already established using the Riemann-Roch theorem:

$$Z(t) = q^{g-1}t^{2g-2}Z\left(\frac{1}{qt}\right).$$

We can compute

$$\begin{aligned}
q^g t^{2g} L\left(\frac{1}{qt}\right) &= q^g t^{2g}\left(1 - \frac{1}{qt}\right)\left(1 - \frac{1}{t}\right) Z\left(\frac{1}{qt}\right) \\
&= q^{g-1}t^{2g-2}(1-t)(1-qt)Z\left(\frac{1}{qt}\right) \\
&= (1-t)(1-qt)Z(t) \\
&:= L(t),
\end{aligned}$$

where we've distributed one q and two ts in the first steps.

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Proof (of d).
Using the functional equation from (c), we can write

$$L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right) = \left(\frac{a_{2g}}{q^g}\right) + \left(\frac{a_{2g-1}}{q^{g-1}}\right) t + \cdots + (a_0 q^g)\, t^{2g},$$

where we're correcting by enough in t but not enough in q and seeing what we get. Equating coefficients, for $0 \le j \le g$ we have

$$a_{2g-j} = q^{g-j} a_j. \tag{1}$$

Using the fact that A_0 is the number of effective degree zero divisors, which is only zero, we have $A_0 = 1$ and we can multiply formal power series to obtain

$$\begin{aligned} L(t) = a_0 + a_1 t + \cdots + a_{2g} t^{2g} &= (1-t)(1-qt) \sum_{n=0}^{\infty} A_n t^n \\ &= \left(1 - (q+1)t + qt^2\right)(1 + A_1 t + A_2 t^2 + \cdots) \\ &= 1 + (A_1 - (q+1))\, t + \cdots . \end{aligned}$$

From this, we can read off

- $L(0) = a_0 = 1$
- $a_1 = A_1 - (q+1) = \Sigma_1(K/k) - (q+1)$
- $a_{2g} = a_{2g-0} = q^{g-0} a_0 = a^g$ by taking $j = 0$ in eq. 1, and thus $\deg L = 2g$.

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Proof (of e (the most interesting!)).
Consider the **reciprocal polynomial**

$$L^{\perp}(t) \coloneqq t^{2g} L\left(\frac{1}{t}\right) = t^{2g} + a_1 t^{2g-1} + \cdots + q^g.$$

The original polynomial had $\mathbb{Z}$ coefficients and constant term 1, so this polynomial is monic and has a nonzero constant term. Thus its roots are patently nonzero algebraic integers in $\overline{\mathbb{Z}}^{\bullet}$.
If $L^{\perp}(t) = \prod_{j=1}^{2g} (t - \alpha_j)$, then

$$L(t) = t^{2g} L^{\perp}\left(\frac{1}{t}\right) = \prod_{j=1}^{2g} (1 - \alpha_j t)$$

and if the roots of $L(t)$ are r_j, then the roots of $L^{\perp}(t)$ are the reciprocal roots $1/r_j$ and vice-versa. This shows the first assertion that $r_j \in \overline{\mathbb{Z}}$ as well.
The most interesting part is what follows. Making the substitution $t = qu$ and using (c) we

get

$$\begin{aligned}
L^{\perp}(t) &= \prod_{j=1}^{2g}(t-\alpha_j) \\
&:= t^{2g} L\left(\frac{1}{t}\right) \\
&= q^{2g}u^{2g} L\left(\frac{1}{qu}\right) && \text{by (c).}
\end{aligned}$$

Using $u = t/q$, we can write

$$\begin{aligned}
q^g L(u) &= q^g \prod_{j=1}^{2g}(1-\alpha_j u) \\
&= q^g \prod_{j=1}^{2g}\left(1-\frac{\alpha_j}{q}t\right) \\
&= q^g \prod_{j=1}^{2g}\frac{\alpha_j}{q}\prod_{j=1}^{2g}\left(t-\frac{1}{\alpha_j}\right) \\
&= \prod_{j=1}^{2g}\left(t-\frac{q}{\alpha_j}\right),
\end{aligned}$$

where we've pulled out a factor of $-\alpha_j/q$ and in the last step we've used that $\prod_{j=1}^{2g}\alpha_j = q^g$. This follows because the α_j are the roots of $L^{\perp}$, which has even degree, so the product of all of the roots is equal to the constant term of $L^{\perp}$, which is the leading term of L, which we showed was q^g.
This says that if we take these roots α_j as a multiset and replace each α_j with q/α_j, we get the same multiset back. I.e., this multiset is stable under the involution

$$\begin{aligned}
\mathbb{C}^{\times} &\to \mathbb{C}^{\times} \\
z &\mapsto \frac{q}{z}.
\end{aligned}$$

This almost pairs up the elements of this finite set of roots, except it may have fixed points. The complex numbers α such that $\alpha = q/\alpha$ are precisely $\pm\sqrt{q}$

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