# Homework 6

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# 1 Homework Problems

## 1.1 Problem 1

Todo

## 1.2 Problem 2

We can note that since $f$ has 4 roots, the Galois group $G$ of its splitting field will be a subgroup of $S_4$. Moreover, $G$ must be a *transitive subgroup* of $S_4$, i.e. the action of $G$ on the roots of $f$ should be transitive. This reduces the possibilities to $G \cong S^4, A^4, D^4, \mathbb{Z}_4, \mathbb{Z}_2^2$.

Since $f$ has exactly 2 real roots and thus a pair of roots that are complex conjugates, the automorphism given by complex conjugation is an element of $G$. But this corresponds to a 2-cycle $\tau = (ab)$, and we can then make the following conclusions:

- Not $A_4$: $A_4$ contains only even cycles, and $\tau$ is odd.
- Not $Z_4$: This subgroup is generated by a single 4-cycle $\sigma$, which up to conjugacy is $(1234)$, and $\sigma^n$ is not a 2-cycle for any $n$.

- Not $\mathbb{Z}_2^2$: In order to be transitive, this subgroup must be $\{e, (12)(34), (13)(24), (14)(23)\}$, which does not contain $\tau$.

The only remaining possibilities are $S^4$ and $D^4$. $\square$

## 1.3 Problem 3

### 1.3.1 Part 1

To see that $\phi(n)$ is even for all $n > 2$, we can take a prime factorization of $n$ and write

$$\phi(n) = \phi\left(\prod_{i=1}^{m} p_i^{k_i}\right) = \prod_{i=1}^{m} \phi(p_i^{k_i}) = \prod_{i=1}^{m} p^{k_i-1}(p-1) = \prod_{i=1}^{m} p^{k_i-1} \prod_{i=1}^{m}(p-1)$$

where each $k_i \geq 1 \implies k_i - 1 \geq 0$. But every prime power is odd, and a product of odd numbers is odd, so the first product is odd. It is also true that $p-1$ is even for every prime $p$, and the second term is a product of even terms and thus even. So $\phi(n)$ is the product of an even and an odd number, which is always even.

### 1.3.2 Part 2

Suppose $\phi(n) = 2$. Take a prime factorization of $n$, so we have

$$2 = \phi(n) = \prod_{i=1}^{m} \phi(p_i^{k_i})$$

Since the only factors of 2 are 1 and 2, we must have $\phi(p_i^{k_i}) = 2$ for exactly one $i$, and the rest must be equal to 1.

Consider the term that equals 2. We have $\phi(p_i^{k_i}) = p^{k_i-1}(p-1) = 2$, so we must have either

- Case 1: $p - 1 = 2$ and $p^{k_i-1} = 1$, so $p = 3$ and $k_i = 1$. So $3 \mid n$, but $3^\ell$ does *not* divide $n$ for any $\ell > 1$.
- Case 2: $p^{k_i-1} = 2$ and $(p-1) = 1$, so $p = 2$ and $k_i = 2$. Thus $2^2$ divides $n$ but $2^\ell$ does not for any $\ell > 2$.

In either case, it remains to check are whether the other factors where $\phi(p_j^{k_j}) = 1$ can contribute any other distinct divisors to $n$. We can note that $\phi(p_j^{k_j}$ iff $p^{k_j-1}(p-1) = 1$, so this forces $p = 2$ and $k_j = 1$. So $n$ may or may not contain a single factor of 2, but by uniqueness of prime factorization, this can only happen in case 1. Note that this also forces $2 \mid n$ but $2^2$ does not divide $n$.

In summary, we've found that $\phi(n) = 2$ implies that

- $3 \mid n, 9 \nmid n$, and
  - $2 \mid n, 4 \nmid n$
  - $2 \nmid n$
- $2^2 \mid n, 2^3 \nmid n$.

This reduces the possibilities to the finite set $n \in \{6, 3, 4\}$, and $\phi(6) = \phi(3) = \phi(4) = 2$. $\square$

### 1.4 Problem 4

Suppose $F = K[\alpha_1, \cdots, \alpha_n]$ where $\alpha_1^{n_1} \in K$ for some $n_1$ and }or each $i$ we have $\alpha_i^{n_i} \in K[\alpha_1, \cdots, \alpha_{i-1}]$ for some powers $n_i$. We want to show that $F = E[\beta_1, \cdots \beta_m]$ where each $\beta_i$ satisfy a similar condition.

Let $A = \{\alpha_i \ni \alpha_i \notin E\}$, then it is since $E \hookrightarrow F$, adjoining all elements of $A$ to $E$ will yield exactly $F$.

### 1.5 Problem 5

### 1.6 Problem 6

## 2 Qual Problems

### 2.1 Problem 1

### 2.2 Problem 2

### 2.3 Problem 3