Title

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Thursday 27th August, 2020

Contents

1 Friday, August 21

Reference: Carter's "Finite Groups of Lie Type".
Reference: Humphrey's "Linear Algebraic Groups" (Springer)

1.1 Intro and Definitions

Definition 1.0.1 (Affine Variety)**.**
Let $k = \bar{k}$ be algebraically closed (e.g. $k = \mathbb{C}, \overline{\mathbb{F}_p}$). A variety $V \subseteq k^n$ is an *affine k-variety* iff V is the zero set of a collection of polynomials in $k[x_1, \cdots, x_n]$.

Here $\mathbb{A}^n := k^n$ with the Zariski topology, so the closed sets are varieties.

Definition 1.0.2 (Affine Algebraic Group)**.**
An *affine algebraic k-group* is an affine variety with the structure of a group, where the

multiplication and inversion maps

$$\mu : G \times G \longrightarrow G$$
$$\iota : G \longrightarrow G$$

are continuous.

Example 1.1.
$G = \mathbb{G}_a \subseteq k$ the *additive group* of k is defined as $\mathbb{G}_a \coloneqq (k, +)$. We then have a *coordinate ring* $k[\mathbb{G}_a] = k[x]/I = k[x]$.

Example 1.2.
$G = \mathrm{GL}(n, k)$, which has coordinate ring $k[x_{ij}, T]/\left\langle \det(x_{ij}) \cdot T = 1 \right\rangle$.

Example 1.3.
Setting $n = 1$ above, we have $\mathbb{G}_m \coloneqq \mathrm{GL}(1, k) = (k^\times, \cdot)$. Here the coordinate ring is $k[x, T]/\left\langle xT = 1 \right\rangle$.

Example 1.4.
$G = \mathrm{SL}(n, k) \leq \mathrm{GL}(n, k)$, which has coordinate ring $k[G] = k[x_{ij}]/\left\langle \det(x_{ij}) = 1 \right\rangle$.

Definition 1.0.3 (Irreducible)**.**
A variety V is *irreducible* iff V can not be written as $V = \cup_{i=1}^n V_i$ with each $V_i \subsetneq V$ a proper subvariety.

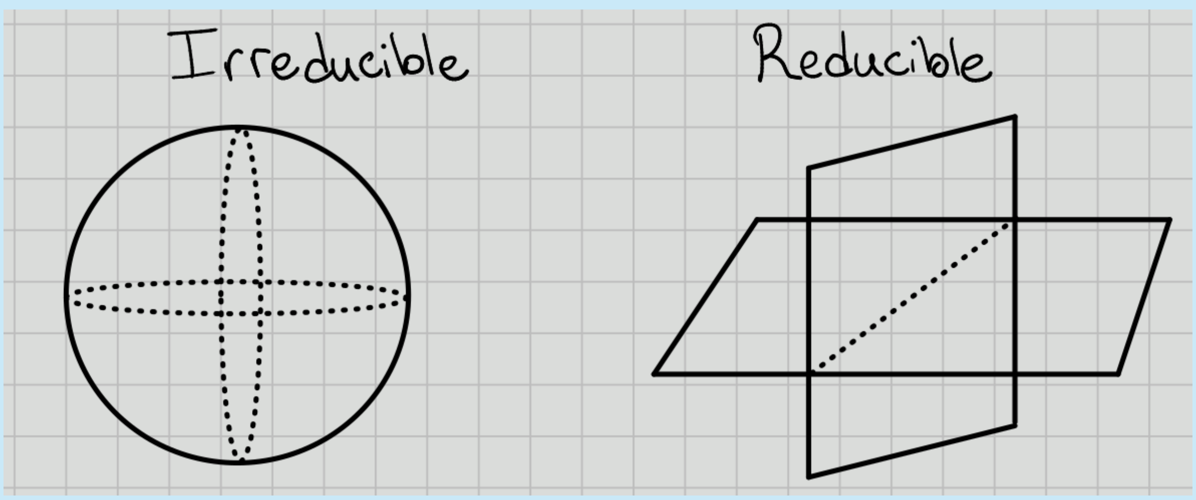

Figure 1: Reducible vs Irreducible

Proposition 1.1 ***(?).***
There exists a unique irreducible component of G containing the identity e. Notation: G^0.

Proposition 1.2 ***(?).***
G is the union of translates of G^0, i.e. there is a decomposition

$$G = \coprod_{g \in \Gamma} g \cdot G^0,$$

where we let G act on itself by left-translation and define Γ to be a set of representatives of distinct orbits.

Proposition 1.3 ***(?).***
One can define solvable and nilpotent algebraic groups in the same way as they are defined for finite groups, i.e. as having a terminating derived or lower central series respectively.

1.2 Jordan-Chevalley Decomposition

Proposition 1.4 ***(Existence and Uniqueness of Radical).***
There is a maximal connected normal solvable subgroup $R(G)$, denoted the *radical of* G.

- $\{e\} \subseteq R(G)$, so the radical exists.
- If $A, B \leq G$ are solvable then AB is again a solvable subgroup.

Definition 1.4.1 (Unipotent).
An element u is *unipotent* $\iff$ $u = 1 + n$ where n is nilpotent $\iff$ its the only eigenvalue is $\lambda = 1$.

Proposition 1.5 ***(JC Decomposition).***
For any G, there exists a closed embedding $G \hookrightarrow \mathrm{GL}(V) = \mathrm{GL}(n, k)$ and for each $x \in G$ a unique decomposition $x = su$ where s is semisimple (diagonalizable) and u is unipotent.

Define $R_u(G)$ to be the subgroup of unipotent elements in $R(G)$. :::{.definition title="Semisimple and Reductive"} Suppose G is connected, so $G = G^0$, and nontrivial, so $G \neq \{e\}$. Then

- G is semisimple iff $R(G) = \{e\}$.
- G is reductive iff $R_u(G) = \{e\}$. :::

Example 1.5.
$G = \mathrm{GL}(n, k)$, then $R(G) = Z(G) = kI$ the scalar matrices, and $R_u(G) = \{e\}$. So G is reductive and semisimple.

Example 1.6.
$G = \mathrm{SL}(n, k)$, then $R(G) = \{I\}$.

Exercise 1.1.
Is this semisimple? Reductive? What is $R_u(G)$?

Definition 1.5.1 (Torus)**.**
A *torus* $T \subseteq G$ in G an algebraic group is a commutative algebraic subgroup consisting of semisimple elements.

Example 1.7.
Let

$$T := \left\langle \begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_n \end{bmatrix} \subseteq \mathrm{GL}(n, k) \right\rangle.$$

Remark 1.
Why are torii useful? For $g = \mathrm{Lie}(G)$, we obtain a root space decomposition

$$g = \left(\bigoplus_{\alpha \in \Phi_-} g_\alpha \right) \oplus t \oplus \left(\bigoplus_{\alpha \in \Phi_+} g_\alpha \right).$$

When G is a simple algebraic group, there is a classification/correspondence:

$$(G, T) \iff (\Phi, W).$$

where Φ is an irreducible root system and W is a Weyl group.

2 Monday, August 24

2.1 Review and General Setup

- $k = \bar{k}$ is algebraically closed
- G is a reductive algebraic group
- $T \subseteq G$ is a *maximal split torus*

Split: $T \cong \bigoplus \mathbb{G}_m$.

We'll associate to this a root system, not necessarily irreducible, yielding a correspondence

$$(G, T) \iff (\Phi, W)$$

with W a Weyl group.

This will be accomplished by looking at $\mathfrak{g} = \mathrm{Lie}(G)$. If G is simple, then $\mathfrak{g}$ is "simple", and Φ irreducible will correspond to a Dynkin diagram.

There is this a 1-to-1 correspondence

$$G \text{ simple}/\sim \iff A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$$

where $\sim$ denotes *isogeny*.

Taking the Zariski tangent space at the identity "linearizes" an algebraic group, yielding a Lie algebra.

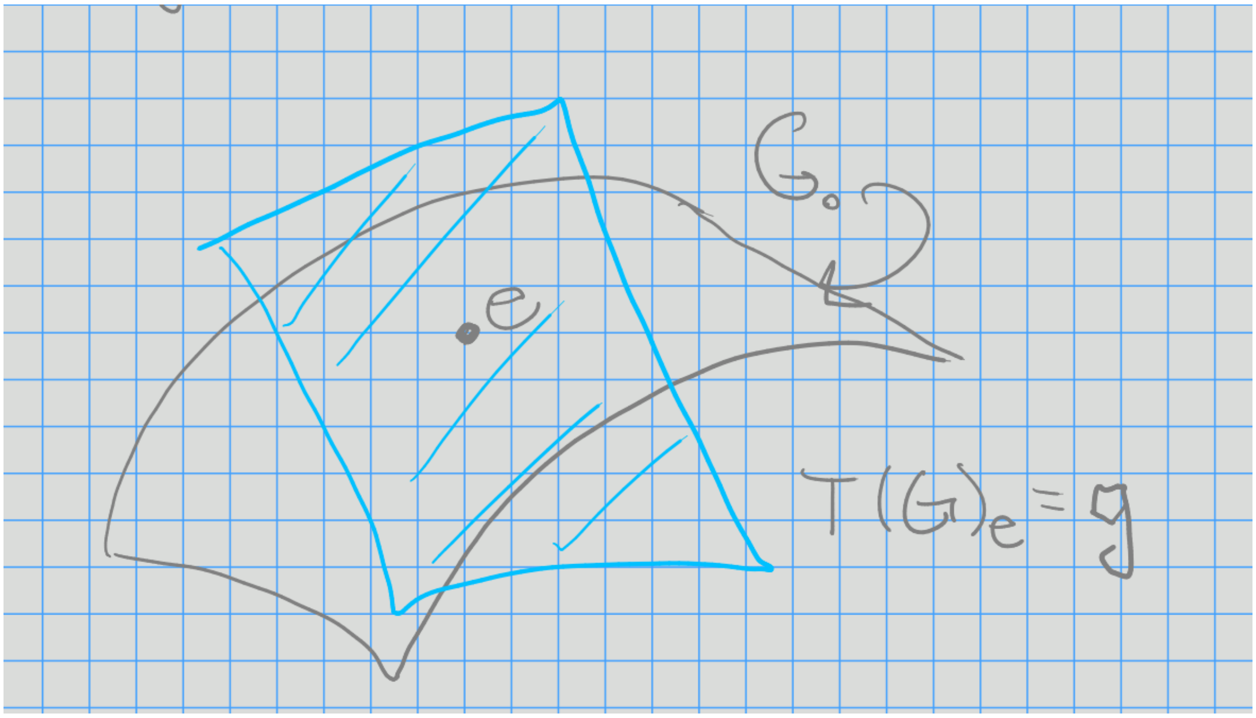

Figure 2: Image

We have the coordinate ring $k[G] = k[x_1, \cdots, x_n]/\mathcal{I}(G)$ where $\mathcal{I}(G)$ is the zero set. This is equal to $\{f : G \longrightarrow k\}$,

2.2 The Associated Lie Algebra

Definition 2.0.1 (The Lie Algebra of an Algebraic Group)**.**
Define *left translation* is

$$\begin{aligned}\lambda_x : k[G] &\longrightarrow k[G] \\ y &\mapsto f(x^{-1}y).\end{aligned}$$

Define *derivations* as

$$\text{Der } k[G] = \left\{ D : k[G] \longrightarrow k[G] \;\middle|\; D(fg) = D(f)g + fD(g) \right\}.$$

We can then realize the Lie algebra as

$$\mathfrak{g} = \text{Lie}(G) = \left\{ D \in \text{Der} k[G] \;\middle|\; \lambda_x \circ D = D \circ \lambda_x \right\},$$

the left-invariant derivations.

Example 2.1.

- $G = \mathrm{GL}(n,k) \implies \mathfrak{g} = \mathfrak{gl}(n,k)$
- $G = \mathrm{SL}(n,k) \implies \mathfrak{g} = \mathfrak{sl}(n,k)$

Let G be reductive and T be a split torus. Then T acts on $\mathfrak{g}$ via an *adjoint action.* (For $\mathrm{GL}_n, \mathrm{SL}_n$, this is conjugation.)

There is a decomposition into eigenspaces for the action of T,

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Phi} g_\alpha \right) \oplus t$$

where $t = \mathrm{Lie}(T)$ and $g_\alpha \coloneqq \left\{ x \in \mathfrak{g} \;\middle|\; t.x = \alpha(t)x \ \forall t \in T \right\}$ with $\alpha : T \longrightarrow K^\times$ a rational function (a *root*).

In general, take $\alpha \in \hom_{\mathrm{AlgGrp}}(T, \mathbb{G}_m)$.

Example 2.2.
Let $G = \mathrm{GL}(n,k)$ and

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \;\middle|\; a_j \in k^\times \right\}.$$

Then check the following action:

$$t.\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1^{-1} & & 0 \\ & \ddots & \\ 0 & & a_n^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_1 a_2^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= a_1 a_2^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 3: Action

which indeed acts by a rational function.

Then

$$g_\alpha = \operatorname{span}\left\{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\} = g_{(1,-1,0)}.$$

For $\mathfrak{g} = \mathfrak{gl}(3,k)$, we have

$$\begin{aligned}\mathfrak{g} = t &\oplus g_{(1,-1,0)} \oplus g_{(-1,1,0)}\\ &\oplus g_{(0,1,-1)} \oplus g_{(0,-1,1)}\\ &\oplus g_{(1,0,-1)} \oplus g_{(-1,0,1)}.\end{aligned}$$

2.3 Representations

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a group homomorphisms, then equivalently V is a (rational) G-module.

For $T \subseteq G$, $T \curvearrowright G$ semisimply, so we can simultaneously diagonalize these operators to obtain a *weight space decomposition* $V = \bigoplus_{\lambda \in X(T)} V_\lambda$, where

$$\begin{aligned}V_\lambda &:= \left\{v \in V \;\middle|\; t.v = \lambda(t)v \;\forall t \in T\right\}\\ X(T) &:= \hom(T, \mathbb{G}_m).\end{aligned}$$

Example 2.3.
Let $G = \mathrm{GL}(n,k)$ and V the n-dimensional natural representation as column vectors,

$$V = \left\{[v_1, \cdots, v_n] \;\middle|\; v_j \in k\right\}.$$

Then

$$T = \left\{\begin{bmatrix}a_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & a_n\end{bmatrix} \;\middle|\; a_j \in k^\times\right\}.$$

Consider the basis vectors $\mathbf{e}_j$, then

$$\begin{bmatrix}a_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & a_n\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = a_j \begin{bmatrix}0\\1\\0\end{bmatrix} = a_1^0 a_2^0 \cdots a_j^0 \cdots a_n^0 \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Here the weights are of the form $\varepsilon_j := [0, 0, \cdots, 1, \cdots, 0]$ with a 1 in the jth spot, so we have

$$V = V_{\varepsilon_1} \oplus V_{\varepsilon_2} \oplus \cdots \oplus V_{\varepsilon_n}.$$

Example 2.4.
For $V = \mathbb{C}$, we have $t.v = (a_1^0 \cdots a_n^0)v$ and $V = V_{(0,0,\cdots,0)}$.

2.4 Classification

Let G be a simple algebraic group (ano closed, connected, normal subgroups other than $\{e\}, G$) that is nonabelian that is nonabelian.

Example 2.5.
Let $G = \mathrm{SL}(3, k)$. Then

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2^{-1} & 0 \\ 0 & 0 & a_2^{-1} \end{bmatrix} \;\middle|\; a_1, a_2 \in k^\times \right\}$$

and

$$t. \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = a_1^2 a_2^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and $\alpha_1 = (2, -1)$.

What is α_1? Note that you can recover the Cartan something here?

Then

$$\mathfrak{g} = \mathfrak{g}_{(2,-1)} \oplus \mathfrak{g}_{(-2,1)} \oplus \mathfrak{g}_{(-1,2)} \oplus \mathfrak{g}_{(1,-2)} \oplus \mathfrak{g}_{(1,1)} \oplus \mathfrak{g}_{(-1,-1)}.$$

Then $\alpha_2 = (-1, 2)$ and $\alpha_1 + \alpha_2 = (1, 1)$.

This gives the root space decomposition for $\mathfrak{sl}_3$:

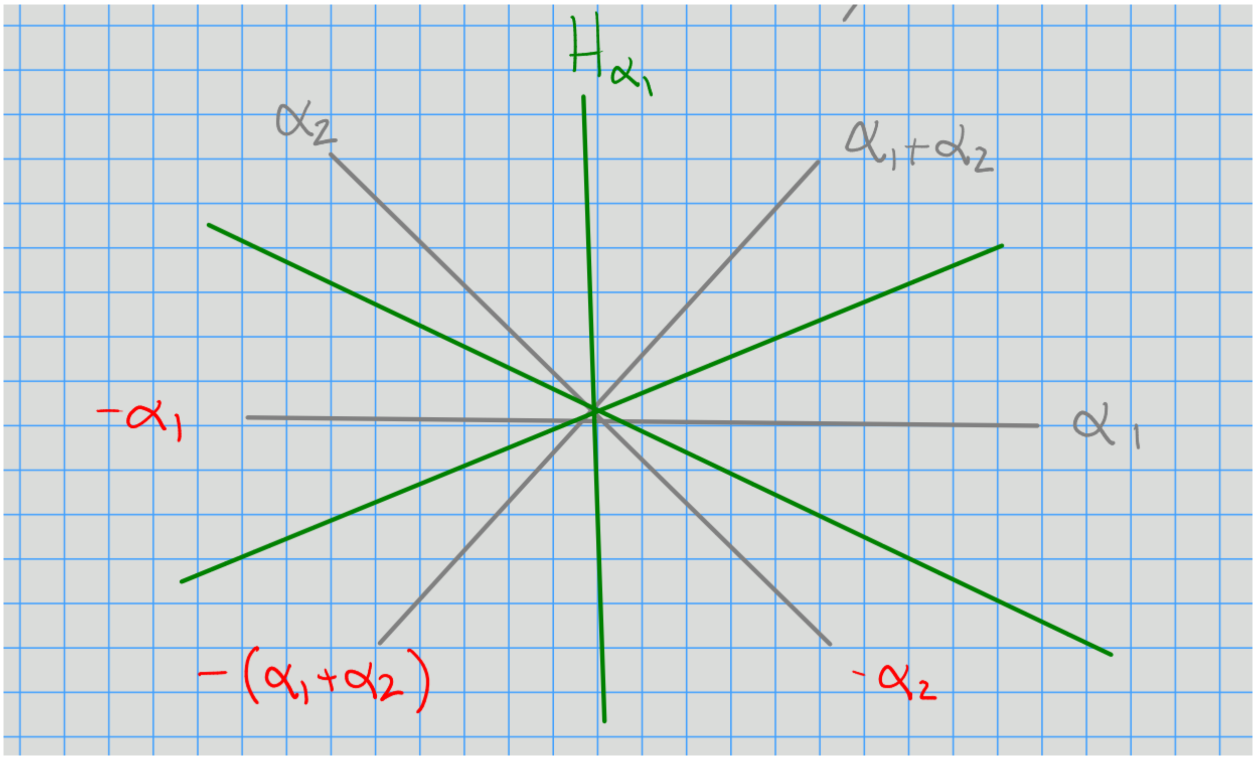

Figure 4: Image

Then the Weyl group will be generated by reflections through these hyperplanes.

3 Wednesday, August 26

3.1 Review

- G a reductive algebraic group over k
- $T = \prod_{i=1}^{n} \mathbb{G}_m$ a maximal split torus
- $\mathfrak{g} = \mathrm{Lie}(G)$
- There's an induced root space decomposition $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$
- When G is simple, Φ is an *irreducible* root system
  - There is a classification of these by Dynkin diagrams

Example 3.1.
A_n corresponds to $\mathfrak{sl}(n+1, k)$ (mnemonic: A_1 corresponds to $\mathfrak{sl}(2)$)

- We have representations $\rho : G \longrightarrow \mathrm{GL}(V)$, i.e. V is a G-module
- For $T \subseteq G$, we have a weight space decomposition: $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ where $X(T) = \hom(T, \mathbb{G}_m)$.

Note that $X(T) \cong \mathbb{Z}^n$, the number of copies of $\mathbb{G}_m$ in T.

3.2 Root Systems and Weights

Example 3.2.
Let $\Phi = A_2$, then we have the following root system:

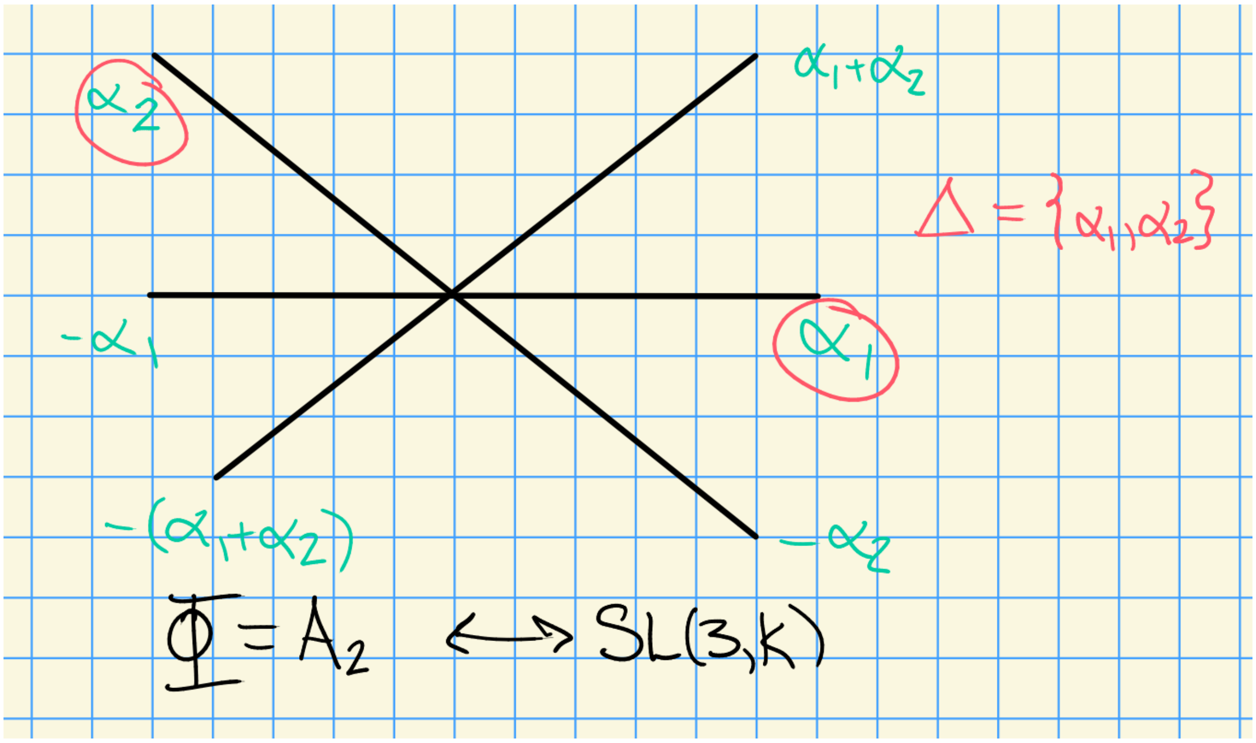

Figure 5: Image

In general, we'll have $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a basis of *simple roots*.

Remark 2.
Every root $\alpha \in I$ can be expressed as either positive integer linear combination (or negative) of simple roots.

For any $\alpha \in \Phi$, let s_α be the reflection across H_α, the hyperplane orthogonal to α. Then define the *Weyl group* $W = \left\{ s_\alpha \mid \alpha \in \Phi \right\}$.

Example 3.3.
Here the Weyl group is S_3:

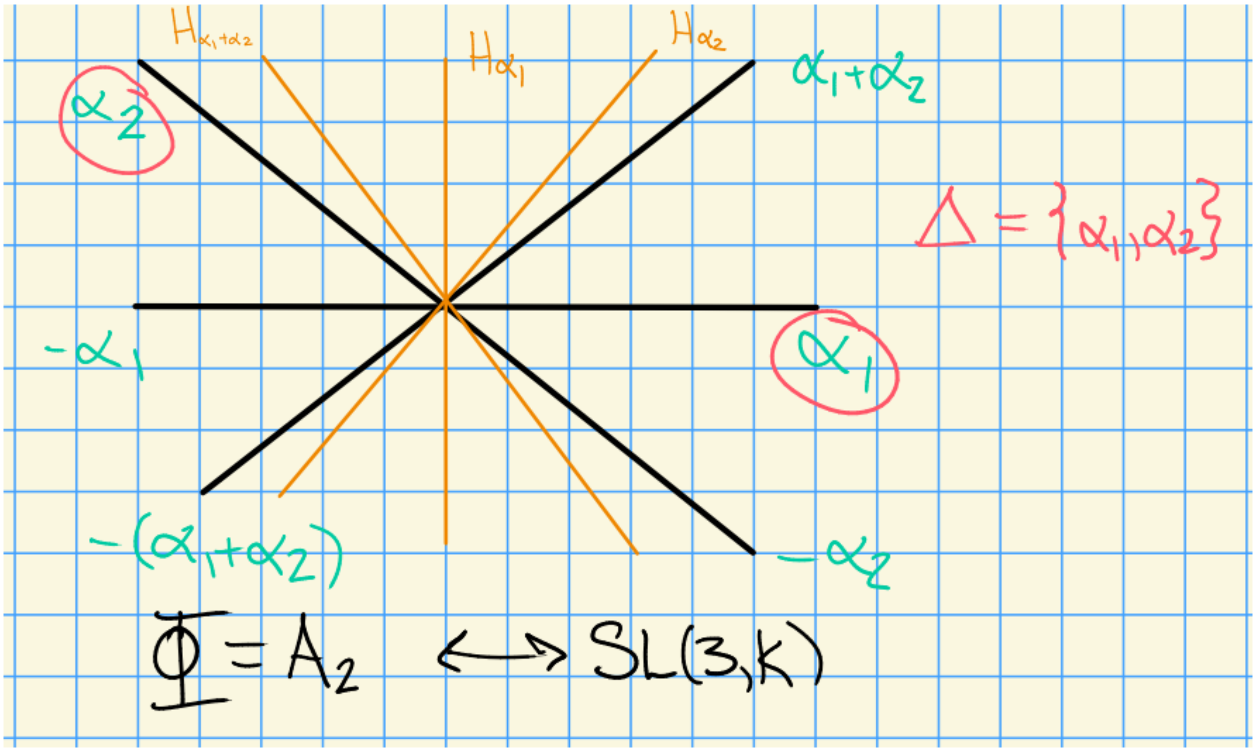

Figure 6: Image

Remark 3.
W acts transitively on bases.

Remark 4.
$X(T) \subseteq \mathbb{Z}\Phi$, recalling that $X(T) = \hom(T, \mathbb{G}_m) = \mathbb{Z}^n$ for some n. Denote $\mathbb{Z}\Phi$ the *root lattice* and $X(T)$ the *weight lattice*.

Example 3.4.
Let $G = \mathfrak{sl}(2, \mathbb{C})$ then $X(T) = \mathbb{Z}\omega$ where $\omega = 1$, $\mathbb{Z}\Phi = \mathbb{Z}\{\alpha\}$ Then there is one weight α, and the root lattice $\mathbb{Z}\Phi$ is just $2\mathbb{Z}$. However, the weight lattice is $\mathbb{Z}\omega = \mathbb{Z}$, and these are not equal in general.

Remark 5.
There is partial ordering on $X(T)$ given by $\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha \in \Delta} n_\alpha \alpha$ where $n_\alpha \geq 0$. (We say λ *dominates* μ.)

Definition 3.0.1 (Fundamental Dominant Weights)**.**
We extend scalars for the weight lattice to obtain $E := X(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, a Euclidean space with an inner product $\langle \cdot, \ \cdot \rangle$.
For $\alpha \in \Phi$, define its *coroot* $\alpha^\vee := \dfrac{2\alpha}{\langle \alpha, \ \alpha \rangle}$. Define the *simple coroots* as $\Delta^\vee := \{\alpha_i^\vee\}_{i=1}^n$, which has a dual basis $\Omega := \{\omega_i\}_{i=1}^n$ the *fundamental weights*. These satisfy $\left\langle \omega_i, \ \alpha_j^\vee \right\rangle = \delta_{ij}$.

What is the notation for fundamental weights? Definitely not Ω usually!

Important because we can index irreducible representations by fundamental weights.

A weight $\lambda \in X(T)$ is *dominant* iff $\lambda \in \mathbb{Z}^{\geq 0}\Omega$, i.e. $\lambda = \sum n_i \omega_i$ with $n_i \in \mathbb{Z}^{\geq 0}$.

If G is simply connected, then $X(T) = \bigoplus \mathbb{Z}\omega_i$.

See Jantzen for definition of simply-connected, $\mathrm{SL}(n+1)$ is simply connected but its adjoint $PGL(n+1)$ is not simply connected.

3.3 Complex Semisimple Lie Algebras

When doing representation theory, we look at the Verma modules $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda \twoheadrightarrow L(\lambda)$.

Theorem 3.1 *(?)***.**
$L(\lambda)$ as a finite-dimensional $U(\mathfrak{g})$-module $\iff$ λ is dominant, i.e. $\lambda \in X(T)_+$.

Thus the representations are indexed by lattice points in a particular region:

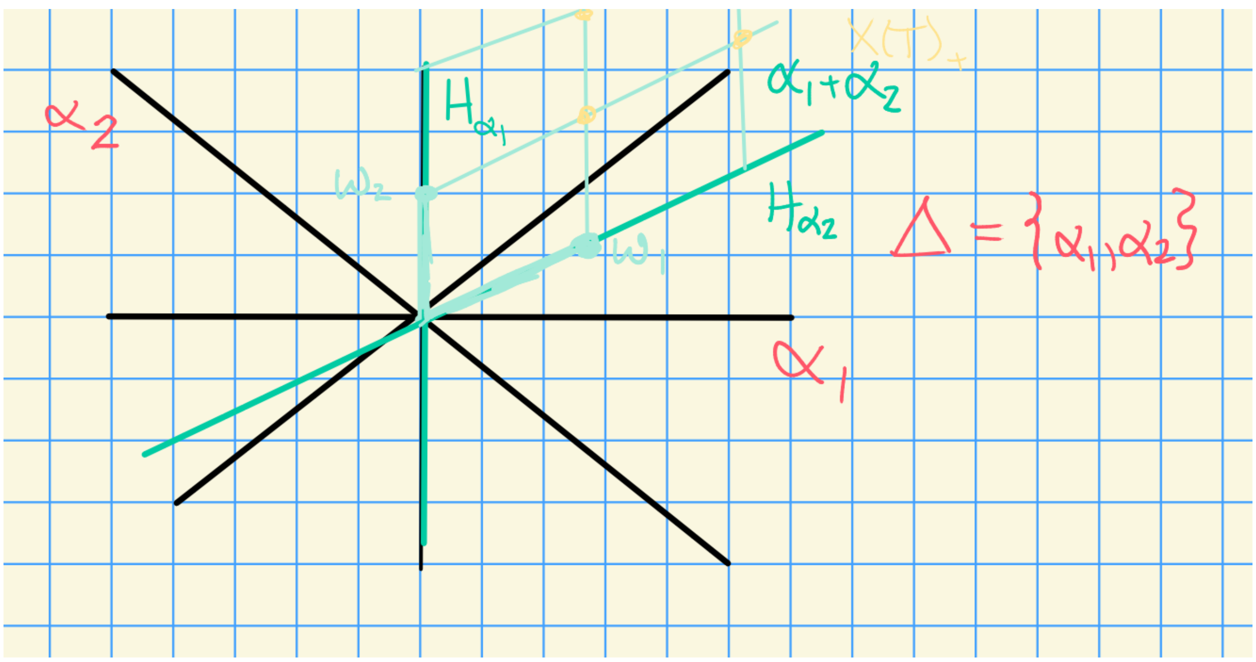

Figure 7: Image

Question 1:

Suppose G is a simple (simply connected) algebraic group. How do you parameterize *irreducible* representations?

For $\rho : G$
$to\mathrm{GL}(V)$, V is a *simple module* (an *irreducible representation*) iff the only proper G-submodules of V are trivial.

Answer 1: They are also parameterized by $X(T)_+$. We'll show this using the induction functor $\mathrm{Ind}_B^G \lambda = H^0(G/B, \mathcal{L}(\lambda))$ (sheaf cohomology of the flag variety with coefficients in some line bundle).

> We'll define what B is later, essentially upper-triangular matrices.

Question 2: What are the dimensions of the irreducible representations for G?

Answer 2: Over $k = \mathbb{C}$ using Weyl's dimension formula.

For $k = \overline{\mathbb{F}_p}$: conjectured to be known for $p \geq h$ (the *Coxeter number*), but by Williamson (2013) there are counterexamples. Current work being done!