Title

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1 Chapter 1

1.1 Within Chapter

Nice mnemonic: Maximal $\implies$ prime $\implies$ radical Field $\implies$ domain $\implies$ reduced

Proposition 1.1: Fix an ideal $\mathfrak{a} \trianglelefteq R$. There is a correspondence

$$\left\{ \mathfrak{b} \mathrel{\Big|} \mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R \right\} \iff \left\{ \tilde{\mathfrak{b}} \trianglelefteq R/\mathfrak{a} \right\}.$$

Proof: Adapted from proof for groups here: `https://math.stackexchange.com/a/955413/147053`.

Let $f : R \to T$ be any ring homomorphism and let $S(R), S(T)$ denote the lattices of subrings of R, T respectively. Then f induces two maps:

$$\begin{aligned} F : S(R) &\to S(T) \\ H &\mapsto f(H) \end{aligned}$$

$$\begin{aligned} F^{-1} : S(T) &\to S(R) \\ K &\mapsto f^{-1}(K). \end{aligned}$$

It follows that

- $H \leq R \implies F(H) \leq \operatorname{im} f$, by the subring test
  - Subring test: contains 1, closed under multiplication/subtraction.
  - Properties of ring homomorphisms: $f(sa+b) = sf(a) + f(b)$ and $f(1) = 1$.

- Thus if f is not surjective, F is not surjective either.
- $K \leq T \implies \ker f \subseteq F^{-1}(K)$.
  - Follows because subrings contain 0, and $H \in \ker F \implies f(H) = 0_T \in K$.
  - Thus if there is any subring H that *doesn't* contain $\ker f$, F^{-1} is not surjective.

The claim is that if you restrict to

- $S'(R) \coloneqq \left\{ H \leq R \;\middle|\; \ker f \subseteq H \right\}$ and
- $S'(T) \coloneqq \left\{ K \leq T \;\middle|\; K \subseteq \text{im } f \right\}$,

this is a bijection.

This follows from the fact that

- $(F \circ F^{-1})(K) = K \bigcap \text{im } f \leq T$
  - No clear motivation for why it's *this* specific thing, but the inclusions are easy to check.
- $(F^{-1} \circ F)(H) = \langle H, \ker f \rangle \leq S$.
  - Inclusions easy to check, need to take subring generated since $F(H)$ is a pushforward/direct image, which don't preserve sub-structures in general.

So we take the projection $f = \pi : R \to R/\mathfrak{a}$, then

- $K \subseteq \text{im } \pi \implies K \bigcap \text{im } \pi = K \implies (F \circ F^{-1})(K) = K$,
- $\ker \pi \subseteq H \implies \langle H, \ker \pi \rangle = H \implies (F^{-1} \circ F)(H) = H$,

so both directions are surjections. Restricting to just those subrings that are ideals preserves this bijection. Moreover, $\ker \pi = \mathfrak{a}$ so $S'(R)$ is the set of ideals containing $\mathfrak{a}$, and $\text{im } \pi = R/\mathfrak{a}$, so $S'(T)$ is the set of ideals of the quotient.

$\blacksquare$

Proposition 1.2: TFAE

1. R is a field
2. R is simple, i.e. the only ideals of R are $0, R$.
3. Every nonzero homomorphism $\phi : R \to S$ for S an arbitrary ring is injective.

Proof:

Lemma: $I \trianglelefteq R$ and $1 \in I \implies I = R$. This is because $RI \subseteq I$, and $r \in R \implies r \cdot 1 \in I \implies r \in I \implies R \subseteq I$.

$1 \implies 2$:

Let $0 \neq I \trianglelefteq R$ for R a field, then pick any $x \in I$, since $x^{-1} \in R$, we have $x^{-1}x = 1 \in I \implies I = R$.

$\not{1} \implies \not{2}$:

If R is not a field, pick a non-unit element r; then $(r) \trianglelefteq R$ is a proper ideal.

$2 \implies 3$:

$\ker \phi \trianglelefteq R$ is an ideal, so $\ker \phi = 0$.

$3 \implies 2$:

Take $\mathfrak{a} \ntrianglelefteq R$ a proper ideal and let $S = R/\mathfrak{a}$ with $\phi : R \to S$ the projection. ϕ is a bijection, since it's always a surjection and assumed injective. So $R \cong S = R/\mathfrak{a}$, forcing $\mathfrak{a} = (0)$.

■

Proposition: If $\mathfrak{m} \trianglelefteq R$ is maximal iff $R/\mathfrak{m}$ is a field.

Proof:

$R/\mathfrak{m}$ is a field $\iff$ $R/\mathfrak{m}$ is simple $\iff$ there are no nontrivial ideals $\mathfrak{a}$ such that $\mathfrak{m} \subset \mathfrak{a}$ (correspondence) $\iff$ $\mathfrak{m}$ is maximal.

Proposition: $\mathfrak{p} \trianglelefteq R$ is prime iff $R/\mathfrak{p}$ is a domain.

Proof:

$\implies$:

WLOG, $(x + \mathfrak{p})(y + \mathfrak{p}) = xy + \mathfrak{p} = 0 \iff xy \in \mathfrak{p} \iff x \in \mathfrak{p} \iff (x + \mathfrak{p}) = 0$.

$\impliedby$:

WLOG, $xy \in \mathfrak{p} \implies (x + \mathfrak{p})(y + \mathfrak{p}) = 0 \implies x + \mathfrak{p} = 0 \implies x \in \mathfrak{p}$.

Proposition: Maximal ideals are prime.

Proof: Let $\mathfrak{m} \trianglelefteq A$ be maximal, then $R/\mathfrak{m}$ is simple and thus a field, so $\mathfrak{m}$ is prime.

Proposition: Prime does not imply maximal in general.

Proof: Take $(0) \in \mathbb{Z}$, then $ab = 0 \implies a = 0$ or $b = 0$, so this is prime. It is not maximal, because $(0) \in (n)$ for any n.

Theorem 1.3: Every ring R has a nontrivial nonzero maximal ideal, and every ideal is contained in a maximal ideal.

Proof: Take the sublattice of the ideal lattice given by proper ideals; every chain has an upper bound given by union, so apply Zorn's lemma. Similarly, for a fixed $\mathfrak{a}$, take the sublattice of ideals containing $\mathfrak{a}$.

Corollary 1.5: Every non-unit of R is contained in a maximal ideal.

Proof: ?

Proposition 1.6: If $A \setminus \mathfrak{m} \subset R^\times$, then A is a local ring with $\mathfrak{m}$ its maximal ideal. If $\mathfrak{m}$ is maximal and $1 + m \in R^\times$ for all $m \in \mathfrak{m}$, then A is a local ring.

Proof: ?

Proposition: If $f \in k[x_1, \cdots x_n]$ is irreducible over k, then (f) is prime.

Proposition: $\mathbb{Z}$ is a PID, and (p) is prime iff p is zero or a prime number, and every such ideal is maximal.

Proposition: $k[\{x_i\}]$ has maximal ideals that are not principal iff $n > 1$.

Exercise: Characterize the maximal and prime ideals of $k[x_1, \cdots, x_n]$? Is this a field, domain, PID, UFD, a local ring, …?

Proposition: Every nonzero prime ideal in a PID is maximal.

Proof: ?

Definition: The set $\mathrm{nil}(A)$ of all nilpotent elements in a ring A is the nilradical of A. The set $J(A) = \bigcap_{\mathfrak{m} \in \mathrm{Spec}_{\max}(A)} \mathfrak{m}$ is the Jacobson radical.,

Proposition 1.7: $\mathrm{nil}(A) \trianglelefteq R$ is an ideal and $A/\mathfrak{R}$ has no nonzero nilpotent elements.

Proof: ?

Proposition 1.8: $\mathrm{nil}(A) = \bigcap \mathfrak{p} \in \mathrm{Spec}(A)\mathfrak{p}$ is the intersection of all prime ideals of A.

Proof: ?

Proposition 1.9: $x \in J(A)$ iff $1 - xa \in A^{\times}$ for all $a \in A$.

Proposition: If $(m), (n) \trianglelefteq \mathbb{Z}$ then $(m) \bigcap (n) = (\gcd(m, n))$ and $(m)(n) = (mn)$.

Exercise: If $\mathfrak{a} \trianglelefteq k[x_1, \cdots, x_m]$, characterize $\mathfrak{a}^n$.

Exercise: Show that $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$ are coprime iff there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$.

Proposition 1.10: Let $\{mfa_i\} \trianglelefteq A$ be a family of ideals and define $\phi : A \to \prod A/\mathfrak{a}_i$.

1. If $\{\mathfrak{a}_i\}$ are pairwise coprime, then $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$
2. ϕ is surjective iff $\{\mathfrak{a}_i\}$ are pairwise coprime.
3. ϕ is injective iff $\bigcap \mathfrak{a}_i = (0)$.

Exercise: Show that the union of ideals is not necessarily an ideal.

Proposition 1.11:

a. Let $\{\mathfrak{p}_i\}$ be a set of prime ideals and let $\mathfrak{a} \in \bigcup \mathfrak{p}$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

b. Let $\{\mathfrak{a}_i\}$ be ideals and $\mathfrak{p} \supseteq \bigcap \mathfrak{a}_i$ be prime. $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i, and if $\mathfrak{p} = \bigcap \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

Exercise: Let $A = \mathbb{Z}$, and characterize the ideal quotient $(m : n)$.

Exercise 1.12:

1. $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$
2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$
3. $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
4. $(\bigcap \mathfrak{a}_i : \mathfrak{b}) = \bigcap (\mathfrak{a}_i : \mathfrak{b})$
5. $(\mathfrak{a} : \sum \mathfrak{b}_i) = \bigcap (\mathfrak{a} : \mathfrak{b}_i)$

Proposition: For $\mathfrak{a} \trianglelefteq A$, $\sqrt{\mathfrak{a}}$ is an ideal.

Exercise 1.13:

1. $\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$
3. $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \bigcap \mathfrak{b}} = \sqrt{\mathfrak{a}} \bigcap \sqrt{\mathfrak{b}}$
4. $\sqrt{\mathfrak{a}} = (1) \iff \mathfrak{a} = (1)$
5. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

6. For $\mathfrak{p}$ prime, $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n \geq 1$.

Proposition 1.14: $\sqrt{\mathfrak{a}} = \bigcap\limits_{\mathfrak{a} \subseteq \mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$

Proposition 1.15: Let D be the set of zero-divisors in A. Then $D = \bigcup\limits_{x \neq 0} \sqrt{\operatorname{Ann}(x)}$.

Exercise: Let $(m) \trianglelefteq \mathbb{Z}$ where $m = \prod p_i^{k_i}$, and show that $\sqrt{(m)} = (p_1 p_2 \cdots) = \bigcap (p_i)$.

Proposition 1.16: If $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$ are coprime then $\mathfrak{a}, \mathfrak{b}$ are coprime.

Exercise: Show that if $f : A \to B$ and $\mathfrak{a} \trianglelefteq A$, it is not necessarily the case that $f(\mathfrak{a}) \trianglelefteq B$.

Exercise: Show that if $\mathfrak{b}$ is prime then $A \cdot f^{-1}(\mathfrak{b})$ is prime, but if $\mathfrak{a}$ is prime then $B \cdot f(\mathfrak{a})$ need not be prime.

Exercise: Write $\mathfrak{a}^e \coloneqq \langle f(\mathfrak{a}) \rangle$ and $\mathfrak{b}^c = \left\langle f^{-1}(\mathfrak{b}) \right\rangle$. Let $f : \mathbb{Z} \to \mathbb{Z}[i]$ be the inclusion, and show that

- $(2)^e = \left\langle (1+i)^2 \right\rangle$, which is not prime in $\mathbb{Z}[i]$
- (Nontrivial) If $p = 1 \mod 4$, then $\mathfrak{p}^e$ is the product of two distinct prime ideals
- If $p = 3 \mod 4$ then $\mathfrak{p}^e$ is prime.

Proposition: Let $C = \left\{ \mathfrak{b}^c \;\middle|\; \mathfrak{b} \trianglelefteq B \right\}$ and $E = \left\{ \mathfrak{a}^e \;\middle|\; \mathfrak{a} \trianglelefteq A \right\}$. Then

1. $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ and $\mathfrak{b} \supset \mathfrak{b}^{ce}$,
2. $\mathfrak{b}^c = \mathfrak{b}^{cec}$ and $\mathfrak{a}^e = \mathfrak{a}^{ece}$
3. $C = \left\{ \mathfrak{a} \trianglelefteq A \;\middle|\; \mathfrak{a}^{ec} = \mathfrak{a} \right\}$ and $E = \left\{ \mathfrak{b} \trianglelefteq B \;\middle|\; \mathfrak{b}^{ce} = \mathfrak{b} \right\}$.
4. The map $\phi : C \to E$ given by $\phi(\mathfrak{a}) = \mathfrak{a}^{ec}$ is a bijection with inverse $\mathfrak{b} \mapsto \mathfrak{b}^c$.
5. If $\mathfrak{a} \in C$ then $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b}^{cec} = \mathfrak{a}^{ec}$, and if $\mathfrak{a} = \mathfrak{a}^{ec}$ then $\mathfrak{a}$ is the contraction of $\mathfrak{a}^e$.

Exercise 1.18:

$$
\begin{array}{ll}
(\mathfrak{a}_1 + \mathfrak{a}_2)^{\mathfrak{e}} = \mathfrak{a}_1^{\mathfrak{e}} + \mathfrak{a}_2^{\mathfrak{e}}, & (\mathfrak{b}_1 + \mathfrak{b}_2)^c \geq \mathfrak{b}_1^{\mathfrak{c}} + \mathfrak{b}_2^{\mathfrak{c}} \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^{\mathfrak{e}} \cap \mathfrak{a}_2^{e}, & (\mathfrak{b}_1 \cap \mathfrak{b}_2)^{\mathfrak{c}} = \mathfrak{b}_1^{\mathfrak{c}} \cap \mathfrak{b}_{\mathfrak{z}}^{\mathfrak{c}} \\
(\mathfrak{a}_1 \mathfrak{a}_2)^{\mathfrak{e}} = \mathfrak{a}_1^{\mathfrak{e}} \mathfrak{a}_2^{\mathfrak{e}}, & (\mathfrak{b}_1 \mathfrak{b}_2)^{\mathfrak{c}} \supseteq \mathfrak{b}_1^{\mathfrak{c}} \mathfrak{b}_2^{\mathfrak{c}} \\
(\mathfrak{a}_1 : \mathfrak{a}_2)^{\mathfrak{e}} \subseteq (\mathfrak{a}_1^{\mathfrak{e}} : \mathfrak{a}_2^{\mathfrak{e}}), & (\mathfrak{b}_1 : \mathfrak{b}_2)^{\mathfrak{c}} \subseteq (\mathfrak{b}_1^{\mathfrak{c}} : \mathfrak{b}_2^{\mathfrak{c}}) \\
r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e), & r(\mathfrak{b})^c = r(\mathfrak{b}^c)
\end{array}
.
$$

1.2 End of Chapter Exercises