

# Title

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References: <https://www.daniellitt.com/etale-cohomology>

Prerequisites:

- Homological Algebra
  - Abelian Categories
  - Derived Functors
  - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
  - Etale morphism
  - Grothendieck topologies
  - The etale topology
  - Etale cohomology and the basis theorems
  - Etale cohomology of curves
  - Comparison theorems to singular cohomology
  - Focused on the case where coefficients are a constructible sheaf.
- Prove the Weil Conjectures (more than one proof)
  - Proving the Riemann Hypothesis for varieties over finite fields
- Topics
  - Weil 2 (Strengthening of RH, used in practice)
  - Formality of algebraic varieties (topological features unique to varieties)
  - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose  $X/\mathbb{C}$  is a quasiprojective variety: a finite type separated integral  $\mathbb{C}$ -scheme.

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If you take the complex points, it naturally has the structure of a complex analytic space  $X(\mathbb{C})^{\text{an}}$ : you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology  $H^i(X(\mathbb{C})^{\text{an}}, \mathbb{Z})$ , which satisfies several nice properties:

- Finitely generated  $\mathbb{Z}$ -modules
- Extra Hodge structure when tensored up to  $\mathbb{C}$  (same as  $\mathbb{C}$  coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general “nice” schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We’ll associate  $X$  a “nice scheme”  $\rightsquigarrow H^i(X_{\text{et}}, \mathbb{Z}/\ell^n \mathbb{Z})$ . Take the inverse limit over all  $n$  to obtain the  $\ell$ -adic cohomology  $H^i(X_{\text{et}}, \mathbb{Z}_\ell)$ . You can tensor with  $\mathbb{Q}$  to get something with  $\mathbb{Q}_\ell$  coefficients. And as in singular cohomology, you can a “twisted coefficient system”.

What are nice schemes:

- $X = \text{Spec } \mathcal{O}_k$ , the ring of integers over a number field.
- $X$  a variety over an algebraically closed field
  - Typical, most analogous to taking a variety over  $\mathbb{C}$ .
  - For  $\mathbb{C}$ - variety,  $H_{\text{sing}}^i$  will vanish above  $i = 2d$ .
- $X$  a variety over a non-algebraically closed field
  - Over a finite field,  $H^i$  will vanish for  $i > 2d + 1$  but generally not vanish for  $i = 2d + 1$ .