Assignment 6: The Fourier Transform

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \lim_{\xi'\to 0} \frac{1}{2} \int_{\mathbb{R}^n} [f(x)) - f(x-\xi')]\, e^{-2\pi i x\cdot\xi}\ dx$$
.

The fact that the limit as $\xi \to \infty$ is equivalent to the limit $\xi' \to 0$ is a direct consequence of computing

$$\lim_{|\xi|\to\infty} \frac{\xi}{2|\xi|^2} = \lim_{|\xi|\to\infty} \frac{1}{2|\xi|}\frac{\xi}{|\xi|} = \mathbf{0},$$

since $\frac{\xi}{|\xi|}$ is a unit vector, and the term $\frac{1}{2|\xi|}$ is a scalar that goes to zero.

But as an immediate consequence, this yields

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \frac{1}{2}\left|\int_{\mathbb{R}^n} [f(x) - f(x-\xi')]\, e^{-2\pi i x\cdot\xi}\ dx\right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')| \left|e^{-2\pi i x\cdot\xi}\right|\ dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\ dx \\
&\to 0,
\end{aligned}$$

which follows from continuity in L^1 since $f(x-\xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds.

Note: Sorry, I couldn't figure out how to prove the hint!!

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$
\begin{aligned}
\widehat{f * g}(\xi) &:= \int \int f(x-y)g(y)\ e^{-2\pi i x \cdot \xi}\ dy\ dx \\
&=_? \int \int f(x-y)g(y)\ e^{-2\pi i x \cdot \xi}\ dx\ dy \\
&= \int \int f(t)e^{-2\pi i (x-y) \cdot \xi}\ g(y)\ e^{-2\pi i y \cdot \xi}\ dx\ dy \\
&\quad (t = x - y,\ dt =\ dx) \\
&= \int \int f(t)e^{-2\pi i t \cdot \xi} g(y) e^{-2\pi i y \cdot \xi}\ dt\ dy \\
&= \int f(t)e^{-2\pi i t \cdot \xi} \left(\int\ g(y)\ e^{-2\pi i y \cdot \xi}\ dy \right)\ dt \\
&= \int f(t)e^{-2\pi i t \cdot \xi}\ \hat{g}(\xi)\ dt \\
&= \hat{g}(\xi) \int f(t)e^{-2\pi i t \cdot \xi}\ dt \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}
$$

To see that this swap is justified, we'll apply Fubini-Tonelli. Note that if $f, g \in L^1(\mathbb{R}^n)$, then the map $(x, y) \mapsto f(x-y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$. Since g is measurable as well, taking the cylinder on g is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. The exponential is continuous, and thus measurable on $\mathbb{R}^n$. Thus the integrand $F(x, y)$ is a product of measurable functions and thus measurable. In particular, $|F| = |fg|$ is measurable, and ths computation shows that one iterated integral is finite. From a previous homework question, we know that $f \in L^1 \implies \hat{f}$ is bounded, and thus $\hat{f}\hat{g}$ is bounded. Since $|F|$ is measurable and one iterated integrable was finite, Fubini-Tonelli applies.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$
\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},
$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$
\widehat{(f * g) * h} = \widehat{f * g}\ \hat{h} = \hat{f}\ \hat{g}\ \hat{h} = \hat{f}\ \widehat{g * h} = f * (g * h).
$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\ \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then

$$\lim_{|\xi|\to\infty} \hat{I}(\xi) \neq 0,$$

which by Problem 1 shows that I can not be in L^1, a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x-y)$. We then have

$$\begin{aligned}
\hat{g}(\xi) &:= \int g(x)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x-y)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x-y)e^{-2\pi i(x-y)\cdot\xi}e^{-2\pi i y\cdot\xi}\ dx \\
&= e^{-2\pi i y\cdot\xi}\int f(x-y)e^{-2\pi i(x-y)\cdot\xi}\ dx \\
&\quad (t = x-y, dt = dx) \\
&= e^{-2\pi i y\cdot\xi}\int f(t)e^{-2\pi i t\cdot\xi}\ dt \\
&= e^{-2\pi i y\cdot\xi}\hat{f}(\xi).
\end{aligned}$$

3.1.2 (ii)

Let $h(x) = e^{2\pi i x\cdot y}f(x)$. We then have

$$\begin{aligned}
\hat{h}(\xi) &:= \int e^{2\pi i x\cdot y}f(x)e^{-2\pi i x\cdot\xi}\ dx \\
&= \int e^{2\pi i x\cdot y-2\pi i x\cdot\xi)f(x}\ dx \\
&= \int f(\xi-y)e^{-2\pi i x\cdot(\xi-y)}\ dx \\
&= \hat{f}(\xi-y).
\end{aligned}$$

3.2 (b)

We'll use the fact that if $\langle\cdot,\ \cdot\rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x},\mathbf{y}\in V$ we have

$$\langle A\mathbf{x},\ \mathbf{y}\rangle = \left\langle \mathbf{x},\ A^T\mathbf{y}\right\rangle$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$\begin{aligned}
\frac{1}{|\det T|}\hat{f}(T^{-T}\xi) &= \frac{1}{|\det T|}\int f(x)e^{-2\pi i x\cdot T^{-T}\xi}\ dx \\
&\quad x \mapsto Tx,\ dx \mapsto |\det T|\ dx \\
&= \frac{1}{|\det T|}\int f(Tx)e^{-2\pi i Tx\cdot T^{-T}\xi}|\det T|\ dx \\
&= \int f(Tx)e^{-2\pi i x\cdot\xi}\ dx \\
&\quad \text{since } Tx\cdot T^{-T}\xi = T^{-1}Tx\cdot\xi = x\cdot\xi \\
&= \widehat{(f\circ T)}(\xi).
\end{aligned}$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial\xi}\hat{f}(\xi) &:= \frac{\partial}{\partial\xi}\int f(x)e^{-2\pi i x\cdot\xi}\ dx \\
&=_? \int f(x)\frac{\partial}{\partial\xi}e^{-2\pi i x\cdot\xi}\ dx \\
&= \int f(x)2\pi i x e^{-2\pi i x\cdot\xi}\ dx \\
&= 2\pi i\int xf(x)e^{-2\pi i x\cdot\xi}\ dx \\
&:= 2\pi i\hat{g}(\xi).
\end{aligned}$$

To see that the interchange is justified, we just note that we can apply the dominated convergence theorem, since $\int \left|f(x)e^{-2\pi i x\cdot\xi}\right| \leq \int |f| < \infty$, where we assumed $f \in L^1$.

4.1.2 (ii)

We have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) e^{-2\pi i x\cdot\xi}\ dx \\
&= f(x) e^{-2\pi i x\cdot\xi}\Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i\xi) e^{-2\pi i x\cdot\xi}\ dx \\
&\qquad \text{(integrating by parts)} \\
&= -\int f(x)(-2\pi i\xi) e^{-2\pi i x\cdot\xi}\ dx \\
&\qquad \text{(since } f(\infty) = f(-\infty) = 0\text{)} \\
&= 2\pi i\xi \int f(x) e^{-2\pi i x\cdot\xi}\ dx \\
&:= 2\pi i\xi \hat{f}(\xi).
\end{aligned}$$

4.2 (b)

Let $G(x) = e^{-\pi x^2}$ and ∂_ξ be the operator that differentiates with respect to ξ.

Then

$$\partial_\xi \left(\frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi)}{G(\xi)^2},$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = 0.$$

A direct computation shows that

$$\partial_\xi G(\xi) = -2\pi\xi G(\xi), \tag{1}$$

and we claim that $\partial_\xi \hat{G}(\xi) = -2\pi\xi \hat{G}(\xi)$ as well, which follows from the following computation:

$$
\begin{aligned}
\partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) e^{-2\pi i x\cdot\xi}\, dx \\
&= \int G(x) \partial_\xi e^{-2\pi i x\cdot\xi}\, dx \\
&= \int G(x)(-2\pi i x) e^{-2\pi i x\cdot\xi}\, dx \\
&= \int G(x)(-2\pi i x) e^{-2\pi i x\cdot\xi}\, dx \\
&= i \int 2\pi x G(x) e^{-2\pi i x\cdot\xi}\, dx \\
&= i \int \partial_x G(x) e^{-2\pi i x\cdot\xi}\, dx \qquad \text{by (1)} \\
&:= i\ \widehat{\partial_x G(x)}(\xi) \\
&= i\ (2\pi i \xi \hat{G}(\xi)) \qquad \text{by part (i)} \\
&= -2\pi\xi \hat{G}(\xi).
\end{aligned}
$$

We can thus write

$$
G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),
$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x)\, dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$
\begin{aligned}
\hat{D}(\xi) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-2\pi i x\xi}\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\xi) + i\sin(-2\pi x\xi)\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\xi)\ dx \\
&\qquad \text{(since sin is odd and the domain is symmetric about 0)} \\
&= 2\int_{0}^{\frac{1}{2}} \cos(-2\pi x\xi)\ dx \\
&\qquad \text{(since cos is even and the domain is symmetric about 0)} \\
&= 2\left(\frac{1}{2\pi\xi}\sin(-2\pi x\xi)\Big|_{x=0}^{x=\frac{1}{2}}\right) \\
&= \frac{\sin(\pi\xi)}{\pi\xi}.
\end{aligned}
$$

5.2 (b)

5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$.

5.2.2 (ii)

Letting $\mathcal{F}$ denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if f is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting F be the box function, F can be seen to be even from its definition. Since $f := \mathcal{F}(F)$ by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that $\hat{f}(x) = F(x)$, the original box function.

5.3 (c)

By a direct computation of the integral in question, we have

$$
\begin{aligned}
I(x) &\coloneqq \int e^{-2\pi|\xi|} e^{2\pi i x\xi}\ d\xi \\
&= \int_{-\infty}^{0} e^{-2\pi(-\xi)} e^{-2\pi i x\xi}\ d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x\xi}\ d\xi \\
&= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x\xi}\ d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x\xi}\ d\xi \\
&\qquad \text{by the change of variables } \xi \mapsto -\xi,\ d\xi \mapsto -d\xi \text{ and swapping integration bounds} \\
&= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x\xi} + e^{2\pi\xi} e^{2\pi i x\xi}\ d\xi \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-u} e^{-ixu} + e^{-u} e^{ixu}\ du \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)}\ du \\
&= \frac{1}{2\pi}\left(\frac{-e^{-u(1+ix)}}{1+ix}\Big|_{u=0}^{u=\infty} + \frac{-e^{-u(1-ix)}}{1+ix}\Big|_{u=0}^{u=\infty} \right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) \\
&= \frac{1}{2\pi}\frac{2}{1+x^2} \\
&= \frac{1}{\pi}\frac{1}{1+x^2},
\end{aligned}
$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$
\begin{aligned}
I(x) = P(x) &= \int \hat{P}(\xi) e^{-2\pi i x\xi}\ dx \\
\implies \int e^{-2\pi|\xi|} e^{2\pi i x\xi} &= \int \hat{P}(\xi) e^{-2\pi i x\xi}\ dx \\
\implies \int e^{-2\pi|\xi|} e^{2\pi i x\xi} - \hat{P}(\xi) e^{-2\pi i x\xi}\ dx &= 0 \\
\implies \int \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right) e^{-2\pi i x\xi}\ dx &= 0 \\
\implies \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right) e^{-2\pi i x\xi} &=_{a.e.} 0 \\
\implies e^{-2\pi|\xi|} &=_{a.e} \hat{P}(\xi),
\end{aligned}
$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

6 Problem 6

We first note that if $G_t(x) \coloneqq t^{-n} e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2|\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \right| \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \, dx \\
&\qquad \text{since the integrand and thus integral is positive.} \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \left(\int_{\mathbb{R}^n} G_t(x) \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, (1) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt,
\end{aligned}$$

which we claim is finite, so $f \in L^1$.

To see that the norm is finite, we note that

$$t \in [0, 1] \implies e^{-\pi t^2} < 1$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$t \in [1, \infty) \implies t^{2\varepsilon - 1} \leq 1.$$

Thus

$$\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_0^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

> Justifying the interchange: we note that the integrand $G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1}$ is non-negative, and we've just showed that one of the iterated integrals is absolutely convergent, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on $\mathbb{R}^n$ and the remaining terms are continuous on $\mathbb{R}$, so they are all measurable on $\mathbb{R}^n$ and $\mathbb{R}$ respectively But then taking cylinders on everything in sight yields measurable functions, and the product of measurable functions is measurable.

If another interchange of integrals is justified, we can compute

$$
\begin{aligned}
\hat{f}(\xi) &\coloneqq \int_{\mathbb{R}^n} \left(\int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \; dt \right) e^{-2\pi i x \cdot \xi} \; dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \; dt \; dx \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \; dx \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \left(\int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} \; dx \right) \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \hat{G}_t(\xi) \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} e^{-\pi t^2 |\xi|^2} \; dt \\
&= \int_0^\infty e^{-\pi t^2 (1 + |\xi|^2)} t^{2\varepsilon - 1} \; dt \\
&= \int_0^\infty e^{-\pi (t\sqrt{1 + |\xi|^2})^2} t^{2\varepsilon - 1} \; dt \\
&\qquad s = t\sqrt{1 + |\xi|^2}, \; ds = \sqrt{1 + |\xi|^2} dt \\
&= \int_0^\infty e^{-\pi s^2} \left(\frac{s}{\sqrt{1 - |\xi|^2}} \right)^{2\varepsilon - 1} \frac{1}{\sqrt{1 + |\xi|^2}} \; ds \\
&= (1 + |\xi|^2)^{-\frac{2\varepsilon - 1}{2}} (1 + |\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon - 1} \; ds \\
&= (1 + |\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon - 1} \; dt \\
&\coloneqq F(\xi) \|f\|_1 .
\end{aligned}
$$

To see that the interchange is justified, note that

$$
\int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} e^{-2\pi i x \cdot \xi} \right| \; dt \; dx = \int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \right| \; dt \; dx,
$$

since $\left| e^{2\pi i x \cdot \xi} \right| = 1$. The integrand appearing is precisely what we showed was measurable when computed $\|f\|_1$ above, so Tonelli applies.

Thus $F(\xi)$ is the Fourier transform of the function $g(x) \coloneqq f(x) / \|f\|_1$. $\square$