An Introduction to Model Categories

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These are some notes on model categories, primarily written to assist me personally in learning the theory, and secondarily as a loose set of notes for a talk. As such they're highly informal and omit all of the relevant proofs. Hopefully they deliver some of the intuition.

I primarily used two sets of notes in writing this: Dwyer and Spalinski's *Homotopy Theories and Model Categories*, and the first two sections of the notes from Vezzosi's seminar *Autour de la Géométrie Algébrique Dérivée*. These notes are very heavily based on the above two documents.

1 Motivation

Model categories were originally developed as an abstraction of homotopy theory. Let's suppose we only care about topological spaces up to some weak form of equivalence. These could be homotopy equivalences or even **weak homotopy equivalences**, maps $X \to Y$ which induce isomorphisms $\pi_n X \to \pi_n Y$. Then we want to invert these weak equivalences in some sense, i.e. view our objects and morphisms 'up to homotopy'. If we invert homotopy equivalences, then we want to end up with with some category of 'homotopy types'.

If we invert the weak homotopy equivalences we expect to obtain the category of CW complexes, since it is a theorem that any (Hausdorff) topological space is weakly homotopy equivalent to a CW complex. CW complexes are a very nice class of spaces: **Whitehead's Theorem** tells us that any weak homotopy equivalence between connected CW complexes is actually a homotopy equivalence. A theme of model category theory is to replace our objects by weakly equivalent objects that are better behaved.

A useful class of maps in homotopy theory is the class of **fibrations**, maps $E \to B$ that satisfy the homotopy lifting property: for any homotopy $H : X \times [0, 1] \to B$, if we can lift $H(x, 0)$ to E then we can lift all of H to E extending the original lift. Fibrations generalise fibre bundles. Dually we have the class of **cofibrations**: maps satisfying the homotopy extension property.

Our strategy in defining a model category will be to define three classes of morphisms: weak equivalences, fibrations, and cofibrations. We'll code up some of their properties in axioms and see what comes out. We'll aim to define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$, which is the category obtained by inverting the weak equivalences.

Model categories have applications beyond topology: the category of chain complexes of R-modules for a fixed ring R can be given a model structure. In this case the homotopy category is the derived category. We can recover a lot of homological algebra via the machinery of model categories. More generally, if we want to invert weak equivalences in an algebraic way, model categories are a useful tool.

2 The axioms

The axioms for a model category (sometimes called a **closed model category** or **Quillen model category**) are simply an abstraction of some properties of homotopy theory. Accordingly, a **model structure** on a category $\mathcal{C}$ is determined by three distinguished classes of morphisms in $\mathcal{C}$, along with some axioms. The distinguished classes are the **weak equivalences**, **fibrations** and **cofibrations**. We stipulate that the distinguished classes are closed under composition, and that every identity map $id_A : A \to A$ is a fibration, a cofibration, and a weak equivalence.

The morphisms we're interested in are the weak equivalences, since we wish to turn these into isomorphisms. The fibrations and cofibrations are in some sense just technical machinery to allow us to do this. We often denote weak equivalences by $\xrightarrow{\sim}$.

When we want to refer to the classes of weak equivalences, fibrations and cofibrations we call them $\mathbf{W}$, $\mathbf{Fib}$ and $\mathbf{Cof}$ respectively. Fibrations that are also weak equivalences are called **acyclic fibrations**, and similarly cofibrations that are weak equivalences are called **acyclic cofibrations**. So the acyclic cofibrations are $\mathbf{W} \cap \mathbf{Cof}$ and the acyclic fibrations are $\mathbf{W} \cap \mathbf{Fib}$.

There are four axioms for the model structure:

Axiom 1 If f, g are composable maps, and if any two of f, g, or fg are weak equivalences then so is the third ("two-out-of-three property"). This is a property of (weak) homotopy equivalences.

Axiom 2 A map f is a **retract** of a map g if there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, "id"] & B \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r, "i'"] \arrow[rr, bend right, "id"'] & B' \arrow[r, "r'"] & A' \end{tikzcd}$$

Intuitively, if we think of this happening in some nice category of topological spaces, then A is a retract of B, A' is a retract of B', and f, g respect the retractions. Axiom 2 says that the distinguished classes are closed under retracts (i.e. if g is a distinguished morphism and f is a retract of g then f is also distinguished).

Axiom 3 The homotopy lifting axiom. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd} \tag{1}$$

where i is a cofibration, p a fibration, and at least one of i and p are weak equivalences, then there exists a map $h : B \to X$ making the diagram commute. The map h is called a **lift** in the diagram.

Axiom 4 The factorisation axiom. This says that every morphism f in $\mathcal{C}$ factors as $f = p \circ i$, where i is a cofibration, p a fibration, and at least one of i and p are weak equivalences. Note that these splittings are not necessarily functorial. However, in many cases, we can choose functorial splittings - an important technique for constructing splittings is the **small object argument**, which (if $\mathcal{C}$ is sufficiently nice) allows us to transfinitely construct factorisations, which we can choose to be functorial.

We say that a category $\mathcal{C}$ with a model structure is a **model category** if it has all finite limits and colimits (in particular, initial and terminal objects exist).

3 Properties of model categories

Duality Note that the axioms are very symmetric: if $\mathcal{C}$ is a model category, then $\mathcal{C}^{op}$ is a model category in a natural way, where the cofibrations in $\mathcal{C}^{op}$ are precisely the fibrations in $\mathcal{C}$ and vice versa. This is the model category analogue of Eckmann-Hilton duality in homotopy theory.

In fact we can say more: if we've already chosen our weak equivalences, then the cofibrations are completely determined by the fibrations and vice versa. A map $i : A \to B$ is said to have the **left lifting property** ("LLP") with respect to a map $p : X \to Y$ if a lift exists in any commutative square of the form (1). Then the cofibrations are precisely the maps that have the LLP with respect to acyclic fibrations.

The right lifting property ("RLP") is defined similarly. The fibrations are precisely the maps that have the RLP with respect to acyclic cofibrations.

Initial and terminal objects Any model category $\mathcal{C}$ has an initial object $\emptyset$ and a terminal object $*$. An object X is said to be **fibrant** if the unique map $X \to *$ is a fibration. X is **cofibrant** if the unique map from $\emptyset$ is a cofibration. Fibrant and cofibrant objects have nice properties with regard to homotopy and will become important later.

Reprise of the axioms Given the facts about the RLP and the LLP above, a quick way to state Axioms 3 and 4 is to say that $\mathcal{C}$ comes equipped with two weak factorisation systems $(\mathbf{W} \cap \mathbf{Cof}, \mathbf{Fib})$ and $(\mathbf{Cof}, \mathbf{W} \cap \mathbf{Fib})$. Moreover, Axiom 2 (the retracts axiom) is a consqeuence of axioms 1,3 and 4, though the proof is nontrivial. So if we were to pick a 'quick' set of axioms we'd specify the two-out-of-three property of weak equivalences and the existence of the weak factorisation systems.

4 Examples of model categories

It's often quite difficult to verify that a given category is in fact a model category. So we skip doing that here, and just state a few examples.

The prototypical example of a model category is **Top**, the category of topological spaces. One model structure is given by letting the weak equivalences be weak homotopy equivalences and fibrations Serre fibrations. If we restrict to compactly generated Hausdorff spaces then the homotopy category is equivalent to the category of CW-complexes and maps up to honmotopy equivalence. In general to get nice topological results we have to restrict to a 'convenient category' of topological spaces since the whole category **Top** is very badly behaved from a homotopy-theoretic perspective.

There's another model structure on **Top**: this time the weak equivalences are the homotopy equivalences, and the fibrations are the (topological) fibrations: maps that satisfy the homotopy lifting property. With respect to this structure, Ho(**Top**) is the category of topological spaces with maps up to homotopy equivalence.

If R is a commutative ring, the category $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ of chain complexes of left R-modules and chain maps admits a model structure: the weak equivalences are the quasi-isomorphisms (maps inducing isomorphisms on homology) and the fibrations are the maps that are modulewise surjective. The cofibrant objects are the chain complexes of projective modules. In this case the homotopy category is the derived category.

The model structure above on $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ is called the **projective model structure**. Dually we also have the **injective model structure**, where the cofibrations are the modulewise injective maps. The category $\mathbf{Ch}^+(R\text{-}\mathbf{Mod})$ of nonnegatively graded chain complexes admits analogous model structures.

If $\mathcal{C}$ is a model category and A an object then the category $A \downarrow \mathcal{C}$ of objects under A inherits a model category structure. As an example, if $*$ is the one-point topological space then the category $* \downarrow \mathbf{Top}$ is just the category $\mathbf{Top}_*$ of pointed topological spaces.

The category **sSet** of simplicial sets is a model category: the weak equivalences are maps f whose geometric realisation $|f|$ is a weak homotopy equivalence, and the cofibrations are the maps g whose components g_n are injective for all n. Note that we're just pulling back the model structure from **Top** via the geometric realisation functor. In this case $\mathrm{Ho}(\mathbf{sSet})$ is equivalent to $\mathrm{Ho}(\mathbf{Top})$, where **Top** is given the first model structure. So **sSet** is a good combinatorial model for homotopy theory in **Top**.

In general, if $\mathcal{C}$ is a category then we have the category $\mathbf{s}\mathcal{C}$ of simplicial objects in $\mathcal{C}$. When $\mathcal{C}$ is a reasonable-looking category with a forgetful functor to **Set** (think of the categories of groups, rings, vector spaces, Lie algebras...) then $\mathbf{s}\mathcal{C}$ has a model category structure induced by pulling back the model structure on **sSet**. In particular if we look at $R\text{-}\mathbf{Mod}$, then $\mathbf{s}R\text{-}\mathbf{Mod}$ is equivalent to $\mathbf{Ch}^+(R\text{-}\mathbf{Mod})$ by the Dold-Kan correspondence, and hence the homotopy theory of $\mathbf{s}R\text{-}\mathbf{Mod}$ is just homological algebra in $R\text{-}\mathbf{Mod}$. In this sense homotopy theory can be said to be homological algebra in **Set**!

A non-interesting example: every category $\mathcal{C}$ with finite limits and colimits can be given the **trivial model structure**, where the weak equivalences are the isomorphisms and every morphism is both a fibration and a cofibration. In this case $\mathrm{Ho}(\mathcal{C})$ is just $\mathcal{C}$ again.

The category **Cat** of all small categories can be given a model structure where the weak equivalences are the equivalences of categories. Moreover there is a unique model structure with this property, called the **canonical model structure**. The cofibrations are the functors injective on objects.

Analogously, the category **Grpd** of all small groupoids admits a unique model structure where the weak equivalences are the equivalences of groupoids.

5 Homotopy

We want to define a notion of homotopy between two maps. This takes a bit of technical machinery. In general we need to define notions of **left homotopy** and **right homotopy**. In the important cases these will turn out to be the same. We need these left and right notions since in **Top** there are two equivalent ways of defining a homotopy between maps $X \to Y$: we can either look at 'left homotopies' $X \times [0,1] \to Y$ or 'right homotopies' $X \to Y^{[0,1]}$. In general we don't expect these two notions to be the same!

For a topological space A the cylinder $A \times [0,1]$ comes equipped with a map $A \amalg A \to A \times [0,1]$ identifying A with the ends of the cylinder. Moreover, $A \times [0,1]$ is homotopy equivalent to A via the quotient map, and the composite of these two maps is the codiagonal $A \amalg A \to A$. Note that a homotopy between two maps $f, g : X \to Y$ is the same thing as a factorisation of the map $f + g : X \amalg X \to Y$ through the map to $X \times [0,1]$. These are more or less the properties that make the cylinder useful in homotopy theory.

With this in mind, a **cylinder object** for an object A in a model category is an object $A \wedge I$ factoring the codiagonal map $id_A + id_A : A \amalg A \to A$ into $A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$. Keep in mind that $A \wedge I$ is just a notation - cylinder objects may not be functorial in any way, and in general are not unique (they do exist, since A is a cylinder object for A). Since we have two canonical maps $n_1, n_2 : A \to A \amalg A$ we get two **structure maps** $i_1 = i \circ n_1$ and $i_2 = i \circ n_2$ from A to $A \wedge I$.

Two maps $f, g : A \to B$ are said to be **left homotopic** if there exists a map $H : A \wedge I \to B$ such that the diagram

$$\begin{tikzcd}
A \arrow[dr, "n_1"] \arrow[rrr, bend left, "f"] & & & \\
& A \amalg A \arrow[r, "i"] & A \wedge I \arrow[r, "H"] & B \\
A \arrow[ur, "n_2"] \arrow[rrru, bend right, "g"'] & & &
\end{tikzcd}$$

commutes. Note that this is the same thing as a factorisation $f + g = H \circ i$.

We can dualise the definition of cylinder object to get the definition of a **path object** B^I, which comes equipped with maps $B \xrightarrow{\sim} B^I \xrightarrow{p} B \times B$ factoring the diagonal map. Path objects are the analogues of path spaces PB in topology.

Right homotopies are defined with path objects in an analogous way: two maps $f, g : A \to B$ are said to be **right homotopic** if there exists a map $H : A \to B^I$ such that the diagram

$$\begin{tikzcd} & & & B \\ A \arrow[r, "H"] \arrow[rrru, bend left, "f"] \arrow[rrrd, bend right, "g"'] & B^I \arrow[r, "p"] & B \times B \arrow[ru, "\pi_1"] \arrow[rd, "\pi_2"] & \\ & & & B \end{tikzcd}$$

commutes. Note that this is the same thing as a factorisation $f \times g = p \circ H$.

If A is cofibrant and B is fibrant then the left and right homotopy relations on the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ agree, and we get an equivalence relation which we call **homotopy**. The set of homotopy equivalence classes of maps from A to B is denoted $\pi(A, B)$.

An object that's both fibrant and cofibrant is called **fibrant-cofibrant**. If A and B are fibrant-cofibrant then we can say even more: a map $f : A \to B$ is a weak equivalence if and only if f is a homotopy equivalence, i.e. there exists a map $g : B \to A$ such that gf and fg are both homotopic to the identity. This is the analogue of Whitehead's Theorem in **Top**, which says that a weak homotopy equivalence between CW complexes is the same thing as a homotopy equivalence.

6 The homotopy category

The above section gives us an idea of how to invert weak equivalences, at least on the full subcategory $\mathcal{C}_{cf}$ of fibrant-cofibrant objects: pass to the appropriate 'homotopy category' $\pi\mathcal{C}_{cf}$ whose morphisms are homotopy classes of maps. Then a weak equivalence in $\mathcal{C}_{cf}$ becomes an isomorphism in $\pi\mathcal{C}_{cf}$.

So to extend this idea to the whole category $\mathcal{C}$, we want first to replace our objects by weakly equivalent fibrant-cofibrant objects. To do this, for each object X apply the factorisation axiom to the map $\emptyset \to X$ to obtain an acyclic fibration $QX \to X$ with QX cofibrant. Similarly we obtain an acyclic cofibration to a fibrant object $X \to RX$.

The assignments $X \mapsto RX$ and $X \mapsto QX$ are functors (recall that the splitting in the factorisation axiom need not be functorial). The functor $F = RQ : \mathcal{C} \to \mathcal{C}_{cf}$ takes an object to a fibrant-cofibrant replacement. In fact F induces a functor F' from $\mathcal{C}$ to $\pi\mathcal{C}_{cf}$. Then we have that $\mathrm{Hom}_{\pi\mathcal{C}_{cf}}(F'X, F'Y) \cong \pi(FX, FY)$.

The **homotopy category** $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$, but $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) := \mathrm{Hom}_{\pi\mathcal{C}_{cf}}(F'X, F'Y)$. So we replace the objects by weakly equivalent fibrant-cofibrant objects, and then consider the class of maps between them up to homotopy. Note that we could have used the functor QR instead of RQ; we'll soon give a symmetric definition of the homotopy category that makes that easier to see.

Denote by γ the functor sending a model category to its homotopy category. Morphisms in $\mathrm{Ho}(\mathcal{C})$ work as we want them to: if f is a morphism in $\mathcal{C}$ then $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence.

One way to invert morphisms in a category is to localise; if W is a class of morphisms in $\mathcal{C}$ then a **localisation** of W is a functor $l : \mathcal{C} \to \mathcal{D}$ such that $l(f)$ is an isomorphism for every $f \in W$, along with the specification that l is universal among such functors (every such functor factors through l). So if localisations exist, any two are naturally isomorphic.

If $\mathcal{C}$ is a model category and W is the set of weak equivalences, then $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ is a localisation of W. This gives us a nice quick definition of the homotopy category, without the model category formalism. So why don't we just work with localisation at weak equivalences and forget the model category framework?

One of the advantages is that we can describe morphisms in the homotopy category more concretely if we work with model categories. Another advantage is that model categories resolve some of the set-theoretic issues around localisation.

7 Adjunctions and equivalences

We want to formulate the right notion of an adjunction or an equivalence between two model categories. This turns out to be the concept of a Quillen adjunction or Quillen equivalence. Let $F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$ be an adjunction between model categories. This is said to be a **Quillen adjunction** if any of the following equivalent conditions are satisfied:

- F preserves cofibrations and G preserves fibrations
- F preserves cofibrations and acyclic cofibrations
- G preserves fibrations and acyclic fibrations
- F preserves acyclic cofibrations and G preserves acyclic fibrations

In this situation we sometimes call F a **left Quillen functor** and G a **right Quillen functor**.

A Quillen adjunction is said to be a **Quillen equivalence** if for every cofibrant X in $\mathcal{C}$ and every fibrant Y in $\mathcal{D}$, a map $X \to G(Y)$ is a weak equivalence if and only if the corresponding map $F(X) \to Y$ is a weak equivalence. We'll see soon that Quillen equivalences induce equivalences of the respective homotopy categories, via their derived functors.

8 Derived functors

Let $\mathcal{C}$ be a model category, $\mathcal{D}$ any category and $F : \mathcal{C} \to \mathcal{D}$ a functor. Ideally, F will descend to a functor $hF : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$, i.e. factor through the map $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$. When it does, we'd like to choose a universal such functor. The left and right derived functors of F are reasonable definitions for this.

Consider the class of pairs (G, t) where $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ is a functor and $t : G\gamma \to F$ is a natural transformation. A **left derived functor** (LF, t) of F is a left-universal pair in the sense that for any element (G, s) there exists a natural transformation $s' : G \to LF$ such that we have a commutative diagram

$$\begin{tikzcd} & F \\ G\gamma \arrow[ur, "s"] \arrow[r, "s'"'] & (LF)\gamma \arrow[u, "t"] \end{tikzcd}$$

A **right derived functor** is a right-universal such functor.

If left derived functors of F exist, then there is only one up to canonical natural isomorphism. But left derived functors may not always exist! They do if F satisfies a few properties. For example if F takes weak equivalences to isomorphisms, its left derived functor LF exists since F will factor through the localisation.

If $\mathcal{D}$ is a model category we may compose F with the functor γ to get a functor $F\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$. With this in mind the **total left derived functor** of F is the functor $\mathbb{L}F := L(F\gamma) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. So total left derived functors allow us to consider our functors 'up to homotopy' in a universal way.

Quillen adjunctions induce adjunctions on the homotopy category. More precisely, let $F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$ be a Quillen adjunction. Then the total left and right derived functors $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \longleftrightarrow \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$ exist and form an adjoint pair. Moreover, if the pair (F, G) is a Quillen equivalence then their total derived functors above are an equivalence of categories.

Example The adjoint pair $|\cdot| : \mathbf{sSet} \longleftrightarrow \mathbf{Top} : \mathrm{Sing}$ is a Quillen equivalence and hence we get an equivalence $\mathrm{Ho}(\mathbf{sSet}) \longleftrightarrow \mathrm{Ho}(\mathbf{Top})$.

Not all equivalences between homotopy categories arise this way! There are examples of model categories $\mathcal{C}$ and $\mathcal{D}$ where $\mathrm{Ho}(\mathcal{C})$ is equivalent to $\mathrm{Ho}(\mathcal{D})$, but $\mathcal{C}$ and $\mathcal{D}$ are not Quillen equivalent.

9 Homotopy limits and colimits

Limits and colimits in general do not respect homotopy equivalences. For example, in **Top** we have a commutative diagram

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

where the vertical maps are homotopy equivalences. The pushout of the top row is S^n and the pushout of the bottom row is $*$. But the induced map $S^n \to *$ is not a homotopy equivalence!

Recall the definition of a colimit: if J is a category then $\mathcal{C}^J$ is the category of diagrams of type J in $\mathcal{C}$ and we have a functor colim: $\mathcal{C}^J \to \mathcal{C}$, left adjoint to the constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^J$. So to take colimits in a homotopy invariant manner we could pass to the total derived functor $\mathbb{L}\mathrm{colim}$.

If we want to take colimits up to homotopy, we'd better have some notion of homotopy of diagrams of type J. That is, we want to put a model structure on $\mathcal{C}^J$; hopefully one that allows us to take total left derived functors to get **homotopy colimits** $\mathbb{L}\mathrm{colim}$ and **homotopy limits** $\mathbb{R}\mathrm{lim}$. Note that $\mathbb{L}\mathrm{colim}$ will be adjoint to $\mathbb{R}\Delta$ and $\mathbb{R}\mathrm{lim}$ will be right adjoint to $\mathbb{L}\Delta$.

Homotopy colimits will not be the same as colimits in the homotopy category: the categories $\mathrm{Ho}(\mathcal{C}^J)$ and $\mathrm{Ho}(\mathcal{C})^J$ have no reason to be equivalent!

A problem is that homotopy colimits in general do not exist. We're going to need some extra structure on either J or $\mathcal{C}$. In what follows we describe one such situation where extra structure on J gives us what we want.

A subcategory R' of a category R is said to be **wide** (sometimes **lluf**) if $\mathrm{Obj}(R') = \mathrm{Obj}(R)$. A **Reedy category** is a category R equipped with two wide subcategories R^+ and R^- and a degree function $d : \mathrm{Obj}(R) \to \mathbb{N}$ such that

- Every nonidentity morphism in R^+ raises the degree
- Every nonidentity morphism in R^- lowers the degree
- Every morphism in R admits a factorisation into a morphism from R^- followed by a morphism from R^+

Many of the categories we'd like to take colimits over are Reedy categories, including

- Any ordinal
- The simplex category Δ
- Many typical 'diagram categories' such as the pushout category $(\bullet \leftarrow \bullet \rightarrow \bullet)$, the coequaliser category $(\bullet \rightrightarrows \bullet)$, and the direct limit category $(\bullet \rightarrow \bullet \rightarrow \cdots)$
- The opposite of any Reedy category

If $\mathcal{C}$ is any model category and J is a Reedy category, then it turns out that $\mathcal{C}^J$ has a natural model structure where homotopy colimits exist.

10 From model categories to $(\infty, 1)$-categories

We often think of model categories as carrying, in addition to the morphisms and homotopies, some kind of higher homotopical data. Passing to the homotopy category discards this higher information. So given a model category, we'd like to define some category that keeps track of the higher homotopical data.

More technically, given a model category we'd like to define an $(\infty, 1)$-category: an ∞-category where all higher morphisms are invertible. This associated $(\infty, 1)$-category should remember the higher homotopical information about our original category. There are lots of ways of making this definition, depending on your precise meaning of $(\infty, 1)$-category.

If you think of $(\infty, 1)$-categories as quasicategories (i.e. certain kinds of simplicial set) then we can build an $(\infty, 1)$-category out of a model category by taking the simplicial nerve of the subcategory of fibrant-cofibrant objects. This quasicategory knows the higher homotopical information about our original model category.

However, we want to think of $(\infty, 1)$-categories as simplicially enriched categories - categories where the hom sets are simplicial sets. Thinking of a simplicial set as basically the same thing as a topological space, we see that for any two objects we want to define a **mapping space** between them.

One of our intuitions for a mapping space $\mathrm{Map}(X, Y)$ is that the connected components of $\mathrm{Map}(X, Y)$ should correspond to the homotopy classes of maps from X to Y. This is because a homotopy from f to g should correspond to a path between them in the space $\mathrm{Map}(X, Y)$.

11 Mapping spaces

Let's go back to thinking about the category **Top** of topological spaces. Let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. Then the functor $[-,-]$ encodes a lot of homotopical information: for example $\pi_n(X) = [S^n, X]$ (we are being a little careless about basepoints here). Note that we can identify homotopies of maps from X to Y with maps from $X \times |\Delta^1|$ to Y.

Setting $X^n = X \times |\Delta^n|$, we see that maps $X^n \to Y$ should record 'n-th order homotopies' between maps $X \to Y$. Note that the X^n together form a cosimplicial object in **Top**. Moreover, $X^0 \cong X$ and $X^n \simeq X$ since $|\Delta^n|$ is contractible.

If cX^* is the constant simplicial object at X we have a natural map $X^* \to cX^*$ where the maps are objectwise weak equivalences. Since the simplex category Δ is a Reedy category, the existence of this map is equivalent to saying that we have a Reedy weak equivalence $X^* \to cX^*$ in $\mathbf{Top}^\Delta$.

We can generalise this to any model category. Let X be an object of a model category $\mathcal{C}$. Then a **cosimplicial resolution** of X is a acyclic cofibration $A^* \to cX^*$ in the model category $\mathcal{C}^\Delta$. Dually a **simplicial resolution** of X is a acyclic fibration $cX^* \to A^*$ in $\mathcal{C}^{\Delta^{op}}$.

If $A^* \to cX^*$ is a cosimplicial resolution, then $A^0 \to X$ is an acyclic cofibration and the map $A^0 \amalg A^0 \to A^1 \to A^0$ is a cylinder object for A^0. This is a good clue that a cosimplicial resolution of X records a lot of the higher homotopical data about X. Loosely we can think of the A^n as higher cylinder objects for X.

Suppose that in our model category we can construct splittings of maps (as in the factorisation axiom) functorially. Then we can also construct cosimplicial resolutions functorially. In what follows we assume we can do this and hence have a cosimplicial resolution functor $r : \mathcal{C} \to \mathcal{C}^\Delta$, along with an analogous simplicial resolution functor $\bar{r}$. Since cosimplicial resolutions are unique up to weak equivalence, choosing a particular functor for good won't affect what we want to do.

Using cosimplicial resolutions we can construct mapping spaces. First note that if X and Y are objects then $\mathrm{Hom}(rX, Y)$ is a simplicial set. This suggests that to get a mapping space we should take a cosimplicial resolution for X. But we should probably also replace Y by a simplicial resolution too.

With this in mind, if X and Y are objects in a model category, then we can form a bisimplicial set $\mathrm{Hom}(rX, \bar{r}Y)$. We can take the diagonal of this bisimplicial set to get a (fibrant) simplicial set $\mathrm{map}_\mathcal{C}(X, Y)$ which we call the **homotopy function complex** from X to Y.

Technically this is a **two-sided** function complex. We could have chosen just a cosimplicial resolution for X or a simplicial resolution for Y to get a **left** (resp. **right**) function complex. In either case we'd have ended up with something weakly equivalent to our original definition.

If we hadn't fixed a cosimplicial resolution functor, we'd have to keep track of the cosimplicial resolutions we chose to define our function complexes with. However, doing this doesn't make things appreciably harder.

One of our intuitions for mapping spaces is that the set $\pi_0 \mathrm{map}_{\mathcal{C}}(X, Y)$ of path components of $\mathrm{map}_{\mathcal{C}}(X, Y)$ should be in bijection with the homotopy classes of maps from X to Y. This does indeed happen with this construction.

Finally, we can say that weakly equivalent objects have weakly equivalent mapping spaces: if $f : X \xrightarrow{\sim} Y$ is a weak equivalence then f induces weak equivalences $\mathrm{map}_{\mathcal{C}}(W, X) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}}(W, Y)$ and $\mathrm{map}_{\mathcal{C}}(X, Z) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}}(Y, Z)$, as long as W is cofibrant and Z is fibrant.

Example In the category of simplicial sets, $\mathrm{map}_{\mathbf{sSet}}(X, Y)$ is the simplicial set that at level n has the set $\mathrm{Hom}_{\mathbf{sSet}}(X' \times \Delta^n, Y')$, where X' and Y' are fibrant-cofibrant replacements for X and Y. This agrees with our intuition that maps $X \times \Delta^n \to Y$ should record 'n-th order homotopies' between maps $X \to Y$.

12 Bousfield localisation

Suppose we have a model category $\mathcal{C}$ and we want to localise the homotopy category $\mathrm{Ho}(\mathcal{C})$. Since $\mathrm{Ho}(\mathcal{C})$ forgets a lot of homotopical information, we'd like some localisation procedure that remembers this information. So rather than working at the level of $\mathrm{Ho}(\mathcal{C})$, we want to add more weak equivalences to $\mathcal{C}$ such that when we pass to the homotopy category we obtain the desired localisation.

As always, we'd like to define a universal such 'homotopy localisation'. With this in mind, suppose $\mathcal{C}$ is a model category and S a class of morphisms of $\mathcal{C}$. Consider the class of pairs $(\mathcal{D}, F)$ where $\mathcal{D}$ is a model category and F is a left Quillen functor (the left adjoint of a Quillen adjunction) such that the total left derived functor $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ takes images of morphisms in S to isomorphisms. Then we define a **left localisation** of $\mathcal{C}$ with respect to S to be a pair $(L_S\mathcal{C}, l)$ universal among such pairs.

Right localisations have a similar definition. Since the theory of right localisations is dual to that of left localisations, we'll only consider left localisations in what follows.

We'd like to characterise the morphisms in $\mathcal{C}$ that map to isomorphisms when we take a left localisation with respect to S. One approach is to look at the objects that make the morphisms in S look like weak equivalences.

With this in mind, say that a fibrant object X is S-local if for every morphism $f : A \to B$ in S the induced morphism of function complexes $f^* : \mathrm{map}_{\mathcal{C}}(B, X) \to \mathrm{map}_{\mathcal{C}}(A, X)$ is a weak equivalence of simplicial sets. Say that a map $f : A \to B$ is an S-local equivalence if for every S-local object X the induced morphism f^* above is a weak equivalence.

Then the S-local equivalences capture abstractly what morphisms map to isomorphisms under localisation. Let $F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$ be a Quillen pair, and S a class of arrows in $\mathcal{C}$. Then $\mathbb{L}F$ takes the images of arrows in S to isomorphisms in $\mathrm{Ho}(\mathcal{D})$ if and only if F takes S-local equivalences between cofibrant objects into weak equivalences.

Let $\mathcal{C}$ be a model category and S a class of morphisms. The **left Bousfield localisation** with respect to S is a new model category $L_S\mathcal{C}$ with the same underlying category as $\mathcal{C}$ but with more distinguished morphisms:

- The weak equivalences of $L_S\mathcal{C}$ are precisely the S-local equivalences of $\mathcal{C}$
- The cofibrations of $L_S\mathcal{C}$ are precisely the cofibrations of $\mathcal{C}$
- The fibrations of $L_S\mathcal{C}$ are precisely the maps with the right lifting property with respect to the acyclic (in $L_S\mathcal{C}$) cofibrations.

So we add in more weak equivalences and keep the cofibrations the same. Since the weak equivalences together with the cofibrations determine the fibrations, we have to add more fibrations too. A **right Bousfield localisation** is defined similarly: the weak equivalences are the same as in the left localisation but we keep the fibrations constant and add cofibrations.

Note that we specify that these new classes of distinguished morphisms must put a model structure on $L_S\mathcal{C}$. This is not always the case if we let $\mathcal{C}$ and S be arbitrary! So left (resp. right) Bousfield localisations do not always exist. But if they do they are left (resp. right) localisations. Perhaps it would be more appropriate to call the category obtained from $\mathcal{C}$ by adding more weak equivalences a left (resp. right) 'prelocalisation', and if the prelocalisation is a model category then we call it the Bousfield localisation.

If $\mathcal{C}$ satisfies some technical hypotheses - namely, if it is **cellular** and **left** (resp. **right**) **proper** then a left (resp. right) Bousfield localisation exists.