Assignment 6: The Fourier Transform

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = \lim_{\xi' \to 0} \frac{1}{2} \int_{\mathbb{R}^n} (f()) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \; dx$$

.

But as an immediate consequence, this yields

$$
\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\int_{\mathbb{R}^n} (f(x) - f(x-\xi')) \exp(-2\pi i x \cdot \xi)\ dx\right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')||\exp(-2\pi i x \cdot \xi)|\ dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\ dx \\
&\to 0,
\end{aligned}
$$

which follows from continuity in L^1 since $f(x-\xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds, and that $\xi' \to 0$ as $\xi \to \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$
\begin{aligned}
\widehat{(f * g)}(\xi) &\coloneqq \int\int f(x-y)g(y)\exp(-2\pi x \cdot \xi)\ dy\ dx \\
&=_{?} \int\int f(x-y)g(y)\exp(-2\pi x \cdot \xi)\ dx\ dy \\
&= \int\int f(t)\exp(-2\pi i(x-y)\cdot\xi)g(y)\exp(-2\pi i y \cdot \xi)\ dx\ dy \\
&\quad (t = x-y,\ dt =\ dx) \\
&= \int\int f(t)\exp(-2\pi i t \cdot \xi)g(y)\exp(-2\pi i y \cdot \xi)\ dt\ dy \\
&= \int f(t)\exp(-2\pi i t \cdot \xi)\left(\int g(y)\exp(-2\pi i y \cdot \xi)\ dy\right)\ dt \\
&= \int f(t)\exp(-2\pi i t \cdot \xi)\hat{g}(\xi)\ dt \\
&= \hat{g}(\xi)\int f(t)\exp(-2\pi i t \cdot \xi)\ dt \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}
$$

It thus remains to show that this swap is justified.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$\widehat{(f * g) * h} = \widehat{f * g}\ \hat{h} = \hat{f}\ \hat{g}\ \hat{h} = \hat{f}\ \widehat{g * h} = f * (g * h).$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\ \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then $\lim_{|\xi|\to\infty} \hat{I}(\xi) \neq 0$, which by Problem 1 shows that I can not be in L^1, a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x - y)$. We then have

$$\begin{aligned}
\hat{g}(\xi) &\coloneqq \int g(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x-y) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x-y) \exp(-2\pi i (x-y) \cdot \xi) \exp(-2\pi i y \cdot \xi)\ dx \\
&= \exp(-2\pi i y \cdot \xi) \int f(x-y) \exp(-2\pi i (x-y) \cdot \xi)\ dx \\
&\quad (t = x - y, dt = dx) \\
&= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi)\ dt \\
&= \exp(-2\pi i y \cdot \xi) \hat{f}(\xi).
\end{aligned}$$

3.1.2 (ii)

Let $h(x) = \exp(2\pi i x\ cdot y) f(x)$. We then have

$$
\begin{aligned}
\hat{h}(\xi) &:= \int \exp(2\pi i x \cdot y) f(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int \exp(2\pi i x \cdot y - 2\pi i x \cdot \xi) f(x)\ dx \\
&= \int f(\xi - y) \exp(-2\pi i x \cdot (\xi - y))\ dx \\
&= \hat{f}(\xi - y).
\end{aligned}
$$

3.2 (b)

We'll use the fact that if $\langle \cdot,\ \cdot \rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$
\langle A\mathbf{x},\ \mathbf{y} \rangle = \left\langle \mathbf{x},\ A^T \mathbf{y} \right\rangle
$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$
\begin{aligned}
\frac{1}{|\det T|} \hat{f}(T^{-T}\xi) &= \frac{1}{|\det T|} \int f(x) \exp(-2\pi i x \cdot T^{-T}\xi)\ dx \\
&\qquad x \mapsto Tx,\ dx \mapsto |\det T|\ dx \\
&= \frac{1}{|\det T|} \int f(Tx) \exp(-2\pi i Tx \cdot T^{-T}\xi) |\det T|\ dx \\
&= \int f(Tx) \exp(-2\pi i x \cdot \xi)\ dx \\
&\qquad \text{since } Tx \cdot T^{-T}\xi = T^{-1}Tx \cdot \xi = x \cdot \xi \\
&= (\widehat{f \circ T})(\xi).
\end{aligned}
$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$
\begin{aligned}
\frac{\partial}{\partial \xi}\hat{f}(\xi) &:= \frac{\partial}{\partial \xi}\int f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&=_? \int f(x)\frac{\partial}{\partial \xi}\exp(-2\pi i x\cdot\xi)\ dx \\
&= \int f(x) 2\pi i x\exp(-2\pi i x\cdot\xi)\ dx \\
&= 2\pi i\int x f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&:= 2\pi i\hat{g}(\xi).
\end{aligned}
$$

It thus remains to show that this interchange is justified. TODO

4.1.2 (ii)

We have

$$
\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&= f(x)\exp(-2\pi i x\cdot\xi)\Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i\xi)\exp(-2\pi i x\cdot\xi)\ dx \\
&\qquad \text{(integrating by parts)} \\
&= -\int f(x)(-2\pi i\xi)\exp(-2\pi i x\cdot\xi)\ dx \\
&\qquad \text{(since } f(\infty)=f(-\infty)=0\text{)} \\
&= 2\pi i\xi\int f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&:= 2\pi i\xi\hat{f}(\xi).
\end{aligned}
$$

4.2 (b)

Let $G(x)=\exp(-\pi x^2)$ and ∂_ξ be the operator that differentiates with respect to ξ.

Then

$$
\partial_\xi\left(\frac{\hat{G}(\xi)}{G(\xi)}\right) = \frac{G(\xi)\partial_\xi\hat{G}(\xi)-\hat{G}(\xi)\partial_\xi G(\xi)}{G(\xi)^2},
$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$
G(\xi)\partial_\xi\hat{G}(\xi)-\hat{G}(\xi)\partial_\xi G(\xi)=0.
$$

Using the following facts,

- $\partial_\xi G(\xi) = -2\pi\xi G(\xi)$ by computing directly,
- $\partial_\xi \hat{G}(\xi) = -2\pi\xi \hat{G}(\xi)$, which follows from the following computation

$$\begin{aligned}
\partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int G(x) \partial_\xi \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int G(x)(-2\pi i x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= i \int 2\pi x G(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= i \int \partial_x G(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&:= i\ \widehat{\partial_x G(x)}(\xi) \\
&= i\ (2\pi i \xi \hat{G}(\xi)) \\
&= -2\pi\xi \hat{G}(\xi),
\end{aligned}$$

we can thus write

$$G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x)\ dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$
\begin{aligned}
\hat{D}(\xi) &\coloneqq \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-2\pi i x\cdot\xi}\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi) + i\sin(-2\pi x\cdot\xi)\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi)\ dx \\
&\qquad \text{(since sin is odd and the domain is symmetric about 0)} \\
&= 2\int_{0}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi)\ dx \\
&\qquad \text{(since cos is even and the domain is symmetric about 0)} \\
&= 2\left(\frac{1}{2\pi\xi}\sin(-2\pi x\cdot\xi)\Big|_{x=0}^{x=\frac{1}{2}}\right) \\
&= \frac{\sin(\pi\xi)}{\pi\xi}.
\end{aligned}
$$

5.2 (b)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2 = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$.

6 Problem 6