

# Title

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# Table of Contents

## Contents

<b>Table of Contents</b>	<b>2</b>
<b>1 Lecture 10</b>	<b>3</b>
1.1 Representability and Local Triviality . . . . .	4
1.1.1 What Hilbert 90 Means . . . . .	5
1.1.2 Geometric Interpretations . . . . .	6

# 1 | Lecture 10

**Remark 1.0.1:** What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of ??; we'll now continue with the proof:

*Proof (of Hilbert 90).*

**Observation 1.0.2:** Let  $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$ , then the data of a  $\text{GL}_n$ -torsor split by a  $\tau$ -cover  $U \rightarrow X$  is the same as descent data for a vector bundle relative to  $U/X$ .

This descent data comes from the following:

$$\begin{array}{c} U \times_X U \\ \pi_1 \downarrow \quad \downarrow \pi_2 \\ U \\ \downarrow \\ X \end{array}$$

That  $U$  trivializes our torsor means that  $\pi^*T = \pi^*G$  as a  $G$ -torsor, where  $G$  acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with  $G$  in both, yielding

$$\begin{array}{ccc} \pi_1^* \pi^* T & \xrightarrow{\sim} & \pi_2^* \pi^* T \\ \downarrow & & \downarrow \\ \pi_1^* \pi^* G & \xrightarrow{\sim} & \pi_2^* \pi^* G \end{array}$$

Both of the bottom objects are isomorphic to  $G|_{U \times U}$ .

**Claim:** The top horizontal map is descent data for  $T$ , and the bottom horizontal map is an automorphism of a  $G$ -torsor and thus is a section to  $G$ . I.e. a section to  $\text{GL}_n$  is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial  $\text{GL}_n$ -torsor on the fppf site, it's also trivial on the other two sites, yielding the desired maps back and forth. Thus  $H^1(X_{\text{ét}}, \text{GL}_n)$  is in bijection with  $n$ -dimensional vector bundles on  $X$ . ■

**Exercise 1.0.3(?):** See if Hilbert 90 is true for groups other than  $\text{GL}_n$ .

## 1.1 Representability and Local Triviality

**Question 1.1.1:** Suppose  $G$  is an affine flat  $X$ -group scheme. Are all  $G$ -torsors representable by a  $X$ -scheme?

**Answer 1.1.2:** Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a  $G$ -torsor flat locally and use fppf descent.

**Question 1.1.3:** Given a  $G$ -torsor  $T$  that is fppf locally trivial, is it étale locally trivial?

**Answer 1.1.4:** In general no, but yes if  $G$  is smooth.

*Proof (Sketch).*

You can take an fppf local trivialization, trivialize by  $p$  itself, then slice to get an étale trivialization. Given a torsor  $T \rightarrow X$ , we can base change it to itself:

$$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow \wr \exists & & \downarrow \\ T & \xrightarrow{f} & X \end{array}$$

The torsor  $T \times_X T \rightarrow T$  is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that  $T \times T \cong T \times G$  by the  $G$ -action map, which is equivalent to triviality here. Here  $f$  is smooth map since  $G$  itself was smooth and the fibers of  $T$  are isomorphic to the fibers of  $G$ . We can thus find some  $U$  such that

$$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow \wr \exists & & \downarrow \\ T & \xrightarrow{f} & X \\ \uparrow \text{closed} & & \uparrow \\ U & \xrightarrow{\exists \text{ét}} & X \end{array}$$

Here “slicing” means finding such a  $U$ , and this can be done using the structure theorem for smooth morphisms. ■

**Example 1.1.5 (non-smooth group schemes):**

- $\alpha_p$ , the kernel of Frobenius on  $\mathbb{A}^1$  or  $\mathbb{G}_a$ ,
- $\mu_p$  in characteristic  $p$ , representing  $p$ th roots of unity, the kernel of Frobenius on  $\mathbb{G}_m$ ,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$ , etc.

### 1.1.1 What Hilbert 90 Means

**Example 1.1.6(?)**: Let  $X = \operatorname{Spec} k, n = 1$ , so we're looking at  $H^*(\operatorname{Spec} k, \mathbb{G}_m)$ .

$$\begin{aligned} H^1((\operatorname{Spec} k)_{\text{zar}}, \mathbb{G}_m) &= 0 \\ &= H^1((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m) \\ &= H^1(\operatorname{Gal}(k^s/k), \bar{k}^\times). \end{aligned}$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.<sup>1</sup>

Let's generalize this observation.

**Example 1.1.7(?)**: Let  $X$  be any scheme and  $n = 1$ , then  $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$ .

**Example 1.1.8(?)**: Let's compute  $H^1(X_{\text{ét}}, \mu_\ell)$  where  $\ell$  is an invertible function on  $X$ . We have a SES of étale sheaves, the **Kummer sequence**,

$$1 \rightarrow \mu_\ell \rightarrow \mathbb{G}_m \xrightarrow{z \mapsto z^\ell} \mathbb{G}_m \rightarrow 1.$$

This is exact in the étale topology since adjoining an  $\ell$ th power of any function gives an étale cover. We get a LES in cohomology

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & \swarrow & & & \\ H^0(X_{\text{ét}}, \mu_\ell) & \longrightarrow & H^0(X_{\text{ét}}, \mathbb{G}_m) & \xrightarrow{z \mapsto z^\ell} & H^0(X_{\text{ét}}, \mathbb{G}_m) & & \\ & & \swarrow & & & & \\ H^1(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \operatorname{Pic}(X) & \xrightarrow{[\ell]} & \operatorname{Pic}(X) & & \\ & & \swarrow & & & & \\ H^2(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \dots & & & & \end{array}$$

We know that  $H^0(X_{\text{ét}}, \mathbb{G}_m)$  are invertible functions on  $X$ , and the red term is what we'd like to compute.

Suppose now  $H^0(X, \mathcal{O}_X) = k = \bar{k}$ , then  $H^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k)$  since it is the kernel of the  $\ell$ th power map. We can also compute  $H^1(X_{\text{ét}}, \mu_\ell)$ , since our diagram reduces to

<sup>1</sup>This is called "90" since Hilbert numbered his theorems in at least one of his books.

$$\begin{array}{ccccc}
& & & & 0 \\
& & \swarrow & & \\
\mu_\ell(k) & \xleftarrow{\quad} & k^\times & \xrightarrow{z \mapsto z^\ell} & k^\times \\
& \searrow \delta & & & \\
H^1(X_{\text{ét}}, \mu_\ell) & \xrightarrow{\quad} & \text{Pic}(X)[\ell] & \xrightarrow{[\ell]} & \text{Pic}(X) \\
& \nwarrow & & & \\
H^2(X_{\text{ét}}, \mu_\ell) & \xrightarrow{\quad} & \dots & & 
\end{array}$$

where surjectivity of  $\delta$  follows from the fact that  $k = \bar{k}$  and thus every element has an  $\ell$ th root, making  $H^1$  the kernel of  $[\ell]$ .

**Example 1.1.9(?):** Let  $X/k$  with  $k = \bar{k}$  with  $\ell$  invertible in  $k$ , then (claim)  $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_\ell$  given by sending a generator to some choice of a primitive  $\ell$ th root of unity. To be explicit, we have a representation  $\mathbb{Z}/\ell\mathbb{Z} = \text{hom}(\cdot, \text{Spec } k[t]/t(t-1)\cdots(t-\ell+1))$  and  $\mu_\ell = \text{Spec } k[t]/t^\ell - 1$ . These are both disjoint unions of points, and hence schemes of dimension zero since  $\ell$  is invertible in the base and the Chinese Remainder Theorem, so one can write down the isomorphism explicitly between the schemes and hence the functors they represent.

**Corollary 1.1.10(?).**

If  $\mu_\ell \subseteq k$ , then

$$H^i(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) = H^i(X_{\text{ét}}, \mu_\ell).$$

Since the isomorphism depends on the choice of a primitive root, this will not be Galois equivariant, which will come up when we talk about Galois actions on étale cohomology. This already happens for  $H^0$ , since  $G \curvearrowright \mathbb{Z}/\ell\mathbb{Z}$  trivially but not on  $\mu_\ell$ .

### 1.1.2 Geometric Interpretations

Let  $X$  be an affine scheme, we now know  $H^1(X_{\text{ét}}, \mathbb{F}_p) = \text{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$ , the Artin-Schreier map, and these are  $\mathbb{F}_p$ -torsors. We also know  $H^1(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$  in terms of the LES if  $k = \bar{k}$  and  $\text{ch}(k) = p$ , and this is a  $\mathbb{Z}/\ell\mathbb{Z}$ -torsor. Being torsors here geometrically means they're covering spaces with those groups as Galois groups.

**Question 1.1.11:** How does one write down these torsors/covering spaces?

**Example 1.1.12(?):** Given

$$[Y] \in H^1(X_{\text{ét}}, \mathbb{F}_p) = \text{coker}(\mathcal{O}_X \rightarrow \mathcal{O}_X)$$

where we write  $[Y]$  to denote thinking of the torsor as some geometric object, how do we write down the covering space? Using Artin-Schreier, we can write  $Y = \{y^p - y = a\}$  for some  $a \in \mathcal{O}_X$ , an **Artin-Schreier covering**.

If  $\ell \neq \text{ch}(k)$  and  $[Z] \in H^1(X_{\text{ét}}, \mu_\ell)$  and assume  $\text{Pic}(X) = 0$ . Then we can write

$$H^1 = \text{coker}(\mathcal{O}_X \xrightarrow{x \mapsto x^\ell} \mathcal{O}_X^\times)$$

In this case,  $Z = \{z^\ell = f\}$  where  $f \in \mathcal{O}_X^\times$  is an element representing the class in cohomology, and  $\mu_\ell \curvearrowright Z$  by multiplication by  $z$ .