Homotopy Groups of Spheres

Graduate Student Seminar

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Summary

Big Points

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

History

- 1860s-1890s: (Roughly) defined by Jordan for complex

Examples

Application: SO3

Application: $\pi_1(\mathrm{SO}(n, \mathbb{R})$, the lie group of rigid rotations in 3-space. The fibration $\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$ yields a LES in homotopy:

which reduces to

and thus $\pi_1(\mathrm{SO}(3, \mathbb{R})) \cong \pi_1(\mathrm{SO}(4, \mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(\mathrm{SO}(3, \mathbb{R}))$. Use the fact that "accidental" homeomorphism in low dimension $\mathrm{SO}(3, \mathbb{R}) \cong_{\mathrm{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

> *Can also use the fact that $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is a double cover from the universal cover.*

Important consequence: $\mathrm{SO}(3, \mathbb{R})$ is not simply connected! See "plate trick", there is a loop of rotations that is not contractible, but squares to the identity. Causes problems in robotics (leads to paths in configuration spaces that encounter singularities) and compute graphics (smoothly interpolating between e.g. quaternions for rotated camera views).

Examples

Example: Knowing Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.
Use the fact that for any fixed group G, the functor

$$\begin{aligned} h_G(\,\cdot\,) : \mathrm{hoTop}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ X &\mapsto . \end{aligned}$$

representable by a space called BG (Brown's representability theorem). I.e., letting $I(G, X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism $I(G, X) \cong [X, BG]$. In general, identify $G = \mathrm{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

> *Note that for a poset of spaces $(M_i, \hookrightarrow)$, the space $M^\infty := \varinjlim M_i$. This are infinite dimensional "Hilbert manifolds".*

Proof:

$$I(\mathbb{R}^1, X) = [X, B(GL(1, \mathbb{R}))] \tag{1}$$

Last Part

Theorems and Definitions

A map $f : X \longrightarrow Y$ is called a *weak homotopy equivalence* if the induced maps $f_i^* : \pi_i(X, x_0) \longrightarrow \pi_i(Y, f(x_0))$ are isomorphisms for every $i \geq 0$. If a map $X \xrightarrow{f} Y$ satisfies $f(X^{(n)}) \subseteq Y^{(n)}$, then f is said to be a *cellular map*. Any map $X \xrightarrow{f} Y$ between CW complexes is homotopic to a cellular map. For every topological space X, there exists a CW complex Y and a weak homotopy equivalence $f : X \longrightarrow Y$. Moreover, if X is n-dimensional, Y may be chosen to be n-connected and is obtained from X by attaching cells of dimension greater than n. **Abbreviated statement**: if X, Y are CW complexes, then any map $f : X \longrightarrow Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence. (Note: f must induce maps on all homotopy groups simultaneously.) If X is an n- connected CW complex, then there are maps $\pi_i X \longrightarrow \pi_{i+1}\Sigma X$ which is an isomorphism for $i \leq 2n$ and a surjection for $i = 2n + 1$.

- Theorem: $\pi_1 S^1 = \mathbb{Z}$