Bott / Tu: Applications of Spectral Sequences

Notation and Remarks

- For M a manifold, $T(M)$ is the unit tangent bundle of M
- For R a ring $R\delta_i$ denotes a copy of R appearing in the ith (co)homological degree
- $S^n \subset \mathbb{R}^{n+1}$ and $S^{2n-1} \subset \mathbb{C}^n$
- Theorem: $F \to E \to B$ a fibration results in
  $E_2^{p,q} = H^p(B, H^q(F; G)) = H^p(B; G) \otimes H^q(F; G)$
  for nice enough spaces X and groups G
  - Corollary: $H^n(X \times Y) = \bigoplus_{p+q=n} H^p(X, H^q(Y))$
- Facts about tensor products
  - $(rm) \otimes n = r(m \otimes n) = m \otimes (rn)$
  - $(r+s)(m \otimes n) = rm \otimes n + sm \otimes n$
  - $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q = \mathbb{Z}/\gcd(p,q)$ and $\gcd(p,q) = 1$ yields 0.
  - Some computations:
    - $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$
    - $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$
    - $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$
    - $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$
    - $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$
    - $R[x] \otimes_R S \cong S[x]$
    - $k \to K$ a field extension: $k[x]/(f) \otimes_k K \cong K[x]/(f)$
  - Symmetric, Associative
  - $(\oplus A_i) \otimes B = \oplus(A_i \otimes B)$
  - $\mathbb{Z} \otimes A = A$
  - $\mathbb{Z}_n \otimes A = \frac{A}{nA}$

List of Results

- A simply connected n-dimensional manifold M_n is orientable
  - Use $S^{n-1} \to T(M_n) \to M_n$
- $H^*(\mathbb{CP}^2) = \mathbb{R}\delta_0 + \mathbb{R}\delta_2 + \mathbb{R}\delta_4$

- Use $S^1 \to S^5 \to \mathbb{CP}^2$
- $H^*(\mathbb{CP}^2) = \frac{\mathbb{R}[x]}{(x^3)}$
  - Use $S^1 \to S^5 \to \mathbb{CP}^2$
- $H^*(\mathbb{CP}^n) = \sum_{i=0}^{n} \mathbb{R}\delta_{2i}$
  - Use $S^1 \to S^{2n+1} \to \mathbb{CP}^n$
- $H^*(\mathbb{CP}^n) = \frac{\mathbb{R}[x]}{(x^{n+1})}$
  - Use $S^1 \to S^{2n+1} \to \mathbb{CP}^n$
- $H^*(SO^3) = \mathbb{Z}\delta_0 + \mathbb{Z}_2\delta_2 + \mathbb{Z}\delta_3$
  - Use $S^1 \to T(S^2) \to S^2$ and identify $T(S^2) = SO^3$
  - Also use $E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$
- $H^*(SO^4) = ?$
  - Use $SO^3 \to SO^4 \to S^3$
- $H^*(U^n) = ?$
  - Use $U^{n-1} \to U^n \to S^{2n-1}$
- $H^*(\Omega S^2) = \sum_{i=0}^{\infty} \mathbb{Z}\delta_i$
  - Use $\Omega S^2 \to PS^2 \to S^2$
  - Also use $E_2^{p,q} = H^p(S^2, H^q(\Omega S^2))$
- $H^*(\Omega S^3) = \sum_{i=0}^{\infty} \mathbb{Z}\delta_{2i}$
  - Use $\Omega S^3 \to PS^3 \to S^3$
- $H^*(\Omega S^n) = \sum_{i=0}^{\infty} \mathbb{Z}\delta_{i(n-1)}$
  - Use $\Omega S^3 \to PS^3 \to S^3$
- $H^*(\Omega S^2) = \frac{\mathbb{Z}[x]}{(x^2)} \otimes \mathbb{Z}\{1, e, \frac{1}{2!}e^2, \cdots\}, \dim x = 1, \dim e = 2$
  - Use $\Omega S^3 \to PS^3 \to S^3$
- $H^*(\Omega S^n) = \frac{\mathbb{Z}[x]}{(x^2)} \otimes \mathbb{Z}\{1, e, \frac{1}{2!}e^2, \cdots\}, \dim x = n-1, \dim e = 2(n-1)$
  - Use $\Omega S^3 \to PS^3 \to S^3$

List of Fibrations

- $S^1 \to S^{2n+1} \to \mathbb{CP}^n$, the Hopf fibration?
- $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ the generalized Hopf fibration? (not used here)
- Hopf Fibrations
  - $S^0 \to S^1 \to S^1$

        - Induced by $S^1 \subset \mathbb{R}^2 \to S^1 = \mathbb{R} \bigcup \infty$
    - $S^1 \to S^3 \to S^2$
        - Induced by $S^3 \subset \mathbb{C}^2 \to S^2 = \mathbb{C} \bigcup \infty$
    - $S^3 \to S^7 \to S^4$
        - Induced by $S^7 \subset \mathbb{H}^2 \to S^4 = \mathbb{H} \bigcup \infty$
    - $S^7 \to S^{15} \to S^8$
        - Induced by $S^{15} \subset \mathbb{O}^2 \to S^8 = \mathbb{O} \bigcup \infty$
- $SO^3 \to SO^4 \to S^3$
- $U^{n-1} \to U^n \to S^{2n-1}$
    - Can compute $H^*(U^n)$
- $\Omega S^n \to PS^n \to S^n$, path-loop fibration
    - $\Omega S^3 \to PS^3 \to S^3$:
        - Can compute $H^*(\Omega S^n)$
- $Y \to X \times Y \to X$ (not used here)

Fibrations

- $SO_{n-1}(R) \to SO_n(R) \to S^{n-1}$
- $S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}$
- $S^1 \to S^{2n+1} \to \mathbb{CP}^n$
- $\Omega B \to PB \to B$
- $K(A, n) \to K(B, n) \to K(C, n)$ for any SES of groups.
- $S^0 \to S^1 \to \mathbb{RP}^1 = S^1$
- $S^1 \to S^3 \to \mathbb{CP}^1 = S^2$
- $S^3 \to S^7 \to \mathbb{HP}^1 = S^4$
- $S^7 \to S^{15} \to \mathbb{OP}^1 = S^8$

Define the Stiefel Manifold:

$$\mathbb{V}(k, n) = \{A \in \mathbb{F}^{nk} \mid A\bar{A}^t = I\}$$

and the Grassmanian

$$G(k, n) = ?$$

Obtained from fiber bundles involving Stiefel Manifold :

- $O^{n-1} \to O^n \to S^{n-1}$
- $SO^{n-1} \to SO^n \to S^{n-1}$

- $U^{n-1} \to U^n \to S^{2n-1}$
- $SU^{n-1} \to SU^n \to S^{2n-1}$
- $Sp^{n-1} \to Sp^n \to S^{4n-1}$
- $SO^n \to O^n \to S^0$
- $SU^n \to U^n \to S^1$
- $\mathbb{V}(k,k) \to \mathbb{V}(k,n) \to \mathbb{G}(k,n)$

Interesting Spaces to Look At:

$O, SO, Spin, U, or Sp$