

# Moduli Spaces

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March 15, 2020

## Contents

|                                |          |
|--------------------------------|----------|
| <b>1 Thursday January 9th</b>  | <b>1</b> |
| 1.1 Representability . . . . . | 2        |
| 1.2 Projective Space . . . . . | 6        |
| <b>2 Tuesday January 14th</b>  | <b>8</b> |

## List of Definitions

|  |    |
|--|----|
| 1.1 Definition . . . . .                                     | 2  |
| 1.2 Definition . . . . .                                     | 3  |
| 2.1 Definition – Equalizer . . . . .                         | 11 |
| 2.2 Definition . . . . .                                     | 12 |
| 2.3 Definition – Zariski Sheaf . . . . .                     | 14 |
| 2.4 Definition – Subfunctors, Open/Closed Functors . . . . . | 14 |
| 2.5 Definition – coverings . . . . .                         | 16 |

## List of Theorems

|                                |    |
|--------------------------------|----|
| 1.1 Theorem – Yoneda . . . . . | 2  |
| 1.1 Proposition . . . . .      | 6  |
| 2.1 Proposition . . . . .      | 17 |

## 1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, . . . Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces

- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

## 1.1 Representability

Last time: Fix an  $S$ -scheme, i.e. a scheme over  $S$ .

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \mathrm{hom}_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where  $T' \xrightarrow{f} T$  is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ T &\mapsto x \longrightarrow \text{triangles} \end{aligned}$$

of the form

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & X \\ & \searrow & \nearrow \\ & T & \end{array} .$$

**Theorem 1.1 (Yoneda).**

$$\mathrm{hom}_{\mathrm{Fun}}(h_x, F) = F(x).$$

**Corollary 1.1.**

$$\mathrm{hom}_{\mathrm{Sch}/S}(x, y) \cong \mathrm{hom}_{\mathrm{Fun}}(h_x, h_y).$$

**Definition 1.1.**

A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

**Definition 1.2.**

A **moduli space** for that “something” appearing above is an  $M \in \text{Obj}(\text{Sch}/S)$  such that  $F \cong h_M$ .

Now fix  $S = \text{Spec}(k)$ .

$h_m$  is the functor of points over  $M$ .

**Remark (1)**  $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{“families over Spec } k\text{”} = F(\text{Spec } k)$ .

**Remark (2)**  $h_M(M) \cong F(M)$  are families over  $M$ , and  $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{U_{\text{niv}}}$  is the universal family

Every family is uniquely the pullback of  $\xi_{U_{\text{niv}}}$ . This makes it much like a classifying space.

For  $T \in \text{Sch}/S$ ,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{U_{\text{niv}}}). \end{aligned}$$

where  $T \xrightarrow{f} M$  and  $f = h_M(f)(\text{id}_M)$ .

**Remark (3)** If  $M$  and  $M'$  both represent  $F$  then  $M \cong M'$  up to unique isomorphism.

$$\xi_M \qquad \qquad \xi_{M'}$$

$$M \xrightarrow{f} M'$$

$$M' \xrightarrow{g} M$$

$$\xi_{M'} \qquad \qquad \xi_M$$

which shows that  $f, g$  must be mutually inverse by using universal properties.

**Example 1.1.**

A length 2 subscheme of  $\mathbb{A}_k^1$  then  $F(S) = \{V(x^2 + bx + c)\} \subset \mathbb{A}'_5$  where  $b, c \in \mathcal{O}_s(s)$ , which is functorially bijective with  $\{b, c \in \mathcal{O}_s(s)\}$  and  $F(f)$  is pullback.

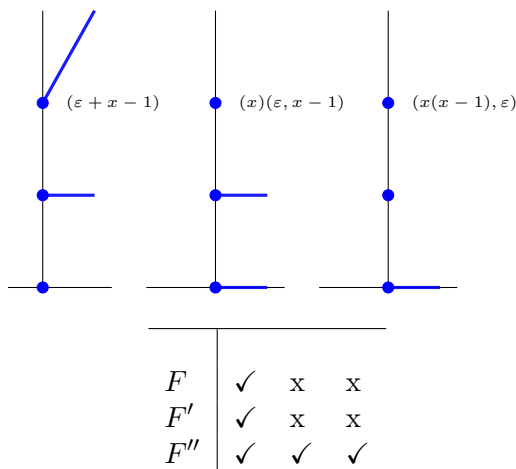
Then  $F$  is representable by  $\mathbb{A}_k^2(b, c)$  and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where  $b, c \in k[b, c]$ .

Moreover,  $F'(S)$  is the set of effective Cartier divisors in  $\mathbb{A}'_5$  which are length 2 for every geometric fiber.  $F''(S)$  is the set of subschemes of  $\mathbb{A}'_5$  which are length 2 on all geometric fibers. In both cases,  $F(f)$  is always given by pullback.

Problem:  $F''$  is not a good moduli functor, as it is not representable. Consider  $\text{Spec } k[\varepsilon]$ .



$$\begin{array}{ccccc}
 \text{Spec } k & \xleftarrow{i} & \text{Spec } k[\varepsilon] & & \\
 & & & \nearrow & \\
 & & & & = F'(\text{Spec } k) \\
 F(\text{Spec } k[\varepsilon]) & \xrightarrow{F(i)} & F(\text{Spec } k) & & \\
 \uparrow \subset & & \uparrow \in & & \\
 T_p F','' & & P = V(x(x-1)) & & \\
 & & & \searrow & \\
 & & & & = F''(\text{Spec } k)
 \end{array}$$

We think of  $T_p F',''$  as the tangent space at  $p$ .

If  $F$  is representable, then it is actually the Zariski tangent space.

$$\begin{array}{ccc}
 M(\text{Spec } k[\varepsilon]) & \longrightarrow & M(\text{Spec } k) \\
 \uparrow \subset & & \uparrow \subset \\
 T_p M & \longrightarrow & p
 \end{array}$$

$$\begin{array}{ccc}
 & \text{Spec } k & \\
 \swarrow & & \searrow \text{?} \\
 \text{Spec } k[\varepsilon] & \xrightarrow{\quad} & \text{Spec } \mathcal{O}_{M,p} \subset M
 \end{array}$$
  

$$\begin{array}{ccc}
 & & k \\
 & \nearrow & \uparrow \\
 \mathcal{O}_{M,p} & \xrightarrow{\quad} & k[\varepsilon]
 \end{array}$$
  

$$\begin{array}{ccc}
 m_p & & (\varepsilon) \\
 \\
 m_p^2 & & 0
 \end{array}$$

Moreover,  $T_p M = (m_p/m_p^2)^\vee$ , and in particular this is a  $k$ -vector space. To see the scaling structure, take  $\lambda \in k$ .

$$\begin{aligned}
 \lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
 \varepsilon &\mapsto \lambda \varepsilon
 \end{aligned}$$

$$\lambda^* : \text{Spec } (k[\varepsilon]) \longrightarrow \text{Spec } (k[\varepsilon])$$

$$\begin{aligned}
 \lambda : M(\text{Spec } (k[\varepsilon])) &\longrightarrow M(\text{Spec } (k[\varepsilon])) \\
 \cup &\quad \cup \\
 T_p M &\longrightarrow T_p M.
 \end{aligned}$$

**Conclusion:** If  $F$  is representable, for each  $p \in F(\text{Spec } k)$  there exists a unique point of  $T_p F$  that are invariant under scaling.

1. If  $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$ , there exists a fiber product

$$\begin{array}{ccc}
 F \times_G F' & \xrightarrow{\quad} & F' \\
 \downarrow & & \downarrow \\
 F & \xrightarrow{\quad} & G
 \end{array}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes  $X \times_T Y$  for  $X, Y \longrightarrow T$ ,

where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If  $F, F', G$  are representable, then so is the fiber product  $F \times_G F'$ .
4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set},$$

for any  $T \xrightarrow{f} S$  there is an induced functor

$$\begin{aligned} F_T : (\text{Sch}/T) &\longrightarrow \text{Set} \\ x &\mapsto F(x). \end{aligned}$$

5.  $F$  is representable by  $M/S$  implies that  $F_T$  is representable by  $M_T = M \times_S T/T$ .

## 1.2 Projective Space

Consider  $\mathbb{P}_{\mathbb{Z}}^n$ , i.e. “rank 1 quotient of an  $n + 1$  dimensional free module”.

### Proposition 1.1.

$\mathbb{P}_{\mathbb{Z}}^n$  represents the following functor

$$\begin{aligned} F : \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ F(S) &= \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 / \sim. \end{aligned}$$

where  $\sim$  identifies diagrams of the following form:

$$\begin{array}{ccccc} \mathcal{O}_s^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow = & & \downarrow \cong & & \\ \mathcal{O}_s^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and  $F(f)$  is given by pullbacks.

**Remark**  $\mathbb{P}_S^n$  represents the following functor:

$$\begin{aligned} F_S : (\text{Sch}/S)^{\text{op}} &\longrightarrow \text{Set} \\ T &\mapsto F_S(T) = \left\{ \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

*Proof (of Theorem).*

Note:  $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$  is the same as giving  $n + 1$  sections  $s_1, \dots, s_n$  of  $L$ , where surjectivity ensures that they are not the zero section.

So

$$F_i(S) = \{ \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 \} / \sim,$$

with the additional condition that  $s_i \neq 0$  at any point.

There is a natural transformation  $F_i \longrightarrow F$  by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$  is a subfunctor iff  $F(s) \hookrightarrow G(s)$ .

**Claim:** It is enough to show that each  $F_i$  and each  $F_{ij}$  are representable, since we have natural transformations:

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \uparrow & & \uparrow \\ F_{ij} & \longrightarrow & F_j \end{array}$$

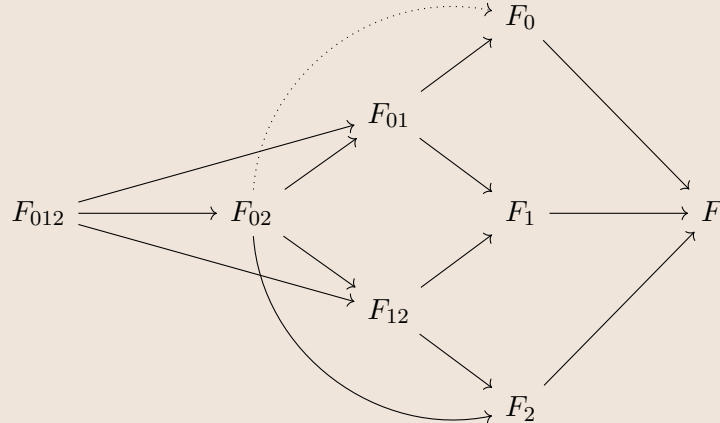
and each  $F_{ij} \longrightarrow F_i$  is an open embedding (on the level of their representing schemes).

**Example .**

For  $n = 1$ , we can glue along open subschemes



For  $n = 2$ , we get overlaps of the following form:



This claim implies that we can glue together  $F_i$  to get a scheme  $M$ . We want to show that  $M$  represents  $F$ .  $F(s)$  (LHS) is equivalent to an open cover  $U_i$  of  $S$  and sections of  $F_i(U_i)$  satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for  $\mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0$ ,  $U_i$  is the locus where  $s_i \neq 0$  and by surjectivity, this gives a cover of  $S$ .

RHS to LHS comes from gluing.

*Proof (of Claim).*

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than  $s_i$ .

So specifying  $F_i(S)$  is equivalent to specifying  $n - 1$  functions  $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$  with  $f_k \neq 0$ . We know this is representable by  $\mathbb{A}^n$ .

We also know  $F_{ij}$  is obviously the same set of sequences, where now  $s_j \neq 0$  as well, so we need to specify  $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$  with  $f_j \neq 0$ . This is representable by  $\mathbb{A}^{n-1} \times \mathbb{G}_m$ , i.e.  $\text{Spec } k[x_1, \dots, \widehat{x_i}, \dots, x_n, x_j^{-1}]$ . Moreover,  $F_{ij} \hookrightarrow F_i$  is open.

What is the compatibility we are using to glue? For any subset  $I \subset \{0, \dots, n\}$ , we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigcap_{i \in I} F_i,$$

and  $F_I \longrightarrow F_J$  when  $I \supset J$ .

## 2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space  $\mathbb{P}_{\mathbb{Z}}^n$  constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}_{\mathbb{Z}}^n} : \mathbb{P}_{\mathbb{Z}}^n \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ s &\mapsto \mathbb{P}_{\mathbb{Z}}^n(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for  $L$  a line bundle, up to isomorphisms of diagrams:

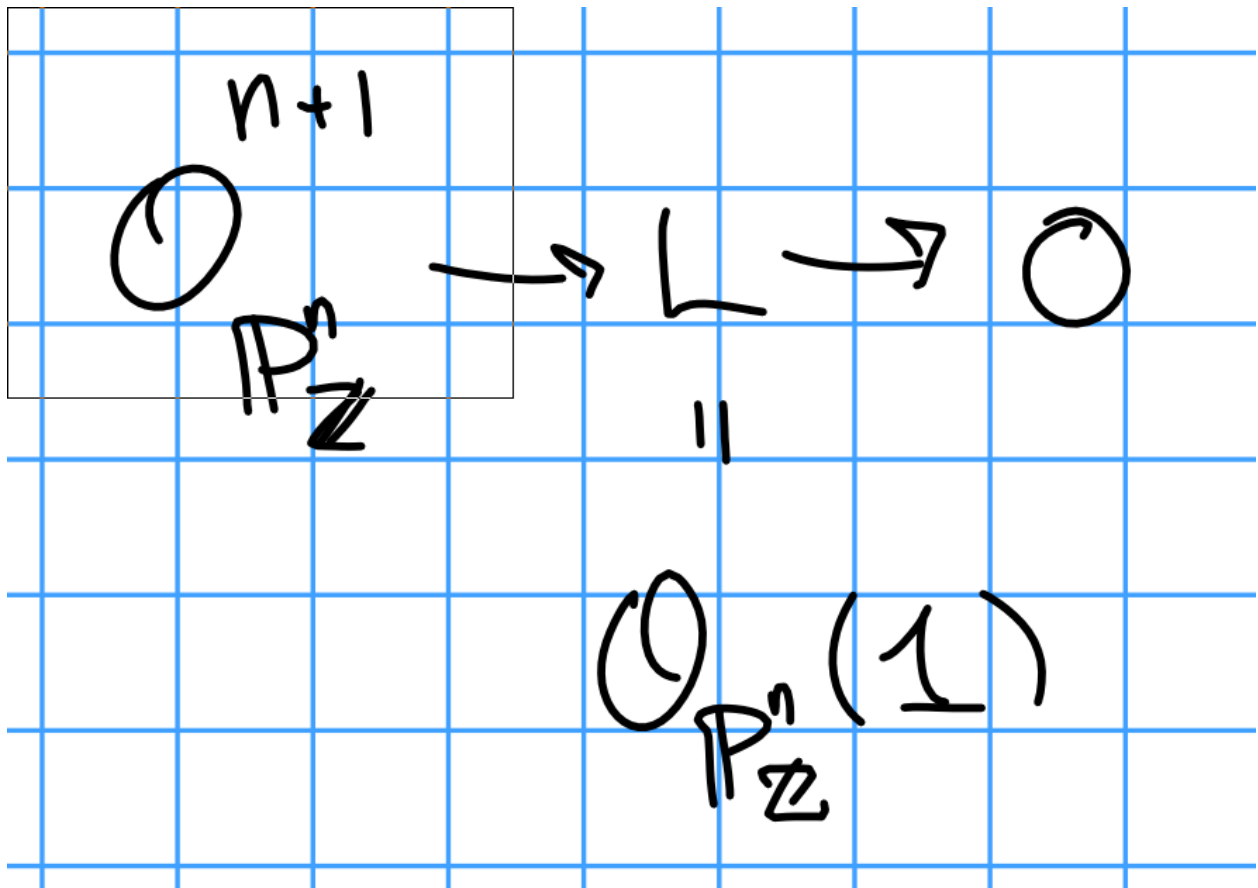




That is, line bundles with  $n + 1$  sections that globally generate it, up to isomorphism.

The point was that for  $F_i \subset \mathbb{P}_{\mathbb{Z}}^n$  where  $F_i(s) = \{ \mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0 \mid s_i \text{ invertible} \}$  are representable and can be glued together, and projective space represents this functor.

Remark: Because projective space represents this functor, there is a universal object:



and other functors are pullbacks of the universal one.

**Example 2.1.**

Show that  $\mathbb{P}^n_{\mathbb{Z}}$  is proper over  $\text{Spec } \mathbb{Z}$ . Use the evaluative criterion, i.e. there is a unique lift



**Definition 2.1** (Equalizer).

For a category  $C$ , we say a diagram  $x \rightarrow y \rightrightarrows z$  is an *equalizer* iff it is universal wrt the property:



(Here  $X$  is the universal object).

**Example 2.2.**

For sets,  $X = \{y \mid f(y) = g(y)\}$  for  $Y \xrightarrow{f,g} Z$ .

**Definition 2.2.**

A coequalizer is the dual notion,

**Example 2.3.**

Take  $C = \text{Sch}/S$ ,  $X/S$  a scheme, and  $X_\alpha \subset X$  an open cover. We can take two fiber products,  $X_{\alpha\beta}, X_{\beta\alpha}$ :

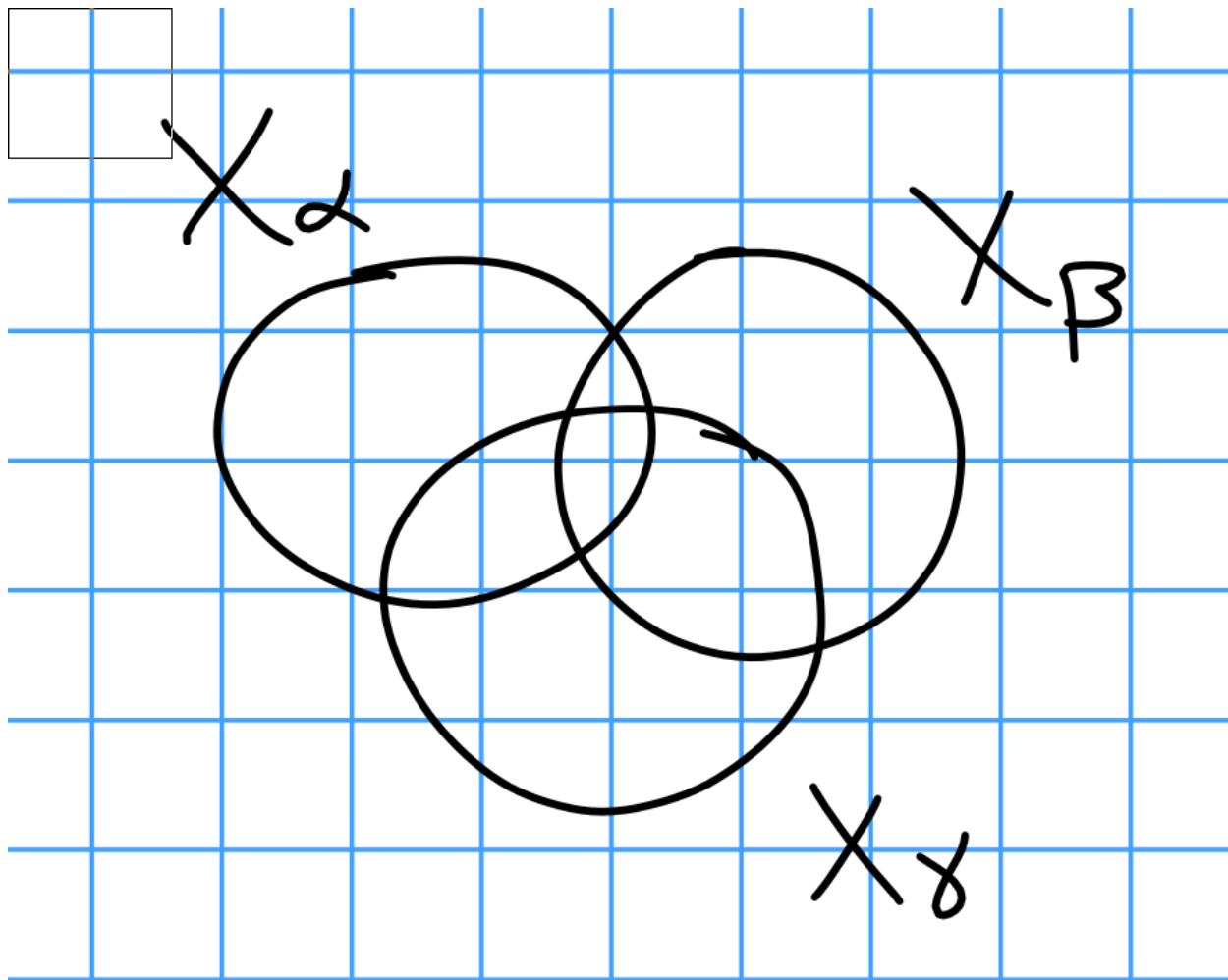


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These are canonically isomorphic.

In  $Sch/S$ , we have  $\coprod_{\alpha\beta} X_{\alpha\beta} \xrightarrow{f_{\alpha\beta}, g_{\alpha\beta}} \coprod X_{\alpha} \longrightarrow X$  where  $f_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_{\alpha}$  and  $g_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_{\beta}$ ; this is a coequalizer.

Conversely, we can glue schemes. Given  $X_{\alpha} \longrightarrow X_{\alpha\beta}$  (schemes over open subschemes), we need to check triple intersections:



Then  $\phi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$  must satisfy the cocycle condition:

1.  $\phi_{\alpha\beta}^{-1}(X_{\beta\alpha} \cap X_{\beta\gamma}) = X_{\alpha\beta} \cap X_{\alpha\gamma}$ , noting that the intersection is exactly the fiber product  $X_{\beta\alpha} \times_{X_{\beta}} X_{\beta\gamma}$ .
2. The following diagram commutes:



Then there exists a scheme  $X/S$  such that  $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \rightarrow X$  is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

**Definition 2.3** (Zariski Sheaf).

A functor  $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$  is a *Zariski sheaf* iff for any scheme  $T/S$  and any open cover  $T_\alpha$ , the following is an equalizer:

$$F(T) \rightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

**Example 2.4.**

Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover  $\coprod_{\alpha\beta} T_{\alpha\beta} \rightarrow \coprod_\alpha T_\alpha \rightarrow T$ , since giving a local map to  $X$  that agrees on intersections is enough to specify a map from  $T \rightarrow X$ .

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

**Definition 2.4** (Subfunctors, Open/Closed Functors).

Suppose we have a morphism  $F' \rightarrow F$  in the category  $\text{Fun}(\text{Sch}/S, \text{Set})$ .

- This is a **subfunctor** if  $\iota(T)$  is injective for all  $T/S$ .
- $\iota$  is **open/closed/locally closed** iff for any scheme  $T/S$  and any section  $\xi \in F(T)$  over

$T$ , then there is an open/closed/locally closed set  $U \subset T$  such that for all maps of schemes  $T' \xrightarrow{f} T$ , we can take the pullback  $f^*\xi$  and  $f^*\xi \in F'(T')$  iff  $f$  factors through  $U$ .

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

**Note** This is the same as asking if the subfunctor  $F'$ , which maps to  $F$  (noting a section is the same as a map to the functor of points), and since  $T \rightarrow F$  and  $F' \rightarrow F$ , we can form the fiber product  $F' \times_F T$ :



and  $F' \times_F T \cong U$ .

Note: this is almost tautological!

Thus  $F' \rightarrow F$  is open/closed/locally closed iff



and  $g$  is open/closed/locally closed.

I.e. base change is representable, and (?).

### Exercise (Tautologous)

1. If  $F' \rightarrow F$  is open/closed/locally closed and  $F$  is representable, then  $F'$  is representable as an open/closed/locally closed subscheme
2. If  $F$  is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

#### Definition 2.5 (coverings).

A collection of open subfunctors  $F_\alpha \subset F$  is an open cover iff for any  $T/S$  and any section  $\xi \in F(T)$ , i.e.  $\xi : T \rightarrow F$ , the  $T_\alpha$  in the following diagram are an open cover of  $T$ :





**Example 2.5.**

Given  $F(s) = \{\mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0\}$  and  $F_i(s)$  given by those where  $s_i \neq 0$  everywhere, the  $F_i \rightarrow F$  are an open cover. Because the sections generate everything, taking the  $T_i$  yields an open cover.

**Proposition 2.1.**

A Zariski sheaf  $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$  with a representable open cover is representable.

*Proof.*

Let  $F_\alpha \subset F$  be an open cover, say each  $F_\alpha$  is representable by  $x_\alpha$ . Form the fiber product  $F_{\alpha\beta} = F_\alpha \times_F F_\beta$ . Then  $x_\beta$  yields a section (plus some openness condition?), so  $F_{\alpha\beta} = x_{\alpha\beta}$  representable. Because  $F_\alpha \subset F$ , the  $F_{\alpha\beta} \rightarrow F_\alpha$  have the correct gluing maps. This follows from Yoneda (schemes embed into functors), and we get maps  $x_{\alpha\beta} \rightarrow x_\alpha$  satisfying the gluing conditions. Call the gluing scheme  $x$ ; we'll show that  $x$  represents  $F$ .

First produce a map  $x \rightarrow F$  from the sheaf axioms. We have a map  $\xi \in \prod_{\alpha} F(x_\alpha)$ , and

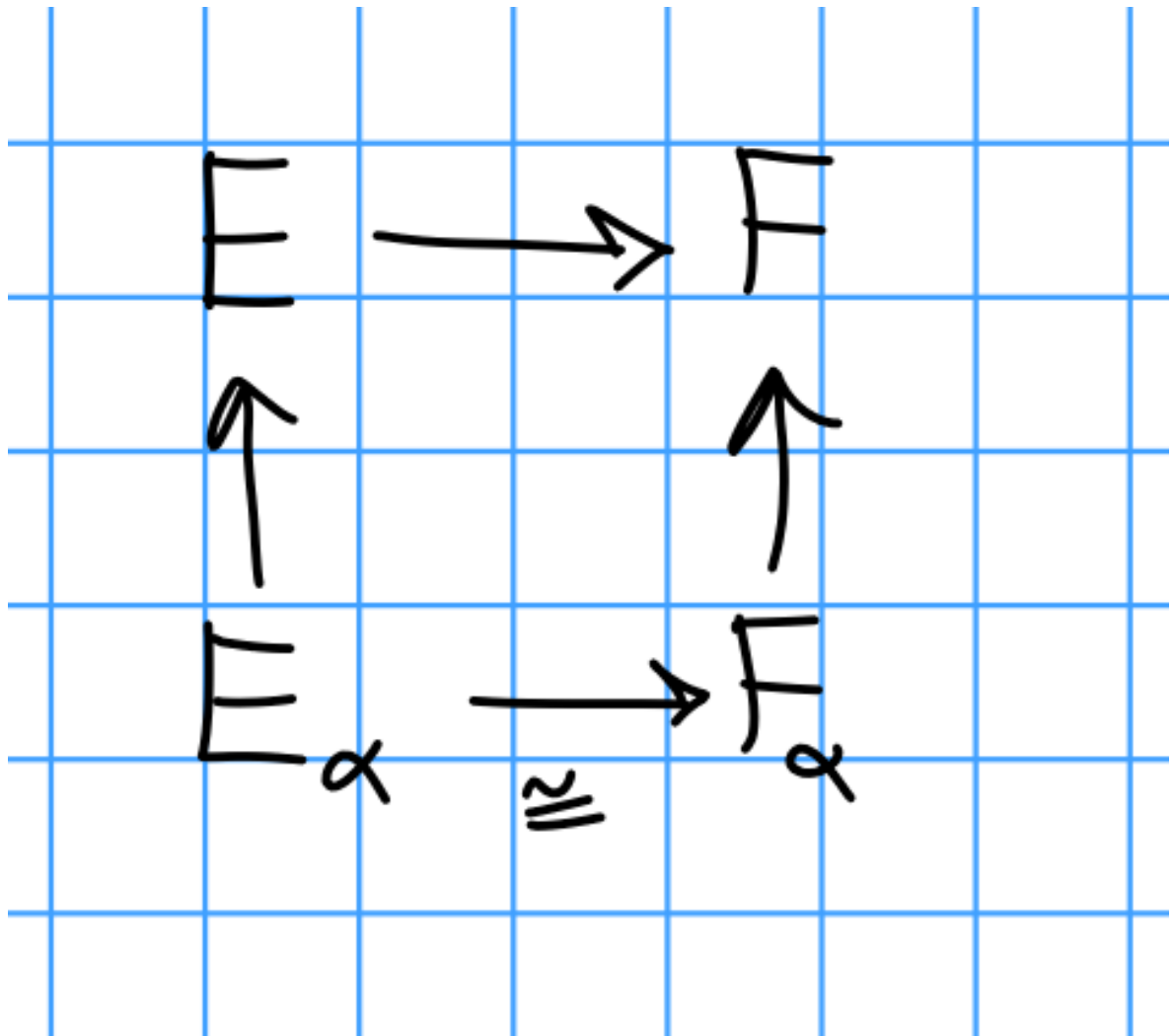
because we can pullback, we get a unique element  $\xi \in F(X)$  coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

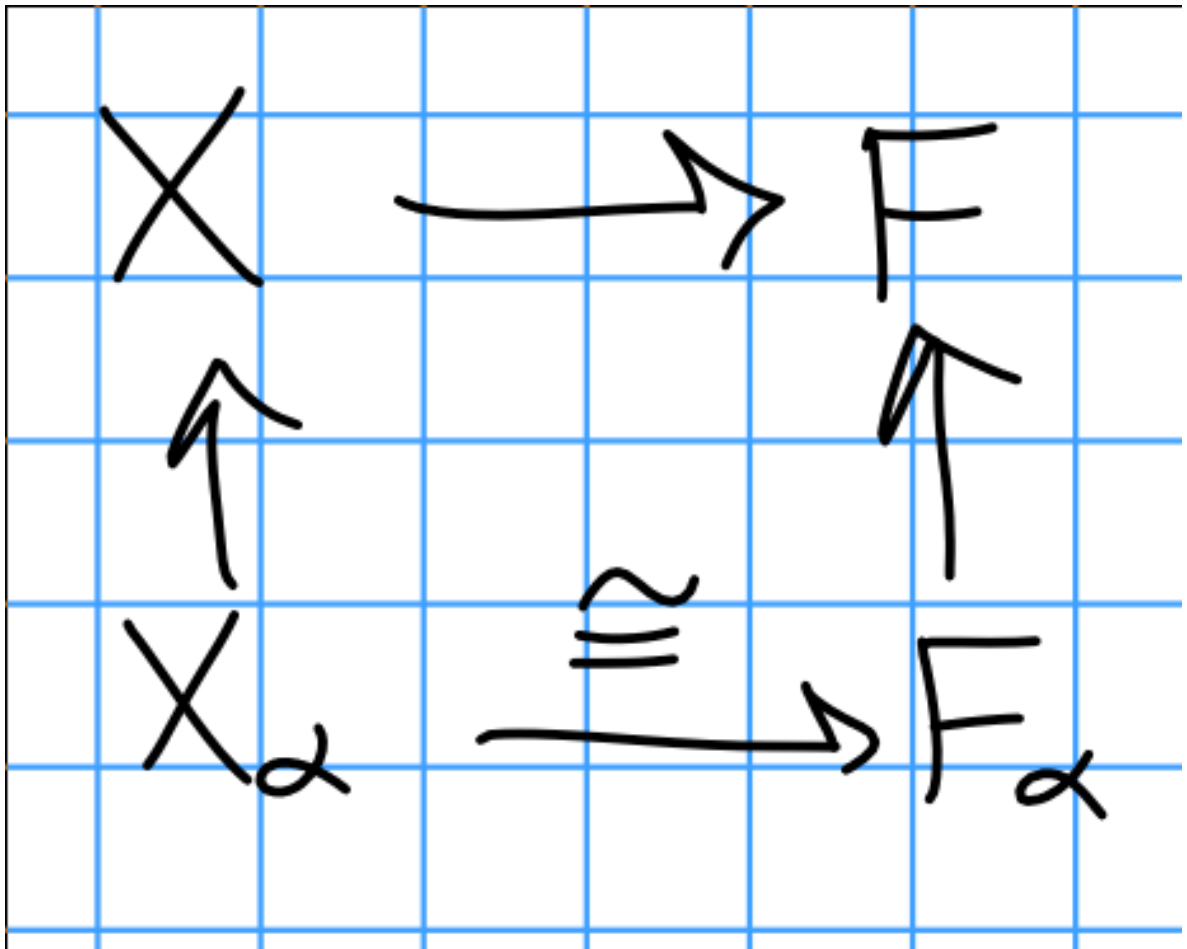
**Lemma 2.1.**

If  $E \rightarrow F$  is a map of functors and  $E, F$  are zariski sheaves, where there are open covers  $E_\alpha \rightarrow E, F_\alpha \rightarrow F$  with commutative diagrams



(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:



**Example 2.6.**

For  $S$  and  $E$  a locally free coherent  $\mathcal{O}_S$  module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} / \sim$$

is a generalization of projectivization, then  $S$  admits a cover  $U_i$  trivializing  $E$ .

Then the restriction  $F_i \longrightarrow \mathbb{P}E$  where  $F_i(T)$  is the above set if  $f$  factors through  $U_i$  and empty otherwise. On  $U_i$ ,  $E \cong \mathcal{O}_{U_i}^{n_i}$ , so  $F_i$  is representable by  $\mathbb{P}_{U_i}^{n_i-1}$  by the proposition. (Note that this is clearly a sheaf.)

**Example 2.7.**

For  $E$  locally free over  $S$  of rank  $n$ , take  $r < n$  and consider the functor  $\text{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} / \sim$  (a Grassmannian) where  $Q$  is locally free of rank  $k$ .

**Exercise**

---

a. Show that this is representable

b. For the plucker embedding  $\mathrm{Gr}(k, E) \rightarrow \mathbb{P} \wedge^k E$ , then a section over  $T$  is given by  $f^*E \rightarrow Q \rightarrow 0$  corresponding to  $\wedge^k f^*E \rightarrow \wedge^k Q \rightarrow 0$ , noting that the left-most term is  $f^* \wedge^k E$ .

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take  $S = \mathrm{Spec} k$ , then  $E$  is a  $k$ -vector space  $V$ , then sections of the Grassmannian are quotients of  $V \otimes \mathcal{O}$  that are free of rank  $n$ .

Take the subfunctor  $G_w \subset \mathrm{Gr}(k, V)$  where

$$G_w(T) = \{\mathcal{O}_T \otimes V \rightarrow Q \rightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting  $V = W \oplus U$ , then  $G_W = \mathbb{A}(\mathrm{hom}(U, W))$ . If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for  $\mathbb{P}^n$  of all flat families of subschemes.