

# Homological Algebra Problem Sets

## Problem Set 3

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# 1 | Wednesday, February 17

**Problem 1.0.1** (Prove Corollary 2.3.2)

For  $R$  a PID, show that an  $R$ -module  $A$  is divisible if and only if  $A$  is injective.

*Recall that a module is divisible if and only if for every  $r \neq 0 \in R$  and every  $a \in A$ , we have  $a = br$  for some  $b \in A$ .*

**Solution:**

Note: we'll assume  $R$  is commutative, and since  $R$  is a domain, it has no nonzero zero divisors and thus all elements  $r \in R$  are left-cancelable.

$\Rightarrow$  : Suppose  $A$  is divisible, we then want to show every  $R$ -module morphism of the following form lifts, where we regard the ideal  $J$  and the ring  $R$  as  $R$ -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since  $R$  is a PID, we have  $J = jR$  for some  $j \in R$ , so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider  $f(j) \in A$ . Since  $A$  is divisible, we have  $A = jA$ , so we can write  $f(j) = j\mathbf{a}'$  for some  $\mathbf{a}' \in A$ . Using  $R$ -linearity and the fact that  $j$  is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending  $R$ -linearly yields a well-defined  $R$ -module morphism. Moreover, the diagram commutes by construction, since  $\iota(1_R) = 1_R$ .

$\Leftarrow$  : Suppose  $A \in R\text{-Mod}$  is injective, where by Baer's criterion we equivalently have a lift of the following form for every  $J \leq R$ :

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let  $j \in R$  be a nonzero element that is not a zero-divisor, we then want to show that  $A = jA$ , i.e. that for every  $\mathbf{a} \in A$ , there is a  $\mathbf{a}' \in A$  such that  $\mathbf{a} = j\mathbf{a}'$ . Fixing  $\mathbf{a} \in A$ , define a map  $f_a : J \rightarrow A$  in the following way: for  $x \in J$ , use the fact that  $\langle j \rangle := jR$  to first write  $x = jr$  for some  $r \in R$ , and then set  $f_a(x) = f_a(jr) := r\mathbf{a}$ . To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion  $jR \hookrightarrow R$  and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an  $r$  on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting  $\mathbf{a}$  range over all elements of  $A$  yields the desired result.

**Problem 1.0.2** (Calculating Ext Groups)

Calculate  $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$  for distinct primes  $p, q$ .

**Solution:**

We'll use the following facts:

- $\varphi : \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\sim} \mathbb{Z}/n$ , where  $\varphi(g) := g(1)$ .
  - That this is an isomorphism follows from
  - Surjectivity: for each  $\ell \in \mathbb{Z}/n$  define a map

$$\begin{aligned} \psi_g : \mathbb{Z} &\rightarrow \mathbb{Z}/n \\ 1 &\mapsto [\ell]_n. \end{aligned}$$

- Injectivity: if  $g(1) = [0]_n$ , then

$$g(x) = xg(1) = x[0]_n = [0]_n.$$

- $\mathbb{Z}$ -module morphism:

$$\varphi(gf) := \varphi(g \circ f) := (g \circ f)(1) = g(f(1)) = f(1)g(1) = \varphi(g)\varphi(f),$$

where we've used the fact that  $\mathbb{Z}/n$  is commutative.

We can start by taking a resolution of  $\mathbb{Z}/p$  by projective  $\mathbb{Z}$ -modules:

$$0 \rightarrow \mathbb{Z} \xrightarrow{m_p} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/p \rightarrow 0$$

$$:= 0 \rightarrow P_{-1} \xrightarrow{m_p} P_0 \xrightarrow{\varepsilon} \mathbb{Z}/p \rightarrow 0,$$

where  $m_p(x) := px$  is multiplication by  $p$ . This is a well-defined resolution since  $\mathbb{Z}$  is a free  $\mathbb{Z}$ -module and hence projective. We now apply the contravariant hom  $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z}/q)$  to the resolution  $0 \rightarrow P_{-1} \rightarrow P_0 \rightarrow 0$  yields

$$0 \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/q) \xrightarrow{m_p^*} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/q) \xrightarrow{\varepsilon^*} 0,$$

and thus

$$\begin{aligned} H_0(\mathbb{Z}/p, \mathbb{Z}/q) &= \ker(m_p^*) \\ H_1(\mathbb{Z}/p, \mathbb{Z}/q) &= \text{coker}(m_p^*). \end{aligned}$$

**Problem 1.0.3 (Weibel 2.3.2)**

For  $A \in \mathbf{Ab}$ , define  $I(A) := \bigoplus_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ , and let  $e_A : A \rightarrow I(A)$ . Show that  $e_A$  is injective.

*Hint: if  $a \in A$ , find a map  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $f(a) \neq 0$  and extend this to a map  $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$ .*

**Problem 1.0.4 (Weibel 2.4.2)**

If  $U : \mathcal{B} \rightarrow \mathcal{C}$  is an exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

*Problem 1.0.5* (Weibel 2.4.3)

If  $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$  is exact with  $P$  projective or  $F$ -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

Show that  $L_{m+1} F(A)$  is the kernel of  $F(M_m) \rightarrow F(P_m)$ . Conclude that if  $P \rightarrow A$  is an  $F$ -acyclic resolution of  $A$ , then  $L_i F(A) = H_i(F(P))$ .

*Problem 1.0.6* (Weibel 2.5.2)

Show that the following are equivalent:

- a.  $A$  is a projective  $R$ -module.
- b.  $\text{Hom}_R(\cdot, A)$  is an exact functor.
- c.  $\text{Ext}_R^{i \neq 0}(A, B) = 0$  and for all  $B$ , i.e.  $A$  is  $\text{Hom}_R(\cdot, B)$ -acyclic for all  $B$ .
- d.  $\text{Ext}_R^1(A, B)$  vanishes for all  $B$ .

*Problem 1.0.7* (Weibel 2.6.4)

Show that  $\text{colim}$  is left adjoint to  $\Delta$ , and conclude that  $\text{colim}$  is right-exact when  $\mathcal{A}$  is abelian and  $\text{colim}$  exists. Show that the pushout, i.e.  $\bullet \leftarrow \bullet \rightarrow \bullet$ , is not an exact functor on  $\mathbf{Ab}$ .