Further Topics in Analysis: Solutions 8

1. Find the pointwise limit of the following sequences of functions on the segment $[0,2]$. Is this convergence uniform on $[0,2]$?

(a) $f_n(x) = \frac{x}{n}$;

(b) $f_n(x) = \frac{x}{nx+1}$;

(c) $f_n(x) = \frac{x^n}{1+x^n}$.

Solution. (a) Fix $x \in [0,2]$. Then

$$\lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus the sequence of functions $f_n(x) = \frac{x}{n}$ converges pointwise on $[0,2]$ to the function $f(x) = 0$.

To verify that $f_n(x) = \frac{x}{n}$ converges to $f(x) = 0$ uniformly on $[0,2]$ we need to check that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in [0,2])(\forall n \in \mathbb{N})[(n \geq N) \Rightarrow (|f_n(x) - f(x)| < \varepsilon)]. \tag{1}$$

Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$. Then for any $x \in [0,2]$ and any $n \geq N$ we obtain

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| = \frac{x}{n} \leq 2\frac{1}{N} < \varepsilon.$$

Thus $f_n(x) = \frac{x}{n}$ converges uniformly on $[0,2]$ to $f(x) = 0$.

(b) Note that

$$0 \leq f_n(x) = \frac{x}{nx+1} \leq \frac{x}{nx} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Then by the sandwich rule we conclude that for each fixed $x \in [0,2]$

$$\lim_{n\to\infty} \frac{x}{nx+1} = 0.$$

So the sequence of functions $\frac{x}{xn+1}$ converges pointwise on $[0,2]$ to the function $f(x) = 0$.

To verify that $f_n(x) = \frac{x}{nx+1}$ converges to $f(x) = 0$ uniformly on $[0,2]$ we need to check (1). Fix $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for any $x \in [0,2]$ and any $n \geq N$ we obtain

$$|f_n(x) - f(x)| = \left|\frac{x}{nx+1} - 0\right| \leq \frac{x}{nx} \leq \frac{1}{N} < \varepsilon.$$

Thus $f_n(x) = \frac{x}{nx+1}$ converges uniformly on $[0,2]$ to $f(x) = 0$.

(c) For $x \in [0,2]$ we compute that

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = f(x), \quad \text{where } f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 1, & \text{if } 1 < x \leq 2. \end{cases}$$

So the sequence of functions $f_n(x) = \frac{x^n}{1+x^n}$ converges pointwise on $[0,2]$ to the function $f(x)$.

Next we show that $f_n(x) = \frac{x^n}{1+x^n}$ does not converge uniformly to $f(x)$ on $[0,2]$. Indeed, all the functions $f_n(x)$ are continuous on $[0,2]$. Assume that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ uniformly on $[0,2]$. Then, by the Weierstrass Theorem on Uniform Convergence, the limit function $f(x)$ must be continuous on $[0,2]$. However $f(x)$ is not continuous on $[0,2]$! We conclude that $f_n(x) = \frac{x^n}{1+x^n}$ does not converge uniformly to $f(x)$ on $[0,2]$.

2. Find the pointwise limit of the following sequences of functions. Is this convergence uniform? Justify your answer.

(a) $f_n(x) = \sqrt[n]{x}$ on the closed segment $[0, 1]$;

(b) $f_n(x) = \frac{x^n - 1}{x^n + 1}$ on the closed segment $[0, 2]$;

(c) $f_n(x) = (1 - x^2)^n$ on the closed segment $[-1, 1]$.

Solution. (a) We see that the pointwise limit of $f_n(x) = \sqrt[n]{x}$ on the interval $[0, 1]$ is

$$\lim_{n\to\infty} \sqrt[n]{x} = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0, 1]. \end{cases}$$

Clearly the pointwise limit is a discontinuous function. Thus by Weierstrass's Theorem on Uniform Convergence (Theorem 12.8) we conclude that the sequences $f_n(x) = \sqrt[n]{x}$ does not converge uniformly on $[0, 1]$.

(b) We see that the pointwise limit of $f_n(x) = \frac{x^n - 1}{x^n + 1}$ on the segment $[0, 2]$ is

$$\lim_{n\to\infty} \frac{x^n - 1}{x^n + 1} = \begin{cases} -1 & \text{if } x \in [0, 1), \\ 0 & \text{if } x = 1, \\ 1 & \text{if } x \in (1, 2]. \end{cases}$$

Clearly the pointwise limit is a discontinuous function. Thus by Weierstrass's Theorem on Uniform Convergence (Theorem 12.8) we conclude that the sequences $f_n(x) = \frac{x^n - 1}{x^n + 1}$ does not converge uniformly on $[0, 2]$.

(c) We see that the pointwise limit of $f_n(x) = (1 - x^2)^n$ on the interval $[-1, 1]$ is

$$\lim_{n\to\infty} (1 - x^2)^n = \begin{cases} 0 & \text{if } x = [-1, 0) \cup (0, 1], \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly the pointwise limit is a discontinuous function. Thus by Weierstrass's Theorem on Uniform Convergence (Theorem 12.8) we conclude that the sequences $f_n(x) = (1 - x^2)^n$ does not converge uniformly on $[-1, 1]$.

3. Find the pointwise limit of the following sequences of functions. Is this convergence uniform? Justify your answer.

(a) $f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{if } x \neq \frac{1}{n} \end{cases}$ on the closed segment $[0, 1]$;

(b) $f_n(x) = \sqrt[n]{x}$ on the open segment $(0, 1)$;

(c) $f_n(x) = \frac{x}{n}$ on the real line $\mathbb{R}$.

Solution. (a) We see that for each $x \in [0, 1]$

$$\lim_{n\to\infty} f_n(x) = 0,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on $[0, 1]$ to the limit function $f(x) = 0$.

We are going to show this convergence is not uniform on $[0, 1]$. To do this we need to check that

$$(\exists\varepsilon > 0)(\forall N \in \mathbb{N})(\exists x \in [0, 1])(\exists n \in \mathbb{N})[(n \geq N) \wedge (|f_n(x) - 0| \geq \varepsilon)].$$

Set $\varepsilon = 1$. For any given $N \in \mathbb{N}$ choose $x_N = \frac{1}{N}$ and $n = N$. Then

$$|f_n(x_N)| = 1 = \varepsilon,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ does not converge to 0 uniformly on $[0, 1]$.

(b) We see that for each $x \in (0, 1)$

$$\lim_{n\to\infty} \sqrt[n]{x} = 1,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on $(0, 1)$ to the limit function $f(x) = 1$.

We are going to show this convergence is not uniform on $[0, 1]$. To do this we need to check that

$$(\exists\varepsilon > 0)(\forall N \in \mathbb{N})(\exists x \in (0, 1))(\exists n \in \mathbb{N})[(n \geq N) \wedge (|f_n(x) - 1| \geq \varepsilon)].$$

Set $\varepsilon = \frac{1}{2}$. For any given $N \in \mathbb{N}$ choose $x_N = \frac{1}{2^N}$ and $n = N$. Observe that $x_N \in (0, 1)$. Then

$$|f_n(x_N) - 1| = |\sqrt[n]{x_N} - 1| = |\frac{1}{2} - 1| = \frac{1}{2} = \varepsilon,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ does not converge to $f(x) = 1$ uniformly on $(0, 1)$.

(c) We see that for each $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = 0,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}$ to the limit function $f(x) = 0$.

We are going to show this convergence is not uniform on $\mathbb{R}$. To do this we need to check that

$$(\exists\varepsilon > 0)(\forall N \in \mathbb{N})(\exists x \in \mathbb{R})(\exists n \in \mathbb{N})[(n \geq N) \wedge (|f_n(x) - 0| \geq \varepsilon)].$$

Set $\varepsilon = 1$. For any given $N \in \mathbb{N}$ choose $x_N = N$ and $n = N$. Then

$$|f_n(x_N)| = 1 = \varepsilon,$$

that is $(f_n(x))_{n\in\mathbb{N}}$ does not converge to 0 uniformly on $[0, 1]$.

Note that in all these examples the limit function is continuous but the convergence is not uniform!

4. Let

$$f_n : [0, 1] \to \mathbb{R} : x \mapsto \frac{n}{n^2 + x^2} \qquad (n \in \mathbb{N}^+).$$

Show that $(f_n)_{n\in\mathbb{N}^+}$ converges uniformly on $[0, 1]$.

Solution. Note that for any $x \in [0, 1]$,

$$f_n(x) = \frac{1/n}{1 + (x/n)^2} \to 0 \text{ as } n \to \infty.$$

Let $f : [0, 1] \to \mathbb{R}$ be the zero function. Then, for any $x \in [0, 1]$,

$$|\, f_n(x) - f(x)\,| = \frac{n}{n^2 + x^2} \leqslant \frac{n}{n^2} = \frac{1}{n} \qquad (n \in \mathbb{N}^+).$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}^+$ such that $N > 1/\varepsilon$. Then, for any $\mathbb{N}^+ \ni n \geqslant N$,

$$|\, f_n(x) - f(x)\,| < \varepsilon \qquad (x \in [0, 1]).$$

That is, $(f_n)_{n\in\mathbb{N}^+}$ converges to f uniformly on $[0, 1]$.

5. Let A be a subset of the real line and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from A to $\mathbb{R}$. Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on A if and only if

$$(\forall\, \varepsilon > 0)(\exists\, N \in \mathbb{N})(\forall\, x \in A)(\forall\, n, m \in \mathbb{N})[(n \geq N) \wedge (m \geq N) \Rightarrow (|f_n(x) - f_m(x)| < \varepsilon)].$$

Hint: Use Cauchy's Theorem (Theorem 10.5).

SOLUTION. Assume that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on A to the limit function $f : A \to \mathbb{R}$. Thus, by Definition 12.4, for any $\varepsilon > 0$

$$(\exists N \in \mathbb{N})(\forall x \in A)(\forall n \in \mathbb{N})[(n \geq N) \Rightarrow (|f_n(x) - f(x)| < \varepsilon/2)].$$

Therefore (compare the proof of Theorem 10.4) by the triangle inequality for any $x \in A$ and for any $n, m \in \mathbb{N}$ we have

$$(n \geq N) \wedge (m \geq N) \Rightarrow (|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f(x)|}_{<\varepsilon/2} + \underbrace{|f_m(x) - f(x)|}_{<\varepsilon/2} < \varepsilon),$$

that is $(*)$ holds.

Assume that $(*)$ holds. Fix $x \in A$. Then $(*)$ means that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers. Therefore by Cauchy's Theorem (Theorem 10.5) $(f_n(x))_{n\in\mathbb{N}}$ converges. Denote the limit of $(f_n(x))_{n\in\mathbb{N}}$ by $f(x)$, i.e. In such a way we defined a function $f : A \to \mathbb{R}$ such that for every fixed $x \in A$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

This means the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on A to the function $f(x)$.

We need to prove that $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ uniformly on A. To do this, we use $(*)$. Fix $\varepsilon > 0$. Then by $(*)$ there exists $N \in \mathbb{N}$ such that for any $x \in A$ and for any $n, m \in \mathbb{N}$ such that $n \geq N$ and $m \geq N$ we have

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

Fix n in the above inequality and let $m \to \infty$. Then

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

Therefore $(f(x)_n)_{n\in\mathbb{N}}$ converges uniformly on A to $f(x)$.