Title

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Remark 1.
There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned}\rho : \mathrm{MCG}(\Sigma) &\to \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_*.\end{aligned}$$

Theorem 1.1 ***(Mapping Class Group of the Torus).***
The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

Proof .

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned}\tilde\sigma : (\mathrm{Map}(X, X), \circ) &\to (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*.\end{aligned}$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff $\mathrm{Map}^0(X, X) \subseteq \ker \tilde\sigma = \tilde\sigma^{-1}(\mathrm{id})$
  - This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.
- So we have a (now injective) map

$$\begin{aligned}\tilde\sigma : \mathrm{MCG}(X) &\to \mathrm{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: $\mathrm{im}(\tilde\sigma) \subseteq \mathrm{SL}(2, \mathbb{Z})$.

- We can thus freely restrict the codomain to define the map

$$\begin{aligned} \sigma : \mathrm{MCG}(X) &\to \mathrm{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

- The claim is now that this is surjective.

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