

# Morse Theory

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## 1 Thursday January 9th

**Recall:** For  $M^n$  a closed smooth manifold, consider a smooth map  $f : M^n \rightarrow \mathbb{R}$ .

**Definition:** A critical point  $p$  of  $f$  is *non-degenerate* iff  $\det(H := \frac{\partial^2 f}{\partial x_i \partial x_j}(p)) \neq 0$  in some coordinate system  $U$ .

**Lemma (The Morse Lemma):** For any non-degenerate critical point  $p$  there exists a coordinate system around  $p$  such that

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - x_2^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2.$$

$\lambda$  is called the *index of  $f$  at  $p$* .

**Lemma:**  $\lambda$  is equal to the number of *negative* eigenvalues of  $H(p)$ .

*Proof:* A change of coordinates sends  $H(p) \rightarrow A^t H(p) A$ , which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that  $A$  is invertible and not necessarily orthogonal. Use the fact that  $A^t H A$  is diagonalizable.

This means that  $f$  can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

*Proof of Morse Lemma:*

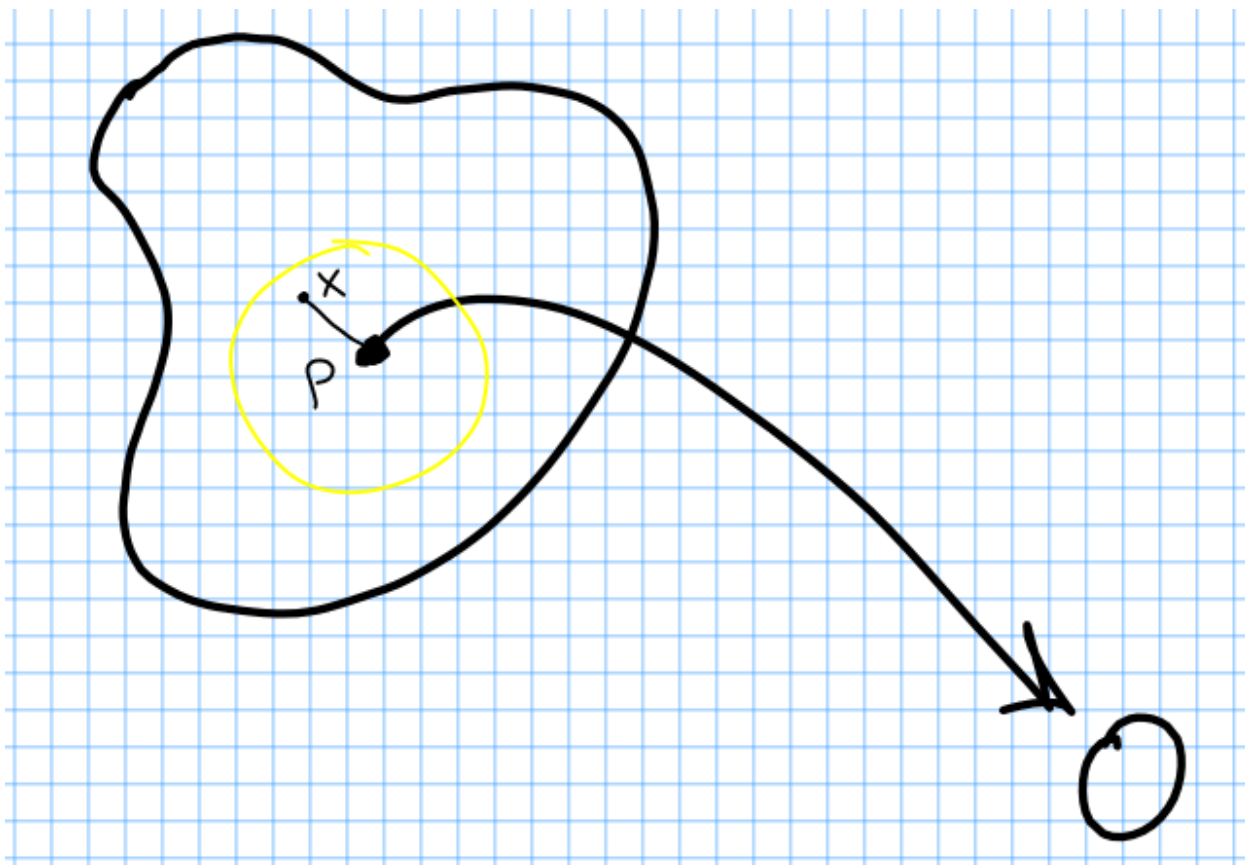
Suppose that we have a coordinate chart  $U$  around  $p$  such that  $p \mapsto 0 \in U$  and  $f(p) = 0$ .

**Step 1 – Claim:** There exists a coordinate system around  $p$  such that

$$f(x) = \sum_{i,j=1}^n x_i x_j h_{ij}(x),$$

where  $h_{ij}(x) = h_{ji}(x)$ .

*Proof:* Pick a convex neighborhood  $V$  of  $0 \in \mathbb{R}^n$ .



Restrict  $f$  to a path between  $x$  and  $0$ , and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \dots, tx_n)}{dt} dt = f(x_1, \dots, x_n) - f(0) = f(x).$$

since  $f(0) = 0$ .

We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \dots + \frac{\partial f}{\partial x_n} x_n dt \implies \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i} dt = f(x).$$

We thus have  $f(x) = \sum_{i=1}^n x_i g_i(x)$  where  $\frac{\partial f}{\partial x_i}(0) = 0$ , and  $\frac{\partial f}{\partial x_i} = x_1 \frac{\partial g_1}{\partial x_i} + \dots + g_i + x_i \frac{\partial g_i}{\partial x_i} + \dots + x_n \frac{\partial g_n}{\partial x_i}$ .

When we plug  $x = 0$  into this expression, the only term that doesn't vanish is  $g_i$ , and thus  $\frac{\partial f}{\partial x_i}(0) = g_i(0)$  and  $g_i(0) = 0$ .

Applying the same result to  $g_i$ , we obtain  $g_i(x) = \sum_{j=1}^n x_j h_{ij}(x)$ , and thus  $f(x) = \sum_{i,j=1}^n x_i x_j h_{ij}(x)$ .

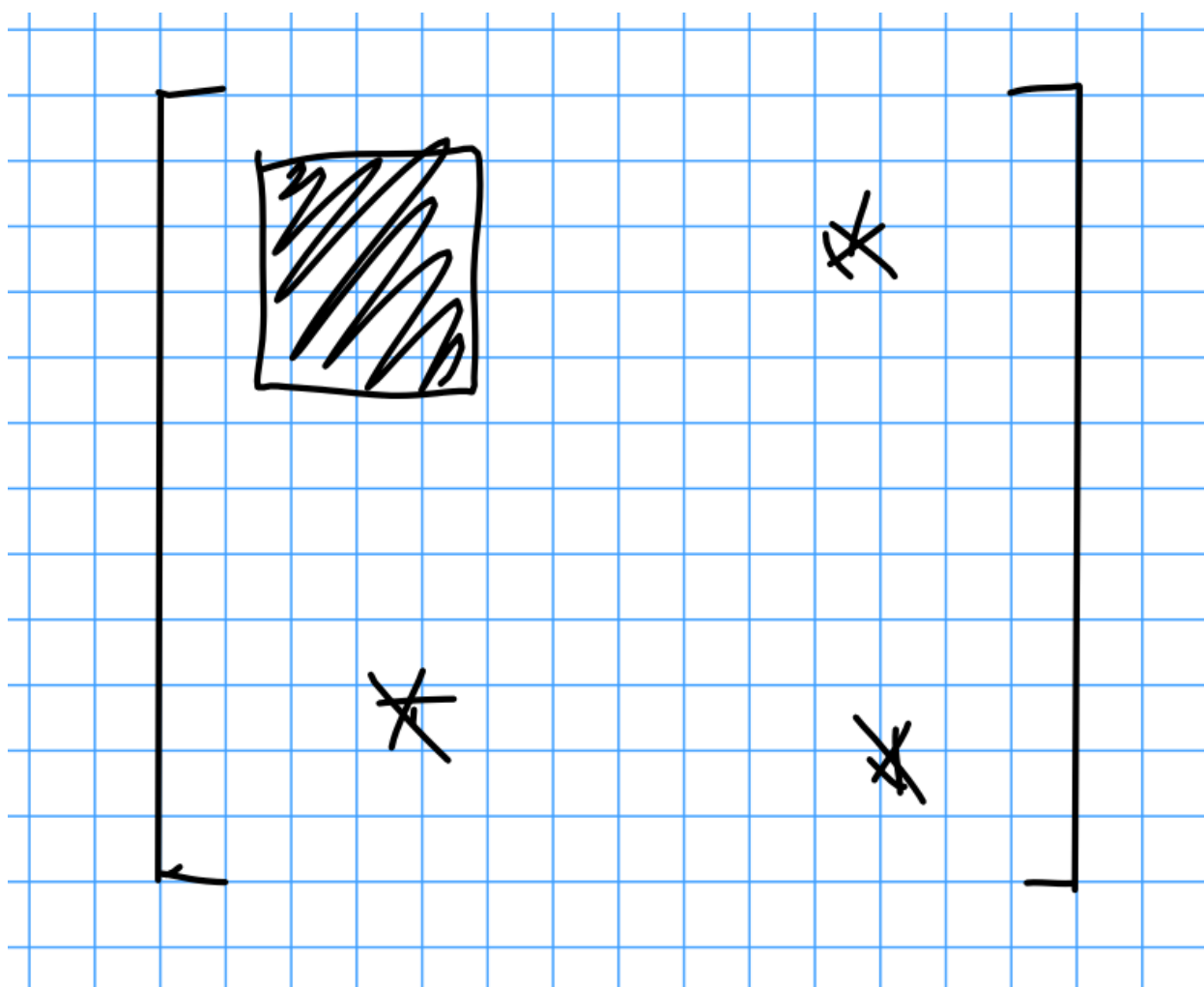
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We still need to show  $h$  is symmetric. For every pair  $i, j$ , there is a term of the form  $x_i x_j h_{ij} + x_j x_i h_{ji}$ . So let  $H_{ij}(x) = \frac{h_{ij}(x) + h_{ji}(x)}{2}$  (i.e. symmetrize/average  $h$ ), then  $f(x) = \sum_{i,j=1}^n x_i x_j H_{ij}(x)$  and this shows claim 1. ■

**Step 2 – Induction:** Assume that in some coordinate system  $U_0$ ,

$$f(y_1, \dots, y_n) = \pm y_1^2 \pm y_2^2 \pm \dots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \dots, y_n).$$

Note that  $H_{rr}(0)$  is given by the top-left block of  $H_{ij}(0)$ , which is thus looks like



Note that this block is symmetric.

Claim 1: There exists a linear change of coordinates such that  $H_{rr}(0) \neq 0$ .

---

We can use the fact that  $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = H_{ij}(0) + H_{ji}(0) = 2H_{ij}(0)$ , and thus  $H_{ij}(0) = \frac{1}{2} \left( \frac{\partial f}{\partial x_i \partial x_j} \right)$ .

Since  $H(0)$  is non-singular, we can find  $A$  such that  $A^t H(0) A$  has nonzero  $rr$  entry, namely by letting the first column of  $A$  be an eigenvector of  $H(0)$ , then  $A = [\mathbf{v}, \dots]$  and thus  $H(0)A = [\lambda \mathbf{v}, \dots]$  and  $A^t[\lambda \mathbf{v}] = [\lambda \|\mathbf{v}\|^2, \dots]$ .

So

$$\begin{aligned}
\sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \dots, y_n) &= y_r^2 H_{rr}(y_1, \dots, y_n) + \sum_{i > r} 2y_i y_r H_{ir}(y_1, \dots, y_n) \\
&= H_{rr}(y_1, \dots, y_n) \left( y_r^2 + \sum_{i > r} 2y_i y_r H_{ir}(y_1, \dots, y_n) / H_{rr}(y_1, \dots, y_n) \right) \\
&= H_{rr}(y_1, \dots, y_n) \left( y_r + \sum_{i > r} y_i H_{ir}(y_1, \dots, y_n) / H_{rr}(y_1, \dots, y_n) \right)^2 \\
&\quad \cdot \sum_{i > r}^n y_i^2 (H_{ir} Y / H_{rr}(Y))^2 \\
&\quad \cdot \sum_{i,j > r}^n H_{ir}(Y) H_{jr}(Y) / H_{rr}(Y)^2 \\
&\quad \text{by completing the square.}
\end{aligned}$$

Note that  $H_{rr}(0) \neq 0$  implies that  $H_{rr} \neq 0$  in a neighborhood of zero as well.

Now define a change of coordinates  $\phi : U \rightarrow \mathbb{R}^n$  by

$$z_i = \begin{cases} y_i & i \neq r \\ \sqrt{H_{rr}(y_1, \dots, y_n)} \left( y_r + \sum_{i > r} y_i H_{ir}(Y) / H_{rr}(Y) \right) & i = r \end{cases}$$

This means that

$$f(z) = \pm z_1^2 \pm \dots \pm z_{r-1}^2 \pm z_r^2 + \sum_{i,j \geq r+1}^n z_i z_j \tilde{H}(z_1, \dots, z_n).$$

Exercise: show that  $d_0 \phi$  is invertible, and by the inverse function theorem, conclude that there is a neighborhood  $U_2 \subset U_1$  of 0 on which  $\phi$  is still invertible. ■

**Corollary:** The nondegenerate critical points of a Morse function  $f$  are isolated.

*Proof:* In some neighborhood around  $p$ , we have

$$f(x) = f(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2,$$

Thus  $\frac{\partial f}{\partial x_i} = 2x_i$ , and so  $\frac{\partial f}{\partial x_i} = 0$  iff  $x_1 = x_2 = \dots = x_n = 0$ .

---

**Corollary:** On a closed (compact) manifold  $M$ , a Morse function has only finitely many critical points.

We will need these facts to discuss the  $h$ -cobordism theorem. For a closed smooth manifold,  $\partial M = \emptyset$ , so  $M$  will define a cobordism  $\emptyset \rightarrow \emptyset$ .

**Definition:** Let  $W$  be a cobordism from  $M_0 \rightarrow M_1$ . A *Morse function* is a smooth map  $f : W \rightarrow [a, b]$  such that

1.  $f^{-1}(a) = M_0$  and  $f^{-1}(b) = M_1$ ,
2. All critical points of  $f$  are non-degenerate and contained in  $\text{int}(W) := W \setminus \partial W$ .

So  $f$  is equal to the endpoints only on the boundary.

Next time: existence of Morse functions. This is a fairly restrictive notion, but they are dense in the  $C^2$  topology on (?).

## 2 Tuesday January 14th

### 2.1 Existence of Morse Functions

Notation: Let  $F(M; \mathbb{R})$  be the space of smooth functions from  $M$  to  $\mathbb{R}$  with the  $C^2$  topology.

Theorem: Morse functions form an open dense subset of  $F(M; \mathbb{R})$  in the  $C^2$  topology.

Recall that the  $C^2$  topology is defined by noting that  $F(M, \mathbb{R})$  is an abelian group under addition, so we'll define open sets near the zero function and define open sets around  $f$  by translation. (I.e., if  $N$  is an open neighborhood of 0, then  $N + f$  is an open neighborhood of  $f$ .)

So we'll define a base of open sets around 0. Take a finite cover of  $M$ , say by coordinate systems  $\{U_\alpha\}$ . Then let  $h_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ . Now (exercise) we can find a compact refinement  $C_\alpha \subset U_\alpha$  with each  $C_\alpha$  compact and  $\bigcup_\alpha C_\alpha = M$ . We can now define  $f_\alpha := f \circ h_\alpha^{-1}$  for any  $f : M \rightarrow \mathbb{R}$

$$\begin{array}{ccc} U_\alpha & \xrightarrow{h_\alpha} & \mathbb{R}^n \\ \downarrow f|_{U_\alpha} & \nearrow f_\alpha & \\ C_\alpha & & \end{array}$$

Now for each  $\delta > 0$ , define

$$N(\delta) = \left\{ f : M \rightarrow \mathbb{R} \mid \left\{ \begin{array}{l} |f_\alpha(p)| < \delta \\ \left| \frac{\partial f_\alpha}{\partial x_i} \right| < \delta \\ \left| \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \right| < \delta \end{array} \right. \quad \forall p \in h_\alpha(C_\alpha), \forall \alpha \right\}.$$

Corollary:  $f + N(\delta)$  (for all  $\delta$ ) is a basis for open neighborhoods around  $f$ .

Lemma: This topology does not depend on the choice of  $\{U_\alpha, h_\alpha\}$ .

Proof: See Milnor 2.

Lemma 1: Let  $f : U \rightarrow \mathbb{R}$  be a  $C^2$  map for  $U \subseteq \mathbb{R}^n$ . For “almost all” linear maps  $L : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f + L$  has only nondegenerate critical points.

Almost all: Note that  $\text{hom}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$ , so the complement of the set of such maps has measure zero in  $\mathbb{R}^n$ .

Proof: Consider  $X = U \times \text{hom}(\mathbb{R}^n, \mathbb{R})$ , which contains a subspace  $M = \{(x, L) \mid \partial_x(f + L) = 0\}$ , i.e.  $x$  is a critical point of  $f$ . If  $\partial_x f + L = 0$ , then  $L = -\partial_x f$ . We thus obtain an identification of  $M$  with  $U$  by sending  $x \in U$  to  $(x, -\partial_x f)$ .

There is also a projection onto the second component, where  $(x, L) \mapsto L$ . So let  $\pi : X \rightarrow \text{hom}(\mathbb{R}^n, \mathbb{R})$  be this projection; then there is a map  $\tilde{\pi} : U \rightarrow \text{hom}(\mathbb{R}^n, \mathbb{R})$  given by  $x \mapsto \partial_x f$ . Note that  $f + L$  has a *degenerate* critical point iff there is an  $x \in U$  such that  $\partial_x(f + L) = 0$  (or equivalently  $L = -\partial_x f$ ), and the second derivative of  $f + L$  is zero. Since  $L$  is linear, this says that the matrix  $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(x)$  is singular. But this says  $x$  is a critical point for  $\tilde{\pi}$ .

This happens iff  $\tilde{\pi}(x) = -\partial_x f = L$ , so  $L$  is a critical value for  $\tilde{\pi}$ . Thus  $f + L$  has a degenerate critical point  $\iff L$  is a critical value for  $\tilde{\pi}$ .

Now Sard’s theorem applies: if  $g : M^n \rightarrow \mathbb{R}^n$  is a map from any manifold to  $\mathbb{R}^n$  that is  $C^1$ , then the set of critical values of  $g$  in  $\mathbb{R}^n$  has measure zero.

Thus the set of critical values of  $\tilde{\pi}$  has measure zero, and thus for almost all  $L$ ,  $f + L$  has no degenerate critical points.

Summary: Consider the map of first derivatives. It has a critical point whenever the 2nd derivative is singular, which is exactly the nondegeneracy condition.

Lemma 2: Let  $K \subset U \subset \mathbb{R}^n$  with  $K$  compact and  $U$  open, and let  $f : U \rightarrow \mathbb{R}$  have only nondegenerate critical points. Then there exists a  $\delta > 0$  such that every  $g : U \rightarrow \mathbb{R}$  that is  $C^2$  which satisfies

1.  $\left| \frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p) \right| < \delta$ , and
2.  $\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p) \right| < \delta$

for all  $i, j$  and  $p \in K$  has only nondegenerate critical points.

Proof:

Define  $|df| = \sqrt{\left|\frac{\partial f}{\partial x_1}\right|^2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|^2}$ . Now note that  $S(f) = |df| + \left|\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right| \geq 0$ . This is an equality iff both terms are zero, and the first term is zero iff  $x$  is a critical point, while the second term is zero iff  $x$  is degenerate.

Since  $f$  has only nondegenerate critical points, this inequality is strictly positive on  $K$ , i.e.  $S(f) > 0$ . Since  $K$  is compact,  $S(f)$  takes on a positive infimum on  $K$ , say  $\mu$ . Then  $S(f) \geq \mu > 0$  on  $K$ .

Thinking of  $S$  as defining a norm, the reverse triangle inequality yields

$$||df| - |dg|| \leq |df - dg| \leq \sqrt{n\delta^2} \leq \frac{\mu}{2},$$

where we can choose  $\delta$  such that  $\sqrt{n\delta^2} < \mu$ .

We can also pick  $\delta$  small enough such that

$$||\det J_f| - |\det(J_g)|| \leq \frac{\mu}{2},$$

where  $J_f = \left( \frac{\partial f}{\partial x_i \partial x_j} \right)$  is shorthand for the matrix of partial derivatives appearing previously, and we just note that picking entries close enough makes the difference of determinant small enough (although there's something to prove there).

Then

$$\begin{aligned} & |df| - |dg| + |\det(J_f)| - |\det J_g| < \mu \\ \implies & 0 \leq |df| + |\det(J_f)| - \mu < |dg| + |\det(J_g)|, \end{aligned}$$

The second inequality follows from just moving terms in the first inequality.

which makes the last term strictly positive, and thus nonzero on  $K$ . Then  $g$  has no degenerate critical points in  $K$ . ■

Proof summary:

1.  $\|f\|_2(x) = 0$  iff  $x$  is a degenerate critical point.
2.  $\|f\|_2(x) \geq \mu > 0$  in  $K$ .
3. We can pick  $\delta$  small enough such that  $\|f\|_2 - \|g\|_2 < \mu$  on  $K$ .
4. This forces  $\|g\|_2 > 0$  on  $K$ , so  $g$  has *no* nondegenerate critical points on  $K$ .

## 2.2 Morse Functions are Open

We still want to show that Morse functions form an open dense subset.

To see that they form an open set, suppose  $f \in F(M, \mathbb{R})$  is Morse. Then take a finite cover of  $M$ , say  $\{(U_i, h_i)\}_{i=1}^k$ . Pick compact  $C_i \subset U_i$  that still covers  $M$ .

Note that any  $g$  satisfying the 2 required conditions where  $|f - g| < \delta$  (?), then  $g \in N(\delta) + f$ .

By lemma 2, there exists a  $\delta > 0$  such that every  $g \in N_1 := f + N(\delta)$  has only nondegenerate points in  $C_1$ . We can pick a  $\delta$  similarly to define an  $N_i$  for every  $i$ . Then taking  $N = \bigcap_{i=1}^k N_i$ , this yields an open neighborhood of  $f$  such that every  $g \in N$  has only nondegenerate critical points on  $C_1 \cup C_2 \cdots \cup C_k = M$ .



## 2.3 More Functions are Dense

We want to show that this set is dense, so we'll fix some open set and show that there exists a Morse function in it.

Let  $f \in N$  for  $N$  an open set; we'll then change  $f$  gradually to make it Morse. Convention: we'll say  $f$  is *good* on  $S \subset M$  iff  $f$  has only nondegenerate critical points in  $S$ .

Pick a smooth bump function  $\lambda : M^n \rightarrow [0, 1]$  such that

- $\lambda \equiv 1$  on an open neighborhood of  $C_1$ , and
- $\lambda \equiv 0$  on an open neighborhood of  $M \setminus U_1$ .

Note: we can do this because  $C_1 \subset U_1$  is closed, and  $M \setminus U_1$  is closed, so we can find disjoint open sets containing each respectively using the fact that  $M^n$  is Hausdorff (?).

Now let  $f_1 = f + \lambda L$  for some linear function  $L : \mathbb{R}^n \rightarrow \mathbb{R}$ , so  $f_1 = f + L$  on an open neighborhood of  $C_1$ . By Lemma 1, for almost every  $L$ ,  $f_1$  is good.

Note that we need  $\lambda$  because  $L$  is only defined on  $\mathbb{R}^n$ , not on  $M$ .

Now  $f_1 - f = \lambda L$  is supported in  $U_1$ . If we pick the coefficients of  $L$  small enough, noting that  $\lambda$  is bounded, then the first and second derivatives of  $f - f_1$  will be bounded, and we can arrange for  $f_1 \in f + N(\varepsilon)$  for  $\varepsilon > 0$  as small as we'd like. For  $\varepsilon$  sufficiently small, we can arrange for  $N(\varepsilon) \subset N_\delta$  for the finitely many  $\delta$ s, and so  $N(\varepsilon) \subset N$ .

By Lemma 2, there exists a neighborhood  $N_1 \subseteq N$  containing  $f_1$  such that every  $g \in N_1$  is good on  $C_1$ . Since  $f_1 \in N_1$ , we can repeat this process to obtain an  $f_2 \in N_2 \subseteq N_1$  and so on inductively. Then since every  $g \in N_2$  is good on  $C_2$  and  $N_2 \subseteq N_1$ , every  $g \in N_2$  is good on  $C_1 \cup C_2$ . This yields an  $f_k \in N_k \subset N_{k-1} \subset \dots \subset N_1 \subset N$ , so  $f_k$  is good on  $\bigcup C_i = M$ .

■

Thursday: We'll show that every pair of critical points can be arranged to take on different values, and then order them. This yields  $f(p_1) < c_1 < f(p_2) < c_2 < \dots < c_{k-1} < f(p_k)$ , and since the  $c_i$  are regular values, the inverse images  $f^{-1}(c_i)$  are smooth manifolds and we can cut along them.

## 3 Thursday January 16th

Theorem: Let  $f : M \rightarrow \mathbb{R}$  be morse with critical points  $p_1, \dots, p_k$ . Then  $f$  can be approximated by a morse function  $g$  such that

1.  $g$  has the same critical points of  $f$
2.  $g(p_i) \neq g(p_j)$  for all  $i \neq j$ .

Idea: Change  $f$  gradually near critical points without actually changing the critical points themselves.

Proof: Suppose  $f(p_1) = f(p_2)$ .

Choose  $\bar{U} \subset N$  open neighborhoods of  $p_1$  such that  $\bar{N}$  doesn't contain  $p_i$  for any  $i$  except for 1. Note that this is possible because the critical points are isolated.

Choose a bump function  $\lambda \equiv 1$  on  $U$  and 0 on  $M \setminus N$ . Now let  $f_1 = f + \varepsilon \lambda$ , where we'll see how to choose  $\varepsilon$  small enough soon.

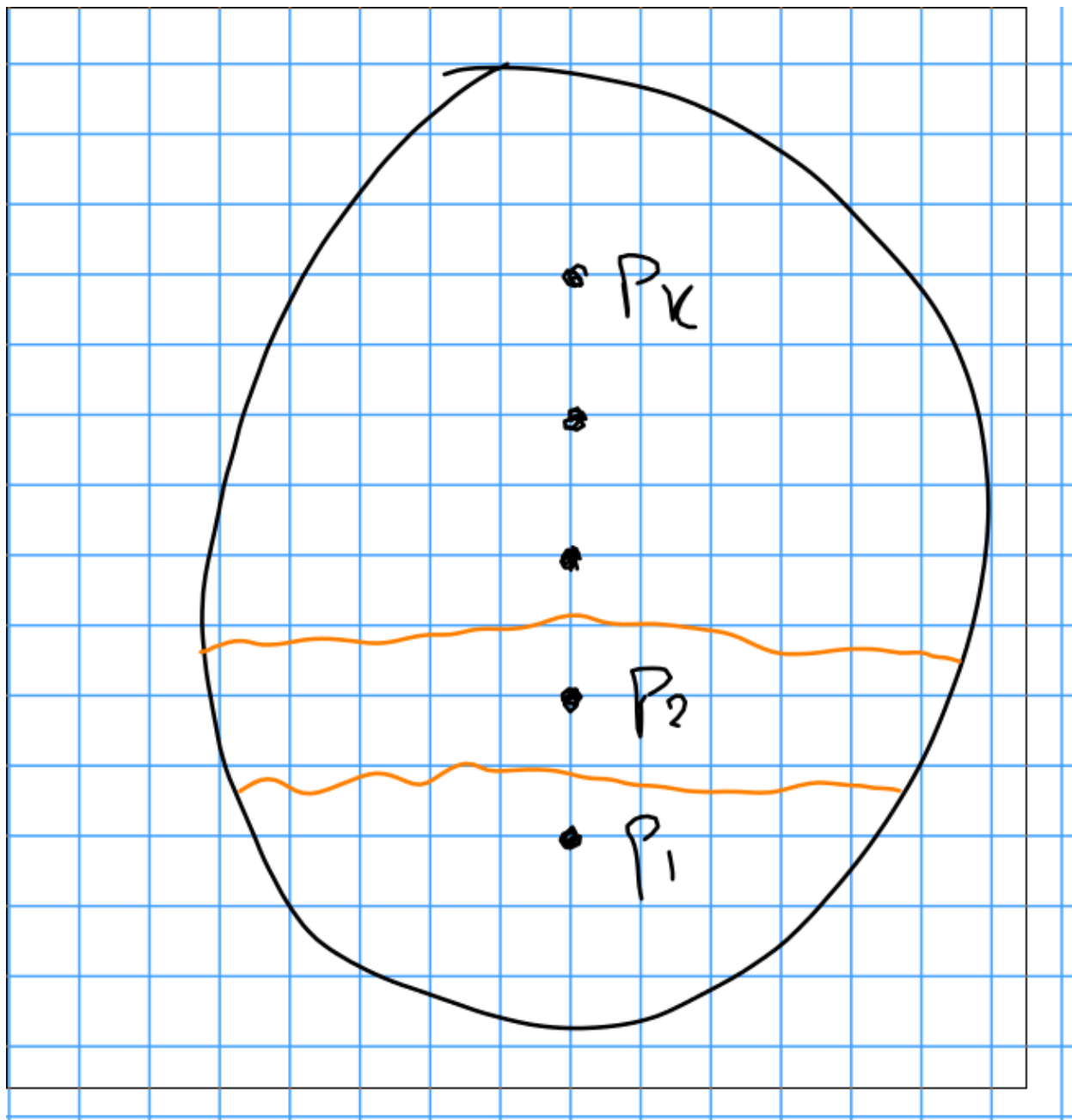


Figure 1: Image

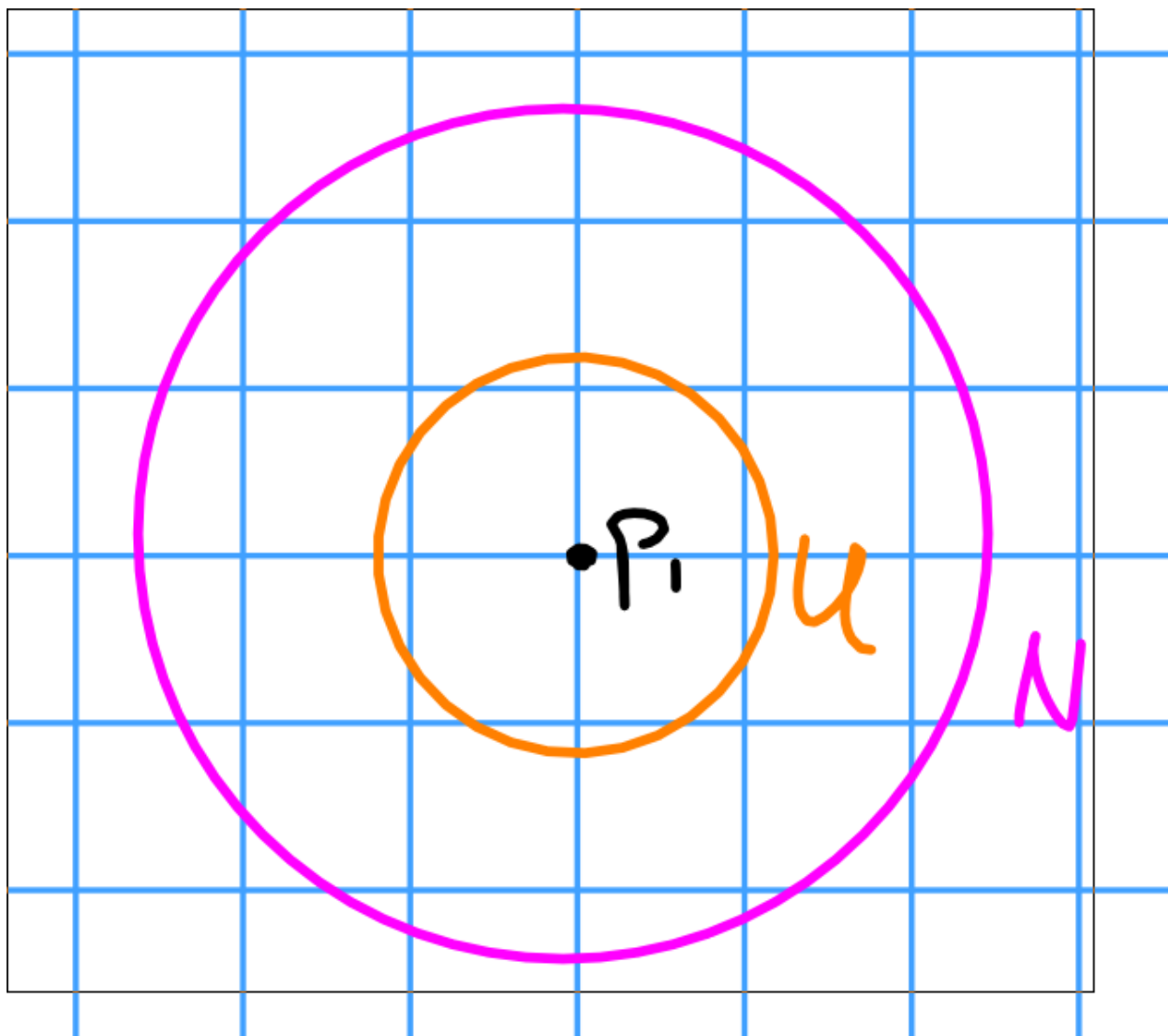


Figure 2: Image

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Let  $K := \{x \mid 0 < \lambda(x) < 1\}$ , which is compact.

Pick a Riemannian metric on  $M$ , then we can talk about gradients. Recall that  $\text{grad}f$  is the vector field that satisfies  $\langle X, \text{grad}f \rangle = X(f)$  for all vector fields  $X$  on  $M$ . Because  $f$  has no critical points in  $K$ ,  $\text{grad}f$  is nonzero for some field  $X$ , so  $\text{grad}f$  is nonzero, noting that  $\text{grad}f$  is only zero at the critical points of  $f$ .

In particular, on  $K$  we have  $0 < c \leq |\text{grad}f|$  for some  $c$ , and  $\text{grad}\lambda \leq c'$  for some  $c'$ . So pick  $0 < \varepsilon < c'/c$  such that  $f_1(p_1) \neq f_1(p_2)$ ,  $f_1(p_1) = f(p_1) + \varepsilon$ , and  $f_1(p_i) = f(p_i)$  for all  $i \neq 1$ . Note that this is possible because there are only finitely many points, so almost every  $\varepsilon$  will work.

**Claim 1:** The critical points of  $f_1$  are exactly the critical points of  $f$ .

In  $K$ , we have

$$\text{grad}f_1 = \text{grad}f + \varepsilon \text{grad}\lambda \implies |\text{grad}f_1| \geq |\text{grad}f| - \varepsilon |\text{grad}\lambda| \geq c - \varepsilon c' > 0.$$

If  $x \notin K$ , we have

1.  $x \in U$ , or
2.  $x \in M \setminus N$

In case 1,  $\lambda$  is constant and  $\text{grad}\lambda = 0$ , so  $\text{grad}f_1 = \text{grad}f$ . In case 2,  $\lambda$  is again constant, so the same conclusion holds. ■

**Claim 2:**  $f_1$  is morse.

Proof: In a neighborhood of  $p_1$ , we have  $f_1 \equiv f + \varepsilon$ . In a neighborhood of  $p_i$ , we have  $f_1 \equiv f$ .

We can then check that  $J_{f_1}(p_i) = J_f(p_i)$ , and since  $f$  is morse,  $f_1$  is morse as well. ■

Recall that this lets us put an order on  $f(p_i)$ . Between every critical value, pick regular values  $c_i$ , i.e.  $f(p_1) < c_1 < f(p_2) < \dots$ . Then  $f^{-1}(c_i)$  is a smooth submanifold of dimension  $n - 1$ , and we have the following schematic:

Moreover,  $f^{-1}[c_i, c_{i+1}]$  is a cobordism from  $f^{-1}(c_i)$  to  $f^{-1}(c_{i+1})$ .

Definition: Recall that for  $(W; M_0, M_1)$  a cobordism, a morse function  $f : W \longrightarrow [a, b]$  is morse iff

1.  $f^{-1}(a) = M_0$  and  $f^{-1}(b) = M_1$ .
2.  $f$  has only nondegenerate critical points and no critical points near  $\partial W = M_1 \coprod M_2$ , i.e. all critical points are in  $W^\circ$  (the interior).

Proof of density of morse functions goes through in the same way, with extra care taken to choose neighborhoods that do not intersect  $\partial W$ .

Theorem:

1. For every cobordism  $(W; M_1, M_2)$  there exists a morse function.
2. The set of such morse functions is dense in the  $C^2$  topology.
3. Any morse function  $f : (W; M_1, M_2) \longrightarrow [a, b]$  can be approximated by another morse function  $g : (W; M_1, M_2) \longrightarrow [a, b]$  such that  $g$  has the same critical points of  $f$  and  $g(p_i) \neq g(p_j)$  for  $i \neq j$  (i.e. the critical points are distinct).

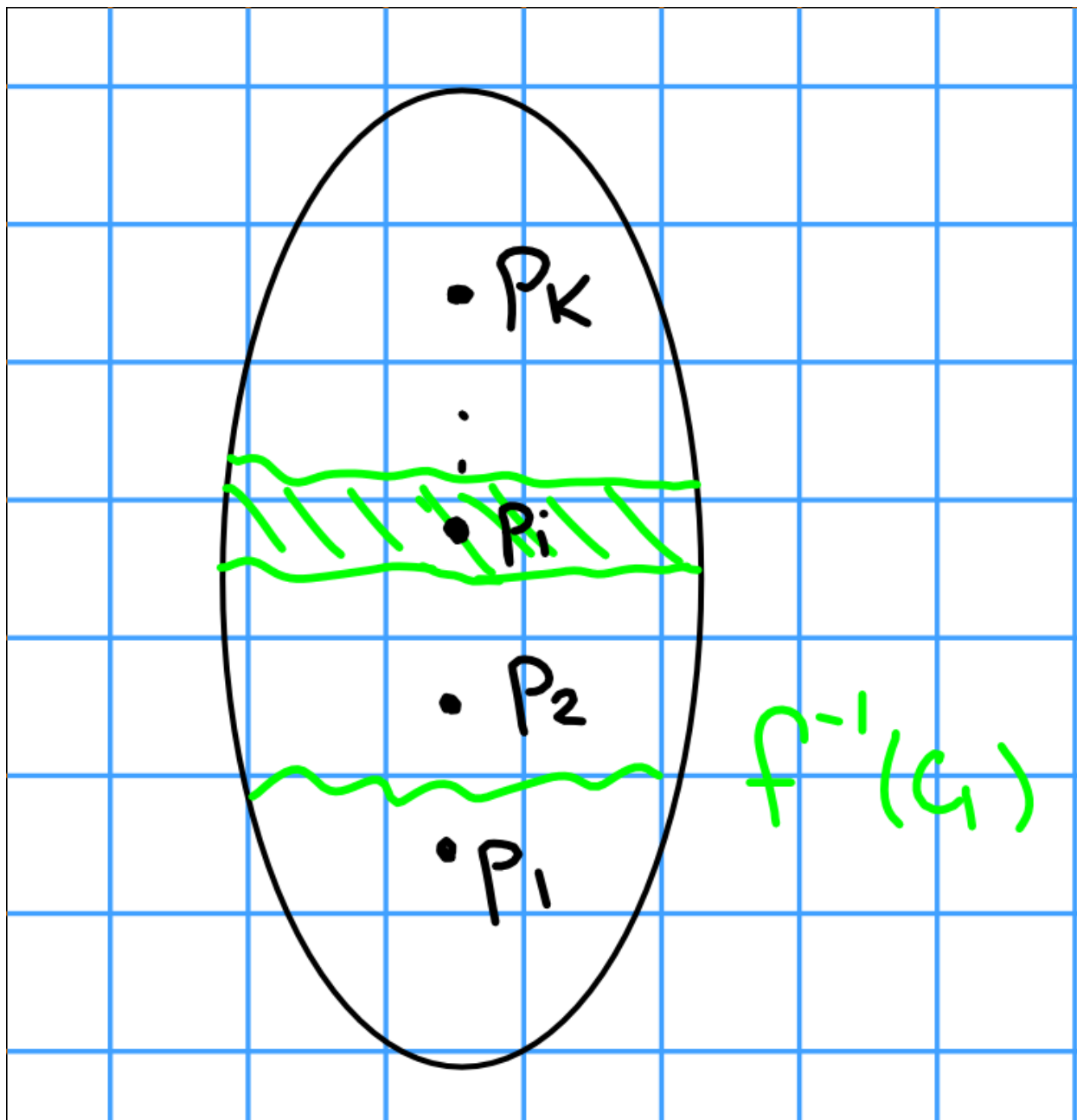


Figure 3: Image

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Note that  $n$ -manifolds are a special cases of cobordisms, namely a manifold  $M$  is a cobordism  $(W; M, \emptyset)$ . So all statements about cobordisms will hold for  $n$ -manifolds.

Definition: The **morse number**  $\mu$  of a cobordism  $(W; M_0, M_1)$  is the minimum of  $\left| \left\{ \text{critical points of } f \mid f \text{ is morse} \right\} \right|$ .

We'll be considering cobordisms with  $\mu = 0$ .

Note: if we take  $X = \text{grad} f$ , we have  $\langle X, \text{grad} f \rangle = \|\text{grad} f\|^2 \geq 0$ , which motivates our next definition.

Definition: Let  $f : W \rightarrow [a, b]$  be a morse function. Then a **gradient-like vector field** for  $f$  is a vector field  $\xi$  on  $W$  such that

1.  $\xi(f) > 0$  on  $W \setminus \text{crit}(f)$ .
2. For every critical point  $p$  there exist coordinates  $(x_1, \dots, x_n)$  on  $U \ni p$  such that

$$f(X) = f(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_n^2,$$

as in the Morse Lemma, where  $\lambda$  is the index, and

$$\xi = (-x_1, -x_2, \dots, -x_\lambda, x_{\lambda+1}, \dots, x_n) \text{ in } U.$$

**Lemma:** Every morse function  $f$  on  $(W; M_0, M_1)$  has a gradient-like vector field.

*Proof:* For simplicity, assume  $f$  has a single critical point  $p$ . Pick coordinate  $(x_1, \dots, x_n)$  on an open set  $U_0$  around  $p$  such that  $f$  has the form given in (1) above. Define  $\xi_0$  on  $U_0$  to be (2) above.

Every point  $q \in W \setminus U_0$  has a neighborhood  $U'$  such that  $df \neq 0$  on  $U'$ . By the implicit function theorem, there is a smaller neighborhood  $U''$  such that  $q \in U'' \subset U$  such that  $f = c_0 + x_1$  on  $U''$  for some constant  $c_0$ .

Exercise: check that this works!

But since  $W \setminus U_0$  is a closed subset of a compact manifold, it is compact, so we can cover it with finitely many  $U_i$  that satisfy

1.  $U_i \cap U = \emptyset$  for some open  $U$  containing  $p$  such that  $U \subset U_0$  and  $\overline{U} \subset U_0$ .
2.  $U_i$  has a coordinate chart  $(x_1^2, \dots, x_n^2)$  such that  $f = c_i + x_1^2$  on  $U_i$  for some constants  $c_i$ .

Thus on  $U_i$  we can set  $\xi_i = (1, 0, \dots, 0) = \frac{\partial}{\partial x_1^2}$  to get local vector fields. We can then take a partition of unity  $\rho_1, \dots, \rho_k$  and set  $\xi = \sum_i \rho_i \xi_i$ .

Now consider  $\xi(f)$ . By definition,  $\xi(f) = \sum_i \rho_i \xi_i(f)$ . Note that  $\rho_i \xi_i(f) = 1$  in  $U_i$ , and  $\rho_0 \xi_0(f) \geq 0$ , so  $\xi(f) \geq 0$ . If  $x$  is not a critical point, then at least 1  $\xi_i(f)(x)$  is positive and thus  $\xi(f)(x) > 0$ .

This is because  $x$  is either in  $U$ , in which case the 0 term is positive, or  $x \in U_i$ , in which case one of the remaining terms is positive.

---

The idea here: if we can make locally gradient-like vector fields, we can use partitions of unity to extend them to global vector fields.

Theorem: Any cobordism  $(W; M_0, M_1)$  with  $\mu = 0$  is a product cobordism, i.e.

$$(W; M_0, M_1) \cong (M_0 \times I; M_0 \times \{0\}, M_0 \times \{1\}).$$

Proof: Let  $f : W \rightarrow I$  be morse with no critical points, and let  $\xi$  be a gradient-like vector field for  $f$ . Then  $\xi(f) > 0$  on  $W$ , so we can normalize to replace  $\xi$  with  $\frac{1}{\xi(f)}\xi$  and assume  $\xi(f) = 1$ . Then consider the integral curves of  $\xi$ , given by  $\phi : [a, b] \rightarrow W$ .

i.e.  $d\phi = \xi$ .

We can thus compute  $\frac{\partial}{\partial t} f \circ \phi(t) = df(\frac{\partial \phi}{\partial t}) = df(\xi) = \xi(f) = 1$ . By the FTC, this implies that  $f \circ \phi(t) = c_0 + t$  for some constant  $c_0$ . So reparameterize by defining  $\psi(s) = \phi(s - c_0)$ , then  $f \circ \psi(s) = s$ . For every  $x \in W$ , there exists a unique maximal integral curve  $\psi_x(s)$  that passes through  $x$ .

Note that this works because maximal curves must intersect the boundary at precisely  $t = 0, 1$  and  $f$  is an increasing function. So for any curve passing through  $x$ , we can extend it to a maximal.

We can then defin

$$\begin{aligned} h : M_0 \times I &\rightarrow W \\ (x, s) &\mapsto \psi_x(s) \\ (\psi_y(0), f(y)) &\mapsto y \\ &\cdot \end{aligned}$$

## 4 Elementary Cobordism

Recall that an elementary cobordism is a cobordism that has a Morse function with exactly one critical point.

Definition: An  $n$ -dimensional  $\lambda$ -handle is a copy of  $D^\lambda \times D^{n-\lambda}$  which is attached to  $\partial M^n$  via an embedding  $\phi : \partial D^\lambda \times D^{n-\lambda} \hookrightarrow \partial M$ .

Example: Let  $\lambda = 1, n = 2, n - \lambda = 1$  and take  $M^2 = D^2$  and we attach  $D^1 \times D^1$ . Note that there's not necessarily a smooth structure on the resulting manifold, so we can “smooth corners”:

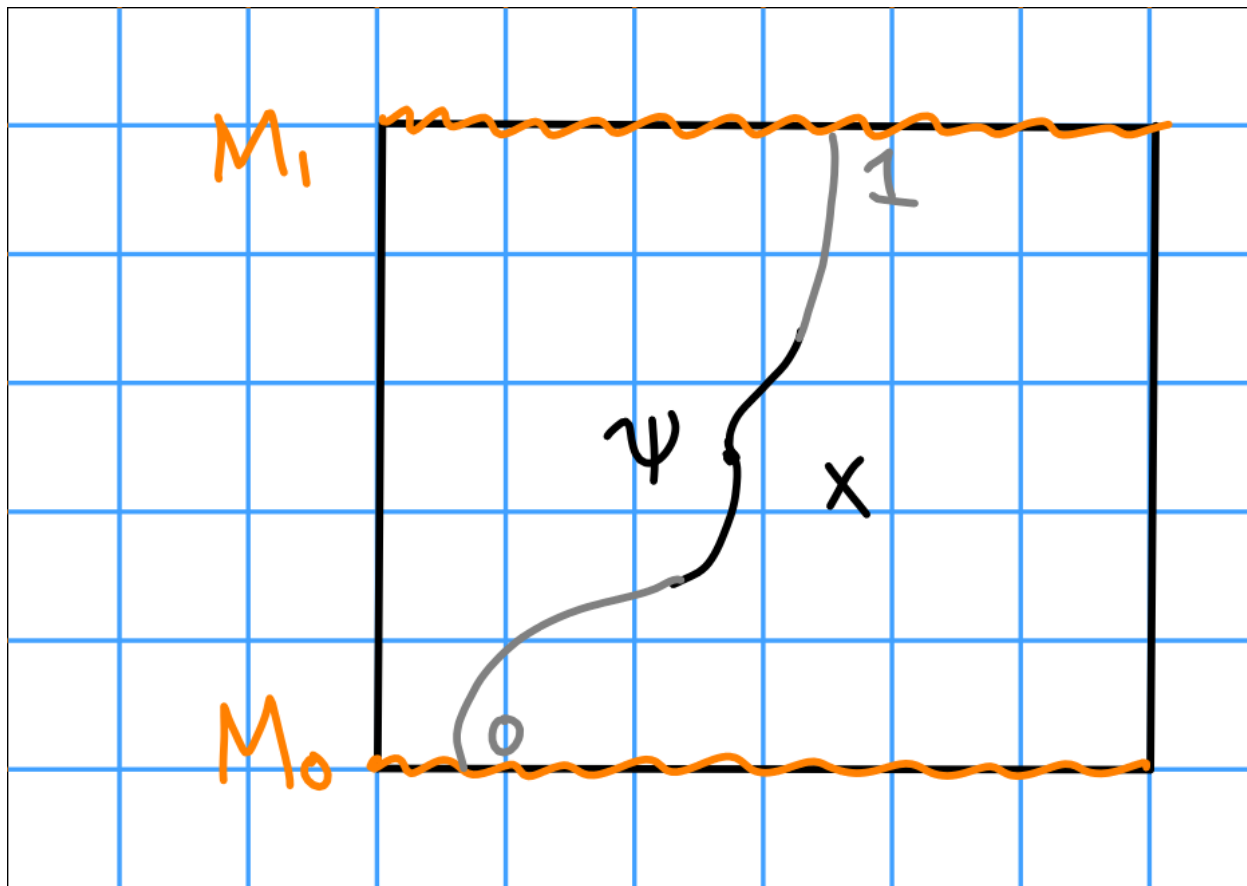
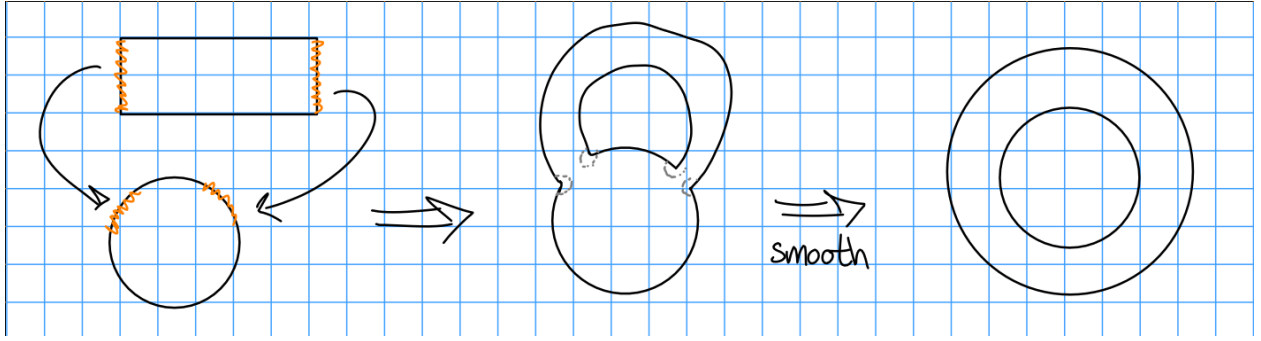
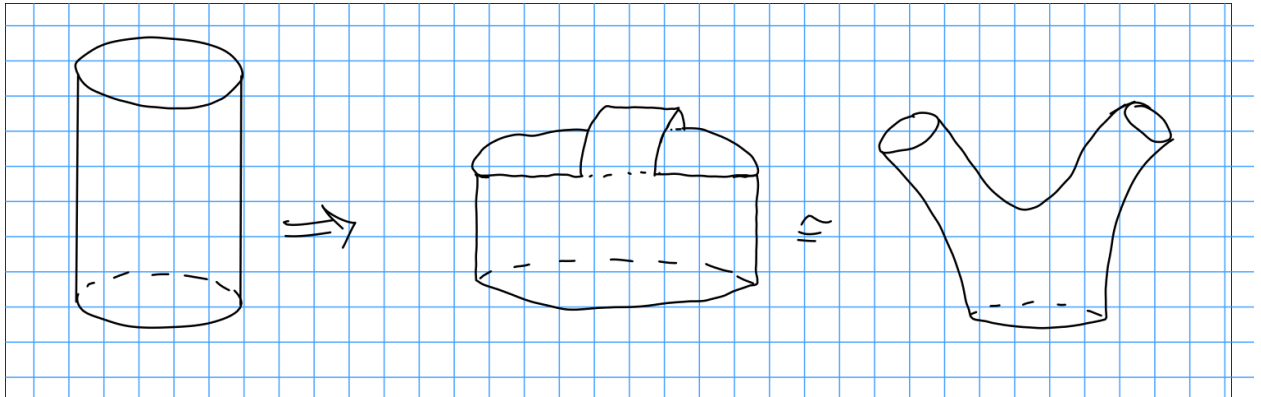


Figure 4: Image





Example:



Note: the above is just a homeomorphism.

Definition: Let  $M$  be an  $n - 1$  dimensional smooth manifold, and  $\rho : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M^{n-1}$  be an embedding.

Then noting that  $\partial D^{n-\lambda} = S^{n-\lambda-1}$ , consider the space

$$X(M, \phi) = (M \setminus \rho(S^{\lambda-1} \times \{0\})) \times (D^\lambda \times S^{n-\lambda-1}) / \langle \rho(u, tv) \sim (tu, v) \mid t \in (0, 1), \forall u \in S^{\lambda-1}, \forall v \in S^{n-\lambda-1} \rangle,$$

where we note that we can parameterize  $D^{n-\lambda} = tv$  where  $v$  is a point on the boundary.

Note that this accomplishes the goal of smoothing, and is referred to as surgery (of type  $\lambda, n - \lambda$ ) on  $M$  along  $\phi$ .

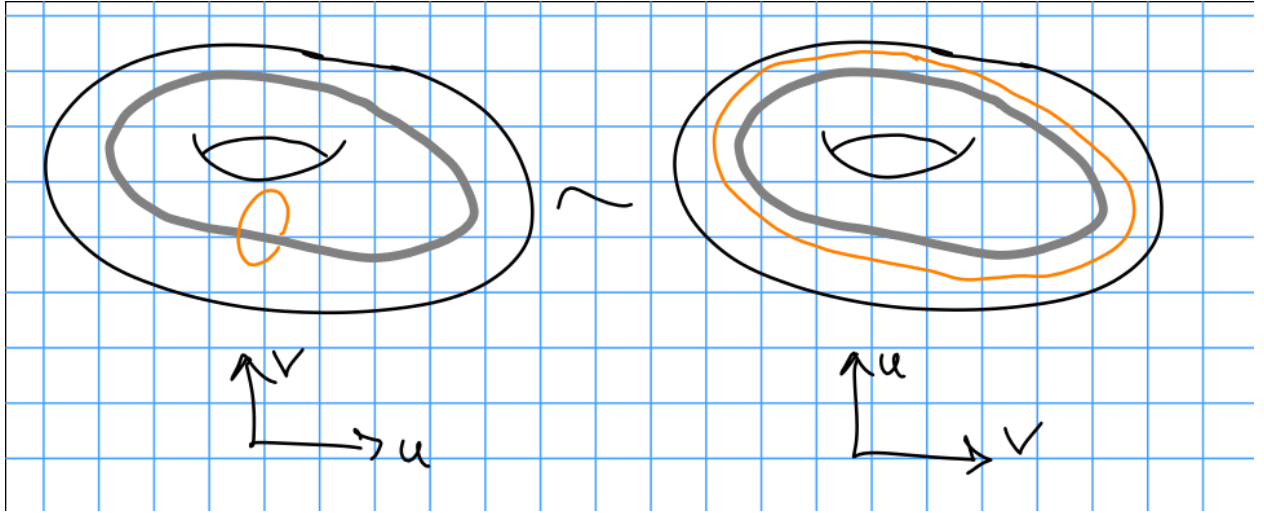
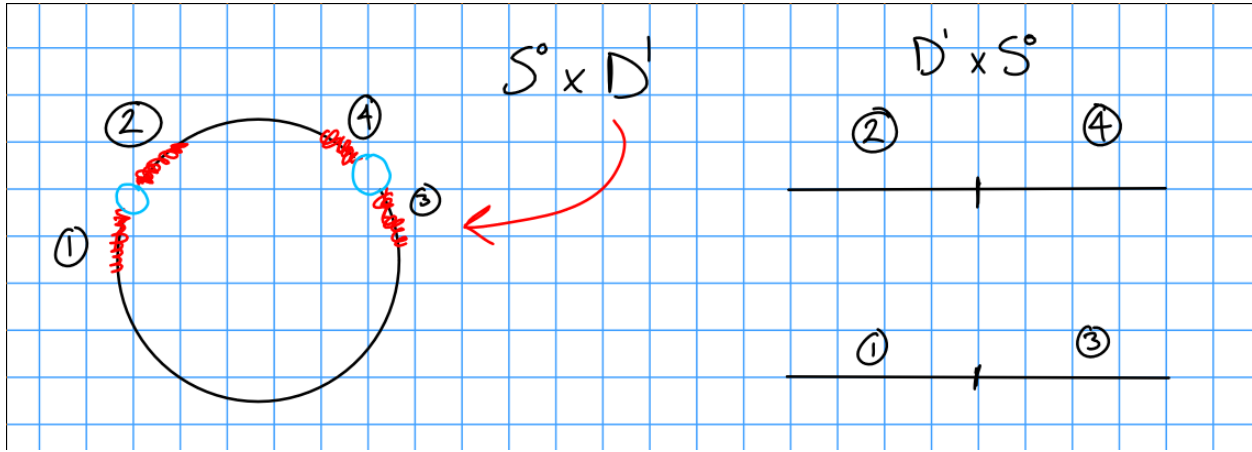


Figure 5: Image



Example:

Example:  $n - 1 = 3$  and  $\lambda = 2$  implies  $\lambda - 1 = 1$ , and take  $\rho : S^1 \times D^2 \rightarrow S^3$ , which has image a tubular neighborhood of a knot. Then  $\phi(S^1 \times \{0\}) = K$  for some knot, and  $(S^3 \setminus K) \amalg (D^2 \times S^1) / \dots$ . Then note that  $\partial\phi(\{u\} \times D^2) = \{u\} \times S^1$ , which no longer bounds a disk since we have removed the core of tube.

Theorem: Suppose  $M' = X(M, \rho)$  is obtained from  $M$  by surgery of type  $\lambda$ . Then there exists an elementary cobordism  $(W; M, M')$  with a Morse function  $f : W \rightarrow [-1, 1]$  with only one index  $\lambda$  critical point.

Example: Let  $M = S^1$  and  $\lambda = 1$ .

Proof: Write  $\mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$ , and  $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$ . Then  $L_\lambda = \{(\mathbf{x}, \mathbf{y}) \mid -1 \leq -\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \leq 1, \|\mathbf{x}\|\|\mathbf{y}\| < \sinh(1)\}$ .

The left boundary is given by  $\partial_L : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = -1$ , and there is a map

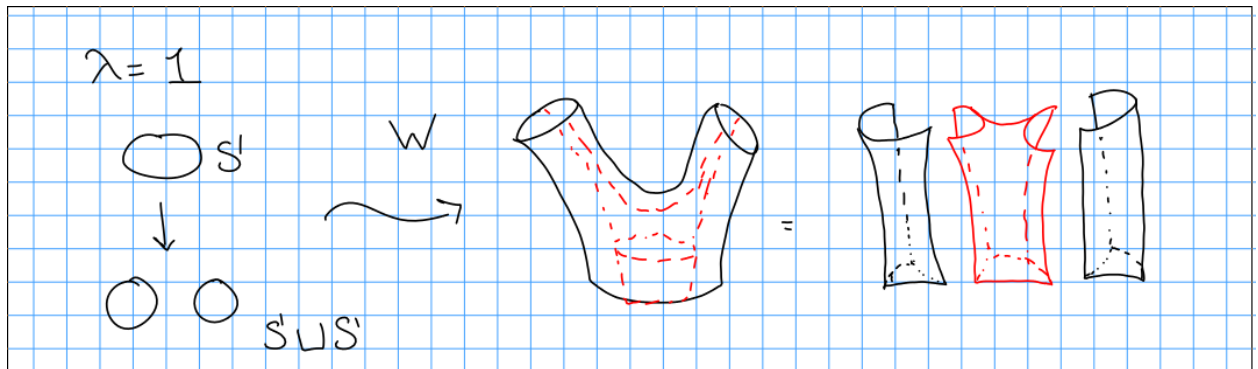


Figure 6: Image

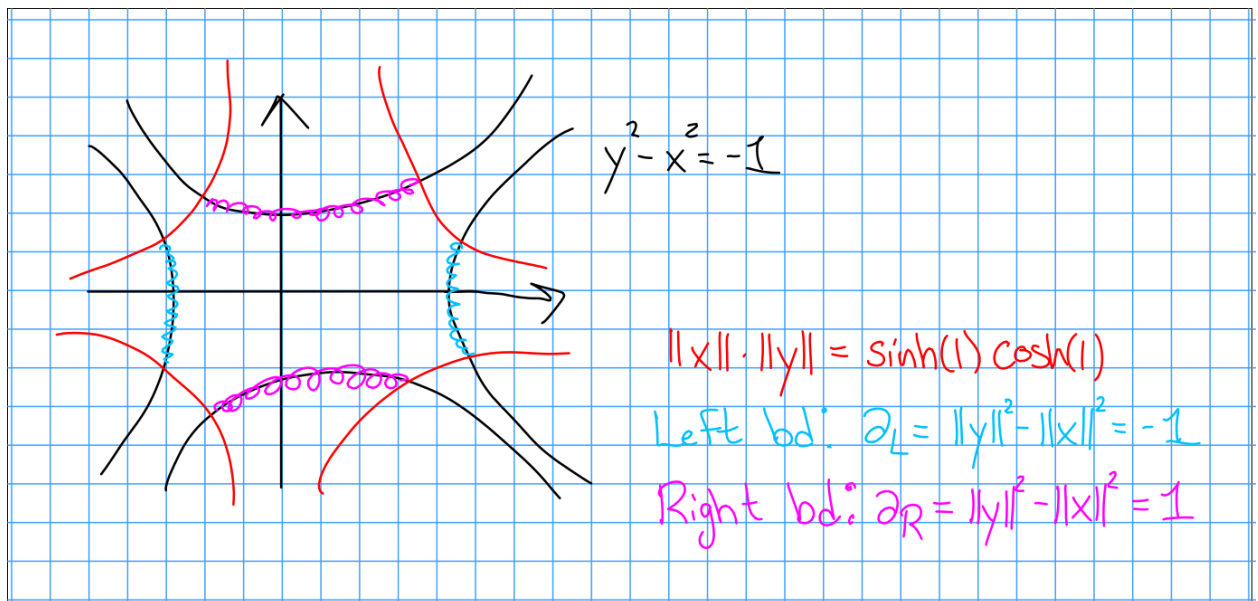


Figure 7: Image

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$$S^{\lambda-1} \times D^{n-\lambda} \xrightarrow{\text{diffeo}} \partial_L$$

$$(u, tv) \mapsto (u \cosh(t), v \sinh(t)) \quad t \in [0, 1),$$

which is clearly invertible.

The right boundary is given by  $\partial_R : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = 1$ , and there is a map

$$S^{\lambda-1} \times D^{n-\lambda} \xrightarrow{\text{diffeo}} \partial_L$$

$$(tu, v) \mapsto (u \sinh(t), v \cosh(t)).$$

In the above picture, we can consider the orthogonal trajectories, which are given by  $y^2 - x^2 = c$ , which has gradient  $(-x, y)$  and  $xy = c$  which has gradient  $(y, x)$ , so these are orthogonal.

Recall that near a point  $p \in M$ , the morse function has the form  $f(\mathbf{x}, \mathbf{y}) = f(p) - \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$  with a gradient-like vector field given by  $\xi = (-\mathbf{x}, \mathbf{y})$ .

The orthogonal trajectories will generally be of the form  $\|\mathbf{x}\|\|\mathbf{y}\| = c$ , which we can parameterize as  $t \mapsto (t\mathbf{x}, \frac{1}{t}\mathbf{y})$ .

Construction of  $W$ : Take

$$W(M, \phi) = ((M \setminus \phi(S^{\lambda-1} \times \{0\})) \times D^1) \amalg L_\lambda / \left\langle \phi(u, tv) \times c \sim (\mathbf{x}, \mathbf{y}) \mid \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = c, (\mathbf{x}, \mathbf{y}) \in \text{orthogonal tra} \right\rangle$$

This amounts to closing up in the following two ways:

This has two boundaries: when  $c = -1$ , we obtain  $M$ , and  $c = 1$  yields  $X(M, \phi)$ . The Morse function is given by  $f : W(M, \phi) \rightarrow [-1, 1]$  where

$$\begin{cases} f(z, c) = c & z \in M \setminus \phi(S^{\lambda-1} \times \{0\}), c \in D^1 \\ f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 & (\mathbf{x}, \mathbf{y}) \in L_\lambda \end{cases}.$$

## 5 Thursday January 23rd

Recall from last time:  $M$  is a closed smooth  $n - 1$  manifold and  $\phi : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M$ , and we used surgery to obtain  $\chi(M, \phi)$  and a cobordism  $W(M, \phi)$  from  $M$  to  $\chi(M, \phi)$ .

This yields a saddle  $L_\lambda \subseteq \mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$ . We construct the cobordism using

$$S^{\lambda-1} \times D^{n-\lambda} \longrightarrow \partial_L$$

$$(u, tv) \mapsto (u \cosh t, v \sinh t).$$

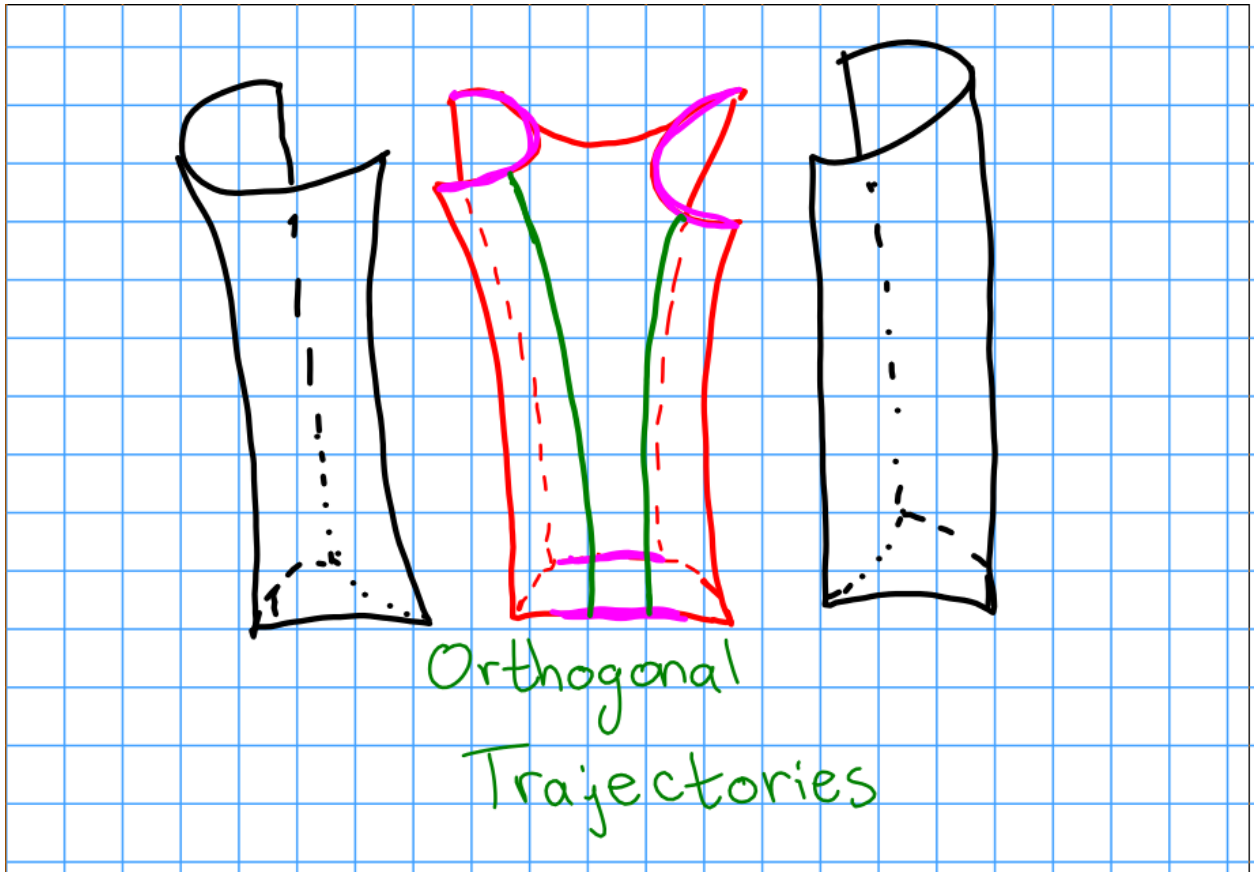


Figure 8: Image

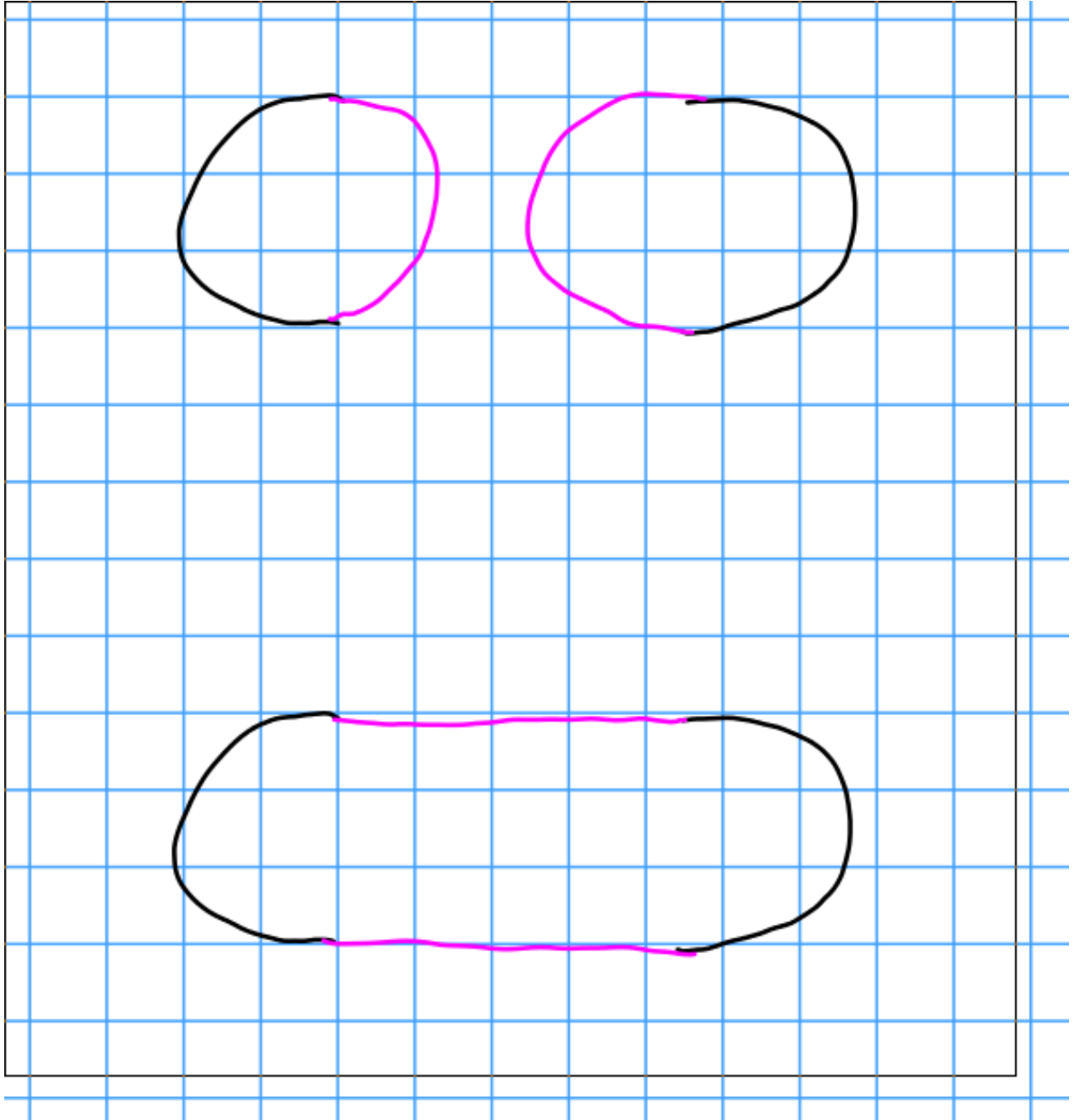
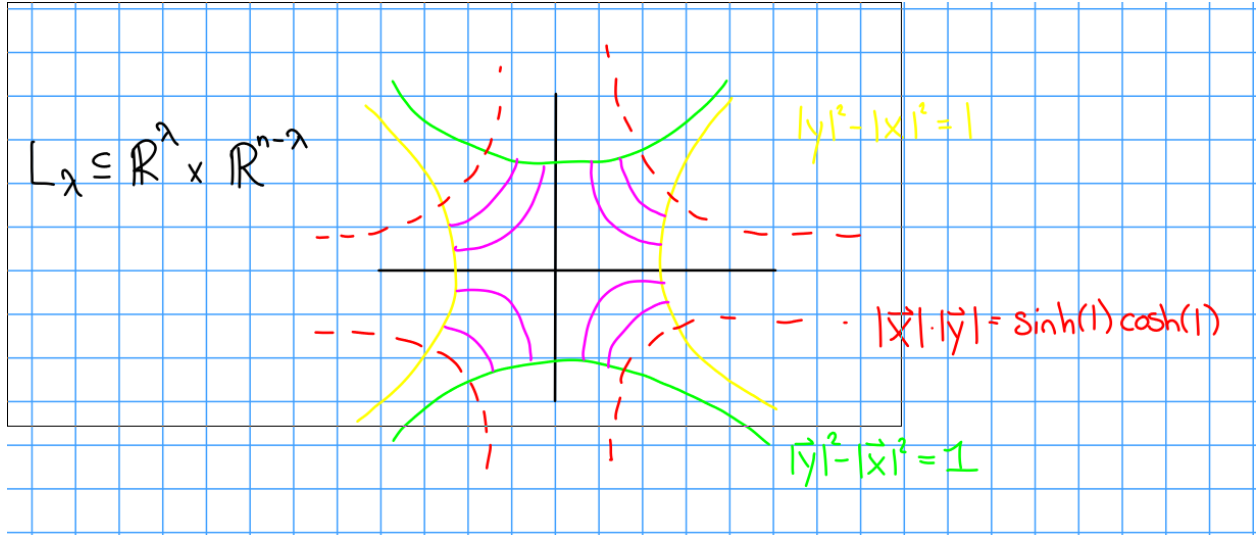


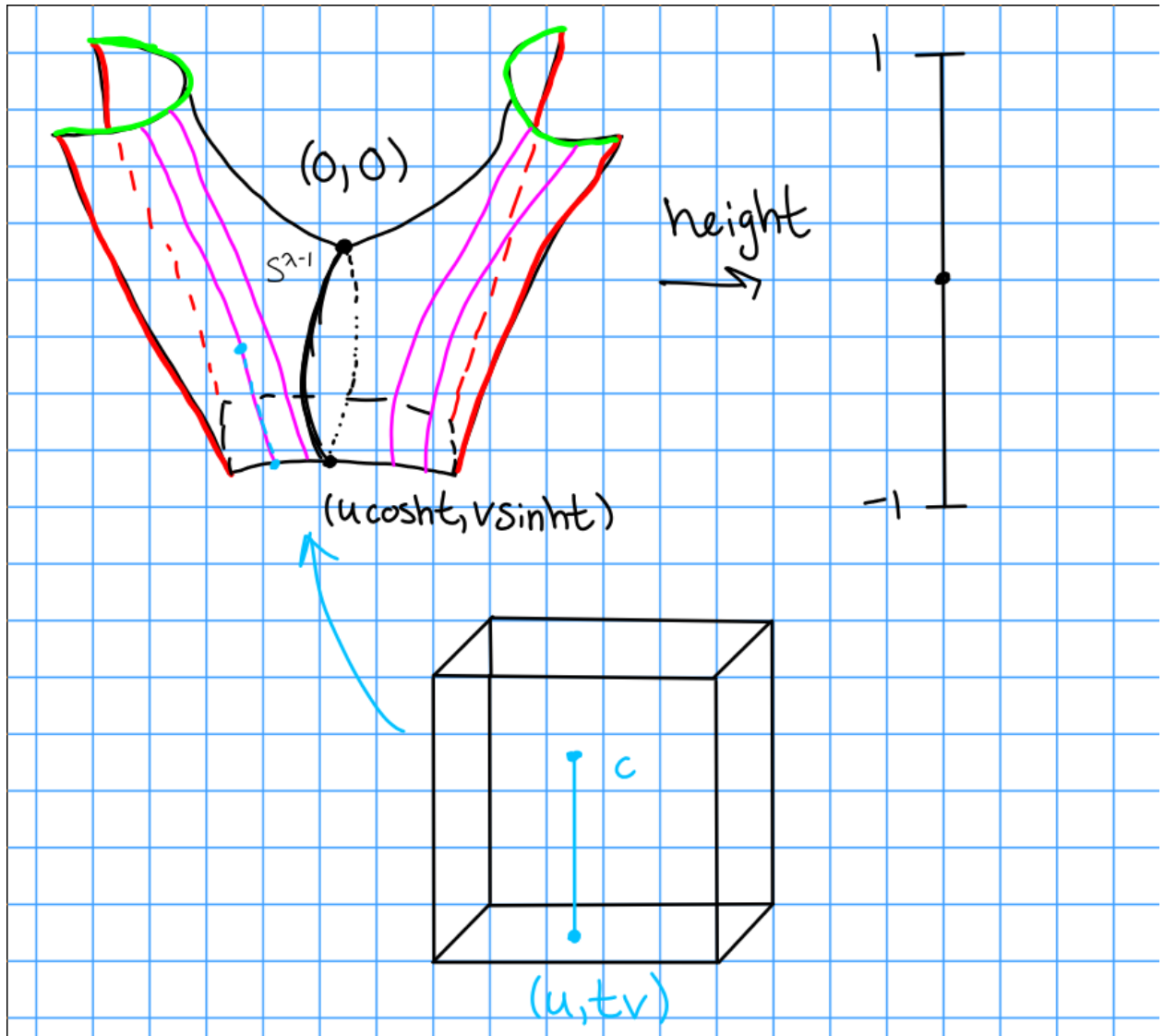
Figure 9: Image

This yields

$$M \setminus \left\{ \phi(S^{\lambda-1}) \times \{0\} \right\} \coprod L_\lambda / \left\langle (u, tv) \times c \sim (\mathbf{x}, \mathbf{y}) \mid \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = c \right\rangle$$

$x, y$  are on a curve that starts from  $(u \cosh t, v \sinh t)$ .





Todo: review!

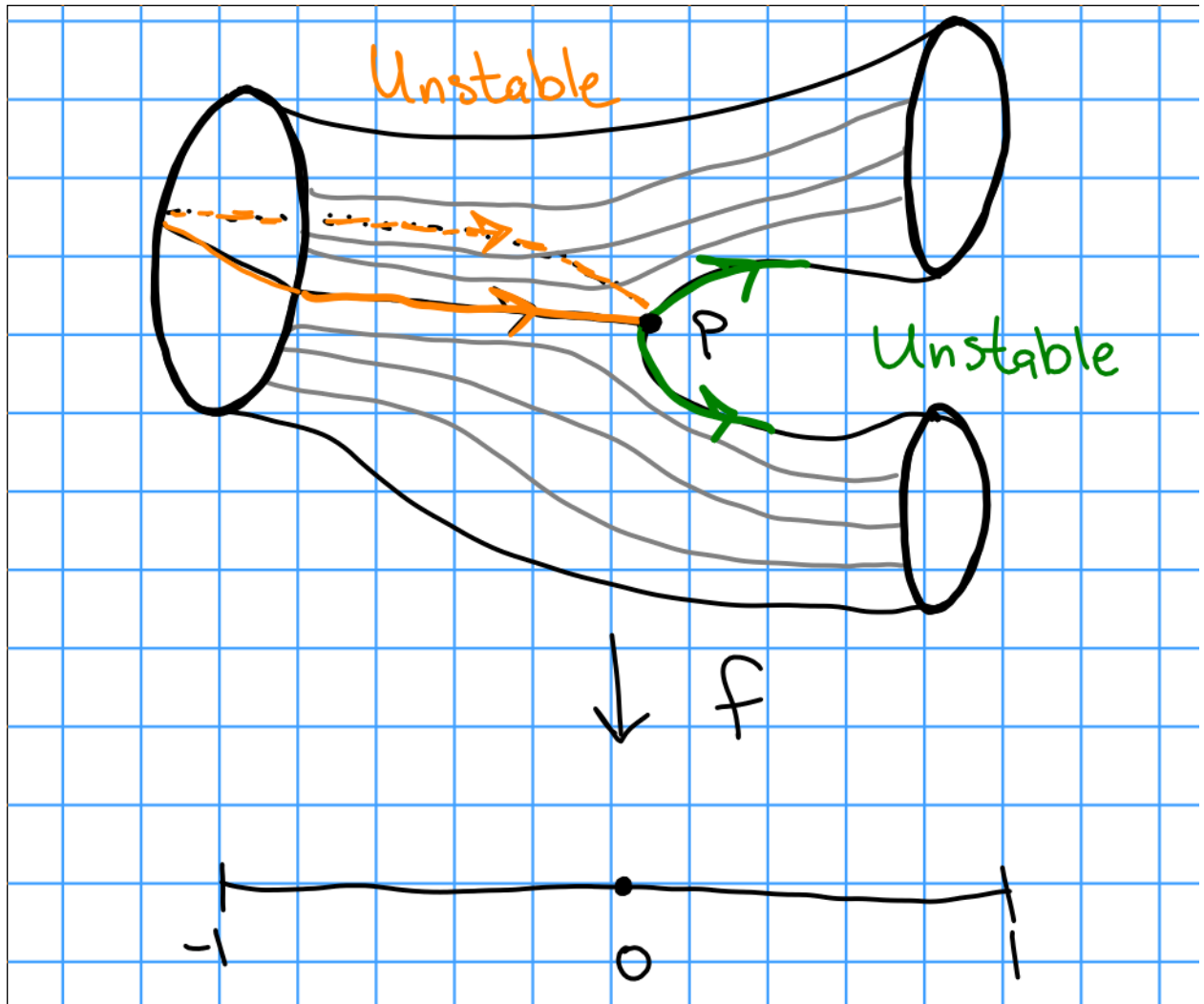
Suppose  $W(; M_0, M_1)$  is an elementary cobordism and  $f : W \rightarrow [-1, 1]$  is a Morse function with one critical point  $p$ , and  $\xi$  a gradient-like vector field for  $f$ .

The goal is to construct  $\phi_L : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M_0$ , the characteristic embedding.

Let  $\psi_x$  be the integral curve of  $\xi$  such that  $\psi_x(0) = x$ , and define  $W^s(p) = \left\{ x \in W \mid \lim_{t \rightarrow \infty} \psi_x(t) = p \right\}$  to be the stable manifold, and  $W^u(p) = \left\{ x \in W \mid \lim_{t \rightarrow -\infty} \psi_x(t) = p \right\}$ .

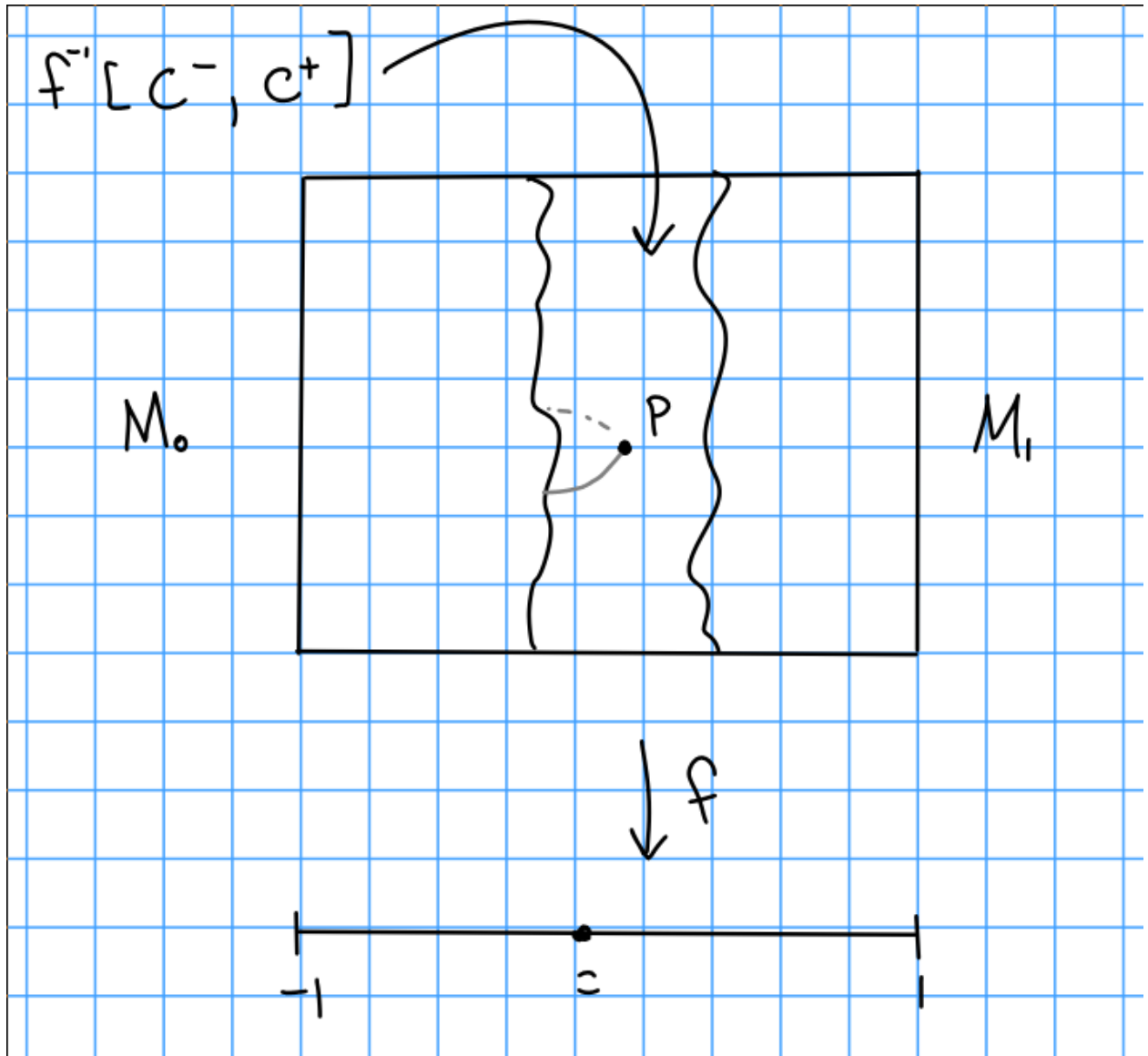
Claim:  $W^s(p), W^u(p)$  are diffeomorphic to disks of dimension  $\lambda$  and  $n - \lambda$  respectively.





Moreover,  $\partial W^s(p) = W^s(p) \cap M_0 \cong S^{\lambda-1}$  (Milnor refers to this as the “left sphere”  $S_L$  and  $\partial W^u(p) = W^u(p) \cap M_1 \cong S^{n-\lambda-1}$  (the “right sphere”  $S_R$ ).

Proof:



Choose an open  $U \ni p$  and a coordinate chart  $h : OD_{2\varepsilon} \rightarrow U$ . Then  $f \circ h(x, y) = c - \|x\|^2 - \|y\|^2$ , which takes on a minimum value of  $c - 4\varepsilon^2$  and a maximum of  $c + 4\varepsilon^2$ , and  $\xi \circ h(x, y) = (-x, y)$ . We thus obtain the inequalities

$$-1 < c - 4\varepsilon^2 < c + 4\varepsilon^2 < 1 \text{ if } 4\varepsilon^2 < c + 1, 1 - c.$$

Now let  $W_\varepsilon = f^{-1}[c - \varepsilon^2, c + \varepsilon^2]$ , and we want a cobordism from  $M_{-\varepsilon} = f^{-1}(c - \varepsilon^2)$  to  $M_{+\varepsilon} = f^{-1}(c + \varepsilon^2)$ . Then

$$M_\mp \cap U = \left\{ h(x, y) \mid c - \|x\|^2 + \|y\|^2 = c_\mp \varepsilon^2 \iff -\|x\|^2 + \|y\|^2 = \mp \varepsilon^2 \right\}.$$

Then  $W^2(p) \cap U = \{h(x, 0)\} \cong D^\lambda$ , and  $W^s(p) \cap M_{-\varepsilon} = \{h(x, 0) \mid \|x\| = \varepsilon\}$ . By flowing along

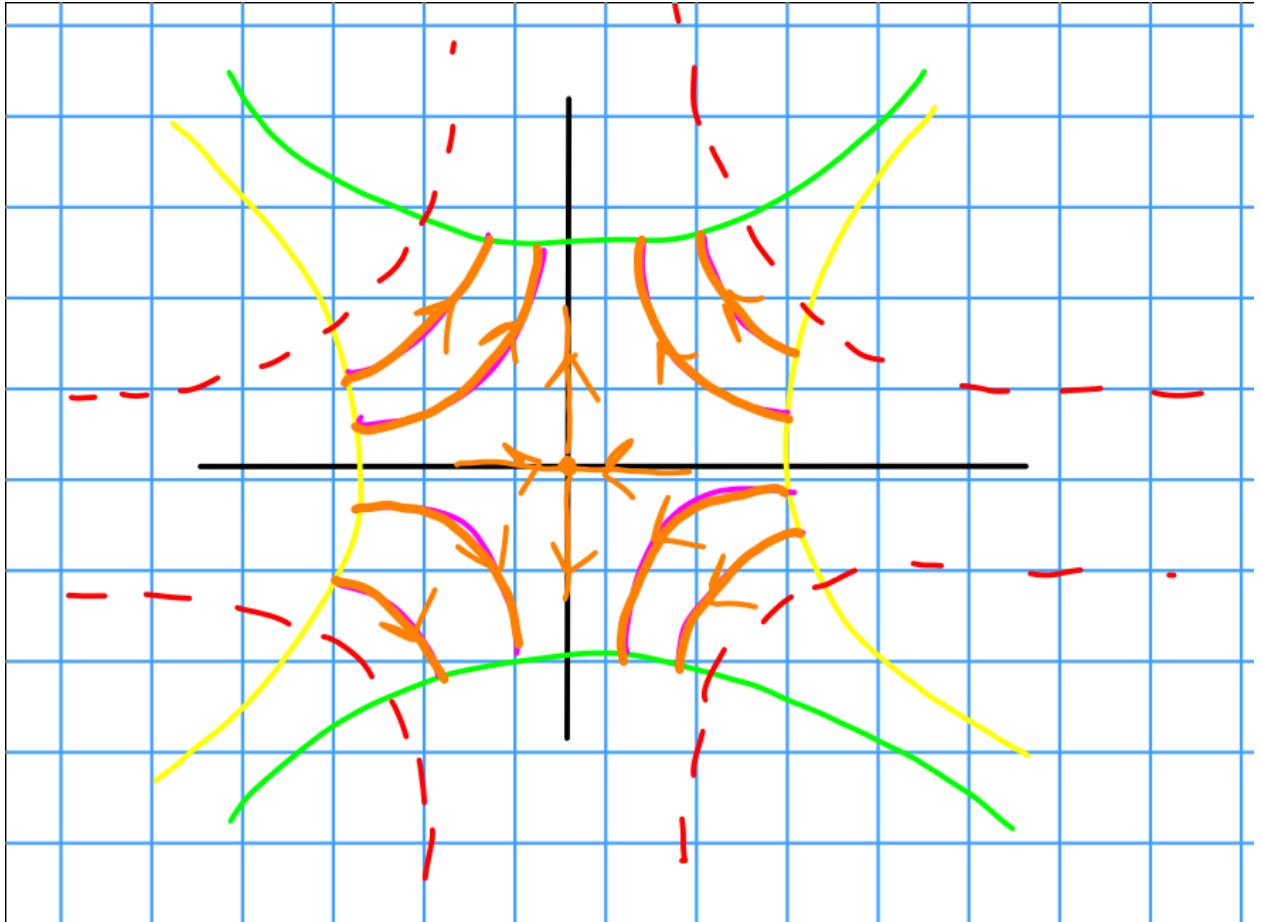


Figure 10: Image

the integral curves of  $\xi$ , we obtain a diffeomorphism  $\Psi_{-\varepsilon} : M_0 \longrightarrow M_{-\varepsilon}$ . Then  $W^s(p) = \{h(x, 0)\} \bigcup \{\text{integral curves of } \xi \text{ passing through points in } W^s(p) \cap M_{-\varepsilon}\}$ .

So  $S_L = \Psi_{-\varepsilon}^{-1}(W^s(p) \cap M_{-\varepsilon})$ , and we can define the embedding  $\phi_L$  by the following composition:

$$\begin{array}{ccc}
 S^{\lambda-1} \times D^{n-\lambda} & \xrightarrow{\quad\quad\quad} & M_0 \\
 & \searrow & \nearrow \Psi_{-\varepsilon}^{-1} \\
 L(u, tv) & \xrightarrow{\quad\quad\quad} & h(\varepsilon u \cosh t, \varepsilon v \sinh t) \\
 & \searrow & \nearrow \\
 & & M_{-\varepsilon}
 \end{array}$$

Similarly, one can show  $W^u(p) \cong D^{n-\lambda}$  and define another embedding  $\phi_R D^\lambda \times S^{n-\lambda-1} \hookrightarrow M_1$ . ■

**Theorem:** Let  $(W; M_0, M_1)$  be an elementary cobordism with  $(f, \xi, p)$  a Morse function with a gradient-like vector field and a critical point as above. Then there is a diffeomorphism of cobordisms

$$(W(M_0, \phi_L); M_0, \chi(M_0, \phi_L)) \cong (W; M_0, M_1),$$

where  $\phi_L : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M_0$  is the characteristic embedding.

*Proof:* Consider  $f^{-1}[-1, c - \varepsilon^2] \bigcup W_\varepsilon \bigcup f^{-1}[c + \varepsilon^2, 1]$ , and note that  $f^{-1}[-1, c - \varepsilon^2], f^{-1}[c + \varepsilon^2, 1]$  is a product (?). Then  $(W; M_0, M_1) \cong (W_\varepsilon; M_{-\varepsilon}, M_\varepsilon)$ . We then have

$$(W(M, \phi_L); M_0, \chi(M_0, \phi_L)) \cong (W(M_{-\varepsilon}, \phi), M_{-\varepsilon}, \chi(M_{-\varepsilon}, \phi)).$$

So we'll define a diffeomorphism from the RHS to the RHS of the former diffeomorphism. Define  $f \longrightarrow [c - \varepsilon^2, c + \varepsilon^2]$  in the former and  $f' \longrightarrow [-1, 1]$  in the latter. Then take  $f \circ k = c + \varepsilon^2 f'$  to match up the domains. Take  $(z, t) \in (M_{-\varepsilon} \setminus \phi(\{S^{\lambda-1} \times \{0\}\})) \times D^1$ . Then  $k(z, t)$  is the point on the integral curve which passes through  $z$  with  $f(k(z, t)) = c + \varepsilon^2 t$ . This map will take flow lines to flow lines. Now define

$$(x, y) \in L_\lambda \longrightarrow h(\varepsilon x, \varepsilon y).$$

*Exercise:* Show that this is a well-defined diffeomorphism.

**Theorem:** For an elementary cobordism  $(W; M_0, M_1)$  and  $(f, \xi, p)$  as above, then  $M_0 \bigcup D_L$  (where  $D_L \cong W^s(p)$ ) is a deformation retraction of  $W$ .

*Corollary:*

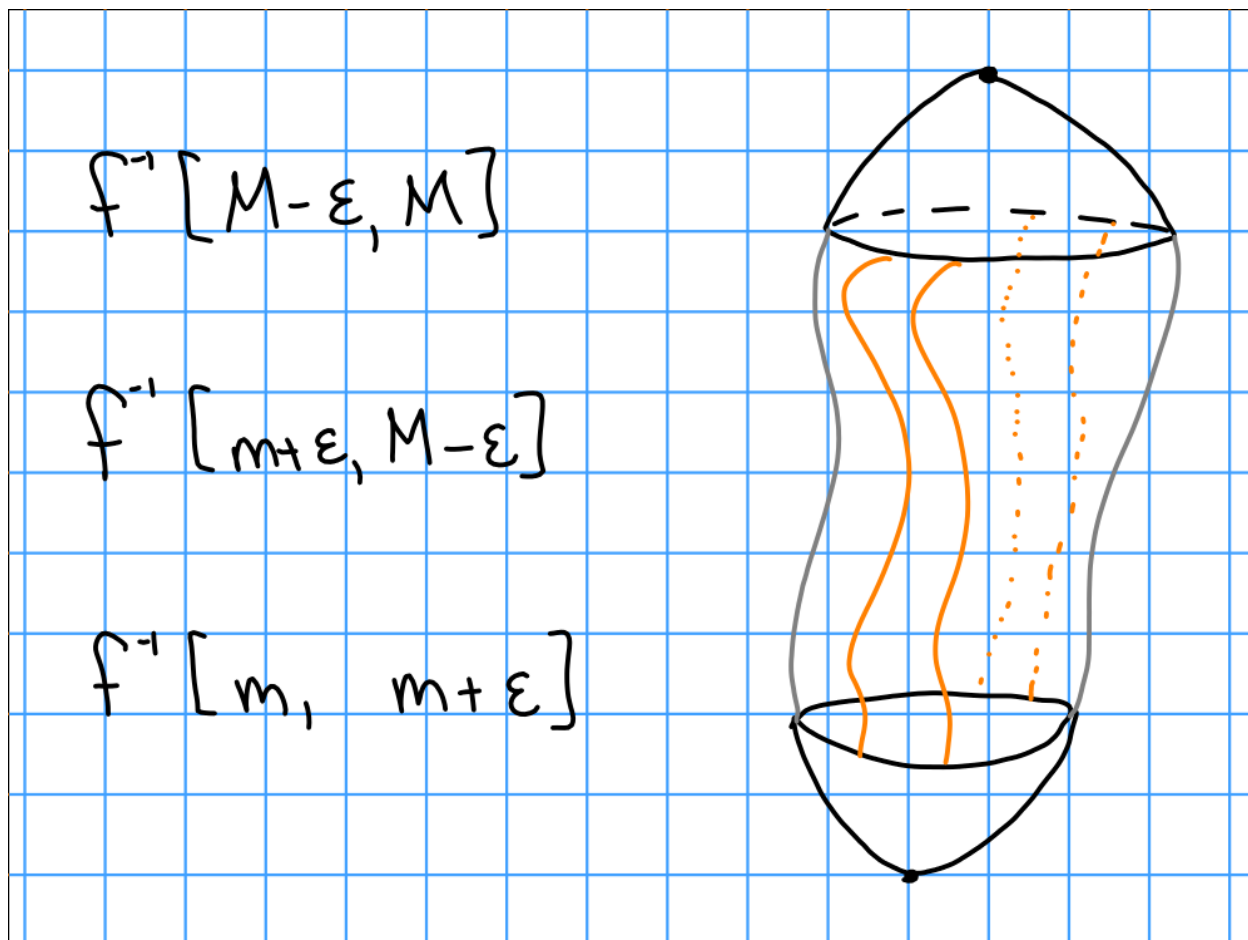


Figure 11: Image

$$H_i(W, M_0) = H_i(M_0 \cup D_L, M_0) \stackrel{\text{excision}}{=} H_i(D_L, S_L) \cong \begin{cases} \mathbb{Z} & i = \lambda \\ 0 & \text{otherwise} \end{cases}.$$

Theorem (Reeb): If a closed  $n$ -manifold  $M$  has a Morse function with exactly 2 critical points, then  $M$  is homeomorphic to  $S^n$ .

Proof: If  $m = \min_{x \in M} f(x) = f(A)$  and  $M = \max_{x \in M} f(x) = f(B)$  where  $A, B$  are critical points. Then points near  $A$  are in the unstable manifold, so  $\text{Ind}(A) = 0$ , and points near  $B$  are in the stable, so  $\text{Ind}(B) = n$ .

The middle piece of the cobordism is a product cobordism, and  $M$  is called a twisted sphere.

Exercise: Every twisted sphere is homeomorphic to  $S^n$ .

## 6 Tuesday January 28th

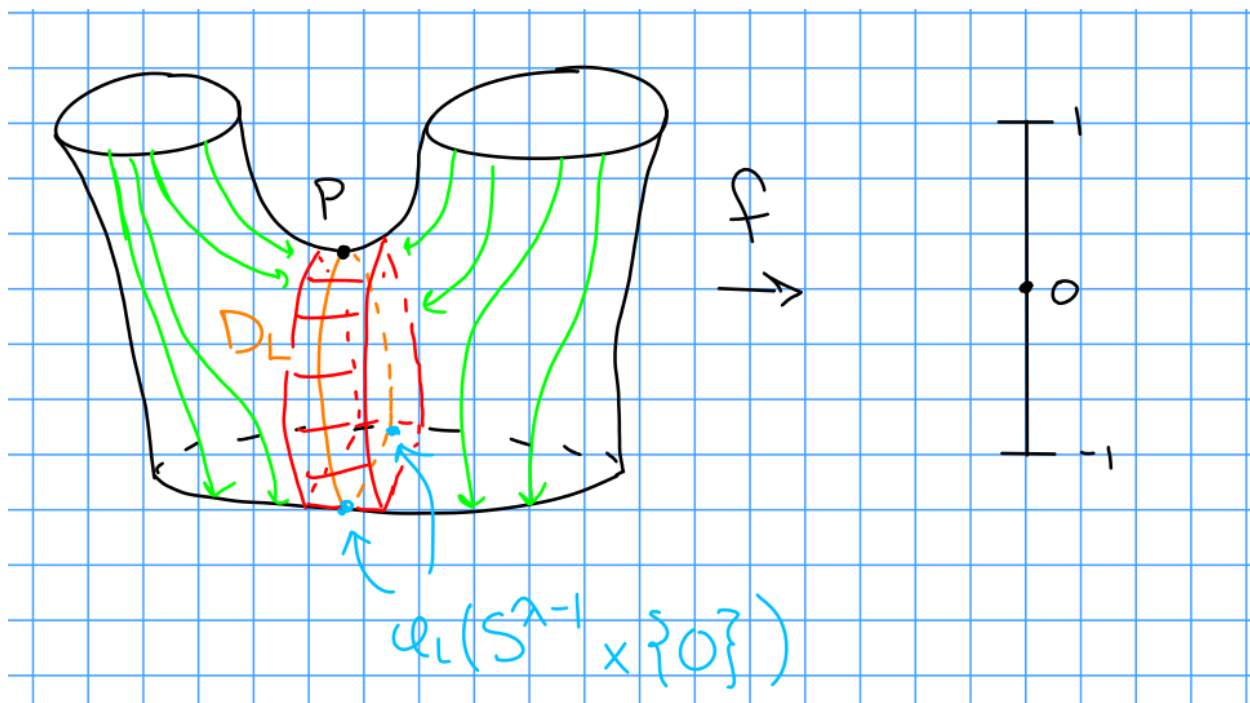
Setup: Fix an elementary cobordism  $(W; M_0, M_1)$ , a Morse function  $f : W \rightarrow [-1, 1]$  with exactly one critical point  $p$  with index  $\text{Ind}(p) = \lambda$ . This yields a gradient-like vector field  $\xi$ , and  $D_L = W^s(p) = \left\{ x \in W \mid \lim_{t \rightarrow \infty} \psi_x(t) = p \right\}$  the stable manifold.

**Theorem:**  $W \cong M_0 \cup D_L$ , a  $\lambda$  dimensional disk, is a homotopy equivalence. More precisely, there is a deformation retract.

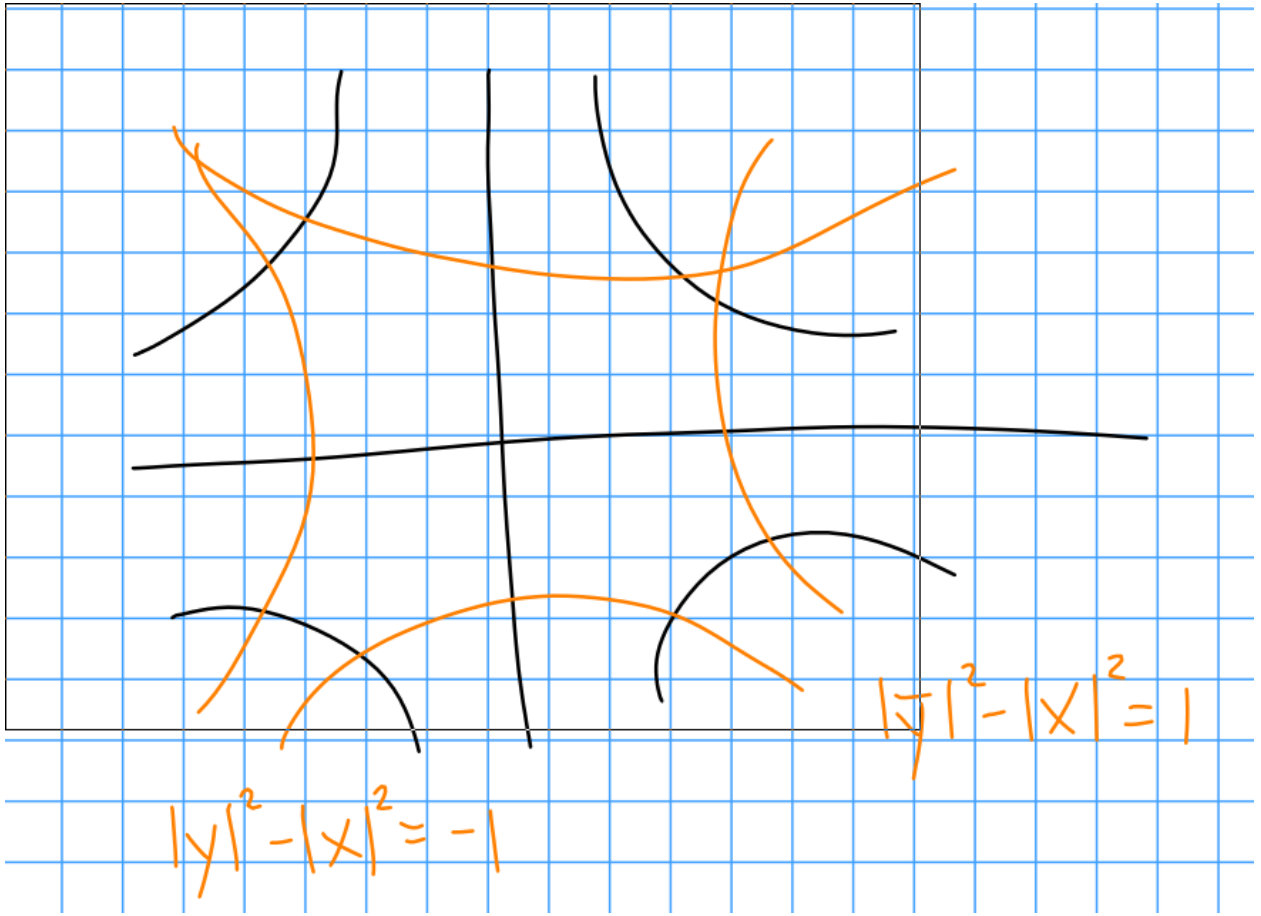
Proof: Take the characteristic embedding  $\phi_L : S^{\lambda-1} \times OD^{n-\lambda} \hookrightarrow M_0$ . We have a cobordism  $(W(M_0, \phi_L); M_0, \chi(M_0, \phi_L)) \cong (W; M_0, M_1)$ .

Recall that the LHS is constructed via  $(M_0 \setminus \phi(S^{\lambda-1} \times 0)) \times D_1 \amalg L_\lambda / \sim$ .

Retraction 1:  $W(M_0, \phi_L) \xrightarrow{r_t} M_0 \cup C$ . We'll construct this retraction. This follows the green integral curves to retract onto the red.



Identify  $D_L = \{(\mathbf{x}, \mathbf{0})\} \subset L_\lambda$  in the local picture:



Define  $C = \left\{ (\mathbf{x}, \mathbf{y}) \mid \|\mathbf{y}\| \leq \frac{1}{10} \right\}$ .

Choose  $(Z, c)$  such that  $z \in M_0 \setminus \phi_L(S^{\lambda-1} \times OD^{n-\lambda})$  and  $c \in [-1, 1]$ . Let  $r_t(z, c) = (z, c + t(-1 - c))$ , note what happens at  $t = -1, 1, 0$ .

We can parameterize the integral curves in the local picture as  $(\mathbf{x}/r, r\mathbf{y})$ .

So for  $(\mathbf{x}, \mathbf{y}) \in L_\lambda$ , we can define

$$r_t(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}, \mathbf{y}) & \|\mathbf{y}\| \leq \frac{1}{10} \iff (\mathbf{x}, \mathbf{y}) \in C \\ ? & ? \\ (\mathbf{x}/\rho(t), \rho(t)\mathbf{y}) & \|\mathbf{y}\| \geq \frac{1}{10} \end{cases}.$$

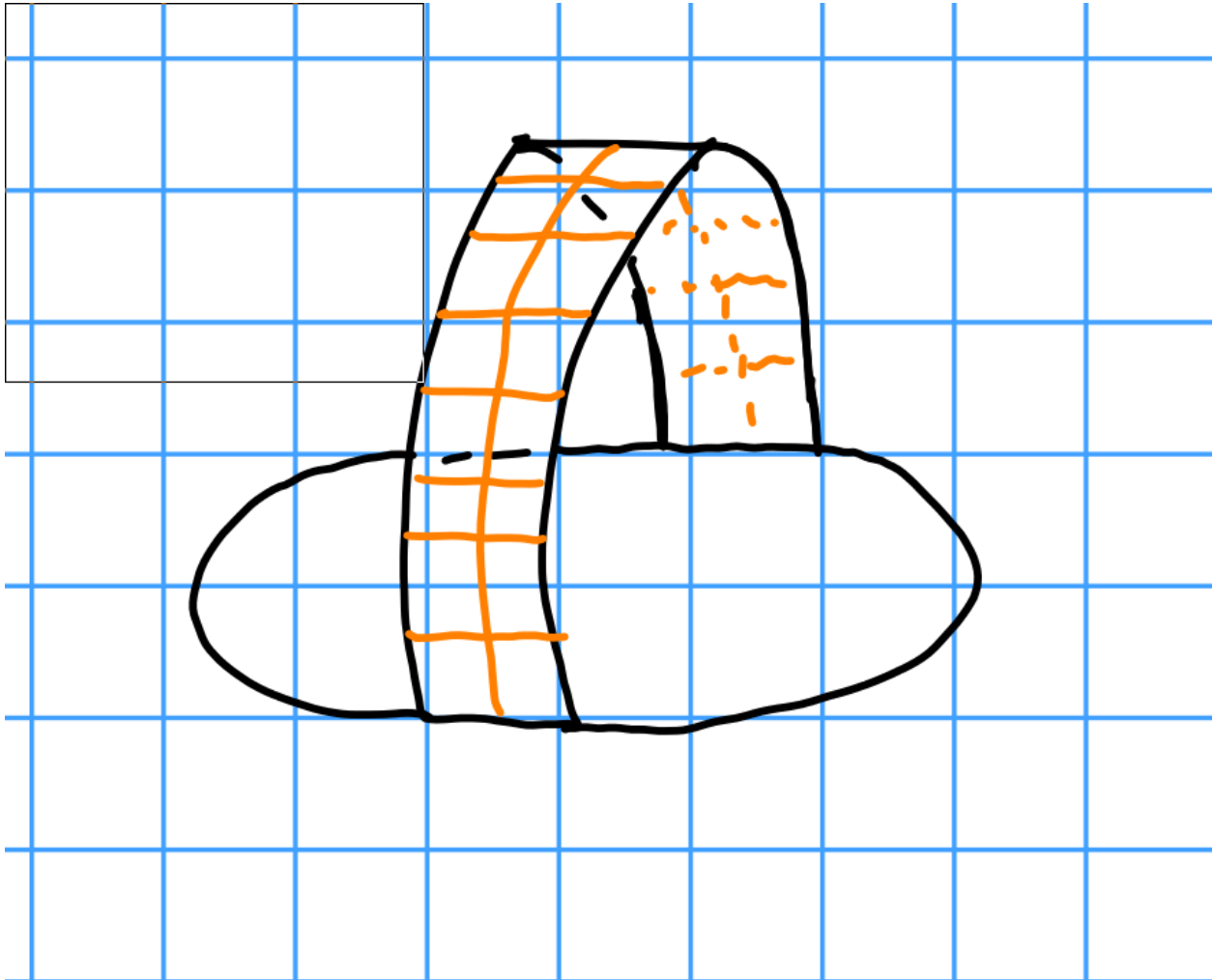
where  $\rho(t)$  is the solution of

$$\begin{aligned} \rho(t)^2 \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 / \rho(t)^2 &= (\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2)(1 - t) - t \\ \rho(t) \|\mathbf{y}\|^2 &\geq \frac{1}{10} \implies \rho(t) \geq \frac{1}{10\|\mathbf{y}\|}. \end{aligned}$$

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So we define  $\rho(t) = \max(\text{positive solutions for the above equation}, \frac{1}{10\|\mathbf{y}\|})$ .

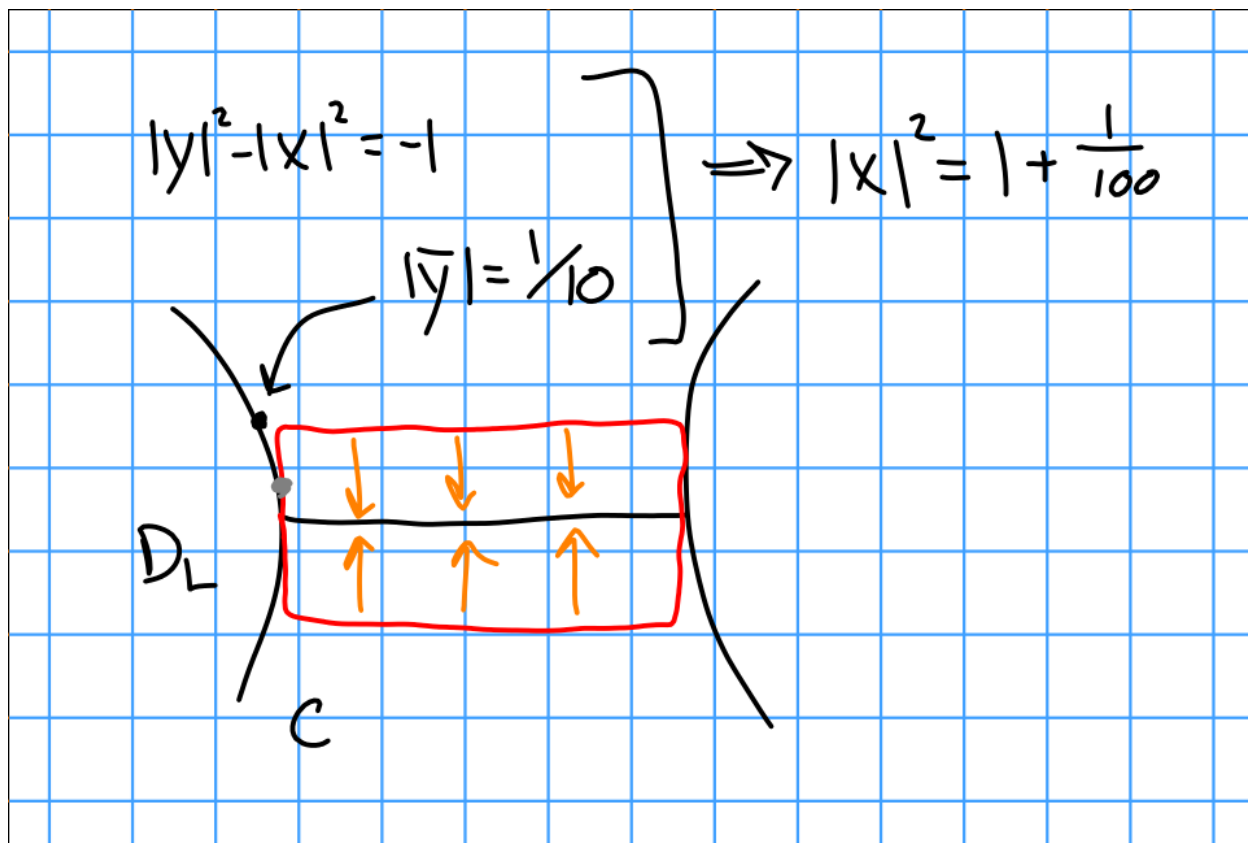
Retraction 2:  $M_0 \bigcup C \xrightarrow{r'_t} M_0 \bigcup D_L$



We want the restriction of  $r'_t$  to  $M_0 \setminus C$  to be the identity, so for  $(\mathbf{x}, \mathbf{y}) \in C$  we define

$$r'_t(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}, (1-t)\mathbf{y}) & \|\mathbf{x}\| \leq 1 \\ (\mathbf{x}, \alpha(t)\mathbf{y}) & 1 \leq \|\mathbf{x}\| \leq \sqrt{1 + \frac{1}{100}} \end{cases}.$$





We define  $\alpha(t)$  at  $t = 0$  to be the identity, and at  $t = 1$  we want  $\|\alpha(t)\mathbf{y}\|^2 - \|x\|^2 = -1$ , and solving yields

$$\alpha(t) = (1 - t) + t \frac{\sqrt{\|x\|^2 - 1}}{\|\mathbf{y}\|}.$$

■

**Corollary:** For  $M$  a closed smooth  $n$ -manifold with a Morse function  $f : M \rightarrow \mathbb{R}$ ,  $M$  is homotopy-equivalent to a CW complex with one  $\lambda$ -cell for each critical point of index  $\lambda$ .

Proof: See Milnor's book (Morse Theory).

## 6.1 Morse Inequalities

Let  $M$  be a closed smooth manifold and  $f : M \rightarrow \mathbb{R}$  Morse, and fix  $F$  a field. Let  $b_i(M)$  denote the  $i$ th Betti number, which is  $\text{rank } H_i(M; F)$ .

Weak Morse Inequalities:

1.  $b_i(M) \leq$  the number of index  $i$  critical points, denoted  $c_i(M)$ .
2.  $\xi(M) = \sum (-1)^i c_i(M)$ .

---

Strong Morse Inequalities:  $b_i(M) - b_{i-1}(M) + \cdots \pm b_0(M) \leq c_i - c_{i-1} + c_{i-2} \cdots \pm c_0$ .

**Lemma:** The weak inequalities are consequences of the strong ones.

*Proof implying (1):* We have  $b_i - \cdots \leq c_i - \cdots$  and separately  $b_{i-1} - \cdots \leq c_{i-1}$ , and adding these inequalities yields  $b_i \leq c_i$ .

*Proof implying (2):* To obtain the equality, multiply through by a negative sign. For  $i > n$ , we have  $b_{i-1} - b_{i-2} + \cdots = c_{i-1} - c_{i-2} + \cdots$ , where the LHS is  $\pm \chi(M)$  and the RHS has matching signs.

*Proof of Strong Morse Inequalities:* Suppose  $f(p_1) < \cdots < f(p_k)$ . We can select points  $a_i$  such that  $a_0 < f(p_1) < a_1 < \cdots$ . Let  $M_i = f^{-1}[a_0, a_i]$ ; we then have  $\emptyset := M_0 \subset M_1 \subset \cdots \subset M_k = M$ .

Using excision, we have

$$\begin{aligned} H_j(M_i, M_{i-1}) &= H_j(f^{-1}[a_{i-1}, a_i], f^{-1}(a_{i-1})) \\ &= \begin{cases} \mathbb{F} & j = \text{Ind}(p_i) = \lambda_i \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

So  $b_j(M_i, M_{i-1}) = 1$  iff  $j = \lambda_i$ , and 0 otherwise.

Sub-lemma: Define  $S_i := b_i(X, Y) - b_{i-1}(X, Y) + \cdots$ , i.e. the LHS of the strong Morse inequality. Then  $S_i$  is subadditive, i.e. if  $X \supset Y \supset Z$  then  $S_i(X, Z) \leq S_i(X, Y) + S_i(Y, Z)$ .

This implies the strong inequality, since  $S_i(M, \emptyset) \leq S_i(M_k, M_{k-1}) + S_i(M_{k-1}, M_{k-2}) + \cdots + S_i(M_1, M_0)$ . The RHS here equals  $\sum_{j=1}^k T_i(M_j, M_{j-1}) = T_i(M)$ , where  $T_i(M) = c_i - c_{i-1} + \cdots$ .

Write down the relative homology exact sequence:

$$\begin{array}{ccccccc} H_{i+1}(X, Y) & \xrightarrow{\partial_i} & H_i(Y, Z) & \xrightarrow{f_i} & H_i(X, Z) & \xrightarrow{g_i} & H_i(X, Y) \\ & & & & & \swarrow \partial_{i-1} & \\ & & H_{i-1}(Y, Z) & & & & \end{array}$$

then  $\text{rank}(\partial_i) = \dim \ker(f_i) = b_i(Y, Z) - \text{rank}(f_i) = b_i(Y, Z) - b_i(X, Z) + \text{rank}(g_i) = \cdots = S_i(Y, Z) - S_i(X, Z) + S_i(X, Y) > 0$  since ranks are positive.

## 7 Thursday January 30th

### 7.1 Morse Inequality Example

Example: Consider  $f : \mathbb{CP}^n \rightarrow \mathbb{R}$  where (recall)  $\mathbb{CP}^n = S^{2n+1} \subset \mathbb{C}^{n+1} / \sim$  where  $\mathbf{z} \sim \lambda \mathbf{z}$  for all  $|\lambda| = 1$  in  $\mathbb{C}^\times$ , where  $f$  is given by  $[z_0 : \cdots : z_n] \mapsto \sum i|z_i|^2$ .

Note that we can take the coefficients to be any  $n+1$  distinct real numbers, here we just take  $1, 2, \dots, n+1$  for simplicity.

Cover  $\mathbb{CP}^n$  with  $n+1$  coordinate charts  $(U_j, h_j)$  where  $U_j = \{z \mid z_j \neq 0\}$  and  $h_j : U_j \rightarrow \mathbb{R}^{2n}$  is given by first defining  $[z_0 : \cdots : z_n] \mapsto (\hat{z}_j z_0, \hat{z}_j z_1, \dots, \hat{z}_j z_n)$  where  $\hat{z}_j = z_j / |z_j|$ .

Denote the image coordinates by  $z_k = x_k + iy_k$ . Then define  $h_j$  by  $[z_0 : \cdots : z_n] \xrightarrow{h_j} (x_0, y_0, \dots, x_{j-1}, y_{j-1}, x_{j+1}, y_{j+1}, \dots, x_n, y_n)$ .

Note that  $|z_j| = 1 - \sum_{i \neq j} |z_i|^2$ , so this is a one-to-one correspondence (i.e. we can recover the magnitude of  $z_j$  from the image point).

So what is  $f$  in these coordinates? We can write

$$\begin{aligned} f \circ h_j^{-1}(x_0, y_0, \dots, \hat{x}_j, \hat{y}_j, \dots, x_n, y_n) &= \sum_{i \neq j} i(x_i^2 + y_i^2) + j|z_j|^2 \\ &= \sum_{i \neq j} i(x_i^2 + y_i^2) + j - \sum_{i \neq j} j(x_i^2 + y_i^2) \\ &= j + \sum_{i \neq j} (i - j)(x_i^2 + y_i^2) \\ &= j + (-j)(x_0^2 + y_0^2) + (-j+1)(x_1^2 + y_1^2) + \cdots + (n-j)(x_n^2 + y_n^2). \end{aligned}$$

So what are the critical points? The derivative is zero iff  $x_i = y_i = 0$  for some  $i \neq j$ . So there is only one critical point,  $p_j = [0 : 0 : \cdots : 0 : 1_j : \cdots : 0]$ . Thus there are  $n+1$  critical points given by  $\text{crit}(f) = \{p_0, \dots, p_n\}$ . Using the above equations, we can find that  $\text{Ind}_f(p_j) = 2j$  (count positive and negative terms).

Note that we had the inequality  $b_i(M) \leq |\{\text{critical points with index } i\}|$ . Noting that  $H^i(\mathbb{CP}^n; \mathbb{F}) = \mathbb{F}$  for  $i = 0, 2, 4, \dots, 2n$  and 0 otherwise, so we have exact equality here.

Note that there are QHS where the inequality has to be strict, but equality can be obtained with  $\mathbb{CP}^n, S^n$ , etc.

## 7.2 Rearrangement

Fix a Morse function  $f : W \rightarrow [0, 1]$ , with  $p, q \in \text{crit}(f)$  and  $f(p) < f(q)$ . Can we change  $f$  to a new Morse function  $g$  such that  $\text{crit}(g) = \text{crit}(f)$  and  $g(p) > g(q)$ , where  $g = f + \text{const.}$  in a neighborhood of  $p$  and a neighborhood of  $q$ ?

Note that we obtain elementary cobordisms in each case:

Pick  $\xi$  a gradient-like vector field for  $f$ , which decomposes  $W^*(p) = W^s(p) \cup W^u(p)$ .

**Lemma:** Let  $f : W \rightarrow I$  be a Morse function with 2 critical points  $p, q$  and  $\xi$  a gradient-like vector field for  $f$  such that  $W^*(p) \cap W^*(q) = \emptyset$ . Then for any two points  $a, b \in I$ , there exists a Morse function  $g$  such that:

1.  $\xi$  is gradient-like for  $g$ ,
2.  $\text{crit}(g) = \text{crit}(f)$ , with  $g(p) = a$  and  $g(q) = b$ .
3.  $g = f$  near  $M_0$  and  $M_1$ , and  $g = f + \text{const.}$  in some neighborhood of  $p$ , and some neighborhood of  $q$ .

So this is stronger: we can modify our Morse function to take on any two real numbers.

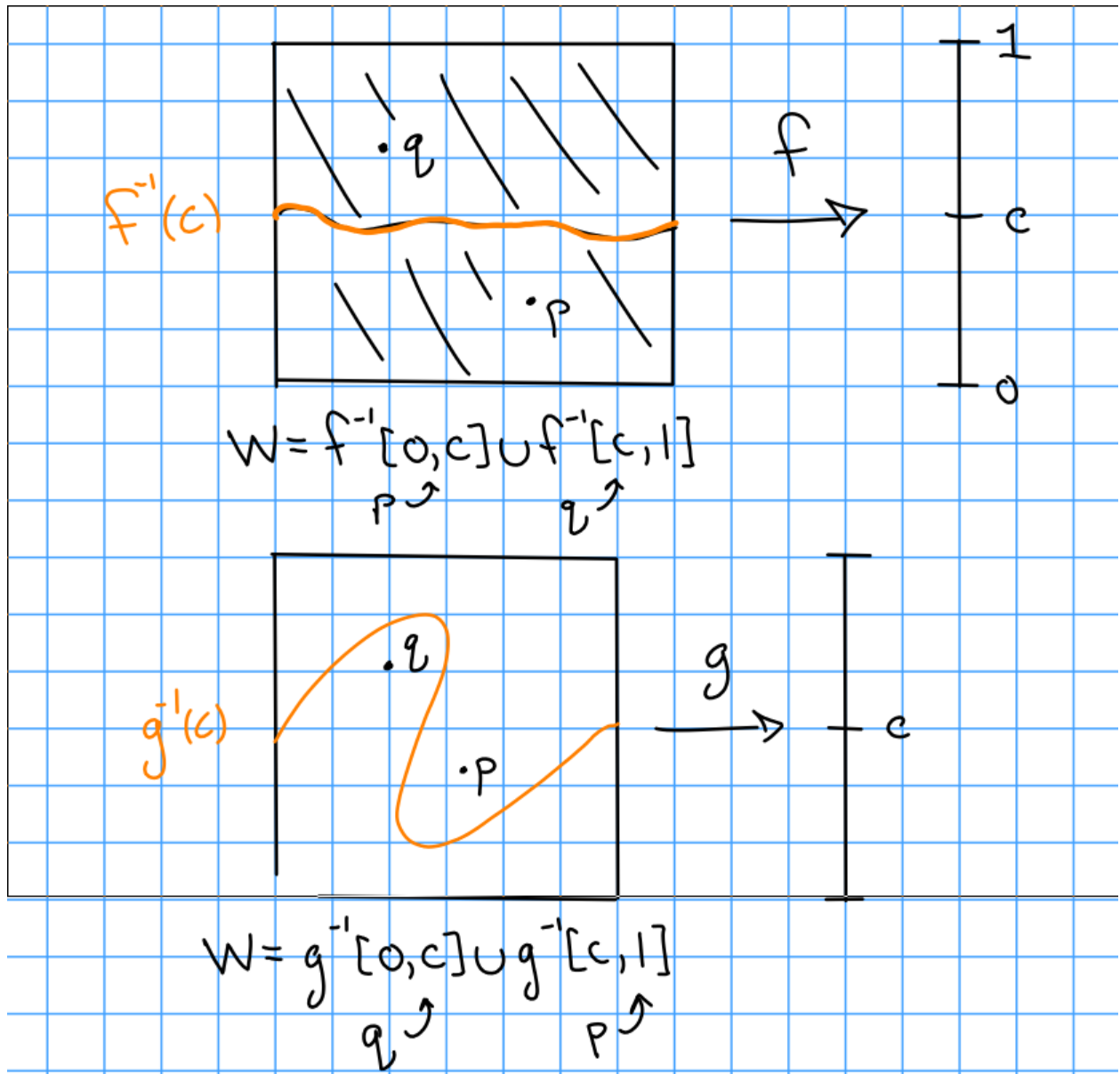


Figure 12: Image

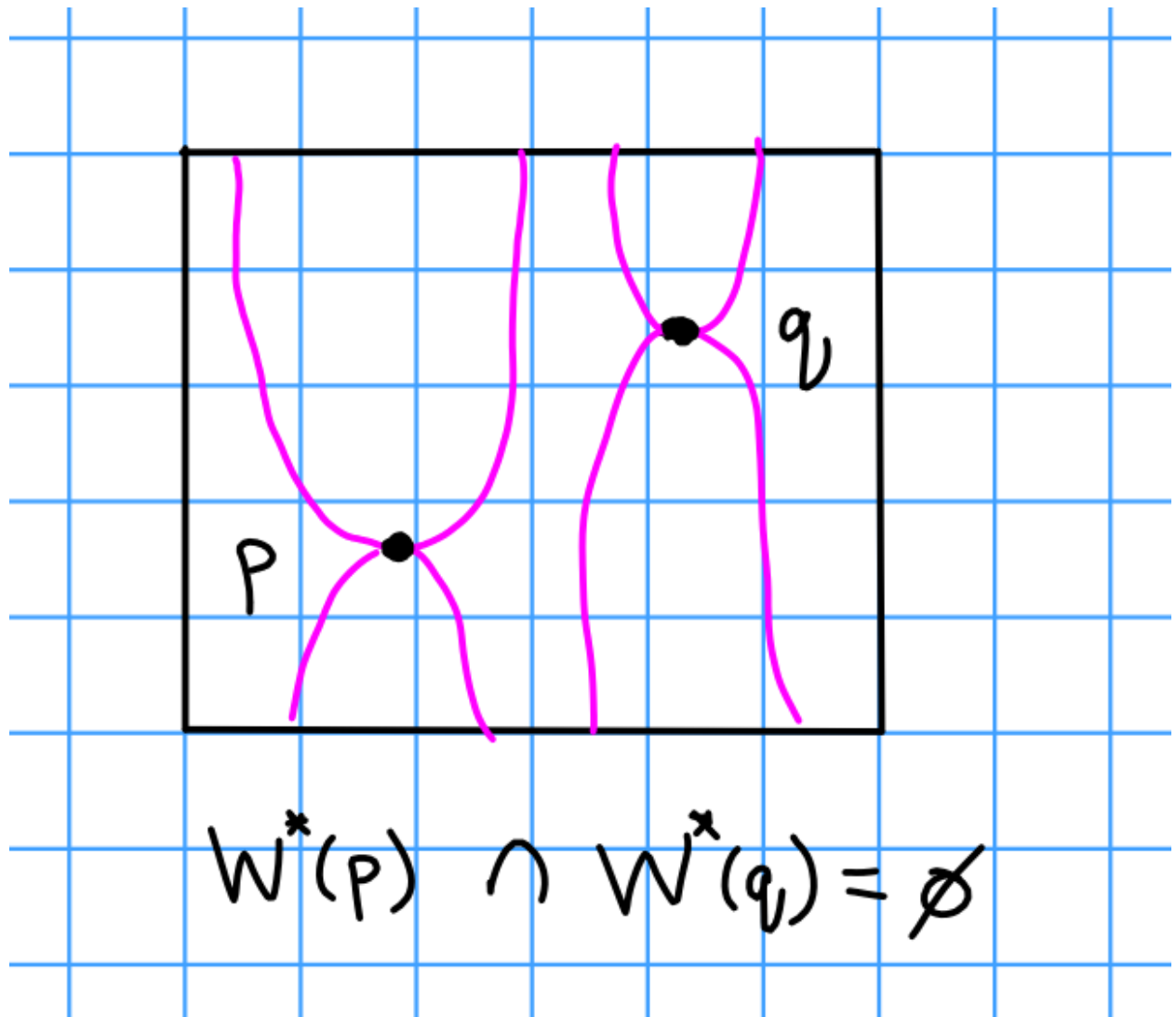


Figure 13: Image

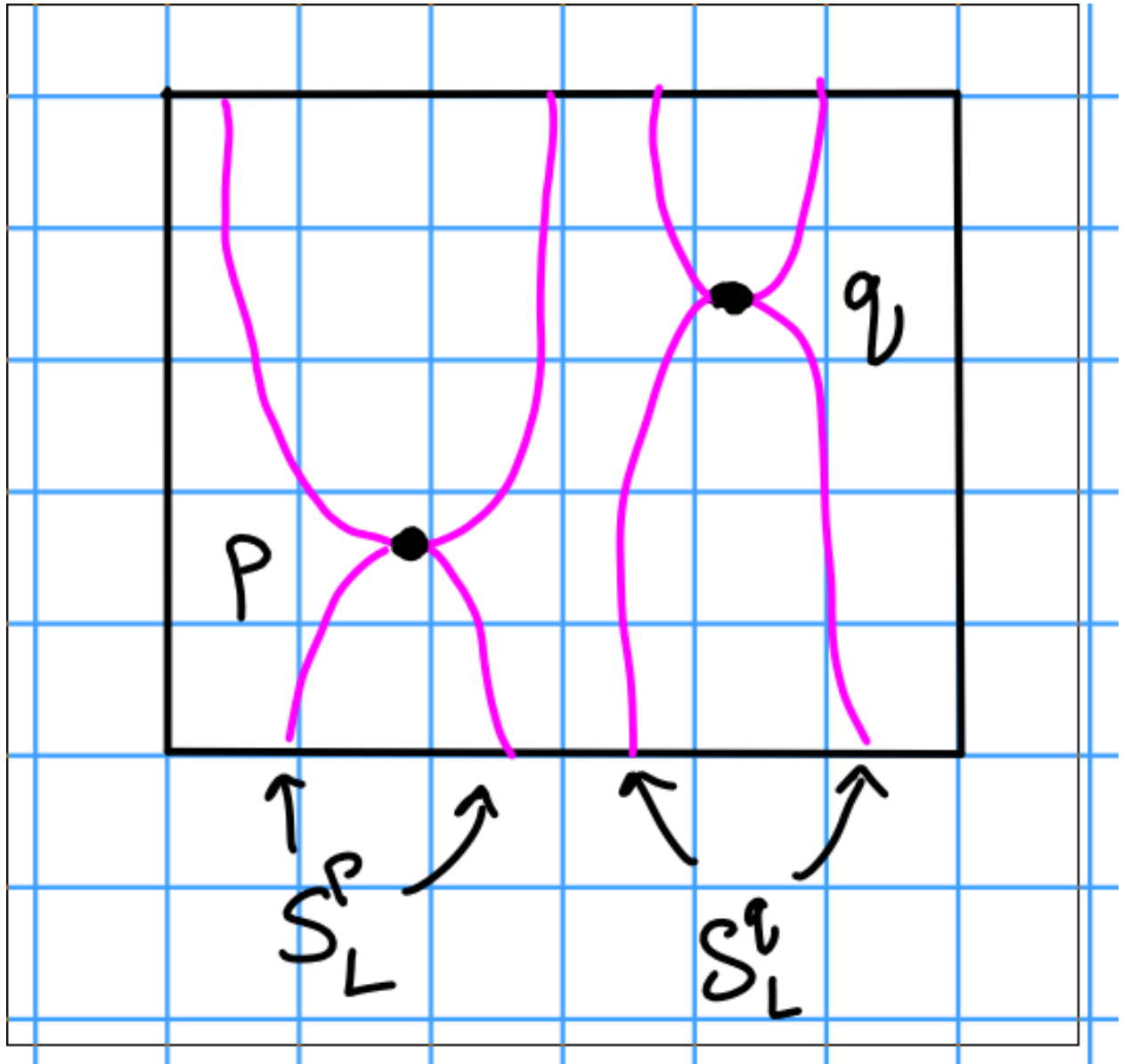


Figure 14: Image

*Idea of proof:* We want the two purple sections here, since we want to modify  $p$  and  $q$  separately:

*Proof:* We can find a  $\mu : M_0 \rightarrow I$  such that  $\mu \equiv 0$  near  $S_L^P$  and  $\mu \equiv 1$  near  $S_L^q$ . So extend  $\mu$  to  $\bar{\mu} : W \rightarrow I$  such that  $\bar{\mu}$  is constant over the integral curves of  $\xi$  and  $\bar{\mu} \equiv 0$  near  $W^*(p)$  and  $\bar{\mu} \equiv 1$  near  $W^*(q)$ .

Here the integral curves are green:

Let  $g(z) = G(f(z), \bar{\mu}(z))$  where  $G : I \times I \rightarrow I$  will be defined as follows:

1. Fix a  $y \in W$  to  $\bar{\mu}$  is constant, then  $G(\cdot, y) : I \rightarrow I$  is increasing (since  $f$  is increasing) and surjective, i.e.  $\frac{\partial G}{\partial x} > 0$  everywhere.
2.  $G(x, y) = x$  whenever  $x$  is near 0 or 1.
3.  $\frac{\partial G}{\partial x}(x, 0) = 1$  for  $x$  near  $f(p)$  and  $\frac{\partial G}{\partial x}(x, 1) = 1$  for  $x$  near  $f(q)$ .

Note that  $a = g(p) = G(f(p), 0)$  and  $b = g(q) = G(f(q), 1)$ , and the slope should be constant near  $a, b$ .

We get something like the following graphs:

■

When can such a function exist? I.e. is this a relatively strong condition? If  $f(p) < f(q)$ , it is possible that  $W^u(p) \cap W^s(q) \neq \emptyset$ :

□

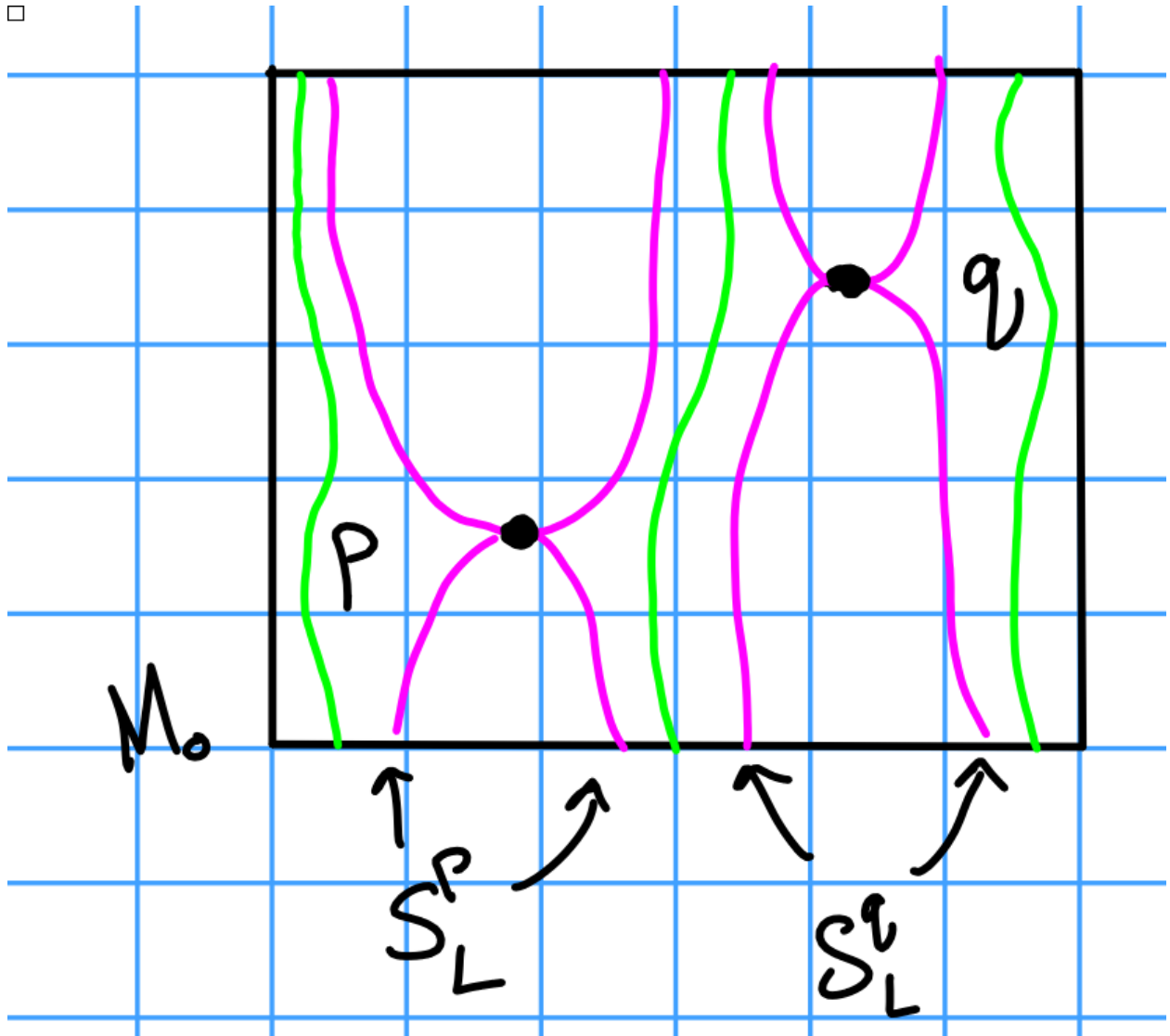


Figure 15: Image



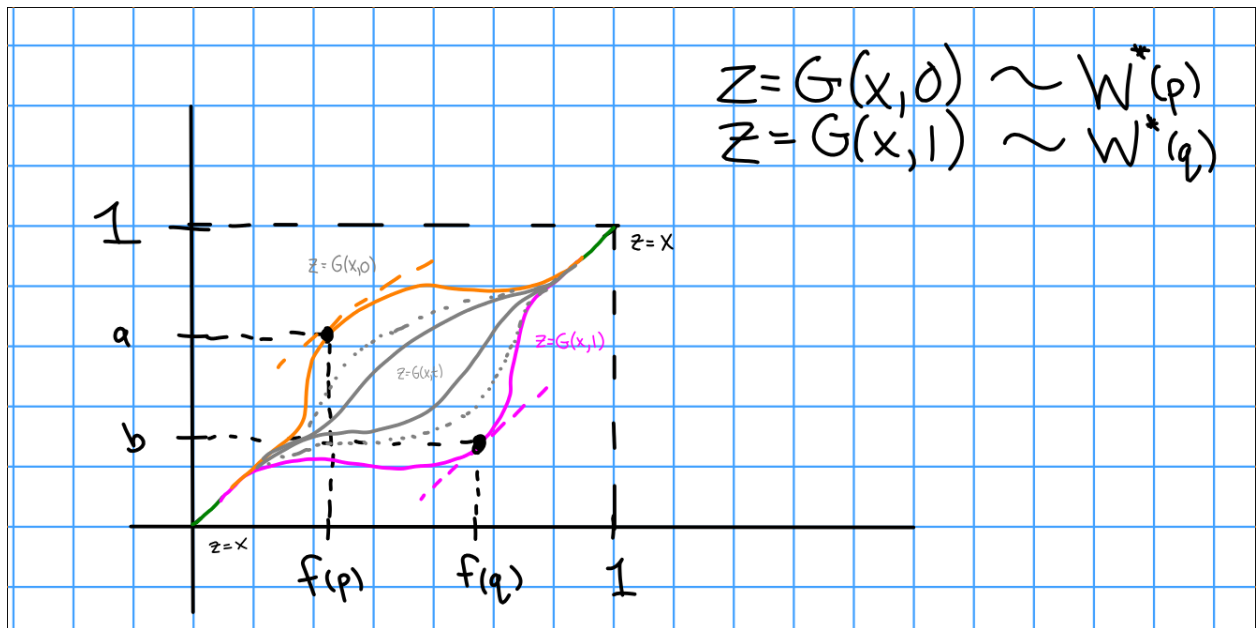
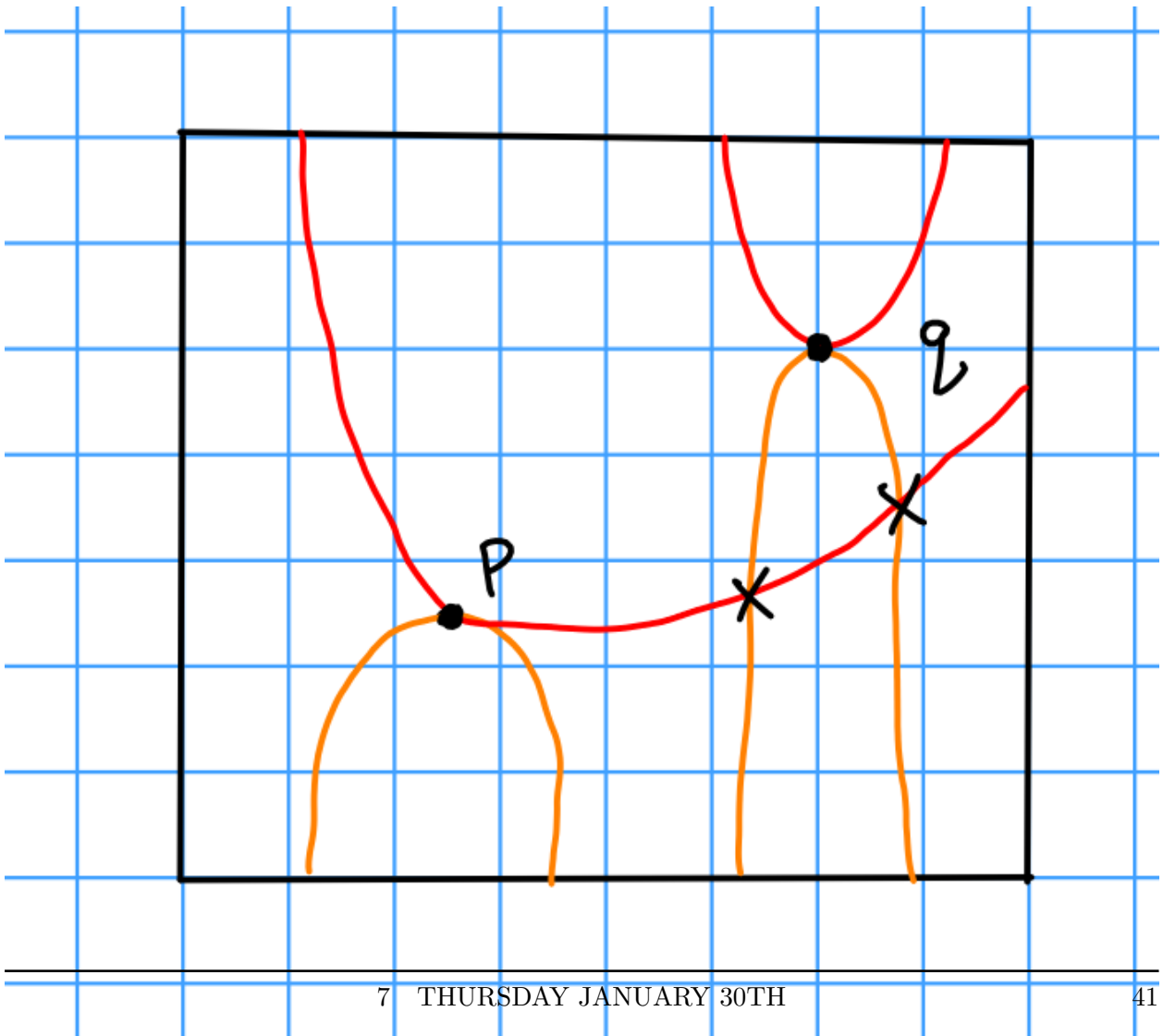
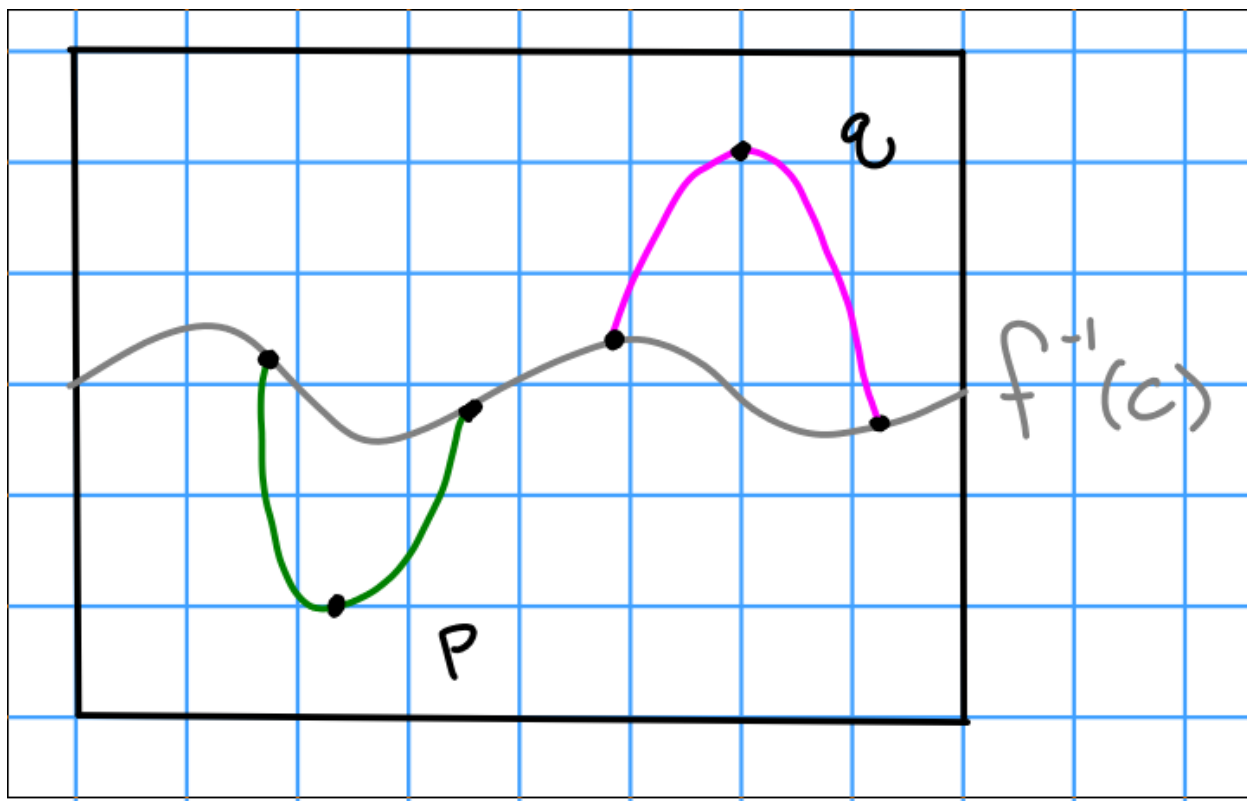


Figure 16: Image



**Theorem:** If  $f(p) < f(q)$  and  $\text{Ind}(p) \geq \text{Ind}(q)$ , then it is possible to change  $\xi$  in a neighborhood of  $f^{-1}(c)$  for some  $f(p) < c < f(q)$  such that  $W_p^u \cap W_q^s = \emptyset$ .

Main Idea:



Note that  $W_p^u \cap W_q^s = \emptyset$  iff  $S_R^c(p) \cap S_L^c(q) = \emptyset$ , where  $S_R^c(p) = W_p^u \cap f^{-1}(c)$  and  $S_L^c(q) = W_q^s \cap f^{-1}(c)$ .

We have the implication

$$\begin{aligned} & \begin{cases} \dim W = n \\ \text{Ind}(p) = \lambda \\ \text{Ind}(q) = \lambda' \end{cases} \\ \implies & \begin{cases} \dim f^{-1}(c) = n - 1 \\ \dim S_R^c(p) = n - \lambda - 1 \\ \dim S_L^c(q) = \lambda' - 1 \end{cases} . \end{aligned}$$

and thus  $\dim S_R^c(p) + \dim S_L^c(q) = n - \lambda - \lambda' + 2 < n - 1 = \dim f^{-1}(c)$ .

**Lemma 1:** If  $M^m, N^n \subset V^v$  are smooth submanifolds and  $m + n < v$  then there exists a diffeomorphism  $h : V \rightarrow V$  which is isotopic to the identity such that  $h(M) \cap N = \emptyset$ .

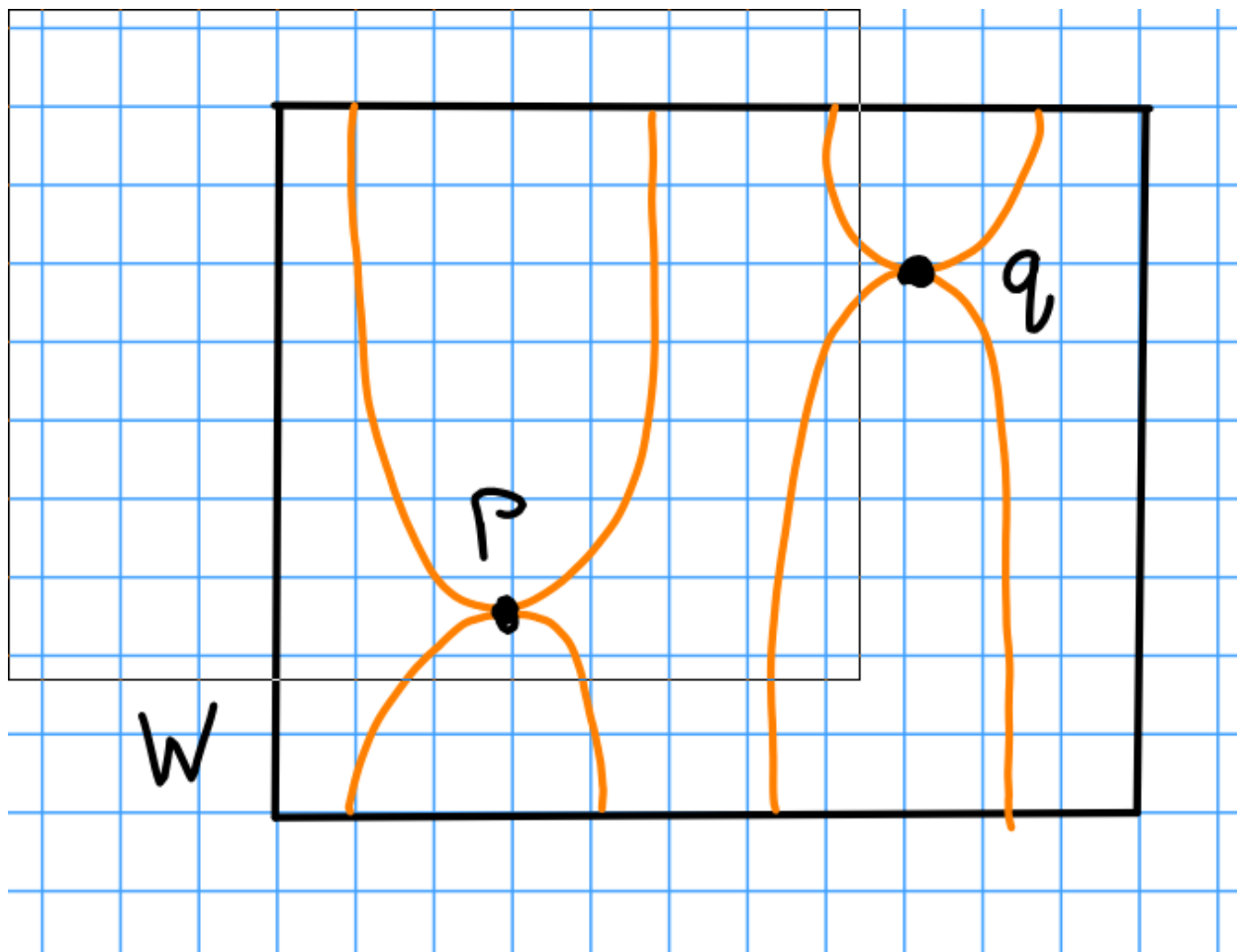
Idea: We'll use this new diffeomorphism to modify the vector field  $\xi$  to make things disjoint.

## 8 Tuesday February 4th

### 8.1 Modifying Vector Fields

Recall:  $f : W \rightarrow I$  Morse,  $\text{crit}(f) = \{p, q\}$  where  $f(p) < f(q)$ , and  $\xi$  a gradient-like vector field for  $f$ .

Theorem: If  $W^*(p) \cap W^*(q) = \emptyset$  then for any  $a, b \in (0, 1)$  we can change  $f$  “nicely” to a new Morse function  $g$  such that  $g(p) = a$  and  $g(q) = b$ .



Note that these are disjoint iff  $W^u(p) \cap W^s(q) = \emptyset$  iff  $S_R^c(p) \cap S_L^c(q) = \emptyset$ . If  $\epsilon(p) \geq \epsilon(q)$  then  $\dim S_R^c(p) = \dim S_L^c(q) < n - 1 = \dim f^{-1}(c)$ .

**Lemma 1:** For  $M^m, N^n \subset V^v$  submanifolds with  $m + n < v$ , there exists a diffeomorphism  $h : V \rightarrow V$  smoothly isotopic to  $\text{id}_V$  such that  $h(M) \cap N = \emptyset$ .

I.e. low enough dimension submanifolds can smoothly be made disjoint.

**Lemma 2:** Let  $f : W \rightarrow I$  be Morse with gradient-like vector field  $\xi$  and regular value  $x \in (0, 1)$ . Let  $h : f^{-1}(c) \rightarrow f^{-1}(c)$  be smoothly isotopic to the identity, and define  $M := f^{-1}(c)$ . Then we

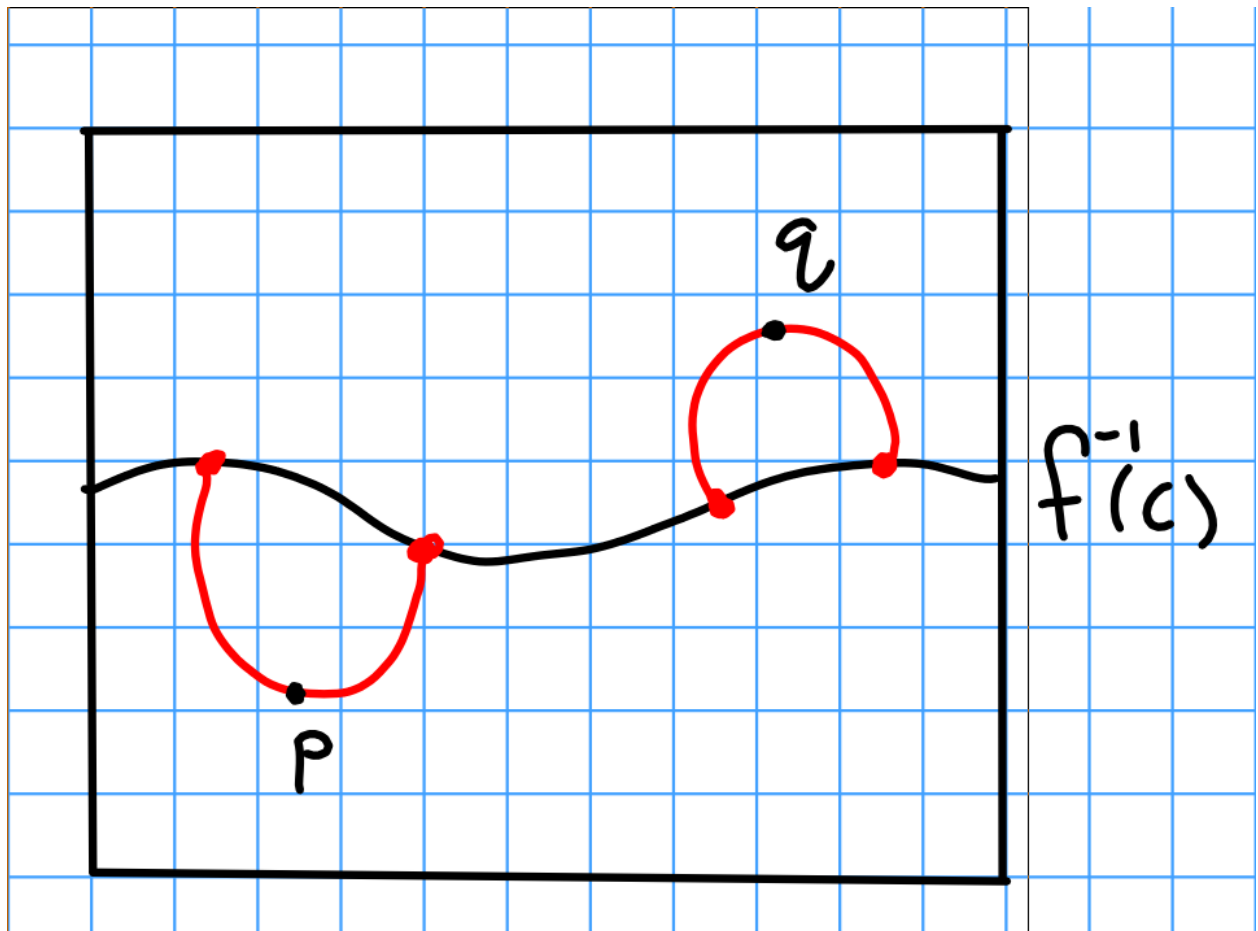


Figure 17: Image

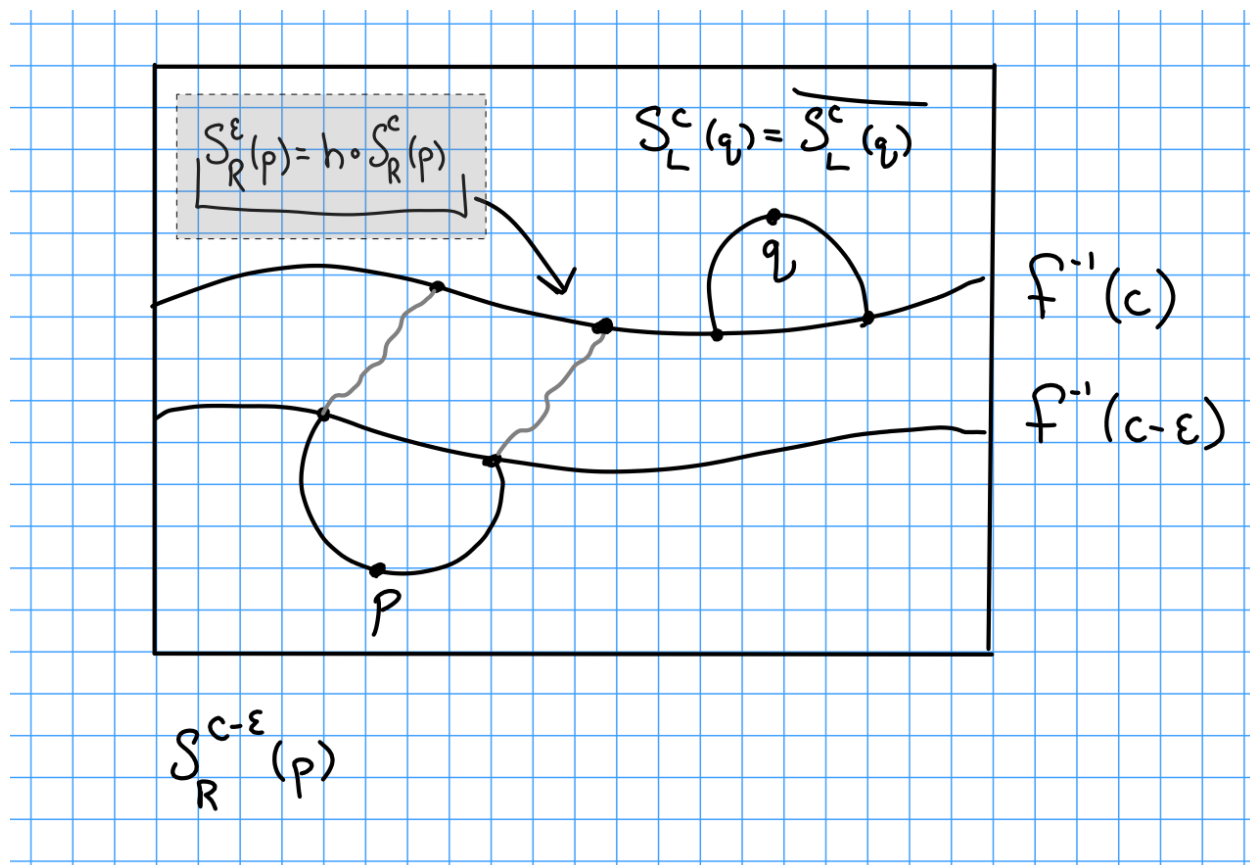
can change  $\xi$  over  $f^{-1}[c - \varepsilon, c]$  to a new gradient-like vector field  $\bar{\xi}$  such that if we let

$$\Phi : f^{-1}(c - \varepsilon) \longrightarrow M$$

be the flow induced by  $\xi$  and

$$\bar{\Phi} : f^{-1}(c - \varepsilon) \longrightarrow M$$

be induced by  $\bar{\xi}$ .



Note that the left/right spheres are defined in terms of gradient-like vector fields, so “bar” here refers to a new gradient-like vector field.

Then picking  $h$  from lemma 1, we can arrange so that  $\bar{S}_L^c(q) \cap \bar{S}_R^c(p) = \emptyset$ .

Then  $\bar{\xi} = h \circ \Phi$ .

*Proof of lemma 2:*

We have  $[c - \varepsilon, c] \times M \xrightarrow{\phi} f^{-1}[c - \varepsilon, c] \xrightarrow{f} [c - \varepsilon, c]$ , which we can factor by projection onto the first component. This satisfies the following properties:

1.  $\phi_*(\frac{\partial}{\partial t}) = \hat{\xi} := \frac{1}{\xi(f)}\xi$
2.  $\phi|_{\{x\} \times M} = \text{id}$

Note: the product cobordism  $[c - \varepsilon, c] \times M$  is easier to work with here, we can then push it to  $f^{-1}[c - \varepsilon, c]$  via  $\phi$ .

We now define  $h_t$  by the properties

- For  $t$  near  $c - \varepsilon$ ,  $h_t = \text{id}$ , and
- For  $t$  near  $x$ ,  $h_t = h$ .

We use this to construct a diffeomorphism

$$H : [c - \varepsilon, c] \times M \longrightarrow [c - \varepsilon, c] \times M$$

$$(t, x) \mapsto (t, h_t(x)).$$

Both the domain and codomain map via  $\phi$  to  $f^{-1}[c - \varepsilon, c]$ , so we can consider

$$H_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial H}{\partial t}(t, x) = \left(1, \frac{\partial h_t}{\partial t}(x)\right) \quad \text{for } t \text{ near } c - \varepsilon \text{ and } c.$$

We then have  $\xi' = (\phi \circ H \circ \phi^{-1})_* \hat{\xi} = (\phi \circ H)_*\left(\frac{\partial}{\partial t}\right)$ . Thus for  $t$  near  $c - \varepsilon$  and  $c$ , we have  $\xi' = \phi_*\left(\frac{\partial}{\partial t}\right) = \hat{\xi}$ . So define

$$\bar{\xi} = \begin{cases} \xi(f) \cdot \xi' & \text{on } f^{-1}[c - \varepsilon, c] \\ \xi & \text{everywhere else} \end{cases}.$$

On  $[c - \varepsilon, c] \times M$ , what are the integral curves of  $H_*\left(\frac{\partial}{\partial t}\right)$ ? Picking a  $t \in [c - \varepsilon, c]$ , we have  $H(t, x) = (t, h_t(x))$  by definition, and thus the integral curves of  $\hat{\xi}$  are given by  $\phi(t, h_t(x))$  for all  $x \in M$ . Then  $\phi(c - \varepsilon, x) = \phi(c - \varepsilon, h_{c - \varepsilon}(x))$  for  $t = c - \varepsilon$ , which is just  $\Phi^{-1}(x)$ . Then for  $t = c$  we get  $\phi(c, h(x)) = h(x)$ . Thus  $\bar{\Phi} \circ \Phi^{-1}(x) = h(x)$ , yielding  $\bar{\Phi} = h \circ \Phi$ .

Corollary: Given any Morse function  $f$  on an  $n$ -dimensional cobordism  $(W^n; M_0, M_1)$  we can get a new Morse function  $g$  such that

- $\text{crit}(g) = \text{crit}(f)$ ,
- $g(p) = \text{Ind}(p)$  (since we can make the critical points take on any values)
- $g^{-1}(-1/2) = M_0$  and  $g^{-1}(n + 1/2) = M_1$

Such a Morse function is called *self-indexing*.

## 8.2 Cancellation

Note that we may have “extraneous” critical points:

Here note that  $W$  is diffeomorphic to a product cobordism.

Let  $f : W \longrightarrow [0, 1]$  be Morse,  $\text{crit}(f) = \{p, q\}$ ,  $\text{Ind}(p) = \lambda$  and  $\text{Ind}(q) = \lambda + 1$  with  $f(p) < f(q)$ .

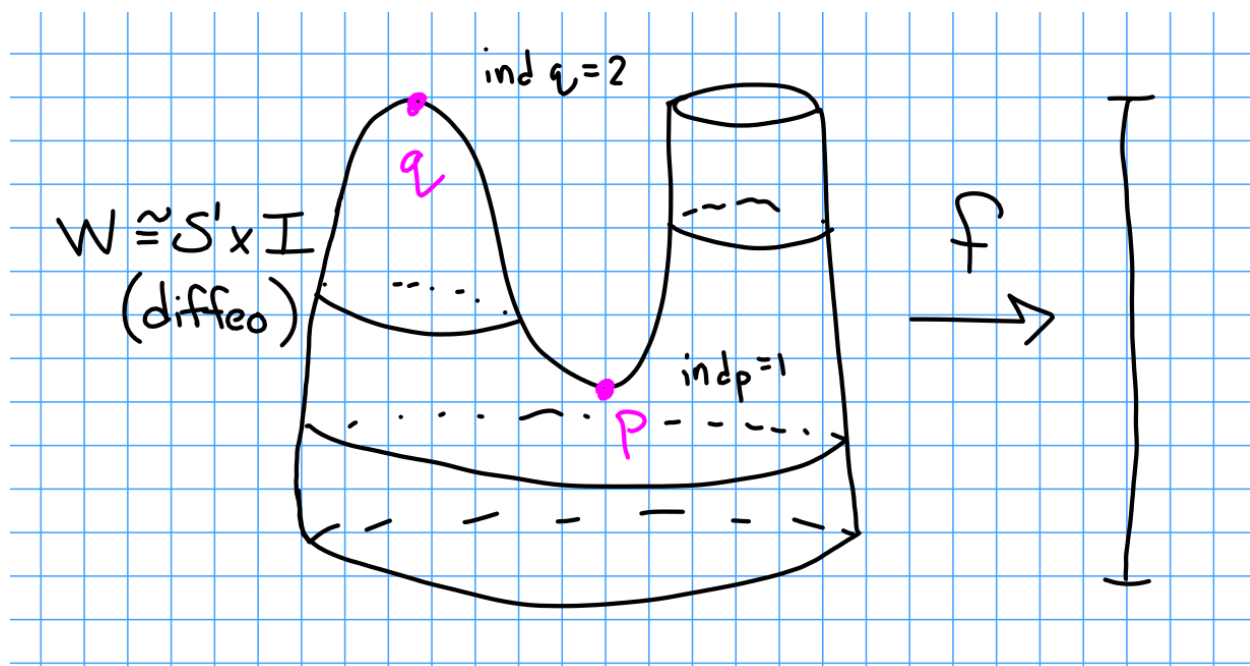


Figure 18: Image

Pick  $c \in [f(p), f(q)]$ , then consider  $S_R^c(p) \cap S_L^c(q)$ . We have  $\dim S_R^c(p) = n - \lambda - 1$  and  $\dim S_L^c(q) = \lambda + 1 - 1$ , and so the dimension of their intersection is  $n - 1$ , i.e.  $\dim f^{-1}(c)$ .

**Definition:** Submanifolds  $M^m, N^n \subset V^v$  are called *transverse* if for any  $p \in M \cap N$ ,  $T_p V \subset \text{span}\{T_p M, T_p N\}$  and we write  $M \pitchfork N$ .

**Example:**

- If  $m + n < v$ , then  $M \pitchfork N$  iff  $M \cap N = \emptyset$ .
- If  $m + n = v$ , then  $M \pitchfork N$  iff  $\dim M \cap N = 0$ .
- In general, if  $M \pitchfork N$  then  $M \cap N$  is a smooth submanifold of dimension  $m + n - v$ .

**Theorem:** For submanifolds  $M_m, N^n \subset V$ , then there exist  $h : V \xrightarrow{\text{diff}} V$  smoothly isotopic to  $\text{id}_V$  such that  $h(M) \pitchfork N$ .

**Corollary:** We can change  $\xi$  in  $f^{-1}[c - \varepsilon, c]$  such that  $S_R^c(p) \pitchfork S_L^c(q)$ , so their intersection consists of finitely many points.

**Lemma (First Cancellation):**

If  $S_R^c(p)$  intersects  $S_L^c(q)$  in exactly one point, then  $W$  is a product cobordism.

**Idea of proof:**

1. Change  $\xi$  in a neighborhood of the integral curve from  $p$  to  $q$  such that the new vector field is nonvanishing.
2. Change  $f$  to  $g$  with no critical point such that the new vector field is gradient-like for  $\xi$ .

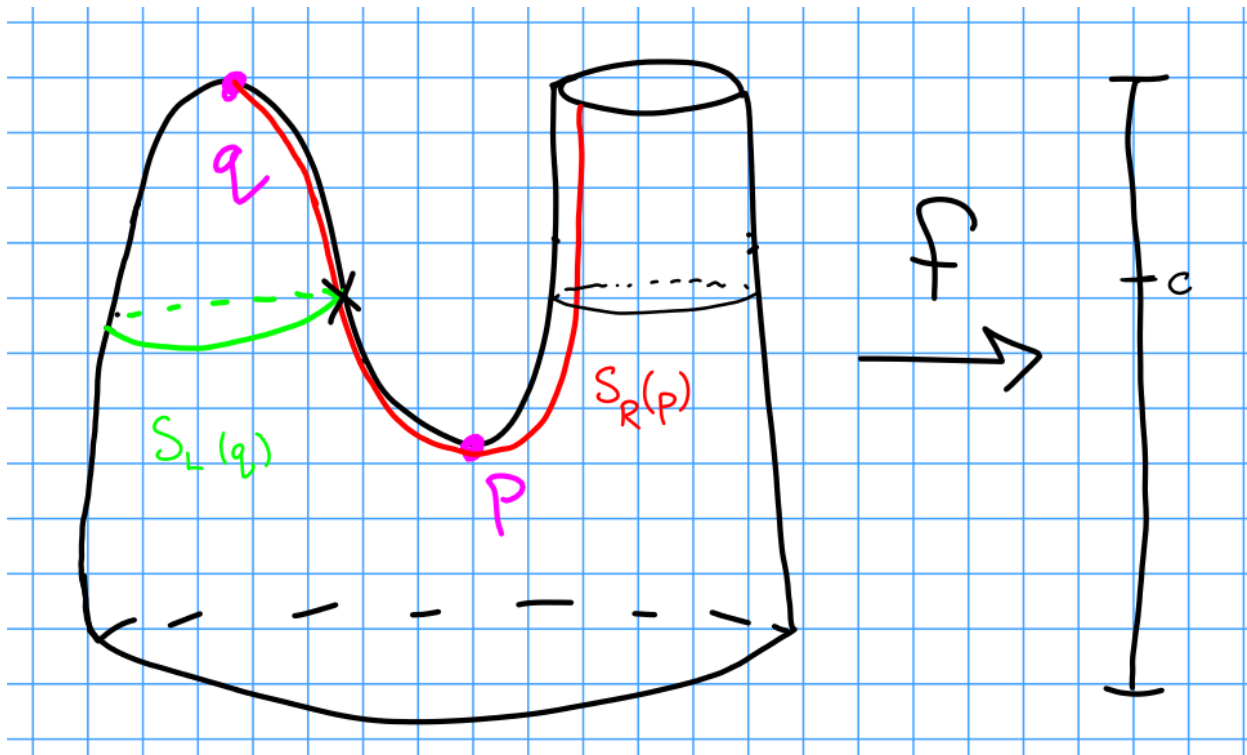


Figure 19: Image

## 9 Thursday February 6th

Cancellation: Let  $f : W \rightarrow I$  be Morse,  $\text{crit}(f) = \{p, q\}$  with  $f(p) < f(q)$  and  $\text{Ind}(p) = \lambda$ ,  $\text{Ind}(q) = \lambda + 1$ . Let  $\xi$  be its gradient-like vector field, then  $S_R^c(p) \cap S_L^c(q) = \{\text{pt}\}$ , so there exists a unique integral curve  $T$  from  $p$  to  $q$ .

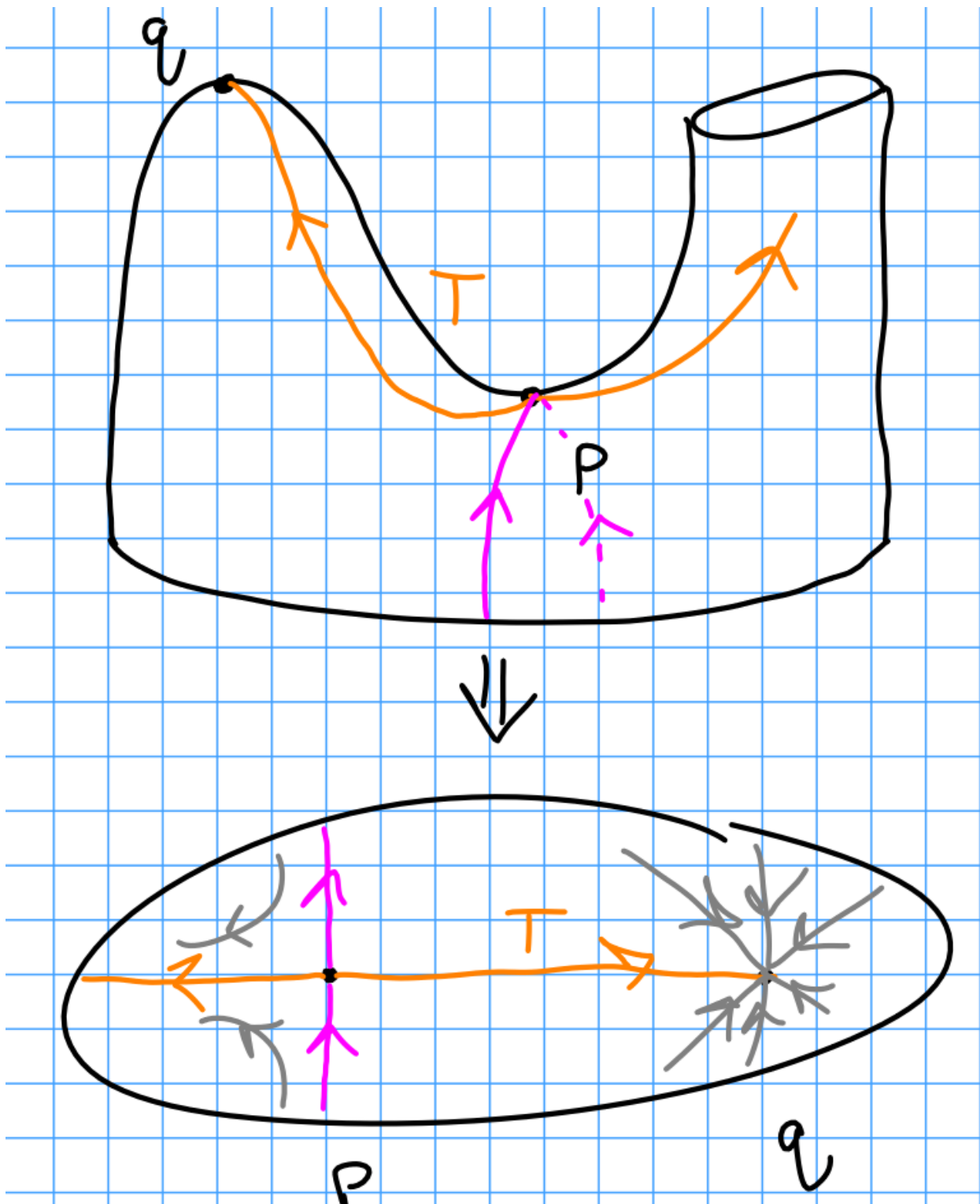
In this situation  $W$  is diffeomorphic to the product cobordism.

We will show

**Theorem 1:** We can change  $\xi$  in a compact neighborhood of  $T$  to get a nonvanishing vector field  $\xi'$  for which the integral curves originate at  $M_0$  and end at  $M_1$ .

Example:





Moreover, it takes a particularly nice standard form:

There exists a neighborhood  $U_T$  and a coordinate chart  $h : U_T \rightarrow \mathbb{R}^n$  such that

1.  $h(p) = (0, \dots, 0)$  and  $h(q) = (1, 0, \dots, 0)$ .

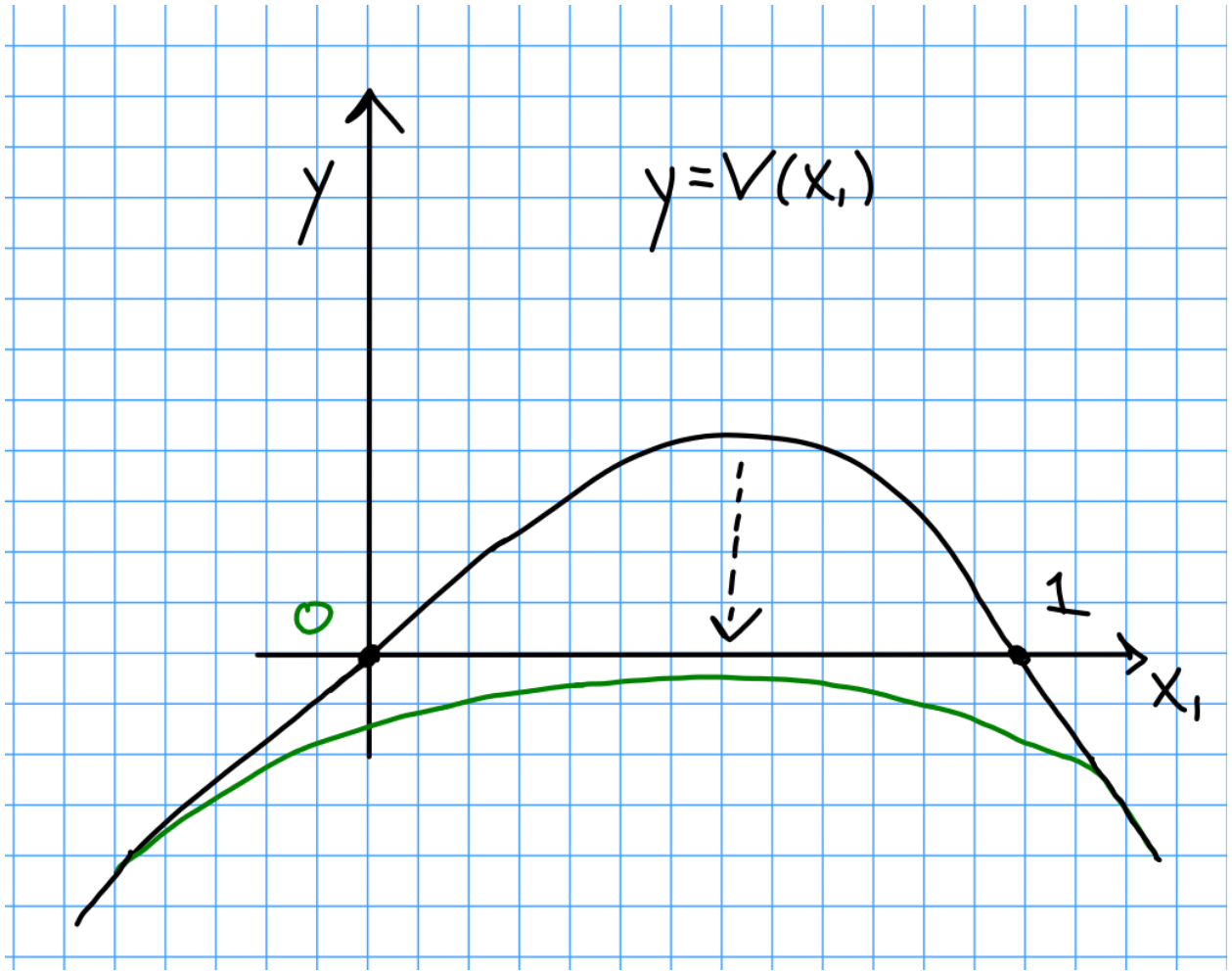


Figure 20: Image

2.  $h_*\xi = (V(x_1), -x_2, -x_3, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots, x_n)$ .
3.  $V(x)$  is smooth and positive over  $(0, 1)$  with  $V(0) = V(1) = 0$ , and  $V(x) < 0$  everywhere else.
4. (Minor)  $|V'(0)| = |V'(1)| = 1$ .

Thus near  $p$  we have  $(x_1, -x_2, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots, x_n)$  and near  $q$ , we have  $(-x_1, -x_2, \dots, -x_{\lambda+1}, x_{\lambda+2}, x_n)$ .

*Proof:*

*Step 1:*

Consider  $(U(x_1, \rho), -x_2, -x_3, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots, x_n)$  where  $\rho(\mathbf{x}) = (x_2^2 + x_3^2 + \dots + x_n^2)^{1/2}$ , which measures the distance between the two curves above. Some facts:

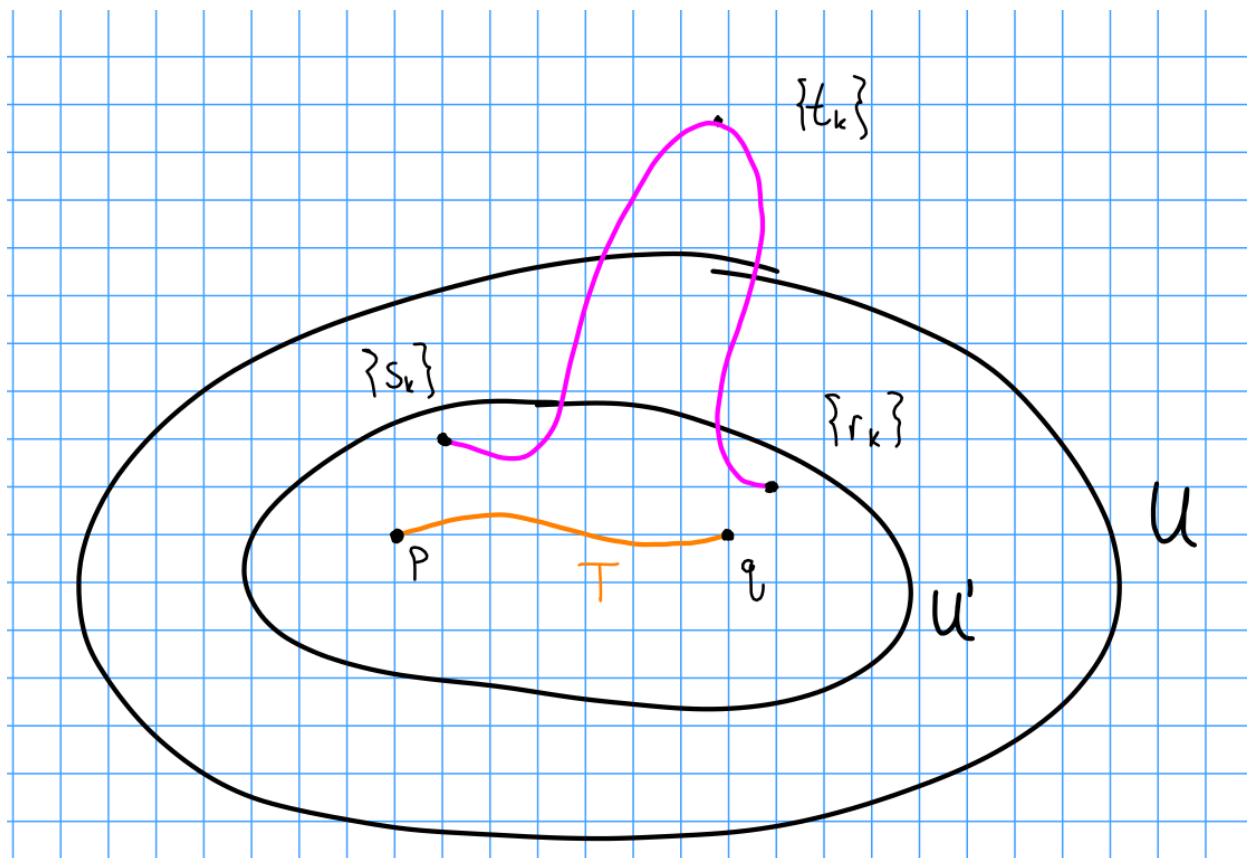
1.  $U(x_1, \phi)$  is equal to  $V(x_1)$  outside of a compact neighborhood of  $h(T)$  in  $h(U_T)$ .
2.  $U(x_1, 0) < 0$  for all  $x_1$ .

Then  $\xi' = h_*(u, -x_2, \dots, x_n)$  in  $U_T$  and  $\xi' = \xi$  everywhere else. Thus  $\xi'$  is nowhere vanishing.

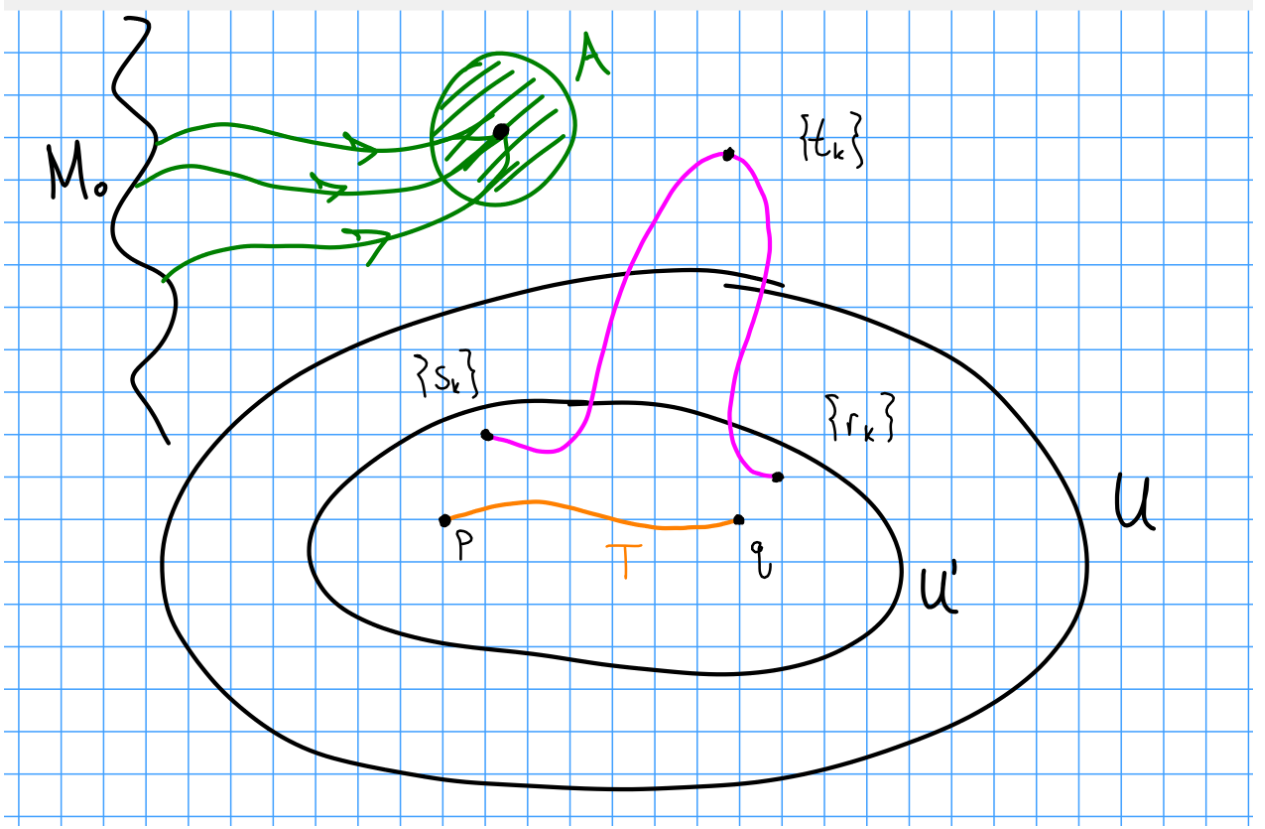
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Step 2:

We want to pick  $U'$  such that  $T \subset U' \subset \bar{U} \subset U_T$  where  $\bar{U}$  is a compact set such that any trajectory of  $\xi$  that exits  $U$  never re-enters  $U'$ .



Suppose such a  $U'$  does not exist. Then there exist sequences of points  $\{s_k\}, \{r_k\} \subset U$  and  $\{t_k\} \subset W \setminus U$  all on the same integral curves  $\gamma_k$  such that  $\{s_k\} \rightarrow T$  and  $\{r_k\} \rightarrow T$ . Since  $W \setminus U$  is compact,  $\{t_k\}$  has a limit point  $A$ . Then consider  $\psi_A(t)$ , which are integral curves that originate from  $M_0$  and end on  $M_1$ .



Then there exists a neighborhood  $A$  such that for each  $a \in A$ , the integral curves (half trajectories) containing  $a$  originate on  $M_0$ . Moreover, for  $k$  large enough, all  $t_k$  are in  $A$ . The union of all of these half trajectories has a positive distance from  $T$ , so there is a small enough  $U$  disjoint from these trajectories, so  $\{s_k\} \not\rightarrow T$ , a contradiction. ■

We now consider the flow lines of  $\xi'$  in  $h(U_T)$ . We have

$$\frac{\partial x_1}{\partial t} = u(x_1, \phi), \frac{\partial x_2}{\partial t} = -x_2, \dots, \frac{\partial x_n}{\partial t} = x_n.$$

Thus  $x_2 = x_2^0 e^{-t}, \dots, x_n = x_n^0 e^t$ .

So if  $x_i \neq 0$  for some  $\lambda + 2 \leq i \leq n$ , the  $|x_i|$  is increasing exponential and thus it will escape  $h(U)$ . The corresponding trajectory will then escape  $U$ , and so it will follow the integral curves of the original  $\xi$  and ends at  $M_1$ . If  $x_{\lambda+2}^0 = \dots = x_n^0 = 0$ , then

$$\phi(\mathbf{x}) = (x_2^2 + \dots + x_{\lambda+1}^2)^{1/2} = e^{-t} (\sum (x_i^0)^2)^{1/2}.$$

Thus  $\phi(x)$  will decrease exponentially.

If it leaves  $U$ , we are in the previous case. Otherwise, if it doesn't leave  $U$ , then there exists an  $\varepsilon > 0$  such that  $u(x_1, \phi) < 0$  for all

$$N_\varepsilon = \left\{ (x, p) \in h(U) \mid \phi < \varepsilon \right\}.$$

---

Thus there exists a  $-\alpha < 0$  such that  $u < -\alpha$  on  $N_\varepsilon$ .

For  $t$  large enough,

$$\phi(\mathbf{x}(t)) \in N_\varepsilon \implies \frac{\partial x_1}{\partial t} = u(x_1, \phi) < -\alpha.$$

Thus  $x_1(t) < -\alpha t + \text{const.}$  for large enough  $t$ , and as  $t$  increases  $\mathbf{x}(t)$  will go out of  $U$ . By the previous argument, it must end at  $M_1$ .

Thus every integral curve of  $\xi$  starts at  $M_0$  and ends at  $M_1$ . ■

**Lemma:**  $\xi'$  gives a diffeomorphism from

$$W' = (M_0 \times I; M_0 \times 0, M_1 \times 1) \longrightarrow W = (W; M_0, M_1).$$

*Proof:* Take  $\pi : W \longrightarrow M_0$  and follow the integral curves backward. Then for all  $x \in M_0$ , there is a  $\tau(x) \in \mathbb{R}^{\geq 0}$  such that  $\psi_X(\tau(x)) \in M_1$ .

So we get a

$$\widehat{\xi} = \tau(\pi(q))^{-1} \xi'_q$$

and we can define  $\phi(x, t) = \widehat{\phi}_X(t)$ . ■

**Theorem 2:**  $\xi'$  is a gradient-like vector field for some Morse function  $g : W \longrightarrow I$  such that  $g$  has no critical points (since  $\xi'$  has no zeros) and  $g = f$  near  $M_0$  and  $M_1$ .

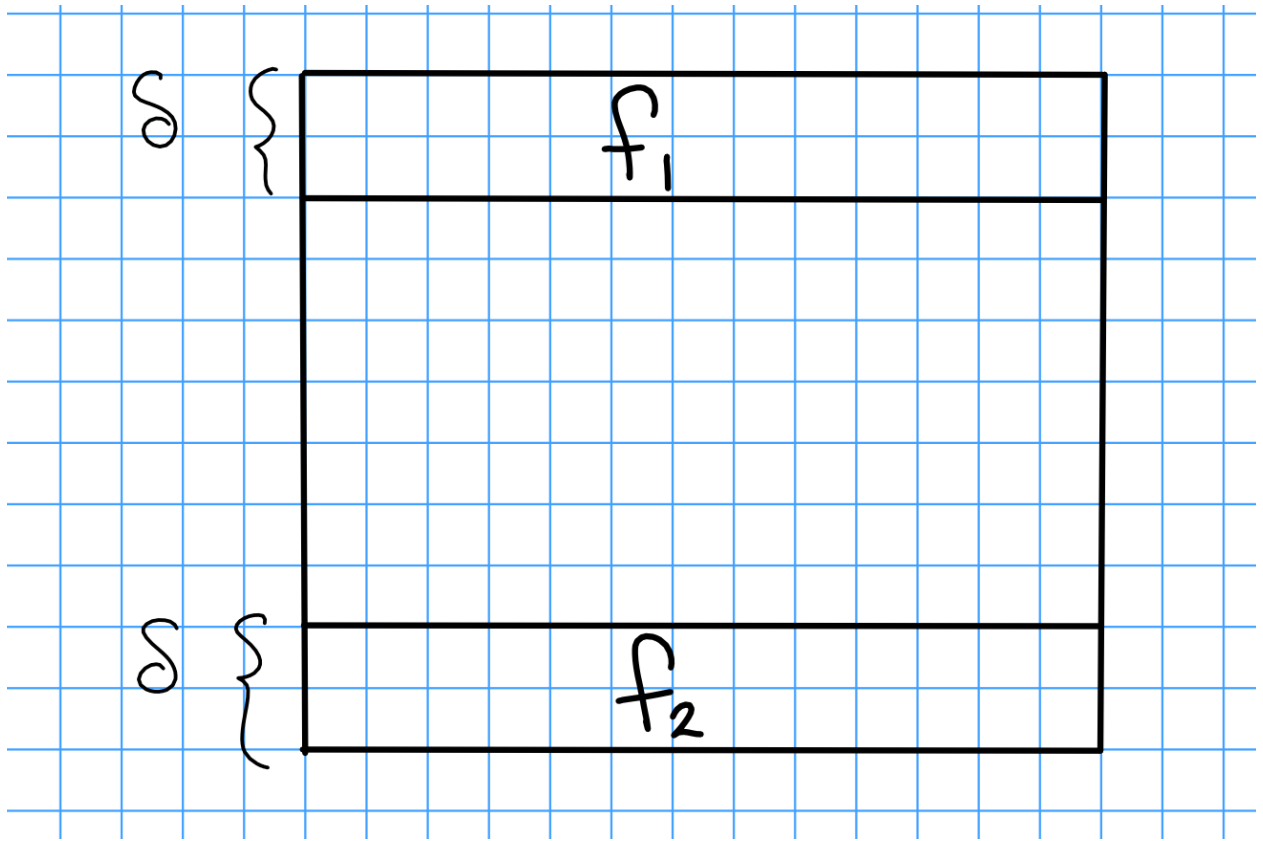
*Proof:* We want to build a  $k : M_0 \times I \longrightarrow I$  such that the following diagram commutes:

$$\begin{array}{ccc} M_0 \times I & \xrightarrow{\quad k \quad} & I \\ \downarrow \phi & \nearrow g & \\ W & & \end{array}$$

This needs to satisfy

1.  $k$  is equal to  $f_1 := f \circ \phi$  near  $M_0 \times 0$  and  $M_0 \times 1$ .
2.  $\frac{\partial k}{\partial t} < 0$ .

Since  $\frac{\partial f_1}{\partial t} > 0$  near  $M_0 \times 0$  and  $M_0 \times 1$ , take  $\delta > 0$  such that  $\frac{\partial f_1}{\partial t} > 0$  on  $M_0 \times [0, \delta)$  and  $M_1 \times (1 - \delta, 1]$ .



So pick  $\lambda : I \rightarrow I$  such that  $\lambda \equiv 1$  near  $t = 0, 1$  and  $\lambda \equiv 0$  on  $[\delta, 1 - \delta]$ .

Then pick any positive  $\bar{K} : M_0 \rightarrow \mathbb{R}$ , and then take

$$K(x, u) := \int_0^u \lambda(t) \frac{\partial f_1}{\partial t} + (1 - \lambda(t)) \bar{K}(x) dt.$$

Then

$$\frac{\partial K}{\partial u} = \lambda(u) \frac{\partial f_1}{\partial u} + (1 - \lambda(u)) \bar{K}(x) > 0$$

since the first term is positive near  $M_0 \times 1$  or  $0$ , and  $\bar{K}$  is positive everywhere.

To see that it satisfies the first property, note that  $\int_0^s \frac{\partial f_1}{\partial t} dt = f_1$  for  $s$  near  $0$ .

To see that property 2, note

$$\begin{aligned} \int_0^1 \lambda(t) \frac{\partial f_1}{\partial t} dt + \bar{K} \int_0^1 (1 - \lambda(t)) dt &= g(x, 1) = f_1(x) \\ \implies \bar{K}(x) &= \frac{f_1(x) - \int_0^1 \lambda(t) \frac{\partial f_1}{\partial t} dt}{\int_0^1 (1 - \lambda(t)) dt}. \end{aligned}$$

■

## 10 Tuesday February 11th

### 10.1 Cancellation

The setup:  $f : W \rightarrow [0, 1]$  a morse function with  $\text{crit}(f) = \{p, q\}$  with  $\text{Ind}(p) = \lambda$  and  $\text{Ind}(q) = \lambda + 1$ , with a gradient-like vector field  $\xi$  such that there exists a *single* flow line  $T$  from  $p$  to  $q$ .

**Lemma:** There exists a gradient-like vector field  $\xi'$  for  $f$  such that

1.  $T$  is still the single flow line from  $p$  to  $q$ .
2.  $\xi'$  is *standard* in a neighborhood  $U_T$  of  $T$ , i.e. there exists  $h : U_T \rightarrow \mathbb{R}^n$  such that

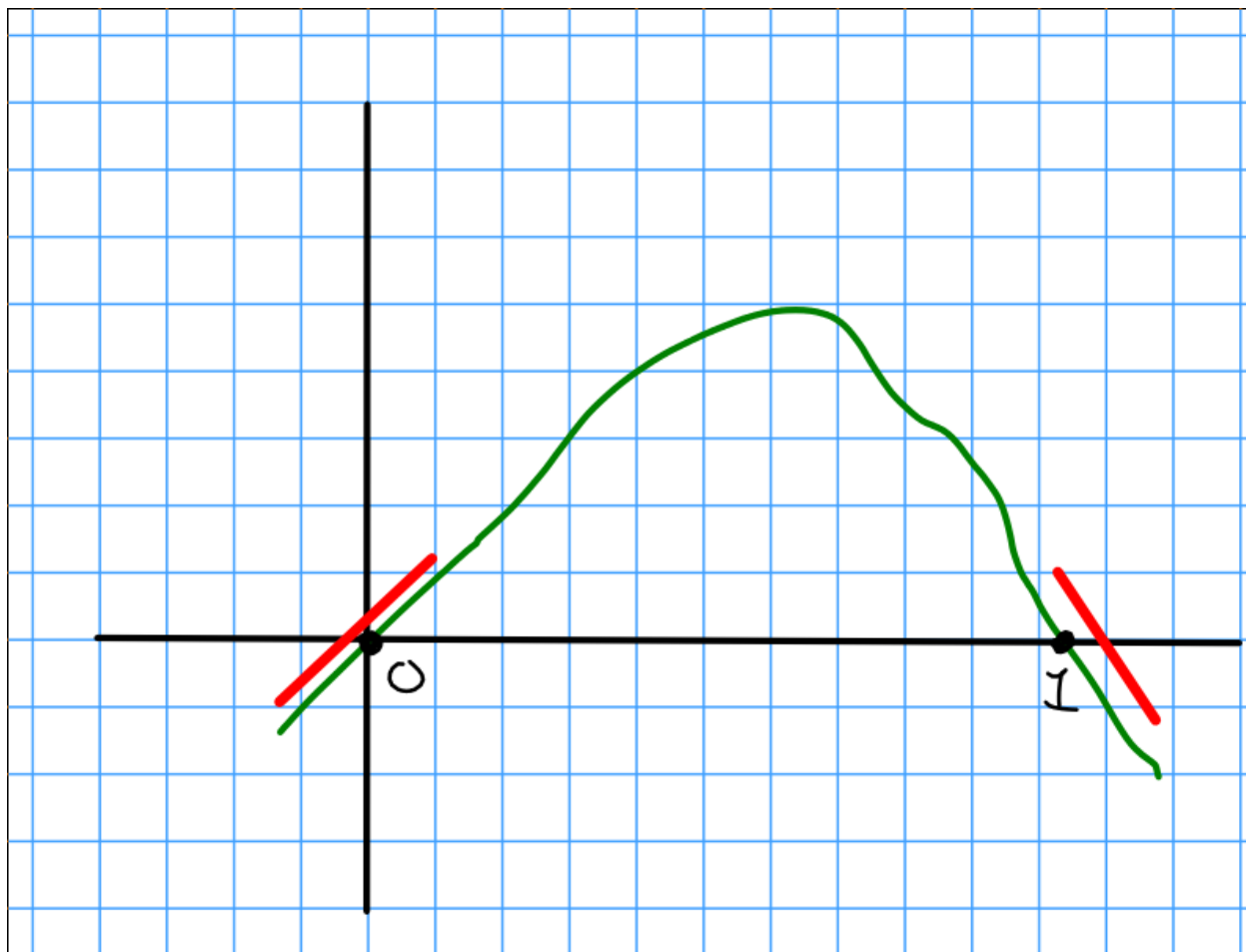
$$h(p) = (0, 0, \dots, 0)$$

$$h(q) = (1, 0, \dots, 0).$$

$h(T)$  is contained in the  $x$ -axis, and

$$h_*\xi' = (V(x_1), -x_2, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots, x_n),$$

where  $V$  satisfies the property that near 0 and 1,  $|V'| = 1$ :



*Proof:*

Let  $\eta = V(x_1)$  from above. Define the following vector field:

$$F(\mathbf{x}) = f(p) + 2 \int_0^{x_1} v(t) dt - x_2^2 - x_3^2 - \cdots - x_{\lambda+1}^2 - \cdots + x_n^2.$$

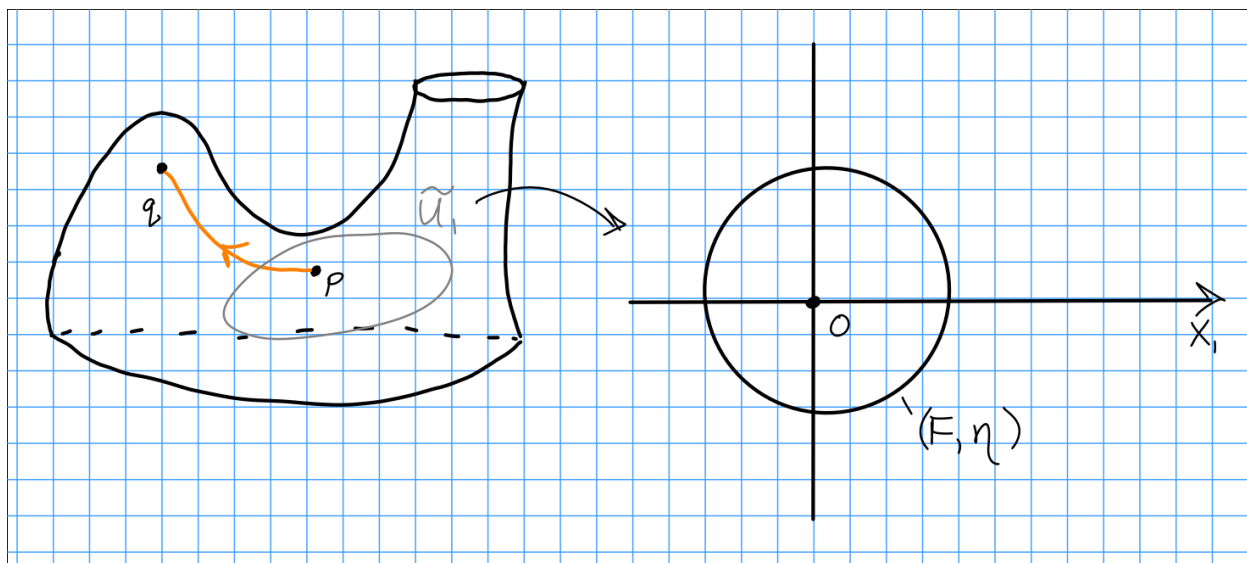
Then  $\eta$  is gradient-like for  $F$ , and we can pick  $v(t)$  such that

$$\begin{aligned} F(1, 0, \dots, 0) &= f(p) + 2 \int_0^1 v(t) dt &= f(q) \\ \implies \int_0^1 v(t) dt &= \frac{1}{2}(f(q) - f(p)). \end{aligned}$$

We know that  $v(t) = t$  near  $(0, 0, \dots, 0)$ , and since  $\int_0^1 t dt = \frac{x_1^2}{2}$ , we have

$$\begin{aligned} F(\mathbf{x}) &= f(p) + 2 \int_0^{x_1} t dt + \cdots + x_n^2 = f(p) + x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + \cdots + x_n^2 \\ \implies \eta(\mathbf{x}) &= (x_1, -x_2, \dots, -x_{\lambda+1}, x_{\lambda+2}, \dots, x_n). \end{aligned}$$

Then there exists a neighborhood  $\tilde{U}_1$  of  $p$  and  $h_1 : \tilde{U}_1 \rightarrow \mathbb{R}^n$  such that  $\tilde{h}_1(p) = (0, 0, \dots, 0)$  with  $F \circ \tilde{h}_1 = f$  and  $\tilde{h}_{1*} = \eta$ .



Similarly, near  $(1, 0, \dots, 0)$  we have  $v(t) = 1 - t$  and since  $\int_0^1 v(t) dt = f(q) - f(p)$ , we have

$$\begin{aligned} F(\mathbf{x}) &= f(p) + 2 \int_0^1 v(t) dt + 2 \int_1^{x_1} (1 - t) dt - \cdots + x_n^2 \\ &= f(q) - (x_1 - 1)^2 - x_2^2 - \cdots + x_n^2, \end{aligned}$$



and there exists a neighborhood  $\tilde{U}_2$  of  $q$  and  $\tilde{h}_2 : \tilde{U}_2 \rightarrow \mathbb{R}^n$  such that  $\tilde{h}_2(q) = (1, 0, \dots, 0)$ ,  $F \circ \tilde{h}_2 = f$ , and  $\tilde{h}_{2*}\xi = \eta$ .

So pick  $(\tilde{U}_1, \tilde{h}_1)$  and  $(\tilde{U}_2, \tilde{h}_2)$  such that  $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$  and  $\tilde{h}_1(\tilde{U}_1) \cap \tilde{h}_2(\tilde{U}_2) = \emptyset$ .

Pick

$$a_1 < f(p) < b_1 < b_2 < f(q) < a_2$$

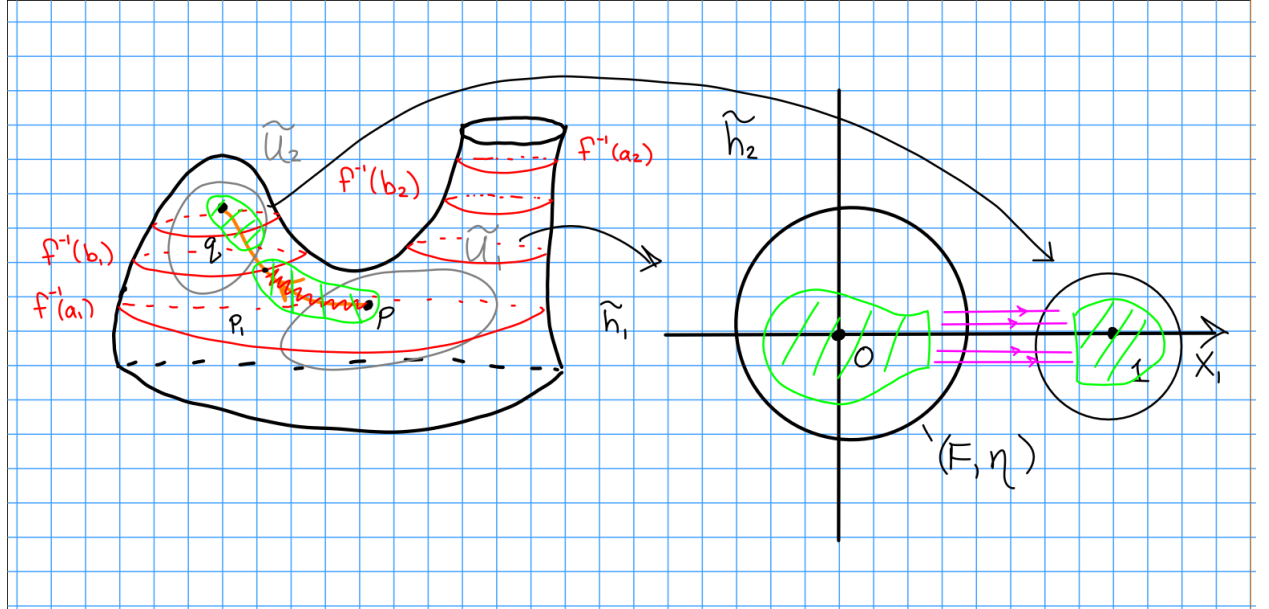
such that

$$f^{-1}[a_1, b_1] \cap T \subset \tilde{U}_1 \quad \text{and} \quad f^{-1}[b_2, a_2] \cap T \subset \tilde{U}_2$$

and set  $p_i = f^{-1}(b_i) \cap T$ .

Let  $U_1$  and  $U_2$  be closed neighborhood of the arc  $p \rightarrow p_1$  in  $\tilde{U}_1 \cap f^{-1}[a_1, b_1]$  and  $q \rightarrow p_2$  in  $\tilde{U}_2 \cap f^{-1}[b_2, a_2]$ .

Let  $h_i = \tilde{h}_i|_{U_i}$ . Then  $\xi$  yields a diffeomorphism  $\psi : f^{-1}(b_1) \rightarrow f^{-1}(b_2)$ .



Fix a small neighborhood  $\lambda$  of  $h_1(p_1)$  in  $h_1(f^{-1}(b_1) \cap U_1)$ , following the flow lines of  $\eta$  yields a diffeomorphism  $\phi : V_1 \rightarrow V_2$  where  $V_2$  is a sufficiently small neighborhood of  $h_2(p_2)$  in  $h_2(f^{-1}(b_2) \cap U_2)$ , the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(b_1) & \xrightarrow{\psi} & f^{-1}(b_2) \\ \uparrow h_1^{-1} & & \uparrow h_2^{-1} \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

If for  $V_1$  small enough we have  $h_2^{-1} \circ \phi \circ h_1$  restricted to  $h_1^{-1}(V_1)$  is equal to  $\psi$ , then we can extend  $(h_1, h_2)$  to a diffeomorphism  $h$  from  $U_1 \cap U_0 \cap U_2$ , where  $U_0$  is a small neighborhood of  $p_1 p_2$  such

that it preserves the trajectories and level sets. We then obtain  $h_*\eta = K\eta$ , where  $K$  is some positive function.

We can extend  $K$  to a positive smooth function over  $W$ , which yields  $\xi' := \frac{1}{K}\xi$  and thus  $h_*\xi' = \eta$ . So  $\xi'$  is a gradient-like vector field for  $f$ .

In case the above inequality does *not* hold, we can use an isotopy to change  $\psi \rightarrow \psi'$  and change  $\xi \rightarrow \xi'$  in  $f^{-1}[a, b]$  such that the integral curves of  $\xi'$  induce  $\psi'$ . So find an isotopy such that  $\psi'$  is equal to  $h_2^{-1}\phi h_1$  near  $p_1$ , and furthermore  $\psi'(S_R^{b_1}(p))$  intersects  $S_L^{b_2}(q)$  transversely in  $p_2$ , i.e.  $p_2$  is the only intersection point.

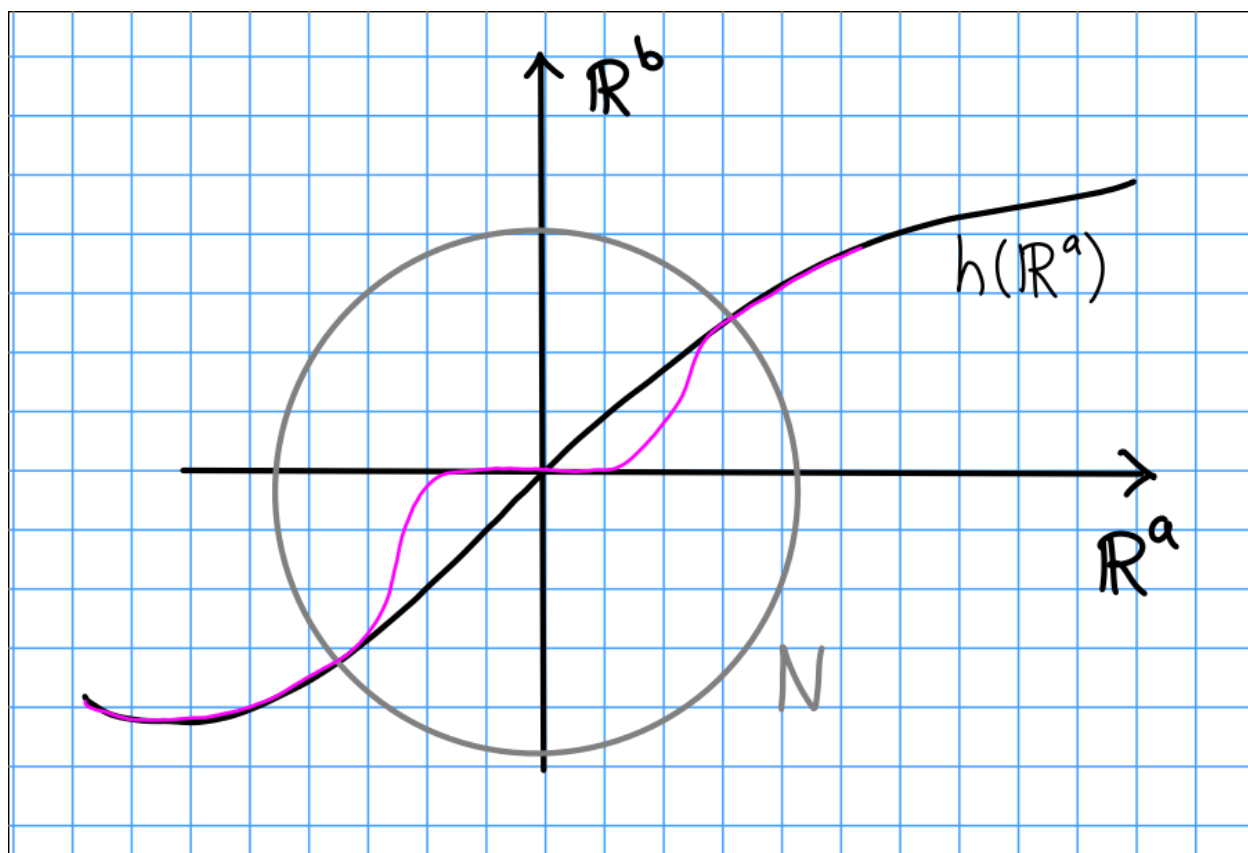
We can do this last step locally. Let  $\phi' = h_2^{-1}\phi h_1$ , then  $(\phi')^{-1}\psi : \tilde{V}_1 \hookrightarrow \tilde{V}_1$  for some small neighborhood  $\tilde{V}_1 \subset V_1$  containing  $p_1$ . Note that if  $S_R^{b_1}(p) \cap S_L^{b_1}(q)$  at  $p_1$ , then  $(\phi')^{-1}\psi S_R^{b_1}(p) \cap (\phi')^{-1}\psi S_L^{b_1}(q)$  at  $p_1$ , and so  $S_R^{b_1}(p) \cap (\phi')^{-1}\psi S_L^{b_1}(q)$ .

**Theorem:** Identify  $\mathbb{R}^n = \mathbb{R}^a \oplus \mathbb{R}^b$  where  $a + b = n$ . Suppose that  $h : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$  is an orientation-preserving embedding such that  $h(0) = 0$  with  $h(\mathbb{R}^a) \cap \mathbb{R}^b$  with *intersection number*  $+1$  at  $\{0\}$ .

Then there exists a smooth isotopy  $h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

1.  $h_0 = h$ ,
2.  $h_t(0) = 0, h_t(x) = x$  for  $x$  outside of a neighborhood  $N$  of zero,
3.  $h_1 = \text{id}$  in a smaller neighborhood of zero in  $N$
4.  $h_1(\mathbb{R}^a) \cap \mathbb{R}^b = \{0\}$ .

Rough idea: modify  $h(\mathbb{R}^a)$  in a neighborhood of 0:



---

Next time: a second cancellation theorem. Suppose  $W, M_0, M_1$  are simply connected and  $2 \leq \lambda < \lambda + 1 \leq n - 3$ . If

$$S_R^c(p) \cdot S_L^c(q) = \pm 1,$$

then  $W \cong M_0 \times [0, 1]$  are diffeomorphic.

We'll briefly review the intersection number later. Also: homological intersection number.

## 11 Tuesday February 18th

### 11.1 Cancellation Theorems

**Theorem (Rearrangement):** If a Morse function  $f$  has 2 critical points with  $f(p) < c < f(q)$  and  $\text{Ind}(p) \geq \text{Ind}(q)$ , then  $\xi$  can be perturbed in a neighborhood of  $f^{-1}(c)$  such that  $W_p^s \cap W_q^u = \emptyset$ .

**First Cancellation:** If  $S_R \cap S_L' = \{\text{pt}\}$ , then the cobordism is diffeomorphic to a product.

**Second Cancellation:** Suppose  $(W, V_0, V_1)$  is a cobordism and  $f : W \rightarrow \mathbb{R}$  has two critical points. If  $S_R \cdot S_L' = \pm 1$ , then  $W^n \cong V_0 \times [0, 1]$ .

**Whitney's Trick:** If  $M, M' = M^m, M^n \subset V^{m+n}$  are closed submanifolds with  $M \cap M'$  such that  $M, \nu(M')$  (the normal bundle) are oriented. Assume  $m + n \geq 5$  and  $n \geq 3$ , and if  $m = 1, 2$  then assume  $\pi_1(V \setminus M') \hookrightarrow \pi_1(V)$ . Let  $p, q$  be in the intersection,  $\varepsilon(p) \cdot \varepsilon(q) = -1$  (local intersection numbers), and there exists a contractible loop  $L \subset V$  such that

- $L = L_0 \cup L_1$
- $L_0$  is smooth in  $M$ ,  $L_1$  is smooth in  $M'$ .
- $L_0, L_1$  go from  $p$  to  $q$ .

And suppose each  $L_i \cap (M \cap M' \setminus \{p, q\}) = \emptyset$ .

Then

1.  $h_0 = \text{id}$ ,
2.  $h_+ = \text{id}$  near  $M \cap M' \setminus \{p, q\}$
3.  $h_1(M) \cap M' = M \cap M' \setminus \{p, q\}$ .

Recall homological intersection number: If  $M, N \subset V$  are closed smooth submanifolds, then for  $[M], [N] \in H_*(V)$ , then  $[M] \cdot [N] = \sum_{p \in M \cap N} \varepsilon(p)$ .

*Sketch of proof:*

1. Given  $L$ , find a  $D^2 \hookrightarrow V$  that it bounds. Note that  $D^2$  may have self-intersections.
2. Continuous maps can be approximated by smooth maps, and smooth intersections can be perturbed to be transverse. This lets the disc be perturbed, and since  $2 + 2 \leq 5$ , the self-intersection can be made zero.
3. Something else.

## 11.2 Facts From Differential Geometry

Let  $M$  be smooth, then there exists a Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $T_p M$  which is symmetric and positive definite.

Given  $p, q \in M$  and a curve  $c(t)$  from  $p$  to  $q$ , we want a parallel transport map  $f_c : T_p M \rightarrow T_q M$ .

The exponential map: something that maps a neighborhood in  $T_p M$  to a neighborhood of  $p$  in  $M$ . Take geodesics starting at  $p$  and evaluate at  $t = 1$ .

Geodesics: curves of global shortest length.

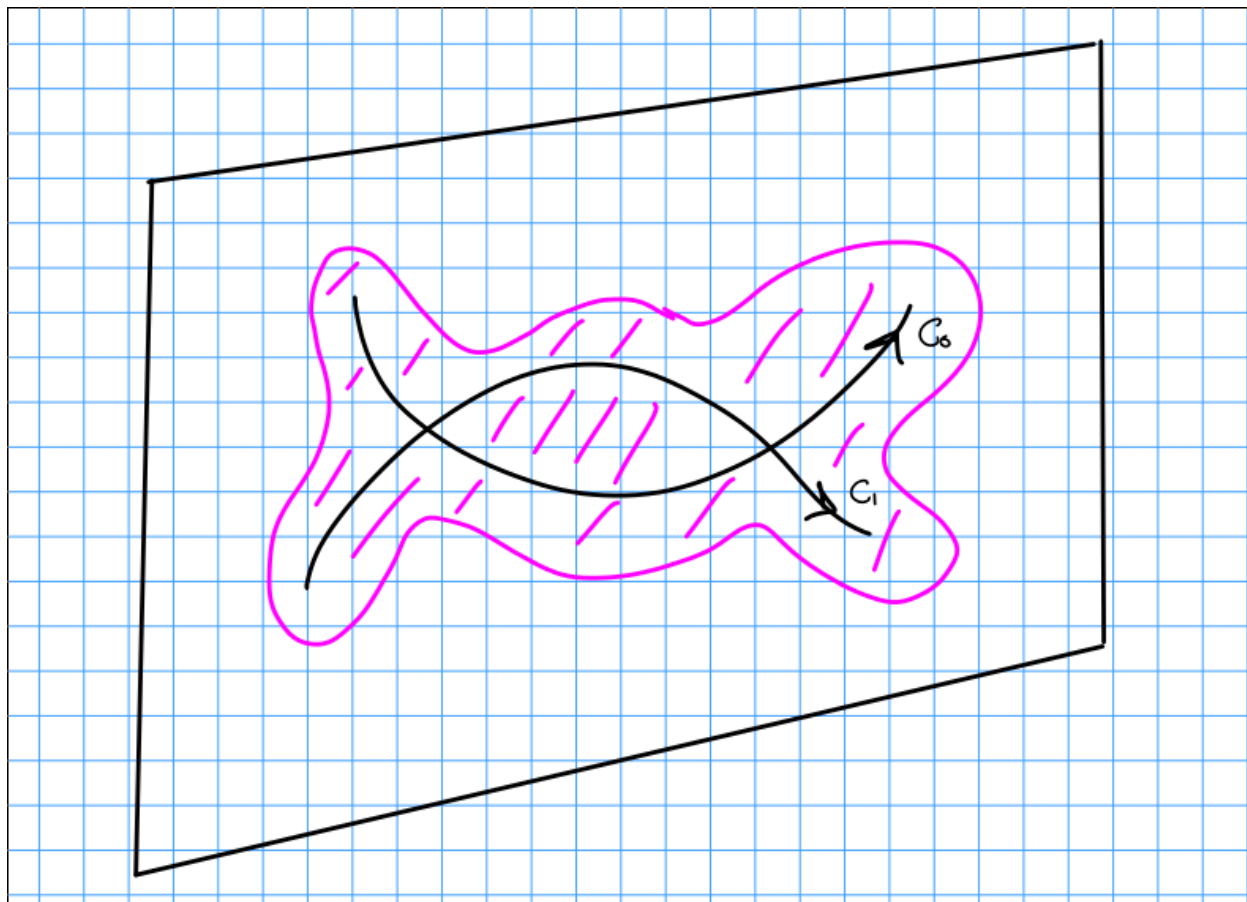
Normal bundle: for  $M \subset V$ , then  $TM \subset TV|_M$  is a subbundle with a metric induced from the metric on  $V$ . The normal bundle is  $TM \perp$ .

Totally geodesic submanifold: If  $M \subset V$  is a submanifold with  $p \in M$  and  $v \in T_p M$ .  $M$  is totally geodesic iff the entire geodesic starting at  $p$  with initial velocity  $v$  is entirely contained in  $M$ .

Any Riemannian metric comes with a canonical connection: the Levi-Cevita connection. Parallel transport along a curve in a totally geodesic submanifold.

## 11.3 Proving Whitney's Trick

Lemma: Let  $L_0, L_1$  be the image of  $C_0, C'_0 \subset \mathbb{R}^2$ . Let  $U$  be a neighborhood of  $C_0 \cup C'_0$  in  $\mathbb{R}^2$ , including the region they bound:



We can extend the maps embedding  $U \cap (C_0 \cap C'_0)$  to  $\phi_1 : U \rightarrow V$  be the embedding, so  $\phi_1|_{\partial D^2} = L$ . We then get a map

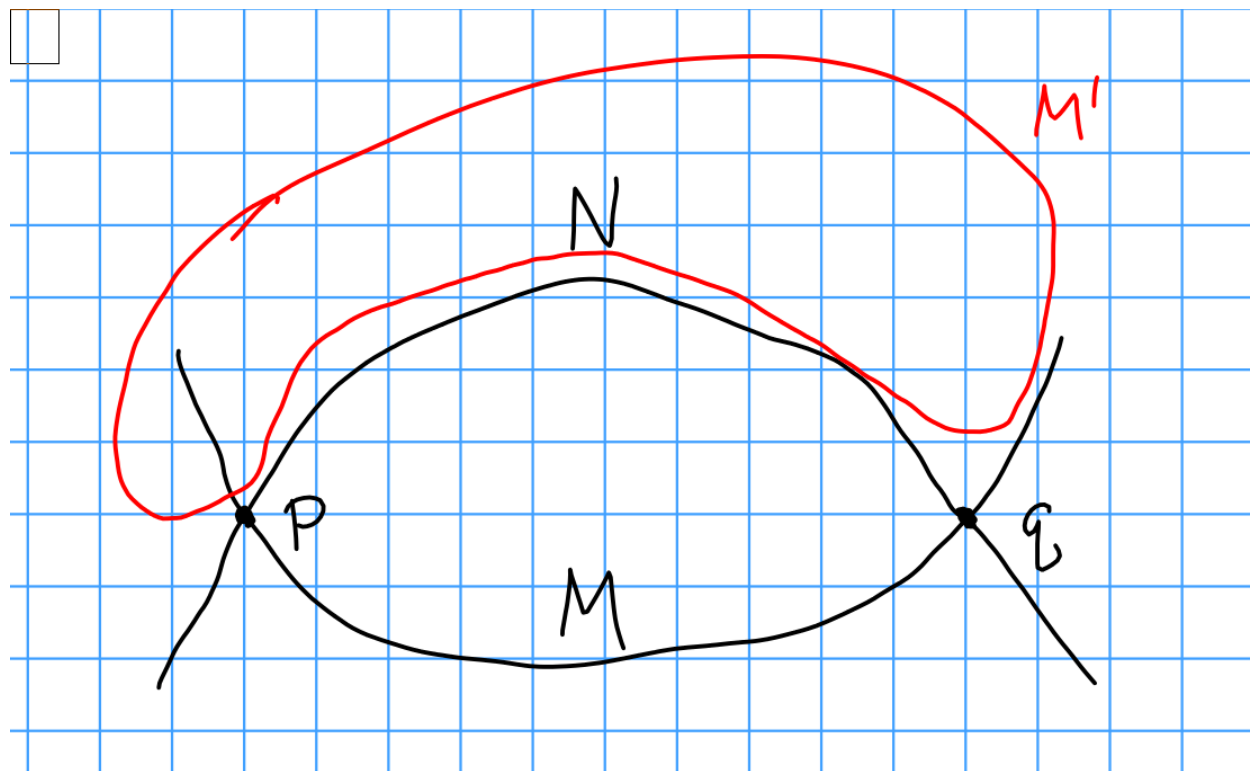
$$\begin{aligned}\phi : U \times \mathbb{R}^{m-1} \times \mathbb{R}^{n-1} &\longrightarrow V \\ \phi^{-1}(M) &= (U \cap C_0) \times \mathbb{R}^{m-1} \times \{0\} \\ \phi^{-1}(M') &= (U \cap C'_0) \times \{0\} \times \mathbb{R}^{m-1}\end{aligned}$$

## 12 Thursday February 27th

Setup:  $M^m, N^n \subset V^{m+n}$  closed submanifolds,  $M \pitchfork N$ ,  $M$  oriented (i.e. an orientation of  $TM$ ) and  $N$  co-oriented (i.e. an orientation  $\nu_N = TM/TV$ ).

Each  $p \in M \cap N$  has a sign  $\varepsilon(p) \in \{-1, 1\}$ . If  $p, q \in M \cap N$  with  $\varepsilon(p) = 1, \varepsilon(q) = -1$ , we would like an isotopy  $(h_t)_{0 \leq t \leq 1}$  of  $V$  such that  $h_0 = \text{id}$  and  $h_1(M) \cap N = (M \cap N) \setminus \{p, q\}$ .

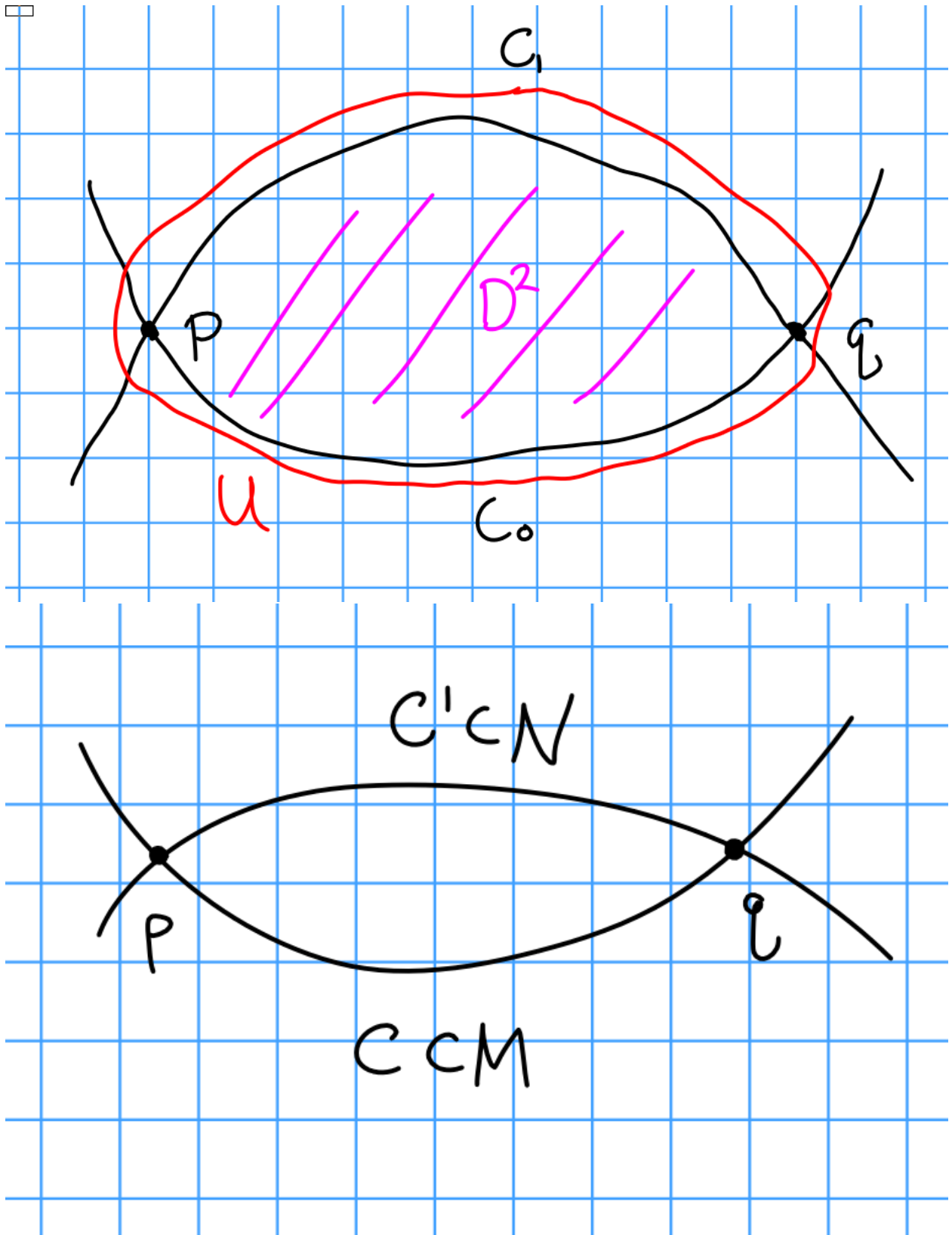
Idea: we want to push  $M$  off of  $N$ :



From last week, assume  $\dim V \geq 5$  and  $\dim N \geq 3$  and  $\pi_1(V \setminus N) = \pi(V) = 0$ .

Why? In the 2-dimensional model above, we want the disc in the middle to be contractible.

□

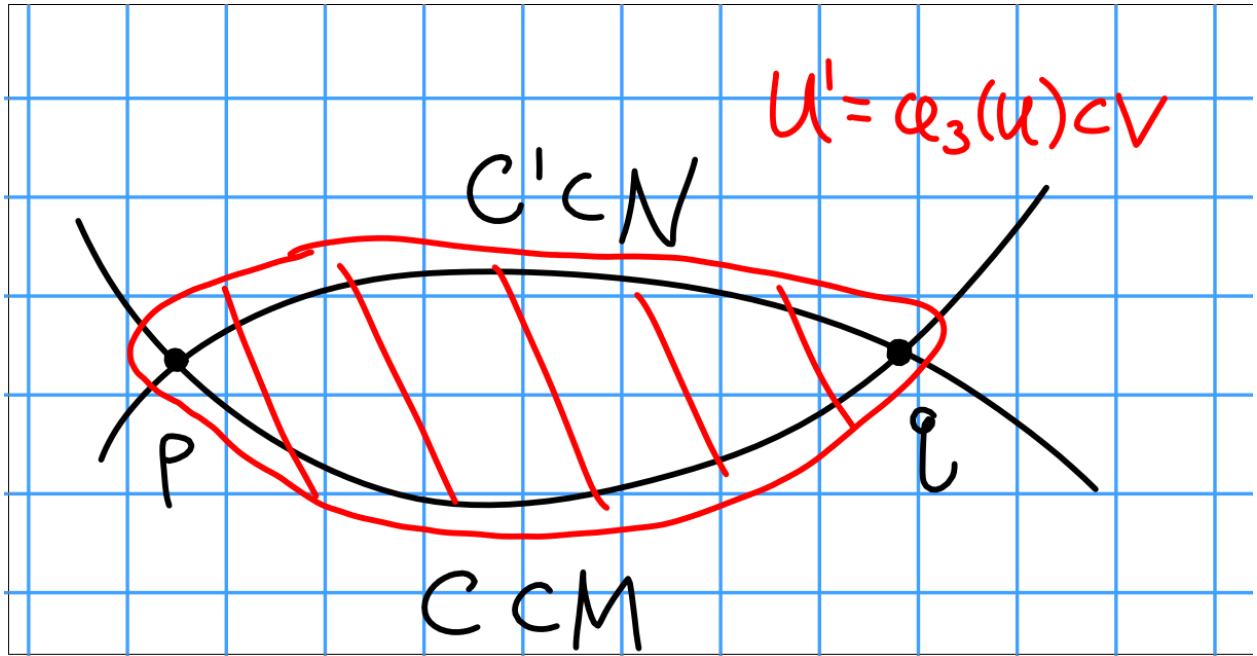


Then there is a smooth embedding  $\phi_3 : U \longrightarrow V$ , 2-dimensional to  $n + 1$  dimensional, sending

$U \cap C_i$  to  $C_i$  and  $U(C_0 \cup C'_0)$  to  $V \setminus (M \cup N)$ .

Goal for today: under the same hypotheses,  $\phi_3$  extends to an embedding  $\phi : U \times \mathbb{R}^{m-1} \times \mathbb{R}^{n-1} \rightarrow V$  such that  $\phi^{-1}(M) = (U \cap C_0) \times \mathbb{R}^{m-1} \times \{0\}$  and  $\phi^{-1}(N) = (U \cap C'_0) \times \{0\} \times \mathbb{R}^{n-1}$ .

Let  $U' = \phi_3(U) \subset V$ :



### Lemma 12.1.

There exist vector fields along  $U'$ ,  $\xi_1, \dots, \xi_{m-1}, \eta_1, \dots, \eta_{n-1}$ , such that

1. These are orthonormal to each other and orthogonal to  $U'$ .

> Note that we'll need a Riemannian metric to make sense of this, and particularly one such

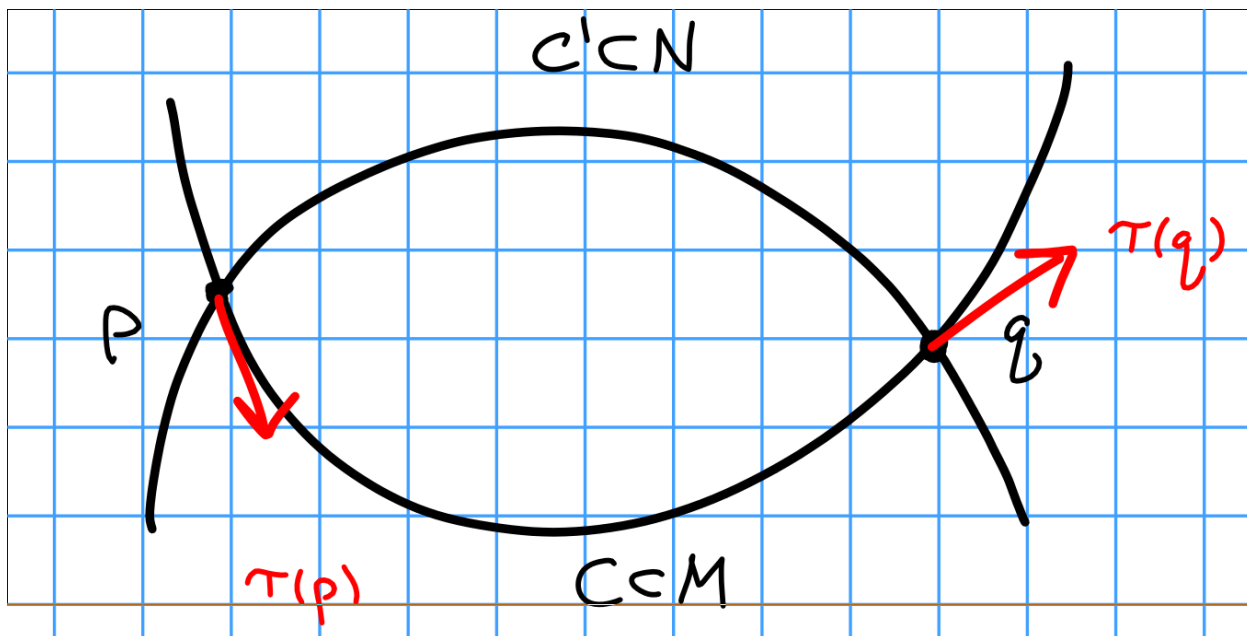
2.  $\xi_1, \dots, \xi_{m-1}$  are tangent to  $M$  along  $C$
3.  $\eta_1, \dots, \eta_{n-1}$  are tangent to  $N$  along  $C'$ .

Given this, we have  $\phi(u, \mathbf{x}, \mathbf{y}) = \exp_{\psi_3(u)} \left( \sum x_i \xi_i(\phi_3(u)) + \sum y_j \eta_j(\phi_r(u)) \right)$ , where the exponential maps is evaluating a geodesic path at time 1.

*Proof (of Lemma).*

Let  $\tau$  be the unit tangent vector field along  $C$ , oriented from  $p$  to  $q$ :

■



Let  $\nu'$  be the unit vector field along  $C'$  normal to  $C'_1$  pointing toward the interior of  $U'$ . Thus  $\nu'(p) = \tau(p)$  and  $\nu'(q) = -\tau(q)$ .

First, complete tangents to an orthonormal basis: choose  $\xi_i(p)$  such that  $\{\tau(p), \xi_1(p), \dots\}$  is an oriented orthonormal basis for  $T_p M$ . Riemannian metrics induce a unique notion of parallel transport, extend  $\xi_i$  to all of  $C$  by parallel transport. This preserves inner products, and in particular we obtain an orthonormal basis for  $T_q M$ .

We can use this to obtain bases for the orthogonal complements, and thus for the normal bundles. Since  $\varepsilon(p) = 1$ , an orientation of  $T_p M$  yields an orientation of  $(\nu_N)_p$ . Thus  $\{\nu'(p), \xi_i(p)\}$  is an oriented basis, and similarly by flipping the sign of the first term, since  $\nu'(q) = -\tau(q)$ ,  $\{\nu'(q), \xi_i(q)\}$  is an oriented basis for  $(\nu_N)_q$ .

Consider the bundle over  $C'$  with fibers equal to orthonormal bases  $\{w_1, \dots, w_{n-1}\} \in (T_x V)^{n-1}$  with each  $w_i$  orthonormal and orthogonal to  $\nu'(x)$ . This has fiber  $O(n-1)$ , and since the base  $C'$  is contractible, this is a trivial bundle.

We have elements in the fiber over  $p$  and  $q$  inducing the same orientation, so are related by an element of  $SO(n-1)$ , which is connected and thus path-connected. This gives a path in the frame bundle connecting  $\{\xi_1(p), \dots, \xi_{m-1}(p)\}$  to  $\{\xi_1(q), \dots, \xi_{m-1}(q)\}$ .

So we extend the  $\xi'$  over all of  $C'$ , remaining orthogonal to  $N$  and  $U'$ . So we have vector fields  $\xi_1, \dots, \xi_m$  along  $C \cup C'$ . We want to show that these can be extended over all of  $U'_1$  remaining orthogonal to  $U'$ .

Consider the bundle over  $U'$  whose fibers are  $(m-1)$ -tuples of vectors in  $T_x V$  that are orthonormal to  $T_x U'$ . Since  $U'$  is contractible, this bundle is trivial, and the fiber is orthonormal  $(m-1)$ -frames in an  $(m+n-2)$ -dimensional vector space, the orthogonal complement of  $T_x U'$ .

Since any orthonormal basis of size  $m+n-2$  will send  $m-1$  frames to other  $m-1$  frames, with some redundancy if the upper-left block is the identity. Thus the fibers are isomorphic to  $O(m+n-2)/O(n-1)$ .



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The construction of  $\xi_1, \dots, \xi_n$  over all of  $U'$  is now reduced to extending the loop on  $O(m+n-2)/O(n-1)$  determined by  $\xi_i$  on  $C \cup C'$  to a disk, i.e.  $U'$ .

In fact,  $\pi_1$  of this space is 0, so this can be done. Once we have  $\xi_i$ , just take  $\nu_i$  to be any orthonormal over  $U'$  such that  $\xi$  is orthogonal to  $TN$  along  $C'$ .

To see why this is, consider the fibrations  $O(n-1) \rightarrow O(n-m-2) \rightarrow Q$  the quotient above and take the LES in homotopy, also consider  $O(n) \rightarrow O(n+1) \rightarrow S^k$ .