Title

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Recall that the Riemann-Zeta function a product expansion

$$\zeta(s) = \sum n^{-s} = \prod_{p \in P} (1 - p^{-s})^{-1}$$

where the product is taken over all primes P.

Let $X = V(\{f_i\}) \coloneqq V(f)$ be the vanishing locus of a family of polynomials in $F = \mathbb{F}_q[x_1, \cdots, x_n]$ for some prime power q.

Let $N_m = \left|\left\{ \mathbf{x} \in X(\mathbb{F}_q) \;\middle|\; f_i(\mathbf{x}) = 0 \right\}\right| = |V(f)| \subset F$, the number of $\mathbb{F}_q$ points, or equivalently just the size of this variety.

Then the Hasse-Weil Zeta function is defined as

$$\zeta_X(t) = \exp \sum_{m \geq 1} \frac{N_m}{m} t^m$$

We immediately make a change of variables and send $t \to q^{-s}$ to obtain

$$\zeta_X(s) = \exp \sum_{m \geq 1} \frac{N_m}{m} \left(q^{-s}\right)^m.$$

> Why? Turns the zeta function into a Dirichlet series in s. Yields $|t| = q^{-\Re(s)}$. Defined for $|t| < \frac{1}{q}$ in $\mathbb{C}$, extended to all of $\mathbb{C}$ as a rational function in x. Converts "All zeros of ζ_X have absolute value $\frac{1}{\sqrt{q}}$" to "All zeros of ζ_X have real part $\frac{1}{2}$".

Explanation of why exponential appears

Rough explanation: Take a bad first approximation and then correct. Let X be a fixed variety, for $p \in X$ define $\|p\|_X = q^n$ where n is the n occurring in the minimal field of definition of p, which is $\mathbb{F}_{q^n}$.

Attempt to define

$$\zeta_{X,q}(s) = \prod_{p \in X} \frac{1}{1 - \|p\|_X^{-s}}.$$

Note that $-\log(x+1) = \sum_{n \geq 1} \frac{x^n}{n}$.

Now fix one $p \in X$ and consider the factor it contributes, and take its logarithm:

$$\begin{aligned}
\log\left(\frac{1}{1 - \|p\|_X^{-s}}\right) &= -\log(1 - \|p\|_X^{-s}) \\
&= -\log(-\|p\|_X^{-s} + 1) \\
&= \sum_{j \geq 1} \frac{\|p\|_X^{-js}}{k} \\
&= \sum_{j \geq 1} \frac{q^{-nks}}{k} \\
&= \sum_{j \geq 1} \frac{n}{nk} (q^{-s})^{nk} \\
(m = nk) \qquad &= \sum_{j \geq 1} \frac{n}{m} (q^{-s})^{m},
\end{aligned}$$