Moduli Spaces

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1 Thursday January 9th

1.1 Representability

Last time: Fix an S-scheme, i.e. a scheme over S.

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{Sch/S}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ T \mapsto x &\longrightarrow \text{triangles} \end{aligned}$$

of the form

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd}$$

Theorem 1.1 ***(Yoneda).***
$\hom_{Fun}(h_x, F) = F(x)$.

Corollary 1.2.
$\hom_{\mathrm{Sch}/S}(x, y) \cong \hom_{\mathrm{Fun}}(h_x, h_y)$.

Definition 1.1.
A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 1.2.
A **moduli space** for that "something" appearing above is an $M \in \mathrm{Obj}(Sch/S)$ such that $F \cong h_M$.

Now fix $S = \mathrm{Spec}(k)$.

h_m is the functor of points over M

Remark (1) $h_m(\mathrm{Spec}(k)) = M(\mathrm{Spec}(k)) \cong$ "families over Spec$k$" $= F(\mathrm{Spec}k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M, and $\mathrm{id}_M \in \mathrm{Mor}_{Sch/S}(M, M) = \xi_{Univ}$ is the universal family

Every family is uniquely the pullback of ξ_{Univ} This makes it much like a classifying space.

For $T \in \mathrm{Sch}/S$,

$$h_M \xrightarrow{\cong} F$$
$$f \in h_M(T) \xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\mathrm{Univ}}).$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\begin{tikzcd} \xi_M & \xi_{M'} \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

$$\begin{tikzcd} M' \arrow[r, "g"] & M \\ \xi_{M'} & \xi_M \end{tikzcd}$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.
A length 2 subscheme of $\mathbb{A}^1_k$ then $F(S) = \left\{V(x^2 + bx + c)\right\} \subset \mathbb{A}'_5$ where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

Then F is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

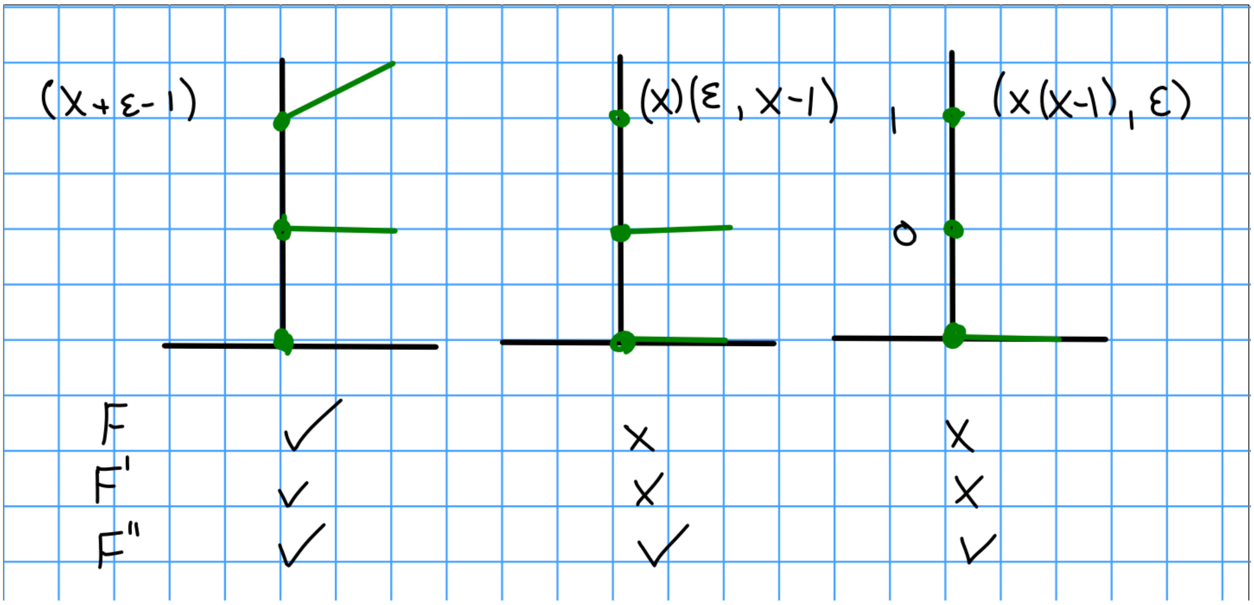
$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber.

$F''(S)$ is the set of subschemes of $\mathbb{A}^{\prime}_{5}$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\mathrm{Spec} k[\varepsilon]$.



$$\begin{tikzcd}
\mathrm{Spec} k \arrow[r, hook, "i"] & \mathrm{Spec} k[\varepsilon] & & = F'(\mathrm{Spec} k) \arrow[dr] & \\
F(\mathrm{Spec} k[\varepsilon]) \arrow[r, "F(i)"] & F(\mathrm{Spec} k) \arrow[urr] & & & = F''(\mathrm{Spec} k) \\
\subset & \in & & & \\
T_pF^{\prime,\prime\prime} & P = V(x(x-1)) & & &
\end{tikzcd}$$

We think of $T_pF^{\prime,\prime\prime}$ as the tangent space at p.

If F is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\mathrm{Spec} k[\varepsilon]) \arrow[r] & M(\mathrm{Spec} k) \\
\subset & \subset \\
T_pM \arrow[r] & p
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Spec} k \arrow[dl, hook'] \arrow[dr, "?"] & \\
\mathrm{Spec} k[\varepsilon] \arrow[rr] & & \mathrm{Spec} \mathcal{O}_{M,p} \subset M
\end{tikzcd}$$

$$\begin{tikzcd}
 & k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
m_p & (\varepsilon) \\
m_p^2 & 0
\end{tikzcd}$$

Moreover, $T_pM = (m_p/m_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned} \lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\ \varepsilon &\mapsto \lambda\varepsilon \end{aligned}$$

$$\lambda^* : \mathrm{Spec}(k[\varepsilon]) \longrightarrow \mathrm{Spec}(k[\varepsilon])$$

$$\begin{aligned} \lambda : M(\mathrm{Spec}(k[\varepsilon])) &\longrightarrow M(\mathrm{Spec}(k[\varepsilon])) \\ \supset T_pM &\longrightarrow T_pM \subset . \end{aligned}$$

Conclusion: If F is representable, for each $p \in F(\mathrm{Spec} k)$ there exists a unique point of T_pF that are invariant under scaling.

1. If $F, F', G \in \mathrm{Fun}((\mathrm{Sch}/S)^{\mathrm{op}}, \mathrm{Set})$, there exists a fiber product

   $$\begin{tikzcd}
   F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\
   F \arrow[r] & G
   \end{tikzcd}$$

   where

   $$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

   $$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}/T) &\longrightarrow \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

Claim: $\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(S) = \mathcal{O}_s^{n+1} &\longrightarrow L \longrightarrow 0/\sim . \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, "="] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $F(f)$ is given by pullbacks.

Remark $\mathbb{P}^n_S$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F_S(T) = \mathcal{O}_T^{n+1} &\longrightarrow L \longrightarrow 0/\sim . \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of claim).
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a

subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

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Claim 2: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{tikzcd}
F_i \arrow[r] & F \\
F_{ij} \arrow[u] \arrow[r] & F_j \arrow[u]
\end{tikzcd}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example 1.2.
For $n = 1$, we can glue along open subschemes

$$\begin{tikzcd}
 & F_0 \\
F_{01} \arrow[ur] \arrow[dr] & \\
 & F_1
\end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd}
 & & & F_0 \arrow[ddr] & \\
 & & F_{01} \arrow[ur] \arrow[dr] & & \\
F_{012} \arrow[urr] \arrow[r] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\
 & & F_{12} \arrow[ur] \arrow[dr] & & \\
 & & & F_2 \arrow[uur] &
\end{tikzcd}$$

This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S.

RHS to LHS comes from gluing.

Proof of (Claim 2)

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec$k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I \; F} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.