

Problem Set 5

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1 Problem 1

We first make the following claim:

$$S := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$$
$$T := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = \sup \left\{ \sum_{(j,k) \in C} a_{kj} \mid C \subset \mathbb{N}^2, |B| < \infty \right\}.$$

It suffices to show the first equality holds, as the other case will follow similarly. Let $S = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}$ and $S' = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$.

Then consider any bounded set $B \subset \mathbb{N}^2$; so $B \subset \{1, \dots, n_1\} \times \{1, \dots, n_2\}$ for some $n_1, n_2 \in \mathbb{N}$. We then have

$$\sum_B a_{jk} \leq \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} a_{jk} \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}.$$

where the first equality holds $a + jk \geq 0$ for all j, k , so the sum can only increase if we add more terms. But this holds for every B and thus holds if we take the supremum over all of them, so $S' \leq S$.

To see that $S \leq S'$, we can just note that

$$S = \lim_{J \rightarrow \infty} \sum_{j=1}^J \left(\lim_{K \rightarrow \infty} \sum_{k=1}^K a_{jk} \right).$$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs (j, k) , we have the following inequality:

$$\sum_{(j,k) \in B} a_{jk} < \sum_{(k,j) \in C} a_{jk} \leq \sum_{(k,j) \in C} a_{jk} \leq T,$$

since T is a supremum over *all* such sets C , and the terms of any finite sum can be rearranged.

But since this holds for every B , we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_B \sum_{(k,j) \in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that $T \leq S$, yielding the desired equality. \square

2 Problem 2

We want to show the following equality:

$$\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx = \int_0^1 f(x) \, dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned}
\int_0^1 \int_x^1 \frac{f(t)}{t} dt dx &= \int_0^1 \int_0^t \frac{f(t)}{t} dx dt \\
&= \int_0^1 \frac{f(t)}{t} \int_0^t dx dt \\
&= \int_0^1 \frac{f(t)}{t} (t - 0) dt \\
&= \int_0^1 f(t) dt,
\end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$\begin{aligned}
F : \mathbb{R}^2 &\rightarrow \mathbb{R} \\
(x, t) &\mapsto \frac{\chi_A(x, t) \hat{f}(x, t)}{t}.
\end{aligned}$$

where $A = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq t \leq 1\}$ and $\hat{f}(x, t) := f(t)$ is the cylinder on f .

This defines a measurable function on \mathbb{R}^2 , since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right| dx dt \\
&= \int_0^1 \int_0^t \frac{|f(t)|}{t} dx dt \quad \text{since } t > 0 \\
&= \int_0^1 \frac{|f(t)|}{t} \int_0^t dx dt \\
&= \int_0^1 |f(t)| < \infty,
\end{aligned}$$

where the switch is justified by Tonelli and the last inequality holds because f was assumed to be measurable.

Since this shows that $F \in L^1(\mathbb{R}^2)$, and we can thus apply Fubini to F to justify the initial switch. \square

3 Problem 3

Let $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$, and define

$$f(x, y) = \frac{x^{1/3}}{(1 + xy)^{3/2}}$$
$$F(x, y) = \chi_A(x, y)f(x, y).$$

Note that F Then, if all iterated integrals exist and a switch of integration order is justified, we would have

$$\begin{aligned} \int_{\mathbb{R}^2} F &= ? \int_0^\infty \int_y^\infty f(x, y) \, dx \, dy \\ &= ? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1 + xy)^{3/2}} \, dy \, dx \\ &= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3} \sqrt{1 + x^2}} \, dx \\ &= 2 \int_0^1 \frac{1}{x^{2/3} \sqrt{1 + x^2}} \, dx + 2 \int_1^\infty \frac{1}{x^{2/3} \sqrt{1 + x^2}} \, dx \\ &\leq \int_0^1 x^{-2/3} \, dx + \int_0^\infty x^{-5/3} \, dx \\ &= 2(3) + 2 \left(\frac{3}{2} \right) < \infty, \end{aligned}$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1 + x^2} \geq \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1 + x^2} \geq \sqrt{1}$.

Since F is non-negative, we have $|F| = F$, and so the above computation would imply that $F \in L^1(\mathbb{R}^2)$. It thus remains to show that $\int F$ is equal to its iterated integrals, and that the switch of integration order is justified

Since F is non-negative, Tonelli can be applied directly if F is measurable in \mathbb{R}^2 . But f is measurable on A , since it is continuous at almost every point in A , and χ_A is measurable, so F is a product of measurable functions and thus measurable.

4 Problem 4

4.1 Part (a)

For any $x \in \mathbb{R}^n$, let $A_x := A \cap (x - B)$.

We can then write $A_t := A \cap (t - B)$ and $A_s := A \cap (s - B)$, and thus

$$\begin{aligned}
g(t) - g(s) &= m(A_t) - m(A_s) \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) \, dx - \int_{\mathbb{R}^n} \chi_{A_s}(x) \, dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_s}(x) \, dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_t}(t - s + x) \, dx \\
&\quad (\text{since } x \in s - B \iff s - x \in B \iff t - (s - x) \in t - B),
\end{aligned}$$

and thus by continuity in L^1 , we have

$$|g(t) - g(s)| \leq \int_{\mathbb{R}^n} |\chi_{A_t}(x) - \chi_{A_t}(t - s + x)| \, dx \rightarrow 0 \quad \text{as } t \rightarrow s$$

which means g is continuous.

To see that $\int g = m(A)m(B)$, if an interchange of integrals is justified, we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} g(t) \, dt &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{A_t}(x) \, dx \, dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x, t) \, dx \, dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x, t) \, dx \, dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_B(t - x) \, dx \, dt \\
&\quad (\text{since } x \in t - B \iff t - x \in B) \\
&= ? \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_B(t - x) \, dt \, dx \\
&= \int_{\mathbb{R}^n} \chi_A(x) \int_{\mathbb{R}^n} \chi_B(t - x) \, dt \, dx \\
&= \int_{\mathbb{R}^n} \chi_A(x) \, m(B) \, dx \\
&\quad (\text{by translation invariance of Lebesgue integral}) \\
&= m(B) \int_{\mathbb{R}^n} \chi_A \, dx \\
&= m(B)m(A).
\end{aligned}$$

To see that this is justified, we note that the map $F(x, t) = \chi_A(x) \chi_B(x - t)$ is non-negative, and we claim is measurable in \mathbb{R}^{2n} .

- The first component is $\chi_A(x)$, which is measurable on \mathbb{R}^n , and thus the cylinder over it will be measurable on \mathbb{R}^{2n} .
- The second component involves $\chi_B(t - x)$, which is $\chi_B(x)$ composed with a reflection (which is still measurable) followed by a translation (which is again still measurable).

- Thus, as a product of two measurable functions, the integrand is measurable.

So Tonelli applies to $|F|$, and thus $\int |F| = m(A)m(B) < \infty$ since A, B were assumed to be bounded. But then F is integrable by Fubini, and the claimed equality holds.

4.2 Part (b)

Supposing that $m(A), m(B) > 0$, we have $\int g(t) dt > 0$, using the fact that $\int g = 0$ a.e. $\iff g = 0$ a.e., we can conclude that if $T = \{t \ni g(t) \neq 0\}$, then $m(T) > 0$. So there is some $t \in \mathbb{R}^n$ such that $g(t) \neq 0$, and since g is continuous, there is in fact some open ball B_t containing t such that $t' \in B_t \implies g(t') \neq 0$. So we have

- $\forall t' \in B_t, A \cap t' - B \neq \emptyset \iff$
- $\forall t' \in B_t, \exists x \in A \cap t' - B \iff$
- $\forall t' \in B_t, \exists x$ such that $x \in A$ and $x \in t' - B \iff$
- $\forall t' \in B_t, \exists x$ such that $x \in A$ and $x = t' - B$ for some $b \in B \iff$
- $\forall t' \in B_t, \exists x$ such that $x \in A$ and $t' = x + B$ for some $b \in B \iff$
- $\forall t' \in B_t, \exists t'$ such that $t' \in A + B$

And thus $B_t \subseteq A + B$.

5 Problem 5

If the iterated integrals exist and are equal (so an interchange of integration order is justified), we have

$$\begin{aligned}
 \int_0^1 F(x)g(x) &:= \int_0^1 \left(\int_0^x f(y) dy \right) g(x) dx \\
 &= \int_0^1 \int_0^x f(y)g(x) dy dx \\
 &= ? \int_0^1 \int_y^1 f(y)g(x) dx dy \\
 &= \int_0^1 f(y) \left(\int_y^1 g(x) dx \right) dy \\
 &= \int_0^1 f(y)(G(1) - G(y)) dy \\
 &= G(1) \int_0^1 f(y) dy - \int_0^1 f(y)G(y) dy \\
 &= G(1)(F(1) - F(0)) - \int_0^1 f(y)G(y) dy \\
 &= G(1)F(1) - \int_0^1 f(y)G(y) dy \quad \text{since } F(0) = 0,
 \end{aligned}$$

which is what we want to show.

To see that this is justified, let $I = [0, 1]$ and note that the integrand can be written as $H(x, y) = \hat{f}(x, y)\hat{g}(x, y)$ where $\hat{f}(x, y) = \chi_I f(y)$ and $\hat{g}(x, y) = \chi_{IG}(x)$ are cylinders over f and g respectively.

Since f, g are in $L^1(I)$, their cylinders are measurable over $\mathbb{R} \times I$, and thus \hat{f}, \hat{g} are measurable on \mathbb{R}^2 as products of measurable functions. Then H is a measurable function as a product of measurable functions as well.

But then $|H|$ is non-negative and measurable, so by Tonelli all iterated integrals will be equal. We want to show that $H \in L^1(\mathbb{R}^2)$ in order to apply Fubini, so we will show that $\int |H| < \infty$.

To that end, noting that $f, g \in L^1$, we have $\int_0^1 f := C_f < \infty$ and $\int_0^1 g := C_g < \infty$. Then,

$$\begin{aligned} \int_{\mathbb{R}^2} |H| &= \int_0^1 \int_0^1 |f(x)g(y)| \, dx \, dy \\ &= \int_0^1 \int_0^1 |f(x)| |g(y)| \, dx \, dy \\ &= \int_0^1 |g(y)| \left(\int_0^1 |f(x)| \, dx \right) \, dy \\ &= \int_0^1 |g(y)| C_f \, dy \\ &= C_f \int_0^1 |g(y)| \, dy \\ &= C_f C_g < \infty, \end{aligned}$$

and thus by Fubini, the original interchange of integrals was justified.

6 Problem 6

6.1 Part (a)

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \left| \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \left(\int_{x-h}^{x+h} |f(y)| \, dy \right) \, dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\
&\stackrel{=?}{=} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, dx \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \int_{y-h}^{y+h} dx \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| ((y+h) - (y-h)) \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} 2h |f(y)| \, dy \\
&= \int_{\mathbb{R}} |f(y)| \, dy < \infty
\end{aligned}$$

since f was assumed to be in $L^1(\mathbb{R})$, where the changed bounds of integration are determined by considering the following diagram:

To justify the change in the order of integration, consider the function $H(x, y) = \frac{1}{2h} \chi_A(x, y) f(y)$ where $A = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x-h \leq x, y \leq x+h\}$. Since f is measurable, the constant function $(x, y) \mapsto \frac{1}{2h}$ is measurable, and characteristic functions are measurable, H is a product of measurable functions and thus measurable.

Thus it makes sense to write $\int |H|$ as an iterated integral by Tonelli, and since $\int_{\mathbb{R}^2} |H| = \int_{\mathbb{R}} |A_h(f)| < \infty$ by the above calculation, we have $H \in L^1(\mathbb{R}^2)$, and Fubini applies.

6.2 Part (b)

Let $\varepsilon > 0$; we then have

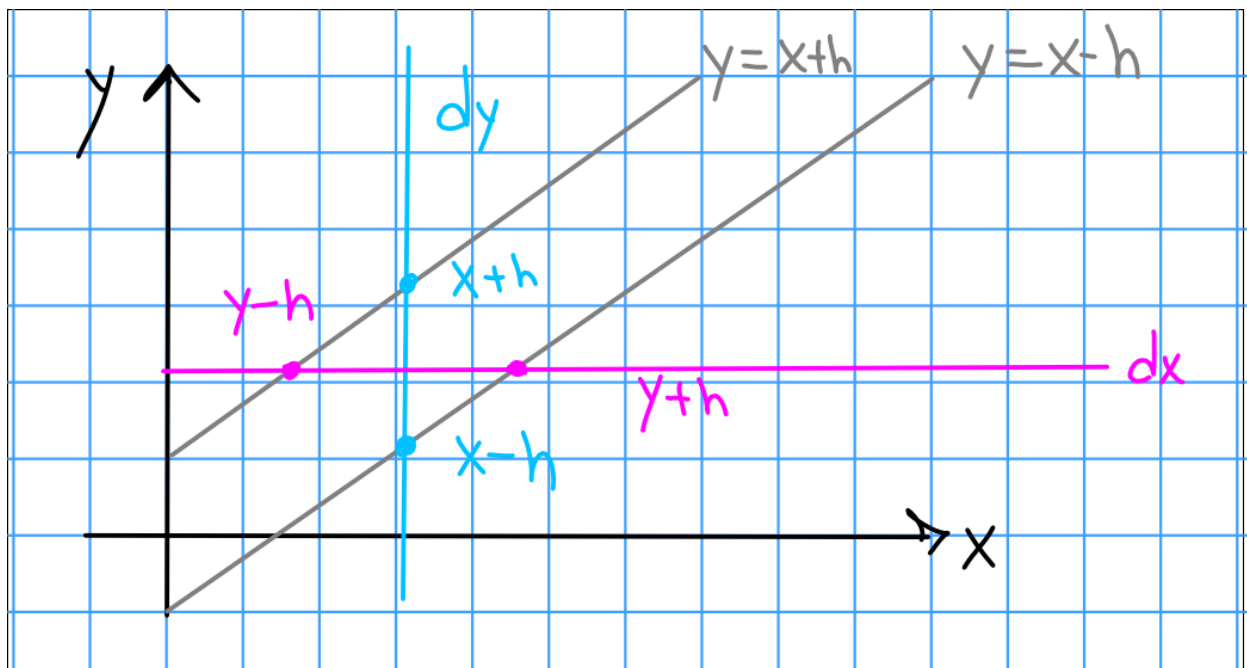


Figure 1: Changing the bounds of integration

$$\begin{aligned}
 \int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
 &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
 &\quad \text{since } \frac{1}{2h} \int_{x-h}^{x+h} f(x) \, dy = \frac{1}{2h} f(x)((x+h) - (x-h)) = \frac{1}{2h} f(x)2h = f(x) \\
 &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{B(h,x)} f(y) - f(x) \, dy \right| \, dx \\
 &\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - f(x)| \, dy \, dx \\
 &\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{-h}^h |f(y-x) - f(x)| \, dy \, dx
 \end{aligned}$$

but since $h \rightarrow 0$ will force $y \rightarrow x$ in the integral, for a fixed x we can let $\tau_x(y) = f(y-x)$ and we have $\|\tau_x - f\|_1 \rightarrow 0$ by continuity in L^1 . Thus $\int_{-h}^h |f(y-x) - f(x)| \rightarrow 0$, forcing $\|A_h(f) - f\|_1 \rightarrow 0$ as $h \rightarrow 0$. \square