

# Title

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# 1 | Lecture 15: The $L$ -Polynomial

Recall that we had  $Z(t) + F(t) + G(t)$ :

$$(q-1)F(t) = \sum_{0 \leq \deg C \leq 2g-2} q^{\ell(C)} t^{\deg(C)}$$

$$(q-1)G(t) = h \left( \frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t} \right).$$

Note that  $F(t)$  is a polynomial of degree at most  $2g-2$ , and clearing denominators in  $G(t)$  yields a polynomial of degree at most  $2g$

**Definition 1.0.1** (The  $L$ -polynomial)

The  $L$ -polynomial is defined as

$$L(t) := (1-t)(1-qt)Z(t) = (1-t)(1-qt) \sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[t].$$

It turns out that the degree bound of  $2g$  is sharp, and the coefficients closer to the middle are most interesting:

**Theorem 1.0.2** (?).

Let  $K/\mathbb{F}_q$  be a function field of genus  $g \geq 1$ , then

- $\deg L = 2g$ .
- $L(1) = h$
- $L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right)$ .
- Writing  $L(t) = \sum_{j=1}^{2g} a_j t^j$ ,
  - $a_0 = 1$  and  $a_{2g} = q^g$ .
  - For all  $0 \leq j \leq g$ , we have  $a_{2g-j} = q^{g-j} a_j$ .
  - $a_1 = |\Sigma(K/\mathbb{F}_q)| - (q+1)$ , which notably does not depend on  $g$ .
  - Write  $L(t) = \prod_{j=1}^{2g} (1 - \alpha_j t) \in \mathbb{C}[t]$  <sup>a</sup>
- The  $\alpha_j \in \bar{\mathbb{Z}}$  <sup>b</sup> (which were *a priori* in  $\mathbb{C}$ ) and can be ordered such that for all  $1 \leq j \leq g$ , we have  $a_j a_{g+j} = q$ . <sup>c</sup>

f. If  $L_r(t) = (1 - t)(1 - q^r t)Z_r(t)$  then  $L_r(t) = \prod_{j=1}^{2g} (1 - \alpha_j^r t)$ , where  $K_r$  is the constant extension  $K\mathbb{F}_{q^r}/\mathbb{F}_{q^r}$

<sup>a</sup>The polynomial isn't monic, but rather has a constant coefficient, so this expansion is somewhat more natural than (say)  $\prod (t - \alpha)$ .

<sup>b</sup> $\bar{\mathbb{Z}}$  denotes the algebraic integers.

<sup>c</sup>This is the first hint at the Riemann hypothesis: if for example they all had the same complex modulus, this would force  $|a_j| = \sqrt{q}$ . Thus proving that they all have the same absolute value is 99% of the content!

Note that the  $\alpha_i$  are reciprocal roots.

*Proof (of a).*

We saw from  $Z(t) = F(t) + G(t)$  that  $\deg L \leq 2g$ . Equality will follow from the proof of (d) part 1, since this would imply that  $a_{2g} = q^g \neq 0$ . ■

*Proof (of b).*

Our formula  $Z(t) =$  ■