Title

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Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{ \text{Radical ideals } k[x_1, \cdots, x_n] \right\} \longrightarrow V(\,\cdot\,) I(\,\cdot\,) \left\{ \text{Affine varieties in } \mathbb{A}^n \right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 1.1 ***(?).***
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

> Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty.

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Returning to the proof that $I(V(J)) \subseteq \sqrt{J}$: let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. But $ft - 1 \neq 0$ if $f = 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

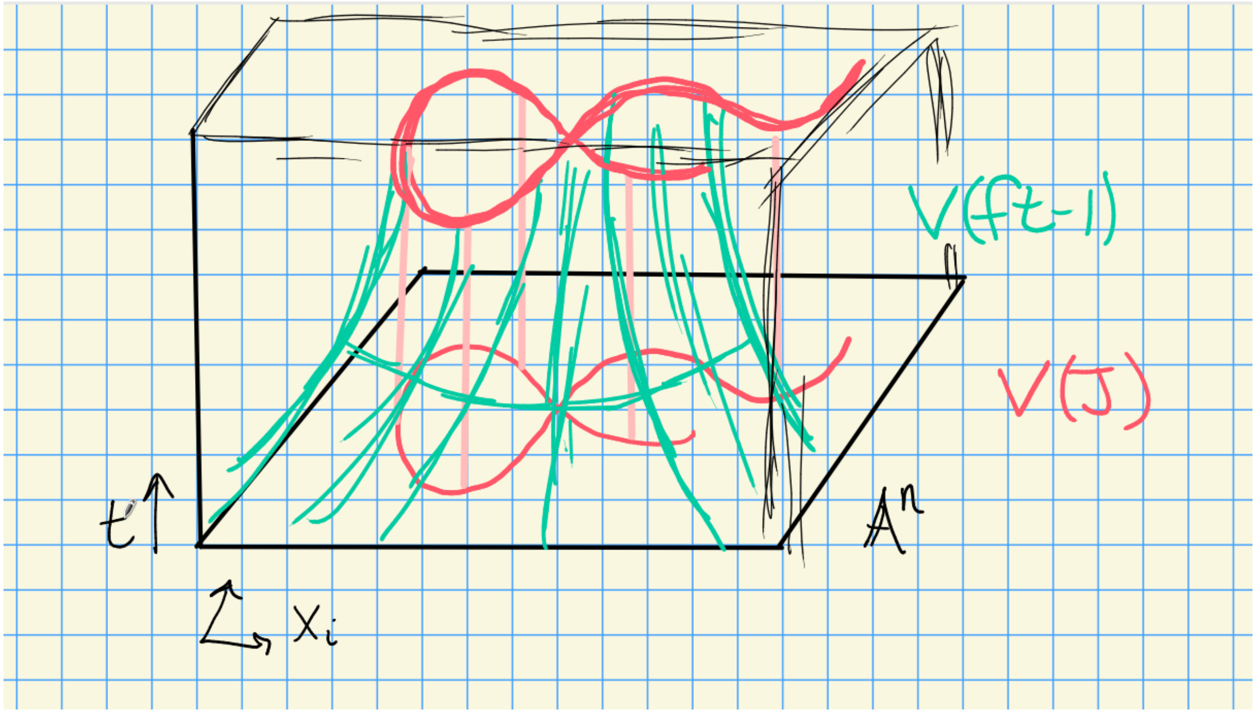

Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so $1 = \langle ft - 1 \rangle \, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$ with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N \coloneqq (ft-1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t.

Now take $k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle \,.$$

Observation: there is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle \,.$$

Exercise 1.1.
Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 1.1.

Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then $I(V(f)) = \langle (x-a_1)(x-a_2) \cdots (x-a_k) \rangle = \sqrt{J} \subsetneq J$. Note that this loses information.

Example 1.2.
Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\left\{\text{Points of } \mathbb{A}^n\right\} \iff \left\{\text{Maximal ideals of } k[x_1, \cdots, x_n]\right\}.$$

Theorem 1.2 ***(Properties of I).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) at the level of V.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

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