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- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \{f : X \rightarrow Y \mid f \text{ is continuous}\}.$$

- Topologize with the *compact-open* topology: $U \in \text{hom}_T(X, X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.

- * If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \{d(f_n(x), f(x)) \mid x \in K\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact}.$$

In words: $f_n \rightarrow f$ uniformly on every compact set.

- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with $d(f, g) = \sup_{x \in X} (f(x), g(x))$.
- So define $\text{Map}(X, Y) = \text{hom}_{\text{Top}}(X, Y)$ equipped with the compact-open topology.
 - Can immediately consider a lot of interesting spaces by considering $\text{Map}(\cdot, Y)$:

$$X = I := [0, 1] \rightsquigarrow P(Y) := \{f : I \rightarrow Y\} = Y^I$$

$$X = S^1 \rightsquigarrow \Omega(Y) = \mathcal{L}(Y; x_0) := \{f : S^1 \rightarrow Y\} = Y^{S^1}.$$

- Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property: there is a homeomorphism

$$\text{Map}(X \times Z, Y) \leftrightarrow \cong \text{Map}(Z, \text{Map}(X, Y))$$

$$H : X \times Z \rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y)$$

$$(x, z) \mapsto H(x, z) \iff z \mapsto H(\cdot, z).$$

Categorically, $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top .

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- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps f, g are homotopic \iff there is a path in $\text{Map}(X, Y)$ connecting them.

* Proof:

$$\text{Map}(I, \text{Map}(X, Y)) \cong \text{Map}(Y \times I, X),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

* Note that we can interpret the RHS as the space of paths

- Now we can bootstrap up to play fun recursive games by applying the $\text{Map}(I, \cdot)$ functor: define

$$\text{Map}_I^1(X, Y) := \text{Map}(I, \text{Map}(X, Y))$$

and then

$$\begin{aligned} \text{Map}_I^2(X, Y) &:= \text{Map}(I, \text{Map}_I^1(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(I, \text{Map}(X, Y))) \\ &\cong \text{Map}(I, \text{Map}(Y \times I, X)). \end{aligned}$$

Interpretation: this is a path in the space of paths

- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.

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