

# Problem Set 1

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## 1 Problem 5

### 1.1 Part 1

Let  $A \in \text{Mat}(n, n)$  be a positive definite  $n \times n$  matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and  $B \in \text{Mat}(n, n)$  be positive semi-definite, so

$$\langle v, Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v, (A + B)v \rangle &= \langle v, Av \rangle + \langle v, Bv \rangle \\ &> \langle v, Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

## 1.2 Part 2

Let  $M$  be a smooth manifold with tangent bundle  $TM$  and a maximal smooth atlas  $\mathcal{A}$ . Choose a covering of  $M$  by charts  $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$  such that  $M \subseteq \bigcup_{i \in I} U_i$ . Then choose a partition of unity  $\{f_i\}_{i \in I}$  subordinate to  $\mathcal{C}$ .

In each copy of  $\phi_i(U_i) \cong \mathbb{R}^n$ , let  $g^i$  be the Euclidean metric given by the identity matrix, i.e.  $g_{jk}^i := \delta_{jk}$ . We then have

$$g^i : T\phi_i(U_i) \times T\phi_i(U_i) \rightarrow \mathbb{R}$$

$$(\partial x_i, \partial x_j) \mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

which is defined for pairs of vectors in  $T\phi_i(U_i) \cong T\mathbb{R}^n \cong \mathbb{R}^n$ , which is spanned by  $\{\partial x_i\}_{i=1}^n$ , and is defined on basis vectors as the Kronecker delta and extended linearly.

Let  $G^i$  be the pullback of  $g^i$  along  $\phi_i$ , so

$$G^i : TU_i \times TU_i \rightarrow \mathbb{R}$$

$$G^i(p, q) := (\phi_i)^* g^i(p, q) = g^i(\phi_i(p), \phi_i(q))$$

Then define the following inner product on  $M$  which is defined by

$$g : TM \rightarrow TM$$

$$(p, q) \mapsto \sum_{i \in I} f_i G^i(p, q)$$

## 2 Problem 6

### 2.1 Part 1

Let  $M = S^2$  as a smooth manifold, and consider a vector field on  $M$ ,

$$X : M \rightarrow TM$$

We want to show that there is a point  $p \in M$  such that  $X(p) = 0$ .

Every vector field on a compact manifold without boundary is complete, and since  $S^2$  is compact with  $\partial S^2 = \emptyset$ ,  $X$  is necessarily a complete vector field.

Thus every integral curve of  $X$  exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\left. \frac{\partial}{\partial s} \phi_s(p) \right|_{s=t} = X(\phi_t(p)),$$

$$\phi_0(p) = p$$

at every point  $p \in M$ .

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular,  $\phi_0 = \text{id}_M$ , and  $\phi_1 \in \text{Diff}(M, M)$ . Moreover  $\phi_0$  is homotopic to  $\phi_1$  via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to  $\phi_0$  and  $\phi_1$ . For an arbitrary map  $f : M \rightarrow M$ , we have

$$\Lambda(f) = \sum_k \text{Tr} \left( f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where  $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of  $M$ .

Since homotopic maps induce equal maps on homology, we also have  $\Lambda(\phi_1) = \chi(M)$ .

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have  $\chi(S^2) = 2 \neq 0$ , and thus  $\phi_1$  has a fixed point  $p_0$ , thus

$$\left. \frac{\partial}{\partial t} \phi_t(p_0) \right|_{t=1} \text{ so}$$

$$\begin{aligned} &\phi_t(p) = p \\ \implies &\frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies &\left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies &X(\phi_1(p_0)) := \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 && \text{by definition of } \phi_1 \end{aligned}$$

so  $X(\phi_1(p_0)) = 0$ , which shows that  $p_0$  is a zero of  $X$ . So  $X$  has at least one zero, as desired.  $\square$

## 2.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\rightarrow S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\rightarrow (\mathbf{x}, [1, 1]) \end{aligned}$$

which is the identity on the  $S^2$  component and assigns the constant vector  $[1, 1]$  to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & TS^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

can *not* have a nowhere vanishing section.  $\square$