Qual Problems

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1.

a) If $\phi: R \to S$ is a ring homomorphism, then $\ker\phi \trianglelefteq R$ is an ideal. But the only ideals of a field are 0 & R, so either ϕ is injective or the zero map.

b) No, consider
$$\begin{aligned}\phi: \mathbb{Z}_p[x] &\to \mathbb{Z}_p \\ f &\mapsto f(1)\end{aligned}$$

Then
$$\left.\begin{aligned} f_1 = x+(p-1)x^2 &\mapsto 1+p-1 = 0 \in \mathbb{Z}_p \\ f_2 = (p-1)x + x^2 &\mapsto p-1+1 = 0 \in \mathbb{Z}_p \end{aligned}\right]$$
where $f_2 \neq f_1$ but $\phi(f_1) = \phi(f_2) = 0$ so $\ker\phi \neq 0$. ▨

2. If F contains a nonzero nilpotent ($x^n = 0$) then F contains a zero divisor ($x \cdot x^{n-1} = 0$) and thus F can't be a field since
$$(x^{n-1})^{-1}(x^{n-1}\cdot x) \begin{cases} = ((x^{n-1})^{-1}\cdot x^{n-1})x = x \\ = (x^{n-1})^{-1}(x^n) = (x^{n-1})^{-1}\cdot 0 = 0 \end{cases} \Rightarrow x = 0. \ \#.$$
But $\mathbb{Z}_{p^k}$ contains a nilpotent iff $k>1$, namely $[p]$ since $[p^k] = 0 \in \mathbb{Z}_{p^k}$.

So if $\mathbb{Z}_{p^k} \leq \mathbb{Z}_n$ for any p and any $k>1$, it contains a nilpotent, which would have to be in one of the terms of a direct sum decomposition, forcing that term to not be a field.

So $\mathbb{Z}_n \cong \bigoplus F_i$, each F_i a field, iff n is square-free, i.e. $n = \prod_{i=1}^m p_i^{\alpha_i}$ with $\alpha_i = 1 \ \forall i$.

Then take each F_i to be $\mathbb{Z}_{p_i}$, and $\mathbb{Z}_n \cong \bigoplus \mathbb{Z}_{p_i}$ by the Chinese Remainder Theorem. ▨

3. Suppose r isn't invertible. Then $r \in I := R \setminus R^\times$. This is an ideal because if r is not invertible, tr can not be invertible for any $t \in R$. It is maximal because if $I \subsetneq J$, then J contains a unit, so $J = R$.

Suppose $r \in I$ for some maximal I. If $r \in R^\times$, then $r^{-1} \in R$, so $r^{-1}r \in I \Rightarrow 1 \in I$, so $I = R$ which contradicts the maximality of I. So r can not be a unit.

4.

a) G is <u>solvable</u> if there is a normal series with abelian quotients, i.e.
$$1 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G, \quad H_i/H_{i-1} \text{ abelian}$$

b) $\#G = 36 = 2^2 \cdot 3^2$, so

$$\begin{array}{ll} n_2 \equiv 1 \bmod 2 & n_3 \equiv 1 \bmod 3 \\ n_2 \mid 9 & n_3 \mid 4 \end{array}$$

So $n_2 \in \{1,3,9\}$ & $n_3 \in \{1,4\}$

If $n_3 = 1$, take

$$\begin{array}{ccccc} & & 9 & & 36 \\ 1 & \trianglelefteq & Q_3 & \trianglelefteq & G \\ & \uparrow & & \uparrow & \\ & 9 & & 4 & \Big] \text{ size of quotients} \\ & \uparrow & & \uparrow & \\ & & \text{Must be abelian} & & \end{array}$$

Otherwise $n_3 = 4$, so define

$$\phi : G \to \mathrm{Sym}(\mathrm{Syl}(3,G)) \cong S_4$$
$$g \mapsto (Q_3 \mapsto g Q_3 g^{-1})$$

Then $\mathrm{im}\,\phi \leq S_4$, and since S_4 is solvable, so is any subgroup.

We claim $\ker\phi$ is solvable as well. It can't be the case that $\ker\phi = 0$, since this would force $|G| \leq |S_4| \Rightarrow 36 \leq 24$. If we write $\mathrm{Syl}(3,G) = \{H_1, \cdots, H_4\}$, we can identify $\ker\phi = \bigcap_{i=1}^{4} N_G(H_i) \trianglelefteq G$. But it also can not be G, since this would force $n_3 = 1$. So $|\ker\phi| > 1$. By sylow 3, we know $[G : N_G(H_i)] = n_p = 4$, so $|N_G(H_i)|$ divides 9 and thus $|\ker\phi| = 3$ or 9. In either case, $\ker\phi$ must be abelian & thus solvable.

So $\ker\phi$ is solvable & $G/\ker\phi \cong \mathrm{im}\,\phi$ is solvable, which implies that G itself is also solvable. ▨