Title

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1.1 Kempf's Theorem

Next topic: Kempf's Vanishing Theorem. Proof in Jantzen's book involving ampleness for sheaves.

Setup:

We have

$$\begin{tikzcd} G & \text{a reductive algebraic group over } k = \bar{k} \\ B \arrow[u, "\subseteq"] & \text{the Borel subgroup} \\ T \arrow[u, "\subseteq"] & \text{its maximal torus} \end{tikzcd}$$

along with the weights $X(T)$.

We can consider derived functors of induction, yielding $R^n \operatorname{Ind}_B^G \lambda = \mathcal{H}^n(G/B, \mathcal{L}(\lambda)) \coloneqq H^n(\lambda)$ where $\mathcal{L}(\lambda)$ is a line bundle and G/B is the flag variety.

Recall that

- $H^0(\lambda) = \operatorname{Ind}_B^G(\lambda)$,
- $\lambda \notin X(T)_+ \implies H^0(\lambda) = 0$
- $\lambda \in X(T)_+ \implies L(\lambda) = \operatorname{Soc}_G H^0(\lambda) \neq 0$.

Theorem 1.1 ***(Kempf).***
If $\lambda \in X(T)_+$ a dominant weight, then $H^n(\lambda) = 0$ for $n > 0$.

Remark 1.
In char $(k) = 0$, $H^n(\lambda)$ is known by the Bott-Borel-Weil theorem. In positive characteristic, this is not know: the characters char $H^n(\lambda)$ is known, and it's not even known if or when they vanish. Wide open problem!

> Could be a nice answer when $p > h$ the Coxeter number.

1.2 Good Filtrations and Weyl Filtrations

We define two classes of distinguished modules for $\lambda \in X(T)_+$:

- $\nabla(\lambda) \coloneqq H^0(\lambda) = \operatorname{Ind}_B^G \lambda$ the costandard/induced modules.
- $\Delta(\lambda) = V(\lambda) \coloneqq H^0(-w_0\lambda) = \operatorname{Ind}_B^G \lambda$ the standard/Weyl modules
  - Here w_0 is the longest element in the Weyl group

We have

$$L(\lambda) \hookrightarrow \nabla(\lambda)\Delta(\lambda) \qquad\qquad \twoheadrightarrow L(\lambda).$$

We define the category Rat-G of rational G-modules. This is a *highest weight category* (as is e.g. Category $\mathcal{O}$).

Definition 1.1.1 (Good Filtrations)**.**
An (possibly infinite) ascending chain of G-modules

$$0 \leq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V$$

is a **good filtration** of V iff

1. $V = \cup_{i \geq 0} V_i$
2. $V_i / V_{i-1} \cong H^0(\lambda_i)$ for some $\lambda_i \in X(T)_+$.

> In characteristic zero, the H^0 are irreducible and this recovers a composition series. Since we don't have semisimplicity in this category, this is the next best thing.

Definition 1.1.2 (Weyl Filtration)**.**
With the same conditions of a good filtration, a chain is a **Weyl filtration** on V iff

1. $V = \cup_{i \geq 0} V_i$
2. $V_i / V_{i-1} \cong V(\lambda_i)$ for some $\lambda_i \in X(T)_+$.

> I.e. the different is now that the quotients are standard modules.

Definition 1.1.3 (Tilting Modules)**.**
V is a **tilting module** iff V has both a good filtration and a Weyl filtration.

Theorem 1.2 ***(Ringel, 1990s).***
Let $\lambda \in X(T)_+$ be a dominant weight. Then there is a unique indecomposable highest weight tilting module $T(\lambda)$ with highest weight λ.

Example 1.1.
We have the following situation for type A_2:

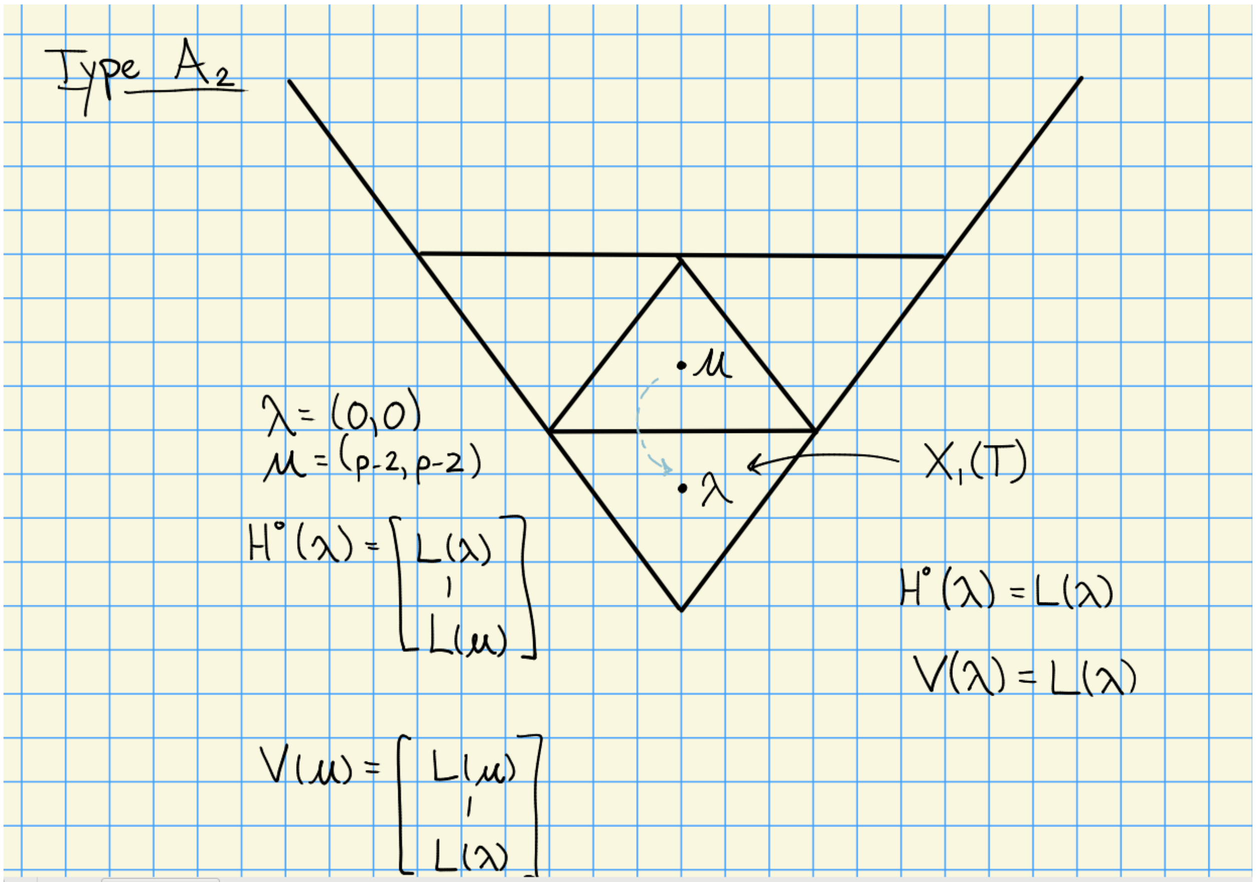

Figure 1: Image

And thus a decomposition:

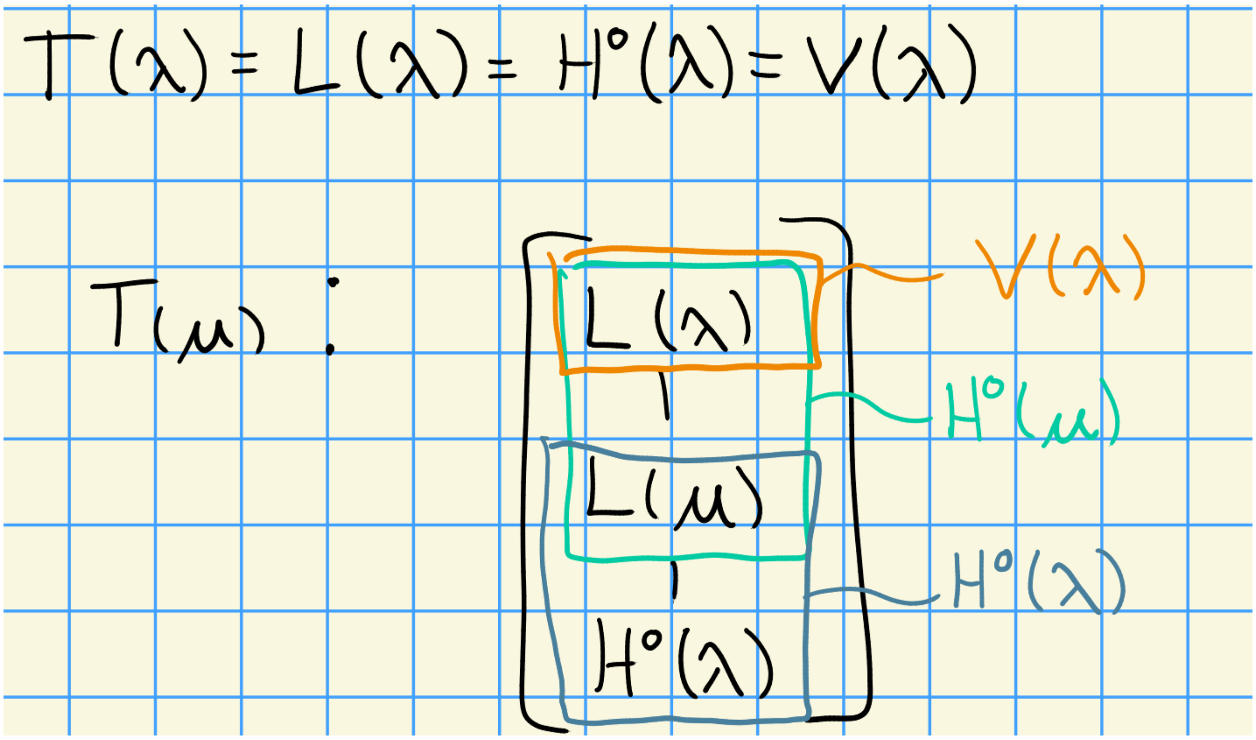

Figure 2: Image

The picture to keep in mind is the following: 4 types of modules, all indexed by dominant weights:

1.3 Cohomological Criteria for Good Filtrations

We'll take cohomology in the following way: let G be an algebraic group scheme, and define

$$H^n(G, M) \coloneqq \mathrm{Ext}G^n(k, M)$$

where to compute $\mathrm{Ext}_G^n(M, N)$ we take an injective resolution $N \hookrightarrow I_*$, apply $\hom_G(M, \cdot)$, and take kernels mod images.

Letting $\lambda \in \mathbb{Z}\Phi$ be integral, so $\lambda_{\alpha \in \Delta} = \sum n_\alpha \alpha$, define the **height**

$$\mathrm{ht}(\lambda) = \sum_{\alpha \in \Delta} n_\alpha.$$

Lemma 1.3 *(?).*
There exists an injective resolution of B-modules

$$0 \to k \to I_0 \to I_1 \to \cdots$$

where

1. I_0 is the injective hull of k,

2. All weights of I_j, say μ satisfy $\mathrm{ht}(\mu) \geq j$.

$$k[u] \text{ an injective } B\text{-module}$$
$$k \hookrightarrow \operatorname{Ind}_T^B k := I_0 = k[u].$$

We thus get a diagram of the form

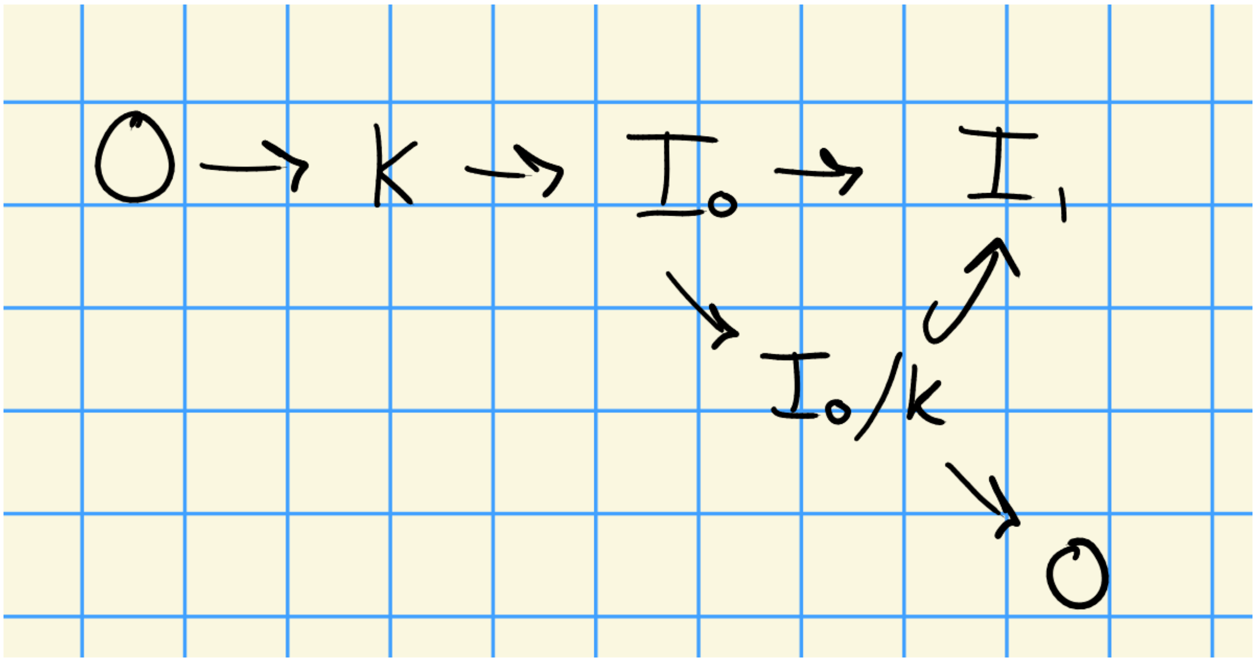

Figure 3: Image

Proposition 1.4 *(?)***.**
Let $H \leq G$, then there exists a spectral sequence

$$E_2^{i,j} = \operatorname{Ext}_G^i(N, R^j \operatorname{Ind}_H^G M) \implies \operatorname{Ext}_H^{i+j}(N, M)$$

for $N \in \operatorname{Mod}(G), M \in \operatorname{Mod}(H)$.

Example 1.2.
Let $H = B$ and take $G = G$ itself, and let $N = k$ the trivial module and $M \in \operatorname{Mod}(G)$ be any rational G-module. We have

$$E_2^{i,j} = \operatorname{Ext}_B^i(k, R^j \operatorname{Ind}_B^G M) \implies \operatorname{Ext}_B^{i+j}(k, M).$$

Observations:

0. $R^0 \operatorname{Ind}_B^G k = \operatorname{Ind}_B^G k = k$.

1. The tensor identity works here, i.e. $R^j \operatorname{Ind}_B^G M = \left(R^j \operatorname{Ind}_B^G k\right) \otimes M$.
2. $R^j \operatorname{Ind}_B^G k = 0$ for $j > 0$ since we have a dominant weight.

The spectral sequence thus collapses on E_2:

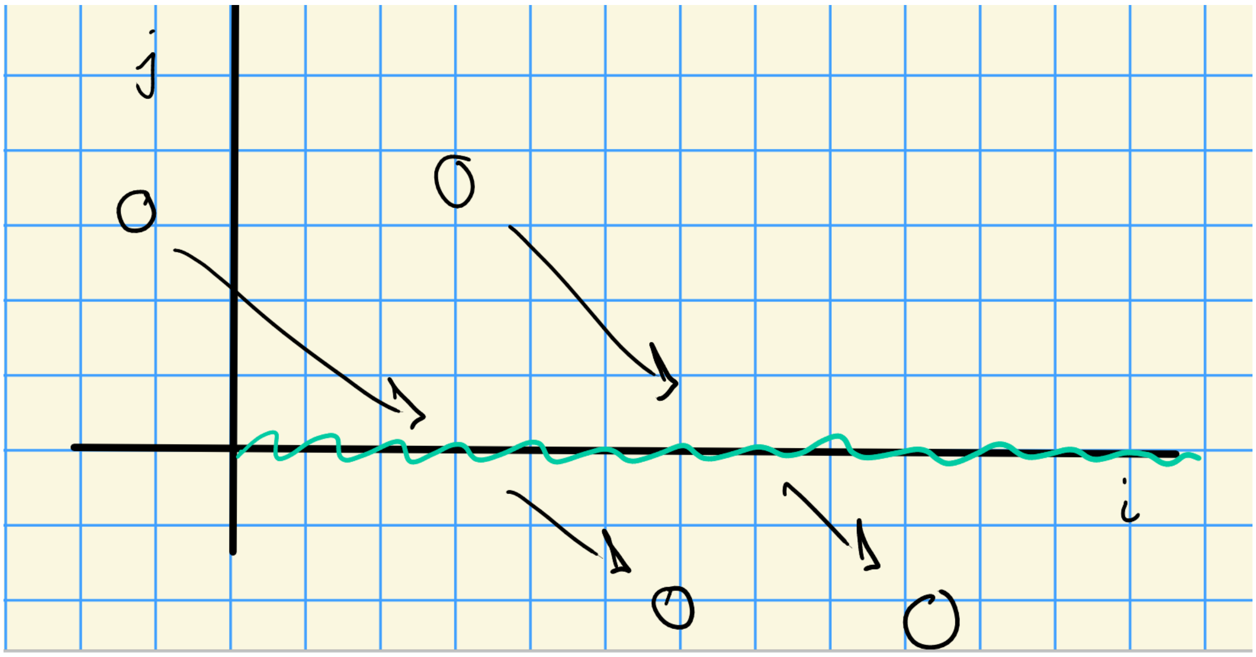

Figure 4: Image

Thus

$$E_2^{i,0} = \operatorname{Ext}_B^i(k, M) = H^i(B, M).$$

Corollary 1.5 ***(?).***
Let $G \supseteq P \supseteq B$ where P is a *parabolic* subalgebra and let M be a rational G-module. Then $H^n(G, M) = H^n(P, M) = H^n(B, M)$ for all $n \geq 0$.

Example 1.3.
Fix a Dynkin diagram and take a subset $J \subseteq \Delta$.

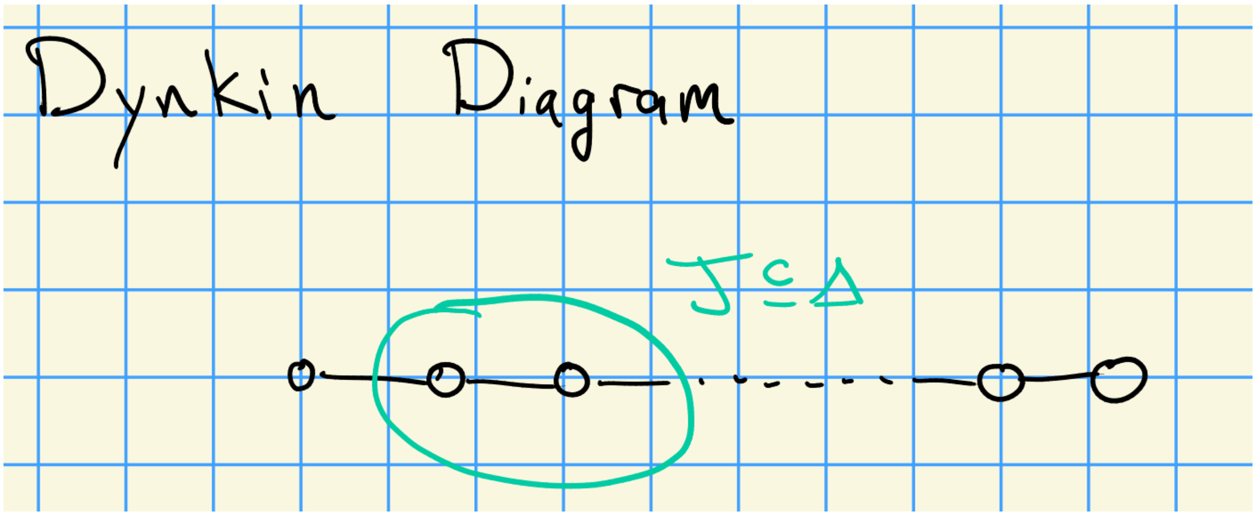

Figure 5: i

Then $L_j \rtimes U_j = P_J = P$, and we have a decomposition like

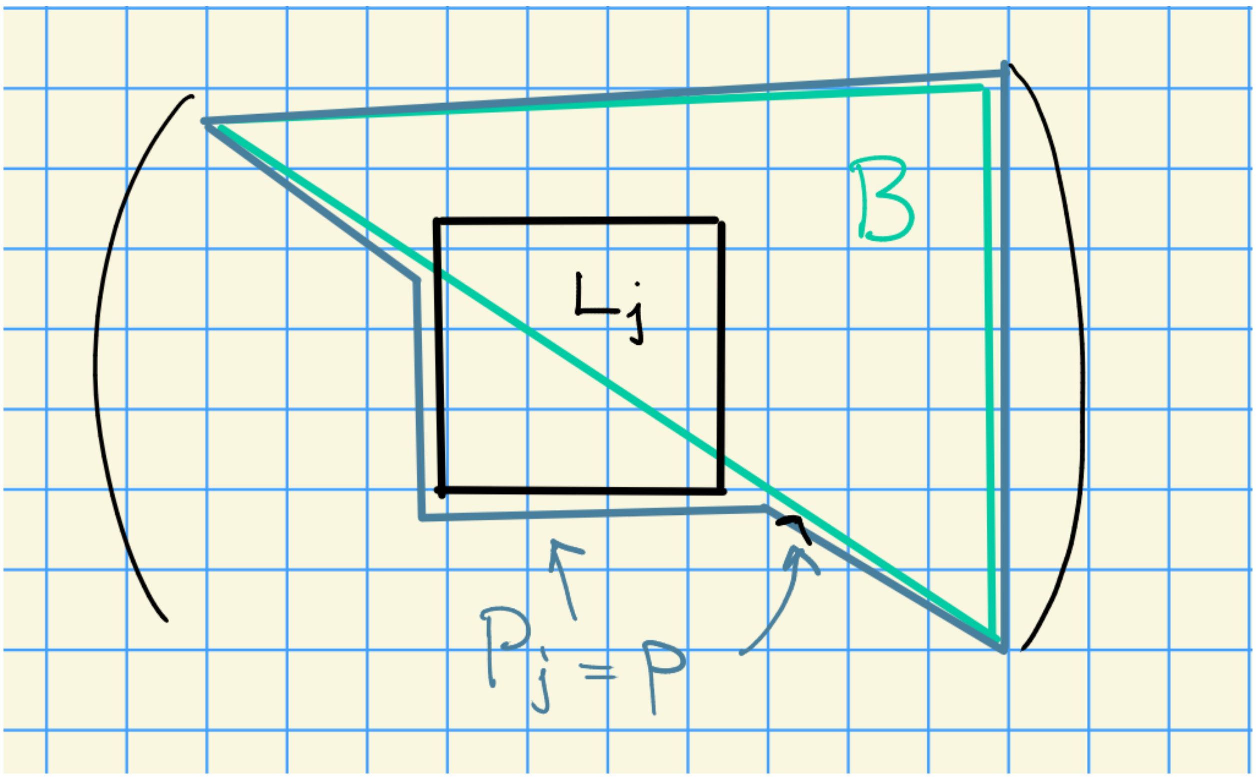

Figure 6: Image

Proposition 1.6 ***(?).***
Let $M \in \mathrm{Mod}(P)$ with $P \supseteq B$.

a. If $\dim M < \infty$ then $\dim H^n(P, M) < \infty$ for all n.
b. If $H^j(P, M) \neq 0$ then there exists λ a weight of M with $-\lambda \in \mathbb{N}\Phi^+$ and $\mathrm{ht}(-\lambda) \geq j$.