Math 8100 Assignment 7 & 8

Due date: Wednesday 3rd of November 2010

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove the following properties of $L^\infty = L^\infty(X, \mathcal{M}, \mu)$.
   (a) If f and g are measurable functions on X, then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.
   (b) $\|\cdot\|_\infty$ is a norm on L^∞.
   (c) L^∞ is a Banach space.
   (d) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.
   (e) The simple functions are dense in L^∞.

2. Let $a = \{a_j\}_{j=-\infty}^{\infty}$ be a sequence of complex numbers, and let

$$\|a\|_p = \Big(\sum_{j=-\infty}^{\infty} |a_j|^p\Big)^{1/p} \quad \text{if } 0 < p < \infty \qquad \text{and} \qquad \|a\|_\infty = \sup_j |a_j| \quad \text{if } p = \infty.$$

   Then, for $0 < p \leq \infty$ we define

$$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \,:\, \|a\|_p < \infty\}.$$

   Prove that if $0 < p < q \leq \infty$, then $\ell^p(\mathbb{Z}) \subseteq \ell^q(\mathbb{Z})$ and $\|a\|_q \leq \|a\|_p$. [*Hint: Consider $q = \infty$ first.*]

3. Let f and g be two non-negative Lebesgue measurable functions on $[0, \infty)$. Suppose that

$$A := \int_0^\infty f(y)\, y^{-1/2} dy < \infty \qquad \text{and} \qquad B := \Big(\int_0^\infty |g(y)|^2 dy\Big)^{1/2} < \infty$$

   Prove that

$$\int_0^\infty \Big(\int_0^x f(y)\, dy\Big) \frac{g(x)}{x}\, dx \leq AB$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < q < \infty$. Prove that if $L^q(X) \subseteq L^p(X)$, then X does not contain sets of arbitrarily large finite measure.
   Note that in the case of Lebesgue measure this conclusion is equivalent to $m(X) < \infty$.

5. Suppose that $0 < p_0 < p_1 \leq \infty$. Find examples of functions f on $(0, \infty)$, such that $f \in L^p$ iff
   (a) $p_0 < p < p_1$
   (b) $p_0 \leq p \leq p_1$
   (c) $p = p_0$ [*Hint: Consider functions of the form $f(x) = x^{-a}|\log x|^b$*]

6. (a) Let

$$F(x) = \Big(\frac{\sin \pi x}{\pi x}\Big)^2 \quad \text{and} \quad G(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

      i. Show that $\widehat{G}(\xi) = F(\xi)$.
      [*Hint: It may help to write $\widehat{G}(\xi) = H(\xi) + H(-\xi)$ where $H(\xi) = e^{2\pi i\xi}\int_0^1 y e^{-2\pi i y\xi} dy$*]
      ii. What is the Fourier transform of the function F? Be sure to explain your answer.

   (b) Give an example (no proof required) of a function $g \notin L^1(\mathbb{R})$, but yet is the Fourier transform of an L^1 function.

7. Show that for each $\varepsilon > 0$ the function $F(\xi) = (1 + |\xi|^2)^{-\varepsilon}$ is the Fourier transform of an $L^1(\mathbb{R}^n)$ function.

[Hint: With $K_\delta(x) = \delta^{-n/2}e^{-\pi|x|^2/\delta}$ consider $f(x) = \int_0^\infty K_\delta(x)e^{-\pi\delta}\delta^{\varepsilon-1}\,d\delta$. Use Fubini/Tonelli to prove that $f \in L^1(\mathbb{R}^n)$, and

$$\widehat{f}(\xi) = \int_0^\infty e^{-\pi\delta|\xi|^2}e^{-\pi\delta}\delta^{\varepsilon-1}\,d\delta.$$

Show that $\widehat{f}(\xi) = \pi^{-\varepsilon}\Gamma(\varepsilon)F(\xi)$, where $\Gamma(s)$ is the gamma function defined by $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$.]

8. (Hilbert's Inequality) Let

$$Tf(x) = \int_0^\infty \frac{f(y)}{x+y}\,dy$$

(a) Show that Tf satisfies the norm inequality

$$\left(\int_0^\infty |Tf(x)|^p dx\right)^{1/p} \le C_p \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}$$

for $1 < p < \infty$, with

$$C_p = \int_0^\infty \frac{1}{x^{1/p}(x+1)}\,dx.$$

(b) Show, without using complex analysis, that

$$C_p \le \frac{p^2}{p-1}.$$

Remark: It is a standard exercise in contour integration to show that in fact $C_p = \pi/\sin(\pi/p)$.

Challenge Problem VII & VIII

Hand these in to me at some point in the semester

VII. (**A Generalized Hölder's Inequality**) Suppose that $1 \le p_j \le \infty$ and $\sum_{j=1}^n 1/p_j = 1/r \le 1$. If $f_j \in L^{p_j}$ for $j = 1, \dots, n$, then $\prod_{j=1}^n f_j \in L^r$ and $\|\prod_{j=1}^n f_j\|_r \le \prod_{j=1}^n \|f_j\|_{p_j}$.

VIII. (**Young's Inequality**) Suppose $1 \le p, q, r \le \infty$ with $p^{-1} + q^{-1} = r^{-1} + 1$. Prove that if $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and

$$\|f * g\|_r \le \|f\|_p\|g\|_q$$

[Hint: Use the above exercise to show that

$$|f * g(x)|^r \le \|f\|_p^{r-p}\|g\|_q^{r-q} \int |f(y)|^p|g(x-y)|^q\,dy.]$$