Homological Algebra Problem Sets

Problem Set 3

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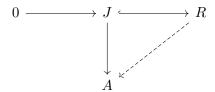*Problem* 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

> *Recall that a module is* divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative.
$\impliedby$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$0 \longrightarrow J \lhook\joinrel\longrightarrow R$$
$$\downarrow \qquad \swarrow$$
$$A$$
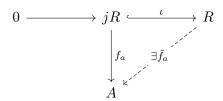
Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$f_a : J = jR \to R$$
$$x = jr \mapsto r\mathbf{a}.$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$0 \longrightarrow jR \overset{\iota}{\lhook\joinrel\longrightarrow} R$$
$$\downarrow{\scriptstyle f_a} \qquad {\scriptstyle \exists \tilde{f}_a}$$
$$A$$

We can now use the fact that

$$r\mathbf{a} = f_a(jr)$$
$$= \tilde{f}_a(\iota(jr))$$
$$= \tilde{f}_a(jr)$$
$$= jr\tilde{f}_a(1_R) \qquad\qquad \text{using } R\text{-linearity and } j, r \in R$$
$$= rj\tilde{f}_a(1_R) \qquad\qquad \text{since } R \text{ is commutative}$$
$$\implies \mathbf{a} = j\tilde{f}_a(1_R) \in jA,$$

where in the last step we have canceled an r on the left, using the fact

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Problem 1.0.2 (Calculating Ext Groups)
Calculate $\text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

Problem 1.0.3 (Weibel 2.3.2)
For $A \in \mathbf{Ab}$, define $I(A) := \bigoplus\limits_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \to I(A)$. Show that e_A is injective.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Problem 1.0.5 (Weibel 2.4.3)
If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} FM \qquad\qquad i \geq 2.$$

Show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

    a. A is a projective R-module.

    b. $\text{Hom}_R(\,\cdot\,, A)$ is an exact functor.

    c. $\text{Ext}^{i \neq 0}_R(A, B) = 0$ and for all B, i.e. A is $\text{Hom}_R(\,\cdot\,, B)$-acyclic for all B.

    d. $\text{Ext}^1_R(A, B)$ vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \to \bullet$, is not an exact functor on $\mathbf{Ab}$.