

# Chapter 9

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Important Theorems:

Important ideas:

- Compactness of  $\mathcal{L}(x, y)$ .
- $\partial^2 = 0$ .
- Using broken trajectories to compactify
- Gluing

## 1 | Background from Chapter 8

- $(M, \omega)$  with  $\omega \in \Omega^2(M)$  is a symplectic manifold with an almost complex structure  $J$ .
- $H \in C^\infty(M; \mathbb{R})$  a Hamiltonian with  $X_H$  the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in  $T_x M$ :

$$\omega_x(\cdot, X_H(x)) = (dH)_x(\cdot).$$

- Zeros of vector field  $X_H$  correspond to critical points of  $H$ :

$$X_H(x) = 0 \iff (dH)_x = 0.$$

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- Take the associated flow  $\psi^t : M \rightarrow M$ , assumed 1-periodic so  $\psi^1(x) = x$ : critical points of  $H$  are periodic trajectories.
  - $u \in C^\infty(\mathbb{R} \times S^1; M)$  is a solution to the Floer equation.
  - The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y\end{aligned}$$

$$Y \in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- $\mathcal{LM}$  is the *free loop space* of  $M$ , i.e. space of contractible loops on  $M$ , i.e.  $C^\infty(S^1; M)$  with the  $C^\infty$  topology
  - Loops in  $\mathcal{LM}$  can be viewed as maps  $S^2 \rightarrow M$ , since they're maps  $I \times S^1 \rightarrow M$  with the boundaries pinched:



Figure 1: Loops in  $\mathcal{LM}$

- Elements  $x \in \mathcal{LM}$  can be viewed as maps  $S^1 \rightarrow M$ .
- Can extend to maps from a closed disc,  $u : \mathbb{D}^2 \rightarrow M$ .
- The action functional is given by

$$\begin{aligned}\mathcal{A}_H : \mathcal{LW} &\rightarrow \mathbb{R} \\ x &\mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) \, dt\end{aligned}$$

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- Example:  $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t dt - p dq)$ .
  - Correspondence between trajectories of the gradient of  $\mathcal{A}_H$  and solutions to Floer equations.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every  $u \in C^\infty(S^2, M)$ ,

$$\int_{S^2} u^* \omega = 0 \quad \text{or equivalently} \quad \langle \omega, \pi_2 M \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every  $u \in C^\infty(S^2; M)$  there exists a symplectic trivialization of the fiber bundle  $u^*TM$ , equivalently

$$\langle c_1 TM, \pi_2 M \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- $x, y$  periodic orbits of  $H$  (nondegenerate, contractible), equivalently critical points of  $\mathcal{A}_H$ .
- Maslov index: used the fact that
  - $\text{Sp}(2n, \mathbb{R})$  retracts onto  $U(n)$ : use a polar decomposition  $S = PQ$  as a PSD times orthogonal, then homotope  $P$  to  $I$ .
  - $\pi_1 U_n = \mathbb{Z}$ : use  $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$  by the determinant, and  $\pi_1 SU(n, \mathbb{C}) = 0$ .
  - Thus every path in  $\gamma : I \rightarrow \text{Sp}(2n, \mathbb{R})$  can be assigned an integer by getting a map  $\tilde{\gamma} : I \rightarrow S^1$  and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$ , the moduli space of contractible finite-energy solutions to the Floer equation connecting  $x, y$ .
  - Showed that after perturbing  $H$  to get transversality, get a manifold of dimension  $\mu(x) - \mu(y)$ .
  - How did we do it: describe as zeros of a section of a vector bundle over  $\mathcal{P}^{1,p}(x, y)$  (Banach manifold modeled on the Sobolev spaces  $W^{1,p}$ ), apply Sard-Smale to show  $\mathcal{M}(x, y)$  is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality. Followed from showing  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .

#### Goals

- Construct Floer homology and prove the Arnold Conjecture (“Symplectic Morse Inequalities?”):

$$\# \{1\text{-Periodic trajectories of } X_H\} \geq \sum_{k \in \mathbb{Z}} HM_k(w; \mathbb{Z}/2\mathbb{Z}).$$

#### Steps

1. Define the action functional  $\mathcal{A}_H$ .
2. Construct the chain complex (graded vector space)  $CF_*$ .
3. Define  $X_H$ , which will be used to define  $\partial$  later.

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4. Count trajectories.
  5. Show finite-energy trajectories connect critical points of  $\mathcal{A}_H$ .
  6. Show compactness property for space of trajectories of finite energy.
  7. Define  $\partial$  (uses a compactness property in 9.1c)
  8. Show space of trajectories is a manifold (plus genericity, “Smale property”)
  9. Show that  $\partial^2 = 0$ .
  10. Show that  $HF_*$  doesn’t depend on  $\mathcal{A}_H$  or  $X_H$
  11. Show  $HF_* \cong HM_*$ , and compare dimensions of the vector spaces  $CM_*$  and  $CF_*$ .

## 2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$$

$$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x, y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting  $x$  to  $y$ :

$$\mathcal{L}(x, y) := \mathcal{M}(x, y)/\mathbb{R}.$$

- Use the quotient topology, define sequentially:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

- When  $|\mu(x) - \mu(y)| = 1$ , get a compact 0-manifold, so the number of trajectories

$$n(x, y) := \#\mathcal{L}(x, y)$$

is well-defined.

- $C_k(H) := \mathbb{Z}/2\mathbb{Z}[S]$  where  $S$  is the set of periodic orbits of  $X_H$  of Maslov index  $k$ .
  - Finitely many since they are nondegeneracy implies they are isolated.
- Defined a differential

$$\begin{aligned} \partial : C_k(H) &\rightarrow C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y)y \end{aligned}$$

$$\begin{aligned} n(x, y) &:= \#\{\text{Trajectories of } \text{grad } \mathcal{A}_H \text{ connecting } x, y\} \pmod{2} \\ &= \#\mathcal{L}(x, y) \pmod{2}. \end{aligned}$$

- Examined  $\partial^2$ :

$$\begin{aligned}
\partial^2 : C_k(H) &\rightarrow C_{k-2}(H) \\
x &\mapsto \partial(\partial(x)) \\
&= \partial \left( \sum_{\mu(y)=\mu(x)-1} n(x, y) y \right) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x, y) \partial(y) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x, y) \left( \sum_{\mu(z)=\mu(y)-1} n(y, z) z \right) \\
&= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x, y) n(y, z) z \\
&= \sum_{\mu(z)=\mu(y)-1} \left( \sum_{\mu(y)=\mu(x)-1} n(x, y) n(y, z) \right) z \quad (\text{finite sums, swap order}),
\end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x, y) n(y, z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When  $\mu(z) = \mu(x) - 2$ ,  $\mathcal{L}(x, z)$  is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get  $\bar{\mathcal{L}}(x, y)$ .
- We'll then have

$$\bar{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial \bar{\mathcal{L}}(x, z), \quad \partial \bar{\mathcal{L}}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z),$$

which “space-ifies” the equation we want.

- We'll show  $\partial \bar{\mathcal{L}}(x, z)$  is a 1-manifold, which must have an even number of points, and thus

$$\sum_{\mu(y)=\mu(x)-1} n(x, y) n(y, z) = \#(\partial \bar{\mathcal{L}}(x, z)) \equiv 0 \pmod{2}.$$

## 2.1 Three Important Theorems

- Shown last time: a sequence of trajectories can converge to a broken trajectory.

**Theorem 2.1 (9.1.7).**

Let  $\{u_n\}$  be a sequence in  $\mathcal{M}(x, y)$ , then there exist

- A subsequence  $\{u_{n_j}\}$
- Critical points  $\{x_0, x_1, \dots, x_{\ell+1}\}$  with  $x_0 = x$  and  $x_{\ell+1} = y$
- Sequences  $\{s_n^1\}, \{s_n^2\}, \dots, \{s_n^\ell\}$ .
- Elements  $u^k \in \mathcal{M}(x_k, x_{k+1})$  such that for every  $0 \leq k \leq \ell$ ,

$$u_n \cdot s_n^k \xrightarrow{n \rightarrow \infty} u^k.$$

**Corollary 2.2 (Compactness).**

$\bar{\mathcal{L}}(x, z)$  is compact.

**Definition 2.2.1** (Regular Pair).

For an almost complex structure  $J$  and a Hamiltonian  $H$ , the pair  $(H, J)$  is **regular** if the Floer map  $\mathcal{F}$  is transverse to the zero section in the following vector bundle:

$$\begin{array}{ccc} E_u := \{\text{Vector fields tangent to } M \text{ along } u\} & \longrightarrow & C^\infty(\mathbb{R} \times S^1; TM) \\ & \searrow \mathcal{F} \quad \downarrow \quad \nearrow \mathbf{0} & \\ & C^\infty(\mathbb{R} \times S^1; M) & \end{array}$$

**Theorem 2.3 (9.2.1).**

Let  $(H, J)$  be a regular pair with  $H$  nondegenerate and  $x, z$  be two periodic trajectories of  $H$  such that

$$\mu(x) = \mu(z) + 2.$$

Then  $\bar{\mathcal{L}}(x, z)$  is a compact 1-manifold with boundary with

$$\partial \bar{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x, z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \quad \text{where} \quad \mathcal{I}(x, z) = \{y \mid \mu(x) < \mu(y) < \mu(z)\}.$$

Note: possibly a typo in the book?

**Remark 1.**

- As a corollary,  $\partial^2 = 0$ .
- Most of chapter 9 is spent proving this theorem.

**Remark 2.**

Some notation:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M}(x, z) \\ & & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

Hats will generally denote maps induced on quotient.

**Theorem 2.4(9.2.3: Gluing).**

Let  $x, y, z$  be three critical points of  $\mathcal{A}_H$  with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a  $\rho_0 > 0$  and a differentiable map

$$\psi : [\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$$

such that  $\hat{\psi}$ , the induced map on the quotient

$$\begin{array}{ccc} [\rho_0, \infty) & \xrightarrow{\psi} & \mathcal{M}(x, z) \\ & \searrow \hat{\psi} & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

is an embedding that satisfies

$$\hat{\psi}(\rho) \xrightarrow{\rho \rightarrow \infty} (\hat{u}, \hat{v}) \in \bar{\mathcal{L}}(x, z).$$

2. For any sequence  $\{\ell_n\} \subseteq \mathcal{L}(x, z)$ ,

$$\ell_n \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v}) \implies \ell_n \in \text{im}(\hat{\psi}) \text{ for } n \gg 0.$$

## 2.2 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$ , so we quotient to define  $\mathcal{L}(x, y) := \mathcal{M}_{x,y}/\mathbb{R}$  with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

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**Proposition 2.5(?)**

$\mathcal{L}(x, y)$  is Hausdorff.

- Want to show  $\mathcal{L}(x, y)$  is a compact 0-dimensional manifold.
- Have a differential

$$\partial : C_k(H) \longrightarrow C_{k-1}(H)$$

$$\partial(x) = \sum_{\text{Ind}(y)=k-1} n(x, y)y.$$

with  $n(x, y)$  the number (mod 2) of trajectories of grad  $\mathcal{A}_H$  connecting  $x, y$ , i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

$$M := \overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

and show that  $M$  is compact with  $\partial M$  equal to the last union.

- Last time: closure of space of trajectories connecting  $x, y$  contains “broken” trajectories.
- Last time: toward proving that  $M$  is compact

## 3 | 9.2

- Wanted to compactify  $\mathcal{L}(x, y)$ , needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

**Theorem 3.1(9.2.1).**

Let  $(H, J)$  be a regular pair with  $H$  nondegenerate.

Let  $x, z$  be two periodic trajectories of  $H$  such that  $\mu(x) = \mu(z) + 2$ .

Then  $\overline{\mathcal{L}}(x, y)$  is a compact 1-manifold with boundary satisfying

$$\partial \overline{\mathcal{L}}(x, y) = \bigcup_{\mu(x) < \mu(y) < \mu(z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

As a corollary,  $\partial^2 = 0$ .

- Know  $\overline{\mathcal{L}}(x, y)$  is compact and  $\mathcal{L}(x, y)$  is a 1-manifold
- Now suffices to study in a neighborhood of boundary points (“gluing theorem”)

### 3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function  $w_p$  which interpolates between  $u$  and  $v$  (not exactly a solution itself, but will be approximated by one later).



2. Constructing  $\psi$  a “true solution” from  $w_p$  using the Newton-Picard method. We’ll have

$$\psi(p) = \exp_{w_p}(\gamma(p)) \quad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p} \mathcal{P}(x, z).$$

where  $\mathcal{P} = ?$ .

3. Get a lift  $\hat{\psi} = \pi \circ \psi$  where  $\pi = ?$  satisfying

- $\hat{\psi}(p) \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v})$
- $\hat{\varphi}$  is an embedding
- $\hat{\psi}$  is unique in the following sense (the last point)

**Theorem 3.2(9.2.3 (Gluing Theorem)).**

Let  $x, y, z$  be critical points of the action functional  $\mathcal{A}_H$  such that  $\mu(x) = \mu(y) + 1 = \mu(z) + 2$ . Let  $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$  be trajectories, inducing  $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$ .

- There exist a differentiable map  $\psi : (\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$  for some  $\rho > 0$  such that
- $\pi \circ \psi : (\rho_0, \infty) \rightarrow \mathcal{L}(x, z)$  is an embedding
- $\hat{\psi} \xrightarrow{\rho \rightarrow \infty} (\bar{u}, \bar{v}) \in \overline{\mathcal{L}(x, z)}$ .
- If  $\ell_n \in \mathcal{L}(x, z)$  with  $\ell_n \xrightarrow{n \rightarrow \infty} (\bar{u}, \bar{v})$ , then for  $n \gg 1$  we have  $\ell \in \mathfrak{F}(\hat{\psi})$ .

## 4 | 9.3: Pre-gluing

- Choose a bump function  $\beta$  on  $\{0\}^c \subset \mathbb{R} \rightarrow [0, 1]$  which is 1 on  $|x| \geq 1$  and 0 on  $|x| < \varepsilon$
- Split into positive and negative parts  $\beta^\pm$ :

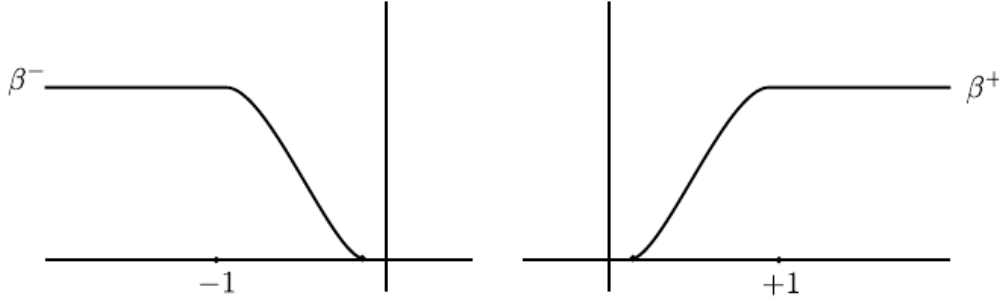


Figure 2: Figure 9.3

- Define the interpolation  $w_\rho$  from  $u$  to  $v$  in the following way:

$$w_\rho(s, t) = \begin{cases} u(s + \rho, t) & \text{if } s \leq -1 \\ \exp_{y(t)} \left( \beta^-(s) \exp_{y(t)}^{-1}(u(s + \rho, t)) + \beta^+(s) \exp_{y(t)}^{-1}(v(s - \rho, t)) \right) & \text{if } s \in [-1, 1] \\ v(s - \rho, t) & \text{if } s \geq 1 \end{cases}$$

- 
- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\}.$$

## 5 | 9.4: Construction of $\psi$ .

- Have constructed  $w_\rho \in C^\infty_\times(x, z)C^\infty(x, z)$  for every  $\rho \geq \rho_0$ , since there is exponential decay.
- Yields  $\psi_\rho \in \mathcal{M}(x, z)$  a true solution (to be defined).
- Need to check that  $\mathcal{F}(\psi_\rho) = 0$  where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- $\psi_\rho$  already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining  $\mathcal{F}_\rho$ .

$$\begin{aligned} W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n}) &\xrightarrow{\mathcal{F}_\rho} L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \\ (y_1, \dots, y_{2n}) &\longmapsto \left[ \left( \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } H_t \right) \left( \exp_{w_\rho} \sum y_i Z_i^\rho \right) \right]_{Z_i} \end{aligned}$$

where  $\mathcal{F}_\rho := \mathcal{F} \circ \exp_{w_\rho}$  written in the bases  $Z_i$ . sd - Newton-Picard method, general idea

- Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(\mathbf{x}_0)}.$$

- Allows finding zeros of  $f$  given an approximate zero  $x_0$ .
- Linearize  $\mathcal{F}_\rho$ .