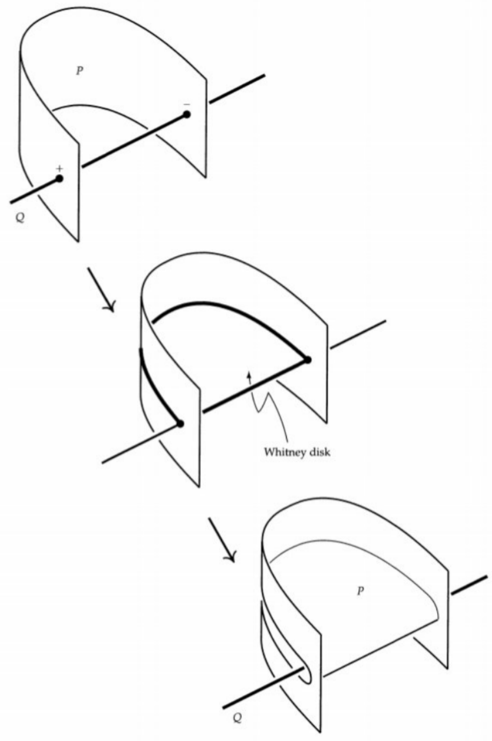


1.26. The Whitney trick

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2021-02-19

Table of Contents

Contents

1 | Tuesday, January 12

1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

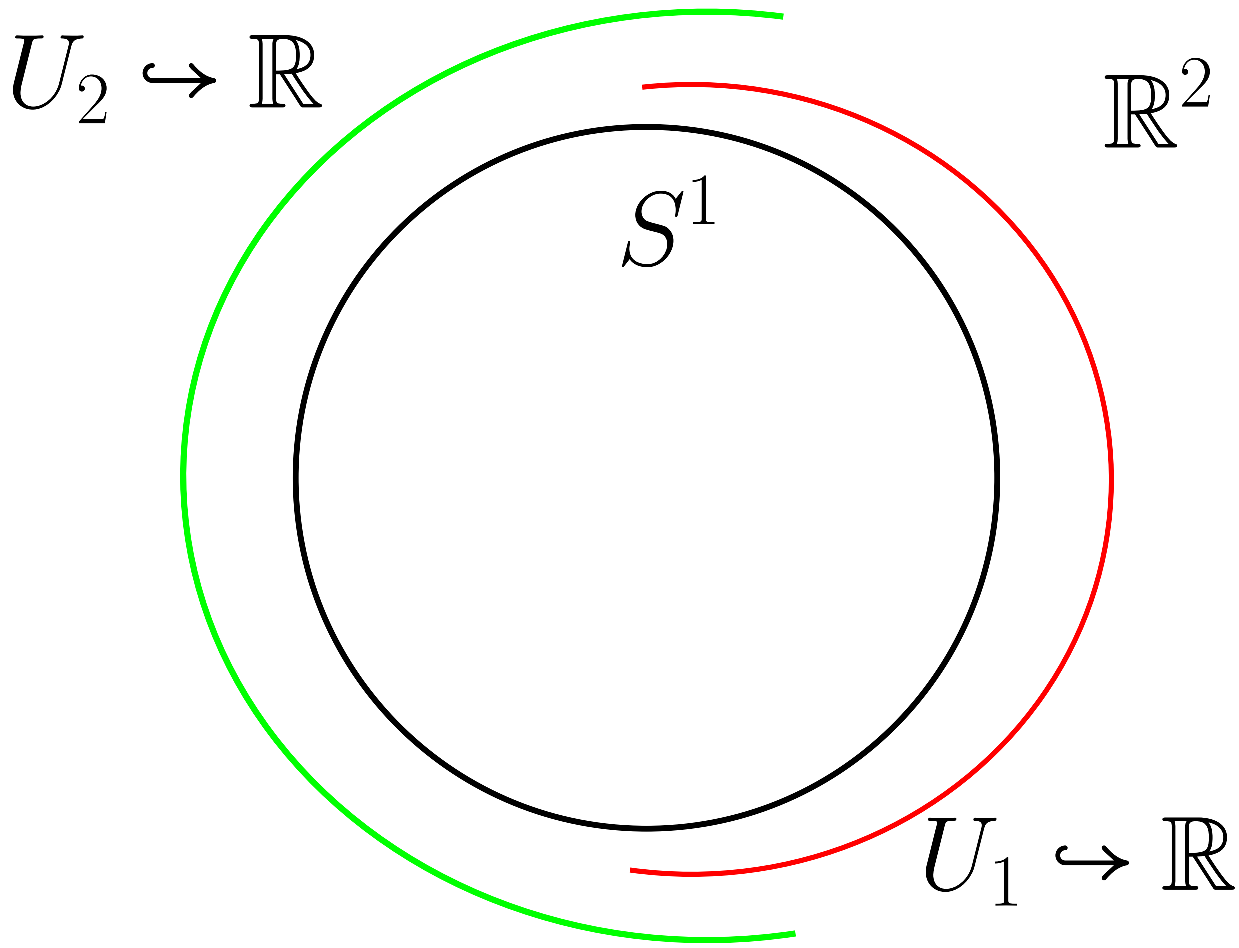
- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
  - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic R^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

  A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

> **Definition 1.2.1** (Topological Manifold)
> Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to $\mathbb{R}^n$ with a countable topological base, so we have charts $\varphi_u : U \to \mathbb{R}^n$ which are homeomorphisms from open sets covering X.

Example 1.2.2 ***(The circle):*** S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} \coloneqq \varphi_V \to \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on $\mathbb{CP}^N$.

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6
Consider $\mathbb{R}^n$ as a topological manifold: are any two smooth structures on $\mathbb{R}^n$ diffeomorphic?

Remark 1.2.7: Fix a copy of $\mathbb{R}$ and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\to \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

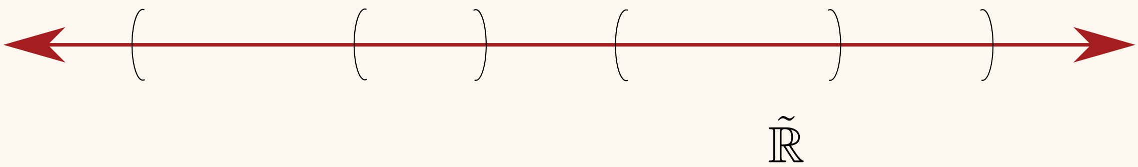
This is also a smooth structure on X, since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

$$\begin{aligned} X &\to X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

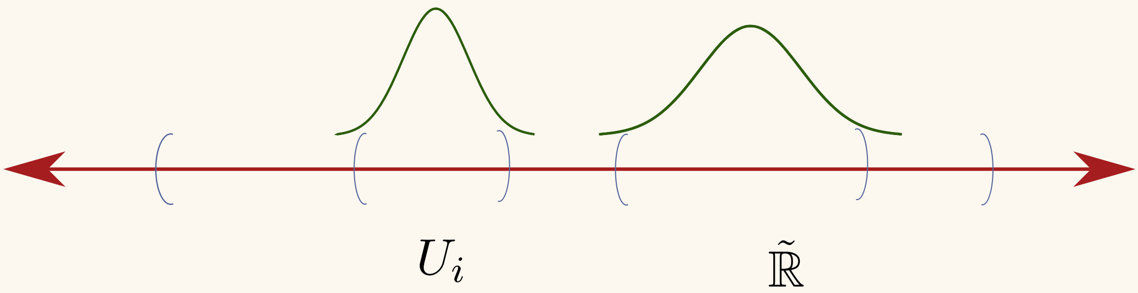
Claim: $\mathbb{R}$ admits a unique smooth structure.

Proof (sketch).
Let $\tilde{\mathbb{R}}$ be some exotic $\mathbb{R}$, i.e. a smooth manifold homeomorphic to $\mathbb{R}$. Cover this by coordinate charts to the standard $\mathbb{R}$:



Fact
There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \to \mathbb{R}$ with $f_i \geq 0$ and $\operatorname{supp} f \subseteq U_i$ such that $\sum f_i = 1$ (*i.e., bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\to \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan

To understand smooth structures on X, we should try to solve differential equations on X.

■

Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2.

Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.[1]

2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.

3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.

4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $\operatorname{ks}(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure.

[1] Note that this doesn't start at C^0, so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if $\mathrm{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.

6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, $\mathbb{R}^n$ for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!

7. (Taubes) $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures.

8. A compact oriented smooth surface Σ, the space of complex-analytic structures is a complex orbifold [2] of dimension $3g - 2$ where g is the genus of Σ, up to biholomorphism (i.e. *moduli*).

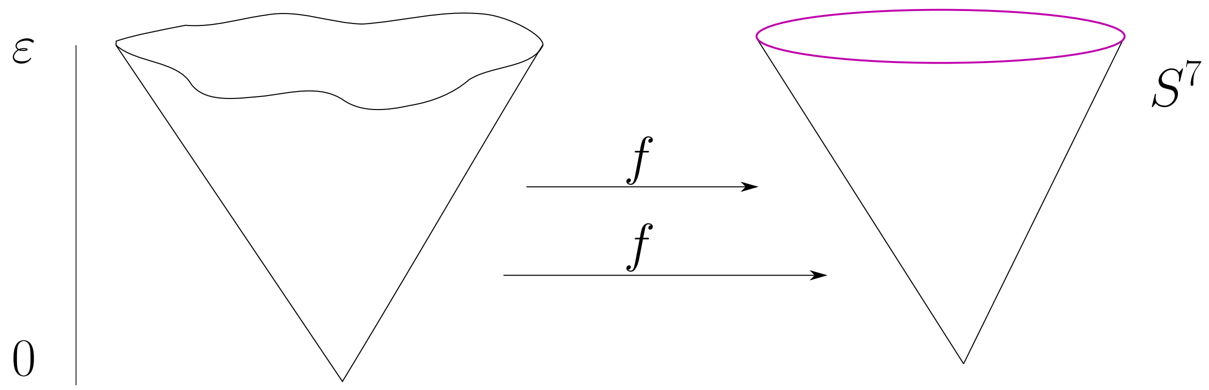
Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group.

2 | Friday, January 15

Remark 2.0.1: Let

$$V \coloneqq \left\{ a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0 \right\} \subseteq \mathbb{C}^5$$
$$S_\varepsilon \coloneqq \left\{ |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 \right\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \cdots, 28$ yields the 28 smooth structures on S^7. Note that V_k is the cone over $V_k \cap S_\varepsilon$.

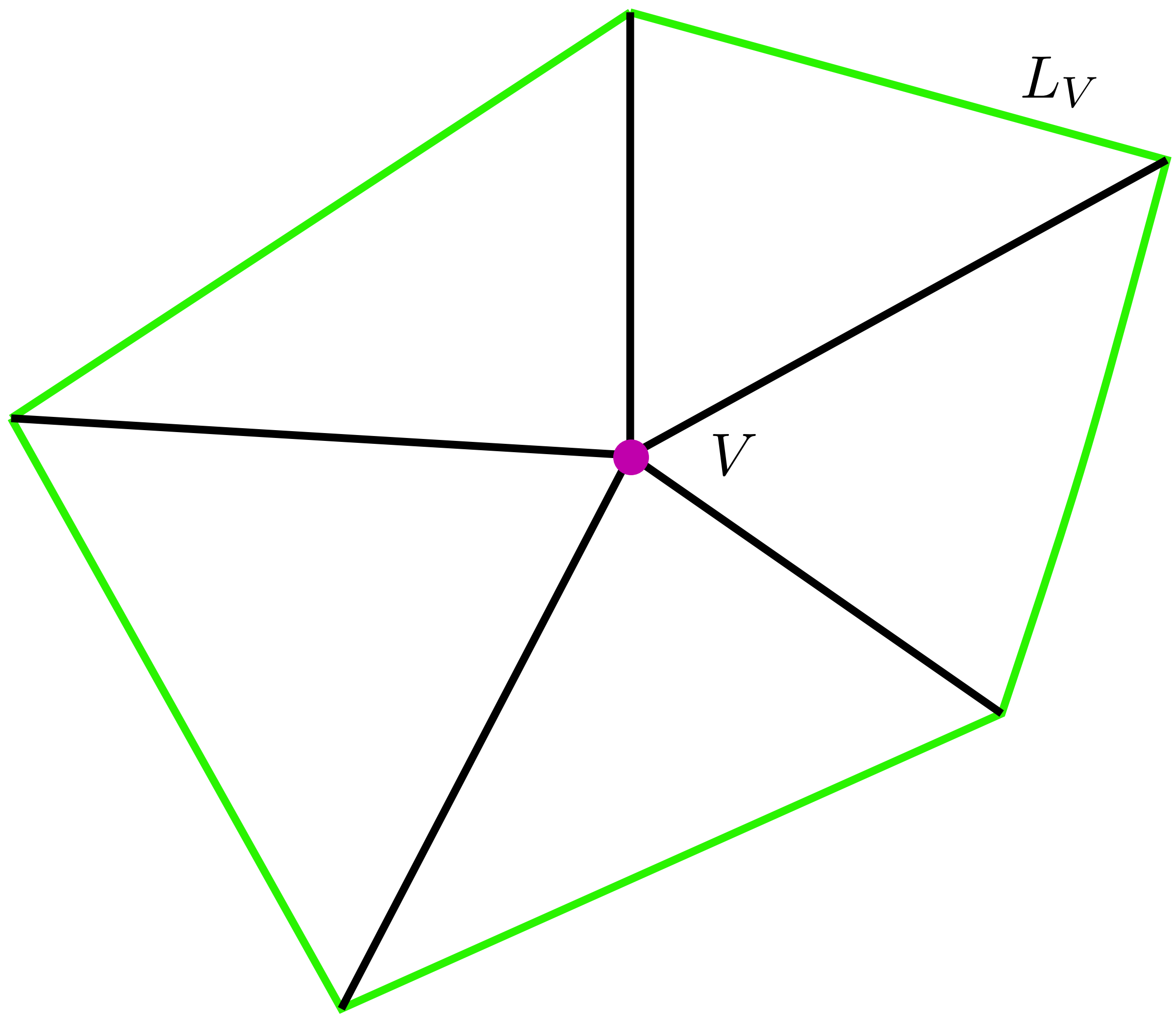


? Admits a smooth structure, and $\overline{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure.

Question 2.0.2
Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

[2] Locally admits a chart to $\mathbb{C}^n/\Gamma$ for Γ a finite group.

Answer 2.0.3
No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V:



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a "PL triangulation" where the links are spheres.

Remark 2.0.4: What's special in dimension 4? Recall the **Kirby-Siebenmann** invariant $\mathrm{ks}(x) \in H^4(X; \mathbb{Z}_2)$ for X a topological manifold where $\mathrm{ks}(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)
Let X be a topological 4-manifold, then

$$\mathrm{ks}(X) \coloneqq \mathrm{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X;\mathbb{Z}_2) = H^4(X \times \mathbb{R};\mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\operatorname{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} \end{cases}.$$

Remark 2.0.7: $\operatorname{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X;\mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha,\ \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 ***(Freedman).***
If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\operatorname{ks}(X) = \operatorname{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha,\ \varphi\beta \rangle = \langle \alpha,\ \beta \rangle$.
Conversely, every **unimodular** bilinear form appears as $H^2(X;\mathbb{Z})$ for some X, i.e. the pairing induces a map

$$\begin{aligned} H^2(X;\mathbb{Z}) &\to H^2(X;\mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha,\ \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for "being an element of the orthogonal group", where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X;\mathbb{Z}) \xrightarrow{PD} H_2(X;\mathbb{Z})$ from Poincaré , where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1,\ \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)
The **signature** of a topological 4- manifold is the signature of Q_X, where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X;\mathbb{R})$ and for some a, b

$$(H^2(X;\mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of $+1$s appearing in the matrix and b is the number of -1s. This is $\mathbb{R}^{ab}$ where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a+1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a\times a} \oplus -I_{b\times b}.$$

Then the signature is $a-b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 ***(Rokhlin's Theorem).***
Suppose $\langle \alpha, \ \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X;\mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\operatorname{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 ***(Donaldson).***
Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X;\mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

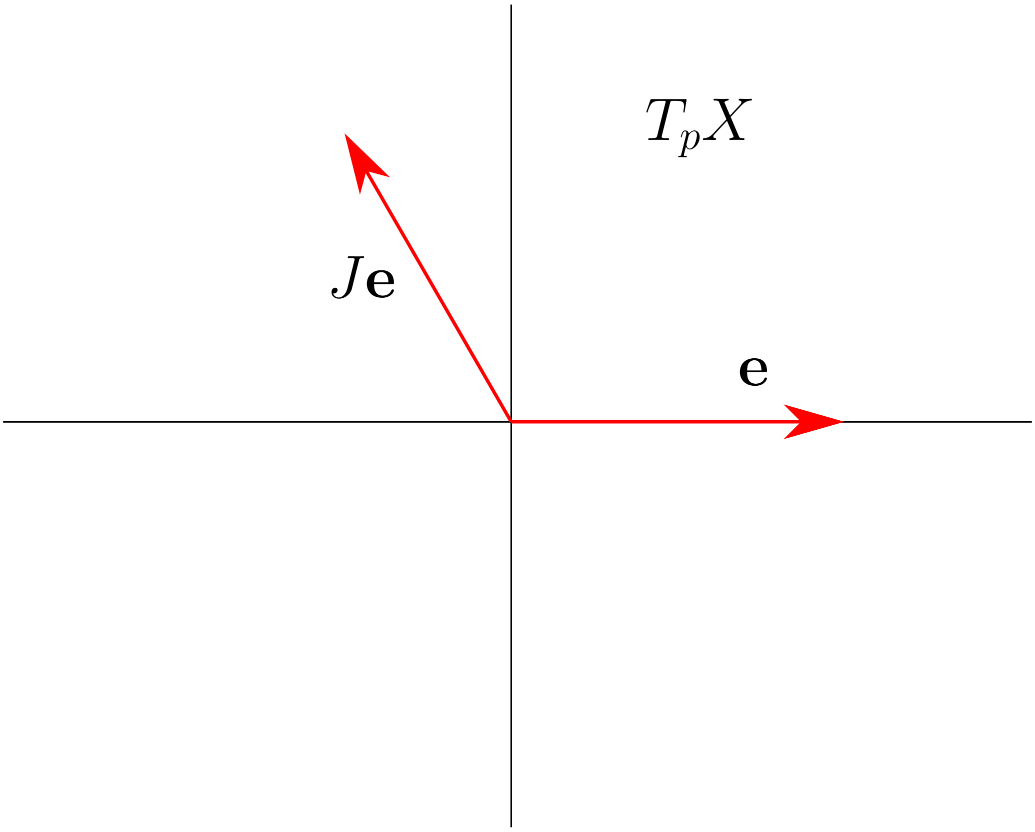
Remark 3.1.1: Last time we showed $\mathbb{R}^1$ had a unique smooth structure, so now we'll do this for $\mathbb{R}^2$. The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)
A **Riemannian metric** $g \in \mathrm{Sym}^2 T^*X$ for X a smooth manifold is a metric on every T_pX given by

$$g_p : T_pX \times T_pX \to \mathbb{R}$$
$$g(v,v) \geq 0, g(v,v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)
An **almost complex structure** is a $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{id}$.

Remark 3.1.4: Let $e \in T_pX$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of T_pX.



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

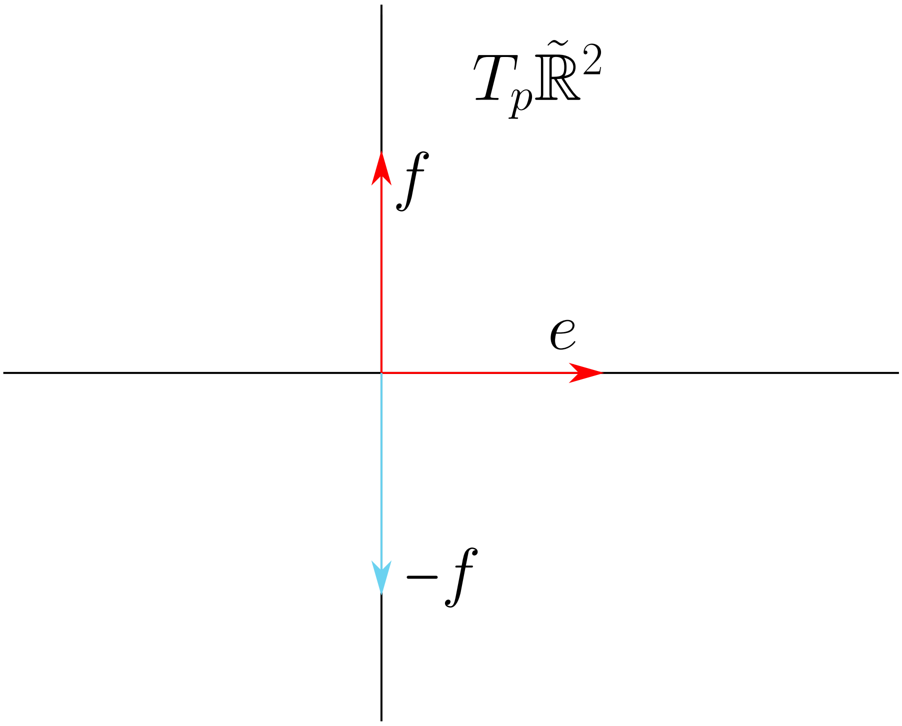
Let $\tilde{\mathbb{R}}^2$ be an exotic $\mathbb{R}^2$.

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g \coloneqq \sum f_I g_i$ with g_i metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e \coloneqq f \in T_p\tilde{\mathbb{R}}^2$ such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$.
- $\{e, f\}$ is an oriented basis of $T_p\tilde{\mathbb{R}}^2$

This is because after choosing e, there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5 *(?).*
Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \to \mathbb{C}$ such that J is multiplication by i.

So $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to $\mathbb{D}$ or $\mathbb{C}$, both of which are diffeomorphic to $\mathbb{R}^2$.

See the Newlander-Nirenberg theorem, a result in complex geometry.

4 Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

4.1 Sheaves

Definition 4.1.1 (Presheaves and Sheaves)
Recall that if X is a topological space, a **presheaf** of abelian groups $\mathcal{F}$ is an assignment $U \to \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

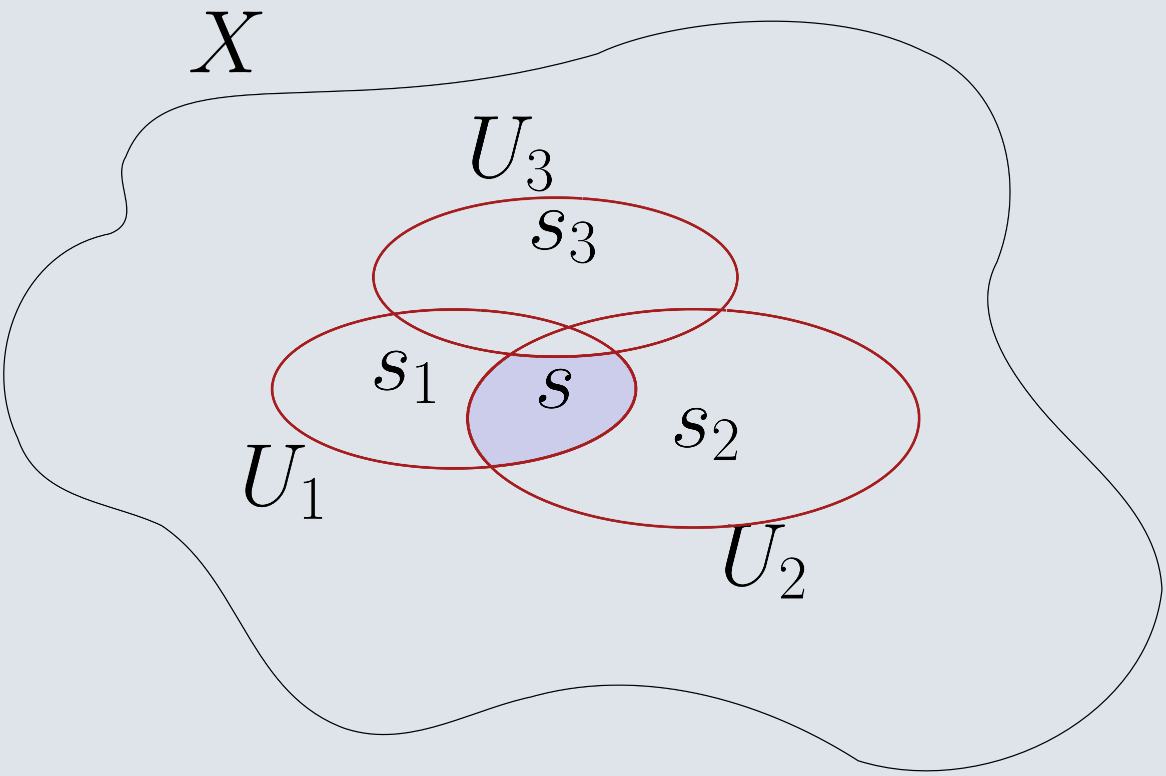
a. $\mathcal{F}(\varnothing) = 0$, the trivial abelian group.

b. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U) = \mathrm{id}_{\mathcal{F}(U)}$

c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say $\mathcal{F}$ is a **sheaf** if additionally:

d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2 *(?):* Let X be a topological manifold, then $\mathcal{F} \coloneqq C^0(\cdot, \mathbb{R})$ the set of continuous functionals form a sheaf. We have a diagram

$$\begin{tikzcd}
U \arrow[r, "\mathcal{F}"] & C^0(U;\mathbb{R}) \arrow[d, hook, dashed, "\text{restrict cts. functions}"] \\
V \arrow[u, hook] \arrow[r, "\mathcal{F}"'] & C^0(V;\mathbb{R})
\end{tikzcd}$$

Link to diagram

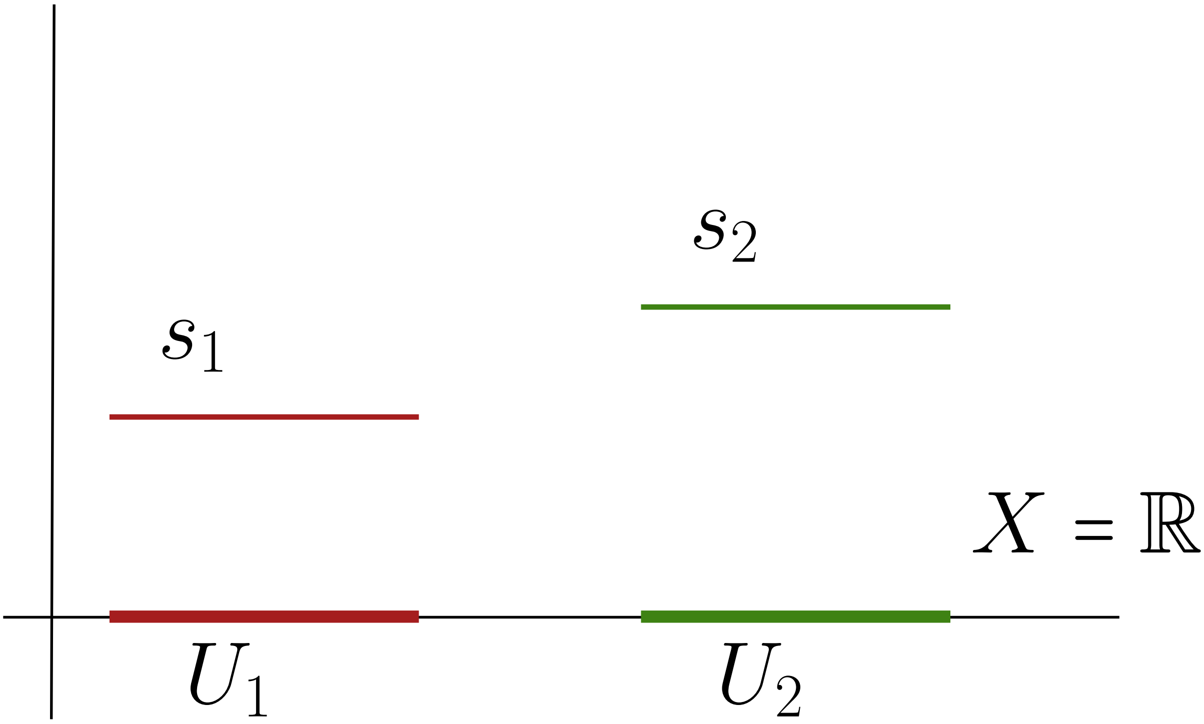
Property (d) holds because given sections $s_i \in C^0(U_i;\mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in C^0(\bigcup_i U_i;\mathbb{R})$ such that $s|_{U_i} = s_i$ for all i – continuous functions glue.

Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote $\mathcal{O}$. For example, $\mathcal{O} \coloneqq C^0(\cdot\,;\mathbb{R})$ for topological manifolds, and $\mathcal{O} \coloneqq C^\infty(\cdot\,;\mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set $\mathcal{O}$ to be the sheaf of holomorphic functions.

Example 4.1.4 ***(Locally Constant Sheaves):*** Let $A \in \mathbf{Ab}$ be an abelian group, then $\underline{A}$ is the sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \to A$. E.g. let $X \in \mathbf{Mfd_{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant $\implies$ globally constant in this case.

⚠ **Warning 4.1.5**

Note that the presheaf of constant functions doesn't satisfy (d)! Take $\mathbb{R}$ and a function with two different values on disjoint intervals:



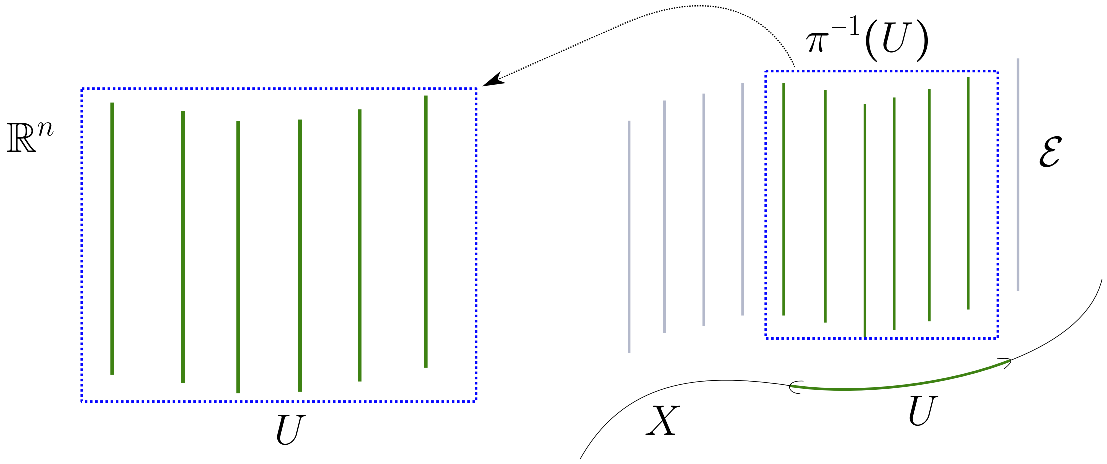
Note that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

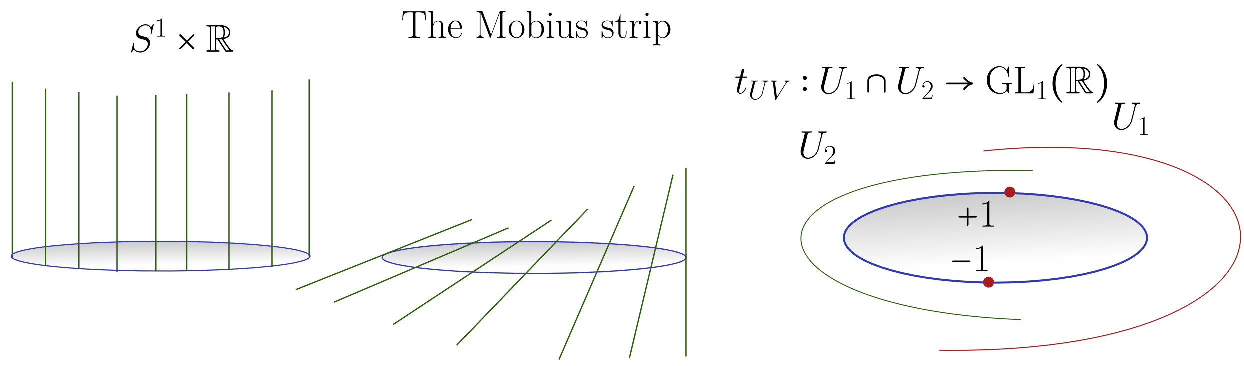
Remark 4.2.1: Let $\pi : \mathcal{E} \to X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \to \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 ***(Bundles over*** S^1***):*** There are two $\mathbb{R}^1$ bundles over S^1:



Note that the Mobius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: $\mathcal{E}$ is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \to \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \mathrm{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4 ***(?):*** The trivial line bundle gives the sheaf $\mathcal{O}$: maps $U \xrightarrow{s} U \times Y$ for $Y = \mathbb{R}, \mathbb{C}$ such that $\pi \circ s = \mathrm{id}$ are the same as maps $U \to Y$.

Definition 4.2.5 ($\mathcal{O}$-modules)
An **$\mathcal{O}$-module** is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6 ***(?):*** If $\mathcal{E}$ is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \curvearrowright s \coloneqq fs$, i.e. just multiplying functions.

Example 4.2.7 ***(Non-example):*** The locally constant sheaf $\underline{\mathbb{R}}$ is not an $\mathcal{O}$-module: there isn't natural action since the sections of $\mathcal{O}$ are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

Definition 4.2.8 (Morphisms of Sheaves)
A **morphism** of sheaves $\mathcal{F} \to \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{F}(V)
\end{tikzcd}$$

Example 4.2.9 ***(An $\mathcal{O}$-module that is not a vector bundle.):*** Let $X = \mathbb{R}$ and define the **skyscraper sheaf** at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) \coloneqq \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}.$$

The $\mathcal{O}(U)$-module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\to \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments.

Example 4.2.10 ***(of morphisms):*** Let $X = \mathbb{R} \in \mathbf{Mfd}_{\mathrm{Sm}}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x\cdot) : \mathcal{O} &\to \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$

for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.11 *(?)*
Check that $\ker \varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \to \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x\cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\operatorname{coker}(x\cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to $\mathbb{R}_0$. So $\mathcal{O} \to \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{(x\cdot)} \mathcal{O} \to \mathbb{R}_0 \to 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles.

5 | Lecture 4 (Friday, January 22)

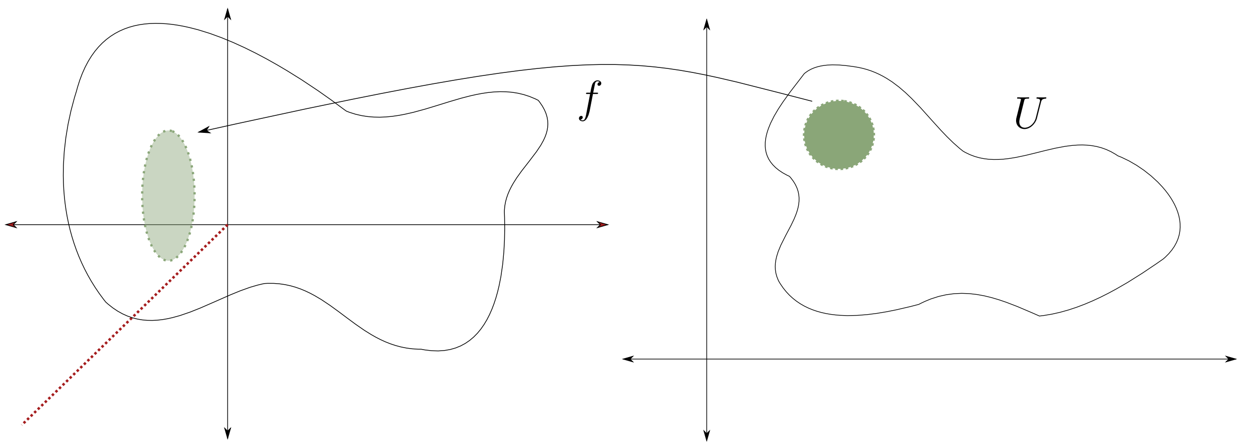
5.1 The Exponential Exact Sequence

Let $X = \mathbb{C}$ and consider $\mathcal{O}$ the sheaf of holomorphic functions and $\mathcal{O}^\times$ the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \to \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \to \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\underline{\mathbb{Z}}$, and we get an exact sequence

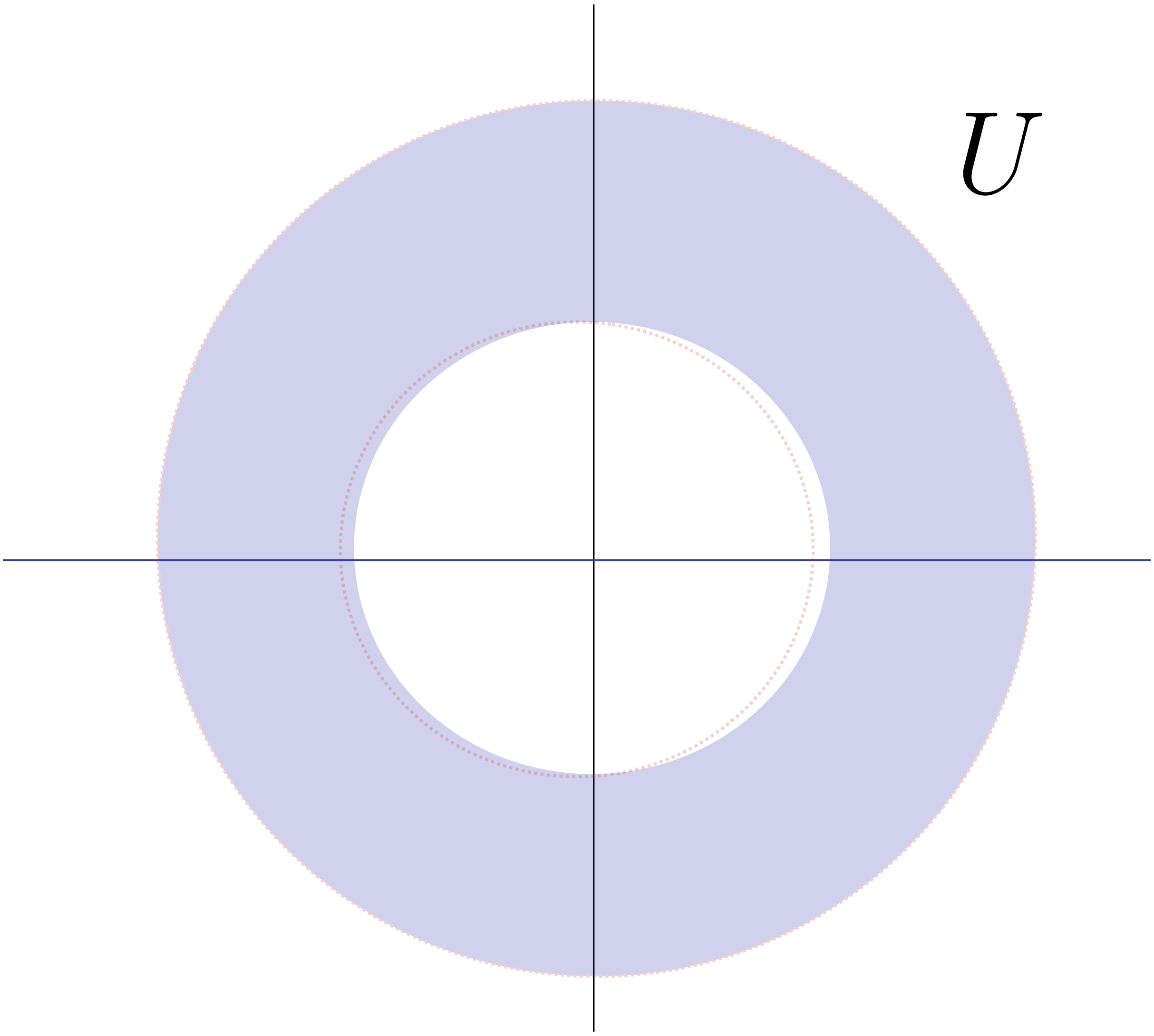
$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to \operatorname{coker}(\exp) \to 0.$$

Question 5.1.1
What is the cokernel sheaf here?

Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}^\times(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U:



Thus on any sufficiently small open set, $\operatorname{coker}(\exp) = 1$. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z, which is nontrivial.

Exercise 5.1.2 (?)
Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property $\operatorname{Hom}_{\mathrm{Presh}}(\mathcal{G}, \mathcal{F}^{\mathrm{Presh}}) = \operatorname{Hom}_{\mathrm{Sh}}(\mathcal{G}, \mathcal{F}^{\mathrm{Sh}})$ for any sheaf $\mathcal{G}$.

Definition 5.1.3 (Global Sections Sheaf)
The **global sections** sheaf of $\mathcal{F}$ on X is given by $H^0(X; \mathcal{F}) = \mathcal{F}(X)$.

Example 5.1.4 ***(?):***

- $C^{\infty}(X) = H^0(X, C^{\infty})$ are the smooth functions on X
- $VF(X) = H^0(X; T)$ are the smooth vector fields on X for T the tangent bundle
- If X is a complex manifold then $\mathcal{O}(X) = H^0(X; \mathcal{O})$ are the globally holomorphic functions on X.
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

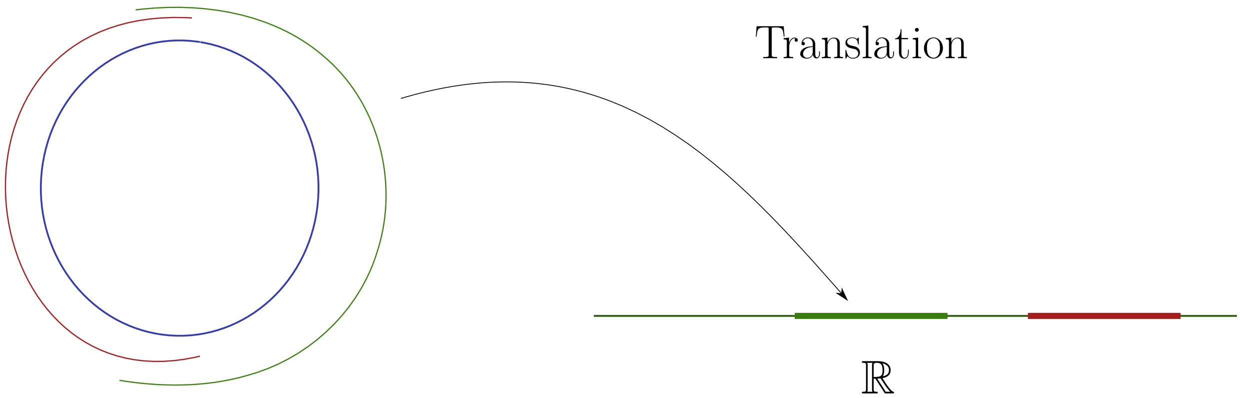
Remark 5.1.5: Given vector bundles V, W, we have constructions $V \oplus W, V \otimes W, V^{\vee}, \mathrm{Hom}(V, W) = V^{\vee} \otimes W, \mathrm{Sym}^n V, \Lambda^p V$, and so on. Some of these work directly for sheaves:

- $\mathcal{F} \oplus \mathcal{G}(U) \coloneqq \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathscr{H}\mathrm{om}(V, W)$ we only get presheaves, so we need to sheafify.

⚠ **Warning 5.1.6**
$\mathrm{Hom}(V, W)$ will denote the *global* homomorphisms $\mathscr{H}\mathrm{om}(V, W)(X)$, which is a sheaf.

Example 5.1.7 ***(?):*** Let $X^n \in \mathbf{Mfd}_{\mathrm{sm}}$ and let Ω^p be the sheaf of smooth p-forms, i.e $\Lambda^p T^{\vee}$, i.e. $\Omega^p(U)$ are the smooth p forms on U, which are locally of the form $\sum f_{i_1, \cdots, i_p}(x_1, \cdots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_p}$ where the $f_{i_1, \cdots, i_p}$ are smooth functions.

Example 5.1.8 ***(Sub-example):*** Take $X = S^1$, writing this as $\mathbb{R}/\mathbb{Z}$, we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x + c) = dx$ for c a constant:

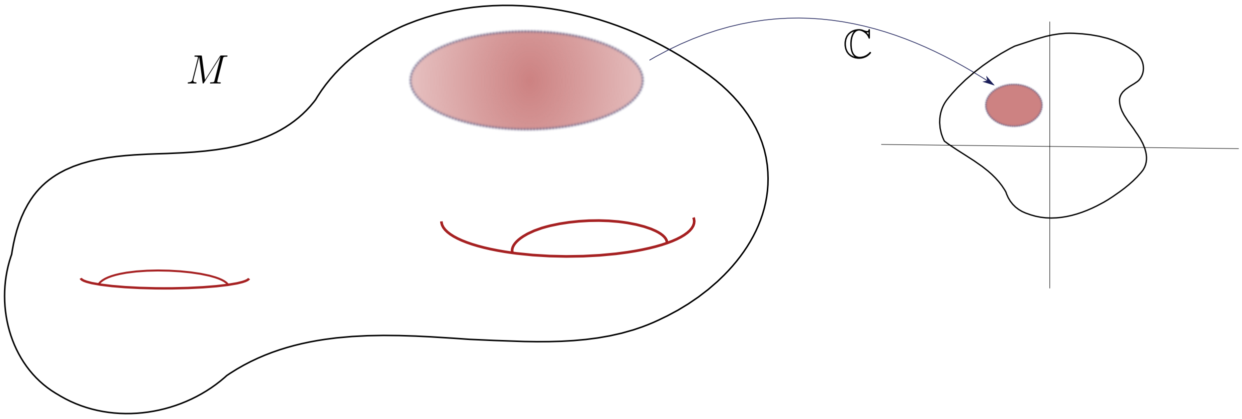


Exercise 5.1.9 ***(?)***
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.1.10: Note that there is a map $d : \Omega^p \to \Omega^{p+1}$ where $\omega \mapsto d\omega$.

⚠ **Warning 5.1.11**
d is **not** a map of $\mathcal{O}$-modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a $\mathbb{C}$-vector bundle.



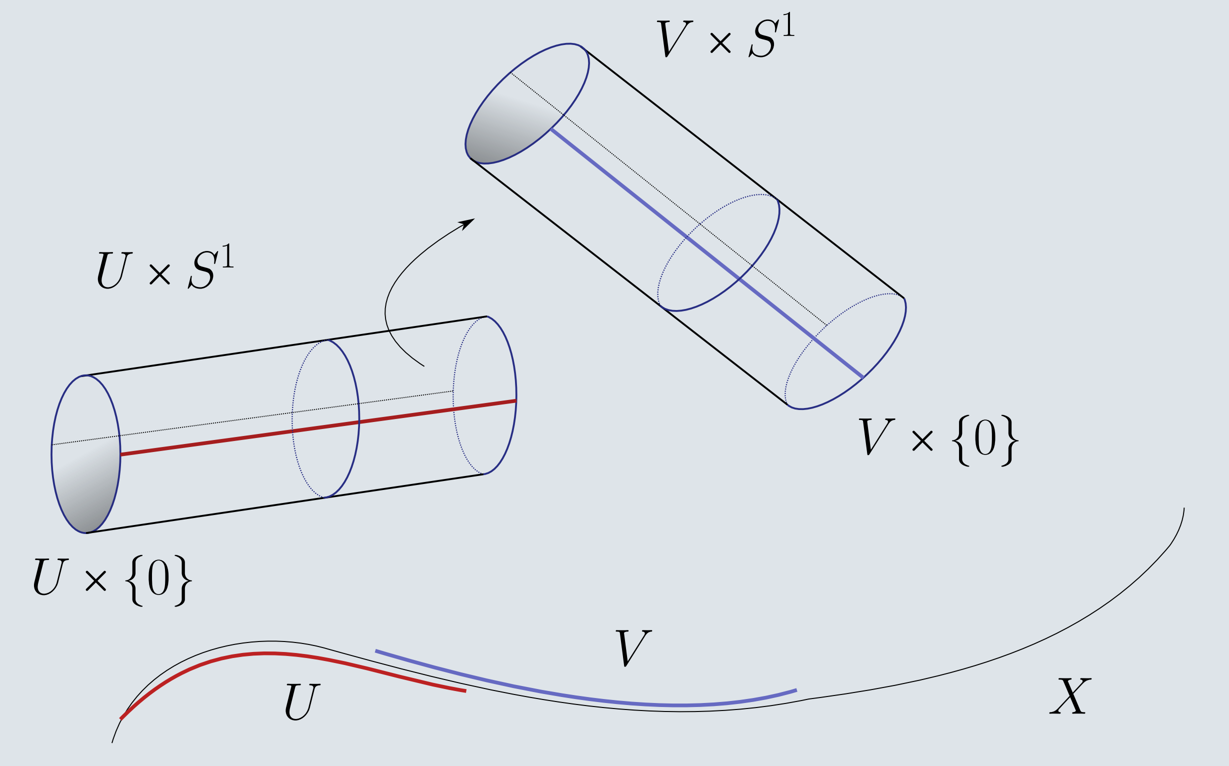
Remark 5.1.12 ***(Subtlety 1):*** Note that Ω^p for complex manifolds is $\Lambda^p T^\vee$, and so if we want to view $X \in \mathbf{Mfd}_{\mathbb{R}}$ we'll write $X_{\mathbb{R}}$. $TX_{\mathbb{R}}$ is then a real vector bundle of rank $2n$.

Remark 5.1.13 ***(Subtlety 2):*** Ω^p will denote *holomorphic* p-forms, i.e. local expressions $\sum f_I(z_1, \cdots, z_n) \Lambda dz_I$. For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$-forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.1.14: Note that $T^\vee X_{\mathbb{R} \otimes \mathbb{C}} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = f dz + g d\bar{z}$ where f, g are smooth. Then $\Omega^d X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p_{\nwarrow} \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas $\Omega^1_{\nwarrow}$ is infinite-dimensional.

6 Principal G-Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)
Let G be a (possibly disconnected) Lie group. Then a **principal G-bundle** $\pi : P \to X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \to G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \to G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \mathbf{Mfd}_{\mathrm{Sm}}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \to \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)
Define $\mathrm{Frame}_p(X) \coloneqq \{\text{bases of } T_p X\}$, and $\mathrm{Frame}X \coloneqq \bigcup_{p \in X} \mathrm{Frame}_p X$.

Remark 6.0.4: More generally, $\mathrm{Frame}\mathcal{E}$ can be defined for any vector bundle $\mathcal{E}$, so $\mathrm{Frame}X \coloneqq \mathrm{Frame}TX$. Note that $\mathrm{Frame}X$ is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle where $n \coloneqq \mathrm{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\mathrm{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 *(Important):* A principal G-bundle admits a G-action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\to P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point**: the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)
The **orthogonal frame bundle** of a vector bundle $\mathcal{E}$ equipped with a metric g is defined as $\mathrm{OFrame}_p\mathcal{E} \coloneqq \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r \coloneqq \mathrm{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \to \{x\}$ of a principal G-bundle are naturally **torsors** over G, i.e. a set with a free transitive G-action.

Definition 6.0.8 (?)
Let $\mathcal{E} \to X$ be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \to \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from $\mathcal{E}$.

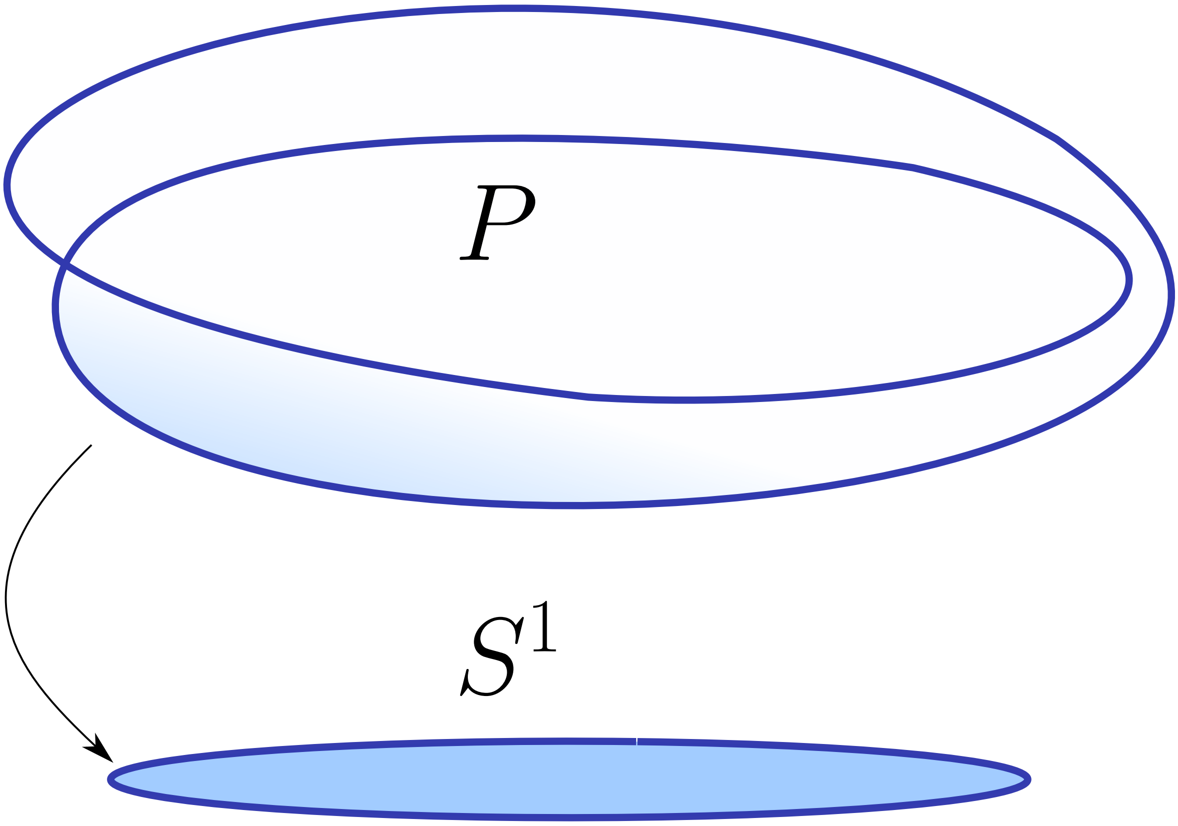
Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{C}^n \\ (z_1, \cdots, z_n) &\mapsto (\overline{z_1}, \cdots, \overline{z_n}). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)
We define the **unitary frame bundle** $\mathrm{UFrame}(\mathcal{E}) \coloneqq \bigcup_p \mathrm{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of $\mathcal{E}_p$ given by $(e_1, \cdots, e_n)$ where $h(e_i, \overline{e_j}) = \delta_{ij}$.

Remark 6.0.11: This is a principal G-bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\overline{A}^t = \mathrm{id}$.

Example 6.0.12 ***(of more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G-bundle:



Example 6.0.13 ***(more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U, i.e.

$$P \coloneqq \left\{ (x, O) \;\middle|\; x \in X,\, O \text{ is an orientation of } T_p X \right\}.$$

Note that P is an oriented manifold, $P \to X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \mathrm{Frame}X / \mathrm{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)
Let $P \to X$ be a principal G-bundle and let $G \to \mathrm{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \left\{ (p, v) \;\middle|\; p \in P,\, v \in V \right\} / \sim \qquad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 ***(?):*** Note that $\mathrm{Frame}(\mathcal{E})$ is a $\mathrm{GL}_r(\mathbb{R})$-bundle and the map $\mathrm{GL}_r(\mathbb{R}) \xrightarrow{\mathrm{id}} \mathrm{GL}(\mathbb{R}^r)$ is a representation. At every fiber, we have $G \times_G V = (p, v)/\sim$ where there is a unique representative of this equivalence class given by (e, pv). So $P \times_G V_p \to \{p\} \cong V_x$.

Exercise 6.0.16 *(?)*
Show that $\mathrm{Frame}(\mathcal{E}) \times_{\mathrm{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \to G$, and so by the equivalence relation, $\operatorname{im} t_{UV} \in \mathrm{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Them $TM \to \mathrm{Frame}(M)$ which is a principal $\mathrm{SO}(3)$-bundle. The universal cover is the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a $\mathbb{C}^2$ bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)
Let $\mathcal{E} \to X$ be a vector bundle, then a **connection** on $\mathcal{E}$ is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\to \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge w + s \otimes d\omega. \end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \to \Omega^{p+1}$. The standard example is

$$\begin{aligned} d : \mathcal{O} &\to \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

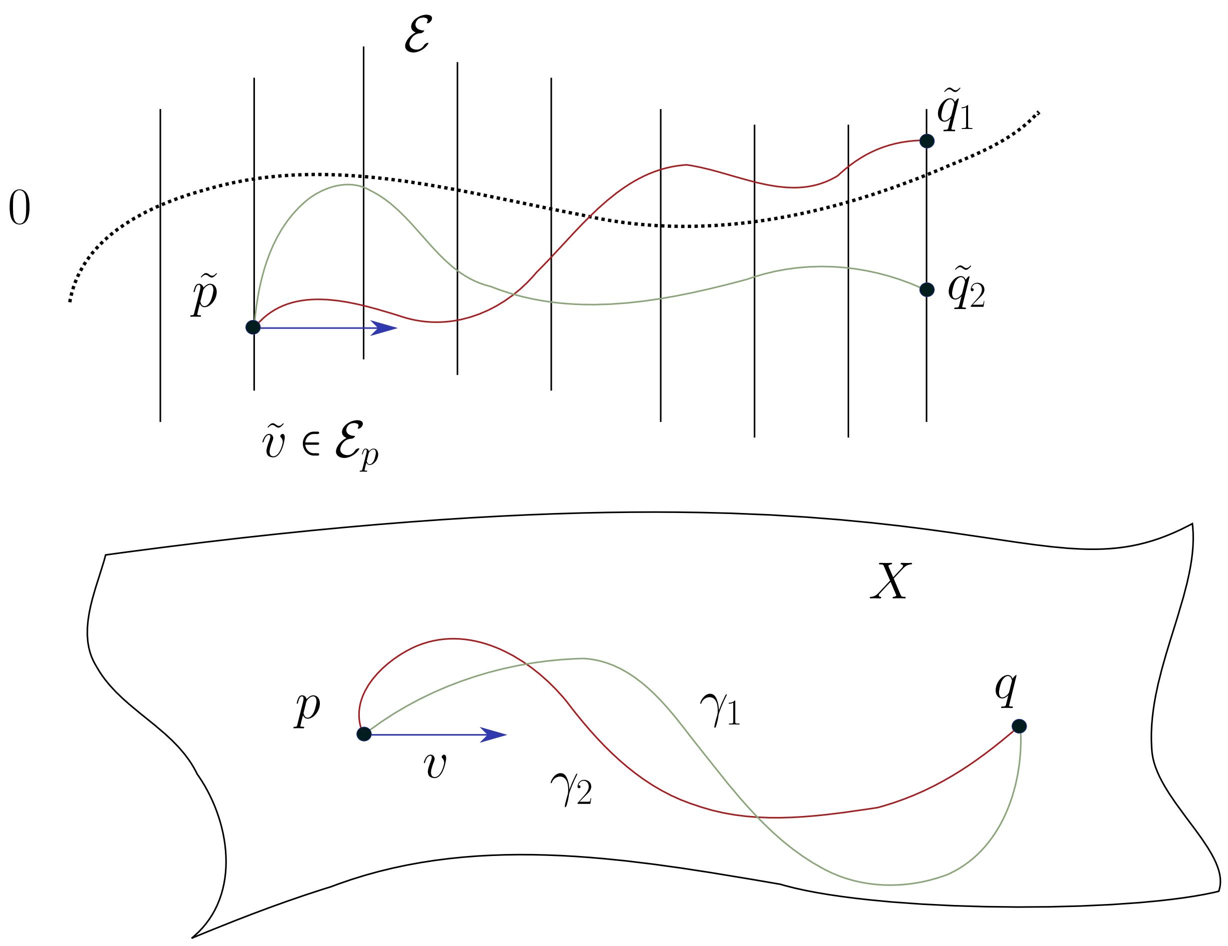
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇, i.e. the map

$$F_\nabla \coloneqq \nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2$$

is $\mathcal{O}$-linear, so $F_\nabla(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E}\Omega^1 \otimes \Omega^p$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

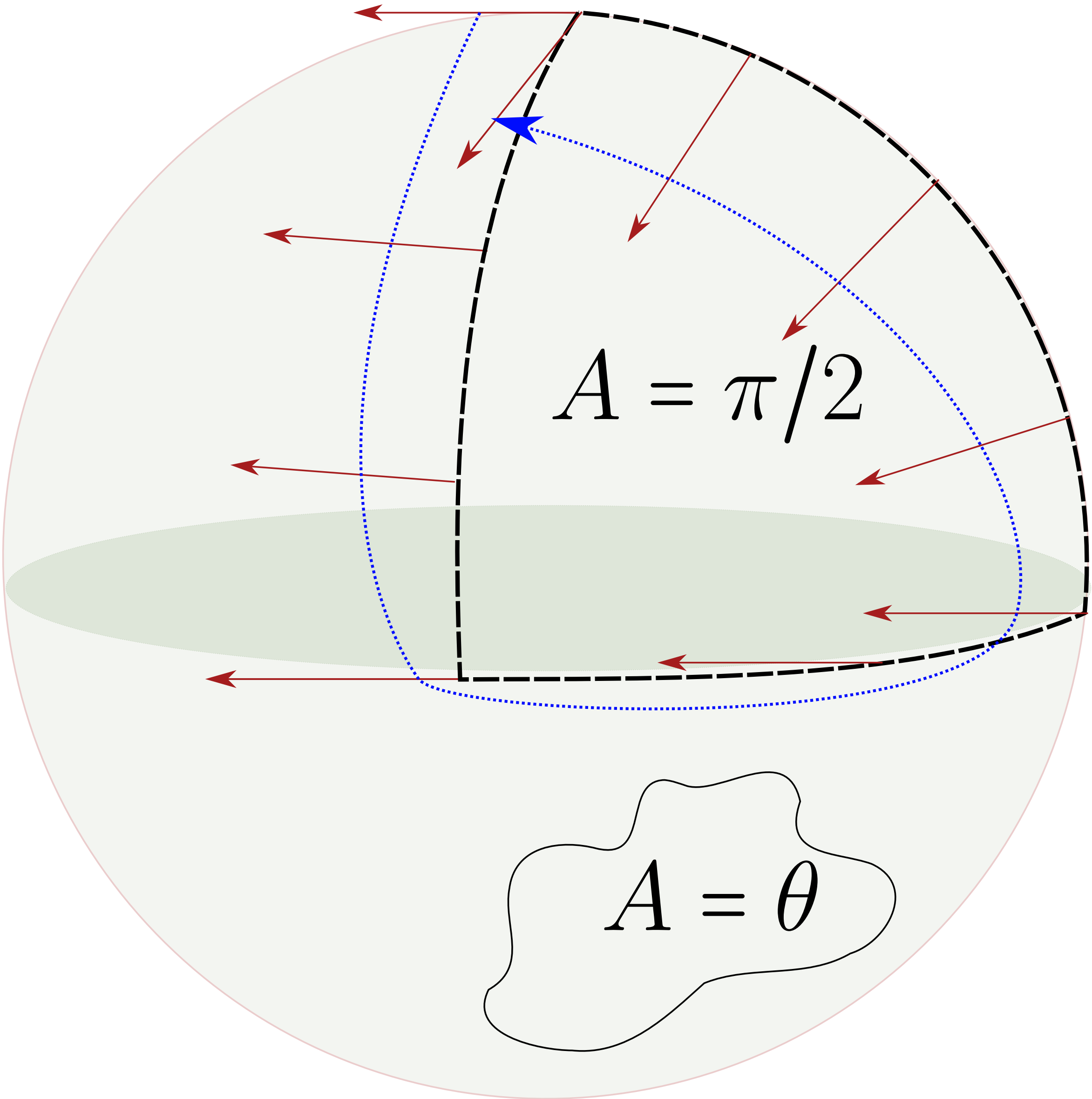


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1, yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ.

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_\nabla = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ.

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 ***(A classic example):*** The Levi-Cevita connection ∇^{LC} on TX, which depends on a metric g. Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)
A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) := \left\{ s \in \mathcal{E}(U) \;\middle|\; \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)
Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}^d}$. Furthermore $\operatorname{rank}(L) = \operatorname{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (?)
Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and let $\mathfrak{U} := \{U_i\} \rightrightarrows X$ be an open cover of X. Let $U_{i_1, \cdots, i_p} := U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C^p_{\mathfrak{U}}(X, \mathcal{F}) := \prod_{i_1 < \cdots < i_p} \mathcal{F}(U_{i_1, \cdots, i_p})$$

with a differential

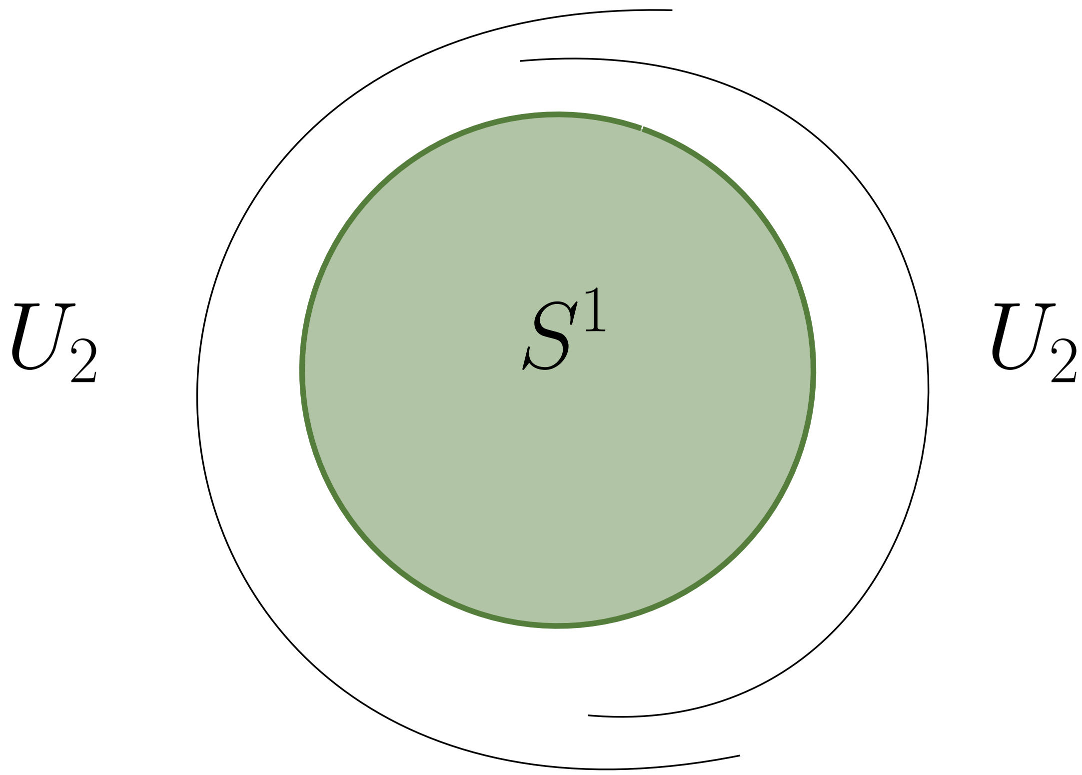
$$\begin{aligned} \partial^p : C^p_{\mathfrak{U}}(X, \mathcal{F}) &\to C^{p+1}_{\mathfrak{U}}(X \mathcal{F}) \\ \sigma &\mapsto (\partial \sigma)_{i_0, \cdots, i_p} := \prod_j (-1)^j \sigma_{i_0, \cdots, \widehat{i_j}, \cdots, i_p} \Big|_{U_{i_0, \cdots, i_p}} \end{aligned}$$

where we've defined this just on one given term in the product, i.e. a p-fold intersection.

Exercise 7.2.2 (?)
Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H^p_{\mathfrak{U}}(X, \mathcal{F})$ with respect to the cover $\mathfrak{U}$ is defined as $\ker \partial^p / \operatorname{im} \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4 *(?):* Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



ere we have

$$C^0(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_1) \oplus \underline{\mathbb{Z}}(U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\substack{\text{double} \\ \text{intersections}}} \underline{\mathbb{Z}}(U_{ij}) \underline{\mathbb{Z}}(U_{12}) = \underline{\mathbb{Z}}(U_1 \cap U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}.$$

We then get

$$\begin{aligned} C^0(S^1, \underline{\mathbb{Z}}) &\xrightarrow{\partial} C^1(S^1, \underline{\mathbb{Z}}) \\ \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields $H^*(S^1, \underline{\mathbb{Z}}) = [\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex $C^p_{\mathfrak{U}}(X, \mathcal{F}) \coloneqq \prod_{i_1, \cdots, i_p} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_p})$ for $\mathfrak{U} \coloneqq \{U_i\}$ is an open cover of X and F is a sheaf of abelian groups.

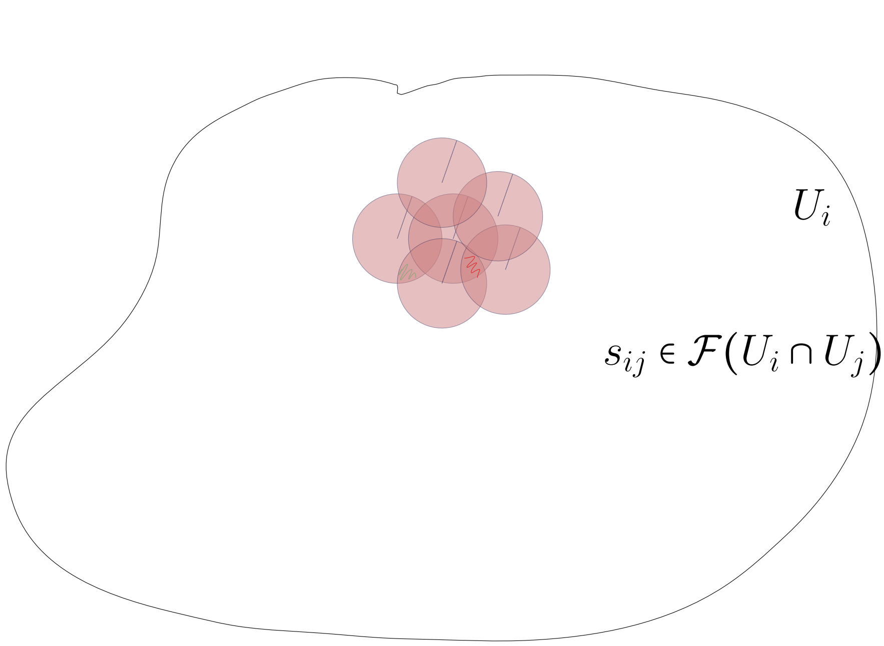
Fact 8.0.1
If $\mathfrak{U}$ is a sufficiently fine cover then $H^p_{\mathfrak{U}}(X, \mathcal{F})$ is independent of $\mathfrak{U}$, and we call this $H^p(X; \mathcal{F})$.

Remark 8.0.2: Recall that we computed $H^p(S^1, \underline{\mathbb{Z}} = [\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

Theorem 8.0.3 *(?).*
Let X be a paracompact and locally contractible topological space. Then $H^p(X, \underline{\mathbb{Z}}) \cong H^p_{\text{Sing}}(X, \underline{\mathbb{Z}})$. This will also hold more generally with $\underline{\mathbb{Z}}$ replaced by $\underline{A}$ for any $A \in \mathbf{Ab}$.

Definition 8.0.4 (Acyclic Sheaves)
We say $\mathcal{F}$ is *acyclic* on X if $H^{>0}(X; \mathcal{F}) = 0$.

Remark 8.0.5: How to visualize when $H^1(X; \mathcal{F}) = 0$:



On the intersections, we have $\operatorname{im} \partial^0 = \left\{ (s_i - s_j)_{ij} \;\middle|\; s_i \in \mathcal{F}(U_i) \right\}$, which are *cocycles*. We have $C^1(X; \mathcal{F})$ are collections of sections of $\mathcal{F}$ on every double overlap. We can check that $\ker \partial^1 = \left\{ (s_{ij}) \;\middle|\; s_{ij} - s_{ik} + s_{jk} = 0 \right\}$, which is the cocycle condition. From the exercise from last class, $\partial^2 = 0$.

Theorem 8.0.6 *((Important!)).*

Let X be a paracompact Hausdorff space and let

$$0 \to \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a SES of sheaves of abelian groups, i.e. $\mathcal{F}_3 = \operatorname{coker}(\varphi)$ and φ is injective. Then there is a LES in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X;\mathcal{F}_1) \arrow[r] & H^0(X;\mathcal{F}_2) \arrow[r] & H^0(X;\mathcal{F}_3) \arrow[dll] \\
& H^1(X;\mathcal{F}_1) \arrow[r] & H^1(X;\mathcal{F}_2) \arrow[r] & H^1(X;\mathcal{F}_3) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

Example 8.0.7 ***(?):*** For X a manifold, we can define a map and its cokernel sheaf:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \to \underline{\mathbb{Z}/2\mathbb{Z}} \to 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X;\mathbb{Z}) \arrow[r] & H^0(X;\mathbb{Z}) \arrow[r] & H^0(X;\mathbb{Z}/2\mathbb{Z}) \arrow[dll] \\
& H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{Z}) \arrow[r] & H^1(X;\mathbb{Z}/2\mathbb{Z}) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

Corollary 8.0.8 ***(of theorem).***

Suppose $0 \to \mathcal{F} \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \cdots$ is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose I_n is acyclic for all $n \geq 0$. This is referred to as an **acyclic resolution**. Then the homology can be computed at $H^p(X;\mathcal{F}) = \ker(I_p(X) \to I_{p+1}(X))/\operatorname{im}(I_{p-1}(X) \to I_p(X))$.

Note that locally having kernels equal images is different than satisfying this globally!

Proof (of corollary).
This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{aligned} 0 \to \mathcal{F} \to I_0 \xrightarrow{d_0} \operatorname{im}(d_0) \to 0 \qquad & \operatorname{im}(d_0) = \ker(d_1) \\ 0 \to \ker(d_1) \hookrightarrow I_1 \to I_1/\ker(d_1) = \operatorname{im}(d_1) \qquad & \operatorname{im}(d_1) = \ker(d_2) \end{aligned}$$

.

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving I_0, we obtain:

$$\begin{tikzcd} & & \cdots \arrow[dll] \\ H^{p-1}(\mathcal{F}) \arrow[r] & H^{p-1}(\mathcal{I}) = 0 \arrow[r] & H^{p-1}(\operatorname{im}()) \arrow[dll, "\cong"] \\ H^p(\mathcal{F}) \arrow[r] & \cdots = 0 & \end{tikzcd}$$

The middle entries vanish since I_* was assumed acyclic, and so we obtain $H^p(\mathcal{F}) \cong H^{p-1}(\operatorname{im} d_0) \cong H^{p-1}(\ker d_1)$. Now taking the LES associated to I_1, we get $H^{p-1}(\ker d_1) \cong H^{p-2}(\operatorname{im} d_1)$. Continuing this inductively, these are all isomorphic to $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$ after the pth step.

■

Corollary 8.0.9 *(of the previous corollary).*
Suppose $\mathfrak{U} \rightrightarrows X$, then if $\mathcal{F}$ is acyclic on each $U_{i_1,\cdots,i_p}$, then $\mathfrak{U}$ is sufficiently fine to compute Čech cohomology, and $H^p_{\mathfrak{U}}(X;\mathcal{F}) \cong H^p(X;\mathcal{F})$.

Proof (?).
See notes.

■

Corollary 8.0.10 *(of corollary).*
Let $X \in \mathbf{Mfd}_{\searrow}$, then $H^p(X,\underline{\mathbb{R}}) = H^p_{\mathrm{dR}}(X;\ RR)$.

Proof (?).
Idea: construct an acyclic resolution of the sheaf $\underline{\mathbb{R}}$ on M. The following exact sequence works:

$$0 \to \underline{\mathbb{R}} \to \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \cdots.$$

So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that $\ker d^p = \operatorname{im} d^{p-1}$ on any contractible coordinate chart. In other words, we want to show that if $d\omega = 0$ for $\omega \in \Omega^p(\mathbb{R}^n)$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$. This is true by integration! Using the previous corollary, $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$.

■

Check Hartshorne to see how injective resolutions line up with derived functors!

9 | Monday, February 01

Remark 9.0.1: Last time $\underline{\mathbb{R}}$ on a manifold M has a resolution by vector bundles:

$$0 \to \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots.$$

This is an exact sequence of sheaves of any smooth manifold, since locally $d\omega = 0 \implies \omega = d\alpha$ (by the *Poincaré d-lemma*). We also want to know that Ω^k is an acyclic sheaf on a smooth manifold.

Exercise 9.0.2 (?)
Let $X \in Top$ and $\mathcal{F} \in \mathrm{Sh}(\mathbf{Ab})_{/X}$. We say $\mathcal{F}$ is **flasque** if and only if for all $U \supseteq V$ the map $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ is surjective. Show that $\mathcal{F}$ is acyclic, i.e. $H^i(X; \mathcal{F}) = 0$. This can also be generalized with a POU.

Example 9.0.3 *(?):* The function $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$, but doesn't extend to a continuous map on $\mathbb{R}$. So the restriction map is not surjective.

Remark 9.0.4: Any vector bundle on a smooth manifold is acyclic. Using the fact that Ω^k is acyclic and the above resolution of $\underline{\mathbb{R}}$, we can write $H^k(X; \mathbb{R}) = \ker(d_k) / \operatorname{im} d_{k-1} \coloneqq H^k_{dR}(X; \mathbb{R})$.

Remark 9.0.5: Now letting $X \in \mathbf{Mfd}_{\mathbb{C}}$, recalling that Ω^p was the sheaf of holomorphic p-forms. Locally these are of the form $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$ where $f_I(\mathbf{z})$ is holomorphic. There is a resolution

$$0 \to \Omega^p \to A^{p,0},$$

where in $A^{p,0}$ we allowed also f_I are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth $(p, 0)$ form to check if it's holomorphic?

Example 9.0.6*(?):* For $p = 0$, we have

$$0 \to \mathcal{O} \to A^{0,0}.$$

where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations.

Definition 9.0.7 (?)
Let $\omega \in A^{p,q}(X)$ where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J \coloneqq \partial + \bar{\partial}$$

with $|I| = p, |J| = q$.

Example 9.0.8*(?):* The function $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$ is smooth, and $df = \bar{z}dz + zd\bar{z}$. This can be checked by writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy_j$, and $\frac{\partial}{\partial \bar{z}} g = 0$ if and only if g is holomorphic. Here we get $\partial\omega \in A^{p+1,q}(X)$ and $\bar{\partial} \in A^{p,q+1}(X)$, and we can write $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$.

Definition 9.0.9 (Cauchy-Riemann Equations)
Recall the Cauchy-Riemann equations: ω is a holomorphic $(p,0)$-form on $\mathbb{C}^n$ if and only if $\bar{\partial}\omega = 0$.

Remark 9.0.10: Thus to extend the previous resolution, we should take

$$0 \to \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \to \cdots.$$

The fact that this is exact is called the *Poincaré $\bar{\partial}$-lemma.*

Remark 9.0.11: There are no bump functions in the holomorphic world, and since Ω^p is a holomorphic bundle, it may not be acyclic. However, the $A^{p,q}$ *are* acyclic (since they are smooth vector bundles and thus admit POUs), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \operatorname{im}(\bar{\partial}_{q-1}).$$

Note the similarity to H_{dR}, using $\bar{\partial}$ instead of d. This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers** $h^{p,q}(X) \coloneqq \dim_{\mathbb{C}} H^q(X; \Omega^p)$. These are analogies:

Smooth	Complex
$\mathbb{R}$	Ω^p
Ω^k	$A^{p,q}$
Betti numbers β_k	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

Theorem 9.0.12 ***(Properties of Singular Cohomology).***
Let $X \in \mathbf{Top}$, then $H^i_{\text{Sing}}(X;\mathbb{Z})$ satisfies the following properties:

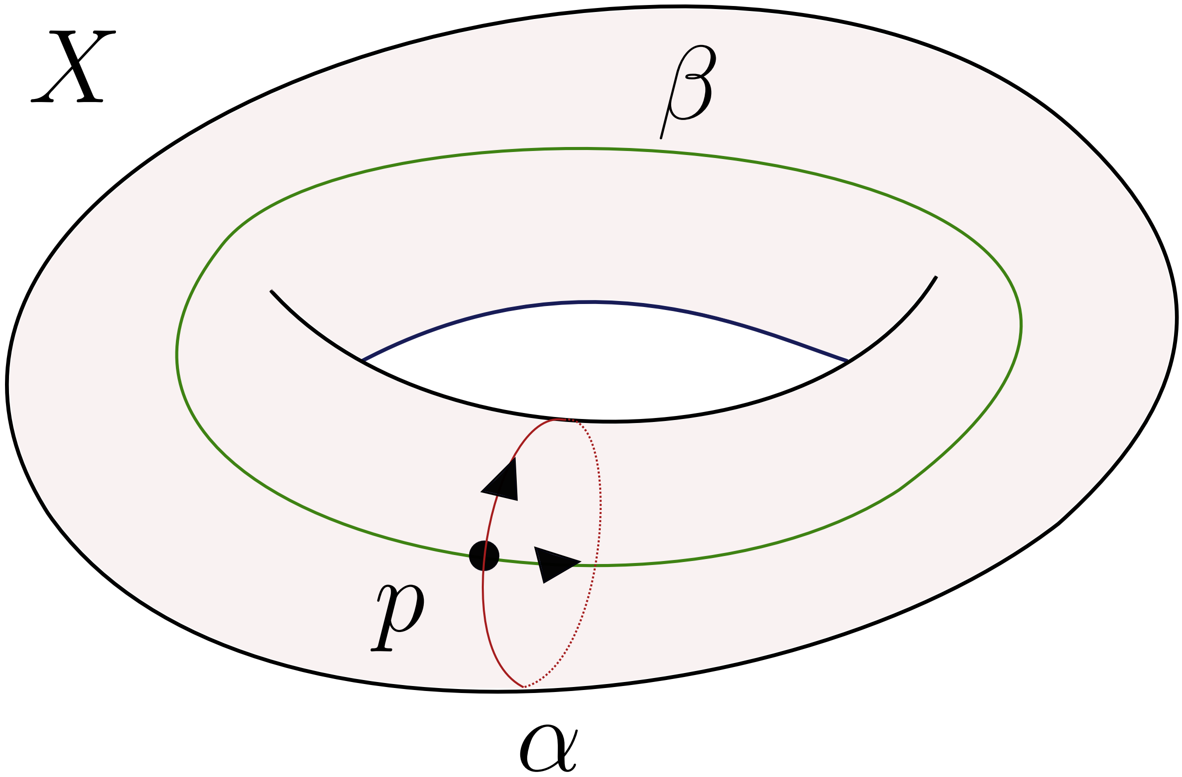
- Functoriality: given $f \in \text{Hom}_{\mathbf{Top}}(X,Y)$, there is a pullback $f^* : H^i(Y;\mathbb{Z}) \to H^i(X;\mathbb{Z})$.
- The cap product: a pairing
$$H^i(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X;\mathbb{Z}) \to H_{j-i}(X;\mathbb{Z})$$
$$\varphi \otimes \sigma \mapsto \varphi\left(\sigma|_{\Delta_{0,\cdots,j}}\right)\sigma|_{\Delta_{i,\cdots,j}}.$$
This makes H_* a module over H^*.
- There is a ring structure induced by the cup product:
$$H^i(X;\mathbb{R}) \times H^j(X;\mathbb{R}) \to H^{i+j}(X;\mathbb{R}) \qquad \alpha \cup \beta = (-1)^{ij}\beta \cup \alpha.$$
- Poincaré Duality: If X is an oriented manifold, there exists a fundamental class $[X] \in H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ and $(\cdot) \cap X : H^i \to H_{n-i}$ is an isomorphism.

Remark 9.0.13: Let $M \subset X$ be a submanifold where X is a smooth oriented n-manifold. Then $M \hookrightarrow X$ induces a pushforward $H_n(M;\mathbb{Z}) \xrightarrow{\iota_*} H_n(X;\mathbb{Z})$ where $\sigma \mapsto \iota \circ \sigma$. Using Poincaré duality, we'll identify $H_{\dim M}(X;\mathbb{Z}) \to H^{\operatorname{codim} M}(X;\mathbb{Z})$ and identify $[M] = PD(\iota_*([M]))$. In this case, if $M \pitchfork N$ then $[M] \cap [N] = [M \cap N]$, i.e. the cap product is given by intersecting submanifolds.

⚠ **Warning 9.0.14**
This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

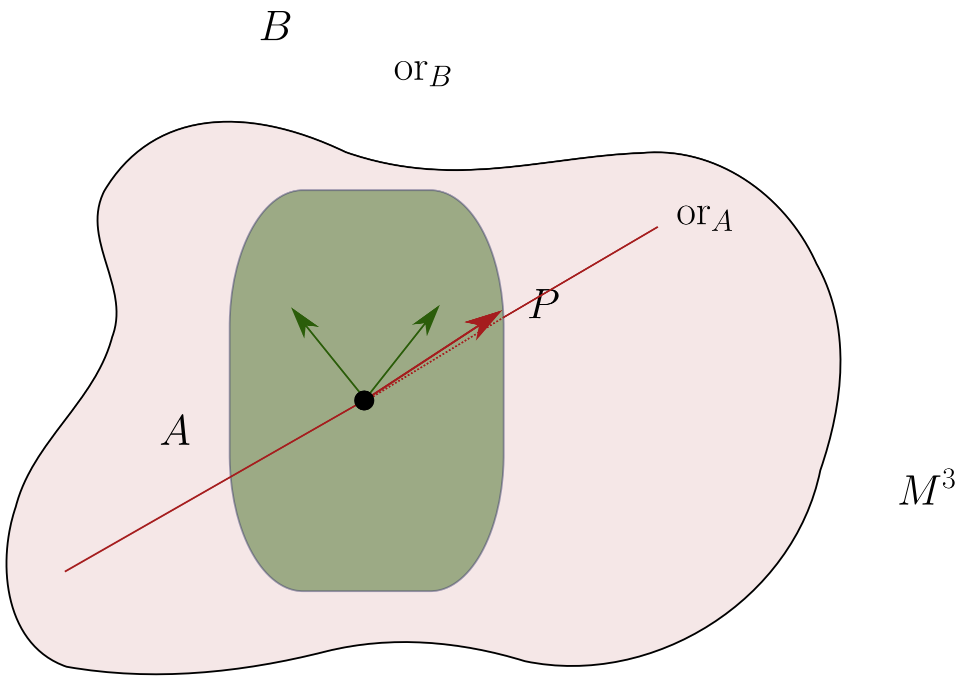


We can then take the fundamental classes of the submanifolds, say $[\alpha], [\beta] \in H^1(X;\mathbb{Z}) \xrightarrow{PD} H^1(X,\mathbb{Z})$. Here $T_p\alpha \oplus T_p\beta = T_pX$, since the intersections are transverse. Since α, β are oriented, let $\{e\}$ be a basis of $T_p\alpha$ (up to $\mathbb{R}^+$) and similarly $\{f\}$ a basis of $T_p\beta$. We can then ask if $\{e, f\}$ constitutes an *oriented* basis of T_pX. If so, we write $\alpha \cdot_p \beta \coloneqq +1$ and otherwise $\alpha \cdot_p \beta = -1$. We thus have

$$[\alpha] \smile [\beta] \in H^2(X;\mathbb{Z}) \xrightarrow{PD} H_0(X;\mathbb{Z}) = \mathbb{Z}$$

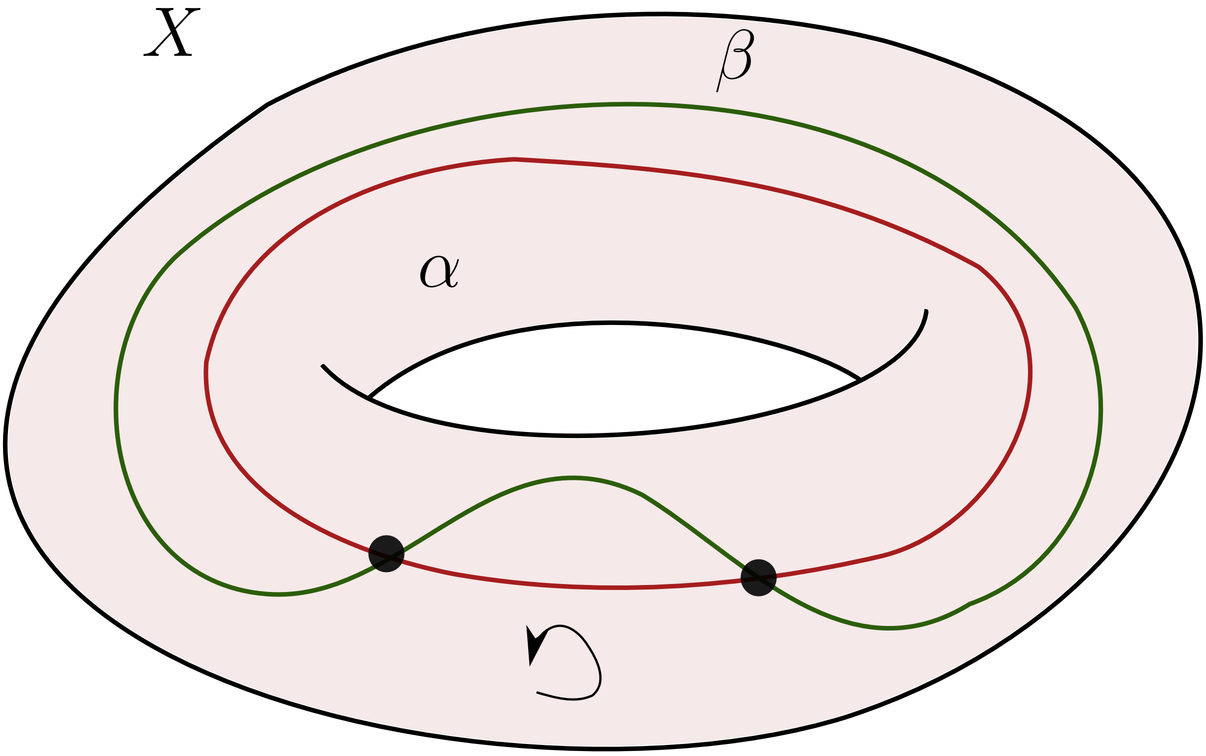
since X is connected. We can thus define the **intersection form** $\alpha \cdot \beta \coloneqq [\alpha] \smile [\beta]$. In general if A, B are oriented transverse submanifolds of M which are themselves oriented, we'll have $[A] \smile [B] = [A \cap B]$. We need to be careful: how do we orient the intersection? This is given by comparing the orientations on A and B as before.

Example 10.0.1 ***(?):*** If $\dim M = \dim A + \dim B$, then any $p \in A \cap B$ is oriented by comparing $\{\mathrm{or}_A, \mathrm{or}_B\}$ to or_M.



Here it suffices to check that $\{e, f_1, f_2\}$ is an oriented basis of T_pM.

Example 10.0.2*(?):* In this case, $[\alpha] \smile [\beta] = 0$ and so $\alpha \cdot \beta = 0$:



Remark 10.0.3: Note that cohomology with $\mathbb{Z}$ coefficients can be defined for any topological space, and Poincaré duality still holds.

Remark 10.0.4: We'll be considering $M = M^4$, smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

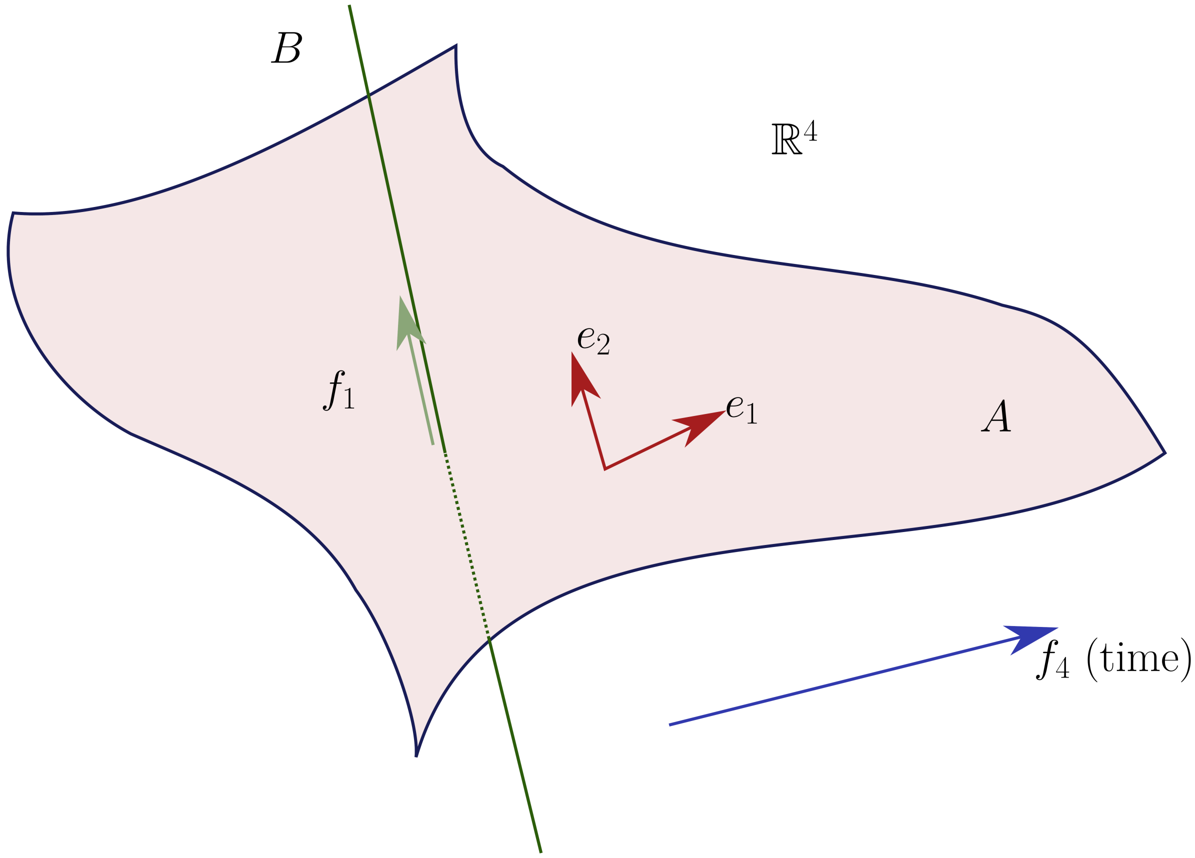


Figure 1: Picking one basis element in the time direction

Here ? is oriented in the "forward time" direction, and this is a surface at time $t = 0$. Where $A \cdot B = +1$, since $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$ is a oriented basis for $\mathbb{R}^4$. For $?^2$, switching the order of α, β no longer yields an oriented basis, but in this case it is ? and $A \cdot B = B \cdot A$. This is because

$$A \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad\qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

Remark 10.0.5: Let M^{2n} be an oriented manifold, then the cup product yields a bilinear map $H^n(M;\mathbb{Z}) \otimes H^n(M;\mathbb{Z}) \to \mathbb{Z}$ which is symmetric when n is odd and antisymmetric (or symplectic) when n is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$\begin{aligned} (H^n(M;\mathbb{Z})/\mathrm{tors})^\vee &\xrightarrow{\sim} H^n(M;\mathbb{Z})/\mathrm{tors} \\ \alpha \smile \cdot &\mapsfrom \alpha, \end{aligned}$$

where the LHS are linear functionals on cohomology.

Remark 10.0.6: Recall the universal coefficients theorem:

$$H^i(X;\mathbb{Z})/\mathrm{tors} \cong (H_i(X;\mathbb{Z})/\mathrm{tors})^\vee .$$

The general theorem shows that $H^i(X;\mathbb{Z})_{\text{tors}} = H_{i-1}(X;\mathbb{Z})_{\text{tors}}$.

Remark 10.0.7: Note that if M is an oriented 4-manifold, then

$$\begin{array}{cccccccc} & \text{tors} & \text{torsionfree} & & & \text{tors} & \text{torsionfree} \\ H^0 & 0 & \mathbb{Z} & & H_0 & 0 & \mathbb{Z} \\ H^1 & 0 & \mathbb{Z}^{\beta_1} & & H_1 & A & \mathbb{Z}^{\beta_1} \\ H^2 & A & \mathbb{Z}^{\beta_2} & \xrightarrow{PD} & H_2 & A & \mathbb{Z}^{\beta_2} \\ H^3 & A & \mathbb{Z}^{\beta_1} & & H_3 & 0 & \mathbb{Z}^{\beta_1} \\ H^4 & 0 & \mathbb{Z} & & H_4 & 0 & \mathbb{Z} \end{array}$$

In particular, if M is simply connected, then $H_1(M) = \mathbf{Ab}(\pi_1(M)) = 0$, which forces $A = 0$ and $\beta_1 = 0$.

Definition 10.0.8 (Lattice)
A **lattice** is a finite-dimensional free $\mathbb{Z}$-module L together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\to \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice $(L, \cdot)$ is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\to L^\vee \\ \ell &\mapsto \ell \cdot (\cdot). \end{aligned}$$

Remark 10.0.9: How to determine if a lattice is unimodular: take a basis $\{e_1, \cdots, e_n\}$ of L and form the *Gram matrix* $M_{ij} \coloneqq (e_i \cdot e_j) \in \mathrm{Mat}(n \times n, \mathbb{Z})^{\mathrm{Sym}}$. Then $(L, \cdot)$ is unimodular if and only if $\det(M) = \pm 1$ if and only if M^{-1} is integral. In this case, the rows of M^{-1} will form a basis of the dual basis.

Definition 10.0.10 (?)
The **index** of a lattice is $|\det M|$.

Exercise 10.0.11 (?)
Prove that $|\det M| = |L^\vee / L|$.

Remark 10.0.12: In general, for M^{4k}, the H^{2k}/tors is unimodular. For M^{4k+2}, the H^{2k+1}/tors is a unimodular *symplectic* lattice, which is obtained by replacing the word "symmetric" with "antisymmetric" everywhere above.

Example 10.0.13 ***(?):*** For the torus, since the dimension is $2 \pmod 4$, you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check!

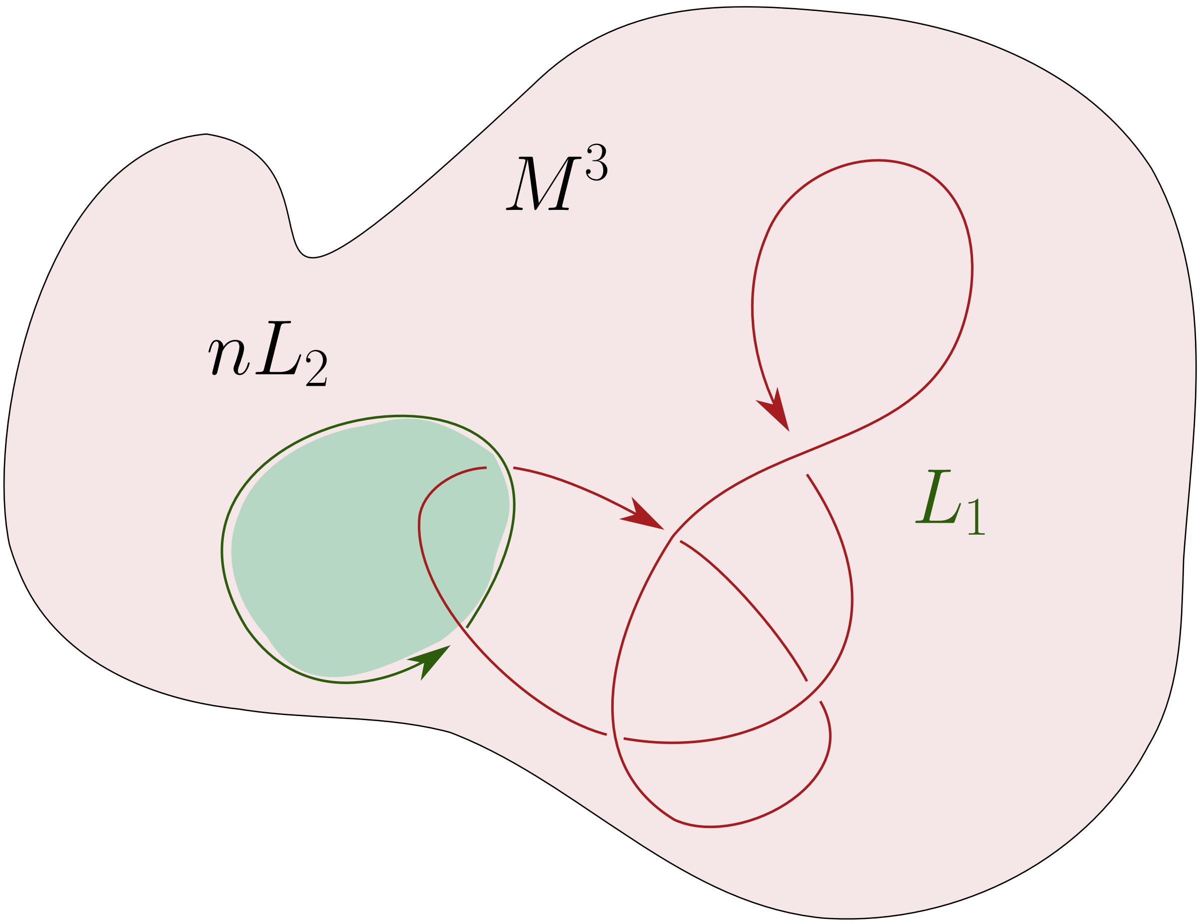
Definition 10.0.14 (?)
A lattice is **nondegenerate** if $\det M \neq 0$.

Definition 10.0.15 (?)
The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space with an $\mathbb{R}$-valued symmetric bilinear form. This allows extending the lattice from $\mathbb{Z}^n$ to $\mathbb{R}^n$.

Remark 10.0.16: If $(L, \cdot)$ is nondegenerate, then Gram-Schmidt will yield an orthonormal basis $\{v_i\}$. The number of positive norm vectors is an invariant, so we obtain $\mathbb{R}^{p,q}$ where p is the number of $+1$s in the Gram matrix and q is the number of -1s. The **signature** of $(L, \cdot)$ is (p, q), or by abuse of notation $p - q$. This is an invariant of the 4-manifold, as is the lattice itself $H^2(X; \mathbb{Z})/\text{tors}$ equipped with the intersection form.

Remark 10.0.17: There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$



Remark 10.0.18: $A \cdot B \coloneqq \sum_{p \in A \cap B} \operatorname{sgn}_p(A, B)$, where $A \pitchfork B$ and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \smile [\omega] = \int_A \omega.$$

11 | Friday, February 05

Remark 11.0.1: Recall that a lattice is **unimodular** if the map $L \to L^\vee \coloneqq \operatorname{Hom}(L, \mathbb{Z})$ is an isomorphism, where $\ell \mapsto \ell \cdot (\,\cdot\,)$. To check this, it suffices to check if the Gram matrix M of a basis $\{e_i\}$ satisfies $|\det M| = 1$.

Example 11.0.2 ***(Determinant 1 Integer Matrices):*** The matrices $[1]$ and $[-1]$ correspond to the lattice $\mathbb{Z}e$ where either $e^2 \coloneqq e \cdot e = 1$ or $e^2 = -1$. If M_1, M_2 both have absolute determinant 1, then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

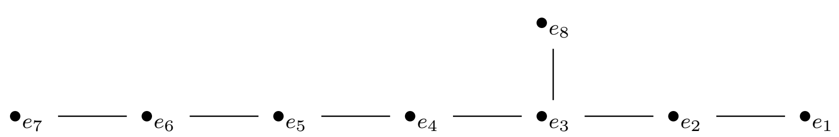
So if L_1, L_2 are unimodular, then taking an orthogonal sum $L_1 \oplus L_2$ also yields a unimodular lattice. So this yields diagonal matrices with p copies of $+1$ and q copies of -1. This is referred to as $rm1_{p,q}$, and is an *odd* unimodular lattice of signature (p, q) (after passing to $\mathbb{R}$). Here *odd* means that there exists a $v \in L$ such that v^2 is odd.

Example 11.0.3 ***(Even unimodular lattices):*** An even lattice must have no vectors of odd norm, so all of the diagonal elements are in $2\mathbb{Z}$. This is because $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i<j} 2n_i, n_j e_i \cdot e_j$. Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as H, sometimes referred to as the *hyperbolic cell* or *hyperbolic plane.*

Example 11.0.4 ***(A harder even unimodular lattice):*** This is built from the E_8 Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \to e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example, $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$. You can check that $\det(e_i \cdot e_j) = 1$, and this is referred to as the E_8 lattice. This is of signature $(0, 8)$, and it's negative definite if and only if $v^2 < 0$ for all $v \neq 0$. One can also negate the intersection form to define $-E_8$. Note that any simply-laced Dynkin diagram yields some lattice. For example, E_{10} is unimodular of signature $(1, 9)$, and it turns out that $E_{10} \cong E_8 \oplus H$.

Definition 11.0.5 (?)
Take

$$\mathbf{II}_{a,a+8b} \coloneqq \bigoplus_{i=1}^{a} H \oplus \bigoplus_{j=1}^{b} E_8,$$

which is an even unimodular lattice since the diagonal entries are all -2, and using the fact that the signature is additive, is of signature $(a, a+8b)$. Similarly,

$$\mathbf{II}_{a+8b,a} \coloneqq \bigoplus_{i=1}^{a} H \oplus \bigoplus_{j=1}^{b} (-E_8),$$

which is again even and unimodular.

Remark 11.0.6: Thus

- $\mathbf{I}_{p,q}$ is odd, unimodular, of signature (p,q).
- $\mathbf{II}_{p,q}$ is even, unimodular, of signature (p,q) only for $p \equiv q \pmod 8$.

Theorem 11.0.7 ***(Serre).***
Every unimodular lattice which is not positive or negative definite is isomorphic to either $\mathbf{I}_{p,q}$ or $\mathbf{II}_{p,q}$ with $8 \mid p-q$.

Remark 11.0.8: So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: E_8
16	2: $E_8^{\oplus 2}, D_{16}^{+}$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$>8 \times 10^{16}$!!!!

Note that the signature of a definite lattice must be divisible by 8.

Remark 11.0.9: There is an isometry: $f : E_8 \to E_8$ where $f \in O(E_8)$, the linear maps preserving the intersection form (i.e. the Weyl group $W(E_8)$, given by $v \mapsto v + (v, e_i)e_i$. The Leech lattice also shows up in the sphere packing problems for dimensions $2, 4, 8, 24$. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of L as a subset of $\mathbb{R}^n$, namely $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$ for example, and the map $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$ is an isometric embedding into $\mathbb{R}^n$ with the standard form. Connection to classification of Lie groups: root lattices.

Remark 11.0.10: If M^4 is a compact oriented 4-manifold and if the intersection form on $H^2(M; \mathbb{Z})$ is indefinite, then the only invariants we can extract from that associated lattice are

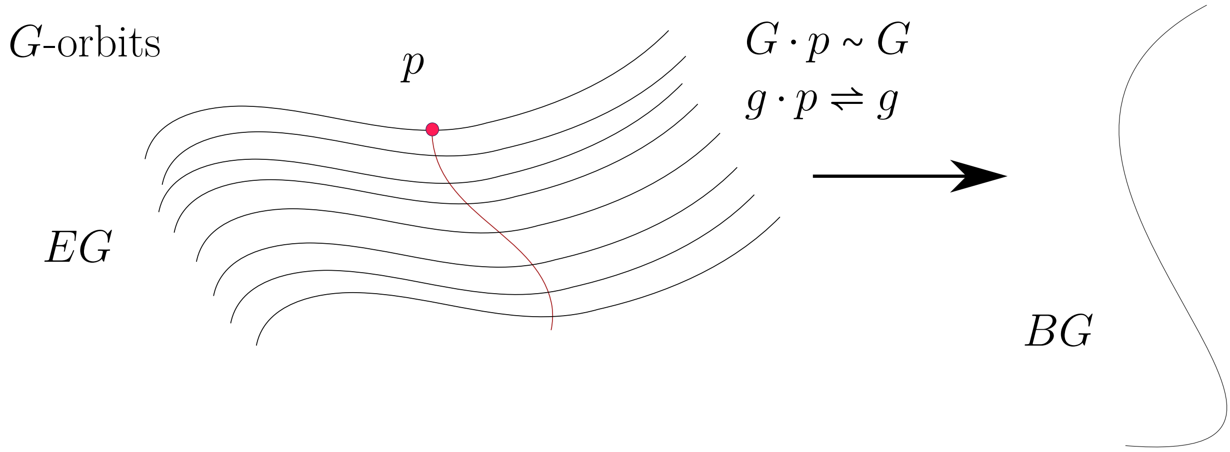
- Whether it's even or odd, and
- Its signature

If the lattice is even, then the signature satisfies $8 \mid p-q$. So Poincaré duality forces unimodularity, and then there are further number-theoretic restrictions. E.g. this prohibits $\beta_2 = 7$, since then the signature couldn't possibly be 8 if the intersection form is even.

11.1 Characteristic Classes

Definition 11.1.1 (?)
Let G be a topological group, then a **classifying space** EG is a contractible topological space admitting a free continuous G-action with a "nice" quotient.

Remark 11.1.2: Thus there is a map $EG \to BG \coloneqq EG/G$ which has the structure of a principal G-bundle.



Here we use a point p depending on U in an orbit to identify orbits $g \cdot p$ with g, and we want to take transverse slices to get local trivializations of $U \in BG$. It suffices to know where $\pi^{-1}(U) \cong U \times G$, and it suffices to consider $U \times \{e\}$. Moreover, $EG \to BG$ is a universal principal G-bundle in the sense that if $P \to X$ is a universal G-bundle, there is an $f : X \to BG$.

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[d] & EG \arrow[d] \\
X \arrow[r, "f"] & BG
\end{tikzcd}$$

Link to Diagram

Here bundles will be classified by homotopy classes of f, so

$$\left\{\text{Principal } G\text{-bundles}_{/X}\right\} \rightleftharpoons [X, BG].$$

Warning 11.1.3
This only works for paracompact Hausdorff spaces! The line $\mathbb{R}$ with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

12 | Monday, February 08

Last time: BG and EG. See Milnor and Stasheff.

Example 12.0.1 *(?):* Let $G \coloneqq \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$, then we can take

$$EG = \mathbb{R}^\infty \coloneqq \left\{ (a_1, a_2, \cdots) \;\middle|\; a_i \in \mathbb{R}, a_{i \gg 0} = 0, a_i \text{ not all zero} \right\}.$$

Then $\mathbb{R}^\times$ acts on EG by scaling, and we can take the quotient $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$, where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in \mathbb{R}^\times$. This yields $\mathbb{RP}^\infty$ as the quotient. You can check that E_G is contractible: it suffices to show that $S^\infty \coloneqq \left\{ \sum |a_i| = 1 \right\}$ is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$. Similarly $B\mathbb{C}^\times \cong \mathbb{CP}^\infty$ with $E\mathbb{C}^\times \coloneqq \mathbb{C}^\infty \setminus \{0\}$.

Example 12.0.2 *(?):* Consider $G = \mathrm{GL}_n(\mathbb{R})$. It turns out that $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$, which is the set of linear subspaces of $\mathbb{R}^\infty$ of dimension d. This is spanned by d vectors $\{e_i\}$ in some large enough $\mathbb{R}^N \subseteq \mathbb{R}^\infty$, since we can take N to be the largest nonvanishing coordinate and include all of the vectors into $\mathbb{R}^\infty$ by setting $a_{>N} = 0$. For any $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$, since $\mathbb{R}^d$ has a standard basis, there is a natural GL_d torsor: the set of ordered bases of linear subspaces. So define

$$EG \coloneqq \{\text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty)\},$$

then any $A \in \mathrm{GL}_d(\mathbb{R})$ acts on EG by sending $(L, \{e_i\}) \mapsto (L, \{Le_i\})$. We can identify EG as d-tuples of linearly independent elements of $\mathbb{R}^\infty$, and there is a map

$$\begin{aligned} EG &\to BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over BGL_d:

$$\begin{tikzcd} \mathcal{E}_L \coloneqq L \arrow[r] & \mathcal{E} \arrow[d] \\ & BGL_d \end{tikzcd}$$

So $\mathcal{E} \subseteq BGL_d \times \mathbb{R}^\infty$, where we can define $\mathcal{E} \coloneqq \left\{ (L, p) \;\middle|\; p \in L \right\}$. In this case, $EG = \mathrm{Frame}(\mathcal{E})$ is the frame bundle of this universal bundle. The same setup applies for $G \coloneqq \mathrm{GL}_d(\mathbb{C})$, except we take $\mathrm{Gr}_d(\mathbb{C}^\infty)$.

Example 12.0.3 *(?):* Consider $G = O_d$, the set of orthogonal transformations of $\mathbb{R}^d$ with the standard bilinear form, and U_d the set of unitary such transformations. To be explicit:

$$U_d \coloneqq \left\{ A \in \operatorname{Mat}(d \times d, \mathbb{C}) \;\middle|\; \langle Av,\ Av \rangle = \langle v,\ v \rangle \right\},$$

where

$$\langle [v_1, \cdots, v_n],\ [v_1, \cdots, v_n] \rangle = \sum |v_i|^2.$$

Alternatively, $A^t A = I$ for O_d and $\overline{A^t} A = I$ for U_d. In this case, $BO_d = \operatorname{Gr}_d(\mathbb{R}^\infty)$ and $BU_d = \operatorname{Gr}_d(\mathbb{C}^\infty)$, but we'll make the fibers smaller: set the fiber over L to be

$$(EO_d)_L \coloneqq \{\text{orthogonal frames of } L\}$$

and similarly $(EU_d)_L$ the unitary frames of L. That there are related comes from the fact that GL_d retracts onto O_d using the Gram-Schmidt procedure.

Remark 12.0.4: Recall that there is a bijective correspondence

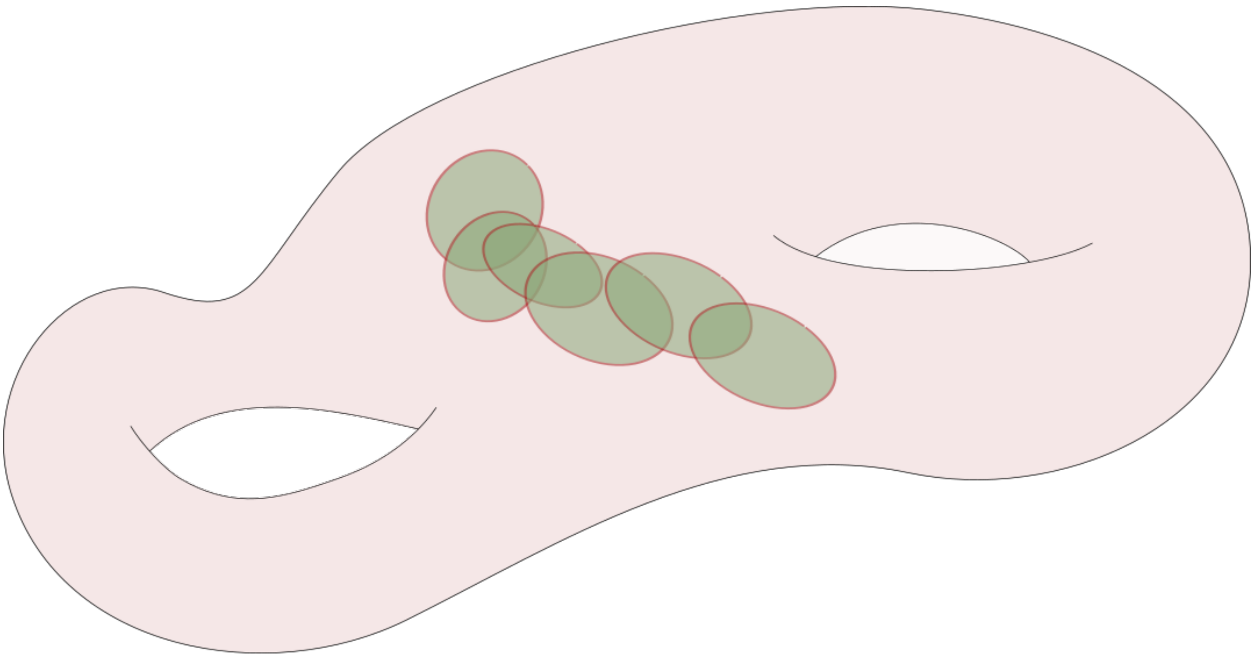
$$\left\{ \substack{\text{Principal } G\text{- bundles} \\ \text{on } X} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \substack{\text{Principal } \operatorname{GL}_d\text{-bundles} \\ \text{on } X} \right\} \rightleftharpoons \left\{ \substack{\text{Principal } O_d\text{-bundles} \\ \text{on } X} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles $\mathcal{E} \to X$ of rank d, and on the RHS we have bundles with a metric. In local trivializations $U \times \mathbb{R}^d \to \mathbb{R}^d$, the metric is the standard one on $\mathbb{R}^d$. This is referred to as a **reduction of structure group**, i.e. a principal GL_d bundle admits possibly different trivializations for which the transition functions lie in the subgroup O_d.

Example 12.0.5 *(?):* Given any trivial principal G-bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.



Remark 12.0.6: We want to compute $H^*(BU_d;\mathbb{Z})$. Why is this important? Given any complex vector bundle $\mathcal{E} \to X$ there is an associated principal U_d bundle by choosing a metric, so we get a homotopy class $[X, BU_d]$. Given any $f \in [X, BU_d]$ and any $\alpha \in H^k(BU_d;\mathbb{Z})$, we can take the pullback $f^*\alpha \in H^k(X;\mathbb{Z})$, which are **Chern classes**.

Exercise 12.0.7 (?)
Show that $H^*(BU_d;\mathbb{Z})$ stabilizes as $d \to \infty$ to an infinitely generated polynomial ring $\mathbb{Z}[c_1, c_2, \cdots]$ with each c_i in cohomological degree $2i$, so $c_i \in H^{2i}(BU_d, \mathbb{Z})$.

Definition 12.0.8 (?)
There is a map $BU_{d-1} \to BU_d$, which we can identify as $\mathrm{Gr}_{d-1}(C^\infty) \to \mathrm{Gr}_d(\mathbb{C}^\infty)$. This is defined by sending a basis $\{v_1, \cdots, v_{d-1}\} \mapsto \operatorname{span}\{(1,0,0,\cdots), sv_1, \cdots, sv_{d-1}\}$ where $s : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is the map that shifts every coordinate to the right by one.

Question: does $\mathrm{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?

This will yield a fiber sequence $S^{2d-1} \to BU_{d-1} \to BU_d$, and using connectedness of the sphere and the LES in homotopy this will identify $H^*(BU_d) = H^*(BU_{d-1})[c_d]$ where $c_d \in H^{2d}(BU_d)$. The **Chern class** of a vector bundle $\mathcal{E}$, denoted $c_k(\mathcal{E})$, will be defined as the pullback f^*c_k.

13 | Wednesday, February 10

Theorem 13.0.1 *(?)*.
As $n \to \infty$, we have

$$H^*(BO_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \cdots] \qquad\qquad w_i \in H^i$$

Definition 13.0.2 (?)
Given any principal O_n-bundle $P \to X$, there is an induced map $X \xrightarrow{f} BO_n$, so we can pull back the above generators to define the **Stiefel-Whitney classes** f^*w_i.

Remark 13.0.3: If $P \coloneqq \mathrm{OFrame}TX$, then f^*w_1 measures whether X has an orientation, i.e. $f^*w_1 = 0 \iff X$ can be oriented. We also have $f^*w_i(P) = w_i(\mathcal{E})$ where $P = \mathrm{OFrame}(\mathcal{E})$. In general, we'll just write w_i for Stiefel-Whitney classes and c_i for Chern classes.

Definition 13.0.4 (Pontryagin Classes)
The **Pontryagin classes** of a real vector bundle $\mathcal{E}$ are defined as

$$p_i(\mathcal{E}) = (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that the complexified bundle above is a complex vector bundle with the same transition functions as $\mathcal{E}$, but has a reduction of structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{R})$.

Observation 13.0.5
$\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ are examples of $K(\pi, n)$ spaces, which are the unique-up-to-homotopy spaces defined

by

$$\pi_k K(\pi, n) = \begin{cases} \pi & k = n \\ 0 & \text{else.} \end{cases}$$

Theorem 13.0.6 *(Brown Representability).*

$$H^n(X; \pi) \cong [X, K(\pi, n)].$$

Example 13.0.7 *(?):*

$$[X, \mathbb{RP}^\infty] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$$
$$[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z}).$$

Proposition 13.0.8 *(?).*
There is a correspondence

$$\{\text{Complex line bundles}\} \rightleftharpoons [X, \mathbb{CP}^\infty] = [X, BC^\times] \rightleftharpoons H^2(X; \mathbb{Z})$$

Importantly, note that for $X \in \mathbf{Mfd}_{\mathbb{C}}$, $H^2(X; \mathbb{Z})$ measures *smooth* complex line bundles and not holomorphic bundles.

Proof (?).
We'll take an alternate direct proof. Consider the exponential exact sequence on X:

$$0 \to \underline{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times.$$

Note that $\underline{\mathbb{Z}}$ consists of locally constant $\mathbb{Z}$-valued functions, $\mathcal{O}$ consists of smooth functions, and $\mathcal{O}^\times$ are ???.

Can't read screenshot! :(

This yields a LES in homology:

$$\begin{tikzcd}
H^0(X; \underline{\mathbb{Z}}) \arrow[r] & H^0(X; \mathcal{O}) \arrow[r] & H^0(X; \mathcal{O}^\times) \arrow[dll, out=0, in=180] \\
H^1(X; \underline{\mathbb{Z}}) \arrow[r] & H^1(X; \mathcal{O}) \arrow[r] & H^1(X; \mathcal{O}^\times) \arrow[dll, out=0, in=180] \\
H^2(X; \underline{\mathbb{Z}}) \arrow[r] & H^2(X; \mathcal{O}) \arrow[r] & H^2(X; \mathcal{O}^\times)
\end{tikzcd}$$

Link to Diagram

Since $\mathcal{O}$ admits a partition of unity, $H^{>0}(X;\mathcal{O}) = 0$ and all of the red terms vanish. For complex line bundles L, $H^1(X,\mathcal{O}^\times) \cong H^2(X;\mathbb{Z})$. Taking a local trivialization $L|_U \cong U \times \mathbb{C}$, we obtain transition functions

$$t_{UV} \in C^\infty(U \cap V, \mathrm{GL}_1(\mathbb{C}))$$

where we can identify $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. We then have

$$(t_{U_{ij}}) \in \prod_{i<j} \mathcal{O}^\times(U_i \cap U_j) = C^1(X;\mathcal{O}^\times).$$

Moreover,

$$\left(t_{U_{ij}} t_{U_{ik}}^{-1} t_{U_{jk}}\right)_{i,j,k} = \partial(t_{U_{ij}})_{i,j} = 0,$$

since transitions functions satisfy the cocycle condition. So in fact $(t_{U_{ij}}) \in Z^1(X;\mathcal{O}^\times) = \ker\partial^1$, and we can take its equivalence class $[(t_{U_{ij}})] \in H^1(X;\mathcal{O}^\times) = \ker\partial^1/\operatorname{im}\partial^0$. Changing trivializations by some $s_i \in \prod_i \mathcal{O}^\times(U_i)$ yields a composition which is a different trivialization of the same bundle:

$$\begin{tikzcd} L|_{U_i} \arrow[r, "h_i"] \arrow[rr, bend right] & U_i \times \mathbb{C} \arrow[r, "\cdot s_i"] & U_i \times \mathbb{C} \end{tikzcd}$$

So the $(t_{U_{ij}}$ change *exactly* by an $\partial^0(s_i)$. Thus the following map is well-defined:

$$L \mapsto [(t_{U_{ij}})] \in H^1(X;\mathcal{O}^\times).$$

There is another construction of the map

$$\begin{aligned} \{L\} * &\to H^2(X;\mathbb{Z}) \\ L &\mapsto c_1(L). \end{aligned}$$

Take a smooth section of L and $s \in H^0(X;L)$ that intersects an $\mathcal{O}$-section of L transversely. Then

$$V(s) \coloneqq \left\{ x \in X \;\middle|\; s(x) = 0 \right\}$$

is a submanifold of real codimension 2 in X, and $c_1(L) = [V(s)] \in H^2(X;\mathbb{Z})$.

∎

Theorem 13.0.9 ***(Splitting Principle for Complex Vector Bundles).***

1. Suppose that $\mathcal{E} = \bigoplus_{i=1}^r L_i$ and let $c(\mathcal{E}) \coloneqq \sum c_i(\mathcal{E}$. Then

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + c_i(L_i)).$$

2. Given any vector bundle $\mathcal{E} \to X$, there exists some Y and a map $Y \to X$ such that $f^* : H^k(X;\mathbb{Z}) \hookrightarrow H^k(Y;\mathbb{Z})$ is injective and $f^*\mathcal{E} = \bigoplus_{i=1}^r L_i$.

Slogan 13.0.10
To verify any identities on characteristic classes, it suffices to prove them in the case where $\mathcal{E}$ splits into a direct sum of line bundles.

Example 13.0.11 ***(?):***

$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}).$$

To prove this, apply the splitting principle. Choose Y, Y' splitting $\mathcal{E}, \mathcal{E}'$ respectively, this produces a space Z and a map $f : Z \to X$ where both split. We can write

$$\begin{aligned} f^*\mathcal{E} &= \bigoplus L_i & \qquad c(f^*\mathcal{E}) &= \prod (1 + c_1(L_i)) \\ f^*\mathcal{F} &= \bigoplus M_j & \qquad c(f^*\mathcal{E}) &= \prod (1 + c_1(M_j)) \end{aligned}$$

.

We thus have

$$\begin{aligned} c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &= \prod (1 + c_1(L_i))\,(1 + c_1(M_j)) \\ &= c(f^*\mathcal{E})c(f^*\mathcal{F}), \end{aligned}$$

and $f^*(c(\mathcal{E} \oplus \mathcal{F}) = f^*(c(\mathcal{E})c(\mathcal{F}))$. Since f^* is injective, this yields the desired identity.

Example 13.0.12 ***(?):*** We can compute $c(\mathrm{Sym}^2 \mathcal{E})$, and really any tensorial combination involving $\mathcal{E}$, and it will always yield some formula in the $c_i(\mathcal{E})$.

14 | Friday, February 12

Remark 14.0.1: Last time: the splitting principle. Suppose we have $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$ and let $x_i \coloneqq c_i(L_i)$. Then $c_k(\mathcal{E})$ is the degree $2k$ part of $\prod_{i=1}^r (1 + x_i)$ where each x_i is in degree 2. This is equal to $e_k(x_1, \cdots, x_r)$ where e_k is the kth elementary symmetric polynomial.

Example 14.0.2 ***(?):*** For example,

- $e_1 = x_1 + \cdots x_r$.
- $e_2 = x_1x_2 + x_1x_3 + \cdots = \sum_{i<j} x_i x_j$

- $e_3 = \sum_{i<j<k} x_i x_j x_k$, etc.

Remark 14.0.3: The theorem is that any symmetric polynomial is a polynomial in the e_i. For example, $p_2 = \sum x_i^2$ can be written as $e_1^2 - 2e_2$. Similarly, $p_3 = \sum x_i^3 = e_1^3 - 3e_1e_2 - 3e_3$ Note that the coefficients of these polynomials are important for representations of S_n, see *Schur polynomials.*

Remark 14.0.4: Due to the splitting principle, we can pretend that $x_i = c_i(L_i)$ exists even when $\mathcal{E}$ doesn't split. If $\mathcal{E} \to X$, the individual symbols x_i don't exist, but we can write '

$$x_1^3 + \cdots + x_r^3 = e_1^3 - 3e_1e_2 - 3e_3 \coloneqq c_1(\mathcal{E})^3 + 3c_1(\mathcal{E})c_2(\mathcal{E}) + \cdots,$$

which is a well-defined element of $H^6(X;\mathbb{Z})$. So this polynomial defines a characteristic class of $\mathcal{E}$, and this can be done for any symmetric polynomial. We can change basis in the space of symmetric polynomials to now define different characteristic classes.

Definition 14.0.5 (Chern Character)
The **Chern character** is defined as

$$\begin{aligned}
\mathrm{ch}(\mathcal{E}) &\coloneqq \sum_{i=1}^{r} e^{x_i} \in H^*(X;\mathbb{Q}) \\
&\coloneqq \sum_{i=1}^{r}\sum_{k=0}^{\infty} \frac{x_i^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{p_k(x_1,\cdots,x_r)}{k!} \\
&= \mathrm{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E}) - c_2(\mathcal{E})}{2!} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) - 3c_3(\mathcal{E})}{3!} + \cdots \\
&\quad \in H^0 + H^2 + H^4 + H^6 \\
&= \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \mathrm{ch}_2(\mathcal{E}) + \cdots, \\
&\quad \mathrm{ch}_i(\mathcal{E}) \in H^{2i}(X;\mathbb{Q}).
\end{aligned}$$

Definition 14.0.6 (Todd Class)
The **total Todd class**

$$\mathrm{td}(\mathcal{E}) \coloneqq \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}.$$

Note that

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \frac{x_i}{2} + \frac{x_i^2}{12} + \frac{x_i^4}{720} + \cdots = 1 + \frac{x_i}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}B_i}{(2i)!} x^{2i}.$$

where L'Hopital shows that the derivative at $x_i = 0$ exists, so it's analytic at zero and the expansion makes sense, and the B_i are Bernoulli numbers.

Remark 14.0.7 ***(Very important and useful!!):*** $\mathrm{ch}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{ch}(\mathcal{E}) + \mathrm{ch}(\mathcal{F})$ and $\mathrm{ch}(\mathcal{E} \otimes \mathcal{F}) = \sum_{i,j} e^{x_i + y_j} = \mathrm{ch}(\mathcal{E})\,\mathrm{ch}(\mathcal{F})$ using the fact that $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$. So ch is a "ring morphism" in the sense that it preserves multiplication $\otimes$ and addition $\oplus$, making the Chern character even better than the total Chern class.

Definition 14.0.8 (Todd Class)
Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, then define the **Todd class** of X as $\mathrm{td}_{\mathbb{C}}(X) \coloneqq \mathrm{td}(TX)$ where TX is viewed as a complex vector bundle. If $X \in \mathbf{Mfd}_{\mathbb{R}}$, define $\mathrm{td}_{\mathbb{R}} = \mathrm{td}(TX \otimes_{\mathbb{R}} \mathbb{C})$.

14.1 Section 5: Riemann-Roch and Generalizations

Remark 14.1.1: Let $X \in \mathbf{Top}$ and let $\mathcal{F}$ be a sheaf of vector spaces. Suppose $h^i(X;\mathcal{F}) \coloneqq \dim H^i(X;\mathcal{F}) < \infty$ for all i and is equal to 0 for $i \gg 0$.

Definition 14.1.2 (Euler Characteristic of a Sheaf)
The **Euler characteristic** of $\mathcal{F}$ is defined as

$$\chi(X;\mathcal{F}) \coloneqq \chi(\mathcal{F}) \coloneqq \sum_{i=0}^{\infty} (-1)^i h_i(X;\mathcal{F}).$$

⚠ **Warning 14.1.3**
This is not always well-defined!

Example 14.1.4 *(?)*: Let $X \in \mathbf{Mfd}_{\mathrm{cpt}}$ and take $\mathcal{F} \coloneqq \underline{\mathbb{R}}$, we then have

$$\chi(X;\underline{\mathbb{R}}) = h^0(X;\mathbb{R}) - h^1(X;\mathbb{R}) + \cdots = b_0 - b_1 + b_2 - \cdots \coloneqq \chi_{\mathbf{Top}}(X).$$

Example 14.1.5 *(?)*: Let $X = \mathbb{C}$ and take $\mathcal{F} \coloneqq \mathcal{O} \coloneqq \mathcal{O}^{\mathrm{holo}}$ the sheaf of holomorphic functions. We then have $h^{>0}(X;\mathcal{O}) = 0$, but $H^0(X;\mathcal{O})$ is the space of all holomorphic functions on $\mathbb{C}$, making $\dim_{\mathbb{C}} h^0(X;\mathcal{O})$ infinite.

Example 14.1.6 *(?)*: Take $X = \mathbb{P}^1$ with $\mathcal{O}$ as above, $h^0(\mathbb{P}^1;\mathcal{O}) = 1$ since $\mathbb{P}^1$ is compact and the maximum modulus principle applies, so the only global holomorphic functions are constant. We can write $\mathbb{P}^1 = \mathbb{C}_1 \cup \mathbb{C}_2$ as a cover and $h^i(\mathbb{C},\mathcal{O}) = 0$, so this is an acyclic cover and we can use it to compute $h^1(\mathbb{P}^1;\mathcal{O})$ using Čech cohomology. We have

- $C^0(\mathbb{P}^1;\mathcal{O}) = \mathcal{O}(\mathbb{C}_1) \oplus \mathcal{O}(\mathbb{C}_2)$
- $C^1(\mathbb{P}^1;\mathcal{O}) = \mathcal{O}(\mathbb{C}_1 \cap \mathbb{C}_2) = \mathcal{O}(\mathbb{C}^\times)$.
- The boundary map is given by

$$\begin{aligned} \partial_0 : C^0 &\to C^1 \\ (f(z), g(z)) &\mapsto g(1/z) - f(z) \end{aligned}$$

  and there are no triple intersections.

Is every holomorphic function on $\mathbb{C}^\times$ of the form $g(1/z) - f(z)$ with f, g holomorphic on $\mathbb{C}$. The answer is yes, by Laurent expansion, and thus $h^1 = 0$. We can thus compute $\chi(\mathbb{P}^1; \mathcal{O}) = 1 - 0 = 1$.

15 | Monday, February 15

Remark 15.0.1: Last time: we saw that $\chi(\mathbb{P}^1, \mathcal{O}) = 1$, and we'd like to generalize to holomorphic line bundles on a Riemann surface. This will be the main ingredient for Riemann-Roch.

Theorem 15.0.2 *(?).*
Let $X \in \mathbf{Mfd}_{\mathbb{C}}$ be compact and let $\mathcal{F}$ be a holomorphic vector bundle on X [a] Then χ is well-defined and

$$h^{>\dim_{\mathbb{C}} X}(X; \mathcal{F}) = 0.$$

[a] Or more generally a finitely-generated $\mathcal{O}$-module, i.e. a coherent sheaf.

Remark 15.0.3: The locally constant sheaf $\underline{\mathbb{C}}$ is not an $\mathcal{O}$-module, i.e. $\underline{\mathbb{C}}(U) \notin \mathcal{O}(U)$-**Mod**. In fact, $h^{2i}(X, \underline{\mathbb{C}}) = \mathbb{C}$ for all i.

Proof (?).
We'll can resolve $\mathcal{F}$ as a sheaf by first mapping to its smooth sections and continuing in the following way:

$$0 \to \mathcal{F} \to C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \to \cdots,$$

where $\bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} \, d\bar{z}_i$. Suppose we have a holomorphic trivialization of $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ and we have sections $(s_1, \cdots, s_r) \in C^\infty \mathcal{F}(U)$, which are smooth functions on U. In local coordinates we have

$$\bar{\partial} s \coloneqq (\bar{\partial} s_1, \cdots, \bar{\partial} s_r),$$

but is this well-defined globally? Given a different trivialization over $V \subseteq X$, the s_i are related by transition functions, so the new sections are $t_{UV}(s_1, \cdots, s_r)$ where $t_{UV} : U \cap V \to \mathrm{GL}_r(\mathbb{C})$. Since t_{UV} are holomorphic, we have

$$\bar{\partial}(t_{UV}(s_1, \cdots, s_r)) = t_{UV} \bar{\partial}(s_1, \cdots, s_r).$$

This makes $\bar{\partial} : C^\infty \mathcal{F} \to F \otimes A^{0,1}$ a well-defined (but not $\mathcal{O}$-linear) map. We can thus continue this resolution using the Leibniz rule:

$$0 \to \mathcal{F} \to C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \xrightarrow{\bar{\partial}} \cdots F \otimes A^{0,2} \xrightarrow{\bar{\partial}} \cdots,$$

which is an exact sequence of sheaves since $(A^{0,\cdot}, \bar{\partial})$ is exact.

Why? Split into line bundles?

We can identify $C^\infty \mathcal{F} = \mathcal{F} \otimes A^{0,0}$, and $\mathcal{F} \otimes A^{0,q}$ is a smooth vector bundle on X. Using partitions of unity, we have that $\mathcal{F} \otimes A^{0,q}$ is acyclic, so its higher cohomology vanishes, and

$$H^i(X; \mathcal{F}) \cong \frac{\ker(\bar{\partial} : \mathcal{F} \otimes A^{0,i} \to \mathcal{F} \otimes A^{0,i+1}}{\operatorname{im}(\bar{\partial} : \mathcal{F} \otimes A^{0,i-1} \to \mathcal{F} \otimes A^{0,i}}.$$

However, we know that $A^{0,p} = 0$ for all $p > n \coloneqq \dim_{\mathbb{C}} X$, since any wedge of $p > n$ forms necessarily vanishes since there are only n complex coordinates.

∎

Warning 15.0.4
This only applies to holomorphic vector bundles or $\mathcal{O}$-modules!

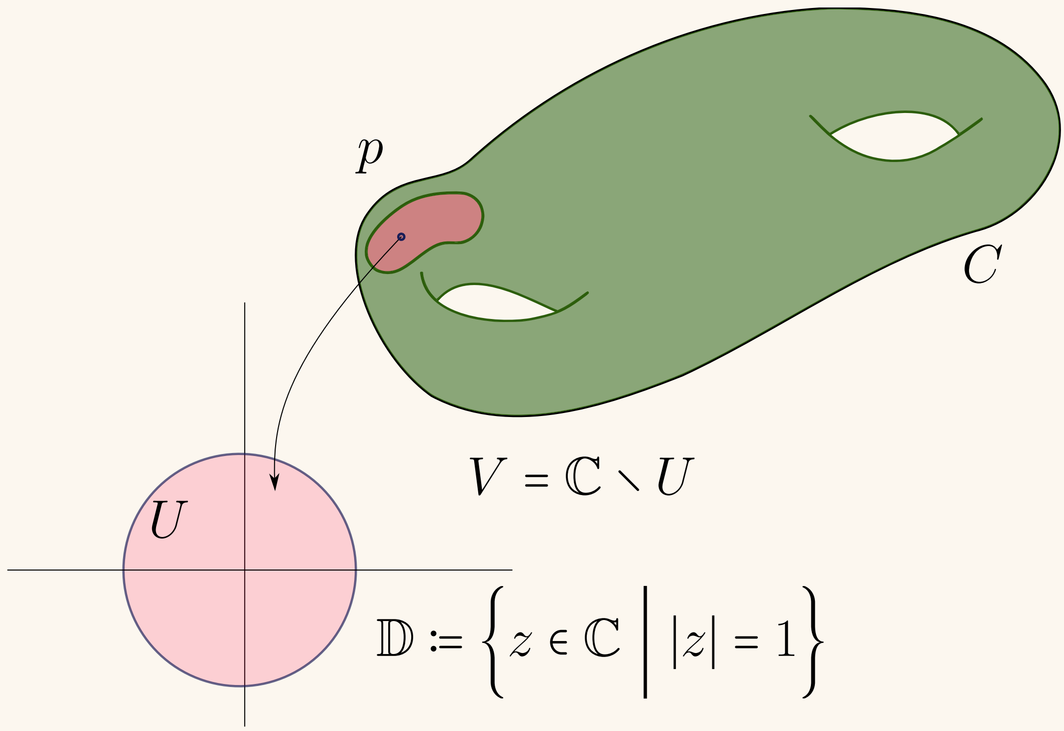
15.1 Riemann-Roch

Theorem 15.1.1 ***(Riemann-Roch).***
Let C be a compact connected Riemann surface, i.e. $X \in \mathbf{Mfd}_{\mathbb{C}}$ with $\dim_{\mathbb{C}}(X) = 1$, and let $\mathcal{L} \to C$ be a holomorphic line bundle. Then

$$\chi(C, \mathcal{L}) = \deg(L) + (1 - g) \qquad \text{where } \int_C c_1(\mathcal{L})$$

and g is the genus of C.

Proof (?).
We'll introduce the notion of a "point bundle", which are particularly nice line bundles, denoted $\mathcal{O}(p)$ for $p \in \mathbb{C}$.



Taking $\mathbb{D}$ to be a disc of radius $1/2$ and V to be its complement, we have $t_{uv}(z) = z^{-1} \in \mathcal{O}^*(U \cap V)$. We can take a holomorphic section $s_p \in H^0(C, \mathcal{O}(p))$, where $s_p|_U = z$ and $s_p|_V = 1$. Then $t_{uv}(s_p|_U) = s_p|_V$ on the overlaps. We have a function which precisely vanishes to first order at p. Recall that $c_1(\mathcal{O}(p))$ is represented by $[V(s)] = [p]$, and moreover $\int_C c_1(\mathcal{O}(p)) = 1$.
We now want to generalize this to a **divisor**: a formal $\mathbb{Z}$-linear combination of points.
Example 15.1.2 ***(?):*** Take $p, q, r \in C$, then a divisor can be defined as something like $D \coloneqq 2[p] - [q] + 3[r]$.
Define $\mathcal{O}(D) \coloneqq \bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}$ for any $D = \sum n_i [p_i]$. Here tensoring by negatives means taking duals, i.e. $\mathcal{O}(-[p]) \coloneqq \mathcal{O}^{\otimes -1} \coloneqq \mathcal{O}(p)^\vee$, the line bundle with inverted transition functions. $\mathcal{O}(D)$ has a meromorphic section given by

$$s_D \coloneqq \prod s_{p_i}^{n_i} \in \mathrm{Mero}(C, \mathcal{O}(D))$$

where we take the sections coming from point bundles. We can compute

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i \coloneqq \deg(D).$$

.
Example 15.1.3 ***(?):***

$$\deg(2[p] - [q] + 3[r]) = 4.$$

Remark 15.1.4: Assume our line bundle L is $\mathcal{O}(D)$, we'll prove Riemann-Roch in this case by induction on $\sum |n_i|$. The base case is $\mathcal{O}$, which corresponds to taking an empty divisor. Then either

- Take $D = D_0 + [p]$ with $\deg(D_0) < \sum |n_i|$ (for which we need some positive coefficient), or
- Take $D_0 = D + [p]$.

Claim: There is an exact sequence

$$0 \to \mathcal{O}(D_0) \to \mathcal{O}(D) \to \mathbb{C}_p \to 0$$
$$s \in \mathcal{O}(D_0)(U) \mapsto s \cdot s_p \in \mathcal{O}(D_0 + [p])(U),$$

where the last term is the skyscraper sheaf at p.

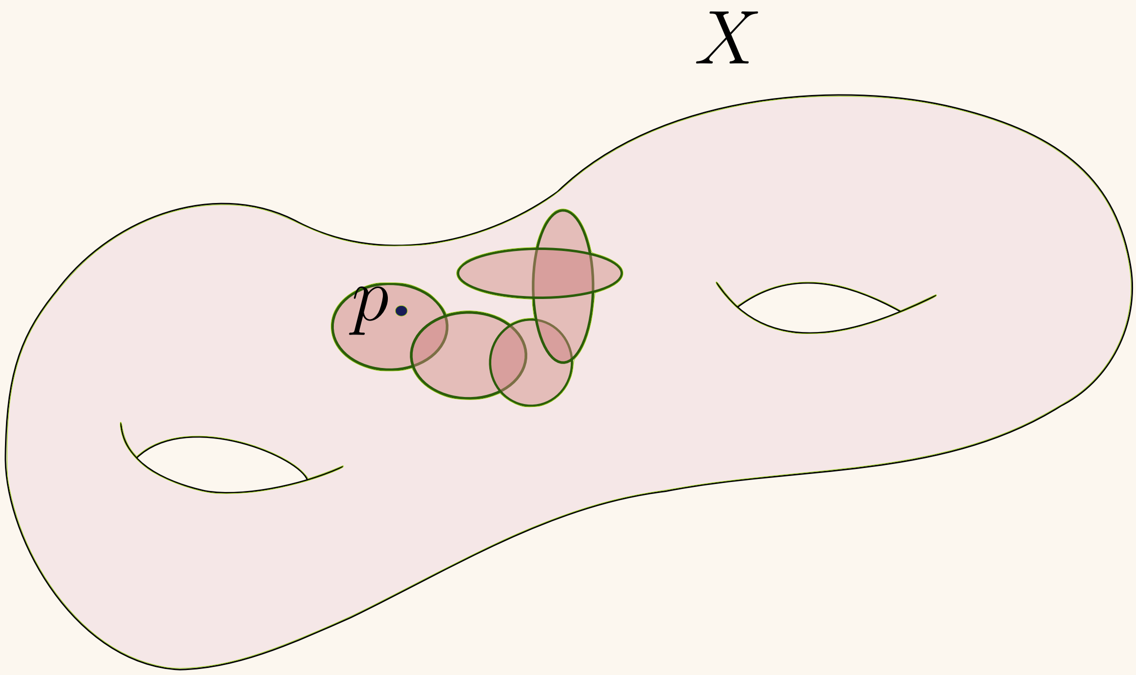
Proof (?).
The given map is $\mathcal{O}$-linear and injective, since $s_p \neq 0$ and $ss_p = 0$ forces $s = 0$. Recall that we looked at $\mathcal{O} \xrightarrow{\cdot z} \mathcal{O}$ on $\mathbb{C}$, and this section only vanishes at p (and to first order). The same situation is happening here. ∎

Thus there is a LES

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}(D_0)) \arrow[r] & H^0(\mathcal{O}(D)) \arrow[r] & H^0(\mathcal{O}(\mathbb{C}_p)) \arrow[dll] \\
& H^1(\mathcal{O}(D_0)) \arrow[r] & H^1(\mathcal{O}(D)) \arrow[r] & H^1(\mathcal{O}(\mathbb{C}_p)) = 0 \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

We also have $h^1(\mathbb{C}_p) = 0$ by taking a sufficiently fine open cover where p is only in one open set. So just check Čech cocycles yields $C^1_U(C, \mathbb{C}_p) \coloneqq \prod_{i<j} \mathbb{C}_p(U_i \cap U_j) = 0$ since p is in no intersection.



We obtain $\chi(\mathcal{O}(D) = \chi(\mathcal{O}(D_0)) + 1$, using that it is additive in SESs

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \implies \qquad \chi(\mathcal{E}_2) = \chi(\mathcal{E}) + \chi(\mathcal{E}_3)$$

and thus

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i = \deg(D) = \deg D_0 + 1.$$

The last step is to show that $\chi(C, \mathcal{O}) = 1 - g$, so just define g so that this is true! ■

Remark 15.1.5: Why is every $L \cong \mathcal{O}(D)$ for some D? Easy to see if L has meromorphic sections: if s is a meromorphic section of L, then the following works:

$$D = \mathrm{Div}(s) = \sum_p \mathrm{Ord}_p(s)[p].$$

Then $\mathcal{O} \cong L \otimes \mathcal{O}(-D)$ has a meromorphic section ss_{-D}, a global nonvanishing section with $\mathrm{Div}(ss_{-D}) = \varnothing$. Proving that every holomorphic line bundle has a meromorphic section is hard!

16 | Friday, February 19

16.1 Applications of Riemann-Roch

Definition 16.1.1 (Curves)
A **curve** is a compact complex manifold of complex dimension 1.

Example 16.1.2*(?)*: Let C be a curve, then Ω_C^1 is the sheaf of holomorphic 1-forms, and $\Omega_C^{>1} = 0$. We also have the sheaves $A^{1,0}, A^{0,1}, A^{1,1}$, the sheaves of smooth (p,q)-forms. Here the only nonzero combinations are $(0,0), (0,1), (1,0), (1,1)$ by dimensional considerations. Let L be a holomorphic line bundle on C, then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg(L) + 1 - g.$$

Remark 16.1.3: In general it can be hard to compute $h^1(L)$, since this is sheaf cohomology (sections over double overlaps, cocycle conditions, etc). On the other hand, h^0 is easy to understand, since $h^0(\Omega_C^1)$ is the dimension of the global holomorphic sections $H^0(C, L) = L(C)$. A key tool here is the following:

Proposition 16.1.4*(Serre Duality).*

$$H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega_C^1)^\vee,$$

noting that these are both global sections of a line bundle.

Proof (?).
Recall that we had a resolution of the sheaf L given by by smooth vector bundles:

$$0 \to L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar\partial} L \otimes A^{0,1} \xrightarrow{\bar\partial} 0.$$

So we know that $H^1(C, L) = H^0(L \otimes A^{0,1}) / \bar\partial H^0(L \otimes A^{0,0})$. Choose a Hermitian metric h on L, i.e. a map $h : L \otimes \overline{L} \to \mathcal{O}$. On fibers, we have $h_p : L_p \otimes \overline{L_p} \to \mathbb{C}$. We'll also choose a metric on C, say g. Since C is a Riemann surface, we have an associated volume form ν on C (essentially the determinant), so we can define a pairing between sections of $L \otimes A^{0,0}$:

$$\langle s,\ t \rangle \coloneqq \int_C h(s, \bar t)\, d\nu.$$

Note that $\langle s,\ s \rangle = \int_C h(s, \bar s\, d\nu \geq 0$ since $h(s, \bar s)(p) = 0 \iff s_p = 0$, and moreover this integral is zero if and only if $s = 0$. So we have an inner product on $H^0(L \otimes A^{0,0})$. We can also define a pairing on sections of $L \otimes A^{0,1}$, say

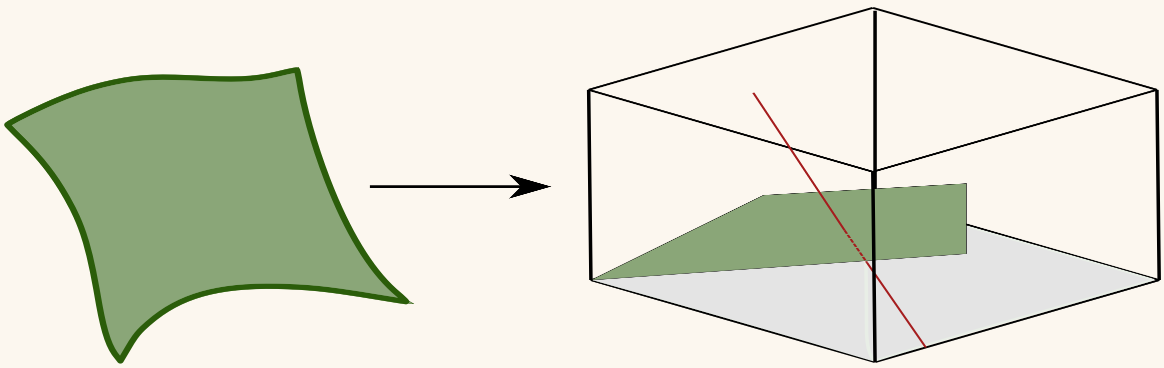
$$\langle s \otimes \alpha,\ t \otimes \beta \rangle = \int_C h(s, \bar t) \alpha \wedge \overline{\beta}.$$

Note that h is a smooth function and $\alpha \wedge \overline{\beta}$ is a $(1,1)$-form. Moreover, this is positive and nondegenerate. We want to understand the cokernel of the linear map

$$H^0(L \otimes A^{0,0}) \xrightarrow{\overline{\partial}} H^0(L \otimes A^{0,1}).$$

To compute $\operatorname{coker}(\overline{\partial})$, we can look at the kernel of the adjoint, and it suffices to find the orthogonal complement of $\operatorname{im}(\overline{\partial})$, i.e.

$$\operatorname{coker}(\overline{\partial}) = \left\{ t \in H^0(L \otimes A^{0,1}) \;\middle|\; \left\langle \overline{\partial} s,\ t \right\rangle = 0 \,\forall s \right\}.$$



So we want to understand sections $t \in H^0(L \otimes A^{0,1})$ such that

$$\int_C (\overline{\partial} s)\overline{t} = 0 \qquad\qquad \forall s \in H^0(L \otimes A^{0,0}),$$

where $\partial C = \varnothing$. We'll basically want to do integration by parts on this. Note that $h(s,t) = hst$ here where we view h as a certain section. Note that $\overline{t} \in H^0(\overline{L} \otimes A^{1,0})$, so we can replace ∂ with $d = \overline{\partial} + \partial$ and apply Stokes' theorem:

$$\begin{aligned}
\int_C sd(h\overline{t}) &= 0 \qquad\qquad \forall s \in H^0(L \otimes A^{0,0}) \\
0 &= \int_C s\overline{\partial}(h\overline{t}) \\
&= \int_C s\frac{\overline{\partial}(h\overline{t})}{d\nu} d\nu \\
&= \left\langle s,\ \overline{\frac{\overline{\partial}(h\overline{t})}{d\nu}} \right\rangle
\end{aligned}$$

where $h \in C^\infty(L^{-1} \otimes \overline{L}^{-1})$ and $h\overline{t} \in C^\infty(L^{-1} \otimes A^{1,0})$. But the right-hand side is in $H^0(L \otimes A^{0,0})$ and by nondegeneracy we can conclude

$$\overline{\frac{\overline{\partial}(h\overline{t})}{d\nu}} = 0 \iff \overline{\partial}(h\overline{t}) = 0.$$

We thus have $h\overline{t} \in H^0(L^{-1} \otimes A^{1,0}$ which is a holomorphic line bundle tensored with $A^{0,0}$. Thus $\operatorname{coker}(\overline{\partial}) \cong_h H^0(L^{-1} \otimes \Omega^1)$.

■

Remark 16.1.5: We showed $\left\langle \bar{\partial} s,\ t \right\rangle = \langle s,\ Y(t) \rangle$ where Y is the adjoint given above. Then the kernel of Y wound up being where $\bar{\partial}$ vanishes, i.e. holomorphic sections of a separate bundle. Here we had

- $t \in H^0(L \otimes A^{0,1})$
- $\bar{t} \in H^0(\overline{L} \otimes A^{1,0})$
- $h \in H^0(L^{-1} \otimes \overline{L^{-1}})$

ToDos

List of Todos

Definitions

Theorems

Exercises

Figures

List of Figures

Bibliography

[1] Danny Calegari. *Notes on 4-manifolds*. https://math.uchicago.edu/~dannyc/courses/4manifolds_2018/4_manifolds_notes.pdf.

[2] Richard Mandelbaum. "Four-dimensional topology: an introduction". In: *Bull. Amer. Math. Soc. (N.S.)* 2.1 (Jan. 1980), pp. 1–159. URL: https://projecteuclid.org:443/euclid.bams/1183545202.

[3] Akhil Matthew. *The Dirac Operator*. https://math.uchicago.edu/~amathew/dirac.pdf.

[4] Yuli Rudyak. *Piecewise Linear Structures on Topological Manifolds*. https://hopf.math.purdue.edu/Rudyak/PLstructures.pdf.

[5] Dietmar Salamon. *Spin Geometry and Seiberg-Witten Invariants*. https://people.math.ethz.ch/~salamon/PREPRINTS/witsei.pdf. 1999.

[6] Tom Weston. *An Introduction to Cobordism Theory*. https://people.math.umass.edu/~weston/oldpapers/cobord.pdf.