

# Homological Algebra Problem Sets

## Problem Set 3

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# 1 | Wednesday, February 17

**Problem 1.0.1** (Prove Corollary 2.3.2)

For  $R$  a PID, show that an  $R$ -module  $A$  is divisible if and only if  $A$  is injective.

*Recall that a module is divisible if and only if for every  $r \neq 0 \in R$  and every  $a \in A$ , we have  $a = br$  for some  $b \in A$ .*

**Solution:**

Note: we'll assume  $R$  is commutative, and since  $R$  is a domain, it has no nonzero zero divisors and thus all elements  $r \in R$  are left-cancelable.

$\Rightarrow$  : Suppose  $A$  is divisible, we then want to show every  $R$ -module morphism of the following form lifts, where we regard the ideal  $J$  and the ring  $R$  as  $R$ -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since  $R$  is a PID, we have  $J = jR$  for some  $j \in R$ , so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider  $f(j) \in A$ . Since  $A$  is divisible, we have  $A = jA$ , so we can write  $f(j) = j\mathbf{a}'$  for some  $\mathbf{a}' \in A$ . Using  $R$ -linearity and the fact that  $j$  is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending  $R$ -linearly yields a well-defined  $R$ -module morphism. Moreover, the diagram commutes by construction, since  $\iota(1_R) = 1_R$ .

$\Leftarrow$  : Suppose  $A \in R\text{-Mod}$  is injective, where by Baer's criterion we equivalently have a lift of the following form for every  $J \leq R$ :

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let  $j \in R$  be a nonzero element that is not a zero-divisor, we then want to show that  $A = jA$ , i.e. that for every  $\mathbf{a} \in A$ , there is a  $\mathbf{a}' \in A$  such that  $\mathbf{a} = j\mathbf{a}'$ . Fixing  $\mathbf{a} \in A$ , define a map  $f_a : J \rightarrow A$  in the following way: for  $x \in J$ , use the fact that  $\langle j \rangle := jR$  to first write  $x = jr$  for some  $r \in R$ , and then set  $f_a(x) = f_a(jr) := r\mathbf{a}$ . To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion  $jR \hookrightarrow R$  and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an  $r$  on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting  $\mathbf{a}$  range over all elements of  $A$  yields the desired result.

**Problem 1.0.2** (Calculating Ext Groups)

Calculate  $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$  for distinct primes  $p, q$ .

The following are several claims that are later used in the actual solution:

**Claim 1:** For any  $m \in \mathbb{Z}$ ,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

*Proof (?)*.

Note that there is an injection

$$1 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \rightarrow \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \rightarrow 1$$

where  $\pi_n$  is the canonical quotient morphism, and applying the left-exact contravariant functor  $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$  yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any  $\mathbb{Z}$ -module morphism  $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$ ,

1. Since  $\mathbb{Z}$  is a free  $\mathbb{Z}$ -module with generator 1,  $f$  is entirely determined by  $f(1)$ , and
2.  $f$  descends to a map  $\tilde{f} : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$  if and only if  $f(n) \in \mathbb{Z}$ , i.e.  $f(n) = [0]$  is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule  $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$  as all of those morphism  $\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  which descend to make the following diagram commute.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z} \\ \pi_n \downarrow & \nearrow \exists \tilde{f} & \\ \mathbb{Z}/n & & \end{array}$$

[Link to Diagram](#)

To characterize these, it suffices to determine all of the possible images  $f(1)$ . Moreover, we can restrict our attention to coset representatives in the interval  $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$ , where we want to find all  $q := f(1) \in [0, 1)$  such that  $nq = 1$ . A complete list of  $n$  such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}.$$

Setting  $f_i(1) := \left[ \frac{i}{n} \right]$  (where we take the equivalence class mod  $\mathbb{Z}$ ) yields  $n$  distinct morphisms  $f_i : \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  that descend to  $\tilde{f}_i : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$ . We can define a map

$$\begin{aligned} \Psi : \mathbb{Z} &\rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i, \end{aligned}$$

and using the fact that if  $i = i' \pmod{n}$ , write  $i' = i + kn$  for some  $k \in \mathbb{Z}$ , then

$$f_{i'}(1) = f_{i+kn}(1) = \left\lfloor \frac{i+kn}{n} \right\rfloor = \left\lfloor \frac{i}{n} + k \right\rfloor = \left\lfloor \frac{i}{n} \right\rfloor = f_i(1),$$

since  $k \in \mathbb{Z}$ , so by the first isomorphism theorem  $\Psi$  descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

**Claim 2:**  $\mathbb{Q}/\mathbb{Z}$  is an injective object in  $\mathbb{Z}$ -modules.

*Proof (?)*.

By the previous exercise, it suffices to show that  $\mathbb{Q}/\mathbb{Z}$  is divisible. More generally, if any group  $G$  is divisible and  $N \trianglelefteq G$  is a normal subgroup, then  $G/N$  will be divisible. This follows from the fact that if  $\bar{a}, \bar{b} \in G/N$  and  $n \in \mathbb{Z}$ , we can write  $\bar{a} = a + N$  and  $\bar{b} = b + N$  for some coset representatives, use divisibility to write  $a = nb$ , and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That  $\mathbb{Q}$  is divisible is a straightforward check: let  $n \in \mathbb{Z}$  and  $a \in \mathbb{Q}$ , we then want a  $b \in \mathbb{Q}$  such that  $a = nb$ , and  $b := \frac{a}{n} \in \mathbb{Q}$  works. Since  $\mathbb{Q}$  is an abelian group,  $\mathbb{Z}$  is automatically normal, and the result follows.

■

**Claim:**

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \quad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

*Proof (?)*.

Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M := \mathbb{Z}/m$ ,
- $A := \mathbb{Z}$ ,
- $I := n\mathbb{Z}$ ,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n(\mathbb{Z}/m)} \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\rightarrow \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
 \ker \varphi &= \{x \in \mathbb{Z} \mid x(1 \otimes 1) = 0\} \\
 &= \{x \in \mathbb{Z} \mid n \mid x \text{ or } m \mid x\} \\
 &= \langle n, m \rangle \\
 &= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
 &:= \langle d \rangle.
 \end{aligned}$$

Now applying the first isomorphism theorem yields the result. ■

### Solution:

We'll follow the procedure outlined in Weibel:

- Define the contravariant functor  $F(\cdot) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \cdot)$ , then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution  $I$  of  $\mathbb{Z}/q$ .
- Write  $F(I)$  as a new (not necessarily exact) chain complex.
- Compute  $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$ .

We can first take the following injective resolution:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbb{Z}/q & \xrightarrow{d^{-1}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & \mathbb{Q}/\mathbb{Z} \xrightarrow{d^1} 1 \\
 & & & & & & \\
 & & [1]_q & \longrightarrow & \begin{bmatrix} 1 \\ q \end{bmatrix} & & 
 \end{array}$$

$$[x] \longrightarrow [qx]$$

[Link to Diagram](#)

This is a chain complex by construction, since  $d^2([1]_q) = \begin{bmatrix} q \left( \frac{1}{q} \right) \end{bmatrix} = [1] = [0]$ . We now delete the augmentation and apply  $F(\cdot)$ :

$$\begin{array}{ccccccc}
1 & \longrightarrow & I^0 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & I^1 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^1} & 1 \\
& & \downarrow F(\cdot) & & & & \\
1 & \longrightarrow & F(I^0) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^0 := F(d^0)} & F(I^1) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^1 := F(d^1)} & 1 \\
\parallel & & \uparrow \Psi \cong & & \uparrow \Psi \cong & & \parallel \\
1 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^0} & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^1} & 1
\end{array}$$

[Link to Diagram](#)

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that  $d^0(x) := qx$  was multiplication by  $q$ , we have  $\partial^0(f) = d^0 \circ f$  is post-composition by the multiplication by  $q$  map, and  $\tilde{\partial}^0$  similarly becomes multiplication by  $q$ .

We now take homology:

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\text{im } \partial^0} = \frac{\mathbb{Z}/p}{q(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where  $d := \gcd(p, q) = 1$  if  $p, q$  are coprime.

**Problem 1.0.3** (Weibel 2.3.2)

Let  $A \in \mathbf{Ab}$ , and show that the following map is injective:

$$\begin{aligned}
\varepsilon_A : A &\rightarrow I(A) := \prod_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\
a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},
\end{aligned}$$

i.e. when looking at the image  $\varepsilon_A(a)$  in the product, the component indexed by  $f$  is an element of  $\mathbb{Q}/\mathbb{Z}$  obtained by evaluating  $f(a)$ .

*Hint: if  $a \in A$ , find a map  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $f(a) \neq 0$  and extend this to a map  $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$ .*

**Solution:**

By contrapositive, we'll suppose  $a \neq 0$  and show  $\varepsilon_A(a) \neq 0$ . Following the hint, we first consider the cyclic subgroup  $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$  and define a map

$$\begin{aligned}
f_a : a\mathbb{Z} &\rightarrow \mathbb{Z} \\
an &\mapsto n.
\end{aligned}$$

We now pick  $\ell > 1 \in \mathbb{Z}$  to be any integer, and define a composition  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ :



$$\begin{array}{ccccccc}
 a\mathbb{Z} & \xrightarrow{f_a} & \mathbb{Z} & \xleftarrow{\iota} & \mathbb{Q} & \xrightarrow{x \mapsto \frac{x}{\ell}} & \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \\
 & \nearrow f & & & & & \nearrow f \\
 & & & & & & \\
 an & \longmapsto & n & \longmapsto & n & \longmapsto & \frac{n}{\ell} \longmapsto \left[ \frac{n}{\ell} \right] \\
 & & & & & & \\
 a & \xrightarrow{\quad\quad\quad} & & & & & \left[ \frac{1}{\ell} \right]
 \end{array}$$

[Link to Diagram](#)

By choice of  $\ell$ , this map satisfies  $f(a) = [1/\ell] \neq 0$ , so the map is nonzero. Since  $\mathbb{Q}/\mathbb{Z}$  is injective, the universal property provides a lift  $\tilde{f}$ :

$$\begin{array}{ccc}
 A & & \\
 \uparrow & \searrow \exists \tilde{f} & \\
 a\mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z}
 \end{array}$$

[Link to Diagram](#)

Since  $\tilde{f}$  lifts  $f$ , it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) := f(a) \neq 0,$$

so the  $f$  component of the image of  $a$  is nonzero and thus  $\mathbf{a} := \varepsilon_A(a) \neq 0$  in the product.

**Problem 1.0.4** (Weibel 2.4.2)

If  $U : \mathcal{B} \rightarrow \mathcal{C}$  is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

**Solution:**

We'll show that  $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$  for every object  $X$ . Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let  $P \rightarrow X$  be a projective resolution of  $X$ . Fixing labeling, we have the following situation:

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 \xrightarrow{\varepsilon} X \xrightarrow{0} 0 \\
 & & & & \downarrow F(\cdot) & & \\
 \cdots & \longrightarrow & FP_2 & \xrightarrow{F(\partial_2)} & FP_1 & \xrightarrow{F(\partial_1)} & FP_0 \xrightarrow{0} 0
 \end{array}$$

[Link to Diagram](#)

We now have by definition

$$L_i F(X) := \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U(L_i F(X)) := U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution  $P \rightarrow X$ , and apply a similar process:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X & \xrightarrow{0} & 0 \\ & & & & & \Downarrow (U \circ F)(\cdot) & & & & & \\ \cdots & \longrightarrow & UFP_2 & \xrightarrow{(UF)(\partial_2)} & UFP_1 & \xrightarrow{(UF)(\partial_1)} & UFP_0 & \xrightarrow{0} & 0 & & \end{array}$$

[Link to Diagram](#)

Again, by definition,

$$(L_i(UF))(X) := \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor  $U$  to the following SES to produce a new SES, from which we'll produce the desired isomorphism  $f$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{im} F(\partial_{i+1}) & \xrightarrow{\iota_i} & \ker F(\partial_i) & \xrightarrow{\pi_i} & \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \longrightarrow 0 \\ & & & & \Downarrow U & & \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{U(\pi_i)} & U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow f \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{\tilde{\pi}_i} & \frac{U(\ker F(\partial_i))}{U(\operatorname{im} F(\partial_{i+1}))} \longrightarrow 0 \end{array}$$

[Link to Diagram](#)

Here  $\tilde{\pi}_i$  is the natural quotient map, whose image is  $\operatorname{coker} U(\iota_i)$ . Finally, the map  $f$  exists in any abelian category, using that whenever  $0 \rightarrow A \xrightarrow{g_1} B \xrightarrow{g_2} C \rightarrow 0$  is exact, there is an isomorphism  $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$ .

*Problem 1.0.5* (Weibel 2.4.3)

If  $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$  is exact with  $P$  projective or  $F$ -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

Show that  $L_{m+1} F(A)$  is the kernel of  $F(M_m) \rightarrow F(P_m)$ . Conclude that if  $P \rightarrow A$  is an  $F$ -acyclic resolution of  $A$ , then  $L_i F(A) = H_i(F(P))$ .

**Solution:**

Let  $P_M \rightarrow M$  and  $P_A \rightarrow A$  be projective resolutions of  $M$  and  $A$  respectively, then applying the Horseshoe Lemma, there is a projective resolution  $P_P \rightarrow P$  of  $P$  such that the following is a short exact sequence of chain complexes:

$$0 \rightarrow P_M \rightarrow P_P \rightarrow P_A \rightarrow 0.$$

*Problem 1.0.6* (Weibel 2.5.2)

Show that the following are equivalent:

- $A$  is a projective  $R$ -module.
- $\text{Hom}_R(\cdot, A)$  is an exact functor.
- $\text{Ext}_R^{i \neq 0}(A, B) = 0$  and for all  $B$ , i.e.  $A$  is  $\text{Hom}_R(\cdot, B)$ -acyclic for all  $B$ .
- $\text{Ext}_R^1(A, B)$  vanishes for all  $B$ .

*Problem 1.0.7* (Weibel 2.6.4)

Show that  $\text{colim}$  is left adjoint to  $\Delta$ , and conclude that  $\text{colim}$  is right-exact when  $\mathcal{A}$  is abelian and  $\text{colim}$  exists. Show that the pushout, i.e.  $\bullet \leftarrow \bullet \rightarrow \bullet$ , is not an exact functor on  $\mathbf{Ab}$ .