

# Title

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Last time: projective varieties  $V(f_i) \subset \mathbb{P}_{/k}^n$  with  $f_i$  homogeneous. We proved the projective nullstellensatz: for any projective variety  $X$ , we have  $V_p(I_p(X))$  and for any homogeneous ideal  $I$  with  $\sqrt{I} \neq I_0$  the irrelevant ideal,  $I_p(V_p(I)) = \sqrt{I}$ . Recall that  $I_0 = \langle x_0, \dots, x_n \rangle$ . We had a notion of a projective coordinate ring,  $S(X) := k[x_1, \dots, x_n]/I_p(X)$ , which is a graded ring since  $I_p(X)$  is a homogeneous ideal.

Note that  $S(X)$  is not a ring of functions on  $X$ : e.g. for  $X = \mathbb{P}^n$ ,  $S(X) = k[x_1, \dots, x_n]$  but  $x_0$  is not a function on  $\mathbb{P}^n$ . This is because  $f([x_0 : \dots : x_n]) = f([\lambda x_0 : \dots : \lambda x_n])$  but  $x_0 \neq \lambda x_0$ . It still makes sense to ask if  $f$  is zero, so  $V_p(f)$  is a well-defined object.

**Definition 1.0.1** (Dehomogenization of functions and ideals).

Let  $f \in k[x_1, \dots, x_n]$  be a homogeneous polynomial, then we define its dehomogenization as

$$f^i := f(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n].$$

For a homogeneous ideal, we define

$$J^i := \{f^i \mid f \in J\}.$$

*Example 1.0.1* : This is usually not homogeneous. Take

$$\begin{aligned} f &= x_0^3 + x_0x_1^2 + x_0x_1x_2 + x_0^2 + x_1 \\ \implies f^i &= 1 + x_1^2 + x_1x_2 + x_1, \end{aligned}$$

where has terms of mixed degrees.

*Remark 1.0.1* :

- $(fg)^i = f^i g^i$ ,
- $(f + g)^i = f^i + g^i$

In other words, evaluating at  $x_0 = 1$  is a ring morphism.

**Definition 1.0.2** (Homogenization of a function).

Let  $f \in k[x_1, \dots, x_n]$ , then the **homogenization** of  $f$  is defined by

$$f^h := x_0^d f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

where  $d := \deg(f)$ .

*Example 1.0.2 (?)*: Let  $f(x_1, x_2) = 1 + x_1^2 + x_1x_2 + x_2^3$ , then

$$f^h(x_0, x_1, x_2) = x_0^3 + x_0x_1^2 + x_0x_1x_2 + x_2^3,$$

which is a homogeneous polynomial of degree 3. Note that  $(f^h)^i = f$ .

*Example 1.0.3 (?)*: It need not be the case that  $(f^i)^h = f$ . Take  $f = x_0^3 + x_0x_1x_2$ , then  $f^i = 1 + x_1x_2$  and  $(f^i)^h = x_0^2 + x_1x_2$ . Note that the total degree dropped, since everything was divisible by  $x_0$ .

*Remark 1.0.2 :*

$$(f^i)^h = f \iff x_0 \nmid f.$$

**Definition 1.0.3** (Homogenization of an ideal).

Given  $J \subset k[x_1, \dots, x_n]$ , define its **homogenization** as

$$J^h := \{f^h \mid f \in J\}.$$

*Example 1.0.4 :* This is not a ring morphism, since  $(f+g)^h \neq f^h + g^h$  in general. Taking  $f = x_0^2 + x_1$  and  $g = -x_0^2 + x_2$ , we have  $f^h + g^h = x_0x_1 + x_0x_2$  while  $(f+g)^h = x_12 + x_2$ .

*Remark 1.0.3 :* What is the geometric significance? Set  $U_0 := \{[x_0 : \dots : x_n] \in \mathbb{P}_k^n \mid x_0 \neq 0\} \cong \mathbb{A}_{/k}^n$  with coordinates  $\left[\frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}\right]$ .

**Proposition 1.0.1 (?)**.

The conclusion is thus that  $U_0$  with the subspace topology is equal to  $\mathbb{A}^n$  with the Zariski topology.

*Proof (?)*.

If we define the Zariski topology on  $\mathbb{P}^n$  as having closed sets  $V_p(I)$ , we would want to check that  $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$  is closed in the subspace topology. This amounts to showing that  $V_p(I) \cap U_0$  is closed in  $\mathbb{A}^n \cong U_0$ . We can check that

$$V_p(f, f \in I) = \{[x_0 : \dots : x_n] \mid f(\mathbf{x}) = 0 \forall f \in I\}.$$

Intersecting with  $U_0$  yields  $\{[x_1 : \dots : x_n] \mid f(\mathbf{x}) = 0, x_0 \neq 0\}$ . Equivalently, we can rewrite this set as

$$\left\{[x_1 : \dots : x_n] \mid f\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = 0, f \text{ homogeneous}\right\}$$

Since these are coordinates on  $\mathbb{A}^1$ , we have  $V_p(I) \cap U_0 = V_a(I^i)$  which is closed.

Conversely, given a closed set  $V(I)$ , we can write this as  $V(I) = U_0 \cap V_p(I^h)$ . ■

**Corollary 1.0.1(?)**.

$\mathbb{P}^n$  is irreducible of dimension  $n$ , where the proof is that its covered by irreducible topological spaces of dimension  $n$  with nonempty intersection combined with a fact from the exercises.

*Example 1.0.5 (?)*: Consider  $f(x_1, x_2) = x_1^2 - x_2^2 - 1$  and consider  $V(f) \subset \mathbb{A}_{\mathbb{C}}^2$ :

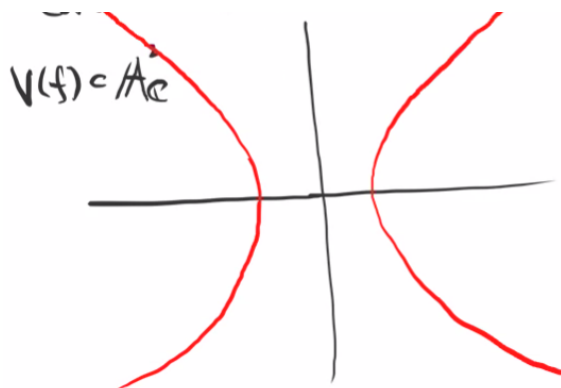


Figure 1: Image

Note that for real projective space, we can view this as a sphere with antipodal points identified. We can thus visualize this in the following way:

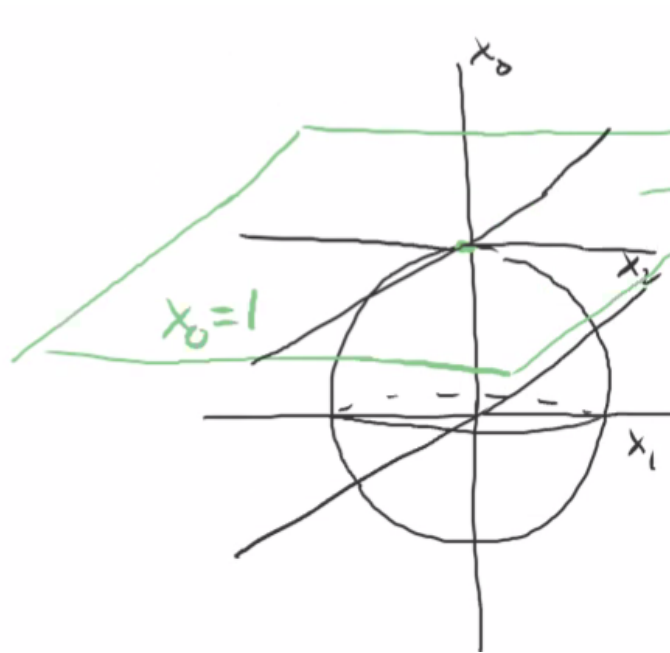


Figure 2: O

We can normalize the  $x_0$  coordinate to one, hence the plane. We can also project  $V(f)$  from the plane onto the sphere, mirroring to antipodal points:

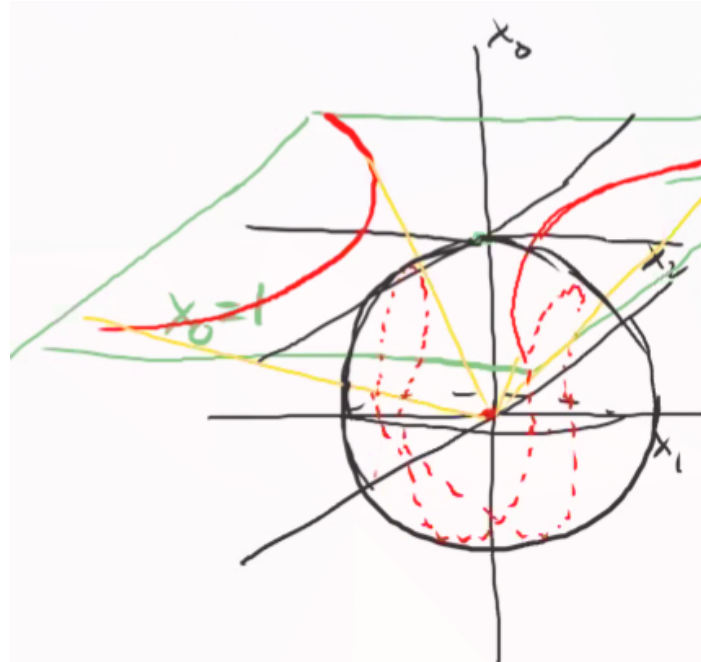


Figure 3: Image

This misses some points on the equator, since we aren't including points where  $x_0 = 0$ . Consider

the homogenization  $V(f^h) \subset \mathbb{P}_{\mathbb{C}}^2$ . It's given by  $f^h = x_1^2 - x_2^2 - x_0^2$ , then

$$V(f^h) \cap V(x_0) = \{[0 : x_1 : x_2] \mid f^h(0, x_1, x_2) = 0\} = \{[0 : 1 : 1], [0 : 1 : -1]\},$$

which can be seen in the picture as the points at infinity:

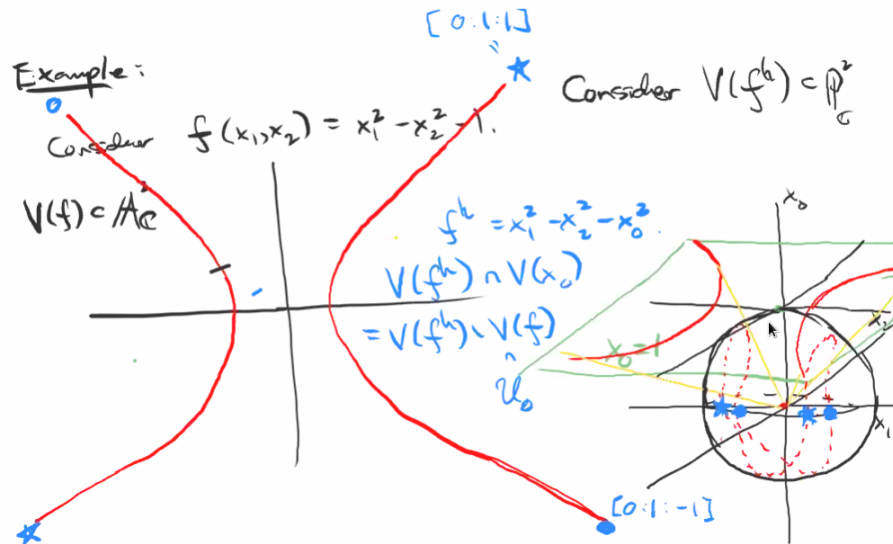


Figure 4: A

Note that the equator is  $V(x_0) = \mathbb{P}_{\mathbb{C}}^2 \setminus U_0 \cong \mathbb{P}^2 \setminus \mathbb{A}^2$ . So we get a circle of points at infinity, i.e.  $V(x_0) = \mathbb{P}^1 = \{[0 : v_1 : v_2]\}$ .

*Example 1.0.6 (?)*: Consider  $V(f)$  where  $f$  is a line in  $\mathbb{A}_{\mathbb{C}}^2$ , say  $f = ax_1 + bx_2 + c$ . This yields  $f^h = ax_1 + bx_2 + cx_0$  and we can consider  $V(f^h) \cong \mathbb{P}_{\mathbb{C}}^2$ . We know  $\mathbb{P}_{\mathbb{C}}^1$  is topologically a sphere and  $\mathbb{A}_{\mathbb{C}}^1$  is a point:



Figure 5:  $\mathbb{P}_{\mathbb{C}}^1$

The points at infinity correspond to

$$V(f^h) = V(f^h) \cap V(x_0) = \{[0 : -b : a]\},$$

which is a single point not depending on  $c$ .

*Remark 1.0.4* :  $\mathbb{P}_{/k}^2$  for any field  $k$  is a **projective plane**, which satisfies certain axioms:

1. There exists a unique line through any two distinct points,
2. Any two distinct lines intersect at a single point.

A famous example is the *Fano plane*:

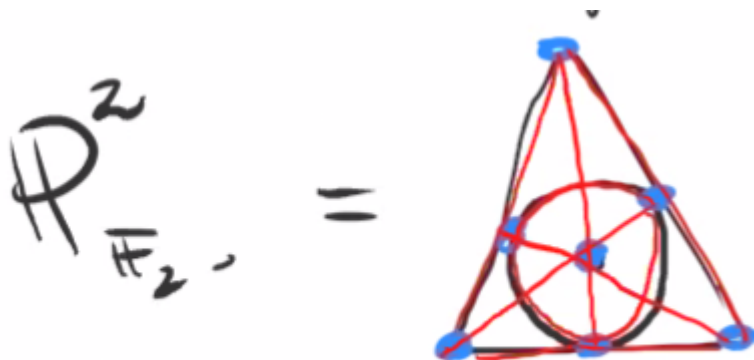


Figure 6: Fano Plane

Why is this true?  $\mathbb{P}_{/k}^2$  is the set of lines in  $k^3$ , and the lines in  $\mathbb{P}_{/k}^2$  are the vanishing loci of homogeneous polynomials and also planes in  $k^3$ , since any two lines determine a unique plane and any two planes intersect at the origin.

**Proposition 1.0.2(?)**.

Let  $J \subset k[x_1, \dots, x_n]$  be an ideal. Let  $X := V_a(J) \subset \mathbb{A}^n$  where we identify  $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$ . Then the closure  $\overline{X} \subset \mathbb{P}^n$  is given by  $\overline{X} = V_p(J^h)$ . In particular,  $V_a(J) = V_p(J^h)$ .

*Proof* (?).

Note that it's clear that  $V_p(J^h)$  is closed and contains  $V_a(J)$ . For the reverse containment, let  $Y \supseteq X$  be closed; we want to show that  $Y \supseteq V_p(J^h)$ . Since  $Y$  is closed,  $Y = V_p(J')$  where  $J'$  is some homogeneous ideal. Any element  $f' \in J'$  satisfies  $f' = x^d f$  for some maximal  $d$  where  $x_0^d f$  vanishes on  $X$ . ■