SURFACES

1 (Fall '05). State the classification theorem for surfaces (compact, without boundary, but not necessarily orientable). For each surface in the classification, indicate the structure of the first homology group and the value of the Euler characteristic. Also, explain briefly how the 2-holed torus and the connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2$ fit into the classification.

2 (Spring '16). Give a list without repetitions of all compact surfaces (orientable or non-orientable and with or without boundary) that have Euler characteristic negative one. Explain why there are no repetitions on your list.

3 (Spring '07). Describe the topological classification of all compact connected surfaces M without boundary having Euler characteristic $\chi(M) \geq -2$. No proof is required.

4 (Spring '09). How many surfaces are there, up to homeomorphism, which are: connected, compact, possibly with boundary, possibly nonorientable, and with Euler characteristic -3? Describe one representative from each class.

5 (Fall '13). Prove that the Euler characteristic of a compact surface with boundary which has k boundary components is $\leq 2 - k$.

6 (Spring '13). What surface is represented by the 6-gon with edges identified according to the symbol $xyzxy^{-1}z^{-1}$?

7 (Spring '15). Let X be the topological space obtained as the quotient space of a regular $2n$-gon ($n \geq 2$) in $\mathbb{R}^2$ by identifying opposite edges via translations in the plane. First show that X is a compact, orientable surface without boundary and then identify its genus as a function of n.

8 (Fall '10).

(a) Show that any compact connected surface with nonempty boundary is homotopy equivalent to a wedge of circles (Hint: you may assume that any compact connected surface without boundary is given by identifying edges of a polygon in pairs.)
(b) For each surface appearing in the classification of compact surfaces with nonempty boundary, say how many circles are needed in the wedge from part (a). (Hint: you should be able to do this even if you have not done part (a).)

9 (Fall '04). Let M_g^2 be the compact oriented surface of genus g. Show that there exists a continuous map $f : M_g^2 \to S^2$ which is not homotopic to a constant map.

10 (Spring '11) Show that $\mathbb{R}P^2 \vee S^1$ is not homotopy equivalent to a compact surface (possibly with boundary).

11 (Fall '14). Identify (with proof, but of course you can appeal to the classification of surfaces) all of the compact surfaces without boundary that have a cell decomposition having exactly one 0-cell and exactly two 1-cells (with no restriction on the number of cells of dimension larger than 1).

12 (Fall '11). For any natural number g let Σ_g denote the (compact, orientable) surface of genus g. Determine, with proof, all valued of g with the property that

there exists a covering space $\pi : \Sigma_5 \to \Sigma_g$. (Hint: How does the Euler characteristic behave for covering spaces?)

13 (Spring '14). Find all surfaces, orientable and non-orientable, which can be covered by a closed surface (i.e. compact with empty boundary) of genus 2. Prove that your answer is correct.

14 (Spring '18).

(a) Write down (without proof) a presentation for $\pi_1(\Sigma_2, p)$ where Σ_2 is a closed, connected, orientable genus 2 surface and p is any point on Σ_2.

(b) Show that $\pi_1(\Sigma_2, p)$ is not abelian by showing that it surjects onto a free group of rank 2.

(c) Show that there is no covering space map from Σ_2 to $S^1 \times S^1$. You may use the fact that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ together with the result in part (b) above.

15 (Fall '16). Give an example, with explanation, of a closed curve in a surfaces which is not nullhomotopic but is nullhomologous.

16 (Fall '17). Let M be a compact orientable surface of genus 2 without boundary. Give an example of a pair of loops $\gamma_0, \gamma_1 : S^1 \to M$ with $\gamma_0(1) = \gamma_1(1)$ such that there is a continuous map $\Gamma : [0,1] \times S^1 \to M$ such that $\Gamma(0,t) = \gamma_0(t), \Gamma(1,t) = \gamma_1(t)$ for all $t \in S^1$, but such that there is **no** such map Γ with the additional property that $\Gamma_s(1) = \gamma_0(1)$ for all $s \in [0,1]$. (You are not required to prove that your example satisfies the stated property.)

17 (Fall '18). Let C be cylinder. Let I and J be disjoint closed intervals contained in ∂C. What is the Euler characteristic of the surface S obtained by identifying I and J? Can all surface with nonempty boundary and with this Euler characteristic be obtained from this construction?

18 (Spring '19). Let Σ be a compact connected surface and let $p_1, \ldots, p_k \in \Sigma$. Prove that $H_2(\Sigma \setminus \cup_{i=1}^k \{p_i\}) = 0$.