# Title

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## Contents

## 1 Wednesday November 20

Last time:

$$\mathbb{Z}\Lambda \iff \{\mathfrak{h}^* \to \mathbb{Z}_{\geq 0} \mid \sim\}$$
$$e(\mu) \mapsto e_\mu$$
$$e(\lambda)e(\mu) = e(\lambda + \mu) \mapsto f \star g(\lambda) = \sum_{a+b=\lambda} f(a)g(b)$$

and $\mathrm{ch}L(\lambda) = \sum_{\mu \in \Lambda} \dim L(\lambda)_\mu e(\mu)$.

We have the Kostant function $p(\lambda) = \#\left\{(k_\alpha)_\alpha \mid -\lambda = \sum_{\alpha \in \Phi^+} k_\alpha \alpha\right\}$ and the Weyl function $q = e_\rho \star \prod_{\alpha \in \Phi^+}(1 - e_{-\alpha}) = \prod_{\alpha \in \Phi^+}(e_{\alpha/2} - e_{-\alpha/2})$.

Lemma: $p \star e_\lambda = \mathrm{ch}M(\lambda)$, so $q \star \mathrm{ch}M(\lambda) = e_{\lambda+\rho}$ and $q \star p = e_\rho$.

### 1.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of $W$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, i.e. a reflection for hyperplanes passing through $-\rho$.

E.g. for type $A2$, where $W(0) = 0$, we have:

Type A2

And for the dot action, we have

Image

where $W \cdot 0 = 0$ and $s(\alpha_1) = -\alpha_1$.

**Theorem (Harish-Chandra):** If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$ for $\mu \leq \lambda$.

Proof: Postponed.

Remark: if we sum over $\mu \leq \lambda$, we obtain

$$\mathrm{ch}M(\lambda) = \sum_{\mu \in W \cdot \lambda} a_{\lambda\mu}\mathrm{ch}L(\mu)$$

$$\mathrm{ch}L(\lambda) = \sum_{\mu \in W \cdot \lambda} b_{\lambda\mu}\mathrm{ch}M(\mu)$$

$$= \sum_{W \cdot \lambda \in \Lambda} c_{\lambda W}\mathrm{ch}M(w \cdot \lambda).$$

**Theorem (Weyl's Character Formula):** If $\lambda \in \Lambda^+$, then

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w \in W}(-1)^{\ell(w)}e(w \cdot \lambda)}{\sum_{w \in W}(-1)^{\ell(w)}e(w \cdot 0)}$$

*Proof*:

We have $\mathrm{ch}L(\lambda) = \sum_w c_{\lambda w}\mathrm{ch}M(w \cdot \lambda)$, and so by the lemma,

$$q * \mathrm{ch}L(\lambda) = \sum c_{\lambda w}q \star \mathrm{ch}M(W(\lambda + \rho) - \rho) = \sum_w c_{\lambda w}e_{W(\lambda+p)}$$

Thus for all $\alpha \in \Phi^+$, we have

$$s_\alpha(q \star \mathrm{ch}L(\lambda)) = \sum_w c_{\lambda, s_\alpha w}e_{w(\lambda+\rho)}$$

On the other hand, by part (c) of the lemma, we have

$$(s_\alpha \star q) \star \mathrm{ch}L(\lambda)$$

which implies that $c_{\lambda, s_\alpha w} = -c_{\lambda, w}$ by comparing term-by-term, and thus $c_{\lambda, w} = (-1)^{\ell(w)}$ because $c_{\lambda e} = 1$.

In particular, $q = q \star e(0) = q \star \mathrm{ch}L(0) = \sum_{w \in W}(-1)^{\ell(w)}e_{w(\rho)}$, and thus

$$\mathrm{ch}L(\lambda) = \frac{\sum_w(-1)^{\ell(w)}e_{w(\lambda+p)}}{\sum_w(-1)^{\ell(w)}e_{w(p)}}$$

$$= \frac{\sum_w(-1)^{\ell(w)}e(w \cdot \lambda)}{\sum_w(-1)^{\ell(w)}e(w \cdot 0)}.$$

$\square$

*Example:* For type $A1$, we have $W = \Sigma_2 = \{\mathbf{1}, s\}$. Take $\lambda = 3$ under

$$\Lambda \equiv \mathbb{Z}$$
$$\alpha_1 \to 2$$
$$w_1 = \rho \to 1,$$

from which we obtain

$$
\begin{aligned}
\mathrm{ch}L(3) &= \frac{e(\mathbf{1} \cdot 3) - e(s \cdot 3)}{e(\mathbf{1} \cdot 0) - e(s \cdot 0)} \\
&= \frac{e(3) - e(-5)}{e(0) - e(-2)} \\
&= e(3) + e(1) + e(-1) + e(-3) \quad \text{by long division.}
\end{aligned}
$$

**Corollary (Kostant's Dimension Formula):**

If $\mu \leq \lambda \in \Lambda^+$, then

$$\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} P(w \cdot \lambda - \mu).$$

*Proof:* $p \star e_\mu(w \cdot \lambda) = \sum_{a+b=w \cdot \lambda} p(a) e_\mu(h) = p(w \cdot \lambda - \mu)$, since this is the only term that survives.

Then $p(w \cdot \lambda - \mu)$ is the coefficient for $e(\mu)$ in $\mathrm{ch}M(w \cdot \lambda) = \dim M(\lambda)_\mu$. Thus $\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} \dim M(w \cdot \lambda)_\mu$.

**Corollary (Weyl's Dimension Formula):**

If $\lambda \in \Lambda^+$, then

$$\dim L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha^\vee)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha^\vee)}$$

*Proof (sketch):*

Define an operator $\partial = \prod_{\alpha \in \Phi^+} \partial_a$, where $\partial_a : e(\mu) \mapsto (u, \alpha^\vee) e(\mu)$. Then $\partial$ is well-defined since $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ for all $\alpha, \beta$, and (exercise) $\partial$ is a derivation.

Define an evaluation homomorphism $\nu : \sum_\mu c_\mu e(\mu) \mapsto \prod_\mu c_\mu$. Note that $\nu(\mathrm{ch}L(\lambda)) = \dim L(\lambda)$, and $\nu(q) = 0$ because $\nu(e_{\alpha_i-1}) = 0$.

Claim:

$$\nu(\partial(q \star \mathrm{ch}L(\mu - \rho))) = |w| \prod_{\alpha \in \Phi^+} (\mu, \alpha^\vee)$$

This is relatively straightforward once you know that you have a derivation and a homomorphism.

With this claim, we have

$$\nu(\partial(q \star \mathrm{ch}L(\lambda))) = \nu(\partial q)\nu(\mathrm{ch}L(\lambda)) + \nu(q)\nu(\partial \mathrm{ch}L(\lambda))$$

where we can identify a number of terms, and then taking ratios yields Weyl's dimension formula.