Title

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Problem 1.

Let C denote the Cantor set.

1. Show that C contains point that is not an endpoint of one of the removed intervals.
2. Show that C is nowhere dense, meager, and has measure zero.
3. Show that C is uncountable.

Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let C_n be the nTh stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:

   $$C_n = [0, \frac{1}{3^n}] \bigcup [\frac{2}{2^n}, \frac{3}{3^n}] \bigcup \cdots \bigcup [\frac{3^n - 1}{n}, 1],$$

   and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in C_n is never removed in any later step, and so all endpoints remaining in C are of this form where we allow $0 \leq n < \infty$.

   Thus, our goal is to produce a number $x \in [0, 1]$ such that $x \neq \frac{k}{3^n}$ for any k or n, but also satisfies $x \in C$. So we will need a general characterization of all of the points in C.

Lemma: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

Proof: By induction on the index k in a_k, first consider note that if $x \in C$ then $x \in C_1 = [0,1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. So if $x \in C_1$, then $x \notin (\frac{1}{3}, \frac{2}{3})$. But note that a_1 is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \leq x < \frac{1}{3}, \\ 1 & \frac{1}{3} \leq x < \frac{2}{3}, \\ 2 & \frac{2}{3} \leq x < 1. \end{cases}$$

Since the interval $(\frac{1}{3}, \frac{2}{3})$ is deleted in C_1, we find that $a_1 = 1 \iff x = \frac{1}{3}$. In this case, however, we claim that we can find a ternary expansion of x that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that $x = x'$, which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left(\sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left(\sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 (\frac{1}{1 - \frac{1}{3}}) - 2 - 2(3^{-1}) \\ &= 2 \left(\frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have $\frac{1}{3} = (0.1)_3 = (0.222\cdots)_3$ as ternary expansions, and a similar proof shows that such an expansion without 1s can be found for any endpoint.

For the inductive step, consider a_n: the claim is that if $a_n = 1$, then $x \notin C_{n+1}$ – that is, it is contained in one of the intervals deleted at the $n+1$st stage. Writing the deleted interval at this stage as (a, b), we find that $a_n = 1$ if and only if $x \in [a, b)$. Since $x \in C$, the only way a_n can be 1 is if x was in fact the endpoint a (since no previous digit was a 1, by hypothesis). However, as shown above, every such endpoint has a ternary expansion containing no 1s. $\square$

Therefore, if we can produce an x that satisfies $x \neq \frac{k}{3^n}$ for any k, n **and** x has no 1s in its ternary expansion, we will have an $x \in C$ that is not an endpoint.

So take

$$x = (0.\overline{02})_3 = (0.020202\cdots)_3.$$

This evidently has no 1s in its ternary expansion, and if we sum the corresponding geometric series, we find $x = \frac{1}{4}$. This is not of the form $\frac{k}{3^n}$ for any k, n, and thus fulfills both conditions.

2. We first show that C is nowhere dense by showing that the interior of its closure is empty, i.e. $(\overline{C})^\circ = \emptyset$.

   To do so, we note that C is itself closed and so $C = \overline{C}$. To see why this is, consider C^c; we'll show that it is open. By construction, C_1^c is the open interval $(\frac{1}{3}, \frac{2}{3})$ that is deleted, and similarly C_n^c is the finite union of the open intervals that are deleted at the nth stage. But then

   $$C^c = \left(\bigcap C_n\right)^c = \bigcup C_n^c$$

   is an infinite union of open sets, which is also open. So C is closed.

   It is also the case that C has empty interior, so $C^\circ = \emptyset$. Towards a contradiction, suppose $x \in C$ is an interior point; then there is some neighborhood $N_\varepsilon(x) \subset C$. Since we are on the real line, we can write this as an interval $(x-\varepsilon, x+\varepsilon)$, which has length $2\varepsilon > 0$. Moreover, we have the containment

   $$(x-\varepsilon, x+\varepsilon) \subset C \subset C_n$$

   for every n.

   Claim: The length of C_n is $(\frac{2}{3})^n$ where we define $C_0 = [0,1]$. Letting L_n be the length of C_n, one easy way to see that this is the case is to note that L_n satisfies the recurrence relation

   $$L_{n+1} = \frac{2}{3} L_n,$$

   since an interval of length $\frac{1}{3}L_n$ is removed at each stage. With the initial conditions $L_0 = 1$, it can be checked that $L_n = \left(\frac{2}{3}\right)^n$ solves this relation.

   Now, since $x \in C = \bigcap C_n$, it is in every C_n. So we can choose n large enough such that

   $$\left(\frac{2}{3}\right)^n \leq 2\varepsilon.$$

   Letting $\mu(X)$ denote the length of an interval, we always have $C \subseteq C_n$ and so $\mu(C) \leq \mu(C_n)$.

   Using the subadditivity of measures, we now have

   $$\begin{aligned} (x-\varepsilon, x+\varepsilon) &\subset C \subset C_n \\ \implies \mu(x-\varepsilon, x+\varepsilon) &\leq \mu(C) \leq \mu(C_n) \\ \implies 2\varepsilon &\leq \left(\frac{2}{3}\right)^n, \end{aligned}$$

   a contradiction. So C has no interior points.

   But this means that

   $$(\overline{C})^\circ = C^\circ = \emptyset,$$

and so C is nowhere dense.

To see that $\mu(C) = 0$, we can use the fact that for any sets, measures are additive over disjoint sets and we have

$$\mu(A) + \mu(X \setminus A) = \mu(X) \implies \mu(X \setminus A) = \mu(X) - \mu(A).$$

Here we will take $X = [0, 1]$, so $\mu(X) = 1$, and $A = C$ the Cantor set.

By tracing through the construction of the Cantor set, letting B_n be the length of the interval that is removed at each stage, we can deduce

$$\begin{aligned} B_1 &= \frac{1}{3} \\ B_2 &= \frac{2}{9} \\ &\cdots \\ B_n &= \frac{2^n}{3^{n+1}}. \end{aligned}$$

We can identify $B_n = \mu(C_n^c)$, and using the fact that $C_n^c \bigcap C_{>n}^c = \emptyset$ and the fact that measures are additive over disjoint sets, we can compute

$$\begin{aligned} \mu(C) &= 1 - \mu(C^c) \\ &= 1 - \mu((\bigcap_{n=0}^{\infty} C_n)^c) \\ &= 1 - \mu(\coprod_{n=0}^{\infty} C_n^c) \\ &= 1 - \sum_{n=0}^{\infty} \mu(C_n^c) \\ &= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{-n}} \\ &= 1 - \frac{1}{3} \sum_{n=0} \left(\frac{2}{3}\right)^n \\ &= 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) \\ &= 1 - \frac{1}{3}(3) = 0, \end{aligned}$$

which is what we wanted to show. $\square$

3. Let $y \in [0, 1]$ be arbitrary, we will construct an element $x \in C$ such that $y = f(x)$. We first note that every number has a binary expansion, and we can write

$$y = \sum_{k=1}^{\infty} y_k 2^{-k} \quad \text{where } y_k \in \{0, 1\}.$$

Now we construct

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k = 2y_k \implies a_k \in \{0, 2\}\,.$$

By the characterization given in part (1), we see that $x \in C$ because it has no 1s in its ternary expansion. Moreover, under f, we have $a_k \mapsto \frac{1}{2}a_k = \frac{1}{2}(2a_k) = a_k$, and so $f(x) = y$ by construction.

This shows that C surjects onto $[0,1]$, and in particular, $\#C \geq \#[0,1]$ holds for the cardinalities of these sets. Since $[0,1]$ is uncountable (say, by Cantor's diagonalization argument), this shows that C is uncountable.

Problem 2.

1. Show that X is G_δ iff X^c is F_σ.
2. Show that X closed $\implies$ X is G_δ and X open $\implies$ X is F_σ.
3. Give an example of an F_σ set that is not G_δ, and a set that is neither.

Solution 2.

1. To show the forward direction, suppose X is a F_σ, so $X = \bigcup_{i\in\mathbb{N}} A_i$ with each A_i an closed set. By definition, each A_i^c is open, and we have

   $$X^c = \left(\bigcup_{i\in\mathbb{N}} A_i \right)^c = \bigcap_{i\in\mathbb{N}} A_i^c,$$

   which exhibits X^c as a countable intersection of closed sets, making it an G_δ.

   The reverse direction proceeds analogously: supposing X^c is G_δ, we can write $X^c = \bigcap_{i\in\mathbb{N}} B_i$ with each B_i open, where B_i^c is closed by definition, and

   $$X = (X^c)^c = (\bigcap B_i)^c = \bigcup B_i^c$$

   which exhibits X as a union of closed sets, and thus an F_σ.

2. Suppose X is closed, we want to then write X as a countable intersection of open sets. For every $x \in X$ and every $n \in \mathbb{N}$, define

   $$\begin{aligned} B_n(x) &= \left\{ y \in \mathbb{R}^n \ni |x-y| \leq \frac{1}{n} \right\}, \\ V_n &= \bigcup_{x\in X} B_n(x), \\ W &= \bigcap_{n\in\mathbb{N}} V_n. \end{aligned}$$

Explicitly, we have

$$W = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in X} B_n(x),$$

and the claim is that W is a G_δ and $W = X$.

To see that the V_n are open, note that n is fixed and each $B_n(x)$ is an open ball around a point x. Any union of open sets is open, and thus so is V_n. By construction, W is then a countable intersection of open sets, and thus W is a G_δ by definition.

We show $W = X$ in two parts. To see that $X \subseteq W$, note that if $x \in X$, then $x \in B_n(x)$ for every n and thus $x \in V_n$ for every n as well. But this means that $x \in \bigcap_n V_n$, and so $x \in W$.

To see that $W \subseteq X$, let $w \in W$ be arbitrary. If $w \in X$, there is nothing to check, so suppose $w \notin X$ towards a contradiction.

Since $w \in \bigcap_n V_n$, it is in V_n for every n. But this means that there is some particular x_0 such that $w \in B_n(x_0)$ for every n as well, and moreover since we assumed $w \notin X$, we have $w \neq x_0$.

Then, letting $N_\varepsilon(w)$ be an arbitrary neighborhood of w, we can find an n large enough such that $B_n(x) \subset N_\varepsilon(w)$. This means that $x_0 \neq w$ can be found in every neighborhood of w, which makes w a limit point of X. However, since we assumed X was closed, it contains all of its limit points, which would force $w \in X$, a contradiction. $\square$

Now suppose X is an open set, we want to show it is an F_σ and can thus be written as a countable union of closed sets. We can use the fact that X^c is closed, and by the previous result, X^c is thus a G_δ. But by an earlier result, X^c is a $G_\delta \iff (X^c)^c = X$ is an F_σ, and we are done.

3. We want to construct a set that can be written as a countable union of closed sets, but not as a countable intersection of open sets. Note that in $\mathbb{R}$ with the usual topology, singletons are closed, and so $\{p\}^c$ is an open set for any point p.

   With this motivation, consider $X = \mathbb{Q}$ and $X^c = \mathbb{R} \setminus \mathbb{Q}$. We can write

   $$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\},$$

   which exhibits X as a countable union of closed sets because $\mathbb{Q}$ itself is countable. So $\mathbb{Q}$ is an F_σ set. Suppose towards a contradiction that $\mathbb{Q}$ is also G_δ, so we have $\mathbb{Q} = \bigcap_{i \in \mathbb{N}} O_i$ with each O_i open. So each O_i covers $\mathbb{Q}$, i.e. $\mathbb{Q} \subseteq O_i$, which (importantly!) forces each O_i to be dense in $\mathbb{R}$.

   But now note that we can also write

   $$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\},$$

   where we can note that $\mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$ for each q. We can appeal to the Baire category theorem twice, which tells us that any countable intersection of dense sets will also be dense. This first tells us that the above intersection, and thus $\mathbb{R} \setminus \mathbb{Q}$, is dense in $\mathbb{R}$. Then, writing

   $$\left(\bigcap_{i \in \mathbb{N}} O_i \right) \bigcap \left(\bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\} \right) = \mathbb{Q} \bigcap \mathbb{R} \setminus \mathbb{Q} = \emptyset,$$

we produce what is still just a countable intersection of dense sets, and by Baire, the result would need to be dense as well. Since the empty set is *not* dense in $\mathbb{R}$, so we arrive at a contradiction.

Problem 3.

1. Let r_n be an enumeration of the rationals, define $f(r_n) = \frac{1}{n}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\lim_{x \to c} f(x) = 0$ for every $c \in I$, and $D_f = \mathbb{Q} \bigcap I$.
2. Show that ω_f is (in general) well-defined, and that f is continuous at $x \iff \omega_f(x) = 0$.
3. Show that for every $\varepsilon > 0$, the set $A(\varepsilon) = \{x \in \mathbb{R} \ni \omega_f(x) > \varepsilon\}$ is closed, and thus D_f is an F_σ set.

Solution 3.

1. We need to show that

$$\forall c,\ \forall \varepsilon > 0,\ \exists \delta \ni \quad |x - c| \leq \delta \implies |f(x)| \leq \varepsilon.$$

To that end, let ε be fixed and $c \in I$ be arbitrary. If $c \in I \setminus \mathbb{Q}$, then $f(c) = 0 < \varepsilon$ and there's nothing to prove. Otherwise, $c \in \mathbb{Q}$ and so $c = r_n$ for some n. Towards a contradiction, suppose there