$\boxed{A1}$

Suppose

$$\left.\begin{array}{ll} \bullet\ A\vec{u} = \lambda\vec{u} & (1) \\ \bullet\ A\vec{v} = \mu\vec{v} & (2) \\ \bullet\ A = A^T & (3) \end{array}\right\} \text{ and } \lambda, \mu \text{ are distinct.}$$

We want to show $\vec{u} \perp \vec{v}$, ie $\langle \vec{u}, \vec{v} \rangle = 0$

How to proceed? Play with $\langle \vec{u}, \vec{v} \rangle$ in such a way that we use all 3 hypotheses:

$$\boxed{\langle \vec{u}, \vec{v} \rangle} = \vec{u}^T \vec{v} = \frac{1}{\mu} \mu \vec{u}^T \vec{v}$$

$$= \frac{1}{\mu} \vec{u}^T (\mu \vec{v}) = \frac{1}{\mu} \vec{u}^T \boxed{(A\vec{v})} \quad \text{by (2)}$$

$$= \frac{1}{\mu} \vec{u}^T \boxed{(A^T \vec{v})} = \frac{1}{\mu} (\vec{u}^T A^T) \vec{v} \quad \text{by (3)}$$

$$= \frac{1}{\mu} (A\vec{u})^T \vec{v} = \frac{1}{\mu} \boxed{(\lambda \vec{u})^T} \vec{v} \quad \text{by (1)}$$

$$= \frac{\lambda}{\mu} \vec{u}^T \vec{v} \boxed{= \frac{\lambda}{\mu} \langle \vec{u}, \vec{v} \rangle}$$

So we have:

$$\langle \vec{v}, \vec{v} \rangle = \frac{\lambda}{\mu} \langle \vec{u}, \vec{v} \rangle$$

$$\Rightarrow \left(1 - \frac{\lambda}{\mu}\right) \langle \vec{u}, \vec{v} \rangle = 0 \in \mathbb{R}.$$

Since $\mathbb{R}$ is a field, it is also an integral domain, where

$$a \cdot b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

So we must h ve

$$\left(1 - \frac{\mu}{\lambda}\right) = 0 \quad \text{or} \quad \langle \vec{u}, \vec{v} \rangle = 0$$

But since $\mu, \lambda$ were distinct, we have $\mu \neq \lambda$, and so $\mu/\lambda \neq 1$ and thus $1 - \mu/\lambda \neq 0$. This lets us conclude that $\langle \vec{u}, \vec{v} \rangle$ must be zero. $\blacksquare$

A2

Suppose

- $A$ is $m \times n$

We want to show

1) $A^T A$ is symmetric
2) $A^T A$ is positive semidefinite

① Let $M = A^T A$. We want to show $M^T = M$.

$$M^T = (A^T A)^T = A^T (A^T)^T = A^T A = M,$$

{ using the fact $(AB)^T = B^T A^T$ }

So $M^T = M$. □

② An $n \times n$ matrix $M$ is PSD iff $\forall \vec{x} \in \mathbb{R}^n$, $\langle \vec{x}, M\vec{x} \rangle \geq 0 \in \mathbb{R}$.

We proceed by computing:

$$\langle \vec{x}, M\vec{x} \rangle = \vec{x}^T M \vec{x} = \vec{x}^T A^T A \vec{x}$$

$$= (A\vec{x})^T A\vec{x}$$

$$= \langle A\vec{x}, A\vec{x} \rangle .$$

But $A\vec{x}$ is just some vector, say $A\vec{x} := \vec{y} \in \mathbb{R}^n$, and one of the properties of an inner product is

$$\left\{ \begin{array}{l} \forall \vec{v} \in \mathbb{R}^n, \boxed{\langle \vec{v}, \vec{v} \rangle \geq 0} \quad \text{and} \\ \langle \vec{v}, \vec{v} \rangle = 0 \text{ iff } \vec{v} = \vec{0} \in \mathbb{R}^n \end{array} \right\}$$

So we can immediately conclude that

$$\langle \vec{x}, M\vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle = \boxed{\langle \vec{y}, \vec{y} \rangle \geq 0,}$$

so $M$ is PSD by definition. ∎

(Alternatively, note $\langle \vec{y}, \vec{y} \rangle = \|y\|_2^2$, and all squares are $\geq 0$ in $\mathbb{R}$.)

Alternative short soln:

You can use the fact that for any square matrix $B$, $B^T$ is the adjoint operator of $B$, i.e.:

$$\forall \vec{x} \in \mathbb{R}^n, \begin{cases} \langle B\vec{x}, \vec{x} \rangle = \langle \vec{x}, B^T\vec{x} \rangle \\ \langle \vec{x}, B\vec{x} \rangle = \langle B^T\vec{x}, \vec{x} \rangle . \end{cases}$$

Then the proof is one line:

$$\langle \vec{x}, A^T A\vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle = \|A\vec{x}\|_2^2 \geq 0. \blacksquare$$

Uses adjoint property for $A^T$ to move $A^T$ to the LHS, then uses $(A^T)^T = A$.