Algebra Qual Prep Week 1: Groups Warmup

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  - Matrix groups $\mathrm{GL}_n(k), \mathrm{O}_n(k), \mathrm{SL}_n(k), \mathrm{SO}_n(k)$
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  - Free groups F_n (and presentations/relations)
- The 4 fundamental isomorphism theorems
- Finite groups of order $\#G \leq 20$
- Structure:
  - Cyclic -> Abelian -> Nilpotent -> Solvable -> All Groups

1.2 Review Exercises

- State the definitions of the following:
  - Group morphism (aka *group homomorphism*)
  - Centralizer
  - Normalizer

  - Conjugacy class
  - Center
  - Inner automorphism
  - Commutator
  - p-group

- Write definitions or presentations for all of the special families of groups appearing above.
- State what it means for a cycle to be even or odd.
- Find a counterexample for the converse of Lagrange's theorem.
- State the 4 fundamental isomorphism theorems

1.3 Unsorted Questions

For everything that follows, assume G is a finite group.

- $H \leq G$ denotes that H is a subgroup of G.
- $\#G$ denotes the *order* of G.
- e or e_G denotes the identity element of G.
- Multiplicative notation is generally used everywhere to denote the (possibly noncommutative) binary operation

1.3.1 Orders

- Prove Lagrange's theorem.
- Prove Cauchy's theorem.
- Prove that if $\#G$ is prime, then G is cyclic
- Prove that for every $g \in G$, the order of g divides the order of G.
- Prove that if $\#G = n$, then $g^n = e$ for every $g \in G$

1.3.2 Cosets

- Let $H \leq G$ be a subgroup (not necessarily normal), and let G/H denote the set of left cosets of G by H. Prove that any two cosets $xH, yH \in G/H$ have the same cardinality.

- Prove the *fundamental theorem of cosets*: for $xH, yH \in G/H$,

$$xH = yH \iff x^{-1}y \in H \iff y^{-1}x \in H$$

- Suppose $\#G = pq$ with $p, q \geq 2$ prime, and let $H \leq G$ be a proper subgroup. Prove that H must be cyclic.

1.3.3 Normal Subgroups

- Let $s \in G$, and state the definition of the **centralizer** of $C_G(s)$ of s in G.
  - Show that $C(s) \leq G$ is a subgroup.
  - Let $\langle s \rangle \subseteq C_G(s)$, where $\langle s \rangle$ is the subgroup of G generated by s.
  - Prove that $\langle s \rangle \trianglelefteq G$ is in fact a **normal** subgroup.
- Let $H \leq G$ be a subgroup and $N \trianglelefteq G$ be a normal subgroup. Show that $NH \leq G$ is a subgroup.
- Let G_1, G_2 be groups and $H_2 \leq G_2$ a subgroup. Suppose $\varphi : G_1 \to G_2$ is a group morphism.
- Suppose $\varphi : G_1 \to G_2$ is a group morphism.
  - Show that the image $\varphi(G_1) \leq G_2$ is a subgroup of G_2
  - Show that the preimage $\varphi^{-1}(H_2) \leq G_1$ is a subgroup of G_1,
  - Show that the kernel $\ker \varphi \trianglelefteq G_1$ is a normal subgroup of G_1.
  - Prove that group morphisms *preserve coset structure* in the following sense:

$$xH_1 = yH_1 \iff \varphi(x)H_2 = \varphi(y)H_2.$$

  - Prove the **first isomorphism theorem**: φ is injective $\iff \ker \varphi = \{e_{G_1}\}$.

1.3.4 Symmetric Groups

- Let $\sigma = (4\,2\,1)(6\,1\,3\,2) \in S_6$ in cycle notation.
  - Write σ as a product of disjoint cycles.
  - Compute the order of σ. What is the general theorem about the order of cycles?
  - Determine if σ is even or odd. What is the general theorem?
- Suppose $\varphi : S_n \to G$ with n even and $\#G = m$ odd.
  - Prove that if $\tau \in S_n$ is a transposition, then $\tau \in \ker \varphi$.
  - Prove that in fact *every* $\sigma \in S_n$ satisfies $\sigma \in \ker \varphi$, so φ is the trivial morphism.
  - Does this hold if n is odd?

1.3.5 Matrix Groups

- Let $\mathbb{F}_p$ be the finite field with p elements, where p is a prime. Show that the centers of $\mathrm{GL}_n(\mathbb{F}_p)$ and $\mathrm{SL}_n(\mathbb{F}_p)$ consist only of scalar matrices.
  - Show that the scalars ζ that appear in scalar matrices $Z(\mathrm{SL}_n(\mathbb{F}_p))$ are roots of unity in $\mathbb{F}_p$, i.e. $\zeta^p = 1$.
- Determine the orders $\#\,\mathrm{GL}_n(\mathbb{F}_p)$ and $\#\mathrm{SL}_n(\mathbb{F}_p)$.

1.4 Warmup Problems

- (**Important**) Prove that if $G/Z(G)$ is cyclic then G is abelian.
- (**Important**) Classify all groups of order p^2.
- (**Important**) Show that if $H \leq G$ and $[G : H] = 2$ then H is normal.
  - Suppose that the same result holds with 2 replaced by p defined as the smallest prime factor of $\#G$
- Prove that if $H \leq G$ is a proper subgroup, then G can not be written as a union of conjugates of H.
  - Use this to prove that if $G = \mathrm{Sym}(X)$ is the group of permutations on a finite set X with $\#X = n$, then there exists a $g \in G$ with no fixed points in X.
- Let $G \leq H$ where H is a finite p-group, and suppose $\varphi : G \to H/[H,H]$ be defined by composing the inclusion $G \hookrightarrow H$ with the natural quotient map $H \to H/[H,H]$.

  Prove that $G = H$ by induction on $\#H$ in the following way:
  - Letting $N \trianglelefteq H$ be any nontrivial normal subgroup of H, use the inductive hypothesis to show that $H = GN$.
  - Let $Z = Z(H)$ be the center of H. Using that $GZ = H$ by (1), show that $G \cap Z \neq \emptyset$. Set $N := G \cap Z$ and apply (1) to conclude.
- Determine all pairs $n, p \in \mathbb{Z}^{\geq 1}$ such that $\mathrm{SL}_n(\mathbb{F}_p)$ is solvable.

1.5 Qual Problems

1.13 Spring 2021 #2

Let $H \trianglelefteq G$ be a normal subgroup of a finite group G, where the order of H is the smallest prime p dividing $|G|$. Prove that H is contained in the center of G.

2.1 Fall 2019 #1 ✨

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G. Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

Relevant concepts omitted.

1.5 Fall 2016 #1 🚩

Let G be a finite group and $s, t \in G$ be two distinct elements of order 2. Show that subgroup of G generated by s and t is a dihedral group.

Recall that the dihedral groups of order $2m$ for $m \geq 2$ are of the form

$$D_{2m} = \left\langle \sigma, \tau \,\middle|\, \sigma^m = 1 = \tau^2, \tau\sigma = \sigma^{-1}\tau \right\rangle.$$

1.6 Fall 2015 #1 🚩

Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H which is of finite index.

1.7 Spring 2015 #1 🚩

For a prime p, let G be a finite p-group and let N be a normal subgroup of G of order p. Prove that N is contained in the center of G.

1.12 Fall 2019 Midterm #5

Let G be a nonabelian group of order p^3 for p prime. Show that $Z(G) = [G, G]$.

1.13 Spring 2021 #2

Let $H \trianglelefteq G$ be a normal subgroup of a finite group G, where the order of H is the smallest prime p dividing $|G|$. Prove that H is contained in the center of G.