4-08-2018 Research Notes

Preliminaries

Definition: n connectivity

A space X is said to be *n-connected* if $\pi_i X = 0$ for $1 \leq i \leq n$.

Definition: Weak Homotopy Equivalence

A map $f: X \to Y$ is called a *weak homotopy equivalence* if the induced maps $f_i^*: \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$ are isomorphisms for every $i \geq 0$.

This is a strictly weaker notion than homotopy equivalence - for example, let L be the long line. Then $\pi_i(L) = 0$ for all i, but L is not contractible, and thus $L \not\simeq \{\mathrm{pt}\}$. However, the inclusion $\{\mathrm{pt}\} \hookrightarrow L$ is a weak homotopy equivalence, which can not be a homotopy equivalence.

Any weak homotopy equivalence induces isomorphisms on all integral co/homology groups, and thus co/homology groups with any coefficients by the UCT.

Definition: Cellular Map

If a map $X \xrightarrow{f} Y$ satisfies $f(X^{(n)}) \subseteq Y^{(n)}$, then f is said to be a *cellular map*.

Theorem: Cellular Approximation

Any map $X \xrightarrow{f} Y$ between CW complexes is homotopic to a cellular map.

Theorem: CW Approximation

For every topological space X, there exists a CW complex Y and a weak homotopy equivalence $f : X \to Y$. Moreover, if X is n-dimensional, Y may be chosen to be n-connected and is obtained from X by attaching cells of dimension greater than n.

Theorem: Whitehead

Abbreviated statement: if X, Y are CW complexes, then any map $f : X \to Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.

(Note: f must induce maps on all homotopy groups simultaneously.)

Full Statement: If $(X, x_0) \xrightarrow{f} (Y, f(x_0))$ such that the induced maps

$$\begin{gathered} f_* : \pi_*(X, x_0) \to \pi_*(Y, y_0) \\ [g] \mapsto [f \circ g] \end{gathered}$$

are all isomorphisms and Y is connected, then f is a homotopy equivalence.

Theorem: Uniqueness of E-M Spaces

If X is a space with one nontrivial homology group G in degree k, so that X satisfies

$$\pi_i(X) = \begin{cases} G, \ i = k \\ 0, \ \text{otherwise} \end{cases}$$

Then $X \simeq K(G, k)$.

(Note: two spaces with isomorphic homotopy groups may *not* be homotopy-equivalent in general - this is one exception.)

Theorem: Hurewicz

Given a space X, define a family of maps

$$
\begin{aligned}
h_k : \pi_k X &\to H_k X \\
[f] &\mapsto f_*(\mu_k)
\end{aligned}
$$

where $H_k X = \langle \mu_k \rangle$.

If X is $n-1$ connected where $n \geq 2$, then h_k is an isomorphism for all $k \leq n$.

In particular, $\pi_n X \cong H_n X$ as groups.

Theorem: Freudenthal Suspension

If X is an n- connected CW complex, then there are maps $\pi_i X \to \pi_{i+1} \Sigma X$ which is an isomorphism for $i \leq 2n$ and a surjection for $i = 2n+1$.

Theorem: Homotopy LES for a Fibration

Theorem: Existence of Postnikov Tower

Theorem: Spectral sequence of a Fibration

Theorem: Existence of Whitehead Tower

Main Stuff

- Theorem: $\pi_1 S^1 = \mathbb{Z}$
    - *Proof*: Covering space theory
- Theorem: $\pi_{1+k} S^1 = 0$ for all $0 < k < \infty$
    - *Proof*: Use universal cover by $\mathbb{R}$
    - Theorem: $\mathbb{R}^n$ is contractible
    - Theorem: R covers S^1
    - Theorem: Covering spaces induce $\pi_i X \cong \pi_i \tilde{X}, i \geq 2$
- Theorem: $\pi_1 S^n = 0$ for $n \geq 2$.
    - S^n is simply connected.
- Theorem: $\pi_n S^n = \mathbb{Z}$
    - *Proof*: The degree map is an isomorphism. [G&M 4.1]
    - Alternatively:
        - LES of Hopf fibration gives $\pi_1 S^1 \cong \pi_2 S^2$
        - Freudenthal suspension: $\pi_k S^k \cong \pi_{k+1} S^{k+1}, k \geq 2$
- Theorem: $\pi_k S^n = 0$ for all $1 < k < n$

    - *Proof*: By cellular approximation: For $k < n$,
        - Approximate $S^k \xrightarrow{f} S^n$ by $\tilde{f}$
        - $\tilde{f}$ maps the k-skeleton to a point,
        - Which forces $\pi_k S^n = 0$?
    - Alternatively: Hurewicz
- Theorem: $\pi_k S^2 = \pi_k S^3$ for all $k > 2$
- Theorem: $\pi_k S^2 \neq 0$ for any $2 < k < \infty$
    - Corollary: $\pi_k S^3 \neq 0$ for any $2 < k < \infty$
- Theorem: $\pi_k S^2 = \pi_k S^3$
    - *Proof*: LES of Hopf fibration
- Theorem: $\pi_3 S^2 = \mathbb{Z}$
    - *Proof*: Method of killing homotopy
- Theorem: $\pi_4 S^2 = \mathbb{Z}_2$
    - *Proof*: Continued method of killing homotopy
- Theorem: $\pi_{n+1} S^n = \mathbb{Z}$ for $n \geq 2$?
    - *Proof*: Freudenthal suspension in stable range?
- Theorem: $\pi_{n+2} S^n = \mathbb{Z}_2$ for $n \geq 2$?
    - *Proof*: Freudenthal suspension in stable range?