Title

Contents

1 Monday February 24th **2**
1.1 Classification of Locally K-Analytic Lie Groups 2

1 | Monday February 24th

1.1 Classification of Locally K-Analytic Lie Groups

Let K be a locally compact field with discrete valuation, R a valuation ring with $\mathfrak{m} = (\pi)$ its maximal ideal where $R/\mathfrak{m} \cong \mathbb{F}_q = \mathbb{F}_{p^a}$. There are two cases:

- If $\operatorname{ch} K = 0$, then $\mathbb{Q}_p \subset K$ and $d = [L : \mathbb{Q}_p]$
- If $\operatorname{ch} K = p > 0$, then $K \cong \mathbb{F}_q((t))$.

Fact (Lie Theoretic) Let G be a compact commutative g-dimensional K-analytic lie group (i.e. locally looks like k^n with transition harts given by convergent power series) which is 2nd countable.

a. There exists a filtration by open subgroups $G = G^0 \supset G^1 \supset \cdots \supset G^n \supset \cdots$ such that

0. G^0/G^i is finite and discrete for all i,
1. $G^i \cong (\mathfrak{m}^i)^g$, with addition given by a g-dimensional formal group law,
2. $\cap G^i = (0)$, so the filtration is exhaustive,
3. G/G^{i+1} is p-torsion,
4. $G^1[\text{tors}] = G^1[p^\infty]$

b. If $\operatorname{ch}(K) = 0$, then there exists an open subgroup U of G such that $U \cong (R^g, +)$ as K-analytic Lie groups.

> Analog in Lie theory: Lie groups with isomorphic Lie algebras yield isomorphic universal covers. Can then recover the formal group from the Lie algebra. Wildly false in characteristic p, since we lose information about the height of the formal group.

Theorem (C-Lacy)

a. If $\operatorname{ch}(K) = 0$ (i.e. in a p-adic field), then $G \cong (R, +)^g \oplus G[\text{tors}]$ as topological groups, where $G[\text{tors}]$ is finite, which is in turn isomorphic to $\mathbb{Z}_p^{dg} \oplus G[\text{tors}]$.

b. If $\operatorname{ch}(K) = p$, then there exists a countable set I such that $G \cong \prod_{i \in I} \mathbb{Z}_p \oplus G[\text{tors}]$ as topological groups, where each of the groups on the RHS are closed subgroups. Moreover, $G[\text{tors}] < \infty \iff G[p] < \infty$, and when these conditions hold, I is infinite.

Lemma If H is a a commutatve torsionfree pro-p group, then $H \cong \prod_{i \in I} \mathbb{Z}_p$. If H is 2nd countable, then I is countable.

> *Proof (of lemma, sketch).*
> We'll take *Pontryagin duals*. Recall that if G is an locally compact abelian (LCA) group, then $G^\vee := \hom(G, \mathbb{R}/\mathbb{Z})$ is an LCA group. Note that the dual of a profinite group (inverse limit) is an ind-finite group (direct limit), which are discrete torsion groups.

$H^\vee$ is a discrete p-primary torsion group. Example: $\mathbb{Z}_p^\vee = \mathbb{Q}_p/\mathbb{Z}_p$, which flips the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \arrow[r] & \mathbb{Q}_p/\mathbb{Z}_p \arrow[r] \arrow[ll, bend left] & 0 \end{tikzcd}$$

Then $H^\vee = [p]H^\vee$, and thus $H^\vee$ is divisible. We then apply the structure theory of divisible group to get a direct sum, then applying duality again yields a direct product, which proves the lemma.

■

Proof (of Theorem) Assume that $G[\text{tors}]$ is finite. We have a SES of commutative profinite groups:

$$0 \to G[\text{tors}] \to G \to G/G[\text{tors}] \to 0,$$

and taking Pontryagin duals yields

$$0 \to (G/G[\text{tors}])^\vee \to G^\vee \to G[\text{tors}]^\vee.$$

Then $G/G[\text{tors}]$ is torsionfree and has a finite index pro-p subgroup, so $G/G[\text{tors}]$ is itself a pro-p group. By the lemma, $G/G[\text{tors}] \cong \prod_{i \in I} \mathbb{Z}_p$, so

$$(G/G[\text{tors}])^\vee \cong \bigoplus_{i \in I} \mathbb{Q}_p/\mathbb{Z}_p.$$

But this is divisible, and hence injective since we're over a PID, so the dual sequence above splits. So the original sequence splits.

We thus have an isomorphism of topological groups

$$G \cong G/G[\text{tors}] \oplus G[\text{tors}] \cong \prod_{i \in I} \mathbb{Z}_p \oplus G[\text{tors}],$$

where $G[\text{tors}]$ was assumed finite.

Suppose $\text{ch}(K) = 0$. We have two open subgroups of G, $\prod_{i \in I} \mathbb{Z}_P \leq G$ (open since its complement is finite) and $(R, +)^g \cong \mathbb{Z}_p^{dg}$ by Serre. It follows that $|I| = dg$.

Suppose instead that $\text{ch}(K) > 0$. The claim is that $[G : pG]$ is infinite and thus $|I|$ is infinite. This is because the cokernel of multiplication by p on $\prod \mathbb{Z}_p \oplus G[\text{tors}]$ is infinite iff I is infinite, so it suffices to check the size of this cokernel.

Consider the formal group law in characteristic p given by

$$[p] \in R[[X_1^p, \cdots, X_g^p]]^g.$$

It suffices to restrict to $G^1 = (t\mathbb{F}_q[[t]]^g, \mathrm{fgl})$. Then $pG_1 \subseteq t\mathbb{F}_q[[t]]^g \cap \mathbb{F}_q[[t^p]]^g$. But $[\mathbb{F}_q[[t]] : \mathbb{F}_q[[t^p]]]$ is infinite, so $[G : pG_1]$ is infinite, so $[G : pG]$ is infinite and thus I is infinite.

If we know the torsion is finite, can we find bounds on their size? We'll need to revisit Néron models (as covered in the abelian varieties course), and introduce Tate curves.