Section 8.6: The Solutions of the Floer Equation are “Somewhere Injective”.

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Contents

0.1 Outline

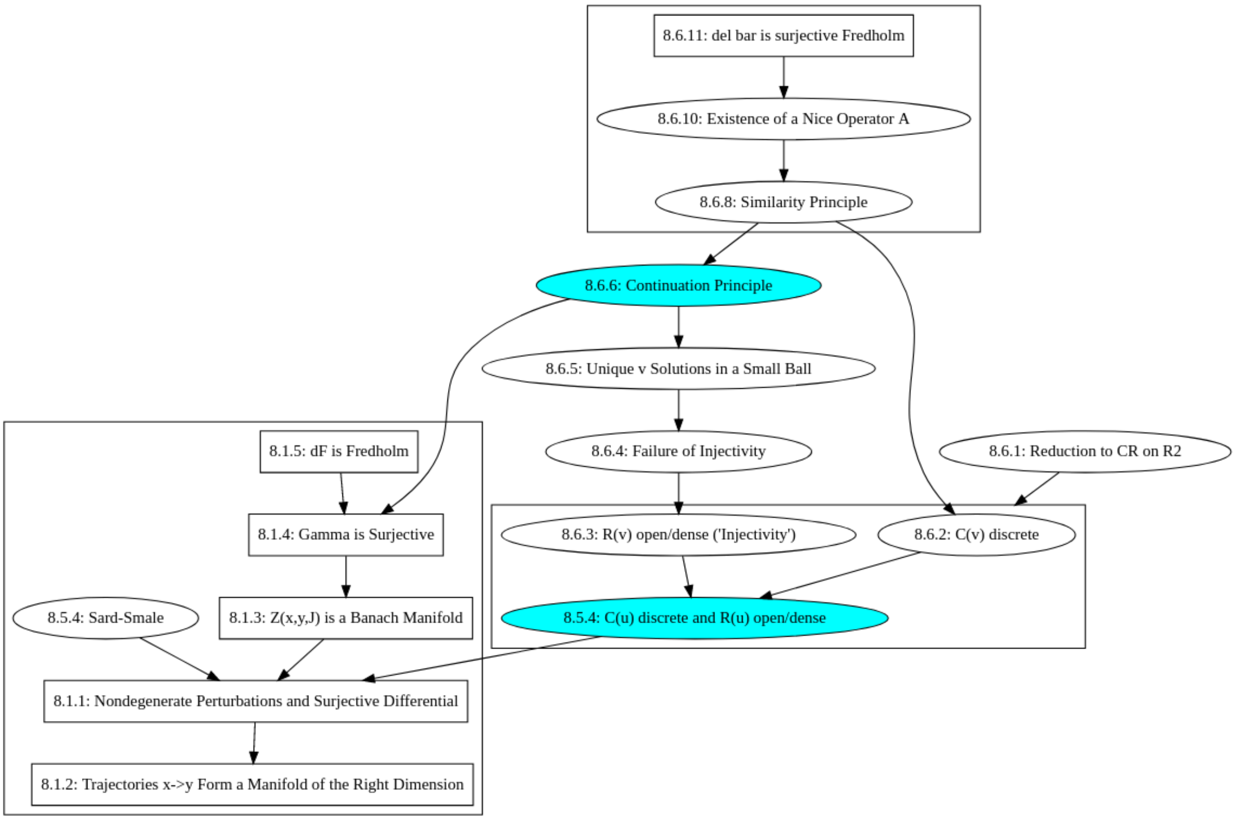
Two Goals:

1. Critical points are discrete and regular points are open/dense.
2. The continuation principle (used elsewhere, see diagram later)

- Idea: For $\mathbb{C}$, a holomorphic function with all derivatives vanishing on a line is identically zero.

0.2 Outline of Statements

$$\begin{tikzcd}
8.1.5 \arrow[r, Rightarrow] & 8.1.4 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & 8.1.3 \\
& 8.1.1 \arrow[r, Rightarrow] & 8.1.2
\end{tikzcd}$$



What we'll try to prove:

- 8.6.1: Reduction to Cauchy-Riemann equations on $\mathbb{R}^2$ (short)
- 8.6.3 (Partial): $R(v)$ is open.

Statements of "big" theorems for the chapter, in reverse order of implication:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- 8.1.4: $\Gamma : W^{1,p} \times C^\infty_\varepsilon \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x, y, J)$ is a Banach manifold
- 8.1.1: For $h \in \mathcal{H}_{\text{reg}}, H_0 + h$ is nondegenerate and $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(H_0 + h, J)$.
- 8.1.2: For $h \in \mathcal{H}_{\text{reg}}$ and all contractible orbits x, y of H_0, $\mathcal{M}(x, y, H_0 + h)$ is a manifold of dimension $\mu(x) - \mu(y)$.

0.3 Notation

- The Floer equation and its linearization:

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y \\ &\quad Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \operatorname{End}(\mathbb{R}^{2n})). \end{aligned}$$

- $X(t, u) : S^1 \times W \longrightarrow W$ is a time-dependent periodic vector field on $\mathbb{R}^{2n}$, J an almost-complex structure, both smooth
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the equation

$$\frac{\partial u}{\partial s} + J(t, u)\left(\frac{\partial u}{\partial t} - X(t, u)\right) = 0$$

Note: not sure why we've replaced $\operatorname{grad}_u(H)$ with $-J(t, u)\cdot X(t, u)$ in the Floer equation.

- $C(u)$ the set of critical points and $R(u)$ the set of regular points of u:

$$\begin{aligned} (s_0, t_0) \in C(u) \subseteq \mathbb{R} \times S^1 &\iff \frac{\partial u}{\partial s}(s_0, t_0) = 0 \\ (s_0, t_0) \in R(u) \subset \mathbb{R} \times S^1 &\iff (s_0, t_0) \notin C(u) \ \&\ s \neq s_0 \implies u(s_0, t_0) \neq u(s, t_0). \end{aligned}$$

0.4 Goal 1: Discrete Critical Points and Dense Regular Points

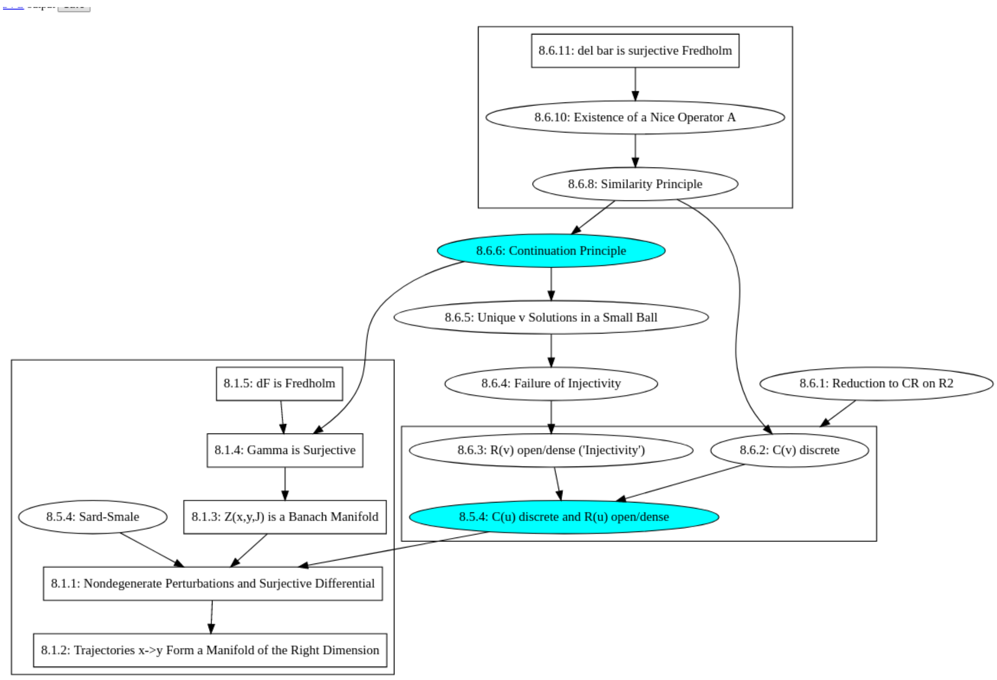
Goal 1: prove the following theorem

Theorem 0.1 *(8.5.4)*.

1. $C(u)$ is discrete and
2. $R(u) \hookrightarrow \mathbb{R} \times S^1$ is open and dense.

Outline of the proof:

- Prove 8.6.1: Reduction to CR
  - (direct, short) which transforms the Floer(?) equation
$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0 \quad \text{where} \quad u \in C^\infty(\mathbb{R} \times S^1; W)$$
    to a Cauchy-Riemann equation on $\mathbb{R}^2$:
$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} = 0 \quad \text{where} \quad v \in C^\infty(\mathbb{R}^2; W)$$
- Reduce 8.5.4 (Discrete/Open/Dense) to two statements
  - 8.6.2: $C(v)$ (and thus $C(u)$) is discrete Proved later using 8.6.8: *Similarity Principle.*
  - 8.6.3 (Injectivity): If v is a smooth periodic solution of CR with $\frac{\partial v}{\partial s} \neq 0$ then $R(v) \subseteq \mathbb{R}^2$ is open and dense.
- Prove 8.6.3 (Injectivity)
  - Show open (easier)
  - Show dense (delicate!)
- Prove 8.6.8: Similarity Principle
  - Use similarity principle to prove 8.6.6: Continuation Principle. Yields theorem.



0.5 8.6.1: Transform to Cauchy-Riemann

Proposition 0.2 ***(8.6.1, Transform to CR-equation on R2).***
If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0.$$

Then there exists
- An almost complex structure J_1
- A diffeomorphism φ on W ?
- A map $v \in C^\infty(\mathbb{R}^2; W)$

satisfying

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$

where
1. $v(s,t+1) = \varphi(v(s,t))$
2. $C(u) = C(v)$, i.e. u, v have the same critical points
3. $R(u) = R(v)$.

Proof

- Recall the vector field was defined as $X(t,u) : S^1 \times W \longrightarrow W$.
- Since $W \times S^1$ is compact, the flow ψ_t of X_t is defined on all of W
  - We thus have a map $\psi_t : W \longrightarrow W$ such that

$$\frac{\partial}{\partial t}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}$$

- Define the (important!) map

$$v(s,t) \coloneqq \left(\psi_t^{-1} \circ u\right)(s,t)$$

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$$v(s,t) \coloneqq \left(\psi_t^{-1} \circ u\right)(s,t)$$

- We can then compute

$$\begin{aligned}\frac{\partial u}{\partial s} &= (d\psi_t)\left(\frac{\partial v}{\partial s}\right) \\ \frac{\partial u}{\partial t} &= (d\psi_t)\left(\frac{\partial v}{\partial t}\right) + X_t(u).\end{aligned}$$

  - Attempt at explanation: rearrange, use chain rule, and known derivative of ψ_t:

$$u(s,t) = (\psi_t \circ v)(s,t) \implies \frac{\partial u}{\partial s}(s,t) = \frac{\partial \psi_t}{\partial s}(v(s,t)) \cdot \frac{\partial v}{\partial s}(s,t)$$

$$? \implies \frac{\partial u}{\partial s} = (d\psi_t) \cdot \left(\frac{\partial v}{\partial s}\right)$$

and

$$\begin{aligned}\frac{\partial u}{\partial t}(s,t) &= \frac{\partial \psi_t}{\partial t}(v(s,t)) \cdot \frac{\partial v}{\partial t}(s,t) \\ &= (X_t \circ \psi_t)(v(s,t)) \cdot \frac{\partial v}{\partial t}(s,t) \\ &= (X_t \circ \psi_t \circ v)(s,t) \cdot \frac{\partial v}{\partial t}(s,t) \\ &= X_t(u(s,t)) \cdot \frac{\partial v}{\partial t}(s,t) \\ &= X_t(u)\left(\frac{\partial v}{\partial t}\right) \cdots ????.\end{aligned}$$

Note sure how to relate partial derivatives $\frac{\partial}{\partial \cdot}\psi_t$ to differential $d\psi_t$. Not sure why we're picking up *addition* in the t derivative.

- Given that result, we can compute,

$$
\begin{aligned}
0 &= \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_t(u)\right) && \text{since } u \text{ is a solution} \\
&= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} - JX_t(u) && \text{expanding terms} \\
&= \left((d\psi_t)\left(\frac{\partial v}{\partial s}\right)\right) + J\left((d\psi_t)\left(\frac{\partial v}{\partial t}\right) + X_t(u)\right) - JX_t(u) && \text{by substitution} \\
&= (d\psi_t)\left(\frac{\partial v}{\partial s}\right) + J(u)\,(d\psi_t)\left(\frac{\partial v}{\partial t}\right) && \text{cancelling} \\
&= (d\psi_t)\left(\frac{\partial v}{\partial s} + (d\psi_t)^{-1}\,J(u)\,(d\psi_t)\left(\frac{\partial v}{\partial t}\right)\right) && \text{collecting terms} \\
&\coloneqq (d\psi_t)\left(\frac{\partial v}{\partial s} + \psi_t^* J(v)\right) && \text{by definition.}
\end{aligned}
$$

- Conclude that v is a solution of

$$\frac{\partial v}{\partial s} + \psi_t^\star J(v)\frac{\partial v}{\partial t} = 0.$$

- Set $\varphi \coloneqq \psi_1$ and $J_1(v) \coloneqq \psi_1^* J(v)$ to obtain

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$

  of which v is still a solution

- Property 1, Periodicity: attempt to check directly, using $\psi_{t+1} = \psi_t \circ \psi_1$:

$$
\begin{aligned}
v(s, t+1) &\coloneqq (\psi_t^{-1} \circ u)(s, t+1) \\
{}_? &= \left(\psi_1 \circ \psi_t^{-1} \circ u\right)(s, t) \\
&= \psi_1(v(s, t)) \\
&\coloneqq \varphi(v(s, t)).
\end{aligned}
$$

? Just a guess.

- Recall definition of v:

$$v(s, t) \coloneqq \psi_t^{-1}(u(s, t))$$

- Verifying that $C(v) = C(u)$: not spelled out. Property of flow?
  - Need to check that

$$\frac{\partial u}{\partial s}(s_0, t_0) = 0 \implies \frac{\partial v}{\partial s}(s_0, t_0) = 0$$

    where

$$\frac{\partial u}{\partial s} = (d\psi_t)\left(\frac{\partial v}{\partial s}\right)$$

  - How: ?

- Verifying that $R(v) = R(u)$
  - Need to check that for $(s_0, t_0) \notin C(u)$ and $s \neq s_0$ we have

$$u(s_0, t_0) \neq u(s, t_0) \implies v(s_0, t_0) \neq v(s, t_0)$$

  - Follows directly:

$$\begin{aligned} v(s_0, t_0) \neq v(s, t_0) &\iff \psi_t^{-1}(u(s_0, t_0)) \neq \psi_t^{-1}(u(s, t_0)) \quad \text{by definition} \\ &\iff \left(\psi_t \circ \psi_t^{-1}\right)(u(s_0, t_0)) \neq \left(\psi_t \circ \psi_t^{-1}\right)(u(s, t_0)) \quad \text{injectivity of } \psi_t \\ &\iff u(s_0, t_0) \neq u(s, t_0). \end{aligned}$$

■

0.6 Splitting the Main Theorem

- The main theorem is equivalent to two upcoming statements

Proposition 0.3 ***(8.6.2: Statement 1, Critical Points are Discrete).***
Let $z = s + it$ where $(s,t) \in \mathbb{R}^1 \times S^1$ (?). There exists a constant $\delta > 0$ such that

$$0 < |z| < \delta \implies (dv)_z \neq 0.$$

Proof .
Postponed to p.264 because it relies on the 8.6.8 (Similarity Principle).
■

For the second statement, we set up some notation/definitions.

- $v \in C^\infty(\mathbb{R}^2; W)$ is a solution satisfying

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$

$$v(s, t+1) = \varphi(v(s,t))$$

$$C(v) = C(u),\ R(v) = R(u).$$

- The **regular points** are given by

$$R(v) = \left\{ (s,t) \in \mathbb{R}^2 \;\middle|\; \frac{\partial v}{\partial s}(s,t) \neq 0, \quad v(s,t) \neq x^\pm(t), \quad v(s,t) \notin v(\mathbb{R} \setminus \{s\}, t) \right\}.$$

> Note: last condition should be equivalent to $s \neq s' \implies v(s,t) \neq v(s',t)$. For reference, also equivalent to $v(s,t) = v(s',t) \implies s = s'$.

- **Multiple points** are defined as follows:
  - Set $\overline{\mathbb{R}} = \mathbb{R} \coprod \{\pm\infty\}$
  - Extend $v : \mathbb{R}^2 \longrightarrow W$ to

$$v : \overline{\mathbb{R}} \times \mathbb{R} \longrightarrow W$$
$$v(\pm\infty, t) = x^\pm(t).$$

  - Define the set of *multiple points* as

$$M(s,t) := \left\{ (s',t) \in \mathbb{R}^2 \;\middle|\; s \neq s' \in \overline{\mathbb{R}}, \quad v(s',t) = v(s,t) \right\}$$

> Note that the same t is used throughout.

- Recast definition of $R(v)$ as "points in the complement of $C(v)$ that do not admit multiples".

– Potentially incorrect formulation:

$$R(v) = C(v)^c \bigcap_{(s,t)\in\overline{\mathbb{R}}\times\mathbb{R}} M(s,t)^c.$$

Proposition 0.4 ***(8.6.3: Regular Points Open/Dense, "Injectivity").***
Let v be a smooth solution of the Cauchy-Riemann equation, then

$$\left.\begin{aligned} & v(s,t+1) = \varphi(v(s,t)) \\ & \frac{\partial v}{\partial s} \neq 0 \end{aligned}\right\} \implies R(v) \subseteq \mathbb{R}^2 \quad \text{is open and dense.}$$

Proof (Long).
Splits into two parts:

- Show $R(v)$ is open (easy)
- Show $R(v)$ is dense (delicate)

■

0.7 Regular Points Are Open

Proving the first part: $R(v)$ is open.

- Want to show $R(v)^c$ is closed.
- Toward a contradiction, suppose otherwise: $R(v)^c$ is *open*.
  - Use limit point definition: $R(v)^c$ is closed $\iff$ it contains all of its limit points
  - So $R(v)^c$ does *not* contain one of its limit points
  - Produces a sequence

$$R(v)^c \supseteq \{(s_n, t_n)\}_{n\in\mathbb{N}} \xrightarrow{n\to\infty} (s,t) \in R(v)$$

- The first two conditions defining $R(v)$ are open conditions:
  - The two conditions:

$$\begin{aligned} \frac{\partial v}{\partial s}(s,t) &\neq 0 && \text{Condition 1} \\ v(s,t) &\neq x^{\pm}(t) && \text{Condition 2.} \end{aligned}$$

  - Thus for $N \gg 1$ we have

$$n \geq N \implies \frac{\partial v}{\partial s}(s_n, t_n) \neq 0, \qquad v(s_n, t_n) \neq x^{\pm}(t)$$

- But $(s_n, t_n) \notin R(v)$ for such n, so they must fail the last condition: injectivity
  - Third condition:

$$s \neq s' \implies v(s,t) \neq v(s',t)$$

- Failing this conditions means:

$$\forall n > N, \; \exists s'_n \in \mathbb{R} \text{ s.t. } s'_n \neq s_n \quad \text{and} \quad v(s_n, t_n) = v(s'_n, t_n).$$

- Produces a sequence $\{s'_n\}_{n\in\mathbb{N}}$, want to show it is bounded.
  - Toward a contradiction, suppose not, then there is a subsequence

$$\{s_{n_k}\}_{n_k\in\mathbb{N}} \overset{n_k \longrightarrow \infty}{\longrightarrow} \pm\infty.$$

  - This implies

$$\begin{aligned} v(s,t) &= \lim_{n_k \longrightarrow \infty} v(s'_{n_k}, t'_{n_k}) \qquad \text{using continuity of } v \\ &= v(\pm\infty, t) \\ &:= x^{\pm}(t). \end{aligned}$$

  - Why? By definition, precisely because we extended v by setting $v(\pm\infty, t) = x^{\pm}(t)$.
  - But condition 2 for points in $R(v)$ says $v(s,t) \neq x^{\pm}(t)$, so this contradicts $(s,t) \in R(v)$.
- Sequence is bounded, so apply Bolzano-Weierstrass to extract a convergent subsequence converging to some limit:

$$\left\{s'_{n_j}\right\} \overset{n_j \longrightarrow \infty}{\longrightarrow} s'.$$

  - Use the fact that injectivity failed:

$$\begin{aligned} \forall n, \; s'_n \neq s_n \quad &\text{and} \quad v(s_n, t_n) = v(s'_n, t_n) \\ \implies \lim_{n_k \longrightarrow \infty} v(s_n, t_n) &= \lim_{n_k \longrightarrow \infty} v(s'_n, t'_n) \\ \iff v(s,t) &= v(s', t) \qquad \text{using continuity.} \end{aligned}$$

  - Use the fact that because $(s,t) \in R(v)$ we must have

$$s = s'.$$

- *(Minor technical point)* Now have $\left\{s'_{n_j}\right\}_{n_j\in\mathbb{N}} \longrightarrow s'$ and $\{s_n\}_{n\in\mathbb{N}} \longrightarrow s$
  - Since the latter sequence is convergent, every subsequence converges to the same limit, so $\left\{s_{n_j}\right\}_{n_j\in\mathbb{N}} \longrightarrow s$.
- Again using failed injectivity, i.e.

$$\begin{aligned} v(s,t) &= v(s', t) \\ \implies v(s,t) - v(s', t) &= 0. \end{aligned}$$

  we have

$$s'_{n_j} \neq s_{n_j} \quad \text{and} \quad v(s_{n_j}, t_{n_j}) = v(s'_{n_j}, t_{n_j})$$

.

- In the last step, we do have equality in the limit, $s = s'$, and $\forall n_j$,
$$v(s_{n_j}, t_{n_j}) - v(s'_{n_j}, t_{n_j}) = 0,$$
$$s_{n_j} - s'_{n_j} \neq 0$$
thus
$$\frac{\partial v}{\partial s}(s,t) = \lim_{n_j \longrightarrow \infty} \frac{v(s_{n_j}, t) - v(s'_{n_j}, t)}{s_{n_j} - s'_{n_j}} = 0.$$
- But $(s,t) \in R(v)$ and so this contradicts Condition 1.

This proves that $R(v)$ is open. ■

Lemma 8.6.4: For every $r > 0$ there exists a $\delta > 0$ such that
$$|t - t_0|,\ |s - s_0| < \delta \implies \exists s' \in B_r(s_j) \text{ s.t. } v(s,t) = v(s',t).$$

Proof: short.

Lemma 8.6.5: Let v_1, v_2 be two solutions of the CR-equation with $X_t \equiv 0$ on $B_\varepsilon(0)$, $v_1(0,0) = v_2(0,0)$ such that $(dv_1)_0, (dv_2)_0 \neq 0$. Also suppose
$$\forall \varepsilon\ \exists \delta \text{ s.t.}$$
$$\forall (s,t) \in B_\delta(0),\ \exists s' \in \mathbb{R} \begin{cases} (s',t) \in B_\varepsilon(0) \\ v_1(s,t) = v_2(s',t) \end{cases}.$$

Then
$$\forall z \in B_\varepsilon(0), \quad v_1(s,t) = v_2(s,t).$$

Take perturbed CR equation:
$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Fix $S \in C^\infty(\mathbb{R}^2; \mathrm{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$ be a solution to the perturbed CR equation and let $p > 2$. Then there exists $0 < \delta < \varepsilon$ and a map $A \in W^{1,p}(B_\delta, \mathrm{GL}(\mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_\delta \longrightarrow \mathbb{C}^n$ such that
$$\forall (s,t) \in B_\delta \quad Y(s,t) = A(s,t)\ \sigma(s+it) \quad \text{and} \quad J_0 A)s,t) = A(s,t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of $\bar\partial$) For every $p > 1$, the following operator is surjective and Fredholm:
$$\bar\partial : W^{1,p}\Big(S^2; \mathbb{C}^n\Big) \longrightarrow L^p(\Lambda^{0,1} T^* S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7

1 Goal 2: Continuation Principle

Goal 2: prove a continuation principle:

Proposition 1.1 ***(8.6.6, Continuation Principle).***
On an open $U \subset \mathbb{R}^2$, let Y be a solution to the perturbed CR equation

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}$ and $S \in C^\infty(\mathbb{R}^2, \mathrm{End}(\mathbb{R}^{2n}))$.
Say that f has an *infinite-order zero* at z_0 iff

$$\forall k \geq 0, \quad \sup_{|z - z_0| \leq t} \frac{|f(z)|}{r^k} \overset{r \to 0}{\longrightarrow} 0.$$

For f smooth, equivalently $f^{(k)}(z_0) = 0$ for all k.

Then the set

$$C := \left\{ (s,t) \in U \;\middle|\; Y \text{ has an infinite order zero at } (s,t) \right\}$$

is clopen. In particular, if U is connected and $Y = 0$ on some nonempty $V \subset U$, then $Y \equiv 0$.

Proposition 1.2 ***(8.1.4,).***

Define

$$\mathcal{Z}(x, y, J) := \left\{ (u, H_0 + h) \,|h \in \mathcal{C}^\infty_\varepsilon(H_0) \text{ and } u \in \mathcal{M}(x, y, J, H) \right\}.$$

If $(u, H_0 + h) \in \mathcal{Z}(x, y)$ then the following map admits a continuous right-inverse and is surjective:

$$\begin{aligned} \Gamma : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \times \mathcal{C}^\infty_\varepsilon(H_0) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ (Y, h) &\longmapsto \left(d\mathcal{F}^{H_0 + h}\right)_u (Y) + \mathrm{grad}_u\, h \end{aligned}$$

where $\mathcal{F}^{H_0 + h}$ is the Floer operator corresponding to $H_+ h$.

Used to show (via the implicit function theorem) that $\mathcal{Z}(x, y, J)$ is a Banach manifold when $x \neq y$.