CRAG

The Weil Conjectures

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Zeta Functions

Definition of Zeta Function

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\, n \in \mathbb{Z}^{\geq 2}$$

Definition (Zeta Function)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic* variety $X \subset \mathbb{P}^N_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ x \in \mathbb{P}^\infty_{\mathbb{F}} \;\middle|\; f_1(x) = \cdots = f_M(x) = 0 \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(x) = \sum_{\substack{I=(i_1,\cdots,i_n) \\ \sum_j i_j = d}} \alpha_I \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot x) = \lambda^d f(x).$$