Qual Problems

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Contents

1 Problem 1

1.1 Part 1

Definition: An element $r \in R$ is *irreducible* if whenever $r = st$, then either s or t is a unit.

Definition: Two elements $r, s \in R$ are *associates* if $r = \ell s$ for some unit ℓ.

A ring R is a *unique factorization domain* iff for every $r \in R$, there exists a set $\left\{p_i \;\middle|\; 1 \leq i \leq n\right\}$ such that $r = u \prod_{i=1}^n p_i$ where u is a unit and each p_i is irreducible.

Moreover, this factorization is unique in the sense that if $r = w \prod_{i=1}^n q_i$ for some w a unit and q_i irreducible elements, then each q_i is an associate of some p_i.

1.2 Part 2

A ring R is a *principal ideal domain* iff whenever $I \trianglelefteq R$ is an ideal of R, there is a single element $r_i \in R$ such that $I = (r_i)$.

1.3 Part 3

An example of a UFD that is not a PID is given by $R = k[x, y]$ for k a field.

That R is a UFD follows from the fact that if k is a field, then k has no prime elements since every non-zero element is a unit. So the factorization condition holds vacuously for k, and k is a UFD. But then we can use the following result:

> **Theorem**: If R is a UFD, then $R[x]$ is a UFD.

Since k is a UFD, the theorem implies that $k[x]$ is a UFD, from which it follows that $k[x][y] = k[x, y]$ is also a UFD.

To see that R is not a PID, consider the ideal $I = (x, y)$, and suppose $I = (g)$ for some single $g \in k[x, y]$.

Note that $I \neq R$, since I contains no degree zero polynomials. Moreover, since $(x) \subset I = (g)$ (and similarly for y), we have $g \mathrel{\Big|} x$ and $g \mathrel{\Big|} y$, which forces $\deg g = 0$.

So in fact $g \in k$ and thus g is invertible, but then $(g) = g^{-1}(g) = (1) = k$, so this forces $I = k \trianglelefteq k[x, y]$. However, $x \notin k$ (nor y), which is a contradiction.

2 Problem 2

Lemma 1: Then A has n distinct eigenvalues $\iff m_A(x) = \chi_A(x)$.

Proof:

We'll use the fact that every eigenvalue is a always root of both $m_A(x)$ and $\chi_A(x)$ (potentially with differing multiplicities), so we can write

$$\begin{aligned} m_A(x) &= \prod_i (x - \lambda_i)^{p_i} \\ \chi_A(x) &= \prod_i (x - \lambda_i)^{q_i} \end{aligned}$$

where $1 \leq p_i \leq q_i$ for every i.

$\implies$: If A has n distinct eigenvalues, then $\chi_A(x) = \prod_{i=1}^n (x - \lambda_i)$ in $\overline{k}[x]$. Noting that every exponent is 1, we have $q_i = 1$ for all i, which forces $p_i = 1$ and thus $m_A(x) = \chi_A(x)$.

$\impliedby$: If $m_A(x) = \chi_A(x)$, then $p_i = q_i$ for all i. If we then consider $JCF(A)$, we have

- The number of Jordan block J_{λ_i} is the dimension of the eigenspace E_{λ_i},
- $q_i =$ the sum of the sizes of all Jordan blocks J_{λ_i}, and
- $p_i =$ the size of the largest Jordan block J_{λ_i}.

Thus $p_i = q_i$ for every $i \iff$ there is one Jordan block for every $\lambda_i \iff \dim E_{\lambda_i} = 1$ for every i.

But $\dim E_{\lambda_i}$ is precisely the multiplicity of λ_i in $\chi_A(x)$, which means that $\chi_A(x) = \prod_i (x - \lambda_i)$. Since $\chi_A(x)$ is a degree n polynomial, this says that χ_A has n distinct linear factors, corresponding to n distinct eigenvalues of A.

$\square$

Lemma 2: Let $k[x] \curvearrowright V$ in the usual way with A to obtain an invariant factor decomposition

$$V = \frac{k[x]}{(f_1)} \oplus \frac{k[x]}{(f_2)} \oplus \cdots \oplus \frac{k[x]}{(f_k)}, \quad f_1 \mid f_2 \mid \cdots \mid f_k.$$

Then it is always the case that

- $m_A(x) = f_k(x)$, i.e. the minimal polynomial is the invariant factor of largest degree,
- $\chi_A(x) = \prod_{i=1}^k f_i(x)$, i.e. the characteristic polynomial is the product of all of the invariant factors.

$\square$

Now to prove the main result:

(1) $\implies$ (2):

Suppose

$$V = \operatorname{span}_k \left\{ \mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \cdots A^{n-1}\mathbf{v} \right\} := \operatorname{span}_k \mathcal{B}$$

where $\dim_k V = n$.

Then $A^n\mathbf{v}$ is necessarily a linear combination of these basis elements, and in particular, there are coefficients c_i (not all zero) such that

$$A^n\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i \mathbf{v}.$$

The consider computing the matrix of A in $\mathcal{B}$ by considering the images of all basis elements under A.

Letting $\mathcal{B} = \left\{ \mathbf{w}_i := A^i\mathbf{v} \;\middle|\; 0 \leq i \leq n-1 \right\}$, we have

$$\begin{aligned}
\mathbf{w}_0 := \mathbf{v} &\mapsto A\mathbf{v} := \mathbf{w}_1 \\
\mathbf{w}_1 := A\mathbf{v} &\mapsto A^2\mathbf{v} := \mathbf{w}_2 \\
\mathbf{w}_2 := A^2\mathbf{v} &\mapsto A^3\mathbf{v} := \mathbf{w}_3 \\
\vdots \quad & \quad \vdots \\
\mathbf{w}_{n-2} := A^{n-2}\mathbf{v} &\mapsto A^{n-1}\mathbf{v} := \mathbf{w}_{n-1} \\
\mathbf{w}_{n-1} := A^{n-1}\mathbf{v} &\mapsto A^n\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i \mathbf{v}_i := \sum_{i=0}^{n-1} c_i \mathbf{w}_i.
\end{aligned}$$

This means that with respect to the basis $\mathcal{B}$, A has the following matrix representation:

$$[A]_{\mathcal{B}} = \begin{bmatrix}
0 & 0 & \dots & 0 & c_0 \\
1 & 0 & \dots & 0 & c_1 \\
0 & 1 & \dots & 0 & c_2 \\
 & & \ddots & & \vdots \\
0 & 0 & \dots & 1 & c_{n-1}
\end{bmatrix}$$

But this is the companion matrix for $p(x) = \sum_{i=0}^{n-1} c_i x^i$, which always satisfy the property that $p(x)$ equals both their characteristic *and* their minimal polynomial.

Thus by lemma 1, the matrix $[A]_{\mathcal{B}}$ has distinct eigenvalues.

$(2) \implies (1)$:

Suppose A has distinct eigenvalues. By Lemma 1, $\chi_A(x) = m_A(x)$, and so we have

$$\chi_A(x) = f_k(x) = \prod_{i=1}^{k} f_i(x) = m_A(x),$$

which can only happen if $f_1(x) = f_2(x) = \cdots = f_{n-1}(x) = 1$, in which case there is only one nontrivial invariant factor.

So we have

$$V \cong \frac{k[x]}{(f_k)}, \quad \mathrm{Ann}(V) = (f_k), \quad \deg f_k = n,$$

which exhibits V as a cyclic $k[x]$-module and thus we have $V = k[x]\mathbf{v}$ for some $\mathbf{v} \in V$.

We can now note that since $\deg f_n = \dim V = m$, we have

$$k[x]/(f_n) = \mathrm{span}_{k[x]} \left\{1, x, \cdots, x^{n-1}\right\} \iff V \cong k[x]\mathbf{v} = \mathrm{span}_{k[x]} \left\{1\mathbf{v}, x\mathbf{v}, x^2\mathbf{v}, \cdots x^{n-1}\mathbf{v}\right\},$$

But then noting that $k[x] \curvearrowright V$ by $\mathbf{w} \mapsto x\mathbf{w}$, so $k[A] \curvearrowright V$ by $\mathbf{w} \mapsto A\mathbf{w}$, which yields

$$V \cong k[x]\mathbf{v} = \mathrm{span}_{k[x]} \left\{1\mathbf{v}, x\mathbf{v}, \cdots x^{n-1}\mathbf{v}\right\} \iff V \cong k[x]\mathbf{v} = \mathrm{span}_{k} \left\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \cdots A^{n-1}\mathbf{v}\right\},$$

$\square$

3 Problem 3

3.1 Part 1

Let $\mathbf{v} = [0, 1, 0]^t$, We compute

$$M\mathbf{v} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(0) + 0(1) + x(0) \\ 0(0) + 1(1) + 0(0) \\ y(0) + 0(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which shows that $\mathbf{v}$ is an eigenvector of M with eigenvalue $\lambda = 1$.

3.2 Part 2

Noting that the rank is the dimension of the column space, we find that

- $\text{rank}(M) \geq 1$, since it is not the zero matrix,
- $\text{rank}(M) \geq 2$, since neither $[1, 0, y]^t$ or $[x, 0, 1]^t$ can be in the span of $[0, 1, 0]^t$, and
- $\text{rank}(M) = 3 \iff \det(M) \neq 0$.

So we compute

$$\det_M(x, y) = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{vmatrix} = 1(1-0) - 0(1-xy) + x(-y) = 1 - xy,$$

and so $\det_M(x, y) = 0 \iff xy = 1$. Thus

$$\text{rank}(M) = \begin{cases} 3 & xy = 1 \\ 2 & \text{else.} \end{cases}$$

3.3 Part 3

Since M is diagonalizable $\iff$ M is full rank, which in this case means $\text{rank}(M) = 3$, we have

$$S = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; M \text{ is diagonalizable } \right\} = \left\{ \left(x, \frac{1}{x} \right) \;\middle|\; x \in \mathbb{R} \setminus \{0\} \right\} \subset \mathbb{R}^2.$$