Complex Analysis

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Contents

1 Definitions

In these notes, C generally denotes some closed contour, $\mathbb{H}$ is the upper half-plane, C_R is a semicircle of radius R in $\mathbb{H}$, f will denote a complex function.

1. **Analytic**

   f is analytic at z_0 if it can be expanded as a convergent power series in some neighborhood of z_0.

2. **Holomorphic**

   A function f is holomorphic at a point z_0 if $f'(z_0)$ exists in a neighborhood of z_0.

   (Note - this is more than just being differentiable at a single point!)

   Big Theorem: f is a holomorphic complex function iff f is analytic.

3. **Meromorphic**

   Holomorphic, except for possibly a finite number of singularities.

4. **Conformal**

   f is conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.

5. **Harmonic**

   A function $u(x, y)$ is harmonic if it satisfies Laplace's equation,

$$\Delta u = u_{xx} + u_{yy} = 0$$

Some other notions to look up:

- Conformal maps
- Analytic
- Theorem: Analytic $\implies$ conformal
-

2 Preliminary Notions

2.1 What is the Complex Derivative?

In small neighborhoods, the derivative of a function at a point rotates it by an angle $\Delta\theta$ and scales it by a real number λ according to

$$\Delta\theta = \arg f'(z_0), \ \lambda = |f'(z_0)|$$

2.2 nth roots of a complex number

The nth roots of z_0 are given by writing $z_0 = re^{i\theta}$, and are

$$\zeta = \left\{ \sqrt[n]{r} \exp\left[i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \mid k = 0, 1, 2, \dots, n-1 \right\}$$

or equivalently

$$\zeta = \left\{ \sqrt[n]{r}\omega_n^k \mid k = 0, 1, 2, \dots, n-1 \right\} \text{ where } \omega_n = e^{\frac{2\pi i}{n}}$$

This can be derived by looking at $\left(re^{i\theta + 2k\pi}\right)^{\frac{1}{n}}$.

It is also useful to immediately recognize that $z^2 + a = (z - i\sqrt{a})(z + i\sqrt{a})$.

2.3 The Cauchy-Riemann Equations

If $f(x+iy) = u(x,y) + iv(x,y)$ or $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$, then f is complex differentiable if u, v satisfy

$$\begin{aligned} u_x &= v_y \qquad & u_y &= -v_x \\ ru_r &= v_\theta \qquad & u_\theta &= -rv_r \end{aligned}$$

In this case,

$$f'(x+iy) = u_x(x,y) + iv_x(x,y)$$

or in polar coordinates,

$$f'(re^{i\theta}) = e^{i\theta}(u_r(r,\theta) + iv_r(r,\theta))$$

3 Integration

3.1 The Residue Theorem

If f is meromorphic inside of a closed contour C, then

$$\oint_C f(z)dz = 2\pi i \sum_{z_k} \operatorname*{Res}_{z=z_k} f(z)$$

where $\operatorname*{Res}_{z=z_k} f(z)$ is the coefficient of z^{-1} in the Laurent expansion of f.

If f is analytic everywhere in the interior of C, then $\oint_C f(z)dz = 0$.

If f is meromorphic inside of a contour C and analytic everywhere else, one can equivalently calculate the residue at infinity

$$\oint_C f(z)dz = 2\pi i \sum_{z_k} \operatorname*{Res}_{z=0} z^{-2} f(z^{-1})$$

3.2 Computing Residues

3.3 Simple Poles

If z_0 is a pole of order m, define $g(z) := (z-z_0)^m f(z)$.

If $g(z)$ is analytic and $g(z_0) \neq 0$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

In the case where $m = 1$, this reduces to

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0)$$

To compute residues this way, attempt to write f in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where ϕ only needs to be analytic at z_0.

3.4 Rational Functions

If $f(z) = \frac{p(z)}{q(z)}$ where

1. $p(z_0) \neq 0$
2. $q(z_0) = 0$
3. $q'(z_0) \neq 0$

then the residue can be computed as

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

3.5 Computing Integrals

When computing real integrals, the following contours can be useful:

One often needs bounds, which can come from the following lemmas

The Arc Length Bound If $|f(z)| \leq M$ everywhere on C, then

$$|\oint_C f(z)dz| \leq ML_C$$

where L_C is the length of C.

Jordan's Lemma: If f is analytic outside of a semicircle C_R and $|f(z)| \leq M_R$ on C_R where $M_R \to 0$, then

$$\int_{C_R} f(z)e^{iaz}dz \to 0$$

.

Can also be used for integrals of the form $\int f(z)\cos azdz$ or $\int f(z)\sin azdz$, just take real/imaginary parts of e^{iaz} respectively.

4 Conformal Maps

1. Linear Fractional Transformations:

$$f(z) = \frac{az + b}{cz + d} \qquad f^{-1}(z) = \frac{-dz + b}{cz - a}$$

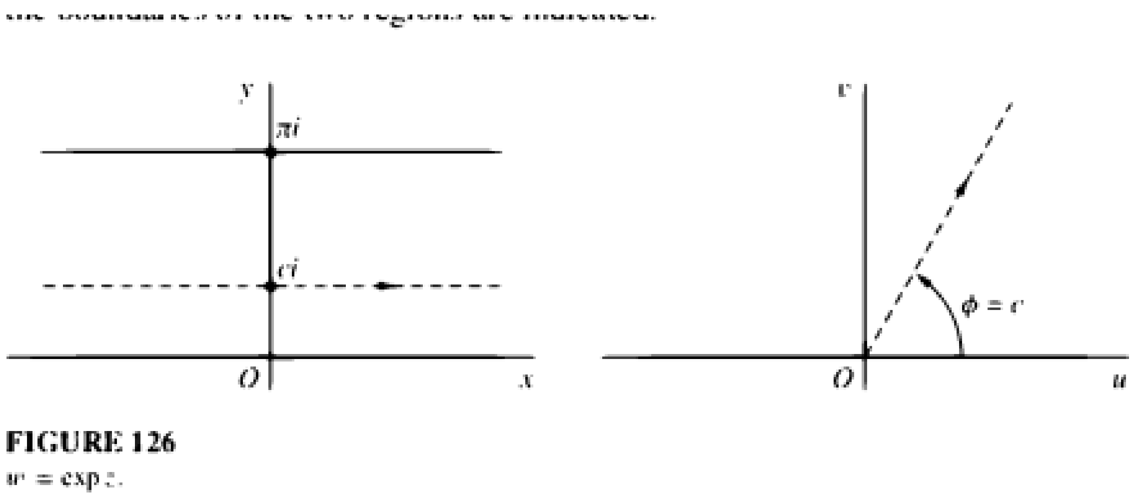


Figure 1: image

2. $[z_1, z_2, z_3] \mapsto [w_1, w_2, w_3]$

   Every linear fractional transformation is determined by its action on three points. Given 3 pairs points $z_i \mapsto w_i$, construct one using the implicit equation

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

3. z^k : Wedge $\mapsto \mathbb{H}$

   Just multiplies the angle by k. If a wedge makes angle θ, use $z^{\frac{\pi}{\theta}}$.

   It is useful to know that $z \mapsto z^2$ is equivalent to $(x, y) \mapsto (x^2 - y^2, 2xy)$.

4. $e^z : \mathbb{C} \mapsto \mathbb{C}$

Horizontal lines	$\mapsto$	rays from origin
Vertical lines	$\mapsto$	circles at origin
Rectangles	$\mapsto$	portions of wedges/sectors

5. $\log : \mathbb{H} \mapsto \mathbb{R} + i[0, \pi]$

   Just the inverse of what the exponential map does.

Rays	$\mapsto$	Horizontal Lines
Wedges	$\mapsto$	Horizontal Strips

6. $\sin : [0, \pi/2] + i\mathbb{R} \mapsto \mathbb{H}_{\mathcal{R}(z)>0}$

   Maps the infinite strip to the first quadrant.

7. $z \mapsto \frac{i-z}{i+z} : \mathbb{H} \mapsto D^\circ$.

$\mathbb{R}_{>0}$	$\mapsto$	Upper half of D°
$\mathbb{R}_{<0}$	$\mapsto$	Bottom half of D°

   Has inverse $w \mapsto i\frac{1-w}{1+w}$

8. $z \mapsto z + z^{-1} : \partial D \mapsto \mathbb{R}$

   Maps the boundary of the circle to the real axis, and the plane to $\mathbb{H}$.

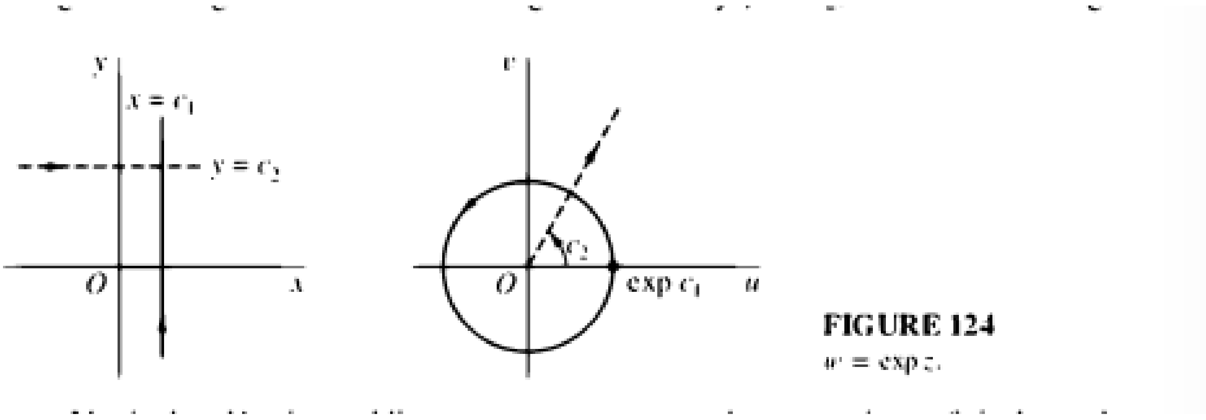


Figure 2: image

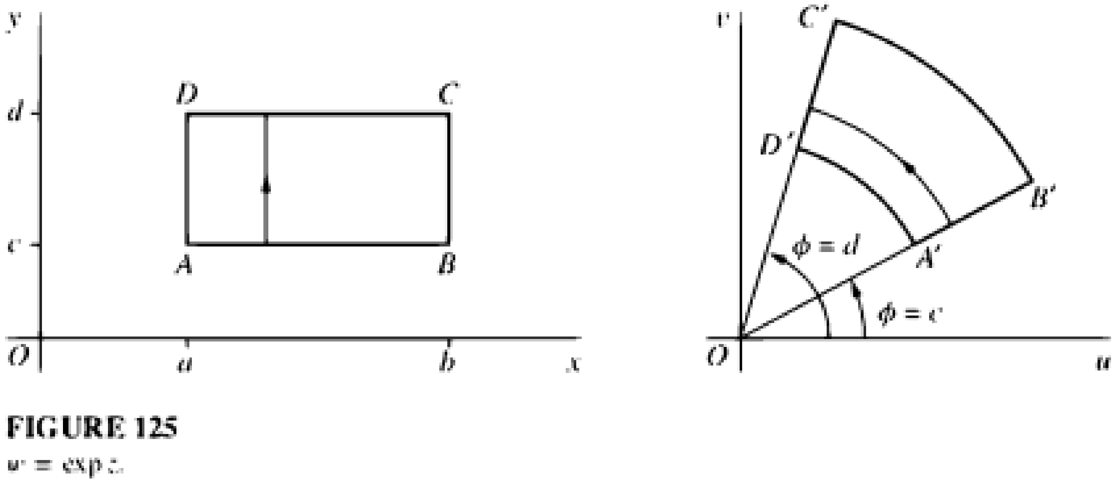


Figure 3: image

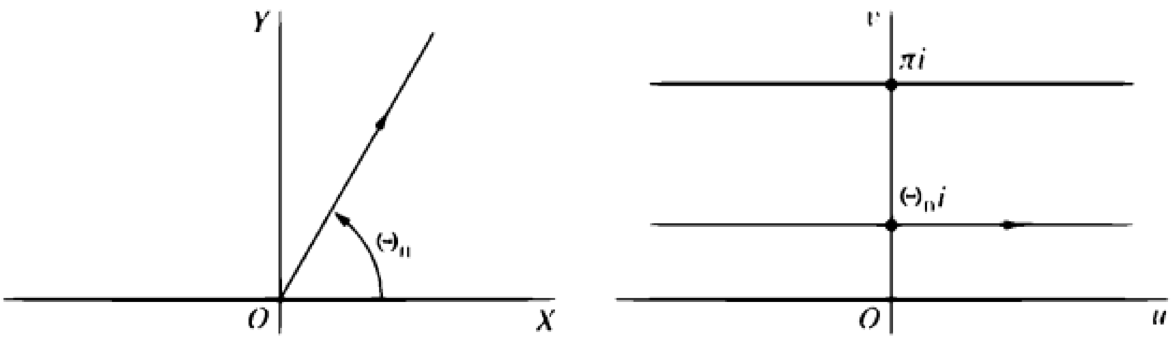


Figure 4: $z \mapsto \log z$

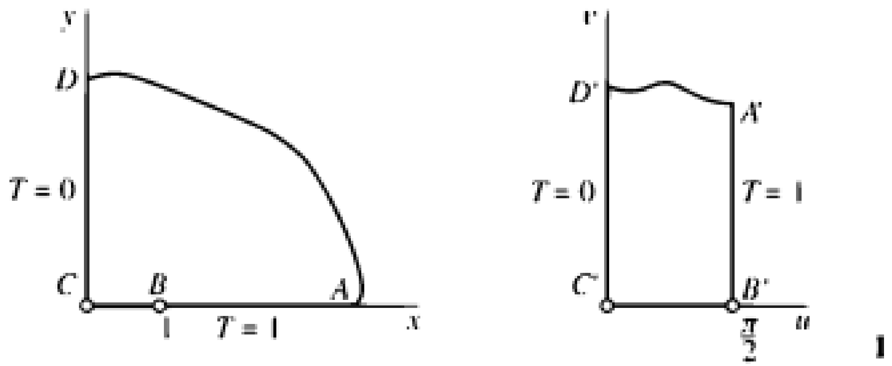


Figure 5: $z \hookleftarrow \sin w$

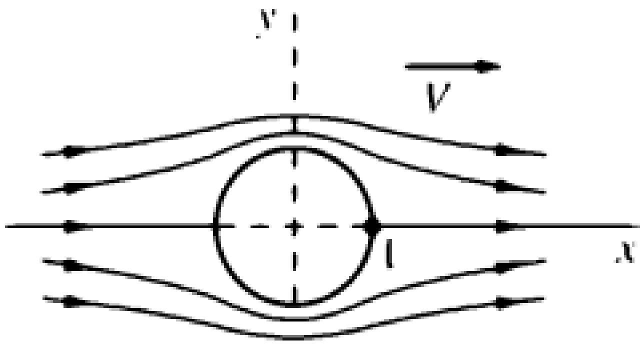
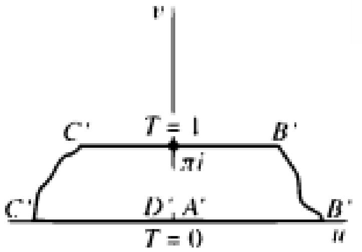


Figure 6: $z \mapsto z + z^{-1}$

The general technique is use solutions to the boundary value problem on a simple domain D, and compose one or several conformal maps to map a given problem into D, then pull back the solution.

4.1.1 Heat Flow: Steady Temperatures

Generally interested in finding a harmonic function $T(x,y)$ which represents the steady-state temperature at any point. Usually given as a Dirichlet problem on a domain D of the form



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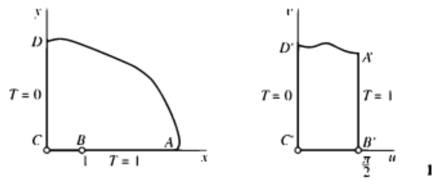
$$\Delta T = 0$$
$$T(\partial D) = f(\partial D)$$

where f is a given function that prescribes values on ∂D, the boundary of D.

Embed this in an analytic function with its harmonic conjugate to yield solutions of the form $F(x+iy) = T(x,y) + iS(x,y)$.

The **isotherms** are given by $T(x,y) = c$.

The **lines of flow** are given by $S(x,y) = c$.



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Any easy solution on the domain $\mathbb{R} \times i[0,\pi]$ in the u,v plane, where

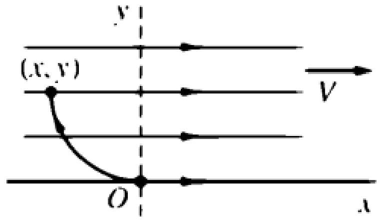
$$T(x,0) = 0$$
$$T(x,\pi) = 1$$

is given by $T(u,v) = \frac{1}{\pi}v$.

It is harmonic, as the imaginary part of the analytic $F(u+iv) = \frac{1}{\pi}(u+iv)$, since every analytic function has harmonic component functions.

Similar methods work with different domains, just pick a smooth interpolation between the boundary conditions.

4.1.2 Fluid Flow



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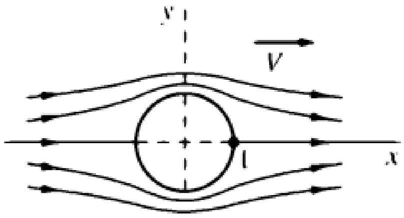
Write $F(z) = \phi(x,y) + i\psi(x,y)$. Then F is the complex potential of the flow, $\overline{F'}$ is the velocity, and setting $\psi(x,y) = c$ yields the streamlines.

A solution in $\mathbb{H}$ is $F(z) = Az$ some some velocity A. Apply conformal mapping appropriately.



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4.2 Theorems

4.2.1 General Theorems

1. **Liouville's Theorem**:

   If f is entire and bounded on $\mathbb{C}$, then f is constant.

2. If f is continuous in a region D, f is bounded in D.

3. If f is differentiable at z_0, f is continuous at z_0.

   Note - the converse need not hold!

4. If $f = u + iv$, where u, v satisfy the Cauchy-Riemann equations **and** have continuous partials, then f is differentiable.

   Note - continuous partials are not enough, consider $f(z) = |z|^2$.

5. Rouché's Theorem

   If $p(z) = f(z) + g(z)$ and $|g(z)| < |f(z)|$ everywhere on C, then f and p have the same number of zeros with C.

6. **The Argument Principle**

   If f is analytic on a closed contour C and meromorphic within C, then

$$W := \frac{1}{2\pi}\Delta_C \arg f(z) = Z - P$$

Proof: Evaluate the integral $\oint_C \frac{f'(z)}{f(z)} dz$ first by parameterizing, changing to polar, and using the FTC, and second by using residues directly from the Laurent series.

7. **The Main Story**: The following are equivalent
   - f is continuous
   - f' exists
   - f is analytic
   - f is conformal
   - f satisfies the Cauchy-Riemann equations

4.2.2 Theorems About Analytic Functions

1. If f is analytic on D, then $\oint_C f(z)dz = 0$ for any closed contour $C \subset D$.

   Note: this does not require f to be f' to be continuous on C.

2. **Maximum Modulus Principle**

   If f is analytic in a region D and not constant, then $|f(z)|$ attains its maximum on ∂D.

3. If f is analytic, then $f^{(n)}$ is analytic for every n. If $f = u(x,y) + iv(x,y)$, then all partials of u, v are continuous.
4. If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.
5. If $f = u + iv$ is analytic, then u, v are harmonic conjugates.
6. If f is holomorphic, f is C_∞ (smooth).
7. If f is analytic, f is holomorphic.

   Proof: Since f has a power series expansion at z_0, its derivative is given by the term-by-term differentiation of this series.

4.3 Some Useful Formulae

$$f_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \dots$$

$$\frac{1}{1-z} = \sum_k z^k$$

$$e^z = \sum_k \frac{1}{k!} z^k$$

$$\left(\sum_i a_i z^i\right)\left(\sum_j b_j z^j\right) = \sum_n \left(\sum_{i+j=n} a_i b_j\right) z^n$$

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) & &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \\
\cosh z &= \frac{1}{2}(e^{z} + e^{-z}) & = \cos iz &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \\
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) & &= z - \frac{z^3}{3!} + \frac{z^4}{4!} - \dots \\
\sinh z &= \frac{1}{2}(e^{z} - e^{-z}) & = -i \sin iz &= z + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots
\end{aligned}$$

Mnemonic: just remember that cosine is an even function, and that the even terms of e^z are kept. Similarly, sine is an odd function, so keep the odd terms of e^z.

Harmonic Conjugate

$$v(x,y) = \int_{(0,0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt$$

The Gamma Function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Useful to know: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

5 Question

1. True or False: If f is analytic and bounded in $\mathbb{H}$, then f is constant on $\mathbb{H}$.

   1=1

   False: Take $f(z) = e^{-z}$, where $|f(z)| \leq 1$ in $\mathbb{H}$.

   1=0

2. Compute $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)} dx$

   1=1

   Two semicircles needed to avoid singularity at zero. Limit equals the residue at zero, solution is $\pi(\frac{1}{a^2} - \frac{e^{-a}}{a^2})$.

   1=0

3. Compute $\int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta$

   1=1

   Cosine sub, solution is $\frac{2\pi}{\sqrt{3}}$

   1=0

4. Find the first three terms of the Laurent expansion of $\frac{e^z+1}{e^z-1}$.

   1=1

   Equals $2z^{-1} + 0 + 6^{-1}z + \ldots$

   1=0

5. Compute $\int_{S_1} \frac{1}{z^2+z-1} dz$

   1=1

   Equals $i\frac{2\pi}{5}$

   1=0

6. True or false: If f is analytic on the unit disk $E = \{z : |z| < 1\}$, then there exists an $a \in E$ such that $|f(a)| \geq |f(0)|$.

   1=1

   True, by the maximum modulus principal. Suppose otherwise. Then $f(0)$ is a maximum of f inside S_1. But by the MMP, f must attain its maximum on ∂S_1.

   1=0

7. Prove that if $f(z)$ and $f(\bar{z})$ are both analytic on a domain D, then f is constant on D

   1=1

   Analytic $\implies$ Cauchy-Riemann equations are satisfied. Also have the identity $f' = u_x + iv_x$, and $f' = 0 \implies f$ is constant.

   1=0