

# Topology Qual Problems

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## 1 Problems

### 1.1 Homotopy

1. Show that any non-surjective map  $f : X \rightarrow S^n$  is homotopic to the constant map.
2. Let  $f, g : X \rightarrow S^n$  be such that  $\forall x \in X, f(x) \neq -g(x)$ . Show that  $f \simeq g$ .
3. Let  $\alpha : S^n \rightarrow S^n, \alpha(p) = -p$  be the antipodal map on  $S^n$ . Show that  $n$  odd  $\implies f \simeq \text{id}$ .

4. Show that  $X$  is homotopy-equivalent to a point  $\iff \text{id}_X \simeq g$  for some constant map  $g$ .
5. Show that  $S^1 \times I \simeq M$ , the Mobius strip.
6. Show that  $\mathbb{R}^3 - S^1 \simeq S^1 \vee S^2$ .
7. Classify the letters of the alphabet up to homeomorphism, and up to homotopy.
8. **REVISIT** Let  $f, g : S^1 \rightarrow X$ ,  $P = X \cup_f B^2 \cong X \amalg B^2 / \sim$ , where  $x \sim f(x)$ ,  $Q = X \cup_g B^2$ . Show that  $f \simeq g \implies P \simeq Q$ .

## 1.2 Fundamental Group

1. Show that  $x, y \in X$  path & simply-connected  $\implies$  all paths from  $x$  to  $y$  are homotopic rel  $\{0, 1\}$ .
2. Show that for  $X$  path connected,  $\pi_1(X) = \mathbb{K} \iff \forall \text{cts. } f : S^1 \rightarrow X \text{ } f, \text{ extends to a continuous map } F : B^2 \rightarrow X$ .
3. Show  $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ .
4. Show  $\pi_1(S^n) = 1$  for  $n \geq 2$ .
5. Show that  $S^2 - \{p_0, p_1\} \simeq S^1$ .
6. Show that  $S^3 - \{p_0, p_1\} \simeq S^2$ .
7. Show that  $S^2 \not\simeq S^3$ .
8. For each of the following  $f : S^1 \rightarrow S^1$ , identify the corresponding  $f_* : \mathbb{Z} \rightarrow \mathbb{Z}$ :
  1.  $z \mapsto z^n$
  2.  $\bar{x} \mapsto -\bar{x}$
  3.  $e^{i\theta} \mapsto e^{2\pi i \sin \theta}$
9. Determine the winding number of the following map:  $f : S^1 \rightarrow \mathbb{C} - \{0\}$ ,  $z \mapsto 8z^4 + 4z^3 + 2z^2 + z^{-1}$
10. Identify  $\pi_1(M, [(1, \frac{1}{2})])$ , and identify the class of  $\partial M$ .
11. Let  $X = S^1 \times S^1$  and  $\gamma$  a loop based at  $x_0$ . What is the induced map  $\gamma_\#$ ?

## 1.3 Group Actions

1. Show that octagon pasting is homeomorphic to the  $T = \mathbb{R}^2 / \mathbb{Z}^2$ .
2. Let  $x_0$  be the image of 0, show that there is an order 6 homeomorphism  $f : T \rightarrow T$  fixing  $x_0$ . Find a representation of  $f_*$  as a matrix, and find its determinant.
3. Show that  $\pi_1(K)$ , the Klein bottle, is given by pairs  $(m, n)$  where  $(m, n) \star (p, q) = (m + (-1)^n p, n + q)$ 
  1. Show this is torsion-free
  2. Show that  $T$  is a double cover of  $K$ .
4. For each of these actions of  $\mathbb{Z}_2$  on  $S^n$ , compute  $\pi_1(S^n / \mathbb{Z}_2)$ 
  1.  $S^1, z \mapsto -z$
  2.  $S^2, (x, y, z) \mapsto (-x, -y, z)$
  3.  $S^3, (z, w) \mapsto (-z, -w)$

## 1.4 Applications

1. Let  $i : \mathbb{RP}^2 \rightarrow \mathbb{RP}^3$ , induced by  $S^2 \hookrightarrow S^3$  as the equator. Show that  $i \not\simeq \text{const}$ .
2. Show that there is no map  $f : S^2 \rightarrow S^1$  that commutes with the antipodal map.
3. Prove that for any  $f : S^2 \rightarrow \mathbb{R}^2$ , there exists  $x \in S^2$  such that  $f(x) = f(-x)$ .
4. Prove the Ham Sandwich theorem.

5. Show that  $K$  can not be a topological group.

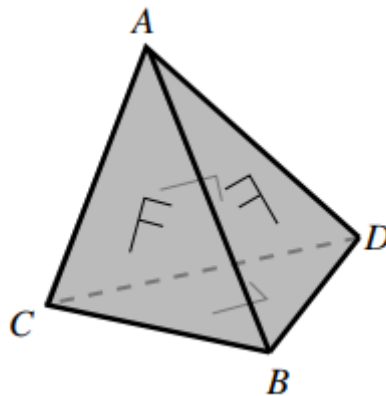
## 1.5 Van Kampen's Theorem

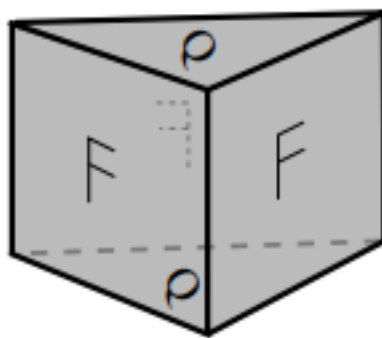
1. Compute a presentation of  $\pi_1(T)$  and prove it is isomorphic to  $\mathbb{Z}_2$ .
2. (Images)
3. Show that  $T - D^1 := X \simeq S^1 \vee S^1$ .
  1. Show there does not exist a retraction  $r : X \rightarrow \partial X$ .
4. Images
5. Images
6. Images
7. Calculate a presentation of  $\pi_1(S^3 - K)$
8. Show that all 3 presentations of  $\pi_1(K)$  are isomorphic
  1. Square with sides glued
  2. Two mobius strips glued along boundary
  3. Multiplication rule
9. Given a group  $G = \langle A : R \rangle$ , show how to construct a CW-complex  $X$  such that  $\pi_1(X) = G$ .
10. Write down the fundamental group of the following spaces:
11.  $\mathbb{R}^2 - \{0, 1\}$
12.  $\mathbb{R}^2 - I$
13. The symbol  $\oplus \in \mathbb{R}^2$
14.  $S^2 - \{p_i\}_{i=1}^4$
15.  $T - \{p_0\}$
16.  $S^2/\mathbb{Z}_2$  via the antipodal map
17.  $S^2/\mathbb{Z}_3$  via a  $2\pi/3$  rotation about the  $z$ -axis.
18.  $S_2 \cup \{(0, 0, z) \mid -1 \leq z \leq 1\}$
19.  $\mathbb{R}^3 - \{(x, y, 0) \mid x^2 + y^2 = 1\}$
20.  $\mathbb{R}^2 - H$ , the Hopf link
21. Prove that the homophony group is trivial.

## 1.6 Mayer Vietoris (Sheet 7)

1. Compute the homology of:
  1.  $\mathbb{RP}^2 = M \cup_{\partial} D^2$
  2.  $T^2 = S^1 \times S^1 = (S^1 \times I) \cup_f (S^1 \times I)$  where  $(x, 0) \sim (x, 1) \sim (\bar{x}, 0) \in \mathbb{C}$
  3.  $S^1 \cup_f B^2$  attached along  $\partial B^2$  using  $z \mapsto z^n$
2. Show  $\tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$ 
  1. Show  $\Sigma S^n \cong S^{n+1}$
3. For  $f : S^n \hookrightarrow \mathbb{C}P^n$ , show  $\deg f = \deg \Sigma f$ 
  1. Conclude  $\pi_n(S^n) = \mathbb{Z}$
4. Let  $\{A_i\}^n \in \mathbf{Ab}$  be finitely generated, show  $\exists X \mid H_i(X) \cong A_i$  for  $i \leq n$  and 0 otherwise.
5. Suppose  $X = \bigcup_i^n A_i$  such that for any  $1 \leq k \leq n$ ,  $\bigcap_i^k A_i$  is either empty or contractible, show  $i \geq n - 1 \implies \tilde{H}_i(X) = 0$  and that this bound is sharp.

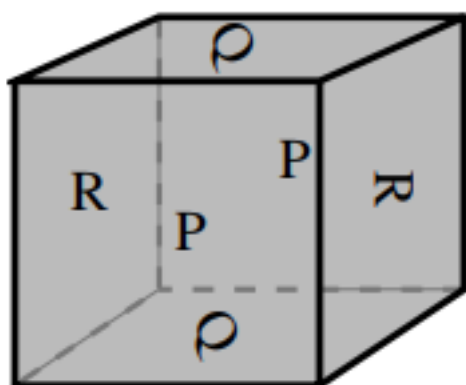
- $$\begin{array}{ccccccc} H_n(X) & \longrightarrow & H_n(A \cap B) & \longrightarrow & H_n(A) \oplus H_n(B) & \longrightarrow & H_{n-1}(X) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_n(Y) & \longrightarrow & H_n(C \cap D) & \longrightarrow & H_n(C) \oplus H_n(D) & \longrightarrow & H_{n-1}(Y) \end{array}$$





8. This identification space:

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9. This identification space: (a natural number) is defined by the

10. Describe a CW complex structure for the lens space  $L(p, 1)$  and compute  $\pi_1, H_*$  for it.

## 1.8 Degree

1. Let  $p(x) = \sum_i^n a_i x^i$ , view  $p : \mathbb{C} \cup \infty \rightarrow \mathbb{C} \cup \infty$  and determine its topological degree
2. Let  $p(z) = \frac{\prod_i^n (z - a_i)}{\prod_j^m (z - b_j)}$  with all  $a_i, b_j$  distinct. What is its topological degree?
3. Show that if  $f : S^m \rightarrow S^n$  and  $\exists U \subset S^m$  such that  $f|_U \cong f(U)$ , then  $m = n$  and  $f$  is surjective.

## 1.9 Universal Coefficient Theorem (Sheet 10)

1. Identify the following groups up to isomorphism
  1.  $\mathbb{Z}_m \otimes \mathbb{Z}_n$
  2.  $\mathbb{Z}_{60}^4 \otimes (\mathbb{Z}_{24}^3 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{120})$
  3.  $\mathbb{Z}_n \otimes \mathbb{Q}$
  4.  $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Q}/\mathbb{Z})$
2. Compute:
  1.  $\text{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$
  2.  $\text{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$
3. Compute the following directly from chain complexes and check using UCT:
  1.  $H_*(\mathbb{RP}^n; \mathbb{Z}_2)$

2.  $H_*(\mathbb{RP}^n, \mathbb{Z}_3)$
3.  $H^*(\mathbb{RP}^n, \mathbb{Z}_6)$
4. For any space  $X$ , show that  $H^1(X)$  is free abelian
5. Show that  $H_*(X; \mathbb{Q}) = H_*(X; \mathbb{Z}) \otimes \mathbb{Q}$   $H^*(X; \mathbb{Z}) = \text{hom}(H_*(X; \mathbb{Z}), \mathbb{Q})$
6. Construct a space  $X$  such that  $H_*(X; \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_{12}, \mathbb{Z} \oplus \mathbb{Z}_4, 0 \cdots)$  Compute  $H^*(X; \mathbb{Z})$
7. Compute  $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}_2)$
8. Compute  $H_*(\Sigma \mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$
9. Compute  $H_*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z})$
10. Let  $G$  be a topological group. Show that  $H_*(G)$  is an algebra. Show that  $G \curvearrowright H_*(G)$ , which factors through the homomorphism  $G \rightarrow \pi_0(G)$  yielding a trivial action if  $G$  is path-connected.

### 1.10 Homological Algebra (Sheet 11)

1. Show that  $\ker A \rightarrow A \otimes \mathbb{Q}$  given by  $a \mapsto a \otimes 1$  is the torsion subgroup of  $A$ .
2. Show that  $A \hookrightarrow B \implies A \otimes \mathbb{Q} \hookrightarrow B \otimes \mathbb{Q}$
3. Find a free resolution of  $\mathbb{Q}$  as a  $\mathbb{Z}$ -module.
4. Compute  $\text{Tor}(\mathbb{Q}, A)$ 
  1. Compute  $\text{Tor}(\mathbb{Q}/\mathbb{Z}, A)$
- 5.
6. Let  $R = \mathbb{Z}[x, y]$ , and  $M = R/(x - y)$ ,  $N = R/(x, y)$ . Construct free resolutions of  $M, N$  to compute:
  - $\text{Ext}_R^*(M, M)$
  - $\text{Ext}_R^*(M, N)$
  - $\text{Ext}_R^*(N, M)$
  - $\text{Ext}_R^*(N, N)$
7. Let  $\Lambda_*$  be the exterior algebra generated by the symbols  $\{dx_i\}^n$  over a field  $k$ . Show that letting  $d = \cdot \vee dx_1$  yields a chain complex  $0 \rightarrow \Lambda^0 \rightarrow \Lambda^1 \rightarrow \cdots \rightarrow \Lambda^n \rightarrow 0$  with trivial homology. Compute what happens when  $dx_1$  is replaced with an arbitrary non-zero element in  $\Lambda^1$ .
8. Define  $M$  as the group ring  $R = \mathbb{Z}[\mathbb{Z}_2]$  with the action  $(\cdot) \times -1$ . Construct a free resolution of  $M$  and compute  $\text{Tor}_R^*(M, M)$ .
9. Show  $\text{Tor}_R^*(\cdot, \cdot)$  is symmetric in the following way: Given  $M, N$ , take free resolutions, view  $M_* \rightarrow M$  as a chain map and tensor with  $N_*$  to get a chain map  $\psi : M_* \otimes_R N_* \rightarrow M \otimes_R N_*$ . Show that  $\psi$  is a quasi-isomorphism using the exact sequence  $0 \rightarrow (Z_n, 0) \rightarrow (N_n, 0) \rightarrow (B_{n-1}, 0) \rightarrow 0$ , then switch the roles of  $M, N$ .
10. Prove that for a SES  $0 \rightarrow A \rightarrow B \rightarrow C$ , the group  $\text{Ext}(C, A)$  classifies extensions of  $C$  by  $A$  up to isomorphism.

### 1.11 Cohomology Ring (Sheet 12)

Todo

## 2 Topology Problems: Solutions

### 2.1 Homotopy

1. **Main Idea:** A linear homotopy projected onto the sphere works.

Let  $f : X \rightarrow S^n \subset \mathbb{R}^{n+1}$  be an arbitrary map that fails to be surjective. Then, by definition, there is at least one point  $s_0 \in S^n - f(X)$ .

Then,  $\forall x \in X$ , since  $f(x) \neq s_0$ , there is a unique geodesic  $C$  connecting  $f(x)$  and  $s_0$ . So a variant of the straight line homotopy will work, by interpolating between  $f(x)$  and  $s_0$  along  $C$ .

So let  $H : X \times I \rightarrow S^n$  be defined by  $H(x, t) = P(ts_0 + (1 - t)f(x))$ , where  $P : \mathbb{R}^{n+1} \rightarrow S^n$  is given by  $P(x) = x/\|x\|$ . This is well defined, since the denominator is zero iff  $f(x) = s_0$ , which by assumption is not the case. This is a homotopy, since  $H(x, 0) = P(f(x)) = f(x)$  (since  $P$  fixes  $S^n$ ) and  $H(x, 1) = P(s_0) = s_0$  (since  $s_0 \in S^n$ ).

2. **Main Idea:** Exact same idea as 1, just a more complicated check.

Take  $H(x, t) = P(tf(x) + (1 - t)g(x))$ . This is well defined; the only case to check is when the denominator is zero. But  $\|x\| = 0$  iff  $x = 0$ , which would imply  $tf(x) + (1 - t)g(x) = 0$  and so  $tf(x) = -(1 - t)g(x)$ .

Taking norms and observing that since  $f, g \in S^n \implies \|f\| = \|g\| = 1$ , this forces  $t = 1 - t$  and thus  $t = 1/2$ . But this would force  $(1/2)f(x) = (-1/2)g(x)$  and thus  $f(x) = -g(x)$ , which we assumed was not the case.

3. **Main Idea:** Linear homotopy fails continuity without the condition from (2), so use complex embedding to avoid the origin at  $t = 1/2$ .

Suppose  $n$  is odd and define  $f : S^n \rightarrow S^n$  to be the antipodal map. Since  $n + 1$  is even, we have  $n + 1 = 2m$  for some  $m \in \mathbb{N}$ , so identify  $S^n = S^{2m-1} \subset \mathbb{R}^{2m} \cong \mathbb{C}^m$ .

Then  $z \in S^n$  can be written as a vector  $z \in \mathbb{C}^m$  such that  $\|z\| = 1$ .

Then define  $P : \mathbb{C}^m \rightarrow \mathbb{C}^m$  by  $P(z) = z/\|z\|$ , the projection onto the complex unit sphere, and define  $H : \mathbb{C}^m \times I \rightarrow \mathbb{C}^m$  by  $H(z, t) = P(e^{i\pi t}z)$ .

This is a homotopy, since  $H(z, 0) = P(z) = z$  (since  $\|z\| = 1$ ), so this is the identity map. We also have  $H(z, 1) = P(-z) = -z$ , the antipodal map.

This is well-defined, since  $e^{i\pi t} > 0$  and  $z \neq 0$ , so the linear homotopy in ambient  $\mathbb{C}^m$  avoids the origin and thus the denominator when taking the projection is never zero.

4.  $\Leftarrow$ : **Main Idea:** Projection and inclusion are homotopy inverses. One composition is equality, the other is just equality *up to homotopy*, but that's all we need!

Suppose  $\text{id}_X$  is nullhomotopic.

Then there exists some constant map  $g : X \rightarrow \{x_0\}$  for some  $x_0 \in X$  where  $g(x) = x_0$  and  $g \simeq \text{id}_X$ .

This means there is some homotopy  $F : X \times I \rightarrow X$  such that  $F(x, 0) = \text{id}_X(x) = x$  and  $F(x, 1) = g(x) = x_0$  for all  $x \in X$ .

So let  $p : X \rightarrow \{x_0\}$  be the projection map sending every point to  $x_0$ , and  $\iota : \{x_0\} \rightarrow X$  be the inclusion. We will show that the two compositions are homotopy inverses, from which it follows

that  $X \simeq \{x_0\}$ . This means that  $X$  is homotopy-equivalent to a point, and thus by definition contractible.

Then  $(p \circ \iota) : \{x_0\} \rightarrow \{x_0\}$  is given by  $p(\iota(x_0)) = p(x_0) = x_0$ , so this is the identity on the target space  $\{x_0\}$ .

Similarly,  $(\iota \circ p) : X \rightarrow X$  is given by  $\iota(p(x)) = \iota(x_0) = x_0$ , so this is the constant map on  $X$  mapping every point from  $X$  to  $x_0$ . But then this map is exactly  $g$ , and by assumption this is homotopic to the identity on  $X$ .

But then we have  $p \circ \iota \simeq \text{id}_{\{x_0\}}$  and  $\iota \circ p \simeq \text{id}_X$ , so they are homotopy inverses.

$\Rightarrow$ : **Main Idea:** One of the homotopy inverses *is* just a constant map.

Suppose  $X \simeq \{x_0\}$ , then there exist a pair of homotopy inverses

$f : X \rightarrow \{x_0\}$  and  $g : \{x_0\} \rightarrow X$  such that  $f \circ g \simeq \text{id}_{\{x_0\}}$  and  $g \circ f \simeq \text{id}_X$ .

Since  $\{x_0\}$  is a single point space,  $f$  is necessarily a constant map (i.e.  $f(x) = x_0$  for every  $x \in X$ .) But then  $(g \circ f)(x) = g(x_0) = y_0$  for some constant  $y_0 \in X$ , so  $g \circ f$  is a constant map. By assumption,  $g \circ f \simeq \text{id}_X$ , so the identity is homotopic to a constant map.

5. **Main Idea:** Deformation retract  $M$  onto its center circle; two spaces that deformation retract onto a common space are themselves homotopy equivalent.

Claim:  $S^1 \times I \simeq S^1 \times \{*\}$  This is because  $I$  is contractible, so  $I \simeq \{*\}$ . (Maybe needs further proof)

Claim:  $M \simeq S^1 \times \{*\}$ .

If both of these claims hold, then we will have  $M \simeq S^1 \times I$  as two spaces that deformation retract onto a common space. Identifying  $M = I \times I / \sim$  where  $(x, 0) \sim (1 - x, 1)$ , fix  $x = 1/2$ .

Then consider the subspace  $U = \{(1/2, y) \mid y \in [0, 1]\} \subset M$ . Claim:  $U \cong \{*\} \times S^1$  for some point  $*$ .

$U$  can be written  $\{1/2\} \times (I / \sim)$ , and since  $(1/2, 0) \sim (1/2, 1)$ , we have  $I / \sim = I / \partial I \cong S^1$ , so  $U \cong \{1/2\} \times S^1$  as desired (taking  $*$  =  $\frac{1}{2}$ ).

However, we can define a homotopy from  $M$  onto  $U$ , in the form of a deformation retract.

Let  $F : M \times I \rightarrow M$  be defined by  $F((x, y), t) = F_t(x, y) = ((1 - t)x + \frac{1}{2}t, y)$ . Then  $F((x, y), 0) = (x, y) = \text{id}_M$ , and  $F((x, y), 1) = (\frac{1}{2}, y) \subseteq U$ . Moreover, if  $(x, y) \in U$ , then  $(x, y) = (\frac{1}{2}, y)$  and  $F((x, y), t) = ((1 - t)\frac{1}{2} + \frac{1}{2}t, y) = (\frac{1}{2} - t\frac{1}{2} + \frac{1}{2}t, y) = (\frac{1}{2}, y) = (x, y)$ , so  $F = \text{id}_U$ . This makes  $F$  a deformation retract from  $M$  onto  $U$ , and so  $M \simeq U$ .

But then, summarizing our results, we have  $S^1 \times I \simeq S^1 \times \{*\} \cong S^1 \times \left\{\frac{1}{2}\right\} = U \simeq M$ , and so  $S^1 \times I \simeq M$  as desired.

6. **Main Idea:** Using a funky deformation retract. See Hatcher, PDF page 55, Example 1.23. Add picture!!

Deformation retract

$R^3 - S^1$  onto  $S^2 - U$ , where  $U$  is a diameter inside  $S^2$  also passing through the middle of  $S^1$  in the interior. This can be done by moving points outside of  $S^2$  towards the surface, and points inside



$S^2$  just move away from the  $S^1$  inside (either towards  $U$  or towards the surface of  $S^2$ , so they don't hit  $S^1$ ).

Then take a geodesic between the endpoints of the diameter on  $S^2$ , pick any point  $p$  on the geodesic, and move both diameter points towards it. This yields  $S^2 \vee S^1$  at the point  $p$ .

7. **Main Idea:** Nothing to it. Homotopy:

$$8. A \simeq \Delta \simeq S^1$$

$$9. a \simeq d \simeq o \simeq S^1$$

$$10. B \simeq 8 \simeq S^1 \vee S^1$$

$$11. b \simeq o \simeq S^1$$

$$12. C \simeq *$$

$$13. c \simeq l \simeq *$$

$$14. D \simeq S^1$$

$$15. d \simeq o \simeq S^1$$

$$16. E \simeq *$$

$$17. e \simeq d \simeq S^1$$

$$18. F \simeq *$$

$$19. f \simeq *$$

$$20. G \simeq *$$

$$21. g \simeq 8 \simeq S^1 \vee S^1$$

$$22. H \simeq *$$

$$23. h \simeq l \simeq *$$

$$24. I \simeq *$$

$$25. i \simeq \{*_1, *_2\}$$

$$26. J \simeq *$$

$$27. j \simeq i \simeq \{*_1, *_2\}$$

$$28. K \simeq *$$

$$1. k \simeq K \simeq *$$

$$29. L \simeq *$$

$$1. l \simeq *$$

$$30. M \simeq *$$

$$1. m \simeq *$$

$$31. N \simeq *$$

$$1. n \simeq *$$

32.  $O \simeq S^1$ 
  1.  $o \simeq S^1$
33.  $P \simeq D \simeq S^1$ 
  1.  $p \simeq P \simeq S^1$
34.  $Q \simeq O \simeq S^1$ 
  1.  $q \simeq p \simeq o \simeq S^1$
35.  $R \simeq D \simeq S^1$ .
  1.  $r \simeq l \simeq S^1$
36.  $S \simeq *$ 
  1.  $s \simeq S \simeq *$
37.  $T \simeq *$ 
  1.  $t \simeq l \simeq *$
38.  $U \simeq *$ 
  1.  $u \simeq U \simeq *$
39.  $V \simeq *$ 
  1.  $v \simeq V \simeq *$
40.  $W \simeq *$ 
  1.  $w \simeq W \simeq *$
41.  $X \simeq *$ 
  1.  $x \simeq X \simeq *$
42.  $Y \simeq *$ 
  1.  $y \simeq v \simeq *$
43.  $Z \simeq *$ 
  1.  $z \simeq Z \simeq *$

This results in a partition of the alphabet into the following homotopy types:

- $\{A, D, O, P, Q, R, S^1\} \cup \{a, b, d, e, g, o, p, q\}$
- $\{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z, *\} \cup \{c, f, h, k, l, m, n, r, s, t, u, v, w, x, y, z\}$
- $\{B, S^1 \vee S^1\}$
- $\{i, j, \{*, *\}\}$

Homeomorphisms: ignore ligatures!!

1.  $\{A, R\}$  Can remove a point to obtain two components homeomorphic to  $\{I, F\}$  respectively.
2.  $\{D, O, S^1\}$  These all have no single point that can be removed to disconnect the space.
3.  $\{B, S^1 \vee S^1\}$  Remove point at junction

4.  $\{C, G, I, J, L, M, N, S, U, V, W, Z, [0, 1]\}$  These all have a point that can be removed to yield **two** components, but no points that yield **three**. (Intuitively, all can be obtained by twisting a straight wire.)
5.  $\{E, F, T, Y, \bigvee_{i=1}^3 [0, 1]\}$  These all have a point that can be removed to yield 3 connected components homeomorphic to  $I$ . This is the “pasting” point in the vee.
6.  $\{H, K, \bigvee_{i=1}^5 [0, 1]\}$  Can remove **two** points to disconnect each into **five** components.
7.  $\{P, Q, S^1 \vee [0, 1]\}$  Both contain a nontrivial loop.
8.  $\{X, \bigvee_{i=1}^4 [0, 1]\}$  Can remove **one** point to separate into **four** components.
9. **Main Idea:** Show that both spaces are a deformation retract of the same space. (See Hatcher, Proposition 0.18, p. 25)

Suppose we have the following maps

$$\begin{aligned} f : S^1 &\rightarrow X \\ g : S^1 &\rightarrow X \end{aligned}$$

where  $f \simeq g$ . Then there exists a homotopy

$$H : S^1 \times I \rightarrow X$$

such that  $H(z, 0) = f(z)$  and  $H(z, 1) = g(z)$ .

Then define

$$\begin{aligned} P &:= X \coprod_f B^2 \\ Q &:= X \coprod_g B^2 \end{aligned}$$

We want to that  $P$  and  $Q$  are homotopy-equivalent. In order to do so, we will construct a larger space which deformation retracts onto both  $P$  and  $Q$ , which is a homotopy equivalence.

With  $H$  in hand, we can define the space  $R = X \coprod_H B^2 \times I$ , where we recognize  $S^1 = \partial B^2$ . In particular,  $S^1$  is a subspace of  $B^2$ .

Claim: Both  $P$  and  $Q$  are subspaces of  $R$ . Since  $H(z, 0) = f(z)$ . So considering  $X \coprod_H B^2 \times \{0\} \cong X \coprod_f B^2 = P$ . A similar argument holds at the point  $1 \in I$ . (*Not a strong argument*)

But note that  $B^2 \times I$  is a solid cylinder, and so can be deformation retracted onto the outer shell plus one of the “lids”. Formally, this would be given by  $S^1 \times I \cup B^2 \times \{p\}$  for some  $p \in [0, 1]$ .

Claim: choosing  $p = 0$  induces a deformation retract of  $R$  onto  $P$ , and choosing  $p = 1$  induces a deformation retract of  $R$  onto  $Q$ .

Proof: ?

## 2.2 Fundamental Group

1. **Main idea:** just algebraic manipulations using the  $\pi_1$  functor and unravelling definitions.

Let  $X$  be path connected and simply connected, and let  $x, y \in X$  be two arbitrary points. Then consider two paths,  $\gamma : I \rightarrow X, \gamma(0) = x, \gamma(1) = y$   $\alpha : I \rightarrow X, \alpha(0) = x, \alpha(1) = y$ .

We would like to show  $\gamma \simeq \alpha$ . Since  $X$  is simply connected, we know that  $\pi_1(X) = 0$ . This means that for any  $a, b \in \pi_1(X), a = b = e$ , the identity element in this group.

So we construct two loops: one as  $\gamma\bar{\alpha}$ , the other as  $\alpha\bar{\gamma}$ . Apply the  $\pi_1$  functor yields  $[\gamma\bar{\alpha}] = e = [c_x] = [\alpha\bar{\gamma}]$ , where  $[c_x]$  is the equivalence class of the constant path at  $x$ , and equivalently the identity element in  $\pi_1(X)$ . Lemma: If  $f \simeq g$ , then  $f \circ h \simeq g \circ h$  for any  $h$ .

But this says  $\gamma\bar{\alpha} \simeq c_x$  and  $\alpha\bar{\gamma} \simeq c_x$ . But  $\gamma \simeq c_x \circ \gamma \simeq (\alpha\bar{\gamma}) \circ \gamma \simeq \alpha \circ (\bar{\gamma} \circ \gamma) \simeq \alpha$ , which is what we desired.

2. **Main Idea** Homotopies on maps  $S^1 \rightarrow X$  are cylinders, find a way to continuously map a cylinder onto a disk given the existence of such a homotopy. Let  $X$  be path connected,  $\pi_1(X) = 0$ , and let  $f : S^1 \rightarrow X$  be arbitrary. Then  $f(S^1) \subseteq X$  is a path in  $X$ , and since  $\pi_1(X) = 0$ , this path is homotopic to a point  $x_0$ . So  $f$  is homotopic to the constant map  $c_{x_0} : S^1 \rightarrow X, z \mapsto x_0$ .

So let  $H : S^1 \times I \rightarrow X$  be this homotopy. We know that  $H(z, 0) = f(z)$  and  $H(z, 1) = c_{x_0}(z) = x_0$ .

Claim: Consider quotient  $\frac{S^1 \times I}{S^1 \times \{1\}}$  with the projection map  $p : S^1 \times I \rightarrow S^1 \times \{1\}$ . Then  $H$  factors through the quotient uniquely (why?), and there exists a unique  $\hat{H}$  making this diagram commute:

This follow from the universal property of the quotient in **Top**, where it is sufficient that  $H$  is constant on  $S^1 \times \{1\}$  - but this is exactly what was deduced above.

However, the quotient object constructed is homeomorphic to  $D^2$ , as per the following diagram

Here, we just recognize that  $S^1 \times I$  is a cylinder, and quotienting at the  $t = 1$  point in  $I$  simply collapses the top portion of the cylinder to a point, forming a cone. We then take the flattening map to just project every point on the cone directly downwards onto the base circle, yielding  $D^2$ .

(Note: I guess this map can be constructed as  $\Phi : S^1 \times I \rightarrow D^2$  where  $\Phi(z, t) = z(1 - t)$ . Since  $t = 1$  on  $S^1 \times \{1\}$ ,  $\Phi(z, 1) = 0$  and this is exactly the kernel of  $\Phi$ . Continuous as product of continuous functions, need to check injective/surjective and show inverse is continuous.)

Need to check injective/surjective, show that kernel is  $S^1 \times 1$ , then use first isomorphism theorem.)

But then  $\hat{H}$  is exactly a continuous map from  $D^2 \rightarrow X$ , as desired.

3.  $\Rightarrow$  Let  $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$  be an arbitrary loop in  $X \times Y$ . Then  $\alpha$  is equivalently a map  $S^1 \rightarrow X \times Y$ . Considering  $S^1$  to be a subset of  $\mathbb{R}^2$ , we can parameterize  $\alpha$  as  $\alpha(z) = \alpha(x + iy) = (\alpha_x(x), \alpha_y(y))$  in components. In particular, since  $\alpha$  is continuous, so are  $\alpha_x, \alpha_y$ . Moreover, since  $\alpha(0) = \alpha(0 + i0) = (x_0, y_0)$ , we have  $\alpha_x(0) = x_0, \alpha_y(0) = y_0$ . (Note: alternatively, given the product, we have projections  $p_X, p_Y$ , so we can define the map  $\alpha \mapsto (p_X \circ \alpha, p_Y \circ \alpha)$ )

But then  $\alpha_x : S^1 \rightarrow X$  and  $\alpha_y : S^1 \rightarrow Y$  are loops entirely in  $X, Y$  at the respective base points, and so we can define the map  $F : \pi_1(X \times Y, (x_0, y_0)) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$  by  $[\alpha] = [(\alpha_x, \alpha_y)] \mapsto ([\alpha_x], [\alpha_y])$

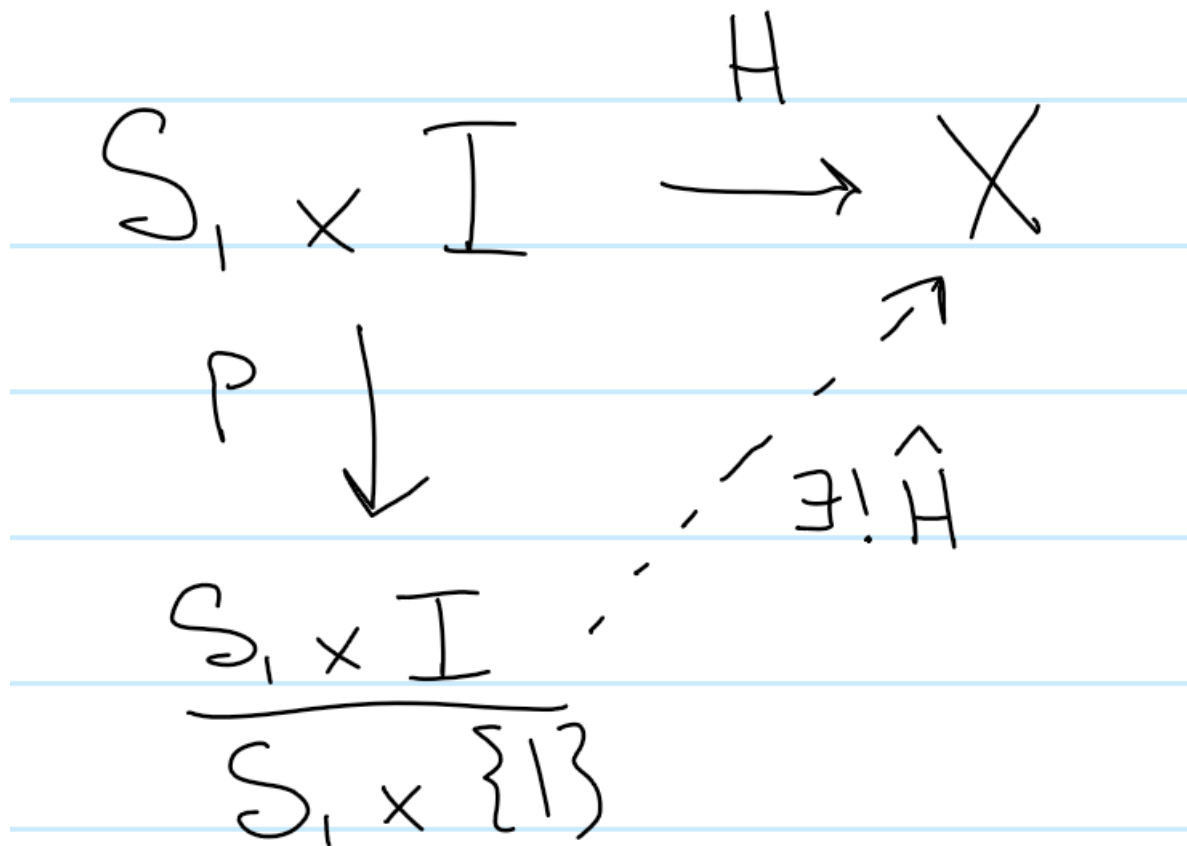


Figure 1: universal1

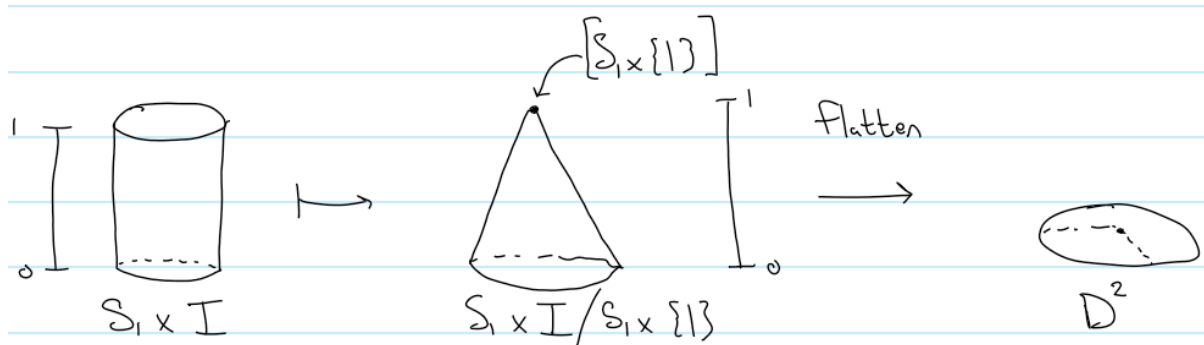


Figure 2: 2017-11-24 14\_59\_29-Untitled page - OneNote

This is injective, since  $([a], [b]) = ([c], [d])$  on the RHS means that  $[a] = [c], [b] = [d]$  in the fundamental groups, and thus  $a \simeq c, b \simeq d$  in the spaces. We want to show that  $[(a, b)] = [(c, d)]$ , which would follow if  $\alpha(x + iy) = (a(x), b(y)) \simeq \beta(x + iy) = (c(x), d(y))$  in  $X \times Y$ . ...?

This is surjective, because if  $([a], [b])$  are elements in the right-hand side, then  $a(0) = a(1) = x_0$  and  $b(0) = b(1) = y_0$ , so we can consider  $(a, b) : I \rightarrow X \times Y$  where  $(a, b)(z) = (a, b)(x + iy) = (a(x), b(y))$ . This is then a loop in  $X \times Y$ , since  $(a, b)(0) = (a(0), b(0)) = (0, 0) = (x_0, y_0)$  and similarly  $(a, b)(1) = (a(1), b(1)) = (x_0, y_0)$ . So this is actually a map  $(a, b) : S^1 \rightarrow X \times Y$ , or in other words, a loop in  $X \times Y$  based at  $(x_0, y_0)$ , which lifts to an element of the fundamental group on the LHS.

Maps in both directions are continuous, since a vector function is continuous iff its component functions are continuous.

This is well-defined, due to the fact that if  $a \simeq b$ , then  $p_X \circ a \simeq p_X \circ b$ , and  $F = (f_x, f_y)$  is a homotopy iff its components functions are homotopies.

4. Let  $A = S^n - \{n_p = \text{North Pole}\}, B = S^n - \{s_p = \text{South Pole}\}$ . Then  $A \cup B = S^n$  and  $A \cap B = S^n - \{n_p, s_p\}$ . Since  $A, B$  are open and path connected, we can apply van Kampen's theorem to obtain  $\pi_1(X) = \pi_1(A) * \pi_1(B)$  amalgamated over  $\pi_1(A \cap B)$ . But  $A \cong \mathbb{R}^n \cong B$  via stereographic projection, and since  $\mathbb{R}^n$  is contractible,  $\pi_1(\mathbb{R}^n) = 0 = \pi_1(A) = \pi_1(B)$ . So  $\pi_1(X) = 0 * 0 = 0$  as desired.

This follow because we can compute  $A \cap B \cong \mathbb{R}^n - \{\text{pt}\} \cong S^{n-1}$ , and so  $\pi_1(A \cap B) = \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}^1) = 0 \times 0 = 0$ , and so has the presentation  $\pi_1(A \cap B) = \langle w \mid w^1 = e \rangle$ . We can then look at the inclusions  $i : A \cap B \rightarrow A, j : A \cap B \rightarrow B$  and the induced homomorphisms  $I : \pi_1(A \cap B) \rightarrow \pi_1(A), J : \pi_1(A \cap B) \rightarrow \pi_1(B)$ . But since both sides in both maps are trivial, these are constant maps between identities. We can then present the group  $0 = \pi_1(A) = \langle a \mid a^1 = e \rangle$  and since  $I(w)J(w)^{-1} = ee^{-1} = e$ , we have  $\pi_1(B) = \langle b \mid b^1 = e \rangle$ , so  $\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B) = \langle a, b \mid a^1 = b^1 = e \rangle$ .

(See [https://en.wikipedia.org/wiki/Seifert%E2%80%93van\\_Kampen\\_theorem](https://en.wikipedia.org/wiki/Seifert%E2%80%93van_Kampen_theorem) for presentation of amalgamated product)

5. WLOG, assume  $p_0, p_1$  are the north and south poles of  $S^2$ . We can then form a deformation retract of  $X$  onto the equator of  $S^2$ , which is equal to  $S^1$ . To do so, just move every point  $x$  along the unique great circle connecting  $x, p_0, p_1$ , and proceed at linear speed towards the equator. This is well defined at every point on  $S^2$  *except* the poles, which are not included in  $X$ , and the equator is fixed at every instant. So this forms a deformation retract. Alternatively, use the fact that  $\mathbb{R}^n - \{\text{pt}\} \cong S^{n-1} \times \mathbb{R}$  via polar coordinates, and  $S^n - \{\text{pt}\} \cong \mathbb{R}^n$  by stereographic projection. So  $S^2 - \{p_0, p_1\} \cong \mathbb{R}^2 - \{p_1\} \cong S^1 \times \mathbb{R}$ . But since  $\mathbb{R}$  is contractible, the last one is homotopic to  $S^1 \times \{0\} \cong S^1$ . **Alternatively:** use the lemma, then  $k = 2$  and so  $S^2 - \{p_1, p_2\} \simeq \bigvee_{i=1}^1 S^1 = S^1$ .
6. Lemma:  $S^n - \{p_i\}_{i=1}^k = \bigvee_{k-1} S^{n-1}$ , i.e.  $S^n$  minus  $k$  points is equal to  $k - 1$  copies of  $S^{n-1}$ . Proof:  $S^n - \{p_1\} \cong \mathbb{R}^n$  by stereographic projection, so  $S^n - \{p_1, p_2, \dots, p_k\} \cong \mathbb{R}^n - \{p_2, \dots, p_k\}$ . WLOG, suppose none of these points are zero (otherwise, take a translation away from zero. This is affine and continuous.) Then fix 0 as the base point, and form  $k - 1$  loops  $\alpha_i$ , where the  $i$ th loop encircles  $p_i$ . Then  $\mathbb{R}^n$  deformation retracts onto  $\bigcup_{i=1}^{k-1} \alpha_i$ , which is homeomorphic to  $\bigvee_{i=1}^{k-1} S^1$ .
7. Theorem:  $\pi_1(\bigvee_{i=1}^k S^1) \cong *_{i=1}^n \mathbb{Z}$ , the free product of  $n$  copies of  $\mathbb{Z}$ . Proof: By induction, using Van-Kampen's theorem. Base case: Take  $i = 1$ , then  $\pi_1(S^1) = \mathbb{Z}$  as proved in Hatcher.

Inductive step: Suppose this holds for all  $k < n$ , then we have  $X = \bigvee^n S^1 = \left(\bigvee^{n-1} S^1\right) \vee S^1$ .

Let  $p$  be the point of common intersection, then let  $U = \bigvee^{n-1} S^1$  and  $V = S^1 \cup \{p\}$

Then  $U \cup V = X$ ,  $U \cap V = \{p\}$ , both  $U, V$  are path-connected. Since we have  $\pi_1(\{pt\}) = 0$ , the amalgamated free product reduces to the usual free product. By the IH, we have  $\pi_1(U) = *^{n-1}\mathbb{Z}$ , so

$$\pi_1(X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) \underset{\text{IH}}{=} (*^{n-1}\mathbb{Z}) * \pi_1(V) = (*^{n-1}\mathbb{Z}) * \mathbb{Z} = *^n\mathbb{Z}.$$

Definition: Let  $F_n := *^n\mathbb{Z}$  be the free abelian group on  $n$  generators. Lemma: If  $n \neq m$ ,  $F_n \not\cong F_m$ . Proof: If  $F_n \cong F_m$ , then  $\mathbb{Z}^n \cong \mathbb{Z}^m$ . But then tensor both sides with  $\mathbb{Z}_2$  over  $\mathbb{Z}$ , yielding  $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}_2$ . But the LHS is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^n$ , while the RHS is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^m$ . (Why?) These are both finite groups - there are 2 elements in  $\mathbb{Z}/2\mathbb{Z}$ , so the first has  $2^n$  elements and the latter has  $2^m$  elements. But if  $2^n = 2^m$ , then  $n = m$ . The lemma follows from the contrapositive.

Now we have all we need - let  $X = S^2 - \{p_1, p_2\}$  and  $Y = S^3 - \{q_1, q_2\}$ . Then by the previous problems,  $X \simeq S^1$  and  $Y \simeq S^2$ , so if  $S^2 \cong S^3$  then  $X \simeq Y$  and  $S^1 \simeq S^2$ . But  $\pi_1(S^1) = \mathbb{Z}$  and  $\pi_1(S^2) = 0$ , so  $S^1 \not\simeq S^2$ , a contradiction.

8. Here we go:

9. Let  $\alpha(t) = e^{2\pi it}$  where  $t \in [0, 1]$ , be a loop in  $S^1$  parameterized by  $t$ , which goes around  $S^1$  exactly once. Then under the map  $f : z \mapsto z^n$ , we obtain  $f(\alpha(t)) = e^{2\pi nit}$  where  $t \in [0, 1]$ . This resulting loop then goes around  $S^1$   $n$  times, so the induced homomorphism on  $\pi_1(S^1) = \mathbb{Z}$  is the map  $f^* : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f^*(a) = na$ .
10. Define  $\alpha$  as above, and define  $f : S^1 \rightarrow S^1$  to be the antipodal map, so  $f(z) = -z$  for  $z \in S^1 \subset \mathbb{C}$ . We then left  $\alpha$  to the fundamental group, and define  $f_*([\alpha]) = [f \circ \alpha]$ . Computing, we have  $(f \circ \alpha)(t) = f(\alpha(t)) = -e^{2\pi it}$ . Where  $\alpha(0) = \alpha(1) = 1 + 0i$ , we have  $(f \circ \alpha)(0) = (f \circ \alpha)(1) = -1 + 0i$ . But note that  $\alpha$  was a counter-clockwise loop in  $S^1$ , and the image of  $\alpha$  is also a counter-clockwise loop. So this maps the generator  $[\alpha] \in \pi_1(S^1, 1)$  to the generator  $[\alpha'] \in \pi_1(S^1, -1)$ . But since  $S^1$  is path-connected, the fundamental groups at these two base points are isomorphic. Alternatively: the antipodal map on  $S^1$  is homotopic to the identity map (since  $n = 1$  is odd), so  $[f \circ \alpha] = [f][\alpha] = [\text{id}][\alpha] = [\alpha]$ , so the induced homomorphism on  $\pi_1(S^1)$  is the identity map.
11. Let  $\alpha(t) = e^{it}$  where  $t \in [0, 2\pi]$  be a counter-clockwise loop in  $S^1$ ; then  $[\alpha]$  generates the fundamental group. Then  $f^*([\alpha]) = [(f \circ \alpha)(t)] = [e^{it} \mapsto e^{2\pi i \sin t}]$ . Then just consider how  $\sin$  behaves in each quadrant. In quadrant 1, as  $t$  ranges from  $0, \pi/2$  then  $\sin t$  ranges from 0 to 1, so  $\alpha$  is exactly traced out. In quadrant two,  $\bar{\alpha}$  is traced out, since  $\sin t$  decreases from 1 to 0. This happens again in the bottom quadrants, so we have  $f^*([\alpha]) = [\alpha \bar{\alpha} \alpha \bar{\alpha}] = [\alpha][\alpha]^{-1}[\alpha][\alpha]^{-1} = [\text{id}]$ . But the identity element in  $\mathbb{Z}$  is 0, so the induced homomorphism on  $\mathbb{Z}$  is  $f^*(a) = 0$ , the homomorphism sending everything to 0.
12. From complex analysis,  $W(f(\alpha(t))) = Z_f - P_f = 4 - 1 = 3$ . No idea how to approach with induced maps on the fundamental group of  $S^1$  or  $\mathbb{C} - \{0\}$ .
13. Let  $M$  be the mobius strip, identified as  $I \times I / (t, 0) \sim (1-t, 1)$ , and let  $x_0 = [(1, \frac{1}{2})] = [(0, \frac{1}{2})]$ . Let  $X$  be the line  $(t, \frac{1}{2})$  for  $t \in I$ ; by the identification of the endpoints this is actually a copy of  $I/\partial I \cong S^1$  inside of  $M$  representing the middle circle of the strip. But then  $M$  deformation

retracts onto  $S^1$  by just moving every point in  $I \times I$  horizontally towards this line, so  $M \simeq S^1$  and  $\pi_1(M) \cong \mathbb{Z}$ , generated by the loop described which we'll call  $\alpha$ .

To see what the boundary curve is, label the corners  $a, b$  with the suitable identification. Then take a path from  $a$  to  $b$  on the right-hand boundary of the square. By sliding this through  $I \times I$ , this is homotopic  $\alpha$ . But similarly, the path from  $b$  to  $a$  on the LHS of the square is also homotopic to  $\alpha$ , so the loop  $a \rightarrow b \rightarrow a \simeq \alpha^2$ , so if  $[\alpha] = 1 \in \pi_1(M)$ , then  $[a \rightarrow b \rightarrow a] = 2$ .

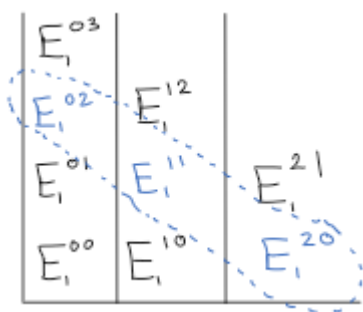
11. First note that  $\pi_1(S^1 \times S^1) \cong F^2$ , the free group on two generators, say  $[\alpha], [\beta]$  corresponding to the two nontrivial loops on the torus - say  $\alpha$  is the longitudinal loop, and  $\beta$  is the meridian. Then if  $\gamma$  is a loop on a torus, then you can just count how many times it winds longitudinally and around the meridian, say  $m$  and  $n$  times respectively. Then  $\gamma$  can be homotoped into  $m$  copies of  $\alpha$  and  $n$  copies of  $\beta$  based at  $x_0$ . So the induced map is  $f_\# : F^2 \rightarrow F^2$  given by  $\alpha \mapsto \alpha^m, \beta \mapsto \beta^n$ . Since  $F^2 \cong \mathbb{Z} \times \mathbb{Z}$ , we equivalently have  $[\alpha] = (1, 0), [\beta] = (0, 1)$ , and then  $f_\# : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is given by  $(1, 0) \mapsto (m, 0)$  and  $(0, 1) \mapsto (0, n)$ .

## 2.3 Group Actions

- 1.

## 2.4 Covering Spaces

1. Any covering map  $p : S^1 \times S^1 \rightarrow \mathbb{RP}^2$  would induce an injection on fundamental groups, but  $\pi_1(T) = \mathbb{Z}^2$  and  $\pi_1(\mathbb{Z}_2)$  - but there are no homomorphisms between these groups. Why? One of them has an element of order 2, the other does not.
2. Theorem: if  $M_g \rightarrow M_h$  is an  $n$ -sheeted covering space, then  $g = n(h - 1) + 1$ .



3. Draw CW square for  $T$  and cut down the center to see two copies of  $K$ .
4. Let  $p : \tilde{G} \rightarrow G$  be such a covering,  $a, b \in \tilde{G}$ , we then want to show that  $p(a)p(b) = p(a \star b)$  for some group operation  $\star$  which we need to construct.

Pick a basepoint  $x \in G$  and any point  $\tilde{x} \in p^{-1}(x)$ . Since  $\tilde{G}$  is path connected, pick two paths  $\alpha, \beta$  from  $\tilde{x}$  to  $a, b$  respectively.

Now define a path  $f : I \rightarrow G$  by  $f(t) = (p \circ \alpha)(t) \cdot (p \circ \beta)(t)$ , that is, evaluating  $f, g$  at a given time in  $\tilde{G}$ , projecting the results down into  $G$ , and multiplying them there. By uniqueness of path lifting, this yields a lift  $\tilde{f} : I \rightarrow \tilde{G}$

Then define  $a \star b = \tilde{f}(1)$ , the endpoint of  $\tilde{f}$  in  $\tilde{G}$ . Then by construction,



$p(a \star b) = p(\tilde{f}(1)) = f(1) = (p \circ \alpha)(1) \cdot (p \circ \beta)(1) = p(a)p(b)$ . (Need to show this is continuous, and doesn't depend on  $\alpha, \beta$ ?)

5. Since  $T^n = \prod_n S^1$ , we have  $\pi_1(T^n) = \prod_n \pi_1(S^1) = \mathbb{Z}^n$ . We can also construct a cover  $p: \mathbb{R}^n \rightarrow T^n$  by just taking  $\mathbb{R} \rightarrow S^1$  the usual cover in each coordinate, yielding the covering space  $\tilde{X} = \mathbb{R}^n$  over  $X = T^n$ .

By Hatcher (prop 4.1), the induced maps  $p_*^i: \pi_i(\tilde{X}) \rightarrow \pi_i(X)$  is an isomorphism for  $i \geq 2$ . But  $\pi_i(\mathbb{R}^n) = 0$  for  $i \neq 0$ , so by this isomorphism  $\pi_i(T^n) = 0$  for  $i \geq 2$ .

6. General construction: construct a tree  $T$  by picking a basepoint in  $G$  and adding a vertex for every non-backtracking walk in  $G$ .

In this case, it's the infinite 3-valent graph (also called the infinite  $k$ -regular tree)

This is the universal cover, because  $T$  is connected and acyclic (i.e. a tree). This means that  $\pi_1(T) = 0$ , so  $T$  is simply connected. Since universal covers are simply connected and unique up to isomorphism, this is it.

7. Generators of the subgroups:

8.  $\langle ab^{-1}, aba^{-2}, a^3b^{-1}a^{-2}, a^3 \rangle$

9.  $\langle b, aba^{-1}, a^2ba^{-2}, a^3 \rangle$

10.  $\langle b^2, ba, a^3, aba^{-1} \rangle$

11.  $\langle b \rangle$

12.  $\langle ba, b^{-1}a \rangle$

Relevant covers:

- 1.
- 2.
- 3.
- 4.

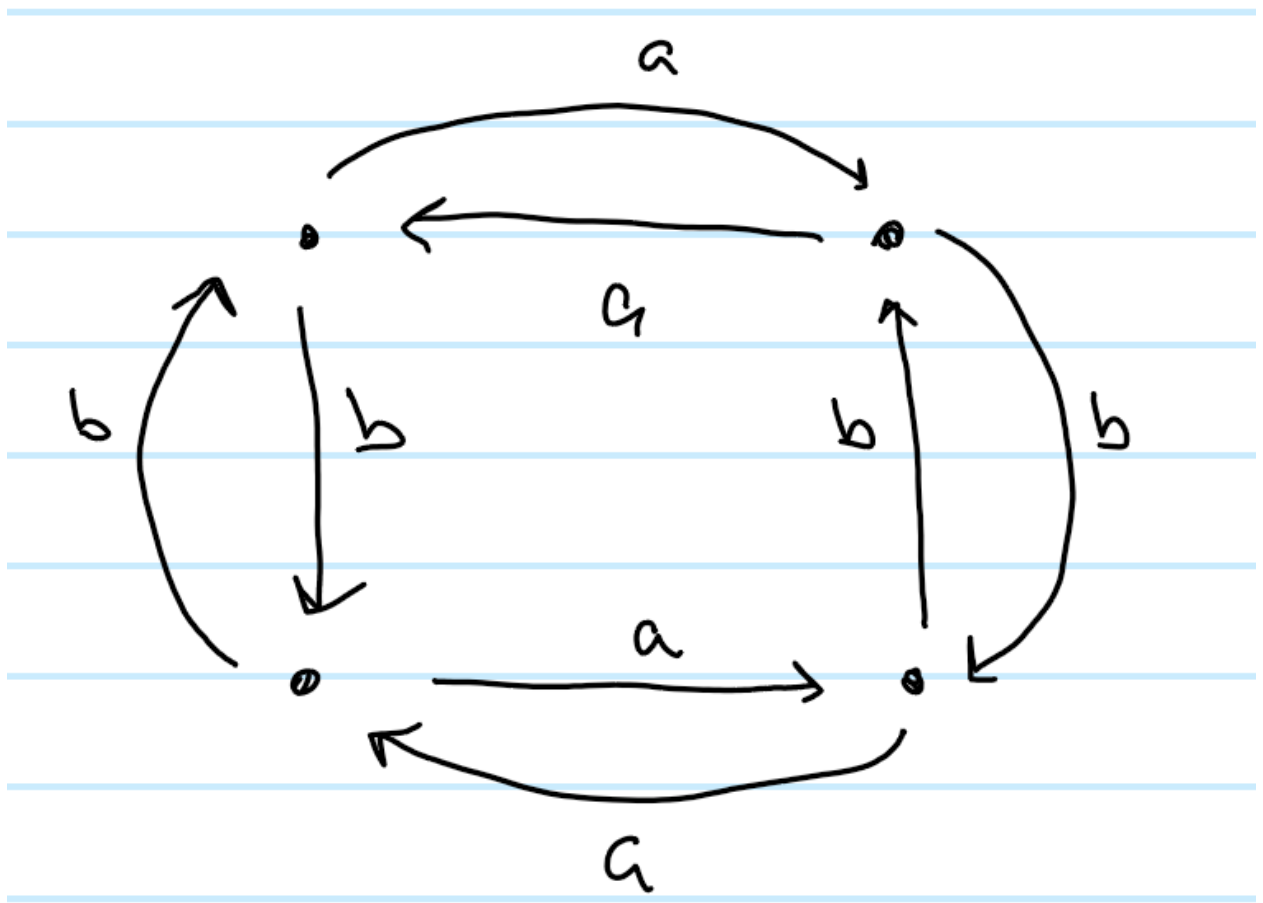


Figure 3: 1512964258737

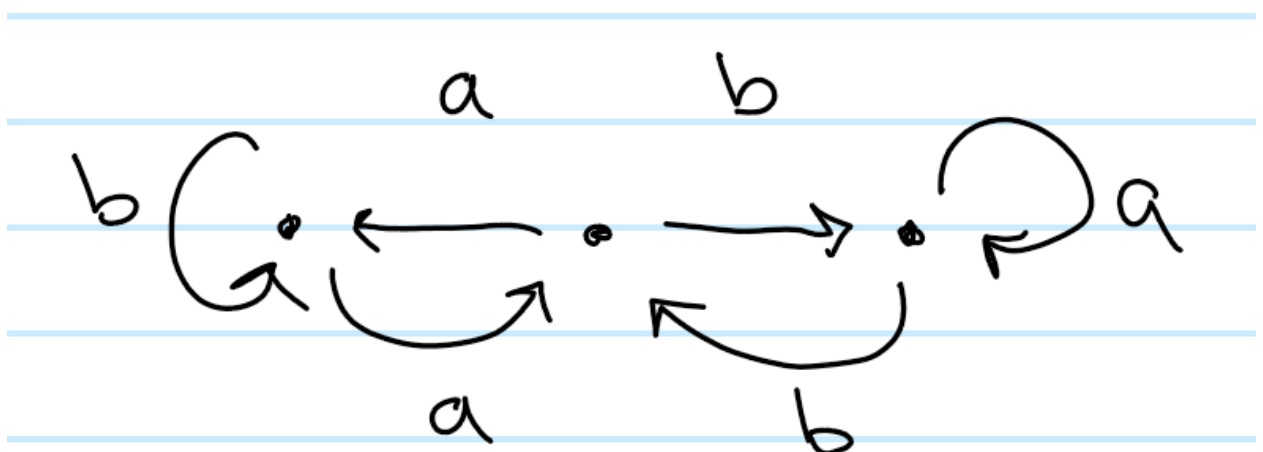


Figure 4: 1512964650272

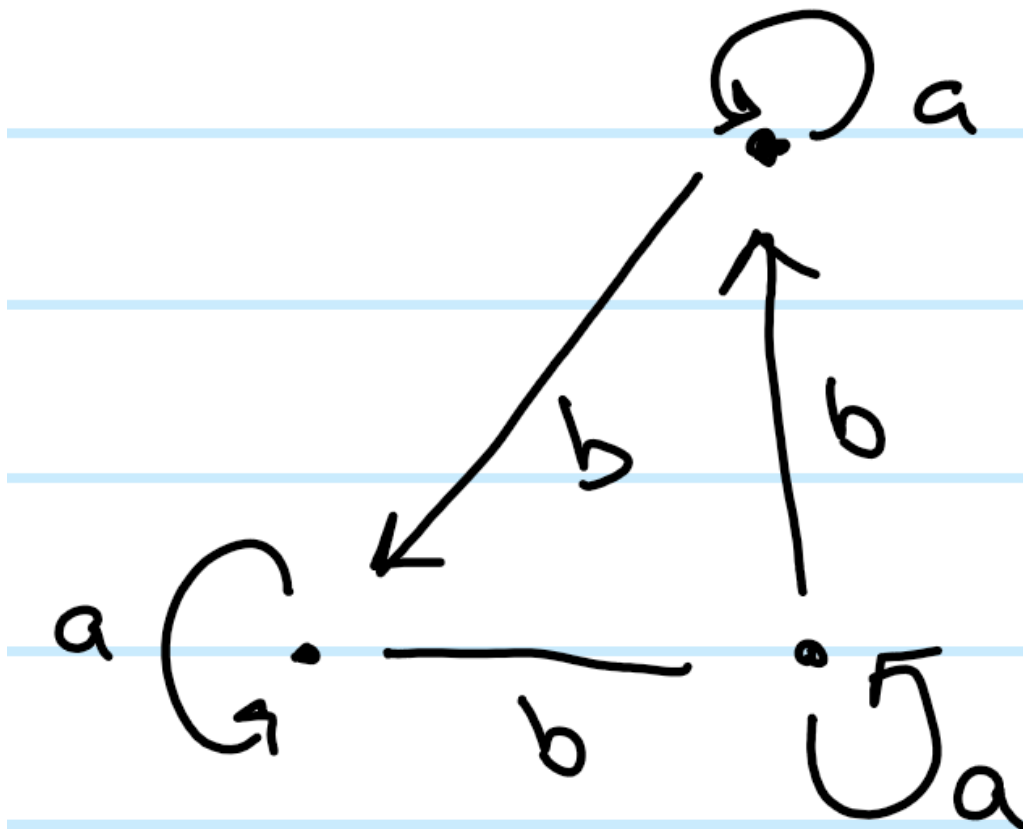


Figure 5: 1512965253808

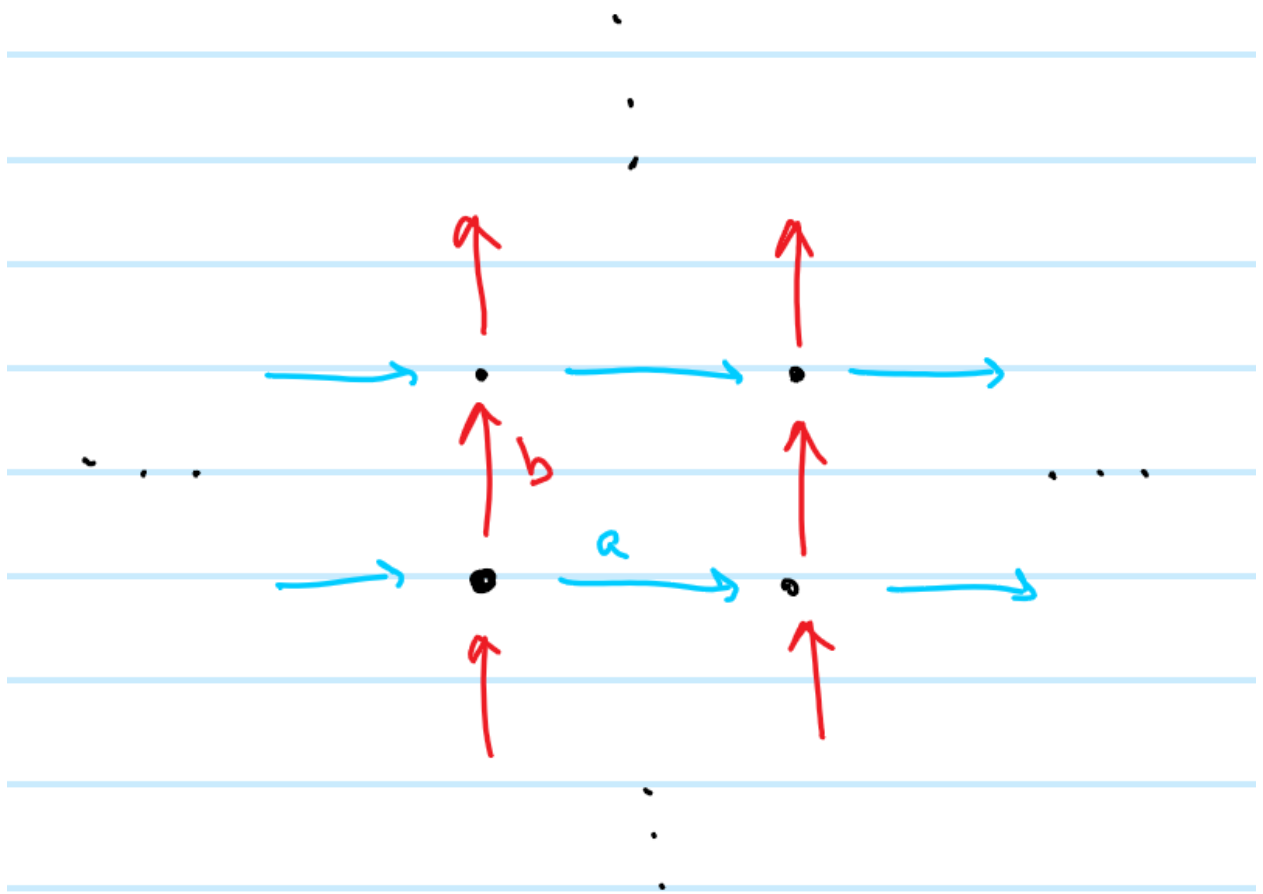
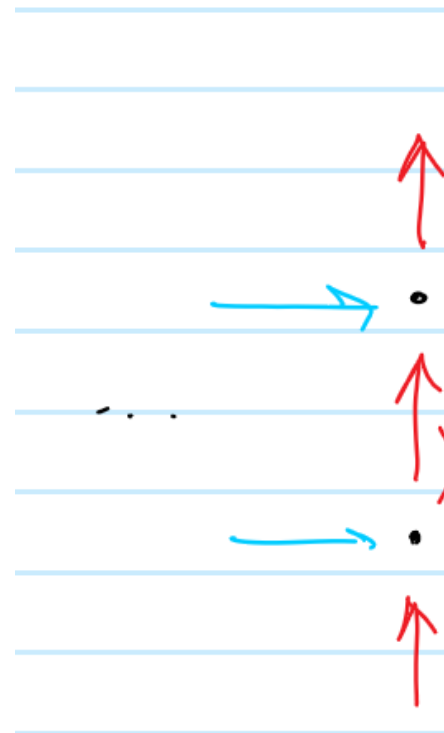


Figure 6: 1512965792844



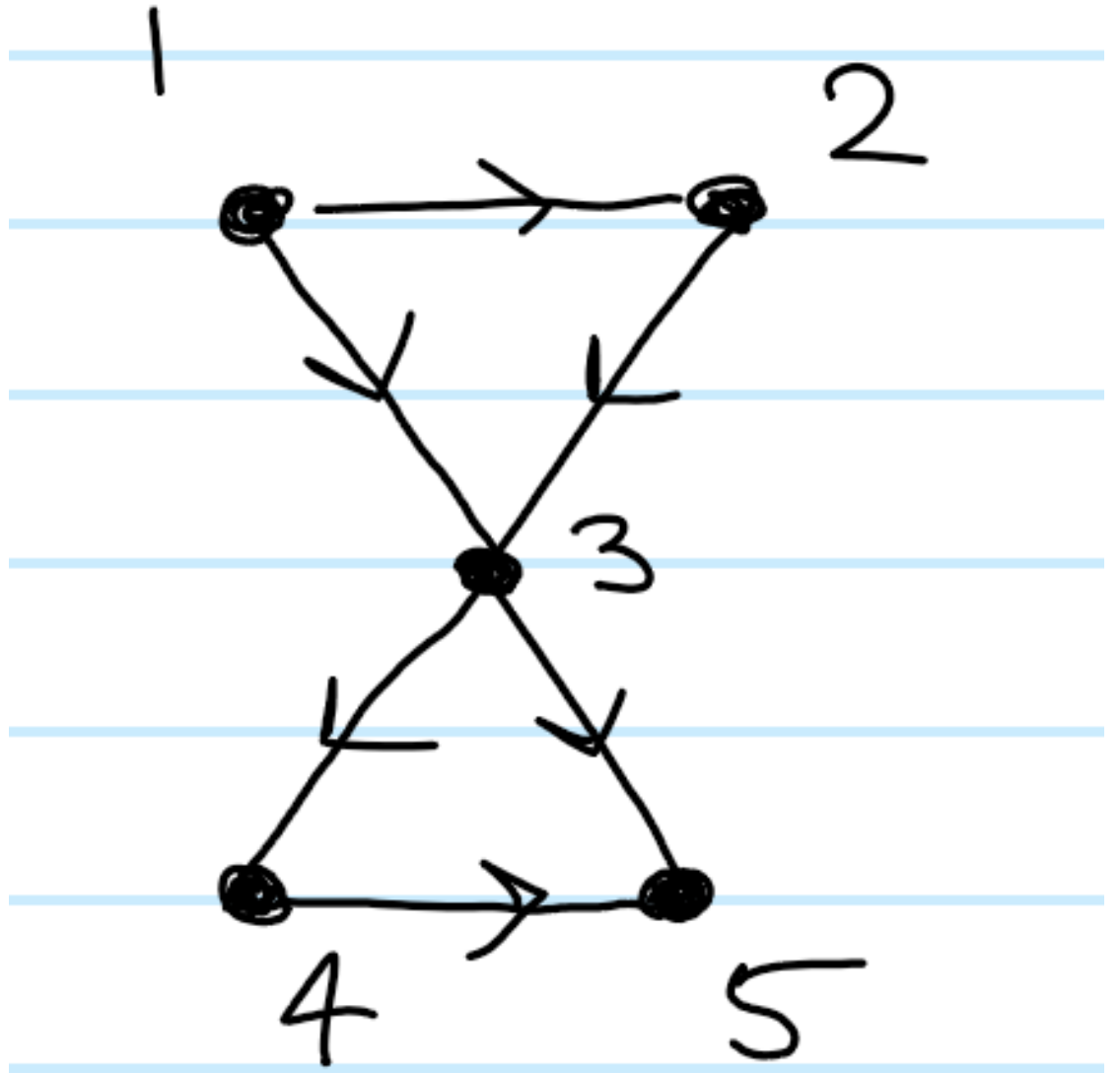
5. Let  $T$  be a copy of the Cayley Tree on two on the two generators  $a, b$ , then:



6. This is just the Cayley graph over  $\mathbb{Z} \times \mathbb{Z}$ , or essentially the integer lattice:
7. It's helpful to note that  $\langle (1, 0), (0, p) \rangle \subset \langle (1, 0), (0, 1) \rangle \cong \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$  is an index  $p$  subgroup.

## 2.5 Simplicial Homology

1. Todo



2. Figure 8

Here we have:  $C_3 = \emptyset$   $C_1 = [12], [23], [13], [34], [35], [45] \cong \mathbb{Z}^6$   $C_0 = [1], [2], [3], [4], [5] \cong \mathbb{Z}^5$

So we have  $C_2 \rightarrow C_1 \rightarrow C_0 \cong 0 \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^5 \xrightarrow{\partial_0} 0$

Computing boundary operators, we have

$$\begin{aligned} \partial_1([12]) &= [2] - [1] & \partial_1([23]) &= [3] - [2] & \partial_1([13]) &= [3] - [1] & \partial_1([34]) &= [4] - [3] & \partial_1([35]) &= [5] - [3] \\ \partial_1([45]) &= [5] - [4] \end{aligned}$$

$$\partial_0 = 0$$

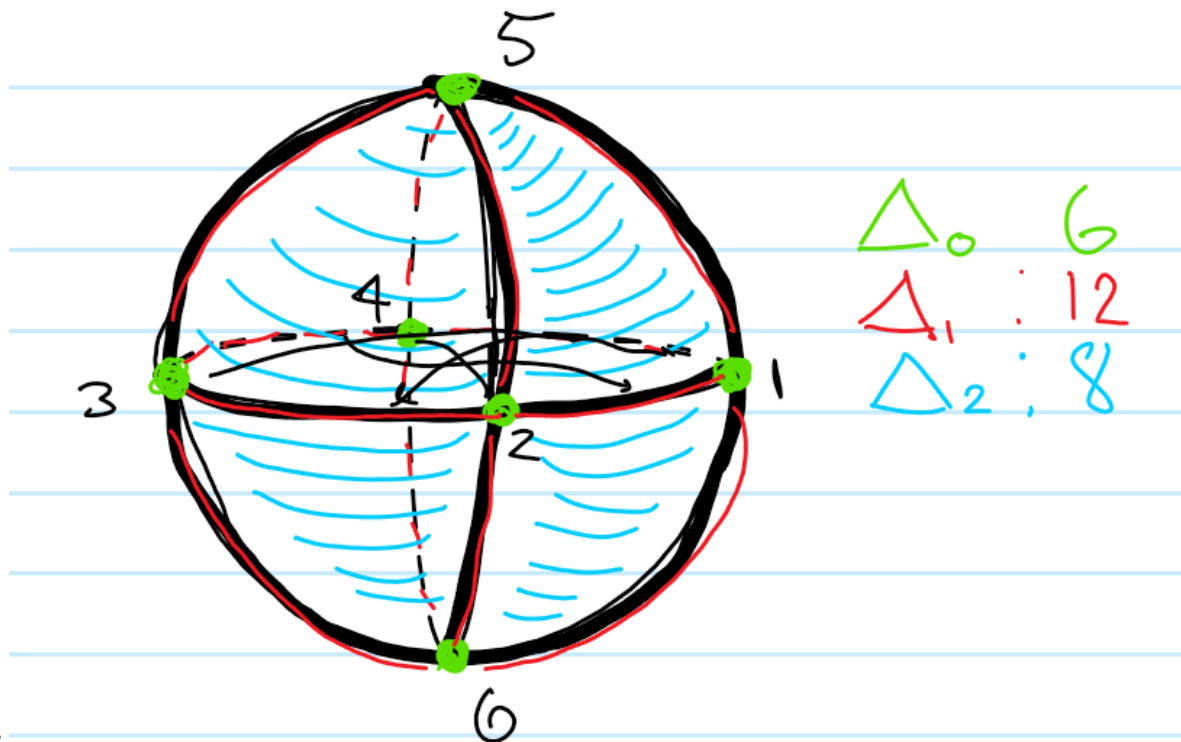
And so  $H_0 = \ker \partial_0 / \text{im } \partial_1 = \frac{C_0}{\langle \partial_1([ij]) \rangle}$ , but from the above calculation we have  $[5] = [4] = [3] = [2] = [1]$  in the quotient, so there is just one generator and  $H_0 \cong \mathbb{Z}$ .

Note that  $\partial_2$  is an injection from 0 into  $C_1$ , since there are no 2-simplices. Moreover, one can generate two 1-cycles, so we have  $H_1 = \frac{\ker \partial_1}{\text{im } \partial_2} = \frac{\langle [23] - [31] + [12], [45] - [35] + [34] \rangle}{0} \cong \mathbb{Z}^2$ .

One way to see that these are the generators is to pretend there are two 2-simplices,  $[123], [345]$

and compute  $\partial_2$  of both of them. Since  $\partial_1\partial_2 = 0$ , anything in the image of  $\partial_2$  would have to go to zero anyways, and would thus be in the kernel of  $\partial_1$ . Since it's not actually the boundary of any 2-chain, it doesn't become trivial in homology.

So we have  $H_2 \rightarrow H_1 \rightarrow H_0 = 0 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}$ .



## 2. $S^2$

So we have  $C_0 = 1, 2, 3, 4, 5, 6$   $C_1 = 12, 14, 15, 16, 23, 25, 26, 34, 35, 36, 45, 46$   $C_2 = 126, 236, 346, 146, 125, 23$

$C_3 = \emptyset$

And  $0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \cong 0 \xrightarrow{\partial_3} \mathbb{Z}^8 \xrightarrow{\partial_2} \mathbb{Z}^{12} \xrightarrow{\partial_1} \mathbb{Z}^6 \xrightarrow{\partial_0} 0$  We have  $\partial_1([ij]) = j - i$  and  $\partial_2([ijk]) = jk - ik + ij$ .

We know in advance we should have  $\prod H_n = (\dots, 0, \mathbb{Z}, 0, \mathbb{Z})$ .

For  $H_0 = \frac{\ker \partial_0}{\text{im } \partial_1} = \frac{C_0}{\langle \{j-i | i < j\} \rangle}$ . In the quotient, we see  $1 = 6 = 3 = 2 = 5 = 4$  by just taking the indicated walk on the graph, so there is one generator in the quotient and  $H_0 \cong \mathbb{Z}$ .

For  $H_1 = \frac{\ker \partial_1}{\text{im } \partial_2}$ , we just note that there are 6 2-cycles, so each are in the kernel of  $\partial_1$ , but each of them comes from a 2-cell, so is in the image of  $\partial_2$ . So both groups in question are  $\mathbb{Z}^8$ , and the quotient is zero. For  $H_3 = \frac{\ker \partial_2}{\text{im } \partial_3}$ , since  $\text{im } \partial_3 = 0$ , we can just look at  $\partial_3([123456]) = 23456 - 13456 + 12456 - 12356 + 12346 - 12345$ . This is an element (and the only one) that goes to zero under  $\partial_2$ , it generates  $\ker \partial_2$ . So there is one generator, and  $H_3 = \mathbb{Z}$ .

## 3. $\mathbb{RP}^2$

4.  $S^2 \cup_f D^2$ , where  $f$  attaches to the equator

5.  $T \cup_f D^2$ , where  $f$  attaches inside the torus



## 2.6 Mayer Vietoris Problems

### 2.6.1 $\mathbb{RP}^2$

We start with a few known facts. Let  $A = M$ , the Mobius strip, and  $B = D^2$ , the solid disk.

- $\mathbb{RP}^2 = M \amalg_{\partial} D^2$
- $H_*(M) = H_*(S^1)$ , by a deformation retract of  $M$  onto its center circle.
- $H_*(D^2) = \mathbb{Z}\delta_0$
- $H_*(S^1) = \mathbb{Z}(\delta_0 + \delta_1)$
- $M \cap D^2 = \partial M = S^1$

From Mayer-Vietoris, we have

$$\begin{array}{c}
 \begin{array}{c} \cdots 0 \\ \delta_3 \\ \hookrightarrow H_2 \partial M \xrightarrow{(i^*, -j^*)_2} H_2 M \oplus H_2 D^2 \xrightarrow{(l^* - r^*)_2} H_2 \mathbb{RP}^2 \end{array} \\
 \delta_2 \\
 \begin{array}{c} \hookrightarrow H_1 \partial M \xrightarrow{(i^*, -j^*)_1} H_1 M \oplus H_1 D^2 \xrightarrow{(l^* - r^*)_1} H_1 \mathbb{RP}^2 \end{array} \\
 \delta_1 \\
 \begin{array}{c} \hookrightarrow H_0 \partial M \xrightarrow{(i^*, -j^*)_0} H_0 M \oplus H_0 D^2 \xrightarrow{(l^* - r^*)_0} H_0 \mathbb{RP}^2 \end{array} \\
 \delta_0 \\
 \hookrightarrow 0
 \end{array}$$

and plugging in what is known yields

$$\begin{array}{c}
\begin{array}{c}
\hookrightarrow 0 \xrightarrow{(i^2, -j^2)} 0 \oplus 0 \xrightarrow{l^2 - r^2} H_2 \mathbb{RP}^2 \hookrightarrow 0 \\
\delta_3
\end{array} \\
\begin{array}{c}
\hookrightarrow \mathbb{Z} \xrightarrow{(i^1, -j^1)} \mathbb{Z} \oplus 0 \xrightarrow{l^1 - r^1} H_1 \mathbb{RP}^2 \hookrightarrow 0 \\
\delta_2
\end{array} \\
\begin{array}{c}
\hookrightarrow \mathbb{Z} \xrightarrow{(i^0, -j^0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{l^0 - r^0} H_0 \mathbb{RP}^2 \hookrightarrow 0 \\
\delta_1
\end{array} \\
\begin{array}{c}
\hookrightarrow 0 \\
\delta_0
\end{array}
\end{array}$$

where  $i : S^1 \rightarrow M$  and  $j : S^1 \rightarrow D^2$ .

We can then identify all of the induced maps:

- $i^2 : H_2 \partial M \rightarrow H_2 M \implies i^2 : 0 \rightarrow 0 \implies i^2 = 0$
- $i^1 : H_1 \partial M \rightarrow H_1 M$ , i.e.  $i^1 : \mathbb{Z} \rightarrow \mathbb{Z}$  where  $1 \mapsto 2$ 
  - Since  $M$  deformation retracts onto its center circle,  $H_1 M \cong H_1 S_M$  where  $S_M$  is the center circle (homotopies induce isomorphisms on homology). But  $H_1 \partial M$  is generated by a cycle of edges which includes into  $\partial M$ , which retracts onto a cycle that double covers  $S_M$ , so this map acts by doubling the generator.
- $i^0 : H_0 \partial M \rightarrow H_0 M$ , i.e.  $i^0 : \mathbb{Z} \rightarrow \mathbb{Z}$
- $j^2 : H_2 \partial M \rightarrow H_2 D^2 \implies j^2 : 0 \rightarrow 0 \implies j^2 = 0$
- $j^1 : H_1 \partial M \rightarrow H_1 D^2 \implies j^1 : \mathbb{Z} \rightarrow 0 \implies j^1 = 0$
- $j^0 : H_0 \partial M \rightarrow H_0 D^2 \implies j_0 : \mathbb{Z} \rightarrow \mathbb{Z}$

So we can that the only nontrivial maps are  $j^0, i^0, i^1$ .

**Claim:**  $H_2(\mathbb{RP}^2) = 0$ :

We consider the portion of the sequence

$$\begin{array}{c}
\cdots 0 \rightarrow H_2 \mathbb{RP}^2 \xrightarrow{\delta_2} H_1 \partial M \xrightarrow{(i^1, -j^1)} H_1 M \oplus H_1 D^2 \cdots \\
\cdots 0 \rightarrow H_2 \mathbb{RP}^2 \xrightarrow{\delta_2} \mathbb{Z} \xrightarrow{(i^1, -j^1)} \mathbb{Z} \oplus 0 \cdots
\end{array}$$

We will show that  $\ker \delta_2 = \text{im } \delta_2 = 0$ . By the first isomorphism theorem, we would then have  $\frac{H_2 \mathbb{RP}^2}{\ker \delta_2} \cong \text{im } \delta_2$  yielding  $\frac{H_2 \mathbb{RP}^2}{0} = H_2 \mathbb{RP}^2 \cong 0$ .

- *Claim:*  $\ker \delta_2 = 0$

This follows because it is on the left tail of an exact sequence, where  $\ker \delta_2 = \text{im } 0 = 0$ .

- *Claim:*  $\text{im } \delta_2 = 0$

$$(i^1, -j^1) : H_1 \partial M \rightarrow H_1 M \oplus H_1 D^2$$

is injective; explicitly, it is the map

$$\begin{aligned} M_2 : \mathbb{Z} &\rightarrow \mathbb{Z} \oplus 0 \\ 1 &\mapsto (2, 0) \end{aligned}$$

From above, know that  $-j^1$  is a zero map, and that  $i^1$  doubles each generator. By this explicit construction, it is injective since 0 maps to 0.

But then  $\ker(i^1, -j^1) = \text{im } \delta_2 = 0$  by exactness.

So now we have:

$$\begin{array}{ccccccc} & & & & 0 & \hookrightarrow & \\ & & & & \nearrow 0 & & \\ \hookrightarrow 0 & \xrightarrow{0 \times 0} & 0 \oplus 0 & \xrightarrow{0} & 0 & \hookrightarrow & \\ & & & & \searrow 0 & & \\ & & & & \hookrightarrow \mathbb{Z} & \xrightarrow{x \mapsto (2x, 0)} & \mathbb{Z} \oplus 0 \xrightarrow{l^1 - r^1} H_1 \mathbb{RP}^2 \\ & & & & \nearrow \delta_1 & & \\ \hookrightarrow \mathbb{Z} & \xrightarrow{(i^0, -j^0)} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{l^0 - r^0} & H_0 \mathbb{RP}^2 & \hookrightarrow & \\ & & & & \searrow \delta_0 & & \\ & & & & \hookrightarrow 0 & & \end{array}$$

**Claim:**  $H_1(\mathbb{RP}^2) = \mathbb{Z}_2$

Here we are examining this portion of the sequence:

$$\begin{aligned} \dots \mathbb{Z} &\xrightarrow{x \mapsto (2x, 0)} H_1 M \oplus H_1 D^2 \xrightarrow{l^1 - r^1} H_1 \mathbb{RP}^1 \xrightarrow{\delta_1} H_0 \partial M \xrightarrow{(i^0, -j^0)} H_0 M \oplus H_0 D^2 \dots \\ &\dots \mathbb{Z} \xrightarrow{x \mapsto (2x, 0)} \mathbb{Z} \oplus 0 \xrightarrow{l^1 - r^1} H_1 \mathbb{RP}^1 \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{(i^0, -j^0)} \mathbb{Z} \oplus \mathbb{Z} \dots \end{aligned}$$

In general, we have the first isomorphism theorem: given any map  $f$  we have  $\frac{\text{dom } f}{\ker f} \cong \text{im } f$ . Here we will take  $f = l^1 - r^1$  and identify the necessary components to apply this theorem.

- Claim:  $\text{im } l^1 - r^1 = H_1 \mathbb{RP}^2$ .
  - We use the fact that the maps  $(i^*, j^*)$  are all injections, so in particular  $0 = \ker(i^0, j^0) = \text{im } \delta_1$  by exactness. Consequently  $\ker \delta_1 = H_1 \mathbb{RP}^1 = \text{im } l^1 - r^1$  by exactness.
- What is  $\ker(l^1 - r^1)$ ?

– By exactness,  $\ker(l^1 - r^1) = \text{im } (x \mapsto (2x, 0)) = 2\mathbb{Z} \oplus 0$

By the first isomorphism theorem, we have  $\text{im } (l^1 - r^1) \cong \frac{\text{dom}(l^1 - r^1)}{\ker(l^1 - r^1)} = \frac{\mathbb{Z} \oplus 0}{2\mathbb{Z} \oplus 0} \cong \mathbb{Z}_2$ .

Note that  $l^1 - r^1$  is a nontrivial homomorphism from  $2\mathbb{Z} \cong \mathbb{Z}$  to  $\mathbb{Z}_2$ , of which there is only one: the natural quotient map  $x \mapsto x \pmod 2$ .

There is also no nontrivial homomorphism from  $\mathbb{Z}_2 \rightarrow \mathbb{Z}$ , so  $\delta_1 = 0$ .

We now have:

$$\begin{array}{ccccccc}
 & & & & 0 & \hookrightarrow & \\
 & & & & \nearrow & & \\
 & & 0 & & & & \\
 & \hookrightarrow & 0 & \xrightarrow{0 \times 0} & 0 \oplus 0 & \xrightarrow{0} & 0 \hookrightarrow \\
 & & & & \searrow & & \\
 & & 0 & & & & \\
 & \hookrightarrow & \mathbb{Z} & \xrightarrow{x \mapsto (2x, 0)} & \mathbb{Z} \oplus 0 & \xrightarrow{x \mapsto x \pmod 2} & \mathbb{Z}_2 \hookrightarrow \\
 & & & & \searrow & & \\
 & & 0 & & & & \\
 & \hookrightarrow & \mathbb{Z} & \xrightarrow{(i^0, -j^0)} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{l^0 - r^0} & H_0\mathbb{RP}^2 \hookrightarrow \\
 & & & & \searrow & & \\
 & & \delta_0 & & & & \\
 & \hookrightarrow & 0 & & & & 
 \end{array}$$

## 2.7 Claim: $H_0(\mathbb{RP}^2) = \mathbb{Z}$

Here we examine

$$\begin{array}{ccccccc}
 H_1\mathbb{RP}^2 & \xrightarrow{\delta_1} & H_0\partial M & \xrightarrow{(i^0, j^0)} & H_0M \oplus H_0D^2 & \xrightarrow{l^0 - r^0} & H_0\mathbb{RP}^2 \xrightarrow{\delta_0} 0 \\
 & & & & \xrightarrow{\delta_1} & & \\
 \mathbb{Z}_2 & \xrightarrow{\delta_1} & \mathbb{Z} & \xrightarrow{(i^0, j^0)} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{l^0 + r^0} & H_0\mathbb{RP}^2 \xrightarrow{\delta_0} 0
 \end{array}$$

Since there is no nontrivial homomorphism from  $\mathbb{Z}_2 \rightarrow \mathbb{Z}$ , we have  $\delta_1 = 0$ .

We also have  $\delta_0 = 0$  and  $\ker \delta_0 = H_0\mathbb{RP}^2 = \text{im } l^0 + r^0$  making  $l^0 + r^0$  surjective, so by the first isomorphism theorem we have  $H_0\mathbb{RP}^2 \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\ker l^0 + r^0} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\text{im } (i^0, j^0)}$

By a similar argument used earlier, the double covering of the boundary circle  $\partial M$  over  $S^1$  yields the map  $(i^0, j^0) : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  given by  $x \mapsto (2x, 2x)$  with

### Summary:

With all of this information, we finally have

Three commutative triangles illustrating the Third Isomorphism Theorem for groups:

- Triangle 1:  $0 \xrightarrow{0 \mapsto (0,0)} 0 \oplus 0 \xrightarrow{(0,0) \mapsto 0} 0$  with a curved arrow  $0 \xrightarrow{0 \mapsto 0} 0$  at the bottom.
- Triangle 2:  $\mathbb{Z} \xrightarrow{x \mapsto (2x,0)} 2\mathbb{Z} \oplus 0 \xrightarrow{(x,0) \mapsto x \bmod 2} \mathbb{Z}_2$  with a curved arrow  $\mathbb{Z} \xrightarrow{x \mapsto x \bmod 2} \mathbb{Z}_2$  at the bottom.
- Triangle 3:  $\mathbb{Z} \xrightarrow{x \mapsto (2x,x)} 2\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(x,y) \mapsto x-y} \mathbb{Z}$  with a curved arrow  $\mathbb{Z} \xrightarrow{x \mapsto x} \mathbb{Z}$  at the bottom.

And so we find  $H_*(\mathbb{RP}^2) = \mathbb{Z}\delta_0 + \mathbb{Z}_2\delta_1$

## 2.8 Cellular Homology

## 2.9 Degree

## 2.10 UCT

## 2.11 Homological Algebra