## **Homological Algebra Problem Sets**

Problem Set 1

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*Problem* 1.0.1 (Weibel 1.1.2)

Show that a morphism  $u: C \to D$  of chain complexes preserves boundaries and cycles respectively, hence inducing a map  $H_n(C) \to H_n(D)$  for each n. Prove that  $H_n: \operatorname{Ch}(R\operatorname{-mod}) \to R\operatorname{-mod}$  is a functor.

## **Solution:**

Claim 1: The chain map u induces the following well-defined maps:

$$Z_n(u): Z_n(C) \to Z_n(D)$$

$$B_n(u): B_n(C) \to B_n(D).$$

Proof (of claim (1)).

We'll use the convention that  $Z_n := \ker d_n$  and  $B_n := \operatorname{im} d_{n+1}$  where we index chain complexes as  $C = \left( \cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots \right)$ . Unraveling definitions, we would like to show the existence of maps

$$Z_n(u) : \ker d_n^C \to \ker d_n^D$$
  
 $B_n(u) : \operatorname{im} d_{n+1}^C \to \operatorname{im} d_{n+1}^D.$ 

It suffices to show

a. 
$$x \in \ker d_n^C \Longrightarrow u_n(x) \in \ker d_n^D$$
, and  
b.  $y \in \operatorname{im} d_{n+1}^C \Longrightarrow u_n(y) \in \operatorname{im} d_{n+1}^D$ .

Since u is a morphism of chain complexes, we have a commuting ladder where  $u_{n-1} \circ d_n^C = d_n^D \circ u_n$ :

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \xrightarrow{d_{n-1}^C} \cdots$$

$$\downarrow u_{n+1} \qquad \downarrow u_n \qquad \downarrow u_{n-1} \qquad \downarrow$$

### Link to Diagram

To see that (a) holds, we compute

$$x \in \ker d_n^C \qquad \leq C_n$$

$$\iff d_n^C(x) = 0_R \qquad \in C_{n-1}$$

$$\iff (u_{n-1} \circ d_n^C)(x) = 0_R \quad \in D_{n-1} \quad \text{and } u_n \text{ is a ring morphism and sends } 0_R \to 0_R$$

$$\iff (d_n^D \circ u_n)(x) = 0_R \quad \in D_{n-1} \quad \text{commutativity}$$

$$\iff x \in \ker(d_n^D \circ u_n) \qquad \leq D_{n-1}$$

$$\iff u_n(x) \in \ker d_n^D \qquad \leq D_n.$$

Similarly, for (b) we have

$$y \in \operatorname{im} d_{n+1}^{C} \iff \exists x \in C_{n+1} \text{ such that } d_{n+1}^{C}(x) = y$$

$$\implies u_{n+1}(x) \in D_{n+1}$$

$$\implies (d_{n+1}^{D} \circ u_{n+1})(x) \in \operatorname{im} d_{n+1}^{D} \leq D_{n}$$

$$\implies (u_{n} \circ d_{n+1}^{C})(x) \in \operatorname{im} d_{n+1}^{D} \leq D_{n} \qquad \text{using commutativity}$$

$$\iff u_{n}(y) \in \operatorname{im} d_{n+1}^{D} \qquad \text{using } d_{n+1}^{C}(x) = y.$$

Now noting that  $H_n(C) := Z_n(C)/B_n(C)$ , since  $u_n$  preserves  $Z_n$  there is a well-defined restriction of each  $u_n : C_n \to D_n$  to  $u_n : Z_n(C) \to Z_n(D)$ . Composing with the projection  $Z_n(D) \to Z_n(D)/B_n(D) := H_n(D)$  yields maps  $u_n : Z_n(C) \to H_n(D)$ .

Problem 1.0.2 (Weibel 1.1.4)

Show that for every  $A \in R$ -mod and  $C \in Ch(R\text{-mod})$  that  $D := \operatorname{Hom}_{R\text{-mod}}(A, C)$  is a chain complex of abelian groups. Taking  $A := Z_n$ , show that  $H_n(D) = 0 \implies H_n(C) = 0$ . Is the converse true?

#### Solution:

We first show that if  $A \in R$ -mod and  $C \in Ch(R\text{-mod})$ , then

$$D_n \coloneqq \operatorname{Hom}_{R\text{-}\operatorname{mod}}(A, C_n).$$

defines a chain complex of abelian groups. Fixing notation, we write

$$C \coloneqq (\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots).$$

1.  $D_n$  is an abelian group for all n: Define an operation

$$+_D: D_n \times D_n \to D_n$$

$$(f,g) \mapsto \begin{cases} f+g: A \to C_n \\ x \mapsto f(x) +_C g(x) \end{cases},$$

where  $+_C$  is the addition on  $C_n$  provided by its structure as an R-module. We can then check that this operation is commutative:

$$(f +_D g)(x) \coloneqq f(x) +_C g(x)$$
  
=  $g(x) +_C f(x)$  since the addition on  $C_n$  is commutative  
=  $(g +_D f)(x)$ ,

The additive inverse of f is -f, there is an identity function  $\mathrm{id}_{C_n}(x) = x$ , and the sum of two functions  $A \to C_n$  is again a function  $A \to C_n$ , making  $D_n$  an abelian group for all n.

2. There exist differentials  $D_n \xrightarrow{d_n^D} D_{n-1}$ : Noting that we have differentials  $C_n \xrightarrow{d_n^C} C_{n-1}$ , we can define

$$d_n^D: D_n \to D_{n-1}$$

$$(A \xrightarrow{f} C_n) \mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1}),$$

i.e. we send  $f \mapsto d_n^C \circ f$  be precomposing with the differential from  $C_*$ .

3.  $(d^D)^2 = 0$ : We can explicitly write

$$(d^D)^2: D_n \to D_{n-2}$$

$$(A \xrightarrow{f} C_n) \mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1} \xrightarrow{d_{n-1}^C} C_{n-2}),$$

and so  $f \mapsto d_{n-1}^C \circ d_n^C \circ f$ . The claim is that this is the zero map, which follows from writing this as  $(d^C)^2 \circ f = 0 \circ f = 0$ , using that  $C_*$  is a chain complex.

Thus

$$D := (\cdots \to D_{n+1} \xrightarrow{d_{n+1}^D} D_n \xrightarrow{d_n^D} D_{n-1} \to \cdots) \in \mathrm{Ch}(\mathrm{Ab}).$$

Writing  $Z_n = Z_n(C) = \ker d_n^C$ , we now show the following:

$$H_n(\operatorname{Hom}_{R\text{-}\operatorname{mod}}(Z_n,C)=0 \implies H_n(C)=0.$$

It suffices to show that  $\ker d_n^C \subseteq \operatorname{im} d_{n+1}^C$ , so let  $y \in \ker d_n^C$ ; we want to produce the following:

$$x \in C_{n+1}, \quad d_{n+1}^C(x) = y.$$

We can start with the inclusion map

$$\iota : \ker d_n^C \hookrightarrow C_n,$$

which by definition is an element of  $D_n := \text{hom}(Z_n, C_n)$ . By assumption, the following complex is exact at n since its homology vanishes at position n:

$$(\cdots \to D_{n+1} \to D_n \to D_{n-1} \to \cdots) :=$$

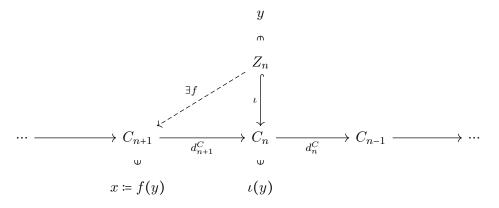
$$\cdots \to \operatorname{Hom}_R(Z_n,C_{n+1}) \xrightarrow{d_{n+1}^D} \operatorname{Hom}_R(Z_n,C_n) \xrightarrow{d_n^D} \operatorname{Hom}_R(Z_n,C_{n-1}) \to \cdots.$$

Claim:  $d_n^D(\iota) = 0$ .

This can be seen by writing this out as the composition

$$d_n^D(\ker d_n^C \xrightarrow{\iota} C_n) = (\ker d_n^C \xrightarrow{\iota} C_n \xrightarrow{d_n^C} C_{n-1}).$$

We can now use the general fact that the  $f(\ker f) = 0$  for any map f, i.e. the image of the kernel is necessarily zero. Taking  $f = d_n^C$  shows that this composition is zero. By exactness,  $\ker d_n^D = \operatorname{im} d_{n+1}^D$  and we can thus pull  $\iota$  back to some  $f \in D_{n+1} := \operatorname{Hom}_R(Z_n, C_{n+1})$ , and since our original  $y \in \ker d_n^C := Z_n$ , it makes sense to consider  $x := f(y) \in C_{n+1}$  and to identity  $y = \iota(y) \in C_n$ :



## Link to Diagram

Importantly, this f satisfies  $\iota = d_{n+1}^D(f) = d_{n+1}^C \circ f$ , and so we can write

$$y = \iota(y) = (d_{n+1}^C \circ f)(y) \coloneqq d_{n+1}(x),$$

which is what we wanted to show.

Problem 1.0.3 (Weibel 1.1.6: Homology of a graph)

Let  $\Gamma$  be a finite graph with vertices  $V = \{v_1, \dots, v_V\}$  and edge  $E = \{e_1, \dots, e_E\}$ . Define the **incidence matrix** of  $\Gamma$  to be the  $V \times E$  matrix A where

$$A_{ij} = \begin{cases} 1 & e_j \text{ starts at } v_i, \\ -1 & e_j \text{ ends at } v_i, \\ 0 & \text{else.} \end{cases}$$

Define a chain complex by taking free R-modules:

$$C := (\cdots \to 0 \to C_1 \to C_0 \to 0 \to \cdots) = (\cdots \to 0 \to R^E \xrightarrow{A} R^V \to 0 \to \cdots).$$

If  $\Gamma$  is connected, show that  $H_0(C)$  and  $H_1(C)$  are free R-modules of dimensions 1 and E-V+1 respectively.

Hint: choose a basis  $\{v_1, v_2 - v_1, \dots, v_V - v_1\}$  and use a path from  $v_1 \rightsquigarrow v_i$  to produce an element  $e \in C_1$  with  $d(e) = v_i - v_1$ .

#### **Solution:**

We first make the following two observations:

1. 
$$H_0(C) = \operatorname{coker}(A) \cong R^V / \operatorname{im} A \Longrightarrow \operatorname{rank} H_0(C) = V - \operatorname{rank} \operatorname{im} A$$
, and

2. 
$$H_1(C) = \ker(A) \implies \operatorname{rank} H_1(C) = \operatorname{rank} \ker A$$

Claim:  $\operatorname{rankim}(A) = V - 1$ .

Given this claim, applying observation (1) we immediately obtain

rank 
$$H_0(C) = V - (V - 1) = 1$$
,

which is the first equality we want to show. For the second equality, we can use the first isomorphism theorem to get a SES of free R-modules

$$0 \to \ker(A) \hookrightarrow R^E \to \operatorname{im}(A) \to 0,$$

and since  $\operatorname{im}(A)$  is free and thus projective, this sequence splits. So  $R^E \cong \ker(A) \oplus \operatorname{im}(A)$ , and taking free ranks yields

$$E = \operatorname{rank} \ker(A) + (V - 1) \implies \operatorname{rank} \ker(A) = E - V + 1,$$

and this yields the second equality by using observation (2) to identify the LHS with rank  $H_1(C)$ .

Proof (of claim). Using the fact that

$$\mathcal{B} \coloneqq \{v_1, \cdots, v_V\}$$

is a basis for  $\mathbb{R}^V$  as a free  $\mathbb{R}$ -module, we can make a change of basis to

$$\mathcal{B}' \coloneqq \{v_1, v_2 - v_1, \dots, v_V - v_1\}.$$

That this is again a basis follows from the fact that the change-of-basis matrix M is upper-triangular with ones on the diagonal and thus satisfies  $\det M = 1_R \in R^{\times}$  (i.e. it's a unit), so M is nonsingular. We can then observe that if  $e_i$  is an edge between two vertices  $v_{i_1} \stackrel{e_i}{\longrightarrow} v_{i_2}$ , then  $d(e_i) := Ae_i = v_{i_2} - v_{i_2}$ . By linearity, if  $e_{i_1}, \dots, e_{i_n}$  is a sequence of edges connecting  $v_1$  to  $v_i$  for any  $1 \le j \le V$ , then

$$d(e_{i_1} + \dots + e_{i_n}) = v_j - v_1.$$

Since  $\Gamma$  is connected, there always exists such a sequence of edges connecting each  $v_j$  to  $v_1$ , and thus  $v_j - v_1$  is in im(A). We can conclude that

$$V-1 \le \operatorname{rankim}(A) \le V$$
.

To see that rank im(A)  $\neq V$ , note that if e is any sequence of edges connecting  $v_1$  to itself in a loop, then  $d(e_1) = v_1 - v_1 = 0$ . Any other path e' must necessarily start or end at some  $v_i \neq v_1$  and satisfies  $d(e') = v_i - v_1 \neq v_1$ , and so  $v_1 \notin \text{im}(A)$ . Thus

$$\operatorname{rankim}(A) = V - 1.$$

Problem 1.0.4 (Weibel 1.1.7: Tetrahedra)

The **tetrahedron** T is a surface with 4 vertices, 6 edges, and 4 faces of dimension 2, and its homology is the homology of the complex

$$C. \coloneqq (\cdots \to 0 \to R^4 \to R^6 \to R^4 \to 0 \to \cdots).$$

Write down the matrices in this complex and computationally verify that

$$H_*(T) = [R, 0, R, 0, \cdots].$$

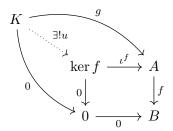
*Problem* 1.0.5 (Weibel 1.2.3)

Let  $\mathcal{A}$  be the category Ch(R-mod) and let f be a chain map. Show that the complex ker f is a (categorical) kernel of f and that coker f is a (categorical) cokernel of f.

#### **Solution:**

For a fixed map  $f: A \to B$ , the *kernel* of f is an object ker f satisfying the following universal property: for any object K with a morphism  $K \xrightarrow{g} A$  making the following outer square

commute, there is a unique morphism  $u: K \to \ker f$  making the entire diagram commute:



We'll use without proof that kernels exist in  $\mathcal{A} = R$ -mod and are given by  $\ker f := \{a \in A \mid f(a) = 0_B\}$  along with an inclusion map  $\iota^f : \ker f \to A$ .

Let  $A, B \in Ch(A)$  be chain complexes and  $f : A \to B$  be a chain map. We will construct ker f as a chain complex and show it satisfies the correct universal property.

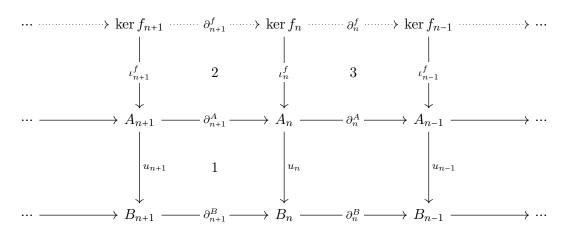
Claim 1: There are unique objects  $\ker f_n \in R$ -mod which can be assembled into a unique chain complex  $(\ker f, \partial^f)$ .

### **Proof of Claim 1:**

Let  $u: A \to B$  be a chain map, so that we have a commuting diagram of the following form:

### Link to Diagram

Appealing to the universal property of kernels in R-mod, we can produce unique objects  $\ker f_n$  and morphisms  $\iota_n^f : \ker f_n \to A_n$  satisfying  $(\ker f_n \to A_n \to B_n) = 0$  for every n. We also claim that there are maps  $\partial_n^f : \ker f_n \to \ker f_{n-1}$ , yielding the following diagram:

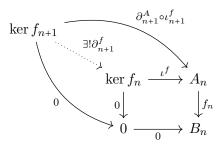


## Link to Diagram

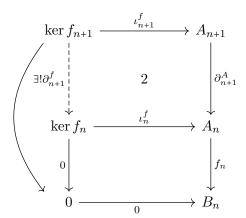
Why the  $\partial_n^f$  exist: this follows from the universal property of kernels in  $\mathcal{A}$ : Using the commutativity of square 1 we have

$$0 = (\ker f_{n+1} \to A_{n+1} \to B_{n+1} \to B_n) = (\ker f_{n+1} \to A_{n+1} \to A_n \to B_n),$$

where we've also used the fact that (ker  $f_{n+1} \to A_{n+1} \to B_{n+1} = 0$ ) from the universal property of ker  $f_{n+1}$ . So we can fit these into an appropriate diagram in  $\mathcal{A}$ , which supplies these differentials:



Why the  $\iota^f$ : ker  $f \to A$  assemble into a chain map: Note that everything here commutes, and we can break the northeast corner of this diagram up and rearrange things slightly to form the following diagram:



## Link to Diagram

Here, square 2 is precisely the square 2 appearing in the original diagram, and commutativity of it for each n is precisely what is required for  $\iota^f$  to be a chain map.

Why  $(\partial^f)^2 = 0$ : Using the commutativity of square 3 and the fact that  $(\partial^A)^2 = 0$ , we have

$$\iota_{n-1}^{f} \circ (\partial^{f})^{2} \coloneqq (\ker f_{n+1} \to \ker f_{n} \to \ker f_{n-1} \to A_{n-1})$$

$$= (\ker f_{n+1} \to A_{n+1} \to A_{n} \to A_{n-1})$$

$$\coloneqq \iota_{n+1}^{f} \circ (\partial^{A})^{2}$$

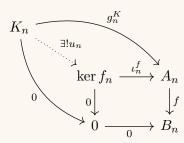
$$= 0.$$

and since  $\iota_{n-1}^f$  is not the zero map, this forces  $(\partial^f)^2=0$ .

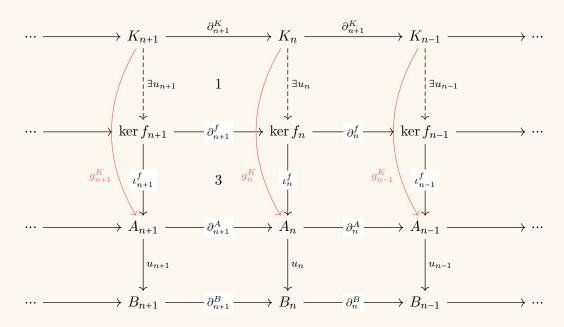
**Claim 2:** The complex ker f satisfies the universal property of kernels in  $Ch(\mathcal{A})$ , i.e. if  $g^K: K \to A$  is a chain map satisfying  $K \to A \to B = 0$ , there is a unique chain map  $u: K \to \ker f$  making the appropriate diagram commute.

## Proof (?).

Again using the universal property of kernels in R- mod, for each n we have a commutative diagram



This results in a diagram of the following form:



## Link to Diagram

It only remains to check that the  $u_n$  assemble to a chain map  $K \to \ker f$ , which would follow from the commutativity of e.g. square (1). However, if (1) were *not* commutative, then the rectangle formed by (1) and (3) together would not be commutative – but  $g^K$  was assumed to be a chain map, so this rectangle commutes, yielding a contradiction.

Note: a proof of a similar flavor seems to work for the cokernel complex by reversing all of the arrows.

#### Problem 1.0.6 (?)

Verify exactness in the Snake Lemma in at least two other positions.

## Solution:

This follows from the construction of the complex ker f above, specifically using the fact that the constructed differential  $\partial^f$  satisfies  $(\partial^f)^2 = 0$ .