Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

where

$$\begin{aligned} S : \mathbb{R} \times S^1 &\longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\ S(s,t) &\xrightarrow{s \longrightarrow \pm\infty} S^{\pm}(t). \end{aligned}$$

Outline

- Took $R^\pm : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$: symplectic paths associated to $S^\pm$
- These paths defined $\mu(x), \mu(y)$
- Section 8.7:

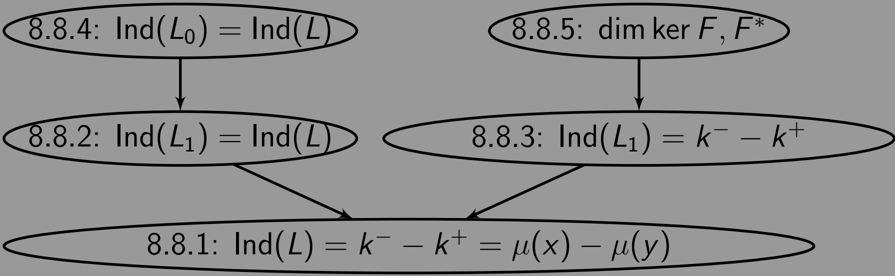
$$R^\pm \in \mathcal{S} := \left\{ R(t) \;\middle|\; R(0) = \mathrm{id},\ \det(R(1) - \mathrm{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\mathrm{Ind}(L) \overset{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know $\mathrm{Ind}(L) = \mathrm{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing $\mathrm{Ind}(L_1)$ by computing kernels.



8.8.5: dim ker F, F^*

Recall

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

$$\begin{aligned} L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Z &\longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z \end{aligned}$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since coker $L_1 \cong$ ker L_1^*, it suffices to compute ker L_1^*
- We have

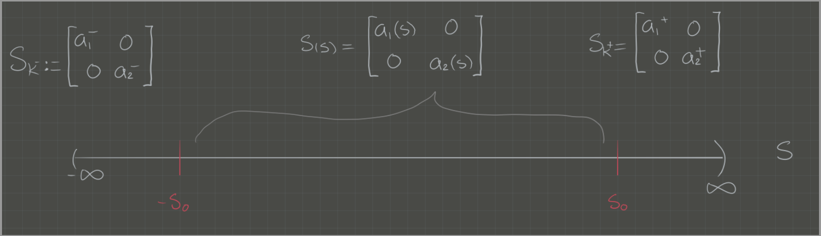
$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^{n} \mathrm{Mat}(2;\mathbb{R}).$$

- This allows us to reduce to the $n = 1$ case.

Setup

L_1 used a path of diagonal matrices constant near ∞:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



Statement of Later Lemma (8.8.5)

Let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

Note: F is L_1 for $n = 1$:

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

Statement of Lemma

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s) Y.$$

Suppose $a_i^{\pm} \notin 2\pi\mathbb{Z}$.

1. Suppose $a_1(s) = a_2(s)$ and set $a^{\pm} := a_1^{\pm} = a_2^{\pm}$. Then

$$\dim \operatorname{Ker} F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \operatorname{Ker} F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

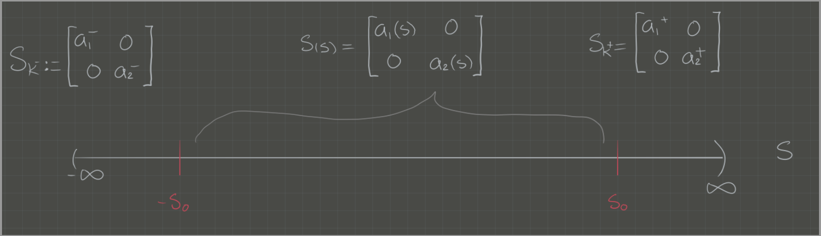
2. Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \# \left\{ i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$
$$\dim \operatorname{Ker} F^* = \# \left\{ i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Statement of Lemma

In words:

1. If $S(s)$ is a scalar matrix, set $a^\pm = a_1^\pm = a_2^\pm$ to the limiting scalars and count the integer multiples of 2π between a^- and a^+.
2. Otherwise, if S is uniformly bounded by 1, count the number of entries the flip from positive to negative as s goes from $-\infty \longrightarrow \infty$.



Proof of Assertion 1

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \mathrm{Ker}\, F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \mathrm{Ker}\, F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) := a_1(s) = a_2(s)$.
- Start with equation on $\mathbb{R}^2$,

$$\mathbf{Y}(s,t) = [Y_1(s,t), Y_2(s,t)].$$

- Replace with equation on $\mathbb{C}$:

$$\mathbf{Y}(s,t) = Y_1(s,t) + iY_2(s,t).$$

Assertion 1, Step 1: Reduce to CR

- Expand definition of the PDE

$$F(\mathbf{Y}) = 0 \rightsquigarrow \bar{\partial}\mathbf{Y} + S\mathbf{Y} = 0$$

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\mathbf{Y} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \mathbf{Y} = 0.$$

- Change of variables: want to reduce to $\bar{\partial}\tilde{Y} = 0$
- Choose $B \in \mathrm{GL}(1, \mathbb{C})$ such that $\bar{\partial}B + SB = 0$
- Set $Y = B\tilde{Y}$, which (?) reduces the previous equation to

$$\bar{\partial}\tilde{Y} = 0.$$

Assertion 1, Step 1: Reduce to CR

Can choose (and then solve)

$$B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix} \quad \text{where} \quad \frac{\partial b}{\partial s} = -a(s)b(s)$$

$$\implies b(s) = \exp\left(\int_0^s -a(\sigma)\, d\sigma\right) := \exp\left(-A(s)\right).$$

Remarks:

- For some constants C_i, we have

$$A(s) = \begin{cases} C_1 + a^- s, & s \leq -\sigma_0 \\ C_2 + a^+ s, & s \geq \sigma_0 \end{cases}.$$

- The new $\tilde{Y}$ satisfies CR, is continuous and L^1_{loc}, so elliptic regularity $\implies C^\infty$.
- The real/imaginary parts of $\tilde{Y}$ are C^∞ and harmonic.

Assertion 1, Step 2: Solve CR

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$
- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of $\tilde{Y}$ in z.
- Deduce that the solutions of the system are given by

$$\tilde{Y}(u) = \sum_{\ell \in \mathbf{Z}} c_\ell u^\ell \implies \tilde{Y}(s + it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $\{c_\ell\}_{\ell \in \mathbb{Z}} \subset \mathbb{C}$ converges for all s, t.

Assertion 1, Step 2: Solve CR

Use $e^{s+it} = e^s(\cos(t) + i\sin(t))$ to write in real coordinates:

$$\tilde{Y}(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi s\ell} \begin{bmatrix} \cos(2\pi\ell t) & -\sin(2\pi\ell t) \\ \sin(2\pi\ell t) & \cos(2\pi\ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

Use

$$Y = B\tilde{Y} = \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \tilde{Y}$$

to write

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi s\ell} \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \begin{bmatrix} \cos(2\pi\ell t) & -\sin(2\pi\ell t) \\ \sin(2\pi\ell t) & \cos(2\pi\ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

For $s \leq s_0$ this yields for some constants K, K':

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi\ell - a^-} \begin{bmatrix} e^{K}(\alpha_\ell\cos(2\pi\ell t) - \beta_\ell\sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell\sin(2\pi\ell t) + \beta_\ell\cos(2\pi\ell t)) \end{bmatrix}.$$

Condition on L^p Solutions

For $s \leq s_0$ we had

$$Y(s,t) = \sum_{\ell \in \mathbb{Z}} e^{\left(2\pi\ell - a^-\right)s} \begin{bmatrix} e^K(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

and similarly for $s \geq s_0$, for some constants C, C' we have:

$$Y(s,t) = \sum_{\ell \in \mathbb{Z}} e^{\left(2\pi\ell - a^+\right)s} \begin{bmatrix} e^C(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

Then

$$Y \in L^p \iff \text{exponential terms} \overset{\ell \to \infty}{\longrightarrow} 0.$$

$$Y(s,t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^{K}(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$

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