# Mapping Class Groups

D. Zack Garza

Wednesday $30^{\text{th}}$ September, 2020

# Contents

# 1 Setup

- All manifolds:
  - Connected
  - Oriented
  - 2nd countable (countable basis)
  - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
  - With boundary (possibly empty)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Curves: simple, closed, oriented
- For $X, Y$ topological spaces, consider

$$Y^X = C(X,Y) = \hom_{\mathrm{Top}}(X,Y) \coloneqq \left\{ f : X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

## 1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
  - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology $\mathcal{O}_{\mathrm{CO}}$:

$$U \in \mathcal{O}_{\mathrm{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

### 1.1.1 Mapping Spaces

- So define

$$\mathrm{Map}(X,Y) \coloneqq (\hom_{\mathrm{Top}}(X,Y), \mathcal{O}_{\mathrm{CO}}) \qquad \text{where } \mathcal{O}_{\mathrm{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
  - $\mathrm{Homeo}(X,Y)$ the subspace of homeomorphisms
  - $\mathrm{Imm}(X,Y)$, the subspace of immersions (injective map on tangent spaces)
  - $\mathrm{Emb}(X,Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
  - $C^k(X,Y)$, the subspace of $k\times$ differentiable maps
  - $C^\infty(X,Y)$ the subspace of smooth maps
  - $\mathrm{Diffeo}(X,Y)$ the subspace of diffeomorphisms
  - $C^\omega(X,Y)$ the subspace of analytic maps
  - $\mathrm{Isom}(X,Y)$ the subspace of isometric maps (for Riemannian metrics)
  - $[X,Y]$ homotopy classes of maps

## 1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty,K} := \sup \left\{ d(f_n(x), f(x)) \;\middle|\; x \in K \right\} \overset{n\to\infty}{\to} 0 \quad \forall K \subseteq X \text{ compact.}$$

  - In words: $f_n \to f$ uniformly on every compact set.
- If $X$ itself is compact and $Y$ is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

### 1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{f_\alpha\} \subset \hom_{\mathrm{Top}}(X, Y)$$

  form a compact subset of $\mathrm{Map}(X, Y)$?
- Essentially answered by:

**Theorem 1.1** ***(Ascoli).***
If $X$ is locally compact Hausdorff and $(Y, d)$ is a metric space, a family $\mathcal{F} \subset \hom_{\mathrm{Top}}(X, Y)$ has compact closure $\iff$ $\mathcal{F}$ is equicontinuous and $F_x := \left\{ f(x) \;\middle|\; f \in \mathcal{F} \right\} \subset Y$ has compact closure.

**Corollary 1.2** ***(Arzela).***
If $\{f_n\} \subset \hom_{\mathrm{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every $X$, it contains a uniformly convergent subsequence.

## 1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
  - Can take $G$ to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} := \hom_{\mathrm{TopGrp}}(G, S^1)$$

    where we consider $S^1 \subset \mathbb{C}$.
- Can integrate with respect to the Haar measure $\mu$, define $L^p$ spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) := \int_G f(x)\overline{\chi(x)} d\mu(x).$$

# 2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\mathrm{Map}(\,\cdot\,, Y)$:

$$
\begin{aligned}
X = \{\mathrm{pt}\} \rightsquigarrow &\quad \mathrm{Map}(\{\mathrm{pt}\}, Y) \cong Y \\
X = I \rightsquigarrow &\quad \mathcal{P}Y \coloneqq \{f : I \to Y\} = Y^I \\
X = S^1 \rightsquigarrow &\quad \mathcal{L}Y \coloneqq \left\{f : S^1 \to Y\right\} = Y^{S^1}.
\end{aligned}
$$

- Adjoint property: there is a homeomorphism

$$
\begin{aligned}
\mathrm{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \mathrm{Map}(Z, Y^X) \\
H : X \times Z \to Y &\iff \tilde{H} : Z \to \mathrm{Map}(X, Y) \\
(x, z) \mapsto H(x, z) &\iff z \mapsto H(\,\cdot\,, z).
\end{aligned}
$$

  - Categorically, $\hom(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top.
  - A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

## 2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space $PY$, the based loop space $\Omega Y$.
- Importance in homotopy theory: the path space fibration

$$
\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y
$$

  - Plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$
\pi_0 \mathrm{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},
$$

  i.e. two maps $f, g$ are homotopic $\iff$ they are connected by a path in $\mathrm{Map}(X, Y)$.

  Picture!

### 2.1.1 Proof

$$
\mathcal{P}\mathrm{Map}(X, Y) = \mathrm{Map}(I, Y^X) \cong \mathrm{Map}(X \times I, Y),
$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \to Y$, the LHS are paths in the space of maps.

## 2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I, \cdot)$:

$$\begin{aligned}
\mathcal{P}\mathrm{Map}(X,Y) &\coloneqq \mathrm{Map}(I, Y^X) \\
\mathcal{P}^2\mathrm{Map}(X,Y) &\coloneqq \mathcal{P}\mathrm{Map}(I, Y^X) = \mathrm{Map}(I, (Y^X)^I) = \mathrm{Map}(I, Y^{XI}) \\
&\vdots \\
\mathcal{P}^n\mathrm{Map}(X,Y) &\coloneqq \mathcal{P}^{n-1}\mathrm{Map}(I, Y^{XI}) = \mathrm{Map}(X, Y^{XI^n}).
\end{aligned}$$

- Can interpret

$$\mathcal{P}^2\mathrm{Map}(X,Y) = \mathcal{P}\mathrm{Map}(X \times I, Y).$$

  as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

# 3 Defining the Mapping Class Group

## 3.1 Isotopy

- Define a homotopy between $f, g : X \to Y$ as a map $F : X \times I \to Y$ restricting to $f, g$ on the ends.
  - Equivalently: a *path*, an element of $\mathrm{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \to Y$ to be homeomorphisms for every $t$.
  - Equivalently: a path in the subspace of homeomorphisms, an element of $\mathrm{Map}(I, \mathrm{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\mathrm{Map}(X, Y)$, subspace of homeomorphisms.

## 3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
  - In a general category $\mathcal{C}$, we can always define the group $\mathrm{Aut}_{\mathcal{C}}(X)$.
    * If the group has a topology, we can consider $\pi_0 \mathrm{Aut}_{\mathcal{C}}(X)$, the set of path components.
    * Since groups have identities, we can consider $\mathrm{Aut}^0_{\mathcal{C}}(X)$, the path component containing the identity.
  - So we make a general definition, the *extended mapping class group*:

$$\mathrm{MCG}^{\pm}_{\mathcal{C}}(X) \coloneqq \mathrm{Aut}_{\mathcal{C}}(X)/\mathrm{Aut}^0_{\mathcal{C}}(X).$$

  - Here the $\pm$ indicates that we take both orientation preserving and non-preserving automorphisms.
  - Has an index 2 subgroup of orientation-preserving automorphisms, $\mathrm{MCG}^+(X)$.
  - Can define $\mathrm{MCG}_\partial(X)$ as those that restrict to the identity on $\partial X$.

Picture: quotienting out by identity component

## 3.3 Definitions in Several Categories

- Now restrict attention to

$$\mathrm{Homeo}(X) \coloneqq \mathrm{Aut}_{\mathrm{Top}}(X) = \left\{ f \in \mathrm{Map}(X, X) \;\middle|\; f \text{ is an isomorphism} \right\}$$
$$\text{equipped with } \mathcal{O}_{\mathrm{CO}}.$$

  - Taking $\mathrm{MCG}^{\pm}_{\mathrm{Top}}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\mathrm{Diffeo}(X) \coloneqq \mathrm{Aut}_{C^\infty}(X).$$

  - Taking $\mathrm{MCG}_{C^\infty}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\mathrm{ho}(X) \coloneqq \{[f] \in [X, Y]\}.$$

  - Taking $\mathrm{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences.*

### 3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
  - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi \mathrm{Aut}(X)$.
- For surfaces: For $\Sigma$ a genus $g$ surface, MCG$(S)$ acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \to \mathrm{MCG}(\Sigma) \to T(\Sigma) \to \mathcal{M}_g \to 0$$

  where the last term is the moduli space of Riemann surfaces homeomorphic to $X$.
- $T(S)$ is the moduli space of complex structures on $S$, up to the action of homeomorphisms that are isotopic to the identity:
  - Points are isomorphism classes of marked Riemann surfaces
  - Equivalently the space of hyperbolic metrics
- Used in the Neilsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
  - Periodic,
  - Reducible (preserves some simple closed curves), or
  - Pseudo-Anosov (has directions of expansion/contraction)

Picture: $\mathcal{M}_g$.

# 4 Examples of MCG

## 4.1 The Plane: Straight Lines

- $\mathrm{MCG}_{\mathrm{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \to \mathbb{R}^2$, take the straight-line homotopy:

$$\begin{aligned} F &: \mathbb{R}^2 \times I \to \mathbb{R}^2 \\ F(x,t) &= tf(x) + (1-t)x. \end{aligned}$$

Picture: parameterize line between $x$ and $f(x)$ and flow along it over time.

## 4.2 The Closed Disc: The Alexander Trick

- $\mathrm{MCG}_{\mathrm{Top}}(\overline{\mathbb{D}}^2) = 1$: for any $f : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ such that $f|_{\partial \overline{\mathbb{D}}^2} = \mathrm{id}$, take

$$F : \overline{\mathbb{D}}^2 \times I \to \overline{\mathbb{D}}^2$$
$$F(x,t) := \begin{cases} tf\left(\frac{x}{t}\right) & \|x\| \in [0,t) \\ x & \|x\| \in [1-t, 1] \end{cases}.$$

  - This is an isotopy from $f$ to the identity.
  - Interpretation: "cone off" your homeomorphism over time:

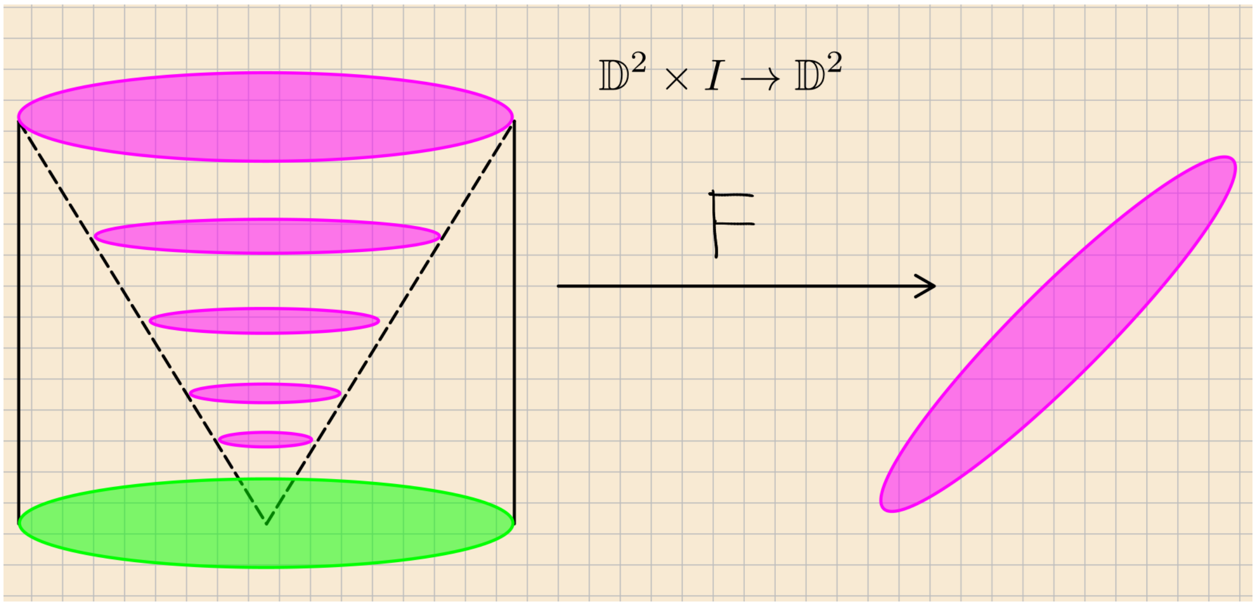


Figure 1: Image

  - Note that this won't work in the smooth category: singularity at origin

## 4.3 Overview of Big Results

- The word problem in $\mathrm{MCG}(\Sigma_g)$ is solvable
- Any finite group is $\mathrm{MCG}(X)$ for some compact hyperbolic 3-manifold $X$.
- For $g \geq 3$, the center of $\mathrm{MCG}(\Sigma_g)$ is trivial and $H_1(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = 1$
  - Why care: same as abelianization of the group.

**Theorem 4.1** ***(Dehn-Neilsen-Baer).***
Let $\Sigma_g$ be compact and oriented with $\chi(\Sigma_g) < 0$. Then

$$\mathrm{MCG}_\partial^+(\Sigma_g) \cong \mathrm{Out}_\partial(\pi_1(\Sigma_g)) \cong_{\mathrm{Grp}} \pi_0 \mathrm{ho}_\partial(\Sigma_g).$$

- For $g \geq 4$, $H_2(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$
  - Why care: used to understand surface bundles

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

  - Find the classifying space $B$Diffeo
  - Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\mathrm{Diffeo}(\Sigma_g)] \cong [S^{n-1}, \mathrm{Diffeo}(\Sigma_g)]$$

  - Theorem (Earle-Ells): For $g \geq 2$, $\mathrm{Diffeo}_0(\Sigma_g)$ is contractible. As a consequence, $\mathrm{Diffeo}(\Sigma_g) \twoheadrightarrow \mathrm{Mod}(\Sigma_g)$ is a homotopy equivalence, and there is a correspondence:

$$\left\{\begin{matrix}\text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B\end{matrix}\right\} \Big/ {\scriptstyle \text{Bundle isomorphism}} \iff \left\{\begin{matrix}\text{Monodromy Representations} \\ \rho{:}\pi_1(B) \to \mathrm{MCG}(\Sigma_g)\end{matrix}\right\} \Big/ {\scriptstyle \text{Conjugacy}}.$$

---

# 5 | Dehn Twists

- $\mathrm{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation

**Claim:** Let $A \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; 1 \leq |z| \leq 2 \right\}$, then $\mathrm{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \exp\left(2\pi i |z|\right) z. \end{aligned}$$

# 6 MCG of the Torus

## 6.1 Setup

**Definition 6.0.1** (Special Linear Group)**.**

$$\mathrm{SL}(n, \Bbbk) = \left\{ M \in \mathrm{GL}(n, \Bbbk) \;\middle|\; \det M = 1 \right\} = \ker \det_{\mathbb{G}_m}.$$

**Definition 6.0.2** (Symplectic Group)**.**

$$\mathrm{Sp}(2n, \Bbbk) = \left\{ M \in \mathrm{GL}(2n, \Bbbk) \;\middle|\; M^t \Omega M = \Omega \right\} \leq \mathrm{SL}(2n, \Bbbk)$$

where $\Omega$ is a nondegenerate skew-symmetric bilinear form on $\Bbbk$.
Example:

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

**Definition 6.0.3** (Algebraic Intersection)**.**
A bilinear antisymmetric form on middle homology:

$$\widehat{\iota} : H_1(\Sigma_g; \mathbb{Z}) \otimes H_1(\Sigma_g; \mathbb{Z}) \to \mathbb{Z}.$$

Note that this is a symplectic pairing.

- There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned} \rho : \mathrm{MCG}(\Sigma) &\to \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_*. \end{aligned}$$

- For a surface of finite genus $g \geq 1$, elements in $\mathrm{im}\rho$ preserve the *algebraic intersection form*
- Thus there is an interesting surjective representation:

$$0 \to \mathrm{Tor}(\Sigma_g) \hookrightarrow \mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}(2g; \mathbb{Z}) \to 0.$$

- Kernel is the *Torelli group*, interesting because the symplectic group is well understood, so questions about MCG reduce to questions about Tor.

**Theorem 6.1** ***(Mapping Class Group of the Torus).***
The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

## 6.2 Proof

- For $f$ any automorphism, the induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Homeo}(X, X), \circ) &\to (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*. \end{aligned}$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff

$$\mathrm{Homeo}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$$

  - This is true here, since any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned} \tilde{\sigma} : \mathrm{MCG}(X) &\to \mathrm{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

**Claim:** $\mathrm{im}(\tilde{\sigma}) \subseteq \mathrm{SL}(2, \mathbb{Z})$.

*Proof .*

- Algebraic intersection numbers in $\Sigma_2$ correspond to determinants
- $f \in \mathrm{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2

■

- We can thus freely restrict the codomain to define the map

$$\begin{aligned} \sigma : \mathrm{MCG}(X) &\to \mathrm{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

**Claim:** $\sigma$ is surjective.

- $\mathbb{R}^2$ is the universal cover of $\Sigma_2$, with deck transformation group $\mathbb{Z}^2$.
- Any $A \in \mathrm{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \mathrm{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

**Claim:** $\tilde{A}$ is equivariant wrt $\mathbb{Z}^2$

*Proof .*

$$\mathrm{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that $S^2 = 1$ and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

- If $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $A \in \mathrm{SL}(2, \mathbb{Z})$, we have $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. this preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \;\middle|\; p, q \in \mathbb{Z} \right\} \cong \left\{ p\omega_1 + q\omega_2 \;\middle|\; p, q \in \mathbb{Z} \right\} \simeq \left\{ p' + q'\tau \;\middle|\; p', q' \in \mathbb{Z} \right\}.$$

  where the $\omega_i, \tau$ come from identifying $\mathbb{R}^2$ with $\mathbb{C}$, and in the last step we've rescaled the lattice by *homothety* to align one vector with the $x$-axis.

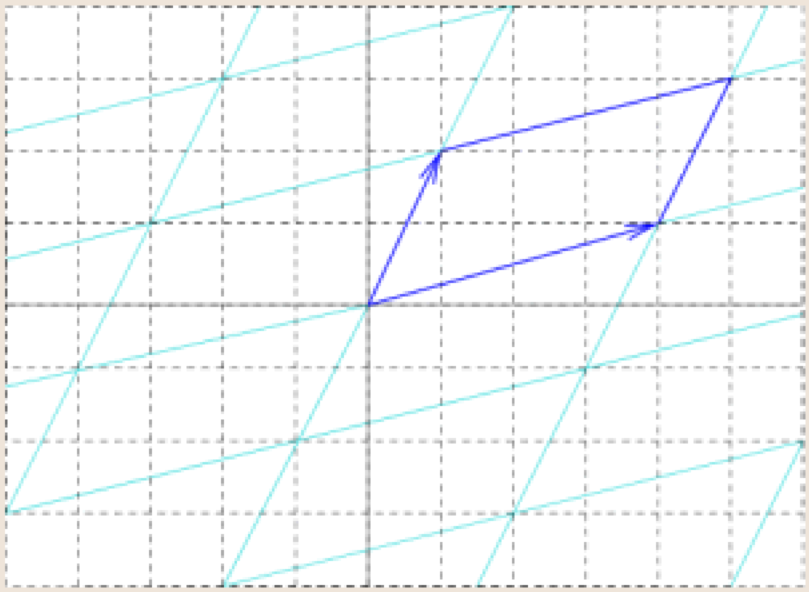

Figure 2: Lattice

■

- So $\tilde{A}$ descends to a well-defined map $\psi_{\tilde{A}}$ on $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$, which is still a linear self-homeomorphism
- There is a correspondence

$$\left\{ \text{Primitive vectors in } \mathbb{Z}^2 \right\} \iff \left\{ \begin{matrix} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{matrix} \right\} / \text{homotopy},$$

  where a vector $\mathbf{v}$ is *primitive* iff
- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

**Claim:** $\sigma$ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.

**Proposition 6.2** ***(Hatcher 1B.9).***
Let $X$ be a connected CW complex and $Y$ a $K(G,1)$. Then there is a map

$$\hom_{\mathrm{Grp}}(\pi_1(X;x_0),\pi_1(Y;y_0)) \to \hom_{\mathrm{Top}}((X;x_0),(Y;y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map. Moreover, the map is unique up to homotopies fixing $x_0$.

- Thus there is a correspondence

$$\left\{\begin{matrix}\text{Homotopy classes of}\\ \text{maps } \Sigma_2 \circlearrowleft\end{matrix}\right\} \iff \left\{\text{Group morphisms } \mathbb{Z}^2 \circlearrowleft\right\}.$$

- Claim: any element $f \in \mathrm{MCG}(\Sigma_2)$ has a representative $\varphi$ which fixes any given basepoint
- So if $f \in \ker\sigma$, then $f \simeq \varphi \simeq \mathrm{id}$ are homotopic, so $\ker\sigma = 1$.