# Assignment 6: The Fourier Transform

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# 1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \lim_{\xi'\to 0} \frac{1}{2} \int_{\mathbb{R}^n} (f()) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx$$

.

But as an immediate consequence, this yields

$$
\begin{aligned}
\left| \hat{f}(\xi) \right| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx \right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \, dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\
&\to 0,
\end{aligned}
$$

which follows from continuity in $L^1$ since $f(x - \xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds, and that $\xi' \to 0$ as $\xi \to \infty$.

## 2 Problem 2

### 2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$
\begin{aligned}
\widehat{(f * g)}(\xi) &:= \int \int f(x - y) g(y) \exp(-2\pi x \cdot \xi) \, dy \, dx \\
&=_? \int \int f(x - y) g(y) \exp(-2\pi x \cdot \xi) \, dx \, dy \\
&= \int \int f(t) \exp(-2\pi i (x - y) \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dx \, dy \\
&\qquad (t = x - y, \ dt = \ dx) \\
&= \int \int f(t) \exp(-2\pi i t \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dt \, dy \\
&= \int f(t) \exp(-2\pi i t \cdot \xi) \left( \int g(y) \exp(-2\pi i y \cdot \xi) \, dy \right) \, dt \\
&= \int f(t) \exp(-2\pi i t \cdot \xi) \hat{g}(\xi) \, dt \\
&= \hat{g}(\xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\
&= \hat{g}(\xi) \hat{f}(\xi).
\end{aligned}
$$

It thus remains to show that this swap is justified.

## 2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

### 2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$\widehat{(f * g) * h} = \widehat{f * g}\ \hat{h} = \hat{f}\ \hat{g}\ \hat{h} = \hat{f}\ \widehat{g * h} = f * (g * h).$$

### 2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\ \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then $\lim_{|\xi| \to \infty} \hat{I}(\xi) \neq 0$, which by Problem 1 shows that $I$ can not be in $L^1$, a contradiction.

# 3 Problem 3

## 3.1 (a)

### 3.1.1 (i)

Let $g(x) = f(x - y)$. We then have

$$\begin{aligned}
\hat{g}(\xi) &:= \int g(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x - y) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x - y) \exp(-2\pi i (x - y) \cdot \xi) \exp(-2\pi i y \cdot \xi)\ dx \\
&= \exp(-2\pi i y \cdot \xi) \int f(x - y) \exp(-2\pi i (x - y) \cdot \xi)\ dx \\
&\quad (t = x - y, dt = dx) \\
&= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi)\ dt \\
&= \exp(-2\pi i y \cdot \xi)\hat{f}(\xi).
\end{aligned}$$

**3.1.2 (ii)**

Let $h(x) = \exp(2\pi i x \ cdot y) f(x)$. We then have

$$\hat{h}(\xi) := \int \exp(2\pi i x \cdot y) f(x) \exp(-2\pi i x \cdot \xi) \ dx$$

$$= \int \exp(2\pi i x \cdot y - 2\pi i x \cdot \xi) f(x) \ dx$$

$$= \int f(\xi - y) \exp(-2\pi i x \cdot (\xi - y)) \ dx$$

$$= \hat{f}(\xi - y).$$

**3.2 (b)**

We'll use the fact that if $\langle \cdot, \ \cdot \rangle$ is an inner product on a vector space $V$ and $A$ is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle A\mathbf{x}, \ \mathbf{y} \rangle = \left\langle \mathbf{x}, \ A^T\mathbf{y} \right\rangle$$

where $A^{-T}$ denotes the transpose of the inverse of $A$ (or $(A^{-1})^*$ if $V$ is complex).

We then have

$$\frac{1}{|\det T|} \hat{f}(T^{-T}\xi) = \frac{1}{|\det T|} \int f(x) \exp(-2\pi i x \cdot T^{-T}\xi) \ dx$$

$$x \mapsto Tx, \ dx \mapsto |\det T| \ dx$$

$$= \frac{1}{|\det T|} \int f(Tx) \exp(-2\pi i Tx \cdot T^{-T}\xi) |\det T| \ dx$$

$$= \int f(Tx) \exp(-2\pi i x \cdot \xi) \ dx$$

$$\text{since } Tx \cdot T^{-T}\xi = T^{-1}Tx \cdot \xi = x \cdot \xi$$

$$= \widehat{(f \circ T)}(\xi).$$

# 4 Problem 4

## 4.1 (a)

### 4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$\frac{\partial}{\partial \xi} \hat{f}(\xi) := \frac{\partial}{\partial \xi} \int f(x) \exp(-2\pi i x \cdot \xi) \, dx$$

$$=_? \int f(x) \frac{\partial}{\partial \xi} \exp(-2\pi i x \cdot \xi) \, dx$$

$$= \int f(x) 2\pi i x \exp(-2\pi i x \cdot \xi) \, dx$$

$$= 2\pi i \int x f(x) \exp(-2\pi i x \cdot \xi) \, dx$$

$$:= 2\pi i \hat{g}(\xi).$$

It thus remains to show that this interchange is justified. TODO

### 4.1.2 (ii)

We have

$$\hat{h}(\xi) := \int \frac{\partial f}{\partial x}(x) \exp(-2\pi i x \cdot \xi) \, dx$$

$$= f(x) \exp(-2\pi i x \cdot \xi) \Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx$$

$$\text{(integrating by parts)}$$

$$= - \int f(x)(-2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx$$

$$\text{(since } f(\infty) = f(-\infty) = 0)$$

$$= 2\pi i \xi \int f(x) \exp(-2\pi i x \cdot \xi) \, dx$$

$$:= 2\pi i \xi \hat{f}(\xi).$$

### 4.2 (b)

Let $G(x) = \exp(-\pi x^2)$ and $\partial_\xi$ be the operator that differentiates with respect to $\xi$.

Then

$$\partial_\xi \left( \frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi)}{G(\xi)^2},$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi) = 0.$$

Using the following facts,

- $\partial_\xi G(\xi) = -2\pi\xi G(\xi)$ by computing directly,
- $\partial_\xi \hat{G}(\xi) = -2\pi\xi \hat{G}(\xi)$, which follows from the following computation

$$
\begin{aligned}
\partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= \int G(x) \partial_\xi \exp(-2\pi i x \cdot \xi) \, dx \\
&= \int G(x)(-2\pi i x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= i \int 2\pi x G(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= i \int \partial_x G(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&:= i \, \widehat{\partial_x G(x)}(\xi) \\
&= i \, (2\pi i \xi \hat{G}(\xi)) \\
&= -2\pi\xi \hat{G}(\xi),
\end{aligned}
$$

we can thus write

$$
G(\xi)\partial_\xi \hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),
$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant $c_0$, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x) \, dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

## 5 Problem 5

### 5.1 (a)

By a direct computation. we have

$$\hat{D}(\xi) := \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 e^{-2\pi i x \cdot \xi} \, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \cdot \xi) + i \sin(-2\pi x \cdot \xi) \, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \cdot \xi) \, dx$$

(since sin is odd and the domain is symmetric about 0)

$$= 2 \int_{0}^{\frac{1}{2}} \cos(-2\pi x \cdot \xi) \, dx$$

(since cos is even and the domain is symmetric about 0)

$$= 2 \left( \frac{1}{2\pi\xi} \sin(-2\pi x \cdot \xi) \Big|_{x=0}^{x=\frac{1}{2}} \right)$$

$$= \frac{\sin(\pi\xi)}{\pi\xi}.$$

## 5.2 (b)

### 5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left( \frac{\sin(\pi\xi)}{\pi\xi} \right)^2$.

### 5.2.2 (ii)

Letting $\mathcal{F}$ denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if $f$ is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting $F$ be the box function, $F$ can be seen to be even from its definition. Since $f := \mathcal{F}(F)$ by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that $\hat{f}(x) = F(x)$, the original box function.

## 5.3 (c)

By a direct computation of the integral in question, we have

$$I(x) := \int e^{-2\pi|\xi|} e^{2\pi i x\xi} \, d\xi$$

$$= \int_{-\infty}^{0} e^{-2\pi(-\xi)} e^{-2\pi i x\xi} \, d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x\xi} \, d\xi$$

$$= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x\xi} \, d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x\xi} \, d\xi$$

by the change of variables $\xi \mapsto -\xi$, $d\xi \mapsto -d\xi$ and swapping integration bounds

$$= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x\xi} + e^{2\pi\xi} e^{2\pi i x\xi} \, d\xi$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-u} e^{-ixu} + e^{-u} e^{ixu} \, du$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)} \, du$$

$$= \frac{1}{2\pi} \left( \frac{-e^{-u(1+ix)}}{1+ix} \Big|_{u=0}^{u=\infty} + \frac{-e^{-u(1-ix)}}{1+ix} \Big|_{u=0}^{u=\infty} \right)$$

$$= \frac{1}{2\pi} \left( \frac{1}{1+ix} + \frac{1}{1-ix} \right)$$

$$= \frac{1}{2\pi} \frac{2}{1+x^2}$$

$$= \frac{1}{\pi} \frac{1}{1+x^2},$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$I(x) = P(x) = \int \hat{P}(\xi) e^{-2\pi i x\xi} \, dx$$

$$\implies \int e^{-2\pi|\xi|} e^{2\pi i x\xi} = \int \hat{P}(\xi) e^{-2\pi i x\xi} \, dx$$

$$\implies \int e^{-2\pi|\xi|} e^{2\pi i x\xi} - \hat{P}(\xi) e^{-2\pi i x\xi} \, dx = 0$$

$$\implies \int \left( e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x\xi} \, dx = 0$$

$$\implies \left( e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x\xi} =_{a.e.} 0$$

$$\implies e^{-2\pi|\xi|} =_{a.e} \hat{P}(\xi),$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

## 6 Problem 6

We first note that if $G_t(x) := t^{-n} e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2 |\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$
\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \right| \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \, dx \\
&\qquad \text{since the integrand and thus integral is positive.} \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \left( \int_{\mathbb{R}^n} G_t(x) \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, (1) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt,
\end{aligned}
$$

which we claim is finite, so $f \in L^1$.

> Justifying the interchange: we note that the integrand $G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1}$ is non-negative, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on $\mathbb{R}^n$ and the remaining terms are continuous on $\mathbb{R}$, so they are all measurable on $\mathbb{R}^n$ and $\mathbb{R}$ respectively But then taking cylinders on them all still yields measurable functions, and the product of measurable functions is measurable.

To see that the norm is finite, we note that

$$
t \in [0,1] \implies e^{-\pi t^2} < 1
$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$
t \in [1, \infty) \implies t^{2\varepsilon - 1} \leq 1.
$$

Thus

$$
\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_0^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}
$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

If an interchange of integrals is justified, we can compute

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} \left( \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \; dt \right) e^{-2\pi i x \cdot \xi} \; dx$$

$$= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \; dt \; dx$$

$$=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \; dx \; dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left( \int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} \; dx \right) \; dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \hat{G}_t(\xi) \; dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} e^{-\pi t^2 |\xi|^2} \; dt$$

$$= \int_0^\infty e^{-\pi t^2 (1+|\xi|^2)} t^{2\varepsilon-1} \; dt$$

$$= \int_0^\infty e^{-\pi (t\sqrt{1+|\xi|^2})^2} t^{2\varepsilon-1} \; dt$$

$$s = t\sqrt{1+|\xi|^2}, \; ds = \sqrt{1+|\xi|^2} dt$$

$$= \int_0^\infty e^{-\pi s^2} \left( \frac{s}{\sqrt{1-|\xi|^2}} \right)^{2\varepsilon-1} \frac{1}{\sqrt{1+|\xi|^2}} \; ds$$

$$= (1+|\xi|^2)^{-\frac{2\varepsilon-1}{2}} (1+|\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon-1} \; ds$$

$$= (1+|\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon-1} \; dt$$

$$:= F(\xi) \|f\|_1.$$

Thus $F(\xi)$ is the Fourier transform of the function $g(x) := f(x)/\|f\|_1$.

To see that the interchange is justified, note that if we compute $\int |\cdots| \; dt$.

$$\int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \right| \; dt \; dx = \int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \right| \; dt \; dx,$$

where the integrand is now precisely what we showed was measurable when compute $\|f\|_1$ above. So Tonelli applies.