Title

D. Zack Garza

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1 Thursday January 9th

Recall: For M^n a closed smooth manifold, consider a smooth map $f : M^n \to \mathbb{R}$.

Definition: A critical point p of f is *non-degenerate* iff $\det(H \coloneqq \frac{\partial^i f}{\partial x_i \partial x_j}(p)) \neq 0$ in some coordinate system U.

Lemma (The Morse Lemma): For any non-degenerate critical point p there exists a coordinate system around p such that

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

λ is called the *index of f at p*.

Lemma: λ is equal to the number of *negative* eigenvalues of $H(p)$.

Proof: A change of coordinates sends $H(p) \to A^t H(p) A$, which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that A is invertible and not necessarily orthogonal.

This means that f can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Proof of Morse Lemma:

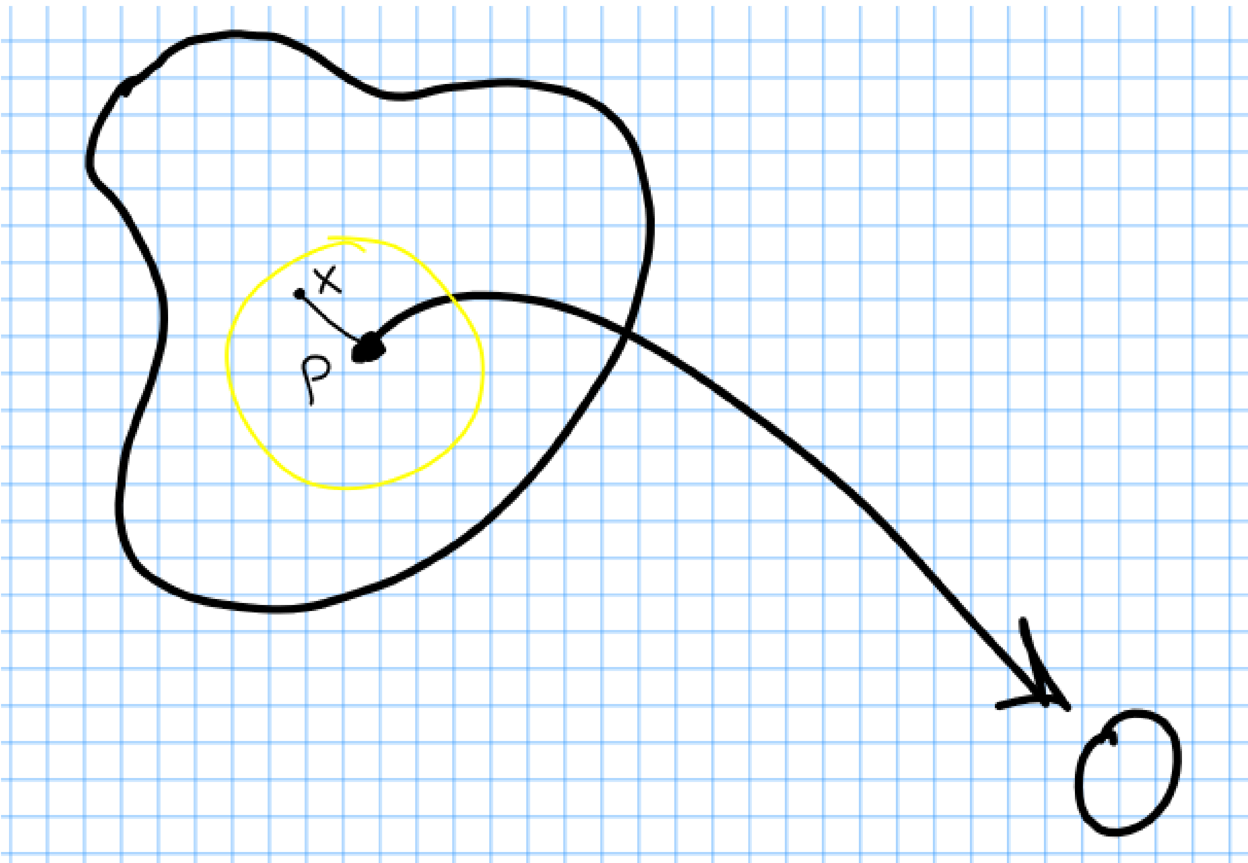
Suppose that we have a coordinate chart U around p such that $p \mapsto 0 \in U$ and $f(p) = 0$.

Step 1 – Claim: There exists a coordinate system around p such that

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

where $h_{ij}(x) = h_{ji}(x)$.

Proof: Pick a convex neighborhood V of $0 \in \mathbb{R}^n$.



Restrict f to a path between x and 0, and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \cdots, tx_n)}{dt}\, dt = f(x_1, \cdots, x_n) - f(0) = f(x_1, \cdots, x_n).$$

since $f(0) = 0$.

We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots \frac{\partial f}{\partial x_n} x_n \, dt \implies \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}\, dt = f(x).$$

We thus have $f(x) = \sum_{i=1}^{n} x_i g_i(x)$ where $\frac{\partial f}{\partial x_i}(0) = 0$, and $\frac{\partial f}{\partial x_i} = x_1 \frac{\partial g_1}{\partial x_i} + \cdots + g_i + x_i \frac{\partial g_i}{\partial x_i} + \cdots + x_n \frac{\partial g_n}{\partial x_i}$.
When we plug $x = 0$ into this expression, the only term that doesn't vanish is g_i, and thus $\frac{\partial f}{\partial x_i}(0) = g_i(0)$ and $g_i(0) = 0$.

Applying the same result to g_i, we obtain $g_i(x) = \sum_{j=1}^{n} x_j h_{ij}(x)$, and thus $f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x)$.

We still need to show h is symmetric. For every pair i, j, there is a term of the form $x_i x_j h_{ij} + x_j x_i h_{ji}$. So let $H_{ij}(x) = \frac{h_{ij}(x) + h_{ji}(x)}{2}$ (i.e. symmetrize/average h), then $f(x) = \sum_{i,j=1}^{n} x_i x_j H_{ij}(x)$ and this shows claim 1.

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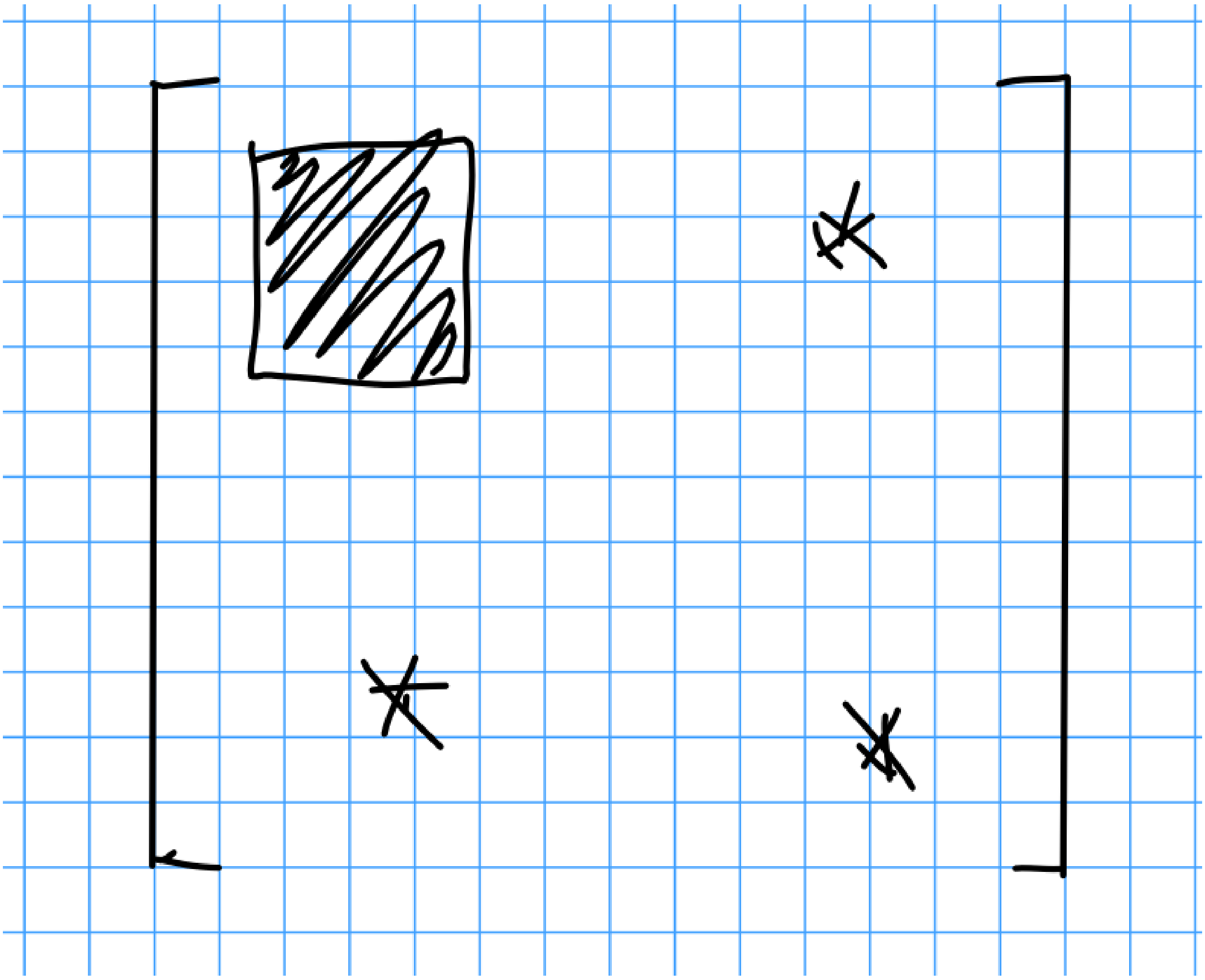
Step 2 – Induction: Assume that in some coordinate system U_0,

$$f(y_1, \cdots, y_n) = \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \cdots, y_n).$$

Note that $H_{rr}(0)$ is given by the top-left block of $H_{ij}(0)$, which is thus looks like



Note that this block is symmetric.

Claim 1: There exists a linear change of coordinates such that $H_{rr}(0) \neq 0$.

We can use the fact that $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = H_{ij}(0) + H_{ji}(0) = 2H_{ij}(0)$, and thus $H_{ij}(0) = \frac{1}{2}\left(\frac{\partial f}{\partial x_i \partial x_j}\right)$.

Since $H(0)$ is non-singular, we can find A such that $A^t H(0) A$ has nonzero rr entry, namely by letting the first column of A be an eigenvector of $H(0)$, then $A = [\mathbf{v}, \cdots]$ and thus $H(0)A = [\lambda \mathbf{v}, \cdots]$ and $A^t[\lambda \mathbf{v}] = [\lambda \|\mathbf{v}\|^2, \cdots]$.

So

$$
\begin{aligned}
\sum_{i,j\geq r} y_i y_j H_{ij}(y_1,\cdots,y_n) &= y_r^2 H_{rr}(y_1,\cdots,y_n) + \sum_{i>r} 2y_i y_r H_{ir}(y_1,\cdots,y_n) \\
&= H_{rr}(y_1,\cdots,y_n)\left(y_r^2 + \sum_{i>r} 2y_i y_r H_{ir}(y_1,\cdots,y_n)/H_{rr}(y_1,\cdots,y_n)\right) \\
&= H_{rr}(y_1,\cdots,y_n)\left(\left(y_r + \sum_{i>r}^{n} y_i H_{ir}(y_1,\cdots,y_n)/)_{rr}(y_1,\cdots,y_n)\right)^2 \sum_{i>r}^{n} y_i^2 \left(H_{ir}Y/H_{rr}(Y)\right.\right.
\end{aligned}
$$

Note that $H_{rr}(0) \neq 0$ implies that $H_{rr} \neq 0$ in a neighborhood of zero as well.

Now define a change of coordinates $\phi : U \to \mathbb{R}^n$ by

$$
z_i = \begin{cases} y_i & i \neq r \\ \sqrt{H_{rr}(y_1,\cdots,y_n)}\left(y_r + \sum_{i>r} y_i H_{ir}(Y)/H_{rr}(Y)\right) & i = r \end{cases}.
$$

This means that $f(z) = \pm z_1^2 \pm \cdots \pm z_{r-1}^2 \pm z_r^2 + \sum_{i,j\geq r+1^n z_i z_j \tilde{H}(z_1,\cdots,z_n)}$.

Exercise: show that $d_0\phi$ is invertible, and by the inverse function theorem, conclude that there is a neighborhood $U_2 \subset U_1$ of 0 on which ϕ is still invertible.

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Corollary: The nondegenerate critical points of a Morse function f are isolated.

Proof: In some neighborhood around p, we have $f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$, Thus $\dfrac{\partial f}{\partial x_i} = 2x_i$, and so $\dfrac{\partial f}{\partial x_i} = 0$ iff $x_1 = x_2 = \cdots = x_n = 0$.

Corollary: On a closed (compact) manifold M, a Morse function has only finitely many critical points.

We will need these facts to discuss the h-cobordism theorem. For a closed smooth manifold, $\partial M = \emptyset$, so M will define a cobordism $\emptyset \to \emptyset$.

Definition: Let W be a cobordism from $M_0 \to M_1$. A *Morse function* is a smooth map $f : W \to [a,b]$ such that

1. $f^{-1}(a) = M_0$ and $f^{-1}(b) = M_1$,
2. All critical points of f are non-degenerate and contained in $\mathrm{int}(W) := W \setminus \partial W$.

So f is equal to the endpoints only on the boundary.

Next time: existence of Morse functions. This is a fairly restrictive notion, but they are dense in the C^2 topology on (?).