# Homework 6

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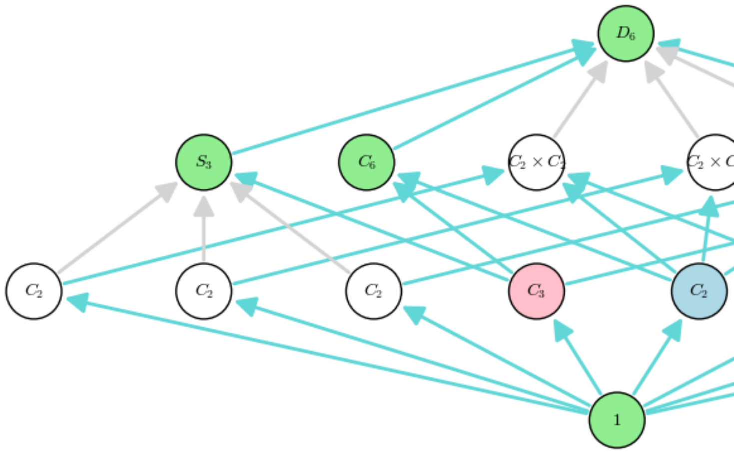
## 1 Homework Problems

### 1.1 Problem 1

The splitting field of this polynomial is $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}, \zeta_3)$ where $\zeta_3$ is a primitive third root of unity.

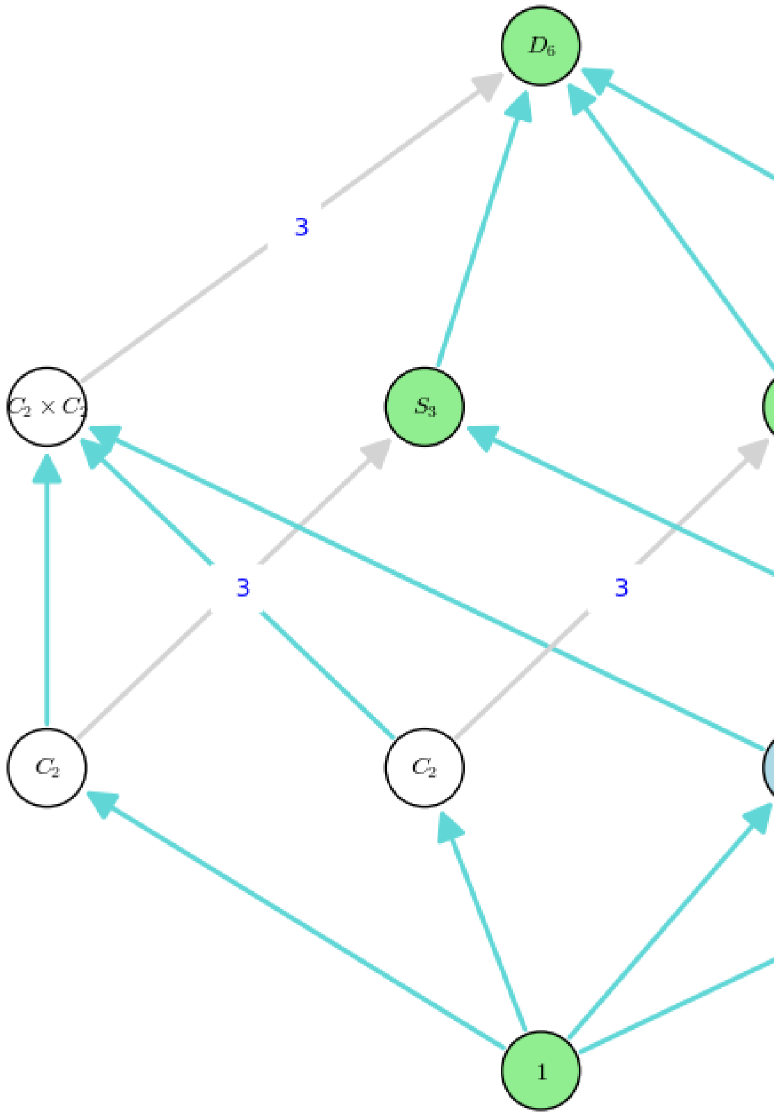
To get the degree of this extension, we extend fields in the indicated order. Since $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ is totally real, the minimal polynomial of $\zeta$ over it still has degree $\phi(3) = 2$. A quick check also shows that $\sqrt{3}$ is not contained in $\mathbb{Q}(\sqrt[3]{2})$, yielding another degree 2 extension, and finally a degree 3 extension.

Thus we have an extension of degree 12, and since we've constructed a Galois extension $L$ (a separable splitting field), if we define $G := \mathrm{Gal}(\mathbb{Q}/L)$, we have $|G| = 12$. Since we know that the splitting field of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ has Galois group $D_3$, we must have $D_3 \leq G$. This reduces the possibilities just $D_3 \times \mathbb{Z}_2 \cong D_6$.




We have the following subgroup diagram (Figure 1).

where we can simplify things by only considering conjugacy classes of subgroups, since these will cor-

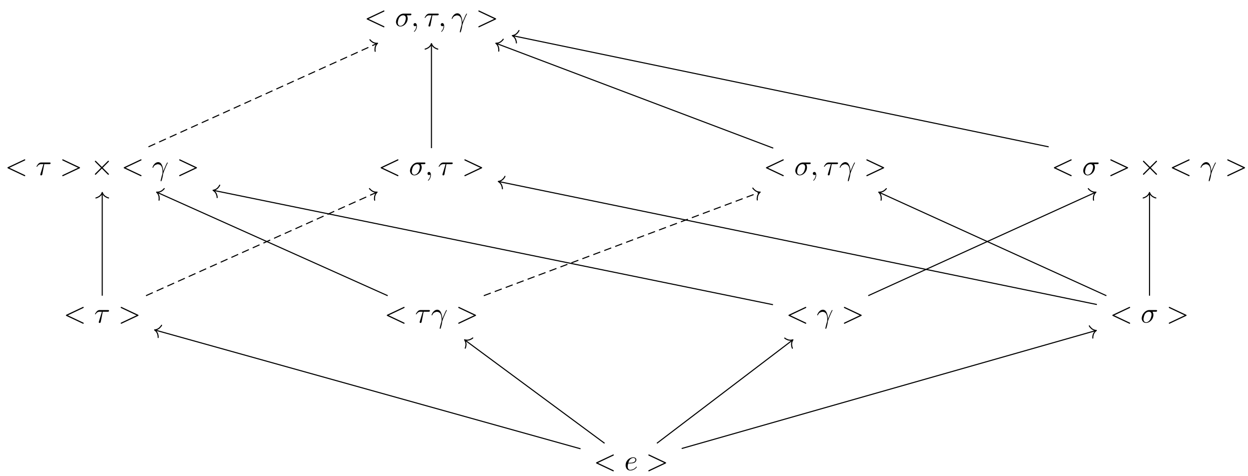


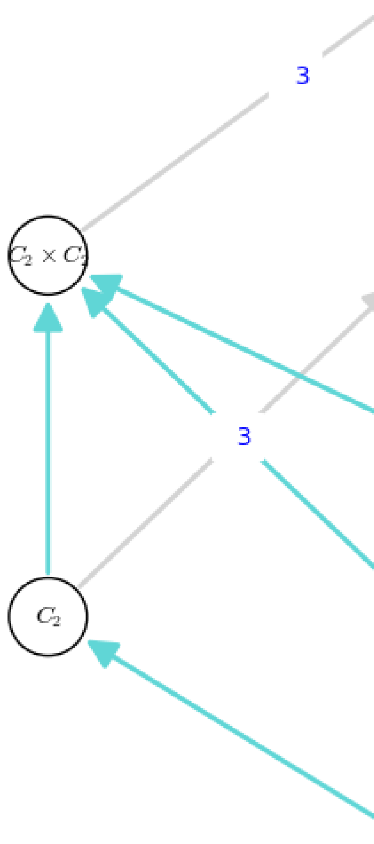

respond to conjugate field extensions (Figure 2).

We can explicitly identify the relevant automorphisms:

$$\sigma : \sqrt[3]{2} \mapsto \zeta_3 \sqrt[3]{2}$$
$$\tau : \zeta_3 \mapsto \zeta_3^2$$
$$\gamma : \sqrt{3} \mapsto -\sqrt{3}.$$

We can then present $G = \langle \sigma, \gamma, \tau \mid \sigma^3 = \tau^2 = \gamma^2 = (\sigma\tau)^2 = [\sigma, \gamma] = [\tau, \gamma] = e \rangle$, and obtain the following lattice:




which, up to conjugacy, fix the following intermediate field extensions (Figure 3).

□

## 1.2 Problem 2

We can note that since $f$ has 4 roots, the Galois group $G$ of its splitting field will be a subgroup of $S_4$. Moreover, $G$ must be a *transitive subgroup* of $S_4$, i.e. the action of $G$ on the roots of $f$ should be transitive. This reduces the possibilities to $G \cong S^4, A^4, D^4, \mathbb{Z}_4, \mathbb{Z}_2^2$.

Since $f$ has exactly 2 real roots and thus a pair of roots that are complex conjugates, the automorphism given by complex conjugation is an element of $G$. But this corresponds to a 2-cycle $\tau = (ab)$, and we can then make the following conclusions:

- Not $A_4$: $A_4$ contains only even cycles, and $\tau$ is odd.
- Not $Z_4$: This subgroup is generated by a single 4-cycle $\sigma$, which up to conjugacy is $(1234)$, and $\sigma^n$ is not a 2-cycle for any $n$.
- Not $\mathbb{Z}_2^2$: In order to be transitive, this subgroup must be $\{e, (12)(34), (13)(24), (14)(23)\}$, which does not contain $\tau$.

The only remaining possibilities are $S^4$ and $D^4$. $\square$

## 1.3 Problem 3

### 1.3.1 Part 1

To see that $\phi(n)$ is even for all $n > 2$, we can take a prime factorization of $n$ and write

$$\phi(n) = \phi\left(\prod_{i=1}^{m} p_i^{k_i}\right) = \prod_{i=1}^{m} \phi(p_i^{k_i}) = \prod_{i=1}^{m} p^{k_i - 1}(p-1) = \prod_{i=1}^{m} p^{k_i - 1} \prod_{i=1}^{m} (p-1)$$

where each $k_i \geq 1 \implies k_i - 1 \geq 0$. But every prime power is odd, and a product of odd numbers is odd, so the first product is odd. It is also true that $p - 1$ is even for every prime $p$, and the second term is a product of even terms and thus even. So $\phi(n)$ is the product of an even and an odd number, which is always even.

### 1.3.2 Part 2

Suppose $\phi(n) = 2$. Take a prime factorization of $n$, so we have

$$2 = \phi(n) = \prod_{i=1}^{m} \phi(p_i^{k_i})$$

Since the only factors of 2 are 1 and 2, we must have $\phi(p_i^{k_i}) = 2$ for exactly one $i$, and the rest must be equal to 1.

Consider the term that equals 2. We have $\phi(p_i^{k_i}) = p^{k_i - 1}(p-1) = 2$, so we must have either

- Case 1: $p - 1 = 2$ and $p^{k_i - 1} = 1$, so $p = 3$ and $k_i = 1$. So $3 \mid n$, but $3^\ell$ does *not* divide $n$ for any $\ell > 1$.
- Case 2: $p^{k_i - 1} = 2$ and $(p - 1) = 1$, so $p = 2$ and $k_i = 2$. Thus $2^2$ divides $n$ but $2^\ell$ does not for any $\ell > 2$.

In either case, it remains to check are whether the other factors where $\phi(p_j^{k_j}) = 1$ can contribute any other distinct divisors to $n$. We can note that $\phi(p_j^{k_j}$ iff $p^{k_j - 1}(p-1) = 1$, so this forces $p = 2$ and $k_j = 1$. So $n$ may or may not contain a single factor of 2, but by uniqueness of prime factorization, this can only happen in case 1. Note that this also forces $2 \;\Big|\; n$ but $2^2$ does not divide $n$.

In summary, we've found that $\phi(n) = 2$ implies that

- $3 \;\Big|\; n$, 9 does not divide $n$, and
  - $2 \;\Big|\; n$, 4 does not divide $n$
  - 2 does not divide $n$
- $2^2 \;\Big|\; n, 2^3$ does not divide n.

This reduces the possibilities to the finite set $n \in \{6, 3, 4\}$, and $\phi(6) = \phi(3) = \phi(4) = 2$. $\square$

## 1.4 Problem 4

Note that since $\zeta(\zeta + \zeta^{-1}) = \zeta^2 + 1$, we have the relation $\zeta^2 - (\zeta + \zeta^{-1})\zeta + 1 = 0$. But then

$$f(x) = x^2 - (\zeta + \zeta^{-1})x + 1$$

is a polynomial in $\mathbb{Q}(\zeta + \zeta^{-1})$ for which $f(\zeta) = 0$. Thus $g = \min(\zeta, \mathbb{Q}(\zeta + \zeta^{-1}))$ divides $f$, but since $\deg f = 2$ and $\mathbb{Q}(\zeta + \zeta^{-1})$ is totally real, $\zeta \notin \mathbb{Q}(\zeta + \zeta^{-1})$. This means that $g$ can not be linear and must have degree at least 2, but the above argument shows that $g$ has degree at *most* 2, so it must be 2. Letting $m = [\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}]$, we have

$$\begin{aligned}
&[\mathbb{Q}(\zeta) : \mathbb{Q}] = [\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta + \zeta^{-1})][\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] \\
&\implies \phi(n) = 2m,
\end{aligned}$$

and so $m = \phi(n)/2$ as desired.

## 1.5 Problem 5

Suppose $F = K[\alpha_1, \cdots, \alpha_n]$ where $\alpha_1^{n_1} \in K$ for some $n_1$ and }or each $i$ we have $\alpha_i^{n_i} \in K[\alpha_1, \cdots, \alpha_{i-1}]$ for some powers $n_i$. We want to show that $F = E[\beta_1, \cdots \beta_m]$ where each $\beta_i$ satisfy a similar condition.

Let $A = \{\alpha_i \ni \alpha_i \notin E\}$, then it is since $E \hookrightarrow F$, adjoining all elements of $A$ to $E$ will yield exactly $F$. Using the order of $\alpha_i$ given by the definition of $F$ as a radical extension, let $\beta_1$ be the $\alpha_i \in A$ with the smallest index $i$. Then by assumption, there is some $m_1$ such that $\beta^{m_1} \in K[\alpha_1, \cdots, \alpha_{i-1} \subset F$, so we can construct $F_1 := E[\beta_1]$ which will be a radical extension.

Inductively letting $A_2 = A \setminus \{\beta_1\}$ and repeating this process to construct $L_2$ will yield radical extensions at every step, and since $A$ is finite, there is some $n$ such that $L_n = L$. But then $L$ is a radical extension over $E$ as desired.

### 1.6 Problem 6

#### 1.6.1 Part 2

The normal closure $L$ of $K$ is defined as the smallest extension of $K$ such that if $\alpha$ is a root of any irreducible polynomial in $K[x]$ and $\alpha \in L$, then all of its conjugates are in $L$ as well. But this means any such polynomial splits in $L$. In particular, if $u \in L$, then $f$ splits in $L$, and so $L$ contains the splitting field $F$.

#### 1.6.2 Part 3

## 2 Qual Problems

### 2.1 Problem 1

#### 2.1.1 Part 1

If $L/K$ is a finite field extension which is both separable and a splitting field of some polynomial in $K[x]$, then $[L : K] = |\mathrm{Gal}|L/K$.

#### 2.1.2 Part 2

The extension $\mathbb{Q}(\zeta_{43})$ is the splitting field of the cyclotomic polynomial $\Phi_{43}(x) = \sum_{i=1}^{4} 2x^i$, which is degree $\phi(43) = 42$ since 43 is prime.

Moreover, the Galois group is isomorphic to $\mathbb{Z}_{43}^\times \cong \mathbb{Z}_{42}$.

#### 2.1.3 Part 3

Since proper subfields will correspond to intermediate extensions which will correspond to subgroups of the Galois group, this problem is reduced to counting the number of distinct subgroups of $\mathbb{Z}_{42}$. This is a cyclic group, so there is exactly one subgroup of order $d$ for each $d$ dividing 42. Since $42 = 2 * 3 * 7$, we have

- A subgroup of order 2, corresponding to a field extension of degree 21,
- A subgroup of order 3, corresponding to a field extension of degree 14,
- A subgroup of order 6, corresponding to a field extension of degree 7,
- A subgroup of order 7, corresponding to a field extension of degree 6,
- A subgroup of order 14, corresponding to a field extension of degree 3,
- A subgroup of order 21, corresponding to a field extension of degree 2.

### 2.2 Problem 2

#### 2.2.1 Part 1

A splitting field of $f$ over $F$ is an extension $L \geq F$ that contains every root of $f$, so that $f$ can be decomposed as a product of linear factors i.e. $f(x) = \prod_{i=1}^{\deg f} (x - \alpha_i)^{m_i}$ in $L[x]$.

### 2.2.2 Part 2

If $E \geq F$ is a finite extension, then it is algebraic and $E = F[\alpha_1, \cdots, \alpha_n]$. So we can let $g(x) = \prod_{i=1}^{n}(x - \alpha_i)$. By construction, each $\alpha_i$ is a root, and so $E$ is a splitting field for $g$.

### 2.2.3 Part 3

Since $E$ was shown to be a splitting field, it only remains to show that it is separable. But this follows from the fact that each $\alpha_i$ is a separable *element*, since their minimal polynomial over $F$ is $g$. So $E$ is a Galois extension.

## 2.3 Problem 3

### 2.3.1 Part 1

False: take $K \leq L \leq M$ as $\mathbb{Q} \leq \mathbb{Q}(\sqrt[3]{2}) \leq \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. Then $M$ is the splitting field of $x_3 - 2$, and in characteristic zero is thus Galois. But $L$ is not the splitting field of any irreducible polynomial in $\mathbb{Q}[x]$, so it is *not* Galois.

### 2.3.2 Part 2

This is true. By the Galois correspondence, it suffices to show that $H \coloneqq \mathrm{Gal}(M/L)$ is a normal subgroup of $G \coloneqq \mathrm{Gal}(M/K)$. To that end, let $\phi \in G$, so $\phi : M \to M$ is a lift of $\mathrm{id}_K$. Then $H \trianglelefteq G$ iff $\phi H \phi^{-1} = H$. Letting $\sigma \in H$, we need to show that

$$(\phi^{-1} \circ \sigma \circ \phi)(L) = L,$$

i.e. that this composition is some automorphism of $M$ that fixes $L$.

Consider how this acts on elements of $L$. If $\ell \in L$, then $\ell = \sum k_i \ell_i$ since $L$ is a finite-degree extension, thus algebraic, thus spanned by some basis $\ell_i \in L$ as a vector space over $K$.

In particular, since $\phi$ is some $M$-automorphism, it restricts to an $L$-automorphism, which must send each $\ell_i$ to some conjugate $\ell_i'$. Similarly, $\phi^{-1}(\ell_i') = \ell_i$.

We thus have

$$\begin{aligned}
(\phi^{-1}\sigma\phi)(a) &= (\phi^{-1}\sigma\phi)(\sum k_i \ell_i) \\
&= (\phi^{-1}\sigma)(\sum k_i \phi(\ell_i)) \\
&= (\phi^{-1}\sigma)(\sum k_i \ell_i') \\
&= (\phi^{-1})(\sum k_i \sigma(\ell_i')) \\
&= (\phi^{-1})(\sum k_i \ell_i') \quad \text{since } \sigma \text{ fixes } L \\
&= \sum k_i \phi^{-1}(\ell_i') \\
&= \sum k_i \ell_i
\end{aligned}$$

,

and so this composite fixes $L$ as desired. This $H \trianglelefteq G$, which is what we wanted to show.