Problem Set 2

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Definition: A space X is connected iff it can not be written as $X = U \coprod V$ with U, V nonempty, disjoint, and open.

1 Problem 19

Prove that the product of two connected spaces is connected.

Solution:

Use the fact that a union of spaces containing a common point is still connected. Fix a point $(a, b) \in X \times Y$. Since the horizontal slice $X_b \coloneqq X \times \{b\}$ is homeomorphic to X which is connected, as are all of the vertical slices $Y_x \coloneqq \{x\} \times Y \cong Y$ (for any x), the "T-shaped" space $T_x \coloneqq X_b \bigcup Y_x$ is connected for each x.

Note that $(a, b) \in T_x$ for every x, so $\displaystyle\bigcup_{x \in X} T_x = X \times Y$ is connected.

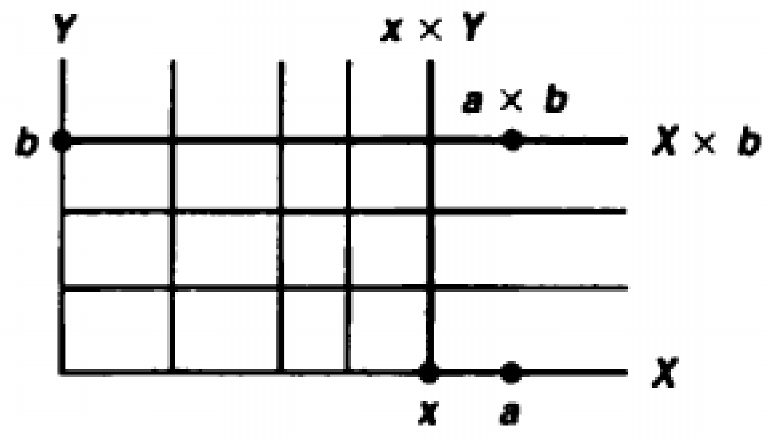

Figure 23.2

2 Problem 22

?

3 Problem 23

Note: this is precisely the cofinite topology.

1. $\mathbb{R} \in \tau$ since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is trivially a finite set, and $\emptyset \in \tau$ by definition.
2. If $U_i \in \tau$ then $(\bigcup\limits_i U_i)^c = \bigcap U_i^c$ is an intersection of finite sets and thus finite, so $\bigcup\limits_i U_i \in \tau$.
3. If $U_i \in \tau$, then $(\bigcap\limits_{i=1}^n U_i)^c = \bigcup\limits_{i=1}^n U_i^c$ is a finite union of finite sets and thus finite, so $\bigcap U_i \in \tau$.

So τ forms a topology.

To see that $(\mathbb{R}, \tau)$ is compact, let $\{U_i\} \rightrightarrows \mathbb{R}$ be an open cover by elements in τ.

Fix any U_α, then $U_\alpha^c = \{p_1, \cdots, p_n\}$ is finite, say of size n. So pick $U_1 \ni p_1, \cdots, U_n \ni p_n$; then $\mathbb{R} \subset U_\alpha \bigcup\limits_{i=1}^n U_i$ is a finite cover.

4 Problem 27

4.1 a

$\Longrightarrow$:

If X is totally bounded, let $\varepsilon = \frac{1}{n}$ for each n, and let $\{x_i\}$ be an arbitrary sequence. For $n = 1$, pick a finite open cover $\{U_i\}_n$ such that $\text{diam}U_i < \frac{1}{n}$ for every i.

Choose V_1 such that there are infinitely many $x_i \in V_1$. (Why?) Note that $\text{diam}V_i < 1$. Now choose $x_i \in V_1$ arbitrarily and define it to be y_1.

Then since V_1 is totally bounded, repeat this process to obtain $V_2 \subseteq V_1$ with $\text{diam}(V_2) < \frac{1}{2}$, and choose $x_i \in V_2$ arbitrarily and define it to be y_2.

This yields a nested family of sets $V_1 \supseteq V_2 \supseteq \cdots$ and a sequence $\{y_i\}$ such that $d(y_i, y_j) < \max(\frac{1}{i}, \frac{1}{j}) \to 0$, so $\{y_i\}$ is a Cauchy subsequence.

$\Longleftarrow$:

Then fix $\varepsilon > 0$ and pick x_1 arbitrarily and define $S_1 = B(\varepsilon, x_1)$. Then pick $x_2 \in S_1^c$ and define $S_2 = S_1 \bigcup B(\varepsilon, x_2)$, and so on. Continue by picking $x_{n+1} \in S_n^c$ (Since X is not totally bounded, this can always be done) and defining $S_{n+1} = S_n \bigcup B(\varepsilon, x_{n+1})$.

Then $\{x_n\}$ is not Cauchy, because $d(x_i, x_j) > \varepsilon$ for every $i \neq j$.

4.2 b

Take $X = C^0([0,1])$ with the sup-norm, then $f_n(x) = x^n$ are all bounded by 1, but $\|f_i - f_j\| = 1$ for every i, j, so no subsequence can be Cauchy, so X can not be totally bounded.

Moreover, $\{f_n\}$ is closed. (Why?)

5 Problem 30

Let $A \subset X$ be compact, and pick a fixed $x \in X \setminus A$. Since X is Hausdorff, for arbitrary $a \in A$, there exists opens $U_a \ni a$ and $U_{x,a} \ni x$ such that $V_a \bigcap U_{x,a} = \emptyset$. Then $\left\{U_a \;\middle|\; a \in A\right\} \rightrightarrows A$, so by compactness there is a finite subcover $\{U_{a_i}\} \rightrightarrows A$.

Now take $U = \bigcup_i U_{a_i}$ and $V_x = \bigcap_i V_{a_i,x}$, so $U \bigcap V = \emptyset$. Note that both U and V_x are open.

But then defining $V \coloneqq \bigcup_{x \in X \setminus A} V_x$, we have $X \setminus A \subset V$ and $V \bigcap A = \emptyset$, so $V = X \setminus A$, which is open and thus A is closed.