

Title

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1 | Lecture 07

Last time: stalks, sheafification, and $\mathrm{Sh}(X_{\text{ét}})$ is abelian. Next up, we're aiming to define sheaf cohomology for $\mathrm{Sh}(X_{\text{ét}})$.

Remark 1.0.1 (Esoteric!): Related to a question asked by a viewer: there is not in fact a morphism from $X_{\text{fppf}} \rightarrow X_{\text{ét}}$, since locally finitely-presented need not be finitely presented (part of the condition for fppf). There is instead a morphism $X_{\text{fppf}} \rightarrow X_{\text{ét,fp}}$ to a corresponding finitely presented site. There is also a map $X_{\text{ét}} \rightarrow X_{\text{ét,fp}}$ inducing an equivalence on the category of sheaves via pushforward. 

Theorem 1.0.2 (Enough injectives).

$\mathrm{Sh}(X_{\text{ét}})$ has enough injectives.

Proof (?).

Given $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ we want an injective sheaf \mathcal{I} and an injection $\mathcal{F} \hookrightarrow \mathcal{I}$. For each $x \in X$, choose a geometric point \bar{x} over x , and let $I(\bar{x})$ be an injective \mathbb{Z} -module with a map $\mathcal{F}_{\bar{x}} \rightarrow I(\bar{x})$. These exist because the category of \mathbb{Z} -modules has enough injectives. The injectives in this category are **divisible** abelian groups.

Claim: The following object works:

$$\mathcal{I} := \prod_{\bar{x}} (\iota_{\bar{x}})_* I(\bar{x}).$$

We need to check

1. There is a map $\mathcal{F} \rightarrow \mathcal{I}$: The RHS is a product, so we map into the components. $\mathcal{F}_{\bar{x}}$ maps into its own associated skyscraper sheaf where the map is sending sections to their germs. Then the skyscraper sheaf for $\mathcal{F}_{\bar{x}}$ maps into the skyscraper sheaf for $I(\bar{x})$ by pushforward.
2. This is a monomorphism: check on stalks.
3. \mathcal{I} is injective: check the lifting property directly.

■

1.1 What Else We Get From Sheafification

Remark 1.1.1: We now know that $\mathrm{Sh}(X_{\text{ét}})$ is abelian with enough injectives. This is true for $\mathrm{Sh}(\tau)$ for any site τ , but this is substantially harder to show.

1.1.1 Inverse Images

For $f : X \rightarrow Y$, we have a map on presheaves

$$f^{-1} : \text{Presh}(Y_{\text{ét}}) \rightarrow \text{Presh}(X_{\text{ét}})$$

$$\mathcal{F}(V \xrightarrow{\text{ét}} X) \mapsto \varinjlim \mathcal{F}(U \rightarrow X),$$

where the limit is over diagrams of the form

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow \text{ét} & & \downarrow \text{ét} \\ X & \longrightarrow & Y \end{array}$$

Fact 1.1.2: f^{-1} is left adjoint to pushforward as functors on presheaves.

Exercise 1.1.3(?): Check this.

Definition 1.1.4 (Inverse Image Sheaf)

$$f^* \mathcal{F} := (f^{-1} \mathcal{F})^a.$$

Theorem 1.1.5(?).

f^* is left adjoint to f_* .

Proof (?).

Sheafification is a left adjoint. ■

Example 1.1.6(?):

- For $\bar{x} \xrightarrow{\iota} X$ a geometric point, we have $\iota^* \mathcal{F} = \mathcal{F}_{\bar{x}}$.
- For $Y \xrightarrow{f} X$, we have $f^* \underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$.
- More generally, for $Y \xrightarrow{f} X$ and any representable functor $\mathcal{F} := \underline{\text{hom}}_X(\cdot, Z)$, we have $f^* \mathcal{F} = \underline{\text{hom}}_Y(\cdot, Y \times_X Z)$.

1.2 Étale Cohomology

See ?? for the definition of étale cohomology. How do we compute $H^i(X_{\text{ét}}, \mathcal{F})$? Choose an injective resolution

$$\mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$$

with the \mathcal{I}^j injectives. From the general theory of derived functors, we obtain

$$H^i(X_{\text{ét}}, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)),$$

where the RHS is a complex of abelian groups. Injective resolutions are difficult to find in general. Suppose $\pi : X_{\text{ét}} \rightarrow Y_{\text{ét}}$ comes from a map of schemes, then we can compute derived functors of other functors such as the pushforward,

$$(R^i \pi_*) \mathcal{F} = H^i(\pi_* \mathcal{I}^\bullet),$$

where the RHS are sheaves on $Y_{\text{ét}}$. Implicit here is the claim that π_* is left-exact. You can also find $(L^{>0} \pi^*) \mathcal{G} = 0$.

Exercise 1.2.1 (?): Check that pullback is exact.

Proposition 1.2.2 (Properties of étale cohomology).

1. $H^0(X_{\text{ét}})$