# Real Analysis

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# 1 Lecture 1 (Thu 15 Aug 2019 11:04)

> See Folland's Real Analysis, definitely a recommEnded reference.

Possible first day question: how can we "measure" a subset of $\mathbb{R}$? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of $\mathbb{R}$. We'll come back to this in a few lectures.

## 1.1 Notions of "smallness" in $\mathbb{R}$

**Definition 1.** Let $E$ be a set, then $E$ is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes $\emptyset, \mathbb{N}$.

**Definition 2.** A set $E$ is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

**Exercise 1.** Show that any finite subset of $\mathbb{R}$ is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in $Y$ iff the closure of $X$ is all of $Y$. So we'll make the following definition.

**Definition 3.** A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval $I$ contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. For example, $\mathbb{Q}$ is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently,

- $A^c$ contains a dense, open set.
- The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than $\varepsilon$.

**Definition 4.** Definition: $E$ is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

**Exercise 2.** Show that a countable union of null sets is null.

We have several relationships

- Countable $\implies$ Meager, but not the converse.
- Countable $\implies$ Null, but not the converse.

**Exercise 3.** Show that the "middle third" Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

**Theorem 1.** Every $E \subseteq \mathbb{R}$ can be written as $E = A \coprod B$ where $A$ is null and $B$ is meager.

> This gives some information about how nullity and meagerness interact – in particular, $\mathbb{R}$ itself is neither meager nor null. Idea: if meager $\implies$ null, this theorem allows you to write $\mathbb{R}$ as the union of two null sets. This is bad!

*Proof.* We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Around each $q_j$, put an interval around it of size $1/2^{j+k}$ where we'll allow $k$ to vary, yielding multiple intervals around $q_j$. To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that $A$ is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick $k$ such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that $A^c$ :
$B$ is meager. Note that $G_k$ covers the rationals, and is a countable union of open sets, so it is dense. So $G_k$ is an open and dense set. By one of the equivalent formulations of meagerness, this means that $G_k^c$ is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager. $\square$

## 1.2 $\mathbb{R}$ is not small

**Theorem 2.**

- A (Cantor): $\mathbb{R}$ is not countable.
- B (Baire): $\mathbb{R}$ is not meager. (Baire Category Theorem)
- C (Borel): $\mathbb{R}$ is not null.

Note that theorems B and C imply theorem A. You can also replace $\mathbb{R}$ with any nonempty interval $I = [a, b]$ where $a < b$. This is a strictly stronger statement – if any subset of $\mathbb{R}$ is not countable, then certainly $\mathbb{R}$ isn't, and so on.

*Proof of (A).* Begin by thinking of $I = [0, 1]$, then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry. □

*A second proof of (A).* Take an interval $I$, and suppose it is countable so $I = \{x_i\}$. Choose $I_1 \subseteq I$ that avoids $x_1$, so $x_1 \notin I_1$. Choose $I_2 \subseteq I_1$ avoiding $x_2$ and so on to produce a nested sequence of closed intervals. Since $\mathbb{R}$ is complete, the intersection $\bigcap_{n=1}^{\infty} I_n$ is nonempty, so say it contains $x$. But then $x \in I_1 \in I$, for example, but $x \neq x_i$ for any $i$, so $x \notin I$, a contradiction. □ □

*Proof of (B).* Suppose $I = \bigcup_{i=1}^{\infty} A_n$ where each $A_n$ is nowhere dense. We'll again construct a nested sequence of closed sets. Let $I_1 \subseteq I$ be a subinterval that misses all of $A_1$, so $A_1 \bigcap I_1 = \emptyset$ using the fact that $A_1$ is nowhere dense. Repeat the same process, let $I_2 \subset I_1 \setminus A_2$. By the nested interval property, there is some $x \in \bigcap A_i$. □

> Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of $\mathbb{R}$ is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

**Exercise 4.** iasdsadsad

Consider the Thomae function: it is continuous on $\mathbb{Q}$, but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Can this be switched to get some function $f$ that is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$? The answer is no. The set of discontinuities of a function is *always* an $F_\sigma$ set, and $\mathbb{R} \setminus \mathbb{Q}$ is not one. Equivalently, the rationals are not a $G_\delta$ set.

Some facts:

- The pointwise limit of continuous functions has a meager set of discontinuities.
- If $f$ is integrable, the set of discontinuities is null.
- If $f$ is monotone, they are countable.
- There is a continuous nowhere differentiable function: let $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$, and in fact *most* functions are like this.
- If $f$ is continuous and monotone, the discontinuities are null.

**Theorem 3.** Let $I = [a, b]$. If $I \subseteq \bigcup_{i=1}^{\infty} I_i$, then $|I| \leq \sum_{i=1}^{\infty} |I_i|$.

*Proof.* The proof is by induction. Assume $I \subseteq \bigcup_{n}^{N+1} I_n$, where wlog we can assume that $a < a_{N+1} < b \leq b_{N+1}$, then $[a, a_{N+1}] \subset \bigcup_{n=1}^{N} I_n$ so the inductive hypothesis applies. But then $b - a \leq b_{N+1} - a = (b_{N+1} - a_{N+1}) + (a_{N+1} - a) \leq \sum_{n=1}^{N+1} |I_n|$. □

> Note that this proves that the reals are uncountable!

# 2 Lecture 2