Solutions to Point-Set Qual Questions

D. Zack Garza

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Let π_X, π_Y denote the canonical projections, which we can note are continuous and preserve open sets.

$\implies$: Suppose $X \times Y$ is compact, and let $\{U_\alpha\}, \{V_\beta\}$ be open covers of X and Y respectively.

Let $T_{\alpha\beta} = U_\alpha \times V_\beta$; then $\{T_{\alpha\beta}\}$ is an open cover of $X \times Y$. So there is a finite subcover $\{T_{ij}\}$, $\{\pi_X(T_{ij})\}$ is an open cover of X, and similarly for Y. So both X, Y are compact.

$\impliedby$: Suppose X and Y are compact, and let $U_\alpha \rightrightarrows X \times Y$ be an open cover. Let $\pi_Y : X \times Y \to Y$ be the canonical projection; then $\{\pi_Y(U_\alpha)\} \rightrightarrows Y$ and by compactness of Y there is a finite subcover of the form $\left\{\pi_Y(U_i) \;\middle|\; 1 \leq i \leq n\right\}$. Then $\{V_{x,i} \coloneqq \{x\} \times U_i\}$ is an open cover of $\{x\} \times Y$ for any fixed x.

So if we fix an $x \in X$, we can let $V_{x,i} \rightrightarrows \{x\} \times Y$ be any finite subcollection covering this slice. By the Tube Lemma, there is an open set W_x such that $\{x\} \times Y \subset W_x \times Y \subset \bigcup V_{x,i} = \{x\} \times Y$.

Then $\{W_x\} \rightrightarrows X$ as x varies is an open cover of X, and by compactness of X, there are finitely many $x_j \in X$ such that $W_{x_j} \rightrightarrows X$. But then $X \times Y = \bigcup_j W_{x_j} \times Y = \bigcup_j \bigcup_i W_{x_j} \times V_{x_j,i} \subset \bigcup_\alpha U_\alpha$ is a finite cover.

Todo: Prove tube lemma.

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Let $X = A \bigcup B$ with $A = \left\{ (0, y) \;\middle|\; y \in [-1, 1] \right\}$ and $B = \left\{ (x, \sin(1/x)) \;\middle|\; x \in (0, 1] \right\}$. Since B is the graph of a continuous function, which is always connected. Moreover, $X = \overline{A}$, and the closure of a connected set is still connected.

> Alternative direct argument: the subspace $X' = B \bigcup \{\mathbf{0}\}$ is not connected. If it were, write $X' = U \coprod V$, where wlog $\mathbf{0} \in U$. Then there is an open such that $\mathbf{0} \in N_r(\mathbf{0}) \subset U$. But any neighborhood about zero intersects B, so we must have $V \subset B$ as a strict inclusion. But then $U \bigcap B$ and V disconnects B, a connected set, which is a contradiction.

To see that X is not path-connected, suppose toward a contradiction that there is a continuous function $f : I \to X \subset \mathbb{R}^2$. In particular, f is continuous at $\mathbf{0}$, and so

$$\forall \varepsilon \quad \exists \delta \;\middle|\; \|\mathbf{x}\| < \delta \implies \|f(\mathbf{x})\| < \varepsilon.$$

where the norm is the standard Euclidean norm.

However, we can pick $\varepsilon < 1$, say, and consider points of the form $\mathbf{x}_n = (\frac{1}{2n\pi}, 0)$. In particular, we can pick n large enough such that $\|\mathbf{x}_n\|$ is as small as we like, whereas $\|f(\mathbf{x}_n)\| = 1 > \varepsilon$ for all n, a contradiction.

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Consider the (continuous) projection $\pi : \mathbb{R}^2 \to \mathbb{RP}^1$ given by $(x, y) \mapsto [y/x, 1]$ in homogeneous coordinates. (I.e. this sends points to lines through the origin with rational slope).

Note that the image of π is $\mathbb{RP}^1 \setminus \{\infty\}$, which is homeomorphic to $\mathbb{R}$.

If we now define $f = \pi|_X$, we have $f(X) \twoheadrightarrow \mathbb{Q} \subset \mathbb{R}$. If X were connected, then $f(X)$ would also be connected, but $\mathbb{Q} \subset \mathbb{R}$ is disconnected, a contradiction.

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Let $X := \bigcup_\alpha X_\alpha$, and let $p \in \bigcap X_\alpha$. Suppose toward a contradiction that $X = A \coprod B$ with A, B nonempty, disjoint, and relatively open as subspaces of X. Wlog, suppose $p \in A$, so let $q \in B$ be arbitrary.

Then $q \in X_\alpha$ for some α, so $q \in B \bigcap X_\alpha$. We also have $p \in A \bigcap X_\alpha$.

But then these two sets disconnect X_α, which was assumed to be connected – a contradiction.

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> Lemma: X is connected iff the only subsets of X that are closed and open are $\emptyset, X$.

If $S \subset X$ is not connected, then there exists a subset $A \subset S$ that is both open and closed in the subspace topology, where $A \neq \emptyset, S$.

Suppose S is not connected, then choose A as above. Then $B = S \setminus A$ yields a pair A, B that disconnects S. Since A is closed in S, $\overline{A} = A$ and thus $\overline{A} \cap B = A \cap B = \emptyset$. Similarly, since A is open, B is closed, and $\overline{B} = B \implies \overline{B} \cap A = B \cap A = \emptyset$.