

# Problem Set 7

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## Contents

<b>1</b>	<b>Problem 1</b>	<b>1</b>
1.1	Part a . . . . .	1
1.2	Part b . . . . .	2
<b>2</b>	<b>Problem 2</b>	<b>3</b>
2.1	Part a: . . . . .	3
2.2	Part b: . . . . .	4
<b>3</b>	<b>Problem 3</b>	<b>4</b>
3.1	Part a . . . . .	4
3.2	Part b . . . . .	5
<b>4</b>	<b>Problem 4</b>	<b>5</b>
<b>5</b>	<b>Problem 5</b>	<b>5</b>
5.1	Part 1 . . . . .	5
5.2	Part b . . . . .	7

## 1 Problem 1

### 1.1 Part a

We want to show that  $\ell^2(\mathbb{N})$  is complete, so let  $\{x_n\} \subseteq \ell^2(\mathbb{N})$  be a Cauchy sequence, so  $\|x^j - x^k\|_{\ell^2} \rightarrow 0$ . We want to produce some  $\mathbf{x} := \lim_{n \rightarrow \infty} x^n$  such that  $x \in \ell^2$ .

To this end, for each fixed index  $i$ , define

$$\mathbf{x}_i := \lim_{n \rightarrow \infty} x_i^n.$$

This is well-defined since  $\|x^j - x^k\|_{\ell^2} = \sum_i |x_i^j - x_i^k|^2 \rightarrow 0$ , and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed  $i$ , the sequence  $|x_i^j - x_i^k|^2$  is a Cauchy sequence of real numbers which necessarily converges in  $\mathbb{R}$ .

We also have  $\|\mathbf{x} - x^j\|_{\ell^2} \rightarrow 0$  since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\| \lim_{k \rightarrow \infty} x^k - x^j \right\|_{\ell^2} = \lim_{k \rightarrow \infty} \|x^k - x^j\|_{\ell^2} \rightarrow 0$$

where the limit can be passed through the norm because the map  $t \mapsto \|t\|_{\ell^2}$  is continuous. So  $x^j \rightarrow \mathbf{x}$  in  $\ell^2$  as well.

It remains to show that  $\mathbf{x} \in \ell^2(\mathbb{N})$ , i.e. that  $\sum_i |\mathbf{x}_i|^2 < \infty$ . To this end, we write

$$\begin{aligned} \|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\ &\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\ &\rightarrow M < \infty, \end{aligned}$$

where  $\|\mathbf{x}_i - x^j\|_{\ell^2} \rightarrow 0$  and the second sum is finite because  $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$ .  $\square$

## 1.2 Part b

Let  $H$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ .

**Lemma:** For any complex number  $z$ , we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, we have

$$\Re(\langle x, iy \rangle) = \Re(-i\langle x, y \rangle) = \Im(\langle x, y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, y \rangle)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, y \rangle)$$

$$\begin{aligned} \|x + iy\|^2 &= \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

$$\begin{aligned} \|x - iy\|^2 &= \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

and summing these all

$$\begin{aligned}\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 &= 4 \Re(\langle x, y \rangle) + 4i \Im(\langle x, y \rangle) \\ &= 4\langle x, y \rangle.\end{aligned}$$

To conclude that a linear map  $U$  is an isometry iff  $U$  is unitary, if we assume  $U$  is unitary then we can write

$$\|x\|^2 := \langle x, x \rangle = \langle Ux, Ux \rangle := \|Ux\|^2.$$

Assuming now that  $U$  is an isometry, by the polarization identity we can write

$$\begin{aligned}\langle Ux, Uy \rangle &= \frac{1}{4} \left( \|Ux + Uy\|^2 + \|Ux - Uy\|^2 + i\|Ux + iUy\|^2 - i\|Ux - iUy\|^2 \right) \\ &= \frac{1}{4} \left( \|U(x + y)\|^2 + \|U(x - y)\|^2 + i\|U(x + iy)\|^2 - i\|U(x - iy)\|^2 \right) \\ &= \frac{1}{4} \left( \|x + y\|^2 + \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right) \\ &= \langle x, y \rangle.\end{aligned}$$

□

## 2 Problem 2

Lemma: The map  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$  is continuous.

Proof:

Let  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then

$$\begin{aligned}|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\ &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \\ &\rightarrow 0 \cdot M + C \cdot 0 < \infty,\end{aligned}$$

where  $\|y_n\| \rightarrow M$  since  $y_n \rightarrow y$  implies that  $\|y_n\|$  is bounded.

### 2.1 Part a:

Using the lemma, letting  $\{e_n\}$  be a sequence in  $E^\perp$ , so  $y \in E \implies \langle e_n, y \rangle = 0$ . Since  $H$  is complete,  $e_n \rightarrow e \in H$ ; we can show that  $e \in E^\perp$  by letting  $y \in E$  be arbitrary and computing

$$\langle e, y \rangle = \left\langle \lim_n e_n, y \right\rangle = \lim_n \langle e_n, y \rangle = \lim_n 0 = 0,$$

so  $e \in E^\perp$ .

## 2.2 Part b:

Let  $S := \text{span}_H(E)$ ; then the smallest closed subspace containing  $E$  is  $\overline{S}$ , the closure of  $S$ . We will proceed by showing that  $E^{\perp\perp} = \overline{S}$ .

$\overline{S} \subseteq E^{\perp\perp}$ :

Let  $\{x_n\}$  be a sequence in  $S$ , so  $x_n \rightarrow x \in \overline{S}$ .

First, each  $x_n$  is in  $E^{\perp\perp}$ , since if we write  $x_n = \sum a_i e_i$  where  $e_i \in E$ , we have

$$y \in E^\perp \implies \langle x_n, y \rangle = \left\langle \sum_i a_i e_i, y \right\rangle = \sum_i a_i \langle e_i, y \rangle = 0 \implies x_n \in (E^\perp)^\perp.$$

It remains to show that  $x \in E^{\perp\perp}$ , which follows from

$$y \in E^\perp \implies \langle x, y \rangle = \left\langle \lim_n x_n, y \right\rangle = \lim_n \langle x_n, y \rangle = 0 \implies x \in (E^\perp)^\perp,$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$ :

For notation convenience, we'll just write  $S$  for  $\overline{S}$ . Let  $x \in E^{\perp\perp}$ . Noting that  $S$  is closed, we can define  $P$ , the operator projecting elements onto  $S$ , and write

$$x = Px + (x - Px) \in S \oplus S^\perp$$

But since  $\langle x, x - Px \rangle = 0$  because  $x - Px \in E^\perp$  and  $x \in (E^\perp)^\perp$ , we can rewrite the first term in this inner product to obtain

$$0 = \langle x, x - Px \rangle = \langle Px + (x - Px), x - Px \rangle = \langle Px, x - Px \rangle + \langle x - Px, x - Px \rangle,$$

where we can note that the first term is zero because  $Px \in S$  and  $x - Px \in S^\perp$ , and the second term is  $\|x - Px\|^2$ .

But this says  $\|x - Px\|^2 = 0$ , so  $x - Px = 0$  and thus  $x = Px \in S$ , which is what we wanted to show.

## 3 Problem 3

### 3.1 Part a

We compute

$$\begin{aligned} \|e_0\|^2 &= \int_0^1 1^2 dx = 1 \\ \|e_1\|^2 &= \int_0^1 3(2x-1)^2 dx = \frac{1}{2}(2x-1)^2 \Big|_0^1 = 1 \\ \langle e_0, e_1 \rangle &= \int_0^1 \sqrt{3}(2x-1) dx = \frac{\sqrt{3}}{4}(2x-1) \Big|_0^1 = 0. \end{aligned}$$

which verifies that this is an orthonormal system.

### 3.2 Part b

We first note that this system spans the degree 1 polynomials in  $L^2([0, 1])$ , since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from  $\{1, x\}$  to  $\{e_0, e_1\}$  which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial  $f$  to  $x^3$  is given by the projection onto this subspace, and since  $\{e_i\}$  is orthonormal this is given by

$$\begin{aligned} f(x) &= \sum_i \langle x^3, e_i \rangle e_i \\ &= \langle x^3, 1 \rangle 1 + \langle x^3, \sqrt{3}(2x-1) \rangle \sqrt{3}(2x-1) \\ &= \int_0^1 x^2 dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1) dx \\ &= \frac{1}{3} + \sqrt{3}(2x-1) \frac{\sqrt{3}}{6} \\ &= x - \frac{1}{6}. \end{aligned}$$

We can also compute

$$\begin{aligned} \|f - g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2 dx \\ &= \frac{1}{180} \\ \implies \|f - g\|_2 &= \frac{1}{\sqrt{180}}. \end{aligned}$$

## 4 Problem 4

Todo

## 5 Problem 5

### 5.1 Part 1

We use the following algorithm: given  $\{v\}_i$ , we set

- $e_1 = v_1$ , and then normalize to obtain  $\hat{e}_1 = e_1 / \|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i, \hat{e}_k \rangle \hat{e}_k$

The result set  $\{\hat{e}_i\}$  is the orthonormalized basis.

We set  $e_1 = 1$ , and check that  $\|e_1\|^2 = 2$ , and thus set  $\hat{e}_1 = \frac{1}{\sqrt{2}}$ .

We then set

$$\begin{aligned} e_2 &= x - \langle x, \hat{e}_1 \rangle \hat{e}_1 \\ &= x - \langle x, 1 \rangle 1 \\ &= x - \int_{-1}^1 \frac{1}{\sqrt{2}} x \, dx \\ &= x - \int \text{odd function} \\ &= x, \end{aligned}$$

and so  $e_2 = x$ . We can then check that

$$\|e_2\| = \left( \int_{-1}^1 x^2 \, dx \right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set  $\hat{e}_2 = \sqrt{\frac{3}{2}}x$ .

We continue to compute

$$\begin{aligned} e_3 &= x^2 - \langle x^2, \hat{e}_1 \rangle \hat{e}_1 - \langle x^2, \hat{e}_2 \rangle \hat{e}_2 \\ &= x^2 - \frac{1}{2} \int_{-1}^1 x^2 \, dx - \frac{3}{2} x \int_{-1}^1 x^3 \, dx \\ &= x^2 - \left( \frac{1}{6} x^3 \right) \Big|_{-1}^1 + \frac{3}{2} x \int_{-1}^1 \text{odd function} \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

We can then check that  $\|e_3\|^2 = \frac{8}{45}$ , so we set

$$\begin{aligned} \hat{e}_3 &= \sqrt{\frac{45}{8}} \left( x^2 - \frac{1}{3} \right) \\ &= \frac{1}{2} \sqrt{\frac{45}{2}} \frac{1}{3} (3x^2 - 1) \\ &= \frac{1}{3} \sqrt{\frac{45}{2}} \left( \frac{3x^2 - 1}{2} \right). \end{aligned}$$

In summary, this yields

$$\begin{aligned}\hat{e}_1 &= \frac{1}{\sqrt{2}} \\ \hat{e}_2 &= x \\ \hat{e}_3 &= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2-1}{2}\right),\end{aligned}$$

which are scalar multiples of the first three Legendre polynomials.

## 5.2 Part b

Let  $p(x) = a + bx + cx^2$ , we are then looking for  $p$  such that  $\|x^3 - p(x)\|_2^2$  is minimized. Noting that

$$p(x) \in \text{span}\{1, x, x^2\} = \text{span}\{P_0(x), P_1(x), P_2(x)\},$$

we can conclude that  $p(x)$  will be the projection of  $x^3$  onto this subspace of  $L^2([0, 1])$ . Thus  $p(x) = \sum_{i=0}^2 \langle x^3, \hat{e}_i \rangle \hat{e}_i$ .