

# Full Notes

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## Contents

1	Friday January 10	1
2	Monday January 13th	2
3	Wednesday January 15th	4
3.1	Topology and Algebra of $\mathbb{C}$	4
4	Friday January 17th	6
5	Wednesday January 22nd	9
6	Friday January 24th	11
7	Appendix	14
7.1	Useful Techniques	15

## 1 Friday January 10

Recall that  $\mathbb{C}$  is a field, where  $z = x + iy \implies \bar{z} = x - iy$ , and if  $z \neq 0$  then  $z^{-1} = \bar{z}/|z|^2$ .

**Lemma (Triangle Inequality):**  $|z + w| \leq |z| + |w|$

*Proof:*

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\bar{w}| - \Re z\bar{w}) \geq 0.$$

**Lemma (Reverse Triangle Inequality):**  $||z| - |w|| \leq |z - w|$ .

*Proof:*

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

**Claim:**  $(\mathbb{C}, |\cdot|)$  is a normed space.

**Definition:**  $\lim z_n = z \iff |z_n - z| \rightarrow 0 \in \mathbb{R}$ .

**Definition:** A disc is defined as  $D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$ , and a subset is open iff it contains a disc. By convention,  $D_r$  denotes a disc about  $z_0 = 0$ .

**Definition:**  $\sum_k z_k$  converges iff  $S_N := \sum_{|k| < N} z_k$  converges.

Note that  $z_n \rightarrow z$  and  $z_n = x_n + iy_n$ , and

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon \implies |x - x_n|, |y - y_n| < \varepsilon.$$

Since  $\mathbb{R}$  is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

Note: This is useful precisely when you don't know the limiting term.

Note that  $\sum_k z_k$  thus converges if  $\left| \sum_{k=m}^n z_k \right| < \varepsilon$  for  $m, n$  large enough, so sums converges iff they have small tails.

**Definition:**  $S_N = \sum_{k=1}^N z_k$  converges absolutely iff  $\tilde{S} := \sum_{k=1}^N |z_k|$  converges.

Note that the partial sums  $\sum_{k=1}^N |z_k|$  are monotone, so  $\tilde{S}_N$  converges iff the partial sums are bounded above.

**Definition:** A sum of the form  $\sum_{k=0}^{\infty} a_k z_k$  is a power series.

*Examples:*

$$\sum x^k = \frac{1}{1-x}$$

$$\sum (-x^2)^k = \frac{1}{1+x^2}.$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at  $x = 1$  and the second as a pole at  $x = i$ .

## 2 Monday January 13th

Recall that  $\sum z_k$  converges iff  $s_n = \sum_{k=1}^n z_k$  converges.

**Lemma:** Absolute convergence implies convergence.

The most interesting series:  $f(z) = \sum a_k z^k$ , i.e. power series.

**Divergence lemma:** If  $\sum z_k$  converges, then  $\lim z_k = 0$ .

*Corollary:* If  $\sum z_k$  converges,  $\{z_k\}$  is uniformly bounded by a constant  $C > 0$ , i.e.  $|z_k| < C$  for all  $k$ .

**Proposition:** If  $\sum a_k z_k$  converges at some point  $z_0$ , then it converges for all  $|z| < |z_0|$ .

The inequality is necessarily strict. For example,  $\sum \frac{z^{n-1}}{n}$  converges at  $z = -1$  (alternating harmonic series) but not at  $z = 1$  (harmonic series).

*Proof:* Suppose  $\sum a_k z_1^k$  converges. The terms are uniformly bounded, so  $|a_k z_1^k| \leq C$  for all  $k$ . Then we have

$$|a_k| \leq C/|z_1|^k$$

, so if  $|z| < |z_1|$  we have

$$|a_k z^k| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k.$$

So if  $|z| < |z_1|$ , the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting  $r = |z|/|z_1|$ , we have

$$\sum |a_k z^k| \leq \sum cr^k = \frac{c}{1-r},$$

and so we have absolute convergence. ■

*Exercise (future problem set):* Show that  $\sum \frac{1}{k} z^{k-1}$  converges for all  $|z| = 1$  except for  $z = 1$ . (Use summation by parts.)

**Definition** The radius of convergence is the real number  $R$  such that  $f(z) = \sum a_k z^k$  converges precisely for  $|z| < R$  and diverges for  $|z| > R$ . We denote a disc of radius  $R$  centered at zero by  $D_R$ .

If  $R = \infty$ , then  $f$  is said to be *entire*.

**Proposition:** Suppose that  $\sum a_k z^k$  converges for all  $|z| < R$ . Then  $f(z) = \sum a_k z^k$  is continuous on  $D_R$ , i.e. using the sequential definition of continuity,  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$  for all  $z_0 \in D_R$ .

Recall that  $S_n(z) \rightarrow S(z)$  uniformly on  $\Omega$  iff  $\forall \varepsilon > 0$ , there exists a  $M \in \mathbb{N}$  such that  $n > M \implies |S_n(z) - S(z)| < \varepsilon$  for all  $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample:  $f_n(x) = x^n$  on  $[0, 1]$ ; then  $f_n \rightarrow \delta(1)$ . Note that it uniformly converges on  $[0, 1 - \varepsilon]$  for any  $\varepsilon > 0$ .

*Exercise:* Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

Proof of proposition: Write  $f(z) = \sum_{k=0}^N a_k z^k + \sum_{k=N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$ . Note that if  $|z| < R$ , then there exists a  $T$  such that  $|z| < T < R$  where  $f(z)$  converges uniformly on  $D_T$ .

Check!

We need to show that  $|R_N(z)|$  is uniformly small for  $|z| < s < T$ . Note that  $\sum a_k z^k$  converges on  $D_T$ , so we can find a  $C$  such that  $|a_k z^k| \leq C$  for all  $k$ . Then  $|a_k| \leq C/T^k$  for all  $k$ , and so

$$\begin{aligned} \left| \sum_{k=N+1}^{\infty} a_k z^k \right| &\leq \sum_{k=N+1}^{\infty} |a_k| |z|^k \\ &\leq \sum_{k=N+1}^{\infty} (C/T^k) s^k \\ &= C \sum_{k=N+1}^{\infty} |s/T|^k \\ &= C \frac{r^{N+1}}{1-r} = C \varepsilon_n \rightarrow 0, \end{aligned}$$

which follows because  $0 < r = s/T < 1$ .

So  $S_N(z) \rightarrow f(z)$  uniformly on  $|z| < s$  and  $S_N(z)$  are all continuous, so  $f(z)$  is continuous.

There are two ways to compute the radius of convergence:

- Root test:  $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$ .
- Ratio test:  $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$ .

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

### 3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

#### 3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function  $f : \Omega \rightarrow \mathbb{C}$
2. Complex-differentiability: For  $\Omega \subset \mathbb{C}$  open and  $z_0 \in \Omega$ , there exists  $\varepsilon > 0$  such that  $D_\varepsilon = \{z \mid |z - z_0| < \varepsilon\} \subset \Omega$ , and  $f$  is **holomorphic** (complex-differentiable) at  $z_0$  iff

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(z_0 + h) - f(z_0))$$

exists; if so we denote it by  $f'(z_0)$ .

*Example:*  $f(z) = z$  is holomorphic, since  $f(z+h) - f(z) = z+h - z = h$ , so  $f'(z_0) = \frac{h}{h} = 1$  for all  $z_0$ .

*Example:* Given  $f(z) = \bar{z}$ , we have  $f(z+h) - f(z) = \bar{h}$ , so the ratio is  $\frac{\bar{h}}{h}$  and the limit doesn't exist. Note that if  $h \in \mathbb{R}$ , then  $\bar{h} = h$  and the ratio is identically 1, while if  $h$  is purely imaginary, then  $\bar{h} = -h$  and the limit is identically  $-1$ .

We say  $f$  is holomorphic on an open set  $\Omega$  iff it is holomorphic at every point, and is holomorphic on a closed set  $C$  iff there exists an open  $\Omega \supset C$  such that  $f$  is holomorphic on  $\Omega$ .

If  $f$  is holomorphic, writing  $h = h_1 + ih_2$ , then the following two limits exist and are equal:

$$\begin{aligned} \lim_{h_1 \rightarrow 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\ \lim_{h_2 \rightarrow 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0) \\ &\implies \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}. \end{aligned}$$

So if we write  $f(z) = u(x, y) + iv(x, y)$ , we have

$$\left. \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} = \frac{1}{i} \left. \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \right|_{(x_0, y_0)},$$

and equating real and imaginary parts yields the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ \iff \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{aligned}$$

The usual rules of derivatives apply:

1.  $(\sum f)' = \sum f'$

Proof: Direct.

2.  $(\prod f)' =$  product rule

Proof: Consider  $(f(z+h)g(z+h) - f(z)g(z))/h$  and use continuity of  $g$  at  $z$ .

3. Quotient rule

Proof: Nice trick, write  $q = \frac{f}{g}$  so  $qg = f$ , then  $f' = q'g + qg'$  and  $q' = \frac{f'}{g} - \frac{fg'}{g^2}$ .

4. Chain rule

Proof: Use the fact that if  $f'(g(z)) = a$ , then

$$f(z+h) - f(z) = ah + r(z, h), \quad |r(z, h)| = o(|h|) \rightarrow 0.$$

Write  $b = g'(z)$ , then

$$f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$$

by considering error terms, and so

$$\frac{1}{h}(f(g(z+h)) - f(g(z))) \rightarrow f'(g(z))g'(z)$$

## 4 Friday January 17th

Reference: See Lang's Complex Analysis, there are plenty of solution manuals.

Let  $f; \Omega \rightarrow \mathbb{C}$  be a complex-valued function. Recall that  $f$  is *complex differentiable* iff the usual ratio/limit exists. Note that  $h = x + iy$  and  $h \rightarrow 0 \iff x, y \rightarrow 0$ .

We can write  $f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$ . This follows from Cauchy-Riemann since  $u_x = v_y$  and  $u_y = -v_x$ .

Definition: We want to define  $\partial, \bar{\partial}$  operators. We have the identities

$$x = \frac{z + \bar{z}}{2} \quad y = \frac{z - \bar{z}}{2i}.$$

We can then write

$$\begin{aligned} dz &= dx + i dy \\ d\bar{z} &= dx - i dy. \end{aligned}$$

We define the dual operators by  $\left\langle \frac{\partial}{\partial z}, dz \right\rangle = 1$  and similarly  $\left\langle \frac{\partial}{\partial \bar{z}}, d\bar{z} \right\rangle = 1$ . By the chain rule, we can write

$$\begin{aligned} f_z &= f_x x_z + f_y y_z \\ &= \frac{1}{2} f_x + f_y \frac{1}{2i} \\ &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f, \end{aligned}$$

and similarly  $f_{\bar{z}} = f_x x_{\bar{z}} + f_y y_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{1}{2i} \frac{\partial}{\partial y} \right) f$ .

We thus find  $\partial_x = \partial_z + \partial_{\bar{z}}$  and  $\partial_y = i(\partial_z - \partial_{\bar{z}})$ , and define

$$\begin{aligned}\partial f &= \frac{\partial f}{\partial z} dz \\ \bar{\partial} f &= \frac{\partial f}{\partial \bar{z}} d\bar{z} \\ df &= f_z dz + f_{\bar{z}} d\bar{z}.\end{aligned}$$

Proposition:  $f$  is holomorphic iff  $f_{\bar{z}} = 0$ .

This means that  $f$  depends on  $z$  alone and not  $\bar{z}$ .

Proof:  $\bar{\partial} f = 0$  iff  $\frac{1}{2}(f_x + if_y) = 0$ , so  $(u_x - v_y) + i(v_x + u_y) = 0$ . ■

Application to PDEs: We can write  $u_{xx} = v_{xy}, u_{yy} = v_{yx}$  and so  $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$ . Thus  $\Delta f = 0$ , and  $f$  satisfies Laplace's equation and is said to be *harmonic*.

Corollary: If  $f$  is analytic, then  $u, v$  are both harmonic functions.

**Theorem (Chain Rule):** Let  $w = f(z)$  and  $g(w) = g(f(z))$ . Then

$$\begin{aligned}h_z &= g_w f_z + g_{\bar{w}} \bar{f}_z \\ h_{\bar{z}} &= g_w f_{\bar{z}} + g_{\bar{w}} \bar{f}_{\bar{z}}.\end{aligned}$$

If  $f, g$  are holomorphic,  $f_{\bar{z}} = g_{\bar{w}} = 0$ , so  $h_{\bar{z}} = 0$  and  $h$  is holomorphic and  $h_z = g_w f_z$ .

Example: Given a power series  $f = \sum a_n(z - z_0)^n$ . Then

1. There exists a radius of convergence  $R$  such that  $f$  converges precisely on  $D_R(z_0)$ .
2.  $f$  is continuous on  $D_R(z_0)^\circ$ .
3. By the root test,  $R = (\limsup |a_n|^{1/n})^{-1} = \liminf |a_n/a_{n+1}| = (\limsup |a_{k+1}/a_k|)^{-1}$ .

Recall the ratio test:  $\sum a_k$  converges absolutely iff  $\limsup |a_{k+1}/a_k| < 1$

**Theorem:** If  $f(z) = \sum_{n=0} a_n z^n$  is holomorphic on  $|z| < R$  for  $R > 0$  then  $f'(z) = \sum_{n=1} a_n n z^{n-1}$ .

*Exercise:* Show  $\lim_n n^{\frac{1}{n}} = 1$ . Also tricky: show  $\lim \sin(n)$  doesn't exist, and  $\sin(n)$  is dense in  $[-1, 1]$ .

Proof: Consider  $\limsup |a_n n|^{\frac{1}{n}}$ .

Remark: An analytic function is holomorphic in its domain of convergence, so analytic implies holomorphic. The converse requires Cauchy's integral formula.

Note: look for 13 equivalent statements, Springer GTM Lipman.

Proof: Given  $|z| < R$ , fix  $r > 0$  such that  $|z| < r < R$ . Suppose that  $|w - z| < r - |z|$ , so  $|w| < r$ .

We want to show

$$|S| = \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1} a_n n z^{n-1} \right| \rightarrow 0 \quad \text{as } w \rightarrow z.$$

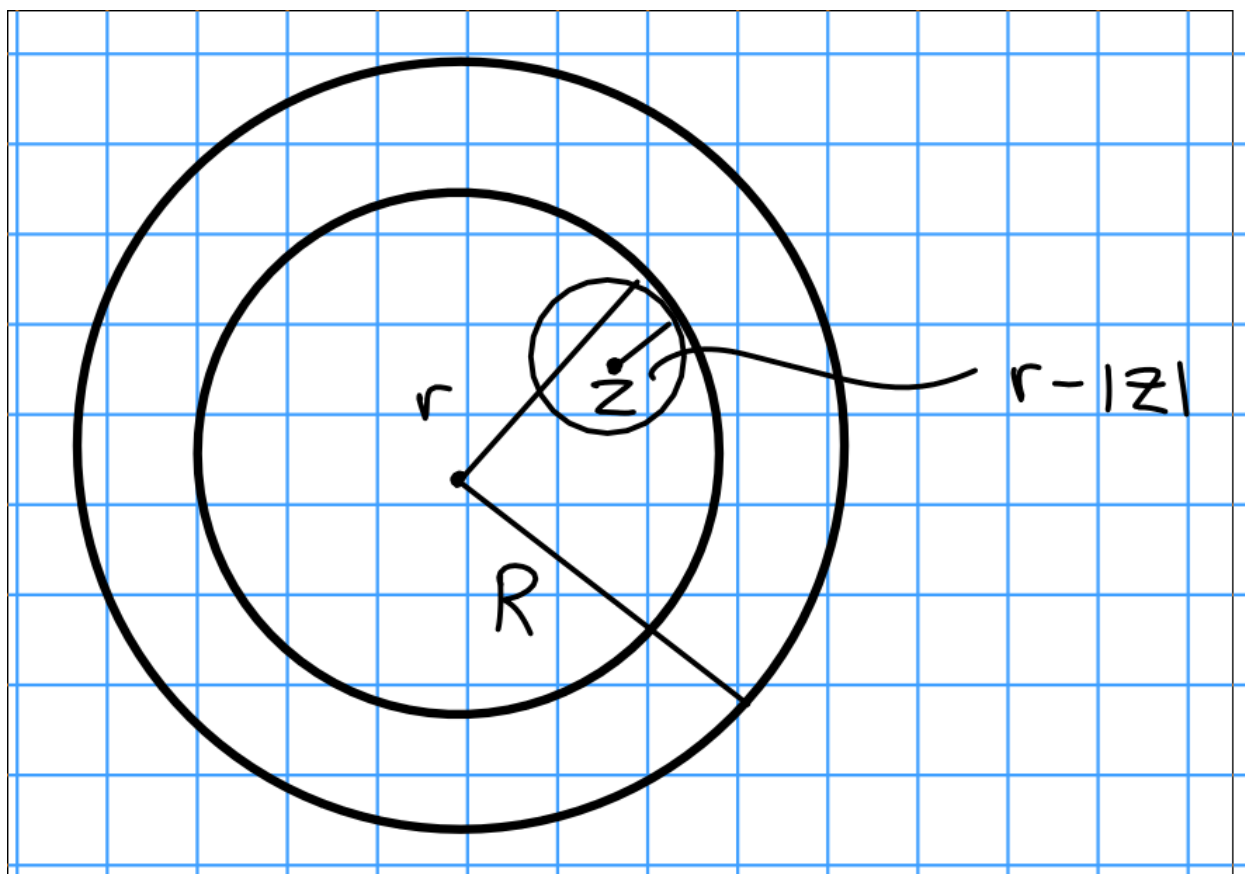


Figure 1: Image



Idea: write everything in terms of power series. Use the fact that  $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots)$ , and so  $\left| \frac{(w^k - z^k)}{(w - z)} \right| \leq kr^{k-1}$ .

$$\begin{aligned}
S &= \sum_{n=1} a_n \left( \frac{w^n - z^n}{w - z} - nz^{n-1} \right) \\
&= \sum_{n=1} a_n (w^{n-1} + w^{n-2}z + \dots + z^{n-1} + nz^{n-1}) \\
&= \sum_{n=1} a_n ((w^{n-1} - z^{n-1}) + (w^{n-2} - z^{n-2})z + \dots + (w - z)z^{n-2}) = \sum_{n=1} a_n (w - z) (\dots + z^{n-2}) \\
&\leq \sum_{n=2} |a_n| \frac{1}{2} n(n-1) r^{n-2} |z - w|.
\end{aligned}$$

■

Next time: trying to prove holomorphic functions are analytic.

## 5 Wednesday January 22nd

Note: multiple complex variables, see Hormander or Steven Krantz

Recall from last time that if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  with  $z_0 \neq 0$  has radius of convergence  $R = (\limsup |a_n|^{1/n})^{-1} > 0$ , then  $f'$  exists and is obtained by differentiating term-by-term. We have  $f$  analytic implies  $f$  holomorphic (and smooth), we want to show the converse. For this, we need integration.

**Definition:** A parameterized curve is a function  $z(t)$  which maps a closed interval  $[a, b] \subset \mathbb{R}$  to  $\mathbb{C}$ .

**Definition:** The curve is said to be smooth iff  $z'$  exists and is continuous on  $[a, b]$ , and  $z'(t) \neq 0$  for any  $t$ . At the boundary  $\{a, b\}$ , we define the derivative by taking one-sided limits.

**Definition:** A curve is said to be piecewise smooth iff  $z(t)$  is continuous on  $[a, b]$  and there are  $a < a_1 < \dots < a_n = b$  with  $z$  smooth on each  $[a_k, a_{k+1}]$ .

Note: may fail to have tangent lines at  $a_i$ .

**Definition:** Two parameterizations  $z : [a, b] \rightarrow \mathbb{C}$ ,  $\tilde{z} : [c, d] \rightarrow \mathbb{C}$  are equivalent iff there exists a  $C^1$  bijection  $s : [c, d] \rightarrow [a, b]$  where  $s \mapsto t(s)$  such that  $s' > 0$  and  $\tilde{z}(s) = z(s(t))$ .

Note that  $s' > 0$  preserves orientation and  $s' < 0$  reverses orientation.

**Definition:**

$$\gamma : [a, b] \rightarrow \mathbb{C} \implies \gamma^- := [a, b] \text{ to } \mathbb{C}, \quad t \mapsto \gamma(a + b - t).$$

**Definition:** A curve is closed iff  $z(a) = z(b)$ , and is simple iff  $z(t) \neq z_{t_1}$  for  $t \neq t_1$ .

**Definition:** For  $C_r(z_0) := \{z \mid |z - z_0| = r\}$ , the positive orientation is given by  $z(t) = z_0 + re^{2\pi it}$  for  $t \in [0, 1]$ .

**Definition:** The integral of  $f$  over  $\gamma$  is defined as

$$\int_{\gamma} f \, dz = \int_a^b f(z(t)) z'(t) \, dt.$$

Note: This doesn't depend on parameterization, since if  $t = t(s)$ , then a change of variables yields

$$\int_{\gamma} f \, dz = \int_c^d f(z(t(s))) z'(t(s)) t'(s) \, ds = \int_c^d f(\tilde{z}(s)) \tilde{z}'(s) \, ds.$$

Definition: The length of  $\gamma$  is defined as  $|\gamma| = \int |z'(t)| \, dt$ .

Proposition:

1. We can extend this definition to piecewise smooth curves by

$$\int_{\gamma} f \, dz = \sum \int_{a_k}^{a_{k+1}} f \, dz$$

2. This integral is linear and  $\int_{\gamma} f = - \int_{\gamma^-} f$ .
3. We have an inequality

$$\left| \int_{\gamma} f \right| \leq \max_{a \leq t \leq b} |f(z(t))| |\gamma|.$$

Definition: A function  $F$  is a primitive for  $f$  on  $\Omega$  iff  $F$  is holomorphic on  $\Omega$  and  $F'(z) = f(z)$  on  $\Omega$ .

Recall that in  $\mathbb{R}$ , we have  $F(x) \int_a^x f(t) \, dt$  as an antiderivative with  $F'(x) = f(x)$ , and  $\int_a^b f = F(b) - F(a)$ .

Theorem: If  $f$  is continuous, has a primitive  $F$  in  $\Omega$ , and  $\gamma$  is a curve beginning at  $w_0$  and ending at  $w_1$ , then  $\int_{\gamma} f = F(w_1) - F(w_0)$ .

Proof: Use definitions, write  $z(t)$  where  $z(a) = w_1, z(b) = w_2$ . Then

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(z(t)) z'(t) \, dt \\ &= \int_a^b F'(z(t)) z'(t) \, dt \\ &= \int_a^b F_t \, dt \\ &= F(z(b)) - F(z(a)) \quad \text{by FTC} \\ &= F(w_1) - F(w_2). \end{aligned}$$

Note that if  $\gamma$  is piecewise smooth, the sum of the integrals telescopes to yield the same conclusion.

**Corollary:** If  $f$  is continuous and  $\gamma$  is a closed curve in  $\Omega$ , and  $f$  has a primitive in  $\Omega$ , then  $\oint f = 0$ .

## 6 Friday January 24th

**Corollary:** If  $\gamma$  is a closed curve on  $\Omega$  an open set and  $f$  is continuous with a primitive in  $\Omega$  (i.e. an  $F$  holomorphic in  $\Omega$  with  $F' = f$ ) then  $\int_{\gamma} f dz = 0$ .

*Proof (easy):*

$$\int_{\gamma} f dz = \int_{\gamma} F' = F'(z)z(t) dt = F(z(b)) - F(z(a)) = 0.$$

Corollary: If  $f$  is holomorphic with  $f' = 0$  on  $\Omega$ , then  $f$  is constant.

*Proof (easy):* Pick  $w_0 \in \Omega$ ; we want to fix  $w_0 \in \Omega$  and show  $f(w) = f(w_0)$  for all  $w \in \Omega$ .

Take any path  $\gamma : w_0 \rightarrow w$ , then

$$0 = \int_{\gamma} f' = f(w) - f(w_0).$$

Example: Let  $f(z) = e^{-z^2}$ , this is holomorphic. Write  $f(z) = \sum (-1)^n z^{2n}/n!$ , so  $\int f = \sum (-1)^n z^{2n+1}/(n!(2n+1))$ . Since  $f$  is entire,  $\int f$  is entire, and  $(\int f)' = f$  so this function has a primitive. Thus  $\int_{\gamma} f(z) = 0$  for *any* closed curve. So take  $\gamma$  a rectangle with vertices  $\pm a, \pm a + ib$ .

So

$$\int_{\gamma} f = \int_{-a}^a e^{-x^2} dx + \int e^{-(a+iy)^2} i dy - \int_{-a}^a e^{-(x+ib)^2} dx - \int_0^b e^{-(a+iy)^2} i dy = 0.$$

We can do some estimates,

$$\begin{aligned} e^{-(a+iy)^2} &= e^{-(a^2+2iaiy-y^2)} = e^{-a^2+y^2} e^{2iaiy} \leq e^{-a^2+y^2} \leq e^{-a^2+b^2} \\ \left| \int_0^b e^{-(a+iy)^2} i dy \right| &\leq e^{-a^2+b^2} \cdot b \\ \int_{-a}^a e^{-(x^2+2ibx)-b^2} &= e^{b^2} \int_{-a}^a e^{-x^2} (\cos(2bx) - i \sin(2bx)) \stackrel{\text{odd fn}}{=} e^{b^2} \int_{-a}^a e^{-x^2} \cos(2bx) dx. \end{aligned}$$

Now take  $a \rightarrow \infty$  to obtain



Figure 2: Image

$$\int_{\mathbb{R}} e^{-x^2} dx - e^{b^2} = \int_{\mathbb{R}} e^{-x^2} \cos(2bx) dx.$$

We can compute

$$\int_{\mathbb{R}} e^{-x^2} = \left[ \left( \int_{\mathbb{R}} e^{-x^2} \right)^2 \right]^{1/2} = \left( \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \right) = \sqrt{\pi}.$$

and then conclude

$$\int_{\mathbb{R}} e^{-x^2} \cos(2bx) = \sqrt{\pi} e^{-b^2}.$$

Make a change of variables  $2b = 2\pi\xi$ , so  $b = \pi\xi$ , then

$$\int_{\mathbb{R}} e^{-x^2} \cos(2\pi\xi x) dx = \sqrt{\pi} e^{-\pi^2\xi^2}.$$

Thus  $\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\pi^2\xi^2}$ , allowing computation of the Fourier transform. Note that this can be used to prove the Fourier inversion formula.

**Exercise:** Show that this is an approximate identity and prove the Fourier inversion formula.

**Exercise:** Show  $\mathcal{F}(e^{-ax^2}) = \sqrt{\pi/a} e^{-\pi^2/a\xi^2}$ , and thus taking  $a = \pi$  makes  $e^{\pi x^2}$  is an eigenfunction of  $\mathcal{F}$  with eigenvalue 1.

Theorem: If  $f$  has a primitive on  $\Omega$  then  $F(z)$  is holomorphic and  $\int_{\gamma} f = 0$ . If  $f$  is holomorphic, then  $\int_{\gamma} f = 0$ .

**Theorem (Green's):** Take  $\Omega \in \mathbb{R}^2$  bounded with  $\partial\Omega$  piecewise smooth. If  $f, g \in C^1\overline{\Omega}$ , then

$$\int_{\partial\Omega} f dx + g dy = \iint_{\Omega} (g_x - f_y) dA.$$

Proof: Not given here!

**Proof of Theorem:** Write  $\gamma = \partial\Omega$ , and noting that  $f_z = f_x = \frac{1}{i} f_y$  implies that  $\frac{\partial f}{\partial \bar{z}}$ , so

$$\begin{aligned}
\int_{\gamma} f \, dz &= \int_{\gamma} f(z) (dx + i dy) \\
&= \int f(z) \, dx + i \int f(z) \, dy \\
&= \iint_{\Gamma} (i f_x - f_y) \, dA \\
&= i \iint_{\Gamma} \left( f_x - \frac{1}{i} f_y \right) \, dA \\
&= i \iint 0 \, dA = 0.
\end{aligned}$$

Next class: We'll prove that this integral over any triangle is zero by a limiting process.

## 7 Appendix

Collection of facts used on problem sets

**Standard forms of conic sections:**

- Circle:  $x^2 + y^2 = r^2$
- Ellipse:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola:  $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ 
  - Rectangular Hyperbola:  $xy = \frac{c^2}{2}$ .
- Parabola:  $-4ax + y^2 = 0$ .

Mnemonic: Write  $f(x, y) = Ax^2 + Bxy + Cy^2 + \dots$ , then consider the discriminant  $\Delta = B^2 - 4AC$ :

- $\Delta < 0 \iff$  ellipse
  - $\Delta < 0$  and  $A = C, B = 0 \iff$  circle
- $\Delta = 0 \iff$  parabola
- $\Delta > 0 \iff$  hyperbola

**Completing the square:**

$$\begin{aligned}
x^2 - bx &= (x - s)^2 - s^2 \quad \text{where } s = \frac{b}{2} \\
x^2 + bx &= (x + s)^2 - s^2 \quad \text{where } s = \frac{b}{2}.
\end{aligned}$$

**Useful Properties**

- $\Re(z) = \frac{1}{2}(z + \bar{z})$  and  $\Im(z) = \frac{1}{2i}(z - \bar{z})$ .
- $z\bar{z} = |z|^2$

- $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
- $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ .

## Useful Series

$$\begin{aligned}\sum_{k=1}^n k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^n k^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

## Cauchy-Riemann Equations

$$\begin{aligned}u_x &= v_y \quad \text{and} \quad u_y = -v_x \\ \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

### 7.1 Useful Techniques

**Showing a function is constant:** Write  $f = u + iv$  and use Cauchy-Riemann to show  $u_x, u_y = 0$ , etc.

**Deriving Polar Cauchy-Riemann:** See walkthrough here. Take derivative along two paths, along a ray with constant angle  $\theta_0$  and along a circular arc of constant radius  $r_0$ . Then equate real and imaginary parts. See problem set 1.

Computing Arguments:  $\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w)$ .

The sum of the interior angles of an  $n$ -gon is  $(n-2)\pi$ , where each angle is  $\frac{n-2}{n}\pi$ .