

⑦ Defn  $A/B$  is normal iff every lift of  $\text{id}_B$  is an  $A$ -automorphism.

We have 
$$\begin{array}{c} E \\ | \\ K \\ | \\ F \end{array}$$
 with  $E/F$  Galois

$\Rightarrow$  Suppose  $K/F$  is normal, then if  $\sigma \in \text{Gal}(E/F)$ , then  $\sigma|_F = \text{id}_F$ , so  $\sigma$  is a lift of  $\text{id}_F$  and thus  $\sigma(K) = K$ . But by the fundamental theorem of Galois theory, we have

$$K \xleftrightarrow{\lambda} \text{Gal}(E/K)$$

$$\sigma(K) \longleftrightarrow \tau \text{Gal}(E/K) \tau^{-1} \text{ for some } \tau \in \text{Gal}(E/F)$$

So if  $\sigma(K) = K$  for every  $\sigma$ , then  $\tau \text{Gal}(E/K) \tau^{-1} = \text{Gal}(E/K)$

$\forall \tau \in \text{Gal}(E/F)$ , so  $\text{Gal}(E/K) \trianglelefteq \text{Gal}(E/F)$

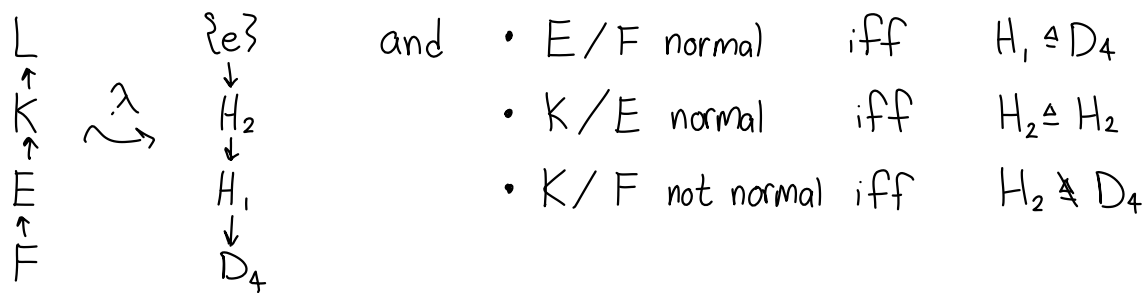
$\Leftarrow$ : Running this argument backwards shows

$$\text{Gal}(E/K) \trianglelefteq \text{Gal}(E/F) \Rightarrow \sigma(K) = K \quad \forall \sigma \in \text{Gal}(E/F)$$

$\Rightarrow$  Every lift of  $\text{id}_F$  is a  $K$ -automorphism

$\Rightarrow K/F$  is normal. ■

⑧ By the Galois correspondence, we have



So writing  $D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$

we can take  $H_1 = \langle \sigma^2, \tau \rangle = \{e, \tau, \sigma^2, \tau\sigma^2\}$ , then  $[D_4 : H_1] = 2$  so  $H_1 \trianglelefteq D_4$ . We can then take  $H_2 = \langle \tau \rangle = \{e, \tau\} \trianglelefteq H_1$ . We have  $H_1 \not\trianglelefteq D_4$ , since e.g. if we write

$$\sigma = (1234), \tau = (24) \in S_n, \sigma\tau\sigma^{-1} = (13) \notin \langle \tau \rangle.$$

But  $H_1 \trianglelefteq H_2$ , since  $H_1 \cong \{e, \tau, \sigma^2, \tau\sigma^2\} = \{(), (24), (13)(24), (13)\}$ , while

- $\tau\tau\tau^{-1} = \tau \in H_1$
- $\sigma^2\tau\sigma^{-2} = (\sigma\tau\sigma^{-1})^2 = (13)(13) = e \in H_1$
- $\tau\sigma^2\tau(\tau\sigma^2)^{-1} = (13)(24)(13) = (24) = \tau \in H_1$

So  $hH_2h^{-1} = H_2 \quad \forall h \in H_1$  and thus  $H_2 \trianglelefteq H_1$ . So taking

$$\begin{array}{l}
 H_1 = \langle \sigma^2, \tau \rangle \\
 H_2 = \langle \tau \rangle
 \end{array}$$

suffices.  $\blacksquare$

⑨ If  $f(x) = x^3 - 7$ , the splitting field of  $f$  is  $\mathbb{Q}(\sqrt[3]{7}, \zeta_3)$  where  $\zeta_3 = e^{2\pi i/3}$ .

1) Since  $\min(\sqrt[3]{7}, \mathbb{Q}) = x^3 - 7$ ,  $\min(\zeta_3, \mathbb{Q}(\sqrt[3]{7})) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$ , we have  $[\mathbb{Q}(\sqrt[3]{7}, \zeta_3) : \mathbb{Q}] = 6$ .

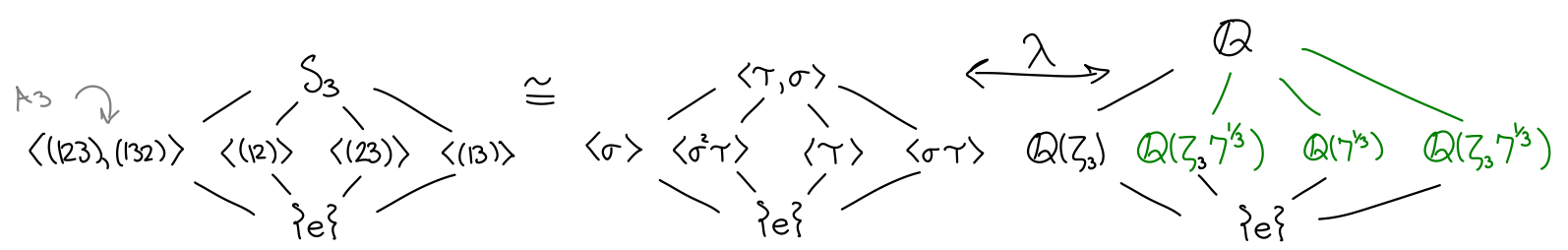
Since  $\mathbb{Q}(\sqrt[3]{7}, \zeta_3)$  is a splitting field, and separable since  $\text{char } \mathbb{Q} = 0$ . Thus

$|\text{Gal}(\mathbb{Q}(\sqrt[3]{7}, \zeta_3)/\mathbb{Q})| = 6$  as well, so  $\text{Gal}(\mathbb{Q}(\sqrt[3]{7}, \zeta_3) \cong \mathbb{Z}_6$  or  $S_3$ . Since  $\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}$  is not

a normal extension, the galois group can not be abelian, so  $\boxed{\text{Gal}(\mathbb{Q}(\sqrt[3]{7}, \zeta_3)/\mathbb{Q} \cong S_3}$ .

We have the correspondence  $\sigma = \begin{cases} \sqrt[3]{7} \mapsto \zeta_3 \sqrt[3]{7} \\ \zeta_3 \mapsto \zeta_3 \end{cases}, \tau = \begin{cases} \sqrt[3]{7} \mapsto \sqrt[3]{7} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases}$  we have

$\searrow (123) \in S_n \qquad \searrow (23) \in S_n$



Noting that  $A_3 \trianglelefteq S_3$  is the only normal subgroup, the green extensions are not galois over  $\mathbb{Q}$ .

2) Since  $\gamma^{1/3} \in \mathbb{R}$ ,  $L = \mathbb{Q}(\zeta_3) \Rightarrow |\text{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})| = \deg \min(\zeta_3, \mathbb{R}) = 2 \Rightarrow \boxed{\text{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{R}) \cong \mathbb{Z}_2}$   
 generated by  $\tau = \{ \zeta_3 \mapsto \bar{\zeta}_3 = \zeta_3^2 \}$ .

3) Since  $f(x)$  is irreducible over  $\mathbb{Z}$ , it is irreducible over  $\mathbb{F}_p$  for all  $p$ . So the splitting field is  $\mathbb{F}_3(\gamma^{1/3}, \zeta_3) := \mathbb{F}_3[x] / (x^2+x+1, x^3-\gamma)$  which is a finite extension & separable since  $\mathbb{F}_3$  is a finite field, and is thus separable. So the galois group is order 6, and by the same argument used in (1),  $\boxed{\text{Gal}(\mathbb{F}_3(\gamma^{1/3}, \zeta_3)/\mathbb{F}_3) \cong S_3}$ .