Title

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- For X, Y topological spaces, consider

$$Y^X = C(X,Y) = \hom_{\text{Top}}(X,Y) \coloneqq \left\{ f: X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

  - Topologize with the *compact-open* topology: $U \in \hom_T(X,X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
    - If $Y = (Y,d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} \coloneqq \sup\left\{ d(f_n(x), f(x)) \;\middle|\; x \in K \right\} \overset{n\to\infty}{\to} 0 \quad \forall K \subseteq X \text{ compact}.$$

      In words: $f_n \to f$ uniformly on every compact set.
- So define $\text{Map}(X,Y) = \hom_{\text{Top}}(X,Y)$ equipped with the compact-open topology.
  - Can immediately consider a lot of interesting spaces:

$$\begin{aligned} X = I \coloneqq [0,1] \rightsquigarrow \quad & P(Y;x_0) \coloneqq \left\{ f: I \to Y \;\middle|\; f(0) = x_0 \right\} = Y^I \\ X = S^1 \rightsquigarrow \quad & \Omega(Y;x_0) = \mathcal{L}(Y;x_0) \coloneqq \left\{ f: S^1 \to Y \right\} = Y^{S^1}. \end{aligned}$$

  - Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences).
  - Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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