Linearization Continued

Section 8.4 Follow-Up

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Review

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:
  For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

- We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

Review

- We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$
\begin{aligned}
\iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\
Y = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right).
\end{aligned}
$$

- We regard $Y(s, t)$ as a tangent vector to W in some Euclidean embedding.

Review

- We seek to compute the composite map in charts:

$$\begin{tikzcd}
W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \arrow[r, "\iota"] \arrow[rrr, bend left, dashed, "\mathcal{F}_u"] & \mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, bend left, dotted, "\bar{\mathcal{F}}"] & L^p\left(\mathbb{R}\times S^1;TW\right) \arrow[r] & L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)
\end{tikzcd}$$

$$u \xrightarrow{\quad\bar{\mathcal{F}}\quad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

$$(y_1,\ldots,y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)$$

Add a Tangent

$$
\begin{aligned}
\mathcal{F}(u) = &\quad \tfrac{\partial u}{\partial s} &\quad +J(u)\tfrac{\partial u}{\partial t} &\quad -J(u)X_t(u) \\
\mathcal{F}(u+Y) = &\quad \tfrac{\partial (u+Y)}{\partial s} &\quad +J(u+Y)\tfrac{\partial (u+Y)}{\partial t} &\quad -J(u+Y)X_t(u+Y)
\end{aligned}
$$

Extract the part that is linear in Y and collect terms:

$$
\begin{aligned}
&(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\qquad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)
\end{aligned}
$$

.

Review

Recall the Leibniz rule

$$(dJ)(Y)\cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}
(d\mathcal{F})_u(Y) &= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right) \\
&= \sum_{i=1}^{2n}\left(\frac{\partial y_i}{\partial s}Z_i + \frac{\partial y_i}{\partial t}J(u)Z_i\right) \\
&\quad + \sum_{i=1}^{2n} y_i\left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t}\right. \\
&\qquad\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t\right).
\end{aligned}$$

Use the fact that this is $O_1 + O_0$ in Y.

Review

Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s,t)$$

where

- *S is linear*
- *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*

$$\frac{\partial S}{\partial s}(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$
\begin{aligned}
O_0 = S(y_1, \cdots, y_{2n}) &= \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right. \\
&\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right] \\
&= \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i) X_t \right) \right. \\
&\qquad \left. + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right].
\end{aligned}
$$

- Claim: the terms in blue and orange vanish in the limit $s \longrightarrow \pm\infty$, so it suffices to prove that the remaining yields a symmetric operator.

Proof

$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since u is a solution and

$$\frac{\partial u}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

This seems to be the full argument for the blue term.

Proof

$$\frac{\partial Z_i}{\partial s} \overset{s \longrightarrow \pm\infty}{0}$$

– Thus

$$\frac{\partial S}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0.$$

– The term in blue vanishes as $s \longrightarrow \pm\infty$
  – Using the fact that u is a solution
  – Uses $\frac{\partial u}{\partial s} \longrightarrow 0$ uniformly (as do its derivatives?)

– Suffices to show the remaining part is symmetric in the limit