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1 Monday, October 26

Problem. *(Gathmann 4.13)*
Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the induced map on coordinate rings. Determine if the following statements are true or false:

a. f is surjective $\iff$ f^* is injective.
b. f is injective $\iff$ f^* is surjective.
c. If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism, then f is *affine linear*, i.e. $f(x) = ax + b$ for some $a, b \in k$.
d. If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism, then f is *affine linear*, i.e. $f(x) = Ax + b$ for some $a \in \operatorname{Mat}(2 \times 2, k)$ and $b \in k^2$.

Solution:

a. **True**. This follows because if $p, q \in A(Y)$, then

$$\begin{aligned} f * p &= f^* q \\ &\implies (p \circ f) = (q \circ f) && \text{by definition} \\ &\implies p = q, \end{aligned}$$

where in the last implication we've used the fact that f is surjective iff f admits a right-inverse.

Problem. *(Gathmann 4.19)*
Which of the following are isomorphic as ringed spaces over $\mathbb{C}$?

(a) $\mathbb{A}^1 \setminus \{1\}$
(b) $V\left(x_1^2 + x_2^2\right) \subset \mathbb{A}^2$
(c) $V\left(x_2 - x_1^2, x_3 - x_1^3\right) \setminus \{0\} \subset \mathbb{A}^3$
(d) $V\left(x_1 x_2\right) \subset \mathbb{A}^2$
(e) $V\left(x_2^2 - x_1^3 - x_1^2\right) \subset \mathbb{A}^2$
(f) $V\left(x_1^2 - x_2^2 - 1\right) \subset \mathbb{A}^2$

Problem. *(Gathmann 5.7)*
Show that

a. Every morphism $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\widehat{f} : \mathbb{A}^1 \to \mathbb{P}^1$.
b. Not every morphism $f : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\widehat{f} : \mathbb{A}^2 \to \mathbb{P}^1$.
c. Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Problem. (Gathmann 5.8)
Show that

a. Every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$f(x) = \frac{ax+b}{cx+d} \qquad\qquad a, b, c, d \in k.$$

where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

b. Given three distinct points $a_i \in \mathbb{P}^1$ and three distinct points $b_i \in \mathbb{P}^1$, there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(a_i) = b_i$ for all i.

Proposition 1.0.1*(?)*.
There is a bijection

$$\{\text{ morphisms } X \to Y\} \xleftrightarrow{1:1} \{K\text{ -algebra homomorphisms } \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)\}$$
$$f \longmapsto f^*$$

Problem. (Gathmann 5.9)
Does the above bijection hold if

a. X is an arbitrary prevariety but Y is still affine?
b. Y is an arbitrary prevariety but X is still affine?