

# Title

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# 1 | Lecture 3

Last of preliminaries. Upcoming: one-variable function fields and their valuation rings.

## 1.1 Polynomials Defining Regular Function Fields

Where's the curve:  $f(x, y) = 0$ .

**Exercise 1.1.1:** Let  $R_1, R_2$  be  $k$ -algebras that are also domains with fraction fields  $K_i$ . Show  $R_1 \otimes_k R_2$  is a domain  $\iff K_1 \otimes_k K_2$  is a domain.<sup>1</sup>

**Definition 1.1.2** (Geometrically Irreducible)

A polynomial of positive degree  $f \in k[t_1, \dots, t_n]$  is **geometrically irreducible** if  $f \in \bar{k}[t_1, \dots, t_n]$  is irreducible as a polynomial.

If  $n = 1$  then  $f$  is geometrically irreducible  $\iff$  it's linear, i.e. of degree 1. Let  $f$  be irreducible, then since polynomial rings are UFDs then  $\langle f \rangle$  is a prime ideal (irreducibles generate principal ideals) and  $k[t_1, \dots, t_n]/\langle f \rangle$  is a domain. Let  $K_f$  be the fraction field.

**Exercise 1.1.3 (an easy one):**

- Above for  $1 \leq i \leq n$  let  $x_i$  be the image of  $t_i$  in  $K_f$ . Show that  $K_f = k(x_1, \dots, x_n)$ .
- Show that if  $K/k$  is generated by  $x_1, \dots, x_n$ , then it is the fraction field of  $k[t_1, \dots, t_n]/\mathfrak{p}$  for some prime ideal  $\mathfrak{p}$  (equivalently, a height 1 ideal).

**Proposition 1.1.4 (?)**.

Suppose that  $f$  is geometrically irreducible.

- The function field  $K/k$  is regular.
- For all  $\ell/k$ ,  $f \in \ell[t_1, \dots, t_n]$  is irreducible.

In this case we say  $f$  is *absolutely irreducible* as a synonym for geometrically irreducible.

*Proof.*

By definition of geometric irreducibility,  $\bar{k}[t_1, \dots, t_n]/\langle f \rangle = k[t_1, \dots, t_n]/\langle f \rangle \otimes_k \bar{k}$  is a domain. The exercise shows that  $K_f \otimes_k k$  is a domain, so  $K_f$  is regular. It follows that for all  $\ell/k$ ,  $K_f \otimes_k \ell$  is a domain, so  $\ell[t_1, \dots, t_n]/\langle f \rangle$  is a domain. ■

<sup>1</sup>Hint: use a denominator clearing argument.

Moral: geometrically irreducible polynomials are good sources of regular function fields.

**Exercise 1.1.5:** Let  $k$  be a field,  $d \in \mathbb{Z}^+$  such that  $4 \nmid d$  and  $p(x) \in k[x]$  be positive degree. Factor  $p(x) = \prod_{i=1}^r (x - a_i)^{\ell_i}$  in  $\bar{k}[x]$ .

a. Suppose that for some  $i$ ,  $d \nmid \ell_i$ . Show that  $f(x, y) := y^d - p(x) \in k[x, y]$  is geometrically irreducible. Conclude that  $K_f := k[x, y] / \langle y^d - p(x) \rangle$  is a regular one-variable function field over  $k$ , and thus elliptic curves yield regular function fields.<sup>2</sup>

b. What happens when  $4 \mid d$ ?

**Exercise 1.1.6 (Nice, Recommended):** Assume  $k$  is a field, if necessary assuming  $\text{ch}(k) \neq 2$ .

a. Let  $f(x, y) = x^2 - y^2 - 1$  and show  $K_f$  is rational:  $K_f = k(z)$ .

b. Let  $f(x, y) = x^2 + y^2 - 1$ . Show that  $K_f$  is again rational.

c. Let  $k = \mathbb{C}$  and  $f(x, y) = x^2 + y^2 + 1$ ,  $K_f$  is rational.

d. Let  $k = \mathbb{R}$ . For  $f(x, y) = x^2 + y^2 + 1$ , is  $K_f$  rational?<sup>3</sup>

**Question 1.1.7:** Can we always construct regular function fields using geometrically irreducible polynomials?

**Answer 1.1.8:** In several variables, no, since not every variety is birational to a hypersurface. In one variable, yes, as the following theorem shows:

**Theorem 1.1.9 (Regular Function Fields in One Variable are Geometrically Irreducible).**

Let  $K/k$  be a one variable function field (finitely generated, transcendence degree one). Then

- If  $K/k$  is separable, then  $K = k(x, y)$  for some  $x, y \in K$ .
- If  $K/k$  is regular (separable + constant subfield is  $k$ , so stronger) then  $K \cong K_f$  for a geometrically irreducible  $f \in k[x, y]$ .

Recall separable implies there exists a separating transcendence basis.

*Proof (of a).*

This means there exists a primitive element  $x \in K$  such that  $K/k(x)$  is finite and separable. By the Primitive Element Corollary (FT 7.2), there exist a  $y \in K$  such that  $K = k(x, y)$ . ■

<sup>2</sup>Referred to as *hyperelliptic* or *superelliptic* function fields. Hint: use FT 9.21 or Lang's Algebra.

<sup>3</sup>This is an example of a non-rational genus zero function field.

*Proof (of b).*

Omitted for now, slightly technical. ■

Importance of last result: a regular function field on one variable corresponds to a nice geometrically irreducible polynomial  $f$ .

**Remark 1.1.10:** Note: the plane curve module may not be smooth, and in fact usually is not possible. I.e.  $k[x, y]/\langle f \rangle$  is a one-dimensional noetherian domain, which need not be integrally closed.

**Question 1.1.11:** Can every one variable function field be 2-generated?

**Answer 1.1.12:** Yes, as long as the ground field is perfect. In positive characteristic, the suspicion is no: there exists finite inseparable extensions  $\ell/k$  that need arbitrarily many generators. However, what if  $K/k$  has constant field  $k$  but is not separable? Riemann-Roch may have something to say about this.

**Example 1.1.13:** Example from earlier lecture:

$$ax^p + b - y^b$$

**Remark 1.1.14:** We can find examples of nice function fields by taking irreducible polynomials in two variables. This will define a one-variable function field. If the polynomial is geometrical reducible, this produces regular function fields.

Next: One variable function fields and their valuations.