Notes on Lee's Manifolds

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Contents

1 Preface: Point Set Review

1.1 Quotients

Definition 1.0.1 (Saturated)**.**
A subset $A \subseteq X$ is *saturated* with respect to $p : X \longrightarrow Y$ if whenever $p^{-1}(\{y\}) \bigcap A \neq \emptyset$, then $p^{-1}(\{y\}) \subseteq A$.
Equivalently, $A = p^{-1}(B)$ for some $B \subseteq Y$, i.e. it is a complete inverse image of some subset of Y, i.e. A is a union of fibers $p^{-1}(b)$.

Definition 1.0.2 (Quotient Map)**.**
A continuous surjective map $p : X \twoheadrightarrow Y$ is a *quotient map* if $U \subseteq Y$ is open **iff** $p^{-1}(U) \subset X$ is open.

> Note that $\implies$ comes from the definition of continuity of p, but $\impliedby$ is a stronger condition.

Equivalently, p maps saturated subsets of X to open subsets of Y.

Definition 1.0.3 (Universal Property of Quotients)**.**
For $\pi : X \longrightarrow Y$ a quotient map, if $g : X \longrightarrow Z$ is a map that is constant on each $p^{-1}(\{y\})$,

then there is a unique map f making the following diagram commute:

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"] & Z \end{tikzcd}$$

Fact: an injective quotient map is a homeomorphism.

Fact: a product of quotient maps need not be a quotient map.

1.2 Subspaces

Definition 1.0.4 (The Subspace Topology).
$U \subset A$ is open iff $U = V \bigcap A$ for some open $V \subseteq X$.

Proposition 1.1 ***(Universal Property of Subspaces).***
If X and $\iota_S : S \hookrightarrow Y$ is a subspace, then every continuous map $f : X \longrightarrow S$ lifts to a continuous map $\tilde{f} : X \longrightarrow Y$ where $\tilde{f} \coloneqq \iota_S \circ f$:

$$\begin{tikzcd} & Y \\ X \arrow[ur, dotted, "\exists! \tilde{f}"] \arrow[r, "f"] & S \arrow[u, hook, "\iota_S"] \end{tikzcd}$$

Note that we can view $\iota_S \coloneqq \mathrm{id}_Y|_S$. The subspace topology is the unique topology for which this property holds.

Some properties of subspace:

- The inclusion ι_S is a topological embedding.
- Restricting a continuous map to a subspace is still continuous.
- A basis for the subspace topology for $A \subset X$ can be obtained by intersecting basis elements of X with A.
- If X is Hausdorff/first/second-countable, then so is A.

1.3 Products

Definition 1.1.1 (The Product Topology).
The coarsest topology such that every projection map $p_\alpha : \prod_\beta X_\beta \longrightarrow X_\alpha$ is continuous, i.e. for every $U_\alpha \subseteq X_\alpha$ open, $p_\alpha^{-1}(U_\alpha) \in \prod X_\beta$ is open. For finite index sets, we can take the box topology: the collection of sets of the form $\prod_{i=1}^N U_i$ with each U_i open in X_i forms a basis for the product topology on $\prod_{i=1}^N X_i$.

Why these differ: in $\mathbb{R}^\infty$, the set $S = \prod(-1, 1)$ is open in the box topology but not the product topology, since $\{0\}^\infty$ is not contained in any basic open neighborhood contained in S.

Some properties of products:

- Projections π_i are continuous by definition.
- A basis for the product topology can be obtained by taking the product of bases.
- A map $f : X \longrightarrow \prod Y_i$ *into* a product is continuous iff each component function $F_i \coloneqq \pi_i \circ f : X \longrightarrow Y_i$ is continuous.
  - I.e. if we have continuous maps $f_i : X \longrightarrow Y_i$ then the composite map $F = [f_1, f_2, \cdots]$ is continuous.
- Separate continuity does not imply joint continuity: A map $f : \prod X_i \longrightarrow Y$ *out* of a product need not be continuous even if (defining $\iota_j : X_j \hookrightarrow \prod X_i$) the map $f \circ \iota_j : X_j \longrightarrow Y$ is continuous for all arbitrary inclusions ι_j.
- Any map of the form $f_{\mathbf{a}_j} : X_j \longrightarrow \prod_{i=1}^n X_i$ where $x \mapsto (a_1, \cdots, a_{j-1}, x, a_{j+1}, \cdots a_n)$ is a topological embedding.
- If X_i are Hausdorff/first/second-countable, then so is $\prod_{i=1}^n X_i$.

1.4 Misc

Definition 1.1.2 (Precompact)**.**
A subset $A \subseteq X$ is *precompact* iff its closure $\mathrm{cl}_X(A)$ is compact in X.

Definition 1.1.3 (Locally Compact)**.**
A space X is *locally compact* iff every $x \in X$ has a neighborhood which is contained in some compact subset of X.

2 Chapter 1: Point-Set Properties of Topological Manifolds

Pages 1- 29.

2.1 Notes

Definition 2.0.1 (Topological Manifold)**.**
A topological space M that satisfies

1. M is Hausdorff, i.e. points can be separated by open sets
2. M is second-countable, i.e. has a countable basis
3. M is locally Euclidean, i.e. every point has a neighborhood homeomorphic to an open subset $\widehat{U}$ of $\mathbb{R}^n$ for some fixed n.

The last property says $p \in M \implies \exists U$ with $p \in U \subseteq M$, $\widehat{U} \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi : U \longrightarrow \widehat{U}$.

Note that second countability is primarily needed for existence of partitions of unity.

Exercise Show that the in the last condition, $\widehat{U}$ can equivalently be required to be an open ball or $\mathbb{R}^n$ itself.

Theorem 2.1 ***(Topological Invariance of Dimension).***
Two nonempty topological manifolds of different dimensions can not be homeomorphic.

Exercise Show that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique.

Exercise Show that subspaces and finite products of Hausdorff (resp. second countable) spaces are again Hausdorff (resp. second countable).

Thus any open subset of a topological manifold with the subspace topology is again a topological manifold.

Exercise Give an example of a connected, locally Euclidean Hausdorff space that is not second countable.

Definition 2.1.1 (Charts)**.**
A chart on M is a pair (U, φ) where $U \subseteq M$ is open and $\varphi : U \longrightarrow \widehat{U}$ is a homeormohpsim from U to $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. If $p \in M$ and $\varphi(p) = 0 \in \overline{U}$, then the chart is said to be *centered* at p. Note that any chart about p can be modified to a chart $(\varphi_1, \widehat{U}_1)$ that is centered at p by defining $\varphi_1(x) = x - \varphi(v)$.

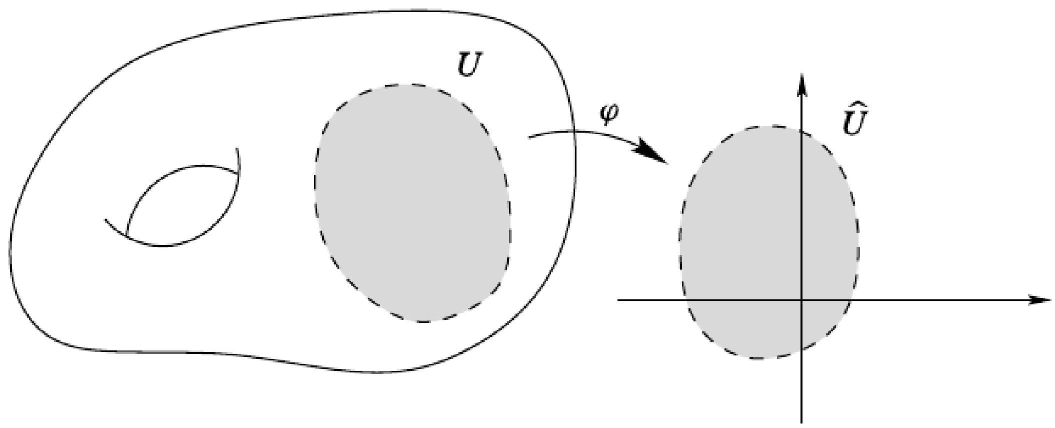


Fig. 1.2 A coordinate chart

U is the *coordinate domain* and φ is the *coordinate map.*

Note that we can write φ in components as $\varphi(p) = \left[x^1(p), \cdots, x^n(p)\right]$ where each x^i is a map $x^i : U \longrightarrow \mathbb{R}$. The component functions x^i are the *local coordinates* on U.

Shorthand notation: $\left[x^i\right] := \left[x^1, \cdots, x^n\right]$.

Example 2.1 (Graphs of Continuous Functions)**.**
Define

$$\Gamma(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \;\middle|\; x \in U,\ y = f(x) \in \widehat{U}\right\}.$$

This is a topological manifold since we can take $\varphi : \Gamma(f) \longrightarrow U$ by restricting $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ to the subspace $\Gamma(f)$. Projections are continuous, restrictions of continuous functions are continuous.

Thus graphs of continuous functions $f : \mathbb{R}^n \to \mathbb{R}^k$ are locally Euclidean?

This is a homeomorphism because the map $g : x \mapsto (x, f(x))$ is continuous and $g \circ \pi_1 = \mathrm{id}_{\mathbb{R}^n}$ is continuous with $\pi_1 \circ g = \mathrm{id}_{\Gamma(f)}$. Note that $U \cong \Gamma(f)$, and thus $(U, \varphi) = (\Gamma(f), \varphi)$ is a single *global* coordinate chart, called the *graph coordinates* of f.

Note that this works in greater generality:: "The same observation applies to any subset of $\mathbb{R}^{n+k}$ by setting *any* k of the coordinates equal to some continuous function of the other n."

Coordinates as numbers vs functions?

Example 2.2 (Spheres)**.**
S^n is a subspace of $\mathbb{R}^{n+1}$ and is thus Hausdorff and second-countable by exercise 2.1.

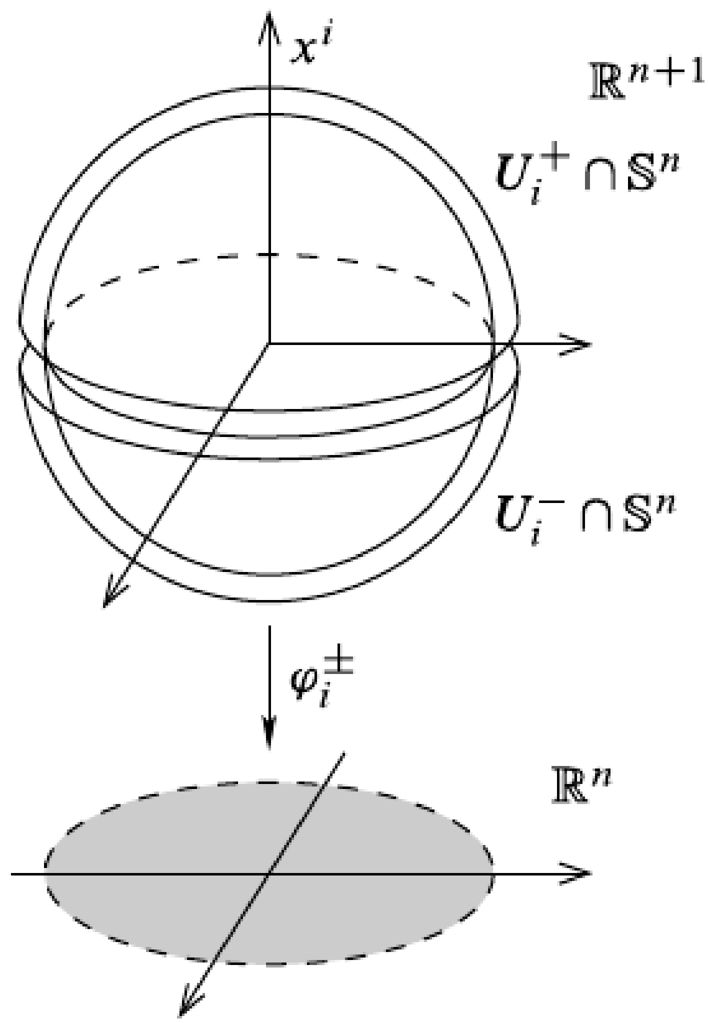


Fig. 1.3 Charts for $\mathbb{S}^n$

To see that it's locally Euclidean, take

$$U_i^+ := \left\{ \left[x^1, \cdots, x^n\right] \in \mathbb{R}^{n+1} \;\middle|\; x^i > 0 \right\} \quad \text{for} \quad 1 \leq i \leq n+1$$
$$U_i^- := \left\{ \left[x^1, \cdots, x^n\right] \in \mathbb{R}^{n+1} \;\middle|\; x^i < 0 \right\} \quad \text{for} \quad 1 \leq i \leq n+1.$$

Define

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R}^{\geq 0} \\ \mathbf{x} &\mapsto \sqrt{1 - \|\mathbf{x}\|^2}. \end{aligned}$$

Note that we immediately need to restrict the domain to $\mathbb{D}^n \subset \mathbb{R}^n$, where $\|x\|^2 \leq 1 \implies 1 - \|x\|^2 \geq 0$, to have a well-defined real function $f : \mathbb{D}^n \longrightarrow \mathbb{R}^{\geq 0}$.

Then (claim)

$$U_i^+ \bigcap S^n \quad \text{is the graph of} \quad x^i = f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1})$$
$$U_i^- \bigcap S^n \quad \text{is the graph of} \quad x^i = -f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}).$$

This is because

$$\begin{aligned} \Gamma(x^i) &:= \{(\mathbf{x}, f(\mathbf{x})) \subseteq \mathbb{R}^n \times \mathbb{R}\} \\ &= \left\{ \left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right], f\left(\left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right]\right) \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \\ &= \left\{ \left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right], \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right)^{\frac{1}{2}} \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \end{aligned}$$

and any vector in this set has norm satisfying

$$\|(\mathbf{x}, y)\|^2 = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 + \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right) = 1$$

and is thus in S^n.

To see that any such point also has positive i coordinate and is thus in U_i^+, we can rearrange (?) coordinates to put the value of f in the ith coordinate to obtain

Seems like f is always the *last* coordinate in the graph

$$\Gamma(x_i) = \left\{ \left[x^1, \cdots, f(x^1, \cdots, \widehat{x^i}, \cdots, x^n), \cdots, x^n\right] \right\}$$

and note that the square root only takes on positive values.

Thus each $U_i^\pm \bigcap S^n$ is the graph of a continuous function and thus locally Euclidean, and we can define chart maps

$$\begin{aligned} \varphi_i^\pm : U_i^\pm \bigcap S^n &\longrightarrow \mathbb{D}^n \\ \left[x^1, \cdots, x^n\right] &\mapsto [x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}] \end{aligned}$$

yield $2(n+1)$ charts that are graph coordinates for S^n.

Example 2.3 (Projective Space)**.**
Define $\mathbb{RP}^n$ as the space of 1-dimensional subspaces of $\mathbb{R}^{n+1}$ with the quotient topology determined by the map

How is this map a quotient map?

$$\begin{aligned} \pi : \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{RP}^n \\ \mathbf{x} &\mapsto \operatorname{span}_{\mathbb{R}} \{\mathbf{x}\}. \end{aligned}$$

Notation: for $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\}$ write $[\mathbf{x}] \coloneqq \pi(\mathbf{x})$, the line spanned by $\mathbf{x}$.

Define charts:

$$\tilde{U}_i \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\} \;\middle|\; x^i \neq 0 \right\}, \quad U_i = \pi(\tilde{U}_i) \subseteq \mathbb{RP}^n$$

and chart maps

$$\begin{aligned} \tilde{\varphi}_i : \tilde{U}_i &\longrightarrow \mathbb{R}^n \\ \left[x^1, \cdots, x^{n+1}\right] &\mapsto \left[\frac{x^1}{x^i}, \cdots \widehat{x^i}, \cdots \frac{x^{n+1}}{x^i}\right]. \end{aligned}$$

Then (claim) this descends to a continuous map $\varphi_i : U_i \longrightarrow \mathbb{R}^n$ by the universal property of the quotient:

$$\begin{tikzcd} \tilde{U}_i \arrow[d, "\pi_U"'] \arrow[dr, "\tilde{\varphi}_i"] & \\ U_i \arrow[r, dotted, "\varphi_i"] & \mathbb{R}^n \end{tikzcd}$$

- The restriction $\pi_U : \tilde{U}_i \longrightarrow U_i$ of π is still a quotient map because $\tilde{U}_i = \pi_U^{-1}(U_i)$ where $U_i \subseteq \mathbb{RP}^n$ is open in the quotient topology and thus $\tilde{U}_i$ is saturated.

  Thus π_U sends saturated sets to open sets and is thus a quotient map.

- $\tilde{\varphi}_i$ is constant on preimages under π_U: fix $y \in U_i$, then $\pi_U^{-1}(\{y\}) = \left\{ \lambda \mathbf{y} \;\middle|\; \lambda \in \mathbb{R} \setminus \{0\} \right\}$, i.e. the point $y \in \mathbb{RP}^n$ pulls back to every nonzero point on the line spanned by $\mathbf{y} \in \mathbb{R}^n$.

  But

$$\begin{aligned} \tilde{\varphi}_i(\lambda \mathbf{y}) &= \varphi_i\left(\left[\lambda y^1, \cdots, \lambda y^i, \cdots, \lambda y^n\right]\right) \\ &= \left[\frac{\lambda y^1}{\lambda y^i}, \cdots, \widehat{\lambda y^i}, \cdots, \frac{\lambda y^{n+1}}{\lambda y^i}\right] \\ &= \left[\frac{y^1}{y^i}, \cdots, \widehat{y^i}, \cdots, \frac{y^{n+1}}{y^i}\right] \\ &= \tilde{\varphi}_i(\mathbf{y}). \end{aligned}$$

So this yields a continuous map

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n.$$

We can now verify that φ is a homeomorphism since it has a continuous inverse given by

$$\begin{aligned}\varphi_i^{-1} : \mathbb{R}^n &\longrightarrow U_i \subseteq \mathbb{RP}^n \\ \mathbf{u} \coloneqq \left[u^1, \cdots, u^n\right] &\mapsto \left[u^1, \cdots, u^{i-1}, 1, u^{i+1}, \cdots, u^n\right].\end{aligned}$$

It remains to check: Exercise

1. The $n+1$ sets $U_1, \cdots, U_{n+1}$ cover $\mathbb{RP}^n$.
2. $\mathbb{RP}^n$ is Hausdorff
3. $\mathbb{RP}^n$ is second-countable.

Exercise (1.6) Show that $\mathbb{RP}^n$ is Hausdorff and second countable.

Exercise (1.7) Show that $\mathbb{RP}^n$ is compact. (Hint: show that π restricted to S^n is surjective.)

Definition 2.1.2 (Topological Embedding)**.**
A continuous map $f : X \longrightarrow Y$ is a *topological embedding* iff it is injective and $\tilde{f} : X \longrightarrow f(X)$ is a homeomorphism.

Example 2.4 (Product Manifolds)**.**
Let $M \coloneqq M_1 \times \cdots \times M_k$ be a product of manifolds of dimensions $n_1, \cdots, n_k$ respectively. A product of Hausdorff/second-countable spaces is still Hausdorff/second-countable, so just need to check that it's locally Euclidean.

- Let $\mathbf{p} \in \prod_{i=1}^{N} M_i$, so $p_i \in M_i$
- Choose a chart (U_i, φ_i) with $p_i \in U_i$ and assymble a product map:

$$\Phi \coloneqq \prod \varphi_i : \prod U_i \longrightarrow \prod R^{n_i} \cong \mathbb{R}^{\Sigma n_i} \coloneqq \mathbb{R}^N.$$

- Claim: Φ is a homeomorphism onto its image in R^N.
  - Each φ_i is a homeomorphism onto $\varphi_i(U_i)$ (by the definition of a chart on M_i)
  - It suffices to show that that Φ^{-1} exists and is continuous, where

$$\Phi^{-1}(V) \coloneqq \left(\prod \varphi_i\right)^{-1}\left(\prod V_i\right).$$

  - Φ is a product of continuous functions and thus continuous.
  - $\Phi^{-1} \coloneqq \left(\prod \varphi_i\right)^{-1} = \prod \varphi_i^{-1}$, which are all assumed continuous since φ_i were homeomorphisms.

Example 2.5 (Torii)**.**
$T^n \coloneqq \prod_{i=1}^{n} S^1$ is a topological n-manifold.

Definition 2.1.3 (Precompact)**.**
A subset $A \subseteq X$ is *precompact* iff its closure $\text{cl}_X(A)$ is compact in X.

Proposition 2.2.
Every topological manifold has a countable basis of precompact coordinate balls.

Proposition 2.3.
Let M be a topological manifold.

- M is locally path-connected.
- M is connected $\iff$ M is path-connected
- The connected components and path components of M coincide.
- $\pi_0(M)$ is countable and each component is open and a connected topological manifold.

Proposition 2.4.
Every topological manifold M is locally compact.

Proof .
M has a basis of precompact open sets.
■

Theorem 2.5 ***(Manifolds are Paracompact).***
Given any open cover $\mathcal{U} \rightrightarrows M$ of a topological manifold and any basis $\mathcal{B}$ for the topology on M, there exists a countable locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Proposition 2.6.
$\pi_1(M)$ is countable.

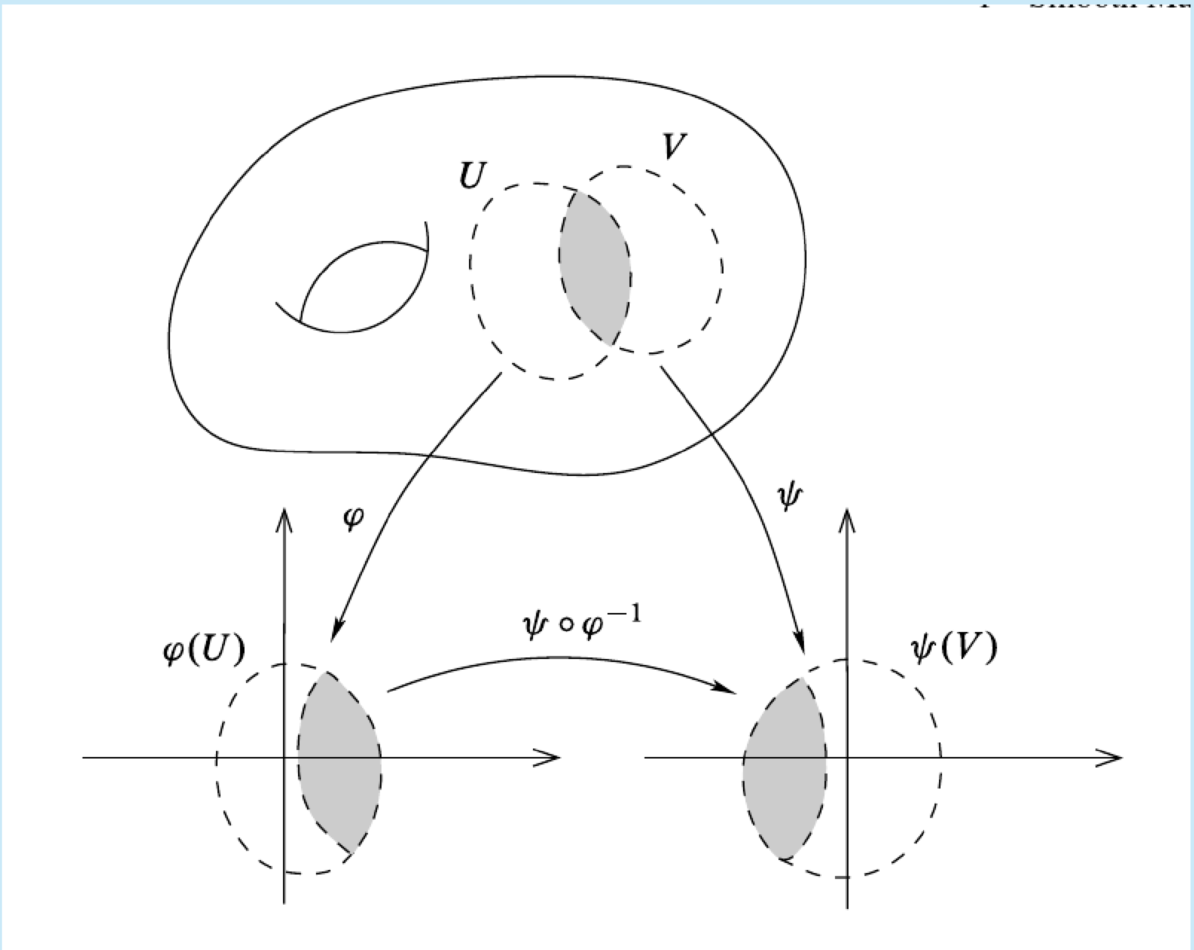
3 Chapter 1: Smooth Manifolds

Definition 3.0.1 (Smooth Functions)**.**
A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $[f_1(\mathbf{x}^n), f_2(\mathbf{x}^n), \cdots, f_m(\mathbf{x}^n)]$ (or any subsets thereof) is said to be C^∞ or **smooth** iff each f_i has continuous partial derivatives of all orders.

Definition 3.0.2 (Diffeomorphism)**.**
A smooth bijective map with a smooth inverse is a *diffeomorphism.*

Remark A diffeomorphism is necessarily a homeomorphism, but not conversely.

Definition 3.0.3 (Transition Maps)**.**
If $(U, \varphi), (V, \psi)$ are two charts on M such that $U \bigcap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \bigcap V) \longrightarrow \psi(U \bigcap V)$ is a function $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and is called the *transition map* from φ to ψ.



Two charts are *smoothly compatible* iff $U \bigcap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Definition 3.0.4.
A collection of charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}$ is an *atlas* for M iff $\{U_\alpha\} \rightrightarrows M$, and is a *smooth atlas* iff all of the charts it contains are pairwise smoothly compatible.

Remark To show an atlas is smooth, it suffices to show that an arbitrary $\psi \circ \varphi^{-1}$ is smooth. This is because this immediately implies that its inverse is smooth, and these these are diffeomorphisms. Alternatively, one can show that $\psi \circ \varphi^{-1}$ is smooth, injective, and has nonsingular Jacobian at each point.

Remark Attempting to define a function $f : M \longrightarrow \mathbb{R}$ to be smooth iff $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth for each φ may not work because many atlases give the "same" smooth structure in the sense that they all determine the same collection of smooth functions on M.

What does "determine the same collection of smooth functions" mean?

For example, take the following two atlases on $\mathbb{R}^n$:

$$\begin{aligned}\mathcal{A}_1 &= \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\} \\ \mathcal{A}_2 &= \left\{ \left(\mathbb{D}_1(\mathbf{x}), \mathrm{id}_{\mathbb{D}_1(\mathbf{x})} \right) \;\middle|\; \mathbf{x} \in \mathbb{R}^n \right\}.\end{aligned}$$

Claim: a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth wrt either atlas iff it is smooth in the usual sense.

Definition 3.0.5 (Maximal or Complete Atlas)**.**
A smooth atlas on M is *maximal* iff it is not properly contained in any larger smooth atlas.

Remark Not every topological manifold admits a smooth structure. See Kervaire's 10-dimensional manifold from 1960.

Definition 3.0.6 (Smooth Structures and Smooth Manifolds)**.**
If M is a topological manifold, a maximal smooth atlas $\mathcal{A}$ is a *smooth structure* on M. The triple $(M, \tau, \mathcal{A})$ where $\mathcal{A}$ is a smooth structure is a *smooth manifold*.

Remark To show that two smooth structures are *distinct*, it suffices to show that they are not smoothly compatible, i.e. one of the transition functions $\psi \circ \varphi^{-1}$ is not smooth. This is because any maximal atlas $\mathcal{A}_1$ must contain ψ and likewise $\mathcal{A}_2$ contains φ^{-1}, but no maximal atlas can contain φ *and* ψ because all charts in a maximal atlas are smoothly compatible by definition.

Proposition 3.1.
Let M be a topological manifold.
1. Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the *smooth structure determined by* $\mathcal{A}$.
2. Two smooth atlases for M determine the same smooth structure $\iff$ their union is a smooth atlas.

Remark That we can place many requirements on the functions $\psi \circ \varphi^{-1}$ and get various other structures: C^k, real-analytic, complex-analytic, etc. C^0 structures recover topological manifolds.

Definition 3.1.1 (Smooth Charts, Maps, Domains)**.**
If $(M,\ \tau, \mathcal{A})$ is a smooth manifold, any chart $(U, \varphi) \in \mathcal{A}$ is a *smooth chart*, where U is a *smooth coordinate domain* and φ is a *smooth coordinate map*. A *smooth coordinate ball* is a smooth coordinate domain U such that $\varphi(U) = \mathbb{D}^n$.

Definition 3.1.2 (Regular Coordinate Ball)**.**
A set $B \subseteq M$ is a *regular coordinate ball* if there is a smooth coordinate ball B' such that $\mathrm{cl}_M(B) \subseteq B'$, and a smooth coordinate map $\varphi : B' \longrightarrow \mathbb{R}^n$ such that for some positive numbers $r < r'$,
- $\varphi(B) = \mathbb{D}_r(\mathbf{0})$,
- $\varphi(B') = \mathbb{D}_{r'}(\mathbf{0})$, and
- $\varphi(\mathrm{cl}_M(B)) = \mathrm{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$.

This says B "sits nicely" insane a larger coordinate ball.

Remark $\mathrm{cl}_M(B) \cong_{\mathrm{Top}} \mathrm{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$ which is closed and bounded and thus compact, so $\mathrm{cl}_M(B)$ is compact. Thus every regular coordinate ball in M is precompact.

Proposition 3.2.
Every smooth manifold has a countable basis of regular coordinate balls.

Remark There is only one 0-dimensional smooth manifold, up to equivalence of smooth structures.

Definition 3.2.1 (Standard Smooth Structure on $\mathbb{R}^n$)**.**
Define the atlas $\mathcal{A}_0 = \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ and take the smooth structure it generates, this is the *standard smooth structure* on $\mathbb{R}^n$.

Proposition 3.3.
There are at least two distinct smooth structures on $\mathbb{R}^n$.

Proof .
Define $\psi(x) = x^3$; then $\mathcal{A}_1 := \{(\mathbb{R}^n, \varphi)\}$ defines a smooth structure.
Then $\mathcal{A}_1 \neq \mathcal{A}_0$, which follows because $\left(\mathrm{id}_{\mathbb{R}^n} \circ \varphi^{-1}\right)(x) = x^{\frac{1}{3}}$, which is not smooth at $\mathbf{0}$. ■

4 Chapter 1 Problems

4.1 Recommended Problems

Exercise (Problem 1.6) Show that if $M^n \neq \emptyset$ is a topological manifold of dimension $n \geq 1$ and M has a smooth structure, then it has uncountably many distinct ones.

Recommended problem

> Hint: show that for any $s > 0$ that $F_s(x) := |x|^{s-1}x$ defines a homeomorphism $F_x : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ which is a diffeomorphism iff $s = 1$.

Solution:

Define

$$
\begin{aligned}
F_s : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\
\mathbf{x} &\mapsto \|\mathbf{x}\|^{s-1}\mathbf{x}.
\end{aligned}
$$

Claim: F_s restricted to $\mathbb{D}^n$ is a continuous map $\mathbb{D}^n \longrightarrow \mathbb{D}^n$.

- Note that if $\|\mathbf{x}\| \leq \varepsilon < 1$ then

$$\|F_s(\mathbf{x})\| = \|\|\mathbf{x}\|^s \widehat{\mathbf{x}}\| = \|\mathbf{x}\|^s \leq \|\mathbf{x}\| \leq \varepsilon < 1,$$

  so $F_s(\mathbb{D}^n) \subseteq \mathbb{D}^n$ and moreover $F_s(\mathbb{D}^n_\varepsilon) \subseteq \mathbb{D}^n_\varepsilon$.
  - We'll use the fact that $F_s^{-1} = F_{\frac{1}{s}}$ is of the same form, and thus $F_s^{-1}(\mathbb{D}^n) \subseteq \mathbb{D}^n$, forcing $F_s(\mathbb{D}^n) = \mathbb{D}^n$.
- This is a continuous function, since it can be written as a composition of continuous functions:

$$
\begin{array}{ccccccccc}
\mathbb{D}^n & \xrightarrow{\Delta} & \mathbb{D}^n \times \mathbb{D}^n & \xrightarrow{(\|\cdot\|,\ \mathrm{id}_{\mathbb{D}^n})} & \mathbb{D}^n \times \mathbb{D}^n & \xrightarrow{((\,\cdot\,)^{s-1},\ \mathrm{id}_{\mathbb{D}^n})} & \mathbb{D}^1 \times \mathbb{D}^n & \xrightarrow{(a,b)\mapsto ab} & \mathbb{D}^n \\
\\
\mathbf{x} & \longrightarrow & (\mathbf{x}, \mathbf{x}) & \longrightarrow & (\|\mathbf{x}\|, \mathbf{x}) & \longrightarrow & (\|\mathbf{x}\|^{s-1}, \mathbf{x}) & \longrightarrow & \|\mathbf{x}\|^{s-1}\mathbf{x}
\end{array}
$$

> Note: These should all be on punctured discs $\mathbb{D}^n \setminus \{0\}$, but continuity at zero follows from the fact that $\|F_s(\mathbf{x})\| \leq \|\mathbf{x}\| \longrightarrow 0$, so $\lim_{\mathbf{x} \longrightarrow \mathbf{0}} F_s(\mathbf{x}) = \mathbf{0}$ and the sequential definition of continuity applies.

Claim: F_s is a bijection $\mathbb{D}^n \setminus \mathbf{0}\circlearrowleft$ that extends to a bijection $\mathbb{D}^n\circlearrowleft$.

We can note that

$$F_s(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|^s \dfrac{\mathbf{x}}{\|\mathbf{x}\|} \coloneqq \|\mathbf{x}\|^s \widehat{\mathbf{x}} & \text{if } \|\mathbf{x}\| \neq 0 \\ \mathbf{0} & \text{if } \|\mathbf{x}\| = 0 \end{cases}$$

This follows because we can construct a two-sided inverse that composes to the identity, namely $F_{\frac{1}{s}}$, for $\mathbf{x} \neq \mathbf{0}$, and note that $F_s(\mathbf{0}) = \mathbf{0}$. Using the fact that $\|t\mathbf{x}\| = t\|\mathbf{x}\|$ for any scalar t, we can check that

$$\begin{aligned} \left(F_s \circ F_{\frac{1}{s}}\right)(\mathbf{x}) &= F_s(\|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}}) \\ &= \left\| \|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}} \right\|^s \cdot \widehat{\|\mathbf{x}\|^{\frac{1}{s}} \mathbf{x}} \\ &= \left(\|\mathbf{x}\|^{\frac{1}{s}}\right)^s \cdot \|\widehat{\mathbf{x}}\|^s \cdot \frac{\|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}}}{\left\| \|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}} \right\|} \\ &= \|\mathbf{x}\| \cdot 1^s \cdot \left(\frac{\|\mathbf{x}\|^{\frac{1}{s}}}{\|\mathbf{x}\|^{\frac{1}{s}}} \right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\ &= \|\mathbf{x}\| \widehat{\mathbf{x}} \\ &= \mathbf{x}. \end{aligned}$$

and similarly

$$\begin{aligned} \left(F_{\frac{1}{s}} \circ F_s\right)(\mathbf{x}) &= F_{\frac{1}{s}}(\|\mathbf{x}\|^s \widehat{\mathbf{x}}) \\ &= \| \|\mathbf{x}\|^s \widehat{\mathbf{x}} \|^{\frac{1}{s}} \cdot \widehat{\|\mathbf{x}\|^s \widehat{\mathbf{x}}} \\ &= (\|\mathbf{x}\|^s)^{\frac{1}{s}} \|\widehat{\mathbf{x}}\|^{\frac{1}{s}} \cdot \frac{\|\mathbf{x}\|^s \widehat{\mathbf{x}}}{\| \|\mathbf{x}\|^s \widehat{\mathbf{x}} \|} \\ &= \|\mathbf{x}\| \cdot 1^{1-s} \cdot \left(\frac{\|\mathbf{x}\|^s}{\|\mathbf{x}\|^s} \right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\ &= \|\mathbf{x}\| \widehat{\mathbf{x}} \\ &= \mathbf{x}. \end{aligned}$$

Claim: F_s is a homeomorphism for all s.

This follows from the fact that the domain $\mathbb{D}^n$ is compact and the codomain $\mathbb{D}^n$ is Hausdorff, and a continuous bijection between such spaces is a homeomorphism.

Claim: F_s is a diffeomorphism iff $s = 1$.

If $s = 1$, $F_s = \mathrm{id}_{\mathbb{D}^n}$ which is clearly a diffeomorphism.

Otherwise, we claim that F_s is not smooth at $\mathbf{x} = 0$.

Exercise (Problem 1.7) Let $N \coloneqq [0, \cdots, 1] \in S^n$ and $S \coloneqq [0, \cdots, -1]$ and define the stereographic projection

Recommended problem

$$\sigma : S^n \setminus N \longrightarrow \mathbb{R}^n$$
$$\left[x^1, \cdots, x^{n+1}\right] \mapsto \frac{1}{1 - x^{n+1}} \left[x^1, \cdots, x^n\right]$$

and set $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus S$ (projection from the South pole)

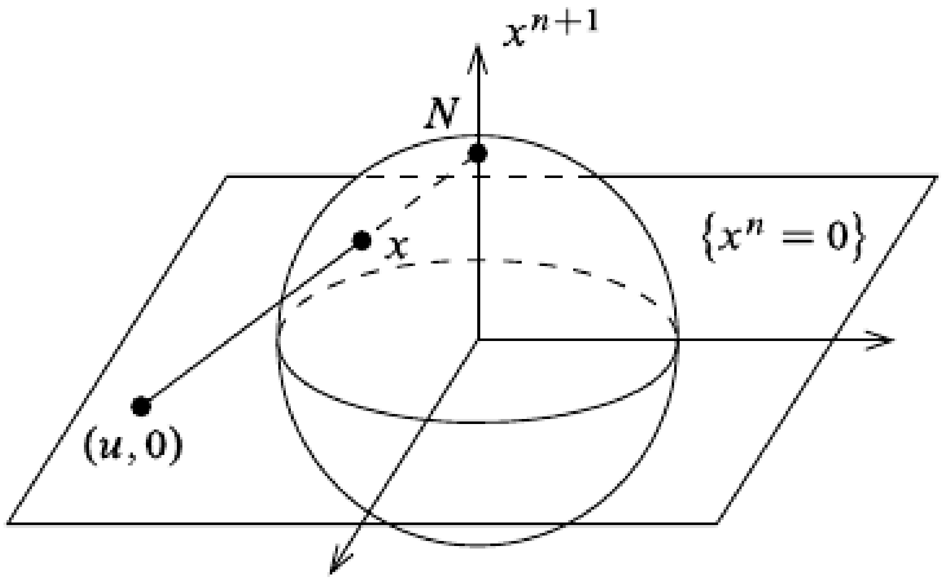


Fig. 1.13 Stereographic projection

1. For any $x \in S^n \setminus N$ show that $\sigma(x) = \mathbf{u}$ where $(\mathbf{u}, 0)$ is the point where the line through N and x intersects the linear subspace $H_{n+1} := \left\{x^{n+1} = 0\right\}$.

   Similarly show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects H_{n+1}.

2. Show that σ is bijective and

$$\sigma^{-1}(\mathbf{u}) = \sigma^{-1}\left(\left[u^1, \cdots, u^n\right]\right) = \frac{1}{\|\mathbf{u}\|^2 + 1} \left[2u^1, \cdots, 2u^n, \|\mathbf{u}\|^2 - 1\right].$$

3. Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas

$$\mathcal{A} := \{(S^n \setminus N, \sigma), (S^n \setminus S, \tilde{\sigma})\}$$

   define a smooth structure on S^n.

4. Show that this smooth structure is equivalent to the standard smooth structure: Put graph coordinates on S^n as outlined in 2.2 to obtain $\left\{(U_i^\pm, \varphi_i^\pm)\right\}$.

   For indices $i < j$, show that

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}\left[u^1, \cdots, u^n\right] = \left[u^1, \cdots, \widehat{u^i}, \cdots, \pm\sqrt{1 - \|\mathbf{u}\|^2}, \cdots u^n\right]$$

where the square root appears in the jth position. Find a similar formula for $i > j$. Show that if $i = j$, then

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1} = \varphi_i^{-} \circ (\varphi_i^{+})^{-1} = \mathrm{id}_{\mathbb{D}^n}.$$

Show that these yield a smooth atlas.

Exercise (Problem 1.8) Define an *angle function* on $U \subset S^1$ as any continuous function $\theta : U \longrightarrow \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

Recommended problem

Show that U admits an angle function iff $U \neq S^1$, and for any such function θ, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

Exercise (Problem 1.9) Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to equip it with a smooth structure, using the correspondence

Recommended problem

$$\begin{aligned} \mathbb{R}^{2n} &\iff \mathbb{C}^n \\ \left[x^1, y^1, \cdots, x^n, y^n\right] &\iff \left[x^1 + iy^1, \cdots, x^n + iy^n\right]. \end{aligned}$$

Todo list