Category $\mathcal{O}$, Problem Set 3

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1 Humphreys 1.10

Prove that the transpose map τ fixes $Z(\mathfrak{g})$ pointwise.

> Check that τ commutes with the Harish-Chandra morphism ξ and use the fact that ξ is injective.

1.1 Solution

We first note that after choosing a PBW basis for $\mathfrak{g}$, τ is defined on $\mathfrak{g}$ in the following way:

$$\begin{aligned} \tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha \end{aligned}$$

which lifts to an anti-involution $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by extending linearly over PBW monomials. We can note that since τ fixes $\mathfrak{h}$ pointwise by definition, its lift also fixes $U(\mathfrak{h})$ pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned} \xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}. \end{aligned}$$

We will use without proof that ξ is injective.

Proposition 1.1.
The following diagram commutes

$$\begin{tikzcd}
Z(\mathfrak{g}) \arrow[r, "\xi"] \arrow[d, "\tau"] & U(\mathfrak{h}) \arrow[d, "\tau"] \\
Z(\mathfrak{g}) \arrow[r, "\xi"] & U(\mathfrak{h})
\end{tikzcd}$$

Proof.
We will show that for all $z \in Z(\mathfrak{g})$, $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$. Expand z in a PBW basis as $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}$. We then make the following computations:

$$\begin{aligned}
(\xi \circ \tau)(z) &= (\xi \circ \tau)\left(\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}\right) \\
&= \xi\left(\prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_j}\right) \\
&= \prod_j h_j^{s_j}
\end{aligned}$$

.

Similarly, we have

$$\begin{aligned}
(\tau \circ \xi)(z) &= \tau\left(\prod_j h_j^{s_j}\right) \\
&= \prod_j h_j^{s_j}
\end{aligned}$$

,

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of ξ, we have

$$\begin{aligned} (\xi \circ \tau z) &= \xi(z) \\ \implies \tau(z) &= z. \end{aligned}$$

■

2 Humphreys 1.12

Fix a central character χ and let $\left\{V^{(\lambda)}\right\}$ be a collection of modules in $\mathcal{O}$ indexed by the weights λ for which $\chi = \chi_\lambda$ satisfying

1. $\dim V^{(\lambda)} = 1$
2. $\mu < \lambda$ for all weights μ of $V^{(\lambda)}$.

Then the symbols $[V^{(\lambda)}]$ form a $\mathbb{Z}$-basis for the Grothendieck group $K(\mathcal{O}_x)$.

For example take $V^{(\lambda)} = M(\lambda)$ or $L(\lambda)$.

3 Humphreys 1.13

Suppose $\lambda \not\in \lambda$, so the linkage class $W \cdot \lambda$ is the disjoint union of its nonempty intersections of various cosets of $\Lambda_r \in \mathfrak{h}^\vee$.

Prove that each $M \in \mathcal{O}_{\chi_\lambda}$ has a corresponding direct sum decomposition $M = \bigoplus M_i$ in which all weights of M_i lie in a single coset.

Recall exercise 1.1b.