# Title

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# Contents



# 1 | Friday, September 25

## 1.1 Compact-Open Topology

## 1.2 Isotopy

- Define a homotopy between $f, g : X \to Y$ as a map $F : S \times I \to S$ restricting to $f, g$ on the ends.
  - Equivalently: a *path* in $\mathrm{Map}(X, Y)$.
- Isotopy: require the partially-applied function $F_t : S \to S$ to be homeomorphisms for every $t$.
  - Equivalently: a path in $\mathrm{Map}(X, Y)$

## 1.3 Self-Homeomorphisms

- In any category, the automorphisms form a group.
  - In a general category $\mathcal{C}$, we can always define the group $\mathrm{Aut}_{\mathcal{C}}(X)$.
    - If the group has a topology, we can consider $\pi_0 \mathrm{Aut}_{\mathcal{C}}(X)$, the set of path components.
    - Since groups have identities, we can consider $\mathrm{Aut}^0_{\mathcal{C}}(X)$, the path component containing the identity.
  - So we make a general definition, the *extended mapping class group*:

$$\mathrm{MCG}^{\pm}_{\mathcal{C}}(X) \coloneqq \mathrm{Aut}_{\mathcal{C}}(X) / \mathrm{Aut}^0_{\mathcal{C}}(X).$$

  - Here the $\pm$ indicates that we take both orientation preserving and non-preserving automorphisms.

- Has an index 2 subgroup of orientation-preserving automorphisms, $\mathrm{MCG}^+(X)$.

- Now restrict attention to

$$\mathrm{Homeo}(X) := \mathrm{Aut}_{\mathrm{Top}}(X) = \left\{ f \in \mathrm{Map}(X, X) \;\middle|\; f \text{ is an isomorphism} \right\}$$
$$\text{equipped with } \mathcal{O}_{\mathrm{CO}}.$$

  - Taking $\mathrm{MCG}^{\pm}_{\mathrm{Top}}(X)$ yields ??

- Similarly, we can do all of this in the smooth category:

$$\mathrm{Diffeo}(X) := \mathrm{Aut}_{C^\infty}(X).$$

  - Taking $\mathrm{MCG}_{C^\infty}(X)$ yields ??

- Similarly, we can do this for the homotopy category of spaces:

$$\mathrm{ho}(X) := \{[f]\}.$$

  - Taking $\mathrm{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences.*

- For topological manifolds: Isotopy classes of homeomorphisms
  - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi \mathrm{Aut}(X)$.

- For surfaces: $\mathrm{MCG}(S)$ on the Teichmuller space $T(S)$, yielding a SES

$$0 \to \mathrm{MCG}(S) \to T(S) \to \widetilde{\mathcal{M}}_g(S) \to 0$$

  where the last term is the moduli space of Riemann surfaces homeomorphic to $X$.

  - $T(S)$ is the moduli space of complex structures on $S$, up to the action of homeomorphisms that are isotopic to the identity:
    * Points are isomorphism classes of marked Riemann surfaces
  - Used in the Neilsen-Thurston Classification (for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of: periodic: reducible (preserves some simple closed curves), or pseudo-Anosov (has directions of expansion/contraction))

- Generated by Dehn twists: a self homeomorphism
- Any finite group is $\mathrm{MCG}(X)$ for some compact hyperbolic 3-manifold $X$.

**Theorem 1.1** ***(Dehn-Neilsen-Baer).***

$$\mathrm{MCG}^{\pm}(\Sigma_g) \cong \mathrm{Out}(\pi_1(\Sigma_g)).$$

## 1.4 Dehn Twists

**Claim:** Let $A := \left\{ z \in \mathbb{C} \;\middle|\; 1 \leq |z| \leq 2 \right\}$, then $\mathrm{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \exp(2\pi i |z|)\, z. \end{aligned}$$