

# Interesting Topological Spaces in Algebraic Geometry

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## 1 Ideas for Spaces

- Curves
  - Elliptic Curves
  - Higher genus
  - Hyperelliptic curves
  - The modular curve
- Surfaces
  - Compact Riemann surfaces
    - \* Bolza Surface (Genus 2)
    - \* Klein Quartic (Genus 3)
    - \* Hurwitz Surfaces
  - Kummer surfaces
- Compact Complex Surfaces
  - Rational ruled
  - Enriques Surfaces
  - $K3$

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- \* Kahler Manifolds
    - Kodaira
    - Toric
    - Hyperelliptic
    - Properly quasi-elliptic
    - General type
    - Type VII
  - Fake projective planes
  - Conics
  - Calabi-Yau manifolds
    - Dimension 1: All elliptic curves (up to homeomorphism)
    - Dimension 2:  $K3$  surfaces
    - Dimension 3 (threefolds): 500 million +, unknown if infinitely many
    - The bananafold
    - Hyperkähler
  - Hurwitz schemes
  - Topological galois groups, e.g.  $G(\bar{F}/F)$  for  $F = \mathbb{Q}, \mathbb{F}_p$ .
  - $\text{Spec}(R)$  for  $R$  a DVR (a Sierpinski space)
  - Quiver Grassmannians
  - Rigid analytic spaces
  - Affine line with two origins
  - Moduli stack of elliptic curves  $\mathcal{M}_{1,1}$ .
  - Abelian Surface
  - Fano Varieties
  - Curves: isomorphic to  $\mathbb{P}^1$
  - Surfaces: Del Pezzo surfaces
  - Weighted projective space
  - Toric Varieties
  - Grassmannian
  - Flag Varieties
  - Moduli Spaces

Due to Kunihiro Kodaira's classification of complex surfaces, we know that any compact hyperkähler 4-manifold is either a  $K3$  surface or a compact torus  $T^4$ . (Every Calabi–Yau manifold in 4 (real) dimensions is a hyperkähler manifold, because  $SU(2)$  is isomorphic to  $Sp(1)$ .)

As was discovered by Beauville, the Hilbert scheme of  $k$  points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension  $4k$ . This gives rise to two series of compact examples: Hilbert schemes of points on a  $K3$  surface and generalized Kummer varieties.

## 2 Intro/Motivation

Ursula Whitcher

Assume the universe is a “space”. Which one is it? What structures does it have? How many possible spaces *could* it be, and how can we test to find out?

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## 3 Analogies

Notation: all dimensions are over  $\mathbb{R}$ .

Impossible goal: pick a category, understand all of the objects and all of the maps. Two main categories with a forgetful functor: **Diff**  $\longrightarrow$  **Top**. Question:

- What's in the “image” of this functor? (Manifolds that admit a differentiable structure.)
- What is the “fiber” above a given topological manifold? (Distinct differentiable structures)

Differentiable Manifolds: classified by geometric structure in low dimensions ( $\leq 4$ ), algebraic data/methods in high dimensions

### 3.1 Topological Category

Identify objects up to homeomorphism

- Dimension 0: The point (terminal object)
- Dimensions 1:  $S^1, \mathbb{R}$
- Dimension 2:  $\langle S, T, \mathbb{RP} \mid S = 0, 3\mathbb{RP} = \mathbb{RP} + T \rangle$ . Classified by  $\pi_1$  (orientability and “genus”). Riemann, Poincare, Klein.
- Dimension 3: Can always be given a unique smooth structure, see uniformization.
- Dimension 4:
- Dimension  $n \geq 5$ :

### 3.2 Smooth Category

Generally expect things to split into more classes.

- 2-manifolds: Homeomorphic  $\iff$  diffeomorphic. Every surface admits a complex structure and a metric. Thus always orientable.
  - Uniformization: Holomorphically equivalent to a quotient of one of three spaces
    - \*  $\mathbb{CP}^1$ , positive curvature (spherical)
    - \*  $\mathbb{C}$ , zero curvature (flat)
    - \*  $\mathbb{H}$  (equiv.  $\mathbb{D}^\circ$ ), negative curvature (hyperbolic)
  - Stratified by genus:
    - \* Genus 0: Only  $\mathbb{CP}^1$
    - \* Genus 1: All of the form  $\mathbb{C}/\Lambda$ , with a distinguished point  $[0]$ , i.e. an elliptic curve. Has a topological group structure!
- 3-manifolds: Thurston's Geometrization
  - Oriented prime 3-manifolds can be decomposed into geometric “pieces” of 8 possible types
  - Geometric structure: a diffeo  $M \cong \tilde{M}/\Gamma$  where  $\Gamma$  is a discrete Lie group acting freely/transitively on  $X$
- 4-manifolds: classified in the topological category by surgery, but not in the smooth category
- $n$ -manifolds,  $n \geq 5$ : classified by surgery

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## 4 Moduli Spaces

$$\begin{array}{ccc} \mathbb{T}^\ell & \longrightarrow & E \\ & & \downarrow \\ & & \mathrm{Gr}(\ell, n)/\mathbb{T} \end{array}$$

## 5 Elliptic Curves

- Equivalently, Riemann surfaces with one marked point.
- Equivalently,  $\mathbb{C}/\Lambda$  a lattice, where homothetic lattices (multiplication by  $\lambda \in \mathbb{C} \setminus \{0\}$ ) are equivalent.
- Parameterized by a moduli space:
  - For  $X = \mathbb{C}/\Lambda$  choose a positively oriented basis  $\{z, w\}$  for  $\Lambda$ 
    - \* Note: push into meridians on a torus, generators of  $H_1(X)$ , and require that their intersection is  $+1$ .
  - Replace  $z \mapsto 1, w \mapsto \frac{w}{z}$ ; the orientation condition forces  $\Im(w) > 0$  so this yields a point  $w \in \mathbb{H}$ .

## 6 Kahlers

- For complex manifold, replace Riemannian metric with a Hermitian metric (positive definite sesquilinear inner product on tangent bundle)
- If skew-symmetric part is symplectic (closed and nondegenerate) then the metric is Kahler
- Includes smooth projective varieties, but not all complex manifolds
- Specialize to Calabi-Yaus: compact and Ricci-flat, or first Chern class vanishes

## 7 Calabi-Yaus

- As manifolds: Ricci-flat, i.e. Ricci curvature tensor vanishes (measures deviation of volumes of “geodesic balls” from Euclidean balls of the same radius).
- Applications: Physicists want to study  $G_2$  manifolds (an exceptional Lie group, automorphisms of octonions), part of  $M$ -theory uniting several superstring theories, but no smooth or complex structures. Indirect approach: compactify an 11-dimension space, one small  $S^1$  dimension  $\longrightarrow$  10 dimensions, 4 spacetime and 6 “small” Calabi-Yau.