More group classification

(see also: pqr!)

4.4 Classify gps of order pq (distinct primes)

Soln

Wlog $\boxed{q<p}$. Cases:

1) $q \nmid p-1$
2) $q \mid p-1$

Unmotivated :(

Case 1: $q \nmid p-1$ WTS $G \cong (\mathbb{Z}/pq\mathbb{Z}, +)$

- Consider $S_p \in Syl_p(G)$, then by Sylow 3
  - $n_p \equiv 1 \mod p$ $\Rightarrow n_p \in \{1, p+1, 2p+1, \dots\}$
  - $n_p \mid q$ $\Rightarrow n_p \in \{1, q\}$
  - $[G : N_G(S_p)] = n_p$ $+\ q<p$

  $\boxed{\Rightarrow n_p = 1.}$

  Note $|S_p| = p \Rightarrow S_p \cong \mathbb{Z}/p$

- Consider $S_q \in Syl_q(G)$

  $q<p$
  - $n_q \equiv 1 \mod q$ $\Rightarrow n_q \in \{1, q, q+1, 2q+1, \dots, kq+1\}$
  - $n_q \mid p$ $\Rightarrow n_q \in \{1, p\}$
  - $[G : N_G(S_q)] = n_q$

  $\therefore n_q = 1$ or $p = \ell q + 1$
  $\Updownarrow$
  $p-1 = \ell q$
  $\Downarrow$
  $q \mid p-1$ #

  why doesn't this work mod p-1...

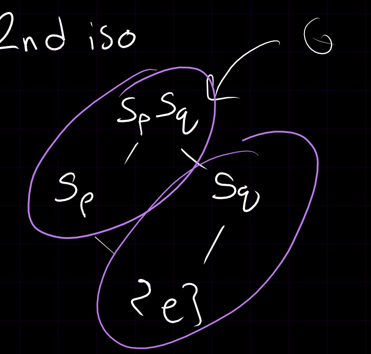
  Sim., $S_q \cong \mathbb{Z}/q$ $\boxed{\Rightarrow n_q = 1.}$

- So $n_p = n_q = 1$. Apply char. of direct products:
  - $S_p, S_q \trianglelefteq G$ (Both normal mnem: $B = \ker(A \times B \xrightarrow{\pi_A} A) \trianglelefteq A \times B$)

    Not in case 2

    - True by above
  - $S_p \cap S_q = \{1_G\}$ $f: S_p \times S_q \to G$, $(x,y) \mapsto xy$
    - True by Coprime order
  - $S_pS_q = G$ 2nd iso
    - $|S_p \cap S_q| = \{1\}$ by coprime order
    - $|S_pS_q| = \dfrac{|S_p| \cdot |S_q|}{|S_p \cap S_q|} = \dfrac{p \cdot q}{1} = pq$
    - $\boxed{S_pS_q \le G \ \&\ |S_pS_q| = |G|} \Rightarrow S_pS_q = G.$

    Relies on finiteness!



$\Rightarrow G \cong S_p \times S_q \cong \mathbb{Z}/p \times \mathbb{Z}/q \cong_{CRT} \mathbb{Z}/pq.$

Case 2:

- S_p argument works, S_q doesn't $\rightsquigarrow$ Semidirect prod

  $\langle a \mid a^p \rangle$ $\langle b \mid b^q \rangle$

  $G \cong S_p \rtimes_\psi S_q \cong (\mathbb{Z}/p, +) \rtimes_\psi (\mathbb{Z}/q, +)$, $\psi: \mathbb{Z}/q \to \underset{Grp}{Aut}(\mathbb{Z}/p)$

  $\hookrightarrow \left\langle a,b \,\middle|\, a^p, b^q, \ bab^{-1} = \psi(a) \right\rangle$

  $(a',b')(a,b) = (\underset{\psi}{?}, b'b)$

- $Aut(\mathbb{Z}/p) \cong (\mathbb{Z}_p^\times, \cdot) \cong (\mathbb{Z}/p-1, +)$

  $\Rightarrow$ Need maps $\mathbb{Z}/q \to \mathbb{Z}/p-1$

- Trivial map $[1]_q \mapsto [1]_{p-1} \rightsquigarrow id_{S_q}$ $\Rightarrow$ direct product.

- Is there a nontrivial map?

  $q \mid p-1$, by Cauchy's Thm, $\exists\, \alpha$ of order $q \in \mathbb{Z}/p-1$

  Then $[1]_q \mapsto [\alpha]_{p-1} \rightsquigarrow \alpha \in Aut(\mathbb{Z}_p)$ of order q (non-triv)

- **Claim**: All choices of α yield iso semidirect prods.

  Have $G \cong \mathbb{Z}/p \rtimes_{\psi_\ell} \mathbb{Z}/q$, $\psi_\ell: (\mathbb{Z}/q, +) \to \underbrace{Aut\, \mathbb{Z}/p}_{cyclic} \overset{\langle \xi \rangle}{=} \cong (\mathbb{Z}/p^\times, \cdot)$

  $\cong \left\langle a,b \,\middle|\, a^p, b^q, \ bab^{-1} = \psi_\ell(a) = a^\ell \right\rangle$

  $\phi_\ell: \xi \mapsto \xi^\ell$ $\quad \xi \to \xi^\ell \to (\xi^\ell)^\ell \to \cdots$

  nontriv Aut $\Rightarrow \ell \not\equiv 1 \mod p$

  order q Aut $\Rightarrow \ell^q \equiv 1 \mod p$

  Use $Aut\, \mathbb{Z}/p = \left\{ \begin{array}{c} \phi_\ell: \mathbb{Z}_p \to \mathbb{Z}_p \\ \xi \mapsto \xi^\ell \end{array} \,\middle|\, \ell = 1, 2, \dots, p-1 \right\}$

  $((\mathbb{Z}/p)^\times, \cdot) = \left\{ \begin{array}{c} [1]_p, [2]_p, \dots, [p-1]_p \\ [x]_p \cdot [y]_p := [xy]_p \end{array} \right\}$

Possibilities for im ψ_ℓ $= \{ \phi_\ell \mid \underbrace{\phi_\ell \circ \phi_\ell \circ \dots \circ \phi_\ell}_{q \text{ times}} = id_{\mathbb{Z}/p} \}$ (order q)

$= \{ [x]_p \in (\mathbb{Z}/p)^\times, \ [x]_p^\ell = [1]_p \}$

$= \left\{ n \in \mathbb{Z}/p \,\middle|\, \begin{array}{c} n \not\equiv 1 \mod p \\ n^\ell \equiv 1 \mod p \end{array} \right\}$ ..... ?

Big idea: Choosing a different ℓ

$\Rightarrow$ Choosing a new generator for $(\mathbb{Z}/p)^\times$

```tikzcd
\mathbb{Z}/q \arrow[r, "\psi_{\ell_0}"] \arrow[rr, bend right, "\psi_\ell"'] & Aut\, \mathbb{Z}/p \arrow[r, "\tau_\ell", "\text{choose new gen}"'] & Aut\, \mathbb{Z}/p
```

All ψ_ℓ arise this way

$\mathbb{Z}/p \rtimes_{\psi_{\ell_0}} \mathbb{Z}/q \cong \mathbb{Z}/p \rtimes_{\tau_\ell \circ \psi_{\ell_0}} \mathbb{Z}/q$

$= \mathbb{Z}/p \rtimes_{\psi_\ell} \mathbb{Z}/q$

All isomorphic.

$\rightsquigarrow$ Common question!

(pqr: maybe difficult!)