Title

D. Zack Garza

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Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on $\mathbb{R}^n$
- Holomorphic functions on $\mathbb{C}$

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on $\mathbb{R}$ a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \left\{ f : U \to \mathbb{R} \;\middle|\; f(x) = c \right\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$. Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0, 1)$ and $f_2 = 2$ on $(2, 3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf $\mathcal{G}$ on $\mathbb{R}$ given by *locally* constant function, i.e. $\left\{ f : U \to \mathbb{R} \;\middle|\; \forall p \in U, \exists U_p \ni p,\ f|_{U_p} \text{ is constant} \right\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 1.1.
Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open. Then

define a sheaf by

$$\begin{aligned}
\emptyset &\mapsto 0 \\
\{p\} &\mapsto R \\
\{q\} &\mapsto S \\
\implies \{p,q\} &\mapsto R \times S,
\end{aligned}$$

where the sheaf condition forces the assignment of the whole space to be the product. Note that the first 3 assignments are automatically compatible, which means that we need a unique $f \in \mathcal{F}(X)$ restricting to R and S. In other words, $\mathcal{F}(X)$ needs to be unique and have maps to R, S, but this is exactly the universal property of the product.

Example 1.2.
Consider the presheaf on X given by $\mathcal{F}(X) = R \times S \times T$. Taking $T = \mathbb{Z}/2\mathbb{Z}$, we can force uniqueness to fail: by projecting to R, S, there are two elements in the fiber, namely $(r, s, 0) \mapsto r, s$ and $(r, s, 1) \mapsto r, s$.

Example 1.3.
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Can check that it's closed under finite intersections and arbitrary unions, so this forms a topology. Now make the assignments

$$\begin{aligned}
\{a\} &\mapsto A \\
\{b\} &\mapsto B \\
\{a,b\} &\mapsto C \\
X &\mapsto ?.
\end{aligned}$$

We have a situation like this:

$$\begin{tikzcd}
 & \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\
B \arrow[dr] & & C \arrow[dl] \\
 & A \arrow[d] & \\
 & \emptyset &
\end{tikzcd}$$

Unique gluing says that given $r \in B, s \in C$ such that $\varphi_B(r) = \varphi_C(s)$, there should exist a unique $t \in \mathcal{F}(X)$ such that $t|_{\{a,b\}} = r$ and $t|_{\{a,c\}} = s$. This recovers exactly the fiber product.

$$B \times_A C := \{\}.$$