Title

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1 Lecture 10

Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of **??**; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\mathrm{zar}}, X_{\text{ét}}, X_{\mathrm{fppf}}$, then the data of a GL_n-torsor split by a τ-cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d] \\ X \end{tikzcd}$$

That U trivializes our torsor means that $\pi^* T = \pi^* G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\begin{tikzcd} \pi_1^* \pi^* T \arrow[r, "\sim"] \arrow[d] & \pi_2^* \pi^* T \arrow[d] \\ \pi_1^* \pi^* G \arrow[r, "\sim"] & \pi_2^* \pi^* G \end{tikzcd}$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n-torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{ét}}, \mathrm{GL}_n)$ is in bijection with n-dimensional vector bundles on X.

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Exercise 1.0.3*(?)*: See if Hilbert 90 is true for groups other than GL_n.

1.1 Representability and Local Triviality

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).
You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:

$$\begin{tikzcd}
T \times_X T \arrow[r] \arrow[d, shift right] & T \arrow[d] \\
T \arrow[u, shift right, bend right, "\exists"'] \arrow[r, "f"] & X
\end{tikzcd}$$

The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that

$$\begin{tikzcd}
& T \times_X T \arrow[r] \arrow[d, shift right] & T \arrow[d] \\
& T \arrow[u, shift right, bend right, "\exists"'] \arrow[r, "f"] & X \\
U \arrow[ur, hook, "\text{closed}"] \arrow[urr, bend right, "\exists\text{ét}"'] & &
\end{tikzcd}$$

Here "slicing" means finding such a U, and this can be done using the structure theorem for smooth morphisms.

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Example 1.1.5 ***(non-smooth group schemes):***

- α_p, the kernel of Frobenius on $\mathbb{A}^1$ or $\mathbb{G}_a$,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on $\mathbb{G}_m$,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$, etc.

1.1.1 What Hilbert 90 Means

Example 1.1.6 *(?):* Let $X = \operatorname{Spec} k, n = 1$, so we're looking at $H^{\cdot}(\operatorname{Spec} k, \mathbb{G}_m)$.

$$\begin{aligned}H^1((\operatorname{Spec} k)_{\text{zar}}, \mathbb{G}_m) &= 0 \\ &= H^1((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m) \\ &= H^1(\operatorname{Gal}(k^s/k), \bar{k}^\times).\end{aligned}$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.[1]

Let's generalize this observation.

Example 1.1.7 *(?):* Let X be any scheme and $n = 1$, then $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

Example 1.1.8 *(?):* Let's compute $H^1(X_{\text{ét}}, \mu_\ell)$ where ℓ is an invertible function on X. We have a SES of étale sheaves, the **Kummer sequence**,

$$1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{z \mapsto z^p} \mathbb{G}_m \to 1.$$

This is exact in the étale topology since adjoining an ℓth power of any function gives an étale cover. We get a LES in cohomology

$$\begin{tikzcd} & & 0 \arrow[dll] \\ H^0(X_{\text{ét}}, \mu_\ell) \arrow[r] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[r, "z \mapsto z^\ell"] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[dll] \\ {\color{red} H^1(X_{\text{ét}}, \mu_\ell)} \arrow[r] & \operatorname{Pic}(X) \arrow[r, "{[\ell]}"] & \operatorname{Pic}(X) \arrow[dll] \\ H^2(X_{\text{ét}}, \mu_\ell) \arrow[r] & \cdots & \end{tikzcd}$$

We know that $H^0(X_{\text{ét}}, \mathbb{G}_m)$ are invertible functions on X, and the red term is what we'd like to compute.

Suppose now $H^0(X, \mathcal{O}_X) = k = \bar{k}$, then $H^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k)$ since it is the kernel of the ℓth power map. We can also compute $H^1(X_{\text{ét}}, \mu_\ell)$, since our diagram reduces to

[1] This is called "90" since Hilbert numbered his theorems in at least one of his books.

$$\begin{tikzcd}
 & & 0 \arrow[dll, hook] \\
\mu_\ell(k) \arrow[r] & k^\times \arrow[r, "z\mapsto z^\ell"] & k^\times \arrow[dll, two heads, "\delta"'] \\
{\color{red}H^1(X_{\text{ét}},\mu_\ell)} \arrow[r] & \operatorname{Pic}(X)[\ell] \arrow[r, "{[\ell]}"] & \operatorname{Pic}(X) \arrow[dll] \\
H^2(X_{\text{ét}},\mu_\ell) \arrow[r] & \cdots &
\end{tikzcd}$$

where surjectivity of δ follows from the fact that $k = \bar{k}$ and thus every element has an ℓth root, making H^1 the kernel of $[\ell]$.

Example 1.1.9*(?)*: Let $X_{/k}$ with $k = \bar{k}$ with ℓ invertible in k, then (claim) $\underline{\mathbb{Z}/\ell\mathbb{Z}} \cong \mu_\ell$ given by sending a generator to some choice of a primitive ℓth root of unity. To be explicit, we have a representation $\underline{\mathbb{Z}/\ell\mathbb{Z}} = \hom(\,\cdot\,, \operatorname{Spec} k[t]/t(t-1)\cdots(t-\ell+1))$ and $\mu_\ell = \operatorname{Spec} k[t]/t^\ell - 1$. These are both disjoint unions of points, and hence schemes of dimension zero since ℓ is invertible in the base and the Chinese Remainder Theorem, so one can write down the isomorphism explicitly between the schemes and hence the functors they represent.

Corollary 1.1.10*(?)*.
If $\mu_\ell \subseteq k$, then

$$H^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) = H^i(X_{\text{ét}}, \mu_\ell).$$

Since the isomorphism depends on the choice of a primitive root, this will not be Galois equivariant, which will come up when we talk about Galois actions on étale cohomology. This already happens for H^0, since $G \curvearrowright \mathbb{Z}/\ell\mathbb{Z}$ trivially but not on μ_ℓ.

1.1.2 Geometric Interpretations

Let X be an affine scheme, we now know $H^1(X_{\text{ét}}, \mathbb{F}_p) = \operatorname{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_x)$, the Artin-Schreier map, and these are $\mathbb{F}_p$-torsors. We also know $H^1(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$ in terms of the LES if $k = \bar{k}$ and $\operatorname{ch}(k) = p$, and this is a $\mathbb{Z}/\ell\mathbb{Z}$-torsor. Being torsors here geometrically means they're covering spaces with those groups as Galois groups.

Question 1.1.11: How does one write down these torsors/covering spaces?

Example 1.1.12*(?)*: Given

$$[Y] \in H^1(X_{\text{ét}}, \mathbb{F}_p) = \operatorname{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$$

where we write $[Y]$ to denote thinking of the torsor as some geometric object, how to we write down the covering space? Using Artin-Schreier, we can write $Y = \{y^p - y = a\}$ for some $a \in \mathcal{O}_X$, an **Artin-Schreier covering**.

If $\ell \neq \mathrm{ch}(k)$ and $[Z] \in H^1(X_{\text{ét}}, \mu_\ell)$ and assume $\mathrm{Pic}(X) = 0$. Then we can write

$$H^1(X_{\text{ét}}, \mu_\ell) = \mathrm{coker}(\mathcal{O}_X \xrightarrow{x \mapsto x^\ell} \mathcal{O}_X{}^\times)$$

In this case, $Z = \left\{z^\ell = f\right\}$ where $f \in \mathcal{O}_X{}^\times$ is an element representing the class in cohomology, and $\mu_\ell \curvearrowright Z$ by multiplication by z.

Remark 1.1.13: The process of explicitly writing down covers is called **explicit geometric class field theory**, which gives a recipe for writing down abelian covers of covers. In general, for $\mathrm{Pic}(X) \neq 0$, the Picard group plays a crucial role.

1.2 Computing the Cohomology of Curves

This is one of Daniel's favorite topics in the entire course!

Theorem 1.2.1 *(?).*
Let $X_{/k}$ be a smooth curve over $k = \bar{k}$, then

$$H^i(X_{\text{ét}}, \mathbb{G}_m) = \begin{cases} \mathcal{O}_X(X)^\times & i = 0 \\ \mathrm{Pic}(X) & i = 1 \\ 0 & \text{else,} \end{cases}$$

noting that $\mathcal{O}_X(X)^\times$ are the global sections of $\mathbb{G}_m$, i.e. invertible functions on X.

The first two cases we've done, $i > 1$ is the hard case.

Corollary 1.2.2 *(?).*
For X a smooth proper connected curve of genus g, $k = \bar{k}$, and $\ell \neq \mathrm{ch}(k)$ is prime,

$$H^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & i = 0 \\ \mathrm{Pic}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g} & i = 1 \\ \mathbb{Z}/\ell^n\mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}.$$

Proof (of corollary).
We'll use some theory of abelian varieties: $\mathrm{Pic}^0(X) = \mathrm{Jac}(X)$, and we have a SES

$$0 \to \mathrm{Jac}(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where we identify the Néron-Severi group as $\mathbb{Z}$.[a] We'll use that $\mathrm{Jac}(X)$ is a g-dimensional abelian variety, and so $\mathrm{Jac}(X)[\ell^n] \cong_{\mathrm{Grp}} (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$.
The Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

yields a LES where we identify $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$:

$$\begin{tikzcd}
 & & 0 \arrow[dll] \\
H^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "{[\ell]}"] & \mathrm{Pic}(X) \arrow[dll] \\
H^2(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So we're just computing the kernel and cokernel of $[\ell]$.

Computing H^1: We'll need one more fact: $\mathrm{Jac}(X)(\bar{k})$ is a divisible group. We can identify

$$H^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \mathrm{Pic}(X)[\ell^n] = \mathrm{Jac}(X) = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}.$$

where the 2nd equality uses the fact that $\mathrm{Pic}(X)$ is an extension of $\mathbb{Z}$ by an abelian variety and $\mathbb{Z}$ has no torsion, and the last equality is general theory of abelian varieties.

Computing H^2: Since $\mathrm{Jac}(X)$ is divisible, we can identify

$$\mathrm{coker}(\mathrm{Pic}(X) \xrightarrow{[\ell^n]} \mathrm{Pic}(X)) \cong \mathrm{coker}(\mathbb{Z} \xrightarrow{[\ell^n]} \mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z}.$$

The vanishing of higher cohomology follows from the vanishing for $\mathbb{G}_m$. So assuming the theorem and the theory of abelian varieties proves this corollary.

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[a] See Hartshorne Ch. 4, or anything that discusses cohomology of curves.

Exercise 1.2.3*(?)*: Check this using the snake lemma after applying multiplication by ℓ to the SES.

Remark 1.2.4: X is a scheme over $\bar{k}$, and if it started over some subfield L then $\mathrm{Gal}(L_{/k}) \curvearrowright X$ and thus the corresponding functors. These isomorphisms will not be Galois equivariant, and the $\mathbb{Z}/\ell^n\mathbb{Z}$ showing up in degree 2 cohomology will admit a Galois action via the cyclotomic character.

1.2.1 Proof of Theorem

Goal: we want to show that $H^{>1}(X_{\text{ét}}, \mathbb{G}_m) = 0$ for X a smooth curve over $k = \bar{k}$. Three ingredients:

1. The Leray spectral sequence,
2. The divisor exact sequence,
3. Brauer groups.

1.3 Pushforwards and the Leray Spectral Sequence

Suppose $X \xrightarrow{f} Y$ is a morphism of schemes, then we get a functor $f_* \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$: given $\mathcal{F} \in Sh(X_{\text{ét}})$, we have $f_* \mathcal{F}(U \to Y) \coloneqq \mathcal{F}(U \times_Y X)$. This is left-exact and thus has right-derived functors $R^{\cdot} f_* : \mathrm{Sh}^{\mathrm{Ab}}(X_{\text{ét}}) \to \mathrm{Sh}^{\mathrm{Ab}}(Y_{\text{ét}})$.

How to think about this:

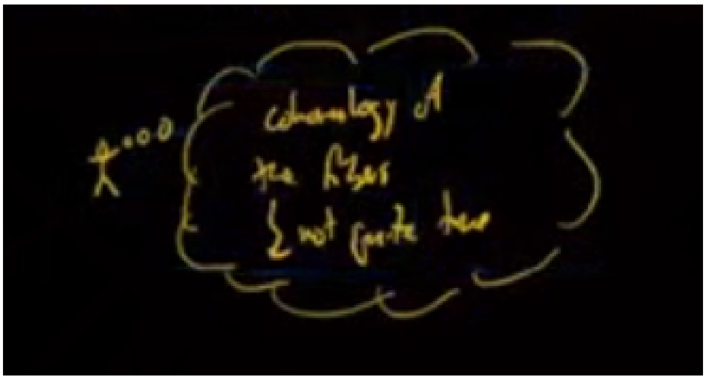

Figure 1: Cohomology of the fibers: but not quite!

This is not quite true, and the obstruction is called **the base change property**, which we'll see later in the course. What's true in general is that $R^i f_* \mathcal{F}$ is the sheafification of the presheaf $V \to H^i(f^{-1}(V), \mathcal{F})$, which is not quite the cohomology of the fibers since sheafification is somewhat brutal.

Proposition 1.3.1 ***(Derived pushforwards for finite morphisms).***
If f is a finite morphism (e.g. a closed immersion) then $R^{>0} f_* = 0$.

Exercise 1.3.2 ***(Proof, must-do!):*** Prove this. The claim is that f_* is right-exact, which in this case shows it is exact. Check on stalks. Compute that the stalk of $f_* \mathcal{F}$ at $\bar{y} \in Y$ is given by

$$f_* \mathcal{F}_{\bar{y}} = \bigoplus_{\bar{x} \in f^{-1}(\bar{y})} \mathcal{F}_{\bar{x}}$$

for f a finite morphism (not necessarily unramified).

Proposition 1.3.3 ***(technical).***
f_* preserves injectives.

Exercise 1.3.4 ***(proof):*** Prove this! You can do this by showing the following fact from category theory: this is true for any functor with an exact left adjoint, which here is f^* and is exact since filtered colimits and sheafification are both exact, or alternatively you can check on stalks, since the stalks of f^{-1} are the stalks of the original functor.

Corollary 1.3.5 ***(The Leray Spectral Sequence).***
Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are morphisms of schemes, then there is a spectral sequence

$$R^i g_* R^j f_* \mathcal{F} \Rightarrow R^{i+j}(g \circ f)_* \mathcal{F}.$$

As a special case, for $Z = \operatorname{Spec} k$ with $k = \bar{k}$, then g_*, f_* are taking global sections so we get

$$H^i(Y, R^j f_* \mathcal{F}) \Rightarrow H^{i+j}(X, \mathcal{F}).$$

Proof (sketch).
There is a general statement (see Tohoku): given two functors between abelian functors where the first preserves injectives, you get such a spectral sequence. How to explicitly compute this: we can take an injective resolution $\mathcal{F} \to \mathcal{I}^\cdot$ and compute

$$R^i f_* \mathcal{F} \mathcal{H}^i(f_* \mathcal{I}^\cdot).$$

$f_* \mathcal{I}$ is a complex of injectives, and we want $\mathcal{H}^{i+j}(g_* f_* \mathcal{I}^\cdot) = R^{i+j}(g \circ f)_* \mathcal{F}$, and the content here is that we don't have to take an additional injective resolution of $f_* \mathcal{I}$. Now take the spectral sequence of the filtered complex $f_* \mathcal{I}^\cdot$ where the filtration is by the truncations $\tau_{\leq p} f_* \mathcal{I}^\cdot$ where you replace the pth term with the kernel of the differential and zero beyond this point. An example of a differential is given by the following: there are SESs

$$0 \to \tau_{\leq p} f_* \mathcal{I}^\cdot \to \tau_{\leq p+1} f_* \mathcal{I}^\cdot \to \mathcal{H}^{p+1}(f_* \mathcal{I}^\cdot) = R^{p+1} f_* \mathcal{F} \to 0,$$

and applying RG_* yields a map

$$R^{p+1} f_* \mathcal{F} \xrightarrow{\delta} R^{q+1} g_* \tau_{\leq p} f_* \mathcal{I}^\cdot,$$

and after some splicing this δ will be the differential. ∎