

# Real Analysis

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## Contents

<b>1 Lecture 1 (Thu 15 Aug 2019 11:04)</b>	<b>1</b>
1.1 Notions of “smallness” in $\mathbb{R}$ . . . . .	1
1.2 $\mathbb{R}$ is not small . . . . .	2

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## 1 Lecture 1 (Thu 15 Aug 2019 11:04)

See Folland’s Real Analysis, definitely a recommended reference.

Possible first day question: how can we “measure” a subset of  $\mathbb{R}$ ? We’d like bigger sets to have a higher measure, we wouldn’t want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of  $\mathbb{R}$ . We’ll come back to this in a few lectures.

### 1.1 Notions of “smallness” in $\mathbb{R}$

Definition: Let  $E$  be a set, then  $E$  is *countable* if it is in a one-to-one correspondence with  $E' \subseteq \mathbb{N}$ , which includes  $\emptyset, \mathbb{N}$ .

Definition:  $E$  is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of  $\mathbb{R}$  is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a  $X \subseteq Y$  is dense in  $Y$  iff the closure of  $X$  is all of  $Y$ . So we’ll make the following definition.

Definition: A set  $A \subseteq \mathbb{R}$  is *nowhere dense* if every interval  $I$  contains a subinterval  $S \subseteq I$  such that  $S \subseteq A^c$ .

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we’re giving a name to such a countable union above. Example:  $\mathbb{Q}$  is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, -  $A^c$  contains a dense, open set. - The interior of the closure is empty.

We’d like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than  $\varepsilon$ .

Definition:  $E$  is a *null set* (or has *measure zero*) if  $\forall \varepsilon > 0$ , there exists a sequence of intervals  $\{I_j\}_{j=1}^\infty$  such that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable  $\implies$  Meager, but not the converse.
- Countable  $\implies$  Null, but not the converse.

Exercise: Show that the “middle third” Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem: Every  $E \subseteq \mathbb{R}$  can be written as  $E = A \cup B$  where  $A$  is null and  $B$  is meager.

This gives some information about how nullity and meagerness interact – in particular,  $\mathbb{R}$  itself is neither meager nor null. Idea: if meager  $\implies$  null, this theorem allows you to write  $\mathbb{R}$  as the union of two null sets. This is bad!

Proof: We can assume  $E = \mathbb{R}$ . Take an enumeration of the rationals, so  $\mathbb{Q} = \{q_j\}_{j=1}^\infty$ . Around each  $q_j$ , put an interval around it of size  $1/2^{j+k}$  where we’ll allow  $k$  to vary, yielding multiple intervals around  $q_j$ . To do this, define  $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 1/2^{j+k})$ . Now let  $G_k = \bigcup_j I_{j,k}$ . Finally, let  $A = \bigcap_k G_k$ ; we claim that  $A$  is null.

Note that  $\sum_j |I_{j,k}| = \frac{1}{2^k}$ , so just pick  $k$  such that  $\frac{1}{2^k} < \varepsilon$ .

Now we need to show that  $A^c := B$  is meager. Note that  $G_k$  covers the rationals, and is a countable union of open sets, so it is dense. So  $G_k$  is an open and dense set. By one of the equivalent formulations of meagerness, this means that  $G_k^c$  is nowhere dense. But then  $B = \bigcup_k G_k^c$  is meager.

## 1.2 $\mathbb{R}$ is not small

Theorem A (Cantor):  $\mathbb{R}$  is not countable.

Theorem B (Baire):  $\mathbb{R}$  is not meager. (Baire Category Theorem)

Theorem C (Borel):  $\mathbb{R}$  is not null.

Note that theorems B and C imply theorem A. You can also replace  $\mathbb{R}$  with any nonempty interval  $I = [a, b]$  where  $a < b$ . which is a strictly stronger statement – if any subset of  $\mathbb{R}$  is not countable, then certainly  $\mathbb{R}$  isn’t, and so on.