Problem Set 4

D. Zack Garza

February 4, 2020

Contents

1 The Fundamental Group

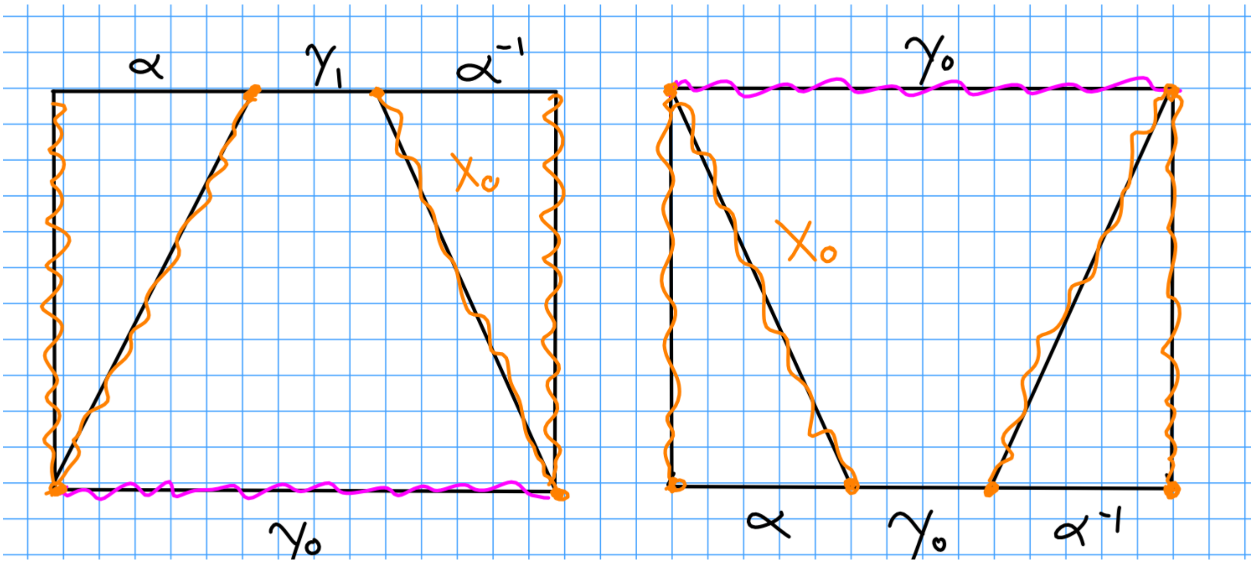
1.1 1

Proposition: $\gamma_1 \simeq \gamma_2 \iff \gamma_1, \gamma_2$ are conjugate in $\pi_1(X, x_0)$, i.e. $\exists [\alpha] \in \pi_1$ such that $[\gamma_1] = [\alpha][\gamma_2][\alpha]^{-1}$.

Proof:

$\implies$: Clear, since $\gamma_1 \sim \gamma_2 \implies [\gamma_1] = [\gamma_2] \in \pi_1(X)$, so take $\alpha(t) = x_0$ the constant loop for all t.

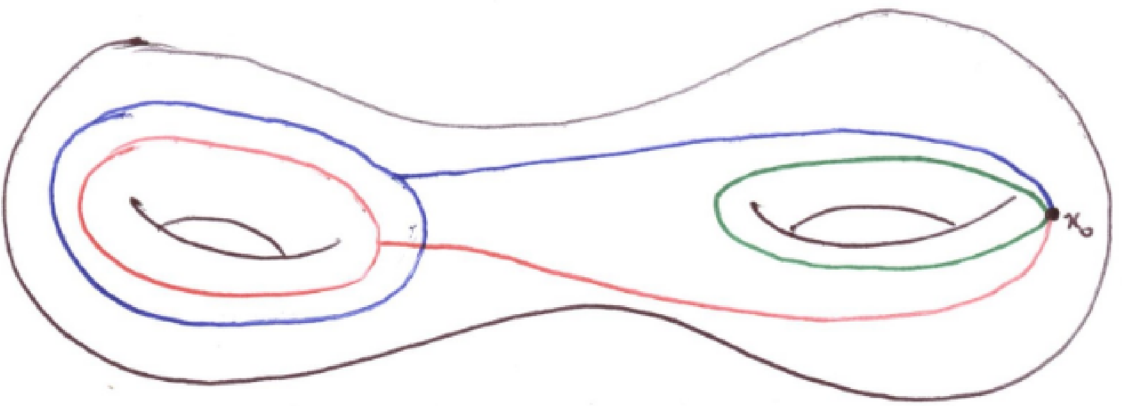
$\impliedby$: ? Forgot how these arguments go.



■

Counterexample where homotopic loops are not equal in π_1, but just conjugate:

It's not a great picture, but the blue and red loops below are freely homotopic, but not homotopic relative to the basepoint x_0. In $\pi_1(X, x_0)$, they are conjugate via the green loop.



2 Covering Spaces

2.1 1b

Homotopy lifting property:

$$
\begin{tikzcd}
 & \tilde{X} \arrow[d, "\pi"] \\
Y \times I \arrow[r, "H"] \arrow[ur, dashed, "\exists \tilde{H}"] & X
\end{tikzcd}
$$

π clearly induces a map p_* on π_1 by functoriality, so we'll show that $\ker p_*$ is trivial. Let $\gamma : S^1 \to \tilde{X} \in \pi_1(\tilde{X})$ and suppose $\alpha := p_*(\gamma) = [e] \in \pi_1(X)$. We'll show $\gamma \simeq [e]$ in $\pi_1(\tilde{X})$.

Since $\alpha = [e]$, $\alpha \simeq$ const. and thus there is a homotopy $H : I \times S^1 \to X$ such that $H_0 = \text{const.}(x_0)$ and $H_1 = \gamma$. By the HLP, this lifts to $\tilde{H} : I \times S^1 \to \tilde{X}$. Noting that $\pi^{-1}(\text{const.}(x_0))$ is still a constant loop, this says that γ is homotopic to a constant loop and thus nullhomotopic.

2.2 1c

Since both spaces are path-connected, the degree o the covering map π is precisely the index of the included fundamental group. This forces π to be a degree 1 covering and hence a homeomorphism.

2.3 6

Note $\pi_1 \mathbb{RP}^2 = \mathbb{Z}/2\mathbb{Z}$, so $\pi_1 X = (\mathbb{Z}/2\mathbb{Z})^2$.

The pullback of any neighborhood of the basepoint needs to be locally homeomorphic to one of

- $S^2 \vee S^2$
- $\mathbb{RP}^2 \vee S^2$

And so *all* possibilities for regular covering spaces are given by

- $\vee_{2k} S^2$ "beads" wrapped into a necklace for any $k \geq 1$
- $\mathbb{RP}^2 \vee (\bigvee^k S^2) \vee \mathbb{RP}^2$
- $\vee^\infty S^2$, the universal cover

To get a threefold cover, we want the basepoint to lift to three preimages, so we can take

- $S^2 \vee S^2 \vee S^2$ wrapped
- $\mathbb{RP}^2 \vee S^2 \vee \mathbb{RP}^2$.

2.4 7

- $\mathbb{RP}_3 \vee S^2 \vee \mathbb{RP}^3$, which has $\pi_2 = 0 * \mathbb{Z} * 0 = \mathbb{Z}$ since $\pi_{i \geq 1} X = \pi_{i \geq 1} \tilde{X}$ and $\tilde{\mathbb{RP}}^3 = S^3$.
- $\mathbb{RP}^2 \vee S^3 \vee \mathbb{RP}^2$, which was $\pi_2 = \mathbb{Z} * 0 * \mathbb{Z} = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}$

2.5 8

Yes,

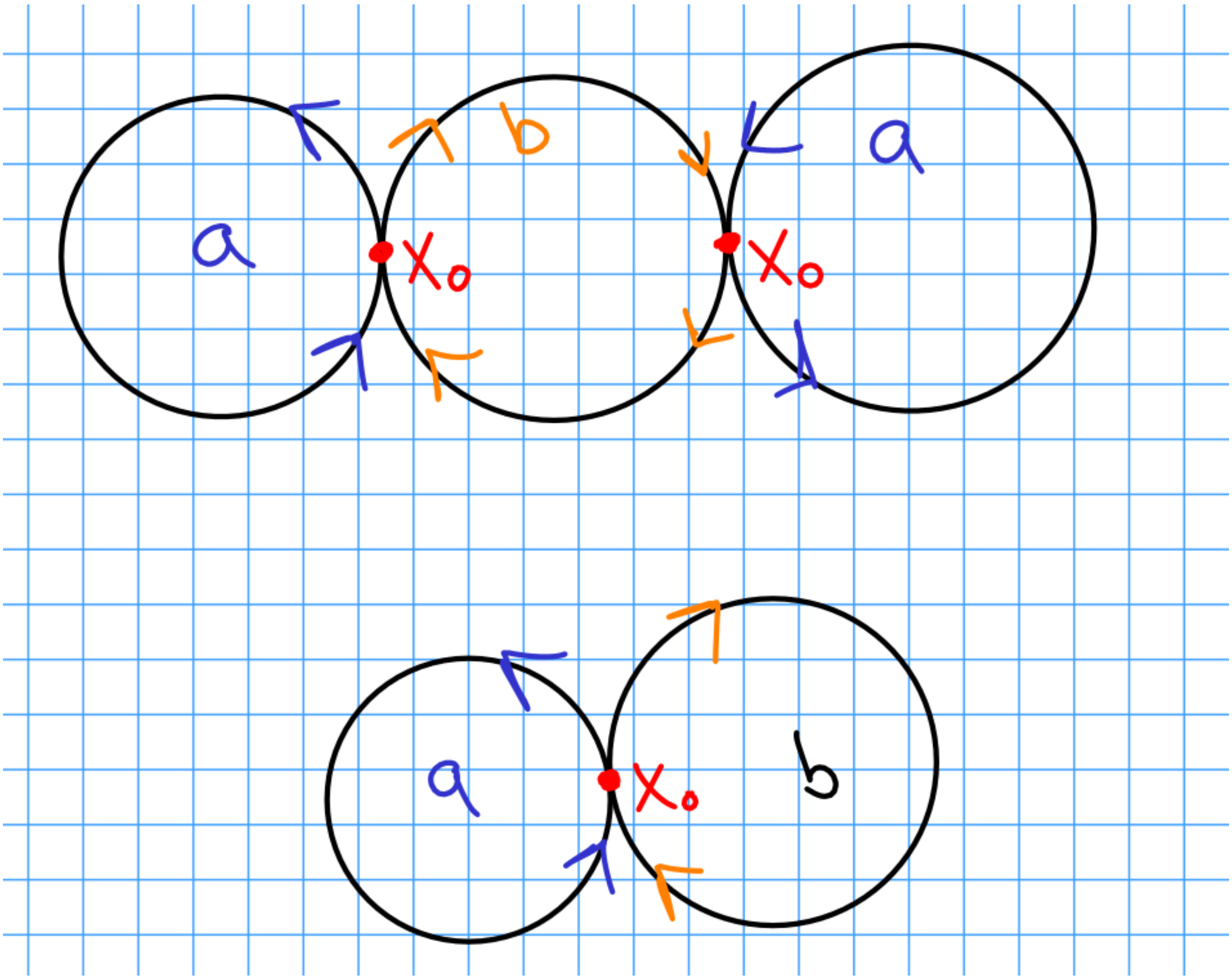


Figure 1: Image