Title

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Last time: projective varieties $V(f_i) \subset \mathbb{P}^n_{/k}$ with f_i homogeneous. We proved the projective nullstellensatz: for any projective variety X, we have $V_p(I_p(X))$ and for any homogeneous ideal I with $\sqrt{I} \neq I_0$ the irrelevant ideal, $I_p(V_p(I)) = \sqrt{I}$. Recall that $I_0 = \langle x_0, \cdots, x_n \rangle$. We had a notion of a projective coordinate ring, $S(X) := k[x_1, \cdots, x_n]/I_p(X)$, which is a graded ring since $I_p(X)$ is a homogeneous ideal.

Note that $S(X)$ is not a ring of functions on X: e.g. for $X = \mathbb{P}^n$, $S(X) = k[x_1, \cdots, x_n]$ but x_0 is not a function on $\mathbb{P}^n$. This is because $f([x_0 : \cdots : x_n]) = f([\lambda x_0 : \cdots : \lambda x_n])$ but $x_0 \neq \lambda x_0$. It still makes sense to ask if f is zero, so $V_p(f)$ is a well-defined object.

Definition 1.0.1 (Dehomogenization of functions and ideals)**.**
Let $f \in k[x_1, \cdots, x_n]$ be a homogeneous polynomial, then we define its dehomogenization as

$$f^i := f(1, x_1, \cdots, x_n) \in k[x_1, \cdots, x_n].$$

For a homogeneous ideal, we define

$$J^i := \left\{ f^i \;\middle|\; f \in J \right\}.$$

Example 1.0.1 : This is usually not homogeneous. Take

$$\begin{aligned} f &= x_0^3 + x_0 x_1^2 + x_0 x_1 x_2 + x_0^2 + x_1 \\ \implies f^i &= 1 + x_1^2 + x_1 x_2 + x_1, \end{aligned}$$

where has terms of mixed degrees.

Remark 1.0.1 :

- $(fg)^i = f^i g^i$,
- $(f+g)^i = f^i + g^i$

In other words, evaluating at $x_0 = 1$ is a ring morphism.

Definition 1.0.2 (Homogenization of a function)**.**
Let $f \in k[x_1, \cdots, x_n]$, then the **homogenization** of f is defined by

$$f^h := x_0^d f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

where $d := \deg(f)$.

Example 1.0.2 (?): Let $f(x_1, x_2) = 1 + x_1^2 + x_1x_2 + x_2^3$, then

$$f^h(x_0, x_1, x_2) = x_0^3 + x_0x_1^2 + x_0x_1x_2 + x_2^3,$$

which is a homogeneous polynomial of degree 3. Note that $(f^h)^i = f$.

Example 1.0.3 (?): It need not be the case that $(f^i)^h = f$. Take $f = x_0^3 + x_0x_1x_2$, then $f^i = 1 + x_1x_2$ and $(f^i)^h = x_0^2 + x_1x_2$. Note that the total degree dropped, since everything was divisible by x_0.

Remark 1.0.2 :

$$(f^i)^h = f \iff x_0 \nmid f.$$

Definition 1.0.3 (Homogenization of an ideal)**.**
Given $J \subset k[x_1, \cdots, x_n]$, define its **homogenization** as

$$J^h := \left\{ f^h \;\middle|\; f \in J \right\}.$$

Example 1.0.4 : This is not a ring morphism, since $(f+g)^h \neq f^h + g^h$ in general. Taking $f = x_0^2 + x_1$ and $g = -x_0^2 + x_2$, we have $f^h + g^h = x_0x_1 + x_0x_2$ while $(f+g)^h = x_12 + x_2$.

Remark 1.0.3 : What is the geometric significance? Set

$$U_0 := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n_{/k} \;\middle|\; x_0 \neq 0 \right\} \cong \mathbb{A}^n_{/k}$$

with coordinates $\left[\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right]$.

Proposition 1.0.1 ***(?).***
The conclusion is thus that U_0 with the subspace topology is equal to $\mathbb{A}^n$ with the Zariski topology.

Proof (?).
If we define the Zariski topology on $\mathbb{P}^n$ as having closed sets $V_p(I)$, we would want to check that $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$ is closed in the subspace topology. This amounts to showing that $V_p(I) \cap U_0$ is closed in $\mathbb{A}^n \cong U_0$. We can check that

$$V_p(f, f \in I) = \left\{ [x_0 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0 \, \forall f \in I \right\}.$$

Intersecting with U_0 yields $\left\{ [x_1 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0,\, x_0 \neq 0 \right\}$. Equivalently, we can rewrite this set as

$$\left\{ [x_1 : \cdots : x_n] \;\middle|\; f\left(\left[1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right]\right) = 0,\ f \text{ homogeneous} \right\}$$

Since these are coordinates on $\mathbb{A}^1$, we have $V_p(I) \cap U_0 = V_a(I^i)$ which is closed. Conversely, given a closed set $V(I)$, we can write this as $V(I) = U_0 \cap V_p(I^h)$.

■

Corollary 1.0.1 ***(?).***
$\mathbb{P}^n$ is irreducible of dimension n, where the proof is that its covered by irreducible topological spaces of dimension n with nonempty intersection combined with a fact from the exercises.

Example 1.0.5 (?): Consider $f(x_1, x_2) = x_1^2 - x_2^2 - 1$ and consider $V(f) \subset \mathbb{A}^2_{/\mathbb{C}}$:

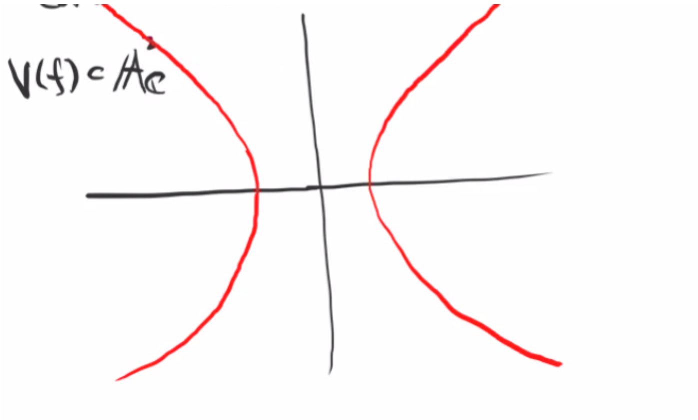

Figure 1: Image

Note that for real projective space, we can view this as a sphere with antipodal points identified. We can thus visualize this in the following way:

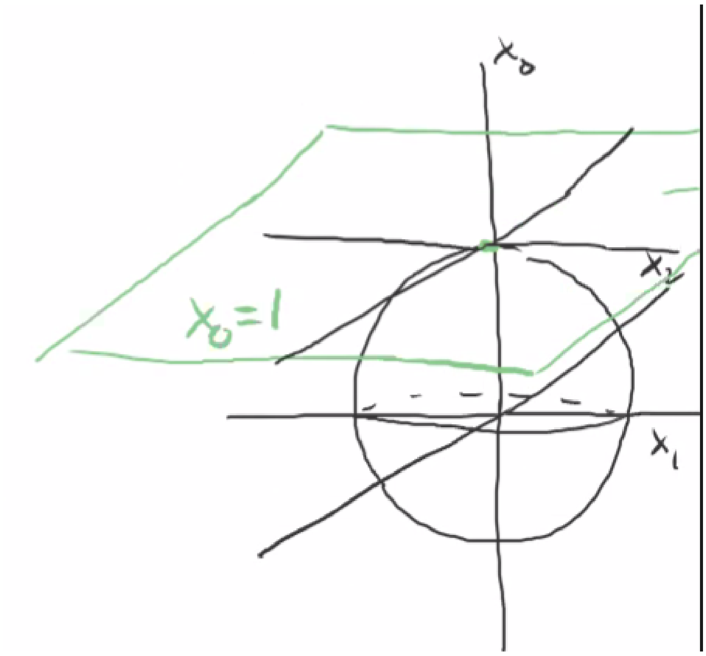

Figure 2: O

We can normalize the x_0 coordinate to one, hence the plane. We can also project $V(f)$ from the plane onto the sphere, mirroring to antipodal points:

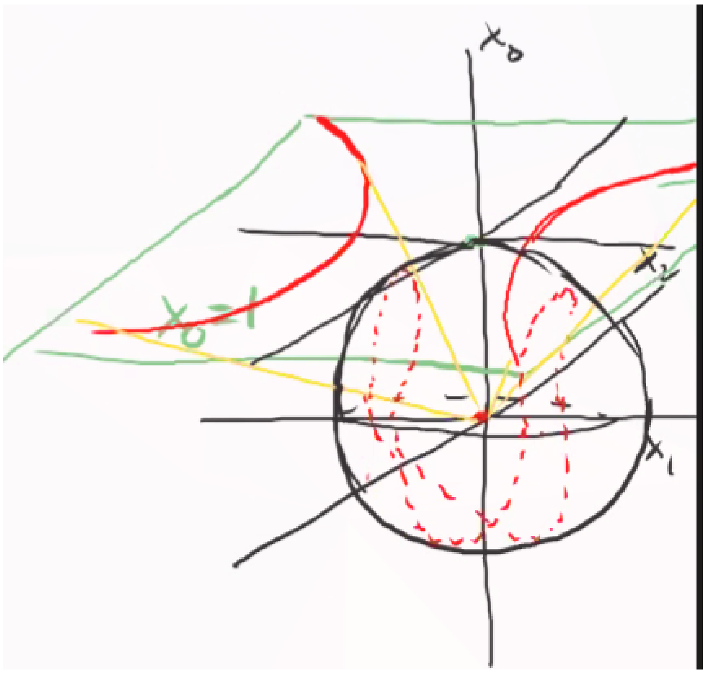

Figure 3: Image

This misses some points on the equator, since we aren't including points where $x_0 = 0$. Consider the homogenization $V(f^h) \subset \mathbb{P}^2_{/\mathbb{C}}$. It's given by $f^h = x_1^2 - x_2^2 - x_0^2$, then

$$V(f^h) \cap V(x_0) = \left\{ [0 : x_1 : x_2] \;\middle|\; f^h(0, x_1, x_2) = 0 \right\} = \{[0 : 1 : 1], [0 : 1 : -1]\},$$

which can be seen in the picture as the points at infinity:

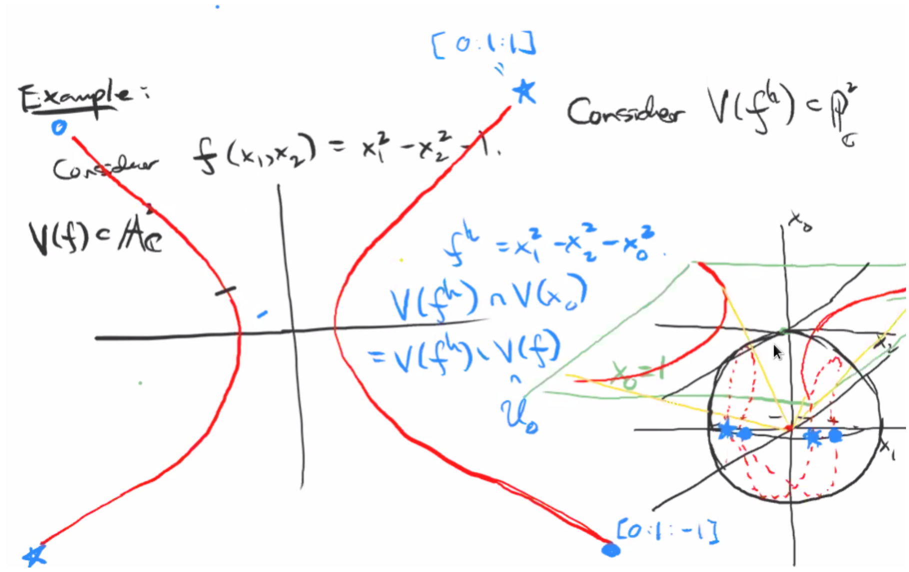

Figure 4: A

Note that the equator is $V(x_0) = \mathbb{P}^2_{/\mathbb{C}} \setminus U_0 \cong \mathbb{P}^2 \setminus \mathbb{A}^2$. So we get a circle of points at infinity, i.e. $V(x_0) = \mathbb{P}^1 = \{[0 : v_1 : v_2]\}$.

Example 1.0.6 (?): Consider $V(f)$ where f is a line in $\mathbb{A}^2_{/\mathbb{C}}$, say $f = ax_1 + bx_2 + c$. This yields $f^h = ax_1 + bx_2 + cx_0$ and we can consider $V(f^h) \cong \mathbb{P}^2_{\mathbb{C}}$. We know $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere and $\mathbb{A}^1_{/\mathbb{C}}$ is a point:

Figure 5: $\mathbb{P}^1_{\mathbb{C}}$

The points at infinity correspond to

$$V(f^h) = V(f^h) \cap V(x_0) = \{[0 : -b : a]\},$$

which is a single point not depending on c.

Remark 1.0.4 : $\mathbb{P}^2_{/k}$ for any field k is a **projective plane**, which satisfies certain axioms:

1. There exists a unique line through any two distinct points,

2. Any two distinct lines intersect at a single point.

A famous example is the *Fano plane*:

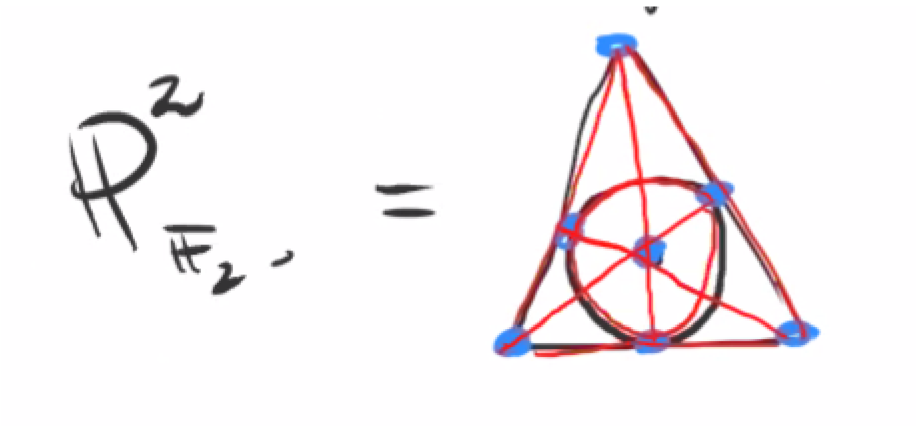

Figure 6: Fano Plane

Why is this true? $\mathbb{P}^2_{/k}$ is the set of lines in k^3, and the lines in $\mathbb{P}^2_{/k}$ are the vanishing loci of homogeneous polynomials and also planes in k^3, since any two lines determine a unique plane and any two planes intersect at the origin.

Proposition 1.0.2 ***(?).***
Let $J \subset k[x_1, \cdots, x_n]$ be an ideal. Let $X \coloneqq V_a(J) \subset \mathbb{A}^n$ where we identify $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$. Then the closure $\overline{X} \subset \mathbb{P}^n$ is given by $\overline{X} = V_p(J^h)$. In particular, $V_a(J) = V_p(J^h)$.

Proof (?).
Note that it's clear that $V_p(J^h)$ is closed and contains $V_a(J)$. For the reverse containment, let $Y \supseteq X$ be closed; we want to show that $Y \supseteq V_p(J^h)$. Since Y is closed, $Y = V_p(J')$ where J' is some homogeneous ideal. Any element $f' \in J'$ satisfies $f' = x^d f$ for some maximal d where $x_0^d f$ vanishes on X. We also have $f = 0$ on X since $X \subset U_0$. We can compute

$$f \in I_a(X) = I_a(V_a(J)) = \sqrt{J},$$

so $f^m \in J$. Then $(f^h)^m \in J^h$ for some m, and this $f^h \in \sqrt{J^h}$.

We can conclude that $J' \subset \sqrt{J}$, which shows that $V_p(J') \supseteq V_p(J^h)$ as desired. ■

Definition 1.0.4 (Projective Closure)**.**
The **projective closure** of $X = V_a(J)$ is the smallest closed subset containing X and is given by

$$\overline{X} = V_p(J^h).$$