Title

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Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on $\mathbb{R}^n$
- Holomorphic functions on $\mathbb{C}$

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on $\mathbb{R}$ a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \left\{ f : U \to \mathbb{R} \;\middle|\; f(x) = c \right\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$. Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0,1)$ and $f_2 = 2$ on $(2,3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf $\mathcal{G}$ on $\mathbb{R}$ given by *locally* constant function, i.e. $\left\{ f : U \to \mathbb{R} \;\middle|\; \forall p \in U, \exists U_p \ni p,\ f|_{U_p} \text{ is constant} \right\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 1.1.
Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open. Then

define a sheaf by

$$\begin{aligned} \emptyset &\mapsto 0 \\ \{p\} &\mapsto R \\ \{q\} &\mapsto S \\ \implies \{p,q\} &\mapsto R \times S, \end{aligned}$$

where the sheaf condition forces the assignment of the whole space to be the product. Note that the first 3 assignments are automatically compatible, which means that we need a unique $f \in \mathcal{F}(X)$ restricting to R and S. In other words, $\mathcal{F}(X)$ needs to be unique and have maps to R, S, but this is exactly the universal property of the product.

Example 1.2.
Consider the presheaf on X given by $\mathcal{F}(X) = R \times S \times T$. Taking $T = \mathbb{Z}/2\mathbb{Z}$, we can force uniqueness to fail: by projecting to R, S, there are two elements in the fiber, namely $(r,s,0) \mapsto r, s$ and $(r,s,1) \mapsto r, s$.

Example 1.3.
Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}\}$. Can check that it's closed under finite intersections and arbitrary unions, so this forms a topology. Now make the assignments

$$\begin{aligned} \{a\} &\mapsto A \\ \{b\} &\mapsto B \\ \{a,b\} &\mapsto C \\ X &\mapsto ?. \end{aligned}$$

We have a situation like this:

$$\begin{tikzcd} & \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & C \arrow[dl] \\ & A \arrow[d] & \\ & \emptyset & \end{tikzcd}$$

Unique gluing says that given $r \in B, s \in C$ such that $\varphi_B(r) = \varphi_C(s)$, there should exist a unique $t \in \mathcal{F}(X)$ such that $t|_{\{a,b\}} = r$ and $t|_{\{a,c\}} = s$. This recovers exactly the fiber product.

$$B \times_A C \coloneqq \left\{ (r,s) \in B \times C \;\middle|\; \varphi_B(r) = \varphi_C(s) \in A \right\}.$$

Example 1.4.
Let X be an affine variety with the Zariski topology and let $\mathcal{F} := \mathcal{O}_X$ be the sheaf of regular functions:

$$\mathcal{O}_X(U) := \left\{ f : U \to k \;\middle|\; \forall p \in U,\, \exists U_p \ni p,\;\; f|_{U_p} = \frac{g_p}{h_p} \right\}.$$

Is this a presheaf? We can check that there are restriction maps:

$$\begin{aligned} \mathcal{O}_X(U) &\to \mathcal{O}_X(V) \\ \{f : U \to K\} &\mapsto \{\, f|_V(x) := f(x) \text{ for } x \in V\}\,. \end{aligned}$$

This makes sense because if $V \subset U$, any $x \in V$ is in the domain of f. Given that f is locally a fraction, say $\rho = g_p/h_p$ on $U_p \ni p$, is $\varphi|_V$ locally a fraction? Yes: for all $p \in V \subset U$, $\varphi = g_p/f_p$ on U_p and this remains true on $U_p \cap V$.

To check that $\mathcal{O}_X$ is a sheaf, given a set of regular functions $\{\varphi_i : U_i \to k\}$ agreeing on intersections, define

$$\begin{aligned} \varphi : \cup U_i \to k \\ \varphi(x) := \varphi_i(x) \text{ if } x \in U_i. \end{aligned}$$

This is well-defined, since if $x \in U_i \cap U_j$, $\varphi_i(x) = \varphi_j(x)$ since both restrict to the same function on $U_i \cap U_j$ by assumption.

Why is φ locally a fraction? We need to check that for all $p \in U := \cup U_i$ there exists a $U_p \ni p$ with $\varphi|_{U_p} = g_p/h_p$. But any $p \in \cup U_i$ implies $p \in U_i$ for some i. Then there exists an open set $U_{i,p} \ni p$ in U_i such that $\varphi|_{U_{i,p}} = g_p/h_p$ by definition of a regular function. So take $U_p = U_{i,p}$ and use the fact that $\varphi|_{U_i} = \varphi_i$ along with compatibility of restriction.

Remark 1.
General observation: any presheaf of functions is a sheaf when the functions are defined by a local property, i..e any property that can be checked at p by considering an open set $U_p \ni p$.

As in the examples of smooth or holomorphic functions, these were local properties. E.g. checking that a function is smooth involves checking on an open set around each point. On the other hand, being a constant function is not a local property.