

# Full Notes

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## 1 Wednesday January 8

Course text: <http://math.uga.edu/~pete/integral2015.pdf>

Summary: The study of commutative rings, ideals, and modules over them.

The chapters we'll cover:

- 1 (Intro),
- 2 (Modules, partial),
- 3 (Ideals, CRT),
- 7 (Localization),
- 8 (Noetherian Rings),
- 11 (Nullstellensatz),
- 12 (Hilbert-Jacobson rings),
- 13 (Spectrum),
- 14 (Integral extensions),
- 17 (Valuation rings),
- 18 (Normalization),
- 19 (Picard groups),
- 20 (Dedekind domains),

- 22 (1-dim Noetherian domains)

In number theory, arises in the study of  $\mathbb{Z}_k$ , the ring of integers over a number field  $k$ , along with *localizations* and *orders* (both preserve the fraction field?).

In algebraic geometry, consider  $R = k[t_1, \dots, t_n]/I$  where  $k$  is a field and  $I$  is an ideal.

Some preliminary results:

1. In  $\mathbb{Z}_k$ , ideals factor uniquely into primes (i.e. it is a Dedekind domain).
2.  $\mathbb{Z}_k$  has an integral basis (i.e. as a  $\mathbb{Z}$ -modules,  $\mathbb{Z}_k \cong \mathbb{Z}^{[k:\mathbb{Q}]}$ ).
3. The Nullstellansatz: there is a bijective correspondence

$$\{\text{Irreducible Zariski closed subsets of } \mathbb{C}^n\} \iff \{\text{Prime ideals in } \mathbb{C}[t_1, \dots, t_n]\}.$$

4. Noether normalization (a structure theorem for rings of the form  $R$  above).

All of these results concern particularly “nice” rings, e.g.  $\mathbb{Z}_k, \mathbb{C}[t_1, \dots, t_n]$ . These rings are

- Domains
- Noetherian
- Finitely generated over other rings
- Finite Krull dimension (supremum of length of chains of prime ideals)
  - In particular,  $\dim \mathbb{Z}_k = 1$  since nonzero prime ideals are maximal in a Dedekind domain
- Regular (nonsingularity condition, can be interpreted in scheme-theoretic language)

Note: schemes will have “local charts” given by commutative rings, analogous to building a manifold from Euclidean  $n$ -space. General philosophy (Grothendieck): Every commutative ring is the ring of functions on some space, so we should study the category of commutative rings as a whole (i.e. let the rings be arbitrary). This does not hold for non-commutative rings! I.e. we can’t necessarily associate a geometric space to every non-commutative ring. A common interesting example:  $k[G]$ , the group ring of an arbitrary group. Good references: Lam, ‘Lectures on Modules and Rings’.

*Example:* Let  $X$  be a topological space and  $C(X)$  be the continuous functions  $f : X \rightarrow \mathbb{R}$ . This is a ring under pointwise addition/multiplication. (This generally holds for the hom set into any commutative ring.)

*Example:* Take  $X = [0, 1]$  and  $C(X)$  as a ring.

### Exercise:

1. Show that  $C(X)$  is not a domain. > Hint: find two nonzero functions whose product is identically zero, e.g. bump functions. > Note that they are not analytic/holomorphic.
2. Show that it is not noetherian (i.e. there is an ideal that is *not* finitely generated).
3. Fix a point  $x \in [0, 1]$  and show that the ideal  $\mathfrak{m}_x = \{f \mid f(x) = 0\}$  is maximal.
4. Are all maximal ideals of this form? > Hint: See textbook chapter 5, or Gilman and Jerison ‘Rings of Continuous Functions’.

Theorem of Swan: A theorem about topological vector bundles over  $C([0, 1])$ , see textbook. There is a categorical equivalence between vector bundles on a compact space and f.g. projective modules over this ring. (So commutative algebra has something to say about other branches of Mathematics!)

**Definition:** A topological space is called *boolean* (or a *Stone space*) iff it is compact, hausdorff, and totally disconnected.

*Example:* A projective variety over  $p$ -adics with  $\mathbb{Q}_p$  points plugged in.

**Definition:** A ring is boolean if every element is idempotent, i.e.  $x \in R \implies x^2 = x$ .

*Exercise:* If  $R$  is a boolean domain, then it is isomorphic to the field with 2 elements.

**Lemma:** There is a categorical equivalence between Boolean spaces, Boolean rings, and so-called “Boolean algebras”.

## 2 Monday January 13

### 2.1 Logistics

Some topics for final projects

- The cardinal Krull dimension of  $\text{Hol}(X)$ .
- Galois connections
- Ordinal filtrations
- Lam-Reyes prime ideal principal
- $C(X)$
- $\text{Hol}(X)$
- Semigroup rings
- Swan’s Theorem
  - Vector bundles on a compact space
- Boolean rings and Stone duality
- More Nullstellansatz
  - Beyond Hilbert’s usual one
- Hochster’s Theorem
  - Characterizes  $\text{Spec}R$  as a topological space, i.e. when is a topological space homeomorphic to the spectrum of some commutative ring.
- Invariant theory (quotients of rings under finite group actions, i.e.  $R^G$  for  $|G| < \infty$ )
  - For  $R = k$  a field, this is Galois theory
  - Easy case of geometric invariant theory, when  $G$  is infinite
- UFDs
  - What conditions does a ring need to have to ensure unique factorization?
- Euclidean rings
- Claborn (Leedham-Green-Clark): Every commutative group is (up to isomorphism) the class group of some Dedekind domain.
  - A type of inverse problem, class group measures deviation from being a UFD
  - Uses ordinal filtrations, transfinite induction
  - See proof in elliptic curves course

### 2.2 Rings of Functions

Let  $k$  be a field,  $X$  a set of cardinality  $|X| \geq 2$ , and define  $k^X := \text{Maps}(X, k) = \{f : X \rightarrow k\}$  is a ring under pointwise addition and multiplication. As a ring, this is a (big!) cartesian product.

*Some facts:*

- $k^X$  is not a domain (exercise), and there are nontrivial idempotents ( $e^2 = e$ ) > Note: it could be worse and have nilpotents.
- $k^X$  is *reduced*, i.e. it has no nonzero nilpotents, where  $z \in R$  is nilpotent iff  $\exists n \geq 1$  such that  $z^n = 0$ .
  - Note: domains are reduced, cartesian products of reduced rings are reduced.
- Every subring of  $k^X$  is reduced. > Moral: should be viewing every ring as functions on some space, but this can't literally be true because of the above restrictions. > Nilpotent elements are “hard to view as functions”.
- For  $X$  a topological space,  $C(X)$  the ring of continuous functionals to  $\mathbb{R}$ , then  $C(X) \subset \mathbb{R}^X$ .

*Exercise:* When is  $C(X)$  a domain? (Note that we can have products of nonzero functions being identically zero.)

*Example:* Let  $R$  be the ring of holomorphic functions  $\mathbb{C} \circledast$ , i.e.  $\text{Hol}(\mathbb{C}, \mathbb{C}) := \{f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is holomorphic}\}$ .

The set of zeros of such an  $f$  must be discrete, the example of bump functions doesn't go through holomorphically.

This is a domain, not Noetherian, not a PID, but every f.g. ideal is principal (thus this is a Bezout domain, a non-Noetherian analog of a PID).

It has infinite Krull dimension: recall that ideals are prime iff  $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$  iff  $R/\mathfrak{p}$  is a domain, and the Krull dimension is the supremum  $S$  of lengths of chains of prime ideals (only when  $S$  is finite).

If  $C \subset (X, \leq)$  is a finite-length chain in a totally ordered set, then the length  $\ell(C) = |C| - 1$  (1 less than the number of elements appearing). The *cardinal Krull dimension* of a ring  $R$  is the actual supremum.

Note: in Noetherian rings, there can still be finite but unbounded length chains.

Letting  $X$  be a complex manifold (i.e. covered by subsets of  $\mathbb{C}^n$  with holomorphic transition functions) and let  $\text{Hol}(X)$  be the holomorphic functionals  $f : X \rightarrow \mathbb{C}$ . Then  $\text{Hol}(X)$  is a domain iff  $X$  is connected.

Note that if  $X$  is disconnected, we can take a function that is constant on one component and zero on another, then switch, then multiply to get a zero function.

If  $X$  is a compact connected projective variety, then  $\text{Hol}(X)$  is just constant functions by the open mapping functions. So  $\text{Hol}(X) = \mathbb{C}$ , and  $\text{carddim}\mathbb{C} = 0$  because for any field there are only two ideals, and here  $(0)$  is prime. Moreover,  $\text{carddim}\text{Hol}(\mathbb{C}) \geq \aleph_0$ .

Note that for complex manifolds,  $X$  is either compact or supports many holomorphic functions.

**Theorem:** If  $X$  is a connected complex manifold which has a nontrivial holomorphic function, i.e.  $\text{Hol}(X) \supset \mathbb{C}$ , then there exists a chain of prime ideals in  $\text{Hol}(X)$  of length  $|\mathbb{R}| > \aleph_0$ , i.e. it has at least the cardinality of the continuum.

Note: the cardinality could be even bigger!

Maximals are prime: equivalent to fields are integral domains.

## 2.3 Rings

Take all rings to be unital, i.e. containing 1. A ring without identity is referred to as an *rng*. In this course, all rings are commutative.

*Example:* This is a fairly special restriction. Take  $(A, +)$  a commutative group and define  $\text{End}(A) = \{f : A \rightarrow A\}$  the ring of group homomorphisms under pointwise addition and composition. This is generally not commutative, i.e.  $\text{End}(\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)) = M_2(\mathbb{Z}/(2))$  the ring of matrices with  $\mathbb{Z}/(2)$  entries, which is not commutative.

*Exercise:* Given  $(A, +)$ , show that  $\text{End}(\bigoplus^n A) = M_n(\text{End}(A))$ .

Generally, if  $R$  is a ring and  $M$  is an  $R$ -module, then  $\text{End}_R(M) = \{f : M \rightarrow M\}$  of  $R$ -module homomorphisms is always a ring under pointwise addition and composition, and is (probably) non-commutative.

## 3 Wednesday January 15th

Cayley's theorem: For  $G$  a group, then there is a canonical injective group homomorphism  $\Phi : G \hookrightarrow \text{Sym}(G) \cong S_n$  for  $n = |G|$ . The map is given by  $g \mapsto g \cdot$ , i.e. multiplying on the left. Is there an analog for rings?

Take a similar map:

$$\begin{aligned} R &\rightarrow \text{End}_{\mathbb{Z}}(R, +) \\ r &\mapsto (x \mapsto rx). \end{aligned}$$

Unfortunately there is no specialization for commutative groups/rings –  $\text{Sym}(G)$  for example is noncommutative when  $|G| \geq 2$ . Similarly, even if  $R$  is commutative,  $\text{End}(R, +)$  is probably not. As per the Grothendieck philosophy, we find that all rings are a ring of functions on something – namely themselves, since this map is injective.

All rings are commutative here, so take  $R^\times = \{x \in R \mid \exists y \text{ s.t. } xy = 1\}$ . For  $R$  a group,  $R^\times$  is a commutative group, so this is an interesting invariant.

Another interesting invariant: the class group.

Notation: Let  $R^\bullet = R \setminus \{0\}$ . An element  $x \in R$  is a zero divisor iff there exists  $y \in R^\bullet$  such that  $xy = 0$ . For  $x, y \in R$  we write  $x \mid y$  iff  $\exists z \in R$  such that  $xz = y$ .

$R$  is a domain iff 0 is the only zero divisor, i.e.  $xy = 0 \implies x = 0$  or  $y = 0$ .  $(R^\bullet, \cdot)$  is a commutative monoid (group without inverses) iff  $R$  is a domain. Observe that  $R$  is a field iff  $R^\bullet = R^\times$ .

For rings  $R, S$  we have the usual definition of ring homomorphism, additionally requiring  $f(1) = 1$ . Note that  $f(0) = 0$  follows from  $f(x+y) = f(x) + f(y)$ , but  $f(1) = 1$  does not. Rings have products  $R_1 \times R_2$  which is again a ring under coordinate-wise operations. Note that there are canonical projections  $\pi_i : R_1 \times R_2 \rightarrow R_i$ . There is a dual inclusion  $\iota_1 : R_1 \rightarrow R_1 \times R_2$  given by  $x \mapsto (x, 0)$ , but these are not ring homomorphisms (although everything is a group homomorphism). This is because  $\iota_1(1) = (1, 0) \neq (1, 1)$ , the identity of  $R_1 \times R_2$ . Note that 1 always has to map to an idempotent

element, i.e.  $e^2 = e$ , and idempotents are always zero divisors. Also note that the map  $x \mapsto 0$  is not a ring homomorphism unless  $S = 0$ .

A ring homomorphism is a map  $f : R \rightarrow S$  is an isomorphism iff it has a two-sided inverse, i.e. there exists a morphism  $g : S \rightarrow R$  with  $g \circ f = \text{id}_R$  and  $f \circ g = \text{id}_S$ .

*Exercise:* Check that this is equivalent to  $f$  being a bijection.

*Exercise:* Check that the zero ring is the final object in the category of rings. Show that  $\mathbb{Z}$  is the initial object in this category?

$R$  is a subring of  $S$  iff  $R \subset S$  and the inclusion  $R \hookrightarrow S$  is a morphism.

Adjoining elements: Suppose  $R \leq S$  is a subring and  $X \subset S$  is just a subset. Then there exists a ring  $R[X]$  such that

- Top-down description:  $R[X] \leq S$  is a subring containing  $R$  and  $X$ , and is minimal with respect to this property (obtained by intersecting all such subrings)
- Bottom-up description: things resembling  $\sum r_i x_i$

*Exercise 1.6:* Take  $R = \mathbb{Z}, S = \mathbb{Q}, P$  a arbitrary set of prime numbers. Let  $\mathbb{Z}_P = \mathbb{Z}[\{\frac{1}{p} \mid p \in P\}]$ .

- When do we have  $\mathbb{Z}_{P_1} \cong \mathbb{Z}_{P_2}$ ? (Hint: take  $P_1 = \{3, 7, 11\}, P_2 = \{5\}$ . Need  $P_1 = P_2$ !)
- Show that every subring  $T$  such that  $\mathbb{Z} \leq T \leq \mathbb{Q}$  is of the form  $\mathbb{Z}_P$  for some unique set of primes  $P$ .

Note that if  $T$  is any intermediate ring between  $R$  and  $S$ , then  $R[T] = T$ .

### 3.1 Ideals and Quotients

For  $f : R \rightarrow S$  a ring homomorphism, define  $I = \ker f = f^{-1}(\{0\})$ . Then  $I$  is a subgroup of  $(R, +)$ , and for all  $i \in I$  and all  $r \in R$  we have  $ri \in I$ , since  $f(ri) = f(r)f(i) = f(r)0 = 0$ . In other words,  $RI \subseteq I$ .

By definition, an ideal  $I$  of  $R$  is an additive subgroup of  $R$  that satisfies these properties. Is every ideal the kernel of a ring homomorphism? The answer is yes, namely the quotient  $\pi : R \rightarrow R/I$ .

**Theorem:** Let  $I \subset (R, +)$ , then TFAE:

- $I$  is an ideal of  $R$ , written  $I \trianglelefteq R$ .
- There exists a ring structure on the quotient group  $R/I$  such that the projection  $r \mapsto r + I$  is a ring morphism.

When these conditions hold, the ring structure on  $R/I$  is *unique* and we refer to this as the *quotient ring*.

## 4 Friday January 17th

For a  $R \subset T$  a subring of a ring, the set of intermediate rings is a large/interesting class of rings. Recall: uncountably many rings between  $\mathbb{Z}$  and  $\mathbb{Q}$ ! Taking  $R$  a PID and  $T$  its fraction field, a similar result will hold.

Define  $I \trianglelefteq R$  as the kernel of a ring morphism. This implies that  $I \subset (R, +)$  with the absorption property  $RI \subset I$ . Conversely, any  $I$  satisfying these two properties is the kernel of a ring morphism: namely  $R \rightarrow R/I$ . This makes sense as a group morphism.

**Exercise:** Define  $xy + I = (x + I)(y + I)$ , need to check well-definedness. Write out  $(x + i_1)(y + i_2) = \dots$ , need to check that  $i_1y + i_2x + i_1i_2 \in I$ , but the absorption property does precisely this.

Note that if we were in a non-commutative setting, this would define a left ideal. These don't have to coincide with right ideals – there are rings where the former satisfy properties that the latter does not.

Example: The subrings of  $R = \mathbb{Z}$  are of the form  $n\mathbb{Z}$  for  $n \geq 0$ , with the usual quotient.

Definition: An ideal  $I \trianglelefteq R$  is *proper* iff  $I \subsetneq R$ .

**Exercise:** An ideal  $I$  is not proper iff  $I$  contains a unit.

**Exercise:**  $R$  is a field iff the only ideals are  $0, R$ .

Definition: Let  $\mathcal{I}(R)$  be the set of all ideals in  $R$ . What structure does it have? It is partially ordered under inclusion. It is a complete lattice, i.e. every element has an infimum (GLB) and a supremum (LUB). Namely, for a family of ideals  $\{I_j\}$ , the infimum is the intersection and supremum is defined as  $\langle I_j \mid j \in J \rangle$ , the smallest ideal containing all of the  $I_j$ , i.e.  $\langle y \rangle = \left\{ \sum_{i=1}^n r_i y_i \mid n \in \mathbb{N}_{>0}, r_i \in R, y_i \in y \right\}$ .

**Exercise:** For  $I_1, I_2 \trianglelefteq R$ , it is the case that  $I_1 + I_2 := \{i_1 + i_2\} = \langle I_1, I_2 \rangle$ .

Theorem: Let  $I \trianglelefteq R$  and  $\phi : R \rightarrow R/I$ , and define  $\ell(I) = \{I \subset J \trianglelefteq R\}$ . Then we can define maps

$$\begin{aligned} \Phi : \ell(R) &\rightarrow \ell(R/I) \\ J &\mapsto \frac{I + J}{J}, \end{aligned}$$

$$\begin{aligned} \Psi : \ell(R/I) &\rightarrow \ell(R) \\ J \trianglelefteq R/I &\mapsto \phi^{-1}(J). \end{aligned}$$

We can check that  $\Psi \circ \Phi(J) = I + J$ , and  $\Phi \circ \Psi(J) = J (= J/I?)$ . So  $\Psi$  has a left inverse and is thus injective. Its image is the collection of ideals that contain  $J$ , and  $\Psi : \ell(R/I) \rightarrow \ell_I(R)$  is a bijection and is in fact a lattice isomorphism with  $\ell_I(R) \subset \ell(R)$ .

Note that this gives us everything above (?) an ideal in the ideal lattice; the dual notion will come from localization.

Remarks:

The ideal lattice  $\ell(R)$  is

- A complete lattice under subset inclusion,
- A commutative monoid under addition

- A commutative monoid under *multiplication*, which we'll define.

Definition: For  $I, J \trianglelefteq R$ , we define  $IJ = \langle ij \mid i \in I, j \in J \rangle$ . Note that we have to take the ideal generated by products here.

For  $\langle x \rangle = (x)$  a principal ideal and  $\langle y \rangle$  principal, we do have  $(x)(y) = (xy)$ . Note that  $IJ \subset I \cap J$ , whereas the sum was larger than  $I, J$ .

**Exercise:** Note that  $(\ell(R), \cdot)$  has an absorbing element, namely  $(0)I = (0)$ . For  $(M, +)$  a commutative monoid and  $M \hookrightarrow G$  a group, then multiplication by  $x$  is injective and so for all  $y \in M$ ,  $xz = yz \implies x = y$ , so  $M$  is cancellative.

Question: what if we consider  $\mathcal{I}^\bullet(R)$  the set of nonzero ideals of  $R$ . Does this help? We will see next time.

## 5 Wednesday January 22nd

Let  $R$  be a ring and let  $\mathcal{I}(R)$  be the set of ideals  $I \trianglelefteq R$ . This algebraic structure is

- Partially ordered under inclusion
- Forms a complete lattice with sup the ideal generated by a family and inf the intersection.
- Forms a commutative monoid under  $I + J$
- Forms a commutative monoid under  $IJ$

For any commutative monoid  $(M, +)$ , there exists a group completion  $G(M)$  such that

- $G(M)$  is a commutative group
- $g : M \rightarrow G(M)$  is a monoid homomorphism
- For any map  $\phi : (M, +) \rightarrow (G, +)$  into a commutative group, we have the following diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\forall \phi} & G \\
 & \searrow g \quad \nearrow \exists! \phi & \\
 & M(G) &
 \end{array}$$

So  $\phi$  factors through  $M(G)$ .

If this exists, it is unique up to unique isomorphism (as are all objects defined by universal properties). It remains to construct it.

**Exercise:** For  $(M, +)$  a commutative monoid, show that TFAE

1. There exists an injective  $\iota : M \hookrightarrow G$  monoid homomorphism for  $G$  some commutative group.
2. The map  $g : M \rightarrow G(M)$  is an injection.
3.  $M$  is cancellative, i.e.  $\forall x, y, z \in M$  we have  $x + z = y + z \implies x = y$ , i.e. the map  $p_z(x) = x + z$  is injective.

The content here is in  $3 \implies 1$ .

A commutative monoid is *reduced* iff  $M^\times = (0)$ , i.e. if " $\forall m \in M \exists n$  such that  $m + n = 0$ "  $\implies m = 0$



Example:  $(\mathbb{N}, +)$  and  $(\mathbb{Z}^+, \cdot)$  are cancellative and reduced.

Definition  $z \in M$  is a zero element iff  $z + x = z$  for all  $x \in M$ .

Remark: If  $M$  has a zero element, then  $G(M) = \{0\}$ .

$(0)$  is a zero element of  $(\mathcal{I}(R), \cdot)$ , so this is not cancellative. If we take  $\mathcal{I}^\bullet$  the set of nonzero ideals with multiplication, then this is a submonoid of  $\mathcal{I}(R)$  iff  $R$  is a domain.

For  $R$  a domain, let  $\mathcal{I}_1(R)$  be the set of nonzero principal ideals of  $R$ , then  $\mathcal{I}_1(R) = R^\bullet / R^\times$ , so this is reduced and cancellative.

What is the group completion? In this case, it will consist of fractional ideals.

If  $R$  is a PID, then  $\mathcal{I}_1^\bullet(R) = \mathcal{I}^\bullet(R)$  is reduced and cancellative.

Example:  $\mathcal{I}^\bullet \cong (\mathbb{Z}^+, \cdot)$ .

Warning: If  $R$  is not a PID, then  $\mathcal{I}^\bullet(R)$  need not be cancellative.

**Exercise:** Take  $R = \mathbb{Z}[\sqrt{-3}]$  and  $p_2 := \langle 1 + \sqrt{-3}, 1 - \sqrt{-3} \rangle$ . Show that  $|R/p_2| = 2$ ,  $|R/(2)| = 4$ , and  $p_2^2 = p_2(2)$  and  $|R/p_2^2| = 8$ . Conclude that  $\mathcal{I}^\bullet(R)$  is not cancellative.

What went wrong here? Take  $K = \mathbb{Q}[\sqrt{-3}]$ , then  $\mathbb{Z}_K[\frac{1 + \sqrt{-3}}{2}]$  is the integral closure of  $\mathbb{Z}$  in  $K$ .  $\mathbb{Z}_K$  is a Dedekind domain, and there are inclusions

$$\mathbb{Z} \subset \mathbb{Z}[\sqrt{-3}] \subsetneq \mathbb{Z}[\frac{1 + \sqrt{-3}}{2}] \subseteq K.$$

Here the problem is that  $\mathbb{Z}[\sqrt{-3}]$  is not a Dedekind domain. If  $R$  is a Dedekind domain, then  $\mathcal{I}^\bullet(R)$  is cancellative.

**Exercise:** Does the converse hold?

Things that are too small to be the full rings of integers, and things tend to wrong.

## 5.1 Pushing / Pulling

Let  $f : R \rightarrow S$  be a ring homomorphism.

We can define a pushforward on the set of ideals  $\mathcal{I}(R)$ :

$$\begin{aligned} f_* : \mathcal{I}_R &\rightarrow \mathcal{I}(S) \\ I &\mapsto \langle f(I) \rangle. \end{aligned}$$

and a pullback

$$\begin{aligned} f^* : \mathcal{I}(S) &\rightarrow \mathcal{I}(R) \\ J &\mapsto f^{-1}(J). \end{aligned}$$

**Exercise:** Show that  $f^{-1}(J) \trianglelefteq R$ .

For  $I \trianglelefteq R$  and  $J \trianglelefteq S$ , then

$$\begin{aligned} f^*f_*(I) &\supseteq I \\ f_*f^*(J) &\subseteq J. \end{aligned}$$

**Exercise:** These are not equal in general, and give examples where equality does and does not hold.

If  $f$  is surjective,  $f_*f^*J = J$ .

Will also hold for localization, which is dual to taking a quotient.

Define  $\bar{I} := f^*f_*(I)$  and  $J^\circ := f_*f^*(J)$ , the closure and interior respectively. Show that these operations are idempotent.

Definition: An ideal  $\mathfrak{p}$  iff  $ab \in \mathfrak{p} \implies a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ .

**Exercise:**  $I$  is prime iff  $R/I$  is a domain.

Definition:  $\text{Spec}(R) = \{\mathfrak{p} \trianglelefteq R\}$  the collection of prime ideals is the spectrum.

**Exercise:** Show that for  $I \trianglelefteq R$ , if we define  $V(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq I\} \subseteq \text{Spec}(R)$ , then  $\{V(I) \mid I \in \mathcal{I}(R)\}$  are the closed sets for a topology on  $\text{Spec}(R)$  (the Zariski topology).

**Exercise:** If  $f : R \rightarrow S$  and  $J \in \text{Spec}(S)$  then  $f^*(J) \in \text{Spec}(R)$ . Show that  $f^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$  is a continuous map. Conclude that  $\text{Spec}(\cdot)$  is a functor.

Definition:  $I \trianglelefteq R$  is maximal iff  $I$  is proper and is not contained in any other proper ideal.

**Exercise:**  $I$  is maximal iff  $R/I$  is a field.

**Exercise:** Show that maximal ideals are prime.

Definition: Let  $\text{Spec}_{\max}(R)$  be the set of maximal ideals and define  $V(I) = \{\mathfrak{m} \in \text{Spec}_{\max}(R) \mid \mathfrak{m} \supseteq I\}$ . Show that these form the closed sets for a topology, and that this is the subspace topology for the Zariski topology.

**Exercise:** Show that if  $f : R \rightarrow S$  and  $\mathfrak{m} \in \text{Spec}_{\max}(S)$  that  $f^*(\mathfrak{m})$  is prime but need not be maximal.

If  $f$  is an integral extension, then maximals do pull back to maximals.