Problem Set 8

D. Zack Garza

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1 Problem 1

1.1 Part a

Define a map

$$\begin{aligned}\phi_{\text{ev}} : \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) &\to A \\ (f : \mathbb{Z}_m \to A) &\mapsto f(1)\end{aligned}$$

Then noting that ϕ_{ev} is a homomorphism, forcing $f(\overline{0}) = 0_A$ (where $\overline{0} : \mathbb{Z}_m \to A$ is the zero map), we must have

$$0 = f(0) = f(m) = mf(1),$$

we must have $mf(1) = 0$ in A. So

$$\text{im } \phi_{\text{ev}} = \{a \in A \mid ma = 0\} := A[m].$$

It is also the case that

$$\ker \phi_{\text{ev}} = \{f \in \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \mid f(1) = 0\} = \{\overline{0}\},$$

which follows from the fact that $\mathbb{Z}_m = \langle 1 \mod m \rangle$ and $A = \langle 1_A \rangle$ as $\mathbb{Z}$-modules, so if $f(1 \mod m) = 0_A$ then

$$f(n \mod m) = nf(1 \mod m) = 0$$

and so f is necessarily the zero map. So $ker \phi = \overline{0}$.

We can then apply the first isomorphism theorem,

$$\frac{\hom_{\mathbb{Z}}(\mathbb{Z}_m, A)}{\ker \phi_{\text{ev}}} \cong \text{im } \phi_{\text{ev}} \implies \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m].$$

1.2 Part 2

The claim is that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{(m,n)}$, from which the result immediately follows by part 1. Expanding definitions, we have

$$\begin{aligned}
\mathbb{Z}_n[m] &= \{x \in \mathbb{Z}_n \ni mx = 0\} \\
&= \left\{x \in \mathbb{Z}_n \ni o(x) \;\middle|\; m \;\text{ and }\; o(x) \;\middle|\; n\right\} \\
&= \left\{x \in \mathbb{Z}_n \ni o(x) \;\middle|\; \gcd(m,n)\right\} \\
&\cong \mathbb{Z}_{\gcd(m,n)}.
\end{aligned}$$