# Title

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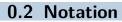
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# **Prologue**





 $\bullet \ \ Gathmann's \ Algebraic \ Geometry \ notes [@Andreas Gathmann 515].$ 





• If a property P is said to hold **locally**, this means that for every point p there is a neighborhood  $U_p \ni p$  such that P holds on  $U_p$ .

Notation	Definition
$\overline{k[\mathbf{x}] = k[x_1, \cdots, x_n]}$	Polynomial ring in $n$ indeterminates
$k(\mathbf{x}) = k(x_1, \cdots, x_n)$	Rational function field in $n$ indeterminates
$\mathcal{U} \rightrightarrows X$	An open cover $\mathcal{U} = \left\{ U_j \; \middle  \; j \in J  ight\}$
$\Delta_X$	The diagonal $\{(x,x) \mid x \in X\} \subseteq X \times X$
$\mathbb{A}^n_{/k}$	Affine <i>n</i> -space $\mathbb{A}^n_{/k} \coloneqq \left\{ \mathbf{a} = [a_1, \cdots, a_n] \mid a_j \in k \right\}$
$\mathbb{P}^n_{/k}$	Projective n-space
,	$\mathbb{P}^n_{/k} := (k^n \setminus \{0\}) / x \sim \lambda x$
	$\mathbb{P}_{/k}^n = \left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \mid p, q, \in k[x_1, \cdots, x_n] \right\}$
$V(J), V_a(J)$	Variety associated to an ideal $J \leq k[x_1, \dots, x_n]$
	$V_a(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0,  \forall f \in J \right\}$
$I(S), I_a(S)$	Ideal associated to a subset $S \subseteq \mathbb{A}_k^n$
	$I_a(S) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \mid f(\mathbf{x}) = 0  \forall \mathbf{x} \in X \right\}$
A(X)	Coordinate ring of a variety, $k[x_1, \dots, x_n]/I(X)$
$V_p(J)$	Projective variety of an ideal
	$V_p(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \mid f(\mathbf{x}) = 0,  \forall f \in J \right\}$
$I_p(S)$	Projective ideal (?) $I_p(S) :=$
	$\{f \in k[x_1, \dots, x_n] \mid f \text{ is homogeneous and } f(x) = 0 \forall x \in S\}$
S(X)	Projective coordinate ring, $S(X) := k[x_1, \dots, x_n]/I_p(X)$
$f^h$	Homogenization, $f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$
$f^i$	Dehomogenization, $f^i := f(1, x_1, \cdots, x_n)$
$J^h$	Homogenization of an ideal, $J^h \coloneqq \left\{ f^j \mid f \in J \right\}$
$\overline{X}$	Projective closure of a subset
	$\overline{X} \coloneqq V_p(J^h) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n \;\middle \; f^h(\mathbf{x}) = 0  orall f \in X  ight\}$

0.2 Notation 3

$\mathcal{O}_X$	Structure sheaf $\mathcal{O}_X(U) := \left\{ f : U \to k \mid f \in k(\mathbf{x}) \text{ locally} \right\}$
D(f)	Distinguished open set, $D(f) := V(f)^c = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$

#### 0.3 Summary of Important Concepts



- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for X = D(f) a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

#### 0.4 Useful Examples



#### 0.4.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$  a hyperbola
- V(x) a coordinate axis
- V(x-p) a point.

#### 0.4.2 Presheaves / Sheaves

- $C^{\infty}(\cdot, \mathbb{R})$ , a sheaf of smooth functions
- $C^0(\cdot,\mathbb{R})$ , a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$ , the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$ , the constant sheaf associated to  $\mathbb{R}$  (locally constant real-valued functions)
- $\operatorname{Hol}(\cdot,\mathbb{C})$ , a sheaf of holomorphic functions
- $K_p$  the skyscraper sheaf:

$$K_p(U) \coloneqq \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

### 0.5 The Algebra-Geometry Dictionary



Let  $k = \bar{k}$ , we're setting up correspondences

Algebra	Geometry
$\frac{1}{k[x_1,\cdots,x_n]}$	$\mathbb{A}^n_{/k}$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \cdots, x_n - p_n$	Points $[a_1, \cdots, a_n]$
Radical ideals $J = \sqrt{J} \le k[x_1, \cdots, x_n]$	V(J) the zero locus
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets
I(S) the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
I + J	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
I(V):I(W)	$\overline{V\setminus W}$
$k[x_1,\cdots,x_n]/I(X)$	A(X) (Functions on $X$ )
A(X) a domain	X is irreducible
A(X) indecomposable	X is connected
Krull dimension $n$ (chaints of primes)	Topological dimension $n$ (chains of irreducibles)