Title

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Last time: Bott-Borel-Weil. Stated for characteristic zero, working toward a generalization.

Let Δ be the set of simple roots, and $\alpha \in \Delta$. We can form a Levi decomposition $P_\alpha \coloneqq L_\alpha \rtimes U_\alpha$:

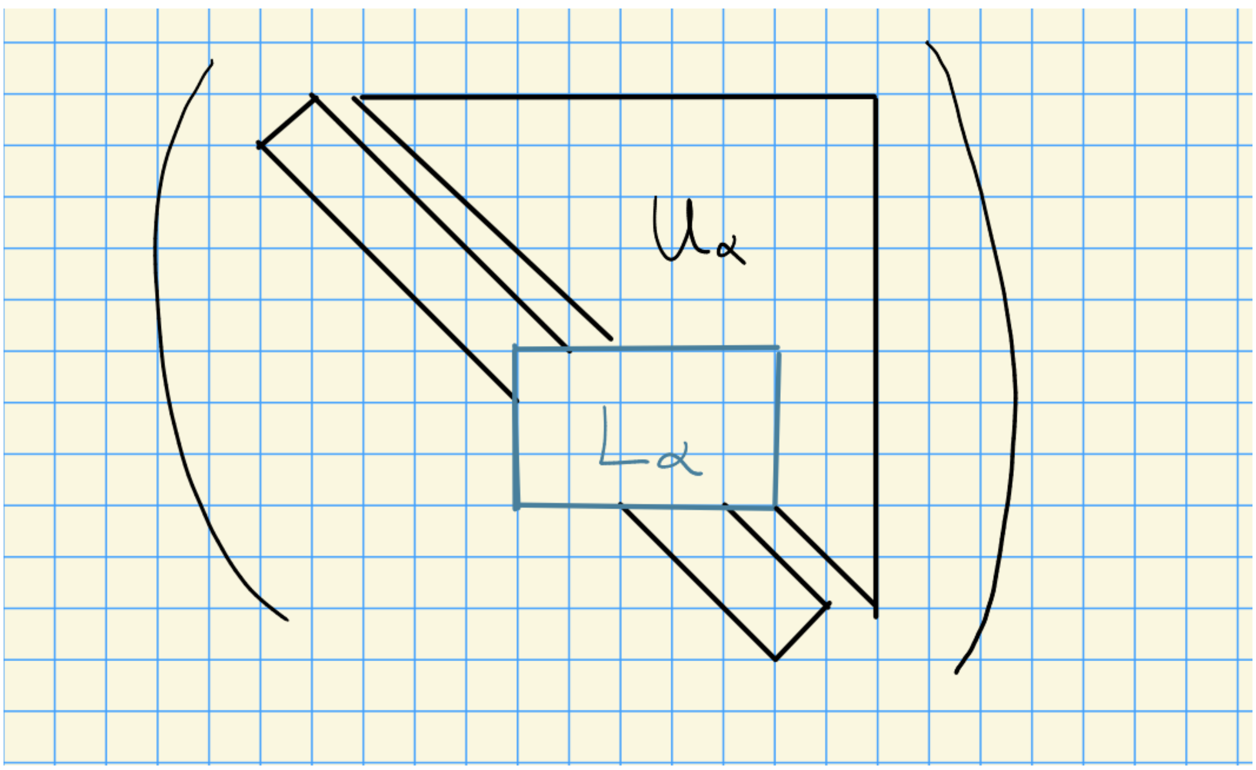

Figure 1: Image

We have $B \subseteq P_\alpha \subseteq G$. The dot action is given by the following: Let W be the Weyl group, then W acts on $X(T)$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} w_n.$$

We obtained a formula

$$S_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho,\ \alpha^\vee \rangle \alpha.$$

1.1 Bott-Borel-Weil Theory

Proposition 1.1.1 ***(?).***
Let $\alpha \in \Delta$ be simple and $\lambda \in X(T)$ be an arbitrary weight. Then

- U_α acts trivially on $\operatorname{Ind}_B^{P_\alpha} \lambda$.
- (Kempf's Vanishing for P_α) If $\langle \lambda,\ \alpha^\vee \rangle = r \geq 0$, then

$$R^i \operatorname{Ind}_B^{P_\alpha} \lambda = 0 \qquad \text{for } i \geq 0,$$

  and $\dim \operatorname{Ind}_B^{P_\alpha} \lambda = r + 1$.
- If $\langle \lambda,\ \alpha^\vee \rangle = -1$, then $R^i \operatorname{Ind}_B^{P_\alpha} \lambda = 0$ for all i.
- If $\langle \lambda,\ \alpha^\vee \rangle \leq -2$, then
  - $R^i \operatorname{Ind}_B^{P_\alpha} \lambda = 0$ for $i \neq 1$, and
  - $\dim R^1 \operatorname{Ind}_B^{P_\alpha} \lambda = r + 1$

Note: we have

$$\begin{aligned} \operatorname{Ind}_B^{P_\alpha} \lambda &= S^r(V) && \text{when } \langle \lambda,\ \alpha^\vee \rangle = r \geq 0 \\ R^1 \operatorname{Ind}_B^{P_\alpha} &= S^r(V)^\vee && \text{where } V \text{ is a 2-dim representation and } \langle \lambda,\ \alpha^\vee \rangle \leq -2 \\ & && \text{and } r = |\langle \lambda,\ \alpha^\vee \rangle| - 1. \end{aligned}$$

This gives us an analog of A_1 or SL_2 theory. Also note that we have Serre duality:

$$H^1(\lambda) = H^0(-(\lambda + 2\rho))^\vee.$$

Corollary 1.1.1 ***(?).***
Let $\alpha \in \Delta$ and $\lambda \in X(T)$, and suppose λ is dominant with respect to α, i.e. $\langle \lambda,\ \alpha^\vee \rangle \geq 0$.

- If char $(k) = 0$ then $\operatorname{Ind}_B^{P_\alpha} \lambda = R^1 \operatorname{Ind}_B^{P_\alpha} s_\alpha \cdot \lambda$
- If char $(k) = p$ and if there exists an s, m with $0 < s < p$ and $\langle \lambda,\ \alpha^\vee \rangle = sp^m - 1$ (Steinberg weights), then

$$\operatorname{Ind}_B^{P_\alpha} \lambda = R^1 \operatorname{Ind}_B^{P_\alpha} s_\alpha \cdot \lambda.$$

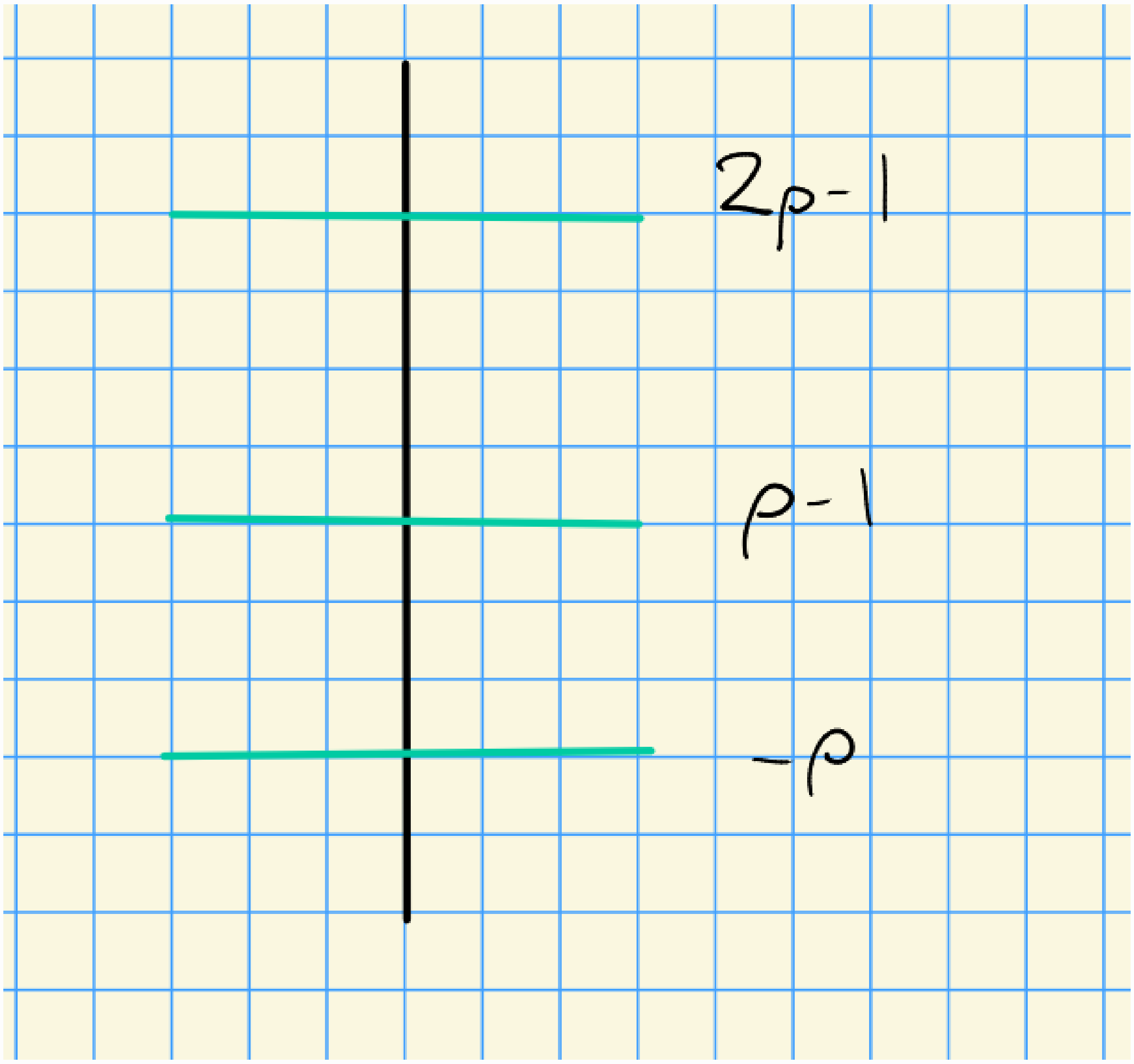

Figure 2: O

The proof of this will use a Grothendieck-type spectral sequence of the form

$$E_2^{i,j} = R^i \operatorname{Ind}_{P_\alpha}^{G} \left(R^j \operatorname{Ind}_{B}^{P_\alpha} \lambda \right) \Rightarrow R^{i+j} \operatorname{Ind}_{B}^{G} \lambda.$$

We'll have a version of *Grothendieck vanishing*:

$$R^j \operatorname{Ind}_{B}^{P_\alpha} \lambda = 0 \qquad \text{for } j > \dim P_\alpha / B = 1.$$

So the resulting spectral sequence will only be supported on the first two lines, and $E_3 = E_\infty$. Note the differential will be of bidegree $\partial_r \rightsquigarrow (r, 1-r)$, and E_2 will look like the following,

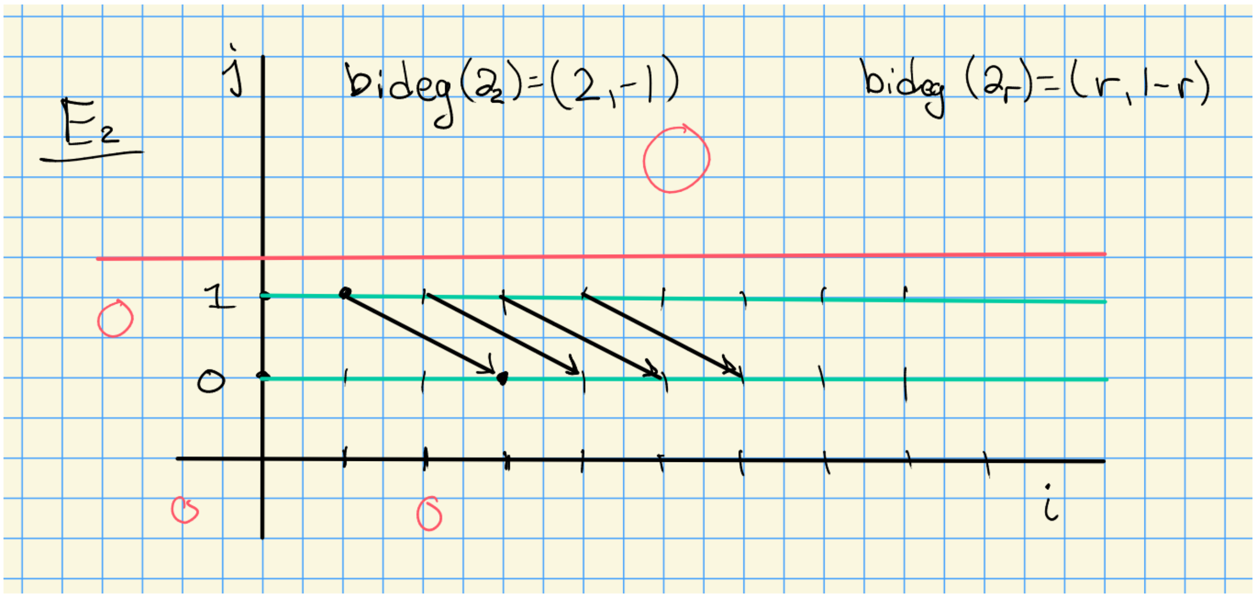

Figure 3: Image

Recall that $R^i \operatorname{Ind}_B^G \lambda \coloneqq H^i(\lambda)$

Proposition 1.1.2 ***(?)*.**
Let $\alpha \in \Delta$ and $\lambda \in X(T)$.

1. If $\langle \lambda, \alpha^\vee \rangle = -1$, then $H^{\cdot}(\lambda) = 0$.
2. If $\langle \lambda, \alpha^\vee \rangle \geq 0$, then $H^i(\lambda) = R^i \operatorname{Ind}_B^{P_\alpha} \lambda$ for all $i \geq 0$.
3. If $\langle \lambda, \alpha^\vee \rangle \leq -2$, then

$$H^i(\lambda) = R^{i-1} \operatorname{Ind}_{P_\alpha}^G \left(R^1 \operatorname{Ind}_B^{P_\alpha} \lambda \right) \qquad \forall i.$$

4. Suppose $\langle \lambda, \alpha^\vee \rangle \geq 0$. If char $(k) = 0$, or char $(k) = p > 0$ and $\langle \lambda, \alpha^\vee \rangle = sp^n - 1$, then

$$H^i(\lambda) = H^{i+1}(s_\alpha \cdot \lambda).$$

Proof (of a).
If $\langle \lambda, \alpha^\vee \rangle = -1$, then $R^{\cdot} \operatorname{Ind}_B^{P_\alpha} \lambda = 0$. But this is what appears as the "coefficients" in the spectral sequence, so $E_2^{\cdot, \cdot} = 0$ and this $R^{\cdot} \operatorname{Ind}_B^{P_\alpha} = 0$.

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Proof (of b).
If $\langle \lambda, \alpha^\vee \rangle = 0$, then $R^j \operatorname{Ind}_B^{P_\alpha} \lambda = 0$ for all $j > 0$. Thus only the bottom line survives, and the spectral sequence degenerates on page 2. Thus $E_2^{1,0} = R^i \operatorname{Ind}_B^G \lambda$, where the LHS is equal to $R^i \operatorname{Ind}_{P_\alpha}^G \left(\operatorname{Ind}_B^{P_\alpha} \lambda \right)$.

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Proof (of c).
If $\langle \lambda,\ \alpha^\vee \rangle = -2$, then $R^i \operatorname{Ind}_B^{P_\alpha} \lambda = 0$ for $i \neq 1$, so only $i = 1$ survives Then

$$R^{i-1} \operatorname{Ind}_{P_\alpha}^G \left(\operatorname{Ind}_B^{PP_\alpha} \alpha \right) = R^i \operatorname{Ind}_B^G \lambda,$$

so there is some dimension shifting. ■

Proof (of d).
If $\langle \lambda,\ \alpha^\vee \rangle \geq 0$, then by (b),

$$\begin{aligned} H^i(\lambda) &= R^i \operatorname{Ind}_{P_\alpha}^G \left(\operatorname{Ind}_B^{P_\alpha} \lambda \right) && \text{by c} \\ &= R^i \operatorname{Ind}_{P_\alpha}^G \left(R^1 \operatorname{Ind}_B^{P_\alpha} s_\alpha \cdot \lambda \right) && \text{by corollary} \\ &= H^{i+1}(s_\alpha \cdot \lambda). \end{aligned}$$

We can then check that

$$\begin{aligned} s_\alpha \cdot \lambda &= \lambda - \langle \lambda + \rho,\ \alpha^\vee \rangle \alpha \\ &= \lambda - (\langle \lambda,\ \alpha^\vee \rangle + 1)\alpha && \text{using } \langle \rho,\ \alpha^\vee \rangle = 1 \end{aligned}$$

$$\begin{aligned} \implies \langle s_\alpha \cdot \lambda,\ \alpha^\vee \rangle &= \langle \lambda,\ \alpha^\vee \rangle - (\langle \lambda,\ \alpha^\vee \rangle + 1)\langle \alpha,\ \alpha^\vee \rangle \\ &= \langle \lambda,\ \alpha^\vee \rangle - (\langle \lambda,\ \alpha^\vee \rangle + 1)2 \\ &= -\langle \lambda,\ \alpha^\vee \rangle - 2 \\ &\leq -2. \end{aligned}$$

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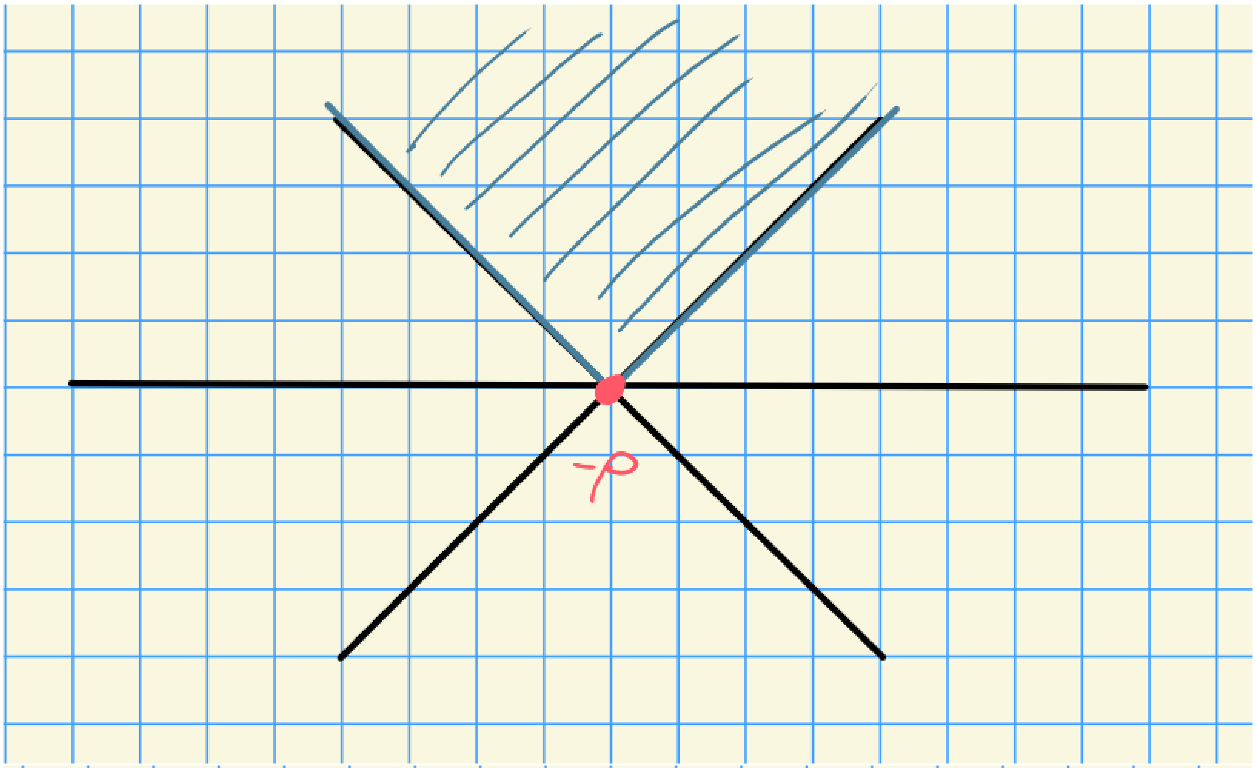
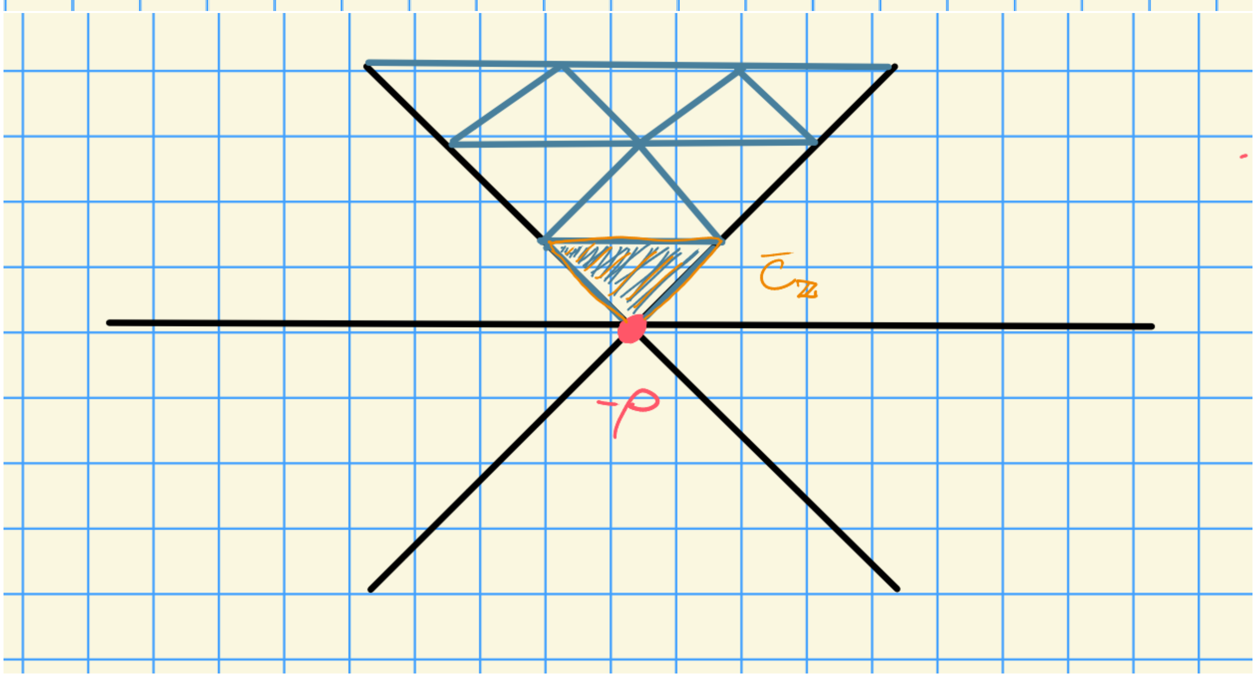
Now define

$$\begin{aligned} \overline{C}_{\mathbb{Z}} &:= \left\{ \lambda \in X(T) \;\middle|\; 0 \leq \langle \lambda + \rho,\ \beta^\vee \rangle \forall \beta \in \Phi^+ \right\} && \text{if char } (k) = 0 \\ &:= \left\{ \lambda \in X(T) \;\middle|\; 0 \leq \langle \lambda + \rho,\ \beta^\vee \rangle \leq \text{char } (k) \forall \beta \in \Phi^+ \right\} && \text{if char } (k) = p. \end{aligned}$$

Idea:





Theorem 1.1.1 ***(Bott-Borel-Weil Generalization, due to Andersen).*** a. If $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\lambda \notin X(T)_+$, then $H^0(w \cdot \lambda) = 0$.

b. If $\lambda \in \overline{C}_{\mathbb{Z}} \cap X(T)_+$, then for all $w \in W$,

$$H^i(w \cdot \lambda) = \begin{cases} H^0(\lambda) & i = \ell(w) \\ 0 & \text{otherwise} \end{cases}.$$

Note that this covers everything in the char $(k) = 0$ case, but only gives the following hexagon in the char $(k) = p$ case:

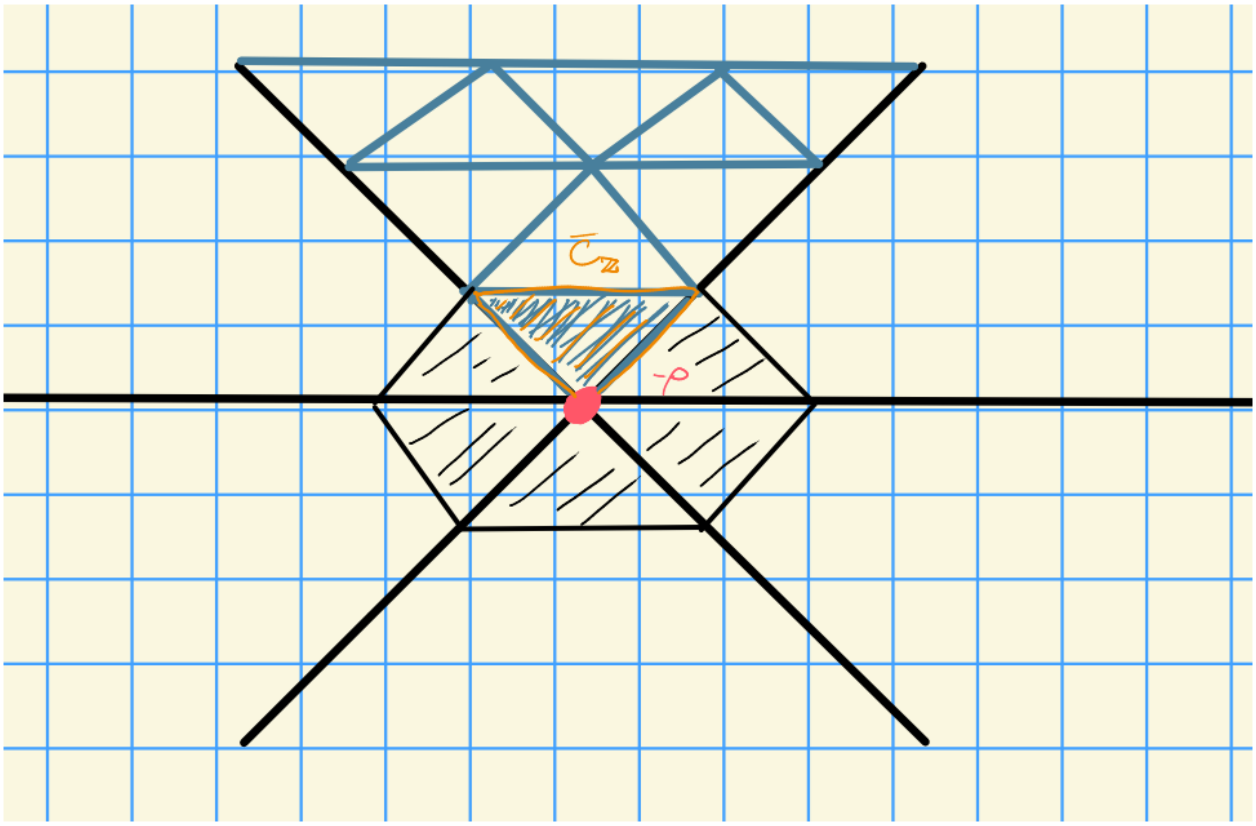


Figure 4: Image

Remark 1.1.1.
Open Problem: Determine char $H^i(\lambda)$ for $\lambda \in X(T)$ in characteristic $p > 0$.

Andersen provided necessary an sufficient conditions for $H^1(\lambda) \neq 0$ and computed $\mathrm{Soc}_G H^1(\lambda)$.