HW#4 Qual Problems

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- D We will use the fact that if $\psi: \mathcal{D}(\alpha) \to F$ is any field homomorphism, then it must be the case that $\psi(\alpha) = \beta$, some conjugate of α in α . In particular, we must have $\beta \in \alpha \cap F$. There are 17 possibilities here: $\beta \in \alpha \cap F$. There are 17 possibilities here: $\beta \in \alpha \cap F$. There are 17 possibilities here: $\beta \in \alpha \cap F$.
 - (a) $\# \{ \sigma : \mathcal{D}(\mathcal{A}) \to \mathbb{C} \} = \# S_{\mathcal{A}} \cap \mathbb{C} = \# S_{\mathcal{A}} = \underline{17} \text{ Since each } \sqrt{27} \, \overline{\mathcal{S}}_{77}^{k} \in \mathbb{C},$

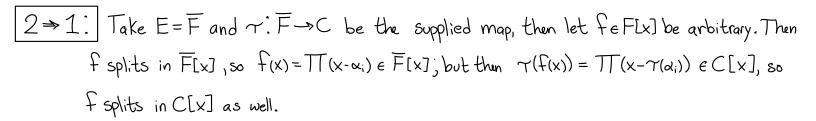
 - \bigcirc # $\{\sigma: \mathcal{D}(\mathcal{A}) \rightarrow \overline{\mathcal{D}}\}$ = # $\mathcal{S}_{\mathcal{A}} \cap \overline{\mathcal{Q}} = \#\mathcal{S}_{\mathcal{A}} = 17$, since $\overline{\mathcal{Q}}$ contains every root of every polynomial in $\mathcal{Q}[x]$.
- (2) Claim C is algebraically closed. Supposing otherwise, there would be some $f(x) \in C[x]$ s.t. f has no roots in C. Let α be one root, then $f(\alpha) = 0$ in $C(\alpha)$. Write $f(x) = \sum_{j=1}^{n} c_j x^j$; then α is also a root in $F(c_1, c_2, \cdots, c_n, \alpha)$, which is a f-inite extension of F(since C itself was an algebraic extension). This means that $g := \min(\alpha, F)$ is actually in F[x]; but then g splits into linear factors in C, so $\alpha \in C$. m
- Let $E \ge F$ be an arbitrary algebraic extension. Then consider $S = \{A, \gamma\} \mid F \le A \le E, \}$ Make S a poset by defining $(A, \gamma) \prec (B, \sigma)$ iff $B \ge A$ and $\sigma|_{A} = \gamma$.

For any chain $(A_i, \gamma_i) \times (A_{i+1}, \gamma_{i+1}) \times \cdots$ define $(A, \gamma) := (UA_i, f)$ where $f: A \to C$ is defined as $x \in A \Rightarrow x \in A_i$ for some $i \Rightarrow f(x) = \gamma_i(x)$. This is an upper bound, so Zorn's lemma applies to yield some maximal elt $(A, \gamma: A \to C)$.

Claim: A = E.

Otherwise, let $\alpha \in E \setminus A$, and consider $f = \min(\alpha, F)$. Then $\underline{\tau(f)} \in C[x]$ splits into linear factors, and has some root $B \in C$. So consider $E \geq \underline{A(\alpha)} \geq A$; we could then define $\tau' : \underline{A(\alpha)} \rightarrow C$ by $X \mapsto B$ if $x = \alpha$ which makes $(A(\alpha), \tau') \uparrow (A, \tau)$, contradicting maximality. T(x) else

So $(A, \tau: A \rightarrow C) = (E, \tau: E \rightarrow C)$, and τ is the desired extension.



Suppose $\dim_k R = n < \infty$, then any basis of R is a linearly indepent spanning set containing m elts. Then suppose that there is no pair $(m, \{c_0, \cdots, c_{m-1}\})$ s.t. $\alpha + \sum_{j=0}^{m-1} c_j \dot{\alpha}^j = 0$. Then, for example, if R has basis $\{v_1, \cdots, v_n\}$, then $\{v_1, \cdots, v_n, \alpha^i, \cdots, \alpha^m\}$ is a linearly independent spanning set of size > m. \gg So some pair $(m, \{c_i\})$ must exist.

(b)
$$a'' + c_n a'' + \cdots + c_1 a + c_0 = 0$$
 iff $a'' + c_n a'' + \cdots + c_1 a = -c_0$ Exists since $c_0 \neq 0$.

iff $a(a'' + c_{n-1}a'' + \cdots + c_1)(-c_0'') = 1$

$$\vdots = a^{-1}$$
So $a \in \mathbb{R}^{\times}$.

© Suppose $a \neq 0$ is not a zero divisor. If $C_0 \neq 0$, we are done by (b), so suppose $C_0 = 0$. Then $a^n + C_n \cdot a^{n-1} + \cdots + C_2 a^2 + C_1 a = 0 \quad \text{iff}$ $a(a^{n-1} + C_{n-1}a^{n-2} + \cdots + C_2 a + C_1) = 0 \quad \text{iff}$ $a^{n-1} + C_{n-1}a^{n-2} + \cdots + C_2 a + C_1 = 0$

Since a is not a zero divisor. Proceeding inductively, there is some smallest j such that $c_j \neq 0$ and $a + \cdots + c_j = 0$. But case (b) again yields $a \in \mathbb{R}^{\times}$.