Problem Set 3

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Exercise 0.1 (Gathmann 2.33)**.**
Define

$$X := \left\{ M \in \operatorname{Mat}(2 \times 3, k) \;\middle|\; \operatorname{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that X is an irreducible variety, and find its dimension.

Solution:
We'll use the following fact from linear algebra:

Definition ***(Matrix Minor).***
For an $m \times n$ matrix, a *minor of order* ℓ is the determinant of a $\ell \times \ell$ submatrix obtained by deleting any $m - \ell$ rows and any $n - \ell$ columns.

Theorem 0.1 ***(Rank is a Function of Minors).***
If $A \in \operatorname{Mat}(m \times n, k)$ is a matrix, then the rank of A is equal to the order of largest nonzero minor.

Thus

$$M_{ij} = 0 \text{ for all } \ell \times \ell \text{ minors } M_{ij} \iff \operatorname{rank}(M) < \ell,$$

following from the fact that if one takes $\ell = \min(m, n)$ and all $\ell \times \ell$ minors vanish, then the largest nonzero minor must be of size $j \times j$ for $j \leq \ell - 1$. But $\det M_{ij}$ is a polynomial f_{ij} in its entries, which means that X can be written as

$$X = V(\{f_{ij}\}),$$

which exhibits X as a variety. Thus

$$M = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \implies X = V(\langle xb - ya, yc - zb, xc - za \rangle) \subset \mathbb{A}^6.$$

Claim: The ideal above is prime, and so the coordinate ring $A(X)$ is a domain and thus X is irreducible.

Claim: $\dim(X) = 4$.

Heuristic: there are three degrees of freedom in choosing the first row x, y, z, and to enforce the rank 1 condition, the second row must be a scalar multiple of the first, yielding one degree of freedom for the scalar.

Exercise 0.2 (Gathmann 2.34).
Let X be a topological space, and show

a. If $\{U_i\} \rightrightarrows X$, then $\dim X = \sup_{i \in I} \dim U_i$.

b. If X is an irreducible affine variety and $U \subset X$ is a nonempty subset, then $\dim X = \dim U$. Does this hold for any irreducible topological space?

Exercise 0.3 (Gathmann 2.36).
Prove the following:

a. Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.

b. A complex affine variety of dimension at least 1 is never compact in the classical topology.

Exercise 0.4 (Gathmann 2.40).
Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1 x_4 - x_2 x_3 \rangle$$

and show the following:

a. R is an integral domain of dimension 3.

b. $x_1, \cdots, x_4$ are irreducible but not prime in R, and thus R is not a UFD.

c. $x_1 x_4$ and $x_2 x_3$ are two decompositions of the same element in R which are nonassociate.

d. $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

Exercise 0.5 (Problem 5).
Consider a set U in the complement of $(0,0) \in \mathbb{A}^2$. Prove that any regular function on U extends to a regular function on all of $\mathbb{A}^2$.