# Title

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## Contents



### 0.1 Exercises

**Problem 1.**

Let $C$ denote the Cantor set.

1. Show that $C$ contains point that is not an endpoint of one of the removed intervals.
2. Show that $C$ is nowhere dense, meager, and has measure zero.
3. Show that $C$ is uncountable.

**Solution 1.**

1. First we will characterize the endpoints of the removed intervals. Let $C_n$ be the $n$th stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:
$$C_n = [0, \frac{1}{3^n}] \bigcup [\frac{2}{2^n}, \frac{3}{3^n}] \bigcup \cdots \bigcup [\frac{3^n - 1}{n}, 1],$$
and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in $C_n$ is never removed in any later step, and so all endpoints remaining in $C$ are of this form where we allow $0 \leq n < \infty$.

   Thus, our goal is to produce a number $x \in [0,1]$ such that $x \neq \frac{k}{3^n}$ for any $k$ or $n$, but also satisfies $x \in C$. So we will need a general characterization of all of the points in $C$.

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Lemma: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.
$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

Proof: By induction on the index $k$ in $a_k$, first consider note that if $x \in C$ then $x \in C_1 = [0,1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. So if $x \in C_1$, then $x \notin (\frac{1}{3}, \frac{2}{3})$. But note that $a_1$ is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \leq x < \frac{1}{3}, \\ 1 & \frac{1}{3} \leq x < \frac{2}{3}, \\ 2 & \frac{2}{3} \leq x < 1. \end{cases}$$

Since the interval $(\frac{1}{3}, \frac{2}{3})$ is deleted in $C_1$, we find that $a_1 = 1 \iff x = \frac{1}{3}$. In this case, however, we claim that we can find a ternary expansion of $x$ that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that $x = x'$, which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left( \sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left( \sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2(\frac{1}{1 - \frac{1}{3}}) - 2 - 2(3^{-1}) \\ &= 2 \left( \frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have $\frac{1}{3} = (0.1)_3 = (0.222 \cdots)_3$ as ternary expansions, and a similar proof shows that such an expansion without 1s can be found for any endpoint.

For the inductive step, consider $a_n$: the claim is that if $a_n = 1$, then $x \notin C_{n+1}$ – that is, it is contained in one of the intervals deleted at the $n + 1$st stage. Writing the deleted interval at this stage as $(a, b)$, we find that $a_n = 1$ if and only if $x \in [a, b)$. Since $x \in C$, the only way $a_n$ can be 1 is if $x$ was in fact the endpoint $a$ (since no previous digit was a 1, by hypothesis). However, as shown above, every such endpoint has a ternary expansion containing no 1s. $\square$

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Therefore, if we can produce an $x$ that satisfies $x \neq \frac{k}{3^n}$ for any $k, n$ **and** $x$ has no 1s in its ternary expansion, we will have an $x \in C$ that is not an endpoint.

So take

$$x = (0.\overline{02})_3 = (0.020202\cdots)_3.$$

This evidently has no 1s in its ternary expansion, and if we sum the corresponding geometric series, we find $x = \frac{1}{4}$. This is not of the form $\frac{k}{3^n}$ for any $k, n$, and thus fulfills both conditions.

2. We first show that $C$ is nowhere dense by showing that the interior of its closure is empty, i.e. $(\overline{C})^\circ = \emptyset$.

   To do so, we note that $C$ is itself closed and so $C = \overline{C}$. To see why this is, consider $C^c$; we'll show that it is open. By construction, $C_1^c$ is the open interval $(\frac{1}{3}, \frac{2}{3})$ that is deleted, and similarly $C_n^c$ is the finite union of the open intervals that are deleted at the $n$th stage. But then

   $$C^c = \left(\bigcap C_n\right)^c = \bigcup C_n^c$$

   is an infinite union of open sets, which is also open. So $C$ is closed.

   It is also the case that $C$ has empty interior, so $C^\circ = \emptyset$. Towards a contradiction, suppose $x \in C$ is an interior point; then there is some neighborhood $N_\varepsilon(x) \subset C$. Since we are on the real line, we can write this as an interval $(x-\varepsilon, x+\varepsilon)$, which has length $2\varepsilon > 0$. Moreover, we have the containment

   $$(x-\varepsilon, x+\varepsilon) \subset C \subset C_n$$

   for every $n$.

   Claim: The length of $C_n$ is $(\frac{2}{3})^n$ where we define $C_0 = [0,1]$. Letting $L_n$ be the length of $C_n$, one easy way to see that this is the case is to note that $L_n$ satisfies the recurrence relation

   $$L_{n+1} = \frac{2}{3} L_n,$$

   since an interval of length $\frac{1}{3}L_n$ is removed at each stage. With the initial conditions $L_0 = 1$, it can be checked that $L_n = \left(\frac{2}{3}\right)^n$ solves this relation.

   Now, since $x \in C = \bigcap C_n$, it is in every $C_n$. So we can choose $n$ large enough such that

   $$\left(\frac{2}{3}\right)^n \leq 2\varepsilon.$$

   Letting $\mu(X)$ denote the length of an interval, we always have $C \subseteq C_n$ and so $\mu(C) \leq \mu(C_n)$.

   Using the subadditivity of measures, we now have

   $$\begin{aligned} &(x-\varepsilon, x+\varepsilon) \subset C \subset C_n \\ \implies &\mu(x-\varepsilon, x+\varepsilon) \leq \mu(C) \leq \mu(C_n) \\ \implies &2\varepsilon \leq \left(\frac{2}{3}\right)^n, \end{aligned}$$

   a contradiction. So $C$ has no interior points.

   But this means that

   $$(\overline{C})^\circ = C^\circ = \emptyset,$$

and so $C$ is nowhere dense.

To see that $\mu(C) = 0$, we can use the fact that for any sets, measures are additive over disjoint sets and we have

$$\mu(A) + \mu(X \setminus A) = \mu(X),$$

since measures are additive over disjoint sets. Rearranging, we