Morse Theory

D. Zack Garza

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Contents

1 Thursday January

Recall: For M^n a closed smooth manifold, consider a smooth map $f : M^n \longrightarrow \mathbb{R}$.

Definition 1.0.1.
A critical point p of f is *non-degenerate* iff $\det(H \coloneqq \dfrac{\partial^i f}{\partial x_i \partial x_j}(p)) \neq 0$ in some coordinate system U.

Proposition 1.1 ***(The Morse Lemma).***
For any non-degenerate critical point p there exists a coordinate system around p such that

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

λ is called the *index of f at p*.

Lemma 1.2.
λ is equal to the number of *negative* eigenvalues of $H(p)$.

Proof .
A change of coordinates sends $H(p) \longrightarrow A^t H(p) A$, which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that A is invertible and not necessarily orthogonal. Use the fact that $A^t HA$ is diagonalizable.

This means that f can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

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1.1 Proof of Morse Lemma

Suppose that we have a coordinate chart U around p such that $p \mapsto 0 \in U$ and $f(p) = 0$.

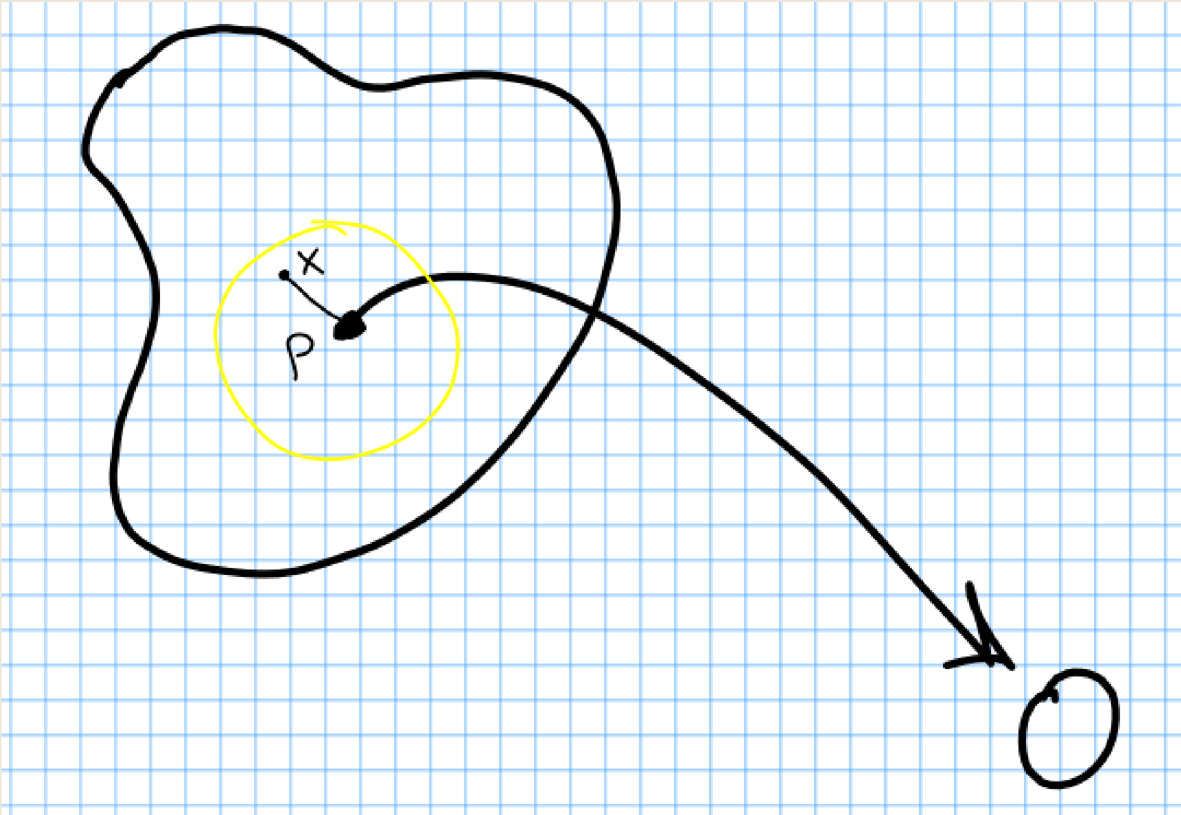
1.1.1 Step 1

Claim 1.
There exists a coordinate system around p such that

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

where $h_{ij}(x) = h_{ji}(x)$.

Proof .
Pick a convex neighborhood V of $0 \in \mathbb{R}^n$.



Restrict f to a path between x and 0, and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \cdots, tx_n)}{dt}\, dt = f(x_1, \cdots, x_n) - f(0) = f(x_1, \cdots, x_n).$$

since $f(0) = 0$.
We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots \frac{\partial f}{\partial x_n} x_n \, dt \implies \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}\, dt = f(x).$$

We thus have $f(x) = \sum_{i=1}^n x_i g_i(x)$ where $\frac{\partial f}{\partial x_i}(0) = 0$, and $\frac{\partial f}{\partial x_i} = x_1 \frac{\partial g_1}{\partial x_i} + \cdots + g_i + x_i \frac{\partial g_i}{\partial x_i} + \cdots + x_n \frac{\partial g_n}{\partial x_i}$.
When we plug $x = 0$ into this expression, the only term that doesn't vanish is g_i, and thus $\frac{\partial f}{\partial x_i}(0) = g_i(0)$ and $g_i(0) = 0$.
Applying the same result to g_i, we obtain $g_i(x) = \sum_{j=1}^n x_j h_{ij}(x)$, and thus $f(x) = \sum_{i,j=1}^n x_i x_j h_{ij}(x)$.
We still need to show h is symmetric. For every pair i, j, there is a term of the form $x_i x_j h_{ij} + x_j x_i h_{ji}$. So let $H_{ij}(x) = \frac{h_{ij}(x) + h_{ji}(x)}{2}$ (i.e. symmetrize/average h), then $f(x) =$

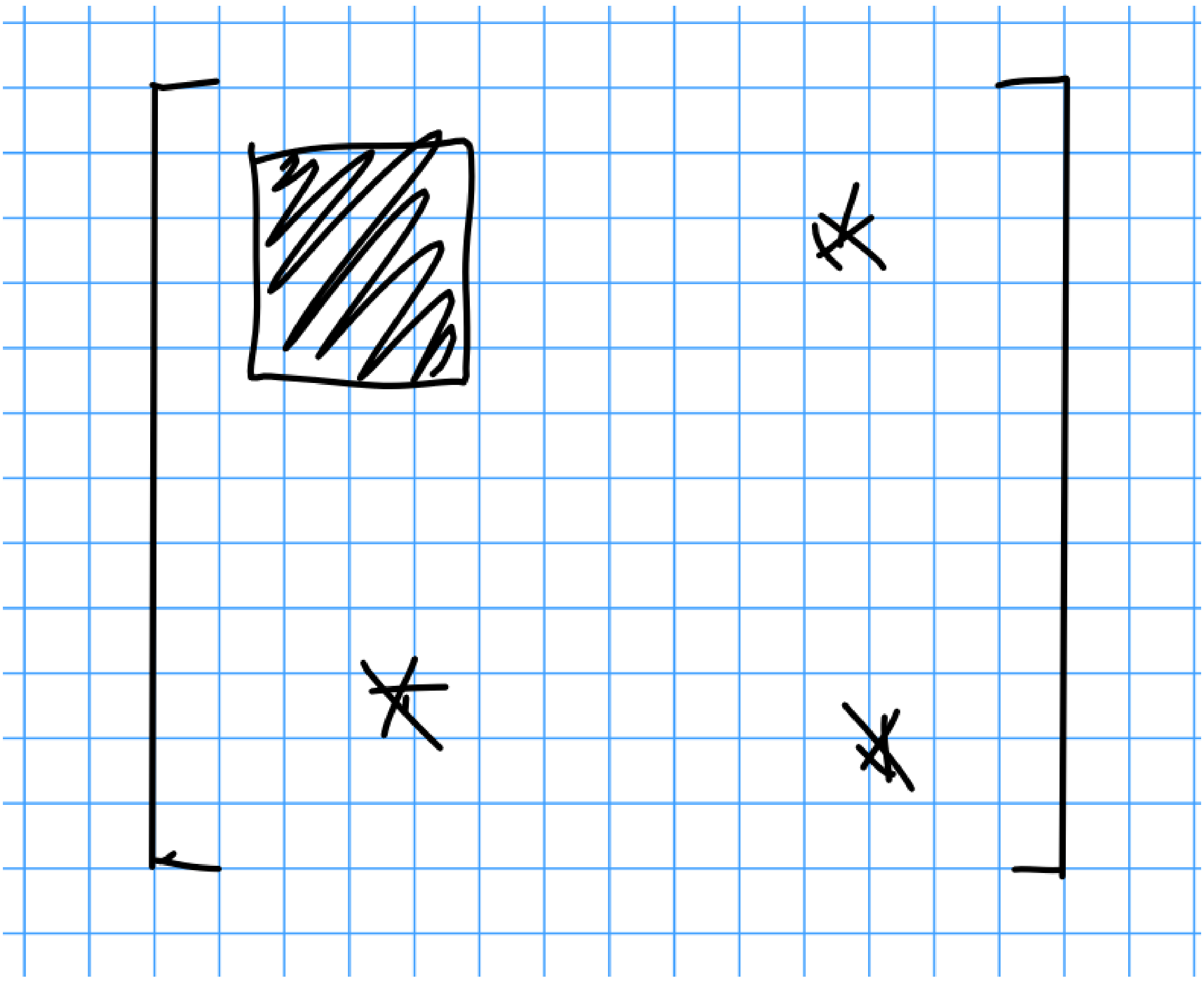
$\sum_{i,j=1}^{n} x_i x_j H_{ij}(x)$ and this shows claim 1. ■

1.1.2 Step 2: Induction

Assume that in some coordinate system U_0,

$$f(y_1, \cdots, y_n) = \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \cdots, y_n).$$

Note that $H_{rr}(0)$ is given by the top-left block of $H_{ij}(0)$, which is thus looks like



Note that this block is symmetric.

Claim 2 (1)**.**
There exists a linear change of coordinates such that $H_{rr}(0) \neq 0$.

We can use the fact that $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = H_{ij}(0) + H_{ji}(0) = 2H_{ij}(0)$, and thus $H_{ij}(0) = \frac{1}{2}\left(\frac{\partial f}{\partial x_i \partial x_j}\right)$.

Since $H(0)$ is non-singular, we can find A such that $A^t H(0) A$ has nonzero rr entry, namely by letting the first column of A be an eigenvector of $H(0)$, then $A = [\mathbf{v}, \cdots]$ and thus $H(0)A = [\lambda \mathbf{v}, \cdots]$ and $A^t[\lambda \mathbf{v}] = [\lambda \|\mathbf{v}\|^2, \cdots]$.

So

$$\begin{aligned}
\sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \cdots, y_n) &= y_r^2 H_{rr}(y_1, \cdots, y_n) + \sum_{i>r} 2 y_i y_r H_{ir}(y_1, \cdots, y_n) \\
&= H_{rr}(y_1, \cdots, y_n)\left(y_r^2 + \sum_{i>r} 2 y_i y_r H_{ir}(y_1, \cdots, y_n)/H_{rr}(y_1, \cdots, y_n)\right) \\
&= H_{rr}(y_1, \cdots, y_n)(\left(y_r + \sum_{i>r}^n y_i H_{ir}(y_1, \cdots, y_n)/H_{rr}(y_1, \cdots, y_n)\right)^2 \\
&\cdot \sum_{i>r}^n y_i^2 \left(H_{ir} Y / H_{rr}(Y)\right)^2 \\
&\cdot \sum_{i,j>r}^n H_{ir}(Y) H_{jr}(Y)/H_{rr}(Y))^2 \\
&\quad \text{by completing the square.}
\end{aligned}$$

Note that $H_{rr}(0) \neq 0$ implies that $H_{rr} \neq 0$ in a neighborhood of zero as well.

Now define a change of coordinates $\phi : U \longrightarrow \mathbb{R}^n$ by

$$z_i = \begin{cases} y_i & i \neq r \\ \sqrt{H_{rr}(y_1, \cdots, y_n)}\left(y_r + \sum_{i>r} y_i H_{ir}(Y)/H_{rr}(Y)\right) & i = r \end{cases}.$$

This means that

$$f(z) = \pm z_1^2 \pm \cdots \pm z_{r-1}^2 \pm z_r^2 + \sum_{i,j \geq r+1}^n z_i z_j \tilde{H}(z_1, \cdots, z_n).$$

Exercise: show that $d_0\phi$ is invertible, and by the inverse function theorem, conclude that there is a neighborhood $U_2 \subset U_1$ of 0 on which ϕ is still invertible.

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Corollary 1.3.
The nondegenerate critical points of a Morse function f are isolated.

Proof .
In some neighborhood around p, we have

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2,$$

Thus $\dfrac{\partial f}{\partial x_i} = 2x_i$, and so $\dfrac{\partial f}{\partial x_i} = 0$ iff $x_1 = x_2 = \cdots = x_n = 0$. ■

Corollary 1.4.
On a closed (compact) manifold M, a Morse function has only finitely many critical points.

We will need these facts to discuss the h-cobordism theorem. For a closed smooth manifold, $\partial M = \emptyset$, so M will define a cobordism $\emptyset \longrightarrow \emptyset$.

Definition 1.4.1 (Morse Function)**.**
Let W be a cobordism from $M_0 \longrightarrow M_1$. A *Morse function* is a smooth map $f : W \longrightarrow [a, b]$ such that

1. $f^{-1}(a) = M_0$ and $f^{-1}(b) = M_1$,
2. All critical points of f are non-degenerate and contained in $\operatorname{int}(W) := W \setminus \partial W$.

So f is equal to the endpoints only on the boundary.

Next time: existence of Morse functions. This is a fairly restrictive notion, but they are dense in the C^2 topology on (?).

2 Tuesday January 14th

2.1 Existence of Morse Functions

Notation Let $F(M; \mathbb{R})$ be the space of smooth functions from M to $\mathbb{R}$ with the C^2 topology.

Theorem 2.1.
Morse functions form an open dense subset of $F(M; \mathbb{R})$ in the C^2 topology.

Recall that the C^2 topology is defined by noting that $F(M, \mathbb{R})$ is an abelian group under addition, so we'll define open sets near the zero function and define open sets around f by translation. (I.e., if N is an open neighborhood of 0, then $N + f$ is an open neighborhood of f.)

So we'll define a base of open sets around 0. Take a finite cover of M, say by coordinate systems $\{U_\alpha\}$. Then let $h_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$. Now (exercise) we can find a compact refinement $C_\alpha \subset U_\alpha$ with each C_α compact and $\bigcup\limits_\alpha C_\alpha = M$. We can now define $f_\alpha := f \circ h_\alpha^{-1}$ for any $f : M \longrightarrow \mathbb{R}$

$$\begin{tikzcd}
U_\alpha \arrow[r, "h_\alpha"] \arrow[d, "f|_{U_\alpha}"'] & \mathbb{R}^n \arrow[dl, "f_\alpha"] \\
C_\alpha &
\end{tikzcd}$$

Now for each $\delta > 0$, define

$$N(\delta) = \left\{ f : M \longrightarrow \mathbb{R} \;\middle|\; \begin{cases} |f_\alpha(p)| < \delta \\ \left|\dfrac{\partial f_\alpha}{\partial x_i}\right| < \delta \\ \left|\dfrac{\partial^2 f_\alpha}{\partial x_i \partial x_j}\right| < \delta \end{cases} \quad \forall p \in h_\alpha(C_\alpha),\ \forall \alpha \right\}.$$

Corollary 2.2.
$f + N(\delta)$ (for all δ) is a basis for open neighborhoods around f.

Lemma 2.3.
This topology does not depend on the choice of $\{U_\alpha, h_\alpha\}$.

Proof .
See Milnor 2.

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Lemma 2.4 (1)**.**
Let $f : U \longrightarrow \mathbb{R}$ be a C^2 map for $U \subseteq \mathbb{R}^n$. For "almost all" linear maps $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f + L$ has only nondegenerate critical points.

Almost all: Note that $\hom(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$, so the complement of the set of such maps has measure zero in $\mathbb{R}^n$.

Proof .
Consider $X = U \times \hom(\mathbb{R}^n, \mathbb{R})$, which contains a subspace $M = \left\{ (x, L) \;\middle|\; \partial_x(f + L) = 0 \right\}$, i.e. x is a critical point of f. If $\partial_x f + L = 0$, then $L = -\partial_x f$. We thus obtain an identification of M with U by sending $x \in U$ to $(x, -\partial_x f)$.
There is also a projection onto the second component, where $(x, L) \mapsto L$. So let $\pi : X \longrightarrow \hom(\mathbb{R}^n, \mathbb{R})$ be this projection; then there is a map $\tilde{\pi} : U \longrightarrow \hom(\mathbb{R}^n, \mathbb{R})$ given by $x \mapsto \partial_x f$.
Note that $f + L$ has a *degenerate* critical point iff there is an $x \in U$ such that $\partial_x(f + L) = 0$ (or equivalently $L = -\partial_x f$), and the second derivative of $f + L$ is zero. Since L is linear, this says that the matrix $\left(\dfrac{\partial f^2}{\partial x_i \partial x_j} \right)(x)$ is singular. But this says x is a critical point for $\tilde{\pi}$.
This happens iff $\tilde{\pi}(x) = -\partial_x f = L$, so L is a critical value for $\tilde{\pi}$. Thus $f + L$ has a degenerate critical point $\iff$ L is a critical value for $\tilde{\pi}$.
Now Sard's theorem applies: if $g : M^n \longrightarrow \mathbb{R}^n$ is a map from any manifold to $\mathbb{R}^n$ that is C^1, then the set of critical values of g in $\mathbb{R}^n$ has measure zero.
Thus the set of critical values of $\tilde{\pi}$ has measure zero, and thus for almost all L, $f + L$ has no

degenerate critical points. ■

Summary: Consider the map of first derivatives. It has a critical point whenever the 2nd derivative is singular, which is exactly the nondegeneracy condition.

Lemma 2.5 (2).
Let $K \subset U \subset \mathbb{R}^n$ with K compact and U open, and let $f : U \longrightarrow \mathbb{R}$ have only nondegenerate critical points. Then there exists a $\delta > 0$ such that every $g : U \longrightarrow \mathbb{R}$ that is C^2 which satisfies

1. $\left|\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)\right| < \delta$, and
2. $\left|\frac{\partial x}{\partial x_i \partial x_j}(p) - \frac{\partial g}{\partial x_i \partial x_j}(p)\right| < \delta$

for all i, j and $p \in K$ has only nondegenerate critical points.

Proof .
Define $|df| = \sqrt{\left|\frac{\partial f}{\partial x_1}\right|^2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|^2}$. Now note that $S(f) = |df| + \left|\det\left(\frac{\partial f^2}{\partial x_i \partial x_j}\right)\right| \geq 0$. This is an equality iff both terms are zero, and the first term is zero iff x is a critical point, while the second term is zero iff x is degenerate.
Since f has only nondegenerate critical points, this inequality is strictly positive on K, i.e. $S(f) > 0$. Since K is compact, $S(f)$ takes on a positive infimum on K, say μ. Then $S(f) \geq \mu > 0$ on K.
Thinking of S as defining a norm, the reverse triangle inequality yields

$$||df| - |dg|| \leq |df - dg| \leq \sqrt{n\delta^2} \leq \frac{\mu}{2},$$

where we can choose δ such that $\sqrt{n\delta^2} < \mu$.
We can also pick δ small enough such that

$$||\det J_f| - |\det(J_g)|| \leq \frac{\mu}{2},$$

where $J_f = \left(\frac{\partial f}{\partial x_i \partial x_j}\right)$ is shorthand for the matrix of partial derivatives appearing previously, and we just note that picking entries close enough makes the difference of determinant small enough (although there's something to prove there).
Then

$$\begin{aligned} &|df| - |dg| + |\det(J_f)| - |\det J_g| < \mu \\ \implies &0 \leq |df| + |\det(J_f)| - \mu < |dg| + |\det(J_g)|, \end{aligned}$$

The second inequality follows from just moving terms in the first inequality.

which makes the last term strictly positive, and thus nonzero on K. Then g has no degenerate critical points in K.

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Proof summary:

1. $\|f\|_2(x) = 0$ iff x is a degenerate critical point.
2. $\|f\|_2(x) \geq \mu > 0$ in K.
3. We can pick δ small enough such that $\|f\|_2 - \|g\|_2 < \mu$ on K.
4. This forces $\|g\|_2 > 0$ on K, so g has *no* nondegenerate critical points on K.

2.2 Proof that Morse Functions are Open

We still want to show that Morse functions form an open dense subset.

To see that they form an open set, suppose $f \in F(M, \mathbb{R})$ is Morse. Then take a finite cover of M, say $\{(U_i, h_i)\}_{i=1}^k$. Pick compact $C_i \subset U_i$ that still covers M.

Note that any g satisfying the 2 required conditions where $|f - g| < \delta$ (?), then $g \in N(\delta) + f$.

By lemma 2, there exists a $\delta > 0$ such that every $g \in N_1 := f + N(\delta)$ has only nondegenerate points in C_1. We can pick a δ similarly to define an N_i for every i. Then taking $N = \bigcap_{i=1}^k N_i$, this yields an open neighborhood of f such that every $g \in N$ has only nondegenerate critical points on $C_1 \bigcup C_2 \cdots \bigcup C_k = M$.

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2.3 Proof that Morse Functions are Dense

We want to show that this set is dense, so we'll fix some open set and show that there exists a Morse function in it.

Let $f \in N$ for N an open set; we'll then change f gradually to make it Morse.

Convention We'll say f is *good* on $S \subset M$ iff f has only nondegenerate critical points in S.

Pick a smooth bump function $\lambda : M^n \longrightarrow [0, 1]$ such that

- $\lambda \equiv 1$ on an open neighborhood of C_1, and
- $\lambda \equiv 0$ on an open neighborhood of $M \setminus U_1$.

Note: we can do this because $C_1 \subset U_1$ is closed, and $M \setminus U_1$ is closed, so we can find disjoint open sets containing each respectively using the fact that M^n is Hausdorff (?).

Now let $f_1 = f + \lambda L$ for some linear function $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, so $f_1 = f + L$ on an open neighborhood of C_1. By Lemma 1, for almost every L, f_1 is good.

Note that we need λ because L is only defined on $\mathbb{R}^n$, not on M.

Now $f_1 - f = \lambda L$ is supported in U_1. If we pick the coefficients of L small enough, noting that λ is bounded, then the first and second derivatives of $f - f_1$ will be bounded, and we can arrange for

$f_1 \in f + N(\varepsilon)$ for $\varepsilon > 0$ as small as we'd like. For ε sufficiently small, we can arrange for $N(\varepsilon) \subset N_\delta$ for the finitely many δs, and so $N(\varepsilon) \subset N$.

By Lemma 2, there exists a neighborhood $N_1 \subseteq N$ containing f_1 such that every $g \in N_1$ is good on C_1. Since $f_1 \in N_1$, we can repeat this process to obtain an $f_2 \in N_2 \subseteq N_1$ and so on inductively. Then since every $g \in N_2$ is good on C_2 and $N_2 \subseteq N_1$, every $g \in N_2$ is good on $C_1 \bigcup C_2$. This yields an $f_k \in N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset N$, so f_k is good on $\bigcup C_i = M$.

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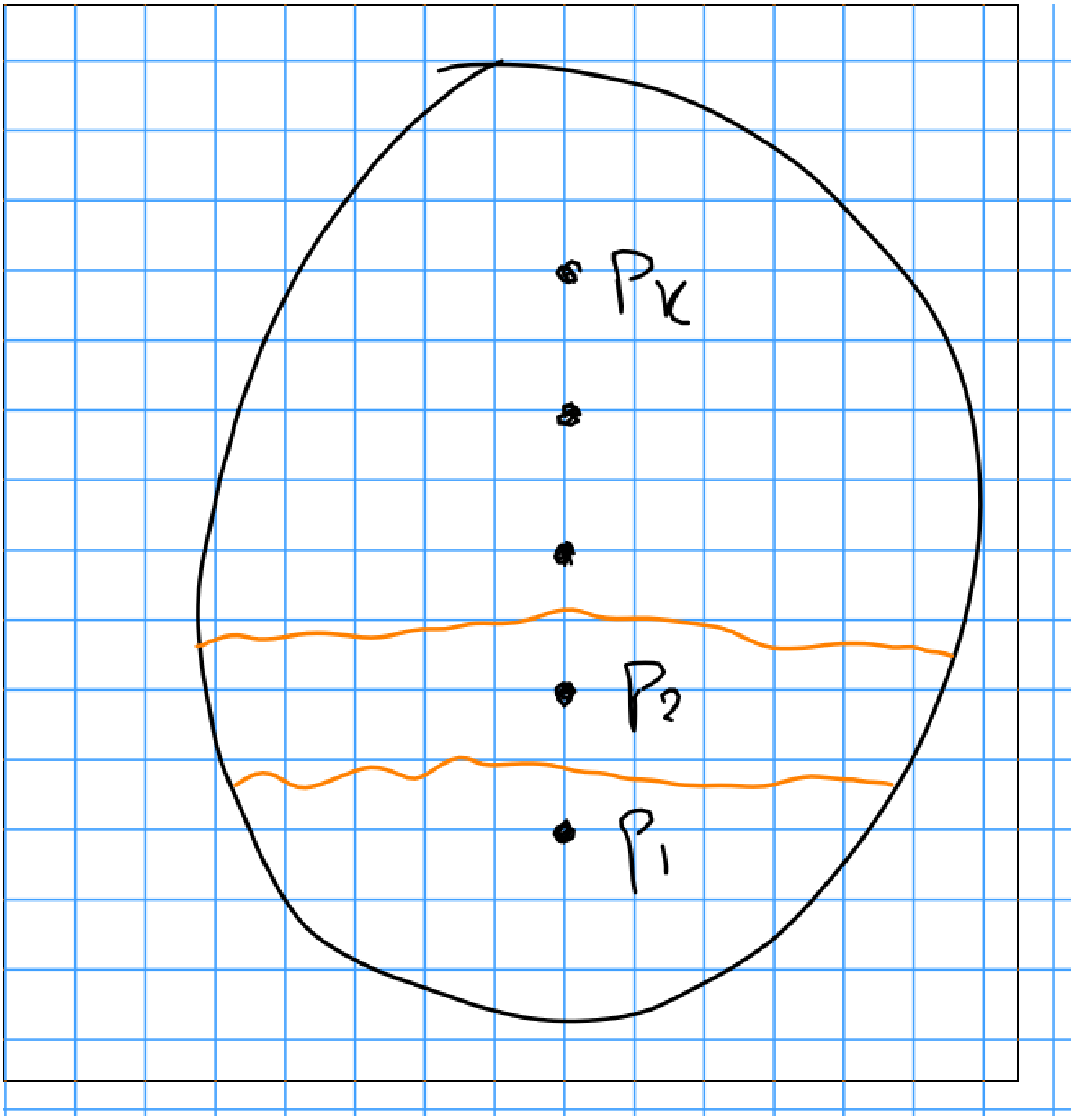
Thursday: We'll show that every pair of critical points can be arranged to take on different values, and then order them. This yields $f(p_1) < c_1 < f(p_2) < c_2 < \cdots c_{k-1} < f(p_k)$, and since the c_i are regular values, the inverse images $f^{-1}(c_i)$ are smooth manifolds and we can cut along them.



3 Thursday January 16th

3.1 Theorem: Approximation with Morse Functions with Distinct Critical Points

Theorem 3.1.
Let $f : M \longrightarrow \mathbb{R}$ be Morse with critical points $p_1, \cdots, p_k$. Then f can be approximated by a Morse function g such that

1. g has the same critical points of f

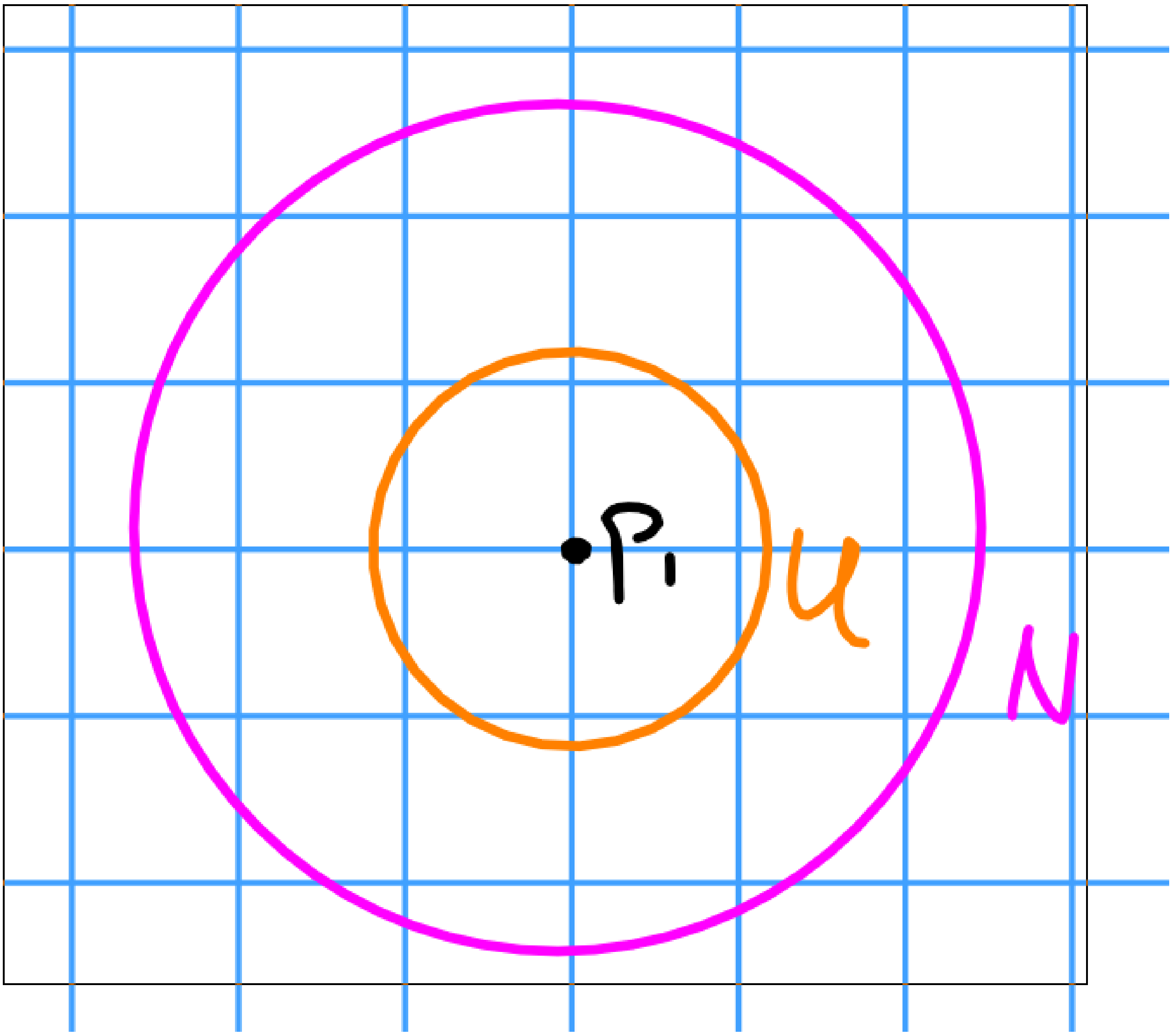
2. $g(p_i) \neq g(p_j)$ for all $i \neq j$.

Idea: Change f gradually near critical points without actually changing the critical points themselves.

3.2 Proof of Theorem

Suppose $f(p_1) = f(p_2)$.

Choose $\overline{U} \subset N$ open neighborhoods of p_1 such that $\overline{N}$ doesn't contain p_i for any i except for 1. Note that this is possible because the critical points are isolated.



Choose a bump function $\lambda \equiv 1$ on U and 0 on $M \setminus N$. Now let $f_1 = f + \varepsilon\lambda$, where we'll see how to choose ε small enough soon.

Let $K := \left\{x \;\middle|\; 0 < \lambda(x) < 1\right\}$, which is compact.

Pick a Riemannian metric on M, then we can talk about gradients. Recall that grad$f$ is the vector field that satisfies $\langle X,\ f\rangle$ for all vector fields X on M. Because f has no critical points in K, $X(f)$ is nonzero for some field X, so $\mathrm{grad} f$ is nonzero, noting that $\mathrm{grad} f$ is only zero at the critical points of f.

In particular, on K we have $0 < c \le |\mathrm{grad} f|$ for some c, and $\mathrm{grad}\lambda \le c'$ for some c'. So pick $0 < \varepsilon < c'/c$ such that $f_1(p_1) \ne f_1(p_2)$, $f_1(p_1) = f(p_1) + \varepsilon$, and$f_1(p_i) = f(p_i)$ for all $i \ne 1$. Note that this is possible because there are only finitely many points, so almost every ε will work.

Claim 1 The critical points of f_1 are exactly the critical points of f.

Proof .
In K, we have

$$\mathrm{grad} f_1 = \mathrm{grad} f + \varepsilon \mathrm{grad}\lambda \implies |\mathrm{grad} f_1| \ge |\mathrm{grad} f| - \varepsilon|\mathrm{grad}\lambda| \ge x - \varepsilon c' > 0.$$

If $x \notin K$, we have

1. $x \in U$, or
2. $x \in M \setminus N$

In case 1, λ is constant and $\mathrm{grad}\lambda = 0$, so $\mathrm{grad} f_1 = \mathrm{grad} f$. In case 2, λ is again constant, so the same conclusion holds.

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Claim 2 f_1 is Morse.

Proof .
In a neighborhood of p_1, we have $f_1 \equiv f + \varepsilon$. In a neighborhood of p_i, we have $f_1 \equiv f$.
We can then check that $J_{f_1}(p_i) = J_f(p_i)$, and since f is Morse, f_1 is Morse as well.

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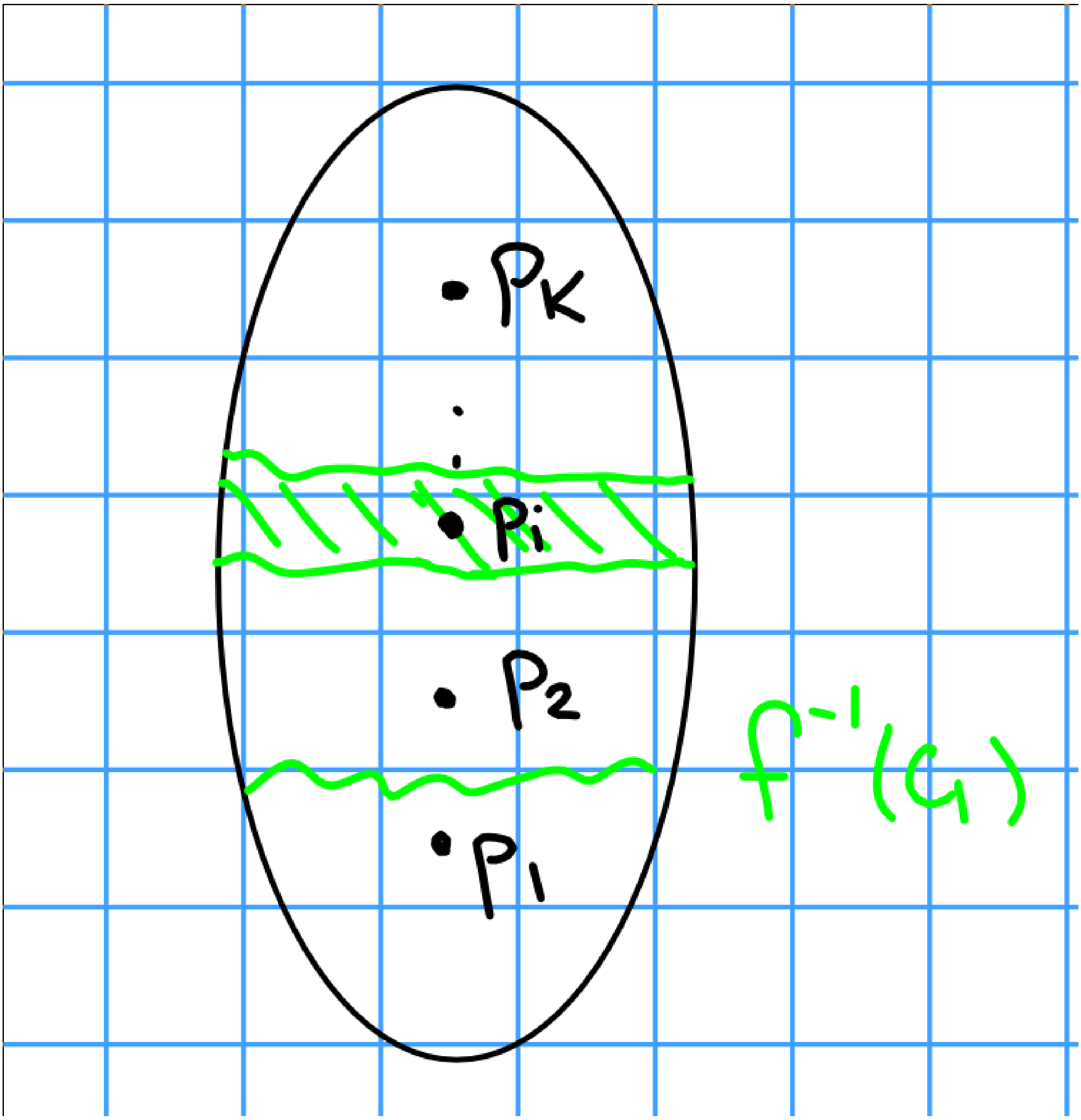
Recall that this lets us put an order on $f(p_i)$. Between every critical value, pick regular values c_i, i.e. $f(p_1) < c_1 < f(p_2) < \cdots$. Then $f^{-1}(c_i)$ is a smooth submanifold of dimension $n-1$, and we have the following schematic:



Moreover, $f^{-1}[c_i, c_{i+1}]$ is a cobordism from $f^{-1}(c_2)$ to $f^{-1}(c_{i+1})$.

Definition 3.1.1.
Recall that for $(W; M_0, M_1)$ a cobordism, a Morse function $f : W \longrightarrow [a, b]$ is Morse iff

1. $f^{-1}(a) = M_0$ and $f^{-1}(b) = M_1$.
2. f has only nondegenerate critical points and no critical points near $\partial W = M_1 \coprod M_2$, i.e. all critical points are in W° (the interior).

Proof of density of Morse functions goes through in the same way, with extra care taken to choose neighborhoods that do not intersect ∂W.

Theorem 3.2. 1. For every cobordism $(W; M_1, M_2)$ there exists a Morse function.
2. The set of such Morse functions is dense in the C^2 topology.
3. Any Morse function $f : (W; M_1, M_2) \longrightarrow [a, b]$ can be approximated by another Morse function $g : (W; M_1, M_2) \longrightarrow [a, b]$ such that g has the same critical points of f and $g(p_i) \neq g(p_j)$ for $i \neq j$ (i.e. the critical points are distinct).

Note that n-manifolds are a special cases of cobordisms, namely a manifold M is a cobordism $(W; M, \emptyset)$. So all statements about cobordisms will hold for n-manifolds.

Definition 3.2.1.
The **Morse number** μ of a cobordism $(W; M_0, M_1)$ is the minimum of $\left| \left\{ \text{critical points of } f \;\middle|\; f \text{ is Morse} \right\} \right|$.

We'll be considering cobordisms with $\mu = 0$.

Note: if we take $X = \operatorname{grad} f$, we have $\langle X, \operatorname{grad} f \rangle = \|\operatorname{grad} f\|^2 \geq 0$, which motivates our next definition.

Definition 3.2.2.
Let $f : W \longrightarrow [a, b]$ be a Morse function. Then a **gradient-like vector field** for f is a vector field ξ on W such that
1. $\xi(f) > 0$ on $W \setminus \operatorname{crit}(f)$.
2. For every critical point p there exist coordinates $(x_1, \cdots, x_n)$ on $U \ni p$ such that

$$f(X) = f(p) - x_1^2 - \cdots - x_{\lambda^2} + x_{\lambda+1}^2 + \cdots + x_n^2,$$

as in the Morse Lemma, where λ is the index, and

$$\xi = (-x_1, -x_2, \cdots, -x_\lambda, x_{\lambda+1}, \cdots, x_n) \text{ in } U.$$

Lemma 3.3.
Every Morse function f on $(W; M_0, M_1)$ has a gradient-like vector field.

3.3 Proof: Every Morse Function has a Gradient-Like Vector Field

For simplicity, assume f has a single critical point p. Pick coordinate $(x_1, \cdots, x_n)$ on an open set U_0 around p such that f has the form given in (1) above. Define ξ_0 on U_0 to be (2) above.

Every point $q \in W \setminus U_0$ has a neighborhood U' such that $df \neq 0$ on U'. By the implicit function theorem, there is a smaller neighborhood U'' such that $q \in U'' \subset U$ such that $f = c_0 + x_1$ on U'' for some constant c_0.

Exercise: check that this works!

But since $W \setminus U_0$ is a closed subset of a compact manifold, it is compact, so we can cover it with finitely many U_i that satisfy

1. $U_i \bigcap U = \emptyset$ for some open U containing p such that $U \subset U_0$ and $\overline{U} \subset U_0$.
2. U_i has a coordinate chart $(x_1^2, \cdots, x_n^2)$ such that $f = c_i + x_1^2$ on U_i for some constants c_i.

Thus on U_i we can set $\xi_i = (1, 0, \cdots, 0) = \dfrac{\partial}{\partial x_1^2}$ to get local vector fields. We can then take a partition of unity $\rho_1, \cdots, \rho_k$ and set $\xi = \displaystyle\sum_i \rho_i \xi_i$.

Now consider $\xi(f)$. By definition, $\xi(f) = \displaystyle\sum_i \rho_i \xi_i(f)$. Note that $\rho_i \xi_i(f) = 1$ in U_i, and $\rho_0 \xi_0(f) \geq 0$, so $\xi(f) \geq 0$. If x is not a critical point, then at least 1 $\xi_i(f)(x)$ is positive and thus $\xi(f)(x) > 0$.

This is because x is either in U, in which case the 0 term is positive, or $x \in U_i$, in which case one of the remaining terms is positive.

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The idea here: if we can make locally gradient-like vector fields, we can use partitions of unity to extend them to global vector fields.

Theorem 3.4.
Any cobordism $(W; M_0, M_1)$ with $\mu = 0$ is a product cobordism, i.e.

$$(W; M_0, M_1) \cong (M_0 \times I; M_0 \times \{0\}, M_0 \times \{1\}).$$

Proof .
Let $f : W \longrightarrow I$ be Morse with no critical points, and let ξ be a gradient-like vector field for f. Then $\xi(f) > 0$ on W, so we can normalize to replace ξ with $\dfrac{1}{\xi(f)}\xi$ and assume $\xi(f) = 1$. Then consider the integral curves of ξ, given by $\phi : [a, b] \longrightarrow W$.

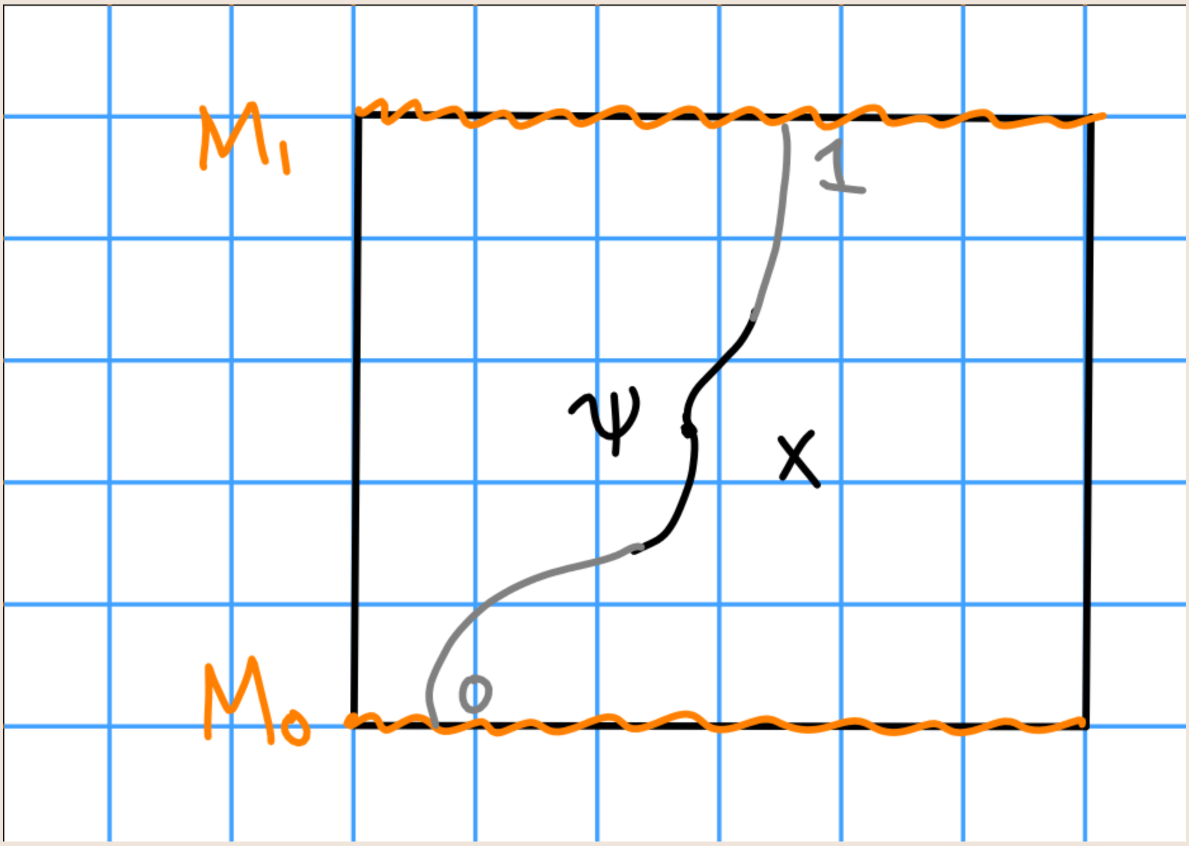
i.e. $d\phi = \xi$.

We can thus compute

$$\frac{\partial}{\partial t} f \circ \phi(t) = df(\frac{\partial \phi}{\partial t}) = df(\xi) = \xi(f) = 1.$$

By the FTC, this implies that $f \circ \phi(t) = c_0 + t$ for some constant c_0. So reparameterize by defining $\psi(s) = \phi(s - c_0)$, then $f \circ \psi(s) = s$. For every $x \in W$, there exists a unique maximal integral curve $\psi_x(s)$ that passes through x.

Note that this works because maximal curves must intersect the boundary at precisely $t = 0, 1$ and f is an increasing function. So for any curve passing through x, we can extend it to a maximal.



We can then define

$$
\begin{aligned}
h : M_0 \times I &\longrightarrow W \\
(x, s) &\mapsto \psi_x(s) \\
(\psi_y(0), f(y)) &\leftarrow\!\shortmid y
\end{aligned}
$$

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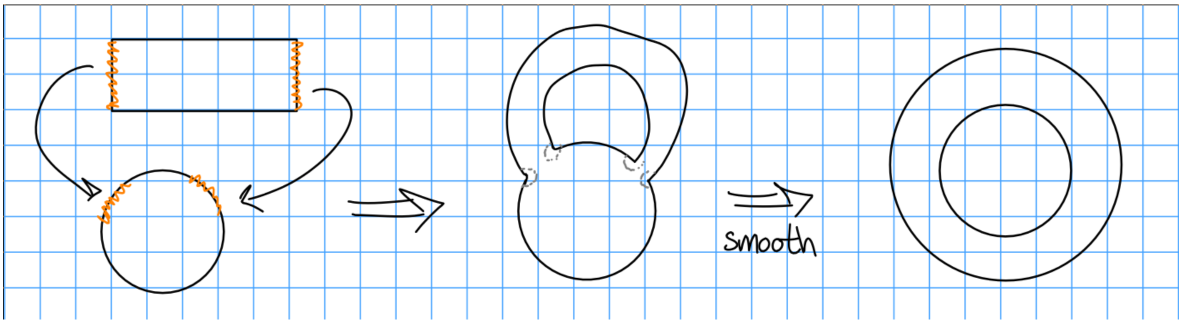
4 January 21st

4.1 Elementary Cobordism

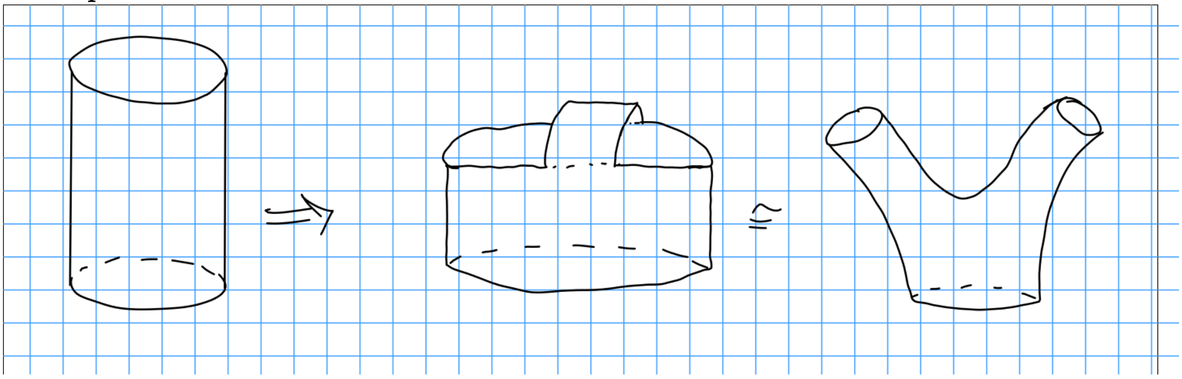
Recall that an elementary cobordism is a cobordism that has a Morse function with exactly one critical point.

Definition 4.0.1.
An n-dimensional λ-handle is a copy of $D^\lambda \times D^{n-\lambda}$ which is attached to ∂M^n via an embedding $\phi : \partial D^\lambda \times D^{n-\lambda} \hookrightarrow \partial M$.

Example 4.1.
Let $\lambda = 1, n = 2, n - \lambda = 1$ and take $M^2 = D^2$ and we attach $D^1 \times D^1$. Note that there's not necessarily a smooth structure on the resulting manifold, so we can "smooth corners":



Example 4.2.



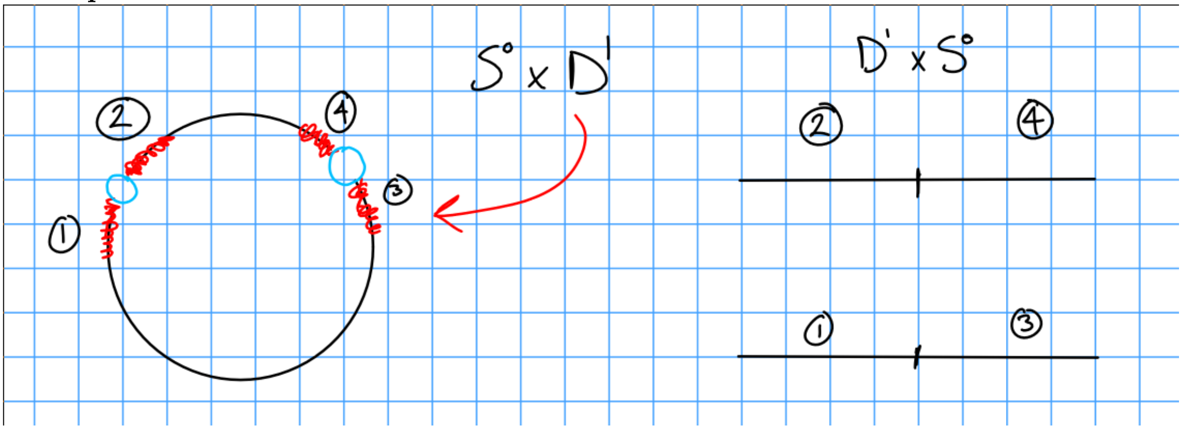
Note: the above is just a homeomorphism.

Definition 4.0.2.
Let M be an $n-1$ dimensional smooth manifold, and $\rho : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M^{n-1}$ be an embedding.
Then noting that $\partial D^{n-\lambda} = S^{n-\lambda-1}$, consider the space

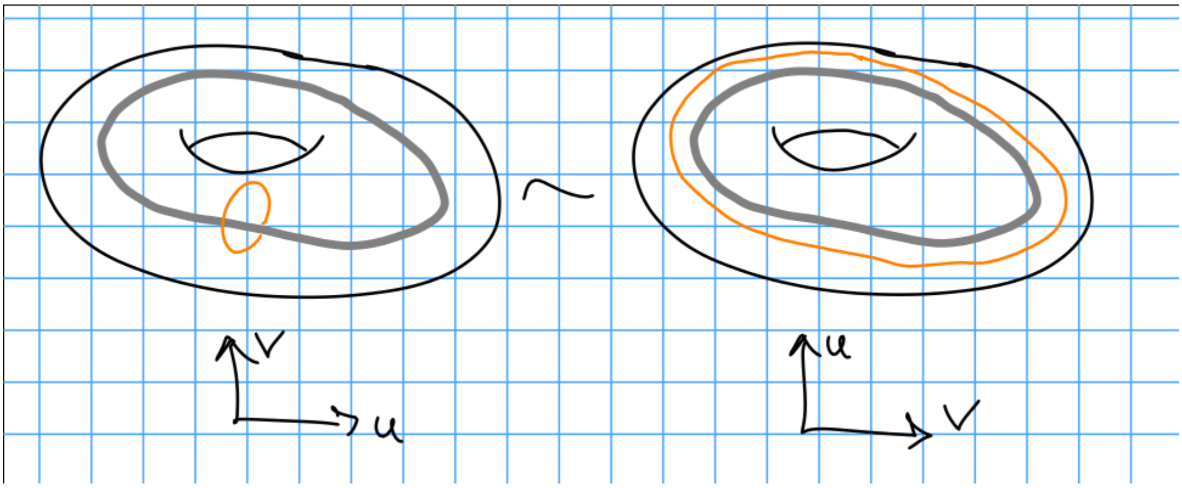
$$\begin{aligned} X(M, \phi) = &(M \setminus \rho(S^{\lambda-1} \times \{0\})) \times (D^\lambda \times S^{n-\lambda-1}) \\ &/ \left\langle \rho(u, tv) \sim (tu, v) \;\middle|\; t \in (0,1), \forall u \in S^{\lambda-1}, \forall v \in S^{n-\lambda-1} \right\rangle, \end{aligned}$$

where we note that we can parameterize $D^{n-\lambda} = tv$ where v is a point on the boundary.
Note that this accomplishes the goal of smoothing, and is referred to as **surgery** (of type $\lambda, n - \lambda$) on M along ϕ.

Example 4.3.

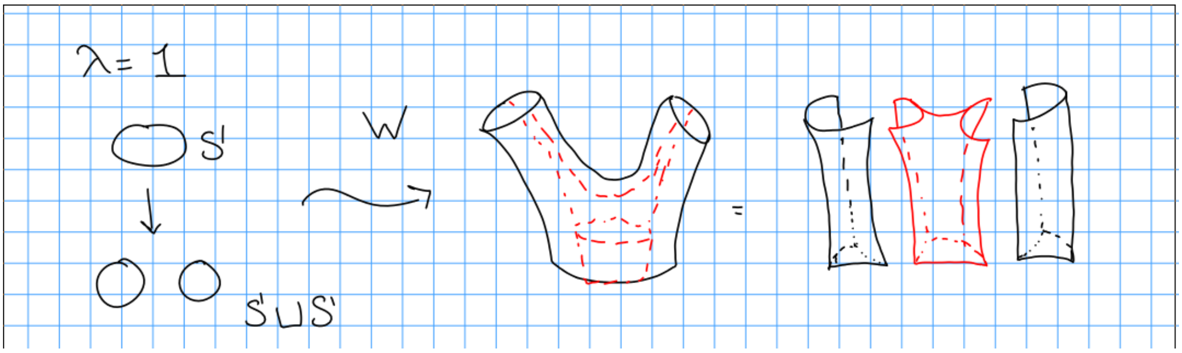


Example 4.4.
$n-1=3$ and $\lambda=2$ implies $\lambda-1=1$, and take $\rho : S^1 \times D^2 \longrightarrow S^3$, which has image a tubular neighborhood of a knot. Then $\phi(S^1 \times \{0\}) = K$ for some knot, and $(S^3 \setminus K)\coprod(D^2 \times S^1)/\cdots$. Then note that $\partial\phi(\{u\} \times D^2) = \{u\} \times S^1$, which no longer bounds a disk since we have removed the core of tube.



Theorem 4.1.
Suppose $M' = X(M,\rho)$ is obtained from M by surgery of type λ. Then there exists an elementary cobordism $(W; M, M')$ with a Morse function $f : W \longrightarrow [-1,1]$ with only one index λ critical point.

Example 4.5.
Let $M = S^1$ and $\lambda = 1$.

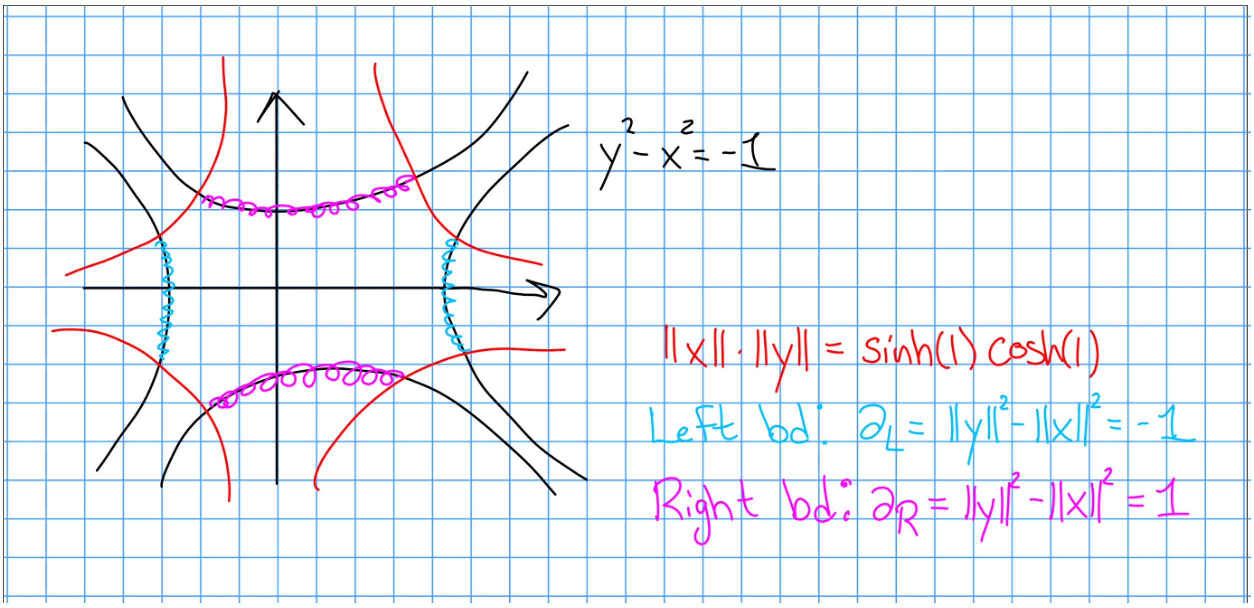


4.1.1 Proof: Surgeries Come From Cobordisms With Special Morse Functions

Write $\mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$, and $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$.

Then

$$L_\lambda = \{(\mathbf{x}, \mathbf{y}) \;\Big|\; -1 \leq -\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \leq 1,\ \|\mathbf{x}\|\|\mathbf{y}\| < \sinh(1)\cosh(1)\}.$$



The left boundary is given by $\partial_L : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = -1$, and there is a map

$$S^{\lambda-1} \times D^{n-\lambda} \xrightarrow{\text{diffeo}} \partial_L$$
$$(u, tv) \mapsto (u\cosh(t), v\sinh(t)) \quad t \in [0, 1),$$

which is clearly invertible.

The right boundary is given by $\partial_R : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = 1$, and there is a map

$$\begin{aligned} S^{\lambda-1} \times D^{n-\lambda} &\xrightarrow{\text{diffeo}} \partial_L \\ (tu, v) &\mapsto (u\sinh(t), v\cosh(t)). \end{aligned}$$

In the above picture, we can consider the orthogonal trajectories, which are given by $y^2 - x^2 = c$, which has gradient $(-x, y)$ and $xy = c$ which has gradient (y, x), so these are orthogonal.

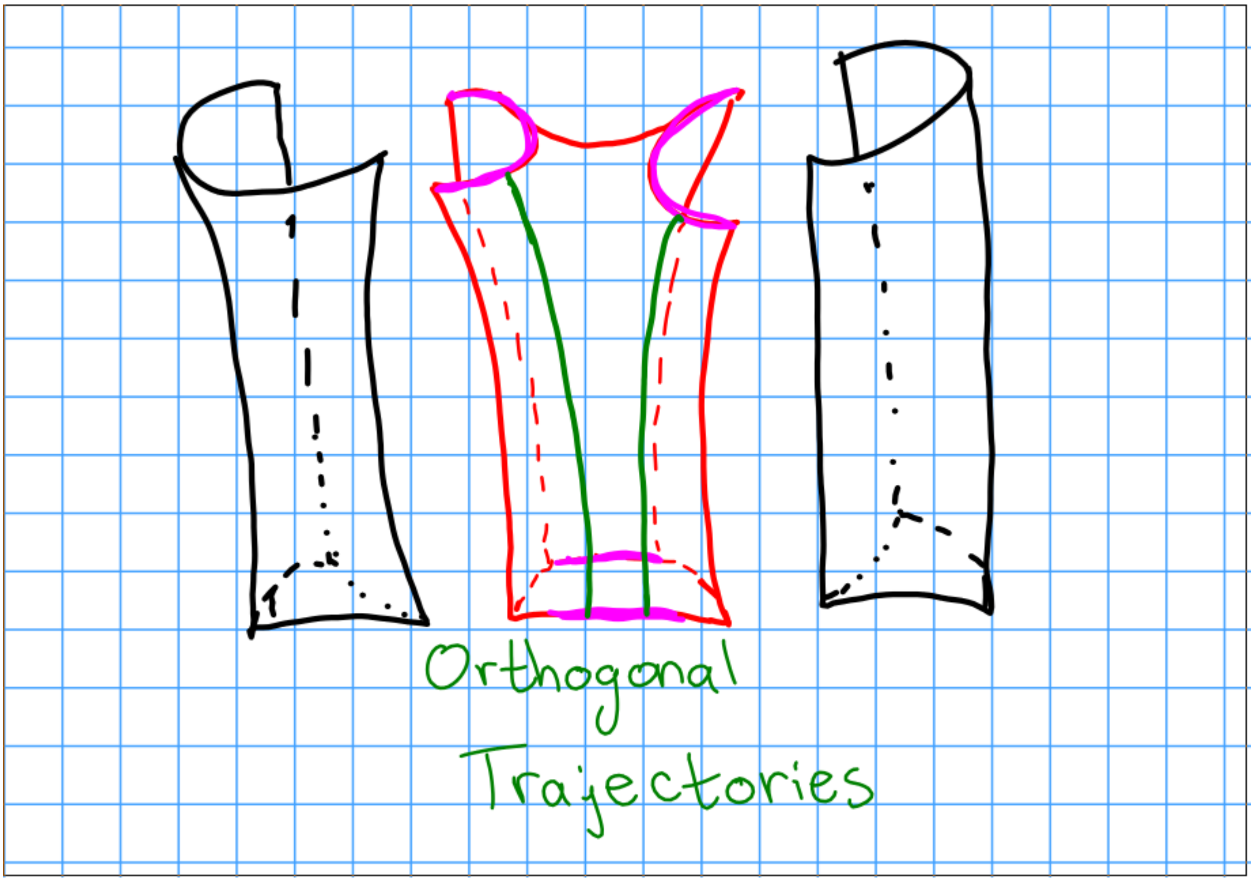
Recall that near a point $p \in M$, the morse function has the form $f(\mathbf{x}, \mathbf{y}) = f(p) - \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ with a gradient-like vector field given by $\xi = (-\mathbf{x}, \mathbf{y})$.

The orthogonal trajectories will generally be of the form $\|\mathbf{x}\|\|\mathbf{y}\| = c$, which we can parameterize as $t \mapsto (t\mathbf{x}, \frac{1}{t}\mathbf{y})$.

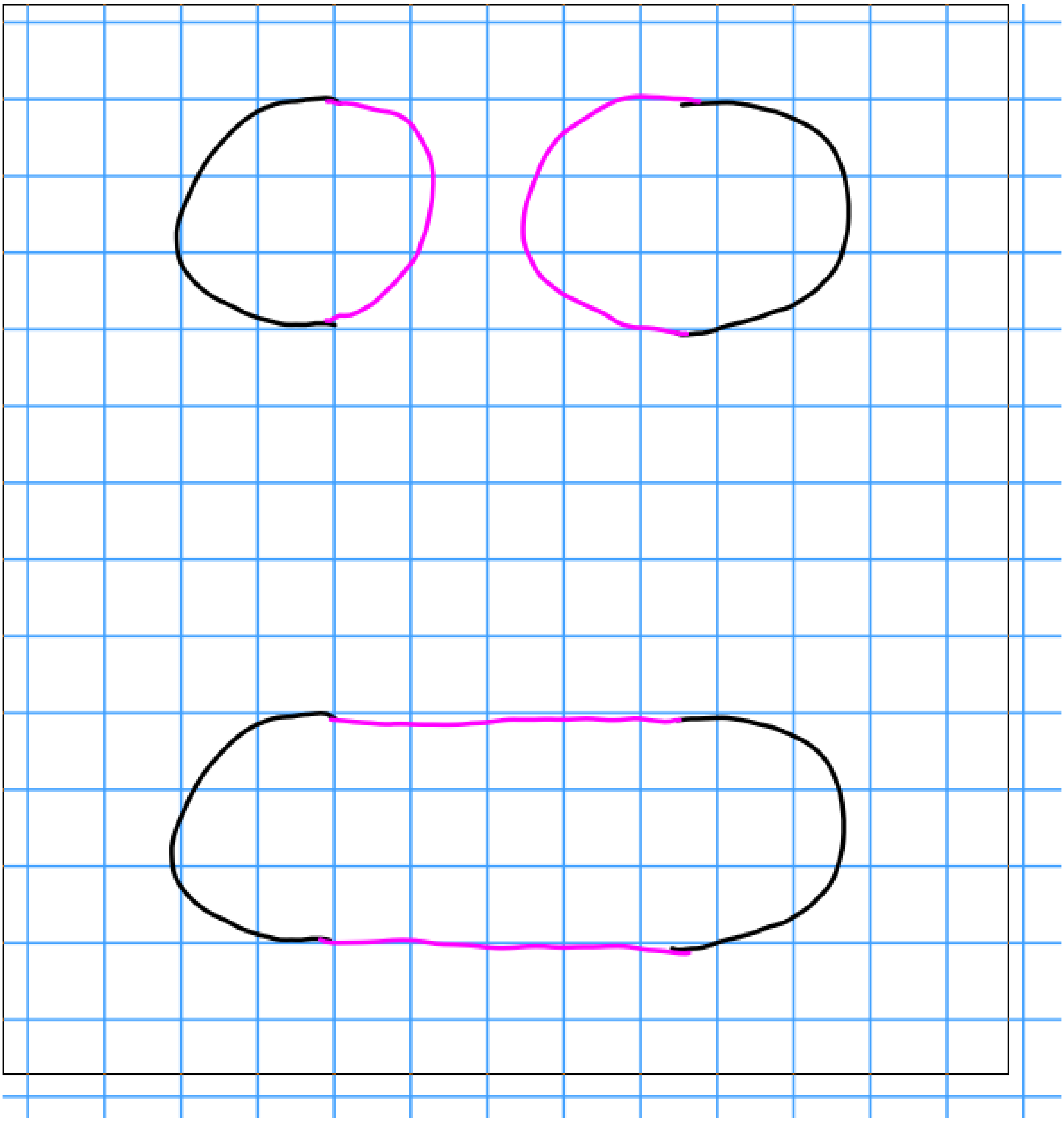
Construction of W:

Take

$$\begin{aligned} W(M,\phi) = &((M \setminus \phi(S^{\lambda-1} \times \{0\})) \times D^1) \coprod L_\lambda \\ &/ \left\langle \phi(u, tv) \times c \sim (\mathbf{x}, \mathbf{y}) \;\middle|\; \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = c, (\mathbf{x}, \mathbf{y}) \in \text{orthogonal trajectories starting from} (u\cosh(t), v\,\text{si} \right. \end{aligned}$$



This amounts to closing up in the following two ways:



This has two boundaries: when $c = -1$, we obtain M, and $c = 1$ yields $X(M, \phi)$. The Morse function is given by $f : W(M, \phi) \longrightarrow [-1, 1]$ where

$$\begin{cases} f(z, c) = c & z \in M \setminus \phi(S^{\lambda-1} \times \{0\}), c \in D^1 \\ f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 & (\mathbf{x}, \mathbf{y}) \in L_\lambda \end{cases}.$$