Math 200A Homework Question Compendium

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Contents

1 One

1. *Given*:

$$\forall x \in G, x^2 = e$$

Show:

$$G \in \mathbf{Ab}$$

2. *Given*:

$$|G| < \infty, |G| = 0 \mod 2$$

Show:

$$\exists g \in G \ni o(g) = 2$$

3. *Given:*

$$G \in \mathbf{Ab}$$

   Show:

$$T(G) \leq G$$

   (where

$$T(G) = \{g \in G : |g| < \infty\}$$

4. *Show*: Every finite group is finitely generated.
   - *Show*:

     $$\mathbb{Z}$$

     is finitely generated
   - *Show*:

     $$H \leq (\mathbb{Q}, +) \implies$$

     $$H$$

     is cyclic
   - *Show*:

     $$\mathbb{Q}$$

     is not finitely generated
5. *Show*:

$$\mathbb{Q}/\mathbb{Z}$$

   has, for each coset, exactly one representative in

$$[0, 1) \cap \mathbb{Q}$$

   - *Show*: Every element of

     $$\mathbb{Q}/\mathbb{Z}$$

     has finite order.
   - *Show*: There are elements in

     $$\mathbb{Q}/\mathbb{Z}$$

     of arbitrarily large order.
   - *Show*:

     $$\mathbb{Q}/\mathbb{Z} = T(\mathbb{R}/\mathbb{Z})$$

- *Show*:

$$\mathbb{Q}/\mathbb{Z} \cong \mathbb{C}^x$$

6. *Given*:

$$G/Z(G)$$

   is cyclic *Show*:

$$G$$

   is abelian

7. *Given*:

$$H \trianglelefteq G, K \trianglelefteq G, H \cap K = e$$

   Show:

$$\forall h \in H, \forall k \in K, hk = kh$$

8. *Given*:

$$|G| < \infty, \quad H \leq G, \quad N \trianglelefteq G, (|H|, [G : N]) = 1$$

   Show:

$$H \leq N$$

9. *Given*:

$$|G| < \infty, N \trianglelefteq G, (|N|, [G : N]) = 1$$

   Show:

$$N$$

   is the unique subgroup of order

$$|N|$$

2 Two

1. *Given*: For every triplet in

$$G$$

   , two elements commute *Show*:

$$G$$

   is abelian

2. *Given*:

$$H_1, H_2, H_3 \leq G, G = H_1 \cup H_2$$

Show:

$$G = H_1 \vee G = H_2$$

3. *Given*:

$$G = H_1 \cup H_2 \cup H_3, G$$

finite *Show*:

$$G = H_i \vee \forall i, [G : H_i] = 2$$

4. *Show*: TFAE;

$$\text{clos}(H)$$

is:

- The smallest normal subgroup of

$$G$$

containing

$$H$$

.
- The subgroup generated by all conjugates of

$$H$$

.
-

$$\bigcap_{H \leq N \trianglelefteq G} N$$

-

$$\phi : G \to -$$

,

$$\phi(H) = e$$

, then

$$\phi$$

factors through

$$G/\text{clos}(H)$$

5. *Given*:

$$H, K \trianglelefteq HK \leq G$$

Show:

$$\frac{HK}{H \cap K} \cong \frac{HK}{H} \times \frac{HK}{K}$$

6. *Given*:

$$H \leq G, N \trianglelefteq G, H \in \mathrm{Hall}(G)$$

Show:

$$H \cap N \in \mathrm{Hall}(N) \wedge \frac{HN}{N} \in \mathrm{Hall}(\frac{G}{N})$$

7. *Given*:

$$|G| = n, G$$

cyclic,

$$\sigma_i : G \to G \ni x \mapsto x^i$$

- Show

$$\sigma_i \in End(G)$$

- Show

$$\sigma_i \in Aut(G)$$

  iff

$$(i, n) = 1$$

-

$$\sigma_i = \sigma_j$$

  iff

$$i = j \mod n$$

-

$$\tau \in Aut(G) \implies \exists i \ni \tau = \sigma_i$$

-

$$\sigma_i \circ \sigma_j = \sigma_{ij}$$

6. The map

$$\begin{aligned}\psi : Z_n^\times &\to Aut(G)\\ i &\mapsto \sigma_i\end{aligned}$$

is an isomorphism.

8. *Given*:

$$G$$

is cyclic *Show*:

$$Aut(G)$$

is abelian of order

$$\phi(n)$$

9. *Show*:

$$D_\infty \cong \langle a, b \mid b^2 = e, ba = a^{-1}b\rangle$$

10. *Show*:

$$Q_8 \cong \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1}\rangle$$

11. *Show*:

$$\langle x, y \mid xy^2 = y^3, yx^2 = x^3y\rangle = \langle e\rangle$$

3 Three

1. *Given*:

$$G \sim X$$

transitively,

$$H \trianglelefteq G$$

- *Show*:

  $$H \sim X$$

  , but possibly not transitively
- *Show*:

  $$G$$

  acts transitively on

  $$\left\{\mathcal{O}_\langle : h \ \in H\right\}$$

- *Show*:

$$\forall i, j, |\mathcal{O}_{h_i}| = |\mathcal{O}_{h_j}|$$

- *Given*:

$$x \in \mathcal{O}_h$$

  Show:

$$|\mathcal{O}_h| = |H : H \cap G_x|$$

- *Show*:

$$|\{\mathcal{O}_h\}_{h \in H}| = [G : HG_x]$$

2. *Given*:

$$\mathcal{K}$$

   a conjugacy class in

$$S_n$$

   ,

$$\{\mathcal{O}_s : s \in S_n\}$$

   orbits of an

$$A_n$$

   -action on

$$S_n$$

   Show:

$$\mathcal{K} = \mathcal{O}_s \vee \mathcal{K} = \mathcal{O}_{s_i} \cup \mathcal{O}_{s_j}$$

   Show: Case 2 occurs iff

$$\{k_i\}$$

   , the cycle lengths in disjoint cycle form, are odd and distinct

3. i:

$$|G| < \infty, H < G$$

- *Show*:

$$\{gHg^{-1} : g \in G\} = [G : N_G(H)]$$

- *Show*:

$$G \neq \bigcup_{g \in G} gHg^{-1}$$

4. Prove Cauchy's Theorem. Given

$$p \mid o(G) < \infty$$

$$X = \left\{ (a_i)_{i=1}^{p} \in G^p \ni \prod_{i=1}^{p} a_i = e \right\}$$

$$asdsadas$$

- *Show*:

$$(a_1 a_2 \cdots a_p) = e \implies (a_2 a_3 \cdots a_p a_1) = e$$

- *Show*:

$$(Z_p, +) \sim X$$

  and

$$\bar{1} \sim (a_1 a_2 \cdots a_p) = (a_2 a_3 \cdots a_p a_1)$$

- *Show*:

$$|X| = |G|^{p-1}$$

- *Show*:

$$\{\mathcal{O}_x : |\mathcal{O}_x| = 1\} > 1$$

  and

$$\exists a \in G \ni a^p = e$$

5. *Given*:

$$G \sim X, \quad |G| < \infty, \quad 1 < |X| < \infty$$

- *Show*:

$$\exists g \ inG \ni \forall x \in X, g \sim x \neq x$$

- *Show*: This holds if

$$|G| = \infty$$

  , but not if

$$|X| = \infty$$

  as well.

6. *Given*:

$$H \leq G$$

. *Show*:

$$\text{core}(H)$$

is

- The largest

$$N \trianglelefteq G, N \subseteq H$$

- Generated by all normal subgroups contained in

$$H$$

- Given by

$$\bigcap_{g \in G} gHg^{-1}$$

- The kernel of

$$G \sim \frac{G}{H} \ni x \sim gH = (xg)H$$

7. *Given*:

$$[H : G] = n < \infty$$

- *Show*:

$$[\text{core}(H) : G]$$

divides

$$n!$$

- *Show*:

$$G$$

simple

$$\implies o(G) \mid n! \wedge |G| < \infty$$

8. *Given*:

$$A_n$$

is simple for

$$n \geq 5$$

Show:

$$\nexists H \in A_n \ni [H : A_n] < n$$

Show:

$$\exists H [H : A_n] = n$$

9. *Given*:

$$r$$

beads of

$$n$$

colors *Show*: How many distinct circular bracelets can be made.

4 Four

1. *Given*:

$$H \text{ char } G$$

Show:

$$H \trianglelefteq G$$

2. *Given*:

$$H \text{ char } K \trianglelefteq G$$

Show:

$$H \trianglelefteq G$$

3. *Given*:

$$K = \langle k \rangle \trianglelefteq G$$

Show:

$$H \leq K \implies H \trianglelefteq G$$

4. *Show*

$$H \trianglelefteq K \trianglelefteq G \not\Longrightarrow H \trianglelefteq G$$

5. *Given*:

$$P \leq H \leq K \leq G < \infty, P \in \mathrm{Syl}_p(G)$$

Show:

$$P, H \trianglelefteq K \implies P \trianglelefteq K$$

6. *Show*:

$$N_G(N_G(P)) = N_G(P)$$

7. *Given*:

$$\sigma \in Aut(G)$$

Show:

$$\sigma Inn(G)\sigma^{-1} = Inn(G)$$

iff

$$\forall g \in G, g^{-1}\sigma(g) \in Z(G)$$

8. *Show*:

$$Inn(G) \text{ char } Aut(G)$$

9. *Given*:

$$H \subseteq G, P \in \mathrm{Syl}_p(G)$$

- *Show*:

$$\exists g \in G \ni gPg^{-1} \in Syl_p(H)$$

- *Given*:

$$H \trianglelefteq G$$

Show:

$$P \cap H \in \mathrm{Syl}_p(H)$$

- *Given*:

$$P \trianglelefteq G$$

Show:

$$P \cap H \in \mathrm{Syl}_p(H)$$

and

$$|\mathrm{Syl}_p(H)| = 1$$

10. *Given*:

$$|G| = pqr, p < q < r$$

Show:

$$\exists P_i \in \mathrm{Syl}_i(G) \trianglelefteq G$$

11. *Given*:

$$|G| = 595$$

Show: All sylow subgroups are normal

12. *Given*:

$$|G| = p(p+1)$$

Show:

$$\exists N \trianglelefteq G$$

where

$$|N| = p$$

or

$$p+1$$

5 Five

1. Given:

$$G = H \rtimes_{\psi} K$$

$$\begin{aligned} \psi : K &\to Aut(H) \\ k &\mapsto \psi(k) \end{aligned}$$

$$\theta \in Aut(H)$$

$$\rho : K \to K$$

$$\begin{aligned} \phi_\theta : Aut(H) &\to Aut(H) \\ \rho &\mapsto \theta \circ \rho \circ \theta^{-1} \end{aligned}$$

$$\begin{aligned} \psi_2 : K &\to Aut(H) \\ k &\mapsto (\phi_\theta \circ \psi)(k) \end{aligned}$$

$$\begin{aligned} \psi_3 : K &\to Aut(H) \\ k &\mapsto (\psi \circ \rho)(k) \end{aligned}$$

Show:

$$H \rtimes_{\psi} K \cong H \rtimes_{\psi_2} K \cong H \rtimes_{\psi_3} K$$

2. Classify groups of order

$$pq, p < q, p \mid q - 1$$

3. Classify groups of order 20.
4. Classify groups of order 75.
5. Show:

$$|G| < 60 \implies G$$

is not simple.

6. Show:

$$|G| < 60 \implies G$$

is solvable

7. Given:

$$|G| < \infty$$

,

$$H \leq G$$

maximal

$$\implies [G : H] = p$$

, a prime. Show:

$$|G|$$

is solvable

- Given:

$$P \in Syl_p(G) \wedge \exists H \ni N_G(P) \leq H \leq G$$

Show:

$$[G : H] = 1 \mod p$$

- Given:

$$p \mid o(G)$$

, the largest such prime Show:

$$\exists P \trianglelefteq G \in Syl_p(G),$$

8.

$$|G| < \infty$$

- Given:

$$G$$

is characteristically simple Show:

$$\exists H \text{ (simple) } \ni G \cong H^n$$

. Show: Whether or not the converse holds
- Given:

$$N \trianglelefteq G$$

minimal Show:

$$N$$

is characteristically simple,

$$N \cong H^n$$

6 Six

1. Given:

$$G$$

is nilpotent Show:

$$H \leq G \implies H, G/H$$

are nilpotent
2. Show:

$$G/Z(G)$$

is nilpotent

$$\implies G$$

is nilpotent
3. Given:

$$|G| < \infty$$

Show:

$$|G|$$

is nilpotent iff

$$a, b \in G, (a, b) = 1 \implies ab = ba$$

4. Show:

$$D_{2n}$$

is nilpotent iff

$$n = 2^i$$

5. Given:

$$|G| < \infty$$

- Show

$$\Phi(G) \text{ char } G$$

- Show

$$\Phi(G)$$

  is nilpotent
- Given:

$$|P| = p^e$$

  Show:

$$P/\Phi(P)$$

  is an elementary abelian p-group Show:

$$N \trianglelefteq P, P/N$$

  is elementary abelian

$$\implies \Phi(P) \subseteq N$$

6. Given:

$$R$$

a commutative ring,

$$x, y \in R$$

nilpotent

- Show:

$$x + y$$

  is nilpotent Show:

$$\{x \in R : x \text{ is nilpotent}\} \trianglelefteq R$$

- Given:

  $$u \in R^{\times}, x \in R$$

  nilpotent Show:

  $$u + x \in R^{\times}$$

- Show: An counterexample to 1 when

  $$R$$

  is noncommutative.

7. Given:

   $$R$$

   a commutative ring,

   $$R[[x]]$$

   its formal power series

   - Show:

     $$\sum_{i=0}^{\infty} a_i x^i \in R[[x]]^{\times} \iff a_0 \in R^{\times}$$

   - Show:

     $$R$$

     a domain

     $$\implies R[[x]]$$

     a domain
   - Given:

     $$R$$

     a field Show:

     $$I = \{r \in R[[x]] : r_0 = 0\}$$

     is a maximal ideal of

     $$R[[x]]$$

     Show:

     $$I$$

     is the unique maximal ideal

8. Given:

$$R$$

a commutative ring,

$$G$$

a finite group,

$$RG$$

a group ring.

- Given:

  $$\mathcal{K} = \{k_1, k_2, \cdots k_m\}$$

  a conjugacy class in

  $$G$$

  Show:

  $$K = \sum_{i=1}^{m} k_i \in RG \implies K \in Z(RG)$$

- Given:

  $$\mathcal{K}_1 \cdots \mathcal{K}_r$$

  distinct conjugacy classes in

  $$G$$

  ,

  $$K_i = \sum_j k_j \ni k_j \in \mathcal{K}_i$$

  Show:

  $$Z(RG) = \{\sum a_l K_l : \forall 1 \leq l \leq r, a_l \in R\}$$

  (All

  $$R$$

  -linear combinations of the

  $$\mathcal{K}_i$$

)

9. Given:

$$R$$

a ring,

$$M_n(R)$$

its matrix ring

- Given:

$$I \trianglelefteq R$$

(two-sided) Show:

$$M_n(I) \trianglelefteq M_n(R)$$

Show:

$$\frac{M_n(R)}{M_n(I)} \cong M_n(\frac{R}{I})$$

- Show:

$$\forall I_M \trianglelefteq M_n(R), I$$

is of the form

$$M_n(I)$$

for some

$$I \trianglelefteq R$$

Show:

$$R$$

a division ring

$$\implies M_n(R)$$

is a simple ring.

7 Seven

8 Eight