Analysis HW 3

Zack Garza

(1a) If $m_*(E)$, take $B = \mathbb{R}^n$, otherwise suppose $m_*(E) < \infty$ and let $\varepsilon > 0$. Choose $\{Q_i\} \rightrightarrows E$ then choose open $\{L_i\}$ s.t. $Q_i \subseteq L_i$ and $|L_i| < (m_*(E) + \varepsilon)/2^i$.

Then define $L(\varepsilon) = \bigcup_{i=1} L_i$; then $L(\varepsilon)$ is open (and thus Borel) and

$$m(L(\varepsilon)) = m_*(L(\varepsilon)) \leq \sum_{i=1}^{\infty} |L_i| < m_*(E) + \varepsilon.$$

So take a sequence $\varepsilon_k = 1/k \to 0$; then let $L^n = \bigcap_{k=1}^{n} L_{1/k}$. We have $L^{k+1} \subseteq L^k \ \forall k$, and $m(L^1) \leq m_*(E) + 1 < \infty$, so $L^n \searrow E$ and by upper continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty} L^n\right) = m\left(\bigcap_{k=1}^{\infty} L_{1/k}\right) \overset{\text{continuity}}{=} \lim_{k\to\infty} m(L_{1/k}) = \lim_{k\to\infty} m_*(E) + 1/k = m_*(E),$$

so take $B = \bigcap_{n=1}^{\infty} L^n$. ■

(1b) Let $\varepsilon > 0$; since $E \in \mathcal{L}(\mathbb{R}^n)$, there exists a closed set K_ε s.t. $m(E \setminus K_\varepsilon) < \varepsilon$. If $m(E) < \infty$, then $m(K_\varepsilon) = m(E) - \varepsilon$, so take the sequence $\varepsilon_n = 1/n$ and let $K^n = \bigcup_{i=1}^{n} K_{1/i}$, then $K^n \subseteq K^{n+1} \ \forall i$ and $K^n \nearrow E$, so by continuity of measure from below,

$$m\left(\bigcup_{n=1}^{\infty} K^n\right) = \lim_{n\to\infty} m(K^n) = \lim_{n\to\infty} m(E) - 1/n = m(E),$$

so take $B = \bigcup_{n=1}^{\infty} K^n$, which is a countable union of closed sets and thus Borel.

If $m(F) = \infty$, let $E_n = E \cap \overline{B(n,0)}$. Then $\exists B_n$ (by the bounded case) such that $B_n \subseteq E_n$ is closed and $m(B_n) = m(E_n)$. But $E_n \nearrow E$, so

$$m(E) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(B_n) = m\left(\bigcup_{n=1}^{\infty} B_n\right),$$

so take $B = \bigcup_{n=1}^{\infty} B_n$, which is borel since each B_n is. ■

(1c) Since $m(E) = m_*(E)$, choose $\{Q_j\} \rightrightarrows E$ closed cubes such that $\sum_{j=1}^{\infty} |Q_j| < m(E) + \varepsilon/2$. Since $\sum_{i=1}^{\infty} |Q_i|$ converges, choose N such that $\sum_{i=N}^{\infty} |Q_i| < \varepsilon/2$, and let $A = \bigcup_{i=1}^{N-1} Q_i$. Then,

$$E \Delta A = \left(E \setminus \bigcup_{i=1}^{N-1} Q_i\right) \bigsqcup \left(\bigcup_{i=1}^{N-1} Q_i \setminus E\right)$$

$$\subseteq \bigcup_{i=N}^{\infty} Q_i \quad \bigsqcup \left(\bigcup_{i=1}^{\infty} Q_i \setminus E\right)$$

$$\Rightarrow m(E \Delta A) \leq m\left(\bigcup_{i=N}^{\infty} Q_i\right) + \left(m\left(\bigcup_{i=1}^{\infty} Q_i\right) - m(E)\right) \leq \varepsilon/2 + \left((m(E) + \varepsilon/2) - m(E)\right) = \varepsilon. ■$$

(2a) Choose an open set $\mathcal{O} \supseteq E$ s.t. $m_*(\mathcal{O}) < (1/1-\varepsilon)\, m_*(E)$, so that $(1-\varepsilon)m_*(\mathcal{O}) < m_*(E)$.
Then write $\mathcal{O} = \bigsqcup_{i=1}^{\infty} Q_i$ with each Q_i a closed cube, then towards a contradiction suppose that $m(E \cap Q_i) < (1-\varepsilon)m(Q_i)\ \forall i$. Then, writing $E = \bigsqcup_{i=1}^{\infty}(E \cap Q_i)$, we have

$$m(E) = \sum_{i=1}^{\infty} m(E \cap Q_i) > \sum_{i=1}^{\infty}(1-\varepsilon)m(Q_i) = (1-\varepsilon)\, m\left(\bigsqcup_{i=1}^{\infty} Q_i\right) = (1-\varepsilon)\, m(\mathcal{O}) \quad \#$$

so we must have $m(E \cap Q_j) \geq (1-\varepsilon)m(Q_j)$ for some j. ■

(2b) Let $\varepsilon > 0$ be arbitrary, and by (a) choose Q such that $m(E \cap Q) \geq (1-\varepsilon)m(Q)$.
Then let $E_0 = E \cap Q \subseteq E$, so $E_0 - E_0 \subseteq E - E$, and supposing towards a contradiction that $E_0 - E_0$ contains no ball around 0, choose $d \ll 1$ such that $d \notin E_0 - E_0$, and thus $E_0 \cap E_0 + d = \emptyset$. Also choose d small enough that $m(Q \cup Q + d) < m(Q) + \varepsilon$.
Then $E_0 \cup E_0 + d = E_0 \sqcup E_0 + d$, so $m(E_0 \cup E_0 + d) = 2m(E_0) \geq 2(1-\varepsilon)m(Q)$
Since $E_0 \cup E_0 + d \subseteq Q \cup Q + d$, we also have $m(E_0 \cup E + d) < m(Q) + \varepsilon$.
But then

$$2(1-\varepsilon)m(Q) \leq m(E_0 \cup E_0 + d) < m(Q) + \varepsilon$$

and taking $\varepsilon \to 0$ yields $2m(Q) < m(Q)$. #
So $E_0 - E_0$ must contain an open ball. ■

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(3) Fix x and let $L = \limsup_{y \to x} f(y) = \lim_{\delta \to 0} \sup_{y \in B_\delta(x)} f(y)$. Then consider $S_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$; we will show every $x \in S_\alpha$ has a ball $B_\delta(x) \subseteq S_\alpha$, making S_α open, and since α is arbitrary, this will show f is Borel measurable. Let $x \in S_\alpha$, so $f(x) < \alpha$. Then since f is upper-semicts, pick δ s.t. $y \in B_\delta(x) \Rightarrow f(y) \leq f(x)$. But then $y \in B_\delta(x) \Rightarrow f(y) \leq f(x) < \alpha \Rightarrow y \in S_\alpha$, so $B_\delta(x) \subseteq S_\alpha$ as desired. ■

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(4) $S = \{x \in \mathbb{R}^n \mid \lim f_n(x) \text{ exists}\} \in \mathcal{M}$ iff $S^c \in \mathcal{M}$, which is what we'll show. Noting that if we let $F(x) = \limsup_{n \to \infty} f_n(x)$, $G(x) = \liminf_{n \to \infty} f_n(x)$, then

$$\begin{aligned} S^c &= \{x \mid F(x) > G(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \mid F(x) > q > G(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \left(\{x \mid F(x) > q\} \cap \{x \mid G(x) < q\}\right) \end{aligned}$$

$= \bigcup_{q\in\mathbb{Q}} (M_q \cap N_q)$ where each M_q, N_q is measurable, thus making S^c a countable union of measurable sets & thus measurable. (E.g., M_q is measurable exactly because if $\{f_n\}$ are measurable, then $\limsup_{n\to\infty} f_n := F$ is measurable, as shown in class.) ■

(5a) f is well-defined because each $x \in C$ has a unique ternary expansion which contains no 1^s, and f is cts as we can write $g_n(x) = (a_k/2)\cdot(\frac{1}{2})^n$, so $f = \sum_{n=1}^{\infty} g_n$, where we have $|g_n(x)| \le 1/2^{n+1}$ which is summable, so f is uniformly cts by the M-test. Moreover,

$(0)_{10} = (0)_3 = (0.000\cdots)_3 \xmapsto{f} (0.000\cdots)_2 = (0)_{10}$, so $f(0)=0$, and

$(1)_{10} = (0.222\cdots)_3 \xmapsto{f} (0.111\cdots)_2 = (1)_{10}$, so $f(1)=1$. ■

(5b) $f \twoheadrightarrow [0,1]$, so consider $f^{-1}([0,1]\cap\mathcal{N})$ for $\mathcal{N}$ the non-measurable set. Since this is a subset of a measure zero set, it is measurable, and so $\underbrace{f^{-1}([0,1]\cap\mathcal{N})}_{\text{measurable}} \underbrace{\xmapsto{f}}_{\text{cts}} \underbrace{[0,1]\cap\mathcal{N}.}_{\text{not measurable}}$ ■

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(6a) Since f is cts, constant fns are cts, and f is a piecewise combination of cts fns that agree on intersections, F is cts. Constant fns are nondecreasing, so it only remains to show f is nondecreasing on C. Let $x = \sum a_n 3^{-n}$, $y = \sum b_n 3^{-n}$, and $x > y$. Then there is some minimal N such that $a_k = b_k \ \forall\, k < N$ and $a_N > b_N$. Then $\frac{1}{2}a_N > \frac{1}{2}b_N$, and $\frac{1}{2}a_k = \frac{1}{2}b_k \ \forall\, k<N$, which means that $f(x) > f(y)$ since

$$f(x)-f(y) = \sum_{n=1}^{\infty}(\tfrac{1}{2}a_n - \tfrac{1}{2}b_n)2^{-n} = \tfrac{1}{2}(a_N-b_N)2^{-N} + \tfrac{1}{2}\sum_{n=N+1}^{\infty}(a_n-b_n)2^{-n} \ge \tfrac{1}{2}(a_N-b_N)2^{-N} > 0.$$

(6b) Since $F(x)$ and $x\mapsto x$ are continuous and nondecreasing, and in fact $x\mapsto x$ is strictly increasing, G is continuous and strictly increasing & thus injective. To see that G is surjective, we just note that $G(0)=0$ and $G(1)=2$, so this follows from the IVT.

(6c₁) Let I be one of the intervals in C^c, then $x,y\in I \Rightarrow F(x)=F(y)$ and so $G(b)-G(a) = b-a = m(I)$. Then $m(I) = m(G(I))$ since G is cts, and so $m(G(C^c)) = m(G(\bigsqcup_{n=1}^{\infty} I_n)) = m(\bigsqcup_{n=1}^{\infty} I_n) = 1$, so

$$m(G(C)) = m([0,2]\setminus G(C^c)) = 2-1 = 1.$$

(6c₂) We have $\mathbb{R} = \bigcup_{q\in\mathbb{Q}}(\mathcal{N}+q)$, so $G(C) = \bigcup_{q\in\mathbb{Q}}(G(C)\cap\mathcal{N}+q)$, so $m(G(C)) \le \sum_{i=1}^{\infty}(G(C)\cap\mathcal{N}+q_i)$.

$$0 < 1 = m(G(C)) \le \sum_{i=1}^{\infty} m_*(G(C)\cap \mathcal{N}+q_i).$$

Note that no term can be measurable, since if we let $E_i = G(C)\cap\mathcal{N}+q_i$, then $x,y\in E_i \Rightarrow x-y\in\mathbb{R}\setminus\mathbb{Q}$, so E_i-E_i can't contain any ball around zero and thus can't be Lebesgue measurable by (2b). But by the inequality, not <u>every</u> term can have $m_*(E_i)=0$, so <u>some</u> $E_i\subseteq G(C)$ is not measurable.

(6c3) Let $\mathcal{N}'=E_i$, then $\mathcal{N}' = G(C)\cap \mathcal{N}+q_i$ for some i, so $G^{-1}(\mathcal{N}')\subseteq C$ and $m(C)=0$ implies $G^{-1}(\mathcal{N}')$ is measurable and $m(G^{-1}(\mathcal{N}'))=0$. But every cts function is Borel measurable, and since $G(G^{-1}(\mathcal{N}'))=\mathcal{N}'$ is not Borel, it can not pull back to a Borel set.

(6d) As shown above, E_i is not measurable and $G^{-1}(E_i)$ is null, so take $\varphi = \chi_{G^{-1}(E_i)}$. Then

$$S_\alpha = \{x\in[0,1] \mid \varphi(x)>\alpha\} = \left\{\begin{matrix} G^{-1}(E_i), & 0\le\alpha<1 \\ [0,1], & \alpha=0 \\ \emptyset, & \text{else}\end{matrix}\right\}$$ both of which are measurable, so $\varphi\in\mathcal{M}$.

But for $\alpha=\frac{1}{2}$, $S_{\frac{1}{2}} = \{x\in[0,2] \mid (\varphi\circ G^{-1})(x) > \frac{1}{2}\} = \{x\in[0,2]\mid G^{-1}(x)\in G^{-1}(E_i)\} = E_i \notin \mathcal{M}$. $\blacksquare$