# Problem Set 5

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## 1 Problem 1

We first make the following claim (TODO):

$$S := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sup\left\{\sum_{(j,k)\in B} a_{jk} \ni B \subset \mathbb{N}^2,\ |B| < \infty\right\}$$

$$T := \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj} = \sup\left\{\sum_{(j,k)\in C} a_{kj} \ni C \subset \mathbb{N}^2,\ |B| < \infty\right\}.$$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs $(j, k)$, we have the following inequality:

$$\sum_{(j,k)\in B} a_{jk} < \sum_{(k,j)\in C} a_{jk} \leq \sum_{(k,j)\in C} a_{jk} \leq T,$$

since $T$ is a supremum over *all* such sets $C$, and the terms of any finite sum can be rearranged.

But since this holds for every $B$, we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_{B} \sum_{(k,j)\in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that $T \leq S$, yielding the desired equality. $\quad\square$

## 1.1 Problem 2

We want to show the following equality:

$$\int_0^1 g(x)\ dx = \int_0^1 f(x)\ dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx = \int_0^1 f(x)\ dx$$

Note that if we can switch the order of integration, we would have

$$\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx =_? \int_0^1 \int_0^t \frac{f(t)}{t}\ dx\ dt$$
$$= \int_0^1 \frac{f(t)}{t} \int_0^t\ dx\ dt$$
$$= \int_0^1 \frac{f(t)}{t}(t-0)\ dt$$
$$= \int_0^1 f(t)\ dt,$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$F : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,t) \mapsto \frac{\chi_A(x,t)\hat{f}(x,t)}{t}.$$

where $A = \{(x,t) \subset \mathbb{R}^2 \ni 0 \le x \le t \le 1\}$ and $\hat{f}(x,t) := f(t)$ is the cylinder on $f$.

This defines a measurable function on $\mathbb{R}^2$, since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\int_{\mathbb{R}^2} |F| = \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right|\ dx\ dt$$
$$= \int_0^1 \int_0^t \frac{|f(t)|}{t}\ dx\ dt \quad \text{since } t > 0$$
$$= \int_0^1 \frac{|f(t)|}{t} \int_0^t\ dx\ dt$$
$$= \int_0^1 |f(t)| < \infty,$$

where the last inequality holds because $f$ was assumed to be measurable. So $F$ is measurable on the product space $\mathbb{R}^2$, and we can thus apply Tonelli to $F$ to justify the initial switch. $\qquad\square$

## 2 Problem 3

Let $A = \{0 \le x \le y\} \subset \mathbb{R}^2$, and define

$$F(x, y) = \chi_A(x, y) \frac{x^{1/3}}{(1+xy)^{3/2}}.$$

Then $F \in L^1(\mathbb{R}^2) \iff f \in L^1(\mathbb{R}^2)$, and if this is true then we would have

$$\int_{\mathbb{R}^2} F = \int_0^\infty \int_y^\infty f(x, y) \; dx \; dy$$

and if an interchange of integrals is justified, this yields

$$
\begin{aligned}
\int_{\mathbb{R}^2} F &=_? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1+xy)^{3/2}} \; dy \; dx \\
&= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3}\sqrt{1+x^2}} \; dx \\
&= 2 \int_0^1 \frac{1}{x^{2/3}\sqrt{1+x^2}} \; dx + 2 \int_1^\infty \frac{1}{x^{2/3}\sqrt{1+x^2}} \; dx \\
&\le \int_0^1 x^{-2/3} \; dx + \int_0^\infty x^{-5/3} \\
&= 2(3) + 2\left(\frac{3}{2}\right) < \infty,
\end{aligned}
$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1+x^2} \ge \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1+x^2} \ge \sqrt{1}$.

Since this implies that $F$ is measurable, it remains to show that this interchange of integrals is justified.

Toward that end, we can note for $(x, y) \in A$, we have $x, y \in [0, \infty)$, and so $F(x, y)$ is a non-negative function on $\mathbb{R}^2$. In particular, $|F(x, y)| = F(x, y)$. But this means that Tonelli applies, and $\int F$ is equal to any iterated integral. Since we computed one such iterated integral and found it to be finite, this means that $\int |F| = \int F < \infty$.