Math 174, HW #7

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1)

Problem 1A, 1B				
i	xi	yi	Trapezoidal	Simpsons
0	0.00	4.00	-	-
1	0.25	3.76	9.71E-01	1.85E+00
2	0.50	3.20	8.71E-01	-
3	0.75	2.56	7.20E-01	-
4	1.00	2.00	5.70E-01	8.80E-01
	Sum		3.13E+00	2.73E+00
	Absolute Error		1.04E-02	4.07E-01
Problem 1C				
i	xi	yi	Trapezoidal	Simpsons
0	0.00	4.00	-	-
1	0.13	3.94	4.96E-01	9.80E-01
2	0.25	3.76	4.81E-01	-
3	0.38	3.51	4.54E-01	-
4	0.50	3.20	4.19E-01	8.46E-01
	0.00	4.00	4.50E-01	8.46E-01
5	0.63	2.88	4.30E-01	-
6	0.75	2.56	3.40E-01	-
7	0.88	2.27	3.02E-01	5.68E-01
8	1.00	2.00	2.67E-01	-
	Sum		3.64E+00	3.24E+00
	Absolute Error		4.97E-01	9.82E-02
	Times Smaller		0.02	4.14

Trapezoidal rule: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\left(f(x_{i-1}) + f(x_i)\right)\Delta x_i$

Composite Simpson's: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n/2} \frac{1}{3}\left(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})\right)\Delta x_i$

2) $\int_0^h f(x)\,dx - hf(h/2) := A$

$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(x)(x-\frac{h}{2})^n$, by Taylor expansion around $a = \frac{h}{2}$.

$$\Rightarrow A = \left(\int_0^h \sum_{n=0}^{\infty} \frac{1}{n!} f^n(\tfrac{h}{2})(x-\tfrac{h}{2})^n dx\right) - hf(\tfrac{h}{2})$$

$$= \left(\sum_{n=0}^{\infty} \int_0^h \frac{1}{n!} f^n(\tfrac{h}{2})(x-\tfrac{h}{2})^n dx\right) - hf(\tfrac{h}{2})$$

$$= \left(\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\tfrac{h}{2}) \int_0^h (x-\tfrac{h}{2})^n dx\right) - hf(\tfrac{h}{2})$$

$$= \left(\frac{1}{1} f(\tfrac{h}{2}) \int_0^h (x-\tfrac{h}{2})^0 dx + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(\tfrac{h}{2}) \int_0^h (x-\tfrac{h}{2})^n dx\right) - hf(\tfrac{h}{2})$$

$$= \left(hf(\tfrac{h}{2}) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(\tfrac{h}{2}) \cdot \frac{1}{n+1} (x-\tfrac{h}{2})^{n+1}\Big|_0^h\right) - hf(\tfrac{h}{2})$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)!} f^{(n)}(\tfrac{h}{2}) \underbrace{\left((\tfrac{h}{2})^{n+1} - (-\tfrac{h}{2})^{n+1}\right)}$$

n odd $\rightarrow$ $n+1$ even $\rightarrow$ this is zero

$$= \frac{1}{2} f'(\tfrac{h}{2})\left((\tfrac{h}{2})^2 - (-\tfrac{h}{2})^2\right) + \frac{1}{6} f''(\tfrac{h}{2})\left((\tfrac{h}{2})^3 - (-\tfrac{h}{2})^3\right) + \frac{1}{24} f'''(\tfrac{h}{2})\left((\tfrac{h}{2})^4 - (-\tfrac{h}{2})^4\right) + \cdots$$

$$= 0 + \frac{1}{6} f''(\tfrac{h}{2}) \cdot 2(\tfrac{h}{2})^3 + 0 + O(h^5)$$

$$= \frac{1}{3} f''(\tfrac{h}{2})(\tfrac{h}{2})^3 + O(h^5)$$

$$= O(h^3). \blacksquare$$

3)

$T = T(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 +$

3)

$$I = T(h) + K_2h^2 + K_4h^4 + K_6h^6 + \dots$$

where $I = \int_a^b f(x)\,dx$

$$T(h) = \sum_{k=1}^{(b-a)/h} \tfrac{1}{2}(f(x_{k-1}) + f(x_k))h$$

and $I - T(h) \approx K_2h^2$

Let $N_1(h) = T(h)$, then

$$N_2(2h) = 2N_1(h) - N_1(2h)$$

$$A = N_1(2h) + 4C_1h^2 + 16C_2h^4 + \dots = M$$
$$B = N_1(h) + C_1h^2 + C_2h^4 + \dots = M$$

$$A - 4B = N_1(2h) - 4N_1(h) + 12C_2h^4 + \dots = M - 4M = -3M$$

$$\rightarrow M = \tfrac{1}{3}(4N_1(h) - N_1(2h)) + 3C_2h^4 + \dots$$

So let $N_2(h) = \frac{1}{3}(4N_1(h) - N_1(2h))$

$$= \tfrac{1}{3}(4T(h) - T(2h))$$

$$= \tfrac{1}{3}\left(\sum_{k=1}^{2N} 2(f(a+(k-1)h) + f(a+kh))h \quad \sum_{k=1}^{N} \tfrac{1}{2}(f(x+2(k-1)h) + f(x+2kh))2h\right)$$

$x_0 = a$

$x_{2N} = b$

where $N = \frac{(a-b)}{h}$. So let $x_k = a + kh$, then this equals

$$= \tfrac{h}{3}(2(f(x_0) + \underline{f(x_1)} + \underline{f(x_1)} + f(x_2) + \dots f(x_{2n})) - (f(x_0) + \underline{f(x_2)} + \underline{f(x_2)} + f(x_4) + \dots + f(x_{2N}))$$

$$= \tfrac{h}{3}(2(f(x_0) + f(x_{2N}) + \underline{2f(x_1)} + 2f(x_2) + \dots + 2f(x_{2N-1})) - (f(x_0) + f(x_{2N}) + \underline{2f(x_2)} + 2f(x_4) + \dots + 2f(x_{2N-2}))$$

$$= \tfrac{h}{3}(f(x_0) + f(x_{2N}) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{2N-2}) + 4f(x_{2N}))$$

$$= \tfrac{h}{3}\left(f(x_0) + f(x_{2N}) + 4\sum_{\substack{1\le j\le 2N \\ j \text{ odd}}} f(x_j) + 2\sum_{\substack{1\le j\le 2N \\ j \text{ even}}} f(x_j)\right),$$

which is Simpson's rule. ∎

4) We have $y' = -2ty$, $y(0) = 2$. Let $h = 0.5$, so $\{t_i\} = \{0, \frac{1}{2}, 1\}$ and $w_2 \approx y(t_2) = y(1)$.

and $f(t,y) = -2ty$

(a)

$w_0 = y(0) = \underline{2}$

Then $w_i = \begin{cases} y(0), & i=0 \\ w_{i-1} + hf(t_{i-1}, w_{i-1}) & \text{else} \end{cases} = \begin{cases} 2, & i=0 \\ w_{i-1}(1-t_{i-1}), & \text{else} \end{cases}$

$$w_1 = w_0 + hf(t_0, w_0)$$
$$= 2 + \tfrac{1}{2}(-2\cdot 0\cdot 2)$$
$$= 2$$

$h = \frac{1}{2}$, $f(t,w) = -2tw$

$$= w_{i-1} + \tfrac{1}{2}(-2\cdot w_{i-1}\cdot t_{i-1})$$
$$= w_{i-1} - w_{i-1}t_{i-1}$$
$$= w_{i-1}(1 - t_{i-1})$$

$w_2 = 2(1 - \frac{1}{2}) = 1$ $\boxed{\text{So } y(1) \approx 1}$

Exact soln: $\frac{2}{e} \approx .7357$)

$\boxed{\rightarrow E(0.5) \approx .26424111}$

Taking $h = \frac{1}{4}$, $\{t_i\} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

$w_i = w_{i-1} + \frac{1}{4}(-2 w_{i-1} t_{i-1})$

$= \omega_{i-1}(1-t_{i-1})$

Taking $h=\frac{1}{4}$, $\{t_i\}=\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

$$\omega_i = \omega_{i-1} + \tfrac{1}{4}(-2\cdot\omega_{i-1}t_{i-1})$$
$$= \underline{\omega_{i-1}(1-\tfrac{1}{2}t_{i-1})}$$

so $\omega_0 = y(0) = 2$

$$\omega_1 = 2(1-\tfrac{1}{2}\tfrac{1}{4}) = 7/4$$
$$\omega_2 = \tfrac{7}{4}(1-\tfrac{1}{2}\tfrac{1}{2}) = 21/16$$
$$\omega_3 = \tfrac{21}{16}(1-\tfrac{1}{2}\cdot\tfrac{3}{4}) = 105/128$$
$$\omega_4 = \tfrac{105}{128}(1-\tfrac{1}{2}\cdot 1) = 105/256$$

(b) $\boxed{\rightarrow y(1) \approx 105/256}$

$\boxed{\rightarrow E(0.25) \approx 0.3256}$

(c) $\boxed{\rightarrow E(\tfrac{1}{2})/E(\tfrac{1}{4}) \approx .8115}$