

Category \mathcal{O} , Problem Set 3

D. Zack Garza

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1 Humphreys 1.10

Prove that the transpose map τ fixes $Z(\mathfrak{g})$ pointwise.

Check that τ commutes with the Harish-Chandra morphism ξ and use the fact that ξ is injective.

1.1 Solution

We first note that after choosing a PBW basis for \mathfrak{g} , τ is defined on \mathfrak{g} in the following way:

$$\begin{aligned}\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha\end{aligned}$$

which lifts to an anti-involution $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by extending linearly over PBW monomials. We can note that since τ fixes \mathfrak{h} pointwise by definition, its lift also fixes $U(\mathfrak{h})$ pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned}\xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}.\end{aligned}$$

Proposition 1.1.

The following diagram commutes

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \\ \downarrow \tau & & \downarrow \tau \\ Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \end{array}$$

Proof.

We will show that for all $z \in Z(\mathfrak{g})$, $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$. Expand z in a PBW basis as $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k}$. We then make the following computations:

$$\begin{aligned} (\xi \circ \tau)(z) &= (\xi \circ \tau) \left(\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} \right) \\ &= \xi \left(\prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_k} \right) \quad \text{since } \tau \text{ is an anti-homomorphism} \\ &= \prod_j h_j^{s_j} \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\tau \circ \xi)(z) &= \tau \left(\prod_j h_j^{s_j} \right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of ξ , we have

$$\begin{aligned} (\xi \circ \tau)z &= \xi(z) \\ \implies \tau(z) &= z. \end{aligned}$$

■

2 Humphreys 1.12

Fix a central character χ and let $\{V^{(\lambda)}\}$ be a collection of modules in \mathcal{O} indexed by the weights λ for which $\chi = \chi_\lambda$ satisfying

1. $\dim V^{(\lambda)} = 1$
2. $\mu < \lambda$ for all weights μ of $V^{(\lambda)}$.

Then the symbols $[V^{(\lambda)}]$ form a \mathbb{Z} -basis for the Grothendieck group $K(\mathcal{O}_x)$.

For example take $V^{(\lambda)} = M(\lambda)$ or $L(\lambda)$.

3 Humphreys 1.13

Suppose $\lambda \neq \mu$, so the linkage class $W \cdot \lambda$ is the disjoint union of its nonempty intersections of various cosets of $\Lambda_r \in \mathfrak{h}^\vee$.

Prove that each $M \in \mathcal{O}_{\chi_\lambda}$ has a corresponding direct sum decomposition $M = \bigoplus M_i$ in which all weights of M_i lie in a single coset.

Recall exercise 1.1b.