Problem Set 8

D. Zack Garza

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1 Regular Problems

1.1 Problem 1

1.1.1 Part a

Define a map

$$\begin{aligned}\phi_{\text{ev}} : \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) &\to A \\ (f : \mathbb{Z}_m \to A) &\mapsto f(1)\end{aligned}$$

Then ϕ_{ev} is a $\mathbb{Z}$-module homomorphism, since

$$\begin{aligned}\phi_{\text{ev}}(nf+g) &= (nf+g)(1) \\ &= nf(1) + g(2) \\ &= n\phi_{\text{ev}}(f) + \phi_{\text{ev}}(g)\end{aligned}$$

But this forces $f(\overline{0}) = 0_A$ (where $\overline{0} : \mathbb{Z}_m \to A$ is the zero map), we have

$$0 = f(0) = f(m) = mf(1),$$

we must have $mf(1) = 0$ in A. So

$$\text{im } \phi_{\text{ev}} = \{a \in A \mid ma = 0\} \coloneqq A[m].$$

It is also the case that

$$\ker \phi_{\text{ev}} = \{f \in \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \mid f(1) = 0\} = \left\{\overline{0}\right\},$$

which follows from the fact that $\mathbb{Z}_m = \langle 1 \mod m \rangle$ and $A = \langle 1_A \rangle$ as $\mathbb{Z}$-modules, so if $f(1 \mod m) = 0_A$ then

$$f(n \mod m) = nf(1 \mod m) = 0$$

and so f is necessarily the zero map. So $\ker \phi = \overline{0}$.

We can then apply the first isomorphism theorem,

$$\frac{\hom_{\mathbb{Z}}(\mathbb{Z}_m, A)}{\ker \phi_{\text{ev}}} \cong \text{im } \phi_{\text{ev}} \implies \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m].$$

1.1.2 Part 2

Lemma: If $x \mid n$ and $x \mid m$ then $x \mid \gcd(m, n)$

Proof: We have $x \mid km + \ell n$ for any integers k, ℓ. So let $d = \gcd(m, n)$, then there exist integers a, b such that $am + bn = d$. But we can now just take $k = a$ and $\ell = b$. $\square$

We claim that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{(m,n)}$, from which the result immediately follows by part 1.

Define a map

$$\begin{aligned} \phi : \mathbb{Z} &\to \mathbb{Z}_n[m] \\ 1 &\mapsto [1] \mod n. \end{aligned}$$

The claim is that this is an isomorphism.

Then ϕ is clearly surjective (since $\mathbb{Z} \to \mathbb{Z}_n$ is a quotient map and $\mathbb{Z}_n[m]$ is a subgroup of $\mathbb{Z}_n$) and if we let $d \coloneqq \gcd(m, n)$, we have

$$
\begin{aligned}
\ker\phi &= \{x \in \mathbb{Z}_n \ni mx = 0\} \\
&= \left\{x \in \mathbb{Z}_n \ni x \;\middle|\; m\right\} \\
&= \left\{x \in \mathbb{Z} \ni x \;\middle|\; n \text{ and } x \;\middle|\; m\right\} \\
&= \left\{x \in \mathbb{Z} \ni x \;\middle|\; d\right\} \quad \text{by the lemma} \\
&= d\mathbb{Z} \ .
\end{aligned}
$$

Then by the first isomorphism theorem, we have

$$
\frac{\mathbb{Z}}{\ker\phi} \cong \text{im } \phi \implies \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}} \coloneqq \mathbb{Z}_{\gcd(m,n)} \cong \mathbb{Z}_n[m].
$$

1.1.3 Part 3

Let $f \in \mathbb{Z}^* = \hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, so $f : \mathbb{Z}_m \to \mathbb{Z}$. These are both $\mathbb{Z}$-modules generated by their identity elements, so such a map is determined by where it send $[1]_{\text{ mod } m}$.

So let $f([1]_{\text{ mod } m}) = n \in \mathbb{Z}$. Since f is a module homomorphism, we have $f([0]_{\text{ mod } m}) = 0$, and in particular we have

$$
\begin{aligned}
0 &= f([0]_{\text{ mod } m}) \\
&= f([m]_{\text{ mod } m}) \\
&= f([1m]_{\text{ mod } m}) \\
&= mf([1]_{\text{ mod } m}),
\end{aligned}
$$

which forces $f([1]) \in \mathbb{Z}[m] = \{0\}$, so f must be the zero map and $\mathbb{Z}^* = 0$.

Note: $\mathbb{Z}[m] = 0$ because $\mathbb{Z}$ is an integral domain, so $mx = 0$ forces $m = 0$ or $x = 0$.

1.1.4 Part 4

To see that $\mathbb{Z}_m$ is a $\mathbb{Z}_{mk}$ module, we define an action

$$
\begin{aligned}
\mathbb{Z}_{mk} &\curvearrowright \mathbb{Z}_m \\
[x]_{mk} &\curvearrowright [y]_m \coloneqq [xy]_m
\end{aligned}
$$

This is a well-defined action:

If $[x_1]_{mk} = [x_2]_{mk}$ are two representatives of the same equivalence class, then

$$
[x_1]_{mk} - [x_2]_{mk} = [x_1 - x_2]_{mk} = [0]_{mk} \implies m \;\middle|\; x_1 - x_2.
$$

But then

$$\begin{aligned}([x_1]_{mk} \curvearrowright [y]_m) - ([x_2]_{mk} \curvearrowright [y]_m) &= [x_1 y]_m - [x_2 y]_m \\ &= [(x_1 - x_2)y]_m \\ &= [0]_m,\end{aligned}$$

which shows that their resulting actions on $\mathbb{Z}_m$ are equal.

This action yields a module structure:

- $r.(x+y) = r.x + r.y$:

$$[r]_{mk} \curvearrowright ([x]_m + [y]_m) = [r]_{mk} \curvearrowright [x+y]_m = [r(x+y)]_m = [rx]_m + [ry]_m.$$

- $(r+s).x = r.x + s.x$:

$$[r]_{mk} + [s]_{mk} \curvearrowright [x]_m = [r+s]_{mk} \curvearrowright [x]_m = [(r+s)x]_m = [rx]_m + [sx]_m.$$

- $(rs).x = r.s.x$:

$$\begin{aligned}[r]_{mk} \cdot [s]_{mk} \curvearrowright [x]_m &= [rs]_{mk} \curvearrowright [x]_m \\ &= [(rs)x]_m \\ &= [r]_{mk} \curvearrowright [sx]_m \\ &= [r]_{mk} \curvearrowright ([s]_{mk} \curvearrowright [x]_m).\end{aligned}$$

- $1.x = x$:

$$[1]_{mk} \curvearrowright [x]_m = [1x]_m = [x]_m.$$

$\mathbb{Z}_m^* \coloneqq \hom_{\mathbb{Z}_{mk}}(\mathbb{Z}_m, \mathbb{Z}_{mk}) \cong \mathbb{Z}_m$:

Define a map

$$\begin{aligned}\phi : \hom_{\mathbb{Z}_{mk}}(\mathbb{Z}_m, \mathbb{Z}_{mk}) &\to \mathbb{Z}_m \\ f &\mapsto [f([1]_m)]_m\end{aligned}$$

ϕ **is a homomorphism**, as

$$\begin{aligned}\phi(f+g) &= [(f+g)([1]_m)]_m = [f([1]_m) + g([1]_m)]_m = [f([1]_m)]_m + [g([1]_m)]_m, \\ \phi([r]_{mk} \curvearrowright f) &= [[r]_{mk} f([1]_m)]_m = [r]_m \cdot [f([1]_m)]_m = [r]_{mk} \curvearrowright \phi(f).\end{aligned}$$

ϕ **is injective**, as $[f([1]_m)]_m = [0]_m$, then for any $1 \leq \ell \leq m$, we have

$$[f([\ell]_m)]_m = [\ell f([1]_m)]_m = \ell [f([1]_m)]_m = \ell [0]_m = [0]_m,$$

so f must be the zero map.

ϕ **is surjective**, since if $[\ell]_m \in \mathbb{Z}_m$, we can define

$$\begin{aligned} f_\ell : \mathbb{Z}_m &\to \mathbb{Z}_{mk} \\ [1]_m &\mapsto [\ell]_{mk} \end{aligned}$$

which makes sense and is well-defined because $\mathbb{Z}_m \hookrightarrow \mathbb{Z}_{mk}$, and the map is defined on the generator.

So we have the desired bijection. $\square$

1.2 Problem 2

We have the map

$$\begin{aligned} \pi : \mathbb{Z} &\to \mathbb{Z}_2 \\ x &\mapsto [x]_2 \end{aligned}$$

which is a surjection and thus an epimorphism in the category $\mathbb{Z}$-Mod, and if we apply the functor $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \cdot\,)$ to π we obtain an induced map

$$\begin{aligned} \overline{\pi} : \hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) &\to \hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ f &\mapsto \pi \circ f. \end{aligned}$$

The claim is that $\overline{\pi}$ is *not* a surjection, and thus not an epimorphism (in the same category).

To see that this is the case, we can simply note that $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = 0$ by part 3 of Problem 1, whereas $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \neq 0$.

For example, one can define $\mathrm{id}_{\mathbb{Z}_2} : \mathbb{Z}_2 \to \mathbb{Z}_2,\ [x]_2 \to [x]_2$, which is a nontrivial module homomorphisms.

So any such f appearing must be the zero map, and thus $\overline{\pi}$ is also the zero map. $\square$

1.3 Problem 3

Let $f : R \to R$ be an endomorphism of R in the category of rings. We can then check that for any $r \in R$, we have $f(r) = f(r1_R) = rf(1_R)$, which says that f is given by right-multiplication by some fixed element $x_f \coloneqq f(1_R)$, i.e.

$$\begin{aligned} f : R &\to R \\ r &\mapsto r \cdot x_f \end{aligned}$$

and so we can attempt to define

$$\begin{aligned}\phi_1 : \hom_R(R,R) &\to R \\ f &\mapsto x_f \coloneqq f(1_R)\end{aligned}$$

We can check that

$$(g \circ f(r)) = g(f(r)) = g(r \cdot x_f) = r \cdot x_f \cdot x_g,$$

which shows that in fact

$$\phi(g \circ f) = x_f \cdot x_g,$$

which reverses the multiplication. So the correct codomain is R^{op}, and we amend the definition:

$$\begin{aligned}\phi_2 : \hom_R(R,R) &\to R^{op} \\ f &\mapsto x_f \coloneqq f(1_R)\end{aligned}$$

By construction, ϕ_s **is a ring homomorphism**. If R is commutative, then $x_f \cdot x_g = x_g \cdot x_f$, which makes ϕ_1 a ring homomorphism as well. It remains to check that it is an isomorphism/

ϕ_1 **is in injective**: We can check that $\ker \phi_1 = 0$ as a ring. To that end, suppose $\phi_1(f) = x_f = 0$. Then $f(r) = r \cdot 0 = 0$, so f can only be the zero map.

ϕ_1 **is surjective**: Let $x \in R$ be arbitrary, then we can define $f : R \to R$ by $f(1_R) = x$, so $f(r) = r \cdot x$. This is an endomorphism of R, and thus an element of $\hom_R(R,R)$.

By the first isomorphism theorem for rings, we thus have $\hom_R(R,R) \cong R$. $\square$

1.4 Problem 4

We have maps

$$\begin{aligned}\theta_A : A &\to (A^\vee)^\vee \\ a &\mapsto (\mathrm{ev}_a : f \mapsto f(a))\end{aligned}$$

$$\begin{aligned}\theta_B : B &\to (B^\vee)^\vee \\ b &\mapsto (\mathrm{ev}_b : g \mapsto g(b))\end{aligned}$$

$$\begin{aligned}f : A &\to B \\ a &\mapsto f(a)\end{aligned}$$

$$\begin{aligned} f^\vee : B^\vee &\to A^\vee \\ g &\mapsto g \circ f \end{aligned}$$

$$\begin{aligned} f^{\vee\vee} : A^{\vee\vee} &\to B^{\vee\vee} \\ h &\mapsto h \circ f^\vee \end{aligned}$$

We can now check that $f^{\vee\vee} \circ \theta_A = \theta_B \circ f$ as maps from A to $B^{\vee\vee}$. Letting $a \in A$, and $h \in B^{\vee\vee}$ (so $h : B^\vee \to R$), we will show that both maps act on h in the same way.

For notational convenience, write $\phi \curvearrowright h \coloneqq h \circ \phi$. We then have

$$\begin{aligned} (f^{\vee\vee} \circ \theta_A)(a) \curvearrowright h &\coloneqq f^{\vee\vee}(\theta_A(a)) \curvearrowright h \\ &\coloneqq f^{\vee\vee}(\mathrm{ev}_a) \curvearrowright h \\ &= (\mathrm{ev}_a \circ f^\vee) \curvearrowright h \\ &\coloneqq h \circ (\mathrm{ev}_a \circ f) \\ &\coloneqq h(f(a)) \\ &= \mathrm{ev}_{f(a)} \curvearrowright h \\ &\coloneqq \theta_B(f(a)) \curvearrowright h \\ &\coloneqq (\theta_B \circ f)(a) \curvearrowright h, \end{aligned}$$

which shows that these actions agree, and thus the diagram commutes.

1.5 Problem 5

Let E be a free module over R an integral domain. Then E has a basis $\{\mathbf{e}_i\} \subseteq F$, so if $x \neq 0 \in E$, we have

$$x = \sum_i r_i \mathbf{e}_i$$

where each $r_i \in R$. Moreover, since $x \neq 0$, at least one $r_i \neq 0$, so let r_j denote one of the nonzero coefficients.

Now suppose x is a torsion element, so $mx = 0$ for some $m \neq 0 \in E$. We can then write

$$mx = m \sum_i r_i \mathbf{e}_i = \sum_i m r_i \mathbf{e}_i = 0$$

But by linear independence, this forces $mr_i = 0$ for all i. In particular, $mr_j = 0$ where $r_j \neq 0$. But this exhibits either m or r_j as a zero divisor, and since the only zero divisor in an integral domain is zero, we must have $m = 0$ or $r_j = 0$, a contradiction.

So x can not be a torsion element. But since $x \in E$ was arbitrary, E must be torsion-free.

For an example of a torsion-free module over an integral domain that is *not* free, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $\mathbb{Q}$ is clearly torsion-free, since it is an integral domain and the same argument as above applies.

But $\mathbb{Q}$ is not free as $\mathbb{Z}$-module. Supposing that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots\} \subset \mathbb{Q}$ was a $\mathbb{Z}$-basis, consider $\mathbf{b}_1 = \frac{p_1}{q_1}$ and $\mathbf{b}_2 = \frac{p_2}{q_2}$. Then $\mathbf{b}_1, \mathbf{b}_2$ can not be linearly independent over $\mathbb{Z}$, which follows from the fact that

$$q_1 p_2 \mathbf{b}_1 + q_2 p_1 \mathbf{b}_2 = p_2 p_1 - p_1 p_2 = 0,$$

while $q_1 p_2,\ q_2 p_1 \neq 0 \in \mathbb{Z}$. $\square$

1.6 Problem 6

If A is a cyclic module over a commutative ring R, so we have $A = Ra$ for some $a \in A$. By Hungerford's definition, the submodule A has order $r \iff$ the element a has order $r \iff$ the order ideal $\mathcal{O}_a \coloneqq \{x \in R \mid xa = 0\} = (r)$.

In particular, $ra = 0$.

1.6.1 Part 1

Since $(r, s) = (1)$, we can find $t_1, t_2 \in R$ such that

$$\begin{aligned}
t_1 r + t_2 s = 1 &\implies t_1 ra + t_2 sa = 1a \\
&\implies t_1(ra) + t_2 sa = a \\
&\implies t_2 sa = a && \text{since } ra = 0 \\
&\implies s(t_2 a) = a && \text{since } R \text{ is commutative,}
\end{aligned}$$

which implies that $a \in sA$ and thus $A \subseteq sA$. However, we always have $sA \subseteq A$ for modules, so this shows that $A = sA$.

To see that $A[s] = \{x \in A \mid sx = 0\} = 0$, let $x \in A[s]$; we will show $x = 0$. Since $x \in A = Ra$, we have $x = r_1 a$, and in particular

$$ra = 0 \implies rx = rr_1 a = r_1(ra) = 0.$$

So we now have $rx = 0$ and $sx = 0$, and we can write

$$\begin{aligned}
x &= (t_1 r + t_2 s)x \\
&= t_1(rx) + t_2(sx) \\
&= t_1 0 + t_2 0 \\
&= 0.
\end{aligned}$$

So $x = 0$ and thus $A[s] = 0$. $\square$

1.6.2 Part 2

Suppose $r = sk$. Toward an application of the first isomorphism theorem, define a map

$$\begin{aligned} \phi : R &\to sA = sRa \\ x &\mapsto sxa. \end{aligned}$$

ϕ is well-defined:

This follows from that fact that $a \in A \implies xA \in A$ for any $x \in R$, so the codomain is in fact sA.

ϕ is an R-module homomorphism:

We have

$$\begin{aligned} t \in R &\implies \phi(tx) = s(tx)a = t(sxa) = t\phi(x) \\ x, y \in R &\implies \phi(x+y) = s(x+y)a = sxa + sya + \phi(x) + \phi(y) \end{aligned}$$

$\ker \phi = (k)$:

Suppose $x \in \ker \phi$ so $sxa = 0_A$; we'd like to show $x \in (k)$.

By definition $sx \in \mathcal{O}_a$, and by assumption $\mathcal{O}_a = (r)$, so $sx = t_1 r$ for some $t_1 \in R$.

$$\begin{aligned} & sxa = 0_A & \\ \implies\ & sx = t_1 r & \text{since } sx \in \mathcal{O}_a \\ \implies\ & sx = t_1(sk) & \text{since } r = sk \text{ by assumption} \\ \implies\ & sx = s(t_1 k) & \text{since elements in } R \text{ and } A \text{ commute} \\ \implies\ & x = t_1 k & \text{since } R \text{ is a domain, so } sm = sn, s \neq 0 \implies m = n, \end{aligned}$$

which exhibits $x = t_1 k \implies x \in (k)$ as desired.

ϕ is surjective:

Since $A = Ra$, we have $sA = sRA$ and thus $x \in sA \implies x = sra$ for some $r \in R$; but then $\phi(r) = sra = x$.

We thus have

$$R/\ker \phi \cong \text{im } \phi \implies R/(k) \cong sA.$$

Similarly, define a map

$$\begin{aligned} \psi : R &\to A[s] \\ x &\mapsto kxa \end{aligned}$$

ψ **is well-defined**:

It suffices to check that im $\psi \subseteq A[s]$ (since we will show surjectivity shortly), i.e. that s annihilates anything in the image. This follows from

$$s(kxa) = (sk)xa = rxa = x(ra) = 0,$$

since $ra = 0$ by assumption.

ψ **is an R-module homomorphism:**

We can check

$$\psi(tr_1 + r_2) = k(tr_1 + r_2)s = tkr_1s + kr_2s = t\psi(r_1) + \psi(r_2)$$

which follows because elements of R commute with those from A under multiplication.

$\ker \psi = (s)$:

Suppose $x \in \ker \psi$, so $kxa = 0$. Then $kx \in \mathcal{O}_a = (r)$, so $kx = rt_1$. Then

$$\begin{aligned}
&kxa = 0_A \\
&\implies kx = rt_1 && \text{since } kx \in \mathcal{O}_a \\
&\implies kx = (sk)t_1 && \text{since } r = sk \\
&\implies kx = k(st_1) && \text{since } R \text{ is commutative} \\
&\implies x = st_1 && \text{since } R \text{ is a domain,}
\end{aligned}$$

and so $x \in (s)$ as desired.

ψ **is surjective:**

Letting $y \in A[s]$ be arbitrary. We have

$$\begin{aligned}
y \in A[s] &\implies x = t_1a, \quad sx = 0 \\
&\implies s(t_1a) = 0 \\
&\implies st_1 \in \mathcal{O}_a \implies \exists x \in R \ni st_1 = xr = x(sk) \\
&\implies st_1 = sxk \\
&\implies t_1 = xk && \text{since } R \text{ is a domain} \\
&\implies y = t_1a = (xk)a = kxa,
\end{aligned}$$

so $\psi(x) = y$.

We can then apply the first isomorphism theorem

$$R/\ker \psi \cong \text{im } \psi \implies R/(s) \cong A[s].$$

□

1.7 Problem 7

Lemma: If M is a cyclic module over a PID, then M has exactly 1 invariant factor.

Lemma: Let A be a cyclic module, so $A = Ra$. If the order of A is r, so $\mathcal{O}_a = (r)$, then $A \cong R/(r)$.

This means that we can write $A = R/(a)$ and $B = R/(b)$, and a, b are the invariant factors of A, B respectively.

We can now write

$$A \oplus B = R/(a) \oplus R/(b)$$