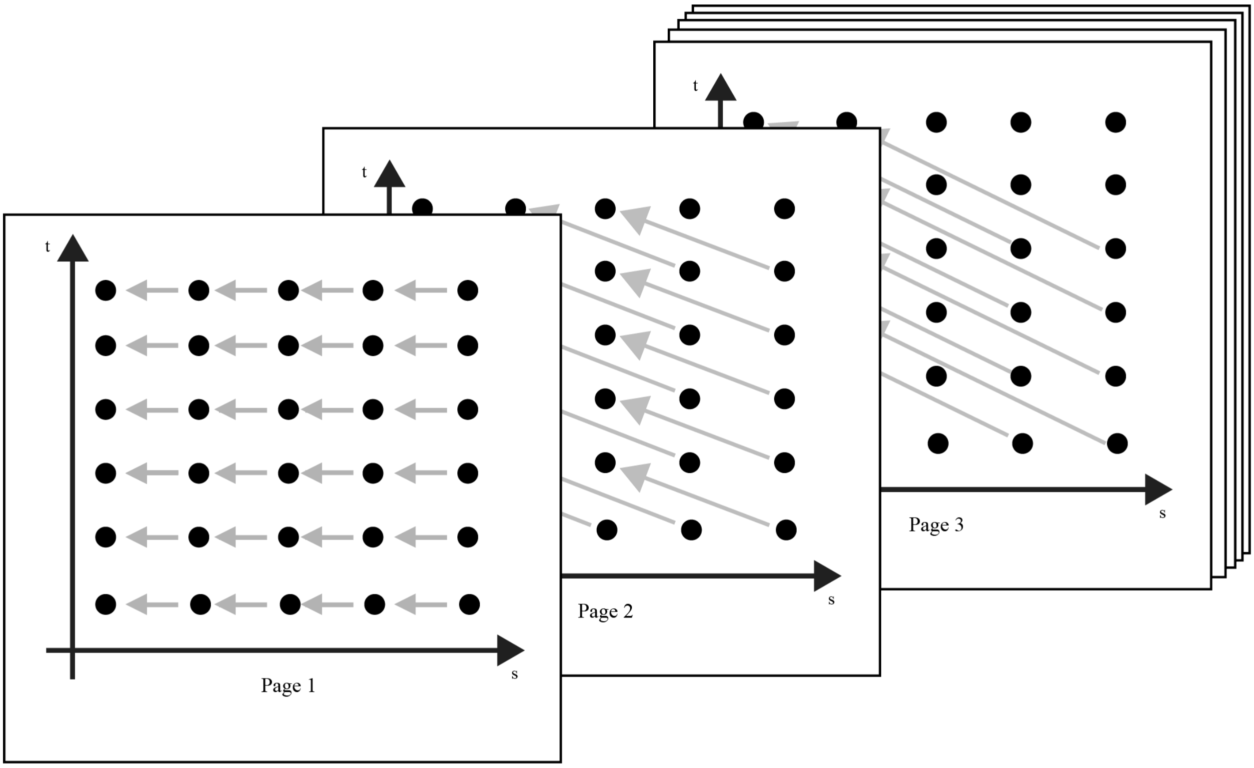

Notes: These are notes live-tex'd from a graduate course in Homological Algebra taught by Brian Boe at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Homological Algebra

Lectures by Brian Boe. University of Georgia, Spring 2021

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Reference:

- The course text is Weibel [1].
- See the many corrections/errata: http://www.math.rutgers.edu/~weibel/Hbook-corrections.html
- Sections we'll cover:
  - 1.1-1.5,
  - 2.2-2.7,
  - 3.4,
  - 3.6,
  - 6.1,
  - 5.1-5.2,
  - 5.4-5.8,
  - 6.8,
  - 6.7,
  - 6.3,
  - 7.1-7.5,
  - 7.7-7.8,
  - Appendix A (when needed)
- Course Website: https://uga.view.usg.edu/d2l/le/content/2218619/viewContent/33763436/View

1.1 Overview

Definition 1.1.1 (Exact complexes)
A **complex** is given by

$$\cdots \xrightarrow{d_{i-1}} M_{i-1} \xrightarrow{d_i} M_i \xrightarrow{d_{i+1}} M_{i+1} \to \cdots .$$

where $M_i \in \mathsf{R\text{-}Mod}$ and $d_i \circ d_{i-1} = 0$, which happens if and only if $\operatorname{im} d_{i-1} \subseteq \ker d_i$. If $\operatorname{im} d_{i-1} = \ker d_i$, this complex is **exact**.

Example 1.1.2*(?):* We can apply a functor such as $\otimes_R N$ to get a new complex

$$\cdots \xrightarrow{d_{i-1}\otimes 1_N} M_{i-1} \otimes_R N \xrightarrow{d_i \otimes 1} M_i \otimes N \to M_{i+1} \xrightarrow{d_{i+1}\otimes 1} \cdots .$$

Example 1.1.3 ***(?):*** Applying $\operatorname{Hom}(N, -)$ similarly yields

$$\operatorname*{Hom}_R(N, M_i) \xrightarrow{d_{i-1}^*} \operatorname*{Hom}_R(N, M_{i+1}),$$

where $d_i^* = d_i \circ (-)$ is given by composition.

Example 1.1.4 ***(?):*** Applying $\operatorname{Hom}(-, N)$ yields

$$\operatorname*{Hom}_R(M_i, N) \xrightarrow{d_i^*} \operatorname*{Hom}_R(M_{i+1}, N)$$

where $d_i^* = (-) \circ d_i$.

Remark 1.1.5: Note that we can also take complexes with arrows in the other direction. For F a functor, we can rewrite these examples as

$$d_i^* \circ d_{i-1}^* = F(d_i) \circ F(d_{i-1}) = F(d_i \circ d_{i-1}) = F(0) = 0,$$

provided F is nice enough and sends zero to zero. This follows from the fact that functors preserve composition. Even if the original complex is exact, the new one may not be, so we can define the following:

Definition 1.1.6 (Cohomology)

$$H^i(M^*) = \ker d_i^* / \operatorname{im} d_{i-1}^*.$$

Remark 1.1.7: These will lead to i**th derived functors**, and category theory will be useful here. See appendix in Weibel. For a category $\mathcal{C}$ we'll define

- $\operatorname{Obj}(\mathcal{C})$ as the objects
- $\operatorname*{Hom}_{\mathcal{C}}(A, B)$ a set of morphisms between them, where a more modern notation might be $\operatorname{Mor}(A, B)$.
- Morphisms compose: $A \xrightarrow{f} B \xrightarrow{g} C$ means that $g \circ f \in \operatorname*{Hom}_{\mathcal{C}}(A, C)$
- Associativity
- Identity morphisms

See the appendix for diagrams defining zero objects and the zero map, which we'll need to make sense of exactness. We'll also needs notions of kernels and images, or potentially cokernels instead of images since they're closely related.

Remark 1.1.8: In the examples, we had $\ker d_i \subseteq M_i$, but this need not be true since the objects in the category may not be sets. Such an example is the category of complexes of R-modules: $\mathsf{Cx}(\mathsf{R\text{-}Mod})$. In this setting, kernels will be subcomplexes but not subsets.

Definition 1.1.9 (Functors)
Recall that **functors** are "functions" between categories $F : \mathcal{C} \to \mathcal{D}$ such that

- Objects are sent to objects,
- Morphisms are sent to morphisms, so $A \xrightarrow{f} B \rightsquigarrow F(A) \xrightarrow{F(f)} F(B)$,
- F respects composition and identities

Example 1.1.10 ***(Hom):*** $\operatorname{Hom}_R(N, -) : \mathsf{R\text{-}Mod} \to \mathsf{Ab}$, noting that the hom set may not have an R-module structure.

Remark 1.1.11: Taking cohomology yields the ith derived functors of F, for example $\operatorname{Ext}^i, \operatorname{Tor}_i$. Recall that functors can be *covariant* or contravariant. See section 1 for formulating simplicial and singular homology (from topology) in this language.

1.2 Chapter 1: Chain Complexes

1.2.1 Complexes of R-modules

Definition 1.2.1 (Exactness)
Let R be a ring with 1 and define $\mathsf{R\text{-}Mod}$ to be the category of *right* R-modules. $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if and only if $\ker g = \operatorname{im} f$, and in particular $g \circ f = 0$.

Definition 1.2.2 (Chain Complex)
A **chain complex** is

$$C_- \coloneqq (C_-, d_-) \coloneqq \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \right)$$

for $n \in \mathbb{Z}$ such that $d_n \circ d_{n+1} = 0$. We drop the n from the notation and write $d^2 \coloneqq d \circ d = 0$.

Definition 1.2.3 (Cycles and boundaries)

- $Z_n = Z_n(C_-) = \ker d_n$ are referred to as n**-cycles**.
- $B_n = B_n(C_-) = \operatorname{im} d_{n+1}$ are the n**-boundaries**.

Definition 1.2.4 (Homology of a chain complex)
Note that if $d^2 = 0$ then $B_n \leq Z_n \leq C_n$. In this case, it makes sense to define the quotient module $H^n(C_-) \coloneqq Z_n / B_n$, the n**th homology** of C_-.

Definition 1.2.5 (Maps of chain complexes)
A map $u : C_- \to D_-$ of chain complexes is a sequence of maps $u_n : C_n \to D_n$ such that all of the following squares commute:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle u_{n+1}} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n-1}} & & \\ \cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \end{array}$$

Link to Diagram

Remark 1.2.6: We can thus define a category $\mathrm{Ch}(\mathsf{R\text{-}Mod})$ where

- The objects are chain complexes,
- The morphisms are chain maps.

Exercise 1.2.7 (Weibel 1.1.2)
A chain complex map $u : C_- \to D_-$ restricts to

$$u_n : Z_n(C_-) \to Z_n(D_-)$$
$$u_n : B_n(D_-) \to B_n(D_-)$$

and thus induces a well-defined map $u_{n,*} : H_n(C_-) \to H_n(D_-)$.

Remark 1.2.8: Each H_n thus becomes a functor $\mathrm{Ch}(\mathsf{R\text{-}Mod}) \to \mathsf{R\text{-}Mod}$ where $H_n(u) \coloneqq u_{*,n}$.

2 | Friday, January 15

2.1 Review

See assignment posted on ELC, due Wed Jan 27

Remark 2.1.1: Recall that a chain complex is C_- where $d^2 = 0$, and a map of chain complex is a ladder of commuting squares

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n-1} & \xrightarrow{d_{n-1}} & C_n & \xrightarrow{d_n} & C_{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle u_{n-1}} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n+1}} & & \\ \cdots & \longrightarrow & D_{n-1} & \xrightarrow{d_{n-1}} & D_n & \xrightarrow[d_n]{} & D_{n+1} & \longrightarrow & \cdots \end{array}$$

Link to diagram

Recall that $u_n : Z_n(C) \to Z_n(D)$ and $u_n : B_n(C) \to B_n(D)$ preserves these submodules, so there are induced maps $u_{-,n} : H_n(D) \to H_n(D)$ where $H_n(C) \coloneqq Z_n(C)/B_n n - 1(C)$. Moreover, taking $H_n(-)$ is a functor from $\mathsf{Ch}(\mathsf{R\text{-}Mod}) \to \mathsf{R\text{-}Mod}$ for any fixed n and on objects $C \mapsto H_n(C)$ and chain maps $u_n \to H_n(u) \coloneqq u_{*,n}$. Note the lower indices denote maps going down in degree.

2.2 Cohomology

Definition 2.2.1 (Quasi-isomorphism)
A chain map $u : C \to D$ is a **quasi-isomorphism** if and only if the induced map $u_{*,n} : H^n(C) \to H^n(D)$ is an isomorphism of R-modules.

Remark 2.2.2: Note that the usual notion of an isomorphism in the categorical sense might be too strong here.

Definition 2.2.3 (Cohomology)
A **cochain complex** is a complex of the form

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \cdots$$

where $d^n \circ d^{n-1} = 0$. We similarly write $Z^n(C) \coloneqq \ker d^n$ and $B^n(C) \coloneqq \operatorname{im} d^{n-1}$ and write the R-module $H^n(C) \coloneqq Z^n/B^n$ for the nth **cohomology** of C.

Remark 2.2.4: There is a way to go back and forth bw chain complexes and cochain complexes: set $C_n \coloneqq C^{-n}$ and $d_n \coloneqq d^{-n}$. This yields

$$C^{-n} \xrightarrow{d^{-n}} C^{-n+1} \iff C_n \xrightarrow{d^n} C_{n-1},$$

and the notions of $d^2 = 0$ coincide.

Definition 2.2.5 (Bounded complexes)
A cochain complex C is **bounded** if and only if there exists an $a \leq b \in \mathbb{Z}$ such that $C_n \neq 0 \iff a \leq n \leq b$. Similarly C^n is bounded above if there is just a b, and **bounded below** for just an a. All of the same definitions are made for cochain complexes.

Remark 2.2.6: See the book for classical applications:

- 1.1.3: Simplicial homology
- 1.1.5: Singular homology

2.3 Operations on Chain Complexes

Remark 2.3.1: Write Ch for $\mathsf{Ch}(\mathsf{R\text{-}Mod})$, then if $f, g : C \to D$ are chain maps then $f + g : C \to D$ can be defined as $(f + g)(x) = f(x) + g(x)$, since D has an addition coming from its R-module structure. Thus the hom sets $\operatorname{Hom}_{\mathsf{Ch}}(C, D)$ becomes an abelian group. There is a distinguished **zero object**[1] 0, defined as the chain complex with all zero objects and all zero maps. Note that we also have a zero map given by the composition $(C \to 0) \circ (0 \to D)$.

Definition 2.3.2 (Products and Coproducts)
If $\{A_\alpha\}$ is a family of complexes, we can form two new complexes:

- The **product** $\left(\prod_\alpha A_\alpha\right)_n := \prod_\alpha A_{\alpha,n}$ with the differential
$$\left(\prod d_\alpha\right)_n : \prod A_{\alpha,n} \xrightarrow{d_{\alpha,n}} \prod A_{\alpha,n-1}.$$
- The **coproduct** $\left(\coprod_\alpha A_\alpha\right)_n := \bigoplus_\alpha A_{\alpha,n}$, i.e. there are only finitely many nonzero entries, with exactly the same definition as above for the differential.

Remark 2.3.3: Note that if the index set is finite, these notions coincide. By convention, finite direct products are written as direct sums.

These structures make Ch into an **additive category**. See appendix for definition: the homs are abelian groups where composition distributes over addition, existence of a zero object, and existence of finite products. Note that here we have arbitrary products.

Definition 2.3.4 (Subcomplexes)
We say B is a **subcomplex** of C if and only if

- $B_n \leq C_n \in \mathsf{R\text{-}Mod}$ for all n,
- The differentials of B_n are the restrictions of the differentials of C_n.

Remark 2.3.5: This can be alternatively stated as saying the inclusion $i : B \to C$ given by $i_n : B_n \to C_n$ is a morphism of chain complexes. Recall that some squares need to commute, and this forces the condition on restrictions.

Definition 2.3.6 (Quotient Complexes)
When $B \leq C$, we can form the quotient complex C/B where
$$C_n/B_n \xrightarrow{\overline{d_n}} C_{n-1}/B_{n-1}.$$
Moreover there is a natural projection $\pi : C \to C/B$ which is a chain map.

[1] See appendix A 1.6 for initial and terminal objects. Note that $\emptyset$ is an initial but non-terminal object in Set, whereas zero objects are both.

Remark 2.3.7: Suppose $f : B \to C$ is a chain map, then there exist induced maps on the levelwise kernels and cokernels, so we can form the **kernel** and **cokernel** complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \ker f_n \arrow[r, dashed, "\exists d_n"] \arrow[d, "i_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "\pi_n"] & C_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{coker} f_n \arrow[r, dashed, "\exists d_n"] & \operatorname{coker} f_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here $\ker f \leq B$ is a subcomplex, and $\operatorname{coker} f$ is a quotient complex of C. The chain map $i : \ker f \to B$ is a categorical kernel of f in Ch, and π is similarly a cokernel. See appendix A 1.6. These constructions make Ch into an **abelian category**: roughly an additive category where every morphism has a kernel and a cokernel.

3 1.2: Chain Complex of Chain Complexes (Wednesday, January 20)

See phone pic for missed first 10m

3.1 Double Complexes

Remark 3.1.1: Consider a double complex:

$$C_{p,\cdot}:$$

$$C_{\cdot,q}:$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v_{p-1,q+1}"] & C_{p,q+1} \arrow[l, "d^h_{p,q+1}"] \arrow[d, "d^v_{p,q+1}"] & C_{p+1,q+1} \arrow[l, "d^h_{p+1,q+1}"] \arrow[d, "d^v_{p+1,q+1}"] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v_{p-1,q}"] & C_{p,q} \arrow[l, "d^h_{p,q}"'] \arrow[d, "d^v_{p,q}"] & C_{p+1,q} \arrow[l, "d^h_{p+1,q}"'] \arrow[d, "d^v_{p+1,q}"] & \cdots \arrow[l] \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d] & C_{p,q+1} \arrow[l, "d^h_{p,q+1}"'] \arrow[d] & C_{p+1,q+1} \arrow[l, "d^h_{p+1,q+1}"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Link to Diagram

All of the individual rows and columns are chain complexes, where $(d^h)^2 = 0$ and $(d^v)^2 = 0$, and the square anticommute: $d^v d^h + d^h d^v - 0$, so $d^v d^h = -d^h d^v$. This is almost a chain complex of chain complexes, i.e. an element of $\mathsf{Ch}(\mathsf{ChR\text{-}Mod})$). It's useful here to consider lines parallel to the line $y = x$.

> **Definition 3.1.2** (Bounded Complexes)
> A double complex $C_{-,-}$ is **bounded** if and only if there are only finitely many nonzero terms along each constant diagonal $p + q = n$.

Example 3.1.3*(?):* A *first quadrant* double complex $\{C_{p,q}\}_{p,q\geq 0}$ is bounded: note that this can still have infinitely many terms, but each diagonal is finite because each will hit a coordinate axis.

Remark 3.1.4*(The sign trick):* The squares anticommute, since the d^v are not chain maps between the horizontal chain complexes. This can be fixed by changing every one out of four signs, defining

$$f_{*,q} : C_{*,q} \to C_{*,q-1}$$
$$f_{p,q} := (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1}.$$

This yields a new double complex where the signs of each column alternate:

$$\begin{tikzcd}
C_{0,q} \arrow[d, "d^v"] & C_{1,q} \arrow[l, "d^h"'] \arrow[d, "-d^v"] & C_{2,q} \arrow[l, "d^h"'] \arrow[d, "d^v"] \\
C_{0,q-1} & C_{1,q-1} \arrow[l, "d^h"'] & C_{2,q-1} \arrow[l, "d^h"']
\end{tikzcd}$$

Now the squares commute and $f_{-,q}$ are chain maps, so this object is an element of $\mathsf{Ch}(\mathsf{ChR\text{-}Mod})$.

3.2 Total Complexes

Remark 3.2.1: Recall that products and coproducts of R-modules coincide when the indexing set is finite.

Definition 3.2.2 (Total Complexes)
Given a double complex $C_{-,-}$, there are two ordinary chain complexes associated to it referred to as **total complexes**:

$$(\mathrm{Tot}^{\Pi} C)_n := \prod_{p+q=n} C_{p,q}$$
$$(\mathrm{Tot}^{\oplus} C)_n := \bigoplus_{p+q=n} C_{p,q}.$$

Writing $\mathrm{Tot}(C)$ usually refers to the former. The differentials are given by

$$d_{p,q} = d^h + d^v : C_{p,q} \to C_{p-1,q} \oplus C_{p,q-1},$$

where $C_{p,q} \subseteq \mathrm{Tot}^{\oplus}(C)_n$ and $C_{p-1,q} \oplus C_{p,q-1} \subseteq \mathrm{Tot}^{\oplus}(C)_{n-1}$. Then you extend this to a differential on the entire diagonal by defining $d = \bigoplus_{p,q} d_{p,q}$.

Exercise 3.2.3 (?)
Check that $d^2 = 0$, using $d^v d^h + d^h d^v = 0$.

Remark 3.2.4: Some notes:

- $\mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\Pi}(C)$ when C is bounded.
- The total complexes need not exist if C is unbounded: one needs infinite direct products and infinite coproducts to exist in $\mathcal{C}$. A category admitting these is called **complete** or **cocomplete**.[2]

[2] Recall that abelian categories are additive and only require *finite* products/coproducts. A counterexample: categories of *finite* abelian groups, where e.g. you can't take infinite sums and stay within the category.

3.3 More Operations

Definition 3.3.1 (Truncation below)
Fix $n \in \mathbb{Z}$, and define the n**th truncation** $\tau_{\geq n}(C)$ by

$$\tau_{\geq n}(C) = \begin{cases} 0 & i < n \\ Z_n & i = n \\ C_i & i > n. \end{cases}.$$

Pictorially:

$$\cdots \longleftarrow 0 \xleftarrow{d_n} Z_n \xleftarrow{d_{n+1}} C_{n+1} \xleftarrow{d_{n+2}} C_{n+2} \longleftarrow \cdots$$

Link to diagram

This is sometimes call the **good truncation of** C **below** n.

Remark 3.3.2: Note that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0 & i < n \\ H_i(C) & i \geq n. \end{cases}.$$

Definition 3.3.3 (Truncation above)
We define the quotient complex

$$\tau_{<n} C := C/\tau_{\geq n} C.$$

which is C_i below n, C_n/Z_n at n. Thus is has homology

$$\begin{cases} H_i(C) & i < n. \\ 0 & i \geq n \end{cases}.$$

Definition 3.3.4 (Translation)
If C is a chain complex and $p \in \mathbb{Z}$, define a new complex $C[p]$ by

$$C[p]_n := C_{n+p}.$$

$$\begin{tikzcd}
\text{Degrees} & -p & & 0 & & p \\
C & C_{-p} & \cdots & C_0 \arrow[dll, dashed] & \cdots & C_p \arrow[dll, dashed] \\
C[p] & C_0 & \cdots & C_p & \cdots & C_{2p}
\end{tikzcd}$$

Link to Diagram

Similarly, if C is a *cochain* complex, we set $C[p]^n := C^{n-p}$:

$$
\begin{tikzcd}
\text{Degrees} & -p & & 0 & & p \\
C & C^{-p} \arrow[r] \arrow[drrr, dashed] & \cdots \arrow[r] & C^0 \arrow[r] \arrow[drr, dashed] & \cdots \arrow[r] & C^p \\
C[p] & C^0 \arrow[r] & \cdots \arrow[r] & C^{-p} \arrow[r] & \cdots \arrow[r] & C^0
\end{tikzcd}
$$

Link to Diagram

Mnemonic: Shift p positions in the same direction as the arrows.

In both cases, the differentials are given by the shifted differential $d[p] := (-1)^p d$. Note that these are not alternating: p is the fixed translation, so this is a constant that changes the signs of all differentials. Thus $H_n(C[p]) = H_{n+p}(C)$ and $H^n(C[p]) = H^{n-p}$.

Exercise 3.3.5
Check that if $C^n := C_{-n}$, then $C[p]^n = C[p]_{-n}$.

Remark 3.3.6: We can make translation into a functor $[p] : \mathsf{Ch} \to \mathsf{Ch}$: given $f : C \to D$, define $f[p] : C[p] \to D[p]$ by $f[p]_n := f_{n+p}$, and a similar definition for cochain complexes changing p to $-p$.

4 | Lecture 4 (Friday, January 22)

4.1 Long Exact Sequences

Remark 4.1.1: Some terminology: in an abelian category $\mathcal{A}$ an example of an **exact complex** in $\mathsf{Ch}(\mathcal{A})$ is

$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \cdots .$$

where *exactness* means ker = im at each position, i.e. $\ker f = 0, \operatorname{im} f = \ker g, \operatorname{im} g = C$. We say f is monic and g epic.

As a special case, if $0 \to A \to 0$ is exact then A must be zero, since the image of the incoming map must be 0. This also happens when every other term is zero. If $0 \to A \xrightarrow{f} B \to 0$, then $A \cong B$ since f is both injective and surjective (say for R-modules).

Theorem 4.1.2 ***(Long Exact Sequences).***
Suppose $0 \to A \to B \to C \to 0$ is a SES in $\mathsf{Ch}(\mathcal{A})$ (note: this is a sequence of *complexes*), then there are natural maps

$$\delta : H_n(C) \to H_{n-1}(A)$$

called **connecting morphisms** which decrease degree such that the following sequence is exact:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "f_*=H_n(f)"] & H_n(B) \arrow[r, "g_*=H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

This is referred to as the **long exact sequence in homology**. Similarly, replacing chain complexes by cochain complexes yields a similar connecting morphism that increases degree.

Note on notation: some books use ∂ for homology and δ for cohomology.

The proof that this sequence exists is a consequence of the *snake lemma*.

Lemma 4.1.3 ***(The Snake Lemma).***
The sequence highlighted in red in the following diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] & \ker(\alpha) \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & \ker(\gamma) \arrow[d] \arrow[ddd, dotted, "\exists\delta", to=4-3] \\
& 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
& 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
& & \operatorname{coker}(\alpha) \arrow[r] & \operatorname{coker}(\beta) \arrow[r] & \operatorname{coker}(\gamma) \arrow[r] & \operatorname{coker}(g') \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proof (of the Snake Lemma: Existence).

- Start with $c \in \ker(\gamma) \leq C$, so $\gamma(c) = 0 \in C'$
- **Choose** $b \in B$ by surjectivity
  - We'll show it's independent of this choice.
- Then $b' \in B'$ goes to $0 \in C'$, so $b' \in \ker(B' \to C')$
- By exactness, $b' \in \ker(B' \to C') = \operatorname{im}(A' \to B')$, and now produce a unique $a' \in A'$ by injectivity
- Take the image $[a'] \in \operatorname{coker} \alpha$
- Define $\partial(c) \coloneqq [a']$.

■

Proof (of the Snake Lemma: Uniqueness).

- We chose b, suppose we chose a different $\tilde{b}$.
- Then $\tilde{b} - b \mapsto c - c = 0$, so the difference is in $\ker g = \operatorname{im} f$.
- Produce an $\tilde{a} \in A$ such that $\tilde{a} \mapsto \tilde{b} - b$
- Then $\bar{a} \coloneqq \alpha(\tilde{a})$, so apply f'.
- Define $\beta(\tilde{b}) = \tilde{b}' \in B$.
- Commutativity of the LHS square forces $\tilde{a}' \mapsto \tilde{b}' - b'$.
- Then $\bar{a} + a' \mapsto \tilde{b}' - b' + b' = \tilde{b}'$.
- So $\tilde{a}' + a'$ is the desired pullback of $\tilde{b}'$
- Then take $[\tilde{a}'] \in \operatorname{coker} \alpha$; are $a', \tilde{a}'$ in the same equivalence class?
- Use that fact that $\tilde{a} = a' + \bar{a}$, where $\bar{a} \in \operatorname{im} \alpha$, so $[\tilde{a}] = [a' + \bar{a}] = [a'] \in \operatorname{coker} \alpha \coloneqq A' / \operatorname{im} \alpha$.

■

A few changes in the middle, redo!

Proof (of the Snake Lemma: Exactness).

- Let's show $g : \ker \beta \to \ker \gamma$.
  - Let $b \in \ker \beta$, then consider $\gamma(g(\beta)) = g'(\beta(b)) = g'(0) = 0$ and so $g(b) \in \ker \gamma$.
- Now we'll show $\operatorname{im}(g|_{\ker \beta}) \subseteq \ker \delta$
  - Let $b \in \ker \beta, c = g(b)$, then how is $\delta(c)$ defined?
  - Use this b, then apply β to get $b' = \beta(b) = 0$ since $b \in \ker \beta$.
  - So the unique thing mapping to it a' is zero, and thus $[a'] = 0 = \delta(c)$.
- $\ker \delta \subseteq \operatorname{im}(g|_{\ker \beta})$
  - Let $c \in \ker \delta$, then $\delta(c) = 0 = [a'] \in \operatorname{coker} \alpha$ which implies that $a' \in \operatorname{im} \alpha$.
  - Write $a' = \alpha(a)$, then $\beta(b) = b' = f'(a') = f'(\alpha(a))$ by going one way around the LHS square, and is equal to $\beta(f(a))$ going the other way.
  - So $\tilde{b} \coloneqq b - f(a) \in \ker \beta$, since $\beta(b) = \beta(f(a))$ implies their difference is zero.

– Then $g(\tilde{b}) = g(b) - g(f(a)) = g(b) = c$, which puts $c \in g(\ker \beta)$ as desired.

■

Exercise 4.1.4 (?)
Show exactness at the remaining places – the most interesting place is at $\operatorname{coker} \alpha$. Also check that all of these maps make sense.

Remark 4.1.5: We assumed that $\mathcal{A} = \mathsf{R\text{-}Mod}$ here, so we could chase elements, but this happens to also be true in any abelian category $\mathcal{A}$ but by a different proof. The idea is to embed $\mathcal{A} \to \mathsf{R\text{-}Mod}$ for some ring R, do the construction there, and pull the results back – but this doesn't quite work! $\mathcal{A}$ can be too big. Instead, do this for the smallest subcategory $\mathcal{A}_0$ containing all of the modules and maps involved in the snake lemma. Then $\mathcal{A}_0$ is small enough to embed into $\mathsf{R\text{-}Mod}$ by the **Freyd-Mitchell Embedding Theorem**.

5 | Lecture 5 (Monday, January 25)

5.1 LES Associated to a SES

Theorem 5.1.1 ***(Every SES of chain complexes induces a LES in homology).***
For every SES of chain complexes, there is a long exact sequence in homology.

Proof (?).
Suppose we have a SES of chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

which means that for every n there is a SES of R-modules. Recall the diagram for the snake lemma, involving kernels across the top and cokernels across the bottom. Applying the snake lemma, by hypothesis $\operatorname{coker} g = 0$ and $\ker f = 0$. There is a SES

$$A_n/dA_{n+1} \to B_n/dB_{n+1} \to C_n/dC_{n+1} \to 0$$

Using the fact that $B_n \subseteq Z_n$, we can use the 1st and 2nd isomorphism theorems to produce

$$\begin{tikzcd}
 & H_n(A) \arrow[r, "f_*"] \arrow[d] & H_n(B) \arrow[r, "g_*"] \arrow[d] & H_n(C) \arrow[d] \arrow[dddll, dotted, bend left, "\exists\delta"] & \\
 & A_n/dA_{n+1} \arrow[r, "f"] \arrow[d, "d_n"] & B/dB_{n+1} \arrow[r, "g"] \arrow[d, "d_n"] & C/dC_{n+1} \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] \arrow[d] & Z_{n-1}(B) \arrow[r] \arrow[d] & Z_{n-1}(C) \arrow[d] & \\
 & \substack{\operatorname{coker} d_n \\ = Z_{n-1}(A)/dA_n \\ = H_{n-1}(A)} \arrow[r, "f_*"'] & H_{n-1}(B) \arrow[r, "g_*"'] & H_{n-1}(C) &
\end{tikzcd}$$

Link to diagram

This yields an exact sequence relating H_n to H_{n-1}, and these can all be spliced together.

- $\ker(A_n/dA_{n-1} \to Z_{n-1}(A) = Z_n(A)/dA_{n+1} \coloneqq H_n(A)$ using the 2nd isomorphism theorem

$\blacksquare$

Remark 5.1.2: Note that d is *natural*, which means the following: there is a category $\mathcal{S}$ whose objects are SESs of chain complexes and whose maps are chain maps:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

There is another full subcategory $\mathcal{L}$ of Ch whose objects are LESs of objects in the original abelian category, i.e. exact chain complexes. The claim is that the LES construction in the theorem defines a functor $\mathcal{S} \to \mathcal{L}$. We've seen how this maps objects, so what is the map on morphisms? Given a morphism as in the above diagram, there is an induced morphism:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "H_n(u_A)"] & H_n(B) \arrow[r] \arrow[d, "H_n(u_B)"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "H_n(u_C)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(u_A)"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

The first two squares commute, and *naturality* means that the third square commutes as well.

Exercise 5.1.3 (?)
Check the details!

Remark 5.1.4: It is sometimes useful to explicitly know how to compute snake lemma boundary elements. See the book for a recipe for computing $\partial(\xi)$:

- Lift ξ to a cycle $c \in Z_n(C) \subseteq C_n$.
- Pull c back to a preimage $b \in B_n$ by surjectivity.
- Apply the differential to get $d(b) \in Z_{n-1}(B)$, using that images are contained in kernels.
- Since this is in kernel of the outgoing map, it's in the kernel of the incoming map and thus there exists an $a \in Z_{n-1}(A)$ such that $f(a) = db$
- So set $\delta(\xi) \coloneqq [a] \in H_{n-1}(A)$.

Remark 5.1.5: Why is naturality useful? Suppose $H_n(B) = 0$, you get isomorphisms, and this allows inductive arguments up the LES. The LES in homology is sometimes abbreviated as an **exact triangle**:

$$\begin{tikzcd}
& H_*(A) \arrow[dr, "f"] & \\
H_*(C) \arrow[ur, squiggly, "\partial"] & & H_*(B) \arrow[ll, "g"]
\end{tikzcd}$$

Here $\partial : H_*(C) \to H_*(A)[1]$ shifts degrees. Note that this motivates the idea of **triangulated categories**, which is important in modern research. See Weibel Ch.10, and exercise 1.4.5 for how to construct these as quotients of Ch.

5.2 1.4: Chain Homotopies

Remark 5.2.1: Assume for now that we're in the situation of R-modules where R is a field, i.e. vector spaces. The main fact/advantage here that is not generally true for R-modules: every subspace has a complement. Since $B_n \subseteq Z_n \subseteq C_n$, we can write $C_n = Z_n \oplus B'_n$ for every n, and $Z_n = B_n \oplus H_n$. This notation is suggestive, since $H_n \cong Z_n/B_n$ as a quotient of vector spaces. Substituting, we get $C_n = B_n \oplus H_n \oplus B'_n$. Consider the projection $C_n \to B_n$ by projecting onto the first factor. Identifying $B_n \coloneqq \operatorname{im}(C_{n+1} \to C_n) \cong C_{n+1}/Z_{n+1}$ by the 1st isomorphism theorem in the reverse direction. But this image is equal to B'_{n+1}, and we can embed this in C_{n+1}, so define $s_n : C_n \to C_{n+1}$ as the composition

$$s_n \coloneqq (C_n \xrightarrow{\text{Proj}} B_n = \operatorname{im}(C_{n+1} \to C_n) \xrightarrow{d_{n+1}^{-1}} C_{n+1}/Z_{n+1} \xrightarrow{\cong} B'_{n+1} \hookrightarrow C_{n+1}.$$

Claim 1: $d_{n+1}s_n d_{n+1} = d_{n+1}$ are equal as maps.

Proof (?).

- Check on the first factor $B'_{n+1} \subseteq C_{n+1}$ directly to get $s_n d_{n+1}(x) = d_{n+1}(x)$ for $x \in B'_{n+1}$, and then applying d_{n+1} to both sides is the desired equality.
- On the second factor Z_{n+1}, both sides give zero since this is exactly the kernel.

$\blacksquare$

Claim 2: $d_{n+1}s_n + s_{n-1}d_n = \mathrm{id}_{C_n}$ if and only if $H_n = 0$, i.e. the complex C is exact at C_n. This map is the sum of taking the two triangle paths in this diagram:

$$
\begin{tikzcd}
 & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d_{n+1}"] & C_n &
\end{tikzcd}
$$

Proof (?).
We again check this on both factors:

- Using the first claim, $s_n = 0$ on B'_n and thus $s_{n-1}d_n = \mathrm{id}_{B'_n}$.
- On H_n, $s_n = 0$ and $d_n = 0$, and so the LHS is $0 = \mathrm{id}_{H_n}$ *if and only if* $H_n = 0$.
- On B_n, and tracing through the definition of s_n yields $d_{n+1}s_n(x) = x$ and this yields id_{B_n}.

$\blacksquare$

Next time: summary of decompositions, start general section on chain homotopies.

6 | Wednesday, January 27

See phone pic for missed first 10m.

6.1 1.4: Chain Homotopies

Definition 6.1.1 (Split Exact)
A complex is called **split** if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. In this case, the maps s_n are referred to as the **splitting maps**, and if C is additionally acyclic, we say C is **split exact**.

Remark 6.1.2: Note that when C is split exact, we have

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d"] & C_n &
\end{tikzcd}$$

Link to Diagram

Example 6.1.3 ***(Not all complexes split):*** Take

$$C = \left(0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{d} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \right).$$

Then $\operatorname{im} d = \{0, 2\} = \ker d$, but this does not split since $\mathbb{Z}/2\mathbb{Z}^2 \not\cong \mathbb{Z}/4\mathbb{Z}$: one has an element of order 4 in the underlying additive group. Equivalently, there is no complement to the image. What might be familiar from algebra is $ds = \mathrm{id}$, but the more general notion is $dsd = d$.

Example 6.1.4 ***(?):*** The following complex is not split exact for the same reason:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \to \cdots.$$

Question 6.1.5
Given $f, g : C \to D$, when do we get equality $f_* = g_* : H_*(C) \to H_*(D)$?

Definition 6.1.6 (Homotopy Terminology for Chains)
A chain map $f : C \to D$ is **nullhomotopic** if and only if there exist maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$:

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}$$

Link to Diagram

The map s is called a **chain contraction**. Two maps are **chain homotopic** (or initially: f is chain homotopic to g, since we don't yet know if this relation is symmetric) if and only if $f - g$ is nullhomotopic, i.e. $f - g = ds + sd$. The map s is called a **chain homotopy** from f to g. A map f is a **chain homotopy equivalence** if both fg and gf are chain homotopic to the identities on C and D respectively.

Lemma 6.1.7 *(?).*
If map $f : C \to D$ is nullhomotopic then $f_* : H_*(C) \to H_*(D)$ is the zero map. Thus if f, g are chain homotopic, then they induce equal maps.

Proof (?).
An element in the quotient $H_n(C)$ is represented by an n-cycle $x \in Z_n(C)$. By a previous exercise, $f(x)$ is a well-defined element of $H_n(D)$, and using that $d(x) = 0$ we have

$$f(x) = (ds + sd)(x) = d(s(x)),$$

and so $f[x] = [f(x)] = [0]$.

x $\qquad$ $d(x) = 0$

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}$$

$d(s(x))$

Link to Diagram

Now applying the first part to $f - g$ to get the second part. ■

See Weibel for topological motivations.

6.2 1.5 Mapping Cones

Remark 6.2.1: Note that we'll skip *mapping cylinders*, since they don't come up until the section on triangulated categories. The goal is to see how any two maps between homologies can be fit into a LES. This helps reduce questions about *quasi-isomorphisms* to questions about split exact complexes.

Definition 6.2.2 (Mapping Cones)
Suppose we have a chain map $f : B \to C$, then there is a chain complex $\operatorname{cone}(f)$, the **mapping cone of** f, defined by

$$\operatorname{cone}(f)_n = B_{n-1} \oplus C_n.$$

The maps are given by the following:

$$\begin{tikzcd}
B_{n-1} \arrow[r, "-d^B"] \arrow[rdd, "-f"'] & B_{n-2} \\
\oplus & \oplus \\
C_n \arrow[r, "d^C"] & C_{n-1}
\end{tikzcd}$$

Link to Diagram

We can write this down: $d(b, c) = (-d(b), -f(b) + d(c))$, or as a matrix

$$\begin{bmatrix} -d^B & 0 \\ -f & d^C \end{bmatrix}.$$

Exercise 6.2.3 (?)
Check that the differential on $\operatorname{cone}(f)$ squares to zero.

Exercise 6.2.4 (Weibel 1.5.1)
When $f = \mathrm{id} : C \to C$, we write $\operatorname{cone}(C)$ instead of $\operatorname{cone}(\mathrm{id})$. Show that $\operatorname{cone}(C)$ is split exact, with splitting map $s(b, c) = (-c, 0)$ for $b \in C_{n-1}, c \in C_n$.

Proposition 6.2.5 ***(LES in homology of a single chain map using the cone).***
Suppose $f : B \to C$ is a chain map, then the induced maps $f_* : H(B) \to H(C)$ fit into a LES. There is a SES of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & \operatorname{cone}(f) & \longrightarrow & B[-1] & \longrightarrow & 0 \\ & & c & \longmapsto & (0,c) & & & & \\ & & & & (b,c) & \longmapsto & -b & & \end{array}$$

Link to Diagram

Exercise 6.2.6 *(?)*
Check that these are chain maps, i.e. they commute with the respective differentials d.

The corresponding LES is given by the following:

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}\operatorname{cone}(f) \arrow[r, "\delta_*"] & H_{n+1}(B[-1]) = H_n(B) \arrow[dll, "\partial"] \\ H_n(C) \arrow[r] & H_n \operatorname{cone}(f) \arrow[r] & H_n(B[-1]) = H_{n-1}(B) \arrow[dll] \\ \cdots & & \end{tikzcd}$$

Link to Diagram

Overflowing :(

Lemma 6.2.7 *(?)***.**
The map $\partial = f_*$

Proof (?).
Letting $b \in B_n$ is an n-cycle.

1. Lift b to anything via δ, say $(-b, 0)$.
2. Apply the differential d to get $(db, fb) = (0, fb)$ since b was a cycle.
3. Pull back to C_n by the map $C \to \operatorname{cone}(f)$ to get fb.
4. Then the connecting morphism is given by $\partial[b] = [fb]$. But by definition of f_*, we have $[fb] = f_*[b]$.

■

7 | Friday, January 29

7.1 Mapping Cones

Remark 7.1.1: Given $f : B \to C$ we defined $\operatorname{cone}(f)_n \coloneqq B_{n-1} \oplus C_n$, which fits into a SES

$$0 \to C \to \operatorname{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

and thus yields a LES in cohomology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(\operatorname{cone}(f)) \arrow[r] & H_n(B) \arrow[dll, "\delta = f_*"] \\
H_n(C) \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r] & H_{n-1}(B) \arrow[dll, "\delta"] \\
\cdots & &
\end{tikzcd}$$

Link to Diagram

Corollary 7.1.2 *(?).*
$f : B \to C$ is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is exact.

Proof (?).
In the LES, all of the maps f_* are isomorphisms, which forces $H_n(\operatorname{cone}(f)) = 0$ for all n. ■

Remark 7.1.3: So we can convert statements about quasi-isomorphisms of complexes into exactness of a single complex.

We'll skip the rest, e.g. mapping cylinders which aren't used until the section on triangulated categories. We'll also skip the section on δ-functors, which is a slightly abstract language.

7.2 Ch. 2: Derived Functors

Remark 7.2.1: Setup: fix $M \in \mathsf{R\text{-}Mod}$, where R is a ring with unit. Note that by an upcoming exercise, $\underset{R}{\mathrm{Hom}}(M, -) : \mathsf{Mod\text{-}R} \to \mathsf{Ab}$ is a *left-exact* functor, but not in general right-exact: given a SES

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad\qquad \in \mathsf{Ch}(\mathsf{Mod\text{-}R}),$$

there is an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \underset{R}{\mathrm{Hom}}(M, A) \arrow[r, "f_* = f \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, B) \arrow[r, "g_* = g \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, C)
\end{tikzcd}$$

Link to Diagram

However, this is not generally surjective: not every $M \to C$ is given by composition with a morphism $M \to B$ (*lifting*). To create a LES here, one could use the cokernel construction, but we'd like to do this functorially by defining a sequence functors F^n that extend this on on the right to form a LES:

$$\begin{tikzcd}
0 \arrow[r] & \underset{R}{\mathrm{Hom}}(M, A) \arrow[r, "f_* = f \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, B) \arrow[r, "g_* = g \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, C) \arrow[dll, out=0, in=180] \\
& F^1(A) \arrow[r] & F^1(B) \arrow[r] & F^1(C) \arrow[dll, out=0, in=180] \\
& F^2(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

It turns out such functors exist and are denoted $F^n(-) := \mathrm{Ext}^n_R(M, -)$:

$$
\begin{tikzcd}
0 \arrow[r] & \underset{R}{\mathrm{Hom}}(M, A) \arrow[r, "f_* = f \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, B) \arrow[r, "g_* = g \circ (-)"] & \underset{R}{\mathrm{Hom}}(M, C) \arrow[dll] \\
& \mathrm{Ext}^1_R(A) \arrow[r] & \mathrm{Ext}^1_R(B) \arrow[r] & \mathrm{Ext}^1_R(C) \arrow[dll] \\
& \mathrm{Ext}^2_R(A) \arrow[r] & \cdots &
\end{tikzcd}
$$

Link to Diagram

By convention, we set $\mathrm{Ext}^0_R(-) \coloneqq \underset{R}{\mathrm{Hom}}(M, -)$. This is an example of a general construction: **right-derived functors** of $\underset{R}{\mathrm{Hom}}(M, -)$. More generally, if $\mathcal{A}$ is an abelian category (with a certain additional property) and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor (where $\mathcal{B}$ is another abelian category) then we can define right-derived functors $R^n F : \mathcal{A} \to \mathcal{B}$. These send SESs in $\mathcal{A}$ to LESs in $\mathcal{B}$:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & FA \arrow[r] & FB \arrow[r] & FC \arrow[dll] \\
& R^1 FA \arrow[r] & R^1 FB \arrow[r] & R^1 FC \arrow[dll] \\
& \cdots & &
\end{tikzcd}
$$

Link to Diagram

Similarly, if F is *right-exact* instead, there are left-derived functors $L^n F$ which form a LES ending with 0 at the right:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & \cdots \\
 & LFA \arrow[r] \arrow[urr, to=1-4, bend right] & LFB \arrow[r] & LFC \\
 & FA \arrow[r] \arrow[to=2-4, bend right] & FB \arrow[r] & FC \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

7.3 2.2: Projective Resolutions

Definition 7.3.1 (Projective Modules)
Let $\mathcal{A} = \mathsf{R\text{-}Mod}$, then $P \in \mathsf{R\text{-}Mod}$ satisfies the following universal property:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Remark 7.3.2: Free modules are projective. Let $F = R^X$ be the free module on the set X. Then consider $\gamma(x) \in C$, by surjectivity these can be pulled back to some elements in B:

$$\begin{tikzcd}
 & X \arrow[d, hook, "\iota_X"] \arrow[dddl, dotted, "\exists\tilde{\beta}"'] & \\
 & F \arrow[d, "\gamma"] \arrow[ddl, dashed, "\exists\beta"'] & \\
 & & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

$$\exists b \in g^{-1}(\gamma(x)) \coloneqq \beta(x) \qquad\qquad \gamma(x)$$

Link to Diagram

This follows from the universal property of free modules:

$$
\begin{tikzcd}
 & \exists F(X) \arrow[d, dashed, "\exists ! f' \in \mathrm{Hom}_R(F(X), X)"] & \\
X \arrow[ur, "\exists g \in \mathrm{Hom}_{\mathsf{Set}}(X, F(X))"] \arrow[r, "f \in \mathrm{Hom}_{\mathsf{Set}}(X, M)"'] & M & \in \mathsf{R\text{-}Mod}
\end{tikzcd}
$$

Link to Diagram

Proposition 7.3.3 ***(Projective if and only if summand of free (for modules)).***
An R-module is projective if and only if it is a direct summand of a free module.

Exercise 7.3.4 (?)
Prove the $\Longleftarrow$ direction!

Proof (?).
$\Longrightarrow$: Assume P is projective, and let $F(P)$ be the free R-module on the underlying set of P. We can start with this diagram:

$$
\begin{tikzcd}
 & F(P) \arrow[d, dashed] \\
P \arrow[ur] \arrow[r, "\mathrm{id}_P"] & P
\end{tikzcd}
$$

Link to Diagram

And rearranging, we get

$$
\begin{tikzcd}
 & & & P \arrow[dl, "\exists \iota"'] \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & \ker \pi \arrow[r, hook] & F(P) \arrow[r, two heads, "\pi"] & P \arrow[r] \arrow[l, dashed, bend left, "\iota"] & 0
\end{tikzcd}
$$

Link to Diagram

Since $\pi \circ \iota$, the SES splits and this $F(P) \cong P \oplus \ker \pi$, making P a direct summand of a free module. ■

Example 7.3.5 *(?):* Not every projective module is free. Let $R = R_1 \times R_2$ a direct product of unital rings. Then $P := R_1 \times \{0\}$ and $P' := \{0\} \times R_2$ are R-modules that are submodules of R. They're projective since R is free over itself as an R-module, and their direct sum is R. However they can not be free, since e.g. P has a nonzero annihilator: taking $(0,1) \in R$, we have $(0,1) \cdot P = \{(0,0)\} = 0_R$. No free module has a nonzero annihilator, since ix $0 \neq r \in R$ then $rR \neq 0$ since $r1_R \in rR$, which implies that $r\left(\bigoplus R\right) \neq 0$.

Example 7.3.6 *(?):* Taking $R = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ admits projective R-modules which are not free.

Example 7.3.7 *(?):* Let F be a field, define the ring $R := \mathrm{Mat}(n \times n, F)$ with $n \geq 2$, and set $V = F^n$ thought of as column vectors. This is left R-module, and decomposes as $R = \bigoplus_{i=1}^{n} V$ corresponding to the columns of R, using that $AB = [Ab_1, \cdots, Ab_n]$. Then V is a projective R-module as a direct summand of a free module, but it is not free. We have vector spaces, so we can consider dimensions: $\dim_F R = n^2$ and $\dim_F V = n$, so V can't be a free R-module since this would force $\dim_F V = kn^2$ for some k.

Example 7.3.8 *(?):* How many projective modules are there in a given category? Let $\mathcal{C} := \mathsf{Ab}^{\mathrm{fin}}$ be the category of *finite* abelian groups, where we take the full subcategory of the category of all abelian groups. This is an abelian category, although it is not closed under *infinite* direct sums or products, which has no projective objects.

Proof (?).
Over a PID, every submodule of a free module is free, and so we have free $\iff$ projective in this case. So equivalently, we can show there are no free $\mathbb{Z}$-modules, which is true because $\mathbb{Z}$ is infinite, and any such module would have to contain a copy of $\mathbb{Z}$. ■

Remark 7.3.9: The definition of projective objects extends to any abelian category, not just R-modules.

8 | Monday, February 01

Recall the universal of projective modules.

$$
\begin{tikzcd}
& P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}
$$

Definition 8.0.1 (Enough Projective)
If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has **enough projectives** if and only if for all $a \in \mathcal{A}$ there exists a projective object $P \in \mathcal{A}$ and a surjective morphism $P \twoheadrightarrow A$.

Example 8.0.2 *(?):* $\mathsf{Mod\text{-}R}$ has enough projectives: for all $A \in \mathsf{Mod\text{-}R}$, one can take $F(A) \twoheadrightarrow A$.

Example 8.0.3 *(?):* The category of finite abelian groups does *not* have enough projectives.

Why?

Lemma 8.0.4 *(?).*
P is projective if and only if $\underset{\mathcal{A}}{\mathrm{Hom}}(P, -)$ is an exact functor.

Exercise 8.0.5 (?)
Prove this!

Definition 8.0.6 ((Key))
Let $M \in \mathsf{Mod\text{-}R}$, then a **projective resolution** of M is an exact complex

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_1 \overset{\varepsilon}{\twoheadrightarrow} M \to 0.$$

We write $P_- \overset{\epsilon}{\twoheadrightarrow} M$.

Lemma 8.0.7 *((Key)).*
Every object $M \in \mathsf{Mod\text{-}R}$ has a projective resolution. This is true in any abelian category with enough projectives.

Proof (?).

- Since there are enough projectives, choose $P_0 \xrightarrow{\epsilon_0} M \to 0$.
- To extend this, set $M_0 := \ker \epsilon_0$, then find a projective cover $P_1 \xrightarrow{\epsilon_1} M_0$
- Use that $d_1 := \iota_0 \circ \epsilon_1$ and $\operatorname{im} d_1 = M_0 = \ker \epsilon_0$
- Then $d_2 := \iota_1 \circ \epsilon_2$ with $\operatorname{im} d_2 = M_1$, and $\ker d_1 = \ker \epsilon_1 = M_1$.
- Continuing in this fashion makes the complex exact at every stage.

$$\begin{tikzcd}
 & & 0 \arrow[dr, hook] & & 0 & & & & \\
 & & & M_1 \arrow[ur] \arrow[dr, hook] & & & & & \\
\cdots \arrow[rr, dashed] & & P_2 \arrow[ur] \arrow[rr, dashed, "\exists d_2"] & & P_1 \arrow[rr, dashed, "\exists d_1"] \arrow[dr, two heads, "\varepsilon_0"] & & P_0 \arrow[r, two heads, "\varepsilon_0"] & M & 0 \\
 & M_2 \arrow[ur, hook] & & & & M_0 \arrow[ur, "\iota_0"] \arrow[dr] & & & \\
0 \arrow[ur, hook] & & & & 0 \arrow[ur, hook] & & 0 & &
\end{tikzcd}$$

Link to Diagram

■

8.1 Comparison Theorem

Theorem 8.1.1 ***(Comparison Theorem).***
Suppose $P_{-} \xrightarrow{\epsilon} M$ is a projective resolution of an object in $\mathcal{A}$ and $(M \xrightarrow{f} N) \in \mathrm{Mor}(\mathcal{A})$ and $Q_{-} \xrightarrow{\eta} N$ a resolution of N. Then there exists a chain map $P \xrightarrow{f} Q$ lifting f which is unique up to chain homotopy:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[d, dashed, "\exists f_2"] & P_1 \arrow[r, "d_1^P"] \arrow[d, dashed, "\exists f_1"] & P_0 \arrow[r, "\varepsilon = d_0^P"] \arrow[d, dashed, "\exists f_0"] & M \arrow[r] \arrow[d, "f_{-1} := f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2^Q"'] & Q_1 \arrow[r, "d_1^Q"'] & Q_0 \arrow[r, "\eta = d_0^Q"'] & N \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Remark 8.1.2: The proof will only use that $P \xrightarrow{\epsilon} M$ is a chain complex of projective objects, i.e. $d^2 = 0$, and that $\epsilon \circ d_1^p = 0$. To make the notation more consistent, we'll write $Z_{-1}(P) := M$ and $Z_{-1}(Q) := N$. Toward an induction, suppose that the f_i have been constructed for $i \leq n$, so $f_{i-1} \circ d = d \circ f_i$.

Proof (Existence).
A fact about chain maps is that they induce maps on the kernels of the outgoing maps, so there is a map $f_n' : Z_n(P) \to Z_n(Q)$. We get a diagram where the top row is not necessarily exact:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d"] \arrow[d, dashed, "\exists f_{n+1}"'] & Z_n(P) \arrow[d, "f_{n'}"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0
\end{tikzcd}$$

Link to Diagram

Using the definition of projective, since P_{n+1} is projective, the map $f_{n+1} : P_{n+1} \to Q_{n+1}$ exists where $d \circ f_{n+1} = f_n' \circ d = f_n \circ d$, since $f_n = f_n'$ on $\operatorname{im} d \subseteq Z_n(P)$. This yields commutativity of the above square.

■

Proof (Uniqueness).
Suppose $g : P \to Q$ is another lift of f', the consider $h \coloneqq f - g$. This is a chain map $P \to Q$ lifting of $f' - f' = 0$. We'll construct a chain contraction $\{s_n :; P_n \to Q_{n+1}\}$ by induction on n:
We have the following diagram:

$$\begin{tikzcd}
 & P_0 \arrow[r, "\varepsilon"] \arrow[d, "h_0 := f_0 - f_0'"'] & M \arrow[d, "f - f' = 0"] \\
Q_1 \arrow[r, "d"'] & Q_0 \arrow[r, "\eta"'] & N
\end{tikzcd}$$

Link to Diagram

Setting $P_{-1} \coloneqq 0$ and $s_{-1} : P_{-1} \to Q_0$ to be the zero map, we have $\eta \circ h_0 = \varepsilon(f' - f') = 0$. Using projectivity of P_0, there exists an s_0 as shown below which satisfies $h_0 = d \circ s_0 = ds_0 + s_{-1}d$ where $s_{-1}d = 0$:

$$\begin{tikzcd}
 & P_0 \arrow[r, "d_0 = 0"] \arrow[dl, dashed, "\exists s_1"'] \arrow[d, "h_0"] & P_{-1} = 0 \arrow[dl, "s_{-1} = 0"] \\
Q_1 \arrow[r, two heads] & d(Q_1) \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proceeding inductively, assume we have maps $s_i : P_i \to Q_{i+1}$ such that $h_{n-1} = ds_{n-1} + s_{n-2}d$, or equivalently $ds_{n-1} = h_{n-1} - s_{n-2}d$. We want to construct s_n in the following diagram:

$$\begin{tikzcd}
 & P_n \arrow[r, "d"] \arrow[dl, dashed, "\exists s_n"'] \arrow[d, "h_n"] & P_{n-1} \arrow[r, "d"] \arrow[dl, "s_{n-1}"'] \arrow[d, "h_{n-1}"] & P_{n-2} \arrow[dl, "s_{n-2}"'] \\
Q_{n+1} \arrow[r, "d"] & Q_n \arrow[r, "d"] & Q_{n-1} &
\end{tikzcd}$$

Link to Diagram

So consider $h_n - s_{n-1}d : P_n \to Q_n$, which we want to equal $d(s_n)$. We want exactness, so we need better control of the image! We have $d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d$. But this is equal to $dh_n - h_{n-1}d = 0$ since h is a chain map. Thus we get $h_n - s_{n-1}d : P_n \to Z_n(Q)$, and thus using projectivity one last time, we obtain the following:

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\exists s_n"'] \arrow[d, "h_n - s_{n-1}d"] & \\ Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0 \end{tikzcd}$$

Link to Diagram

Since P_n is projective, there exists an $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$. ■

9 Tuesday, February 02

todo

10 Wednesday, February 03

Remark 10.0.1: All rings have 1 in this course!

10.1 Horseshoe Lemma

Proposition 10.1.1 ***(Horseshoe Lemma).***
Suppose we have a diagram like the following, where the columns are exact and the rows are projective resolutions:

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}
$$

Link to Diagram

Note that if the vertical sequence were split, one could sum together to two resolutions to get a resolution of the middle. This still works: there is a projective resolution of P of A given by

$$P_n \coloneqq P_n' \oplus P_n''$$

which lifts the vertical column in the above diagram to an exact sequence of complexes

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0,$$

where $\iota_n : P_n' \hookrightarrow P_n$ is the natural inclusion and $\pi_i : P_n \twoheadrightarrow P_n''$ the natural projection.

10.1.1 Proof of the Horseshoe Lemma

We can construct this inductively:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"'] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
\ker(\varepsilon) \arrow[r] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"'] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \\
\ker(\varepsilon'') \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] \arrow[ur, dashed, "\eta''"'] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

Link to Diagram

- P_0'' projective and π_A surjective implies ε'' lifts to $\eta'' : P_0'' \to A$
- Composing yields $\eta' \coloneqq \iota_A \circ \eta' : P_0' \to A$
- Get $\varepsilon \coloneqq \eta' \oplus \eta'' : P_0 \coloneqq P_0' \oplus P_0'' \to A$.

Flipping the diagram, we can apply the snake lemma to the two columns:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[uurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

Link to Diagram

We can now conclude that

- $\operatorname{coker} \varepsilon = 0$
- $\partial = 0$ since it lands on the zero moduli

So append a zero onto the far left column:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"'] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"'] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[d] \arrow[uurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

Link to Diagram

Thus the LHS column is a SES, and we have the first step of a resolution. Proceeding inductively, at the next step we have

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r, "d_1'"] & \ker(\varepsilon') \arrow[r] \arrow[d] & 0 \\
 & & \ker(\varepsilon) \arrow[d] & \\
\cdots \arrow[r] & P_1'' \arrow[r, "d_1''"] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}
$$

Link to Diagram

However, this is precisely the situation that appeared before, so the same procedure works.

Exercise 10.1.2 (?)
Check that the middle complex is exact! Follows by construction.

10.2 Injective Resolutions

Definition 10.2.1 (Injective Objects)
Let $\mathcal{A}$ be an abelian category, then $I \in \mathcal{A}$ is **injective** if and only if it satisfies the following universal property: A is projective if and only if for every monic $\alpha : A \to I$, any map $f : A \to B$ lifts to a map $B \to I$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \\
& I \arrow[ur, dashed, "\exists\beta"'] &
\end{tikzcd}$$

Link to Diagram

We say $\mathcal{A}$ **has enough injectives** if and only if for all A, there exists $A \hookrightarrow I$ where I is injective.

Slogan 10.2.2
Maps on subobjects extend.

Proposition 10.2.3 ***(Products of Injectives are Injective).***
If $\{I_\alpha\}$ is a family of injectives and $I := \prod_\alpha I_\alpha \in A$, then I is again injective.

Proof (?).
Use the universal property of direct products. ■

10.3 Baer's Criterion

Proposition 10.3.1 *(Baer's Criterion).*
An object $E \in \mathsf{R\text{-}Mod}$ is injective if and only if for every right ideal $J \trianglelefteq R$, every map $J \to E$ extends to a map $R \to E$. Note that J is a right R-submodule.

Proof (?).
$\implies$: This is essentially by definition. Instead of taking arbitrary submodules, we're just taking R itself and *its* submodules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\
 & E &
\end{tikzcd}$$

Link to Diagram

$\impliedby$: Suppose we have the following:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \\
 & E &
\end{tikzcd}$$

Link to Diagram

Let $\mathcal{E} := \left\{ \alpha' : A' \to E \;\middle|\; A \leq A' \leq B \right\}$, i.e. all of the intermediate extensions:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, red] \arrow[d, "\alpha"] & \color{red}{A'} \arrow[r, red] & B \\
 & E & &
\end{tikzcd}$$

Link to Diagram

Add a partial order to $\mathcal{E}$ where $\alpha' \leq \alpha''$ if and only if α'' extends α'. Applying Zorn's lemma (and abusing notation slightly), we can produce a maximal $\alpha' : A' \to E$. The claim is that $A' = B$. Supposing not, then A' is a proper submodule, so choose a $b \in B \setminus A'$. Then define the set $J := \left\{ r \in R \;\middle|\; br \in A' \right\}$, this is a right ideal of R since A' was a right R-module. Now applying the assumption of Baer's condition on E, we can produce a map $f : R \to E$:C

$$
\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, "b\cdot -"'] & R \arrow[ddl, dashed, "\exists f"] \\
& A' \arrow[d, "\alpha'"'] & \\
& E &
\end{tikzcd}
$$

Link to Diagram

Now let $A'' := A' + bR \leq B$, and provisionally define

$$
\begin{aligned}
\alpha'' : A'' &\to E \\
a + br &\mapsto \alpha'(a) + f(r).
\end{aligned}
$$

Remark 10.3.2: Is this well-defined? Consider overlapping terms, it's enough to consider elements of the form $br \in A'$. In this case, $r \in J$ by definition, and so $\alpha'(br) = f(r)$ by commutativity in the previous diagram, which shows that the two maps agree on anything in the intersection.

Note that α'' now extends α', but $A' \subsetneq A''$ since $b \in A'' \setminus A'$. But then A'' strictly contains A', contradicting its maximality from Zorn's lemma.

■

Remark 10.3.3: Big question: what *are* injective modules really? These are pretty nonintuitive objects.

11 | Friday, February 05

See missing first 10m Recall the definition of injectives.

Remark 11.0.1: Over a PID, divisible is equivalent (?) to injective as a module.

Example 11.0.2 ***(?):*** $\mathbb{Q}$ is divisible, and thus an injective $\mathbb{Z}$-module. Similarly $\mathbb{Q}/\mathbb{Z} \rightleftharpoons [0,1) \cap \mathbb{Q}$.

Example 11.0.3 ***(?):*** Let $p \in \mathbb{Z}$ be prime, then $\mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{Q}$ has elements of the form $\sum \frac{a_i}{p^{n_i}}$, and is not divisible. On the other hand, $\mathbb{Z}_{p^\infty} := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \rightleftharpoons \mathbb{Z}[\frac{1}{p}] \cap [0,1)$ is divisible since $p^n \left(\frac{a}{p^n} \right) = a \in \mathbb{Z}$, which equals zero in $\mathbb{Z}_{p^\infty}$. To solve $xr = a/p^n$ with $r, a \in \mathbb{Z}$ and $r \neq 0$, first assume $\gcd(r, p) = 1$ by just dividing through by any common powers of p. This amounts to solving $1 = srtp^n$ where

$s, t \in \mathbb{Z}$:

$$\begin{aligned}\frac{a}{p^n} &= sr\left(\frac{a}{p^n}\right) + tp^n\left(\frac{a}{p^n}\right) \\ &= \left(\frac{sa}{p^n}\right) r \\ &:= xr \in \mathbb{Z}_{p^\infty}.\end{aligned}$$

Fact 11.0.4
Every injective abelian group is isomorphic to a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ for various primes p.

Example 11.0.5 ***(?):*** $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}$. To prove this, do induction on the number of prime factors in the denominator.

Exercise 11.0.6 (2.3.2)
$\mathsf{Ab} = \mathbb{Z}\text{-}\mathsf{Mod}$ has enough injectives.

Remark 11.0.7: As a consequence, $\mathsf{Mod}\text{-}\mathsf{R}$ has enough injectives for *any* ring R.

11.1 Transferring Injectives Between Categories

Next we'll use our background in projectives to deduce analogous facts for injectives.

Definition 11.1.1 (Opposite Category)
Let $\mathcal{A}$ be any category, then there is an opposite/dual category $\mathcal{A}^{\mathrm{op}}$ defined in the following way:

- $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$
- $A \to B \in \mathrm{Mor}(\mathcal{A}) \implies B \to A \in \mathrm{Mor}(\mathcal{A}^{\mathrm{op}})$, so

$$\begin{aligned}\operatorname*{Hom}_{\mathcal{A}}(A, B) &\rightleftharpoons \operatorname*{Hom}_{\mathcal{A}^{\mathrm{op}}}(B, A) \\ f &\rightleftharpoons f^{\mathrm{op}}.\end{aligned}$$

- We require that if $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, then $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$ where $C \xrightarrow{g^{\mathrm{op}}} B \xrightarrow{f^{\mathrm{op}}} A$.
- $\mathbb{1}_A^{\mathrm{op}} = \mathbb{1}_A$ in $\mathcal{A}^{\mathrm{op}}$.

⚠ **Warning 11.1.2**
Thinking of these as functions won't quite work! For $f : A \to B$, there may not be any map $B \to A$ – you'd need it to be onto to even define such a thing, and if it's not injective there are many choices.

Note that initials and terminals are swapped, and since 0 is both. Counterintuitively, $A \to 0 \to B$ is 0, which maps to $B \to 0 \to A = 0^{\mathrm{op}}$.

Remark 11.1.3: Note that $(-)^{\mathrm{op}}$ switches

- Monics and epis,
- Initial and terminal objects,
- Kernels and cokernels.

Moreover, $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{\mathrm{op}}$ is abelian.

Definition 11.1.4 (Contravariant Functors)
A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

$$C_1 \xrightarrow{f} C_2 \qquad C_2 \xrightarrow{f^{\mathrm{op}}} C_1 \qquad FC_2 \xrightarrow{F(f)} FC_1$$

$$\mathcal{C} \rightsquigarrow \mathcal{C}^{\mathrm{op}} \qquad \mathcal{C}^{\mathrm{op}} \rightsquigarrow \mathcal{D}$$

In particular, $F(\mathbb{1}) = \mathbb{1}$ and $F(gf) = F(f)F(g)$

Link to Diagram

Example 11.1.5 *(?):* $\operatorname{Hom}_R(-, A) : \mathsf{Mod\text{-}R} \to \mathsf{Ab}$ is a contravariant functor in the first slot.

Definition 11.1.6 (Left-Exact Functors)
A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is **left exact** if and only if the corresponding covariant functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$: That is, SESs in $\mathcal{A}$ get mapped to long left-exact sequences in $\mathcal{B}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, squiggly, "F(-)"] & C \arrow[r] & 0 \\
0 \arrow[r] & FC \arrow[r] & FB \arrow[r] & FA \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 11.1.7 *(?).*
If $\mathcal{A}$ is abelian and $A \in \mathcal{A}$, then the following are equivalent:

- A is injective in $\mathcal{A}$.
- A is projective in $\mathcal{A}^{\mathrm{op}}$.
- The contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, A)$ is exact.

Lemma 11.1.8 *(?).*
If an abelian category $\mathcal{A}$ has enough injectives, then every $M \in \mathcal{A}$ has an injective resolution:

$$0 \to M \to I^0 \to I^1 \to \cdots .$$

which is an exact cochain complex with each I^n injective. There is a version of the comparison lemma that is proved in roughly the same way as for projective resolutions.

Next up: how to transport injective resolutions in $\mathbb{Z}$-Mod to R-Mod.

Observation 11.1.9
If $A \in \mathsf{Ab}$ and $N \in \mathsf{R\text{-}Mod}$ then $\underset{\mathsf{Ab}}{\mathrm{Hom}}(N, A) \in \mathsf{Mod\text{-}R}$ in the following way: taking $f : N \to A$ and $r \in R$, define a right action $(f \cdot r)(n) := f(rn)$.

Exercise 11.1.10 (?)
Check that this is a morphism of abelian groups, that this yields a module structure, along with other details. For noncommutative rings, it's crucial that the r is on the right in the action and on the left in the definition.

Lemma 11.1.11 *(?).*
If $M \in \mathsf{Mod\text{-}R}$, then the following natural map τ is an isomorphism of abelian groups for each $A \in \mathsf{Ab}$:

$$\begin{aligned} \tau : \underset{\mathsf{Ab}}{\mathrm{Hom}}(\mathrm{Forget}(M), A) &\to \underset{\mathsf{Mod\text{-}R}}{\mathrm{Hom}}(M, \underset{\mathsf{Ab}}{\mathrm{Hom}}(R, A)) \\ f &\mapsto \tau(f)(m)(r) := f(mr), \end{aligned}$$

where $m \in M$ and $r \in R$ and $\mathrm{Forget} : \mathsf{Mod\text{-}R} \to \mathsf{Mod\text{-}\mathbb{Z}}$ is a forgetful functor. Note that R is a left R-module, so the hom in the RHS is a right R-module and the hom makes sense.

Exercise 11.1.12 *(?)*
Check the details here, particularly that the module structures all make sense.

There is a map μ going the other way: $\mu(g)(m) = g(m)(1_R)$ for $m \in M$.

Remark 11.1.13: A quick look at why these maps are inverses:

$$\begin{aligned} \mu(\tau(f)) &= (\tau f)(m)(1_R) \\ &= f(m \cdot 1) \\ &= f(m). \end{aligned}$$

Conversely,

$$\begin{aligned} \tau(\mu(g))(m)(r) &= (\mu g)(mr) \\ &= g(mr)(1) \\ &= g(m \cdot r) && \text{since } g \in \mathrm{Mor}_{\mathsf{R\text{-}Mod}} \\ &= g(m)(r \cdot 1) && \text{by observation earlier} \\ &= g(m)(r). \end{aligned}$$

Remark 11.1.14: The ? functor in the lemma will be the forgetful functor applied to M, yielding an adjoint pair.

12 | Monday, February 08

12.1 Transporting Injectives

Remark 12.1.1: Last time: we had a lemma that for any $M \in \mathsf{Mod\text{-}R}$ and $A \in \mathsf{Ab}$ there is an isomorphism

$$\operatorname*{Hom}_{\mathsf{Ab}}(F(M), A) \cong \operatorname*{Hom}_{\mathsf{Mod\text{-}R}}(M, \operatorname*{Hom}_{\mathsf{Ab}}(R, A)),$$

where $F : \mathsf{Mod\text{-}R} \to \mathsf{Ab}$ is the forgetful functor.

Definition 12.1.2 (Adjoints)
A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are **adjoint** is there are natural bijections

$$\tau_{AB} : \operatorname*{Hom}_{B}(L(A), B) \xrightarrow{\sim} \operatorname*{Hom}_{A}(A, R(B)) \qquad\qquad \forall A \in A, B \in B,$$

where *natural* means that for all $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ there is a diagram

$$\begin{tikzcd}
\operatorname*{Hom}_{B}(LA', B) \arrow[r, "(Lf)^*"] \arrow[d, "\tau"] & \operatorname*{Hom}_{B}(LA, B) \arrow[r, "g_*"] \arrow[d, "\tau"] & \operatorname*{Hom}_{B}(LA, B') \arrow[d, "\tau"] \\
\operatorname*{Hom}_{A}(A', RB) \arrow[r, "f^*"] & \operatorname*{Hom}_{A}(A, RB) \arrow[r, "(Rg)_*"] & \operatorname*{Hom}_{A}(A, RB')
\end{tikzcd}$$

Link to Diagram

In this case we say L is **left adjoint** to R and R is **right adjoint** to L and write $\mathcal{A} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$.

Remark 12.1.3: The lemma thus says that $\operatorname*{Hom}_{\mathsf{Ab}}(R, -) : \mathsf{Ab} \to \mathsf{Mod\text{-}R}$ (using that $R \in \mathsf{R\text{-}Mod}$ is a left R-module) is right adjoint to the forgetful functor $\mathsf{Mod\text{-}R} \to \; Ab$.

Remark 12.1.4: Recall that F is **additive** if $\operatorname*{Hom}_{\mathcal{B}}(B', B) \to \operatorname*{Hom}_{\mathcal{A}}(FB', FB)$ is a morphism of abelian groups for all $B, B' \in \mathcal{B}$.

Proposition 12.1.5 ***(Right adjoints to exact functors preserve injectives, left adjoints preserve projectives).***
If $R : \mathcal{B} \to \mathcal{A}$ is an additive functor and right adjoint to an exact functor $L : \mathcal{A} \to \mathcal{B}$, then

$I \in \mathcal{B}$ injective implies $R(I) \in \mathcal{A}$ is injective. Dually, if $\mathcal{L} : A \to B$ is additive and left adjoint to an exact functor $R : \mathcal{B} \to \mathcal{A}$, then $P \in \mathcal{A}$ projective implies $L(P) \in \mathcal{B}$ is projective.

Corollary 12.1.6 ***(?).***
If $I \in \mathsf{Ab}$ is injective, then $\operatorname{Hom}_{\mathsf{Ab}}(R, I) \in \mathsf{Mod\text{-}R}$ is injective.

Proof (?).
This follows from the previous lemma: $\operatorname{Hom}_{\mathsf{Ab}}(R, -)$ is right adjoint to the forgetful functor $\mathsf{R\text{-}Mod} \to \mathsf{Ab}$ which is certainly exact. This follows from the fact that kernels and images don't change, since these are given in terms of set maps and equalities of sets.

■

Exercise 12.1.7 (2.3.5, 2.3.2)
Show that $\mathsf{Mod\text{-}R}$ has enough injectives, using that Ab has enough injectives.

Proof (of proposition).
It suffices to show that the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(-, RI)$ is exact. We know it's left exact, so we'll show surjectivity. Suppose we have a SES $0 \to A \xrightarrow{f} A'$ which is exact in $\mathcal{A}$. Then $0 \to LA \xrightarrow{Lf} LA'$ is exact, and we can apply hom to obtain the exact sequence

$$\operatorname{Hom}_{\mathcal{B}}(LA', I) \xrightarrow{(LF)^*} \operatorname{Hom}_{\mathcal{B}}(LA, I) \to 0.$$

Applying τ yields

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{B}}(LA', I) \arrow[r, "(Lf)^*"] \arrow[d, "\tau\sim"] & \operatorname{Hom}_{\mathcal{B}}(LA, I) \arrow[r] \arrow[d, "\tau\sim"] & 0 \arrow[d, dashed] \\
\operatorname{Hom}_{\mathcal{A}}(A', RI) \arrow[r, "f^*"] & \operatorname{Hom}_{\mathcal{A}}(A, RI) \arrow[r, dashed] & 0
\end{tikzcd}$$

Link to Diagram

- The top sequence is exact since I is injective in $\mathcal{B}$
- Therefore the bottom map is onto (diagram chase)

■

12.2 2.4: Left Derived Functors

Remark 12.2.1: Goal: define left derived functors of a right exact functor F, with applications the bifunctor $- \otimes_R -$, which is right exact and covariant in each variable. We're ultimately interested in Hom functors and Ext, but this is slightly more technical since it's covariant in one slot and contravariant in the other, so focusing on this functor makes the theory slightly easier to develop. This can be fixed by switching $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ once in a while. Everything for left derived functors will have a dual for right derived functors.

Remark 12.2.2: Let $\mathcal{A}, \mathcal{B}$ be abelian categories where $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor (which implicitly assumes F is additive). We want to define $L_i F : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$.

Definition 12.2.3 (Left Derived Functors)
For $A \in \mathcal{A}$, fix once and for all a projective resolution $P \xrightarrow{\varepsilon} A$, where $P_{<0} = 0$. Then define $FP = (\cdots \to F(P_1) \xrightarrow{Fd_1} F(P_0) \to 0$, noting that A no longer appears in this complex. We can write $H_0(FP) = FP_0/(Fd_1)(FP_1)$, and define

$$(L_i F)(A) \coloneqq H_i(FP).$$

Remark 12.2.4: Note that $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$ is exact, and since F is right exact, it can be shown that the following is a SES: $FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{F\varepsilon} FA \to 0$. We can use this to compute the original homology, despite it not having any homology itself! From this, we can extra $L_0(A) \coloneqq FP_0/(Fd_1)(FP_1) = FP_0/\ker F(\varepsilon)$ using exactness at FP_0, and by the 1st isomorphism theorem this is isomorphic to the image FA using surjectivity. So $L_0 F \cong F$.

Theorem 12.2.5 ***(Left-derived functors are additive).***
$L_i F : \mathcal{A} \to \mathcal{B}$ are additive functors.

Proof (?).
First, $\mathbb{1}_P : P \to P$ lifts $\mathbb{1}_A : A \to A$ since it yields a commuting ladder, and $F(\mathbb{1}) = \mathbb{1}$, so $(L_i f)(\mathbb{1}) = \mathbb{1}$. Then in the following diagram, the outer rectangle commutes since the inner squares do:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\tilde{f}"] & A \arrow[d, "f"] \\
P' \arrow[r] \arrow[d, "\tilde{g}"] & A' \arrow[d, "g"] \\
P'' \arrow[r] & A''
\end{tikzcd}$$

Link to Diagram

So $\tilde{g} \circ \tilde{f}$ lifts $g \circ f$ and therefore $g_* f_* = (gf)_*$. Thus $L_i F$ is a functor. That they are additive comes from checking the following diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, shift right, "\tilde{g}"'] \arrow[d, shift left, "\tilde{f}"] & A \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \\
Q \arrow[r] & B
\end{tikzcd}$$

Link to Diagram

Then $\tilde{f} + \tilde{g}$ lifts $f + g$, and H_i is an additive functor: $(F\tilde{f} + F\tilde{g})_* = (F\tilde{f})_* + (F\tilde{g})_*$. Thus $L_i F$ is additive.

■

13 | Wednesday, February 10

Remark 13.0.1: Setup: Let $\mathcal{A}, \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ has enough projectives. Let $P \xrightarrow{\varepsilon} A \in \mathcal{A}$ be a projective resolution, we want to define the left derived functors $L_i F(A) := H_i(FP)$.

Lemma 13.0.2 *(?).*
$L_i F(A)$ is well-defined up to natural isomorphism, i.e. if $Q \to A$ is a projective resolution, then there are canonical isomorphism $H_i(FP) \xrightarrow{\sim} H_i(FQ)$. In particular, changing projective resolutions yields a new functor $\widehat{L}_i F$ which are naturally isomorphic to F.

Proof (?).
We can set up the following situation

$$\begin{tikzcd}
P \arrow[r, "\varepsilon_P"] \arrow[d, "\exists f"] & A \arrow[r] \arrow[d, "\mathbb{1}_A"] & 0 \\
Q \arrow[r, "\varepsilon_Q"] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here f exists by the comparison theorem, and thus there are induced maps $f_* : H_*(FP) \to H_*(FQ)$ by abuse of notation – really, this is more like $(f_*)_i = H_u(Ff)$. We're using that both F and H_i are both additive functors. Note that the lift f of $\mathbb{1}_A$ is not unique, but any other lift is chain homotopic to f, i.e. $f - f' = ds + sd$ where $s : P \to Q[1]$. So they induce the same maps on homology, i.e. $f'_* = f_*$. Thus the isomorphism is canonical in the sense that it doesn't depend on the choice of lift.
Similarly there exists a $g : Q \to P$ lifting $\mathbb{1}_A$, and so gf and $\mathbb{1}_P$ are both chain maps lifting $\mathbb{1}_A$, since it's the composition of two maps lifting $\mathbb{1}_A$. So they induce the same map on homology by the same reasoning above. We can write

$$g_* f_* = (gf)_* = (\mathbb{1}_{FP})_* = \mathbb{1}_{H_*(FP)},$$

and similarly $f_* g_* = \mathbb{1}_{H_*(FQ)}$, making f_* an isomorphism.

■

Corollary 13.0.3 *(?)*.
If A is projective, then $L^{>0}FA = 0$.

Proof (?).
Use the projective resolution $\cdots \to 0 \to A \xrightarrow{\mathbb{1}_A} A \to 0 \to \cdots$. In this case $H_{>0}(FP) = 0$. ■

Remark 13.0.4: This is an interesting result, since it doesn't depend on the functor! Short aside on F-acyclic objects – we don't need something as strong as a *projective* resolution.

Definition 13.0.5 (F-acyclic objects)
An object $Q \in \mathcal{A}$ is **F-acyclic** if $L_{>0}FQ = 0$.

Remark 13.0.6: Note that projective implies F-acyclic for every F, but not conversely. For example, flat R-modules are acyclic for $- \otimes_R -$. In general, flat does not imply projective, although projective implies flat.

Definition 13.0.7 (F-acyclic resolutions)
An **F-acyclic resolution** of A is a left resolution $Q \to A$ for which every Q_i is F-acyclic.

Remark 13.0.8: One can compute $L_iF(A) \cong H_i(FQ)$ for any F-acyclic resolution. For the L_iF to be functors, we need to define them on maps!

Lemma 13.0.9 *(?)*.
If $f : A \to A'$, there is a natural associated morphism $L_iF(f) : L_iF(A) \to L_iF(A')$.

Proof (?).
Again use the comparison theorem:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\exists \tilde{f}"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
P' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Then there is an induced map $\tilde{f}_* : H_*(FP) \to H_*(FP')$, noting that one first needs to apply F to the above diagram. As before, this is independent of the lift using the same argument as before, using the additivity of F and H_* and the chain homotopy is pushed through F appropriately. So set $(L_iF)(f) := (\tilde{f}_*)_i$. ■

Remark 13.0.10: We can now pick up the theorem from the end of last time:

Theorem 13.0.11 ***(Left-derived functors are additive).***
$L_iF : \mathcal{A} \to \mathcal{B}$ are additive functors.

Proof (?).
Done last time! ■

Theorem 13.0.12 ***(Existence of connecting maps for left-derived functors).***
Using the same assumptions as before, given a SES

$$0 \to A' \to A \to A'' \to 0$$

there are natural connecting maps δ yielding a LES

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_{i+1}"'] \\
L_iF(A') \arrow[r] & L_iF(A) \arrow[r] & L_iF(A'') \arrow[dll, "\delta_i"'] \\
L_{i-1}F(A') \arrow[r, dashed] & \cdots \arrow[r, dashed] & \cdots \arrow[dll, "\delta_1"'] \\
FA' \arrow[r] & FA \arrow[r] & FA'' \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proof (?).
Using the Horseshoe lemma, we can obtain the following map:

$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
P' \arrow[r] \arrow[d] & A' \arrow[d] & \\
P \arrow[r, dashed, "\exists"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
P'' \arrow[r] & A'' \arrow[d] & \\
 & 0 &
\end{tikzcd}
$$

Link to Diagram

So we get a SES of complexes over $\mathcal{A}$, $0 \to P' \to P \to P'' \to 0$. One can use that $P = P' \oplus P''$, or alternatively that each P''_n is a projective R-module, to see that there are splittings

$$
\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, "f"'] & P \arrow[l, dashed, bend right, "f'"'] \arrow[r, "g"'] & P'' \arrow[l, dashed, bend right, "g'"'] \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Note that this can be phrased in terms of $g'g = \mathbb{1}, f'f = \mathbb{1}$, or $g'g + f'f = \mathbb{1}$. Since F is additive, it preserves all of these relations, particularly the ones that define being split exact. So additive functors preserve split exact sequences. Thus $0 \to FP' \to FP \to FP'' \to 0$ is still split exact, even though F is only right exact. Now take homology and use the LES in homology to get the desired LES above, and δ is the connecting morphism that comes from the snake lemma.
Proving naturality: we start with the following setup.

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "g'"] & A \arrow[r] \arrow[d, "g"] & A'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}
$$

Naturality of δ will be showing that the following square commutes:

$$
\begin{tikzcd}
L_{i+1}F(A'') \arrow[r, "\delta"] \arrow[d] & L_iF(A') \arrow[d] \\
L_{i+1}F(B'') \arrow[r, "\delta"] & L_iF(B')
\end{tikzcd}
$$

We now apply the horseshoe lemma several times:

$$
\begin{tikzcd}
0 & & P' \arrow[rr] \arrow[dr] \arrow[ddd, dotted, "\exists G'"'] & & P \arrow[rr] \arrow[dr, dashed] \arrow[ddd, dashed, "\exists G"'] & & P'' \arrow[dr] \arrow[ddd, dotted, "\exists G''"'] & & 0 & \\
& 0 \arrow[rr] & & A' \arrow[rr] \arrow[d, "g'"] & & A \arrow[rr] \arrow[d, "g"] & & A'' \arrow[rr] \arrow[d, "g''"] & & 0 \\
& 0 \arrow[rr] & & B' \arrow[rr] & & B \arrow[rr] & & B'' \arrow[rr] & & 0 \\
0 & & Q' \arrow[rr] \arrow[ur] & & Q \arrow[rr] \arrow[ur, dashed] & & Q'' \arrow[ur] & & 0 &
\end{tikzcd}
$$

It turns out (details omitted see Weibel p. 46) that G can be chosen such that we get a commutative diagram of chain complexes with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "G'"] & P \arrow[r] \arrow[d, "G"] & P'' \arrow[r] \arrow[d, "G''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

We proved naturality of the connecting maps ∂ in the corresponding LES in homology in general (see prop. 1.3.4). This translates to naturality of the maps $\delta_i : L_i(A'') \to L_{i-1}(A')$. ■

Remark 13.0.13: See exercise 2.4.3 for "dimension shifting". This is a useful tool for inductive arguments.

14 | Friday, February 12

Remark 14.0.1: Last time: right-exact functors have left-derived functors where a SES induces a LES. The functors are *natural* with respect to the connecting morphisms in the sense that certain squares commute. Weibel refers to $\{L_i F\}_{i \geq 0}$ as a **homological δ-functor**, i.e. anything that takes SESs to LESs which are natural with respect to connecting morphism.

14 Friday, February 12

14.1 Aside: Natural Transformations

Definition 14.1.1 (Natural Transformation)
Given functors $F, G, \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \implies G$ is the following data:

- For all $C \in \mathcal{C}$ there is a map $F(C) \xrightarrow{\eta_C} G(C) \in \mathrm{Mor}(\mathcal{D})$, sometimes referred to as $\eta(C)$.
- If $C \xrightarrow{f} C' \in \mathrm{Mor}(\mathcal{C})$, there is a diagram

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\eta_C"] & FC' \arrow[d, "\eta_{C'}"] \\
GC \arrow[r, "Gf"] & GC'
\end{tikzcd}$$

Link to Diagram

- η is a **natural isomorphism** if all of the η_C are isomorphisms, and we write $F \cong C$.

Definition 14.1.2 (Equivalence of Categories)
A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if and only if there exists a $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathbb{1}_{\mathcal{C}}$ and $FG \cong \mathbb{1}_{\mathcal{D}}$.

Example 14.1.3 ***(?):*** A category $\mathcal{C}$ is **small** if $\mathrm{Ob}(\mathcal{C})$ is a set, then take $\mathcal{C} := \mathsf{Cat}$ whose objects are categories and morphisms are functors. Note that in all categories, all collections of morphisms should be sets, and the small condition guarantees it. In this case, natural transformations $\eta : F \to G$ is an additional structure yielding morphisms of morphisms. These are called **2-morphisms**, and in this entire structure is a **2-category**, and our previous notion is referred to as a **1-category**.

Theorem 14.1.4 ***(Left-derived functors of a right-exact functor form a universal δ-functor).***
Assume $\mathcal{A}, \mathcal{B}$ are abelian and $F : \mathcal{A} \to \mathcal{B}$ is a right-exact additive functor where $\mathcal{A}$ has enough projectives. Then the family $\{L_i F\}_{i \geq 0}$ is a *universal δ-functor* where $L_0 F \cong F$ is a natural isomorphism.

Remark 14.1.5: Here *universal* means that if $\{T_i\}_{i \geq 0}$ is also a δ-functor with a natural *transformation* (not necessarily an isomorphism) $\varphi_0 : T_0 \to F$, then there exist unique morphism of δ-functors $\{\varphi_i : T_i \to L_i F\}_{i \geq 0}$, i.e. a family of natural transformations that commute with the respective δ maps coming from both the T_i and the $L_i F$, which extend φ_0. This will be important later on when we try to show Ext and Tor are functors in either slot.

Proof (?).
Assume $\{T_i\}_{i\geq 0}$ and φ_0 are given, and assume inductively that $n > 0$ and we've defined $\varphi_i : T_i \to F$ for $0 \leq i < n$ which commute with the δ maps. Step 1: given $A \in \mathcal{A}$, fix a reference exact sequence: pick a projective mapping onto A and its kernel to obtain

$$0 \to K \to P \to A \to 0.$$

Applying the functors T_i and $L_i F$ yields

$$\begin{tikzcd}
 & x \arrow[r, dotted] & k \arrow[r, dotted] \arrow[ddd, dotted, bend right] & 0 \arrow[ddd, dotted, bend right] \\
 & T_n A \arrow[r, "\delta"] \arrow[d, dashed, "\exists\varphi_{n-1}(A)"] & T_{n-1}K \arrow[r] \arrow[d, "\varphi_{n-1}(K)"] & T_{n-1}P \arrow[d, "\varphi_{n-1}(P)"] \\
L_n FP = 0 \arrow[r] & L_n FA \arrow[r, hook, "\delta"'] & L_{n-1}FK \arrow[r] & L_{n-1}FP \\
 & \exists! y \coloneqq \varphi_{n-1}(x) \arrow[r, dotted] & \ell \arrow[r, dotted] & 0
\end{tikzcd}$$

Link to Diagram

So define $\varphi_n(A)(x) \coloneqq y$, which makes the LHS square commute by construction. Note that $L_n FP$ vanishes (as do all its higher derived functors) since P is projective.

Warning 14.1.6

The map $\varphi_n(A)$ could depend on the choice of P!

We now want to show that φ_n is a natural transformation. Supposing $f : A' \to A$, we need to show φ_n commutes with f.

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, dashed, "\exists h"] & P' \arrow[r] \arrow[d, dashed, "\exists g"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Since P' is projective, we can lift f to $P' \to P$, and then define h to be the restriction of g to $K' \to K$.

$$\begin{tikzcd}
T_nA' \arrow[rrr, "T_nf"] \arrow[dr, "\delta'"] \arrow[ddd, "\varphi_n(A')"] & & & T_nA \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
& T_{n-1}K' \arrow[r, "T_{n-1}h"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}K \arrow[d, "\varphi_{n-1}"] & \\
& L_{n-1}FK' \arrow[r, "L_{n-1}Fh"] & L_{n-1}FK & \\
L_nFA' \arrow[ur, "\delta'"] \arrow[rrr, "L_nFf"] & & & L_nF(A) \arrow[ul, "\delta"']
\end{tikzcd}$$

Link to Diagram

Note that all of the quadrilaterals here commute. The middle top and bottom come from naturality of T_*, L_*F with respect to δ, the RHS/LHS due to the construction of the φ_n, and φ_{n-1} is natural by the inductive hypothesis. Now in order to traverse $T_nA' \to T_nA \to L_nF(A)$, we can pass the path through one commuting square at a time to make it equal to $T_nA' \to L_nFA' \to L_nFA$, so the outer square commutes. We have

$$\delta\varphi_n(A)T_nF = \delta L_nFf\varphi_n(A'),$$

and since δ is monic (using the previous vanishing due to projectivity), so we can cancel on the left and this yields the definition of naturality.

Corollary 14.1.7 ***(?).***
The definition of $\varphi_n(A)$ does not depend on the choice of P. Taking $A' = A$ in the previous argument with $f = \mathbb{1}$, suppose $P' \neq P$. Then $T_nf = \mathbb{1} = L_nFf$ and setting $\varphi_n'(A)$ to be the map coming from P', we get $\varphi_n'(A) = \varphi_n(A)$ using the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, dashed, "\exists h"] & P' \arrow[r] \arrow[d, dashed, "\exists g"] & A \arrow[r] \arrow[d, "\mathbb{1}"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

So the $\varphi_n(A)$ are uniquely defined. We now want to show that φ_n commutes with the δ_n coming from an *arbitrary* SES instead of a fixed reference SES.

$$\begin{tikzcd}
T_nA \arrow[rr, "\delta_{(2)}"'] \arrow[d, "="] \arrow[dr, phantom, "T_* \text{ a } \delta \text{ functor}"] & & T_{n-1}C \arrow[ddd, "\varphi_{n-1}"] \\
T_nA \arrow[r, "\delta_{(2)}"] \arrow[d, "\varphi_n"] \arrow[dr, phantom, "\text{reference}"] & T_{n-1}K \arrow[ur, "T_{n-1}h"] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, phantom, "\varphi_{n-1}\text{natural}"] & \\
L_nFA \arrow[r, "\delta_{(1)}"] \arrow[dr, phantom, "L_*F \text{ a } \delta \text{ functor}"] & L_{n-1}FA \arrow[dr, "L_{n-1}Fh"] & \\
L_nFA \arrow[u, "="] \arrow[rr, "\delta_{(1)}"'] & & L_{n-1}FC
\end{tikzcd}$$

Link to Diagram

This diagram commutes for the reasons indicated, and commutativity of the outside square implies that φ_n commutes with the δ coming from any SES.

See section 2.4 in Weibel.

■

15 | Monday, February 15

15.1 2.5: Right-Derived Functors

Remark 15.1.1: Today: right-derived functors of a left-exact functor. Luckily we can use some opposite category tricks which save us some work of re deriving everything.

Definition 15.1.2 (Right Derived Functors)
Let $F : \mathcal{A} \to \mathcal{B}$ be left-exact where $\mathcal{A}$ has enough injectives. Given $A \in \mathcal{A}$, fix an injective resolution $0 \to A \xrightarrow{\varepsilon} I$ and define

$$R^i\mathcal{F} := H^i(FA) \qquad i \geq 0.$$

Remark 15.1.3: Then

$$0 \to FA \xrightarrow{F\varepsilon} FI^0 \xrightarrow{Fd^0} FI^1$$

is exact, and

$$R^0FA = \ker F(d^0) / \langle 0 \rangle = \operatorname{im} F\varepsilon \cong FA,$$

and so there is naturally an isomorphism $R^0F \cong F$. Observe that F yields a right-exact functor $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$, where we note that $F^{\mathrm{op}}(f^{\mathrm{op}}) = F(f)^{\mathrm{op}}$. Note that taking the opposite category sends injectives to projectives and so $\mathcal{A}^{\mathrm{op}}$ has enough projectives. This means that L_iF^{op} are defined using the projective resolution I, so we have

$$R^iF(A) = (L_iF^{\mathrm{op}})^{\mathrm{op}}.$$

Thus all results about left-derived functors translate to right-derived functors:

- R_iF is independent of the choice of injective resolution, up to a natural isomorphism.
- If A is injective, then $R^{i>0}F(A) = 0$.
- The collection $\left\{R^iF\right\}_{i\geq 0}$ forms a universal cohomological δ-functor for F.
- An object $Q \in \mathcal{A}$ is **F-acyclic** if $R^{>0}F(Q) = 0$.
- R^iF can be computed using F-acyclic objects instead of injective resolutions.

Definition 15.1.4 (?)
Fix a right R-module $M \in \mathsf{Mod\text{-}R}$, then $F \coloneqq \underset{\mathsf{Mod\text{-}R}}{\operatorname{Hom}}(M, -) : \mathsf{Mod\text{-}R} \to \mathsf{Ab}$ is a left-exact functor. Its right-derived functors are **ext functors** and denoted $\operatorname{Ext}^i_{\mathsf{Mod\text{-}R}}(M, -)$.

Example 15.1.5 ***(?):***

$$\operatorname{Ext}^i_{\mathsf{Mod\text{-}R}}(M, A) = (R^iF)(A) = [R^i \underset{\mathsf{Mod\text{-}R}}{\operatorname{Hom}}(M, -)](A).$$

Remark 15.1.6: Exercises 2.5.1, 2.5.2 are important extensions of our existing characterizations of injectives and projectives in $\mathsf{Mod\text{-}R}$. These upgrade the characterization involving Hom to one involving Ext. [3]

Remark 15.1.7: Fix $B \in \mathsf{Mod\text{-}R}$ and consider $G \coloneqq \underset{\mathsf{Mod\text{-}R}}{\operatorname{Hom}}(-, B) : \mathsf{Mod\text{-}R} \to \mathsf{Ab}$. Then G is still left-exact, but is now *contravariant*. We can regard it as a covariant functor left-exact functor $G : \mathsf{Mod\text{-}R}^{\mathrm{op}} \to \mathsf{Ab}$. So we define $R^iG(A)$ by an injective resolution of A in $\mathcal{A}^{\mathrm{op}}$, and this is the same as a projective resolution of A in $\mathcal{A}$. So apply G and take cohomology. It turns out that

$$R^i \underset{\mathsf{Mod\text{-}R}}{\operatorname{Hom}}(-, B) \cong R^i \underset{\mathsf{Mod\text{-}R}}{\operatorname{Hom}}(A, -)(B) \coloneqq \operatorname{Ext}^i_{\mathsf{R\text{-}Mod}}(A, B),$$

so we can use the same notation $\operatorname{Ext}^i_R(-, B)$ for both cases.

[3] Note the typo in 2.5.1.3, it should say the following: "B is $\underset{R}{\operatorname{Hom}}(A, -)$ is acyclic for all A."

15.2 2.6: Adjoint Functors and Left/Right Exactness

Slogan 15.2.1
− adjoints are $-^{\text{op}}$ exact, since − adjoints have −-derived functors.

Theorem 15.2.2 ***(Exactness of adjoint functors).***
Let

$$\mathcal{A} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{B}$$

be an adjoint pair of functors. Then there exists a natural isomorphism

$$\tau_{AB} : \underset{\mathcal{B}}{\mathrm{Hom}}(LA, B) \xrightarrow{\sim} \underset{\mathcal{A}}{\mathrm{Hom}}(A, RB) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Moreover,

- L is right exact, and
- B is left exact.

Proposition 15.2.3 ***(1.6: Yoneda).***
A sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact in $\mathcal{A}$ if and only if for all $M \in \mathrm{Ob}(\mathcal{A})$, the sequence

$$\underset{\mathcal{A}}{\mathrm{Hom}}(M, A) \xrightarrow{\alpha^* := \alpha \circ -} \underset{\mathcal{A}}{\mathrm{Hom}}(M, B) \xrightarrow{\beta^* := \beta \circ -} \underset{\mathcal{A}}{\mathrm{Hom}}(M, C)$$

is exact.

Proof (?).

1. Take $M = A$, then $0 = \beta^* \alpha^* (\mathbb{1}_A) = \beta \alpha \mathbb{1} = \beta \alpha$. Thus $\operatorname{im} \alpha \subseteq \ker \beta$.
2. Take $M = \ker \beta$ and consider the inclusion $\iota : \ker M \hookrightarrow B$, then $\beta^*(\iota) = \beta \iota = 0$ and thus $\iota \in \ker \beta^* = \operatorname{im} \alpha^*$. So there exists $\sigma \in \mathrm{Hom}(\ker \beta, A)$ such that $\iota = \alpha^*(\sigma) := \alpha \sigma$, and thus $\ker \beta = \operatorname{im} \iota \subset \operatorname{im} \alpha$.

Thus $\ker \beta = \operatorname{im} \alpha$, yielding exactness of the bottom sequence.

■

Proof (of theorem).

We'll first prove that R is left-exact. Take a SES in B, say

$$0 \to B' \to B \to B'' \to 0.$$

Apply the left-exact covariant functor $\underset{\mathcal{B}}{\operatorname{Hom}}(LA, -)$ followed by τ:

$$\begin{tikzcd}
0 \arrow[r] & \underset{\mathcal{B}}{\operatorname{Hom}}(LA, B') \arrow[r] \arrow[d, rightsquigarrow, "\tau_{AB}"] & \underset{\mathcal{B}}{\operatorname{Hom}}(LA, B) \arrow[r] \arrow[d, rightsquigarrow, "\tau_{AB}"] & \underset{\mathcal{B}}{\operatorname{Hom}}(LA, B'') \arrow[d, rightsquigarrow, "\tau_{AB}"] \\
0 \arrow[r, dashed] & \underset{\mathcal{B}}{\operatorname{Hom}}(A, RB') \arrow[r] & \underset{\mathcal{B}}{\operatorname{Hom}}(A, RB) \arrow[r] & \underset{\mathcal{B}}{\operatorname{Hom}}(A, RB'')
\end{tikzcd}$$

Link to Diagram

The bottom sequence is exact by naturality of τ. Now applying the Yoneda lemma, we obtain an exact sequence

$$0 \to \underset{\mathcal{A}}{\operatorname{Hom}}(A, RB') \to \underset{\mathcal{A}}{\operatorname{Hom}}(A, RB) \to \underset{\mathcal{A}}{\operatorname{Hom}}(A, RB'').$$

So R is left exact. Now $L^{\mathrm{op}} : \mathcal{A} \to \mathcal{B}$ is right adjoint to R^{op}, so L^{op} is left exact and thus L is right exact.

$\blacksquare$

15.3 Tensor Product Functors and Tor

Remark 15.3.1: Let

- $R, S \in \mathsf{Ring}$,
- $B \in \mathsf{(R, S)\text{-}biMod}$,
- $C \in \mathsf{S\text{-}Mod}$.

Then $\underset{S}{\operatorname{Hom}}(B, C) \in \mathsf{Mod\text{-}R}$ in a natural way: given $f : B \to C$, define $(f \cdot r)(b) = f(rb)$.

Exercise 15.3.2 (?)
Check that this is a well-defined morphism of right S-modules.

Remark 15.3.3: We saw this structure earlier with $S = \mathbb{Z}$, see p.41.

Proposition 15.3.4 ***(Tensor-Hom adjunction).***
Fix R, S and ${}_R B_S$ as above. Then for every $A \in \mathsf{Mod\text{-}R}$ and $C \in$ --$\mathsf{Mod}S$ there is a natural

isomorphism

$$\begin{aligned} \tau : \operatorname*{Hom}_S(A \otimes_R B, C) &\xrightarrow{\sim} \operatorname*{Hom}_R(A, \operatorname*{Hom}_S(B, C)) \\ f &\mapsto g(a)(b) = f(a \otimes b) \\ f(a \otimes b) = g(a)(b) &\leftarrow g. \end{aligned}$$

Note that the tensor product is a right S-module, and the hom on the right is a right R-module, so these expressions make sense. Here B is fixed, so A and C are variables and this is a statement about bifunctors

$$- \otimes_R B : \textsf{Mod-R} \to \textsf{Mod-S},$$

which is left adjoint to

$$\operatorname*{Hom}_S(B, -) : \textsf{Mod-S} \to \textsf{Mod-R}.$$

So the former is a left adjoint and the latter is a right adjoint, so by the theorem, $- \otimes_R B$ is right exact.

Remark 15.3.5: If B is only a left R-module, we can always take $S = \mathbb{Z}$, which makes this into a functor

$$- \otimes_R B : \textsf{Mod-R} \to \textsf{Ab}.$$

Since this is a right exact functor from a category with enough injectives, we can define left-derived functors.

Definition 15.3.6 (?)
Let $B \in \textsf{(R, S)-biMod}$ and let

$$T(-) := - \otimes_R B : \textsf{Mod-R} \to \textsf{Mod-S}.$$

Then define $\operatorname{Tor}_n^R(A, B) := L_n T(A)$.

Remark 15.3.7: Note that these are easier to work with, since they're covariant in both variables.

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Remark 16.0.1: We looked at $B \in \textsf{(R, S)-biMod}$ and showed $- \otimes_R B : \textsf{R-Mod} \to \textsf{S-Mod}$ is left adjoint to hom, and has left-derived functors $\operatorname{Tor}_n^R(-, B) := L_n(- \otimes_R B)$.

$$\begin{tikzcd} \textsf{R-Mod} \arrow[r, shift left, "- \otimes_R B"] \arrow[r, phantom, "\perp"] & \textsf{S-Mod}. \arrow[l, shift left, "\operatorname{Hom}_S(B, -)"] \end{tikzcd}$$

Note that $\operatorname{Tor}_0^R(A, B) \cong A \otimes_R B$.

Remark 16.0.2: $A \otimes_R -$ is also right exact, and it turns out that

$$L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A).$$

So unambiguously denote either of this left derived functors as $\mathrm{Tor}_n(A, B)$.

16.1 Limits and Colimits

Definition 16.1.1 (Functor Category)
Given categories $\mathcal{I}, \mathcal{A}$, define a **functor category** $\mathcal{A}^{\mathcal{I}}$ by

- $\mathrm{Ob}(\mathcal{A}^{\mathcal{I}})$: functors $A : \mathcal{I} \to \mathcal{A}$.
- $\mathrm{Mor}(\mathcal{A}^{\mathcal{I}})$: natural transformations $\eta : A \to B$ between functors.

$\mathcal{I}$ is thought of as an index category, and we'll write $A_i := A(i) \in \mathcal{A}$ for $i \in \mathcal{I}$. If $\alpha : i \to j$ is a morphism in I, then denote $A(\alpha) := \alpha_*$, which is the morphism defined by the following:

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_*"] \arrow[d, "\eta_i"] & A_j \arrow[d, "\eta_j"] \\
B_i \arrow[r, "\alpha_*"] & B_j
\end{tikzcd}$$

Link to Diagram

Composition is defined by $A \xrightarrow{\eta} B \xrightarrow{\zeta} C$ is given by $(\zeta\eta)_i = \zeta_i \circ \eta_i$. We need the collection of morphisms to be sets, so we'll require $\mathcal{I}$ to be a *small category* (i.e. the class of objects forms a set).

Example 16.1.2 ***(Poset Category):*** Take $(I, \leq)$ a poset (which is reflexive, antisymmetric, transitive, but not every two elements are comparable), define a category by

- $\mathrm{Ob}(\mathcal{I}) = I$
- $\left|\underset{\mathcal{I}}{\mathrm{Hom}}(i, j)\right| \leq 1$, and $i \to j \iff i \leq j$

Note that if $i \not\leq j$, then $\underset{\mathcal{I}}{\mathrm{Hom}}(i, j) = \emptyset$.

Remark 16.1.3: Both $\mathcal{A}, \mathcal{A}^{\mathcal{I}}$ are small, so we can consider functors between them.

Definition 16.1.4 (Diagonal Functor)
The **diagonal functor** is defined as $\Delta : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ where for $B \in \mathcal{A}$ the functor $\Delta(B)$ is the constant functor, i.e. $\Delta(B)_i = B$ for all $i \in \mathcal{I}$. All morphism are sent to the identity,

i.e. $i \overset{\alpha}{\to} j \xrightarrow{\Delta(B)} B \xrightarrow{\mathbb{1}_B} B$.

Work out how morphisms work here with respect to natural transformations.

Definition 16.1.5 (Colimit)
The **colimit** of a functor $A : \mathcal{I} \to \mathcal{A}$ is an object $C \in \mathcal{A}$ which we'll denote $\operatorname{colim}_{i \in \mathcal{I}} A_i$, along with a natural transformation $\eta : A \to \Delta(C)$ which is universal among natural transformations of the form $\theta : A \to \Delta(B)$ for $B \in \mathcal{A}$. The unique map in the universal property is from $C \to B$, and we have the following situation:

$$\begin{tikzcd}
\mathcal{I} & & \\
i \arrow[d, "\alpha"] & A_i \arrow[r, "\eta_i"'] \arrow[d, "\alpha_*"] & C \arrow[d, equal] \\
j & A_j \arrow[r, "\eta_j"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A_i \arrow[dd, "\alpha_*"] \arrow[dr, "\eta_i"] \arrow[drr, bend left, "\theta_i"] & & \\
& C \arrow[r, dashed, "\exists!\gamma"] & B \\
A_j \arrow[ur, "\eta_j"] \arrow[urr, bend right, "\theta_j"'] & &
\end{tikzcd}$$

Link to Diagram

Example 16.1.6*(?):* Let $(I, \leq)$ be a poset and take $\mathcal{I}$ its poset category. Then there are morphisms $i \to j \iff i \leq j$, and we have a diagram

$$\begin{tikzcd}
A_i \arrow[dr, "\eta_i"] \arrow[drr, bend left, "\theta_i"] & & \\
& C \arrow[r, dashed, "\exists!\gamma"] & D \\
A_j \arrow[ur, "\eta_j"'] \arrow[urr, bend right, "\theta_j"'] & &
\end{tikzcd}$$

Link to Diagram

This is the **direct limit**. Note that for a poset of category of subsets, this ends up being the union.

Example 16.1.7 ***(?):*** Let $\mathrm{Ob}(\mathcal{I}) = \{1, 2\}$, and take two maps, one of which we'll label by "0":

$$\begin{tikzcd}
1 \arrow[d, shift right] \arrow[d, shift left, "0"] \\
2
\end{tikzcd}$$

Link to Diagram

Suppose now that $\mathcal{A}$ is an abelian category, and suppose we're given a morphism $A_1 \xrightarrow{f} A_2$ in $\mathcal{A}$. Define $A \in \mathcal{A}^{\mathcal{I}}$, and define a functor

$$\begin{tikzcd}
1 \arrow[d, shift right] \arrow[d, shift left, "0"] & A_1 \arrow[d, shift right, "f"'] \arrow[d, shift left, "0"] \arrow[rr, bend left, "\theta_1"] & & B \\
2 & A_2 \arrow[rru, bend right, "\theta_2"] & &
\end{tikzcd}$$

Link to Diagram

By commutativity,

- $\theta_2 \circ 0 = \theta_1 \implies \theta_1 = 0$
- $\theta_2 \circ f = \theta_1 = 0.$

So suppose there was a colimit C, then it'd fit into this diagram as follows:

$$\begin{tikzcd}
1 \arrow[dd, shift right] \arrow[dd, shift left, "0"] & A_1 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "0"] \arrow[dr, "0"] \arrow[drr, bend left, "\theta_1"] & & \\
& & C \arrow[r, dashed] & B \\
2 & A_2 \arrow[ur, "p"] \arrow[urr, bend right, "\theta_2"'] & &
\end{tikzcd}$$

Link to Diagram

Note that C is precisely the cokernel of f!

Remark 16.1.8: Think about this last diagram: what happens when you mod out by larger modules?

Exercise 16.1.9 (Colimits always exist)
Suppose I is a discrete category, i.e. $\mathrm{Hom}(i,j) = \emptyset$ unless $i = j$, in which case $\mathrm{Hom}(i,i) = \{\mathbb{1}_i\}$. Supposing that $A : I \to \mathcal{A}$, show that $\mathrm{colim}_{i \in \mathcal{I}} = \coprod_i A_i$.

Definition 16.1.10 (?)
A category $\mathcal{A}$ is **cocomplete** if every colimit $\mathrm{colim}_{i \in \mathcal{I}} A_i$ exists for every $A \in \mathcal{A}^{\mathcal{I}}$ and all small categories $\mathcal{I}$.

Exercise 16.1.11 (Taking colimits defines a functor for cocomplete categories)
Show that when $\mathcal{A}$ is cocomplete, $\mathrm{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ defines a functor.

Exercise 16.1.12 (Weibel 2.6.4)
Show that the functor colim is left-adjoint to the diagonal functor Δ, so there is an adjunction

$$\begin{tikzcd} \mathcal{A}^{\mathcal{I}} \arrow[r, shift left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "\Delta"] \end{tikzcd}.$$

Thus when $\mathcal{A}$ is abelian and colim exists, it is right-exact (since left-adjoints are always right-exact). Note that it's not exact in general.

Proposition 16.1.13 ***(Cocomplete iff all coproducts exist).***
For any abelian category $\mathcal{A}$, the following are equivalent:

1. $\coprod A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$ (*set-indexed coproducts*).
2. $\mathcal{A}$ is cocomplete.

Remark 16.1.14: We'll prove this next time, note that $2 \implies 1$ since coproducts are special cases of limits.

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17.1 Colimits and Adjoints

Proposition 17.1.1 ***(Characterizations of cocomplete categories).***
Assume $\mathcal{A}$ is abelian so we have cokernels for maps. TFAE:

1. $\bigoplus A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$.

2. $\mathcal{A}$ is cocomplete, i.e. $\operatorname{colim}_{i \in I} A_i$ exists for every functor $\mathcal{I} \to \mathcal{A}$ with $\mathcal{I}$ small.

Proof (?).
Note that (1) is a special case of (2), so it suffices to show $1 \implies 2$. Given a functor $A : \mathcal{I} \to \mathcal{A}$ and let $f : \bigoplus_{\alpha i \to j} A_i \to \bigoplus_{i \in \mathcal{I}} A_i$ where $i, j \in \mathcal{I}$.

$$\begin{tikzcd} i \arrow[r, "\alpha"] \arrow[d, squiggly, "A"] & j \arrow[d, squiggly, "A"] & & \mathcal{I} \arrow[d, squiggly, "A"] \\ A_i \arrow[r, "\alpha_*"] & A_j & & \mathcal{A} \end{tikzcd}$$

Link to Diagram

Then the map $f(a_{i,\alpha}) = \alpha_*(a_i) - a_i \in A_j - A_i$, so this is $\alpha_* - \mathbb{1}$. Let $C \coloneqq \operatorname{coker} f \coloneqq \bigoplus_{i \in I} A_i / \operatorname{im}(f)$, and we'll denote elements in this quotient with a bar.

Claim: $C = \operatorname{colim}_{i \in I} A_i$ with

$$\begin{aligned} \eta_i : A_i &\to C \\ a_i &\mapsto \overline{a_i}, \end{aligned}$$

where we first embed A_i into the direct sum and then take the quotient.

Exercise *(?)*
Use the universal property of cokernels in $\mathcal{A}$. Check that the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[dr, "\eta_i"] \arrow[dd, "\alpha_*"'] & \\ & C \\ A_j \arrow[ur, "\eta_j"'] & \end{tikzcd}$$

This essentially follows from the fact that $\overline{\alpha_*(a_i)} = \overline{a_i}$.

■

Remark 17.1.3: Mod-R satisfies (1), since direct sums of R-modules still have an R-module structure. Thus Mod-R is cocomplete.

Definition 17.1.4 (Limits)
The **limit** of a functor $A : \mathcal{I} \to \mathcal{A}$ is the colimit of the dual functor $A^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.

Remark 17.1.5: Note that this amounts to reversing arrows in the conditions of a colimit. Many of the results for colimits go through with arrows reversed. Examples: kernels, direct products. If

I is a poset, then limits are referred to as **inverse limits**, using $\varprojlim_{i \in I} A_i$.

Definition 17.1.6 (Complete Categories)
$\mathcal{A}$ is **complete** if and only if $\lim_{i \in I} A_i$ exists whenever $\mathcal{I}$ is small and $A : \mathcal{I} \to \mathcal{A}$.

Theorem 17.1.7 ***(The Adjoint-Limit Theorem).***
Let $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ be an adjoint pair, where now $\mathcal{A}, \mathcal{B}$ are now arbitrary categories (not necessarily abelian). Then

- The **left adjoint** L preserves **colimits** (direct sums, cokernels, etc). I.e. if $A : \mathcal{I} \to \mathcal{A}$ has a colimit, then so does $(L \circ A) : \mathcal{I} \to \mathcal{B}$, and $L(\operatorname{colim} A_i) = \operatorname{colim}(LA_i)$.
- The **right adjoint** R preserves **limits** (direct products, kernels, etc).

Proof (?).
Not given in the book! See MacLane's *Categories for the Working Mathematician.* ■

Remark 17.1.8: Recall left adjoints are right-exact and have left-derived functors.

Corollary 17.1.9 ***(?).***
If $\mathcal{A}$ is a cocomplete abelian category with enough projectives and $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$. Then for every set-indexed collection of objects $\{A_i\}$,

$$(L_* F)\left(\bigoplus_{i \in I} A_i\right) = \bigoplus_{i \in I} L_* F(A_i)),$$

so left-derived functors commute with direct sums.

Proof (?).
Let P_i be the projective resolution of A_i, so $P_i \to A_i$, then $\bigoplus P_i \to \bigoplus A_i$ is a projective resolution, and by definition

$$\begin{aligned}
(L_* F)\left(\bigoplus A_i\right) &= H_*\left(F\left(\bigoplus P_i\right)\right) \\
&= H_*\left(\bigoplus F P_i\right) \quad \text{by the theorem} \\
&\cong \bigoplus H_*(F P_i) \text{homology commutes with } \oplus \in \mathsf{Ch}(\mathcal{A}) \\
&= \bigoplus_i L_* F(A_i).
\end{aligned}$$

■

Corollary 17.1.10 *(?).*
For $A_i \in \mathsf{R\text{-}Mod}, B \in \mathsf{Mod\text{-}R}$,

$$\mathrm{Tor}_*^R\left(\bigoplus_{i\in I} A_i, B\right) \cong \bigoplus_{i\in I} \mathrm{Tor}_*^R(A_i, B).$$

Proof (of corollary).

$$\mathrm{Tor}_*^R(-, B) = L_* F, \qquad F := (- \otimes_R B),$$

and F is a left-adjoint by the tensor-hom adjunction. ■

Remark 17.1.11: One can also show directly from the definition that

$$\mathrm{Tor}_*^R(A, \bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I} \mathrm{Tor}_*^R(A, B_i).$$

This uses the fact that $P \otimes_R (\bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I} (P \otimes B_i)$.

Remark 17.1.12: We'll skip the rest of this section, we (hopefully) won't need filtered colimits.

17.2 Balancing Tor and Ext

Remark 17.2.1: Idea: their derived functors with either variable fixed will essentially be the same. We'll start by showing that the two left-derived functors of $- \otimes_R -$ give the same results, and similarly for the two right-derived functors $\underset{R}{\mathrm{Hom}}(-,-)$. We'll use double complexes!

17.2.1 Tensor Product Complexes

Remark 17.2.2: Suppose we have two chain complexes $(P)_R \in \mathsf{Ch}(\mathsf{Mod\text{-}R}), {}_R(Q) \in \mathsf{Ch}(\mathsf{R\text{-}Mod})$. Then there is a double complex where i, j indexes rows and columns: $P \otimes_R Q = \{P_i \otimes_R Q_j\}_{i,j}$, the **tensor product double complex** of P and Q. We use the sign trick from 1.2.5:

- $d^h := d^P \otimes \mathbb{1}$
- $d^v := (-1)^i 1 \otimes d^Q$

Taking the direct sum totalization $\mathrm{Tor}^{\oplus}(P \otimes_R Q)$ is the **total tensor product chain complex** of P and Q. Note that this has a single differential! The big theorem from this section:

Theorem 17.2.3 *(Tor is balanced).*

$$L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A) := \operatorname{Tor}_n^R(A, B).$$

Remark 17.2.4: Note that this makes the right-hand side notation unambiguous.

Proof (?).
Choose projective resolutions $P \xrightarrow{\varepsilon} A \in \mathsf{Mod}\text{-}\mathsf{R}$ and $Q \xrightarrow{\eta} B \in \mathsf{R}\text{-}\mathsf{Mod}$. We'll form 3 tensor product double complexes.

- $P \otimes Q$: A first quadrant double complex, since the projective resolutions have nonnegative indices.
- $A \otimes Q$, embedding $A \hookrightarrow \mathsf{Ch}(\mathcal{A})$ as a complex concentrated in degree 0 (so one column)
- $P \otimes B$ (one row).

There are several maps of double complexes among these induced by ϵ, η:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A \otimes Q_2 & P_0 \otimes Q_2 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_1 \otimes Q_2 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_2 \otimes Q_2 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & \cdots \arrow[l] \\
A \otimes Q_1 & P_0 \otimes Q_1 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_1 \otimes Q_1 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_2 \otimes Q_1 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & \cdots \arrow[l] \\
A \otimes Q_0 & P_0 \otimes Q_0 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes \eta"] & P_1 \otimes Q_0 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes \eta"] & P_2 \otimes Q_0 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes \eta"] & \cdots \arrow[l] \\
 & P_0 \otimes B & P_1 \otimes B & P_2 \otimes B &
\end{tikzcd}$$

Link to Diagram

We'll show there are two maps:

$$A \otimes Q = \operatorname{Tot}(A \otimes Q) \xleftarrow{\varepsilon \otimes \mathbb{1}} \operatorname{Tor}(P \otimes Q) \xrightarrow{1\otimes\eta} \operatorname{Tor}(P \otimes B) = P \otimes B,$$

using that totalizing a one-row or one-column complex is summing along diagonals where each has one term, yielding actual equality of the first and last terms respectively above. Moreover, we'll show these are **quasi-isomorphisms**, and so

$$L_*(A \otimes -) \xleftarrow{\varepsilon\otimes\mathbb{1}} H_*(\mathrm{Tor}(P \otimes Q)) \xrightarrow{\mathbb{1}\otimes\eta} L_*(- \otimes B)(A).$$

We'll continue with the proof of this next time. ■

18 | Wednesday, February 24

18.1 Finishing the Proof of Balancing Tor

We were trying to prove that taking the left derived functors of the two slots in Tor yield the same thing.

See the diagram from last time!

Proof (?).
We'll need the following:

Claim: This induces a *quasi-isomorphism*

$$P \otimes B \xleftarrow{\mathbb{1}\otimes\eta} \mathrm{Tor}(P \otimes Q) \xrightarrow{\varepsilon\otimes\mathbb{1}} \mathrm{Tot}(A \otimes Q) = A \otimes Q,$$

i.e. it is a morphism that induces an isomorphism on homology.

Recall that by Corollary 1.5, a chain complex is a quasi-isomorphism if and only if the cone complex is acyclic/exact. In degree n of the total complex, the nth piece is the nth diagonal and we have

$$(P_n \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_n).$$

where $P_0 \xrightarrow{\varepsilon\otimes\mathbb{1} A\otimes Q_n}$. Recall that for a map $B_n \xrightarrow{f} C_n$, the cone complex was given by

$$\begin{tikzcd}
B_{n-1} \arrow[d, "-d^B"] \arrow[drr, "-f"] & \oplus & C_n \arrow[d, "d^C"] \\
B_{n-2} & \oplus & C_{n-1}
\end{tikzcd}$$

Link to Diagram

Writing one term out explicitly, we have

$$\begin{tikzcd}
(P_{n-1} \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_{n-1}) \oplus Q_n \arrow[d, "-d^{\oplus}"] \arrow[drr, "-(\varepsilon \otimes \mathbb{1})"'] & \oplus & A \otimes Q_n \arrow[d, "\mathbb{1} \otimes d"] \\
(P_{n-1} \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_{n-2}) \oplus Q_{n-1} & \oplus & A \otimes Q_{n-1}
\end{tikzcd}$$

Link to Diagram

Call this complex (2).

Fix spacing

On the other hand, consider the double complex obtained from $P \otimes Q$ by adjoining the shifted complex $(A \otimes Q)[1, 0]$[a] in column $i = -1$. This has the effect of keeping the same complex but relabeling left-most column "in degree 0" into "degree -1. Note that this negatives the leftmost vertical differentials $A \otimes Q_n \to A \otimes Q_{n-1}$. Now call everything above the dotted line C.

Consider $\mathrm{Tot}(C)[-1]$, which in degree n is $(\mathrm{Tot}(C))_{n-1}$ and since this was an odd shift, negates all of the signs of differential. So in degree n, this explicitly looks like

$$n : \quad (P_{n-1} \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_{n-1}) \oplus (A \otimes Q_n)$$

$$n : \quad (P_{n-1} \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_{n-1}) \oplus (A \otimes Q_n)$$

.

and we have

$$\begin{tikzcd}
(P_{n-1} \otimes Q_0) \oplus \cdots \arrow[d, "-d^{\oplus}"] & \oplus (P_0 \otimes Q_{n-1}) \arrow[drr, "-(\varepsilon \otimes \mathbb{1})"'] & \oplus & A \otimes Q_n \arrow[d, "\mathbb{1} \otimes d"] \\
(P_{n-1} \otimes Q_0) \oplus \cdots \oplus (P_0 \otimes Q_{n-2}) & & \oplus & A \otimes Q_{n-1}
\end{tikzcd}$$

Link to Diagram

Calling this complex (3), we have (3) = (2), so it suffices to show (2) is exact, i.e. $\mathrm{Tot}(C)$ is acyclic. This follow from the next result we'll prove, the *acyclic assembly lemma*. Note that if Q_j is projective, then it's an algebra fact that $- \otimes_R Q_j$ is exact (not just right exact) since projective implies flat. This implies that the rows of C are exact, since this is taking a project resolution (which is exact) and tensoring with a flat module. Using that C is supported on the upper half-plane and has exact rows, by this part (3) of the acyclic assembly lemma, $\mathrm{Tot}^{\oplus}(C)$ will be acyclic. A similar argument will go through to show that $\mathbb{1} \otimes \eta$ is also a quasi-isomorphism by adjoining $(P \otimes B)$ as the -1st row and applying a version of the lemma for right half-plane complexes with exact columns.

■

[a] The book may have the sign incorrect here.

18.2 Acyclic Assembly Lemma

Proposition 18.2.1 ***(Acyclic Assembly Lemma).***
Let C be a double complex in $\mathsf{Mod\text{-}R}$, then

- $\operatorname{Tot}^{\prod}(C)$ is acyclic if either
  1. C is upper half-plane with exact columns, or
  2. C is right half-plane with exact rows.
- $\operatorname{Tot}^{\oplus}(C)$ is acyclic if either
  3. C is upper half-plane with exact rows[a], or
  4. C is right half-plane with exact columns.

[a]This is the part we used previously, and (4) is the one used for the other half of the argument.

Remark 18.2.2: It suffices to prove (1). Interchanging rows and columns by reflecting along the line $i = j$ interchanges the types showing up in (1) and (2), and doesn't change the total complex. This similarly switches (3) and (4), so we have $1 \implies 2$ and $4 \implies 3$, so we'll show that $1 \implies 4$. Let $\tau_n C$ be the double complex obtained taking a *good truncation* of C at level n:

$$(\tau_n C)_{ij} \coloneqq \begin{cases} C_{ij} & j > n \\ \ker(d^v : C_{i,n} \to C_{i,n-1} & j = n. \end{cases}.$$

Up to translation $\tau_n C$ is a 1st quadrant complex, and since we're in case (4), we're assuming the columns are exact. Now using (1), $\operatorname{Tot}^{\oplus}(\tau_n C) = \operatorname{Tot}^{\prod}(\tau_n C)$ since we now have a first quadrant complex and all diagonals are finite, and we can conclude both are exact. This implies that $\operatorname{Tot}^{\oplus} C$ is acyclic since every cycle in $\operatorname{Tot}^{\oplus}(C)$ is nonzero in only finitely many terms. Thus each such cycle is a cycle in $\operatorname{Tot}(\tau_n C)$ for some $n \ll 0$, and hence a boundary by the previous argument.

Remark 18.2.3: Note that this argument does not go through for the direct product, since then there may be infinitely many nonzero terms on any diagonal, and not every cycle would be represented after some finite truncation and shift.

Proof (of proposition).
By translating C left or right, it's enough to prove that $H_0 \operatorname{Tot}^{\prod} C = 0$. We can write

$$(\operatorname{Tot}^{\prod} C)_0 = \prod_{j \geq 0} C_{-j,j} \ni c \coloneqq (\cdots, c_{-j,j}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0}),$$

letting the latter element by a 0-cycle. By inducting on j, we'll construct an element b such that $b_{-j,j+1} \in C_{-j,j+1} \subseteq (\operatorname{Tot}^{\prod} C)_1$ such that

$$d^v(b_{-j,j+1}) + d^h(b_{-j+1,j}) = c_{-j,j},$$

which will make c a boundary.

■

19 | Friday, February 26

Today: trying to prove acyclic assembly lemma

Proof (Of acyclic assembly lemma).
We reduced to proving one case, where C is a double complex upper half-plane with exact columns $\implies$ $\mathrm{Tot}^{\prod}(C)$ is acyclic. It's enough to check in degree 0 by shifting. Fix a 0-cycle $\mathbf{c} = (\cdots, c_{-j,j}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0})$. Find $b \in \prod_{j \leq 0} C_{-j,j+1}$\$ such that $d(b) = c$, so $c_{-j,j} = d^v(b_{-j,j+1}) + d^h(b_{-j+1,j})$.

$$\begin{tikzcd}
b_{-j,j+1} & & & & & & \\
c_{-j,j} \arrow[d, "d^v"] & b_{-j+1,j} & & & & & \\
0 & c_{-j+1,j-1} \arrow[l, "d^h"'] & b_{-j+2,j-1} & & & & \\
& & & \ddots & & & \\
& & & & c_{-2,2} & b_{-1,2} & \\
& & & & & c_{-1,1} & b_{0,1} \\
& & & & & & c_{0,0} \arrow[dd] \\
\bullet & & & & & & \\
& & & & & & 0
\end{tikzcd}$$

Link to Diagram

Construct by induction on j: set $b_{1,0} = 0$ and need $c_{0,0} = d^v(b_{0,1})$. Since $d^v c_{0,0} = 0$ and the columns are exact, we can lift this to some $b_{0,1}$ such that $d^v b_{0,1} = c_{0,0}$. Inductively, we want $d^v(b_{-j,j+1}) = c_{j,-j} - d^h(b_{-j+1,j})$. Then

$$\begin{aligned}
d^v(c_{j,-j} - d^h b_{-j+1,j}) &= d^v c_{j,-j} + d^h d^v b_{-j+1,j} \\
&= d^v c_{j,-j} + d^h \left(c_{-j+1,j-1} - d^h b_{-j+2,j-1}\right) \\
&= d^v c_{j.-j} + d^h c_{-j+1,j-1} \\
&= 0 \text{ since } d^{\prod} = 0.
\end{aligned}$$

By exactness of column j, we can lift to $b_{-j,j+1}$, making c a boundary. ■

Remark 19.0.1: This proves that $- \otimes_R -$ is balanced, i.e. taking the derived functors in either variable with the same pair (A, B) results in the same thing. To prove a similar result for hom and ext, we want to consider $\underset{R}{\mathrm{Hom}}(A, -)$ which requires injective resolutions, and $\underset{R}{\mathrm{Hom}}(-, B)$ is contravariant and left-exact, so we take an injective resolution in $\mathcal{C}^{\mathrm{op}}$, i.e. a projective resolution in $\mathcal{C}$. So take a projective resolution $P \to A$ and an injective resolution $B \to I$ and make a first quadrant double complex $C_{i,j} \coloneqq \mathrm{Hom}(P_i, I^j)$ for $i, j \geq 0$. Define the differentials using the following sign convention:

$$
\begin{tikzcd}
(-1)^{i+j+1} d_I f(p) & \mathrm{Hom}(P_i, I^{j+1}) & \\
& \mathrm{Hom}(P_i, I^j) \arrow[u, "d^v"] \arrow[r, "d^h"'] & \mathrm{Hom}(P_{i+1}, I^j) \\
f(p) \arrow[uu, mapsto] \arrow[rr, mapsto] & & f(d^P p)
\end{tikzcd}
$$

Link to Diagram

Now applying a dual argument as the one for tor yields a "dual acyclic assembly lemma".

Remark 19.0.2: We'll skip the first 3 sections of chapter 3. It's worth looking at 3.2 on tor and flatness. There's a slightly circular statement that projective implies flat in the book, since we used this to show that certain rows were exact, so refer to a good algebra book for alternative proofs.

19.1 Ext^1 and Extensions

Definition 19.1.1 (Module Extensions)
Let $A, B \in \textsf{Mod-R}$, then an **extension of A by B** is a SES

$$\xi : 0 \to B \to X \to A \to 0.$$

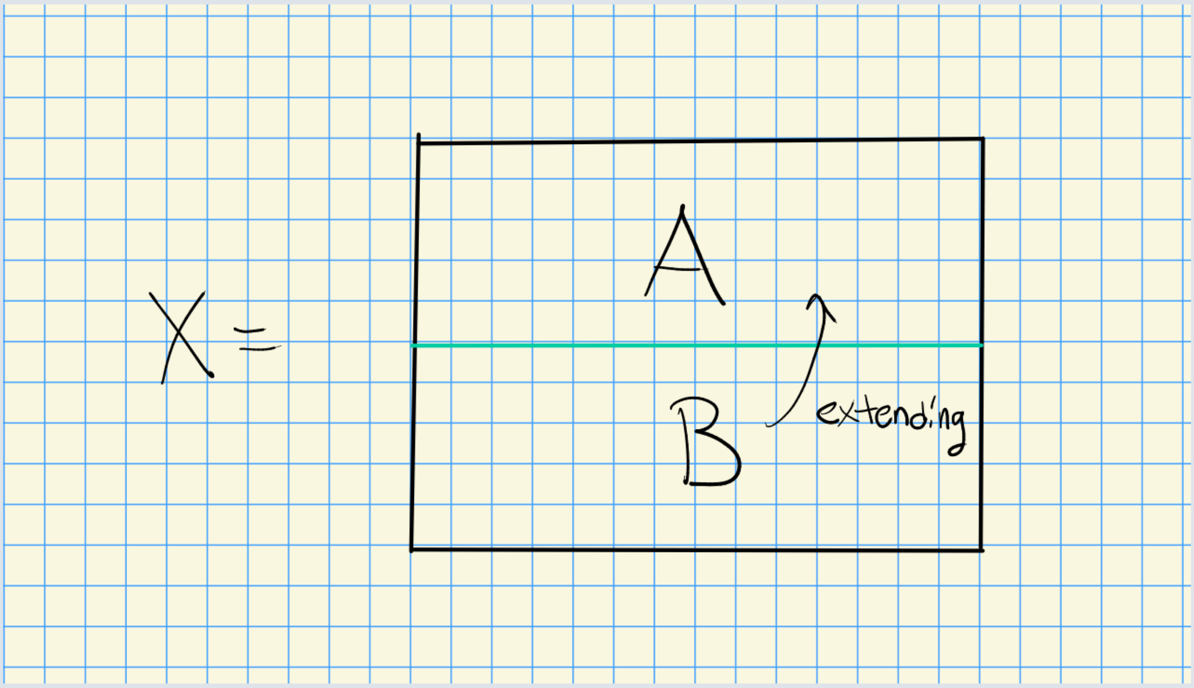


Figure 1: image_2021-02-26-09-41-27

We say two extensions ξ, ξ' are equivalent and write $\xi \sim \xi'$ iff

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "\exists"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

An extension is **split** if and only if it is equivalent to

$$0 \to B \overset{\iota}{\hookrightarrow} A \oplus B \to A \overset{\pi}{\twoheadrightarrow} A \to 0.$$

⚠ Warning 19.1.2

Note that a SES as above is related to $\operatorname{Ext}(A, B)$, which reverses the order!

Lemma 19.1.3 *(?)*.
If $\operatorname{Ext}^1(A, B) = 0$ then every extension of A by B is split.

⚠ Warning 19.1.4

There are lots of corrections needed to this proof in Weibel!

Proof (of lemma).

Given an extension ξ, look at the LES associated to $\overset{*}{\mathrm{Hom}}(-,B)$:

$$
\begin{tikzcd}
 & & \cdots & \mathrm{Ext}^1(A,B) \arrow[l] \\
 & \mathrm{Hom}(B,B) \arrow[rru, bend left] & \mathrm{Hom}(X,B) \arrow[l] & \mathrm{Hom}(A,B) \arrow[l] \\
 & \mathbb{1}_B & \sigma \arrow[l, mapsto]
\end{tikzcd}
$$

Link to Diagram

However, this gives a splitting:

$$
\begin{tikzcd}
 & B & & & \\
0 \arrow[r] & B \arrow[u, "\mathbb{1}_B"] \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed, "\exists"] \arrow[lu, dashed, "\exists\sigma"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Todo: label $(X,B) \to (B,B)$ as f_*.
This is one of the many equivalent criteria for a SES of modules to be split. ■

Remark 19.1.5: More generally, given ξ, let $\Theta(\xi) := \partial(\mathbb{1}_B) \in \mathrm{Ext}^1(A,B)$. Thus TFAE:

- ξ is split
- $\mathbb{1} - B$ lifts to some $\sigma \in \mathrm{Hom}(X,B)$
- $\mathbb{1}_B \in \mathrm{im}\, f_* = \ker \partial$
- $\Theta(\xi) = 0$, even if $\mathrm{Ext}^1(A,B) \neq 0$.

Then $\Theta(\xi)$ is an *obstruction* to ξ being split.

Remark 19.1.6: If $\xi' \sim \xi$ then $\partial'(\mathbb{1}_B) = \partial(\mathbb{1}_B) \in \mathrm{Ext}^1(A,B)$ by naturality of the connecting morphisms. So equivalent extensions have the same obstruction, i.e. Θ only depends only on the equivalence class $[\xi]$ of the SES.

Theorem 19.1.7 *(Module extensions correspond to Ext groups).*
Given $A, B \in \mathsf{Mod\text{-}R}$ (or an abelian category with enough projectives and injectives), there is a correspondence

$$\left\{ 0 \to B \to X \to A \to 0 \right\}_{/\sim} \underset{\Theta}{\overset{\Psi}{\rightleftharpoons}} \operatorname{Ext}^1(A, B)$$

Note that this is a bijection of sets, but we'll upgrade it to a bijection of abelian groups.

20 | Monday, March 01

Remark 20.0.1: Last time: we looked at group extensions. Given $\xi : 0 \to B \to X \to A \to 0$, we had a canonical element in $\operatorname{Ext}^1(A, B)$, namely $\Theta(\xi) = \delta(\mathbb{1}_B)$. This only depends on the equivalence class of ξ.

Theorem 20.0.2 *(Module extensions biject with Ext groups).*
Given $A, B \in \mathsf{Mod\text{-}R}$, there is a bijection

$$\left\{ \text{Extensions of } A \text{ by } B \right\} \underset{\Theta}{\overset{\Phi}{\rightleftharpoons}} \operatorname{Ext}^1_R(A, B)$$

Proof (?).

Claim: Θ is surjective.
Fix a SES

$$0 \to M \xrightarrow{j} P \xrightarrow{\pi} A \to 0$$

with P projective, and take the LES resulting from applying $\operatorname{Hom}(-, B)$:

$$\begin{tikzcd}
0 \arrow[d] & & \\
\operatorname{Hom}(A,B) \arrow[r] & \operatorname{Hom}(P,B) \arrow[r] & \operatorname{Hom}(M,B) \arrow[dll, "\partial"] \\
\operatorname{Ext}^1(A,B) \arrow[r] & \operatorname{Ext}^1(P,B) = 0 & \\
x & &
\end{tikzcd}$$

Link to Diagram

Letting $x \in \operatorname{Ext}^1(A, B)$ and choose $\beta \in \operatorname{Hom}(M, B)$ with $\partial\beta = x$ using that P is projective and thus $\operatorname{Ext}^1(P, B)$ vanishes. Now let X be the **pushout** of $j : M \to P$ and $\beta : M \to B$. Note that we can apply the universal property of cokernels to get a map of the following form:

$$\begin{tikzcd}
M \arrow[r, "{g=(j,-\beta)}"] \arrow[dr, dotted, "\therefore 0"'] & P \oplus B \arrow[r] \arrow[d, "\pi\oplus 0"] & X = \operatorname{coker} g \arrow[r] \arrow[dl, dashed, "\exists!\mu"] & 0 \\
& A & &
\end{tikzcd}$$

Link to Diagram

Taking the pushout yields a diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r, "j"'] \arrow[d, "\beta"'] & P \arrow[r, "\pi"'] \arrow[d, "\sigma"] \arrow[dl, phantom, "\ulcorner", very near start] & A \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & B \arrow[r, "\iota"'] & X \arrow[r, "\mu"'] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Exercise *(?)*
Check that this diagram commutes and that the new row is exact.

Taking the LES for $\operatorname{Hom}(-, B)$ yields

$$\begin{tikzcd}
& & \beta \arrow[r, mapsto] \arrow[ddd, bend left, mapsto] & x \arrow[ddd, bend left, dashed, mapsto] \\
\cdots \arrow[r] & \operatorname{Hom}(P,B) \arrow[r] & \operatorname{Hom}(P,B) \arrow[r] & \operatorname{Ext}^1(A,B) \arrow[d, dash] \\
\cdots \arrow[r] & \operatorname{Hom}(X,B) \arrow[r] \arrow[u, "\sigma_*"] & \operatorname{Hom}(B,B) \arrow[r] \arrow[u, "\beta_*"] & \operatorname{Ext}^1(A,B) \\
& & \mathbb{1}_B \arrow[r, mapsto, "\partial"] & \Theta(\xi)
\end{tikzcd}$$

$(*)$ Link to Diagram

So we

- Started with x
- Took a reference SES
- Produce the cokernel
- Took a pushout and found β.
- Showed that $\beta \mapsto x$.

Review video: 9:28 AM!

This shows surjectivity, but depended on choice of β.

Claim: Θ is injective.

Note that the previous construction there is a way to associate to $x \in \operatorname{Ext}^1(A,B)$ an extension of A by B. To see that this gives a well-defined map Ψ, so $\Psi(x) = [\xi]$ as well, suppose $\beta' \in \operatorname{Hom}(M,B)$ is another lift of x. Note that although $\operatorname{Ext}^1(P,B) = 0$, the fact that $\ker \partial = \operatorname{Hom}(M,B) \neq 0$, there are many such choices of lifts. Using exactness of diagram $(*)$, there exists an $f \in \operatorname{Hom}(P,B)$ such that $\beta' = \beta + fj$, recalling that $j: M \to P$. Now taking the pushout X' of j and β', the maps $i: B \to X$ and $\sigma + if : P \to X$ induce an isomorphism $X' \xrightarrow{\sim} X$ and thus an equivalence $\xi \xrightarrow{\sim} \xi'$.

Exercise *(?)*
Check this isomorphism.

Moreover, given any extension ξ, we can fit it into a diagram of the following form:

$$\begin{tikzcd}
& 0 \arrow[r] & M \arrow[r] \arrow[d, dashed, "\exists \beta"] & P \arrow[r] \arrow[d, dashed, "\exists \sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi: & 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

First we use projectivity of P to get $\sigma : P \to X$. Then restricting σ to the kernels of π, μ respectively makes $\beta : M \to B$, so this diagram commutes

Exercise *(?)*
Check that X is the pushout of j and β.

It follows that $\Psi(\Theta(\xi)) = \xi$ and thus Θ is injective, making it a bijection.

■

Remark 20.0.6: Note the importance of the reversed directions after taking the Hom!

Remark 20.0.7: How can we upgrade this to a group homomorphism? One way is to pull back the group structure from the right-hand side to the left-hand side, but it turns out that Baer worked out an intrinsic group structure around 1934. We can construct the "smallest" extension such that A is a quotient and B is a submodule.

Definition 20.0.8 (Baer Sum (1934))
Suppose we have two extensions of A by B:

$$\begin{aligned}
\xi &: 0 \to B \xrightarrow{i} X \xrightarrow{\pi} A \to 0 \\
\xi' &: 0 \to B \xrightarrow{i'} X' \xrightarrow{\pi'} A \to 0
\end{aligned}$$

.

Let X'' be the **pullback** of π, π', defined by

$$X'' := \left\{ (x, x') \in X \times X' \;\middle|\; \pi(x) = \pi'(x') \in A \right\},$$

which identifies the two copies of A. This fits into a cartesian square

$$\begin{tikzcd}
X'' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "\pi'"] \\
X \arrow[r, "\pi"] & A
\end{tikzcd}$$

Link to Diagram

Note that X'' contains 3 copies of B:

- $B \times 0$, or really $i(B) \times \{0\} \subset X''$ (using exactness).
- $0 \times B$, i.e. $\{0\} \times i'(B) \subseteq X''$ (using exactness).
- $\tilde{\Delta} = \left\{(-b, b) \;\middle|\; b \in B\right\}$, the **skew diagonal**. One can check that $\pi i(-b) = 0 = \pi' i'(b)$.

Note that we're identifying B with $i(B), i'(B)$. Set $Y := X''/\tilde{\Delta}$, then $(b, 0) + (-b, b) = (0, b)$ where $(-b, b) \in \tilde{\Delta}$, so $B \times 0$ and $0 \times B$ have the same image in Y, since

$$(B \times 0) \cap \tilde{\Delta} = \{(0,0)\} = (0 \times B) \cap \tilde{\Delta}.$$

In fact this image in Y is isomorphic to B, by construction of what we're quotienting out by. Denoting this subgroup of Y by B, we get a SES

$$\varphi : 0 \to B \to Y \to Y/B \to 0.$$

What is Y/B? We can write this as

$$Y/B = \frac{X''/\tilde{\Delta}}{(0 \times B)/\tilde{\Delta}} \cong \frac{X''}{(0 \times B) + \tilde{\Delta}} \cong \frac{X''/0 \times B}{(\tilde{\Delta} + (0 \times B))/(0 \times B)}.$$

But the numerator is isomorphic to X by π_1, and the denominator is isomorphic to B by π_1. So φ is an extension of A by B called the **Baer sum** of ξ, ξ'.

Corollary 20.0.9 *(?)*.
The equivalence classes of extensions of A by B is an abelian group under Baer sums, where zero is the class of split extensions. Moreover, the map Θ from the previous theorem is an isomorphism of abelian groups.

Remark 20.0.10: Next time we'll check this by showing $\Theta(\varphi) = \Theta(\xi) + \Theta(\xi')$.

21 | Wednesday, March 03

21.1 Baer Sum and Higher Exts

Last time: Baer sum.

Remark 21.1.1:

$$\begin{tikzcd}
\xi' : & 0 \arrow[r] & B \arrow[r, "\iota'"] & X' \arrow[r, "\pi'"] & A \arrow[r] & 0 \\
\text{Ref} : & 0 \arrow[r] & M \arrow[r, "j"] \arrow[u, "\beta'"] & P \arrow[r, "\pi"] \arrow[u, "\sigma'"'] & A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Link to Diagram

$$\begin{tikzcd}
 & & \mathbb{1}_B \arrow[r, mapsto] \arrow[dd, bend left, mapsto] & \Theta(\xi') \arrow[dd, bend left, mapsto] \\
\cdots \arrow[r] & \mathrm{Hom}(X', B) \arrow[r] \arrow[d, "\sigma'_*"] & \mathrm{Hom}(B, B) \arrow[r] \arrow[d, "\beta'_*"] & \mathrm{Ext}^1_R(A, B) \arrow[d, equal] \\
\cdots \arrow[r] & \mathrm{Hom}(P, B) \arrow[r] & \mathrm{Hom}(M, B) \arrow[r] & \mathrm{Ext}^1_R(A, B) \\
 & & \beta' \arrow[r, mapsto] & \partial(\beta') = \Theta(\xi')
\end{tikzcd}$$

Link to Diagram

We want to define $\xi' \oplus \xi''$, An important takeaway is that Θ can alternatively be defined as a map induced by the original boundary map coming from the SES, i.e. $\partial(\beta') = \Theta(\xi')$. This fits into the diagram as follows:

$$\begin{tikzcd}
\xi' : & 0 \arrow[r] & B \arrow[r, "\iota'"] & X' \arrow[r, "\pi'"] & A \arrow[r] & 0 \\
\mathrm{Ref} : & 0 \arrow[r] & M \arrow[r, "j"] \arrow[u, "\beta'"] \arrow[d, "\beta''"'] & P \arrow[r, "\pi"] \arrow[u, "\sigma'"] \arrow[d, "\sigma''"'] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
\xi'' : & 0 \arrow[r] & B \arrow[r, "\iota''"] & X'' \arrow[r, "\pi''"] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

We define

$$\tilde{X} \coloneqq \left\{ (x', x'') \in X' \times X'' \;\middle|\; \pi'(x') = \pi''(x'') \right\} \twoheadrightarrow Y,$$

and note that we had a skew diagonal $\tilde{\Delta} \subseteq \tilde{X}$. This yields a YES

$$\varphi : 0 \to B \to Y \to Y/B \cong A \to 0.$$

Corollary 21.1.2 *(?).*
The set of equivalence classes of extensions of A by B is an abelian group under the Baer sum, where

$$[\xi] \oplus [\xi'] \coloneqq [\varphi],$$

where the identity element 0 is the class of split extensions. The map Θ is an isomorphism of abelian groups.

Remark 21.1.3: One should check that this is well-defined since we're using equivalence classes. There is a fast way to do both at once, i.e. showing Θ is well-defined and also a group morphism.

Proof (?).
We'll show that

$$\Theta(\varphi) = \Theta(\xi) + \Theta(\xi'') \in \operatorname{Ext}^1_R(A, B),$$

which will make it a group isomorphism since Θ was already a set bijection. Considering commutativity in the 3-row diagram, we can get a well-defined map

$$\sigma \coloneqq \sigma' \oplus \sigma'' : P \to \tilde{X}.$$

So let $\bar{\sigma} : P \to Y$ be the induced map. The restriction of $\bar{\sigma}$ to M is induced by the map

$$\beta' + \beta'' : M \to (B \times 0) + (0 \times B) \subseteq \tilde{X}.$$

These both map to B in Y under the SES $0 \to B \to Y \to Y/B \to 0$. This gives a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\beta'+\beta''"'] & P \arrow[r] \arrow[d, "\bar{\sigma}"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

We then have $\Theta(\varphi) = \partial(\beta' + \beta'') = \partial(\beta') + \partial(\beta'')$ using that $\partial \in \text{Mor}(\textsf{R-Mod})$. But this is equal to $\Theta(\xi') + \Theta(\xi'')$, which is what we wanted to show.

■

Remark 21.1.4: What about the 0 element for split SESs? Recall that additive functors preserve split exact sequences, since these are just in terms of sums of maps composing to the identity. Then applying the hom functor to the original SES produces another SES, which in particular has no Ext correction term.

Remark 21.1.5: Similarly, $\text{Ext}^n(A, B)$ is identified with equivalence classes of longer sequences with $n+2$ terms, and an equivalence is a sequence of maps that result in commuting squares:

$$
\begin{tikzcd}
\xi : & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_n \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] \arrow[d, dashed] & X_1 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi' : & 0 \arrow[r] & B \arrow[r] & X'_n \arrow[r] & \cdots \arrow[r] & X'_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Note that if $P_* \to A \to 0$ is a projective resolution, then the comparison theorem yields maps and a commutative diagram

$$
\begin{tikzcd}
\varphi : & 0 \arrow[r] & M \arrow[r] \arrow[d, dashed, "\exists\beta"] & P_{n-1} \arrow[r] \arrow[d, dashed, "\exists"] & \cdots \arrow[r] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed, "\exists"] & A \arrow[r] \arrow[d, equal, "\mathbb{1}_A"] & 0 \\
\xi' : & 0 \arrow[r] & B \arrow[r] & X'_n \arrow[r] & \cdots \arrow[r] & X'_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Then the dimension shifting theorem (Exc. 2.4.3) and its proof yields an exact sequence

$$
\text{Hom}(P_{n-1}, B) \to \text{Hom}(M, B) \xrightarrow{\partial} \text{Ext}^n(A, B) \to 0,
$$

and the asserted bijection is then given by $\Theta(\xi) := \partial(\beta)$.

21.2 3.6: Kunneth and Universal Coefficient Theorems

Observation 21.2.1
If R is a field F then $\mathrm{Tor}_n^F(A,B)=0$ for all $n>0$, i.e. every module over a field is a complex space, hence free, hence projective, hence flat, and so $A\otimes_F -$ is exact.

Question 21.2.2
If $P_* \in \mathsf{Ch}(\textsf{Mod-R})$ is a complex of of right R-modules and $M \in \textsf{R-Mod}$ is a left R-module, how is the homology of P_* and that of $P_* \otimes_R M$ related?

Lemma 21.2.3 *(?)*.
Given a 5-term exact sequence

$$A_1 \xrightarrow{\alpha} A_2 \xrightarrow{f} B \xrightarrow{g} C_1 \xrightarrow{\gamma} C_2,$$

there is a corresponding SES

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\bar{f}"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

$$\substack{A_2/\ker f = A_2/\operatorname{im}\alpha \\ = \operatorname{coker}\alpha} \qquad\qquad \operatorname{im} g = \ker f$$

Link to Diagram

In particular, we can always take $A=\operatorname{coker}\alpha$ and $C=\ker\gamma$ in any abelian category.

Theorem 21.2.4 *(The Kunneth Formula)*.
Let $P_* \in \mathsf{Ch}(\textsf{Mod-R})$ be a chain complex of flat right R-modules such that each boundary module dP_n is again flat. Then for every $M \in \textsf{R-Mod}$ and all N, there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_n(P_*)\otimes_R M \arrow[r] & H_n(P_*\otimes_R M) \arrow[r] & \mathrm{Tor}_R^1(H_{n-1}(P_*), M) \arrow[r] & 0 \end{tikzcd}$$

Link to Diagram

Remark 21.2.5: Note that the correction term vanishes if R is a field.

Proof (?).
Let $Z_n := Z_n(P_*)$, there there is a SES

$$0 \to Z_n \to P_n \xrightarrow{d} dP_n \to 0.$$

Since P_n, dP_n are flat by assumption, by Exc. 3.2.2, Z_n is also flat. Taking the LES from applying $- \otimes_R M$, noting that M is arbitrary yields

$$\begin{tikzcd}
 & & 0 \\
 Z_n \otimes_R M \arrow[r] & P_n \otimes_R M \arrow[r] & dP_n \otimes_R M \arrow[u] \\
 & \cdots \arrow[r] & \mathrm{Tor}_1(dP_n, M) \arrow[llu, out=0, in=180]
\end{tikzcd}$$

Link to Diagram

Here $\mathrm{Tor}_1(dP_n, M) = 0$ since dP_n is flat, noting that one could also apply $\mathrm{Tor}(dP_n, -)$ to get a similar LES. So this lifts to a SES of complexes

$$0 \to Z_* \otimes M \to P_* \otimes M \to dP_* \otimes M \to 0,$$

where we can consider $d \otimes \mathbb{1}$ in the middle. We'll pick this up next time! ■

22 | Friday, March 05

See first 10m

Observation 22.0.1
For a SES

$$A_1 \xrightarrow{\alpha} A_2 \xrightarrow{f} B \xrightarrow{g} C_1 \xrightarrow{\gamma} C_2,$$

one can obtain an exact sequence

$$0 \to \operatorname{coker} \alpha \xrightarrow{\bar{f}} B \xrightarrow{g} \ker \gamma \to 0.$$

Observation 22.0.2
For a SES

$$0 \to Y \xrightarrow{i} Z \xrightarrow{\pi} \frac{Z}{Y} \to 0$$

there is an induced exact sequence

Some missed stuff here.

Proof (of Kunneth Formula (continued)).
Note that

$$0 \to Z_* \otimes M \to P_* \otimes M \to dP_* \otimes M \to 0,$$

where the differentials for the end terms are zero, and the homology will recover the original complex.

$$\begin{tikzcd}
 & & & H_{n+} \arrow[dlll, out=0, in=180] \\
 & H_n(Z \otimes M) = Z \otimes M \arrow[r] & H_n(P \otimes M) \arrow[r] & H_n \arrow[dlll, out=0, in=180] \\
 & H_{n-1}(Z \otimes M) = Z_{n-1} \otimes M & &
\end{tikzcd}$$

Link to Diagram

By using the explicit formula for ∂, it turns out that $\partial = (dP_{i+1} \overset{i}{\hookrightarrow} Z) \otimes \mathbb{1}M$. By observation one, we get a SES

$$0 \to \frac{Z_n \otimes M}{dP_{n+1} \otimes M} \to H_n(P \otimes M) \to \ker i(\otimes \mathbb{1}_M) \to 0.$$

By observation 1, the first term equals $H_n(P_*) \otimes M$. From this, we get a flat resolution of $H_{n-1}(P)$:

$$\begin{tikzcd}
\text{deg}: & 2 & 1 & 0 & & \\
0 \arrow[r] & 0 \arrow[r] & dP_n \arrow[r] & Z_{n-1} \arrow[r] & H_{n-1}(P) \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

So we can use this to compute $\operatorname{Tor}(H_{n-1}(P), M)$ by taking homology:

$$\begin{tikzcd}
\text{deg} & 2 & 1 & 0 & \\
0 \arrow[r] & 0 \arrow[r] & dP_n \otimes M \arrow[r, "i \otimes \mathbb{1}"] & Z_{n-1} \otimes M \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Thus

$$\ker(i \otimes \mathbb{1}_M) = \operatorname{Tor}_1(H_{n-1}(P), M) \cong \ker(dP_m \xrightarrow{\partial} Z_{n-1} \otimes M).$$

■

Theorem 22.0.3 ***(Universal Coefficient Theorem).***
Let P_* be a chain complex of free abelian groups. For every abelian groups M and every n, the Kunneth sequence splits non-canonically as

$$H_n(P_* \otimes M) \cong (H_n(P_*) \otimes M) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M).$$

Remark 22.0.4: In optimal situations the tor term vanishes, e.g. if either term is torsionfree (so no elements of finite order).

Fact 22.0.5
Every subgroup of a free abelian group is free (hence projective, hence flat).

Proof (?).
Since $dP_n \leq dP_{n-1}$, we can conclude dP_n is free. Thus the following SES splits:

$$0 \to Z_n \to P_n \xrightarrow{d} dP_n \to 0.$$

So any lift of the identity map on dP_n gives an isomorphic copy of the last term in the middle term, yielding $P_n \cong Z_n \oplus dP_n$. Now tensoring with M and using that it distributes over direct sums yields

$$P_n \otimes M \cong (Z_n \otimes M) \oplus (dP_n \otimes M).$$

The left-hand side contains a copy of $\ker(d_n \otimes \mathbb{1} : P_n \otimes M \to P_{n-1} \otimes M)$, which itself contains a copy of $Z_n \otimes M$. So by a linear algebra exercise, we have $\ker(d_n \otimes \mathbb{1}) \cong (Z_n \otimes M) \oplus A$ for some unknown A, and since $dP_{n+1} \otimes M = \operatorname{im}(d_{n+1} \otimes \mathbb{1})$ is contained in the first term, we can use the partial exactness of tensoring to preserve quotients and obtain

$$H_n(P \otimes M) = (H_n(P) \otimes M) \oplus C'$$

for some C'. Now applying the Kunneth formula we find that $C' = \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$, yielding the claimed direct sum. ∎

Remark 22.0.6: The following is a generalization for both.

Theorem 22.0.7 ***(Kunneth formula for complexes).***
Let $P, Q \in \mathsf{Ch}(\mathsf{R\text{-}Mod})$ be complexes, then

$$P \otimes Q := \operatorname{Tot}^{\oplus}(P \otimes Q)_n := \bigoplus_{p+q=n} P_p \otimes Q_q$$

with differential[a]

$$d(a \otimes b) = (da) \otimes b + (-1)^p a \otimes (db).$$

If P_n, dP_n are flat for all n, then there exists a SES

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_q(Q) \to H_n(P \otimes Q) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(P), H_q(Q)) \to 0.$$

[a] Recall that the squares would commute if we took the usual differentials, so we use a sign trick to get $d^2 = 0$.

Proof (?).
Omitted here, but uses same ideas as the previous proofs. Hint: take Q to have M in degree 0.

■

22.1 Applications to Topology

Definition 22.1.1 (Simplicial Homology)
See some applications in section 1 of Weibel, e.g. simplicial and singular homology. The setup: $X \in \mathsf{Top}, R \in \mathsf{Ring}$ unital, and for $k \geq 0$ let $S_k = S_k(X)$ be the free R-module on $\operatorname{Hom}_{\mathsf{Top}}(\Delta_k, X)$ where Δ_k is the standard simplex By ordering the vertices, this induces an ordering on the faces by taking lexicographic ordering. Then the restriction of a map $\Delta_k \to X$ to the ith face of Δ_k gives a map $\Delta_{k-1} \to X$, which induces an R-module morphism $\partial_i : S_k \to S_{k-1}$ By summing these we can define $d := \sum_{i=0}^{k} (-1)^i \partial_i : S_k \to S_{k-1}$ and it turns out that $d^2 = 0$. So we can define a complex

$$\cdots \to S_2 \xrightarrow{d} S_1 \to S_0 \to 0 \in \mathsf{Ch}(\mathsf{R\text{-}Mod}).$$

Taking it homology yields the **simplicial homology** of the complex $H_n(X; R) := H_n(S_*(X))$.

Remark 22.1.2: Taking $R = \mathbb{Z}$ makes $S_k(X)$ a free abelian group. If M is any abelian group, we can define $H_n(X; M) := H_n(S_*(X) \otimes_{\mathbb{Z}} M)$, the homology with **coefficients** in M. If no coefficients are specified, we write $H_n(X) := H_n(X; \mathbb{Z})$. There is then a universal coefficient theorem in topology:

$$H_n(X; M) \cong (H_n(X) \otimes_{\mathbb{Z}} M) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), M).$$

Remark 22.1.3: Next week: group cohomology, spectral sequences next week. This will give us some objects to apply spectral sequences.

23 | Monday, March 08

23.1 3.6: Universal Coefficients Theorem

Remark 23.1.1: Let $X \in \mathsf{Top}$ and $S_k(X)$ be the free $\mathbb{Z}$-module on $\operatorname{Hom}_{\mathsf{Top}}(\Delta_k, X)$, which assemble into a chain complex $S(X)$. For $M \in \mathsf{Ab}$, we defined $H^n(X; M) := H^n(\operatorname{Hom}(S(X), M))$ and write $H^n(X) := H^n(X; \mathbb{Z})$. The universal coefficient theorem states

$$H^n(X; M) \cong \operatorname{Hom}_{\mathbb{Z}}(H_n(X), M) \oplus \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), M).$$

Warning 23.1.2
Note that this is homology on the RHS, not cohomology!

Theorem 23.1.3 ***(Universal Coefficients Theorem for Cohomology).***
Let P_* be a chain complex of projective R-modules. Assume dP_n is also projective for all n. For $M \in \mathsf{R\text{-}Mod}$, there is a split SES

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(P), M) \to H^n(\operatorname{Hom}_R(P_*, M)) \to \operatorname{Hom}_R(H_n(P), M) \to 0.$$

Ask about naturality!

Proof (Sketch).
As in the last lecture with free abelian groups, since the dP_n are projective we can split $P_n \cong Z_n \oplus dP_n$ since $Z_n = \ker d$. Applying homs, since it's an additive functor this yields a new split exact sequence

$$0 \to \operatorname{Hom}(dP_n, M) \to \operatorname{Hom}(P_n, M) \to \operatorname{Hom}(Z_n, M) \to 0.$$

Now running the proof for the original Kunneth formula and replacing tensor products to homs, these assemble into a split exact sequence of complexes and this yields the desired SES. Using the strategy of the proof of the UCF for free abelian groups to see that the sequence splits (although non-canonically). ■

Remark 23.1.4: Note that flat is weaker than projective for tensor products, but in an asymmetric situation, there's nothing weaker than projective for the hom functors to be exact (since this is an iff).

23.2 Ch. 6: Group Homology and Cohomology

23.2.1 Definitions and Properties

Definition 23.2.1 (Modules of Groups)
Let $G \in \mathsf{Grp}$ be any group, finite or infinite, and let $A \in \mathsf{G\text{-}Mod}$ be a left G-module, i.e. an abelian group on which G acts by additive maps on the left, written $g.a$ or ga for $g \in G, a \in A$. Here *additive* means that $g.(a_1 + a_2) = g.a_1 + g.a_2$. Note that this implies $g.0 = 0, -g.a = -(g.a), g_1(g_2.a) = (g_1g_2).a, 1_G.a = a$. Writing $\underset{R}{\operatorname{End}}(A) := \underset{R}{\operatorname{Hom}}(A, A)$, we have a group morphism

$$\begin{aligned} G &\to \underset{\mathbb{Z}}{\operatorname{End}}(A) \\ g &\mapsto g.(-). \end{aligned}$$

Definition 23.2.2 (Equivariant Maps)
If $B \in \mathsf{G\text{-}Mod}$ is another left G-module, then

$$\underset{G}{\operatorname{Hom}}(A, B) = \left\{ f \in \underset{\mathbb{Z}}{\operatorname{Hom}}(A, B) \;\middle|\; f(g.a) = g(f(a)) \quad \forall a \in A, \forall g \in G \right\},$$

which are G**-equivariant maps**.

Definition 23.2.3 (Integral Group Ring)
We define

$$\mathbb{Z}G := \left\{ \sum_{i=1}^{N} m_i g_i \;\middle|\; m_i \in \mathbb{Z}, g_i \in G, n \in \mathbb{N} \right\}.$$

We can equip this with a ring structure using $(mg)(m'g') = mm'gg'$ and extending $\mathbb{Z}$-linearly.

Remark 23.2.4: There is an equality of categories $\mathsf{G\text{-}Mod} = \mathbb{Z}\mathsf{G\text{-}Mod}$. This is also the same as the functor category $\mathsf{Ab}^{\mathcal{G}}$ (a category of the form $\mathcal{A}^{\mathcal{I}}$) where $\mathcal{G}$ is the category with one object whose morphisms are the elements of G. In other words, $\operatorname{Ob}(\mathcal{G}) := \{1\}$ and $\underset{\mathcal{G}}{\operatorname{Hom}}(1, 1) = G$. Note that every morphism is invertible since G is a group.

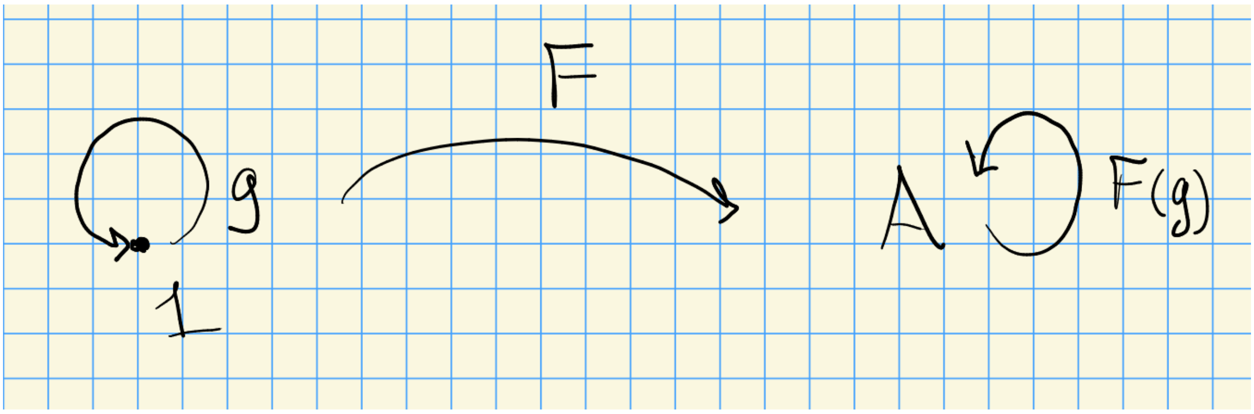


Figure 2: image_2021-03-08-09-36-58

The right-hand side yields a G-module since $F(g)(a) = g.a$.

Definition 23.2.5 (Trivial modules)
An object $A \in \mathsf{G\text{-}Mod}$ is a **trivial** module if and only if $g.a = a$ for all $g \in G$.

Remark 23.2.6: Any $G \in \mathsf{Ab}$ can be viewed as a trivial G-module in this way. This yields a functor $\mathrm{Triv} : \mathsf{Ab} \to \mathsf{G\text{-}Mod}$. There is a distinguished trivial G-module, namely $A := \mathbb{Z}$ with the trivial G-action. There are two natural functors $\mathsf{G\text{-}Mod} \to \mathsf{Ab}$:

- $A^G := \left\{ a \in A \;\middle|\; g.a = a \forall g \in G \right\}$, the **invariant subgroup** of A.
- $A_G := A / \left\langle ga - a \;\middle|\; g \in G, a \in A \right\rangle$, where we take the G-module generated by the relation in the denominator, which are the **coinvariants** of A.

Exercise 23.2.7 (6.1.1)

1. A^G is the maximal trivial submodule of A, so the functor $(-)^G$ is right-adjoint to Triv. These should both be easy checks! So this is left-exact and has right-derived functors (similar to ext).

2. A_G is the largest G-trivial quotient of A, and $(-)_G$ is left-adjoint to Triv. Thus it is right-exact and has left-derived functors (similar to tor).

Lemma 23.2.8 *(?)*.
Let $A \in \mathsf{G\text{-}Mod}$ and $\mathbb{Z}$ be the trivial G-module. Then

1. $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$, and
2. $A^G \cong \underset{G}{\mathrm{Hom}}(\mathbb{Z}, A)$ (**important!!**)

⚠ **Warning 23.2.9**

Number 2 above is important to remember!

Proof (of 1).
Viewing $\mathbb{Z} =_{\mathbb{Z}} \mathbb{Z}_{\mathbb{Z}G} \in (\mathbb{Z}, \mathbb{Z}\mathsf{G})\text{-}\mathsf{biMod}$ with the trivial structure, recall[a] that we have a functor

$$\underset{(}{\mathrm{Hom}}\,\mathbb{Z}, -) : \mathbb{Z}\text{-}\mathsf{Mod} \to \mathbb{Z}\mathsf{G}\text{-}\mathsf{Mod}$$

where $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, A)$ has an action $(g.f)(x) := f(x.g)$ for $x \in \mathbb{Z}g \in G$. Since $x.g = x$ for all x, g, we have $g.f = f$ and thus $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, A)$ is a trivial G-module, and there is an isomorphism in Ab:

$$\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, A) \xrightarrow[\mathsf{Ab}]{\sim} A$$
$$f \mapsto f(a).$$

Thus $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, -) \cong \mathrm{Triv}(-)$. By prop 2.6.3, the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} (-)$ is left-adjoint to $\underset{\mathbb{Z}}{\mathrm{Hom}}({}_{\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}G}, -)$. Now applying exercise 6.1.1 part 2, $(-)_G \cong \mathrm{Triv}(-)$. Since left-derived functors are universal δ-functors, we have a natural isomorphism $(-)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} (-)$ since they're both left-adjoint to the same functor.

■

[a] See Weibel p. 41.

Proof (of 2).
Taking $f(1)$, we have $A^G \cong \underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, A^G)$. Using the adjoint property from exercise 6.1.1 part 1, this is isomorphic to $\underset{G}{\mathrm{Hom}}(\mathrm{Triv}(\mathbb{Z}), A)$. Thus $(-)^G \cong \underset{G}{\mathrm{Hom}}(\mathbb{Z}, -)$.

■

Remark 23.2.10: The exts here will classify extensions in the category of left $\mathbb{Z}$-modules. Note the switched order on the hom functor however!

24 | Ch. 6: Group Homology and Cohomology (Wednesday, March 10)

Lemma 24.0.1 *(?).*
Last time: started setting up group homology. For G a group and $A \in \mathsf{G}\text{-}\mathsf{Mod}$, we think of $\mathbb{Z}$ as a trivial G-module and

1. $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$, the G-coinvariants.
2. $A^G \cong \underset{\mathbb{Z}G}{\mathrm{Hom}}(\mathbb{Z}, A)$. the G-invariants, this is the largest G-trivial submodule of A

Definition 24.0.2 (?)
For $A \in \mathsf{G\text{-}Mod}$,

1. $H_*(G; A) := L_*(-))G(A)$ are the **homology groups of G with coefficients in A**. It is isomorphic to $\operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$ by (1) in the lemma above. In particular, $H_0(G; A) \cong A_G$.
2. $H^*(G; A) := R^*(-)^G(A)$ is the **cohomology of G with coefficients in A**. It is isomorphic to $\operatorname{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, A)$ by (2) in the lemma. In particular, $H^0(G; A) \cong A^G$.

Ask about contructing resolutions: take any "augmentation" map and iterate kernels? Different resolution lengths?

Example 24.0.3*(?):* For $G = \{1\}$, for any $A \in \mathsf{G\text{-}Mod}$ we have $A^G = A = A_G$. Forgetful functors are usually exact, and in this case $(-)^G, (-)_G : \mathsf{G\text{-}Mod} \to \mathsf{Ab}$ is really a forgetful functor and thus exact. Here $H_n(G; A) = 0 = H^n(G; A)$ for $n > 0$.

Example 24.0.4*(?):* Let G be infinite cyclic, which we'll write multiplicatively to prevent the notation from conflicting with the addition on $\mathbb{Z}G$, so $G := T = \langle t \rangle = \left\{ t^n \mathrel{\Big|} n \in \mathbb{Z} \right\}$. Then $\mathbb{Z}G = \mathbb{Z}[t, t^{-1}]$ are integral Laurent polynomials, since we're taking integer linear combinations of various t^n. Computing $H_*(T, A) \cong \operatorname{Tor}_*^{\mathbb{Z}T}(\mathbb{Z}, A)$ and $H^*(T; A) \cong \operatorname{Ext}_{\mathbb{Z}T}^*(\mathbb{Z}, A)$ using a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}T$-module, since the first slot Ext requires an injective resolution in the opposite category. It suffices to take a free resolution:

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0 := \cdots \to 0 \to \mathbb{Z}T \xrightarrow{\times(t-1)} \mathbb{Z}T \xrightarrow{\mathrm{ev}_1} \mathbb{Z} \to 0.$$

Note that the resolution ends here because the multiplication $\times(t-1)$ is injective on polynomials rings. Thus $H_{>\geq 2}(T; A) = H^{\geq 2}(T; A) = 0$. The zeroth terms are invariants/coinvariants. For Tor, we apply $- \otimes_{\mathbb{Z}T} A$ to this resolution to obtain

$$\begin{aligned} 0 \to FP_1 \to FP_0 \to 0 &:= 0 \to \mathbb{Z}T \otimes_{\mathbb{Z}T} A \xrightarrow{(t-1)\otimes \mathbb{1}} \mathbb{Z}T \otimes_{\mathbb{Z}T} A \to 0 \\ &= 0 \to A \xrightarrow{(t-1)\otimes \mathbb{1}} A \to 0. \end{aligned}$$

One can check that

- $\ker(t-1) \otimes \mathbb{1} = A^T = H_1(T; A)$ is equal to the invariants and
- $\operatorname{coker}(t-1) \otimes \mathbb{1} = A_T = H_0(T; A)$ is equal to the coinvariants.

The second fact had to be true, but the first is surprising!

For Ext^*, we apply the contravariant $\operatorname*{Hom}_{\mathbb{Z}T}(-, A)$ to obtain

$$0 \to \operatorname*{Hom}_{\mathbb{Z}T}(\mathbb{Z}T, A) \xrightarrow{-\circ(t-1)} \operatorname*{Hom}_{\mathbb{Z}T}(\mathbb{Z}T, A) \to 0.$$

One checks

- $\operatorname{coker}(- \circ (t-1)) = A_T = H^1(T; A)$ (surprising!) and
- $\ker(- \circ (t-1)) = A^T = H^0(T; A)$

Remark 24.0.5: See exercise 6.1.2 for kG-modules for $k \in \mathsf{Ring}$ arbitrary.

Question 24.0.6
What can we say about H_0 and H^0 for more general groups?

24.1 H_0 for Groups

Definition 24.1.1 (Augmentation Maps)
Define the **augmentation map**

$$\begin{aligned} \varepsilon : \mathbb{Z}G &\to \mathbb{Z} \\ \sum n_i g_i &\mapsto \sum n_i, \end{aligned}$$

which is a ring morphism. Define $\mathcal{I} := \ker \varepsilon$ to be the **augmentation ideal**.

Observation 24.1.2
There is a basis of $\mathbb{Z}G$ as a $\mathbb{Z}$-module given by

$$\mathcal{B} := B_1 \cup B_2 := \{1\} \cup \left\{ g - 1 \;\middle|\; 1 \neq g \in G \right\}.$$

Note that $\varepsilon(g-1) = 0$, so $\mathcal{I}$ is a free $\mathbb{Z}$-module with basis B_2. Here the kernel should be expected to have codimension 1! We also have $\mathbb{Z}G/\mathcal{I} \cong \mathbb{Z}$ as rings, where the left-hand side is a G-module. Letting $\overline{}$ denote coset/equivalence class representatives, we have

$$g\bar{1} = \overline{g1} = \bar{g} = \bar{1},$$

and so the action $G \curvearrowright \mathbb{Z}G/\mathcal{I}$ is trivial.

Fact 24.1.3
For R a ring and $\mathcal{I} \trianglelefteq R$ a (left? right?) ideal and $M \in \mathsf{R\text{-}Mod}$,

$$R/I \otimes_R M \cong M/IM.$$

So for any $A \in \mathsf{G\text{-}Mod}$ we have

$$\begin{aligned} H_0(G; A) &= A_G \\ &\cong \mathbb{Z} \otimes_{\mathbb{Z}G} A \\ &= \operatorname{Tor}_0^{\mathbb{Z}G}(\mathbb{Z}; A) \\ &= \mathbb{Z}G/\mathcal{I} \otimes_{\mathbb{Z}G} A \\ &\cong A/\mathcal{I}A. \end{aligned}$$

Example 24.1.4 ***(?):***

- $H_0(G;\mathbb{Z}) \cong \mathbb{Z}/\mathcal{I}\mathbb{Z} \cong \mathbb{Z}$, where $\mathcal{I}\mathbb{Z} = 0$ since $\mathbb{Z}$ is the trivial G-module and $(g-1)a = ga - 1a = a - a = 0$.

- $H_0(G;\mathbb{Z}G) \cong \mathbb{Z}G/\mathcal{I} \cong \mathbb{Z}$.

- $H_0(G;\mathcal{I}) \cong \mathcal{I}/\mathcal{I}^2$.

Example 24.1.5 ***(?):*** Noting that $A = \mathbb{Z}G$ is projective in $\mathbb{Z}\mathsf{G}\text{-}\mathsf{Mod}$, so $H_n(G;\mathbb{Z}G) = 0$ for $n > 0$, using that this was a version of Tor and projective implies flat.

24.2 H^0 for Groups

Definition 24.2.1 (Norm Element)
Let G be a finite group, then the **norm element** is defined by

$$N = \sum_{g\in G} g \in \mathbb{Z}G.$$

Remark 24.2.2: For $h \in G$,

$$hN = \sum_{g} hg = \sum_{g'\in G} g' = N,$$

and so $N \in (\mathbb{Z}G)^g$. Similarly $Nh = N$ and so $Z(\mathbb{Z}G)$ is in the center.

Note the two different Zs here!

Lemma 24.2.3 ***(?).***
Let G be finite, then

$$H^0(G;\mathbb{Z}G) = (\mathbb{Z}G)^G = \mathbb{Z}N,$$

which is a two-sided ideal of $\mathbb{Z}G$ that is isomorphic to $\mathbb{Z}$.

Proof (?).
The inclusion $\mathbb{Z}N \subseteq (\mathbb{Z}G)^G$ is clear from the previous remark, so it remains to show the other inclusion. Suppose

$$a \in \sum_{g\in G} n_g g \in (\mathbb{Z}G)^G.$$

Then for all $h \in G$, we have

$$a = ha = \sum n_g h_g.$$

Now note that the g are a free $\mathbb{Z}$-basis for $\mathbb{Z}G$, so we can equate coefficients of h to find that $n_h = n_1$. Since h was arbitrary, we have $a = n_1 N \in \mathbb{Z}N$.

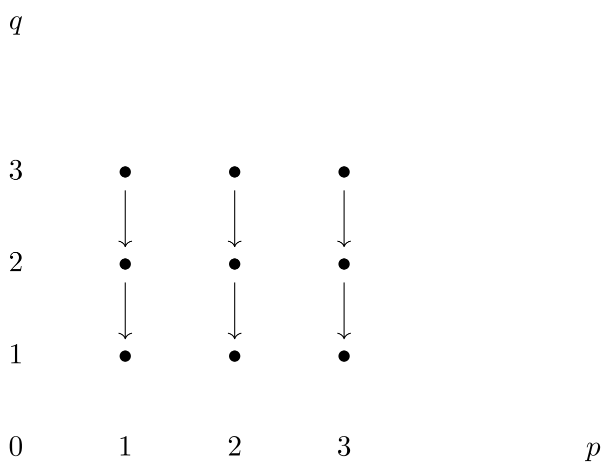
$\blacksquare$

Remark 24.2.4: Exercise 6.1.3 shows that $H^0(G;\mathbb{Z}G) = 0$ when G is infinite, in which case $\mathcal{I} = \left\{ a \in \mathbb{Z}G \;\middle|\; Na = 0 \right\}$ is the annihilator of the norm element. Next class we'll start on spectral sequences.

25 | Spectral Sequences (Monday, March 15)

25.1 Motivation

Remark 25.1.1: Invented by John Leray, 1946 while a prisoner of war in Austria, as an algorithmic way to compute homology of chain complexes. Start with a first-quadrant double complex $\left\{ E_{p,q} \;\middle|\; p,q \geq 0 \right\}$, say of R-modules. Let $T_n := \bigoplus_p$ be the total complex (direct sum or product, since the diagonals are finite) where $d := d^b + d^h$. Suppose one could compute the homology of each "piece" of the differential separately and independently. First forget d^h, and let this complex be $E^0_{p,q}$ (where the 0 superscript denotes a "zeroth approximation").



Link to Diagram

Now let $E^1 p,q := H_q(E^0_{p,q})$ be the homology obtained from the vertical complexes, i.e. $E^1_{p,q} := \ker d^v_{p,q} / \operatorname{im} d^v_{p,q-1}$. Recall that by convention we require anticommutativity, so $d^v d^h + d^h d^v = 0$, so this is not quite a complex of complexes. So these won't quite give a chain map, but $d^v d^h = -d^h d^v$ is enough to induce well-defined maps on $E^1_{*,*}$ since they will preserve kernels and images. So E^1 has horizontal differentials $d^h : E^1_{*,*} \to E^1_{*-1,*}$:

$$
\begin{tikzcd}
E^1_{p,q}: & q & & & & \\
& 3 & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \\
& 2 & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \\
& 1 & \bullet & \bullet \arrow[l] & \bullet \arrow[l] & \\
& 0 & 1 & 2 & 3 & p
\end{tikzcd}
$$

Link to Diagram

We can now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{*,q})$ at the p, q spot. We've done the horizontal and vertical homology separately, how close is $\left\{ E^2_{p,q} \;\middle|\; p+q=n \right\}$ to giving us information about the total homology?

Exercise 25.1.2 (5.1.1)
If $E^0_{*,*}$ consists of only two columns p and $p-1$, then there is a SES

$$
0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E^2_{p,q} \to 0.
$$

$$
\begin{tikzcd}
n \arrow[dr] & & & & & \\
& \vdots \arrow[dr] & & & & \\
q+1 & & \bullet \arrow[dr] & & & \\
n-p=q & & & \bullet \arrow[dr] & & \\
& & & & \vdots \arrow[dr] & \\
& & p-1 & p & & n
\end{tikzcd}
$$

Link to Diagram

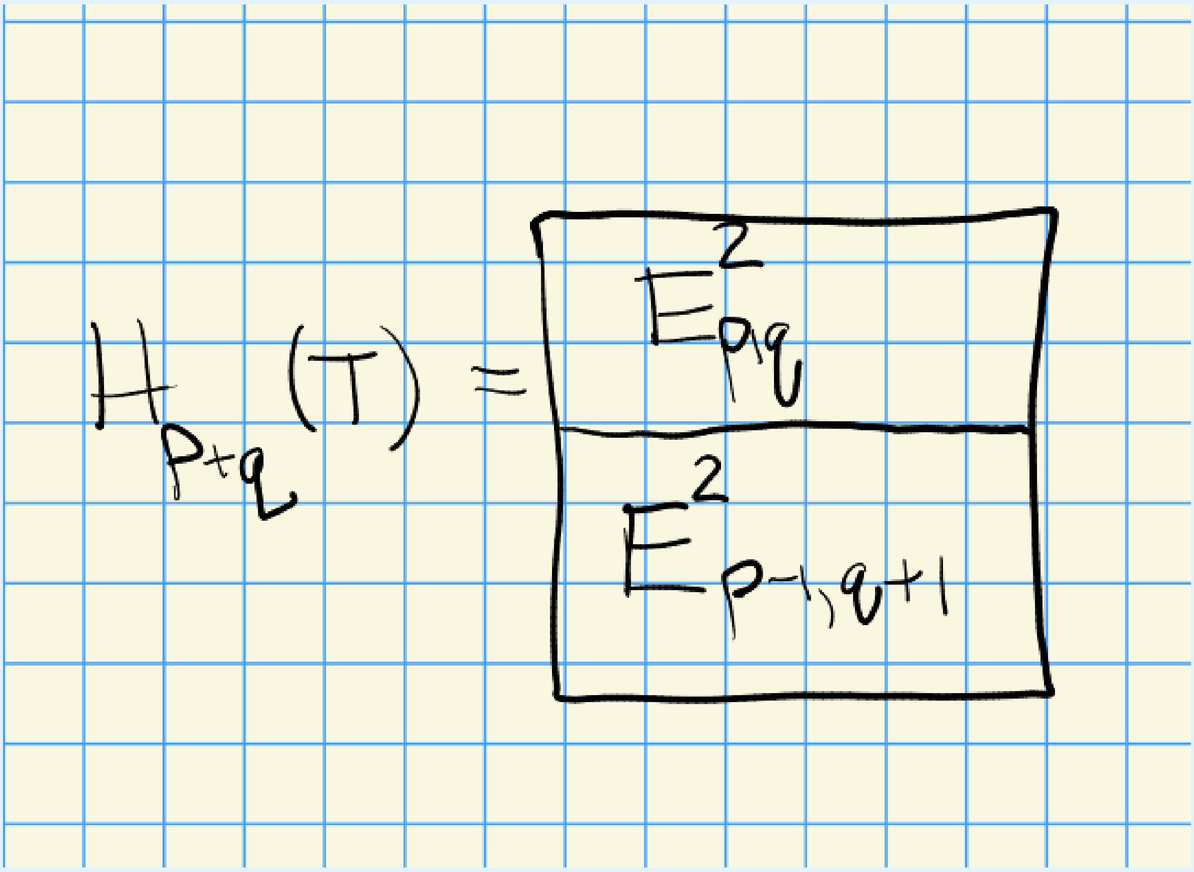
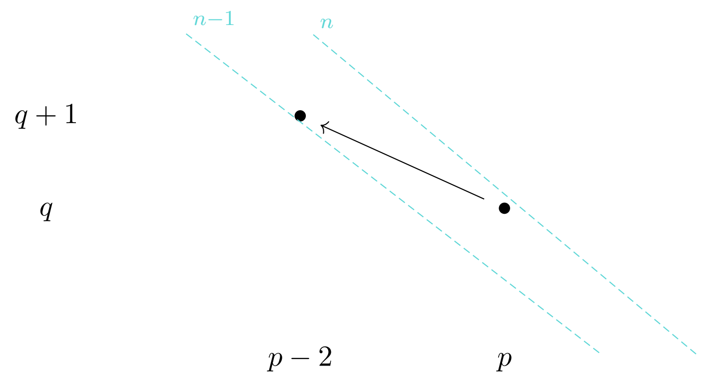


Figure 3: image_2021-03-15-09-29-09

So in general, $H_*(T)$ is determined up to extensions.

Exercise 25.1.3 (5.1.2)
We view $E^2_{*,*}$ as a 2nd order approximation to $H_*(T_*)$. We've used both differentials, so how do we continue? There are well-defined maps $d^2_{p,q}: E^2_{p,q} \to E^2_{p-2,q+1}$ such that $d^2_{*,*} \circ d^2_{*,*} = 0$ (noting that these are superscripts, not squaring).

Remark 25.1.4: This yields differentials on E^2 on lines of slope $-1/2$ which move from the nth diagonal to the $n-1$st diagonal:



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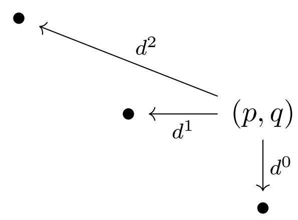


Link to Diagram

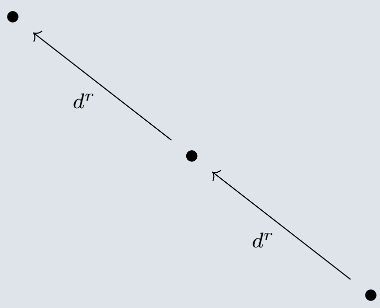
So we let E^3 be the homology, and it turns out there are differentials $d^3 : E^3_{p,q} \to E^3_{p-3,q+2}$ which go from diagonal n to $n-1$.

25.2 Setup

Definition 25.2.1 (Homology Spectral Sequences)
A **homology spectral sequence** starting with E^a for $a \in \mathbb{Z}$ in an abelian category $\mathcal{A}$ consists of the following data:

a. Pages: For all $r \geq a$ and all $p, q \in \mathbb{Z}$, a family $\left\{E^r_{p,q}\right\}$ of objects in $\mathcal{A}$ (some of which my be zero), where typically $a = 1, 2$.

b. Differentials: A family of maps $\left\{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\right\}$ with $d^r \circ d^r = 0$ of slope $-\dfrac{r-1}{r}$ in that lattice $E^r_{*,*}$ the form chain complexes. We take the convention that the differentials go to the left:



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c. Structure Maps: Isomorphisms $E^{r+1}_{p,q} \cong \ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$.

We denote $E^r_{*,*}$ to be the r**th page** of the sequence, and the **total degree** of an entry $E^r_{p,q}$ is $p+q$.

Remark 25.2.2: The term $E^{r+1}_{p,q}$ is a *subquotient*, i.e. a submodule of a quotient, of $E^r_{p,q}$, and hence inductively a subquotient of $E^a_{p,q}$ by transitivity of "being a subquotient". The terms of total degree n lie on a line of slope -1, and each differential $d^r_{p,q}$ decreases the total degree by 1.

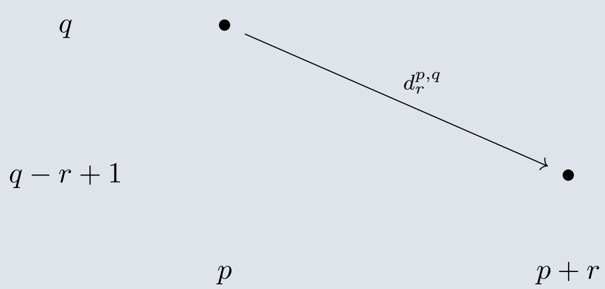
Remark 25.2.3: There is a category of homology spectral sequences over a fixed abelian category $\mathcal{A}$. The objects consist of the above data of pages, differentials, and structure maps from the above definition The morphisms $f : E \to \tilde{E}$ are families of maps

$$f^r_{p,q} : E^r_{p,q} \to \tilde{E}^r_{p,q}$$

for all $r \geq \max\{a, \tilde{a}\}$ with $\tilde{d}^r f^r = f^r d^r$ such that $f^{r+1}_{p,q}$ is the map on homology induced by $f^r_{p,q}$.

Definition 25.2.4 (Cohomology Spectral Sequence)
A **cohomology** spectral sequence is defined dually: we'll write this as $E^{p,q}_r, d^{p,q}_r$, where the differentials go down and to the right, and increase the total degree by 1:

$$d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r.$$



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There is similarly a category of these.

Lemma 25.2.5 ***(Mapping Lemma).***
Let $f : E \to \tilde{E}$ be a morphism of spectral sequences (homology or cohomology) such that for some fixed r, the map $f^r : E^r_{p,q} \to \tilde{E}^r_{p,q}$ is an isomorphism for all p, q. Then all $f^s_{p,q}$ are isomorphisms for all $s \geq r$ and all p, q.

Proof (of the mapping lemma).
There is a commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & B^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}(\sim)"'] & Z^r_{p,q} \arrow[r] \arrow[d, "f^r_{p,q}(\sim)"'] & E^{r+1}_{p,q} \arrow[r] \arrow[d, "f^{r+1}_{p,q}"'] & 0 \arrow[r] \arrow[d, no head] & 0 \arrow[d, no head] \\
0 \arrow[r] & \tilde{B}^r_{p,q} \arrow[r] & \tilde{Z}^r_{p,q} \arrow[r] & \tilde{E}^r_{p,q} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Extending the right-hand side as indicate, we can apply the Five Lemma to conclude that $f^{r+1}_{p,q}$ is an isomorphism. Now do induction on r.

■

26 | Wednesday, March 17

26.1 5.2: Spectral Sequences

Remark 26.1.1: Recall that we had

- $\left\{ E^r_{p,q} \;\middle|\; r \geq a, p, q \in \mathbb{Z} \right\}$ for some a.
- $d^r_{p,q} : E^r_{p,q} \to E_{p-r,q+r-1}$ with $d^2 = 0$.
- $E^{r+1}_{p,q} \cong \ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$.

Example 26.1.2 ***(First quadrant spectral sequences):*** A **first quadrant** (homology) spectral sequence is one with $E^r_{p,q} = 0$ for $p, q < 0$. Note that for a fixed p, q, there is an $r \gg 0$ such that the differential entering and leaving $E^r_{p,q}$ will be zero. The domain will be in quadrant 2 and the codomain in quadrant 4. In this case $E^r_{p,q} \cong E^{r+1}_{p,q}$ and we call this "stable" module $E^\infty_{p,q}$. Note that $r = r(p,q)$ can generally depend on p, q.

Definition 26.1.3 (Bounded)
We say a spectral sequence is **bounded** if there are only finitely many nonzero terms of total degree n. If so, there exists some uniform r_0 such that for $r \geq r_0$, we have $E^r_{p,q} \cong E^{r+1}_{p,q} \cong E^\infty_{p,q}$.

See video for image.

Remark 26.1.4: For the rest of this course, we'll restrict our attention to bounded spectral sequences.

Definition 26.1.5 (Convergence of a homology spectral sequences)
A bounded spectral sequences E **converges** to H_* if we are given

1. A family of objects $\{H_n\}_{n\in\mathbb{Z}}$
2. For each n, a finite (here increasing) filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_p H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

   where each $F_i H_n$ is a subobject of H_n

3. Isomorphisms

$$E^{\infty}_{p,q} \cong \frac{F_p H_{p+q}}{F_{p-1}H_{p+q}},$$

   or equivalently

$$E^{\infty}_{p,n-p} \cong \frac{F_p H_n}{F_{p-1}H_n},$$

   which are the $t-s$ **successive quotients** (or **sections**) of the filtration, which depend on n. We refer to $t-s$ as the **length** of the filtration

In this case we write

$$E^{a}_{p,q} \Rightarrow H_{p+q},$$

thinking of $a \to \infty$.

Remark 26.1.6: We saw a case where the length of the filtration was 2, when we had 2 columns. Recall that this only yields information up to extensions, since this only computes quotients.

Remark 26.1.7: We can form a similar definition for a cohomology spectral sequence. The conditions change slightly:

(2') We have a *decreasing* filtration

$$H^n = F^s H^n \supseteq \cdots \supseteq F^p H^n \supseteq F^{p+1}H^n \supseteq \cdots \supseteq F^t H^n = 0.$$

In this case we have

$$E_{\infty}^{p,q} \cong \frac{F^p H^{p+q}}{F^{p+1}H^{p+q}}.$$

Then each H_n will have a filtration of length $n+1$ by explicitly counting terms on the diagonal, so we obtain

$$0 = F_{-1}H_n \subset F_0 H_n \subseteq \cdots \subseteq F_{n-1}H_n \subseteq F_n H_n = H_n.$$

Then

$$\begin{aligned} E_{0,n} &\cong F_0 H_n \hookrightarrow H_n \\ E_{p,n-p} &\cong \frac{F_p H_n}{F_{p-1} H_n} \\ H_n \twoheadrightarrow E_{n,0} &\cong \frac{H_n}{F_{n-1} H_n}. \end{aligned}$$

See video for remarks!

Definition 26.1.8 (Edge maps)
Assume that $a \geq 1$. Provided $a \geq 1$, note that $E^r_{0,n}$ is a quotient of $E^a_{0,n}$ for all r, since the outgoing (?) differentials are all zero. Similarly, $E^r_{n,0}$ is a subobject of $E^a_{n,0}$ for all r. We thus have maps

$$\begin{aligned} E^a_{0,n} \twoheadrightarrow E^\infty_{0,n} \hookrightarrow H_n \\ H_n \twoheadrightarrow E^\infty_{n,0} \hookrightarrow E^a_{n,0}. \end{aligned}$$

These compositions are referred to as the **edge maps**.

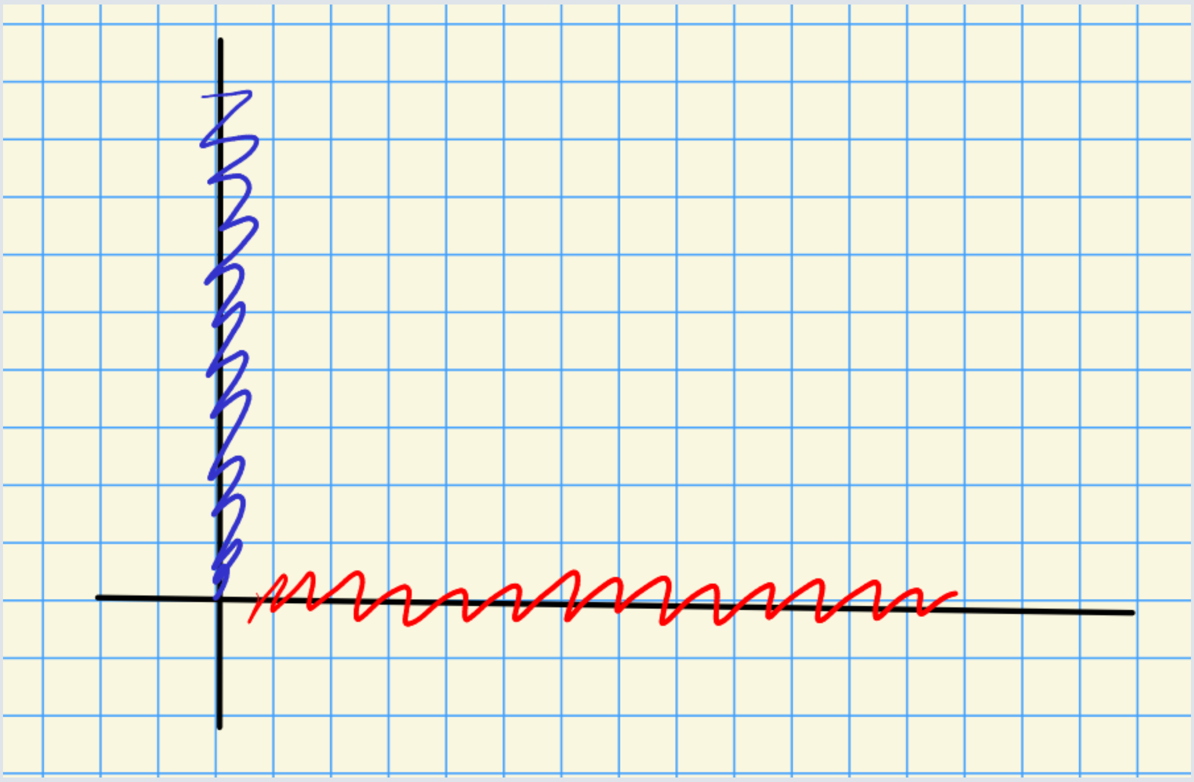


Figure 4: Edges of a spectral sequence

Remark 26.1.9: For a first quadrant *cohomological* spectral sequence, the edge maps are

$$E_a^{n,0} \twoheadrightarrow E_\infty^{n,0} \hookrightarrow H^n$$
$$H^n \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E_a^{0,n}.$$

Definition 26.1.10 (Collapsing of a spectral sequence)
A spectral sequence E **collapses** at E^r if there is exactly one nonzero row (or column) in $E^r_{*,*}$.

Remark 26.1.11: This implies that $E^r_{p,q} = E^\infty_{p,q}$ at this point. In this case, we can read off the single nonzero section:

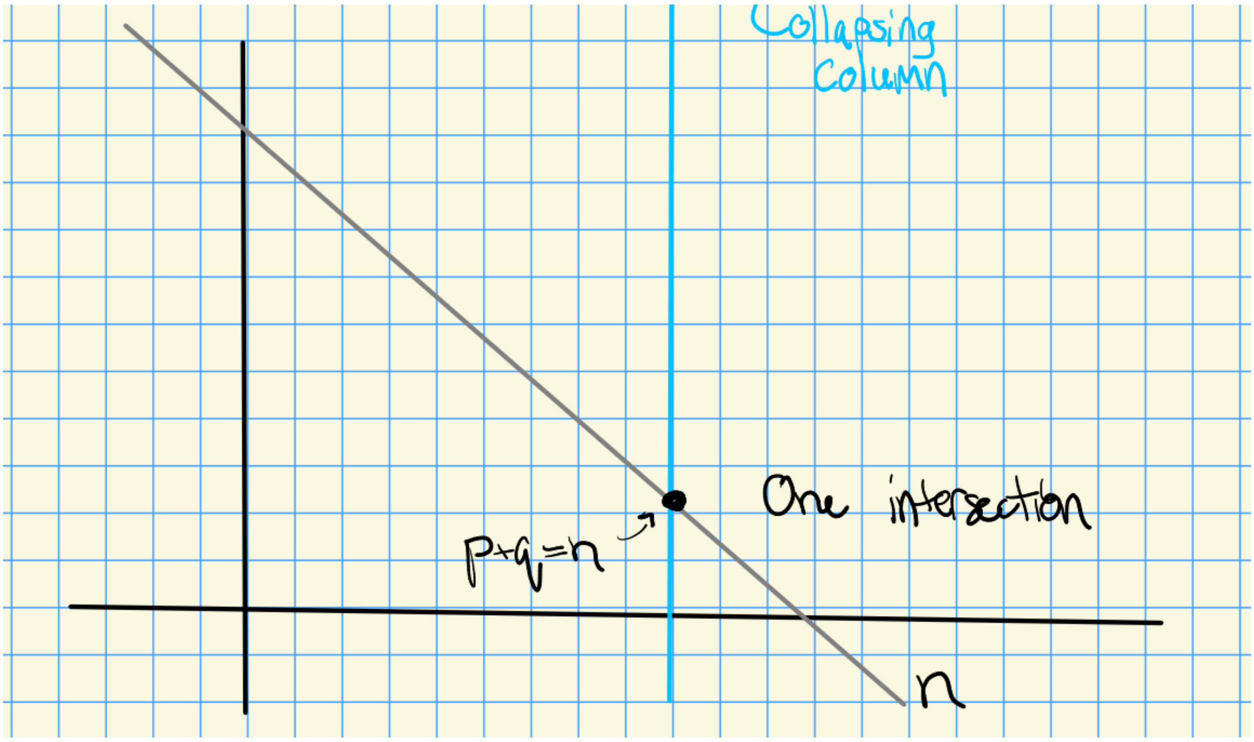


Figure 5: image_2021-03-17-09-55-34

Here we'll have

$$E^\infty_{p,q} \cong \frac{F_p H_n}{F_{p-1} H_n} \cong \frac{H_n}{0} \cong H_n.$$

Remark 26.1.12: A more common definition of a spectral sequence **collapsing at** r is that for all p, q, the differentials $d^r_{p,q} = 0$. Note that this implies stabilization at r, but doesn't allow for such a simple statement about the diagonals since they may intersect multiple nonzero objects.

Remark 26.1.13: Some things we're skipping from the book, around the last part of 5.2:

- Definitions pertaining to unbounded spectral sequences.
- Weak convergence.
- Filtrations that are infinite in on or both filtrations.
- Filtrations that don't limit to a union equal to H_n or intersection to 0.
- Abutment, which is convergence when the filtration is not finite.

We'll skip 5.3 on the Leray spectral sequence and jump to 5.4, constructing a spectral sequence.

27 | Friday, March 19

27.1 Spectral Sequence of a Filtration

Definition 27.1.1 (?)
A **filtration** of a chain complex C is an ordered family of subcomplexes

$$F \coloneqq \cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots \subseteq C \qquad\qquad p \in \mathbb{Z}$$

such that there are commutative diagrams

$$\begin{tikzcd}
F_pC_n \arrow[r, hook] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
F_pC_{n-1} \arrow[r, hook] & C_{n-1}
\end{tikzcd}$$

Link to Diagram

A filtration is **exhaustive** if $\bigcup_{p\in\mathbb{Z}} F_pC_n = C_n$ for all n.

Remark 27.1.2: The construction of the spectral sequence will show that C and $\bigcup_p F_pC$ give rise to the same spectral sequence. So we will assume that all filtrations are exhaustive.

Theorem 27.1.3 *(Construction of the spectral sequence of a filtration).*
A filtration F of $C \in \mathsf{Ch}(\mathsf{R\text{-}Mod})$ determines a spectral sequence starting with

$$E^0_{p,q} \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} \qquad\qquad E^1_{p,q} = H_{p+q}(E^0_{p,*}).$$

Since d preserves numerators and denominators, we get well-defined differentials $\bar{d}$ on the quotients:

$$\begin{tikzcd}
 & & E^0_{p-1,q+1} \arrow[dr] & & \\
 & F_{p-1}C_{p+q+1} \arrow[r] \arrow[d, "d"] & F_pC_{p+q+1} \arrow[r] \arrow[d, "d"] & E^0_{p,q+1} \arrow[d, "\bar{d}"] \arrow[dr] & \\
F_{p-2}C_{p+q} \arrow[r] & F_{p-1}C_{p+q} \arrow[r] \arrow[d, "d"] & F_pC_{p+q} \arrow[r] \arrow[d, "d"] & E^0_{p,q} \arrow[d, "\bar{d}"] & \ddots \\
 & F_{p-1}C_{p+q-1} \arrow[r] & F_pC_{p+q-1} \arrow[r] & E^0_{p,q-1} &
\end{tikzcd}$$

Link to Diagram

Taking vertical homology of the E^0 terms on the right yields $E^1_{p,q}$. Note that the blue terms contribute to the same diagonal $p + q = n$.

Definition 27.1.4 (Bounded Filtrations)
A filtration F on a chain complex C is **bounded** if for each n there are $s < t \in \mathbb{Z}$ such that $F_sC_n = 0$ and $F_tC_n = C_n$.

Remark 27.1.5: Note that this implies that each diagonal of total degree n has only finitely many nonzero terms, so the spectral sequence will again be bounded. We'll next show that this spectral sequence converges to $H_*(C)$.

Definition 27.1.6 (Canonically Bounded Filtrations)
A filtration F is **canonically bounded** if and only if $F_{-1}C_n = 0$ and $F_nC_n = C_n$ for all n. In this case,

$$E^0_{p,q} := \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} = \begin{cases} 0 & p < 0 \\ 0 & q < 0 \quad (p > n, p-1 \geq n). \end{cases}$$

So E becomes a first quadrant spectral sequence.

Remark 27.1.7: Note that all elements on all pages are subquotients of E^0 elements, so they can only get smaller, and terms that become 0 on some page stay 0 for all remaining pages.

27.2 Construction of the Spectral Sequence of a Filtration

Remark 27.2.1: For ease of notation, we'll suppress the subscript q since it can always be recovered as $q = n - p$. Define the canonical quotients

$$\eta_p : F_pC \to F_pC/F_{p-1}C = E_p^0.$$

Define

$$A_p^r \coloneqq \left\{ c \in F_pC \;\middle|\; d(c) \in F_{p-r}(C) \right\},$$

which are elements of F_pC which are cycles modulo $F_{p-r}C$, the **approximate cycles**. Note that any actual cycle is in all A^r. This differential takes things r columns to the left, so we'll want to define a differential that associates the following terms

$$\begin{tikzcd}
 & & F_{p-1}C_{n+1} \arrow[r, hook] \arrow[d, "d"] & F_pC_{n+1} \arrow[d, "d"] & \\
 & & F_{p-1}C_n \arrow[r, hook] \arrow[d, "d"] & F_pC_n \arrow[d, "d"] & c \arrow[d, mapsto] \\
F_{p-r}C \arrow[r, hook] & \cdots & F_{p-1}C_{n-1} \arrow[r, hook] & F_pC_{n-1} & dc
\end{tikzcd}$$

Link to Diagram

Similarly, define

$$\begin{aligned}
Z_p^r &\coloneqq \eta_p(A_p^r \subseteq E_p^0 \\
B_p^r &\coloneqq \eta_p(dA_{p+r-1}^{r-1}) \subseteq \eta_p(F_pC) \subseteq E_p^0.
\end{aligned}$$

Observation 27.2.2
Some key observations:

1. $F_pC = A_p^0 = A_p^{-1} = A_p^{-2} = \cdots$

2. $A_p^{r+1} \subseteq A_p^r$

3. $A_p^r \cap F_{p-1}C = A_{p-1}^{r-1}$.

> **Exercise 27.2.3** (?)
> Work through these facts using the diagram above.

Remark 27.2.4: Some consequences:

(1) $\implies Z_p^0 = E_p^0$ (taking $r = 0$ in the quotient map η_p).

(2) $\implies Z_p^{r+q} \subseteq Z_p^r$, since these are images of subgroups

(3) $\implies A_{p+r-1}^{r-1} \subseteq A_{p+r}^r$, replacing $p \mapsto p + r$ in the intersection formula. Then applying d yields $B_p^r \subseteq D_p^{r+1}$.

(1) $\implies B_p^0 = \eta_p(dA_{p-1}^{-1}) \subseteq \eta_p(F_{p-1}C) = 0$, since this occurs in the denominator for η_p and d preserves filtration degree.

So the Z_p get smaller and the B_p get bigger. What happens in the middle?

> **Proposition 27.2.5** ***(All boundaries are contained in all cycles in a spectral sequence).***
> $B_p^r \subseteq Z_p^s$ for all $r, s \geq 0$.

> *Proof (?).*
> A sequence of implications:
>
> $$\begin{aligned} B_p^r \ni x = \eta_p(dc) \text{ for some } c &\implies d(dc) = 0 \in F_{p-s}C \,\forall s \\ &\implies dc \in A_p^s \\ &\implies \eta_p(dc) \in Z_p^s. \end{aligned}$$
>
> ■

Remark 27.2.6: Set $B_p^\infty \coloneqq \cup_{r\geq 1} B_p^r \subseteq Z_p^\infty \coloneqq \bigcap_{s\geq 1} Z_p^s$, which follows from a set theory exercise.

Remark 27.2.7: Combining and summarizing these results: for every $p \geq 0$, we have a tower of groups:

$$0 = B_p^0 \hookrightarrow B_p^1 \hookrightarrow \cdots \hookrightarrow B_p^r \hookrightarrow \cdots \hookrightarrow B_p^\infty \hookrightarrow Z_p^\infty \hookrightarrow \cdots \hookrightarrow Z_p^r \hookrightarrow \cdots \hookrightarrow Z$$

Link to Diagram

Remark 27.2.8: Note that using standard isomorphism theorems, we have

$$Z_p^r \cong \frac{A_p^r}{A_p^r \cap F_{p-1}CC} \overset{(3)}{=} \frac{A_p^r}{A_{p-1}^{r-1}}.$$

So set

$$E_p^r := Z_p^r / B_p^r \cong \frac{A_p^r + F_{p-1}C}{dA_{p+r-1}^{r-1} + F_{p-1}C} \cong \frac{A_p^r}{dA_{p+r-q}^{r-1} + A_{p-1}^{r-1}},$$

making E_p^r a quotient of A_p^r. Using a similar calculation, one can show

$$\frac{Z_p^{r+1}}{B_p^r} \cong \frac{A_p^{r+1} + A_{p-1}^{r-1}}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}}.$$

Remark 27.2.9: There will be an induced differential on this quotient, which will follow from checking that the different preserves the numerator and denominator.

28 | Monday, March 22

28.1 5.4: Spectral Sequence of a Filtration

Remark 28.1.1: We have an increasing filtration $F_pC \subseteq F_{p+1}C$, where we defined

$$E_{p,q}^0 = \frac{F_pC_{p+q}}{F_{p-1}C_{p+1}} \qquad\qquad E_{p,q}^1 = H_{p+q}E_{p,*}^0.$$

1. We have a map

   $$\eta_p : F_pC \twoheadrightarrow \frac{F_pC}{F_{p-1}C} = E_p^0,$$

   where we've dropped the q from notation.

2. $$A_{p,q}^r = \left\{ c \in C_pC \;\middle|\; dc \in F_{p-1}C \right\},$$

   the eventual cycles. We defined $Z_p^r = \eta_p A_p^r$ and $B_p^r = \eta_p dA_{p+r-1}^{r-1}$, and wrote $A_p^r \cap F_{p-1}C = A_{p-1}^{r-1}$.

3. We had the chain of inclusions

   $$0 = B_p^r \subseteq \cdots \subseteq B_p^\infty \subset Z_p^\infty \subset \cdots \subseteq Z_p^1 = E_p^).$$

4. We also have $E_p^r = Z_p^r / B_p^r = A_p^r / dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}$

5. $Z_p^{r+1}/B_p r \cong \dfrac{A_p^{r+1} + A_{p-1}^{r-1}}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}}$.

6. $dA_p^r \cap F_{p-r-1}C = dA_P^{r+1}$.

See video for missed spoken details!

Obviously we have

$$d : A_p^r \to A_{p-r}^r$$
$$d : A_{p-1}^r \to dA_{p-1}^{r-1},$$

so d induces a well-defined map $d_p^r : E_p^r \to E_{p-r}^r$, which of course squares to zero, which goes r columns to the left and decreases the total degree n by 1 since the original d did on C_n. This is what we need to set up a spectral sequence, since we now have pages and differentials, and it just remains to show that $E^{r+1} \cong H_*(E^r, d^r)$.

Lemma 28.1.2 *(?).*
d determines isomorphisms $Z_p^r / Z_p^{r+1} \xrightarrow{\sim} B_{p-r}^{r+1} / B_{p-r}^r$.

Proof (?).
Unwind definitions! Note that we have $B_{p-r}^{r+1} = \eta_{p-r} dA_p^r$, using that the lower index on B and upper index on A should sum to the lower index on A. This is equal to $dA_p^r / dA_p^r \cap F_{p-r-1}C$, where the latter term is $\ker \eta_{p-r}$ and $B_{p-r}^r = \eta_{p-r} dA_{p-1}^{r-1}$. This yields

$$\frac{B_{p-r}^{r+1}}{B_{p-r}^r} \cong \frac{dA_p^r}{dA_{p-1}^{r-1} + (dA_p^r \cap F_{p-r-1}C)}.$$

Similarly,

$$\frac{Z_p^r}{Z_p^{r+1}} := \frac{\eta_p A_p^r}{\eta_p A_p^{r+1}} \cong \frac{A_p^r}{A_p^{r+1} + (A_p^r \cap F_{p-1}C)} \overset{(3)}{\cong} \frac{A_p^r}{A_p^{r+1} + A_{p-1}^{r-1}}.$$

Now applying the map induced by $d : A_p^r \to F_{p-r}C$ to this quotient, we have $\ker d|_{A_p^r} \subseteq A_p^{r+1}$. These go down r steps, but everything in the kernel goes down as far as you'd like! So d kills one of the denominator terms, and thus induces an injective map on the quotient. Thus $\dfrac{Z_p^r}{Z_p^{r+1}} \xrightarrow{\sim} \dfrac{dA_p^r}{dA_p^{r+1} + dA_{p-1}^{r-1}}$, which is exactly the previous expression with the order switched, so this is isomorphic to B_{p-r}^{r+1}/B_{p-r}^r.

■

Proposition 28.1.3 *(The $r+1$st page is the homology of the rth page).*

$$\frac{\ker d_p^r}{\operatorname{im} d_{p+r}^r} \cong E_p^{r+1} := \frac{Z_p^{r+1}}{B_p^{r+1}}.$$

Proof (?).
Recall that $d_p^r : E_p^r \to E_{p-r}^r$ and by (4), $E_p^r \cong \dfrac{A_p^r}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}}$. Substituting $p \hookleftarrow p - r$, we have

$$\ker d_p^r = \frac{\left\{ z \in A_p^r \;\middle|\; dz \in dA_{p-1}^{r-1} + A_{p-r-1}^{r-1} \right\}}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}} = \frac{A_{p-1}^{r-1} + A_p^{r+1}}{dA_{p+r-1}^{r-1} + A_{p-1}^{r-1}} \overset{(5)}{\cong} \frac{Z_p^{r+1}}{B_p^r} \qquad \text{which is (6).}$$

Here we've used that $x \in F_pC \implies dx \in F_{p-r-1}C \implies dx \in A_{p-r-1}^?$. What is the image of d_p^r in general? Note that later we can replace $p \hookleftarrow p + r$. By the 1st isomorphism theorem, we have

$$d_p^r : E_p^r = Z_p^r / B_p^r \xrightarrow{\sim} \frac{Z_p^r / B_p^r}{Z_p^{r+1} / B_p^r} \xrightarrow{\sim} \frac{Z_p^r}{Z_p^{r+1}} \xrightarrow{d} \frac{B_{p-r}^{r+1}}{B_{p-r}^r} \hookrightarrow \frac{Z_{p-r}^r}{B_{p-r}^r} = E_{p-r}^r,$$

where we've applied the lemma from last time, and we've used the fact that in the last map, all of the B are contained in all of the Z, so we can choose any superscript we want. These are all isomorphisms up until the last part, so

$$\operatorname{im} d_p^r \cong B_{p-r}^{r+1} / B_{p-r}^{r+1}.$$

. Replacing $p \hookleftarrow p + r$, we get a 7th fact
Fact *(7)*

$$\operatorname{im} d_{p+r}^r \cong B_p^{r+1} / B_p^{r+1}.$$

Now combining (6) and (7), we have

$$\frac{\ker d_p^r}{\operatorname{im} d_{p+r}^r} \xrightarrow{\sim} \frac{Z_p^{r+1} / B_p^r}{B_p^{r+1} / B_p^r} \cong \frac{Z_p^{r+1}}{B_p^{r+1}} = E_p^{r+1}.$$

■

28.2 5.5: Convergence of the Spectral Sequence of a Filtration

Remark 28.2.1: We'll restrict our attention to bounded complexes.

Remark 28.2.2: A filtration F on a chain complex C induces a filtration on the homology H_*C, where $H_pH_nC = \operatorname{im}(H_nF_pC \to H_nC)$:

$$
\begin{tikzcd}
F_pC_{n+1} \arrow[r, hook] \arrow[d, "d"] & C_{n+1} \arrow[d, "d"] \\
F_pC_n \arrow[r, hook] \arrow[d, "d"] & C_n \arrow[d, "d"] \\
F_pC_{n-1} \arrow[r, hook] & C_{n-1}
\end{tikzcd}
$$

Link to Diagram

See video for missed details.

These inclusions induce a map from the homology of the subcomplex to the homology of the total complex.

Remark 28.2.3: If the filtration on C is bounded, say $0 = F_sC_n \subseteq \cdots \subseteq F_tC_n = C_n$ for some $s < t$, then so is the induced filtration on H_nC. Also note that $F_tH_n = H_n$ and $F_sH_n = 0$.

Theorem 28.2.4 ***(Classical Convergence Theorem).***
Assume F is a bounded filtration on C, then the spectral sequence is bounded and converges to H_*C, so

$$E^1_{p,q} = H_{p+q}\left(\frac{F_pC}{F_{p-1}C}\right) \Rightarrow H_{p+q}C.$$

Remark 28.2.5: Need to check next time that the $E^\infty_{p,q}$ terms give the proper quotients.

29 | Wednesday, March 24

Remark 29.0.1: Last time: we're trying to prove the classical convergence theorem in the bounded case. We have

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}C.$$

We'd like this converge, i.e. the E^∞ page will be the sections of $H_{p+q}C$. Writing $C'_n := F_pC_n$ for the filtered pieces, we have

$$\begin{tikzcd}
C'_n \arrow[r, hook] \arrow[d, hook] & C_n \arrow[d, hook] \\
Z'_n \arrow[r, hook] \arrow[d, hook] & Z_n \arrow[d, hook] \\
B'_n \arrow[r, hook] & B_n
\end{tikzcd}$$

Link to Diagram

Then the induced filtration on homology is

$$\begin{aligned}
H'_n \coloneqq \frac{Z'_n}{B'_n} &\hookrightarrow H_n \coloneqq \frac{Z_n}{B_n} \\
z' + B'_n &\mapsto z' + B_n.
\end{aligned}$$

Proof (of classical convergence theorem).
As discussed, we have a natural bounded filtration on each H_nC. Fixing p, n and writing $q = n - p$, we have

$$A_p^r = \left\{ c \in F_pC_n \;\middle|\; d(c) \in F_{p-r}C_{n-1} \right\}.$$

This stabilizes for large r, namely whenever $F_{p-r}C_{n-1} = 0$ (which happens since the complex is bounded). Call the stabilized object $A_p^\infty \coloneqq \left\{ c \in F_pC_n \;\middle|\; d(c) = 0 \right\}$, which is $\ker d$ in the pth filtered piece. Some facts:

0. $Z_p^r = \eta_p(A_p^r)$ where

$$\eta_p : F_pC_n \to \frac{F_pC_n}{F_{p-1}C_n}$$

where $Z_p^\infty = \eta_p(A_p^\infty)$.

1. $A_p^\infty \coloneqq \ker(F_pC_n \xrightarrow{d} F_pC_{n-1})$, which is the "numerator" of F_pH_nC.

2. $d(C_{n+1}) \cap F_pC_n = \bigcup_{r \in \mathbb{Z}} d(A_{p+r}^r)$:

$$\begin{tikzcd}
& A_{p+r} \arrow[dl, "d"'] \arrow[d, hook] & \\
F_pC_n & F_{p+r}C_{n+1} \arrow[r, hook] & C_{n+1}
\end{tikzcd}$$

Link to Diagram

3. Recall that we defined $B_p^r := \eta_p(dA_{p+r-1}^{r-1})$. We can write $B_p^\infty = \eta_p(\cup_r dA_{p+r}^r)$, where the left-hand side and the inner term on the right-hand side are equal to $\bigcup_{r\geq 1} B_p^r$.

4. $A_{p-1}^\infty = A_p^\infty \cap F_{p-1}C_n = \ker(A_p^\infty \xrightarrow{\eta_p} E_p^0)$.

Now to assemble this, note that

$$
\begin{aligned}
\frac{F_pH_nC}{F_{p-1}H_nC} &\cong \frac{A_p^\infty}{A_{p-1}^\infty + \bigcup_r dA_{p+r}^r} && \text{by 1 and 2} \\
&\cong \frac{\eta_p(A_p^\infty)}{\eta_p\left(\bigcup_{r\geq 0} dA_{p+r}^r\right)} && \text{by 4} \\
&= \frac{Z_p^\infty}{B_p^\infty} && \text{by 0, 3} \\
&= E_p^\infty.
\end{aligned}
$$

where we've used that $A_{p-1}^\infty + \bigcup_{r>0} dA_{p+r}^r \subseteq \ker \eta_p = F_{p-1}C$.

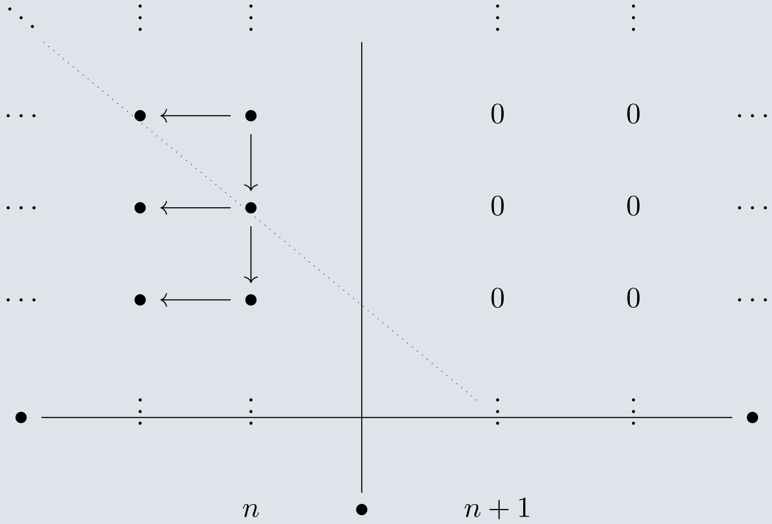
■

29.1 Applications: Two Spectral Sequences of a Double Complex

Remark 29.1.1: Consider two different filtrations of the total complex $\mathrm{Tot}(C)$ (either sum or product) of a double complex $C_{*,*}$. We know there is an spectral sequence associated to each and play them off of each other to get extra information about cohomology.

Definition 29.1.2 (Filtration I: by columns (of a double complex))
Let ${}^IF_n \,\mathrm{Tot}(C)$ be the total subcomplex obtain by applying truncation functors:

$$
\left({}^I\tau_{\leq n}C\right)_{p,q} := \begin{cases} C_{p,q} & p \leq n \\ 0 & p > n. \end{cases}.
$$



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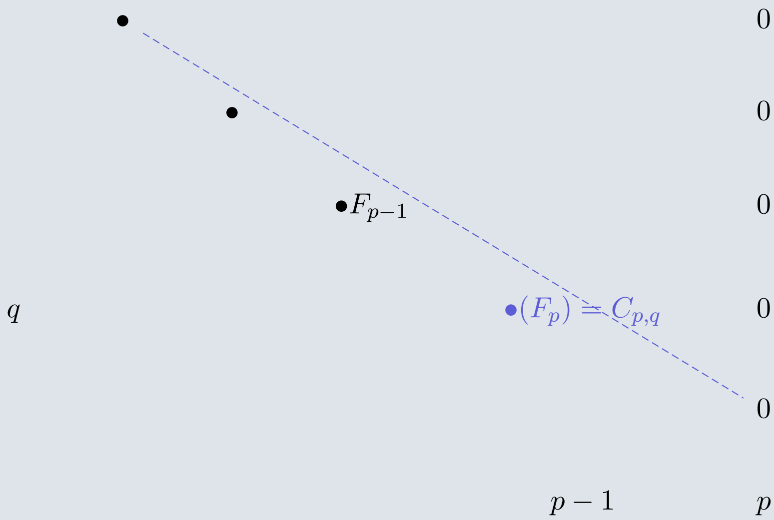
We still have $d = d^v + d^h : {}^I F_n \to {}^I F_n$. By the construction theorem, there is a spectral sequence $\left\{ {}^I E^r_{p,q} \right\}$ starting with ${}^I E^0_{p,q} = C_{p,q}$ and

$$
{}^I E^0_{p,q} = \frac{F_p \operatorname{Tot}(C)_{p+q}}{F_{p-1} \operatorname{Tot}(C)_{p+q}}.
$$



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Recall that $d^r_p : E^r_p \to E^r_{p-r}$ (going r columns to the left, where we've suppressed q) is the map induced from $d : \operatorname{Tot}(C)_n \to \operatorname{Tot}(C)_{n-1}$. So for $r = 0$, we have $d^0_{p,q} : E^0_{p,q} \to E^0_{p,q-1}$. But the left-hand side is $C_{p,q}$ and the right-hand side is $C_{p,q-1}$, so it's perhaps not surprising that this coincides with the original d^v from $C_{*,*}$.

Thus ${}^I E^1_{pq} = H^v_q(C_{p,*})$ by taking homology in the vertical direction. For the differential, we want $d^1_{pq} : E^1_{pq} \to E^1_{p-1,q}$, and these will just be the maps induced on the vertical homology by d^h. So we write ${}^I E^2_{p,q} = H^h_p H^v_q(C_{**})$.

If C is a first quadrant complex, the filtration is canonically bounded since $F_{-1}\operatorname{Tot}(C) = 0$ and $F_n \operatorname{Tot}(C)_n = \operatorname{Tot}(C)_n$. So we get the spectral sequence that we started constructing in section 5.1, and we now know it converges to $H_* \operatorname{Tot}(C)$ by the classical convergence theorem. So

$$ {}^I E^2_{p,q} = H^h_p H^v_q(C) \Rightarrow H_{p+q} \operatorname{Tot}(C). $$

Remark 29.1.3: We can say something about the unbounded case. Suppose C is 4th quadrant, then $F_{-1}\operatorname{Tot}(C) = 0$, so the first filtration ${}^I F$ is bounded below. The diagonals are infinite, so we take $\operatorname{Tot}(C) \coloneqq \operatorname{Tot}^{\oplus}(C)$. Every element of $(\operatorname{Tot}(C))_n$ lives in $\bigoplus_{p=0}^{N} C_{p,n-p}$ for some finite N and the filtration is exhaustive, i.e. $\operatorname{Tot}^{\oplus} C = \bigcup_{p\geq 0} F_p \operatorname{Tot}^{\oplus} C$. A version of the classical convergence theorem will yield

$$ {}^I E^r_{pq} \Rightarrow H_{p+q} \operatorname{Tot}^{\oplus} C. $$

However, this will not hold for Tot^{Π}.

Remark 29.1.4: Next time: a second filtration and its spectral sequence, and how to play them off of each other.

30 | Friday, March 26

30.1 5.6: Two Spectral Sequences on Total Complexes

Remark 30.1.1: Recall that we had two filtrations on a total complex: the first was fixing a vertical line and replacing everything to the right with zeros, which was given by ${}^I E^0_{p,q} = F_p(\operatorname{Tot})/F_{p-1}(\operatorname{Tot}) = C_{p,q}$. Taking homology with the vertical differentials yielded ${}^I E^1_{p,q} = H^v_q(C_{p,*})$, and ${}^I E^2_{p,q} = H^h_p H^v_q(C_{*,*})$. Applying the classical convergence theorem when this is 1st quadrant yields some spectral sequence with these as the pages which converges to $H_{p+q}(\operatorname{Tot}(C))$.

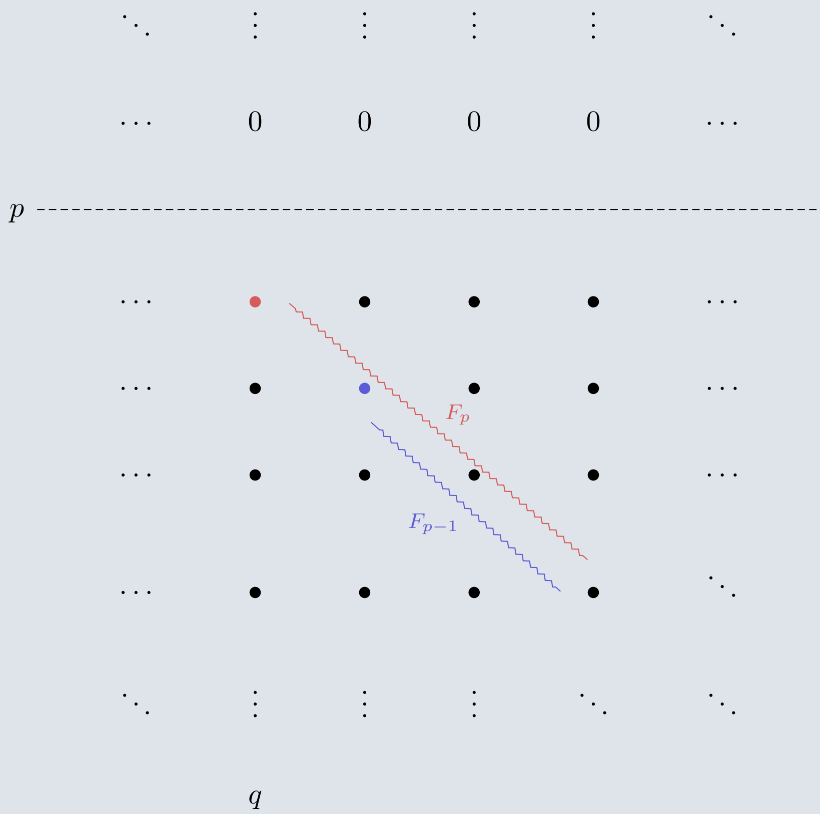
Definition 30.1.2 (The second filtration)
We'll define a filtration by rows: let ${}^{II}F_n \operatorname{Tot}(C)$ be the total complex of the double complex

$$ ({}^{II}\tau_{\leq n} C)_{p,q} = \begin{cases} C_{p,q} & p, q \leq \\ 0 & p, q > n. \end{cases} $$

This is the complex gotten by replacing everything below the nth row with zeros. We define the 0th page

$$ {}^{II}E^0_{p,q} = \frac{{}^{II}F_p \operatorname{Tot}(C)_{p+q}}{{}^{II}F_{p-1} \operatorname{Tot}(C)_{p+q}} = C_{q,p}, $$

which follows from the fact that we are modding out a full diagonal by a diagonal with one fewer elements:



Link to Diagram

Warning 30.1.3
Note the switched order!

Remark 30.1.4: Note that the differential is

$$\begin{aligned} d^0 &: E^0_{p,q} \to E^0_{p,q-1} \\ = d^h &: C_{q,p} \to C_{q-1,p}. \end{aligned}$$

We similarly have ${}^{I}IE^{I}_{p,q} = H^h_q(C_{*,p})$, again noting the switched indices, with differential

$$\begin{aligned} d^1 &: E^1_{p,q} \to E^1_{p-1,q} \\ &= H^h(C_{q,p}) \to H^h(C_{*,p-1}) \end{aligned}$$

which comes from the original differential inducing a map on horizontal homology. Then ${}^{II}E^2_{p,q} = H^v_p H^h_q(C)$.

Remark 30.1.5: Note that transposing everything about the line $p = q$ interchanges filtrations I and II, and thus the two spectral sequences ${}^{I}E_{p,q} \rightleftharpoons {}^{II}E_{q,p}$. Using that first quadrant sequences are canonically bounded, we can apply the classical convergence theorem to ${}^{II}E$ to obtain

$$ {}^{II}E^2_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(C)). $$

Transposing sends QIV to QII and thus ${}^{II}E \Rightarrow H_{p+q}\,\mathrm{Tot}^{\oplus}(C)$. Note that this does not guarantee anything about $\mathrm{Tot}^{\Pi}(C)$.

Remark 30.1.6: In particular, if we have a QI double complex, both filtrations converge to the homology of the total complex.

30.2 Application: Balancing Tor

Remark 30.2.1: Our proof in 2.7 that $\mathrm{Tor}_*^R(A, B)$ could be computed either by a projective resolution $P_* \twoheadrightarrow A$ or a projective resolution $Q_* \twoheadrightarrow B$ was a disguised spectral sequence argument. So we'll go recover it using the actual spectral sequence.

Remark 30.2.2: We have a QI double complex C given by $C_{p,q} \coloneqq (P \otimes Q)_{p,q} = P_p \otimes Q_q$, and we now have two spectral sequences converging to $H_*(\mathrm{Tot}(P \otimes Q))$. Taking the first filtration, we can write

$$ H_q^v(\mathrm{Tot}(C)) = H_q(P_p \otimes Q_q) = P_p \otimes H_q(Q). $$

Using that P is an exact complex, and noting that we delete the augmentation when taking homology, we have

$$ H_1^v(\mathrm{Tot}(C)) = \begin{cases} 0 & q > 0 \\ P_p \otimes B & q = 0. \end{cases} $$

Thus

$$ E^2_{p,q} = \begin{cases} H_p^h(P_* \otimes B) & q = 0 \\ 0 & 1 > 0, \end{cases} $$

meaning that this collapses at E^2 and we have

$$ H_p(\mathrm{Tot}(P \otimes Q)) \cong L_p(- \otimes B)(A) \coloneqq \mathrm{Tor}_p^R(A, B). $$

Now consider taking the second filtration, which yields

$$ {}^{II}E^1_{p,q} = H_q^h(P_q \otimes Q_p) = H_q(P_*) \otimes Q_p = \begin{cases} A \otimes Q_p & q = 0 \\ 0 & q > 0. \end{cases} $$

The second pages comes from taking the vertical homology, so

$$ {}^{II}E^2_{p,q} = H_p^v H_q^h(P_q \otimes Q_p) = \begin{cases} H_p^v(A \otimes Q) & q = 0 \\ 0 & q > 0. \end{cases}, $$

which is $L_p(A\otimes -)(B)$ in $q=0$. Since ${}^{II}E^2_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(P\otimes Q)) = L_p(-\otimes B)(A)$, and we thus have

$$L_p(A\otimes -)(B) \cong L_p(-\otimes B)(A).$$

Remark 30.2.3: See the this section of Weibel for other applications in the exercises: the Kunneth formula, the Universal Coefficient Theorem, and the Acyclic Assembly Lemma.

30.3 Hypercohomology

Remark 30.3.1: We'd like to compute derived functors acting on chain complexes instead of just objects.

Definition 30.3.2 (Cartan-Eilenberg Resolutions)
Let $\mathcal{A}$ be an abelian category with enough projectives and let $A_* \in \mathsf{Ch}(\mathcal{A})$. A (left) **Cartan-Eilenberg resolution** (a CE resolution) $P_{*,*}$ of A_* is an upper half-plane complex (so $P_{p,q}=0$ when $q<0$) of projective objects and an augmentation chain map $P_{*,0} \xrightarrow{\varepsilon} A_*$ such that

1. If $A_p = 0$ then the entire column $P_{p,*}$ is zero.
2. The augmentation induces maps on boundaries and in homology which are projective resolutions in $\mathcal{A}$:

$$B_p(P, d^h) \xrightarrow{B_p(\varepsilon)} B_p(A)$$
$$H_p(P, d^h) \xrightarrow{H_p(\varepsilon)} H_p(A).$$

Remark 30.3.3: So we have the following situation

$$\begin{tikzcd}
q: & \cdots & P_{p+1,q} \arrow[l] & P_{p,q} \arrow[l] & P_{p-1,q} \arrow[l] & \cdots \arrow[l] \\
 & & \vdots & \vdots & \vdots & \\
 & \cdots & P_{p+1,1} \arrow[l] \arrow[d] & P_{p,1} \arrow[l] \arrow[d] & P_{p-1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & P_{p+1,0} \arrow[l] \arrow[d] & P_{p,0} \arrow[l] \arrow[d] & P_{p-1,0} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \cdots & A_{p-1} \arrow[l] & A_p \arrow[l] & A_{p-1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Link to Diagram

The situation in row q will be:

$$
\begin{tikzcd}
\cdots & P_{p+1,q} \arrow[l] & P_{p,q} \arrow[l] & P_{p-1,q} \arrow[l] & \cdots \arrow[l] \\
 & & Z_p(P, d^h) \arrow[u, hook] & & \\
 & & & H_p(P, d^h)_q & \\
 & & B_p(P, d^h) \arrow[uu, hook] & &
\end{tikzcd}
$$

Link to Diagram

Here when we take the homology of the complex along the rows p, we'll obtain

$$H_q(P, d^h) = \frac{Z_p(P, d^h)_q}{B_p(P, d^h)_q},$$

and since the induces maps preserve cycles and boundaries, we get induced maps on homology.

Exercise 5.7.1 shows that $P_{p,*} \xrightarrow{\varepsilon} A_p$ will be a projective resolution in $\mathcal{A}$ and so $Z_p(P, d^h)_* \to Z_p(A)$.

Lemma 30.3.4 ***(?).***
Every A_* has a CE resolution $P_{*,*} \xrightarrow{\varepsilon} A$.

Proof (?).
Choose a levelwise resolution and use the horseshoe lemma:

$$
\begin{tikzcd}
0 \arrow[r] & B_p(A) \arrow[r] & Z_p(A) \arrow[r] & H_p(A) \arrow[r] & 0 \\
0 \arrow[r] & P^B_{p,*} \arrow[u] \arrow[r, dashed] & P^Z_{p,*} \arrow[u, dashed] \arrow[r, dashed] & P^H_{p,*} \arrow[u] \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Recall that this involved a direct sum construction. Now do a similar thing for the following SES:

$$\begin{tikzcd}
0 \arrow[r] & Z_p(A) \arrow[r] & A_p \arrow[r, "d_p"] & B_{p-1}(A) \arrow[r] & 0 \\
0 \arrow[r] & P^Z_{p,*} \arrow[u] \arrow[r, dashed] & P^A_{p,*} \arrow[u, dashed] \arrow[r, dashed, "\tilde{d}_p"] & P^B_{p-1,*} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

We use the fact that we have the two side resolutions from the previous step. So set $P_{p,q} \coloneqq P^A_{p,q}$ assembled into a double complex using the sign trick: $d^v \coloneqq (-1)^p d$ where we used the differential d from $P^A_{p,*}$. We can now define

$$d^h : P^A_{p+1,*} \xrightarrow{\tilde{d}_{p+1}} P^B_{p,*} \hookrightarrow P^Z_{p,*} \hookrightarrow P^A_{p,*}.$$

One then checks that $B_p(\varepsilon)$ and $H_p(\varepsilon)$ are indeed projective resolutions.

■

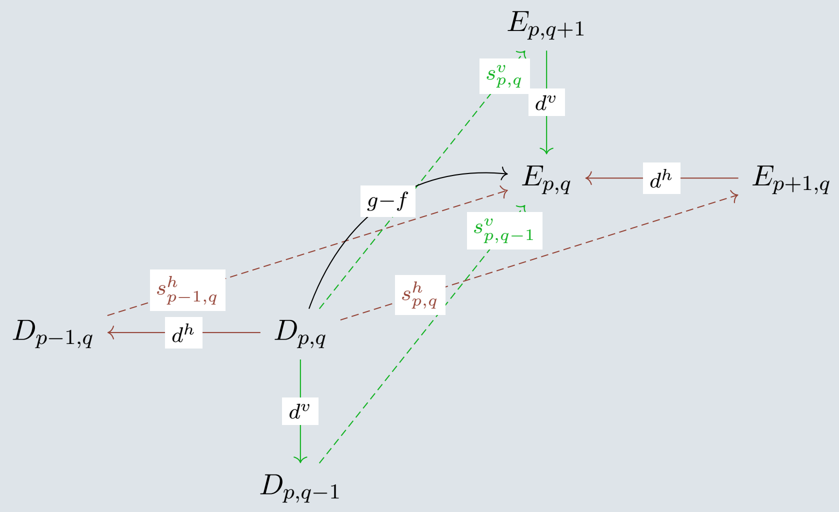
31 | Monday, March 29

31.1 Maps of Double Complexes

Remark 31.1.1: Last time: we talked about hypercohomology. We're doing this so we can set up a Grothendieck spectral sequence.

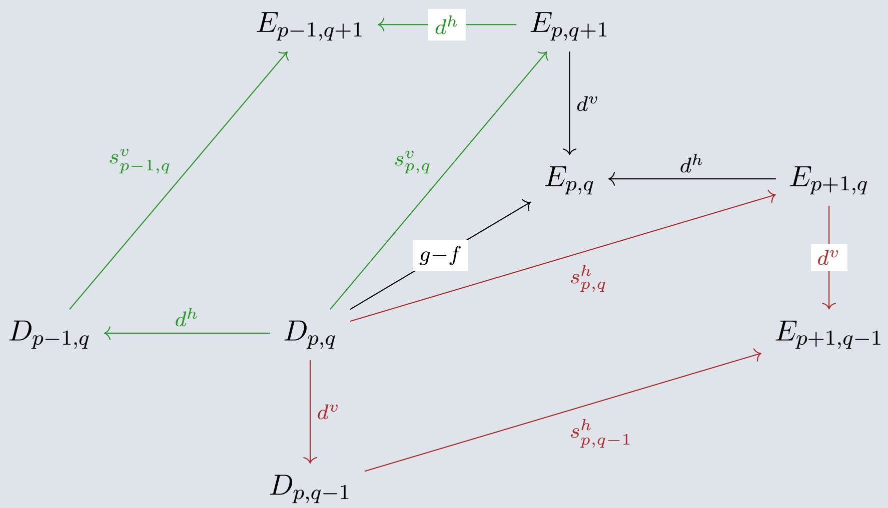
Definition 31.1.2 (Chain homotopies of double complexes)
Let $f, g : D \to E$ be two maps between double complexes. A **chain homotopy** from f to g consists of $s^h_{p,q} : D_{p,q} \to E_{p+1,q}$ and $s^v_{p,q} : D_{p,q} \to E_{p,q+1}$ for all p, q satisfying the following conditions:

1. All of the possible maps $D_{p,q} \to E_{p,q}$ summed should be equal to $g - f$, i.e. $g - f = (d^h s^h + s^h d^h) + (d^v s^v + s^v d^v)$:



Link to Diagram

2. The two rectangles below should be zero, i.e. $s^v d^h + d^h s^v = 0 = s^h d^v + d^v s^h$:



Link to Diagram

Remark 31.1.3: The definition is set up so that $s^h + s^v : \mathrm{Tot}(D)_n \to \mathrm{Tot}(E)_{n+1}$ is a chain homotopy $\mathrm{Tot}^{\oplus}(D) \to \mathrm{Tot}^{\oplus}(E)$.

Remark 31.1.4: Exercises 5.7.2 and 5.7.3 show:

1. If $f : A \to B$ is a chain map and $P \to A, Q \to B$ are CE resolutions, then there is a map of double complexes $\tilde{f} : P \to Q$ lifting f.

2. If $f, g : A \to B$ are chain homotopic, then $\tilde{f}, \tilde{g}$ are chain homotopic in the sense just defined.

3. Any two CE resolutions P, P' of A are chain homotopy equivalent, as are $\operatorname{Tot}^{\oplus}(F(P))$ and $\operatorname{Tot}^{\oplus}(F(P'))$ for any additive functor F.

Remark 31.1.5: This last remark shouldn't be too hard to believe: chain homotopies are defined in terms of addition.

31.2 Hypercohomology

Definition 31.2.1 (Hyper Left-Derived Functors)
Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor where $\mathcal{A}$ has enough projectives and $\mathcal{B}$ is cocomplete (closed under direct sums/coproducts). If $A \in \mathsf{Ch}(\mathcal{A})$ is a chain complex and $P \to A$ a CE resolution, define

$$\mathbb{L}_i F(A) \coloneqq H_i \operatorname{Tot}^{\oplus} F(P) : \mathsf{Ch}(\mathcal{A}) \to \mathcal{B}.$$

If $f : A \to B$ is a chain map in $\mathsf{Ch}(\mathcal{A})$ and $\tilde{f} : P \to Q$ where P, Q are CE resolutions of A, B resp., define $L_i F(f)$ to be the map

$$H_i \operatorname{Tot}(F\tilde{f}) \to \mathbb{L}_i F(B).$$

This yields a functor

$$\mathbb{L}_i F : \mathsf{Ch}(\mathcal{A}) \to \mathcal{B},$$

the **hyper left-derived functor** of F.

Remark 31.2.2: Recall that chain homotopy yields a notion of equivalence, and chain homotopic maps induce the same map on homology. The same is true for double complexes. There is a lemma that shows a SES of double complexes induces a LES in homology.

Proposition 31.2.3 ***(Convergence of spectral sequences and filtration comparison).***

a. There is always a convergent spectral sequence

$$^{II}E^2_{p,q}(L_p F)(H_q(A)) \Rightarrow \mathbb{L}_{p+q} F(A).$$

b. If A is bounded below complex, so there exists a p_0 such that $A_p = 0$ for $p < p_0$, then there is another spectral sequence

$$^{I}E^2_{p,q} = H_p L_q F(A) \Rightarrow \mathbb{L}_{p+q} F(A).$$

Proof (of (a)).
These are the spectral sequences associated to the upper half-plane double complex $FP_{*,*}$. Recall that $^{II}E^2_{p,q} = H^v_p H^h_q(FP) \Rightarrow H_{p+q} \operatorname{Tot}^{\oplus} FP \coloneqq \mathbb{L}_{p+q} F(A)$. The filtration by rows is exhaustive since we are taking the direct sum, so any cycle or boundary is supported in some

finite row. So what we want to show is that

$$^{II}E^2_{p,q}(L_pF)(H_qA) = H^v_p H^h_q(FP).$$

The main claim is the following: $H^h_q(FP) = FH^h_q(P)$.
Fix a row p of the double complex so we can drop p and h from the notation. We have the following situation:

$$\begin{tikzcd}
\cdots & P_{q-1} \arrow[l, "d"'] & P_q \arrow[l, "d"'] & P_{q+1} \arrow[l, "d"'] & \cdots \arrow[l, "d"'] \\
& & Z_q \arrow[u, hook] & & \\
& & B_q \arrow[u, hook] & &
\end{tikzcd}$$

Link to Diagram

We have a SES

$$0 \to B_q \to Z_q \to H_q \to 0,$$

which induces a LES

$$\begin{tikzcd}
\cdots \arrow[r] & L_2FH_q \arrow[r] & L_1FH_q \arrow[dll, hook] & \\
FB_q \arrow[r] & FZ_q \arrow[r] & FH_q \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

We have $L_1FH_q = 0$, since in the CE resolution we assume that $H_q(P, d^h)$ is projective.
The second SES we have is

$$0 \to Z_q \to P_q \xrightarrow{d} B_{q-1}$$

inducing the LES

$$\begin{tikzcd}
\cdots \arrow[r] & L_2FP_q \arrow[r] & L_1FB_{q-1} \arrow[dll, hook] & \\
FZ_q \arrow[r] & FP_q \arrow[r] & FB_{q-1} \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here $L_iFB_{q-1} = 0$ since $B_{p-q}(P, d^h)$ was projective. Putting these together, we have

$$H_q(FP) = \frac{\ker Fd : FP_q \to FP_{q-1}}{\operatorname{im} Fd : FP_{q+1} \to FP_q} \cong \frac{FZ_q}{FB_q} \cong FH_q(P_{*,*}).$$

Now what is its vertical homology? The map $H_q(P_{*,*}) \to H_q(A)$ is a projective resolution, so apply F to the source – it's no longer exact, and you get $FH_q(P)$ from above, and taking homology yields the left-derived functors applied to the source. Thus

$$H_p^v FH_q^h(P) = L_pF(H_q(A)),$$

and the left-hand side is equal to $H_p^v H_q^h(FP)$.

■

Exercise 31.2.4 (Prove (b))
Prove part (b) of the proposition.

Remark 31.2.5: There is a cohomology variant of this: everything dualizes to $R^iF(A)$ for a left exact functor $F : \mathcal{A} \to \mathcal{B}$ where $A \in \mathsf{Ch}(\mathcal{A})$, $\mathcal{A}$ has enough injectives, and B is complete. Using a right CE resolution $I^{*,*}$ of injective objects in A yields an upper half-plane complex with $A^* \to I^{*,0}$ such that the induces maps on cohomology are themselves injective resolutions of $B^p(A^*)$ and $H^p(A^*)$. In this case

$$R^iF(A^*) = H^i \operatorname{Tot}^{\Pi} F(I^{*,*}).$$

We can prove dual version of all of the results about left hyper-derived functors, although there are some slight convergence issues to worry about due to the direct product.

32 | Wednesday, March 31

Remark 32.0.1: Last time we talked about hypercohomology and hyper derived functors, and we proved that two spectra sequences converging to $\mathbb{L}_{p+q}F(A)$.

32.1 Grothendieck Spectral Sequences

Remark 32.1.1: We'll focus on the cohomological version, which gives a spectral sequence from a composition of functors. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories with enough injectives, and let $G : \mathcal{A} \to \mathcal{B}$, $F : \mathcal{B} \to \mathcal{C}$ be left exact functors. By a previous result, $FG : \mathcal{A} \to \mathcal{C}$ is left exact, which follows from checking that it preserves 4-term exact sequences. Recall that $B \in \mathcal{B}$ is F-acyclic if $R^iF(B) = 0$ for all $i > 0$.

Theorem 32.1.2 *(Grothendieck Spectral Sequence).*
Assume the above setup, and that G sends injectives in $\mathcal{A}$ to F-acyclic objects in $\mathcal{B}$. Then there is a convergent QI spectral sequence for each $A \in \mathcal{A}$:

$$E_2^{p,q} = (R^p F)(R^q G)(A) \Rightarrow R^{p+q}(FG)(A).$$

The edge maps are the natural maps

$$\begin{gathered}(R^p F)(GA) \to R^p(FG)(A) \\ R^q(FG)(A) \to F(R^q G(A)).\end{gathered}$$

The exact sequences of the low-degree terms are

$$0 \to (R^j F)(GA) \to R^j(FG)(A) \to F(R^j G(A)) \to (R^j F)(GA) \to R^j(FG)(A).$$

Proof (?).
Choose an injective resolution $A \hookrightarrow I$ in $\mathcal{A}$ and apply G to form the cochain complex $G(I) \in \mathcal{B}$. Using a first quadrant CE resolution of $G(I)$, form the hyper right-derived functors $\mathbb{R}^i F(G(I))$. We have the two spectral sequences that converge to this, since the complex is bounded below:

$${}^{I}E_1^{p,q} = H^p R^q F(GI) \Rightarrow (\mathbb{R}^{p+q} F)(GI).$$

By hypothesis I^p is injective in $\mathcal{A}$, and thus $G(I^p)$ is F-acyclic in $\mathcal{B}$, so this spectral sequence collapses onto the horizontal axis at the 2nd page. So $(\mathbb{R}^p F)(GI) = H^p(FG(I))$, which is by definition $R^p(FG)(A)$, and this holds for all $p > 0$. This follows because only one term survives on each diagonal, and the associated graded is just to those terms, so it lifts to just being the actual homology.
The second spectral sequence converges to the same thing, and so by reindexing the previous limiting term $p \mapsto p + q$, we can write

$${}^{II}E_2^{p,q} = (R^p F)(H^q(GI)) \Rightarrow R^{p+q}(FG)(A).$$

But this is $(R^p F)(R^q G)(A)$ by definition.
By example 5.2.6, the edge maps from the p-axis are

$$E_2^{p,0} \to E_\infty^{p,0} \hookrightarrow H^p,$$

and composing these yields $(R^p F)(GA) \to R^p(FG)(A)$. We also have $H^q \twoheadrightarrow E_\infty^{p,0} \hookrightarrow E_2^{0,q}$. ■

Remark 32.1.3: We're skipping the section on sheaf cohomology and 5.9, so we'll move into chapter 6.

32.2 6.8: The Lyndon-Hochschild-Serre Spectral Sequence

Remark 32.2.1: Let $H \trianglelefteq G$ and $A \in \mathsf{G\text{-}Mod}$, then $A_H, A^H \in \mathsf{G/H\text{-}Mod}$. The canonical projection $p : G \to G/H$ induces a forgetful functor $p^* : \mathsf{G/H\text{-}Mod} \to \mathsf{G\text{-}Mod}$ given by pullback. Note that G/H-modules are essentially G-modules where H acts trivially, so this functor forgets the trivial H action. Generally, this works a bit like the Frobenius map, which yields a representation that can be pulled back.

Lemma 32.2.2 *(?).*
The invariant functor $(-)_H$ has a left adjoint and the coinvariant functor $(-)^H$ has a right adjoint.

Proof (?).
A G/H-module is a G-module with a trivial H action, so both A_H, A^H are G/H-modules. One needs to check that although H preserves these submodules, so does G. The universal property of $A^H \hookrightarrow A$ as the largest trivial submodule and $A \to A_H$ as the largest trivial quotient imply that there are natural isomorphisms: for $A \in \mathsf{G\text{-}Mod}$ and $B \in \mathsf{G/H\text{-}Mod}$,

$$\begin{aligned} \operatorname*{Hom}_G(p^*B, A) &\xrightarrow{\sim} \operatorname*{Hom}_{G/H}(B, A^H) \\ f &\mapsto f \end{aligned}$$

which is well-defined since $f(b) = f(hb) = hf(b) = f(b)$, putting $f(b) \in A^H$. We also have

$$\begin{aligned} \operatorname*{Hom}_G(A, p^\sharp B) &\xrightarrow{\sim} \operatorname*{Hom}_{G/H}(A_H, B) \\ (\tilde{f} : A \xrightarrow{\pi} A_H \xrightarrow{f} B) &\leftarrow f, \end{aligned}$$

and these give the required adjunction. ■

Theorem 32.2.3 *(Lyndon-Hochschild-Serre Spectral Sequence).*
Let $H \trianglelefteq G$ for $A \in \mathsf{G\text{-}Mod}$, then there are two QI spectral sequences:

$$\begin{aligned} E^2_{p,q} &= H_p(G/H, H_q(H, A)) \\ E_2^{p,q} &= H^p(G/H, H^q(H, A)). \end{aligned}$$

Remark 32.2.4: Note that we can identify the functors

$$(-)^H, (-)_H : \mathsf{G\text{-}Mod} \to \mathsf{G/H\text{-}Mod},$$

whose derived functors are group homology/cohomology. The idea will be that G-invariants can be written as a composition of other functors, and we can apply the Grothendieck spectral sequence construction.

33 | Friday, April 02

33.1 Review: The Lyndon-Hochschild-Serre Spectral Sequence

Remark 33.1.1: We're trying to prove the Lyndon-Hochschild-Serre spectral sequence for $H \trianglelefteq G$.

Lemma 33.1.2 *(?).*
Let $H \trianglelefteq G$ and $A \in \mathsf{G\text{-}Mod}$ with

$$\rho : G \to \frac{G}{H}.$$

Then A_H, A^H are in $\frac{\mathsf{G}}{\mathsf{H}}$-$\mathsf{Mod}$ and $(-)^H$ (respectively $(-)_H$) are right (respectively left) adjoin to

$$\varphi^{\#} : \frac{\mathsf{G}}{\mathsf{H}}\text{-}\mathsf{Mod} \to \mathsf{G\text{-}Mod}.$$

Theorem 33.1.3 *(Lyndon-Hochschild-Serre Spectral Sequence).*
Let $H \trianglelefteq G$ and $A \in \mathsf{G\text{-}Mod}$, then there exist two $Q1$ spectral sequences:

$$E^2_{p,q} = H_p\left(\frac{G}{H}, H_q(H; A)\right) \Rightarrow H_{p+q}(G; A)$$
$$E_2^{p,q} = H^p\left(\frac{G}{H}, H_q(H; A)\right) \Rightarrow H^{p+q}(G; A).$$

Proof (?).
We want to write this as a composition of functors:

$$\begin{tikzcd}
\mathsf{G\text{-}Mod} \arrow[r, "(-)^H"] \arrow[dr, dashed, "(-)^G"'] & \mathsf{G/H\text{-}Mod} \arrow[d, "(-)^{\frac{G}{H}}"] \\
 & \mathsf{Ab}
\end{tikzcd}$$

Link to Diagram

We can write

$$\begin{aligned}(A^H)^{G/H} &= \left\{a \in A \;\middle|\; ha = a \forall h \in H\right\} \\ &= \left\{a \in A^H \;\middle|\; \bar{g}a = a \forall \bar{g} \in G/H\right\} \\ &= \left\{a \in A \;\middle|\; \alpha = a \forall g \in G\right\} \\ &= A^G.\end{aligned}$$

By the lemma, $(-)^H$ is right adjoint to $\rho^\#$, which is exact. By prop 2.3.10, it sends injectives to injectives, and injectives are F-acyclic for $F(-) = (-)^{\frac{G}{H}}$. So this is a valid setup for the Grothendieck spectral sequence.

■

33.2 Application: Bootstrapping Homology of Cyclic Groups

Example 33.2.1 ***(?):*** Let C_m be cyclic of order m, and suppose we have the results from section 6.2:

1. If m is odd,

$$H_q(C_m; \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/m & q \text{ odd} \\ 0 & q \text{ even.} \end{cases}$$

2. If $H \leq Z(G)$ and A is a trivial G-module, then $G/H \curvearrowright H_*(H; A)$ trivially as well. [4]

3. If A is a trivial C_2-module and let $\times 2 : A \to A$ be multiplication, then

$$H_p(C_2; A) = \begin{cases} A & p = 0 \\ \operatorname{coker}(\times 2) = A/2A & p \text{ odd} \\ \ker(\times s) = \left\{a \in A \;\middle|\; 2a = 0\right\} & p \text{ even.} \end{cases}$$

   Note that the previous fact was a special case of multiplication by m.

Using the SES

$$0 \to C_m \to C_{2m} \to C_2 \to 0,$$

we can use the LHS spectral sequence to compute

$$E^2_{p,q} = H_p(C_2; H_q(C_m; \mathbb{Z})) \Rightarrow H_{p+q}(C_{2m}; \mathbb{Z}).$$

Let $A = H_q(C_m; \mathbb{Z})$, then by fact (2) we'll get a trivial C_2-module, and we can then use fact (3).

[4]Note that this can be phrased in terms of the image of the functor lying in trivial modules.

- For $q = 0$ we have

$$E^2_{p,0} = H_p(C_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}/2 & p \text{ odd} \\ 0 & p \text{ even} \end{cases} \qquad \text{by (3).}$$

- For $p = 0$ we have

$$E^2_{0,q} = H_q(C_m;\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}/m & p \text{ odd} \\ 0 & p \text{ even.} \end{cases}$$

- For $q > 0$ odd and $p > 0$ odd, note that $\mathbb{Z}/m \xrightarrow{\times 2} \mathbb{Z}/m$ is a bijection for odd m, so

$$E^2_{p,q} = H_p(C_2;\mathbb{Z}/m) = 0 \qquad \text{since } \frac{\mathbb{Z}/m}{2\mathbb{Z}/m} = 0.$$

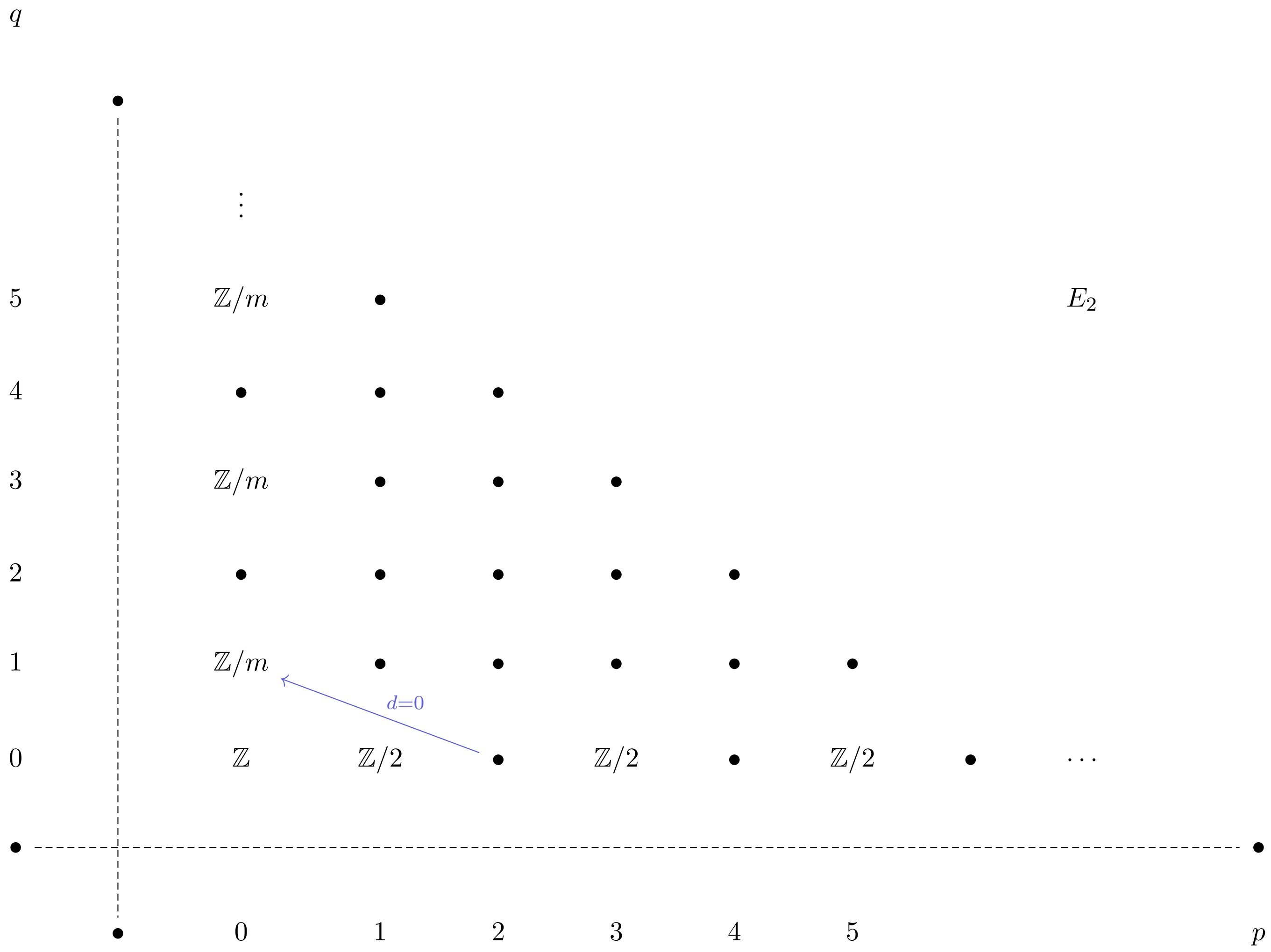
- For $q > 0$ odd and $p > 0$ even,

$$E^2_{p,q} = H_p(C_2;\mathbb{Z}/m) = 0.$$

- For $q > 0$ even and $p > 0$,

$$H_q(C_m;\mathbb{Z}) = 0 \implies E^2_{p,q} = 0.$$

Thus the E_2 page of the LHS spectral sequence looks like the following, where there is only one possible nontrivial differential which is forced to be zero:



Link to Diagram

Note that each diagonal only has (at most) two nonzero terms along the axes, and so we'll get a 2-term filtration. Recall that in general we get $\{F_i H_n\}_{i=1}^n$ where $F_{\leq -1} H_n = 0$ and $F_{\geq n} H_n = H_n$. Here $E_{0,n}^\infty$ comes from F_{-1}, F_0 and $E_{n,0}^\infty$ comes from F_{n-1}, F_n. So we have

$$
\begin{aligned}
H_0(C_{2m}; \mathbb{Z}) &= \mathbb{Z} \\
H_n(C_{2m}; \mathbb{Z}) &= 0 \text{ for } n \text{ even.}
\end{aligned}
$$

For n odd, we get a SES

$$
0 \to \mathbb{Z}/m \to H_n(C_{2m}; \mathbb{Z}) \to \mathbb{Z}/2 \to 0.
$$

Letting $B \in \mathsf{Ab}$ be the middle term, its order is $2m$, the product of the two outer elements. By Cauchy's theorem, since $2 \mid \# B$, there is an element $y \in B$ of order 2. So send the generator of $\mathbb{Z}/2$ to y to form the splitting. Thus

$$
B \cong \mathbb{Z}/m \oplus \mathbb{Z}/2 \cong \mathbb{Z}/m \times \mathbb{Z}/2 \cong \mathbb{Z}/2m,
$$

where we've now used the $\gcd(2, m) = 1$. So

$$H_n(C_{2m}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2m & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Question 33.2.2
Can you get the group homology of any cyclic group this way? Similar formulas likely hold, see section 6.2.

33.3 Restriction and Inflation

Remark 33.3.1: The exact sequence of low degree terms in the cohomological LHS spectral sequence are of the form

$$\begin{tikzcd} 0 \arrow[r] & H^1(G/H; A^H) \arrow[r, "\text{inflation}"] & H^1(G; A) \arrow[r, "\text{restriction}"] & H^1(H; A) \arrow[dll, "d_2"] \\ & H^2(G/H; A^H) \arrow[r, "\text{inflation}"] & H^2(G; A) & \end{tikzcd}$$

Link to Diagram

Note that these maps have particular name, **inflation** and **restriction**.

Remark 33.3.2: We thought of homology as a functor of the module A, but here we see it's varying. Can this be thought of as a functor of the group instead?

Setup: let $\rho : H \to G$ be a group morphism, then recall that any G-module becomes an H-module by composition with ρ, which yields an exact functor

$$\rho^\# : \mathsf{G\text{-}Mod} \to \mathsf{H\text{-}Mod}.$$

Letting $A \in \mathsf{G\text{-}Mod}$, set

- $T_n(A) := H_n(G; A)$
- $T^n(A) := H^n(G; A)$
- $S_n(A) := H_n(G; \rho^\# A)$
- $S^n(A) := H^n(G; \rho^\# A)$

34 | Monday, April 05

34.1 Restriction and Inflation

Definition 34.1.1 (Restriction and Corestriction)
Let $\rho : H \to G$ be a group morphism, this induces an exact functor $\rho^\sharp : \mathsf{G\text{-}Mod} \to \mathsf{H\text{-}Mod}$. We define

- $T_n(A) := H_n(G; A)$
- $T^n(A) := H^n(G; A)$
- $S_n(A) := H_n(\rho^\sharp G; A)$
- $S^n(A) := H^n(\rho^\sharp G; A)$

These are all functors $\mathsf{G\text{-}Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$. As in section 2.1, H_n defines a homological δ-functor, and since $\rho^\sharp$ is exact, T_n, S_n are homological δ-functors as well. We have a map

$$\begin{aligned} A^G &\hookrightarrow (\rho^\sharp A)^H \\ T^0 A &\to S^0 A. \end{aligned}$$

and similarly

$$\begin{aligned} (\rho^\sharp A)_H &\to A_G \\ S_0 A &\to T_0 A. \end{aligned}$$

These maps on the 0th terms extend to morphisms of δ-functors.
There thus exist two maps

$$\begin{aligned} \operatorname{res}_H^G H^*(G; A) &\to H^*(G; \rho^\sharp A) && \text{restriction} \\ \operatorname{cores}_H^G H_*(G; \rho^\sharp A) &\to H_*(G; A) && \text{corestriction.} \end{aligned}$$

Remark 34.1.2: A special case is when $H \leq G$ is a subgroup and $\rho : H \hookrightarrow G$ is the inclusion. Then we define a capital Res as

$$\rho^\sharp = \operatorname{Res}_H^G : \mathsf{G\text{-}Mod} \to \mathsf{H\text{-}Mod},$$

which is a restriction of the action to a subgroup and thus a type of forgetful functor.

Remark 34.1.3: Note that $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module with basis being any set of coset representatives, thus any projective G-module restricts to a projective H-module, using the characterization of projective modules as direct summands of free modules.

Remark 34.1.4: Recall that

$$H_*(G;A) \cong \operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$$
$$H_*(G;A) \cong \operatorname{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, A).$$

We can compute both using a $\mathbb{Z}G$-projective resolution $P_* \to \mathbb{Z}$. This is also a $\mathbb{Z}H$-projective resolution, so we can use this to compute $H^*(H;-)$ and $H_*(H;-)$ as well.

Fact 34.1.5

1. There's a natural chain map induced by the forgetful functor:

$$\beta : \operatorname*{Hom}_G(P_*, A) \to \operatorname*{Hom}_H(P^*, A).$$

2. There is an induced map

$$H^*(\beta) : \operatorname{Ext}_G^*(\mathbb{Z}, A) \to \operatorname{Ext}_H^*(\mathbb{Z}, A),$$

which is equal to the map

$$\operatorname{res}_H^G : H^*(G;A) \to H^*(H;A),$$

giving a way to calculate res from something just coming from restriction of functions.

3. There is a chain map

$$\alpha : P_* \otimes_{\mathbb{Z}H} A \to P_* \otimes \mathbb{Z}H P_* \otimes_{\mathbb{Z}G} A$$
$$p \otimes a \mapsto p \otimes a,$$

which induces

$$H(\alpha) : \operatorname{Tor}_*^H(\mathbb{Z}, A) \to \operatorname{Tor}_*^G(\mathbb{Z}, A)$$

which is equal to

$$\operatorname{cores}_H^G : H_*(H;A) \to H_*(G;A).$$

So this can be computed from tensor products.

Definition 34.1.6 (Inflation and Coinflation)
Now consider quotient groups instead: assume $H \trianglelefteq G$ and let $\rho : G \to G/H$. By precomposing with ρ, we get a map $\rho^\sharp : \frac{\mathsf{G}}{\mathsf{H}}\text{-}\mathsf{Mod} \to \mathsf{G}\text{-}\mathsf{Mod}$. Given a G-module, taking H invariants yields a G/H-module, so $H^*(G/H; A^H) \in \frac{\mathsf{G}}{\mathsf{H}}\text{-}\mathsf{Mod}$. We form the following composition:

$$H^*\left(\frac{G}{H}; A^H\right) \xrightarrow{\text{res}} H^*(G; A^H) \xrightarrow{H^*(G;-)(A^H \hookrightarrow A)} H^*(G;A)$$

Link to Diagram

We'll refer to this as **inflation**. We similarly define **coinflation** as the following composition:

$$H_*(G;A) \xrightarrow{H_*(G;-)(A \twoheadrightarrow A_H)} H(G;A_H) \xrightarrow{\quad\text{cores}\quad} H_*\left(\frac{G}{H}, A_H\right)$$

Link to Diagram

Remark 34.1.7: When $* = 0$, we can write

$$\inf : (A^H)^{\frac{G}{H}} \to (A^H)^G \to A^G,$$

and note that this is exactly the functor composition we needed to get the LHS spectral sequence. Similarly there is a LHS for homology, and an isomorphism

$$\text{coinf} : A_G \to (A_H)_G \to (A_H)_{\frac{G}{H}}.$$

Remark 34.1.8: When $A \in \mathsf{H\text{-}Mod}^{\text{Triv}}$, $A_H \hookrightarrow A$ is the identity, so $A^H = A = A_H$. In this case $\inf = \text{res}$ and $\text{coinf} = \text{cores}$.

Remark 34.1.9: Back to the LHS spectral sequence, the five-term exact sequence yields

$$0 \to E_2^{1,0} \to H^1(T) \to E_2^{0,1} \xrightarrow{d_2} E_{2,0} \to H^2(T),$$

which we can identify as

$$0 \to H^1\left(\frac{G}{H}; A^H\right) \xrightarrow{\inf} H^1(G;A) \xrightarrow{\text{res}} H^1(H;A)^{\frac{G}{H}} \xrightarrow{d_2} H^2\left(\frac{G}{H}; A^H\right) \xrightarrow{\inf} H^2(G;A).$$

There is a similar story in homology with coinflation and corestriction.

34.2 Shapiro's Lemma, Induced/Coinduced Modules

Definition 34.2.1 (Induced and Coinduced Modules)
Let $H \leq G$ and $B \in \mathbb{Z}\mathsf{H\text{-}Mod}$. Define the **induced G-module** (or tensor-induced G-module)

$$\text{Ind}_H^G(B) := \mathbb{Z}G \otimes_{\mathbb{Z}H} B \in \mathbb{Z}\mathsf{G\text{-}Mod}.$$

This is a $\mathbb{Z}G$-module with an action on the first tensor factor. Similarly define the **coinduced** or **hom-induced G-module**.

$$\text{coInd}_H^G(B) := \underset{H}{\text{Hom}}(\mathbb{Z}G, B) \in \mathbb{Z}\mathsf{G\text{-}Mod}.$$

Here the action is $(g.f)(g') := f(gg')$.

Lemma 34.2.2 *(Shapiro's Lemma (Frobenius Reciprocity)).*

$$H_*(G; \operatorname{Ind}_H^G B) \cong H_*(H; B) \quad (1)$$
$$H^*(G; \operatorname{coInd}^G B) \cong H^*(H; B) \quad (2).$$

Remark 34.2.3: So this provides a way of computing homology on subgroups when the coefficients are in these induced/coinduced modules.

35 | Wednesday, April 07

35.1 6.3: Shapiro's Lemma, (co)Induced Modules (cont)

Remark 35.1.1: Recall that we had two ways of inducing an H-module up to a G-module for $H \leq G$ a subgroup. In this case, we can take cohomology with coefficients in any $B \in \mathbb{Z}\mathsf{H}\text{-}\mathsf{Mod}$. Shapiro's lemma (or Frobenius Reciprocity) allowed compute homology and cohomology when the coefficients are in induced or coinduced modules:

$$H_*(G; \operatorname{Ind}_H^G B) \cong H_*(H; B) \quad (1)$$
$$H^*(G; \operatorname{coInd}^G B) \cong H^*(H; B) \quad (2).$$

Proof (of Shapiro's lemma).
Let $P_* \to \mathbb{Z}$ be a right $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$ module, these are still projective over $\mathbb{Z}H$. Then take

$$P_* \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} B) \cong P_* \otimes_{\mathbb{Z}H} B.$$

The homology of the left-hand side computes $\operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, \operatorname{Ind}_H^G B)$. On the other hand, we can consider P_* to be a projective resolution in $\mathbb{Z}H$ and thus the homology of the right-hand side is $\operatorname{Tor}_*^{\mathbb{Z}H}(\mathbb{Z}, B)$, which is $H_*(H; B)$.
For (2), use the tensor-hom adjunction. [a]

■

[a] See proposition 2.6.3 in Weibel.

Theorem 35.1.2 *(Adjoints of Restriction are Induction and Coinduction).*

For $H \leq G, A \in \mathbb{Z}\mathsf{G}\text{-}\mathsf{Mod}, B \in \mathbb{Z}\mathsf{H}\text{-}\mathsf{Mod}$,

$$\operatorname{Ext}^*_G(\operatorname{Ind}^G_h B, A) \cong \operatorname{Ext}^*_H(B, \operatorname{Res}^G_H A) \tag{1}$$
$$\operatorname{Ext}^*_G(A, \operatorname{coInd}^G_H B) \cong \operatorname{Ext}^*_H(\operatorname{Res}^G_H A, B) \tag{1}.$$

Remark 35.1.3: Taking $A = \mathbb{Z} \in \mathbb{Z}\mathsf{G}\text{-}\mathsf{Mod}^{\mathrm{Triv}}$, one gets result (2) in Shapiro's lemma. This shows that Ind is left adjoint to Res and coInd is right adjoint to it, so these will have derived functors. A special case is when $H = \{1\}$ is the trivial group, in which case any H-module B is an abelian group such that $B^H = B = B_H$. So $(-)^H, (-)_H$ are exact, and thus their higher derived functors are zero, i.e. $H_n(H, B) = 0 = H^n(H; B)$ for $n > 0$. Moreover

$$H_n(G; \mathbb{Z}G \otimes_{\mathbb{Z}} B) \cong H^n(G, \underset{\mathbb{Z}}{\operatorname{Hom}}(\mathbb{Z}G, B)) \cong \begin{cases} B & n = 0 \\ 0 & n > 0. \end{cases}$$

Lemma 35.1.4 *(?).*
If the index $[G : H]$ (i.e. the number of left or right cosets) is finite, then

$$\operatorname{Ind}^G_H B \cong \operatorname{coInd}^G_H B \qquad \in \mathsf{G}\text{-}\mathsf{Mod}.$$

Proof (?).
Let X be a set of left coset representatives for G/H, where we'll take the convention that left cosets are of the form gH. Then X is a free $\mathsf{Mod}\text{-}\mathbb{Z}\mathsf{H}$-basis of $\mathbb{Z}G$, so

$$\operatorname{Ind}^G_H B \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} B \cong \bigoplus_{x \in X} x \otimes B \qquad \in \mathbb{Z}\text{-}\mathsf{Mod}.$$

How does $g \curvearrowright x \otimes b$ for $g \in G$? We have $gx \in yH$ for some $y \in X$, so for some $h \in H$ we have

$$gx = gh.$$

We can then compute

$$\begin{aligned} g(x \otimes b) &= gx \otimes b \\ &= yh \otimes b \\ &= y \otimes hb. \end{aligned}$$

since $h \in \mathbb{Z}H$. Now $X^{-1} \coloneqq \left\{ x^{-1} \;\middle|\; x \in X \right\}$ is a set of coset representatives for ${}_H\backslash^G$ and hence a left $\mathbb{Z}H$-basis for $\mathbb{Z}G$. We can thus write

$$\begin{aligned} \operatorname{coInd}^G_H B &\cong \underset{\mathbb{Z}H}{\operatorname{Hom}}(\mathbb{Z}G, B) \\ &\cong \underset{\mathbb{Z}H}{\operatorname{Hom}}\left(\bigoplus_{x \in X} \mathbb{Z}Hx^{-1}, B \right) \\ &= \prod_{x \in X} \underset{\mathbb{Z}H}{\operatorname{Hom}}(\mathbb{Z}Hx^{-1}, B) && \text{by exc. A.1.4} \\ &= \prod_{x \in X} \pi_x(A), \end{aligned}$$

where each term is a copy of A. This follows because we can specify such a module hom by specifying the image of a basis. So here for $b \in B$, $\pi_x(B)$ for a fixed x is the H-module morphism $\mathbb{Z}G \to B$ where $x^{-1} \mapsto b$ and $z^{-1} \mapsto 0$ for $z \neq x$.
How does G act on these homs? Using equation (??)

Sort out which equation this was!

we have

$$y^{-1}g = hx^{-1},$$

and thus

$$\begin{aligned}
(g \cdot \pi_x(b))(y^{-1}) &= (\pi_x(b))(y^{-1}g) \\
&= (\pi_x(b))(hx^{-1}) \\
&= h(\pi_x(b))(x^{-1}) \\
&= hb,
\end{aligned}$$

and y^{-1} is the only one that lights up for the G-action, i.e. $(g \cdot \pi_x(b))(z^{-1}) = 0$ for $y \neq z$, and thus

$$g \cdot \pi_x(b) = \pi_y(hb).$$

Thus we have a G-module map

$$\begin{aligned}
\operatorname{Ind}_H^G B &\xrightarrow{\sim} \operatorname{coInd}_H^G B \\
\bigoplus_{x \in X} x \otimes B &\xrightarrow{\sim} \bigoplus_{x \in X} \pi_x B \\
x \otimes B &\mapsto \pi_x(b),
\end{aligned}$$

which is an isomorphism since

$$g \cdot (x \otimes b) = y \otimes hb \mapsto \pi_y(hb) = g \cdot \pi_x(b).$$

■

Corollary 35.1.5 *(?)*.
If G is a finite group, then for any $A \in \mathsf{G\text{-}Mod}$,

$$H^{>0}(G; \mathbb{Z}G \otimes_{\mathbb{Z}} A) = 0.$$

Proof (?).
We think of A as a module for the trivial subgroup, and so

$$\begin{aligned}
H^n(G; \mathbb{Z}G \otimes_{\mathbb{Z}} A) &\cong H^n(G, \operatorname{Ind}_1^G A) \\
&\cong H^n(G; \operatorname{coInd}_1^G A) && \text{by the lemma} \\
&= H^n(1; A) && \text{by Shapiro's lemma} \\
&= 0,
\end{aligned}$$

for $n > 0$, since these are the higher derived functors of taking fixed points, and everything is fixed by 1.

$\blacksquare$

35.2 Lie Algebra (Co)homology

Remark 35.2.1: Motivation and historical background: if G is a Lie group, $G \in \mathsf{Grp} \cap \mathsf{Mfd}(C^\infty)$, i.e. the group operations are smooth maps. Usually these are real manifolds, they were introduced in the late 1800s by Sophus Lie who studied differential equations on such objects. Taking the tangent space at the identity, we write $\mathfrak{g} = T_e G$, which is a **Lie Algebra**. Lie showed that this is isomorphic to the vector space of left G-invariant vector fields (1st order differential operators) on G, which enjoys a **bracket** operation:

$$[X, Y](f) = X(Yf) - Y(Xf) \qquad\qquad f \in C^\infty.$$

This turns out to again be a 1st order operator, despite looking like it might be 2nd order. This led to the study of abstract Lie algebras.

36 Section 7.1: Lie Algebras (Friday, April 09)

36.1 Definitions

Definition 36.1.1 (k-algebras)
Let $k \in \mathsf{CRing}$, e.g. a field. An **algebra** over k is a k-module with a bilinear product $A^{\otimes 2} \to A$.

Remark 36.1.2: The product need not be associative, and A need not have 1, so $A = 0$ is an algebra.

Definition 36.1.3 (Lie Algebra Definitions)
A **Lie algebra** $\mathfrak{g}$ is a k-algebra whose product (denoted $[-,-]$) is called the **Lie bracket**, which satisfies

1. $[xx] = 0$ for all $x \in \mathfrak{g}$, and skew-symmetry: $[xy] = -[yx]$ for all $x, y \in \mathfrak{g}$.
2. The Jacobi identity:

$$[x[yz]] + [y[zx]] + [z[xy]] = 0 \iff [x[yz]] = [[xy]z] = [y[xz]],$$

so the product behaves like a derivation.

- There is an **adjoint** map $\mathrm{ad}_x := [-, x] : \mathfrak{g}\circlearrowleft$.
- A **(2-sided) ideal** $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is a k-submodule absorbing under the bracket, so $[xh] \in \mathfrak{h} \forall x \in \mathfrak{g}, h \in \mathfrak{h}$ In particular, $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra.
- A **morphism** $\rho : \mathfrak{g} \to \mathfrak{g}'$ of Lie algebras is a k-module map which preserves the bracket, so $[\rho(x)\rho(y)] = \rho([xy])$, so we get a category $\mathsf{Lie\text{-}Alg}_{/k}$.
- If $\mathfrak{h} \trianglelefteq \mathfrak{g}$, there is a **quotient** Lie algebra $\mathfrak{g}/\mathfrak{h}$ consisting of additive coset $x + \mathfrak{h}$, and a SES

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0.$$

- A Lie algebra $\mathfrak{g}$ is **abelian** if $[xy] = 0$ for all $x, y \in \mathfrak{g}$. Any k-module can be made into an abelian Lie algebra by setting $[xy] := 0$.
- The **derived subalgebra** of $\mathfrak{g}$ is $[\mathfrak{g}\mathfrak{g}] \trianglelefteq \mathfrak{g}$, the k-submodule of $\mathfrak{g}$ generated by all brackets $[xy]$. The largest abelian quotient of $\mathfrak{g}$ is given by $\mathfrak{g}/[\mathfrak{g}\mathfrak{g}]$.

36.2 Examples

Example 36.2.1 ***(?):*** Let A be any associative k-algebra, not necessarily with 1, and let $\mathfrak{g} := \mathrm{Lie}(A)$ be the same k-module with a bracket defined as $[xy] := xy - yx$. One can check that this satisfies the Jacobi identity. So there is a functor

$$\mathrm{Lie} : \mathsf{Alg}_{/k}(\mathsf{Assoc}) \to \mathsf{Lie\text{-}Alg}_{/k}.$$

In particular, for $A \in \mathsf{Alg}_{/k}(\mathsf{Assoc})$ (e.g. $A = k$), the ring $\mathrm{Mat}(m \times m; A) \in \mathsf{Alg}_{/k}(\mathsf{Assoc})$ and can be mapped into Lie Algebras. We write

$$\mathfrak{gl}_m(A) := \mathrm{Lie}(\mathrm{Mat}(m \times m; A)) \cong \mathrm{Lie}(\underset{k}{\mathrm{End}}(A^m)),$$

and often omit notation to write $\mathfrak{gl}_m := \mathfrak{gl}_m(k)$ where $[xy] := xy - yx$ as the **general linear Lie algebra** over A.

Example 36.2.2 ***(Important special cases):*** Let $A \in \mathsf{Alg}_{/k}(\mathsf{Assoc}, \mathsf{Comm})$ be an associative commutative k-algebra, then

- $\mathfrak{sl}_m(A)$ is the **special linear Lie algebra**, which consists of all trace zero matrices in $\mathfrak{gl}_m(A)$.
- $\mathfrak{o}_m(A)$ is the **orthogonal algebra** of all skew-symmetric matrices, i.e. $x^t = -x$.
- $\mathfrak{t}_m(A)$ is the **upper triangular matrices**, so $x_{ij} = 0$ if $i > j$.
- $\mathfrak{n}_m(A)$ are the **strictly upper triangular matrices**.
- $\mathfrak{d}(A)$ are the **diagonal matrices**, so $x_{ij} = 0$ if $i \neq j$.[5]

[5] Note that this is referred to as $\mathfrak{h}$ or sometimes $\mathfrak{t}$, since it's the torus.

Definition 36.2.3 (Derivation Algebras)
Let $A \in \mathsf{Alg}_{/k}$, not necessarily associative. A **derivation** D of A (or from A to A) is a k-module endomorphism of A satisfying the **Leibniz rule**:

$$D(ab) = (Da)b + a(Db) \qquad \forall a, b \in A.$$

We write $\operatorname{Der}(A) \leq \underset{k}{\operatorname{End}}(A)$ as the k-submodule of all derivations. One can check that $[D_1, D_2]$ is again a derivation for derivations D_i, so $\operatorname{Der}(A) \in \mathsf{Lie\text{-}Alg}$ called the **derivation algebra** of A.

Definition 36.2.4 (Nilpotent Algebras)
Let $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}$, and define a decreasing sequence of ideals

$$\mathfrak{g}^0 := \mathfrak{g}, \quad \mathfrak{g}^1 := [\mathfrak{g}\mathfrak{g}], \quad \cdots \mathfrak{g}^n := [\mathfrak{g}^{n-1}\mathfrak{g}].$$

This yields the **lower central series**

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^n \supseteq \cdots,$$

and $\mathfrak{g}$ is said to be **nilpotent** if $\mathfrak{g}^n = 0$ for some n.

Example 36.2.5*(?)*: For $\mathfrak{g} := \mathfrak{n}_m(A)$ the strictly upper triangular matrices, we have $x \in \mathfrak{g}^n \iff x_{ij} = 0$ unless $j \geq i + (n+1)$. So we get $n+1$ diagonals of all zeros:

$$\begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot \\ \vdots & 0 & 0 & \cdot & \cdot \\ \vdots & \ddots & 0 & 0 & \cdot \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \ddots & 0 \end{bmatrix}$$

Definition 36.2.6 (Solvable Algebras)
Define

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(1)} := [\mathfrak{g}^{(0)}\mathfrak{g}^{(0)}], \quad \mathfrak{g}^{(n+1)} := [\mathfrak{g}^{(n)}\mathfrak{g}^{(n)}],$$

which yields a decreasing sequence of ideals, the **derived series**,

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(0)} \supseteq \cdots \supseteq \mathfrak{g}^{(0)} \supseteq \cdots,$$

$\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(n)} = 0$ for some n.

Remark 36.2.7: Note that nilpotent implies solvable, since one can show by induction that $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$.

Example 36.2.8 ***(?):*** For $\mathfrak{g} = \mathfrak{t}_m(A)$ for A commutative, the diagonal of the product is the product along the diagonals, so

- $\mathfrak{g}^{(1)}$ are matrices with zeros on the diagonal,
- $\mathfrak{g}^{(2)}$ are matrices with zeros on 2 diagonals,
- $\mathfrak{g}^{(3)}$ are matrices with zeros on 4 diagonals,

and so on, so $\mathfrak{g}$ is solvable. On the other hand, taking brackets with one diagonal of zeros doesn't introduce new zero diagonals, and $\mathfrak{g}^2 = \mathfrak{g}^1$. So $\mathfrak{g}$ is not nilpotent, provided $m \geq 2$

Remark 36.2.9: Next time: $\mathfrak{g}$-modules.

37 | Monday, April 12

37.1 Lie Algebra Homology

Remark 37.1.1: Last time: Lie algebras. Fix a cocommutative ring k, usually a field, then a Lie algebra $\mathfrak{g}$ over k is a k-module with a bilinear product called the bracket such that

- $[xx] = 0$, so $[xy] = -[yx]$
- The Jacobi identity holds.

Definition 37.1.2 (Modules over Lie algebras)
A left $\mathfrak{g}$-module M is a k-module with a k-bilinear product

$$\begin{aligned} \cdot : \mathfrak{g} \otimes_k M &\to M \\ x \otimes m &\mapsto x \cdot m \end{aligned}$$

which is compatible with the bracket in the following sense:

$$[xy]m = x(ym) - y(xm) \quad \forall x, y \in \mathfrak{g}, m \in M, \tag{1}$$

i.e. there is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(M) := \mathrm{Lie}(\underset{k}{\mathrm{End}}(M))$, the Lie algebra of the endomorphism algebra.

Example 37.1.3 ***(Algebra Commutators):*** For $A \in \mathsf{Alg}_{/k}(\mathsf{Assoc})$ and $\mathfrak{g} \in \mathrm{Lie}(A)$, then any $M \in$ A-Mod (so the action is associative) can be made into an $M' \in \mathfrak{g}$-Mod by the formula equation (1).

Example 37.1.4 ***(Adjoint Representations):*** Any Lie algebra $\mathfrak{g}$ is a module over itself by the **adjoint representation**, where $\mathrm{ad}_x(-) := [x, -]$.

Example 37.1.5 ***(Trivial Modules):*** Any $M \in \mathsf{k}\text{-}\mathsf{Mod}$ becomes a trivial $\mathfrak{g}$-module by defining $xm = 0$ for all $x \in \mathfrak{g}, m \in M$. Note that this is acting by zero instead of the identity: this is motivated from Lie algebras obtained from Lie groups by taking tangent spaces at the identity. A trivial group action on the elements would be the identity, but then taking its derivative acting on tangent vectors to curves would be zero.

There is a *unique* trivial $\mathfrak{g}$-module, namely k with this trivial action.

Definition 37.1.6 (Morphisms of Lie algebra modules)
A morphism $M \xrightarrow{f} N$ of $\mathfrak{g}$-modules is a morphism of k-modules commuting with the module action, so $f(xm) = x(fm)$ for $x \in \mathfrak{g}, m \in M$. This yields $\operatorname{Hom}_{\mathfrak{g}}(M, N) \leq \operatorname{Hom}_{k}(M, N)$ as a k-submodule.

Remark 37.1.7: This yields a category $\mathfrak{g}\text{-}\mathsf{Mod} \leq \mathsf{k}\text{-}\mathsf{Mod}$ which is a subcategory of k-modules, and this is in fact an abelian category. So we have notions of (co)kernels, injectives and projectives, etc. There is also a category $\mathsf{Mod}\text{-}\mathfrak{g}$, but these can be sent to left $\mathfrak{g}$-modules by defining $x \cdot m := -mx$ which makes $\mathfrak{g}$ anticommutative. Thus there is an equivalence of categories

$$\mathfrak{g}\text{-}\mathsf{Mod} \xrightarrow{\sim} \mathsf{Mod}\text{-}\mathfrak{g},$$

and so we usually just refer to left modules.

Remark 37.1.8: We'll want to take homology and cohomology. There are some relevant functors:

- The trivial module functor:

  $$\operatorname{Triv} : \mathsf{k}\text{-}\mathsf{Mod} \to \mathfrak{g}\text{-}\mathsf{Mod},$$

  which sends M to itself, adding the structure of a trivial $\mathfrak{g}$-action.

- $\mathfrak{g}$-invariants:

  $$\begin{aligned} (-)^{\mathfrak{g}} : \mathfrak{g}\text{-}\mathsf{Mod} &\to \mathsf{k}\text{-}\mathsf{Mod} \\ M &\mapsto M^{\mathfrak{g}} := \left\{ x \in M \;\middle|\; xm = 0 \forall\, x \in \mathfrak{g} \right\}. \end{aligned}$$

  - This yields the largest $\mathfrak{g}$-trivial submodule, and similarly $(-)^{\mathfrak{g}}$ is right-adjoint to Triv.

    $$\mathsf{k}\text{-}\mathsf{Mod} \underset{(-)^{\mathfrak{g}}}{\overset{\operatorname{Triv}}{\underset{\perp}{\rightleftarrows}}} \mathfrak{g}\text{-}\mathsf{Mod}.$$

  - There is an isomorphism

    $$\begin{aligned} \operatorname{ev}_1 : \operatorname{Hom}_{\mathfrak{g}}(k, M) &\xrightarrow{\sim} M^{\mathfrak{g}} \\ f &\mapsto f(1_k). \end{aligned}$$

    where k is the trivial $\mathfrak{g}$-module.

- $\mathfrak{g}$-coinvariants:

$$\begin{aligned}(-)_{\mathfrak{g}} : \mathfrak{g}\text{-Mod} &\to \text{k-Mod} \\ M &\mapsto M/\mathfrak{g}M.\end{aligned}$$

  - This is the largest $\mathfrak{g}$-trivial *quotient* of M, so this is left-adjoint to Triv:

$$\mathfrak{g}\text{-Mod} \underset{\text{Triv}}{\overset{(-)^{\mathfrak{g}}}{\underset{\perp}{\rightleftarrows}}} \text{k-Mod}.$$

We might expect this is related to some tensor product, but it may not be clear what ring one should tensor over.

Remark 37.1.9: Assume that $\mathfrak{g}$-Mod has enough projectives, which we'll see is true in a later section by identifying this with a category R-Mod of modules over a ring.

Definition 37.1.10 (Cohomology of Lie algebras)
Define the **(co)homology of $\mathfrak{g}$ with coefficients in** M as

$$\begin{aligned}H_n(\mathfrak{g}; M) &:= \mathbb{L}(-)_{\mathfrak{g}}(M) \\ H^n(\mathfrak{g}; M) &:= \mathbb{R}(-)^{\mathfrak{g}}(M).\end{aligned}$$

Example 37.1.11*(?)*: If $\mathfrak{g} = \{0\}$, then $M^{\mathfrak{g}} = M = M_{\mathfrak{g}}$ and these functors are exact (and are essentially the identity) and thus their higher derived functors are zero. So $H^n(0; M) = 0 = H_n(0; M)$.

37.2 The Universal Enveloping Algebra

Remark 37.2.1: A better name might be the universal *associative* algebra. This plays an analogous role to the group algebra $\mathbb{Z}G$ of a group. We'll assign an associative algebra $\mathcal{U}(\mathfrak{g})$ to $\mathfrak{g}$, and there will be an equivalence of categories

$$\mathfrak{g}\text{-Mod} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})\text{-Mod},$$

where we'll know that the latter has enough projectives and injectives, allowing us to compute homology and cohomology with injective and projective resolutions.

Definition 37.2.2 (Tensor Algebra)
For $k \in$ CRing and $M \in$ k-Mod, and **tensor algebra** is defined as

$$T(M) := \bigoplus_{i \geq 0} M^{\otimes_k n} := k \otimes \bigoplus_{n \geq 1} M^{\otimes_k n}.$$

Remark 37.2.3: Note that $T(M) \in$ k-Mod by extending the k-action over sums and tensor products in the obvious way, and in fact $T(M) \in \mathsf{gr}(\mathsf{Alg}_{/k})$ where tensors in different degrees are juxtaposed.

Explicitly, for $m \in M^{\otimes n}$ and $m' \in M^{\otimes n'}$, we write $m \otimes m' \in M^{\otimes(n+n')}$, which is what it means to be a *graded* algebra.

Remark 37.2.4: There is an inclusion map

$$M = M^{\otimes 1} \overset{\iota}{\hookrightarrow} T(M)_1 \hookrightarrow T(M).$$

where $T(M)_j := \bigoplus_{n \geq j} M^{\otimes n}$, and in fact $T(M)$ is generated as a k-algebra by $\iota(M)$. For example, for $m, m' \in M$, we have $\iota(m) \otimes \iota(m') \in T(M)_2$. This yields a functor

$$T : \mathsf{k\text{-}Mod} \to \mathsf{Alg}_{/k}(\mathsf{Assoc}, \mathsf{Unital}),$$

as well as a forgetful functor

$$\text{Forget} : \mathsf{Alg}_{/k} \to \mathsf{k\text{-}Mod}.$$

The pair (T, i) is a **universal** associative algebra in the following sense: if $M \in \mathsf{k\text{-}Mod}$ and $A \in \mathsf{Alg}_{/k}(\mathsf{Assoc})$, then there is a k-module morphism $M \to \text{Forget}(A)$ making the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, "\iota"] \arrow[dr, "f"'] & T(M) \arrow[d, dashed, "\exists! \tilde{f} \in \mathsf{Alg}_{/k}"] \\
& A
\end{tikzcd}$$

Link to Diagram

Note that the red portion of the diagram happens in $\mathsf{k\text{-}Mod}$, while the blue portion is in $\mathsf{Alg}_{/k}$, so this allows lifting module morphisms to algebra morphisms. Commuting here means that

$$f(m_1)f(m_2) = \tilde{f}(m_1 m_2) := f(\iota(m_1) \otimes \iota(m_2)).$$

There is thus a natural isomorphism

$$\underset{\mathsf{k\text{-}Mod}}{\text{Hom}}(M, \text{Forget}(A)) \xrightarrow{\sim} \underset{\mathsf{Alg}_{/k}}{\text{Hom}}(T(M), A).$$

38 Universal Enveloping Algebras (Wednesday, April 14)

Remark 38.0.1: Continuing section 7.3 on universal enveloping algebras.: Letting $k \in \mathsf{CRing}$, $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}_{/k}$, $M \in \mathsf{k\text{-}Mod}$, we defined the tensor algebra $T(M) := k \oplus \bigoplus_{i \geq 1} M^{\otimes n} \in \mathsf{gr}\,\mathsf{Alg}_{/k}(\mathsf{Assoc}, \mathsf{Unital})$ and noted that it was universal for maps from M to k-algebras.

Definition 38.0.2 (Universal Enveloping Algebra)
Let $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}_{/k}$, then define the **universal enveloping algebra** of $\mathfrak{g}$ as

$$\mathcal{U}(\mathfrak{g}) := \frac{T(\mathfrak{g})}{\left\langle xy - yx - [xy] \,\middle|\, x, y \in \mathfrak{g} \right\rangle}.$$

Remark 38.0.3: There is an injection $k \hookrightarrow \mathcal{U}(\mathfrak{g})$, so $\mathcal{U}(\mathfrak{g})$ is unital. The relations guarantee that there is a Lie algebra morphism $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$. Note that we do not know if this is injective yet! Thus there is a functor

$$\mathcal{U} : \mathsf{Lie\text{-}Alg}_{/k} \to \mathsf{Alg}_{/k},$$

and it turns out that this is adjoint to the Lie functor.

Fact 38.0.4
There is an adjunction

$$\mathsf{Lie\text{-}Alg} \underset{\text{Lie}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathsf{Alg}_{/k}.$$

Thus for every $f : \mathfrak{g} \to \mathrm{Lie}(A)$ for $A \in \mathsf{Alg}_{/k}(\mathsf{Assoc})$, we have a commuting diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "f\in\mathsf{Lie\text{-}Alg}"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists f\in\mathsf{Alg}_{/k}"] \\
& A
\end{tikzcd}$$

Link to Diagram

Thus there is a natural isomorphism

$$\operatorname{Hom}_{\mathsf{Lie\text{-}Alg}}(\mathfrak{g}, \mathrm{Lie}(A)) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Alg}_{/k}}(\mathcal{U}(\mathfrak{g}), A).$$

Theorem 38.0.5 *(?).*
There is an equivalence of categories

$$\mathfrak{g}\text{-}\mathsf{Mod} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})\text{-}\mathsf{Mod},$$

where we use the fact that $\mathcal{U}(\mathfrak{g})$ has an underlying ring structure.

Concretely, if $M \in \mathfrak{g}\text{-}\mathsf{Mod}$ and $\prod x_i \in \mathcal{U}(\mathfrak{g})$, then setting $(x_1 \cdots x_n)m = x_1(\cdots x_n m)$ (and similarly for every i) for $m \in M$ makes m into a $\mathcal{U}(\mathfrak{g})$-module. Conversely, if $M \in \mathcal{U}(\mathfrak{g})\text{-}\mathsf{Mod}$, we can set $xm := \iota(x)m$ for $x \in \mathfrak{g}$ to make M into a $\mathfrak{g}$-module.

Proof (?).
Let $M \in \mathsf{k\text{-}Mod}$ and set $E := \underset{k}{\operatorname{End}}(M) \in \mathsf{Alg}_{/k}$. Note that a $\mathfrak{g}$-module is a k-module M with a morphism of Lie algebras $\mathfrak{g} \to \operatorname{Lie}(E)$. Using the adjunction, we can map such a morphism to $\tilde{f} : \mathcal{U}(\mathfrak{g}) \to E$, and by definition a $\mathcal{U}(\mathfrak{g})$-module is a k-module M with a k-algebra morphism $\mathcal{U}(\mathfrak{g}) \to \underset{k}{\operatorname{End}}(M) = E$.

■

Corollary 38.0.6 *(?).*
The category $\mathfrak{g}\text{-}\mathsf{Mod}$ has enough projectives and injectives.

Remark 38.0.7: We'll now set up an analog of the augmentation for group algebras, $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$.

Definition 38.0.8 (Augmentation Ideal for Lie Algebras)
There is a unique surjective morphism $\varepsilon \in \mathsf{Alg}_{/k}(\mathcal{U}(\mathfrak{g}), k)$ where $\varepsilon \circ \iota(\mathfrak{g}) = 0$. The kernel $I := \ker \varepsilon$ is defined as the **augmentation ideal**, and is a two-sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\iota(\mathfrak{g})$ and write $\mathfrak{g}\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})\mathfrak{g}$, i.e. those elements which contain at least one tensor factor.

Remark 38.0.9: We can identify the coinvariants:

$$k \cong \mathcal{U}(\mathfrak{g})/\mathfrak{g} = \mathcal{U}(\mathfrak{g})/\mathfrak{g}\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})_{\mathfrak{g}}.$$

Corollary 38.0.10 *(?).*

1. $H_*(\mathfrak{g}; M) \cong \operatorname{Tor}_*^{\mathcal{U}(\mathfrak{g})}(k, M)$,
2. $H^*(\mathfrak{g}; M) \cong \operatorname{Ext}_{\mathcal{U}(\mathfrak{g})}^*(k, M)$,

Proof (?).
To show that two derived functors are isomorphic, it's enough to show that their underlying functors (the degree 0 parts) are isomorphic. Starting with (2), we observed that $M^g \cong \underset{\mathfrak{g}}{\operatorname{Hom}}(k, M) \cong \underset{\mathcal{U}(\mathfrak{g})}{\operatorname{Hom}}(k, M)$.

For (1), we can write

$$k \otimes_{\mathcal{U}(\mathfrak{g})} M \cong \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{I}} \right) \otimes_{\mathcal{U}(\mathfrak{g})} M \cong M/\mathcal{I}M \cong M/\mathfrak{g}M = M_{\mathfrak{g}},$$

so $k \otimes_{\mathcal{U}(\mathfrak{g})} (-) = (-)_{\mathfrak{g}}$.

■

Remark 38.0.11: So Lie algebra (co)homology is just a special case of the usual Tor and Ext we've already looked at. We'll next find a basis for $\mathcal{U}(\mathfrak{g})$:

Theorem 38.0.12 ***(Poincaré-Birkhoff-Witt (PBW) Theorem).***
Let $\mathfrak{g}$ be free in $\mathsf{k\text{-}Mod}$ and fix a k-basis, so $\mathfrak{g} \in \mathsf{Vect}_{/k}$. Note that this makes $\iota : \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ an injection. Let $\{x_\alpha\}_{\alpha \in A}$ be a fixed totally ordered k-basis for $\mathfrak{g}$. If $I = (\alpha_1, \cdots, \alpha_p) \in A^p$,

we'll write monomials as $x_I \coloneqq x_{\alpha_1} \cdots x_{\alpha_p} \in \mathcal{U}(\mathfrak{g})$, where we'll suppress writing $\iota(x_{\alpha_j})$. We'll say I is (weakly) increasing if $\alpha_1 \leq \cdots \leq \alpha_p \in A$. Noting that the empty sequence $\emptyset \in A^0$ is increasing, set $x_\emptyset \coloneqq 1 \in \mathcal{U}(\mathfrak{g})$, and if $I = (\alpha) \in A^1$ is a single index, then we'll write $x_\alpha \in \mathfrak{g}$ and $x_{(\alpha)} \in \mathcal{U}(\mathfrak{g})$.

Then if $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}_{/k}$ is a free k-module, a k-basis for $\mathcal{U}(\mathfrak{g})$ is given by the monomials x_I as I ranges over finite increasing sequences from A.

Proof (?).
Omitted. ■

Remark 38.0.13: To at least see why these are a spanning set, suppose $\beta > \alpha$. We can commute elements:

$$x_\beta x_\alpha = x_\alpha x_\beta + [x_\beta x_\alpha].$$

However, note that the commutator here has lower degree (here, the other factors are degree 2 and the commutator is degree 1). This decreases the number of misorders as well, so induction roughly works. The fact that these are linearly independent is harder and uses some actual representation theory.

39 | Friday, April 16

39.1 The Enveloping Algebra (Continued)

Remark 39.1.1: Last time: the PBW theorem. Let $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}$ and free as a k-module with k-basis $\{x_\alpha\}_{\alpha \in A}$. Then $\mathcal{U}(\mathfrak{g})$ has a k-basis $\{x_I\}$ where $I = (\alpha_1, \cdots, \alpha_p)$ is a finite increasing sequence from A

Example 39.1.2*(?)*: If k is a field and $\dim_k \mathfrak{g}$ is finite with basis $\{x_1, \cdots, x_n\}$. Take $I = (1, \cdots, 1, 2 \cdots, 2, n \cdots, n)$ where each i occurs a_i times. Then a basis for $\mathcal{U}(\mathfrak{g})$ is $\left\{ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \;\middle|\; a_i \geq 0 \right\}$

Corollary 39.1.3*(?)*.
The map $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective, so we can identify $\iota(\mathfrak{g})$ with $\mathfrak{g}$.

Proof (?).
The elements $x_{(\alpha)} \coloneqq \iota(x_\alpha) \in \mathcal{U}(\mathfrak{g})$ are k-linearly independent. ■

Corollary 39.1.4 *(?).*
If $\mathfrak{h} \leq \mathfrak{g}$ is a subalgebra and k is a field, then $\mathcal{U}(\mathfrak{g})$ is free as a $\mathcal{U}(\mathfrak{h})$-module.

Proof (?).
Choose an ordered basis for $\mathfrak{h}$ first and then extend this to an ordered basis for $\mathfrak{g}$ – that one can do this is a fact from linear algebra. Then the x_I where $I = (\alpha_1, \cdots, \alpha_p)$ is increasing and no $x_{\alpha_i} \in \mathfrak{h}$ will be a basis for $\mathcal{U}(\mathfrak{g})$ over $\mathcal{U}(\mathfrak{h})$. ■

Example 39.1.5 *(?):* If $\dim_k \mathfrak{g} < \infty$ and $\{x_1, \cdots, x_k\}$ is a basis for $\mathfrak{h}$ and $\{x_1, \cdots, x_k, x_{k+1} \cdots, x_n\}$ is a basis for $\mathfrak{g}$, then the PBW basis is given by $\left\{ x_1^{a_1} \cdots x_k^{a_k} x_{k+1}^{a_{k+1}} \cdots x_n^{a_n} \;\middle|\; a_i \geq 0 \right\}$. Then $\left\{ x_{k+1]}^{a_{k+1} \cdots x_n^{a_n}} \right\}$ form a free left $\mathcal{U}(\mathfrak{h})$-module basis for $\mathcal{U}(\mathfrak{g})$.

Exercise 39.1.6 (?)
Some suggested exercises:

- 7.3.4
- 7.3.6
- 7.3.7 for working with $\mathcal{U}(\mathfrak{h})$ as a Hopf algebra.
- 7.3.9 for representations of Lie algebras in characteristic p.

39.2 H^1 for Lie Algebras (Weibel 7.4)

Remark 39.2.1: Recall that we have an augmentation ideal $\mathcal{I} \trianglelefteq \mathcal{U}(\mathfrak{h})$ and a SES

$$0 \to \mathcal{I} \to \mathcal{U}(\mathfrak{g}) \to k \to 0.$$

Applying the functor $\operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(-, M)$ for a fixed $M \in \mathfrak{g}\text{-}\mathsf{Mod}$ yields a LES:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(k, M) \arrow[r] & \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) \arrow[r] & \operatorname{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{I}, M) \arrow[dll, "\delta"] \\
& \operatorname{Ext}^1_{\mathcal{U}(\mathfrak{g})}(k, M) = H^1(\mathfrak{g}; M) \arrow[r] & {\color{red}\operatorname{Ext}^1_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) = 0} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here the red term vanishes since $\mathcal{U}(\mathfrak{g})$ is free and this projective as a $\mathfrak{g}$-module. Note that for $n \geq 2$, we have the following situation:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}^{n-1}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) = 0 \arrow[r] & \mathrm{Ext}^{n-1}_{\mathcal{U}(\mathfrak{g})}(\mathcal{I}, M) \arrow[dll, "\delta", "\cong"'] \\
\mathrm{Ext}^1_{\mathcal{U}(\mathfrak{g})}(k, M) = H^1(\mathfrak{g}; M) \arrow[rr] & & \mathrm{Ext}^{n}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) = 0 \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Thus we get a degree shifting isomorphism

$$H^n(\mathfrak{g}; M) \cong \mathrm{Ext}^{n-1}_{\mathcal{U}(\mathfrak{g})}(\mathcal{I}, M).$$

Remark 39.2.2: We thus have

$$H^1(\mathfrak{g}; M) \cong \operatorname*{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{I}, M) / \operatorname{im}\left(\operatorname*{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) \cong M \to \operatorname{Hom}(\mathcal{I}, M) \right).$$

Next goal: to more concretely express all of the terms here as M-valued derivations on $\mathfrak{g}$.

Definition 39.2.3 (Derivations of an algebra)
Let $M \in \mathfrak{g}\text{-}\mathsf{Mod}$, then a **derivation from $\mathfrak{g}$ into M** is a k-linear map $D : \mathfrak{g} \to M$ satisfying the Leibniz rule:

$$D([xy]) = x \cdot (Dy) - y \cdot (Dx) \qquad\qquad x, y \in \mathfrak{g}.$$

Remark 39.2.4: The set of all such maps $\mathrm{Der}(\mathfrak{g}, M) \leq_{k\text{-}\mathsf{Mod}} \operatorname*{Hom}_{k\text{-}\mathsf{Mod}}(\mathfrak{g}, M)$ is a k-submodule. A special case is taking $M := \mathfrak{g}$, regarded as a $\mathfrak{g}$-module using the adjoint representation. In fact, for any k-algebra (not necessarily associative), we get

$$D(ab) = (Da) \cdot b + a \cdot (Db).$$

When $A := \mathfrak{g} \in \mathsf{Lie\text{-}Alg}$ with the adjoint action, we obtain

$$\begin{aligned}
D([xy]) &= [x, Dy] + [Dx, y] \\
&= [x, Dy] - [y, Dx] \\
&= x \cdot Dy - y \cdot (Dx),
\end{aligned}$$

recovering the previous definition.

Definition 39.2.5 (Inner Derivations)
For $M \in \mathfrak{g}\text{-}\mathsf{Mod}$, fix an $m \in M$. We then define

$$\begin{aligned}
D_m : \mathfrak{g} &\to M \\
x &\mapsto x \cdot m.
\end{aligned}$$

Any derivation of this form is said to be an **inner derivation**, and this yields a k-submodule

$$\operatorname{Inn}(\mathfrak{g}, M) \leq_{\mathsf{k\text{-}Mod}} \operatorname{Der}(\mathfrak{g}, M).$$

Remark 39.2.6: Note that this is indeed a derivation:

$$D_m([xy])][xy] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) = x \cdot (D_m y) - y \cdot (D_m x).$$

It also turns out that any inner derivation is of this form, bracketing against a fixed element.

Proposition 39.2.7 ***(?)*.**

$$\operatorname{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathcal{I}, M) \cong \operatorname{Der}(\mathfrak{g}, M).$$

Proof (?).

Claim: There exists such a map.
Say $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(\mathcal{I}, M)$ and set

$$\begin{aligned} D_\varphi : \mathfrak{g} &\to M \\ x &\mapsto \varphi(x). \end{aligned}$$

Then D_φ is a derivation, so we have

$$\begin{aligned} D_\varphi([xy]) &:= \varphi([xy]) \\ &= \varphi(xy - yx) \\ &= x\varphi(y) - y\varphi(x) && \text{since } \varphi \text{ is } \mathfrak{g}\text{-linear} \\ &= xD_\varphi(y) - yD_\varphi(x). \end{aligned}$$

Claim: This map is a natural isomorphism, in the sense that it doesn't depend on any choices:

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}(\mathcal{I}, M) &\to \operatorname{Der}(\mathfrak{g}, M) \\ \varphi &\mapsto D_\varphi. \end{aligned}$$

Proof (of surjectivity).
Recall that we can write $\mathcal{I} = \mathcal{U}(\mathfrak{g})\,\mathfrak{g}$, so the following product map is a surjection:

$$\begin{aligned}\theta : \mathcal{U}(\mathfrak{g}) \otimes_k \mathfrak{g} &\twoheadrightarrow \mathcal{U}(\mathfrak{g})\,\mathfrak{g} = \mathcal{I} \\ x \otimes y &\mapsto xy.\end{aligned}$$

One checks that the kernel is given by

$$\ker(\theta) = \left\{ u \otimes [xy] - (ux \otimes y - uy \otimes x) \;\middle|\; x, y \in \mathfrak{g}, u \in \mathcal{U}(\mathfrak{g}) \right\}.$$

Now given $D \in \operatorname{Der}(\mathfrak{g}, M)$, consider the map

$$\begin{aligned}f : \mathcal{U}(\mathfrak{g}) \otimes_k \mathfrak{g} &\to M \\ f(u \otimes x) &= u \cdot Dx.\end{aligned}$$

One can compute the following, using that D is a derivation:

$$\begin{aligned}f\left(u \otimes [xy] - ux \otimes y - uy \otimes x\right) &= uD([xy]) - (ux) \cdot D(y) + (uy) \cdot D(x) \\ &= u(x \cdot Dy - y \cdot Dx) - u \cdot (x \cdot Dy) + u \cdot (y \cdot Dx) \\ &= 0.\end{aligned}$$

So f induces a well-defined morphism of $\mathfrak{g}$-modules, and descends to a map

$$\varphi : \mathcal{U}(\mathfrak{g})\,\mathfrak{g} = \mathcal{I} \to M,$$

which is clearly also a morphism of $\mathfrak{g}$-modules. So $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(\mathcal{I}, M)$ and $D_\varphi(x) = \varphi(x) = \varphi(1 \cdot x) = f(1 \cdot x) = 1 \cdot Dx = Dx$, and so $D = D_\varphi$.

■

Might as well find-and-replace "map" with "morphism"!

Proof (of injectivity).
Suppose over D that we have D_ψ for some $\psi \in \operatorname{Hom}_{\mathfrak{g}}(\mathcal{I}, M)$. We then have

$$\varphi(ux) = uD(x) = u\Psi(x) = \Psi(ux) \qquad \forall u \in \mathcal{U}(\mathfrak{g}), x \in \mathfrak{g}.$$

Since $\mathcal{I} = \mathcal{U}(\mathfrak{g})\,\mathfrak{g}$ and $\varphi = \Psi$, yielding a 1-to-1 map.

■

■

40 Lie Algebra Cohomology (Monday, April 19)

40.1 Identification of H^1 as Derivations

Remark 40.1.1: Let $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}_{/k}$ and $M \in \mathfrak{g}\text{-}\mathsf{Mod}$, we were showing

$$H^1(\mathfrak{g}; M) \cong \frac{\mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathcal{I}, M)}{\mathrm{im}\left(\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{U}(\mathfrak{g}), M) \to \mathrm{Hom}_{\mathfrak{g}}(\mathcal{I}, M)\right)},$$

where the source in the denominator is isomorphic to M, given by the map ev_1. We found a map

$$\begin{aligned} \underset{\mathfrak{g}}{\mathrm{Hom}}(\mathcal{I}, M) &\xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}, M) \\ \varphi &\mapsto (D_\varphi : x \mapsto \varphi(x)). \end{aligned}$$

We also defined inner derivations as those given by maps $D_m(x) := mx$ for some $m \in M$.

Theorem 40.1.2 *(?).*

$$H^1(\mathfrak{g}; M) \cong \frac{\mathrm{Der}(\mathfrak{g}, M)}{\mathrm{Inn}(\mathfrak{g}, M)}.$$

Proof (?).
In the formula, we already know that the numerator is isomorphic to $\mathrm{Der}(\mathfrak{g}, M)$, so it remains to look at the denominator. The map appearing there is restriction to $\mathcal{I}$, i.e. $\varphi \mapsto \varphi|_{\mathcal{I}}$. The associated derivation is given by

$$D_\varphi(x) = \varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = xm := D_m(x),$$

and so $D_\varphi = D_m$. Conversely, given an m, we get a derivation D_m, and thus the image is precisely all inner derivations.

■

40.2 LHS Spectral Sequences

Remark 40.2.1: If $\mathfrak{h} \trianglelefteq \mathfrak{g}$, there is a SES in $\mathsf{Lie\text{-}Alg}$:

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0.$$

Theorem 40.2.2 *(LHS Spectral Sequence).*
Let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ and $M \in \mathfrak{g}\text{-}\mathsf{Mod}$, then there are first quadrant spectral sequences

$$E^2_{p,q} = H_p(\mathfrak{g}/\mathfrak{h}; H_q(\mathfrak{h}; M)) \Rightarrow H_{p+q}(\mathfrak{g}; M)$$
$$E^2_{p,q} = H^p(\mathfrak{g}/\mathfrak{h}; H_q(\mathfrak{h}; M)) \Rightarrow H^{p+q}(\mathfrak{g}; M).$$

Remark 40.2.3: This comes from a similar application of the Grothendieck spectral sequence. The exact sequences in low-degree terms are given as usual[6] and similar inflation and restriction maps appear here. This is useful because it allows computing homology of "smaller" algebras, which one might have control over by induction.

40.2.1 7.7: Chevalley-Eilenberg (Koszul) Complex

Remark 40.2.4: A computationally efficient way of compute Lie algebra cohomology using a projective resolution of the trivial $\mathfrak{g}$-module $k \in \mathfrak{g}\text{-}\mathsf{Mod}$, recalling that this involves acting by zero. [7]

We're going to define a chain complex

$$V_*(\mathfrak{g}) \xrightarrow{\varepsilon} k,$$

which will turn out to be supported in finitely many degrees when $\dim_k \mathfrak{g} < \infty$.

Remark 40.2.5: We'll assume $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}_{/k}(\mathsf{Free})$, which happens e.g. if $k \in \mathsf{Field}$. Recall that the exterior algebra was a graded algebra defined as the quotient of the tensor algebra:

$$\bigwedge^* \mathfrak{g} := \frac{T(\mathfrak{g})}{\left\langle x^{\otimes 2} \;\middle|\; x \in \mathfrak{g} \right\rangle} = \bigoplus_{p \geq 0} \bigwedge^p \mathfrak{g}.$$

We write $x_1 \wedge x_2 \wedge \cdots \wedge x_p$ for the image of $\mathfrak{g} \hookrightarrow \bigwedge^* \mathfrak{g}$ Note that this is a 2-sided homogeneous ideal, and since $x \wedge x = 0$ we have $x \wedge y = -y \wedge x$.

Remark 40.2.6: If $\{x_\alpha\}$ is an ordered basis for $\mathfrak{g}$, then there is an ordered basis for $\bigwedge^p \mathfrak{g}$:

$$\left\{ x_{\alpha 1}, x_{\alpha 2}, \cdots, x_{\alpha p} \;\middle|\; \alpha_1 < \cdots < \alpha_p \right\},$$

where we note that the indices are strictly increasing like the sequences I we had previously. One can always arrange this by commuting things to organize the sequence properly. We also have $\bigwedge^0 \mathfrak{g} \cong k$ with a basis of 1_k, and $\bigwedge^1 \mathfrak{g} \cong \mathfrak{g}$. In particular, if $\dim \mathfrak{g} = n < \infty$, then $\bigwedge^p \mathfrak{g} = 0$ for all $p > n$, and in this case $\bigwedge^n \mathfrak{g} \cong k$.

[6] See Weibel p.233.
[7] See VIGRE project at UGA: programmed this resolution in GAP to compute Lie algebra cohomology!

> **Definition 40.2.7** (The Chevalley-Eilenberg (or Koszul) Complex)
> Define
>
> $$V_p(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^p \mathfrak{g},$$
>
> where the maps are given below.

Fact 40.2.8
$V_p(\mathfrak{g})$ is free in $\mathcal{U}(\mathfrak{g})$-Mod, since we've constructed a free basis, and so in particular it is projective. The maps in the complex are given by the following:

$$\begin{tikzcd}
 & & & \mathcal{I} = \ker \varepsilon = \mathcal{U}(\mathfrak{g})\,\mathfrak{g} \arrow[dr, hook] & \\
 & & & & \mathcal{U}(\mathfrak{g}) \arrow[d, dash, "\cong"] \arrow[dr, dash] & \\
V_p(\mathfrak{g}) \arrow[r, "d_p"] \arrow[d, dash] & V_{p-1}(\mathfrak{g}) \arrow[r] \arrow[d, dash] & \cdots \arrow[r] & V_1(\mathfrak{g}) \arrow[r, "d_1"] \arrow[d, dash] & V_0(\mathfrak{g}) \arrow[r, dashed, dash] \arrow[d, dash] & {} \\
\mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^p \mathfrak{g} \arrow[r, "d_p"] & \mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^{p-1} \mathfrak{g} & & \mathcal{U}(\mathfrak{g}) \otimes_k \mathfrak{g} \arrow[r, "d_1"] & \mathcal{U}(\mathfrak{g}) & \\
u \otimes \bigwedge_j x_{i_j} \arrow[r, mapsto] & \theta_1 + \theta_2 & & u \otimes x \arrow[r, mapsto] & ux &
\end{tikzcd}$$

Link to Diagram

Here we define

$$\theta_1 := \sum_{i=1}^{p} u x_i \otimes x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p$$
$$\theta_2 := \sum_{i<j}^{p} (-1)^{i+j} u \otimes [x_i x_j] \otimes x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_p,$$

where the hat denotes omitting a term. Note that $\operatorname{im} d_1 = \mathcal{U}(\mathfrak{g})\,\mathfrak{g} = \mathcal{I} = \ker \varepsilon$, so we get exactness at the first position, and exercise 7.7.1 shows that $d^2 = 0$.

Example 40.2.9 ***(?):*** For $p = 2$, we have

$$d(u \otimes x \otimes y) = (ux \otimes y - uy \otimes x) + (-u \otimes [xy]) .$$

Remark 40.2.10: We want this to be a projective resolution, so not just that ker $\subset$ im, but rather we want exactness everywhere so ker $=$ im. We'll proceed by showing its homology vanishes.

Theorem 40.2.11 ***(Koszul Resolution).***
The Koszul complex $V_*(\mathfrak{g}) \xrightarrow{\varepsilon} k$ is a projective resolution in $\mathfrak{g}$-Mod.

Proof (of theorem).
Choose an ordered basis $\{e_\alpha\}_{\alpha \in \Omega}$, where Ω some totally ordered index set, for $\mathfrak{g}$ over k. By the PBW theorem, $V_n := V_n(\mathfrak{g})$ has a free k-basis given by

$$e_I \otimes (e_{\alpha_1} \otimes \cdots e_{\alpha_n}) . \tag{2}$$

for $I = [\beta_1, \cdots, \beta_m]$ some weakly increasing sequence from Ω. This gives a filtration, so we're heading toward using the spectral sequence of a filtered complex. The filtered pieces are given by $F_p V_n$ defined as the k-module generated by elements of the form given in equation (2) where $m + n \le p$. Looking at the formula for d, we will get a differential

$$d_n F_p V_n \to F_p V_{n-1}.$$

■

41 Exactness of the Chevalley-Eilenberg Resolution (Wednesday, April 21)

Remark 41.0.1: Recall that $\mathfrak{g}$ was free over k with an ordered basis $\left\{ e_\alpha \;\middle|\; \alpha \in \Omega \right\}$. We defined

$$V_n(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^n \mathfrak{g}$$

with a differential $d = \theta_1 + \theta_2$. We claimed that $V_n(\mathfrak{g}) \xrightarrow{\varepsilon} k$ is a projective resolution, and we were showing that V_* was an exact complex.

Proof (of theorem, continued).
We define a filtration

$$F_p V_n := k \left\langle e_I \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_n} \;\middle|\; I = [\beta_1, \beta_2, \cdots, \beta_m], \alpha_1 \le \cdots \le \alpha_n,\, m + n \le p \right\rangle .$$

Note that $d : F_p V_n \to F_p V_{n-1}$, and in fact θ_2 maps into $F_{p-1} V_{n-1}$. We'll focus on θ_1 for simplicity. It lands in the same complex since we can rearrange elements in the sum defining the differential to express everything in terms of the given basis, where every expression will be of length one less. The commutation relation was

$$e_\beta e_\alpha = e_\alpha e_\beta + [e_\beta e_\alpha],$$

where the left-hand side is degree 2, and the right-hand side is a degree 2 term plus a degree 1 term. Moreover d preserves the filtration: when rearranging, the degree u term in θ_1 will decrease to $m-1$, the expression following it may increase to $n+1$, and $(m-1)+(n+1) = m+n$. So $F_p V_*$ is a subcomplex of V_*, and we have

$$0 = F_{-1}V_* \subseteq F_0 V_* \subseteq \cdots \subseteq F_p V_* \subseteq V_* = \bigcup_{p \geq 0} F_p V_*,$$

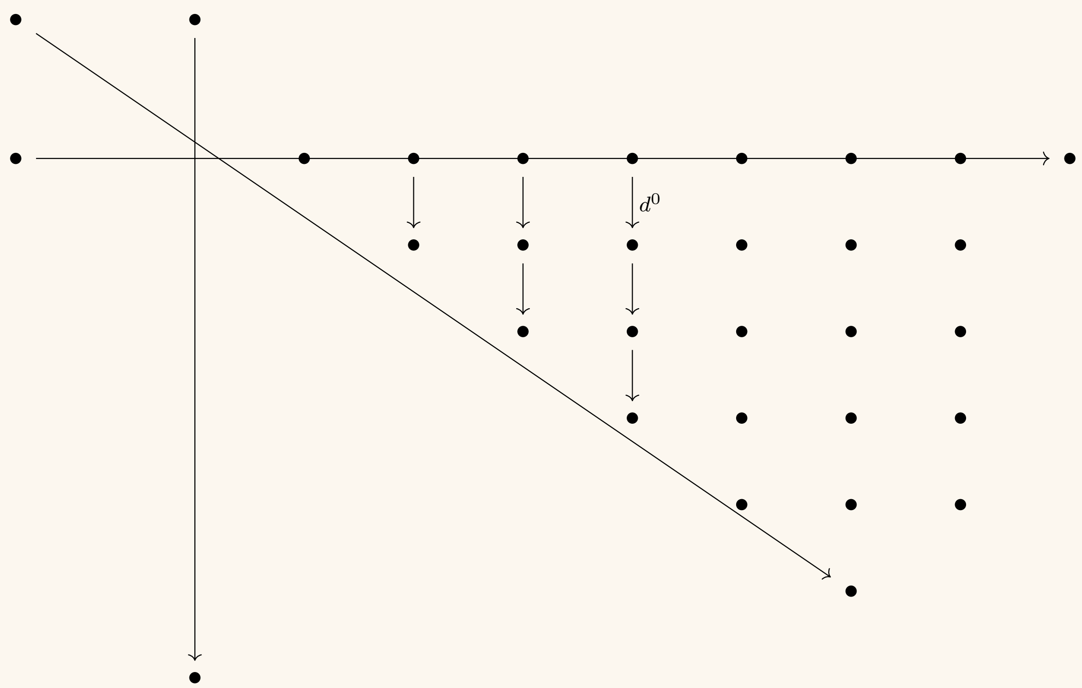
which is not a finite filtration, but is bounded below and exhaustive. So by the canonical convergence theorem (Weibel 5.5.1), there is a convergent spectral sequence

$$E^0_{p,q} \coloneqq \frac{F_p V_{p+q}}{F_{p-1} V_{p+q}} \Rightarrow H_{p+q}(V_*(\mathfrak{g})).$$

We have $E^0_{p,q} = 0$ unless

- $p \geq 0$, since the exterior algebra is only graded in positive degrees.
- $p + q = n \geq 0$
- $q \leq 0$, which requires some explanation. We have $m + n \leq p$, and so if $p + q = n \leq p - m$ for $m \geq 0$, $q \leq -m \leq 0$.

So this is a 4th quadrant spectral sequence that is supported above the line $y = -x$. Recall that $E^1_{p,q} = H^v_q(E^0_{p,*})$.



Link to Diagram

Note that

$$E^0_\infty := F_0V_0/F_{-1}V_0 = k \otimes_k k \cong k,$$

since we take expressions with length zero in each factor defining F_pV_n. Moreover this position is already stable provided the $E^1_{p,0} = 0$ for all p, the first and third quadrants are all zeros, and thus all differentials will be trivial from E^1 onward.

Claim: For $p > 0$, $E^0_{p,*}$ is exact, and thus the spectral sequence collapses at E^1.
Note that turning the page yields

$$E^1_{p,q} = \begin{cases} k & (p,q) = (0,0) \\ 0 & \text{else.} \end{cases}$$

Thus $H_n(V_*(\mathfrak{g})) = k$ in $n = 0$ and zero elsewhere, which proves the result.

Proof (Sketch).
For $q \gg 0$, define $A_q \coloneqq k\left\langle e_I \;\middle|\; I = [\beta_1, \beta_2, \cdots, \beta_q] \text{ increasing}\right\rangle \subseteq \mathcal{U}(\mathfrak{g})$. So A_q is the qth graded piece of the standard increasing filtration by degree,

$$k = U_0 \subset U_1 \subset \cdots \subseteq \mathcal{U}(\mathfrak{g}).$$

Note that this is a section standard filtration of $\mathcal{U}(\mathfrak{g})$ by degree with respect to the PBW basis[a]. We have $A_q \cong F_q V_0 / F_{q-1} V_0$ and

$$E^0_{p,q} = \frac{F_p V_{p+q}}{F_{p-1} V_{p+q}} \cong A_{-q} \otimes_k \bigwedge^{p+q} \mathfrak{g}.$$

The negative sign is introduced since this is nonzero precisely when $-p \leq q \leq 0$ so q is negative and $-q$ is positive. Now using the definition of $d : V_n \to V_{n-1}$, d^0 is vertical and

$$d^0 : E^0_{p,q} \to E^0_{p,q-1} \qquad n = p+q$$

$$\cong d^0 : A_{-1} \otimes_k \bigwedge^{n} \mathfrak{g} \to A_{-q+1} \otimes_k \bigwedge^{n-1} \mathfrak{g} \qquad n = p+q.$$

Recalling how d^0 was defined, note that we're modding out by lower order terms and thus brackets get killed when we commute elements to order them.

By Weibel 7.3.6, $A \coloneqq \bigoplus_{q \geq 0} A_q$ is in fact a graded algebra, and $A \cong \operatorname{gr} \mathcal{U}(\mathfrak{g})$, the associated graded of $\mathcal{U}(\mathfrak{g})$. This turns out to be a polynomial ring on the indeterminates $\mathbf{x} = \{e_\alpha\}_{\alpha \in \Omega}$, i.e. $A \cong k[\mathbf{x}]$. In Weibel section 4.5, Weibel studies the *Koszul* complex and the map $A \otimes_k \bigwedge^{*} \mathfrak{g} \to A$. By comparing the formula for d between these two complexes, one observes that the Koszul complex differentials are equal to the d^0 here. So we have an equality of complexes

$$A \otimes_k \bigwedge^{*} \mathfrak{g} = \bigoplus_{p \geq 0} E^0_{p,*}.$$

Weibel section 4.5 shows that when $A \in \mathsf{CRing}$ with no zero divisors, e.g. a polynomial ring, then

$$H_n\left(A \otimes_k \bigwedge^{*} \mathfrak{g}\right) = \begin{cases} k & n = 0 \\ 0 & \text{else.} \end{cases}$$

On the other hand, we have

$$\begin{aligned} H_n\left(A \otimes_k \bigwedge^{*} \mathfrak{g}\right) &= \bigoplus H^v_{n-p}(E^0_{p,*}) \qquad\qquad p+q = n \implies q = n-p \\ &= \bigoplus_{p \geq 0} E^1_{p,n-p}. \end{aligned}$$

But we've already shown that $E^1_{0,0} = k$, so all of the other E^1 terms must be zero. ∎

[a] See exercise 7.3.6.

■

Remark 41.0.2: See section 4.5 on Koszul complex. We'll do 7.8 next time.

42 | Friday, April 23

42.1 Applications Chevalley-Eilenberg Complex

Remark 42.1.1: Last time: $V_n(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^n \mathfrak{g} \xrightarrow{\varepsilon} k$ is a projective resolution in $\mathfrak{g}$-Mod. Note that we can introduce negative signs to easily interchange $\mathfrak{g}$-Mod and Mod-$\mathfrak{g}$.

Corollary 42.1.2 ***(Chevalley-Eilenberg).***
Let $M \in$ Mod-$\mathfrak{g}$, then

$$H_*(\mathfrak{g}; M) \cong \operatorname{Tor}_*^{\mathcal{U}(\mathfrak{g})}(M, k)$$

is the homology of the following complex:

$$M \otimes_{\mathcal{U}(\mathfrak{g})} V_*(\mathfrak{g}) := M \otimes_{\mathcal{U}(\mathfrak{g})} \mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^* \mathfrak{g} \cong M \otimes_k \bigwedge^* \mathfrak{g},$$

where we have a concrete differential d on $\bigwedge^* \mathfrak{g}$ and we can define $\partial := \mathbb{1} \otimes d$. If $M \in \mathfrak{g}$-Mod) (which is more convenient for cohomology), then

$$H^*(\mathfrak{g}; M) \cong \operatorname{Ext}_{\mathcal{U}(\mathfrak{g})}(k, M)$$

is the cohomology of the cochain complex

$$\underset{\mathcal{U}(\mathfrak{g})}{\operatorname{Hom}}(V_*(\mathfrak{g}), M) := \underset{\mathcal{U}(\mathfrak{g})}{\operatorname{Hom}}(\mathcal{U}(\mathfrak{g}) \otimes_k \bigwedge^* \mathfrak{g}, M) \cong \underset{k}{\operatorname{Hom}}(\bigwedge^* \mathfrak{g}, M).$$

Remark 42.1.3: This is very concrete! Standard trick for exterior algebras: any n-cochain $f \in \underset{k}{\operatorname{Hom}}(\bigwedge^n \mathfrak{g}, M)$ can be viewed as an alternating k-multilinear function $f(x_1, \cdots, x_n) : \mathfrak{g} \to M$. The cochain differential should increase degree, so we define

$$\Theta_1 f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot f(x_1, \cdots, \widehat{x_i}, \cdots, x_n) + \sum_{i<j} (-1)^{i+1} f([x_i x_j], x_1, \cdots, \widehat{x_i}, \cdots, \widehat{x_j}, \cdots, x_n).$$

Note that the tor definition has the arguments switched compared to the original definition. This is to set up the tensor cancellation of $\cdots \otimes_{\mathcal{U}(\mathfrak{g})} \mathcal{U}(\mathfrak{g}) \cdots$. Swapping factors and introducing signs makes this work for left $\mathfrak{g}$-modules.

Corollary 42.1.4 *(?).*
If k is a field and $\dim_k \mathfrak{g} = n$, then for any $M \in \mathfrak{g}\text{-}\mathsf{Mod}$,

$$H^i(\mathfrak{g}; M) = 0 = H_i(\mathfrak{g}; M) \qquad \forall i \geq n+1.$$

Proof (?).
This follows from the fact that $\bigwedge^{\geq n+1} \mathfrak{g} = 0$. ■

Example 42.1.5 *(?):* Take $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then $\dim_{\mathbb{C}} \mathfrak{g} = 3$ (4 dimensions and one linear condition). Then $H^i(\mathfrak{g}; m) = 0$ for all $i > 3$.

42.2 Brief Intro to Semisimple Lie Algebras (Weibel 7.8)

Remark 42.2.1: Public service section since we won't have a Lie algebras course next Fall. Semisimples: the most important and interesting classes of Lie algebras! These occur frequently and we can prove a lot about them. We'll assume $\mathfrak{g}$ is a finite dimensional Lie algebra over a field k, where we'll soon assume $\operatorname{ch}(k) = 0$.

Definition 42.2.2 (Simple Lie Algebras)
A Lie algebra $\mathfrak{g}$ is **simple** if it has no ideals other than 0 and $\mathfrak{g}$ and $[\mathfrak{g}\mathfrak{g}] \neq 0$ (i.e. $\mathfrak{g}$ is not abelian).

Remark 42.2.3: Recall that $\mathsf{Lie\text{-}Alg}^{\mathsf{Ab}} \approx \mathsf{k\text{-}Mod}$ are vector spaces, and so if $\mathfrak{g}$ is abelian it automatically has a chain of ideals by just taking vector subspaces. These are closed under brackets since bracketing is zero. So if $\dim_k \mathfrak{g} \geq 2$, there are nontrivial ideals, so the abelian condition rules out all 1-dimensional Lie algebras – they're all abelian by taking a generator, bracketing it with itself, and noting you get zero. So there's only one 1-dimensional Lie algebra over any field k: the abelian one.

Remark 42.2.4: The derived algebra $[\mathfrak{g}\mathfrak{g}] \trianglelefteq \mathfrak{g}$ is a subalgebra and always an ideal, so if $\mathfrak{g}$ is simple then $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$. So $\mathfrak{gl}_n(\mathbb{C})$ is not simple, since $[\mathfrak{gl}_n \mathfrak{gl}_n] = \mathfrak{sl}_n$ by taking traces.

Remark 42.2.5: The vector space sum of any two solvable ideals is again a solvable ideal. Note that this works for products of solvable subgroups $N, H \leq G$ with N normal. Use 2-out-of-3 property for solvable groups and quotient by N. By finite-dimensionality, we can find a maximal solvable ideal:

Definition 42.2.6 (?)
For $\dim_k \mathfrak{g} < \infty$, define the **radical** to be $\operatorname{rad}\mathfrak{g} := \sum I_j$ be the sum of all solvable ideals $I_j \trianglelefteq \mathfrak{g}$. We say $\mathfrak{g}$ is **semisimple** if $\operatorname{rad}\mathfrak{g} = 0$.

Lemma 42.2.7 *(?).*
Simple implies semisimple.

Lemma 42.2.8 *(?).*
$\mathfrak{g}/\mathrm{rad}\mathfrak{g}$ is always semisimple.

Remark 42.2.9: There shouldn't be any solvable ideals in this quotient, otherwise you could lift. Next up, our most powerful tool for semisimple Lie algebras:

Definition 42.2.10 (?)
Recall that for $x \in \mathfrak{g}$ we can define $\mathrm{ad}_x \in \underset{k}{\mathrm{End}}(\mathfrak{g})$ where $\mathrm{ad}_x(y) \coloneqq [x, y]$. It has a well-defined (and basis-independent) trace, so define the **Killing form**[a]:

$$\kappa(x, y) \coloneqq \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) \in k \qquad\qquad x, y \in \mathfrak{g}.$$

[a] Named for a mathematician *named* Killing.

Remark 42.2.11: This is a symmetric bilinear form since traces don't depend on the order of products. It has another nice property, $\mathfrak{g}$-invariance:

$$\kappa([xy], z] = \kappa(x, [yz]).$$

Proposition 42.2.12 *(Cartan's Criterion).*
Let $\mathrm{ch}(k) = 0$ and $\dim_k \mathfrak{g} < \infty$. Then $\mathfrak{g}$ is semisimple $\iff$ κ is nondegenerate.

Proof (?).
Omitted, see Humphreys. ∎

Theorem 42.2.13 *(?).*
Let $\mathrm{ch}(k) = 0$, then $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g} \cong \bigoplus_{i=1}^{r} \mathfrak{g}_i$ as a direct sum/product of simple ideals, so $[\mathfrak{g}_i \mathfrak{g}_j] = 0$ for $i \neq j$ and $[\mathfrak{g}_i \mathfrak{g}_i] = \mathfrak{g}_i$. In particular, every ideal of $\mathfrak{g}$ is a sum of sum of certain $\mathfrak{g}_i$'s, and $\mathfrak{g} = [\mathfrak{g}\mathfrak{g}]$.

Remark 42.2.14: These are like "orthogonal" ideals. So we can study semisimple Lie algebras by just studying simple Lie algebras.

Observation 42.2.15
Reminder: if $M \in \mathfrak{g}\text{-}\mathsf{Mod}(\mathrm{Triv})$, then any derivation $D \in \mathrm{Der}(\mathfrak{g}, M)$ satisfies $D([xy]) = 0$ for all $x, y \in \mathfrak{g}$. This follows from expanding the Leibniz rule and using trivial modules act by zero. There is an isomorphism

$$\mathrm{Der}(\mathfrak{g}, M) \cong \underset{k\text{-}\mathsf{Mod}}{\mathrm{Hom}}(\mathfrak{g}^{\mathrm{ab}}, M) \qquad\qquad \mathfrak{g}^{\mathrm{ab}} \coloneqq \mathfrak{g}/[\mathfrak{g}\mathfrak{g}].$$

Recall that H^1 is related to derivations.

Corollary 42.2.16 *(?).*
Let $\mathfrak{g} \in \mathfrak{g}\text{-}\mathsf{Mod}(,,)$ with $\dim_k \mathfrak{g} < \infty$, then

$$H^1(\mathfrak{g}; k) = 0 = H_1(\mathfrak{g}; k).$$

Proof (?).
Since $[\mathfrak{g}\mathfrak{g}] = \mathfrak{g}$, we have $\mathfrak{g}^{\text{ab}} = 0$. By Weibel theorem 7.4.1, one can check that $H_1(\mathfrak{g}; k) \cong \mathfrak{g}^{\text{ab}} = 0$. We also had $\text{Der}(\mathfrak{g}, k) \twoheadrightarrow H^1(\mathfrak{g}; k)$ (it was outer derivations), the left-hand side is isomorphic to $\underset{k}{\text{Hom}}(\mathfrak{g}^{\text{ab}}; k)$. ■

Theorem 42.2.17 *(?).*
Let $\mathfrak{g} \in \mathsf{Lie\text{-}Alg}(,,)$ with $\dim_k \mathfrak{g} < \infty$ and $\text{ch}(k) = 0$. Then if $k \neq M$ is a simple $\mathfrak{g}$-module (where simple means no proper nontrivial $\mathfrak{g}$-invariant submodules), then

$$H^i(\mathfrak{g}; M) = 0 = H_i(\mathfrak{g}; M).$$

Proof (?).
Omitted. This uses the Casimir operator for M, which is in the center $Z(\mathcal{U}(\mathfrak{g}))$. ■

43 | Appendix: Extra Definitions

Definition 43.0.1 (Acyclic)
A chain complex C is **acyclic** if and only if $H_*(C) = 0$.

44 | Extra References

- https://www.math.wisc.edu/~csimpson6/notes/2020_spring_homological_algebra/notes.pdf

45 Useful Facts

Proposition 45.0.1 ***(Algebra Facts).***

- Free $\implies$ projective $\implies$ flat $\implies$ torsionfree (for finitely-generated R-modules)
  - Over R a PID: free $\iff$ torsionfree

Remark 45.0.2: Notational conventions:

- Finite direct products: $\bigoplus$
- Cohomological indexing: C^i, ∂^i
- Homological indexing: C_i, ∂_i
- Right-derived functors R^iF.
  - Come from left-exact functors
  - Require *injective* resolutions
  - Extend to the right: $0 \to F(A) \to F(B) \to F(C) \to L_1F(A) \cdots$
- Left-derived functors L_iF.
  - Come from right-exact functors
  - Require *projective* resolutions
  - Extend to the left: $\cdots L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$
- Colimits:
  - Examples: coproducts, direct limits, cokernels, initial objects, pushouts
  - Commute with left adjoints, i.e. $L(\operatorname{colim} F_i) = \operatorname{colim} LF_i$.
- Examples of limits:
  - Products, inverse limits, kernels, terminal objects, pullbacks
  - Commute with right adjoints. i.e. $R(\operatorname{colim} F_i) = \operatorname{colim} RF_i$.

45.1 Hom and Ext

Proposition 45.1.1 ***(Basic properties of Hom).***

- $\operatorname{Hom}_R(A, -)$ is:
  - Covariant
  - Left-exact
  - Is a functor that sends $f : X \to Y$ to $f_* : \operatorname{Hom}(A, X) \to \operatorname{Hom}(A, Y)$ given by $f_*(h) = f \circ h$.
  - Has right-derived functors $\operatorname{Ext}^i_R(A, B) := R^i \operatorname{Hom}_R(A, -)(B)$ computed using *injective* resolutions.

- $\operatorname*{Hom}_R(-, B)$ is:
  - Contravariant
  - Right-exact
  - Is a functor that sends $f : X \to Y$ to $f^* : \operatorname{Hom}(Y, B) \to \operatorname{Hom}(X, B)$ given by $f^*(h) = h \circ f$.
  - Has left-derived functors $\operatorname{Ext}_R^i(A, B) := L_i \operatorname*{Hom}_R(-, B)(A)$ computed using *projective* resolutions.
- For $N \in (\mathsf{R}, \mathsf{S}')\text{-}\mathsf{biMod}$ and $M \in (\mathsf{R}, \mathsf{S})\text{-}\mathsf{biMod}$, $\operatorname*{Hom}_R(M, N) \in (\mathsf{S}, \mathsf{S}')\text{-}\mathsf{biMod}$.
  - Mnemonic: the slots of $\operatorname*{Hom}_R$ use up a left R-action. In the first slot, the right S-action on M becomes a left S-action on Hom. In the second slot, the right S'-action on N becomes a right S'-action on Hom.

Proposition 45.1.2 ***(Basic Properties of Ext).***

- $\operatorname{Ext}^{>1}(A, B) = 0$ for any A projective or B injective.

Fact 45.1.3
A maps $A \xrightarrow{f} B$ in $\mathsf{R}\text{-}\mathsf{Mod}$ is injective if and only if $f(a) = 0_B \implies a = 0_A$. Monomorphisms are injective maps in $\mathsf{R}\text{-}\mathsf{Mod}$.

Proposition 45.1.4 ***(Recipe for computing*** Ext_R^i***).***
Write $F(-) := \operatorname*{Hom}_R(A, -)$. This is left-exact and thus has right-derived functors $\operatorname{Ext}_R^i(A, B) := R^i F(B)$. To compute this:

- Take an *injective* resolution:
$$1 \to B \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots .$$
- Remove the augmentation ε and just keep the complex
$$I^- := \left(1 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots\right).$$
- Apply $F(-)$ to get a new (and usually **not exact**) complex
$$F(I)^- := \left(1 \xrightarrow{\partial^{-1}} F(I^0) \xrightarrow{\partial^0} F(I^1) \xrightarrow{\partial^1} \cdots\right),$$
where $\partial^i := F(d^i)$.
- Take homology, i.e. kernels mod images:
$$R^i F(B) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Note that $R^0F(B) \cong F(B)$ canonically:

- This is defined as $\ker \partial^0 / \operatorname{im} \partial^{-1} = \ker \partial^0 / 1 = \ker \partial^0$.
- Use the fact that $F(-)$ is left exact and apply it to the *augmented* complex to obtain

$$1 \to F(B) \xrightarrow{F(\varepsilon)} F(I^0) \xrightarrow{\partial^0} F(I^1) \xrightarrow{\partial^1} \cdots .$$

- By exactness, there is an isomorphism $\ker \partial^0 \cong F(B)$.

Proposition 45.1.5 ***(Computing** $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n)$**).***
$\varphi : \underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\sim} \mathbb{Z}/n$, where $\varphi(g) := g(1)$.

- That this is an isomorphism follows from
- Surjectivity: for each $\ell \in \mathbb{Z}/n$ define a map

$$\begin{aligned} \psi_y : \mathbb{Z} &\to \mathbb{Z}/n \\ 1 &\mapsto [\ell]_n. \end{aligned}$$

- Injectivity: if $g(1) = [0]_n$, then

$$g(x) = xg(1) = x[0]_n = [0]_n.$$

- $\mathbb{Z}$-module morphism:

$$\varphi(gf) := \varphi(g \circ f) := (g \circ f)(1) = g(f(1)) = f(1)g(1) = \varphi(g)\varphi(f),$$

  where we've used the fact that $\mathbb{Z}/n$ is commutative.

Proposition 45.1.6 ***(Common Hom Groups).***

- $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}/m, \mathbb{Z}) = 0$.
- $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d$.
- $\underset{\mathbb{Z}}{\mathrm{Hom}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$.

Proposition 45.1.7 ***(Common Ext Groups).***

- $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/m, G) \cong G/mG$
  - Use $1 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/m \to 1$ and apply $\underset{\mathbb{Z}}{\mathrm{Hom}}(-, \mathbb{Z})$.
- $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d$.

Slogan 45.1.8

- In Ab, direct colimits commute with finite limits. Inverse limits do not generally commute with finite colimits.
- Left adjoints are right-exact with left-derived functors. Right adjoints are left-exact with

right-derived functors.

- Left adjoints commute with colimits: $L(\operatorname{colim} F) = \operatorname{colim}(L \circ F)$

Proposition 45.1.9 ***(Characterizations of Splittings).***
TFAE in R-Mod:

- A SES $0 \to A \to B \to C \to 0$ is split.
- ?

45.2 Tensor and Tor

Proposition 45.2.1 ***(Basic Properties of the Tensor Product).***

- $A \otimes_R -$ is:
  - Covariant
  - Right-exact
  - Left-exact
  - Has left-derived functors $\operatorname{Ext}_R^i(A, B) \coloneqq L_i \operatorname{Hom}_R(-, B)(A)$ computed using *projective* resolutions.
- $- \otimes_R B$ is:
  - Covariant
  - Right-exact
  - Has left-derived functors $\operatorname{Ext}_R^i(A, B) \coloneqq L_i \operatorname{Hom}_R(-, B)(A)$ computed using *projective* resolutions.
- Tensor commutes with colimits: $(\operatorname{colim} A_i) \otimes_R M = \operatorname{colim}(A_i \otimes_R M)$.

Proposition 45.2.2 ***(Basic Properties of Tor).***

- $\operatorname{Tor}_n^R(A, B) = 0$ for either A or B flat.

Fact 45.2.3
The most useful SES for proofs here:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n \to 0.$$

Proposition 45.2.4 ***(Common Tensor Products).***

- $\mathbb{Z}/n \otimes_{\mathbb{Z}} G \cong G/nG$

- $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/d$.
- $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong 0$.

Proposition 45.2.5 ***(Common Tor Groups).***

- $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, G) \cong \left\{ h \in H \;\middle|\; nh = e \right\}$
- $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}) \cong 0$.
- $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/d$.

45.3 Universal Properties

Proposition 45.3.1 ***(Universal Property of the Quotient for Groups).***
If $f : G \to K$ and $H \trianglelefteq G$ (so that G/H is defined), then the map f descends to the quotient if and only if $H \subseteq \ker(f)$.

Proposition 45.3.2 ***(Kernels as pullbacks and cokernels as pushouts).***
The kernel $\ker f$ of a morphism f can be characterized as a cartesian square, and the cokernel $\operatorname{coker} f$ as a cocartesian square:

$$\begin{tikzcd}
K \arrow[drr, bend left] \arrow[dr, dashed, "\exists!"'] \arrow[ddr, bend right] & & & & \\
& \ker f \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[d, dotted] \arrow[ddr, bend left] & \\
& 0 \arrow[r] & B \arrow[r, dotted] \arrow[drr, bend right, "0"] & \operatorname{coker} f \arrow[dr, dashed, "\exists!"'] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & & & C
\end{tikzcd}$$

Link to Diagram

45.4 Adjunctions

Definition 45.4.1 (Adjoints)

todo

Proposition 45.4.2 ***(Tensor-Hom Adjunction).***

For a fixed $M \in$ (R, S)-biMod, there is an adjunction

$$\text{Mod-R} \underset{\text{Hom}_S(M,-)}{\overset{-\otimes_R M}{\rightleftarrows}} \text{Mod-S},$$

so for $Y \in$ (A, R)-biMod and $Z \in$ (B, S)-biMod, there is a (natural) isomorphism in (B, A)-biMod:

$$\underset{S}{\text{Hom}}(X \otimes_R M, Z) \xrightarrow{\sim} \underset{R}{\text{Hom}}(X, \underset{S}{\text{Hom}}(M, Z)).$$

Proposition 45.4.3 ***(Forgetful Adjunctions).***
Let F : R-Mod $\to \mathbb{Z}$-Mod be the forgetful functor, then there are adjunctions

$$\text{R-Mod} \underset{\text{Hom}_{\mathbb{Z}}(R,-)}{\overset{F}{\rightleftarrows}} \mathbb{Z}\text{-Mod}$$

$$\mathbb{Z}\text{-Mod} \underset{F}{\overset{R\otimes_{\mathbb{Z}}-}{\rightleftarrows}} \text{R-Mod}.$$

ToDos

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Bibliography

[1] Charles A. Weibel. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.