# Category $\mathcal{O}$, Problem Set 3

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## Contents



# 1 Humphreys 1.10

Prove that the transpose map $\tau$ fixes $Z(\mathfrak{g})$ pointwise.

> Check that $\tau$ commutes with the Harish-Chandra morphism $\xi$ and use the fact that $\xi$ is injective.

## 1.1 Solution

We first note that after choosing a PBW basis for $\mathfrak{g}$, $\tau$ is defined on $\mathfrak{g}$ in the following way:

$$\begin{aligned}
\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\
x_\alpha &\mapsto y_\alpha \\
h_\alpha &\mapsto h_\alpha \\
y_\alpha &\mapsto x_\alpha
\end{aligned}$$

which lifts to an anti-involution $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by extending linearly over PBW monomials. We can note that since $\tau$ fixes $\mathfrak{h}$ pointwise by definition, its lift also fixes $U(\mathfrak{h})$ pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned} \xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}. \end{aligned}$$

**Proposition 1.1.**
The following diagram commutes

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\xi"] \arrow[d, "\tau"] & U(\mathfrak{h}) \arrow[d, "\tau"] \\ Z(\mathfrak{g}) \arrow[r, "\xi"] & U(\mathfrak{h}) \end{tikzcd}$$

*Proof.*
We will show that for all $z \in Z(\mathfrak{g})$, $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$. Expand $z$ in a PBW basis as $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}$. We then make the following computations:

$$\begin{aligned} (\xi \circ \tau)(z) &= (\xi \circ \tau)\left(\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}\right) \\ &= \xi\left(\prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_j}\right) \quad \text{since } \tau \text{ is an anti-homomorphism} \\ &= \prod_j h_j^{s_j} \end{aligned}$$

.

Similarly, we have

$$\begin{aligned} (\tau \circ \xi)(z) &= \tau\left(\prod_j h_j^{s_j}\right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

,

where we note that the two resulting expressions are equal. ∎

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of $\xi$, we have

$$\begin{aligned}(\xi \circ \tau)(z) &= \xi(z) \\ \implies \tau(z) &= z.\end{aligned}$$

■

## 2 Humphreys 1.12

Fix a central character $\chi$ and let $\left\{V^{(\lambda)}\right\}$ be a collection of modules in $\mathcal{O}$ indexed by the weights $\lambda$ for which $\chi = \chi_\lambda$ satisfying

1. $\dim V^{(\lambda)} = 1$
2. $\mu < \lambda$ for all weights $\mu$ of $V^{(\lambda)}$.

Then the symbols $[V^{(\lambda)}]$ form a $\mathbb{Z}$-basis for the Grothendieck group $K(\mathcal{O}_\chi)$.

For example take $V^{(\lambda)} = M(\lambda)$ or $L(\lambda)$.

### 2.1 Solution

Following a similar proof outlined here.

Fix a $\lambda_0$ such that $\chi = \chi_{\lambda_0}$ by Harish-Chandra's theorem, fix some order on the Weyl group $W = \left\{w_j \;\middle|\; 1 \leq j \leq |W| < \infty\right\}$, and note that $\chi_{\lambda_0} = \chi_{w\cdot\lambda_0}$ for each $w \in W$.

**Proposition 2.1.**
The simple modules $\left\{L(w \cdot \lambda_0) \;\middle|\; w \in W\right\}$ form a $\mathbb{Z}$-basis for $\mathcal{O}_\chi$.

*Proof.*
Write $\mathcal{L} = \operatorname{span}_{\mathbb{Z}} \left\{[L(w_j \cdot \lambda_0)] \;\middle|\; 1 \leq j \leq |W|\right\} \subset K(\mathcal{O}_\chi)$.
**Spanning**: Let $M \in \mathcal{O}_\chi$ be arbitrary, and consider $[M] \in K(\mathcal{O}_\chi)$. By Humphreys Theorem 1.11, $M$ has a finite composition series

$$M = M_1 > M_2 > \cdots > M_n$$

with simple quotients $M^{i+1}/M^i \cong L(\lambda_i)$ for some $\lambda_i \in \mathfrak{h}^\vee$. By collecting terms, we can write

$$[M] = \sum_{i=1}^{n} [L(\lambda_i)] = \sum_{i=1}^{n'} c_i [L(\lambda_i)] \in K(\mathcal{O}_\chi),$$

where each $c_i$ is the multiplicity of $L(\lambda_i)$ in the above composition series.
By definition, $M \in \mathcal{O}_\chi \iff L(\lambda_i) \in \mathcal{O}_\chi$, i.e. $M$ is in this block precisely when all of its composition factors are. But this forces each $L(\lambda_i) = L(w_j \cdot \lambda_0)$ for some $j$, and so we have

$$[M] = \sum_{i=j}^{n'} c_j [L(w_j \cdot \lambda_0)] \in \mathcal{L}.$$

**Linear Independence**: Define a family of maps

$$\begin{aligned} r_j : \mathcal{O}_\chi &\longrightarrow \mathbb{Z}^{\geq 0} \\ M &\mapsto \left| \left\{ M^{i+1}/M^i \;\middle|\; M^{i+1}/M^i \cong L(w_j \cdot \lambda_0) \right\} \right|, \end{aligned}$$

i.e. the map that counts the multiplicity of $L(w_j \cdot \lambda_0)$ appearing in any composition series of $M$ for a fixed $j$.
This lifts to a group morphism $r_j : K(\mathcal{O}_\chi) \longrightarrow \mathbb{Z}^{\geq 0}$ which satisfies

$$r_j(L(w_i \cdot \lambda_0)) = \delta_{ij},$$

i.e. it takes the value 1 on the Verma modules in $\mathcal{L}$ precisely when $i = j$ and zero otherwise.
Now suppose $\sum_{i=1}^{n} a_i [L(w_i \cdot \lambda_0)] = [0]$ in $K(\mathcal{O}_\chi)$. For each fixed $j$, we can then apply the above group morphism to obtain

$$\begin{aligned} r_j \left( \sum_{i=1}^{n} a_i [L(w_i \cdot \lambda_0)] \right) &= \sum_{i=1}^{n} a_i r_j([L(w_i \cdot \lambda_0)]) \\ &= \sum_{i=1}^{n} a_i r_j \delta_{ij} \\ &= a_j. \end{aligned}$$

Since group morphisms preserve equalities and $r_j([0]) = 0 \in \mathbb{Z}$, this forces $a_j = 0$ for each $j$. ■

**Proposition 2.2.**
An arbitrary set of the stated form $\mathcal{V} = \left\{ V^{(\lambda_i)} \;\middle|\; 1 \leq i < N < \infty \right\}$ is also a $\mathbb{Z}$-basis of $K(\mathcal{O}_\chi)$.

*Proof.*
We first note that we can similarly write $V^{(\lambda_i)} = V^{(w_j \cdot \lambda_0)}$ for some $j$, so wlog we reindex the $\lambda_i$ to $\lambda_j$s. Similarly, fixing a $V^{\lambda_j}$, for $\mu < \lambda_j$, there is an $i$ such that $\mu = w_i \cdot \lambda_0$, so we reindex all lower weights accordingly as well.
By the previous proposition, for each fixed $V^{(\lambda_i)}$, we can write

$$[V^{(\lambda_j)}] = [L(w_j \cdot \lambda_0] + \sum_{\mu_i < \lambda_j} a_{ij}[L(w_i \cdot \lambda_0)].$$

The matrix $A = (a_{ij})$ is then strictly upper-triangular with ones on the diagonal, and is thus invertible, and so expresses a change of basis matrix $\mathcal{L} \longrightarrow \mathcal{V}$.

$\blacksquare$

# 3 Humphreys 1.13

Suppose $\lambda \not\in \Lambda$, so the linkage class $W \cdot \lambda$ is the disjoint union of its nonempty intersections of various cosets of $\Lambda_r \in \mathfrak{h}^\vee$.

Prove that each $M \in \mathcal{O}_{\chi_\lambda}$ has a corresponding direct sum decomposition $M = \bigoplus M_i$ in which all weights of $M_i$ lie in a single coset.

Recall exercise 1.1b.

## 3.1 Solution

Fix a nonintegral $\lambda \in \mathfrak{h}^\vee \setminus \Lambda$ and $M \in \mathcal{O}_{\chi_\lambda}$, and write

$$\mathfrak{h}^\vee/\Lambda = \left\{ \lambda_i + \Lambda \;\middle|\; i \in I \right\} = \left\{ [\lambda_i] \;\middle|\; i \in I \right\}$$

for some indexing set $I$. As in exercise 1.1, for each $i$ we can define

$$M_i = M^{[\lambda_i]} := \sum_{\mu \in [\lambda_i]} M_\mu,$$

the sum of weight spaces $M_\mu$ for which $\mu \in [\lambda_i]$. Note that by construction, all of the weights of $M_i$ lie in the single coset $[\lambda_i]$.

By the result of that exercise, $M = \bigoplus_{i=1}^{N} M^{[\lambda_i]}$ is a finite direct sum of such modules.

$\blacksquare$

Note: letting $W \cdot \lambda$ be the orbit of $\lambda$ under the action of $W$, i.e. the linkage class of $\lambda$, when $\lambda \in \Lambda$ is integral, then $W \cdot \lambda$ intersects only one coset $[\lambda_i]$, and there is only one term in the above sum.