Problem Set 8

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1 Problem 1

1.1 Part a

It follows from the definition that $||f||_{\infty} = 0 \iff f = 0$ almost everywhere, and if $||f||_{\infty}$ is the best upper bound for f almost everywhere, then $||cf||_{\infty}$ is the best upper bound for cf almost everywhere.

So it remains to show the triangle inequality. Suppose that $|f(x)| \leq ||f||_{\infty}$ a.e. and $|g(x)| \leq ||g||_{\infty}$ a.e., then by the triangle inequality for the $|\cdot|_{\mathbb{R}}$ we have

$$|(f+g)(x)| \le |f(x)| + |g(x)|$$
 a.e.
 $\le ||f||_{\infty} + ||g||_{\infty}$ a.e.,

which means that $\|f+g\|_{\infty} \leq \|f\|_{\infty} + \|g\|_{\infty}$ as desired.

1.2 Part b

 \Longrightarrow : Suppose $||f_n - f||_{\infty} \to 0$, then for every ε , N_{ε} can be chosen large enough such that $|f_n(x) - f(x)| < \varepsilon$ a.e., which precisely means that there exist sets E_{ε} such that $x \in E_{\varepsilon} \Longrightarrow |f_n(x) - f(x)|$ and $m(E_{\varepsilon}^c) = 0$.

But then taking the sequence $\varepsilon_n := \frac{1}{n} \to 0$, we have $f_n \rightrightarrows f$ uniformly on $E := \bigcap_n E_n$ by definition, and $E^c = \bigcup_n E_n^c$ is still a null set.

 \Leftarrow : Suppose $f_n \rightrightarrows f$ uniformly on some set E and $m(E^c) = 0$. Then for any ε , we can choose N large enough such that $|f_n(x) - f(x)| < \varepsilon$ on E; but then ε is an upper bound for $f_n - f$ almost everywhere, so $||f_n - f||_{\infty} < \varepsilon \to 0$.

1.3 Part c

To see that simple functions are dense in $L^{\infty}(X)$, we can use the fact that $f \in L^{\infty}(X) \iff$ there exists a g such that f = g a.e. and g is bounded.

Then there is a sequence s_n of simple functions such that $||s_n - g||_{\infty} \to 0$, which follows from a proof in Folland:

Proof. (a) For
$$n = 0, 1, 2, ...$$
 and $0 \le k \le 2^{2n} - 1$, let
$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \text{and} \quad F_n = f^{-1}((2^n, \infty]),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} k 2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that $\phi_n \leq \phi_{n+1}$ for all n, and $0 \leq f - \phi_n \leq 2^{-n}$ on the set where $f \leq 2^n$. The result therefore follows.



However, $C_c^0(X)$ is dense $L^{\infty}(X) \iff$ every $f \in L^{\infty}(X)$ can be approximated by a sequence $\{g_k\} \subset C_c^0(X)$ in the sense that $\|f - g_n\|_{\infty} \to 0$. To see why this can *not* be the case, let f(x) = 1, so $\|f\|_{\infty} = 1$ and let $g_n \to f$ be an arbitrary sequence of C_c^0 functions converging to f pointwise.

Since every g_n has compact support, say $\sup(g_n) := E_n$, then $g_n|_{E_n^c} \equiv 0$ and $m(E_n^c) > 0$. In particular, this means that $||f - g_n||_{\infty} = 1$ for every n, so g_n can not converge to f in the infinity norm.

2 Problem 2

2.1 Part a

2.1.1 Part i

Lemma: $||1||_p = m(X)^{1/p}$

This follows from $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$ and taking pth roots. \square

By Holder with p = q = 2, we can now write

$$\begin{split} \|f\|_1 &= \|1 \cdot f\|_1 \leq \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\ \Longrightarrow \|f\|_1 \leq m(X)^{1/2} \|f\|_2. \end{split}$$

Letting $M \coloneqq \|f\|_{\infty}$, We also have

$$\begin{split} \|f\|_2^2 &= \int_X |f|^2 \le \int_X |M|^2 = M^2 \int_X 1 = M^2 m(X) \\ \Longrightarrow \|f\|_2 \le m(X)^{1/2} \|f\|_\infty \\ \Longrightarrow m(X)^{1/2} \|f\|_2 \le m(X) \|f\|_\infty, \end{split}$$

and combining these yields

$$||f||_1 \le m(X)^{1/2} ||f||_2 \le m(X) ||f||_{\infty},$$

from which it immediately follows

$$m(X) < \infty \implies L^{\infty}(X) \subseteq L^{2}(X) \subseteq L^{1}(X).$$

The Inclusions Are Strict:

1.
$$\exists f \in L^1(X) \setminus L^2(X)$$
:

Let X = [0, 1] and consider $f(x) = x^{-\frac{1}{2}}$. Then

$$||f||_1 = \int_0^1 x^{-\frac{1}{2}} < \infty$$
 by the *p* test,

while

$$||f||_2^2 = \int_0^1 x^{-1} \to \infty$$
 by the *p* test.

2. $\exists f \in L^2(X) \setminus L^\infty(X)$:

Take X = [0, 1] and $f(x) = x^{-\frac{1}{4}}$. Then

$$||f||_2^2 = \int_0^1 x^{-\frac{1}{4}} < \infty$$
 by the *p* test,

while $||f||_{\infty} > M$ for any finite M, since f is unbounded in neighborhoods of 0, so $||f||_{\infty} = \infty$.

2.1.2 Part ii

1. $\exists f \in L^2(X) \setminus L^1(X)$ when $m(X) = \infty$:

Take $X = [1, \infty)$ and let $f(x) = x^{-1}$, then

$$||f||_2^2 = \int_0^\infty x^{-2} < \infty$$
 by the p test, $||f||_1 = \int_0^\infty x^{-1} \to \infty$ by the p test,.

2. $\exists f \in L^{\infty}(X) \setminus L^{2}(X)$ when $m(X) = \infty$:

Take $X = \mathbb{R}$ and f(x) = 1. Then $||f||_{\infty} = 1 < \infty$ but $||f||_{2} = \int_{\mathbb{R}} 1 = \infty$.

$$L^2(X) \subseteq L^1(X) \implies m(X) < \infty$$
:

Let $f = \chi_X$, by assumption we can find a constant M such that $\|\chi_X\|_2 \leq M \|\chi_X\|_1$.

Then pick a sequence of sets $E_k \nearrow X$ such that $m(E_k) < \infty$ for all $k, \chi_{E_k} \nearrow \chi_X$, and thus $\|\chi_{E_k}\|_p \le M \|\chi_E\|_p$. By the lemma, $\|\chi_{E_k}\|_p = m(E)^{1/p}$, so we have

$$\|\chi_{E_k}\|_2 \le M \|\chi_{E_k}\|_1 \implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \le M$$

$$\implies \frac{m(E_k)^{1/2}}{m(E_k)} \le M$$

$$\implies m(E_k)^{-1/2} \le M$$

$$\implies m(E_k) \le M^2 < \infty.$$

and by continuity of measure, we have $\lim_K m(E_k) = m(X) \le M^2 < \infty$.

- 2.2 Part b
- 3 Problem 3
- 4 Problem 4
- 5 Problem 5
- 6 Problem 6