Problem Set 7

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Contents

1 Regular Problems

1.1 Problem 1

Note that if either $p = 1$ or $q = 1$, G is a p-group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2 q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$\begin{aligned} n_p &\equiv 1 \mod p & n_q &\equiv 1 \mod q \\ n_p &\mid q & n_q &\mid p^2, \end{aligned}$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \left\{1, p, p^2\right\}.$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. This forces $n_p = q$, and we proceed by cases:

1.1.1 Case 1: $p = q$.

Then $\#G = p^3$ and G is a p-group. But every p-group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.1.2 Case 2: $p > q$.

Here, since $n_p \mid q$, we must have $n_p < q$. But if $n_p < q < p$ and $n_p = 1 \mod p$, then $n_p = 1$.

1.1.3 Case 3: $q > p$.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q:

- $n_q = p$: If $n_q = p = 1 \mod q$ and $p < q$, this forces $p = 1$.
- $n_q = p^2$: We will reach a contradiction by showing that this forces

$$\left| P := \bigcup_{S_p \in \mathrm{Syl}(p,G)} S_p \setminus \{e\} \right| + \left| Q := \bigcup_{S_q \in \mathrm{Syl}(q,G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

  We have

$$\begin{aligned}
|P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\
&= p^2(q-1) + q(p^2-1) + 1 \\
&= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 \qquad \text{(since } q > 1\text{)} \\
&= (p^2q - p^2) + (p^2-1) + (q-1)(p^2-1) + 1 \\
&= p^2q + (q-1)(p^2-1) \\
&\geq p^2q + (2-1)(2^2-1) \qquad \text{(since } p, q \geq 2\text{)} \\
&= p^2q + 3 \\
&> p^2q = |G|,
\end{aligned}$$

  which is a contradiction. $\square$

1.2 Problem 2

We'll use the fact that $H \trianglelefteq N(H)$ for any subgroup H (following directly from the closure axioms for a subgroup), and thus

$$P \trianglelefteq N(P) \quad \text{and} \quad N(P) \trianglelefteq N^2(P).$$

Since it is then clear that $N(P) \subseteq N^2(P)$, it remains to show that $N^2(P) \subseteq N(P)$.

So if we let $x \in N^2(P)$, so x normalizes $N(P)$, we need to show that x normalizes P as well, i.e. $xPx^{-1} = P$.

However, supposing that $|G| = p^k m$ where $(p, m) = 1$, we have

$$P \leq N(P) \leq G \implies p^k \,\Big|\, |N(P)| \,\Big|\, p^k m,$$

so in fact $P \in \mathrm{Syl}(p, N(P))$ since it is a maximal p-subgroup.

Then $P' := xPx^{-1} \in \mathrm{Syl}(p, N(P))$ as well, since all conjugates of Sylow p-subgroups are also Sylow p-subgroups.

But since $P \trianglelefteq N(P)$, there is only *one* Sylow p- subgroup of $N(P)$, namely P. This forces $P = P'$, i.e. $P = xPx^{-1}$, which says that $x \in N(P)$ as desired. $\square$

1.3 Problem 3

By definition, G is simple iff it has no non-trivial subgroups, so we will show that if $|G| = 148$ then it must contain a normal subgroup.

Noting that $248 = p^2 q$ where $p = 2, q = 37$, we find that (for example) $n_2 \,\Big|\, 37$ but $n \equiv 1 \mod 2$; but the only odd divisor of 7 is 1, forcing $n_2 = 1$. So G has a normal Sylow 2-subgroup and we are done.

1.4 Problem 4

Let $\tau := (i, j)$ denote the transposition and $\sigma = (s_1, s_2 \cdots, s_p)$ denote the p-cycle. Since there is some power σ^k that sends j to 1, we can assume $\tau = (1, j)$ without loss of generality by conjugating the original τ by σ^k. We can also safely assume $s_1 = 1$ by shifting the entries of σ in cycle notation. Moreover, since σ contains all p integers between 1 and p, we also have $j = s_k$ for some k.

All in all, we can assume

$$\tau = (1, s_j) \quad \sigma = (1, s_2, s_3 \cdots s_j, \cdots s_p).$$

Let $S = \langle \tau, \sigma \rangle$.

...

So we have $\tau := (1, 2), \sigma := (1, 2, \cdots, p) \in S$.

We can then get all adjacent transpositions: noting that $\sigma^k(i) = i + k \mod p$, we have $\sigma^k \tau \sigma^{-k} = (\sigma^k(1), \sigma^k(2)) = (k + 1 \mod p, k + 2 \mod p)$ for every $1 \leq k \leq p$. So if $\tau_i = (i, i + 1 \mod p)$, we have $\langle \tau_i \rangle \subset S$.

But this also gives us all transpositions of the form $(1, j)$ for each $2 \leq j \leq p$:

$$\begin{aligned}
(1, 3) &= (2, 3)^{-1}(1, 2)(2, 3) \\
(1, 4) &= (3, 4)^{-1}(1, 3)(3, 4) \\
&\cdots \\
(1, j) = (j - 1, j)^{-1}(1, j - 1)(j-), j).
\end{aligned}$$

Thus we have $\langle (1,j) \ni 2 \leq j \leq p \rangle \subseteq S$.

But now if $\gamma = (g_1, g_2, \cdots, g_k) \in S_p$ is an arbitrary cycle, we can write

$$\gamma = (g_1, g_2, \cdots, g_k) = (1, g_1)(1, g_2), \cdots (1, g_k).$$

Then writing any arbitrary permutation as a product of disjoint cycles, we find that $S_p \in \langle (1,j) \ni 2 \leq j \leq p \rangle \subseteq S$, and so $S_p \subseteq S$ as desired.

2 Qual Problems