Category $\mathcal{O}$, Problem Set 4

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1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^{\vee}$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Show that the induced $U(\mathfrak{g})$-module structure $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

> Hence $M \notin \mathcal{O}$.

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

2.1 Solution

We first note that $M \in \mathcal{O} \implies M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ where each M_λ is a finite-dimensional weight space.

Moreover, $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$ is defined to be a direct sum of duals of weight spaces, which are still finite-dimensional.

So let $M, N \in \mathcal{O}$; we will proceed by showing that both $(M \otimes_\mathbb{C} L)^\vee$ and $M^\vee \otimes_\mathbb{C}^\vee$ have identical direct sum decompositions.

We first have

$$(M \otimes_\mathbb{C} L)^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} (M \otimes_\mathbb{C} L)_\lambda^\vee, \quad \text{the } \lambda \text{ weight space of } M \otimes_\mathbb{C} L$$

$$= \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha + \beta = \lambda} (M_\alpha \otimes_\mathbb{C} L_\beta) \right)^\vee \quad \text{by an exercise on the weight spaces of a tensor product.}$$

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ.

3.1 Solution

We would like to show the following two propositions:

1. $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. In general, the associated simple system $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \bigcap \Delta$.

3.1.1 Proof of Proposition 1

We'll use the definition that for an abstract root system Φ, a positive system Φ^+ is defined by picking a hyperplane H not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of Φ^+ is on one side of H, then any subset satisfies this property as well, thus $\Phi_{[\lambda]} \bigcap \Phi^+$ consists only of positive roots and thus forms a positive system.

3.1.2 Proof of Proposition 2

Concretely, we can realize Φ and Δ as subsets of $\mathbb{R}^2$ in the following way:

$$\Phi = P_1 \coprod P_2 := \{[1,0],[0,1],[-1,0],[0,-1]\} \coprod \{[1,1],[-1,1],[1,-1],[-1,-1]\}$$
$$\Delta := \{\alpha,\beta\} := \{[1,0],[-1,1]\},$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\Phi_{[\lambda]} := \left\{\gamma \in \Phi \;\middle|\; \langle \lambda,\; \gamma^\vee \rangle \in \mathbb{Z}\right\}, \qquad \gamma^\vee := \frac{2}{\|\gamma\|^2}\,\gamma,$$

$$\Delta_{[\lambda]} := \left\{\gamma \in \Delta \;\middle|\; \langle \lambda,\; \gamma^\vee \rangle \in \mathbb{Z}\right\}.$$

Now choosing $\lambda := \dfrac{\alpha}{2} = \left[\dfrac{1}{2},0\right]$, we now consider the inner products $\langle \lambda,\; \gamma^\vee \rangle$ for $\gamma \in \Phi$:
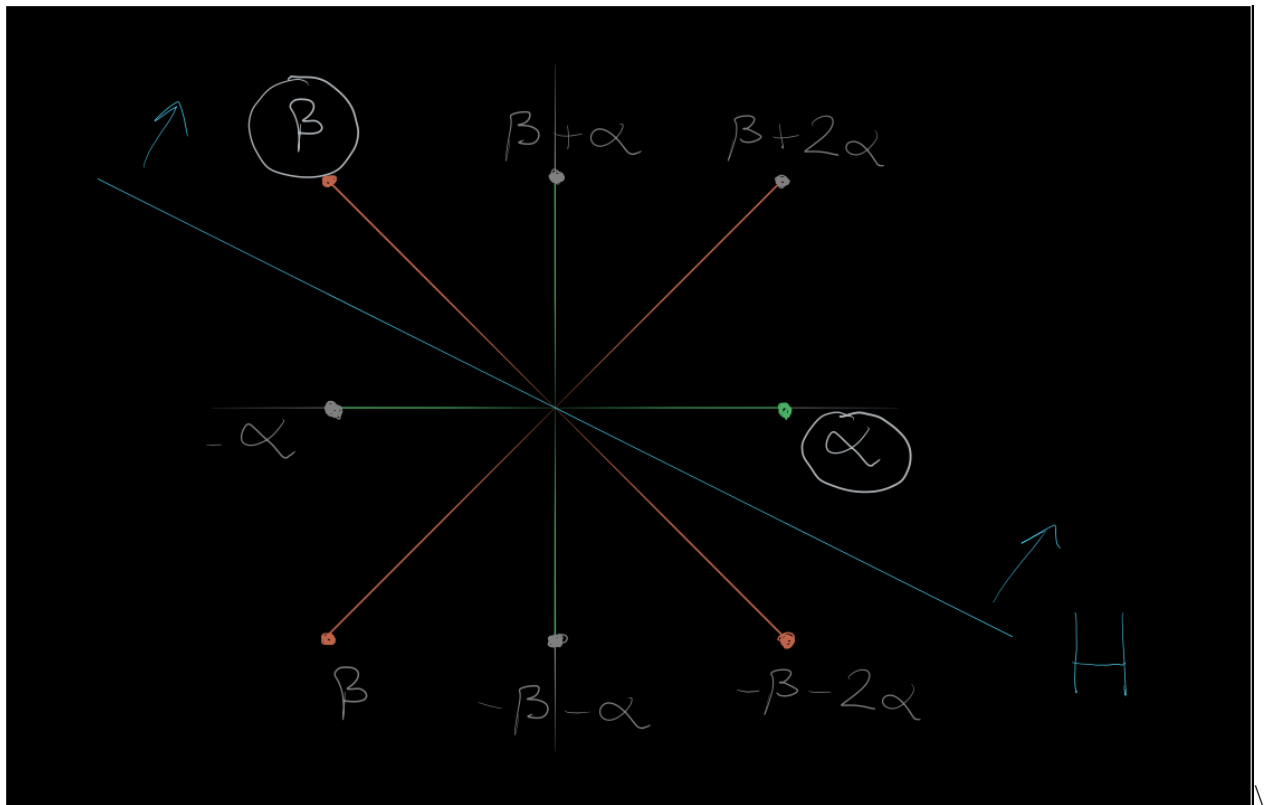
Thus

$$\gamma_1 \in P_1 \implies \left\langle \left[\frac{1}{2},0\right],\; 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right)\langle[1,0],\; \gamma_1\rangle = (\gamma_1)_1 \in \{0,\pm1\} \in \mathbb{Z}$$

$$\gamma_2 \in P_2 \implies \langle \lambda,\; \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2},0\right],\; \frac{2}{\left(\sqrt{2}\right)^2}[\pm1,\pm1] \right\rangle = \pm\frac{1}{2} \notin \mathbb{Z}$$

where $(\gamma_1)_1$ denotes the first component of γ_1.

We thus find that

$$\Phi_{[\lambda]} = P_1 \qquad\qquad \text{the short roots}$$
$$\Delta_{[\lambda]} = \Phi_{[\lambda]} \bigcap \Delta = \{\alpha\} \qquad\qquad \text{the single short simple root.}$$

Choosing the following hyperplane H not containing any root, we can choose a positive system:

$$\Phi^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$$

where we can note that $\Phi^+ \bigcap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi_{[\lambda]}^+ = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$. But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \neq \{\alpha\} = \{\alpha, \alpha + \beta\} \bigcap \{\alpha, \beta\} = \Phi_{[\lambda]}^+ \bigcap \Delta,$$

which is what we wanted to show.

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that coker ϕ has a standard filtration.