

# Title

D. Zack Garza

August 21, 2019

## Contents

0.1 Exercises . . . . .	1
-------------------------	---

### 0.1 Exercises

#### Problem 1.

Let  $C$  denote the Cantor set.

1. Show that  $C$  contains point that is not an endpoint of one of the removed intervals.
2. Show that  $C$  is nowhere dense, meager, and has measure zero.
3. Show that  $C$  is uncountable.

#### Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let  $C_n$  be the  $n$ th stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:

$$C_n = [0, \frac{1}{3^n}] \cup [\frac{2}{2^n}, \frac{3}{3^n}] \cup \cdots \cup [\frac{3^n - 1}{n}, 1],$$

and so the endpoints are precisely the numbers of the form  $\frac{k}{3^n}$  where  $0 \leq k \leq 3^n$ . Moreover, any endpoint appearing in  $C_n$  is never removed in any later step, and so all endpoints remaining in  $C$  are of this form where we allow  $0 \leq n < \infty$ .

Thus, our goal is to produce a number  $x \in [0, 1]$  such that  $x \neq \frac{k}{3^n}$  for any  $k$  or  $n$ , but also satisfies  $x \in C$ .

Claim: If  $x \in C$ , then one can take its ternary expansion and find that all of its digits are either 0 or 2, i.e.

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \text{ where } a_k \in \{0, 2\}.$$

Proof: Towards a contradiction suppose that  $x \in C$  and contains a 1 in its ternary expansion, so  $a_k = 1$  for some  $k$ .