

Problem Set 5

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1 Problem 1

We first make the following claim (TODO):

$$S := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$$
$$T := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = \sup \left\{ \sum_{(j,k) \in C} a_{kj} \mid C \subset \mathbb{N}^2, |C| < \infty \right\}.$$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs (j, k) , we have the following inequality:

$$\sum_{(j,k) \in B} a_{jk} < \sum_{(k,j) \in C} a_{jk} \leq \sum_{(k,j) \in C} a_{kj} \leq T,$$

since T is a supremum over *all* such sets C , and the terms of any finite sum can be rearranged.

But since this holds for every B , we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_B \sum_{(k,j) \in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that $T \leq S$, yielding the desired equality. \square

1.1 Problem 2

We want to show the following equality:

$$\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx = \int_0^1 f(x) \, dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned} \int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx &= \int_0^1 \int_0^t \frac{f(t)}{t} \, dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} \int_0^t dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} (t - 0) \, dt \\ &= \int_0^1 f(t) \, dt, \end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$\begin{aligned} F : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x, t) &\mapsto \frac{\chi_A(x, t) \hat{f}(x, t)}{t}. \end{aligned}$$

where $A = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq t \leq 1\}$ and $\hat{f}(x, t) := f(t)$ is the cylinder on f .

This defines a measurable function on \mathbb{R}^2 , since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\begin{aligned} \int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right| \, dx \, dt \\ &= \int_0^1 \int_0^t \frac{|f(t)|}{t} \, dx \, dt \quad \text{since } t > 0 \\ &= \int_0^1 \frac{|f(t)|}{t} \int_0^t dx \, dt \\ &= \int_0^1 |f(t)| \, dt < \infty, \end{aligned}$$

where the last inequality holds because f was assumed to be measurable. So F is measurable on the product space \mathbb{R}^2 , and we can thus apply Tonelli to F to justify the initial switch. \square

2 Problem 3

Let $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$, and define

$$F(x, y) = \frac{\chi_A(x, y)x^{1/3}}{(1 + xy)^{3/2}}.$$

Then $F \in L^1(\mathbb{R}^2) \iff f \in L^1(\mathbb{R}^2)$, and if this is true then we would have

$$\int_{\mathbb{R}^2} F = \int_0^\infty \int_y^\infty f(x, y) \, dx \, dy$$

and if an interchange of integrals is justified, this yields

$$\begin{aligned} \int_{\mathbb{R}^2} F &= \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1 + xy)^{3/2}} \, dy \, dx \\ &= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3}\sqrt{1 + x^2}} \, dx \\ &= 2 \int_0^1 \frac{1}{x^{2/3}\sqrt{1 + x^2}} \, dx + 2 \int_1^\infty \frac{1}{x^{2/3}\sqrt{1 + x^2}} \, dx \\ &\leq \int_0^1 x^{-2/3} \, dx + \int_0^\infty x^{-5/3} \, dx \\ &= 2(3) + 2\left(\frac{3}{2}\right) < \infty, \end{aligned}$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1 + x^2} \geq \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1 + x^2} \geq \sqrt{1}$.

Since this implies that F is measurable, it remains to show that this interchange of integrals is justified.

Toward that end, ...