An essay on spectral sequences

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This is primarily an essay on the Serre spectral sequence for the homology/cohomology of a fibration. But we view the Serre spectral sequence as a special case of the spectral sequence of a filtered space or filtered chain complex, which has many other important cases: the spectral sequence(s) of a double complex (Grothendieck's composite functor spectral sequence in homological algebra is obtained this way); the Atiyah-Hirzebruch spectral sequence for generalized homology theories; the Rothenberg-Steenrod spectral sequence for the homology of classifying spaces; and more.

The goal is to explain the ideas behind the spectral sequence and to illustrate how it is used; technical details are largely omitted, and occasionally even precise statements are lacking. For details see e.g. McCleary's "User's guide to spectral sequences", Hatcher's chapter on spectral sequences (available online), or for a purely algebraic viewpoint, Weibel's text on homological algebra. There is an elegant abstract construction of spectral sequences based on the notion of an "exact couple". I won't discuss exact couples here—in fact, I won't even give the abstract definition of a spectral sequence—because the abstract view is easy (I'll leave it to you to read about it), whereas really learning spectral sequences requires working with the details in specific cases.

The essay is not supposed to be read in linear order. For instance, when reading the section on the spectral sequence of a filtered space, you will want to simultaneously read about the Serre spectral sequence to see an example.

It's essential to draw diagrams of the E_2-page of your spectral sequence, for example dots corresponding to integer lattice points in the first quadrant, with each dot representing a module over some base ring R (for us, R is a principal ideal domain) and with arrows between dots representing the differentials of the spectral sequence. I haven't had time to include such diagrams here, but will provide plenty of them in class.

1 Long exact sequences as a prototype

Suppose given a space X and subspace A (or a chain complex and a subcomplex, if you want a purely algebraic example). We hope to compute H_*X using the long exact sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \longrightarrow H_nA \longrightarrow H_nX \longrightarrow H_n(X,A) \longrightarrow H_{n-1}A \longrightarrow \ldots$$

Three steps are required to carry out this procedure:

- Compute H_*A and $H_*(X,A)$.

- Compute the boundary maps $\partial_n : H_n(X, A) \longrightarrow H_{n-1}A$.
- Recover H_nX from the extension $0 \longrightarrow \mathit{coker}\, \partial_{n+1} \longrightarrow H_nX \longrightarrow \mathit{ker}\, \partial_n \longrightarrow 0$.

If we are lucky, step 1 alone may suffice. For example, take $X = \mathbb{C}P^n$ and $A = \mathbb{C}P^{n-1}$ (where we assume the latter has already been computed). Using one form or another of excision, we know that $H_*(X, A) = \tilde{H}_*S^{2n}$. In particular, the groups H_nA, $H_n(X, A)$ are zero for n odd, so the boundary maps of step 2 are all zero and the extensions of step 3 are as trivial as trivial can be, since one or the other of the two end terms is zero. Hence we obtain $H_*\mathbb{C}P^n$ with having to open the "black box" at all, i.e. we don't even need to know how the boundary maps are defined.

When we are not so lucky, we have to open the black box and look inside. For example, consider the inductive calculation for real projective spaces. In the case n even there is the boundary map $\partial_n : \mathbb{Z} = H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \longrightarrow H_{n-1}\mathbb{R}P^{n-1} = \mathbb{Z}$; to get past step 2 we need to compute it. It turns out to be multiplication by ± 2, and with this information in hand the "extension problem" of step 3 is trivial as before.

One can often avoid the additive extension problem by using field coefficients. Much more common is the *multiplicative* extension problem in cohomology. As an illustration, consider the inductive computation of $H^*\mathbb{C}P^n$. At the inductive step we have an exact sequence of the form

$$0 \longrightarrow \langle z \rangle \longrightarrow H^*\mathbb{C}P^n \longrightarrow \mathbb{Z}[y]/y^n \longrightarrow 0,$$

where $y \in H^2$, z generates H^{2n} and the subgroup it generates is an ideal; note $z^2 = 0$. It follows that $y^n = kz$ for some $k \in \mathbb{Z}$, but without additional input (e.g. from Poincaré duality or the Gysin sequence) we cannot determine k. Taking field coefficients doesn't help. This is the multiplicative extension problem.

2 The spectral sequence of a filtered space

Now suppose we are given a space X and a filtration $X_0 \subset X_1 \subset \ldots$ with $\cup X_n = X$ and $H_*X \cong \mathit{colim}_n H_*X_n$ (the direct limit; I have in mind a CW-complex with the X_i's subcomplexes, but any such filtration will do). The filtration on X yields a filtration on homology: $F_nH_kX = Im\,(H_kX_n \longrightarrow H_kX)$. If you want a purely algebraic example, take a chain complex given as the ascending union of subchain complexes. Now suppose we know $H_*(X_n, X_{n-1})$ for all n. How can we recover H_*X from this data?

The obvious approach is to consider the long exact sequences of the pairs (X_n, X_{n-1}) inductively. More generally, one could try to make use of all the long exact sequences of pairs (X_n, X_{n-k}), which fit together to form a vast, interlocking maze of exact sequences. Abstractly speaking, what a spectral sequence does is to organize this data in a useable form. So in a sense, a spectral sequence is just a glorified book-keeping device. This is a misleading assertion, however, as the interesting part lies in the details of a particular spectral sequence.

2.1 Construction of the spectral sequence: an intuitive view

I'll display the data in a "tower" that I would normally write vertically, but for easier typesetting will write horizontally here (rotate counterclockwise 90 degrees to get my preferred display). To avoid clutter, as well as to emphasize the abstract structure of the tower, I'll set $A_n = H_*X_n$ and $B_n = H_*(X_n, X_{n-1})$. The maps i, j, ∂ below are all implicitly indexed by their group of origin: $j_n : A_n \longrightarrow B_n$ and so on. The indices are omitted in the diagram to avoid clutter.

$$
\begin{tikzcd}
A_{n-2} \arrow[r, "i"] \arrow[d, "j"'] & A_{n-1} \arrow[r, "i"] \arrow[d, "j"'] & A_n \arrow[r, "i"] \arrow[d, "j"'] & A_{n+1} \arrow[r, "i"] \arrow[d, "j"'] & A_{n+2} \arrow[d, "j"'] \\
B_{n-2} & B_{n-1} \arrow[ul, dotted, "\partial"] & B_n \arrow[ul, dotted, "\partial"] & B_{n+1} \arrow[ul, dotted, "\partial"] & B_{n+2} \arrow[ul, dotted, "\partial"]
\end{tikzcd}
$$

Each triangle with dotted arrow forms an exact sequence, where the dotted arrow lowers dimension by 1. The solid arrows preserve dimension. The complete diagram starts on the left with A_0, B_0, and continues indefinitely to the right.

Now consider a nonzero $\alpha \in H_k(X_n, X_{n-1})$. It aspires to represent a nonzero element of H_kX, but to achieve its dream it must pass two tests: first, it must come from some $\beta \in H_kX_n$; second, β must push forward to a nonzero $\gamma \in H_kX$. The following preliminary discussion is deliberately loose, and will be tightened up a bit afterward.

Consider the first test. Such a β exists if and only if $\partial\alpha = 0$. But we don't yet know A_{n-1}, so instead of trying to compute $\partial\alpha$ directly we push it into the known group B_{n-1} via j. If $j\partial\alpha = 0$ then $\partial\alpha$ comes from A_{n-2}. We then push this new element into B_{n-2}; if it is zero then it comes from A_{n-3} and so on. The process stops after a finite number of steps. If at the end we find that $\partial\alpha$ comes all the way from $A_{-1} := 0$, then $\partial\alpha = 0$ and the first test has been passed.

Now consider the second test. Thus we suppose there is a β with $j_n(\beta) = \alpha$. Then $i(\beta) \neq 0 \in A_{n+1} \Leftrightarrow \beta \notin Im\,\partial_{n+1}$. But we know the B_n's and not yet the A_n's, so we focus on $Im\, j_n\partial_{n+1}$. If α is in its image then the corresponding β pushes forward to zero in A_{n+1}, so α has failed immediately in its quest to represent a nonzero element of H_*X. Continuing this process, suppose we push β far ahead to A_{n+k} where it is still nonzero. Once again, if at this point it is in the image of ∂ then at the next step β will become zero in A_{n+k+1} and α has again failed the test.

At this point we observe there is a qualitative difference between the first and second tests. In the first test, the final answer is guaranteed to be forthcoming after a finite number of steps. In the second test, it isn't clear that we will ever determine α's fate, because the sequence of A_n's may well be infinite. To avoid this difficulty we make the following simplifying assumption. The assumption holds in a number of important cases, including the Serre spectral sequence. Recall at this point that each of the B_n's is a graded module, namely $H_*(X_n, X_{n-1})$.

Connectivity hypothesis. $H_k(X_n, X_{n-1}) = 0$ for $k < n$.

The key point is that the connectivity (i.e. the range in which $H_k(X_n, X_{n-1}) = 0$) tends to infinity with n, but this extra generality would only be a distraction at present. In any case, with the connectivity hypothesis we see that the outcome of the second test can be determined in a finite number of steps. Incoming ∂'s are trying to kill β as it marches ahead from one A_{n+k} to the next, but if α has dimension i, then at every step any would-be assassins have dimension $i+1$, and therefore don't exist for $n+k \geq i$.

The preceding discussion is imprecise, to say the least. We have blithely ignored numerous choices being made; for example: Suppose $j\partial\alpha = 0$ and we choose an element $\alpha_{n-2} \in A_{n-2}$ with $i_{n-2}\alpha_{n-2} = \partial_n\alpha$. We push α_{n-2} into B_{n-2} and obtain a nonzero element γ. Does this mean α has failed? No! Perhaps we just made a bad choice of α_{n-2}. It could be that γ was already hit at the previous stage, that is, $\gamma = j_{n-2}\partial_{n-1}x$ for some $x \in B_{n-1}$. Replacing α_{n-2} by $\alpha_{n-2} - \partial_{n-1}x$, we still have an element mapping to $\partial_n\alpha$, but now it goes to zero in B_{n-2}.

Continuing our imprecise discussion, what remains at B_n after the smoke has cleared and the dust has settled? In a fixed $H_k(X_n, X_{n-1})$, what remains are the "permanent cycles"—that is, the elements that pass the first test—modulo the "eventual boundaries"—that is, the elements that fail the second test. This module is in fact the n-th graded quotient of H_kX. To see this (with lots of handwaving), suppose $\gamma \in F_nH_kX$. This means that γ comes from an $\alpha \in H_kX_n$. Pushing it into $H_k(X_n, X_{n-1})$, we get a permanent cycle. If α' is another such element, then $\alpha - \alpha'$ goes to zero in some H_kX_{n+j} (this uses the connectivity hypothesis), so that $\alpha - \alpha'$ is an "eventual boundary". It follows that the module of "(permanent cycles)/(eventual boundaries)" in $H_k(X_n, X_{n-1})$ is isomorphic to $F_nH_kX/F_{n-1}H_kX$ as claimed.

The rigorous construction of the spectral sequence makes all this precise. In the next section we start over and give a precise statement of its properties, leaving the construction inside the black box.

2.2 Precise formulation of the spectral sequence

We continue to assume the connectivity hypothesis. It is then convenient to re-index in the following way:

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}).$$

Thus the initial data (i.e. the part we are supposing "known") is displayed in the first quadrant. This is called the E^1-term or the E^1-page. Along the vertical q-axis we have simply H_qX_0, while along the horizontal p-axis we have $H_p(X_p, X_{p-1})$—the first possibly non-vanishing homology of these $(p-1)$-connected pairs. The index p is the *filtration degree*, while $p+q$ is the *topological degree*.

The one straightforward fact we have is that the maps $d^1 := j\partial$ in the previous section satisfy $d^1d^1 = 0$. Thus E^1, d^1 is a chain complex, with $d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$. The homology of this chain complex is the E^2-term, $E^2_{p,q}$. The spectral sequence is often viewed as starting with E^2, and consists of a sequence of bigraded chain complexes $E^r_{*,*}$ with the following properties:

- The differential d^r has the form $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ (to the left r, up $r-1$).
- E^{r+1} is the homology of E^r, d^r.
- The spectral sequence *converges* to H_*X in the sense that for each fixed (p,q), $E^r_{p,q}$ eventually stabilizes to the p-th graded quotient of $H_{p+q}X$.

Note all differentials d^r lower the topological degree by 1.

To use the spectral sequence, we need to carry out three steps, just as in the case of long exact sequences. The stable values achieved in the third step give the E^∞-term.

- Compute the E_1-term, or in practice, the E_2-term.
- Compute all the differentials and hence the E^∞-term.
- Recover H_*X from its associated graded module (the latter coinciding with E^∞).

The last step is known as the "additive extension problem". If all the associated graded modules are free, for example if our coefficient ring is a field, then there is no extension problem; the last step is immediate.

In the cohomology version, however, there may well be a "multiplicative extension problem", even with field coefficients. By the "cohomology version" I simply mean apply cohomology instead of homology to our filtered space, or consider a filtered cochain complex. In the case of a filtered space, this entails reversing all the arrows in the diagram of §2.1, and replacing the increasing filtration F_n with a decreasing filtration $F^nH^kX = Ker\,(H_kX \longrightarrow H_kX_{n-1})$ (where X_{-1} is interpreted as the empty set, so that $F^0H^*X = H^*X$). Note that now each element of $H^k(X_n, X_{n-1})$ has infinitely many chances to support a differential, but only finitely many chances to be hit by a differential. In the most general setting there is the further complication that inverse limits aren't as well-behaved as direct limits, but we'll confine our attention to cases satisfying the Connectivity Hypothesis. This guarantees that the inverse limit is achieved at a finite stage, with the upshot being that the cohomology version is essentially just the dual of the homology version. In any case, the multiplicative extension problem results from the fact that the ring structure on the associated graded ring of a filtration does not, in the absence of further data or special hypotheses, determine the original ring structure. We'll see many examples of this below.

3 The Serre spectral sequence

3.1 Construction of the spectral sequence

We will discuss the Serre spectral sequence for Serre fibrations $\pi : X \longrightarrow B$ with B a connected CW-complex (I'd like to use E in place of X, but there are too many E's floating around). Since every space is weakly equivalent to a CW-complex, there is no real loss of generality. Moreover we can assume B has a unique zero-cell, since we can always achieve this by contracting a maximal tree in the 1-skeleton to a point. We take this 0-cell b_0 as our basepoint. (See below under "spectral sequence of a double complex" for another approach that avoids the CW assumption.) Let F denote the fiber over b_0.

The skeletal filtration B^n on B induces a filtration $X^n = \pi^{-1}B^n$ on X. The Serre spectral sequence is the spectral sequence of this filtered space. It satisfies the connectivity hypothesis and:

- $E^2_{p,q} = H_p(B; \underline{H}_q F)$, where $\underline{H}$ indicates a local coefficient system.
- The spectral sequence converges to H_*X.
- From the E_2-term on, the spectral sequence is natural with respect to pullbacks of fibrations.
- Analogous statements hold for cohomology, with the additional property that the differentials are derivations (and the description of $E_2^{*,*}$ is as bigraded algebras).

We won't define homology with local coefficients here, although a brief discussion is given later. Let's instead indicate what's involved in identifying the E_2-term, and give conditions under which the local coefficient system is just an ordinary coefficient module H_qF. For purposes of illustration, I'll assume the fibration is a fiber bundle.

Consider a characteristic map for a cell $\phi : D^p \longrightarrow B$. The pullback of E over D^p is trivial, since D^p is contractible. From this it would appear that the E^1-term is given by

$$E^1_{p,q} = H_{p+q}(D^p \times F, S^{p-1} \times F) \cong \tilde{H}_{p+q}(S^p \wedge F_+) \cong H_qF.$$

Moreover, thinking about how cellular homology works, it is plausible that d^1 is just the cellular boundary map, thereby yielding $E^2_{p,q} = H_p(B; H_qF)$. And indeed this esssentially works, except for one technical problem: "The" fiber F is the fiber over the chosen basepoint, while the displayed formula involves all the fibers, or at least the fibers over the center point of each cell. These fibers are not canonically identified with F unless the bundle is trivial, so the displayed equation is too naive. However, all we need is that the *homology* of each fiber is canonically identified with H_*F. We attempt to do this by choosing a path from the center point of the cell to the basepoint and using a trivialization of the pullback over the interval. But this still doesn't quite work, because the construction might depend on the homotopy class of the path. This is what leads to local coefficient systems. In many cases, however, the local coefficient system is trivial, a question that involves only the fundamental group of B.

The upshot is that $E^2_{p,q} = H_p(B; H_qF)$ (ordinary homology with coefficients) provided that any of the following conditions hold:

- B is simply-connected.
- X is pulled back from a bundle over a simply-connected base space.
- The fibration is a fiber bundle with path-connected structure group G.
- Every homomorphism $\pi_1 B \longrightarrow Aut\, H_*F$ is trivial.

Note the third condition follows from the first two, since then BG is simply-connected. But it can also be proved more directly (once we define what we're talking about!). Note that the fourth condition holds if the nonzero H_qF's are all $\mathbb{F}_2$'s, for example in mod 2 cohomology with F a sphere or projective space. In any case, for the time being we will assume that each of the above conditions implies the simpler description of E_2.

To simplify matters we often make the following further, more modest assumption:

Homological Condition: $H_*(B; H_*F) = H_*B \otimes_R H_*F$, and similarly in cohomology.

In the homology case, this condition holds if either H_*B or H_*F is free over R; of course this is automatic if R is a field. In the cohomology it will hold provided H_*B is free of finite type, in which case $E^2 = H^*B \otimes H^*F$ as R-algebras.

For simplicity, until further notice we always assume:

Standard Hypothesis: The local coefficient system is trivial, and the Homological Condition holds. Thus $E^2_{*,*} = H_*B \otimes_R H_*F$ in the homology case, and similarly for cohomology.

3.2 Multiplicative structure in the cohomology spectral sequence

In this section we assume the Standard Hypothesis—but only to simplify the discussion; it isn't always needed. We continue to write $F \longrightarrow X \longrightarrow B$ for the fibration.

The cohomology Serre spectral sequence is compatible with the ring structure—a very powerful aid to computation. By "compatible" we mean it is a spectral sequence of R-algebras, in the following sense:

- Each E_r-term is a (bi)graded R-algebra, and d_r is a derivation (in the graded sense).
- The E_∞-term is the associated graded algebra of H^*X.
- The isomorphism $E_2^{*,*} \cong H^*B \otimes H^*F$ as algebras.

Needless to say, the proof is highly technical; we certainly won't give it here. Note that in cohomology, all differentials on the horizontal axis H^*B vanish. Thus the spectral sequence is in particular a spectral sequence of H^*B-modules, meaning that each E_r-term is an H^*B-module, the differentials are module homomorphisms, and so on. This can vastly simplify the calculation of differentials; see the next section.

3.3 Collapse and edge homomorphisms

In this section we work in cohomology, and assume the Standard Hypothesis. Now the E_2-term has two distinguished "edges", i.e. the vertical axis $E_2^{0,*}$ and the horizontal axis $E_2^{*,0}$. Remember that in cohomology, the differentials d_r go to the right r and down $r-1$.

3.3.1 The vertical edge and collapse of the spectral sequence

The spectral sequence is said to *collapse* if $E_2 = E_\infty$, i.e. all differentials are zero. In this case H^*X has an associated graded ring $H^*B \otimes H^*F$; in other words, at least additively it looks the same as a product bundle $B \times F$. Now the differentials are all derivations, and H^*B automatically consists of permanent cycles, so the spectral sequence is a spectral sequence of H^*B-modules (meaning that the differentials are module maps). We conclude that $H^*X \cong H^*B \otimes H^*F$ as H^*B-modules. If H^*F is R-free then H^*X is a free R-module. Finally, note that by the derivation property the spectral sequence collapses if and only if the vertical axis H^*F consists entirely of permanent cycles.

Let's take a closer look at the vertical axis. Note that there are no incoming differentials, so as we pass from one page of the spectral sequence to the next, we are simply passing to a (possibly) smaller subgroup of H^qF. Eventually we arrive at $E_\infty^{0,q} \subset H^qF$, the subgroup of permanent cycles. Moreover, the construction of the spectral sequence shows that $E_\infty^{0,q}$ is the top graded quotient F^0/F^1 of H^qX, where $F^1 = Ker\,(H^qE \longrightarrow H^qF)$. Thus the restriction $i^*H^qE \longrightarrow H^qF$ factors as

$$H^qE \longrightarrow E_\infty^{0,q} \subset E_2^{0,q} = H^qF,$$

where the first map is surjective. The term "edge homomorphism" refers to this factorization.Thus the permanent cycles are precisely the image of i^*; in particular the spectral sequence collapses if and only if i^* is surjective.

3.3.2 The horizontal edge

The horizontal edge is just H^*B. There are no outgoing differentials, so as we pass from one page of the spectral sequence to the next, we are simply passing to a (possibly) smaller quotient of H^*B. Eventually we arrive at $E_\infty^{p,0}$, the quotient by all eventual boundaries. Moreover, the construction of the spectral sequence shows that the subgroup of eventual boundaries is the kernel of $H^pB \longrightarrow H^pX$, and hence $E_\infty^{p,0}$ is the quotient of H^pB by this kernel or equivalently the image of H^pB in H^pX. Also (by definition) $E_\infty^{p,0}$ is the p-th and last filtration of H^pX. Thus $\pi^* : H^*B \longrightarrow H^*X$ factors as

$$H^pB = E_2^{p,0} \longrightarrow E_\infty^{p,0} \subset H^pX$$

where the first map is surjective.

Behaviour at the horizontal edge has relatively little influence on (possible) collapse of the spectral sequence, but is always worth examining. For example, suppose the fibration has a section $B \longrightarrow X$. Then π^* is injective, and hence no differentials can hit the horizontal axis.

3.4 The transgression

We continue to work in cohomology, and assume the Standard Hypothesis. The *transgression* is the "last chance" differential $d_r : E_r^{0,r-1} \longrightarrow E_r^{r,0}$ going from the vertical axis to the horizontal axis (one for each $r \geq 2$). Its source is a submodule of $H^{r-1}F$, while its target is a quotient of H^rB. The transgression is sometimes more manageable than a general differential, because it has a concrete description independent of the spectral sequence.

As we just saw, $\alpha \in H^{r-1}F$ is a permanent cycle if and only if it is in the image of $i^* : H^* \longrightarrow H^*F$. This is equivalent to $\delta\alpha = 0$, where $\delta : H^{r-1}F \longrightarrow H^r(X, F)$ is the coboundary map for the long exact sequence of the pair (X, F). Let $\rho : (X, F) \longrightarrow (B, b_0)$ denote the natural map induced by $\pi : X \longrightarrow B$. By examining the construction of the spectral sequence, one can show:

1. The domain of the transgression is $\{\alpha \in H^{r-1}F : \delta\alpha \in Im\, \rho^*\}$. Such an element is called *transgressive*.

Next recall that $\beta \in H^rB$ is an eventual boundary if $\beta \in Ker\, \pi^*$. One can show:

2. The target of the transgression is $H^rB/Ker\, \rho^*$.

And finally, using τ as a generic notation for transgression (r is determined by the source), τ is computed as follows:

3. If α is as in (1), choose β with $\delta\alpha = \rho^*\beta$. Then $\tau\alpha = \beta \bmod Ker\, \rho^*$.

Proving the above is a nice exercise in understanding the construction of the spectral sequence. On the other hand, as advertised, one can compute the transgression directly from (1)-(3) in many cases. In particular, using the recipe of (1)-(3) it is easy to show that in mod p cohomology, τ commutes with Steenrod operations. This last fact, by the way, can help to resolve a puzzling question that may have occured to you: How on earth would you know that a given α is transgressive, without first working through the entire spectral sequence? Steenrod operations preserve transgressive elements, allowing one to work upward from low-dimensional α's already known to be transgressive. For example, take $p = 2$ and consider the fibration

$$\mathbb{R}P^\infty \longrightarrow BSpin(n) \longrightarrow BSO(n).$$

The generator $x \in H^1\mathbb{R}P^\infty$ is transgressive for a trivial reason: Since $r = 2$ here, it is only required that x survive from E_2 to E_2! Hence x^{2^k} is transgressive for all k, since $Sq^1x = x^2$, $Sq^2x^2 = x^4$, etc. In fact $d_2x = w_2$, where w_2 is the universal second Stiefel-Whitney class. Then the transgressions of the x^{2^k}'s can be computed as iterated Steenrod operations on w_2, interpreted in the appropriate quotients of $H^*BSO(n)$. Actually carrying out this idea to compute $H^*(BSpin(n); \mathbb{F}_2)$ is quite difficult, and was first done by Quillen around 1970.

An earlier famous example of the use of transgression and Steenrod operations is in the computation of the mod p cohomology of mod p Eilenberg-Maclane spaces; see below for a brief introduction.

Finally, here is the analogous description of the transgression in homology:

1. The domain of the transgression is $Im\,(\rho_* : H_r(E, F) \longrightarrow H_rB)$. These elements are called *transgressive*.

2. The target of the transgression is $H_{r-1}F/\partial(Ker\,\rho_*)$, where $\partial : H_r(E, F) \longrightarrow H_{r-1}F$ is the boundary map for the pair.

And finally, using τ as a generic notation for transgression (r is determined by the source), τ is computed as follows:

3. If α is as in (1), choose β with $\rho_*\beta = \alpha$. Then $\tau\alpha = \partial\beta$.

4 Sample calculations with the Serre spectral sequence

4.1 Stiefel manifolds, unitary and orthogonal groups

4.1.1 Cohomology

Our first example could also be done using the Gysin sequence, since it involves sphere bundles of vector bundles. In fact for sphere bundles the Gysin sequence and the Serre spectral sequence are essentially equivalent. However, we'll proceed as though we didn't know about the Gysin sequence; the point is to start with a simple example for which Steps 2 (computation of differentials) and 3 (the additive/multiplicative extension problems) are trivial. This is the nicest case; we don't even have to open the black box!

Let $V_k\mathbb{C}^n$ denote the Stiefel manifold of orthonormal k-frames in $\mathbb{C}^n$. Note that $V_1\mathbb{C}^n = S^{2n-1}$ and $V_n\mathbb{C}^n = U(n)$. We'll take $R = \mathbb{Z}$ as our coefficients, although any coefficients would do.

Proposition 4.1 *$H^*V_k\mathbb{C}^n$ is an exterior algebra on generators $x_n, x_{n-1}, ..., x_{n-k+1}$ with $|x_i| = 2i-1$. In particular, the unitary group has $H^*U(n) \cong \mathbb{Z}\langle x_1, ..., x_n\rangle$.*

Proof: We proceed by induction on k. The projection $V_{k+1}\mathbb{C}^n \longrightarrow V_k\mathbb{C}^n$ is a fiber bundle with fiber $S^{2n-2k-1}$, and we will use the Serre spectral sequence of this fibration, proceeding in a standard sequence of steps.

1. The local coefficient system is trivial: From the long exact sequence on homotopy groups, we find by induction that $V_k\mathbb{C}^n$ is simply-connected for $k < n$.

2. The Homological Condition holds: This is clear since the fiber is a sphere, and the finite generation condition is automatic since $V_k\mathbb{C}^n$ is a compact manifold (or by induction). Thus

$$E_2 = H^*V_k\mathbb{C}^n \otimes H^*S^{2n-2k-1} = \mathbb{Z}\langle x_n, x_{n-1}, ...x_{n-k+1}\rangle \otimes H^*S^{2n-2k-1}.$$

3. The spectral sequence collapses: Note that there are only two nonzero rows, $q = 0$ and $q = 2n-2k-1$. Therefore the only possible differential is the one going from the top row to the bottom row, namely d_{2n-2k}. Furthermore, d_{2n-2k} is a homomorphism of $H^*V_k\mathbb{C}^n$-modules and is therefore determined by what it does on

$$E_{2n-2k}^{0,2n-2k-1} = E_2^{0,2n-2k-1} = H^{2n-2k-1}S^{2n-2k-1} = R.$$

But the target of this differential is $E_{2n-2k}^{p,2n-2k} = H^{2n-2k}V_k\mathbb{C}^n = 0$, where the last equality holds because by our inductive hypothesis $\tilde{H}^i V_k\mathbb{C}^n = 0$ for $i < |x_{n-k+1}| = 2n-2k+1$. So this differential is zero for the trivial reason that it has a zero target.

4. There are no additive extensions: Since E_2 is a free $\mathbb{Z}$-module, so is E_∞ by the collapse. It is very easy to show that any module with an increasing filtration (even an infinite one) by submodules with free quotients is itself free. Here we have a decreasing filtration, but since it is a finite filtration it can just as well be regarded as increasing; hence H^*V_{k+1} is a free R-module.

5. There are no multiplicative extensions. Note that E_2 is an exterior algebra of the desired form, hence so is E_∞. This tells us that H^*V_{k+1} has an associated graded algebra that is exterior of the desired form. So why is H^*V_{k+1} itself exterior? The short answer is that exterior algebras over $\mathbb{Z}$ on generators of odd degree are free objects in the category of commutative graded algebras, and this precludes multiplicative extension problems much as the free $\mathbb{Z}$-modules in (4) precluded additive extensions problems. Making this statement precise and proving it is a good exercise, and completes the proof.

It's instructive to constrast the preceeding example with the case of real Stiefel manifolds and $SO(n)$. The real case is much harder and has a more complicated answer. We'll give only a brief discussion here, just to show where the problems lie.

Let's try to apply the same inductive method, using the fibrations $V_{k+1}\mathbb{R}^n \longrightarrow V_k\mathbb{R}^n$, having fiber S^{n-k-1}. Steps 1 and 2 work fine as before, but in Step 3 we run into trouble even when $k=1$: The E_2-term is an exterior algebra on generators in $E_2^{0,n-2}$ and $E_2^{n-1,0}$, but now there is a possible differential

$$d_{n-1} : E_{n-1}^{0,n-2} \longrightarrow E_{n-1}^{n-1,0}.$$

If n is even then $V_2\mathbb{R}^n \longrightarrow V_1\mathbb{R}^n$ has a section (it is the same thing as a unit length vector field on S^{n-1}). Hence no differential can hit the bottom row (see above under "edge homomorphisms") and the spectral sequence collapses. But if n is odd there is no easy way to evaluate the differential; one has to open the black box to see that d_{n-1} is nonzero and in fact hits twice a generator (the factor of 2 can be traced to the Euler characteristic of S^{n-1}). In fact, let's specialize further to the case $n=3$; it can be very enlightening to look at cases where you already know the answer.

When $n=3$ we have $V_2\mathbb{R}^3 = SO(3) \cong \mathbb{R}P^3$. Knowing $H^*\mathbb{R}P^3$ in advance, it is then clear that the d_2 has to hit twice a generator, as the spectral sequence collapses after that and this is the only way to get $H^1 = 0$ and $H^2 = \mathbb{Z}/2$. The appearance of torsion also suggests that things are going to get worse as we move up in the induction. As a fallback position we could take coefficients in $\mathbb{F}_2$ instead. Then the spectral sequence in our $SO(3)$ example collapses, and no additive extensions are possible since we are over a field. But now there is definitely a non-trivial multiplicative extension. The E_∞ term, i.e. the associated graded, thinks that $H^*(SO(3),\mathbb{F}_2) \cong \mathbb{F}_2[x,y]/(x^2,y^2)$, where $|x|=1$, $|y|=2$. But in fact $H^*SO(3) \cong \mathbb{F}_2[x]/x^4$. The point here is that the spectral sequence only tells us that if $x \in H^1$ is a generator, then $x^2 = 0$ *modulo higher filtrations*. To find the actual height of x we have to get outside help.

4.1.2 The homology ring of $U(n)$

We conclude this section with a discussion of the homology ring of $U(n)$. Since $U(n)$ is highly non-commutative, there is no reason to expect the homology ring to be graded commutative. But as it happens it is an exterior algebra on primitive generators; moreover, these generators have an explicit geometric description.

We will define a map $\phi : S^1 \wedge (\mathbb{C}P^{n-1}_+) \longrightarrow U(n)$. Here the "+" denotes a disjoint basepoint. Note that in general $X \wedge (Y_+)$ is the "half-smash" product $(X \times Y)/(x_0 \times Y)$, where $x_0 \in X$ is the basepoint. First define $\rho : S^1 \times \mathbb{C}P^{n-1} \longrightarrow U(n)$ by taking the pair (z, L) to the unitary transformation given by multiplication by z on the line L and by the identity on $L^\perp$. This map is easily seen to be continuous, and sends $1 \times \mathbb{C}P^{n-1}$ to the identity. Hence it factors through a continuous map ϕ as desired.

Now let b_i denote the standard generator of $H_{2i}\mathbb{C}P^{n-1}$, $0 \leq i \leq n-1$, and let $x_i = \phi_* \Sigma b_i$. These elements are primitive because all positive-dimensional homology classes in a suspension are primitive (this is dual to the fact that in the cohomology of a suspension, all cup products of positive-dimensional classes are zero).

Proposition 4.2 $H_*U(n) \cong R\langle x_1, ..., x_n \rangle$.

A sketch of the proof: We proceed by induction on n, using the Serre spectral sequence of the fibration $U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}$. The following key lemma makes an interesting exercise:

Lemma 4.3 *The composite* $(S^1 \wedge (\mathbb{C}P^{n-1}_+) \longrightarrow U(n) \longrightarrow S^{2n-1}$ *induces an isomorphism on* H_{2n-1}.

Now one can show that for general reasons the spectral sequence is of modules over $H_*U(n-1)$, where H_*S^{2n-1} has the trivial module structure. By the lemma $H_{2n-1}S^{2n-1} = E^2_{2n-1,0}$ survives to E^∞, so the spectral sequence collapses. It follows that $H_*U(n)$ is generated by the primitive x_i's. However, at this point we don't know that the ring is graded commutative, so we can't automatically conclude that $x_n^2 = 0$ and that x_n anti-commutes with the other x_i's. Nevertheless, this follows from a Hopf algebra argument, completing the proof.

Here's a convenient term for describing the phenomenon illustrated in Proposition 4.2:

Definition: If X is a homotopy-associative H-space, a *generating map* or *generating complex* for H_*X is a a space A and a map $f : A \longrightarrow X$ (in practice, A is often a subcomplex) such that $f_*(H_*A)$ generates H_*X as a ring.

Thus $S^1 \wedge (\mathbb{C}P^{n-1}_+)$ is a generating complex for $U(n)$. For future reference, we record what happens for $SU(n)$. Let $L_0 \in \mathbb{C}P^{n-1}$ denote the standard basepoint. Then $\rho(z, L)\rho(z, L_0)^{-1}$ maps into $SU(n)$ and factors through $S^1 \wedge \mathbb{C}P^{n-1}$. Then $S^1 \wedge \mathbb{C}P^{n-1}$ is a generating complex for $SU(n)$, and $H_*SU(n) \cong R\langle x_2, ..., x_n \rangle$.

4.2 Flag manifolds

Let $X = F_{1,2}\mathbb{C}^n$ be the manifold of pairs $V_1 \subset V_2$ of linear subspaces of $\mathbb{C}^n$ with $dim\, V_i = i$, an example of a "partial complete flag manifold". Then there is a fiber bundle

$$\mathbb{C}P^{n-2} \xrightarrow{i} X \xrightarrow{\pi} \mathbb{C}P^{n-1},$$

where $\pi(V_1 \subset V_2) = V_1$. Then the local coefficient system is trivial (since the base is simply-connected), and the E_2-term of the cohomology spectral sequence is given by

$$E_2^{p,q} = H^p\mathbb{C}P^{n-1} \otimes H^q\mathbb{C}P^{n-2}.$$

Assume for a moment that we take field coefficients. Even if you haven't read any of the previous sections of this essay, you can interpret the displayed E_2-term as follows: It gives an upper bound on the size of H^*X; more specifically $dim\, H^*X \leq n(n-1)$ (the dimension of the tensor product on the right). Better, it gives an upper bound on the size of each H^nX; in fact it tells us that $|H^*X| \leq |H^*\mathbb{C}P^{n-1} \otimes H^*s\mathbb{C}P^{n-2}|$, where $|M|$ is the Poincaré polynomial (or series) of a graded vector space of finite type, and $\leq$ just means inequality for each coefficient (these being non-negative integers). Note this upper bound would be the actual answer if the bundle was a product bundle.

The next step is to compute the differentials in the spectral sequence. Luckily, we don't need to open the black box here; all differentials are zero because the E_2-term is zero in odd topological degrees, i.e. when $p + q$ odd. Since the differentials go from $p + q = n$, to $p + q = n - 1$, no differentials are possible. Thus $E_2 = E_\infty$ and our E_2 is in fact the associated graded ring of a filtration on H^*X. In fact this all works even with integer coefficients; the point is that there are no additive extension problems: if an increasing filtration on an abelian group has free graded quotient, then the group itself is free. Here we have a decreasing filtration, but since it is finite in each topological degree, we can just as well regard it as increasing.

The final step is to solve the "multiplicative extension problem", in other words, to recover the ring structure of H^*X from that of the associated graded ring. If we're very lucky, these coincide, but that turns out not to be the case in the present example. The collapse of the spectral sequence ensures that as abelian groups π^* is a split injection and i^* is a split surjection. In fact as $H^*\mathbb{C}P^{n-1}$-modules, H^*X is a free module $H^*X \cong H^*\mathbb{C}P^{n-1} \otimes H^*\mathbb{P}^{n-2}$. But there is no reason that i^* should split as rings. There is an element $y \in E_\infty^{0;2}$ (the cohomology of the fiber) with $y^{n-1} = 0$. We can choose $z \in H^2X$ restricting to it, but all we know about z is that $z^{n-1} = 0$ modulo elements of higher filtration. This situation is exactly analogous to the "long exact sequence prototype" example given earlier. For the time being, then, we don't know the ring structure exactly, but we have a lot of information. For example, we know that the ring is generated by two elements of dimension 2.

Continuing using an induction argument (that I'll leave to the reader as an interesting and instructive exercise), we get a result for the complete flag manifold $\mathcal{F}_n$ of flags $V_1 \subset V_2 ... \subset V_{n-1} \subset \mathbb{C}^n$ with dim $V_i = i$. Note that $\mathcal{F}_n$ can be thought of as ordered n-tuples of orthogonal lines in $\mathbb{C}^n$, yielding a natural embedding $g : \mathcal{F}_n \subset (\mathbb{C}P^{n-1})^n$.

Proposition 4.4 *$H^*\mathcal{F}_n$ has an associated graded ring isomorphic to $\otimes_{i=1}^{n-1} H^*\mathbb{C}P^i$. Moreover it is generated by n elements of dimension 2, namely the images under g^* of the natural generators $y_1, ..., y_n$ of $H^*(\mathbb{C}P^{n-1})^n$.*

In particular, $H^*\mathcal{F}_n$ is concentrated in even degrees, and the Euler characteristic

$$\chi(\mathcal{F}_n) = n!.$$

In fact the Poincaré polynomial is

$$|H^*\mathcal{F}_n| = \prod_{i=0}^{n-1}(1 + t^2 + ... + t^{2i}) = \prod_{i=1}^{n} \frac{1 - t^{2i}}{1 - t^2}.$$

How to compute the actual ring structure? This depends on what you mean by "compute". The manifold $\mathcal{F}_n$, which is in fact a smooth projective variety, has its famous Schubert cell decomposition, which gives a basis for the cohomology. One can then ask for an explicit description of the multiplication in this basis—the so-called "Schubert calculus", a vast subject in itself. But from a topological perspective, this is perhaps more than we want or need. We know that $H^*\mathcal{F}_n$ is a quotient ring of $\mathbb{Z}[y_1, ..., y_n]$ by some ideal I, so we may be satisfied with a description of I, such as a set of generators for it. In fact there is a very elegant answer:

Theorem 4.5 *I is the ideal generated by the elementary symmetric functions $\sigma_1, ..., \sigma_n$ in the y_i's. Thus*

$$H^*\mathcal{F}_n \cong \mathbb{Z}[y_1, ..., y_n]/(\sigma_1, ..., \sigma_n).$$

In invariant theory, the algebra on the right in the theorem is known as the "coinvariant algebra". Note that it has the alternative more functorial description: Abbreviating $A := \mathbb{Z}[y_1, ..., y_n]$ and letting A^{S_n} denote the ring of invariants of the symmetric group, we have

$$A/(\sigma_1, ...\sigma_n) \cong A \otimes_{A^{S_n}} \mathbb{Z},$$

where $\mathbb{Z}$ is an A^{S_n}-module via the augmentation.

We will prove the theorem using another Serre spectral sequence. Indeed this approach even yields a new calculation of $H^*BU(n)$, and hence a construction of the Chern classes; for now, however, we'll take $H^*BU(n)$ as known: $H^*BU(n) \cong \mathbb{Z}[c_1, ..., c_n]$, where $|c_i| = 2i$.

We need one fact from the theory of classifying spaces. Let $\mathcal{F}_{n,\infty} \subset (\mathbb{C}P^\infty)^n$ denote the subspace of n-tuples of orthogonal lines. Then the inclusion is a homotopy equivalence. To see this, let $V_n\mathbb{C}^\infty$ denote the orthonormal n-frames. Then $V_n\mathbb{C}^\infty$ is known to be contractible, and $V_n\mathbb{C}^\infty \longrightarrow \mathcal{F}_{n,\infty}$ is a principal T^n-bundle (where $T^n = (S^1)^n$). This shows that $\mathcal{F}_{n,\infty}$ is a model for BT^n and that the inclusion is an equivalence as claimed. The upshot is that we have a fiber bundle $q : \mathcal{F}_{n,\infty} \longrightarrow BU(n) = G_n\mathbb{C}^\infty$ with fiber $i : \mathcal{F}_n \subset \mathcal{F}_{n,\infty}$, where q maps an n-tuple of orthogonal lines to its direct sum (this is just an explicit manifestation of the fact that $BT^n \longrightarrow BU(n)$ has homotopy fiber $U(n)/T^n$). Thus

$$H^*\mathcal{F}_{n,\infty} \cong \mathbb{Z}[y_1, ..., y_n],$$

where the generators y_i are pulled back from $(\mathbb{C}P^\infty)^n$ in the evident way.

Now let's consider the Serre spectral sequence of this fibration. The local coefficient system is trivial, since $BU(n)$ is simply-connected, and $E_2 = H^*BU(n) \otimes H^*\mathcal{F}_n$. Moreover the spectral sequence collapses, as can be seen in three different, independent ways:

- The E_2-term is concentrated in even topological degrees.
- i^* is surjective, by Proposition 4.4.
- The E_2-term has the same Poincaré series as H^*BT^n.

The first way is the simplest, but it doesn't generalize to the case of mod 2 cohomology of real flag manifolds and $BO(n)$, whereas the second and third ways do. I expand on the third method: We compute the Poincaré series for cohomology with rational coefficients. Then

$$|E_2| = |H^*BU(n)| \cdot |H^*\mathcal{F}_n| = \frac{1}{(1-t^2)^n} = |H^*BT^n|,$$

where we have used the known Poincaré series of $H^*BU(n)$ as well as that of $H^*\mathcal{F}_n$ computed earlier. Since $|E_\infty| = |H^*BT^n|$ (the former is an associated graded vector space of a filtration on the latter), clearly there are no differentials in the rational cohomology spectral sequence. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, and the integral E_2-term is free abelian, there are no differentials in the integral cohomology spectral sequence either.

Proof of the theorem: The collapse of the spectral sequence implies that H^*BT^n is a free $H^*BU(n)$-module (this can be regarded as a topological proof of a result from invariant theory; A is free over A^{S_n} of rank $n!$). Or as a better way of saying it,

$$H^*BT^n = H^*BU(n) \otimes_{\mathbb{Z}} H^*\mathcal{F}_n.$$

It follows that $H^*BT^n \otimes_{H^*BU(n)} \mathbb{Z}$ is abstractly isomorphic to $H^*\mathcal{F}_n$. But the composite $H^*BU(n) \longrightarrow H^*BT^n \longrightarrow H^*\mathcal{F}_n$ is trivial, so we get a homomorphism

$$\phi : H^*BT^n \otimes_{H^*BU(n)} \mathbb{Z} \longrightarrow H^*\mathcal{F}_n$$

Since ϕ is a surjective homomorphism of finitely-generated free abelian groups of the same rank, it is an isomorphism as desired.

You can now check that e.g. for $n = 3$ the actual ring structure is not the same as that of the associated graded; there are no nonzero elements in H^2 with square zero.

Finally, it's interesting to note that the same spectral sequence can be used to show from scratch that $H^*BU(n) \longrightarrow H^*BT^n$ is an isomorphism onto the symmetric group invariants. Then one can define the universal Chern classes as the elements corresponding to the elementary symmetric functions. We won't give the argument here, but to get started, note that the second of the three listed "collapse conditions" follows without knowing anything about $BU(n)$ other than that its homology is of finite type.

5 More applications and computations

The main idea of this section is to study the homology Serre spectral sequence of the path-loop fibration $\Omega X \longrightarrow PX \longrightarrow X$. Since PX is contractible, it is a spectral sequence converging to zero (ignoring $E^2_{0,0}$). If we know H_*X we can, with luck, determine $H_*\Omega X$ (or in some cases, vice-versa). Let's proceed without delay to a few of the innumerable examples.

5.1 The Hurewicz theorem and the homology version of Whitehead's Theorem

5.1.1 The Hurewicz theorem

In this section homology groups have integer coefficients.

We get a new proof of the Hurewicz theorem. Suppose X is $(n-1)$-connected, where $n \geq 2$ or $n = 1$ and $\pi_1 X$ is abelian. We want to show the Hurewicz map $h : \pi_n X \longrightarrow H_n X$ is an isomorphism, starting from the known case $n = 1$. Since $\pi_n X \cong \pi_{n-1}\Omega X \cong \pi_{n-2}\Omega^2 X \cong \ldots \cong \pi_1 \Omega^{n-1} X$, the idea is to obtain similar isomorphisms on homology, compatible with the Hurewicz map. Then the Hurewicz theorem for X reduces to $\pi_1\Omega^{n-1}X \cong H_1\Omega^{n-1}X$!

So for $n \geq 2$, consider the path-loop fibration of X. On the bottom row the first nonzero group (ignoring $E^2_{0,0}$, as usual) is $E^2_{n,0}$, and columns 1 through $n-1$ are identically zero. Hence the only possible differential on $E^2_{n,0}$ is d_n (the transgression). Moreover, since the spectral sequence is converging to zero, this d_n must be an isomorphism onto $E^n_{0,n-1} = H_{n-1}\Omega X$. One can show the isomorphism is compatible with the Hurewicz map, so the Hurewicz theorem for X is equivalent to the Hurewicz theorem for ΩX in dimension $n-1$. Continuing in this way we get the desired reduction, completing the proof.

5.1.2 The homology version of Whitehead's theorem

In this section homology groups have integer coefficients.

Let $f : X \longrightarrow Y$ be a map of spaces. Recall that Whitehead's theorem states that if X, Y are CW-complexes (or at least homotopy-equivalent to CW-complexes) and f is a weak equivalence, then f is a homotopy equivalence. In many situations, however, the only readily available information is that f induces an isomorphism on homology groups. Hence we want conditions sufficient to imply that a homology isomorphism is a weak equivalence. This is false in general, but with the fundamental group as the sole source of trouble. The simplest homology version is:

Theorem 5.1 *Let $f : X \longrightarrow Y$ be a map of simply-connected spaces. Then if H_*f is an isomorphism, f is a weak equivalence.*

Proof: Let F be the homotopy-fiber of f. In other words, convert f to a fibration and take the actual fiber; thus we may assume f is a fibration. Let $i : F \longrightarrow X$ be the inclusion. From the long exact sequence on homotopy groups, it is equivalent to show that F is weakly contractible. By the Hurewicz theorem, this in turn is equivalent to F simply-connected and $\tilde{H}_*F = 0$. Since X is simply-connected, $\pi_2 Y \longrightarrow \pi_1 F$ is surjective, so $\pi_1 F$ is abelian and hence isomorphic to $H_1 F$. So it remains to show $\tilde{H}_*F = 0$.

Since Y is simply-connected, the local coefficient system of the fibration is trivial. In particular the vertical edge of the spectral sequence is $H_0(B; H_*F) = H_*F$. Now suppose $H_qF \neq 0$ for some $q > 0$, and take the minimal such q. Then the only possible differential hitting H_qF is $d^{q+1} : H_{q+1}Y \longrightarrow H_qF$. But f_* is surjective, so (see "edge homomorphisms") the horizontal edge consists of permanent cycles and $d^{q+1} = 0$. Hence (*loc. cit.*) i_* is injective. But f_* is injective and (obviously) $Im\, i_* \subset Ker\, f_*$, forcing $Im\, i_* = 0$ and $H_qF = 0$, contradiction.

The most general version of the theorem assumes only that X, Y are "nilpotent spaces"; i.e. the fundamental group acts nilpotently on the higher homotopy groups. In particular the theorem holds for "simple spaces"; i.e. the fundamental group acts trivially on the higher homotopy groups, and therefore it holds when X, Y are H-spaces.

5.1.3 Variants of the Hurewicz and Whitehead theorems

Here's an example of the kind of variant I have in mind:

Theorem 5.2 *Rational Hurewicz Theorem.*
Let X be a path-connected, strongly simple space (e.g. simply-connected or an H-space). Then $\pi_k X \otimes \mathbb{Q} = 0$ for $k < n$ if and only if $H_*(X;\mathbb{Z}) \otimes \mathbb{Q} = 0$ for $k < n$, in which case the Hurewicz map induces an isomorphism $\pi_n X \otimes \mathbb{Q} \xrightarrow{\cong} H_n X \otimes \mathbb{Q}$.

The next theorem is not the most general possible, but at least it can be deduced directly from the rational Hurewicz theorem.

Theorem 5.3 *Rational Whitehead Theorem.*
Let $f : X \longrightarrow Y$ be a map of simply-connected spaces, and assume the homotopy fiber is strongly simple. Then f induces an isomorphism on rational homotopy groups if and only if it induces an isomorphism on rational homology groups.

For a brief indication of how to prove the rational Hurewicz theorem and related results, see below under "Finite generation of homotopy groups". The rational Whitehead theorem as stated above is an exercise, applying the Serre spectral sequence and the rational Hurewicz theorem (to the fiber).

5.2 Loop spaces on a suspension, the James construction and stable homotopy

5.2.1 Loop spaces on spheres

Consider the path-loop fibration of a space X, with H_*X free over our coefficient principal ideal domain R. Note that the Hopf group (=group object in the homotopy category) ΩX has a homotopy-action on PX (paths beginning at the basepoint) given by precomposing paths with loops. One can show that this makes the Serre spectral sequence into a spectral sequence of $H_*\Omega X$-modules, where the E^2-term is the free module $H_*\Omega X \otimes H_*X$. Now, this might seem a rather useless structure, since PX is contractible and we typically know H_*X, not $H_*\Omega X$. Far from it:

Proposition 5.4 *Suppose $n \geq 1$. For any coefficient ring R, $H_*\Omega S^{n+1} \cong R[x]$, where $|x| = n$.*

Given the multiplicative structure, this is an easy and enlightening exercise, so I won't ruin it by giving the proof. Worthy of note here is that the proposition holds even for n odd. In the graded world commutative rings have $2x^2 = 0$ for any odd degree generator. So, paradoxically, a polynomial algebra on such a generator is not a commutative ring (except in characteristic 2). On the other hand, this isn't surprising; loop spaces need not be homotopy-commutative as H-spaces, and therefore the case n even of the proposition is the more surprising of the two.

5.2.2 Loops on a suspension and the James construction

In general the relation between H_*X, and $H_*\Omega X$ is very complicated, but there is one situation in which the preceding proposition has an elegant generalization: The loop space of a suspension ΣX, where X is path-connected. Let $j : X \longrightarrow \Omega\Sigma X$ be adjoint to the identity of ΣX. Let $T(\tilde{H}_*X)$ denote the tensor algebra. Then by the universal property of tensor algebras, we get a natural homomorphism $\phi : T(\tilde{H}_*X) \cong H_*\Omega\Sigma X$.

Theorem 5.5 *$\phi : T(\tilde{H}_*X) \longrightarrow H_*\Omega\Sigma X$ is an isomorphism.*

By the natural map we mean start from j_* and then apply the universal property of a tensor algebra. To prove the theorem, we would like to use the same method we used for the special case of the loops on a sphere. This requires the key lemma that depends on the fact that our loop space is the loops on a suspension.

Lemma 5.6 *All elements of $H_*\Sigma X$ are transgressive. In other words, $H_*(P\Sigma X, \Omega\Sigma X) \longrightarrow H_*\Sigma X$ is surjective.*

The proof is an instructive exercise. Now let H_kX be the first non-vanishing homology group with $k > 0$. Then the first differential is the transgression $d_{k+1} : H_{k+1}\Sigma X \longrightarrow H_k\Omega\Sigma X$, and as in the sphere case we conclude that (1) d_{k+1} is an isomorphism; and by using the multiplicative structure (2) $H_*\Omega\Sigma X$ contains the tensor algebra $T(H_kX)$. By the lemma all elements of the horizontal edge beyond $k+1$ are cycles for d_{k+1}. So the E_{k+2}-term looks just like the E_{k+1}-term, except that now the horizontal and vertical edges are zero up to degrees $k+2$ and $k+1$ respectively. In other words, we can repeat the process and show by induction that $T(\tilde{H}_{i\leq m}X) \subset H_*\Omega\Sigma X$. Letting $m \longrightarrow \infty$ yields the theorem.

Now let X be a connected CW-complex, and let JX denote the James construction, i.e. the free topological monoid on X (see Hatcher). If H_*X is free over the coefficient ring R, then $H_*JX \cong T(\tilde{H}_*X)$ (see Hatcher, Proposition 3C8). There is a natural map $i : X \longrightarrow \Omega\Sigma X$ that sends x to the loop in the reduced suspension given by a "longitude" through x. Now any loop space can be made strictly associative by using the "Moore loops". Hence i extends uniquely to a map of topological monoids $\phi : JX \longrightarrow \Omega\Sigma X$. Moreover ϕ is compatible with the natural maps $X \longrightarrow JX$ and $X \longrightarrow \Omega\Sigma X$. It follows that ϕ_* is an isomorphism, in effect the identity of $T(\tilde{H}_*X)$. Since ΣX is simply-connected by assumption, from Whitehead's theorem we conclude:

Theorem 5.7 *Let X be a connected CW-complex. Then $JX \longrightarrow \Omega\Sigma X$ is a weak equivalence (hence a homotopy equivalence, by a theorem of Milnor asserting that the loop space of a CW-complex has the homotopy type of a CW-complex).*

In particular ΩS^{n+1} is equivalent to a CW-complex with one cell in each dimension divisible by n.

5.2.3 The Freudenthal suspension theorem and stable homotopy groups

Let X, Y be pointed CW-complexes, and let $[X, Y]_*$ denote pointed homotopy classes of pointed maps. Then we can form the suspension sequence

$$[X, Y]_* \longrightarrow [\Sigma X, \Sigma Y]_* \longrightarrow [\Sigma^2 X, \Sigma^2 Y] \longrightarrow ...$$

where each arrow takes a map f to its suspension. In fact from the second term on these sets are groups (abelian from the third term on), and suspension is a group homomorphism. Now, assuming that X is a *finite* complex, we define the stable homotopy classes of maps $\{X, Y\}$ as the direct limit of this sequence.

Caution. The direct limit is defined whether X is finite or not. But it turns out that for infinite complexes this is the "wrong" definition of stable maps, so we exclude them here. The correct definition involves the category of spectra; see the classic text by J.F. Adams.

In particular, the stable homotopy groups of X are defined by $\pi_n^S X = \{S^n, X\} = colim_k \pi_{n+k} \Sigma^k X$.

Theorem 5.8 *Freudenthal suspension theorem*

Let X be an $(n-1)$-connected CW-complex, where $n \geq 2$. Then suspension $\pi_m X \longrightarrow \pi_{m+1} \Sigma X$ is an isomorphism for $m < 2n-1$ and an epimorphism for $m = 2n-1$.

Proof: The suspension is induced by the inclusion $X \longrightarrow \Omega\Sigma X \cong JX$. So it is enough to show that $X \longrightarrow JX$ is a $(2n-1)$-equivalence. Recall that JX is filtered by subspaces $J_k X$ such that $J_k X / J_{k-1} X = \wedge^k X$, where $J_1 X = X$. Hence the inclusions $J_{k-1} X \longrightarrow J_k X$ are $(kn-1)$-equivalences, and hence $X \longrightarrow JX$ is a $(2n-1)$-equivalence as desired.

As a corollary we see that for any X, the direct limit defining $\pi_n^S X$ eventually stabilizes. Explicitly, the suspension is an isomorphism for $k > n+1$, the point being that even if X is not even assumed path-connected, $\Sigma^k X$ will be $(k-1)$-connected. In particular the stable homotopy groups of spheres, defined as $\pi_n^S S^0$, are given by the common value of $\pi_{n+k} S^k$ for $k > n+1$.

For example, we know that $\pi_3 S^2 \cong \mathbb{Z}$, generated by the Hopf map (using the Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$). This is the group just before the stable range; the stable value of $\pi_1^S S^0$, which is achieved from $\pi_4 S^3$ on, is $\mathbb{Z}/2$. The stable group is therefore generated by suspensions of the Hopf map.

5.3 Finite generation of homotopy groups

We start by recalling that the homotopy groups of a finite complex need not be finitely-generated. The simplest example is $X = S^1 \vee S^2$. Its universal cover is an infinite string of balloons; using the Hurewicz theorem one deduces that $\pi_2 X$ is free abelian of countably infinite rank. It is clear that the fundamental group is the culprit here, so one can still hope for finite-generation in the simply-connected case. This turns out to be true, and in fact we can do better.

Call a pointed, path-connected space X *strongly simple* (Spanier's term) if $\pi_1 X$ acts trivially on $[Y, X]_*$ (pointed homotopy classes of maps) for all well-pointed Y (e.g. Y a CW-complex). In particular, $\pi_1 X$ is abelian. For example, any simply-connected space is strongly simple, and any H-space is strongly simple. The proof of the latter fact is identical to the proof that H-spaces have abelian fundamental group. The space $S^1 \vee S^2$ is not strongly simple, indeed $\pi_1 \cong \mathbb{Z}$ acts shifting the string of balloons in the evident way.

The following theorem is due to Serre:

Theorem 5.9 *Let X be a strongly simple finite complex. Then the homotopy groups of X are finitely-generated.*

This is certainly true for π_1, so consider π_k for $k \geq 2$. The first idea of the proof is to "loop down". We know that $\pi_k X \cong \pi_{k-1} \Omega X$, where ΩX is the loop space. Iterating, we have $\pi_k X \cong \pi_1 \Omega^{k-1} X \cong H_1 \Omega^{k-1} X$, where the second isomorphism uses e.g. the fact that $\Omega^{k-1} X$ is an H-space. So it would be enough to know that finite-generation of homology groups is preserved under looping. The key result here is the following, where we take our coefficients as usual in a principal ideal domain R; for immediate purposes $R = \mathbb{Z}$ is the relevant case.

Lemma 5.10 *Let $F \longrightarrow E \longrightarrow B$ be a Serre fibration, and assume the local coefficient system is trivial so that $E^2_{p,q} = H_p(B; H_q F)$. Then if any two of F, E, B have finitely-generated homology groups (as R-modules), so does the third.*

Note this fits nicely with the long exact sequence prototype from the beginning of the essay. The easiest case is when F, B are assumed to have finitely-generated homology. Then check that the homology groups of B with coefficients in any finitely-generated module are finitely-generated (easy). It follows that $\oplus^n_{p+q} E^2_{p,q}$ is finitely-generated for each n, hence the same is true for E^∞. Finally if an R-module has a finite filtration with finitely-generated quotients, then it is finitely-generated. (You'll need to use the fact that R is noetherian in these arguments, to ensure that submodules of finitely-generated modules are finitely-generated.)

The case we actually need for the theorem is when E, B are assumed finitely-generated. Then the argument is similar in flavor, but run backwards and hence a little more complicated. We show by induction on q that $H_q F$ is finitely-generated. In a nutshell, we know that $E^\infty_{0,q}$ is finitely-generated since it is a submodule of $H_q E$. On the other hand, it is also a quotient of $H_q F$, obtained by modding out the images of successive differentials. But these differentials all originate from various $E^2_{p,k}$ with $k < q$, which are known to be finitely-generated by the inductive hypothesis and the "check" from the previous paragraph.

The upshot is we modded out a finite sequence of finitely-generated submodules and in the end obtained a finitely-generated module; hence the module we started with was finitely-generated.

Now the idea is to apply the lemma to path-loop fibrations whose base space has finitely-generated homology groups. Since the path-space is contractible, it certainly has finitely-generated homology groups! Then we can, we hope, conclude that the same is true for the fiber. The fly in the ointment is the assumption of a trivial local coefficient system in the lemma. Even if we started with a simply-connected X, looping down lowers the connectivity and we will likely arrive at a non-simply-connected loop space before we ever get to $\Omega^{k-1}X$ in the proposed argument.

To deal with this, first note that since homotopy groups are defined in terms of pointed maps, when we pass from a pointed space Y to ΩY, we might as well take instead $\Omega_0 Y$, the component of the constant loop that serves as the basepoint of ΩY. Now, it is easy to show that $\Omega_0 Y$ is homeomorphic to $\Omega\tilde{Y}$, where $\tilde{Y}$ is the universal cover (assuming, as we do, that Y has such a cover). Hence in our inductive "looping down" construction, each step breaks into two: Instead of passing directly to the loop space, we take the universal cover first and then take the loop space. Since the preceeding lemma works fine for simply-connected spaces, what remains to be shown is:

Lemma 5.11 *Suppose Y is a strongly simple space with finitely-generated homology. Then the universal cover $\tilde{Y}$ also has finitely-generated homology.*

A brief outline of the proof:

Step 1.: There is a fiber sequence $\tilde{Y} \longrightarrow Y \longrightarrow K(\pi_1 Y, 1)$ (this step doesn't use the strongly simple hypothesis).

Step 2: The local coefficient system of this fibration is trivial.

Step 3: $K(\pi_1 Y, 1)$ has finitely-generated homology. (Note that $\pi_1 Y$ is a finitely-generated abelian group.)

Step 4: Now apply the first lemma to the fiber sequence of Step 1.

Working out the details of the proof of Serre's theorem makes a very instructive (albeit long) exercise.

Remark. Examining the proofs above, one finds that the only properties of the class $\mathcal{C}$ of finitely-generated abelian groups that we used were the following:

1. It is closed under isomorphisms, and given a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, if any two of A, B, C are in $\mathcal{C}$ then so is the third. Such a class is called a *Serre class* of abelian groups.
2. If A, B are in $\mathcal{C}$, then so are $A \otimes B$ and $Tor(A, B)$ (this was implicitly used when considering homology with coefficients, via the universal coefficient theorem).
3. If A is in $\mathcal{C}$, then so is $H_* K(A, 1)$.

Other examples include torsion groups and p-torsion groups for a fixed prime p. Hence the same method applies to a much broader class of theorems of the form "the homotopy groups of X are in $\mathcal{C}$ if and only if the homology groups are" (with the same fundamental group restrictions, of course). Similarly one can prove "mod $\mathcal{C}$' Hurewicz and Whitehead theorems, making use of Postnikov towers in addition to looping down. For example, the rational Hurewicz and Whitehead theorems mentioned earlier correspond to the case where $\mathcal{C}$ is the class of torsion abelian groups. See Spanier for details of this theory, or Serre's original paper (if you read French).

5.4 Cohomology of Eilenberg-Maclane spaces

Fix a prime p and let H^* denote $H^*(-,\mathbb{F}_p)$. Let $K_n = K(\mathbb{Z}/p,n)$, the Eilenberg-Maclane space that represents the functor H^n on the homotopy category of CW-complexes. We want to compute H^*K_n, because this will determine all natural transformations or "cohomology operations" on mod p cohomology. Since $\Omega K_n \cong K_{n-1}$, the idea is to use induction and the Serre spectral sequence of the path-loop fibration

$$K_{n-1} \longrightarrow PK_n \longrightarrow K_n.$$

Since PK_n is contractible, we have a spectral sequence converging to zero (ignoring H^0 which is represented by $E_2^{0,0}$). We know the cohomology of the fiber and wish to deduce the cohomology of the base. To illustrate, let's consider the smallest interesting case: $p = n = 2$.

The local coefficient system is trivial since K_2 is simply-connected. Moreover H_*K_2 is finite dimensional in each degree (why?), so $E_2 = H^*K_1 \otimes H^*K_2$. Since $K_1 = \mathbb{R}P^\infty$, we know $H^*K_1 \cong \mathbb{F}_2[x]$, where $|x| = 1$. None of these elements on the vertical axis can survive, since the spectral sequence is converging to zero. In particular, we must have $H^1K_2 = 0$ and $H^2K_2 = \mathbb{F}_2$ (well, this much was obvious without the spectral sequence, using the Hurewicz theorem); moreover $d_2 : H^1K_1 \longrightarrow H^2K_2$ must be an isomorphism. Let $y_0 = d_2x$. Then convince yourself of the following:

1. H^*K_2 contains a polynomial algebra on y_0 (i.e. y_0 has infinite height).
2. Since d_2 annihilates squares by the derivation property, d_2 is a map of modules over $\mathbb{F}_2[x^2] \otimes H^*K_2$.
3. Therefore d_2 is completely determined by d_2x, with the result that the E_3-term has all odd rows identically zero, and even rows copies of $H^*K_2/(y_0)$.
4. Thus the picture at E_3 is just a sparser replica of E_2. Since the spectral sequence converges to zero, there must be an element $y_1 \in E_3^{3,0}$ such that $d_3x_1 = y_1$, where x_1 is the element formerly known as x^2 at the E_2-level. I've renamed it to emphasize that in E_3 it is no longer a square. Again y_1 has infinite height.
5. E_4 is an even sparser replica of E_2, of the form $\mathbb{F}_2[x_2] \otimes H^*K_2/(y_1, y_2)$. Here x_2 was once known as x^4, changed its name to x_1^2 at E_3, and is not a square in E_4. Moreover $E_4 = E_5$ (i.e. d_4 is identically zero) and $d_5x_2 = y_2 \in E_2^{5,0}$ once again has infinite height.
6. The pattern continues, with d_r nonzero only when $r = 2^k + 1$, in which case d_r on x_k (the element formerly known as x^{2^k}) is an element $y_k \in E_r^{2^k+1,0}$ of infinite height.
7. Conclude that $H^*K_2 \cong \mathbb{F}_2[y_1, y_2, ...]$, where $|y_k| = 2^{k+1}$.

Note that the elements $x^{2^k} \in E_2$ are transgressive. In fact this can be see in advance, since the transgression commutes with Steenrod operations and $Sq^{2^k}x^{2^k} = x^{2^{k+1}}$. It follows that we can take y_k (abuse of notation here; I mean an element of H^*K_2 corresponding to the y_k above) to be

$$y_k = Sq^{2^{k-1}} Sq^{2^{k-2}} ... Sq^4 Sq^2 Sq^1 y_0.$$

Moreover a very general result—Borel's transgression theorem—tells us that H^*K_2 has to be a polynomial algebra on the transgressions of the x^{2^k}'s. With Borel's theorem in hand, one can continue this method to inductively compute H^*K_n for all p and all n; needless to say, the book-keeping gets much more complicated.

5.5 Loop spaces of unitary groups and the Bott periodicity theorem

There is a rich, beautiful theory of loop spaces of compact Lie groups G, beginning with the pioneering work of Bott in the 1950's. Here I'll consider the unitary and special unitary groups, but before diving in, two general remarks are in order. First, since G is itself the loop space of its classifying space BG, ΩG is in fact a double-loop space and hence a homotopy-commutative H-space. In particular, the homology ring $H_*\Omega G$ is a graded commutative ring. Second, for very general reasons we have $\Omega G \cong \Omega_0 G \times \pi_1 G$, where $\Omega_0 G$ is the component of the trivial loop; moreover $\Omega_0 G \cong \Omega\tilde{G}$, where $\tilde{G}$ is the universal covering group. Hence we can reduce to studying simply-connected groups.

In the case of $U(n)$, recall that the determinant induces an isomorphism $\pi_1 U(n) \cong \pi_1 S^1 \cong \mathbb{Z}$. We say that a loop has *degree* k if the degree of its determinant is k. Thus

$$\Omega U(n) = \Omega_0 U(n) \times \mathbb{Z} = \coprod_k \Omega_k U(n),$$

where $\Omega_k U(n)$ denotes the loops of degree k. Since the universal cover of $U(n)$ is $SU(n) \times \mathbb{R}$, we have $\Omega_0 U(n) \cong \Omega SU(n)$. Thus the results below for $SU(n)$ have immediate consequences for $U(n)$. We first compute the homology ring of $\Omega SU(n)$. Let $j : \mathbb{C}P^{n-1} \longrightarrow \Omega SU(n)$ be adjoint to the map $S^1 \wedge \mathbb{C}P^{n-1} \longrightarrow SU(n)$ considered earlier (see the section on Stiefel manifolds etc.). Abusing notation, we will write b_i for $j_* b_i$ as well.

Proposition 5.12 *For any coefficient ring R, $H_*\Omega SU(n) \cong R[b_1, ..., b_{n-1}]$. In particular, $\mathbb{C}P^{n-1}$ is a generating complex for $\Omega SU(n)$.*

Proof: By induction on n. Looping the fibration $SU(n) \longrightarrow SU(n+1) \longrightarrow S^{2n+1}$ yields a fibration on the loop spaces, in which both maps are maps of H-spaces. One can show that in this situation the Serre spectral sequence is a spectral sequence of algebras. Now it is trivial to check that the Standard Hypothesis holds, and since E_2 is concentrated in even topological degrees, the spectral sequence collapses. Since $H_*\Omega S^{2n+1} \cong R[x]$ with $|x| = 2n$, it follows by induction that the associated graded ring E^∞ is a polynomial algebra on generators of the required degrees. Since these are all graded commutative rings, we conclude as in earlier examples that $H_*\Omega SU(2n+1)$ itself is polynomial of the same form. In order to show that we can take the b_i's as the generators, by induction we need only show that b_n maps to a generator of $H_{2n}\Omega S^{2n+1}$. This can be deduced from the "adjoint" fact proved earlier for the homology of unitary groups.

Writing SU for the direct limit (union) of the $SU(n)$'s, we get $H_*\Omega SU = R[b_1, b_2, ...]$.

The next result is the complex Bott periodicity theorem. It has had a huge influence not only on algebraic topology, but on many other areas such as algebraic K-theory and index theory (e.g. the Atiyah-Singer index theorem).

Theorem 5.13 *There is a homotopy equivalence $BU \times \mathbb{Z} \cong \Omega U$.*

Corollary 5.14 $\Omega^2(BU \times \mathbb{Z}) \cong (BU \times \mathbb{Z})$, *and* $\Omega^2 U \cong U$.

Proof: Since $\Omega BU \cong U$, this follows immediately from the theorem. Hence the term "periodicity"; under repeated looping we get a sequence of spaces with period 2.

The periodicity theorem has many different proofs. The one I'll sketch here is the "topological" proof, meaning that it is based on standard methods of topology, such as the Serre spectral sequence, rather than differential geometry, index theory or other areas.

Bearing in mind our earlier discussion, it is enough to show that $BU \cong \Omega SU$. Our first task is to construct a plausible candidate $\phi : BU \longrightarrow \Omega SU$; thus we want a connection between Grassmannians and loops in unitary groups. In fact there is a very simple such connection. Consider a finite Grassmannian $G_k\mathbb{C}^n$. For each $W \in G_k\mathbb{C}^n$ and each $z \in S^1$, we form the unitary transformation given by multiplication by z on W and the identity on $W^\perp$. This construction is easily seen to define an embedding $\phi_{k,n} : G_k\mathbb{C}^n \longrightarrow \Omega U(n)$. Moreover these maps are compatible with direct sums, in the sense that the following diagram commutes:

$$\begin{tikzcd}
G_j\mathbb{C}^m \times G_k\mathbb{C}^n \arrow[r] \arrow[d] & G_{j+k}\mathbb{C}^{m+n} \arrow[d] \\
\Omega U(j) \times \Omega U(k) \arrow[r] & \Omega U(j+k)
\end{tikzcd}$$

Here the horizontal maps are the evident "direct sum" maps. Now, as it stands $\phi_{k,n}$ maps into $\Omega_k U(n)$, but we can multiply by a fixed loop of degree $-k$ to get back into $\Omega_0 U(n) \cong \Omega SU(n)$. Then (omitting many details here!) one can arrange things so that if BU is expressed as the direct limit of the $G_n\mathbb{C}^{2n}$'s, these maps fit together to produce $\phi : BU \longrightarrow \Omega SU$. Moreover ϕ is a map of H-spaces; this fact traces back to the above commutative diagram.

I claim that ϕ is a weak equivalence (hence a homotopy equivalence, as noted above). Since both spaces are simply-connected, by Whitehead's theorem it suffices to show ϕ induces an isomorphism on integral homology. Since the two homology rings are already known to be abstractly isomorphic, and free of finite type, it suffices to show ϕ_* is surjective. But ϕ is also compatible with the generating complexes, i.e. the following diagram homotopy commutes:

$$\begin{tikzcd}
\mathbb{C}P^\infty \arrow[r, "i"] \arrow[d, "j"'] & \Omega SU \\
BU \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where i and j are the standard generating complexes. It is very plausible that the diagram commutes, since by definition j is just the $BU(1)$ version of ϕ, and indeed the proof is easy once the precise definition of ϕ is sorted out. Since the image of j_* generates $H_*\Omega SU$ as a ring, and ϕ_* is a ring homomorphism, this completes the proof of my claim and of the periodicity theorem.

5.6 Rational homotopy groups of spheres

We know that $\pi_{n+k}S^n$ is a finitely-generated abelian group, hence the first question to ask is: what is the rank?

Theorem 5.15 *rank $\pi_{n+k}S^n = 1$ if $k = 0$ or n is even and $k = n-1$, and is zero otherwise.*

Corollary 5.16 *The stable homotopy groups of spheres π_k^S are finite for $k > 0$.*

To prove the theorem, we first compute the rational cohomology of rational Eilenberg-MacLane spaces.

Theorem 5.17

$$H^*(K(\mathbb{Q},n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q}\langle x\rangle & \text{if } n \text{ odd} \\ \mathbb{Q}[x] & \text{if } n \text{ even} \end{cases}$$

(Of course $|x| = n$.)

Here is a sketch of the proof; the details make a good exercise (or see the references) First show that $K(\mathbb{Q},1)$ is the mapping telescope (cf. Hatcher) of the sequence

$$S^1 \xrightarrow{2} S^1 \xrightarrow{3} S^1 \xrightarrow{4} \ldots$$

and deduce the case $n = 1$. Then proceed by induction, using the path-loop fibration $K(\mathbb{Q},n) \longrightarrow PK(\mathbb{Q},n+1) \longrightarrow K(\mathbb{Q},n+1)$.

Now we sketch the proof of Theorem 5.15; again the details make a good exercise. If n is odd, let $\alpha : S^n \longrightarrow K(n,\mathbb{Q})$ be any non-trivial map (or to be specific, take the map corresponding to $1 \in \mathbb{Q}$). Then α is a rational homology isomorphism and hence by Theorem 5.17 and the rational Whitehead theorem is a rational homotopy isomorphism. If n is even, let $f : K(\mathbb{Q},n) \longrightarrow K(\mathbb{Q},2n)$ represent x^2, and let F be the homotopy fiber. Show that S^n is rationally equivalent to F and deduce the result.

The corollary follows immediately because for n even, $\pi_{2n-1}S^n$ is just outside the stable range.

For n even, the Hopf invariant $H : \pi_{2n-1}S^n \longrightarrow \mathbb{Z}$ is a group homomorphism and is nonzero; hence any element with nonzero Hopf invariant represents an element of infinite order and is a generator of the rational homotopy as rational vector space. One quick way to produce an element with nonzero Hopf invariant is as follows: Let $T(\tau_n)$ denote the Thom space of the tangent bundle of S^n. It is a cell complex with one n-cell and one $2n$-cell, with the Thom class u generating H^n. Let f denote the attaching map of the top cell. Since $u^2 = eu$, where e is the Euler class, and e is twice a generator of H^nS^n, it follows from the definition of H that $H(f) = \pm 2$. It turns out that H is onto for $n = 2, 4, 8$ and has image $2\mathbb{Z}$ otherwise; this is the content of Adams' famous "Hopf invariant one" theorem.

6 Local coefficient systems and fibrations

To appear. This will be an overview of homology/cohomology with local coefficients and how it fits with the Serre spectral sequence.

7 Other examples of spectral sequences

To appear. Brief overview of the spectral sequence of a double complex, and perhaps others.