Algebraic Topology 2: Smooth Manifolds

D. Zack Garza

August 16, 2019

Contents

1 Lecture 1

The key point of this class will be a discussion of *smooth structures.* As you may recall, a sensational result of Milnor's exhibited exotic spheres with smooth structures – i.e., a differentiable manifold M which is homeomorphic but *not* diffeomorphic to a sphere.

Summary of this result: Look at bundles $S^3 \to X \to S^4$, then one can construct some $X \cong S^7 \in \mathbf{Top}$ but $X \ncong S^7 \in \mathbf{Diff}^\infty$. There are in fact 7 distinct choices for X.

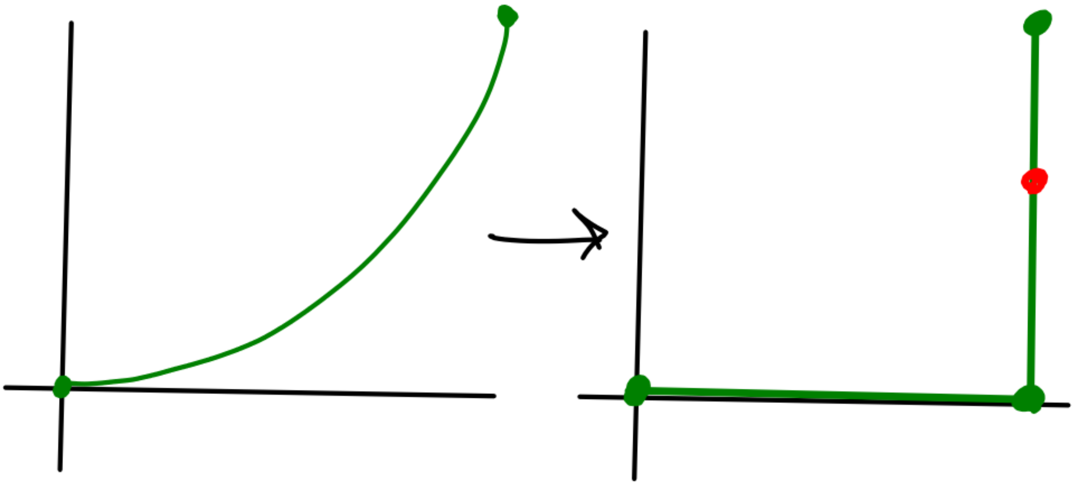
> It is not known if there are exotic smooth structures on S^4. The Smooth Poincare' conjecture is that these do not exist; this is believed to be false.

The other key point of this course is to show that $X \in \mathbf{Diff}^\infty \implies X \hookrightarrow \mathbb{R}^n$ for some n, and is in fact a topological subspace.

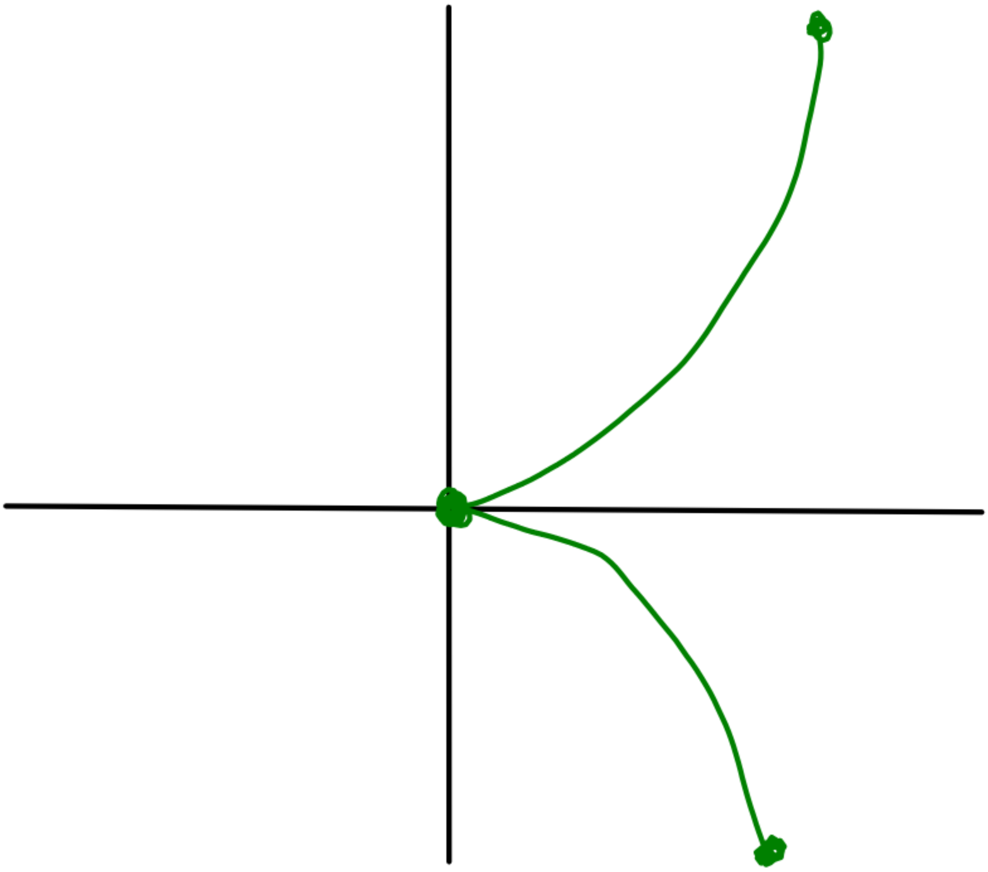
A short list of words/topics we hope to describe:

- Differentiable manifolds
- Local charts
- Submanifolds
- Projective spaces
- Lie groups
- Tangent spaces
- Vector fields
- Cotangent spaces
- Differentials of smooth mapsG
- Differential forms
- de Rham's theorem

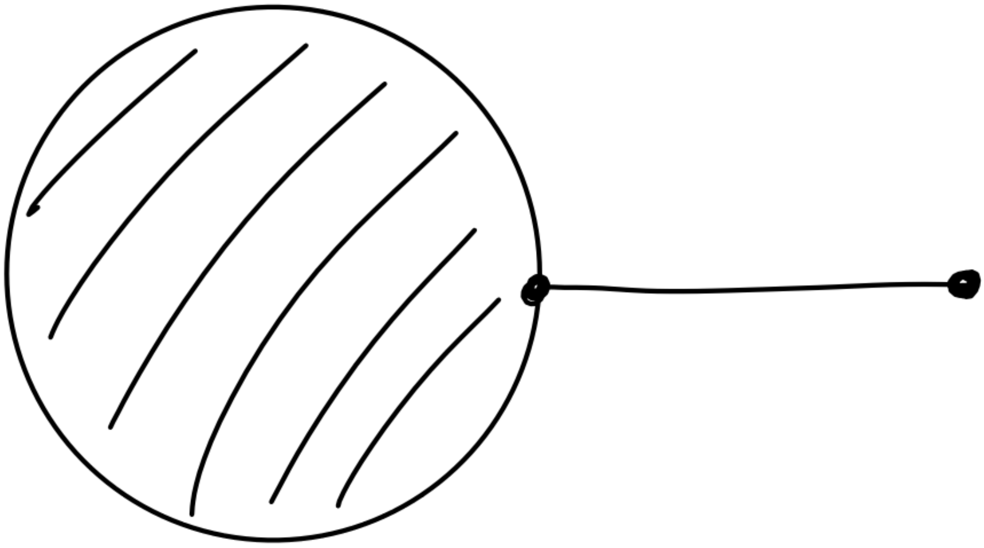
We'd like a notion of "convergence" for, say, curves in $\mathbb{R}^2$. Consider the following examples.



Note the problematic point on the bottom right, as well as the fact that neither of the usual notions of pointwise or uniform convergence will yield a point on the LHS that converges to the red point on the RHS.



Note the problematic point at the origin.



Note the problematic point in the middle, for which all neighborhoods of it are not homeomorphic to either a 2-dimensional nor a 1-dimensional space.

Definition 1. A topological space M is said to be a **topological manifolds** when

- M is Hausdorff, so $p \neq q \in M \implies \exists N(p), N(q)$ such that $N(p) \bigcap N(q) = \emptyset$.
- asdsa
- dsadas