Interesting Topological Spaces in Algebraic Geometry

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1	Ideas for Spaces	
	 Curves Elliptic Curves Higher genus Hyperelliptic curves The modular curve Surfaces Compact Riemann surfaces * Bolza Surface (Genus 2) * Klein Quartic (Genus 3) * Hurwizt Surfaces Kummer surfaces Compact Complex Surfaces Rational ruled Enriques Surfaces K3 	

- * Kahler Manifolds
- Kodaira
- Toric
- Hyperelliptic
- Properly quasi-elliptic
- General type
- Type VII
- Fake projective planes
- Conics
- Calabi-Yau manifolds
 - Dimension 1: All elliptic curves (up to homeomorphism)
 - Dimension 2: K3 surfaces
 - Dimension 3 (threefolds): 500 million +, unknown if infinitely many
 - The bananafold
 - Hyperkähler
- Hurwitz schemes
- Topological galois groups, e.g. $G(\overline{F}/F)$ for $F = \mathbb{Q}, \mathbb{F}_p$.
- Spec (R) for R a DVR (a Sierpinski space)
- Quiver Grassmannians
- Rigid analytic spaces
- Affine line with two origins
- Moduli stack of elliptic curves $\mathcal{M}_{1,1}$.
- Abelian Surface
- Fano Varieties
- Curves: isomorphic to \mathbb{P}^1
- Surfaces: Del Pezzo surfaces
- Weighted projective space
- Toric Varieties
- Grassmannian
- Flag Varieties
- Moduli Spaces

Due to Kunihiko Kodaira's classification of complex surfaces, we know that any compact hyperkähler 4-manifold is either a K3 surface or a compact torus T^{4} . (Every Calabi–Yau manifold in 4 (real) dimensions is a hyperkähler manifold, because SU(2) is isomorphic to Sp(1).)

As was discovered by Beauville, the Hilbert scheme of k points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension 4k. This gives rise to two series of compact examples: Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

2 Intro/Motivation

Ursula Whitcher

Assume the universe is a "space". Which one is it? What structures does it have? How many possible spaces *could* it be, and how can we test to find out?

3 Analogies

Notation: all dimensions are over \mathbb{R}

Impossible goal: pick a category, understand all of the objects and all of the maps. Two main categories with a forgetful functor: $\mathbf{Diff} \longrightarrow \mathbf{Top}$. Question:

- What's in the "image" of this functor? (Manifolds that admit a differentiable structure.)
- What is the "fiber" above a given topological manifold? (Distinct differentiable structures)

Differentiable Manifolds: classified by geometric structure in low dimensions (≤ 4), algebraic data/methods in high dimensions

3.1 Topological Category

Identify objects up to homeomorphism

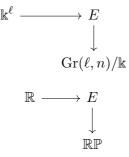
- Dimension 0: The point (terminal object)
- Dimeions 1: S^1 , \mathbb{R}
- Dimension 2: $\langle \mathbb{S}, \mathbb{T}, \mathbb{RP} \mid \mathbb{S} = 0, 3\mathbb{RP} = \mathbb{RP} + \mathbb{T} \rangle$. Classified by π_1 (orientability and "genus"). Riemann, Poincare, Klein.
- Dimension 3: Can always be given a unique smooth structure, see uniformization.
- Dimension 4:
- Dimension $n \geq 5$:

3.2 Smooth Category

Generally expect things to split into more classes.

- 2-manifolds: Homeomorphic \iff diffeomorphic. Every surface admits a complex structure and a metric. Thus always orientable.
 - Uniformization: Holomorphically equivalent to a quotient of one of three spaces
 - * \mathbb{CP}^1 , positive curvature (spherical)
 - * C, zero curvature (flat)
 - * \mathbb{H} (equiv. \mathbb{D}°), negative curvature (hyperbolic)
 - Stratified by genus:
 - * Genus 0: Only \mathbb{CP}^1
 - * Genus 1: All of the form \mathbb{C}/Λ , with a distinguished point [0], i.e. an elliptic curve. Has a topological group structure!
- 3-manifolds: Thurston's Geometrization
 - Oriented prime 3-manifolds can be decomposed into geometric "pieces" of 8 possible types
 - Geometric structure: a diffeo $M\cong \tilde{M}/\Gamma$ where Γ is a discrete Lie group acting freely/transitively on X
- 4-manifolds: classified in the topological category by surgery, but not in the smooth category
- n-manifolds, $n \geq 5$: classified by surgery

4 Moduli Spaces



5 Elliptic Curves

- Equivalently, Riemann surfaces with one marked point.
- Equivalently, \mathbb{C}/Λ a lattice, where homothetic lattices (multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$) are equivalent.
- Parameterized by a moduli space:
 - For $X = \mathbb{C}/\Lambda$ choose a positively oriented basis $\{z, w\}$ for Λ
 - * Note: push into meridians on a torus, generators of $H_1(X)$, and require that their
 - intersection is +1.

 Replace $z \mapsto 1, w \mapsto \frac{w}{z}$; the orientation condition forces $\Im(w) > 0$ so this yields a point

6 Kahlers

- For complex manifold, replace Riemannian metric with a Hermitian metric (positive definite sesquilinear inner product on tangent bundle)
- If skew-symmetric part is symplectic (closed and nondegenerate) then the metric is Kahler
- Includes smooth projective varieties, but not all complex manifolds
- Specialize to Calabi-Yaus: compact and Ricci-flat, or first Chern class vanishes

7 Calabi-Yaus

- As manifolds: Ricci-flat, i.e. Ricci curvature tensor vanishes (measures deviation of volumes of "geodesic balls" from Euclidean balls of the same radius).
- Applications: Physicists want to study G_2 manifolds (an exceptional Lie group, automorphisms of octonions), part of M-theory uniting several superstring theories, but no smooth or complex structures. Indirect approach: compactify an 11-dimension space, one small S^1 dimension \longrightarrow 10 dimensions, 4 spacetime and 6 "small" Calabi-Yau.