

Problem Set 1

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1 Problem 6

1.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M ,

$$X : M \rightarrow TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \text{id}_M$, and $\phi_1 \in \text{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H : M \times I \rightarrow M$$

$$(p, t) \mapsto \phi_t(p).$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f : M \rightarrow M$, we have

$$\Lambda(f) = \sum_k \text{Tr} \left(f_* \big|_{H_k(X; \mathbb{Q})} \right).$$

where $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of M .

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0 , thus

$$\left. \frac{\partial}{\partial t} \phi_t(p_0) \right|_{t=1} \text{ so}$$

$$\begin{aligned} &\phi_t(p) = p \\ \implies &\frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies &\left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies &X(\phi_1(p_0)) := \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 && \text{by definition of } \phi_1 \end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X . So X has at least one zero, as desired. \square

1.2 Part 2