Chapter 9

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Contents

Important Theorems:

Important ideas:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

1 | Background from Chapter 8

- (M, ω) with $\omega \in \Omega^2(M)$ is a symplectic manifold with an almost complex structure J.
- $H \in C^\infty(M; \mathbb{R})$ a Hamiltonian with X_H the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in $T_x M$:
  $$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$
  - Zeros of vector field X_H correspond to critical points of H:
  $$X_H(x) = 0 \iff (dH)_x = 0.$$
  - Take the associated flow $\psi^t : M \to M$, assumed 1-periodic so $\psi^1(x) = x$: critical points of H are periodic trajectories.

- $u \in C^\infty(\mathbb{R} \times S^1; M)$ is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y\end{aligned}$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \operatorname{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}M$ is the *free loop space* of M, i.e. space of contractible loops on M, i.e. $C^\infty(S^1; M)$ with the C^∞ topology
  - Loops in $\mathcal{L}M$ can be viewed as maps $S^2 \to M$, since they're maps $I \times S^1 \to M$ with the boundaries pinched:

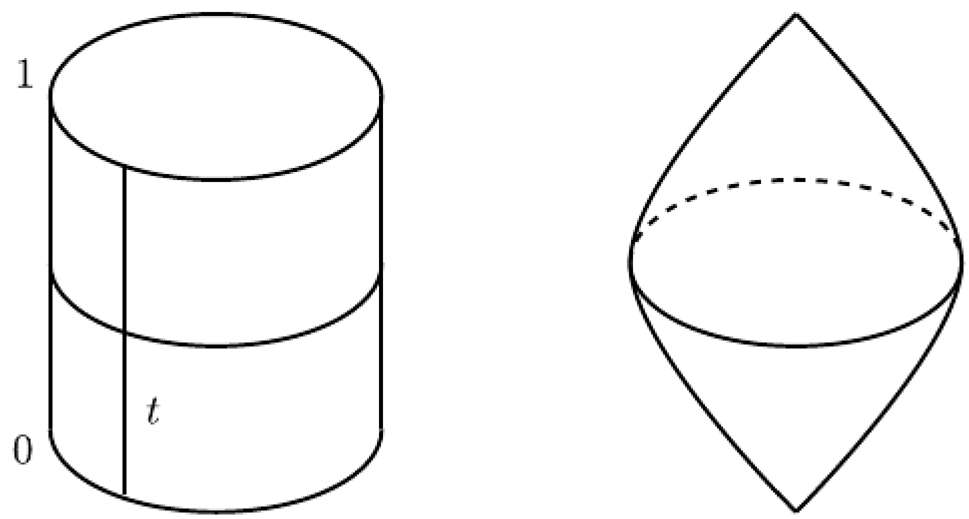

Figure 1: Loops in $\mathcal{L}M$

  - Elements $x \in \mathcal{L}M$ can be viewed as maps $S^1 \to M$.
  - Can extend to maps from a closed disc, $u : \overline{\mathbb{D}}^2 \to M$.
- The action functional is given by

$$\begin{aligned}\mathcal{A}_H &: \mathcal{L}W \to \mathbb{R} \\ &x \mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt\end{aligned}$$

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq).$

- Correspondence between trajectories of the gradient of $\mathcal{A}_H$ and solutions to Floer equations.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, M)$,

$$\int_{S^2} u^*\omega = 0 \quad \text{or equivalently} \quad \langle \omega,\ \pi_2 M \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM, equivalently

$$\langle c_1 TM,\ \pi_2 M \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of $\mathcal{A}_H$.

- Maslov index: used the fact that
  - $\mathrm{Sp}(2n, \mathbb{R})$ retracts onto $U(n)$: use a polar decomposition $S = PQ$ as a PSD times orthogonal, then homotope P to I.
  - $\pi_1 U_n = \mathbb{Z}$: use $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$ by the determinant, and $\pi_1 SU(n, \mathbb{C}) = 0$.
  - Thus every path in $\gamma : I \to \mathrm{Sp}(2n, \mathbb{R})$ can be assigned an integer by getting a map $\tilde{\gamma} : I \to S^1$ and taking (approximately) its winding number.

- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y.
  - Showed that after perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
  - How did we do it: describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$ (Banach manifold modeled on the Sobolev spaces $W^{1,p}$), apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality. Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

Goals

- Construct Floer homology and prove the Arnold Conjecture ("Symplectic Morse Inequalities?"):

$$\#\left\{\text{1-Periodic trajectories of } X_H\right\} \geq \sum_{k \in \mathbb{Z}} HM_k(w;\ \mathbb{Z}/2\mathbb{Z}).$$

Steps

1. Define the action functional $\mathcal{A}_H$.
2. Construct the chain complex (graded vector space) CF_*.
3. Define X_H, which will be used to define ∂ later.
4. Count trajectories.

5. Show finite-energy trajectories connect critical points of $\mathcal{A}_H$.
6. Show compactness property for space of trajectories of finite energy.
7. Define ∂ (uses a compactness property in 9.1c)
8. Show space of trajectories is a manifold (plus genericity, “Smale property”)
9. Show that $\partial^2 = 0$.
10. Show that HF_* doesn’t depend on $\mathcal{A}_H$ or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_*.

2 | 9.1 and Review

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}.$$

- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k.
  - Finitely many since they are nondegeneracy implies they are isolated.
- Trying to define a differential

$$\begin{aligned} \partial : C_k(H) &\to C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y) y \end{aligned}$$

$$\begin{aligned} n(x, y) &\coloneqq \# \{\text{Trajectories of grad } \mathcal{A}_H \text{ connecting } x, y\} \mod 2 \\ &= \dim \mathcal{L}(x, y) \mod 2, \end{aligned}$$

  where $\mathcal{L}(x, y)$ will be the space of such trajectories
  - Want to show $\mathcal{L}(x, y)$ is a compact 0-manifold.
- Three important theorems:
- 9.1.7

Theorem 2.1 ***(9.1.7).***
Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, y)$, then there exist
- A subsequence $\left\{u_{n_j}\right\}$
- Critical points $\{x_0, x_1, \cdots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = y$
- For $0 \leq k \leq \ell$, sequences $\left\{s_n^k\right\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1}$ such that for every $0 \leq k \leq \ell$,

$$u_n \cdot s_n^k \xrightarrow{n\to\infty} u^k.$$

- 9.2.1
- 9.2.3

2.1 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$, so we quotient to define $\mathcal{L}(x,y) =:= \mathcal{M}_{x,y}/\mathbb{R}$ with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \overset{n\to\infty}{\to} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n+s,\,\cdot\,) \overset{n\to\infty}{\to} u(s,\cdot).$$

Proposition 2.2 ***(?).***
$\mathcal{L}(x,y)$ is Hausdorff.

- Want to show $\mathcal{L}(x,y)$ is a compact 0-dimensional manifold.
- Have a differential

$$\begin{aligned} \partial : C_k(H) &\longrightarrow C_{k-1}(H) \\ \partial(x) &= \sum_{\mathrm{Ind}(y)=k-1} n(x,y)y. \end{aligned}$$

with $n(x,y)$ the number (mod 2) of trajectories of grad $\mathcal{A}_H$ connecting x, y, i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

$$M := \overline{\mathcal{L}}(x,z) = \mathcal{L}(x,z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x,y) \times \mathcal{L}(y,z).$$

  and show that M is compact with ∂M equal to the last union.
- Last time: closure of space of trajectories connecting x, y contains "broken" trajectories.
- Last time: toward proving that M is compact

3 | 9.2

- Wanted to compactify $\mathcal{L}(x,y)$, needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

Theorem 3.1 ***(9.2.1).***
Let (H,J) be a regular pair with H nondegenerate.
Let x, z be two periodic trajectories of H such that $\mu(x) = \mu(z) + 2$.
Then $\overline{\mathcal{L}}(x,y)$ is a compact 1-manifold with boundary satisfying

$$\partial\overline{\mathcal{L}}(x,y) = \bigcup_{\mu(x)<\mu(y)<\mu(z)} \mathcal{L}(x,y) \times \mathcal{L}(y,z).$$

As a corollary, $\partial^2 = 0$.

- Know $\overline{\mathcal{L}}(x,y)$ is compact and $\mathcal{L}(x,y)$ is a 1-manifold
- Now suffices to study in a neighborhood of boundary points ("gluing theorem")

3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function w_p which interpolates between u and v (not exactly a solution itself, but will be approximated by one later).
2. Constructing ψ a "true solution" from w_p using the Newton-Picard method. We'll have
$$\psi(p) = \exp_{w_p}(\gamma(p)) \qquad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x,z).$$
where $\mathcal{P} =?$.
3. Get a lift $\widehat{\psi} = \pi \circ \psi$ where $\pi =?$ satisfying

- $\widehat{\psi}(p) \overset{n\to\infty}{\to} (\widehat{u}, \widehat{v})$
- $\widehat{\varphi}$ is an embedding
- $\widehat{\psi}$ is unique in the following sense (the last point)

Theorem 3.2 ***(9.2.3 (Gluing Theorem)).***
Let x, y, z be critical points of the action functional $\mathcal{A}_H$ such that $\mu(x) = \mu(y) + 1 = \mu(z) + 2$. Let $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ be trajectories, inducing $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$.

- There exist a differentiable map $\psi : (\rho_0, \infty) \to \mathcal{M}(x, z)$ for some $\rho > 0$ such that
- $\pi \circ \psi : (\rho_0, \infty) \to \mathcal{L}(x, z)$ is an embedding
- $\widehat{\psi} \overset{\rho\to\infty}{\to} (\bar{u}, \bar{v}) \in \overline{\mathcal{L}(x, z)}$.
- If $\ell_n \in \mathcal{L}(x, z)$ with $\ell_n \overset{n\to\infty}{\to} (\bar{u}, \bar{v})$, then for $n \gg 1$ we have $\ell \in \Im(\widehat{\psi})$.

4 | 9.3: Pre-gluing

- Choose a bump function β on $\{0\}^c \subset \mathbb{R} \to [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^{\pm}$:

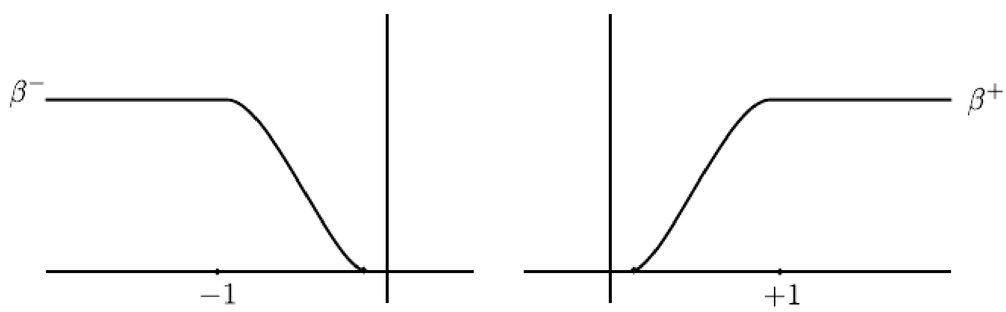

Figure 2: Figure 9.3

- Define the interpolation w_ρ from u to v in the following way:

$$w_\rho(s,t) = \begin{cases} u(s+\rho,t) & \text{if } s \le -1 \\ \exp_{y(t)}\left(\beta^-(s)\exp^{-1}_{y(t)}(u(s+\rho,t)) \ +\beta^+(s)\exp^{-1}_{y(t)}(v(s-\rho,t))\right) & \text{if } s \in [-1,1] \\ v(s-\rho,t) & \text{if } s \ge 1 \end{cases}$$

- Why does this make sense?

$$|s| \le 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \ \middle| \ \sup_{t \in S^1} \|Y(t)\| \le r_0 \right\}.$$

5 | 9.4: Construction of ψ.

- Have constructed $w_\rho \in C^\infty_\searrow(x,z)C^\infty(x,z)$ for every $\rho \ge \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x,z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } Hx$$

  in the weak sense.
- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining $\mathcal{F}_\rho$.

$$\begin{aligned} W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\xrightarrow{\mathcal{F}_\rho} L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ (y_1, \ldots, y_{2n}) &\longmapsto \left[\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } H_t\right)\left(\exp_{w_\rho} \sum y_i Z_i^\rho\right)\right]_{Z_i} \end{aligned}$$

where $\mathcal{F}_\rho \coloneqq \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i. sd - Newton-Picard method, general idea

- Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

- Allows finding zeros of f given an approximate zero x_0.
- Linearize $\mathcal{F}_\rho$.