18-02-14: Adjoint and Classifying Spaces

In general, we define the classiyfing space $K(G,n)$ (also known as an Eilenberg-MacLane space) to be a space X such that $\pi_n(X) = G$ and for $k \neq n,\ \pi_k(X) = 0$.

Note: in my notation, I will simply write this as $\pi_*(X) = G\delta_n$

It is worth mentioning here that there are nice Serre spectral sequences for this family of fibrations:

$$K(\mathbb{Z}, n-1) \to \{\text{pt}\} \to K(\mathbb{Z}, n)$$

By examining an appropriate spectral sequence, we were able to find that $H_*(\mathbb{RP}^\infty) = \mathbb{Z}_2\delta_1$, which makes $\mathbb{RP}^\infty$ an geometric model of the classifying space $K(\mathbb{Z}_2, 1)$.

Recall that $\mathbb{CP}^\infty$ is defined as the limit of the sequence of inclusions

$$\mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \mathbb{CP}^3 \subset \dots$$

together with the weak limit topology.

There are a handful of easily recognizable geometric models for a few other types of classifying spaces.

$G \setminus n$	1	2	3
$\mathbb{Z}$	S^1	$\mathbb{CP}^\infty$	No good model!
$\mathbb{Z}_2$	$\mathbb{RP}^\infty$	.	.
$\mathbb{Z}_p$	$L(\infty, p)$	.	.
$*_n\mathbb{Z}$	$\bigvee_n S^1$	.	.

Note: $*_n\mathbb{Z}$ is the free group on n generators. Also, these spaces can all be constructed as a CW complex for any given G - just start with some $\bigvee S^1$ and add cells to kill off all higher homotopy.

Using spectral sequences, we also found that $K(\mathbb{Z}, 3)$ was a space that, although simple from the point of view of homotopy, had a more complicated structure in homology. It was a number of odd properties- it has torsion in infinitely many dimensions, doesn't satisfy Poincare duality (even in a truncated sense).

Consider the fibration

$$S^1 \to S^{2\infty+1} \to \mathbb{CP}^\infty$$

where these infinite-dimensional spaces are defined using the weak topology.

There is a perfectly good filtration arising from the inclusions in this diagram:

$$\begin{tikzcd}
S^3 \arrow[r, "\subseteq"] \arrow[d] & S^5 \arrow[r, "\subseteq"] \arrow[d] & S^7 \arrow[r, "\subseteq"] \arrow[d] & \cdots \\
\mathbb{CP}^1 \arrow[r, "\subseteq"] & \mathbb{CP}^3 \arrow[r, "\subseteq"] & \mathbb{CP}^5 \arrow[r, "\subseteq"] & \cdots
\end{tikzcd}$$

So we can apply the usual spectral sequence to this filtration. We know that E_∞ can only contain $\mathbb{Z}$ in dimension zero, and we obtain the following E_2 page:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rrd, "d_2 \cong"] & 0 & 0 \arrow[rrd, "d_2 \cong"] & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$

Since d_2 is an isomorphism, it must take generators to generators, and so we can deduce the following facts:

- $d_2(\alpha \otimes 1) = 1 \otimes \beta$
- $d_2(1 \otimes \beta) = 0$

We can now compute

$$\begin{aligned} d_2(\alpha \otimes \beta) &= d_2(\alpha \otimes 1) \cup (1 \otimes \beta) + 0 \\ &= 1 \otimes \beta^2 \end{aligned}$$

And using the cup product structure on cohomology, we can fill out the following diagram that summarizes these results:

$$\begin{tikzcd}
\alpha \otimes 1 \arrow[rrd, "d_2 \cong"] & \cdot & \alpha \otimes \beta \arrow[rrd, "d_2 \cong"] & \cdot & \cdot \\
\cdot & \cdot & 1 \otimes \beta \arrow[u, "\cup"] & \cdot & 1 \otimes \beta^2
\end{tikzcd}$$

Thus, just from knowing that d_2 is an isomorphism, we can conclude that $H^4(\mathbb{CP}^\infty) = \mathbb{Z} < \beta^2 >$. Alternatively, we'll write this as $H^4(\mathbb{CP}^\infty) = \mathbb{Z}.\, \beta^2$

By a repeated application of this argument, we find that $H^{2n}(\mathbb{CP}^\infty) = \mathbb{Z}.\, \beta^n$, allowing us to conclude that

$$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[\beta_{(2)}]. \qquad \blacksquare$$

If we know $H^*(\mathbb{CP}^\infty)$, which is the easiest case, we can then use the inclusion $\mathbb{CP}^n \xrightarrow{i} \mathbb{CP}^\infty$ (as a cellular map) to induce

$$\begin{gathered} H^*(\mathbb{CP}^n) \xrightarrow{i^*} H^*(\mathbb{CP}^\infty) \\ \beta \mapsto \beta \end{gathered}$$

which is a actually a *ring* homomorphism instead of just a group homomorphism. This presents a good argument for the use of cohomology, due to its extra ring structure.

This is an isomorphism on low-dimensional (co)homology, which reflects the idea encapsulated in the weak limit that these should be approximately equal for large enough n.

This is indicative of a general principle: if X is a CW complex and X^n is its n-skeleton, then the inclusion $X^n \xrightarrow{i} X$ induces an isomorphism $H_k(X^n) \cong H_k(X)$ for $k < n$. (Note that this may or may not be an isomorphism for $k = n$.)

In particular, it is again a ring homomorphism, and so carries true relations/equations to true relations/equations.

Dually, homology does have *some* type of ring structure, however it is slightly unnatural and onerous to define and use. There is a natural coproduct on $H_*(X)$ for any space X, which has a "one in, two out" type and takes this form:

$$\begin{gathered} H_*(X) \xrightarrow{\Delta} H_*(X) \times H_*(X) \\ a \mapsto \sum a' \otimes a'' \end{gathered}$$

This coproduct satisfies a form of coassociativity, i.e. if we have

$$\begin{aligned} \Delta(a) &= \sum b_i \otimes c_i \\ (\Delta \otimes 1)\Delta(a) &= \sum_{i,j} (d^i_j \otimes e^i_j) \otimes c_i \\ (1 \otimes \Delta)\Delta(a) &= \sum_{i,j} b_i \otimes (f^i_k \otimes g^i_k) \end{aligned}$$

then the “structure coefficients” agree, i.e. we have $b_i = \sum_j (d^i_j \otimes e^i_j)$ and $c_i = \sum_k (f^i_k \otimes g^i_k)$.

In other words, just note that each element on the right hand side of these equations is an element of $H_*^{\otimes 3}$, and so coassociativity simply requires that they are the same element of this space.

We can specialize by looking at the case where V is a vector space, with a coproduct $V \xrightarrow{\Delta} V \otimes V$. Then pick a basis $\{e_i\}_{i \in I}$, and write

$$\Delta(e_i) = \sum_{j,k} \Delta_i^{j,k}(e_j \otimes e_k)$$

where $\Delta_i^{j,k} \in k$, the ground field of V. Then coassociativity requires that we have

$$\sum_{j,k,l,m} \Delta_i^{j,k} \Delta_j^{l,m}(e_l \otimes e_m \otimes e_k) = \sum_{j,k} \Delta_i^{j,k} e_j \otimes \Delta_k^{p,q}(e_p \otimes e_q)$$

or in other words, that

$$\sum_j \Delta_i^{j,k} \Delta_j^{l,m} = \sum_i \Delta_i^{l,r} \Delta_r^{m,k} \qquad \forall k, l, m$$

It is worth noting that there is also a version of the universal coefficient theorem for homology, which comes in the form

$$0 \to \mathrm{Ext}(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H_n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \to 0$$

One question that comes up here is whether or not there is a sense in which Ext and Hom are “duals” of each other. In some way, this is case, using the “Frobenius duality” of $\cdot \otimes R$ and $\mathrm{Hom}(\cdot, S)$.

Aside: Frobenius duality occurs in algebras A over some field k possessing a nondegenerate bilinear form $A \times A \xrightarrow{\sigma} k$ satisfying $\sigma(ab, c) = \sigma(a, bc)$. Such a \sigma$ is called a Frobenius norm. A simple example is the trace of a matrix, another example is any Hopf algebra.

This kind of duality comes in the form of something like

$$\mathrm{Hom}(M \otimes N, P) = \mathrm{Hom}(M, \mathrm{Hom}_{\mathrm{in}}(N, P))$$

where $\mathrm{Hom}_{\mathrm{in}}$ is an "internal hom", which is actually an object in the category whose underlying set is the usual Hom. One might also call this "map", and denote it $[N, P]$, then the above statement translates to the condition that if $N, P \in \mathcal{C}$ for some category, then $\mathrm{Hom}_{\mathrm{in}}(N, P) = [N, P] \in \mathcal{C}$ is also an object in the same category. (This might also be denoted $\mathcal{H}om$.)

For an analogy, let $\mathcal{C} = \mathbf{Top}$, and $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ be the set of continuous maps from X to Y. Then notice that we can put a topology on this space, say $\mathcal{T}$, so define $\mathrm{Map}(X, Y) = (\mathrm{Hom}_{\mathbf{Top}}(X, Y), \mathcal{T})$, which is in fact an **object** in $\mathbf{Top}$. This becomes the aforementioned "internal hom".

Then, the previous adjunction becomes

$$\mathrm{Hom}_{\mathbf{Top}}(X \times Y, Z) = \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{Map}(X, Y)) \qquad (\in \mathbf{Set})$$

More generally, consider what happens in categories of R modules, where R is generally non-commutative. We can then take objects like $M_R \in \textbf{mod-R}$ and ${}_R N_S \in \textbf{R-mod-S}$. We can then form the tensor product $M_R \otimes_R {}_R N_S$, and the adjunction becomes

$$\mathrm{Hom}_{\textbf{mod-S}}(M_R \otimes_R {}_R N_S, P_S) = \mathrm{Hom}_{\textbf{mod-R}}(M_R, \mathrm{Hom}_{\textbf{mod-S}}({}_R N_S, P_S)) \qquad (\in \mathbf{Ab})$$

Again, in the second argument of the right-hand side, we identify this as an internal hom - this works because the object $\mathrm{Hom}_{\textbf{mod-S}}({}_R N_S, P_S)$ actually becomes a right R-module by precomposition.

In some ways, this resembles the kind of adjunction that occurs in an inner product space - for example, given a matrix A, it may have an "adjoint" matrix A^* that satisfies

$$\langle , Av \rangle w = \langle , w \rangle A^* v$$

and so we can think of Hom like a Hermitian inner product of this form, which is contravariant (re: conjugate) in the first argument. Note that the choice of which argument is contrvariant varies! In Physics, the second argument is often conjugate-linear, while the first is linear.

We can also look at this as an almost-commuting of the following diagram

$$\begin{tikzcd}
\textbf{mod-R} \arrow[d] \arrow[d, bend right, dashed, leftrightarrow, "\cdot \otimes_R N_s"'] & \times & \textbf{mod-R} \arrow[r, "\langle \cdot , \cdot \rangle"] & \mathbf{Ab} \arrow[d, leftrightarrow, "\cong"] \\
\textbf{mod-S} & \times & \textbf{mod-S} \arrow[u] \arrow[u, bend right, dashed, "\mathrm{hom}_R(N_S, \cdot)"'] \arrow[r, "\langle \cdot , \cdot \rangle"] & \mathbf{Ab}
\end{tikzcd}$$

where we can simplify by choosing elements, yielding

$$\begin{tikzcd}
M \arrow[d] & \times & \mathrm{hom}_R(N, P) \arrow[r, "\mathrm{hom}_R"] & \mathbf{Ab} \arrow[d, leftrightarrow, "\cong"] \\
M \otimes_R N & \times & P \arrow[u] \arrow[r, "\mathrm{hom}_S"] & \mathbf{Ab}
\end{tikzcd}$$

In this framework, we can now talk about pairs of adjoint functors $\mathcal{C} \overset{R}{\underset{L}{\leftrightarrow}} \mathcal{D}$ between categories, which satisfy

$$\mathrm{Hom}_{\mathcal{C}}(LA, X) = \mathrm{Hom}_{\mathcal{D}}(A, RX)$$

for every $A \in \mathcal{D}, X \in \mathcal{C}$, plus a few more properties concerning how these act under natural transformations.

Then L is said to be left adjoint to R, and R is right adjoint to L, which is sometimes denoted $L \vdash R$.

Example: Free and forgetful functors.

Work in $\mathbf{Grp}$ and $\mathbf{Set}$, then let F be the free group functor and U by the forgetful functor. Then we have

$$\mathrm{Hom}_{\mathbf{Grp}}(F(S), G) \cong \mathrm{Hom}_{\mathbf{Set}}(S, U(G))$$

Example: The classifying space functor.

Define the classifying space functor $\mathbf{Cat} \xrightarrow{B} \mathbf{Set}$, denoted $|\cdot|$. As an input, it takes a category $\mathcal{C}$, then define a simplicial complex where the

- The vertices (0-simplices) are the objects,
- The edges (1-simplices) are the morphisms,
- The 2-simplices are triangles

$$\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "fg"] & & C
\end{tikzcd}$$

where the inside is considered “filled in” to denote the equivalence between the bottom fg and the top "f then g" path.

- The 3-simplices are the tetrahedra

$$\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[drr] & D \arrow[dr, "h"] & \\
A \arrow[ur, "f"] \arrow[urr] \arrow[rrr, "fgh"] & & & C
\end{tikzcd}$$

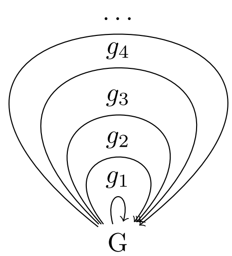
with the interior space filled in similarly.

(Note that we only label the outer morphisms, because the rest can be named as concatenations of others.)

- And so on, etc.

This produces a CW complex, and hence a topological space, from the input category.

Example: Let G be a discrete group of order n – it is equivalently a category with one object and n morphisms.



Then BG is called *the classifying space of* G. $H_*(BG, \mathbb{Z})$ is denoted the homology of the group, and we have

- $\pi_0(BG) = \{\mathrm{pt}\}$
- $\pi_1(BG) = G$
- $\pi_k(BG) = 0$ for $k \geq 2$.

Some concrete examples of these are:

- $B\mathbb{Z}_2 = \mathbb{RP}^\infty$
- $B\mathbb{Z} = S^1$
- $BS_3 = ?$

This construction can be carried out for *topological* groups as well, with the following sequence of gluings:

- A point
- $G \times I$
- $G \times G \times \Delta^2$
- $G \times G \times G \times \Delta^3$
- $\cdots$ etc

A concrete example of this is $BS^1 = \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$. This is related to the homogeneous space fibration

$$H \to G \xrightarrow{g \mapsto g.p} G/H$$

for a chosen basepoint $p \in G/H$ such that H stabilizes p.

On a different note, it is worth mentioning some of the fibrations to which a spectral sequence might apply. One that comes up is

$$U_{n-k} \times U_k \to U_n \to Gr_{\mathbb{C}}(n,k)$$

where $Gr_{\mathbb{C}}(n,k)$ is the set of k-planes in $\mathbb{C}^n$. Here, it is worth noting that $U_n \simeq GL_n(\mathbb{C})$ and $O_n \simeq GL_n(\mathbb{R})$.

From this, it can be concluded that $G_{\mathbb{C}}(k,n) = \frac{U_n}{U_{n-k} \times U_k}$, and further that if we take $\lim_{n\to\infty}$ we obtain $Gr_{\mathbb{C}}(k,\infty) = \frac{X}{U_k}$, where X is some contractible space, and we thus find that $Gr_{\mathbb{C}}(k,\infty) = BU_k$, the classifying space for U_k.

It can further be shown that there is another fibration

$$U_k \to EU_k \to BU_k$$

where EU_k is a contractible space on which U_k acts and BU_k is the above quotient. We can then find interesting structure here arising from the fact that $H^*(Gr_{\mathbb{C}}(k,\infty)) = H^*(BU_k)$.