Zack Garza

1a) Note that if $x \in C$ is an endpoint of a removed interval, then $x = k/3^n$ for some integers $n \geq 1$ and $0 \leq k \leq 3^n$. So we just need a real number $x \in (0,1)$ satisfying

a) x has <u>some</u> ternary expansion
$$x = \sum_{i=1}^{\infty} a_i 3^{-i} \quad \text{where } a_i \neq 1 \text{ for any } i, \text{ and}$$

b) $x \neq k/3^n$ for any $k, n \in \mathbb{N}^{>0}$,

then we will have $x \in C$ by (a) and x not an endpoint by (b).

<u>Claim</u>: $x = (0.\overline{02})_3 = (0.020202\cdots)_3$ works.

Base 3

<u>Pf</u>: By construction, x satisfies

(a) $$x = \sum_{i=0}^{\infty} a_i 3^{-i}, \quad a_i \in \{0, 2\}$$

So no $a_i = 1$ and thus $x \in C$.

(b) To see that x satisfies (b), we can compute

$$x = (0.020202\cdots)_3$$
$$= 0\cdot 3^{-1} + 2\cdot 3^{-2} + 0\cdot 3^{-3} + 2\cdot 3^{-4} + \cdots$$
$$= \sum_{i=1}^{\infty} 2\cdot 3^{-2i} = 2\sum_{i=1}^{\infty} 3^{-2i} = 2\sum_{i=1}^{\infty} \left(\tfrac{1}{9}\right)^i$$
$$= 2\left(-1 + \sum_{i=0}^{\infty} \left(\tfrac{1}{9}\right)^i\right)$$
$$= 2\left(-1 + \frac{1}{1-\frac{1}{9}}\right) \boxed{= 1/4,}$$

where $4 \neq 3^n$ for any integer n. $\blacksquare$

(1b) If a set X is <u>nowhere dense</u> in a topological space, it equivalently satisfies

$$(\overline{X})^{\circ} = \emptyset$$

(i.e., the interior of the closure is empty.)

It then suffices to show that

a) C is closed, so $\overline{C} = C$, and

b) C has no interior points, so $C^{\circ} = \emptyset$.

(a) To see that C is closed, we will show $C^c := [0,1] \setminus C$ is open. An arbitrary union of open sets is open, so the claim is that $C^c = \bigcup_{j \in J} A_j$ for some collection of open sets $\{A_j\}_{j \in J}$.

Consider C_n, the n^{th} stage of the process used to construct the Cantor set, so $C = \bigcap_{i=1}^{\infty} C_n$.

But by induction, C_n^c is a union of open sets.

In particular, $C_1^c = (\frac{1}{3}, \frac{2}{3})$, and

$$C_n^c = \underbrace{\left(\bigcup_{i=1}^{n-1} C_i^c \right)}_{\text{Open by hypothesis}} \cup \underbrace{\left(\begin{array}{c}\text{Exactly } n \text{ open intervals} \\ \text{that were deleted}\end{array} \right)}_{\text{open by construction}},$$

So C_n^c is open for each n. But then

$$C^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c = \bigcup_{i=1}^{\infty} C_n^c$$

is a union of open sets and thus open. So C is closed.

(b) To see that $C^\circ = \emptyset$, suppose towards a contradiction that $x \in C^\circ$, so there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) := (x-\varepsilon, x+\varepsilon) \subsetneq C$. Letting $\mu(I)$ denote the length of an interval, we have $\mu(N_\varepsilon(x)) = 2\varepsilon > 0$.

<u>Claim</u>: Let $L_n := \mu(C_n)$, then $L_n = \left(\frac{2}{3}\right)^n$.

This follows immediately by noting that L_n satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3} L_n, \qquad L_0 = 1$$

Since an interval of length $\frac{1}{3} L_{n-1}$ is removed at the n^{th} stage, which has the unique claimed solution.

But if $I_1 \subseteq I_2$ are real intervals, we must have

$\underbrace{\mu(I_1) \leq \mu(I_2)}_{\text{Uses subadditivity of measure}}$, whereas if we choose n large enough such that $\left(\frac{2}{3}\right)^n < 2\varepsilon$, we have

$$(x-\varepsilon, x+\varepsilon) \subsetneq C = \bigcap_{i=1}^{\infty} C_i \Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq C_n \text{, but}$$

$$\mu((x-\varepsilon, x+\varepsilon)) = 2\varepsilon > \left(\tfrac{2}{3}\right)^n = \mu(C_n) \text{, a contradiction.}$$

So such an $x \in C^\circ$ can't exist, and $C^\circ = \emptyset$.

Thus $(\overline{C})^\circ = C^\circ = \emptyset$, and C is nowhere dense,

and since a meager set is a countable union of nowhere dense sets, C is meager. $\square$

Claim: C is measure zero.

Measures are additive over disjoint sets, i.e.

$$A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B),$$

And if $A \subseteq B$, we have

$$\mu(B) = \mu(B \sqcup (B \setminus A)) = \mu(B) + \mu(B \setminus A)$$
$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A).$$

Now let B_n be the union of the intervals that are deleted at the n^{th} step. We have

$$\mu(B_0) = 0$$

$$\mu(B_1) = 1/3$$

$$\mu(B_2) = 2\left(\tfrac{1}{9}\right) = 2/9$$

$$\mu(B_3) = 4\left(\tfrac{1}{27}\right) = 4/27$$

$$\vdots$$

$$\mu(B_n) = 2^{n-1}/3^n$$

Moreover, if $i \neq j$, then $B_i \cap B_j = \emptyset$, and

$$C^c := [0,1] - C = \bigsqcup_{i=1}^{\infty} B_i .$$

We thus have

$$\begin{aligned}
\mu(C) &= \mu([0,1]) - \mu(C^c) \\
&= 1 - \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) \\
&= 1 - \sum_{n=1}^{\infty} \mu(B_n) \\
&= 1 - \sum_{n=1}^{\infty} 2^{n-1}/3^n
\end{aligned}$$

$$= 1 - (1/3) \sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n$$

$$= 1 - (1/3)\left(1/_{1-2/3}\right)$$

$$= 0. \qquad \blacksquare$$

1c Let $y \in [0,1]$ be arbitrary, we will produce an $x \in C$ such that $f(x) = c$.

Write $y = (a_1\, a_2 \cdots)_2 = \sum_{i=1}^{\infty} a_i 2^{-i}$ where $a_i \in \{0, 1\}$

Now define

$$x = (2a_1\, 2a_2 \cdots)_3 = \sum_{i=1}^{\infty} (2a_i) 3^{-i} := \sum_{i=1}^{\infty} b_i 3^{-i}$$

Since $a_i \in \{0,1\}$, $b_i = 2a_i \in \{0,2\}$, meaning x has no 1^s in its ternary expansion and so $x \in C$.

Moreover, under f we have

$$\left.\begin{array}{ccc} b_i & \mapsto & \frac{1}{2} b_i \\ \parallel & & \parallel \\ 2a_i & \mapsto & \frac{1}{2}(2a_i) = a_i \end{array}\right\} \text{ So } b_i \mapsto a_i \text{ and thus } f(x) = y.$$

So $C \twoheadrightarrow [0,1]$, which is uncountable, thus so is C. $\blacksquare$

(2a) $(\Rightarrow)$ Suppose X is G_δ, so $X=\bigcup_{n=1}^{\infty} A_i$ with each A_i closed. Then A_i^c is open by definition, and so

$$X^c=\left(\bigcup_{i=1}^{\infty} A_i\right)^c=\bigcap_{i=1}^{\infty} A_i^c$$

is a countable intersection of open sets, and thus F_σ.

$(\Leftarrow)$ Suppose X^c is an F_σ, so $X^c=\bigcap_{i=1} B_i$ with each B_i open. Then each B_i^c is closed by definition, and

$$X=(X^c)^c=\left(\bigcap_{i=1}^{\infty} B_i\right)^c=\bigcup_{i=1}^{\infty} B_i^c$$

is a countable union of closed sets, and thus G_δ.

(2b) Suppose X is closed, we will show $X=\bigcap_{n=1}^{\infty} C_n$ with each C_n open. For each $x\in X$ and $n\in\mathbb{N}$, define

- $B_n(x)=\{y\in\mathbb{R}^n \mid d(x,y)<\frac{1}{n}\}$
- $C_n=\bigcup_{x\in X} B_n(x)$
- $W=\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty}\bigcup_{x\in X} B_n(x)$

Since each $B_n(x)$ is open by construction and C_n is a union of opens, each C_n is open.

Claim: $W = X$.

$X \subseteq W$: If $x \in X$, then $x \in B_n(x) \subseteq C_n$ for all n, and so

$$x \in \bigcap_{n=1}^{\infty} C_n = W.$$

$W \subseteq X$: Suppose there is some $w \in W \setminus X$ (so $w \neq x$ for any $x \in X$) towards a contradiction.

Since $w \in \bigcap_{i=1}^{n} C_n$, $w \in C_n$ for every n. So $w \in \bigcup_{x \in X} B_n(x)$ for every n. But then there is some particular $x_0 \in X$ such that $w \in B_n(x_0)$ for every n (otherwise we could take N large enough so that $w \notin B_N(x)$ for any $x \in X$, so $x \notin \bigcup_{x \in X} B_N(x)$) where $w \neq x_0$.

But then if $N_\varepsilon(x)$ is an arbitrary neighborhood of x, we can take $\frac{1}{n} < \varepsilon$ to obtain $w \in B_n(x) \subsetneq N_\varepsilon(x)$, which makes w a limit point of X. But since X is closed, it contains its limit points, forcing the contradiction $w \in X$.

So X is a countable intersection of open sets, and thus a G_δ set. $\blacksquare$

Now suppose X is open. Then X^c is closed, and thus a G_δ set. But then $(X^c)^c = X$ is an F_σ set by problem (2a). $\blacksquare$

(2c) Using the fact that singletons are closed in metric spaces, we can write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ as a countable union of closed sets, so $\mathbb{Q}$ is an F_δ set. Suppose $\mathbb{Q}$ was also a G_δ set, so $\mathbb{Q} = \bigcap_{i=1}^{\infty} A_i$ with each A_i open. Then for any fixed n, $\mathbb{Q} \subseteq A_i$, so A_i is dense in $\mathbb{R}$ for every i.

However, it is also true that $\{q\}^c := \mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$, and we can write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$$

as in intersection of open dense sets; since $\mathbb{R}$ is a Baire space, countable intersections of open dense sets are dense.

But then $\left(\bigcap_{i=1}^{\infty} A_i\right) \cap \left(\bigcap_{q \in \mathbb{Q}} \{q\}^c\right) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$

must be dense in $\mathbb{R}$, which is absurd.

Note that this argument also works when $\mathbb{R}$ is replaced with any open interval I and $\mathbb{Q}$ is replaced with $\mathbb{Q} \cap I$.

For a set that is neither G_δ nor F_δ, consider

$A = \mathbb{Q} \cap (0, \infty)$, positive rationals

$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0)$, negative irrationals

A is F_σ but not G_δ , using above argument, and

dually B is G_δ but not F_σ.

<u>Claim</u>: $X = A \cup B$ is neither G_δ nor F_σ.

Suppose X is G_δ. Then $X \cap \overbrace{(0,\infty)}^{\text{open}} = A$ is G_δ as well. $\#$

Suppose X is F_σ. Then X^c is G_δ, but

$$X^c = (A \cup B)^c = A^c \cap B^c = (\mathbb{Q} \cap (-\infty, 0)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, \infty))$$

and thus $X^c \cap \overbrace{(-\infty, 0)}^{\text{open}} = A$ is G_δ. $\#$

So X is neither G_δ or F_σ. $\blacksquare$

3a) **Claim:** $c \in [0,1] \Rightarrow \lim_{x \to c} f(x) = 0$.

This holds iff $\forall c \in I$, $\forall \varepsilon$, $\exists \delta$ s.t. $|x-c| < \delta \Rightarrow |f(x)| < \varepsilon$,

so let $\varepsilon > 0$ be arbitrary. Consider the set

$S = \{ n \in \mathbb{N} \mid \frac{1}{n} \geq \varepsilon \}$, which is a <u>finite</u> set, and so

$S_q = \{ r_n \in \mathbb{Q} \mid \frac{1}{n} \geq \varepsilon \}$ is <u>finite</u> as well.

So choose $\delta < \min_{r_n \in S_q} d(c, r_n)$ so $N_\delta(c) \cap S_q = \emptyset$

Then $|x-c| < \delta \Rightarrow \begin{cases} \bullet\ f(x) = 0 \text{ if } x \in I \setminus \mathbb{Q}, \text{ or} \\ \bullet\ x = r_m \in (\mathbb{Q} \setminus S_q) \cap I \text{ for some } m \text{ such that} \\ \quad 1/m < \varepsilon \text{ by construction.} \end{cases}$

But then $|f(x)| = |1/m| < \varepsilon$ as desired. $\square$

So
- $c \in I \setminus \mathbb{Q} \Rightarrow f(c) = 0 = \lim_{x \to c} f(x)$,
- $c = r_n \in I \cap \mathbb{Q} \Rightarrow f(c) = \frac{1}{n} \neq 0 = \lim_{x \to c} f(x)$

and f is discontinuous on $I \cap \mathbb{Q}$. $\blacksquare$

(3b.1) Claim: ω_f is well defined

This amounts to showing that the sup and limit exist in

$$\omega_f(x) = \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

Let $x \in \mathbb{R}$ be arbitrary and δ fixed.

Since f is bounded, there is some M such that

$$\forall y \in \mathbb{R}, \ |f(y)| < M \quad , \text{ and so}$$

$$y,z \in \mathbb{R} \Rightarrow |f(y) - f(z)| = |f(y) + (-f(z))| \leq |f(y)| + |-f(z)|$$

$$= |f(y)| + |f(z)| < 2M,$$

which holds for $y,z \in B_\delta(x) \subseteq \mathbb{R}$ as well.

And so $\{ |f(y) - f(z)| \text{ s.t. } y,z \in B_\delta(x) \}$ is bounded above and thus has a least upper bound, and thus the following supremum exists.

$$S(\delta, x) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

To see that the $\lim_{\delta \to 0} S(\delta, x)$ exists, note that

$$\delta_1 \leq \delta_2 \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$$

and so for a fixed x, $S(\delta, x)$ is a monotonically decreasing function of δ that is bounded below by 0, which converges by the monotone convergence theorem. $\square$

Claim: f is continuous at x iff $\omega_f(x) = 0$.

($\Leftarrow$) Suppose $\omega_f(x) = 0$ and let $\varepsilon > 0$ be arbitrary; we will produce a δ to use in the definition of continuity.

Since $\omega_f(x) = \lim_{d \to 0^+} S(d,x) = 0$, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x)| < \varepsilon, \quad \text{which means}$$

$$d < \delta \Rightarrow \sup_{y,z \in B_d(x)} |f(y) - f(z)| < \varepsilon$$

So fix $z = x$ and let y vary, yielding

$$d < \delta \Rightarrow \sup_{y \in B_d(x)} |f(y) - f(x)| < \varepsilon$$

But now for an arbitrary $t \in B_\delta(x)$, we have $|x-t| < \delta$ and

$$|f(x) - f(t)| \leq \sup_{y \in B_\delta(x)} |f(x) - f(y)| < \varepsilon,$$

which exactly says $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$. $\square$

($\Rightarrow$) Suppose f is continuous at x and let $\varepsilon > 0$ be arbitrary; we will show $\omega_f(x) < \varepsilon$.

Since f is continuous, choose δ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

We then have

$$y, z \in B_\delta(x) \Rightarrow |x-y| < \delta \quad \text{and} \quad |x-z| < \delta,$$

$$\Rightarrow |f(x)-f(y)| < \tfrac{\varepsilon}{2} \quad \text{and} \quad |f(x)-f(z)| < \tfrac{\varepsilon}{2}$$

$$\Rightarrow |f(y)-f(z)| \leq |f(y)-f(x)| + |f(x)-f(z)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

and so

$$y, z \in B_\delta(x) \Rightarrow |f(y)-f(z)| < \varepsilon \quad \Rightarrow \sup_{y,z \in B_\delta(x)} |f(y)-f(z)| \leq \varepsilon$$

$$\Rightarrow S(\delta, x) \leq \varepsilon,$$

and since $S(d,x)$ is monotonically decreasing in d,

$$\omega_f(x) = \lim_{d \to 0^+} S(d,x) \leq S(\delta, x) \leq \varepsilon$$

as desired. $\blacksquare$

3b.2 We will show that

$$A_\varepsilon^c = \{ x \in \mathbb{R} \mid \omega_f(x) < \varepsilon \}$$

is open by showing every point is an interior point.

Fix $\varepsilon > 0$ and let $x \in A_\varepsilon^c$ be arbitrary. We want to produce a δ such that

$$B_\delta(x) \subsetneq A_\varepsilon^c \quad \text{or equivalently} \quad |y-x| < \delta \Rightarrow \omega_f(y) < \varepsilon.$$

Write $\omega_f(x) = \lim_{d \to 0^+} S(d,x)$; since $\omega_f(x) < \varepsilon$ and this limit exists, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x) - 0| < \varepsilon \Rightarrow |S(d,x)| < \varepsilon.$$

Now suppose $y \in B_\delta(x)$, so $|y-x| < \delta$. Then there exists some δ' such that $B_{\delta'}(y) \subset B_\delta(x)$, and we claim that

$$S(\delta', y) \le S(\delta, x)$$

Note that if this is true, then

S is monotonically decreasing in d

$$\omega_f(y) = \lim_{d \to 0} S(d,y) \le S(\delta', y) \le S(\delta, x) < \varepsilon.$$

To see why this is true, we just note that

$$a,b \in B_{\delta'}(y) \subset B_\delta(x) \Rightarrow a,b \in B_\delta(x)$$

$$\Rightarrow \sup_{a,b \in B_{\delta'}(y)} |f(y)-f(z)| \leq \sup_{y,z \in B_\delta(x)} |f(y)-f(z)|,$$

since the supremum can only <u>increase</u> over a larger set.

So $\omega_f(y) < \varepsilon$ as desired. $\blacksquare$

Finally, note that if $D_f = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$,

then $D_f = \{x \in \mathbb{R} \mid \omega_f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid \omega_f(x) \geq \frac{1}{n}\}$

$$= \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

is a countable union of closed sets and thus F_σ. $\blacksquare$

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4. Claim: f is increasing, i.e. $x \le y \Rightarrow f(x) \le f(y)$

Fix $x \in \mathbb{R}$, and define

$$A_x := \{ t \in X \mid x > t \}, \quad A_x^c := \{ t \in X \mid x \le t \}.$$

(Note that $t \in A_x$ or $t \in A_x^c \Rightarrow t = x_n$ for some n, and $X = A_x \sqcup A_x^c$.)

Then noting that

$$x_n \in A_x \Rightarrow f_n(x) \equiv 1$$

and

$$x_n \in A_x^c \Rightarrow f_n(x) \equiv 0,$$

We can write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \cdot 1 + \sum_{\{n \mid x_n \in A_x^c\}} \frac{1}{n^2} \cdot 0$$

$$= \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2}.$$

Now if $y \ge x$, then $y \ge t$ for every $t \in A_x$, so $A_y \supseteq A_x$.

But then

$$f(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \leq \sum_{\{n \mid x_n \in A_y\}} \frac{1}{n^2} = f(y),$$

where the inequality holds because

$$A_x \subseteq A_y \Rightarrow \{n \mid x_n \in A_x\} \subseteq \{n \mid x_n \in A_y\}$$
$$\Rightarrow \left|\{n \mid x_n \in A_x\}\right| \leq \left|\{n \mid x_n \in A_y\}\right|,$$

so the latter sum has at least as many terms and everything is positive. So $f(x) \leq f(y)$.

Claim: f is continuous on $\mathbb{R} \setminus X$ since $\sum f_n \xrightarrow{u} f$ and each f_n is continuous there.

Since $|f_n(x)| \leq 1$ by definition, and $|f_n(x)/n^2| \leq |1/n^2| := M_n$ where $\sum M_n < \infty$, $\sum f_n \xrightarrow{u} f$ by the M test.

Note that for a fixed n, $D_{f_n} = \{x_n\}$. This is because if we take a sequence $\{y_i\} \to x_n$ with each $y_i > x_n$, then $f(y_i) = 1$ for every i, and

$$\lim_{i\to\infty} f(y_i) = \lim_{i\to\infty} 1 = 1 \neq f(\lim_{i\to\infty} y_i) = f(x_n) = 0$$

So f_n is not continuous at $x = x_n$. Otherwise, either $x > x_n$ or $x < x_n$, in which case we can let ε be arbitrary and choose $\delta < |x - x_n|$ to get

$$y \in B_\delta(x) \Rightarrow \begin{cases} y > x_n \Rightarrow |f(y) - f(x)| = |0-0| < \varepsilon \\ y < x_n \Rightarrow |f(y) - f(x)| = |1-1| < \varepsilon . \end{cases}$$

Letting $F_N = \sum_{n=1}^{N} f_n$, we find that

$$\underbrace{F_N = f_1 + f_2 + \dots + f_N}_{\text{discontinuous at: } \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_N\}}$$

So F_N is continuous on $\mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$.

and since $\mathbb{R} \setminus X \subseteq \mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$, F_N is continuous there too.

But then $f = \text{uniform limit}(F_N)$ is continuous on $\mathbb{R} \setminus X$. $\blacksquare$

5a) Let $X = (C(I), \|\cdot\|_\infty)$ where $I = [0,1]$,

$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$, and

$$d(f,g) = \|f-g\|_\infty = \sup_{x\in I} |f(x)-g(x)|.$$

Claim: X is a metric space.

1) $d(f,g)=0 \Rightarrow f=g$

If $\sup\limits_{x\in I}|f(x)-g(x)| = 0$ then $|f(x)-g(x)|=0 \ \forall x \in \mathbb{R}$,

so $f(x)=g(x) \ \forall x\in\mathbb{R}$ and $f=g$.

2) $d(f,g)=d(g,f)$

We have $d(f,g) = \sup\limits_{x\in I}|f(x)-g(x)|$

$\sup\limits_{x\in I}|g(x)-f(x)|$

$= d(g,f).$

3) $d(f,h) \le d(f,g) + d(g,h)$

We have $d(f,g) = \sup\limits_{x\in I}|f(x)-g(x)|$

$= \sup\limits_{x\in I}|f(x)-h(x)+h(x)-g(x)|$

$$\leq \sup_{x\in I}\Big(|f(x)-h(x)|+|h(x)-g(x)|\Big) \quad \leftarrow \Delta\text{-ineq in } \mathbb{R}$$

$$= \sup_{x\in I}|f(x)-h(x)| + \sup_{x\in I}|h(x)-g(x)|$$

$$= d(f,h)+d(h,g).$$

So X is a metric space. $\square$

<u>Claim</u>: X is complete.

Let $\{f_i\}$ be a Cauchy sequence in X, we will show that it converges in X. Since $\{f_i\}$ is Cauchy in X, we have

$$\forall \varepsilon>0,\ \exists N_0 \mid n\geq m\geq N_0 \Rightarrow \|f_n-f_m\|_\infty<\varepsilon$$

First we will define a candidate limit function f, then show $f\in X$.

1) Define $f := \lim_{n\to\infty} f_n$ by $f(x)=\lim_{n\to\infty} f_n(x)$.

This is well-defined; let $S_x=\{f_i(x)\}\subseteq\mathbb{R}$ for a fixed x, and we claim S_x is Cauchy in $\underline{\mathbb{R}}$, which is complete.

This follows because if $\{f_i\}$ is Cauchy in X, then

$$|f_n(x)-f_m(x)|\leq \sup_{x\in I}|f_n(x)-f_m(x)| = \|f_n-f_m\|_\infty \to 0.$$

2) $f \in X$, for which it suffices to show f is continuous.

Let $\varepsilon > 0$, and since $\{f_i\}$ is Cauchy, choose N_0 large s.t.

$$n \geq N_0 \Rightarrow \|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Now fix $n \geq N_0$; since <u>f_n is continuous,</u> choose δ such that

$$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

Then

$$\begin{aligned}
|x-y| < \delta \Rightarrow |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq \sup_{x \in I} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + \sup_{y \in I} |f_n(y) - f(y)| \\
&= \|f - f_n\|_\infty + |f_n(x) - f_n(y)| + \|f_n - f\|_\infty \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

So f is continuous, $f = \lim f_n \in X$, and X is complete. ∎

(5b) Let $B = \{f \in X \mid \|f\|_\infty \leq 1\}$

Claim: B is closed.

Let f be a limit point of B, so there is some sequence $f_n \to f$ in X with each $f_n \in B$ so $\|f_n\|_\infty \leq 1 \; \forall n$.

Let $\varepsilon > 0$, and since $f_n \to f$ in X, choose N_0 such that

$$n \geq N_0 \Rightarrow \|f_n - f\| < \varepsilon$$

Then,

$$\begin{aligned} \|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &< \varepsilon + 1, \end{aligned}$$

and taking $\varepsilon \to 0$ yields $\|f\|_\infty \leq 1$. ∎

Claim: B is bounded

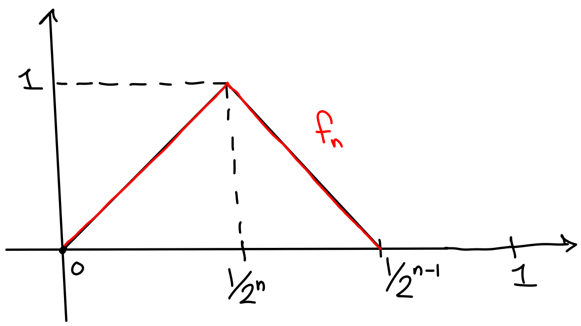
A subset $B \subseteq X$ is bounded iff there is some $x \in X$ and some $r > 0$ in $\mathbb{R}$ where $B \subset N(r, x) = \{y \in X \mid d(y,x) < r\}$.

Choose $x = 0$, $r = 2$, then $f \in B \Rightarrow d(f, 0) = \|f - 0\|_\infty = 1 < 2$, so $f \in N(2, 0)$.

Claim: B is not compact.

Since B is a metric space, B is compact iff B is sequentially compact.

Define f_n as the triangle:



Then $f_n \xrightarrow{\mathbb{R}} f$ where $f(x) = \begin{cases} 1, & x=0 \\ 0, & x \in (0,1] \end{cases}$,

Pointwise in $\mathbb{R}$

and so $\forall n$, $\|f_n - f\|_\infty = 1$, attained at $x=0$. So $\lim_{n\to\infty} \|f_n - f\|_\infty \neq 0$,

and $\{f_n\}$ does not converge in X, nor can any subsequence. $\blacksquare$

Claim: B is not totally bounded.

If it were, $\forall \varepsilon$ there would exist a finite collection $\{g_i\}_{i=1}^N \subseteq B$ such that $B \subseteq \bigcup_{i=1}^N N(\varepsilon, g_i)$ where

$$N(\varepsilon, g_i) = \{h \in B \mid \|h - g_i\| < \varepsilon\}.$$

Note that if $h_1, h_2 \in N(\varepsilon, g_i)$ then $\|h_1 - h_2\| \leq \|h_1 - g\| + \|g - h_2\| < 2\varepsilon$.

So choose $\varepsilon = \frac{1}{2}$, and consider the collection $\{f_n\}_{n=1}^{\infty}$.

Since $\|f_n - f_m\| = 1$, each $N(\varepsilon, g_i)$ can contain at most one f_n, since $f_n, f_m \in N(\varepsilon, g_i)$ for $n \neq m$ would imply $\|f_n - f_m\|_\infty < 2\varepsilon = 2(\frac{1}{2}) = 1$. But there are finitely many $N(\varepsilon, g_i)$ and infinitely many f_n, so if this is a cover of B, so $N(\varepsilon, g_i)$ must contain at least 2 f_n^s. $\#$

(6a) Claim: If $\sum g_n \xrightarrow{U} G$, then $g_n \xrightarrow{U} 0$.

Let $G_N = \sum_{n=1}^{N} g_n$ and $G = \lim_{N \to \infty} G_N$.

Suppose $G_N \xrightarrow{u} G$, then choose N large enough so that

$$\forall x \in X,\ n \geq N \Rightarrow |G_n(x) - G(x)| < \frac{\varepsilon}{2}$$

Then letting $n > n-1 > N$, we have

$$\begin{aligned}
|g_n(x)| &= \left| \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{n-1} g_i(x) \right| \\
&= \left| \left(\sum_{i=1}^{n} g_i(x) - G(x) \right) - \left(\sum_{i=1}^{n-1} g_i - G(x) \right) \right| \\
&\leq \left| \sum_{i=1}^{n} g_i(x) - G(x) \right| + \left| \sum_{i=1}^{n-1} g_i - G(x) \right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

So $\forall x \in X, |g_n(x)| < \varepsilon \Rightarrow g_n \xrightarrow{u} 0$. $\square$

Now let $g_n = 1/1+n^2x$, we'll show g_n does <u>not</u> converge to 0 uniformly.

Note $g_n \xrightarrow{u} g$ iff $\forall \varepsilon, \exists N_0 \mid \forall x, n \geq N_0 \Rightarrow |g_n(x) - g(x)| < \varepsilon$,

so let $\varepsilon < \frac{1}{2}$, N_0 be arbitrary, and choose $x_0 < 1/N_0^2$. Then,

$$|g_{N_0}(x_0)| = \frac{1}{|1+N_0^2 x|} = \frac{1}{|1+N_0^2(1/N_0^2)} = \frac{1}{2} > \varepsilon. \quad \blacksquare$$

<u>Claim</u>: g is continuous on $(0, \infty)$.

Let $x \in (0, \infty)$ be arbitrary, and choose $a < x$. We will show g converges uniformly on $[a, \infty)$, and since each g_n is continuous on $[a, \infty)$ as well, g will be the uniform limit of continuous functions and thus continuous itself.

We can use the M-test. Since $x > a$,

$$\left|1/1+n^2x\right| \leq \left|1/n^2x\right| \leq \left|1/n^2a\right| = \frac{1}{a}\left|\frac{1}{n^2}\right|,$$

where $$\sum_{n=1}^{\infty} \frac{1}{a}\frac{1}{n^2} = \frac{1}{a}\sum \frac{1}{n^2} < \infty,$$

So g converges uniformly on $[a, \infty)$.

(6b) <u>Claim:</u> g is differentiable on $(0,\infty)$.

If $g'(x)$ exists, we have

$$g'(x) = \lim_{a\to x} (x-a)^{-1}(g(x)-g(a))$$

$$= \lim_{a\to x} (x-a)^{-1} \sum_{n=1}^{\infty} \frac{-n^2(x-a)}{(1+n^2x)(1+n^2a)}$$

$$= \lim_{a\to x} \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2x)(1+n^2a)}$$

$$= \sum (-n^2)/(1+n^2x)^2 ,$$

which exists because it converges uniformly on $[a,\infty)$, as

$$\left|\frac{-n^2}{(1+n^2x)^2}\right| \le \left|\frac{n^2}{(n^2x)^2}\right| = \left|\frac{1}{n^2x^2}\right| \le \left|\frac{1}{a^2n^2}\right| := M_n$$

where $\sum M_n = \sum \frac{1}{a^2n^2} = \frac{1}{a^2}\sum \frac{1}{n^2} < \infty$.

So g is <u>continuously differentiable</u> on $(0,\infty)$. ∎

7a) Claim: $h_n \xrightarrow{u} 0$ on $[0, \infty)$

Note that $h_n'(x) = \dfrac{1-nx}{(1+x)^n} \Rightarrow h_n' = 0$ iff $x = 1/n$ and

$$h_n''(x) = \frac{1+x+nx}{nx^2(1+x)^{n-1}} \quad \text{and} \quad h_n''(\tfrac{1}{n}) < 0,$$

so $x = \frac{1}{n}$ is a global maximum and thus

$$\forall x, \ |h_n(x)| \le |h_n(\tfrac{1}{n})| = \left| \frac{1/n}{(1+\frac{1}{n})^n} \right| = \frac{1}{n(1+\frac{1}{n})^n} \le \frac{1}{2n} \quad \text{for } n > 1$$

so $\sup_{x \in [0,\infty)} |h_n(x)| = |h_n(\frac{1}{n})| = O(\frac{1}{n}) \to 0$, thus $\|h_n\|_\infty \to 0$

and $h_n \to 0$ uniformly.

7b) Let $h(x) = \sum_{n=1}^{\infty} h_n(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1}$

i) Demonstrably, $h(0) = 0$, and for a fixed x we have

$$h(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1} = (x/1+x) \sum_{n=1}^{\infty} (1/1+x)^n$$

$$= \frac{x}{1+x}\left(\frac{1}{1-(1/1+x)}\right) \quad \text{since } x > 0 \Rightarrow (1/1+x) < 1$$

$$= 1. \quad \square$$

ii) It can not converge uniformly on $[0,\infty)$, otherwise h would be the uniform limit of continuous functions, but h is discontinuous.

7c) Let $a>0$ and $X=[a,\infty)$.

Claim: $\sum h_n \xrightarrow{u} h$ on X.

$(1+x)^{n+1} = \sum_{j=1}^{n} \binom{n}{j} x^j \geq 1+nx+n^2x^2$

$x>a>0$, so positive terms.

Since $x>a$, we have

$$|h_n(x)| = \left|\frac{x}{(1+x)^{n+1}}\right| \leq \left|\frac{x}{1+nx+n^2x^2}\right| \leq \left|\frac{a}{1+na+n^2a^2}\right| \leq \left|\frac{a}{n^2a^2}\right| = \left|\frac{1}{n^2a}\right|$$

So let $M_n = 1/an^2$, then $\sum M_n < \infty \Rightarrow \sum h_n \xrightarrow{u} h$ by the M test. ■