Problem Set 5

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Proposition 1.1.
Suppose $\lambda + \rho \in \Lambda^+$. Then $M(w \cdot \lambda) \subset M(\lambda)$ for all $w \in W$. Thus all $[M(\lambda) : L(w \cdot \lambda)] > 0$.

More precisely, if $w = s_n \cdots s_1$ is a reduced expression for w in terms of simple reflections corresponding to roots α_i, then there is a sequence of embeddings:

$$M(w \cdot \lambda) = M\left(\lambda_n\right) \subset M\left(\lambda_{n-1}\right) \subset \cdots \subset M\left(\lambda_0\right) = M(\lambda)$$

Here

$$\begin{aligned} \lambda_0 := \lambda, \lambda_k := s_k \cdot \lambda_{k-1} = (s_k \dots s_1) \cdot \lambda \implies \lambda_n = s_n \cdot \lambda_{n-1} = w \cdot \lambda \\ w \cdot \lambda = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0 = \lambda \text{with} \quad \langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+ \text{ for } k = 0, \cdots, n-1. \end{aligned}$$

Assume $\lambda + \rho \in \Lambda^+$.

a. Prove that the unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_\diamond \cdot \lambda)$, where $w_\diamond$ is the longest element of W.

b. In case $\lambda \in \Lambda^+$, show that the inclusions obtained in the above proposition are all proper.

2 4.6

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