Math 8100 Assignment 9

Due date: Friday 19th of November 2010

1. For each $k \in \mathbb{Z}$, define $\{u_k\} \in \ell^2(\mathbb{Z})$ by $u_k(j) = 1$ if $j = k$, $u_k(j) = 0$ otherwise. Verify that the set $\{u_k\}_{k\in\mathbb{Z}}$ forms a complete orthonormal system in $\ell^2(\mathbb{Z})$.

2. In $L^2(0,1)$ let $e_0(x) = 1$, $e_1(x) = \sqrt{3}(2x-1)$ for all $x \in (0,1)$.
   (a) Show that e_0, e_1 is an orthonormal system in $L^2(0,1)$.
   (b) Show that the polynomial of degree 1 which is closest with respect to the norm of $L^2(0,1)$ to the function $f(x) = x^2$ is given by $g(x) = x - 1/6$. What is $\|f-g\|_2$?

3. Let E be a subset of a Hilbert space H.
   (a) Show that $E^\perp$ is a closed subspace of H.
   (b) Show that $(E^\perp)^\perp$ is the smallest closed subspace of H that contains E.

4. (a) The first three Legendre polynomials are
$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2-1)/2.$$
   Show that the orthonormal system in $L^2(-1,1)$ obtained by applying the Gram-Schmidt process to $1, x, x^2$ are scalar multiples of these.
   (b) Compute
$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$
   (c) Find
$$\max \int_{-1}^{1} x^3 g(x) \, dx$$
   where g is subject to the restrictions
$$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} xg(x)\,dx = \int_{-1}^{1} x^2 g(x)\,dx = 0\,; \quad \int_{-1}^{1} |g(x)|^2\,dx = 1.$$

5. (a) Verify that the following systems are orthogonal in $L^2(E)$:
      i. $\{1/2, \cos x, \sin x, \ldots, \cos kx, \sin kx, \ldots\}$, when E is any interval of length 2π.
      ii. $\{e^{2\pi ikx/(b-a)}\}_{k=-\infty}^{\infty}$, when $E = (a,b)$.
   (b) Let $f \in L^1(0, 2\pi)$.
      i. Show that for any $\epsilon > 0$ we can write $f = g + h$, where $g \in L^2$ and $\|h\|_1 < \epsilon$.
      ii. Use this decomposition of f to prove the Riemann-Lebesgue lemma:
$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0$$

6. Prove that every closed convex set K in a Hilbert space has a unique element of minimal norm.
   Hint: If $0 \in K$, then the result is trivial; otherwise adapt the proof of Theorem 5.24 in Folland.

Challenge Problem IX

Hand this in to me at some point in the semester

IX. **The Mean Ergodic Theorem:**
Let U be a unitary operator on a Hilbert space H, $M = \{x : Ux = x\}$, P be the orthogonal projection onto M and $S_N = \frac{1}{N}\sum_{n=0}^{N-1} U^n$. Prove that $\|S_N x - Px\| \to 0$ as $N \to \infty$ for all $x \in H$.