Floer Talk

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Contents

1 Background and Notation

- $(W, \omega \in \Omega_2(W))$ is a (compact?) symplectic manifold
- $C^\infty(A, B)$ is the space of smooth maps with the C^∞ topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C^\infty_{\text{Loc}}(A, B)$ is the space with the C^∞ uniform convergence topology on compact subsets of A
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$ given by $H_t \in C^\infty(W; \mathbb{R})$ is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

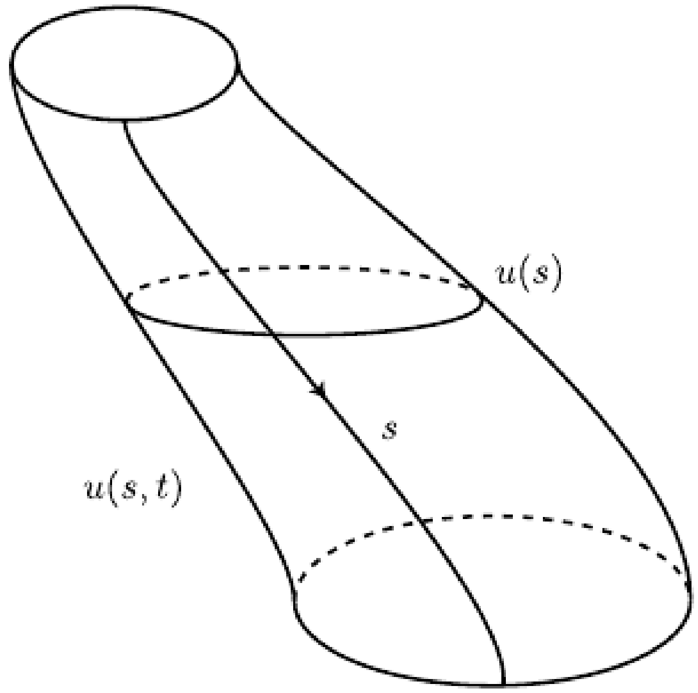
  where $\mathcal{L}W$ is the contractible loop space of W, $u : \mathbb{D} \longrightarrow W$ is an extension of $x : S^1 \longrightarrow W$ to the disc with $u(\exp(2\pi i t)) = x(t)$.

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq)$.
- Critical points of the action functional $\mathcal{A}_H$ are given by contractible loops $x, y \in \mathcal{L}W$
- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be J-holomorphic curves.

- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
  - They correspond to "embedded cylinders" with sides u and contractible caps x, y regarded as loops in W.
  - They also correspond to paths in $\mathcal{L}W$ from $x \longrightarrow y$ (precisely: trajectories of the vector field $-\mathrm{grad}\mathcal{A}_H$)



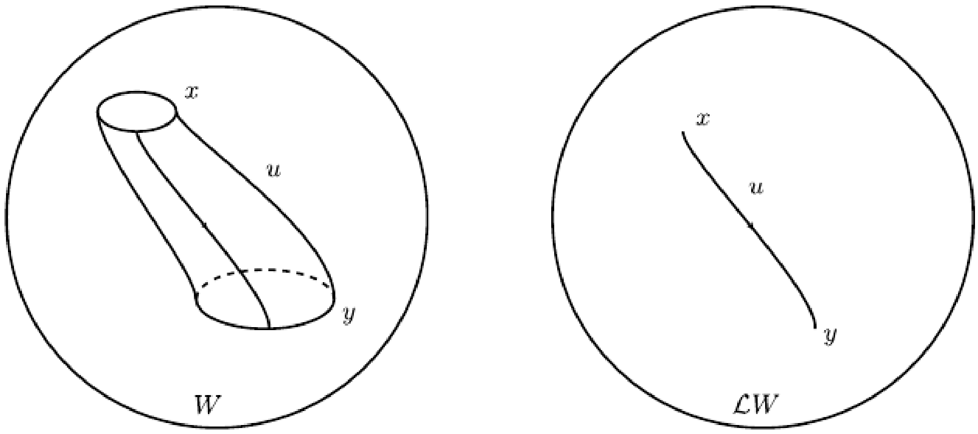


Fig. 6.5

Here $u(s) \in \mathcal{L}W$ is a loop with value at time t given by $u(s,t)$, and $\lim_{s \longrightarrow -\infty} u_s(t) = x$, $\lim_{s \longrightarrow \infty} u_s(t) = y$.

- The energy of a solution is $E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 \, dsdt$.
- $\mathcal{M} = \left\{ u \in C^\infty(\mathbb{R}; \mathcal{L}W) \;\middle|\; E(u) < \infty \right\}$ (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x,y) = ?$

For x, y contractible loops in W that are critical points of $\mathcal{A}_H$,

$$C_{\searrow}(x,y) := \left\{ u \in C^\infty \mathbb{R} \times S^1; W; \lim_{s \longrightarrow -\infty} u(s,t) = x(t), \lim_{s \longrightarrow \infty} u(s,t) = y(t), \left| \frac{\partial u}{\partial s}(s,t) \right| \leq K e^{-\delta |s|} \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(s \right.$$

where $K, \delta > 0$ are constants depending on u. So $|\partial_s u(s,t)|, |\partial_t u(s,t) - X_H(u)| \sim e^{|s|}$.

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator: $\dim \ker - \dim \operatorname{coker}$.
- Fredholm operators: those for which the index makes sense, i.e. $\dim \ker < \infty, \dim \operatorname{coker} < \infty$.
- Elliptic operators: generalize the Laplacian Δ, coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset
- Sobolev spaces: in dimension 1, define $\|u(t)\|_{s,p} = \sum_{i=0}^{s} \left\| \partial_t^i u(t) \right\|_{L^p}$ on $C^\infty(\overline{U})$, then take the completion and denote $W^{s,p}(\overline{U})$. Yields a distribution space, elements are functions with weak derivatives.

2 Talk

Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of $\mathcal{F}$.

What is $\mathcal{F}$?

We started with the unadorned Floer map:

$$\begin{aligned}\mathcal{F} : \mathcal{C}^\infty\left(\mathbf{R}\times S^1; W\right) &\longrightarrow \mathcal{C}^\infty\left(\mathbf{R}\times S^1; TW\right)\\ u &\mapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u\left(H_t\right)\end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x,y) &\longrightarrow \mathcal{L}^p(x,y)\\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u).\end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned}\mathcal{P}^{1,p}(x,y) : ? &\longrightarrow ?\\ (s,t) &\mapsto \exp_{w(s,t)} Y(s,t).\end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C^\infty_\searrow(x,y)$.

3 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian H, perturb (without modifying the periodic orbits) so that $\mathcal{M}(x,y)$ are manifolds of the right dimension.

Start by construction $\mathcal{C}^\infty_\varepsilon(H) \subset \mathcal{C}^\infty$, the space of perturbations of H. Idea: define a norm $\|\cdot\|_\varepsilon$ and take the subspace of finite-norm elements.

$$\begin{aligned}\|h\|_\varepsilon &= \sum_{k\geq 0} \varepsilon k \sup_{(x,t)\in W\times S^1} \left|d^k h(x,t)\right|\\ &= \sum_{k\geq 0} \varepsilon k \sup_{(x,t)\in W\times S^1} \sup_{i, z\in B(0,1)} \left|d^k (h\circ \Psi_i^{-1})(z)\right|.\end{aligned}$$

Where $\{\varepsilon_k\} \subset \mathbb{R}$ is chosen such that $\mathcal{C}^\infty_\varepsilon \hookrightarrow \mathcal{C}^\infty(W\times S^1)$ is dense for the C^∞ topology, and the $\Psi_i : B_i \longrightarrow \overline{B(0,1)}$ is a fixed finite sequence of diffeomorphisms where $\bigcup_i B_i^\circ = W \times S^1$.

> Note that we'll only use density for the C^1 topology in our case.

Proposition 3.1.
Such a sequence $\{\varepsilon_k\}$ can be chosen.

Proof .
Show that $C^\infty(W \times S^1)$ is separable, yielding a sequence $(f_n) \subset C^\infty(W \times S^1)$ that is dense in the C^1 topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k \leq n} \|f_k\| C^n(W \times S^1)}$$

where the diffeomorphisms Ψ_i are used to compute these norms.

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Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$ and are nondegenerate.

3.1 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

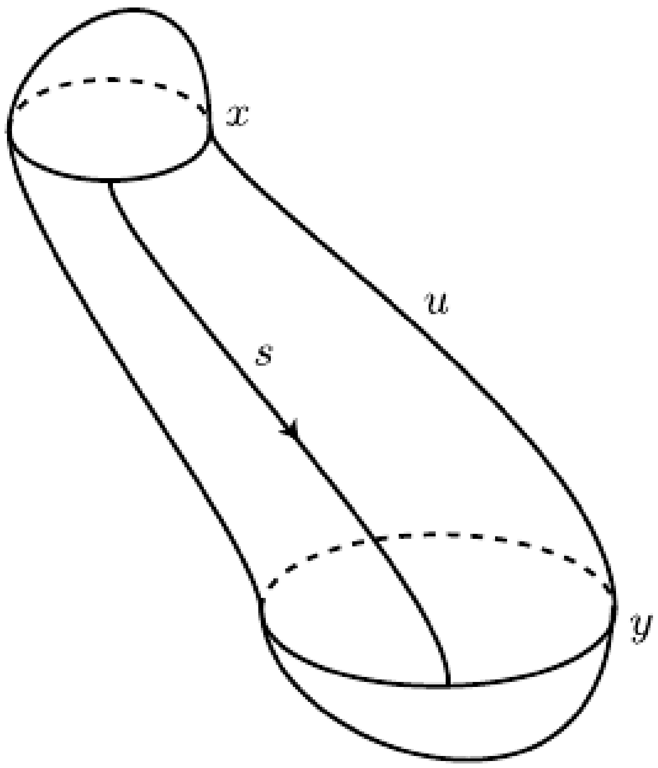
We can then identify $\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R} \times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$ (i.e. solutions to the Floer equation), and $C_\searrow(x, y)$ was the set of maps $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions.

Fix a solution $u \in \mathcal{M}(x, y) \subset C^\infty_{\mathrm{Loc}}(\mathbb{R} \times S^1; W)$.

We lift each map to $\tilde{u} : S^2 \longrightarrow W$ in the following way: the loops x, y are contractible, so they bound discs. So we extend according to:



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Recall assumption 6.22: every smooth map $w : S^2 \longrightarrow W$ yields a symplectic trivialization of w^*TW (e.g. when $\pi_2(W) = 0$, so every map from S^2 extends to B^3).

Trivialize the symplectic fiber bundle $\tilde{u}^*TW$ to obtain an orthonormal unitary frame $\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$ depending smoothly on $(s,t) \in S^2$, where $\lim\limits_{s \longrightarrow \infty} Z_i$ exists for each i. We also require that $\partial_s Z_i, \partial_s^2 Z_i, \partial_s \partial_t Z_i \overset{s \longrightarrow \pm\infty}{\Longrightarrow} 0$ for each i.

This frame defines a chart about u of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y) \\ \mathbf{y} = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

Since $(d\exp)_0 = \mathrm{id}$, we have $(d\iota)_0(\mathbf{y}) = \sum\limits_i y_i Z_i$.

We'll now consider and compute the differential of

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R} \times S^1; TW\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right) \\ u &\longmapsto \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right). \end{aligned}$$

Take the vector $Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$, where we view Y as a vector in $\mathbb{R}^m$ tangent to W, given by $Y = \sum y_i Z_i$.

We write

$$\mathcal{F}(u+Y) = \frac{\partial(u+Y)}{\partial s} + J(u+Y)\frac{\partial(u+Y)}{\partial t} - J(u+Y)X_t(u+Y)$$

and extract the part that is linear in Y:

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y).$$

Lemma 3.2 ***(Acting by Derivation).***
For any $J \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and $Y, v :? \longrightarrow \mathbb{R}^m$ we have

$$(dj)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

There is a proof.

For every such smooth map $u : \mathbb{R} \times S^1 \longrightarrow W$, $(d\mathcal{F})_u(Y) = O_1 + O_0$ where O_i are differential operators of order i, and in fact O_1 can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$ where $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^n)$ is linear of order 0, and in fact we have

Proposition 3.3.
If u solves Floer's equation, then $(d\mathcal{F})_u = \bar{\partial} + S(s,t)$ where S is linear, tends to a symmetric operator as $s \longrightarrow \pm\infty$, and $\lim \partial_t S = 0$ uniformly in t.

There is a very long computational proof.

Denote the order 0 part of $(d\mathcal{F})_u$ as $Y \mapsto S \cdot Y$ so $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and define $S^\pm := \lim\limits_{s \longrightarrow \pm\infty} S(s, \cdot)$.

Proposition 3.4.
The equation $\partial_t Y = J_0 S^\pm Y$ linearizes Hamilton's equation $\dot{z} = X_t(z)$ at $x = \lim\limits_{s \longrightarrow \pm\infty} u$ for S^+ and S^- respectively.

Proof: uses previous proposition.

Given a solution u, the product

$$\begin{aligned} u \cdot s :? &\longrightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and $\mathcal{F}(u \cdot s) = 0$ for all s.

Punchline:

Thus $\dfrac{\partial u}{\partial s}$ is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s}\mathcal{F}(u \cdot s) = (d\mathcal{F})_u\left(\frac{\partial u}{\partial s}\right).$$

Along any nonconstant solution connecting x and y, $\dim\ker(d_{\mathcal{F}})_u \geq 1$.