

# Category $\mathcal{O}$ , Problem Set 4

D. Zack Garza

Sunday 26<sup>th</sup> April, 2020

## Contents

<b>1</b>	<b>Humphreys 3.1</b>	<b>1</b>
1.1	Solution . . . . .	1
<b>2</b>	<b>Humphreys 3.2</b>	<b>2</b>
2.1	Solution . . . . .	2
<b>3</b>	<b>Humphreys 3.4</b>	<b>2</b>
3.1	Solution . . . . .	2
3.1.1	Proof of Proposition 1 . . . . .	2
3.1.2	Proof of Proposition 2 . . . . .	2
<b>4</b>	<b>Humphreys 3.7</b>	<b>3</b>
4.1	a . . . . .	3
4.2	b . . . . .	4

## 1 Humphreys 3.1

Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and identify  $\lambda \in \mathfrak{h}^\vee$  with a scalar. Let  $N$  be a 2-dimensional  $U(\mathfrak{b})$ -module defined by letting  $x$  act as 0 and  $h$  act as  $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ .

Show that the induced  $U(\mathfrak{g})$ -module structure  $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$  fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

### 1.1 Solution

Reference 1 Reference 2

Hence  $M \notin \mathcal{O}$ .

---

## 2 Humphreys 3.2

Show that for  $M \in \mathcal{O}$  and  $\dim L < \infty$ ,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

### 2.1 Solution

By theorem 3.2d, we have

$$M, N \in \mathcal{O} \implies (M \oplus N)^\vee \cong M^\vee \oplus N^\vee$$

and by definition,  $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$  is the direct sum of the duals of various weight spaces.

## 3 Humphreys 3.4

Show that  $\Phi_{[\lambda]} \cap \Phi^+$  is a positive system in the root system  $\Phi_{[\lambda]}$ , but the corresponding simple system  $\Delta_{[\lambda]}$  may be unrelated to  $\Delta$ .

For a concrete example, take  $\Phi$  of type  $B_2$  with a short simple root  $\alpha$  and a long simple root  $\beta$ . If  $\lambda := \alpha/2$ , check that  $\Phi_{[\lambda]}$  contains just the four short roots in  $\Phi$ .

### 3.1 Solution

We would like to show the following two propositions:

1.  $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \cap \Phi^+$  is a positive system in  $\Phi_{[\lambda]}$ ,
2. In general, the associated simple system  $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \cap \Delta$ .

#### 3.1.1 Proof of Proposition 1

We'll use the definition that for an abstract root system  $\Phi$ , a positive system  $\Phi^+$  is defined by picking a hyperplane  $H$  not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of  $\Phi^+$  is on one side of  $H$ , then any subset satisfies this property as well, thus  $\Phi_{[\lambda]} \cap \Phi^+$  consists only of positive roots.

Finally, letting  $N = \text{rank } \Phi_\lambda$ , since  $\text{rank } \Phi_{[\lambda]} \cap \Phi^+$  contains at least  $N$  elements, these still form a spanning set.

#### 3.1.2 Proof of Proposition 2

Concretely, we can realize  $\Phi$  and  $\Delta$  as subsets of  $\mathbb{R}^2$  in the following way:

$$\begin{aligned} \Phi &= P_1 \amalg P_2 := \{[1, 0], [0, 1], [-1, 0], [0, -1]\} \amalg \{[1, 1], [-1, 1], [1, -1], [-1, -1]\} \\ \Delta &:= \{\alpha, \beta\} := \{[1, 0], [-1, 1]\}, \end{aligned}$$

---

where we note that  $P_1$  consists of short roots (of norm 1) and  $P_2$  of long roots (of norm  $\sqrt{2}$ ) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\begin{aligned}\Phi_{[\lambda]} &:= \left\{ \gamma \in \Phi \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}, & \gamma^\vee &:= \frac{2}{\|\gamma\|^2} \gamma, \\ \Delta_{[\lambda]} &:= \left\{ \gamma \in \Delta \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}.\end{aligned}$$

Now choosing  $\lambda := \frac{\alpha}{2} = \left[ \frac{1}{2}, 0 \right]$ , we now consider the inner products  $\langle \lambda, \gamma^\vee \rangle$  for  $\gamma \in \Phi$ :

Thus

$$\begin{aligned}\gamma_1 \in P_1 &\implies \left\langle \left[ \frac{1}{2}, 0 \right], 2\gamma_1 \right\rangle = 2 \left( \frac{1}{2} \right) \langle [1, 0], \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z} \\ \gamma_2 \in P_2 &\implies \langle \lambda, \gamma_2^\vee \rangle = \left\langle \left[ \frac{1}{2}, 0 \right], \frac{2}{(\sqrt{2})^2} [\pm 1, \pm 1] \right\rangle = \pm \frac{1}{2} \notin \mathbb{Z}\end{aligned}$$

where  $(\gamma_1)_1$  denotes the first component of  $\gamma_1$ .

We thus find that

$$\begin{aligned}\Phi_{[\lambda]} &= P_1 && \text{the short roots} \\ \Delta_{[\lambda]} &= \{\alpha\} && \text{the single short simple root.}\end{aligned}$$

Choosing the following green hyperplane not containing any root, we can choose a positive system

$$\Phi^+ = \{[1, 0], [0, 1], [1, 1], [-1, -1]\} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$$

where we can note that  $\Phi^+ \cap \Delta = \Delta$ , since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi_{[\lambda]}^+ = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take  $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$ . But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \neq \{\alpha\} = \{\alpha, \alpha + \beta\} \cap \{\alpha, \beta\} = \Phi_{[\lambda]}^+ \cap \Delta,$$

which is what we wanted to show.

## 4 Humphreys 3.7

### 4.1 a

If a module  $M$  has a standard filtration and there exists an epimorphism  $\phi : M \longrightarrow M(\lambda)$ , prove that  $\ker \phi$  admits a standard filtration.

**4.2 b**

Show by example that when  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  that the existence of a monomorphism  $\phi : M(\lambda) \longrightarrow M$  where  $M$  has a standard filtration fails to imply that  $\text{coker } \phi$  has a standard filtration.