

Fall 2019 Algebra

UGA

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1 Problem Set One

1.1 Exercises

Problem 1 (Hungerford 1.6.3).

If $\sigma = (i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$, then show that $\tau \sigma \tau^{-1} = (\tau(i_1) \tau(i_2) \cdots \tau(i_r))$.

Problem 2 (Hungerford 1.6.4).

Show that $S_n \cong \langle (12), (123 \cdots n) \rangle$ and also that $S_n \cong \langle (12), (23 \cdots n) \rangle$

Problem 3 (Hungerford 2.2.1).

Let G be a finite abelian group that is not cyclic. Show that G contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p .

Problem 4 (Hungerford 2.2.12.b).

Determine all abelian groups of order n for $n \leq 20$.

Problem 5 (Hungerford 2.4.1).

Let G be a group and $A \trianglelefteq G$ be a normal abelian subgroup. Show that G/A acts on A by conjugation and construct a homomorphism $\varphi : G/A \rightarrow \text{Aut}(A)$.

Problem 6 (Hungerford 2.4.9).

Let $Z(G)$ be the center of G . Show that if $G/Z(G)$ is cyclic, then G is abelian.

Note that Hungerford uses the notation $C(G)$ for the center.

Problem 7 (Hungerford 2.5.6).

Let G be a finite group and $H \trianglelefteq G$ a normal subgroup of order p^k . Show that H is contained in every Sylow p -subgroup of G .

Problem 8 (Hungerford 2.5.9).

Let $|G| = p^n q$ for some primes $p > q$. Show that G contains a unique normal subgroup of index q .

1.2 Qual Problems

Problem 9.

Let G be a finite group and p a prime number. Let X_p be the set of Sylow- p subgroups of G and n_p be the cardinality of X_p . Let $\text{Sym}(X)$ be the permutation group on the set X_p .

1. Construct a homomorphism $\rho : G \rightarrow \text{Sym}(X_p)$ with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that if G is simple then the order of G divides $n_p!$.
3. Show that for any $1 \leq a \leq 4$ and any prime power p^k , no group of order ap^k is simple.

Problem 10.

Let G be a finite group and let $N \trianglelefteq G$, and let p be a prime number and Q a subgroup of G such that $N \subset Q$ and Q/N is a Sylow p -subgroup of G/N .

1. Prove that Q contains a Sylow p -subgroup of G .
2. Prove that every Sylow p -subgroup of G/N is the image of a Sylow p -subgroup of G .

Problem 11.

Let G be a finite group and $H < G$ a subgroup. Let n_H be the number of subgroups of G that are conjugate to H . Show that n_H divides the order of G .

Problem 12.

Let $G = S_5$, the symmetric group on 5 elements. Identify all conjugacy classes of elements in G , provide a representative from each class, and prove that this list is complete.