Title

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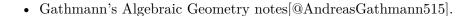
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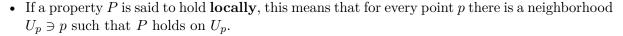
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Prologue





0.2 Notation



$$k[\mathbf{x}] \coloneqq k[x_1, \cdots, x_n] \qquad \text{The polynomial ring in n indeterminates} \\ k(\mathbf{x}) \coloneqq k(x_1, \cdots, x_n) \qquad \text{The rational function field} \\ \mathbb{A}^n \qquad k(\mathbf{x}) \coloneqq \left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \ \middle| \ p, q, \in k[x_1, \cdots, x_n] \right\} \\ V(J), V_a(J) \qquad \text{The variety associated to an ideal $J \leq k[x_1, \cdots, x_n]$} \\ V(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \ \middle| \ f(\mathbf{x}) = 0, \, \forall f \in J \right\} \\ I(S), I_a(S) \qquad \text{The ideal associated to a subset $S \subseteq \mathbb{A}^n_k$} \\ I(S) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \ \middle| \ f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in X \right\} \\ A(X) \coloneqq k[x_1, \cdots, x_n]/I(X) \qquad \text{The coordinate ring of a variety} \\ \mathcal{O}_X \qquad \text{The structure sheaf $\left\{ f : U \to k \ \middle| \ f \in k(\mathbf{x}) \text{ locally} \right\}$} \\ D(f) \qquad \qquad \text{A distinguished open set} \\ D(f) \coloneqq V(f)^c = \left\{ x \in \mathbb{A}^n \ \middle| \ f(x) \neq 0 \right\} \\ \Delta_X \qquad \qquad \text{The diagonal $\left\{ (x, x) \ \middle| \ x \in X \right\} \subseteq X \times X$} \\ \mathcal{U} \rightrightarrows X \qquad \qquad \text{An open cover} \\ V_p(J) \qquad \qquad \text{The projective variety of an ideal} \\ V_p(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \ \middle| \ f(\mathbf{x}) = 0, \, \forall f \in J \right\}.$$

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \le k[x_1, \dots, x_n]$	V(J) the zero locus
I(S) the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
I + J	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$

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Algebra	Geometry
$\overline{I \cap J, IJ}$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
I(V):I(W)	$\overline{V\setminus W}$
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
 - The Zariski topology?
 - Irreducibility?
 - Connectedness?
 - Dimension?
- What is the definition of a presheaf?
 - What are some examples and counterexamples?
- What is the definition of sheaf?
 - What are some examples?
 - What are some presheaves that are not sheaves?
- What is the definition of \mathcal{O}_X , the sheaf of regular functions?
 - How does one compute \mathcal{O}_X for X = D(f) a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
 - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
 - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples



- $V(xy-1) \subseteq \mathbb{A}^2$ a hyperbola
- V(x) a coordinate axis
- V(x-p) a point.

0.4.2 Presheaves / Sheaves

- $C^{\infty}(\cdot,\mathbb{R})$, a sheaf of smooth functions
- $C^0(\cdot,\mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$, the constant sheaf associated to \mathbb{R} (locally constant real-valued functions)
- $\operatorname{Hol}(\cdot,\mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.5 Useful Algebra Facts

Fact 0.5.1:

- $\mathfrak{p} \leq R$ is prime $\iff R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \leq R$ is maximal $\iff R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \leq R$ is generated by an irreducible element f, then $\langle f \rangle$ is maximal

Proposition 0.5.2 (Finitely generated polynomial rings are Noetherian).

A polynomial ring $k[x_1, \dots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \subseteq k[x_1, \dots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \dots, f_m \rangle$.

Proof(?).

A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.5.5), $k[x_1, \dots, x_n]$ is thus Noetherian.

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Proposition 0.5.3 (Properties and Definitions of Ideal Operations).

$$\begin{split} I+J &\coloneqq \left\{f+g \ \middle| \ f \in I, \ g \in J\right\} \\ IJ &\coloneqq \left\{\sum_{i=1}^N f_i g_i \ \middle| \ f_i \in I, \ g_i \in J, N \in \mathbb{N}\right\} \\ I+J &= \langle 1 \rangle \implies I \cap J = IJ \end{split} \qquad \text{(coprime or comaximal)} \ \langle a \rangle + \langle b \rangle = \langle a,b \rangle \ . \end{split}$$

Theorem 0.5.4 (Noether Normalization).

Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist t_1, \dots, t_ℓ such that k_2 is finite over $k_1(t_1, \dots, t_\ell)$.

Theorem 0.5.5 (Hilbert's Basis Theorem).

If R is a Noetherian ring, then R[x] is again Noetherian.

0.6 The Algebra-Geometry Dictionary

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Let $k = \bar{k}$, we're setting up correspondences

Ring Theory Geometry/Topology of Affine Varieties
Polynomial functions Affine space

$$k[x_1, \cdots, x_n]$$
 $\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$

Maximal ideals
$$\langle x_1 - a_1, \cdots, x_n - a_n \rangle$$
 Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$

Radical ideals
$$I \leq k[x_1, \dots, x_n]$$
 Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomia

$$I\mapsto V(I)\coloneqq \Big\{a\ \Big|\ f(a)=0 \forall f\in I\Big\}$$

$$I(X)\coloneqq \Big\{f\ \Big|\ f|_X=0\Big\} \leftrightarrow X$$

Radical ideals containing
$$I(X)$$
, i.e. ideals in $A(X)$ closed subsets of X, i.e. affine subvarieties

$$A(X)$$
 is a domain X irreducible

$$A(X)$$
 is not a direct sum X connected
Prime ideals in $A(X)$ Irreducible closed subsets of X

Krull dimension
$$n$$
 (longest chain of prime ideals) dim $X = n$, (longest chain of irreducible closed subsets