

# Problem Set 5

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## 1 4.3

### Proposition 1.1.

Suppose  $\lambda + \rho \in \Lambda^+$ . Then  $M(w \cdot \lambda) \subset M(\lambda)$  for all  $w \in W$ . Thus all  $[M(\lambda) : L(w \cdot \lambda)] > 0$ .

More precisely, if  $w = s_n \cdots s_1$  is a reduced expression for  $w$  in terms of simple reflections corresponding to roots  $\alpha_i$ , then there is a sequence of embeddings:

$$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_0) = M(\lambda)$$

Here

$$\begin{aligned} \lambda_0 &:= \lambda, \lambda_k := s_k \cdot \lambda_{k-1} = (s_k \cdots s_1) \cdot \lambda \implies \lambda_n = s_n \cdot \lambda_{n-1} = w \cdot \lambda \\ w \cdot \lambda = \lambda_n &\leq \lambda_{n-1} \leq \cdots \leq \lambda_0 = \lambda \text{ with } \langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+ \text{ for } k = 0, \dots, n-1. \end{aligned}$$

Assume  $\lambda + \rho \in \Lambda^+$ .

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- a. Prove that the unique simple submodule of  $M(\lambda)$  is isomorphic to  $M(w_\diamond \cdot \lambda)$ , where  $w_\diamond$  is the longest element of  $W$ .
  - b. In case  $\lambda \in \Lambda^+$ , show that the inclusions obtained in the above proposition are all proper.

## 2 4.6

### Theorem 2.1 (Verma).

Let  $\lambda \in \mathfrak{h}^\vee$ . Given  $\alpha > 0$ , suppose  $\mu := s_\alpha \cdot \lambda \leq \lambda$ . Then there exists an embedding  $M(\mu) \subset M(\lambda)$ .

Work through the steps of Verma's Theorem in the special case discussed in the previous problem

### 2.1 Solution

Let  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$  and identify its root system  $A_2$  with  $\Delta = \{\alpha, \beta\}$  and  $\Phi^+ = \{\alpha, \beta, \gamma := \alpha + \beta\}$ . We can also identify the Weyl group as  $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\gamma\}$  where there is a reduced expression  $s_\gamma = w_0 = s_\alpha s_\beta s_\alpha$ .

We can begin by letting  $\lambda \in \Lambda$  be an arbitrary integral weight and let  $\mu \neq \lambda$  be an arbitrary weight linked to  $\lambda$ , where WLOG apply some Weyl group element to  $\mu$  to place it in the dominant chamber and assume

$$\mu := s_\alpha \cdot \lambda < \lambda$$

(where the inequality is strict).

#### 2.1.1 Part 1

Since  $\mu$  is assumed integral, we can find some  $w \in W$  such that

$$\mu' := w^{-1} \cdot \mu \in \Lambda^+ - \rho.$$

**Claim:**  $w = s_\alpha s_\beta$ , so  $w^{-1} = s_\beta s_\alpha$  and thus

$$\mu' = s_\beta s_\alpha \cdot \mu$$

As in Proposition 4.3, we then write

$$\begin{aligned} \mu_0 &= \mu' \\ \mu_1 &= s_\beta \cdot \mu' \\ \mu_2 &= s_\alpha s_\beta \cdot \mu' = w \cdot \mu' = \mu \end{aligned}$$

which satisfies

$$\begin{aligned} \mu &= \mu_2 \leq \mu_1 \leq \mu_0 = \mu' \\ \mu &= s_\alpha s_\beta \cdot \mu' \leq s_\beta \mu' \leq \mu'. \end{aligned}$$

which (by the proposition) gives a sequence of embeddings

$$\begin{aligned} M(\mu) &= M(\mu_2) \hookrightarrow M(\mu_1) \hookrightarrow M(\mu_0) = M(\mu') \\ &\quad \text{i.e.} \\ M(\mu) &= M(s_\alpha s_\beta \cdot \mu') \hookrightarrow M(s_\beta \cdot \mu') \hookrightarrow M(\mu'). \end{aligned}$$

### 2.1.2 Step 2

We now define

$$\lambda' := w^{-1}\lambda = s_\beta s_\alpha \cdot \lambda$$

and the parallel list of weights

$$\begin{aligned} \lambda_0 &= \lambda' \\ \lambda_1 &= s_\beta \cdot \lambda' \\ \lambda_2 &= s_\alpha s_\beta \cdot \lambda' := \lambda. \end{aligned}$$

We can similarly use the fact that  $\lambda \neq \mu \implies \mu_k \neq \lambda_k$  for any  $k$ .

### 2.1.3 Step 3

To relate  $\mu_k$  to  $\lambda_k$ , We now define  $w_k = s_n \cdots s_{k+1}$ :

$$\begin{aligned} w_0 &= s_\alpha s_\beta \\ w_1 &= s_\alpha \\ w_2 &:= 1 \end{aligned}$$

and using the calculation

$$\mu_k = w_k^{-1} s_\alpha w_k \cdot \lambda_k = s_{\beta_k} \cdot \lambda_k$$

we compute

$$\begin{aligned} s_{\beta_0} &= (s_\alpha s_\beta)^{-1} s_\alpha (s_\alpha s_\beta) = s_\gamma \\ s_{\beta_1} &= s_\alpha^{-1} s_\alpha s_\alpha = s_\alpha \\ s_{\beta_2} &:= s_\alpha \end{aligned}$$

and thus obtain

$$\begin{aligned} \mu_0 &= s_\alpha \cdot \lambda_0 \\ \mu_1 &= s_\alpha \cdot \lambda_1 \\ \mu_2 &= s_\gamma \cdot \lambda_2. \end{aligned}$$

**2.1.4 Step 4**

We have  $\mu_0 \geq \mu_1 \geq \mu_2$  with  $\lambda_0 < \mu_0$  but  $\lambda_2 > \mu_2$ , so we now look for where the inequality switches. It suffices to check how  $\mu_1$  and  $\lambda_1$  are related, and we find  $\mu_1 < \lambda_1$ .

**2.1.5 Step 5**

From the last step, we fix  $k = 0$  and now want to show  $M(\mu_{k+i}) \subset M(\mu_{k+i})$  for  $i = 1, 2$ , since the  $i = 2$  case yields the desired  $M(\mu) \subset M(\lambda)$ .

**2.1.6 Step 6**

We first want to show  $M(\mu_1) \subset M(\lambda_1)$ . We write

$$\mu_1 - \lambda_1 = s_1\mu_0 - s_1\lambda_0.$$

We then note that

$$\begin{aligned}\mu_1 - \lambda_1 &= c_1\beta_1 \\ s_\alpha\mu_0 - s_1\lambda_0 &= s_\alpha(\mu_0 - \lambda_0) = d_1\beta_0\end{aligned}$$

where  $c_1$  is negative and  $b_1$  is positive, and we already know that  $\beta_1 = \beta_0 = \alpha$  by a direct computation. Thus we have  $\mu_1 = s_\alpha\lambda_1$ , and applying Proposition 1.4,

$$M(s_\alpha \cdot \lambda_1) \hookrightarrow M(\lambda_1) \implies M(\mu_1) \hookrightarrow M(\lambda_1).$$

**2.1.7 Step 7**

We thus have embeddings

$$M(\mu) = M(\mu_2) = M(s_\alpha \cdot \mu_1) \hookrightarrow M(\mu_1) \hookrightarrow M(\lambda_1), \tag{1}$$

and we then apply Proposition 4.5: set  $n = \langle \lambda + \rho, \alpha^\vee \rangle$  (which is integral by assumption).

Claim: regardless of the sign of  $n$ , we obtain

$$M(\mu) = M(\mu_2) \subset M(s_\beta \cdot \lambda_1) \hookrightarrow M(\lambda_2) = M(\lambda),$$

which is what we wanted to show.

This follows from analyzing the following cases:

- If  $n \leq 0$ , we have

$$M(\lambda_1) \hookrightarrow M(s_\beta \cdot \lambda_1) = M(\lambda),$$

and since  $M(\mu_1) \hookrightarrow M(\lambda_1)$  by step 6, composing this with the RHS of the embeddings in equation (1) yields the desired embedding.

- Otherwise,  $n \geq 0$ , and we instead obtain

$$M(s_\alpha \cdot \mu_1) \hookrightarrow M(s_\alpha \cdot \lambda_1) \hookrightarrow M(\lambda_1)$$

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### 3 4.11

In the case of  $\mathfrak{sl}(3, \mathbb{C})$ , what can be said at this point about Verma modules with a singular integral highest weight?

Aside from the trivial case  $-\rho$ , a typical linkage class has 3 elements. For example, if  $\lambda$  lies in the  $\alpha$  hyperplane and is antidominant, the linked weights are  $\lambda, s_\beta \cdot \lambda, s_\alpha s_\beta \cdot \lambda$ .