

# Linearization and Transversality

## Sections 8.3 and 8.4

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Linearization and  
Transversality

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Review 8.2

Section 8.3: The  
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 $H$

Section 8.4:  
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## Section 8.3: The Space of Perturbations of $H$

# Goal

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**Goal:** Given a fixed Hamiltonian  $H \in C^\infty(W \times S^1; \mathbb{R})$ , perturb it (without modifying the periodic orbits) so that  $\mathcal{M}(x, y)$  are manifolds of the expected dimension.

# Goal

Start by trying to construct a subspace  $C_\varepsilon^\infty(H) \subset C^\infty(W \times S^1; \mathbb{R})$ , the space of perturbations of  $H$  depending on a certain sequence  $\varepsilon = \{\varepsilon_k\}$ , and show it is a dense subspace.

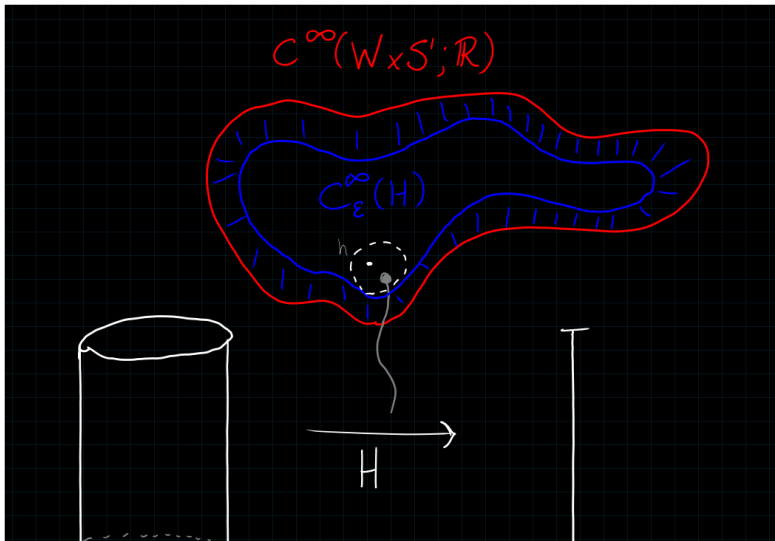
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# Define an Absolute Value

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Idea: similar to how you build  $L^2(\mathbb{R})$ , define a norm  $\|\cdot\|_\varepsilon$  on  $C_\varepsilon^\infty(H)$  and take the subspace of finite-norm elements.

- Let  $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$  denote a perturbation of  $H$ .
- Fix  $\varepsilon = \{\varepsilon_k \mid k \in \mathbb{Z}^{\geq 0}\} \subset \mathbb{R}^{>0}$  a sequence of real numbers, which we will choose carefully later.
- For a fixed  $\mathbf{x} \in W$ ,  $t \in \mathbb{R}$  and  $k \in \mathbb{Z}^{\geq 0}$ , define

$$|d^k h(\mathbf{x}, t)| = \max \left\{ d^\alpha h(\mathbf{x}, t) \mid |\alpha| = k \right\},$$

the maximum over all sets of multi-indices  $\alpha$  of length  $k$ .

*Note: I interpret this as*

$$d^{\alpha_1, \alpha_2, \dots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

*the partial derivatives wrt the corresponding variables.*

# Define a Norm

- Define a norm on  $C^\infty(W \times S^1; \mathbb{R})$ :

$$\|h\|_\infty = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} |d^k h(x, t)|.$$

- Since  $W \times S^1$  is assumed compact (?), fix a finite covering  $\{B_i\}$  of  $W \times S^1$  such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

- Choose them in such a way we obtain charts

$$\psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \quad (?).$$

- Obtain the computable form

$$\|h\|_\infty = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \psi_i^{-1})(z)|.$$

# Define a Banach Space

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- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \mid \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

- Show that the sequence  $\{\varepsilon_k\}$  can be chosen so that  $C_\varepsilon^\infty$  is a *dense* subspace for the  $C^\infty$  topology, and in particular for the  $C^1$  topology.

## Theorem

*Such a sequence  $\{\varepsilon_k\}$  can be chosen.*

## Lemma

*$C^\infty(W \times S^1; \mathbb{R})$  with the  $C^1$  topology is separable as a topological space (contains a countable dense subset).*



# Sketch Proof of Theorem

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- By the lemma, produce a sequence  $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$  dense for the  $C^1$  topology.
- Using the norm on  $C^n(W \times S^1; \mathbb{R})$  for the  $f_n$ , define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \mid k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable.

*Why does this imply density? I don't know.*



# Modified Theorem

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The next proposition establishes a version of this theorem with compact support:

## Theorem

*For any  $(\mathbf{x}, t) \in U \in W \times S^1$  there exists a  $V \subset U$  such that every  $h \in C^\infty(W \times S^1; \mathbb{R})$  can be approximated in the  $C^1$  topology by functions in  $C_\varepsilon^\infty$  supported in  $U$ .*

Then fix a time-dependent Hamiltonian  $H_0$  with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \mid h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then  $\text{supp}(h)$  is “far” from  $\text{Per}(H_0)$ , so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

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## Section 8.4: Linearizing the Floer Equation: The Differential of $F$

# Goal

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Choose  $m > n = \dim(W)$  and embed  $TW \hookrightarrow \mathbb{R}^m$  to identify tangent vectors (such as  $Z_i$ , tangents to  $W$  along  $u$  or in a neighborhood  $B$  of  $u$ ) with actual vectors in  $\mathbb{R}^m$ .

*Why? Bypasses differentiating vector fields and the Levi-Cevita connection.*

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential  $d\mathcal{F}$ .

*We've just replaced the codomain here.*

# Definitions

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Recall that

- $x, y$  are contractible loops in  $W$  that are nondegenerate critical points of the action functional  $\mathcal{A}_H$ ,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$  denotes a fixed solution to the Floer equation,
- $C_{\searrow}(x, y) \subset \{u \in C^\infty(\mathbb{R} \times S^1; W)\}$  is the set of smooth solutions  $u : \mathbb{R} \times S^1 \rightarrow W$  satisfying some conditions:

$$\lim_{s \rightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \rightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left| \frac{\partial u}{\partial t}(s, t) \right|, \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

# Compactify to Sphere

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Fix a solution

$$u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^{\infty}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way:

The loops  $x, y$  are contractible, so they bound discs. So we extend by pushing these discs out slightly:

# Lift to 2-Sphere

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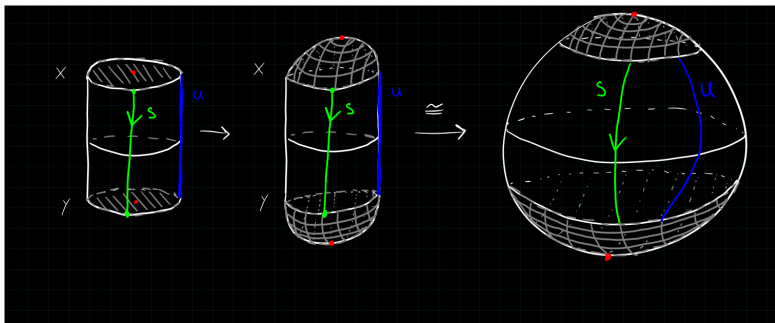
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$$u \in C^\infty(S^1 \times \mathbb{R}; W) \mapsto \tilde{u} \in C^\infty(S^2; W)$$



# Trivial the Pullback

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From earlier in the book, we have

## Assumption (6.22):

For every  $w \in C^\infty(S^2, W)$  there exists a symplectic trivialization of the fiber bundle  $w^*TW$ , i.e.  $\langle c_1(TW), \pi_2(W) \rangle = 0$  where  $c_1$  denotes the first Chern class of the bundle  $TW$ .

*Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:*

$$c_1(TW) = e(\wedge^1(TW)) \in H^2(W; \mathbb{Z})$$

*Assumption is satisfied when all maps  $S^2 \rightarrow W$  lift to  $B^3 \iff \pi_2(W) = 0$ .*

We have a pullback that is a symplectic fiber bundle:

$$\begin{array}{ccc} \tilde{u}^*TW & \xrightarrow{d\tilde{u}} & TW \\ \downarrow & \lrcorner & \downarrow \\ S^2 & \xrightarrow{\tilde{u}} & W \end{array}$$



# Choose a Frame

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- Using the assumption, trivialize the pullback  $\tilde{u}^*TW$  to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

where

- The frame depends smoothly on  $(s, t) \in S^2$ ,
- $\lim_{s \rightarrow \pm\infty} Z_i$  exists for each  $i$ .
- 

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \partial t} \quad \curvearrowright \quad Z_i \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{for each } i$$

Claim: such trivializations exist, “using cylinders near the spherical caps in the figure”.

# Define “Banach Manifold Charts”

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Recall we had  $W^{1,p}(x, y)$  a completion of  $C^\infty$

$$\mathcal{M}(x, y) \subset C_{\searrow}^\infty(x, y) \subset \mathcal{P}^{1,p}(x, y) \underset{\text{defn}}{\subset} \left\{ (s, t) \xrightarrow{\varphi} \exp_{w(s,t)} Y(s, t) \right\}.$$

where we restrict to

- $Y \in W^{1,p}(w^*TW)$ ,
- $w \in C_{\searrow}^\infty(x, y)$

Use the chosen frame  $\{Z_i\}$  to define a chart centered at  $u$  of  $\mathcal{P}^{1,p}(x, y)$  given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\longmapsto \exp_u \left( \sum y_i Z_i \right). \end{aligned}$$

- Note that the derivative at zero is  $\sum_{i=1}^{2n} y_i Z_i$ .



# Add a Tangent

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- Take the vector

$$Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

- View  $Y$  as a vector in  $\mathbb{R}^m$  tangent to  $W$ , given by  $Y = \sum_{i=1}^{2n} y_i Z_i$ .
- Plug  $u + Y$  into the equation for  $\mathcal{F}$ , directly yielding

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$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u) \\ \mathcal{F}(u + Y) &= \frac{\partial(u+Y)}{\partial s} + J(u+Y) \frac{\partial(u+Y)}{\partial t} - J(u+Y) X_t(u+Y)\end{aligned}$$

# Extract Linear Part

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Extract the part that is linear in  $Y$  and collect terms:

$$\begin{aligned}(d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right)\end{aligned}$$

- This is a sum of two differential operators:
  - One of order 1, one of order 0 (Perspective 1)
  - The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)

# Leibniz Rule

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- Now compute in charts. Need a lemma:

## Lemma (Leibniz Rule)

*For any source space  $X$  and any maps*

$$\begin{aligned} J : X &\longrightarrow \text{End}(\mathbb{R}^m) \\ Y, v : X &\longrightarrow \mathbb{R}^m \end{aligned}$$

*we have*

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

# Sketch: Proof of Leibniz Rule

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Differentiate the map

$$\begin{aligned} J \cdot v : X &\longrightarrow \mathbb{R}^m \\ x &\mapsto J(x) \cdot v(x) \end{aligned}$$

to obtain

$$\begin{aligned} &J(x + Y)v(x + y) \\ &= (J(x) + (dJ)_x(Y)) \cdot (v(x) + (dv)_x(Y)) + \cdots \\ &= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x) \\ &\quad + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots \\ &\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y). \end{aligned}$$



# Decompose by Order

Using the chart  $\iota$  defined by  $\{Z_i\}$  to write  $Y = \sum_{i=1}^{2n} y_i Z_i$  and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where  $O_0$  are order 0 terms ("they do not differentiate the  $y_i$ ") and the  $O_1$  are order 1 terms:

$$O_1 = \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$

$$O_0 = \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

# Order One

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- Study  $O_1$  first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \dots, y_{2n}).$$

where  $J_0$  is the standard complex structure on  $\mathbb{R}^{2n} = \mathbb{C}^n$

- The second equality follows from the assumption that the  $Z_i$  are symplectic and orthonormal.
- Note that this writes  $(d\mathcal{F})_u(Y) = O_0 + O_{CR}$ , a sum of an order zero and a Cauchy-Riemann operator.

# Recap

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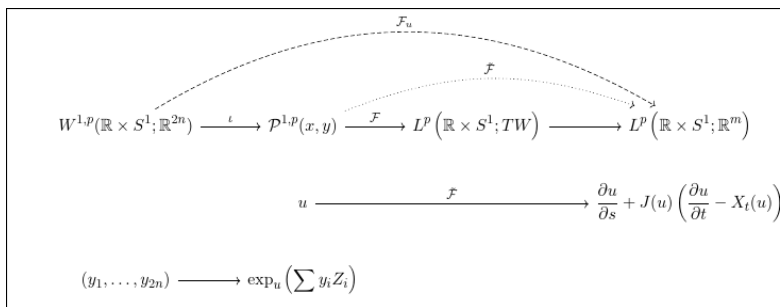
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Note that since we've computed in charts, we have actually computed the differential of  $\mathcal{F}_u$  in the following diagram



So we've technically computed  $(dF_\mu)_0$ .

# Order 0 Term is Linear

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$$\begin{aligned}(d\mathcal{F})_u &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \\ &:= \bar{\partial} Y + SY\end{aligned}$$

where  $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^n))$  is a linear operator of order 0.

# Order 0 Symmetry in the Limit

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## Theorem (8.4.4, CR + Symmetric in the Limit)

*If  $u$  solves Floer's equation, then*

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

*where*

- *$S$  is linear*
- *$S$  tends to a symmetric operator as  $s \rightarrow \pm\infty$ , and*
- 

$$\frac{\partial S}{\partial s}(s, t) \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{uniformly in } t$$

# Proof

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Omitted –  $S$  is exactly  $O_0$  from before:

$$\begin{aligned} O_0 &= \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right) \\ &= \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i) X_t \right) + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right) \end{aligned}$$

- The term in blue vanishes as  $s \rightarrow \pm\infty$ 
  - Using the fact that  $u$  is a solution
  - Uses  $\frac{\partial u}{\partial s} \rightarrow 0$  uniformly (as do its derivatives?)
- Suffices to show the remaining part is symmetric in the limit, i.e. write as

$$A(y_1, \dots, y_{2n}) = \dots \implies A_{ij} = A_{ji}$$

using inner product calculations