Math 8100 Assignment 6

Due date: Friday 8th of October 2010

1. Prove the following:

   (a)
   $$\int_{\{x\in\mathbb{R}^n\,:\,|x|\le 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p < n.$$

   (b)
   $$\int_{\{x\in\mathbb{R}^n\,:\,|x|\ge 1\}} |x|^{-p}\,dx < \infty \quad \text{if and only if} \quad p > n.$$

2. Suppose that $f \in L^1(\mathbb{R}^n)$. Show that
   $$\int_{\mathbb{R}^n} |f(x)|\,dx = \int_0^\infty m(\{x\in\mathbb{R}^n \,:\, |f(x)| > t\})\,dt.$$

3. Recall that the Fourier transform of an integrable function f on $\mathbb{R}^n$ may be defined by
   $$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$$
   and the convolution of two integrable functions f and g on $\mathbb{R}^n$ may be defined by
   $$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$
   Let $f, g, h \in L^1(\mathbb{R}^n)$.

   (a) Prove that for each $\xi \in \mathbb{R}^n$ one has $\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

   (b) i. Show that $f * g = g * f$.

   ii. Show that $(f*g)*h = f*(g*h)$.

   (c) Show that there does not exist $I \in L^1(\mathbb{R}^n)$ such that $f * I = f$ for all $f \in L^1(\mathbb{R}^n)$.

4. (a) Let $f \in L^1(\mathbb{R})$.

   i. Let $g(x) = xf(x)$. Show that if $g \in L^1$, then $\widehat{f}$ is differentiable and $\frac{d}{d\xi}\widehat{f}(\xi) = -2\pi i\,\widehat{g}(\xi)$.

   ii. Suppose f is C^1 and vanishes at infinity. Let $h(x) = \frac{d}{dx}f(x)$.
   Show that if $h \in L^1$, then $\widehat{h}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.

   (b) Let $G(x) = e^{-\pi x^2}$. By considering the derivative of $\widehat{G}(\xi)/G(\xi)$, show that $\widehat{G}(\xi) = G(\xi)$.

5. Suppose that F is a closed subset of $\mathbb{R}$ whose complement has finite measure. Let $\delta(x)$ denote the distance from x to F, namely
   $$\delta(x) = d(x,F) = \inf\{|x-y| \,:\, y \in F\}$$
   and
   $$I_F(x) = \int_{-\infty}^{\infty} \frac{\delta(y)}{|x-y|^2}\,dy.$$

   (a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition $|\delta(x)-\delta(y)| \le |x-y|$.

   (b) Show that $I_F(x) = \infty$ if $x \notin F$.

   (c) Show that $I_F(x) < \infty$ for a.e. $x \in F$, by showing that $\int_F I_F(x)\,dx < \infty$.

Challenge Problem VI

Hand this in to me at some point in the semester

(a) Prove that if $A, B \in \mathcal{M}(\mathbb{R})$, then $A \times B \in \mathcal{M}(\mathbb{R}^2)$ with $m(A \times B) = m(A)m(B)$.

(b) i. The *continuum hypothesis* asserts that whenever S is an infinite subset of $\mathbb{R}$, then either S is countable, or S has the cardinality of $\mathbb{R}$. Accepting the validity of the continuum hypothesis show that there exists an ordering $\prec$ of $\mathbb{R}$ with the property that for each $y \in \mathbb{R}$ the set $\{x \in \mathbb{R} : x \prec y\}$ is at most countable.

ii. Given the ordering $\prec$ from part (i) we define

$$E = \{(x, y) \in [0, 1] \times [0, 1] : x \prec y\}.$$

Show that E is <u>not</u> measurable, even though the slices

$$E_x = \{y \in \mathbb{R} : (x, y) \in E\} \quad \text{and} \quad E^y = \{x \in \mathbb{R} : (x, y) \in E\}$$

are both measurable with $m(E_x) = 1$ and $m(E^y) = 0$ for each $x, y \in [0, 1]$.

[*Hint for part (i): Let* $\prec$ *denote a well-ordering of* $\mathbb{R}$*, and define*

$$X = \{y \in \mathbb{R} : \textit{the set } \{x : x \prec y\} \textit{ is not countable}\}.$$

If X *is empty we are done. Otherwise, consider the smallest element* y' *in* X*, and use the continuum hypothesis.*]