

# Real Analysis

D. Zack Garza

August 27, 2019

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## 1 Summary

Measure and integration theory with relevant examples from Lebesgue integration, Hilbert spaces (only with regard to  $L^2$ ),  $L^p$  spaces and the related Riesz representation theorem. Hahn, Jordan and Lebesgue decomposition theorems, Radon-Nikodym Theorem and Fubini’s Theorem.

*Texts* - Real Analysis, by E. M. Stein and R. Shakarchi - Real Analysis, by G. B. Folland - An introduction to measure theory, by Terrence Tao - Real and Complex Analysis, by W. Rudin

An old course page

### 1.1 Big Theorems

- **Bolzano-Weierstrass:** every bounded sequence has a convergent subsequence.
- **Heine-Borel:** in  $\mathbb{R}^n$ ,  $X$  is compact  $\iff X$  is closed and bounded.

## 1.2 Big Counterexamples

- Differentiability  $\implies$  continuity but not the converse: The Weierstrass function is continuous but nowhere differentiable: Weierstrass function.
- $f$  continuous does not imply  $f'$  is continuous:  $f(x) = x^2 \sin(x)$ .
- Limit of derivatives may not equal derivative of limit:  $f(x) = \frac{\sin(nx)}{n^c}$  where  $0 < c < 1$ .
  - Also shows that a sum of differentiable functions may not be differentiable.
- Limit of integrals may not equal integral of limit:  $\sum \mathbb{1}_{[x = q_n \in \mathbb{Q}]}$ .
- A sequence of continuous functions converging to a discontinuous function:  $f(x) = x^n$  on  $[0, 1]$ .

## 2 Lecture 1 (Thu 15 Aug 2019 11:04)

See Folland's Real Analysis, definitely a recommended reference.

Possible first day question: how can we “measure” a subset of  $\mathbb{R}$ ? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of  $\mathbb{R}$ . We'll come back to this in a few lectures.

### 2.1 Notions of “smallness” in $\mathbb{R}$

**Definition 1.** Let  $E$  be a set, then  $E$  is *countable* if it is in a one-to-one correspondence with  $E' \subseteq \mathbb{N}$ , which includes  $\emptyset, \mathbb{N}$ .

**Definition 2.** A set  $E$  is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

**Exercise 1.** Show that any finite subset of  $\mathbb{R}$  is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a  $X \subseteq Y$  is dense in  $Y$  iff the closure of  $X$  is all of  $Y$ . So we'll make the following definition:

**Definition 3.** A set  $A \subseteq \mathbb{R}$  is *nowhere dense* if every interval  $I$  contains a subinterval  $S \subseteq I$  such that  $S \subseteq A^c$ .

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. For example,  $\mathbb{Q}$  is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently,

- $A^c$  contains a dense, open set.
- The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than  $\varepsilon$ .

**Definition 4.** Definition:  $E$  is a *null set* (or has *measure zero*) if  $\forall \varepsilon > 0$ , there exists a sequence

of intervals  $\{I_j\}_{j=1}^\infty$  such that

$$E \subseteq \bigcup_{j=1}^\infty I_j \text{ and } \sum |I_j| < \varepsilon.$$

**Exercise 2.** Show that a countable union of null sets is null.

We have several relationships

- Countable  $\implies$  Meager, but not the converse.
- Countable  $\implies$  Null, but not the converse.

**Exercise 3.** Show that the “middle third” Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

**Theorem 1.** Every  $E \subseteq \mathbb{R}$  can be written as  $E = A \cup B$  where  $A$  is null and  $B$  is meager.

This gives some information about how nullity and meagerness interact – in particular,  $\mathbb{R}$  itself is neither meager nor null. Idea: if meager  $\implies$  null, this theorem allows you to write  $\mathbb{R}$  as the union of two null sets. This is bad!

*Proof.* We can assume  $E = \mathbb{R}$ . Take an enumeration of the rationals, so  $\mathbb{Q} = \{q_j\}_{j=1}^\infty$ . Around each  $q_j$ , put an interval around it of size  $1/2^{j+k}$  where we’ll allow  $k$  to vary, yielding multiple intervals around  $q_j$ . To do this, define  $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 1/2^{j+k})$ . Now let  $G_k = \bigcup_j I_{j,k}$ . Finally, let  $A = \bigcap_k G_k$ ; we claim that  $A$  is null.

Note that  $\sum_j |I_{j,k}| = \frac{1}{2^k}$ , so just pick  $k$  such that  $\frac{1}{2^k} < \varepsilon$ .

Now we need to show that  $A^c$ :

$B$  is meager. Note that  $G_k$  covers the rationals, and is a countable union of open sets, so it is dense. So  $G_k$  is an open and dense set. By one of the equivalent formulations of meagerness, this means that  $G_k^c$  is nowhere dense. But then  $B = \bigcup_k G_k^c$  is meager.  $\square$

## 2.2 $\mathbb{R}$ is not small

**Theorem 2** (A, Cantor).  $\mathbb{R}$  is not countable.

**Theorem 3** (B, Baire).  $\mathbb{R}$  is not meager. (Baire Category Theorem)

**Theorem 4** (C, Borel).  $\mathbb{R}$  is not null.

Note that theorems B and C imply theorem A. You can also replace  $\mathbb{R}$  with any nonempty interval  $I = [a, b]$  where  $a < b$ . This is a strictly stronger statement – if any subset of  $\mathbb{R}$  is not countable, then certainly  $\mathbb{R}$  isn’t, and so on.

*Proof of (A).* Begin by thinking of  $I = [0, 1]$ , then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry.  $\square$

A second proof of (A). Take an interval  $I$ , and suppose it is countable so  $I = \{x_i\}$ . Choose  $I_1 \subseteq I$  that avoids  $x_1$ , so  $x_1 \notin I_1$ . Choose  $I_2 \subseteq I_1$  avoiding  $x_2$  and so on to produce a nested sequence of closed intervals. Since  $\mathbb{R}$  is complete, the intersection  $\bigcap_{n=1}^{\infty} I_n$  is nonempty, so say it contains  $x$ . But then  $x \in I_1 \subseteq I$ , for example, but  $x \neq x_i$  for any  $i$ , so  $x \notin I$ , a contradiction.  $\square$

*Proof of (B).* Suppose  $I = \bigcup_{i=1}^{\infty} A_n$  where each  $A_n$  is nowhere dense. We'll again construct a nested sequence of closed sets. Let  $I_1 \subseteq I$  be a subinterval that misses all of  $A_1$ , so  $A_1 \cap I_1 = \emptyset$  using the fact that  $A_1$  is nowhere dense. Repeat the same process, let  $I_2 \subset I_1 \setminus A_2$ . By the nested interval property, there is some  $x \in \bigcap A_i$ .  $\square$

Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of  $\mathbb{R}$  is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

Consider the Thomae function: It is continuous on  $\mathbb{Q}$ , but discontinuous on  $\mathbb{R} \setminus \mathbb{Q}$ . Can this be switched to get some function  $f$  that is continuous on  $\mathbb{R} \setminus \mathbb{Q}$  and discontinuous on  $\mathbb{Q}$ ? The answer is no. The set of discontinuities of a function is *always* an  $F_{\sigma}$  set, and  $\mathbb{R} \setminus \mathbb{Q}$  is not one. Equivalently, the rationals are not a  $G_{\delta}$  set.

Some facts:

- The pointwise limit of continuous functions has a meager set of discontinuities.
- If  $f$  is integrable, the set of discontinuities is null.
- If  $f$  is monotone, they are countable.
- There is a continuous nowhere differentiable function: let  $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$ , and in fact *most* functions are like this.
- If  $f$  is continuous and monotone, the discontinuities are null.

**Theorem 5.** Let  $I = [a, b]$ . If  $I \subseteq \bigcup_{i=1}^{\infty} I_i$ , then  $|I| \leq \sum_{i=1}^{\infty} |I_i|$ .

*Proof.* The proof is by induction. Assume  $I \subseteq \bigcup_{n=1}^{N+1} I_n$ , where wlog we can assume that  $a < a_{N+1} < b \leq b_{N+1}$ , then  $[a, a_{N+1}] \subset \bigcup_{n=1}^N I_n$  so the inductive hypothesis applies. But then  $b - a \leq b_{N+1} - a = (b_{N+1} - a_{N+1}) + (a_{N+1} - a) \leq \sum_{n=1}^{N+1} |I_n|$ .  $\square$

Note that this proves that the reals are uncountable!

## 3 Lecture 2

(Find notes for first 15 mins)

- Countable  $\implies$  Cantor, all intervals are not countable
- Meager  $\implies$  Baire, all intervals are not meager
- Null  $\implies$  Borel, all intervals are not null.

Exercise: Verify that  $f$  is continuous at  $x$  iff  $\lim f(x_n) = f(x)$  for every sequence  $\{x_n\} \rightarrow x$ .

Definition: If  $f : X \rightarrow \mathbb{R}$ , the \*oscillation of  $f$  at  $x \in X$  is given as

$$\omega_f(x) = \lim_{\delta \rightarrow 0} \sup_{y \in B_\delta(x)} |f(y) - f(x)|.$$

Exercise: Show that  $f$  is continuous at  $x \iff \omega_f(x) = 0$ .

We can then define points of discontinuity as

$$D_f = \{x \in X \mid \omega_f(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in X \mid \omega_f(x) \geq \frac{1}{n}\right\}$$

Exercise: show that  $D_f$  is closed.

Theorem 1:  $f$  is monotone  $\implies D_f$  is countable.

Hint: we can't cover  $\mathbb{R}$  by uncountable many disjoint intervals.

Theorem 2:  $D_f$  is always an  $F_\sigma$  set.

$\mathbb{R} - \mathbb{Q}$  is not an  $F_\sigma$  set, i.e. one can not construct a function that is discontinuous on exactly this set.

Theorem 3:  $f$  is "1st class"  $\implies D_f$  is meager.

$f$  is first class if  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  pointwise and each  $f_n$  is continuous.

Theorem 4 (Lebesgue Criterion): Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded, then  $f$  is Riemann integrable iff  $D_f$  is null.

So the Dirichlet function is not Riemann integrable.

Exercise: Prove theorems 1 and 2.

### 3.1 Proof of Theorem 3

Proof of theorem 3: Want to show that  $D_f$  is meager. We know it's some countable union of some sets, and it suffices to show that they are nowhere dense.

So let  $F_n = \{x \mid \omega_f(x) \geq \frac{1}{n}\}$  for some fixed  $n$ . Let  $I$  be an arbitrary closed interval, we will show that there exists a subinterval  $J \subseteq I$  with  $J \subseteq F_n^c$ .

Consider

$$E_k = \bigcap_{i,j \leq k} \left\{x \mid |f_i(x) - f_j(x)| \leq \frac{1}{5n}\right\}$$

Motivation: this comes from working backwards from 4-5 triangle inequalities that will appear later.

Some observations:  $E_k$  is closed by the continuity of the  $f_i$  (good exercise). We also have  $E_k \subseteq E_{k+1}$ . Moreover,  $\bigcup_k E_k = \mathbb{R}$  because the  $f_i \rightarrow f$  are Cauchy.

We'll now look for an interval entirely contained in the complement. Let  $I \subset \mathbb{R}$  be an interval, then write  $I = \bigcup_k (I \cap E_k)$ . Baire tells us that  $I$  is not meager, so at least one term appearing in

this union is *not* nowhere dense, i.e. there is some  $k$  for which  $I \cap E_k$  is not nowhere dense, i.e. it contains an open interval (it has a nonempty interior, and its already closed, and thus it contains an interval).

So let  $J$  be this open interval. We want to show that  $J \subseteq F_n^c$ . If  $x \in J$ , then  $x \in E_k$  as well, and so  $|f_i(x) - f_j(x)| \leq 1/5n$  for all  $i, j \geq k$ . So let  $i \rightarrow \infty$ , so  $|f(x) - f_j(x)| \leq 1/5n$  for all  $j \geq k$ .

Now for any  $x \in J$ , there exists some interval  $I(x) \subseteq J$  depending on  $x$  such that  $|f(y) - f_k(x)| \leq 2/5n$ . (where we will rewrite this as  $f(y) - f_k(y) + f_k(y) - f_k(x)$ ).

This implies that  $\omega_f(x) \leq 4/5n$ .  $\square$ .

### 3.2 Proof of Theorem 4

Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Recall that  $f$  is Riemann integrable iff for any  $\varepsilon$  there exists a partition  $P_\varepsilon = \{a = x_1 \leq x_2 \leq \dots \leq x_n = b\}$  of  $[a, b]$  such that  $U(f, P_\varepsilon) - L(f, \varepsilon) \leq \varepsilon$ , where this expression is equal to

$$\sum_n \sup_{y, z \in [x_n, x_{n+1}]} |f(y) - f(z)|(x_{n+1} - x_n)$$

( $\Rightarrow$ ): Let  $\varepsilon > 0$  and  $n$  be fixed, and produce a partition  $P_\varepsilon$  so that this sum is less than  $\varepsilon/n$ .

Recall that we want to show that  $F_n$  is null.

Now exclude from this sum all intervals that miss  $F_n$ , making it no bigger. We also know that in  $F_n$ , the sups are no greater than  $1/n$ ,

$$\varepsilon/n \geq \sum \text{stuff} \geq \sum \frac{1}{n}(x_{n+1} - x_n)$$

( $\Leftarrow$ ): Suppose  $D_f$  is null and let  $\varepsilon > 0$  be arbitrary, we want to construct  $P_\varepsilon$ . Choose  $n > 1/\varepsilon$  and  $F_n \subseteq D_f$  is closed and bounded and thus compact. But a compact measure zero interval can in fact be covered by *finitely* many open intervals. So  $F_n$  is covered by finitely many intervals  $\{I_n\}^N$  with  $\sum |I_n| \leq \varepsilon$ .

Now if  $x \notin F_n$ , then  $\exists \delta(x) > 0$  where  $\sup_{y, z \in B_\delta(x)} |f(y) - f(z)| < \frac{1}{n} < \varepsilon$

Since  $(\bigcup_j I_j)^c$  is compact, there's a finite cover  $I_{N+1}, \dots, I_{N'}$  covering  $F_n^c$ .

## 4 Lecture n

Recall Baire's theorem:  $\mathbb{R}$  can not be written as a countable union of nowhere dense sets. A subset  $A \subseteq \mathbb{R}$  is nowhere dense if every interval contains a subinterval which lies entirely in  $A^c$ , iff  $\overline{A}$  has empty interior iff  $\overline{A}$  contains no open intervals.

Exercise: show that these definitions are equivalent.

Corollary:  $\mathbb{R} \setminus \mathbb{Q}$  is not an  $F_\sigma$  set. Suppose it was, so  $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n \in \mathbb{N}} F_n$  with  $A_n$  closed. Then  $\mathbb{R} = \bigcup_n F_n \cup \bigcup_i \{q_i\}$  where  $\mathbb{Q} = \bigcup \{q_i\}$ . This exhibits  $\mathbb{R}$  as a countable union of closed sets. But the  $F_n$  are nowhere dense, since if they contained in interval they'd also contain a rational.

Exercise: Showing that  $F_n$  are nowhere dense by constructing a sequence of elements in  $F_n$  that converges to an element in  $F_n^c \subset \mathbb{Q}$ .

## 4.1 Riemann Integration

Some good properties:

- Good for approximation (vertical strips)
- Many functions are in  $\mathcal{R}$ , e.g. continuous functions.
- $\mathcal{R}([a, b])$  is a vector space
- The integral is an element of  $\mathcal{R}^\vee$ .
- FTC
- $\mathcal{R}$  is closed under uniform convergence.

Some bad properties:

- The Dirichlet function  $\mathbb{1}_{[x \in \mathbb{Q}]}$  is not in  $\mathcal{R}$  (Exercise!)
  - **Exercise:** show that  $D_f = \mathbb{R}$  (use sequential continuity)
  - It is in  $\mathcal{L}$  (Lebesgue integral).
- $\mathcal{R}$  is not closed under pointwise convergence.
  - Example:  $g_n(x) = \mathbb{1}_{[x \in \mathbb{Q}, x = \frac{p}{q}, q \leq n]} \in \mathcal{R}$ , but  $g_n \not\rightarrow g$ . (Exercise)

In fact, there exists a sequence of *continuous* functions  $\{f_n\}$  such that

- $0 \leq f_n(x) \leq 1$  for all  $x, n$ .
- $f_n(x)$  is decreasing as  $n \rightarrow \infty$  for all  $x$ .
- $f := \lim f_n \notin \mathcal{R}$ .

This seems disturbing! The Lebesgue integral fixes this particular problem. Letting

$$\mathcal{L} = \{f \mid f \text{ is Lebesgue integrable}\},$$

we have the following theorem:

Theorem (Dominated Convergence, Special Case):

Let  $\{f_n : [a, b] \rightarrow \mathbb{R}\} \subset \mathcal{L}$ , such that  $\forall n \in \mathbb{N}, \forall x, |f_n(x)| \leq M$ , if  $f_n \rightarrow f$  pointwise then  $\int f_n \rightarrow \int f$ .

## 4.2 Measure Theory

Can we assign a “measure” to all subsets of  $\mathbb{R}^n$ ?

This should be a function  $m : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathbb{R}^{\geq 0} \cup \infty = [0, \infty]$  with some properties (see handout).

- If  $\{E_i\}_{i \in \mathbb{N}}$  are disjoint, then  $m(\coprod_{i \in \mathbb{N}} E_i) = \sum_{i \in \mathbb{N}} m(E_i)$ .
- If  $E \simeq F$  by translation/rotation/reflection, then  $m(E) = m(F)$ .
- $m(Q) = 1$  if  $Q = [0, 1]^n$ .

But so far, this is impossible. There exists a non-measurable set: Define an equivalence relation  $x \sim y \iff x - y \in \mathbb{Q}$  on  $[0, 1]$ . Note that each equivalence class bijects with  $\mathbb{Q}$ , so each class is countable and there must be an uncountable number of classes. Use the axiom of choice to construct a set  $N$  by choosing exactly one element from each equivalence class.

Now let  $\mathbb{Q} \cap [-1, 1] = \{q_j\}$ , and define  $N_j = N + q_j$ . Note  $j \neq k \implies N_j \cap N_k = \emptyset$ . By translation invariance,  $m(N_j) = m(N)$ , and  $[0, 1] \subseteq \bigcup_j N_j \subseteq [-1, 2]$ . But then by taking measures,  $1 \leq \sum_j m(N_j) \leq 3$ . Using  $m(N_i) = m(N)$ .

But then  $m(N) = 0 \implies 1 > m(N)$ , and if  $m(N) = \varepsilon$  then  $\sum m(N) = \sum \varepsilon > 3$ .

Any open set in  $\mathbb{R}$  can be written as a *countable* union of intervals (Exercise). But how do you go to closed sets, or to  $\mathbb{R}^n$ ? Any open set in  $\mathbb{R}^n$  can be written as an *almost disjoint* union of closed cubes. We can then attempt to ascribe measure by approximating an open set from the inside by cubes, although it's not clear that this is unique (although it is).

## 5 Lecture n+1

Today: 1.2 in Stein, the Lebesgue Outer Measure

Some preliminary results about open sets:

Theorem 1.3 (Stein): Every open subset of  $\mathbb{R}$  can be written as a countable union of disjoint open intervals. Moreover, this representation is unique.

Theorem 1.4 (Stein): As a partial analog of 1.3, every subset of  $\mathbb{R}^n$  can be written as a countable union of *almost disjoint* closed cubes.

$A, B$  are **almost disjoint** iff  $A^\circ \cap B^\circ = \emptyset$ .

We'll now attempt to assign a preliminary notion of measure for all subsets of  $\mathbb{R}$  which extends the notion of volume.

Definition (Outer Measure): If  $E \subseteq \mathbb{R}^n$ , then the **Lebesgue outer measure** of  $E$ , denoted  $m_*(E)$ , is defined as follows:

$$m_*(E) = \inf_{\bigcup_{i \in \mathbb{N}} Q_i \supseteq E, Q_i \text{ closed}} \sum_{i \in \mathbb{N}} |Q_i|$$

i.e. we take the infimum over all coverings of  $E$  by countably many closed cubes.

Remarks:

- $m_*$  is well-defined for *all* subsets  $E \subseteq \mathbb{R}^n$ .
- $m_*(E) \in [0, \infty]$
- For all  $\varepsilon > 0$ , there exists a covering  $E \subseteq \bigcup_{i \in \mathbb{N}} Q_i$  such that  $\sum_{i \in \mathbb{N}} |Q_i| \leq m_*(E) + \varepsilon$ .
- We would not want to merely require coverings by *finite* collections of closed cubes. (See challenge problem and *Jordan content* of sets)

Examples: If  $E$  is countable, then  $m_*(E) = 0$ .

This follows from the fact that any point is a closed cube with zero volume

Example: If  $E \subset \mathbb{R}$ , then  $E$  is null  $\iff m_*(E) = 0$ .  $\implies$  : We can cover by open intervals with lengths summing to  $< \varepsilon$ , so just close them (which doesn't increase the length).

$\impliedby$  : Easy exercise. Increase the length of the  $n$ th open interval by  $\varepsilon/2^n$ .

Example: If  $Q$  is a closed cube, the  $m_*(Q) = |Q|$ , the usual volume.

Since  $Q \subseteq Q$ ,  $Q$  covers itself and  $m_*(Q) \leq |Q|$ . For the other direction, fix  $\varepsilon > 0$ ; we will show  $|Q| \leq m_*(Q) + \varepsilon$  for every  $\varepsilon$ .

Let  $\{Q_j\}$  be an arbitrary covering of  $Q$  by closed cubes. Idea: enlarge the cubes a bit so they're open, and use compactness to get a finite subcover.



Let  $S_j$  denote an open cube with the property that  $Q_j \subseteq S_j$  and

$$|Q_j| \leq |S_j| \leq (1 + \varepsilon)|Q_j|$$

Since  $Q$  is compact, there is a finite  $N$  such that  $E \subseteq \bigcup_{j=1}^N S_j$ , and the claim is that  $|Q| \leq \sum_{j=1}^N |S_j|$  (Lemma 1.2, Stein).

Recall 1-dimensional setting, we did the same thing to prove that  $\mathbb{R}$  was not null.

We then have

$$|Q| \leq \sum_{j=1}^N |S_j| \leq (1 + \varepsilon) \sum_{j=1}^N |Q_j| \leq (1 + \varepsilon) \sum_{j=1}^{\infty} |Q_j|,$$

which is what we wanted to show.

Exercise: Let  $Q$  be open, show  $m_*(Q) = |Q|$ .

4.  $m_*(\mathbb{R}^n) = \infty$  This would follow if we could show that  $|Q| \leq m_*(\mathbb{R}^n)$  for any  $Q$ , and we can take  $Q$  to be arbitrarily large.

This is because any covering of  $\mathbb{R}^n$  is also a covering of  $Q$ .

Properties of Outer Measure

1. Monotonicity: If  $E_1 \subseteq E_2$  then  $m_*(E_1) \leq m_*(E_2)$ .
2. Countable Subadditivity: If  $E = \bigcup_{i \in \mathbb{N}} E_i$  for any countable union, then

$$m_*(\bigcup E_i) \leq \sum m_*(E_i)$$

3. If  $E \subseteq \mathbb{R}^n$  then  $\forall \varepsilon > 0$  there exists an *open* set  $G \supseteq E$  such that

$$m_*(E) \leq m_*(G) \leq m_*(E) + \varepsilon.$$

4. If  $E_1, E_2 \subseteq \mathbb{R}^n$  and

$$\text{dist}(E_1, E_2) := \inf_{x \in E_1, y \in E_2} |x - y| > 0 \implies$$

$$m_*(E_1 \bigcup E_2) = m_*(E_1) + m_*(E_2).$$

5. If  $E = \bigcup_{j \in \mathbb{N}} Q_j$  with the  $Q_j$  *almost disjoint*, then  $m_*(E) = \sum |Q_j|$ .

Remark: Property 4 does not hold in general if we merely assume that  $E_1 \cap E_2 = \emptyset$ . It *will* be true if we restrict the collection of sets we consider to be “measurable”, so any counterexample will have to involve a pathological set.

Warning: Any  $E_j$  could have  $m_*(E) = \infty$ , so we have to be careful with our assumptions and how we work with inequalities, particularly when subtracting measures.

Proof:

1. Obvious since any covering of  $E_2$  is also a covering of  $E_1$ . We're taking infimums over *large* collections of sets, so it could only get smaller.
2. If  $m_*(E_j) = \infty$  for any  $j$ , this is vacuous, so assume  $m_*(E_j) < \infty$  for every  $j$ . Let  $\varepsilon > 0$ . For each  $j$ , there exists a covering  $E_j \subseteq \bigcup_k Q_{j,k}$  where  $\sum_k |Q_{j,k}| \leq m_*(E_j) + \varepsilon/2^j$ .

But now  $E \subseteq \bigcup_{j,k} Q_{j,k}$ , so

$$m_*(E) \leq \sum_{j,k} |Q_{j,k}| = \sum_j \sum_k |Q_{j,k}| \leq \sum_j (m_*(E_j) + \varepsilon/2^j) = \varepsilon + \sum_j m_*(E_j).$$

3. Idea: enlarge open sets in a summable way?

Let  $\varepsilon > 0$ . Then there exists a covering  $E \subseteq \bigcup_{j \in \mathbb{N}} Q_j$  such that  $\sum |Q_j| \leq m_*(E) + \varepsilon/2$ . Let  $S_j$  be an open cube such that  $Q_j \subset S_j$  and  $|S_j| \leq |Q_j| + \varepsilon/2^{j+1}$ .

So define  $G = \bigcup S_j$ , which is open. Now using subadditivity, we have

$$m_*(G) \leq \sum_j |S_j| = \sum_j (|Q_j| + \varepsilon/2^{j+1}) \leq m_*(E) + \varepsilon.$$

4. Just need to show  $m_*(E_1 \cup E_2) \leq m_*(E_1) + m_*(E_2)$ , since the reverse direction follows from (2). Key idea: by subdividing cubes, we can assume that no cube intersects both sets.

Remark: It is possible to construct closed disjoint subsets of  $\mathbb{R}^n$  such that the distance between them is still zero. Take  $X = \mathbb{N}$  and  $Y = \{n + \frac{1}{2n} : n \in \mathbb{N}\}$ .