Title

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Last time:

- The Čech-to-derived spectral sequence,
- The Mayer Vietoris LES,
  - Computes the étale cohomology of a scheme using a Zariski open cover.
- Étale cohomology of quasicoherent sheaves,
  - Agrees with Zariski cohomology, first legitimate computation!
  - Use this to compute:
- Étale cohomology of $\underline{\mathbb{F}_p}$ in characteristic p.

Last time we had a scheme $X_{/\mathbb{F}_p}$ and the *Artin-Schreier* exact sequence of sheaves of $X_{\text{ét}}$:

$$0 \to \underline{\mathbb{F}_p} \to \mathcal{O}_X^{\text{ét}} \xrightarrow{t \mapsto t^p - t} \mathcal{O}_X^{\text{ét}} \to 0.$$

The map appearing here is referred to as the *Artin-Schreier* map f This works over arbitrary fields of characteristic p, with a modified definition replacing t^p.

Exercise 1.0.1 ***(?):*** Check that this is an additive homomorphism of abelian sheaves. This follows from the fact that Frobenius itself is.

Remark 1.0.2: From here onward, H^i will denote $H^i_{\text{ét}}$.

Recall that we had a theorem last time showing that the étale cohomology of quasicoherent sheaves is equivalent to the usual Zariski cohomology. From this we got a long exact sequence:

$$\begin{tikzcd}
H^i(X_{\text{ét}}, \underline{\mathbb{F}_p}) \arrow[r] & H^i(X, \mathcal{O}_X) \arrow[r, "f"] & H^i(X, \mathcal{O}_X) \\
\cdots \arrow[r] & H^{i-1}(X, \mathcal{O}_X) \arrow[ul, "\delta"] &
\end{tikzcd}$$

We don't know how to compute $H^i(X_{\text{ét}}, \mathbb{F}_p)$ generally, but the affine case is easy. For X affine, $H^{>0}(X, \mathcal{O}_X) = 0$, which in facts holds for any quasicoherent sheave replacing $\mathcal{O}_X$, and $H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|}$ where the exponent is the number of connected components of X. So we get an exact sequence

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[r] & 0 & \\
H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|} \arrow[r] & \mathcal{O}_X(X) \arrow[r, "f"'] & \mathcal{O}_X(X) \arrow[ull] \\
& & 0 \arrow[ull]
\end{tikzcd}$$

Remark 1.0.3: $H^1(X, \mathcal{O}_X)$ is not finitely generated in general, e.g. take $X \coloneqq \mathbb{A}^1$, then $\operatorname{coker}(t \mapsto t^p - t)$ as a map $k[t] \to k[t]$ is generally finite dimensional as a k-vector space. So in characteristic p, cohomology with $\mathbb{F}_p$ coefficients is ill-behaved: a nice cohomology theory would assign to every scheme a complex of finite dimensional vector spaces.

Remark 1.0.4: An aside: $\mathbb{G}_a$ is the representing object for $\mathcal{O}_X^{\text{ét}}$.

Remark 1.0.5: If X is proper, $H^i(X_{\text{ét}}, \mathbb{F}_p)$ is finite dimensional. Why? It follows from the exact sequence: by proper pushforward for coherent cohomology, the terms we're interested in are sandwiched between finite dimensional objects. However, these groups still won't have the expected dimension.