Homological Algebra Problem Sets

Problem Set 3

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Problem 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.
$\implies$: Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
 & A &
\end{tikzcd}$$

Link to Diagram

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
 & A &
\end{tikzcd}$$

Link to Diagram

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R-linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned}
\tilde{f} : R &\to A \\
1_R &\mapsto \mathbf{a}',
\end{aligned}$$

and extending R-linearly yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

$\Longleftarrow$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\
& A &
\end{tikzcd}$$

Link to Diagram

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\to R \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f_a"] & R \arrow[dl, dashed, "\exists \tilde{f}_a"] \\
& A &
\end{tikzcd}$$

Link to Diagram

We can now use the fact that

$$\begin{aligned}
r\mathbf{a} &= f_a(jr) \\
&= \tilde{f}_a(\iota(jr)) \\
&= \tilde{f}_a(jr) \\
&= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\
&= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\
\implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA,
\end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting $\mathbf{a}$ range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).
Note that there is an injection

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \to \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \to 1$$

where π_m is the canonical quotient morphism, and applying the left-exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any $\mathbb{Z}$-module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "f"] \arrow[d, "\pi_m"'] & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/n \arrow[ur, dashed, "\exists \tilde{f}"'] &
\end{tikzcd}$$

Link to Diagram

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0, 1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n} \right\}.$$

Setting $f_i(1) := \left[\frac{i}{n} \right]$ (where we take the equivalence class mod $\mathbb{Z}$) yields n distinct morphisms $f_i : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\begin{aligned} \Psi : \mathbb{Z} &\to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i, \end{aligned}$$

and using the fact that if $i = i' \pmod n$, write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left[\frac{i+kn}{n}\right] = \left[\frac{i}{n} + k\right] = \left[\frac{i}{n}\right] = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

Claim 2: $\mathbb{Q}/\mathbb{Z}$ is an injective object in $\mathbb{Z}$-modules.

Proof (?).
By the previous exercise, it suffices to show that $\mathbb{Q}/\mathbb{Z}$ is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That $\mathbb{Q}$ is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b := \frac{a}{n} \in \mathbb{Q}$ works. Since $\mathbb{Q}$ is an abelian group, $\mathbb{Z}$ is automatically normal, and the result follows.

■

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \qquad\qquad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).
Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\textbf{-Mod},$$

and taking

- $M := \mathbb{Z}/m$,
- $A := \mathbb{Z}$,
- $I := n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n\,(\mathbb{Z}/m)} \qquad\qquad \in \mathbb{Z}\textbf{-Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\to \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
\ker \varphi &= \left\{ x \in \mathbb{Z} \;\middle|\; x(1 \otimes 1) = 0 \right\} \\
&= \left\{ x \in \mathbb{Z} \;\middle|\; n \;\middle|\; x \text{ or } m \;\middle|\; x \right\} \\
&= \langle n, m \rangle \\
&= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
&\coloneqq \langle d \rangle \,.
\end{aligned}$$

Now applying the first isomorphism theorem yields the result. ■

Solution:
We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\,\cdot\,) \coloneqq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \,\cdot\,)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of $\mathbb{Z}/q$.
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) \coloneqq R^i F(\mathbb{Z}/q) \coloneqq H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/q \arrow[r, "d^{-1}"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^0"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
& {[1]_q} \arrow[r] & \left[\frac{1}{q}\right] & & \\
& & {[x]} \arrow[r] & {[qx]} &
\end{tikzcd}$$

Link to Diagram

This is a chain complex by construction, since $d^2([1]_q) = \left[q \left(\frac{1}{q} \right) \right] = [1] = [0]$. We now delete the augmentation and apply $F(\,\cdot\,)$:

$$\begin{tikzcd}
1 \arrow[r] & I^0 := \mathbb{Q}/\mathbb{Z} \arrow[rr, "d^0"] & & I^1 := \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
& & {} \arrow[d, Rightarrow, "F(\,\cdot\,)"] & & \\
1 \arrow[r] \arrow[d, equal] & F(I^0) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[rr, "\partial^0 := F(d^0)"] & & F(I^1) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial^1 := F(d^1)"] & 1 \arrow[d, equal] \\
1 \arrow[r] & \mathbb{Z}/p \arrow[u, "\Psi \cong"] \arrow[rr, "\tilde{\partial}^0"] & & \mathbb{Z}/p \arrow[u, "\Psi \cong"] \arrow[r, "\tilde{\partial}^1"] & 1
\end{tikzcd}$$

Link to Diagram

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q, we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q.
We now take homology:

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^1F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\operatorname{im} \partial^0} = \frac{\mathbb{Z}/p}{q\,(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)
Let $A \in \mathbf{Ab}$, and show that the following map is injective:

$$\begin{aligned} \varepsilon_A : A \to I(A) &:= \prod_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\ a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

i.e. when looking at the image $\varepsilon_A(a)$ in the product, the component indexed by f is an element of $\mathbb{Q}/\mathbb{Z}$ obtained by evaluating $f(a)$.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Solution:
By contrapositive, we'll suppose $a \neq 0$ and show $\varepsilon_A(a) \neq 0$. Following the hint, we first consider the cyclic subgroup $a\mathbb{Z} = \left\{ an \;\middle|\; n \in \mathbb{Z} \right\}$ and define a map

$$\begin{aligned} f_a : a\mathbb{Z} &\to \mathbb{Z} \\ an &\mapsto n. \end{aligned}$$

We now pick $\ell > 1 \in \mathbb{Z}$ to be any integer, and define a composition $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$:

$$\begin{tikzcd}
a\mathbb{Z} \arrow[r, "f_a"] \arrow[rrrr, dashed, bend left, "f"] & \mathbb{Z} \arrow[r, hook, "\iota"] & \mathbb{Q} \arrow[r, "x \mapsto \frac{x}{\ell}"] & \mathbb{Q} \arrow[r, "\pi"] & \mathbb{Q}/\mathbb{Z} \\
an \arrow[r, mapsto] & n \arrow[r, mapsto] & n \arrow[r, mapsto] & \frac{n}{\ell} \arrow[r, mapsto] & \left[\frac{n}{\ell}\right] \\
a \arrow[rrrr, dashed] & & & & \left[\frac{1}{\ell}\right]
\end{tikzcd}$$

Link to Diagram

By choice of ℓ, this map satisfies $f(a) = [1/\ell] \neq 0$, so the map is nonzero. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the universal property provides a lift $\tilde{f}$:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists \tilde{f}"] & \\
a\mathbb{Z} \arrow[u, hook] \arrow[r, "f"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Link to Diagram

Since $\tilde{f}$ lifts f, it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) := f(a) \neq 0,$$

so the f component of the image of a is nonzero and thus $\mathbf{a} := \varepsilon_A(a) \neq 0$ in the product.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Solution:
We'll show that $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$ for every object X. Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let $P \to X$ be a projective resolution of X. Fixing labeling, we have the following situation:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, Rightarrow, "F(\,\cdot\,)"] & P_0 \arrow[r, "\varepsilon"] & X \arrow[r, "0"] & 0 \\
\cdots \arrow[r] & FP_2 \arrow[r, "F(\partial_2)"'] & FP_1 \arrow[r, "F(\partial_1)"'] & FP_0 \arrow[r, "0"'] & 0 &
\end{tikzcd}$$

Link to Diagram

We now have by definition

$$L_iF(X) \coloneqq \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U\left(L_iF(X)\right) \coloneqq U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution $P \to X$, and apply a similar process:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, Rightarrow, "(U\circ F)(\,\cdot\,)"] & X \arrow[r, "0"] & 0 \\
\cdots \arrow[r] & UFP_2 \arrow[r, "(UF)(\partial_2)"'] & UFP_1 \arrow[r, "(UF)(\partial_1)"'] & UFP_0 \arrow[r, "0"'] & 0 &
\end{tikzcd}$$

Link to Diagram

Again, by definition,

$$(L_i(UF))(X) \coloneqq \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor U to the following SES to produce a new SES, from which we'll produce the desired isomorphism f:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} F(\partial_{i+1}) \arrow[r, "\iota_i"] & \ker F(\partial_i) \arrow[r, "\pi_i"] \arrow[d, Rightarrow, "U"] & \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \arrow[r] & 0 \\
0 \arrow[r] & U\left(\operatorname{im} F(\partial_{i+1})\right) \arrow[r, "U(\iota_i)"] \arrow[d, equal] & U\left(\ker F(\partial_i)\right) \arrow[r, "U(\pi_i)"] \arrow[d, equal] & U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & U\left(\operatorname{im} F(\partial_{i+1})\right) \arrow[r, "U(\iota_i)"] & U\left(\ker F(\partial_i)\right) \arrow[r, "\tilde{\pi}_i"] & \frac{U(\ker F(\partial_i))}{U(\operatorname{im} F(\partial_{i+1}))} \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here $\tilde{\pi}_i$ is the natural quotient map, whose image is $\operatorname{coker} U(\iota_i)$. Finally, the map f exists an any abelian category, using that whenever $0 \to A \xrightarrow{g_1} B \xrightarrow{g_2} C \to 0$ is exact, there is an isomorphism $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$.

Problem 1.0.5 (Weibel 2.4.3)

- If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} FM \qquad i \geq 2.$$

- Show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with P_i projective or F-acyclic, then

$$L_i F(A) \cong L_{i-m-1} F(M_m) \qquad i \geq m+2.$$

  - Moreover show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$.

- Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Solution:

Claim:

$$L_i F(A) \cong L_{i-1} FM \qquad i \geq 2.$$

Proof (of claim).
Following the proof of Weibel Theorem 2.4.6, let $P_M \to M$ and $P_A \to A$ be projective resolutions of M and A respectively. Then applying the Horseshoe Lemma, there is a projective resolution $P_P \to P$ of P such that the following is a short exact sequence of chain complexes:

$$0 \to P_M \to P_P \to P_A \to 0,$$

where in fact in each degree n piece, this is induces a *split* exact sequence. Using that F is additive and additive functors preserve split exact sequences, the following is a SES for every n:

$$0 \to FP_M^n \to FP_P^n \to FP_A^n \to 0,$$

which implies that there is a SES of chain complexes

$$0 \to FP_M \to FP_P \to FP_A \to 0.$$

Thus there is an associated LES of derived functors:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial_{i+1}"] \\
L_iFM \arrow[r] & \color{red}{L_iFP} \arrow[r] & L_iFA \arrow[dll, "\partial_i"] \\
L_{i-1}FM \arrow[r] & \color{red}{L_{i-1}FP} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Using that P is F-acyclic, the middle terms $L_iFP = 0$ for all $i > 0$, and thus this splits into a collection of SESs:

$$0 \to L_2FA \xrightarrow{\partial_2} L_1FM \to 0$$
$$0 \to L_3FA \xrightarrow{\partial_3} L_2FM \to 0$$
$$\vdots$$
$$0 \to L_iFA \xrightarrow{\partial_3} L_{i-1}FM \to 0.$$

This makes every ∂_i for $i \geq 2$ an isomorphism. ■

Claim:

$$L_iFA \cong \ker(FM \to FP).$$

Proof (?).
Using the same argument as above, consider the lower order terms of the associated LES:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial_2"'] \\
L_1FM \arrow[r] & L_1FP = 0 \arrow[r] & L_1FA \arrow[dll, "\partial_1"'] \\
L_0FM \arrow[r] & L_0FP \arrow[r] & L_0FA \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Noting that $L_1FP = 0$ by F-acyclicity, the highlighted portion forms a four term exact sequence. We can form another exact sequence and compare the two:

$$\begin{tikzcd}
0 \arrow[r] & L_1FA \arrow[r, "\partial_1"] \arrow[d, dashed] & L_0FM \arrow[r] \arrow[d, equal] & L_0FP \arrow[r] \arrow[d, equal] & L_0FA \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(L_0FM \to L_0FP) \arrow[r] & L_0FM \arrow[r] & L_0FP \arrow[r] & L_0FA \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

That the map indicated by the dotted line exists and is an isomorphism holds in any abelian category, using that fact that whenever $0 \to A \to B \xrightarrow{f} C \to 0$ is a SES we have $A \cong \ker f$.

■

Claim: If $P \to A$ is an F-acyclic resolution of A, then there is an isomorphism

$$L_iFA \cong H_i(FP).$$

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

a. A is a projective R-module.

b. $\mathrm{Hom}_R(A, \cdot\,)$ is an exact functor.

c. $\mathrm{Ext}_R^{i \geq 1}(A, B) = 0$ and for all B, i.e. A is $\mathrm{Hom}_R(\,\cdot\,, B)$-acyclic for all B.

d. $\mathrm{Ext}_R^1(A, B)$ vanishes for all B.

Solution:
We'll show

- $a \iff b$
- $b \implies c$
- $c \iff d$:
- $d \implies b$

Proof ($a \iff b$).
Let ξ be the following SES:

$$\xi : 0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

and define the functor $F(\,\cdot\,) \coloneqq \mathrm{Hom}_R(A, \,\cdot\,)$. This is a covariant left-exact functor, and so applying it to the above sequence yields

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] \arrow[d, Rightarrow, "F(\,\cdot\,)\coloneqq\mathrm{Hom}_R(A,\,\cdot\,)"] & M_3 \arrow[r] & 0 \\
0 \arrow[r] & FM_1 \arrow[r, "F(f):\lambda\mapsto f\circ\lambda"'] & FM_2 \arrow[r, "F(g):\lambda\mapsto g\circ\lambda"'] & FM_3 &
\end{tikzcd}$$

Link to Diagram

$\implies$:
For F to be exact, it suffices to show it is right-exact, i.e. that $F(g)$ is surjective. This amounts to asking that every $\varphi \in FM_3 \coloneqq \mathrm{Hom}_R(A, M_3)$ lifts to a preimage $\tilde{\varphi} \in FM_2 \coloneqq \mathrm{Hom}_R(A, M_2)$ satisfying $F(g)(\tilde{\varphi}) = \varphi$. Unwinding definitions, this requires that $g \circ \tilde{\varphi} = \varphi$, which is precisely the lift required for the universal property of projective objects:

$$\begin{tikzcd}
& A \arrow[dl, dashed, "\exists\tilde{\varphi}"'] \arrow[d, "\varphi"] & \\
M_2 \arrow[r, "g"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

If A is projective, this lift always exists, so $\mathrm{Hom}_R(A, \,\cdot\,)$ is an exact functor. Conversely, if $\mathrm{Hom}_R(A, \,\cdot\,)$ is exact, this lift always exists, so A satisfies the universal property of a projective object.

■

Proof ($b \implies c$).
Suppose $F(\,\cdot\,) \coloneqq \mathrm{Hom}_R(A, \,\cdot\,)$ is exact, then since F is left-exact covariant it has right-derived functors $\mathrm{Ext}_R^i(A, B) \coloneqq R^i F(B)$ which are computed in the following way

1. Taking an *injective* resolution of

$$B \to I \coloneqq (I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots).$$

2. Applying $\mathrm{Hom}_R(A, \cdot\,)$ to get the complex

$$FI \coloneqq (0 \to \mathrm{Hom}_R(A, I_0) \xrightarrow{F(\partial_0)} \mathrm{Hom}_R(A, I_1) \xrightarrow{F(\partial_1)} \cdots).$$

3. Defining

$$R^i F(B) \coloneqq \ker F(\partial_i) / \operatorname{im} F(\partial_{i-1}).$$

Note that in step (2), if $\mathrm{Hom}_R(A, \cdot\,)$ is an exact functor, then since I is an acyclic complex, FI is again acyclic and so $\ker F(\partial_i) = \operatorname{im} F(\partial_{i-1}) = 0$ for $i \geq 1$. So

$$\mathrm{Ext}_R^{\geq 1}(A, B) \coloneqq R^{\geq 1} F(B) = 0.$$

■

Proof (c $\iff$ d).
$\implies$: This direction is clear, since if $\mathrm{Ext}_R^i(A, B) = 0$ for all B, then taking $i = 1$ is the statement of (d).

$\impliedby$: This follows from the dimension-shifting isomorphism in a previous exercise. Let $F(\,\cdot\,) \coloneqq \mathrm{Hom}_R(A, \cdot\,)$ and suppose $\mathrm{Ext}_R^1(A, B) \coloneqq L_1 F(B) = 0$ for all B. Let B' be arbitrary, it then suffices to show that $\mathrm{Ext}^i(A, B') \coloneqq L_i(B') = 0$ for all $i > 1$, since we can take B' as one such B in the assumption for the $i = 1$ case.
The dimension shifting results states that if P_i are F-acyclic, then for every exact sequence

$$0 \to M_m \to P_m \to \cdots \to P_0 \to B' \to 0$$

we obtain an isomorphism

$$L_i F(B') \cong L_{i-m-1} F(M_m) \iff L_i F(M_m) \cong L_{i+m+1} F(B').$$

So take any F-acyclic resolution of P, say

$$B' \xrightarrow{\partial_{-1}} I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots,$$

then consider truncating it at the mth stage:

$$0 \to B' \xrightarrow{\partial_{i-1}} I_0 \to I_1 \to \cdots \xrightarrow{\partial_{m-1}} I_m \to M_m \coloneqq \operatorname{coker} \partial_{m-1} \to 0$$

By assumption, we have $L_1F(M_m) = 0$ for every m, and thus

$$\begin{aligned}
0 &= L_1F(B') \quad \text{by assumption} \\
0 &= L_1F(M_0) \cong L_2F(B') \\
0 &= L_1F(M_1) \cong L_3F(B') \\
0 &= L_1F(M_2) \cong L_4F(B') \\
&\vdots \\
0 &= L_1(M_m) \cong L_{m+2}(B') \quad \forall m \geq 0.
\end{aligned}$$

and so $L_i(B') = 0$ for all $i \geq 1$.

■

Proof ($d \implies b$).
Take an arbitrary SES

$$\xi : 0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$$

and consider applying the left-exact covariant functor $F(\,\cdot\,) := \mathrm{Hom}_R(A, \,\cdot\,)$ and taking the associated LES:

$$\begin{tikzcd}
0 \arrow[d] \\
\mathrm{Hom}_R(A, B') \arrow[r, "f^*"] & \mathrm{Hom}_R(A, B) \arrow[r, "g^*"] & \mathrm{Hom}_R(A, B'') \arrow[dll, "\delta_0"] \\
{\color{red}\mathrm{Ext}^1_R(A, B')} \arrow[r] & \mathrm{Ext}^1_R(A, B) \arrow[r] & \mathrm{Ext}^1_R(A, B'') \arrow[dll, "\delta_1"] \\
\cdots
\end{tikzcd}$$

Link to Diagram

By assumption, all of the higher Ext terms vanish, and in particular the red term $\mathrm{Ext}^1_R(A, B') = 0$. This implies that g^* is surjective, making the following sequence exact:

$$0 \to \mathrm{Hom}_R(A, B') \xrightarrow{f^*} \mathrm{Hom}_R(A, B) \xrightarrow{g^*} \mathrm{Hom}_R(A, B'') \to 0,$$

making $\mathrm{Hom}_R(A, \,\cdot\,)$ an exact functor.

■

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is

abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on $\mathbf{Ab}$.

Solution:
Fixing some index category $\mathcal{I}$ and a functor $F : \mathcal{I} \to \mathcal{A}$, so $F \in \mathcal{A}^{\mathcal{I}}$, write $\widehat{A} \coloneqq \operatorname{colim}_{i \in I} F(i)$. We want to show that $\mathcal{A}^{\mathcal{I}} \underset{\Delta}{\overset{\operatorname{colim}}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}$ defines an adjoint pair, so that colim is a left-adjoint and Δ is a right-adjoint. By definition, this is equivalent to showing the existence of natural bijections of sets

$$\tau_{FX} : \operatorname{Hom}_{\mathcal{A}}(\widehat{F}, X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \quad \forall X \in \mathcal{A}, F \in \mathcal{A}^{\mathcal{I}}.$$

We first note that the data of $\widehat{F}$ is equivalent to the following universal property:

$$\begin{tikzcd}
\mathcal{I} & & \mathcal{A} & & & \\
i \arrow[dd, "\varepsilon_{ij}"'] & & F(i) \arrow[dd, "F\varepsilon_{ij}"'] \arrow[dr, "\psi_i"] \arrow[drrr, dotted, bend left, "\varphi_i"] & & & \\
 & {} \arrow[r, Rightarrow, "F(\cdot)"'] & {} & \widehat{F} \arrow[rr, dashed, "\exists ! \eta_{ij}"] & & X \\
j & & F(j) \arrow[ur, "\psi_j"'] \arrow[urrr, dotted, bend right, "\varphi_j"'] & & &
\end{tikzcd}$$

Link to Diagram

That is, $\widehat{F} \in \operatorname{Ob}(\mathcal{A})$ is an object equipped with structure maps ψ_i for every object $F(i)$ in the image of F such that the solid triangle commutes, and for any object X with maps $\varphi_i : F(i) \to X, \varphi_j : F(j) \to X$ making the outer triangle commute, there is a unique map $\eta_{ij} : \widehat{F} \to X$ making the entire diagram commute. We can rewrite this condition in a more suggestive way:

$$\begin{tikzcd}
\mathcal{I} & & \mathcal{A} & & \\
i \arrow[d, "\varepsilon_{ij}"'] & & F(i) \arrow[d, "F\varepsilon_{ij}"'] \arrow[r, "\psi_i"] \arrow[rr, dotted, bend left, "\varphi_i"] & \widehat{F} \arrow[r, dotted, "\exists \eta_i"] \arrow[d, dash, "\mathbb{1}_{\widehat{F}}"] & X \arrow[d, dash, "\mathbb{1}_X"] \\
j \arrow[r, phantom, "\overset{}{\Longrightarrow}_{F(\cdot)}"] & & F(j) \arrow[r, "\psi_j"'] \arrow[rr, dotted, bend right, "\varphi_j"'] & \widehat{F} \arrow[r, dashed, "\exists \eta_j"] & X
\end{tikzcd}$$

Link to Diagram

Applying the Δ functor, we can view this as a simpler universal property in $\mathcal{A}^{\mathcal{I}}$, since the above data is precisely the data of a natural transformation:

$$\begin{tikzcd} & \Delta \widehat{F} \\ F \arrow[ur, "\Psi"] \arrow[r, "\Phi"'] & \Delta X \arrow[u, dashed, "\exists ! \eta"'] \end{tikzcd}$$

Link to Diagram

That is, the functor $\Delta \widehat{F}$ is equipped with structure maps $\Phi : F \to \Delta \widehat{F}$ which assemble into a natural transformation (i.e. a morphism in $\mathcal{A}^{\mathcal{I}}$) such that any other natural transformation from F to a diagonal object ΔX produces a unique natural transformation $\eta : \Delta X \to \Delta F$. This provides exactly the data needed to specify τ:

$$\begin{aligned} \tau_{FX} : \operatorname{Hom}_{\mathcal{A}}(\widehat{F}, X) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \\ \left(\widehat{F} \xrightarrow{f} X\right) &\mapsto \left(F \xrightarrow{\Psi} \Delta \widehat{F} \xrightarrow{\Delta(f)} \Delta X\right), \end{aligned}$$

i.e. we take the image $\Delta(f)$ and pre-compose with the structure morphism Ψ.

Claim: This is a bijection of sets.

Proof (?).
This is surjective by the universal property: any morphism $F \xrightarrow{g} \Delta X$ in $\mathcal{A}^{\mathcal{I}}$ factors through $\Delta \widehat{F}$, and so all such morphisms are of this form.
That this is injective: ???

■

Claim: This is a natural isomorphism, i.e. for all $X \xrightarrow{f} Y \in \operatorname{Mor}(\mathcal{A})$ and all $F \xrightarrow{\eta} G \in \operatorname{Mor}(\mathcal{A}^{\mathcal{I}})$, there is a commuting diagram

$$\begin{tikzcd} \operatorname{Hom}(\widehat{G}, X) \arrow[r, "\widehat{\eta}_*"] \arrow[d, "\tau_{GX}"] & \operatorname{Hom}(\widehat{F}, X) \arrow[r, "f_*"] \arrow[d, "\tau_{FX}"] & \operatorname{Hom}(\widehat{F}, Y) \arrow[d, "\tau_{FY}"] \\ \operatorname{Hom}(G, \Delta X) \arrow[r, "\eta_*"] & \operatorname{Hom}(F, \Delta X) \arrow[r, "(\Delta f)_*"] & \operatorname{Hom}(F, \Delta Y) \end{tikzcd}$$

Link to Diagram

Proof (?).
Todo.

■

Claim: If $\mathcal{A}$ is abelian and $\mathcal{I}$ is an index category such that $\operatorname{colim}_{i \in \mathcal{I}} F(i)$ exists for all $F \in \mathcal{A}^{\mathcal{I}}$, then the functor $\operatorname{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ is right-exact.

Proof (Sketch).
A sketch of the proof proceeds by showing every right adjoint is left-exact:

- Since $\mathrm{Hom}(LA, \cdot)$ is left-exact, we can apply it to a SES $0 \to B' \to B \to B'' \to 0$.
- Applying the natural isomorphisms coming from the adjunction, this is isomorphic to a sequence involving terms $\mathrm{Hom}(\cdot, RB)$.
- This sequence is exact, so applying Yoneda yields an exact sequence

  $$0 \to RB' \to RB \to RB'',$$

  making R left-exact.

Finally, if L is a left adjoint out of $\mathcal{A}$, then L^{op} is a right adjoint out of $\mathcal{A}^{\mathrm{op}}$. Thus L^{op} is left-exact by the above argument, making L right-exact. ∎

Claim: Let $\mathcal{I} \coloneqq (\bullet \leftarrow \bullet \rightarrow \bullet)$ and define the pushout as $\operatorname{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$. Then taking $\mathcal{A} \coloneqq \mathbf{Ab}$, the pushout does not define an exact functor $\mathcal{A}^{\mathcal{I}} \to \mathcal{A}$.

Proof (?).
We proceed by constructing a counterexample. Unwinding definitions, we first note that an exact sequence of objects in $\mathcal{A}^{\mathcal{I}}$ corresponds precisely to an exact sequence of diagrams. For pushouts, writing X_i for the pushout of $A_i \hookleftarrow P_i \hookrightarrow B_i$, this gives an exact sequence of diagrams which in turn corresponds to an exact sequence of the pushout objects X_i:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & & & 0 \arrow[dr] & & & & \\
0 \arrow[dr] & P_1 \arrow[r] \arrow[d] \arrow[dr] & B_1 \arrow[dr] & & & X_1 \arrow[dr] & & & \\
& A_1 \arrow[dr] & P_2 \arrow[r] \arrow[d] \arrow[dr] & B_2 \arrow[dr] & {} \arrow[r, Rightarrow] & {} & X_2 \arrow[dr] & & \\
& & A_2 \arrow[dr] & P_3 \arrow[r] \arrow[d] \arrow[dr] & B_3 \arrow[dr] & & & X_3 \arrow[dr] & \\
& & & A_3 \arrow[dr] & 0 & 0 & & & 0 \\
& & & & 0 & & & &
\end{tikzcd}
$$

Link to Diagram

If we let $f_i : P_i \to B_i$ be arbitrary maps between abelian groups and push out along $A_i = 0$, we recover to cokernels of the f_i:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & & & 0 \arrow[dr] & & & & \\
0 \arrow[dr] & P_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr] & B_1 \arrow[dr] & & & \operatorname{coker} f_1 \arrow[dr] & & & \\
& 0 \arrow[dr] & P_2 \arrow[r, "f_2"] \arrow[d] \arrow[dr] & B_2 \arrow[dr] & {} \arrow[r, Rightarrow] & {} & \operatorname{coker} f_2 \arrow[dr] & & \\
& & 0 \arrow[dr] & P_3 \arrow[r, "f_3"] \arrow[d] \arrow[dr] & B_3 \arrow[dr] & & & \operatorname{coker} f_3 \arrow[dr] & \\
& & & 0 \arrow[dr] & 0 & 0 & & & 0 \\
& & & & 0 & & & &
\end{tikzcd}
$$

Link to Diagram

However, the sequence of cokernels appearing on the right is not exact in general, since this precisely fits into the diagram used in the snake lemma:

$$
\begin{tikzcd}
0 \arrow[r] & \ker f_1 \arrow[r] & \ker f_2 \arrow[r] & \ker f_3 & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_1 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & P_1 \arrow[r] & P_1 \arrow[r] & 0
\end{tikzcd}
$$