

Title

D. Zack Garza

November 26, 2019

Contents

1 Wednesday November 20	1
1.1 Weyl's Character Formula (24.2-3)	1

1 Wednesday November 20

Last time:

$$\begin{aligned}\mathbb{Z}\Lambda &\iff \{\mathfrak{h}^* \rightarrow \mathbb{Z}_{\geq 0} \mid \sim\} \\ e(\mu) &\mapsto e_\mu \\ e(\lambda)e(\mu) = e(\lambda + \mu) &\mapsto f \star g(\lambda) = \sum_{a+b=\lambda} f(a)g(b)\end{aligned}$$

and $\text{ch}L(\lambda) = \sum_{\mu \in \Lambda} \dim L(\lambda)_\mu e(\mu)$.

We have the Kostant function $p(\lambda) = \#\{(k_\alpha)_\alpha \mid -\lambda = \sum_{\alpha \in \Phi^+} k_\alpha \alpha\}$ and the Weyl function $q = e_\rho \star \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha}) = \prod_{\alpha \in \Phi^+} (e_{\alpha/2} - e_{-\alpha/2})$.

Lemma: $p \star e_\lambda = \text{ch}M(\lambda)$, so $q \star \text{ch}M(\lambda) = e_{\lambda+\rho}$ and $q \star p = e_\rho$.

1.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of W is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, i.e. a reflection for hyperplanes passing through $-\rho$.

E.g. for type A_2 , where $W(0) = 0$, we have:

Type A_2

And for the dot action, we have

Image

where $W \cdot 0 = 0$ and $s(\alpha_1) = -\alpha_1$.

Theorem (Harish-Chandra): If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$ for $\mu \leq \lambda$.

Proof: Postponed.

ch are characters, $L(\lambda)$ is a Verma module.

Remark: if we sum over $\mu \leq \lambda$, we obtain

$$\begin{aligned} \text{ch}M(\lambda) &= \sum_{\mu \in W \cdot \lambda} a_{\lambda\mu} \text{ch}L(\mu) \\ \text{ch}L(\lambda) &= \sum_{\mu \in W \cdot \lambda} b_{\lambda\mu} \text{ch}M(\mu) \\ &= \sum_{W \cdot \lambda \in \Lambda} c_{\lambda W} \text{ch}M(w \cdot \lambda). \end{aligned}$$

Theorem (Weyl's Character Formula): If $\lambda \in \Lambda^+$, then

$$\text{ch}L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)}$$

Proof:

We have $\text{ch}L(\lambda) = \sum_w c_{\lambda w} \text{ch}M(w \cdot \lambda)$, and so by the lemma,

$$q * \text{ch}L(\lambda) = \sum c_{\lambda w} q * \text{ch}M(W(\lambda + \rho) - \rho) = \sum_w c_{\lambda w} e_{W(\lambda + \rho)}$$

Thus for all $\alpha \in \Phi^+$, we have

$$s_\alpha(q * \text{ch}L(\lambda)) = \sum_w c_{\lambda, s_\alpha w} e_{w(\lambda + \rho)}$$