Title

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1 | Thursday, November 19

Why use projective varieties? For e.g. a manifold, there is a well-defined intersection pairing, and the same way that $[\mu] \in H^1(T, \mathbb{Z}) = 1$ in the torus, we have $[L]^2 = 1$ in $\mathbb{P}^2_{/\mathbb{C}}$, so every two lines intersect in a unique point. Also, Bezout's theorem: any two curves of degrees d, e in projective space intersect in $d \cdot e$ points. Also note that we have a notion of compactness that works in the projective setting but not for affine varieties.

Last time: we saw the Segre embedding $(\mathbf{x}, \mathbf{y}) \mapsto [x_i y_j]$, which was an isomorphism onto its image $X = V(z_{ij} z_{kl} - z_{ik} z_{kj})$, which exhibits $\mathbb{P}^n \times \mathbb{P}^m$ as a projective variety.

Example 1.0.1(?): For $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, its image is $X = V_p(xy - zw)$, which is a quadric (vanishing locus of a degree 4 polynomial).

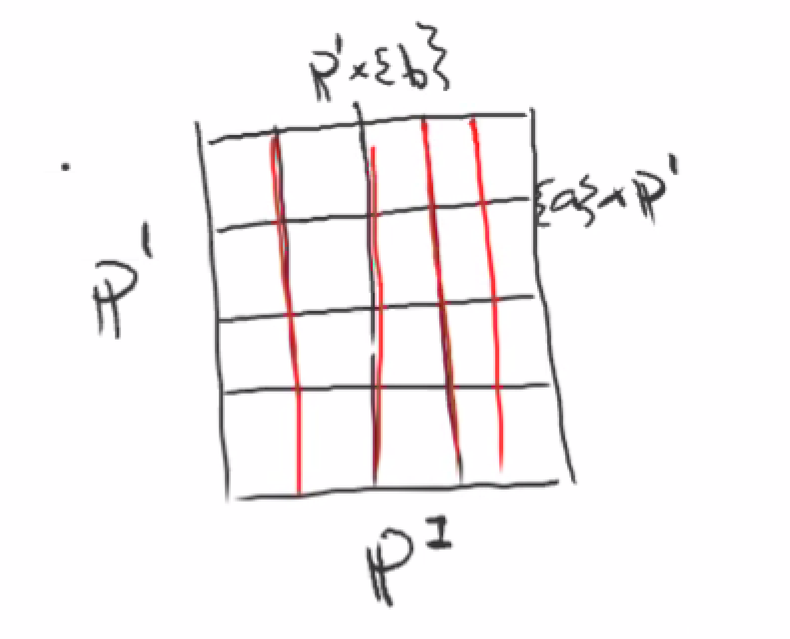

Figure 1: Image

The projection map has fibers, which induce a *ruling* (a family of $\mathbb{P}^1$s), which we can see from the real points:

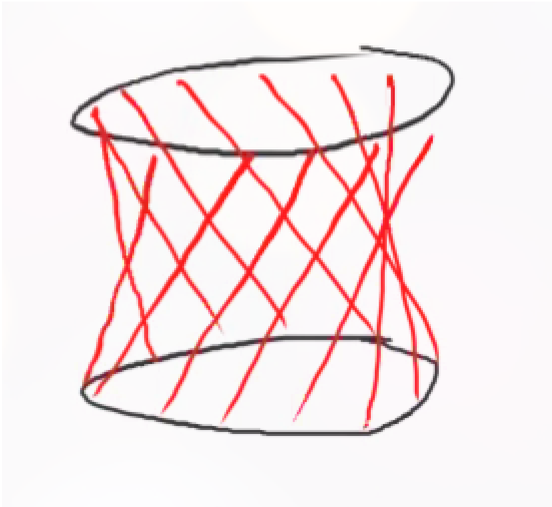

Figure 2: Image

Corollary 1.0.1 *(?).*
Every projective variety is a separated prevariety, and thus a variety.

Proof (?).
It suffices to show that $\Delta_X \subset X \times X$ is closed. We can write

$$\Delta_{\mathbb{P}^n} = \left\{ [x_0 : \cdots : x_n], [y_0 : \cdots : y_n] \;\middle|\; x_i y_j - x_j y_i = 0 \, \forall i, j \right\}.$$

This says that $\mathbf{x}, \mathbf{y}$ differ by scaling. We know that $\Delta_{\mathbb{P}^n} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^n$, which is isomorphic to the Segre variety S_V in $\mathbb{P}^{(n+1)^2 - 1}$, and we can write $z_{ij} = x_i y_j$ and thus

$$\Delta_{\mathbb{P}^n} = S_V \cap V(z_{ij} - z_{ji}).$$

Note that the Segre variety is closed.
The conclusion is that $\mathbb{P}^n$ is a variety, and any closed subprevariety of a variety is also a variety by taking $\Delta_{\mathbb{P}^n} \cap (X \times X) = \Delta_X$, which is closed as the intersection of two closed subsets. ■

Definition 1.0.1 (Closed Maps)
Recall that a map $f : X \to Y$ is topological spaces is **closed** if whenever $U \subset X$ is closed, then $f(U)$ is closed in Y.

Definition 1.0.2 (Complete Varieties)
A variety X is **complete** if the projection $\pi_Y : X \times Y \twoheadrightarrow Y$ is a closed map for any Y.

Slogan: analog of compactness.

Proposition 1.0.1 *(?).*
The projection $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed.

Proof (?).
Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$, and write $Z = V(f_i)$ with $f_i \in S(S_V)$. Note that if the f_i are homogeneous of degree d in z_{ij}, the pulling back only the isomorphism $\mathbb{P}^n \times \mathbb{P}^m \to S_V$ yields $z_{ij} = x_i y_j$ and polynomials h_i which are homogeneous polynomials in x_i, y_j which have degree d in both the x and y variables individually. Consider $a \in \mathbb{P}^m$, we want to determine if $a \in \pi(Z)$ and show that this is a closed condition. Note that $a \notin \pi(Z)$

- $\iff$ there does not exists an $x \in \mathbb{P}^n$ such that $(x, a) \in Z$
- $\iff$ $V_p(f_i(x,a))_{i=1}^r = \emptyset$
- $\iff$ $\sqrt{\langle f_i(x,a)\rangle_{i=1}^r} = \langle 1 \rangle$ or the irrelevant ideal I_0
- $\iff$ there exist $k_i \in \mathbb{N}$ such that $x_i^{k_i} \in \langle f_i(x,a)\rangle_{i=1}^r$
- $\iff$ $k[x_1, \cdots, x_n]_k \subset \langle f_i(x,a)\rangle_{i=1}^r$ (where this is the degree k part)
- $\iff$ the map

$$\begin{aligned}\Phi_a : k[x_1, \cdots, x_n]_{d-\deg f_2} \oplus \cdots \oplus k[x_1, \cdots, x_n]_{d-\deg f_r} &\to k[x_1, \cdots, x_n]_d \\ (g_1, \cdots, g_r) &\mapsto \sum f_i(x,a) g_i(x,a)\end{aligned}$$

  is surjective.

Recap: we have a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$, want to know its projection is closed. We looked at points not in the closed set, this happens iff the degree d part of the polynomial is not contained in the part where we evaluate by a. This reduces to a linear algebra condition: taking arbitrary linear combinations yields a surjective map.

Thus $a \in \pi(Z)$ iff Φ_a is *not* surjective.

Expanding in a basis, we can write Φ_a as a matrix whose entries are homogeneous polynomials in the coordinates of a. Moreover, Φ_a is not surjective iff all $d \times d$ determinants of Φ_a are nonzero (since this may not be square). This is a polynomial condition, so $a \in \pi(Z)$ iff a bunch of homogeneous polynomials vanish, making $\pi(Z)$ is closed. ■

Corollary 1.0.2 *(?).*
The projection $\pi : \mathbb{P}^n \times Y \to Y$ is closed for any variety Y, making $\mathbb{P}^n$ complete.

Proof (?).
How to prove anything for varieties: use the fact that they're glued from affine varieties, so prove in that special case. So first suppose Y is affine. Let $Z \subset \mathbb{P}^n \times Y$ be closed,

and consider $\overline{Y} ss\mathbb{P}^m$ and $\overline{Z} \subset \mathbb{P}^n \times \overline{Y} \subset \mathbb{P}^n \times \mathbb{P}^m$ as a closed subset. Then we know that the projection $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ is closed, so $\pi(\overline{Z}) \subset \mathbb{P}^m$ is closed. But we can write $\pi(Z) = \pi(\overline{Z} \cap \mathbb{P}^n \times Y) = \pi(\overline{Z}) \cap Y$ which is closed. So $\pi(Z)$ is closed in Y, which proves this for affine varieties.

Supposing now that Y is instead glued from affines, it suffices to check that the set is closed in an open cover. So $Z \subset X$ is closed if when we let $X = \cup U_i$, we can show $Z \cap U_i$ is closed. But this essentially follows from above.

■

Corollary 1.0.3 ***(?).***
Any projective variety is complete.

Proof (?).
If $X \subset \mathbb{P}^n$ is closed and if $\mathbb{P}^n \times Y \to Y$ is a closed map for all Y, then restricting to $X \times Y \to Y$ again yields a closed map.

■

Corollary 1.0.4 ***(?).***
Let $f : X \to Y$ be a morphism of (importantly) *varieties* and suppose X is complete. Then $f(X)$ is closed in Y.

Proof (?).
Consider the graph of f, $\Gamma_f = \{(x, f(x))\} \subset X \times Y$. From a previous proof, we know Γ_f is closed when Y is a variety (by pulling back a diagonal). So Γ_f is closed in $X \times Y$, and thus $\pi_Y(\Gamma_f) = f(X)$ is closed because X is complete.

■

Corollary 1.0.5 ***(?).***
Let X be complete, then $\mathcal{O}_X(X) = k$, i.e. every global regular function is constant.

Note: this is an analog of the maximum modulus principle: if X is a compact complex manifold, then any function that is holomorphic on all of X is constant.

Proof (?).
Suppose $\varphi X \to \mathbb{A}^1$ is a regular function. Since $\mathbb{A}^1 \subset \mathbb{P}^1$, extend φ to a morphism $\widehat{\varphi} : X \to PP^1$. By a previous corollary, $\varphi(X)$ is closed, but $\infty \notin \varphi(X)$ implies $\varphi(X) \neq \mathbb{P}^2$, so $\varphi(X)$ is finite. Since X is connected, $\varphi(X)$ is a point, making φ a constant map.

■