Homological Algebra Problem Sets

Problem Set 2

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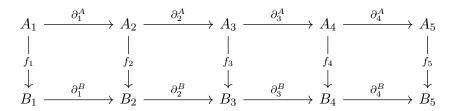
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1 | Tuesday, January 26

Problem 1.0.1 (Weibel 1.3.3)

Prove the 5-lemma. Suppose the following rows are exact:



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- a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
- b. Show that if f_2 , f_4 are epi and f_5 is monic, then f_3 is an epi.
- c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):

Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with " \rightarrow " and the epis with " \rightarrow ":

$$A_{1} \xrightarrow{\partial_{1}^{A}} A_{2} \xrightarrow{\partial_{2}^{A}} A_{3} \xrightarrow{\partial_{3}^{A}} A_{4}$$

$$\downarrow \qquad \downarrow$$

$$\downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow$$

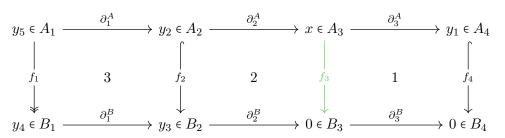
$$B_{1} \xrightarrow{\partial_{1}^{B}} B_{2} \xrightarrow{\partial_{2}^{B}} B_{3} \xrightarrow{\partial_{3}^{B}} B_{4}$$

Link to Diagram

It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2 .
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \rightarrow A_2 \rightarrow A_3$, so x = 0.



Link to Diagram

Solution (Part (b)):

Similarly, it suffices to consider the following portion of the diagram:

Link to Diagram

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

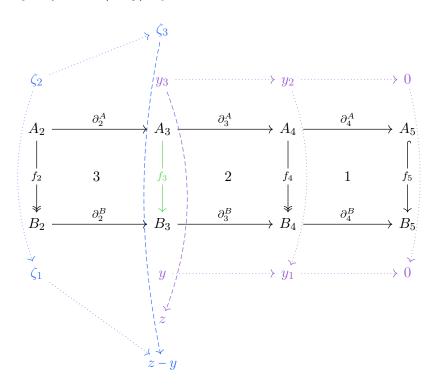
Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4 , pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \rightarrow A_5 \rightarrow B_5$.
- By injectivity of f_5 , the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z y \mapsto y_1 y_1 = 0$ under $B_3 \to B_4$, using that d(z y) = d(z) d(y) since these are R-module morphisms.

- By exactness of the bottom row, pull z y back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2 , pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z y$ under $A_3 \to B_3$.
- But then $y_3 \zeta_3 \mapsto z (z y) = y$ under $A_3 \to B_3$ as desired.



Link to Diagram

Solution (Part (c)):

Given the previous two result, if the outer maps are isomorphisms then f_3 is both monic and epi. Using a technical fact that monic epis are isomorphisms in a category \mathcal{C} if and only if \mathcal{C} is balanced and that all abelian categories are balanced, f_3 is isomorphism.

Problem 1.0.2 (Weibel 1.4.2)

Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$C_n \cong Z_n \oplus B'_n$$

 $Z_n = B_n \oplus H'_n$.

Show that C is split exact if and only if $H'_n = 0$.

Solution:

For this problem, we'll use the fact that if d is an epimorphism, it satisfies the right-cancellation property: if $f \circ d = g \circ d$, then f = g. In particular, if $d_n = d_n s_{n-1} d_n$ with $d_n : C_n \to C_{n-1}$

surjective and $s_{n-1}: C_{n-1} \to C_n$, we can conclude $\mathbb{1}_{C_n} = d_n s_{n-1}$. We'll also use the fact that if we have a SES in any abelian category \mathcal{A} , then the following are equivalent:

- The sequence is split on the left.
- The sequence is split on the right.
- The middle term is isomorphic to the direct sum of the outer terms.

Fixing notation, we'll write $C := (\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots)$, and we'll use concatenation fg to denote function composition $f \circ g$.

⇒ :

Suppose C is split, so we have maps $\{s_n\}$ such that $\partial_n = \partial_n s_{n-1} \partial_n$.

Claim: The short exact sequence

$$0 \to Z_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

admits a right-splitting $f: B_{n-1} \to C_n$, and thus there is an isomorphism

$$C_n \cong Z_n \oplus B'_n = Z_n \oplus B_{n-1}$$
.

Proof (?).

That this sequence is exact follows from the fact that it can be written as

$$0 \to \ker \partial_n \hookrightarrow C_n \stackrel{\partial_n}{\twoheadrightarrow} \operatorname{im} \partial_n \to 0.$$

We proceed by constructing the splitting f. Noting that $s_{n-1}: C_{n-1} \to C_n$ and $B_{n-1} \le C_{n-1}$, the claim is that its restriction $f := s_{n-1}|_{B_{n-1}}$ works. It suffices to show that $(C_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{f} C_n)$ composes to the identity map $\mathbbm{1}_{C_n}$. This follows from the splitting assumption, along with right-cancellability since ∂_n is surjective onto its image:

$$\partial_n = \partial_n s_{n-1} \partial_n \overset{\text{right-cancel } \partial_n}{\Longrightarrow} \mathbb{1}_{C_n} = \partial_n s_{n-1} \coloneqq \partial_n f.$$

Claim: The SES

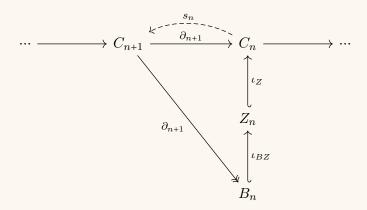
$$0 \to B_n \stackrel{\iota_{BZ}}{\hookrightarrow} Z_n \stackrel{\pi}{\twoheadrightarrow} \frac{Z_n}{B_n} \to 0$$

admits a left-splitting $f: \mathbb{Z}_n \to \mathbb{B}_n$, and thus there is an isomorphism

$$Z_n \cong B_n \oplus H'_n \coloneqq B_n \oplus H_n(C) \coloneqq B_n \oplus \frac{Z_n}{B_n}.$$

Proof (?).

We proceed by again constructing the splitting $f: \mathbb{Z}_n \to \mathbb{B}_n$. The situation is summarized in the following diagram:



Link to Diagram

So a natural candidate for the map f is the composition

$$Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} B_n$$

so $f \coloneqq \partial_{n+1} s_n \iota_Z$. We can simplify this slightly by regarding $Z_n \le C_n$ as a submodule to suppress ι_Z , and identify s_n with its restriction to Z_n to write $f \coloneqq \partial_{n+1} s_n$. The claim is then that $f\iota_{BZ} = \mathbbm{1}_{B_n}$. Anticipating using the fact that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$, we post-compose with ∂_{n+1} and compute:

$$f\iota_{BZ}\partial_{n+1} = \left(C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\iota_{Z}} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_n\right)$$

$$= \left(C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{s_n|_{B_n}} C_{n+1} \xrightarrow{\partial_{n+1}}\right)$$

$$= \partial_{n+1}s_n\partial_{n+1}$$

$$= \partial_{n+1},$$

where in the last step we've used the splitting hypothesis and the fact that it remains true when everything is restricted to the submodule $B_n \leq C_n$. Using surjectivity of ∂_{n+1} onto B_n , we can now conclude as before:

$$f\iota_{BZ}\partial_{n+1}=\partial_{n+1}\overset{\text{right-cancel }\partial_n}{\Longrightarrow}f\iota_{BZ}=\mathbbm{1}_{B_n}.$$

Problem 1.0.3 (Weibel 1.4.3)

Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

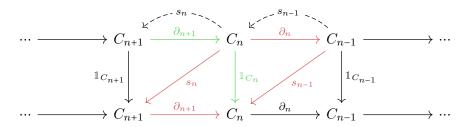
Solution:

⇐=:

C is split: Suppose $\mathbb{1}_C$ is nullhomotopic, so that there exist maps

$$s_n: C_n \to C_{n+1} \qquad \qquad \mathbb{1}_{C_n} = \partial_{n+1} s_n + s_{n-1} \partial_n.$$

We then have the following situation:



Link to Diagram

Here the nullhomotopy is outlined in red, and the map relevant to the splitting in green. Note that $s_n: C_n \to C_{n+1}$ is a candidate for a splitting, we just need to show that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$. We can proceed by post-composing the LHS with the identity $\mathbb{1}_C$, which allows us to substitute in the nullhomotopy:

$$\partial_{n+1} = \mathbb{1}_{C_n} \partial_{n+1}$$

$$= (\partial_{n+1} s_n + s_{n-1} \partial_n) \partial_{n+1}$$

$$= \partial_{n+1} s_n \partial_{n+1} + s_{n-1} \partial_n \partial_{n+1}$$

$$= \partial_{n+1} s_n \partial_{n+1} + s_{n-1} \mathbf{0}$$

$$= \partial_{n+1} s_n \partial_{n+1}.$$
since $\partial^2 = 0$

$$= \partial_{n+1} s_n \partial_{n+1}.$$

C is exact: This follows from the fact that since $\mathbb{1}_C = \partial s + s\partial$ are equal as maps of chain complexes, the images $D_1 := \mathbb{1}_C(C)$ and $D_2 := (\partial s + s\partial)(C)$ are equal as chain complexes and have equal homology. Evidently $D_1 = C$, and on the other hand, each graded piece $(D_2)_n$ only consists of boundaries coming from various pieces of C, since $\partial s + s\partial$ necessarily lands in the images of the maps ∂_n . Thus $C_n(D_2) \subseteq B_n(D_2) = \emptyset$, i.e. there are no chains (or cycles) in D_2 which are not boundaries, and thus $H_n(D_2) := Z_n(D_2)/B_n(D_2) = 0$ for all n. We can thus conclude that $C = D_2 \implies H(C) = H(D_2) = 0$, so C must be exact.

 \implies : Suppose C is split. Then by exercise 1.4.2, we have R-module decompositions

$$C_n \cong Z_n \oplus B_{n-1}$$
$$Z_n \cong B_n \oplus H_n$$

$$\implies C_n \cong B_n \oplus B_{n-1} \oplus H_n.$$

Supposing further that C is exact, we have $H_n = 0$, and thus $C_n \cong B_n \oplus B_{n-1}$. We first note that in this case, we can explicitly write the differential ∂_n . Letting $\mathbb{1}_n$ denote the identity on C_n , where by abuse of notation we also write this for its restriction to any submodules, we have:

$$C_n \xrightarrow{\partial_n} C_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$B_n \xrightarrow{-0} \xrightarrow{-0} B_{n-1}$$

$$\oplus \qquad \oplus$$

$$B_{n-1} \xrightarrow{-0} \xrightarrow{-0} B_{n-2}$$

Link to Diagram

We can thus write ∂_n as the matrix

$$\partial_n = \begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

Similarly using this decomposition, we can construct a map $s_n: C_n \to C_{n+1}$:

$$C_n \xrightarrow{s_n} C_{n+1}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$B_n \xrightarrow{-0} \xrightarrow{-0} B_{n+1}$$

$$\oplus \qquad \oplus$$

$$B_{n-1} \xrightarrow{-0} \xrightarrow{-0} B_n$$

Link to Diagram

We can write this as the following matrix:

$$s_n = \begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix}.$$

We can now verify that s_n is a nullhomotopy from a direct computation:

$$\begin{split} \partial_{n+1} s_n + s_{n-1} \partial_n &= \begin{bmatrix} 0 & \mathbbm{1}_{n+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbbm{1}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbbm{1}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbbm{1}_n \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbbm{1}_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbbm{1}_{n-1} \end{bmatrix} \\ &= \mathbbm{1}_{C_n}, \end{split}$$

expressed as a map $B_n \oplus B_{n-1} \to B_n \oplus B_{n-1}$.

Problem 1.0.4 (Weibel 1.4.5)

Show that chain homotopy classes of maps form a quotient category K of Ch(R-mod) and that the functors H_n factor through the quotient functor $Ch(R\text{-mod}) \to K$ using the following steps:

- 1. Show that chain homotopy equivalence is an equivalence relation on $\{f: C \to D \mid f \text{ is a chain map}\}$. Define $\operatorname{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
- 2. Let $f \simeq g: C \to D$ be two chain homotopic maps. If $u: B \to C, v: D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
- 3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $Ch(R\text{-mod}) \to K$ is an additive functor.
- 4. Is K an abelian category? Explain.

Try at least two parts.

Solution (Part 1):

Claim 1: Chain homotopy equivalence defines an equivalence relation on $\operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A,B)$.

Proof (of claim 1).

We recall that for morphisms $f, g \in \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A, B)$, we have $f \simeq g \iff f - g \simeq 0 \iff \exists s : A \to B[1]$ such that $\partial^B s + s \partial^A = f - g$.

- **Reflexive:** We want to show that $f \simeq f$, i.e. $f f = 0 \simeq 0$. Producing the map s = 0 works, since $\partial s + s \partial = \partial 0 + 0 \partial = 0$.
- **Symmetric**: Suppose $f \simeq g$, so there exists an s such that $\partial s + s\partial = f g$. Then taking s' := -s produces a chain homotopy $g f \simeq 0$, since we can write

$$\partial s' + s' \partial = \partial (-s) + (-s) \partial$$

$$= -\partial s - s \partial$$

$$= -(\partial s + s \partial)$$

$$= -(f - g)$$

$$= g - f.$$

• **Transitive**: Suppose $f \simeq g$ and $g \simeq h$, we want to show $f \simeq h$. By assumption we have maps s, s' such that $\partial s + s \partial = f - g$ and $\partial s' + s' \partial = g - h$, so set $s'' \coloneqq s + s'$. We can then check that this is a chain homotopy from f to h:

$$\partial s'' + s'' \partial = \partial (s + s') + (s + s') \partial$$

$$= (\partial s + s \partial) + (\partial s' + s' \partial)$$

$$= (f - g) + (g - h)$$

$$= f - h,$$

where we've used that abelian categories are enriched over abelian groups, so we have a commutative and associative addition on homs.

Claim 2: $(\text{Hom}_K(A, B), \oplus) \in Ab$, where we define an addition on equivalence classes by

$$[f] \oplus [g] \coloneqq [f+g].$$

Proof (of claim 2).

We check the group axioms directly:

- Closure under operation: We can check that $[f] \oplus [g] = [f+g] = [g']$ makes sense, since $g' = f + g \in \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A, B)$, making [g'] a well-defined equivalence class of maps in $\operatorname{Hom}_K(A, B)$.
- Two-sided Identities: The equivalence class $[0] := \{ f \in \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A, B) \mid f \simeq 0 \}$ serves as an identity, where we take the zero map as a representative. This follows from the fact that

$$[0] \oplus [f] \coloneqq [0+f] = [f] = [f] \oplus [0].$$

• **Associativity**: This again follows from the abelian group structure on the original hom:

$$[f] \oplus ([g] \oplus [h]) = [f + (g+h)] = [(f+g) + h] = ([f] \oplus [g]) \oplus [h].$$

• Two-sided inverses: Let $\ominus[f] := [-f]$ be a candidate for the inverse with respect to \oplus . To see that this works, we have

$$[f] \oplus (\ominus[f]) \coloneqq [f] \oplus [-f] = [f-f] = [0],$$

and a similar calculation goes through to show it's also a left-sided inverse.

• Well-definedness: We need to show that if [f] = [f'] and [g] = [g'], the sums agree, so $[f] \oplus [g] = [f'] \oplus [g']$. We'll use the fact that $[f] = [f'] \iff f - f' \approx 0$ (and similarly for g), so we can compute:

$$([f] \oplus [g]) \ominus ([f'] \oplus [g']) \coloneqq [f+g] \ominus [f'-g']$$

$$\coloneqq [(f+g) - (f'-g')]$$

$$= [(f-f') + (g-g')]$$

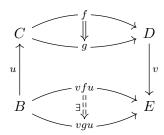
$$= [f-f'] \oplus [g-g']$$

$$= [0] \oplus [0] \qquad \text{using } f \simeq f', \ g \simeq g'$$

$$\coloneqq [0].$$

Solution (Part 2):

We have the following situation, where the double-arrows denote chain homotopies:



Link to Diagram

We want to show that $vgu \simeq vfu$, or equivalently that $vgu - vfu \simeq 0$. This is immediate:

$$vgu - vfu = v(gu - fu) = v(f - g)u \simeq v0u = 0.$$

Alternatively, as an explicit computation, if we assume $f \simeq g$ then there is a nullhomotopy s for f - g, in which case the map $s' \coloneqq vsu$ works as a nullhomotopy for vfu - vgu:

$$vfu - vgu = v(f - g)u = v(\partial s + s\partial)u$$

 $= v\partial su + vs\partial u$
 $= \partial vsu + vsu\partial$ since u, v are chain maps
 $:= \partial s' + s'\partial$.

We can now define a composition map on \mathcal{K} :

$$\circ: \operatorname{Hom}_{K}(A_{1}, A_{2}) \times \operatorname{Hom}_{K}(A_{2}, A_{3}) \to \operatorname{Hom}_{K}(A_{1}, A_{3})$$
$$([f], [g]) \mapsto [f \circ g].$$

The previous argument then precisely says:

- 1. If $u: B \to C$ and $f, gC \to D$ with [f] = [g], the composition $[u] \circ [f] = [u] \circ [g] : B \to D$ is well-defined, and
- 2. If $v: D \to E$ and $f, g: C \to D$ with [f] = [g], the composition $[f] \circ [v] = [g] \circ [v]$ is well-defined.

So this composition on K is well-defined. Moreover,

- There is an identity morphism $[\mathbb{1}_A] \in \operatorname{Hom}_K(A, A)$ coming from the class of $\mathbb{1}_C \in \operatorname{Hom}_{\operatorname{Ch}(A)}(A, A)$,
- It is associative, since $[h] \circ [gf] \coloneqq [hgf] \coloneqq [hg] \circ [f]$, and
- It is unital, in the sense that $[\mathbbm{1}_B] \circ [f] \coloneqq [\mathbbm{1}_A \circ f] = [f] = [f \circ \mathbbm{1}_A] \coloneqq [f] \circ [\mathbbm{1}_B]$ for any $f: A \to B$.

So this data satisfies all of the axioms of a category (Weibel A.1.1).

Solution (Part 3):

To see that $f_0 + f_1 \simeq g_0 + g_1$, we have

$$(f_0 + f_1) - (g_0 + g_1) = (f_0 - g_0) + (f_1 - g_1)$$

 $\approx 0 + 0 = 0.$

To be explicit, if s_i are chain homotopies for f_i, g_i , we can take $s = s_0 + s_1$:

$$(f_0 + f_1) - (g_0 + g_1) = (f_0 - g_0) + (f_1 - g_1)$$

$$= (\partial s_0 + s_0 \partial) + (\partial s_1 + s_1 \partial)$$

$$= (\partial s_0 + \partial s_1) + (s_0 \partial + s_1 \partial)$$

$$= \partial (s_0 + s_1) + (s_0 + s_1) \partial$$

$$= \partial s + s \partial.$$

That K forms an additive category is a consequence of the following facts:

- K has products, since A had products and Ob(K) := Ob(A).
- K has a zero object, for the same reason.
- Composition distributes over addition, i.e.

$$[f]([g] \oplus [h])$$

$$\coloneqq [f][g+h]$$

$$\coloneqq [f(g+h)]$$

$$\coloneqq [fg+fh]$$

$$= [fg] \oplus [fh].$$

Moreover, the quotient functor $\operatorname{Ch}(R\operatorname{-mod}) \to K$ is an additive functor, since the maps $Q: \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A,B) \to \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A,B)/\sim$ are morphisms of abelian groups, using the fact that Q commutes with both additions:

$$Q(f+g) = [f+g] := [f] \oplus [g] = Q(f) + Q(g).$$

Alternatively, we can note that the set of all $f \in \operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A,B)$ which are nullhomotopic form a subgroup H, and since everything is abelian we can form the quotient $\operatorname{Hom}_{\operatorname{Ch}(\mathcal{A})}(A,B)/H$ observe that this is isomorphic as a group to $\operatorname{Hom}_K(A,B)$.

Solution (Part 4):

We can first note that K is an additive category, since we still have a zero object and products inherited from Ch(A).

Note: I don't see a great way to prove that any particular category is abelian or not! Checking the axioms listed in Appendix A.4 seems quite difficult.

Problem 1.0.5 (Weibel 1.5.1)

Let $cone(C) := cone(\mathbb{1}_C)$, so

$$cone(C)_n = C_{n-1} \oplus C_n.$$

Show that cone(C) is split exact, with splitting map given by s(b,c) := (-c,0).

Solution:

Fixing notation, let

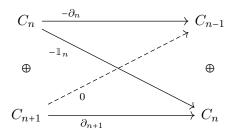
- ∂_n be the *n*th differential on C,
- $\mathbb{1}_C$ be the identity chain map on C,
- $\mathbb{1}_n: C_n \to C_n$ be the *n*th graded component of $\mathbb{1}_C$,
- $\widehat{C} = \operatorname{cone}(C) = \operatorname{cone}(\mathbb{1}_C),$
- $\widehat{\partial}_n$ be the *n*th differential on \widehat{C} , and
- $\widehat{\mathbb{I}}$ be the identity on \widehat{C} ,
- $\widehat{\mathbb{1}}_n : \widehat{C}_n \to \widehat{C}_n$ be the *n*th component of $\widehat{\mathbb{1}}$.

From exercise 1.4.2, it suffices to show that $\widehat{\mathbb{I}}$ is nullhomotopic. Since we have a direct sum decomposition cone $(C)_n = C_{n-1} \oplus C_n$, we can write $\widehat{\mathbb{I}}$ as a block matrix

$$\begin{bmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & \mathbb{1}_n \end{bmatrix}.$$

We can similarly write down the differential on cone(C) in block form:

$$\operatorname{cone}(C)_{n+1} \xrightarrow{\widehat{\partial}_{n+1}} \operatorname{cone}(C)_n$$



Link to Diagram

This yields

$$\widehat{\partial}_n \coloneqq \begin{bmatrix} -\partial_{n-1} & 0 \\ -\mathbb{1}_{n-1} & \partial_n \end{bmatrix}.$$

Similarly, the map $s_n(b,c) = (-c,0)$ can be written as

$$s_n = \begin{bmatrix} 0 & -\mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

We can thus proceed by a direct computation:

$$\begin{split} \partial_{n+1} s_n + s_{n-1} \partial_n &= \begin{bmatrix} -\partial_n & 0 \\ -\mathbbm{1}_n & \partial_{n+1} \end{bmatrix} \begin{bmatrix} 0 & -\mathbbm{1}_n \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\mathbbm{1}_{n-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\partial_{n-1} & 0 \\ -\mathbbm{1}_{n-1} & \partial_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & \partial_n \mathbbm{1}_n \\ 0 & \mathbbm{1}_n \mathbbm{1}_n \end{bmatrix} + \begin{bmatrix} \mathbbm{1}_{n-1} \mathbbm{1}_{n-1} & -\mathbbm{1}_{n-1} \partial_n \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbbm{1}_{n-1} \mathbbm{1}_{n-1} & \partial_n \mathbbm{1}_n - \mathbbm{1}_{n-1} \partial_n \\ 0 & \mathbbm{1}_n \mathbbm{1}_n \end{bmatrix} \\ &= \begin{bmatrix} \mathbbm{1}_{n-1} \mathbbm{1}_{n-1} & 0 \\ 0 & \mathbbm{1}_n \mathbbm{1}_n \end{bmatrix} \\ &= \widehat{\mathbbm{1}}_n. \end{split}$$

Problem 1.0.6 (Weibel 1.5.2)

Let $f: C \to D \in \text{Mor}(\text{Ch}(A))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(-s, f) : \operatorname{cone}(C) \to D.$$

Solution:

Remark 1.0.1: As a notational convention for this problem, I'll take vectors v to be column vectors, and v^t will denote a row vector. I've also written $f \simeq 0$ to denote that f is nullhomotopic.

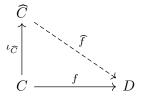
 \implies : Suppose that $f \simeq 0$, so there are maps s_n such that the following diagrams commute for every n:

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n^C} C_{n-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow D_{n+1} \longrightarrow \partial_{n+1}^D \longrightarrow D_n \xrightarrow{s_{n-1}} \cdots$$

Link to Diagram

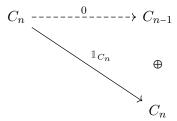
Write $\widehat{C} := \operatorname{cone}(C) := \operatorname{cone}(\mathbb{1}_C) := C[1] \oplus C$, we then want to construct a lift \widehat{f} of f such that the following diagram commutes:



Link to Diagram

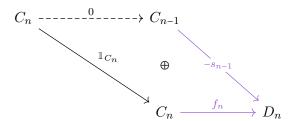
where $\iota_{\widehat{C}}$ is the following inclusion of C into its cone:





Link to Diagram

We define the map \widehat{f} in the following way:



Link to Diagram

That this is a lift follows from computing the composition, which can be done in block matrices:

$$\widehat{f}_n \circ \iota_{\widehat{C},n} = \begin{bmatrix} -s_{n-1} \\ f_n \end{bmatrix}^t \begin{bmatrix} 0 \\ \mathbb{1}_{C_n} \end{bmatrix} = [f_n \mathbb{1}_{C_n}] = f_n,$$

where the first matrix acts as a row vector. It only remains to check that \widehat{f} defines a chain

map, which follows from the following computation:

$$\partial_{n}^{D}\widehat{f}_{n} - \widehat{f}_{n-1}\widehat{\partial}_{n} = \left[\partial_{n}^{D}\right] \begin{bmatrix} -s_{n-1} \\ f_{n} \end{bmatrix}^{t} - \begin{bmatrix} -s_{n-2} \\ f_{n-1} \end{bmatrix} \begin{bmatrix} -\partial_{n}^{C} & 0 \\ -\mathbb{1}_{C_{n}} & \partial_{n}^{C} \end{bmatrix}$$

$$= \begin{bmatrix} \partial_{n}^{D}(-s_{n-1}) - s_{n-2}\partial_{n}^{C} + f_{n-1}\mathbb{1}_{C_{n}} \\ \partial_{n}^{D}f_{n} - f_{n-1}\partial_{n}^{C} \end{bmatrix}^{t}$$

$$= \begin{bmatrix} f_{n-1} - (\partial_{n}^{D}s_{n-1} + s_{n-2}\partial_{n}^{C}) \\ 0 \end{bmatrix}^{t} \qquad \text{since } f \text{ a chain map } \Longrightarrow \partial f = f\partial$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{since } f \simeq 0 \Longrightarrow \partial s + s\partial = f$$

$$= 0.$$

 \iff : Suppose we have a lift $\widehat{f}:\widehat{C}\to D$; then define the following maps as the proposed splittings:

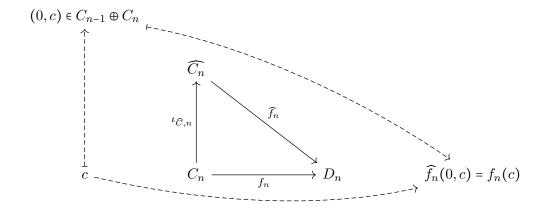
$$s_n: C_{n-1} \to D_n$$

 $c \mapsto \widehat{f_n}(-c, 0).$

There are two relevant facts to observe:

1. We have $f = \tilde{f}\iota_{\widehat{C}}$ where $\iota_{\widehat{C}}(c) \coloneqq (0,c) \in \widehat{C}$ is inclusion into the second direct summand, and in particular

$$f_n(c) = \widehat{f}_n \iota_{\widehat{C},n}(c) = \widehat{f}_n(0,c).$$



Link to Diagram

2. Since \widehat{f} is a chain map, we have for each n

$$\partial_n^D \widehat{f}_n(x,y) = \widehat{f}_{n+1} \widehat{\partial}_n(x,y)$$
 as maps $\widehat{C}_n \to D_n$.

We now proceed to compute at the level of elements that s defines a splitting:

$$\begin{split} \partial_{n+1}^D s_n(c) + s_{n-1} \partial_n^D(c) &\coloneqq \partial_{n+1}^D \widehat{f}_{n+1}(c,0) + \widehat{f}_n \partial_n^C(c,0) \\ &= \partial_{n+1}^D \widehat{f}_{n+1}(c,0) + \widehat{f}_n(\partial_n^C(c),0) \\ &= \widehat{f}_n \widehat{\partial}_{n+1}(c,0) + \widehat{f}_n(\partial_n^C(c),0) & \text{by (2)} \\ &= \widehat{f}_n \left(\begin{bmatrix} -\partial_n^C & 0 \\ -\mathbbmss{1}^C & \partial_{n+1}^C \end{bmatrix} \begin{bmatrix} c \\ 0 \end{bmatrix} \right) + \widehat{f}_n(\partial_n^C(c),0) \\ &= \widehat{f}_n \left(\begin{bmatrix} -\partial_n^C(c) \\ c \end{bmatrix} \right) + \widehat{f}_n(\partial_n^C(c),0) \\ &= \widehat{f}_n(-\partial_n^C(c),c) + \widehat{f}_n(\partial_n^C(c),0) \\ &= \widehat{f}_n(-\partial_n^C(c)+\partial_n^C(c),c) & \text{since } f_n \text{ is an } R\text{-mod morphism} \\ &= \widehat{f}_n(0,c) \\ &= f_n(c) & \text{by (1)}. \end{split}$$

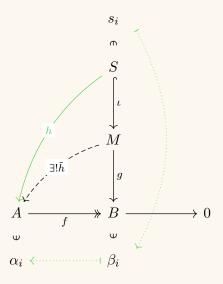
Problem 1.0.7 (Extra)

- a. Show that free implies projective.
- b. Show that $\operatorname{Hom}_R(M, \cdot)$ is left-exact.
- c. Show that P is projective if and only if $\operatorname{Hom}_R(P,\cdot)$ is exact.

Solution:

Proof (of (a)).

Suppose M is free, so we have a set S and an injection $S \to M$ such that every map in $\hom_{\operatorname{Set}}(S,Y)$ for $Y \in R$ -mod lifts to a unique map in $\hom_{R\operatorname{-mod}}(M,Y)$. Suppose further that we have the following situation; we seek to construct a lift $\tilde{h}: M \to A$:



Link to Diagram

This lift exists by first considering $s_i \in S$ and noting that since $\beta_i := \iota g(s_i) \in B$ and f is surjective, there exist *some* elements α_i in A such that $f(\alpha_i) = \beta_i$ for each i. So define the map

$$h: S \to A$$

 $s_i \mapsto \alpha_i$.

By the universal property of free modules, this lifts to a map $\tilde{h}: M \to A$, so M is projective.

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 $Proof\ (of\ (b)).$

Suppose we have a SES

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

The claim is that $\operatorname{Hom}_R(M,\cdot)$ yields an exact sequence

$$0 \to \operatorname{Hom}_R(M,A) \xrightarrow{f_*} \operatorname{Hom}_R(M,B) \xrightarrow{g_*} \operatorname{Hom}_R(M,C) \to \cdots,$$

where $f_*(\alpha) = f \circ \alpha$ for $\alpha : M \to A$ and similarly $g_*(\beta) = g \circ \beta$ for $\beta : M \to B$. To show that this is exact, it suffices to show three things:

- 1. $\ker f_* = 0$,
- 2. im $f_* \subseteq \ker g_*$, and
- 3. $\ker g_* \subseteq \operatorname{im} f_*$.

Proceeding with each part:

- 1. By definition, if $\beta: M \to B$ satisfies $\beta \in \ker f_*$, and thus $f\beta = 0$. Since the original sequence was exact, f is injective, thus a monomorphism, thus satisfies the left-cancellation property. So we can immediately conclude that $\beta = 0$.
- 2. Let $\beta: M \to B$ be in im f_* , so there exists some $\alpha: M \to A$ with $\beta = f\alpha$. We want to show that $\beta \in \ker g_*$, so we can apply g_* to obtain $g_*(\beta) = g_*(f\alpha) := gf\alpha$. But by exactness of the first sequence, gf = 0, so gfa = 0.
- 3. Let $\beta' \in \ker g_*$, so $g\beta' = 0$. In order to show $\beta' \in \operatorname{im} f_*$, we want to construct some $\alpha : M \to A$ such that $\beta' = f_*(\alpha) \coloneqq f\alpha$. Considering $m \in M$, we know $g\beta'(m) = 0$ and thus $\beta'(m) \in \ker g = \operatorname{im} f$ by exactness of the first sequence. So there exists some $a_m \in A$ with $f(a_m) = \beta'(m)$, and we can define a map

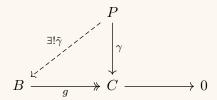
$$\alpha: M \to A$$
$$m \mapsto a_m.$$

By construction, we then have $f\alpha(m) = f(a_m) := \beta'(m)$ for every $m \in M$, so $f\alpha = \beta'$.

Proof (of(c)).

Assume the same setup as (b).

 \implies : Suppose P is projective, so it satisfies the following universal property:



Link to Diagram

Using the results of (b), it suffices to check exactness at $\operatorname{Hom}_R(P,C)$ in the following sequence:

$$0 \to \operatorname{Hom}_R(P,A) \xrightarrow{f_*} \operatorname{Hom}_R(P,B) \xrightarrow{g_*} \operatorname{Hom}_R(P,C) \to 0,$$

or equivalently that g_* is surjective. Using that $B \xrightarrow{g} C \to 0$ is exact if and only if g is surjective, the universal property above means that every $\gamma \in \operatorname{Hom}_R(P,C)$ lifts to a map $\tilde{\gamma} \in \operatorname{Hom}_R(P,B)$ where $g\tilde{\gamma} = \gamma$. Since $g_*(\tilde{\gamma}) \coloneqq g\tilde{\gamma}$, this precisely means that $\gamma \in \operatorname{im} g_*$.

 \Leftarrow : Reversing the above argument, if $\operatorname{Hom}_R(P,\cdot)$ is exact, then every $P \xrightarrow{\gamma} C$ has a preimage under g_* , which is precisely a lift $P \xrightarrow{\tilde{\gamma}} B$. So P satisfies the universal property of projective modules.

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