Real Analysis

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1 Lecture 1 (Thu 15 Aug 2019 11:04)

> See Folland's Real Analysis, definitely a recommended reference.

Possible first day question: how can we "measure" a subset of $\mathbb{R}$? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of $\mathbb{R}$. We'll come back to this in a few lectures.

1.1 Notions of "smallness" in $\mathbb{R}$

Definition: Let E be a set, then E is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes $\emptyset, \mathbb{N}$.

Definition: E is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of $\mathbb{R}$ is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in Y iff the closure of X is all of Y. So we'll make the following definition.

Definition: A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval I contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. Example: $\mathbb{Q}$ is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, - A^c contains a dense, open set. - The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than ε.

Definition: E is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable $\implies$ Meager, but not the converse.
- Countable $\implies$ Null, but not the converse.

Exercise: Show that the "middle third" Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem: Every $E \subseteq \mathbb{R}$ can be written as $E = A \coprod B$ where A is null and B is meager.

> This gives some information about how nullity and meagerness interact – in particular, $\mathbb{R}$ itself is neither meager nor null. Idea: if meager $\implies$ null, this theorem allows you to write $\mathbb{R}$ as the union of two null sets. This is bad!

Proof: We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Around each q_j, put an interval around it of size $1/2^{j+k}$ where we'll allow k to vary, yielding multiple intervals around q_j. To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that A is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick k such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that $A^c := B$ is meager. Note that G_k covers the rationals, and is a countable union of open sets, so it is dense. So G_k is an open and dense set. By one of the equivalent formulations of meagerness, this means that G_k^c is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager.

1.2 $\mathbb{R}$ is not small

Theorem (Cantor): $\mathbb{R}$ is not countable.

Theorem (Baire): $\mathbb{R}$ is not meager. (Baire Category Theorem) Theorem (Borel): $\mathbb{R}$ is not null.