Problem Set 7

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1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence, so $\|x^j - x^k\|_{\ell^2} \to 0$. We want to produce some $\mathbf{x} := \lim\limits_{n\to\infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i := \lim_{n\to\infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left|x_i^j - x_i^k\right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left|x_i^j - x_i^k\right|^2$ is a Cauchy sequence of real numbers which necessarily converges in $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\|\lim_{k\to\infty} x^k - x^j\right\|_{\ell^2} = \lim_{k\to\infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$\begin{aligned} \|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\ &\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\ &\to M < \infty, \end{aligned}$$

where $\|\mathbf{x}_i - x^j\|_{\ell^2} \to 0$ and the second sum is finite because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \ \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, we have

$$\Re(\langle x, \ iy \rangle) = \Re(-i\langle x, \ y \rangle) = \Im(\langle x, \ y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2 \ \Re\left(\langle x, \ y \rangle\right)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \ \Re\left(\langle x, \ y \rangle\right)$$

$$\begin{aligned} \|x + iy\|^2 &= \|x\|^2 + \|y\|^2 + 2 \ \Re\left(\langle x, \ iy \rangle\right) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle) \end{aligned}$$

$$\begin{aligned} \|x - iy\|^2 &= \|x\|^2 + \|y\|^2 - 2 \ \Re\left(\langle x, \ iy \rangle\right) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle) \end{aligned}$$

and summing these all

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x + iy\| &= 4 \ \Re\left(\langle x, \ y \rangle\right) + 4i \ \Im\left(\langle x, \ y \rangle\right) \\ &= 4\langle x, \ y \rangle. \end{aligned}$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 \coloneqq \langle x,\ x\rangle = \langle Ux,\ Ux\rangle \coloneqq \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned}\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2 + \|Ux-Uy\|^2 + i\|Ux+Uy\|^2 - i\|Ux+Uy\|^2\right)\\ &= \frac{1}{4}\left(\|U(x+y)\|^2 + \|U(x-y)\|^2 + i\|U(x+y)\|^2 - i\|U(x+y)\|^2\right)\\ &= \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2 + i\|x+y\|^2 - i\|x+y\|^2\right)\\ &= \langle x,\ y\rangle.\end{aligned}$$

□

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$\begin{aligned}|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle|\\ &= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle|\\ &\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|\\ &\to 0\cdot M + C\cdot 0 < \infty,\end{aligned}$$

where $\|y_n\| \to M$ since $y_n \to y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$\langle e,\ y\rangle = \left\langle \lim_n e_n,\ y\right\rangle = \lim_n \langle e_n,\ y\rangle = \lim_n 0 = 0,$$

so $e \in E^\perp$.

2.2 Part b:

Let $S := \operatorname{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n \to x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$y \in E^{\perp} \implies \langle x_n,\ y\rangle = \left\langle \sum_i a_i e_i,\ y \right\rangle = \sum_i a_i \langle e_i,\ y\rangle = 0 \implies x_n \in (E^{\perp})^{\perp}.$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$y \in E^{\perp} \implies \langle x,\ y\rangle = \left\langle \lim_n x_n,\ y \right\rangle = \lim_n \langle x_n,\ y\rangle = 0 \implies x \in (E^{\perp})^{\perp},$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notation convenience, we'll just write S for $\overline{S}$. Let $x \in E^{\perp\perp}$. Noting that S is closed, we can define P, the operator projecting elements onto S, and write

$$x = Px + (x - Px) \in S \oplus S^{\perp}$$

But since $\langle x,\ x - Px\rangle = 0$ because $x - Px \in E^{\perp}$ and $x \in (E^{\perp})^{\perp}$, we can rewrite the first term in this inner product to obtain

$$0 = \langle x,\ x - Px\rangle = \langle Px + (x - Px),\ x - Px\rangle = \langle Px,\ x - Px\rangle + \langle x - Px,\ x - Px\rangle,$$

where we can note that the first term is zero because $Px \in S$ and $x - Px \in S^{\perp}$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\begin{aligned}
\|e_0\|^2 &= \int_0^1 1^2\ dx = 1 \\
\|e_1\|^2 &= \int_0^1 3(2x-1)^2 = \frac{1}{2}(2x-1)^2\Big|_0^1 = 1 \\
\langle e_0,\ e_1\rangle &= \int_0^1 \sqrt{3}(2x-1)\ dx = \frac{\sqrt{3}}{4}(2x-1)\Big|_0^1 = 0.
\end{aligned}$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0,1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$\begin{aligned}
f(x) &= \sum_i \left\langle x^3,\ e_i \right\rangle e_i \\
&= \left\langle x^3,\ 1 \right\rangle 1 + \left\langle x^3,\ \sqrt{3}(2x-1) \right\rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2\ dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1)\ dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1)\frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}$$

We can also compute

$$\begin{aligned}
\|f-g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2\ dx \\
&= \frac{1}{180} \\
\implies \|f-g\|_2 &= \frac{1}{\sqrt{180}}.
\end{aligned}$$

4 Problem 5

5 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e_1} = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i,\ \hat{e_i} \rangle \hat{e_i}$

The result set $\{\hat{e_i}\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$
\begin{aligned}
e_2 &= x - \langle x,\ \hat{e}_1\rangle \hat{e}_1 \\
&= x - \langle x,\ 1\rangle 1 \\
&= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x \ dx \\
&= x - \int \text{odd function} \\
&= x,
\end{aligned}
$$

and so $e_2 = x$. We can then check that

$$
\|e_2\| = \left(\int_{-1}^{1} x^2 \ dx\right)^{1/2} = \sqrt{\frac{2}{3}},
$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}}x$.

We continue to compute

$$
\begin{aligned}
e_3 &= x^2 - \left\langle x^2,\ \hat{e}_1 \right\rangle \hat{e}_1 - \left\langle x^2,\ \hat{e}_2 \right\rangle \hat{e}_2 \\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2 \ dx - \frac{3}{2}x \int_{-1}^{1} x^3 \ dx \\
&= x^2 - \left(\frac{1}{6}x^3\right)\Big|_{-1}^{1} + \frac{3}{2}x \int_{-1}^{1} \text{odd function} \\
&= x^2 - \frac{1}{3}.
\end{aligned}
$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$
\begin{aligned}
\hat{e}_3 &= \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \\
&= \frac{1}{2}\sqrt{\frac{45}{2}}\frac{1}{3}(3x^2 - 1) \\
&= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right).
\end{aligned}
$$

In summary, this yields

$$
\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{2}} \\
\hat{e}_2 &= x \\
\hat{e}_3 &= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right),
\end{aligned}
$$

which are scalar multiples of the first three Legendre polynomials.