Math 8100 Assignment 10

Due date: Friday 3rd of December 2010

1. Let ν and μ be signed measures. Prove that if $\nu \perp \mu$ and $\nu \ll |\mu|$, then $\nu = 0$.

2. Let $f \in L^1(\mathbb{R}^n)$ with $f \neq 0$.

   (a) Prove there exists $c > 0$ such that $Hf(x) \geq c(1 + |x|)^{-n}$.

   (b) Conclude that $Hf \notin L^1(\mathbb{R}^n)$. Moreover, show that there exists $c' > 0$ such that

   $$m(\{x \in \mathbb{R}^n \,:\, Hf(x) > \alpha\}) \geq c'/\alpha$$

   for all sufficiently small $\alpha > 0$.

3. Consider the function on $\mathbb{R}$ defined by

   $$f(x) = \begin{cases} \frac{1}{|x|(\log 1/|x|)^2} & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Show that f is integrable.

   (b) Show that there exists $c > 0$ such that

   $$Hf(x) \geq \frac{c}{|x|(\log 1/|x|)}$$

   for all $|x| \leq 1/2$. Conclude that the maximal function Hf is not locally integrable.

4. Let $f \in L^1(\mathbb{R})$. Let $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$ denote the upper half plane. For $(x, y) \in \mathcal{U}$ define

   $$u(x, y) = f * P_y(x) \quad \text{where} \quad P_y(t) = \frac{1}{\pi}\frac{y}{t^2 + y^2}.$$

   (a) Prove that there exists a constant C independent of f so that for all $x \in \mathbb{R}$

   $$\sup_{y>0} |u(x, y)| \leq CHf(x).$$

   [*Hint: Write* $u(x, y) = \int_{|t|<y} f(x - t)P_y(t)\,dt + \sum_{k=0}^{\infty} \int_{2^k y \leq |t| < 2^{k+1} y} f(x - t)P_y(t)\,dt$ *and then estimate each term in the sum.*]

   (b) Prove that for $f \in L^1(\mathbb{R})$ and almost every $x \in \mathbb{R}$

   $$\lim_{y \to 0} u(x, y) = f(x).$$

   [*Hint: Follow the proof of the Lebesgue differentiation theorem given in class.*]

5. Let E be a Lebesgue measurable subset of $\mathbb{R}$.

   (a) Suppose 0 is a point of Lebesgue density of E. Show that there is an infinite sequence of points $\{x_k\}_{k=1}^{\infty}$, with $x_k \neq 0$ and $x_k \to 0$, such that $\{-x_k, x_k\} \subseteq E$ for all k.

   [Recall that x is said to be a *point of Lebesgue density of* E if $\lim_{h \to 0} \dfrac{m(E \cap (x - h, x + h))}{2h} = 1$.]

   (b) Prove that for almost every $x \in E$, there is an infinite sequence of points $\{x_k\}_{k=1}^{\infty}$, with $x_k \neq 0$ and $x_k \to 0$, such that $\{x - x_k, x, x + x_k\} \subseteq E$ for all k.

Challenge Problem X

Hand this in to me at some point in the semester

Let $\mathcal{R}$ denote the set of all rectangles in $\mathbb{R}^2$ that contain the origin, and with sides parallel to the coordinate axis. Consider the maximal operator associated to this family, namely

$$M_{\mathcal{R}}f(x) = \sup_{R\in\mathcal{R}} \frac{1}{m(R)} \int_R |f(x-y)|\, dy.$$

(a) Show that $f \mapsto M_{\mathcal{R}}f$ does not satisfy the weak type inequality

$$m(\{x \in \mathbb{R}^2 \,:\, M_{\mathcal{R}}f(x) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1$$

for all $\alpha > 0$, all $f \in L^1$ and some $C > 0$.

(b) Prove that there exist $f \in L^1(\mathbb{R}^2)$ such that for $R \in \mathcal{R}$

$$\limsup_{\operatorname{diam}(R)\to 0} \frac{1}{m(R)} \int_R f(x-y)\, dy = \infty$$

for almost every $x \in \mathbb{R}^2$.