Title

D. Zack Garza

Contents

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p.

$k[\mathbf{x}] \coloneqq k[x_1, \cdots, x_n]$	The polynomial ring in n indeterminates
$k(\mathbf{x}) \coloneqq k(x_1, \cdots, x_n)$	The rational function field
$\mathbb{A}^n_{/k}$	Affine n-space
	$\mathbb{A}^n_{/k} \coloneqq \left\{ [k_1, \cdots, k_n] \;\middle\vert\; k_j \in k \right\}$
$\mathbb{P}^n_{/k}$	Projective n-space
	$\mathbb{P}^n_{/k} \coloneqq (k^n \setminus \{0\}) / x \sim \lambda x$
	$\left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \;\middle\vert\; p, q, \in k[x_1, \cdots, x_n] \right\}$
$V(J), V_a(J)$	Variety associated to an ideal $J \trianglelefteq k[x_1, \cdots, x_n]$
	$\coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \;\middle\vert\; f(\mathbf{x}) = 0, \forall f \in J \right\}$
$I(S), I_a(S)$	Ideal associated to a subset $S \subseteq \mathbb{A}^n_k$
	$\coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle\vert\; f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in X \right\}$
$A(X)$	Coordinate ring of a variety
	$\coloneqq k[x_1, \cdots, x_n]/I(X)$
$\mathcal{O}_X$	Structure sheaf $\left\{ f : U \to k \;\middle\vert\; f \in k(\mathbf{x}) \text{ locally} \right\}$
$D(f)$	Distinguished open set
	$\coloneqq V(f)^c = \left\{ x \in \mathbb{A}^n \;\middle\vert\; f(x) \neq 0 \right\}$
Δ_X	The diagonal $\left\{ (x, x) \;\middle\vert\; x \in X \right\} \subseteq X \times X$
$\mathcal{U} \rightrightarrows X$	An open cover
$V_p(J)$	Projective variety of an ideal
	$\coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \;\middle\vert\; f(\mathbf{x}) = 0, \forall f \in J \right\}$
$I_p(S)$	Projective ideal?
	$\coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle\vert\; f \text{ is homogeneous and } f(x) = 0 \, \forall x \in S \right\}$
$S(X)$	Projective coordinate ring
	$\coloneqq k[x_1, \cdots, x_n]/I_p(X)$
f^h	Homogenization
	$\coloneqq x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)$
f^i	Dehomogenization
J^h for $J \trianglelefteq k[x_1, \cdots, x_n]$	Homogenization of an ideal
	$\coloneqq f(1, x_1, \cdots, x_n)$
$\overline{X}$	Projective closure of a subset
	$\coloneqq V_p(J^h) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n \;\middle\vert\; f^h(\mathbf{x}) = 0 \, \forall f \in X \right\}.$

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\,\cdot\,)$ and $I(\,\cdot\,)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples

0.4.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x-p)$ a point.

0.4.2 Presheaves / Sheaves

- $C^\infty(\,\cdot\,,\mathbb{R})$, a sheaf of smooth functions
- $C^0(\,\cdot\,,\mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\,\cdot\,)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\,\cdot\,)$, the constant sheaf associated to $\mathbb{R}$ (locally constant real-valued functions)
- $\mathrm{Hol}(\,\cdot\,,\mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.5 Useful Algebra Facts

Fact 0.5.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff$ $R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff$ $R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f, then $\langle f \rangle$ is maximal

Proposition 0.5.2 ***(Finitely generated polynomial rings are Noetherian).***
A polynomial ring $k[x_1, \cdots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \cdots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \cdots, f_m \rangle$.

Proof (?).
A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.5.5), $k[x_1, \cdots, x_n]$ is thus Noetherian.

■

Proposition 0.5.3 ***(Properties and Definitions of Ideal Operations).***

$$
\begin{aligned}
I + J &:= \left\{ f + g \;\middle|\; f \in I,\, g \in J \right\} \\
IJ &:= \left\{ \sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\, g_i \in J, N \in \mathbb{N} \right\}
\end{aligned}
$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ \qquad \text{(coprime or comaximal)} \; \langle a \rangle + \langle b \rangle = \langle a, b \rangle \, .$$

Theorem 0.5.4 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Theorem 0.5.5 ***(Hilbert's Basis Theorem).***
If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.6 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

$$
\begin{aligned}
\text{Ring Theory} &\qquad \text{Geometry/Topology of Affine Varieties} \\
\text{Polynomial functions} &\qquad \text{Affine space} \\
k[x_1, \cdots, x_n] &\qquad \mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\} \\
\text{Maximal ideals } \langle x_1 - a_1, \cdots, x_n - a_n \rangle &\qquad \text{Points } [a_1, \cdots, a_n] \in \mathbb{A}^n/k \\
\text{Radical ideals } I \trianglelefteq k[x_1, \cdots, x_n] &\qquad \text{Affine varieties } X \subset \mathbb{A}^n/k, \text{ vanishing locii of polynomi} \\
&I \mapsto V(I) := \left\{ a \;\middle|\; f(a) = 0 \forall f \in I \right\} \\
I(X) := \left\{ f \;\middle|\; f|_X = 0 \right\} &\hookleftarrow X \\
\text{Radical ideals containing } I(X), \text{ i.e. ideals in } A(X) &\qquad \text{closed subsets of } X, \text{ i.e. affine subvarieties} \\
A(X) \text{ is a domain} &\qquad X \text{ irreducible} \\
A(X) \text{ is not a direct sum} &\qquad X \text{ connected} \\
\text{Prime ideals in } A(X) &\qquad \text{Irreducible closed subsets of } X \\
\text{Krull dimension } n \text{ (longest chain of prime ideals)} &\qquad \dim X = n, \text{ (longest chain of irreducible closed subsets}
\end{aligned}
$$