Title

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1 Lecture 07

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Last time: stalks, sheafification, and $Sh(X_{\text{\'et}})$ is abelian. Next up, we're aiming to define sheaf cohomology for $Sh(X_{\text{\'et}})$.

Remark 1.0.1 (Esoteric!): Related to a question asked by a viewer: there is not in fact a morphism from $X_{\text{fppf}} \to X_{\text{\'et}}$, since locally finitely-presented need not be finitely presented (part of the condition for fppf). There is instead a morphism $X_{\text{fppf}} \to X_{\text{\'et},\text{fp}}$ to a corresponding finitely presented site. There is also a map $X_{\text{\'et}} \to X_{\text{\'et},\text{fp}}$ inducing an equivalence on the category of sheaves via pushforward.

Theorem 1.0.2 (Enough injectives).

 $Sh(X_{\text{\'et}})$ has enough injectives.

Proof(?).

Given $\mathcal{F} \in \operatorname{Sh}(X_{\operatorname{\acute{e}t}})$ we want an injective sheaf \mathcal{I} and an injection $\mathcal{F} \hookrightarrow \mathcal{I}$. For each $x \in X$, choose a geometric point \bar{x} over x, and let $I(\bar{x})$ be an injective \mathbb{Z} -module with a map $\mathcal{F}_{\bar{x}} \to I(\bar{x})$. These exist because the category of \mathbb{Z} -modules has enough injectives. The injectives in this category are **divisible** abelian groups.

Claim: The following object works:

$$\mathcal{I} \coloneqq \prod_{\bar{x}} (\iota_{\bar{x}})_* I(\bar{x}).$$

We need to check

- 1. There is a map $\mathcal{F} \to \mathcal{I}$: The RHS is a product, so we map into the components. $\mathcal{F}_{\bar{x}}$ maps into its own associated skyscraper sheaf where the map is sending sections to their germs. Then the skyscraper sheaf for $\mathcal{F}_{\bar{x}}$ maps into the skyscraper sheaf for $I(\bar{x})$ by pushforward.
- 2. This is a monomorphism: check on stalks.
- 3. \mathcal{I} is injective: check the lifting property directly.

1.1 What Else We Get From Sheafification

Remark 1.1.1: We now know that $Sh(X_{\text{\'et}})$ is abelian with enough injectives. This is true for $Sh(\tau)$ for any site τ , but this is substantially harder to show.

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1.1.1 Inverse Images

For $f: X \to Y$, we have a map on presheaves

$$f^{-1}: \operatorname{Presh}(Y_{\operatorname{\acute{e}t}}) \to \operatorname{Presh}(X_{\operatorname{\acute{e}t}})$$

$$\mathcal{F}(V \xrightarrow{\operatorname{\acute{e}t}} X) \mapsto \varprojlim \mathcal{F}(U \to X),$$

where the limit is over diagrams of the form

$$\begin{array}{ccc} V & \longrightarrow & U \\ & \downarrow \text{\'et} & & \downarrow \text{\'et} \\ X & \longrightarrow & Y \end{array}$$

Fact 1.1.2: f^{-1} is left adjoint to pushforward as functors on presheaves.

Exercise 1.1.3(?): Check this.

Definition 1.1.4 (Inverse Image Sheaf)

$$f^*\mathcal{F} \coloneqq \left(f^{-1}\mathcal{F}\right)^a$$
.

Theorem 1.1.5(?).

 f^* is left adjoint to f_* .

Proof (?).

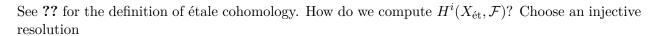
Sheafification is a left adjoint.

Example 1.1.6(?):

- For $\bar{x} \stackrel{\iota}{\hookrightarrow} X$ a geometric point, we have $\iota^* \mathcal{F} = \mathcal{F}_{\bar{x}}$.
- For $Y \xrightarrow{f} X$, we have $f^* \mathbb{Z}/\ell \mathbb{Z} = \mathbb{Z}/\ell \mathbb{Z}$.
- More generally, for $Y \xrightarrow{f} X$ and any representable functor $\mathcal{F} := \underline{\hom}_X(\cdot, Z)$, we have $f^*\mathcal{F} = \underline{\hom}_Y(\cdot, Y \times_X Z)$.

1 Lecture 07

1.2 Étale Cohomology



$$\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

with the \mathcal{I}^j injectives. From the general theory of derived functors, we obtain

$$H^{i}(X_{\operatorname{\acute{e}t}},\mathcal{F}) = H^{i}(\Gamma(X,\mathcal{I}^{\cdot})),$$

where the RHS is a complex of abelian groups. Injective resolutions are difficult to find in general. Suppose $\pi: X_{\text{\'et}} \to Y_{\text{\'et}}$ comes from a map of schemes, then we can compute derived functors of other functors such as the pushforward,

$$\left(R^{i}\pi_{*}\right)\mathcal{F}=H^{i}\left(\pi_{*}\mathcal{I}^{\cdot}\right),$$

where the RHS are sheaves on $Y_{\text{\'et}}$. Implicit here is the claim that π_* is left-exact. You can also find $\left(L^{>0}\pi^*\right)\mathcal{G}=0$.

Exercise 1.2.1(?): Check that pullback is exact.

Proposition 1.2.2(Properties of étale cohomology).

- 1. $H^0(X_{\text{\'et}}, \mathcal{F}) = \mathcal{F}(X)$, aka the global sections $\Gamma(X, \mathcal{F})$.
- 2. $H^{>0}(\mathcal{I}) = 0$ for \mathcal{I} injective.
- 3. Given a SES

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