

# Mapping Class Groups

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# 1 | Setup

- All manifolds:
  - Connected
  - Oriented
  - 2nd countable (countable basis)
  - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
  - With boundary (possibly empty)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Curves: simple, closed, oriented
- For  $X, Y$  topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \left\{ f : X \rightarrow Y \mid f \text{ is continuous} \right\}.$$

## 1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
  - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology  $\mathcal{O}_{\text{CO}}$ :

$$U \in \mathcal{O}_{\text{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

### 1.1.1 Mapping Spaces

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{\text{CO}}) \quad \text{where } \mathcal{O}_{\text{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
  - $\text{Homeo}(X, Y)$  the subspace of homeomorphisms
  - $\text{Imm}(X, Y)$ , the subspace of immersions (injective map on tangent spaces)
  - $\text{Emb}(X, Y)$ , the subspace of embeddings (immersion + diffeomorphic onto image)
  - $C^k(X, Y)$ , the subspace of  $k \times$  differentiable maps
  - $C^\infty(X, Y)$  the subspace of smooth maps
  - $\text{Diffeo}(X, Y)$  the subspace of diffeomorphisms
  - $C^\omega(X, Y)$  the subspace of analytic maps
  - $\text{Isom}(X, Y)$  the subspace of isometric maps (for Riemannian metrics)
  - $[X, Y]$  homotopy classes of maps

## 1.2 Aside on Analysis

- If  $Y = (Y, d)$  is a metric space, this is the topology of “uniform convergence on compact sets”: for  $f_n \rightarrow f$  in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact.}$$

– In words:  $f_n \rightarrow f$  uniformly on every compact set.

- If  $X$  itself is compact and  $Y$  is a metric space,  $C(X, Y)$  can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

### 1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{f_\alpha\} \subset \text{hom}_{\text{Top}}(X, Y)$$

form a compact subset of  $\text{Map}(X, Y)$ ?

- Essentially answered by:

**Theorem 1.1 (Ascoli).**

If  $X$  is locally compact Hausdorff and  $(Y, d)$  is a metric space, a family  $\mathcal{F} \subset \text{hom}_{\text{Top}}(X, Y)$  has compact closure  $\iff \mathcal{F}$  is equicontinuous and  $F_x := \{f(x) \mid f \in \mathcal{F}\} \subset Y$  has compact closure.

**Corollary 1.2 (Arzela).**

If  $\{f_n\} \subset \text{hom}_{\text{Top}}(X, Y)$  is an equicontinuous sequence and  $F_x := \{f_n(x)\}$  is bounded for every  $x$ , it contains a uniformly convergent subsequence.

## 1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
  - Can take  $G$  to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} := \text{hom}_{\text{TopGrp}}(G, S^1)$$

where we consider  $S^1 \subset \mathbb{C}$ .

- Can integrate with respect to the Haar measure  $\mu$ , define  $L^p$  spaces, and for  $f \in L^p(G)$  define a Fourier transform  $\widehat{f} \in L^p(\widehat{G})$ .

$$\widehat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

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## 2 | Path Spaces

- Can immediately consider some interesting spaces via the functor  $\text{Map}(\cdot, Y)$ :

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\ X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\ X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}. \end{aligned}$$

- Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, Y^X) \\ H : X \times Z &\rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\ (x, z) &\mapsto H(x, z) \iff z \mapsto H(\cdot, z). \end{aligned}$$

- Categorically,  $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$  form an adjoint pair in  $\text{Top}$ .
- A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

### 2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space  $PY$ , the based loop space  $\Omega Y$ .
- Importance in homotopy theory: the path space fibration

$$\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y$$

- Plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences.
- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps  $f, g$  are homotopic  $\iff$  they are connected by a path in  $\text{Map}(X, Y)$ .

Picture!

#### 2.1.1 Proof

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, Y^X) \cong \text{Map}(X \times I, Y),$$

and just check that  $\gamma(0) = f \iff H(x, 0) = f$  and  $\gamma(1) = g \iff H(x, 1) = g$ .

- Interpretation: the RHS contains homotopies for maps  $X \rightarrow Y$ , the LHS are paths in the space of maps.

## 2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor*  $\text{Map}(I, \cdot)$ :

$$\begin{aligned}
\mathcal{P}\text{Map}(X, Y) &:= \text{Map}(I, Y^X) \\
\mathcal{P}^2\text{Map}(X, Y) &:= \mathcal{P}\text{Map}(I, Y^X) = \text{Map}(I, (Y^X)^I) = \text{Map}(I, Y^{XI}) \\
&\vdots \\
\mathcal{P}^n\text{Map}(X, Y) &:= \mathcal{P}^{n-1}\text{Map}(I, Y^{XI}) = \text{Map}(X, Y^{XI^n}).
\end{aligned}$$

- Can interpret

$$\mathcal{P}^2\text{Map}(X, Y) = \mathcal{P}\text{Map}(X \times I, Y).$$

as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

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# 3 | Defining the Mapping Class Group

## 3.1 Isotopy

- Define a homotopy between  $f, g : X \rightarrow Y$  as a map  $F : X \times I \rightarrow Y$  restricting to  $f, g$  on the ends.
  - Equivalently: a *path*, an element of  $\text{Map}(I, C(X, Y))$ .
- Isotopy: require the partially-applied function  $F_t : X \rightarrow Y$  to be homeomorphisms for every  $t$ .
  - Equivalently: a path in the subspace of homeomorphisms, an element of  $\text{Map}(I, \text{Homeo}(X, Y))$

Picture: picture of homotopy, paths in  $\text{Map}(X, Y)$ , subspace of homeomorphisms.

### 3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
  - In a general category  $\mathcal{C}$ , we can always define the group  $\text{Aut}_{\mathcal{C}}(X)$ .
    - \* If the group has a topology, we can consider  $\pi_0\text{Aut}_{\mathcal{C}}(X)$ , the set of path components.
    - \* Since groups have identities, we can consider  $\text{Aut}_{\mathcal{C}}^0(X)$ , the path component containing the identity.
  - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X)/\text{Aut}_{\mathcal{C}}^0(X).$$

- Here the  $\pm$  indicates that we take both orientation preserving and non-preserving automorphisms.
- Has an index 2 subgroup of orientation-preserving automorphisms,  $\text{MCG}^+(X)$ .
- Can define  $\text{MCG}_{\partial}(X)$  as those that restrict to the identity on  $\partial X$ .

Picture: quotienting out by identity component

### 3.3 Definitions in Several Categories

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with  $\mathcal{O}_{\text{CO}}$ .

- Taking  $\text{MCG}_{\text{Top}}^{\pm}(X)$  yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking  $\text{MCG}_{C^{\infty}}(X)$  yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f] \in [X, Y]\}.$$

- Taking  $\text{MCG}(X)$  here yields *homotopy classes of self-homotopy equivalences*.



### 3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
  - In the compact-open topology, two maps are isotopic iff they are in the same component of  $\pi\text{Aut}(X)$ .
- For surfaces: For  $\Sigma$  a genus  $g$  surface,  $\text{MCG}(S)$  acts on the Teichmuller space  $T(S)$ , yielding a SES

$$0 \rightarrow \text{MCG}(\Sigma) \rightarrow T(\Sigma) \rightarrow \mathcal{M}_g \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to  $X$ .

- $T(S)$  is the moduli space of complex structures on  $S$ , up to the action of homeomorphisms that are isotopic to the identity:
  - Points are isomorphism classes of marked Riemann surfaces
  - Equivalently the space of hyperbolic metrics
- Used in the Nielsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
  - Periodic,
  - Reducible (preserves some simple closed curves), or
  - Pseudo-Anosov (has directions of expansion/contraction)

Picture:  $\mathcal{M}_g$ .

# 4 | Examples of MCG

## 4.1 The Plane: Straight Lines

- $\text{MCG}_{\text{Top}}(\mathbb{R}^2) = 1$ : for any  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , take the straight-line homotopy:

$$F : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$$

$$F(x, t) = tf(x) + (1 - t)x.$$

Picture: parameterize line between  $x$  and  $f(x)$  and flow along it over time.

## 4.2 The Disc: The Alexander Trick

- $\text{MCG}_{\text{Top}}(\mathbb{D}^2) = 1$ : for any  $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$  such that  $f|_{\partial\mathbb{D}^2} = \text{id}$ , take

$$F : \mathbb{D}^2 \times I \rightarrow \mathbb{D}^2$$

$$F(x, t) := \begin{cases} tf\left(\frac{x}{t}\right) & \|x\| \in [0, t) \\ x & \|x\| \in [1 - t, 1] \end{cases}.$$

- This is an isotopy from  $f$  to the identity.
- Interpretation: “cone off” your homeomorphism over time:

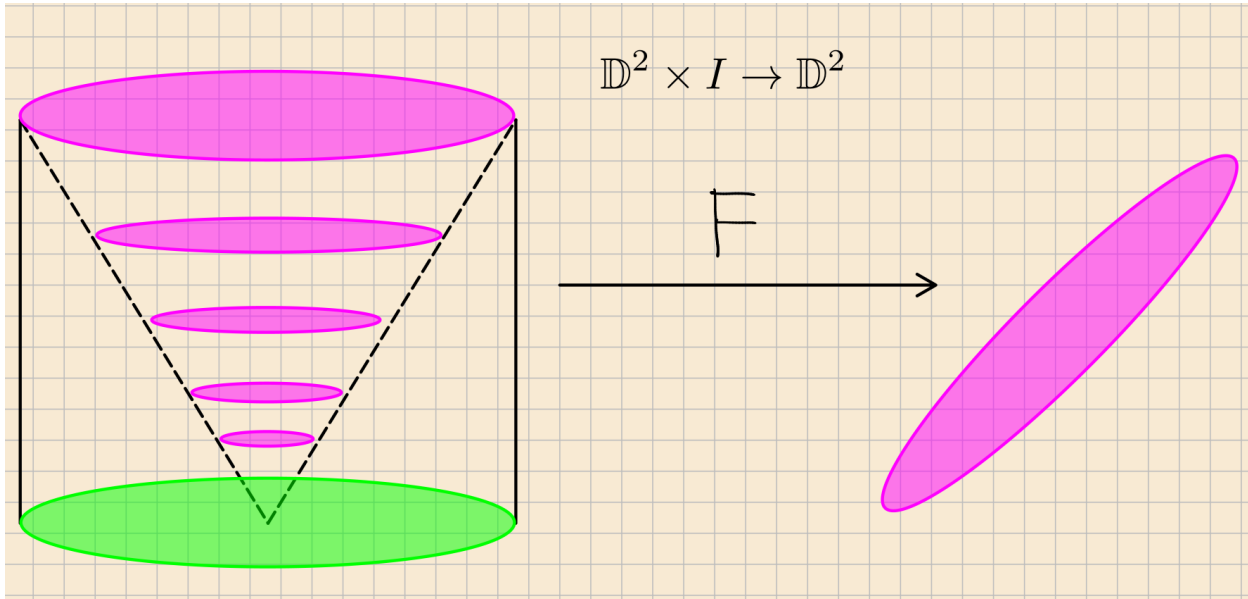


Figure 1: Image

- Note that this won't work in the smooth category: singularity at origin

### 4.3 Overview of Big Results

- The word problem in  $\text{MCG}(\Sigma_g)$  is solvable
- Any finite group is  $\text{MCG}(X)$  for some compact hyperbolic 3-manifold  $X$ .
- For  $g \geq 3$ , the center of  $\text{MCG}(\Sigma_g)$  is trivial and  $H_1(\text{MCG}(\Sigma_g); \mathbb{Z}) = 1$ 
  - Why care: same as abelianization of the group.

**Theorem 4.1 (Dehn-Neilsen-Baer).**

Let  $\Sigma_g$  be compact and oriented with  $\chi(\Sigma_g) < 0$ . Then

$$\text{MCG}_{\partial}^+(\Sigma_g) \cong \text{Out}_{\partial}(\pi_1(\Sigma_g)) \cong_{\text{Grp}} \pi_0 \text{ho}_{\partial}(\Sigma_g).$$

- For  $g \geq 4$ ,  $H_2(\text{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$ 
  - Why care: used to understand surface bundles

$$\begin{array}{ccc} \Sigma_g & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

- Find the classifying space  $B\text{Diffo}$
- Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\text{Diffo}(\Sigma_g)] \cong [S^{n-1}, \text{Diffo}(\Sigma_g)]$$

- Theorem (Earle-Ells): For  $g \geq 2$ ,  $\text{Diffo}_0(\Sigma_g)$  is contractible. As a consequence,  $\text{Diffo}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$  is a homotopy equivalence, and there is a correspondence:

$$\left\{ \begin{array}{c} \text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B \end{array} \right\} / \text{Bundle isomorphism} \iff \left\{ \begin{array}{c} \text{Monodromy Representations} \\ \rho: \pi_1(B) \rightarrow \text{MCG}(\Sigma_g) \end{array} \right\} / \text{Conjugacy}.$$

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# 5 | Dehn Twists

- $\text{MCG}(\Sigma_g)$  is generated by finitely many **Dehn twists**, and always has a finite presentation

**Claim:** Let  $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ , then  $\text{MCG}(A) \cong \mathbb{Z}$ , generated by the map

$$\begin{aligned}\tau_0 : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto \exp(2\pi i|z|)z.\end{aligned}$$

# 6 | MCG of the Torus

**Remark 1.**

There is a natural action of  $\text{MCG}(\Sigma)$  on  $H_1(\Sigma; \mathbb{Z})$ , i.e. a *homology representation* of  $\text{MCG}(\Sigma)$ :

$$\begin{aligned} \rho : \text{MCG}(\Sigma) &\rightarrow \text{Aut}_{\text{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_* . \end{aligned}$$

- For a surface of finite genus  $g \geq 1$ , elements in  $\text{im} \rho$  preserve the *algebraic intersection form*, which is a symplectic pairing.
- Thus there is a surjective representation

$$\rho : \text{MCG}(\Sigma_g) \twoheadrightarrow \text{Sp}(2g; \mathbb{Z}).$$

- Kernel is the *Torelli group*.
- Every homology class in  $H_1$  can be represented by a (possibly non-simple) loop.
- Algebraic intersection: a bilinear antisymmetric form  $\hat{\iota}$  on  $H_1(\Sigma_g; \mathbb{Z})$ 
  - $x$  is isotropic iff  $\iota(x, \cdot) = 0$ .

**Definition 6.0.1** (Special Linear Group).

$$\text{SL}(n, \mathbb{k}) = \left\{ M \in \text{GL}(n, \mathbb{k}) \mid \det M = 1 \right\} = \ker \det_{\mathbb{G}_m} .$$

**Remark 2.**

$$\text{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle .$$

Note that  $S^2 = 1$  and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Moreover, if  $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$  and  $A \in \text{SL}(2, \mathbb{Z})$ , we have  $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$ , i.e. this preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \mid p, q \in \mathbb{Z} \right\} \cong \left\{ p\omega_1 + q\omega_2 \mid p, q \in \mathbb{Z} \right\} \simeq \left\{ p' + q'\tau \mid p', q' \in \mathbb{Z} \right\} .$$

where the  $\omega_i, \tau$  come from identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ , and in the last step we've rescaled the lattice by *homothety* to align one vector with the  $x$ -axis.

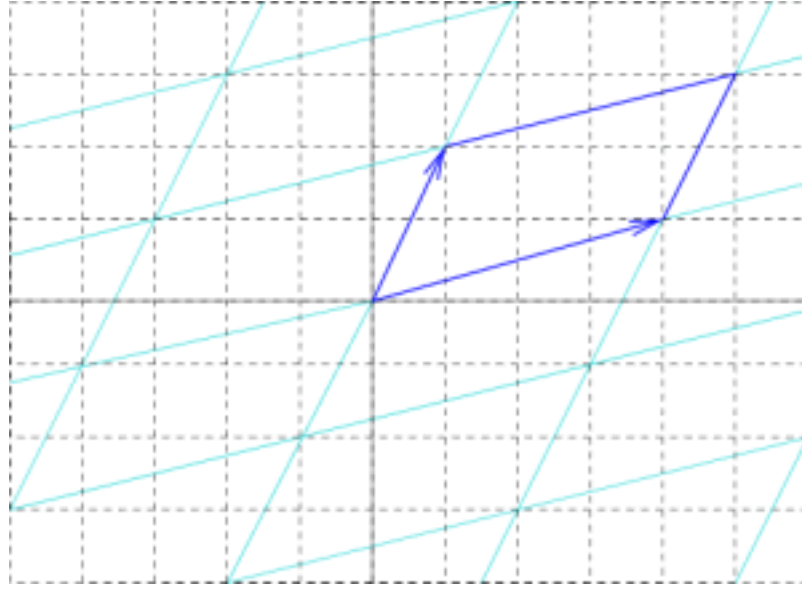


Figure 2: Lattice

**Remark 3.**

For any finite-index subgroup  $G \leq \mathrm{SL}(2, \mathbb{Z})$ , the orbits/left-quotient  $G \backslash \mathbb{H}$  yields a complex curve (i.e. a torus).

**Theorem 6.1 (Mapping Class Group of the Torus).**

The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

**6.1 Proof**

- For  $f$  any automorphism, the induced map  $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Homeo}(X, X), \circ) &\rightarrow (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*. \end{aligned}$$

- This will descend to the quotient  $\mathrm{MCG}(X)$  iff

$$\mathrm{Homeo}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$$

- This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned}\tilde{\sigma} : \text{MCG}(X) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

**Claim:**  $\text{im}(\tilde{\sigma}) \subseteq \text{SL}(2, \mathbb{Z})$ .

*Proof .*

- Algebraic intersection numbers in  $\Sigma_2$  correspond to determinants
- $f \in \text{Homeo}^+(X)$  preserve algebraic intersection numbers.
- See section 1.2

■

- We can thus freely restrict the codomain to define the map

$$\begin{aligned}\sigma : \text{MCG}(X) &\rightarrow \text{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

**Claim:**  $\sigma$  is surjective.

- $\mathbb{R}^2$  is the universal cover of  $\Sigma_2$ , with deck transformation group  $\mathbb{Z}^2$ .
- Any  $A \in \text{SL}(2, \mathbb{Z})$  extends to  $\tilde{A} \in \text{GL}(2, \mathbb{R})$ , a linear self-homeomorphism of the plane that is orientation-preserving.

**Claim:**  $\tilde{A}$  is equivariant wrt  $\mathbb{Z}^2$

*Proof .*

?

■

- So  $\tilde{A}$  descends to a well-defined map  $\psi_{\tilde{A}}$  on  $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$ , which is still a linear self-homeomorphism
- There is a correspondence

$$\{\text{Primitive vectors in } \mathbb{Z}^2\} \iff \left\{ \frac{\text{Oriented simple closed curves in } \Sigma_2}{\text{homotopy}} \right\}.$$

- Thus  $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$ , and we have surjectivity.

**Claim:**  $\sigma$  is injective.

- Useful fact:  $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$ .

**Proposition 6.2(Hatcher 1B.9).**

Let  $X$  be a connected CW complex and  $Y$  a  $K(G, 1)$ . Then there is a map

$$\text{hom}_{\text{Grp}}(\pi_1(X; x_0), \pi_1(Y; y_0)) \rightarrow \text{hom}_{\text{Top}}((X; x_0), (Y; y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map.

Moreover, the map is unique up to homotopies fixing  $x_0$ .

- Thus there is a correspondence

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } \Sigma_2 \hookrightarrow \end{array} \right\} \iff \left\{ \text{Group morphisms } \mathbb{Z}^2 \hookrightarrow \right\}.$$

- Claim: any element  $f \in \text{MCG}(\Sigma_2)$  has a representative  $\varphi$  which fixes any given basepoint
- So if  $f \in \ker \sigma$ , then  $f \simeq \varphi \simeq \text{id}$  are homotopic, so  $\ker \sigma = 1$ .