

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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1 | Tuesday, January 12

1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
 - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic \mathbb{R}^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

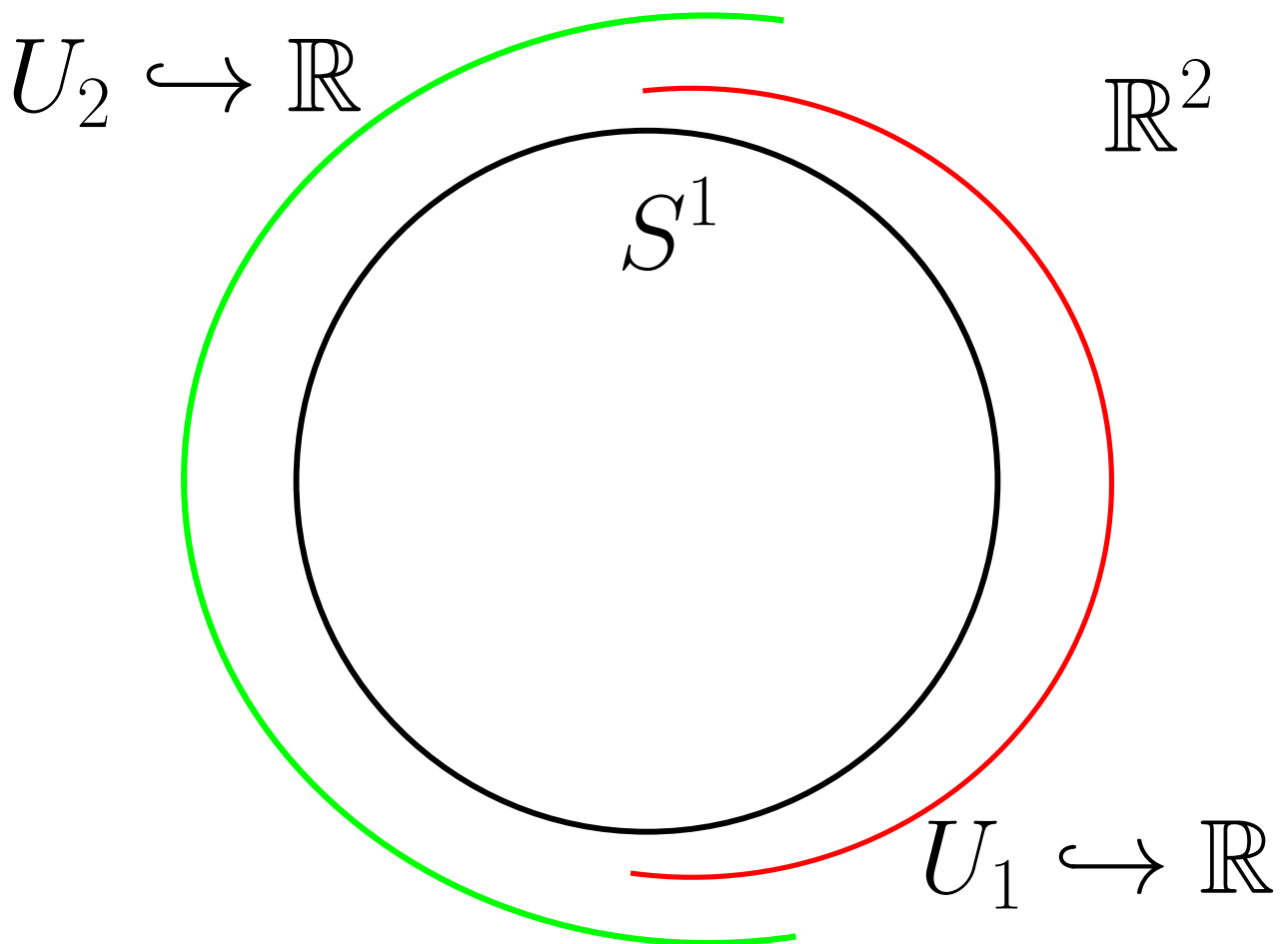
A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

Definition 1.2.1 (Topological Manifold)

Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to \mathbb{R}^n with a countable topological base, so we have charts $\varphi_u : U \rightarrow \mathbb{R}^n$ which are homeomorphisms from open sets covering X .

Example 1.2.2 (The circle): S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on \mathbb{CP}^N .

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6

Consider \mathbb{R}^n as a topological manifold: are any two smooth structures on \mathbb{R}^n diffeomorphic?

Remark 1.2.7: Fix a copy of \mathbb{R} and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\rightarrow \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on X , since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

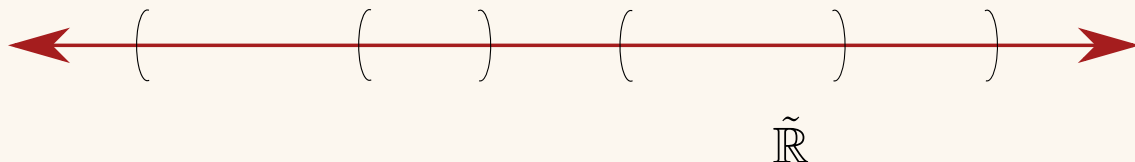
$$\begin{aligned} X &\rightarrow X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

Claim: \mathbb{R} admits a unique smooth structure.

Proof (sketch).

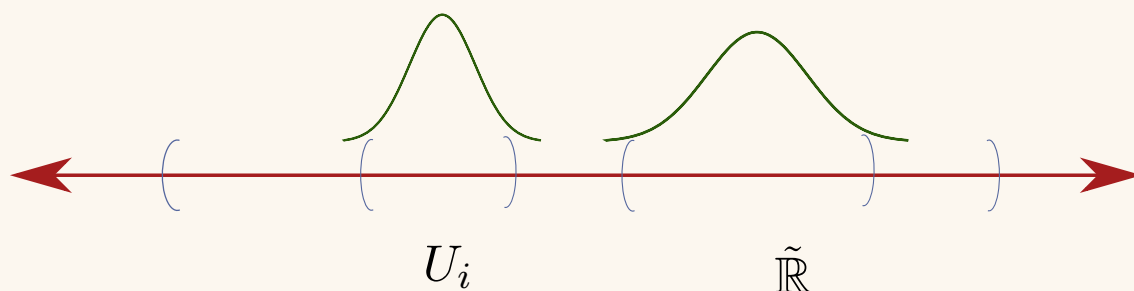
Let $\tilde{\mathbb{R}}$ be some exotic \mathbb{R} , i.e. a smooth manifold homeomorphic to \mathbb{R} . Cover this by coordinate charts to the standard \mathbb{R} :



Fact

There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \rightarrow \mathbb{R}$ with $f_i \geq 0$ and $\text{supp } f \subseteq U_i$ such that $\sum f_i = 1$ (i.e., *bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\rightarrow \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan

To understand smooth structures on X , we should try to solve differential equations on X .



Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2 .

Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.¹
2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $ks(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure.

¹Note that this doesn't start at C^0 , so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if $\text{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.
6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, \mathbb{R}^n for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes) \mathbb{R}^4 admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface Σ , the space of complex-analytic structures is a complex orbifold² of dimension $3g - 2$ where g is the genus of Σ , up to biholomorphism (i.e. *moduli*).

Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group. 

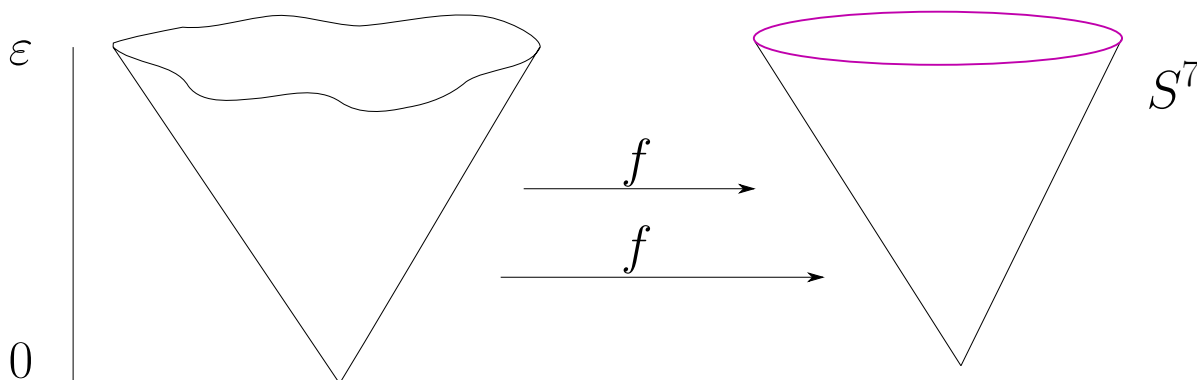
2 | Friday, January 15

Remark 2.0.1: Let

$$V := \{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\} \subseteq \mathbb{C}^5$$

$$S_\varepsilon := \{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \dots, 28$ yields the 28 smooth structures on S^7 . Note that V_k is the cone over $V_k \cap S_\varepsilon$.



? Admits a smooth structure, and $\bar{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure. 

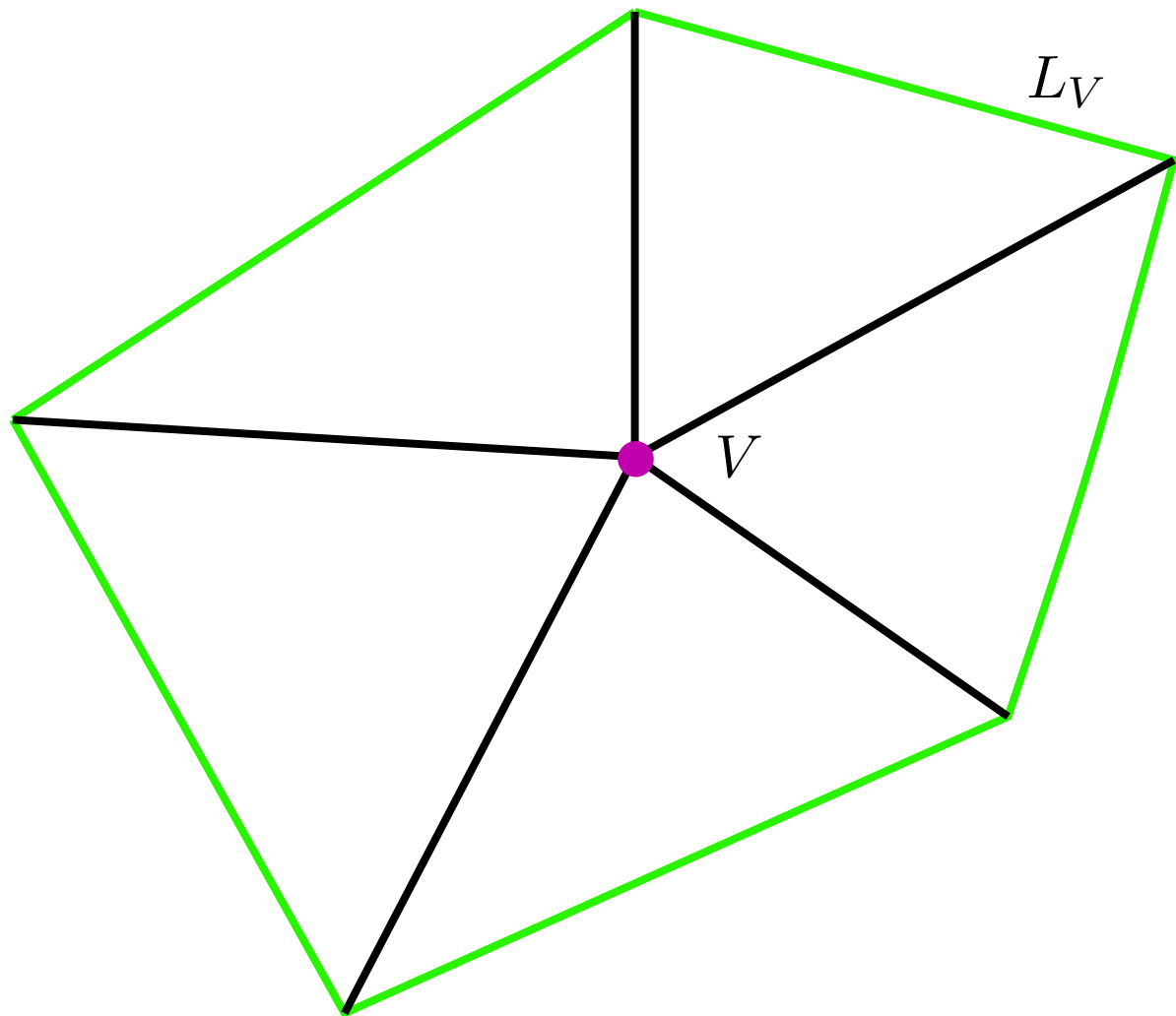
²Locally admits a chart to \mathbb{C}^n/Γ for Γ a finite group.

Question 2.0.2

Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

Answer 2.0.3

No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V :



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a “PL triangulation” where the links are spheres.

Remark 2.0.4: What’s special in dimension 4? Recall the **Kirby-Siebenmann** invariant $ks(x) \in H^4(X; \mathbb{Z}_2)$ for X a topological manifold where $ks(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)

Let X be a topological 4-manifold, then

$$\text{ks}(X) := \text{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2,$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\text{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures.} \end{cases}$$

Remark 2.0.7: $\text{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X; \mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha, \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 (Freedman).

If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\text{ks}(X) = \text{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha, \varphi\beta \rangle = \langle \alpha, \beta \rangle$.

Conversely, every **unimodular** bilinear form appears as $H^2(X; \mathbb{Z})$ for some X , i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\rightarrow H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for “being an element of the orthogonal group”, where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X; \mathbb{Z}) \xrightarrow{PD} H_2(X; \mathbb{Z})$ from Poincaré, where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1, \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \cap \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)

The **signature** of a topological 4-manifold is the signature of Q_X , where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X; \mathbb{R})$ and for some a, b

$$(H^2(X; \mathbb{R}), Q_X) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of +1s appearing in the matrix and b is the number of -1s. This is \mathbb{R}^{ab} where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a + 1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 (Rokhlin's Theorem).

Suppose $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X; \mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\text{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 (Donaldson).

Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X; \mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: \mathbb{R}^2 Has a Unique Smooth Structure

Remark 3.1.1: Last time we showed \mathbb{R}^1 had a unique smooth structure, so now we'll do this for \mathbb{R}^2 . The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)

A **Riemannian metric** $g \in \text{Sym}^2 T^*X$ for X a smooth manifold is a metric on every T_pX given by

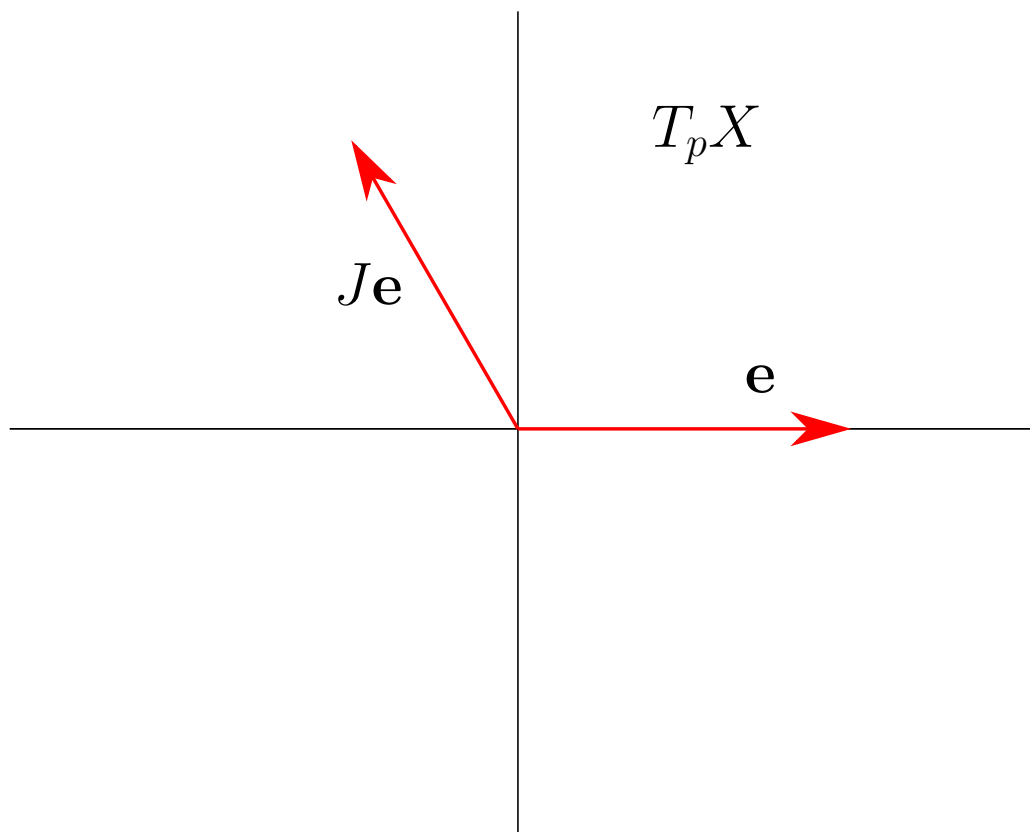
$$g_p : T_pX \times T_pX \rightarrow \mathbb{R}$$

$$g(v, v) \geq 0, g(v, v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)

An **almost complex structure** is a $J \in \text{End}(TX)$ such that $J^2 = -\text{id}$.

Remark 3.1.4: Let $e \in T_pX$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of T_pX .



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

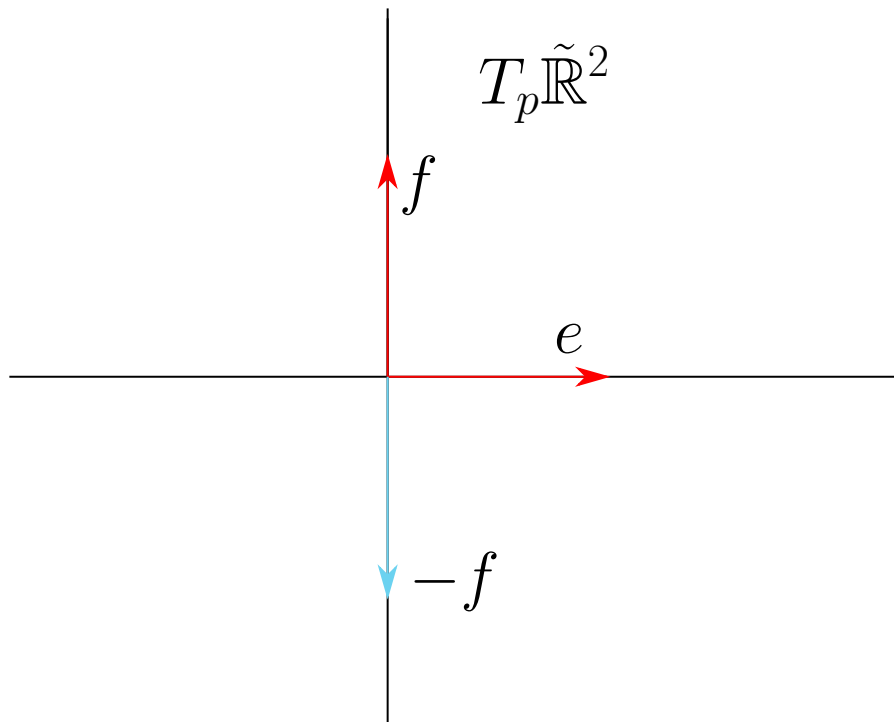
Let $\tilde{\mathbb{R}}^2$ be an exotic \mathbb{R}^2 .

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g := \sum f_I g_I$ with g_I metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e := f \in T_p \tilde{\mathbb{R}}^2$ such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$.
- $\{e, f\}$ is an oriented basis of $T_p \tilde{\mathbb{R}}^2$

This is because after choosing e , there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5(?).

Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \rightarrow \mathbb{C}$ such that J is multiplication by i .

So $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to \mathbb{D} or \mathbb{C} , both of which are diffeomorphic to \mathbb{R}^2 .

See the Newlander-Nirenberg theorem, a result in complex geometry.

4 | Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

4.1 Sheaves

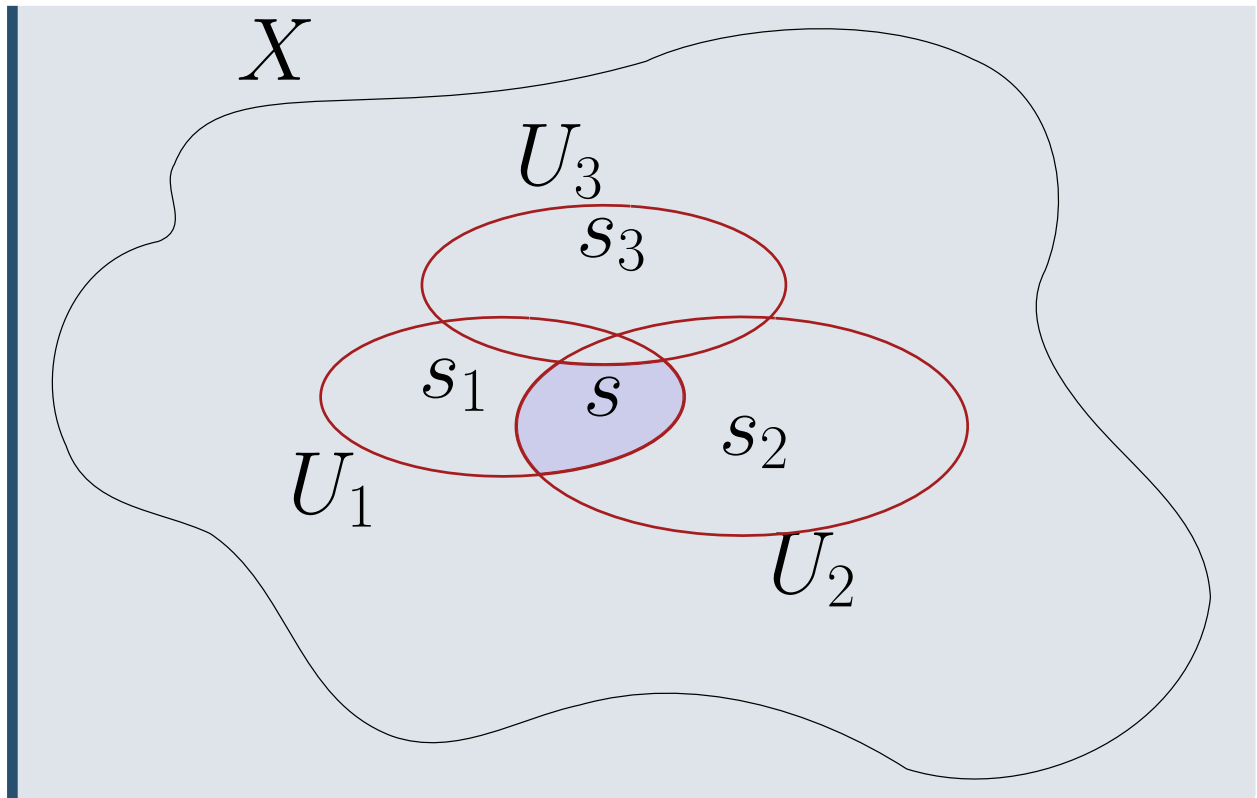
Definition 4.1.1 (Presheaves and Sheaves)

Recall that if X is a topological space, a **presheaf** of abelian groups \mathcal{F} is an assignment $U \rightarrow \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

- a. $\mathcal{F}(\emptyset) = 0$, the trivial abelian group.
- b. $\rho_{UU} : \mathcal{F}(U) \rightarrow \mathcal{F}(U) = \text{id}_{\mathcal{F}(U)}$
- c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say \mathcal{F} is a **sheaf** if additionally:

- d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2(?): Let X be a topological manifold, then $\mathcal{F} := C^0(\cdot, \mathbb{R})$ the set of continuous functionals form a sheaf. We have a diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\mathcal{F}} & C^0(U; \mathbb{R}) \\
 \uparrow & & \downarrow \text{restrict cts. functions} \\
 V & \xrightarrow{\mathcal{F}} & C^0(V; \mathbb{R})
 \end{array}$$

[Link to diagram](#)

Property (d) holds because given sections $s_i \in C^0(U_i; \mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in C^0(\bigcup_i U_i; \mathbb{R})$ such that $s|_{U_i} = s_i$ for all i – continuous functions glue.

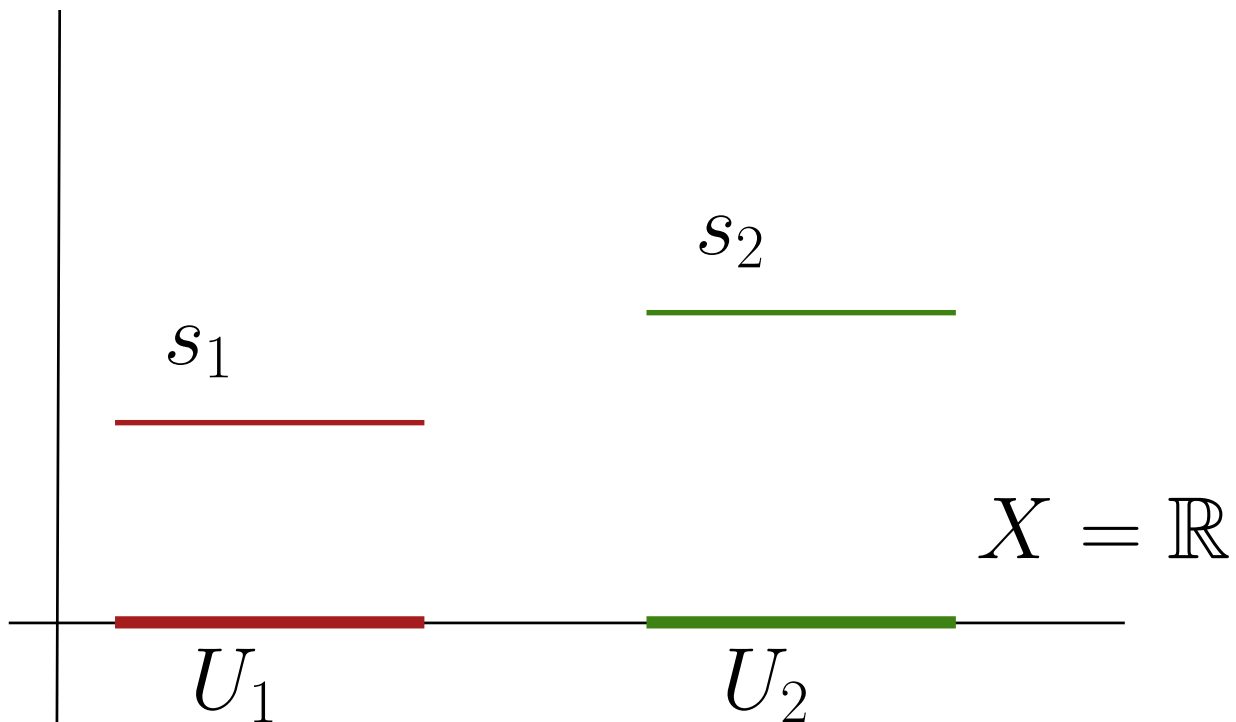
Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote \mathcal{O} . For example, $\mathcal{O} := C^0(\cdot; \mathbb{R})$ for topological manifolds, and $\mathcal{O} := C^\infty(\cdot; \mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set \mathcal{O} to be the sheaf of holomorphic functions.

Example 4.1.4(Locally Constant Sheaves): Let $A \in \mathbf{Ab}$ be an abelian group, then \underline{A} is the

sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \rightarrow A$. E.g. let $X \in \mathbf{Mfd}_{\mathbf{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant \implies globally constant in this case.

Warning 4.1.5

Note that the presheaf of constant functions doesn't satisfy (d)! Take \mathbb{R} and a function with two different values on disjoint intervals:



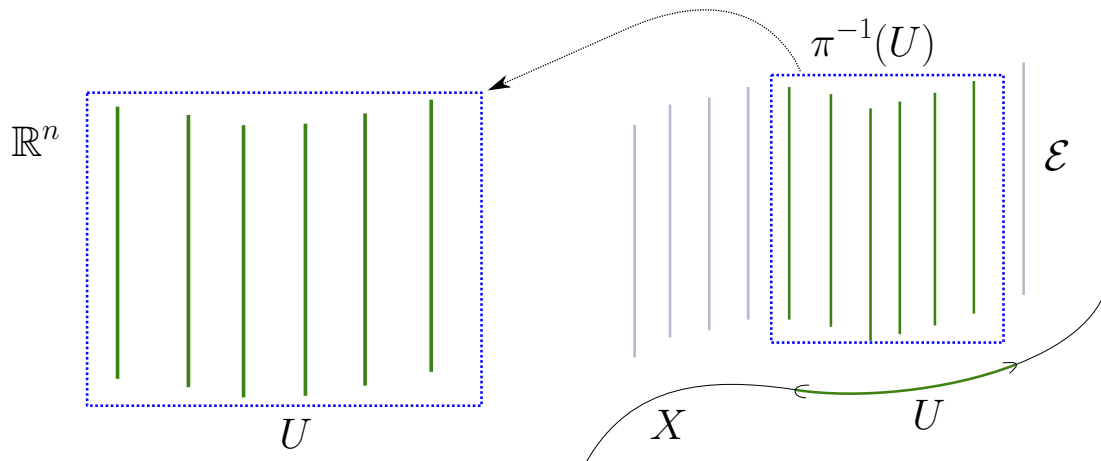
Note that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

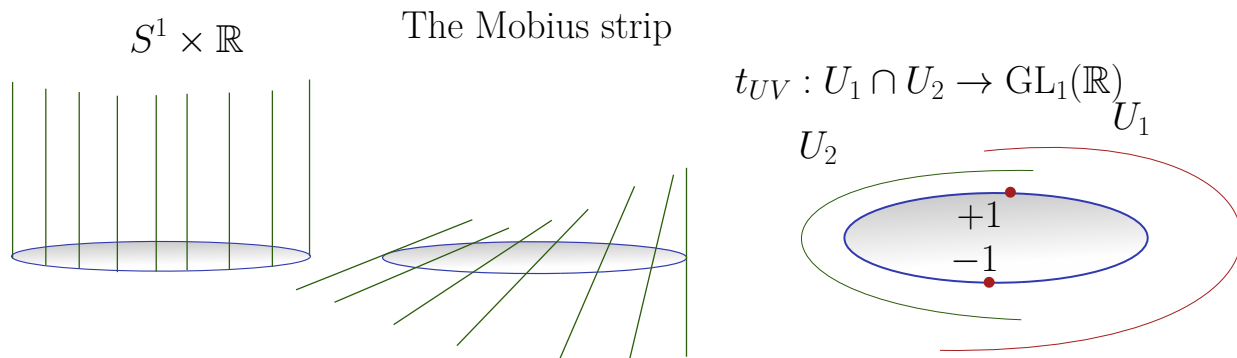
Remark 4.2.1: Let $\pi : \mathcal{E} \rightarrow X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \rightarrow \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 (Bundles over S^1): There are two \mathbb{R}^1 bundles over S^1 :



Note that the Möbius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: \mathcal{E} is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \rightarrow \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \text{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4(?): The trivial line bundle gives the sheaf $\mathcal{O} : \text{maps } U \xrightarrow{s} U \times Y \text{ for } Y = \mathbb{R}, \mathbb{C}$ such that $\pi \circ s = \text{id}$ are the same as maps $U \rightarrow Y$.

Definition 4.2.5 (\mathcal{O} -modules)

An \mathcal{O} -module is a sheaf \mathcal{F} such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6(?): If \mathcal{E} is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \curvearrowright s := fs$, i.e. just multiplying functions.

Example 4.2.7 (Non-example): The locally constant sheaf $\underline{\mathbb{R}}$ is not an \mathcal{O} -module: there isn't natural action since the sections of \mathcal{O} are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

Definition 4.2.8 (Morphisms of Sheaves)

A **morphism** of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:


$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Example 4.2.9 (*An \mathcal{O} -module that is not a vector bundle.*): Let $X = \mathbb{R}$ and define the skyscraper sheaf at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) := \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}$$

The $\mathcal{O}(U)$ -module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\rightarrow \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments. 

Example 4.2.10 (*of morphisms*): Let $X = \mathbb{R} \in \mathbf{Mfd}_{\text{Sm}}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x \cdot) : \mathcal{O} &\rightarrow \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$


for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.11 (?)

Check that $\ker \varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x \cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\text{coker}(x \cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to \mathbb{R}_0 . So $\mathcal{O} \rightarrow \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{(x \cdot)} \mathcal{O} \rightarrow \mathbb{R}_0 \rightarrow 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles. 

5 | Lecture 4 (Friday, January 22)

5.1 The Exponential Exact Sequence

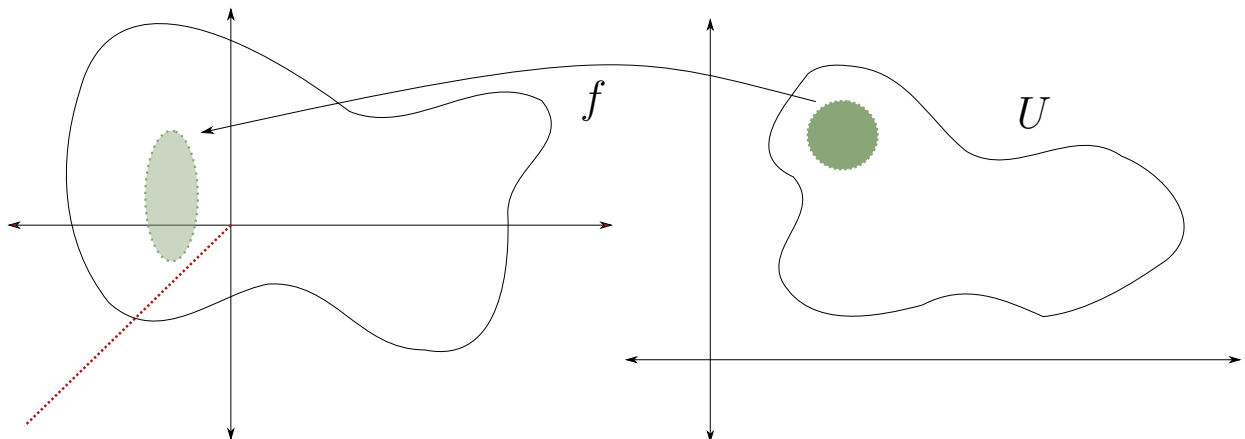
Let $X = \mathbb{C}$ and consider \mathcal{O} the sheaf of holomorphic functions and \mathcal{O}^\times the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \rightarrow \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \rightarrow \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\mathbb{Z}$, and we get an exact sequence

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \rightarrow \text{coker}(\exp) \rightarrow 0.$$

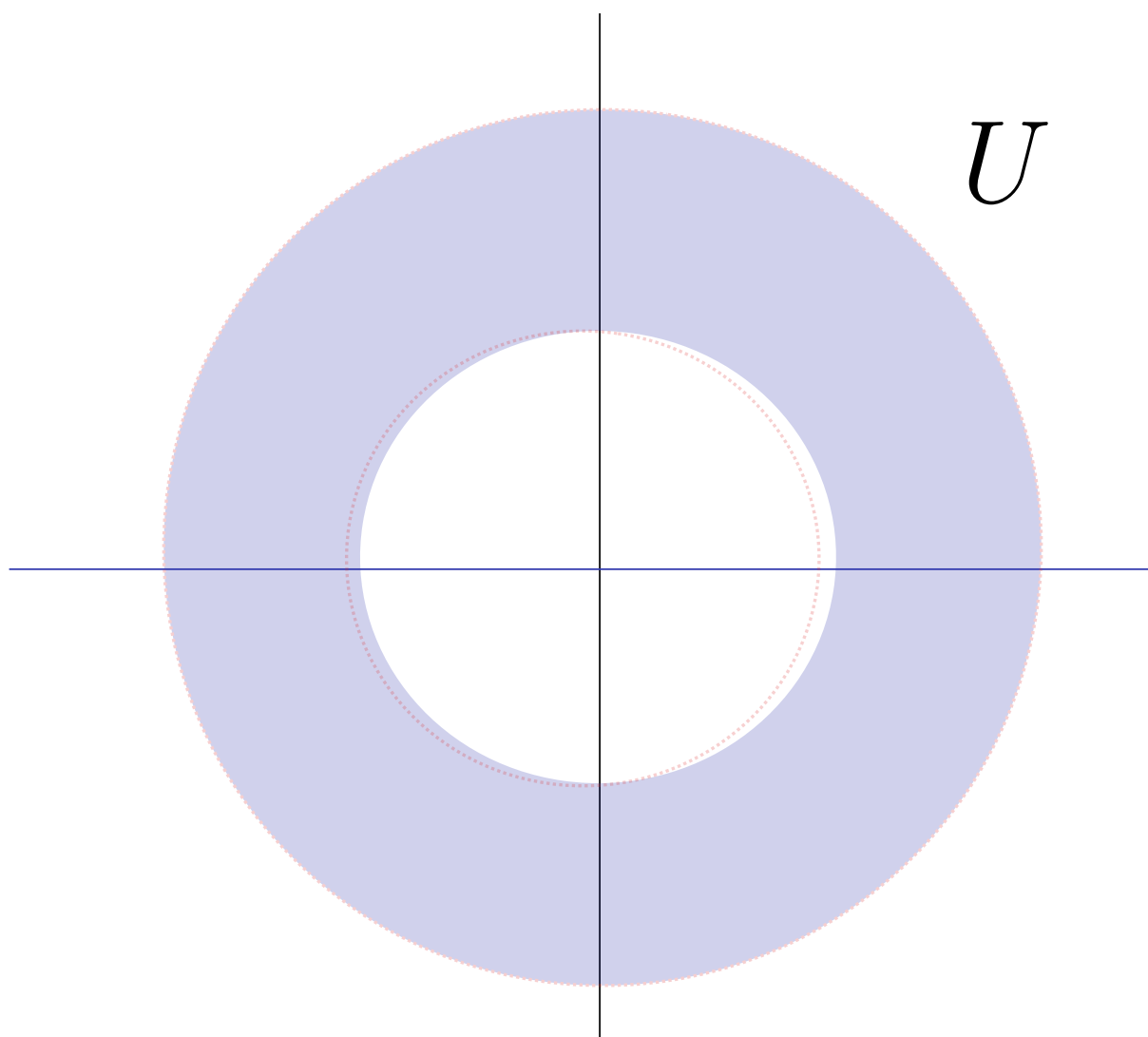
Question 5.1.1

What is the cokernel sheaf here?

Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U :



Thus on any sufficiently small open set, $\text{coker}(\exp) = 1$. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z , which is nontrivial.

Exercise 5.1.2 (?)

Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property $\text{Hom}_{\text{Presheaf}}(\mathcal{G}, \mathcal{F}^{\text{Presheaf}}) = \text{Hom}_{\text{Sheaf}}(\mathcal{G}, \mathcal{F}^{\text{Sh}})$ for any sheaf \mathcal{G} .

Definition 5.1.3 (Global Sections Sheaf)

The **global sections** sheaf of \mathcal{F} on X is given by $H^0(X; \mathcal{F}) = \mathcal{F}(X)$.

Example 5.1.4 (?):

- $C^\infty(X) = H^0(X, C^\infty)$ are the smooth functions on X
- $VF(X) = H^0(X; T)$ are the smooth vector fields on X for T the tangent bundle

- If X is a complex manifold then $\mathcal{O}(X) = H^0(X; \mathcal{O})$ are the globally holomorphic functions on X .
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

Remark 5.1.5: Given vector bundles V, W , we have constructions $V \oplus W, V \otimes W, V^\vee, \text{Hom}(V, W) = V^\vee \otimes W, \text{Sym}^n V, \bigwedge^p V$, and so on. Some of these work directly for sheaves:

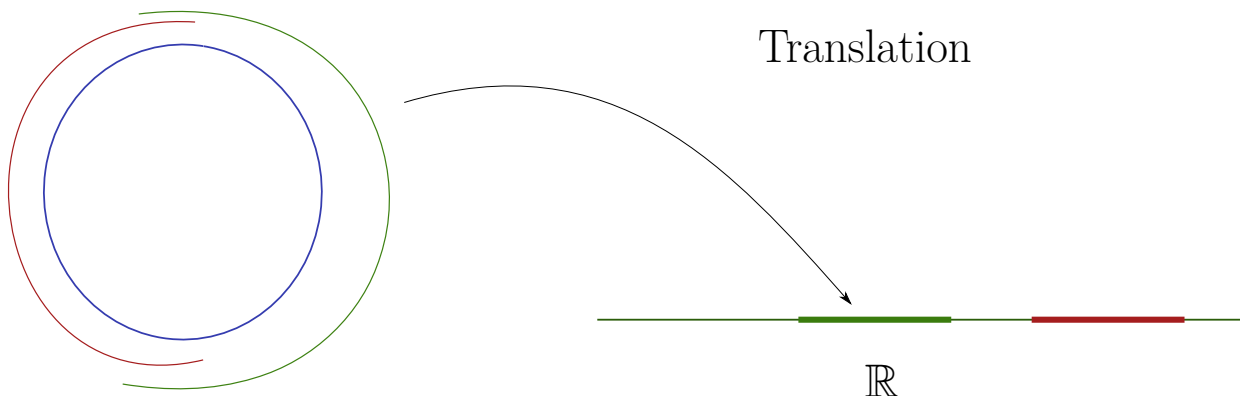
- $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathcal{H}\text{om}(V, W)$ we only get presheaves, so we need to sheafify.

Warning 5.1.6

$\text{Hom}(V, W)$ will denote the *global* homomorphisms $\mathcal{H}\text{om}(V, W)(X)$, which is a sheaf.

Example 5.1.7(?): Let $X^n \in \mathbf{Mfd}_{\text{sm}}$ and let Ω^p be the sheaf of smooth p -forms, i.e. $\bigwedge^p T^\vee$, i.e. $\Omega^p(U)$ are the smooth p forms on U , which are locally of the form $\sum f_{i_1, \dots, i_p}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_p}$ where the f_{i_1, \dots, i_p} are smooth functions.

Example 5.1.8(Sub-example): Take $X = S^1$, writing this as \mathbb{R}/\mathbb{Z} , we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x+c) = dx$ for c a constant:



Exercise 5.1.9(?)

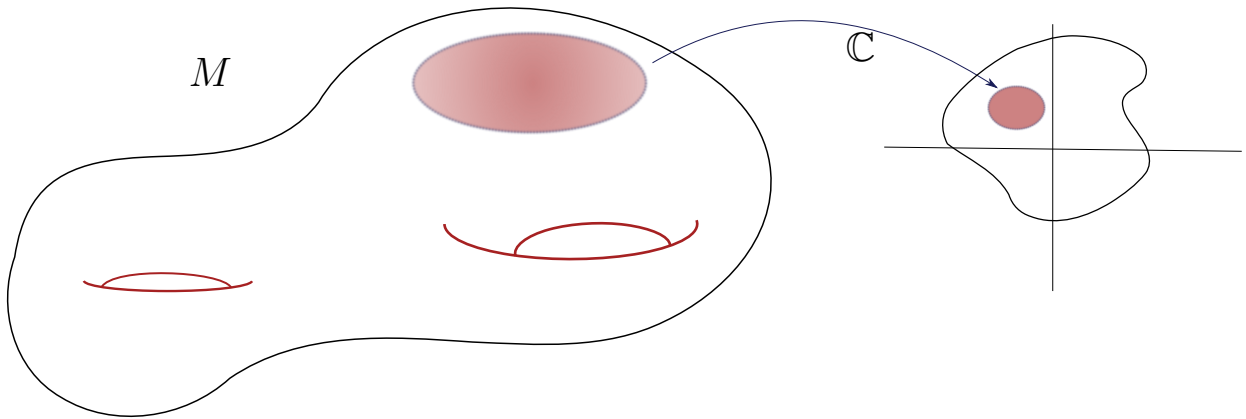
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.1.10: Note that there is a map $d : \Omega^p \rightarrow \Omega^{p+1}$ where $\omega \mapsto d\omega$.

Warning 5.1.11

d is **not** a map of \mathcal{O} -modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a \mathbb{C} -vector bundle.



Remark 5.1.12 (Subtlety 1): Note that Ω^p for complex manifolds is $\bigwedge^p T^\vee$, and so if we want to view $X \in \mathbf{Mfd}_\mathbb{R}$ we'll write $X_\mathbb{R}$. $TX_\mathbb{R}$ is then a real vector bundle of rank $2n$.

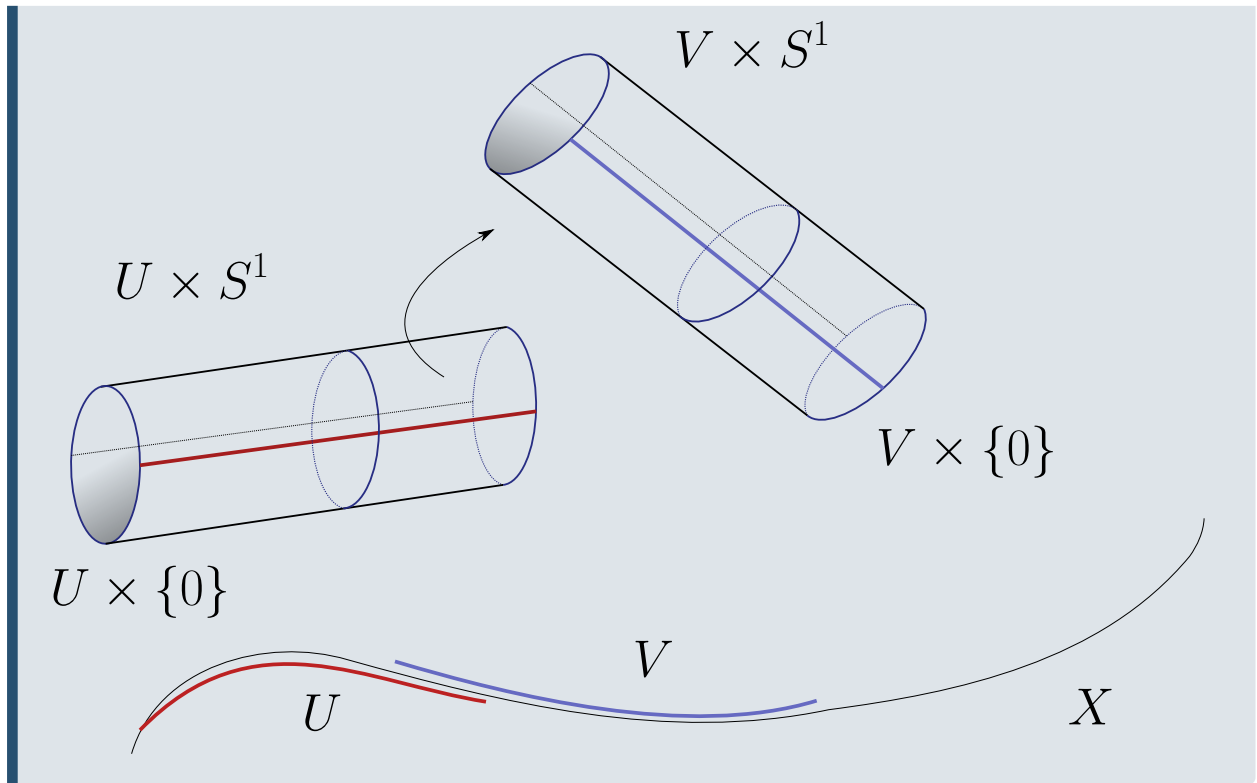
Remark 5.1.13 (Subtlety 2): Ω^p will denote *holomorphic* p -forms, i.e. local expressions $\sum f_I(z_1, \dots, z_n) \bigwedge dz_I$. For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$ -forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.1.14: Note that $T^\vee X_\mathbb{R} \otimes_\mathbb{C} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = f dz + g d\bar{z}$ where f, g are smooth. Then $\Omega^d X_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas Ω^1 is infinite-dimensional.

6 | Principal G -Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)

Let G be a (possibly disconnected) Lie group. Then a **principal G -bundle** $\pi : P \rightarrow X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \rightarrow G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \rightarrow G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \mathbf{Mfd}_{\text{Sm}}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \rightarrow \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)

Define $\text{Frame}_p(X) := \{\text{bases of } T_p X\}$, and $\text{Frame}X := \bigcup_{p \in X} \text{Frame}_p X$.

Remark 6.0.4: More generally, $\text{Frame}\mathcal{E}$ can be defined for any vector bundle \mathcal{E} , so $\text{Frame}X := \text{Frame}TX$. Note that $\text{Frame}X$ is a principal $\text{GL}_n(\mathbb{R})$ -bundle where $n := \text{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\text{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 (Important): A principal G -bundle admits a G -action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\rightarrow P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point:** the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)

The **orthogonal frame bundle** of a vector bundle \mathcal{E} equipped with a metric g is defined as $\text{OFrame}_p \mathcal{E} := \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r := \text{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \rightarrow \{x\}$ of a principal G -bundle are naturally **torsors** over G , i.e. a set with a free transitive G -action.

Definition 6.0.8 (?)

Let $\mathcal{E} \rightarrow X$ be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \rightarrow \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from \mathcal{E} .

Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

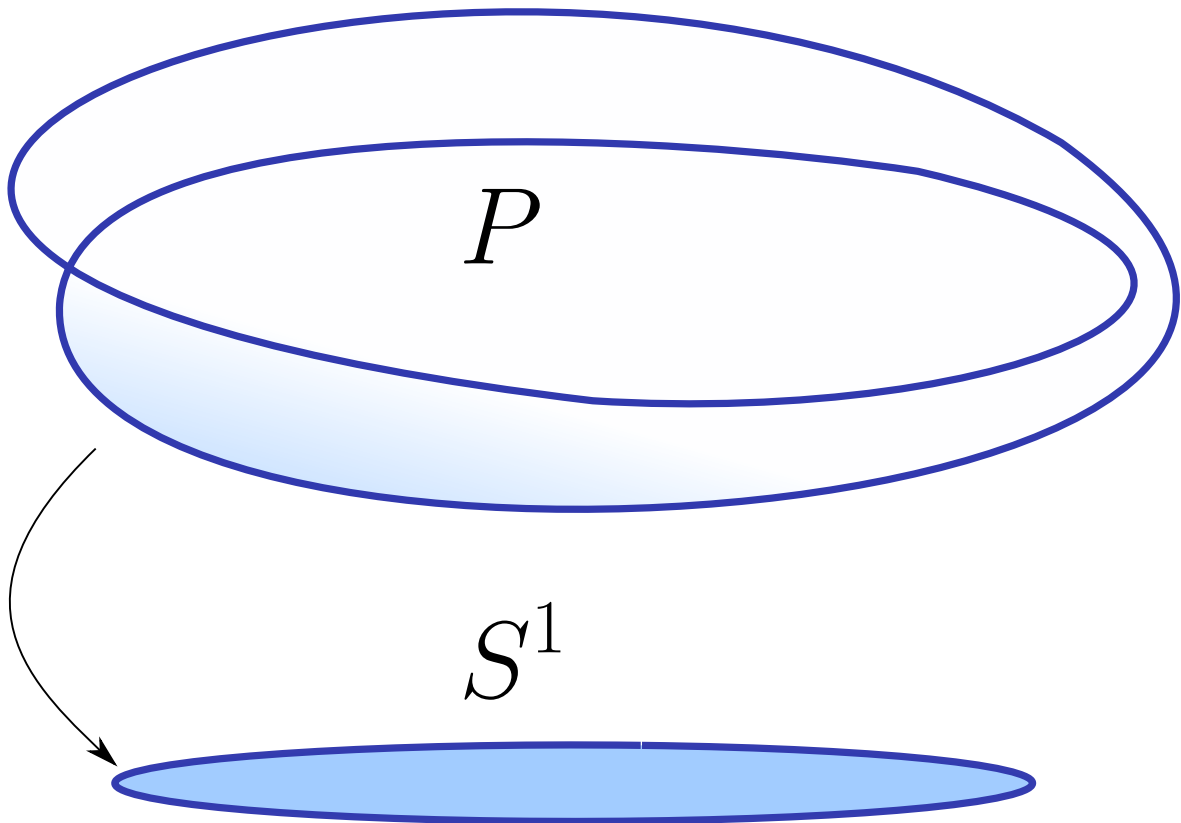
$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\mapsto (\bar{z}_1, \dots, \bar{z}_n). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)

We define the **unitary frame bundle** $\text{UFrame}(\mathcal{E}) := \bigcup_p \text{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of \mathcal{E}_p given by (e_1, \dots, e_n) where $h(e_i, \bar{e}_j) = \delta_{ij}$.

Remark 6.0.11: This is a principal G -bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\bar{A}^t = \text{id}$.

Example 6.0.12 (of more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G -bundle:



Example 6.0.13 (more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U , i.e.

$$P := \{(x, O) \mid x \in X, O \text{ is an orientation of } T_p X\}.$$

Note that P is an oriented manifold, $P \rightarrow X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \text{Frame}X / \text{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)

Let $P \rightarrow X$ be a principal G -bundle and let $G \rightarrow \text{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \{(p, v) \mid p \in P, v \in V\} / \sim \quad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 (?): Note that $\text{Frame}(\mathcal{E})$ is a $\text{GL}_r(\mathbb{R})$ -bundle and the map $\text{GL}_r(\mathbb{R}) \xrightarrow{\text{id}} \text{GL}(\mathbb{R}^r)$ is

a representation. At every fiber, we have $G \times_G V = (p, v) / \sim$ where there is a unique representative of this equivalence class given by (e, pv) . So $P \times_G V_p \rightarrow \{p\} \cong V_x$.

Exercise 6.0.16(?)

Show that $\text{Frame}(\mathcal{E}) \times_{\text{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \rightarrow G$, and so by the equivalence relation, $\text{im } t_{UV} \in \text{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Then $TM \rightarrow \text{Frame}(M)$ which is a principal $\text{SO}(3)$ -bundle. The universal cover is the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a \mathbb{C}^2 bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)

Let $\mathcal{E} \rightarrow X$ be a vector bundle, then a **connection** on \mathcal{E} is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\rightarrow \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge \omega + s \otimes d\omega. \end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \rightarrow \Omega^{p+1}$. The standard example is

$$\begin{aligned} d : \mathcal{O} &\rightarrow \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

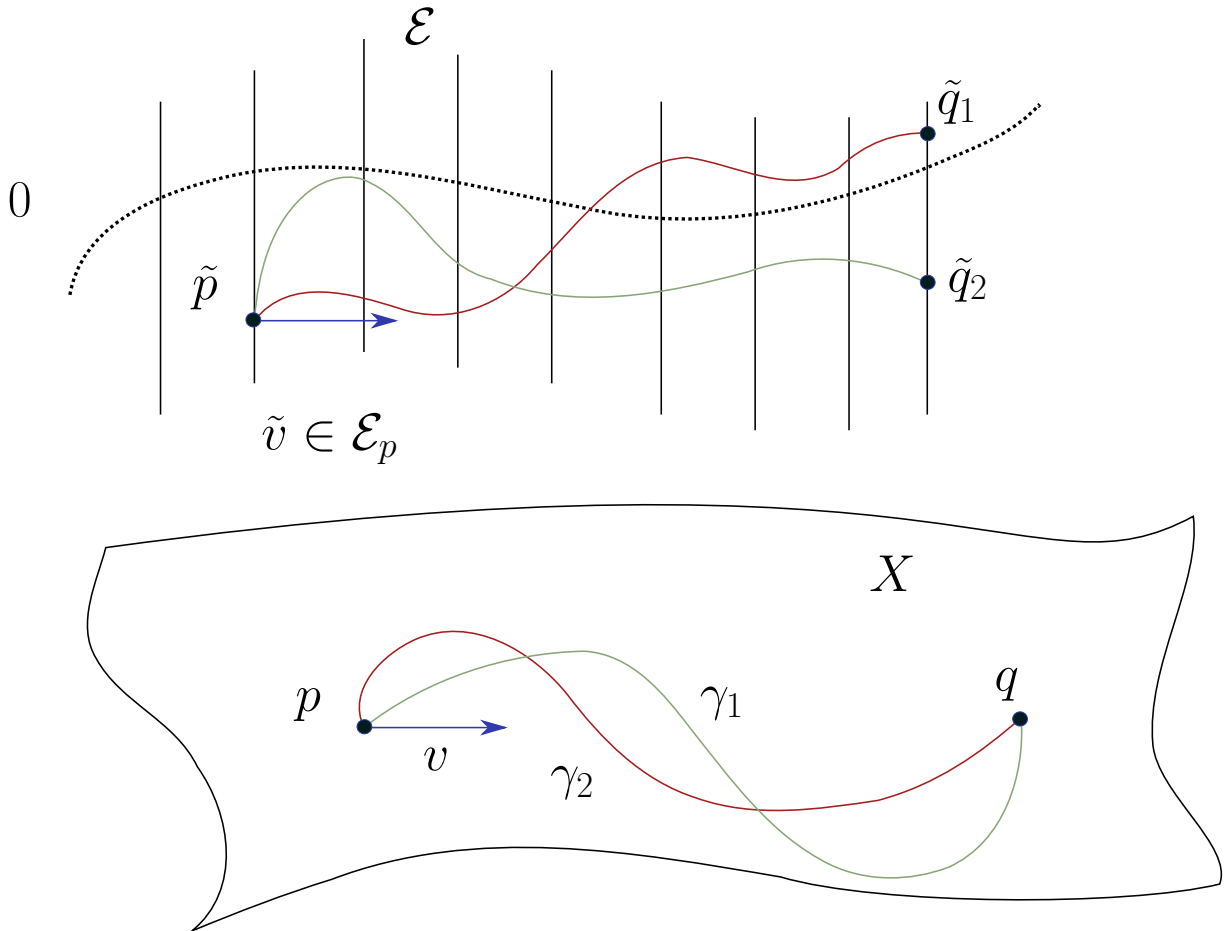
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇ , i.e. the map

$$F_{\nabla} := \nabla \circ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^2$$

is \mathcal{O} -linear, so $F_{\nabla}(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E} \otimes \Omega^{p+1}$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

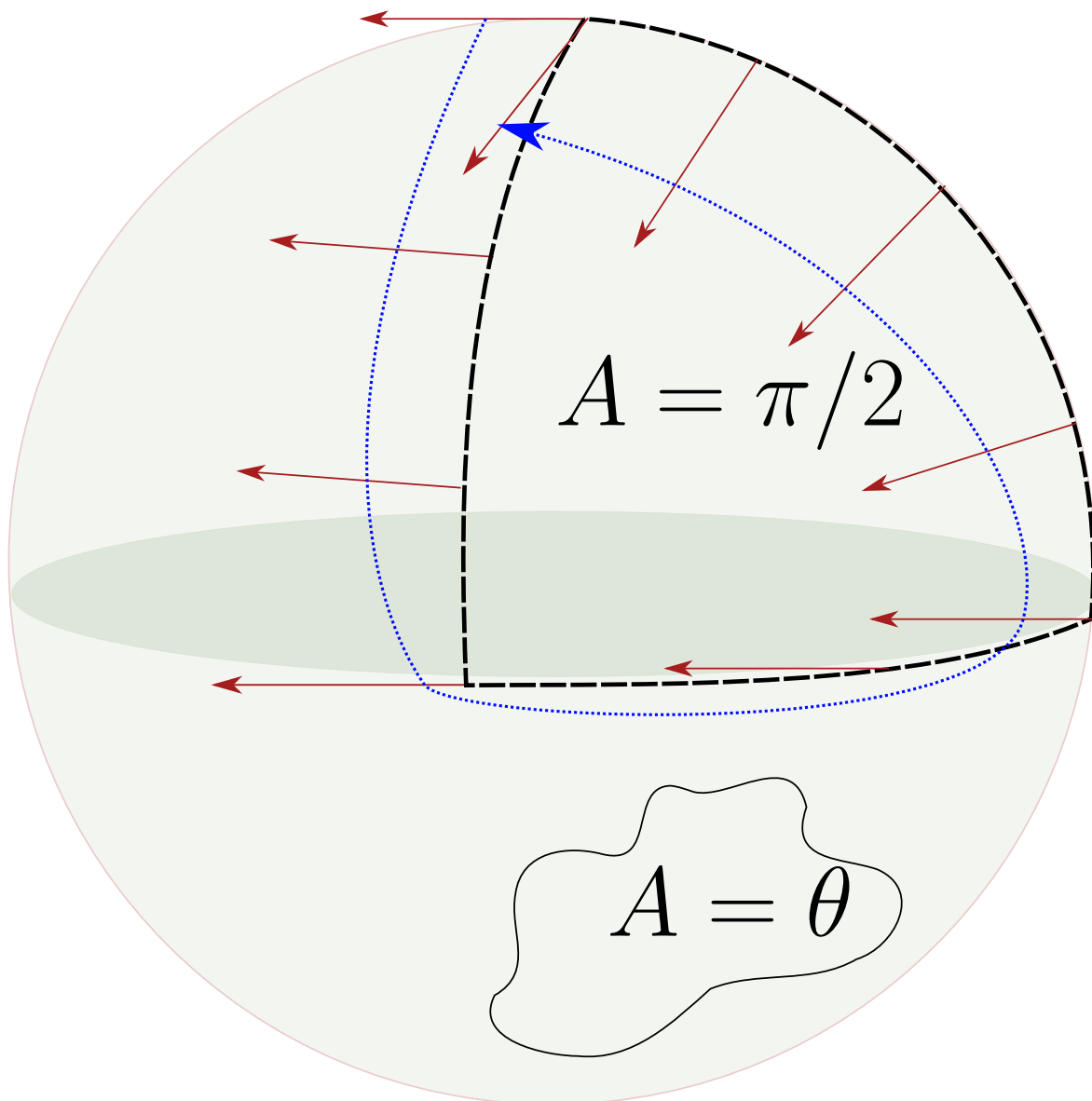


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1 , yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ .

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_{\nabla} = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ .

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 (A classic example): The Levi-Cevita connection ∇^{LC} on TX , which depends on a metric g . Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)

A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) := \left\{ s \in \mathcal{E}(U) \mid \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)

Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}}^d$. Furthermore $\text{rank}(L) = \text{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (?)

Let \mathcal{F} be a sheaf of abelian groups on a topological space X , and let $\mathfrak{U} := \{U_i\} \rightrightarrows X$ be an open cover of X . Let $U_{i_1, \dots, i_p} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1 < \dots < i_p} \mathcal{F}(U_{i_1, \dots, i_p})$$

with a differential

$$\begin{aligned} \partial^p : C_{\mathfrak{U}}^p(X, \mathcal{F}) &\rightarrow C_{\mathfrak{U}}^{p+1}(X, \mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \dots, i_p} := \prod_j (-1)^j \sigma_{i_0, \dots, \widehat{i_j}, \dots, i_p} \Big|_{U_{i_0, \dots, i_p}} \end{aligned}$$

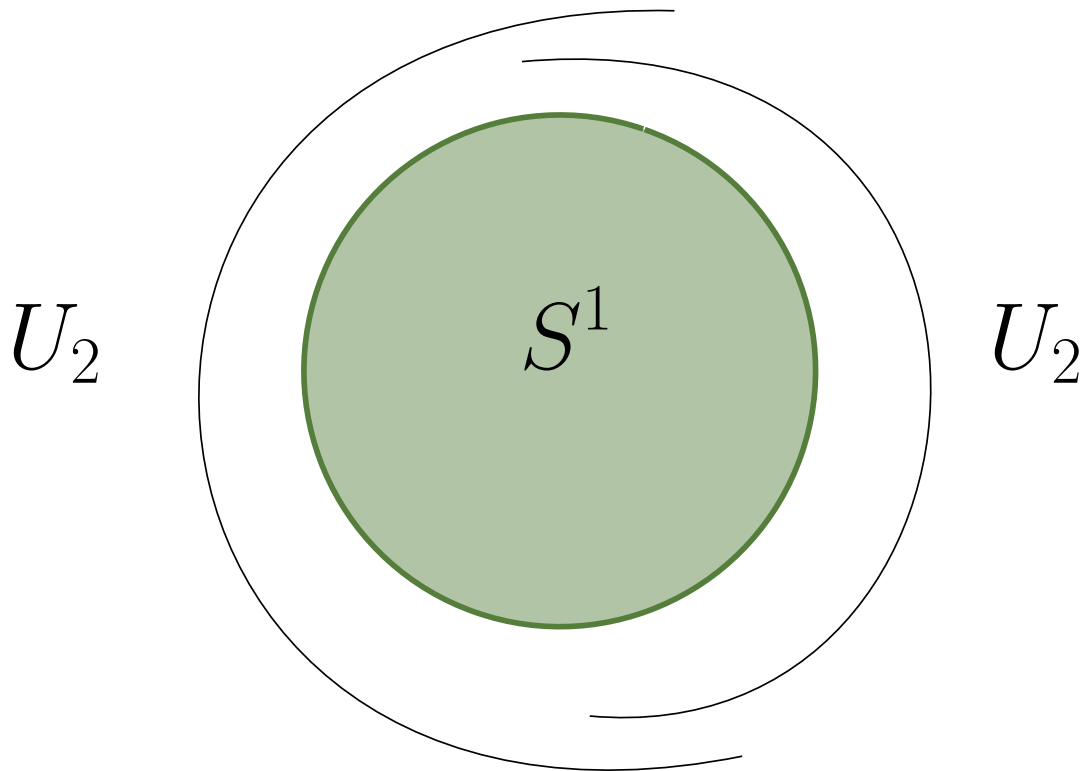
where we've defined this just on one given term in the product, i.e. a p -fold intersection.

Exercise 7.2.2 (?)

Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H_{\mathfrak{U}}^p(X, \mathcal{F})$ with respect to the cover \mathfrak{U} is defined as $\ker \partial^p / \text{im } \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4(?): Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



ere we have

$$C^0(S^1, \mathbb{Z}) = \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) = \mathbb{Z} \oplus \mathbb{Z},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\text{double intersections}} \mathbb{Z}(U_{ij})\mathbb{Z}(U_{12}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}.$$

We then get

$$\begin{aligned} C^0(S^1, \mathbb{Z}) &\xrightarrow{\partial} C^1(S^1, \mathbb{Z}) \\ \mathbb{Z} \oplus \mathbb{Z} &\rightarrow \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields $H^*(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex $C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1, \dots, i_p} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_p})$ for $\mathfrak{U} := \{U_i\}$ is an open cover of X and \mathcal{F} is a sheaf of abelian groups.

Fact 8.0.1

If \mathfrak{U} is a sufficiently fine cover then $H_{\mathfrak{U}}^p(X, \mathcal{F})$ is independent of \mathfrak{U} , and we call this $H^p(X; \mathcal{F})$.

Remark 8.0.2: Recall that we computed $H^p(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

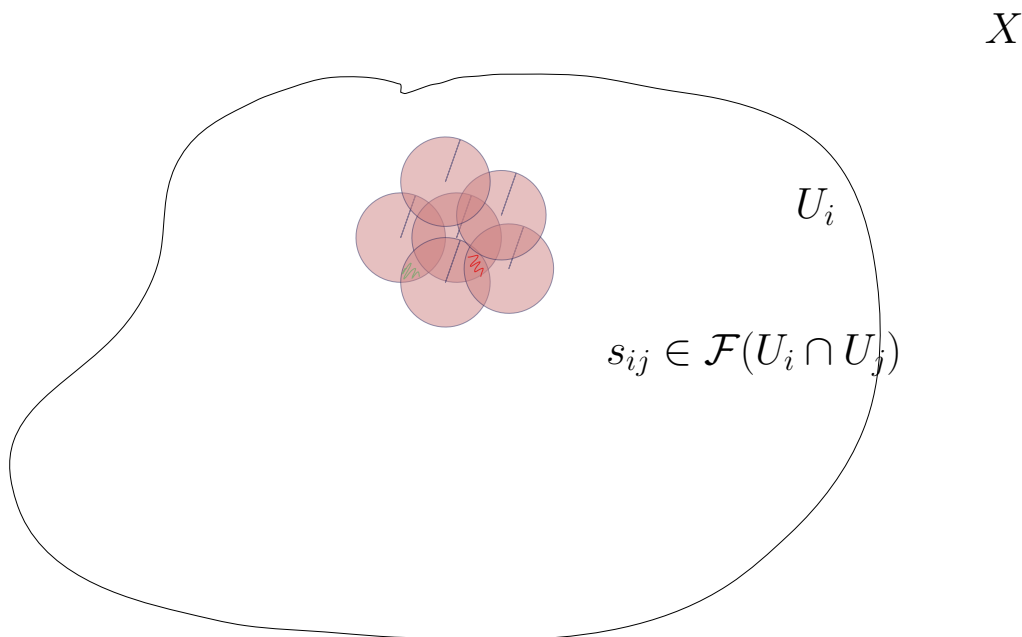
Theorem 8.0.3(?)

Let X be a paracompact and locally contractible topological space. Then $H^p(X, \mathbb{Z}) \cong H_{\text{Sing}}^p(X, \mathbb{Z})$. This will also hold more generally with \mathbb{Z} replaced by \underline{A} for any $A \in \mathbf{Ab}$.

Definition 8.0.4 (Acyclic Sheaves)

We say \mathcal{F} is *acyclic* on X if $H^{>0}(X; \mathcal{F}) = 0$.

Remark 8.0.5: How to visualize when $H^1(X; \mathcal{F}) = 0$:



On the intersections, we have $\text{im } \partial^0 = \{(s_i - s_j)_{ij} \mid s_i \in \mathcal{F}(U_i)\}$, which are *cocycles*. We have $C^1(X; \mathcal{F})$ are collections of sections of \mathcal{F} on every double overlap. We can check that $\ker \partial^1 = \{(s_{ij}) \mid s_{ij} - s_{ik} + s_{jk} = 0\}$, which is the cocycle condition. From the exercise from last class, $\partial^2 = 0$.

Theorem 8.0.6 (*Important!*).

Let X be a paracompact Hausdorff space and let

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a SES of sheaves of abelian groups, i.e. $\mathcal{F}_3 = \text{coker}(\varphi)$ and φ is injective. Then there is a LES in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \mathcal{F}_1) & \longrightarrow & H^0(X; \mathcal{F}_2) & \longrightarrow & H^0(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & H^1(X; \mathcal{F}_1) & \longrightarrow & H^1(X; \mathcal{F}_2) & \longrightarrow & H^1(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

Example 8.0.7(?): For X a manifold, we can define a map and its cokernel sheaf:

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

Corollary 8.0.8(of theorem).

Suppose $0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$ is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose I_n is acyclic for all $n \geq 0$. This is referred to as an **acyclic resolution**. Then the homology can be computed at $H^p(X; \mathcal{F}) = \ker(I_p(X) \rightarrow I_{p+1}(X)) / \text{im}(I_{p-1}(X) \rightarrow I_p(X))$.

Note that locally having kernels equal images is different than satisfying this globally!

Proof (of corollary).

This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{array}{ll} 0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} \text{im}(d_0) \rightarrow 0 & \text{im}(d_0) = \ker(d_1) \\ 0 \rightarrow \ker(d_1) \hookrightarrow I_1 \rightarrow I_1/\ker(d_1) = \text{im}(d_1) & \text{im}(d_1) = \ker(d_2) \\ & \cdot \end{array}$$

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving I_0 , we obtain:

$$\begin{array}{ccccc} & & & & \dots \\ & & & \nearrow & \\ H^{p-1}(\mathcal{F}) & \xleftarrow{\quad} & H^{p-1}(\mathcal{I}_i) = 0 & \xrightarrow{\quad} & H^{p-1}(\text{im}(\lceil \cdot \rceil)) \\ & & \searrow \cong & & \\ & & H^p(\mathcal{F}) & \xleftarrow{\quad} & \dots = 0 \end{array}$$

The middle entries vanish since I_* was assumed acyclic, and so we obtain $H^p(\mathcal{F}) \cong H^{p-1}(\text{im } d_0) \cong H^{p-1}(\ker d_1)$. Now taking the LES associated to I_1 , we get $H^{p-1}(\ker d_1) \cong H^{p-2}(\text{im } d_1)$. Continuing this inductively, these are all isomorphic to $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$ after the p th step. ■

Corollary 8.0.9 (of the previous corollary).

Suppose $\mathfrak{U} \rightrightarrows X$, then if \mathcal{F} is acyclic on each U_{i_1, \dots, i_p} , then \mathfrak{U} is sufficiently fine to compute Čech cohomology, and $H_{\mathfrak{U}}^p(X; \mathcal{F}) \cong H^p(X; \mathcal{F})$.

Proof (?).

See notes. ■

Corollary 8.0.10 (of corollary).

Let $X \in \mathbf{Mfd}_{\setminus}$, then $H^p(X, \mathbb{R}) = H_{\text{dR}}^p(X; \mathbb{R}R)$.

Proof (?).

Idea: construct an acyclic resolution of the sheaf \mathbb{R} on M . The following exact sequence works:

$$0 \rightarrow \underline{\mathbb{R}} \rightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \rightarrow \dots$$

So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that $\ker d^p = \operatorname{im} d^{p-1}$ on any contractible coordinate chart. In other words, we want to show that if $d\omega = 0$ for $\omega \in \Omega^p(\mathbb{R}^n)$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$. This is true by integration! Using the previous corollary, $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$. ■

Check Hartshorne to see how injective resolutions line up with derived functors!

9 | Monday, February 01

Remark 9.0.1: Last time $\underline{\mathbb{R}}$ on a manifold M has a resolution by vector bundles:

$$0 \rightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

This is an exact sequence of sheaves of any smooth manifold, since locally $d\omega = 0 \implies \omega = d\alpha$ (by the *Poincaré d-lemma*). We also want to know that Ω^k is an acyclic sheaf on a smooth manifold. ✍

Exercise 9.0.2 (?)

Let $X \in \mathbf{Top}$ and $\mathcal{F} \in \mathbf{Sh}(\mathbf{Ab})_X$. We say \mathcal{F} is **flasque** if and only if for all $U \supseteq V$ the map $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ is surjective. Show that \mathcal{F} is acyclic, i.e. $H^i(X; \mathcal{F}) = 0$. This can also be generalized with a POU.

Example 9.0.3(?): The function $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$, but doesn't extend to a continuous map on \mathbb{R} . So the restriction map is not surjective. ✍

Remark 9.0.4: Any vector bundle on a smooth manifold is acyclic. Using the fact that Ω^k is acyclic and the above resolution of $\underline{\mathbb{R}}$, we can write $H^k(X; \mathbb{R}) = \ker(d_k) / \operatorname{im} d_{k-1} := H_{dR}^k(X; \mathbb{R})$. ✍

Remark 9.0.5: Now letting $X \in \mathbf{Mfd}_{\mathbb{C}}$, recalling that Ω^p was the sheaf of holomorphic p -forms. Locally these are of the form $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$ where $f_I(\mathbf{z})$ is holomorphic. There is a resolution

$$0 \rightarrow \Omega^p \rightarrow A^{p,0},$$

where in $A^{p,0}$ we allowed also f_I are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth $(p, 0)$ form to check if it's holomorphic? ✍

Example 9.0.6 (?): For $p = 0$, we have

$$0 \rightarrow \mathcal{O} \rightarrow A^{0,0}.$$

where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations.

Definition 9.0.7 (?)

Let $\omega \in A^{p,q}(X)$ where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J := \partial + \bar{\partial}$$

with $|I| = p, |J| = q$.

Example 9.0.8 (?): The function $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$ is smooth, and $df = \bar{z}dz + z d\bar{z}$. This can be checked by writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, and $\frac{\partial}{\partial \bar{z}} g = 0$ if and only if g is holomorphic. Here we get $\partial\omega \in A^{p+1,q}(X)$ and $\bar{\partial} \in A^{p,q+1}(X)$, and we can write $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$.

Definition 9.0.9 (Cauchy-Riemann Equations)

Recall the Cauchy-Riemann equations: ω is a holomorphic $(p,0)$ -form on \mathbb{C}^n if and only if $\bar{\partial}\omega = 0$.

Remark 9.0.10: Thus to extend the previous resolution, we should take

$$0 \rightarrow \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \rightarrow \dots$$

The fact that this is exact is called the *Poincaré $\bar{\partial}$ -lemma*.

Remark 9.0.11: There are no bump functions in the holomorphic world, and since Ω^p is a holomorphic bundle, it may not be acyclic. However, the $A^{p,q}$ are acyclic (since they are smooth vector bundles and thus admit POU's), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \text{im}(\bar{\partial}_{q-1}).$$

Note the similarity to H_{dR} , using $\bar{\partial}$ instead of d . This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers** $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Omega^p)$. These are analogies:

Smooth	Complex
\mathbb{R}	Ω^p
Ω^k	$A^{p,q}$
Betti numbers β_k	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

Theorem 9.0.12 (Properties of Singular Cohomology).

Let $X \in \mathbf{Top}$, then $H_{\text{Sing}}^i(X; \mathbb{Z})$ satisfies the following properties:

- Functoriality: given $f \in \text{Hom}_{\mathbf{Top}}(X, Y)$, there is a pullback $f^* : H^i(Y; \mathbb{Z}) \rightarrow H^i(X; \mathbb{Z})$.
- The cap product: a pairing

$$H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X; \mathbb{Z}) \rightarrow H_{j-i}(X; \mathbb{Z})$$

$$\varphi \otimes \sigma \mapsto \varphi \left(\sigma|_{\Delta_{0, \dots, j}} \right) \sigma|_{\Delta_{i, \dots, j}}.$$

This makes H_* a module over H^* .

- There is a ring structure induced by the cup product:

$$H^i(X; \mathbb{R}) \times H^j(X; \mathbb{R}) \rightarrow H^{i+j}(X; \mathbb{R}) \quad \alpha \cup \beta = (-1)^{ij} \beta \cup \alpha.$$

- Poincaré Duality: If X is an oriented manifold, there exists a fundamental class $[X] \in H_n(X; \mathbb{Z}) \cong \mathbb{Z}$ and $(\cdot) \cap X : H^i \rightarrow H_{n-i}$ is an isomorphism.

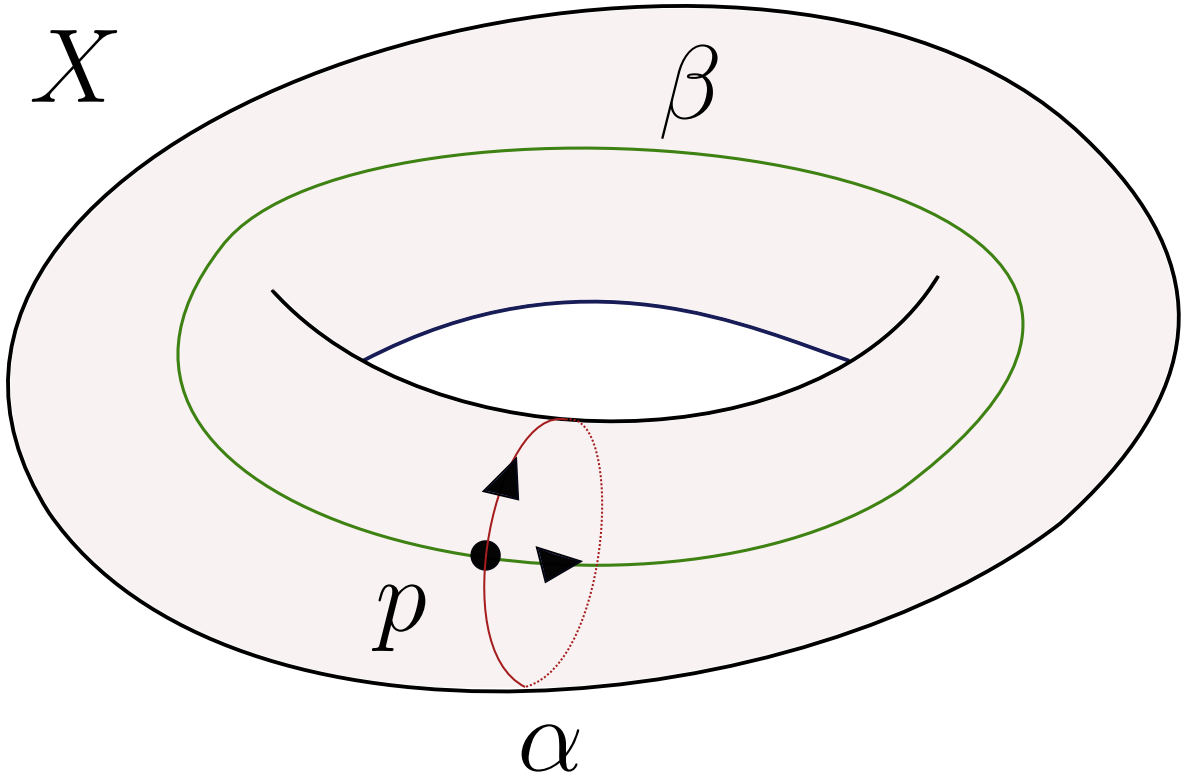
Remark 9.0.13: Let $M \subset X$ be a submanifold where X is a smooth oriented n -manifold. Then $M \hookrightarrow X$ induces a pushforward $H_n(M; \mathbb{Z}) \xrightarrow{\iota_*} H_n(X; \mathbb{Z})$ where $\sigma \mapsto \iota \circ \sigma$. Using Poincaré duality, we'll identify $H_{\dim M}(X; \mathbb{Z}) \rightarrow H^{\text{codim } M}(X; \mathbb{Z})$ and identify $[M] = PD(\iota_*([M]))$. In this case, if $M \pitchfork N$ then $[M] \cap [N] = [M \cap N]$, i.e. the cap product is given by intersecting submanifolds.

Warning 9.0.14

This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

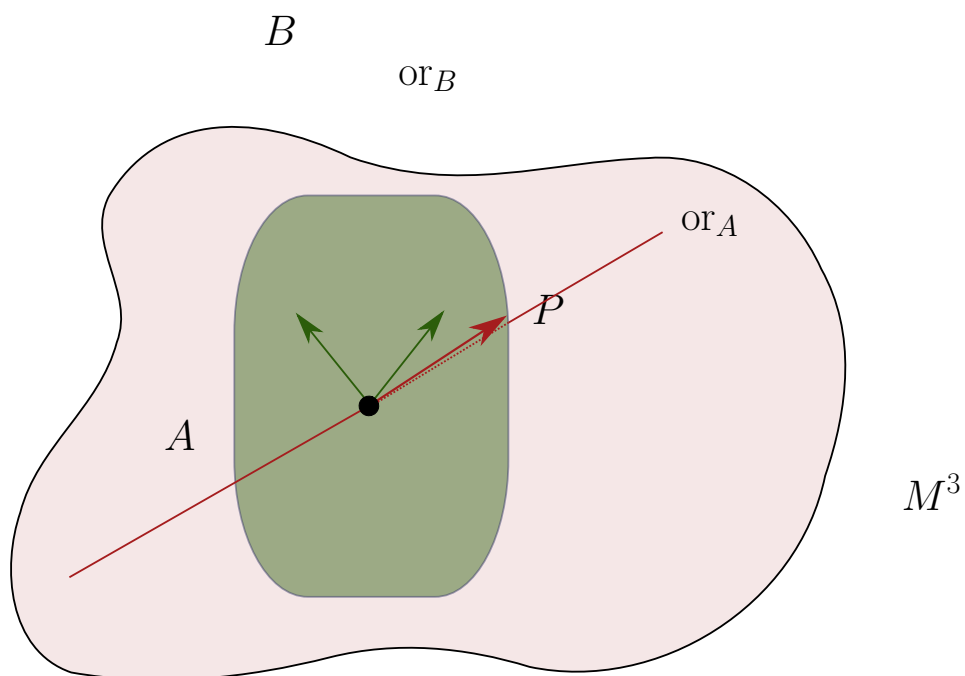


We can then take the fundamental classes of the submanifolds, say $[\alpha], [\beta] \in H^1(X; \mathbb{Z}) \xrightarrow{PD} H^1(X, \mathbb{Z})$. Here $T_p\alpha \oplus T_p\beta = T_pX$, since the intersections are transverse. Since α, β are oriented, let $\{e\}$ be a basis of $T_p\alpha$ (up to \mathbb{R}^+) and similarly $\{f\}$ a basis of $T_p\beta$. We can then ask if $\{e, f\}$ constitutes an *oriented* basis of T_pX . If so, we write $\alpha \cdot_p \beta := +1$ and otherwise $\alpha \cdot_p \beta = -1$. We thus have

$$[\alpha] \smile [\beta] \in H^2(X; \mathbb{Z}) \xrightarrow{PD} H_0(X; \mathbb{Z}) = \mathbb{Z}$$

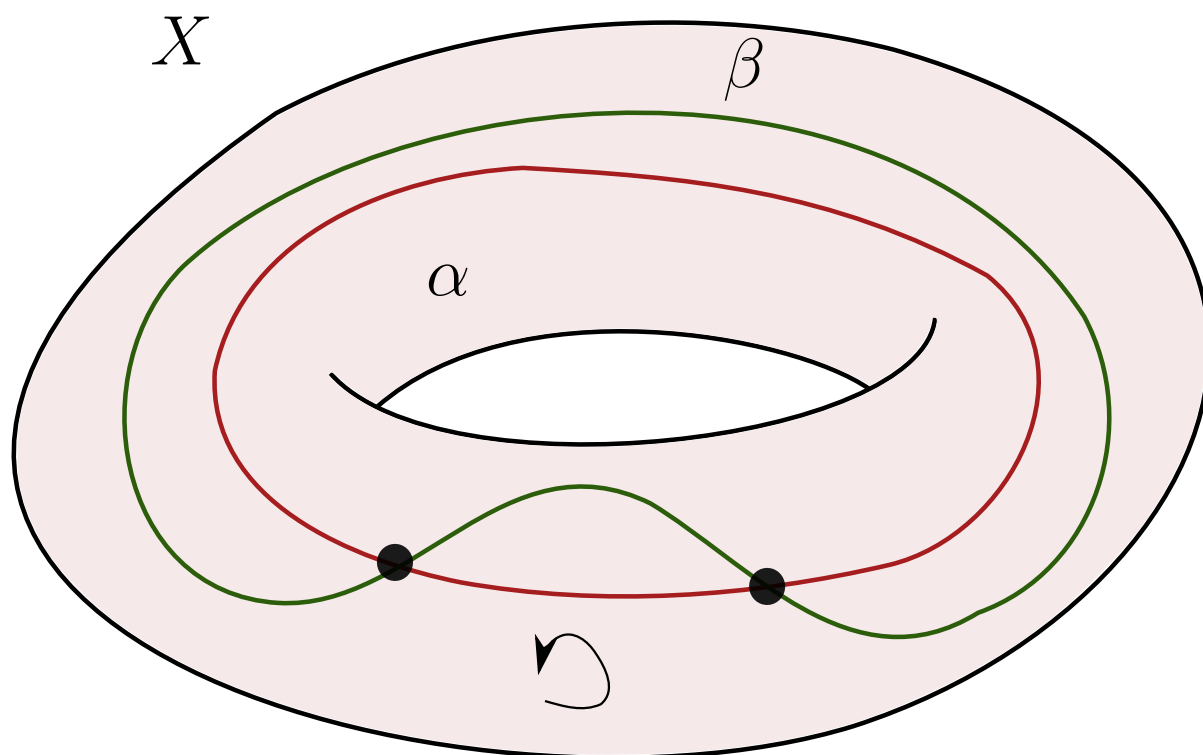
since X is connected. We can thus define the **intersection form** $\alpha \cdot \beta := [\alpha] \smile [\beta]$. In general if A, B are oriented transverse submanifolds of M which are themselves oriented, we'll have $[A] \smile [B] = [A \cap B]$. We need to be careful: how do we orient the intersection? This is given by comparing the orientations on A and B as before.

Example 10.0.1(?): If $\dim M = \dim A + \dim B$, then any $p \in A \cap B$ is oriented by comparing $\{\text{or}_A, \text{or}_B\}$ to or_M .



Here it suffices to check that $\{e, f_1, f_2\}$ is an oriented basis of $T_p M$.

Example 10.0.2(?): In this case, $[\alpha] \smile [\beta] = 0$ and so $\alpha \cdot \beta = 0$:



Remark 10.0.3: Note that cohomology with \mathbb{Z} coefficients can be defined for any topological space, and Poincaré duality still holds.

Remark 10.0.4: We'll be considering $M = M^4$, smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

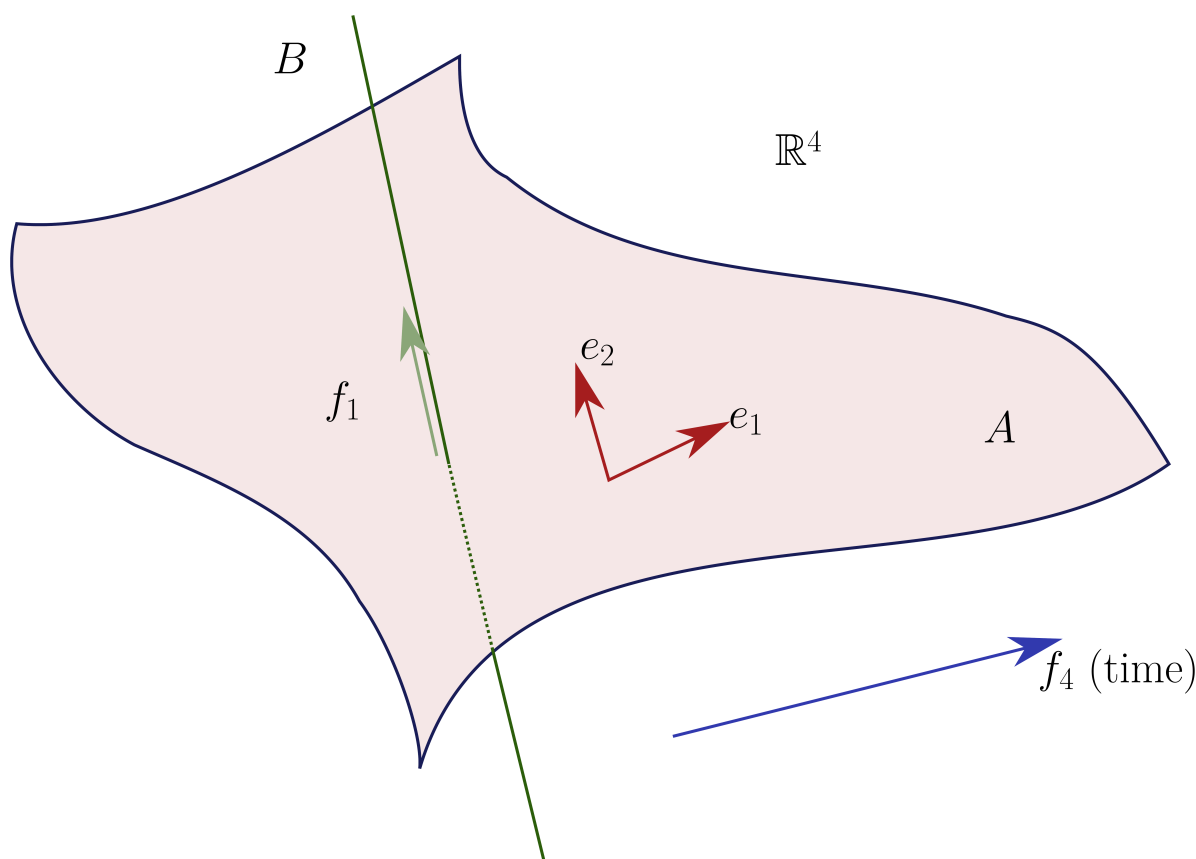


Figure 1: Picking one basis element in the time direction

Here α is oriented in the “forward time” direction, and this is a surface at time $t = 0$. Where $A \cdot B = +1$, since $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$ is a oriented basis for \mathbb{R}^4 . For α^2 , switching the order of α, β no longer yields an oriented basis, but in this case it is α and $A \cdot B = B \cdot A$. This is because

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

Remark 10.0.5: Let M^{2n} be an oriented manifold, then the cup product yields a bilinear map $H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ which is symmetric when n is odd and antisymmetric (or symplectic) when n is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$(H^n(M; \mathbb{Z})/\text{tors})^\vee \xrightarrow{\sim} H^n(M; \mathbb{Z})/\text{tors}$$

$$\alpha \smile \cdot \mapsto \alpha,$$

where the LHS are linear functionals on cohomology.

Remark 10.0.6: Recall the universal coefficients theorem:

$$H^i(X; \mathbb{Z})/\text{tors} \cong (H_i(X; \mathbb{Z})/\text{tors})^\vee.$$

The general theorem shows that $H^i(X; \mathbb{Z})_{\text{tors}} = H_{i-1}(X; \mathbb{Z})_{\text{tors}}$.

Remark 10.0.7: Note that if M is an oriented 4-manifold, then

	tors	torsionfree		tors	torsionfree	
H^0	0	\mathbb{Z}		H_0	0	\mathbb{Z}
H^1	0	\mathbb{Z}^{β_1}		H_1	A	\mathbb{Z}^{β_1}
H^2	A	\mathbb{Z}^{β_2}	\xrightarrow{PD}	H_2	A	\mathbb{Z}^{β_2}
H^3	A	\mathbb{Z}^{β_1}		H_3	0	\mathbb{Z}^{β_1}
H^4	0	\mathbb{Z}		H_4	0	\mathbb{Z}

In particular, if M is simply connected, then $H_1(M) = \mathbf{Ab}(\pi_1(M)) = 0$, which forces $A = 0$ and $\beta_1 = 0$.

Definition 10.0.8 (Lattice)

A **lattice** is a finite-dimensional free \mathbb{Z} -module L together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\rightarrow \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice (L, \cdot) is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\rightarrow L^\vee \\ \ell &\mapsto \ell \cdot (\cdot). \end{aligned}$$

Remark 10.0.9: How to determine if a lattice is unimodular: take a basis $\{e_1, \dots, e_n\}$ of L and form the *Gram matrix* $M_{ij} := (e_i \cdot e_j) \in \text{Mat}(n \times n, \mathbb{Z})^{\text{Sym}}$. Then (L, \cdot) is unimodular if and only if $\det(M) = \pm 1$ if and only if M^{-1} is integral. In this case, the rows of M^{-1} will form a basis of the dual basis.

Definition 10.0.10 (?)

The **index** of a lattice is $|\det M|$.

Exercise 10.0.11 (?)

Prove that $|\det M| = |L^\vee/L|$.

Remark 10.0.12: In general, for M^{4k} , the H^{2k}/tors is unimodular. For M^{4k+2} , the H^{2k+1}/tors is a unimodular *symplectic* lattice, which is obtained by replacing the word “symmetric” with “antisymmetric” everywhere above.

Example 10.0.13(?): For the torus, since the dimension is 2 (mod 4), you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check!

Definition 10.0.14 (?)

A lattice is **nondegenerate** if $\det M \neq 0$.

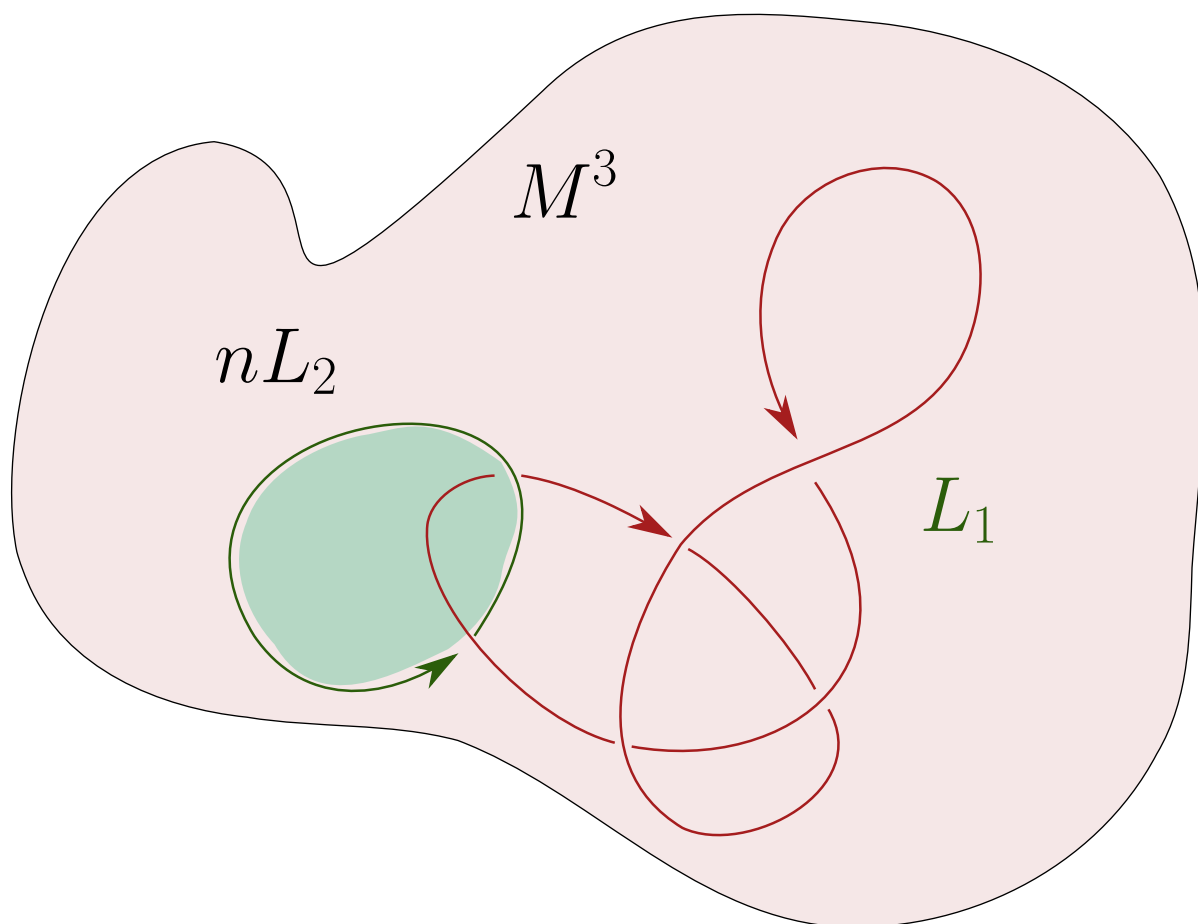
Definition 10.0.15 (?)

The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space with an \mathbb{R} -valued symmetric bilinear form. This allows extending the lattice from \mathbb{Z}^n to \mathbb{R}^n .

Remark 10.0.16: If (L, \cdot) is nondegenerate, then Gram-Schmidt will yield an orthonormal basis $\{v_i\}$. The number of positive norm vectors is an invariant, so we obtain $\mathbb{R}^{p,q}$ where p is the number of +1s in the Gram matrix and q is the number of -1s. The **signature** of (L, \cdot) is (p, q) , or by abuse of notation $p - q$. This is an invariant of the 4-manifold, as is the lattice itself $H^2(X; \mathbb{Z})/\text{tors}$ equipped with the intersection form.

Remark 10.0.17: There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$



Remark 10.0.18: $A \cdot B := \sum_{p \in A \cap B} \text{sgn}_p(A, B)$, where $A \pitchfork B$ and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \smile [\omega] = \int_A \omega.$$

11 | Friday, February 05

Remark 11.0.1: Recall that a lattice is **unimodular** if the map $L \rightarrow L^\vee := \text{Hom}(L, \mathbb{Z})$ is an isomorphism, where $\ell \mapsto \ell \cdot (\cdot)$. To check this, it suffices to check if the Gram matrix M of a basis $\{e_i\}$ satisfies $|\det M| = 1$.

Example 11.0.2 (Determinant 1 Integer Matrices): The matrices $[1]$ and $[-1]$ correspond to the lattice $\mathbb{Z}e$ where either $e^2 := e \cdot e = 1$ or $e^2 = -1$. If M_1, M_2 both have absolute determinant 1,

then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

So if L_1, L_2 are unimodular, then taking an orthogonal sum $L_1 \oplus L_2$ also yields a unimodular lattice. So this yields diagonal matrices with p copies of $+1$ and q copies of -1 . This is referred to as $rm1_{p,q}$, and is an *odd* unimodular lattice of signature (p, q) (after passing to \mathbb{R}). Here *odd* means that there exists a $v \in L$ such that v^2 is odd.

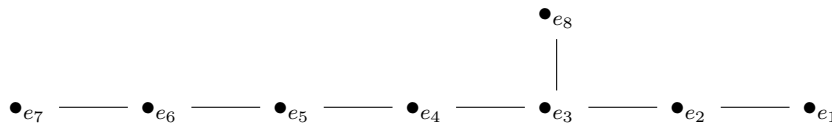
Example 11.0.3 (Even unimodular lattices): An even lattice must have no vectors of odd norm, so all of the diagonal elements are in $2\mathbb{Z}$. This is because $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i < j} 2n_i n_j e_i \cdot e_j$.

Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as H , sometimes referred to as the *hyperbolic cell* or *hyperbolic plane*.

Example 11.0.4 (A harder even unimodular lattice): This is built from the E_8 Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \rightarrow e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example, $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$. You can check that $\det(e_i \cdot e_j) = 1$, and this is referred to as the E_8 lattice. This is of signature $(0, 8)$, and it's negative definite if and only if $v^2 < 0$ for all $v \neq 0$. One can also negate the intersection form to define $-E_8$. Note that any simply-laced Dynkin diagram yields some lattice. For example, E_{10} is unimodular of signature $(1, 9)$, and it turns out that $E_{10} \cong E_8 \oplus H$.

Definition 11.0.5 (?)

Take

$$\Pi_{a,a+8b} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b E_8,$$

which is an even unimodular lattice since the diagonal entries are all -2 , and using the fact

that the signature is additive, is of signature $(a, a + 8b)$. Similarly,

$$\mathbf{II}_{a+8b,a} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b (-E_8),$$

which is again even and unimodular.

Remark 11.0.6: Thus

- $\mathbf{I}_{p,q}$ is odd, unimodular, of signature (p, q) .
- $\mathbf{II}_{p,q}$ is even, unimodular, of signature (p, q) only for $p \equiv q \pmod{8}$.

Theorem 11.0.7 (Serre).

Every unimodular lattice which is not positive or negative definite is isomorphic to either $\mathbf{I}_{p,q}$ or $\mathbf{II}_{p,q}$ with $8 \mid p - q$.

Remark 11.0.8: So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: E_8
16	2: $E_8^{\oplus 2}, D_{16}^+$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$> 8 \times 10^{16}!!!!$

Note that the signature of a definite lattice must be divisible by 8.

Remark 11.0.9: There is an isometry: $f : E_8 \rightarrow E_8$ where $f \in O(E_8)$, the linear maps preserving the intersection form (i.e. the Weyl group $W(E_8)$, given by $v \mapsto v + (v, e_i)e_i$. The Leech lattice also shows up in the sphere packing problems for dimensions 2, 4, 8, 24. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of L as a subset of \mathbb{R}^n , namely $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$ for example, and the map $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$ is an isometric embedding into \mathbb{R}^n with the standard form. Connection to classification of Lie groups: root lattices.

Remark 11.0.10: If M^4 is a compact oriented 4-manifold and if the intersection form on $H^2(M; \mathbb{Z})$ is indefinite, then the only invariants we can extract from that associated lattice are

- Whether it's even or odd, and
- Its signature

If the lattice is even, then the signature satisfies $8 \mid p - q$. So Poincaré duality forces unimodularity,

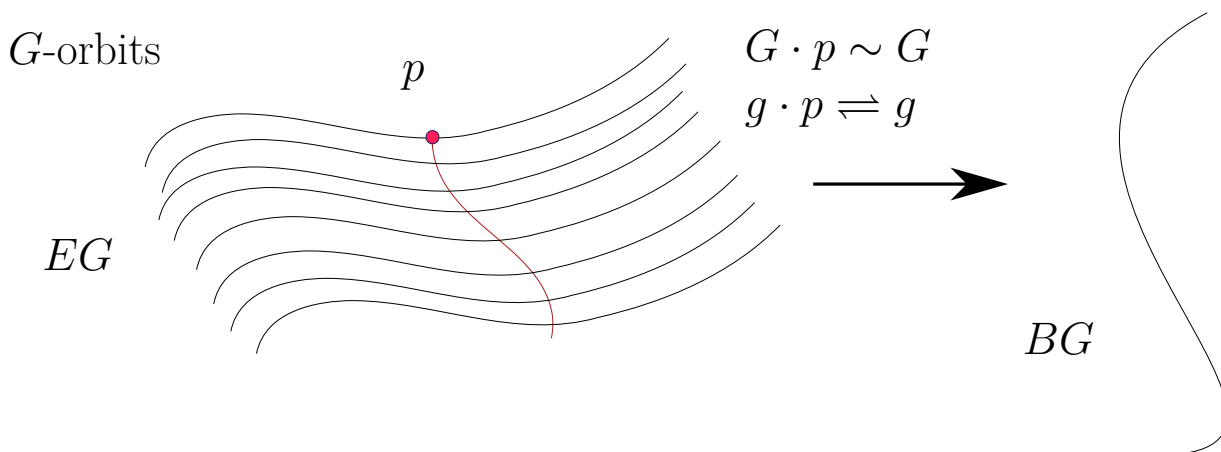
and then there are further number-theoretic restrictions. E.g. this prohibits $\beta_2 = 7$, since then the signature couldn't possibly be 8 if the intersection form is even.

11.1 Characteristic Classes

Definition 11.1.1 (?)

Let G be a topological group, then a **classifying space** EG is a contractible topological space admitting a free continuous G -action with a “nice” quotient.

Remark 11.1.2: Thus there is a map $EG \rightarrow BG := EG/G$ which has the structure of a principal G -bundle.



Here we use a point p depending on U in an orbit to identify orbits $g \cdot p$ with g , and we want to take transverse slices to get local trivializations of $U \in BG$. It suffices to know where $\pi^{-1}(U) \cong U \times G$, and it suffices to consider $U \times \{e\}$. Moreover, $EG \rightarrow BG$ is a universal principal G -bundle in the sense that if $P \rightarrow X$ is a universal G -bundle, there is an $f : X \rightarrow BG$.

$$\begin{array}{ccc} P & \xrightarrow{\quad\quad\quad} & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & BG \end{array}$$

[Link to Diagram](#)

Here bundles will be classified by homotopy classes of f , so

$$\{\text{Principal } G\text{-bundles}/_X\} \cong [X, BG].$$

⚠ Warning 11.1.3

This only works for paracompact Hausdorff spaces! The line \mathbb{R} with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

12 | Monday, February 08

Last time: BG and EG . See Milnor and Stasheff.

Example 12.0.1(?): Let $G := \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$, then we can take

$$EG = \mathbb{R}^\infty := \left\{ (a_1, a_2, \dots) \mid a_i \in \mathbb{R}, a_i \gg 0 = 0, a_i \text{ not all zero} \right\}.$$

Then \mathbb{R}^\times acts on EG by scaling, and we can take the quotient $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$, where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in \mathbb{R}^\times$. This yields \mathbb{RP}^∞ as the quotient. You can check that EG is contractible: it suffices to show that $S^\infty := \left\{ \sum |a_i| = 1 \right\}$ is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$. Similarly $BC^\times \cong \mathbb{CP}^\infty$ with $EC^\times := \mathbb{C}^\infty \setminus \{0\}$.

Example 12.0.2(?): Consider $G = \mathrm{GL}_n(\mathbb{R})$. It turns out that $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$, which is the set of linear subspaces of \mathbb{R}^∞ of dimension d . This is spanned by d vectors $\{e_i\}$ in some large enough $\mathbb{R}^N \subseteq \mathbb{R}^\infty$, since we can take N to be the largest nonvanishing coordinate and include all of the vectors into \mathbb{R}^∞ by setting $a_{>N} = 0$. For any $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$, since \mathbb{R}^d has a standard basis, there is a natural GL_d torsor: the set of ordered bases of linear subspaces. So define

$$EG := \{\text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty)\},$$

then any $A \in \mathrm{GL}_d(\mathbb{R})$ acts on EG by sending $(L, \{e_i\}) \mapsto (L, \{Le_i\})$. We can identify EG as d -tuples of linearly independent elements of \mathbb{R}^∞ , and there is a map

$$\begin{aligned} EG &\rightarrow BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over $B\mathrm{GL}_d$:

$$\begin{array}{ccc} \mathcal{E}_L := L & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & B\mathrm{GL}_d \end{array}$$

So $\mathcal{E} \subseteq B\mathrm{GL}_d \times \mathbb{R}^\infty$, where we can define $\mathcal{E} := \{(L, p) \mid p \in L\}$. In this case, $EG = \mathrm{Frame}(\mathcal{E})$ is the frame bundle of this universal bundle. The same setup applies for $G := \mathrm{GL}_d(\mathbb{C})$, except we take $\mathrm{Gr}_d(\mathbb{C}^\infty)$.

Example 12.0.3(?): Consider $G = O_d$, the set of orthogonal transformations of \mathbb{R}^d with the standard bilinear form, and U_d the set of unitary such transformations. To be explicit:

$$U_d := \left\{ A \in \text{Mat}(d \times d, \mathbb{C}) \mid \langle Av, Av \rangle = \langle v, v \rangle \right\},$$

where

$$\langle [v_1, \dots, v_n], [v_1, \dots, v_n] \rangle = \sum |v_i|^2.$$

Alternatively, $A^t A = I$ for O_d and $\overline{A}^t A = I$ for U_d . In this case, $BO_d = \text{Gr}_d(\mathbb{R}^\infty)$ and $BU_d = \text{Gr}_d(\mathbb{C}^\infty)$, but we'll make the fibers smaller: set the fiber over L to be

$$(EO_d)_L := \{\text{orthogonal frames of } L\}$$

and similarly $(EU_d)_L$ the unitary frames of L . That there are related comes from the fact that GL_d retracts onto O_d using the Gram-Schmidt procedure.

Remark 12.0.4: Recall that there is a bijective correspondence

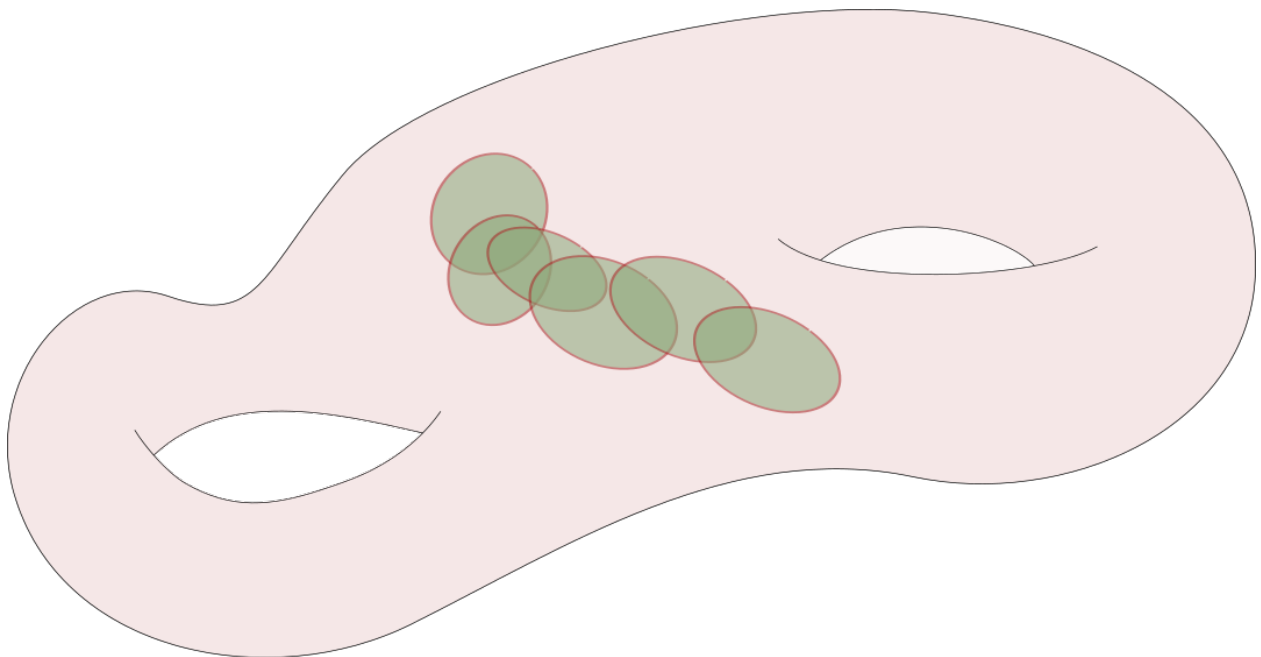
$$\left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \begin{array}{c} \text{Principal } \text{GL}_d\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Principal } O_d\text{-bundles} \\ \text{on } X \end{array} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles $\mathcal{E} \rightarrow X$ of rank d , and on the RHS we have bundles with a metric. In local trivializations $U \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, the metric is the standard one on \mathbb{R}^d . This is referred to as a **reduction of structure group**, i.e. a principal GL_d bundle admits possibly different trivializations for which the transition functions lie in the subgroup O_d .

Example 12.0.5(?): Given any trivial principal G -bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.



Remark 12.0.6: We want to compute $H^*(BU_d; \mathbb{Z})$. Why is this important? Given any complex vector bundle $\mathcal{E} \rightarrow X$ there is an associated principal U_d bundle by choosing a metric, so we get a homotopy class $[X, BU_d]$. Given any $f \in [X, BU_d]$ and any $\alpha \in H^k(BU_d; \mathbb{Z})$, we can take the pullback $f^*\alpha \in H^k(X; \mathbb{Z})$, which are **Chern classes**.

Exercise 12.0.7 (?)

Show that $H^*(BU_d; \mathbb{Z})$ stabilizes as $d \rightarrow \infty$ to an infinitely generated polynomial ring $\mathbb{Z}[c_1, c_2, \dots]$ with each c_i in cohomological degree $2i$, so $c_i \in H^{2i}(BU_d, \mathbb{Z})$.

Definition 12.0.8 (?)

There is a map $BU_{d-1} \rightarrow BU_d$, which we can identify as

$$\begin{aligned} \text{Gr}_{d-1}(C^\infty) &\rightarrow \text{Gr}_d(\mathbb{C}^\infty) \\ \{v_1, \dots, v_{d-1}\} &\mapsto \text{span}\{(1, 0, 0, \dots), sv_1, \dots, sv_{d-1}\}. \end{aligned}$$

This is defined by sending a basis where $s: \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$ is the map that shifts every coordinate to the right by one.

Question: does $\text{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?

This will yield a fiber sequence

$$S^{2d-1} \rightarrow BU_{d-1} \rightarrow BU_d$$

and using connectedness of the sphere and the LES in homotopy this will identify

$$H^*(BU_d) = H^*(BU_{d-1})[c_d] \quad \text{where } c_d \in H^{2d}(BU_d).$$

The **Chern class** of a vector bundle \mathcal{E} , denoted $c_k(\mathcal{E})$, will be defined as the pullback f^*c_k .

13 | Wednesday, February 10

Theorem 13.0.1 (?)

As $n \rightarrow \infty$, we have

$$H^*(BO_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \dots] \quad w_i \in H^i.$$

Definition 13.0.2 (?)

Given any principal O_n -bundle $P \rightarrow X$, there is an induced map $X \xrightarrow{f} BO_n$, so we can pull back the above generators to define the **Stiefel-Whitney classes** f^*w_i .

Remark 13.0.3: If $P := \text{OFrame}TX$, then f^*w_1 measures whether X has an orientation, i.e. $f^*w_1 = 0 \iff X$ can be oriented. We also have $f^*w_i(P) = w_i(\mathcal{E})$ where $P = \text{OFrame}(\mathcal{E})$. In general, we'll just write w_i for Stiefel-Whitney classes and c_i for Chern classes.

Definition 13.0.4 (Pontryagin Classes)

The **Pontryagin classes** of a real vector bundle \mathcal{E} are defined as

$$p_i(\mathcal{E}) = (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that the complexified bundle above is a complex vector bundle with the same transition functions as \mathcal{E} , but has a reduction of structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{R})$.

Observation 13.0.5

\mathbb{RP}^∞ and \mathbb{CP}^∞ are examples of $K(\pi, n)$ spaces, which are the unique-up-to-homotopy spaces defined by

$$\pi_k K(\pi, n) = \begin{cases} \pi & k = n \\ 0 & \text{else.} \end{cases}$$

Theorem 13.0.6 (*Brown Representability*).

$$H^n(X; \pi) \cong [X, K(\pi, n)].$$

Example 13.0.7 (?):

$$[X, \mathbb{RP}^\infty] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$$

$$[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z}).$$

Proposition 13.0.8 (?).

There is a correspondence

$$\{\text{Complex line bundles}\} \rightleftharpoons [X, \mathbb{CP}^\infty] = [X, BC^\times] \rightleftharpoons H^2(X; \mathbb{Z})$$

Importantly, note that for $X \in \mathbf{Mfd}_{\mathbb{C}}$, $H^2(X; \mathbb{Z})$ measures *smooth* complex line bundles and not holomorphic bundles.

Proof (?).

We'll take an alternate direct proof. Consider the exponential exact sequence on X :

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times.$$

Note that $\underline{\mathbb{Z}}$ consists of locally constant \mathbb{Z} -valued functions, \mathcal{O} consists of smooth functions, and \mathcal{O}^\times are ???.

Can't read screenshot! :(

This yields a LES in homology:

$$\begin{array}{ccccccc}
H^0(X; \mathbb{Z}) & \longrightarrow & H^0(X; \mathcal{O}) & \longrightarrow & H^0(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^1(X; \mathbb{Z}) & \longrightarrow & H^1(X; \mathcal{O}) & \longrightarrow & H^1(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^2(X; \mathbb{Z}) & \longrightarrow & H^2(X; \mathcal{O}) & \longrightarrow & H^2(X; \mathcal{O}^\times) & \longrightarrow &
\end{array}$$

[Link to Diagram](#)

Since \mathcal{O} admits a partition of unity, $H^{>0}(X; \mathcal{O}) = 0$ and all of the red terms vanish. For complex line bundles L , $H^1(X; \mathcal{O}^\times) \cong H^2(X; \mathbb{Z})$. Taking a local trivialization $L|_U \cong U \times \mathbb{C}$, we obtain transition functions

$$t_{UV} \in C^\infty(U \cap V, \mathrm{GL}_1(\mathbb{C}))$$

where we can identify $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. We then have

$$(t_{U_{ij}}) \in \prod_{i < j} \mathcal{O}^\times(U_i \cap U_j) = C^1(X; \mathcal{O}^\times).$$

Moreover,

$$(t_{U_{ij}} t_{U_{ik}}^{-1} t_{U_{jk}})_{i,j,k} = \partial(t_{U_{ij}})_{i,j} = 0,$$

since transition functions satisfy the cocycle condition. So in fact $(t_{U_{ij}}) \in Z^1(X; \mathcal{O}^\times) = \ker \partial^1$, and we can take its equivalence class $[(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times) = \ker \partial^1 / \mathrm{im} \partial^0$. Changing trivializations by some $s_i \in \prod_i \mathcal{O}^\times(U_i)$ yields a composition which is a different trivialization of the same bundle:

$$\begin{array}{ccccc}
L|_{U_i} & \xrightarrow{h_i} & U_i \times \mathbb{C} & \xrightarrow{\cdot s_i} & U_i \times \mathbb{C} \\
& & \searrow & & \nearrow \\
& & & &
\end{array}$$

[Link to Diagram](#)

So the $(t_{U_{ij}})$ change *exactly* by an $\partial^0(s_i)$. Thus the following map is well-defined:

$$L \mapsto [(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times).$$

There is another construction of the map

$$\begin{aligned}
\{L\} &\rightarrow H^2(X; \mathbb{Z}) \\
L &\mapsto c_1(L).
\end{aligned}$$

Take a smooth section of L and $s \in H^0(X; L)$ that intersects an \mathcal{O} -section of L transversely. Then

$$V(s) := \{x \in X \mid s(x) = 0\}$$

is a submanifold of real codimension 2 in X , and $c_1(L) = [V(s)] \in H^2(X; \mathbb{Z})$. ■

Theorem 13.0.9 (Splitting Principle for Complex Vector Bundles).

1. Suppose that $\mathcal{E} = \bigoplus_{i=1}^r L_i$ and let $c(\mathcal{E}) := \sum c_i(\mathcal{E})$. Then

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + c_i(L_i)).$$

2. Given any vector bundle $\mathcal{E} \rightarrow X$, there exists some Y and a map $Y \rightarrow X$ such that $f^* : H^k(X; \mathbb{Z}) \hookrightarrow H^k(Y; \mathbb{Z})$ is injective and $f^*\mathcal{E} = \bigoplus_{i=1}^r L_i$.

Slogan 13.0.10

To verify any identities on characteristic classes, it suffices to prove them in the case where \mathcal{E} splits into a direct sum of line bundles.

Example 13.0.11 (?) :


$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}).$$


To prove this, apply the splitting principle. Choose Y, Y' splitting $\mathcal{E}, \mathcal{F}'$ respectively, this produces a space Z and a map $f : Z \rightarrow X$ where both split. We can write

$$\begin{aligned} f^*\mathcal{E} &= \bigoplus L_i & c(f^*\mathcal{E}) &= \prod (1 + c_1(L_i)) \\ f^*\mathcal{F} &= \bigoplus M_j & c(f^*\mathcal{F}) &= \prod (1 + c_1(M_j)). \end{aligned}$$


We thus have

$$\begin{aligned} c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &= \prod (1 + c_1(L_i)) (1 + c_1(M_j)) \\ &= c(f^*\mathcal{E})c(f^*\mathcal{F}), \end{aligned}$$

and $f^*(c(\mathcal{E} \oplus \mathcal{F})) = f^*(c(\mathcal{E})c(\mathcal{F}))$. Since f^* is injective, this yields the desired identity. 

Example 13.0.12 (?) : We can compute $c(\text{Sym}^2 \mathcal{E})$, and really any tensorial combination involving \mathcal{E} , and it will always yield some formula in the $c_i(\mathcal{E})$. 

14 | Friday, February 12

Remark 14.0.1: Last time: the splitting principle. Suppose we have $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$ and let $x_i := c_1(L_i)$. Then $c_k(\mathcal{E})$ is the degree $2k$ part of $\prod_{i=1}^r (1 + x_i)$ where each x_i is in degree 2. This is equal to $e_k(x_1, \dots, x_r)$ where e_k is the k th elementary symmetric polynomial. 

Example 14.0.2(?): For example,

- $e_1 = x_1 + \cdots x_r$.
- $e_2 = x_1x_2 + x_1x_3 + \cdots = \sum_{i<j} x_ix_j$
- $e_3 = \sum_{i<j<k} x_ix_jx_k$, etc.

Remark 14.0.3: The theorem is that any symmetric polynomial is a polynomial in the e_i . For example, $p_2 = \sum x_i^2$ can be written as $e_1^2 - 2e_2$. Similarly, $p_3 = \sum x_i^3 = e_1^3 - 3e_1e_2 - 3e_3$. Note that the coefficients of these polynomials are important for representations of S_n , see *Schur polynomials*.

Remark 14.0.4: Due to the splitting principle, we can pretend that $x_i = c_i(L_i)$ exists even when \mathcal{E} doesn't split. If $\mathcal{E} \rightarrow X$, the individual symbols x_i don't exist, but we can write '

$$x_1^3 + \cdots + x_r^3 = e_1^3 - 3e_1e_2 - 3e_3 := c_1(\mathcal{E})^3 + 3c_1(\mathcal{E})c_2(\mathcal{E}) + \cdots,$$

which is a well-defined element of $H^6(X; \mathbb{Z})$. So this polynomial defines a characteristic class of \mathcal{E} , and this can be done for any symmetric polynomial. We can change basis in the space of symmetric polynomials to now define different characteristic classes.

Definition 14.0.5 (Chern Character)

The **Chern character** is defined as

$$\begin{aligned} \text{ch}(\mathcal{E}) &:= \sum_{i=1}^r e^{x_i} \in H^*(X; \mathbb{Q}) \\ &:= \sum_{i=1}^r \sum_{k=0}^{\infty} \frac{x_i^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{p_k(x_1, \dots, x_r)}{k!} \\ &= \text{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 - c_2(\mathcal{E})}{2!} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) - 3c_3(\mathcal{E})}{3!} + \cdots \\ &\in H^0 + H^2 + H^4 + H^6 \\ &= \text{ch}_0(\mathcal{E}) + \text{ch}_1(\mathcal{E}) + \text{ch}_2(\mathcal{E}) + \cdots, \\ &\text{ch}_i(\mathcal{E}) \in H^{2i}(X; \mathbb{Q}). \end{aligned}$$

Definition 14.0.6 (Todd Class)

The **total Todd class**

$$\text{td}(\mathcal{E}) := \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}.$$

Note that

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \frac{x_i}{2} + \frac{x_i^2}{12} + \frac{x_i^4}{720} + \cdots = 1 + \frac{x_i}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x_i^{2i}.$$

where L'Hopital shows that the derivative at $x_i = 0$ exists, so it's analytic at zero and the expansion makes sense, and the B_i are Bernoulli numbers.

Remark 14.0.7 (Very important and useful!!): $\text{ch}(\mathcal{E} \oplus \mathcal{F}) = \text{ch}(\mathcal{E}) + \text{ch}(\mathcal{F})$ and $\text{ch}(\mathcal{E} \otimes \mathcal{F}) = \sum_{i,j} e^{x_i + y_j} = \text{ch}(\mathcal{E}) \text{ch}(\mathcal{F})$ using the fact that $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$. So ch is a “ring morphism” in the sense that it preserves multiplication \otimes and addition \oplus , making the Chern character even better than the total Chern class.

Definition 14.0.8 (Todd Class)

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, then define the **Todd class** of X as $\text{td}_{\mathbb{C}}(X) := \text{td}(TX)$ where TX is viewed as a complex vector bundle. If $X \in \mathbf{Mfd}_{\mathbb{R}}$, define $\text{td}_{\mathbb{R}} = \text{td}(TX \otimes_{\mathbb{R}} \mathbb{C})$.

14.1 Section 5: Riemann-Roch and Generalizations

Remark 14.1.1: Let $X \in \mathbf{Top}$ and let \mathcal{F} be a sheaf of vector spaces. Suppose $h^i(X; \mathcal{F}) := \dim H^i(X; \mathcal{F}) < \infty$ for all i and is equal to 0 for $i \gg 0$.

Definition 14.1.2 (Euler Characteristic of a Sheaf)

The **Euler characteristic** of \mathcal{F} is defined as

$$\chi(X; \mathcal{F}) := \chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h_i(X; \mathcal{F}).$$

 **Warning 14.1.3**

This is not always well-defined!

Example 14.1.4(?): Let $X \in \mathbf{Mfd}_{\text{cpt}}$ and take $\mathcal{F} := \underline{\mathbb{R}}$, we then have

$$\chi(X; \underline{\mathbb{R}}) = h^0(X; \underline{\mathbb{R}}) - h^1(X; \underline{\mathbb{R}}) + \cdots = b_0 - b_1 + b_2 - \cdots := \chi_{\mathbf{Top}}(X).$$


Example 14.1.5(?): Let $X = \mathbb{C}$ and take $\mathcal{F} := \mathcal{O} := \mathcal{O}^{\text{holo}}$ the sheaf of holomorphic functions. We then have $h^{>0}(X; \mathcal{O}) = 0$, but $H^0(X; \mathcal{O})$ is the space of all holomorphic functions on \mathbb{C} , making $\dim_{\mathbb{C}} h^0(X; \mathcal{O})$ infinite.

Example 14.1.6(?): Take $X = \mathbb{P}^1$ with \mathcal{O} as above, $h^0(\mathbb{P}^1; \mathcal{O}) = 1$ since \mathbb{P}^1 is compact and the maximum modulus principle applies, so the only global holomorphic functions are constant. We can write $\mathbb{P}^1 = \mathbb{C}_1 \cup \mathbb{C}_2$ as a cover and $h^i(\mathbb{C}, \mathcal{O}) = 0$, so this is an acyclic cover and we can use it to compute $h^1(\mathbb{P}^1; \mathcal{O})$ using Čech cohomology. We have


- $C^0(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1) \oplus \mathcal{O}(\mathbb{C}_2)$
- $C^1(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1 \cap \mathbb{C}_2) = \mathcal{O}(\mathbb{C}^\times)$.
- The boundary map is given by

$$\begin{aligned} \partial_0 : C^0 &\rightarrow C^1 \\ (f(z), g(z)) &\mapsto g(1/z) - f(z) \end{aligned}$$

and there are no triple intersections.

Is every holomorphic function on \mathbb{C}^\times of the form $g(1/z) - f(z)$ with f, g holomorphic on \mathbb{C} . The answer is yes, by Laurent expansion, and thus $h^1 = 0$. We can thus compute $\chi(\mathbb{P}^1; \mathcal{O}) = 1 - 0 = 1$. 

15 | Monday, February 15


Remark 15.0.1: Last time: we saw that $\chi(\mathbb{P}^1, \mathcal{O}) = 1$, and we'd like to generalize to holomorphic line bundles on a Riemann surface. This will be the main ingredient for Riemann-Roch. 

Theorem 15.0.2(?).

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$ be compact and let \mathcal{F} be a holomorphic vector bundle on X .^a Then χ is well-defined and

$$h^{>\dim_{\mathbb{C}} X}(X; \mathcal{F}) = 0.$$

^aOr more generally a finitely-generated \mathcal{O} -module, i.e. a coherent sheaf.

Remark 15.0.3: The locally constant sheaf $\underline{\mathbb{C}}$ is not an \mathcal{O} -module, i.e. $\underline{\mathbb{C}}(U) \notin \mathcal{O}(U)\text{-Mod}$. In fact, $h^{2i}(X, \underline{\mathbb{C}}) = \mathbb{C}$ for all i . 

Proof (?).

We'll resolve \mathcal{F} as a sheaf by first mapping to its smooth sections and continuing in the following way:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \rightarrow \dots,$$

where $\bar{\partial}f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$. Suppose we have a holomorphic trivialization of $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ and we have sections $(s_1, \dots, s_r) \in C^\infty \mathcal{F}(U)$, which are smooth functions on U . In local coordinates we have

$$\bar{\partial}s := (\bar{\partial}s_1, \dots, \bar{\partial}s_r),$$

but is this well-defined globally? Given a different trivialization over $V \subseteq X$, the s_i are related by transition functions, so the new sections are $t_{UV}(s_1, \dots, s_r)$ where $t_{UV} : U \cap V \rightarrow \mathrm{GL}_r(\mathbb{C})$.

Since t_{UV} are holomorphic, we have

$$\bar{\partial}(t_{UV}(s_1, \dots, s_r)) = t_{UV}\bar{\partial}(s_1, \dots, s_r).$$

This makes $\bar{\partial} : C^\infty \mathcal{F} \rightarrow F \otimes A^{0,1}$ a well-defined (but not \mathcal{O} -linear) map. We can thus continue this resolution using the Leibniz rule:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \xrightarrow{\bar{\partial}} \dots F \otimes A^{0,2} \xrightarrow{\bar{\partial}} \dots,$$

which is an exact sequence of sheaves since $(A^{0,\cdot}, \bar{\partial})$ is exact.

Why? Split into line bundles?

We can identify $C^\infty \mathcal{F} = \mathcal{F} \otimes A^{0,0}$, and $\mathcal{F} \otimes A^{0,q}$ is a smooth vector bundle on X . Using partitions of unity, we have that $\mathcal{F} \otimes A^{0,q}$ is acyclic, so its higher cohomology vanishes, and

$$H^i(X; \mathcal{F}) \cong \frac{\ker(\bar{\partial} : \mathcal{F} \otimes A^{0,i} \rightarrow \mathcal{F} \otimes A^{0,i+1})}{\operatorname{im}(\bar{\partial} : \mathcal{F} \otimes A^{0,i-1} \rightarrow \mathcal{F} \otimes A^{0,i})}.$$

However, we know that $A^{0,p} = 0$ for all $p > n := \dim_{\mathbb{C}} X$, since any wedge of $p > n$ forms necessarily vanishes since there are only n complex coordinates. ■

⚠ Warning 15.0.4

This only applies to holomorphic vector bundles or \mathcal{O} -modules!



15.1 Riemann-Roch

Theorem 15.1.1 (Riemann-Roch).

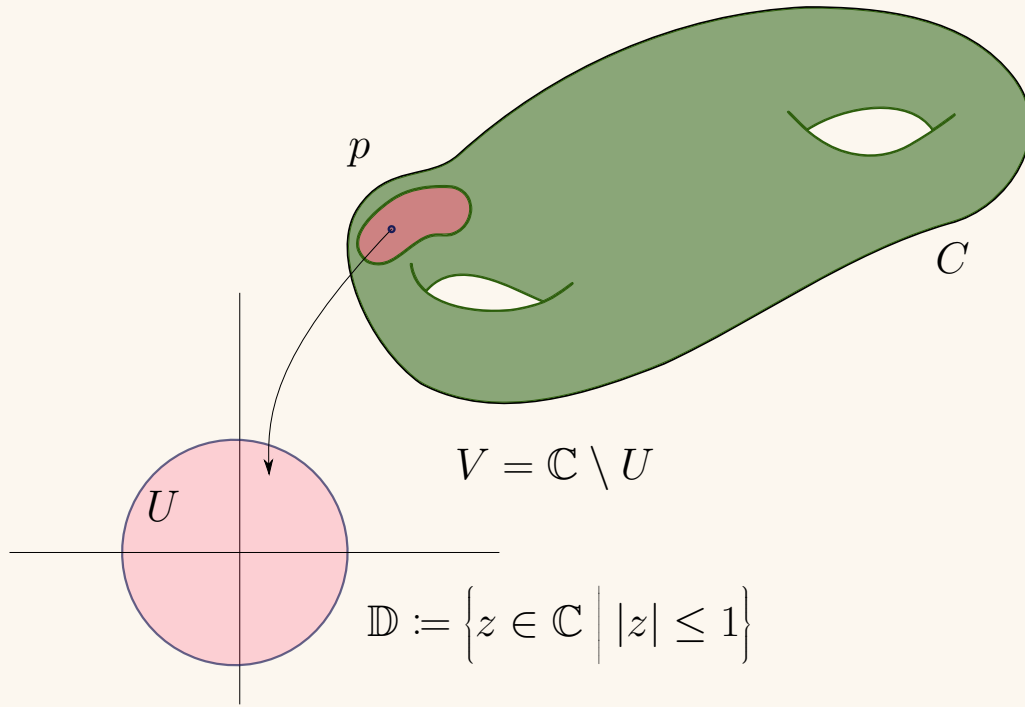
Let C be a compact connected Riemann surface, i.e. $X \in \mathbf{Mfd}_{\mathbb{C}}$ with $\dim_{\mathbb{C}}(X) = 1$, and let $\mathcal{L} \rightarrow C$ be a holomorphic line bundle. Then

$$\chi(C, \mathcal{L}) = \deg(L) + (1 - g) \quad \text{where } \int_C c_1(\mathcal{L})$$

and g is the genus of C .

Proof (?).

We'll introduce the notion of a “point bundle”, which are particularly nice line bundles, denoted $\mathcal{O}(p)$ for $p \in \mathbb{C}$.



Taking \mathbb{D} to be a disc of radius $1/2$ and V to be its complement, we have $t_{uv}(z) = z^{-1} \in \mathcal{O}^*(U \cap V)$. We can take a holomorphic section $s_p \in H^0(C, \mathcal{O}(p))$, where $s_p|_U = z$ and $s_p|_V = 1$. Then $t_{uv}(s_p|_U) = s_p|_V$ on the overlaps. We have a function which precisely vanishes to first order at p . Recall that $c_1(\mathcal{O}(p))$ is represented by $[V(s)] = [p]$, and moreover $\int_C c_1(\mathcal{O}(p)) = 1$.

We now want to generalize this to a **divisor**: a formal \mathbb{Z} -linear combination of points.

Example 15.1.2(?): Take $p, q, r \in C$, then a divisor can be defined as something like $D := 2[p] - [q] + 3[r]$.

Define $\mathcal{O}(D) := \bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}$ for any $D = \sum n_i [p_i]$. Here tensoring by negatives means taking duals, i.e. $\mathcal{O}(-[p]) := \mathcal{O}^{\otimes -1} := \mathcal{O}(p)^\vee$, the line bundle with inverted transition functions. $\mathcal{O}(D)$ has a meromorphic section given by

$$s_D := \prod s_{p_i}^{n_i} \in \text{Mero}(C, \mathcal{O}(D))$$

where we take the sections coming from point bundles. We can compute

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i := \deg(D).$$

Example 15.1.3(?):

$$\deg(2[p] - [q] + 3[r]) = 4.$$

Remark 15.1.4: Assume our line bundle L is $\mathcal{O}(D)$, we'll prove Riemann-Roch in this case by induction on $\sum |n_i|$. The base case is \mathcal{O} , which corresponds to taking an empty divisor. Then either

- Take $D = D_0 + [p]$ with $\deg(D_0) < \sum |n_i|$ (for which we need some positive coefficient),
or
- Take $D_0 = D + [p]$.

Claim: There is an exact sequence

$$0 \rightarrow \mathcal{O}(D_0) \rightarrow \mathcal{O}(D) \rightarrow \mathbb{C}_p \rightarrow 0$$

$$s \in \mathcal{O}(D_0)(U) \mapsto s \cdot s_p \in \mathcal{O}(D_0 + [p])(U),$$

where the last term is the skyscraper sheaf at p .

Proof (?).

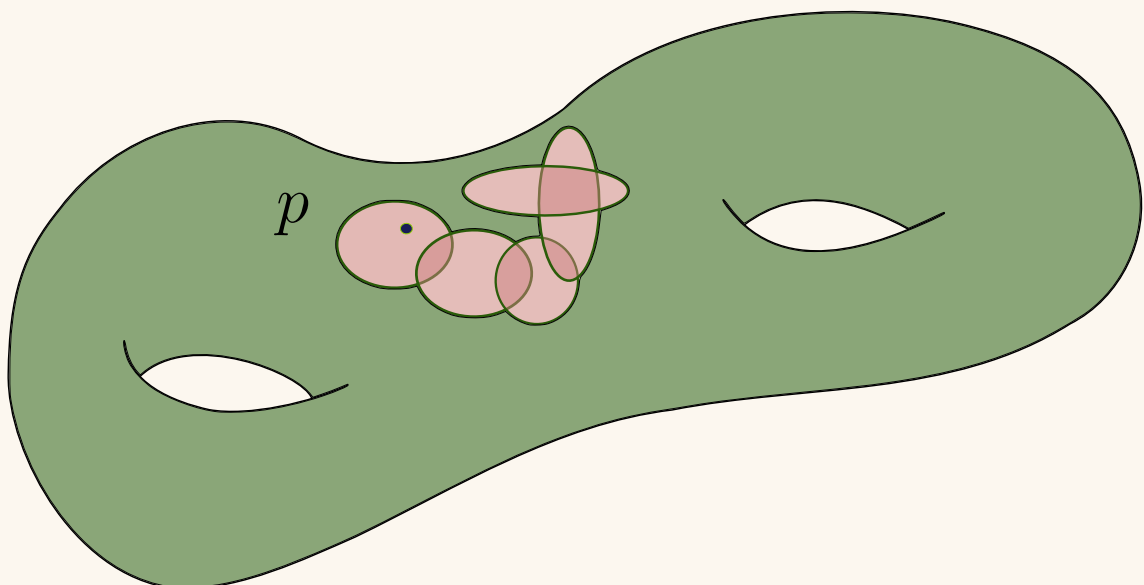
The given map is \mathcal{O} -linear and injective, since $s_p \neq 0$ and $ss_p = 0$ forces $s = 0$. Recall that we looked at $\mathcal{O} \xrightarrow{z} \mathcal{O}$ on \mathbb{C} , and this section only vanishes at p (and to first order). The same situation is happening here. ■

Thus there is a LES

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 \hookrightarrow & H^0(\mathcal{O}(D_0)) & \longrightarrow & H^0(\mathcal{O}(D)) & \longrightarrow & H^0(\mathcal{O}(\mathbb{C}_p)) & \longrightarrow \\
 & \searrow & & \searrow & & \searrow & \\
 \hookrightarrow & H^1(\mathcal{O}(D_0)) & \longrightarrow & H^1(\mathcal{O}(D)) & \longrightarrow & H^1(\mathcal{O}(\mathbb{C}_p)) = 0 & \longrightarrow \\
 & \searrow & & \searrow & & \searrow & \\
 & & & & & & 0
 \end{array}$$

[Link to Diagram](#)

We also have $h^1(\mathbb{C}_p) = 0$ by taking a sufficiently fine open cover where p is only in one open set. So just checking Čech cocycles yields $C_U^1(C, \mathbb{C}_p) := \prod_{i < j} \mathbb{C}_p(U_i \cap U_j) = 0$ since p is in no intersection.



X

p

We obtain $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D_0)) + 1$, using that it is additive in SESs

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0 \implies \chi(\mathcal{E}_2) = \chi(\mathcal{E}_\infty) + \chi(\mathcal{E}_3)$$


and thus

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i = \deg(D) = \deg D_0 + 1.$$

The last step is to show that $\chi(C, \mathcal{O}) = 1 - g$, so just define g so that this is true! ■

Remark 15.1.5: Why is every $L \cong \mathcal{O}(D)$ for some D ? Easy to see if L has meromorphic sections: if s is a meromorphic section of L , then the following works:

$$D = \text{Div}(s) = \sum_p \text{Ord}_p(s)[p].$$

Then $\mathcal{O} \cong L \otimes \mathcal{O}(-D)$ has a meromorphic section ss_{-D} , a global nonvanishing section with $\text{Div}(ss_{-D}) = \emptyset$. Proving that every holomorphic line bundle has a meromorphic section is hard! 

16 | Friday, February 19

16.1 Applications of Riemann-Roch

Definition 16.1.1 (Curves)

A **curve** is a compact complex manifold of complex dimension 1.

Example 16.1.2(?): Let C be a curve, then Ω_C^1 is the sheaf of holomorphic 1-forms, and $\Omega_C^{>1} = 0$. We also have the sheaves $A^{1,0}, A^{0,1}, A^{1,1}$, the sheaves of smooth (p, q) -forms. Here the only nonzero combinations are $(0, 0), (0, 1), (1, 0), (1, 1)$ by dimensional considerations. Let L be a holomorphic line bundle on C , then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg(L) + 1 - g.$$

Remark 16.1.3: In general it can be hard to compute $h^1(L)$, since this is sheaf cohomology (sections over double overlaps, cocycle conditions, etc). On the other hand, h^0 is easy to understand, since $h^0(\Omega_C^1)$ is the dimension of the global holomorphic sections $H^0(C, L) = L(C)$. A key tool here is the following:

Proposition 16.1.4 (Serre Duality).

$$H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega_C^1)^\vee,$$

noting that these are both global sections of a line bundle.

Proof (?).

Recall that we had a resolution of the sheaf L given by smooth vector bundles:

$$0 \rightarrow L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar{\partial}} L \otimes A^{0,1} \xrightarrow{\bar{\partial}} 0.$$

So we know that

$$H^1(C, L) = H^0(L \otimes A^{0,1}) / \bar{\partial} H^0(L \otimes A^{0,0}).$$

Choose a Hermitian metric h on L , i.e. a map $h : L \otimes \bar{L} \rightarrow \mathcal{O}$. On fibers, we have $h_p : L_p \otimes \bar{L}_p \rightarrow \mathbb{C}$. We'll also choose a metric on C , say g . Since C is a Riemann surface, we have an associated volume form ν on C (essentially the determinant), so we can define a pairing between sections of $L \otimes A^{0,0}$:

$$\langle s, t \rangle := \int_C h(s, \bar{t}) d\nu.$$

Note that

$$\langle s, s \rangle = \int_C h(s, \bar{s}) d\nu \geq 0 \quad \text{since } h(s, \bar{s})(p) = 0 \iff s_p = 0,$$

and moreover this integral is zero if and only if $s = 0$. So we have an inner product on $H^0(L \otimes A^{0,0})$. We can also define a pairing on sections of $L \otimes A^{0,1}$, say

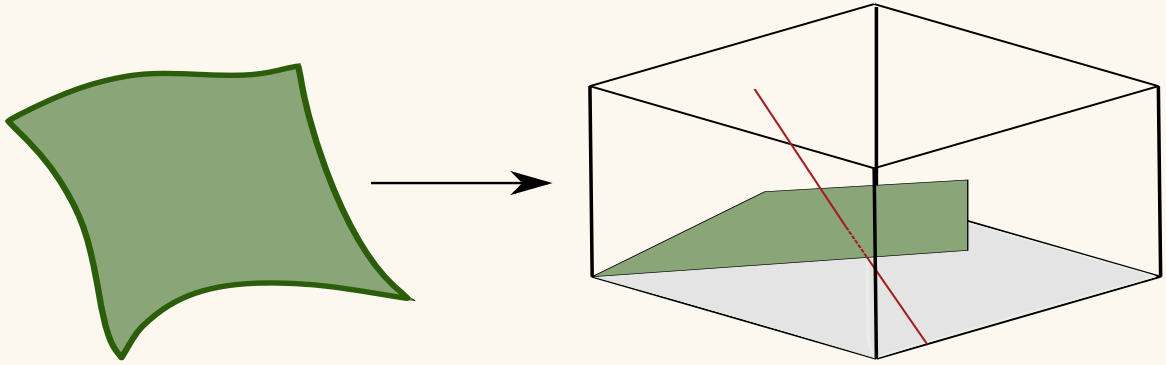
$$\langle s \otimes \alpha, t \otimes \beta \rangle = \int_C h(s, \bar{t}) \alpha \wedge \bar{\beta}.$$

Note that h is a smooth function and $\alpha \wedge \bar{\beta}$ is a $(1, 1)$ -form. Moreover, this is positive and nondegenerate. We want to understand the cokernel of the linear map

$$H^0(L \otimes A^{0,0}) \xrightarrow{\bar{\partial}} H^0(L \otimes A^{0,1}).$$

To compute $\text{coker}(\bar{\partial})$, we can look at the kernel of the adjoint, and it suffices to find the orthogonal complement of $\text{im}(\bar{\partial})$, i.e.

$$\text{coker}(\bar{\partial}) = \left\{ t \in H^0(L \otimes A^{0,1}) \mid \langle \bar{\partial}s, t \rangle = 0 \forall s \right\}.$$



So we want to understand sections $t \in H^0(L \otimes A^{0,1})$ such that

$$\int_C (\bar{\partial}s) \bar{t} = 0 \quad \forall s \in H^0(L \otimes A^{0,0}),$$

where $\partial C = \emptyset$. We'll basically want to do integration by parts on this. Note that $h(s, t) = hst$ here where we view h as a certain section. Note that $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$, so we can replace ∂ with $d = \bar{\partial} + \partial$ and apply Stokes' theorem:

$$\begin{aligned} \int_C sd(h\bar{t}) &= 0 & \forall s \in H^0(L \otimes A^{0,0}) \\ 0 &= \int_C s\bar{\partial}(h\bar{t}) \\ &= \int_C s \frac{\bar{\partial}(h\bar{t})}{d\nu} d\nu \\ &= \left\langle s, \frac{\bar{\partial}(h\bar{t})}{d\nu} \right\rangle \end{aligned}$$

where $h \in C^\infty(L^{-1} \otimes \bar{L}^{-1})$ and $h\bar{t} \in C^\infty(L^{-1} \otimes A^{1,0})$. But the right-hand side is in $H^0(L \otimes A^{0,0})$ and by nondegeneracy we can conclude

$$\frac{\bar{\partial}(h\bar{t})}{d\nu} = 0 \iff \bar{\partial}(h\bar{t}) = 0.$$

We thus have $h\bar{t} \in H^0(L^{-1} \otimes A^{1,0})$ which is a holomorphic line bundle tensored with $A^{0,0}$. Thus $\text{coker}(\bar{\partial}) \cong_h H^0(L^{-1} \otimes \Omega^1)$. ■

Remark 16.1.5: We showed $\langle \bar{\partial}s, t \rangle = \langle s, Y(t) \rangle$ where Y is the adjoint given above. Then the kernel of Y wound up being where $\bar{\partial}$ vanishes, i.e. holomorphic sections of a separate bundle. Here we had

- $t \in H^0(L \otimes A^{0,1})$
- $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$
- $h \in H^0(L^{-1} \otimes \bar{L}^{-1})$

17 | Monday, February 22

Remark 17.0.1: Last time: Serre duality, and we'll review Riemann-Roch. Recall that this depended on the statement that every holomorphic line bundle $L \rightarrow C$ for C a complex curve is of the form $L = \mathcal{O}(D)$ for some divisor D . Then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg L + 1 - g, \quad \deg L = \int_C c_1(L),$$

Serre duality said that the space of sections $H^1(C; L)$ is naturally isomorphic to $H^0(C, L^{-1} \otimes \Omega_C^1)^\vee$. Notation: given $X \in \mathbf{Mfd}_{\mathbb{C}}^n$ of complex, dimension n , the **canonical bundle** is written $K_X := \Omega_X^n$ and is the sheaf of holomorphic n -forms. Serre duality will generalize: if $\mathcal{E} \rightarrow X$ is a holomorphic vector bundle, then $H^i(X; \mathcal{E}) \cong H^{n-i}(X; \mathcal{E}^\vee \otimes K_X)^\vee$. Note that only H^0, H^1 are the only nontrivial degrees for a curve. For 4-manifolds, we'll have an H^2 as well.

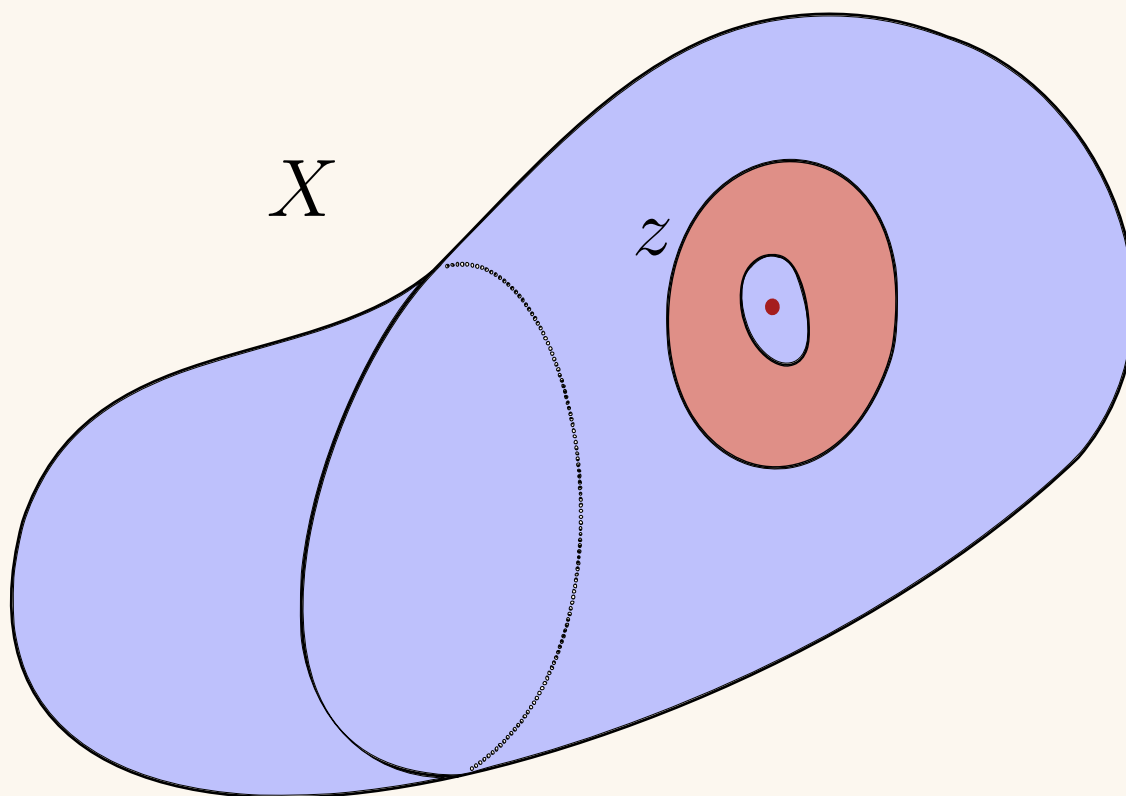
17.1 Applications of Riemann-Roch

Proposition 17.1.1 (?).

There is a unique complex $X \in \mathbf{Mfd}_{\mathbb{C}}$ diffeomorphic to S^2 .

Proof (of proposition).

Note existence is clear, since we can take $\mathbb{CP}^1 := (\mathbb{C}^2 \setminus \{0\})/\mathbf{x} \sim \lambda \mathbf{x}$ for $\lambda \in \mathbb{C}^\times$, which is identified as the set of complex lines through 0 in \mathbb{C}^2 . This decomposes as $\mathbb{C} \cup \mathbb{C} = \{[1, *]\} \cup \{[*], 1\}$. We now want to show that any two such complex manifolds are biholomorphic. Let $X \in \mathbf{Mfd}_{\mathbb{C}}^1$ with $X \cong_{C^\infty} S^2$, and consider for $p \in X$ the point bundle $\mathcal{O}(p) \rightarrow X$. The defining property was that there exists a section $s_p \in H^0(X; \mathcal{O}(p))$ which vanishes at first order at p :



We have

$$\chi(X; \mathcal{O}(p)) = \deg \mathcal{O}(p) + 1 - g(x) = 1 + 1 - 0 = 2.$$

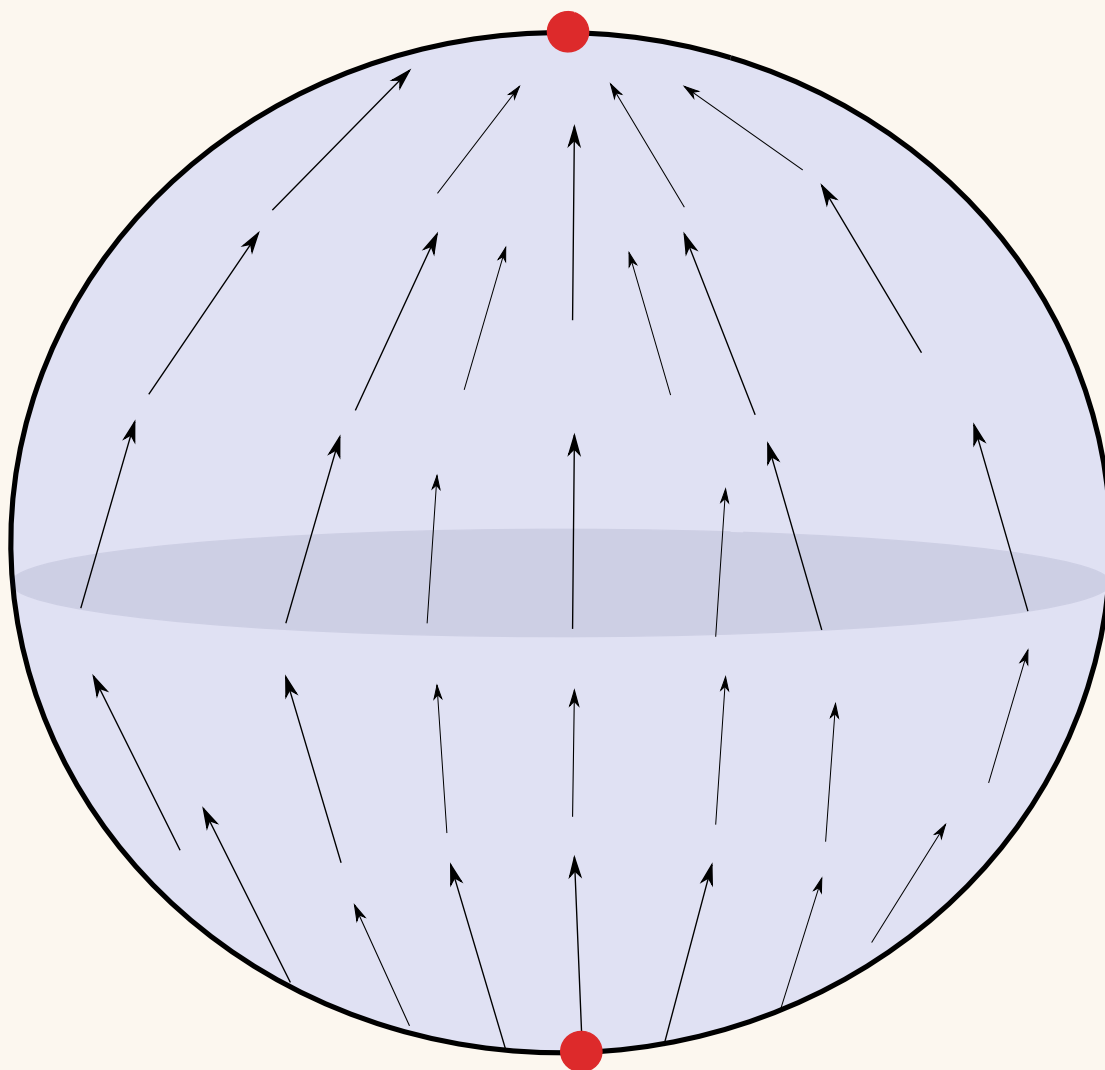
Exercise (?)

Check that $\deg \mathcal{O}(p) = 1$.

On the other hand we have

$$\chi(X; \mathcal{O}(p)) = h^0(\mathcal{O}(p)) - h^1(\mathcal{O}(p)).$$

We have $h^1(\mathcal{O}(p)) = H^0(K \otimes \mathcal{O}(-p))$, and $K_X = \Omega_X^1 = T^\vee X$, so the question is: what is the degree of TX for $X \cong S^2$? We need to compute $\int_X c_1(TX)$. How many zeros does a vector field on the sphere have? You can take the gradient vector field for a height function to get 2, noting that the two zeros come in with a positive orientation



In coordinates on \mathbb{CP}^1 , the coordinate is given by z and $z \frac{\partial}{\partial z} \mapsto -2 \frac{\partial}{\partial w}$ for the coordinate $w = 1/z$. We get $\int_X c_1(TX) = 2$ and thus $\deg K_X = -2$ by dualizing.

Fact

$\deg K_X = 2g - 2$. Use the existence of a smooth vector field on X .

Lemma 17.1.4(?).

If $\deg L < 0$ on C the $h^0(C, L) = 0$.

Proof (?).

If $s \in H^0(C, L)$ is nonzero, then since s is a holomorphic section,

$$0 \leq \sum_{p \in C} \text{Ord}_p(s) = \deg L.$$

■

By this lemma, $h^1(\mathcal{O}(p)) = 0$. We have $H^0(X; \mathcal{O}(p)) = \mathbb{C}s_p \oplus \mathbb{C}s$ for our specific section s_p and some other section $s \neq \lambda s_p$. Note that s/s_p is a meromorphic section of $\mathcal{O}(p) \times \mathcal{O}(-p) = \mathcal{O}$, so we have a map

$$\varphi : \frac{s}{s_p} : X \rightarrow \mathbb{P}^1.$$

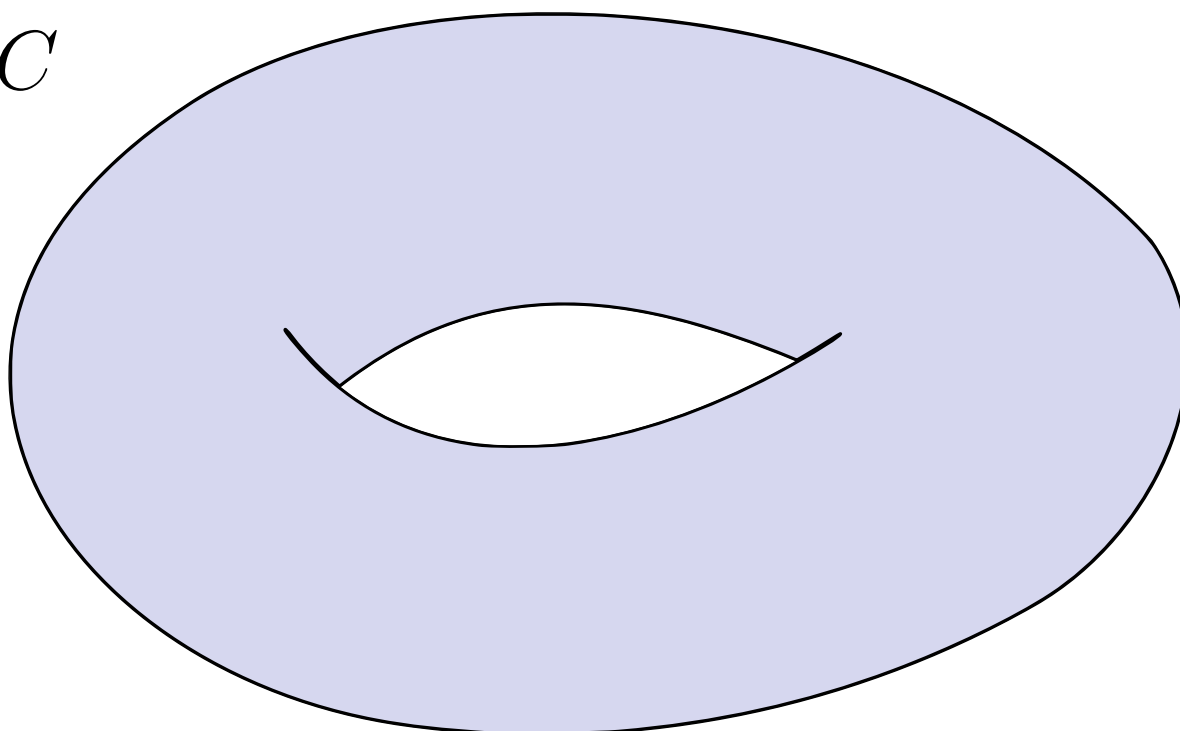
Note that $P \mapsto \infty \in \mathbb{P}^1$ under this φ , and it's only the ratio that is well-defined. We have $\varphi^{-1}(u) = \{s/s_p = u\} = \{s - us_p = 0\}$ which is a single point. So φ is a degree 1 map, and X is biholomorphic to \mathbb{P}^1 via φ .

■

Remark 17.1.5: So there is only one genus 0 Riemann surface. What about genus 1?

$$g(C) = 1$$

C



By Riemann-Roch we know

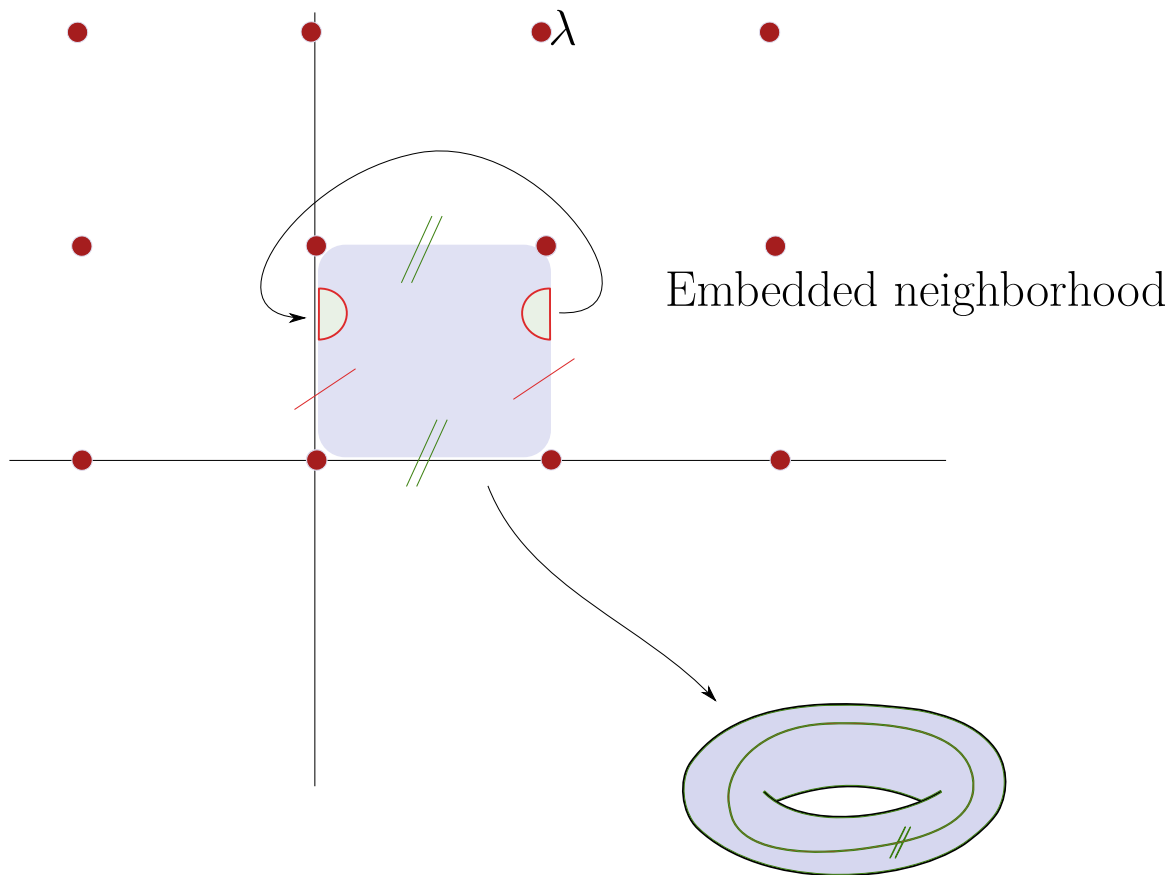
$$\chi(C; \mathcal{O}) = \deg \mathcal{O} + l - 1 = 0 = h^0(\mathcal{O}) - h^1(\mathcal{O}).$$

We know $h^0(\mathcal{O}) = 1$ by the maximum modulus principle and $h^1(C; \mathcal{O}) = 1$. By Serre duality, $h^0(C, K) = 1$, and since $\deg K = 2g - 2 = 0$. So let $s \in H^0(C, K)$ by a nonzero section, which we know exists. We then get $\text{Ord}_p s = 0$ for all p , so s vanishes nowhere. But then we get an isomorphism of sheaves, since s everywhere nonvanishing implies trivial cokernel:

$$\mathcal{O} \xrightarrow{s} K.$$

So $K_C = \mathcal{O}_C$ if $g(C) = 1$, and such a Riemann surface is an **elliptic curve**.

Example 17.1.6(?): Let $C := \mathbb{C}/\Lambda$ for Λ some lattice.



All transition functions are of the form $z \mapsto z + \lambda$ for some $\lambda \in \Lambda$. What is a nonvanishing section of K_C , i.e. a holomorphic one form $\omega := f(z)dz$ on \mathbb{C} that descends to \mathbb{C}/Λ . We would need $f(z)dz = f(z + \lambda)d(z + \lambda)$ for all λ . Something like $f = 1$ works, so $\omega = dz$ descends. In fact, f must be constant, since $H^0(\mathbb{C}/\Lambda, \mathcal{O}) = \mathbb{C}dz$ by the maximum modulus principle. Now let $p, q \in C$

and apply Riemann-Roch to the line bundle $\mathcal{O}(p+q)$ yields

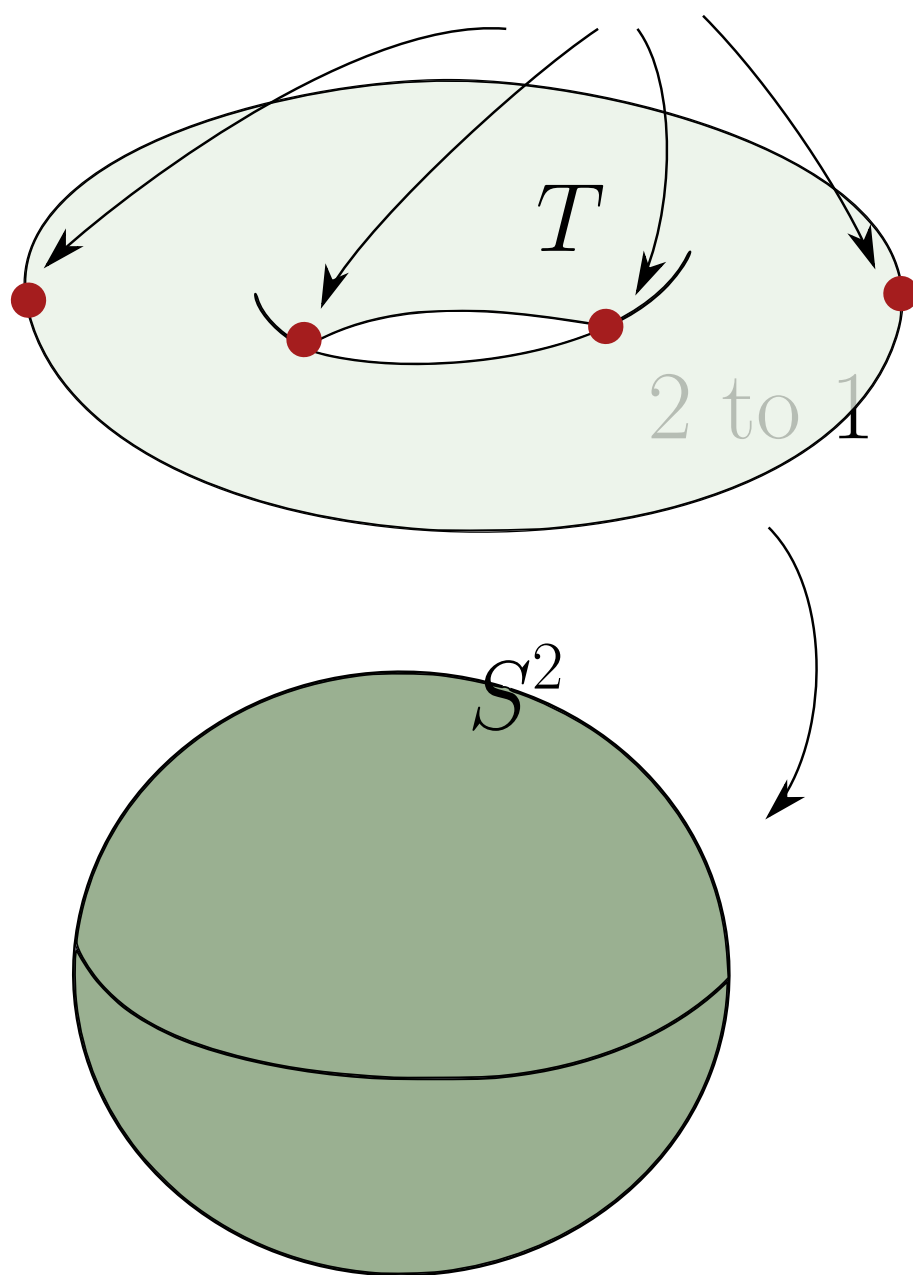
$$\begin{aligned}\chi(\mathcal{O}(p+q)) &= h^0(\mathcal{O}(p+q)) - h^1(\mathcal{O}(-p-q)) \\ &= h^0(\mathcal{O}(p+q)) - 0 \\ &= \deg \mathcal{O}(p+q) + 1 - 1 \\ &= 2.\end{aligned}$$

Thus there is a section $s_{p+q} \in H^0(\mathcal{O}(p+q)) \ni s$ that vanishes at $p+q$, and similarly a map

$$\frac{s}{s_{p+q}} : C \xrightarrow{\varphi} \mathbb{P}^1.$$

We can check $\varphi^{-1}(\infty) = p+q$ and $\deg \varphi = 2$. Thus genus 1 surfaces have a generically 2-to-1 map to \mathbb{P}^1 .

Branch Points



Note that homothetic lattices define an isomorphism between the elliptic curves, and lattices mod homothety are in correspondence of elliptic curves. By acting $\mathrm{PGL}_2(C) \curvearrowright \mathbb{P}^1$ since GL_2 acts on lines since scaling an element fixes a line. This is dimension 3. So elliptic curves are also in correspondence with $\{4 \text{ points on } \mathbb{P}^1\} / \mathrm{PGL}_2(\mathbb{C})$ since this is now dimension 1. Note that by applying homothety, the two basis vectors for a lattice can be rescaled so one is length 1 and the other is a complex number τ , and we can identify this space with $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

Exercise 17.1.7 (?)

Show that any $g(C) = 2$ curve has a degree 2 map to \mathbb{P}^1 .

Remark 17.1.8: Similarly $g(C) = 3$ are usually a curve of degree 4 in \mathbb{CP}^2 . Severi proof in the 50s: false! issues with building moduli space for $g \geq 23$. Need to use orbifold structure to take into account automorphisms.

18 | Wednesday, February 24

Last time:

$$\begin{aligned}\chi(C, L) &= h^0(C, L) - h^1(C, L) \\ &= h^0(C, L) - h^0(C, L^{-1} \otimes K_C) \\ &= \deg L + 1 - g,\end{aligned}$$

which is determined by purely topological information. We can generalize this to arbitrary ranks of the bundle and arbitrary dimensions of manifold:

Theorem 18.0.1 (Hirzebruch-Riemann-Roch (HRR) Formula).

Let X be a compact complex manifold and let $\mathcal{E} \rightarrow X$ be a holomorphic vector bundle. Then

$$\chi(\mathcal{E}) = \int_C \mathrm{ch}(\mathcal{E}) \mathrm{td}(X).$$

The constituents here:

- The **Chern character**, summed over R the *Chern roots*, which is in mixed cohomological degree.

$$\mathrm{ch}(\mathcal{E}) := \sum_{x_i \in R} e^{x_i} = \mathrm{ch}_0(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E}) + \cdots + \mathrm{ch}_i(\mathcal{E}) \in H^{2i}(X; \mathbb{Q}).$$

- The **Todd class**, defined as

$$\mathrm{td}(F) := \prod_{x_i \in R} \frac{x_i}{1 - e^{-x_i}}$$

where $\mathrm{td}(X) := \mathrm{td}(TX)$ is viewed as a complex vector bundle, which is again in mixed cohomological degree.

Remark 18.0.2: Note that integrating over cohomology classes in mixed degree is just equal to the integral over the top degree terms. Applying this to $X = C$ a curve and $\mathcal{E} := \mathcal{O}$, we obtain

$$\chi(C, \mathcal{O}) = \int_C \text{ch}(\mathcal{O}) \text{td}(C).$$

We have

- $\text{ch}(\mathcal{O}) = e^{c_1(\mathcal{O})} = e^0 = 1$
- $\text{td}(C) := \text{td}(TC) = c_1(TC)/(1 - e^{-c_1(TC)})$, whose Taylor coefficients are the Bernoulli numbers. We can expand $x/(1 - e^{-x}) = 1 + (x/2) + (x^2/12) - x^4/720 + \dots$, and since terms above degree 2 vanish, we have

$$\begin{aligned} \dots &= \int_C 1 + \left(1 + \frac{c_1(TC)}{2}\right) \\ &= \int_C \left(\frac{c_1(TC)}{2}\right) \\ &= \frac{1}{2} \chi_{\text{Top}}(C) && \text{Chern-Gauss-Bonnet} \\ &= \frac{2 - 2g}{2} \\ &= 1 - g. \end{aligned}$$

We thus obtain

$$\begin{aligned} \chi(C, L) &= \int_C \text{ch}(L) \text{td}(C) \\ &= \int_C (1 + c_1(L)) \left(1 + \frac{c_1(TC)}{2}\right) \\ &= \int_C c_1(L) + \frac{c_1(TC)}{2} \\ &= \deg L + 1 - g. \end{aligned}$$

Remark 18.0.3: Note that this is a better definition of genus than the previous one, which was just the correction term in Riemann-Roch. Here we can define it as $g := h^1/2$.

Exercise 18.0.4 (?)

Try to state and prove a Riemann-Roch formula for vector bundles on curves.

Proposition 18.0.5 (?).

Let S be a compact complex surface, i.e. $S \in \mathbf{Mfd}_{\mathbb{C}}^2$. An example might be $C \times D$ for C, D two complex curves, or \mathbb{CP}^2 . Let $L \rightarrow S$ be a holomorphic vector bundle. Then

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2} (L^2 - L \cdot K).$$

Note that $L^2 := \int_S c_1(L)c_1(L)$ is just shorthand for taking the intersection of L with itself. Recall that $K := \Omega_S^2$ is the space of holomorphic top forms.

Proof (?).

Let x_1, x_2 be the Chern roots of TS . By HRR, we have

$$\begin{aligned}
 \chi(L) &= \int_S \text{ch}(L) \text{td}(S) \\
 &= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!} \right) \left(\frac{x_1}{1 - e^{-x_1}} \frac{x_2}{1 - e^{-x_2}} \right) \\
 &= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!} \right) \left(1 + \frac{x_1}{2} + \frac{x_1^2}{12} \right) \left(1 + \frac{x_2}{2} + \frac{x_2^2}{12} \right) \\
 &= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!} \right) \left(1 + \frac{x_1 + x_2}{2} + \frac{x_1^2 + x_2^2 + 3x_1x_2}{12} \right) \\
 &= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!} \right) \left(1 + \frac{c_1(x_1, x_2)}{2} + \frac{c_1(x_1, x_2)^2 + c_2(x_1, x_2)}{12} \right) \\
 &= \int_S \left(1 + c_1(L) + \frac{c_1(L)^2}{2!} \right) \left(1 + \frac{c_1(T)}{2} + \frac{c_1(T)^2 + c_2(T)}{2} \right) \\
 &= \int_S \frac{c_1(L)^2}{2} + \frac{c_1(L)c_1(T)}{2} + \frac{c_1(T)^2}{2} + \frac{c_2(T)}{12} \quad \text{Take deg 4} \\
 &= \int_S \left(\frac{c_1(L)^2 + c_1(L)c_1(T)}{2} \right) + \chi(\mathcal{O}_S) \quad \text{HRR on last two terms.}
 \end{aligned}$$

where we've applied HRR to \mathcal{O}_S . It remains to show that $c_1(T) = -c_1(K)$. We have

$$K = \Omega_S^2 = \bigwedge^2 T^\vee.$$

Note that $\bigwedge^{\text{top}} \mathcal{E} := \det(\mathcal{E})$ for any bundle \mathcal{E} since this is a 1-dimensional bundle. We have $c_1(T) = -c_1(T^\vee)$ since the Chern roots of T^\vee are $-x_1, -x_2$. So it suffices to show $c_1(T^\vee) = c_1(K)$, but there is a general result that $c_1(\mathcal{E}) = c_1(\det \mathcal{E})$. This uses the splitting principle $\mathcal{E} = \bigoplus_{i=1}^r L_i$ with $x_i = c_1(L_i)$. We have $c_1(\mathcal{E}) = \sum x_i$ and $\det \mathcal{E} = \bigotimes_{i=1}^r L_i$, so $\sum x_i = c_1(L_1 \otimes \cdots \otimes L_r)$. ■

Remark 18.0.6: We want to use the following formula:

$$\chi(S, L) = \chi(\mathcal{O}_S) = \frac{1}{2}(L^2 - L \cdot K).$$

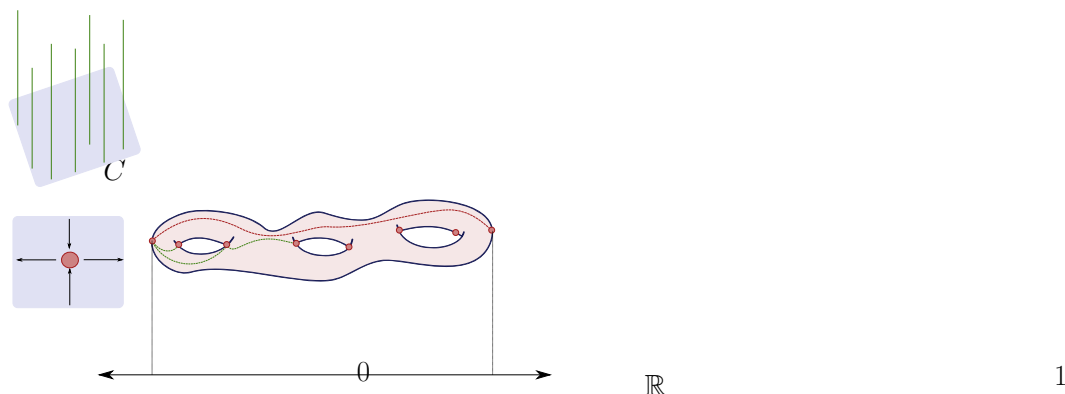
This requires knowing $\chi(\mathcal{O}_S)$. Applying HRR yields

$$\begin{aligned}\chi(\mathcal{O}_S) &= \int_S \frac{c_1(T)^2 + c_2(T)}{12} \\ &= \int_S \frac{(-c_1(K))^2 + c_2(T)}{12} \\ &= \frac{K^2 + \int_S c_2(T)}{12},\end{aligned}$$

so we just need to understand $\int_S c_2(T)$. But for $n = \text{rank } \mathcal{E}$, $c_n(\mathcal{E})$ (the top Chern class) is the fundamental class of a zero locus of a section of \mathcal{E} . Note that $S \in \mathbf{Mfd}_{\mathbb{R}}^4$ is oriented, so $\int_S c_2(T)$ is the signed number of zeros of a smooth vector field.

TC

∇f has 8 zeros



Looking at the tangent bundle of the surface, the local sign of an intersection will be the number of incoming directions (mod 2), i.e. the index of the critical point. Then the signed number of zeros here yields $1 - 6 + 1 = -4 = \chi_{\text{Top}}(C)$. More generally, we have

$$\chi_{\text{Top}}(M^n) = \int_C c_n(TM),$$

the **Chern-Gauss-Bonnet** formula. We can thus write

$$\chi(\mathcal{O}_S) = \frac{K^2 + \chi_{\text{Top}}(S)}{12}.$$

ToDos

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