

# Category $\mathcal{O}$

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## 1 Wednesday January 8

Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category  $\mathcal{O}$ ".

Course Website: <https://faculty.franklin.uga.edu/brian/math-8030-spring-2020>

### 1.1 Chapter Zero: Review

Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase  $\mathfrak{g}$  characters!

In this course, we'll take  $k = \mathbb{C}$ .

Recall that a Lie Algebra is a vector space  $\mathfrak{g}$  with a bracket  $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying

- $[xx] = 0$  for all  $x \in \mathfrak{g}$ 
  - Exercise: this implies  $[xy] = -[yx]$ .
- $[x[yz]] = [[xy]z] + [y[xz]]$  (The Jacobi identity)
  - This says  $x$  acts as a derivation.

Hint: Consider  $[x + y, x + y]$ . Note that the converse holds iff  $\text{char } k \neq 2$ .

Exercise: This implies Lie Algebras never have an identity.

**Definition:**  $\mathfrak{g}$  is *abelian* iff  $[xy] = 0$  for all  $x, y \in \mathfrak{g}$ .

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
  - A vector subspace that is closed under brackets.
- Homomorphisms
  - I.e. a linear transformation  $\phi$  that commutes with the bracket, i.e.  $\phi([xy]) = [\phi(x)\phi(y)]$ .
- Ideals

*Exercise:* Given a vector space (possibly infinite-dimensional) over  $k$ , then (exercise)  $\mathfrak{gl}(V) := \text{End}_k(V)$  is a Lie algebra when equipped with  $[fg] = f \circ g - g \circ f$ .

**Definition:** A *representation* of  $\mathfrak{g}$  is a homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  for some  $V$ .

*Example:* The adjoint representation is  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , where  $\text{ad}(x)(y) := [xy]$ .

Representations give  $\mathfrak{g}$  the structure of a module over  $V$ , where  $x \cdot v := \phi(x)(v)$ . All of the usual module axioms hold, where now  $[xy] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ .

*Example:* The trivial representation  $V = k$  where  $x \cdot a = 0$ .

**Definition:**  $V$  is *irreducible* (or *simple*) iff  $V$  has exactly two  $\mathfrak{g}$ -invariant subspaces, namely  $0, V$ .

**Definition:**  $V$  is *completely reducible* iff  $V$  is a direct sum of simple modules, and *indecomposable* iff  $V$  can not be written as  $V = M \oplus N$ , a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragradient/dual*  $V^\vee := \text{hom}_k(V, k)$  where  $(x \cdot f) = -f(x \cdot v)$  for  $f \in V^\vee, x \in \mathfrak{g}, v \in V$ .
- The direct sum  $V \oplus W$  where  $x \cdot (v, w) = (x \cdot v, x \cdot w)$  and  $x \cdot (v + w) = x \cdot v + x \cdot w$ .
- The tensor product where  $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$ .
- $\text{hom}_k(V, W)$  where  $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$ .  
– Note that if we take  $W = k$  then the first term vanishes and this recovers the dual.

## 1.2 Semisimple Lie Algebras

**Definition:** The derived ideal is given by  $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \text{span}_k(\{[xy] \mid x, y \in \mathfrak{g}\})$ .

This is the analog of the commutator subgroup.

**Lemma:**  $\mathfrak{g}$  is abelian iff  $\mathfrak{g}^{(1)} = \{0\}$ , and 1-dimensional algebras are always abelian.

This follows because if  $[xy] := xy - yx$  then  $[xy] = 0 \iff xy = yx$ .

**Definition:** A lie group  $\mathfrak{g}$  is *simple* iff the only ideals of  $\mathfrak{g}$  are  $0, \mathfrak{g}$  and  $\mathfrak{g}^{(1)} \neq \{0\}$ .

Note that this rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

**Definition:** The derived series is defined by  $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$ , continuing inductively.  $\mathfrak{g}$  is said to be solvable if  $\mathfrak{g}^{(n)} = 0$  for some  $n$ .

**Lemma:** Abelian implies solvable.

Review definition of nilpotent algebras.

**Definition:**  $\mathfrak{g}$  is semisimple (s.s.) iff  $\mathfrak{g}$  has no nonzero solvable ideals.

Exercise: Simple implies semisimple.

Some remarks:

1. Semisimple algebras  $\mathfrak{g}$  will usually have solvable subalgebras.
2.  $\mathfrak{g}$  is semisimple iff  $\mathfrak{g}$  has no nonzero abelian ideals.

**Definition:** The Killing form is given by  $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow k$  where  $\kappa(x, y) = \text{Tr}(\text{ad } x \text{ ad } y)$ , which is a symmetric bilinear form.

**Lemma:**  $\kappa([xy], z) = \kappa(x, [yz])$ .

Recall that if  $\beta : V^{\otimes 2} \rightarrow k$  is any symmetric bilinear form, then its radical is defined by

$$\text{rad}\beta = \left\{ v \in V \mid \beta(v, w) = 0 \ \forall w \in V \right\}.$$

**Definition:** A bilinear form  $\beta$  is nondegenerate iff  $\text{rad}\beta = 0$ .

**Lemma:**  $\text{rad}\kappa \trianglelefteq \mathfrak{g}$  is an ideal, which follows by the above associative property.

**Theorem:**  $\mathfrak{g}$  is semisimple iff  $\kappa$  is nondegenerate.

Example: The standard example of a semisimple lie algebra is  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{Tr}(x) = 0 \right\}$ .

Note: from now on,  $\mathfrak{g}$  will denote a semisimple lie algebra over  $\mathbb{C}$ .

**Theorem (Weyl):** Every finite dimensional representation of a semisimple  $\mathfrak{g}$  is completely reducible.

I.e., the category of finite-dimensional representations is relatively uninteresting – there are no extensions, everything is a direct sum, so once you classify the simple algebras (which isn't terribly difficult) then you have complete information.

## 2 Friday January 10th

Let  $\mathfrak{g}$  be a finite dimensional semisimple lie algebra over  $\mathbb{C}$ .

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form  $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ , the trace zero  $n \times n$  matrices.

Let  $\mathfrak{h}$  be a maximal toral subalgebra, where  $x \in \mathfrak{g}$  is *toral* if  $x$  is semisimple, i.e.  $\text{ad } x$  is semisimple (i.e. diagonalizable).

*Example:*  $\mathfrak{h}$  is the diagonal matrices in  $\mathfrak{sl}(n, \mathbb{C})$ .

**Fact:**  $\mathfrak{h}$  is abelian, so  $\text{ad } \mathfrak{h}$  consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

where  $\mathfrak{g}_{\alpha} = \left\{ x \in \mathfrak{g} \mid [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$  where  $\alpha \in \mathfrak{h}^{\vee}$  is a linear functional.

Here  $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$ , so  $[hx] = 0$  corresponds to zero eigenvalues, and (fact) it turns out that  $\mathfrak{h}$  is its own centralizer.

We then obtain a set of roots of  $\mathfrak{h}, \mathfrak{g}$  given by  $\Phi = \left\{ \alpha \in \mathfrak{h}^{\vee} \mid \alpha \neq 0, \mathfrak{g}_{\alpha} \neq \{0\} \right\}$ .

*Example:*  $\mathfrak{g}_{\alpha} = \mathbb{C}E_{ij}$  for some  $i \neq j$ , the matrix with a 1 in the  $i, j$  position and zero elsewhere.

**Fact:** The restriction  $\kappa|_{\mathfrak{h}}$  is nondegenerate, so we can identify  $\mathfrak{h}, \mathfrak{h}^\vee$  via  $\kappa$  (can always do this with vector spaces with a nondegenerate bilinear form), where  $\kappa$  maps to another bilinear form  $(\cdot, \cdot)$ .

$$\mathfrak{h}^\vee \ni \lambda \iff t_\lambda \in \mathfrak{h}$$

$$\lambda(h) = \kappa(t_\lambda, h) \quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

## 2.1 Facts About $\Phi$ and Root Spaces

Let  $\alpha, \beta \in \Phi$  be roots.

1.  $\Phi$  spans  $\mathfrak{h}^\vee$  and does not contain zero.
2. If  $\alpha \in \Phi$  then  $-\alpha \in \Phi$ , but no other scalar multiple of  $\alpha$  is in  $\Phi$ .

*Aside:*

- $\dim \mathfrak{g}_\alpha = 1$ .
- If  $0 \neq x_\alpha \in \mathfrak{g}_\alpha$  then there exists a unique  $y_\alpha \in \mathfrak{g}_{-\alpha}$  such that  $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$  spans a 3-dimensional subalgebra in  $\mathfrak{sl}_2$ , given by  $x_\alpha = [0, 1; 0, 0], y_\alpha = [0, 0; 1, 0], h_\alpha = [1, 0; 0, -1]$ .
- Under the correspondence  $\mathfrak{h} \iff \mathfrak{h}^\vee$  induced by  $\kappa$ ,  $h_\alpha \iff \alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$ . Thus for all  $\lambda \in \mathfrak{h}^\vee$ ,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

- If  $\alpha + \beta \neq 0$ , then  $\kappa(g_\alpha, g_\beta) = 0$ .

3.  $(\beta, \alpha^\vee) \in \mathbb{Z}$
4.  $S_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in \Phi$ .

If  $\alpha + \beta \in \Phi$ , then  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ . Example: If  $\alpha = E_{ij}, \beta = E_{jk}$  where  $k \neq i$ , then  $[E_{ij}, E_{jk}] = E_{ik}$ .

- $\mathfrak{g}$  is generated as an algebra by the root spaces  $\mathfrak{g}_\alpha$
- Root strings: If  $\beta \neq \pm\alpha$ , then the roots of the form  $\alpha + k\beta$  for  $k \in \mathbb{Z}$  form an unbroken string  $\alpha - r\beta, \dots, \alpha - \beta, \alpha, \alpha + \beta, \dots, \alpha + \ell\beta$  consisting of at most 4 roots where  $r - s = (\alpha, \beta^\vee)$ .

*Example:* The circled roots below form the root string for  $\beta$ :



In general, a subset  $\Phi$  of a real euclidean space  $E$  satisfying conditions (1) through (4) is an (*abstract*) *root system*.

When  $\Phi$  comes from a  $\mathfrak{g}$ ,  $E := \mathbb{R}\Phi$ .

### 2.1.1 The Root System

There exists a subset  $\Delta \subseteq \Phi$  such that

- $\Delta$  is a  $\mathbb{C}$ -basis for  $\mathfrak{g}^\vee$
- $\beta \in \Phi$  implies that  $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$  with either
  - All  $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$  or  $\beta < 0$ .
  - All  $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$  or  $\beta > 0$ .

$\Delta$  is called a *simple system*. If  $\Delta = \{a_1, \dots, a_\ell\}$  then  $\Phi^+$  are the *positive roots*, and  $\Phi^+ \ni \beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ ,

then the *height* of  $\beta$  is defined as  $\sum c_\alpha \in \mathbb{Z}_{>0}$ .

Note that  $\mathbb{Z}\Phi := \Lambda_r$  is a lattice, and is referred to as the *root lattice*, and  $\Lambda_r \subset E = \mathbb{R}\Phi$ . We also have  $\Phi^+ = \{\beta^\vee \mid \beta \in \Phi\}$ , the *dual root system*, is a root system with simple system  $\Delta^\vee$ .

Important subalgebras of  $\mathfrak{g}$ :

- Upper triangular with zero diagonal  $\mathfrak{n} = \mathfrak{n}^+ = \sum_{\beta > 0} \mathfrak{g}_\beta$
- Lower triangular with zero diagonal  $\mathfrak{n}^- = \sum_{\beta > 0} \mathfrak{g}_{-\beta}$

- Upper triangular,  $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$  a Borel subalgebra
- Lower triangular,  $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$ .

There is thus a triangular (Cartan) decomposition,  $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ .

**Fact:** If  $\beta \in \Phi^+ \setminus \Delta$ , and if  $\alpha \in \Delta$  such that  $(\beta, \alpha^\vee) > 0$ , then  $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$  has height strictly less than the height of  $\beta$ .

By root strings,  $\beta - \alpha \in \Phi^+$  is positive root of height one less than  $\beta$ , yielding a way to induct on heights (useful technique).

### 2.1.2 Weyl Groups

For  $\alpha \in \Phi$ , define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\rightarrow \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in  $E$  perpendicular to  $\alpha$ :

Note that  $S_\alpha^2 = \text{id}$ .

Define  $W$  as the subgroup of  $\text{gl}(E)$  generated by all  $s_\alpha$  for  $\alpha \in \Phi$ , this is the *Weyl group* of  $\mathfrak{g}$  or  $\Phi$ , which is finite and  $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$  is generated by simple reflections.

By (4),  $W$  leaves  $\Phi$  invariant. In fact  $W$  is a finite Coxeter group with generators  $S = \{s_\alpha \mid \alpha \in \Delta\}$  and defining relations  $(s_\alpha s_\beta)^{m(\alpha, \beta)} = 1$  for  $\alpha, \beta \in \Delta$  where  $m(\alpha, \beta) \in \{2, 3, 4, 6\}$  when  $\alpha \neq \beta$  and  $m(\alpha, \alpha) = 1$ .

Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group*.

## 3 Monday January 13th

### 3.1 Lengths

Recall that we have a root space decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$  for finite dimensional semisimple lie algebras over  $\mathbb{C}$ . We have  $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$ , for  $\lambda \in \mathfrak{h}^\vee$  and the Weyl group  $W = \langle s_\beta \mid \beta \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Delta \rangle$  where  $\Delta = \{a_i\}$  are the simple roots. For  $w \in W$ , we can take the reduced expression for  $w$  by writing  $w = s_1 \cdots s_n$  with  $s_i$  simple and  $n$  minimal. The length is uniquely determined, but not the expression. So we define  $\ell(w) := n$  where  $\ell(1) := 0$ .

*Facts:*

1.  $\ell(w)$  is the size of the set  $\{\beta \in \Phi^+ \mid w\beta < 0\}$ 
  - The above set is equal to  $\Phi^+ \cap w^{-1}\Phi^-$ .



Figure 1: Image



- In particular, for  $\beta \in \Phi^+$ ,  $\beta$  is simple (i.e.  $\beta \ni \Delta$  iff  $\ell(s_\beta) = 1$ ).
  - Note:  $\alpha$  is the only root that  $s_\alpha$  sends to a negative root, so  $s_\alpha(\beta) > 0$  for all  $\beta \in \Phi^+ \setminus \{\alpha\}$ .
2.  $\ell(w) = \ell(w^{-1})$  for all  $w \in W$ , so  $\ell(w)$  is also the size of  $\Phi \cap w\Phi$  (replacing  $w^{-1}$  with  $w$ )
  3. There exists a unique  $w_0 \in W$  with  $\ell(w_0)$  maximal such that  $\ell(w_0) = |\Phi^+|$  and  $w_0(\Phi^+) = \Phi^-$ .
- Also  $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For  $\alpha \in \Phi^+$ ,  $w \in W$ , we have either

$$\begin{aligned}\ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0\end{aligned}$$

Taking inverses yields  $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$ .

### 3.2 Bruhat Order

Let  $S$  be the set of simple reflections, i.e.  $S = \{s_\alpha \mid \alpha \in \Delta\}$ . Then define

$$T := \bigcup_{w \in W} wSw^{-1} = \{s_\beta \mid \beta \in \Phi^+\}.$$

This is the set of *all* reflections in  $W$  through hyperplanes in  $E$ .

We'll write  $w' \xrightarrow{t} w$  means  $w = tw'$  and  $\ell(w') < \ell(w)$ . Note that in the literature, it's also often assumed that  $\ell(w') = \ell(w) - 1$ . In this case, we say  $w'$  covers  $w$ , and refer to this as “the covering relation”. So  $w' \rightarrow w$  means that  $w' \xrightarrow{t} w$  for some  $t \in T$ . We extend this to a partial order:  $w' < w$  means that there exists a  $w$  such that  $w' = w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_n = w$ . This is called the **Bruhat-Chevalley order** on  $W$ .

*Corollary:*  $w' < w \implies \ell(w') < \ell(w)$ , so  $1 \in W$  is the unique minimal element in  $W$  under this order.

It turns out that if we set  $w = w't$  instead, this results in the same partial order.

If you restrict  $T$  to simple reflections, this yields the *weak Bruhat order*. In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

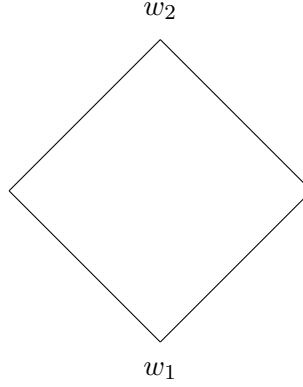
Recall that lie algebras yield finite crystallographic coxeter groups.

*Properties:* For  $(W, S)$  a coxeter group,

- a.  $w' \leq w$  iff  $w'$  occurs as a subexpression/subword of every reduced expression  $s_1 \dots s_n$  for  $w$ , where a subexpression is any subcollection of  $s_i$  in the same order.

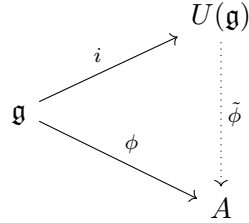
Note that this implies that  $1$  is not only a minimal element in this order, but an infimum.

- b. Adjacent elements  $w', w$  (i.e.  $w' < w$  and there does not exist a  $w''$  such that  $w' < w'' < w$ ) in the Bruhat order differ in length by 1.
- c. If  $w' < w$  and  $s \in S$ , then  $w's \leq w$  or  $w's \leq ws$  (or both). i.e., if  $\ell(w_1) = 2 = \ell(w_2)$ , then the size of  $\{w \in W \mid w_1 < w < w_2\}$  is either 0 or 2.



### 3.3 Properties of Universal Enveloping Algebras

Let  $\mathfrak{g}$  be any lie algebra, and  $\phi : \mathfrak{g} \rightarrow A$  be any map into an associative algebra. Then there exists an object  $U(\mathfrak{g})$  and a map  $i$  such that the following diagram commutes:



Note that  $\tilde{\phi}$  is a map in the category of associative algebras.

Moreover any lie algebra homomorphism  $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  induces a morphism of associative algebras  $U(\mathfrak{g}_1) \rightarrow U(\mathfrak{g}_2)$ , where  $\mathfrak{g}$  generates  $U(\mathfrak{g})$  as an algebra.

$U(\mathfrak{g})$  can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle [x, y] - x \otimes y - y \otimes x \mid x, y \in \mathfrak{g} \rangle.$$

Note that this ideal is not necessarily homogeneous.

*Properties:*

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$  by the extension of the adjoint action,  $(\text{ad } x)(u) = xu - ux$  for  $x \in \mathfrak{g}, u \in U(\mathfrak{g})$ .

**Big Theorem (Poincaré-Birkhoff-Witt, i.e. PBW):** If  $\{x_1, \dots, x_n\}$  is a basis for  $\mathfrak{g}$ , then  $\{x_1^{t_1}, \dots, x_n^{t_n} \mid t_i \in \mathbb{Z}^+\}$  (noting that  $x^n = x \otimes x \otimes \dots \otimes x$  and  $\mathbb{Z}^+$  includes 0) is a basis for  $U(\mathfrak{g})$ .

*Corollary:*  $i : \mathfrak{g} \rightarrow U(\mathfrak{g})$  is injective, so we can think of  $\mathfrak{g} \subseteq U(\mathfrak{g})$ .

If  $\mathfrak{g}$  is semisimple, then it admits a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$  and choose a compatible basis for  $\mathfrak{g}$ , then  $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$ .

If  $\phi : \mathfrak{g} \rightarrow \text{gl}(V)$  is any lie algebra representation, it induces an algebra representation  $U(\mathfrak{g})$  of  $U(\mathfrak{g})$  on  $V$  and vice-versa. It satisfies  $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$  for all  $x, y \in \mathfrak{g}$  and  $v \in V$ . Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

*Notation:*  $\mathfrak{Z}(\mathfrak{g})$  denotes the center of  $U(\mathfrak{g})$ .

### 3.4 Integral Weights

We have a Euclidean space  $E = \mathbb{R}\Phi^+$ , the  $\mathbb{R}$ -span of the roots. We also have the **integral weight lattice**

$$\Lambda = \left\{ \lambda \in E \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in \Phi (\text{or } \Phi^+ \text{ or } \Delta) \right\}.$$

There is a sublattice  $\Lambda_r \subseteq \Lambda$ , which is an additive subgroup of finite index.

There is a partial order of  $\Lambda$  on  $E$  and  $\mathfrak{h}^\vee$ . We write  $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+ \Delta = \mathbb{Z}^+ \Phi^+$ . For a basis  $\Delta = \{\alpha_1, \dots, \alpha_n\}$ , define a dual basis  $(w_i, \alpha_j^\vee) = \delta_{ij}$ . The fundamental weights are given by a  $\mathbb{Z}$ -basis for  $\Lambda$ . Then  $\Lambda$  is a free abelian group of rank  $\ell$ , and  $\Lambda^+ = \mathbb{Z}^+ w_1 + \dots + \mathbb{Z}^+ w_\ell$  are the **dominant integral weights**.

Note that in Jantzen's book,  $X$  is used for  $\Lambda$  and  $X^+$  correspondingly.

## 4 Wednesday January 15th

### 4.1 Review

The Weyl vector is given by  $\rho = \bar{w}_1 + \dots + \bar{w}_\ell = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \Lambda^+$ .

- If  $\alpha \in \Delta$  then  $(\rho, \alpha^\vee) = 1$
- $s_\alpha(\rho) = \rho - \alpha$ .

Let  $\lambda \in \Lambda^+$ ; a few facts:

1. The size of  $\left\{ \mu \in \Lambda^+ \mid \mu \leq \lambda \right\}$  (with the partial order from last time) is finite.
2.  $w\lambda < \lambda$  for all  $w \in W$ .

The Weyl chamber (for a fixed root,  $E = \text{Euclidean space}$ ) is  $C = \left\{ \lambda \in E \mid (\lambda, \alpha) > 0 \forall \alpha \in \Delta \right\}$  (Note that the hyperplane splits  $E$  into connected components, we mark this component as distinguished.)

- A connected component of the union of hyperplanes is orthogonal to roots
- They're in bijection with  $\Delta$
- They're permuted simply transitively by  $W$ .

And  $\bar{C}$  denotes the fundamental domain.

## 4.2 Weight Representations

For  $\lambda \in \mathfrak{h}^\vee$ , we let  $M_\lambda = \{v \in M \mid h \cdot v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$  denote a *weight space* of  $M$  if  $M_\lambda \neq 0$ . In this case,  $\lambda$  is a *weight* of  $M$ . The dimension of  $M_\lambda$  is the *multiplicity* of  $\lambda$  in  $M$ , and we define the set of weights as  $\Pi(M) = \{\lambda \in \mathfrak{h}^\vee \mid M_\lambda \neq 0\}$ .

Example if  $M = \mathfrak{g}$  under the adjoint action, then  $\Pi(M) = \Phi \cup \{0\}$ .

Remark: The weight vectors for distinct weights are linearly independent. Thus there is a  $\mathfrak{g}$ -submodule given by  $\sum_\lambda M_\lambda$ , which is in fact a direct sum.

Note: It may not be the case that  $\sum_\lambda M_\lambda = M$ , and can in fact be zero, although this is an odd situation. See Humphreys #1, #20.2, p. 110.

In our case, we'll have a *weight module*  $M = \bigoplus_\lambda M_\lambda$ , so  $\mathfrak{h} \curvearrowright M$  semisimply.

## 4.3 Finite-dimensional Modules

Recall Weyl's complete reducibility theorem, which implies that any finite dimensional  $\mathfrak{g}$ -module is a weight module. In fact,  $\mathfrak{n}, \mathfrak{n}^- \curvearrowright M$  nilpotently.

Some facts:

- $\Pi(M) \subset \Lambda$  is a subset of the integral lattice.
- $\Pi(M)$  is  $W$ -invariant.
- $\dim M_\lambda = \dim M_{w\lambda}$  for any  $\lambda \in \Pi(M), w \in W$ .

## 4.4 Simple Finite Dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules

Fix the standard basis  $\{x, h, y\}$  of  $\mathfrak{sl}(2, \mathbb{C})$  with  $[hx] = 2x, [hy] = -2y, [xy] = h$ . Since  $\dim \mathfrak{h} = 1$ , there is a bijection  $\mathfrak{h}^\vee \leftrightarrow \mathbb{C}$ ,  $\Lambda \leftrightarrow \mathbb{Z}$ , and  $\Lambda_r \leftrightarrow 2\mathbb{Z}$  with  $\alpha \rightarrow 2$  and  $\rho \rightarrow 1$ .

There is a correspondence between weights and simple modules:

$$\begin{aligned} \{\text{Isomorphism classes of simple modules}\} &\iff \Lambda^+ = \{0, 1, 2, 3, \dots\} \\ L(\lambda) &\iff \lambda. \end{aligned}$$

Moreover,  $L(\lambda)$  has a 1-dimensional weight spaces with weights  $\lambda, \lambda - 2, \dots, -\lambda$  and thus  $\dim L(\lambda) = \lambda + 1$ .

Examples:

- $L(0) = \mathbb{C}$ , the trivial representation,
- $L(1) = \mathbb{C}^2$ , the natural representation where  $\mathfrak{sl}(2, \mathbb{C})$  acts by matrix multiplication,
- $L(2) = \mathfrak{g}$ , the adjoint representation.

Choose a basis  $\{v_1, \dots, v_\lambda\}$  for  $L(\lambda)$  so that  $\mathbb{C}v_0 = M_\lambda, \mathbb{C}v_1 = M_{\lambda-2}, \dots, \mathbb{C}v_\lambda M_{-\lambda}$ . Then



Figure 2: Image

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$ , where  $v_{-1} := 0$
- $y \cdot v_i = (i + 1)v_{i+1}$  where  $v_{\lambda+1} := 0$ .

We then say  $L(\lambda)$  is a highest weight module, since it is generated by a highest weight vector  $\lambda$ . Then  $W = \{1, s_\alpha\}$ , where  $s_\alpha$  is reflection through 0 by the identification  $\alpha = 2$ .

## 5 Chapter 1: Category $\mathcal{O}$ Basics

The category of  $U(\mathfrak{g})$ -modules is too big. Motivated by work of Verma in 60s, started by Bernstein-Gelfand-Gelfand in the 1970s. Used to solve the Kazhdan-Lusztig conjecture.

### 5.1 Axioms and Consequences

Definition:  $\mathcal{O}$  is the full subcategory of  $U(\mathfrak{g})$  modules consisting of  $M$  such that

1.  $M$  is finitely generated as a  $U(\mathfrak{g})$ -module.
2.  $M$  is  $\mathfrak{h}$ -semisimple, i.e.  $M$  is a weight module  $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ .
3.  $M$  is locally  $n$ -finite, i.e. the dimension of  $U(\mathfrak{n})v < \infty$  for all  $v \in M$ .



Figure 3: Image

Example: If  $\dim M < \infty$ , then  $M$  is  $\mathfrak{h}$ -semisimple, and axioms 1, 3 are obvious.

Lemma: Let  $M \in \mathcal{O}$ , then

4.  $\dim M_\mu < \infty$  for all  $\mu \in \mathfrak{h}^\vee$ .
5. There exist  $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^\vee$  such that  $\Pi(M) \subset \bigcup_{i=1}^r (\lambda_i - \mathbb{Z}^+ \Phi^+)$ .

Proof: By axiom 2, every  $v \in M$  is a finite sum of weight vectors in  $M$ . We can thus assume that our finite generating set consists of weight vectors. We can then reduce to the case where  $M$  is generated by a single weight vector  $v$ . So consider  $U(\mathfrak{g}) \cdot v$ . By the PBW theorem, there is a triangular decomposition  $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$ .

By axiom 3,  $U(\mathfrak{n}) \cdot v$  is finite dimensional, so there are finitely many weights of finite multiplicity in the image. Then  $U(\mathfrak{h})$  acts by scalar multiplication, and  $U(\mathfrak{n}^-)$  produces the “cones” that result in the tree structure:

A weight of the form  $\mu = \lambda_i - \sum n_i \alpha_i$  can arise from  $y_{n_1}^{n_1} \dots$ .

## 6 Friday January 17th

Let  $M$

1. be finitely generated,
2. semisimple  $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ ,
3. locally finite
4.  $\dim M_\mu < \infty$  for all  $\mu \in \mathfrak{h}^\vee$ ,
5. satisfy the forest condition for weights.



Figure 4: Image

**Theorem:**

- a.  $\mathcal{O}$  is Noetherian (ascending chain condition on submodules, i.e. no infinite filtrations by submodules)
- b.  $\mathcal{O}$  is closed under quotients, submodules, finite direct sums
- c.  $\mathcal{O}$  is abelian (similar to a category of modules)
- d. If  $M \in \mathcal{O}$ ,  $\dim L < \infty$ , then  $L \otimes M \in \mathcal{O}$  and the endofunctor  $M \mapsto L \otimes M$  is exact
- e. If  $M \in \mathcal{O}$  is locally  $Z(\mathfrak{g})$ -finite (recall: this is the center of  $U(\mathfrak{g})$ ), i.e.  $\dim \text{span} Z(\mathfrak{g}) \cdot v < \infty$  for all  $v \in M$ .
- f.  $M \in \mathcal{O}$  is finitely generated module. (?)

*Proofs of a and b:* See BA II, page 103.

*Proof of c:* Implied by (b), BA II Page 330.

*Proof of d:* Can check that  $L \otimes M$  satisfies 2 and 3 above. Need to check first condition. Take a basis  $\{v_i\}$  for  $L$  and  $\{w_j\}$  a finite set of generators for  $M$ . The claim is that  $B = \{v_i \otimes w_j\}$  generates  $L \otimes M$ . Let  $N$  be the submodule generated by  $B$ .

For any  $v \in V$ ,  $v \otimes w_j \in N$ . For arbitrary  $x \in \mathfrak{g}$ , compute  $x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + x \otimes (v \cdot w_j)$ . Since the LHS is in  $N$  and the first term on the RHS is in  $N$ , the entire RHS is in  $N$ . By iterating, we find that  $v \otimes (u \cdot w_j) \in N$  for all PBW monomials  $u$ . So  $L \otimes M \in \mathcal{O}$ .

*Proof of e:* Since  $v \in M$  is a sum of weight vectors, wlog we can assume  $v \in M_\lambda$  is a weight vector (where  $\lambda \in \mathfrak{h}^\vee$ ). For any central element  $z \in Z(\mathfrak{g})$ , we can compute  $h \cdot (z \cdot v) = z \cdot (h \cdot v) = z \cdot \lambda(h)v = \lambda(h)z \cdot v$ . Thus  $z \cdot v \in M_\lambda$ . By (4), we know that  $\dim M_\lambda < \infty$ , so  $\dim \text{span} Z(\mathfrak{g}) \cdot v < \infty$  as well.

*Proof of f:* By 5,  $M$  is generated by a finite dimensional  $U(\mathfrak{b})$  submodule  $N$ . Since we have a triangular decomposition  $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{b})$ , there is a basis of weight vectors for  $N$  that generates  $M$  as a  $U(\mathfrak{n}^-)$  module.

## 6.1 Highest Weight Modules

**Definition:** A maximal vector  $v^+ \in M \in \mathcal{O}$  is a nonzero vector such that  $\mathfrak{n} \cdot v^+ = 0$ .

Note: By 2 and 3, every nonzero  $M \in \mathcal{O}$  has a maximal vector.

**Definition:** A highest weight module  $M$  of highest weight  $\lambda$  is a module generated by a maximal vector of weight  $\lambda$ , i.e.  $M = U(\mathfrak{g})v^+ = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})v^+ = U(\mathfrak{n}^-)v^+$ .

**Theorem:** Let  $M = U(\mathfrak{n}^-)v^+$  be a highest weight module, where  $v^+ \in M_\lambda$ . Fix  $\Phi^+ = \{\beta_1, \dots, \beta_n\}$  with root vectors  $y_i \in \mathfrak{g}_{\beta_i}$ .

- a.  $M$  is the  $\mathbb{C}$ -span of PBW monomials  $\langle y_1^{t_1} \cdots y_m^{t_m} \rangle$  of weight  $\lambda - \sum t_i \beta_i$ . Thus  $M$  is a module.
- b. All weights  $\mu$  of  $M$  satisfy  $\mu \leq \lambda$
- c.  $\dim M_\mu < \infty$  for all  $\mu \in T(M)$ , and  $\dim M_\lambda = 1$ . In particular, property (3) holds and  $M \in \mathcal{O}$ .
- d. Every nonzero quotient of  $M$  is a highest-weight module of highest weight  $\lambda$ .
- e. Every submodule of  $M$  is a weight module, and any submodule generated by a maximal vector with  $\mu < \lambda$  is proper. If  $M$  is semisimple, then the set of maximal weight vectors equals  $\mathbb{C}^\times v^+$ .
- f.  $M$  has a unique maximal submodule  $N$  and a unique simple quotient  $L$ , thus  $M$  is indecomposable.



- g. All simple highest weight modules of highest weight  $\lambda$  are isomorphic. For such  $M$ ,  $\dim \text{End}(M) = 1$ . (Category  $\mathcal{O}$  version of Schur's Lemma, generalizes to infinite dimensional case)

*Proofs of a to e:* Either obvious or follows from previous results. First few imply  $M$  is in  $\mathcal{O}$ , and we know the latter hold for such modules.

*Proof of f:*  $N$  is a sum of submodules, so  $N = \sum M_i$ , proper submodules of  $M$ . So take  $L = M/N$ . To see indecomposability, there exists a better proof in section 1.3.

*Proof of g:* Let  $M_1 = U(\mathfrak{n}^-)v_1^+$  and  $M_2$  be define similarly, where the  $v_i \in (M_i)_\lambda$  have the same weight. Then  $M_0 := M_1 \oplus M_2$  implies that  $v^+ := (v_1^+, v_2^+)$  is a maximal vector for  $M_0$ . So  $N := U(\mathfrak{n}^-)v^+$  is a highest weight module of highest weight  $\lambda$ .

We have the following diagram:



and since e.g.  $N \rightarrow M_1$  is not the zero map, it is a surjection.

By (f),  $N$  is a unique simple quotient, so this forces  $M_1 \cong M_2$ . Since  $M$  is simple, any nonzero  $\mathfrak{g}$ -endomorphism  $\phi$  must be an isomorphism, and so we take  $v^+ \mapsto cv^+$  for some  $c \neq 0$ . Note that since  $\phi$  is also a  $\mathfrak{h}$ -morphism, we have  $\dim M_\lambda = 1$ .

Since  $v^+$  generates  $M$  and  $\phi(u \cdot v^+) = u\phi(v^+) = cu \cdot v^+$ , thus  $\phi$  is multiplication by a constant.

## 7 Wednesday January 22nd

Note: Try problems 1.1 and 1.3\*.

**Recall:** In category  $\mathcal{O}$ , we have finite dimensional, semisimple modules over  $\mathbb{C}$  with triangular decompositions.

If  $M$  is any  $U(\mathfrak{g})$  module than a  $v^+ \in M_\lambda$  a weight vector (so  $\lambda \in \mathfrak{h}^\vee$ ) is primitive iff  $\mathfrak{n} \cdot v^+ = 0$ . Note: it doesn't have to be of maximal weight.  $M$  is a highest weight module of highest weight  $\lambda$  iff it's generated over  $U(\mathfrak{g})$  as an associative algebra by a maximal vector  $v^+$  of weight  $\lambda$ . Then  $M = U(\mathfrak{g}) \cdot v^+$ .

See structure of highest weight modules, and irreducibility.

**Corollary:** If  $0 \neq M \in \mathcal{O}$ , then  $M$  has a finite filtration with quotients highest weight modules, i.e.  $M_0 \subset M_1 \subset \dots \subset M_n = M$  with  $M_i/M_{i-1}$  highest weight modules. Note that the quotients are not necessarily simple, so this isn't a composition series, although we'll show such a series exists later.

**Theorem:** Let  $V$  be the  $\mathfrak{n}$  submodule of  $M$  generated by a finite set of weight vectors which generate  $M$  as a  $U(\mathfrak{g})$  module. I.e. take the finite set of weight vectors and act on them by  $U(\mathfrak{n})$ .

Then  $\dim_{\mathbb{C}} V < \infty$  since  $M$  is locally  $\mathfrak{n}$ -finite.

*Proof:* Induction on  $n = \dim V$ . If  $n = 1$ ,  $M$  itself is a highest weight module.

Note that  $\mathfrak{n}$  increases weights.

For  $n > 1$ , choose a weight vector  $v_1 \in V$  of weight  $\lambda$  which is maximal among all weights of  $V$ . Set  $M_1 := U(\mathfrak{g})v_1$ ; this is a highest weight submodule of  $M$  of highest weight  $\lambda$ . ( $\mathfrak{n}$  has to kill  $v_1$ , otherwise it increases weight and  $v_1$  wouldn't be maximal.)

Let  $\overline{M} = M/M_1 \in \mathcal{O}$ , this is generated by the image of  $\overline{V}$  of  $V$  and thus  $\dim \overline{V} < \dim V$ . By the IH,  $\overline{M}$  has the desired filtration, say  $0 \subset \overline{M}_2 \subset \overline{M}_{n-1} \subset \overline{M}_n = \overline{M}$ . Let  $\pi : M \rightarrow M/M_1$ , then just take the preimages  $\pi^{-1}(\overline{M}_i)$  to be the filtration on  $M$ .

Note: by isomorphism theorems, the quotients in the series for  $M$  are isomorphic to the quotients for  $\overline{M}$ .

## 7.1 Verma and Simple Modules

Constructing *universal* highest weight modules using “algebraic induction”. Start with a nice subalgebra of  $\mathfrak{g}$  then “induce” via  $\otimes$  to a module for  $\mathfrak{g}$ .

Recall  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ , here  $\mathfrak{h} \oplus \mathfrak{n}$  is the Borel subalgebra  $\mathfrak{b}$ , and  $\mathfrak{n}$  corresponds to a fixed choice of positive roots in  $\Phi^+$  with basis  $\Delta$ . Then  $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ . Given any  $\lambda \in \mathfrak{h}^\vee$ , let  $\mathbb{C}_\lambda$  be the 1-dimensional  $\mathfrak{h}$ -module (i.e. 1-dimensional  $\mathbb{C}$ -vector space) on which  $\mathfrak{h}$  acts by  $\lambda$ .

Let  $\{1\}$  be the basis for  $\mathbb{C}$ , so  $h \cdot 1 = \lambda(h)1$  for all  $h \in \mathfrak{h}$ . Then there is a map  $\mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ , so make  $\mathbb{C}_\lambda$  a  $\mathfrak{b}$ -module via this map. This “inflate”  $\mathbb{C}_\lambda$  into a 1-dimensional  $\mathfrak{b}$ -module.

Note that  $\mathfrak{h}$  is solvable, and by Lie’s Theorem, every finite dimensional irreducible  $\mathfrak{b}$ -module is of the form  $\mathbb{C}_\lambda$  for some  $\lambda \in \mathfrak{h}^\vee$ .

**Definition:**  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda := \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$  is the *Verma module of highest weight  $\lambda$* . This process is called algebraic/tensor induction. This is a  $U(\mathfrak{g})$  module via left multiplication, i.e. acting on the first tensor factor.

Since  $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$ , we have  $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ , but at what level of structure?

- As a vector space (clear)
- As a  $\mathfrak{n}^-$ -module via left multiplication
- As a  $\mathfrak{h}^-$ -module via the  $\otimes$  action.

In particular,  $M(\lambda)$  is a *free*  $U(\mathfrak{n}^-)$ -module of rank 1.

Note: this always happens when tensoring with a vector space?

Consider  $v^+ := 1 \otimes 1 \in M(\lambda)$  (note that  $U(\mathfrak{n}^-)$  is not homogeneous, so not graded, but does have a filtration). Then  $v^+$  is nonzero, and freely generates  $M(\lambda)$  as a  $U(\mathfrak{n}^-)$ -module. Moreover  $\mathfrak{n} \cdot v^+ = 0$  since for  $x \in \mathfrak{g}_\beta$  for  $\beta \in \Phi^+$ , we have

$$\begin{aligned}
x(1 \otimes 1) &= x \otimes 1 \\
&= 1 \otimes x \cdot 1 \quad \text{since } x \in \mathfrak{b} \\
&= 1 \otimes 0 \implies x \in \mathfrak{n} \\
&= 0,
\end{aligned}$$

and for  $h \in \mathfrak{h}$ ,

$$\begin{aligned}
h(1 \otimes 1) &= h1 \otimes 1 \\
&= 1 \otimes h1 \\
&= 1 \otimes \lambda(h)1 \\
&= \lambda(h)v^+.
\end{aligned}$$

So  $M(\lambda)$  is a highest weight module of highest weight  $\lambda$ , and thus  $M(\lambda) \in \mathcal{O}$ .

**Observation:** Any weight  $\lambda \in \mathfrak{h}^\vee$  is the highest weight of some  $M \in \mathcal{O}$ . Let  $\Pi(M)$  denote the set of weights of a module, then  $\Pi(M(\lambda)) = \lambda - \mathbb{Z}^+ \Phi^+$ .

By PBW, we can obtain a basis for  $M(\lambda)$  as  $\{y_1^{t_1} \cdots y_m^{t_m} v^+ \mid t_i \in \mathbb{Z}^+\}$ . Taking a fixed ordering  $\{\beta_1, \dots, \beta_m\} = \Phi^+$ , then  $0 \neq y_i \in \mathfrak{g}_{-\beta_i}$ . Then every weight of this form is a weight of some  $M(\lambda)$ , and every weight of  $M(\lambda)$  is of this form:  $\lambda - \sum t_i \beta_i$ .

**Remark:** The functor  $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \cdot$  from the category of finite-dimensional  $\mathfrak{g}$ -semisimple  $\mathfrak{b}$ -modules to  $\mathcal{O}$  is an exact functor, since it is naturally isomorphic to  $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \cdot$  (which is clearly exact?)

Alternate construction of  $M(\lambda)$ : Let  $I$  be a left ideal of  $U(\mathfrak{g})$  which annihilates  $v^+$ , so  $I = \langle \mathfrak{n}, h - \lambda(h) \cdot 1 \mid h \in \mathfrak{h} \rangle$ . Since  $v^+$  generates  $M(\lambda)$  as a  $U(\mathfrak{g})$ -module, then (by a standard ring theory result)  $M(\lambda) = U(\mathfrak{g})/I$ , since  $I$  is the annihilator of  $M(\lambda)$ .

**Theorem (Universal property of  $M(\lambda)$ ):** Let  $M$  be any highest weight module of highest weight  $\lambda$  generated by  $v$ . Then  $I \cdot v = 0$ , so  $I$  is the annihilator of  $v$  and thus  $M$  is a quotient of  $M(\lambda)$ . Thus  $M(\lambda)$  is universal in the sense that every other highest weight module arises as a quotient of  $M(\lambda)$ .

By theorem 1.2,  $M(\lambda)$  has a unique maximal submodule  $N(\lambda)$  (nonstandard notation) and a unique simple quotient  $L(\lambda)$  (standard notation).

**Theorem:** Every simple module in  $\mathcal{O}$  is isomorphic to  $L(\lambda)$  for some  $\lambda \in \mathfrak{h}^\vee$  and is determined uniquely up to isomorphism by its highest weight. Moreover, there is an analog of Schur's lemma:  $\dim \text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$ , i.e. it's 1 iff  $\lambda = \mu$  and 0 otherwise.

Note: up to isomorphism, we've found all of the simple modules in  $\mathcal{O}$ , and most are finite-dimensional.

*Proof:* Next class.

## 8 Friday January 24th

A standard theorem about classifying simple modules in category  $\mathcal{O}$ :

**Theorem:** Every simple module in  $\mathcal{O}$  is isomorphic to  $L(\lambda)$  for some  $\lambda \in \mathfrak{h}^\vee$ , and is determined uniquely up to isomorphism by its highest weight. Moreover,  $\dim \text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}$ .

*Proof:* Let  $L \in \mathcal{O}$  be irreducible. As observed in 1.2,  $L$  has a maximal vector  $v^+$  of some weight  $\lambda$ .

Recall: can increase weights and reach a maximal in a finite number of steps.

Since  $L$  is irreducible,  $L$  is generated by that weight vector, i.e.  $LU(\mathfrak{g}) \cdot v^+$ , so  $L$  must be a highest weight module.

Standard argument: use triangular decomposition.

By the universal property,  $L$  is a quotient of  $M(\lambda)$ . But this means  $L \cong L(\lambda)$ , the unique irreducible quotient of  $M(\lambda)$ .

By Theorem 1.2 part g (see last Friday),  $\dim \text{End}_{\mathcal{O}}(L(\lambda)) = 1$  and  $\text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = 0$  since both entries are irreducible.

■

*Proof of Theorem 1.2 f:*

Statement: A highest weight module  $M$  is indecomposable, i.e. can't be written as a direct sum of two nontrivial proper submodules.

Suppose  $M = M_1 \oplus M_2$  where  $M$  is a highest weight module of highest weight  $\lambda$ . Category  $\mathcal{O}$  is closed under submodules, so  $M_i$  are weight modules and have weight-space decompositions. But  $M_\lambda$  is 1-dimensional (triangular decomposition, only  $\mathbb{C}$  acts), and thus  $M_\lambda \subset M_1$ . Since  $M_\lambda$  is a highest weight module, it generates the entire module, so  $M \subset M_1$ . The reverse holds as well, so  $M = M_1$  and this forces  $M_2 = 0$ .

■

### 8.1 1.4: Maximal Vectors in Verma Modules

1.5: Examples in the case  $\mathfrak{sl}(2)$ , over  $\mathbb{C}$  as usual.

First, some review from lie algebras.

Let  $\mathfrak{g}$  be any lie algebra, and take  $u, v \in U(\mathfrak{g})$ . Recall that we have the formula  $uv = [uv] + vu$ , where we use the definition  $[uv] = uv - vu$ .

Let  $x, y_1, y_2$  be in  $\mathfrak{g}$ , what is  $[x, y_1 y_2]$ ? Use the fact that  $\text{ad } x(y_1, y_2)$  acts as a derivation, and so  $[x, y_1 y_2] = [x y_1] y_2 + y_1 [x y_2]$ , which is a bracket entirely in the Lie algebra. This extends to an action on  $U(\mathfrak{g})$  by the product rule.

Recall that  $\mathfrak{sl}(2)$  is spanned by  $y = [0, 0; 1, 0], h = [1, 0; 0, -1], x = [0, 1; 0, 0]$ , where each basis vector spans  $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}$  respectively. Then  $[xy] = h, [hx] = 2x, [hy] = -2y$ , so  $E_{ij}E_{kl} = \delta_{jk}E_{il}$  (should be able to compute easily!).

Then  $\mathfrak{h} = \mathbb{C}$ , so  $\mathfrak{h}^\vee \cong \mathbb{C}$  where  $\lambda \mapsto \lambda(h)$ . So we identify  $\lambda$  with a complex number, this is kind of like a bundle of Verma modules over  $\mathbb{C}$ .

Consider  $M(1)$ , then  $\lambda = 1$  will denote  $\lambda(h) = 1$ . As in any Verma module,  $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda}$ . We can think of  $v^+$  as  $1 \otimes 1$ , with the action  $yv^+ = y1 \otimes 1$ . Note that  $y$  has weight  $-2$ .

Weight	Basis
1	$v^+$
-1	$yv^+$
-3	$y^2v^+$
-5	$y^3v^+$

Consider how  $x \curvearrowright y^2v^+$ . Note that  $x$  has weight  $+2$ . We have

$$\begin{aligned}
x \cdot y^2v^+ &= x \cdot y^2 \otimes 1_{\lambda} \\
&= ([xy^2] + y^2x) \otimes 1 \\
&= ([xy]y + y[xy]) \otimes 1 + y^2 \otimes x \cdot 1 \quad \text{moving } x \text{ across the tensor because ?} \\
&= ([xy]y + y[xy]) \otimes 1 + 0 \quad \text{since } x \text{ is maximal} \\
&= (hy + yh) \otimes 1 \\
&= hy \otimes 1 + y \otimes h \cdot 1 \\
&= hy \otimes 1 + \lambda(h)1 \\
&= hy \otimes 1 + 1 \\
&= ([xy] + yh) \otimes 1 + y \otimes 1 \\
&= -2y \otimes 1 + y \otimes 1 + y \otimes 1 \\
&= 0.
\end{aligned}$$

So  $y$  moves us downward through the table, and  $x$  moves upward, except when going from  $-3 \rightarrow -1$  in which case the result is zero!

Thus there exists a morphism  $\phi : M(-3) \rightarrow M(1)$ , with image  $U(\mathfrak{g})y^2v^+ = U(\mathfrak{n}^-)y^2v^+$ . So the image of  $\phi$  is everything spanned by the bases in the rows  $-3, -5, \dots$ , which is exactly  $M(-3)$ . So  $M(-3) \hookrightarrow M(1)$  as a submodule.

Motivation for next section: we want to find Verma modules which are themselves submodules of Verma modules.

It turns out that  $\text{im } \phi \cong N(1)$ . We should have  $M(1)/N(1) \cong L(1)$ . What is the simple module of weight 1 for  $\mathfrak{sl}(2)$ ? The weights of  $L(n)$  are  $n, n-2, n-4, \dots, -n$ , so the representations are parameterized by  $n \in \mathbb{Z}^+$ . These are the Verma modules for  $\mathfrak{sl}(2)$ . What happens is that  $y \curvearrowright -n \rightarrow -n-2$  gives a maximal vector, so the calculation above roughly goes through the same way. So we'll have a similar picture with  $L(n)$  at the top.

## 8.2 Back to 1.4

*Question 1:* What are the submodules of  $M(\lambda)$ ?

*Question 2:* What are the Verma submodules  $M(\mu) \subset M(\lambda)$ ? Equivalently, when do maximal vectors of weight  $\mu < \lambda$  (the interesting case) lie in  $M(\lambda)$ ?

*Question 3:* As a special case, when do maximal vectors of weight  $\lambda - k\alpha$  for  $\alpha \in \Delta$  lie in  $M(\lambda)$  for  $k \in \mathbb{Z}^+$ ?

Fix a Chevalley basis for  $\mathfrak{g}$  (see section 0.1)  $h_1, \dots, h_\ell \in \mathfrak{h}$  and  $x_\alpha \in \mathfrak{g}_\alpha$  and  $y_\alpha \in \mathfrak{g}_{-\alpha}$  for  $\alpha \in \Phi^+$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$  and let  $x_i = x_{\alpha_i}, y_i = y_{\alpha_i}$  be chosen such that  $[x_i y_i] = h_i$ .

**Lemma:** For  $k \geq 0$  and  $1 \leq i, j \leq \ell$ , then

- a.  $[x_j, y_i^{k+1}] = 0$  if  $j \neq i$
- b.  $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$ .
- c.  $[x_i, y_i^{k+1}] = -(k+1)y_i(k \cdot 1 - h_i)$ .

*Proof (sketch):*

Both easy to prove by induction since  $[x_j, y_i] \rightarrow \alpha_j - \alpha_i \notin \Phi$  is a difference of simple roots.

For  $k = 0$ , all identities are easy. For  $k > 0$ , an inductive formula that uses the derivation property, which we'll do next class.

## 9 Monday January 27th

### 9.1 Section 1.4

Fix  $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ ,  $x_i \in \mathfrak{g}_{\alpha_i}$  and  $y_i \in \mathfrak{g}_{-\alpha_i}$  with  $h_i = [x_i y_i]$ .

**Lemma:** For  $k \geq 0$  and  $1 \leq i, j \leq \ell$ ,

- a.  $[x_j y_i^{k+1}] = 0$  if  $j \neq i$
- b.  $[h_j y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$
- c.  $[x_i y_i^{k+1}] = (k+1)y_i^k(k \cdot 1 - h_i)$ .

*Sketch of proof for (c):*

By induction, where  $k = 0$  is clear.

$$\begin{aligned}
 [x + i y_i^{k+1}] &= [x_i y_i] y_i^k + y_i [x_i y_i^k] \\
 &= h_i y_i^k + y_i (-k y_i^{k-1} ((k-1)1 - h_i)) \quad \text{by I.H.} \\
 &= (k+1) y_i^k h_i - (k^2 - k + 2k) y_i^k \\
 &= -(k+1) y_i^k (k \cdot 1 - h_i)
 \end{aligned}$$

■

**Proposition:** Suppose  $\lambda \in \mathfrak{h}^\vee$ ,  $\alpha \in \Delta$ , and  $n := (\lambda, \alpha^\vee) \in \mathbb{Z}^+$ . Then in  $M(\lambda)$ ,  $y_\alpha^{n+1} v^+$  is a maximal weight vector of weight  $\mu := \lambda - (n+1)\alpha < \lambda$ .

Note this is free as an  $U(\mathfrak{n}^-)$ -module, so  $v^+ \neq 0$ . Note that  $n = \lambda(h_\alpha)$ .

By the universal property, there is a nonzero homomorphism  $M(\mu) \rightarrow M(\lambda)$  with image contained in  $N(\lambda)$ , the unique maximal proper submodules of  $M(\lambda)$ .

*Proof:* Say  $\alpha = \alpha_i$ . Fix  $j \neq i$ .

$$\begin{aligned} x_i y_\alpha^{n+1} \otimes 1 &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes x_j \cdot 1 \\ &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes 0 \quad \text{by a} \\ &= 0. \end{aligned}$$

$$\begin{aligned} x_i y_i^{n+1} \otimes 1 &= [x_i y_i^{n+1} \otimes 1] \\ &= -(n+1) y_i^n (n \cdot 1 - h_i) \otimes 1 \\ &= -(n+1)(n - \lambda(h_i)) 1 \otimes 1 \\ &:= -(n+1)(\lambda(h_i) - \lambda(h_i)) 1 \otimes 1 \\ &= 0. \end{aligned}$$

Since  $g_{\alpha_j}$  generate  $\mathfrak{n}$  as a Lie algebra, since  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ . This shows that  $\mathfrak{n} \cdot y_i^{n+1} v^+ = 0$ , and the weight of  $y_i^{n+1} v^+$  is  $\lambda - (n+1)\alpha_i$ . So  $y_i^{n+1}$  is a maximal vector of weight  $\mu$ . The universal property implies there is a nonzero map  $M(\mu) \rightarrow M(\lambda)$  sending highest weight vectors to highest weight vectors and preserving weights. The image is proper since all weights of  $M_\mu$  are less than or equal to  $\mu < \lambda$ . ■

Consider  $\mathfrak{sl}(2)$ , then  $M(1) \supset M(-3)$ . Note that reflecting through 0 doesn't send 1 to -3, but shifting the origin to -1 and reflecting about that with  $s_\alpha \cdot$  fixes this problem. Note that  $L(1)$  is the quotient.

For  $\lambda \in \mathfrak{h}^\vee$  and  $\alpha \in \Delta$ , we can compute  $s_\alpha \cdot \lambda := s_\alpha(\lambda + \rho) - \rho$  where  $\rho = \sum_{j=1}^{\ell} e_i$ . Then  $(w_j, \alpha_i^\vee) = \delta_{ij}$  and  $(\rho, \alpha_i^\vee) = 1$ .

$$\begin{aligned} s_\alpha \cdot \lambda &= s_\alpha(\lambda + \rho) - \rho \\ &= (\lambda + \rho) - (\lambda + \rho, \alpha^\vee) \alpha - \rho \\ &= \lambda + \rho - ((\lambda < \alpha^\vee) + 1) \alpha - \rho \\ &= \lambda - (n+1) \alpha \\ &= \mu. \end{aligned}$$

So this gives a well-defined, nonzero map  $M(s_\alpha \cdot \lambda) \rightarrow M(\lambda)$  for  $s_\alpha \cdot \lambda < \lambda$ .

**Corollary:** Let  $\lambda, \alpha, n$  be as in the above proposition. Let  $\bar{v}^+$  now be a maximal vector of weight  $\lambda$  in  $L(\lambda)$ . Then  $y_\alpha^{n+1} \bar{v}^+ = 0$ .

*Proof:* If not, then this would be a maximal vector, since it's the image of the vector  $y_i^{n+1} v^+ \in M(\lambda)$  under the map  $M(\lambda) \rightarrow L(\lambda)$  of weight  $\mu < \lambda$ . Then it would generate a proper submodules of  $L(\lambda)$ , but this is a contradiction since  $L(\lambda)$  is irreducible.



Figure 5: Image

## 9.2 Section 1.5

Example:  $\mathfrak{sl}(2)$ . What do Verma modules  $M(\lambda)$  and their simple quotients  $L(\lambda)$  look like?

Fix a Chevalley basis  $\{y, h, x\}$  and let  $\lambda \in \mathfrak{h}^\vee \cong \mathbb{C}$ .

**Fact 1:** For  $v^+ = 1 \otimes 1_\lambda$ , we have  $M(\lambda) = U(\mathfrak{n}^-)v^+ = \mathbb{C} \langle y^i v^+ \mid i \in \mathbb{Z}^+ \rangle$  is a basis for  $M(\lambda)$  with weights  $\lambda - 2i$  where  $\alpha$  corresponds to 2. So the weights of  $M(\lambda)$  are  $\lambda, \lambda - 2, \lambda - 4, \dots$  each with multiplicity 1.

Letting  $v_i = \frac{1}{i!} y^i v^+$  for  $i \in \mathbb{Z}^+$ ; this is a basis for  $M(\lambda)$ . Using the lemma, we have

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i \\ y \cdot v_i &= (i + 1)v_{i+1} \\ x \cdot v_i &= (\lambda - i + 1)v_{i-1}. \end{aligned}$$

Note that these are the same for *finite-dimensional*  $\mathfrak{sl}(2)$ -modules, see section 0.9.

**Fact 2:** We know from the proposition that if  $\lambda \in \mathbb{Z}^+$ , i.e.  $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$ , then  $M(\lambda)$  has a maximal vector of weight  $\lambda - (n + 1)\alpha = \lambda - (\lambda + 1)2 = -\lambda - 2 = s_\alpha \cdot \lambda$ .

*Exercise:* Check that this maximal vector generates the maximal proper submodule  $N(\lambda) = M(-\lambda - 2)$ .



So the quotient  $L(\lambda) = M(\lambda)/N(\lambda) = M(\lambda)/M(-\lambda - 2)$  has weights  $\lambda, \lambda - 2, \dots, -\lambda + 2, -\lambda$ . So when  $\lambda \in \mathbb{Z}^+$ ,  $L(\lambda)$  is the familiar simple  $\mathfrak{sl}(2)$ -module of highest weight  $\lambda$ .

**Fact 3:** When  $\lambda \notin \mathbb{Z}^+$ ,

- $N(\lambda) = \{0\}$ ,
- $M(\lambda) = L(\lambda)$ ,
- $M(\lambda)$  is irreducible
- $L(\lambda)$  is infinite dimensional.

*Proof:* Argue by contradiction. If not,  $M(\lambda) \supset M \neq 0$  is a proper submodule. So  $M \in \mathcal{O}$ , and thus  $M$  has a maximal weight vector  $w^+$ , and by the restriction of weights for modules in  $\mathcal{O}$ , we know  $w^+$  has height  $\lambda - 2m$  for some  $m \in \mathbb{Z}^+$ . Then  $w^+ = cv_i$  where  $0 \neq c \in \mathbb{C}$ , and taking  $v_{-1} := 0$  and  $x \cdot v_i = (\lambda - i + 1)v_{i-1}$  for  $i \geq 1$ , so  $\lambda = i - 1 \implies \lambda \in \mathbb{Z}^+$ .

## 10 Friday January 31st

(3) A useful formula:  $L(\lambda)^\vee \cong L(-w_0)$

*Proof:*  $L(\lambda)^\vee$  is a finite dimensional module, and  $(x \cdot f)(v) = -f(x \cdot v)$ , so  $L(\lambda)^\vee \cong L(\nu)$  for some  $\nu \in \Lambda^+$ . The weights of  $L(\lambda)^\vee$  are the negatives of the weight of  $L(\lambda)$ . Thus the lowest weight of  $L(\lambda)$  is  $w_0\lambda$ , since  $w_0$  reverses the partial order on  $\mathfrak{h}^\vee$ , i.e.  $w_0\Phi^+ = \Phi^-$ .

Then  $\mu \in \Pi(L(\mu)) \implies w_0\mu \in \Pi(L(\lambda)) \implies w_0\mu \leq \lambda$ . This shows that the lowest weight of  $L(\lambda)$  is  $w_0\lambda$ , thus the highest weight  $L(\lambda)^\vee$  is  $-w_0\lambda$  by reversing this inequality.

The inner product is  $W$  invariant and is its own inverse, so we can move it to the other side.

■

### 10.1 1.7: Action of $Z(\mathfrak{g})$

Next big goal: Every module in  $\mathcal{O}$  has a *finite* composition series (Jordan-Holder series, the quotients are simple). Leads to Kazhdan-Lusztig conjectures from 1979/1980, which were solved, but are still open in characteristic  $p$  case.

The technique we'll use the Harish-Chandra homomorphism, which identifies  $\mathcal{Z}(\mathfrak{g})$  explicitly.

It's commutative, subalgebra of a Noetherian algebra, no zero divisors – could be a quotient, but then it'd have zero divisors, so this seems to force it to be a polynomial algebra on some unknown. Also note that  $\mathcal{Z}(\mathfrak{g}) := Z(U(\mathfrak{g}))$ .

Recall:  $\mathcal{Z}(\mathfrak{g})$  acts locally finitely on any  $M \in \mathcal{O}$  – this is by theorem 1.1e, i.e.  $v \in M_\mu$  and  $z \in \mathcal{Z}(\mathfrak{g})$  implies that  $zv \in M_\mu$ . (The calculation just follows by computing the weight and commuting things through.)

Let  $\lambda \in \mathfrak{h}^\vee$  and  $M = U(\mathfrak{g})v^+$  a highest weight module of highest weight  $\lambda$ . Then for  $z \in \mathcal{Z}(\mathfrak{g})$ ,  $z \cdot v^+ \in M_\lambda$  which is 1-dimensional. Thus  $z$  acts by scalar multiplication here, and  $z \cdot v^+ = \chi_\lambda(z)v^+$ . Now if  $u \in U(\mathfrak{u}^-)$ , we have  $z \cdot (u \cdot v^+) = u \cdot (z \cdot v^+) = u(\chi_\lambda(z)v^+) = \chi_\lambda(z)u \cdot v^+$ . Thus  $z$  acts on *all* of  $M$  by the scalar  $\chi_\lambda(z)$ .

*Exercise:* Show that  $\chi_\lambda$  is a nonzero additive and multiplicative function, so  $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$  is linear and thus a morphism of algebras. Conclude that  $\ker \chi_\lambda$  is a maximal ideal of  $\mathcal{Z}(\mathfrak{g})$ .

Note: this is called the infinitesimal character.

Note that  $\chi_\lambda$  doesn't depend on which highest weight module  $M_\lambda$  was chosen, since they're all quotients of  $M(\lambda)$ . In fact, every submodule and subquotient of  $M(\lambda)$  is the same infinitesimal character.

**Definition:**  $\chi_\lambda$  is called the *central (or infinitesimal) character*, and  $\widehat{\mathcal{Z}}(\mathfrak{g})$  denotes the set of all central characters. More generally, any algebra morphism  $\chi : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$  is referred to as a central character. Central characters are in one-to-one correspondence with maximal ideals of  $\mathcal{Z}(\mathfrak{g})$ , where  $\chi \iff \ker \chi$  and  $\mathbb{C}[x_1, \dots, x_n] \iff \langle x_1 - a_1, \dots, x_n - a_n \rangle$  where  $(\mathbf{a}_i) \in \mathbb{C}^n$ .

Next goal: Describe  $\chi_\lambda(z)$  more explicitly.

Using PBW, we can write  $z \in \mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$ . Some observations:

1. Any PBW monomial in  $z$  ending with a factor in  $\mathfrak{n}$  will kill  $v^+$ , and hence can not contribute to  $\chi_\lambda(z)$ .
2. Any PBW monomial in  $z$  beginning with a factor in  $\mathfrak{n}^-$  will send  $v^+$  to a lower weight space, so it also can't contribute.

So we only need to see what happens in the  $\mathfrak{h}$  part. A relevant decomposition here is

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^+).$$

**Exercise:** Why is this sum direct?

Let  $\text{pr} : U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$  be the projection onto the first factor. Then  $\chi_\lambda(z) = \lambda(\text{pr}z)$  for all  $z \in \mathcal{Z}(\mathfrak{g})$ . Then if  $\text{pr}(z) = h_1^{m_1} \cdots h_\ell^{m_\ell}$ , we can extend the action on  $\mathfrak{h}$  to all polynomials in elements of  $\mathfrak{h}$  (which is in fact evaluation on these monomials), and thus  $\chi_\lambda(z) = \lambda(h_1)^{m_1} \cdots \lambda(h_\ell)^{m_\ell}$ .

Note that for  $\lambda \in \mathfrak{h}^\vee$ , we've extended this to the "evaluation map"  $\lambda : U(\mathfrak{g}) \cong S(\mathfrak{h})$ , the symmetric algebra on  $\mathfrak{h}$ . Why is this the correct identification? The RHS is  $T(\mathfrak{h}) / \langle x \otimes y - y \otimes x - [xy] \rangle$ , but notice that the bracket vanishes since  $\mathfrak{h}$  is abelian, and this is the exact definition of the symmetric algebra.

Thus  $\chi_\lambda = \lambda \circ \text{pr}$ .

Observation:

$$\begin{aligned} \lambda(\text{pr}(z_1 z_2)) &= \chi_\lambda(z_1 z_2) \\ &= \chi_\lambda(z_1) \chi_\lambda(z_2) \\ &= \dots \\ &= \lambda(\text{pr}(z_1) \text{pr}(z_2)). \end{aligned}$$

**Exercise:** Show  $\bigcap_{\lambda \in \mathfrak{h}^\vee} \ker \lambda = \{0\}$ .

**Definition:** Let  $\xi = \text{pr}|_{\mathcal{Z}(\mathfrak{g})} : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h})$ .

**Lemma/Definition:**  $\xi$  is an algebra morphism, and is referred to as the *Harish-Chandra homomorphism*.

See page 23 for interpretation of  $\xi$  without reference to representations.

Questions:

1. Is  $\xi$  injective?
2. What is  $\text{im } \xi \subset U(\mathfrak{h})$ ?

When does  $\chi_\lambda = \chi_\mu$ ? Proved last time: we introduced the  $\cdot$  action and proved that  $M(s_\alpha \cdot \lambda) \subset M(\lambda)$  where  $\alpha \in \Delta$ . It'll turn out that the statement holds for all  $\lambda \in W$ .

Wednesday: Section 1.8.

## 11 Wednesday February 5th

Recall the Harish-Chandra morphism  $\xi$ :

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xhookrightarrow{\quad} & U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}) \\ & \searrow \xi & \downarrow \text{pr} \\ & & U(\mathfrak{h}) \end{array}$$

If  $M$  is a highest weight module of highest weight  $\lambda$  then  $z \in \mathcal{Z}(\mathfrak{g})$  acts on  $M$  by scalar multiplication. Note that if we have  $\chi_\lambda(z)$  where  $z \cdot v = \chi_\lambda(z)v$  for all  $v \in M$ , we can identify  $\lambda(\text{pr}(z)) = \lambda(\xi(z))$ .

### 11.1 Central Characters and Linkage

The  $\chi_\lambda$  are not all distinct – for example, if  $M(\mu) \subset M(\lambda)$ , then  $\chi_\mu = \chi_\lambda$ . More generally, if  $L(\mu)$  is a subquotient of  $M(\lambda)$  then  $\chi_\mu = \chi_\lambda$ . So when do we have equality  $\chi_\mu = \chi_\lambda$ ?

Given  $\mathfrak{g} \supset \mathfrak{h}$  with  $\Phi \supset \Phi^+ \supset \Delta$ , then define  $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathfrak{h}^\vee$ . Note that  $\alpha \in \Delta \implies s_\alpha \rho = \rho - \alpha$ .

**Definition:** The *dot action* of  $W$  on  $\mathfrak{h}^\vee$  is given by  $w \cdot \lambda = w(\lambda + \rho) - \rho$ , which implies  $(\rho, \alpha^\vee) = 1$  for all  $\alpha \in \Delta$ . Then  $\rho = \sum_{i=1}^{\ell} w_i$ .

**Exercise:** Check that this gives a well-defined group action.

**Definition:**  $\mu$  is *linked* to  $\lambda$  iff  $\mu = w \cdot \lambda$  for some  $w \in W$ . Note that this is an equivalence relation, with equivalence classes/orbits where the orbit of  $\lambda$  is  $\{w \cdot \lambda \mid w \in W\}$  is called the *linkage class* of  $\lambda$ .

Note that this is a finite subset, since  $W$  is finite.

Note that orbit-stabilizer applies here, so bigger stabilizers yield smaller orbits and vice-versa.

*Example:*  $w \cdot (-\rho) = w(-\rho + \rho) - \rho = -\rho$ , so  $-\rho$  is in its own linkage class.

**Definition:**  $\lambda \in \mathfrak{h}^\vee$  is *dot-regular* iff  $|W \cdot \lambda| = |W|$ , or equivalently if  $(\lambda + \rho, \beta^\vee) \neq 0$  for all  $\beta \in \Phi$ .

To think about: does this hold if  $\Phi$  is replaced by  $\Delta$ ?

We also say  $\lambda$  is *dot-singular* if  $\lambda$  is not dot-regular, or equivalently  $\text{Stab}_W \cdot \lambda \neq \{1\}$ .

I.e. lying on root hyperplanes.

**Exercise:** If  $0 \in \mathfrak{h}^\vee$  is regular, then  $-\rho$  is singular.



**Proposition:** If  $\lambda \in \Lambda$  and  $\mu \in W \cdot \lambda$ , then  $\chi_\mu = \chi_\lambda$ .

*Proof:* Start with  $\alpha \in \Delta$  and consider  $\mu = s_\alpha \cdot \lambda$ . Since  $\lambda \in \Lambda$ , we have  $n := (\lambda, \alpha^\vee) \in \mathbb{Z}$  by definition. There are three cases:

1.  $n \in \mathbb{Z}^+$ , then  $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ . By Proposition 1.4, we have  $\chi_\mu = \chi_\lambda$ .
2. For  $n = -1$ ,  $\mu = s_\alpha \cdot \lambda = \lambda + \rho - (\lambda + \rho, \alpha^\vee)\alpha - \rho = \lambda + n + 1 = \lambda + 0$ . So  $\mu = \lambda$  and thus  $M_\mu = M_\lambda$ .
3. For  $n \leq -2$ ,

$$\begin{aligned}
(\mu, \alpha^\vee) &= (s_\alpha \cdot \lambda, \alpha^\vee) \\
&= (\lambda i(n+1)\alpha, \alpha^\vee) \\
&= n - 2(n+1) \\
&= -n - 2 \\
&\geq 0,
\end{aligned}$$

so  $\chi_\mu = \chi_{s_\alpha \cdot \mu} = \chi_{s_\alpha \cdot (s_\alpha \cdot \lambda)} = \chi_\lambda$ . Since  $W$  is generated by simple reflections and the linkage property is transitive, the result follows by induction on  $\ell(w)$ . ■

**Exercise 1.8 (do but don't turn in):** See book, show that certain properties of the dot action hold (namely nonlinearity).

## 11.2 1.9: Extending the Harish-Chandra Morphism

We want to extend the previous proposition from  $\lambda \in \Lambda$  to  $\lambda \in \mathfrak{h}^\vee$ . We'll use a density argument from affine algebraic geometry, and switch to the Zariski topology on  $\mathfrak{h}^\vee \subset \mathbb{C}^n$ .

Fix a basis  $\Delta = \{a_1, \dots, a_\ell\}$  and use the Killing form to identify these with a basis for  $\mathfrak{h} = \{h_1, \dots, h_\ell\}$ . Similarly, take  $\{w_1, \dots, w_\ell\}$  as a basis for  $\mathfrak{h}^\vee$ , and we'll use the identification

$$\begin{aligned}
\mathfrak{h}^\vee &\iff \mathbb{A}^\ell \\
\lambda &\iff (\lambda(h_1), \dots, \lambda(h_\ell)).
\end{aligned}$$

We identify  $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[h_1, \dots, h_\ell]$  with  $P(\mathfrak{h}^\vee)$  which are polynomial functions on  $\mathfrak{h}^\vee$ . Fix  $\lambda \in \mathfrak{h}^\vee$ , extended  $\lambda$  to be a multiplicative function on polynomials. For  $f \in \mathbb{C}[h_1, \dots, h_\ell]$ , we defined  $\lambda(f)$ . Under the identification, we send this to  $\tilde{f}$  where  $\tilde{f}(\lambda) = \lambda(f)$ .

Note: we'll identify  $f$  and  $\tilde{f}$  notationally going forward and drop the tilde everywhere.

Then  $W$  acts on  $P(\mathfrak{h}^\vee)$  by the dot action:  $(w \cdot \tilde{f})(\lambda) = \tilde{f}(w^{-1} \cdot \lambda)$ .

*Exercise:* Check that this is a well-defined action.

Under this identification, we have

$$\begin{aligned}
\mathfrak{h}^\vee &\iff \mathbb{A}^\ell \\
\Lambda &\iff \mathbb{Z}^\ell.
\end{aligned}$$

Note that  $\Lambda$  is discrete in the analytic topology, but is *dense* in the Zariski topology.

**Proposition:** A polynomial  $f$  on  $\mathbb{A}^\ell$  vanishing on  $\mathbb{Z}^\ell$  must be identically zero.

*Proof:* For  $\ell = 1$ : A nonzero polynomial in one variable has only finitely many zeros, but if  $f$  vanishes on  $\mathbb{Z}$  it has infinitely many zeros.

For  $\ell > 1$ : View  $f \in \mathbb{C}[h_1, \dots, h_{\ell-1}][h_\ell]$ . Substituting any fixed integers for the  $h_i$  for  $i \leq \ell - 1$  yields a polynomial in one variable which vanishes on  $\mathbb{Z}$ . By the first case,  $f \equiv 0$ , so the coefficients must all be zero and the coefficient polynomials in  $\mathbb{C}[h_1, \dots, h_{\ell-1}]$  vanish on  $\mathbb{Z}^{\ell-1}$ . By induction, these coefficient polynomials are identically zero.

■

**Corollary:** The only Zariski-closed subset of  $\mathbb{A}^\ell$  containing  $\mathbb{Z}^\ell$  is  $\mathbb{A}^\ell$  itself, so the Zariski closure  $\overline{\mathbb{Z}^\ell} = \mathbb{A}^\ell$  and  $\mathbb{Z}^\ell$  is dense in  $\mathbb{A}^\ell$ .

## 12 Friday February 7th

So far, we have  $\chi_\lambda = \chi_{w \cdot \lambda}$  if  $\lambda \in \Lambda$  and  $w \in W$ . We have  $\mathfrak{h}^\vee \supset \Lambda$  which is topologically equivalent to  $\mathbb{A}^\ell \supset \mathbb{Z}^\ell$ , where  $\mathbb{Z}^\ell$  is dense in the Zariski topology.

For  $z \in \mathcal{Z}(\mathfrak{g})$ , we have  $\chi_\lambda(z) = \chi_{w \cdot \lambda}(z)$  and so  $\lambda(\xi(z)) = (w \cdot \lambda)(\xi(z))$  where  $\xi : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h}) \cong P(\mathfrak{h}^\vee)$  where we send  $\lambda(f)$  to  $f(\lambda)$ .

Then  $\xi(z)(\lambda) = \xi(z)(w \cdot \lambda)$  for all  $\lambda \in \Lambda$ , and so  $\xi(z) = w^{-1}\xi(z)$  on  $\Lambda$ . But both sides here are polynomials and thus continuous, and  $\Lambda \subset \mathfrak{h}^\vee$  is dense, so  $\xi(z) = w^{-1}\xi(z)$  on all of  $\mathfrak{h}^\vee$ . I.e.,  $\chi_\lambda = \chi_{w \cdot \lambda}$  for all  $\lambda \in \mathfrak{h}^\vee$ .

This in fact shows that the image of  $\mathcal{Z}(\mathfrak{g})$  under  $\xi$  consists of  $W$ -invariant polynomials.

It's customary to state this in terms of the natural action of  $W$  on polynomials without the row-shift. We do this by letting  $\tau_\rho : S(\mathfrak{h}) \xrightarrow{\cong} S(\mathfrak{h})$  be the algebra automorphism induced by  $f(\lambda) \mapsto f(\lambda - \rho)$ . This is clearly invertible by  $f(\lambda) \mapsto f(\lambda + \rho)$ . We then define  $\psi : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\xi} S(\mathfrak{h}) \xrightarrow{\tau_\rho} S(\mathfrak{h})$  as this composition; this is referred to as the Harish-Chandra (HC) homomorphism.

*Exercise:* Show  $\chi_\lambda(z) = (\lambda + \rho)(\psi(z))$  and  $\chi_{w \cdot \lambda}(w(\lambda + \rho))(\psi(z))$ , where  $w(\cdot)$  is the usual  $w$ -action. Replacing  $\lambda$  by  $\lambda + \rho$  and  $w$  by  $w^{-1}$ , we get

$$w\psi(z) = \psi(z)$$

for all  $z \in \mathcal{Z}(\mathfrak{g})$  and all  $w \in W$  where  $(w\psi(z))(\lambda) = \psi(z)(w^{-1}\lambda)$ .

We've proved that

**Theorem 1.9:**

- a. If  $\lambda, \mu \in \mathfrak{h}^\vee$  that are linked, then  $\chi_\lambda = \chi_\mu$ .
- b. The image of the twisted HC homomorphism  $\psi : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h})$  lies in the subalgebra  $S(\mathfrak{h})^W$ .

*Example:* Let  $\mathfrak{g} = \mathfrak{sl}_2$ . Recall from finite-dimensional representations there is a canonical element  $c \in \mathcal{Z}(\mathfrak{g})$  called the Casimir element. For  $\mathcal{O}$ , we need information about the full center  $\mathcal{Z}(\mathfrak{g})$  (hence introducing infinitesimal characters).

Expressing  $c$  in the PBW basis yields  $c = h^2 + 2h + 4yx$ , where  $h^2 + 2h \in U(\mathfrak{h})$  and  $4yx \in \mathfrak{n}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$ .

Enveloping algebra convention:  $xs, hs, ys$

Then  $\xi(c) = \mathfrak{p}(c) = h^2 + 2h$ , and under the identification  $\mathfrak{h}^\vee \iff \mathbb{C}$  where  $\lambda \iff \lambda(h)$ , we can identify  $\rho \iff \rho(h) = 1$ . The row shift is given by  $\psi(c) = (h-1)^2 + 2(h-1) = h^2 - 1$ . This is  $W$ -invariant, since  $s_{\alpha_h} = -h$ . But  $W = \langle s_\alpha, 1 \rangle$ , so  $s_\alpha$  generates  $W$ .

We also have  $\chi_\lambda(c) = (\lambda + \rho)(\psi(c)) = (\lambda + 1)^2 - 1$ . Then

$$\chi_\lambda(c) = \chi_\mu(c) \iff (\lambda + 1)^2 - 1 = (\mu + 1)^2 \iff \mu = \lambda \text{ or } \mu = -\lambda - 2$$

But  $\lambda = 1 \cdot \lambda$  and  $-\lambda - 2 = s_\alpha \cdot \lambda$ , so  $\mathcal{Z}(\mathfrak{g}) = \langle c \rangle := \mathbb{C}[c]$  as an algebra. So these characters are equal iff  $\mu = w \cdot \lambda$  for  $w \in W$ .

## 13 Section 1.10: Harish-Chandra's Theorem

Goal: prove the converse of the previous theorem.

**Theorem (Harish-Chandra):** Let  $\psi : \mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})$  be the twisted HC homomorphism. Then

- a.  $\psi$  is an *isomorphism* of  $\mathcal{Z}(\mathfrak{g})$  onto  $S(\mathfrak{h})^W$ .
- b. For all  $\lambda, \mu \in \mathfrak{h}^\vee$ ,  $\chi_\lambda = \chi_\mu$  iff  $\mu = w \cdot \lambda$  for some  $w \in W$ .
- c. Every central character  $\chi : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$  is a  $\chi_\lambda$ .

*Sketch of proof:*

*Part (a):* Relies heavily on the *Chevalley Restriction Theorem* (which we won't prove here).

Initially we have a restriction map on polynomial functions  $\theta : P(\mathfrak{g}) \rightarrow P(\mathfrak{h})$ . We identified  $P(\mathfrak{g}) = S(\mathfrak{g}^\vee)$ , the formal polynomials on  $\mathfrak{g}^\vee$ . However, for  $\mathfrak{g}$  semisimple, we can identify  $S(\mathfrak{g}^\vee) \cong S(\mathfrak{g})$  via the Killing form.

By the Chinese Remainder Theorem,  $\theta : S(\mathfrak{g})^G \rightarrow S(\mathfrak{h})^W$  is an isomorphism, where the subgroup  $G \leq \text{Aut}(\mathfrak{g})$  is the *adjoint group* generated by  $\left\{ \exp \text{ad } x \mid x \text{ is nilpotent} \right\}$ .

It turns out that  $S(\mathfrak{g})^G$  is very close to  $\mathcal{Z}(\mathfrak{g})$  – it is the associated graded of a natural filtration on  $\mathcal{Z}(\mathfrak{g})$ . This is enough to show that  $\psi$  is a bijection. ■

*Part (b):* We'll prove the contrapositive of the converse.

Suppose  $W \cdot \lambda \cap W \cdot \mu = \emptyset$  and both are in  $\mathfrak{h}^\vee$ . Since these are finite sets, Lagrange interpolation yields a polynomial that is 1 on  $W \cdot \lambda$  and 0 on  $W \cdot \mu$ . Let  $g = \frac{1}{|W|} \sum_{w \in W} w \cdot f$ .

Note: definitely the dot action here, may be a typo in the book.

Then  $g$  is a  $W \cdot$  invariant polynomial with the same properties. By part (a), we can get rid of the row shift to obtain an isomorphism  $\xi : \mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})^{(W \cdot)}$ , the  $W \cdot$  invariant polynomials. Choose  $z \in \mathcal{Z}(\mathfrak{g})$  such that  $\xi(z) = g$ , then  $\chi_\lambda(z) = \lambda(\xi(z)) = \lambda(g) = g(\lambda) = 1$ . So  $\chi_\mu(z) = 0$  similarly, and  $\chi_\lambda = \chi_\mu$ . ■

*Part (c):* This follows from some commutative algebra, we won't say much here. Look at maximal ideals in  $\mathbb{C}[x, y, \dots]$  which correspond to evaluating on points in  $\mathbb{C}^\ell$ .

■

*Remark:* Chevalley actually proved that  $S(\mathfrak{h})^W \cong \mathbb{C}(p_1, \dots, p_\ell)$  where the  $p_i$  are homogeneous polynomials of degrees  $d_1 \leq \dots \leq d_\ell$ . These numbers satisfy some remarkable properties:  $\prod d_i = |W|$  and  $d_1 = 2$  (these are called the *degrees of  $W$* )

## 14 Section 1.11

Theorem: Category  $\mathcal{O}$  is *artinian*, i.e. every  $M \in \mathcal{O}$  is Artinian (DCC) and  $\dim \text{hom}_{\mathfrak{g}}(M, N) < \infty$  for every  $M, N$ .

Recall that  $\mathcal{O}$  is known to be Noetherian from an earlier theorem. This will imply that every  $M$  has a composition/Jordan-Holder series, so we can take composition factors and multiplicities. Most interesting question: what are the factors/multiplicities of the simple modules and Verma modules?