

Title

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Friday 21st August, 2020

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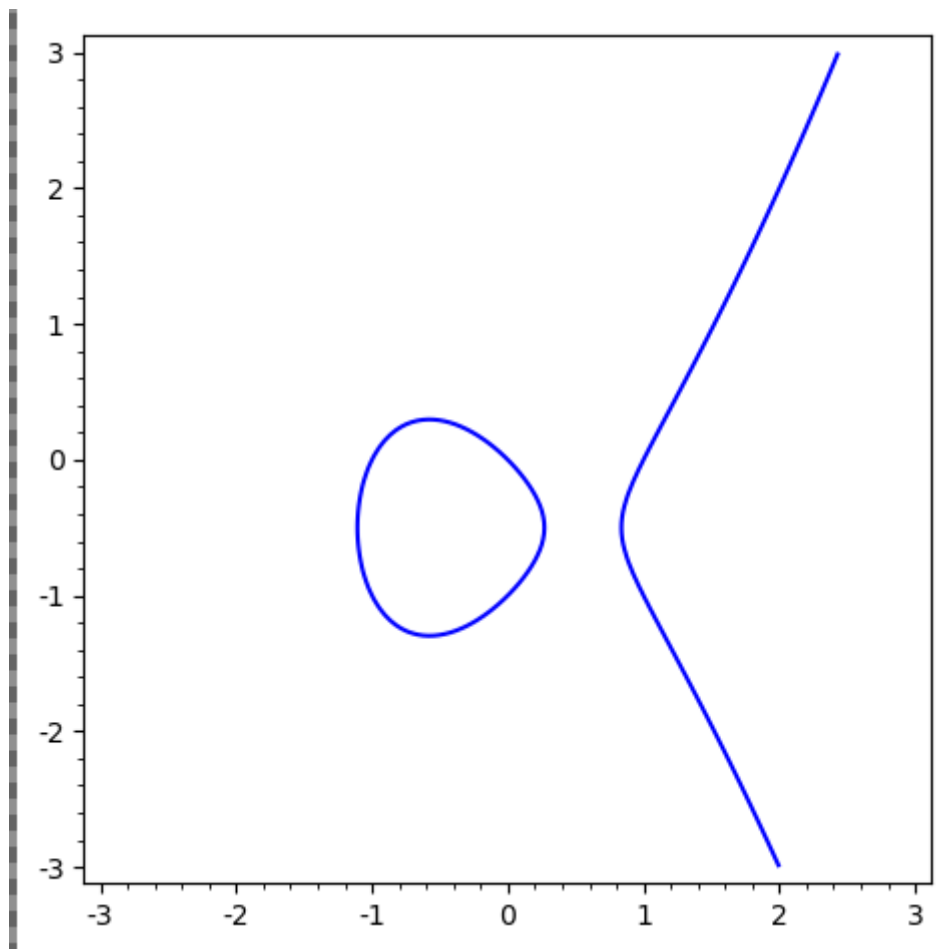
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Reference:
<https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf>

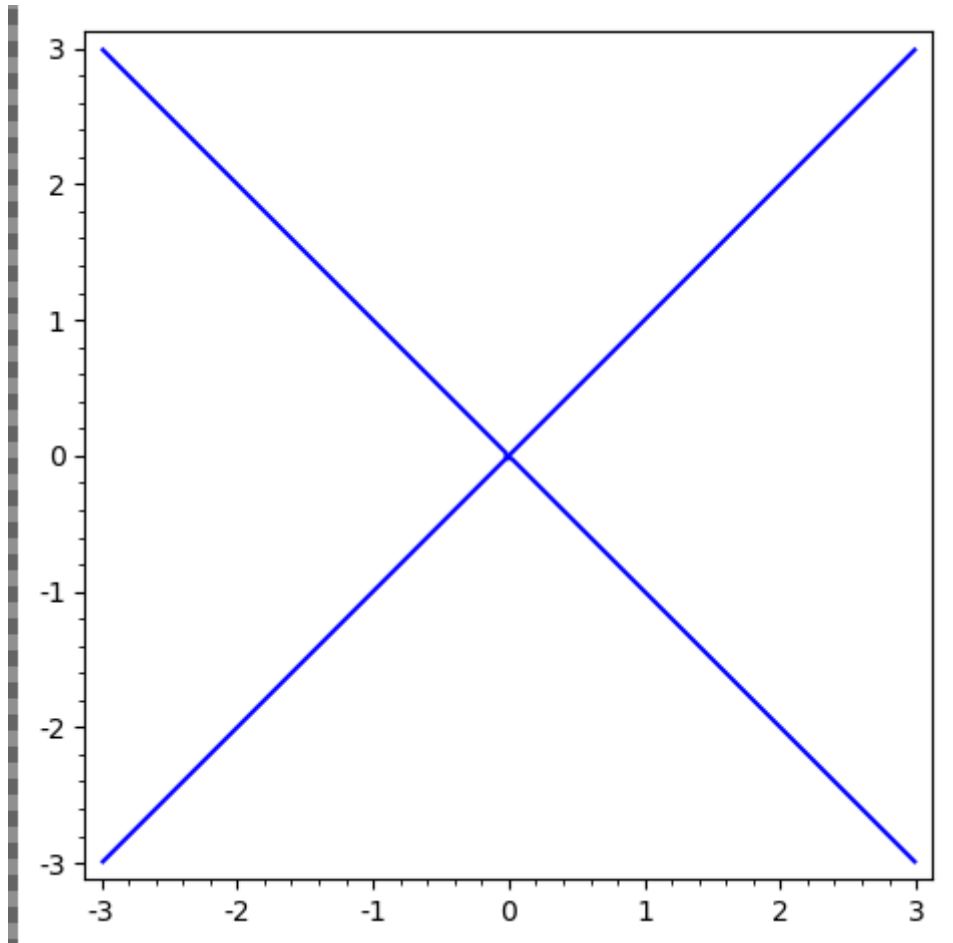
General idea: functions a coordinate ring $R[x_1, \dots, x_n]/I$ will correspond to the geometry of the variety cut out by I .

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over \mathbb{R}
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n = 1$: does it even contain a \mathbb{Q} -point? (Fermat's Last Theorem)
- $x^2 + 1$, which has no \mathbb{R} -points.
- $x^2 + y^2 + 1/\mathbb{R}$ has vanishes nowhere, so ring of functions is not $\mathbb{R}[x, y]/\langle x^2 + y^2 + 1 \rangle$ (problem: \mathbb{R} is not algebraically closed)
- $x^2 - y^2 = 0$ over \mathbb{C} is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$, which has 3 points over \mathbb{F}_3^2 , but $f(x, y) = (x^3 - x)(y^3 - y)$ vanishes at every point
 - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in \mathbb{C} ?)
 - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x, y]/\langle f \rangle$ (addressed by scheme theory)

Theorem 1.1 (Harnack Curve Theorem).

If $f \in \mathbb{R}[x, y]$ is of degree d , then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.

Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2y$

Exercise 1.1.

Show that the vanishing locus of the first two equations above is $X \cup L$ for L a line.

Compare to linear algebra: codimension d iff cut out by exactly d equations.