Chapter 9

D. Zack Garza

Monday 21st September, 2020

Contents

Important Theorems:

Important ideas:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

1 | Background from Chapter 8

- (M, ω) with $\omega \in \Omega^2(M)$ is a symplectic manifold with an almost complex structure J.
- $H \in C^\infty(M; \mathbb{R})$ a Hamiltonian with X_H the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in $T_x M$:

$$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$

  - Zeros of vector field X_H correspond to critical points of H:

$$X_H(x) = 0 \iff (dH)_x = 0.$$

- Take the associated flow $\psi^t : M \to M$, assumed 1-periodic so $\psi^1(x) = x$: critical points of H are periodic trajectories.

- $u \in C^\infty(\mathbb{R} \times S^1; M)$ is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \text{grad}\,_u(H) = 0$$
$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}M$ is the *free loop space* of M, i.e. space of contractible loops on M, i.e. $C^\infty(S^1; M)$ with the C^∞ topology
  - Loops in $\mathcal{L}M$ can be viewed as maps $S^2 \to M$, since they're maps $I \times S^1 \to M$ with the boundaries pinched:

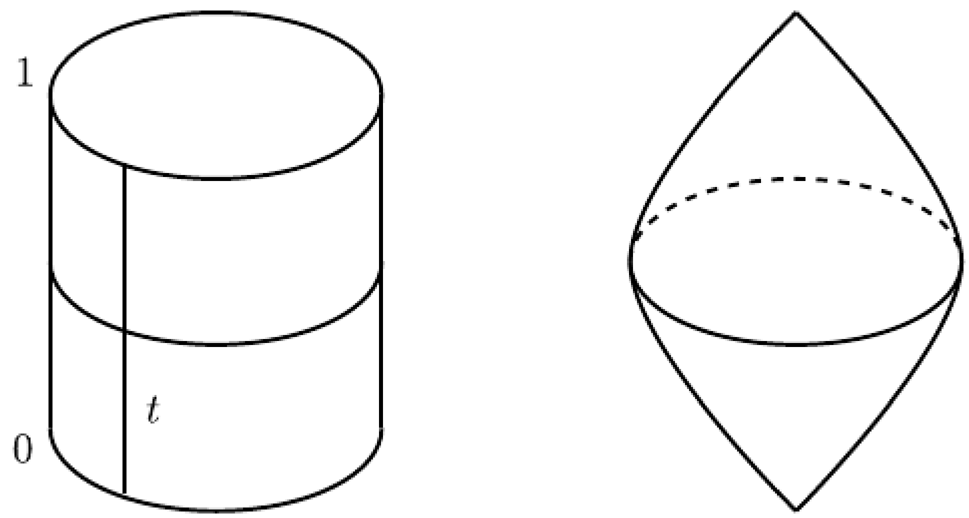

Figure 1: Loops in $\mathcal{L}M$

  - Elements $x \in \mathcal{L}M$ can be viewed as maps $S^1 \to M$.
  - Can extend to maps from a closed disc, $u : \overline{\mathbb{D}}^2 \to M$.

- The action functional is given by

$$\mathcal{A}_H : \mathcal{L}W \to \mathbb{R}$$
$$x \mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt$$

- Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq)$.
- Correspondence between trajectories of the gradient of $\mathcal{A}_H$ and solutions to Floer equations.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, M)$,

$$\int_{S^2} u^*\omega = 0 \quad \text{or equivalently} \quad \langle \omega,\ \pi_2 M\rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM, equivalently

$$\langle c_1 TM,\ \pi_2 M\rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of $\mathcal{A}_H$.
- Maslov index: used the fact that
  - $\mathrm{Sp}(2n, \mathbb{R})$ retracts onto $U(n)$: use a polar decomposition $S = PQ$ as a PSD times orthogonal, then homotope P to I.
  - $\pi_1 U_n = \mathbb{Z}$: use $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$ by the determinant, and $\pi_1 SU(n, \mathbb{C}) = 0$.
  - Thus every path in $\gamma : I \to \mathrm{Sp}(2n, \mathbb{R})$ can be assigned an integer by getting a map $\tilde{\gamma} : I \to S^1$ and taking (approximately) its winding number.

- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y.
  - Showed that after perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
  - How did we do it: describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$ (Banach manifold modeled on the Sobolev spaces $W^{1,p}$), apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality. Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

Goals

- Construct Floer homology and prove the Arnold Conjecture ("Symplectic Morse Inequalities?"):

$$\#\left\{\text{1-Periodic trajectories of } X_H\right\} \geq \sum_{k \in \mathbb{Z}} HM_k(w;\ \mathbb{Z}/2\mathbb{Z}).$$

Steps

1. Define the action functional $\mathcal{A}_H$.
2. Construct the chain complex (graded vector space) CF_*.
3. Define X_H, which will be used to define ∂ later.

4. Count trajectories.
5. Show finite-energy trajectories connect critical points of $\mathcal{A}_H$.
6. Show compactness property for space of trajectories of finite energy.
7. Define ∂ (uses a compactness property in 9.1c)
8. Show space of trajectories is a manifold (plus genericity, "Smale property")
9. Show that $\partial^2 = 0$.
10. Show that HF_* doesn't depend on $\mathcal{A}_H$ or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_*.

2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

  $\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$

  $$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x,y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting x to y:

  $$\mathcal{L}(x, y) \coloneqq \mathcal{M}(x, y)/\mathbb{R}.$$

  - Use the quotient topology, define sequentially:

    $$\tilde{u}_n \overset{n\to\infty}{\longrightarrow} \tilde{u} \quad \iff \quad \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \overset{n\to\infty}{\longrightarrow} u(s, \cdot).$$

  - When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

    $$n(x, y) \coloneqq \#\mathcal{L}(x, y)$$

    is well-defined.

- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k.
  - Finitely many since they are nondegeneracy implies they are isolated.
- Defined a differential

  $$\begin{aligned} \partial : C_k(H) &\to C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y) y \end{aligned}$$

  $$\begin{aligned} n(x, y) &\coloneqq \# \{\text{Trajectories of grad } \mathcal{A}_H \text{ connecting } x, y\} \mod 2 \\ &= \#\mathcal{L}(x, y) \mod 2. \end{aligned}$$

- Examined ∂^2:

$$
\begin{aligned}
\partial^2 : C_k(H) &\to C_{k-2}(H) \\
x &\mapsto \partial(\partial(x)) \\
&= \partial\left(\sum_{\mu(y)=\mu(x)-1} n(x,y)y \right) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x,y)\partial(y) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x,y) \left(\sum_{\mu(z)=\mu(y)-1} n(y,z)z \right) \\
&= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x,y)n(y,z)\, z \\
&= \sum_{\mu(z)=\mu(y)-1} \left(\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) \right) z \qquad \text{(finite sums, swap order)},
\end{aligned}
$$

so it suffices to show

$$
\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.
$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x,z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\overline{\mathcal{L}}(x,y)$.
- We'll then have

$$
\overline{\mathcal{L}}(x,z) = \mathcal{L}(x,z) \cup \partial\overline{\mathcal{L}}(x,z), \qquad \partial\overline{L}(x,z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x,y) \times \mathcal{L}(y,z),
$$

which "space-ifies" the equation we want.

- We'll show $\partial\overline{\mathcal{L}}(x,z)$ is a 1-manifold, which must have an even number of points, and thus

$$
\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = \#\left(\partial\overline{\mathcal{L}}(x,z)\right) \equiv 0 \mod 2.
$$

2.1 Three Important Theorems

- Shown last time: a sequence of trajectories can converge to a broken trajectory.

Theorem 2.1 ***(9.1.7).***
Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, y)$, then there exist

- A subsequence $\left\{u_{n_j}\right\}$
- Critical points $\{x_0, x_1, \cdots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = y$
- Sequences $\left\{s_n^1\right\}, \left\{s_n^2\right\}, \cdots, \left\{s_n^\ell\right\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_n \cdot s_n^k \xrightarrow{n\to\infty} u^k.$$

Definition 2.1.1 (Regular Pair)**.**
For an almost complex structure J and a Hamiltonian H, the pair (H, J) is **regular** if the Floer map $\mathcal{F}$ is transverse to the zero section in the following vector bundle:

$$\begin{tikzcd}
\{?\} \arrow[r] & C^\infty(\mathbb{R} \times S^1; TM) \arrow[d] \\
& C^\infty(\mathbb{R} \times S^1; M) \arrow[u, bend left, dotted, "\mathcal{F}"] \arrow[u, bend right, dotted, "\mathbf{0}"']
\end{tikzcd}$$

Theorem 2.2 ***(9.2.1).***
Let (H, J) be a regular pair with H nondegenerate and x, z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\overline{\mathcal{L}}(x, z)$ is a compact 1-manifold with boundary with

$$\partial\overline{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x,z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \qquad \text{where} \quad \mathcal{I}(x, z) = \left\{y \;\middle|\; \mu(x) < \mu(y) < \mu(z)\right\}.$$

Note: possibly a typo in the book?

Remark 1.

- As a corollary, $\partial^2 = 0$.
- Most of chapter 9 is spent proving this theorem.

Remark 2.
Some notation:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathcal{M}(x,z) \arrow[d, "\pi"] \\ & \mathcal{L}(x,z) \end{tikzcd}$$

Hats will generally denote maps induced on quotient.

Theorem 2.3 ***(9.2.3: Gluing).***
Let x, y, z be three critical points of $\mathcal{A}_H$ with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\widehat{u}, \widehat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

   $$\psi : [\rho_0, \infty) \to \mathcal{M}(x, z)$$

   such that $\widehat{\psi}$, the induced map on the quotient

   $$\begin{tikzcd} {[\rho_0, \infty)} \arrow[r, "\psi"] \arrow[dr, dotted, "\widehat{\psi}"'] & \mathcal{M}(x,z) \arrow[d, "\pi"] \\ & \mathcal{L}(x,z) \end{tikzcd}$$

   is an embedding that satisfies

   $$\widehat{\psi}(\rho) \xrightarrow{\rho \to \infty} (\widehat{u}, \widehat{v}) \in \overline{\mathcal{L}}(x, z).$$

2. For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x, z)$,

   $$\ell_n \xrightarrow{n \to \infty} (\widehat{u}, \widehat{v}) \quad \Longrightarrow \quad \ell_n \in \operatorname{im}(\widehat{\psi}) \text{ for } n \gg 0.$$

2.2 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$, so we quotient to define $\mathcal{L}(x, y) =:= \mathcal{M}_{x,y}/\mathbb{R}$ with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \overset{n \to \infty}{\to} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n + s, \,\cdot\,) \overset{n \to \infty}{\to} u(s, \cdot).$$

Proposition 2.4 ***(?).***
$\mathcal{L}(x, y)$ is Hausdorff.

- Want to show $\mathcal{L}(x,y)$ is a compact 0-dimensional manifold.
- Have a differential

$$\begin{aligned}\partial : C_k(H) &\longrightarrow C_{k-1}(H)\\ \partial(x) &= \sum_{\mathrm{Ind}(y)=k-1} n(x,y)y.\end{aligned}$$

with $n(x,y)$ the number (mod 2) of trajectories of grad $\mathcal{A}_H$ connecting x, y, i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

$$M := \overline{\mathcal{L}}(x,z) = \mathcal{L}(x,z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x,y) \times \mathcal{L}(y,z).$$

  and show that M is compact with ∂M equal to the last union.
- Last time: closure of space of trajectories connecting x, y contains "broken" trajectories.
- Last time: toward proving that M is compact

3 | 9.2

- Wanted to compactify $\mathcal{L}(x,y)$, needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

Theorem 3.1 ***(9.2.1).***
Let (H, J) be a regular pair with H nondegenerate.
Let x, z be two periodic trajectories of H such that $\mu(x) = \mu(z) + 2$.
Then $\overline{\mathcal{L}}(x,y)$ is a compact 1-manifold with boundary satisfying

$$\partial\overline{\mathcal{L}}(x,y) = \bigcup_{\mu(x)<\mu(y)<\mu(z)} \mathcal{L}(x,y) \times \mathcal{L}(y,z).$$

As a corollary, $\partial^2 = 0$.

- Know $\overline{\mathcal{L}}(x,y)$ is compact and $\mathcal{L}(x,y)$ is a 1-manifold
- Now suffices to study in a neighborhood of boundary points ("gluing theorem")

3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function w_p which interpolates between u and v (not exactly a solution itself, but will be approximated by one later).
2. Constructing ψ a "true solution" from w_p using the Newton-Picard method. We'll have

$$\psi(p) = \exp_{w_p}(\gamma(p)) \qquad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x,z).$$

   where $\mathcal{P} =?$.

3. Get a lift $\widehat{\psi} = \pi \circ \psi$ where $\pi =$? satisfying

- $\widehat{\psi}(p) \overset{n\to\infty}{\to} (\widehat{u}, \widehat{v})$
- $\widehat{\varphi}$ is an embedding
- $\widehat{\psi}$ is unique in the following sense (the last point)

Theorem 3.2 ***(9.2.3 (Gluing Theorem)).***
Let x, y, z be critical points of the action functional $\mathcal{A}_H$ such that $\mu(x) = \mu(y) + 1 = \mu(z) + 2$. Let $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ be trajectories, inducing $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$.

- There exist a differentiable map $\psi : (\rho_0, \infty) \to \mathcal{M}(x, z)$ for some $\rho > 0$ such that
- $\pi \circ \psi : (\rho_0, \infty) \to \mathcal{L}(x, z)$ is an embedding
- $\widehat{\psi} \overset{\rho\to\infty}{\to} (\bar{u}, \bar{v}) \in \overline{\mathcal{L}(x, z)}$.
- If $\ell_n \in \mathcal{L}(x, z)$ with $\ell_n \overset{n\to\infty}{\to} (\bar{u}, \bar{v})$, then for $n \gg 1$ we have $\ell \in \Im(\widehat{\psi})$.

4 | 9.3: Pre-gluing

- Choose a bump function β on $\{0\}^c \subset \mathbb{R} \to [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^{\pm}$:

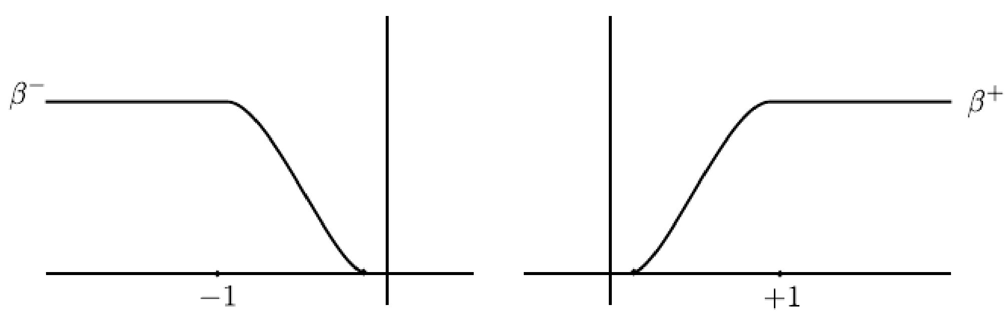

Figure 2: Figure 9.3

- Define the interpolation w_ρ from u to v in the following way:

$$w_\rho(s, t) = \begin{cases} u(s + \rho, t) & \text{if } s \leq -1 \\ \exp_{y(t)} \left(\beta^-(s) \exp^{-1}_{y(t)}(u(s + \rho, t)) \ + \beta^+(s) \exp^{-1}_{y(t)}(v(s - \rho, t)) \right) & \text{if } s \in [-1, 1] \\ v(s - \rho, t) & \text{if } s \geq 1 \end{cases}$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \;\middle|\; \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\}.$$

5 | 9.4: Construction of ψ.

- Have constructed $w_\rho \in C^\infty_\searrow(x,z)C^\infty(x,z)$ for every $\rho \geq \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x,z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } Hx$$

  in the weak sense.
- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining $\mathcal{F}_\rho$.

$$\begin{aligned} W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\xrightarrow{\mathcal{F}_\rho} L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ (y_1, \ldots, y_{2n}) &\longmapsto \left[\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } H_t\right)\left(\exp_{w_\rho} \sum y_i Z_i^\rho\right)\right]_{Z_i} \end{aligned}$$

where $\mathcal{F}_\rho \coloneqq \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i. sd - Newton-Picard method, general idea

- Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

- Allows finding zeros of f given an approximate zero x_0.
- Linearize $\mathcal{F}_\rho$.