

# Title

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## Contents

<b>1 Thursday, September 17</b>	<b>1</b>
1.1 Regular Functions . . . . .	1

# 1 | Thursday, September 17

## 1.1 Regular Functions

See chapter 3 in the notes.

Some examples:

- $X$  a manifold or an open set in  $\mathbb{R}^n$  has a ring of  $C^\infty$  functions.
- $X \subset \mathbb{C}$  has a ring of holomorphic functions.
- $X \subset \mathbb{R}$  has a ring of real analytic functions

These all share a common feature: it suffices to check if a function is a member on an arbitrary open set about a point, i.e. they are *local*.

### Definition 1.0.1 (?).

Let  $X$  be an affine variety and  $U \subseteq X$  open. A **regular function** on  $U$  is a function  $\varphi : U \rightarrow k$  such that  $\varphi$  is “locally a fraction”, i.e. a ratio of polynomial functions.

More formally, for all  $p \in U$  there exists a  $U_p$  with  $p \in U_p \subseteq U$  such that  $\varphi(x) = g(x)/f(x)$  for all  $x \in U_p$  with  $f, g \in A(X)$ .

### Example 1.1.

For  $X$  an affine variety and  $f \in A(X)$ , consider the open set  $U := V(f)^c$ . Then  $\frac{1}{f}$  is a regular function on  $U$ , so for  $p \in U$  we can take  $U_p$  to be all of  $U$ .

### Example 1.2.

For  $X = \mathbb{A}^1$ , take  $f = x - 1$ . Then  $\frac{x}{x-1}$  is a regular function on  $\mathbb{A}^1 \setminus \{1\}$ .

**Example 1.3.**

Let  $X = V(x_1x_4 - x_2x_3)$  and

$$U := X \setminus V(x_2, x_4) = \{[x_1, x_2, x_3, x_4] \mid x_1x_4 = x_2x_3, x_2 \neq 0 \text{ or } x_4 \neq 0\}.$$

Define

$$\begin{aligned} \varphi : U &\rightarrow K \\ [x_1, x_2, x_3, x_4] &\mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}. \end{aligned}$$

This is well-defined on  $\{x_2 \neq 0\} \cap \{x_4 \neq 0\}$ , since  $\frac{x_1}{x_2} = \frac{x_3}{x_4}$ . Note that this doesn't define an element of  $k$  at  $[0, 0, 0, 1] \in U$ . So this is not globally a fraction.

Notation: we'll let  $\mathcal{O}_X(U)$  is the ring of regular function on  $U$ .

**Proposition 1.1(?)**

Let  $U \subset X$  be an affine variety and  $\varphi \in \mathcal{O}_X(U)$ . Then  $V(\varphi) := \{x \in U \mid \varphi(x) = 0\}$  is closed in the subspace topology on  $U$ .

*Proof .*

For all  $a \in U$  there exists  $U_a \subset U$  such that  $\varphi = g_a/f_a$  on  $U_a$  with  $f_a, g_a \in A(X)$  with  $f_a \neq 0$  on  $U_a$ .

Then

$$\{x \in U_a \mid \varphi(x) \neq 0\} = U_a \setminus V(g_a) \cap U_a$$

is an open subset of  $U_a$ , so taking the union over  $a$  again yields an open set. But this is precisely  $V(\varphi)^c$ . ■

**Proposition 1.2.**

Let  $U \subset V$  be open in  $X$  an *irreducible* affine variety. If  $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$  agree on  $U$ , then they are equal.

*Proof .*

$V(\varphi_1 - \varphi_2)$  contains  $U$  and is closed in  $V$ . It contains  $\bar{U} \cap V$ , by an earlier lemma,  $X$  irreducible implies that  $\bar{U} = X$  and so  $V(\varphi_1 - \varphi_2) = V$ . ■

Compare and contrast: Let  $U \subset V \subset \mathbb{R}^n$  be open. If  $\varphi_1, \varphi_2 \in C^\infty(V)$  such that  $\varphi_1, \varphi_2$  are equal when restricted  $U \subset V$ . Does this imply  $\varphi_1 = \varphi_2$ ?

For  $\mathbb{R}^n$ , no, there exist smooth bump functions. For  $\mathbb{C}$  and holomorphic function