Title

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1 | Friday, November 06

1.1 Good (p,r)- Filtrations

Last time: G_rT and G_rB modules. We roughly know the category of G_r modules, and we think of G_rT as graded G_r-modules. We defined

$$\widehat{Z}'_r(\lambda) := \operatorname{Ind}_B^{G_rB}(\lambda)$$
$$\widehat{Z}'_r(\lambda) := \operatorname{Coind}_{B^+}^{G_rB^+}(\lambda).$$

We can use these for classification since we have a correspondence

$$\left\{ \text{Simple } G_rT\text{-modules} \right\} \iff \left\{ X(T) \right\}$$
$$\widehat{L}_r(\lambda) = \widehat{L}_r(\lambda_0) \otimes p^r\lambda \leftarrow\!\shortmid \lambda = \lambda_0 + p^r\lambda_1,$$

where $\widehat{L}_r(\lambda_0)$ is a simple G_r-module and $\lambda_0 \in X_r(T)$.

Proposition 1.1.1 ***(?).***
For each $\lambda \in X(T)$ and $i \in \mathbb{N}$, there exists an isomorphism of G-modules

$$\left\{ X(T) \right\} \quad \widehat{L}_H^i(\lambda) = R^i \operatorname{Ind}_{G_rB}^G \widehat{Z}'_r(\lambda).$$

Proof (?).
We can compose the two functors to get a Grothendieck-type spectral sequence

$$E_2^{m,n} = R^m \operatorname{Ind}_{G_rB}^G \left(R^n \operatorname{Ind}_B^{G_rB}(\lambda) \right) \Rightarrow R^{m+n} \operatorname{Ind}_B^G(\lambda),$$

which follows from induction being transitive. Note that $\operatorname{Ind}_B^{G_rB}(\,\cdot\,)$ is exact, since coinduction is given by $\operatorname{dist}(G_rB) \otimes_{\operatorname{Dist}(B)} \lambda \cong \operatorname{Dist}(U_r^+) \otimes_k \lambda$ is tensoring over a field, and this is dual to induction. Thus $R^{>0} \operatorname{Ind}_B^{G_rB}(\lambda) = 0$ and the spectral sequence collapses to yield

$$R^m \operatorname{Ind}_{G_rB}^G R^0 \operatorname{Ind}_B^{G_rB} = R^m \operatorname{Ind}_{G_rB}^G \operatorname{Ind}_B^{G_rB} = R^m \operatorname{Ind}_B^G(\lambda),$$

where we can just note that $\operatorname{Ind}_B^{G_rB}(\lambda) = \widehat{Z}'_r(\lambda)$.

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Recall *Kempf's vanishing theorem*: if $\lambda \in X(T)_+$ is a dominant weight, then $H^{>0}(\lambda) = 0$.

Definition 1.1.1 (p-filtration, due to Steve Donkin)**.**
Let $M \in G\text{-}\operatorname{mod}$, then M has a (good) (p, r)**-filtration** iff there exists a sequence of G-modules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M$$

such that $M_i / M_{i+1} \cong L(\lambda_0) \otimes H^0(\lambda_1)^{(r)}$ where $\lambda_0 \in X_r(T)$ (so the first time is irreducible) and $\lambda_1 \in X(T)_+$, so the second term is twisted.

Remark 1.1.1 : Question due to Jantzen: let $\lambda \in X(T)_+$. Does $H^0(\lambda)$ have a good (p, r)-filtration?

This question was open for a while, until the following was found:

Proposition 1.1.2 ***(Parshall-Scott, 2013).***
If $p \geq 2(h-1)$ and Lusztig's character formula holds for G, then $H^0(\lambda)$ has a good (p, r) filtration.

Proposition 1.1.3 ***(Bendell-Nakano-Pillen-Sobaje, 2019).***
There are counterexamples to Jantzen's question. Example: $\Phi = G_2$ and $p = 2$.

Later: we'll see how to construct these filtrations by factoring induction into intermediate inductions.

Theorem 1.1.1 ***(?).***
Let $\lambda \in X(T)_+$ and assume every composition factor of the baby Verma $\widehat{Z}_r'(\lambda)$ has the form $\widehat{L}_r(\mu_0 + p^r\mu_1) = \widehat{L}_r(\mu_0) \otimes p^r\mu_1$ where $\mu_0 \in X_r(T)$ and $\mu_1 \in X(T)$ is any weight. Suppose further that $\langle \mu_1 + \rho,\ \beta^\vee \rangle \geq 0$ for all $\beta \in \Delta$ (so it's "pretty dominant"). Then $H^0(\lambda)$ has a good (p, r) filtration, and moreover

$$[\widehat{Z}_r'(\lambda) : \widehat{L}_r(\mu_0) \otimes p^r\mu_1] = [H^0(\lambda) : L(\mu_0) \otimes H^0(\mu_1)^{(r)}].$$

Proof (?).
Suppose $\widehat{L}_r(\mu_0 + p^r\mu_1)$ is a composition factor of $\widehat{Z}_r'$. Then since we have G-modules, we can use the tensor identity to write

$$\begin{aligned} R^i \operatorname{Ind}_{G_rB}^G L_r(\mu_0) \otimes p^r\mu_1 &= L_r(\mu_0) \otimes R^i \operatorname{Ind}_{G_rB}^G p^r\mu_1 \\ &= L_r(\mu_0) \otimes H^i(\mu_1)^{(r)}, \end{aligned}$$

where the last equality follows from a theorem we won't prove here. We can set $i = 0$ to yield

$$\operatorname{Ind}_{G_rB}^G L_r(\mu_0) \otimes p^r\mu_1 \cong L_r(\mu_0) \otimes H^0(\mu_1)^{(r)}.$$

Recall that $H^0(\lambda) = \operatorname{Ind}_{G_rB}^G \widehat{Z}_r'(\lambda)$, so we'll take a composition series for $\widehat{Z}_r'(\lambda)$ and apply the induction functor to it. So let such a composition series be given by

$$0 \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq N_s = \widehat{Z}_r'(\lambda),$$

where $N_i/N_{i-1} \cong L(\mu_0) \otimes p^r \mu_1$ for some $\mu_0 \in X_r(T)$ and $\mu_1 \in X(T)$. Now apply the functor $\operatorname{Ind}_{G_r B}^{G}(\,\cdot\,)$ which yields

$$0 \subseteq \cdots \subset \operatorname{Ind}_{G_r B}^{G} N_i \subseteq \cdots \subseteq H^0(\lambda).$$

Question: is this a good (p, r) filtration?
⚠ *Warning 1.1:* Note that if we have

$$0 \to N_1 \to N_2 \to N_2/N_1 \to 0$$

this yields

$$0 \to \operatorname{Ind} N_1 \to \operatorname{Ind} N_2 \to \operatorname{Ind}(N_2/N_1) \to R^1 \operatorname{Ind} N_1 \to \cdots .$$

Here we need $\operatorname{Ind}(N_2/N_1) \cong \operatorname{Ind} N_2/\operatorname{Ind} N_1$, so we need to show $R^1 \operatorname{Ind} N_1 = 0$.
Using the tensor identity we can write

$$\begin{aligned} R^1 \operatorname{Ind} N_1 &= R^1 \operatorname{Ind}_{G_r B}^{G} L_r(\sigma_0) \otimes p^r \sigma_0 \\ &= L_r(\sigma_0) \otimes \left(R^1 \operatorname{Ind}_{G_r B}^{G} \sigma_1\right)^{(r)} \end{aligned}$$

and $\langle \sigma_1 + \rho,\ \beta^\vee \rangle \geq 0$, so $R^1 \operatorname{Ind}_{G_r B}^{G} \sigma_1 = 0$. Thus we can extend the region from Kempf's vanishing slightly:

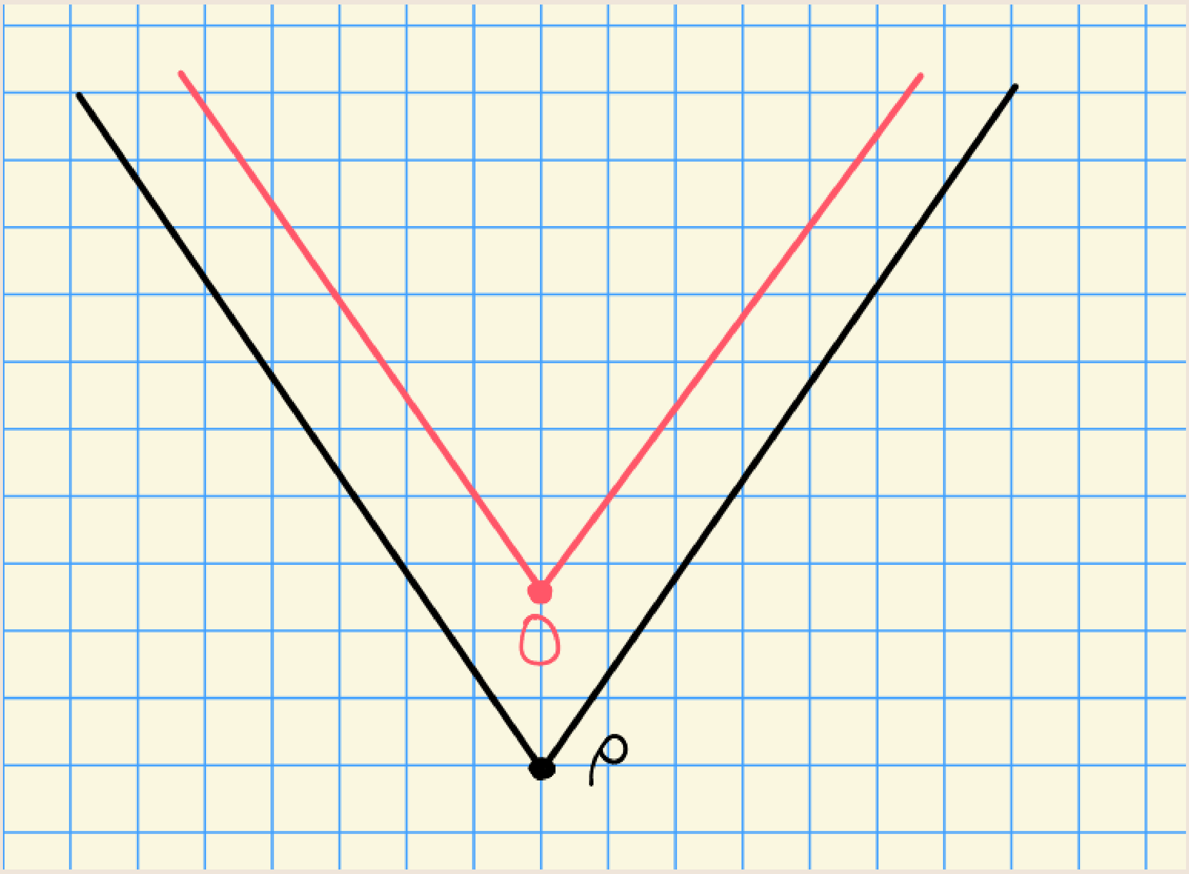


Figure 1: Image

Finding composition factors for the $\widehat{Z}'_r$ is in general a hard problem: if we had this, we'd have the characters of the irreducibles. Some combinatorics can be used here.

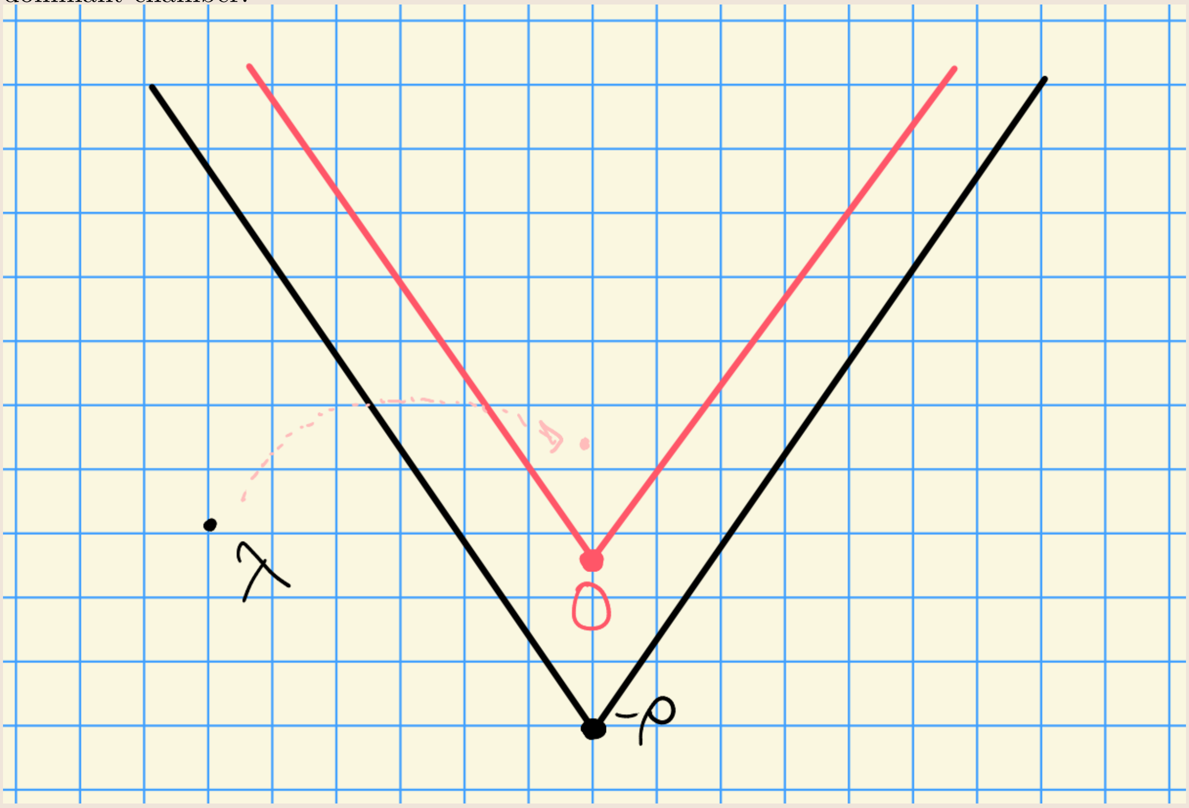
1.2 Strong Linkage

Note that strong linkage for $H^0(\lambda)$ implies strong linkage for $\widehat{Z}'_r(\lambda)$.

Theorem 1.2.1 *(?).*
Let $\lambda, \mu \in X(T)$, then if $[\widehat{Z}'_r(\lambda) : \widehat{L}(\mu)] \neq 0$, then $\mu \uparrow \lambda$ and $\mu \in W_p \cdot \lambda + p^r X(T)$.

Proof (?).
Note that $\widehat{Z}'_r(\lambda)$ is finite dimensional. Idea: tensor by a 1d rep to make all composition factors

dominant. Then for any weight λ, we can find a large enough weight that moves λ into the dominant chamber:



I.e., we can tensor by $p^r v$ for v large so that $\widehat{Z}_r'(\lambda) \otimes p^r v$ has composition factors if the form $L(\mu_0) \otimes p^r \mu_1$ with $\langle \mu_1 + \rho, \ \beta^\vee \rangle \geq 0$ for all $\beta \in \Delta$.
Then $\mu + p^r v \uparrow \lambda p^r v$, which implies $\mu \uparrow \lambda$, and so using strong linkage we have a p-filtration on $H^0(\lambda + p^r v)$.

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Next time: extensions in $G_r T$-mod and the Steinberg module (very important in representation theory, has some nice properties).