8.8 Part 2, Computing the Index of L

D. Zack Garza

Monday 25th May, 2020

Contents

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$
\begin{aligned}
L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\
Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y
\end{aligned}
$$

  where

$$
\begin{aligned}
S : \mathbb{R} \times S^1 &\longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\
S(s,t) &\overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t).
\end{aligned}
$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
  - *(not the same kernel/cokernel)*
- Compute $\operatorname{Ind} L_1$: explicitly describe $\ker L_1, \operatorname{coker} L_1$.
- Replace in two steps:
  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s.
    * Replace $S(s,t)$ by a matrix

$$
\tilde{S}(s,t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.
$$

    * Idea: approximate by cylinders at infinity.
    * Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    * Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\operatorname{Ind}(L) = \mu\left(R^-(t)\right) - \mu\left(R^+(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

  where $S:\mathbb{R}\longrightarrow \operatorname{Mat}(2n;\mathbb{R})$ is a path of diagonal matrices depending on $\operatorname{Ind}(R^{\pm}(t))$; then

$$\operatorname{Ind}(L) = \operatorname{Ind}(L_1) = \operatorname{Ind}(R^-(t)) - \operatorname{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing L_1 to $R^{\pm}$. Let $k^{\pm} := \operatorname{Ind}(R^{\pm})$; then $\operatorname{Ind}(L_1) = k^- - k^+$.
- Lemma 8.8.4: $\operatorname{Ind}(L_0) = \operatorname{Ind}(L)$.
- Han's Talk:
  - Prop 8.8.3, using Lemma 8.8.5
- Me
  - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\operatorname{Ind}(L_1) = K^- - k^+$.

Statement: let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2}\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^{\pm} := a_1^{\pm} = a_2^{\pm}$. Then

$$\begin{aligned} \dim\operatorname{Ker} F &= 2\cdot\#\left\{\ell\in\mathbf{Z} | a^- < 2\pi\ell < a^+\right\} \\ \dim\operatorname{Ker} F^{\star} &= 2\cdot\#\left\{\ell\in\mathbf{Z} | a^+ < 2\pi\ell < a^-\right\}. \end{aligned}$$

2. Suppose $\sup_{s\in\mathbb{R}}\|S(s)\| < 1$, then

$$\begin{aligned} \dim\operatorname{Ker} F &= \#\left\{i\in\{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0\right\} \\ \dim\operatorname{Ker} F^{\star} &= \#\left\{i\in\{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0\right\}. \end{aligned}$$