Title

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1.1 Singularities

Recall that there are three types of singularities:

- Removable
- Poles
- Essential

Theorem 1.1 ***(3.2).***
An isolated singularity z_0 of f is a pole $\iff \lim_{z \longrightarrow z_0} f(z) = \infty$.

Theorem 1.2 ***(3.3, Casorati-Weierstrass).***
If f is holomorphic in $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity z_0, then there exists a radius r such that $f(D_r(\{z_0\}) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof .
Proceed by contradiction. Suppose there exists a $w \in \mathbb{C}$ and a $\delta > 0$ such that

$$D_\delta(w) \bigcap f(D_r(\{z_0\}) \setminus \{z_0\}) = \emptyset.$$

If $z \in D_r(w) \setminus z_0$, then $|f(z) - w| > \delta$. Define $g(z) = \dfrac{1}{f(z) - w}$ on $D_r(z_0) \setminus \{z_0\}$; then $|g(z)| < \dfrac{1}{\delta}$. Then $g(z)$ has a removable singularity at $z = z_0$ by theorem 3.1.
If $g(z_0) \neq 0$, then $f(z) - w$ is holmorphic at z_0, contradicting the fact that z_0 is an essential singularity.
If instead $g(z_0) = 0$, then z_0 is a pole, again a contradiction.

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Note: revisit why this is a contradiction.