Zack
Garza

① Suppose E is bounded, so $\operatorname{diam}(E) \leq M$ for some fixed M. In particular, if $Q_i \subseteq E$ is an interval, then $|Q_i| \leq M$. Let $\varepsilon > 0$, and choose $\underbrace{\{Q_i\} \rightrightarrows E}$ s.t.

i.e. $E \subseteq \bigcup_i E_i$

$$\text{for each } i, \quad |Q_i| \leq \varepsilon/2M$$

Then let $L_i = Q_i^2$. We then have

$$\begin{aligned} |L_i| \leq |b^2 - a^2| = |b-a| \cdot |b+a| &= |Q_i| \cdot |b+a| \\ &\leq |Q_i| \cdot 2M \\ &\leq (\varepsilon/2^{i+1} M) 2M \\ &= \varepsilon/2^i, \end{aligned}$$

so $\sum_{i=1}^\infty |L_i| \leq \sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$, and $\{L_i\} \rightrightarrows E^2$, so

$$m_*(E^2) < \varepsilon \longrightarrow 0.$$

Claim: It suffices to consider the bounded case.

Ball of radius n around 0

Pf If E is not bounded, consider $F_n = E \cap B(n, 0)$.

Then F_n is bounded (by n), and since $F_n \subseteq E \Rightarrow m_*(F_n) \leq m_*(E) = 0$ by subadditivity, $m_*(F_n^2) = 0$ by the bounded case.

But then $E^2 = \bigcup_{n=1}^{\infty} F_n^2 \Rightarrow m_*(E^2) = m\left(\bigcup_{n=1}^{\infty} F_n^2\right) \leq \sum_{n=1}^{\infty} m_*(F_n^2) = 0$

by countable subadditivity. ■

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② Note

1) $E_1 = E_1 \setminus E_2 \sqcup E_1 \cap E_2$

2) $E_2 = E_2 \setminus E_1 \sqcup E_1 \cap E_2$

3) $E_1 \triangle E_2 = E_2 \setminus E_1 \sqcup E_1 \setminus E_2$

4) $E_1 \cup E_2 = \underbrace{(E_1 \triangle E_2) \sqcup (E_1 \cap E_2)}$

All disjoint unions, so we can freely apply measures and use countable additivity.

so

$$\left.\begin{aligned} m(E_1) + m(E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) \\ &\quad + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \end{aligned}\right\} \text{by (1), (2)}$$

$$= \underbrace{m(E_1 \triangle E_2) + m(E_1 \cap E_2)} + m(E_1 \cap E_2) \quad \} \text{by (3)}$$

$$= \underbrace{m(E_1 \cup E_2)} + m(E_1 \cap E_2). \quad \} \text{by (4)}$$

■

3a) Suppose $m(A) = m(B) < \infty$.

Since $A \subseteq E \subseteq B$, we have $E \setminus A \subseteq B \setminus A$. However,

$$B = A \sqcup (B \setminus A) \Rightarrow m(B) = m(A) + m(B \setminus A)$$

$$\Rightarrow m(B) - m(A) = m(B \setminus A)$$

(since $m(A) < \infty$)

$$\Rightarrow m(B \setminus A) = 0$$

(since $m(B) = m(A)$)

So $m_*(E \setminus A) = 0$ by subadditivity.

But then

$E = A \sqcup (E \setminus A)$, where A is measurable by assumption and $E \setminus A$ is an outer measure 0 set and thus measurable.

So E is measurable, and

$$m(E) = m(A) + m(E \setminus A)$$

$$= m(A) + 0$$

$$\Rightarrow m(E) = m(A) = m(B) < \infty.$$

3b) Idea: $[0,1] \subseteq \mathcal{N} \subseteq [-1,2]$, so take

- $A = (-\infty, 0)$
- $E = A \cup (\mathcal{N}+1)$, where $\mathcal{N}$ is the non-measurable set, and $\mathcal{N}+1 = \{x+1 \mid x \in \mathcal{N}\}$ is non-measurable by the same argument used for $\mathcal{N}$.
- $B = \mathbb{R}$

<u>Claim</u>: E is not measurable.

Supposing it were, note that A^c is measurable,

and countable intersections of measurable sets are

measurable, so

$$E \cap A^c = (A \cup (\mathcal{N}+1)) \cap A^c = \mathcal{N}+1$$

must be measurable. $\times\!\!\!\times$

4) Let A, B be fixed, and define

$$E_t := \{x \in \mathbb{R}^n \mid \inf_{a \in A} |x-a| \leq t\} \cap B$$
$$= \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \leq t\} \cap B$$

and

$$f : \mathbb{R} \to \mathbb{R}$$
$$t \mapsto \mu(E_t)$$

Note that $E_0 = A$, so $f(0) = \mu(A)$, and since B is compact and thus bounded, there is some $t = T$ such that $B \subseteq E_T$.

So f maps $[0,T]$ to $[\mu(A), \mu(B) + M]$ for some M.

<u>Claim:</u> f is cts, and for all $t \in [0,T']$ for some T', $A \subseteq E_t \subseteq B$ and each E_t is compact.

Note that if this is true, we can first apply the intermediate value theorem to find a T' such that $f(T') = m(B)$, then restrict f to map $[0,T']$ to $[m(A), m(B)]$. We can apply it again to pull back any $c \in [m(A), m(B)]$ to a t satisfying $c = f(t) = \mu(E_t)$, in which case $A \subseteq E_t \subseteq B$ and $\mu(A) \leq c = \mu(E_t) \leq \mu(B)$ as desired.

- <u>f is cts:</u> We'll show that the 2-sided limit $\lim_{t_i \to t} f(t_i)$ exists and is equal to $f(t)$, using the fact that $a \leq b \Rightarrow E_a \subseteq E_b$.

If $t_i \nearrow t$, then $E_{t_1} \subseteq E_{t_2} \subseteq \cdots \subseteq E_t$, and $\bigcup_{i \in \mathbb{N}} E_{t_i} = E$, so by continuity of measure from below, we have $\lim_{i\to\infty} \mu(E_{t_i}) = \mu(E)$, so

$$\lim_{t_i \to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

Similarly, if $t_i \searrow t$, noting that $t_i \leq T' \Rightarrow t_1 \leq T' \Rightarrow \mu(E_{t_1}) \leq \mu(B) < \infty$,

and $E_{t_1} \supseteq E_{t_2} \supseteq \cdots \supseteq E$, so

we can apply continuity of measure from above to obtain

$$\lim_{t_i \to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

So f is cts. ■

- E_t is compact:

Since $E_t \subseteq B$ which is compact and thus bounded, it suffices to show that E_t is closed. But letting $N_t = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) < t\}$, we have $E_t = \overline{N_t \cap B}$, where N_t is open because $N_t = \bigcup_{a\in A} \underbrace{\{x \in \mathbb{R}^n \mid \operatorname{dist}(x,a) < r\}}_{\text{Open ball around } a}$, and $N_t \subseteq B \Rightarrow N_t \cap B$ is still open. But the closure of any open set is closed. ■

- $t \in [0, T'] \Rightarrow A \subseteq E_t \subseteq B$:

$E_0 = A$ and $t \leq s \Rightarrow E_t \subseteq E_s$, so $A \subseteq E_t$ for all t.

But $E_t = \overline{N_t \cap B} \subseteq \overline{B} = B$ since B is closed, so $E_t \subseteq B$ for all t as well. ■

5a) Recalling that $\mathcal{N}$ is constructed by considering $\frac{\mathbb{R}\cap[0,1)}{\mathbb{Q}\cap[0,1)}$ and taking exactly one element from each equivalence class, we can note that if $E\subseteq\mathcal{N}$, then E contains a choice of <u>at most</u> one element from each equivalence class. We can then take a similar enumeration $\mathbb{Q}\cap[-1,1]=\{q_i\}_{i=1}^{\infty}$ and define $E_j := E+q_j$.

Then $E\subseteq\mathcal{N} \Rightarrow \bigsqcup_{j\in\mathbb{N}} E_j \subseteq \bigsqcup_{j\in\mathbb{N}} \mathcal{N}_j \subseteq [-1,2]$, and since E is measurable, we must have

$$\mu(E) = \mu\Big(\bigsqcup_{j\in\mathbb{N}} E_j\Big) = \sum_{j\in\mathbb{N}} \mu(E_j) = \sum_{j\in\mathbb{N}} \mu(E) \leq 3,$$

which can only hold if $m(E)=0$. $\square$

5b) Suppose $\mu(I\setminus\mathcal{N})<1$, so $m(I\setminus\mathcal{N})=1-2\varepsilon$ for some $\varepsilon>0$. Then choose an open $G\supseteq I\setminus\mathcal{N}$ such that $\mu(G)=\mu(I\setminus\mathcal{N})+\varepsilon = 1-\varepsilon$. Then $I\setminus G\subseteq\mathcal{N}$, and so by (1) we must have $\mu(I \setminus G) = 0$. But then

$$I = G \sqcup I \setminus G \Rightarrow \mu(I) = \mu(G) + \mu(I \setminus G)$$

$$\Rightarrow 1 = 1 - \varepsilon < 1, \text{ a contradiction. } \square$$

5c) Let

$$\left.\begin{aligned} E_1 &= \mathcal{N} \\ E_2 &= I \setminus \mathcal{N} \end{aligned}\right\} \Rightarrow I = E_1 \sqcup E_2$$

but $m_*(E_1) = m_*(\mathcal{N}) > 0$, otherwise $\mathcal{N}$ would be measurable so $m_*(E_1 \sqcup F_2) = 1$ but $m_*(E_1) + m_*(E_2) = 1 + \varepsilon$ for some $\varepsilon > 0$. $\blacksquare$

6a) Claim: E is a countable union of a countable intersection of measurable sets, and thus measurable.

Proof: Write $E = \{x \mid x \in E_j \text{ for infinitely many } j\}$, the claim is that

$$E = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j.$$

- $E \subseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose x is in infinitely many E_j. Then for any fixed k, there is some $M \geq k$ such that $x \in E_M \subseteq \bigcup_{j=k}^{\infty} E_j := S_k$. But this happens for every k, So $x \in \bigcap_{k=1}^{\infty} S_k$. $\square$

- $E \supseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose $x \in \bigcup_{j=k}^{\infty} E_j$ for every k. Then if x were in only finitely many E_j, we could pick a maximal E_M such that $k \geq M \Rightarrow x \notin E_k$, and so $x \notin \bigcup_{j=M}^{\infty} E_j$ – a contradiction. $\square$

<u>Claim:</u> $m(E)=0$

We'll use the fact that $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{j\to\infty} \sum_{n=j}^{\infty} a_n = 0$, i.e. the tails of a convergent sum must become arbitrarily small.

Since $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$, $E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k. So $m(E) \leq \sum_{j=k}^{\infty} E_j \to 0$, forcing $m(E)=0$. $\blacksquare$

(6b) Fix x and let $E_{p,j} = \{ x \in \mathbb{R} \mid |x - p/j| \leq 1/j^3 \}$

and $E_j = \bigcup_{\substack{p \text{ coprime} \\ \text{to } j}} E_{p,j} \subseteq \bigcup_{p=1}^{j} E_{p,j}$, and since $E_{p,j} \subseteq B(1/j^3, p/j)$, $m(E_{p,j}) \leq 2/j^3$ and thus $m(E_j) \leq q(2/j^3) = 2/j^2$.

But then $\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} 2/j^2 < \infty$. Moreover,

$$E=\bigcap_{j=1}^{\infty}\bigcup_{j=k}^{\infty} E_j = \{x\in\mathbb{R} \mid \text{there are infinitely many } j\text{'s such that there exists a } p \text{ coprime to } j \text{ s.t. } |x-p/j|\leq 1/j^3\},$$

which is precisely the set we want. So by (1), $m(E)=0$. ■