

Homotopy Groups of Spheres

Graduate Student Seminar

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Introduction

Outline

Homotopy
Groups of
Spheres

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Introduction

Spheres

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

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Homotopies of paths:



- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \mathbf{hom}_{\mathbf{Top}}(I, X)$$

$$H \in \mathbf{hom}_{\mathbf{Top}}(I \times I, X).$$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

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- Why care about path homotopies? Historically: contour integrals in \mathbb{C}



- By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_{\gamma} f(z) dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \text{hom}_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \text{hom}_{\text{Top}}(X \times I, Y)$$

such that $F(0) = f, F(1) = g$.

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] := \text{hom}_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \text{hom}(X, Y) \\ g \in \text{hom}(Y, X) \end{cases} \quad \text{such that} \quad \begin{cases} f \circ g \sim \text{id}_Y \\ g \circ f \sim \text{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \text{Top} \mid Y \sim X \right\}.$$

The Fundamental Group

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- $\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

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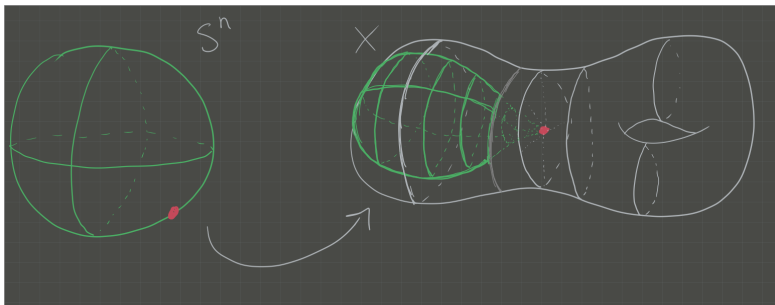
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- Can now generalize to define

$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look \mathbb{A}^1 or \mathbb{P}^1 .

Classification

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- Holy grail: understand the topological category completely
 - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category $DTop$, its interplay with $hoTop$ is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of \mathbb{R}^1 -bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G , the functor

$$\begin{aligned} h_G(\cdot) : \text{hoTop}^{\text{op}} &\longrightarrow \text{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

is representable by a space called BG (Brown's representability theorem).

- I.e., let $\text{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism

$$\text{Bun}_G(X) \cong [X, BG]$$

- In general, identify $G = \text{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n , take $G = GL(n, \mathbb{R})$.

Example: Homotopy Equivalence is Useful

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Note that for a poset of spaces (M_i, \hookrightarrow) , the space $M^\infty := \varinjlim M_i$. These are infinite dimensional “Hilbert manifolds”.

Proof:

$$\begin{aligned}\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, \mathrm{BGL}(1, \mathbb{R})] \\ &= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\ &= [X, \mathbb{RP}^\infty] \\ &= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\ &= H^1(X; \mathbb{Z}/2\mathbb{Z})\end{aligned}$$

Work being swept under the rug: identifying the homotopy type of the representing object.

Example: Homotopy Equivalence is Useful

Corollary: There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.



Example: Higher Homotopy Groups are Useful

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- Application: computing $\pi_1(\mathrm{SO}(n, \mathbb{R}))$ (rigid rotations in \mathbb{R}^n).
- The fibration $\mathrm{SO}(n, \mathbb{R}) \rightarrow \mathrm{SO}(n+1, \mathbb{R}) \rightarrow S^n$ yields a LES in homotopy:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(S^n) \\ & & & & & \swarrow & \\ & & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(S^n) \end{array}$$

Uses of Higher Homotopy

Knowing $\pi_k S^n$, this reduces to

$$\begin{array}{ccccccc} \cdots 0 & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & 0 \end{array}$$

- Thus $\pi_1(SO(3, \mathbb{R})) \cong \pi_1(SO(4, \mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(SO(3, \mathbb{R}))$.
- Use the fact that “accidental” homeomorphism in low dimension $SO(3, \mathbb{R}) \cong_{\text{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

Can also use the fact that $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is a double cover from the universal cover.

Uses of Higher Homotopy

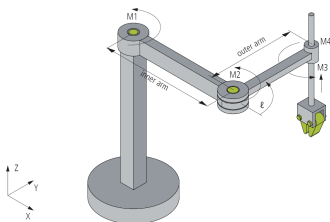
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- Important consequence: $SO(3, \mathbb{R})$ is not simply connected!
 - See “plate trick”: there is a loop of rotations that is not contractible, but squares to the identity.
- Robotics: paths in configuration spaces with singularities
- Computer graphics: smoothly interpolating between quaternions for rotated camera views



Rotation $R_{u,\theta}$:
axis u , angle θ

$$\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0q_1 - 2q_2q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_0q_3 - 2q_1q_2 \\ 2q_0q_2 - 2q_1q_3 & 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

Unit quaternion:

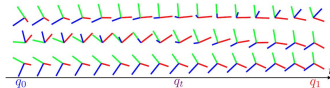
$$q = \cos(\theta/2) + (u_x i + u_y j + u_z k) \sin(\theta/2).$$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

Spherical Linear Interpolation (SLERP):

$$q_t \stackrel{\text{def}}{=} \frac{\sin((1-t)\omega)q_0 + \sin(t\omega)q_1}{\sin(\omega)}$$

\mathbb{R}^4



Spheres

Setup

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- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate:
 $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:

$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
\vdots											

Sphere 1

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