Homological Algebra

Problem Set 7

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Problem 1.0.1 (Weibel 7.2.1)

1. Let $f : M \to N$ be a morphism of $\mathfrak{g}$-modules over a field k. Show that the k-modules $\ker(f), \operatorname{im}(f), \operatorname{coker}(f)$ are the kernel, image, and cokernel respectively of f in the category $\mathfrak{g}$-Mod

2. Show that a monic (resp. epi) in $\mathfrak{g}$-Mod is also a monic (resp. epi) in k-Mod. Use (1) to show that $\mathfrak{g}$-Mod is an abelian category.

Solution:

Proof (of 1).
Note that there is an inclusion of sets

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N) &:= \left\{ f \in \operatorname{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M,N) \;\middle|\; f(gm) = gf(m)\ \forall g \in \mathfrak{g},\ \forall m \in M \right\} \\ &\subseteq \operatorname{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M,N),\end{aligned}$$

where we can regard M, N as k-modules by applying a forgetful functor $\mathfrak{g}$-$\mathsf{Mod} \to \mathsf{k}$-$\mathsf{Mod}$. This is in fact a k-submodule: if $f_1, f_2 \in \operatorname{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N)$ and $t \in k$, we have

$$(tf_1 + f_2)(gm) := tf_1(gm) + f_2(gm) = g \cdot tf_1(m) + g \cdot f_2(m) = g \cdot (tf_1(m) + f_2(m),$$

which shows $tf_1 + f_2 \in \operatorname{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N)$ and the one-step submodule test applies.

Moreover, kernels exist in $\mathfrak{g}$-Mod since they exist in k-Mod: given $f \in \operatorname{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N)$, using the submodule structure above we can identify $f \in \operatorname{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M,N)$ and produce $\ker f$ as a k-submodule of M. Using the kernels are set inclusions in categories of R-modules, we can define a $\mathfrak{g}$-module structure on $\ker f \leq M$ by restricting the $\mathfrak{g}$-action on M. Then the k-module inclusion $\iota : \ker f \hookrightarrow M$ is a morphism of $\mathfrak{g}$-modules, since $\iota(\ell) = \ell$ for $\ell \in \ker f$, so it is product-preserving:

$$g\iota(\ell) = g\ell = \iota(g\ell).$$

Similarly, $\operatorname{im}(f)$ in $\mathfrak{g}$-Mod is gotten by setting $\operatorname{im}(f) := \operatorname{im}_{\mathsf{k}\text{-}\mathsf{Mod}}(f)$ and restricting the $\mathfrak{g}$-action from N to $\operatorname{im}(f)$, and the cokernel is obtained as the quotient $\operatorname{coker}(f) := N/\operatorname{im}(f)$ with a $\mathfrak{g}$-module structure induced by the canonical quotient map. ■

Proof (of 2).
To see that monics in $\mathfrak{g}$-Mod are also monics in k-Mod, first consider the forgetful functor

$$F : \mathfrak{g}\text{-}\mathsf{Mod} \to \mathsf{k}\text{-}\mathsf{Mod}.$$

This is adjoint to the trivial $\mathfrak{g}$-module functor, yielding an adjunction

$$\mathfrak{g}\text{-}\mathsf{Mod} \underset{\mathrm{Triv}}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathsf{k}\text{-}\mathsf{Mod}.$$

We need to check that if $f : A \to B$ in $\mathfrak{g}$-Mod is monic, then its image $F(f) : F(A) \to F(B)$ is monic in k-Mod. Note that being a monomorphism is equivalent to having the following injections on hom sets for all $Z \in \mathfrak{g}\text{-}\mathsf{Mod}$:

$$\begin{aligned} f_* : \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(Z, A) &\hookrightarrow \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(Z, B) \\ h_i &\mapsto f \circ h_i. \end{aligned}$$

So the content of the problem is to check that for all $W \in \mathsf{k}\text{-}\mathsf{Mod}$, the following map is an injection:

$$\begin{aligned} F(f)_* : \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(W, F(A)) &\hookrightarrow \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(W, F(B)) \\ g_i &\mapsto F(f) \circ g_i. \end{aligned}$$

Using the adjunction, we have natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(W, F(A)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathrm{Triv}(W), A) \\ \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(W, F(B)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathrm{Triv}(W), B), \end{aligned}$$

and by assumption,

$$f_* : \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathrm{Triv}(W), A) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(\mathrm{Triv}(W), B),$$

since we can take $Z := \mathrm{Triv}(W)$. Since f_* is an injection, pushing it through the isomorphism shows that $F(f)_*$ is an isomorphism, and so any monic $\mathfrak{g}$-module morphism descends to a monic k-module morphism.

■

Problem 1.0.2 (Weibel 7.2.2)
For $M \in \mathsf{k}\text{-}\mathsf{Mod}$, let $E := \mathrm{End}_{\mathsf{k}\text{-}\mathsf{Mod}}(M) \in \mathsf{Alg}_{/k}$ be the associative algebra of k-module endomorphisms of M. Show that there is a correspondence

$$\left\{ \substack{\text{Maps } \mathfrak{g} \otimes M \to M \\ \text{making } M \text{ a } \mathfrak{g}\text{-module}} \right\} \rightleftharpoons \left\{ \substack{\text{Lie algebra morphisms} \\ \mathfrak{g} \to \mathrm{Lie}(E)} \right\}$$

Conclude that a $\mathfrak{g}$-module may also be described as an $M \in \mathsf{k}\text{-}\mathsf{Mod}$ together with a morphism of Lie algebras

$$\mathfrak{g} \to \mathrm{Lie}\left(\mathrm{End}_{\mathsf{k}\text{-}\mathsf{Mod}}(M)\right).$$

Solution:
Define maps

$$\{f : \mathfrak{g} \otimes_k M \to M \in \mathsf{k\text{-}Mod}\} \underset{\Psi}{\overset{\Theta}{\rightleftharpoons}} \{\mathfrak{g} \to \mathrm{Lie}\left(\mathrm{End}_{\mathsf{k\text{-}Mod}}(M)\right) \in \mathsf{Lie\text{-}Alg}\}$$
$$f \mapsto \Theta_f := (g \mapsto f(g \otimes -))$$
$$\Psi_h := (g \otimes m \mapsto h_g(m)) \leftarrow\!\shortmid h,$$

where $h_g(m) := h(g)(m)$, and we have $\Theta_f(g)(m) := f(g, m)$ and $\Psi_h(g, m) := h(g)(m)$.

Claim: Let $f : \mathfrak{g} \otimes_k M \to M$ be a k-module morphism. Θ_f defines a morphism of Lie algebras.

Proof (?).
Write $[-,-]_{\mathfrak{g}}$ for the bracket on $\mathfrak{g}$ and $[-,-]_{\mathrm{End}}$ for the bracket on $\mathrm{Lie}(\mathrm{End}_{\mathsf{k\text{-}Mod}}(M))$ defined by $[x, y]_{\mathrm{End}} := x \circ y - y \circ x$, it then suffices to check the following identity:

$$[\Theta_f(x), \Theta_f(y)]_{\mathrm{End}} = \Theta_f\left([x, y]_{\mathfrak{g}}\right).$$

Expanding the right-hand side, we have

$$\begin{aligned}
[\Theta_f(x), \Theta_f(y)]_{\mathrm{End}} &:= \Theta_f(x) \circ \Theta_f(y) - \Theta_f(y) \circ \Theta_f(x) \\
&= f(x \otimes -) \circ f(y \otimes -) - f(y \otimes -) \circ f(x \otimes -) \\
&= f(x \otimes f(y \otimes -)) - f(y \otimes f(x \otimes -)),
\end{aligned}$$

and now using multiplicative notation to write $gm := f(g, m)$, evaluating this on an element $m \in M$ yields

$$\left(f(x \otimes f(y \otimes -)) - f(y \otimes f(x \otimes -))\right)(m) = x(ym) - y(xm).$$

For the right-hand side, we have

$$\begin{aligned}
\Theta_f\left([x, y]_{\mathfrak{g}}\right)(m) &:= f([x, y]_{\mathfrak{g}} \otimes m) \\
&= [x, y]_{\mathfrak{g}}(m) \\
&:= x(ym) - y(xm),
\end{aligned}$$

where in the last step we've used that f is a structure map that makes M into a $\mathfrak{g}$-module. So the two sides agree on every element of M, and are thus equal as k-module endomorphisms of M.

■

Claim: Let $h : \mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_{\mathsf{k\text{-}Mod}}(M))$ be a morphism of Lie algebras. Ψ_h defines a morphism of k-modules.

Proof (?).
It suffices to check $\Psi_h(rx+y) = r\left(\Psi_h(x) + \Psi_h(y)\right)$ for x, y in the domain $\mathfrak{g} \otimes M$ and $r \in k$. This follows from a computation: on elementary tensors, we have

$$\begin{aligned}
\Psi_h(r(g_1 \otimes m_1) + (g_2 \otimes m_2)) &= \Psi_h(r(g_1 + g_2) \otimes (m_1 + m_2)) \\
&:= h(r(g_1 + g_2))(m_1 + m_2) \\
&= r\left(h(g_1) + h(g_2)\right)(m_1 + m_2) \quad \text{using } r\text{-linearity of } h \\
&= r\left(h(g_1) + h(g_2)\right)(m_1 + m_2) \\
&= r\left(h(g_1) + h(g_2) \otimes m_1 + m_2\right) \\
&= r\left((h(g_1) \otimes m_1) + (h(g_2) \otimes m_2)\right) \\
&= r\left(\Psi_h(g_1 \otimes m_1) + \Psi_h(g_2 \otimes m_2)\right),
\end{aligned}$$

and extending by linearity shows that Ψ_h is k-linear.

■

Claim: Ψ_h makes M into a $\mathfrak{g}$-module.

Proof (?).
In multiplicative notation, the condition we need to check is the following:

$$[x,y]m = x(ym) - y(xm) \qquad \forall x, y \in \mathfrak{g},\ m \in M.$$

Writing this explicitly, we want

$$\Theta_h([xy], m) = \Theta_h(x, \Theta_h(y, m)) - \Theta_h(y, \Theta_h(x, m)).$$

This follows from a computation:

$$\begin{aligned}
\Theta_h([xy]_{\mathfrak{g}}, m) &:= h([xy]_{\mathfrak{g}})(m) \\
&= \left([h(x), h(y)]_{\mathrm{End}}\right)(m) \quad * \\
&:= \left(h(x) \circ h(y) - h(y) \circ h(x)\right)(m) \\
&= h(x)(h(y)(m)) - h(y)(h(x)(m)) \\
&:= \Theta_h(x, \Theta_h(y, m)) - \Theta_h(y, \Theta_h(x, m)),
\end{aligned}$$

where in the line marked $*$ we've used that h was a Lie algebra morphism and thus we can commute the brackets.

■

Claim: Θ and Ψ are mutually inverse.

Proof (?).
Starting with $f : \mathfrak{g} \otimes M \to M$, we obtain $\Theta_f := (g \mapsto f(g \otimes -))$. Applying Ψ yields

$$\begin{aligned}\Psi_{\Theta_f} &:= (g \otimes m \mapsto \Theta_f(g)(m)) \\ &:= (g \otimes m \mapsto f(g \otimes -)(m)) \\ &:= (g \otimes m \mapsto f(g \otimes m)),\end{aligned}$$

and so this recovers f.

Similarly, starting now with $h : \mathfrak{g} \to \mathfrak{gl}(M)$ and letting $h_g := h(g)$, applying Ψ yields $\Psi_h := (g \otimes m \mapsto h_g(m))$, and applying Θ yields

$$\begin{aligned}\Theta_{\Psi_h} &:= (g \mapsto \Psi_h(g \otimes -)) \\ &:= (g \mapsto (g \otimes - \mapsto h_g(-))) \\ &:= (g \mapsto h_g),\end{aligned}$$

which recovers h.

■

The main result now follows.

Problem 1.0.3 (Weibel 7.3.2)
Given an $M \in \mathsf{k}\text{-}\mathsf{Mod}$, consider the Lie algebra $\operatorname{Lie}(T(M))$ underlying the tensor algebra $T(M)$. Let $\mathfrak{f}$ denote the Lie subalgebra generated by M, so elements of $\mathfrak{f}$ are iterated brackets of elements:

$$x \in \mathfrak{f} \implies x = \sum [x_1\, [x_2\, [\cdots x_n]]] \qquad\qquad x_i \in M.$$

Show that $\mathfrak{f}$ satisfies the universal property of a free Lie algebra of M (see 7.1.5).

Solution:
It suffices to show that the following map is an isomorphism of hom sets:

$$\begin{aligned}\operatorname{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M, \operatorname{Forget}(\mathfrak{g})) &\xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Lie}\text{-}\mathsf{Alg}}(\mathfrak{f}, \mathfrak{g}) \\ f &\mapsto \tilde{f} \\ \bar{g} &\leftarrow\!\shortmid g,\end{aligned}$$

where $\bar{g}$ is the restriction of g to only those sums with a single term and no iterated brackets. To define $\tilde{f}$, first write $[-,-] := [-,-]_{\mathfrak{g}}$ for the bracket on $\mathfrak{g}$. Using the compatibility of f with sums and brackets on $\mathfrak{g}$, given an element $x := \sum [x_1, [x_2, \cdots, x_n]] \in \mathfrak{f}$, we can pass f

through each iterated bracket inductively:

$$\begin{aligned} f(x) &:= \tilde{f}\left(\sum [x_1, [x_2, \cdots, x_n]]\right) \\ &= \sum f\left([x_1, [x_2, \cdots, x_n]]\right) \\ &= \sum [f(x_1), f\left([x_2, \cdots, x_n]\right)] \\ &= \sum [f(x_1), [f(x_2), f\left(\cdots, x_n\right)]] \\ &= \sum [f(x_1), [f(x_2), \cdots, f(x_n)]]. \end{aligned}$$

So we can extend $f : M \to \mathfrak{g}$ to $\tilde{f} : \mathfrak{f} \to \mathfrak{g}$ by linearity and bracketing to make the following definition:

$$\tilde{f}(x) := \tilde{f}\left(\sum [x_1, [x_2, \cdots, x_n]]\right) := \sum [f(x_1), [f(x_2), \cdots, f(x_n)]].$$

Since $\tilde{f} : \mathfrak{f} \to \mathfrak{g}$ is a morphism on the underlying k-modules, it remains to check that it defines a morphism of Lie algebras. For this to be true, we need that

$$\tilde{f}\left([x, y]_{\mathfrak{f}}\right) = [\tilde{f}(x), \tilde{f}(y)],$$

where since $\mathfrak{f} \leq \mathrm{Lie}(T(M))$ is a subalgebra of the tensor algebra, its bracket is defined by $[x, y]_{\mathfrak{f}} := [xy - yx]$. Since both brackets are bilinear, it suffices to check this on sums with a single term and extend by linearity. Moreover since $\tilde{f}$ is defined inductively, it suffices to check on a single iteration of bracketing, in which case we have

Remark 1.0.1: Note: I don't see a clear way to get this result. Since f *is* a morphism of k-modules, one can easily get something like

$$f([x, y]_{\mathfrak{f}}) := f(xy - yx) = f(x)f(y) - f(y)f(x) = [f(x), f(y)]_{\mathfrak{g}'},$$

where $\mathfrak{g}'$ is the same underlying k-module as $\mathfrak{g}$ but with the bracket defined as $[a, b]_{\mathfrak{g}'} := ab - ba$. However, this isn't a priori related to the original bracket $[a, b]_{\mathfrak{g}}$.

Problem 1.0.4 (Weibel 7.3.4)
Let $M, N \in \mathfrak{g}\text{-}\mathsf{Mod}$ and make $\mathrm{Hom}_{\mathsf{k\text{-}Mod}}(M, N)$ into a $\mathfrak{g}$-module via the action

$$(xf)(m) := xf(m) - f(xm) \qquad x \in \mathfrak{g}, m \in M.$$

Show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{k\text{-}Mod}}(M, N)^{\mathfrak{g}} \qquad \in \mathfrak{g}\text{-}\mathsf{Mod}.$$

Solution:
The first claim is that these are equal as sets. This follows because for all $x \in \mathfrak{g}$, for f to be a

morphism of $\mathfrak{g}$-modules it must be product-preserving, and so we have:

$$\begin{aligned} f \in \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N) &\iff xf(m) = f(xm) && \forall m \in M \\ &\iff xf(m) - f(xm) = 0_M && \forall m \in M \\ &\iff x \cdot f = 0 \\ &\iff f \in \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M,N)^{\mathfrak{g}}, \end{aligned}$$

where we've used the definition $A^{\mathfrak{g}} := \left\{ a \in A \;\middle|\; ga = 0_A \ \forall g \in \mathfrak{g} \right\}$. So if we define a map

$$\begin{aligned} \tilde{F} : \mathrm{Hom}_{\mathfrak{g}\text{-}\mathsf{Mod}}(M,N) &\to \mathrm{Hom}_{\mathsf{k}\text{-}\mathsf{Mod}}(M,N)^{\mathfrak{g}} \\ f &\mapsto \tilde{F}_f := F(f) \end{aligned}$$

where $F : \mathfrak{g}\text{-}\mathsf{Mod} \to \mathsf{k}\text{-}\mathsf{Mod}$ is the forgetful functor, the above argument shows that this is a bijection. It only remains to check that $\tilde{F}$ is a morphism of $\mathfrak{g}$-modules, but this follows from the fact that

$$f(m) = F(f)(m) := \tilde{F}_f(m),$$

i.e. f and $F(f)$ are pointwise defined in precisely the same way on the underlying sets. So $\tilde{F}(xf) = x\tilde{F}(f)$, since this already holds for f, making $\tilde{F}$ product-preserving.

Problem 1.0.5 (Weibel 7.7.1)
In the following construction, verify that $d^2 = 0$ and conclude that $V_* \in \mathsf{Ch}(\mathfrak{g}\text{-}\mathsf{Mod})$.

Throughout this section $\mathfrak{g}$ will denote a Lie algebra over k that is free as a k-module. We shall construct the $U\mathfrak{g}$-module chain complex $V_*(\mathfrak{g})$ originally used by C. Chevalley and S. Eilenberg [ChE] in 1948 to define $H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$.

Let $\Lambda^p\mathfrak{g}$ denote the p^{th}-exterior product of the k-module $\mathfrak{g}$, which is generated by monomials $x_1 \wedge \cdots \wedge x_p$ with $x_i \in \mathfrak{g}$; see 4.5.1 above. Our chain complex has $V_p(\mathfrak{g}) = U\mathfrak{g} \otimes_k \Lambda^p\mathfrak{g}$; since $\Lambda^p\mathfrak{g}$ is a free k-module, $V_p(\mathfrak{g})$ is free as a left $U\mathfrak{g}$-module. By convention, $\Lambda^0\mathfrak{g} = k$ and $\Lambda^1\mathfrak{g} = \mathfrak{g}$, so $V_0 = U\mathfrak{g}$ and $V_1 = U\mathfrak{g} \otimes_k \mathfrak{g}$. We define $\varepsilon : V_0(\mathfrak{g}) = U\mathfrak{g} \to k$ to be the augmentation 7.3.5 and $d : V_1(\mathfrak{g}) \to V_0(\mathfrak{g})$ to be the product map $d(u \otimes x) = ux$ from $U\mathfrak{g} \otimes \mathfrak{g}$ to $U\mathfrak{g}$ whose image is the augmentation ideal $\mathfrak{J}$. By 7.3.5, we have an exact sequence

$$V_1(\mathfrak{g}) \xrightarrow{d} V_0(\mathfrak{g}) \xrightarrow{\varepsilon} k \to 0.$$

Definition 7.7.1 For $p \geq 2$, let $d: V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ be given by the formula $d(u \otimes x_1 \wedge \cdots \wedge x_p) = \theta_1 + \theta_2$, where (for $u \in U\mathfrak{g}$ and $x_i \in \mathfrak{g}$):

$$\theta_1 = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p;$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [x_i x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p.$$

(The notation $\hat{x}_i$ indicates an omitted term.) For example, if $p = 2$, then

$$d(u \otimes x \wedge y) = ux \otimes y - uy \otimes x - u \otimes [xy].$$

$V_*(\mathfrak{g})$ with this differential is called the *Chevalley-Eilenberg complex*. It is sometimes also called the *standard complex*.

Hint: write $d(\theta_i) = \theta_{i,1} + \theta_{i,2}$ and show that $-\theta_{i,1}$ is the $i = 1$ part of $\theta_{2,1}$ and $\theta_{2,2} = 0$.
Then show that $-\theta_{1,2}$ is the $i > 1$ part of $\theta_{2,1}$.

Solution:

Todo: start writing calculation.