Problem Set 5

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1 4.3

Proposition 1.1.
Suppose $\lambda+\rho \in \Lambda^{+}$. Then $M(w \cdot \lambda) \subset M(\lambda)$ for all $w \in W$. Thus all $[M(\lambda): L(w \cdot \lambda)]>0$.

More precisely, if $w=s_{n} \cdots s_{1}$ is a reduced expression for w in terms of simple reflections corresponding to roots α_{i}, then there is a sequence of embeddings:

$$M(w \cdot \lambda)=M\left(\lambda_{n}\right) \subset M\left(\lambda_{n-1}\right) \subset \cdots \subset M\left(\lambda_{0}\right)=M(\lambda)$$

Here

$$\begin{gathered}\lambda_{0}:=\lambda, \lambda_{k}:=s_{k} \cdot \lambda_{k-1}=\left(s_{k} \ldots s_{1}\right) \cdot \lambda \Longrightarrow \lambda_{n}=s_{n} \cdot \lambda_{n-1}=w \cdot \lambda \\ w \cdot \lambda=\lambda_{n} \leq \lambda_{n-1} \leq \cdots \leq \lambda_{0}=\lambda \text{with} \quad\left\langle\lambda_{k}+\rho, \alpha_{k+1}^{\vee}\right\rangle \in \mathbb{Z}^{+} \text { for } k=0, \cdots, n-1 .\end{gathered}$$

Assume $\lambda+\rho \in \Lambda^{+}$.

a. Prove that the unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_{\diamond} \cdot \lambda)$, where $w_{\diamond}$ is the longest element of W.

b. In case $\lambda \in \Lambda^+$, show that the inclusions obtained in the above proposition are all proper.

2 4.6

Theorem 2.1 ***(Verma).***
Let $\lambda \in \mathfrak{h}^\vee$. Given $\alpha > 0$, suppose $\mu \coloneqq s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Work through the steps of Verma's Theorem in the special case discussed in the previous problem

2.1 Solution

Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and identify its root system A_2 with $\Delta = \{\alpha, \beta\}$ and $\Phi^+ = \{\alpha, \beta, \gamma \coloneqq \alpha + \beta\}$ We can also identify the Weyl group as $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\gamma\}$ where there is a reduced expression $s_\gamma = w_0 = s_\alpha s_\beta s_\alpha$.

We can begin by letting $\lambda \in \Lambda$ be an arbitrary integral weight and let $\mu \neq \lambda$ be an arbitrary weight linked to λ, where WLOG apply some Weyl group element to μ to place it in the dominant chamber and assume

$$\mu \coloneqq s_\alpha \cdot \lambda < \lambda$$

(where the inequality is strict).

2.1.1 Part 1

Since μ is assumed integral, we can find some $w \in W$ such that

$$\mu' \coloneqq w^{-1} \cdot \mu \in \Lambda^+ - \rho.$$

Claim: $w = s_\alpha s_\beta$, so $w^{-1} = s_\beta s_\alpha$ and thus

$$\mu' = s_\beta s_\alpha \cdot \mu$$

As in Proposition 4.3, we then write

$$\begin{aligned}
\mu_0 &= \mu' \\
\mu_1 &= s_\beta \cdot \mu' \\
\mu_2 &= s_\alpha s_\beta \cdot \mu' = w \cdot \mu' = \mu
\end{aligned}$$

which satisfies

$$\mu = \mu_2 \leq \mu_1 \leq \mu_0 = \mu'$$
$$\mu = s_\alpha s_\beta \cdot \mu' \leq s_\beta \mu' \leq \mu'.$$

which (by the proposition) gives a sequence of embeddings

$$M(\mu) = M(\mu_2) \hookrightarrow M(\mu_1) \hookrightarrow M(\mu_0) = M(\mu')$$
i.e.
$$M(\mu) = M(s_\alpha s_\beta \cdot \mu') \hookrightarrow M(s_\beta \cdot \mu') \hookrightarrow M(\mu').$$

2.1.2 Step 2

We now define

$$\lambda' := w^{-1}\lambda = s_\beta s_\alpha \cdot \lambda$$

and the parallel list of weights

$$\begin{aligned}
\lambda_0 &= \lambda' \\
\lambda_1 &= s_\beta \cdot \lambda' \\
\lambda_2 &= s_\alpha s_\beta \cdot \lambda' := \lambda.
\end{aligned}$$

We can similarly use the fact that $\lambda \neq \mu \implies \mu_k \neq \lambda_k$ for any k.

2.1.3 Step 3

To relate μ_k to λ_k, We now define $w_k = s_n \cdots s_{k+1}$:

$$\begin{aligned}
w_0 &= s_\alpha s_\beta \\
w_1 &= s_\alpha \\
w_2 &:= 1
\end{aligned}$$

and using the calculation

$$\mu_k = w_k^{-1} s_\alpha w_k \cdot \lambda_k = s_{\beta_k} \cdot \lambda_k$$

we compute

$$\begin{aligned}
s_{\beta_0} &= (s_\alpha s_\beta)^{-1} s_\alpha (s_\alpha s_\beta) = s_\gamma \\
s_{\beta_1} &= s_\alpha^{-1} s_\alpha s_\alpha = s_\alpha \\
s_{\beta_2} &:= s_\alpha
\end{aligned}$$

and thus obtain

$$\begin{aligned}
\mu_0 &= s_\alpha \cdot \lambda_0 \\
\mu_1 &= s_\alpha \cdot \lambda_1 \\
\mu_2 &= s_\gamma \cdot \lambda_2.
\end{aligned}$$

2.1.4 Step 4

We have $\mu_0 \geq \mu_1 \geq \mu_2$ with $\lambda_0 < \mu_0$ but $\lambda_2 > \mu_2$, so we now look for where the inequality switches. It suffices to check how μ_1 and λ_1 are related, and we find $\mu_1 < \lambda_1$.

2.1.5 Step 5

From the last step, we fix $k = 0$ and now want to show $M(\mu_{k+i}) \subset M(\mu_{k+i})$ for $i = 1, 2$, since the $i = 2$ case yields the desire $M(\mu) \subset M(\lambda)$.

2.1.6 Step 6

We first want to show $M(\mu_1) \subset M(\lambda_1)$. We write

$$\mu_1 - \lambda_1 = s_1\mu_0 - s_1\lambda_0.$$

We then note that

$$\begin{aligned} \mu_1 - \lambda_1 &= c_1\beta_1 \\ s_\alpha\mu_0 - s_1\lambda_0 = s_\alpha(\mu_0 - \lambda_0) &= d_1\beta_0 \end{aligned}$$

where c_1 is negative and b_1 is positive, and we already know that $\beta_1 = \beta_0 = \alpha$ by a direct computation. Thus we have $\mu_1 = s_\alpha\lambda_1$, and applying Proposition 1.4,

$$M(s_\alpha \cdot \lambda_1) \hookrightarrow M(\lambda_1) \implies M(\mu_1) \hookrightarrow M(\lambda_1).$$

3 4.11

In the case of $\mathfrak{sl}(3, \mathbb{C})$, what can be said at this point about Verma modules with a singular integral highest weight?

> Aside from the trivial case $-\rho$, a typical linkage class has 3 elements. For example, if λ lies in the α hyperplane and is antidominant, the linked weights are $\lambda,\ s_\beta \cdot \lambda,\ s_\alpha s_\beta \cdot \lambda$.