Title

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Last time: we saw the Leray spectral sequence, but no examples yet, so that's what we'll do now. We had $X \xrightarrow{f} Y \xrightarrow{g} Z$ to which we associated the spectral sequence $R^i f_* R^j f_*(\cdot) \Rightarrow R^{i+j}(g \circ f)_*(\cdot)$. To deduce existence we used that pushforwards preserve injectives, and we looked at some E_2 differentials.

Example 1.0.1 ***(?):*** Let $X \xrightarrow{\pi} Z := \operatorname{Spec} k$, where $k \neq \bar{k}$ necessarily. The spectral sequence for the functors π_*, Γ yields the Leray spectral sequence $H^i(k, R^j \pi_* \mathcal{F}) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathcal{F})$. The LHS is the étale cohomology of $\operatorname{Spec} k$, i.e. Galois cohomology. The Galois module corresponding to $R^j \pi_* \mathcal{F}$ is $H^j(X_{k^s}, \mathcal{F})$ by taking the $\bar{k}$ points of this functor So the Leray spectral sequence yields

$$H^i(k, H^j(X_{k^s, \text{ét}}, \mathcal{F})) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathcal{F}).$$

Consider k a finite field and $X_{/k}$ a smooth projective variety. Then the Galois cohomology is given by

$$H^i(k, V) = \begin{cases} V^G & i = 0 \qquad \text{the invariants} \\ V_G & i = 1 \qquad \text{the coinvariants.} \end{cases}$$

This follows from computing the cohomology of $\widehat{\mathbb{Z}}$. Supposing we knew that the cohomological dimension of a smooth projective variety was $2n$ over $\bar{k}$ (e.g. taking $\mathcal{F} := \mathbb{Z}/\ell\mathbb{Z}$ above), then the cohomological dimension of X would be $2n + 1$. This follows from E_2 vanishing for $i > 1$.