

Homotopy Groups of Spheres

Graduate Student Seminar

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Introduction

Outline

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Introduction

Spheres

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

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Homotopies of paths:



- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \mathbf{hom}_{\mathbf{Top}}(I, X)$$

$$H \in \mathbf{hom}_{\mathbf{Top}}(I \times I, X).$$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

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- Why care about path homotopies? Historically: contour integrals in \mathbb{C}



- By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_{\gamma} f(z) dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \text{hom}_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \text{hom}_{\text{Top}}(X \times I, Y)$$

such that $F(0) = f, F(1) = g$.

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] := \text{hom}_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \text{hom}(X, Y) \\ g \in \text{hom}(Y, X) \end{cases} \quad \text{such that } \begin{cases} f \circ g \sim \text{id}_Y \\ g \circ f \sim \text{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \text{Top} \mid Y \sim X \right\}.$$

The Fundamental Group

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- $\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

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- Can now generalize to define

$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look \mathbb{A}^1 or \mathbb{P}^1 .

Classification

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- Holy grail: understand the topological category completely
 - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category $D\text{Top}$, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of \mathbb{R}^1 -bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G , the functor

$$\begin{aligned} h_G(\cdot) : \text{hoTop}^{\text{op}} &\longrightarrow \text{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

is representable by a space called BG (Brown's representability theorem).

- I.e., let $\text{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism

$$\text{Bun}_G(X) \cong [X, BG]$$

- In general, identify $G = \text{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n , take $G = GL(n, \mathbb{R})$.

Example: Homotopy Equivalence is Useful

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Note that for a poset of spaces (M_i, \hookrightarrow) , the space $M^\infty := \varinjlim M_i$. These are infinite dimensional “Hilbert manifolds”.

Proof:

$$\begin{aligned}\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, \mathrm{BGL}(1, \mathbb{R})] \\ &= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\ &= [X, \mathbb{RP}^\infty] \\ &= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\ &= H^1(X; \mathbb{Z}/2\mathbb{Z})\end{aligned}$$

Work being swept under the rug: identifying the homotopy type of the representing object.

Example: Homotopy Equivalence is Useful

Corollary: There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.



Example: Higher Homotopy Groups are Useful

- Application: computing $\pi_1(\mathrm{SO}(n, \mathbb{R}))$ (rigid rotations in \mathbb{R}^n).
- The fibration

$$\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$$

yields a LES in homotopy:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(S^n) \\ & & & & & \swarrow & \\ & & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(S^n) \end{array}$$

Uses of Higher Homotopy

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Knowing $\pi_k S^n$, this reduces to

$$\begin{array}{ccccccc} \cdots 0 & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & 0 \end{array}$$

- Thus $\pi_1(SO(3, \mathbb{R})) \cong \pi_1(SO(4, \mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(SO(3, \mathbb{R}))$ (stabilization)
- Use the fact that “accidental” homeomorphism in low dimension $SO(3, \mathbb{R}) \cong_{\text{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

Can also use the fact that $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is a double cover from the universal cover.

Uses of Higher Homotopy

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- Important consequence: $SO(3, \mathbb{R})$ is not simply connected!
- See "plate trick": non-contractible loop of rotations that squares to the identity.
- Robotics: paths in configuration spaces with singularities
- Computer graphics: smoothly interpolating between quaternions for rotated camera views



Rotation $R_{u,\theta}$:

axis u , angle θ

$$\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0q_1 - 2q_2q_3 & 2q_0q_2 + 2q_1q_3 & 2q_0q_3 - 2q_1q_2 \\ 2q_1q_0 + 2q_2q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & q_1^2 - q_2^2 - q_0^2 + q_3^2 \\ 2q_2q_0 - 2q_1q_3 & 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2q_2q_1 - 2q_0q_2 \\ 2q_3q_0 + 2q_1q_2 & q_1^2 - q_2^2 - q_0^2 + q_3^2 & 2q_0q_2 - 2q_1q_3 & q_0^2 + q_1^2 - q_2^2 - q_3^2 \end{pmatrix}$$

Unit quaternion:

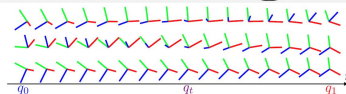
$$q = \cos(\theta/2) + (u_x i + u_y j + u_z k) \sin(\theta/2).$$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

Spherical Linear Interpolation (SLERP):

$$q_t \stackrel{\text{def}}{=} \frac{\sin((1-t)\omega)q_0 + \sin(t\omega)q_1}{\sin(\omega)}$$

\mathbb{R}^4



Spheres

Setup

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- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate:
 $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:

$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
\vdots											

But Wait!

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The corresponding picture in homology is very easy:

$H_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$	\mathbb{Z}										
2		\mathbb{Z}									
3			\mathbb{Z}								
4				\mathbb{Z}							
5					\mathbb{Z}						
6						\mathbb{Z}					
7							\mathbb{Z}				
8								\mathbb{Z}			
9									\mathbb{Z}		
10										\mathbb{Z}	
\vdots											

Slogan: "conservation/duality of complexity"

History

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- 1895: Poincare, *Analysis situs* (“the analysis of position”) in analogy to Euler *Geometria situs* in 1865 on the Kongisberg bridge problem
 - Studies spaces arising from gluing polygons, polyhedra, etc (surfaces!), first use of “algebraic invariant theory” for spaces by introducing π_1 and homology.
- 1920s: Rigorous proof of classification of surfaces (Klein, Möbius, Clifford, Dehn, Heegard)
 - Captured entirely by π_1 (equivalently, by genus and orientability).
- **1931: Hopf discovers a nontrivial (not homotopic to identity) map $S^3 \longrightarrow S^2$**

History

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- 1932/1935: Cech (indep. Hurewicz) introduce higher homotopy groups, gives map relating $\pi_* \rightarrow H_*$, shows $\pi_n X$ are **abelian** groups for $n \geq 2$.
 - Withdrew his paper because of this theorem!
- 1951: Serre uses spectral sequences to show that **all groups $\pi_k S^n$ are torsion except,**
 - $k = n$, since $\pi_n S^n = \mathbb{Z}$
 - $k \equiv 3 \pmod{4}, n \equiv 0 \pmod{2}$, then $\mathbb{Z} \oplus T$
 - Tight bounds on where p -torsion can occur.
- 1953: Whitehead shows the homotopy groups of spheres split into stable and unstable ranges.

Today: We know $\pi_{n+k} S^n$ for

- $k \leq 64$ when $n \geq k + 2$ (stable range)
- $k \leq 19$ when $n < k + 2$ (unstable range)
- We *only* have a complete list for S^0 and S^1 , and know *no* patterns beyond this!
 - Open for ~ 80 years.

Spheres

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We'll fill out as much of this table as is easily known:

		$\pi_k(S^n)$										
		$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$												
2												
3												
4												
5												
6												
7												
8												
9												
10												
:												

$$k < n$$

Claim: $[S^k, S^n] = 0$ for $k < n$.



This follows easily from CW approximation:

Any map $X \xrightarrow{f} Y$ between CW complexes is homotopic to a cellular map.

$k < n$: CW Complexes

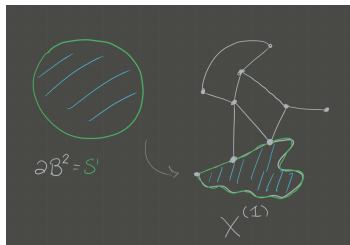
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- Analogy from analysis: C^1 functions dense in L^2 .
 - If you're just computing homotopy groups, *any* space can be replaced with a *weakly equivalent* CW complex.



$k < n$: CW Complexes

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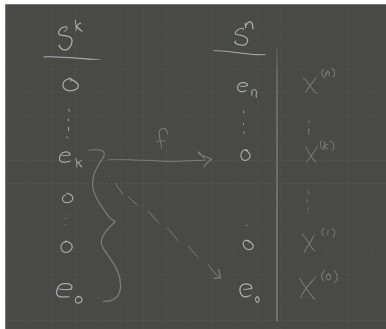
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AT1 can show that spheres have a simple cell decomposition

$$S^k = e_0 \amalg_f e_k$$

Thus any map $f : S^k \rightarrow S^n$ must send the k -skeleton of S^k to the k -skeleton of S^n , which is just a point:



$k \geq 1, n = 1$: Covering Space Theory

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Claim: $\pi_1 S^1 = \mathbb{Z}$ and $\pi_{\geq 2} S^1 = 0$.

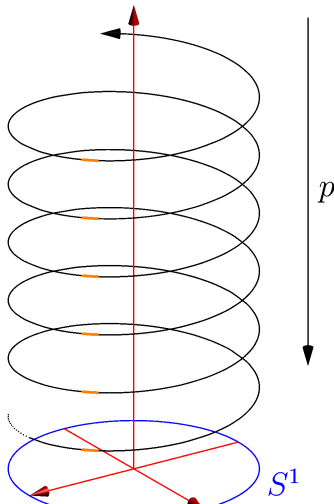
$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
...											

$k \geq 1, n = 1$: Covering Space Theory

- Use the fact that $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$ is a covering space and $\mathbb{Z} \curvearrowright \mathbb{R}$ freely.

\mathbb{R}



$k \geq 1, n = 1$: Covering Space Theory

Theorem: If $F \longrightarrow E \longrightarrow B$ is a *Serre Fibration* then there is a LES in homotopy

$$\begin{array}{ccccccc} & & & \dots & \longrightarrow & \pi_{k+1}(B) & \\ & & & & \swarrow & & \\ \pi_k(F) & \longrightarrow & \pi_k(E) & \longrightarrow & \pi_k(B) & & \\ & & & \nwarrow & & & \\ \pi_{k-1}(F) & \longrightarrow & \dots & & \dots & & \end{array}$$

- If $\tilde{X} \longrightarrow X$ is a universal cover then $\pi_{\geq 2}(X) \cong \pi_{\geq 2}\tilde{X}$.
 - Proof coming up!

Misc: Serre Fibrations

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Claim: $\pi_2 S^2 = \pi_3 S^2 = \pi_3 S^3 = \mathbb{Z}$.

$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	\dots
$n = 1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
\vdots											

Misc: Serre Fibrations

Use the Hopf fibration: $S^1 \longrightarrow S^3 \longrightarrow S^2$ and the fact that $\pi_{\geq 2} S^1 = 0$:

$$\begin{array}{ccccc} \pi_3(S^1) & \longrightarrow & \pi_3(S^3) & \longrightarrow & \pi_3(S^2) \\ & & \swarrow & & \\ \pi_2(S^1) & \longrightarrow & \pi_2(S^3) & \longrightarrow & \pi_2(S^2) \\ & & \swarrow & & \\ \pi_1(S^1) & \longrightarrow & \pi_1(S^3) & \longrightarrow & \pi_1(S^2) \end{array}$$

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_4(S^3) & \longrightarrow & \pi_4(S^2) \\ & & \swarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3(S^2) \\ & & \swarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \pi_2(S^2) \\ & & \swarrow & & \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Note that this works whenever the fiber is contractible (e.g. universal covers, fibers are discrete)

- Hopf Fibration Visualizer

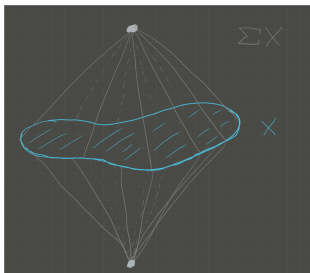
$n = k$: Stabilization

- Theorem (1937, Freudenthal): For $k \gg 0$,

$$[\Sigma^k X, \Sigma^k Y] \cong [\Sigma^{k+1} X, \Sigma^{k+1} Y]$$

- Use the fact that $\Sigma S^k \cong S^{k+1}$, then in some *stable range*

$$\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1}$$



Fun note: corresponds to “smash with a sphere”

$n=k$: Stabilization

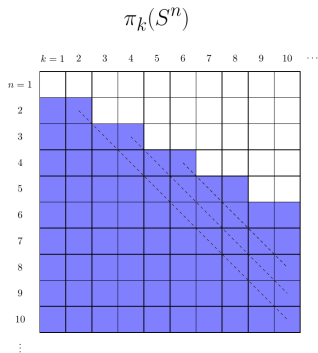
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Stable range:



$n = k$: Stabilization

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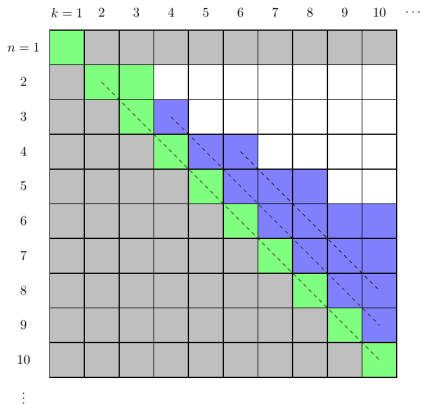
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We can thus suspend things we already know:

$$\pi_k(S^n)$$



$$k = 4, n = 3$$

- Construct a map $S^3 \rightarrow K(\mathbb{Z}, 2)$ by “killing off homotopy” (identify K)
- Convert to a fibration and take the homotopy fiber to get $F \rightarrow S^2 \rightarrow \mathbb{CP}^\infty$
- By LES, $\pi_{\geq 3} F \cong \pi_{\geq 3} S^2$, by Hurewicz $\pi_3 F \cong H_3 F$, which we can compute
- Kill homotopy again and *iterated* homotopy fiber to get $G \rightarrow F \rightarrow K(\mathbb{Z}, 3)$
- By LES, $\pi_{\geq 4} G \cong \pi_{\geq 4} S^2$, by Hurewicz $\pi_4 G \cong H_4 G$.

$$k=4, n=3$$

- Look at Serre's spectral sequence for $G \rightarrow F \rightarrow K(\mathbb{Z}, 3)$.

$$H_5 G$$

$$H_4 G$$

$$0$$

$$0$$

$$0$$

$$\mathbb{Z}$$

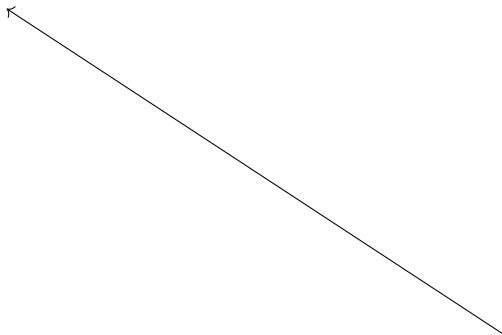
$$0$$

$$0$$

$$\mathbb{Z}$$

$$?$$

$$?$$



$$k=4, n=3$$

- Want target of arrow. Need to know
 - $H^4 F, H^5 F$ (total cohomology)
 - $H^5 K(\mathbb{Z}, 3)$ (source of arrow)
- Use Serre SS on $F \rightarrow S^2 \rightarrow K(\mathbb{Z}, 2)$ to deduce $H^5 F = H^6 F = \mathbb{Z}$
- Use Serre SS on $\Omega K(\mathbb{Z}, 3) \rightarrow \{\text{pt}\} \rightarrow K(\mathbb{Z}, 3)$:

$$k=4, n=3$$

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$$\alpha^3$$

$$0$$

$$\alpha^2$$

$$0$$

$$\alpha$$

$$0$$

$$1$$

$$0$$

$$0$$

$$\mathbb{Z}$$

$$A$$

$$B$$

$$C$$