Problem Set One

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1 Humphreys 1.1

1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee / \Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$.

Proposition: $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M

Proof:

Proposition: M is the direct sum of finitely many submodules of the form $M^{[\lambda]}$.

Proof:

1.2 b

Proposition: The weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee / \Lambda_r$.

2 Humphreys 1.3*

Proposition: For any $M \in \mathcal{O}$, $M(\lambda)$ satisfies the following property:

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right).$$

Proof:

Noting that

- $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$,
- $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$ is an identification of the $\mathfrak{g}$-module M has a $\mathfrak{b}$- module by restricting the action of $\mathfrak{g}$,

consider the following two maps:

$$F : \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) \to \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M)$$
$$\phi \mapsto (F\phi : z \mapsto \phi(1 \otimes z)),$$

and

$$G : \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) \to \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M)$$
$$\psi \mapsto (G\psi : g \otimes v \mapsto g \cdot \psi(v)).$$

It suffices to show that these maps are well-defined and mutually inverse.

To see that F is well-defined, let $\phi : U(\mathfrak{g}) \otimes C_\lambda \to M$ be fixed; we will show that the set map $F\phi : \mathbb{C}_\lambda \to M$ is $U(\mathfrak{b})$-linear. Let $b \in U(\mathfrak{b})$, then

- $b \curvearrowright F\phi(v) = b \curvearrowright (z \mapsto \phi(1 \otimes z))(v) = b \curvearrowright \phi(1 \otimes v) = \phi((b \curvearrowright 1) \otimes v).$