

Title

D. Zack Garza

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Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p .

| | |
|---------------------------------------|---|
| $k[\mathbf{x}] := k[x_1, \dots, x_n]$ | The polynomial ring in n indeterminates |
| $k(\mathbf{x}) := k(x_1, \dots, x_n)$ | The rational function field $\{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), p, q, \in k[x_1, \dots, x_n]\}$ |
| $V(J), V_a(J)$ | The variety associated to an ideal $J \leq k[x_1, \dots, x_n]$ |
| | $V(J) := \{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$ |
| $I(S), I_a(S)$ | The ideal associated to a subset $S \subseteq \mathbb{A}_k^n$ |
| | $I(S) := \{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in S\}$ |
| $A(X) := k[x_1, \dots, x_n]/I(X)$ | The coordinate ring of a variety |
| \mathcal{O}_X | The structure sheaf $\{f : U \rightarrow k \mid f \in k(\mathbf{x}) \text{ locally}\}$. |

Lots of notation to fill in.

| Algebra | Geometry |
|---|-------------------------------------|
| Radical ideals $J = \sqrt{J} \leq k[x_1, \dots, x_n]$ | $V(J)$ the zero locus |
| $I(S)$ the ideal of a set | $S \subseteq \mathbb{A}^n$ a subset |
| $I + J$ | $V(I) \cap V(J)$ |
| $\sqrt{I(V) + I(W)}$ | $V \cap W$ |
| $I \cap J, IJ$ | $V(I) \cup V(J)$ |
| $I(V) \cap I(W), \sqrt{I(V)I(W)}$ | $V \cup W$ |
| $I(V) : I(W)$ | $\overline{V \setminus W}$ |
| Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$ | Irreducible subsets |

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
 - The Zariski topology?
 - Irreducibility?
 - Connectedness?
 - Dimension?
- What is the definition of a presheaf?
 - What are some examples and counterexamples?
- What is the definition of sheaf?
 - What are some examples?
 - What are some presheaves that are not sheaves?
- What is the definition of \mathcal{O}_X , the sheaf of regular functions?
 - How does one compute \mathcal{O}_X for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
 - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
 - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples

0.4.1 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\cdot, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$, the constant sheaf associated to \mathbb{R} (locally constant real-valued functions)
- $\text{Hol}(\cdot)$

0.5 Useful Algebra Facts

Fact 0.5.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f , then $\langle f \rangle$ is maximal

Proposition 0.5.2 (*Finitely generated polynomial rings are Noetherian*).

A polynomial ring $k[x_1, \dots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \dots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \dots, f_m \rangle$.

Proof (?).

A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.5.5), $k[x_1, \dots, x_n]$ is thus Noetherian. ■

Proposition 0.5.3 (*Properties and Definitions of Ideal Operations*).

$$I + J := \{f + g \mid f \in I, g \in J\}$$

$$IJ := \left\{ \sum_{i=1}^N f_i g_i \mid f_i \in I, g_i \in J, N \in \mathbb{N} \right\}$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ \quad (\text{coprime or comaximal}) \quad \langle a \rangle + \langle b \rangle = \langle a, b \rangle.$$

Theorem 0.5.4 (*Noether Normalization*).

Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist t_1, \dots, t_ℓ such that k_2 is finite over $k_1(t_1, \dots, t_\ell)$.

Theorem 0.5.5 (Hilbert's Basis Theorem).

If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.6 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

| Ring Theory | Geometry/Topology of Affine Varieties |
|--|---|
| Polynomial functions | Affine space |
| $k[x_1, \dots, x_n]$ | $\mathbb{A}^n/k := \{[a_1, \dots, a_n] \in k^n\}$ |
| Maximal ideals $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ | Points $[a_1, \dots, a_n] \in \mathbb{A}^n/k$ |
| Radical ideals $I \trianglelefteq k[x_1, \dots, x_n]$ | Affine varieties $X \subset \mathbb{A}^n/k$, vanishing loci of polynomials |
| | $I \mapsto V(I) := \{a \mid f(a) = 0 \forall f \in I\}$ |
| | $I(X) := \{f \mid f _X = 0\} \leftarrow X$ |
| Radical ideals containing $I(X)$, i.e. ideals in $A(X)$ | closed subsets of X , i.e. affine subvarieties |
| $A(X)$ is a domain | X irreducible |
| $A(X)$ is not a direct sum | X connected |
| Prime ideals in $A(X)$ | Irreducible closed subsets of X |
| Krull dimension n (longest chain of prime ideals) | $\dim X = n$, (longest chain of irreducible closed subsets) |