

Floer Talk

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Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of \mathcal{F} .

What is \mathcal{F} ?

We started with the unadorned Floer map:

$$\begin{aligned}\mathcal{F} : \mathcal{C}^\infty(\mathbf{R} \times S^1; W) &\longrightarrow \mathcal{C}^\infty(\mathbf{R} \times S^1; TW) \\ u &\longmapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H_t)\end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u).\end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned}\mathcal{P}^{1,p}(x, y) & \stackrel{?}{\longrightarrow} ? \\ (s, t) & \mapsto \exp_{w(s,t)} Y(s, t).\end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C_\infty^\infty(x, y)$.

1 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian H , perturb (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the right dimension.

Start by construction $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty$, the space of perturbations of H . Idea: define a norm $\|\cdot\|_\varepsilon$ and take the subspace of finite-norm elements.

$$\begin{aligned}\|h\|_\varepsilon &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} |d^k h(x,t)| \\ &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \Psi_i^{-1})(z)|.\end{aligned}$$

Where $\{\varepsilon_k\} \subset \mathbb{R}$ is chosen such that $\mathcal{C}_\varepsilon^\infty \hookrightarrow \mathcal{C}^\infty(W \times S^1)$ is dense for the C^∞ topology, and the $\Psi_i : B_i \rightarrow \overline{B(0,1)}$ is a fixed finite sequence of diffeomorphisms where $\bigcup_i B_i^\circ = W \times S^1$.

Note that we'll only use density for the C^1 topology in our case.

Proposition 1.1.

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Proof.

Show that $C^\infty(W \times S^1)$ is separable, yielding a sequence $(f_n) \subset C^\infty(W \times S^1)$ that is dense in the C^1 topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k \leq n} \|f_k\| C^n(W \times S^1)}$$

where the diffeomorphisms Ψ_i are used to compute these norms. ■

Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\text{Per}(H_0 + h) = \text{Per}(H_0)$ and are nondegenerate.

1.1 8.4: Linearizing the Floer equation: The Differential of \mathcal{F}

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as Z_i , tangents to W along u or in a neighborhood B of u) with actual vectors in \mathbb{R}^m .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

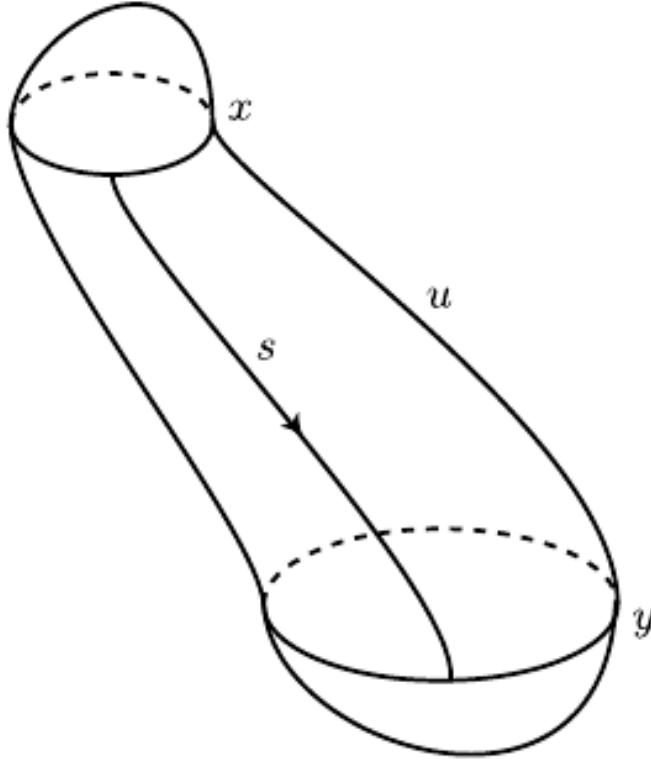
We can then identify $\text{im } \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R} \times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that x, y are contractible loops in W that are nondegenerate critical points of the action functional \mathcal{A}_H (i.e. solutions to the Floer equation), and $C_{\searrow}(x, y)$ was the set of maps $u : \mathbb{R} \times S^1 \rightarrow W$ satisfying some conditions.

Fix a solution $u \in \mathcal{M}(x, y) \subset C_{\text{Loc}}^\infty(\mathbb{R} \times S^1; W)$.

We lift each map to $\tilde{u} : S^2 \rightarrow W$ in the following way: the loops x, y are contractible, so they bound discs. So we extend according to:



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Recall assumption 6.22: every smooth map $w : S^2 \rightarrow W$ yields a symplectic trivialization of w^*TW (e.g. when $\pi_2(W) = 0$, so every map from S^2 extends to B^3).

Trivialize the symplectic fiber bundle \tilde{u}^*TW to obtain an orthonormal unitary frame $\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$ depending smoothly on $(s, t) \in S^2$, where $\lim_{s \rightarrow \pm\infty} Z_i$ exists for each i . We also require that $\partial_s Z_i, \partial_s^2 Z_i, \partial_s \partial_t Z_i \xrightarrow{s \rightarrow \pm\infty} 0$ for each i .

This frame defines a chart about u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\rightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\mapsto \exp_u \left(\sum y_i Z_i \right). \end{aligned}$$

Since $(d\exp)_0 = \text{id}$, we have $(d\iota)_0(\mathbf{y}) = \sum_i y_i Z_i$.

We'll now consider and compute the differential of

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\xrightarrow{\mathcal{F}} L^p(\mathbb{R} \times S^1; TW) \rightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\ u &\mapsto \frac{\partial u}{\partial s} + J(u) \left(\frac{\partial u}{\partial t} - X_t(u) \right). \end{aligned}$$

Take the vector $Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$, where we view Y as a vector in \mathbb{R}^m tangent to W , given by $Y = \sum y_i Z_i$.

We write

$$\mathcal{F}(u + Y) = \frac{\partial(u + Y)}{\partial s} + J(u + Y) \frac{\partial(u + Y)}{\partial t} - J(u + Y) X_t(u + Y)$$

and extract the part that is linear in Y :

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y).$$

Lemma 1.2 (Acting by Derivation).

For any $J \rightarrow \text{End}(\mathbb{R}^m)$ and $Y, v : ? \rightarrow \mathbb{R}^m$ we have

$$(dj)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

There is a proof.

For every such smooth map $u : \mathbb{R} \times S^1 \rightarrow W$, $(d\mathcal{F})_u(Y) = O_1 + O_0$ where O_i are differential operators of order i , and in fact O_1 can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$ where $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^n)$ is linear of order 0, and in fact we have

Proposition 1.3.

If u solves Floer's equation, then $(d\mathcal{F})_u = \bar{\partial} + S(s, t)$ where S is linear, tends to a symmetric operator as $s \rightarrow \pm\infty$, and $\lim \partial_t S = 0$ uniformly in t .

There is a very long computational proof.

Denote the order 0 part of $(d\mathcal{F})_u$ as $Y \mapsto S \cdot Y$ so $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^m)$ and define $S^\pm := \lim_{s \rightarrow \pm\infty} S(s, \cdot)$.

Proposition 1.4.

The equation $\partial_t Y = J_0 S^\pm Y$ linearizes Hamilton's equation $\dot{z} = X_t(z)$ at $x = \lim_{s \rightarrow \pm\infty} u$ for S^+ and S^- respectively.

Proof: uses previous proposition.

Given a solution u , the product

$$\begin{aligned} u \cdot s : ? &\rightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and $\mathcal{F}(u \cdot s) = 0$ for all s .

Punchline:

Thus $\frac{\partial u}{\partial s}$ is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s} \mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left(\frac{\partial u}{\partial s} \right).$$

Along any nonconstant solution connecting x and y , $\dim \ker(d\mathcal{F})_u \geq 1$.