Title

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Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on $\mathbb{R}^n$
- Holomorphic functions on $\mathbb{C}$

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on $\mathbb{R}$ a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \left\{ f : U \to \mathbb{R} \,\middle|\, f(x) = c \right\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$. Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0, 1)$ and $f_2 = 2$ on $(2, 3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf $\mathcal{G}$ on $\mathbb{R}$ given by *locally* constant function, i.e. $\left\{ f : U \to \mathbb{R} \,\middle|\, \forall p \in U, \exists U_p \ni p,\ f|_{U_p} \text{ is constant} \right\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 1.1.
Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open.