# Title

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Chapter 10: From Floer to Morse

- To be precise with notation, define
  - $CM_*(H,J)$ will be the Morse complex associated with a Morse function $H$, its vector field $\nabla H$ the gradient for the metric defined by $J, \omega$.
  - $CF_*(H,J)$ will be the Floer complex

**Theorem 1.0.1*(Main Goal).***
There exists a nondegenerate Hamiltonian that is sufficiently small in the $C^2$ topology for which both the Floer and Morse complexes are well-defined, and

$$CF_*(H,J) \cong CM_{*+n}(H,J) = CM_*(H,J)[n].$$

- Can start with an $H_0$ and rescale to define $H := H_0/k$

Why?

- When sufficiently small, periodic trajectories are constant
  - Thus $\operatorname{crit}(\mathcal{A}_H) = \operatorname{crit}(H)$
  - Implies that $H$ is a Morse function
  - Implies that for the Hessian $\operatorname{Spec}(\nabla^2 H) \cap 2\pi\mathbb{Z} = \emptyset$

- Allows comparing Morse index of critical point to Maslov index of corresponding constant trajectory using

$$\mathrm{Ind}_H(x) = \mu(x) + n.$$

- Gives an isomorphism of vector spaces, up to a dimension shift.
- Next need to show both differentials $\partial_M, \partial_F$ can be defined, and they coincide
- Defining $\partial_M$:
  - Need a vector field $X \in \Gamma(T * M)$ adapted to $H$
  - $X$ needs to satisfy Smale condition

What is the Smale condition?

- Then relate trajectories of $X$ to solutions of Floer equation, i.e. relate

$$\left\{\begin{matrix}\text{Solutions to}\\ \frac{\partial u}{\partial s}+X(u)=0\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Solutions to}\\ \frac{\partial u}{\partial s}+J(u)\frac{\partial u}{\partial t}+\nabla H(u)=0\end{matrix}\right\}.$$

In other words: want $X = \nabla H$ for the metric induced by $J, \omega$.

**Theorem 1.0.2*(Theorem to Prove)*.**
Let $H$ be Morse on $(W, \omega)$. Then there exists a dense subset $\mathcal{J}_{\mathrm{reg}}(H)$ of almost complex structures $J$ calibrated by $\omega$ such that $(H, -JX_H)$ is Morse-Smale.

Note: transversality result analogous to ones in 8.5

What is Morse-Smale?

- Big idea: running ideas backwards, getting theorems for Morse functions similar to what we did when linearizing the Floer operator
- Proof in two steps:
  - Step 1: Morse Side, arbitrary morse functions
    Linearize the Morse equation $\frac{\partial u}{\partial s} + X(u) = 0$ of the flow of $-X$ along one of its solutions $L_u Y = 0$.
    Show that whenever $H$ is Morse and $u$ is a trajectory connecting critical points, $L_u$ is Fredholm and $\mathrm{Ind}(L_u) = \mathrm{Ind}_H(y) - \mathrm{Ind}_H(x)$.
    Show that for $H$ a nondegenerate Hamiltonian and $u$ a trajectory of $JX_H$, the operators $(d\mathcal{F})_u$ and $L_u$ are Fredholm of equal index.
    Show that $X$ is Smale $\iff$ $L_u$ is surjective.
  - Step 2: Floer Side, specific case of Hamiltonian
    Prove the actual result.
- Now fix an almost complex structure to obtain a Smale vector field $X$
- Compare solutions to Floer equation and trajectories of $X$
  - Solutions to Floer equation that *do not* depend on $t$ are precisely trajectories of $X = -\nabla H$.

- Next show that elements in $\ker(d\mathcal{F}_u)$ do not depend on $t$.
- Corollary: $d\mathcal{F}_u$ is surjective along every trajectory of $\nabla H$.
- Then show that replacing $H_k := H/k$ for $k \gg 0$ preserves all critical points and all indices
- Punch line: all the solutions of the Floer equation that we need are time-independent.
  - Statement: For $k \gg 0$, solutions to the Floer equation for $H_k$ connecting $x \to y$ with $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$ are independent of $t$.

Goal by end of Ch. 10:

- Show that all Floer solutions connecting two consecutive critical points are *also* Morse trajectories, and $d\mathcal{F}_v$ is surjective along these trajectories
- Yields equality of complexes

### 1.1 Summary

What is $X_H$

- Take $H_k$ for $k \gg 0$ and $J \in \mathcal{J}_{\mathrm{reg}}$ (dense)
- Then when $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$, trajectories of Floer equation for $(H, J)$ connecting critical points $x, y$ are trajectories of the Smale vector field $X = -JX_H$.
  - $x, y$ will be critical points for both $H$ and $\mathcal{A}_H$
- Regularity? The linearized Floer operator is surjective along these trajectories
- Implies that $\mathcal{M}^{(H,J)}(x, y)$ is a manifold, so $CF_*$ can be defined.
- Claim: this shows the differentials coincide, and we're done.

### 1.2 Linearizing the Morse Equation

- Let $f$ be morse on $V \hookrightarrow \mathbb{R}^m$ $(m \gg 0)$ with adapted pseudo-gradient field $X$, then

$$\left\{\begin{array}{c}\text{Trajectories}\\ \text{of } X\end{array}\right\} \iff \left\{\begin{array}{c}\text{Solutions of}\\ \frac{\partial u}{\partial s} + X(u(s)) = 0\end{array}\right\}.$$

- Fix a metric $g$ on $V$ such that $X = \nabla_g f$.
- Define the space of solutions of finite energy:

$$\begin{aligned} E(u) &:= \int_{\mathbb{R}} \left\| \frac{\partial u}{\partial s} \right\|^2 ds \\ \mathcal{M} &:= \left\{ u \in C^\infty(\mathbb{R}, V) \;\middle|\; \frac{\partial u}{\partial s} + \nabla f = 0, \quad E(u) < \infty \right\}. \end{aligned}$$

- Then $\mathcal{M}$ is compact and equal to $\cup_{x,y} \mathcal{M}(x, y)$, using the fact that if $V$ is compact, *all* trajectories are of finite energy

- Now go to coordinates and linearize the equation of the flow along the solution $u$ to get a linear differential equation
- Yields an equation

$$\begin{aligned} L_u \,:\, W^{1,2}(\mathbb{R}, \mathbb{R}^n) &\to L^2(\mathbb{R}, \mathbb{R}^n) \\ Y &\mapsto \frac{\partial Y}{\partial s} + A(s)Y \,:= L_u Y, \end{aligned}$$

  where $A$ is a matrix limiting to $\nabla_y^2 f$ and $\nabla_x^2 f$ at $s = \pm\infty$
  - Limiting to Hessians of nondegenerate critical points will yield symmetric invertible matrices
  - We then consider $\ker L_u \subseteq \ker(d\mathcal{F}_u)$. Note: we have exponential decay.
- Note: the space of solutions to equation linearized at $u$ is $T_u\mathcal{M}(x, y)$.

### 1.2.1 Showing $L_u$ is Fredholm

- Bootstrapping: $Y \in \ker(L_u)$ in $W^{1,2}$ is continuous, thus $C^1$, this $C^\infty$ and form a finite-dimensional vector space.
- Behavior at infinity: reduces to $L_u Y = 0 \iff \frac{\partial Y}{\partial s} = -AY$ where $A$ is a constant diagonal matrix
  - This is a linear system, so solutions are

$$Y(s) = e^{-As}Y(0) \quad \text{i.e.} \quad y_i(s) = y_i e^{-\lambda_i s}.$$

- Will prove that if $u$ is a trajectory of $\nabla f$ connecting $x \to y$ then $L_u$ is Fredholm
  - Proof: involves bounding $W^{1,2}$ norm of $Y$ by $L^2$ norms of $Y, L_u Y$.
  - Lots of integral estimates: Fourier transform, Plancherel, Cauchy-Schwarz
- Integral bound yields: $\dim\ker L_u < \infty$ and $\operatorname{im}(L)_u$ is closed.
- Lemma: $\dim\operatorname{coker} < \infty$.
  - Proof: computer kernel of adjoint $L_u^* = -\frac{\partial}{\partial s} + A^*$ where the matrix is transposed.
  - Use the fact that $Z \in \operatorname{coker}(L_u) \iff Z \in \ker(L_u^*)$, i.e. $L_u^* Z = 0$ in the sense of distributions

### 1.2.2 Computing $\operatorname{Ind} L_u$

- Unsurprisingly, will show $\operatorname{Ind}(L_u) = \operatorname{Ind}_f(x) - \operatorname{Ind}_f(y)$.
- Ideas in proof:
  - Will choose two real numbers $\sigma, s$ to plug into $u$, and consider *resolvent*: map between tangent spaces to $V$ at $u(\sigma), u(s)$.
  - Look at the tangent spaces at $u(\sigma)$ of the stable and unstable manifolds will be the Floer complex

$$\begin{aligned} E^{\mathrm{u}}(\sigma) &:= T_{u(\sigma)}W^{\mathrm{u}}(x) \\ E^{\mathrm{s}}(\sigma) &:= T_{u(\sigma)}W^{\mathrm{s}}(x) \end{aligned}$$

.

- Then $\ker L_u$ is isomorphic to the intersection for all $\sigma$.

### 1.2.3 Smale Condition

- Recall $X = \nabla_g f$ for $g$ a metric.
- Statement: the vector field $X$ satisfies the Smale condition $\iff$ all $L_u$ are surjective.

*Proof*.

- $L_u$ is surjective $\iff$ $\operatorname{coker}(L_u) = 0 \iff \ker(L_u^*)$ is injective
- This is equivalent to

$$T_{u(\sigma)} W^{\mathrm{u}}(x) + T_{u(\sigma)} W^{\mathrm{s}}(x) = T_{u(\sigma)} V.$$

- This is exactly the transversality condition for the stable and unstable manifolds
  - We want this for all critical points

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## 1.3 10.4: Morse and Floer Trajectories Coincide

### 1.3.1 Comparing Kernels

- Note $\ker(L_u) \subset \ker(d\mathcal{F}_u)$ since

$$\left(\frac{\partial}{\partial s} + S(s)\right) Y = 0 \implies \left(\frac{\partial}{\partial s} J \frac{\partial}{\partial t} + S(s)\right) Y = 0,$$

so just need to show reverse inclusion.

- Use a lemma: for $f : [0,1] \to \mathbb{R}$,

$$\|f\|_{L^p([0,1])} \leq \left\| \frac{\partial f}{\partial t} \right\|_{L^p([0,1])},$$

then apply this to $f(t) := Y(s,t)$ and $p = 2$.

- Yields an equation

$$\|\partial_s Y\|_{L^2}^2 + \|\partial_t Y\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\mathrm{op}}^2 \|Y\|_{L^2}^2 \implies \|Y\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\mathrm{op}} \left\| Y_{L^2}^2 \right\|$$

where the sup term being small forces $Y = 0$.

### 1.3.2 Trajectories are Independent of $t$

- WTS: trajectories of $H_k$ appearing in the Floer complex are exactly those appearing in the Morse complex.
  - I.e. proving 10.1.9

Idea of proof:

- Contradiction: suppose there exists a sequence $n_k \to \infty$ with time-dependent solutions $u_{n_k}$ connecting $x \to y$ which solve the Floer equation
- Consider case where indices differ by 1: using broken trajectories theorem, extract a subsequence converging to some $v \in \mathcal{M}(x, y, H)$.
  - Show $v$ doesn't depend on $t$
  - Since $d\mathcal{F}_v$ is surjective, $v$ is in a 1-dim component, and thus an isolated point of $\mathcal{L}(x, y)$
  - Get a contradiction from taking $k \gg 0$ and using $v_{n_k}(s, t) = v(s + \sigma_k, t) = v(s + \sigma_k)$, which does *not* depend on time
- Consider case where indices differ by 2
  - Use Smale property of the gradient $-JX_H$ of $H$: trajectories $x \to y$ form a 2-manifold
  - Since trajectories are also in $\mathcal{M}(x, y, H)$, parameterizes a submanifold in a neighborhood of $v$.
- Show that convergence toward broken orbits in Morse setting corresponds to converges toward broken trajectories in Floer setting
- Use gluing from last chapter: $\hat{v}_{n_k} \in \mathrm{im}(()\hat{\phi})$ for $k \gg 0$, contradicting the fact that $v_{n_k}$ doesn't depend on $t$