

# 4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2020

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# 1 | Tuesday, January 12

## 1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
  - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic  $\mathbb{R}^4$  was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

## 1.2 Introduction

### Definition 1.2.1 (Topological Manifold)

Recall that a **topological manifold** (or  $C^0$  manifold)  $X$  is a Hausdorff topological space *locally homeomorphic* to  $\mathbb{R}^n$  with a countable topological base, so we have charts  $\varphi_u : U \rightarrow \mathbb{R}^n$  which are homeomorphisms from open sets covering  $X$ .

**Example 1.2.2 (The circle):**  $S^1$  is covered by two charts homeomorphic to intervals:

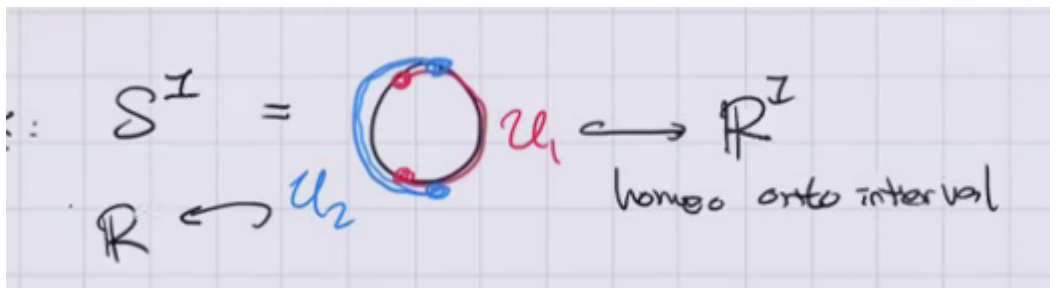


Figure 1: image\_2021-01-13-14-02-19

**Remark 1.2.3:** Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V).$$

**Definition 1.2.4** (Restricted Structures on Manifolds)

- We say  $X$  is a **PL manifold** if and only if  $t_{UV}$  are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say  $X$  is a  $C^k$  **manifold** if they are  $k$  times continuously differentiable, and **smooth** if infinitely differentiable.
- We say  $X$  is **real-analytic** if they are locally given by convergent power series.
- We say  $X$  is **complex-analytic** if under the identification  $\mathbb{R}^n \cong \mathbb{C}^{n/2}$  if they are holomorphic, i.e. the differential of  $t_{UV}$  is complex linear.
- We say  $X$  is a **projective variety** if it is the vanishing locus of homogeneous polynomials on  $\mathbb{CP}^N$ .

**Remark 1.2.5:** Is this a strictly increasing hierarchy? It's not clear e.g. that every  $C^k$  manifold is PL.

**Question 1.2.6**

Consider  $\mathbb{R}^n$  as a topological manifold: are any two smooth structures on  $\mathbb{R}^n$  diffeomorphic?

**Remark 1.2.7:** Fix a copy of  $\mathbb{R}$  and form a single chart  $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$ . There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\rightarrow \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on  $X$ , since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise

there would be a transition function of the form  $t_{VU} : t \mapsto t^{1/3}$ , which is not smooth at zero. However, the map

$$\begin{aligned} X &\rightarrow X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

**Claim:**  $\mathbb{R}$  admits a unique smooth structure.

*Proof (sketch).*

Let  $\tilde{\mathbb{R}}$  be some exotic  $\mathbb{R}$ , i.e. a smooth manifold homeomorphic to  $\mathbb{R}$ . Cover this by coordinate charts to the standard  $\mathbb{R}$ :

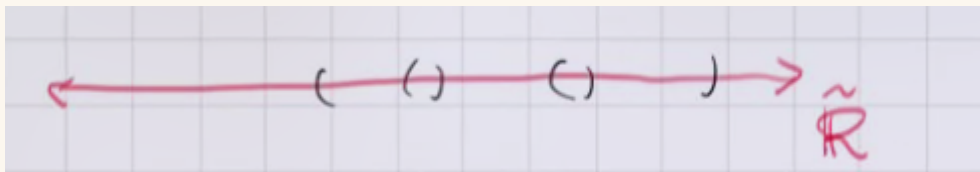


Figure 2: image\_2021-01-13-14-22-18

### Fact

There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions  $f_i : U_i \rightarrow \mathbb{R}$  with  $f_i \geq 0$  and  $\text{supp} f \subseteq U_i$  such that  $\sum f_i = 1$  (i.e., *bump functions*). It is also a purely topological fact that  $\tilde{\mathbb{R}}$  is orientable.

So we have bump functions:

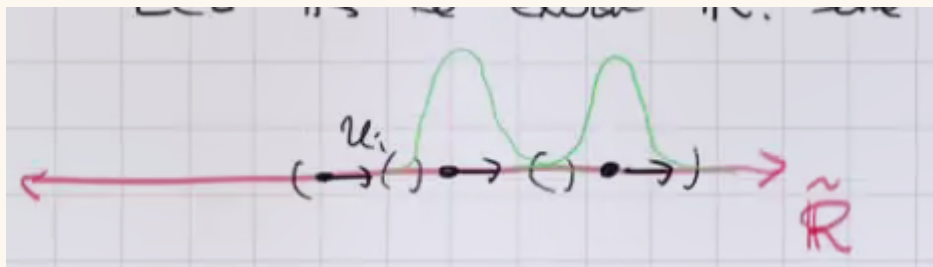


Figure 3: image\_2021-01-13-14-25-30

Take a smooth vector field  $V_i$  on  $U_i$  everywhere aligning with the orientation. Then  $\sum f_i V_i$  is a smooth nowhere vector field on  $X$  that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\rightarrow \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that  $\varphi'(t) = V(\varphi(t))$ . Then  $\varphi$  is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

**Slogan**

To understand smooth structures on  $X$ , we should try to solve differential equations on  $X$ .



**Remark 1.2.10:** Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g.  $S^2$ .

**Question 1.2.11**

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

**Answer 1.2.12**

1. (Munkres) Every  $C^1$  structure gives a unique  $C^k$  and  $C^\infty$  structure.<sup>1</sup>
2. (Grauert) Every  $C^\infty$  structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in  $\dim X \leq 7$ , and it's unique in  $\dim X \leq 6$ , and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let  $X$  be a topological manifold of  $\dim X \geq 5$ , then there exists a cohomology class  $ks(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$  which is 0 if and only if  $X$  admits a PL structure. Moreover, if  $ks(X) = 0$ , then (up to concordance) the set of PL structures is given by  $H^3(X; \mathbb{Z}/2\mathbb{Z})$ .
5. (Moise) Every topological manifold in  $\dim X \leq 3$  admits a unique smooth structure.
6. (Smale et al.): In  $\dim X \geq 5$ , the number of smooth structures on a topological manifold  $X$  is finite. In particular,  $\mathbb{R}^n$  for  $n \neq 4$  has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes)  $\mathbb{R}^4$  admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface  $\Sigma$ , the space of complex-analytic structures is a complex orbifold<sup>2</sup> of dimension  $3g - 2$  where  $g$  is the genus of  $\Sigma$ , up to biholomorphism (i.e. *moduli*).

**Remark 1.2.13:** Kervaire–Milnor:  $S^7$  admits 28 smooth structures, which form a group.

<sup>1</sup>Note that this doesn't start at  $C^0$ , so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

<sup>2</sup>Locally admits a chart to  $\mathbb{C}^n/\Gamma$  for  $\Gamma$  a finite group.

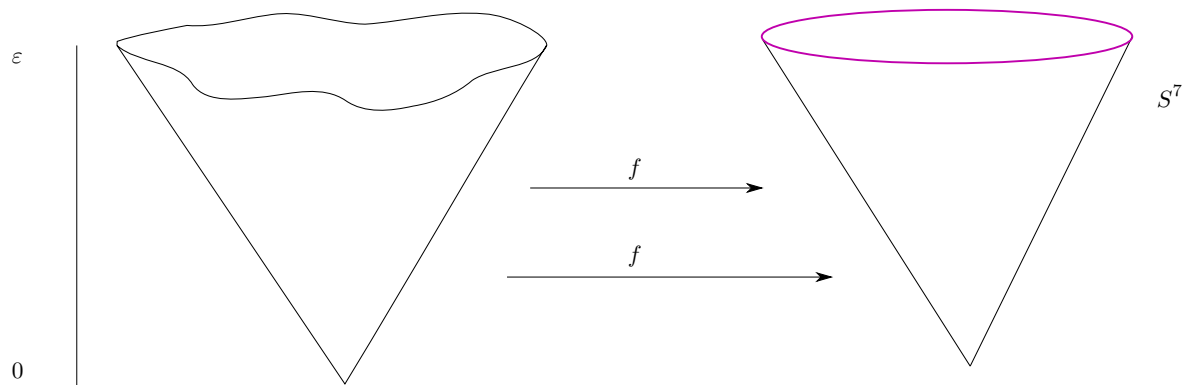
## 2 | Friday, January 15

**Remark 2.0.1:** Let

$$V := \{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\} \subseteq \mathbb{C}^5$$

$$S_\varepsilon := \{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\}.$$

Then  $V_k \cap S_\varepsilon \cong S^7$  is a homeomorphism, and taking  $k = 1, 2, \dots, 28$  yields the 28 smooth structures on  $S^7$ . Note that  $V_k$  is the cone over  $V_k \cap S_\varepsilon$ .



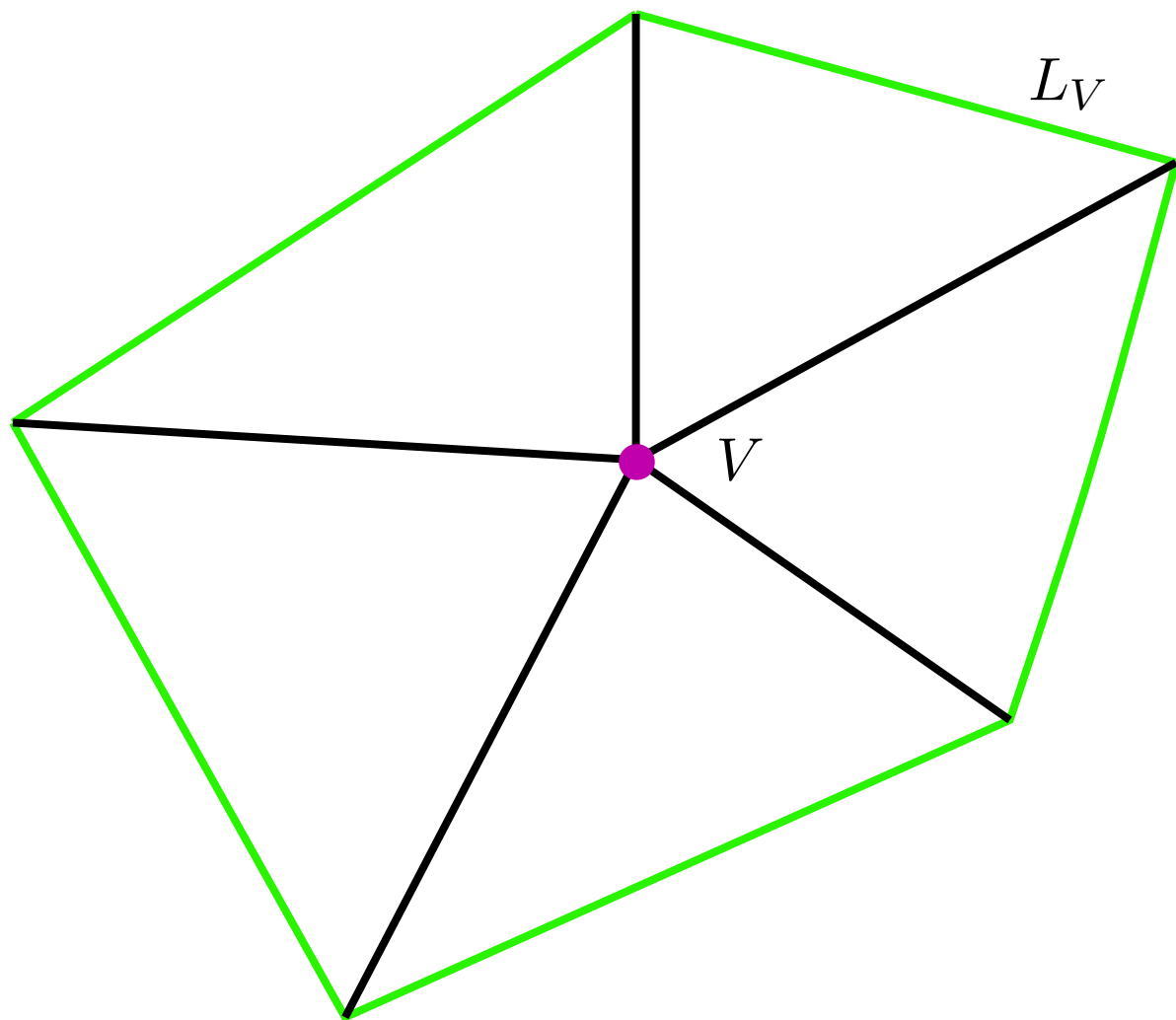
? Admits a smooth structure, and  $\bar{V}_k \subseteq \mathbb{CP}^5$  admits no smooth structure.

### Question 2.0.2

Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

### Answer 2.0.3

No! Given a simplicial complex, there is a notion of the **combinatorial link** of a vertex.



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a “PL triangulation” where the links are spheres.

**Remark 2.0.4:** What’s special in dimension 4? Recall the **Kirby-Siebenmann** invariant  $ks(x) \in H^4(X; \mathbb{Z}_2)$  for  $X$  a topological manifold where  $ks(X) = 0 \iff X$  admits a PL structure, with the caveat that  $\dim X \geq 5$ . We can use this to cook up an invariant of 4-manifolds.

**Definition 2.0.5** (Kirby-Siebenmann Invariant of a 4-manifold)

Let  $X$  be a topological 4-manifold, then

$$ks(X) := ks(X \times \mathbb{R}).$$

**Remark 2.0.6:** Recall that in  $\dim X \geq 7$ , every PL manifold admits a smooth structure, and we



can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\text{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures.} \end{cases}$$

**Remark 2.0.7:**  $\text{ks}(X) \neq 0$  implies that  $X$  has no smooth structure, since  $X \times \mathbb{R}$  doesn't. Note that it was not known if this invariant was nonzero for a while!

**Remark 2.0.8:** Note that  $H^2(X; \mathbb{Z})$  admits a symmetric bilinear form  $Q_X$  defined by

$$\langle \alpha, \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where  $[X]$  is the fundamental class.

## 3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

**Theorem 3.0.1 (Freedman).**

If  $X, Y$  are compact oriented topological 4-manifolds, then  $X \cong Y$  are homeomorphic if and only if  $\text{ks}(X) = \text{ks}(Y)$  and  $Q_X \cong Q_Y$  are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that  $\langle \varphi\alpha, \varphi\beta \rangle = \langle \alpha, \beta \rangle$ .

Conversely, every **unimodular** bilinear form appears as  $H^2(X; \mathbb{Z})$  for some  $X$ , i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\rightarrow H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

**Remark 3.0.2:** Note that preservation of a bilinear form is a stand-in for “being an element of the orthogonal group”, where we only have a lattice instead of a full vector space.

**Remark 3.0.3:** There is a map  $H^2(X; \mathbb{Z}) \xrightarrow{PD} H_2(X; \mathbb{Z})$  from Poincaré, where we can think of elements in the latter as closed surfaces  $[\Sigma]$ , and

$$\langle \Sigma_1, \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

**Definition 3.0.4 (Signature)**

The **signature** of a topological 4-manifold is the signature of  $Q_X$ , where we note that  $Q_X$  is a symmetric nondegenerate bilinear form on  $H^2(X; \mathbb{R})$  and for some  $a, b$

$$(H^2(X; \mathbb{R}), Q_X) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where  $a$  is the number of +1s appearing in the matrix and  $b$  is the number of -1s. This is  $\mathbb{R}^{ab}$  where  $e_i^2 = 1, i = 1 \dots a$  and  $e_i^2 = -1, i = a + 1, \dots b$ , and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is  $a - b$ , the dimension of the positive-definite space minus the dimension of the negative-definite space.

**Theorem 3.0.5 (Rokhlin's Theorem).**

Suppose  $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$  and  $\alpha \in H^2(X; \mathbb{Z})$  and  $X$  a simply connected **smooth** 4-manifold. Then 16 divides  $\text{sig}(X)$ .

**Remark 3.0.6:** Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

**Theorem 3.0.7 (Donaldson).**

Let  $X$  be a smooth simply-connected 4-manifold. If  $a = 0$  or  $b = 0$ , then  $Q_X$  is diagonalizable and there exists an orthonormal basis of  $H^2(X; \mathbb{Z})$ .

**Remark 3.0.8:** This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

### 3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

**Remark 3.1.1:** Last time we showed  $\mathbb{R}^1$  had a unique smooth structure, so now we'll do this for  $\mathbb{R}^2$ . The strategy of solving a differential equation, we'll now sketch the proof.

**Definition 3.1.2** (Riemannian Metrics)

A **Riemannian metric**  $g \in \text{Sym}^2 T^*X$  for  $X$  a smooth manifold is a metric on every  $T_p X$  given by

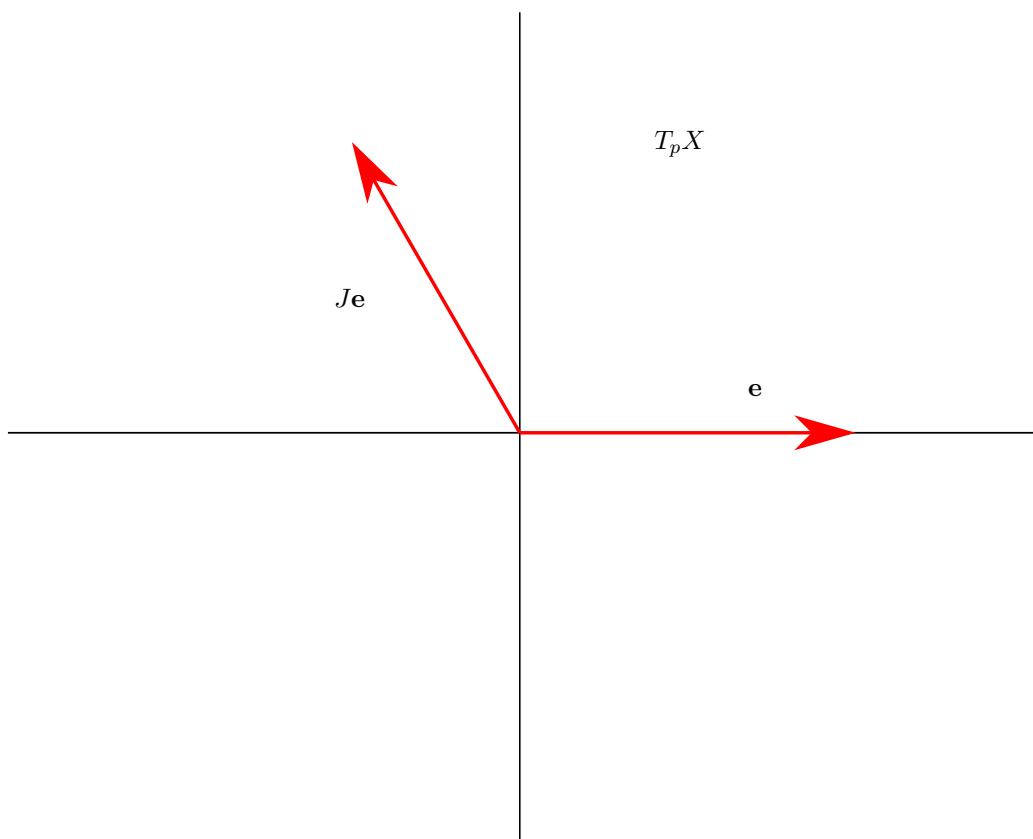
$$g_p : T_p X \times T_p X \rightarrow \mathbb{R}$$

$$g(v, v) \geq 0, g(v, v) = 0 \iff v = 0.$$

**Definition 3.1.3** (Almost complex structure)

An **almost complex structure** is a  $J \in \text{End}(TX)$  such that  $J^2 = -\text{id}$ .

**Remark 3.1.4:** Let  $e \in T_p X$  and  $e \neq 0$ , then if  $X$  is a surface then  $\{e, Je\}$  is a basis of  $T_p X$ .



This is a basis because if  $Je$  and  $e$  are parallel, then ??? In particular,  $J_p$  is determined by a point in  $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

**3.1.1 Sketch of Proof**

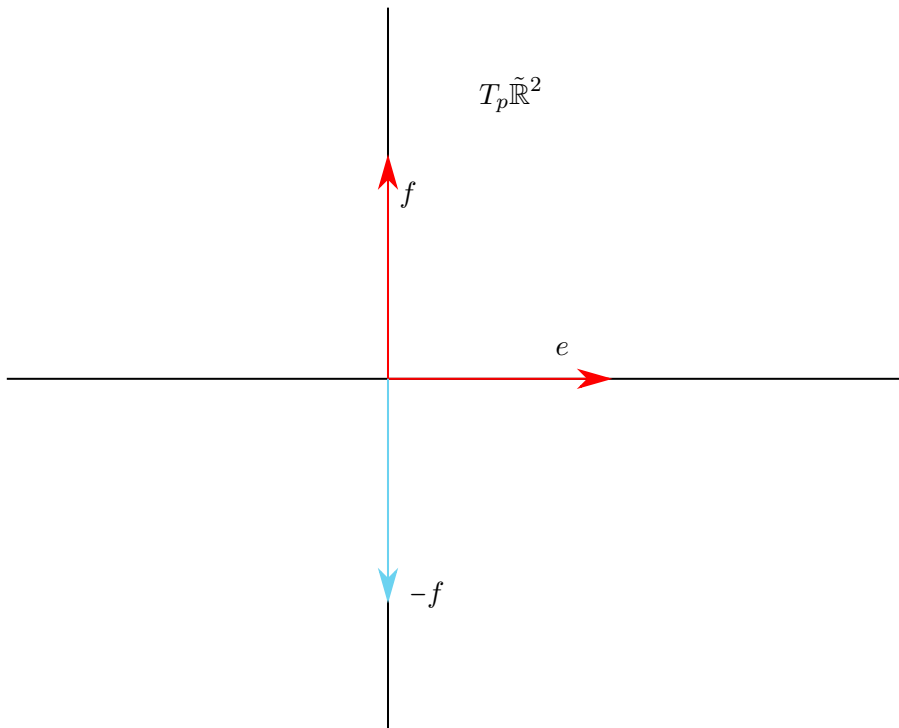
Let  $\tilde{\mathbb{R}}^2$  be an exotic  $\mathbb{R}^2$ .

**Step 1** Choose a metric on  $\tilde{\mathbb{R}}^2$   $g := \sum f_I g_i$  with  $g_i$  metrics on coordinate charts  $U_i$  and  $f_i$  a partition of unity.

**Step 2** Find an almost complex structure on  $\tilde{\mathbb{R}}^2$ . Choosing an orientation of  $\tilde{\mathbb{R}}^2$ ,  $g$  defines a unique almost complex structure  $J_p e := f \in T_p \tilde{\mathbb{R}}^2$  such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$ .
- $\{e, f\}$  is an oriented basis of  $T_p \tilde{\mathbb{R}}^2$

This is because after choosing  $e$ , there are two orthogonal vectors, but only one choice yields an *oriented* basis.



**Step 3** We then apply a theorem:

**Theorem 3.1.5(?).**

Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts  $\varphi_i : U_i \rightarrow \mathbb{C}$  such that  $J$  is multiplication by  $i$ .

So  $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$ , and  $(\tilde{\mathbb{R}}^2, J)$  is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to  $\mathbb{D}$  or  $\mathbb{C}$ , both of which are diffeomorphic to  $\mathbb{R}^2$ .

*See the Newlander-Nirenberg theorem, a result in complex geometry.*

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