

CRAG

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Zeta  
Functions

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## The Weil Conjectures

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## Zeta Functions

Fix  $q$  a prime and  $\mathbb{F} := \mathbb{F}_q$  the (unique) finite field with  $q$  elements, along with its (unique) degree  $n$  extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}}_q \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

## Definition (Zeta Function)

Let  $J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$  be an ideal, then a *projective algebraic variety*  $X \subset \mathbb{P}_{\mathbb{F}}^n$  can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^n \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where  $J$  is generated by *homogeneous* polynomials in  $n + 1$  variables, i.e. there is a fixed  $d = \deg f_i \in \mathbb{Z}^{\geq 1}$  such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{l}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{l}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

- For a fixed variety  $X$ , we can consider its  $\mathbb{F}$ -points  $X(\mathbb{F})$ .
  - Note that  $\#X(\mathbb{F}) < \infty$  is an integer
- For any  $L/\mathbb{F}$ , we can also consider  $X(L)$ 
  - In particular, we can consider  $X(\mathbb{F}_{q^n})$  for any  $n \geq 2$ .
  - We again have  $\#X(\mathbb{F}_{q^n}) < \infty$  and are integers for every such  $n$ .
- So we can consider the sequence

$$[N_1, N_2, \dots, N_n, \dots] := [\#X(\mathbb{F}), \#X(\mathbb{F}_{q^2}), \dots, \#X(\mathbb{F}_{q^n}), \dots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \dots.$$

# Why Generating Functions?

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Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of  $F$ : we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left( \frac{\partial}{\partial z} \right)^n F(z) \Big|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} dz = N_n.$$

*The latter form is very amenable to computer calculation.*

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An OGF is an infinite series – in nice situations, we can hope for a closed-form representation, e.g.  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ .

Recall that by integrating a geometric series we can derive

$$\begin{aligned}\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n &&= 1 + z + z^2 + \dots \\ \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n &&= \sum_{n=0}^{\infty} \int z^n = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots \\ \Rightarrow -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n}.\end{aligned}$$

and so

$$\begin{aligned}Z_{\{\text{pt}\}}(z) &= \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \dots\right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}.\end{aligned}$$