Title

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Contents

0.1 Exercises

Problem 1 (Hungerford 1.6.3)**.**
If $\sigma = (i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$, then show that $\tau\sigma\tau^{-1} = (\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

Solution 1. Let

$$\sigma = (s_1 s_2 \cdots s_r) \in S_n$$

in cycle notation, and $\tau \in S_n$ be arbitrary. Define $t_j = \tau(s_j)$; we would then like to show that

$$\tau\sigma\tau^{-1} = (t_1 t_2 \cdots t_r) \coloneqq (\tau(s_1)\tau(s_2)\cdots\tau(s_r))$$

To this end, it suffices to show that t_i maps to $t_{i+1 \mod r}$, under $\tau\sigma\tau^{-1}$, which is to say

$$\tau\sigma\tau^{-1}(t_i) = \begin{cases} t_{i+1} & i+1 \leq r, \\ t_1 & i = r \end{cases}.$$

Bearing this in mind, we will immediately suppress notation and take all indices $\mod r$ for the rest of this problem.

The following then follows simply by definitions:

$$\begin{aligned} \tau\sigma\tau^{-1}(t_i) &= \tau\sigma(s_i) \\ &= \tau(s_{i+1}) \\ &= t_{i+1}. \end{aligned}$$

$\square$

Problem 2 (Hungerford 1.6.4)**.**
Show that $S_n \cong \langle (12), (123\cdots n) \rangle$ and also that $S_n \cong \langle (12), (23\cdots n) \rangle$

Solution 2. Let $\sigma = (12)$ and $\tau = (123 \cdots n)$.

Claim: S_n is generated by *swaps* $F = \{f_{i,k} = (i \quad i+k) \mid 1 \leq i, k \leq n\} = \langle f_{i,k} \rangle$, and moreover each such swap can be written as a product in σ and τ, and thus

$$S_n = F \subseteq \langle \sigma, \tau \rangle \subseteq S_n$$

which forces $\langle \sigma, \tau \rangle = S_n$ as desired.

To see that $F = S_n$, let $(s_1 s_2 \cdots s_r) \in S_n$ be arbitrary. We then construct the swaps $(s_1 s_2), (s_1, s_3), \cdots (s_1 s_r)$, and note that taking their product yields

$$(s_1 s_2)(s_1, s_3) \cdots (s_1 s_r) = (s_1 s_2 s_3 \cdots s_r).$$

To see that $F \subseteq \langle \sigma, \tau \rangle$, we produce a way to write any swap as a product of powers of these generators. We can first note that for $1 \leq i \leq n$, we have $\sigma(i) = i + 1$ and $\sigma^k(i) = i + k$ (where again everything is taken $\mod r$).

By problem (1), we have

$$\sigma \tau \sigma^{-1} = \sigma \ (12) \ \sigma^{-1} = (\sigma(1) \ \sigma(2)) = (23),$$

and in general,

$$\sigma^k \tau \sigma^{-k} = (\sigma^k(1) \ \sigma^k(2)) = (k \quad k+1).$$

So the cycles $(k \quad k+1)$ are products of powers of τ, σ and thus contained in the group they generate, and we have $F \subseteq \langle \sigma, \tau \rangle$.

If we then define the cycle $\gamma_k = (k \quad k+1)$, we can observe

$$\begin{aligned} \gamma_k \gamma_{k+1} \gamma_k^{-1} &= (k \quad k+1) \ (k+1 \quad k+2) \ (k+1 \quad k) \\ &= (k \quad k+2), . \end{aligned}$$

and so $\langle \gamma_k \rangle$ also contains all cycles of the form $(k \quad k+i)$ for any i. In particular, any swap can be written as such a cycle – explicitly, given a swap $(s_1 s_2)$ (where without loss of generality $s_1 \leq s_2$), let $k = s_1$ and $i = s_2 - s_1$.

This implies that

Problem 3 (Hungerford 2.2.1)**.**
Let G be a finite abelian group that is not cyclic. Show that G contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p.

Problem 4 (Hungerford 2.2.12.b)**.**
Determine all abelian groups of order n for $n \leq 20$.

Problem 5 (Hungerford 2.4.1)**.**
Let G be a group and $A \trianglelefteq G$ be a normal abelian subgroup. Show that G/A acts on A by conjugation and construct a homomorphism $\varphi : G/A \to \mathrm{Aut}(A)$.

Problem 6 (Hungerford 2.4.9)**.**
Let $Z(G)$ be the center of G. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Note that Hungerford uses the notation $C(G)$ for the center.

Problem 7 (Hungerford 2.5.6)**.**
Let G be a finite group and $H \trianglelefteq G$ a normal subgroup of order p^k. Show that H is contained in every Sylow p-subgroup of G.

Problem 8 (Hungerford 2.5.9)**.**
Let $|G| = p^n q$ for some primes $p > q$. Show that G contains a unique normal subgroup of index q.

0.2 Qual Problems

Problem 9.
Let G be a finite group and p a prime number. Let X_p be the set of Sylow-p subgroups of G and n_p be the cardinality of X_p. Let $\mathrm{Sym}(X)$ be the permutation group on the set X_p.

1. Construct a homomorphism $\rho : G \to \mathrm{Sym}(X_p)$ with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that G is simple and the order of G divides $n_p!$.
3. Show that for any $1 \leq a \leq 4$ and any prime power p^k, no group of order ap^k is simple.

Problem 10.
Let G be a finite group and $H < G$ a subgroup. Let n_H be the number of subgroups of G that are conjugate to H. Show that n_H divides the order of G.

Problem 11.
Let $G = S_5$, the symmetric group on 5 elements. Identify all conjugacy classes of elements in G, provide a representative from each class, and prove that this list is complete.