

Linearization and Transversality

Sections 8.3 and 8.4

D. Zack Garza

April 2020

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
 H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Review 8.2

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
 H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Section 8.3: The Space of Perturbations of H

Goal

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Goal: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Goal

Linearization and
Transversality

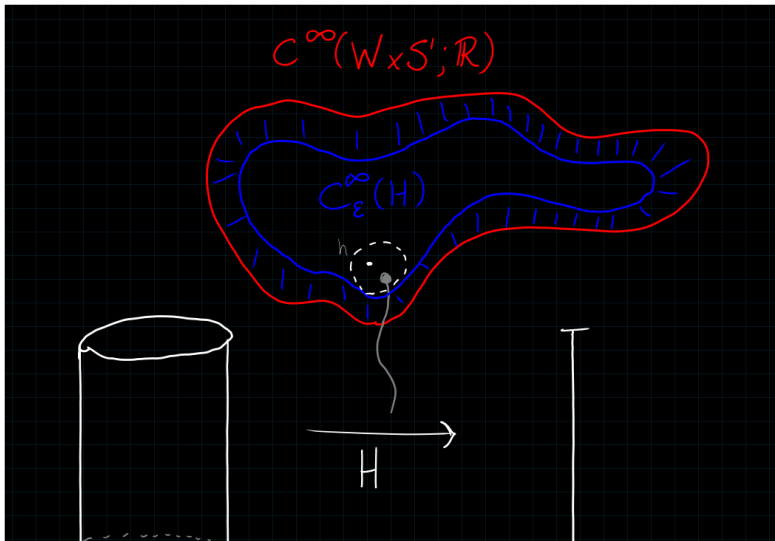
D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
 H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Start by trying to construct a subspace $C_\varepsilon^\infty(H) \subset C^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Define an Absolute Value

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C_\varepsilon^\infty(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$ denote a perturbation of H .
- Fix $\varepsilon = \{\varepsilon_k \mid k \in \mathbb{Z}^{\geq 0}\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$|d^k h(\mathbf{x}, t)| = \max \left\{ d^\alpha h(\mathbf{x}, t) \mid |\alpha| = k \right\},$$

the maximum over all sets of multi-indices α of length k .

Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \dots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

Define a Norm

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{\infty} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} |d^k h(x, t)|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

- Choose them in such a way we obtain charts

$$\psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \quad (?).$$

- Obtain the computable form

$$\|h\|_{\infty} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \psi_i^{-1})(z)|.$$

Define a Banach Space

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \mid \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Theorem

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma

$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Sketch Proof of Theorem

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n , define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \mid k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable.

Why does this imply density? I don't know.



Modified Theorem

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

The next proposition establishes a version of this theorem with compact support:

Theorem

For any $(\mathbf{x}, t) \in U \in W \times S^1$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the C^1 topology by functions in C_ε^∞ supported in U .

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \mid h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\text{supp}(h)$ is “far” from $\text{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
 H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Section 8.4: Linearizing the Floer Equation: The Differential of F

Goal

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Choose $m > n = \dim(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as Z_i , tangents to W along u or in a neighborhood B of u) with actual vectors in \mathbb{R}^m .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify

$$\mathrm{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the codomain here.

Definitions

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional \mathcal{A}_H ,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$ denotes a fixed solution to the Floer equation,
- $C_{\searrow}(x, y) \subset \{u \in C^\infty(\mathbb{R} \times S^1; W)\}$ is the set of smooth solutions $u : \mathbb{R} \times S^1 \rightarrow W$ satisfying some conditions:

$$\lim_{s \rightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \rightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left| \frac{\partial u}{\partial t}(s, t) \right|, \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

Compactify to Sphere

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Fix a solution

$$u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^{\infty}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way:

The loops x, y are contractible, so they bound discs. So we extend by pushing these discs out slightly:

Lift to 2-Sphere

Linearization and
Transversality

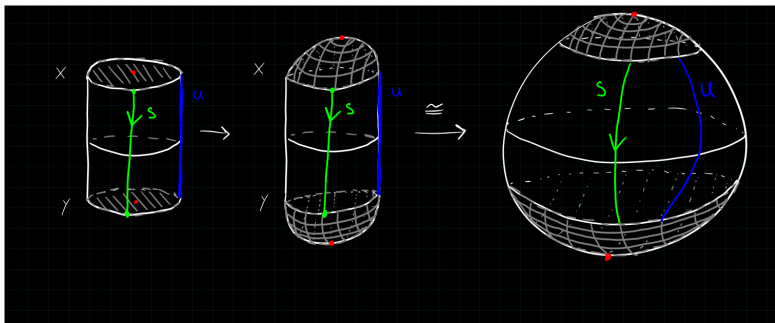
D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \mapsto \tilde{u} \in C^\infty(S^2; W)$$



Trivial the Pullback

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

From earlier in the book, we have

Assumption (6.22):

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle w^*TW , i.e. $\langle c_1(TW), \pi_2(W) \rangle = 0$ where c_1 denotes the first Chern class of the bundle TW .

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\wedge^1(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps $S^2 \rightarrow W$ lift to $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$\begin{array}{ccc} \tilde{u}^*TW & \xrightarrow{d\tilde{u}} & TW \\ \downarrow & \lrcorner & \downarrow \\ S^2 & \xrightarrow{\tilde{u}} & W \end{array}$$

Choose a Frame

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Using the assumption, trivialize the pullback \tilde{u}^*TW to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

where

- The frame depends smoothly on $(s, t) \in S^2$,
- $\lim_{s \rightarrow \pm\infty} Z_i$ exists for each i .
-

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \partial t} \quad \curvearrowright \quad Z_i \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{for each } i$$

Claim: such trivializations exist, “using cylinders near the spherical caps in the figure”.

Define “Banach Manifold Charts”

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Recall we had $W^{1,p}(x, y)$ a completion of C^∞

$$\mathcal{M}(x, y) \subset C_{\searrow}^\infty(x, y) \subset \mathcal{P}^{1,p}(x, y) \underset{\text{defn}}{\subset} \left\{ (s, t) \xrightarrow{\varphi} \exp_{w(s,t)} Y(s, t) \right\}.$$

where we restrict to

- $Y \in W^{1,p}(w^*TW)$,
- $w \in C_{\searrow}^\infty(x, y)$

Use the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\longmapsto \exp_u \left(\sum y_i Z_i \right). \end{aligned}$$

- Note that the derivative at zero is $\sum_{i=1}^{2n} y_i Z_i$.

Add a Tangent

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Take the vector

$$Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

- View Y as a vector in \mathbb{R}^m tangent to W , given by $Y = \sum_{i=1}^{2n} y_i Z_i$.
- Plug $u + Y$ into the equation for \mathcal{F} , directly yielding

Add a Tangent

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u) \\ \mathcal{F}(u + Y) &= \frac{\partial(u+Y)}{\partial s} + J(u+Y) \frac{\partial(u+Y)}{\partial t} - J(u+Y) X_t(u+Y)\end{aligned}$$

Extract Linear Part

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Extract the part that is linear in Y and collect terms:

$$\begin{aligned}(d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right)\end{aligned}$$

- This is a sum of two differential operators:
 - One of order 1, one of order 0 (Perspective 1)
 - The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)

Leibniz Rule

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Now compute in charts. Need a lemma:

Lemma (Leibniz Rule)

For any source space X and any maps

$$\begin{aligned} J : X &\longrightarrow \text{End}(\mathbb{R}^m) \\ Y, v : X &\longrightarrow \mathbb{R}^m \end{aligned}$$

we have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

Sketch: Proof of Leibniz Rule

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Differentiate the map

$$\begin{aligned} J \cdot v : X &\longrightarrow \mathbb{R}^m \\ x &\mapsto J(x) \cdot v(x) \end{aligned}$$

to obtain

$$\begin{aligned} &J(x + Y)v(x + y) \\ &= (J(x) + (dJ)_x(Y)) \cdot (v(x) + (dv)_x(Y)) + \cdots \\ &= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x) \\ &\quad + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots \\ &\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y). \end{aligned}$$

Decompose by Order

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Using the chart ι defined by $\{Z_i\}$ to write $Y = \sum_{i=1}^{2n} y_i Z_i$ and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where O_0 are order 0 terms ("they do not differentiate the y_i ") and the O_1 are order 1 terms:

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$

$$O_0 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

Order One

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \dots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

- The second equality follows from the assumption that the Z_i are symplectic and orthonormal.
- Note that this writes $(d\mathcal{F})_u(Y) = O_0 + O_{CR}$, a sum of an order zero and a Cauchy-Riemann operator.

Recap

Linearization and
Transversality

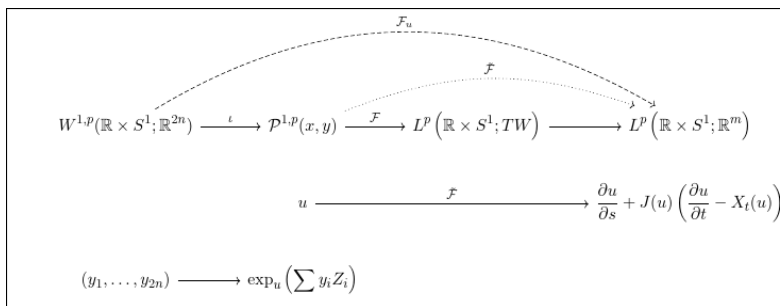
D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of \mathcal{F}

Note that since we've computed in charts, we have actually computed the differential of \mathcal{F}_u in the following diagram



So we've technically computed $(dF_\mu)_0$.

Order 0 Term is Linear

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

$$\begin{aligned}(d\mathcal{F})_u &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \\ &:= \bar{\partial} Y + SY\end{aligned}$$

where $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

- *S is linear*
- *S tends to a symmetric operator as $s \rightarrow \pm\infty$, and*
-

$$\frac{\partial S}{\partial s}(s, t) \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{uniformly in } t$$

Proof

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Omitted – S is exactly O_0 from before:

$$\begin{aligned} O_0 &= \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right) \\ &= \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left(\frac{\partial u}{\partial t} - (Z_i) X_t \right) \right. \\ &\quad \left. + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right). \end{aligned}$$

- The term in blue vanishes as $s \rightarrow \pm\infty$
 - Using the fact that u is a solution
 - Uses $\frac{\partial u}{\partial s} \rightarrow 0$ uniformly (as do its derivatives?)
- Suffices to show the remaining part is symmetric in the limit

Proof

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

- Write the remaining part as

$$A(y_1, \dots, y_{2n}) = \dots \implies A_{ij} = A_{ji}$$

using inner product calculations

- Uses the fact the Z_i needed to be chosen to be unitary and symplectic.



Write O_1 as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm := \lim_{s \rightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

$$\partial_t Y = J_0 S^\pm Y$$

linearizes Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad \text{at} \quad \begin{cases} x = \lim_{s \rightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \rightarrow \infty} u & \text{for } S^+ \end{cases} \quad \text{respectively.}$$

Image

Reminder the x, y were the top/bottom pieces of the original cylinder/sphere:

