Title

D. Zack Garza

Contents

1 Lecture 2

1.1 Review

From last time: we want to prove the following theorem of Serre, a complex analog of the Weil conjectures. After this, we'll talk about étale morphisms, the étale topology, and possibly the definition of étale cohomology.

Theorem 1.1.1 *(Serre).*
Let $X_{/\mathbb{C}}$ be a smooth projective variety and $[H] \in H^2(X;\mathbb{Z})$ be a hyperplane class[a] and an endomorphism $F : X \to X$ a map satisfying $F^*[H] = q[H]$ for some $q \in \mathbb{Z}_{\geq 1}$. Then the eigenvalues of F^* on $H^i(X;\mathbb{C})$ all have absolute value $q^{\frac{i}{2}}$.

[a]Intersection with a hyperplane in projective space.

Note that the same q is appearing in both parts of the theorem. Why prove this theorem? Later on, to prove the Riemann hypothesis for varieties over finite fields, we'll prove that the Frobenius acts in this way on the étale cohomology. There is in fact a *reason* this is true, coming from some special properties of the behaviors of the cohomology of varieties which aren't manifested in random topological spaces.

⚠ *Warning 1.1:* The proof here will not look at all like Deligne's proof of the Riemann hypothesis for varieties over finite fields. We'll see shadows of it, but use a lot of things that are true for complex varieties that are still not known for varieties over finite fields.

Fact 1.1 : There is a cup product map

$$\begin{aligned} L : H^i(X;\mathbb{C}) &\to H^{i+2}(X;\mathbb{C}) \\ \alpha &\mapsto \alpha \smile [H]. \end{aligned}$$

Thus taking the direct sum $\bigoplus_i H^i(X;\mathbb{C})$ yields an algebra.

Theorem 1.1.2 *(Hard Lefschetz).*
Each $H^i(X;\mathbb{C}) \cong \mathrm{im}(L) \oplus H^i_{\mathrm{prim}}$, which is an isomorphism that depends on a choice of hyperplane class $[H]$ but is then canonically defined. Moreover, there is a Hodge decomposition $H^i_{\mathrm{prim}} = \bigoplus_{p+q=i} H^{p,q}_{\mathrm{prim}}$.

Theorem 1.1.3 *(Hodge Index Theorem).*
If $\alpha, \beta \in H^k(X)_{\mathrm{prim}}$, then there is a natural pairing

$$\langle a,\ b\rangle = i^* \int_X a \wedge \overline{\beta} \wedge [H]^{n-k},$$

where we've used the fact that the integrand is a top form and can thus be integrated. Moreover, this is a *definite* bilinear form on $H^{p,q}_{\text{prim}}$, i.e. a nonzero element paired with itself is again nonzero.

The moral of the story here is that cohomology breaks up into pieces, where $\operatorname{im} L$ comes from lower degrees and can perhaps be controlled inductively, and the higher dimensional pieces carry a canonical definite bilinear form.

1.2 Sketch proof of Serre's analog of the Riemann hypothesis

As a reminder, we want to show that the eigenvalues of F^* acting on $H^k(X;\mathbb{C})$ have absolute value $q^{\frac{k}{2}}$ where q is the scalar associated to F acting on $[H]$.

Claim: It suffices to do this for H^k_{prim}.

Why is this true? If we have an eigenvector $\alpha \in H^{k-2}(X;\mathbb{C})$, then by induction on k we can assume the eigenvalue has absolute value $q^{\frac{k-2}{2}}$. Then $F^*(\alpha \smile [H]) = F^*\alpha \smile F^*[H] = \lambda\alpha \smile q[H] = q\lambda(\alpha \smile [H])$, so this is an eigenvector of absolute values $qq^{\frac{k-2}{2}} = q^{\frac{k}{2}}$.

Now for the primitive part, let $\alpha \in H^k_{\text{prim}}$ be an F^* eigenvector. Since F^* preserves $H^{p,q}$, we can assume $\alpha \in H^{p,q}_{\text{prim}}$ for some $p+q=k$. Consider

$$\langle F^*\alpha,\ F^*\alpha\rangle.$$

On one hand, this is equal to $|\lambda|^2\langle\alpha,\ \alpha\rangle$ by sesquilinearity, pulling out a λ and a $\bar{\lambda}$. On the other hand, it is equal to

$$\begin{aligned}
\cdots &= i^* \int F^*\alpha \wedge F^*\bar{\alpha} \wedge [H]^{n-k} \\
&= \frac{i^k}{q^{n-k}} \int F^*\left(\alpha \wedge \bar{\alpha} \wedge [H]^{n-k}\right) \\
&= \frac{1^n i^k}{q^{n-k}} \int \alpha \wedge \bar{\alpha} \wedge H^{n-k} \\
&= q^k\langle\alpha,\ \alpha\rangle.
\end{aligned}$$

Exercise 1.2.1 (?): Using the Lefschetz hyperplane theorem or Poincaré duality, F^* acts on $H^{2n}(X;\mathbb{C})$ via q^n.

So we're done if $\langle\alpha,\ \alpha\rangle \neq 0$, since this implies $|\lambda|^2 = q^k$ and thus $|\lambda| = q^{\frac{k}{2}}$. Why is this true? This is the statement of the Hodge index theorem.

Remark 1.2.1 (Slogan): The structures on cohomology imply this complex analog of the Riemann hypothesis, and we'll want to use something similar for varieties over a finite field. This will be hard! Deligne doesn't quite accomplish this: there's no analog of the Hodge decomposition and we don't know the Hodge index theorem.

This is the proof that will motivate much of the rest of what we'll do in the course.

1.3 Setting Things Up: Étale Morphisms

This is a property of morphism of schemes, see Hartshorne.

Definition 1.3.1 (Étale Morphism)**.**
Suppose $f : X \to Y$ is a morphism of schemes. Then f is **étale** is it is locally of finite presentation, flat, and unramified.

Definition 1.3.2 (Unramified)**.**
f is **unramified** if $\Omega_{X/Y}1 = 0$ (the relative Kahler differentials). Equivalently, all residue field extensions are separable, i.e. given a point in Y with a point in X above it, the residue fields of these points gives a field extension, and we require it to be separable.

Definition 1.3.3 (Formally Etale)**.**
Suppose we have a nilpotent ideal I, so $I^n = 0$ for some n, then $f : X \to Y$ is **formally étale** if there is a unique lift in the following diagram:

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dotted, "\exists!"] & Y
\end{tikzcd}$$

Remark 1.3.1 : This is supposed to resemble a covering space map: We have $\mathrm{Spec}(A) \in Y$ with a nilpotent thickening and a map from A/I, which you may think of as a reduced subscheme. This thus says that tangent vectors downstairs can be lifted in a unique way to tangent vectors upstairs:

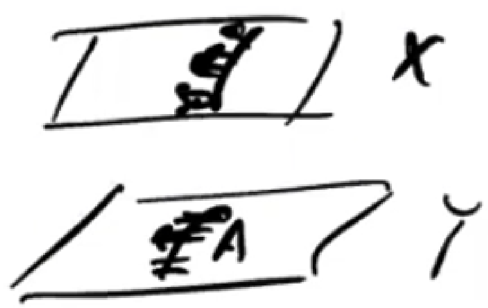

Figure 1: Image

Remark 1.3.2 : There are some equivalent definitions of a morphism being étale:

- Smooth of relative dimension zero
- Locally finite presentation and *formally étale*
- Locally *standard étale*, i.e. for each $x \in X$ with $y := f(x)$, there exists a $U \ni x, V \ni y$ such that $f(U) \subseteq V$ and $V = \operatorname{Spec}(R), U = \operatorname{Spec}\left(R[x]_h/g\right)$ (where we localize at h) such that the derivative g' is a unit in $R[x]_h$ and g is monic.

For this last definition, thinking of $\operatorname{Spec}(R[x])$ as $R \times \mathbb{A}^n$, what happens when modding out by a polynomial g? This yields a curve cutting out the roots of g. Inverting h deletes the locus where h vanishes, and g' being a unit means that the g has no double roots in the fibers. In other word, the delete locus passes through all double roots:

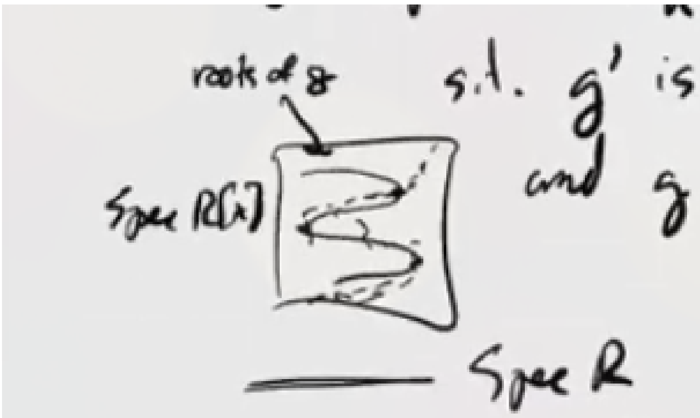

Figure 2: Image

Exercise 1.3.1 (?): Check that standard étale morphisms are étale, and try to understand the proof that all étale morphisms are locally standard étale.

Let's do some examples!

Example 1.3.1 (Example of an étale morphism):

- Multiplication by $[n]$ on an elliptic curve if $n \in \mathbb{Z}$ is invertiable in the base field.
- Take $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, and the map

$$\begin{aligned} \mathbb{G}_m &\to \mathbb{G}_m \\ t^m &\mapsfrom t, \end{aligned}$$

  where n is prime to $\operatorname{ch}(k)$.[1]

  - Note that this is in fact finite étale.

[1] Here we use the convention that everything is prime to zero.

Exercise 1.3.2 (?): Show that the last map above is étale. Hint: use the fact that $\dfrac{\partial}{\partial t}(t^n) = nt^{n-1}$, which is a unit.

Example 1.3.2 (?): Consider the map

$$\begin{aligned} \mathbb{G}_m &\hookrightarrow \mathbb{A}^1 \\ k[t, t^{-1}] &\mapsfrom k[t]. \end{aligned}$$

We need to check 3 things:

- Locally finite presentation,
  - This is a finitely presented ring map, since you just need to adjoin an inverse of t, one element and one relation.
- Flat,
  - Since open embeddings are flat,
- $\Omega^1_{\mathbb{G}_m/\mathbb{A}^1} = 0$,
  - True for a Zariski open embedding.

Note that this is finite onto its image.

Proposition 1.3.1 *(?).*
Any open immersion is étale.[a]

[a] Note that we actually already checked this!

Example 1.3.3 (An étale morphism that is not finite onto its image): Use the fact that $\mathbb{G}_m$ is $\mathbb{A}^1 \setminus \{\mathbf{0}\}$, so take $\mathbb{G}_m \setminus \{1\}$ and the map

$$\begin{aligned} \mathbb{G}_m \setminus \{1\} &\to \mathbb{G}_m \\ t^2 &\mapsfrom t. \end{aligned}$$

What's the picture? For the squaring map, there are two square roots:

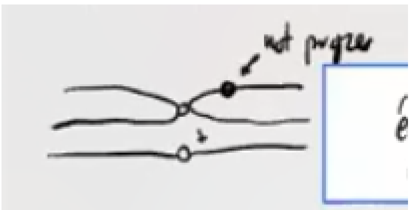

Figure 3: Image

This is an étale surjection but not finite étale, since it is not proper. This also gives a counterexample to étale morphisms always looking like covering spaces, since here that may be true locally but doesn't hold globally.

⚠ *Warning 1.2:* This is an important example to keep in mind, because you'll often see arguments that treat étale maps as though they were finite onto their image.

Example 1.3.4 (?): Take a finite separable field extension, taking Spec of it yields an étale map.

Now for some non-examples:

Example 1.3.5 (A finite map which is not etale): Take $X = \operatorname{Spec} k[x,y]/xy$, which looks like the following:

Figure 4: X

Then the normalization $\tilde{X} \to X$ is not étale, since it is not flat.

Example 1.3.6 (A finite flap map which is not etale): Take the map

$$\begin{aligned} \mathbb{A}^1 &\to \mathbb{A}^1 \\ t^2 &\hookleftarrow t. \end{aligned}$$

The picture is the following:

Figure 5: Image

This is note étale since it is ramified at zero. We can compute

$$\Omega^1_f = k[t]\,dt/d(t^2) = k[t]dt/2t\,dt,$$

where $2t\,dt$ does not generate this module. This is supported at $t = 0$ if $\mathrm{ch} \neq 2$.