Math 174 HW #6

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1)

	x_i	y_i	x_i^2	$x_i y_i$
	-1	-1.1	1	1.1
	-0.5	-1	$-.25$	$.25$
	0	0.9	0	0
	0.5	1.8	$.25$	$.9$
	1	3.2	1	3.2
Σ	$0 = \Sigma x_i$	$3.8 = \Sigma y_i$	$2 = \Sigma x_i^2$	$5.45 = \Sigma x_i y_i$

a)

$$a_0 = (\Sigma x_i^2)(\Sigma y_i) - (\Sigma x_i y_i)(\Sigma x_i) / m\Sigma x_i^2 - (\Sigma x_i)^2$$

$$= 2\cdot 3.8 - \cancel{5.45\cdot 0} / 5\cdot 2 - \cancel{0^2}$$

$$= 7.6/10 = \underline{0.76}$$

$$a_1 = m\Sigma x_i y_i - \Sigma x_i \Sigma y_i / m\Sigma x_i^2 - (\Sigma x_i)^2$$

$$= 5\cdot 5.45 - \cancel{0\cdot 3.8} / 5\cdot 2 - 0^2$$

$$= 27.25/10 = \underline{2.725}$$

b)

$$\boxed{\rightarrow L(x) = a_1 x + a_0 = 2.725x + 0.76}$$

$\underline{n=2, m=5}$

$j=0$

$$a_0 m + a_1 \Sigma x_i + a_2 \Sigma x_i^2 = \Sigma y_i$$

$$a_0 \Sigma x_i + a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 = \Sigma x_i y_i$$

$$a_0 \Sigma x_i^2 + a_1 \Sigma x_i^3 + a_2 \Sigma x_i^4 = \Sigma x_i^2 y_i$$

c)

$$\rightarrow \begin{bmatrix} m & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \end{bmatrix} \qquad \boxed{\text{Eqn (1)}}$$

$$\rightarrow \begin{bmatrix} 5 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2.125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 5.45 \\ 2.3 \end{bmatrix}$$

2) Let $P(x) = a_0$, then

$$E = \sum_{i=1}^{n} (y_i - P(x_i))^2$$

$$= \sum_{i=1}^{n} (y_i - a_0)^2$$

$$= \sum_{i=1}^{n} y_i^2 - 2a_0 \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a_0^2 = \Sigma y_i^2 - 2a_0 \Sigma y_i + na_0^2$$

$$= \sum_{i=1}^{n} y_i^2 - 2a_0 \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a_0^2 = \sum y_i^2 - 2a_0 \sum y_i + na_0^2$$

$$\rightarrow \partial E / \partial a_0 = -2 \sum_{i=1}^{n} y_i + 2na_0$$

And $\partial E/\partial a_0 = 0 \implies \sum_{i=1}^{n} y_i = na_0$ $\boxed{\Rightarrow a_0 = \frac{1}{n} \sum_{i=1}^{n} y_i}$

So a_0 is the average/mean of the y_i.

3) Using eqn 1

$\sum x_i = 1$ $\quad$ $\sum y_i = 5.3$

$\sum x_i^2 = 0.3$ $\quad$ $\sum x_i y_i = 1.1$

$\sum x_i^3 = 0.1$ $\quad$ $\sum x_i^2 y_i = 0.3354$

$\sum x_i^4 = 0.0354$

$$\rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & .3 \\ 1 & .3 & .1 \\ .3 & .1 & .0354 \end{bmatrix}^{-1} \begin{bmatrix} 5.3 \\ 1.1 \\ .3354 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \varepsilon \\ 1 \end{bmatrix} \text{ where } \varepsilon \approx 1.4211 \cdot 10^{-14}$$

$$\boxed{\rightarrow P(x) = x^2 + \varepsilon x + 1 \ (\approx x^2 + 1)}$$

4) Let $P(x) = \sum_{i=0}^{n} a_i (1+x)^i$, then $P_n(\bar{x}) = A\bar{x}$ for some A

Then

$$P(x_i) = y_i \Rightarrow a_0 + a_1(1+x_i) + a_2(1 + x_i + x_i^2) = y_i$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 1+x_1 & 1+x_1+x_1^2 \\ 1 & 1+x_2 & 1+x_2+x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & 1+x_n & 1+x_n+x_n^2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\bar{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\bar{y}}$$

$\rightarrow$ Solve $X\bar{a} = \bar{y}$ for $\bar{a}$.

So define $r(\bar{a}) = \bar{y} - X\bar{a}$, and let us minimize $\|r(\bar{a})\|^2 = \|X\bar{a} - \bar{y}\|^2$. $\quad$ where $\langle \bar{v}, \bar{w} \rangle_c = \sum_{i=1}^{n} c_i v_i w_i$ by defn

We then have $\|r(\bar{a})\|_c^2 = \langle r(\bar{a}), r(\bar{a}) \rangle_c = \langle C r(\bar{a}), r(\bar{a}) \rangle$ $\quad$ and $\|\bar{v}\|_c = \sqrt{\langle \bar{v}, \bar{v} \rangle_c}$

$= \langle C^{1/2} r(\bar{a}), C^{1/2} r(\bar{a}) \rangle$ $\quad$ where $C_{ij} = \begin{cases} c_i & \text{if } i=j \\ 0 & \text{else} \end{cases} = \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \; c_n \end{bmatrix}$, note $C = C^T$

$= \|C^{1/2} r(\bar{a})\|_2^2$ $\quad$ where $C^{1/2}_{ij} = \sqrt{C_{ij}}$

$= \|C^{1/2}(X\bar{a} - \bar{y})\|_2^2$

which is minimized when $X^T C^{1/2} X \hat{a} = X^T C^{1/2} \bar{y}$

$$\boxed{\rightarrow \hat{a} = (X^T C^{1/2} X)^{-1} X^T C^{1/2} \bar{y}}$$

5)

a)

x_i	y_i	Back	Center	Forward
0	1	x	x	.05
0.1	1.01	.05	.2	.15
0.2	1.04	.15	.4	.25
0.3	1.09	.25	.6	.35
0.4	1.16	.35	x	x

$(h = 0.1, 2h = 0.2)$

Note the central dif has error in $O(h^2)$, so it's used when possible.

b) $f(x) = 1 + x^2 \to \frac{\partial}{\partial x} f(x) = 2x$

x_i	$f'(x_i)$	Δ_x	$\lvert \Delta_x - f'(x) \rvert$
0	0	05	.05
0.1	.2	.2	0
0.2	.4	.4	0
0.3	.6	.6	0
0.4	.8	.35	.45

c) $f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h} = (1/2h)\left(f(x_i+h) - f(x_i - h)\right)$

$= (1/2h)\left(\sum_{n=0}^{\infty} \frac{1}{n!} h^n f^{(n)}(x_i) - \sum_{n=0}^{\infty} \frac{1}{n!}(-h)^n f^{(n)}(x_i)\right)$

$= (1/2h) \sum_{n=0}^{\infty} \left(\frac{1}{n!} h^n f^{(n)}(x_i)\right) + (-1)^{n+1} h^n f^{(n)}(x_i)\Big]$ when n is even, $n+1$ is odd, and the terms cancel

$= (1/2h) \sum_{n \text{ odd}} 2 \cdot \frac{1}{n!} h^n f^{(n)}(x_i)$

$= \frac{1}{h}\left(h f'(x) + \frac{1}{6} h^3 f^{(3)}(x_i) + \ldots\right)$

$= f'(x) + \underbrace{\frac{1}{6} h^2 f^{(3)}(x_i) + \ldots}$

But $f^{(3)}(x) = \frac{\partial^3}{\partial x^3} 1 + x^2$

$= \frac{\partial^2}{\partial x^2} 2x$

$= \frac{\partial}{\partial x} 2 \quad = 0$ for all x, so these terms are all zero

and $\Delta_x = f'(x) + 0$
$= f(x)$

d) Using $f''(x_i) \approx \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2}$

$\longrightarrow f''(.2) = \frac{1.01 - 2(1.04) + 1.09}{(.1)^2} \boxed{= 2}$

6) $h = 0.1$

$f'(1) \approx \frac{f(1+.1) + f(1-.1)}{.2} = \frac{f(1.1) - f(.9)}{2} \approx .539402252$

$h = 0.05$

$f'(1) \approx \frac{f(1.05) - f(.95)}{.1} \approx .540077208$

$h = .025$ $\quad f'(1) \approx \frac{f(1.025) - f(.975)}{.05} \approx .540246026$

b) $\frac{\partial}{\partial x} \sin(x) = \cos(x)$

$h = 0.1$ $\quad E = \lvert \Delta_x - \cos(1) \rvert \approx 8.798 \times 10^{-4}$

$h = .05$ $\quad E = 2.251 \times 10^{-4}$

$h = .025$ $\quad E = 5.627 \times 10^{-5}$

c) $h = 0.1 \longrightarrow E(h)/E(h/2) \approx 3.903$

$h = .05 \longrightarrow E(h)/E(h/2) \approx 4.0004$

$\rightsquigarrow$ Possibly converging to 4.

d) $E(x,h) \approx O(h^2)$, so choose h such that

$h^2 \le 10^{-10}$

$\to 2 \log_{10} h \le -10$

$\to \log_{10} h \le -5$

$\to \boxed{h \le 10^{-5}}$

7) $(1/h^2)(f(x+h) - 2f(x) + f(x-h)) = (1/h^2)\left(\sum_{n=0}^{\infty} \frac{1}{n!} h^n f^{(n)}(x) - 2\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) + \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n h^n f^{(n)}(x) \right)$

$$= (1/h^2) \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) (h^n - 2 + (-1)^n h^n)$$

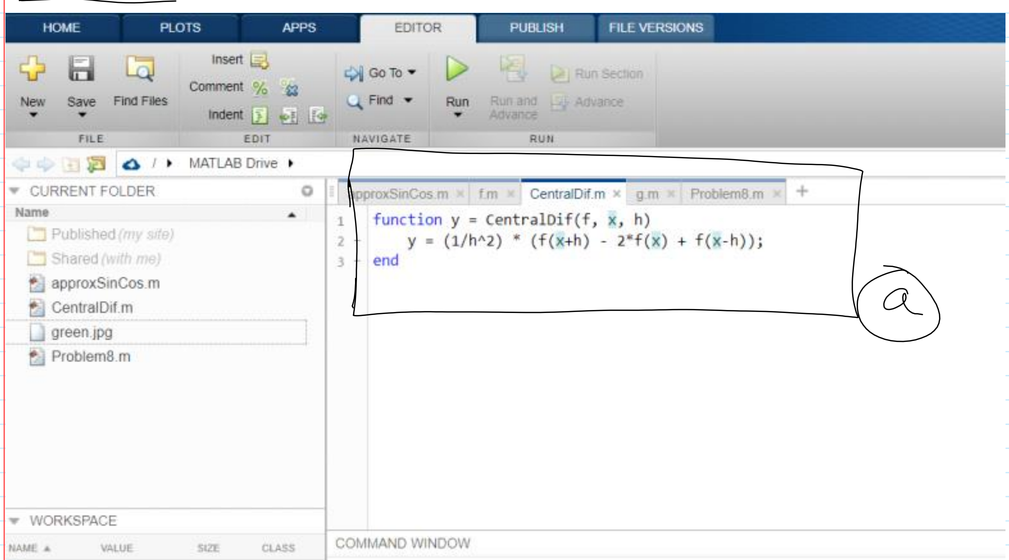
$$= (1/h^2)\left[0 - 2f'(x) + \tfrac{1}{2}(2h^2-2) f''(x) - \tfrac{1}{3} f'''(x) + \tfrac{1}{24}(2h^4-2) f^{(4)}(x) - \dots \right]$$

$$= (-2/h^2) f'(x) + \frac{h^2-1}{h^2} f''(x) - \frac{1}{3h^2} f'''(x) + \frac{1}{12} \frac{h^4-1}{h^2} f^{(4)}(x) - \dots$$

$$\to \left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| \le \left| \frac{-2}{h^2} f'(x) - \frac{1}{h^2} f''(x) - \frac{1}{3h^2} f'''(x) + \underbrace{\frac{h^2 - 1/h^2}{12} f^{(4)}(x)}_{\text{First positive power of } h} \dots \right|$$

$$\approx O(h^2)$$

8) See attached.



b

```
h = 0.05;

f = @(x) exp(x)*sin(x^2);
arrayfun(@(x) CentralDif(f, x, h), [0,1,2])

f = @(x) 1/(1+9*x^2);
arrayfun(@(x) CentralDif(f, x, h), [-1, 0, 1])
```

```
ans =

    2.0025    1.8998   35.9420


ans =

    0.4693  -17.6039    0.4693

>>
```