

# Problem Set 1

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Source: Section 1 of Gathmann

## 1 Exercises

**Exercise 1.1** (Gathmann 1.19).

Prove that every affine variety  $X \subset \mathbb{A}^n/k$  consisting of only finitely many points can be written as the zero locus of  $n$  polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in  $X$  have different  $x_1$ -coordinates.

**Solution:**

Let  $X = \{\mathbf{p}_1, \dots, \mathbf{p}_d\} = \{\mathbf{p}_j\}_{j=1}^d$ , where each  $\mathbf{p}_j \in \mathbb{A}^n$  can be written in coordinates

$$\mathbf{p}_j := [p_j^1, p_j^2, \dots, p_j^n].$$

**Remark.**

Proof idea: for some fixed  $k$  with  $2 \leq k \leq n$ , consider the pairs  $(p_j^1, p_j^k) \in \mathbb{A}^2$ . Letting  $j$  range over  $1 \leq j \leq d$  yields  $d$  points of the form  $(x, y) \in \mathbb{A}^2$ , so construct an interpolating polynomial such that  $f(x) = y$  for each tuple. Then  $f(x) - y$  vanishes at every such tuple.

Doing this for each  $k$  (keeping the first coordinate always of the form  $p_j^1$  and letting the second coordinate vary) yields  $n - 1$  polynomials in  $k[x_1, x_k] \subseteq k[x_1, \dots, x_n]$ , then adding in the polynomial  $p(x) = \prod_j (x - p_j^1)$  yields a system that vanishes precisely on  $\{\mathbf{p}_j\}$ .

**Claim:** Without loss of generality, we can assume all of the first components  $\{p_j^1\}_{j=1}^d$  are distinct.

Todo: follows from "rotation of axes"?

We will use the following fact:

**Theorem 1.1 (Lagrange).**

Given a set of  $d$  points  $\{(x_i, y_i)\}_{i=1}^d$  with all  $x_i$  distinct, there exists a unique polynomial of degree  $d$  in  $f \in k[x]$  such that  $f(x_i) = y_i$  for every  $i$ .

This can be explicitly given by

$$\tilde{f}(x) = \sum_{i=1}^d y_i \left( \prod_{\substack{0 \leq m \leq d \\ m \neq i}} \left( \frac{x - x_m}{x_i - x_m} \right) \right).$$

Equivalently, there is a polynomial  $f$  defined by  $f(x_i) = \tilde{f}(x_i) - y_i$  of degree  $d$  whose roots are precisely the  $x_i$ .

Using this theorem, we define a system of  $n$  polynomials in the following way:

- Define  $f_1 \in k[x_1] \subseteq k[x_1, \dots, x_n]$  by

$$f_1(x) = \prod_{i=1}^d (x - p_i^1).$$

Then the roots of  $f_1$  are precisely the first components of the points  $p$ .

- Define  $f_2 \in k[x_1, x_2] \subseteq k[x_1, \dots, x_n]$  by considering the ordered pairs

$$\{(x_1, x_2) = (p_j^1, p_j^2)\},$$

then taking the unique Lagrange interpolating polynomial  $\tilde{f}_2$  satisfying  $\tilde{f}_2(p_j^1) = p_j^2$  for all  $1 \leq j \leq d$ . Then set  $f_2 := \tilde{f}_2(x_1) - x_2 \in k[x_1, x_2]$ .

- Define  $f_3 \in k[x_1, x_3] \subseteq k[x_1, \dots, x_n]$  by considering the ordered pairs

$$\{(x_1, x_3) = (p_j^1, p_j^3)\},$$

then taking the unique Lagrange interpolating polynomial  $\tilde{f}_3$  satisfying  $\tilde{f}_3(p_j^1) = p_j^3$  for all  $1 \leq j \leq d$ . Then set  $f_3 := \tilde{f}_3(x_1) - x_3 \in k[x_1, x_3]$ .

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Continuing in this way up to  $f_n \in k[x_1, x_n]$  yields a system of  $n$  polynomials.

**Proposition 1.2.**

$$V(f_1, \dots, f_n) = X.$$

*Proof .*

**Claim:**  $X \subseteq V(f_i)$ :

This is essentially by construction. Letting  $p_j \in X$  be arbitrary, we find that

$$f_1(p_j) = \prod_{i=1}^d (p_j^1 - p_i^1) = (p_j^1 - p_j^1) \prod_{\substack{i \leq d \\ i \neq j}} (p_j^1 - p_i^1) = 0.$$

Similarly, for  $2 \leq k \leq n$ ,

$$f_k(p_j) = \tilde{f}_k(p_j^1) - p_j^k = 0,$$

which follows from the fact that  $\tilde{f}_k(p_j^1) = p_j^k$  for every  $k$  and every  $j$  by the construction of  $\tilde{f}_k$ .

**Claim:**  $X^c \subseteq V(f_i)^c$ :

This follows from the fact the polynomials  $f$  given by Lagrange interpolation are unique, and thus the roots of  $\tilde{f}$  are unique. But if some other point was in  $V(f_i)$ , then one of its coordinates would be another root of some  $\tilde{f}$ . ■

**Exercise 1.2** (Gathmann 1.21).

Determine  $\sqrt{I}$  for

$$I := \langle x_1^3 - x_2^6, x_1x_2 - x_2^3 \rangle \trianglelefteq \mathbb{C}[x_1, x_2].$$

**Solution:**

For notational purposes, let  $\mathcal{I}, \mathcal{V}$  denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider  $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$ , the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 &= 0 \\ xy - y^3 &= 0. \end{cases}$$

In the second equation, we have  $(x - y^2)y = 0$ , and since  $\mathbb{C}[x, y]$  is an integral domain, one term must be zero.

1. If  $y = 0$ , then  $x^3 = 0 \implies x = 0$ , and thus  $(0, 0) \in \mathcal{V}(I)$ , i.e. the origin is contained in this vanishing locus.
2. Otherwise, if  $x - y^2 = 0$ , then  $x = y^2$ , with no further conditions coming from the first equation.

Combining these conditions,

$$P := \{(t^2, t) \mid t \in \mathbb{C}\} \subset \mathcal{V}(I).$$

where  $I = \langle x^3 - y^6, xy - y^3 \rangle$ .

We have  $P = \mathcal{V}(I)$ , and so taking the ideal generated by  $P$  yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \langle y - x^2 \rangle \in \mathbb{C}[x, y]$$

and thus  $\sqrt{I} = \langle y - x^2 \rangle$ .

**Exercise 1.3** (Gathmann 1.22).

Let  $X \subset \mathbb{A}^3/k$  be the union of the three coordinate axes. Compute generators for the ideal  $I(X)$  and show that it can not be generated by fewer than 3 elements.

**Solution:**

**Claim:**

$$I(X) = \langle x_2x_3, x_1x_3, x_1x_2 \rangle.$$

We can write  $X = X_1 \cup X_2 \cup X_3$ , where

- The  $x_1$ -axis is given by  $X_1 := V(x_2x_3) \implies I(X_1) = \langle x_2x_3 \rangle$ ,
- The  $x_2$ -axis is given by  $X_2 := V(x_1x_3) \implies I(X_2) = \langle x_1x_3 \rangle$ ,
- The  $x_3$ -axis is given by  $X_3 := V(x_1x_2) \implies I(X_3) = \langle x_1x_2 \rangle$ .

Here we've used, for example, that

$$I(V(x_2x_3)) = \sqrt{\langle x_2x_3 \rangle} = \langle x_2x_3 \rangle$$

by applying the Nullstellensatz and noting that  $\langle x_2x_3 \rangle$  is radical since it is generated by a squarefree monomial.

We then have

$$\begin{aligned} I(X) &= I(X_1 \cup X_2 \cup X_3) \\ &= I(X_1) \cap I(X_2) \cap I(X_3) \\ &= \sqrt{I(X_1) + I(X_2) + I(X_3)} \\ &= \sqrt{\langle x_2, x_3 \rangle + \langle x_1x_3 \rangle + \langle x_1x_2 \rangle} \\ &= \sqrt{\langle x_2x_3, x_1x_3, x_1x_2 \rangle} && \text{since } \langle a \rangle + \langle b \rangle = \langle a, b \rangle \\ &= \langle x_2x_3, x_1x_3, x_1x_2 \rangle, \end{aligned}$$

where in the last equality we've again used the fact that an ideal generated by squarefree monomials is radical.

**Claim:**  $I(X)$  can not be generated by 2 or fewer elements.

Let  $J := I(X)$  and  $R := k[x_1, x_2, x_3]$ , and toward a contradiction, suppose  $J = \langle r, s \rangle$ . Define  $\mathfrak{m} := \langle x, y, z \rangle$  and a quotient map

$$\pi : J \longrightarrow J/\mathfrak{m}J$$

and consider the images  $\pi(r), \pi(s)$ .

Note that  $J/\mathfrak{m}J$  is an  $R/\mathfrak{m}$ -module, and since  $R/\mathfrak{m} \cong k$ ,  $J/\mathfrak{m}J$  is in fact a  $k$ -vector space. Since  $\pi(r), \pi(s)$  generate  $J/\mathfrak{m}J$  as a  $k$ -module,

$$\dim_k J/\mathfrak{m}J \leq 2.$$

But this is a contradiction, since we can produce 3  $k$ -linearly independent elements in  $J/\mathfrak{m}J$ : namely  $\pi(x_1x_2), \pi(x_1x_3), \pi(x_2x_3)$ . We have

$$\alpha_1\pi(x_1x_2) + \alpha_2\pi(x_1x_3) + \alpha_3\pi(x_2x_3) = 0 \in J/\mathfrak{m}J \implies \alpha_1(x_1x_2) + \alpha_2\pi(x_1x_3) + \alpha_3\pi(x_2x_3) \in \mathfrak{m}J.$$

**Exercise 1.4** (Gathmann 1.23: Relative Nullstellensatz).

Let  $Y \subset \mathbb{A}^n/k$  be an affine variety and define  $A(Y)$  by the quotient

$$\pi : k[x_1, \dots, x_n] \longrightarrow A(Y) := k[x_1, \dots, x_n]/I(Y).$$

- Show that  $V_Y(J) = V(\pi^{-1}(J))$  for every  $J \trianglelefteq A(Y)$ .
- Show that  $\pi^{-1}(I_Y(X)) = I(X)$  for every affine subvariety  $X \subseteq Y$ .
- Using the fact that  $I(V(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq k[x_1, \dots, x_n]$ , deduce that  $I_Y(V_Y(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq A(Y)$ .

Conclude that there is an inclusion-reversing bijection

$$\left\{ \begin{array}{c} \text{Affine subvarieties} \\ \text{of } Y \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{in } A(Y) \end{array} \right\}.$$

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**Exercise 1.5** (Extra).

Let  $J \trianglelefteq k[x_1, \dots, x_n]$  be an ideal, and find a counterexample to  $I(V(J)) = \sqrt{J}$  when  $k$  is not algebraically closed.

**Solution:**

Take  $J = \langle x^2 + 1 \rangle \trianglelefteq \mathbb{R}[x]$ , noting that  $J$  is nontrivial and proper but  $\mathbb{R}$  is not algebraically closed. Then  $V(J) \subseteq \mathbb{R}$  is empty, and thus  $I(V(J)) = I(\emptyset)$ .

**Claim:**  $I(V(J)) = \mathbb{R}[x]$ .

Checking definitions, for any set  $X \subset \mathbb{A}^n/k$  we have

$$I(X) = \left\{ f \in \mathbb{R}[x] \mid \forall x \in X, f(x) = 0 \right\}$$

and so we vacuously have

$$I(\emptyset) = \left\{ f \in \mathbb{R}[x] \mid \forall x \in \emptyset, f(x) = 0 \right\} = \{f \in \mathbb{R}[x]\} = \mathbb{R}[x].$$

**Claim:**  $\sqrt{J} \neq \mathbb{R}[x]$ .

This follows from the fact that maximal ideals are radical, and  $\mathbb{R}[x]/J \cong \mathbb{C}$  being a field implies that  $J$  is maximal. In this case  $\sqrt{J} = J \neq \mathbb{R}[x]$ .

That maximal ideals are radical follows from the fact that if  $J \trianglelefteq R$  is maximal, we have  $J \subset \sqrt{J} \subset R$  which forces  $\sqrt{J} = J$  or  $\sqrt{J} = R$ .

But if  $\sqrt{J} = R$ , then

$$1 \in \sqrt{J} \implies 1^n \in J \text{ for some } n \implies 1 \in J \implies J = R,$$

contradicting the assumption that  $J$  is maximal and thus proper by definition.