Review for the second midterm, Math 427
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Any past quiz or homework problem is fair game for the test.

The second midterm will mostly cover ring theory. However, you should not forget about the class equation, Cauchy's theorem, p-subgroups and semi-direct products. Nor should you foget about modules.

PRACTICE PROBLEMS

1 (a) Prove that S_n is a semidirect product of $A_n := \{\sigma \in S_n \mid \text{sign}(\sigma) = 1\}$ and a group of order 2.

(b) Consider the group $O(2)$ of 2×2 real orthogonal matrices. Prove that $O(2)$ is a semi-direct product of $SO(2) := \{A \in O(2) \mid \det A = 1\}$ and a group of order 2.

2 Suppose G is a finite group of order 20. Does it have to have an element of order 2? of order 5? Explain. Hint: Cauchy's theorem.
Does it have to have an element of order 4? Hint: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.

3 Suppose R is a commutative ring with 1, $p \in R$ is irreducible and $u \in R$ a unit. Prove that up is irreducible.

4 Suppose $\phi : G \to H$ and $\psi : H \to K$ are two ring homomorphisms. Prove that their composite $\psi \circ \phi : G \to K$ is also a ring homomorphism.

5 Consider the ideal (x) in the ring of polynomials $\mathbb{R}[x, y]$ in two variables. Prove that the quotient ring $\mathbb{R}[x, y]/(x)$ is isomorphic to the ring $\mathbb{R}[y]$. What does it tell you about the ideal (x)? Is it a maximal ideal? Is it a prime ideal?

6 Let F be a field, $p \in F[x]$ a polynomial of degree $n \geq 1$. Prove that the quotient ring $F[x]/(p)$ is a vector space over F of dimension n. Hints: Proving that the quotient is a vector space over F should be easy. The hard part is the dimension. Show that the set $\{1 + (p), x + (p), \ldots, x^{n-1} + (p)\}$ is a basis of $F[x]/(p)$.

7 Give an example of a principal ideal that is not prime.
Give an example of prime ideal that's not principal.
Give an example of a prime ideal that is not maximal.

8 Prove that $m, n \in \mathbb{Z}$ are relatively prime if and only if $m\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$.

9 Let R be a commutative ring with 1. Suppose that R has no proper ideals, that is, the only ideals in R are 0 and R. Prove that R is a field. Hint: what's $R/0$?

10 Let R be a ring with a subring A and let $I \subset R$ be an ideal. Prove that

(i) $A + I := \{a + i \in R \mid a \in A, i \in I\}$ is a subring of R.

(ii) $A \cap I$ is an ideal in A.

(ii) I is an ideal in $A + I$.

(iv) $(A + I)/I$ is isomorphic to $A/(A \cap I)$.

11 Let R be a ring, $K, I \subset R$ ideals with $K \subset I$. Prove that $I/K := \{i + K \mid i \in I\}$ is an ideal in R/K and that $(R/K)/(I/K)$ is isomorphic to R/I.

12 (i) Does the product ring $\mathbb{Z} \times \mathbb{Z}$ have a maximal ideal? If yes, find one.

(ii) Does the product ring $\mathbb{Z} \times \mathbb{Z}$ have zero divisors? If yes, find two.

13 Let M be a cyclic module over a PID R. Prove that there is $a \in R$ so that M is isomorphic (as an R module) to $R/(a)$.

14 Prove that the quotient ring $\mathbb{R}[x]/(x - 2)$ is isomorphic to to $\mathbb{R}$.

15 Let M be a cyclic R module generated by $x \in M$, $x \neq 0$. Let N be another R module and $v \in N$ an element with $v \neq 0$. Prove that if R is a field, there exists a module homomorphism $\varphi : M \to N$ with $\varphi(x) = v$. Give an example to show that this is not true if $R = \mathbb{Z}$: there may be no module homomorphism φ with $\varphi(x) = v$. Hint: consider $M = \mathbb{Z}_n$, $N = \mathbb{Z}$.

16 Let F be a field. We have seen that an $F[x]$ module M is a vector space over F together with a linear map $T : M \to M$ [$T(m) = xm$ for all $m \in M$]. What is an $F[x]$ module homomorphism $\varphi : M \to N$ in terms of vector spaces and linear maps?