

# Homotopy Groups of Spheres

## Graduate Student Seminar

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# Introduction

# Outline

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Introduction

Spheres

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

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## Homotopies of paths:



- Regard paths  $\gamma$  in  $X$  and homotopies of paths  $H$  as morphisms

$$\gamma \in \mathbf{hom}_{\mathbf{Top}}(I, X)$$

$$H \in \mathbf{hom}_{\mathbf{Top}}(I \times I, X).$$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write  $[\gamma]$  to denote a homotopy class of paths.

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- Why care about path homotopies? Historically: contour integrals in  $\mathbb{C}$



- By the residue theorem, for a meromorphic function  $f$  with simple poles  $P = \{p_i\}$  we know that

$$\oint_{\gamma} f(z) \, dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

# Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \text{hom}_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \text{hom}_{\text{Top}}(X \times I, Y)$$

such that  $F(0) = f, F(1) = g$ .

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] := \text{hom}_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \text{hom}(X, Y) \\ g \in \text{hom}(Y, X) \end{cases} \quad \text{such that} \quad \begin{cases} f \circ g \sim \text{id}_Y \\ g \circ f \sim \text{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \text{Top} \mid Y \sim X \right\}.$$

# The Fundamental Group

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- $\pi_1(X)$  is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$



# Higher Homotopy Groups

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- Can now generalize to define

$$\pi_k(X) := [S^k, X]$$



*Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look  $\mathbb{A}^1$  or  $\mathbb{P}^1$ .*



# Classification

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- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



*Also have the derived category  $D\text{Top}$ , its interplay with  $\text{hoTop}$  is the subject of e.g. the Poincare conjecture(s).*

- Any representative from a green box: a *homotopy type*.

# Example: Homotopy Equivalence is Useful

**Proposition:** Let  $B$  be a CW complex; then isomorphism classes of  $\mathbb{R}^1$ -bundles over  $B$  are given by  $H^1(X, \mathbb{Z}/2\mathbb{Z})$ .

- Use the fact that for any fixed group  $G$ , the functor

$$\begin{aligned} h_G(\cdot) : \text{hoTop}^{\text{op}} &\longrightarrow \text{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

is representable by a space called  $BG$  (Brown's representability theorem).

- I.e., let  $\text{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$ , there is an isomorphism

$$\text{Bun}_G(X) \cong [X, BG]$$

- In general, identify  $G = \text{Aut}(F)$  the automorphism group of the fibers – for vector bundles of rank  $n$ , take  $G = GL(n, \mathbb{R})$ .

# Example: Homotopy Equivalence is Useful

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Note that for a poset of spaces  $(M_i, \hookrightarrow)$ , the space  $M^\infty := \varinjlim M_i$ . These are infinite dimensional “Hilbert manifolds”.

Proof:

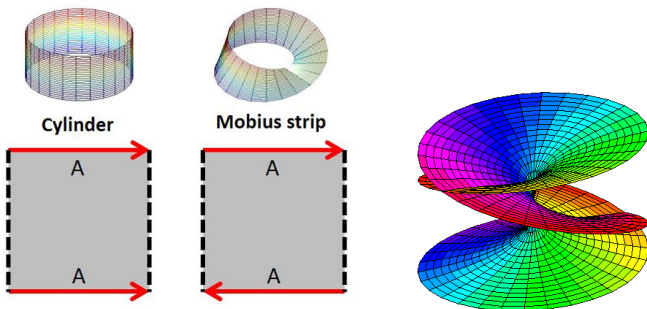
$$\begin{aligned}\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, \mathrm{BGL}(1, \mathbb{R})] \\ &= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\ &= [X, \mathbb{RP}^\infty] \\ &= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\ &= H^1(X; \mathbb{Z}/2\mathbb{Z})\end{aligned}$$

Work being swept under the rug: identifying the homotopy type of the representing object.

# Example: Homotopy Equivalence is Useful

**Corollary:** There are 2 distinct line bundles over  $X = S^1$  (the cylinder and the mobius strip), since  $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Corollary:** A Riemann surface  $\Sigma_g$  satisfies  $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$  and thus there are  $2^{2g}$  distinct real line bundles over it.



# Example: Higher Homotopy Groups are Useful

- Application: computing  $\pi_1(\mathrm{SO}(n, \mathbb{R}))$  (rigid rotations in  $\mathbb{R}^n$ ).
- The fibration

$$\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$$

yields a LES in homotopy:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_2(S^n) \\ & & & & & \swarrow & \\ & & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(\mathrm{SO}(n, \mathbb{R})) & \longrightarrow & \pi_1(S^n) \end{array}$$

# Uses of Higher Homotopy

Knowing  $\pi_k S^n$ , this reduces to

$$\begin{array}{ccccccc} \cdots 0 & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & \pi_2(SO(n, \mathbb{R})) & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & \pi_1(SO(n, \mathbb{R})) & \longrightarrow & 0 \end{array}$$

- Thus  $\pi_1(SO(3, \mathbb{R})) \cong \pi_1(SO(4, \mathbb{R})) \cong \cdots$  and it suffices to compute  $\pi_1(SO(3, \mathbb{R}))$  (stabilization)
- Use the fact that “accidental” homeomorphism in low dimension  $SO(3, \mathbb{R}) \cong_{\text{Top}} \mathbb{RP}^3$ , and algebraic topology I yields  $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$ .

*Can also use the fact that  $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$  is a double cover from the universal cover.*

# Uses of Higher Homotopy

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- Important consequence:  $SO(3, \mathbb{R})$  is not simply connected!
- See "plate trick": non-contractible loop of rotations that squares to the identity.
- Robotics: paths in configuration spaces with singularities
- Computer graphics: smoothly interpolating between quaternions for rotated camera views



Rotation  $R_{u,\theta}$ :

axis  $u$ , angle  $\theta$

$$\begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0q_1 - 2q_2q_3 & 2q_0q_2 + 2q_1q_3 & 2q_0q_3 - 2q_1q_2 \\ 2q_1q_0 + 2q_2q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & q_1^2 - q_2^2 - q_0^2 + q_3^2 \\ 2q_2q_0 - 2q_1q_3 & 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2q_2q_1 - 2q_0q_2 \\ 2q_3q_0 + 2q_1q_2 & q_1^2 - q_2^2 - q_0^2 + q_3^2 & 2q_0q_2 - 2q_1q_3 & q_0^2 + q_1^2 - q_2^2 - q_3^2 \end{pmatrix}$$

Unit quaternion:

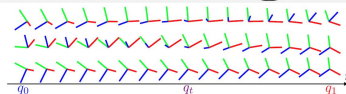
$$q = \cos(\theta/2) + (u_x i + u_y j + u_z k) \sin(\theta/2).$$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

Spherical Linear Interpolation (SLERP):

$$q_t \stackrel{\text{def}}{=} \frac{\sin((1-t)\omega)q_0 + \sin(t\omega)q_1}{\sin(\omega)}$$

$\mathbb{R}^4$



# Spheres



# Setup

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- Defining  $\pi_k(X) = [S^k, X]$ , the simplest objects to investigate:  
 $X = S^n$
- Can consider the bigraded group  $\pi_S := [S^k, S^n]$ :

$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
$\vdots$											

# But Wait!

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The corresponding picture in homology is very easy:

$H_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$	$\mathbb{Z}$										
2		$\mathbb{Z}$									
3			$\mathbb{Z}$								
4				$\mathbb{Z}$							
5					$\mathbb{Z}$						
6						$\mathbb{Z}$					
7							$\mathbb{Z}$				
8								$\mathbb{Z}$			
9									$\mathbb{Z}$		
10										$\mathbb{Z}$	
$\vdots$											

*Slogan: "conservation/duality of complexity"*

# History

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- 1895: Poincare, *Analysis situs* (“the analysis of position”) in analogy to Euler *Geometria situs* in 1865 on the Kongisberg bridge problem
  - Studies spaces arising from gluing polygons, polyhedra, etc (surfaces!), first use of “algebraic invariant theory” for spaces by introducing  $\pi_1$  and homology.
- 1920s: Rigorous proof of classification of surfaces (Klein, Möbius, Clifford, Dehn, Heegard)
  - Captured entirely by  $\pi_1$  (equivalently, by genus and orientability).
- **1931: Hopf discovers a nontrivial (not homotopic to identity) map  $S^3 \longrightarrow S^2$**

# History

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- 1932/1935: Cech (indep. Hurewicz) introduce higher homotopy groups, gives map relating  $\pi_* \rightarrow H_*$ , shows  $\pi_n X$  are **abelian** groups for  $n \geq 2$ .
  - Withdrew his paper because of this theorem!
- 1951: Serre uses spectral sequences to show that **all groups  $\pi_k S^n$  are torsion except,**
  - $k = n$ , since  $\pi_n S^n = \mathbb{Z}$
  - $k \equiv 3 \pmod{4}, n \equiv 0 \pmod{2}$ , then  $\mathbb{Z} \oplus T$
  - Tight bounds on where  $p$ -torsion can occur.
- 1953: Whitehead shows the homotopy groups of spheres split into stable and unstable ranges.

Today: We know  $\pi_{n+k} S^n$  for

- $k \leq 64$  when  $n \geq k + 2$  (stable range)
- $k \leq 19$  when  $n < k + 2$  (unstable range)
- We *only* have a complete list for  $S^0$  and  $S^1$ , and know *no* patterns beyond this!
  - Open for  $\sim 80$  years.

# Spheres

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We'll fill out as much of this table as is easily known:

		$\pi_k(S^n)$										
		$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$												
2												
3												
4												
5												
6												
7												
8												
9												
10												
:												

$$k < n$$

**Claim:**  $[S^k, S^n] = 0$  for  $k < n$ .



This follows easily from CW approximation:

*Any map  $X \xrightarrow{f} Y$  between CW complexes is homotopic to a cellular map.*

# $k < n$ : CW Complexes

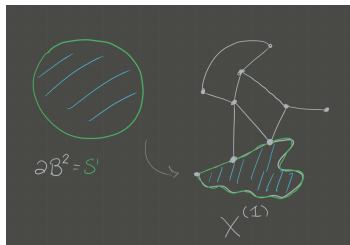
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- Analogy from analysis:  $C^1$  functions dense in  $L^2$ .
  - If you're just computing homotopy groups, *any* space can be replaced with a *weakly equivalent* CW complex.



# $k < n$ : CW Complexes

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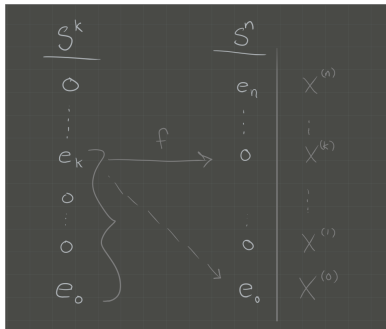
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AT1 can show that spheres have a simple cell decomposition

$$S^k = e_0 \amalg_f e_k$$

Thus any map  $f : S^k \rightarrow S^n$  must send the  $k$ -skeleton of  $S^k$  to the  $k$ -skeleton of  $S^n$ , which is just a point:





# $k \geq 1, n = 1$ : Covering Space Theory

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**Claim:**  $\pi_1 S^1 = \mathbb{Z}$  and  $\pi_{\geq 2} S^1 = 0$ .

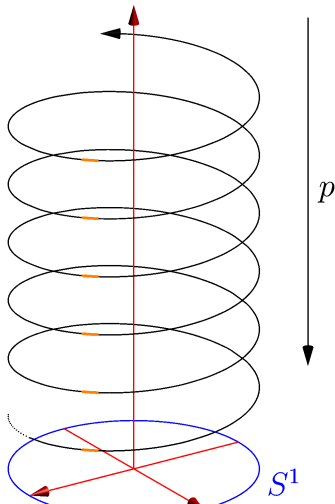
$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	...
$n = 1$											
2											
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...											

# $k \geq 1, n = 1$ : Covering Space Theory

- Use the fact that  $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$  is a covering space and  $\mathbb{Z} \curvearrowright \mathbb{R}$  freely.

$\mathbb{R}$



# $k \geq 1, n = 1$ : Covering Space Theory

Theorem: If  $F \longrightarrow E \longrightarrow B$  is a *Serre Fibration* then there is a LES in homotopy

$$\begin{array}{ccccccc} & & \dots & \longrightarrow & \pi_{k+1}(B) & & \\ & \swarrow & & & \searrow & & \\ \pi_k(F) & \longrightarrow & \pi_k(E) & \longrightarrow & \pi_k(B) & & \\ & \nwarrow & & & \swarrow & & \\ \pi_{k-1}(F) & \longrightarrow & \dots & & \dots & & \end{array}$$

- If  $\tilde{X} \longrightarrow X$  is a universal cover then  $\pi_{\geq 2}(X) \cong \pi_{\geq 2}\tilde{X}$ .
  - Proof coming up!

# Misc: Serre Fibrations

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Claim:  $\pi_2 S^2 = \pi_3 S^2 = \pi_3 S^3 = \mathbb{Z}$ .

$\pi_k(S^n)$

	$k = 1$	2	3	4	5	6	7	8	9	10	$\dots$
$n = 1$											
2											
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10											
$\vdots$											

# Misc: Serre Fibrations

Use the Hopf fibration:  $S^1 \longrightarrow S^3 \longrightarrow S^2$  and the fact that  $\pi_{\geq 2} S^1 = 0$ :

$$\begin{array}{ccccc} \pi_3(S^1) & \longrightarrow & \pi_3(S^3) & \longrightarrow & \pi_3(S^2) \\ & & \swarrow & & \\ \pi_2(S^1) & \longrightarrow & \pi_2(S^3) & \longrightarrow & \pi_2(S^2) \\ & & \swarrow & & \\ \pi_1(S^1) & \longrightarrow & \pi_1(S^3) & \longrightarrow & \pi_1(S^2) \end{array}$$

$$\begin{array}{ccccc} 0 & \longrightarrow & \pi_4(S^3) & \longrightarrow & \pi_4(S^2) \\ & & \swarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3(S^2) \\ & & \swarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \pi_2(S^2) \\ & & \swarrow & & \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

*Note that this works whenever the fiber is contractible (e.g. universal covers, fibers are discrete)*

- Hopf Fibration Visualizer

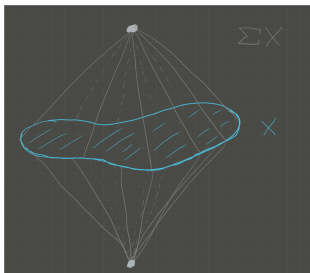
# $n = k$ : Stabilization

- Theorem (1937, Freudenthal): For  $k \gg 0$ ,

$$[\Sigma^k X, \Sigma^k Y] \cong [\Sigma^{k+1} X, \Sigma^{k+1} Y]$$

- Use the fact that  $\Sigma S^k \cong S^{k+1}$ , then in some *stable range*

$$\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1}$$



*Fun note: corresponds to “smash with a sphere”*

# $n=k$ : Stabilization

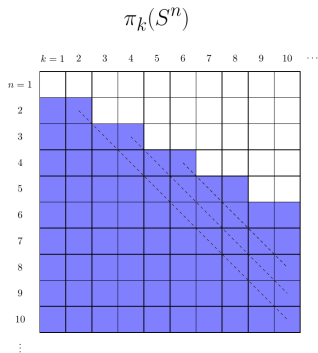
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Stable range:



# $n = k$ : Stabilization

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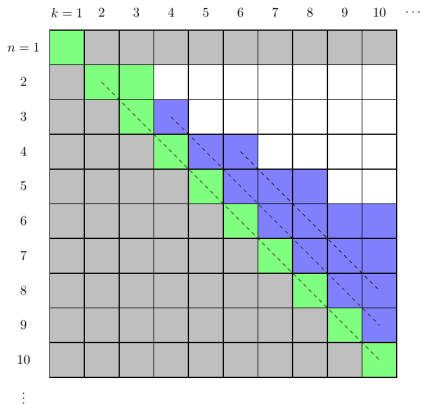
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We can thus suspend things we already know:

$$\pi_k(S^n)$$





$$k = 4, n = 3$$

- Construct a map  $S^3 \rightarrow K(\mathbb{Z}, 2)$  by “killing off homotopy” (identify  $K$ )
- Convert to a fibration and take the homotopy fiber to get  $F \rightarrow S^2 \rightarrow \mathbb{CP}^\infty$
- By LES,  $\pi_{\geq 3} F \cong \pi_{\geq 3} S^2$ , by Hurewicz  $\pi_3 F \cong H_3 F$ , which we can compute
- Kill homotopy again and *iterated* homotopy fiber to get  $G \rightarrow F \rightarrow K(\mathbb{Z}, 3)$
- By LES,  $\pi_{\geq 4} G \cong \pi_{\geq 4} S^2$ , by Hurewicz  $\pi_4 G \cong H_4 G$ .

$$k=4, n=3$$

- Look at Serre's spectral sequence for  $G \rightarrow F \rightarrow K(\mathbb{Z}, 3)$ .

$$H_5 G$$

$$H_4 G$$

$$0$$

$$0$$

$$0$$

$$\mathbb{Z}$$

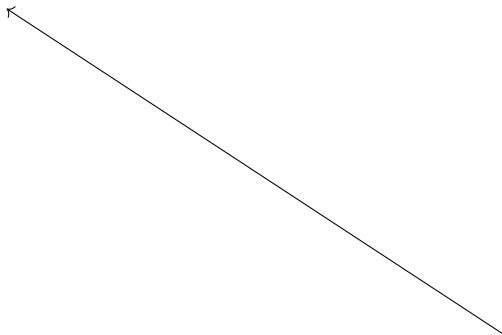
$$0$$

$$0$$

$$\mathbb{Z}$$

$$?$$

$$?$$

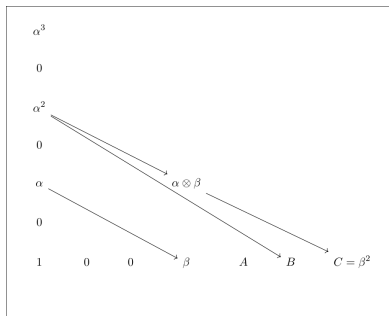


$$k=4, n=3$$

- Want target of arrow. Need to know
  - $H^4 F, H^5 F$  (total cohomology)
  - $H^5 K(\mathbb{Z}, 3)$  (source of arrow)
- Use Serre SS on  $F \rightarrow S^2 \rightarrow K(\mathbb{Z}, 2)$  to deduce  $H^5 F = H^6 F = \mathbb{Z}$
- Use Serre SS on  $\Omega K(\mathbb{Z}, 3) \rightarrow \{\text{pt}\} \rightarrow K(\mathbb{Z}, 3)$ :

$$k=4, n=3$$

Relevant quantity will be  $C$  term in SSS:



- Everything must go! Converges to homology of a point.
- Work out  $d\alpha = \beta$
- Work out  $d(\alpha^2) = 2\alpha \otimes \beta$
- Work out  $\beta^2 \neq 0$
- Work out  $\ker(\alpha^2 \rightarrow \alpha \otimes \beta) = 2\langle \alpha \otimes \beta \rangle$
- Conclude  $A = B = 0$  and  $C \cong \mathbb{Z}/2\mathbb{Z}$ .