Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Linearization and Transversality

Sections 8.3 and 8.4

D. Zack Garza

April 2020

Review 8.2

Section 8.3: The Space of Perturbations of H

Goal: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.
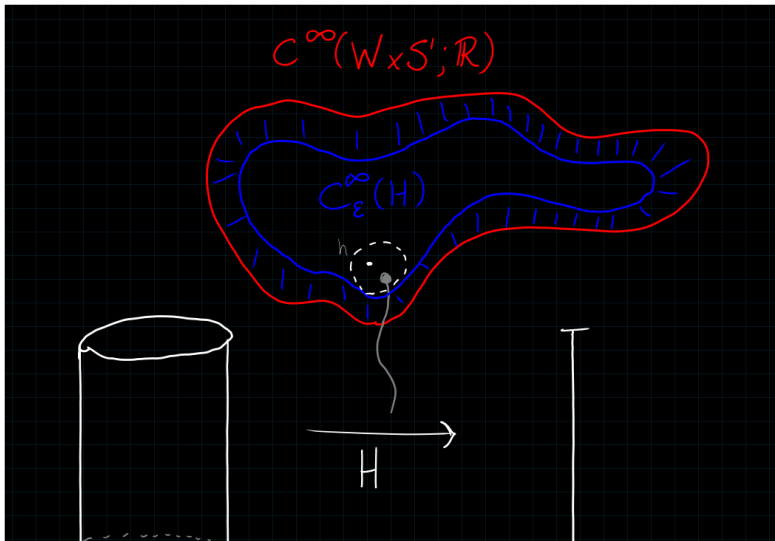
Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Goal

Start by trying to construct a subspace $\mathcal{C}^\infty_\varepsilon(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.

Define an Absolute Value

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\| \cdot \|_\varepsilon$ on $C_\varepsilon^\infty(H)$ and take the subspace of finite-norm elements.

   – Let $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$ denote a perturbation of H.

   – Fix $\varepsilon = \left\{ \varepsilon_k \ \middle| \ k \in \mathbb{Z}^{\geq 0} \right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.

   – For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \ \middle| \ |\alpha| = k \right\},$$

the maximum over all sets of multi-indices α of length k.
Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

Define a Norm

– Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left| d^k h(x, t) \right|.$$

– Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

– Choose them in such a way we obtain charts
$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \ (?).$$

– Obtain the computable form

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} \left| d^k(h \circ \Psi_i^{-1})(z) \right|.$$

Define a Banach Space

– Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \ \Big| \ \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

– Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Theorem

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma

$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Sketch Proof of Theorem

– By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.

– Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n, define

$$\frac{1}{\varepsilon_n} = 2^n \max\left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable.

∎

Why does this imply density? I don't know.

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

The next proposition establishes a version of this theorem with compact support:

Theorem

For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the C^1 topology by functions in C_ε^∞ supported in U.

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \ \Big| \ h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\operatorname{supp}(h)$ is "far" from $\operatorname{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \operatorname{Per}(H_0 + h) = \operatorname{Per}(H_0)$$

and are both nondegenerate.

Section 8.4: Linearizing the Floer Equation: The Differential of F

Choose $m > n = \dim(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify
tangent vectors (such as Z_i, tangents to W along u or in a
neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

*Why? Bypasses differentiating vector fields and the Levi-
Cevita connection.*

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the codomain here.

Definitions

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$ denotes a fixed solution to the Floer equation,
- $C_\searrow(x, y) \subset \left\{ u \in C^\infty(R \times S^1; W) \right\}$ is the set of smooth solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions:

$$\lim_{s \longrightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left| \frac{\partial u}{\partial t}(s, t) \right|, \ \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

Fix a solution

$$u \in \mathcal{M}(x, y) \subset C^{\infty}_{\text{loc}}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way:

The loops x, y are contractible, so they bound discs. So we extend by pushing these discs out slightly:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

Trivial the Pullback

From earlier in the book, we have

Assumption (6.22):

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW), \pi_2(W) \rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps $S^2 \longrightarrow W$ lift to $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$
\begin{array}{ccc}
\tilde{u}^* TW & \xrightarrow{d\tilde{u}} & TW \\
\downarrow & \lrcorner & \downarrow \\
S^2 & \xrightarrow{\tilde{u}} & W
\end{array}
$$

Choose a Frame

Linearization and
Transversality

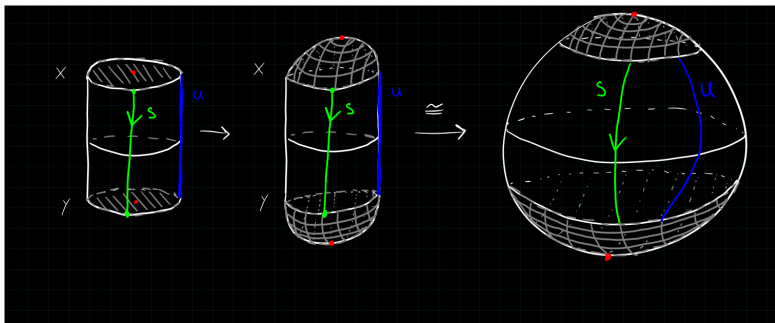D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

– Using the assumption, trivialize the pullback $\tilde{u}^* TW$ to obtain
an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

where
   – The frame depends smoothly on $(s, t) \in S^2$,
   – $\lim_{s \to \infty} Z_i$ exists for each i.
   –

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s\, \partial t} \quad \curvearrowright Z_i \stackrel{s \to \pm\infty}{\Longrightarrow} 0 \quad \text{for each } i$$

Claim: such trivializations exist, "using cylinders near the spherical
caps in the figure".

Define "Banach Manifold Charts"

Recall we had $W^{1,p}(x,y)$ a completion of C^∞

$$\mathcal{M}(x,y) \subset C_\searrow^\infty(x,y) \subset \mathcal{P}^{1,p}(x,y) \underset{\text{defn}}{\subset} \left\{ (s,t) \xrightarrow{\varphi} \exp_{w(s,t)} Y(s,t) \right\}.$$

where we restrict to

- $Y \in W^{1,p}(w^* TW)$,
- $w \in C_\searrow^\infty(x,y)$

Use the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x,y)$ given by

$$\iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow \mathcal{P}^{1,p}(x,y)$$

$$\mathbf{y} = (y_1, \ldots, y_{2n}) \longmapsto \exp_u\left(\sum y_i Z_i\right).$$

- Note that the derivative at zero is $\sum_{i=1}^{2n} y_i Z_i$.

Define the Floer Map in Charts

Define and compute the differential of the composite map $\tilde{\mathcal{F}}$ defined as follows:

$$
\begin{array}{ccc}
 & \tilde{\mathcal{F}} & \\
\mathcal{P}^{1,p}(x,y) \xrightarrow{\ \mathcal{F}\ } & L^p\left(\mathbb{R}\times S^1; TW\right) \longrightarrow & L^p\left(\mathbb{R}\times S^1; \mathbb{R}^m\right) \\
 & & \\
u \xrightarrow{\hspace{3cm}\tilde{\mathcal{F}}\hspace{3cm}} & & \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)
\end{array}
$$

– From now on, let $\mathcal{F}$ denote $\tilde{\mathcal{F}}$.

Add a Tangent

– Take the vector

$$Y(s, t) := (y_1(s, t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

– View Y as a vector in $\mathbb{R}^m$ tangent to W, given by $Y = \sum_{i=1}^{2n} y_i Z_i$.

– Plug $u + Y$ into the equation for $\mathcal{F}$, directly yielding

Add a Tangent

$$\mathcal{F}(u) = \qquad \frac{\partial u}{\partial s} \qquad +J(u)\frac{\partial u}{\partial t} \qquad -J(u)X_t(u)$$

$$\mathcal{F}(u + Y) = \quad \frac{\partial(u+Y)}{\partial s} \quad +J(u+Y)\frac{\partial(u+Y)}{\partial t} \quad -J(u+Y)X_t(u+Y)$$

Extract Linear Part

Extract the part that is linear in Y and collect terms:

$$(d\mathcal{F})_u(Y)$$

$$= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y)$$

$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right)$$

$$+ \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y) \right)$$

.

- – This is a sum of two differential operators:
    - – One of order 1, one of order 0 (Perspective 1)
    - – The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)

Leibniz Rule

– Now compute in charts. Need a lemma:

Lemma (Leibniz Rule)

For any source space X and any maps

$$J : X \longrightarrow \mathsf{End}(\mathbb{R}^m)$$
$$Y, v : X \longrightarrow \mathbb{R}^m$$

we have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

Sketch: Proof of Leibniz Rule

Differentiate the map

$$J \cdot v : X \longrightarrow \mathbb{R}^m$$
$$x \mapsto J(x) \cdot v(x)$$

to obtain

$$J(x + Y)v(x + y)$$
$$= (J(x) + (dJ)_x(Y)) \cdot (v(x) + (dv)_x(Y)) + \cdots$$
$$= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x)$$
$$\quad + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots$$

$$\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y).$$

■

Decompose by Order

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Using the chart ι defined by $\{Z_i\}$ to write $Y = \sum_{i=1}^{2n} y_i Z_i$ and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where O_0 are order 0 terms ("they do not differentiate the y_i") and the O_1 are order 1 terms:

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$

$$O_0 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right.$$

$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

Order One

– Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

– The second equality follows from the assumption that the Z_i are symplectic and orthonormal.

– Note that this writes $(d\mathcal{F})_u(Y) = O_0 + O_{CR}$, a sum of an order zero and a Cauchy-Riemann operator.
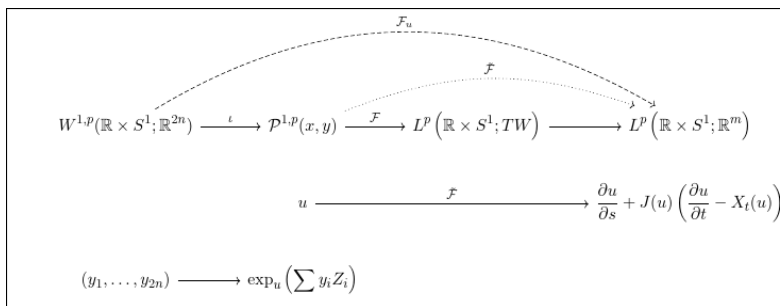
Recap

Note that since we've computed in charts, we have actually computed the differential of $\mathcal{F}_u$ in the following diagram

$$W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \xrightarrow{\iota} \mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R} \times S^1; TW\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right)$$

$$u \xrightarrow{\quad \mathcal{F} \quad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

$$(y_1, \ldots, y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)$$

So we've technically computed $(dF_\mu)_0$.

Order 0 Term is Linear

$$(d\mathcal{F})_u = \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right)$$

$$+ \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y) \right)$$

$$:= \bar{\partial}Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

- *S is linear*
- *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
-

$$\frac{\partial S}{\partial s}(s, t) \stackrel{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \textit{uniformly in t}$$

Proof

Omitted – S is exactly O_0 from before:

$$O_0 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right.$$

$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right)$$

$$= \sum_{i=1}^{2n} y_i \left( )\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t \right) \right.$$

$$\left. + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right).$$

- The term in blue vanishes as $s \longrightarrow \pm\infty$
  - Using the fact that u is a solution
  - Uses $\frac{\partial u}{\partial s} \longrightarrow 0$ uniformly (as do its derivatives?)
- Suffices to show the remaining part is symmetric in the limit

Proof

– Write the remaining part as

$$A(y_1, \cdots, y_{2n}) = \cdots \implies A_{ij} = A_{ji}$$

using inner product calculations

– Uses the fact the Z_i needed to be chosen to be unitary and symplectic.

■

Linearization and
Transversality

D. Zack Garza

Review 8.2

Section 8.3: The
Space of
Perturbations of
H

Section 8.4:
Linearizing the
Floer Equation:
The Differential
of F

Write O_1 as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

$$\partial_t Y = J_0 S^\pm Y$$

linearizes Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad at \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \longrightarrow \infty} u & \text{for } S^+ \end{cases} \quad respectively.$$

Image

Linearization and Transversality

D. Zack Garza

Review 8.2

Section 8.3: The Space of Perturbations of H

Section 8.4: Linearizing the Floer Equation: The Differential of F

Reminder the x, y were the top/bottom pieces of the original cylinder/sphere: