

# Title

# Contents

<b>1 Tuesday, October 20</b>	<b>2</b>
1.1 Gluing Two Opens . . . . .	2
1.2 More General Gluing . . . . .	3

## 1 | Tuesday, October 20

### 1.1 Gluing Two Opens

Recall that a *prevariety* is a ringed space that is locally isomorphic to an affine variety, where we recall that  $(X, \mathcal{O}_X)$  is *locally isomorphic* to an affine variety iff there exists an open cover  $U_i \rightrightarrows X$  such that  $(U_i, \mathcal{O}_{U_i})$ .

We found one way of producing these: the gluing construction. Given two ringed spaces  $(X_1, \mathcal{O}_{X_1})$  and  $(X_2, \mathcal{O}_{X_2})$  and open sets  $U_{12} \in X_1$  and  $U_{21} \in X_2$  and an isomorphism  $(U_{12}, \mathcal{O}_{U_{12}}) \xrightarrow{f} (U_{21}, \mathcal{O}_{U_{21}})$ , we defined

- The topological space as  $X_1 \coprod_f X_2$
- The sheaf of rings as  $\mathcal{O}_X = \left\{ \varphi : U \rightarrow k \mid \varphi|_{U \cap X_i} \text{ is regular for } i = 1, 2 \right\}$ .

**Example 1.1.1.**

$\mathbb{P}^1/k = X_1 \cup X_2$  where  $X_1 \cong \mathbb{A}^1, X_2 \cong \mathbb{A}^2$ . Take  $U_{12} = D(x)$  and  $U_{21} = D(y)$  with

$$f : U_{12} \rightarrow U_{21}$$

$$x \mapsto \frac{1}{x} = y.$$

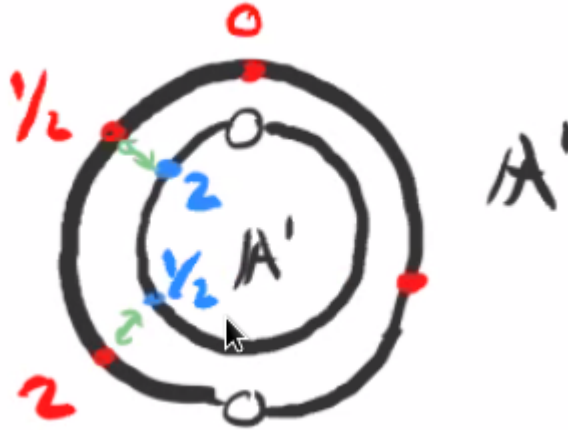


Figure 1: Supposing  $\text{char}(k) \neq 2$ . Note that for  $\mathbb{C}$  this recovers  $S^2$  in the classical topology.

**Example 1.1.2.**

Let  $X_i = \mathbb{A}^1$  and  $U_{12} = D(x), U_{21} = D(y)$  with

$$\begin{aligned} f : U_{12} &\rightarrow U_{21} \\ x &\mapsto x = y. \end{aligned}$$

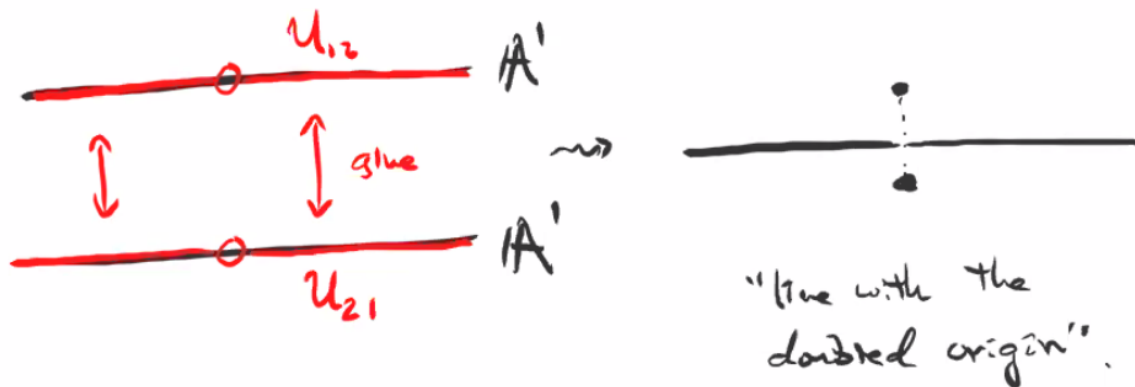


Figure 2: Line with the doubled origin.

Then  $\mathcal{O}_X = \left\{ \varphi : X \rightarrow k \mid \varphi|_{X_i} \text{ is regular} \right\} \cong k[x]$ .

**1.2 More General Gluing**

Now we want to glue more than two open sets. Let  $I$  be an indexing set for prevarieties  $X_i$ . Suppose that for an ordered pair  $(i, j)$  we have open sets  $U_{ij} \subset X_i$  and isomorphisms  $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$  such that

- a.  $f_{ji} = f_{ij}^{-1}$
- b.  $f_{jk} \circ f_{ij} = f_{ik}$  (cocycle condition)



Figure 3: Opens with isomorphisms.

Then the gluing construction is given by

1.  $X := \coprod X_i / \sim$  where  $x \sim f_{ij}(x)$  for all  $i, j$  and all  $x \in U_{ij}$ .
2.  $\mathcal{O}_x(U) := \left\{ \varphi : U \rightarrow k \mid \varphi|_{U \cap X_i} \in \mathcal{O}_{X_i} \right\}$ .

Every prevariety arises from the gluing construction applied to  $X_i$  affine varieties, since a prevariety  $(X, \mathcal{O}_X)$  by definition has an open affine cover  $X_i \rightrightarrows X$  and  $X$  is the result of gluing the  $X_i$ s by the identity.

**Example 1.2.1.**

Let  $X_1 = X_2 = X_3 = \mathbb{A}^2/k$ . Glue by the following instructions:

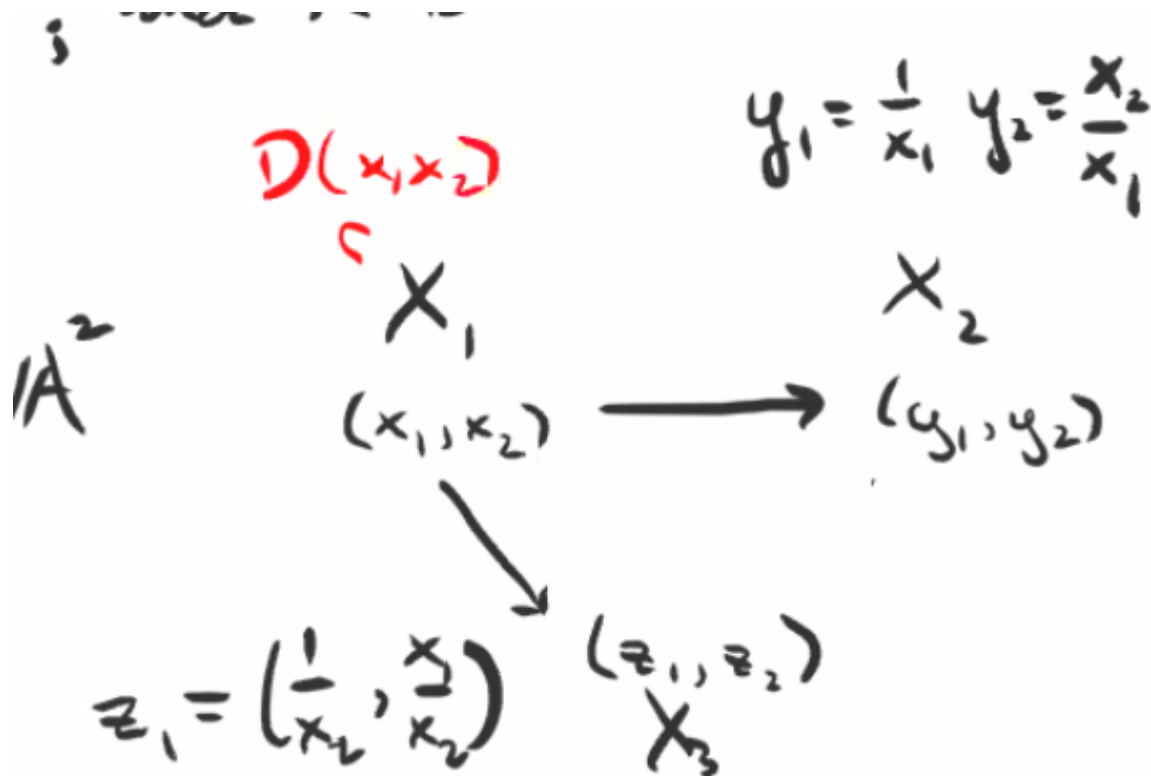


Figure 4: The map not shown is whatever formula is necessary to make the diagram commute.

Here

- $(y_1, y_2) = (1/x_1, x_2/x_1)$
- $(z_1, z_2) = (1/x_2, x_1/x_2)$

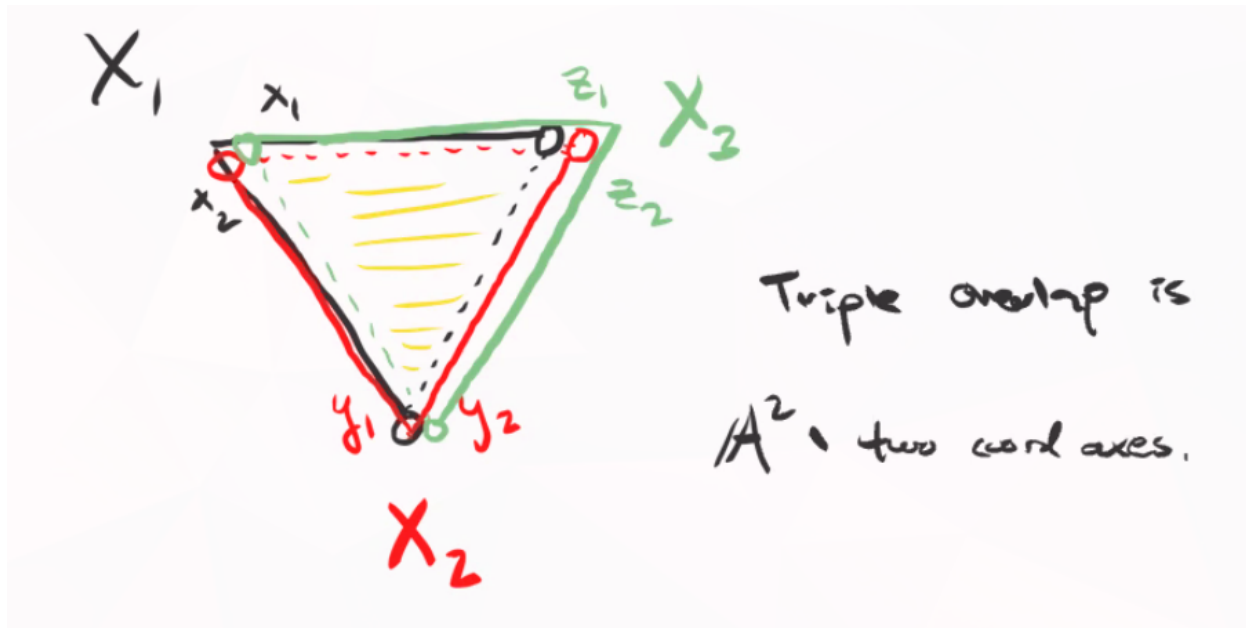


Figure 5: Yields  $\mathbb{P}^2$

Here  $X_1 = [1 : y/x : z/x]$