Algebraic Groups

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Friday 28th August, 2020

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These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Dan Nakano at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Friday 28th August, 2020
02:18

1 Friday, August 21

> Reference: Carter's "Finite Groups of Lie Type".
> Reference: Humphrey's "Linear Algebraic Groups" (Springer)

1.1 Intro and Definitions

> **Definition 1.0.1** (Affine Variety)**.**
> Let $k = \bar{k}$ be algebraically closed (e.g. $k = \mathbb{C}, \overline{\mathbb{F}_p}$). A variety $V \subseteq k^n$ is an *affine k-variety* iff V is the zero set of a collection of polynomials in $k[x_1, \cdots, x_n]$.

Here $\mathbb{A}^n := k^n$ with the Zariski topology, so the closed sets are varieties.

Definition 1.0.2 (Affine Algebraic Group)**.**
An *affine algebraic k-group* is an affine variety with the structure of a group, where the multiplication and inversion maps

$$\mu : G \times G \longrightarrow G$$
$$\iota : G \longrightarrow G$$

are continuous.

Example 1.1.
$G = \mathbb{G}_a \subseteq k$ the *additive group* of k is defined as $\mathbb{G}_a \coloneqq (k, +)$. We then have a *coordinate ring* $k[\mathbb{G}_a] = k[x]/I = k[x]$.

Example 1.2.
$G = \mathrm{GL}(n, k)$, which has coordinate ring $k[x_{ij}, T]/\left\langle \det(x_{ij}) \cdot T = 1 \right\rangle$.

Example 1.3.
Setting $n = 1$ above, we have $\mathbb{G}_m \coloneqq \mathrm{GL}(1, k) = (k^\times, \cdot)$. Here the coordinate ring is $k[x, T]/\left\langle xT = 1 \right\rangle$.

Example 1.4.
$G = \mathrm{SL}(n, k) \leq \mathrm{GL}(n, k)$, which has coordinate ring $k[G] = k[x_{ij}]/\left\langle \det(x_{ij}) = 1 \right\rangle$.

Definition 1.0.3 (Irreducible)**.**
A variety V is *irreducible* iff V can not be written as $V = \cup_{i=1}^n V_i$ with each $V_i \subseteq V$ a proper subvariety.

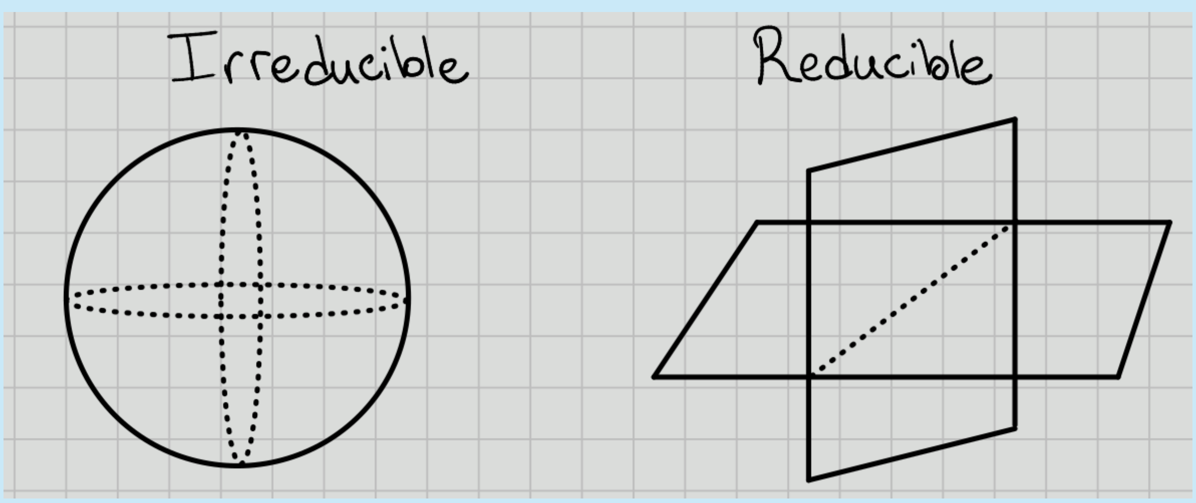

Figure 1: Reducible vs Irreducible

Proposition 1.1 ***(?).***
There exists a unique irreducible component of G containing the identity e. Notation: G^0.

Proposition 1.2 ***(?).***
G is the union of translates of G^0, i.e. there is a decomposition

$$G = \coprod_{g \in \Gamma} g \cdot G^0,$$

where we let G act on itself by left-translation and define Γ to be a set of representatives of distinct orbits.

Proposition 1.3 ***(?).***
One can define solvable and nilpotent algebraic groups in the same way as they are defined for finite groups, i.e. as having a terminating derived or lower central series respectively.

1.2 Jordan-Chevalley Decomposition

Proposition 1.4 ***(Existence and Uniqueness of Radical).***
There is a maximal connected normal solvable subgroup $R(G)$, denoted the *radical of* G.

- $\{e\} \subseteq R(G)$, so the radical exists.
- If $A, B \leq G$ are solvable then AB is again a solvable subgroup.

Definition 1.4.1 (Unipotent).
An element u is *unipotent* $\iff$ $u = 1 + n$ where n is nilpotent $\iff$ its the only eigenvalue is $\lambda = 1$.

Proposition 1.5 ***(JC Decomposition).***
For any G, there exists a closed embedding $G \hookrightarrow \mathrm{GL}(V) = \mathrm{GL}(n, k)$ and for each $x \in G$ a unique decomposition $x = su$ where s is semisimple (diagonalizable) and u is unipotent.

Define $R_u(G)$ to be the subgroup of unipotent elements in $R(G)$. :::{.definition title="Semisimple and Reductive"} Suppose G is connected, so $G = G^0$, and nontrivial, so $G \neq \{e\}$. Then

- G is semisimple iff $R(G) = \{e\}$.
- G is reductive iff $R_u(G) = \{e\}$. :::

Example 1.5.
$G = \mathrm{GL}(n, k)$, then $R(G) = Z(G) = kI$ the scalar matrices, and $R_u(G) = \{e\}$. So G is reductive and semisimple.

Example 1.6.
$G = \mathrm{SL}(n, k)$, then $R(G) = \{I\}$.

Exercise 1.1.
Is this semisimple? Reductive? What is $R_u(G)$?

Definition 1.5.1 (Torus)**.**
A *torus* $T \subseteq G$ in G an algebraic group is a commutative algebraic subgroup consisting of semisimple elements.

Example 1.7.
Let

$$T := \left\langle \begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_n \end{bmatrix} \subseteq \mathrm{GL}(n, k) \right\rangle .$$

Remark 1.
Why are torii useful? For $g = \mathrm{Lie}(G)$, we obtain a root space decomposition

$$g = \left(\bigoplus_{\alpha \in \Phi_-} g_\alpha \right) \oplus t \oplus \left(\bigoplus_{\alpha \in \Phi_+} g_\alpha \right).$$

When G is a simple algebraic group, there is a classification/correspondence:

$$(G, T) \iff (\Phi, W).$$

where Φ is an irreducible root system and W is a Weyl group.

2 Monday, August 24

2.1 Review and General Setup

- $k = \bar{k}$ is algebraically closed
- G is a reductive algebraic group
- $T \subseteq G$ is a *maximal split torus*

Split: $T \cong \bigoplus \mathbb{G}_m$.

We'll associate to this a root system, not necessarily irreducible, yielding a correspondence

$$(G, T) \iff (\Phi, W)$$

with W a Weyl group.

This will be accomplished by looking at $\mathfrak{g} = \mathrm{Lie}(G)$. If G is simple, then $\mathfrak{g}$ is "simple", and Φ irreducible will correspond to a Dynkin diagram.

There is this a 1-to-1 correspondence

$$G \text{ simple}/\sim \iff A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$$

where $\sim$ denotes *isogeny*.

Taking the Zariski tangent space at the identity "linearizes" an algebraic group, yielding a Lie algebra.

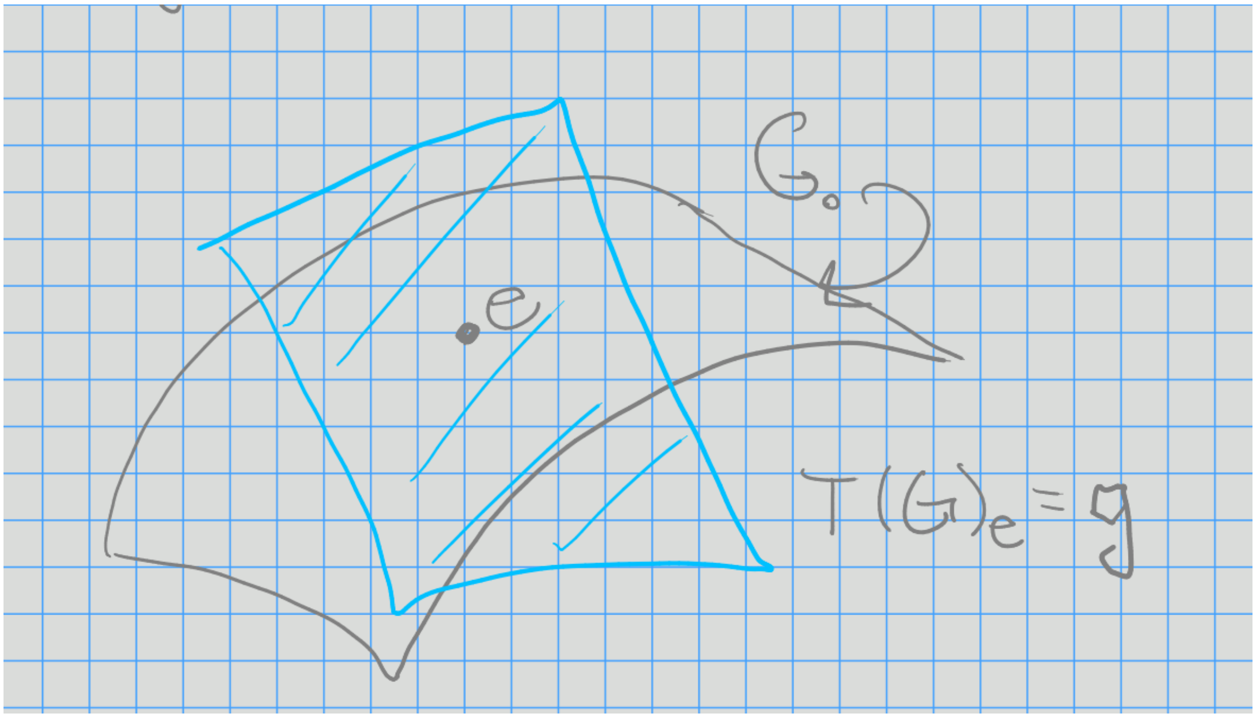

Figure 2: Image

We have the coordinate ring $k[G] = k[x_1, \cdots, x_n]/\mathcal{I}(G)$ where $\mathcal{I}(G)$ is the zero set. This is equal to $\{f : G \longrightarrow k\}$,

2.2 The Associated Lie Algebra

Definition 2.0.1 (The Lie Algebra of an Algebraic Group)**.**
Define *left translation* is

$$\begin{aligned} \lambda_x : k[G] &\longrightarrow k[G] \\ y &\mapsto f(x^{-1}y). \end{aligned}$$

Define *derivations* as

$$\text{Der } k[G] = \left\{ D : k[G] \longrightarrow k[G] \;\middle|\; D(fg) = D(f)g + fD(g) \right\}.$$

We can then realize the Lie algebra as

$$\mathfrak{g} = \text{Lie}(G) = \left\{ D \in \text{Der}k[G] \;\middle|\; \lambda_x \circ D = D \circ \lambda_x \right\},$$

the left-invariant derivations.

Example 2.1.

- $G = \mathrm{GL}(n,k) \implies \mathfrak{g} = \mathfrak{gl}(n,k)$
- $G = \mathrm{SL}(n,k) \implies \mathfrak{g} = \mathfrak{sl}(n,k)$

Let G be reductive and T be a split torus. Then T acts on $\mathfrak{g}$ via an *adjoint action.* (For $\mathrm{GL}_n, \mathrm{SL}_n$, this is conjugation.)

There is a decomposition into eigenspaces for the action of T,

$$\mathfrak{g} = \left(\bigoplus_{\alpha \in \Phi} g_\alpha \right) \oplus t$$

where $t = \mathrm{Lie}(T)$ and $g_\alpha := \left\{ x \in \mathfrak{g} \;\middle|\; t.x = \alpha(t)x \; \forall t \in T \right\}$ with $\alpha : T \longrightarrow K^\times$ a rational function (a *root*).

In general, take $\alpha \in \hom_{\mathrm{AlgGrp}}(T, \mathbb{G}_m)$.

Example 2.2.
Let $G = \mathrm{GL}(n,k)$ and

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \;\middle|\; a_j \in k^\times \right\}.$$

Then check the following action:

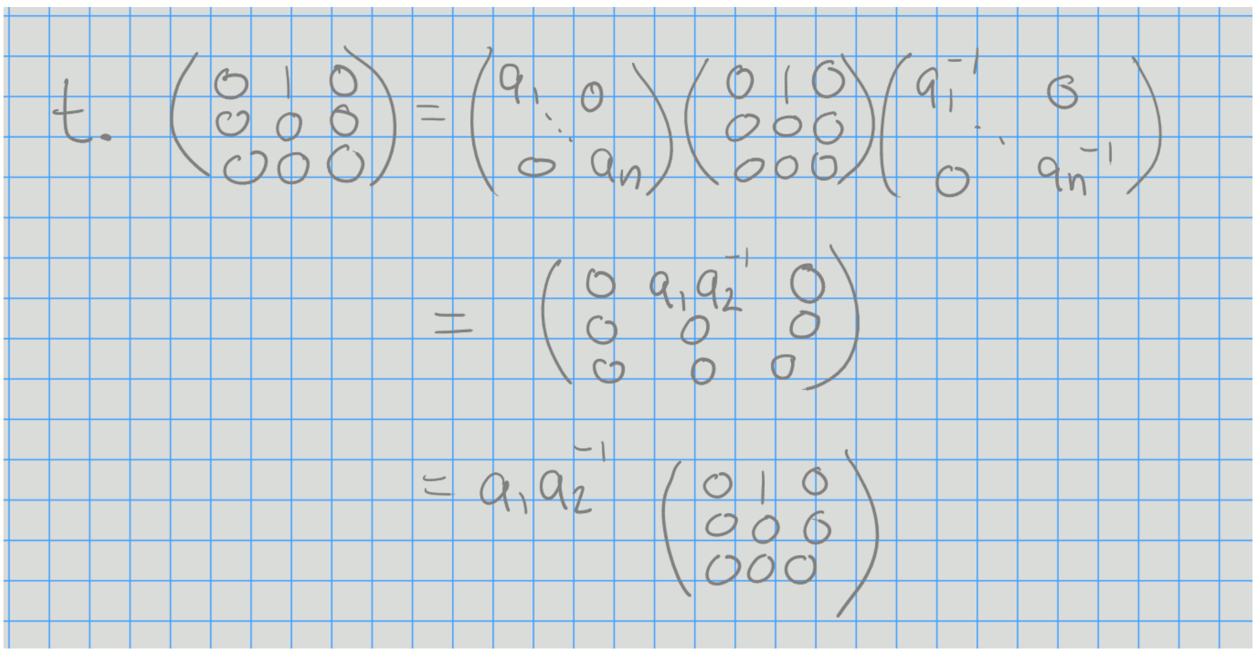

Figure 3: Action

which indeed acts by a rational function.

Then

$$g_\alpha = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = g_{(1,-1,0)}.$$

For $\mathfrak{g} = \mathfrak{gl}(3,k)$, we have

$$\begin{aligned} \mathfrak{g} = t &\oplus g_{(1,-1,0)} \oplus g_{(-1,1,0)} \\ &\oplus g_{(0,1,-1)} \oplus g_{(0,-1,1)} \\ &\oplus g_{(1,0,-1)} \oplus g_{(-1,0,1)}. \end{aligned}$$

2.3 Representations

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a group homomorphisms, then equivalently V is a (rational) G-module.

For $T \subseteq G$, $T \curvearrowright G$ semisimply, so we can simultaneously diagonalize these operators to obtain a *weight space decomposition* $V = \bigoplus_{\lambda \in X(T)} V_\lambda$, where

$$\begin{aligned} V_\lambda &\coloneqq \left\{ v \in V \;\middle|\; t.v = \lambda(t)v \ \forall t \in T \right\} \\ X(T) &\coloneqq \hom(T, \mathbb{G}_m). \end{aligned}$$

Example 2.3.
Let $G = \mathrm{GL}(n,k)$ and V the n-dimensional natural representation as column vectors,

$$V = \left\{ [v_1, \cdots, v_n] \;\middle|\; v_j \in k \right\}.$$

Then

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \;\middle|\; a_j \in k^\times \right\}.$$

Consider the basis vectors $\mathbf{e}_j$, then

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_j \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_1^0 a_2^0 \cdots a_j^0 \cdots a_n^0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Here the weights are of the form $\varepsilon_j \coloneqq [0, 0, \cdots, 1, \cdots, 0]$ with a 1 in the jth spot, so we have

$$V = V_{\varepsilon_1} \oplus V_{\varepsilon_2} \oplus \cdots \oplus V_{\varepsilon_n}.$$

Example 2.4.
For $V = \mathbb{C}$, we have $t.v = (a_1^0 \cdots a_n^0)v$ and $V = V_{(0,0,\cdots,0)}$.

2.4 Classification

Let G be a simple algebraic group (ano closed, connected, normal subgroups other than $\{e\}, G$) that is nonabelian that is nonabelian.

Example 2.5.
Let $G = \mathrm{SL}(3, k)$. Then

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2^{-1} & 0 \\ 0 & 0 & a_2^{-1} \end{bmatrix} \;\middle|\; a_1, a_2 \in k^\times \right\}$$

and

$$t. \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = a_1^2 a_2^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and $\alpha_1 = (2, -1)$.

What is α_1? Note that you can recover the Cartan something here?

Then

$$\mathfrak{g} = \mathfrak{g}_{(2,-1)} \oplus \mathfrak{g}_{(-2,1)} \oplus \mathfrak{g}_{(-1,2)} \oplus \mathfrak{g}_{(1,-2)} \oplus \mathfrak{g}_{(1,1)} \oplus \mathfrak{g}_{(-1,-1)}.$$

Then $\alpha_2 = (-1, 2)$ and $\alpha_1 + \alpha_2 = (1, 1)$.

This gives the root space decomposition for $\mathfrak{sl}_3$:

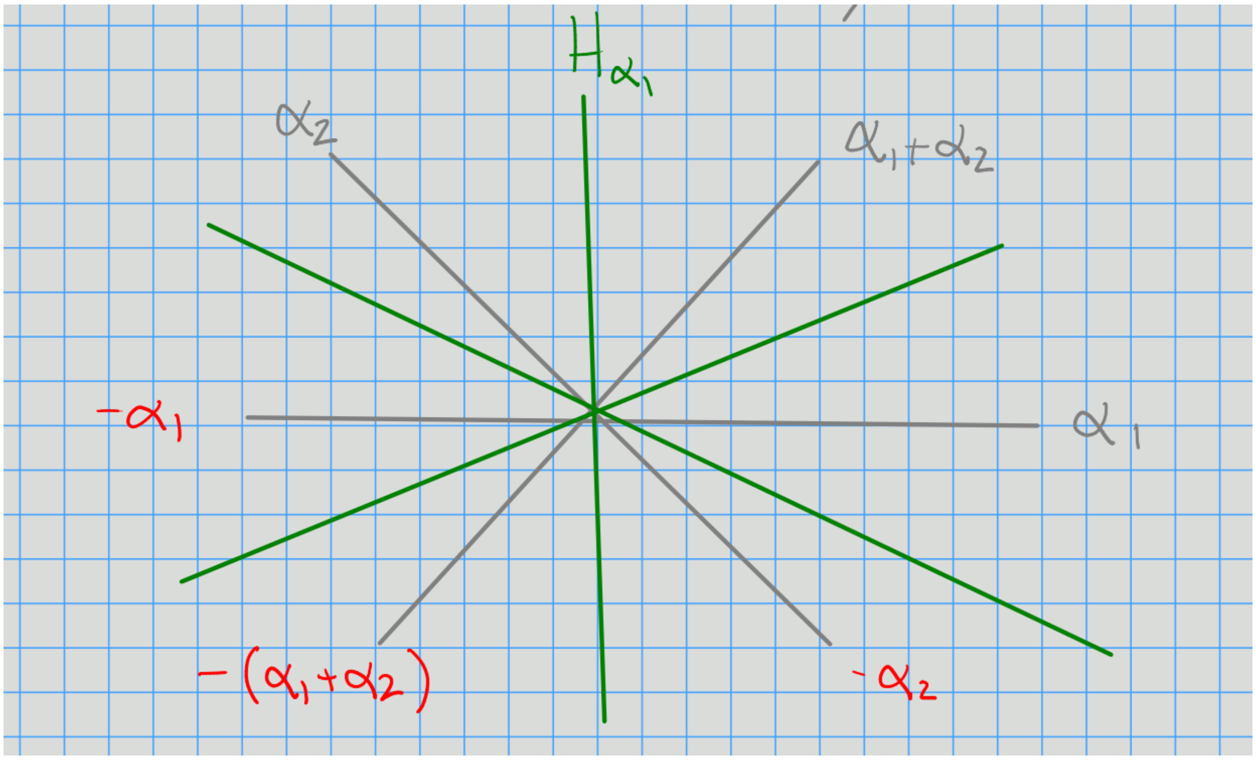

Figure 4: Image

Then the Weyl group will be generated by reflections through these hyperplanes.

3 Wednesday, August 26

3.1 Review

- G a reductive algebraic group over k
- $T = \prod_{i=1}^{n} \mathbb{G}_m$ a maximal split torus
- $\mathfrak{g} = \text{Lie}(G)$
- There's an induced root space decomposition $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$
- When G is simple, Φ is an *irreducible* root system
  - There is a classification of these by Dynkin diagrams

Example 3.1.
A_n corresponds to $\mathfrak{sl}(n+1, k)$ (mnemonic: A_1 corresponds to $\mathfrak{sl}(2)$)

- We have representations $\rho : G \longrightarrow \text{GL}(V)$, i.e. V is a G-module
- For $T \subseteq G$, we have a weight space decomposition: $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ where $X(T) = \hom(T, \mathbb{G}_m)$.

Note that $X(T) \cong \mathbb{Z}^n$, the number of copies of $\mathbb{G}_m$ in T.

3.2 Root Systems and Weights

Example 3.2.
Let $\Phi = A_2$, then we have the following root system:

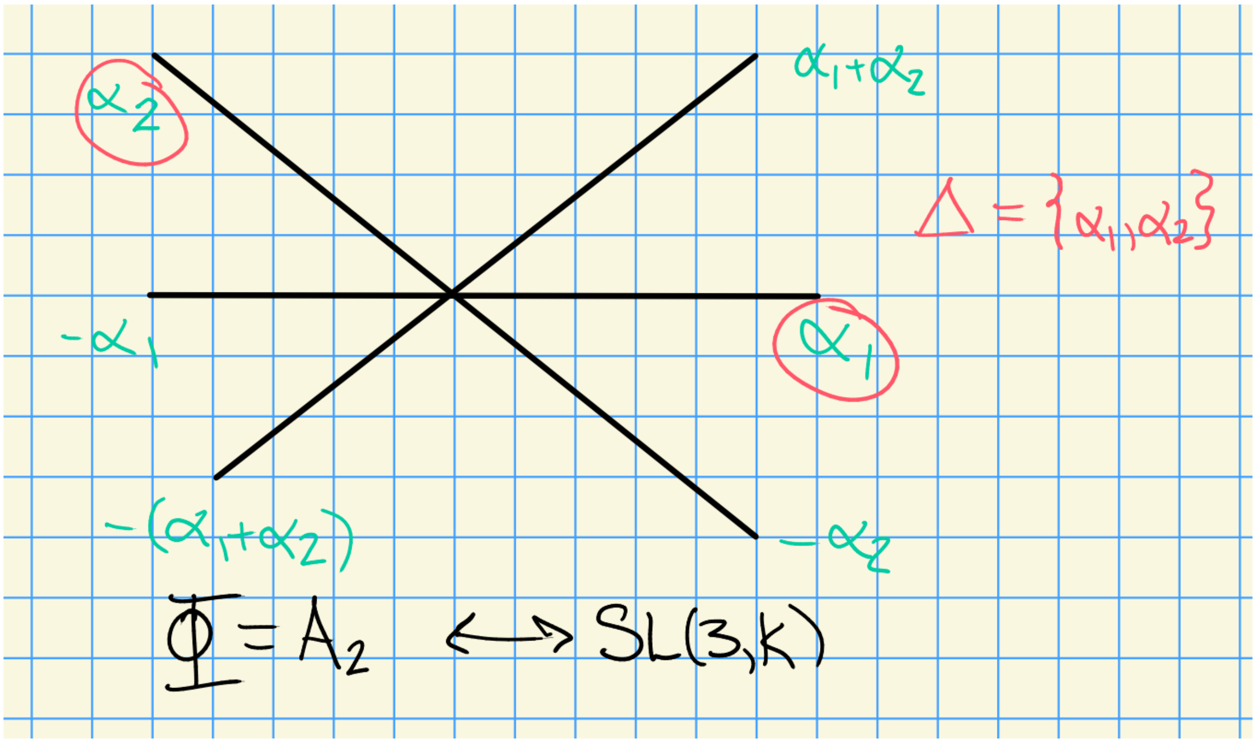

Figure 5: Image

In general, we'll have $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a basis of *simple roots*.

Remark 2.
Every root $\alpha \in I$ can be expressed as either positive integer linear combination (or negative) of simple roots.

For any $\alpha \in \Phi$, let s_α be the reflection across H_α, the hyperplane orthogonal to α. Then define the *Weyl group* $W = \left\{ s_\alpha \;\middle|\; \alpha \in \Phi \right\}$.

Example 3.3.
Here the Weyl group is S_3:

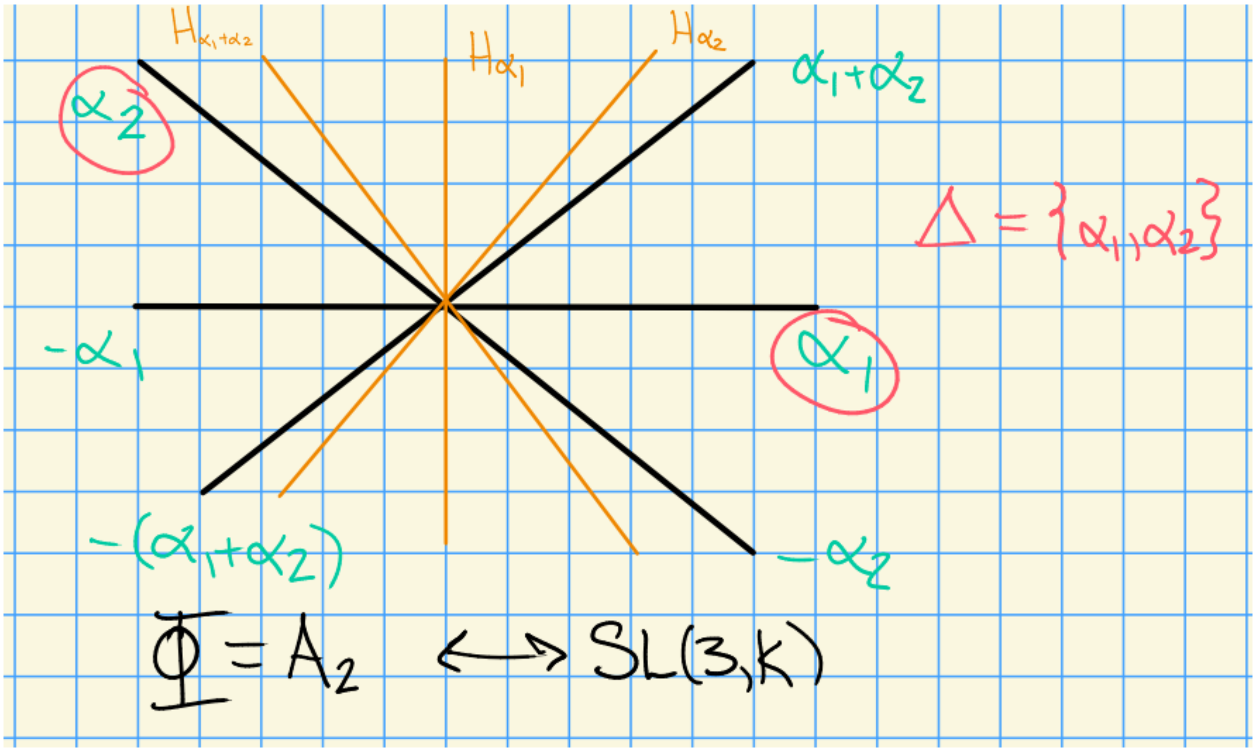

Figure 6: Image

Remark 3.
W acts transitively on bases.

Remark 4.
$X(T) \subseteq \mathbb{Z}\Phi$, recalling that $X(T) = \hom(T, \mathbb{G}_m) = \mathbb{Z}^n$ for some n. Denote $\mathbb{Z}\Phi$ the *root lattice* and $X(T)$ the *weight lattice*.

Example 3.4.
Let $G = \mathfrak{sl}(2, \mathbb{C})$ then $X(T) = \mathbb{Z}\omega$ where $\omega = 1$, $\mathbb{Z}\Phi = \mathbb{Z}\{\alpha\}$ Then there is one weight α, and the root lattice $\mathbb{Z}\Phi$ is just $2\mathbb{Z}$. However, the weight lattice is $\mathbb{Z}\omega = \mathbb{Z}$, and these are not equal in general.

Remark 5.
There is partial ordering on $X(T)$ given by $\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha \in \Delta} n_\alpha \alpha$ where $n_\alpha \geq 0$. (We say λ *dominates* μ.)

Definition 3.0.1 (Fundamental Dominant Weights)**.**
We extend scalars for the weight lattice to obtain $E := X(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, a Euclidean space with an inner product $\langle \cdot, \ \cdot \rangle$.
For $\alpha \in \Phi$, define its *coroot* $\alpha^\vee := \dfrac{2\alpha}{\langle \alpha, \ \alpha \rangle}$. Define the *simple coroots* as $\Delta^\vee := \{\alpha_i^\vee\}_{i=1}^n$, which has a dual basis $\Omega := \{\omega_i\}_{i=1}^n$ the *fundamental weights*. These satisfy $\left\langle \omega_i, \ \alpha_j^\vee \right\rangle = \delta_{ij}$.

What is the notation for fundamental weights? Definitely not Ω usually!

Important because we can index irreducible representations by fundamental weights.

A weight $\lambda \in X(T)$ is *dominant* iff $\lambda \in \mathbb{Z}^{\geq 0}\Omega$, i.e. $\lambda = \sum n_i \omega_i$ with $n_i \in \mathbb{Z}^{\geq 0}$.

If G is simply connected, then $X(T) = \bigoplus \mathbb{Z}\omega_i$.

See Jantzen for definition of simply-connected, $\mathrm{SL}(n+1)$ is simply connected but its adjoint $PGL(n+1)$ is not simply connected.

3.3 Complex Semisimple Lie Algebras

When doing representation theory, we look at the Verma modules $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda \twoheadrightarrow L(\lambda)$.

Theorem 3.1 *(?)***.**
$L(\lambda)$ as a finite-dimensional $U(\mathfrak{g})$-module $\iff$ λ is dominant, i.e. $\lambda \in X(T)_+$.

Thus the representations are indexed by lattice points in a particular region:

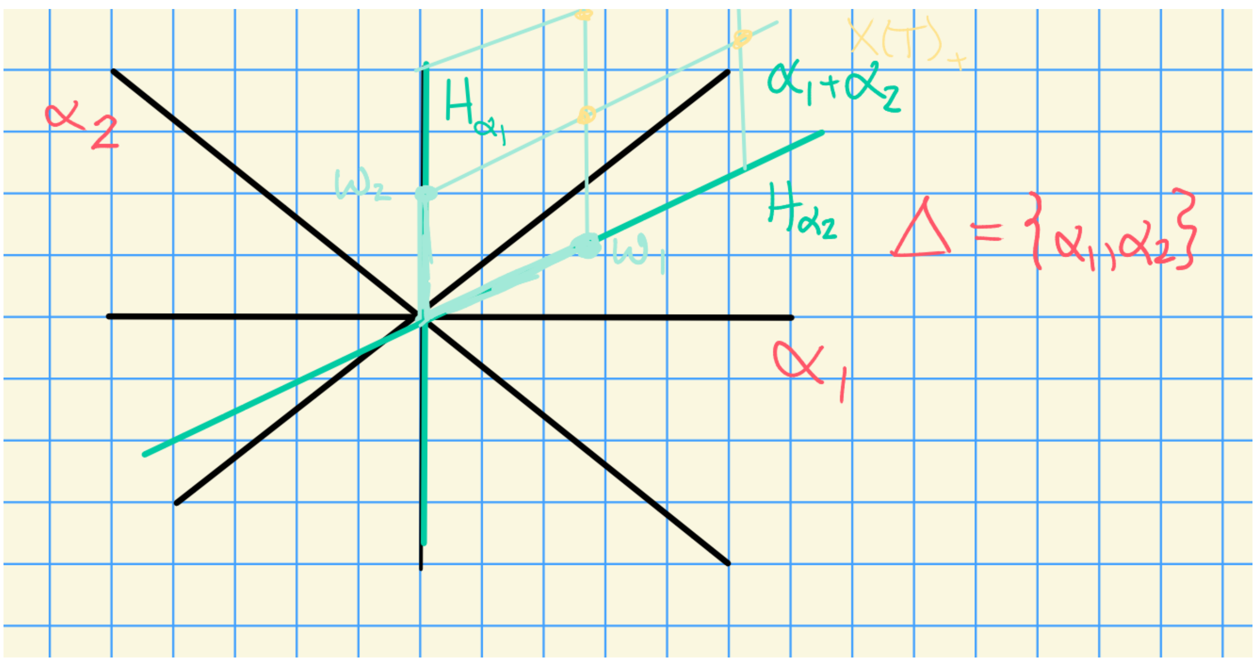

Figure 7: Image

Question 1:

Suppose G is a simple (simply connected) algebraic group. How do you parameterize *irreducible* representations?

For $\rho : G$
$to\mathrm{GL}(V)$, V is a *simple module* (an *irreducible representation*) iff the only proper G-submodules of V are trivial.

Answer 1: They are also parameterized by $X(T)_+$. We'll show this using the induction functor $\mathrm{Ind}_B^G \lambda = H^0(G/B, \mathcal{L}(\lambda))$ (sheaf cohomology of the flag variety with coefficients in some line bundle).

> We'll define what B is later, essentially upper-triangular matrices.

Question 2: What are the dimensions of the irreducible representations for G?

Answer 2: Over $k = \mathbb{C}$ using Weyl's dimension formula.

For $k = \overline{\mathbb{F}_p}$: conjectured to be known for $p \geq h$ (the *Coxeter number*), but by Williamson (2013) there are counterexamples. Current work being done!