

Chapter 9

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1 | Background, Notation, Setup

Important Ideas for This Chapter:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

Goals

- Construct Floer homology and prove the Arnold Conjecture (“Symplectic Morse Inequalities?”):

$$\# \{1\text{-Periodic trajectories of } X_H\} \geq \sum_{k \in \mathbb{Z}} HM_k(W; \mathbb{Z}/2\mathbb{Z}).$$

Here $HM_*(W)$ is the Morse homology.

Strategy:

1. Define the action functional \mathcal{A}_H .
2. Construct the chain complex (graded vector space) CF_* .
3. Define X_H , which will be used to define ∂ later.
4. Count trajectories.
5. Show finite-energy trajectories connect critical points of \mathcal{A}_H .
6. Show compactness property for space of trajectories of finite energy.
7. Define ∂ (uses a compactness property in 9.1c)
8. Show space of trajectories is a manifold (plus genericity, “Smale property”)
9. **Show that $\partial^2 = 0$.**
10. Show that HF_* doesn’t depend on \mathcal{A}_H or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_* .

Ingredients:

- (W, ω) with $\omega \in \Omega^2(W)$ is a symplectic manifold with an almost complex structure J .
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H the corresponding symplectic gradient.
 - Defined by how it acts on tangent vectors in $T_x M$:
$$\omega_x(\cdot, X_H(x)) = (dH)_x(\cdot).$$
 - Zeros of vector field X_H correspond to critical points of H :
$$X_H(x) = 0 \iff (dH)_x = 0.$$
 - Take the associated flow $\psi^t : W \rightarrow W$, assumed 1-periodic so $\psi^1(x) = x$: critical points of H are periodic trajectories.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation.

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- The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y\end{aligned}$$

$$Y \in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}W$ is the *free loop space* on W , i.e. space of contractible loops on W , i.e. $C^\infty(S^1; W)$ with the C^∞ topology
 - Elements $x \in \mathcal{L}W$ can be viewed as maps $S^1 \rightarrow W$.
 - Can extend to maps from a closed disc, $u : \mathbb{D}^2 \rightarrow M$.
 - Loops in $\mathcal{L}W$ can be viewed as maps $S^2 \rightarrow W$, since they're maps $I \times S^1 \rightarrow W$ with the boundaries pinched:

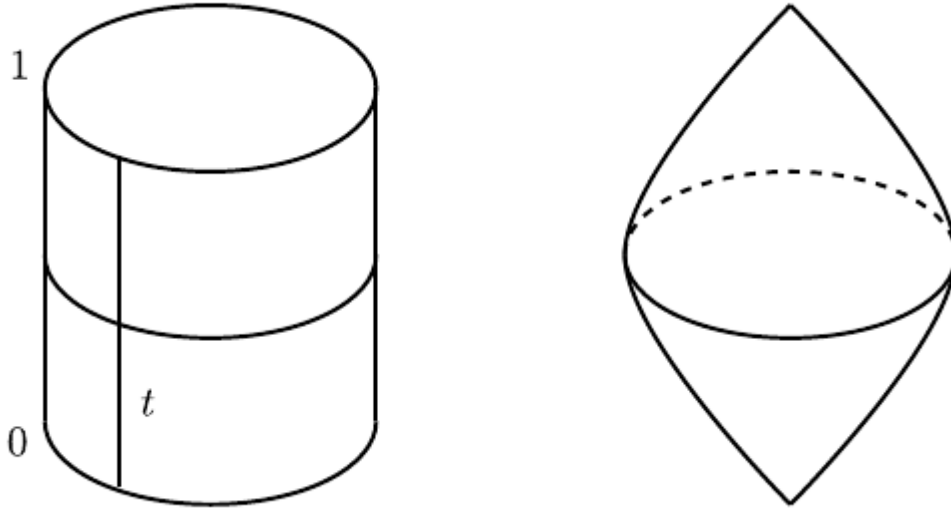


Figure 1: Loops in $\mathcal{L}W$

- The action functional is given by

$$\begin{aligned}\mathcal{A}_H : \mathcal{L}W &\rightarrow \mathbb{R} \\ x &\mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) \, dt\end{aligned}$$

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- Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t dt - p dq)$.
 - Correspondence between trajectories of the gradient of \mathcal{A}_H and solutions to Floer equations.
 - x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of \mathcal{A}_H .
 - Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, W)$,

$$\int_{S^2} u^* \omega = 0 \quad \text{or equivalently} \quad \langle \omega, \pi_2 W \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM , equivalently

$$\langle c_1 TW, \pi_2 W \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- Maslov index: used the fact that
 - $\text{Sp}(2n, \mathbb{R})$ retracts onto $U(n)$: use a polar decomposition $S = PQ$ as a PSD times orthogonal, then homotope P to I .
 - $\pi_1 U_n = \mathbb{Z}$: use $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$ by the determinant, and $\pi_1 SU(n, \mathbb{C}) = 0$.
 - Thus every path in $\gamma : I \rightarrow \text{Sp}(2n, \mathbb{R})$ can be assigned an integer by getting a map $\tilde{\gamma} : I \rightarrow S^1$ and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y .
 - After perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
 - How we did it:
 - * Describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$ (Banach manifold modeled on the Sobolev spaces $W^{1,p}$),
 - * Apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.

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- Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality.
 - * Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$$

$$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x, y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting x to y :

$$\mathcal{L}(x, y) := \mathcal{M}(x, y) / \mathbb{R}.$$

- Use the quotient topology, define sequentially:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

- When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

$$n(x, y) := \#\mathcal{L}(x, y)$$

is well-defined.

- $C_k(H) := \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k .
 - Finitely many since they are nondegeneracy implies they are isolated.

Remark 1.

Some notation:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M}(x, z) \\ & & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

Hats will generally denote maps induced on quotient.

- Defined a differential

$$\begin{aligned} \partial : C_k(H) &\rightarrow C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y)y \end{aligned}$$

$$\begin{aligned} n(x, y) &:= \# \{ \text{Trajectories of } \text{grad } \mathcal{A}_H \text{ connecting } x, y \} \pmod{2} \\ &= \# \mathcal{L}(x, y) \pmod{2}. \end{aligned}$$

- Examined ∂^2 :

$$\begin{aligned} \partial^2 : C_k(H) &\rightarrow C_{k-2}(H) \\ x &\mapsto \partial(\partial(x)) \\ &= \partial \left(\sum_{\mu(y)=\mu(x)-1} n(x, y)y \right) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x, y) \partial(y) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x, y) \left(\sum_{\mu(z)=\mu(y)-1} n(y, z)z \right) \\ &= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x, y)n(y, z) z \\ &= \sum_{\mu(z)=\mu(y)-1} \left(\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) \right) z \quad (\text{finite sums, swap order}), \end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x, z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\bar{\mathcal{L}}(x, y)$.
- We'll then have

$$\bar{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial\bar{\mathcal{L}}(x, z), \quad \partial\bar{\mathcal{L}}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z),$$

which “space-ifies” the equation we want.

- We'll show $\partial\bar{\mathcal{L}}(x, z)$ is a 1-manifold, which must have an even number of points, and thus

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = \#(\partial\bar{\mathcal{L}}(x, z)) \equiv 0 \pmod{2}.$$

3 | Three Important Theorems

3.1 First Theorem: Convergence to Broken Trajectories

- Recall: *broken trajectories* are unions of intermediate trajectories connecting intermediate critical points.
- Shown last time: a sequence of trajectories can converge to a broken trajectory, i.e. there are broken trajectories in the closure of $\mathcal{L}(x, z)$.
- This theorem describes their behavior:

Theorem 3.1 (9.1.7: Convergence to Broken Trajectories).

Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, z)$, then there exist

- A subsequence $\{u_{n_j}\}$
- Critical points $\{x_0, x_1, \dots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = z$
- Sequences $\{s_n^1\}, \{s_n^2\}, \dots, \{s_n^\ell\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_{n_j} \cdot s_n^k \xrightarrow{n \rightarrow \infty} u^k.$$

- Upshots:
 - Every sequence upstairs has a subsequence which (after reparameterizing) converges
 - This descends to actual convergence after quotienting by \mathbb{R} ?
 - Yields uniqueness of limits in $\mathcal{L}(x, z)$, thus a separated topology
 - Sequentially compact \iff compact since $\mathcal{L}(x, z)$ is a metric space?

Corollary 3.2 (Compactness).

$\bar{\mathcal{L}}(x, z)$ is compact.

3.2 Second Theorem: Compactness of $\bar{\mathcal{L}}(x, z)$

Definition 3.2.1 (Regular Pair).

For an almost complex structure J and a Hamiltonian H , the pair (H, J) is **regular** if the Floer map \mathcal{F} is transverse to the zero section in the following vector bundle:

$$\begin{array}{ccc}
 E_u := \{\text{Vector fields tangent to } M \text{ along } u\} & \longrightarrow & C^\infty(\mathbb{R} \times S^1; TM) \\
 & \searrow \mathcal{F} \quad \swarrow \mathbf{0} & \\
 & C^\infty(\mathbb{R} \times S^1; M) &
 \end{array}$$

Most of chapter 9 is spent proving this theorem:

Theorem 3.3(9.2.1).

Let (H, J) be a regular pair with H nondegenerate and x, z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\bar{\mathcal{L}}(x, z)$ is a compact 1-manifold with boundary with

$$\partial \bar{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x, z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \quad \text{where} \quad \mathcal{I}(x, z) = \left\{ y \mid \mu(x) < \mu(y) < \mu(z) \right\}.$$

Note: possibly a typo in the book? Has x, y on the LHS.

Corollary 3.4.

$$\partial^2 = 0.$$

3.3 Third Theorem: Gluing

- We already know that $\bar{\mathcal{L}}(x, z)$ is compact and $\mathcal{L}(x, z)$ is a 1-manifold, so we look at neighborhoods of boundary points.

Theorem 3.5 (9.2.3: Gluing).

Let x, y, z be three critical points of \mathcal{A}_H with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

$$\psi : [\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$$

such that $\hat{\psi}$, the induced map on the quotient

$$\begin{array}{ccc} [\rho_0, \infty) & \xrightarrow{\psi} & \mathcal{M}(x, z) \\ & \searrow \hat{\psi} & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

is an embedding that satisfies

$$\hat{\psi}(\rho) \xrightarrow{\rho \rightarrow \infty} (\hat{u}, \hat{v}) \in \bar{\mathcal{L}}(x, z).$$

2. (“Uniqueness”) For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x, z)$,

$$\ell_n \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v}) \quad \implies \quad \ell_n \in \text{im}(\hat{\psi}) \text{ for } n \gg 0.$$

4 | Gluing Theorem

Broken into three steps:

1. Pre-gluing:

- Get a function w_ρ which interpolates between u and v in the parameter ρ .

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- Not exactly a solution itself, just an “approximation”.

2. Newton’s Method:

- Apply the Newton-Picard method to w_p to construct a true solution

$$\begin{aligned}\psi : [-\rho, \infty) &\rightarrow \mathcal{M}(x, z) \\ \rho &\mapsto \exp_{w_p}(\gamma(p))\end{aligned}$$

for some $\gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p} \mathcal{P}(x, z)$

where $\mathcal{P} = ?$.

- General idea: guess, intersect x -axis with tangent of graph, use that as a new guess and iterate.
- GIF of Newton’s Method

3. Project and Verify Properties:

- Check that the projection $\hat{\psi} = \pi \circ \psi$ satisfies the conditions from the theorem.

5 | 9.3: Pre-gluing, Construction of w_ρ

- Choose (once and for all) a bump function β on $B_\varepsilon(0)^c \subset \mathbb{R} \rightarrow [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^\pm(s)$:

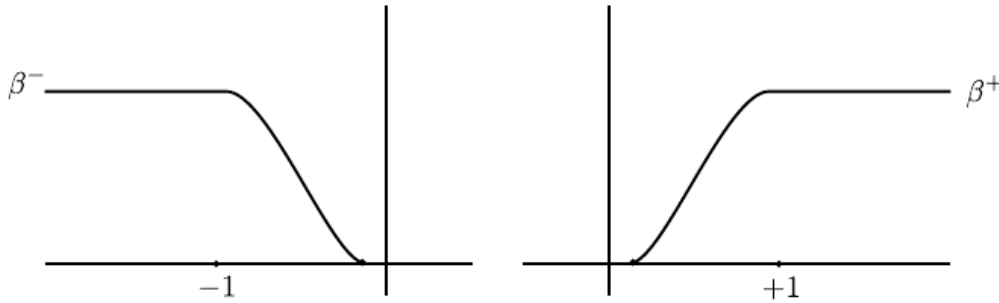


Figure 2: Bump away from zero

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- Define an interpolation w_ρ from u to v in the following way: let

- $\exp[\cdot] := \exp_{y(t)}(\cdot)$ and
- $\ln(\cdot) := \exp_{y(t)}^{-1}(\cdot)$,

then

$$w_\rho : x \rightarrow z$$

$$w_\rho(s, t) := \begin{cases} u(s + \rho, t) & s \in (-\infty, -1] \\ \exp[\beta^-(s) \ln(u(s + \rho, t)) + \beta^+(s) \ln(u(s - \rho, t))] & s \in [-1, 1] \\ u(s - \rho, t) & s \in [1, \infty) \end{cases}.$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\} \subseteq \text{im } \exp_{y(t)}(\cdot),$$

so we can apply $\exp_{y(t)}^{-1}(\cdot)$.

- Can make $|s| \leq 1$ for large ρ , since

$$\begin{aligned} u(s, t) &\xrightarrow{s \rightarrow \infty} y(t) \\ v(s, t) &\xrightarrow{s \rightarrow -\infty} y(t). \end{aligned}$$

- So pick a ρ_0 such that this holds for $\rho > \rho_0$.
- Might have to increase ρ_0 later in the proof, so $\rho > \rho_0$ just means $\rho \gg 0$.

- Some properties:

- $w_\rho \in C^\infty(x, z)$ and is differentiable in ρ .
- $s \in [-\varepsilon, \varepsilon] \implies w_\rho(s, t) = y(t)$.

$$w_\rho(s - \rho, t) \xrightarrow{\rho \rightarrow \infty} u(s, t) \quad \text{in } C_{\text{loc}}^\infty$$

$$w_\rho(s, t) \xrightarrow{\rho \rightarrow \infty} y(t) \quad \text{in } C_{\text{loc}}^\infty.$$

- Now carry out the linearized version on tangent vectors

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- Let $Y \in T_u\mathcal{P}(x, y)$
 - Let $Z \in T_v\mathcal{P}(x, y)$
 - Replace w_ρ with the interpolation

$$Y \#_\rho Z \in T_{w_\rho}\mathcal{P}(x, y) = W^{1,p}(w_\rho^*TW).$$

defined by

$$(Y \#_\rho Z)(s, t) = \begin{cases} Y(s + \rho, t) & s \in (-\infty, -1] \\ \exp_T \left[\beta^-(s) \ln_T(Y(s + \rho, t)) + \beta^+(s) \ln_T(Z(s - \rho, t)) \right] & s \in [-1, 1] \\ Z(s - \rho, t) & s \in [1, \infty) \end{cases},$$

where the subscript T indicates taking tangents of the exponential maps at appropriate points.

6 | 9.4: Construction of ψ .

6.1 Summary

- \mathcal{F}_ρ will be $F \circ \exp w_\rho$ expanded in the bases Z_i coming from a trivialization.
- $L_\rho = (d\mathcal{F}_\rho)_0$ will be the linearization of the Floer operator at zero.
- Newton-Picard method, general idea:
 - Allows finding zeros of f given an approximate zero x_0 , using the extra information of the 1st derivative f' .
 - Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(\mathbf{x}_0)}.$$

- Second variant more useful: only need to know the derivative at one point.

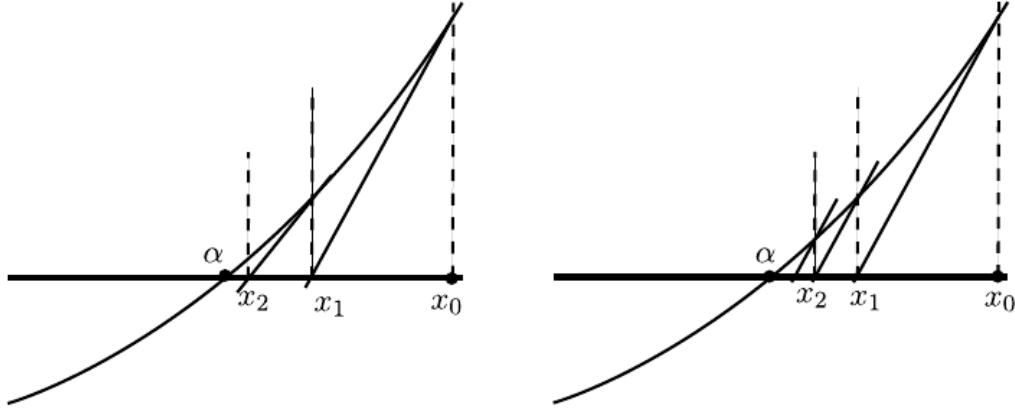


fig. 9.6

Figure 3: Newton Method Variants

- Adapting Newton-Picard to operators:
 - $\frac{1}{f'(x_0)}$ will correspond to L_ρ^{-1} , but it won't be invertible on entire space.
 - Decompose

$$T_{w_\rho} \mathcal{P}(x, z) = W^{1,p}(w_\rho^* T W) = W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) = \ker(L_\rho) \oplus W_\rho^\perp,$$

where L_ρ will have a right inverse on $W_\rho^\perp \subseteq T_{w_\rho} \mathcal{P}(x, z)$.

- Find that $x_0 = 0$ is “approximately” a solution and run Newton-Picard in W_ρ^\perp
- Where does W_ρ^\perp come from? Essentially the kernel of some linear functional given by an integral:

$$W_\rho^\perp := \left\{ Y \in W^{1,p} \mid \int_{\mathbb{R} \times S^1} \langle Y, \dots \rangle ds dt = 0, \text{ plus conditions} \right\}.$$

- Will obtain for every $\rho \geq \rho_0$ an element $\gamma(\rho) \in W_\rho^\perp$ with

$$\mathcal{F}_\rho(\gamma(\rho)) = 0.$$

- Where does γ come from? Intersection-theoretic interpretation on page

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$$\begin{aligned} (\exp_{w_\rho})^{-1} \mathcal{M}(x, z) \cap W_\rho^\perp &\subseteq T_{w_\rho} \mathcal{P}(x, z) && \rightsquigarrow \gamma \\ \mathcal{M}(x, z) \cap \left\{ \exp_{w_\rho} W_\rho^\perp \mid \rho \geq \rho_0 \right\} &\subseteq \mathcal{P}(x, z) && \rightsquigarrow \psi(\rho), \end{aligned}$$

which we get by exponentiating.

- * This gives a codimension 1 subspace in $\mathcal{M}(x, z)$, which we take to be $\psi(\rho)$:

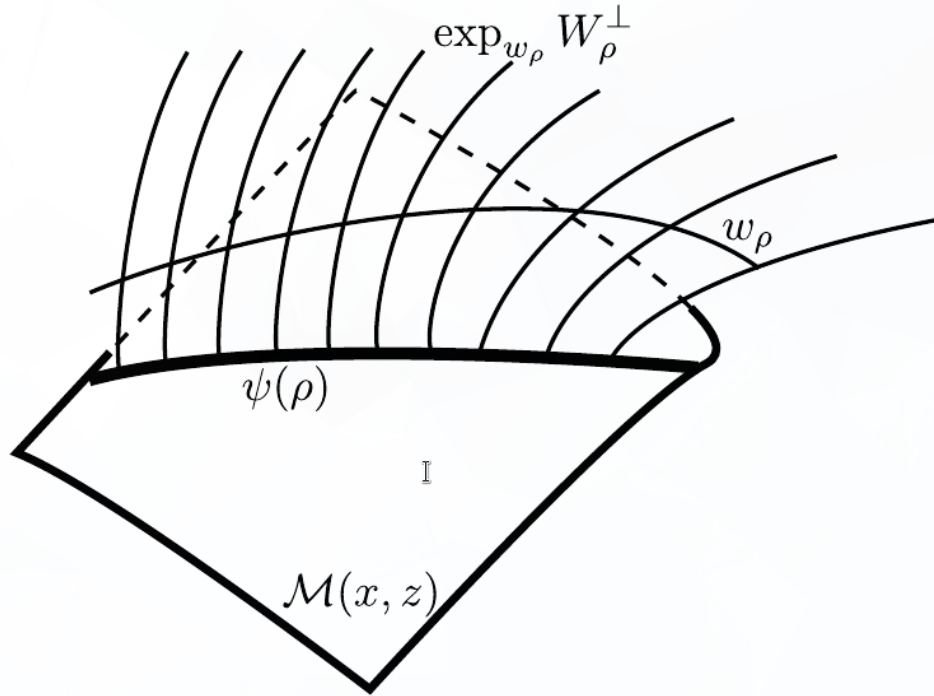


Figure 4: Intersection interpretation

- Apply the implicit function theorem to show differentiability of γ in ρ .
- Use a trivialization Z_i^ρ to get a vector field along w_ρ (also called $\gamma(\rho)$)
 - This is exactly an element of $T_{w_\rho} \mathcal{P}(x, z)$
- Exponentiate to get an element of $\mathcal{M}(x, z)$:

$$\psi(\rho) := \exp_{w_\rho}(\gamma(\rho)).$$

- Project onto $\mathcal{L}(x, z)$ to get $\hat{\psi}$.
- Prove $\hat{\psi}$ is a proper injective immersion and thus an embedding.
- Show that the broken trajectory (\hat{u}, \hat{v}) is the endpoint of an embedded interval in $\bar{\mathcal{L}}(x, z)$.
 - Then show that any other sequence converging to (\hat{u}, \hat{v}) must approach via this interval, otherwise could have cuspidal points:

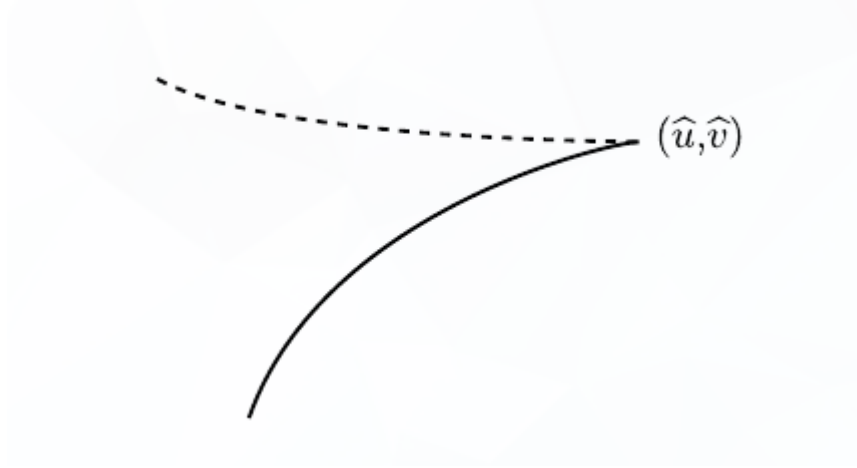


Figure 5: Cuspidal Point on Boundary

6.2 Section 9.4

- Have constructed $w_\rho \in C^\infty_{\searrow}(x, z)C^\infty(x, z)$ for every $\rho \geq \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x, z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining \mathcal{F}_ρ .

$$\begin{aligned}
W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n}) &\xrightarrow{\mathcal{F}_\rho} L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \\
(y_1, \dots, y_{2n}) &\mapsto \left[\left(\frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } H_t \right) (\exp_{w_\rho} \sum y_i Z_i^\rho) \right]_{Z_i}
\end{aligned}$$

where $\mathcal{F}_\rho := \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i . sd

- Linearize \mathcal{F}_ρ .