Homological Algebra

Problem Set 4

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Problem 1.0.1 (Problem 1)
Show that abelianization is left-adjoint to the inclusion $\mathsf{Ab} \to \mathsf{Grp}$.

Solution:
We want to show that there is an adjunction $\mathsf{Grp} \underset{\iota}{\overset{\mathrm{ab}}{\underset{\hookleftarrow}{\overset{\perp}{\rightarrow}}}} \mathsf{Ab}$ where $\mathrm{ab}(A) := A^{\mathrm{ab}}$ is the abelianization functor (claimed to be a left adjoint) and ι is the inclusion of a subcategory (claimed to be a right adjoint). Let $A \in \mathsf{Grp}, B \in \mathsf{Ab}$ and write $A^{\mathrm{ab}} := A/[AA]$ for the abelianization of A.

Claim: There is a bijection of sets given by the map

$$\begin{gathered}
\tau_{AB} : \mathrm{Hom}_{\mathsf{Grp}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Ab}}(A^{\mathrm{ab}}, B) \\
(A \xrightarrow{f} B) \mapsto (A^{\mathrm{ab}} \xrightarrow{\bar{f}} B) \\
(A \xrightarrow{\pi_A} A^{\mathrm{ab}} \xrightarrow{g} B) \leftarrow\!\shortmid (A^{\mathrm{ab}} \xrightarrow{g} B)
\end{gathered}$$

where π_A and $\bar{f}$ are the maps involved in the universal property of the quotient of groups: for $A \xrightarrow{f} B$ with $[AA] \subseteq \ker f$, there is a unique map $\bar{f}$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\pi_A"'] & B \\
A^{\mathrm{ab}} \arrow[ur, dashed, "\bar{f}"'] &
\end{tikzcd}$$

Link to Diagram

Proof (?).
That the forward map τ_{AB} is well-defined follows from the uniqueness of $\overline{f}$ supplied by the universal property. That B is abelian is sufficient to descend a map $f : A \to B$ to the quotient: writing $[AA] := \left\langle ghg^{-1}h^{-1} \;\middle|\; g, h \in B \right\rangle$, we have

$$\begin{aligned} f(ghg^{-1}h^{-1}) &= f(g)f(h)f(g^{-1})f(h^{-1}) \\ &= f(g)f(h)f(g)^{-1}f(h)^{-1} \\ &= f(g)f(g)^{-1}f(h)f(h)^{-1} && \text{using commutativity in } B \\ &= e_B, \end{aligned}$$

the identity element of B, and so $[AA] \subseteq \ker f$.
That the inverse map is well-defined follows from the fact that the canonical quotient map π_A exists for any group A, and compositions in a category are unique when they exist since (axiomatically) the composition pairing must form a well-defined set map on morphisms:

$$\circ : \mathrm{Mor}_{\mathcal{C}}(A_1, A_2) \times \mathrm{Mor}_{\mathcal{C}}(A_2, A_3) \to \mathrm{Mor}_{\mathcal{C}}(A_1, A_3).$$

We can compute the composition

$$(A \xrightarrow{f} B) \xrightarrow{\tau_{AB}} (A^{\mathrm{ab}} \xrightarrow{\overline{f}} B) \xrightarrow{\tau_{AB}^{-1}} (A \xrightarrow{\pi_A} A^{\mathrm{ab}} \xrightarrow{\overline{f}} B) = (A \xrightarrow{f} B),$$

where the last equality follows from commutativity of the above diagram, i.e. $f = \overline{f} \circ \pi_A$.
Computing the composition in the other direction yields

$$(A^{\mathrm{ab}} \xrightarrow{g} B) \xrightarrow{\tau_{AB}^{-1}} (A \xrightarrow{\pi_A} A^{\mathrm{ab}} \xrightarrow{g} B) \xrightarrow{\tau_{AB}} (A \xrightarrow{\overline{g \circ \pi_A}} B) = g,$$

where the last equality follows from the fact that this map is defined to make the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\tau_{AB}^{-1}(g) := g \circ \pi_A"] \arrow[d, "\pi_A"'] & B \\ A^{\mathrm{ab}} \arrow[ur, "g"'] & \end{tikzcd}$$

Link to Diagram

It remains to show naturality, i.e. that for all $f \in \mathrm{Mor}_{\mathsf{Grp}}(A_1, A_0)$ and $g \in \mathrm{Mor}_{\mathsf{Ab}}(B_0, B_1)$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Ab}}(A_0^{\mathrm{ab}}, B_0) \arrow[r, "\mathrm{ab}(f^*)"] & \mathrm{Hom}_{\mathsf{Ab}}(A_1^{\mathrm{ab}}, B_0) \arrow[r, "g_*"] & \mathrm{Hom}_{\mathsf{Ab}}(A_1^{\mathrm{ab}}, B_1) \\ \mathrm{Hom}_{\mathsf{Grp}}(A_0, B_0) \arrow[r, "f^*"'] \arrow[u, "\tau_{A_0 B_0}"'] & \mathrm{Hom}_{\mathsf{Grp}}(A_1, B_0) \arrow[r, "(\iota g)_*"'] \arrow[u, "\tau_{A_1 B_0}"'] & \mathrm{Hom}_{\mathsf{Grp}}(A_1, B_1) \arrow[u, "\tau_{A_1 B_1}"'] \end{tikzcd}$$

Link to Diagram

Explicitly, these maps are given by

- In the vertical direction: as defined above, $f : A_i \to B_k$ is sent to the induced quotient map $\overline{f} : A_i^{\mathrm{ab}} \to B_k$.

Problem 1.0.2 (Weibel Theorem 2.6.10, Part 1)
Let $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B}$ are arbitrary categories. Show that L preserves all colimits, i.e. if $F : \mathcal{I} \to \mathcal{A}$ has a colimit, then so does $LF : \mathcal{I} \to \mathcal{B}$, and

$$L\left(\operatorname*{colim}_{i \in I} F_i\right) = \operatorname*{colim}_{i \in I} L(F_i).$$

Solution:
Suppose $F : \mathcal{I} \to \mathcal{A}$ has a colimit. Starting in the category $\mathcal{B}$, suppose X is an object which admits a collection of commuting diagrams of the following form; we then want to show that the data highlighted in blue exists:

$$\begin{tikzcd}
\mathcal{I} & \mathcal{B} & & \\
i \arrow[dd, "\varepsilon_{ij}"] & LF(i) \arrow[dd, "LF\varepsilon_{ij}"'] \arrow[dr, dashed, "\exists\psi_i"] \arrow[drr, bend left, "\varphi_i"] & & \\
& & \exists\widehat{LF} \arrow[r, dashed, "\exists!\eta_{ij}"] & X \\
j & LF(j) \arrow[ur, dashed, "\exists\psi_j"] \arrow[urr, bend right, "\varphi_j"'] & &
\end{tikzcd}$$

Link to Diagram

Using the adjunction isomorphism, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(LF(i), X) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(F(i), RX) \\
\varphi_i &\mapsto R\varphi_i,
\end{aligned}$$

which yields a commuting diagram in $\mathcal{A}$:

$$\begin{tikzcd}
\mathcal{I} & \mathcal{A} & & \\
i \arrow[dd, "\varepsilon_{ij}"] & F(i) \arrow[dd, "F\varepsilon_{ij}"'] \arrow[dr, dashed, "\exists\tilde{\psi}_i"] \arrow[drr, bend left, "R\varphi_i"] & & \\
& & \exists\widehat{F} \arrow[r, dashed, "\exists!\tilde{\eta}_{ij}"] & RX \\
j & F(j) \arrow[ur, dashed, "\exists\tilde{\psi}_j"] \arrow[urr, bend right, "R\varphi_j"'] & &
\end{tikzcd}$$

Link to Diagram

Here, the existence of the indicated maps is supplied by the fact that F has a colimit $\widehat{F}$ in $\mathcal{A}$ by assumption. Applying the adjunction isomorphism to the maps $\tilde{\eta}_{ij}$ and using functoriality of L, we obtain maps

$$\tilde{\eta}_{ij} \in \mathrm{Hom}_{\mathcal{A}}(\widehat{F}, RX) \xrightarrow{\sim} \eta_{ij} \in \mathrm{Hom}_{\mathcal{B}}(L\widehat{F}, X)$$
$$\tilde{\psi}_{ij} \in \mathrm{Hom}_{\mathcal{A}}(F(i), \widehat{F}) \xrightarrow{L} \psi_{ij} \in \mathrm{Hom}_{\mathcal{B}}(LF(i), L\widehat{F}).$$

This provides all of the necessary blue data in the first diagram. However, this means that taking the object $\widehat{LF} := L\widehat{F}$ satisfies the universal property of the colimit of LF, making them equal by uniqueness. Unwinding notation, we have

$$L\left(\operatorname*{colim}_{i\in I} F_i\right) := L\widehat{F} \cong \widehat{LF} := \operatorname*{colim}_{i\in I} L(F_i).$$

Problem 1.0.3 (Weibel 2.7.3)
Let $I^{\cdot} \in \mathsf{Ch}(\mathsf{Ab})$ be a cochain complex of abelian groups and $P^{\cdot}, Q^{\cdot} \in \mathsf{Ch}(\mathsf{R\text{-}Mod})$. Show that there is a natural isomorphism of double complexes:

$$\mathrm{Hom}_{\mathsf{Ab}}\left(\mathrm{Tot}^{\oplus}(P^{\cdot} \otimes_R Q^{\cdot}), I^{\cdot}\right) \cong \mathrm{Hom}_R\left(P^{\cdot}, \mathrm{Tot}^{\Pi}\left(\mathrm{Hom}_{\mathsf{Ab}}(Q^{\cdot}, I^{\cdot})\right)\right).$$

Solution:
We proceed by comparing the term in bigrade (p, q) on both sides Expanding the left-hand side first, we have

$$\begin{aligned}
\left(\mathrm{Hom}_{\mathsf{Ab}}\left(\mathrm{Tot}^{\oplus}(P \otimes_R Q), I\right)\right)_{p,q} &:= \mathrm{Hom}_{\mathsf{Ab}}\left(\mathrm{Tot}^{\oplus}(P \otimes_R Q)_p, I_q\right) \\
&:= \mathrm{Hom}_{\mathsf{Ab}}\left(\bigoplus_{i+j=p} P_i \otimes_R Q_j, I_q\right) \\
&= \bigoplus_{i+j=p} \mathrm{Hom}_{\mathsf{Ab}}\left(P_i \otimes_R Q_j, I_q\right).
\end{aligned}$$

Similarly expanding the right-hand side we have

$$\begin{aligned}
\left(\mathrm{Hom}_R\left(P, \mathrm{Tot}^{\Pi}\left(\mathrm{Hom}_{\mathsf{Ab}}(Q, I)\right)\right)\right)_{p,q} &:= \mathrm{Hom}_R\left(P_p, \mathrm{Tot}^{\Pi}\left(\mathrm{Hom}_{\mathsf{Ab}}(Q, I)\right)_q\right) \\
&:= \mathrm{Hom}_R\left(P_p, \prod_{i+j=q} \mathrm{Hom}_{\mathsf{Ab}}(Q_i, I_j)\right) \\
&= \prod_{i+j=q} \mathrm{Hom}_R\left(P_p, \mathrm{Hom}_{\mathsf{Ab}}(Q_i, I_j)\right) \\
&= \prod_{i+j=q} \mathrm{Hom}_{\mathsf{Ab}}\left(P_p, \mathrm{Hom}_{\mathsf{Ab}}(Q_i, I_j)\right) \\
&= \prod_{i+j=q} \mathrm{Hom}_{\mathsf{Ab}}\left(P_p \otimes_R Q_i, I_j\right) \\
&\cong \bigoplus_{i+j=q} \mathrm{Hom}_{\mathsf{Ab}}\left(P_p \otimes_R Q_i, I_j\right),
\end{aligned}$$

where we've used

- To justify switching Hom_R to $\mathrm{Hom}_{\mathsf{Ab}}$: that an R-module morphism into a $\mathbb{Z}$-module is determined by the image of 1, and thus is determined the underlying $\mathbb{Z}$-module structure,
- The tensor-hom adjunction $\mathsf{R\text{-}Mod} \rightleftharpoons \mathbb{Z}\text{-}\mathsf{Mod}$, and
- That finite direct products are isomorphic to direct sums.

Remark 1.0.1: These two are not equal as-is, since e.g. the first only involves I_q, but the second involves I_j for $1 \leq j \leq q$. To fix this, it seems like both results need to have a direct product taken over all p and all q, yielding

$$\prod_p \prod_q \bigoplus_{i+j=p} \mathrm{Hom}_{\mathsf{Ab}}\left(P_i \otimes_R Q_j, I_q\right) \cong \prod_p \prod_q \bigoplus_{i+j=q} \mathrm{Hom}_{\mathsf{Ab}}\left(P_p \otimes_R Q_i, I_j\right).$$

Problem 1.0.4 (Weibel 3.2.1)
Show that the following are equivalent for any $B \in \mathsf{R\text{-}Mod}$:

1. B is flat as a left R-module.
2. $\mathrm{Tor}_n^R(A, B) = 0$ for $n \neq 0$ and all A.
3. $\mathrm{Tor}_1^R(A, B) = 0$ for all A.

Solution:
We proceed by showing $2 \implies 3 \implies 1 \implies 2$. By definition, $B \in (\mathsf{R}, \mathbb{Z})\text{-}\mathsf{biMod}$ is **flat** $\iff$ the functor

$$\cdot \otimes_R B : (\mathbb{Z}, \mathsf{R})\text{-}\mathsf{biMod} \to (\mathbb{Z}, \mathbb{Z})\text{-}\mathsf{biMod}$$

is exact. Note that $2 \implies 3$ is immediate, and that $3 \implies 1$ follows by applying $\cdot \otimes_R B$ to a SES and taking the associated LES (noting that tensoring is covariant, right-exact, and so the tor terms occur as left-derived functors):

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] & A \arrow[r] \arrow[d, Rightarrow, "\cdot \otimes_R B"] & A' \arrow[r] & 0 \\
 & & {} & 0 & \\
 & A'' \otimes_R B \arrow[r] & A \otimes_R B \arrow[r] & A' \otimes_R B \arrow[u] & \\
 & \mathrm{Tor}_1^R(A'', B) & \mathrm{Tor}_1^R(A, B) \arrow[l] & \color{red}{\mathrm{Tor}_1^R(A', B)} \arrow[l] \arrow[ull] & \\
 & & & \cdots \arrow[ull] &
\end{tikzcd}$$

Link to Diagram

If the red term Tor_1^R vanishes, then this induces an exact sequence

$$0 \to A'' \otimes_R B \to A \otimes_R B \to A' \otimes_R B \to 0,$$

making $\cdot \otimes_R B$ an exact functor.
Now to see that $1 \implies 2$, we'll use the results of exercise 2.4.3 on dimension shifting. Letting $A \in \textsf{Mod-R}$ be an arbitrary right R-module, if we define the functor $F(A) := A \otimes_R B$, we can take a projective resolution $P^\cdot \xrightarrow{\varepsilon} A$ and (as a result of the exercise) compute

$$\operatorname{Tor}_i^R(A, B) := L_i F(A) \cong H_i(FP)$$

as the homology of the complex FP. However, by assumption, F is an exact functor. and since the resolution $P^\cdot := \cdots \to P_1 \to P_0 \to 0$ is exact, the resulting complex $FP := \cdots \to FP_1 \to FP_0 \to 0$ is again exact. Thus $H_i(FP) = 0$ for every $i \geq 1$, so $\operatorname{Tor}_i^R(A, B) = 0$ for every $i \geq 1$ as well.

Problem 1.0.5 (Weibel 3.2.2)
Show that if the following is a SES

$$0 \to A \to B \to C \to 0$$

with both B and C flat, then A is flat.

Solution:
We want to show that the functor $\cdot \otimes_R A$ is flat for $A \in \textsf{R-Mod}$, and by the previous exercise, it suffices to show $\operatorname{Tor}_1^R(X, A) = 0$ for all $X \in \textsf{Mod-R}$. Start by letting X be arbitrary and applying the functor $X \otimes_R \cdot$ to the above SES to obtain

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$\Downarrow X \otimes_R \cdot$$

$$\begin{tikzcd}
 & & 0 \\
X \otimes_R A \arrow[r] & X \otimes_R B \arrow[r] & X \otimes_R C \arrow[u] \\
\operatorname{Tor}_1^R(X, A) \arrow[r] & {\color{red}\operatorname{Tor}_1^R(X, B)} \arrow[r] & {\color{red}\operatorname{Tor}_1^R(X, C)} \arrow[ull] \\
\cdots \arrow[r] & {\color{red}\operatorname{Tor}_2^R(X, B)} \arrow[r] & {\color{red}\operatorname{Tor}_2^R(X, C)} \arrow[ull]
\end{tikzcd}$$

Link to Diagram

Since B and C are flat, by the previous exercise, $\operatorname{Tor}_n^R(X, B) = \operatorname{Tor}_n^R(X, C) = 0$ for all $n \geq 1$, and so all of the red terms vanish. This forces $\operatorname{Tor}_n^R(X, A) = 0$ for all $n \geq 1$, since each one fits into an exact sequence $0 \to \operatorname{Tor}_n^R(X, A) \to 0$.

Problem 1.0.6 (Weibel 3.4.1)
Show that if p is prime, there are exactly p equivalence classes of extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$ in Ab, the split extension along with the extensions

$$0 \to \mathbb{Z}/p \xrightarrow{x \mapsto px} \mathbb{Z}/p^2 \xrightarrow{x \mapsto ix} \mathbb{Z}/p \to 0 \qquad\qquad 1 \leq i \leq p-1.$$

Solution:
We first note that in any SES $0 \to A \to B \to C \to 0$ we have $C \cong B/A$, we must also have $|B| = |A| \cdot |C|$, so the middle term in any extension must have order p^2. There are only two groups of order p^2, namely $(\mathbb{Z}/p)^{\oplus 2}$ and $\mathbb{Z}/p^2$, and the former yields an extension

$$0 \to \mathbb{Z}/p \xrightarrow{\iota_1} \mathbb{Z}/p \oplus \mathbb{Z}/p \xrightarrow{\pi_2} \mathbb{Z}/p \to 0,$$

where ι_1 is inclusion of the first factor and π_2 is projection onto the second factor. This has a left-section π_1 projecting onto the first factor, so this is a split extension in $\mathbb{Z}$-modules.
For the non-split extensions, we can see exactness from the following facts:

- Exactness in the first position: $x \mapsto px$ is injective, since its kernel is precisely $[p] = [0] \in \mathbb{Z}/p$
- Exactness in the middle: tracing the indicated maps yields $[x] \mapsto [px] \mapsto [ipx] = [0] \in \mathbb{Z}/p$,
- Exactness at the last position: if $1 \leq i \leq p-1$, then the equivalence class $[i]_p$ is a generator of $\mathbb{Z}/p$, and is in the image since $[1]_{p^2} \mapsto [i]_p$.

So for each $1 \leq i \leq p-1$ we do get a SES and thus an extension. None of these extensions can be isomorphic to the split extension, since the middle terms are not isomorphic, and so it remains to show that none of the non-split extensions are isomorphic either. We can proceed by taking any two such extensions, say with $i \neq j \pmod p$, and considering commuting diagrams of the following form and considering what equalities commutativity forces:

$$
\begin{tikzcd}
 & & x \arrow[r, mapsto, dashed] \arrow[ddddd, dashed, bend left] & ix \arrow[ddddd, dashed, bend left] & \\
 & y \arrow[r, mapsto, dashed] \arrow[ddd, dashed, bend right] & py \arrow[ddd, dashed, bend right] & & \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "p"] \arrow[d, equal, "\mathbb{1}"'] & \mathbb{Z}/p^2 \arrow[r, "i"] \arrow[d, "f"] & \mathbb{Z}/p \arrow[r] \arrow[d, equal, "\mathbb{1}"] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "p"] & \mathbb{Z}/p^2 \arrow[r, "j"] & \mathbb{Z}/p \arrow[r] & 0 \\
 & y \arrow[r, mapsto, dashed] & py = f(py) & & \\
 & & f(x) \arrow[r, mapsto, dashed] & jf(x) &
\end{tikzcd}
$$

Link to Diagram

By considering $x \in \mathbb{Z}/p^2$ and traversing the right-hand side square, we obtain

$$ix \equiv jf(x) \pmod p \implies f(x) \equiv j^{-1}ix \pmod p,$$

since $\mathbb{Z}/p$ is a field and i is thus invertible. Since $j^{-1}i$ is a fixed number, this completely determines f.
Now considering $y \in \mathbb{Z}/p$ and traversing the left-hand side square yields

$$\begin{aligned} py \equiv f(py) \pmod{p^2} &\implies py \equiv pf(y) \pmod{p^2} \\ &\implies y \equiv f(y) \pmod p, \end{aligned}$$

where we've used that f is $\mathbb{Z}$-module morphism to pull out the integer p and also that if $p^2 \mid pa - pb$ then $p \mid a - b$ for the second equality. In particular, taking $y \coloneqq \pmod p$ here and using the previous formula for f, we obtain

$$1 = f(1) \coloneqq j^{-1}i \cdot 1 \pmod p \implies i \equiv j \pmod p,$$

which is a contradiction.
This yields p distinct classes of extensions, and since one can compute $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$, using the correspondence from the next exercise, these exhaust all possible extensions.

Problem 1.0.7 (Weibel Theorem 3.4.3, omitted details)
Given $A, B \in \mathsf{R\text{-}Mod}$, there is a 1-to-1 correspondence

$$\begin{aligned} \left\{\text{Extensions of } A \text{ by } B\right\}_{/\sim} &\underset{\Psi}{\overset{\Theta}{\rightleftharpoons}} \operatorname{Ext}^1_R(A, B) \\ \xi &\mapsto \partial(\mathbb{1}_B) \\ \operatorname{pushout}(B \leftarrow M \rightarrow P) &\mapsfrom x \end{aligned}$$

Solution:
We proceed by following Weibel's construction in Theorem 3.4.3 and filling in the details.

Claim: Θ is surjective.
We fix a SES resolving A and apply $\mathrm{Hom}(\cdot, B)$ to get a LES:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "j"] & P \arrow[r, "f"] & A \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d] & & \\ \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(P,B) \arrow[r] & \mathrm{Hom}(M,B) \arrow[dll, "\partial"] \\ \mathrm{Ext}^1(A,B) \arrow[r] & \mathrm{Ext}^1(P,B) = 0 \arrow[r] & \cdots \end{tikzcd}$$

Link to Diagram

Here the red term vanishes since P is projective and thus F-acyclic for $F = \mathrm{Hom}$. Fixing $x \in \mathrm{Ext}^1(A,B)$, by surjectivity of ∂ we pull this back to $\beta \in \mathrm{Hom}(A,B)$ where $\partial\beta = x$. We define X by forming the following pushout:

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner", very near end] & P \arrow[d, dashed] \\ B \arrow[r, dotted] & \exists X \end{tikzcd}$$

Link to Diagram

From this, we can produce a map $X \xrightarrow{\varphi} A$ by forming a second commuting square and using the universal property of the pushout to extend this to a commuting diagram:

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\ulcorner", very near end] & P \arrow[d, "f"] \arrow[ddr, dashed, bend left, "\sigma"] & \\ B \arrow[r, "0"'] \arrow[drr, dashed, bend right, "i"] & A & \\ & & X \arrow[ul, dashed, "\exists!\varphi"] \end{tikzcd}$$

Diagram A (Link to Diagram)

Here we've defined a map $B \to A$ by just sending everything to zero, and the diagram commutes because the composition $M \to P \to A$ is zero, as this was the original SES resolving A. Thus A receives a unique map φ from X, and we can assemble a commuting ladder:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[r, "f"] \arrow[d, "\sigma"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "i"] \arrow[ur, phantom, "\ulcorner"] & X \arrow[r, "\varphi"] & A \arrow[r] & 0
\end{tikzcd}$$

Diagram B (Link to Diagram)

The first square commutes since it comes from a pushout, and the second square commutes since it occurs in Diagram (A) in the triangle involving P, X, A. We can now consider the induced map on long exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(P,B) \arrow[r] & \mathrm{Hom}(M,B) \arrow[r, "\partial"] & \mathrm{Ext}^1(A,B) \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] \arrow[u] & \mathrm{Hom}(X,B) \arrow[r] \arrow[u] & \mathrm{Hom}(B,B) \arrow[r, "\partial'"] \arrow[u, "\beta^*"'] & \mathrm{Ext}^1(A,B) \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

Link to Diagram

By naturality of ∂, all three squares are commutative, and so we have $\mathbb{1}_{\mathrm{Ext}(A,B)}\partial' = \partial\beta^*$ and in particular

$$\begin{aligned}
\Theta(\xi) &:= \partial'(\mathbb{1}_B) \\
&= \mathbb{1}_{\mathrm{Ext}(A,B)}\partial'(\mathbb{1}_B) \\
&= \partial\beta^*(\mathbb{1}_B) && \text{by naturality} \\
&:= \partial(\mathbb{1}_B \circ \beta) \\
&= \partial(\beta) \\
&= x,
\end{aligned}$$

so Θ is surjective.

Claim: The bottom sequence $0 \to B \xrightarrow{i} X \xrightarrow{\varphi} A \to 0$ in Diagram (B) is exact.

We want to show that

- i is injective,
- φ is surjective, and
- $\operatorname{im} i = \ker \varphi$.

Proceeding,

- **i is injective:**

  The map i is given by the formula[a]

$$\begin{aligned}
i : B &\to X \\
b &\mapsto \pi(0, b),
\end{aligned}$$

where π denotes the canonical quotient map $\pi : P \oplus B \twoheadrightarrow X$ coming from alternative characterization of $X = \operatorname{coker}(F)$ for F defined as

$$\begin{aligned} F : M &\to P \oplus B \\ m &\mapsto \begin{bmatrix} j(m) \\ -\beta(m) \end{bmatrix}. \end{aligned}$$

Using the definition of what it means to be zero in a quotient, we thus have

$$b \in \ker i \iff \pi\left(\begin{bmatrix} 0 \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} j(m) \\ -\beta(m) \end{bmatrix} \qquad \text{for some } m \in M.$$

Now since j is injective and $j(m) = 0$, we must have $m = 0$, and since β is an R-module morphism we also have $\beta(m) = 0$. So $(0, b) = (0, 0)$ and $b = 0$, making i injective.

- φ **is surjective**:

  f is surjective, since it occurs at the end of SES, and commutativity of the second square yields $\mathbb{1}_A f = \varphi\sigma \implies f = \varphi\sigma$. If φ failed surjectivity, there would be an $a \in A$ with $a \notin \operatorname{im} \varphi$, which would force $a \notin \operatorname{im}(\varphi\sigma) = \operatorname{im}(f)$, a contradiction.

$\operatorname{im} i = \ker \varphi$:
Let $x \in \ker \varphi \subseteq X$, then using the cokernel characterization of X we can write $x := (p, b) + F(M)$ as some coset representative in the quotient $X = (P \oplus B)/F(M)$ where $p \in P$ and $b \in B$. We can make explicit the maps appearing by revisiting the pushout diagram:

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[d, dashed, "\sigma(p)=\pi(p,0)"] \arrow[dr, dashed, bend left, "p\mapsto(p,0)"] & & \\
B \arrow[r, dashed, "i(b)=\pi(0,b)"] \arrow[rr, dashed, bend right, "b\mapsto(0,b)"'] & X = \operatorname{coker} F \arrow[ul, phantom, "\ulcorner", very near start] & P \oplus B \arrow[l, "\pi"'] & M \arrow[l, "F"']
\end{tikzcd}$$

Link to Diagram

Claim: The following diagram chase produces an element $m \in M$ with $j(m) = p$ and $\beta(m) \in B$ such that $(p, b) \equiv (0, b - \beta(m)) \bmod F(M)$, where the latter is in the image of i.

$$
\begin{tikzcd}
 & \color{green}{\exists m} \arrow[r, dashed, mapsto, green] \arrow[ddd, dashed, mapsto, bend right, green] & \color{red}{p} \arrow[r, dashed, mapsto, red] \arrow[ddd, dashed, mapsto, bend right, red] & \color{red}{0_A} & \\
0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[r, "f"] \arrow[d, "\sigma"'] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r, "i"] & X \arrow[ur, "\varphi"] & & \\
 & \color{green}{\beta(m)} \arrow[dr, dashed, mapsto, green, "i"] & \color{red}{x \coloneqq (p, b) + F(M)} \arrow[uuur, dashed, mapsto, bend right, red] & & \\
 & & \color{green}{(0, \beta(m))} & &
\end{tikzcd}
$$

Link to Diagram

Observing that

$$
\begin{aligned}
\left\{(\tilde{p}, 0) + F(M) \;\middle|\; \tilde{p} \in P\right\} &\subseteq \operatorname{im}\sigma \\
\left\{(0, \tilde{b}) + F(M) \;\middle|\; \tilde{b} \in B\right\} &\subseteq \operatorname{im} i,
\end{aligned}
$$

to show $x \in \operatorname{im} i$ it suffices to show that x can be written in the form $x = (0, \tilde{b}) + F(M)$ for some $\tilde{b} \in B$. We'll take $\tilde{b} \coloneqq b - \beta(m)$ in the diagram, and the result will follow from the calculation

$$
\begin{aligned}
(p, b) &\equiv (0 + p, b) \\
&\equiv (0 + j(m), b) \\
&\equiv (0, b - \beta(m)) \\
&\coloneqq (0, \tilde{b}) \in \operatorname{im} i,
\end{aligned}
$$

where moving the j to the β is justified by the following:

$$
\begin{aligned}
(a + j(m), b) \equiv (a, b - \beta(m)) &\iff ((a + j(m)) - a, b - (b - \beta(m))) \in F(M) \\
&\iff (j(m), \beta(m)) \in F(M),
\end{aligned}
$$

which is true.

Proof (of claim, using a diagram chase).

- We start with $x \coloneqq (p,b)+F(M) \in X$, and since $(p,0)+F(M) \in \operatorname{im}\sigma$, we can pull the first coordinate back to $p \in P$.
- Since $\varphi(x)=0$ and the triangle commutes, $f(p)=0_A$, completing the red part of the diagram.
- Starting the green part, exactness of the top row yields an $m \in M$ such that $j(m)=p$.
- We can take its image under β to get an element $\beta(m) \in B$.
- By commutativity of the square, we have $i(\beta(m)) = \sigma(j(m))$,
- We have
  - $i(\beta(m)) \coloneqq (0,\beta(m)) \pmod{F(M)}$
  - $\sigma(j(m)) \coloneqq (j(m),0) \pmod{F(M)}$
- Since $j(m)=p$, we can write

$$
\begin{aligned}
& (p,0) \equiv (0,\beta(m)) \pmod{F(M)} \\
\implies & (p,b) \equiv (p,0)+(0,b) \\
& \qquad = (j(m),0)+(0,b) \\
& \qquad \equiv (0,-\beta(m))+(0,b) \\
& \qquad \equiv (0,b-\beta(m)) \pmod{F(M)}.
\end{aligned}
$$

■

Returning to Weibel's construction:

Claim: The association $x \mapsto \xi$ where ξ is the extension occurring as the bottom row in Diagram (B) defines a map $\Psi : \operatorname{Ext}^1(A,B) \to \{\text{Extensions of } A \text{ by } B\}_{/\sim}$. In particular, if β' is any other choice of a lift of $x \in \operatorname{Ext}^1(A,B)$ and X' the corresponding pushout with ξ' the corresponding extension, we have $X' \cong X$ and $\xi' \cong \xi$.

Proof (That $X' \cong X$).
We first recall the original construction:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "j"] & P \arrow[r, "f"] \arrow[d, Rightarrow, "\mathrm{Hom}(\,\cdot\,,B)"] & A \arrow[r] & 0 \\
& 0 \arrow[d] & & & \\
& \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(P,B) \arrow[r, "j^*"] & \mathrm{Hom}(M,B) \arrow[dll, "\partial"] & \\
& \mathrm{Ext}^1(A,B) \arrow[r] & 0 \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

Repeating this with a new lift β' with $\partial(\beta') = x$, we first note that $\partial(\beta' - \beta) = 0$ and so $\beta' - \beta \in \ker \partial = \operatorname{im} j^*$. We can thus write

$$\beta' - \beta = j^*(g) \coloneqq gj \implies \beta' = \beta + gj \tag{1}$$

for some $g \in \mathrm{Hom}(P, B)$. Constructing the pushouts, we have

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[d, dashed, "\sigma"] \\
B \arrow[r, dashed, "i"'] & X \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta'"'] & P \arrow[d, dashed, "\sigma'"] \\
B \arrow[r, dashed, "i'"'] & X' \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

Link to Diagram

The claim is now that we can take the maps $i : B \to X$ and $\sigma + ig : P \to X$ to induce an isomorphism $X \cong X'$. By a quick computation, we have

$$\begin{aligned}
(\sigma + ig)j &= \sigma j + igj \\
&= i\beta + igj && \text{by commutativity of the left square} \\
&= i(\beta + gj) \\
&= i\beta' && \text{by equation 1,}
\end{aligned}$$

and similarly

$$\begin{aligned}
(\sigma' - i'g)j &= \sigma' j - i'gj \\
&= i'\beta' - i'gj && \text{by commutativity of the right square} \\
&= i'(\beta' - gj) \\
&= i'\beta && \text{by equation 1,}
\end{aligned}$$

and we have the following commuting squares, where the existence of the indicated maps is provided by the universal properties of X and X':

$$\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[d, "\sigma' - i'g"] \\
{} & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "j"] \arrow[d, "\beta'"'] & P \arrow[d, "\sigma + ig"] \\
{} & {}
\end{tikzcd}$$

[a]I'm not sure how to actually obtain/prove the above formula for $i : B \to X$, this comes from looking it up elsewhere.