

# Title

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## 1 | Monday, November 09

### 1.1 Chapter 1

Let  $k$  be a field, not necessarily algebraically closed.

**Definition 1.1.1** (Algebraic Function Field).

An one variable **algebraic function field**  $F/K$  is a field extension  $F$  of  $K$  which factors as



where  $x \in \bar{k}$  is some element that is not algebraic over  $k$ .

**Definition 1.1.2** (Field of Constants).

The subfield

$$\tilde{k} := \{z \in F \cap K^{\text{alg}}\} \leq F,$$

consisting of elements that are algebraic over  $F$  is denoted the **field of constants**.

**Definition 1.1.3** (Algebraically Closed).

If  $\tilde{k} = k$ , we say that  $k$  is **algebraically closed** in  $F$ .

**Definition 1.1.4** (Rational Function Field).

An extension  $F/k$  is **rational** iff  $F = k(y)$  for some  $y \in k^{\text{transc}}$  which is transcendental over  $k$ .

**Definition 1.1.5** (Valuation Ring).

A ring  $\mathcal{O} \subseteq F$  is a **valuation ring** for  $F$  iff  $k \subset \mathcal{O} \subseteq F$  and  $z \in F \implies z \in \mathcal{O}$  or  $z^{-1} \in \mathcal{O}$ .

**Definition 1.1.6** (Discrete Valuation Ring).

A ring local  $R$  (thus with a unique maximal ideal) which is a PID but not a field is a **discrete valuation ring**.

**Definition 1.1.7** (Place).

A **place** of a function field  $F/K$  is the maximal ideal of a valuation ring of  $F/K$ .

**Definition 1.1.8** (Discrete Valuation).

A **discrete valuation** of  $F/k$  is a function

$$v : F \rightarrow \mathbb{Z} \cup \{\infty\}$$

that is

1. Nondegenerate:  $v(x) = \infty$  iff  $x = 0$ .
2. Multiplicative:  $v(xy) = v(x) + v(y)$ .
3. Ultrametric triangle inequality:  $v(x + y) \geq \min(v(x), v(y))$ .
4. Fiber over one: there exist a  $z \in F$  with  $v(z) = 1$ .
5.  $v|_k = 0$ .

**Definition 1.1.9** (Rational Place).

A place of degree one is said to be a **rational place**.

**Definition 1.1.10** (Valuation Ring of a Place).

The **valuation ring of a place** is defined by

$$\mathcal{O}_P := \left\{ z \in F \mid z^{-1} \notin P \right\}.$$

**Definition 1.1.11** (Degree of a Place).

The **degree** of a place  $P$  is defined by

$$\deg(P) := [F_P : k],$$

where  $F_P = \mathcal{O}_P/P$ .

**Definition 1.1.12** (Discrete Valuation of a Place).

To any place  $P$  we associate the function

$$v_P : F \rightarrow \mathbb{Z} \cup \{\infty\}$$

defined by choosing any prime  $t \in P$ , writing any  $x \in F$  as  $x = t^n u$  with  $u \in \mathcal{O}_P^\times$ , and setting

$$v_P(x) = \begin{cases} n & \text{if } x = t^n u \\ 0 & x = \infty. \end{cases}$$