Real Analysis (Sp. 2021 #2)

Check

$y = x/n$

Compute

$$L := \lim_{n\to\infty} \int_0^n \frac{\cos(x/n)}{x^2 + \cos(x/n)}\,dx$$

$$= \lim_{n\to\infty} \int_0^\infty \chi_{(0,n)}(x) \cdot \left(\frac{\cos(x/n)}{x^2 + \cos(x/n)} \right) dx$$

Nike!

? $$= \int_0^\infty \lim_{n\to\infty} \chi_{(0,n)}(x) \cdot \left(\frac{\cos(x/n)}{x^2 + \cos(x/n)} \right) dx$$

$$= \int_0^\infty \lim_{n\to\infty} \left(\frac{\cos(x/n)}{x^2 + \cos(x/n)} \right) dx$$

Aside

$$\lim_{n\to\infty} \left(\frac{\cos(x/n)}{x^2 + \cos(x/n)} \right) = \frac{\cos(0)}{x^2 + \cos(0)}$$

$$= \frac{1}{x^2+1}$$

$$= \int_0^\infty \underbrace{\frac{1}{x^2+1}}_{g(x)}\,dx$$

$f \le g$

Doesn't cause problem at 0 or ∞ in integral

$$= \arctan(x)\Big|_0^\infty$$

Derive?

$$= \pi/2 - 0 = \pi/2!$$ ✓

$$\left(\frac{\cos(x/n)}{x^2+\cos(x/n)}\right) \leftarrow \int_0^1 \lim_{n\to\infty}\left(\frac{n\cdot\cos(y)}{n^2y^2+\cos(y)}\right) \quad \begin{matrix} y = x/n \\ dy \end{matrix}$$

$$= \int_0^1 \lim_{n\to\infty} \frac{\cos(y)}{\underbrace{ny}_{\to\infty} + \underbrace{(\frac{1}{n})\cos(y)}_{\to 0}} \, dy$$

$y \neq 0$

$$= \underbrace{\int_0^\varepsilon \ldots \, dy}_{\text{No DCT!}} + \underbrace{\int_\varepsilon^1 \ldots \, dy}_{\to 0}$$

Thm (DCT)

- $|f_n| \leq g \quad (\Rightarrow f_n \in L^1)$
- $g \in L_1((0,\infty)) \subseteq L_1(\mathbb{R})$

$\Rightarrow \lim_{n\to\infty} f_n \in L_1((0,\infty))$

$\Rightarrow \lim_{n\to\infty} \int f_n = \int f$

1) On $(1+\varepsilon, \infty)$

$$\left|\frac{\cos(x/n)}{x^2+\cos(x/n)}\right| \leq \left|\frac{1}{x^2+\cos(x/n)}\right|$$

$$(-1 \leq \cos(x/n) \leq 1)$$

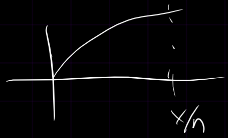
$$\leq \left|\frac{1}{x^2-1}\right| := g =$$

$$\int_{1+\varepsilon}^\infty \frac{1}{x^2-1}\,dx \overset{\pm}{=} \frac{1}{2}\int_{1+\varepsilon}^\infty \frac{1}{x+1} - \frac{1}{x-1}\,dx$$

$$\overset{\pm}{=} \frac{1}{2}\left(\ln\left(\frac{x+1}{x-1}\right)\right)\Big|_{1+\varepsilon}^\infty$$

$$\approx \ln(1) - \text{const} < \infty$$

2) On $(0, 1+\varepsilon)$ $\quad \frac{(x)}{}$



$n >> 1$

Taylor expand ?

$\approx 1 - \frac{x^2}{n^2}$

$n>>1$?

$$\frac{\cos(x/n)}{x^2+\cos(x/n)} \leq 1$$

$$\Rightarrow \int_0^{1+\varepsilon} \ldots \leq \int_0^{1+\varepsilon} 1 \, dx$$

$$= 1+\varepsilon < \infty$$