

The cohomological proof of Brouwer's fixed point theorem

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Abstract

The simplest version of Brouwer's fixed point theorem (discovered by others before him, of course) is as follows: Every continuous map from the unit disc in the plane to itself has a fixed point. I will explain the 'cohomological' proof of this theorem, using divergence and curl of vector fields. The generalization of this technique to higher dimensions is called De Rham cohomology.

1 Introduction

Let us write

$$D^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\} \quad \text{and} \quad S^{n-1} = \partial D^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$$

for the n -dimensional ball in Euclidean n -space and its boundary, the $(n-1)$ -dimensional sphere respectively. Then we have the following famous result.

Brouwer fixed point theorem (Poincaré 1886, ..., Brouwer 1912). Every continuous map $f: D^n \rightarrow D^n$ has a fixed point.

As indicated above, this theorem goes back to Poincaré and maybe even others. The reason Brouwer's name has "stuck" is probably simply because of the amusing story that he came across this theorem while pondering his morning cup of coffee (see Figure 1). Interestingly enough, Brouwer's proof was nonconstructive, even though he was a staunch intuitionist.

We will analyze this theorem in the two-dimensional setting. There is in fact a simple homotopical proof of the theorem: the fundamental group of the circle is nontrivial while that of the disc is trivial, which implies the theorem. But we will ignore this simpler proof and rather use the occasion as an excuse to introduce De Rham cohomology!

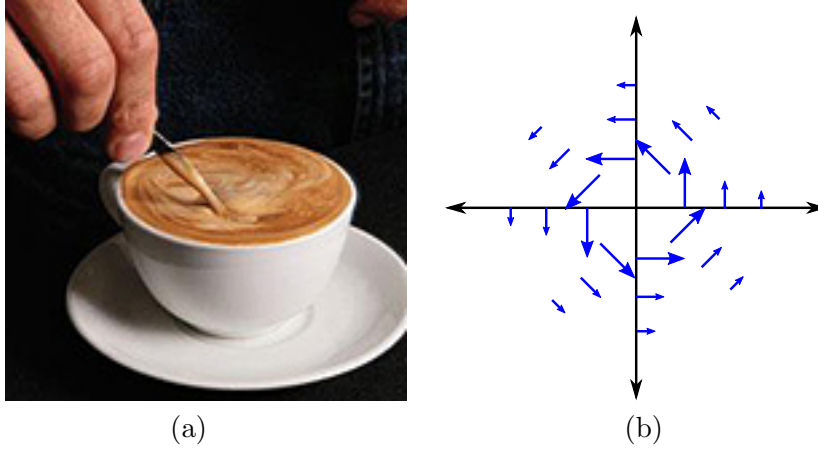


Figure 1: (a) Brouwer reasoned that at each instant while stirring his morning cup of coffee, there must exist a particle on the surface which is in the same position as it was before the stirring commenced. (b) The vector field \mathbf{V} is locally (but not globally!) the gradient of the argument function on the plane.

2 Motivation: planar vector fields

Let U be an open set in the plane, and $\mathbf{V} = \langle P(x, y), Q(x, y) \rangle$ a smooth vector field on U .

Question: Is $\mathbf{V} = \text{grad } f$ for some function f on U ?

The question asks if we can find a function f on U such that

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}.$$

If this is the case, then since

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \Rightarrow$$

it means we must require

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (\text{equivalently, } \text{curl } \mathbf{V} = 0), \tag{1}$$

otherwise there will be no hope of a solution. So now suppose that the vector field \mathbf{V} satisfies $\text{curl } \mathbf{V} = 0$. Is this sufficient to guarantee the existence of f ?

The answer is no. Consider the following vector field (see Figure 1) on $U = \mathbb{R}^2 \setminus \{(0, 0)\}$,

$$\mathbf{V} = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

One verifies that this vector field is indeed curl-free. But if $\mathbf{V} = \text{grad } f$, then

$$\int_C \mathbf{V} \cdot d\mathbf{r} = \int_C \text{grad } f = 0$$

by the fundamental theorem for line integrals, but

$$\begin{aligned} \int_C \mathbf{V} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle -\sin \theta, \cos \theta \rangle \cdot \langle -\sin \theta, \cos \theta \rangle d\theta \\ &= 2\pi, \end{aligned}$$

a contradiction!

3 De Rham cohomology

Motivated by this example, given an open set $U \subseteq \mathbb{R}^2$, we consider the following sequence of linear maps:

$$0 \longrightarrow C^\infty(U) \xrightarrow{\text{grad}} \text{Vect}(U) \xrightarrow{\text{curl}} C^\infty(U) \longrightarrow 0 \quad (2)$$

Here I am writing $C^\infty(U)$ for the smooth functions on U , $\text{Vect}(U)$ for the vector fields on U , and the curl of a vector field $\mathbf{V} = \langle P, Q \rangle$ in the plane is being interpreted as a function,

$$\text{curl } \mathbf{V} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}.$$

For an open set $U \subseteq \mathbb{R}^3$, we have a similar sequence:

$$0 \longrightarrow C^\infty(U) \xrightarrow{\text{grad}} \text{Vect}(U) \xrightarrow{\text{curl}} \text{Vect}(U) \xrightarrow{\text{div}} C^\infty(U) \longrightarrow 0 \quad (3)$$

where we have now introduced the *divergence* of a vector field as a new ingredient.

Staring at the sequences (??) and (??), one sees that they have the fundamental property that the *image of each map lies in the kernel of the next*. This draws on the vector calculus identities

$$\text{grad}(\text{curl } \mathbf{V}) = 0, \quad \text{div}(\text{curl } \mathbf{V}) = 0.$$

The fancy way to say this is that they form a 'chain complex of vector spaces'.

Mathematicians have found a beautiful way to generalize the vector calculus notions of 'curl', 'gradient' and 'divergence' to all dimensions. It is called the *exterior derivative*. For an open set $U \subseteq \mathbb{R}^n$, we have the *De Rham complex*

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \longrightarrow 0.$$

The vector calculus identities (??) translate into the single equation

$$d^2 = 0.$$

In dimensions 2 and 3, this simply reduces to the complexes in (??) and (??) respectively.

We define the p th De Rham cohomology group of an open set $U \subset \mathbb{R}^n$ as the quotient vector space

$$H^p(U) := \frac{\text{Ker}(d: \Omega^p(U) \rightarrow \Omega^{p+1}(U))}{\text{Im}(d: \Omega^{p-1}(U) \rightarrow \Omega^p(U))}.$$

For instance, when $U \subset \mathbb{R}^2$, then this boils down to

$$H^1(U) = \frac{\{\text{vector fields } \mathbf{V} \text{ on } U \text{ satisfying } \text{curl } \mathbf{V} = 0\}}{\{\text{vector fields of the form } \text{grad } f\}}.$$

We can now summarize our counterexample from the previous section in a fancy way: it showed that $H^1(\mathbb{R}^2 \setminus \{(0,0)\}) \neq 0$.

4 Homotopy

We say that two continuous maps

$$f, g: X \rightarrow Y$$

between topological spaces are *homotopic*, and write $f \sim g$, if there exists a continuous map

$$H: X \times [0, 1] \rightarrow Y$$

with

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x)$$

for all $x \in X$. This is an equivalence relation on the space of continuous maps from X to Y (check this!).

We say that a topological space X is *contractible* if there exists a homotopy H from the constant map on X (which sends every point in X to some preassigned basepoint $*$ in X) to the identity map on X .

Theorem 1. *If the open set $U \subseteq \mathbb{R}^n$ is contractible, then $H^p(U) \cong \mathbb{R}$ if $p = 0$ and is zero otherwise.*

Proof. We will prove the theorem in case $p = 1$ and $U \subseteq \mathbb{R}^2$. Suppose that U is contractible.

We need to show that $H^1(U)$ is zero, in other words we need to show that every curl-free vector field \mathbf{V} on U is the gradient of a function f . Given $\mathbf{x} \in U$, we define $f(\mathbf{x})$ as the line integral

$$f(\mathbf{x}) := \int_{\gamma} \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r},$$

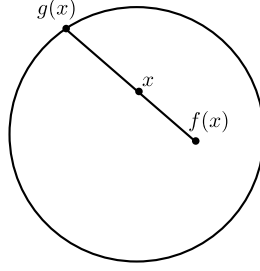


Figure 2: The retraction g of the disc onto the circle.

where γ is any path from some chosen basepoint $*$ to \mathbf{x} . The fact that U is contractible implies that such a path exists. This definition is independent of the path chosen, since if γ' is another path from $*$ to \mathbf{x} , then the path $\gamma'^{-1} \circ \gamma$ is the boundary of a closed region $D \subseteq U$ (again, because U is contractible) and hence, by Green's theorem,

$$\begin{aligned} f_{\gamma'}(\mathbf{x}) - f_{\gamma}(\mathbf{x}) &= \int_{\gamma'^{-1} \circ \gamma} \mathbf{V}(\mathbf{r}) \cdot d\mathbf{r} \\ &= \int_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dA \\ &= 0 \end{aligned}$$

since \mathbf{V} is curl-free.

□

5 Proof of Brouwer's theorem

We can now tie all our machinery together to give a proof of Brouwer's theorem, in the planar case.

Theorem 2. *Every continuous map $f: D^2 \rightarrow D^2$ has a fixed point.*

Proof. Suppose that f does not have a fixed point. (This is the nonconstructive step - we are going to argue by contradiction!). Then we can define a continuous map from the unit disc to the circle (see Figure 2),

$$\begin{aligned} g: D^2 &\rightarrow S^1 \\ x &\mapsto \text{intersection of line segment from } f(x) \text{ through } x \text{ with } S^1 \end{aligned}$$

which restricts to the identity on the unit circle,

$$g|_{S^1} = \text{id}.$$

(Note how we needed the absence of fixed points of f to make g well-defined.)

But such a map g (a continuous ‘retraction’ of the unit disc onto the unit circle) is impossible. For let $U = \mathbb{R}^2 \setminus \{(0,0)\}$, and consider

$$\begin{aligned} r : U &\rightarrow U \\ \mathbf{x} &\mapsto \frac{\mathbf{x}}{\|\mathbf{x}\|}. \end{aligned}$$

Then r is homotopic to the identity map on U . Hence U is contractible: first move a point $\mathbf{x} \in U$ along a radial line until it hits its representative $g(\mathbf{x})$ on the unit circle, and then contract the unit circle down to the point $*$:= $g(0) \in S^1$ by postcomposing with the homotopy

$$H(x, t) = g((1 - t)r(\mathbf{x})), \quad 0 \leq t \leq 1.$$

But the De Rham cohomology group $H^1(U)$ is not zero, and hence by Theorem ?? we know that U cannot be contractible. \square

The proof in the general case where $U \subseteq \mathbb{R}^n$ is identical. The only change is that one needs a separate argument that $H^1(U)$ is nonzero.

6 Wrap-up

I hope that this example has shown the philosophy behind De Rham cohomology: *try to measure the topological properties of a space by investigating the vector fields which can live on it*. One can build up a beautiful machinery in this regard. Two introductory texts are:

- Madsen and Tornehave, From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes, Cambridge University Press 1997.

This is the text I follow in my Honours course.

- Bott and Tu, Differential Forms in Algebraic Topology, Springer Graduate Texts in Mathematics (1982).

This text is a bit more advanced in some ways, and also includes more information on homotopy theory and spectral sequences.