Title

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Tuesday 13th October, 2020

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Recall that *good filtration* is a chain $\{0\} \subseteq V_1 \subseteq \cdots \subseteq V$ satisfying $V = \cup V_i$ and $V_i/V_{i-1} \cong H^0(\lambda_i)$ for λ_i some weight of V.

Lemma 1.1 ***(?).***
Let V be a G-module and $\lambda \in X(T)_+$ with $\hom_G(L(\lambda), V)$. If $\hom_G(L(\mu), V) = 0$ for any $\mu < \lambda$ and $\operatorname{Ext}^1_G(V(\mu), V) = 0$ for *all* $\mu \in X(T)_+$, then V contains a submodule isomorphic to $H^0(\lambda)$.

That is, we have a lift of the following form:

$$
\begin{tikzcd}
L(\lambda) \arrow[r, hook] \arrow[d, hook] & V \\
H^0(\lambda) \arrow[ur, dotted, hook, "\exists"] &
\end{tikzcd}
$$

Theorem 1.0.1 ***(Cohomological Condition for Good Filtrations).***
Let V be a G-module.

1. If V admits a good filtration, then the number of factors isomorphic to $H^0(\lambda)$, denoted $[V : H^0(\lambda)]$, is equal to $\dim \hom_G(V(\lambda), V)$.

   Analog of Jordan-Holder. Note that $H^0(\lambda)$ may not by irreducible, but changing the filtration can not change the number of composition factors.

2. Suppose $\hom_G(V(\lambda), V) < \infty$, then TFAE:

- V admits a good filtration.
- $\operatorname{Ext}^i_G(V(\lambda), V) = 0$ for all $\lambda \in X(T)_+$ and all $i > 0$.
- $\operatorname{Ext}^1_G(V(\lambda), V) = 0$ for all $\lambda \in X(T)_+$.

  Much like measuring projectivity: can check all exts, or just the first.

Proof (Part a).
Suppose V has a good filtration. Idea: induct on the filtration.
Suppose $V = H^0(\lambda_1)$, then

$$[V : H^0(\mu)] = \begin{cases} 0 & \mu \neq \lambda_1 \\ 1 & \mu = \lambda_1 \end{cases} = \dim \hom_G(V(\lambda_1), V),$$

since we know the dimensions of these hom spaces from a previous result.
Suppose now that we have

$$0 \to H^0(\mu_1) \to V H^0(\mu_2) \to 0.$$

Applying $F \coloneqq \hom_G(V(\lambda), \cdot)$, we find that Ext^1_G vanishes. So this leads a SES, and the dimensions are thus additive. The result follows since F is additive.

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Proof (Part b).
$1 \implies 2$: Use the fact that $\mathrm{Ext}^i_G(V(\lambda), H^0(\mu)) = 0$ for all $i > 0$ and all μ.
$2 \implies 3$: Clear!
$3 \implies 1$: Choose a total ordering of weights $\lambda_0, \lambda_1, \cdots \in X(T)$ such that if $\lambda_i < \lambda_j$ then $i < j$. Since $V \neq 0$, there exists a dominant weight $\lambda \in X(T)_+$ such that $\hom_G(V(\lambda), V) \neq 0$, so choose i minimally in this order to produce such a λ_i. Idea: use this to start a filtration. Then $\hom(L(\lambda_i), V) \neq 0$, and we have

$$V(\lambda_i) \twoheadrightarrow L(\lambda_i) \hookrightarrow V.$$

We know that

$$\begin{aligned} \hom_G(V(\mu), V) = 0 \quad & \forall \mu < \lambda_i \\ \hom_G(L(\mu), V) = 0 \quad & \forall \mu < \lambda_i \\ \mathrm{Ext}^1_G(L(\mu), V) = 0 \quad & \forall \mu \in X(T)_+ \text{ by assumption.} \end{aligned}$$

So the following map must be an injection, since there is no socle:

$$\begin{tikzcd} & L(\lambda_i) \arrow[r, hook] \arrow[d, hook] & V \\ 0 \arrow[r] & H^0(\lambda_i) \arrow[ur, hook] & \end{tikzcd}$$

Set $V_1 = H^0(\lambda_i)$, so $V_1 \subseteq V$. We then have a SES

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

Applying $\hom(V(\lambda), \cdot)$ we obtain

$$\to \mathrm{Ext}^1(\cancel{V(\lambda)}, V_1) \to \mathrm{Ext}^1_G(\cancel{V(\lambda)}, V) \to \underline{\mathrm{Ext}^1_G(V(\lambda), V/V_1)}$$

$$\to \mathrm{Ext}^2_G(\cancel{V(\lambda)}, V_1) \to \cdots$$

Figure 1: Cancellation in LES

Now iterate this process to obtain a chain $V_1 \subseteq V_2 \subseteq \cdots \subseteq V$, and set $V' := \cup_{i>0} V_i$. Then $\dim \hom_G(V(\lambda), V') = \dim \hom_G(V(\lambda), V)$ since $\dim \hom_G(V(\lambda), V) < \infty$. But then taking the SES

$$0 \to V' \to V \to V/V' \to 0$$

and applying $\mathrm{Hom}(V(\lambda), \cdot\,)$, we have $\mathrm{Hom}(V(\lambda), V/V') = 0$ and we get an isomorphism of homs. But then $\hom(V(\lambda), V/V') = 0$ for all $\lambda \in X(T)_+$, forcing $V/V' = 0$ and $V = V'$.

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Corollary 1.0.1 ***(?).***
Let $0 \to V_1 \to V \to V_2 \to 0$ be a SES of G-modules with $\dim \hom_G(V(\lambda), V_2) < \infty$ for all $\lambda \in X(T)_+$. If V_1, V have good filtrations, then V_2 also has a good filtration.

Note: this is likely difficult to prove without cohomology! But here we can apply the ext criterion.

Proof .
Let $\lambda \in X(T)_+$, then

$$\to \mathrm{Ext}^1_G(\cancel{V(\lambda)}, V_1) \to \mathrm{Ext}^1_G(\cancel{V(\lambda)}, V) \to \mathrm{Ext}^1_G(V(\lambda), V_2)$$

$$\to \mathrm{Ext}^2_G(\cancel{V(\lambda)}, V_1) \to \cdots$$

Figure 2: Image

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For $\lambda \in X(T)_+$, let $I(\lambda)$ be the injective hull of $L(\lambda)$, so we have

$$0 \to L(\lambda) \hookrightarrow I(\lambda).$$

Theorem 1.0.2 ***(?).***
Let $\lambda \in X(T)_+$ and $I(\lambda)$ be the injective hull of $L(\lambda)$.

a. $I(\lambda)$ has a good filtration.
b. The multiplicity $[I(\lambda) : H^0(\mu)]$ is equal to $[H^0(\mu) : L(\lambda)]$, the composition factor multiplicity.

Brauer-Humphreys Reciprocity. Same idea as in category $\mathcal{O}$: multiplicity of Vermas equals multiplicity of irreducibles.

Proof (of a).
How to check that it has a good filtration? The cohomological criterion! So consider $\operatorname{Ext}^1_G(V(\sigma), I(\lambda))$ for all $\sigma \in X(T)_+$. We want to show it's zero, but this follows because $I(\lambda)$ is injective.

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Proof (of b).
By the previous result, we have

$$\begin{aligned}[I(\lambda) : H^0(\mu)] &= \dim \hom_G(V(\mu), I(\lambda)) \\ &= [V(\mu) : L(\lambda)].\end{aligned}$$

Why does this second equality hold? The functor $\hom_G(\,\cdot\,, I(\lambda))$ is exact, and $\hom_G(L(\mu), I(\lambda)) = \delta_{\lambda,\mu}$. If $\lambda = \mu$ there's only one morphism, since $L(\lambda) \hookrightarrow I(\lambda)$ and $\operatorname{Soc}_G I(\lambda) = L(\lambda)$. This means that they have the same character, $\operatorname{char} H^0(\lambda) = \operatorname{char} V(\lambda)$, and this implies that they have the same composition factors.

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Theorem 1.0.3 ***(Cohomological Criterion for Weyl Filtrations).***
Let V be a G-module.

a. If V admits a Weyl filtration, then

$$[V : V(\lambda)] = \dim \hom_G(V, H^0(\lambda))$$

b. Suppose that $\dim \hom_G(V(\lambda), H^0(\lambda)) < \infty$ for all $\lambda \in X(T)_+$. Then TFAE

- V has a Weyl filtration.
- $\operatorname{Ext}^i_G(V, H^0(\lambda)) = 0$ for all $\lambda \in X(T)_+$ and $i > 0$.
- $\operatorname{Ext}^1_G(V, H^0(\lambda)) = 0$ for all $\lambda \in X(T)_+$.