Title

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Recall that a sheaf of rings on a topological space X is a ring $\mathcal{F}(U)$ for all open sets $U \subset X$ satisfying four properties:

1. The empty set is mapped to zeor.
2. The morphism $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity.
3. Given $W \subset V \subset U$ we have
4. Gluing: given sections $s_i \in \mathcal{F}(U_i)$ which agree on overlaps (restrict to the same function on $U_i \cap U_j$), there is a unique $s \in \mathcal{F}(\cup U_i)$.

Example 1.1.
If X is an affine variety with the zariski topology, $\mathcal{O}_X$ is a sheaf of regular functions, where we recall $\mathcal{O}_X(U)$ are the functions $\varphi : U \to k$ that are locally a fraction.

Recall that the *stalk* of a sheaf $\mathcal{F}$ at a point $p \in X$, is defined as

$$\mathcal{F}_p \coloneqq \left\{ (U, \varphi) \;\middle|\; p \in U \text{ open }, \varphi \in \mathcal{F}(U) \right\} / \sim .$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists a $p \in W \subset U \cap U'$ such φ, φ' restricted to W are equal.

Recall that a *local ring* is a ring with a unique maximal ideal $\mathfrak{m}$. Given a prime ideal $\mathfrak{p} \in R$, so $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$, the complement $R \setminus P$ is closed under multiplication. So we can localize to obtain $R_{\mathfrak{p}} = \left\{ a/s \;\middle|\; s \in R \setminus P, a \in R \right\} / \sim$ where $a'/s' \sim a/s$ iff there exists a $t \in R \setminus P$ such that $t(a's - as') = 0$.

⚠ **Warning:** Note that R_f is localizing at the powers of f, whereas $R_{\mathfrak{p}}$ is localizing at the *complement* of $\mathfrak{p}$.

Since maximal ideals are prime, we can localize any ring R at a maximal ideal $R_{\mathfrak{m}}$, and this will be a local ring. Why? The ideals in $R_{\mathfrak{m}}$ biject with ideals in R contained in $\mathfrak{m}$. Thus all ideals in $R_{\mathfrak{m}}$ are contained in the maximal ideal generated by $\mathfrak{m}$, i.e. $\mathfrak{m}R_{\mathfrak{m}}$.

Lemma 1.1 ***(?).***
Let X be an affine variety. The stalk of the sheaf of regular functions $\mathcal{O}_{X,p} \coloneqq (\mathcal{O}_X)_p$ is isomorphic to the localization $A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p \coloneqq I(\{p\})$.