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## 1 Friday January 10

Recall that  $\mathbb{C}$  is a field, where

$$z = x + iy \implies \bar{z} = x - iy$$

and if  $z \neq 0$  then

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

**Lemma 1.1** (Triangle Inequality).

$$|z + w| \leq |z| + |w|.$$

*Proof .*

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\bar{w}| - \Re z\bar{w}) \geq 0.$$

---

**Lemma 1.2** (Reverse Triangle Inequality).

$$||z| - |w|| \leq |z - w|.$$

*Proof .*

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

**Fact**  $(\mathbb{C}, |\cdot|)$  is a normed space.

**Definition 1.1** (Limits of Complex Sequences).

$$\lim z_n = z \iff |z_n - z| \longrightarrow 0 \in \mathbb{R}.$$

**Definition 1.2** (Complex Discs).

A *disc* is defined as  $D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$ , and a subset is open iff it contains a disc. By convention,  $D_r$  denotes a disc about  $z_0 = 0$ .

**Definition 1.3** (Convergence in).

$\sum_k z_k$  converges iff  $S_N := \sum_{|k| < N} z_k$  converges.

Note that  $z_n \longrightarrow z$  and  $z_n = x_n + iy_n$ , and

$$|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon \implies |x - x_n|, |y - y_n| < \varepsilon.$$

Since  $\mathbb{R}$  is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

Note: This is useful precisely when you don't know the limiting term.

Note that  $\sum_k z_k$  thus converges if  $\left| \sum_{k=m}^n z_k \right| < \varepsilon$  for  $m, n$  large enough, so sums converges iff they have small tails.

**Definition 1.4** (Absolute Convergence).

$S_N = \sum_k^N z_k$  converges absolutely iff  $\tilde{S} := \sum_k^N |z_k|$  converges.

---

Note that the partial sums  $\sum_{k=0}^N |z_k|$  are monotone, so  $\tilde{S}_N$  converges iff the partial sums are bounded above.

**Definition 1.5** (Power Series).

A sum of the form  $\sum_{k=0}^{\infty} a_k z_k$  is a *power series*.

*Examples:*

$$\sum x^k = \frac{1}{1-x}$$

$$\sum (-x^2)^k = \frac{1}{1+x^2}.$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at  $x = 1$  and the second as a pole at  $x = i$ .

## 2 Monday January 13th

Recall that  $\sum z_k$  converges iff  $s_n = \sum_{k=1}^n z_k$  converges.

**Lemma 2.1.**

Absolute convergence implies convergence.

The most interesting series:  $f(z) = \sum a_k z^k$ , i.e. power series.

**Lemma 2.2** (Divergence).

If  $\sum z_k$  converges, then  $\lim z_k = 0$ .

**Corollary 2.3.**

If  $\sum z_k$  converges,  $\{z_k\}$  is uniformly bounded by a constant  $C > 0$ , i.e.  $|z_k| < C$  for all  $k$ .

**Proposition:** If  $\sum a_k z_k$  converges at some point  $z_0$ , then it converges for all  $|z| < |z_0|$ .

Note that this inequality is necessarily strict. For example,  $\sum \frac{z^{n-1}}{n}$  converges at  $z = -1$  (alternating harmonic series) but not at  $z = 1$  (harmonic series).

*Proof .*

Suppose  $\sum a_k z_1^k$  converges. The terms are uniformly bounded, so  $|a_k z_1^k| \leq C$  for all  $k$ . Then we have

$$|a_k| \leq C/|z_1|^k$$

, so if  $|z| < |z_1|$  we have

$$|a_k z^k| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k.$$

So if  $|z| < |z_1|$ , the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting  $r = |z|/|z_1|$ , we have

$$\sum |a_k z^k| \leq \sum cr^k = \frac{c}{1-r},$$

and so we have absolute convergence. ■

**Exercise (future problem set)** Show that  $\sum \frac{1}{k} z^{k-1}$  converges for all  $|z| = 1$  except for  $z = 1$ . (Use summation by parts.)

**Definition 2.1** (Radius of Convergence).

The *radius of convergence* of a series is the real number  $R$  such that  $f(z) = \sum a_k z^k$  converges precisely for  $|z| < R$  and diverges for  $|z| > R$ .

We denote a disc of radius  $R$  centered at zero by  $D_R$ . If  $R = \infty$ , then  $f$  is said to be *entire*.

**Proposition 2.4.**

Suppose that  $\sum a_k z^k$  converges for all  $|z| < R$ . Then  $f(z) = \sum a_k z^k$  is continuous on  $D_R$ , i.e. using the sequential definition of continuity,  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$  for all  $z_0 \in D_R$ .

Recall that  $S_n(z) \rightarrow S(z)$  uniformly on  $\Omega$  iff  $\forall \varepsilon > 0$ , there exists a  $M \in \mathbb{N}$  such that

$$n > M \implies |S_n(z) - S(z)| < \varepsilon$$

for all  $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample:  $f_n(x) = x^n$  on  $[0, 1]$ ; then  $f_n \rightarrow \delta(1)$ . This uniformly converges on  $[0, 1 - \varepsilon]$  for any  $\varepsilon > 0$ .

**Exercise** Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

*Proof (of proposition).*

Write  $f(z) = \sum_{k=0}^N a_k z^k + \sum_{N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$ . Note that if  $|z| < R$ , then there exists a

$T$  such that  $|z| < T < R$  where  $f(z)$  converges uniformly on  $D_T$ .

Check!

We need to show that  $|R_N(z)|$  is uniformly small for  $|z| < s < T$ . Note that  $\sum a_k z^k$  converges on  $D_T$ , so we can find a  $C$  such that  $|a_k z^k| \leq C$  for all  $k$ . Then  $|a_k| \leq C/T^k$  for all  $k$ , and so

$$\begin{aligned} \left| \sum_{k=N+1}^{\infty} a_k z^k \right| &\leq \sum_{k=N+1}^{\infty} |a_k| |z|^k \\ &\leq \sum_{k=N+1}^{\infty} (C/T^k) s^k \\ &= C \sum_{k=N+1}^{\infty} |s/T|^k \\ &= C \frac{r^{N+1}}{1-r} = C \varepsilon_n \rightarrow 0, \end{aligned}$$

which follows because  $0 < r = s/T < 1$ .

So  $S_N(z) \rightarrow f(z)$  uniformly on  $|z| < s$  and  $S_N(z)$  are all continuous, so  $f(z)$  is continuous. ■

There are two ways to compute the radius of convergence:

- Root test:  $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$ .
- 
- Ratio test:  $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$ .

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

### 3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

#### 3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function  $f : \Omega \rightarrow \mathbb{C}$
2. Complex-differentiability: For  $\Omega \subset \mathbb{C}$  open and  $z_0 \in \Omega$ , there exists  $\varepsilon > 0$  such that  $D_\varepsilon = \{z \mid |z - z_0| < \varepsilon\} \subset \Omega$ , and  $f$  is **holomorphic** (complex-differentiable) at  $z_0$  iff

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(z_0 + h) - f(z_0))$$

exists; if so we denote it by  $f'(z_0)$ .

**Example 3.1.**

$f(z) = z$  is holomorphic, since  $f(z+h) - f(z) = z+h - z = h$ , so  $f'(z_0) = \frac{h}{h} = 1$  for all  $z_0$ .

**Example 3.2.**

Given  $f(z) = \bar{z}$ , we have  $f(z+h) - f(z) = \bar{h}$ , so the ratio is  $\frac{\bar{h}}{h}$  and the limit doesn't exist.

Note that if  $h \in \mathbb{R}$ , then  $\bar{h} = h$  and the ratio is identically 1, while if  $h$  is purely imaginary, then  $\bar{h} = -h$  and the limit is identically  $-1$ .

We say  $f$  is *holomorphic on an open set*  $\Omega$  iff it is holomorphic at every point, and is holomorphic on a closed set  $C$  iff there exists an open  $\Omega \supset C$  such that  $f$  is holomorphic on  $\Omega$ .

**Fact** If  $f$  is holomorphic, writing  $h = h_1 + ih_2$ , then the following two limits exist and are equal:

$$\begin{aligned} \lim_{h_1 \rightarrow 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\ \lim_{h_2 \rightarrow 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0) \\ &\implies \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}. \end{aligned}$$

So if we write  $f(z) = u(x, y) + iv(x, y)$ , we have

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \Big|_{(x_0, y_0)} = \frac{1}{i} \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \Big|_{(x_0, y_0)},$$

and equating real and imaginary parts yields **the Cauchy-Riemann equations**:

$$\begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ \iff \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{aligned}$$

The usual rules of derivatives apply:

$$1. (\sum f)' = \sum f'$$

*Proof.*  
Direct. ■

$$2. (\prod f)' = \text{product rule}$$



---

*Proof .*

Consider  $(f(z+h)g(z+h) - f(z)g(z))/h$  and use continuity of  $g$  at  $z$ . ■

### 3. Quotient rule

*Proof .*

Nice trick, write

$$q = \frac{f}{g}$$

so  $qg = f$ , then  $f' = q'g + qg'$  and  $q' = \frac{f'}{g} - \frac{fg'}{g^2}$ . ■

### 4. Chain rule

*Proof .*

Use the fact that if  $f'(g(z)) = a$ , then

$$f(z+h) - f(z) = ah + r(z, h), \quad |r(z, h)| = o(|h|) \rightarrow 0.$$

Write  $b = g'(z)$ , then

$$f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$$

by considering error terms, and so

$$\frac{1}{h}(f(g(z+h)) - f(g(z))) \rightarrow f'(g(z))g'(z)$$

.

■

## 4 Friday January 17th

### 4.1 Antiholomorphic Derivative

Reference: See Lang's Complex Analysis, there are plenty of solution manuals. Note: look for 13 statements equivalent to holomorphic: Springer GTM Lipman.

Let  $f; \Omega \rightarrow \mathbb{C}$  be a complex-valued function. Recall that  $f$  is *complex differentiable* iff the usual ratio/limit exists. Note that  $h = x + iy$  and  $h \rightarrow 0 \iff x, y \rightarrow 0$ .

We can write

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

This follows from Cauchy-Riemann since  $u_x = v_y$  and  $u_y = -v_x$ .

We want to define  $\partial, \bar{\partial}$  operators. We have the identities

$$x = \frac{z + \bar{z}}{2} \quad y = \frac{z - \bar{z}}{2i}.$$

We can then write

$$\begin{aligned} dz &= dx + i dy \\ d\bar{z} &= dx - i dy. \end{aligned}$$

We define the dual operators by  $\left\langle \frac{\partial}{\partial z}, dz \right\rangle = 1$  and similarly  $\left\langle \frac{\partial}{\partial \bar{z}}, d\bar{z} \right\rangle = 1$ .

By the chain rule, we can write

$$\begin{aligned} f_z &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial z} \\ &= \frac{1}{2} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{1}{2i} \\ &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f, \end{aligned}$$

and similarly

$$\begin{aligned} f_{\bar{z}} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}} \\ &= \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{1}{2i} \frac{\partial}{\partial y} \right) f. \end{aligned}$$

We thus find  $\partial_x = \partial_z + \partial_{\bar{z}}$  and  $\partial_y = i(\partial_z - \partial_{\bar{z}})$ , so define

$$\begin{aligned} \partial f &:= \frac{\partial f}{\partial z} dz \\ \bar{\partial} f &:= \frac{\partial f}{\partial \bar{z}} d\bar{z} \\ \implies df &= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \end{aligned}$$

**Definition 4.1** (Holomorphic and Antiholomorphic Derivatives).

$$\begin{aligned} \partial f &= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f \\ \bar{\partial} f &= \left( \frac{\partial}{\partial x} - \frac{1}{2i} \frac{\partial}{\partial y} \right) f. \end{aligned}$$

**Proposition 4.1** (*Holomorphic Functions have vanishing antiholomorphic derivatives*).  
 $f$  is holomorphic iff  $\bar{\partial}f = 0$ .

This means that  $f$  depends on  $z$  alone and not  $\bar{z}$ .

*Proof.*

$$\bar{\partial}f = 0 \text{ iff } \frac{1}{2}(f_x + if_y) = 0, \text{ so } (u_x - v_y) + i(v_x + u_y) = 0.$$

■

Application to PDEs: we can write

$$u_{xx} = v_{xy} \quad u_{yy} = v_{yx}$$

and so

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}.$$

Thus  $\Delta f = 0$ , so  $f$  satisfies Laplace's equation and is said to be *harmonic*.

**Corollary 4.2** (Holomorphic Functions Have Harmonic Components).

If  $f$  is analytic, then  $u, v$  are both harmonic functions.

**Theorem 4.3** (*Chain Rule*).

Let  $w = f(z)$  and  $g(w) = g(f(z))$ . Then

$$h_z = g_w f_z + g_{\bar{w}} \bar{f}_z$$

$$h_{\bar{z}} = g_w \bar{f}_z + g_{\bar{w}} \bar{f}_{\bar{z}}.$$

If  $f, g$  are holomorphic,  $\bar{f}_z = g_{\bar{w}} = 0$ , so  $h_{\bar{z}} = 0$  and  $h$  is holomorphic and

$$h_z = g_w f_z.$$

**Example 4.1.**

Given a power series  $f = \sum a_n(z - z_0)^n$ . Then

1. There exists a radius of convergence  $R$  such that  $f$  converges precisely on  $D_R(z_0)$ .
2.  $f$  is continuous on  $D_R(z_0)^\circ$ .
3. By the root test,  $R = (\limsup |a_n|^{1/n})^{-1} = \liminf |a_n/a_{n+1}| = (\limsup |a_{k+1}/a_k|)^{-1}$ .

Recall the **ratio test**:

$$\sum |a_k| < \infty \iff \limsup |a_{k+1}/a_k| < 1$$

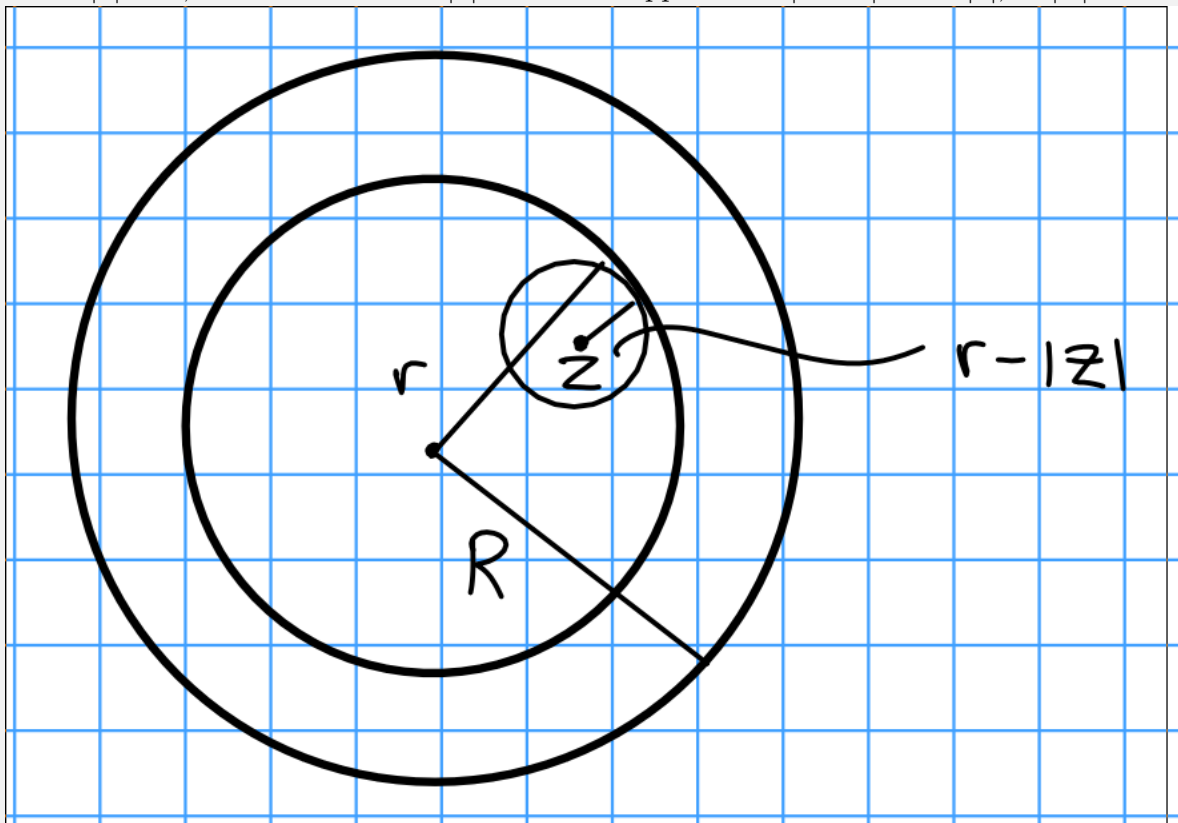
**Theorem 4.4** (*Holomorphic series can be differentiated term-by-term*).

If  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is holomorphic on  $|z| < R$  for  $R > 0$  then

$$f'(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}.$$

*Proof.*

Given  $|z| < R$ , fix  $r > 0$  such that  $|z| < r < R$ . Suppose that  $|w - z| < r - |z|$ , so  $|w| < r$ .



We want to show

$$|S| = \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1}^{\infty} a_n n z^{n-1} \right| \rightarrow 0 \quad \text{as } w \rightarrow z.$$

Idea: write everything in terms of power series. Use the fact that  $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots)$ , and so  $|(w^k - z^k)/(w - z)| \leq kr^{k-1}$ .

$$\begin{aligned}
S &= \sum_{n=1} a_n \left( \frac{w^n - z^n}{w - z} - nz^{n-1} \right) \\
&= \sum_{n=1} a_n (w^{n-1} + w^{n-2}z + \cdots + z^{n-1} + nz^{n-1}) \\
&= \sum_{n=1} a_n ((w^{n-1} - z^{n-1}) + (w^{n-2} - z^{n-2})z + \cdots + (w - z)z^{n-2}) = \sum_{n=1} a_n (w - z) (\cdots + z^{n-2}) \\
&\leq \sum_{n=2} |a_n| \frac{1}{2} n(n-1) r^{n-2} |z - w|.
\end{aligned}$$

■

**Exercise** Show  $\lim_n n^{\frac{1}{n}} = 1$ .

Also tricky: show  $\lim \sin(n)$  doesn't exist, and  $\sin(n)$  is dense in  $[-1, 1]$ .

*Proof.*

Consider  $\limsup |a_n n|^{\frac{1}{n}}$ .

■

Note that an analytic function is holomorphic in its domain of convergence, so analytic implies holomorphic. The converse requires Cauchy's integral formula.

Next time: trying to prove holomorphic functions are analytic.

## 5 Wednesday January 22nd

### 5.1 Parameterized Curves

Note: multiple complex variables, see Hormander or Steven Krantz

Recall from last time that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with  $z_0 \neq 0$  has radius of convergence

$$R = (\limsup |a_n|^{1/n})^{-1} > 0$$

then  $f'$  exists and is obtained by differentiating term-by-term.

We know that  $f$  analytic  $\implies f$  holomorphic (and smooth), and we want to show the converse. For this, we need integration.

**Definition 5.1** (Parameterized Curves).

A *parameterized curve* is a function  $z(t)$  which maps a closed interval  $[a, b] \subset \mathbb{R}$  to  $\mathbb{C}$ .

**Definition 5.2** (Smooth Curves).

The curve is said to be *smooth* iff  $z'$  exists and is continuous on  $[a, b]$ , and  $z'(t) \neq 0$  for any  $t$ . At the boundary  $\{a, b\}$ , we define the derivative by taking one-sided limits.

**Definition 5.3** (Piecewise Smooth Curves).

A curve is said to be *piecewise smooth* iff  $z(t)$  is continuous on  $[a, b]$  and there are  $a < a_1 < \dots < a_n = b$  with  $z$  smooth on each  $[a_k, a_{k+1}]$ .

Note that such a curve may fail to have tangent lines at  $a_i$ .

**Definition 5.4** (Equivalent Parameterizations).

Two parameterizations  $z : [a, b] \rightarrow \mathbb{C}$ ,  $\tilde{z} : [c, d] \rightarrow \mathbb{C}$  are *equivalent* iff there exists a  $C^1$  bijection  $s : [c, d] \rightarrow [a, b]$  where  $s \mapsto t(s)$  such that  $s' > 0$  and  $\tilde{z}(s) = z(s(t))$ .

Note that  $s' > 0$  preserves orientation and  $s' < 0$  reverses orientation.

**Definition 5.5** (Orientations of Curves).

A curve in *reverse orientation* is defined by

$$\gamma : [a, b] \rightarrow \mathbb{C} \implies \gamma^- : [a, b] \rightarrow \mathbb{C} \\ t \mapsto \gamma(a + b - t).$$

**Definition 5.6** (Closed Curves).

A curve is *closed* iff  $z(a) = z(b)$ , and is *simple* iff  $z(t) \neq z(t_1)$  for  $t \neq t_1$ .

**Definition 5.7** (Positively Oriented Curves).

For  $C_r(z_0) := \{z \mid |z - z_0| = r\}$ , the *positive orientation* is given by  $z(t) = z_0 + re^{2\pi it}$  for  $t \in [0, 1]$ .

## 5.2 Definition of the Integral

**Definition 5.8.**

The *integral* of  $f$  over  $\gamma$  is defined as

$$\int_{\gamma} f \, dz = \int_a^b f(z(t)) z'(t) \, dt.$$

Note: this doesn't depend on parameterization, since if  $t = t(s)$ , then a change of variables yields

$$\int_{\gamma} f \, dz = \int_c^d f(z(t(s))) z'(t(s)) t'(s) \, ds = \int_c^d f(\tilde{z}(s)) \tilde{z}'(s) \, ds.$$

**Definition 5.9.**

The *length* of  $\gamma$  is defined as  $|\gamma| = \int |z'(t)| \, dt$ .

**Proposition 5.1.** 1. We can extend this definition to piecewise smooth curves by

$$\int_{\gamma} f \, dz = \sum \int_{a_k}^{a_{k+1}} f \, dz$$

2. This integral is linear and  $\int_{\gamma} f = -\int_{\gamma^-} f$ .
3. We have an inequality

$$\left| \int_{\gamma} f \right| \leq \max_{a \leq t \leq b} |f(z(t))| |\gamma|.$$

**Definition 5.10.**

A function  $F$  is a *primitive* for  $f$  on  $\Omega$  iff  $F$  is holomorphic on  $\Omega$  and  $F'(z) = f(z)$  on  $\Omega$ .

Recall that in  $\mathbb{R}$ , we have

$$F(x) = \int_a^x f(t) dt$$

as an antiderivative with  $F'(x) = f(x)$ , and  $\int_a^b f = F(b) - F(a)$ .

**Theorem 5.2 (Evaluating Integrals with Primitives).**

If  $f$  is continuous, has a primitive  $F$  in  $\Omega$ , and  $\gamma$  is a curve beginning at  $w_0$  and ending at  $w_1$ , then  $\int_{\gamma} f = F(w_1) - F(w_0)$ .

*Proof.*

Use definitions, write  $z(t)$  where  $z(a) = w_0, z(b) = w_1$ . Then

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(z(t)) z'(t) dt \\ &= \int_a^b F'(z(t)) z'(t) dt \\ &= \int_a^b F_t dt \\ &= F(z(b)) - F(z(a)) \quad \text{by FTC} \\ &= F(w_1) - F(w_0). \end{aligned}$$

Note that if  $\gamma$  is piecewise smooth, the sum of the integrals telescopes to yield the same conclusion. ■

**Corollary 5.3** (Functions with Primitives Integrate to Zero Along Loops).

If  $f$  is continuous and  $\gamma$  is a closed curve in  $\Omega$ , and  $f$  has a primitive in  $\Omega$ , then

$$\oint f = 0.$$

---

## 6 Friday January 24th

### Corollary 6.1.

If  $\gamma$  is a closed curve on  $\Omega$  an open set and  $f$  is continuous with a primitive in  $\Omega$  (i.e. an  $F$  holomorphic in  $\Omega$  with  $F' = f$ ) then  $\int_{\gamma} f dz = 0$ .

*Proof (easy).*

$$\int_{\gamma} f dz = \int_{\gamma} F' = F'(z)z(t) dt = F(z(b)) - F(z(a)) = 0.$$

■

### Corollary 6.2.

If  $f$  is holomorphic with  $f' = 0$  on  $\Omega$ , then  $f$  is constant.

*Proof (easy).*

Pick  $w_0 \in \Omega$ ; we want to fix  $w_0 \in \Omega$  and show  $f(w) = f(w_0)$  for all  $w \in \Omega$ .

Take any path  $\gamma : w_0 \rightarrow w$ , then

$$0 = \int_{\gamma} f' = f(w) - f(w_0).$$

■

## 6.1 Integral and Fourier Transform of $e^{-x^2}$

### Example 6.1.

Let  $f(z) = e^{-z^2}$ , this is holomorphic. Write

$$f(z) = \sum \frac{(-1)^n z^{2n}}{n!},$$

so

$$\int f = \sum \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

Since  $f$  is entire,  $\int f$  is entire, and  $(\int f)' = f$  so this function has a primitive. Thus  $\int_{\gamma} f(z) = 0$  for any closed curve. So take  $\gamma$  a rectangle with vertices  $\pm a, \pm a + ib$ .





So

$$\int_{\gamma} f = \int_{-a}^a e^{-x^2} dx + \int e^{-(a+iy)^2} i dy - \int_{-a}^a e^{-(x+ib)^2} dx - \int_0^b e^{-(a+iy)^2} i dy = 0.$$

We can do some estimates,

$$\begin{aligned} e^{-(a+iy)^2} &= e^{-(a^2+2iaiy-y^2)} \\ &= e^{-a^2+y^2} e^{2iaiy} \\ &\leq e^{-a^2+y^2} \\ &\leq e^{-a^2+b^2}, \end{aligned}$$

$$\left| \int_0^b e^{-(a+iy)^2} i dy \right| \leq e^{-a^2+b^2} \cdot b$$

$$\int_{-a}^a e^{-(x^2+2ibx)-b^2} = e^{b^2} \int_{-a}^a e^{-x^2} (\cos(2bx) - i \sin(2bx))$$

$$\stackrel{\text{odd fn}}{=} e^{b^2} \int_{-a}^a e^{-x^2} \cos(2bx) dx.$$

Now take  $a \rightarrow \infty$  to obtain

$$\int_{\mathbb{R}} e^{-x^2} dx = e^{b^2} \int_{\mathbb{R}} e^{-x^2} \cos(2bx) dx.$$

We can compute

$$\int_{\mathbb{R}} e^{-x^2} = \left[ \left( \int_{\mathbb{R}} e^{-x^2} \right)^2 \right]^{1/2} = \left( \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \right) = \sqrt{\pi}.$$

and then conclude

$$\int_{\mathbb{R}} e^{-x^2} \cos(2bx) = \sqrt{\pi} e^{-b^2}.$$

Make a change of variables  $2b = 2\pi\xi$ , so  $b = \pi\xi$ , then

$$\int_{\mathbb{R}} e^{-x^2} \cos(2\pi\xi x) dx = \sqrt{\pi} e^{-\pi^2\xi^2}.$$

Thus  $\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\pi^2\xi^2}$ , allowing computation of the Fourier transform. Note that this can be used to prove the Fourier inversion formula.

**Exercise** Show that this is an approximate identity and prove the Fourier inversion formula.

**Exercise** Show  $\mathcal{F}(e^{-ax^2}) = \sqrt{\pi/a} e^{-\pi^2/a \cdot \xi^2}$ , and thus taking  $a = \pi$  makes  $e^{\pi x^2}$  is an eigenfunction of  $\mathcal{F}$  with eigenvalue 1.

**Theorem 6.3 (Holomorphic Integrals Vanish).**

If  $f$  has a primitive on  $\Omega$  then  $F(z)$  is holomorphic and  $\int_{\gamma} f = 0$ . If  $f$  is holomorphic, then

$$\int_{\gamma} f = 0.$$

**Theorem 6.4 (Green's).**

Take  $\Omega \in \mathbb{R}^2$  bounded with  $\partial\Omega$  piecewise smooth. If  $f, g \in C^1\overline{\Omega}$ , then

$$\int_{\partial\Omega} f dx + g dy = \iint_{\Omega} (g_x - f_y) dA.$$

*Proof .*  
Omitted. ■

*Proof (that holomorphic integrals vanish).*

Write  $\gamma = \partial\Gamma$ , and noting that  $f_z = f_x = \frac{1}{i}f_y$  implies that  $\frac{\partial f}{\partial \bar{z}}$ , so

$$\begin{aligned}\int_{\gamma} f \, dz &= \int_{\gamma} f(z) (dx + i dy) \\ &= \int f(z) \, dx + i \int f(z) \, dy \\ &= \iint_{\Gamma} (i f_x - f_y) \, dA \\ &= i \iint_{\Gamma} \left( f_x - \frac{1}{i} f_y \right) \, dA \\ &= i \iint_{\Gamma} 0 \, dA \\ &= 0.\end{aligned}$$
 ■

Next up, we'll prove that this integral over any triangle is zero by a limiting process.

## 7 Monday January 27th

Open question: does a PDE involving analytic functions always have solutions? Or does this hold with analytic replaced by smooth?

### 7.1 Green's Theorem

Fix a connected domain  $\Omega$  which is bounded with a piecewise  $C^1$  boundary.

**Theorem 7.1 (Green's).**

Given  $f, g \in C^1\bar{\Omega}$ , we can take a vector field  $F = \langle f, g \rangle$  and have

$$\begin{aligned}\int_{\partial\Omega} f \, dx + g \, dy &= \iint_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dA \\ \int_{\partial\Omega} -f \, dx + g \, dy &= \iint_{\Omega} \left( \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \right) \, dA \\ \int_{\partial\Omega} f \, dy - g \, dx &= \iint_{\Omega} \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) \, dA \\ \int_{\partial\Omega} F \cdot \mathbf{n} \, ds &= \iint_{\Omega} \nabla \cdot F \, dA \\ \int_{\partial\Omega} \text{curl}(F) \, ds &= \iint_{\Omega} \text{div}(F) \, dA,\end{aligned}$$

where we take  $\mathbf{n}$  to be orthogonal to  $\partial\Omega$ . The quantities appearing on the RHS are referred to as the flux.

For  $f(z) \in C^1(\Omega)$  holomorphic, we can then write

$$\begin{aligned} \int_{\partial\Omega} f \, dz &= \int_{\partial\Omega} f (dx + i dy) \\ &= \int_{\partial\Omega} f \, dx + i f \, dy \\ &= \iint_{\Omega} (i f_x - f_y) \, dA \\ &= 0, \end{aligned}$$

which follows since  $f$  holomorphic, we can write

$$f'(z) = f_x = \frac{1}{i} f_y,$$

so  $i f_x = f_y$  and thus  $\frac{\partial f}{\partial \bar{z}} = 0$ .

See Taylor's Introduction to Complex Analysis

**Theorem 7.2 (Cauchy's Integral Formula):**

If  $f \in C^1(\bar{\Omega})$  and  $f$  is holomorphic, then for any  $z \in \Omega$

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{d(\xi)}{\xi - z} \, d\xi.$$

*Proof.*

Since  $z \in \Omega$  an open set, we can find some  $r > 0$  such that  $D_r(z) \subset \Omega$ . Then  $\frac{f(\xi)}{\xi - z}$  is holomorphic on  $\Omega \setminus D_r(z)$ . Let  $C_r = \partial D_r(z)$ .

**Claim:**

$$\int_{\partial\Omega} \frac{f(\xi)}{\xi - z} \, d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z} \, d\xi.$$

If we can differentiate through the integral, we can obtain

$$\frac{\partial}{\partial z} f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^2} \, d\xi.$$

and thus inductively

$$(D_z)^n f(z) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(\xi) \, d\xi}{(\xi - z)^{n+1}}.$$

To prove rigorously, need to write

$$\begin{aligned} \Delta_h f(z) &= \frac{1}{h}(f(z+h) - f(z)) \\ &= \frac{1}{2\pi i h} \int_{\partial\Omega} f(\xi) \left( \frac{1}{\xi - (z+h)} - \frac{1}{\xi - z} \right) d\xi = \frac{1}{2\pi i h} \int_{\partial\Omega} f(\xi) \left( \frac{1}{(\xi - z - h)(\xi - z)} \right) d\xi, \end{aligned}$$

and show the integrand converges uniformly, where

$$\frac{1}{(\xi - z - h)(\xi - z)} \xrightarrow{u} \frac{1}{(\xi - z)^2}.$$

Continuing inductively yields the integral formula. ■

*Proof (of claim used in main proof).*

Use the parameterization of  $C_r$  given by  $\xi = z + re^{i\theta}$ . Then

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi)}{\xi - z} d\xi &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}} i r d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta \\ &\xrightarrow{r \rightarrow 0} \frac{1}{2\pi} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z}. \end{aligned}$$

where we use the fact that

$$f(z + re^{i\theta}) = f(z) + f'(z)re^{i\theta} + o(r) \xrightarrow{r \rightarrow 0} f(z)$$

Letting

$$F(\xi) = \frac{f(\xi)}{\xi - z},$$

this is holomorphic on  $\Omega \setminus D_r(z)$ . Let  $\Omega_r = \partial\Omega \cup (-C_r)$ . Take the following path integral:



Then

$$0 = \int_{\partial\Omega_r} F(\xi) d\xi = \int_{\partial\Omega} F(\xi) d\xi - \int_{C_r} F(\xi) d\xi,$$

which forces these integrals to be equal. ■

**Corollary 7.3** (implies smooth).

If  $f$  is holomorphic, then  $f \in C^1(\Omega)$  implies that  $f \in C^\infty(\Omega)$ .

**Theorem 7.4 (Holomorphic implies analytic).**

If  $f$  is holomorphic in  $\Omega$ , then  $f$  is equal to its Taylor series (i.e.  $f(z_0)$  is analytic.)

*Proof.*

Fix  $z_0 \in \Omega$  and let  $r = |z - z_0|$ .

$$\begin{aligned}
\frac{1}{\xi - z} &= \frac{1}{\xi - z_0 - (z - z_0)} \\
&= \frac{1}{\xi - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\xi - z_0}\right)} \\
&= \frac{1}{\xi - z_0} \sum_n \left(\frac{z - z_0}{\xi - z_0}\right)^n \quad \text{for } |z - z_0| < |\xi - z_0|.
\end{aligned}$$

Note that  $\sum z^n$  converges uniformly for any  $|z| < \delta < 1$ .  
Thus

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\xi \in \partial\Omega} f(\xi) \sum \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} d\xi \\
&= \sum \left( \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right) (z - z_0)^n \\
&= \sum \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.
\end{aligned}$$

■

### Corollary 7.5.

$f$  is holomorphic iff  $f$  is analytic.

Counterexample to keep in mind:

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

In the case of  $\mathbb{R}$ , smooth and analytic are very different categories of functions.

## 8 Wednesday January 29th

### 8.1 Cauchy's Integral Formula

#### **Theorem 8.1 (Cauchy's Integral Formula).**

Let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic, so  $f \in C^1(\overline{\Omega})$ . Then for any  $z \in \Omega$ ,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi.$$

In general,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

This implies that  $f$  is analytic, i.e.

$$f(z) = \sum a_n(z - z_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Thus  $f$  is holomorphic iff  $f$  is analytic,

and

$$\int_{\partial\Omega} f = 0 \implies \int_{\partial\Omega_\gamma} \frac{f(\xi)}{\xi - z} d\xi = 0.$$

where  $\Omega_r = \Omega \setminus D_r(z)$ , and  $\partial\Omega_r = \partial\Omega \cup (-\partial D_r)$ .

We can thus shrink integrals:

$$\int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z} d\xi.$$

**Proposition 8.2 (Homotopy Invariance).**

Let  $f \in C^1(\Omega)$  be holomorphic on  $\Omega$ . Let  $\gamma_s(t)$  be a family of smooth curves in  $\Omega$ ; then  $\int_{\gamma_s} f$  is independent of  $s$ .

*Proof .*

Write

$$\gamma_s(t) = \gamma(s, t) : [a, b] \times [0, 1] \longrightarrow \Omega.$$

We have  $\gamma_s(0) = \gamma_s(1)$  so  $\frac{\partial\gamma}{\partial s}(s, 0) = \frac{\partial\gamma}{\partial s}(s, 1)$ . Then

$$\begin{aligned} \frac{\partial\gamma}{\partial s} &= \int_0^1 \left( f'(\gamma(s, t)) \frac{\partial r}{\partial s} \frac{\partial r}{\partial t} + f(\gamma(s, t)) \frac{\partial^2 \gamma}{\partial s \partial t} \right) dt \\ &= \int_0^1 \left( f'(\gamma(s, t)) \frac{\partial r}{\partial s} \frac{\partial r}{\partial t} + f(\gamma(s, t)) \frac{\partial^2 \gamma}{\partial \mathbf{t} \partial \mathbf{s}} \right) dt \\ &= \int_0^1 \frac{\partial}{\partial t} (f(\gamma(s, t)) \gamma_s) \\ &= f(\gamma(s, 1)) \gamma_s(s, 1) - f(\gamma(s, 0)) \gamma_s(s, 0) \\ &= 0. \end{aligned}$$



where we can just take the paths  $\gamma(s, t) = z_0 \in \Omega$  for all  $s, t$ . ■

**Proposition 8.3** (*Pointwise Limit of Locally Uniform is Locally Uniform*).

Let  $\Omega \subset \mathbb{C}$  be open and  $f_v : \Omega \rightarrow \mathbb{C}$ . Suppose that each  $f_v$  is holomorphic,  $f_v \rightarrow f$  pointwise, and *locally uniform*, i.e.  $f_v \rightarrow f$  uniformly on every compact  $K \subset \Omega$ . Then  $f$  is holomorphic in  $\Omega$  and  $f$  is locally uniform.

*Proof.*

Given a compact set  $K \subset \Omega$ , pick an  $O$  with smooth boundary such that  $K \subset O \subset \bar{O} \subset \Omega$ . We have

$$\begin{aligned} f_v(z) &= \frac{1}{2\pi i} \int_{\partial O} \frac{f_v(\xi)}{\xi - z} d\xi \\ f_v^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\partial O} \frac{f_v(\xi)}{(\xi - z)^{n+1}} d\xi \end{aligned}$$

Then on  $\partial O$ , we have uniform convergence

$$\frac{f_v(\xi)}{(\xi - z)^{n+1}} \xrightarrow{u} \frac{f(\xi)}{(\xi - z)^{n+1}}.$$

By moving the limits inside, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial O} \frac{f(\xi)}{\xi - z} d\xi \\ f^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\partial O} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \end{aligned}$$

■

**Theorem 8.4** (*Cauchy's Inequality*).

Given  $z_0 \in \Omega$ , pick the largest disc  $D_R(z_0) \subset \Omega$  and let  $C_R = \partial D_R$ . Using the integral formula, defining  $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

**Corollary 8.5** (Liouville's Theorem).

---

If  $f$  is entire and bounded, then  $f$  is constant.

*Proof .*

For all  $z_0 \in \mathbb{C}$ , there exists an  $M$  such that  $|f(z)| \leq M$ . Then  $|f'(z_0)| \leq \frac{M}{R}$  for any  $R > 0$ . Taking  $R \rightarrow \infty$  yields  $f'(z_0) = 0$ , so  $f$  is constant. ■

**Corollary 8.6** (Weak Fundamental Theorem of Algebra).

Every non-constant polynomial  $p(z) = a_0 + a_1z + \cdots + a_nz^n$  has a root in  $\mathbb{C}$ .

Remark: A general proof technique is when proving something for  $f(z)$ , consider  $\frac{1}{f(z)}$  and  $f(\frac{1}{z})$ .

*Proof .*

Suppose  $p$  is nonconstant and does not have a root,  $\frac{1}{p}$  is entire. Assume that  $a_n \neq 0$ , then

$$\frac{p(z)}{z^n} = a_n \left( \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) := a_n + y$$

We can note that  $\lim_{z \rightarrow \infty} \frac{a_{n-k}}{z^k} \rightarrow 0$ , so there exists an  $R > 0$  such that

$$\begin{aligned} \left| \frac{p(z)}{z^n} \right| &\geq \frac{1}{2} |a_n| \quad \text{for } |z| > R \\ \implies |p(z)| &\geq \frac{1}{2} |a_n| |z|^n \geq \frac{1}{2} |a_n| R^n. \end{aligned}$$

Since  $p(z)$  is continuous and has no root in the disc  $|z| \leq R$ ,  $|p(z)|$  is bounded from below in this disc. Since  $p(z)$  is continuous on a compact set, it attains a minimum, and so  $|p(z)| \geq \min_{|z| \leq R} |p(z)| = c_2 \neq 0$ . Then  $|p(z)| \geq A = \min(c_2, \frac{1}{2} |a_n| R^n)$ , so  $\frac{1}{p}$  is bounded. Then  $f$  is constant, a contradiction. ■

## 9 Friday January 31st

### 9.1 Fundamental Theorem of Algebra

Recall that if  $f$  is holomorphic, we have Cauchy's integral formula.

**Corollary 9.1** (Weak Fundamental Theorem of Algebra).

If  $P(z)$  is a polynomial in  $\mathbb{C}$  then  $P$  has a root in  $\mathbb{C}$ .

*Proof .*

See previous notes. ■

**Corollary 9.2** (Fundamental Theorem of Algebra).

Every polynomial of degree  $n$  has precisely  $n$  roots in  $\mathbb{C}$ .

*Proof .*

By induction on the degree of  $P$ . From the first corollary,  $P$  has a root  $w_1$ , so write  $z = z - w_1 + w_1$ . Then

$$\begin{aligned}
 p(z) &= p(z - w_1 + w_1) \\
 &= \sum_k^n a_k (z - w_1 + w_1)^k \\
 &= \sum_k^n a_k \sum_j^k \binom{k}{j} w_1^{k-j} (z - w_1)^j \\
 &= \sum_k^n \sum_j^k a_k \binom{k}{j} w_1^{k-j} (z - w_1)^j \\
 &= \sum_j^n \left( \sum_{k \geq j} a_k \binom{k}{j} \right) (z - w_1)^j \\
 &= b_0 + b_1(z - w_1) + \cdots + b_n(z - w_1)^n.
 \end{aligned}$$

Since  $P(w_1) = 0$ , we must have  $b_0 = 0$ , and thus this equals

$$\begin{aligned}
 b_1(z - w_1) + \cdots + b_n(z - w_1)^n &= (z - w_1) \left( b_1 + \cdots + b_n(z - w_1)^{n-1} \right) \\
 &:= (z - w_1) \phi(z),
 \end{aligned}$$

where  $\phi(z)$  is degree  $n - 1$ , which has  $n - 1$  roots by induction. ■

**Definition 9.1.**

For a sequence  $\{z_n\}$ , TFAE

1.  $z$  is a limit point.
2. There exists a subsequence  $\{z_{n_k}\}$  converging to  $z$ .
3. For every  $\varepsilon > 0$ , there are infinitely many  $z_i$  in  $D_\varepsilon(z)$ .

**Theorem 9.3.**

Suppose  $f$  is holomorphic on a bounded connected region  $\Omega$  and  $f$  vanishes on a sequence of

distinct points with a limit point in  $\Omega$ .

*Proof.*

WLOG by restricting to a subsequence, suppose that  $\{w_k\} \in \Omega$  with  $f(w_k) = 0$  for all  $k$  and  $z_0$  is a limit point of  $\{w_k\}$ . Let  $U = \{z \in \Omega \mid f(z) = 0\}$ . Then

1.  $U$  is nonempty since  $f(w_k) = f(z_0) = 0$ .
2. Since holomorphic functions are continuous, if  $w_k \rightarrow z$  then  $z \in U$ , so  $U$  is closed.
3. (To prove)  $U$  is open.

Since  $U$  is closed and open,  $U = \Omega$ .

We will first show that  $f(z) \equiv 0$  in a disk containing  $z_0$ . Choose a disc  $D$  containing  $z_0$  and contained in  $\Omega$ . Since  $f$  is holomorphic on  $D$ , we can write

$$f(z) = \sum a_n (z - z_0)^n.$$

Since  $f(z_0) = 0$ , we have  $a_0 = 0$ .

Suppose  $f \not\equiv 0$ . Then there exists a smallest  $n \in \mathbb{Z}^+$  such that  $a_n \neq 0$ , so  $f(z) = a_n (z - z_0)^n + \dots$ . Since  $a_n \neq 0$ , we can factor this as  $a_n (z - z_0)^n (1 + g(z - z_0))$  where

$$g(z - z_0) = \sum_{k=n+1}^{\infty} \frac{a_k}{a_n} (z - z_0)^{k-n}.$$

Note that  $g$  is holomorphic, and  $g(z_0 - z_0) = 0$ .

Choose some  $w_k$  such that  $f(w_k) = 0$  and  $|g(w_k - z_0)| \leq \frac{1}{2}$  by continuity of  $g$ . Then

$$|1 + g(w_k - z_0)| > 1 - \frac{1}{2} = \frac{1}{2}.$$

So

$$|f(w_k)| = |a_n (w_k - z_0)^n (1 + g(w_k - z_0))| > |a_n| |w_k - z_0|^n \frac{1}{2} > 0,$$

a contradiction. So  $U$  is open, closed, and nonempty, so  $U = \Omega$ . ■

#### Corollary 9.4.

Suppose  $f, g$  are holomorphic in a region  $\Omega$  with  $f(z_k) = g(z_k)$  where  $\{z_k\}$  has a limit point. Then  $f(z) \equiv g(z)$ .

#### Theorem 9.5 (Mean Value).

Let  $z_0$  be a point in  $\Omega$  and  $C_\gamma$  the boundary of  $D_r(z_0)$ . Then

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \int_{C_\gamma} f(z)/(z - z_0) dz \\
&= \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta})/re^{i\theta} rie^{i\theta} d\theta \quad \text{by } z = z_0 + re^{i\theta} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{i\theta}) r d\theta \\
&= \frac{1}{|C_\gamma|} \int_0^{2\pi} f(z) ds,
\end{aligned}$$

which is the average value of  $f$  on the circle.

Note that there is another formula that averages over the disc (see book for derivation?)

$$f(z_0) = \frac{1}{D_s(z_0)} \int_{P_s} \int_{D_s} f(z) dA.$$

These imply the maximum modulus principle, since the average can not be the max or min unless  $f$  is constant. Note that  $|f(z)|$  is continuous!

Next time: maximum modulus principle.

## 10 Monday February 3rd

### 10.1 Mean Value Theorem

**Theorem 10.1 (Mean Value for Holomorphic functions).**

$$f(z_0) = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z) dA$$

*Proof (of MVT?).*

Let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic where  $\Omega$  is open and connected. Then by Cauchy's integral

formula, we have  $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$  for any  $z_0 \in \Omega$ .

We can consider  $D_r(z_0)$ , in which case we have for all  $0 < s < r$ ,

■

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + se^{i\theta}) d\theta \\
\implies s \cdot f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} s \cdot f(z_0 + se^{i\theta}) d\theta \\
\implies f(z_0) \int_0^r s ds &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^r f(z_0 + se^{i\theta}) \cdot s ds d\theta \\
\implies \frac{1}{2} r^2 f(z_0) &= \frac{1}{2\pi} \iint_{D_r(z_0)} f(z) dA \\
\implies f(z_0) &= \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z) dA \\
\implies f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.
\end{aligned}$$

**Proposition 10.2 (Maximum in Interior Implies Constant).**

Let  $f$  be holomorphic on  $\Omega$  be open and connected, and suppose that there is a  $z_0 \in \Omega$  such that

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)|,$$

i.e.  $z_0$  is a maximal point of  $f$ . Then  $f$  is constant on  $\Omega$ .

If  $\Omega$  is additionally **bounded**, then  $f$  is continuous on  $\overline{\Omega}$ , then

$$\sup_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \overline{\Omega}} |f(z)|.$$

*Proof .*

Since  $|f|$  is continuous and  $\overline{\Omega}$  is compact,  $|f|$  attains a maximum at some point in  $\overline{\Omega}$ . We want to show that if  $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$ , then  $f$  is constant.

Assume that there exists a  $z_0 \in \Omega$  such that  $f(z) = f(z_0)$ . Let  $O = \{\xi \in \Omega \mid f(\xi) = f(z_0)\}$ .

**Claim .** 1.  $O$  is not empty, since  $z_0 \in O$ .

2.  $O$  is closed, since if  $\xi_n \rightarrow \xi$  then  $f(\xi_n) = f(z_0)$  implies  $f(\xi) = f(z_0)$  since  $f$  is continuous.

3. **(Claim)**  $O$  is open.

Suppose  $\xi_0 \in O$ , then there exists a disc  $D_\rho(\xi_0) \subset \Omega$  such that

$$f(\xi_0) = \frac{1}{\pi \rho^2} \int_{D_\rho(\xi_0)} f(z) dA.$$

Then (claim)  $|f(\xi_0)| \geq |f(z)|$  for all  $z \in D_\rho(\xi_0)$ , which forces  $f(z) = f(\xi_0)$  for all  $z \in D_\rho(\xi_0)$ . ■

*Proof (of the claim):*.

Suppose that  $\sup_{a \in \Omega} |f(z)| = |f(\xi_0)|$  and write  $f(\xi_0) = Be^{i\alpha}$  for  $B > 0$  and  $\alpha \in \mathbb{R}$ . Then define  $g(z) = f(z)e^{-i\alpha}$ ; then  $g(\xi_0) = B$  is real, and thus

$$0 = g(\xi_0) - B = \frac{1}{\pi\rho^2} \iint_{D_\rho(\xi_0)} \Re(g(z) - B) \, dA.$$

Note that  $\Re(g(z) - B) \leq 0$  implies that  $\Re(g(z) - B) \equiv 0$  on  $D_\rho(\xi_0)$ , so we can write  $g(z) = B + iI(z)$  for some real-valued function  $I$ .

But then  $|g(z)|^2 = B^2 + I(z)^2 = B^2$  by the previous statement, and so  $I(z) = 0$ , forcing  $g(z) = B$  and thus  $f(z) = Be^{i\alpha}$ . This shows that  $O$  is open, and thus  $O = \Omega$ . ■

**Proposition 10.3** (*Stein 2.1, Biholomorphisms of the Open Disc are Contractions*).

Suppose  $f$  is holomorphic on  $D_1(0)$  and  $|f(z)| \leq 1$  for all  $|z| < 1$  with  $f(0) = 0$ . Then  $|f(z)| \leq |z|$  for all  $|z| < 1$ .

Moreover, there is a point  $z_0 \in D_1(0)$  such that  $|f(z_0)| = |z_0|$  iff  $f(z) = cz$  for some  $c \in S^1$ .

*Proof .*

Define

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}.$$

Then  $g$  is holomorphic on  $D_1(0)$  and  $|g(z)| \leq \frac{1}{\rho}$  for all  $|z| < \rho < 1$ . Now apply the maximum principle: since this is true for all  $\rho < 1$ , consider the limit  $\rho \rightarrow 1^-$ .

Then  $|g(z)| \leq 1$ , so  $\left| \frac{f(z)}{z} \right| \leq 1$  and  $|f(z)| \leq |z|$ . If  $|f(z_0)| = |z_0|$  for any point, then  $|g(z_0)| = 1$  implies  $g(z_0) = c$  and  $c \in S^1$ .

Thus  $f(z) = cz$  for some  $c \in S^1$ . ■

**Corollary 10.4** (Characterization of Biholomorphisms of the Disc).

Recall that

$$\Phi_a(z) := \frac{z - a}{1 - \overline{a}z}.$$

If  $f : D_1(0) \rightarrow D_1(0)$  is a biholomorphism, then

$$f(z) = c\Phi_a(z) = e^{i\theta}\Phi_a(z)$$

So every such function is a rotated form of  $\Phi_a$ .

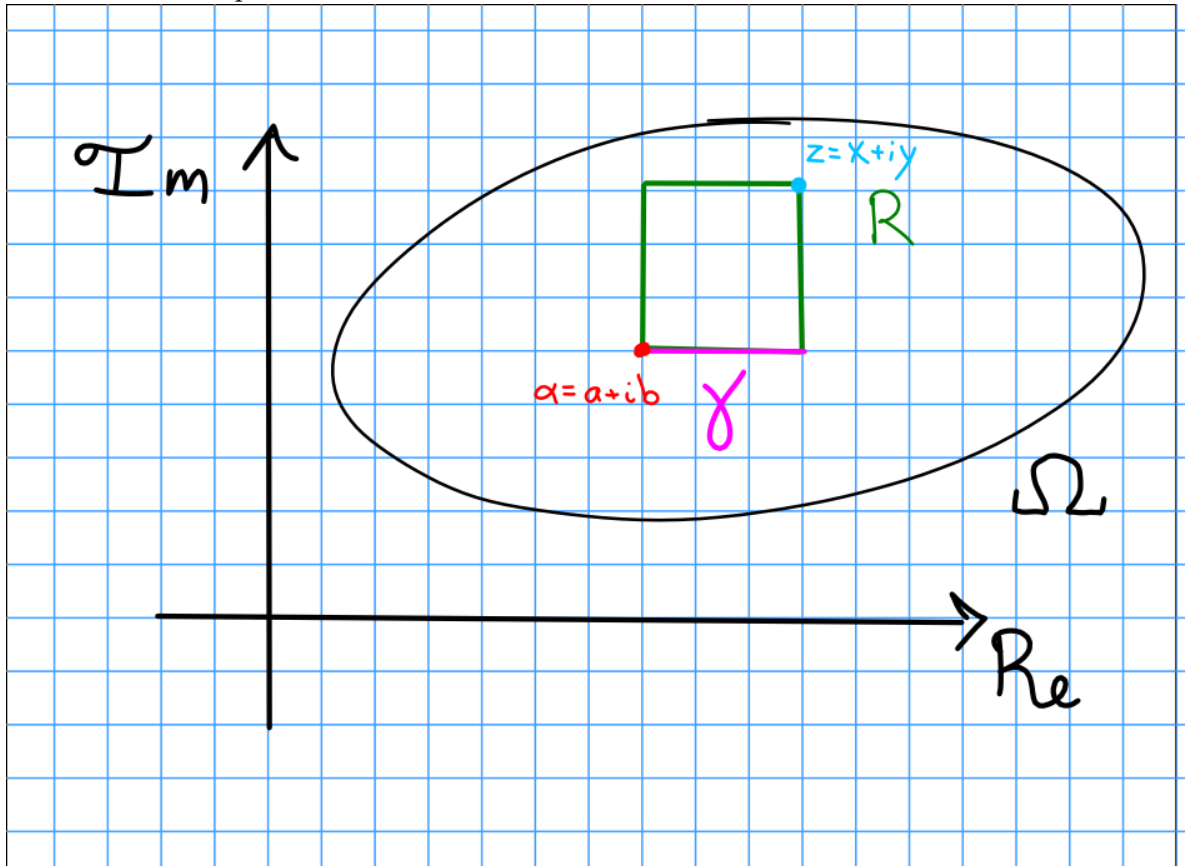
Let  $\Omega$  be a connected open domain and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic with  $f \in C^1$ . Then

$$\int_{\gamma} f(z) dz = 0$$

for every closed curve  $\gamma \subset \Omega$ , which implies that  $f^{(k)}(z)$  exists for all  $k \in \mathbb{N}$  and  $f$  is smooth/holomorphic.

**Theorem 10.5 (Morera, Partial Converse to Cauchy's Integral Theorem):**

Suppose  $g : \Omega \rightarrow \mathbb{C}$  is continuous and  $\int_{\gamma} g(z) dz = 0$  whenever  $\gamma = \partial R$  for some rectangle  $R \subset \Omega$  with sides parallel to the axes:



Then  $g(z)$  is holomorphic in  $\Omega$ .

*Proof .*

Fix a point  $\alpha = a + ib$  and given  $z = x + iy$ , construct a rectangle  $R$  containing  $z$ . Then by assumption,  $\int_{\partial R} g(z) dz = 0$ . Let  $\gamma_{az}$  be the path given by traversing the bottom edge of  $R$ , and  $\sigma_{az}$  by the top path.





Let

$$\begin{aligned} f(z) &= \int_{\gamma_{\alpha z}} g(z) \, dz \\ &= \int_a^x g(s + ib) \, ds + i \int_b^y g(x + it) \, dt. \end{aligned}$$

Since

$$\int_{\partial R} g(z) \, dz = 0 = \int_{\gamma_{\alpha z}} \cdots - \int_{\sigma_{\alpha z}} \cdots,$$

we have

$$\begin{aligned} f(z) &= \int_{\sigma_{\alpha z}} g(z) \, dz \\ &= i \int_b^y g(a + it) \, dt + \int_x^a g(s + iy) \, ds. \end{aligned}$$

Exercise: Apply  $\frac{\partial}{\partial y}$  to the first identity and  $\frac{\partial}{\partial x}$  to the second.

This yields

$$\frac{\partial f}{\partial x} = g(z) \quad \text{and} \quad \frac{\partial f}{\partial y} = ig(z) = i \frac{\partial f}{\partial x}$$

by applying the FTC, which are precisely the Cauchy-Riemann equations for  $f$ . So  $f$  is holomorphic, and thus  $f(z) = g(z)$ . ■

## 11 Wednesday February 5th

### 11.1 Cauchy/Morera Theorems

Recall last time: We have Cauchy's theorem, which says that if  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic then

$$\int_{\gamma} f dz = 0.$$

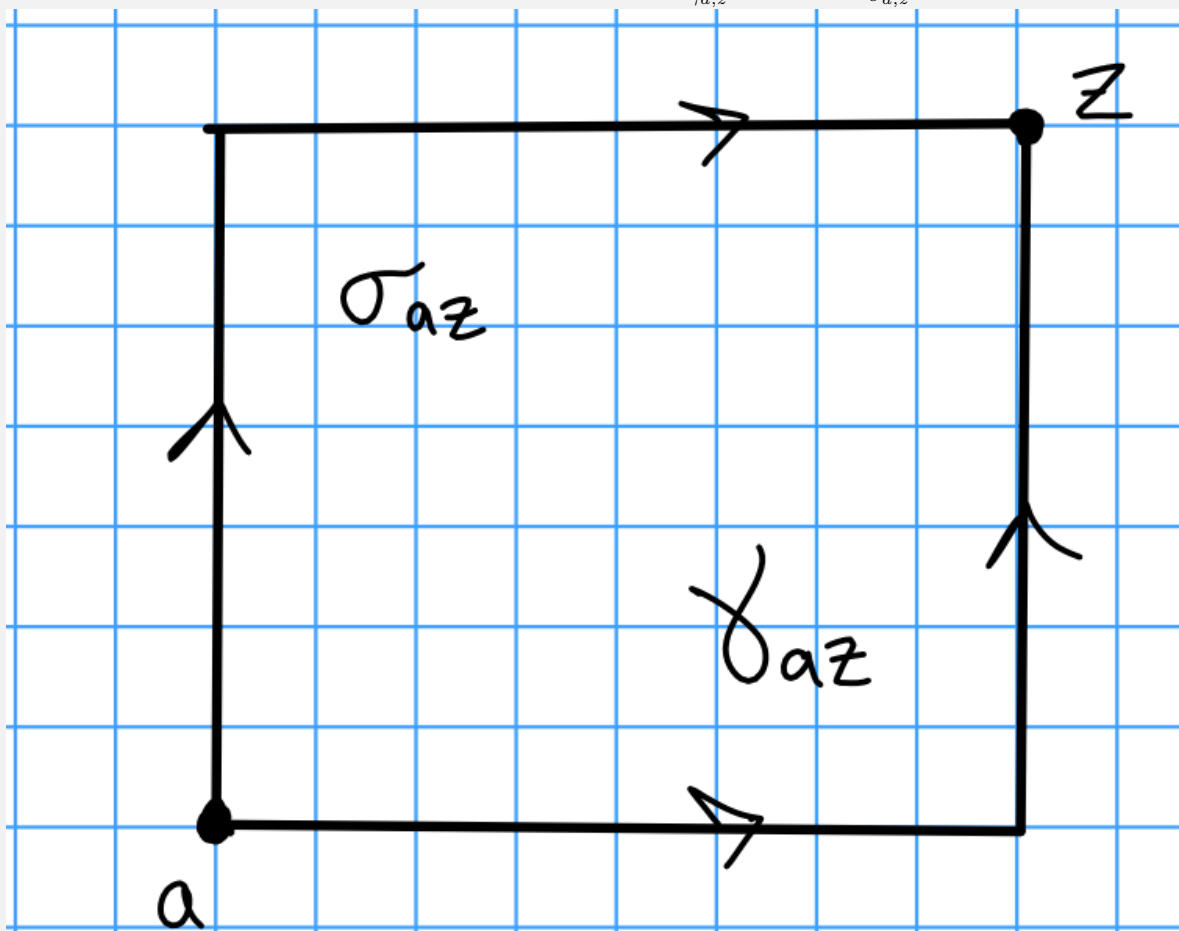
We have a partial converse:

**Theorem 11.1 (Morera).**

If  $g : \Omega \rightarrow \mathbb{C}$  is continuous and  $\int_R g dz = 0$  for every rectangle  $R \subset \Omega$  with sides parallel to the axes, then  $g$  is holomorphic.

*Proof (Morera).*

Fix a point  $a \in \Omega$ , then for any  $z \in \Omega$  define  $f(z) = \int_{\gamma_{a,z}} g(\xi) d\xi = \int_{\sigma_{a,z}} g(\xi) d\xi$ .



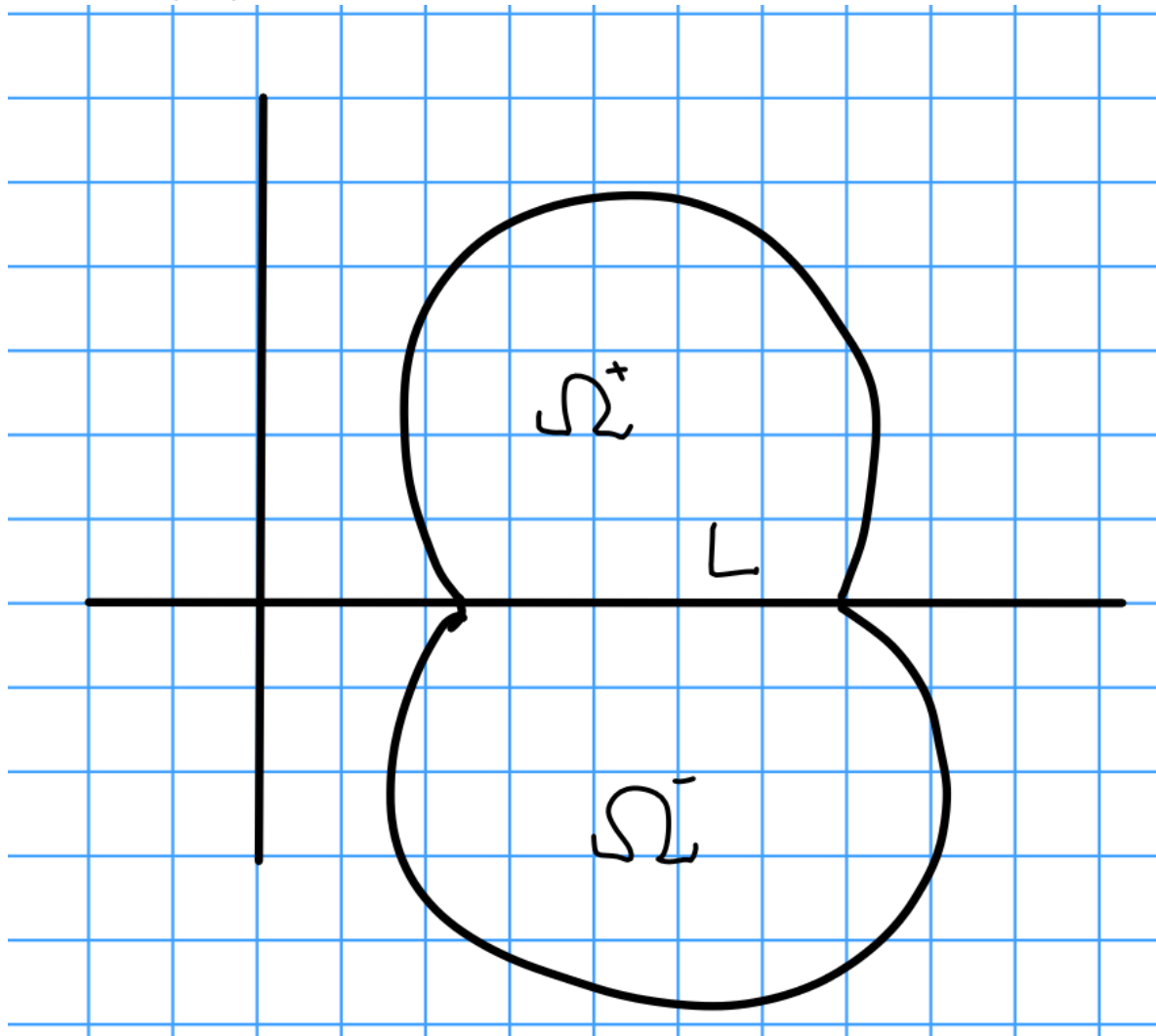
Then  $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = g(z)$ , making  $g$  holomorphic.

■

## 11.2 Schwarz Reflection

**Theorem 11.2** (*Schwarz Reflection, Extending Holomorphic Functions Across Reflected Regions*).

Let  $\Omega = \Omega^+ \cup L \cup \Omega^-$  be a region of the following form:



I.e.,  $L = \{z \in \Omega \mid \operatorname{im} z = 0\}$ ,  $\Omega^\pm = \{\pm \operatorname{im} z > 0\}$  where  $\Omega$  is symmetric about the real axis, i.e.  $z \in \Omega \implies \bar{z} \in \Omega$ .

Assume that  $f : \Omega^+ \cup L \rightarrow \mathbb{C}$  is continuous and holomorphic in  $\Omega^+$  and real-valued on  $L$ . Define

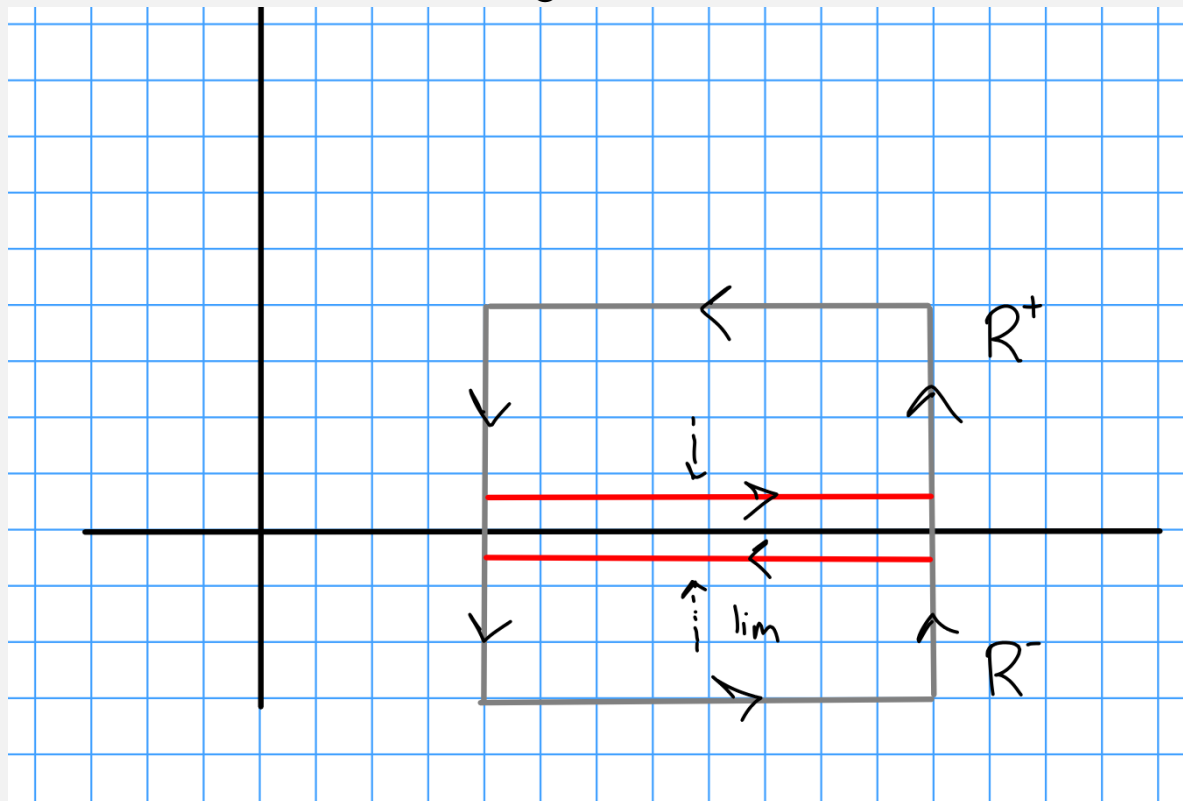
$$g(z) = \begin{cases} f(z) & z \in \Omega^+ \cup L \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}.$$

Then  $g(z)$  is defined and holomorphic on  $\Omega$ .

*Proof (Schwarz Reflection).*

Since  $g$  is  $C^1$  in  $\Omega^-$ , check that  $g$  satisfies the Cauchy-Riemann equations on  $\Omega^-$  and thus holomorphic there. To see that  $g$  is holomorphic on all of  $\Omega$ , we'll show the integral over every rectangle is zero.

It's clear that if  $R \subset \Omega^\pm$ ,  $\int_R g = 0$  since  $g$  is holomorphic there, so it suffices to check rectangles intersecting the real axis. Write  $R = R^+ \cup R^-$ :



We then have  $R^+ = \lim_{\varepsilon \rightarrow 0} R_\varepsilon$  and  $R^- = \lim_{\varepsilon \rightarrow 0} R_{-\varepsilon}$ , and  $\int_{R_{\pm\varepsilon}} g = 0$  for all  $\varepsilon > 0$ . By continuity of  $f$  on  $L$ , we have  $\lim \int_{R_\varepsilon} g(z) dz = 0$ . ■

### 11.3 Goursat's Theorem

**Theorem 11.3 (Goursat, implies smooth).**

If  $f : \Omega \rightarrow \mathbb{C}$  is complex differentiable at each point of  $\Omega$ , then  $f$  is holomorphic. I.e.,

$$f \in C^1(\Omega) \implies f \in C^\infty(\Omega).$$

*Proof (Goursat).*

We have  $\int_R f dz = 0$  for all rectangles  $R$ . Write  $I = \int_R f dz$ . Break  $R$  into 4 sub-rectangles:



Then rewriting the integral and applying the triangle inequality yields

$$I = \int_R f = \sum_{j=1}^4 \int_{R_j} f = \sum_{j=1}^4 I_j \implies |I| \leq \sum_j |I_j|.$$

So for at least one  $j$ , we have  $|I_j| \geq \frac{1}{4}|I|$ ; wlog call it  $R_1$ . By continuing to subdivide, we can write

$$|I| \leq 4|I_k| = 4 \left| \int_{R_1} f \right| \leq 4 \left( 4 \left| \int_{R_2} f \right| \right) \cdots \leq 4^k \left| \int_{R_k} f \right|.$$

This is a sequence of nested compact intervals, so there is some  $z_0 \in \bigcap R_k$ .

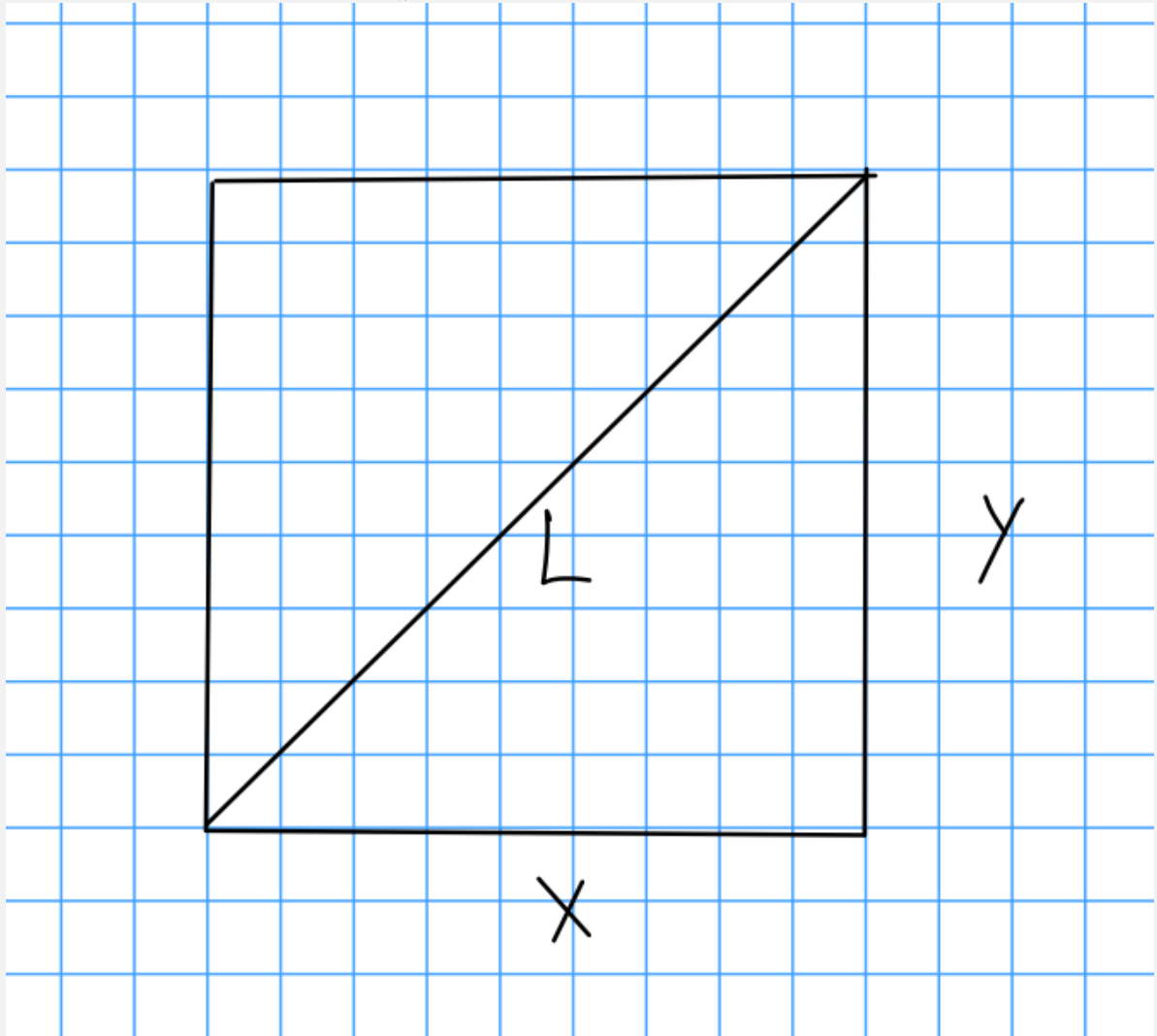
Write  $f(z) = f(z_0) + f'(z_0)(z - z_0) + \delta(z, z_0)$ , and since

$$\lim_{z \rightarrow z_0} \frac{|\delta(z, z_0)|}{|z - z_0|} = 0,$$

we have  $\delta(z, z_0) = o(z - z_0)$ . Then  $|I| \leq 4^k \frac{1}{2^k} |R|$ . We then try to estimate the integral using the fact that  $|\delta(z, z_0)| \leq \delta_k |z - z_0|$  for some constant  $\delta_k \rightarrow 0$  as  $k \rightarrow \infty$ .

$$\begin{aligned}
\int_{R_k} f i &= \int f(z_0) + f'(z_0)(z - z_0) + \delta(z, z_0) \\
&= \int_{R_k} \delta(z, z_0) \quad \text{since the first two terms are holomorphic} \\
&\leq \frac{1}{2^k} |R| \delta_k \frac{C}{2^k} |R| \\
&= c/4^k |R|^2 \delta_k \\
&\xrightarrow{k \rightarrow \infty} 0,
\end{aligned}$$

where we use the fact that in  $R_k$  we have



$$\begin{aligned}
R_k = 2(x + y) &\implies R^2/4 = x^2 + y^2 + x + y \leq_{CS} x^2 + y^2 + x^2 + y^2 = 2(x^2 + y^2) \\
&\implies x^2 + y^2 \leq R^2/8 \implies L = \sqrt{x^2 + y^2} \leq R/\sqrt{8} \\
&\implies |z - z_0| \leq \sqrt{x^2 + y^2} \leq R_k/2\sqrt{2} \text{ and } R_k = \frac{1}{2^k}|R|.
\end{aligned}$$

Note that triangles implies rectangles, but think about how to use triangles to prove it for rectangles (note that sides should be parallel to axes!) ■

## 12 Friday February 7th

**Theorem 12.1** (*The Uniform Limit of Holomorphic Functions is Holomorphic*).

Suppose  $\{f_n\} \rightarrow f$  is a sequence of holomorphic functions converging uniformly on any compact subset  $K \subset \Omega$ . Then  $f$  is holomorphic.

*Proof.*

Let  $D$  be any disc such that  $\overline{D} \subset \Omega$ . For any rectangle  $R \subset D$ , we have

$$\int_R f_n dz = 0.$$

Since  $f_n \rightarrow f$  uniformly,  $\int_R f dz = 0$  and thus  $f$  is holomorphic in  $D$ . ■



**Theorem 12.2 (Uniform Convergence of Derivatives).**

Under the same hypotheses,  $f'_n \rightarrow f$  uniformly on any compact subset  $K \subset \Omega$ .

*Proof.*  
See Stein.



**Corollary 12.3** (When Functions Defined by Integrals are Holomorphic).

Suppose  $F(z, s) : \Omega \times [a, b] \rightarrow \mathbb{C}$  and

1.  $F(z, s)$  is holomorphic in  $z$  for each fixed  $s \in [a, b]$ .
2.  $F(z, s)$  is continuous in  $\Omega \times [a, b]$ .

Then  $f(z) = \int_a^b F(z, s) ds$  is holomorphic on  $\Omega$ .



*Proof.*

Define  $f_n(z) = \left( \sum_{k=1}^n F(z, s_k) \right) \frac{b-a}{n}$  where each  $s_k = a + \frac{b-a}{n}k \in [a, b]$ . Need to show  $f_n(z)$  converges uniformly on any compact  $K \subset \Omega$ , i.e. it's uniformly Cauchy. Fix  $K$  compact, then by a theorem in topology  $K \times [a, b]$  is again compact.

Using the fact that  $F$  is continuous on a compact set and thus uniformly continuous, fix  $\varepsilon > 0$  and find  $\delta > 0$  such that  $\max_{z \in K} |F(z, s) - F(z, t)| < \varepsilon$  for all  $s, t \in [a, b]$  with  $|t - s| < \delta$ .

Thus if  $\frac{b-a}{n} < \delta$  and  $z \in K$ , we have an estimate

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^n \int_{s_{k-1}}^{s_k} F(z, s_k) - F(z, s) \, ds \right| \\ &= \sum_{k=1}^n \int_{s_{k-1}}^{s_k} |F(z, s_k) - F(z, s)| \, ds \\ &\leq \varepsilon(b-a). \end{aligned}$$

Thus  $f_n \xrightarrow{u} f$ . ■

Remark: this is useful for showing

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} \, ds$$

is holomorphic for  $\Re z > 0$ .

**Question:** can every function be uniformly approximated by polynomials?

**Answer:** in general, no. Take  $f(z) = \frac{1}{z}$ , which is holomorphic on  $\mathbb{C} \setminus 0$ , but  $\int_\gamma P_N(z) = 0$  for any polynomial (since they are entire) for any loop  $\gamma$  around 0, but  $\int_\gamma \frac{1}{z} = 2\pi i$ .

**Theorem 12.4(5.2, Uniform Approximation by Polynomials).**

If  $f_n$  is a sequence of holomorphic functions converging uniformly on any compact subset  $K$  of  $\Omega$  then  $f$  is holomorphic in  $\Omega$  and if  $f(z) = \sum a_n(z - z_0)^n$  then  $P_N(z) = \sum_{n=0}^N a_n(z - z_0)^n$ .

**Theorem 12.5(5.7, Uniform Approximation by Rational Functions).**

Any holomorphic function in a neighborhood of a compact set  $K$  can be approximated by a rational function with singularities only in  $K^c$ . If  $K^c$  is connected, it can be approximated by a polynomial.

**Lemma 12.6 (5.8, ???).**

Suppose  $f$  is holomorphic in an open set  $\Omega$  with  $K \subset \Omega$  compact. Then there exist finitely many segments  $\{\gamma_i\}_{i=1}^N$  in  $\Omega \setminus K$  such that for all  $z \in K$ , ???.

*Proof (of Lemma, Idea).*

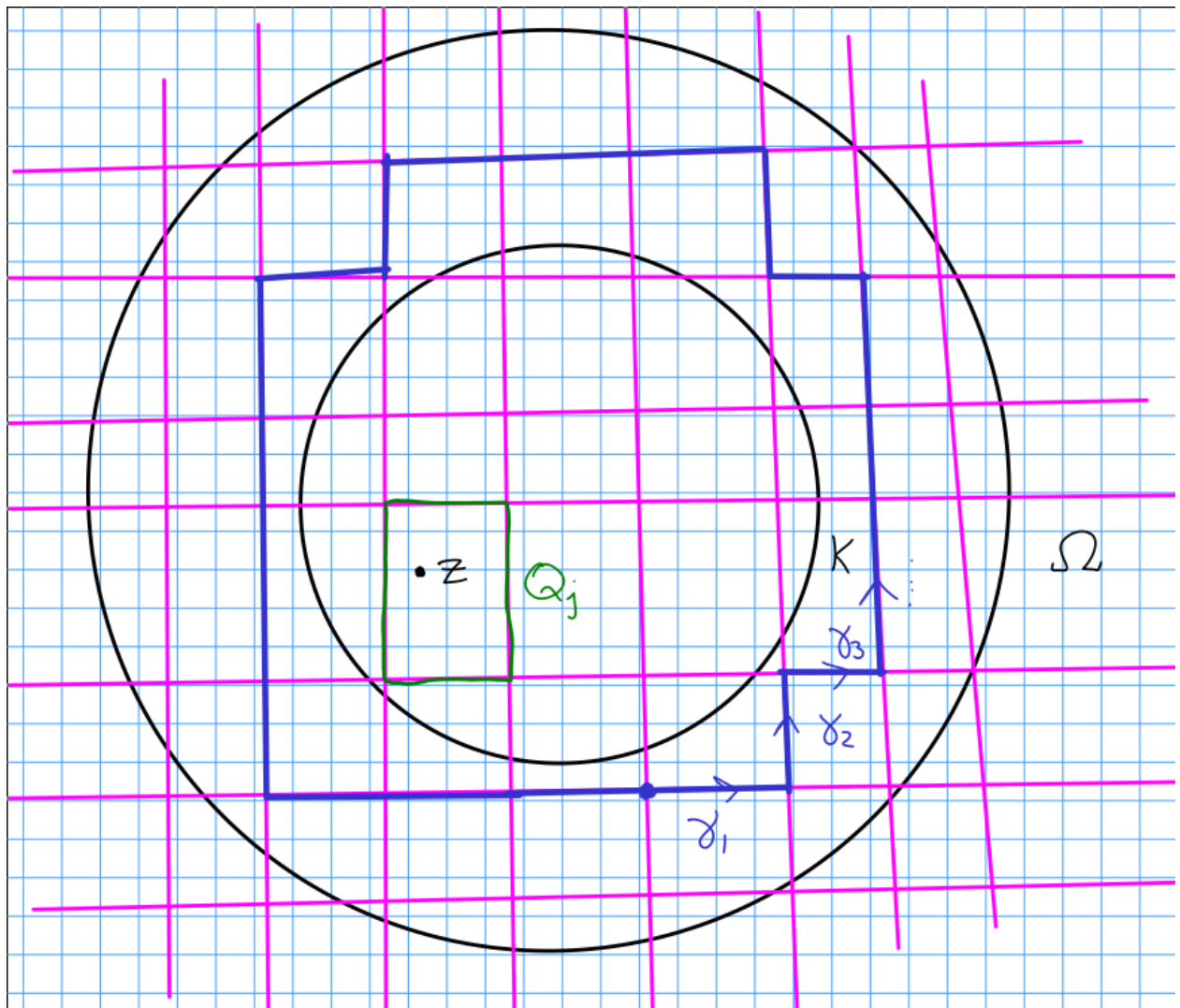
Divide region into squares, take  $\gamma_i$  to be line segments such that they enclose  $K$ .

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=1}^N \int_{\omega_n} \frac{f(\xi)}{z - \xi} d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{z - \xi} d\xi. \end{aligned}$$

where we can rewrite

$$\int_{\gamma_n} \dots = \int_0^1 \frac{f(\gamma_n(t))}{\gamma_n(t) - z_0} \gamma_n'(t) dt = \int_0^1 F(z, s) ds$$

The idea is that we can then write  $\frac{1}{\xi - z} = \frac{1}{\xi} \frac{1}{1 - \frac{z}{\xi}} = \xi^{-1} \sum_k \left(\frac{z}{\xi}\right)^k$ , which allows uniform approximation by polynomials. ■



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## 13 Wednesday February 12th

### 13.1 Singularities

Let  $f(z)$  be holomorphic on  $\Omega$ , then we have Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi.$$

*Example:* Note that  $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus 0$ .

Let  $\Omega$  be an open set containing a disk  $D$  and  $\Omega \setminus p$  be a punctured domain.

#### Definition 13.1.

We say  $f$  has an *isolated singularity* at  $p$  iff  $f$  is defined and holomorphic on some deleted neighborhood of  $p$ .

Classification of singularities:

1. **Removable:**  $|f(z)|$  is bounded on some  $D_r(p) \setminus p$ .

*Example:*  $f(z) = \sin(z)/z$ .

2. **Poles:**  $\lim_{z \rightarrow p} |f(z)| = \infty$ .

*Example:*  $f_n(z) = \frac{1}{z^n}$  at  $p = 0$

3. **Essential:** neither 1 nor 2.

*Example:*  $f(z) = e^{\frac{1}{z}}$  at  $z = 0$ .

Note that for singularities at  $\infty$ , we can just make the change of variables  $z \mapsto \frac{1}{z}$ . Defining  $F(z) = f(\frac{1}{z})$ , the singularities at 0 of  $f$  correspond to singularities at infinity for  $F$ .

### 13.2 Spherical Projection

We can solve for a spherical projection map  $S^2 \rightarrow \mathbb{C}$ . Let  $(0, 0, 1)$  be the North pole of the sphere; then to map to  $(x, y, 0)$  on the plane we can take the parameterization  $\ell : (tx, ty, 1 - t)$ . This yields

$$t \mapsto \left( \frac{2\Re(z)}{1 + |z|^2}, \frac{2\Im(z)}{1 + |z|^2}, 1 - \frac{2}{1 + |z|^2} \right).$$



From this we can induce a spherical metric:

$$\phi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{|z_1|^2 + 1} \sqrt{|z_2|^2 + 1}}.$$

**Proposition 13.1** (*Continuous Extension Over Removable Singularities*).

Let  $p$  be a removable singularity of  $f$ . Then

1.  $\lim_{z \rightarrow p} f(z)$  exists.
2. The function

$$\tilde{f}(x) = \begin{cases} f(z) & z \neq p \\ \lim_{z \rightarrow p} f(z) & z = p \end{cases}.$$

is holomorphic on  $D_r(p)$ .

**Example 13.1.**

Consider

$$\frac{\sin(z)}{z} \xrightarrow{z \rightarrow 0} 1.$$

*Proof (of Proposition).*

Take  $p = 0$  and consider  $g(z) = z^2 f(z)$ . We can verify directly that  $g$  satisfies the Cauchy-Riemann equations on  $D_r(0)$ . Then  $g$  is holomorphic on  $D_r(0)$  and vanishes to order 2 at  $z = 0$ , and

$$f(z) = \frac{g(z)}{z^2}$$

is holomorphic on  $D_r(0)$ .

If  $f(z)$  has a pole at  $z_0$ , then  $\lim_{z \rightarrow z_0} |f(z)| \rightarrow \infty$  by definition, iff  $\lim_{z \rightarrow z_0} \frac{1}{|f(z)|} = 0$  and thus the reciprocal has a zero at  $z = z_0$ . If  $z_0$  is a zero of a nontrivial holomorphic function  $f$ , then  $z_0$  is isolated, i.e. there exists a punctured disc  $D_r(z_0) \setminus z_0$  on which  $f$  is nonzero. ■

**Theorem 13.2(???)**

If  $f$  is holomorphic in a connected domain  $\Omega$  with a zero  $z_0$ , then there exists a non-vanishing holomorphic function  $g(z)$  and some  $n \in \mathbb{N}$  such that

$$f(z) = (z - z_0)^n g(z)$$

.

*Proof .*

Since  $f$  is holomorphic, expand its power series  $f(z) = \sum a_k(z - z_0)^k$ . Since  $f(z_0) = 0$ , we have  $a_0 = 0$ . Choose the smallest  $n$  such that  $a_n \neq 0$ , so

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\ &= (z - z_0)^n(a_n + \cdots) \\ &:= (z - z_0)^n g(z). \end{aligned}$$

Then  $g(z_0) \neq 0$ , so by continuity there exists an  $r$  such that  $|g(z)| \geq |a_n|/2$ . ■

**Definition 13.2.**

A function  $f$  defined on a deleted neighborhood of  $z_0$  has a pole at  $z_0$  if the function  $F = \frac{1}{f}$  with  $F(z_0) := 0$  is holomorphic in a full neighborhood of  $z_0$ .

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## 14 Friday February 14th

### 14.1 Defining Residues

Interesting open problems: dynamical systems on  $\mathbb{C}^2$ .

If  $f$  is holomorphic in  $\Omega$  with  $f(z_0) = 0$  then there exists a disc on which  $f(z) = \sum a_n(z - z_0)^n$  where  $a_0 = f(z_0) = 0$ . There is then a minimal  $k$  such that  $f(z) = (z - z_0)^k g(z)$  where  $g(z_0) \neq 0$ ; this  $k$  is the *order* of the zero  $a_0$ .

#### Definition 14.1.

A function defined in a deleted neighborhood of  $z_0$  has a *pole* at  $z_0$  iff  $F = \frac{1}{f}$  with  $F(z_0) := 0$  is holomorphic in a full neighborhood of  $z_0$ .

#### Theorem 14.1 (*Extraction of Holomorphic Part*).

If  $f$  has a pole at  $z_0$ , then there exists a holomorphic function  $h$  and a unique  $k$  such that  $f(z) = (z - z_0)^{-k} h(z)$ .

*Proof.*

Write

$$\frac{1}{f} = (z - z_0)^k g(z)$$

with  $g(z_0) \neq 0$ . Then there is an  $r$  such that  $|g(z)| \geq \frac{1}{2}|g(z_0)|$  in a disc about  $z_0$ . Then

$$f(z) = \frac{1}{(z - z_0)g(z)} := (z - z_0)^{-k} h(z)$$

where  $h = 1/g$ .

We can then write

$$f(z) = \left( \sum_{i=0}^{k-1} b_i (z - z_0)^{-i} \right) + b_k + \sum_{i=1}^{\infty} b_{k+i} (z - z_0)^i$$

for some fixed  $k$ , where  $\sum b_i (z - z_0)^i$  is the power series expansion of  $h$ . Write this as  $P(z) + G(z)$  where  $G(z) = \sum_{i=0}^{\infty} b_{i+k} (z - z_0)^i$ . Denote  $P$  the *principal part* of  $f$  at the pole  $z = z_0$ .

Note that

$$\int_{D_r(z_0)} f = \int_{D_r(z_0)} P(z) = 2\pi i a_{-1}.$$

■

#### Definition 14.2.

The coefficient  $a_{-1}$  is referred to as the *residue* of  $f$  at  $z = z_0$ .

## 14.2 Residues

Note that

$$\int \frac{1}{(z - z_0)^k} = \begin{cases} 2\pi i & k = 1 \\ 0 & \text{else} \end{cases}.$$

Residues can be computed using the following formula:

$$a_{-1} = \frac{1}{2\pi i} \int_{D_r(z_0)} f. \quad (1)$$

Theorem (Residue Formula) :

$$\text{Res}_{z=z_0} f = \lim_{z \rightarrow z_0} \frac{1}{(k-1)!} \left( \frac{\partial}{\partial z} \right)^{k-1} (z - z_0)^k f(z).$$

*Proof .*

Expand in power series, direct check. ■

A useful special case: if  $z_0$  is a pole of order 1, then

$$\text{Res}_{z=z_0} f = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

A useful formula:

$$\frac{1}{2\pi i} \int_{\Gamma(z_0)} f = \text{Res}_{z=z_0} f.$$

### Theorem 14.2 (Integral Residue Theorem).

Suppose that  $f$  is holomorphic in an open set containing a toy contour  $\gamma$  and its interior except for finitely many poles  $\{z_i\}$ . Then

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum \text{Res}_{z=z_i} f(z).$$

*Proof .*

Omitted to cover some material needed for homework. ■

Note that if  $f$  has a pole of order  $k$ , we can expand it in *Laurent series* as

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$$\sum_{n=-k}^1 a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^k.$$

How to determine the radius of convergence of a Laurent series:

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n \in \mathbb{N}} c_n z^n + \sum_{n \in \mathbb{N}} d_n z^{-n}.$$

Applying the root test,

$$\begin{aligned} \limsup_n |c_n(z-a)|^{1/n} &< 1 \\ \iff \limsup_n |c_n|^{1/n} |z-a| &< 1 \\ \iff |z-a| &\leq \frac{1}{\limsup_n |c_n|^{1/n}} := \rho_1. \end{aligned}$$

Similarly, we need

$$\rho_2 := \limsup_n |d_n|^{1/n} < |z-a|.$$

If  $\rho_1 > \rho_2$ , this will converge on an annulus.

## 15 Monday February 17th

See Hans Lewy 1957 Annals, Folland and Stein 1973. Does a linear system of PDEs with analytic functions have an analytic solution? What about just  $C^\infty$ ?

### 15.1 Getting a Holomorphic Function from a Laurent Series

We can write a formal series

$$\begin{aligned} f(z) &= \sum_{n \in \mathbb{Z}} a_n(z-a)^n \\ &= \sum_{n \geq 0} a_n(z-z_0)^n + \sum_{n \leq -1} a_n(z-z_0)^n \\ &:= A(z) + B(z). \end{aligned}$$

Part  $A$  converges for

$$|z-a| < R_1 = \left( \limsup_n |a_n|^{1/n} \right)^{-1}.$$



Part  $B$  converges for

$$|z - a| > R_2 = \limsup |c_{-n}|^{1/n}.$$

If  $R_1 < R_2$ , this does not converge. Note that if  $R_1 > R_2$ , then  $f$  converges and defines a holomorphic function on the annulus  $R_2 < |z - a| < R_1$ . Moreover,  $f$  converges uniformly on any compact subset of this annulus, so it can be differentiated term-by-term, and the derivative has the same region of convergence.

Note that if  $f$  equals its Laurent expansion, then

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} dz$$

where  $\gamma$  is contained in the annulus of convergence, and  $c_{n \leq -1} = 0$ .

We also have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^m} dz &= \sum_{n \in \mathbb{Z}} \frac{c_n}{2\pi i} \int_{\gamma} \frac{1}{(z - a)^{m-n}} dz \\ &= c_{m-1}, \end{aligned}$$

since

$$\int_{\gamma} \frac{1}{(z - a)^k} dz = \begin{cases} 2\pi i & k = 1 \\ 0 & \text{else} \end{cases},$$

we have the following formula for the coefficients:

$$c_m = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{m+1}} dz. \quad (2)$$

So we can start with a series and get a holomorphic function on some region.

## 15.2 Obtaining a Laurent Series from a Holomorphic Function

We can also start with a holomorphic function and get a Laurent series. Suppose  $f$  is holomorphic on an annulus  $R_2 < |z| < R_1$ . We can then write

$$f(z) = \frac{1}{2\pi i} \int_{|w-a|=R_1} \frac{f(w)}{w - z} dw - \int_{|w-z|=R_2} \frac{f(w)}{w - z} dw$$



Since  $|z - a|/|w - a| < 1$ , we have

$$\begin{aligned}
 \frac{1}{2\pi i} \int_{|w-a|=R_1} \frac{f(w)}{w-z} dz &= \frac{1}{2\pi i} \int_{|w-a|=R_1} \frac{f(w)}{(w-a) - (z-a)} dz \\
 &= \frac{1}{2\pi i} \int_{|w-a|=R_1} \frac{f(w)}{(w-a)} \sum_{n \in \mathbb{N}} \frac{(z-a)^n}{(w-a)^n} dz \\
 &= \sum_{n \in \mathbb{N}} (z-a)^n \frac{1}{2\pi i} \int_{|w-a|=R_1} \frac{f(w)}{(w-a)^{n+1}} dw \\
 &= \sum_{n \in \mathbb{N}} c_n (z-a)^n.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 -\frac{1}{2\pi i} \int_{|w-a|=R_2} \frac{f(w)}{w-z} dw &= -\frac{1}{2\pi i} \int_{|w-a|=R_2} \frac{f(w)}{(w-a)-(z-a)} dw \\
 &= -\frac{1}{2\pi i} \frac{1}{z-a} \int_{|w-a|=R_2} \frac{f(w)}{\frac{w-a}{z-a} - 1} dw \\
 &= \frac{1}{2\pi i} \frac{1}{z-a} \int_{|w-a|=R_2} \frac{f(w)}{1 - \frac{w-a}{z-a}} dw \\
 &= \frac{1}{2\pi i} \frac{1}{z-a} \int_{|w-a|=R_2} f(w) \sum_{n \in \mathbb{N}} \frac{(w-a)^n}{(z-a)^n} dw \\
 &= \sum_{n \in \mathbb{N}} \frac{1}{2\pi i} \frac{1}{(z-a)^{n+1}} \int_{|w-a|=R_2} f(w)(w-a)^n dw \\
 &= \sum_{n=-\infty}^{-1} c_n (z-a)^n.
 \end{aligned}$$

This yields a formula

$$c_m = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{m+1}} dz. \quad (3)$$

In practice, we don't use this formula for extracting coefficients.

**Example 15.1.**

Let  $f(z) = \frac{1}{z(z-1)}$ . This has four Laurent series.

Let  $C(a, R_1, R_2)$  be the annulus centered at  $a$ . Then at  $C(0, 0, 1) = \mathbb{D} \setminus \{0\}$ , we have

$$f(z) = \frac{1}{z} \frac{1}{1-z} = -\frac{1}{z} \sum_{k \in \mathbb{N}} z^k.$$

In  $C(1, 1, 0) = \mathbb{D}(1, 1) \setminus \{1\}$ , we have

$$\begin{aligned}
 f(z) &= \frac{1}{z-1} \frac{1}{z} \\
 &= \frac{1}{z-1} \frac{1}{1+(z-1)} \\
 &= \frac{1}{z-1} \sum_{k \in \mathbb{N}} (-1)^k (z-1)^k.
 \end{aligned}$$

In  $C(0, 1, \infty)$ , we can write

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$$\begin{aligned} f(z) &= \frac{1}{z^2} \frac{1}{1 - \frac{1}{z}} \\ &= \frac{1}{z^2} \sum_{k \in \mathbb{N}} \frac{1}{z^k}. \end{aligned}$$

And in  $C(1, 1, \infty)$  we have

$$f(z) = \frac{1}{z-1} \frac{1}{z-1+1}.$$

## 16 Wednesday February 26th

### 16.1 Argument Principle and Application

Let  $f$  be holomorphic in  $\Omega$  which is open, simple, and connected. Then  $f(z_0) = 0$  implies there exists an integer  $m$  such that  $f(z) = (z - z_0)^m g(z)$  where  $g(z_0) \neq 0$ .

Let  $N_\Omega(f)$  be the number of zeros of  $f$  inside  $\Omega$ , and  $N_\Omega(f, a)$  be the number of zeros of  $f - a$  in  $\Omega$ . Writing  $f = f_1 f_2$  where  $f_1 = (z - z_0)^m$  and  $f_2 = g(z)$ , we have

$$\begin{aligned} \frac{f'}{f} &= \frac{f'_1}{f_1} + \frac{f'_2}{f_2} \\ &= \frac{m}{z - z_0} + \frac{g'}{g}. \end{aligned}$$

Now integrating both sides yields

$$\frac{1}{2\pi i} \int_{D_r(z_0)} \frac{f'(z)}{f(z)} dz = m,$$

so the integral of this function counts the number of zeros of  $f$  in  $D_r(z_0)$ .

**Proposition 16.1 (Argument Principle).**

Let  $f$  be holomorphic in a neighborhood of  $\overline{D_r(z_0)}$  and suppose that  $f$  is non-vanishing on all of  $\partial D_r(z_0)$ . Then

$$\frac{1}{2\pi i} \int_{D_r(z_0)} \frac{f'(z)}{f(z)} dz = N_{D_r(z_0)}(f).$$

More generally, if  $q(z)$  is another holomorphic function in a neighborhood of  $\overline{D_r(z_0)}$  and  $z_1, \dots, z_k$  are the distinct zeros of  $f$  in  $D_r(z_0)$  with orders  $m_1, \dots, m_k$ , then

$$\frac{1}{2\pi i} \int_{D_r(z_0)} q(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^k q(z_j) m_j.$$

*Proof.*

Write  $f(z) = \prod_{j=1}^k (z - z_j)^{m_j} g(z)$ . By Leibniz's rule, if  $h = f_1 \cdots f_\ell$ , then

$$\begin{aligned} \frac{h'}{h} &= \sum_{j=1}^{\ell} \frac{f'_j}{f_j} \\ \implies q \frac{f'}{f} &= 1 \frac{g'}{g} + \sum_{j=1}^k \frac{m_j q}{z - z_j}. \end{aligned}$$

Since  $\frac{g'}{g}$  is holomorphic in the closed disc, integrating both sides yields the desired formula.

Note that if we replace  $f$  by a family  $f_t$  of continuous functions, an integer-valued continuous function must be constant.

**Corollary 16.2.**

Let  $f_t(z)$  for  $0 \leq t \leq 1$  be a family of holomorphic functions on  $D_{r+\varepsilon}(z_0)$  for some  $\varepsilon > 0$ . Suppose  $f_t(z)$  is continuous for all  $z$  in this disc, uniformly in  $z$ , and for all  $t$ ,  $f_t(z)$  is nonvanishing on the boundary.

Then the following integral is independent of  $t$ :

$$\frac{1}{2\pi i} \int_{D_r(z_0)} \frac{f'_t(z)}{f_t(z)} dz.$$

**Theorem 16.3 (Rouché's).**

Let  $f, g$  be holomorphic in a neighborhood of  $\overline{D_r(z_0)}$  and suppose that on  $\partial D_r(z_0)$  we have

$$|f(z) - g(z)| < |f(z)| + |g(z)|.$$

Then  $f$  and  $g$  are nonvanishing on  $\partial D_r(z_0)$  and

$$N_{D_r(z_0)}(f) = N_{D_r(z_0)}(g). \quad (4)$$

*Proof .*

If  $f(w) = 0$  for some  $w \in \partial D_r(z_0)$ , then  $|-g(w)| = |g(w)|$ , but this contradicts condition ??.

■

So let  $t \in [0, 1]$  with  $f_t(z) = (1 - t)f(z) + tg(z)$ . Then (claim)  $f_t$  is nonvanishing on the boundary, so we can apply the previous corollary.

Suppose otherwise that there exists  $w$  on the boundary such that  $f_t(w) = 0$  for some  $t$ , so  $(1 - t)f(w) + tg(w) = 0$ . Then rearranging terms yields

$$\begin{aligned} f(w) &= t(g(w) - f(w)) \\ g(w) &= (1 - t)(g(w) - f(w)). \end{aligned}$$

But then

$$\begin{aligned} |f(w) + g(w)| &= t|g(w) - f(w)| + (1 - t)|g(w) - f(w)| \\ &= |g(w) - f(w)|, \end{aligned}$$

which contradicts condition ??

By the corollary, the integral is continuous in  $t$  and integer-valued, and thus constant.

**Corollary 16.4** (Fundamental Theorem of Algebra).

Let  $p(z) = \sum_{j=1}^n a_j z^j$  be a polynomial of degree  $n$ , so  $a_n \neq 0$ . Let  $f(z) = a_n z^n$  and  $g(z) = p(z)$ . If

$$|z| > \frac{|a_0| + \cdots + |a_{n-1}|}{|a_n|} > 1$$

then

$$\begin{aligned} |f(z) - g(z)| &= |a_0 + \cdots + a_{n-1}z^{n-1}| \\ &\leq |z|^{n-1}(|a_0| + \cdots + |a_{n-1}|) \\ &< |a_n||z|^n \\ &= |f| \\ &\leq |f| + |g|. \end{aligned}$$

Note that this is useful because it tells you where the zeros are, namely in the disc  $|z| < \frac{\sum |a_i|}{|a_n|}$ .

**Example 16.1.**

Let  $p(z) = 9 - 8z + 20z^2$ , then all of the zeros are in a disc of radius  $r = \frac{7}{4}$ .

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Qual alert: problems about power series, Rouché's, linear mapping, integration.

**Example 16.2.**

Let  $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ .

How many zeros are in the unit disc? Take  $g(z) = -8z$ , the largest term. Then  $|f(z) - g(z)| \leq 1 + 2 + 1 + 2 = 6 < |f| + |g| = 8$ , so condition ?? is satisfied. Thus they both have the same number of zeros, but  $g$  has exactly one zero.

What about  $|z| = 2$ ? Then set  $g(z) = z^9$ , then check  $|f(z) - z^9| \leq 150 < 152$ , so all 9 zeros lie in this disc.

**Exercise** Let  $g(z) = z^4 - 4z - 5$ , how many zeros are in  $|z| \leq 1$ ? Note the root on the boundary.

## 17 Friday February 28th

**Theorem 17.1 (Stein).**

Suppose  $f, g$  are holomorphic in  $D_r(z_0)$  and  $|f(z)| > |g(z)|$  on  $\partial D_r(z_0)$ . Then  $f$  and  $f + g$  have the same number of zeros in  $D_r(z_0)$ .

*Proof.*

Let  $f_t(z) = f(z) + tg(z)$  and use the argument principle. ■

**Theorem 17.2 (Stein 4.4: Open Mapping).**

If  $f$  is holomorphic and nonconstant then  $f$  is an open map.

*Proof.*

Let  $w_0 \in \text{im}(f)$  and say  $f(z_0) = w_0$ . We want to show that all  $w$  near  $w_0$  are also in  $\text{im}(f)$ . Define  $g(z) = f(z) - w = f(z) - w_0 + w_0 - w := F(z) + G(z)$ . ■

Now choose  $\delta > 0$  such that  $D_\delta(z_0) \subset \Omega$  and  $f(z) \neq w_0$  on  $\partial D_\delta(z_0)$ . We then select  $\delta$  such that  $|f(z) - w_0| \geq \varepsilon > 0$  on  $\partial D_\delta(z_0)$ . We have  $|F(z)| = |f(z) - w_0| \geq \varepsilon$ .

Now choose  $w$  such that  $|G(z)| = |w - w_0| < \varepsilon$ , noting that  $G(z)$  is a constant function (?). Then  $|F(z)| \geq \varepsilon > |w - w_0| = |G(z)|$ . So apply Rouché's theorem and conclude that there exists  $z \in D_\delta(z_0)$  such that  $f(z) = w$ .

---

Qual alert, some questions related to the Open Mapping Theorem.

**Theorem 17.3 (Stein 4.5: Maximum Modulus).**

If  $f$  is holomorphic and nonconstant on  $\Omega$ , then  $|f|$  can not attain a maximum in  $\Omega$ .

*Proof.*

Suppose toward a contradiction that  $|f|$  attains a maximum in  $\Omega$ , say at  $z_0$ . Since  $f$  is

holomorphic, it is an open mapping, and therefore if  $D_\delta(z_0) \subset \Omega$  then  $f(D_\delta(z_0))$  contains a disc. Thus there exists a  $z \in D_\delta(z_0)$  such that  $|f(z)| > |f(z_0)|$ . But this contradicts maximality of  $f$  at  $z_0$ . ■

**Corollary 17.4.**

If  $|f(z)| = 0$  on  $\partial U$  and  $f$  is nonconstant, then  $f$  has a zero in  $U$ .

*Proof.*

Let  $c = \max_{z \in \partial U} |f(z)|$ . Suppose that  $f(z)$  has no zeros in  $U$ . Then  $g(z) = \frac{1}{f(z)}$  is continuous and holomorphic in  $U$ . Then for all  $z_0 \in U$ ,  $|g(z)| = \frac{1}{|f(z)|} = \frac{1}{|f(z_0)|} > \frac{1}{c}$ , since  $c = \max_{z \in \partial U} |f(z)|$  implies  $|f(z_0)| < c$ . But this contradicts the maximum principle. ■

Proof technique: use the fact that the reciprocal is holomorphic. Note that this is stronger than  $f$  just being smaller in the interior, the modulus actually takes on the smallest value.

**17.1 The Complex Logarithm**

For  $x > 0$ , we define  $\log(x) = \int_1^x \frac{1}{t} dt$ , which is the inverse of  $e^x$ . For  $z \neq 0$ , we'd like to define  $\log(z) = \log(re^{i\theta}) = \log(r) + i\theta$ , but the argument  $\theta$  is not uniquely defined.

**Theorem 17.5.**

Suppose  $\Omega$  is simply connected with  $1 \in \Omega$  and  $0 \notin \Omega$ . Then in  $\Omega$ , there is a branch of the logarithm  $F(z) = \log_\Omega(z)$  such that

1.  $F(z)$  is holomorphic on  $\Omega$ .
2.  $e^{F(z)} = z$  for all  $z \in \Omega$
3.  $F(r) = \log(r)$  for all  $r > 0 \in \mathbb{R}$  near 1.

*Proof.*

**Part 1:**

We define  $F(z)$  as a primitive of the function

$$F(z) = \int_\gamma \frac{1}{w} dw.$$

where  $\gamma$  is any curve in  $\Omega$  connecting 1 and  $z$ . We have

$$\frac{dF}{dz} = \frac{1}{z} = \lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h}.$$

Noting that  $F(z+h) - F(z) = \int_\eta \frac{1}{w} dw$ , we can parameterize  $\eta$  as  $w = (1-s)z + s(z+h) = z + sh$ .



Then

$$\begin{aligned}
 \int \eta \frac{1}{w} dw &= \int_0^1 \frac{1}{z+sh} h ds \\
 \Rightarrow \frac{1}{h} \int_0^1 \frac{1}{w} dw &= \int_0^1 \frac{1}{z+sh} ds \\
 &= \int_0^1 \left( \frac{1}{z} + \frac{1}{z+sh} - \frac{1}{z} \right) ds \\
 &= \frac{1}{z} - \frac{h}{z} \int_0^1 \frac{d}{z+sh} ds \\
 &\xrightarrow{h \rightarrow 0} \frac{1}{z}.
 \end{aligned}$$

**Part 2:**

Note that  $(ze^{F(z)})' = e^{F(z)} + ze^{-F(z)}\left(-\frac{1}{z}\right) = 0$ .

**Part 3:**

To do. ■

Next time: once we have the log we can say more about the argument principle.

## 18 Friday March 6th

Recall  $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} dx$ . Define  $\mathcal{F}_a = ??$ .

**Definition 18.1** (Decay).

$f \in \mathcal{F}_a$  iff 1.  $f$  is holomorphic in the strip  $S_a = \{z = x + iy \mid |y| < a\}$ . 2. There exists an  $A > 0$  such that  $|f(x + iy)| \leq \frac{A}{1 + x^2}$ .

Examples:

- $e^{-z^2} \in \mathcal{F}_a$  for all  $a$
- $\frac{1}{c^2 + z^2} \in \mathcal{F}_a$  for all  $a > c$
- $\frac{1}{\cosh(\pi z)} \in \mathcal{F}_a$  for  $a < \frac{1}{2}$ .

**Lemma 18.1.**

If  $f \in \mathcal{F}_a$ , then  $f^{(n)}(z) \in \mathcal{F}_b$  for all  $b < a$ .

**Theorem 18.2.**

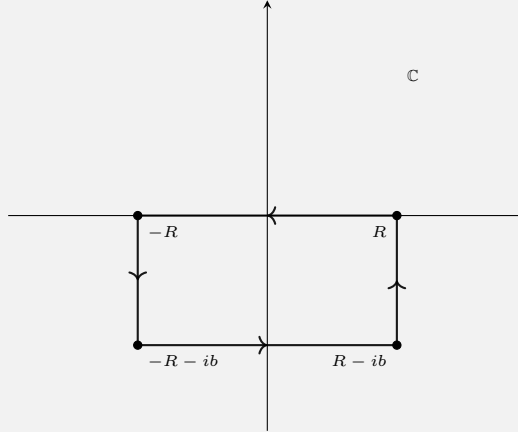
If  $f \in \mathcal{F}_a$ , then  $|\widehat{f}(\xi)| \leq B e^{-2\pi b|\xi|}$  for some constants  $b, B$ .

*Proof.*

If  $\xi = 0$ ,

$$\begin{aligned} |\widehat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} \right| \\ &\leq \int_{\mathbb{R}} |f(x)| \, dx \\ &\leq A \int_{\mathbb{R}} \frac{1}{1+x^2} \, dx \\ &= A\pi. \end{aligned}$$

For  $\xi > 0$ , integrate over the box  $[-R, R] \times i[-b, 0]$ :



Define  $g(z) = f(z)e^{-2\pi i z \cdot \xi}$ . The integral over the rectangle is zero, since  $g$  is holomorphic, so we can equate

$$\int_R^{R-ib} f(z) e^{-2\pi i z \cdot \xi} \, dz = \int_0^b f(R - it) e^{-2\pi i (R - it) \cdot \xi} (-i) \, dt$$

We can use the estimate in  $\mathcal{F}_a$  to obtain

$$\begin{aligned} \int_0^b \dots &\leq \int_0^b \frac{A}{1+R^2} e^{-2\pi s \xi} \, ds \\ &\leq O(R^{-2}). \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} \, d\xi &= \int_{-\infty - ib}^{\infty - ib} \dots \, dz \\ &= \int_{\mathbb{R}} f(x - ib) e^{2\pi i (x - ib) \cdot \xi} \, dx \\ &\leq \int_{\mathbb{R}} \frac{A}{1+x^2} e^{-2\pi b \xi} \, dx \\ &= A\pi e^{-2\pi b \xi}, \end{aligned}$$

so we can take  $B = A\pi$ .

For  $\xi > 0$ , the same argument works with the rectangle above the axis. ■

**Theorem 18.3.**

If  $f \in \mathcal{F}_a$ , then  $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$ .

*Proof.*

Letting  $L_1 = \{x - ib\}$  and  $L_2 = \{x + ib\}$

$$\begin{aligned}
 I &= \int_0^\infty \widehat{f} \cdots + \int_{-\infty}^0 \widehat{f} \cdots \\
 &= \int_0^\infty e^{2\pi i x \cdot \xi} \left( \int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi + \int_{-\infty}^0 e^{2\pi i x \cdot \xi} \left( \int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi \\
 &= \int_{L_1} \int_0^\infty e^{2\pi i x \xi - 2\pi i (s-ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i x \cdot \xi - 2\pi i (s+ib)\xi} d\xi ds \\
 &\quad \text{by absolute convergence, where } z = s - ib \\
 &= \int_{L_1} f(z) \int_0^\infty e^{2\pi i (x-s+ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i (x-s+ib)\xi} d\xi ds \\
 &= \int_{L_1} f(z) \frac{1}{2\pi i (x - i + ib)} ds + \int_{L_2} f(z) \frac{1}{2\pi i (x - s - ib)} \\
 &= \frac{1}{2\pi i} \int \frac{f(z)}{z - x} dz \\
 &= f(x),
 \end{aligned}$$

noting that

$$\int_0^\infty e^{as} ds = \frac{1}{a} \quad \text{for } \Re(a) > 0.$$

■

Note the similar trick: for  $\xi < 0$ , move up, and  $\xi > 0$  move down to form a rectangle. Use the fact that integration along the vertical edges is zero.

## 19 Appendix

$$dz = dx + i dy$$

$$d\bar{z} = dx - i dy$$

$$f_z = f_x = i^{-1} f_y$$

$$\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}.$$

- Holomorphic: once complex differentiable in neighborhoods of every point.

- Analytic: equal to its Taylor series expansion

**Cauchy Inequality:** Given  $z_0 \in \Omega$ , pick the largest disc  $D_R(z_0) \subset \Omega$  and let  $C_R = \partial D_R$ . Using the integral formula, defining  $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

Collection of facts used on problem sets

**Standard forms of conic sections:**

- Circle:  $x^2 + y^2 = r^2$
- Ellipse:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola:  $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ 
  - Rectangular Hyperbola:  $xy = \frac{c^2}{2}$ .
- Parabola:  $-4ax + y^2 = 0$ .

Mnemonic: Write  $f(x, y) = Ax^2 + Bxy + Cy^2 + \dots$ , then consider the discriminant  $\Delta = B^2 - 4AC$ :

- $\Delta < 0 \iff$  ellipse
  - $\Delta < 0$  and  $A = C, B = 0 \iff$  circle
- $\Delta = 0 \iff$  parabola
- $\Delta > 0 \iff$  hyperbola

**Completing the square:**

$$\begin{aligned} x^2 - bx &= (x - s)^2 - s^2 \quad \text{where } s = \frac{b}{2} \\ x^2 + bx &= (x + s)^2 - s^2 \quad \text{where } s = \frac{b}{2}. \end{aligned}$$

**Useful Properties**

- $\Re(z) = \frac{1}{2}(z + \bar{z})$  and  $\Im(z) = \frac{1}{2i}(z - \bar{z})$ .
- $z\bar{z} = |z|^2$
- $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$
- $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ .

**Useful Series**

$$\begin{aligned}\sum_{k=1}^n k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^n k^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

**Cauchy-Riemann Equations**

$$\begin{aligned}u_x &= v_y \quad \text{and} \quad u_y = -v_x \\ \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

**19.1 Useful Techniques**

**Showing a function is constant:** Write  $f = u + iv$  and use Cauchy-Riemann to show  $u_x, u_y = 0$ , etc.

**Deriving Polar Cauchy-Riemann:** See walkthrough here. Take derivative along two paths, along a ray with constant angle  $\theta_0$  and along a circular arc of constant radius  $r_0$ . Then equate real and imaginary parts. See problem set 1.

**Computing Arguments:**  $\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w)$ .

The sum of the interior angles of an  $n$ -gon is  $(n-2)\pi$ , where each angle is  $\frac{n-2}{n}\pi$ .

**19.2 Residues**

If  $p$  is a simple pole,  $\text{Res}(p, f) = \lim_{z \rightarrow p} (z-p)f(z)$ . Example: Let  $f(z) = \frac{1}{1+z^2}$ , then  $\text{Res}(i, f) = \frac{1}{2i}$ .

Green's Theorem: Todo

$$\frac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j.$$