

1a) Note that if $x \in C$ is an endpoint of a removed interval, then $x = k/3^n$ for some integers $n \geq 1$ and $0 \leq k \leq 3^n$. So we just need a real number $x \in (0, 1)$ satisfying

a) x has some ternary expansion

$$x = \sum_{i=1}^{\infty} a_i 3^{-i} \quad \text{where } a_i \neq 1 \text{ for any } i, \text{ and}$$

b) $x \neq k/3^n$ for any $k, n \in \mathbb{N}^{>0}$,

then we will have $x \in C$ by (a) and x not an endpoint by (b).

Claim: $x = (0.\overline{02})_3 = (0.020202\cdots)_3$ works.



Pf: By construction, x satisfies

(a)
$$x = \sum_{i=1}^{\infty} a_i 3^{-i}, \quad a_i \in \{0, 2\}$$

So no $a_i = 1$ and thus $x \in C$.

(b) To see that x satisfies (b), we can compute

$$\begin{aligned}x &= (0.020202 \dots)_3 \\&= 0 \cdot 3^{-1} + 2 \cdot 3^{-2} + 0 \cdot 3^{-3} + 2 \cdot 3^{-4} + \dots \\&= \sum_{i=1}^{\infty} 2 \cdot 3^{-2i} = 2 \sum_{i=1}^{\infty} 3^{-2i} = 2 \sum_{i=1}^{\infty} \left(\frac{1}{9}\right)^i \\&= 2 \left(-1 + \sum_{i=0}^{\infty} \left(\frac{1}{9}\right)^i\right) \\&= 2 \left(-1 + \frac{1}{1 - \frac{1}{9}}\right) = 1/4,\end{aligned}$$

where $4 \nmid 3^n$ for any integer n .



(1b) If a set X is nowhere dense in a topological space, it equivalently satisfies

$$(\overline{X})^\circ = \emptyset$$

(i.e., the interior of the closure is empty.)

It then suffices to show that

a) C is closed, so $\overline{C} = C$, and

b) C has no interior points, so $C^\circ = \emptyset$.

(a) To see that C is closed, we will show $C^c := [0, 1] \setminus C$ is open. An arbitrary union of open sets is open, so the claim is that $C^c = \bigcup_{j \in J} A_j$ for some collection of open sets $\{A_j\}_{j \in J}$.

Consider C_n , the n^{th} stage of the process used to construct the Cantor set, so $C = \bigcap_{i=1}^{\infty} C_n$.

But by induction, C_n^c is a union of open sets.

In particular, $C_1^c = (\frac{1}{3}, \frac{2}{3})$, and

$$C_n^c = \underbrace{\left(\bigcup_{i=1}^{n-1} C_i^c \right)}_{\text{Open by hypothesis}} \cup \underbrace{\left(\text{Exactly } n \text{ open intervals that were deleted} \right)}_{\text{open by construction}},$$

So C_n^c is open for each n . But then

$$C^c = \left(\bigcap_{n=1}^{\infty} C_n \right)^c = \bigcup_{i=1}^{\infty} C_n^c$$

is a union of open sets and thus open. So C is closed.

(b) To see that $C^\circ = \emptyset$, suppose towards a contradiction that $x \in C^\circ$, so there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon) \not\subseteq C$. Letting $\mu(I)$ denote the length of an interval, we have $\mu(N_\varepsilon(x)) = 2\varepsilon > 0$.

Claim: Let $L_n := \mu(C_n)$, then $L_n = \left(\frac{2}{3}\right)^n$.

This follows immediately by noting that L_n satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3}L_n, \quad L_0 = 1$$

Since an interval of length $\frac{1}{3}L_{n-1}$ is removed at the n^{th} stage, which has the unique claimed solution.

But if $I_1 \subseteq I_2$ are real intervals, we must have

$$\mu(I_1) \leq \mu(I_2), \text{ whereas if we choose } n \text{ large}$$

Uses subadditivity
of measure

enough such that $(\frac{2}{3})^n < 2\varepsilon$, we have

$$(x-\varepsilon, x+\varepsilon) \not\subseteq C = \bigcap_{i=1}^{\infty} C_i \Rightarrow \underline{(x-\varepsilon, x+\varepsilon) \subseteq C_n}, \text{ but}$$

$$\mu((x-\varepsilon, x+\varepsilon)) = \underline{2\varepsilon} > \underline{(\frac{2}{3})^n} = \mu(C_n), \text{ a contradiction.}$$

So such an $x \in C^\circ$ can't exist, and $C^\circ = \emptyset$.

Thus $(\bar{C})^\circ = C^\circ = \emptyset$, and C is nowhere dense,

and since a meager set is a countable union of nowhere dense sets, C is meager. \square

Claim: C is measure zero.

Measures are additive over disjoint sets, i.e.

$$A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B),$$

And if $A \subseteq B$, we have

$$\begin{aligned} \mu(B) &= \mu(B \sqcup (B \setminus A)) = \mu(B) + \mu(B \setminus A) \\ &\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A). \end{aligned}$$

Now let B_n be the union of the intervals that are deleted at the n^{th} step. We have

$$\mu(B_0) = 0$$

$$\mu(B_1) = 1/3$$

$$\mu(B_2) = 2(1/9) = 2/9$$

$$\mu(B_3) = 4(1/27) = 4/27$$

\vdots

$$\mu(B_n) = 2^{n-1}/3^n$$

Moreover, if $i \neq j$, then $B_i \cap B_j = \emptyset$, and

$$C^c := [0, 1] - C = \bigcup_{i=1}^{\infty} B_i.$$

We thus have

$$\mu(C) = \mu([0, 1]) - \mu(C^c)$$

$$= 1 - \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= 1 - \sum_{n=1}^{\infty} \mu(B_n)$$

$$= 1 - \sum_{n=1}^{\infty} 2^{n-1}/3^n$$

$$\begin{aligned}
&= 1 - (1/3) \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\
&= 1 - (1/3) (1/1 - 2/3) \\
&= 0. \quad \blacksquare
\end{aligned}$$

(1c) Let $y \in [0, 1]$ be arbitrary, we will produce an $x \in C$ such that $f(x) = y$.

Write $y = (a_1 a_2 \dots)_2 = \sum_{i=1}^{\infty} a_i 2^{-i}$ where $a_i \in \{0, 1\}$

Now define

$$x = (2a_1 2a_2 \dots)_3 = \sum_{i=1}^{\infty} (2a_i) 3^{-i} := \sum_{i=1}^{\infty} b_i 3^{-i}$$

Since $a_i \in \{0, 1\}$, $b_i = 2a_i \in \{0, 2\}$, meaning x has no 1^s in its ternary expansion and so $x \in C$.

Moreover, under f we have

$$\left. \begin{array}{l} b_i \mapsto \frac{1}{2} b_i \\ \parallel \quad \parallel \\ 2a_i \mapsto \frac{1}{2} (2a_i) = a_i \end{array} \right\} \begin{array}{l} \text{So } b_i \mapsto a_i \text{ and} \\ \text{thus } f(x) = y. \end{array}$$

So $C \twoheadrightarrow [0, 1]$, which is uncountable, thus so is C . \blacksquare

(2a) (\Rightarrow) Suppose X is G_δ , so $X = \bigcup_{n=1}^{\infty} A_n$ with each A_n closed. Then A_n^c is open by definition, and so

$$X^c = \left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

is a countable intersection of open sets, and thus F_σ .

(\Leftarrow) Suppose X^c is an F_σ , so $X^c = \bigcup_{n=1}^{\infty} B_n$ with each

B_n open. Then each B_n^c is closed by definition, and

$$X = (X^c)^c = \left(\bigcup_{n=1}^{\infty} B_n \right)^c = \bigcap_{n=1}^{\infty} B_n^c$$

is a countable intersection of closed sets, and thus G_δ .

(2b) Suppose X is closed, we will show $X = \bigcap_{n=1}^{\infty} C_n$ with each C_n open. For each $x \in X$ and $n \in \mathbb{N}$, define

- $B_n(x) = \left\{ y \in \mathbb{R}^n \mid d(x, y) < \frac{1}{n} \right\}$

- $C_n = \bigcup_{x \in X} B_n(x)$

- $W = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in X} B_n(x)$

Since each $B_n(x)$ is open by construction and C_n is a union of opens, each C_n is open.

Claim: $W = X$.

$X \subseteq W$: If $x \in X$, then $x \in B_n(x) \subseteq C_n$ for all n , and so
$$x \in \bigcap_{n=1}^{\infty} C_n = W.$$

$W \subseteq X$: Suppose there is some $w \in W \setminus X$ (so $w \neq x$ for any $x \in X$) towards a contradiction.

Since $w \in \bigcap_{i=1}^{\infty} C_n$, $w \in C_n$ for every n . So $w \in \bigcup_{x \in X} B_n(x)$ for every n . But then there is some particular $x_0 \in X$ such that $w \in B_n(x_0)$ for every n (otherwise we could take N large enough so that $w \notin B_N(x)$ for any $x \in X$, so $w \notin \bigcup_{x \in X} B_N(x)$ where $w \neq x_0$).

But then if $N_\varepsilon(x)$ is an arbitrary neighborhood of x , we can take $\frac{1}{n} < \varepsilon$ to obtain $w \in B_n(x) \subseteq N_\varepsilon(x)$, which makes w a limit point of X . But since X is closed, it contains its limit points, forcing the contradiction $w \in X$.

So X is a countable intersection of open sets, and thus a G_δ set.



Now suppose X is open. Then X^c is closed, and thus a G_δ set. But then $(X^c)^c = X$ is an F_σ set by problem (2a). \blacksquare

(2c) Using the fact that singletons are closed in metric spaces, we can write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ as a countable union of closed sets, so \mathbb{Q} is an F_δ set. Suppose \mathbb{Q} was also a G_δ set, so $\mathbb{Q} = \bigcap_{i=1}^{\infty} A_i$ with each A_i open. Then for any fixed n , $\mathbb{Q} \subseteq A_i$, so A_i is dense in \mathbb{R} for every i .

However, it is also true that $\{q\}^c := \mathbb{R} \setminus \{q\}$ is an open, dense subset of \mathbb{R} , and we can write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$$

as an intersection of open dense sets; since \mathbb{R} is a

Baire space, countable intersections of open dense sets are dense.

$$\text{But then } \left(\bigcap_{i=1}^{\infty} A_i \right) \cap \left(\bigcap_{q \in \mathbb{Q}} \{q\}^c \right) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$$

must be dense in \mathbb{R} , which is absurd. \otimes

Note that this argument also works when \mathbb{R} is replaced with any open interval I and \mathbb{Q} is replaced with $\mathbb{Q} \cap I$.

For a set that is neither G_δ nor F_σ , consider

$$A = \mathbb{Q} \cap (0, \infty) \quad , \quad \text{positive rationals}$$

$$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0) \quad , \quad \text{negative irrationals}$$

A is F_σ but not G_δ , using above argument, and

dually B is G_δ but not F_σ .

Claim: $X = A \cup B$ is neither G_δ nor F_σ .

Suppose X is G_δ . Then $X \cap \overbrace{(0, \infty)}^{\text{open}} = A$ is G_δ as well. #

Suppose X is F_σ . Then X^c is G_δ , but

$$X^c = (A \cup B)^c = A^c \cap B^c = (\mathbb{Q} \cap (-\infty, 0)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, \infty))$$

and thus $X^c \cap \overbrace{(-\infty, 0)}^{\text{open}} = A$ is G_δ . #

So X is neither G_δ or F_σ .



3a) Claim: $c \in [0, 1] \Rightarrow \lim_{x \rightarrow c} f(x) = 0$.

This holds iff $\forall c \in I, \forall \varepsilon, \exists \delta$ s.t. $|x - c| < \delta \Rightarrow |f(x)| < \varepsilon$,

so let $\varepsilon > 0$ be arbitrary. Consider the set

$S = \{n \in \mathbb{N} \mid \frac{1}{n} \geq \varepsilon\}$, which is a finite set, and so

$S_q = \{r_n \in \mathbb{Q} \mid \frac{1}{n} \geq \varepsilon\}$ is finite as well.

So choose $\delta < \min_{r_n \in S_q} d(c, r_n)$ so $N_\delta(c) \cap S_q = \emptyset$

Then $|x - c| < \delta \Rightarrow \begin{cases} \cdot f(x) = 0 \text{ if } x \in I \setminus \mathbb{Q}, \text{ or} \\ \cdot x = r_m \in (\mathbb{Q} \setminus S_q) \cap I \text{ for some } m \text{ such that} \\ \quad \frac{1}{m} < \varepsilon \text{ by construction.} \end{cases}$

But then $|f(x)| = \frac{1}{m} < \varepsilon$ as desired. \square

So $\cdot c \in I \setminus \mathbb{Q} \Rightarrow f(c) = 0 = \lim_{x \rightarrow c} f(x)$,

$\cdot c = r_n \in I \cap \mathbb{Q} \Rightarrow f(c) = \frac{1}{n} \neq 0 = \lim_{x \rightarrow c} f(x)$

and f is discontinuous on $I \cap \mathbb{Q}$. \blacksquare

3b.1 Claim: w_f is well-defined

This amounts to showing that the sup and limit exist in

$$w_f(x) = \lim_{\delta \rightarrow 0^+} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|$$

Let $x \in \mathbb{R}$ be arbitrary and δ fixed.

Since f is bounded, there is some M such that

$$\forall y \in \mathbb{R}, |f(y)| < M, \text{ and so}$$

$$\begin{aligned} y, z \in \mathbb{R} \Rightarrow |f(y) - f(z)| &= |f(y) + (-f(z))| \leq |f(y)| + |-f(z)| \\ &= |f(y)| + |f(z)| < 2M, \end{aligned}$$

which holds for $y, z \in B_\delta(x) \subseteq \mathbb{R}$ as well.

And so $\{|f(y) - f(z)| \text{ s.t. } y, z \in B_\delta(x)\}$ is bounded above and thus has a least upper bound, and thus the following supremum exists.

$$S(\delta, x) = \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|$$

To see that the $\lim_{\delta \rightarrow 0} S(\delta, x)$ exists, note that

$$\delta_1 \leq \delta_2 \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$$

and so for a fixed x , $S(\delta, x)$ is a monotonically

decreasing function of δ that is bounded below by 0, which converges by the monotone convergence theorem. \square

Claim: f is continuous at x iff $\omega_f(x) = 0$.

(\Leftarrow) Suppose $\omega_f(x) = 0$ and let $\varepsilon > 0$ be arbitrary; we will produce a δ to use in the definition of continuity.

Since $\omega_f(x) = \lim_{\delta \rightarrow 0^+} S(\delta, x) = 0$, we can choose δ such that

$$\delta < \delta \Rightarrow |S(\delta, x)| < \varepsilon, \quad \text{which means}$$

$$\delta < \delta \Rightarrow \sup_{y, z \in B_\delta(x)} |f(y) - f(z)| < \varepsilon$$

So fix $z = x$ and let y vary, yielding

$$\delta < \delta \Rightarrow \sup_{y \in B_\delta(x)} |f(y) - f(x)| < \varepsilon$$

But now for an arbitrary $t \in B_\delta(x)$, we have $|x - t| < \delta$ and

$$|f(x) - f(t)| \leq \sup_{y \in B_\delta(x)} |f(x) - f(y)| < \varepsilon,$$

which exactly says $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$. \square

(\Rightarrow) Suppose f is continuous at x and let $\varepsilon > 0$ be arbitrary; we will show $\omega_f(x) < \varepsilon$.

Since f is continuous, choose δ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

We then have

$$y, z \in B_\delta(x) \Rightarrow |x - y| < \delta \quad \text{and} \quad |x - z| < \delta,$$

$$\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x) - f(z)| < \frac{\varepsilon}{2}$$

$$\Rightarrow |f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so

$$y, z \in B_\delta(x) \Rightarrow |f(y) - f(z)| < \varepsilon \quad \Rightarrow \sup_{y, z \in B_\delta(x)} |f(y) - f(z)| \leq \varepsilon$$

$$\Rightarrow S(\delta, x) \leq \varepsilon,$$

and since $S(d, x)$ is monotonically decreasing in d ,

$$\omega_f(x) = \lim_{d \rightarrow 0^+} S(d, x) \leq S(\delta, x) \leq \varepsilon$$

as desired. 

3b.2

We will show that

$$A_\varepsilon^c = \{x \in \mathbb{R} \mid \omega_f(x) < \varepsilon\}$$

is open by showing every point is an interior point.

Fix $\varepsilon > 0$ and let $x \in A_\varepsilon^c$ be arbitrary. We want to produce a δ such that

$$B_\delta(x) \subseteq A_\varepsilon^c \quad \text{or equivalently} \quad |y-x| < \delta \Rightarrow \omega_f(y) < \varepsilon.$$

Write $\omega_f(x) = \lim_{d \rightarrow 0^+} S(d, x)$; since $\omega_f(x) < \varepsilon$ and this limit exists, we can choose δ such that

$$d < \delta \Rightarrow |S(d, x) - 0| < \varepsilon \Rightarrow |S(d, x)| < \varepsilon.$$

Now suppose $y \in B_\delta(x)$, so $|y-x| < \delta$. Then there exists some δ' such that $B_{\delta'}(y) \subset B_\delta(x)$, and we claim that

$$S(\delta', y) \leq S(\delta, x)$$

Note that if this is true, then

$$\omega_f(y) = \lim_{d \rightarrow 0} S(d, y) \leq S(\delta', y) \leq S(\delta, x) < \varepsilon.$$

S is monotonically decreasing in d

To see why this is true, we just note that

$$a, b \in B_{\delta'}(y) \subset B_{\delta}(x) \Rightarrow a, b \in B_{\delta}(x)$$


$$\Rightarrow \sup_{a, b \in B_{\delta'}(y)} |f(y) - f(z)| \leq \sup_{y, z \in B_{\delta}(x)} |f(y) - f(z)|,$$

Since the supremum can only increase over a larger set.

So $w_f(y) < \varepsilon$ as desired. 

Finally, note that if $D_f = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$,

$$\begin{aligned} \text{then } D_f = \{x \in \mathbb{R} \mid w_f(x) \neq 0\} &= \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid w_f(x) \geq \frac{1}{n}\} \\ &= \bigcup_{n=1}^{\infty} A_{\frac{1}{n}} \end{aligned}$$

is a countable union of closed sets and thus F_{σ} . 

④ Claim: f is increasing, i.e. $x \leq y \Rightarrow f(x) \leq f(y)$

Fix $x \in \mathbb{R}$, and define

$$A_x := \{ t \in X \mid x > t \}, \quad A_x^c := \{ t \in X \mid x \leq t \}.$$

(Note that $t \in A_x$ or $t \in A_x^c \Rightarrow t = x_n$ for some n , and $X = A_x \sqcup A_x^c$.)

Then noting that

$$\begin{aligned} x_n \in A_x &\Rightarrow f_n(x) \equiv 1 \\ &\text{and} \\ x_n \in A_x^c &\Rightarrow f_n(x) \equiv 0, \end{aligned}$$

We can write

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \cdot 1 + \sum_{\{n \mid x_n \in A_x^c\}} \frac{1}{n^2} \cdot 0 \\ &= \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2}. \end{aligned}$$

Now if $y \geq x$, then $y \geq t$ for every $t \in A_x$, so $A_y \supseteq A_x$.

But then

$$f(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \leq \sum_{\{n \mid x_n \in A_y\}} \frac{1}{n^2} = f(y),$$

where the inequality holds because

$$\begin{aligned} A_x \subseteq A_y &\Rightarrow \{n \mid x_n \in A_x\} \subseteq \{n \mid x_n \in A_y\} \\ &\Rightarrow |\{n \mid x_n \in A_x\}| \leq |\{n \mid x_n \in A_y\}|, \end{aligned}$$

so the latter sum has at least as many terms and everything is positive. So $f(x) \leq f(y)$.

Claim: f is continuous on $\mathbb{R} \setminus X$ since

$$\sum f_n \xrightarrow{u} f \text{ and each } f_n \text{ is continuous there.}$$

Since $|f_n(x)| \leq 1$ by definition, and

$$|f_n(x)/n^2| \leq 1/n^2 := M_n \text{ where } \sum M_n < \infty,$$

$$\sum f_n \xrightarrow{u} f \text{ by the M test.}$$

Note that for a fixed n , $D_{f_n} = \{x_n\}$. This is

because if we take a sequence $\{y_i\} \rightarrow x_n$ with each $y_i > x_n$, then $f(y_i) = 1$ for every i , and

$$\lim_{i \rightarrow \infty} f(y_i) = \lim_{i \rightarrow \infty} 1 = 1 \neq f(\lim_{i \rightarrow \infty} y_i) = f(x_n) = 0$$

So f_n is not continuous at $x = x_n$. Otherwise, either

$x > x_n$ or $x < x_n$, in which case we can let ε be arbitrary and choose $\delta < |x - x_n|$ to get

$$y \in B_\delta(x) \Rightarrow \begin{cases} y > x_n \Rightarrow |f(y) - f(x)| = |0 - 0| < \varepsilon \\ y < x_n \Rightarrow |f(y) - f(x)| = |1 - 1| < \varepsilon. \end{cases}$$

Letting $F_N = \sum_{n=1}^N f_n$, we find that

$$F_N = \underset{\substack{\uparrow \\ \text{discontinuous at: } \{x_1\} \cup \{x_2\} \cup \dots \cup \{x_N\}}}{f_1} + \underset{\substack{\uparrow}}{f_2} + \dots + \underset{\substack{\uparrow}}{f_N} \quad \left\{ \begin{array}{l} \text{So } F_N \text{ is continuous on} \\ \mathbb{R} \setminus \bigcup_{i=1}^N \{x_N\}. \end{array} \right.$$

and since $\mathbb{R} \setminus X \subseteq \mathbb{R} \setminus \bigcup_{i=1}^N \{x_N\}$, F_N is continuous there too.

But then $f = \text{uniform limit } (F_N)$ is continuous on $\mathbb{R} \setminus X$. \blacksquare

5a) Let $X = (C(I), \|\cdot\|_\infty)$ where $I = [0, 1]$,

$C(I) = \{f: I \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$, and

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in I} |f(x) - g(x)|.$$

Claim: X is a metric space.

1) $d(f, g) = 0 \Rightarrow f = g$

If $\sup_{x \in I} |f(x) - g(x)| = 0$ then $|f(x) - g(x)| = 0 \quad \forall x \in \mathbb{R}$,

so $f(x) = g(x) \quad \forall x \in \mathbb{R}$ and $f = g$.

2) $d(f, g) = d(g, f)$

We have $d(f, g) = \sup_{x \in I} |f(x) - g(x)|$

$$\sup_{x \in I} |g(x) - f(x)|$$

$$= d(g, f).$$

3) $d(f, h) \leq d(f, g) + d(g, h)$

We have $d(f, g) = \sup_{x \in I} |f(x) - g(x)|$

$$= \sup_{x \in I} |f(x) - h(x) + h(x) - g(x)|$$

$$\begin{aligned}
&\leq \sup_{x \in I} (|f(x) - h(x)| + |h(x) - g(x)|) \quad \leftarrow \Delta\text{-ineq in } \mathbb{R} \\
&= \sup_{x \in I} |f(x) - h(x)| + \sup_{x \in I} |h(x) - g(x)| \\
&= d(f, h) + d(h, g).
\end{aligned}$$

So X is a metric space. \square

Claim: X is complete.

Let $\{f_i\}$ be a Cauchy sequence in X , we will show that it converges in X . Since $\{f_i\}$ is Cauchy in X , we have

$$\forall \varepsilon > 0, \exists N_0 \mid n \geq m \geq N_0 \Rightarrow \|f_n - f_m\|_\infty < \varepsilon$$

First we will define a candidate limit function f , then show $f \in X$.

1) Define $f := \lim_{n \rightarrow \infty} f_n$ by $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

This is well-defined; let $S_x = \{f_i(x)\} \subseteq \mathbb{R}$ for a fixed x , and we claim S_x is Cauchy in $\underline{\mathbb{R}}$, which is complete.

This follows because if $\{f_i\}$ is Cauchy in X , then

$$|f_n(x) - f_m(x)| \leq \sup_{x \in I} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty \rightarrow 0.$$

2) $f \in X$, for which it suffices to show f is continuous.

Let $\varepsilon > 0$, and since $\{f_i\}$ is Cauchy, choose N_0 large s.t.


$$n \geq N_0 \Rightarrow \|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Now fix $n \geq N_0$; since f_n is continuous,
choose δ such that

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

Then

$$\begin{aligned} |x - y| < \delta &\Rightarrow |f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \sup_{x \in I} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + \sup_{y \in I} |f_n(y) - f(y)| \\ &= \|f - f_n\|_\infty + |f_n(x) - f_n(y)| + \|f_n - f\|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

So f is continuous, $f = \lim f_n \in X$, and X is complete. 

5b Let $B = \{f \in X \mid \|f\|_\infty \leq 1\}$

Claim: B is closed.

Let f be a limit point of B , so there is some sequence

$f_n \rightarrow f$ in X with each $f_n \in B$ so $\|f_n\|_\infty \leq 1 \forall n$.

Let $\varepsilon > 0$, and since $f_n \rightarrow f$ in X , choose N_0 such that

$$n \geq N_0 \Rightarrow \|f_n - f\| < \varepsilon$$

Then,

$$\begin{aligned} \|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &< \varepsilon + 1, \end{aligned}$$

and taking $\varepsilon \rightarrow 0$ yields $\|f\|_\infty \leq 1$. \square

Claim: B is bounded

A subset $B \subseteq X$ is bounded iff there is some $x \in X$ and

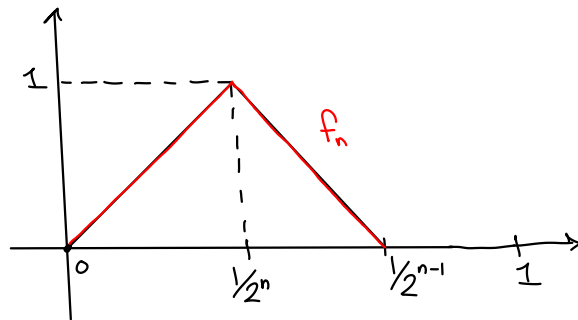
some $r > 0$ in \mathbb{R} where $B \subset N(r, x) = \{y \in X \mid d(y, x) < r\}$.

Choose $x=0$, $r=2$, then $f \in B \Rightarrow d(f, 0) = \|f-0\|_\infty = 1 < 2$, so $f \in N(2, 0)$.

Claim: B is not compact.

Since B is a metric space, B is compact iff B is sequentially compact.

Define f_n as the triangle:



Then $f_n \xrightarrow{\mathbb{R}} f$ where $f(x) = \begin{cases} 1, & x=0 \\ 0, & x \in (0, 1] \end{cases}$,
Pointwise in \mathbb{R}

and so $\forall n$, $\|f_n - f\|_\infty = 1$, attained at $x=0$. So $\lim_{n \rightarrow \infty} \|f_n - f\|_\infty \neq 0$,

and $\{f_n\}$ does not converge in X , nor can any subsequence. ■

Claim: B is not totally bounded.

If it were, $\forall \varepsilon$ there would exist a finite collection

$\{g_i\}_{i=1}^N \subseteq B$ such that $B \subseteq \bigcup_{i=1}^N N(\varepsilon, g_i)$ where

$$N(\varepsilon, g_i) = \{h \in B \mid \|h - g_i\| < \varepsilon\}.$$

Note that if $h_1, h_2 \in N(\varepsilon, g_i)$ then $\|h_1 - h_2\| \leq \|h_1 - g_i\| + \|g_i - h_2\| < 2\varepsilon$.

So choose $\varepsilon = \frac{1}{2}$, and consider the collection $\{f_n\}_{n=1}^{\infty}$.

Since $\|f_n - f_m\| = 1$, each $N(\varepsilon, g_i)$ can contain at most one

f_n , since $f_n, f_m \in N(\varepsilon, g_i)$ for $n \neq m$ would

imply $\|f_n - f_m\|_{\infty} < 2\varepsilon = 2(\frac{1}{2}) = 1$. But there are finitely

many $N(\varepsilon, g_i)$ and infinitely many f_n , so if this is

a cover of B , so $N(\varepsilon, g_i)$ must contain at least 2 f_n . \nexists

(6a) Claim: If $\sum g_n \xrightarrow{u} G$, then $g_n \xrightarrow{u} 0$.

Let $G_N = \sum_{n=1}^N g_n$ and $G = \lim_{N \rightarrow \infty} G_N$.

Suppose $G_N \xrightarrow{u} G$, then choose N large enough so that

$$\forall x \in X, n \geq N \Rightarrow |G_n(x) - G(x)| < \frac{\varepsilon}{2}$$

Then letting $n > n-1 > N$, we have

$$\begin{aligned} |g_n(x)| &= \left| \sum_{i=1}^n g_i(x) - \sum_{i=1}^{n-1} g_i(x) \right| \\ &= \left| \left(\sum_{i=1}^n g_i(x) - G(x) \right) - \left(\sum_{i=1}^{n-1} g_i(x) - G(x) \right) \right| \\ &\leq \left| \sum_{i=1}^n g_i(x) - G(x) \right| + \left| \sum_{i=1}^{n-1} g_i(x) - G(x) \right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

So $\forall x \in X, |g_n(x)| < \varepsilon \Rightarrow g_n \xrightarrow{u} 0. \quad \square$

Now let $g_n = 1/(1+n^2x)$, we'll show g_n does not converge to 0 uniformly.

Note $g_n \xrightarrow{u} g$ iff $\forall \varepsilon, \exists N_0 \mid \forall x, n \geq N_0 \Rightarrow |g_n(x) - g(x)| < \varepsilon$,

so let $\varepsilon < \frac{1}{2}$, N_0 be arbitrary, and choose $x_0 < 1/N_0^2$. Then,

$$|g_{N_0}(x_0)| = \frac{1}{|1 + N_0^2 x|} = \frac{1}{|1 + N_0^2 (1/N_0^2)|} = \frac{1}{2} > \varepsilon. \quad \square$$

Claim: g is continuous on $(0, \infty)$.

Let $x \in (0, \infty)$ be arbitrary, and choose $a < x$. We will show

g converges uniformly on $[a, \infty)$, and since each g_n is continuous

on $[a, \infty)$ as well, g will be the uniform limit of continuous

functions and thus continuous itself.

We can use the M-test. Since $x > a$,

$$|1/(1+n^2x)| \leq |1/n^2x| \leq |1/n^2a| = \frac{1}{a} \left| \frac{1}{n^2} \right|,$$

$$\text{where } \sum_{n=1}^{\infty} \frac{1}{a} \frac{1}{n^2} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

So g converges uniformly on $[a, \infty)$.

6b) Claim: g is differentiable on $(0, \infty)$.

If $g'(x)$ exists, we have

$$\begin{aligned} g'(x) &= \lim_{a \rightarrow x} (x-a)^{-1} (g(x) - g(a)) \\ &= \lim_{a \rightarrow x} (x-a)^{-1} \sum_{n=1}^{\infty} \frac{-n^2 (x-a)}{(1+n^2 x)(1+n^2 a)} \end{aligned}$$

$$= \lim_{a \rightarrow x} \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2 x)(1+n^2 a)}$$

$$= \sum (-n^2) / (1+n^2 x)^2,$$

which exists because it converges uniformly on $[a, \infty)$, as

$$\left| \frac{-n^2}{(1+n^2 x)^2} \right| \leq \left| \frac{n^2}{(n^2 x)^2} \right| = \left| \frac{1}{n^2 x^2} \right| \leq \left| \frac{1}{a^2 n^2} \right| := M_n$$

$$\text{where } \sum M_n = \sum \frac{1}{a^2 n^2} = \frac{1}{a^2} \sum \frac{1}{n^2} < \infty.$$

So g is continuously differentiable on $(0, \infty)$. \blacksquare

7a) Claim: $h_n \xrightarrow{u} 0$ on $[0, \infty)$

Note that $h'_n(x) = \frac{1-nx}{(1+x)^n} \Rightarrow h'_n = 0$ iff $x = 1/n$ and

$$h''_n(x) = \frac{1+x+nx}{nx^2(1+x)^{n+1}} \quad \text{and} \quad h''_n\left(\frac{1}{n}\right) < 0,$$

so $x = \frac{1}{n}$ is a global maximum and thus

$$\forall x, |h_n(x)| \leq |h_n\left(\frac{1}{n}\right)| = \left| \frac{1/n}{(1+1/n)^n} \right| = \frac{1}{n(1+1/n)^n} \leq \frac{1}{2n} \quad \text{for } n > 1$$

so $\sup_{x \in [0, \infty)} |h_n(x)| = |h_n(1/n)| = O(1/n) \rightarrow 0$, thus $\|h_n\|_\infty \rightarrow 0$

and $h_n \rightarrow 0$ uniformly.

7b) Let $h(x) = \sum_{n=1}^{\infty} h_n(x) = \sum_{n=1}^{\infty} \frac{x}{(1+x)^{n+1}}$

i) Demonstrably, $h(0) = 0$, and for a fixed x we have

$$\begin{aligned} h(x) &= \sum_{n=1}^{\infty} \frac{x}{(1+x)^{n+1}} = \left(\frac{x}{1+x} \right) \sum_{n=1}^{\infty} \left(\frac{1}{1+x} \right)^n \\ &= \frac{x}{1+x} \left(\frac{1}{1 - (1/(1+x))} \right) \quad \text{since } x > 0 \Rightarrow (1/(1+x)) < 1 \\ &= 1. \quad \square \end{aligned}$$

ii) It can not converge uniformly on $[0, \infty)$, otherwise h would be the uniform limit of continuous functions, but h is discontinuous.

7c) Let $a > 0$ and $X = [a, \infty)$.

Claim: $\sum h_n \xrightarrow{u} h$ on X .

Since $x > a$, we have

$$(1+x)^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} x^j \geq 1 + nx + n^2 x^2$$

$x > a > 0$, so positive terms.

$$|h_n(x)| = \left| \frac{x}{(1+x)^{n+1}} \right| \leq \left| \frac{x}{1+nx+n^2x^2} \right| \leq \left| \frac{a}{1+na+n^2a^2} \right| \leq \left| \frac{a}{n^2a^2} \right| = \left| \frac{1}{n^2a} \right|$$

So let $M_n = 1/n^2a$, then $\sum M_n < \infty \Rightarrow \sum h_n \xrightarrow{u} h$

by the M test. \blacksquare

Zack
Garza

① Suppose E is bounded, so $\text{diam}(E) \leq M$ for some fixed M . In particular, if $Q_i \subseteq E$ is an interval, then $|Q_i| \leq M$. Let $\varepsilon > 0$, and choose $\{Q_i\} \Rightarrow E$ s.t.
i.e. $E \subseteq \bigcup_i Q_i$
for each i , $|Q_i| \leq \varepsilon/2M$

Then let $L_i = Q_i^2$. We then have

$$|L_i| \leq |b^2 - a^2| = |b-a| \cdot |b+a| = |Q_i| \cdot |b+a|$$

$$\leq |Q_i| \cdot 2M$$

$$\leq (\varepsilon/2^{i+1}M) 2M$$

$$= \varepsilon/2^i,$$

so $\sum_{i=1}^{\infty} |L_i| \leq \sum_{i=1}^{\infty} \varepsilon/2^i = \varepsilon$, and $\{L_i\} \Rightarrow E^2$, so

$$m_*(E^2) < \varepsilon \rightarrow 0.$$

Claim: It suffices to consider the bounded case.

Ball of
radius n
around 0

Pf If E is not bounded, consider $F_n = E \cap \overbrace{B(n,0)}$.

Then F_n is bounded (by n), and since $F_n \subseteq E \Rightarrow m_*(F_n) \leq m_*(E) = 0$

by subadditivity, $m_*(F_n^2) = 0$ by the bounded case.

But then $E^2 = \bigcup_{n=1}^{\infty} F_n^2 \Rightarrow m_*(E^2) = m(\bigcup_{n=1}^{\infty} F_n^2) \leq \sum_{n=1}^{\infty} m_*(F_n^2) = 0$

by countable subadditivity. \blacksquare

② Note

$$1) E_1 = E_1 \setminus E_2 \sqcup E_1 \cap E_2$$

$$2) E_2 = E_2 \setminus E_1 \sqcup E_1 \cap E_2$$

$$3) E_1 \Delta E_2 = E_2 \setminus E_1 \sqcup E_1 \setminus E_2$$

$$4) E_1 \cup E_2 = (E_1 \Delta E_2) \sqcup (E_1 \cap E_2)$$

All disjoint unions, so we can freely apply measures and use countable additivity.

so

$$m(E_1) + m(E_2) = m(E_1 \setminus E_2) + m(E_1 \cap E_2)$$

$$+ m(E_2 \setminus E_1) + m(E_1 \cap E_2)$$

$$= m(E_1 \Delta E_2) + m(E_1 \cap E_2) + m(E_1 \cap E_2) \quad \left. \begin{array}{l} \text{by (1), (2)} \\ \text{by (3)} \end{array} \right\}$$

$$= m(E_1 \cup E_2) + m(E_1 \cap E_2). \quad \left. \begin{array}{l} \text{by (4)} \end{array} \right\}$$

\blacksquare

3a) Suppose $m(A) = m(B) < \infty$.

Since $A \subseteq E \subseteq B$, we have $E \setminus A \subseteq B \setminus A$. However,

$$B = A \sqcup (B \setminus A) \Rightarrow m(B) = m(A) + m(B \setminus A)$$

$$\Rightarrow m(B) - m(A) = m(B \setminus A)$$

(since $m(A) < \infty$)

$$\Rightarrow m(B \setminus A) = 0$$

(since $m(B) = m(A)$)

So $m_*(E \setminus A) = 0$ by subadditivity.

But then

$E = A \sqcup (E \setminus A)$, where A is measurable by assumption and $E \setminus A$ is an outer measure 0 set and thus measurable.

So E is measurable, and

$$m(E) = m(A) + m(E \setminus A)$$

$$= m(A) + 0$$

$$\Rightarrow m(E) = m(A) = m(B) < \infty.$$

3b) Idea: $[0,1] \subseteq \mathcal{N} \subseteq [-1,2]$, so take

- $A = (-\infty, 0)$

- $E = A \cup (\mathcal{N} + 1)$, where \mathcal{N} is the non-measurable set, and $\mathcal{N} + 1 = \{x+1 \mid x \in \mathcal{N}\}$ is non-measurable by the same argument used for \mathcal{N} .

- $B = \mathbb{R}$

Claim: E is not measurable.

Supposing it were, note that A^c is measurable,

and countable intersections of measurable sets are measurable, so

$$E \cap A^c = (A \cup (\mathcal{N} + 1)) \cap A^c = \mathcal{N} + 1$$

must be measurable. ~~XX~~

4) Let A, B be fixed, and define

$$\begin{aligned} E_t &:= \{x \in \mathbb{R}^n \mid \inf_{a \in A} |x-a| \leq t\} \cap B \\ &= \{x \in \mathbb{R}^n \mid \text{dist}(x, A) \leq t\} \cap B \end{aligned}$$

and

$$\begin{aligned} f: \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto \mu(E_t) \end{aligned}$$

Note that $E_0 = A$, so $f(0) = \mu(A)$, and since B is compact and thus bounded, there is some $t = T$ such that $B \subseteq E_T$.

So f maps $[0, T]$ to $[\mu(A), \mu(B) + M]$ for some M .

Claim: f is cts, and for all $t \in [0, T']$ for some T' , $A \subseteq E_t \subseteq B$ and each E_t is compact.

Note that if this is true, we can first apply the intermediate value theorem to find a T' such that

$f(T') = \mu(B)$, then restrict f to map $[0, T']$

to $[\mu(A), \mu(B)]$. We can apply it again to pull back any

$c \in [\mu(A), \mu(B)]$ to a t satisfying $c = f(t) = \mu(E_t)$, in

which case $A \subseteq E_t \subseteq B$ and $\mu(A) \leq c = \mu(E_t) \leq \mu(B)$ as desired.

• f is cts: We'll show that the 2-sided limit $\lim_{t_i \rightarrow t} f(t_i)$ exists and

is equal to $f(t)$, using the fact that $a \leq b \Rightarrow E_a \subseteq E_b$.

If $t_i \nearrow t$, then $E_{t_1} \subseteq E_{t_2} \subseteq \dots \subseteq E_t$, and $\bigcup_{i \in \mathbb{N}} E_{t_i} = E_t$, so

by continuity of measure from below, we have $\lim_{i \rightarrow \infty} \mu(E_{t_i}) = \mu(E)$, so

$$\lim_{t_i \rightarrow t} f(t_i) = \lim_{i \rightarrow \infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

Similarly, if $t_i \searrow t$, noting that $t_i \leq T' \Rightarrow t_i \leq T' \Rightarrow \mu(E_{t_i}) \leq \mu(B) < \infty$,

$$\text{and } E_{t_1} \supseteq E_{t_2} \supseteq \dots \supseteq E, \text{ so}$$

we can apply continuity of measure from above to obtain

$$\lim_{t_i \rightarrow t} f(t_i) = \lim_{i \rightarrow \infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

So f is cts. \square

• E_t is compact:

Since $E_t \subseteq B$ which is compact and thus bounded, it suffices to show that

E_t is closed. But letting $N_t = \{x \in \mathbb{R}^n \mid \text{dist}(x, A) < t\}$, we have

$E_t = \overline{N_t \cap B}$, where N_t is open because $N_t = \bigcup_{a \in A} \underbrace{\{x \in \mathbb{R}^n \mid \text{dist}(x, a) < r\}}_{\text{Open ball around } a}$, and

$N_t \subseteq B \Rightarrow N_t \cap B$ is still open. But the closure of any open set is closed. \square

• $t \in [0, T'] \Rightarrow A \subseteq E_t \subseteq B$:

$E_0 = A$ and $t \leq s \Rightarrow E_t \subseteq E_s$, so $A \subseteq E_t$ for all t .

But $E_t = \overline{N_t \cap B} \subseteq \overline{B} = B$ since B is closed, so $E_t \subseteq B$ for all t as well. \square

5a) Recalling that \mathcal{N} is constructed by considering $\frac{\mathbb{R} \cap [0,1)}{\mathbb{Q} \cap [0,1)}$ and taking exactly one element from each equivalence class, we can note that if $E \subseteq \mathcal{N}$, then E contains a choice of at most one element from each equivalence class. We can then take a similar enumeration $\mathbb{Q} \cap [-1,1] = \{q_i\}_{i=1}^{\infty}$ and define $E_j := E + q_j$. Then $E \subseteq \mathcal{N} \Rightarrow \bigcup_{j \in \mathbb{N}} E_j \subseteq \bigcup_{j \in \mathbb{N}} \mathcal{N}_j \subseteq [-1,2]$, and since E is measurable, we must have

$$\mu(E) = \mu\left(\bigcup_{j \in \mathbb{N}} E_j\right) = \sum_{j \in \mathbb{N}} \mu(E_j) = \sum_{j \in \mathbb{N}} \mu(E) \leq 3,$$

which can only hold if $\mu(E) = 0$. \square

5b) Suppose $\mu(I \setminus \mathcal{N}) < 1$, so $\mu(I \setminus \mathcal{N}) = 1 - 2\varepsilon$ for some $\varepsilon > 0$. Then choose an open $G \supseteq I \setminus \mathcal{N}$ such that $\mu(G) = \mu(I \setminus \mathcal{N}) + \varepsilon = 1 - \varepsilon$. Then $I \setminus G \subseteq \mathcal{N}$,

and so by (1) we must have $\mu(I \setminus G) = 0$. But then

$$I = G \sqcup I \setminus G \Rightarrow \mu(I) = \mu(G) + \mu(I \setminus G)$$

$$\Rightarrow 1 = 1 - \varepsilon < 1, \text{ a contradiction. } \square$$

5c) Let

$$\left. \begin{array}{l} E_1 = N \\ E_2 = I \setminus N \end{array} \right\} \Rightarrow I = E_1 \sqcup E_2$$

but $m_*(E_1) = m_*(N) > 0$, otherwise N would be

measurable so $m_*(E_1 \sqcup E_2) = 1$ but

$$m_*(E_1) + m_*(E_2) = 1 + \varepsilon \text{ for some } \varepsilon > 0. \quad \blacksquare$$

6a) Claim: E is a countable union of a countable intersection of measurable sets, and thus measurable.

Proof: Write $E = \{x \mid x \in E_j \text{ for infinitely many } j\}$, the claim is that

$$E = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k.$$

$\cdot E \subseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k$: Suppose x is in infinitely many E_j . Then for any fixed

k , there is some $M \geq k$ such that $x \in E_M \subseteq \bigcup_{j=k}^{\infty} E_j := S_k$. But this happens for every k ,

So $x \in \bigcap_{k=1}^{\infty} S_k$. \square

$E \supseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$. Suppose $x \in \bigcup_{j=k}^{\infty} E_j$ for every k . Then if x were in only finitely

many E_j , we could pick a maximal E_M such that $k \geq M \Rightarrow x \notin E_k$, and so

$x \notin \bigcup_{j=M}^{\infty} E_j$ - a contradiction. \square

Claim: $m(E) = 0$

We'll use the fact that $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{j \rightarrow \infty} \sum_{n=j}^{\infty} a_n = 0$, i.e. the tails

of a convergent sum must become arbitrarily small.

Since $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$, $E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k . So $m(E) \leq \sum_{j=k}^{\infty} m(E_j) \rightarrow 0$,

forcing $m(E) = 0$. \blacksquare

(6b) Fix x and let $E_{p,j} = \{x \in \mathbb{R} \mid |x - \frac{p}{j}| \leq \frac{1}{j^3}\}$

and $E_j = \bigcup_{\substack{p \text{ coprime} \\ \text{to } j}} E_{p,j} \subseteq \bigcup_{p=1}^j E_{p,j}$, and since $E_{p,j} \subseteq B(\frac{1}{j^3}, \frac{p}{j})$,

$m(E_{p,j}) \leq \frac{2}{j^3}$ and thus $m(E_j) \leq \sum_{p=1}^j \frac{2}{j^3} = \frac{2}{j^2}$.

But then $\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} \frac{2}{j^2} < \infty$. Moreover,

$$E = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j = \{x \in \mathbb{R} \mid \text{there are infinitely many } j\text{'s such that there exists a } p \text{ coprime to } j \text{ s.t. } |x - p/j| \leq 1/j^3\},$$

which is precisely the set we want. So by (1), $m(E) = 0$. \blacksquare