Category $\mathcal{O}$, Problem Set 4

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1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right)$.

Show that the induced $U(\mathfrak{g})$-module structure $M \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

Hence $M \notin \mathcal{O}$.

We first unpack all definitions in terms of tensor products, using the fact that $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(\lambda) & \longrightarrow & M & \longrightarrow & M(\lambda) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & 0
\end{array}$$

$$1 \otimes 1 \xmapsto{\psi} 1 \otimes \mathbf{u} \xmapsto{\phi} 1 \otimes 0$$

$$1 \otimes \mathbf{v} \longmapsto 1 \otimes 1$$

where $N = \operatorname{span}_{\mathbb{C}} \{\mathbf{u}, \mathbf{v}\}$.

We make the following claims:

1. The $U(\mathfrak{b})$ action defined on N lifts to a $U(\mathfrak{g})$-action on M.
2. This is an exact sequence of $U(\mathfrak{g})$-modules.
3. $M \not\cong M(\lambda) \oplus M(\lambda)$, showing that this sequence can not split.

Claim 1: We choose the basis

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and note that in the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, we have

$$\begin{aligned}
\mathfrak{n}^- &= \mathbb{C} \cdot x \\
\mathfrak{h} &= \mathbb{C} \cdot h \\
\mathfrak{n}^+ &= \mathbb{C} \cdot y \\
&.
\end{aligned}$$

Since the action is defined over $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and x acts by zero, we obtain a $\mathfrak{g}$-action on N which thus extends uniquely to a $U(\mathfrak{g})$- action.

Claim 2: We first note that since the submodule $\mathbb{C} \cdot \mathbf{u} < M$ is closed under the action of h (since h acts by $u \mapsto \lambda u$) and is equal to the image of ψ, we can identify $\mathbb{C} \cdot \mathbf{u} \cong \mathbb{C}_\lambda$ as $U(\mathfrak{b})$-modules and identify $M(\lambda)$ as a submodule of N. Since submodules of N lift to submodules of $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N$, the map ψ is an injection. Moreover, the map ϕ is a surjection, since the generator $1 \otimes 1$ of $M(\lambda)$ is precisely the image of one of the generators of M.

To see that the sequence is exact in the middle, we first note that $\operatorname{im} \psi \subset \ker \phi$ by construction, since we explicitly map the aforementioned submodule $\mathbb{C} \cdot \mathbf{u}$ to 0. To see that $\ker \phi \subset \operatorname{im} \psi$, we note that by choosing a PBW basis of $\mathfrak{sl}(2, \mathbb{C})$ and a basis $\{\mathbf{u}, \mathbf{v}\}$ for N, we can obtain a basis of M of the form $\left\{ y^j \otimes \mathbf{u}, y^k \otimes \mathbf{v} \;\middle|\; j, k \in \mathbb{Z}^{\geq 0} \right\}$. But since we know

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

2.1 Solution

We first note that $M \in \mathcal{O} \implies M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ where each M_λ is a finite-dimensional weight space.

Moreover, $M^\vee \coloneqq \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$ is defined to be a direct sum of duals of weight spaces, which are still finite-dimensional.

So let $M, N \in \mathcal{O}$; we will proceed by showing that both $(M \otimes_{\mathbb{C}} L)^\vee$ and $M^\vee \otimes_{\mathbb{C}}^\vee$ have identical direct sum decompositions.

We first have

$$
\begin{aligned}
(M \otimes_{\mathbb{C}} L)^\vee &\coloneqq \bigoplus_{\lambda \in \mathfrak{h}^\vee} (M \otimes_{\mathbb{C}} L)_\lambda^\vee, && \text{the } \lambda \text{ weight spaces of } M \otimes_{\mathbb{C}} L \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta) \right)^\vee && \text{by an exercise on the weight spaces of a tensor product} \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta)^\vee \right) && \text{since the inner term is a finite sum} \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} \left(M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee \right) \right) && \text{since the weight spaces are finite-dimensional,}
\end{aligned}
$$

where we've repeatedly used the fact that $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$ for finite-dimensional vector spaces, which inductively holds for any finite direct sum of vector spaces.

On the other hand, using the fact that

$$
\begin{aligned}
(A \oplus B) \otimes (C \oplus D) &= ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \\
&= (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \\
\implies \left(\bigoplus_{j \in J} A_i \right) \otimes \left(\bigoplus_{k \in K} B_k \right) &= \bigoplus_{j \in J} \bigoplus_{k \in K} (A_j \otimes B_k) \quad \text{by induction} \quad .
\end{aligned}
$$

we can write

$$
\begin{aligned}
M^\vee \otimes_{\mathbb{C}} L^\vee &\coloneqq \left(\bigoplus_{\alpha \in \mathfrak{h}^\vee} M_\alpha^\vee \right) \otimes_{\mathbb{C}} \left(\bigoplus_{\beta \in \mathfrak{h}^\vee} L_\beta^\vee \right) \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} \left(M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee \right) \right),
\end{aligned}
$$

which equals what was obtained above.

This exhibits the isomorphism as $\mathbb{C}$-vector spaces, to see that this is in fact as isomorphism of $U(\mathfrak{g})$-modules we can use the fact that for $M \in \mathcal{O}$, a twisted $\mathfrak{g}$-action was defined as

$$
\mathbf{v} \in M, \; f \in M^\vee, \; g \in \mathfrak{g} \implies (g \cdot f)(\mathbf{v}) = f(\tau(g) \cdot \mathbf{v})
$$

for the transpose map τ. This action can be "linearly extended" over direct products and tensor products by taking the action component-wise, and is thus preserved by all of the isomorphisms appearing above.

Since the final terms $\bigoplus_{\lambda \in \mathfrak{h}} \bigoplus_{\alpha+\beta=\lambda} M_\alpha^\vee \otimes L_\beta^\vee$ are identical, they carry the same action, and since they are preserved by the isomorphisms, working backwards shows that the actions on $(M \otimes L)^\vee$ and $M^\vee \otimes L^\vee$ must also agree, yielding the desired isomorphism.

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ.

> For a concrete example, take Φ of type B_2 with a short simple root α and a long simple root β. If $\lambda \coloneqq \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in Φ.

3.1 Solution

We would like to show the following two propositions:

1. $\Phi_{[\lambda]}^+ \coloneqq \Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. In general, the associated simple system $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \bigcap \Delta$.

3.1.1 Proof of Proposition 1

We'll use the definition that for an abstract root system Φ, a positive system Φ^+ is defined by picking a hyperplane H not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of Φ^+ is on one side of H, then any subset satisfies this property as well, thus $\Phi_{[\lambda]} \bigcap \Phi^+$ consists only of positive roots and thus forms a positive system.

3.1.2 Proof of Proposition 2

Concretely, we can realize Φ and Δ as subsets of $\mathbb{R}^2$ in the following way:

$$\begin{aligned}
&\Phi = P_1 \coprod P_2 \coloneqq \{[1,0],[0,1],[-1,0],[0,-1]\} \coprod \{[1,1],[-1,1],[1,-1],[-1,-1]\} \\
&\Delta \coloneqq \{\alpha, \beta\} \coloneqq \{[1,0],[-1,1]\},
\end{aligned}$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\begin{aligned}
\Phi_{[\lambda]} &\coloneqq \left\{ \gamma \in \Phi \;\middle|\; \langle \lambda,\ \gamma^\vee \rangle \in \mathbb{Z} \right\}, \qquad \gamma^\vee \coloneqq \frac{2}{\|\gamma\|^2}\, \gamma, \\
\Delta_{[\lambda]} &\coloneqq \left\{ \gamma \in \Delta \;\middle|\; \langle \lambda,\ \gamma^\vee \rangle \in \mathbb{Z} \right\}.
\end{aligned}$$

Now choosing $\lambda := \dfrac{\alpha}{2} = \left[\dfrac{1}{2}, 0\right]$, we now consider the inner products $\langle \lambda,\ \gamma^\vee \rangle$ for $\gamma \in \Phi$:

Thus

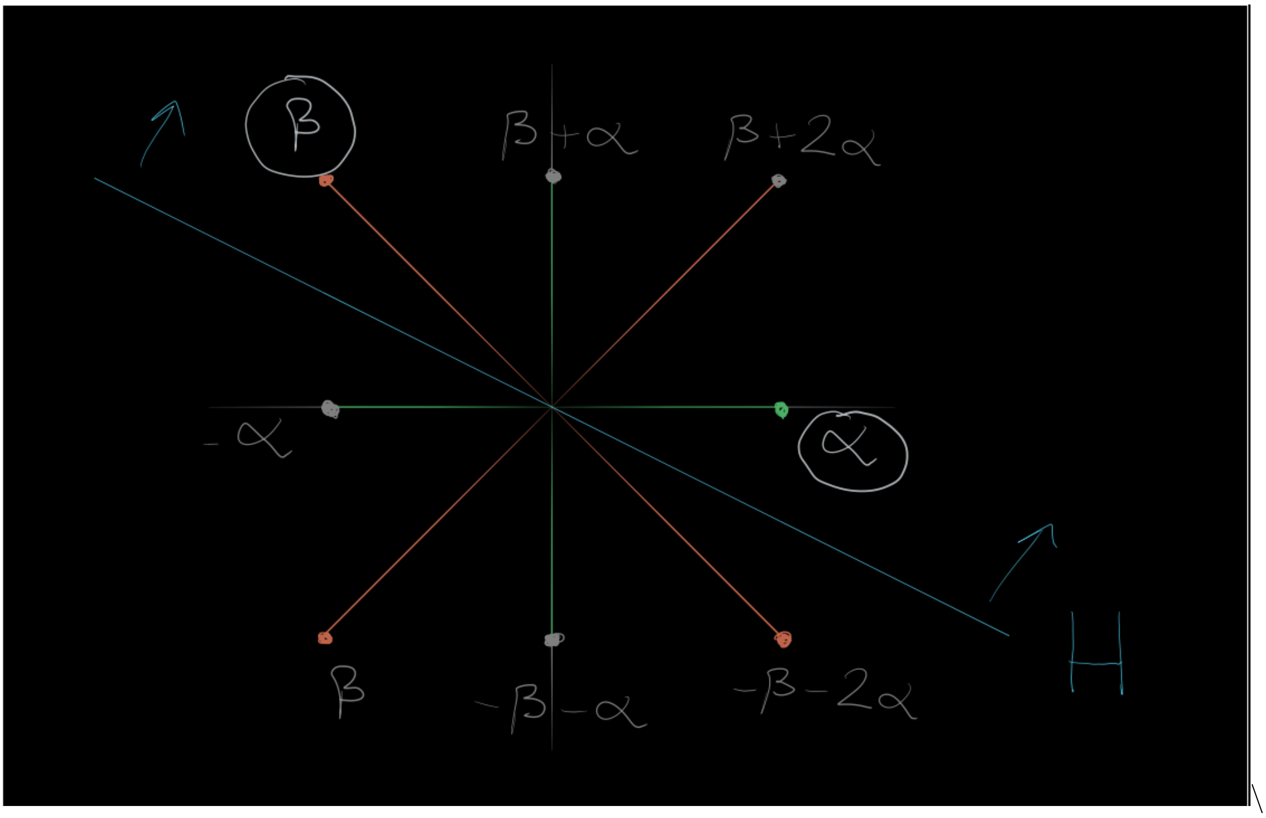
$$\gamma_1 \in P_1 \implies \left\langle \left[\frac{1}{2}, 0\right],\ 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right)\langle [1,0],\ \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z}$$
$$\gamma_2 \in P_2 \implies \langle \lambda,\ \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right],\ \frac{2}{\left(\sqrt{2}\right)^2}[\pm 1, \pm 1] \right\rangle = \pm\frac{1}{2} \notin \mathbb{Z}$$

where $(\gamma_1)_1$ denotes the first component of γ_1.

We thus find that

$$\Phi_{[\lambda]} = P_1 \qquad \text{the short roots}$$
$$\Delta_{[\lambda]} = \Phi_{[\lambda]} \bigcap \Delta = \{\alpha\} \qquad \text{the single short simple root.}$$

Choosing the following hyperplane H not containing any root, we can choose a positive system:



$$\Phi^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$$

where we can note that $\Phi^+ \bigcap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha+\beta, -\alpha-\beta\} \implies \Phi^+_{[\lambda]} = \{\alpha, \alpha+\beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take $\Delta_{[\lambda]} = \{\alpha, \alpha+\beta\}$. But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha+\beta\} \neq \{\alpha\} = \{\alpha, \alpha+\beta\} \bigcap \{\alpha, \beta\} = \Phi^+_{[\lambda]} \bigcap \Delta,$$

which is what we wanted to show.

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that $\operatorname{coker} \phi$ has a standard filtration.