Title

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1 Lecture 07

Last time: stalks, sheafification, and $\mathrm{Sh}(X_{\text{ét}})$ is abelian. Next up, we're aiming to define sheaf cohomology for $\mathrm{Sh}(X_{\text{ét}})$.

Remark 1.0.1 ***(Esoteric!):*** Related to a question asked by a viewer: there is not in fact a morphism from $X_{\text{fppf}} \to X_{\text{ét}}$, since locally finitely-presented need not be finitely presented (part of the condition for fppf). There is instead a morphism $X_{\text{fppf}} \to X_{\text{ét,fp}}$ to a corresponding finitely presented site. There is also a map $X_{\text{ét}} \to X_{\text{ét,fp}}$ inducing an equivalence on the category of sheaves via pushforward.

Theorem 1.0.2 ***(Enough injectives).***
$\mathrm{Sh}(X_{\text{ét}})$ has enough injectives.

Proof (?).
Given $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ we want an injective sheaf $\mathcal{I}$ and an injection $\mathcal{F} \hookrightarrow \mathcal{I}$. For each $x \in X$, choose a geometric point $\bar{x}$ over x, and let $I(\bar{x})$ be an injective $\mathbb{Z}$-module with a map $\mathcal{F}_{\bar{x}} \to I(\bar{x})$. These exist because the category of $\mathbb{Z}$-modules has enough injectives. The injectives in this category are **divisible** abelian groups.

Claim: The following object works:

$$\mathcal{I} := \prod_{\bar{x}} (\iota_{\bar{x}})_* I(\bar{x}).$$

We need to check

1. There is a map $\mathcal{F} \to \mathcal{I}$: The RHS is a product, so we map into the components. $\mathcal{F}_{\bar{x}}$ maps into its own associated skyscraper sheaf where the map is sending sections to their germs. Then the skyscraper sheaf for $\mathcal{F}_{\bar{x}}$ maps into the skyscraper sheaf for $I(\bar{x})$ by pushforward.

2. This is a monomorphism: check on stalks.

3. $\mathcal{I}$ is injective: check the lifting property directly.

■

1.1 What Else We Get From Sheafification

Remark 1.1.1: We now know that $\mathrm{Sh}(X_{\text{ét}})$ is abelian with enough injectives. This is true for $\mathrm{Sh}(\tau)$ for any site τ, but this is substantially harder to show.

1.1.1 Inverse Images

For $f: X \to Y$, we have a map on presheaves

$$\begin{aligned} f^{-1} &: \mathrm{Presh}(Y_{\text{ét}}) \to \mathrm{Presh}(X_{\text{ét}}) \\ \mathcal{F}(V &\xrightarrow{\text{ét}} X) \mapsto \varinjlim \mathcal{F}(U \to X), \end{aligned}$$

where the limit is over diagrams of the form

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\text{ét}"] & U \arrow[d, "\text{ét}"] \\ X \arrow[r] & Y \end{tikzcd}$$

Fact 1.1.2: f^{-1} is left adjoint to pushforward as functors on presheaves.

Exercise 1.1.3 ***(?):*** Check this.

Definition 1.1.4 (Inverse Image Sheaf)

$$f^*\mathcal{F} := \left(f^{-1}\mathcal{F}\right)^a.$$

Theorem 1.1.5 ***(?).***
f^* is left adjoint to f_*.

Proof (?).
Sheafification is a left adjoint. ■

Example 1.1.6 ***(?):***

- For $\bar{x} \xhookrightarrow{\iota} X$ a geometric point, we have $\iota^*\mathcal{F} = \mathcal{F}_{\bar{x}}$.
- For $Y \xrightarrow{f} X$, we have $f^*\underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$.
- More generally, for $Y \xrightarrow{f} X$ and any representable functor $\mathcal{F} := \underline{\hom}_X(\,\cdot\,, Z)$, we have $f^*\mathcal{F} = \underline{\hom}_Y(\,\cdot\,, Y \times_X Z)$.

1.2 Étale Cohomology

See **??** for the definition of étale cohomology. How do we compute $H^i(X_{\text{ét}}, \mathcal{F})$? Choose an injective resolution

$$\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots .$$

with the $\mathcal{I}^j$ injectives. From the general theory of derived functors, we obtain

$$H^i(X_{\text{ét}}, \mathcal{F}) = H^i\left(\Gamma(X, \mathcal{I}^{\cdot})\right),$$

where the RHS is a complex of abelian groups. Injective resolutions are difficult to find in general. Suppose $\pi : X_{\text{ét}} \to Y_{\text{ét}}$ comes from a map of schemes, then we can compute derived functors of other functors such as the pushforward,

$$\left(R^i\pi_*\right)\mathcal{F} = H^i\left(\pi_*\mathcal{I}^{\cdot}\right),$$

where the RHS are sheaves on $Y_{\text{ét}}$. Implicit here is the claim that π_* is left-exact. You can also find $\left(L^{>0}\pi^*\right)\mathcal{G} = 0$.

Exercise 1.2.1 ***(?):*** Check that pullback is exact.

Proposition 1.2.2 ***(Properties of étale cohomology).***

1. $H^0(X_{\text{ét}}, \mathcal{F}) = \mathcal{F}(X)$, aka the global sections $\Gamma(X, \mathcal{F})$.
2. $H^{>0}(\mathcal{I}) = 0$ for $\mathcal{I}$ injective.
3. Given a SES of sheaves in $\mathrm{Sh}(X_{\text{ét}})$

$$0 \to A \to B \to C \to 0$$

there is a LES

$$\cdots \to H^{i+1}(X_{\text{ét}}, C) \xrightarrow{\delta} H^i(X_{\text{ét}}, A) \to \cdots .$$

Example 1.2.3 ***(?):*** Suppose k is a field, not necessarily algebraically closed, and consider $\mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}})$. Let $G := \mathrm{Gal}(k^s/k)$ for a choice of separable closure k^s/k.

Claim: There is a functor from $\mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}})$ to discrete G-modules[1] inducing an equivalence of categories.

Note that when thinking of Galois representations, $\mathbb{Z}_\ell$ is not an example of this, but a representation over a finite field works. E.g. the Tate module (the inverse limit of torsion) of an elliptic curve is not a discrete G-module since the Galois action is not continuous in the discrete topology (although it is in the ℓ-adic topology).

[1] G is a topological group in the inverse limit topology, so a discrete G-module is a module with the discrete topology where the G-action is continuous. In particular, the action on any element factors through a finite quotient of G.

To prove this claim, the map is given by

$$\iota : \mathrm{Sh}((\operatorname{Spec} k)_{\text{ét}}) \to \text{Discrete } G\text{-modules}$$
$$\mathcal{F} \mapsto \varprojlim_{k \subset L \subset k^s} \mathcal{F}(\operatorname{Spec} L).$$

The idea here: you want to evaluate $\mathcal{F}$ on k^s, which doesn't make sense because k^s is not locally finitely-presented, so we take a limit instead. The claim is that the image is a discrete G-module and this is an equivalence. This follows because each term is, and taking limits preserves this property.

Corollary 1.2.4*(?)*.
$H^i((\operatorname{Spec} K)_{\text{ét}}, \mathcal{F}) = H^i(G, \iota\mathcal{F})$, which is the Galois cohomology.

Why? Derived functors only depend on the ambient category, so it suffices to check H^0.

Proof (of claim).
We get a G-module since G acts on the entire diagram and thus its limit.
Exercise 1.2.5*(?)*: Check that this is a discrete G-module.
There is an inverse functor: given $V \to \operatorname{Spec} k$ an étale map, by the classification of étale k-algebras we have $V = \coprod_{kinK'} \operatorname{Spec} k'$ where K' is the set of all finite separable k'/k. Given a discrete G-module M, send it to the Galois fixed points $V \to \prod M^{G'_s}$ where $G'_s := \mathrm{Gal}(k^s/k')$.
Exercise 1.2.6*(Check)*: Check that this is an inverse, it follows from Galois descent. ■

Proof (of corollary).
$\Gamma(\operatorname{Spec} k, \mathcal{F}) = (\iota F)^G$, taking the G-invariants. So $H^0 \xrightarrow{\iota}$ to taking invariants, and thus the higher derived functors agree, where the RHS is group cohomology. ■

Remark 1.2.7: Right now we're only talking about things that look like $\mathbb{Z}/\ell\mathbb{Z}^n$, but the goal when proving the Weil conjectures will be using $\mathbb{Z}_\ell$. We'll be trying to count some number by taking traces, but if we take these in a ring where some prime is zero, this only gives a congruence class. So when we define ℓ-adic cohomology, we'll take some inverse limit.