

# Problem Set One

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## 1 Humphreys 1.1

### 1.1 a

If  $M \in \mathcal{O}$  and  $[\lambda] = \lambda + \Lambda_r$  is any coset of  $\mathfrak{h}^\vee / \Lambda_r$ , let  $M^{[\lambda]}$  be the sum of weight spaces  $M_\mu$  for which  $\mu \in [\lambda]$ .

**Proposition:**  $M^{[\lambda]}$  is a  $U(\mathfrak{g})$ -submodule of  $M$

*Proof:*

Proposition:  $M$  is the direct sum of finitely many submodules of the form  $M^{[\lambda]}$ .

*Proof:*

### 1.2 b

**Proposition:** The weights of an indecomposable module  $M \in \mathcal{O}$  lie in a single coset of  $\mathfrak{h}^\vee / \Lambda_r$ .

## 2 Humphreys 1.3\*

**Proposition:** For any  $M \in \mathcal{O}$ ,  $M(\lambda)$  satisfies the following property:

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M) \cong \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M).$$

*Proof:*

Noting that

- $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{\lambda} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}$ ,
- $\text{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$  is an identification of the  $\mathfrak{g}$ -module  $M$  has a  $\mathfrak{b}$ -module by restricting the action of  $\mathfrak{g}$ ,

consider the following two maps:

$$\begin{aligned} F : \text{hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}, M) &\rightarrow \text{hom}_{U(\mathfrak{b})}(\mathbb{C}_{\lambda}, M) \\ \phi &\mapsto (F\phi : z \mapsto \phi(1 \otimes z)), \end{aligned}$$

and

$$\begin{aligned} G : \text{hom}_{U(\mathfrak{b})}(\mathbb{C}_{\lambda}, M) &\rightarrow \text{hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}, M) \\ \psi &\mapsto (G\psi : g \otimes v \mapsto g \cdot \psi(v)). \end{aligned}$$

It suffices to show that these maps are well-defined and mutually inverse.

To see that  $F$  is well-defined, let  $\phi : U(\mathfrak{g}) \otimes \mathbb{C}_{\lambda} \rightarrow M$  be fixed; we will show that the set map  $F\phi : \mathbb{C}_{\lambda} \rightarrow M$  is  $U(\mathfrak{b})$ -linear. Let  $b \in U(\mathfrak{b})$ , then

$$\begin{aligned} b \curvearrowright F\phi(v) &= b \curvearrowright (z \mapsto \phi(1 \otimes z))(v) \\ &= b \curvearrowright \phi(1 \otimes v) \\ &= \phi((b \curvearrowright 1) \otimes v) \quad \text{since } \phi \text{ is } U(\mathfrak{g})\text{-linear and } b \in U(\mathfrak{g}) \\ &= \phi(1 \otimes (b \curvearrowright v)) \quad \text{since } \mathbb{C}_{\lambda} \text{ is a } \mathfrak{b}\text{-module} \\ &= (z \mapsto \phi(1 \otimes z))(b \curvearrowright v) \\ &= F\phi(b \curvearrowright v). \end{aligned}$$