

# Title

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## 1 Wednesday November 20

Last time:

$$\begin{aligned}\mathbb{Z}\Lambda &\iff \{\mathfrak{h}^* \rightarrow \mathbb{Z}_{\geq 0} \mid \sim\} \\ e(\mu) &\mapsto e_\mu \\ e(\lambda)e(\mu) = e(\lambda + \mu) &\mapsto f \star g(\lambda) = \sum_{a+b=\lambda} f(a)g(b)\end{aligned}$$

and  $\text{ch}L(\lambda) = \sum_{\mu \in \Lambda} \dim L(\lambda)_\mu e(\mu)$ .

We have the Kostant function  $p(\lambda) = \#\{(k_\alpha)_\alpha \mid -\lambda = \sum_{\alpha \in \Phi^+} k_\alpha \alpha\}$  and the Weyl function  $q = e_\rho \star \prod_{\alpha \in \Phi^+} (1 - e_{-\alpha}) = \prod_{\alpha \in \Phi^+} (e_{\alpha/2} - e_{-\alpha/2})$ .

Lemma:  $p \star e_\lambda = \text{ch}M(\lambda)$ , so  $q \star \text{ch}M(\lambda) = e_{\lambda+\rho}$  and  $q \star p = e_\rho$ .

### 1.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of  $W$  is given by  $w \cdot \lambda = w(\lambda + \rho) - \rho$ , i.e. a reflection for hyperplanes passing through  $-\rho$ .

E.g. for type  $A_2$ , where  $W(0) = 0$ , we have:

Type  $A_2$

And for the dot action, we have

Image

where  $W \cdot 0 = 0$  and  $s(\alpha_1) = -\alpha_1$ .

**Theorem (Harish-Chandra):** If  $L(\mu)$  is a composition factor of  $M(\lambda)$ , then  $\mu \in W \cdot \lambda$  for  $\mu \leq \lambda$ .

Proof: Postponed.

ch are characters,  $L(\lambda)$  is a Verma module.

Remark: if we sum over  $\mu \leq \lambda$ , we obtain

$$\begin{aligned} \text{ch}M(\lambda) &= \sum_{\mu \in W \cdot \lambda} a_{\lambda\mu} \text{ch}L(\mu) \\ \text{ch}L(\lambda) &= \sum_{\mu \in W \cdot \lambda} b_{\lambda\mu} \text{ch}M(\mu) &= \sum_{W \cdot \lambda \in \Lambda} c_{\lambda W} \text{ch}M(w \cdot \lambda). \end{aligned}$$

**Theorem (Weyl's Character Formula):** If  $\lambda \in \Lambda^+$ , then

$$\text{ch}L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)}$$