# Moduli Spaces

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March 15, 2020

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## List of Definitions



## List of Theorems

# 1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, . . . Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

## 1.1 Representability

Last time: Fix an $S$-scheme, i.e. a scheme over $S$.

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form} \end{aligned}$$

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd}.$$

**Theorem 1.1** ***(Yoneda).***

$$\hom_{\mathrm{Fun}}(h_x, F) = F(x).$$

**Corollary 1.2.**

$$\hom_{\mathrm{Sch}/S}(x, y) \cong \hom_{\mathrm{Fun}}(h_x, h_y).$$

**Definition 1.2.1** (Moduli Functor)**.**
A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

**Definition 1.2.2** (Moduli Space)**.**
A **moduli space** for that "something" appearing above is an $M \in \mathrm{Obj}(\mathrm{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \mathrm{Spec}\ (k)$.

$h_m$ is the functor of points over $M$.

**Remark (1)** $h_m(\mathrm{Spec}\ (k)) = M(\mathrm{Spec}\ (k)) \cong$ "families over $\mathrm{Spec}\ k$" $= F(\mathrm{Spec}\ k)$.

**Remark (2)** $h_M(M) \cong F(M)$ are families over $M$, and $\mathrm{id}_M \in \mathrm{Mor}_{\mathrm{Sch}/S}(M, M) = \xi_{Univ}$ is the universal family.

Every family is uniquely the pullback of $\xi_{\mathrm{Univ}}$. This makes it much like a classifying space.

For $T \in \mathrm{Sch}/S$,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\mathrm{Univ}}). \end{aligned}$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

**Remark (3)** If $M$ and $M'$ both represent $F$ then $M \cong M'$ up to unique isomorphism.

$$\begin{array}{ccc} \xi_M & & \xi_{M'} \\ \\ M & \xrightarrow{\quad f \quad} & M' \\ \\ \\ M' & \xrightarrow{\quad g \quad} & M \\ \\ \xi_{M'} & & \xi_M \end{array}$$

which shows that $f, g$ must be mutually inverse by using universal properties.

**Example 1.1.**
A length 2 subscheme of $\mathbb{A}^1_k$ (??) then

$$F(S) = \left\{ V(x^2 + bx + c) \right\} \subset \mathbb{A}'_5$$

where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

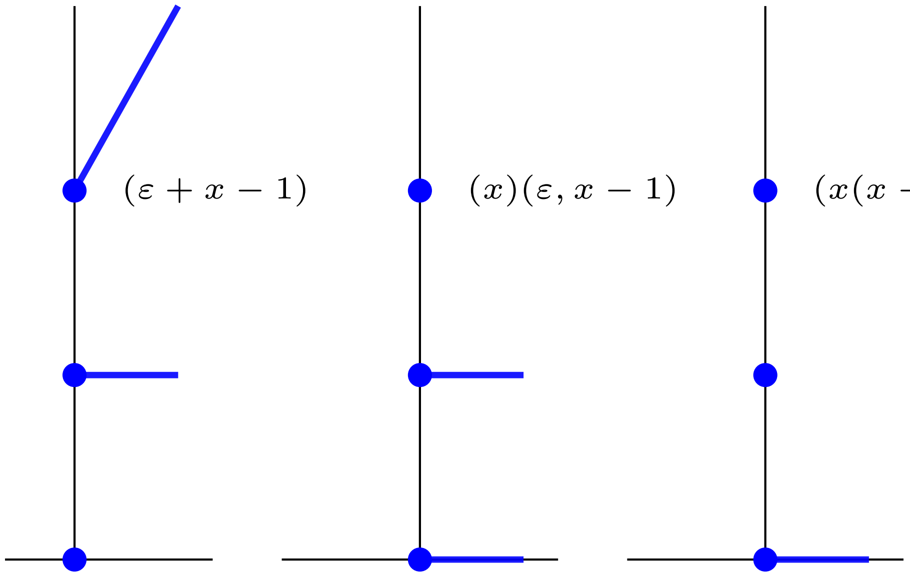
Then $F$ is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: $F''$ is not a good moduli functor, as it is not representable. Consider Spec $k[\varepsilon]$.




| | | | |
|---|---|---|---|
| $F$ | ✓ | x | x |
| $F'$ | ✓ | x | x |
| $F''$ | ✓ | ✓ | ✓ |

$$\begin{tikzcd}
\text{Spec } k \arrow[r, hook, "i"] & \text{Spec } k[\varepsilon] & = F'(\text{Spec } k) \arrow[dr] & \\
F(\text{Spec } k[\varepsilon]) \arrow[r, "F(i)"] & F(\text{Spec } k) \arrow[ur] & & = F''(\text{Spec } k) \\
T_p F^{',''} \arrow[u, hook, "\subset"] & P = V(x(x-1)) \arrow[u, "\in"] & &
\end{tikzcd}$$

We think of $T_pF^{\prime,\prime\prime}$ as the tangent space at $p$.

If $F$ is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\text{Spec } k[\varepsilon]) \arrow[r] & M(\text{Spec } k) \\
T_pM \arrow[u, hook, "\subset"] \arrow[r] & p \arrow[u, hook, "\subset"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{Spec } k \arrow[dl, hook] \arrow[dr, "?"] & \\
\text{Spec } k[\varepsilon] \arrow[rr] & & \text{Spec } \mathcal{O}_{M,p} \subset M
\end{tikzcd}$$

$$\begin{tikzcd}
 & k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
\mathfrak{m}_p \arrow[u, hook] & (\varepsilon) \arrow[u, hook] \\
\mathfrak{m}_p^2 \arrow[u, hook] & 0 \arrow[u, hook]
\end{tikzcd}$$

Moreover, $T_pM = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, and in particular this is a $k$-vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda\varepsilon
\end{aligned}$$

$$\lambda^* : \text{Spec } (k[\varepsilon]) \longrightarrow \text{Spec } (k[\varepsilon])$$

$$\begin{array}{ccc}
\lambda : M(\text{Spec } (k[\varepsilon])) & \longrightarrow & M(\text{Spec } (k[\varepsilon])) \\
\cup & & \cup \\
T_pM & \longrightarrow & T_pM.
\end{array}$$

**Conclusion**: If $F$ is representable, for each $p \in F(\text{Spec } k)$ there exists a unique point of $T_pF$ that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$\begin{tikzcd}
F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\
F \arrow[r] & G
\end{tikzcd}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If $F, F', G$ are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}/T) &\longrightarrow \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. $F$ is representable by $M/S$ implies that $F_T$ is representable by $M_T = M \times_S T/T$.

## 1.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

**Proposition 1.3.**
$\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(S) = \mathcal{O}_s^{n+1} &\longrightarrow L \longrightarrow 0/\sim. \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd}
\mathcal{O}_s^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

and $F(f)$ is given by pullbacks.

**Remark** $\mathbb{P}^n_S$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ T \mapsto F_S(T) &= \left\{ \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim . \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

*Proof (of Proposition).*
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of $L$, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

**Claim:** It is enough to show that each $F_i$ and each $F_{ij}$ are representable, since we have natural transformations:

$$\begin{tikzcd} F_i \arrow[r] & F \\ F_{ij} \arrow[u] \arrow[r] & F_j \arrow[u] \end{tikzcd}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

**Example .**
For $n = 1$, we can glue along open subschemes

$$\begin{tikzcd} & F_0 \\ F_{01} \arrow[ur] \arrow[dr] & \\ & F_1 \end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd}
 & & & F_0 \arrow[ddr] & \\
 & & F_{01} \arrow[ur] \arrow[dr] & & \\
F_{012} \arrow[urr] \arrow[r] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\
 & & F_{12} \arrow[ur] \arrow[dr] & & \\
 & & & F_2 \arrow[uur] &
\end{tikzcd}$$

This claim implies that we can glue together $F_i$ to get a scheme $M$. We want to show that $M$ represents $F$. $F(s)$ (LHS) is equivalent to an open cover $U_i$ of $S$ and sections of $F_i(U_i)$ satisfying the gluing (RHS).
Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, $U_i$ is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of $S$.
RHS to LHS comes from gluing.

■

*Proof (of Claim).*

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than $s_i$.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.
We also know $F_{ij}$ is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec $k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.
What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \mathop{\times}_{i \in I}{}_F \, F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

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## 2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space $\mathbb{P}^n_{\mathbb{Z}}$ constructed via a functor of points, i.e.

$$h_{\mathbb{P}^n_{\mathbb{Z}}} : \mathbb{P}^n_{\mathbb{Z}}\mathrm{Sch}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$
$$s \mapsto \mathbb{P}^n_{\mathbb{Z}}(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}.$$

for $L$ a line bundle, up to isomorphisms of diagrams:

$$\begin{tikzcd}
\mathcal{O}_s^{n+1} \arrow[r] \arrow[d, Rightarrow] & L \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}^n_{\mathbb{Z}}$ where

$$F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \;\middle|\; s_i \text{ is invertible} \right\}$$

are representable and can be glued together, and projective space represents this functor.

**Remark** Because projective space represents this functor, there is a universal object:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\
& \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) &
\end{tikzcd}$$

and other functors are pullbacks of the universal one. (Moduli Space)

**Exercise** Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\operatorname{Spec} \mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

**Definition 2.0.1** (Equalizer)**.**
For a category $C$, we say a diagram $X \longrightarrow Y \rightrightarrows Z$ is an *equalizer* iff it is universal with respect to the property:

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r, shift left] \arrow[r, shift right] & Z \\
& S \arrow[ul, dashed, "\exists!"] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Note that $X$ is the universal object here.

**Example 2.1.**
For sets, $X = \left\{ y \;\middle|\; f(y) = g(y) \right\}$ for $Y \xrightarrow{f,g} Z$.

**Definition 2.0.2** (Coequalizer)**.**
A **coequalizer** is the dual notion,

$$\begin{tikzcd}
 & S & \\
Z \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[u] \arrow[r] & X \arrow[ul, dashed, "\exists!"']
\end{tikzcd}$$

**Example 2.2.**
Take $C = \mathrm{Sch}/S$, $X/S$ a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$\begin{tikzcd}
X_\alpha \arrow[r] & X & & X_\beta \arrow[r] & X \\
X_{\alpha\beta} \arrow[u] \arrow[r] & X_\beta \arrow[u] & & X_{\beta\alpha} \arrow[u] \arrow[r] & X_\alpha \arrow[u]
\end{tikzcd}$$

These are canonically isomorphic.

In $\mathrm{Sch}/S$, we have

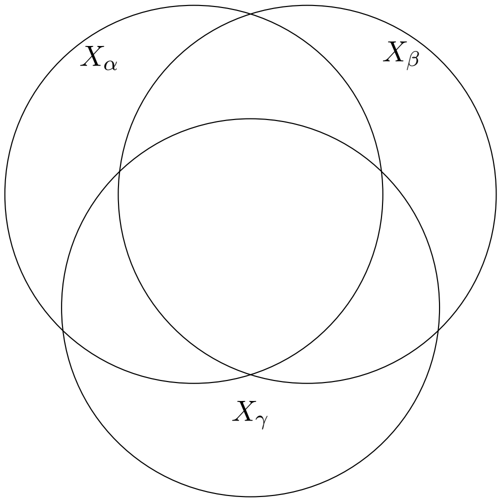
$$\begin{tikzcd}
\coprod_{\alpha\beta} X_{\alpha\beta} \arrow[r, shift left, "f_{\alpha\beta}"] \arrow[r, shift right, "g_{\alpha\beta}"'] & \coprod_\alpha X_\alpha \arrow[r] & X
\end{tikzcd}$$

where

$$\begin{aligned}
f_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\alpha \\
g_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\beta;
\end{aligned}$$

this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:




Then $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the **cocycle condition**:

1.
$$\varphi_{\alpha\beta}^{-1}\left(X_{\beta\alpha} \bigcap X_{\beta\gamma}\right) = X_{\alpha\beta} \bigcap X_{\alpha\gamma},$$
   noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

$$\begin{tikzcd}
X_{\alpha\beta} \bigcap X_{\alpha\gamma} \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\alpha\beta}"'] & & X_{\gamma\alpha} \bigcap X_{\gamma\beta} \\
& X_{\beta\alpha} \bigcap X_{\beta\gamma} \arrow[ur, "\varphi_{\beta\gamma}"'] &
\end{tikzcd}$$

Then there exists a scheme $X/S$ such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \longrightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

**Definition 2.0.3** (Zariski Sheaf)**.**
A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a *Zariski sheaf* iff for any scheme $T/S$ and any open cover $T_\alpha$, the following is an equalizer:

$$F(T) \longrightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

**Example 2.3.**
Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus

if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_\alpha T_\alpha \longrightarrow T,$$

since giving a local map to $X$ that agrees on intersections if enough to specify a map from $T \longrightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

**Definition 2.0.4** (Subfunctors, Open/Closed Functors)**.**
Suppose we have a morphism $F' \longrightarrow F$ in the category $\mathrm{Fun}(\mathrm{Sch}/S, \mathrm{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all $T/S$.
- $\iota$ is **open/closed/locally closed** iff for any scheme $T/S$ and any section $\xi \in F(T)$ over $T$, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff $f$ factors through $U$.

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

**Note** This is the same as asking if the subfunctor $F'$, which maps to $F$ (noting a section is the same as a map to the functor of points), and since $T \longrightarrow F$ and $F' \longrightarrow F$, we can form the fiber product $F' \times_F T$:

$$\begin{tikzcd}
F' \arrow[r] & F \\
F' \times_F T \arrow[u] \arrow[r, "g"] & T \arrow[u, "\xi"']
\end{tikzcd}$$

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \longrightarrow F$ is open/closed/locally closed iff $F' \times_F T$ is representable and $g$ is open/closed/locally closed.

I.e. base change is representable, and (?).

**Exercise (Tautologous)**

1. If $F' \longrightarrow F$ is open/closed/locally closed and $F$ is representable, then $F'$ is representable as an open/closed/locally closed subscheme
2. If $F$ is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

**Definition 2.0.5** (Open Covers)**.**
A collection of open subfunctors $F_\alpha \subset F$ is an **open cover** iff for any $T/S$ and any section $\xi \in F(T)$, i.e. $\xi : T \longrightarrow F$, the $T_\alpha$ in the following diagram are an open cover of $T$:

$$\begin{tikzcd} F_\alpha \arrow[r] & F \\ T_\alpha \arrow[u] \arrow[r] & T \arrow[u, "\xi"'] \end{tikzcd}$$

**Example 2.4.**
Given

$$F(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}$$

and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \longrightarrow F$ are an open cover. Because the sections generate everything, taking the $T_i$ yields an open cover.

**Proposition 2.1.**
A Zariski sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ with a representable open cover is representable.

*Proof .*
Let $F_\alpha \subset F$ be an open cover, say each $F_\alpha$ is representable by $x_\alpha$. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then $x_\beta$ yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \longrightarrow F_\alpha$ have the correct gluing maps.

This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \longrightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme $x$; we'll show that $x$ represents $F$.
First produce a map $x \longrightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

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**Lemma 2.2.**
If $E \longrightarrow F$ is a map of functors and $E, F$ are zariski sheaves, where there are open covers $E_\alpha \longrightarrow E, F_\alpha \longrightarrow F$ with commutative diagrams

$$\begin{tikzcd} E \arrow[r] & F \\ E_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$\begin{tikzcd} X \arrow[r] & F \\ X_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

**Example 2.5.**
For $S$ and $E$ a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} / \sim$$

is a generalization of projectivization, then $S$ admits a cover $U_i$ trivializing $E$.

Then the restriction $F_i \longrightarrow \mathbb{P}E$ were $F_i(T)$ is the above set if $f$ factors through $U_i$ and empty otherwise. On $U_i$, $E \cong \mathcal{O}_{U_i}^{n_i}$, so $F_i$ is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

**Example 2.6.**
For $E$ locally free over $S$ of rank $n$, take $r < n$ and consider the functor $\mathrm{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} / \sim$ (a Grassmannian) where $Q$ is locally free of rank $k$.

**Exercise**

a. Show that this is representable

b. For the Plucker embedding

$$\mathrm{Gr}(k, E) \longrightarrow \mathbb{P} \wedge^k E,$$

a section over $T$ is given by $f^*E \longrightarrow Q \longrightarrow 0$ corresponding to

$$\wedge^k f^*E \longrightarrow \wedge^k Q \longrightarrow 0,$$

noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \operatorname{Spec} k$, then $E$ is a $k$-vector space $V$, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank $n$.

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \longrightarrow Q \longrightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

> Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for $\mathbb{P}^n$ of all flat families of subschemes.

# 3 Thursday January 16th

## 3.1 Subfunctors

A functor $F' \subset F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is **open** iff for all $T \xrightarrow{\xi} F$ where $T = h_T$ and $\xi \in F(T)$.

We can take fiber products:

$$\begin{tikzcd}
F' \arrow[r] & F \\
\begin{array}{c} F' \times_F T \\ \text{Representable} \end{array} \arrow[u] \arrow[r, "\text{Open}"] & T \arrow[u]
\end{tikzcd}$$

So we can think of "inclusion in $F$" as being an *open condition*: for all $T/S$ and $\xi \in F(T)$, there exists an open $U \subset T$ such that for all covers $f : T' \longrightarrow T$, we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff $f$ factors through $U$.

Suppose $U \subset T$ in $\mathrm{Sch}/T$, we then have

$$h_{U/T}(T') = \begin{cases} \emptyset & T' \longrightarrow T \text{ doesn't factor} \\ \{\mathrm{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is $h_{U/T}(T') = \hom_T(T', U)$.

By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{ (a, b) \in F'(T) \times T(T) \;\middle|\; \xi(b) = \iota(a) \text{ in } F(T) \right\},$$

where $F' \xrightarrow{\iota} F$ and $T \xrightarrow{\xi} F$.

So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of $F/T'$ as sections of $F$ over $T/T'$ (?).

$$\begin{tikzcd}
F' \arrow[r, "\iota"] & F & \\
F' \times_F T \arrow[u] \arrow[r] & T \arrow[u, "\xi"] & \\
& & T' \arrow[ul, "f"] \arrow[uul, bend right, "f\circ\xi"']
\end{tikzcd}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U/S}(T'),$$

and so for $U \subset T$ in $\mathrm{Sch}/S$ we have $h_{U/S} \subset h_{T/S}$ is the functor of maps that factor through $U$. We just identify $h_{U/S}(T') = \hom_S(T', U)$ and $h_{T/S}(T') = \hom_S(T', T)$.

**Example 3.1.**
$\mathbb{G}_m, \mathbb{G}_a$. $\mathbb{G}_a$ represents giving a global function, $\mathbb{G}_m$ represents giving an invertible function.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{G}_a \\
T' \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & T \arrow[u, "f\in\mathcal{O}_T(T)"']
\end{tikzcd}$$

where $T' = \{f \neq 0\}$ and $\mathcal{O}_T(T)$ are global functions.

## 3.2 Actual Geometry: Hilbert Schemes

The best moduli space!

Want to parameterize families of subschemes over a fixed object. Fix $k$ a field, $X/k$ a scheme; we'll parameterize subschemes of $X$.

**Definition 3.0.1** (Hilbert Functor)**.**
The hilbert functor is given by

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

which sends $T$ to closed subschemes $Z \subset X \times_S T \longrightarrow T$ which are flat over $T$.

Here flatness replaces the Cartier condition.

**Definition 3.0.2** (Flatness)**.**
For $X \xrightarrow{f} Y$ and $\mathbb{F}$ a coherent sheaf on $X$, $f$ is flat over $Y$ iff for all $x \in X$ the stalk $F_x$ is a flat

$\mathcal{O}_{y,f(x)}$-module.

Note that $f$ is flat if $\mathcal{O}_x$ is.

Flatness corresponds to varying continuously.

**Warning**: Unless otherwise stated, assume schemes are Noetherian.

Note that everything works out if we only path with finite covers.

**Remark** If $X/k$ is projective, so $X \subset \mathbb{P}_k^n$, we have line bundles $\mathcal{O}_x(1) = \mathcal{O}(1)$. For any sheaf $F$ over $X$, there is a hilbert polynomial $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$. ( i.e. we twist by $\mathcal{O}(1)$ $n$ times.)

The cohomology of $F$ isn't changed by the pushforward into $\mathbb{P}_n$ since it's a closed embedding, i.e.

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

**Fact (First)** For $n \gg 0$, $\dim_k H^0 = \dim M_n$, the $n$th graded piece of $M$, which is a graded module over the homogeneous coordinate ring whose $i_* F = \tilde{M}$.

In general, for $L$ ample of $X$ and $F$ coherent on $X$, we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

**Proposition 3.1.**
For $f : X \longrightarrow S$ projective, i.e. there is a factorization:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^n \times S \ni \mathcal{O}(1) \arrow[d] \\ & S \end{tikzcd}$$

If $S$ is reduced, irreducible, locally Noetherian, then $f$ is flat $\iff$ $P_{\mathcal{O}_{x_s}}$ is constant for all $s \in S$.

To be more precise, look the base change to $X_1$, and the pullback of the fiber? $\mathcal{O}\Big|_{x_i}$ ?

Note: not using the word "integral" here! $S$ is flat $\iff$ the hilbert polynomial over the fibers are constant.

**Example 3.2.**
The zero-dimensional subschemes $Z \in \mathbb{P}_k^n$, then $P_Z$ is the length of $Z$, i.e. $\dim_k(\mathcal{O}_Z)$, and

$$P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z).$$

For two closed points in $\mathbb{P}^2$, $P_Z = 2$.

Consider the affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$, which is given by

$$\text{Spec } k[x,y]/(y,x^2) \cong k[x]/(x^2)$$

and $P_Z = 2$. I.e. in flat families, it has to record how the tangent directions come together.

**Example 3.3.**
Consider the flat family $xy = 1$ (flat because it's an open embedding) over $k[x]$, here we have points running off to infinity.

**Proposition 3.2** ***(Modified Characterization of Flatness for Sheaves).***
A sheaf $F$ is flat iff $P_{F_S}$ is constant.

### 3.2.1 Proof

Assume $S = \text{Spec } A$ for $A$ a local Noetherian domain.

**Lemma 3.3.**
For $F$ a coherent sheaf on $X/A$ is flat, we can take the cohomology via global sections $H^0(X; F(n))$. This is an $A$-module, and is a free $A$-module for $n \gg 0$.

*Proof (of Lemma).*
Assumed $X$ was projective, so just take $X = \mathbb{P}^n_A$ and let $F$ be the pushforward. There is a correspondence sending $F$ to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module $\sum_{n \gg 0} H^0(\Pi^m_A; F(n))$ This is equal to $\oplus_{n \gg 0} H^0(\mathbb{P}^m_A; F(n))$ and taking the associated sheaf ($Y \mapsto \tilde{Y}$, as per Hartshorne's notation) which is free, and thus $F$ is free.

See tilde construction in Hartshorne, essentially amounts to localizing free tings.

Conversely, take an affine cover $U_i$ of $X$. We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume $H^i(\mathbb{P}^m; F(n)) = 0$ for $n \gg 0$, and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^m; F(n)) \longrightarrow \mathcal{C}^0(\underline{U}; F(n)) \longrightarrow \cdots \longrightarrow C^m(\underline{U}; F(n)) \longrightarrow 0.$$

Assuming $F$ is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right.
Finally, local Noetherian + finitely generated flat implies free.

∎

By the lemma, we want to show $H^0(\mathbb{P}^m; F(n))$ is free for $n \gg 0$ iff the hilbert polynomials on the fibers $P_{F_S}$ are all constant.

**Claim 1** (1)**.**
It suffices to show that for each point $s \in \text{Spec } A$, we have

$$H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$$

for $k(S)$ the residue field, for $n \gg 0$.

Note that $P_{F_s}$ measures the rank of the LHS.

$\implies$: The dimension of RHS is constant, whereas the LHS equals $P_{F_S}(n)$.

$\impliedby$: If the dimension of the RHS is constant, so the LHS is free.

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank.

For $M$ a finitely generated module over $A$, find $0 \longrightarrow A^n \longrightarrow M \longrightarrow Q$ is surjective after tensoring with $\text{Frac}(A)$, and tensoring with $k(S)$ for a closed point, if $\dim A^n = \dim M$ then $Q = 0$.

*Proof (of Claim 1).*
By localizing, we can assume $s$ is a closed point. Since $A$ is Noetherian, its ideal is f.g. and we have

$$A^m \longrightarrow A \longrightarrow k(S) \longrightarrow 0.$$

We can tensor with $F$ (viewed as restricting to fiber) to obtain

$$F(n)^m \longrightarrow F(n) \longrightarrow F_S(n) \longrightarrow 0.$$

Because $F$ is flat, this is still exact.

We can take $H^*(x, \cdot)$, and for $n \gg 0$ only $H^0$ survives. This is the same as tensoring with $H^0(x, F(n))$.

■

**Definition 3.3.1** (Hilbert Polynomial Subfunctor)**.**
Given a polynomial $P \in \mathbb{Z}[n]$ for $X/S$ projective, we define a subfunctor by picking only those with Hilbert polynomial $p$ fiberwise as $\text{Hilb}^P_{X/S} \subset \text{Hilb}_{X/S}$. This is given by $Z \subset X \times_S T$ with $P_Z = P$.

**Theorem 3.4** ***(Grothendieck).***
If $S$ is Noetherian and $X/S$ projective, then $\text{Hilb}^P_{X/S}$ is representable by a projective $S$-scheme.

See cycle spaces in analytic geometry.

# 4 Thursday January 23

Some facts about the Hilbert polynomial:

1. For a subscheme $Z \subset \mathbb{P}^n_k$ with $\deg P_z = \dim Z = n$, then

$$p_z(t) = \deg z t^n/(n!) + O(t^{n-1}).$$

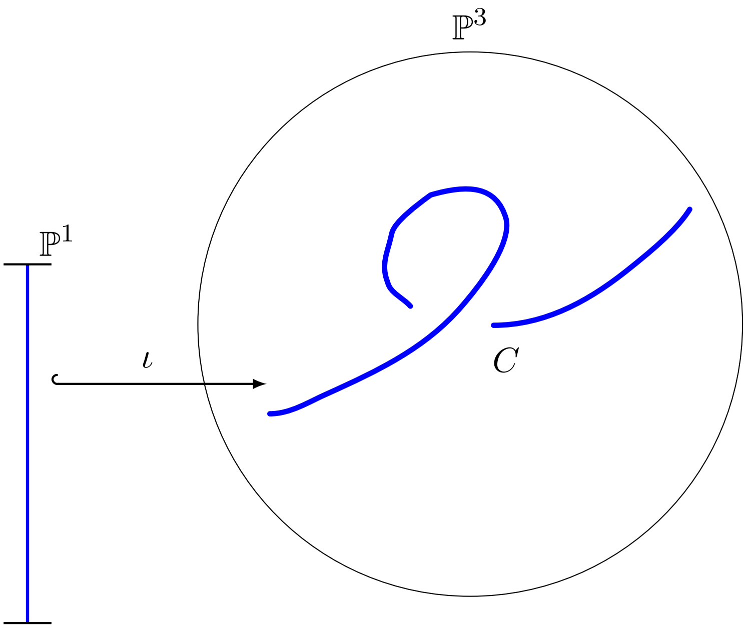
2. We have $p_z(t) = \chi(\mathcal{O}_z(t))$, consider the sequence

$$0 \longrightarrow I_z(t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{(t)} \longrightarrow \mathcal{O}_z^{(t)} \longrightarrow 0,$$

then $\chi(I_z(t)) = \dim H^0(\mathbb{P}^n, J_z(t))$ for $t \gg 0$, and $p_z(0)$ is the Euler characteristic of $\mathcal{O}_Z$.

Serre vanishing, Riemann-Roch, ideal sheaf.

**Example 4.1** (Good to keep in mind)**.**
The twisted cubic:




Then

$$p_C(t) = (\deg C)t + \chi(\mathcal{O}_{\mathbb{P}^1}) = 3t + 1.$$

### 4.0.1 Hypersurfaces

Recall that length 2 subschemes of $\mathbb{P}^1$ are the same as specifying quadratics that cut them out, each such $Z \subset \mathbb{P}^1$ satisfies $Z = V(f)$ where $\deg f = d$ and $f$ is homogeneous. So we'll be looking at $\mathbb{P}H^0(\mathbb{P}^n_k, \mathcal{O}(d))^\vee$, and the guess would be that this is $\mathrm{Hilb}_{\mathbb{P}^n_k}$

Resolve the structure sheaf

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

so we can twist to obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

Then

$$\chi(\mathcal{O}_D(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(t-d)),$$

which is

$$\binom{n+t}{n} - \binom{n+t-d}{n} = \frac{dt^{n-1}}{(n-1)!} + O(t^{n-2}).$$

**Lemma 4.1.**
Anything with the Hilbert polynomial of a degree $d$ hypersurface is in fact a degree $d$ hypersurface.

We want to write a morphism of functors

$$\mathrm{Hilb}_{\mathbb{P}^n_k}^{P_{n,d}} \longrightarrow \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee.$$

which sends flat families to families of equations cutting them out.

Want

$$Z \subset \mathbb{P}^n \times S \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee \longrightarrow L \longrightarrow 0.$$

This happens iff

$$0 \longrightarrow L^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$$

with torsion-free quotient.

Note that we use $L^\vee$ instead of $\mathcal{O}_s$ because of scaling.

We have

$$\begin{aligned} &0 \longrightarrow I_z \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S} \longrightarrow \mathcal{O}_z \longrightarrow 0 \\ &0 \longrightarrow I_z(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \mathcal{O}_z(d) \longrightarrow 0 \quad \text{by twisting.} \end{aligned}$$

We then consider $\pi_s : \mathbb{P}^n \times S \longrightarrow S$, and apply the pushforward to the above sequence noting that it is not right-exact.

$$= 0 \longrightarrow L^\vee = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow \text{locally free.}$$

$$0 \longrightarrow \pi_{s*} I_z(d) \longrightarrow \pi_{s*}\mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \pi_{s*}\mathcal{O}_z(d) \longrightarrow 0$$

This equality follows from flatness, cohomology, and base change. In particular, we need the following facts

The scheme-theoretic fibers, given by $H^0(\mathbb{P}^n, I_z(d))$ and $H^0(\mathbb{P}^n, \mathcal{O}_z(d))$, are all the same dimension.

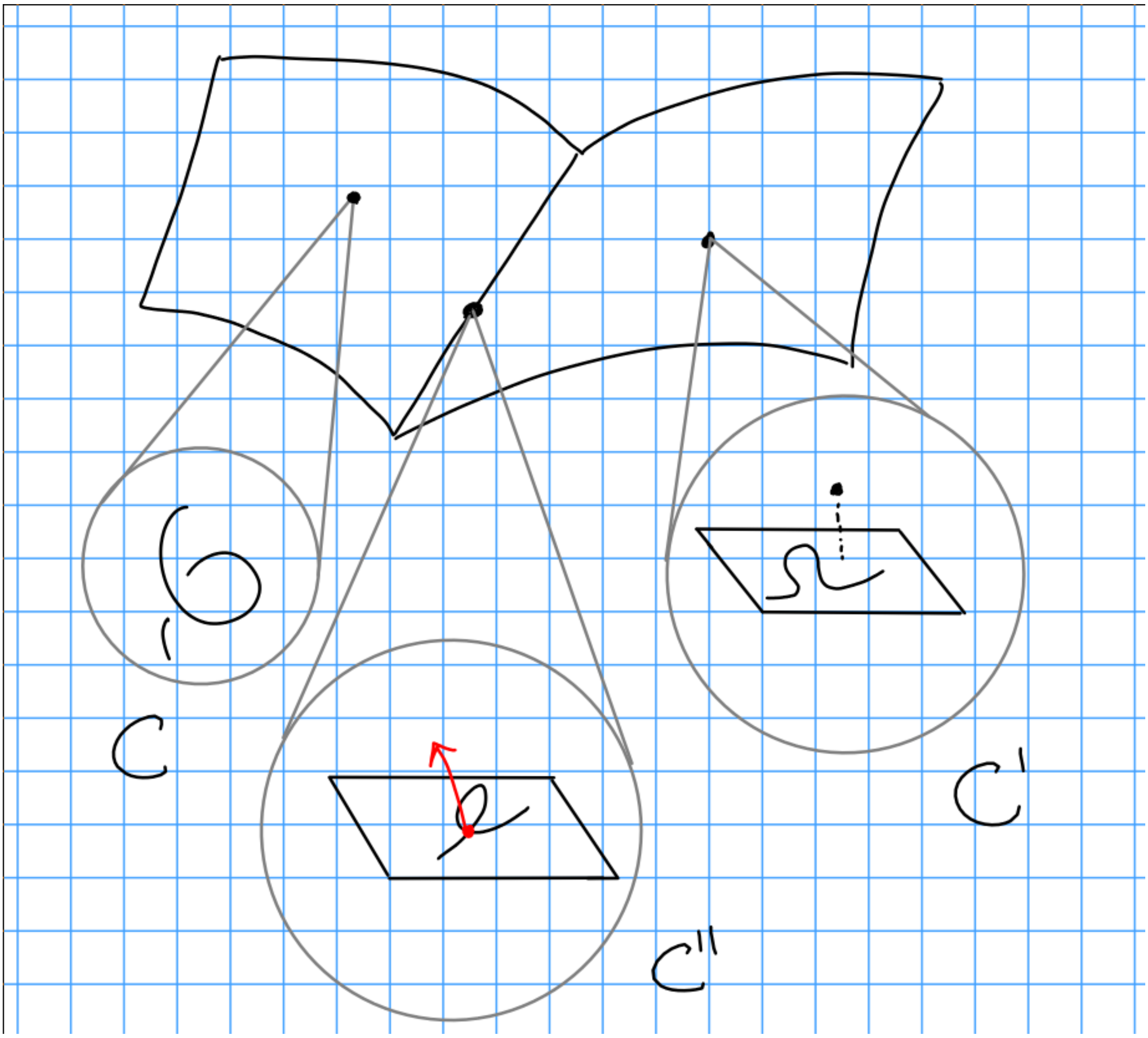
Using

1. Cohomology and base change, i.e. for $X \xrightarrow{f} Y$ a map of Noetherian schemes (or just finite-type) and $F$ a sheaf on $X$ which is flat over $Y$, there is a natural maop (not usually an isomorphism) $R^i f_* f \otimes k(y) \longrightarrow H^i(x_y, F|_{x_y})$, but is an isomorphism if $\dim H^i(x_y, F|_{x_y})$ is constant, in which case $R^i f_* f$ is locally free.
2. If $Z \subset \mathbb{P}^n_k$ is a degree $d$ hypersurface, then independently we know $\dim H^0(\mathbb{P}^n, I_z(d)) = 1$ and $\dim H^0(\mathbb{P}^n, \mathcal{O}_z(d)) = \binom{d+n}{n} - 1$.

To get a map going backwards, we take the universal degree 2 polynomial and form $V(a_{00}x_0^2 + a_{11}x_1^2 + a_{12}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2) \subset \mathbb{P}^2 \times \mathbb{P}^5$.

Next example: twisted cubics.

Consider a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ obtained by taking a basis of a homogeneous cubic polynomial. The canonical example is $(x, y) \longrightarrow (x^3, x^2y, xy^2, y^3)$. Then $P_C(t) = 3t+1$, and $\text{Hilb}^{3t+1}_{\mathbb{P}^3_k}$ has a component with generic point a twisted cubic, and another component with points a curve disjoint union a point, and the overlap are nodal curves with a "fat" 3-dimensional point:




Then $P_{C'} = 1 + \tilde{P}$, the hilbert polynomial of just the base without the disjoint point, so this equals $1+P_{2,3} = 1+(3t+0) = 3t+1$. For $P_{C''}$, we take the sequence $0 \longrightarrow k \longrightarrow \mathcal{O}_{C''} \longrightarrow \mathcal{O}_{C''\text{reduced}} \longrightarrow 0$, so $P_{C''} = 1 + P_{C''\text{red}} = 3t + 1$.

Note: flat families have to have the same constant Hilbert polynomial.

Note that we can get paths in this space from $C \longrightarrow C''$ and $C' \longrightarrow C''$ by collapsing a twisted cubic onto a plane, and sending a disjoint point crashing into the node on a nodal cubic.

We're mapping $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, and there is a natural action of $\mathbb{P}\mathrm{GL}(4) \curvearrowright \mathbb{P}^3$, so we get a map

$$\mathbb{P}\mathrm{GL}(4) \times \mathbb{P}^3 \longrightarrow \mathbb{P}^3.$$

Let $c \in \mathbb{P}^3$ and let $\mathcal{C}$ be the preimage. This induces (?) a map $\mathbb{P}\mathrm{GL}(4) \longrightarrow \mathrm{Hilb}_{\mathbb{P}^3}^{3t+1}$ where the fiber over $[C]$ in the latter is $\mathbb{P}\mathrm{GL}(2) = \mathrm{Aut}(\mathbb{P}^1)$. By dimension counting, we find that the dimension of the twisted cubic component is $15 - 3 = 12$.

The 15 in the other component comes from 3-dim choices of plane, 3-dim choices of a disjoint point, and $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee \cong \mathbb{P}^9$, yielding 15 dimensions.

To show that these are actually different components, we use Zariski tangent spaces. Let $T_1$ be the tangent space of the twisted cubic component, then $\dim T_1 \mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_k} = 12$, and similarly the dimension of the tangent space over the $C'$ component is 15.

Algebra fact: Let $A$ be Noetherian and local, then the dimension of the Zariski tangent space, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$, the Krull dimension. If this is an equality, then $A$ is regular.

Thus dimensions of the tangent spaces give an upper bound.

Proposition: If $X/k$ is projective and $P$ is a Hilbert polynomial, then $[Z] \in \mathrm{Hilb}^P_{X/k}$, i.e. a closed subscheme of $X$ with hilbert polynomial $p$ (note there's an ample bundle floating around) then the tangent space is $\hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.