Zack Garza

(1a) Note that if $x \in C$ is an endpoint of a removed interval, then $x = k/3^n$ for some integers $n \geq 1$ and $0 \leq k \leq 3^n$. So we just need a real number $x \in (0,1)$ satisfying

a) x has <u>some</u> ternary expansion

$$x = \sum_{i=1}^{\infty} a_i 3^{-i} \text{ where } a_i \neq 1 \text{ for any } i, \text{ and}$$

b) $x \neq k/3^n$ for any $k, n \in \mathbb{N}^{>0}$,

then we will have $x \in C$ by (a) and x not an endpoint by (b).

<u>Claim</u>: $x = (0.\overline{02})_3 = (0.020202\cdots)_3$ works.

Base 3

<u>Pf</u>: By construction, x satisfies

(a) $$x = \sum_{i=0}^{\infty} a_i 3^{-i}, \quad a_i \in \{0, 2\}$$

So no $a_i = 1$ and thus $x \in C$.

(b) To see that x satisfies (b), we can compute

$$x = (0.020202\cdots)_3$$

$$= 0\cdot 3^{-1} + 2\cdot 3^{-2} + 0\cdot 3^{-3} + 2\cdot 3^{-4} + \cdots$$

$$= \sum_{i=1}^{\infty} 2\cdot 3^{-2i} = 2\sum_{i=1}^{\infty} 3^{-2i} = 2\sum_{i=1}^{\infty}\left(\frac{1}{9}\right)^i$$

$$= 2\left(-1 + \sum_{i=0}^{\infty}\left(\frac{1}{9}\right)^i\right)$$

$$= 2\left(-1 + \frac{1}{1-\frac{1}{9}}\right) \boxed{= 1/4,}$$

where $4 \neq 3^n$ for any integer n. ▨

(1b) If a set X is nowhere dense in a topological space, it equivalently satisfies

$$(\overline{X})^\circ = \emptyset$$

(i.e., the interior of the closure is empty.)

It then suffices to show that

a) C is closed, so $\overline{C} = C$, and

b) C has no interior points, so $C^\circ = \emptyset$.

(a) To see that C is closed, we will show $C^c := [0,1] \setminus C$ is open. An arbitrary union of open sets is open, so the claim is that $C^c = \bigcup_{j \in J} A_j$ for some collection of open sets $\{A_j\}_{j \in J}$.

Consider C_n, the n^{th} stage of the process used to construct the Cantor set, so $C = \bigcap_{i=1}^{\infty} C_n$.

But by induction, C_n^c is a union of open sets.

In particular, $C_1^c = (\frac{1}{3}, \frac{2}{3})$, and

$$C_n^c = \underbrace{\left(\bigcup_{i=1}^{n-1} C_i^c \right)}_{\text{Open by hypothesis}} \cup \underbrace{\left(\begin{array}{c} \text{Exactly } n \text{ open intervals} \\ \text{that were deleted} \end{array} \right)}_{\text{open by construction}},$$

So C_n^c is open for each n. But then

$$C^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c = \bigcup_{i=1}^{\infty} C_n^c$$

is a union of open sets and thus open. So C is closed.

(b) To see that $C^\circ = \emptyset$, suppose towards a contradiction that $x \in C^\circ$, so there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) := (x-\varepsilon, x+\varepsilon) \subsetneq C$. Letting $\mu(I)$ denote the length of an interval, we have $\mu(N_\varepsilon(x)) = 2\varepsilon > 0$.

<u>Claim</u>: Let $L_n := \mu(C_n)$, then $L_n = \left(\frac{2}{3}\right)^n$.

This follows immediately by noting that L_n satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3} L_n, \qquad L_0 = 1$$

Since an interval of length $\frac{1}{3} L_{n-1}$ is removed at the n^{th} stage, which has the unique claimed solution.

But if $I_1 \subseteq I_2$ are real intervals, we must have

$\underbrace{\mu(I_1) \leq \mu(I_2)}$, whereas if we choose n large

Uses subadditivity of measure

enough such that $\left(\frac{2}{3}\right)^n < 2\varepsilon$, we have

$$(x-\varepsilon, x+\varepsilon) \subsetneq C = \bigcap_{i=1}^{\infty} C_i \Rightarrow \underline{(x-\varepsilon, x+\varepsilon) \subseteq C_n}, \text{ but}$$

$$\mu((x-\varepsilon, x+\varepsilon)) = \underline{2\varepsilon > \left(\tfrac{2}{3}\right)^n} = \mu(C_n), \text{ a contradiction.}$$

So such an $x \in C^\circ$ can't exist, and $C^\circ = \emptyset$.

Thus $(\overline{C})^\circ = C^\circ = \emptyset$, and C is nowhere dense,

and since a meager set is a countable union of nowhere dense sets, C is meager. $\square$

<u>Claim</u>: C is measure zero.

Measures are additive over disjoint sets, i.e.

$$A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B),$$

And if $A \subseteq B$, we have

$$\mu(B) = \mu(B \sqcup (B \setminus A)) = \mu(B) + \mu(B \setminus A)$$
$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A).$$

Now let B_n be the union of the intervals that are deleted at the n^{th} step. We have

$$\mu(B_0) = 0$$

$$\mu(B_1) = 1/3$$

$$\mu(B_2) = 2\left(\tfrac{1}{9}\right) = 2/9$$

$$\mu(B_3) = 4\left(\tfrac{1}{27}\right) = 4/27$$

$$\vdots$$

$$\mu(B_n) = 2^{n-1}/3^n$$

Moreover, if $i \neq j$, then $B_i \cap B_j = \emptyset$, and

$$C^c := [0,1] - C = \bigsqcup_{i=1}^{\infty} B_i .$$

We thus have

$$\begin{aligned}\mu(C) &= \mu([0,1]) - \mu(C^c)\\ &= 1 - \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right)\\ &= 1 - \sum_{n=1}^{\infty} \mu(B_n)\\ &= 1 - \sum_{n=1}^{\infty} 2^{n-1}/3^n\end{aligned}$$

$$= 1 - (1/3) \sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n$$

$$= 1 - (1/3)\left(1/_{1-2/3}\right)$$

$$= 0. \qquad \blacksquare$$

1c Let $y \in [0,1]$ be arbitrary, we will produce an $x \in C$ such that $f(x) = c$.

Write $y = (a_1\, a_2 \cdots)_2 = \sum_{i=1}^{\infty} a_i 2^{-i}$ where $a_i \in \{0,1\}$

Now define

$$x = (2a_1\, 2a_2 \cdots)_3 = \sum_{i=1}^{\infty} (2a_i) 3^{-i} := \sum_{i=1}^{\infty} b_i 3^{-i}$$

Since $a_i \in \{0,1\}$, $b_i = 2a_i \in \{0,2\}$, meaning x has no 1^s in its ternary expansion and so $x \in C$.

Moreover, under f we have

$$\left.\begin{array}{ccc} b_i & \mapsto & \frac{1}{2} b_i \\ \| & & \| \\ 2a_i & \mapsto & \frac{1}{2}(2a_i) = a_i \end{array}\right\} \text{ So } b_i \mapsto a_i \text{ and thus } f(x) = y.$$

So $C \twoheadrightarrow [0,1]$, which is uncountable, thus so is C. $\blacksquare$

(2a) ($\Rightarrow$) Suppose X is G_δ, so $X = \bigcup_{n=1}^{\infty} A_i$ with each A_i closed. Then A_i^c is open by definition, and so

$$X^c = \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

is a countable intersection of open sets, and thus F_σ.

($\Leftarrow$) Suppose X^c is an F_σ, so $X^c = \bigcap_{i=1}^{\infty} B_i$ with each B_i open. Then each B_i^c is closed by definition, and

$$X = (X^c)^c = \left(\bigcap_{i=1}^{\infty} B_i\right)^c = \bigcup_{i=1}^{\infty} B_i^c$$

is a countable union of closed sets, and thus G_δ.

(2b) Suppose X is closed, we will show $X = \bigcap_{n=1}^{\infty} C_n$ with each C_n open. For each $x \in X$ and $n \in \mathbb{N}$, define

- $B_n(x) = \{ y \in \mathbb{R}^n \mid d(x,y) < \frac{1}{n} \}$
- $C_n = \bigcup_{x \in X} B_n(x)$
- $W = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in X} B_n(x)$

Since each $B_n(x)$ is open by construction and C_n is a union of opens, each C_n is open.

Claim: $W = X$.

$X \subseteq W$: If $x \in X$, then $x \in B_n(x) \subseteq C_n$ for all n, and so

$$x \in \bigcap_{n=1}^{\infty} C_n = W.$$

$W \subseteq X$: Suppose there is some $w \in W \setminus X$ (so $w \neq x$ for any $x \in X$) towards a contradiction.

Since $w \in \bigcap_{i=1}^{n} C_n$, $w \in C_n$ for every n. So $w \in \bigcup_{x \in X} B_n(x)$ for every n. But then there is some particular $x_0 \in X$ such that $w \in B_n(x_0)$ for every n (otherwise we could take N large enough so that $w \notin B_N(x)$ for any $x \in X$, so $x \notin \bigcup_{x \in X} B_N(x)$) where $w \neq x_0$.

But then if $N_\varepsilon(x)$ is an arbitrary neighborhood of x, we can take $\frac{1}{n} < \varepsilon$ to obtain $w \in B_n(x) \subsetneq N_\varepsilon(x)$, which makes w a limit point of X. But since X is closed, it contains its limit points, forcing the contradiction $w \in X$.

So X is a countable intersection of open sets, and thus a G_δ set. $\blacksquare$

Now suppose X is open. Then X^c is closed, and thus a G_δ set. But then $(X^c)^c = X$ is an F_σ set by problem (2a). ■

(2c) Using the fact that singletons are closed in metric spaces, we can write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ as a countable union of closed sets, so $\mathbb{Q}$ is an F_δ set. Suppose $\mathbb{Q}$ was also a G_δ set, so $\mathbb{Q} = \bigcap_{i=1}^{\infty} A_i$ with each A_i open. Then for any fixed n, $\mathbb{Q} \subseteq A_i$, so A_i is dense in $\mathbb{R}$ for every i.

However, it is also true that $\{q\}^c := \mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$, and we can write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$$

as in intersection of open dense sets; since $\mathbb{R}$ is a Baire space, countable intersections of open dense sets are dense.

But then $\left(\bigcap_{i=1}^{\infty} A_i\right) \cap \left(\bigcap_{q \in \mathbb{Q}} \{q\}^c\right) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ must be dense in $\mathbb{R}$, which is absurd. ※

Note that this argument also works when $\mathbb{R}$ is replaced with any open interval I and $\mathbb{Q}$ is replaced with $\mathbb{Q} \cap I$.

For a set that is neither G_δ nor F_δ, consider

$A = \mathbb{Q} \cap (0, \infty)$, positive rationals

$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0)$, negative irrationals

A is F_σ but not G_δ , using above argument, and

dually B is G_δ but not F_σ.

<u>Claim</u>: $X = A \cup B$ is neither G_δ nor F_σ.

Suppose X is G_δ. Then $X \cap \overbrace{(0,\infty)}^{\text{open}} = A$ is G_δ as well. #

Suppose X is F_σ. Then X^c is G_δ, but

$$X^c = (A \cup B)^c = A^c \cap B^c = (\mathbb{Q} \cap (-\infty, 0)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, \infty))$$

and thus $X^c \cap \overbrace{(-\infty, 0)}^{\text{open}} = A$ is G_δ. #

So X is neither G_δ or F_σ.

$\blacksquare$

3a) <u>Claim</u>: $c \in [0,1] \Rightarrow \lim_{x \to c} f(x) = 0.$

This holds iff $\forall c \in I$, $\forall \varepsilon$, $\exists \delta$ s.t. $|x-c| < \delta \Rightarrow |f(x)| < \varepsilon$,

so let $\varepsilon > 0$ be arbitrary. Consider the set

$S = \{ n \in \mathbb{N} \mid \frac{1}{n} \geq \varepsilon \}$, which is a <u>finite</u> set, and so

$S_q = \{ r_n \in \mathbb{Q} \mid \frac{1}{n} \geq \varepsilon \}$ is <u>finite</u> as well.

So choose $\delta < \min_{r_n \in S_q} d(c, r_n)$ so $N_\delta(c) \cap S_q = \emptyset$

Then $|x-c| < \delta \Rightarrow \begin{cases} \bullet\ f(x) = 0 \text{ if } x \in I \setminus \mathbb{Q}, \text{ or} \\ \bullet\ x = r_m \in (\mathbb{Q} \setminus S_q) \cap I \text{ for some } m \text{ such that} \\ \quad 1/m < \varepsilon \text{ by construction.} \end{cases}$

But then $|f(x)| = |1/m| < \varepsilon$ as desired. □

So
- $c \in I \setminus \mathbb{Q} \Rightarrow f(c) = 0 = \lim_{x \to c} f(x)$,
- $c = r_n \in I \cap \mathbb{Q} \Rightarrow f(c) = \frac{1}{n} \neq 0 = \lim_{x \to c} f(x)$

and f is discontinuous on $I \cap \mathbb{Q}$. ■

(3b.1) Claim: ω_f is well defined

This amounts to showing that the sup and limit exist in

$$\omega_f(x) = \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

Let $x \in \mathbb{R}$ be arbitrary and δ fixed.

Since f is bounded, there is some M such that

$\forall y \in \mathbb{R}, \ |f(y)| < M$, and so

$$y,z \in \mathbb{R} \Rightarrow |f(y) - f(z)| = |f(y) + (-f(z))| \leq |f(y)| + |-f(z)|$$

$$= |f(y)| + |f(z)| < 2M,$$

which holds for $y,z \in B_\delta(x) \subseteq \mathbb{R}$ as well.

And so $\{|f(y) - f(z)| \text{ s.t. } y,z \in B_\delta(x)\}$ is bounded above and thus has a least upper bound, and thus the following supremum exists.

$$S(\delta, x) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

To see that the $\lim_{\delta \to 0} S(\delta, x)$ exists, note that

$$\delta_1 \leq \delta_2 \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$$

and so for a fixed x, $S(\delta, x)$ is a monotonically decreasing function of δ that is bounded below by 0, which converges by the monotone convergence theorem. $\square$

Claim: f is continuous at x iff $\omega_f(x)=0$.

($\Leftarrow$) Suppose $\omega_f(x)=0$ and let $\varepsilon>0$ be arbitrary; we will produce a δ to use in the definition of continuity.

Since $\omega_f(x)=\lim_{d\to 0^+} S(d,x)=0$, we can choose δ such that

$$d<\delta \Rightarrow |S(d,x)|<\varepsilon \quad , \quad \text{which means}$$

$$d<\delta \Rightarrow \sup_{y,z\in B_d(x)} |f(y)-f(z)|<\varepsilon$$

So fix $z=x$ and let y vary, yielding

$$d<\delta \Rightarrow \sup_{y\in B_d(x)} |f(y)-f(x)|<\varepsilon$$

But now for an arbitrary $t\in B_\delta(x)$, we have $|x-t|<\delta$ and

$$|f(x)-f(t)| \le \sup_{y\in B_\delta(x)} |f(x)-f(y)|<\varepsilon,$$

which exactly says $|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon$. $\square$

($\Rightarrow$) Suppose f is continuous at x and let $\varepsilon > 0$ be arbitrary; we will show $\omega_f(x) < \varepsilon$.

Since f is continuous, choose δ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

We then have

$$y, z \in B_\delta(x) \Rightarrow |x-y| < \delta \quad \text{and} \quad |x-z| < \delta,$$

$$\Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x) - f(z)| < \frac{\varepsilon}{2}$$

$$\Rightarrow |f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so

$$y, z \in B_\delta(x) \Rightarrow |f(y) - f(z)| < \varepsilon \quad \Rightarrow \sup_{y,z \in B_\delta(x)} |f(y) - f(z)| \leq \varepsilon$$

$$\Rightarrow S(\delta, x) \leq \varepsilon,$$

and since $S(d, x)$ is monotonically decreasing in d,

$$\omega_f(x) = \lim_{d \to 0^+} S(d, x) \leq S(\delta, x) \leq \varepsilon$$

as desired. $\blacksquare$

(3b.2) We will show that

$$A_\varepsilon^c = \{ x \in \mathbb{R} \mid \omega_f(x) < \varepsilon \}$$

is open by showing every point is an interior point.

Fix $\varepsilon > 0$ and let $x \in A_\varepsilon^c$ be arbitrary. We want to produce a δ such that

$$B_\delta(x) \subsetneq A_\varepsilon^c \quad \text{or equivalently} \quad |y-x| < \delta \Rightarrow \omega_f(y) < \varepsilon.$$

Write $\omega_f(x) = \lim_{d \to 0^+} S(d,x)$; since $\omega_f(x) < \varepsilon$ and this limit exists, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x) - 0| < \varepsilon \Rightarrow |S(d,x)| < \varepsilon.$$

Now suppose $y \in B_\delta(x)$, so $|y-x| < \delta$. Then there exists some δ' such that $B_{\delta'}(y) \subset B_\delta(x)$, and we claim that

$$S(\delta', y) \leq S(\delta, x)$$

Note that if this is true, then

S is monotonically decreasing in d

$$\omega_f(y) = \lim_{d \to 0} S(d,y) \leq S(\delta', y) \leq S(\delta, x) < \varepsilon.$$

To see why this is true, we just note that

$$a,b \in B_{\delta'}(y) \subset B_\delta(x) \Rightarrow a,b \in B_\delta(x)$$

$$\Rightarrow \sup_{a,b \in B_{\delta'}(y)} |f(y) - f(z)| \leq \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|,$$

since the supremum can only increase over a larger set.

So $\omega_f(y) < \varepsilon$ as desired. ∎

Finally, note that if $D_f = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$,

then $$D_f = \{x \in \mathbb{R} \mid \omega_f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid \omega_f(x) \geq \tfrac{1}{n}\}$$

$$= \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

is a countable union of closed sets and thus F_σ. ∎

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④ <u>Claim</u>: f is increasing, i.e. $x \leq y \Rightarrow f(x) \leq f(y)$

Fix $x \in \mathbb{R}$, and define

$$A_x := \{ t \in X \mid x > t \}, \quad A_x^c := \{ t \in X \mid x \leq t \}.$$

(Note that $t \in A_x$ or $t \in A_x^c \Rightarrow t = x_n$ for some n, and $X = A_x \sqcup A_x^c$.)

Then noting that

$$x_n \in A_x \Rightarrow f_n(x) \equiv 1$$

and

$$x_n \in A_x^c \Rightarrow f_n(x) \equiv 0,$$

We can write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \cdot 1 + \sum_{\{n \mid x_n \in A_x^c\}} \frac{1}{n^2} \cdot 0$$

$$= \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2}.$$

Now if $y \geq x$, then $y \geq t$ for every $t \in A_x$, so $A_y \supseteq A_x$.

But then

$$f(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \leq \sum_{\{n \mid x_n \in A_y\}} \frac{1}{n^2} = f(y),$$

where the inequality holds because

$$A_x \subseteq A_y \Rightarrow \{n \mid x_n \in A_x\} \subseteq \{n \mid x_n \in A_y\}$$
$$\Rightarrow \left|\{n \mid x_n \in A_x\}\right| \leq \left|\{n \mid x_n \in A_y\}\right|,$$

so the latter sum has <u>at least</u> as many terms and everything is positive. So $f(x) \leq f(y)$.

<u>Claim</u>: f is continuous on $\mathbb{R} \setminus X$ since $\sum f_n \xrightarrow{u} f$ and each f_n is continuous there.

Since $|f_n(x)| \leq 1$ by definition, and $|f_n(x)/n^2| \leq |1/n^2| := M_n$ where $\sum M_n < \infty$, $\sum f_n \xrightarrow{u} f$ by the M test.

Note that for a fixed n, $D_{f_n} = \{x_n\}$. This is because if we take a sequence $\{y_i\} \to x_n$ with each $y_i > x_n$, then $f(y_i) = 1$ for every i, and

$$\lim_{i\to\infty} f(y_i) = \lim_{i\to\infty} 1 = 1 \neq f(\lim_{i\to\infty} y_i) = f(x_n) = 0$$

So f_n is not continuous at $x = x_n$. Otherwise, either $x > x_n$ or $x < x_n$, in which case we can let ε be arbitrary and choose $\delta < |x - x_n|$ to get

$$y \in B_\delta(x) \Rightarrow \begin{cases} y > x_n \Rightarrow |f(y) - f(x)| = |0-0| < \varepsilon \\ y < x_n \Rightarrow |f(y) - f(x)| = |1-1| < \varepsilon. \end{cases}$$

Letting $F_N = \sum_{n=1}^{N} f_n$, we find that

$$F_N = \underset{\substack{\uparrow}}{f_1} + \underset{\substack{\uparrow}}{f_2} + \dots + \underset{\substack{\uparrow}}{f_N}$$

discontinuous at: $\{x_1\} \cup \{x_2\} \cup \dots \cup \{x_N\}$

So F_N is continuous on $\mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$.

and since $\mathbb{R} \setminus X \subseteq \mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$, F_N is continuous there too.

But then $f = \text{uniform limit}(F_N)$ is continuous on $\mathbb{R} \setminus X$. ∎

5a) Let $X = (C(I), \|\cdot\|_\infty)$ where $I = [0,1]$,

$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$, and

$$d(f,g) = \|f-g\|_\infty = \sup_{x \in I} |f(x) - g(x)|.$$

<u>Claim</u>: X is a metric space.

1) $d(f,g) = 0 \Rightarrow f = g$

If $\sup_{x\in I} |f(x)-g(x)| = 0$ then $|f(x)-g(x)| = 0 \ \forall x \in \mathbb{R}$,

so $f(x) = g(x) \ \forall x \in \mathbb{R}$ and $f = g$.

2) $d(f,g) = d(g,f)$

We have $d(f,g) = \sup_{x\in I} |f(x)-g(x)|$

$\sup_{x\in I} |g(x)-f(x)|$

$= d(g,f)$.

3) $d(f,h) \leq d(f,g) + d(g,h)$

We have $d(f,g) = \sup_{x\in I} |f(x) - g(x)|$

$= \sup_{x\in I} |f(x) - h(x) + h(x) - g(x)|$

$$\leq \sup_{x\in I}\left(|f(x)-h(x)|+|h(x)-g(x)|\right) \quad \text{← } \Delta\text{-ineq in } \mathbb{R}$$

$$= \sup_{x\in I}|f(x)-h(x)| + \sup_{x\in I}|h(x)-g(x)|$$

$$= d(f,h)+d(h,g).$$

So X is a metric space. □

Claim: X is complete.

Let $\{f_i\}$ be a Cauchy sequence in X, we will show that it converges in X. Since $\{f_i\}$ is Cauchy in X, we have

$$\forall \varepsilon>0,\ \exists N_0 \mid n\geq m\geq N_0 \Rightarrow \|f_n-f_m\|_\infty<\varepsilon$$

First we will define a candidate limit function f, then show $f\in X$.

1) Define $f:=\lim_{n\to\infty} f_n$ by $f(x)=\lim_{n\to\infty} f_n(x)$.

This is well-defined; let $S_x=\{f_i(x)\}\subseteq\mathbb{R}$ for a fixed x, and we claim S_x is Cauchy in $\mathbb{R}$, which is complete.

This follows because if $\{f_i\}$ is Cauchy in X, then

$$|f_n(x)-f_m(x)|\leq \sup_{x\in I}|f_n(x)-f_m(x)| = \|f_n-f_m\|_\infty \to 0.$$

2) $f \in X$, for which it suffices to show f is continuous.

Let $\varepsilon > 0$, and since $\{f_i\}$ is Cauchy, choose N_0 large s.t.

$$n \geq N_0 \Rightarrow \|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Now fix $n \geq N_0$; since f_n is continuous, choose δ such that

$$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

Then

$$\begin{aligned}
|x-y| < \delta \Rightarrow |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq \sup_{x \in I} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + \sup_{y \in I} |f_n(y) - f(y)| \\
&= \|f - f_n\|_\infty + |f_n(x) - f_n(y)| + \|f_n - f\|_\infty \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

So f is continuous, $f = \lim f_n \in X$, and X is complete. ∎

5b) Let $B = \{ f \in X \mid \|f\|_\infty \leq 1 \}$

Claim: B is closed.

Let f be a limit point of B, so there is some sequence $f_n \to f$ in X with each $f_n \in B$ so $\|f_n\|_\infty \leq 1 \ \forall n$.

Let $\varepsilon > 0$, and since $f_n \to f$ in X, choose N_0 such that

$$n \geq N_0 \Rightarrow \|f_n - f\| < \varepsilon$$

Then,

$$\begin{aligned} \|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &< \varepsilon + 1, \end{aligned}$$

and taking $\varepsilon \to 0$ yields $\|f\|_\infty \leq 1$. ∎

Claim: B is bounded

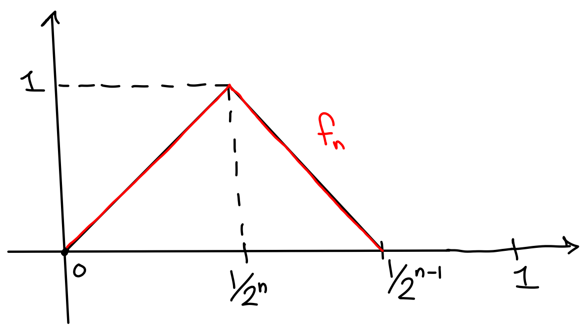
A subset $B \subseteq X$ is bounded iff there is some $x \in X$ and some $r > 0$ in $\mathbb{R}$ where $B \subset N(r, x) = \{ y \in X \mid d(y,x) < r \}$.

Choose $x = 0$, $r = 2$, then $f \in B \Rightarrow d(f, 0) = \|f - 0\|_\infty = 1 < 2$, so $f \in N(2,0)$.

<u>Claim</u>: B is not compact.

Since B is a metric space, B is compact iff B is sequentially compact.

Define f_n as the triangle:



Then $f_n \xrightarrow{\mathbb{R}} f$ where $f(x) = \begin{cases} 1, & x=0 \\ 0, & x \in (0,1] \end{cases}$,

Pointwise in $\mathbb{R}$

and so $\forall n$, $\|f_n - f\|_\infty = 1$, attained at $x=0$. So $\lim_{n\to\infty} \|f_n - f\|_\infty \neq 0$,

and $\{f_n\}$ does not converge in X, nor can any subsequence. $\blacksquare$

<u>Claim</u>: B is not totally bounded.

If it were, $\forall \varepsilon$ there would exist a finite collection $\{g_i\}_{i=1}^N \subseteq B$ such that $B \subseteq \bigcup_{i=1}^N N(\varepsilon, g_i)$ where

$$N(\varepsilon, g_i) = \{h \in B \mid \|h - g_i\| < \varepsilon\}.$$

Note that if $h_1, h_2 \in N(\varepsilon, g_i)$ then $\|h_1 - h_2\| \leq \|h_1 - g\| + \|g - h_2\| < 2\varepsilon$.

So choose $\varepsilon = \frac{1}{2}$, and consider the collection $\{f_n\}_{n=1}^{\infty}$.

Since $\|f_n - f_m\| = 1$, each $N(\varepsilon, g_i)$ can contain at most one f_n, since $f_n, f_m \in N(\varepsilon, g_i)$ for $n \neq m$ would imply $\|f_n - f_m\|_\infty < 2\varepsilon = 2(\frac{1}{2}) = 1$. But there are finitely many $N(\varepsilon, g_i)$ and infinitely many f_n, so if this is a cover of B, so $N(\varepsilon, g_i)$ must contain at least 2 f_n^s. #

(6a) Claim: If $\sum g_n \xrightarrow{U} G$, then $g_n \xrightarrow{U} 0$.

Let $G_N = \sum_{n=1}^{N} g_n$ and $G = \lim_{N\to\infty} G_N$.

Suppose $G_N \xrightarrow{u} G$, then choose N large enough so that

$$\forall x \in X,\ n \geq N \Rightarrow |G_n(x) - G(x)| < \frac{\varepsilon}{2}$$

Then letting $n > n-1 > N$, we have

$$|g_n(x)| = \left|\sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{n-1} g_i(x)\right|$$

$$= \left|\left(\sum_{i=1}^{n} g_i(x) - G(x)\right) - \left(\sum_{i=1}^{n-1} g_i - G(x)\right)\right|$$

$$\leq \left|\sum_{i=1}^{n} g_i(x) - G(x)\right| + \left|\sum_{i=1}^{n-1} g_i - G(x)\right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

So $\forall x \in X$, $|g_n(x)| < \varepsilon \Rightarrow g_n \xrightarrow{u} 0$. $\square$

Now let $g_n = 1/1+n^2x$, we'll show g_n does <u>not</u> converge to 0 uniformly.

Note $g_n \xrightarrow{u} g$ iff $\forall \varepsilon, \exists N_0 \mid \forall x, n \geq N_0 \Rightarrow |g_n(x) - g(x)| < \varepsilon$,

so let $\varepsilon < \frac{1}{2}$, N_0 be arbitrary, and choose $x_0 < 1/N_0^2$. Then,

$$|g_{N_0}(x_0)| = \frac{1}{|1+N_0^2 x|} = \frac{1}{|1+N_0^2(1/N_0^2)} = \frac{1}{2} > \varepsilon.$$

<u>Claim</u>: g is continuous on $(0, \infty)$.

Let $x \in (0, \infty)$ be arbitrary, and choose $a < x$. We will show g converges uniformly on $[a, \infty)$, and since each g_n is continuous on $[a, \infty)$ as well, g will be the uniform limit of continuous functions and thus continuous itself.

We can use the M-test. Since $x > a$,

$$\left|1/1+n^2x\right| \leq \left|1/n^2x\right| \leq \left|1/n^2a\right| = \frac{1}{a}\left|\frac{1}{n^2}\right|,$$

$$\text{where } \sum_{n=1}^{\infty} \frac{1}{a}\frac{1}{n^2} = \frac{1}{a}\sum \frac{1}{n^2} < \infty,$$

So g converges uniformly on $[a, \infty)$.

(6b) <u>Claim:</u> g is differentiable on $(0,\infty)$.

If $g'(x)$ exists, we have

$$g'(x) = \lim_{a\to x} (x-a)^{-1}(g(x)-g(a))$$

$$= \lim_{a\to x} (x-a)^{-1} \sum_{n=1}^{\infty} \frac{-n^2(x-a)}{(1+n^2x)(1+n^2a)}$$

$$= \lim_{a\to x} \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2x)(1+n^2a)}$$

$$= \sum (-n^2)/(1+n^2x)^2 ,$$

which exists because it converges uniformly on $[a,\infty)$, as

$$\left|\frac{-n^2}{(1+n^2x)^2}\right| \le \left|\frac{n^2}{(n^2x)^2}\right| = \left|\frac{1}{n^2x^2}\right| \le \left|\frac{1}{a^2n^2}\right| := M_n$$

where $\sum M_n = \sum \frac{1}{a^2n^2} = \frac{1}{a^2}\sum\frac{1}{n^2} < \infty$.

So g is <u>continuously differentiable</u> on $(0,\infty)$. ∎

7a) <u>Claim</u>: $h_n \xrightarrow{u} 0$ on $[0,\infty)$

Note that $h_n'(x) = \dfrac{1-nx}{(1+x)^n} \Rightarrow h_n' = 0$ iff $x = 1/n$ and

$$h_n''(x) = \frac{1+x+nx}{nx^2(1+x)^{n-1}} \quad \text{and} \quad h_n''(\tfrac{1}{n}) < 0,$$

so $x = \frac{1}{n}$ is a global maximum and thus

$$\forall x,\ |h_n(x)| \leq |h_n(\tfrac{1}{n})| = \left|\frac{1/n}{(1+\frac{1}{n})^n}\right| = \frac{1}{n(1+\frac{1}{n})^n} \leq \frac{1}{2n} \quad \text{for } n>1$$

so $\sup\limits_{x\in[0,\infty)} |h_n(x)| = |h_n(\tfrac{1}{n})| = O(\tfrac{1}{n}) \to 0$, thus $\|h_n\|_\infty \to 0$

and $h_n \to 0$ uniformly.

7b) Let $h(x) = \sum\limits_{n=1}^{\infty} h_n(x) = \sum\limits_{n=1}^{\infty} x/(1+x)^{n+1}$

i) Demonstrably, $h(0)=0$, and for a fixed x we have

$$h(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1} = (x/1+x)\sum_{n=1}^{\infty}(1/1+x)^n$$

$$= \frac{x}{1+x}\left(\frac{1}{1-(1/1+x)}\right) \quad \text{since } x>0 \Rightarrow (1/1+x)<1$$

$$= 1. \quad \square$$

ii) It can not converge uniformly on $[0,\infty)$, otherwise h would be the uniform limit of continuous functions, but h is discontinuous.

(7c) Let $a>0$ and $X=[a,\infty)$.

Claim: $\sum h_n \xrightarrow{u} h$ on X.

$(1+x)^{n+1} = \sum_{j=1}^{n} \binom{n}{j} x^j \geq 1 + nx + n^2x^2$

$x>a>0$, so positive terms.

Since $x>a$, we have

$$|h_n(x)| = \left|\frac{x}{(1+x)^{n+1}}\right| \leq \left|\frac{x}{1+nx+n^2x^2}\right| \leq \left|\frac{a}{1+na+n^2a^2}\right| \leq \left|\frac{a}{n^2a^2}\right| = \left|\frac{1}{n^2a}\right|$$

So let $M_n = 1/an^2$, then $\sum M_n < \infty \Rightarrow \sum h_n \xrightarrow{u} h$ by the M test. ∎

Zack
Garza

① Suppose E is bounded, so $\operatorname{diam}(E) \leq M$ for some fixed M. In particular, if $Q_i \subseteq E$ is an interval, then $|Q_i| \leq M$. Let $\varepsilon > 0$, and choose $\underbrace{\{Q_i\} \rightrightarrows E}_{}$ s.t.

i.e. $E \subseteq \bigcup_i E_i$

$$\text{for each } i, \quad |Q_i| \leq \varepsilon/2M$$

Then let $L_i = Q_i^2$. We then have

$$\begin{aligned}
|L_i| \leq |b^2 - a^2| = |b-a| \cdot |b+a| &= |Q_i| \cdot |b+a| \\
&\leq |Q_i| \cdot 2M \\
&\leq (\varepsilon/2^{i+1}M) 2M \\
&= \varepsilon/2^i,
\end{aligned}$$

so $\sum_{i=1}^\infty |L_i| \leq \sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$, and $\{L_i\} \rightrightarrows E^2$, so

$$m_*(E^2) < \varepsilon \longrightarrow 0.$$

Claim: It suffices to consider the bounded case.

Ball of radius n around 0

Pf If E is not bounded, consider $F_n = E \cap B(n,0)$.

Then F_n is bounded (by n), and since $F_n \subseteq E \Rightarrow m_*(F_n) \leq m_*(E) = 0$ by subadditivity, $m_*(F_n^2) = 0$ by the bounded case.

But then $E^2 = \bigcup_{n=1}^{\infty} F_n^2 \Rightarrow m_*(E^2) = m\left(\bigcup_{n=1}^{\infty} F_n^2\right) \leq \sum_{n=1}^{\infty} m_*(F_n^2) = 0$

by countable subadditivity. $\blacksquare$

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② Note

1) $E_1 = E_1 \setminus E_2 \sqcup E_1 \cap E_2$

2) $E_2 = E_2 \setminus E_1 \sqcup E_1 \cap E_2$

3) $E_1 \triangle E_2 = E_2 \setminus E_1 \sqcup E_1 \setminus E_2$

4) $E_1 \cup E_2 = \underbrace{(E_1 \triangle E_2) \sqcup (E_1 \cap E_2)}$

All disjoint unions, so we can freely apply measures and use countable additivity.

So

$$\begin{aligned} m(E_1) + m(E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) \\ &\quad + m(E_2 \setminus E_1) + m(E_1 \cap E_2) && \text{by (1), (2)} \\ &= \underbrace{m(E_1 \triangle E_2) + m(E_1 \cap E_2)} + m(E_1 \cap E_2) && \text{by (3)} \\ &= \underbrace{m(E_1 \cup E_2)} + m(E_1 \cap E_2). && \text{by (4)} \end{aligned}$$

$\blacksquare$

3a) Suppose $m(A)=m(B)<\infty$.

Since $A\subseteq E\subseteq B$, we have $\underline{E\setminus A\subseteq B\setminus A}$. However,

$$B=A\sqcup(B\setminus A) \Rightarrow m(B)=m(A)+m(B\setminus A)$$

$$\Rightarrow m(B)-m(A)=m(B\setminus A)$$

(since $m(A)<\infty$)

$$\Rightarrow m(B\setminus A)=0$$

(since $m(B)=m(A)$)

So $m_*(E\setminus A)=0$ by subadditivity.

But then

$$E=A\sqcup(E\setminus A),$$

where A is measurable by assumption and $E\setminus A$ is an outer measure 0 set and thus measurable.

So E is measurable, and

$$m(E)=m(A)+m(E\setminus A)$$

$$=m(A)+0$$

$$\Rightarrow m(E)=m(A)=m(B)<\infty.$$

3b) Idea: $[0,1] \subseteq \mathcal{N} \subseteq [-1,2]$, so take

- $A = (-\infty, 0)$
- $E = A \cup (\mathcal{N}+1)$, where $\mathcal{N}$ is the non-measurable set, and $\mathcal{N}+1 = \{x+1 \mid x \in \mathcal{N}\}$ is non-measurable by the same argument used for $\mathcal{N}$.
- $B = \mathbb{R}$

<u>Claim</u>: E is not measurable.

Supposing it were, note that A^c is measurable, and countable intersections of measurable sets are measurable, so

$$E \cap A^c = (A \cup (\mathcal{N}+1)) \cap A^c = \mathcal{N}+1$$

must be measurable. $\times$

4) Let A, B be fixed, and define

$$E_t := \{x \in \mathbb{R}^n \mid \inf_{a \in A} |x-a| \leq t\} \cap B$$
$$= \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,A) \leq t\} \cap B$$

and

$$f: \mathbb{R} \to \mathbb{R}$$
$$t \mapsto \mu(E_t)$$

Note that $E_0 = A$, so $f(0) = \mu(A)$, and since B is compact and thus bounded, there is some $t = T$ such that $B \subseteq E_T$.

So f maps $[0,T]$ to $[\mu(A), \mu(B) + M]$ for some M.

<u>Claim:</u> f is cts, and for all $t \in [0,T']$ for some T', $A \subseteq E_t \subseteq B$ and each E_t is compact.

Note that if this is true, we can first apply the intermediate value theorem to find a T' such that $f(T') = m(B)$, then restrict f to map $[0,T']$ to $[m(A), m(B)]$. We can apply it again to pull back any $c \in [m(A), m(B)]$ to a t satisfying $c = f(t) = \mu(E_t)$, in which case $A \subseteq E_t \subseteq B$ and $\mu(A) \leq c = \mu(E_t) \leq \mu(B)$ as desired.

• <u>f is cts:</u> We'll show that the 2-sided limit $\lim_{t_i \to t} f(t_i)$ exists and is equal to $f(t)$, using the fact that $a \leq b \Rightarrow E_a \subseteq E_b$.

If $t_i \nearrow t$, then $E_{t_1} \subseteq E_{t_2} \subseteq \cdots \subseteq E_t$, and $\bigcup_{i \in \mathbb{N}} E_{t_i} = E$, so by continuity of measure from below, we have $\lim_{i\to\infty} \mu(E_{t_i}) = \mu(E)$, so

$$\lim_{t_i\to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

Similarly, if $t_i \searrow t$, noting that $t_i \leq T' \Rightarrow t_1 \leq T' \Rightarrow \mu(E_{t_1}) \leq \mu(B) < \infty$,

and $E_{t_1} \supseteq E_{t_2} \supseteq \cdots \supseteq E$, so

we can apply continuity of measure from above to obtain

$$\lim_{t_i\to t} f(t_i) = \lim_{i\to\infty} \mu(E_{t_i}) = \mu(E_t) = f(t).$$

So f is cts. ∎

- E_t is compact:

Since $E_t \subseteq B$ which is compact and thus bounded, it suffices to show that E_t is closed. But letting $N_t = \{x \in \mathbb{R}^n \mid \text{dist}(x, A) < t\}$, we have $E_t = \overline{N_t \cap B}$, where N_t is open because $N_t = \bigcup_{a\in A} \underbrace{\{x \in \mathbb{R}^n \mid \text{dist}(x,a) < r\}}_{\text{Open ball around } a}$, and $N_t \subseteq B \Rightarrow N_t \cap B$ is still open. But the closure of any open set is closed. ∎

- $t \in [0, T'] \Rightarrow A \subseteq E_t \subseteq B$:

$E_0 = A$ and $t \leq s \Rightarrow E_t \subseteq E_s$, so $A \subseteq E_t$ for all t.

But $E_t = \overline{N_t \cap B} \subseteq \overline{B} = B$ since B is closed, so $E_t \subseteq B$ for all t as well. ∎

5a) Recalling that $\mathcal{N}$ is constructed by considering $\dfrac{\mathbb{R} \cap [0,1)}{\mathbb{Q} \cap [0,1)}$ and taking exactly one element from each equivalence class, we can note that if $E \subseteq \mathcal{N}$, then E contains a choice of <u>at most</u> one element from each equivalence class. We can then take a similar enumeration $\mathbb{Q} \cap [-1,1] = \{q_i\}_{i=1}^{\infty}$ and define $E_j := E + q_j$.

Then $E \subseteq \mathcal{N} \Rightarrow \bigsqcup_{j \in \mathbb{N}} E_j \subseteq \bigsqcup_{j \in \mathbb{N}} \mathcal{N}_j \subseteq [-1,2]$, and since E is measurable, we must have

$$\mu(E) = \mu\Big(\bigsqcup_{j \in \mathbb{N}} E_j\Big) = \sum_{j \in \mathbb{N}} \mu(E_j) = \sum_{j \in \mathbb{N}} \mu(E) \leq 3,$$

which can only hold if $m(E) = 0$. $\square$

5b) Suppose $\mu(I \setminus \mathcal{N}) < 1$, so $m(I \setminus \mathcal{N}) = 1 - 2\varepsilon$ for some $\varepsilon > 0$. Then choose an open $G \supseteq I \setminus \mathcal{N}$ such that $\mu(G) = \mu(I \setminus \mathcal{N}) + \varepsilon = 1 - \varepsilon$. Then $I \setminus G \subseteq \mathcal{N}$, and so by (1) we must have $\mu(I \setminus G) = 0$. But then

$$I = G \sqcup I \setminus G \Rightarrow \mu(I) = \mu(G) + \mu(I \setminus G)$$

$$\Rightarrow 1 = 1 - \varepsilon < 1, \text{ a contradiction. } \square$$

5c) Let

$$\left.\begin{aligned} E_1 &= N \\ E_2 &= I \setminus N \end{aligned}\right\} \Rightarrow I = E_1 \sqcup E_2$$

but $m_*(E_1) = m_*(N) > 0$, otherwise N would be measurable so $m_*(E_1 \sqcup F_2) = 1$ but

$m_*(E_1) + m_*(E_2) = 1 + \varepsilon$ for some $\varepsilon > 0$. $\blacksquare$

6a) Claim: E is a countable union of a countable intersection of measurable sets, and thus measurable.

Proof: Write $E = \{x \mid x \in E_j \text{ for infinitely many } j\}$, the claim is that

$$E = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j.$$

• $E \subseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose x is in infinitely many E_j. Then for any fixed k, there is some $M \geq k$ such that $x \in E_M \subseteq \bigcup_{j=k}^{\infty} E_j := S_k$. But this happens for every k, so $x \in \bigcap_{k=1}^{\infty} S_k$. $\square$

- $E \supseteq \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_j$: Suppose $x \in \bigcup_{j=k}^{\infty} E_j$ for every k. Then if x were in only finitely many E_j, we could pick a maximal E_M such that $k \geq M \Rightarrow x \notin E_k$, and so $x \notin \bigcup_{j=M}^{\infty} E_j$ – a contradiction. $\square$

<u>Claim</u>: $m(E)=0$

We'll use the fact that $\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{j \to \infty} \sum_{n=j}^{\infty} a_n = 0$, i.e. the tails of a convergent sum must become arbitrarily small.

Since $E = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$, $E \subseteq \bigcup_{j=k}^{\infty} E_j$ for all k. So $m(E) \leq \sum_{j=k}^{\infty} E_j \to 0$, forcing $m(E)=0$. $\blacksquare$

(6b) Fix x and let $E_{p,j} = \{ x \in \mathbb{R} \mid |x - p/j| \leq 1/j^3 \}$

and $E_j = \bigcup_{\substack{p \text{ coprime} \\ \text{to } j}} E_{p,j} \subseteq \bigcup_{p=1}^{j} E_{p,j}$, and since $E_{p,j} \subseteq B(1/j^3, p/j)$, $m(E_{p,j}) \leq 2/j^3$ and thus $m(E_j) \leq q(2/j^3) = 2/j^2$.

But then $\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} 2/j^2 < \infty$. Moreover,

$$E = \bigcap_{j=1}^{\infty} \bigcup_{j=k}^{\infty} E_j = \{ x \in \mathbb{R} \mid \text{there are infinitely many } j\text{'s such that there exists a } p \text{ coprime to } j \text{ s.t. } |x - p/j| \le 1/j^3 \},$$

which is precisely the set we want. So by (1), $m(E) = 0$. ∎