

# Floer Talk

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## 1 Background and Notation

From the text:

- $(W, \omega \in \Omega_2(W))$  is a (compact?) symplectic manifold
- $C^\infty(A, B)$  is the space of smooth maps with the  $C^\infty$  topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C_{\text{loc}}^\infty(A, B)$  is the space with the  $C^\infty$  uniform convergence topology on compact subsets of  $A$
- $H \in C^\infty(W; \mathbb{R})$  a Hamiltonian with  $X_H$  its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$  given by  $H_t \in C^\infty(W; \mathbb{R})$  is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned}\mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) \, dt\end{aligned}$$

where  $\mathcal{L}W$  is the contractible loop space of  $W$ ,  $u : \mathbb{D} \longrightarrow W$  is an extension of  $x : S^1 \longrightarrow W$  to the disc with  $u(\exp(2\pi it)) = x(t)$ .

$$- \text{ Example: } W = \mathbb{R}^{2n} \implies \mathcal{A}_H(x) = \int_0^1 (H_t \, dt - p \, dq).$$

- Critical points of the action functional  $\mathcal{A}_H$  are given by orbits, i.e. contractible loops  $x, y \in \mathcal{L}W$
- In general,  $x, y$  are two periodic orbits of  $H$  of period 1.

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- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u) = 0.$$

This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be  $J$ -holomorphic curves.

- Solutions are functions  $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$ 
    - They correspond to “embedded cylinders” with sides  $u$  and contractible caps  $x, y$  regarded as loops in  $W$ .
    - They also correspond to paths in  $\mathcal{L}W$  from  $x \rightarrow y$  (precisely: trajectories of the vector field  $-\text{grad} \mathcal{A}_H$ )
- 





**Fig. 6.5**

Here  $u(s) \in \mathcal{L}W$  is a loop with value at time  $t$  given by  $u(s, t)$ , and  $\lim_{s \rightarrow -\infty} u_s(t) = x$ ,  $\lim_{s \rightarrow \infty} u_s(t) = y$ .

- The energy of a solution is  $E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds dt$ .
- $\mathcal{M} = \{u \in C^\infty(\mathbb{R}; \mathcal{L}W) \mid E(u) < \infty\}$  (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x, y)$  is the space of solutions of the Floer equation connecting orbits  $x$  and  $y$ .
- $C_{\searrow}(x, y)$ :

$$C_{\searrow}(x, y) := \{u \in C^\infty(\mathbb{R} \times S^1; W) \mid \lim_{s \rightarrow -\infty} u(s, t) = x(t), \lim_{s \rightarrow \infty} u(s, t) = y(t), \\ \left| \frac{\partial u}{\partial s}(s, t) \right| \leq K e^{-\delta|s|}, \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \leq K e^{-\delta|s|}\}$$

where  $K, \delta > 0$  are constants depending on  $u$ . So

$$|\partial_s u(s, t)|, |\partial_t u(s, t) - X_H(u)| \sim e^{|s|}.$$

From the Appendices

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator:  $\dim \ker - \dim \operatorname{coker}$ .
- Fredholm operators: those for which the index makes sense, i.e.  $\dim \ker < \infty, \dim \operatorname{coker} < \infty$ .
- Elliptic operators: generalize the Laplacian  $\Delta$ , coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset

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- Sobolev spaces: in dimension 1, define  $\|u(t)\|_{s,p} = \sum_{i=0}^s \|\partial_t^i u(t)\|_{L^p}$  on  $C^\infty(\bar{U})$ , then take the completion and denote  $W^{s,p}(\bar{U})$ . Yields a distribution space, elements are functions with weak derivatives.
  - Distribution:  $C_c^\infty(U)^\vee$ , the dual of the space of smooth compactly supported functions on an open set  $U \subset \mathbb{R}^n$ .

## 2 Talk

Overview: Analyze the space  $\mathcal{M}(x, y)$  of solutions to the Floer equation connecting two orbits  $x, y$  of  $H$ . Show  $\mathcal{M}(x, y)$  is in fact a manifold of dimension  $\mu(x) - \mu(y)$ .

Strategy:

1. Describe  $\mathcal{M}(x, y)$  as the zero set of a section of a vector bundle over the Banach manifold  $\mathcal{P}(x, y)$ .
2. Apply the Sard-Smale theorem: perturb  $H$  to make  $\mathcal{M}(x, y)$  the inverse image of a regular value of some map.
3. Show that the tangent maps (?) are Fredholm operators of index  $\mu(x) - \mu(y) = \dim \mathcal{M}(x, y)$ .

Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of  $\mathcal{F}$ .

### 2.1 8.3: The Space of Perturbations of $H$

Goal: given a fixed Hamiltonian  $H \in C^\infty(W \times S^1; \mathbb{R})$ , perturb it (without modifying the periodic orbits) so that  $\mathcal{M}(x, y)$  are manifolds of the expected dimension.

Start by trying to construct a subspace  $\mathcal{C}_\varepsilon^\infty(H) \subset C^\infty(W \times S^1; \mathbb{R})$ , the space of perturbations of  $H$  depending on a certain sequence  $\varepsilon = \{\varepsilon_k\}$ , and show it is a dense subspace.



Idea: similar to how you build  $L^2(\mathbb{R})$ , define a norm  $\|\cdot\|_\varepsilon$  on  $C^\infty_\varepsilon(H)$  and take the subspace of finite-norm elements.

- Let  $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$  denote a perturbation of  $H$ .
- Fix  $\varepsilon = \{\varepsilon_k \mid k \in \mathbb{Z}^{\geq 0}\} \subset \mathbb{R}^{>0}$  a sequence of real numbers, which we will choose carefully later.
- For a fixed  $\mathbf{x} \in W, t \in \mathbb{R}$  and  $k \in \mathbb{Z}^{\geq 0}$ , define

$$|d^k h(\mathbf{x}, t)| = \max \left\{ |d^\alpha h(\mathbf{x}, t)| \mid |\alpha| = k \right\},$$

the maximum over all sets of multi-indices  $\alpha$  of length  $k$ .

Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \dots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

- Define a norm on  $C^\infty(W \times S^1; \mathbb{R})$ :

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} |d^k h(x,t)|.$$

- Since  $W \times S^1$  is assumed compact (?), fix a finite covering  $\{B_i\}$  of  $W \times S^1$  such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

- Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}$$

- Obtain the computable form

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \Psi_i^{-1})(z)|.$$

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \mid \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

- Show that the sequence  $\{\varepsilon_k\}$  can be chosen so that  $C_\varepsilon^\infty$  is a *dense* subspace for the  $C^\infty$  topology, and in particular for the  $C^1$  topology.

**Proposition 2.1.**

Such a sequence  $\{\varepsilon_k\}$  can be chosen.

**Lemma 2.2.**

$C^\infty(W \times S^1; \mathbb{R})$  with the  $C^1$  topology is separable as a topological space (contains a countable dense subset).

*Proof (of Lemma, Sketch).*

First prove for  $C^0$ :

- **Idea:** reduce to polynomials in  $\mathbb{R}^m$ .
- Embed  $W \times S^1 \hookrightarrow [-M, M]^m \cong I^m \subset \mathbb{R}^m$  for some large  $m$ , reduces to proving it for  $C^\infty(I^m; \mathbb{R})$ .
- Recall Stone-Weierstrass:
 

For  $A \leq C^0(X; \mathbb{R})$  a subalgebra with  $X$  compact Hausdorff and  $A$  containing a nonzero constant function,  $A$  is dense iff it separates points (for all  $a \neq b \in X$  there exists  $f \in A$  such that  $f(a) \neq f(b)$ )
- Apply to  $A = \mathbb{Q}[x_1, \dots, x_m]$  the subalgebra of polynomial functions, the nonzero constant function  $c(x) = 1$ , and show it separates points via  $f(x) = x - a$ , then  $f(a) = 0$  and  $f(b) = a - b \neq 0$  by assumption.

- Thus  $A$  is a countable dense subset.

Then prove for  $C^1$ :

- **Idea:** Take polynomials convolved with a countable sequence of bump functions, which is still a countable dense subset.
- Choose a smooth bump function  $\chi$  supported on  $B(0, 1)$
- Define the sequence  $\chi_k(x) := k^m \chi(kx)$ .
- Prove that  $(f * \chi_k) \xrightarrow{k \rightarrow \infty} f$  in the  $C_{\text{loc}}^0$  sense (?)
- Show that for a fixed  $k$ , any other sequence  $g_\ell \rightarrow f$  in  $C_{\text{loc}}^\infty$ , we have  $g_\ell * \chi_k \rightarrow f * \chi_k$  in the  $C_{\text{loc}}^0$  sense using

$$|g_\ell - f| \rightarrow 0 \implies \sup_K \left| \frac{\partial}{\partial x_i} (g_\ell - f) * \chi_k \right| \leq \sup_k |g_\ell - f| \cdot (\dots) \rightarrow 0 \quad \forall i$$

- Conclude  $\lim_\ell \lim_k g_\ell * \chi_k = f$ .
- Taking  $g_\ell$  to be polynomial approximations, the following subset is countable and dense:

$$\bigcup_{k \in \mathbb{Z}^{\geq 0}} \{P * \chi_k \mid P \in \mathbb{Q}[x_1, \dots, x_m]\}$$

which are pushed through the charts  $\Psi_i$  to actually compute. ■

The second part of this proof generalizes to  $C^\infty$ .

*Proof (of Proposition, Sketch).*

- By the lemma, produce a sequence  $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$  dense for the  $C^1$  topology.
- Using the norm on  $C^n(W \times S^1; \mathbb{R})$  for the  $f_n$ , define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \mid k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable. ■

Why does this imply density? I don't know.

The next proposition establishes a version of this theorem with compact support:

**Proposition 2.3.**

For any  $(\mathbf{x}, t) \in U \in W \times S^1$  there exists a  $V \subset U$  such that every  $h \in C^\infty(W \times S^1; \mathbb{R})$  can be approximated in the  $C^1$  topology by functions in  $C_\varepsilon^\infty$  supported in  $U$ .

Then fix a time-dependent Hamiltonian  $H_0$  with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \mid h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then  $\text{supp}(h)$  is “far” from  $\text{Per}(H_0)$ , so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

## 2.2 Review 8.2

What is  $\mathcal{F}$ ?

We started with the unadorned Floer map:

$$\begin{aligned} \mathcal{F} : \mathcal{C}^\infty(\mathbb{R} \times S^1; W) &\longrightarrow \mathcal{C}^\infty(\mathbb{R} \times S^1; TW) \\ u &\mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H_t) \end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u). \end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned} \mathcal{P}^{1,p}(x, y) & \text{ :? } \longrightarrow ? \\ (s, t) & \mapsto \exp_{w(s,t)} Y(s, t). \end{aligned}$$

where  $Y \in W^{1,p}(w^*TW)$  and  $w \in C_\infty^\infty(x, y)$ .

### 2.3 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Choose  $m > n = \dim(W)$  and embed  $TW \hookrightarrow \mathbb{R}^m$  to identify tangent vectors (such as  $Z_i$ , tangents to  $W$  along  $u$  or in a neighborhood  $B$  of  $u$ ) with actual vectors in  $\mathbb{R}^m$ .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify

$$\text{im } \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential  $d\mathcal{F}$ .

We’ve just replaced the codomain here.

Recall that

- $x, y$  are contractible loops in  $W$  that are nondegenerate critical points of the action functional  $\mathcal{A}_H$ ,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$  denotes a fixed solution to the Floer equation,
- $C_\infty(x, y)$  was the set of solutions  $u : \mathbb{R} \times S^1 \longrightarrow W$  satisfying some conditions.



Recall:

$$C_{\searrow}(x, y) := \{u \in C^\infty(\mathbb{R} \times S^1; W) \mid \lim_{s \rightarrow -\infty} u(s, t) = x(t), \lim_{s \rightarrow \infty} u(s, t) = y(t)\} \\ \left| \frac{\partial u}{\partial t}(s, t) \right| \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

Fix a solution

$$u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way: the loops  $x, y$  are contractible, so they bound discs. So we extend by pushing these discs out slightly::



From earlier in the book, we have

**Assumption (6.22):**

For every  $w \in C^\infty(S^2, W)$  there exists a symplectic trivialization of the fiber bundle  $w^*TW$ , i.e.  $\langle c_1(TW), \pi_2(W) \rangle = 0$  where  $c_1$  denotes the first Chern class of the bundle  $TW$ .

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps  $S^2 \longrightarrow W$  lift to  $B^3 \iff \pi_2(W) = 0$ .

We have a pullback that is a symplectic fiber bundle:

$$\begin{array}{ccc} \tilde{u}^*TW & \xrightarrow{d\tilde{u}} & TW \\ \downarrow & \lrcorner & \downarrow \\ S^2 & \xrightarrow{\tilde{u}} & W \end{array}$$

- Using the assumption, trivialize the pullback  $\tilde{u}^*TW$  to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

where

- The frame depends smoothly on  $(s, t) \in S^2$ ,
- $\lim_{s \rightarrow \pm\infty} Z_i$  exists for each  $i$ .
- 

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \partial t} \quad \curvearrowright \quad Z_i \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{for each } i$$

Claim: such trivializations exist, “using cylinders near the spherical caps in the figure”.

Recall what  $\mathcal{P}^{1,p}(x, y)$ ,  $J$ ,  $X_t$  are here.

- Use this frame to define a chart centered at  $u$  of  $\mathcal{P}^{1,p}(x, y)$  given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

- Define and compute the differential of the composite map  $\tilde{\mathcal{F}}$  defined as follows:

$$\begin{array}{ccc} & \tilde{\mathcal{F}} & \\ & \curvearrowright & \\ \mathcal{P}^{1,p}(x, y) & \xrightarrow{\mathcal{F}} L^p(\mathbb{R} \times S^1; TW) & \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\ & & \\ u & \xrightarrow{\tilde{\mathcal{F}}} & \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} - X_t(u) \right) \end{array}$$

- From now on, let  $\mathcal{F}$  denote  $\tilde{\mathcal{F}}$ .

- Take the vector

$$Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

- View  $Y$  as a vector in  $\mathbb{R}^m$  tangent to  $W$ , given by  $Y = \sum_{i=1}^{2n} y_i Z_i$ .

- Plug  $u + Y$  into the equation for  $\mathcal{F}$ , directly yielding

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u) \\ \implies \mathcal{F}(u + Y) &= \frac{\partial(u + Y)}{\partial s} + J(u + Y) \frac{\partial(u + Y)}{\partial t} - J(u + Y) X_t(u + Y) \end{aligned}$$

- Extract the part that is linear in  $Y$  and collect terms:

$$\begin{aligned} (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \end{aligned}$$

- This is a sum of two differential operators:
  - \* One of order 1, one of order 2 (Perspective 1)
  - \* The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)
- Now compute in charts. Need a lemma:

**Lemma 2.4 (Leibniz Rule).**

For any  $J, Y, v$  with

- $\text{im } J \subset \text{End}(\mathbb{R}^m)$
- $\text{im } Y, \text{im } v \subset \mathbb{R}^m$ ,

We have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

*Proof .*

Differentiate the map  $x \mapsto J(x)v(x)$ :

$$J(x+Y)v(x+y) = (J(x) + (dJ)_x(Y)) \cdot (v(x) + (dv)_x(Y)).$$

■

For every such smooth map  $u : \mathbb{R} \times S^1 \longrightarrow W$ ,  $(d\mathcal{F})_u(Y) = O_1 + O_0$  where  $O_i$  are differential operators of order  $i$ , and in fact  $O_1$  can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose  $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$  where  $S : \mathbb{R} \times S^1 \longrightarrow \text{End}(\mathbb{R}^n)$  is linear of order 0, and in fact we have

**Proposition 2.5.**

If  $u$  solves Floer's equation, then  $(d\mathcal{F})_u = \bar{\partial} + S(s, t)$  where  $S$  is linear, tends to a symmetric operator as  $s \longrightarrow \pm\infty$ , and  $\lim \partial_t S = 0$  uniformly in  $t$ .

There is a very long computational proof.

Denote the order 0 part of  $(d\mathcal{F})_u$  as  $Y \mapsto S \cdot Y$  so  $S : \mathbb{R} \times S^1 \longrightarrow \text{End}(\mathbb{R}^m)$  and define  $S^\pm := \lim_{s \longrightarrow \pm\infty} S(s, \cdot)$ .

**Proposition 2.6.**

The equation  $\partial_t Y = J_0 S^\pm Y$  linearizes Hamilton's equation  $\dot{z} = X_t(z)$  at  $x = \lim_{s \longrightarrow \pm\infty} u$  for  $S^+$  and  $S^-$  respectively.

Proof: uses previous proposition.

Given a solution  $u$ , the product

$$\begin{aligned} u \cdot s &: ? \longrightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and  $\mathcal{F}(u \cdot s) = 0$  for all  $s$ .

**Punchline:**

Thus  $\frac{\partial u}{\partial s}$  is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s} \mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left( \frac{\partial u}{\partial s} \right).$$

Along any nonconstant solution connecting  $x$  and  $y$ ,  $\dim \ker(d\mathcal{F})_u \geq 1$ .