

Title

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1 | Lecture 09

Last time:

- The Čech-to-derived spectral sequence,
- The Mayer Vietoris LES,
 - Computes the étale cohomology of a scheme using a Zariski open cover.
- Étale cohomology of quasicoherent sheaves,
 - Agrees with Zariski cohomology, first legitimate computation!
 - Use this to compute:
- Étale cohomology of \mathbb{F}_p in characteristic p .

Last time we had a scheme X/\mathbb{F}_p and the *Artin-Schreier* exact sequence of sheaves of $X_{\text{ét}}$:

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}_X^{\text{ét}} \xrightarrow{t \mapsto t^p - t} \mathcal{O}_X^{\text{ét}} \rightarrow 0.$$

The map appearing here is referred to as the *Artin-Schreier* map f . This works over arbitrary fields of characteristic p , with a modified definition replacing t^p .

Exercise 1.0.1 (?): Check that this is an additive homomorphism of abelian sheaves. This follows from the fact that Frobenius itself is.

Remark 1.0.2: From here onward, H^i will denote $H_{\text{ét}}^i$.

Recall that we had a theorem last time showing that the étale cohomology of quasicoherent sheaves is equivalent to the usual Zariski cohomology. From this we got a long exact sequence:

$$\begin{array}{ccccc} H^i(X_{\text{ét}}, \mathbb{F}_p) & \longrightarrow & H^i(X, \mathcal{O}_X) & \xrightarrow{f} & H^i(X, \mathcal{O}_X) \\ & \nwarrow \delta & & & \\ & & \cdots & \longrightarrow & H^{i-1}(X, \mathcal{O}_X) \end{array}$$

We don't know how to compute $H^i(X_{\text{ét}}, \mathbb{F}_p)$ generally, but the affine case is easy. For X affine, $H^{>0}(X, \mathcal{O}_X) = 0$, which in fact holds for any quasicoherent sheaf replacing \mathcal{O}_X , and $H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|}$ where the exponent is the number of connected components of X . So we get an exact sequence

$$\begin{array}{ccccc}
 H^1(X, \mathcal{O}_X) & \xrightarrow{\quad} & 0 & & \\
 & \nwarrow & & & \\
 H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|} & \xrightarrow{\quad} & \mathcal{O}_X(X) & \xrightarrow{f} & \mathcal{O}_X(X) \\
 & \nwarrow & & & \\
 & & 0 & &
 \end{array}$$

Remark 1.0.3: $H^1(X, \mathcal{O}_X)$ is not finitely generated in general, e.g. take $X := \mathbb{A}^1$

