

# Title

D. Zack Garza

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## Contents

<b>1 Thursday, September 10</b>	<b>1</b>
1.1 Proof of Dimension Proposition . . . . .	1
1.1.1 Proof That $P_1$ is Principle . . . . .	2
1.2 Using Dimension Theory . . . . .	3

## 1 Thursday, September 10

Recall that the dimension of a ring  $R$  is the length of the longest chain of prime ideals. Similarly, for an affine variety  $X$ , we defined  $\dim X$  to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking  $R = A(X)$ , i.e.  $\dim \mathbb{A}^n/k = n$ .

### Proposition 1.1 (*Dimensions*).

Let  $k = \bar{k}$ .

- a. The dimension of  $k[x_1, \dots, x_n]$  is  $n$ .
- b. All maximal chains of prime ideals have length  $n$ .

### 1.1 Proof of Dimension Proposition

The case for  $n = 0$  is trivial, just take  $P_0 = \langle 0 \rangle$ . For  $n = 1$ , easy to see since the only prime ideals in  $k[x]$  are  $\langle 0 \rangle$  and  $\langle x - a \rangle$ , since any polynomial factors into linear factors.

Let  $P_0 \subsetneq \dots \subsetneq P_m$  be a maximal chain of prime ideals in  $k[x_1, \dots, x_n]$ ; we then want to show that  $m = n$ . Assume  $P_0 = \langle 0 \rangle$ , since we can always extend our chain to make this true (using maximality). Then  $P_1$  is a minimal prime and  $P_m$  is a maximal ideal (and maximals are prime).

**Claim:**  $P_1$  is principle, i.e.  $P_1 = \langle f \rangle$  for some irreducible  $f$ .

1.1.1 Proof That  $P_1$  is Principle

**Claim:**  $k[x_1, \dots, x_n]$  is a unique factorization domain. This follows since  $k$  is a UFD since it's a field, and  $R$  a UFD  $\implies R[x]$  is a UFD for any  $R$ .

See Gauss' lemma.

**Claim:** In a UFD, minimal primes are principal. Let  $r \in P$ , and write  $r = u \prod p_i^{n_i}$  with  $p_i$  irreducible and  $u$  a unit. So some  $p_i \in P$ , and  $p_i$  irreducible implies  $\langle p_i \rangle$  is prime. Since  $0 \subsetneq \langle p_i \rangle \subset P$ , but  $P$  was prime and assumed minimal, so  $\langle p_i \rangle = P$ .

The idea is to now transfer the chain  $P_0 \subsetneq \dots \subsetneq P_m$  to a maximal chain in  $k[x_1, \dots, x_{n-1}]$ . The first step is to make a linear change of coordinates so that  $f$  is monic in the variable  $x_n$ .

**Example 1.1.**

Take  $f = x_1x_2 + x_3^2x_4$  and map  $x_3 \mapsto x_3 + x_4$ .

So write

$$f(x_1, \dots, x_n) = x_n^d + f_1(x_1, \dots, x_{n-1})x_n^{d-1} + \dots + f_d(x_1, \dots, x_{n-1}).$$

We can then descend to  $k[x_1, \dots, x_n]$  to  $k[x_1, \dots, x_n]/\langle f \rangle$ :

$$\begin{array}{ccccccc} P_0 & \longrightarrow & P_1 & \longrightarrow & \dots & \longrightarrow & P_m \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P_1/P_1 & \longrightarrow & \dots & \longrightarrow & P_m/P_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & P_1/P_1 \cap k[x_1, \dots, x_{n-1}] & \longrightarrow & \dots & \longrightarrow & (P_m/P_1) \cap k[x_1, \dots, x_{n-1}] \end{array}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

**Definition 1.1.1** (Integral Extension).

An *integral* ring extension  $R \hookrightarrow R'$  of  $R$  is one such that all  $r' \in R'$  satisfying a monic polynomial with coefficients in  $R$ , where  $R'$  is finitely generated.

In this case, also implies that  $R'$  is a finitely-generated  $R$  module.

In this case,  $k[x_1, \dots, x_{n-1}] \hookrightarrow k[x_1, \dots, x_n]/\langle f \rangle$  is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

**Lemma 1.2.**

Suppose  $P', Q' \subset R'$  are distinct prime ideals with  $R \hookrightarrow R'$  an integral extension. Then if  $P' \cap R = Q' \cap R$ , neither contains the other, i.e.  $P' \not\subset Q'$  and  $Q' \not\subset P'$ .

*Proof.*

Toward a contradiction, suppose  $P' \subset Q'$ , we then want to show that  $Q' \supset P'$ . Let  $a \in Q' \setminus P'$

(again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.

Then  $\bar{a} \neq 0$  in  $R'/P'$ , and there exists a monic polynomial of minimal degree that  $\bar{a}$  satisfies,  $p(x) = x^n + \sum_{i=2}^n \bar{c}_i x^{n-i}$ . This implies  $\bar{c}_n \in Q'/P'$  (which will contradict  $c_n \in P'$ ), since if  $\bar{c}_n = 0$  then factoring out  $x$  yields a lower degree polynomial that  $\bar{a}$  satisfies. But then  $\bar{a}_n \in Q' \cap R$ , so ???

■

Question: Given  $R \hookrightarrow R'$  is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the “Going Up” Theorem: given  $P \subset R$  prime, there exists  $P' \subset R'$  prime such that  $P' \cap R = P$ . Furthermore, we can lift  $P_1 \subset P_2$  to  $P'_1 \subset P'_2$ , as well as “lifting sandwiches”:

Figure 1: Image

In this process, the length of the chain decreased since  $\langle 0 \rangle$  was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies. ■

## 1.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if  $R \hookrightarrow R'$  is integral then  $\dim R = \dim R'$ .

**Claim:** Any affine variety has finite dimension.

*Proof.*

We have  $\dim X = \dim A(X)$ , where  $A(X) := k[x_1, \dots, x_n]/I$  for some  $I(X) = \sqrt{I(X)}$ .

The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated  $k$ -algebra is an integral extension of some polynomial ring  $k[y_1, \dots, y_d]$ . I.e., the following extension is integral:

$$k[y_1, \dots, y_d] \hookrightarrow k[x_1, \dots, x_n]/I.$$

We can conclude that  $\dim A(X) = d < \infty$ .

■

**Proposition 1.3(?)**.

Let  $X, Y$  be irreducible affine varieties. Then

- a.  $\dim X \times Y = \dim X + \dim Y$ .
- b.  $Y \subset X \implies \dim X = \dim Y + \operatorname{codim}_X Y$ .
- c. If  $f \in A(X)$  is nonzero, then any component of  $V(f)$  has codimension 1.

*Proof .*

**Remark .**

Why is  $X \times Y$  again an affine variety? If  $X \subset \mathbb{A}^n/k$ ,  $Y \subset \mathbb{A}^m/k$  with  $X = V(I)$ ,  $Y = V(J)$ , then  $X \times Y \subset \mathbb{A}^n/k \times \mathbb{A}^m/k = \mathbb{A}^{n+m}/k$  can be given by taking  $I+J \trianglelefteq k[x_1, \dots, x_n, y_1, \dots, y_m]$  using the natural inclusions of  $k[x_1, \dots, x_\ell]$ .

Note that we can write

$$k[x_1, \dots, x_n, y_1, \dots, y_m] = k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m]$$

where we think of  $x_i = x_i \otimes 1$ ,  $y_j = 1 \otimes y_j$ . We thus map  $I, J$  to  $I \otimes 1 + 1 \otimes J$  and obtain  $V(I \otimes 1 + 1 \otimes J) = X \times Y$  and  $A(X \times Y) = A(X) \otimes_k A(Y)$ .

In general, for  $k$ -algebras  $R, S$ ,

$$R/I \otimes_k S/J \cong R \otimes_k S / \langle I \otimes 1 + 1 \otimes J \rangle.$$

**Remark .**

For  $R, S$  finitely generated  $k$ -algebras,  $\dim R \otimes_k S = \dim R + \dim S$ .

Part (a) is proved by the above remarks.

For part (b), the statement is equivalent to  $P \subset A(X)$  with  $I(Y) \subset P$  is a member of some maximal chain, along with the statement that all maximal chains are the same length. ■