

# Category $\mathcal{O}$ , Problem Set 4

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## 1 Humphreys 3.1

Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and identify  $\lambda \in \mathfrak{h}^\vee$  with a scalar. Let  $N$  be a 2-dimensional  $U(\mathfrak{b})$ -module defined by letting  $x$  act as 0 and  $h$  act as  $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ .

Show that the induced  $U(\mathfrak{g})$ -module structure  $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$  fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

### 1.1 Solution

Reference 1 Reference 2

Hence  $M \notin \mathcal{O}$ .

We first unpack all definitions in terms of tensor products, using the fact that  $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\lambda$ :

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$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(\lambda) & \longrightarrow & M & \longrightarrow & M(\lambda) & \longrightarrow & 0 \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
0 & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & 0
\end{array}$$

$$1 \otimes 1 \xrightarrow{\psi} 1 \otimes \mathbf{u} \xrightarrow{\phi} 1 \otimes 0$$

$$1 \otimes \mathbf{v} \xrightarrow{\quad} 1 \otimes 1$$

where  $N = \text{span}_{\mathbb{C}} \{\mathbf{u}, \mathbf{v}\}$ .

We make the following claims:

1. The  $U(\mathfrak{b})$  action defined on  $N$  lifts to a  $U(\mathfrak{g})$ -action on  $M$ .
2. This is an exact sequence of  $U(\mathfrak{g})$ -modules.
3.  $M \not\cong M(\lambda) \oplus M(\lambda)$ , showing that this sequence can not split.

**Claim 1:** We choose the basis

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and note that in the triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ , we have

$$\begin{aligned}
\mathfrak{n}^- &= \mathbb{C} \cdot x \\
\mathfrak{g} &= \mathbb{C} \cdot h \\
\mathfrak{n}^+ &= \mathbb{C} \cdot y
\end{aligned}$$

Since the action is defined over  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$  and  $x$  acts by zero, we obtain a  $\mathfrak{g}$ -action on  $N$  which thus extends uniquely to a  $U(\mathfrak{g})$ -action.

We note that by using a PBW basis for  $\mathfrak{sl}(2, \mathbb{C})$ ,  $M$  is spanned by infinitely many basis elements of the form  $\{y^k \otimes \mathbf{u}, y^j \otimes \mathbf{v} \mid j, k \in \mathbb{Z}^{\geq 0}\}$ . We also note that identifying the image  $\psi(M(\lambda)) = \mathbb{C} \cdot \mathbf{u} \cong \mathbb{C}_\lambda$  (as  $U(\mathfrak{b})$ -modules) we obtain a submodule of  $N$ , which lifts to a submodule of the induced module  $M$ . Similarly,

## 2 Humphreys 3.2

Show that for  $M \in \mathcal{O}$  and  $\dim L < \infty$ ,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

## 2.1 Solution

We first note that  $M \in \mathcal{O} \implies M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$  where each  $M_\lambda$  is a finite-dimensional weight space.

Moreover,  $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$  is defined to be a direct sum of duals of weight spaces, which are still finite-dimensional.

So let  $M, N \in \mathcal{O}$ ; we will proceed by showing that both  $(M \otimes_{\mathbb{C}} L)^\vee$  and  $M^\vee \otimes_{\mathbb{C}} L^\vee$  have identical direct sum decompositions.

We first have

$$\begin{aligned}
 (M \otimes_{\mathbb{C}} L)^\vee &:= \bigoplus_{\lambda \in \mathfrak{h}^\vee} (M \otimes_{\mathbb{C}} L)_{\lambda}^\vee, && \text{the } \lambda \text{ weight spaces of } M \otimes_{\mathbb{C}} L \\
 &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left( \bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta) \right)^\vee && \text{by an exercise on the weight spaces of a tensor product} \\
 &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left( \bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta)^\vee \right) && \text{since the inner term is a finite sum} \\
 &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left( \bigoplus_{\alpha+\beta=\lambda} (M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee) \right) && \text{since the weight spaces are finite-dimensional,}
 \end{aligned}$$

where we've repeatedly used the fact that  $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$  for finite-dimensional vector spaces, which inductively holds for any finite direct sum of vector spaces.

On the other hand, using the fact that

$$\begin{aligned}
 (A \oplus B) \otimes (C \oplus D) &= ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \\
 &= (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \\
 \implies \left( \bigoplus_{j \in J} A_j \right) \otimes \left( \bigoplus_{k \in K} B_k \right) &= \bigoplus_{j \in J} \bigoplus_{k \in K} (A_j \otimes B_k) \quad \text{by induction} \quad .
 \end{aligned}$$

we can write

$$\begin{aligned}
 M^\vee \otimes_{\mathbb{C}} L^\vee &:= \left( \bigoplus_{\alpha \in \mathfrak{h}^\vee} M_\alpha^\vee \right) \otimes_{\mathbb{C}} \left( \bigoplus_{\beta \in \mathfrak{h}^\vee} L_\beta^\vee \right) \\
 &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left( \bigoplus_{\alpha+\beta=\lambda} (M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee) \right),
 \end{aligned}$$

which equals what was obtained above.

This exhibits the isomorphism as  $\mathbb{C}$ -vector spaces, to see that this is in fact an isomorphism of  $U(\mathfrak{g})$ -modules we can use the fact that for  $M \in \mathcal{O}$ , a twisted  $\mathfrak{g}$ -action was defined as

$$\mathbf{v} \in M, f \in M^\vee, g \in \mathfrak{g} \implies (g \cdot f)(\mathbf{v}) = f(\tau(g) \cdot \mathbf{v})$$

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for the transpose map  $\tau$ . This action can be “linearly extended” over direct products and tensor products by taking the action component-wise, and is thus preserved by all of the isomorphisms appearing above.

Since the final terms  $\bigoplus_{\lambda \in \mathfrak{h}} \bigoplus_{\alpha+\beta=\lambda} M_\alpha^\vee \otimes L_\beta^\vee$  are identical, they carry the same action, and since they are preserved by the isomorphisms, working backwards shows that the actions on  $(M \otimes L)^\vee$  and  $M^\vee \otimes L^\vee$  must also agree, yielding the desired isomorphism.

### 3 Humphreys 3.4

Show that  $\Phi_{[\lambda]} \cap \Phi^+$  is a positive system in the root system  $\Phi_{[\lambda]}$ , but the corresponding simple system  $\Delta_{[\lambda]}$  may be unrelated to  $\Delta$ .

For a concrete example, take  $\Phi$  of type  $B_2$  with a short simple root  $\alpha$  and a long simple root  $\beta$ . If  $\lambda := \alpha/2$ , check that  $\Phi_{[\lambda]}$  contains just the four short roots in  $\Phi$ .

#### 3.1 Solution

We would like to show the following two propositions:

1.  $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \cap \Phi^+$  is a positive system in  $\Phi_{[\lambda]}$ ,
2. In general, the associated simple system  $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \cap \Delta$ .

##### 3.1.1 Proof of Proposition 1

We’ll use the definition that for an abstract root system  $\Phi$ , a positive system  $\Phi^+$  is defined by picking a hyperplane  $H$  not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of  $\Phi^+$  is on one side of  $H$ , then any subset satisfies this property as well, thus  $\Phi_{[\lambda]} \cap \Phi^+$  consists only of positive roots and thus forms a positive system.

##### 3.1.2 Proof of Proposition 2

Concretely, we can realize  $\Phi$  and  $\Delta$  as subsets of  $\mathbb{R}^2$  in the following way:

$$\begin{aligned} \Phi &= P_1 \amalg P_2 := \{[1, 0], [0, 1], [-1, 0], [0, -1]\} \amalg \{[1, 1], [-1, 1], [1, -1], [-1, -1]\} \\ \Delta &:= \{\alpha, \beta\} := \{[1, 0], [-1, 1]\}, \end{aligned}$$

where we note that  $P_1$  consists of short roots (of norm 1) and  $P_2$  of long roots (of norm  $\sqrt{2}$ ) and we’ve chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\begin{aligned} \Phi_{[\lambda]} &:= \left\{ \gamma \in \Phi \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}, & \gamma^\vee &:= \frac{2}{\|\gamma\|^2} \gamma, \\ \Delta_{[\lambda]} &:= \left\{ \gamma \in \Delta \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}. \end{aligned}$$

### 3.1 Solution

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Now choosing  $\lambda := \frac{\alpha}{2} = \left[\frac{1}{2}, 0\right]$ , we now consider the inner products  $\langle \lambda, \gamma^\vee \rangle$  for  $\gamma \in \Phi$ :

Thus

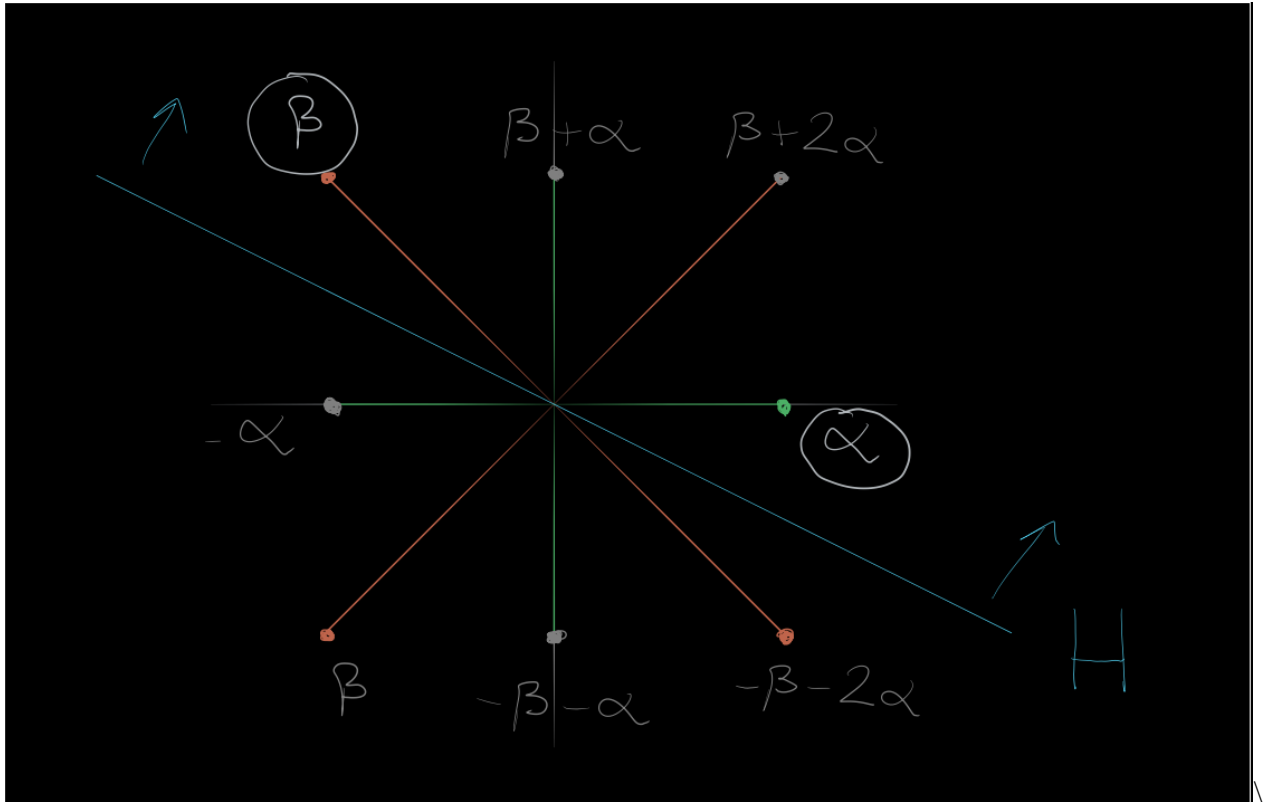
$$\begin{aligned}\gamma_1 \in P_1 &\implies \left\langle \left[\frac{1}{2}, 0\right], 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right) \langle [1, 0], \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z} \\ \gamma_2 \in P_2 &\implies \langle \lambda, \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right], \frac{2}{(\sqrt{2})^2} [\pm 1, \pm 1] \right\rangle = \pm \frac{1}{2} \notin \mathbb{Z}\end{aligned}$$

where  $(\gamma_1)_1$  denotes the first component of  $\gamma_1$ .

We thus find that

$$\begin{aligned}\Phi_{[\lambda]} &= P_1 && \text{the short roots} \\ \Delta_{[\lambda]} &= \Phi_{[\lambda]} \cap \Delta = \{\alpha\} && \text{the single short simple root.}\end{aligned}$$

Choosing the following hyperplane  $H$  not containing any root, we can choose a positive system:



$$\Phi^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$$

where we can note that  $\Phi^+ \cap \Delta = \Delta$ , since we've placed both simple roots on the positive side of this hyperplane by construction.

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But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi_{[\lambda]}^+ = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take  $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$ . But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \neq \{\alpha\} = \{\alpha, \alpha + \beta\} \cap \{\alpha, \beta\} = \Phi_{[\lambda]}^+ \cap \Delta,$$

which is what we wanted to show.

## 4 Humphreys 3.7

### 4.1 a

If a module  $M$  has a standard filtration and there exists an epimorphism  $\phi : M \rightarrow M(\lambda)$ , prove that  $\ker \phi$  admits a standard filtration.

### 4.2 b

Show by example that when  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  that the existence of a monomorphism  $\phi : M(\lambda) \rightarrow M$  where  $M$  has a standard filtration fails to imply that  $\operatorname{coker} \phi$  has a standard filtration.