

# Problem Set 10

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## 1 Problem 1

Let  $\phi$  be an  $n$ -form. It suffices to show these statements for  $n = 2$ .

$\implies$  : Suppose  $\phi$  is alternating, then  $\phi(b, b) = 0$  for all  $b \in B$ .

Letting  $a, b \in B$  be arbitrary, we then have

$$\begin{aligned}\phi(a + b, a + b) &= \phi(a, a + b) + \phi(b, a + b) \\ &= \phi(a, a) + \phi(a, b) + \phi(b, a) + \phi(b, b) \\ &= \phi(a, b) + \phi(b, a) \\ \implies \phi(a, b) &= -\phi(b, a),\end{aligned}$$

which shows that  $\phi$  is skew-symmetric.

$\Leftarrow$  Suppose  $\phi$  is skew-symmetric, so  $\phi(a, b) = -\phi(b, a)$  for all  $a, b \in B$ . Then  $\phi(b, b) = -\phi(b, b)$  by transposing the terms, which says that  $\phi(b, b) = 0$  for all  $b \in B$  and thus  $\phi$  is alternating.

## 2 Problem 2

Let  $f(x) = \det(P + xQ) \in R[x]$ , then  $f$  is a polynomial in  $x$  which is not identically zero.

To see that  $f \not\equiv 0$ , we can use that fact that  $P$  is invertible to evaluate  $f(0) = \det(P) \neq 0$ .

We can now note that  $f$  has finite degree, and thus finitely many zeroes in  $R$ .

### 3 Problem 3

Letting  $k[x] \curvearrowright_\phi E$  to yield a  $k[x]$ -module structure on  $E$  and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \mid q_2 \mid \cdots \mid q_t$$

where  $E_i = k[x]/(q_i)$ . Then  $q_t = q$ , the minimal polynomial of  $E$ .

In particular,  $E_t$  is a  $\phi$ -invariant subspace of  $E$ , and if  $\deg q_t = m$ , then  $E_t$  is in fact an  $m$ -dimensional cyclic module with basis  $\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \dots, \phi^{m-1}(\mathbf{v})\}$  for some  $\mathbf{v} \in E_t$ .

But since  $E_t \leq E$  is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

### 4 Problem 4