

# Problem Set 7

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## 1 Problem 1

### 1.1 Part a

We want to show that  $\ell^2(\mathbb{N})$  is complete, so let  $\{x_n\} \subseteq \ell^2(\mathbb{N})$  be a Cauchy sequence, so  $\|x^j - x^k\|_{\ell^2} \rightarrow 0$ . We want to produce some  $\mathbf{x} := \lim_{n \rightarrow \infty} x^n$  such that  $x \in \ell^2$ .

To this end, for each fixed index  $i$ , define

$$\mathbf{x}_i := \lim_{n \rightarrow \infty} x_i^n.$$

This is well-defined since  $\|x^j - x^k\|_{\ell^2} = \sum_i |x_i^j - x_i^k|^2 \rightarrow 0$ , and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed  $i$ , the sequence  $|x_i^j - x_i^k|^2$  is a Cauchy sequence of real numbers which necessarily converges by completeness of  $\mathbb{R}$ .

We also have  $\|\mathbf{x} - x^j\|_{\ell^2} \rightarrow 0$  since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\| \lim_{k \rightarrow \infty} x^k - x^j \right\|_{\ell^2} = \lim_{k \rightarrow \infty} \|x^k - x^j\|_{\ell^2} \rightarrow 0$$

where the limit can be passed through the norm because the map  $t \mapsto \|t\|_{\ell^2}$  is continuous. So  $x^j \rightarrow \mathbf{x}$  in  $\ell^2$  as well.

It remains to show that  $\mathbf{x} \in \ell^2(\mathbb{N})$ , i.e. that  $\sum_i |\mathbf{x}_i|^2 < \infty$ . To this end, we write

$$\begin{aligned} \|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\ &\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\ &\rightarrow M < \infty, \end{aligned}$$

where  $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$  by the previous argument, and the second term is bounded because  $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$ .  $\square$

## 1.2 Part b

Let  $H$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ .

**Lemma:** For any complex number  $z$ , we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on  $H$  takes values in  $\mathbb{C}$ , we have

$$\Re(\langle x, iy \rangle) = \Re(-i\langle x, y \rangle) = \Im(\langle x, y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, y \rangle)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, y \rangle)$$

$$\begin{aligned} \|x + iy\|^2 &= \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

$$\begin{aligned} \|x - iy\|^2 &= \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

and summing these all

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 &= 4 \Re(\langle x, y \rangle) + 4i \Im(\langle x, y \rangle) \\ &= 4\langle x, y \rangle. \end{aligned}$$

To conclude that a linear map  $U$  is an isometry iff  $U$  is unitary, if we assume  $U$  is unitary then we can write

$$\|x\|^2 := \langle x, x \rangle = \langle Ux, Ux \rangle := \|Ux\|^2.$$

Assuming now that  $U$  is an isometry, by the polarization identity we can write

$$\begin{aligned} \langle Ux, Uy \rangle &= \frac{1}{4} \left( \|Ux + Uy\|^2 + \|Ux - Uy\|^2 + i\|Ux + iUy\|^2 - i\|Ux - iUy\|^2 \right) \\ &= \frac{1}{4} \left( \|U(x + y)\|^2 + \|U(x - y)\|^2 + i\|U(x + iy)\|^2 - i\|U(x - iy)\|^2 \right) \\ &= \frac{1}{4} \left( \|x + y\|^2 + \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right) \\ &= \langle x, y \rangle. \end{aligned}$$

□

## 2 Problem 2

**Lemma:** The map  $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$  is continuous.

*Proof:*

Let  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
&= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\
&\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \\
&\rightarrow 0 \cdot M + C \cdot 0 < \infty,
\end{aligned}$$

where  $\|y_n\| \rightarrow M$  since  $y_n \rightarrow y$  implies that  $\|y_n\|$  is bounded.

## 2.1 Part a:

We want to show that sequences in  $E^\perp$  converge to elements of  $E^\perp$ . Using the lemma, letting  $\{e_n\}$  be a sequence in  $E^\perp$ , so  $y \in E \implies \langle e_n, y \rangle = 0$ . Since  $H$  is complete,  $e_n \rightarrow e \in H$ ; we can show that  $e \in E^\perp$  by letting  $y \in E$  be arbitrary and computing

$$\langle e, y \rangle = \left\langle \lim_n e_n, y \right\rangle = \lim_n \langle e_n, y \rangle = \lim_n 0 = 0,$$

so  $e \in E^\perp$ .

## 2.2 Part b:

Let  $S := \text{span}_H(E)$ ; then the smallest closed subspace containing  $E$  is  $\overline{S}$ , the closure of  $S$ . We will proceed by showing that  $E^{\perp\perp} = \overline{S}$ .

$\overline{S} \subseteq E^{\perp\perp}$ :

Let  $\{x_n\}$  be a sequence in  $S$ , so  $x_n \rightarrow x \in \overline{S}$ .

First, each  $x_n$  is in  $E^{\perp\perp}$ , since if we write  $x_n = \sum a_i e_i$  where  $e_i \in E$ , we have

$$y \in E^\perp \implies \langle x_n, y \rangle = \left\langle \sum_i a_i e_i, y \right\rangle = \sum_i a_i \langle e_i, y \rangle = 0 \implies x_n \in (E^\perp)^\perp.$$

It remains to show that  $x \in E^{\perp\perp}$ , which follows from

$$y \in E^\perp \implies \langle x, y \rangle = \left\langle \lim_n x_n, y \right\rangle = \lim_n \langle x_n, y \rangle = 0 \implies x \in (E^\perp)^\perp,$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$ :

For notation convenience, we'll just write  $S$  for  $\overline{S}$ . Let  $x \in E^{\perp\perp}$ . Noting that  $S$  is closed, we can define  $P$ , the operator projecting elements onto  $S$ , and write

$$x = Px + (x - Px) \in S \oplus S^\perp$$

But since  $\langle x, x - Px \rangle = 0$  because  $x - Px \in E^\perp$  and  $x \in (E^\perp)^\perp$ , we can rewrite the first term in this inner product to obtain

$$0 = \langle x, x - Px \rangle = \langle Px + (x - Px), x - Px \rangle = \langle Px, x - Px \rangle + \langle x - Px, x - Px \rangle,$$

where we can note that the first term is zero because  $Px \in S$  and  $x - Px \in S^\perp$ , and the second term is  $\|x - Px\|^2$ .

But this says  $\|x - Px\|^2 = 0$ , so  $x - Px = 0$  and thus  $x = Px \in S$ , which is what we wanted to show.

### 3 Problem 3

#### 3.1 Part a

We compute

$$\begin{aligned}\|e_0\|^2 &= \int_0^1 1^2 dx = 1 \\ \|e_1\|^2 &= \int_0^1 3(2x-1)^2 dx = \frac{1}{2}(2x-1)^2 \Big|_0^1 = 1 \\ \langle e_0, e_1 \rangle &= \int_0^1 \sqrt{3}(2x-1) dx = \frac{\sqrt{3}}{4}(2x-1) \Big|_0^1 = 0.\end{aligned}$$

which verifies that this is an orthonormal system.

#### 3.2 Part b

We first note that this system spans the degree 1 polynomials in  $L^2([0, 1])$ , since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from  $\{1, x\}$  to  $\{e_0, e_1\}$  which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial  $f$  to  $x^3$  is given by the projection onto this subspace, and since  $\{e_i\}$  is orthonormal this is given by

$$\begin{aligned}
f(x) &= \sum_i \langle x^3, e_i \rangle e_i \\
&= \langle x^3, 1 \rangle 1 + \langle x^3, \sqrt{3}(2x-1) \rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2 dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1) dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1) \frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}$$

We can also compute

$$\begin{aligned}
\|f - g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2 dx \\
&= \frac{1}{180} \\
\Rightarrow \|f - g\|_2 &= \frac{1}{\sqrt{180}}.
\end{aligned}$$

## 4 Problem 4

### 4.1 Part a

#### 4.1.1 i

We can first note that  $\langle 1/\sqrt{2}, \cos(2\pi nx) \rangle = \langle 1/\sqrt{2}, \sin(2\pi mx) \rangle = 0$  for any  $n$  or  $m$ , since this involves integrating either sine or cosine over an integer multiple of its period.

Letting  $m, n \in \mathbb{Z}$ , we can then compute

$$\begin{aligned}
\langle \cos(2\pi nx), \sin(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx) \sin(2\pi mx) dx \\
&= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x) dx \\
&= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2} \int_0^1 \sin(2\pi(n-m)x) dx \\
&= 0,
\end{aligned}$$

which again follows from integration of sine over a multiple of its period (where we use the fact that  $m+n, m-n \in \mathbb{Z}$ ).

Similarly,

$$\begin{aligned}
\langle \cos(2\pi nx), \cos(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx) \cos(2\pi mx) \, dx \\
&= \frac{1}{2} \int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x) \, dx \\
&= \begin{cases} \frac{1}{2} \int_0^1 \cos(4\pi nx) + 1 \, dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}$$

$$\begin{aligned}
\langle \sin(2\pi nx), \sin(2\pi mx) \rangle &= \int_0^1 \sin(2\pi nx) \sin(2\pi mx) \, dx \\
&= \frac{1}{2} \int_0^1 \cos(2\pi(m-n)x) - \cos(2\pi(m+n)x) \, dx \\
&= \begin{cases} \frac{1}{2} \int_0^1 1 - \cos(4\pi nx) \, dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

#### 4.1.2 ii

We have

$$\begin{aligned}
\langle e^{2\pi kx}, e^{-2\pi i\ell x} \rangle &= \int_0^1 e^{2\pi ikx} \overline{e^{-2\pi i\ell x}} \, dx \\
&= \int_0^1 e^{2\pi ikx} e^{-2\pi i\ell x} \, dx \\
&= \int_0^1 e^{2\pi i(k-\ell)x} \, dx \\
&= \int_0^1 1 \, dx = 1 \quad \text{if } k = \ell, \text{ otherwise:} \\
&= \left. \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)} \right|_0^1 \\
&= \frac{e^{2\pi i(k-\ell)} - 1}{2\pi i(k-\ell)} \\
&= 0,
\end{aligned}$$

since  $e^{2\pi ik} = 1$  for every  $k \in \mathbb{Z}$ , and  $k - \ell \in \mathbb{Z}$ . Thus this set is orthonormal.

## 4.2 Part b

### 4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials  $P_n(x)$  such that  $\|f - P_n\|_\infty \rightarrow 0$ , i.e. the  $P_n$  uniformly approximate  $f$  on  $[0, 1]$ .

Letting  $\varepsilon > 0$ , we can thus choose a  $P$  such that  $\|f - P\|_\infty < \varepsilon$ , which necessarily implies that  $\|f - P\|_{L^1} < \varepsilon$  since we have

$$\int_0^1 |f(x) - P(x)| \, dx \leq \int_0^1 \varepsilon \, dx = \varepsilon.$$

Thus we can write

$$f(x) = P(x) + (f(x) - P(x))$$

where  $h(x) := f(x) - P(x)$  satisfies  $\|h\|_{L^1} < \varepsilon$ . It only remains to show that  $P \in L^2([0, 1])$ , but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say  $|P(x)| \leq M < \infty$  for all  $x \in [0, 1]$ , and thus

$$\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2 \, dx \leq \int_0^1 M^2 \, dx = M^2 < \infty.$$

It follows that we can let  $g = P$  and  $h = f - P$  to obtain the desired result.

#### 4.2.2 ii

By part (i), the claim is that it suffices to show this is true for  $f \in L^2$ . In this case, we can identify

$$\begin{aligned} \int_0^1 f(x) \cos(2\pi kx) \, dx &:= \Re(\hat{f}(k)) \\ \int_0^1 f(x) \sin(2\pi kx) \, dx &:= \Im(\hat{f}(k)), \end{aligned}$$

the real and imaginary parts of the  $k$ th Fourier coefficient of  $f$  respectively.

By Bessel's inequality, we know that  $\{\hat{f}(k)\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$ , and so  $\sum_k |\hat{f}(k)| < \infty$ .

But this is a convergent sequence of real numbers, which necessarily implies that  $|\hat{f}(k)| \rightarrow 0$ . In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have  $f \in L^1$ , write  $f = g + h$  where  $g \in L^2$  and  $\|h\|_{L^1} \rightarrow 0$ . Then

$$\begin{aligned} \left| \int_0^1 f(x) \cos(2\pi kx) \, dx \right| &= \left| \int_0^1 (g(x) + h(x)) \cos(2\pi kx) \, dx \right| \\ &\leq \left| \int_0^1 g(x) \cos(2\pi kx) \, dx \right| + \left| \int_0^1 h(x) \cos(2\pi kx) \, dx \right| \\ &\leq \left| \int_0^1 g(x) \cos(2\pi kx) \, dx \right| + \int_0^1 |h(x)| |\cos(2\pi kx)| \, dx \\ &= |\hat{g}(k)| + \varepsilon \\ &\rightarrow 0, \end{aligned}$$

with a similar computation for  $\int f(x) \sin(2\pi kx)$ .  $\square$



## 5 Problem 5

### 5.1 Part 1

We use the following algorithm: given  $\{v\}_i$ , we set

- $e_1 = v_1$ , and then normalize to obtain  $\hat{e}_1 = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i, \hat{e}_k \rangle \hat{e}_k$

The result set  $\{\hat{e}_i\}$  is the orthonormalized basis.

We set  $e_1 = 1$ , and check that  $\|e_1\|^2 = 2$ , and thus set  $\hat{e}_1 = \frac{1}{\sqrt{2}}$ .

We then set

$$\begin{aligned} e_2 &= x - \langle x, \hat{e}_1 \rangle \hat{e}_1 \\ &= x - \langle x, 1 \rangle 1 \\ &= x - \int_{-1}^1 \frac{1}{\sqrt{2}} x \, dx \\ &= x - \int \text{odd function} \\ &= x, \end{aligned}$$

and so  $e_2 = x$ . We can then check that

$$\|e_2\| = \left( \int_{-1}^1 x^2 \, dx \right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set  $\hat{e}_2 = \sqrt{\frac{3}{2}}x$ .

We continue to compute

$$\begin{aligned} e_3 &= x^2 - \langle x^2, \hat{e}_1 \rangle \hat{e}_1 - \langle x^2, \hat{e}_2 \rangle \hat{e}_2 \\ &= x^2 - \frac{1}{2} \int_{-1}^1 x^2 \, dx - \frac{3}{2} x \int_{-1}^1 x^3 \, dx \\ &= x^2 - \left( \frac{1}{6} x^3 \right) \Big|_{-1}^1 + \frac{3}{2} x \int_{-1}^1 \text{odd function} \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

We can then check that  $\|e_3\|^2 = \frac{8}{45}$ , so we set

$$\begin{aligned}
\hat{e}_3 &= \sqrt{\frac{45}{8}} \left( x^2 - \frac{1}{3} \right) \\
&= \frac{1}{2} \sqrt{\frac{45}{2}} \frac{1}{3} (3x^2 - 1) \\
&= \frac{1}{3} \sqrt{\frac{45}{2}} \left( \frac{3x^2 - 1}{2} \right).
\end{aligned}$$

In summary, this yields

$$\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{2}} \\
\hat{e}_2 &= x \\
\hat{e}_3 &= \frac{1}{3} \sqrt{\frac{45}{2}} \left( \frac{3x^2 - 1}{2} \right),
\end{aligned}$$

which are scalar multiples of the first three Legendre polynomials.

## 5.2 Part b

Let  $p(x) = a + bx + cx^2$ , we are then looking for  $p$  such that  $\|x^3 - p(x)\|_2^2$  is minimized. Noting that

$$p(x) \in \text{span} \{1, x, x^2\} = \text{span} \{P_0(x), P_1(x), P_2(x)\} := S,$$

we can conclude that  $p(x)$  will be the projection of  $x^3$  onto  $S$ . Thus  $p(x) = \sum_{i=0}^2 \langle x^3, \hat{e}_i \rangle \hat{e}_i$ .

Proceeding to compute the terms in this expansion, we can note that  $\langle x^3, f \rangle$  for any  $f$  that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$\langle x^3, x \rangle x = x \int_{-1}^1 x^4 dx = \frac{2}{5}x$$

And thus  $p(x) = \frac{2}{5}x$  is the minimizer.

## 5.3 Part c

The first three conditions necessitate  $g \in S^\perp$  and  $\|g\| = 1$ . Since  $S$  is a closed subspace, we can write  $x^3 = p(x) + (x^3 - p(x)) \in S \oplus S^\perp$ , and so  $x^3 - p(x) \in S^\perp$ .

The claim is that  $g(x) := x^3 - p(x)$  is a scalar multiple of the desired maximizer. This follows from the fact that

$$|\langle x^3 - p, g \rangle| \leq \|x^3 - p\| \|g\|$$

by Cauchy-Schwarz, with equality precisely when  $g = \lambda(x^3 - p)$  for some scalar  $\lambda$ . However, the restriction  $\|g\| = 1$  forces  $\lambda = \|x^3 - p\|^{-1}$ .

A computation shows that

$$\|x^3 - p\|^2 = \int_0^1 (x^3 - \frac{2}{5}x)^2 dx = \frac{19}{525},$$

and so we can take

$$g(x) := \frac{25}{\sqrt{19}} \left( x^3 - \frac{2}{5}x \right).$$

## **6 Problem 6**

### **6.1 Part a**

### **6.2 Part b**