Title

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Tuesday 22nd September, 2020

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1.1 Review: Regular Functions

Given an affine variety X and $U \subseteq X$ open, a *regular function* $\varphi : U \to k$ is one locally (wrt the zariski topology) a fraction. We write the set of regular functions as $\mathcal{O}_X$.

Example 1.1.
$X = V(x_1x_4 - x_2x_3)$ on $U = V(x_2, x_4)^c$, the following function is regular:

$$\begin{aligned} \varphi : U &\to k \\ x &\mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0 \\ \\ \frac{x_3}{x_4} & x_4 \neq 0 \end{cases}. \end{aligned}$$

Note that this is not globally a fraction.

Definition 1.0.1 (Distinguished Open Sets)**.**
A *distinguished open set* $D(f) \subseteq X$ for some $f \in A(X)$ is $V(f)^c := \left\{x \in X \;\middle|\; f(x) \neq 0\right\}$.

These are useful because the $D(f)$ form a base for the zariski topology.

Proposition 1.1*(?)*.
For X an affine variety, $f \in A(X)$, we have

$$\mathcal{O}_X(D(f)) = \left\{\frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N}\right\}.$$

Proof .
The first reduction we made was that $\varphi \in \mathcal{O}_X(D(f))$ is expressible as $\frac{g_a}{f_a}$ on distinguished opens $D(f_a)$ covering $D(f)$. We also noted that

$$\frac{g_a}{f_a} = \frac{g_b}{f_b} \text{ on } D(f_a) \cap D(f_b) \implies f_b g_a = f_a g_b \text{ in } A(X).$$

The second step was writing $D(f) = \cup D(f_a)$, and so $V(f) = \cap_a V(f_a)$ implies that $f \in I(V(\left\{f_a \mid a \in U\right\}))$. By the Nullstellensatz, $f \in \sqrt{\left\langle f_a \mid a \in U\right\rangle}$, so $f^N = \sum k_a f_a$ for some N. So construct $g = \sum k_a g_a$, then compute

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b \sum k_a f_a = g_b f^N.$$

Thus $g/f^N = g_b/f_b$ for all b, and we can thus conclude

$$\varphi := \left\{\frac{g_b}{f_b} \text{ on } D(f_b)\right\} = g/f^N.$$

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Corollary 1.2 *(?)*.
For X an affine variety, $\mathcal{O}_X(X) = A(X)$.

Warning: For k not algebraically closed, the proposition and corollary are both false. Take $X = \mathbb{A}^1/\mathbb{R}$, then $\dfrac{1}{x^2+1} \in \mathbb{R}(x)$, but $\mathcal{O}_X(X) \neq A(X) = \mathbb{R}[x]$.

Definition 1.2.1 (Localization).
Let R be a ring and S a set closed under multiplication, then the localization at S is defined by

$$R_S := \left\{r/s \mid r \in R, s \in S\right\}/\sim.$$

where $r_1/s_1 \sim r_2/s_2 \iff s_3(s_2 r_1 - s_1 r_2) = 0$ for some $s_3 \in S$.

Example 1.2.
Let $f \in R$ and take $S = \left\{f^n \mid n \geq 1\right\}$, then $R_f := R_S$.

Corollary 1.3 *(?)*.
$\mathcal{O}_X(D(f)) = A(X)_f$ is the localization of the coordinate ring.

These requires some proof, since the LHS literally consists of functions on the topological space $D(f)$ while the RHS consists of formal symbols.

Proof .
Consider the map

$$\begin{aligned} A(X)_f &\to \mathcal{O}_X(D(f)) \\ \text{“}g/f^n\text{”} &\mapsto g/f^n : D(f) \to k. \end{aligned}$$

By definition, there exists a $k \geq 0$ such that

$$f^k(f^m g - f^n g') = 0 \implies f^k(f^m g - f^n g') = 0 \text{ as a function on } D(f).$$

Since $f^k \neq 0$ on $D(f)$, we have $f^m g = f^n g'$ as a function on $D(f)$, so $g/f^n = g'/g^m$ as functions on $D(f)$.

Surjectivity: By the proposition, we have surjectivity, i.e. any element of $|OO_x(D(f))$ can be represented by some g/f^n.

Injectivity: Suppose g/f^n defines the zero function on $D(f)$, then $g = 0$ on $D(f)$ implies that $fg = 0$ on X (i.e. $fg = 0 \in A(X)$), and we can write $f(g \cdot 1 - f^n \cdot 0) = 0$. Then $g/f^n \sim 0/1 \in A(X)_f$, which forces $g/f^n = 0 \in A(X)_f$.

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1.2 Sheaves

Idea: spaces on functions on topological spaces.

Definition 1.3.1 (Presheaf)**.**
A *presheaf* (of rings) $\mathcal{F}$ on a topological space is

1. For every open set $U \subset X$ a ring $\mathcal{F}(U)$.
2. For any inclusion $U \subset V$ a restriction map $\mathrm{Res}_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying

a. $F(\emptyset) = 0$.
b. $\mathrm{Res}_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.

Example 1.3.
The smooth functions on $\mathbb{R}$ with the standard topology, $\mathcal{F} = C^\infty$ where $C^\infty(U)$ is the set of smooth functions $U \to \mathbb{R}$. It suffices to check the restriction condition, but the restriction of a smooth function is smooth: if f is smooth on U, it is smooth at every point in U, i.e. all derivatives exist at all points of U. So if $V \subset U$, all derivatives of f will exist at points $x

Note that this also works with continuous functions.

Definition 1.3.2 (Sheaf)**.**
A *sheaf* is a presheaf satisfying an additional gluing property: given $\varphi_i \in \mathcal{F}(U_i)$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there exists a unique $\varphi \in \mathcal{F}(\cup_i U_i)$ such that $\varphi|_{U_i} = \varphi_i$.