

CRAG

The Weil Conjectures

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Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}}_q \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}_{\mathbb{F}}^n$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^n \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n + 1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{l}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{l}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

Problem: count points $N_n := X(\mathbb{F}_{q^n})$ for all n .

Definition

Define the *local zeta function* of X as the following formal power series:

$$Z_X(z) = \exp \left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$