

Problem Set 7

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October 26, 2019

Contents

1	Regular Problems	1
1.1	Problem 1	1
1.1.1	Case 1: $p = q$	2
1.1.2	Case 2: $p > q$	2
1.1.3	Case 3: $q > p$	2
1.2	Problem 2	2
2	Qual Problems	3

1 Regular Problems

1.1 Problem 1

Note that if either $p = 1$ or $q = 1$, G is a p -group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$\begin{array}{ll} n_p \equiv 1 \pmod{p} & n_q \equiv 1 \pmod{q} \\ n_p \mid q & n_q \mid p^2, \end{array}$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \{1, p, p^2\}.$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. This forces $n_p = q$, and we proceed by cases:

1.1.1 Case 1: $p = q$.

Then $\#G = p^3$ and G is a p -group. But every p -group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.1.2 Case 2: $p > q$.

Here, since $n_p \mid q$, we must have $n_p < q$. But if $n_p < q < p$ and $n_p \equiv 1 \pmod{p}$, then $n_p = 1$.

1.1.3 Case 3: $q > p$.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q :

- $n_q = p$: If $n_q = p \equiv 1 \pmod{q}$ and $p < q$, this forces $p = 1$.
- $n_q = p^2$: We will reach a contradiction by showing that this forces

$$\left| P := \bigcup_{S_p \in \text{Syl}(p, G)} S_p \setminus \{e\} \right| + \left| Q := \bigcup_{S_q \in \text{Syl}(q, G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

We have

$$\begin{aligned} |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\ &= p^2(q-1) + q(p^2-1) + 1 \\ &= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 \quad (\text{since } q > 1) \\ &= (p^2q - p^2) + (p^2 - 1) + (q-1)(p^2-1) + 1 \\ &= p^2q + (q-1)(p^2-1) \\ &\geq p^2q + (2-1)(2^2-1) \quad (\text{since } p, q \geq 2) \\ &= p^2q + 3 \\ &> p^2q = |G|, \end{aligned}$$

which is a contradiction. \square

1.2 Problem 2

We'll use the fact that $H \trianglelefteq N(H)$ for any subgroup H (following directly from the closure axioms for a subgroup), and thus

$$P \trianglelefteq N(P) \quad \text{and} \quad N(P) \trianglelefteq N^2(P).$$

Since it is then clear that $N(P) \subseteq N^2(P)$, it remains to show that $N^2(P) \subseteq N(P)$.

So if we let $x \in N^2(P)$, so x normalizes $N(P)$, we need to show that x normalizes P as well, i.e. $xPx^{-1} = P$.

However, supposing that $|G| = p^k m$ where $(p, m) = 1$, we have

$$P \leq N(P) \leq G \implies p^k \mid |N(P)| \mid p^k m,$$

so in fact $P \in \text{Syl}(p, N(P))$.

Then $P' := xPx^{-1} \in \text{Syl}(p, N(P))$ as well, since all conjugates of Sylow p -subgroups are also Sylow p -subgroups.

But since $P \trianglelefteq N(P)$, there is only *one* Sylow p -subgroup of $N(P)$, namely P . This forces $P = P'$, i.e. $P = xPx^{-1}$, which says that $x \in N(P)$ as desired. \square

2 Qual Problems