Title

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Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined $\dim X$ to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking $R = A(X)$, i.e. $\dim \mathbb{A}^n/k = n$.

Proposition 1.1 ***(Dimensions).***
Let $k = \bar{k}$.

a. The dimension of $k[x_1, \cdots, x_n]$ is n.
b. All maximal chains of prime ideals have length n.

Proof .
The case for $n = 0$ is trivial, just take $P_0 = \langle 0 \rangle$. For $n = 1$, easy to see since the only prime ideals in $k[x]$ are $\langle 0 \rangle$ and $\langle x - a \rangle$, since any polynomial factors into linear factors.

Let $P_0 \subsetneq \cdots \subsetneq P_m$ be a maximal chain of prime ideals in $k[x_1, \cdots, x_n]$; we then want to show that $m = n$. Assume $P_0 = \langle 0 \rangle$, since we can always extend our chain to make this true (using maximality). Then P_1 is a minimal prime and P_m is a maximal ideal (and maximals are prime).

Claim: P_1 is principle, i.e. $P_1 = \langle f \rangle$ for some irreducible f.

Proof .

Claim: $k[x_1, \cdots, x_n]$ is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD $\implies R[x]$ is a UFD for any R.

See Gauss' lemma.

Claim: In a UFD, minimal primes are principal. Let $r \in P$, and write $r = u \prod p_i^{n_i}$ with p_i irreducible and u a unit. So some $p_i \in P$, and p_i irreducible implies $\langle p_i \rangle$ is prime. Since $0 \subsetneq \langle p_i \rangle \subset P$, but P was prime and assumed minimal, so $\langle p_i \rangle = P$.

The idea is to now transfer the chain $P_0 \subsetneq \cdots \subsetneq P_m$ to a maximal chain in $

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