Algebraic Geometry

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Sunday 4^{th} October, 2020

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These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Philip Engel at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Sunday 4th October, 2020
02:18

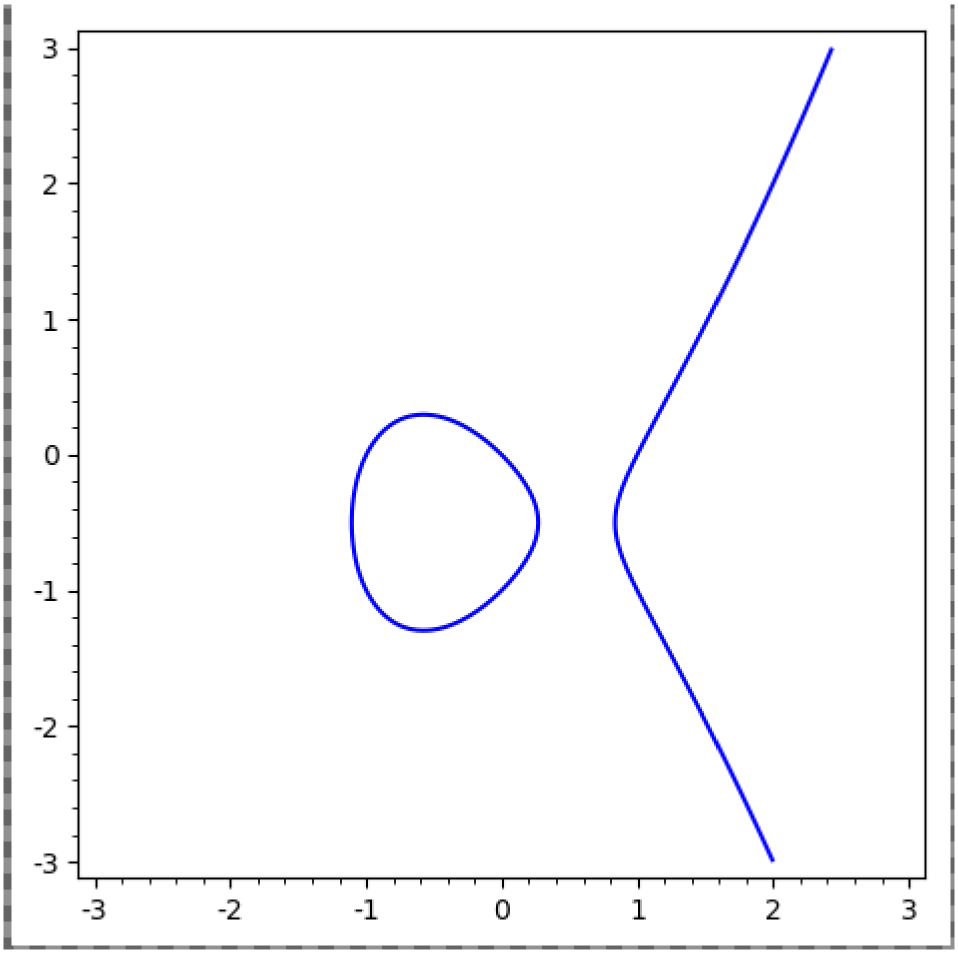
1 | Friday, August 21

> Reference:
> `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

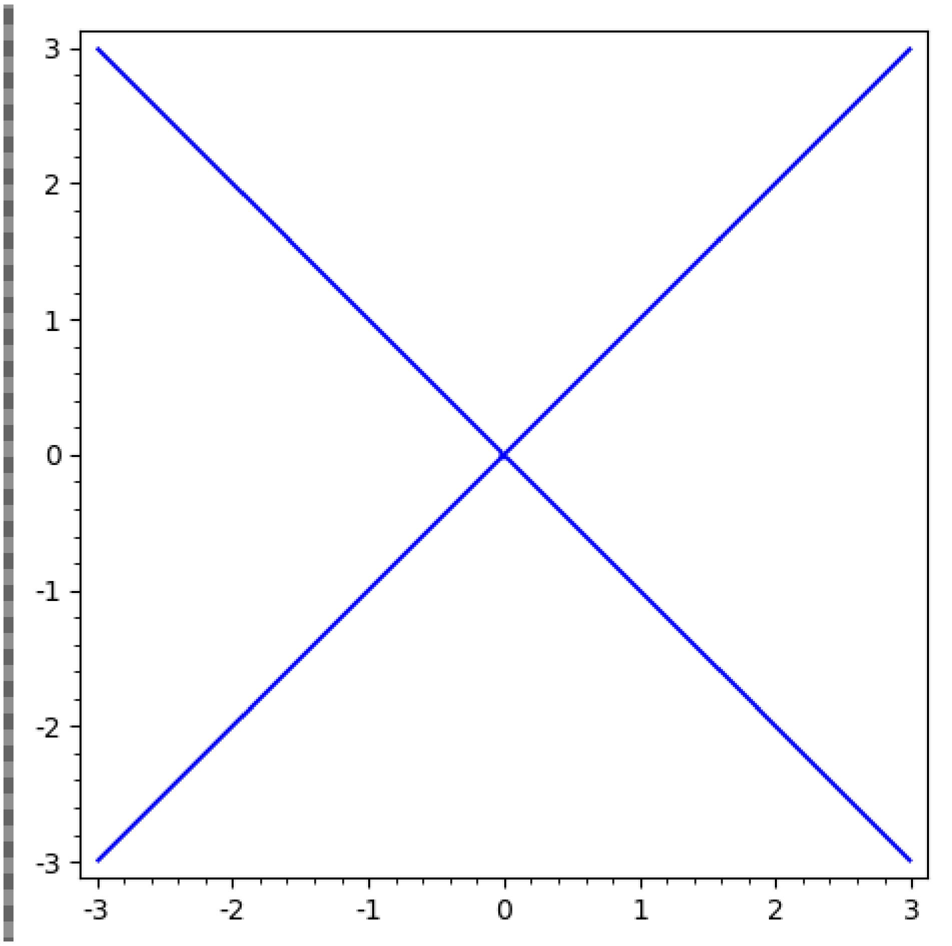
General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)
- $x^2 + 1$, which has no $\mathbb{R}$-points.
- $x^2+y^2+1/\mathbb{R}$ vanishes nowhere, so its ring of functions is not $\mathbb{R}[x,y]/\left\langle x^2+y^2+1 \right\rangle$ (problem: $\mathbb{R}$ is not algebraically closed)
- $x^2 - y^2 = 0$ over $\mathbb{C}$ is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$, which has 3 points over $\mathbb{F}_3^2$, but $f(x, y) = (x^3 - x)(y^3 - y)$ vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in $\mathbb{C}$?)
  - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x, y]/\langle f\rangle$ (addressed by scheme theory)

Theorem 1.1 ***(Harnack Curve Theorem).***
If $f \in \mathbb{R}[x, y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.
Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.1.
Show that the vanishing locus of the first two equations above is $X \cup L$ for L a line.

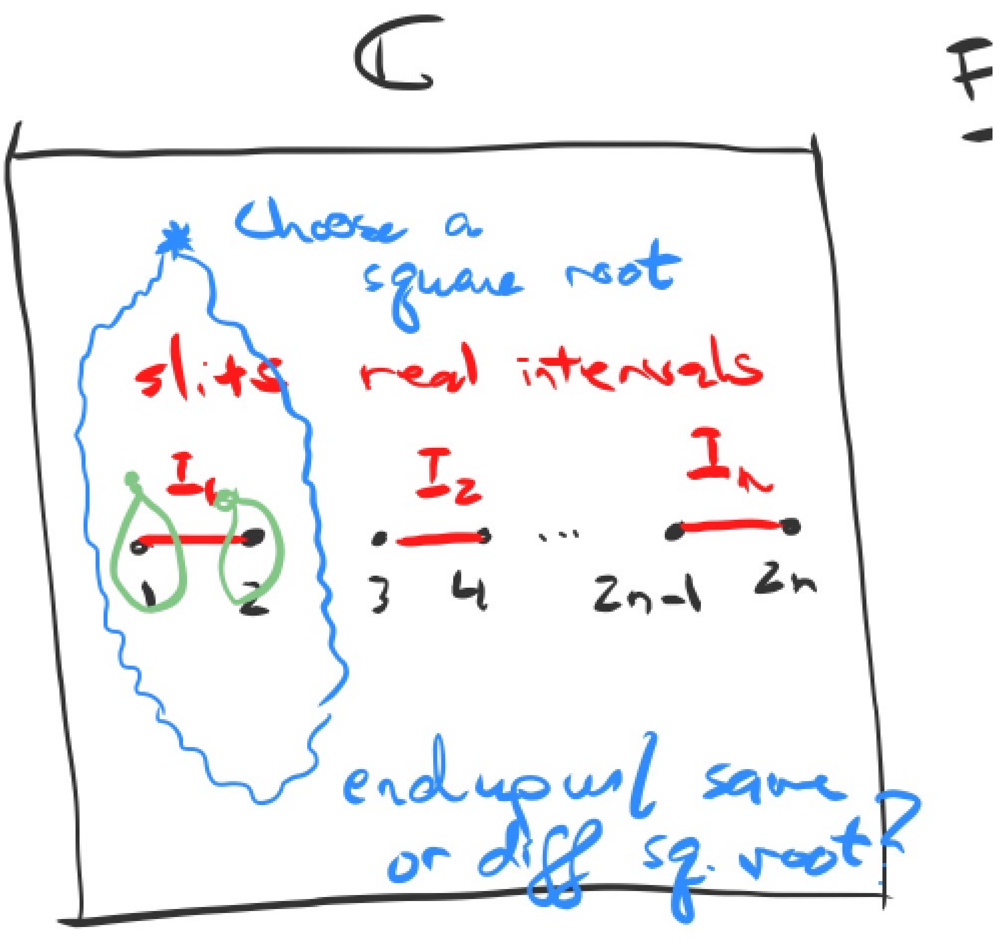
Compare to linear algebra: codimension d iff cut out by exactly d equations.

Example 1.3.
Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

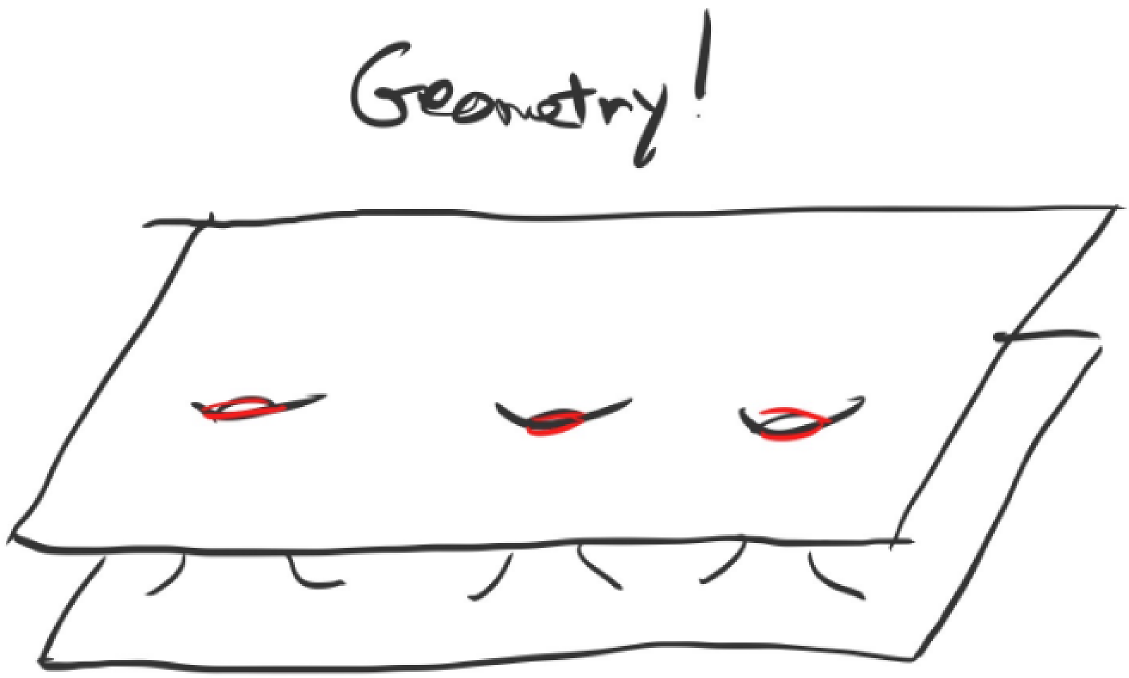
how to visualize the solution set?

Fact: on $\mathbb{C}$ with some slits, you can consistently choose a square root of the RHS.



Away from $x = 1, \cdots, 2n$, there are two solutions for y given x.

After gluing along strips, obtain:



2 | Tuesday, August 25

Let $k = \bar{k}$ and R a ring containing ideals I, J.

Definition 2.0.1 (Radical)**.**
Recall that the *radical* of I is defined as

$$\sqrt{I} = \left\{ r \in R \;\middle|\; r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

Example 2.1.
Let $I = (x_1, x_2^2) \subset \mathbb{C}[x_1, x_2]$, so $I = \left\{ f_1 x_1 + f_2 x_2 \;\middle|\; f_1, f_2 \in \mathbb{C}[x_1, x_2] \right\}$. Then $\sqrt{I} = (x_1, x_2)$, since $x_2^2 \in I \implies x_2 \in \sqrt{I}$.

Given $f \in k[x_1, \cdots, x_n]$, take its value at $a = (a_1, \cdots, a_n)$ and denote it $f(a)$. Set $\deg(f)$ to be the largest value of $i_1 + \cdots + i_n$ such that the coefficient of $\prod x_j^{i_j}$ is nonzero.

Example 2.2.
$\deg(x_1 + x_2^2 + x_1 x_2^3 = 4)$

Definition 2.0.2 (Affine Variety)**.**

1. Affine n-space $\mathbb{A}^n = \mathbb{A}^n_k$ is defined as $\left\{(a_1, \cdots, a_n) \;\middle|\; a_i \in k\right\}$.

   Remark: not k^n, since we won't necessarily use the vector space structure (e.g. adding points).

2. Let $S \subset k[x_1, \cdots, x_n]$ to be a set of polynomials. Then define $V(S) = \left\{x \in \mathbb{A}^n \;\middle|\; f(x) = 0\right\} \subset \mathbb{A}^n$ to be an *affine variety.*

Example 2.3.

- $\mathbb{A}^n = V(0)$.
- For any point $(a_1, \cdots, a_n) \in \mathbb{A}^n$, then $V(x_1 - a_1, \cdots, x_n - a_n) = \{a_1, \cdots, a_n\}$ uniquely determines the point.
- For any finite set $r_1, \cdots, r_k \in \mathbb{A}^1$, there exists a polynomial $f(x)$ whose roots are r_i.

Remark 1.
We may as well assume S is an ideal by taking the ideal it generates, $S \subseteq \langle S \rangle = \left\{\sum g_i f_i \;\middle|\; g_i \in k[x_1, \cdots, x_n],\, f_i \in S\right\}$. Then $V(\langle S \rangle) \subset V(S)$.

Conversely, if f_1, f_2 vanish at $x \in \mathbb{A}^n$, then $f_1 + f_2, gf_1$ also vanish at x for all $g \in k[x_1, \cdots, x_n]$. Thus $V(S) \subset V(\langle S \rangle)$.

Proposition 2.1 ***(Properties and Definitions of Ideal Operations).***

- $I + J := \left\{f + g \;\middle|\; f \in I,\, g \in J\right\}$.
- $IJ := \left\{\sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\, g_i \in J, N \in \mathbb{N}\right\}$.
- If $I + J = \langle 1 \rangle$ then $I \cap J = IJ$ (coprime or comaximal)

Note that if $I = \langle a \rangle$ and $J = \langle b \rangle$, then $I + J = \langle a \rangle + \langle b \rangle = \langle a, b \rangle$.

Proposition 2.2 ***(Properties of V).***

1. If $S_1 \subseteq S_2$ then $V(S_1) \supseteq V(S_2)$.
2. $V(S_1) \cup V(S_2) = V(S_1 S_2) = V(S_1 \cap S_2)$.
3. $\bigcap V(S_i) = V\left(\bigcup S_i\right)$.

We thus have a map

$$V : \{\text{Ideals in } k[x_1, \cdots, x_n]\} \to \{\text{Affine varieties in } \mathbb{A}^n\}.$$

Definition 2.2.1 (The Ideal of a Set)**.**
Let $X \subset \mathbb{A}^n$ be any set, then *the ideal of* X is defined as

$$I(X) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X \right\}.$$

Example 2.4.
Let X be the union of the x_1 and x_2 axes in $\mathbb{A}^2$, then $I(X) = (x_1 x_2) = \left\{ x_1 x_2 g \;\middle|\; g \in k[x_1, x_2] \right\}$.

Note that if $X_1 \subset X_2$ then $I(X_1) \subset I(X_2)$.

Proposition 2.3*(The Image of V is Radical).*
$I(X)$ is a radical ideal, i.e. $I(X) = \sqrt{I(X)}$.
This is because $f(x)^k = 0 \forall x \in X$ implies $f(x) = 0$ for all $x \in X$, so $f^k \in I(X)$ and thus $f \in I(X)$.

Our correspondence is thus

$$\left\{\text{Ideals in } k[x_1, \cdots, x_n]\right\} \xrightarrow{V} \left\{\text{Affine Varieties}\right\}$$
$$\left\{\text{Radical Ideals}\right\} \xleftarrow{I} \left\{?\right\}.$$

Proposition 2.4*(Hilbert Nullstellensatz (Zero Locus Theorem)).*

a. For any affine variety X, $V(I(X)) = X$.
b. For any ideal $J \subset k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

Thus there is a bijection between radical ideals and affine varieties.

2.1 Proof of Nullstellensatz

Remark 2.
Recall the Hilbert Basis Theorem: any ideal in a finitely generated polynomial ring over a field is again finitely generated.

We need to show 4 inclusions, 3 of which are easy.

a: $X \subset V(I(X))$:

- If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$.
- So $x \in V(I(X))$, since every $f \in I(X)$ vanishes at x.

b: $\sqrt{J} \subset I(V(J))$:

- If $f \in \sqrt{J}$ then $f^k \in J$ for some k.
- Then $f^k(x) = 0$ for all $x \in V(J)$.

- So $f(x) = 0$ for all $x \in V(J)$.
- Thus $f \in I(V(J))$.

c: $V(I(X)) \subset X$:

- Need to now use that X is an affine variety.
  - Counterexample: $X = \mathbb{Z}^2 \subset \mathbb{C}^2$, then $I(X) = 0$. But $V(I(X)) = \mathbb{C}^2$, but $\mathbb{C}^2 \not\subset \mathbb{Z}^2$.
- By (b), $I(V(J)) \supset \sqrt{J} \supset J$.
- Since $V(\,\cdot\,)$ is order-reversing, taking V of both sides reverses the containment.
- So $V(I(V(J))) \subset V(J)$, i.e. $V(I(X)) \subset X$.

d: $I(V(J)) \subset \sqrt{J}$ (hard direction)

Theorem 2.5 ***(1st Version of Nullstellensatz).***
Suppose k is algebraically closed and uncountable (still true in countable case by a different proof).
Then the maximal ideals in $k[x_1, \cdots, x_n]$ are of the form $(x_1 - a_1, \cdots, x_n - a_n)$.

Proof .
Let $\mathfrak{m}$ be a maximal ideal, then by the Hilbert Basis Theorem, $\mathfrak{m} = \langle f_1, \cdots, f_r \rangle$ is finitely generated.
Let $L = \mathbb{Q}[\{c_i\}]$ where the c_i are all of the coefficients of the f_i if char $(K) = 0$, **or** $\mathbb{F}_p[\{c_i\}]$ if char $(k) = p$. Then $L \subset k$.
Define $\mathfrak{m}_0 = \mathfrak{m} \cap L[x_1, \cdots, x_n]$. Note that by construction, $f_i \in \mathfrak{m}_0$ for all i, and we can write $\mathfrak{m} = \mathfrak{m}_0 \cdot k[x_1, \cdots, x_n]$.
Claim: $\mathfrak{m}_0$ is a maximal ideal.
If it were the case that

$$\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq L[x_1, \cdots, x_n],$$

then

$$\mathfrak{m}_0 \cdot k[x_1, \cdots, x_n] \subsetneq \mathfrak{m}_0' \cdot k[x_1, \cdots, x_n] \subsetneq k[x_1, \cdots, x_n].$$

So far: constructed a smaller polynomial ring and a maximal ideal in it.

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is a field that is finitely generated over either $\mathbb{Q}$ or $\mathbb{F}_p$.

Theorem 2.6 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Note: this theorem is perhaps more important than the Nullstellensatz!

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is finite over some $\mathbb{Q}(t_1, \cdots, t_n)$, and since k is uncountable, there exists an embedding $\mathbb{Q}(t_1, \cdots, t_n) \hookrightarrow k$.

Use the fact that there are only countably many polynomials over a countable field.

This extends to an embedding of $\varphi : L[x_1, \cdots, x_n]/\mathfrak{m}_0 \hookrightarrow k$ since k is algebraically closed.
Letting a_i be the image of x_i under φ, then $f(a_1, \cdots, a_n) = 0$ by construction, $f_i \in (x_i - a_i)$ implies that $\mathfrak{m} = (x_i - a_i)$ by maximality.

■

3 | Thursday, August 27

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\{\text{Radical ideals } k[x_1, \cdots, x_n]\} \to V(\,\cdot\,)I(\,\cdot\,)\,\{\text{Affine varieties in } \mathbb{A}^n\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 3.1 ***(?).***
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

> Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty. ■

Returning to the proof that $I(V(J)) \subseteq \sqrt{J}$: let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. But $ft - 1 \neq 0$ if $f = 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

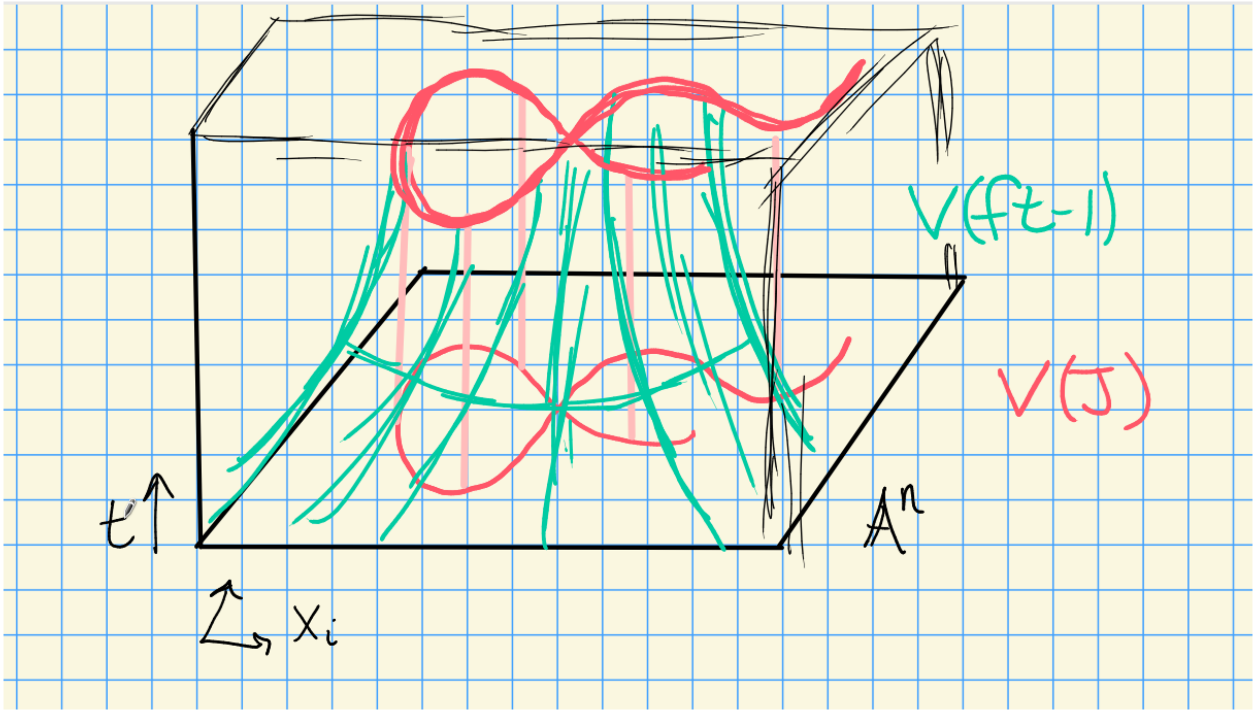


Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so $1 = \langle ft - 1 \rangle\, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$ with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N \coloneqq (ft - 1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t.

Now take $k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle\,.$$

Observation: there is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle\,.$$

Exercise 3.1.
Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 3.1.

Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then $I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J$. Note that this loses information.

Example 3.2.
Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\left\{\text{Points of } \mathbb{A}^n\right\} \iff \left\{\text{Maximal ideals of } k[x_1, \cdots, x_n]\right\}.$$

Theorem 3.2 ***(Properties of I).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) on the variety side.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

∎

Example 3.3.
Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

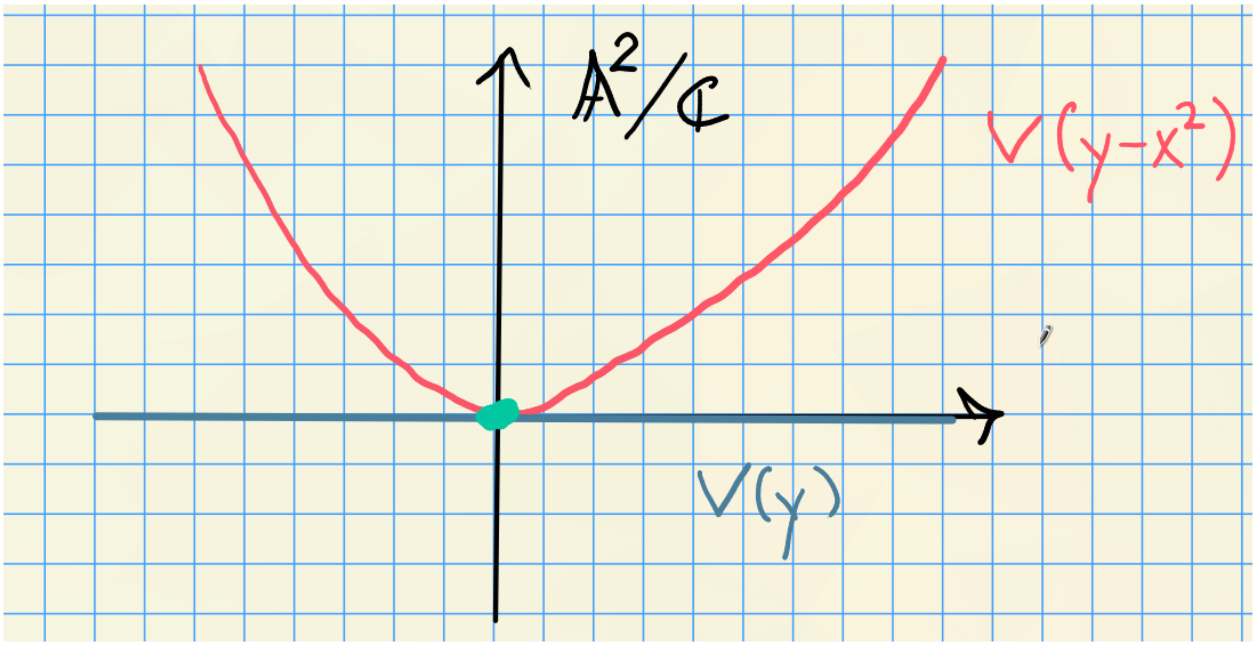


Figure 2: Image

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but $I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$.

Proposition 3.3 ***(?)*****.**
If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof .
Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$. ■

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

Definition 3.3.1 (Coordinate Ring)**.**
For an affine variety X, the *coordinate ring of* X is

$$A(X) \coloneqq k[x_1, \cdots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation: The constructions $V(\,\cdot\,), I(\,\cdot\,)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \left\{ x \in Y \;\middle|\; f(x) = 0 \;\forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \left\{ f \in A(Y) \;\middle|\; f(x) = 0 \;\forall x \in X \right\} \subseteq A(Y).$$

Example 3.4.
For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$\begin{tikzcd} A(\mathbb{A}^n) \arrow[r, two heads, "\cdot/I(Y)"] \arrow[rr, bend left, two heads, "\cdot/I(X)"] & A(Y) \arrow[r, two heads, "\cdot/I(X)"] & A(X) \end{tikzcd}$$

.

Theorem 3.4*(?)*.
Let $X \subset Y$ be an affine subvariety, then

a. $A(X) = A(Y)/I_Y(X)$
b. There is a correspondence

$$\left\{ \text{Affine subvarieties of } Y \right\} \iff \left\{ \text{Radical ideals in } A(Y) \right\}$$
$$X \mapsto I_Y(X)$$
$$V_Y(J) \mapsfrom J.$$

Proof .
Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ■

Example 3.5.
Let $Y = V(y - x^2) \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x-1, y-1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \langle x-1, y-1 \rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$\begin{tikzcd} A(\mathbb{A}^n) \arrow[r] \arrow[d, equal] & A(Y) \arrow[r] \arrow[d, equal] & A(X) \arrow[d, equal] \\ k[x,y] \arrow[r] & k[x,y]/\left\langle y - x^2 \right\rangle \arrow[r] & k[x,y]/\langle x-1, y-1 \rangle \end{tikzcd}$$

4 | Tuesday, September 01

Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \,\forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 4.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i \right)$.

Remark 3.
There are few closed sets, so this is a "weak" topology.

Example 4.1.
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A \coloneqq \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 4.2.
Let $f : \mathbb{A}^1/k \to \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 4.3.
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\bigcup_{i=1}^{n} U_i \times V_i$ with U_i, V_i open in X, Y respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta := V(x-y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

Example 4.4.
Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1 x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 := V(x_1)$ and $X_2 := V(x_2)$. The Zariski topology can detect these decompositions.

Definition 4.0.2 (Irreducibility and Connectedness)**.**
Let X be a topological space.

a. X is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible*.
b. X is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, X is said to be *connected*.

Example 4.5.
$V(x_1 x_2)$ is reducible but connected.

Remark 4.
$\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

Proposition 4.1 ***(?).***
Let X be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof .
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1 \emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle \, .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \to A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

■

Proposition 4.2 ***(?).***
An affine variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof .
$\Longrightarrow$: By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\Longleftarrow$: Suppose X is reducible with $X = X_1 \cup X_2$ with X_i proper and closed. Define $J_i \coloneqq I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so f_i vanishes on X_i. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain.

■

Example 4.6.
Let $X = \{p_1, \cdots, p_d\}$ be a finite set in $\mathbb{A}^n$. The Zariski topology on X is the discrete topology, and $X = \coprod \{p_i\}$. So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^{d} A(\{p_i\}) = \prod_{i=1}^{d} k[x_1, \cdots, x_n] / \left\langle x_j - a_j(p_i) \right\rangle_{j=1}^{d}.$$

Example 4.7.
Set $V(x_1 x_2) = X$, then $A(X) = k[x_1, x_2] / \langle x_1 x_2 \rangle$. This not being a domain (since $x_1 x_2 = 0$) corresponds to $X = V(x_1) \cup V(x_2)$ not being irreducible.

Example 4.8.
$\mathbb{A}^2 / k$ is irreducible since $k[x_1, \cdots x_n]$ is a domain.

Example 4.9.
Let X_1 be the xy plane and X_2 be the line parallel to the y-axis through $[0, 0, 1]$, and let $X = X_1 \coprod X_2$. Then $X_1 = V(z)$ and $X_2 = V(x, z-1)$, and $I(X) = \langle z \rangle \cdots \langle x, z-1 \rangle = \left\langle xz, z^2 - z \right\rangle$.

Then the coordinate ring is given by $A(X) = \mathbb{C}[x, y, z] / \left\langle xz, z^2 - z \right\rangle = \mathbb{C}[x, y, z] / \langle z \rangle \oplus \mathbb{C}[x, y, z] / \langle x, z-1 \rangle$.

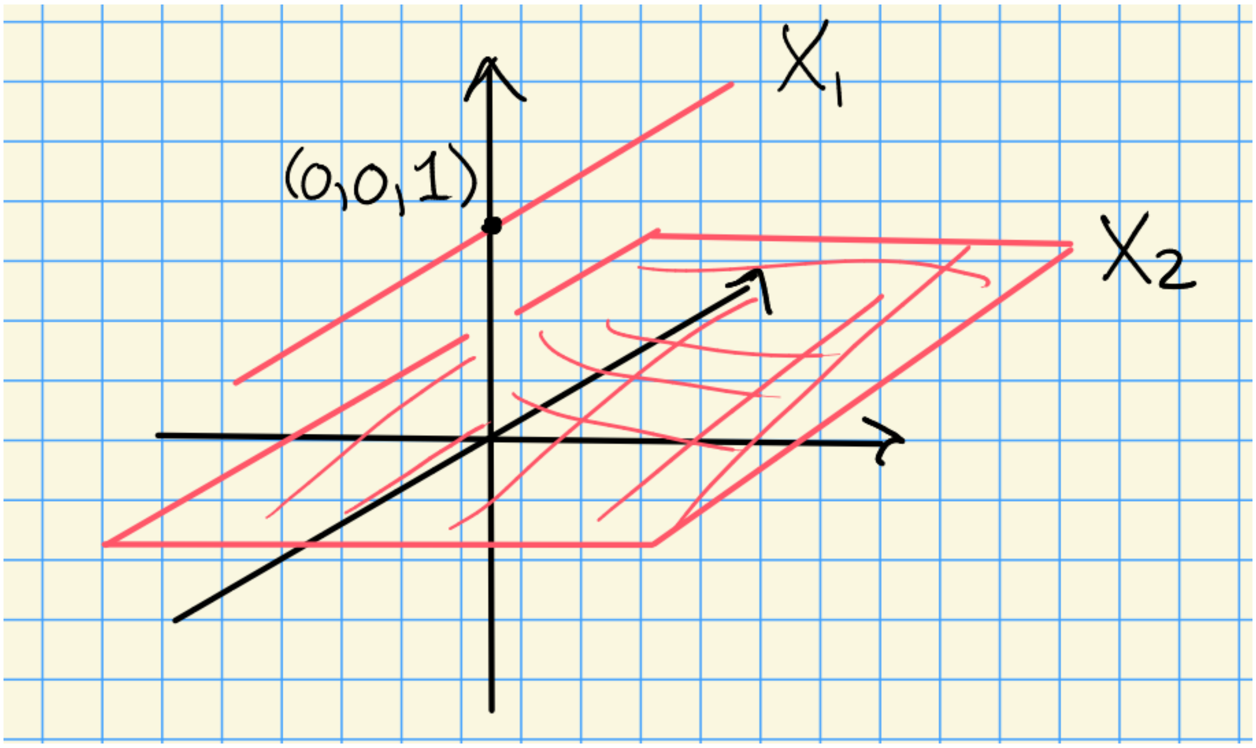


Figure 3: Image

5 | Thursday, September 03

Recall that the Zariski topology is defined on an affine variety $X = V(J)$ with $J \trianglelefteq k[x_1, \cdots, x_n]$ by describing the closed sets.

Proposition 5.1 *(?).*
X is irreducible if its coordinate ring $A(X)$ is a domain.

Proposition 5.2 *(?).*
There is a 1-to-1 correspondence

$$\left\{ \substack{\text{Irreducible subvarieties} \\ \text{of } X} \right\} \iff \left\{ \substack{\text{Prime ideals} \\ \text{in } A(X)} \right\}.$$

Proof .
Suppose $Y \subset X$ is an affine subvariety. Then

$$A(X)/I_X(Y) = A(Y).$$

By NSS, there is a bijection between subvarieties of X and radical ideals of $A(X)$ where $Y \mapsto I_X(Y)$. A quotient is a domain iff quotienting by a prime ideal, so $A(Y)$ is a domain iff $I_X(Y)$ is prime.

Recall that $\mathfrak{p} \trianglelefteq R$ is prime when $fg \in \mathfrak{p} \iff f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus $\bar{f}\bar{g} = 0$ in $R/\mathfrak{p}$ implies $\bar{f} = 0$ or $\bar{g} = 0$ in $R/\mathfrak{p}$, i.e. $R/\mathfrak{p}$ is a domain.

Finally note that prime ideals are radical (easy proof).

Example 5.1.
Consider $\mathbb{A}^2/\mathbb{C}$ and some subvarieties C_i:

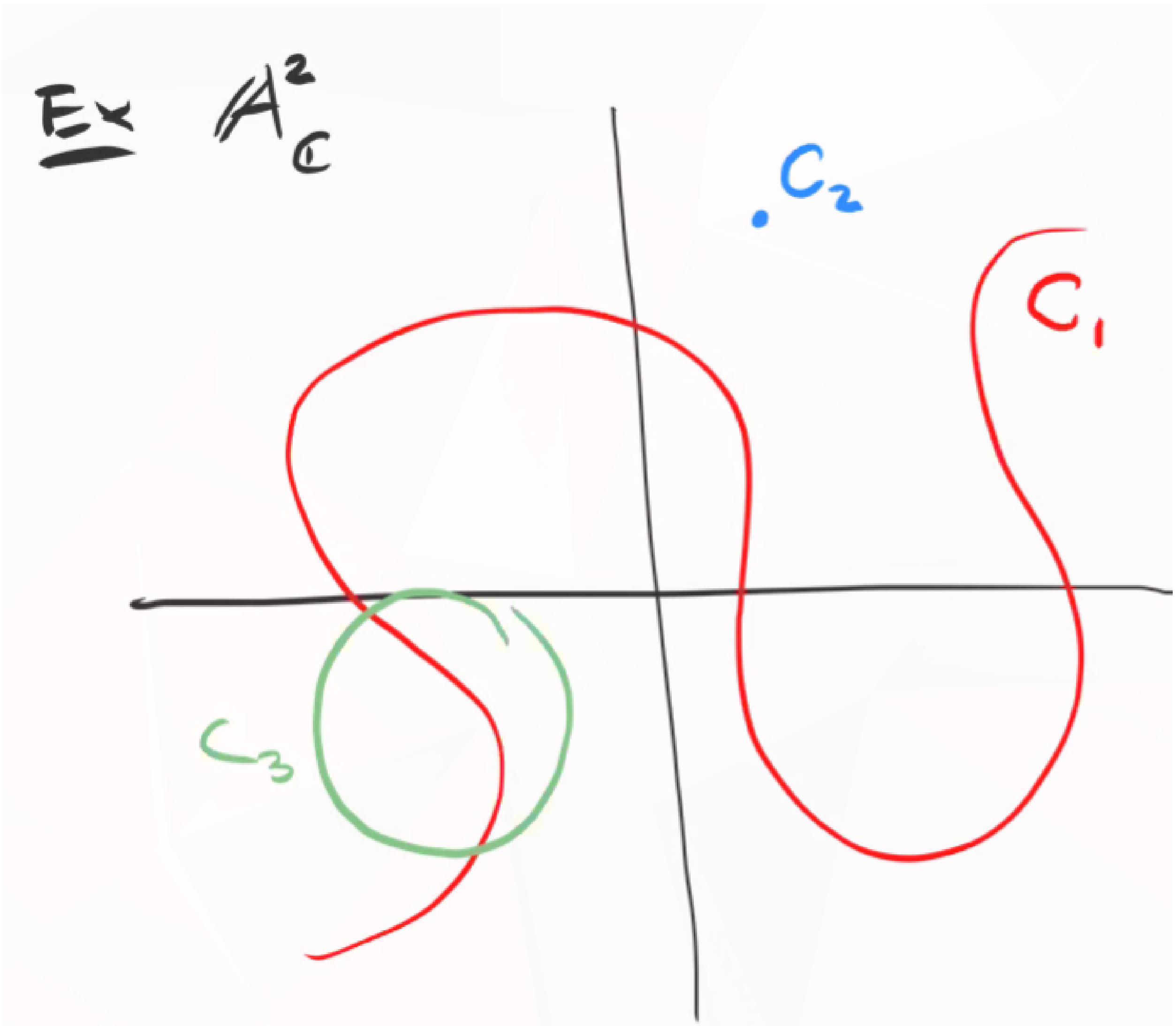


Figure 4: Subvarieties

Then irreducible subvarieties correspond to prime ideals in $\mathbb{C}[x, y]$. Here C_1, C_3 correspond to $V(f), V(g)$ for f, g irreducible polynomials, whereas C_2 corresponds to a maximal ideal, i.e. $V(x_1 - a_1, x_2 - a_2)$.

Note that $I(C_1 \cup C_2 \cup C_3)$ is not a prime ideal, since the variety is reducible as the union of 3 closed subsets.

Example 5.2.
A finite set is irreducible iff it contains only one point.

Example 5.3.
Any irreducible topological space is connected, since irreducible requires a union but connectedness requires a *disjoint* union.

Example 5.4.
$\mathbb{A}^n/k$ is irreducible: by prop 2.8, its irreducible iff the coordinate ring is a domain. However $A(\mathbb{A}^n) = k[x_1, \cdots, x_n]$, which is a domain.

Example 5.5.
$V(x_1 x_2)$ is not irreducible, since it's equal to $V(x_1) \cup V(x_2)$.

Definition 5.2.1 (Noetherian Space)**.**
A *Noetherian* topological space X is a space with no infinite strictly decreasing sequence of closed subsets.

Proposition 5.3*(?)*.
An affine variety X with the zariski topology is a noetherian space.

Proof .
Let $X_0 \supsetneq X_1 \supsetneq \cdots$ be a decreasing sequence of closed subspaces. Then $I(X_0) \subsetneq I(X_1) \subsetneq$. Note that these containments are strict, otherwise we could use $V(I(X_1)) = X_1$ to get an equality in the original chain.
Recall that a ring R is Noetherian iff every ascending chain of ideals terminates. Thus it suffices to show that $A(X)$ is Noetherian.
We have $A(X) = k[x_1, \cdots, x_n]/I(X)$, and if this had an infinite chain $I_1 \subsetneq I_2 \subsetneq \cdots$ lifts to a chain in $k[x_1, \cdots, x_n]$, which is Noetherian. A useful fact: R noetherian implies that $R[x]$ is noetherian, and fields are always noetherian.

■

Remark 5.
Any subspace $A \subset X$ of a noetherian space is noetherian. To see why, suppose we have a chain of closed sets in the subspace topology,

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq \cdots.$$

Then $X_0 \supsetneq X_1 \supsetneq \cdots$ is a strictly decreasing chain of closed sets in X. Why strictly decreasing: $\cap^n X_i = \cap^{n+1} X_i \implies A \cap^n X_i = A \cap^{n+1} X_i$, a contradiction.

Proposition 5.4 ***(Important).***

Every noetherian space X is a finite union of irreducible closed subsets, i.e. $X = \bigcup_{i=1}^{k} X_i$. If we further assume $X_i \not\subset X_j$ for all i, j, then the X_i are unique up to permutation.

Remark 6.
The X_i are the **components** of X. In the previous example $C_1 \cup C_2 \cup C_3$ has three components.

Proof .
If X is irreducible, then $X = X$ and this holds.
Otherwise, write $X = X_1 \cup X_2$ with X_i proper closed subsets. If X_1 and X_1' are irreducible, we're done, so otherwise suppose wlog X_1' is not irreducible.
Then we can express $X = X_1 \cup (X_2 \cup X_2')$ with $X_2, X_2' \subset X_1'$ closed and proper.
Thus we can obtain a tree whose leaves are proper closed subsets:

Figure 5: Image

This tree terminates because X is Noetherian: if it did not, this would generate an infinite decreasing chain of subspaces.
We now want to show that the decomposition is unique if no two components are contained in the other.
Suppose

$$X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=1}^{\ell} X_j'.$$

Note that $X_i \subset X$ implies that $X_i = \bigcup_{j=1}^{\ell} X_i \cap X_j'$. But X_i is irreducible and this would express X_i as a union of proper closed subsets, so some $X_i \cap X_j'$ is *not* a proper closed subset.
Thus $X_i = X_i \cap X_j'$ for some j, which forces $X_i \subset X_j'$. Applying the same argument to X_j' to obtain $X_j' \subset X_k$ for some k.
Then $X_i \subset X_j' \subset X_k$, but $X_i \not\subset X_j$ when $j \neq i$. Thus $X_i = X_j' = X_k$, forcing the X_i to be unique up to permutation.

■

Recall from ring theory: for $I \subset R$ and R noetherian, I has a *primary decomposition* $I = \bigcap_{i=1}^{k} Q_i$ with $\sqrt{Q_i}$ prime. Assuming the Q_i are minimal in the sense that $\sqrt{Q_i} \not\subset \sqrt{Q_j}$ for any i, j, this decomposition is unique.

Applying this to $I(X) \trianglelefteq k[x_1, \cdots, x_n] = R$ yields

$$I(X) = \bigcap_{i=1}^{k} Q_i \implies X = V(I(X)) = \bigcup_{i=1}^{k} V(Q_i).$$

Letting $P_i = \sqrt{Q_i}$, noting that the P_i are prime and thus radical, we have $V(Q_i) = V(P_i)$. Writing $X = \bigcup V(P_i)$, we have $I(V(P_i)) = P_i$ and thus $A(V(P_i)) = R/P_i$ is a domain, meaning $V(P_i)$ are irreducible affine varieties.

Conversely, if we express $X = \bigcup X_i$, we have $I = I\left(\bigcup X_i\right) = \bigcap I(X_i) = \bigcap P_i$ which are irreducible since they are prime.

Remark 7.
There is a correspondence

$$\left\{ \begin{matrix} \text{Irreducible components} \\ \text{of } X \end{matrix} \right\} \iff \left\{ \begin{matrix} \text{Minimal prime ideals} \\ \text{in } A(X) \end{matrix} \right\},$$

where here *minimal* is the condition that no pair of ideals satisfies a subset containment.

Remark 8.
Let X be an irreducible topological space.

Proposition 5.5 ***(1).***
The intersection of nonempty two open sets is *never* empty.

Proof .
Let U, U' be open and $X \setminus U, X \setminus U'$ closed. Then $U \cap U' = \emptyset \iff (X \setminus U) \cup (X \setminus U') = X$, but this is not possible since X is irreducible.

■

Irreducible iff any two nonempty open sets intersect.

Proposition 5.6 *(?)*.
Any nonempty open set is dense, i.e. if $U \subset X$ is open then its closure $\mathrm{cl}_X(U)$ is dense in X.

Proof .
Write $X = \mathrm{cl}_X(U) \cup (X \setminus U)$. Since $X \setminus U \neq X$ and X is irreducible, we have $\mathrm{cl}_X(U) = X$. ■

6 | Tuesday, September 08

Review: we discussed irreducible components. Recall that the *Zariski topology* on an affine variety X has affine subvarieties as closed sets, and a *noetherian space* has no infinitely decreasing chains of closed subspaces.

We showed that any noetherian space has a decomposition into irreducible components $X = \cup X_i$ with X_i closed, irreducible, and unique such that no two are subsets of each other. Applying this to affine varieties, a descending chain of subspaces $X_0 \supsetneq X_1 \cdots$ in X corresponds to an increasing chain of ideals $I(X_0) \subsetneq I(X_1) \cdots$ in $A(X)$. Since $k[x_1, \cdots, x_n]$ is a noetherian ring, this chain terminates, so affine varieties are noetherian.

6.1 Dimension

Definition 6.0.1 (Dimensions)**.**
Let X be a topological space.

1. The *dimension* $\dim X \in \mathbb{N} \cup \{\infty\}$ is either ∞ or the length n of the longest chain of **irreducible** closed subsets $\emptyset \neq Y_0 \subsetneq \cdots \subsetneq Y_n \subset X$ where Y_n need not be equal to X.
2. The *codimension* of Y in X, $\mathrm{codim}_X(Y)$, for an irreducible subset $Y \subseteq X$ is the length of the longest chain $Y \subset Y_0 \subsetneq Y_1 \cdots \subset X$.

Example 6.1.
Consider $\mathbb{A}^1/k$, what are the closed subsets? The finite sets, the empty set, and the entire space.

What are the irreducible closed subsets? Every point is a closed subset, so sets with more than one point are reducible. So the only irreducible closed subsets are $\{a\}, \mathbb{A}^1/k$, since an affine variety is irreducible iff its coordinate ring is a domain and $A(\mathbb{A}^1/k) = k[x]$. We can check

$$\emptyset \subseteq Y_0 = \{a\} \subseteq Y_1 = \mathbb{A}^1/k,$$

which is of length 1, so $\dim(\mathbb{A}^1/k) = 1$.

Note that we count the number of nontrivial strict subset containments in this chain.

Example 6.2.
Consider $V(x_1x_2) \subset \mathbb{A}^2/k$, the union of the x_i axes. Then the closed subsets are $V(x_1), V(x_2)$, along with finite sets and their unions. What is the longest chain of irreducible closed subsets?

Note that $k[x_1, x_2]/\langle x_1 \rangle \cong k[x_2]$ is a domain, so $V(x_i)$ are irreducible. So we can have a chain

$$\emptyset \subsetneq \{a\} \subsetneq V(x_1) \subset X,$$

where a is any point on the x_2-axis, so $\dim(X) = 1$.

The only closed sets containing $V(x_1)$ are $V(x_1) \cup S$ for S some finite set, which can not be irreducible.

Remark 9.
You may be tempted to think that if X is noetherian then the dimension is finite. However, finite dimension requires a bounded length on descending/ascending chains, whereas noetherian only requires "termination", which may not happen in a bounded number of steps. So this is **false**!

Example 6.3.
Take $X = \mathbb{N}$ and define a topology by setting closed subsets be the sets $\{0, \cdots, n\}$ as n ranges over $\mathbb{N}$, along with $\mathbb{N}$ itself. Is X noetherian? Check descending chains of closed sets:

$$\mathbb{N} \supsetneq \{0, \cdots, N\} \supsetneq \{0, \cdots, N-1\} \cdots,$$

which has length at most N, so it terminates and X is noetherian.

But note that all of these closed subsets $X_N \coloneqq \{0, \cdots, N\}$ are irreducible. Why? If $X_n = X_i \cup X_j$ then one of i, j is equal to N, i.e $X_i, X_j = X_N$.

So for every N, there exists a chain of irreducible closed subsets of length N, implying that $\dim(\mathbb{N}) = \infty$.

Remark 10.
Let X be an affine variety. There is a correspondence

$$\left\{ \substack{\text{Chains of irreducible closed subsets} \\ Y_0 \subsetneq \cdots \subsetneq Y_n \text{ in } X} \right\} \left\{ \substack{\text{Chains of prime ideals} \\ P_0 \supsetneq \cdots \supsetneq P_n \text{ in } A(X)} \right\}.$$

Why? We have a correspondence between closed subsets and radical ideals. If we specialize to irreducible, we saw that these correspond to radical ideals $I \subset A(X)$ such that $A(Y) \coloneqq A(X)/I$ is a domain, which precisely correspond to prime ideal in $A(X)$.

We thus make the following definition:

Definition 6.0.2 (Krull Dimension)**.**
The *krull dimension* of a ring R is the length n of the longest chain of prime ideals

$$P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n.$$

Remark 11.
This uses the key fact from commutative algebra: a finitely generated k-algebra M satisfies

1. M has finite k-dimension
2. If M is a domain, every maximal chain has the same length.

Remark 12.
From scheme theory: for any ring R, there is an associated topological space $\operatorname{Spec} R$ given by the set of prime ideals in R, where the closed sets are given by

$$V(I) = \left\{ \text{Prime ideals } \mathfrak{p} \trianglelefteq R \;\middle|\; I \subseteq \mathfrak{p} \right\}.$$

If R is a noetherian ring, then $\operatorname{Spec}(R)$ is a noetherian space.

Example 6.4.
Using the fact above, let's compute $\dim \mathbb{A}^n/k$. We can take the following chain of prime ideals in $k[x_1, \cdots, x_n]$:

$$0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \cdots \subsetneq \langle x_1, \cdots, x_n \rangle.$$

By applying $V(\,\cdot\,)$ we obtain

$$\mathbb{A}^n/k \supsetneq \mathbb{A}^{n-1}/k \cdots \supsetneq \mathbb{A}^0/k = \{0\} \supsetneq \emptyset,$$

where we know each is irreducible and closed, and it's easy to check that these are maximal:

If there were an ideal $\langle x_1, x_2 \rangle \subset P \subset \langle x_1, x_2, x_3 \rangle$, then take $P \cap k[x_1, x_2, x_3] / \langle x_1, x_2 \rangle$ which would yield a polynomial ring in $k[x_1]$. But we know the only irreducible sets in $\mathbb{A}^1/k$ are a point and the entire space.

So this is a chain of maximal length, implying $\dim \mathbb{A}^n/k = n$.

7 | Thursday, September 10

Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined $\dim X$ to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking $R = A(X)$, i.e. $\dim \mathbb{A}^n/k = n$.

Proposition 7.1 *(Dimensions).*
Let $k = \bar{k}$.

a. The dimension of $k[x_1, \cdots, x_n]$ is n.
b. All maximal chains of prime ideals have length n.

7.1 Proof of Dimension Proposition

The case for $n = 0$ is trivial, just take $P_0 = \langle 0 \rangle$. For $n = 1$, easy to see since the only prime ideals in $k[x]$ are $\langle 0 \rangle$ and $\langle x - a \rangle$, since any polynomial factors into linear factors.

Let $P_0 \subsetneq \cdots \subsetneq P_m$ be a maximal chain of prime ideals in $k[x_1, \cdots, x_n]$; we then want to show that $m = n$. Assume $P_0 = \langle 0 \rangle$, since we can always extend our chain to make this true (using maximality). Then P_1 is a minimal prime and P_m is a maximal ideal (and maximals are prime).

Claim: P_1 is principle, i.e. $P_1 = \langle f \rangle$ for some irreducible f.

7.1.1 Proof That P_1 is Principle

Claim: $k[x_1, \cdots, x_n]$ is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD $\implies$ $R[x]$ is a UFD for any R.

> See Gauss' lemma.

Claim: In a UFD, minimal primes are principal. Let $r \in P$, and write $r = u \prod p_i^{n_i}$ with p_i irreducible and u a unit. So some $p_i \in P$, and p_i irreducible implies $\langle p_i \rangle$ is prime. Since $0 \subsetneq \langle p_i \rangle \subset P$, but P was prime and assumed minimal, so $\langle p_i \rangle = P$.

The idea is to now transfer the chain $P_0 \subsetneq \cdots \subsetneq P_m$ to a maximal chain in $k[x_1, \cdots, x_{n-1}]$. The first step is to make a linear change of coordinates so that f is monic in the variable x_n.

Example 7.1.
Take $f = x_1 x_2 + x_3^2 x_4$ and map $x_3 \mapsto x_3 + x_4$.

So write

$$f(x_1, \cdots, x_n) = x_n^d + f_1(x_1, \cdots, x_{n-1}) x_n^{d-1} + \cdots + f_d(x_1, \cdots, x_{n-1}).$$

We can then descend to $k[x_1, \cdots, x_n]$ to $k[x_1, \cdots, x_n] / \langle f \rangle$:

$$\begin{tikzcd}
P_0 \arrow[r] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m \arrow[d] \\
& P_1/P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m/P_1 \arrow[d] \\
& P_1/P_1 \cap k[x_1, \cdots, x_{n-1}] \arrow[r] & \cdots \arrow[r] & (P_m/P_1) \cap k[x_1, \cdots, x_{n-1}]
\end{tikzcd}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

> **Definition 7.1.1** (Integral Extension).
> An *integral* ring extension $R \hookrightarrow R'$ of R is one such that all $r' \in R'$ satisfying a monic polynomial with coefficients in R, where R' is finitely generated.
>
> > In this case, also implies that R' is a finitely-generated R module.

In this case, $k[x_1, \cdots, x_{n-1}] \hookrightarrow k[x_1, \cdots, x_n]/\langle f \rangle$ is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

Lemma 7.2.
Suppose $P', Q' \subset R'$ are distinct prime ideals with $R \hookrightarrow R'$ an integral extension. Then if $P' \cap R = Q' \cap R$, neither contains the other, i.e. $P' \not\subset Q'$ and $Q' \not\subset P'$.

Proof .
Toward a contradiction, suppose $P' \subset Q'$, we then want to show that $Q' \supset P'$. Let $a \in Q' \setminus P'$ (again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.
Then $\bar{a} \neq 0$ in R'/P', and there exists a monic polynomial of minimal degree that $\bar{a}$ satisfies, $p(x) = x^n + \sum_{i=2}^{n} \bar{c}_i x^{n-i}$. This implies $\bar{c}_n \in Q'/P'$ (which will contradict $c_n \in P'$), since if $\bar{c}_n = 0$ then factoring out x yields a lower degree polynomial that $\bar{a}$ satisfies.
But then $\bar{a}_n \in Q' \cap R$, so ???

■

Question: Given $R \hookrightarrow R'$ is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the "Going Up" Theorem: given $P \subset R$ prime, there exists $P' \subset R'$ prime such that $P' \cap R = P$. Furthermore, we can lift $P_1 \subset P_2$ to $P_1' \subset P_2'$, as well as "lifting sandwiches":

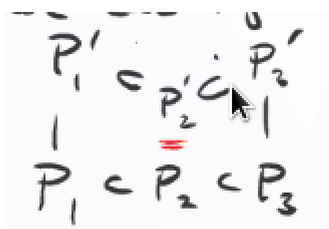


Figure 6: Image

In this process, the length of the chain decreased since $\langle 0 \rangle$ was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies. ■

7.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if $R \hookrightarrow R'$ is integral then $\dim R = \dim R'$.

Claim: Any affine variety has finite dimension.

Proof .
We have $\dim X = \dim A(X)$, where $A(X) := k[x_1, \cdots, x_n]I$ for some $I(X) = \sqrt{I(X)}$.

The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated k-algebra is an integral extension of some polynomial ring $k[y_1, \cdots, y_d]$. I.e., the following extension is integral:

$$k[y_1, \cdots, y_d] \hookrightarrow k[x_1, \cdots, x_n]/I.$$

We can conclude that $\dim A(X) = d < \infty$.

■

Proposition 7.3 ***(?).***
Let X, Y be irreducible affine varieties. Then

a. $\dim X \times Y = \dim X + \dim Y$.
b. $Y \subset X \implies \dim X = \dim Y + \operatorname{codim}_X Y$.
c. If $f \in A(X)$ is nonzero, then any component of $V(f)$ has codimension 1.

Proof .

Remark .
Why is $X \times Y$ again an affine variety? If $X \subset \mathbb{A}^n/k$, $Y \subset \mathbb{A}^m/k$ with $X = V(I), Y = V(J)$, then $X \times Y \subset \mathbb{A}^n/k \times \mathbb{A}^m/k = \mathbb{A}^{n+m}/k$ can be given by taking $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$ using the natural inclusions of $k[x_1, \cdots, x_\ell]$.
Note that we can write

$$k[x_1, \cdots, x_n, y_1, \cdots, y_m] = k[x_1, \cdots, x_n] \otimes_k k[y_1, \cdots, y_n]$$

where we think of $x_i = x_i \otimes 1, y_j = 1 \otimes y_j$. We thus map I, J to $I \otimes 1 + 1 \otimes J$ and obtain $V(I \otimes 1 + 1 \otimes J) = X \times Y$ and $A(X \times Y) = A(X) \otimes_k A(Y)$.
In general, for k-algebras R, S,

$$R/I \otimes_k S/J \cong R \otimes_k S/ \langle I \otimes 1 + 1 \otimes J \rangle .$$

Remark .
For R, S finitely generated k-algebras, $\dim R \otimes_k S = \dim R + \dim S$.

Part (a) is proved by the above remarks.
For part (b), the statement is equivalent to $P \subset A(X)$ with $I(Y) \subset P$ is a member of some maximal chain, along with the statement that all maximal chains are the same length.

■

8 Tuesday, September 15

8.1 Review

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k \coloneqq \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k,$ vanishing locii of polynomials
	$I \mapsto V(I) \coloneqq \left\{ a \;\middle\vert\; f(a) = 0 \forall f \in I \right\}$
$I(X) \coloneqq \left\{ f \;\middle\vert\; f\vert_X = 0 \right\} \hookleftarrow X$	
Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets).

> Recall that we defined the coordinate ring $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, which contained no nilpotents.

We had some results about dimension

1. $\dim X < \infty$ and $\dim \mathbb{A}^n = n$.
2. $\dim Y + \operatorname{codim}_X Y = \dim X$ when $Y \subset X$ is irreducible.
3. Only over $\bar{k} = k$, $\operatorname{codim}_X V(f) = 1$.

Example 8.1.
Take $V(x^2 + y^2) \subset \mathbb{A}^2/\mathbb{R}$

Definition 8.0.1 (?).
An affine variety Y of
- $\dim Y = 1$ is a **curve**,
- $\dim Y = 2$ is a **surface**,
- $\operatorname{codim}_X Y = 1$ is a **hypersurface in** X

Question: Is every hypersurface the vanishing locus of a *single* polynomials $f \in A(X)$?

Answer: This is true iff $A(X)$ is a UFD.

Definition 8.0.2 (Codimension in a Ring).
$\operatorname{codim}_R \mathfrak{p}$ is the length of the longest chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}.$$

Recall that f is irreducible if $f = f_1 f_2 \implies f_i \in R^\times$ for one i, and f is prime iff $\langle f \rangle$ is a prime ideal, or equivalently $f \mathrel{\Big|} ab \implies f \mathrel{\Big|} a$ or $f \mathrel{\Big|} b$.

Note that prime implies irreducible, since f divides itself.

Proposition 8.1 ***(?).***
Let R be a Noetherian domain, then TFAE

a. All prime ideals of codimension 1 are principal.
b. R is a UFD.

Proof .
$a \implies b$:
Let f be a nonzero non-unit, we'll show it admits a prime factorization. If f is not irreducible, then $f = f_1 f_1'$, both non-units. If f_1' is not irreducible, we can repeat this, to get a chain

$$\langle f \rangle \subsetneq \langle f_1' \rangle \subsetneq \langle f_2' \rangle \subsetneq \cdots,$$

which must terminate.
This yields a factorization $f = \prod f_i$ with f_i irreducible. To show that R is a UFD, it thus suffices to show that the f_i are prime. Choose a minimal prime ideal containing f. We'll use Krull's Principal Ideal Theorem: if you have a minimal prime ideal $\mathfrak{p}$ containing f, its codimension $\operatorname{codim}_R \mathfrak{p}$ is one. By assumption, this implies that $\mathfrak{p} = \langle g \rangle$ is principal. But $g \mathrel{\Big|} f$ with f irreducible, so f, g differ by a unit, forcing $\mathfrak{p} = \langle f \rangle$. So $\langle f \rangle$ is a prime ideal.

$b \implies a$:
Let $\mathfrak{p}$ be a prime ideal of codimension 1. If $\mathfrak{p} = \langle 0 \rangle$, it is principal, so assume not. Then there exists some nonzero non-unit $f \in \mathfrak{p}$, which by assumption has a prime factorization since R is assumed a UFD. So $f = \prod f_i$.
Since $\mathfrak{p}$ is a prime ideal and $f \in \mathfrak{p}$, some $f_i \in \mathfrak{p}$. Then $\langle f_i \rangle \subset \mathfrak{p}$ and $\mathfrak{p}$ minimal implies $\langle f_i \rangle = \mathfrak{p}$, so $\mathfrak{p}$ is principal.

$\blacksquare$

Corollary 8.2 ***(?).***
Every hypersurface $Y \subset X$ is cut out by a single polynomial, so $Y = V(f)$, iff $A(X)$ is a UFD.

Example 8.2.
Apply this to $R = A(X)$, we find that there is a bijection

$$\text{codim1 prime ideals} \iff \text{codim1 closed irreducible subsets } Y \subset X, \text{ i.e. hypersurfaces.}$$

Taking $A(X) = \mathbb{C}[x, y, z] / \left\langle x^2 + y^2 - z^2 \right\rangle$, whose real points form a cone:

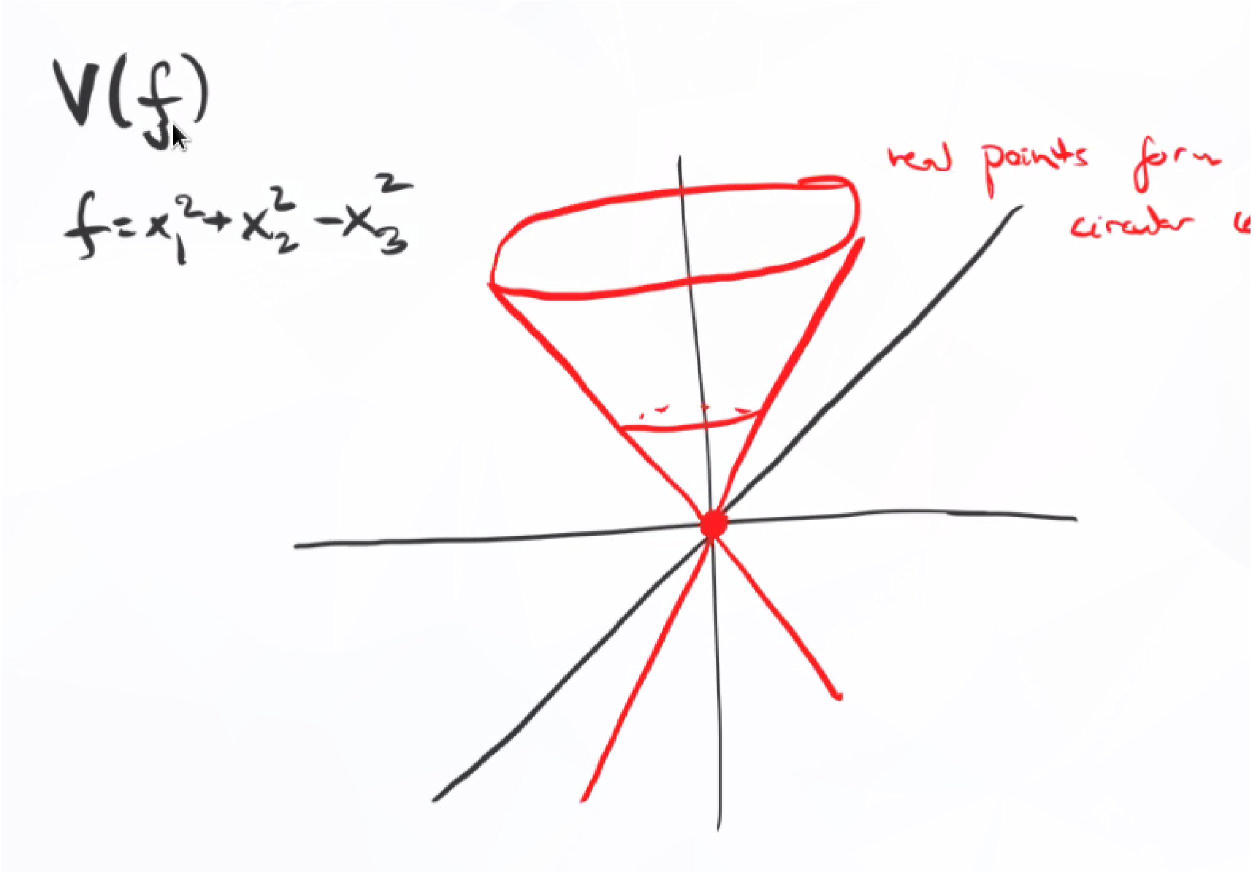


Figure 7: Image

Note that $x^2 + y^2 = (x - iy)(x + iy) = z^2$ in this quotient, so this is not a UFD.

Then taking a line through its surface is a codimension 1 subvariety not cut out by a single polynomial. Such a line might be given by $V(x + iy, z)$, which is 2 polynomials, so why not codimension 2?

Note that $V(z)$ is the union of the lines

- $z = 0, x + iy = 0,$
- $z = 0, x - iy = 0.$

Note that it suffices to show that this ring has an irreducible that is not prime. Supposing $z = f_1 f_2$, some f_i is a unit, then z is not prime because $z \mid xy$ but divides neither of x, y.

Example 8.3.
Note that $k[x_1, \cdots, x_n]$ is a UFD since k is a UFD. Applying the corollary, every hypersurface in $\mathbb{A}^n$ is cut out by a single irreducible polynomial.

Definition 8.2.1 (?).
An affine variety X is of **pure dimension** d iff every irreducible component X_i is of dimension d.

Note that X is a Noetherian space, so has a unique decomposition $X = \cup X_i$.

Given $X \subset \mathbb{A}^n/k$ of pure dimension $n-1$, $X = \cup X_i$ with X_i hypersurfaces with $I(X_j) = \langle f_j \rangle$, $I(X) = \langle f \rangle$ where $f = \prod f_i$.

Definition 8.2.2 (?).
Given such an X, define the **degree of a hypersurface** as the degree of f where $I(X) = \langle f \rangle$.

9 | Thursday, September 17

9.1 Regular Functions

See chapter 3 in the notes.

Some examples:

- X a manifold or an open set in $\mathbb{R}^n$ has a ring of C^∞ functions.
- $X \subset \mathbb{C}$ has a ring of holomorphic functions.
- $X \subset \mathbb{R}$ has a ring of real analytic functions

These all share a common feature: it suffices to check if a function is a member on an arbitrary open set about a point, i.e. they are *local*.

Definition 9.0.1 (?).
Let X be an affine variety and $U \subseteq X$ open. A **regular function** on U is a function $\varphi : U \to k$ such that φ is "locally a fraction", i.e. a ratio of polynomial functions.
More formally, for all $p \in U$ there exists a U_p with $p \in U_p \subseteq U$ such that $\varphi(x) = g(x)/f(x)$ for all $x \in U_p$ with $f, g \in A(X)$.

Example 9.1.
For X an affine variety and $f \in A(X)$, consider the open set $U \coloneqq V(f)^c$. Then $\frac{1}{f}$ is a regular function on U, so for $p \in U$ we can take U_p to be all of U.

Example 9.2.
For $X = \mathbb{A}^1$, take $f = x - 1$. Then $\frac{x}{x-1}$ is a regular function on $\mathbb{A}^1 \setminus \{1\}$.

Example 9.3.
Let $X + V(x_1 x_4 - x_2 x_3)$ and

$$U \coloneqq X \setminus V(x_2, x_4) = \left\{ [x_1, x_2, x_3, x_4] \;\middle|\; x_1 x_4 = x_2 x_3, x_2 \neq 0 \text{ or } x_4 \neq 0 \right\}.$$

Define

$$\varphi : U \to K$$
$$[x_1, x_2, x_3, x_4] \mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}.$$

This is well-defined on $\{x_2 \neq 0\} \cap \{x_4 \neq 0\}$, since $\frac{x_1}{x_2} = \frac{x_3}{x_4}$. Note that this doesn't define an element of k at $[0,0,0,1] \in U$. So this is not globally a fraction.

Notation: we'll let $\mathcal{O}_X(U)$ is the ring of regular function on U.

Proposition 9.1 ***(?)*.**
Let $U \subset X$ be an affine variety and $\varphi \in \mathcal{O}_X(U)$. Then $V(\varphi) := \left\{ x \in U \mid \varphi(x) = 0 \right\}$ is closed in the subspace topology on U.

Proof .
For all $a \in U$ there exists $U_a \subset U$ such that $\varphi = g_a / f_a$ on U_a with $f_a, g_a \in A(X)$ with $f_a \neq 0$ on U_a.
Then

$$\left\{ x \in U_a \mid \varphi(x) \neq 0 \right\} = U_a \setminus V(g_a) \cap U_a$$

is an open subset of U_a, so taking the union over a again yields an open set. But this is precisely $V(\varphi)^c$.

■

Proposition 9.2.
Let $U \subset V$ be open in X an *irreducible* affine variety. If $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ agree on U, then they are equal.

Proof .
$V(\varphi_1 - \varphi_2)$ contains U and is closed in V. It contains $\overline{U} \cap V$, by an earlier lemma, X irreducible implies that $\overline{U} = X$ and so $V(\varphi_1 - \varphi_2) = V$.

■

Compare and contrast: Let $U \subset V \subset \mathbb{R}^n$ be open. If $\varphi_1, \varphi_2 \in C^\infty(V)$ such that φ_1, φ_2 are equal when restricted $U \subset V$. Does this imply $\varphi_1 = \varphi_2$?

For $\mathbb{R}^n$, no, there exist smooth bump functions. You can make a bump function on $V \setminus U$ and extend by zero to U. For $\mathbb{C}$ and holomorphic functions, the answer is yes, by the uniqueness of analytic continuation.

Definition 9.2.1 ((Important) Distinguished Opens)**.**

A **distinguished open set** in an affine variety is one of the form

$$D(f) := X \setminus V(f) = \left\{ x \in X \;\middle|\; f(x) = 0 \right\}.$$

Proposition 9.3.
The distinguished open sets form a base of the zariski topology.

Proof .
Given $f, g \in A(X)$, we can check:

1. Closed under finite intersections: $D(f) \cap D(g) = D(fg)$.
2. $$U = X \setminus V(f_1, \cdots, f_k) = V \setminus \bigcap V(f_i) = \bigcup D(f_i),$$

   and any open set is a *finite* union of distinguished opens by the Hilbert basis theorem.

■

Proposition 9.4*(?)*.
The regular functions on $D(f)$ are given by

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N} \right\} = A(X)_{\langle f \rangle},$$

the localization of $A(X)$ at $\langle f \rangle$.

Note that if $f = 1$, then $\mathcal{O}_X(X) = A(X)$.

Proposition 9.5*(?)*.
Note that $\frac{g}{f^n} \in \mathcal{O}_X(D(f))$ since $f^n \neq 0$ on $D(f)$. Let $\varphi : D(f) \to k$ be a regular function. By definition, for all $a \in D(f)$ there exists a local representation as a fraction $\varphi = g_a/f_a$ on $U_a \ni a$. Note that U_a can be covered by distinguished opens, one of which contains a. Shrink U_a if necessary to assume it is a distinguished open set $U_a = D(h_a)$.

Now replace

$$\varphi = \frac{g_a}{f_a} = \frac{g_a h_a}{f_a h_a},$$

which makes sense because $h_a \neq 0$ on U_a. We can assume wlog that $h_a = f_a$. Why? We have $\varphi = \frac{g_a}{f_a}$ on $D(f_a)$. Since f_a doesn't vanish on U_a, we have $V(f_a h_a) = V(h_a)$ since $V(f_a) \subset D(h_a)^c = V(h_a)$.
Consider $U_a = D(f_a)$ and $U_b = D(f_b)$, on which $\varphi = \frac{g_a}{f_a}$ and $\varphi = \frac{g_b}{f_b}$ respectively. On $U_a \cap U_b = D(f_a f_b)$, these are equal, i.e. $f_b g_a = f_a g_b$ in the coordinate ring $A(X)$.

Then $D(f) = \bigcup_a D(f_a)$, so take the component $V(f) = \cap V(f_a)$ by the Nullstellensatz $f \in$

$I(V(f_a)) = I(V(g_a, a \in D_f)) = \sqrt{f_a \mid a \in D_f}$.

Then there exists an expression $f^n = \sum k_a f_a$ as a finite sum, so set $g - \sum g_a k_a$.

Claim: $\varphi = g/f^n$ on $D(f)$.

This follows because on $D(f_b)$, we have $\varphi = \dfrac{g_b}{f_b}$, and so $gf_b = \sum k_a g_a f_b$.

Finish next class

10 | Tuesday, September 22

10.1 Review: Regular Functions

Given an affine variety X and $U \subseteq X$ open, a *regular function* $\varphi : U \to k$ is one locally (wrt the zariski topology) a fraction. We write the set of regular functions as $\mathcal{O}_X$.

Example 10.1.
$X = V(x_1x_4 - x_2x_3)$ on $U = V(x_2, x_4)^c$, the following function is regular:

$$\begin{aligned} \varphi &: U \to k \\ x &\mapsto \begin{cases} \dfrac{x_1}{x_2} & x_2 \neq 0 \\ \\ \dfrac{x_3}{x_4} & x_4 \neq 0 \end{cases}. \end{aligned}$$

Note that this is not globally a fraction.

Definition 10.0.1 (Distinguished Open Sets)**.**
A *distinguished open set* $D(f) \subseteq X$ for some $f \in A(X)$ is $V(f)^c := \left\{ x \in X \;\middle|\; f(x) \neq 0 \right\}$.

These are useful because the $D(f)$ form a base for the zariski topology.

Proposition 10.1 ***(?).***
For X an affine variety, $f \in A(X)$, we have

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N} \right\}.$$

Proof .
The first reduction we made was that $\varphi \in \mathcal{O}_X(D(f))$ is expressible as $\frac{g_a}{f_a}$ on distinguished opens $D(f_a)$ covering $D(f)$. We also noted that

$$\frac{g_a}{f_a} = \frac{g_b}{f_b} \text{ on } D(f_a) \cap D(f_b) \implies f_b g_a = f_a g_b \text{ in } A(X).$$

The second step was writing $D(f) = \cup D(f_a)$, and so $V(f) = \cap_a V(f_a)$ implies that $f \in I(V(\left\{ f_a \mid a \in U \right\}))$. By the Nullstellensatz, $f \in \sqrt{\left\langle f_a \mid a \in U \right\rangle}$, so $f^N = \sum k_a f_a$ for some N. So construct $g = \sum k_a g_a$, then compute

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b \sum k_a f_a = g_b f^N.$$

Thus $g/f^N = g_b/f_b$ for all b, and we can thus conclude

$$\varphi := \left\{ \frac{g_b}{f_b} \text{ on } D(f_b) \right\} = g/f^N.$$

■

Corollary 10.2 *(?).*
For X an affine variety, $\mathcal{O}_X(X) = A(X)$.

⚠**Warning:** For k not algebraically closed, the proposition and corollary are both false. Take $X = \mathbb{A}^1/\mathbb{R}$, then $\dfrac{1}{x^2+1} \in \mathbb{R}(x)$, but $\mathcal{O}_X(X) \neq A(X) = \mathbb{R}[x]$.

Definition 10.2.1 (Localization)**.**
Let R be a ring and S a set closed under multiplication, then the localization at S is defined by

$$R_S := \left\{ r/s \mid r \in R, s \in S \right\} / \sim.$$

where $r_1/s_1 \sim r_2/s_2 \iff s_3(s_2 r_1 - s_1 r_2) = 0$ for some $s_3 \in S$.

Example 10.2.
Let $f \in R$ and take $S = \left\{ f^n \mid n \geq 1 \right\}$, then $R_f := R_S$.

Corollary 10.3 *(?).*
$\mathcal{O}_X(D(f)) = A(X)_f$ is the localization of the coordinate ring.

These requires some proof, since the LHS literally consists of functions on the topological space $D(f)$ while the RHS consists of formal symbols.

Proof .
Consider the map

$$\begin{aligned} A(X)_f &\to \mathcal{O}_X(D(f)) \\ \text{“} g/f^n \text{”} &\mapsto g/f^n : D(f) \to k. \end{aligned}$$

By definition, there exists a $k \geq 0$ such that

$$f^k(f^m g - f^n g') = 0 \implies f^k(f^m g - f^n g') = 0 \text{ as a function on } D(f).$$

Since $f^k \neq 0$ on $D(f)$, we have $f^m g = f^n g'$ as a function on $D(f)$, so $g/f^n = g'/g^m$ as functions on $D(f)$.

Surjectivity: By the proposition, we have surjectivity, i.e. any element of $|OO_x(D(f))$ can be represented by some g/f^n.

Injectivity: Suppose g/f^n defines the zero function on $D(f)$, then $g = 0$ on $D(f)$ implies that $fg = 0$ on X (i.e. $fg = 0 \in A(X)$), and we can write $f(g \cdot 1 - f^n \cdot 0) = 0$. Then $g/f^n \sim 0/1 \in A(X)_f$, which forces $g/f^n = 0 \in A(X)_f$.

■

10.2 Sheaves

Idea: spaces on functions on topological spaces.

Definition 10.3.1 (Presheaf)**.**
A *presheaf* (of rings) $\mathcal{F}$ on a topological space is

1. For every open set $U \subset X$ a ring $\mathcal{F}(U)$.
2. For any inclusion $U \subset V$ a restriction map $\text{Res}_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying
   a. $F(\emptyset) = 0$.
   b. $\text{Res}_{UU} = \text{id}_{\mathcal{F}(U)}$.
   c. $\text{Res}_{VW} \circ \text{Res}_{UV} = \text{Res}_{UW}$.

Example 10.3.
The smooth functions on $\mathbb{R}$ with the standard topology, $\mathcal{F} = C^\infty$ where $C^\infty(U)$ is the set of smooth functions $U \to \mathbb{R}$. It suffices to check the restriction condition, but the restriction of a smooth function is smooth: if f is smooth on U, it is smooth at every point in U, i.e. all derivatives exist at all points of U. So if $V \subset U$, all derivatives of f will exist at points $x \in V$, so f will be smooth on V.

Note that this also works with continuous functions.

Definition 10.3.2 (Sheaf)**.**
A *sheaf* is a presheaf satisfying an additional gluing property: given $\varphi_i \in \mathcal{F}(U_i)$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there exists a unique $\varphi \in \mathcal{F}(\cup_i U_i)$ such that $\varphi|_{U_i} = \varphi_i$.

11 | Thursday, September 24

Recall that we defined the *regular functions* $\mathcal{O}_X(U)$ on an open set $U \subset X$ an affine variety as the set of functions $\varphi : U \to k$ such that φ is locally a fraction, i.e. for all $p \in U$ there exists a neighborhood of p, say $U_p \subset U$, such that φ restricted to U_p is given by $\frac{g_p}{f_p}$ for some $f_p, g_p \in A(X)$.

We proved that on a distinguished open set $D(f) = V(f)^c$, we have $\mathcal{O}_X(D(f)) = A(X)_f$. An important example was that $\mathcal{O}_X(X) = A(X)$.

Question: If X is a variety over $\mathbb{C}$, does $A(X) = \mathrm{Hol}(X)$? The answer is no, since taking $\mathbb{A}^1/\mathbb{C} \cong \mathbb{C} = X$ we obtain $A(X) = \mathbb{C}[x]$ but for example $e^z \in \mathrm{Hol}(X)$.

On the other hand, if you require that $f \in \mathrm{Hol}(X)$ is meromorphic at ∞, i.e. $f(\dfrac{1}{z})$ is meromorphic at zero, then you do get $\mathbb{C}[z]$. This is an example of GAGA!

Review: what is a category?

Review: what is a presheaf?

12 | Tuesday, September 29

Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on $\mathbb{R}^n$
- Holomorphic functions on $\mathbb{C}$

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on $\mathbb{R}$ a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \left\{ f : U \to \mathbb{R} \;\middle|\; f(x) = c \right\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$. Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0,1)$ and $f_2 = 2$ on $(2,3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf $\mathcal{G}$ on $\mathbb{R}$ given by *locally* constant function, i.e. $\left\{ f : U \to \mathbb{R} \;\middle|\; \forall p \in U, \exists U_p \ni p,\ f|_{U_p} \text{ is constant} \right\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 12.1.
Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open. Then

define a sheaf by

$$
\begin{aligned}
\emptyset &\mapsto 0 \\
\{p\} &\mapsto R \\
\{q\} &\mapsto S \\
\implies \{p,q\} &\mapsto R \times S,
\end{aligned}
$$

where the sheaf condition forces the assignment of the whole space to be the product. Note that the first 3 assignments are automatically compatible, which means that we need a unique $f \in \mathcal{F}(X)$ restricting to R and S. In other words, $\mathcal{F}(X)$ needs to be unique and have maps to R, S, but this is exactly the universal property of the product.

Example 12.2.
Consider the presheaf on X given by $\mathcal{F}(X) = R \times S \times T$. Taking $T = \mathbb{Z}/2\mathbb{Z}$, we can force uniqueness to fail: by projecting to R, S, there are two elements in the fiber, namely $(r, s, 0) \mapsto r, s$ and $(r, s, 1) \mapsto r, s$.

Example 12.3.
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Can check that it's closed under finite intersections and arbitrary unions, so this forms a topology. Now make the assignments

$$
\begin{aligned}
\{a\} &\mapsto A \\
\{b\} &\mapsto B \\
\{a,b\} &\mapsto C \\
X &\mapsto ?.
\end{aligned}
$$

We have a situation like this:

$$
\begin{tikzcd}
 & \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\
B \arrow[dr] & & C \arrow[dl] \\
 & A \arrow[d] & \\
 & \emptyset &
\end{tikzcd}
$$

Unique gluing says that given $r \in B, s \in C$ such that $\varphi_B(r) = \varphi_C(s)$, there should exist a unique $t \in \mathcal{F}(X)$ such that $t|_{\{a,b\}} = r$ and $t|_{\{a,c\}} = s$. This recovers exactly the fiber product.

$$
B \times_A C \coloneqq \left\{ (r, s) \in B \times C \;\middle|\; \varphi_B(r) = \varphi_C(s) \in A \right\}.
$$

Example 12.4.
Let X be an affine variety with the Zariski topology and let $\mathcal{F} \coloneqq \mathcal{O}_X$ be the sheaf of regular functions:

$$\mathcal{O}_X(U) \coloneqq \left\{ f : U \to k \;\middle|\; \forall p \in U,\, \exists U_p \ni p,\;\; f|_{U_p} = \frac{g_p}{h_p} \right\}.$$

Is this a presheaf? We can check that there are restriction maps:

$$\begin{aligned} \mathcal{O}_X(U) &\to \mathcal{O}_X(V) \\ \{f : U \to K\} &\mapsto \{f|_V(x) \coloneqq f(x) \text{ for } x \in V\}. \end{aligned}$$

This makes sense because if $V \subset U$, any $x \in V$ is in the domain of f. Given that f is locally a fraction, say $\rho = g_p/h_p$ on $U_p \ni p$, is $\varphi|_V$ locally a fraction? Yes: for all $p \in V \subset U$, $\varphi = g_p/f_p$ on U_p and this remains true on $U_p \cap V$.

To check that $\mathcal{O}_X$ is a sheaf, given a set of regular functions $\{\varphi_i : U_i \to k\}$ agreeing on intersections, define

$$\begin{aligned} \varphi : \cup U_i &\to k \\ \varphi(x) &\coloneqq \varphi_i(x) \text{ if } x \in U_i. \end{aligned}$$

This is well-defined, since if $x \in U_i \cap U_j$, $\varphi_i(x) = \varphi_j(x)$ since both restrict to the same function on $U_i \cap U_j$ by assumption.

Why is φ locally a fraction? We need to check that for all $p \in U \coloneqq \cup U_i$ there exists a $U_p \ni p$ with $\varphi|_{U_p} = g_p/h_p$. But any $p \in \cup U_i$ implies $p \in U_i$ for some i. Then there exists an open set $U_{i,p} \ni p$ in U_i such that $\varphi|_{U_{i,p}} = g_p/h_p$ by definition of a regular function. So take $U_p = U_{i,p}$ and use the fact that $\varphi|_{U_i} = \varphi_i$ along with compatibility of restriction.

Remark 15.
General observation: any presheaf of functions is a sheaf when the functions are defined by a local property, i..e any property that can be checked at p by considering an open set $U_p \ni p$.

As in the examples of smooth or holomorphic functions, these were local properties. E.g. checking that a function is smooth involves checking on an open set around each point. On the other hand, being a constant function is not a local property.

Definition 12.0.1 (Restriction of a (Pre)sheaf)**.**
Given a sheaf $\mathcal{F}$ on X and an open set $U \subset X$, we can define a sheaf $\mathcal{F}|_U$ on U (with the subspace topology) by defining $\mathcal{F}|_U(V) \coloneqq \mathcal{F}(V)$ for $U \subseteq V$.

Definition 12.0.2 (Stalks)**.**
Let $\mathcal{F}$ be a sheaf on X and $p \in X$ a point. The *stalk* of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$ for $p \in U$, is

defined by

$$\mathcal{F}_p := \left\{ (U, \varphi) \;\middle|\; \varphi \in \mathcal{F}(U) \right\} / \sim$$

where $(U, \varphi) \sim (V, \varphi')$ iff there exists a $W \subset U \cap V$ and $p \in W$ such that $\varphi|_W = \varphi|_W{}'$.

Example 12.5.
What is the stalk of $\mathrm{Hol}(\mathbb{C})$ at $p = 0$?

Examples of equivalent elements in this stalk:

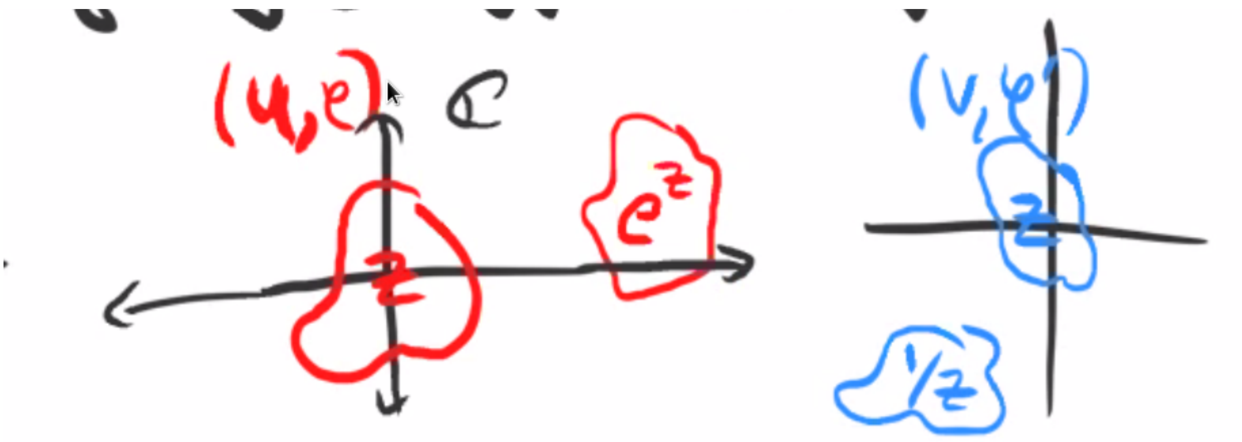


Figure 8: O

In this case

$$\mathrm{Hol}(\mathbb{C})_0 = \left\{ \varphi = \sum_{i>0} c_i z^i \;\middle|\; \varphi \text{ has a positive radius of convergence} \right\}.$$

Definition 12.0.3 (Sections)**.**
An element $f \in \mathcal{F}(U)$ is called a *section* over U, and elements of the stalk $f \in \mathcal{F}_p$ are called *germs* at p.

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13.1 Stalks and Localizations

Recall that a sheaf of rings on a topological space X is a ring $\mathcal{F}(U)$ for all open sets $U \subset X$ satisfying four properties:

1. The empty set is mapped to zeor.

2. The morphism $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity.
3. Given $W \subset V \subset U$ we have
4. Gluing: given sections $s_i \in \mathcal{F}(U_i)$ which agree on overlaps (restrict to the same function on $U_i \cap U_j$), there is a unique $s \in \mathcal{F}(\cup U_i)$.

Example 13.1.
If X is an affine variety with the zariski topology, $\mathcal{O}_X$ is a sheaf of regular functions, where we recall $\mathcal{O}_X(U)$ are the functions $\varphi : U \to k$ that are locally a fraction.

Recall that the *stalk* of a sheaf $\mathcal{F}$ at a point $p \in X$, is defined as

$$\mathcal{F}_p := \left\{ (U, \varphi) \;\middle|\; p \in U \text{ open }, \varphi \in \mathcal{F}(U) \right\} / \sim .$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists a $p \in W \subset U \cap U'$ such φ, φ' restricted to W are equal.

Recall that a *local ring* is a ring with a unique maximal ideal $\mathfrak{m}$. Given a prime ideal $\mathfrak{p} \in R$, so $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$, the complement $R \setminus P$ is closed under multiplication. So we can localize to obtain $R_{\mathfrak{p}} = \left\{ a/s \;\middle|\; s \in R \setminus P, a \in R \right\} / \sim$ where $a'/s' \sim a/s$ iff there exists a $t \in R \setminus P$ such that $t(a's - as') = 0$.

Warning: Note that R_f is localizing at the powers of f, whereas $R_{\mathfrak{p}}$ is localizing at the *complement* of $\mathfrak{p}$.

Since maximal ideals are prime, we can localize any ring R at a maximal ideal $R_{\mathfrak{m}}$, and this will be a local ring. Why? The ideals in $R_{\mathfrak{m}}$ biject with ideals in R contained in $\mathfrak{m}$. Thus all ideals in $R_{\mathfrak{m}}$ are contained in the maximal ideal generated by $\mathfrak{m}$, i.e. $\mathfrak{m}R_{\mathfrak{m}}$.

Lemma 13.1 *(?).*
Let X be an affine variety. The stalk of the sheaf of regular functions $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$ is isomorphic to the localization $A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p := I(\{p\})$.

Proof .
We can write

$$A(X)_{\mathfrak{m}_p} := \left\{ \frac{g}{f} \;\middle|\; g \in A(X),\, f \in A(X) \setminus \mathfrak{m}_p \right\} / \sim$$
$$\text{where } g_1/f_1 \sim g_2/f_2 \iff \exists h(p) \neq 0 \text{ where } 0 = h(f_2 g_1 - f_1 g_2).$$

where the f are regular functions on X such that $f(p) = 0$.
We can also write

$$\mathcal{O}_{X,p} := \left\{ (U, \varphi) \;\middle|\; p \in U,\, \varphi \in \mathcal{O}_X(U) \right\} / \sim$$
$$\text{where } (U, \varphi) \sim (U', \varphi') \iff \exists p \in W \subset U \cap U' \text{ s.t. } \varphi|_W = \varphi'|_W.$$

So we can define a map

$$\begin{aligned} \Phi : A(X)_{\mathfrak{m}_p} &\to \mathcal{O}_{X,p} \\ \frac{g}{f} &\mapsto \left(D_f, \frac{g}{f} \right). \end{aligned}$$

Step 1: There are equivalence relations on both sides, so we need to check that things are well-defined.
We have

$$\begin{aligned} g/f \sim g'/f' &\iff \exists g \text{ such that } h(p) \neq 0,\, h(gf' - g'f) = 0 \in A(X) \\ &\iff \text{the functions } \frac{g}{f}, \frac{g'}{f'} \text{ agree on } W := D(f) \cap D(f') \cap D(h) \\ &\implies (D_f, g/f) \sim (D_{f'}, g'/f'), \end{aligned}$$

since there exists a $W \subset D_f \cap D_{f'}$ such that $g/f, g'/f'$ are equal.

Step 2: Surjectivity, since this is clearly a ring map with pointwise operations.
Any germ can be represented by (U, φ) with $\varphi \in \mathcal{O}_X(U)$. Since the sets D_f form a base for the topology, there exists a $D_f \subset U$ containing p. By definition, $(U, \varphi) = (D_f, \varphi|_{D_f})$ in $\mathcal{O}_{X,p}$. Using the proposition that $\mathcal{O}_X(D(f)) = A(X)_f$, this implies that $\varphi|_{D_f} = g/f^n$ for some n and $f(p) \neq 0$, so (U, φ) is in the image of Φ.

Step 3: Injectivity. We want to show that $g/f \mapsto 0$ implies that $g/f = 0 \in A(X)_{\mathfrak{m}_p}$.
Suppose that $(D_f, g/f) = 0 \in \mathcal{O}_{X,p}$ and $(U, \varphi) = 0 \in \mathcal{O}_{X,p}$, then there exists an open $W \subset D_f$ containing p such that after passing to some distinguished open $D_h \ni p$ such that $\varphi = 0$ on D_h. Wlog we can assume $\varphi = 0$ on U, since we could shrink U (staying in the same equivalence class) to make this true otherwise. Then $\varphi = g/f$ on D_h, using that $\mathcal{O}_X(D_f) = A(X)_f$, so $g/f = 0$ here. So there exists a k such that $f^k(g \cdot 1 - 0 \cdot f) = 0$ in $A(X)$, so $f^k g = 0 \in A(X)_{\mathfrak{m}_p}$.
■

Conclusion:

$$\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}.$$

Example 13.2.
Let $X = \{p, q\}$ with the discrete topology with the sheaf $\mathcal{F}$ given by $p \mapsto R, q \mapsto S, X \mapsto R \times S$.

Then $\mathcal{F}_p = R$, since if U is open and $p \in U$ then either $U = \{p\}$ or $U = X$. We can check that for (r, s) a section of $\mathcal{F}$, we have an equivalence of germs $(X, (r, s)) \sim (\{p\}, r)$ since $\{p\} \subset X \cap \{p\}$. Here X plays the role of U, $\{p\}$ of U', and the last $\{p\}$ the role of $W \subset U \cap U'$.

$$\begin{aligned} \mathcal{O}_{X,p} &\to A(X) \\ (\{p\}, r) &\mapsto r \\ \mathcal{F}_p &\cong R. \end{aligned}$$

Example 13.3.
Let M be a manifold and consider the sheaf C^∞ of smooth functions on M. Then the stalk C_p^∞ at p is defined as the set of smooth functions in a neighborhood of p modulo functions being equivalent if they agree on a small enough ball $B_\varepsilon(p)$. This contains a maximal ideal $\mathfrak{m}_p$, the smooth functions vanishing at p.

Then $\mathfrak{m}_p^2$ is again an ideal, equal to the set $\left\{ f \,\middle|\, \partial_i \partial_j f \Big|_p = 0, \forall i, j \right\}$. Thus $\mathfrak{m}_p / \mathfrak{m}_p^2 \cong \{\partial_v\}^\vee$, the dual of the set of directional derivatives.

13.2 What's the Point!

Problem: what should a map of affine varieties be? A bad definition would be just taking the continuous maps: for example, any bijection $\mathbb{A}^1_{\mathbb{C}}$ is a homeomorphism in the zariski topology. Why? This coincides with the cofinite topology, and the preimage of a cofinite set is cofinite.

How do we fix this?

1. $f : X \to Y$ is continuous, i.e. $f^{-1}(U)$ is open whenever U is open.
2. Given $U \subset Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the function $\varphi \circ f : f^{-1}(U) \to k$ is regular.

We'll take this to be the definition of a morphism $X \to Y$.

Example 13.4.
For smooth manifolds, we also require that there is a pullback that preserves smooth functions:

$$f^* : C^\infty(U) \to C^\infty(f^{-1}(U)).$$

14 Misc Unsorted

algebra	$\leftrightarrow$	geometry
radical ideal $I = \sqrt{I}$	$\to$	$V(I)$ variety
$I(V)$ ideal of a set	$\leftarrow$	solution set V
sum of ideals $I + J$	$\to$	$V(I) \cap V(J)$ intersection of varieties
$\sqrt{I(V) + I(W)}$ radical of sum	$\leftarrow$	intersection of sets $V \cap W$
product of ideals IJ	$\to$	$V(I) \cup V(J)$ union of varieties
$\sqrt{I(V)I(W)}$ radical of product	$\leftarrow$	union of sets $V \cup W$
intersection of ideals $I \cap J$	$\to$	$V(I) \cup V(J)$ union of varieties
$I(V) \cap I(W)$	$\leftarrow$	union of sets $V \cup W$
quotient of ideals $I : J$	$\to$	$\overline{V(I) - V(J)}$ difference of varieties
$I(V) : I(W)$	$\leftarrow$	difference of sets $\overline{V - W}$
elimination $\sqrt{I \cap \mathbb{C}[x_{k+1}, \ldots, x_n]}$	$\leftrightarrow$	$\overline{\pi_k(V(I))}$ projection of varieties

Figure 9: Image