

# Problem Set 3

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**Exercise 0.1** (Gathmann 2.33).

Define

$$X := \left\{ M \in \text{Mat}(2 \times 3, k) \mid \text{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that  $X$  is an irreducible variety, and find its dimension.

## Solution:

We'll use the following fact from linear algebra:

### Definition (*Matrix Minor*).

For an  $m \times n$  matrix, a *minor of order  $\ell$*  is the determinant of a  $\ell \times \ell$  submatrix obtained by deleting any  $m - \ell$  rows and any  $n - \ell$  columns.

### Theorem 0.1 (*Rank is a Function of Minors*).

If  $A \in \text{Mat}(m \times n, k)$  is a matrix, then the rank of  $A$  is equal to the order of largest nonzero minor.

Thus

$$M_{ij} = 0 \text{ for all } \ell \times \ell \text{ minors } M_{ij} \iff \text{rank}(M) < \ell,$$

following from the fact that if one takes  $\ell = \min(m, n)$  and all  $\ell \times \ell$  minors vanish, then the largest nonzero minor must be of size  $j \times j$  for  $j \leq \ell - 1$ . But  $\det M_{ij}$  is a polynomial  $f_{ij}$  in its entries, which means that  $X$  can be written as

$$X = V(\{f_{ij}\}),$$

which exhibits  $X$  as a variety. Thus

$$M = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \implies X = V(\langle xb - ya, yc - zb, xc - za \rangle) \subset \mathbb{A}^6.$$

**Claim:** The ideal above is prime, and so the coordinate ring  $A(X)$  is a domain and thus  $X$  is irreducible.

**Claim:**  $\dim(X) = 4$ .

Heuristic: there are three degrees of freedom in choosing the first row  $x, y, z$ . To enforce the rank 1 condition, the second row must be a scalar multiple of the first, yielding one degree of freedom for the scalar.

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Note: I looked at this for a couple of hours, but I don't know how to prove either of these statements with the tools we have so far!

**Exercise 0.2** (Gathmann 2.34).

Let  $X$  be a topological space, and show

- If  $\{U_i\}_{i \in I} \Rightarrow X$ , then  $\dim X = \sup_{i \in I} \dim U_i$ .
- If  $X$  is an irreducible affine variety and  $U \subset X$  is a nonempty subset, then  $\dim X = \dim U$ . Does this hold for any irreducible topological space?

**Solution:**

We first note that if  $U \subseteq V$ , then  $\dim U \leq \dim V$ . If this were not the case, one could find a chain  $\{I_j\}$  of closed irreducible subsets of  $V$  of length  $n > \dim U$ . But then  $I'_j := I_j \cap U$  would again be a closed irreducible set, yielding a chain of length  $n$  in  $U$ . Thus  $\dim X \geq \dim U_i$ , and it remains true that  $\dim X \geq \sup \dim U_i$ , so it suffices to show that  $\dim X \leq \sup \dim U_i$ . Set  $S := \sup_i \dim U_i$  and  $n := \dim X$ , we want to show that  $s \geq n$ . Let  $\{I_j\}_{j \leq n}$  be a maximal chain of length  $n$  of closed irreducible subsets, so we have

$$\emptyset \subsetneq I_1.$$

**Exercise 0.3** (Gathmann 2.36).

Prove the following:

- Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.
- A complex affine variety of dimension at least 1 is never compact in the classical topology.

**Exercise 0.4** (Gathmann 2.40).

Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1x_4 - x_2x_3 \rangle$$

and show the following:

- $R$  is an integral domain of dimension 3.
- $x_1, \dots, x_4$  are irreducible but not prime in  $R$ , and thus  $R$  is not a UFD.
- $x_1x_4$  and  $x_2x_3$  are two decompositions of the same element in  $R$  which are nonassociate.
- $\langle x_1, x_2 \rangle$  is a prime ideal of codimension 1 in  $R$  that is not principal.

**Exercise 0.5** (Problem 5).

Consider a set  $U$  in the complement of  $(0, 0) \in \mathbb{A}^2$ . Prove that any regular function on  $U$  extends to a regular function on all of  $\mathbb{A}^2$ .