

# Qualifying Exams

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## 1 Fall 2019

### 1.1 1

Let  $G$  be a finite group with  $n$  distinct conjugacy classes. Let  $g_1 \cdots g_n$  be representatives of the conjugacy classes of  $G$ .

Prove that if  $g_i g_j = g_j g_i$  for all  $i, j$  then  $G$  is abelian.

### 1.2 2

Let  $G$  be a group of order 105 and let  $P, Q, R$  be Sylow 3, 5, 7 subgroups respectively.

- (a) Prove that at least one of  $Q$  and  $R$  is normal in  $G$ .
- (b) Prove that  $G$  has a cyclic subgroup of order 35.
- (c) Prove that both  $Q$  and  $R$  are normal in  $G$ .
- (d) Prove that if  $P$  is normal in  $G$  then  $G$  is cyclic.

### 1.3 3

Let  $R$  be a ring with the property that for every  $a \in R, a^2 = a$ .

- (a) Prove that  $R$  has characteristic 2.
- (b) Prove that  $R$  is commutative.

### 1.4 4

Let  $F$  be a finite field with  $q$  elements.

Let  $n$  be a positive integer relatively prime to  $q$  and let  $\omega$  be a primitive  $n$ th root of unity in an extension field of  $F$ .

Let  $E = F[\omega]$  and let  $k = [E : F]$ .

- (a) Prove that  $n$  divides  $q^k - 1$ .

- (b) Let  $m$  be the order of  $q$  in  $\mathbb{Z}/n\mathbb{Z}$ . Prove that  $m$  divides  $k$ .
- (c) Prove that  $m = k$ .

## 1.5 5

Let  $R$  be a ring and  $M$  an  $R$ -module.

Recall that the set of torsion elements in  $M$  is defined by

$$\text{Tor}(M) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}.$$

- (a) Prove that if  $R$  is an integral domain, then  $\text{Tor}(M)$  is a submodule of  $M$ .
- (b) Give an example where  $\text{Tor}(M)$  is not a submodule of  $M$ .
- (c) If  $R$  has zero-divisors, prove that every non-zero  $R$ -module has non-zero torsion elements.

## 1.6 6

Let  $R$  be a commutative ring with multiplicative identity. Assume Zorn's Lemma.

- (a) Show that

$$N = \{r \in R \mid r^n = 0 \text{ for some } n > 0\}$$

is an ideal which is contained in any prime ideal.

- (b) Let  $r$  be an element of  $R$  not in  $N$ . Let  $S$  be the collection of all proper ideals of  $R$  not containing any positive power of  $r$ . Use Zorn's Lemma to prove that there is a prime ideal in  $S$ .
- (c) Suppose that  $R$  has exactly one prime ideal  $P$ . Prove that every element  $r$  of  $R$  is either nilpotent or a unit.

## 1.7 7

Let  $\zeta_n$  denote a primitive  $n$ th root of 1 in  $\mathbb{Q}$ . You may assume the roots of the minimal polynomial  $p_n(x)$  of  $\zeta_n$  are exactly the primitive  $n$ th roots of 1.

Show that the field extension  $\mathbb{Q}(\zeta_n)$  over  $\mathbb{Q}$  is Galois and prove its Galois group is  $(\mathbb{Z}/n\mathbb{Z})^\times$ .

How many subfields are there of  $\mathbb{Q}(\zeta_{20})$ ?

## 1.8 8

Let  $\{e_1, \dots, e_n\}$  be a basis of a real vector space  $V$  and let

$$\Lambda := \left\{ \sum r_i e_i \mid r_i \in \mathbb{Z} \right\}$$

Let  $\cdot$  be a non-degenerate ( $v \cdot w = 0$  for all  $w \in V \iff v = 0$ ) symmetric bilinear form on  $V$  such that the Gram matrix  $M = (e_i \cdot e_j)$  has integer entries.

Define the dual of  $\Lambda$  to be

$$\Lambda^\vee := \{v \in V \mid v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

- (a) Show that  $\Lambda \subset \Lambda^\vee$ .
- (b) Prove that  $\det M \neq 0$  and that the rows of  $M^{-1}$  span  $\Lambda^\vee$ .
- (c) Prove that  $\det M = |\Lambda^\vee/\Lambda|$ .

## 2 Spring 2019

### 2.1 1.

Let  $A$  be a square matrix over the complex numbers. Suppose that  $A$  is nonsingular and that  $A^{2019}$  is diagonalizable over  $\mathbb{C}$ .

Show that  $A$  is also diagonalizable over  $\mathbb{C}$ .

### 2.2 2.

Let  $F = \mathbb{F}_p$ , where  $p$  is a prime number.

- (a) Show that if  $\pi(x) \in F[x]$  is irreducible of degree  $d$ , then  $\pi(x)$  divides  $x^{p^d} - x$ .
- (b) Show that if  $\pi(x) \in F[x]$  is an irreducible polynomial that divides  $x^{p^n} - x$ , then  $\deg \pi(x)$  divides  $n$ .

### 2.3 3.

How many isomorphism classes are there of groups of order 45?

Describe a representative from each class.

### 2.4 4.

For a finite group  $G$ , let  $c(G)$  denote the number of conjugacy classes of  $G$ .

- (a) Prove that if two elements of  $G$  are chosen uniformly at random, then the probability they commute is precisely

$$\frac{c(G)}{|G|}.$$

- (b) State the class equation for a finite group.
- (c) Using the class equation (or otherwise) show that the probability in part (a) is at most

$$\frac{1}{2} + \frac{1}{2[G : Z(G)]}.$$

Here, as usual,  $Z(G)$  denotes the center of  $G$ .

### 2.5 5.

Let  $R$  be an integral domain. Recall that if  $M$  is an  $R$ -module, the *rank* of  $M$  is defined to be the maximum number of  $R$ -linearly independent elements of  $M$ .

- (a) Prove that for any  $R$ -module  $M$ , the rank of  $\text{Tor}(M)$  is 0.
- (b) Prove that the rank of  $M$  is equal to the rank of  $M/\text{Tor}(M)$ .
- (c) Suppose that  $M$  is a non-principal ideal of  $R$ .

Prove that  $M$  is torsion-free of rank 1 but not free.

### 2.6 6.

Let  $R$  be a commutative ring with 1.

Recall that  $x \in R$  is nilpotent iff  $x^n = 0$  for some positive integer  $n$ .

- (a) Show that every proper ideal of  $R$  is contained within a maximal ideal.
- (b) Let  $J(R)$  denote the intersection of all maximal ideals of  $R$ .  
Show that  $x \in J(R) \iff 1 + rx$  is a unit for all  $r \in R$ .
- (c) Suppose now that  $R$  is finite. Show that in this case  $J(R)$  consists precisely of the nilpotent elements in  $R$ .

### 2.7 7.

Let  $p$  be a prime number. Let  $A$  be a  $p \times p$  matrix over a field  $F$  with 1 in all entries except 0 on the main diagonal.

Determine the Jordan canonical form (JCF) of  $A$

- (a) When  $F = \mathbb{Q}$ ,
- (b) When  $F = \mathbb{F}_p$ .

Hint: In both cases, all eigenvalues lie in the ground field. In each case find a matrix  $P$  such that  $P^{-1}AP$  is in JCF.

### 2.8 8.

Let  $\zeta = e^{2\pi i/8}$ .

- (a) What is the degree of  $\mathbb{Q}(\zeta)/\mathbb{Q}$ ?
- (b) How many quadratic subfields of  $\mathbb{Q}(\zeta)$  are there?
- (c) What is the degree of  $\mathbb{Q}(\zeta, \sqrt[4]{2})$  over  $\mathbb{Q}$ ?

### 3 Fall 2018

#### 3.1 1.

Let  $G$  be a finite group whose order is divisible by a prime number  $p$ . Let  $P$  be a normal  $p$ -subgroup of  $G$  (so  $|P| = p^c$  for some  $c$ ).

- (a) Show that  $P$  is contained in every Sylow  $p$ -subgroup of  $G$ .
- (b) Let  $M$  be a maximal proper subgroup of  $G$ . Show that either  $P \subseteq M$  or  $|G/M| = p^b$  for some  $b \leq c$ .

#### 3.2 2.

- (a) Suppose the group  $G$  acts on the set  $X$ . Show that the stabilizers of elements in the same orbit are conjugate.
- (b) Let  $G$  be a finite group and let  $H$  be a proper subgroup. Show that the union of the conjugates of  $H$  is strictly smaller than  $G$ , i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

- (c) Suppose  $G$  is a finite group acting transitively on a set  $S$  with at least 2 elements. Show that there is an element of  $G$  with no fixed points in  $S$ .

#### 3.3 3.

Let  $F \subset K \subset L$  be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.

- (a) If  $L/F$  is Galois, then so is  $K/F$ .
- (b) If  $L/F$  is Galois, then so is  $L/K$ .
- (c) If  $K/F$  and  $L/K$  are both Galois, then so is  $L/F$ .

#### 3.4 4.

Let  $V$  be a finite dimensional vector space over a field (the field is not necessarily algebraically closed).

Let  $\phi : V \rightarrow V$  be a linear transformation. Prove that there exists a decomposition of  $V$  as  $V = U \oplus W$ , where  $U$  and  $W$  are  $\phi$ -invariant subspaces of  $V$ ,  $\phi|_U$  is nilpotent, and  $\phi|_W$  is nonsingular.

#### 3.5 5.

Let  $A$  be an  $n \times n$  matrix.

- (a) Suppose that  $v$  is a column vector such that the set  $\{v, Av, \dots, A^{n-1}v\}$  is linearly independent. Show that any matrix  $B$  that commutes with  $A$  is a polynomial in  $A$ .

- (b) Show that there exists a column vector  $v$  such that the set  $\{v, Av, \dots, A^{n-1}v\}$  is linearly independent  $\iff$  the characteristic polynomial of  $A$  equals the minimal polynomial of  $A$ .

### 3.6 6.

Let  $R$  be a commutative ring, and let  $M$  be an  $R$ -module. An  $R$ -submodule  $N$  of  $M$  is maximal if there is no  $R$ -module  $P$  with  $N \subsetneq P \subsetneq M$ .

- (a) Show that an  $R$ -submodule  $N$  of  $M$  is maximal *iff*  $M/N$  is a simple  $R$ -module: i.e.,  $M/N$  is nonzero and has no proper, nonzero  $R$ -submodules.
- (b) Let  $M$  be a  $\mathbb{Z}$ -module. Show that a  $\mathbb{Z}$ -submodule  $N$  of  $M$  is maximal  $\iff \#M/N$  is a prime number.
- (c) Let  $M$  be the  $\mathbb{Z}$ -module of all roots of unity in  $\mathbb{C}$  under multiplication. Show that there is no maximal  $\mathbb{Z}$ -submodule of  $M$ .

### 3.7 7.

Let  $R$  be a commutative ring.

- (a) Let  $r \in R$ . Show that the map

$$\begin{aligned} r\bullet : R &\rightarrow R \\ x &\mapsto rx. \end{aligned}$$

is an  $R$ -module endomorphism of  $R$ .

- (b) We say that  $r$  is a **zero-divisor** if  $r\bullet$  is not injective. Show that if  $r$  is a zero-divisor and  $r \neq 0$ , then the kernel and image of  $R$  each consist of zero-divisors.
- (c) Let  $n \geq 2$  be an integer. Show: if  $R$  has exactly  $n$  zero-divisors, then  $\#R \leq n^2$ .
- (d) Show that up to isomorphism there are exactly two commutative rings  $R$  with precisely 2 zero-divisors.

You may use without proof the following fact: every ring of order 4 is isomorphic to exactly one of the following:

$$\frac{\mathbb{Z}}{4\mathbb{Z}}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 + t + 1)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2 - t)}, \quad \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[t]}{(t^2)}.$$

## 4 Spring 2018

### 4.1 1.

- (a) Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any  $p$ -group (a group whose order is a positive power of a prime integer  $p$ ) has a nontrivial center.
- (b) Prove that any group of order  $p^2$  (where  $p$  is prime) is abelian.

(c) Prove that any group of order  $5^2 \cdot 7^2$  is abelian.

(d) Write down exactly one representative in each isomorphism class of groups of order  $5^2 \cdot 7^2$ .

#### 4.2 2.

Let  $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[x]$ .

(a) Find the splitting field  $K$  of  $f$ , and compute  $[K : \mathbb{Q}]$ .

(b) Find the Galois group  $G$  of  $f$ , both as an explicit group of automorphisms, and as a familiar abstract group to which it is isomorphic.

(c) Exhibit explicitly the correspondence between subgroups of  $G$  and intermediate fields between  $\mathbb{Q}$  and  $k$ .

#### 4.3 3.

Let  $K$  be a Galois extension of  $\mathbb{Q}$  with Galois group  $G$ , and let  $E_1, E_2$  be intermediate fields of  $K$  which are the splitting fields of irreducible  $f_i(x) \in \mathbb{Q}[x]$ .

Let  $E = E_1 E_2 \subset K$ .

Let  $H_i = \text{Gal}(K/E_i)$  and  $H = \text{Gal}(K/E)$ .

(a) Show that  $H = H_1 \cap H_2$ .

(b) Show that  $H_1 H_2$  is a subgroup of  $G$ .

(c) Show that

$$\text{Gal}(K/(E_1 \cap E_2)) = H_1 H_2.$$

#### 4.4 4.

Let

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 2 & -4 \end{bmatrix} \in M_3(\mathbb{C})$$

(a) Find the Jordan canonical form  $J$  of  $A$ .

(b) Find an invertible matrix  $P$  such that  $P^{-1}AP = J$ .

You should not need to compute  $P^{-1}$ .

#### 4.5 5.

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$$



over a commutative ring  $R$ , where  $b$  and  $x$  are units of  $R$ . Prove that

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \implies MN = 0.$$

#### 4.6 6.

Let

$$M = \{(w, x, y, z) \in \mathbb{Z}^4 \mid w + x + y + z \in 2\mathbb{Z}\},$$

and

$$N = \{(w, x, y, z) \in \mathbb{Z}^4 \mid (w - x), 4 \mid (x - y), 4 \mid (y - z)\}.$$

- (a) Show that  $N$  is a  $\mathbb{Z}$ -submodule of  $M$ .
- (b) Find vectors  $u_1, u_2, u_3, u_4 \in \mathbb{Z}^4$  and integers  $d_1, d_2, d_3, d_4$  such that

$$\{u_1, u_2, u_3, u_4\}$$

is a free basis for  $M$ , and

$$\{d_1 u_1, d_2 u_2, d_3 u_3, d_4 u_4\}$$

is a free basis for  $N$ .

- (c) Use the previous part to describe  $M/N$  as a direct sum of cyclic  $\mathbb{Z}$ -modules.

#### 4.7 7.

Let  $R$  be a PID and  $M$  be an  $R$ -module. Let  $p$  be a prime element of  $R$ . The module  $M$  is called  $\langle p \rangle$ -primary if for every  $m \in M$  there exists  $k > 0$  such that  $p^k m = 0$ .

- (a) Suppose  $M$  is  $\langle p \rangle$ -primary. Show that if  $m \in M$  and  $t \in R$ ,  $t \notin \langle p \rangle$ , then there exists  $a \in R$  such that  $atm = m$ .
- (b) A submodule  $S$  of  $M$  is said to be *pure* if  $S \cap rM = rS$  for all  $r \in R$ . Show that if  $M$  is  $\langle p \rangle$ -primary, then  $S$  is pure if and only if  $S \cap p^k M = p^k S$  for all  $k \geq 0$ .

#### 4.8 8.

Let  $R = C[0, 1]$  be the ring of continuous real-valued functions on the interval  $[0, 1]$ . Let  $I$  be an ideal of  $R$ .

- (a) Show that if  $f \in I$ ,  $a \in [0, 1]$  are such that  $f(a) \neq 0$ , then there exists  $g \in I$  such that  $g(x) \geq 0$  for all  $x \in [0, 1]$ , and  $g(x) > 0$  for all  $x$  in some open neighborhood of  $a$ .
- (b) If  $I \neq R$ , show that the set  $Z(I) = \{x \in [0, 1] \mid f(x) = 0 \text{ for all } f \in I\}$  is nonempty.
- (c) Show that if  $I$  is maximal, then there exists  $x_0 \in [0, 1]$  such that  $I = \{f \in R \mid f(x_0) = 0\}$ .

## 5 Fall 2017

### 5.1 1.

Suppose the group  $G$  acts on the set  $A$ . Assume this action is faithful (recall that this means that the kernel of the homomorphism from  $G$  to  $\text{Sym}(A)$  which gives the action is trivial) and transitive (for all  $a, b$  in  $A$ , there exists  $g$  in  $G$  such that  $g \cdot a = b$ .)

- (a) For  $a \in A$ , let  $G_a$  denote the stabilizer of  $a$  in  $G$ . Prove that for any  $a \in A$ ,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

- (b) Suppose that  $G$  is abelian. Prove that  $|G| = |A|$ . Deduce that every abelian transitive subgroup of  $S_n$  has order  $n$ .

### 5.2 2.

- (a) Classify the abelian groups of order 36.

For the rest of the problem, assume that  $G$  is a non-abelian group of order 36.

You may assume that the only subgroup of order 12 in  $S_4$  is  $A_4$  and that  $A_4$  has no subgroup of order 6.

- (b) Prove that if the 2-Sylow subgroup of  $G$  is normal,  $G$  has a normal subgroup  $N$  such that  $G/N$  is isomorphic to  $A_4$ .
- (c) Show that if  $G$  has a normal subgroup  $N$  such that  $G/N$  is isomorphic to  $A_4$  and a subgroup  $H$  isomorphic to  $A_4$  it must be the direct product of  $N$  and  $H$ .
- (d) Show that the dihedral group of order 36 is a non-abelian group of order 36 whose Sylow-2 subgroup is not normal.

### 5.3 3.

Let  $F$  be a field. Let  $f(x)$  be an irreducible polynomial in  $F[x]$  of degree  $n$  and let  $g(x)$  be any polynomial in  $F[x]$ . Let  $p(x)$  be an irreducible factor (of degree  $m$ ) of the polynomial  $f(g(x))$ .

Prove that  $n$  divides  $m$ . Use this to prove that if  $r$  is an integer which is not a perfect square, and  $n$  is a positive integer then every irreducible factor of  $x^{2n} - r$  over  $\mathbb{Q}[x]$  has even degree.

### 5.4 4.

- (a) Let  $f(x)$  be an irreducible polynomial of degree 4 in  $\mathbb{Q}[x]$  whose splitting field  $K$  over  $\mathbb{Q}$  has Galois group  $G = S_4$ .

Let  $\theta$  be a root of  $f(x)$ . Prove that  $\mathbb{Q}[\theta]$  is an extension of  $\mathbb{Q}$  of degree 4 and that there are no intermediate fields between  $\mathbb{Q}$  and  $\mathbb{Q}[\theta]$ .

- (b) Prove that if  $K$  is a Galois extension of  $\mathbb{Q}$  of degree 4, then there is an intermediate subfield between  $K$  and  $\mathbb{Q}$ .

### 5.5 5.

A ring  $R$  is called *simple* if its only two-sided ideals are  $0$  and  $R$ .

- (a) Suppose  $R$  is a commutative ring with  $1$ . Prove  $R$  is simple if and only if  $R$  is a field.
- (b) Let  $k$  be a field. Show the ring  $M_n(k)$ ,  $n \times n$  matrices with entries in  $k$ , is a simple ring.

### 5.6 6.

For a ring  $R$ , let  $U(R)$  denote the multiplicative group of units in  $R$ . Recall that in an integral domain  $R$ ,  $r \in R$  is called *irreducible* if  $r$  is not a unit in  $R$ , and the only divisors of  $r$  have the form  $ru$  with  $u$  a unit in  $R$ .

We call a non-zero, non-unit  $r \in R$  *prime* in  $R$  if  $r \mid ab \implies r \mid a$  or  $r \mid b$ . Consider the ring  $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ .

- (a) Prove  $R$  is an integral domain.
- (b) Show  $U(R) = \{\pm 1\}$ .
- (c) Show  $3$ ,  $2 + \sqrt{-5}$ , and  $2 - \sqrt{-5}$  are irreducible in  $R$ .
- (d) Show  $3$  is not prime in  $R$ .
- (e) Conclude  $R$  is not a PID.

### 5.7 7.

Let  $F$  be a field and let  $V$  and  $W$  be vector spaces over  $F$ .

Make  $V$  and  $W$  into  $F[x]$ -modules via linear operators  $T$  on  $V$  and  $S$  on  $W$  by defining  $X \cdot v = T(v)$  for all  $v \in V$  and  $X \cdot w = S(w)$  for all  $w \in W$ .

Denote the resulting  $F[x]$ -modules by  $V_T$  and  $W_S$  respectively.

- (a) Show that an  $F[x]$ -module homomorphism from  $V_T$  to  $W_S$  consists of an  $F$ -linear transformation  $R : V \rightarrow W$  such that  $RT = SR$ .
- (b) Show that  $V_T \cong W_S$  as  $F[x]$ -modules  $\iff$  there is an  $F$ -linear isomorphism  $P : V \rightarrow W$  such that  $T = P^{-1}SP$ .
- (c) Recall that a module  $M$  is *simple* if  $M \neq 0$  and any proper submodule of  $M$  must be zero. Suppose that  $V$  has dimension 2. Give an example of  $F$ ,  $T$  with  $V_T$  simple.
- (d) Assume  $F$  is algebraically closed. Prove that if  $V$  has dimension 2, then any  $V_T$  is not simple.