Classifying Space

Usually look at this in the context of a topological group G, and denote BG the classifying space of G. It is the quotient of some contractible space EG by a free action of G, so we have something that looks like $G \to EG \to BG$ and $BG = EG/G$.

For a discrete group G, we have $BG = K(G,1)$, so that $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for $k \neq 1$.

Question: what is a principal bundle? According to wikipedia, any G-principal bundle is a pullback of $EG \to BG$.

Note that contractibility of EG shows that BG is $K(G,1)$.

Examples

Note that EG is always a contractible space upon which G acts freely.

We also have $BX \simeq \Omega X$

- $G \to EG \to BG = EG/G$
- $\mathbb{Z} \to \mathbb{R} \to S^1$
- $\mathbb{Z}^n \to \mathbb{R}^n \to T^n$
- $\mathbb{Z}^{*n} \to ??? \to \bigvee_n S^1$
- $\mathbb{Z}_2 \to S^\infty \to \mathbb{RP}^\infty$
- $\mathbb{Z}_n \to S^\infty \to L_n^\infty$
- $S^0 \to S^\infty \to \mathbb{RP}^\infty$
- $S^1 \to S^\infty \to \mathbb{CP}^\infty$
- $S^3 \to S^\infty \to \mathbb{HP}^\infty$
- NOT TRUE: $S^7 \to S^\infty \to \mathbb{OP}^\infty$
- $T^n \to ? \to (\mathbb{CP}^\infty)^n$
- $O_n \to V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$
- $GL_n(\mathbb{R}) \to V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$
- $SO_n \to ? \to ?$
- $Gr_n(\mathbb{R}^\infty) \to ? \to Gr_n(\mathbb{R}^\infty)$
- $\pi_1(\Sigma_g) \to ? \to \Sigma_g$
- $S_n \to ??? \to \{U \subset \mathbb{R}^\infty,\ |U| = n\}$

Note that $V_n(X)$ is the Stiefel manifold of dimension n orthonormal frames in X.

Also, $\pi_1(\Sigma_g) =< \{a_i, b_i\}_i^n >| \prod_i^g [a_i, b_i] >$

A principal G bundle is a locally trivial free G-space with orbit space B. If G is discrete, then a principal G-bundle over X with total space $\tilde{X}$ is equivalent to a regular covering map with $Aut(\tilde{X}) = G$. Under some hypothesis, there exists a classifying space BG such that $\{\text{isomorphism classes of G-bundles over X}\} \cong [X, BG]$, i.e. bundles of G's over X are equivalent to maps from X into the classifying space, i.e.

$$\mathrm{Hom}(X, BG) \cong \{G\text{-bundles over } X\}$$

It is useful to think of BG as a space whose points are copies of G, so the classifying map $X \xrightarrow{f} BG$ assigns each $x \in X$ to the fiber above x, which is a G.

There is a standard procedure in homotopy theory for constructing a classifying space for every group. One starts by constructing a 2-complex with the given fundamental group, and then one inductively attaches higher dimensional cells to kill all higher homotopy groups. Each element $c \in \pi_n(X_{n-1})$ is represented by some continuous map $\gamma_c : S^n \to X_{n-1}$ with image in the n--skeleton. Let X_n be obtained from X_{n-1} by attaching an $(n+1)$-cell along γ_c, for each $c \in \pi_n(X_{n-1})$.

Conjecture: $B(G \times H) = BG \times BH$
Proof outline: $EG \times EH$ is contractible, and $G \times H$ acts freely on it with quotient equal to the RHS.

Conjecture: $B(G * H) = BG \vee BH$

Unknown: $B(G \otimes H) = BG \otimes BH$

Unknown: $B(G \rtimes_\phi H) =?$

Paper on Chow Rings

Recent result: Chow Rings computed in 2005 for $BGL_n, BSL_n, BSp_n, BO_n, BSO_n$
Cohomology for classifying spaces of linear algebraic groups (equivalently compact Lie Groups) have an algebraic analog: Chow rings of the classifying spaces. For a finite abelian group, the chow ring is the symmetric algebra on the group of characters.

There is a map from the Chow ring back into cohomology, which in general fails surjectivity and injectivity. Tensoring this map with $\mathbb{Q}$ creates an isomorphism, though. In this case, both hve the ring structure of invariants under the Weyl group in the symmetric algebra of the ring of characters of a maximal torus. (Classical result, Leray and Borel.)

Chow rings have not been computed for PGL_n. Need to know about Chern classes, Euler classes,

A_* known for all O_n and SO_n for n odd in 80s, general result for SO_n 2004. PGL_n case is much harder. Understood for $n = 2$, since $PGL_2 \cong SO_3$. Other bits that have been computed: $H^*(BPGL_3, \mathbb{Z}_3), H^*(BPGL_n, \mathbb{Z}_2)$ for $n = 2 \mod 4$ in 70s/80s, incomplete results for $H^*(BPGL_p, \mathbb{Z}_p)$ in 2003.

Term "equivariant" pops up a lot, symplectic forms, schemes, stacks

Further Reading

Characteristic classes are elements of $H^*(BG)$, can be used to define char. classes for bundles.

Connected covers can kill higher homotopy?

You can realize any Eilenberg-MacLane space as a classifying space.

Claim: $\pi_{i+k} B^k G = \pi_i G$.

Proof: If G is a topological group, there is a universal principal G-bundle $EG \to BG$ which induces a LES in homotopy. Since EG is contractible, $\pi_i EG = \pi_{i+1} EG = 0$, so $\pi_{i+1} BG \cong \pi_i G$. When G is an E_2 space, BG is a topological group, and so $\pi_{i+2}(B^2 G) = \pi_{i+2}(B(BG)) = \pi_{i+1}(BG) = \pi_i(G)$ and we conclude the result.

Corollary: If G is a discrete group, $B^k G = K(G, n)$.
Proof: Then $\pi_0 G = G$ and $\pi_i G = 0$ for $i > 0$, so $\pi_k B^k G = G$.

It's possible to take classifying spaces of stacks. E.g. there is a stack that classifies principal bundles *with connections*, but it has issues: it is not a presentable stack, i.e. not covered by a manifold, so an associated sheaf is not representable.

Stable homotopy of BG: same sort of techniques as in S^n, break into components.

EG can be constructed as $\bigcup_n G * G * \cdots * G$, where $*$ is join of two spaces: the suspension of the smash product. For example, $G = \mathbb{Z}_2$ implies $EG = \bigcup_n \mathbb{Z}_2 * \cdots = \bigcup_n S^{n-1} = S^\infty$.