Title

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1.1 Kempf's Theorem

Next topic: Kempf's Vanishing Theorem. Proof in Jantzen's book involving ampleness for sheaves.

Setup:

We have

$$\begin{array}{cc} G & \text{a reductive algebraic group over } k = \bar{k} \\ \uparrow{\scriptstyle\subseteq} & \\ B & \text{the Borel subgroup} \\ \uparrow{\scriptstyle\subseteq} & \\ T & \text{its maximal torus} \end{array}$$

along with the weights $X(T)$.

We can consider derived functors of induction, yielding $R^n \operatorname{Ind}_B^G \lambda = \mathcal{H}^n(G/B, \mathcal{L}(\lambda)) \coloneqq H^n(\lambda)$ where $\mathcal{L}(\lambda)$ is a line bundle and G/B is the flag variety.

Recall that

- $H^0(\lambda) = \operatorname{Ind}_B^G(\lambda)$,
- $\lambda \notin X(T)_+ \implies H^0(\lambda) = 0$
- $\lambda \in X(T)_+ \implies L(\lambda) = \operatorname{Soc}_G H^0(\lambda) \neq 0$.

Theorem 1.1 ***(Kempf).***
If $\lambda \in X(T)_+$ a dominant weight, then $H^n(\lambda) = 0$ for $n > 0$.

Remark 1.
In char $(k) = 0$, $H^n(\lambda)$ is known by the Bott-Borel-Weil theorem. In positive characteristic, this is not know: the characters char $H^n(\lambda)$ is known, and it's not even known if or when they vanish. Wide open problem!

Could be a nice answer when $p > h$ the Coxeter number.

1.2 Good Filtrations and Weyl Filtrations

We define two classes of distinguished modules for $\lambda \in X(T)_+$:

- $\nabla(\lambda) \coloneqq H^0(\lambda) = \operatorname{Ind}_B^G \lambda$ the costandard/induced modules.
- $\Delta(\lambda) = V(\lambda) \coloneqq H^0(-w_0\lambda) = \operatorname{Ind}_B^G \lambda$ the standard/Weyl modules
  - Here w_0 is the longest element in the Weyl group

We have

$$L(\lambda) \hookrightarrow \nabla(\lambda)\Delta(\lambda) \qquad\qquad \twoheadrightarrow L(\lambda).$$

We define the category Rat-G of rational G-modules. This is a *highest weight category* (as is e.g. Category $\mathcal{O}$).

Definition 1.1.1 (Good Filtrations)**.**
An (possibly infinite) ascending chain of G-modules

$$0 \leq V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V$$

is a **good filtration** of V iff

1. $V = \cup_{i \geq 0} V_i$
2. $V_i/V_{i-1} \cong H^0(\lambda_i)$ for some $\lambda_i \in X(T)_+$.

In characteristic zero, the H^0 are irreducible and this recovers a composition series. Since we don't have semisimplicity in this category, this is the next best thing.

Definition 1.1.2 (Weyl Filtration)**.**
With the same conditions of a good filtration, a chain is a **Weyl filtration** on V iff

1. $V = \cup_{i \geq 0} V_i$
2. $V_i/V_{i-1} \cong V(\lambda_i)$ for some $\lambda_i \in X(T)_+$.

I.e. the different is now that the quotients are standard modules.

Definition 1.1.3 (Tilting Modules)**.**
V is a **tilting module** iff V has both a good filtration and a Weyl filtration.

Theorem 1.2 ***(Ringel, 1990s).***
Let $\lambda \in X(T)_+$ be a dominant weight. Then there is a unique indecomposable highest weight tilting module $T(\lambda)$ with highest weight λ.

Example 1.1.
We have the following situation for type A_2:

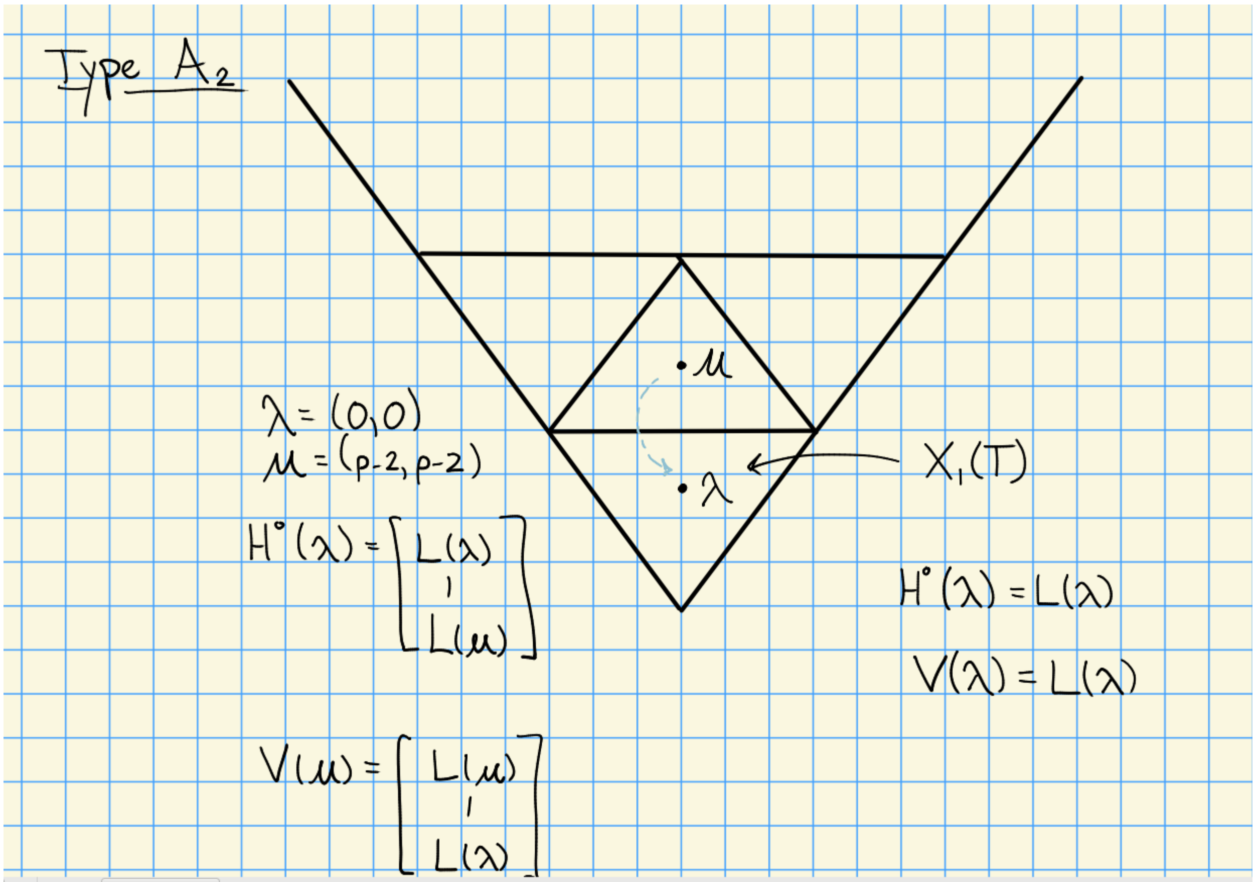

Figure 1: Image

And thus a decomposition:

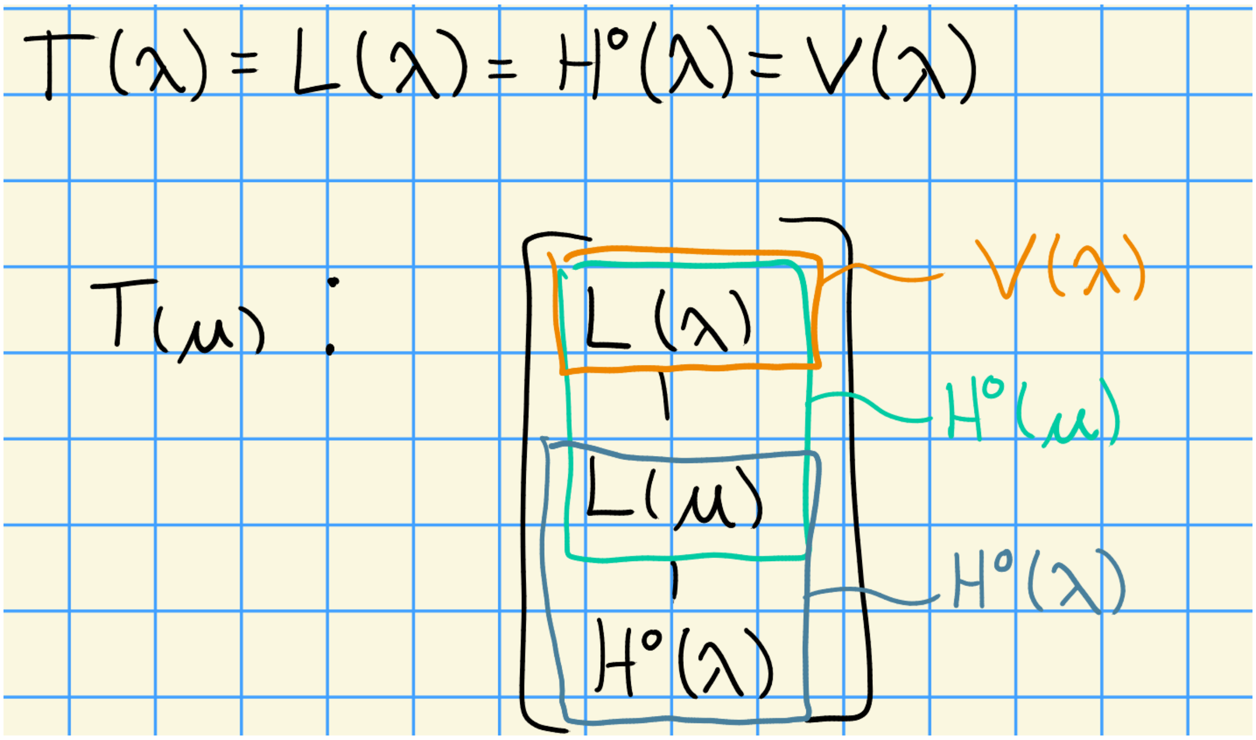

Figure 2: Image

The picture to keep in mind is the following: 4 types of modules, all indexed by dominant weights:

$$
\begin{tikzcd}
 & H^0(\lambda) & \\
L(\lambda) \arrow[ur, hook] & & T(\lambda) \arrow[ul] \\
 & V(\lambda) \arrow[ul] \arrow[ur, hook] &
\end{tikzcd}
$$

1.3 Cohomological Criteria for Good Filtrations

We'll take cohomology in the following way: let G be an algebraic group scheme, and define

.