Title

D. Zack Garza

Contents

1 | Monday, November 09

1.1 Strong Linkage

We have two categories:

- G_rT, with a notion of *strong linkage*, and
- G_r, which instead only has *linkage*.

We'll restate a few theorems.

Theorem 1.1.1 *(?)*.
Let $\lambda, \mu \in X(T)$.

1. If $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_rT} \neq 0$, then $\mu \uparrow \lambda$ are strongly linked.
2. If $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$, then $\mu \in W_p \cdot \lambda + p^r X(T)$.

Example 1.1.1 (?): In the case of $\Phi = A_2$, we'll consider the two different categories.

We have the following picture for $\widehat{Z}$:

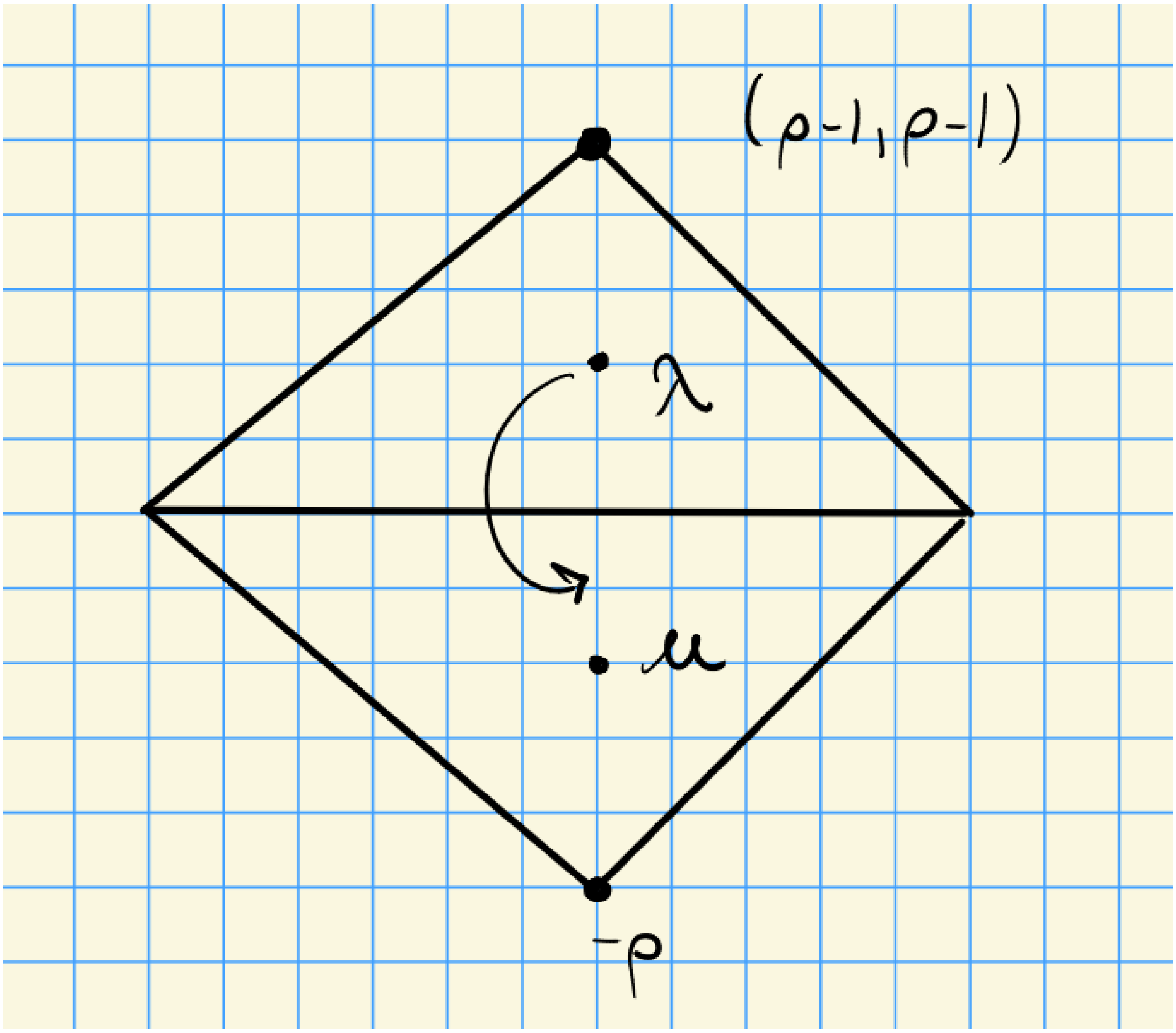

Figure 1: Image

Considering $X_1(T)$ and $[\widehat{Z}_1(\lambda) : \widehat{L}_1(\mu)] \neq 0$, and $\widehat{Z}_1(\lambda)$ has 6 composition factors as G_1T-modules.

On the other hand, for Z, we have the following:

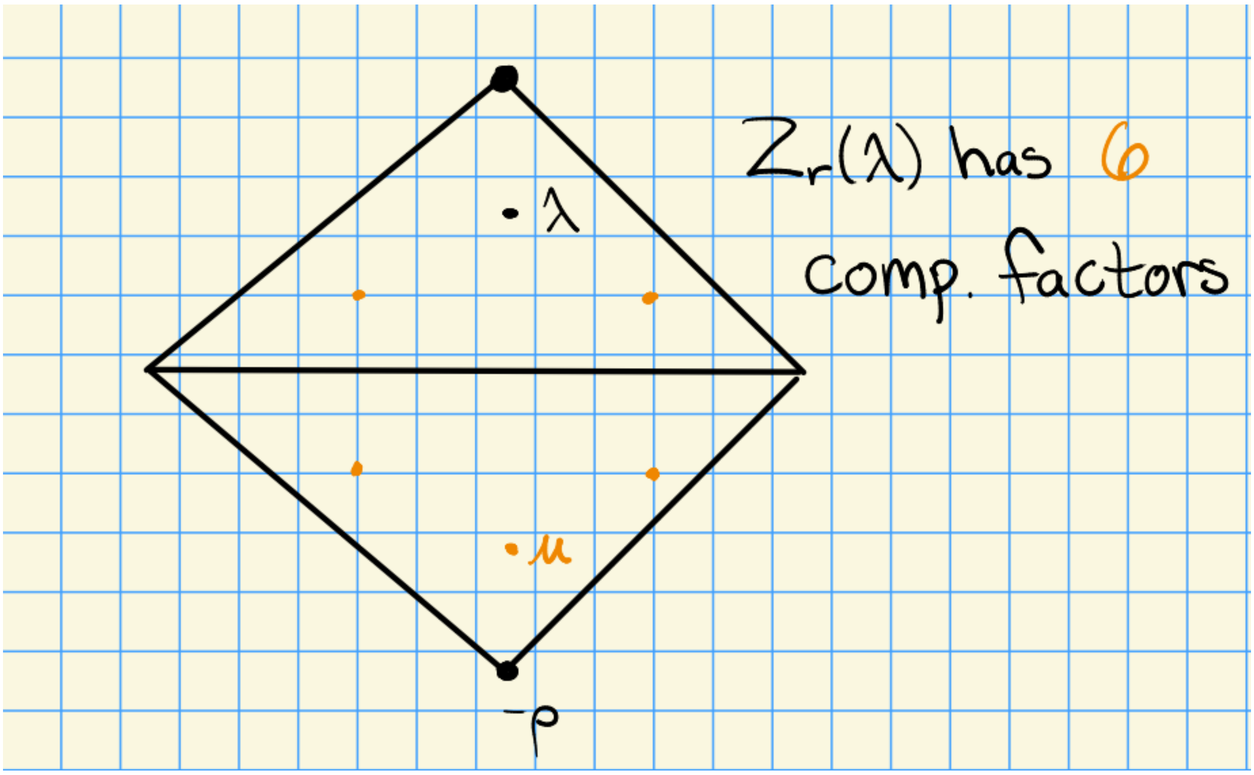

Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

1.2 Extensions

Let $\lambda, \mu \in X(T)$. We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\operatorname{Ext}^j_{G_rT}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) = \operatorname{Ext}^j_{G_r}\left(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)\right) \qquad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\operatorname{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right) = \bigoplus_{\gamma \in X(T)} \operatorname{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right)_\gamma$$

where we are taking the fixed points under the torus T action on the first factor (for which T_r acts trivially). We can write this as

$$\begin{aligned}
\cdots &= \bigoplus_{\gamma \in X(T)} \mathrm{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu) \otimes \gamma\right) \\
&= \bigoplus_{\gamma \in X(T)} \mathrm{Ext}^j_{G_rT}\left(L_r(\lambda), L_r(\mu) \otimes p^r v\right) \\
&= \bigoplus_{v \in X(T)} \mathrm{Ext}^j_{G_rT}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)\right).
\end{aligned}$$

So if we know extensions in the G_r category, we know them in the G_rT category.

There is an isomorphism

$$\mathrm{Ext}^1_{G_rT}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) \cong \mathrm{Hom}_{G_RT}\left(\mathrm{rad}_{G_rT}\widehat{Z}_r(\lambda), \widehat{L}_r(\mu)\right).$$

Finally, for $\lambda, \mu \in X(T)$, if the above Ext^1 vanishes, then $\lambda \in W_p \cdot \mu$ (i.e. λ and μ are linked).

1.3 The Steinberg Modules

Example 1.3.1 (Steinberg): Consider A_2:

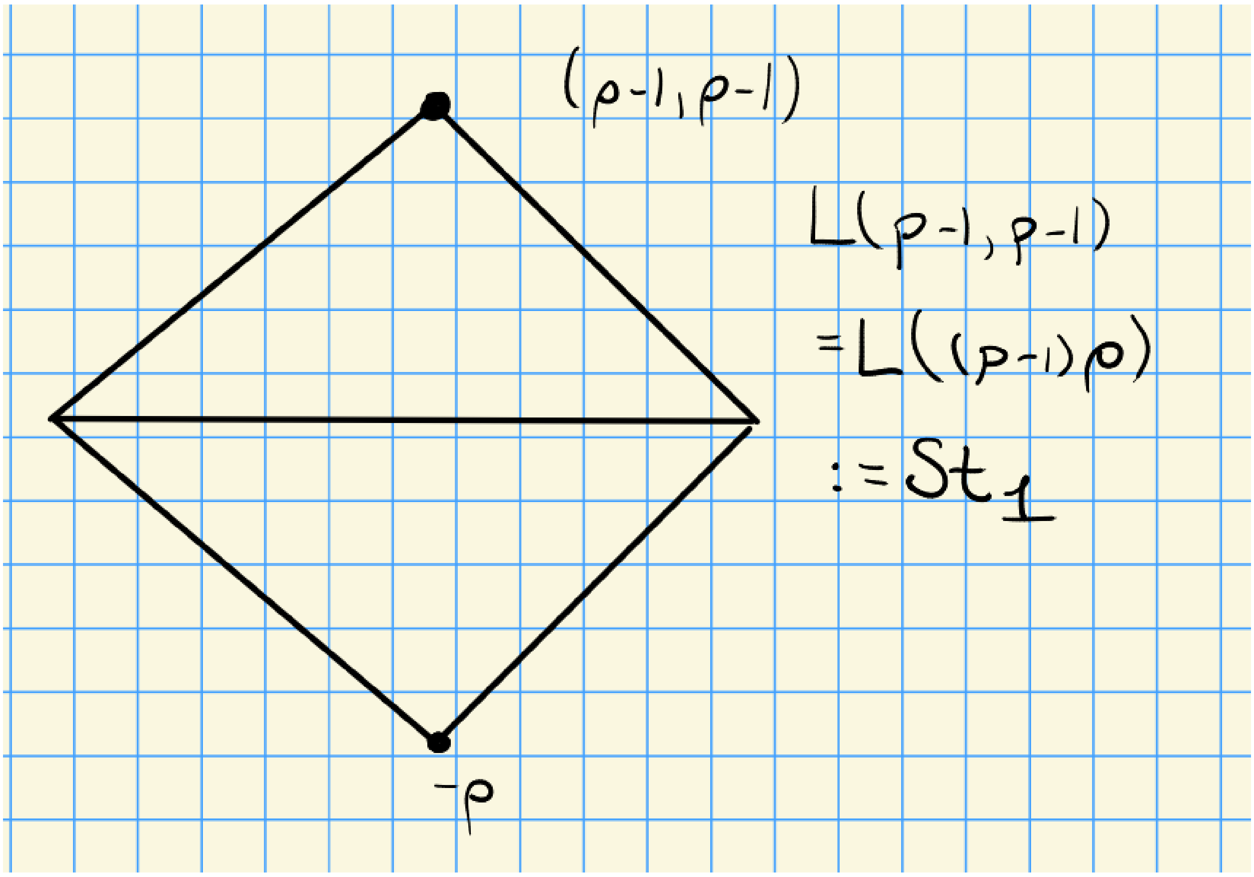

Figure 3: Image

Taking the representation corresponding to $(p-1, p-1)$ yields the "first Steinberg module"

$$L(p-1, p-1) = L((p-1)\rho) := \mathrm{St}_1 .$$

In this case, we have an equality of many modules:

$$H^0((p-1)\rho) = L((p-1)\rho) = V((p-1)\rho) = T((p-1)\rho).$$

Definition 1.3.1 (Steinberg Modules)**.**
The rth **Steinberg module** is defined to be $L((p^r-1)\rho)$.

Remark 1.3.1 : In general, we have

$$L((p^r-1)\rho) = H^0((p^r-1)\rho) = V((p^r-1)\rho).$$

We also have

$$\widehat{Z}_r((p^r-1)\rho) \cong L((p^r-1)\rho) \downarrow_{G_rT} .$$

Theorem 1.3.1 ***(?).***
The Steinberg module is both injective and projective as both a G_r-module and a G_rT-module.

Proof (?).
It suffices to prove that St_r is projective over G_rT, then by a previous theorem, it will also be projective over G_r. Let $\widehat{L}_r(\mu)$ be a simple G_rT-module, and consider

$$\mathrm{Ext}^1_{G_rT}(\mathrm{St}_r, \widehat{L}_r(\mu)) = \mathrm{Ext}^1_{G_rT}(\widehat{L}_r((p^r-1)\rho), \widehat{L}_r(\mu)).$$

If we show this is zero for every simple module, the result will follow.
Suppose $(p^r-1)\rho \not< \mu$. In this case, the RHS above is zero.

Missed why: something to do with radical of the first term?

Otherwise, we have

$$\mathrm{Ext}^1_{G_rT}(\widehat{L}_r(\mu), \mathrm{St}_r) = \mathrm{Hom}_{G_rT}(\mathrm{rad}(\widehat{Z}_r(\mu)), \mathrm{St}_r).$$

Suppose that the RHS is nonzero. Then $\mathrm{rad}\left(\widehat{Z}_r(\mu)\right) \twoheadrightarrow \mathrm{St}_r$, and thus

$$\dim \mathrm{rad}\left(\widehat{Z}_r(\mu)\right) \geq \dim \mathrm{St}_r = p^{r|\Phi^+|}$$

But we know that

$$\dim \mathrm{rad}\left(\widehat{Z}_r(\mu)\right) < \dim \widehat{Z}_r(\mu) = p^{r|\Phi^+|},$$

so we've reached a contradiction and the hom must be zero. ■

Proposition 1.3.1 ***(Open Conjecture, Donkin, MSRI 1990: 'DFilt Conjecture').***
Let G be a reductive group and M a finite-dimensional G-module. Then M has a good (p,r)-filtration iff $\mathrm{St}_r \otimes M$ has a good filtration.

Remark 1.3.2 : See NK (Nakano-Kildetoft, 2015) and BNPS (Bendel-Nakano-Pillen-Subaje, 2018-).

Remark 1.3.3 (Important! What we've been working toward stating): The forward direction is equivalent to the statement that $\mathrm{St}_r \otimes L(\lambda)$ has a good filtration for $\lambda \in X_r(T)$.

Proposition 1.3.2 ***(Conjecture).***
The Dfilt conjecture in the forward direction holds for all p.

Remark 1.3.4 : BNPS has shown that this holds for all rank 2 groups, which is strong evidence.

The reverse implication is **not** true: BNPS-Crelle 2020 shows that for $\Phi = G_2, p = 2$, there exists an $H^0(\lambda)$ that does not have a good (p, r)-filtration.