Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

where

$$\begin{gathered} S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\ S(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} S^\pm(t). \end{gathered}$$

Outline

- Took $R^\pm : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$: symplectic paths associated to $S^\pm$
- These paths defined $\mu(x), \mu(y)$
- Section 8.7:

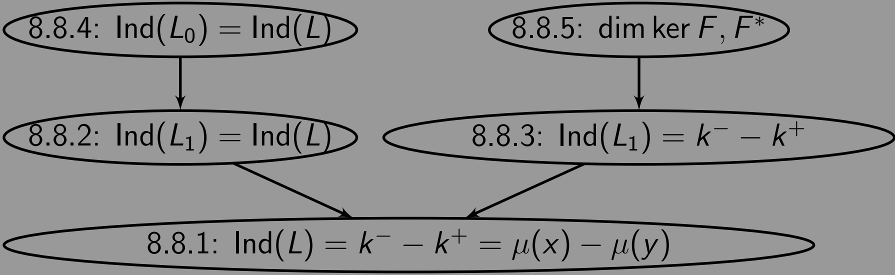
$$R^\pm \in \mathcal{S} := \left\{ R(t) \;\middle|\; R(0) = \mathrm{id},\ \det(R(1) - \mathrm{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\mathrm{Ind}(L) \overset{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know $\mathrm{Ind}(L) = \mathrm{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing $\mathrm{Ind}(L_1)$ by computing kernels.



8.8.5: $\dim\ker F$, F^*

Recall

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since coker $L_1 \cong \ker L_1^*$, it suffices to compute $\ker L_1^*$
- We have

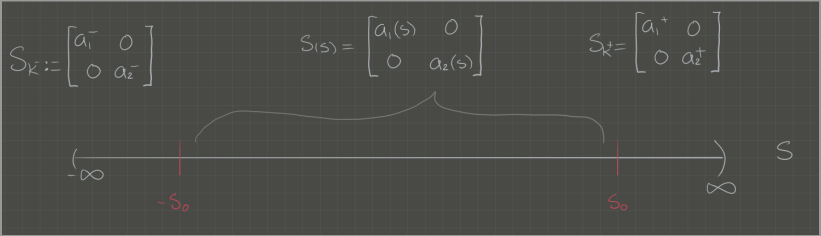
$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^{n} \mathrm{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the $n = 1$ case.

Setup

L_1 used a path of diagonal matrices constant near ∞:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



Statement of Later Lemma (8.8.5)

Let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

Note: F is L_1 for $n = 1$:

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

Statement of Lemma

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose $a_i^\pm \notin 2\pi\mathbb{Z}$.

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \operatorname{Ker} F = 2 \cdot \#\left\{\ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R}\right\}$$
$$\dim \operatorname{Ker} F^* = 2 \cdot \#\left\{\ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R}\right\}.$$

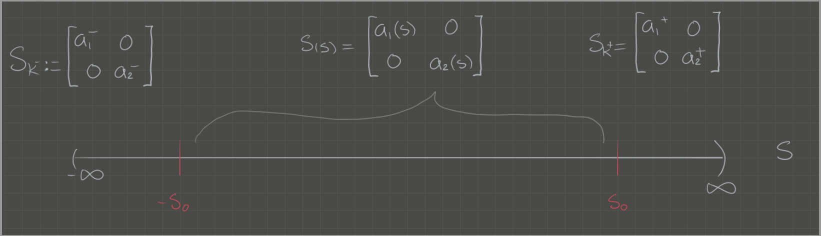
2. Suppose $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \#\left\{i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0\right\}$$
$$\dim \operatorname{Ker} F^* = \#\left\{i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0\right\}.$$

Statement of Lemma

In words:

1. If $S(s)$ is a scalar matrix, set $a^\pm = a_1^\pm = a_2^\pm$ to the limiting scalars and count the integer multiples of 2π between a^- and a^+.
2. Otherwise, if S is uniformly bounded by 1, count the number of entries the flip from positive to negative as s goes from $-\infty \longrightarrow \infty$.



Proof of Assertion 1

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \mathrm{Ker}\, F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \mathrm{Ker}\, F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) := a_1(s) = a_2(s)$.
- Start with equation on $\mathbb{R}^2$,

$$\mathbf{Y}(s,t) = [Y_1(s,t), Y_2(s,t)].$$

- Replace with equation on $\mathbb{C}$:

$$\mathbf{Y}(s,t) = Y_1(s,t) + iY_2(s,t).$$

Assertion 1, Step 1: Reduce to CR

- Expand definition of the PDE

$$F(\mathbf{Y}) = 0 \rightsquigarrow \bar{\partial}\mathbf{Y} + S\mathbf{Y} = 0$$

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\mathbf{Y} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \mathbf{Y} = 0.$$

- Change of variables: want to reduce to $\bar{\partial}\tilde{Y} = 0$
- Choose $B \in \mathrm{GL}(1, \mathbb{C})$ such that $\bar{\partial}B + SB = 0$
- Set $Y = B\tilde{Y}$, which (?) reduces the previous equation to

$$\bar{\partial}\tilde{Y} = 0.$$

Assertion 1, Step 1: Reduce to CR

Can choose (and then solve)

$$B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix} \quad \text{where} \quad \frac{\partial b}{\partial s} = -a(s)b(s)$$

$$\implies b(s) = \exp\left(\int_0^s -a(\sigma)\, d\sigma\right) := \exp\left(-A(s)\right).$$

Remarks:

- For some constants C_i, we have

$$A(s) = \begin{cases} C_1 + a^- s, & s \leq -\sigma_0 \\ C_2 + a^+ s, & s \geq \sigma_0 \end{cases}.$$

- The new $\tilde{Y}$ satisfies CR, is continuous and L^1_{loc}, so elliptic regularity $\implies C^\infty$.
- The real/imaginary parts of $\tilde{Y}$ are C^∞ and harmonic.

Assertion 1, Step 2: Solve CR

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$
- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of $\tilde{Y}$ in z.
- Deduce that the solutions of the system are given by

$$\tilde{Y}(u) = \sum_{\ell \in \mathbf{Z}} c_\ell u^\ell \implies \tilde{Y}(s+it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $\{c_\ell\}_{\ell \in \mathbb{Z}} \subset \mathbb{C}$ converges for all s, t.

Assertion 1, Step 2: Solve CR

Use $e^{s+it} = e^s(\cos(t) + i\sin(t))$ to write in real coordinates:

$$\tilde{Y}(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi s\ell} \begin{bmatrix} \cos(2\pi\ell t) & -\sin(2\pi\ell t) \\ \sin(2\pi\ell t) & \cos(2\pi\ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

Use

$$Y = B\tilde{Y} = \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \tilde{Y}$$

to write

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi s\ell} \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \begin{bmatrix} \cos(2\pi\ell t) & -\sin(2\pi\ell t) \\ \sin(2\pi\ell t) & \cos(2\pi\ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

For $s \leq s_0$ this yields for some constants K, K':

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{2\pi\ell - a^-} \begin{bmatrix} e^K(\alpha_\ell\cos(2\pi\ell t) - \beta_\ell\sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell\sin(2\pi\ell t) + \beta_\ell\cos(2\pi\ell t)) \end{bmatrix}.$$

Condition on L^p Solutions

For $s \leq s_0$ we had

$$Y(s,t) = \sum_{\ell \in \mathbb{Z}} e^{\left(2\pi\ell - a^-\right)s} \begin{bmatrix} e^K(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

and similarly for $s \geq s_0$, for some constants C, C' we have:

$$Y(s,t) = \sum_{\ell \in \mathbb{Z}} e^{\left(2\pi\ell - a^+\right)s} \begin{bmatrix} e^C(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

Then

$$Y \in L^p \iff \text{exponential terms} \overset{\ell \to \infty}{\longrightarrow} 0.$$

Condition on L^p Solutions: Small Tails

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^K(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

- $\ell \neq 0$: Need $\alpha_\ell = \beta_\ell = 0$ **or** $2\pi\ell > a^-$
- $\ell = 0$: Need both
  - $\alpha_0 = 0$ or $a^- < 0$ and
  - $\beta_0 = 0$ or $a^- < 0$.

$$Y(s,t) = \sum_{\ell\in\mathbb{Z}} e^{(2\pi\ell - a^+)s} \begin{bmatrix} e^C(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

- $\ell \neq 0$: Need $\alpha_\ell = \beta_\ell = 0$ **or** $2\pi\ell < a^+$
- $\ell = 0$: Need both
  - $\alpha_0 = 0$ or $a^+ > 0$ and
  - $\beta_0 = 0$ or $a^+ > 0$.

Counting Solutions

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } 2\pi\ell \in (a^-, a^+) & \ell \neq 0 \\ (\alpha_0 = 0 \text{ or } 0 \in (a^-, a^+)) \text{ and } (\beta_0 = 0 \text{ or } 0 \in (a^-, a^+)) & \ell = 0 \end{cases}.$$

- Finitely many such ℓ that satisfy these conditions
- Sufficient conditions for $Y(s,t) \in W^{1,p}$.

Compute dimension of space of solutions:

$$\begin{aligned} \dim \operatorname{Ker} F &= 2 \cdot \# \left\{ \ell \in \mathbb{Z}^* \;\middle|\; 2\pi\ell \in (a^-, a^+) \right\} + 2 \cdot \mathbb{1}\left[0 \in (a^-, a^+)\right] \\ &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a^+) \right\}. \end{aligned}$$

Note: not sure what $\mathbb{Z}^*$ is: most likely $\mathbb{Z} \setminus \{0\}$.

Counting Solutions

Use this to deduce $\dim\ker F^*$:

- $Y \in \ker F^* \iff Z(s,t) := Y(-s,t)$ is in the kernel of the operator

$$\tilde{F} : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Z \mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y.$$

- Obtain $\ker F^* \cong \ker \tilde{F}$.
- Formula for $\dim\ker \tilde{F}$ almost identical to previous formula, just swapping a^- and a^+.

$\blacksquare_1$

Assertion 2

Assertion 2: Suppose $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \#\left\{ i \in \{1,2\} \;\middle|\; a_i^- < 0 < a_i^+ \right\}$$
$$\dim \operatorname{Ker} F^* = \#\left\{ i \in \{1,2\} \;\middle|\; a_i^+ < 0 < a_i^- \right\}.$$

We use the following:

- Lemma 8.8.7:

$$\sup_{s\in\mathbb{R}} \|S(s)\| < 1 \implies \text{the elements in } \ker F, \ \ker F^* \text{ are independent of } t.$$

- Proof: in subsection 10.4.a.

Proof of Assertion 2

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^2\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^2\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

- Given as a fact:

$$\mathbf{Y}\in \ker F \implies \frac{\partial}{\partial s}\mathbf{Y} = \mathbf{a}(s)\mathbf{Y} \qquad := \begin{bmatrix} -a_1(s) & 0 \\ 0 & -a_2(s)\end{bmatrix}\mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c}_0 \exp\left(-\mathbf{A}(s)\right) \quad \text{where} \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma)\ d\sigma.$$

Proof of Assertion 2

– Explicitly in components:

$$\begin{cases} \dfrac{\partial Y_1}{\partial s} &= -a_1(s) Y_1 \\ \dfrac{\partial Y_s}{\partial s} &= -a_2(s) Y_2 \end{cases} \implies \quad Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma)\, d\sigma.$$

– As before, for some constants $C_{j,i}$,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

– Thus

$$Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+),$$

establishing

$$\dim \ker F = \#\left\{ i \in \{1,2\} \;\middle|\; 0 \in (a_i^-, a_i^+) \right\}.$$

$\blacksquare_2$

8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$

Statement and Outline

Statement: let $k^{\pm} := \mathrm{Ind}(R^{\pm})$; then $\mathrm{Ind}(L_1) = k^- - k^+$.

- Consider four cases, depending on parity of $k^{\pm} - n$
- Show all 4 lead to $\mathrm{Ind}(L_1) = k^- - k^+$

1. $k^- \equiv k^+ \equiv n \mod 2.$
2. $k^- \equiv n, k^+ \equiv n - 1 \mod 2$
3. $k^- \equiv n - 1, k^+ \equiv n \mod 2.$
4. $k^- \equiv k^+ \equiv n - 1 \mod 2$

k^-	k^+	n
✓	✓	✓
✓		✓
	✓	✓
✓	✓	

Case 1: $k^+ \equiv k^- \equiv n \mod 2$

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^+)\pi & \\ & & & & & & (n-1-k^+)\pi \end{bmatrix}.$$

Case 1: $k^- \equiv k^+ \equiv n \mod 2$

- Take $a_1(s) = a_2(s)$ so $a_1^\pm = a^\pm$
- Apply the proved lemma to obtain

$$\begin{aligned}\dim\ker L_1 &= 2 \cdot \#\left\{\ell \in \mathbb{Z} \;\middle|\; 2\ell \in (n-1-k^-, n-1-k^+)\right\} \\ &= \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & \text{else} \end{cases}.\end{aligned}$$

Case 2: $k^+ \not\equiv k^- \equiv n \mod 2$

$$S_{k^-} = \begin{bmatrix} -\pi &&&&&&& \\ & -\pi &&&&&& \\ && \ddots &&&&& \\ &&& -\varepsilon\pi &&&& \\ &&&& -\varepsilon\pi &&& \\ &&&&& (n-1-k^-)\pi && \\ &&&&&& (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi &&&&&&& \\ & -\pi &&&&&& \\ && \ddots &&&&& \\ &&& \varepsilon &&&& \\ &&&& -\varepsilon &&& \\ &&&&& (n-2-k^+)\pi && \\ &&&&&& (n-2-k^+)\pi \end{bmatrix}.$$