Category $\mathcal{O}$, Problem Set 4

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1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Show that the induced $U(\mathfrak{g})$-module structure $M \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

Hence $M \notin \mathcal{O}$.

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

2.1 Solution

By theorem 3.2d, we have

$$M, N \in \mathcal{O} \implies (M \oplus N)^\vee \cong M^\vee \oplus N^\vee$$

and by definition, $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$ is the direct sum of the duals of various weight spaces.

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ.

For a concrete example, take Φ of type B_2 with a short simple root α and a long simple root β. If $\lambda := \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in Φ.

3.1 Solution

We would like to show the following two propositions:

1. $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. The simple system $\Delta_{[\lambda]}$ corresponding to $\Phi_{[\lambda]}^+$ is *not* generally given by $\Delta_{[\lambda]} = \Phi_{[\lambda]} \bigcap \Delta$, where Δ is the simple system corresponding to Φ.

We proceed by first showing (2) using the hinted counterexample when Φ is of type B_2 with $\Delta = \{\alpha, \beta\}$ with α a short root and β a long root.

Concretely, we can realize Φ and Δ as subsets of $\mathbb{R}^2$ in the following way:

$$\begin{aligned}
\Phi &= P_1 \coprod P_2 := \{[1,0],[0,1],[-1,0],[0,-1]\} \coprod \{[1,1],[-1,1],[1,-1],[-1,-1]\} \\
\Delta &:= \{\alpha, \beta\} := \{[1,0],[-1,1]\},
\end{aligned}$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\begin{aligned}
\Phi_{[\lambda]} &:= \left\{ \gamma \in \Phi \;\middle|\; \langle \lambda, \; \gamma^\vee \rangle \in \mathbb{Z} \right\}, \qquad \gamma^\vee := \frac{2}{\|\gamma\|^2} \gamma, \\
\Delta_{[\lambda]} &:= \left\{ \gamma \in \Delta \;\middle|\; \langle \lambda, \; \gamma^\vee \rangle \in \mathbb{Z} \right\}.
\end{aligned}$$

Now choosing $\lambda := \frac{\alpha}{2} = \left[\frac{1}{2}, 0\right]$, we now consider the inner products $\langle \lambda, \; \gamma^\vee \rangle$ for $\gamma \in \Phi$:

Thus

$$\gamma_1 \in P_1 \implies \left\langle \left[\frac{1}{2}, 0\right],\ 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right)\langle [1,0],\ \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z}$$
$$\gamma_2 \in P_2 \implies \langle \lambda,\ \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right],\ \frac{2}{\left(\sqrt{2}\right)^2}[\pm 1, \pm 1] \right\rangle = \pm\frac{1}{2} \not\in \mathbb{Z}$$

where $(\gamma_1)_1$ denotes the first component of γ_1.

We thus find that

$$\Phi_{[\lambda]} = P_1 \qquad \text{the short roots}$$
$$\Delta_{[\lambda]} = \{\alpha\} \qquad \text{the single short simple root.}$$

Choosing the following green hyperplane not containing any root, we can choose a positive system

$$\Phi^+ = \{[1,0], [0,1], [1,1], [-1,-1]\} = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$$

where we can note that $\Phi^+ \bigcap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking the corresponding roots in the various other systems on the positive side of this plane, we have

$$\Phi_\lambda = P_1 \implies \Phi^+_{[\lambda]} = \{[0,1], [1,0]\}$$
$$\Phi^+_{[\lambda]} \bigcap \Delta = \{\alpha\}$$
$$\Delta_{[\lambda]} = \{[1,0]\} \qquad = \{\alpha\}$$
$$.$$

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that $\operatorname{coker} \phi$ has a standard filtration.