

# Title

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# 1 | Lecture 10

**Remark 1.0.1:** What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of ??; we'll now continue with the proof:

*Proof (of Hilbert 90).*

**Observation 1.0.2:** Let  $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$ , then the data of a  $\text{GL}_n$ -torsor split by a  $\tau$ -cover  $U \rightarrow X$  is the same as descent data for a vector bundle relative to  $U/X$ .

This descent data comes from the following:

$$\begin{array}{c} U \times_X U \\ \pi_1 \downarrow \quad \downarrow \pi_2 \\ U \\ \downarrow \\ X \end{array}$$

That  $U$  trivializes our torsor means that  $\pi^*T = \pi^*G$  as a  $G$ -torsor, where  $G$  acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with  $G$  in both, yielding

$$\begin{array}{ccc} \pi_1^* \pi^* T & \xrightarrow{\sim} & \pi_2^* \pi^* T \\ \downarrow & & \downarrow \\ \pi_1^* \pi^* G & \xrightarrow{\sim} & \pi_2^* \pi^* G \end{array}$$

Both of the bottom objects are isomorphic to  $G|_{U \times U}$ .


**Claim:** The top horizontal map is descent data for  $T$ , and the bottom horizontal map is an automorphism of a  $G$ -torsor and thus is a section to  $G$ . I.e. a section to  $\text{GL}_n$  is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.


Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial  $\text{GL}_n$ -torsor on the fppf site, it's also trivial on the other two sites, yielding the desired maps back and forth. Thus  $H^1(X_{\text{ét}}, \text{GL}_n)$  is in bijection with  $n$ -dimensional vector bundles on  $X$ . ■

**Exercise 1.0.3(?):** See if Hilbert 90 is true for groups other than  $\text{GL}_n$ .

## 1.1 Representability and Local Triviality

**Question 1.1.1:** Suppose  $G$  is an affine flat  $X$ -group scheme. Are all  $G$ -torsors representable by a  $X$ -scheme? 

**Answer 1.1.2:** Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a  $G$ -torsor flat locally and use fppf descent. 

**Question 1.1.3:** Given a  $G$ -torsor  $T$  that is fppf locally trivial, is it étale locally trivial? 

**Answer 1.1.4:** In general no, but yes if  $G$  is smooth. 

**Example 1.1.5 (*a non-smooth group scheme*):** The kernel of Frobenius on  $\mathbb{A}^1$  or  $\mathbb{G}_a$ , aka  $\alpha_p$ . 