Title

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1 Sunday, August 30

Last of preliminaries. Upcoming: one-variable function fields and their valuation rings.

1.1 Polynomials Defining Regular Function Fields

Where's the curve: $f(x, y) = 0$.

Exercise 1.1.
Let R_1, R_2 be k-algebras that are also domains with fraction fields K_i. Show $R_1 \otimes_k R_2$ is a domain $\iff K_1 \otimes_k K_2$ is a domain.

> Denominator-clearing argument.

> **Definition 1.0.1** (Geometrically Irreducible).
> A polynomial of positive degree $f \in k[t_1, \cdots, t_n]$ is *geometrically irreducible* if $f \in \bar{k}[t_1, \cdots, t_n]$ is irreducible as a polynomial.

If $n = 1$ then f is geometrically irreducible $\iff$ it's linear, i.e. of degree 1.

Let f be irreducible, then since polynomial rings are UFDs then $\langle f \rangle$ is a prime ideal (irreducibles generate principal ideals) and $k[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. Let K_f be the fraction field.

Exercise 1.2.
Easy:

     a. Above for $1 \le i \le n$ let x_i be the image of t_i in K_f. Show that $K_f = k(x_1, \cdots, x_n)$.

     b. Show that if K/k is generated by $x_1, \cdots, x_n$, then it is the fraction field of $k[t_1, \cdots, t_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ (equivalently, a height 1 ideal).

> **Proposition 1.1*(?)*.**
> Suppose that f is geometrically irreducible.
>   a. The function field K/k is regular.
>   b. For all ℓ/k, $f \in \ell[t_1, \cdots, t_n]$ is irreducible.
>   In this case we say f is *absolutely irreducible* as a synonym for geometrically irreducible.

> *Proof* .
> By definition of geometric irreducibility, $\bar{k}[t_1, \cdots, t_n]/\langle f \rangle = k[t_1, \cdots, t_n]/\langle f \rangle \otimes_k \bar{k}$ is a domain.
> The exercise shows that $K_f \otimes_k k$ is a domain, so K_f is regular.
> It follows that for all ℓ/k, $K_f \otimes_k \ell$ is a domain, so $\ell[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. ∎

Moral: geometrically irreducible polynomials are good sources of regular function fields.

Exercise 1.3.

Let k be a field, $d \in \mathbb{Z}^+$ such that $4 \nmid d$ and $p(x) \in k[x]$ be positive degree. Factor $p(x) = \displaystyle\prod_{i=1}^{r}(x - a_i)^{\ell_i}$

in $\bar{k}[x]$.

  a. Suppose that for some i, $d \nmid \ell_i$. Show that $f(x, y) := y^d - p(x) \in k[x, y]$ is geometrically irreducible. Conclude that $K_f := ff\left(k[x,y]/\left\langle y^d - p(x)\right\rangle\right)$ is a regular one-variable function field over k, and thus elliptic curves yield regular function fields.

    Referred to as *hyperelliptic* or *superelliptic* function fields. Hint: use FT 9.21 or Lang's Algebra.

  b. What happens when $4 \mid d$?

Exercise 1.4 (Nice, Recommended)**.**
Assume k is a field, if necessary assuming char $(k) \neq 2$.

  a. Let $f(x, y) = x^2 - y^2 - 1$ and show K_f is is rational: $K_f =