Title

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Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined $\dim X$ to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking $R = A(X)$, i.e. $\dim \mathbb{A}^n/k = n$.

Proposition 1.1 ***(Dimensions).***
Let $k = \bar{k}$.

a. The dimension of $k[x_1, \cdots, x_n]$ is n.
b. All maximal chains of prime ideals have length n.

1.1 Proof of Dimension Proposition

The case for $n = 0$ is trivial, just take $P_0 = \langle 0 \rangle$. For $n = 1$, easy to see since the only prime ideals in $k[x]$ are $\langle 0 \rangle$ and $\langle x - a \rangle$, since any polynomial factors into linear factors.

Let $P_0 \subsetneq \cdots \subsetneq P_m$ be a maximal chain of prime ideals in $k[x_1, \cdots, x_n]$; we then want to show that $m = n$. Assume $P_0 = \langle 0 \rangle$, since we can always extend our chain to make this true (using maximality). Then P_1 is a minimal prime and P_m is a maximal ideal (and maximals are prime).

Claim: P_1 is principle, i.e. $P_1 = \langle f \rangle$ for some irreducible f.

1.1.1 Proof That P_1 is Principle

Claim: $k[x_1, \cdots, x_n]$ is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD $\implies R[x]$ is a UFD for any R.

> See Gauss' lemma.

Claim: In a UFD, minimal primes are principal. Let $r \in P$, and write $r = u \prod p_i^{n_i}$ with p_i irreducible and u a unit. So some $p_i \in P$, and p_i irreducible implies $\langle p_i \rangle$ is prime. Since $0 \subsetneq \langle p_i \rangle \subset P$, but P was prime and assumed minimal, so $\langle p_i \rangle = P$.

The idea is to now transfer the chain $P_0 \subsetneq \cdots \subsetneq P_m$ to a maximal chain in $k[x_1, \cdots, x_{n-1}]$. The first step is to make a linear change of coordinates so that f is monic in the variable x_n.

Example 1.1.
Take $f = x_1 x_2 + x_3^2 x_4$ and map $x_3 \mapsto x_3 + x_4$.

So write

$$f(x_1, \cdots, x_n) = x_n^d + f_1(x_1, \cdots, x_{n-1}) x_n^{d-1} + \cdots + f_d(x_1, \cdots, x_{n-1}).$$

We can then descend to $k[x_1, \cdots, x_n]$ to $k[x_1, \cdots, x_n] / \langle f \rangle$:

$$\begin{tikzcd}
P_0 \arrow[r] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m \arrow[d] \\
& P_1/P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m/P_1 \arrow[d] \\
& P_1/P_1 \cap k[x_1, \cdots, x_{n-1}] \arrow[r] & \cdots \arrow[r] & (P_m/P_1) \cap k[x_1, \cdots, x_{n-1}]
\end{tikzcd}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

> **Definition 1.1.1** (Integral Extension)**.**
> An *integral* ring extension $R \hookrightarrow R'$ of R is one such that all $r' \in R'$ satisfying a monic polynomial with coefficients in R, where R' is finitely generated.
>
> > In this case, also implies that R' is a finitely-generated R module.

In this case, $k[x_1, \cdots, x_{n-1}] \hookrightarrow k[x_1, \cdots, x_n] / \langle f \rangle$ is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

> **Lemma 1.2.**
> Suppose $P', Q' \subset R'$ are distinct prime ideals with $R \hookrightarrow R'$ an integral extension. Then if $P' \cap R = Q' \cap R$, neither contains the other, i.e. $P' \not\subset Q'$ and $Q' \not\subset P'$.

> *Proof .*
> Toward a contradiction, suppose $P' \subset Q'$, we then want to show that $Q' \supset P'$. Let $a \in Q' \setminus P'$

(again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.
Then $\bar{a} \neq 0$ in R'/P', and there exists a monic polynomial of minimal degree that $\bar{a}$ satisfies, $p(x) = x^n + \sum_{i=2}^{n} \bar{c}_i x^{n-i}$. This implies $\bar{c}_n \in Q'/P'$ (which will contradict $c_n \in P'$), since if $\bar{c}_n = 0$ then factoring out x yields a lower degree polynomial that $\bar{a}$ satisfies.
But then $\bar{a}_n \in Q' \cap R$, so ???

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Question: Given $R \hookrightarrow R'$ is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the "Going Up" Theorem: given $P \subset R$ prime, there exists $P' \subset R'$ prime such that $P' \cap R = P$. Furthermore, we can lift $P_1 \subset P_2$ to $P_1' \subset P_2'$, as well as "lifting sandwiches":

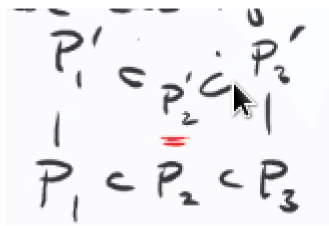

Figure 1: Image

In this process, the length of the chain decreased since $\langle 0 \rangle$ was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies. ■

1.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if $R \hookrightarrow R'$ is integral then $\dim R = \dim R'$.

Claim: Any affine variety has finite dimension.

Proof .
We have $\dim X = \dim A(X)$, where $A(X) := k[x_1, \cdots, x_n]I$ for some $I(X) = \sqrt{I(X)}$.
The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated k-algebra is an integral extension of some polynomial ring $k[y_1, \cdots, y_d]$. I.e., the following extension is integral:

$$k[y_1, \cdots, y_d] \hookrightarrow k[x_1, \cdots, x_n]/I.$$

We can conclude that $\dim A(X) = d < \infty$.

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Proposition 1.3 *(?)*.
Let X, Y be irreducible affine varieties. Then

a. $\dim X \times Y = \dim X + \dim Y$.
b. $Y \subset X \implies \dim X = \dim Y + \operatorname{codim}_X Y$.
c. If $f \in A(X)$ is nonzero, then any component of $V(f)$ has codimension 1.

Remark 1.
Why is $X \times Y$ again an affine variety? If $X \subset \mathbb{A}^n/k$, $Y \subset \mathbb{A}^m/k$ with $X = V(I), Y = V(J)$, then $X \times Y \subset \mathbb{A}^n/k \times \mathbb{A}^m/k = \mathbb{A}^{n+m}/k$ can be given by taking $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$ using the natural inclusions of $k[x_1, \cdots, x_\ell]$.

Note that we can write

$$k[x_1, \cdots, x_n, y_1, \cdots, y_m] = k[x_1, \cdots, x_n] \otimes_k k[y_1, \cdots, y_n]$$

where we think of $x_i = x_i \otimes 1, y_j = 1 \otimes y_j$. We thus map I, J to $I \otimes 1 + 1 \otimes J$ and obtain $V(I \otimes 1 + 1 \otimes J) = X \times Y$ and $A(X \times Y) = A(X) \otimes_k A(Y)$.

In general, for k-algebras R, S,

$$R/I \otimes_k S/J \cong R \otimes_k S / \langle I \otimes 1 + 1 \otimes J \rangle .$$

Remark 2.

Proof . ■