Linearization and Transversality

Sections 8.3 and 8.4

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Review 8.2

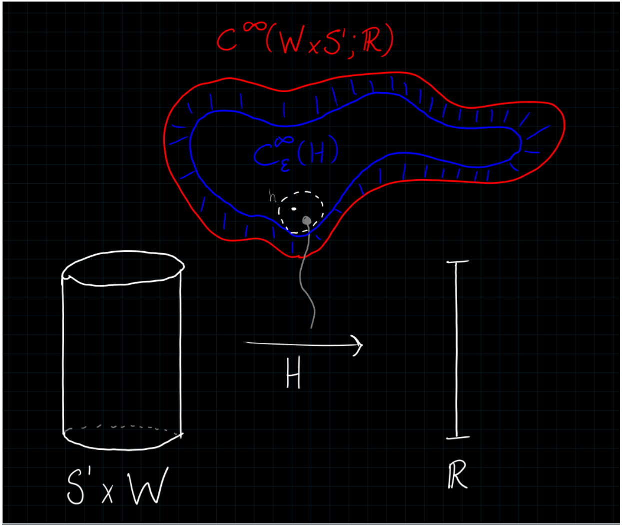
Section 8.3: The Space of Perturbations of H

Goal

Goal: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Goal

Start by trying to construct a subspace $\mathcal{C}^\infty_\varepsilon(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Define an Absolute Value

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C^\infty_\varepsilon(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$ denote a perturbation of H.
- Fix $\varepsilon = \left\{ \varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0} \right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k \right\},$$

the maximum over all sets of multi-indices α of length k.
Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

Define a Norm

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{\text{"}} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left| d^k h(x,t) \right|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$