Morse Theory

D. Zack Garza

March 18, 2020

Contents

List of Definitions

List of Theorems

1 Thursday January

Recall: For M^n a closed smooth manifold, consider a smooth map $f : M^n \longrightarrow \mathbb{R}$.

Definition 1.0.1 (Non-degenerate Critical Points)**.**
A critical point p of f is *non-degenerate* iff $\det(H := \dfrac{\partial^i f}{\partial x_i \partial x_j}(p)) \neq 0$ in some coordinate system U.

Proposition 1.1 ***(The Morse Lemma).***
For any non-degenerate critical point p there exists a coordinate system around p such that

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

λ is called the *index of f at p*.

Lemma 1.2 ***(Relating Index to Eigenvalues).***
λ is equal to the number of *negative* eigenvalues of $H(p)$.

Proof .
A change of coordinates sends $H(p) \longrightarrow A^t H(p) A$, which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that A is invertible and not necessarily orthogonal. Use the fact that $A^t H A$ is diagonalizable.

This means that f can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

■

1.1 Proof of Morse Lemma

Suppose that we have a coordinate chart U around p such that $p \mapsto 0 \in U$ and $f(p) = 0$.

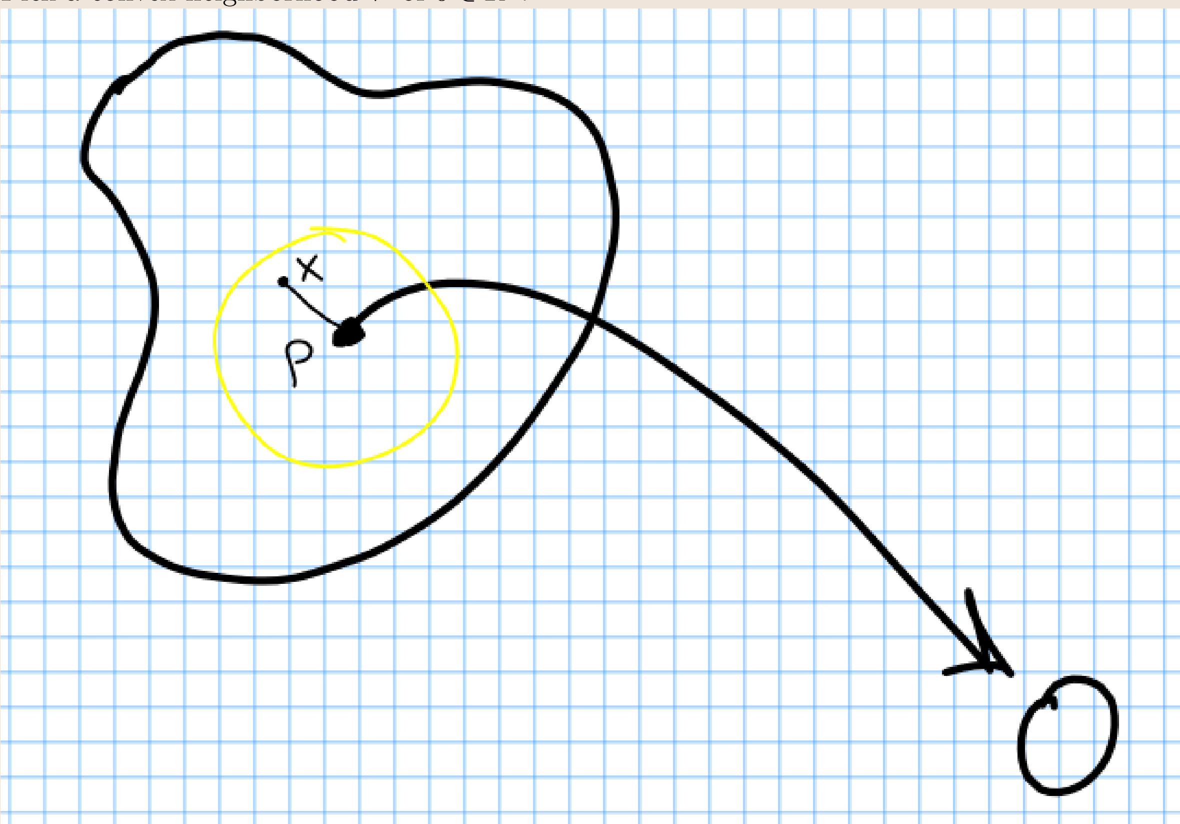
1.1.1 Step 1

Claim 1.
There exists a coordinate system around p such that

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

where $h_{ij}(x) = h_{ji}(x)$.

Proof .
Pick a convex neighborhood V of $0 \in \mathbb{R}^n$.



Restrict f to a path between x and 0, and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \cdots, tx_n)}{dt}\, dt = f(x_1, \cdots, x_n) - f(0) = f(x_1, \cdots, x_n).$$

since $f(0) = 0$.

We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots \frac{\partial f}{\partial x_n} x_n \; dt \implies \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i} \; dt = f(x).$$

We thus have $f(x) = \sum_{i=1}^{n} x_i g_i(x)$ where $\frac{\partial f}{\partial x_i}(0) = 0$, and $\frac{\partial f}{\partial x_i} = x_1 \frac{\partial g_1}{\partial x_i} + \cdots + g_i + x_i \frac{\partial g_i}{\partial x_i} + \cdots + x_n \frac{\partial g_n}{\partial x_i}$.

When we plug $x = 0$ into this expression, the only term that doesn't vanish is g_i, and thus $\frac{\partial f}{\partial x_i}(0) = g_i(0)$ and $g_i(0) = 0$.

Applying the same result to g_i, we obtain $g_i(x) = \sum_{j=1}^{n} x_j h_{ij}(x)$, and thus $f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x)$.

We still need to show h is symmetric. For every pair i, j, there is a term of the form $x_i x_j h_{ij} + x_j x_i h_{ji}$. So let $H_{ij}(x) = \frac{h_{ij}(x) + h_{ji}(x)}{2}$ (i.e. symmetrize/average h), then $f(x) = \sum_{i,j=1}^{n} x_i x_j H_{ij}(x)$ and this shows claim 1.

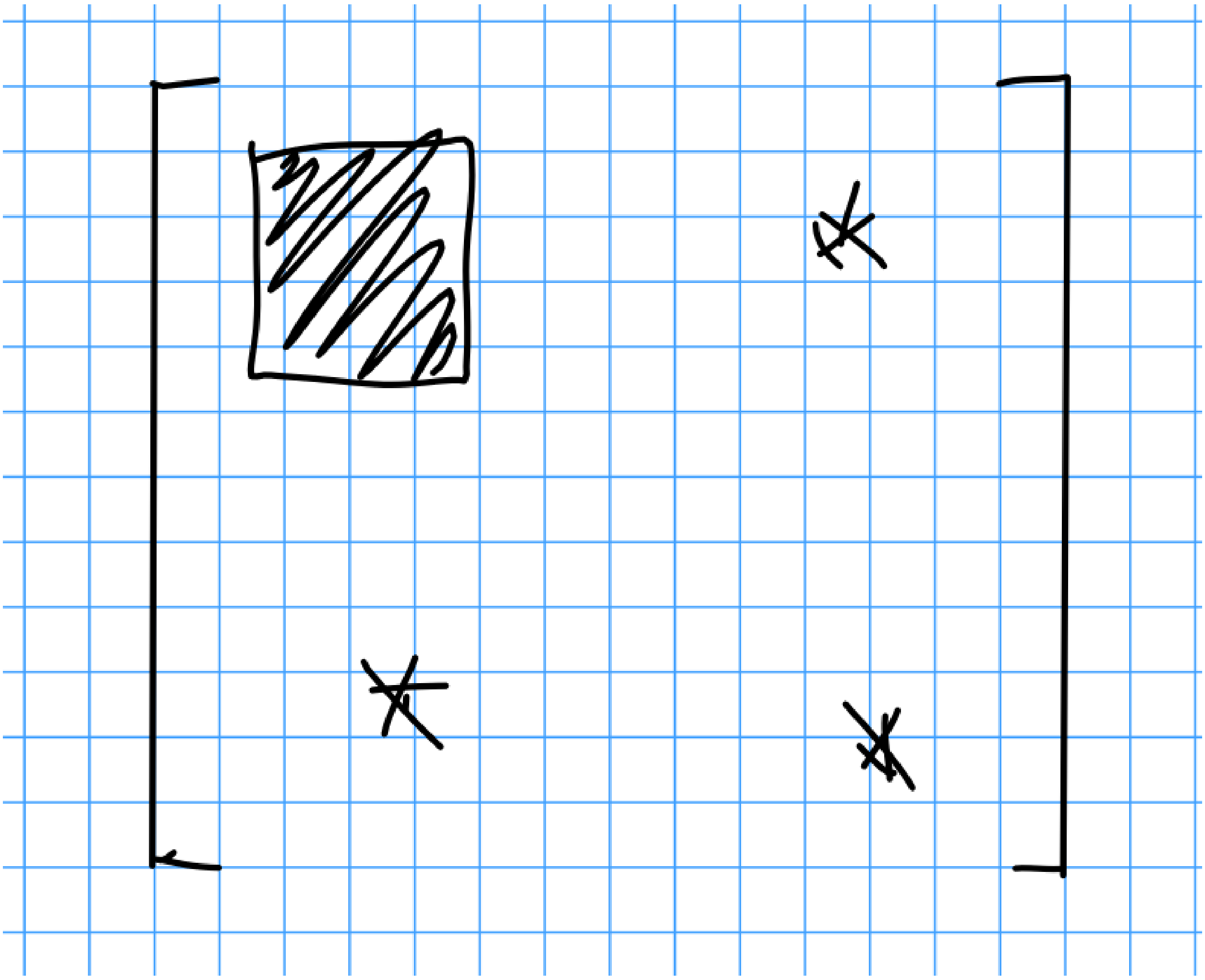
■

1.1.2 Step 2: Induction

Assume that in some coordinate system U_0,

$$f(y_1, \cdots, y_n) = \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \cdots, y_n).$$

Note that $H_{rr}(0)$ is given by the top-left block of $H_{ij}(0)$, which is thus looks like



Note that this block is symmetric.

Claim 2 (1)**.**
There exists a linear change of coordinates such that $H_{rr}(0) \neq 0$.

We can use the fact that $\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = H_{ij}(0) + H_{ji}(0) = 2H_{ij}(0)$, and thus $H_{ij}(0) = \frac{1}{2}\left(\frac{\partial f}{\partial x_i \partial x_j}\right)$.

Since $H(0)$ is non-singular, we can find A such that $A^t H(0) A$ has nonzero rr entry, namely by letting the first column of A be an eigenvector of $H(0)$, then $A = [\mathbf{v}, \cdots]$ and thus $H(0)A = [\lambda \mathbf{v}, \cdots]$ and $A^t[\lambda \mathbf{v}] = [\lambda \|\mathbf{v}\|^2, \cdots]$.

So

$$\begin{aligned}
\sum_{i,j\geq r} y_i y_j H_{ij}(y_1,\cdots,y_n) &= y_r^2 H_{rr}(y_1,\cdots,y_n) + \sum_{i>r} 2y_i y_r H_{ir}(y_1,\cdots,y_n) \\
&= H_{rr}(y_1,\cdots,y_n)\left(y_r^2 + \sum_{i>r} 2y_i y_r H_{ir}(y_1,\cdots,y_n)/H_{rr}(y_1,\cdots,y_n)\right) \\
&= H_{rr}(y_1,\cdots,y_n)(\left(y_r + \sum_{i>r}^{n} y_i H_{ir}(y_1,\cdots,y_n)/H_{rr}(y_1,\cdots,y_n)\right)^2 \\
&\cdot \sum_{i>r}^{n} y_i^2 \left(H_{ir}Y/H_{rr}(Y)\right)^2 \\
&\cdot \sum_{i,j>r}^{n} H_{ir}(Y)H_{jr}(Y)/H_{rr}(Y))^2 \\
&\text{by completing the square.}
\end{aligned}$$

Note that $H_{rr}(0) \neq 0$ implies that $H_{rr} \neq 0$ in a neighborhood of zero as well.

Now define a change of coordinates $\phi : U \longrightarrow \mathbb{R}^n$ by

$$z_i = \begin{cases} y_i & i \neq r \\ \sqrt{H_{rr}(y_1,\cdots,y_n)}\left(y_r + \sum_{i>r} y_i H_{ir}(Y)/H_{rr}(Y)\right) & i = r \end{cases}.$$

This means that

$$f(z) = \pm z_1^2 \pm \cdots \pm z_{r-1}^2 \pm z_r^2 + \sum_{i,j\geq r+1}^{n} z_i z_j \tilde{H}(z_1,\cdots,z_n).$$

Exercise: show that $d_0\phi$ is invertible, and by the inverse function theorem, conclude that there is a neighborhood $U_2 \subset U_1$ of 0 on which ϕ is still invertible.

■

Corollary 1.3.
The nondegenerate critical points of a Morse function f are isolated.

Proof .
In some neighborhood around p, we have

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2,$$

Thus $\dfrac{\partial f}{\partial x_i} = 2x_i$, and so $\dfrac{\partial f}{\partial x_i} = 0$ iff $x_1 = x_2 = \cdots = x_n = 0$.

■

Corollary 1.4.
On a closed (compact) manifold M, a Morse function has only finitely many critical points.

We will need these facts to discuss the h-cobordism theorem. For a closed smooth manifold, $\partial M = \emptyset$, so M will define a cobordism $\emptyset \longrightarrow \emptyset$.

Definition 1.4.1 (Morse Function)**.**
Let W be a cobordism from $M_0 \longrightarrow M_1$. A *Morse function* is a smooth map $f : W \longrightarrow [a, b]$ such that

1. $f^{-1}(a) = M_0$ and $f^{-1}(b) = M_1$,
2. All critical points of f are non-degenerate and contained in $\operatorname{int}(W) := W \setminus \partial W$.

So f is equal to the endpoints only on the boundary.

Next time: existence of Morse functions. This is a fairly restrictive notion, but they are dense in the C^2 topology on (?).

2 Tuesday January 14th

2.1 Existence of Morse Functions

Notation Let $F(M; \mathbb{R})$ be the space of smooth functions from M to $\mathbb{R}$ with the C^2 topology.

Theorem 2.1 ***(Morse Functions Are Dense).***
Morse functions form an open dense subset of $F(M; \mathbb{R})$ in the C^2 topology.

Recall that the C^2 topology is defined by noting that $F(M, \mathbb{R})$ is an abelian group under addition, so we'll define open sets near the zero function and define open sets around f by translation. (I.e., if N is an open neighborhood of 0, then $N + f$ is an open neighborhood of f.)

So we'll define a base of open sets around 0. Take a finite cover of M, say by coordinate systems $\{U_\alpha\}$. Then let $h_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$. Now (exercise) we can find a compact refinement $C_\alpha \subset U_\alpha$ with each C_α compact and $\bigcup_\alpha C_\alpha = M$. We can now define $f_\alpha := f \circ h_\alpha^{-1}$ for any $f : M \longrightarrow \mathbb{R}$

$$\begin{tikzcd} U_\alpha \arrow[r, "h_\alpha"] \arrow[d, "f|_{U_\alpha}"'] & \mathbb{R}^n \arrow[dl, "f_\alpha"] \\ C_\alpha & \end{tikzcd}$$

Now for each $\delta > 0$, define

$$N(\delta) = \left\{ f : M \longrightarrow \mathbb{R} \;\middle|\; \begin{cases} |f_\alpha(p)| < \delta \\ \left| \dfrac{\partial f_\alpha}{\partial x_i} \right| < \delta \\ \left| \dfrac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \right| < \delta \end{cases} \quad \forall p \in h_\alpha(C_\alpha),\ \forall \alpha \right\}.$$

Corollary 2.2.
$f + N(\delta)$ (for all δ) is a basis for open neighborhoods around f.

Lemma 2.3.
This topology does not depend on the choice of $\{U_\alpha, h_\alpha\}$.

Proof .
See Milnor 2.

■

Lemma 2.4 *(1)***.**
Let $f : U \longrightarrow \mathbb{R}$ be a C^2 map for $U \subseteq \mathbb{R}^n$. For "almost all" linear maps $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f + L$ has only nondegenerate critical points.

Almost all: Note that $\hom(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$, so the complement of the set of such maps has measure zero in $\mathbb{R}^n$.

Proof .
Consider $X = U \times \hom(\mathbb{R}^n, \mathbb{R})$, which contains a subspace $M = \left\{(x, L) \;\middle|\; \partial_x(f + L) = 0\right\}$, i.e. x is a critical point of f. If $\partial_x f + L = 0$, then $L = -\partial_x f$. We thus obtain an identification of M with U by sending $x \in U$ to $(x, -\partial_x f)$.
There is also a projection onto the second component, where $(x, L) \mapsto L$. So let $\pi : X \longrightarrow \hom(\mathbb{R}^n, \mathbb{R})$ be this projection; then there is a map $\tilde{\pi} : U \longrightarrow \hom(\mathbb{R}^n, \mathbb{R})$ given by $x \mapsto \partial_x f$.
Note that $f + L$ has a *degenerate* critical point iff there is an $x \in U$ such that $\partial_x(f + L) = 0$ (or equivalently $L = -\partial_x f$), and the second derivative of $f + L$ is zero. Since L is linear, this says that the matrix $\left(\dfrac{\partial f^2}{\partial x_i \partial x_j}\right)(x)$ is singular. But this says x is a critical point for $\tilde{\pi}$.
This happens iff $\tilde{\pi}(x) = -\partial_x f = L$, so L is a critical value for $\tilde{\pi}$. Thus $f + L$ has a degenerate critical point $\iff$ L is a critical value for $\tilde{\pi}$.
Now Sard's theorem applies: if $g : M^n \longrightarrow \mathbb{R}^n$ is a map from any manifold to $\mathbb{R}^n$ that is C^1, then the set of critical values of g in $\mathbb{R}^n$ has measure zero.
Thus the set of critical values of $\tilde{\pi}$ has measure zero, and thus for almost all L, $f + L$ has no degenerate critical points.

■

Summary: Consider the map of first derivatives. It has a critical point whenever the 2nd derivative is singular, which is exactly the nondegeneracy condition.

Lemma 2.5 *(2)***.**
Let $K \subset U \subset \mathbb{R}^n$ with K compact and U open, and let $f : U \longrightarrow \mathbb{R}$ have only nondegenerate critical points. Then there exists a $\delta > 0$ such that every $g : U \longrightarrow \mathbb{R}$ that is C^2 which satisfies

1. $\left|\dfrac{\partial f}{\partial x_i}(p) - \dfrac{\partial g}{\partial x_i}(p)\right| < \delta$, and
2. $\left|\dfrac{\partial x}{\partial x_i \partial x_j}(p) - \dfrac{\partial g}{\partial x_i \partial x_j}(p)\right| < \delta$

for all i, j and $p \in K$ has only nondegenerate critical points.

Proof .
Define $|df| = \sqrt{\left|\frac{\partial f}{\partial x_1}\right|^2 + \cdots + \left|\frac{\partial f}{\partial x_n}\right|^2}$. Now note that $S(f) = |df| + \left|\det\left(\frac{\partial f^2}{\partial x_i \partial x_j}\right)\right| \geq 0$. This is an equality iff both terms are zero, and the first term is zero iff x is a critical point, while the second term is zero iff x is degenerate.
Since f has only nondegenerate critical points, this inequality is strictly positive on K, i.e. $S(f) > 0$. Since K is compact, $S(f)$ takes on a positive infimum on K, say μ. Then $S(f) \geq \mu > 0$ on K.
Thinking of S as defining a norm, the reverse triangle inequality yields

$$||df| - |dg|| \leq |df - dg| \leq \sqrt{n\delta^2} \leq \frac{\mu}{2},$$

where we can choose δ such that $\sqrt{n\delta^2} < \mu$.
We can also pick δ small enough such that

$$||\det J_f| - |\det(J_g)|| \leq \frac{\mu}{2},$$

where $J_f = \left(\frac{\partial f}{\partial x_i \partial x_j}\right)$ is shorthand for the matrix of partial derivatives appearing previously, and we just note that picking entries close enough makes the difference of determinant small enough (although there's something to prove there).
Then

$$\begin{aligned} |df| - |dg| + |\det(J_f)| - |\det J_g| &< \mu \\ \implies 0 \leq |df| + |\det(J_f)| - \mu &< |dg| + |\det(J_g)|, \end{aligned}$$

The second inequality follows from just moving terms in the first inequality.
which makes the last term strictly positive, and thus nonzero on K. Then g has no degenerate critical points in K.

■

Proof summary:

1. $\|f\|_2(x) = 0$ iff x is a degenerate critical point.
2. $\|f\|_2(x) \geq \mu > 0$ in K.
3. We can pick δ small enough such that $\|f\|_2 - \|g\|_2 < \mu$ on K.
4. This forces $\|g\|_2 > 0$ on K, so g has *no* nondegenerate critical points on K.

2.2 Proof that Morse Functions are Open

We still want to show that Morse functions form an open dense subset.

To see that they form an open set, suppose $f \in F(M, \mathbb{R})$ is Morse. Then take a finite cover of M, say $\{(U_i, h_i)\}_{i=1}^k$. Pick compact $C_i \subset U_i$ that still covers M.

Note that any g satisfying the 2 required conditions where $|f-g|<\delta$ (?), then $g \in N(\delta)+f$.

By lemma 2, there exists a $\delta>0$ such that every $g \in N_1 := f+N(\delta)$ has only nondegenerate points in C_1. We can pick a δ similarly to define an N_i for every i. Then taking $N = \bigcap_{i=1}^{k} N_i$, this yields an open neighborhood of f such that every $g \in N$ has only nondegenerate critical points on $C_1 \bigcup C_2 \cdots \bigcup C_k = M$.

■

2.3 Proof that Morse Functions are Dense

We want to show that this set is dense, so we'll fix some open set and show that there exists a Morse function in it.

Let $f \in N$ for N an open set; we'll then change f gradually to make it Morse.

Convention We'll say f is *good* on $S \subset M$ iff f has only nondegenerate critical points in S.

Pick a smooth bump function $\lambda : M^n \longrightarrow [0,1]$ such that

- $\lambda \equiv 1$ on an open neighborhood of C_1, and
- $\lambda \equiv 0$ on an open neighborhood of $M \setminus U_1$.

> Note: we can do this because $C_1 \subset U_1$ is closed, and $M \setminus U_1$ is closed, so we can find disjoint open sets containing each respectively using the fact that M^n is Hausdorff (?).

Now let $f_1 = f + \lambda L$ for some linear function $L : \mathbb{R}^n \longrightarrow \mathbb{R}$, so $f_1 = f + L$ on an open neighborhood of C_1. By Lemma 1, for almost every L, f_1 is good.

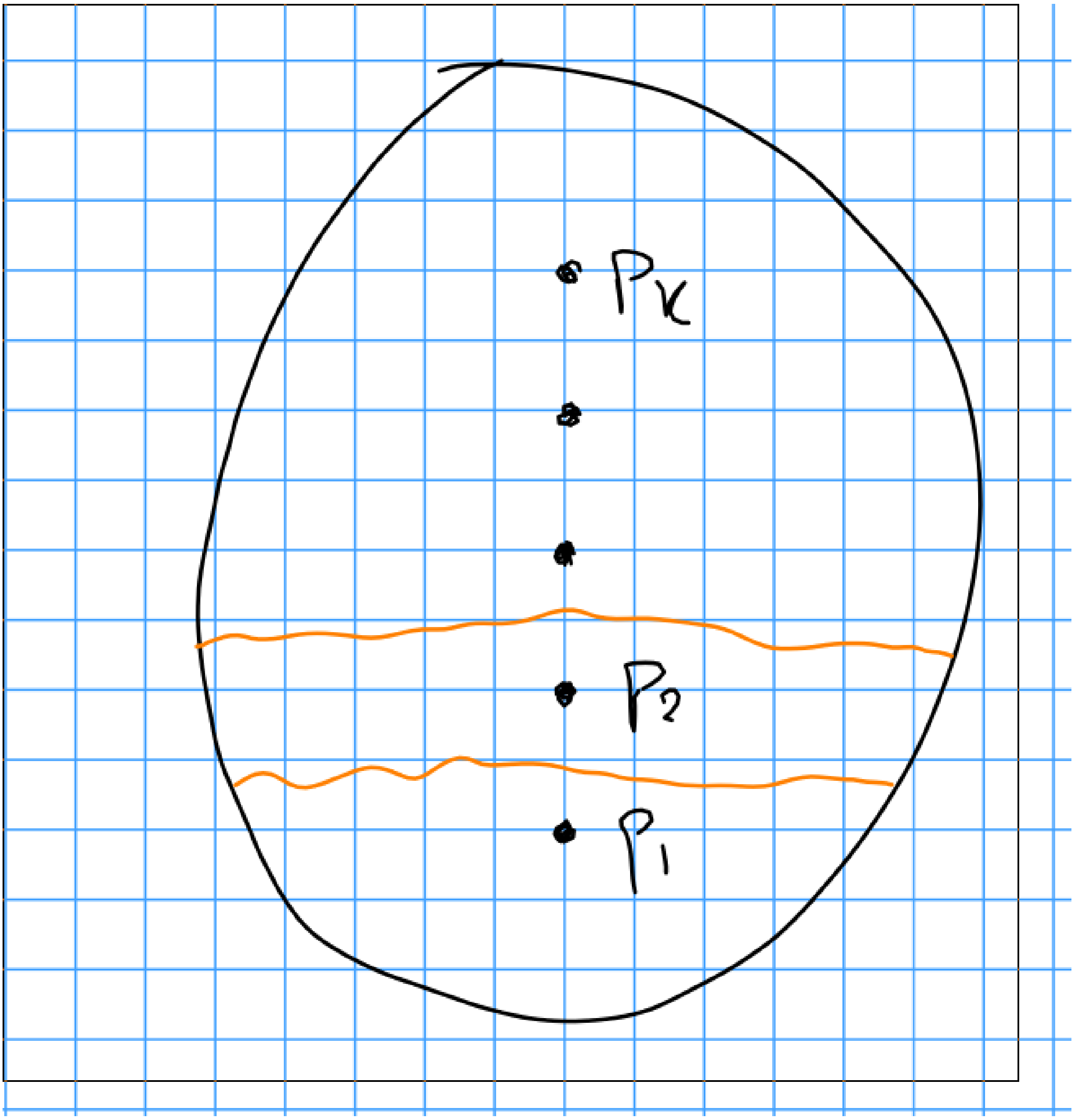
> Note that we need λ because L is only defined on $\mathbb{R}^n$, not on M.

Now $f_1 - f = \lambda L$ is supported in U_1. If we pick the coefficients of L small enough, noting that λ is bounded, then the first and second derivatives of $f - f_1$ will be bounded, and we can arrange for $f_1 \in f + N(\varepsilon)$ for $\varepsilon > 0$ as small as we'd like. For ε sufficiently small, we can arrange for $N(\varepsilon) \subset N_\delta$ for the finitely many δs, and so $N(\varepsilon) \subset N$.

By Lemma 2, there exists a neighborhood $N_1 \subseteq N$ containing f_1 such that every $g \in N_1$ is good on C_1. Since $f_1 \in N_1$, we can repeat this process to obtain an $f_2 \in N_2 \subseteq N_1$ and so on inductively. Then since every $g \in N_2$ is good on C_2 and $N_2 \subseteq N_1$, every $g \in N_2$ is good on $C_1 \bigcup C_2$. This yields an $f_k \in N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset N$, so f_k is good on $\bigcup C_i = M$.

■

Thursday: We'll show that every pair of critical points can be arranged to take on different values, and then order them. This yields $f(p_1) < c_1 < f(p_2) < c_2 < \cdots c_{k-1} < f(p_k)$, and since the c_i are regular values, the inverse images $f^{-1}(c_i)$ are smooth manifolds and we can cut along them.



3 Thursday January 16th

3.1 Approximation with Morse Functions with Distinct Critical Points

Theorem 3.1 ***(Morse Functions and Distinct Critical Points).***
Let $f : M \longrightarrow \mathbb{R}$ be Morse with critical points $p_1, \cdots, p_k$. Then f can be approximated by a Morse function g such that

1. g has the same critical points of f

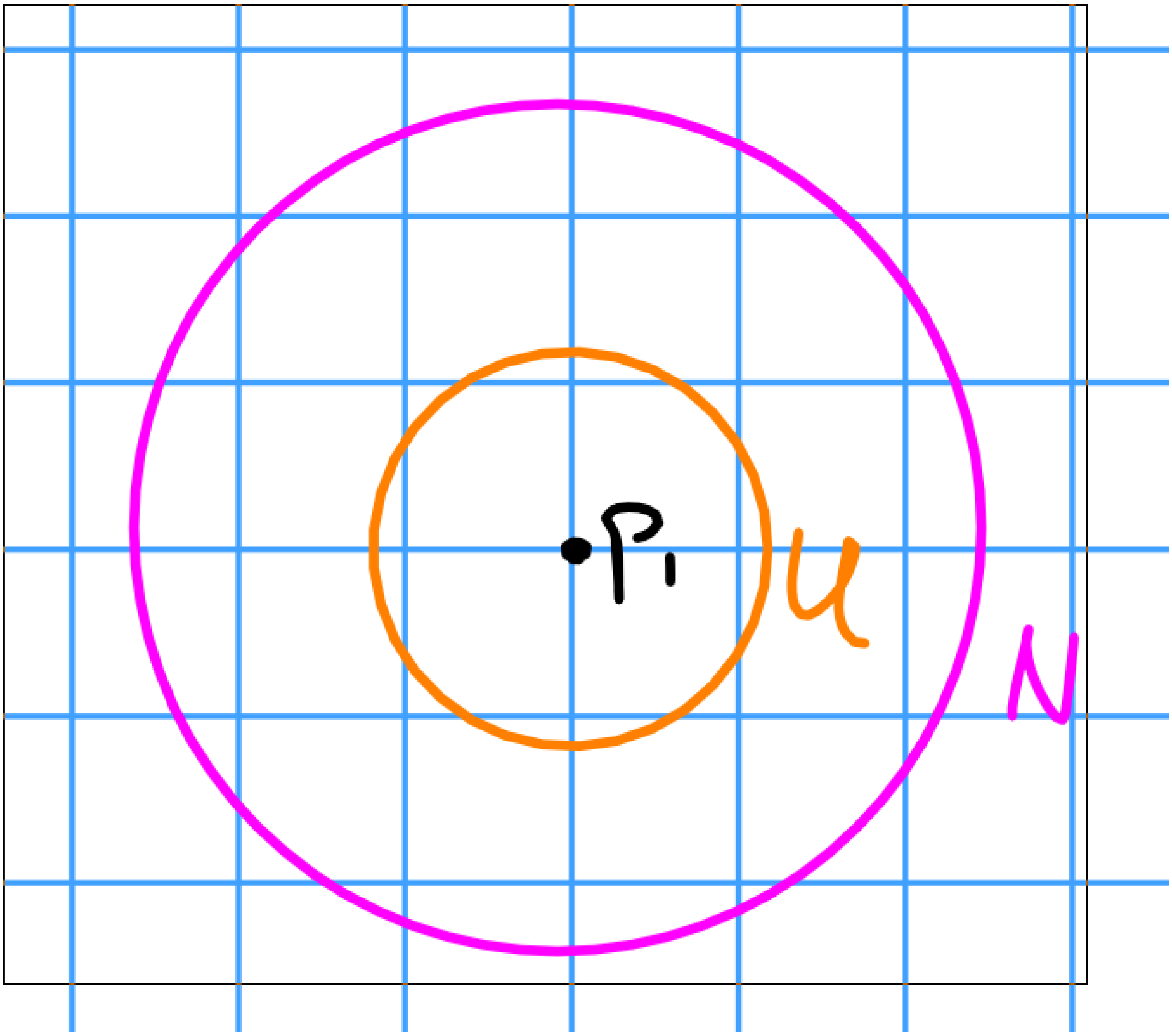
2. $g(p_i) \neq g(p_j)$ for all $i \neq j$.

Idea: Change f gradually near critical points without actually changing the critical points themselves.

3.2 Proof of Theorem

Suppose $f(p_1) = f(p_2)$.

Choose $\overline{U} \subset N$ open neighborhoods of p_1 such that $\overline{N}$ doesn't contain p_i for any i except for 1. Note that this is possible because the critical points are isolated.



Choose a bump function $\lambda \equiv 1$ on U and 0 on $M \setminus N$. Now let $f_1 = f + \varepsilon\lambda$, where we'll see how to choose ε small enough soon.

Let $K := \left\{x \,\middle|\, 0 < \lambda(x) < 1\right\}$, which is compact.

Pick a Riemannian metric on M, then we can talk about gradients. Recall that grad$f$ is the vector field that satisfies $\langle X,\ f\rangle$ for all vector fields X on M. Because f has no critical points in K, $X(f)$ is nonzero for some field X, so grad$f$ is nonzero, noting that grad$f$ is only zero at the critical points of f.

In particular, on K we have $0 < c \leq |\text{grad} f|$ for some c, and $\text{grad}\lambda \leq c'$ for some c'. So pick $0 < \varepsilon < c'/c$ such that $f_1(p_1) \neq f_1(p_2)$, $f_1(p_1) = f(p_1) + \varepsilon$, and$f_1(p_i) = f(p_i)$ for all $i \neq 1$. Note that this is possible because there are only finitely many points, so almost every ε will work.

Claim 1 The critical points of f_1 are exactly the critical points of f.

Proof .
In K, we have

$$\text{grad} f_1 = \text{grad} f + \varepsilon \text{grad}\lambda \implies |\text{grad} f_1| \geq |\text{grad} f| - \varepsilon|\text{grad}\lambda| \geq x - \varepsilon c' > 0.$$

If $x \notin K$, we have

1. $x \in U$, or
2. $x \in M \setminus N$

In case 1, λ is constant and $\text{grad}\lambda = 0$, so $\text{grad} f_1 = \text{grad} f$. In case 2, λ is again constant, so the same conclusion holds.

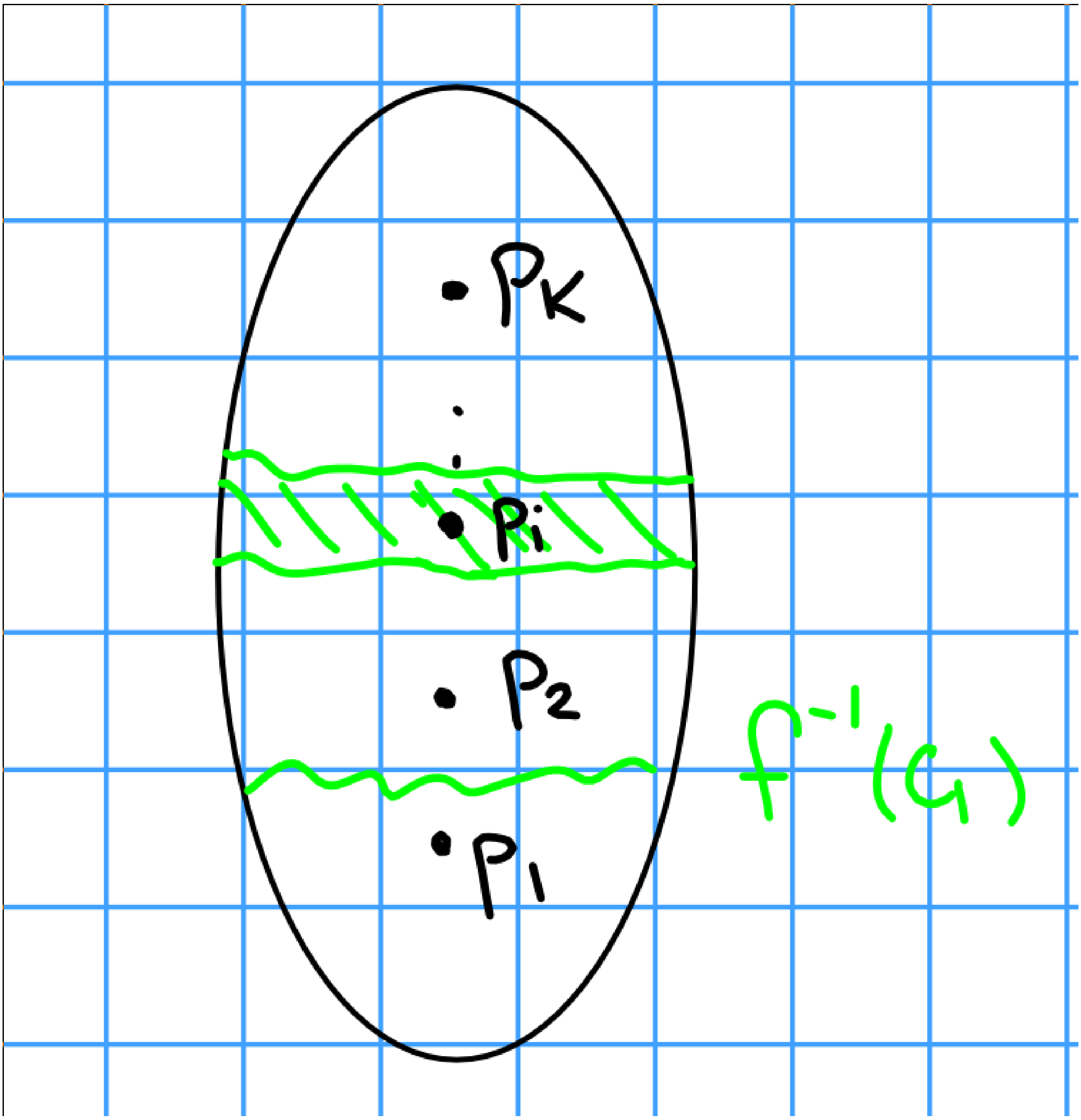
■

Claim 2 f_1 is Morse.

Proof .
In a neighborhood of p_1, we have $f_1 \equiv f + \varepsilon$. In a neighborhood of p_i, we have $f_1 \equiv f$.
We can then check that $J_{f_1}(p_i) = J_f(p_i)$, and since f is Morse, f_1 is Morse as well.

■

Recall that this lets us put an order on $f(p_i)$. Between every critical value, pick regular values c_i, i.e. $f(p_1) < c_1 < f(p_2) < \cdots$. Then $f^{-1}(c_i)$ is a smooth submanifold of dimension $n-1$, and we have the following schematic:



Moreover, $f^{-1}[c_i, c_{i+1}]$ is a cobordism from $f^{-1}(c_2)$ to $f^{-1}(c_{i+1})$.

Definition 3.1.1 (Morse Functions for Cobordisms)**.**
Recall that for $(W; M_0, M_1)$ a cobordism, a Morse function $f : W \longrightarrow [a, b]$ is Morse iff

1. $f^{-1}(a) = M_0$ and $f^{-1}(b) = M_1$.
2. f has only nondegenerate critical points and no critical points near $\partial W = M_1 \coprod M_2$, i.e. all critical points are in W° (the interior).

Proof of density of Morse functions goes through in the same way, with extra care taken to choose neighborhoods that do not intersect ∂W.

Theorem 3.2 ***(Cobordisms, Morse Functions, Distinct Critical Points).***

1. For every cobordism $(W; M_1, M_2)$ there exists a Morse function.
2. The set of such Morse functions is dense in the C^2 topology.
3. Any Morse function $f : (W; M_1, M_2) \longrightarrow [a, b]$ can be approximated by another Morse function $g : (W; M_1, M_2) \longrightarrow [a, b]$ such that g has the same critical points of f and $g(p_i) \neq g(p_j)$ for $i \neq j$ (i.e. the critical points are distinct).

Note that n-manifolds are a special cases of cobordisms, namely a manifold M is a cobordism $(W; M, \emptyset)$. So all statements about cobordisms will hold for n-manifolds.

Definition 3.2.1 (Morse Number)**.**
The **Morse number** μ of a cobordism $(W; M_0, M_1)$ is the minimum of $\left|\left\{\text{critical points of } f \;\middle|\; f \text{ is Morse}\right\}\right|$.

We'll be considering cobordisms with $\mu = 0$.

Note: if we take $X = \text{grad} f$, we have $\langle X, \text{ grad} f\rangle = \|\text{grad} f\|^2 \geq 0$, which motivates our next definition.

Definition 3.2.2 (Gradient-Like Vector Fields)**.**
Let $f : W \longrightarrow [a, b]$ be a Morse function. Then a **gradient-like vector field** for f is a vector field ξ on W such that

1. $\xi(f) > 0$ on $W \setminus \text{crit}(f)$.
2. For every critical point p there exist coordinates $(x_1, \cdots, x_n)$ on $U \ni p$ such that

$$f(X) = f(p) - x_1^2 - \cdots - x_{\lambda^2} + x_{\lambda+1}^2 + \cdots + x_n^2,$$

as in the Morse Lemma, where λ is the index, and

$$\xi = (-x_1, -x_2, \cdots, -x_\lambda, x_{\lambda+1}, \cdots, x_n) \text{ in } U.$$

Lemma 3.3 ***(Morse Functions Have Gradient-Like Vector Fields).***
Every Morse function f on $(W; M_0, M_1)$ has a gradient-like vector field.

3.3 Proof: Every Morse Function has a Gradient-Like Vector Field

For simplicity, assume f has a single critical point p. Pick coordinate $(x_1, \cdots, x_n)$ on an open set U_0 around p such that f has the form given in (1) above. Define ξ_0 on U_0 to be (2) above.

Every point $q \in W \setminus U_0$ has a neighborhood U' such that $df \neq 0$ on U'. By the implicit function theorem, there is a smaller neighborhood U'' such that $q \in U'' \subset U$ such that $f = c_0 + x_1$ on U'' for some constant c_0.

Exercise: check that this works!

But since $W \setminus U_0$ is a closed subset of a compact manifold, it is compact, so we can cover it with finitely many U_i that satisfy

1. $U_i \bigcap U = \emptyset$ for some open U containing p such that $U \subset U_0$ and $\overline{U} \subset U_0$.
2. U_i has a coordinate chart $(x_1^2, \cdots, x_n^2)$ such that $f = c_i + x_1^2$ on U_i for some constants c_i.

Thus on U_i we can set $\xi_i = (1, 0, \cdots, 0) = \dfrac{\partial}{\partial x_1^2}$ to get local vector fields. We can then take a partition of unity $\rho_1, \cdots, \rho_k$ and set $\xi = \sum_i \rho_i \xi_i$.

Now consider $\xi(f)$. By definition, $\xi(f) = \sum_i \rho_i \xi_i(f)$. Note that $\rho_i \xi_i(f) = 1$ in U_i, and $\rho_0 \xi_0(f) \geq 0$, so $\xi(f) \geq 0$. If x is not a critical point, then at least 1 $\xi_i(f)(x)$ is positive and thus $\xi(f)(x) > 0$.

This is because x is either in U, in which case the 0 term is positive, or $x \in U_i$, in which case one of the remaining terms is positive.

■

The idea here: if we can make locally gradient-like vector fields, we can use partitions of unity to extend them to global vector fields.

Theorem 3.4 ***(The Morse Number Detects Product Cobordisms).***
Any cobordism $(W; M_0, M_1)$ with $\mu = 0$ is a product cobordism, i.e.

$$(W; M_0, M_1) \cong (M_0 \times I; M_0 \times \{0\}, M_0 \times \{1\}).$$

Proof (of Theorem).
Let $f : W \longrightarrow I$ be Morse with no critical points, and let ξ be a gradient-like vector field for f. Then $\xi(f) > 0$ on W, so we can normalize to replace ξ with $\dfrac{1}{\xi(f)}\xi$ and assume $\xi(f) = 1$. Then consider the integral curves of ξ, given by $\phi : [a, b] \longrightarrow W$.

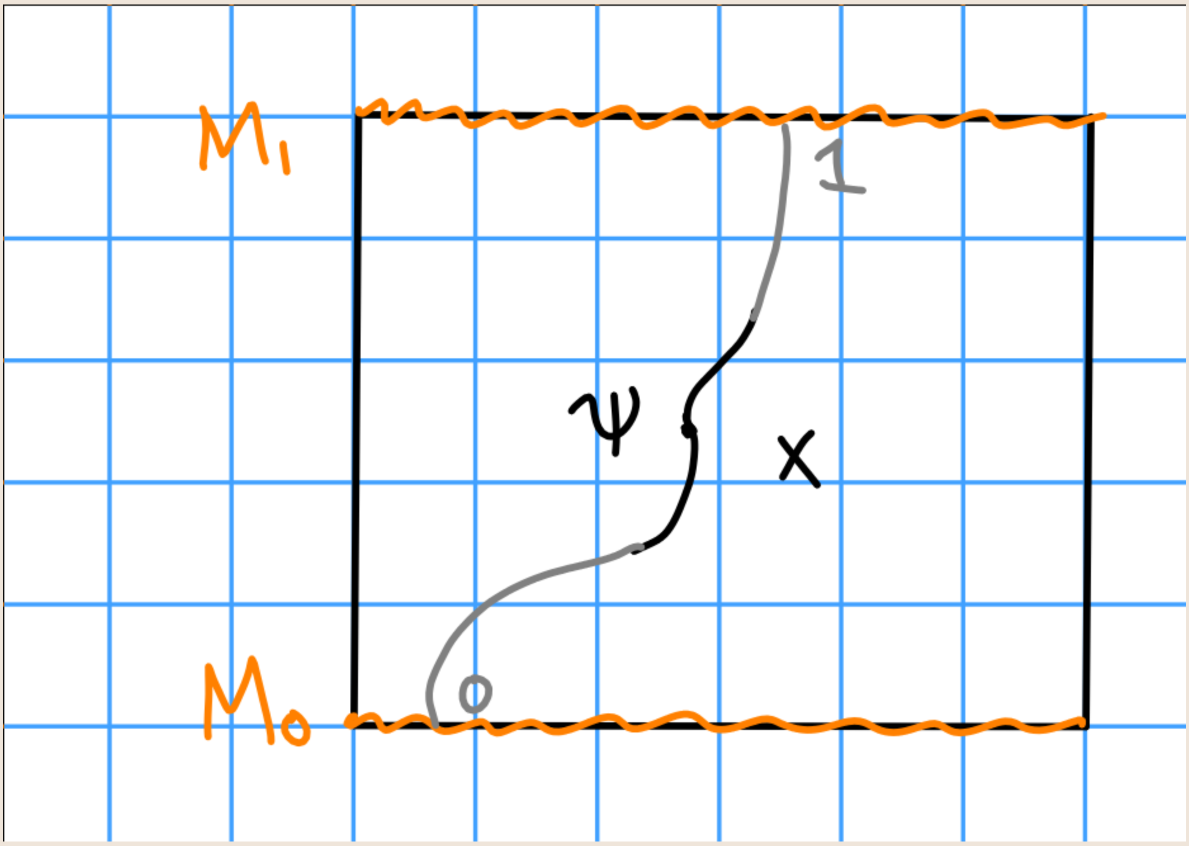
i.e. $d\phi = \xi$.

We can thus compute

$$\frac{\partial}{\partial t} f \circ \phi(t) = df\left(\frac{\partial \phi}{\partial t}\right) = df(\xi) = \xi(f) = 1.$$

By the FTC, this implies that $f \circ \phi(t) = c_0 + t$ for some constant c_0. So reparameterize by defining $\psi(s) = \phi(s - c_0)$, then $f \circ \psi(s) = s$. For every $x \in W$, there exists a unique maximal integral curve $\psi_x(s)$ that passes through x.

Note that this works because maximal curves must intersect the boundary at precisely $t = 0, 1$ and f is an increasing function. So for any curve passing through x, we can extend it to a maximal.



We can then define

$$
\begin{aligned}
h : M_0 \times I &\longrightarrow W \\
(x, s) &\mapsto \psi_x(s) \\
(\psi_y(0), f(y)) &\leftarrow\!\shortmid y
\end{aligned}
$$
.

■

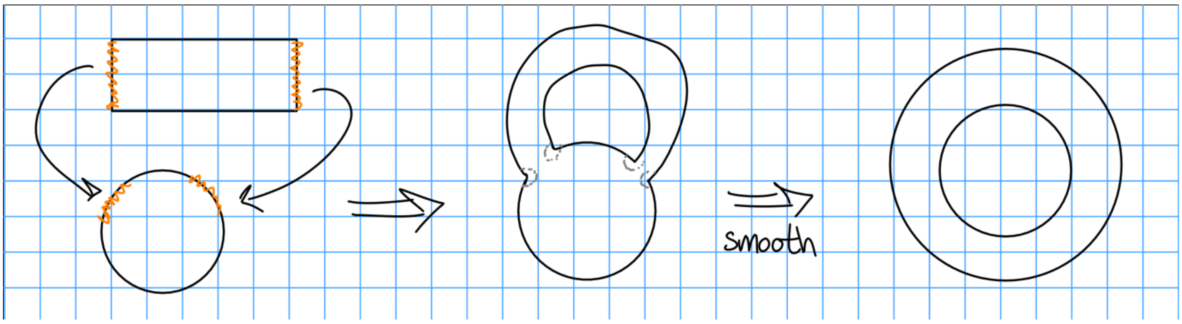
4 January 21st

4.1 Elementary Cobordism

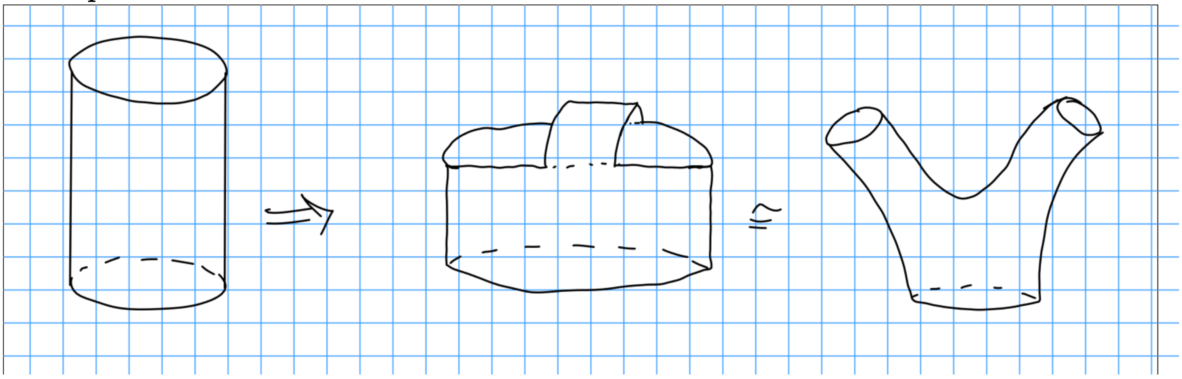
Recall that an elementary cobordism is a cobordism that has a Morse function with exactly one critical point.

Definition 4.0.1 (Handles).
An n-dimensional λ-handle is a copy of $D^\lambda \times D^{n-\lambda}$ which is attached to ∂M^n via an embedding $\phi : \partial D^\lambda \times D^{n-\lambda} \hookrightarrow \partial M$.

Example 4.1.
Let $\lambda = 1, n = 2, n - \lambda = 1$ and take $M^2 = D^2$ and we attach $D^1 \times D^1$. Note that there's not necessarily a smooth structure on the resulting manifold, so we can "smooth corners":



Example 4.2.



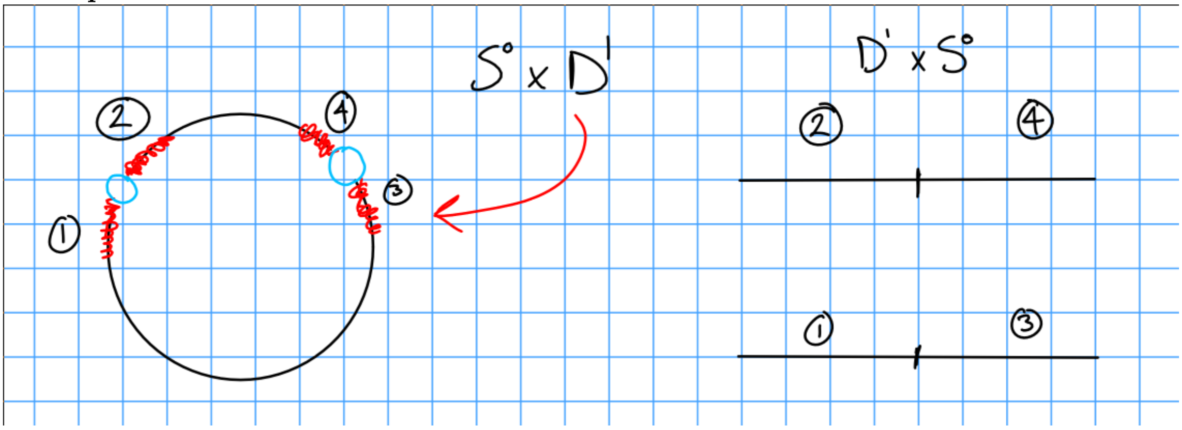
Note: the above is just a homeomorphism.

Definition 4.0.2 (Surgery)**.**
Let M be an $n-1$ dimensional smooth manifold, and $\rho : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M^{n-1}$ be an embedding.
Then noting that $\partial D^{n-\lambda} = S^{n-\lambda-1}$, consider the space

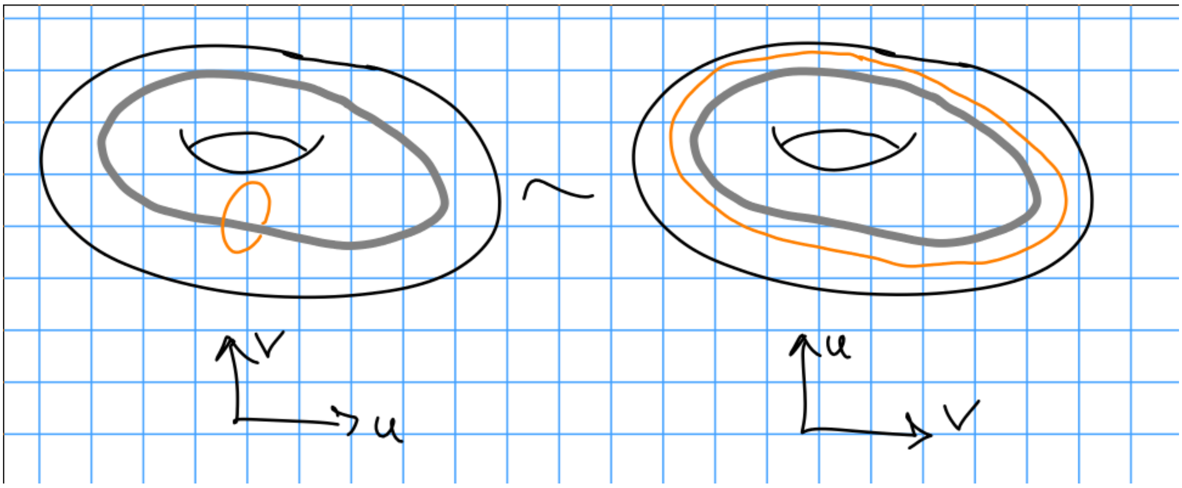
$$\begin{aligned} X(M, \phi) = &(M \setminus \rho(S^{\lambda-1} \times \{0\})) \times (D^{\lambda} \times S^{n-\lambda-1}) \\ &/ \left\langle \rho(u, tv) \sim (tu, v) \;\middle|\; t \in (0,1), \forall u \in S^{\lambda-1}, \forall v \in S^{n-\lambda-1} \right\rangle, \end{aligned}$$

where we note that we can parameterize $D^{n-\lambda} = tv$ where v is a point on the boundary.
Note that this accomplishes the goal of smoothing, and is referred to as **surgery** (of type $\lambda, n-\lambda$) on M along ϕ.

Example 4.3.

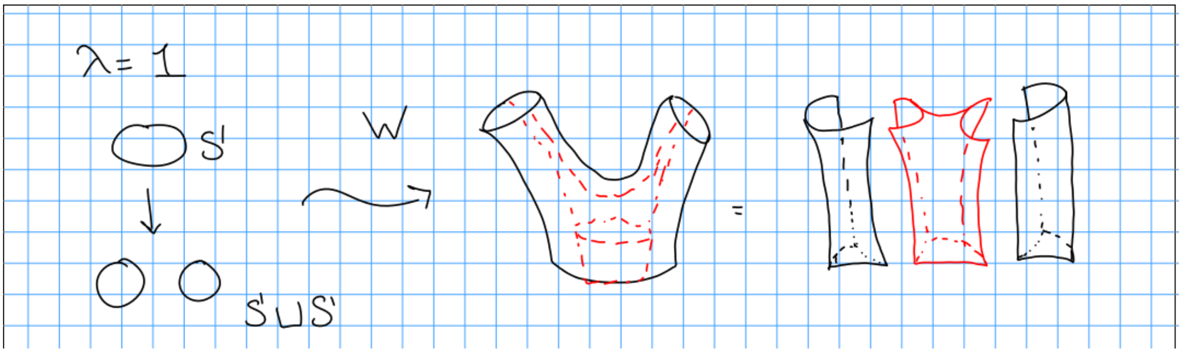


Example 4.4.
$n-1=3$ and $\lambda = 2$ implies $\lambda - 1 = 1$, and take $\rho : S^1 \times D^2 \longrightarrow S^3$, which has image a tubular neighborhood of a knot. Then $\phi(S^1 \times \{0\}) = K$ for some knot, and $(S^3 \setminus K)\coprod(D^2 \times S^1)/\cdots$. Then note that $\partial\phi(\{u\} \times D^2) = \{u\} \times S^1$, which no longer bounds a disk since we have removed the core of tube.



> **Theorem 4.1** ***(Cobordism and Morse Function Induced by Surgery).***
> Suppose $M' = X(M, \rho)$ is obtained from M by surgery of type λ. Then there exists an elementary cobordism $(W; M, M')$ with a Morse function $f : W \longrightarrow [-1, 1]$ with only one index λ critical point.

Example 4.5.
Let $M = S^1$ and $\lambda = 1$.

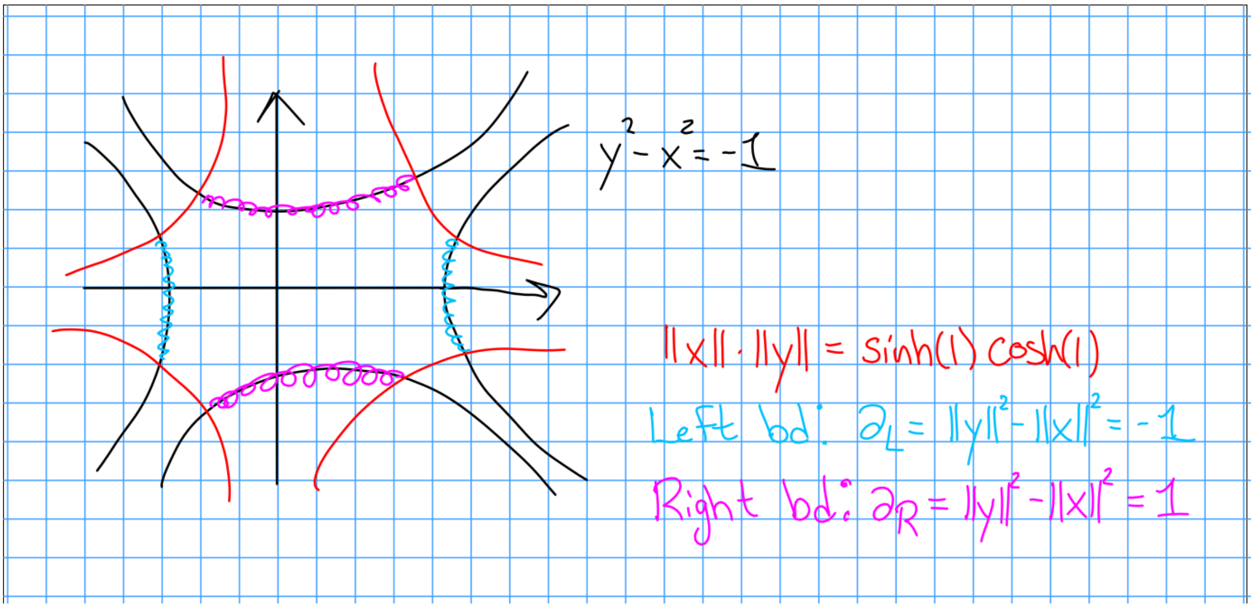


4.1.1 Proof: Surgeries Come From Cobordisms With Special Morse Functions

Write $\mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$, and $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$.

Then

$$L_\lambda = \{(\mathbf{x}, \mathbf{y}) \;\Big|\; -1 \leq -\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \leq 1,\ \|\mathbf{x}\|\|\mathbf{y}\| < \sinh(1)\cosh(1)\}.$$



The left boundary is given by $\partial_L : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = -1$, and there is a map

$$\begin{aligned} S^{\lambda-1} \times D^{n-\lambda} &\xrightarrow{\text{diffeo}} \partial_L \\ (u, tv) &\mapsto (u\cosh(t), v\sinh(t)) \quad t \in [0,1), \end{aligned}$$

which is clearly invertible.

The right boundary is given by $\partial_R : \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 = 1$, and there is a map

$$\begin{aligned} S^{\lambda-1} \times D^{n-\lambda} &\xrightarrow{\text{diffeo}} \partial_L \\ (tu, v) &\mapsto (u \sinh(t), v\cosh(t)). \end{aligned}$$

In the above picture, we can consider the orthogonal trajectories, which are given by $y^2 - x^2 = c$, which has gradient $(-x, y)$ and $xy = c$ which has gradient (y, x), so these are orthogonal.

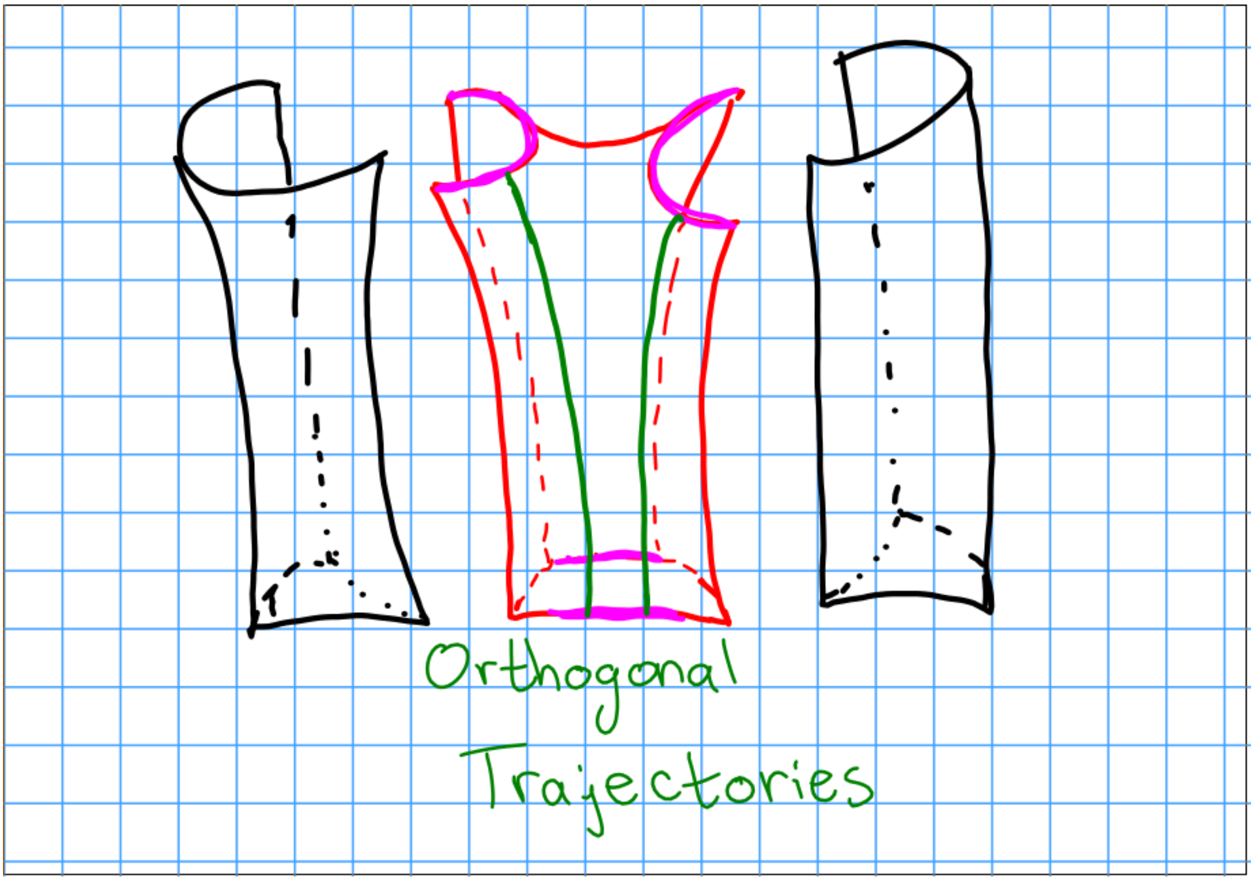
Recall that near a point $p \in M$, the morse function has the form $f(\mathbf{x}, \mathbf{y}) = f(p) - \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ with a gradient-like vector field given by $\xi = (-\mathbf{x}, \mathbf{y})$.

The orthogonal trajectories will generally be of the form $\|\mathbf{x}\|\|\mathbf{y}\| = c$, which we can parameterize as $t \mapsto (t\mathbf{x}, \frac{1}{t}\mathbf{y})$.

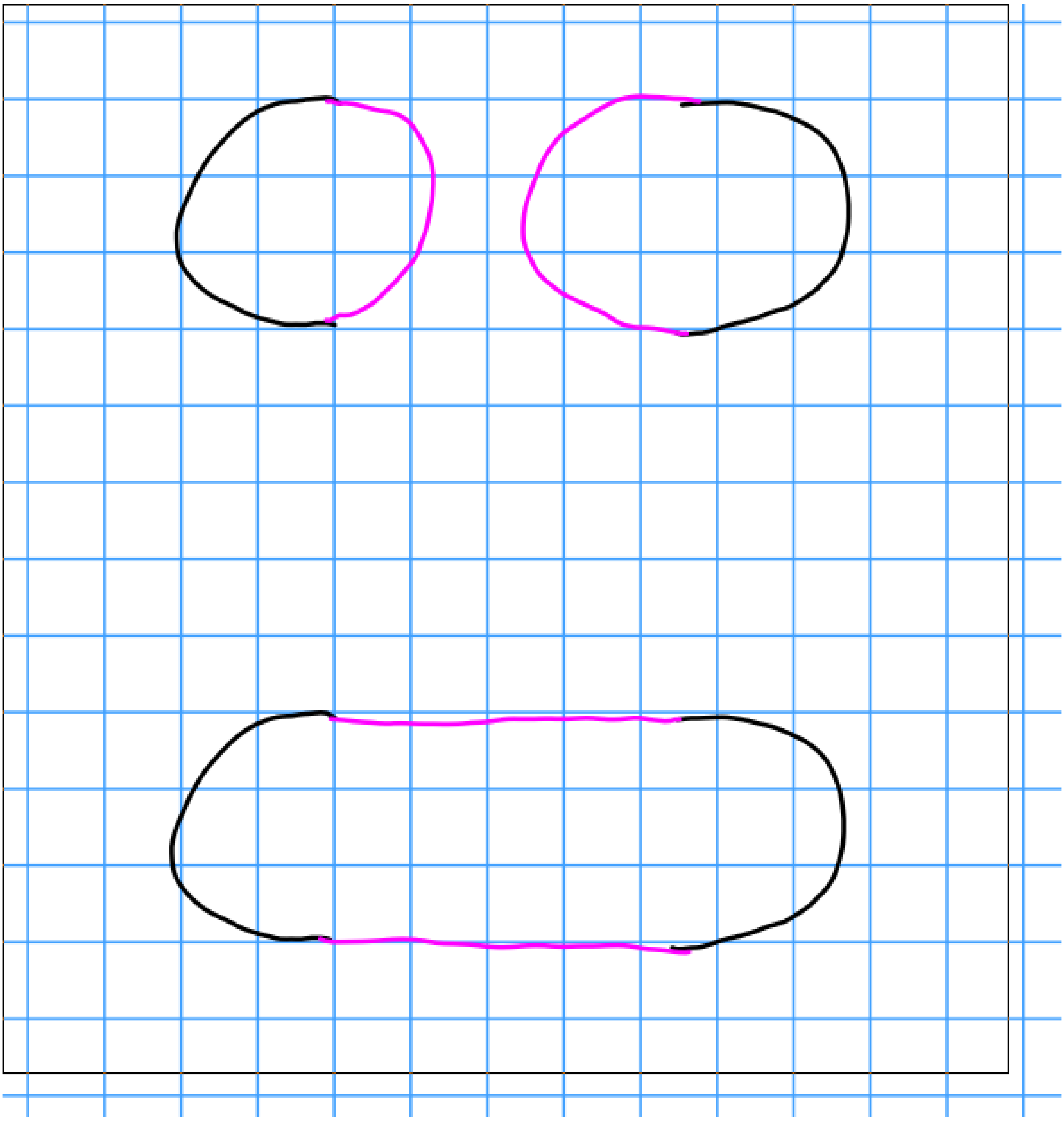
Construction of W:

Take

$$\begin{aligned} &W(M, \phi) = ((M \setminus \phi(S^{\lambda-1} \times \{0\})) \times D^1) \coprod L_\lambda \\ &\qquad / \left\langle \phi(u, tv) \times c \sim (\mathbf{x}, \mathbf{y}) \;\middle|\; \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = c, (\mathbf{x}, \mathbf{y}) \in \text{orth. traj. starting from}(u\cosh(t), v\sinh(t)) \right\rangle. \end{aligned}$$



This amounts to closing up in the following two ways:



This has two boundaries: when $c = -1$, we obtain M, and $c = 1$ yields $X(M, \phi)$. The Morse function is given by $f : W(M, \phi) \longrightarrow [-1, 1]$ where

$$\begin{cases} f(z, c) = c & z \in M \setminus \phi(S^{\lambda-1} \times \{0\}), c \in D^1 \\ f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 & (\mathbf{x}, \mathbf{y}) \in L_\lambda \end{cases}.$$

5 Thursday January 23rd

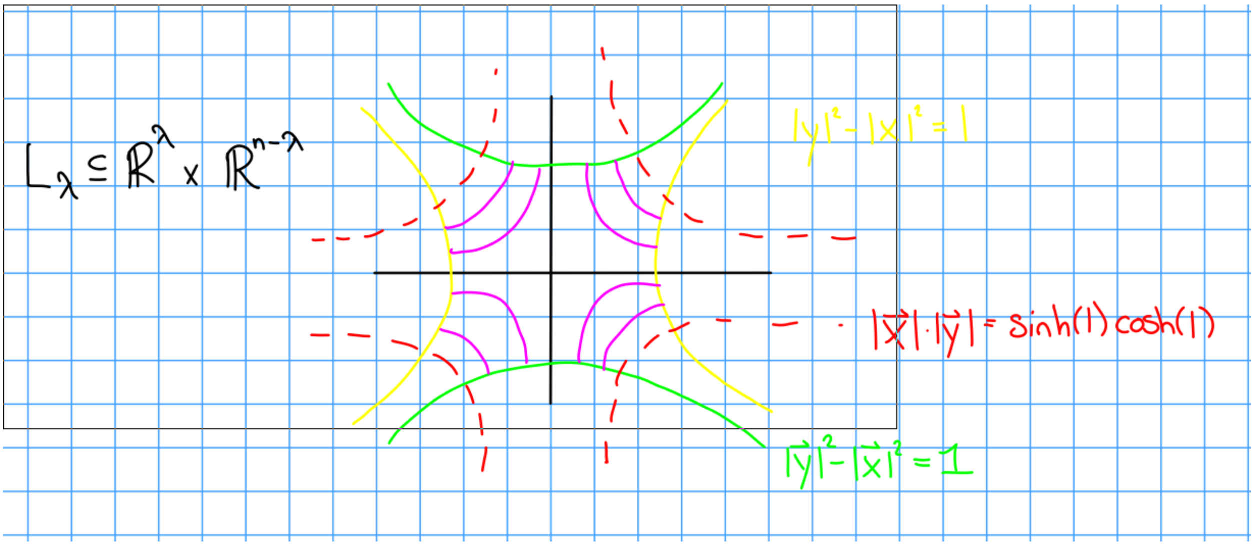
Recall from last time: M is a closed smooth $n-1$ manifold and $\phi : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M$, and we used surgery to obtain $\chi(M, \phi)$ and a cobordism $W(M, \phi)$ from M to $\chi(M, \phi)$.

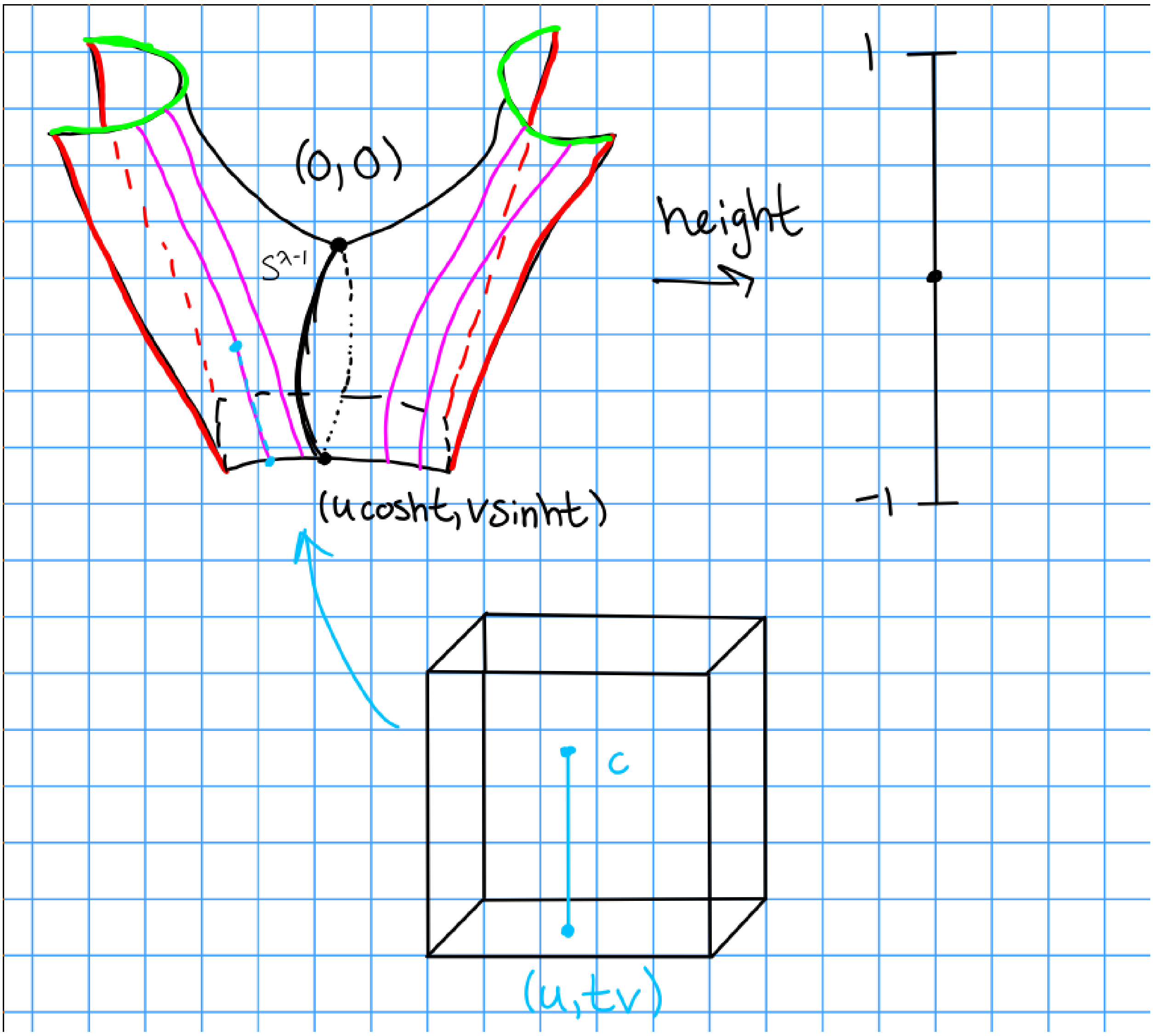
This yields a saddle $L_\lambda \subseteq \mathbb{R}^n = \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda}$. We construct the cobordism using

$$\begin{aligned} S^{\lambda-1} \times D^{n-\lambda} &\longrightarrow \partial_L \\ (u, tv) &\mapsto (u \cosh t, v \sinh t). \end{aligned}$$

This yields

$$M \setminus \left\{ \phi(S^{\lambda-1}) \times \{0\} \right\} \coprod L_\lambda / \left\langle (u, tv) \times c \sim (\mathbf{x}, \mathbf{y}) \;\middle|\; \|x\|^2 + \|y\|^2 = c \right\rangle$$
$$x, y \text{ are on a curve that starts from } (u \cosh t . v \sinh t).$$





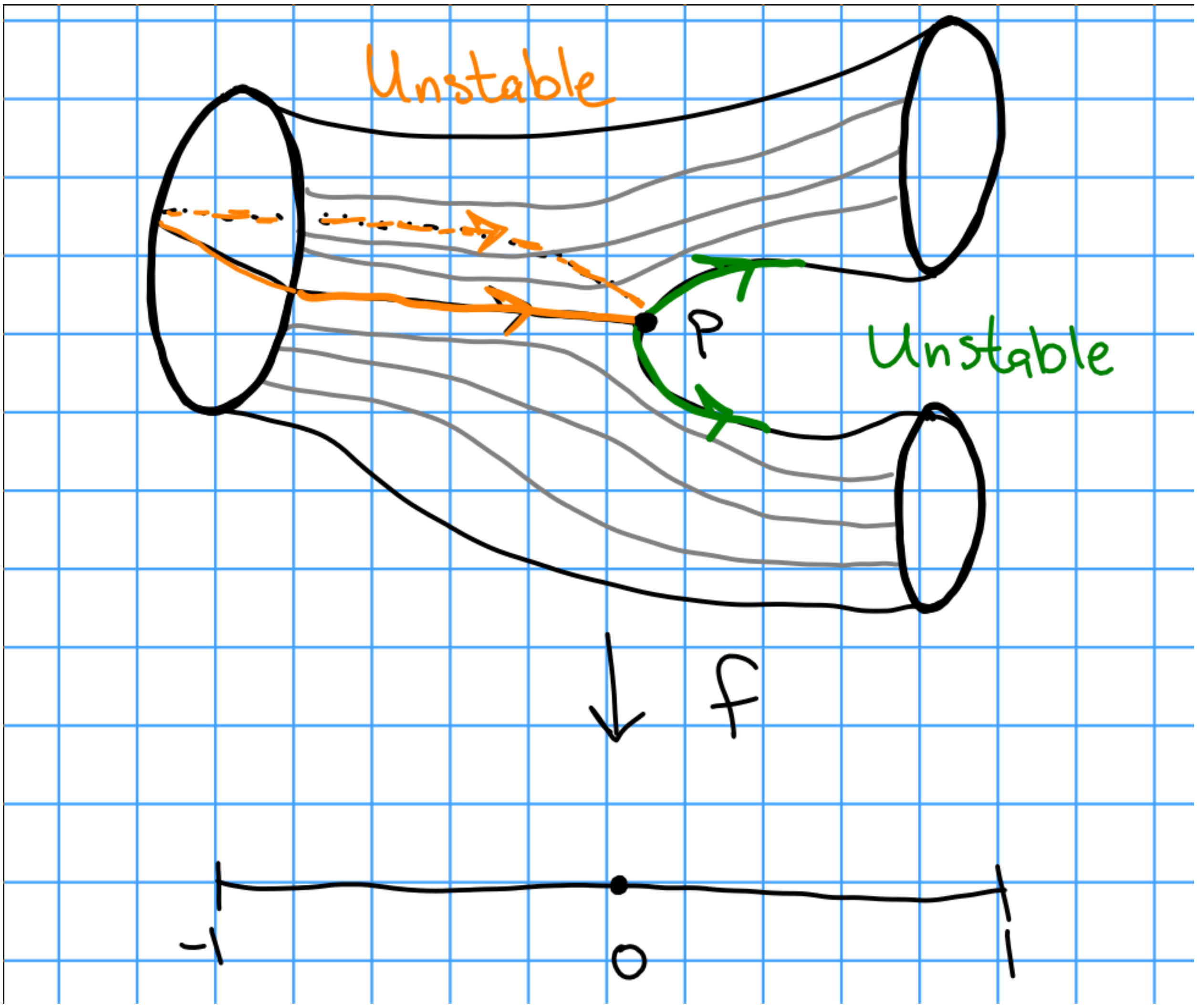
Todo: review!

Suppose $W(; M_0, M_1)$ is an elementary cobordism and $f : W \longrightarrow [-1, 1]$ is a Morse function with one critical point p, and ξ a gradient-like vector field for f.

The goal is to construct $\phi_L : S^{\lambda-1} \times D^{n-\lambda} \hookrightarrow M_0$, the characteristic embedding.

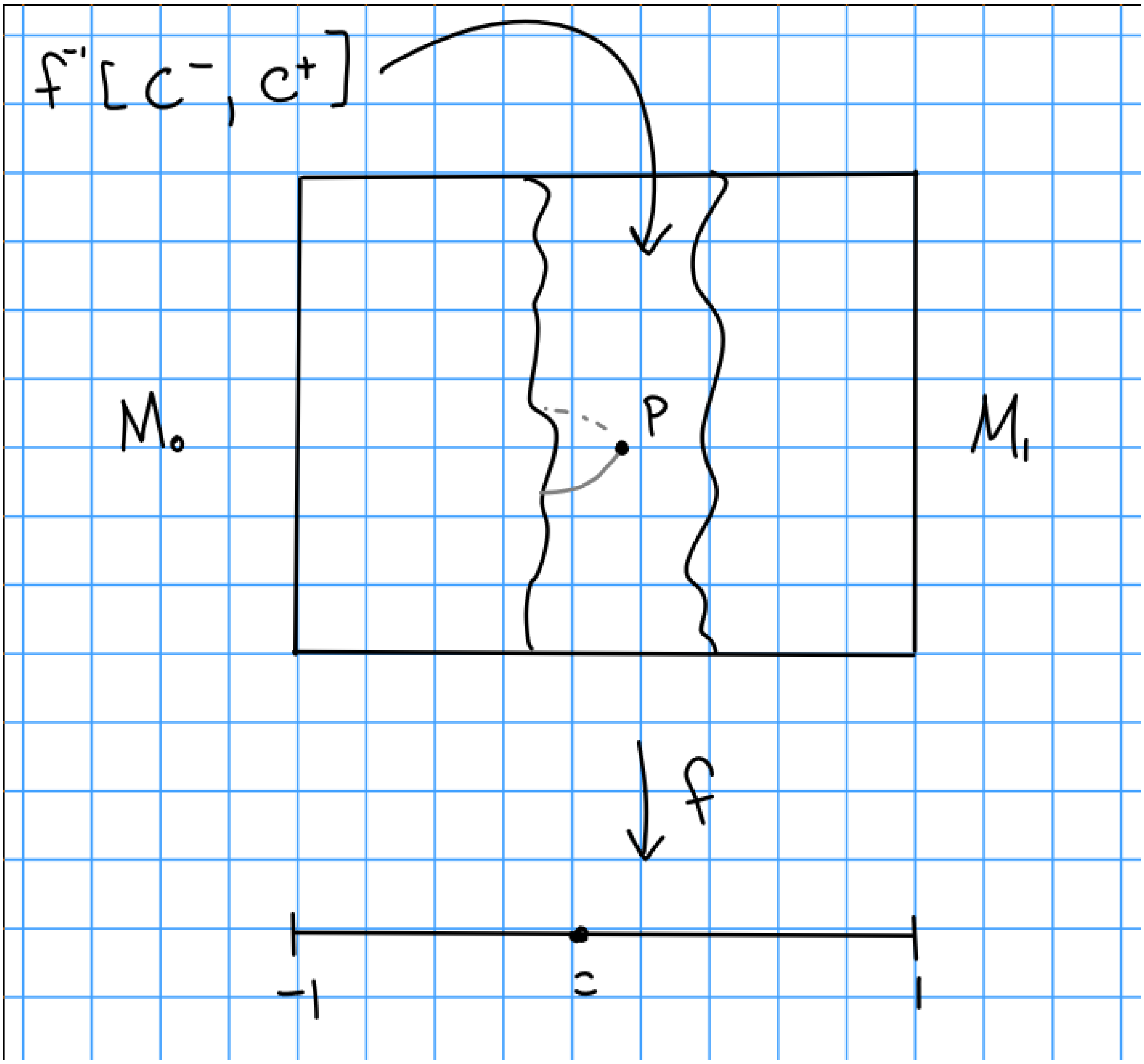
Let ψ_x be the integral curve of ξ such that $\psi_x(0) = x$, and define $W^s(p) = \left\{ x \in W \;\middle|\; \lim_{t \longrightarrow \infty} \psi_x(t) = p \right\}$ to be the stable manifold, and $W^u(p) = \left\{ x \in W \;\middle|\; \lim_{t \longrightarrow -\infty} \psi_x(t) = p \right\}$.

Claim: $W^s(p), W^u(p)$ are diffeomorphic to disks of dimension λ and $n - \lambda$ respectively.



Moreover, $\partial W^s(p) = W^s(p) \bigcap M_0 \cong S^{\lambda-1}$ (Milnor refers to this as the "left sphere" S_L and $\partial W^u(p) = W^u(p) \bigcap M_1 = S^{n-\lambda-1}$ (the "right sphere" S_R).

5.0.1 Proof

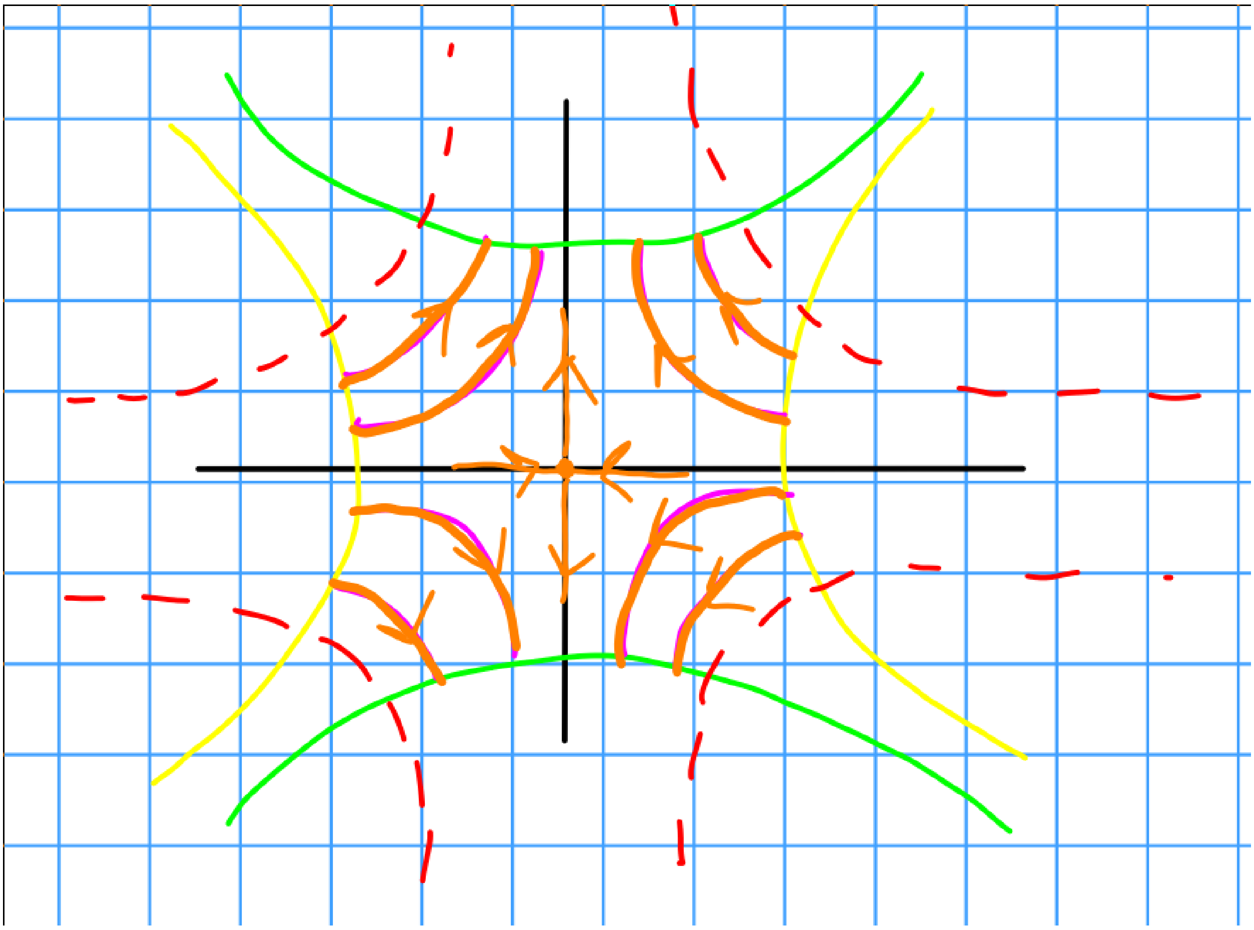


Choose an open $U \ni p$ and a coordinate chart $h : OD_{2\varepsilon} \longrightarrow U$. Then $f \circ h(x,y) = c - \|x\|^2 - \|y\|^2$, which takes on a minimum value of $c - 4\varepsilon^2$ and a maximum of $c + 4\varepsilon^2$, and $\xi \circ h(x,y) = (-x, y)$. We thus obtain the inequalities

$$-1 < c - 4\varepsilon^2 < c + 4\varepsilon^2 < 1 \text{ if } 4\varepsilon^2 < c + 1, 1 - c.$$

Now let $W_\varepsilon = f^{-1}[c - \varepsilon^2, c + \varepsilon^2]$, and we want a cobordism from $M_{-\varepsilon} = f^{-1}(c - \varepsilon^2)$ to $M_{+\varepsilon} = f^{-1}(c + \varepsilon^2)$. Then

$$M_\mp \bigcap U = \left\{ h(x,y) \;\middle|\; c - \|x\|^2 + \|y\|^2 = c_\mp \varepsilon^2 \iff -\|x\|^2 + \|y\|^2 = \mp\varepsilon^2 \right\}.$$



Then $W^2(p)\bigcap U = \{h(x,0)\} \cong D^\lambda$, and $W^s(p)\bigcap M_{-\varepsilon} = \Big\{h(x,0)\ \Big|\ \|x\| = \varepsilon\Big\}$. By flowing along the integral curves of ξ, we obtain a diffeomorphism $\Psi_{-\varepsilon} : M_0 \longrightarrow M_{-\varepsilon}$. Then $W^s(p) = \{h(x,0)\}\bigcup\Big\{\text{integral curves of } \xi \text{ passing through points in } W^s(p)\bigcap M_{-\varepsilon}\Big\}$.

So $S_L = \Psi_{-\varepsilon}^{-1}(W^s(p)\bigcap M_{-\varepsilon})$, and we can define the embedding ϕ_L by the following composition:

$$\begin{tikzcd}
 & S^{\lambda-1}\times D^{n-\lambda} \arrow[rr] \arrow[drr] & & M_0 \\
L(u,tv) \arrow[dr] & & & \\
 & h(\varepsilon u\cosh t, \varepsilon v\sinh t) & M_{-\varepsilon} \arrow[uur, "\Psi_{-\varepsilon}^{-1}"'] &
\end{tikzcd}$$

Similarly, one can show $W^u(p) \cong D^{n-\lambda}$ and define another embedding $\phi_R D^\lambda \times S^{n-\lambda-1} \hookrightarrow M_1$.

■

Theorem 5.1 *(?).*
Let $(W;M_0,M_1)$ be an elementary cobordism with (f,ξ,p) a Morse function with a gradient-like vector field and a critical point as above. Then there is a diffeomorphism of cobordisms

$$(W(M_0,\phi_L);M_0,\chi(M_0,\phi_L))\cong(W;M_0,M_1),$$

where $\phi_L : S^{\lambda-1}\times D^{n-\lambda}\hookrightarrow M_0$ is the characteristic embedding.

Proof .
Consider $f^{-1}[-1,c-\varepsilon^2]\bigcup W_\varepsilon\bigcup f^{-1}[c+\varepsilon^2,1]$, and note that $f^{-1}[-1,c-\varepsilon^2], f^{-1}[c+\varepsilon^2,1]$ is a product (?). Then $(W;M_0,M_1)\cong(W_\varepsilon;M_{-\varepsilon},M_\varepsilon)$. We then have

$$(W(M,\phi_L);M_0,\chi(M_0,\phi_L))\cong(W(M_{-\varepsilon},\phi),M_{-\varepsilon},\chi(M_{-\varepsilon},\phi)).$$

So we'll define a diffeomorphism from the RHS to the RHS of the former diffeomorphism. Define $f\longrightarrow[c-\varepsilon^2,c+\varepsilon^2]$ in the former and $f'\longrightarrow[-1,1]$ in the latter. Then take $f\circ k=c+\varepsilon^2 f'$ to match up the domains. Take $(z,t)\in(M_{-\varepsilon}\setminus\phi(\left\{S^{\lambda-1}\times\{0\}\right\}))\times D^1$. Then $k(z,t)$ is the point on the integral curve which passes through z with $f(k(z,t))=c+\varepsilon^2 t$. This map will take flow lines to flow lines. Now define

$$(x,y)\in L_\lambda\longrightarrow h(\varepsilon x,\varepsilon y).$$

■

Exercise Show that this is a well-defined diffeomorphism.

Theorem 5.2 *(Cobordisms Retract Onto Surgery Components?).*
For an elementary cobordism $(W;M_0,M_1)$ and (f,ξ,p) as above, then $M_0\bigcup D_L$ (where $D_L\cong W^s(p)$) is a deformation retraction of W.

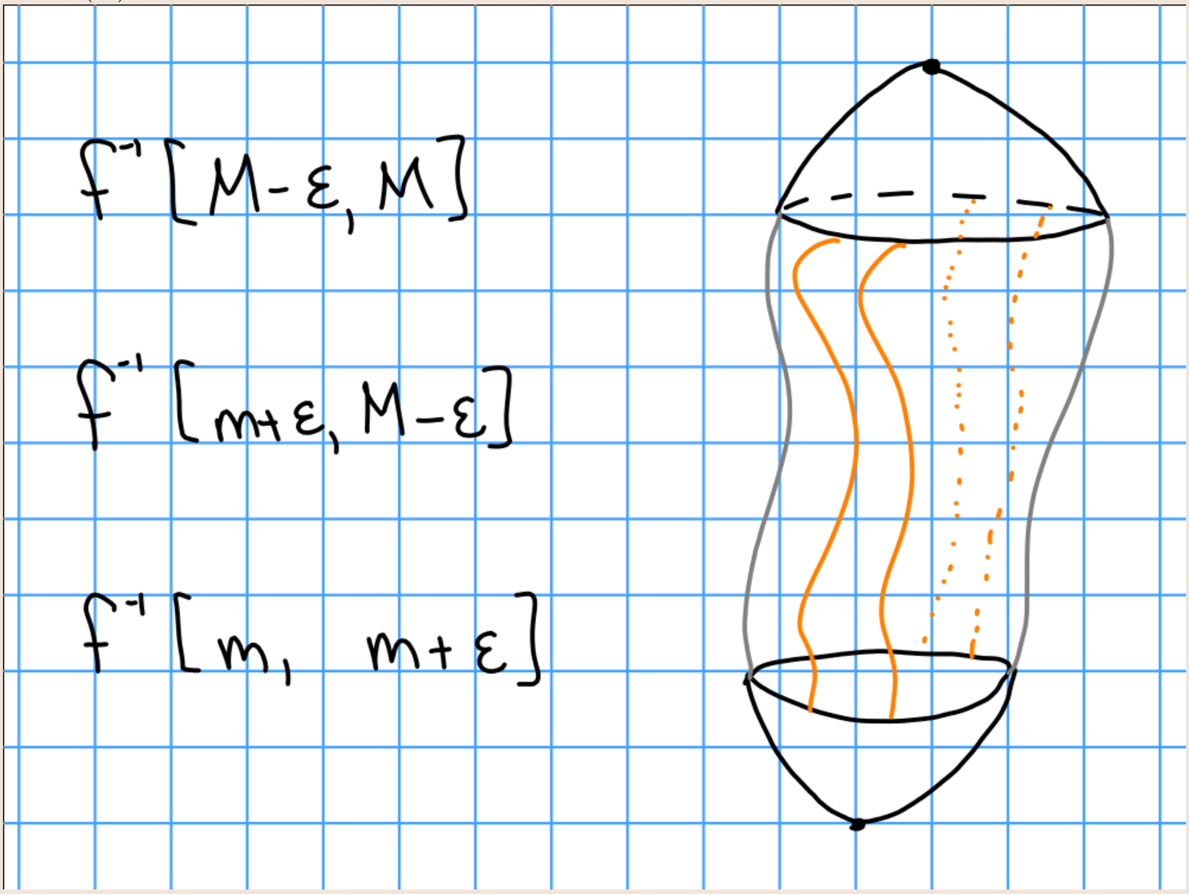
Corollary 5.3.

$$H_i(W,M_0)=H_i(M_0\bigcup D_L,M_0)\overset{\text{excision}}{=}H_i(D_L,S_L)\cong\begin{cases}\mathbb{Z} & i=\lambda\\ 0 & \text{otherwise}\end{cases}.$$

Theorem 5.4 *(Reeb).*
If a closed n-manifold M has a Morse function with exactly 2 critical points, then M is homeomorphic to S^n.

Proof .
If $m=\min\limits_{x\in M}f(x)=f(A)$ and $M=\max\limits_{x\in M}f(x)=f(B)$ where A,B are critical points. Then

points near A are in the unstable manifold, so $\mathrm{Ind}(A) = 0$, and points near B are in the stable, so $\mathrm{Ind}(B) = n$.



The middle piece of the cobordism is a product cobordism, and M is called a twisted sphere. ■

Exercise Every twisted sphere is homeomorphic to S^n.

6 Tuesday January 28th

Setup: Fix an elementary cobordism $(W; M_0, M_1)$, a Morse function $f : W \longrightarrow [-1, 1]$ with exactly one critical point p with index $\mathrm{Ind}_{(} p) = \lambda$. This yields a gradient-like vector field ξ, and $D_L = W^s(p) = \left\{ x \in W \;\middle|\; \lim_{t \longrightarrow \infty} \psi_x(t) = p \right\}$ the stable manifold.

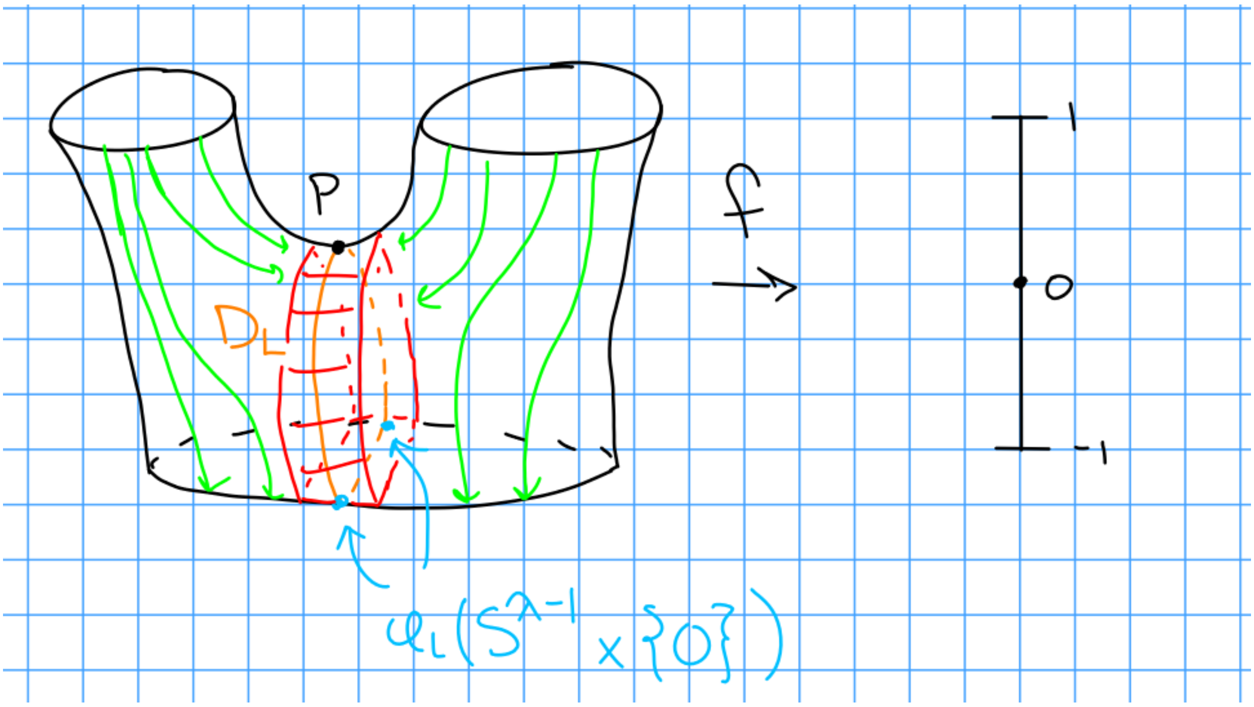
Theorem 6.1 ***(Deformation Retract of Cobordism Onto ??).***
$W \cong M_0 \bigcup D_L$, a λ dimensional disk, is a homotopy equivalence. More precisely, there is a deformation retract.

6.0.1 Proof

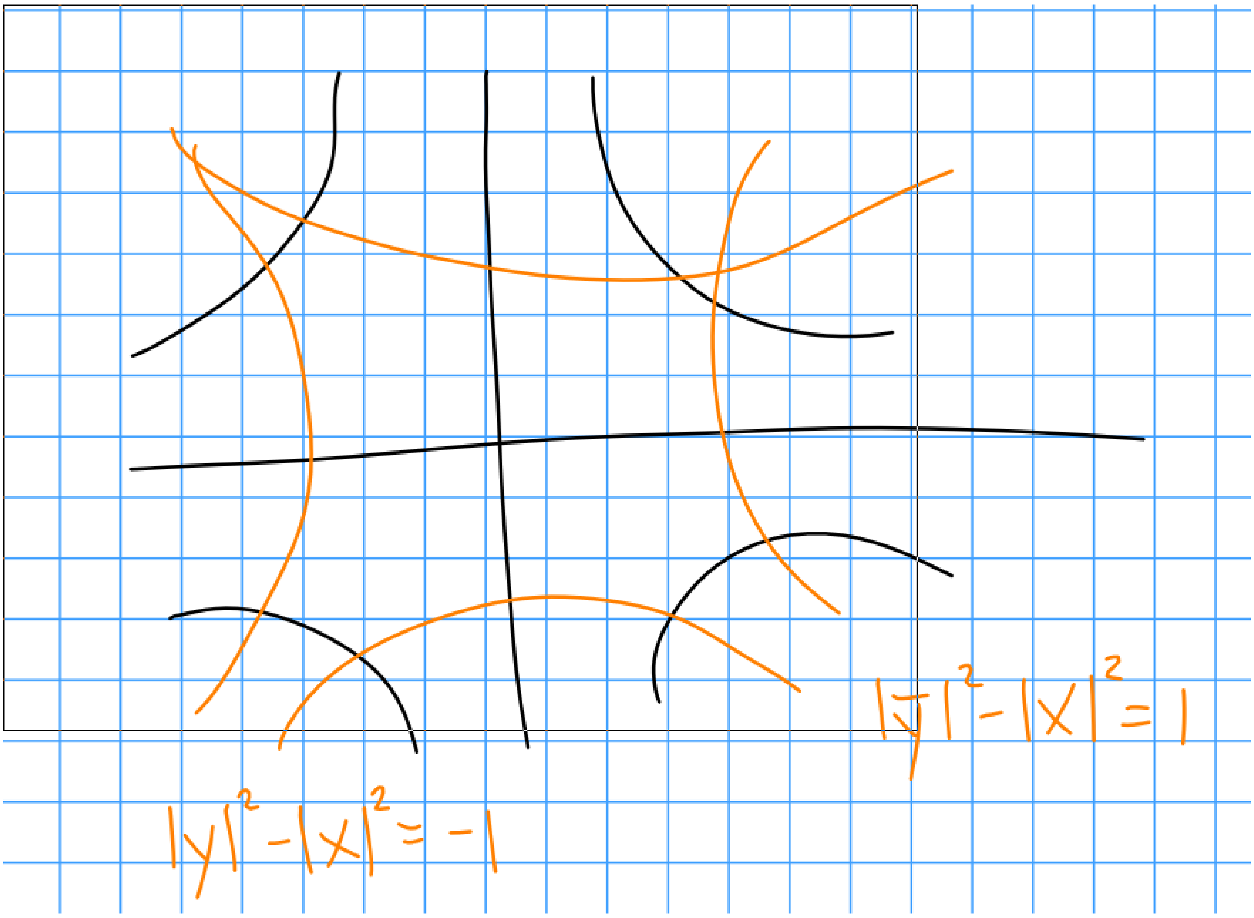
Take the characteristic embedding $\phi_L : S^{\lambda-1} \times OD^{n-\lambda} \hookrightarrow M_0$. We have a cobordism $(W(M_0, \phi_L); M_0, \chi(M_0, \phi_L))$ $(W; M_0, M_1)$.

Recall that the LHS is constructed via $(M_0 \setminus \phi(S^{\lambda-1} \times 0)) \times D_1 \coprod L_\lambda / \sim$.

Retraction 1: $W(M_0, \phi_L) \xrightarrow{r_t} M_0 \bigcup C$. We'll construct this retraction. This follows the green integral curves to retract onto the red.



Identify $D_L = \{(\mathbf{x}, \mathbf{0})\} \subset L_\lambda$ in the local picture:



Define $C = \left\{ (\mathbf{x}, \mathbf{y}) \;\middle|\; \|\mathbf{y}\| \leq \frac{1}{10} \right\}$.

Choose (Z, c) such that $z \in M_0 \setminus \phi_L(S^{\lambda-1} \times OD^{n-\lambda})$ and $c \in [-1, 1]$. Let $r_t(z, c) = (z, c + t(-1-c))$, note what happens at $t = -1, 1, 0$.

We can parameterize the integral curves in the local picture as $(\mathbf{x}/r, r\mathbf{y})$.

So for $(\mathbf{x}, \mathbf{y}) \in L_\lambda$, we can define

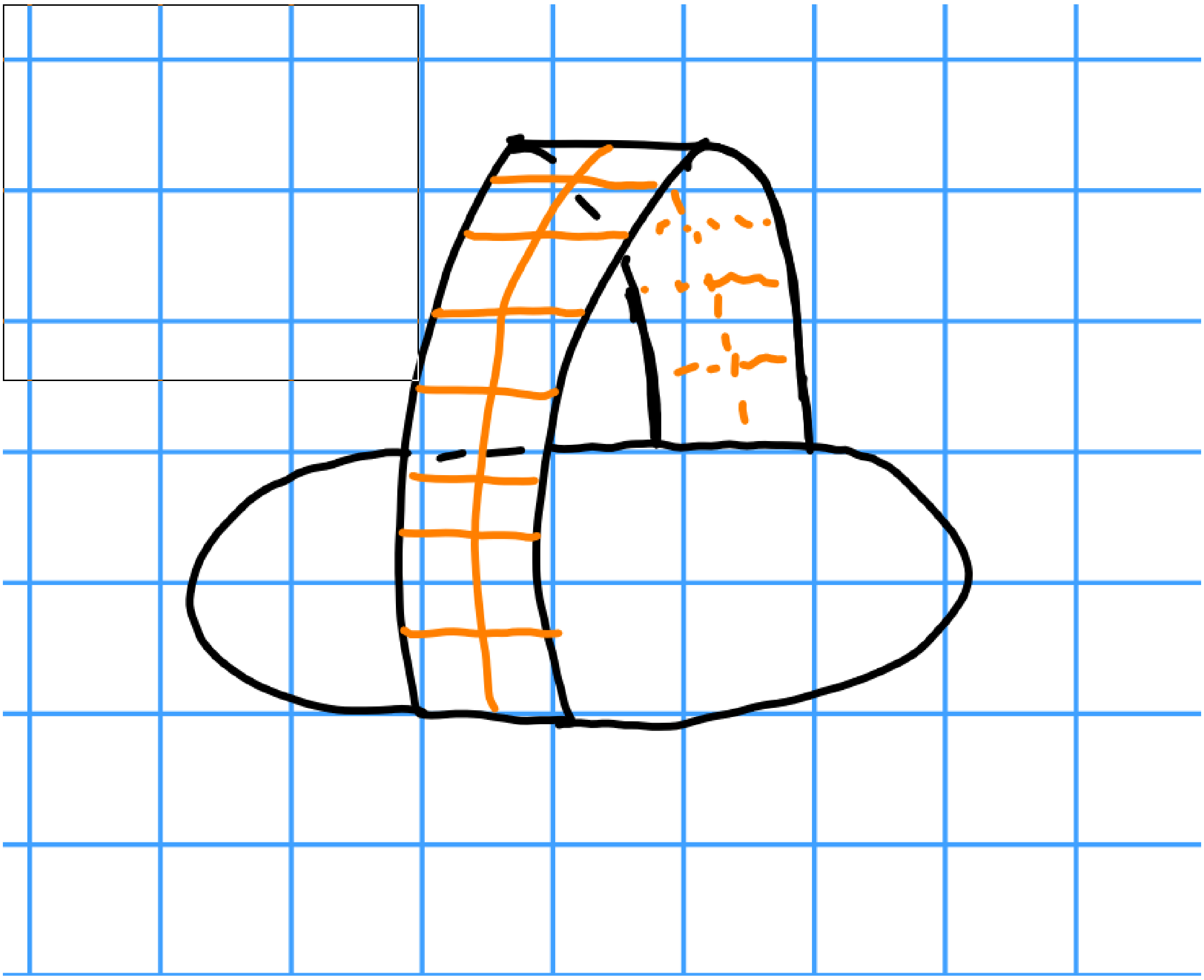
$$r_t(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}, \mathbf{y}) & \|\mathbf{y}\| \leq \frac{1}{10} \iff (\mathbf{x}, \mathbf{y}) \in C \\ ? & ? \\ (\mathbf{x}/\rho(t), \rho(t)\mathbf{y}) & \|\mathbf{y}\| \geq \frac{1}{10} \end{cases}.$$

where $\rho(t)$ is the solution of

$$\rho(t)^2 \|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 / \rho(t)^2 = \left(\|\mathbf{y}\|^2 - \|\mathbf{x}\|^2 \right)(1 - t) - t$$
$$\rho(t)\|\mathbf{y}\|^2 \geq \frac{1}{10} \implies \rho(t) \geq \frac{1}{10\|\mathbf{y}\|}.$$

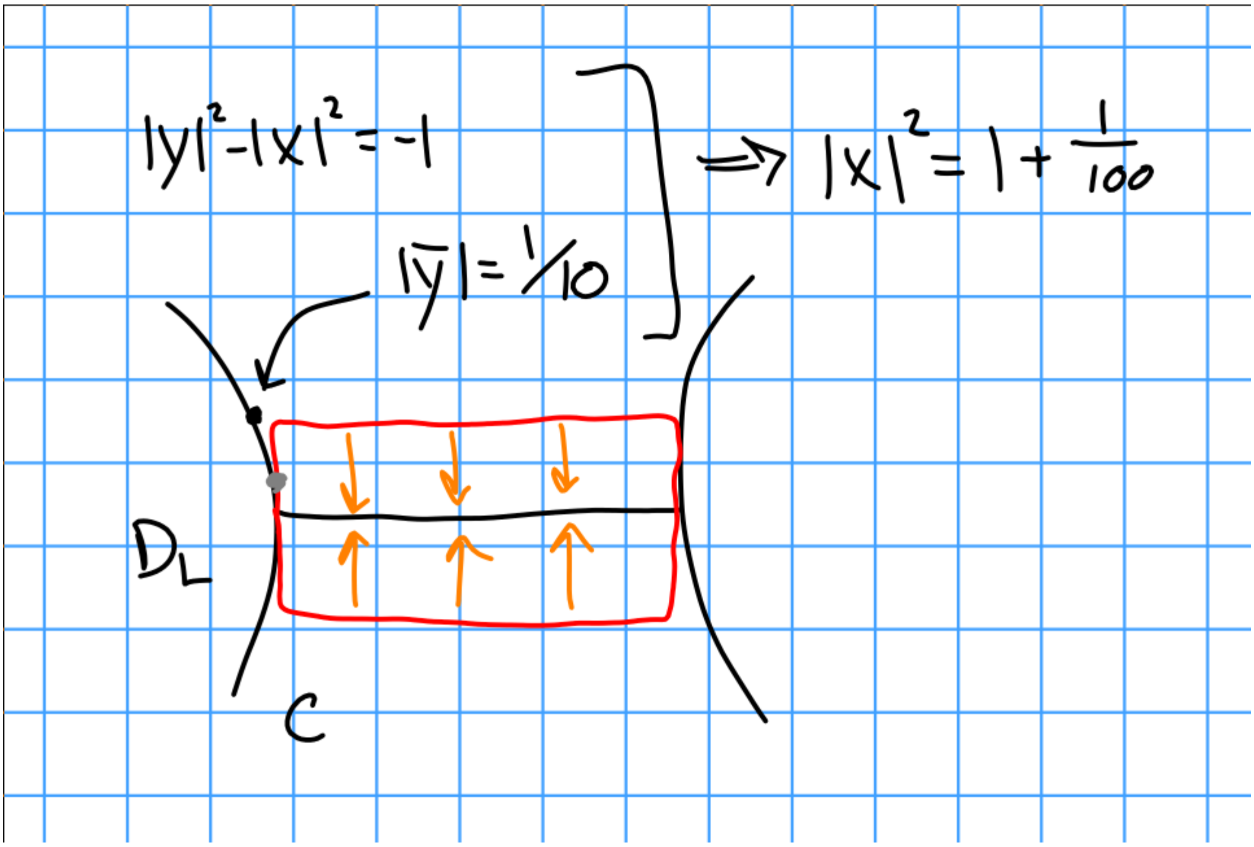
So we define $\rho(t) = \max(\text{positive solutions for the above equation}, \frac{1}{10\|\mathbf{y}\|})$.

Retraction 2: $M_0 \bigcup C \xrightarrow{r'_t} M_0 \bigcup D_L$



We want the restriction of r'_t to $M_0 \setminus C$ to be the identity, so for $(\mathbf{x}, \mathbf{y}) \in C$ we define

$$r'_t(\mathbf{x}, \mathbf{y}) = \begin{cases} (\mathbf{x}, (1-t)\mathbf{y}) & \|\mathbf{x}\| \leq 1 \\ (\mathbf{x}, \alpha(t)\mathbf{y}) & 1 \leq \|\mathbf{x}\| \leq \sqrt{1 + \frac{1}{100}} \end{cases}.$$



We define $\alpha(t)$ at $t=0$ to be the identity, and at $t=1$ we want $\|\alpha(t)\mathbf{y}\|^2-\|x\|^2=-1$, and solving yields

$$\alpha(t)=(1-t)+t\frac{\sqrt{\|x\|^2-1}}{\|\mathbf{y}\|}.$$

■

Corollary 6.2.
For M a closed smooth n-manifold with a Morse function $f:M\longrightarrow\mathbb{R}$, M is homotopy-equivalent to a CW complex with one λ-cell for each critical point of index λ.

Proof: See Milnor's book (Morse Theory).

6.1 Morse Inequalities

Let M be a closed smooth manifold and $f:M\longrightarrow\mathbb{R}$ Morse, and fix F a field. Let $b_i(M)$ denote the ith Betti number, which is rank $H_i(M;F)$.

Weak Morse Inequalities:

1. $b_i(M)\leq$ the number of index i critical points, denoted $c_i(M)$.

2. $\xi(M) = \sum(-1)^i c_i(M)$.

Strong Morse Inequalities: $b_i(M) - b_{i-1}(M) + \cdots \pm b_0(M) \leq c_i - c_{i-1} + c_{i-2} \cdots \pm c_0$.

Lemma 6.3.
The weak inequalities are consequences of the strong ones.

Proof (implying (1)).
We have $b_i - \cdots \leq c_i - \cdots$ and separately $b_{i-1} - \cdots \leq c_{i-1} \cdot$, and adding these inequalities yields $b_i \leq c_i$. ∎

Proof (implying (2)).
To obtain the equality, multiply through by a negative sign. For $i > n$, we have $b_{i-1} - b_{i-2} + \cdots = c_{i-1} - c_{i-2} + \cdots$, where the LHS is $\pm\chi(M)$ and the RHS has matching signs. ∎

6.1.1 Proof of Strong Morse Inequalities

Suppose $f(p_1) < \cdots < f(p_k)$. We can select points a_i such that $a_0 < f(p_1) < a_1 < \cdots$. Let $M_i = f^{-1}[a_0, a_i]$; we then have $\emptyset := M_0 \subset M_1 \subset \cdots M_k = M$.

Using excision, we have

$$\begin{aligned} H_j(M_i, M_{i-1}) &= H_j(f^{-1}[a_{i-1}, a_i], f^{-1}(a_{i-1})) \\ &= \begin{cases} \mathbb{F} & j = \operatorname{Ind}(p_i) = \lambda_i \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

So $b_j(M_i, M_{i-1}) = 1$ iff $j = \lambda_i$, and 0 otherwise.

Lemma 6.4 ***(Sublemma).***
Define $S_i := b_i(X, Y) - b_{i-1}(X, Y) + \cdots$, i.e. the LHS of the strong Morse inequality. Then S_i is *subadditive*, i.e. if $X \supset Y \supset Z$ then $S_i(X, Z) \leq S_i(X, Y) + S_i(Y, Z)$.

This implies the strong inequality, since

$$S_i(M, \emptyset) \leq S_i(M_k, M_{k-1}) + S_i(M_{k-1}, M_{k-2}) + \cdots + S_i(M_1, M_0).$$

The RHS here equals $\sum_{j=1}^{k} T_i(M_j, M_{j-1}) = T_i(M)$, where $T_i(M) = c_i - c_{i-1} + \cdots$.

Write down the relative homology exact sequence:

$$\begin{tikzcd}
H_{i+1}(X,Y) \arrow[r, "\partial_i"] & H_i(Y,Z) \arrow[r, "f_i"] & H_i(X,Z) \arrow[r, "g_i"] & H_i(X,Y) \arrow[dll, "\partial_{i-1}"] \\
& H_{i-1}(Y,Z) & &
\end{tikzcd}$$

then

$\text{rank}(\partial_i) = \dim\ker(f_i) = b_i(Y,Z) - \text{rank}(f_i) = b_i(Y,Z) - b_i(X,Z) + \text{rank}(g_i) = \cdots = S_i(Y,Z) - S_i(X,Z) + S_i$

since ranks are positive.

7 Thursday January 30th

7.1 Morse Inequality Example

Example: Consider $f : \mathbb{CP}^n \longrightarrow \mathbb{R}$ where (recall) $\mathbb{CP}^n = S^{2n+1} \subset \mathbb{C}^{n+1}/\sim$ where $\mathbf{z} \sim \lambda\mathbf{z}$ for all $|\lambda| = 1$ in $\mathbb{C}^\times$, where f is given by $[z_0 : \cdots : z_n] \mapsto \sum i|z_i|^2$.

> Note that we can take the coefficients to be any $n+1$ distinct real numbers, here we just take $1, 2, \cdots, n+1$ for simplicity.

Cover $\mathbb{CP}^n$ with $n+1$ coordinate charts (U_j, h_j) where $U_j = \left\{\mathbf{z} \,\middle|\, z_j \neq 0\right\}$ and $h_j : U_j \longrightarrow \mathbb{R}^{2n}$ is given by first defining $[z_0 : \cdots : z_n] \mapsto (\widehat{z}_j z_0, \widehat{z}_j z_1, \cdots, \widehat{z}_j z_n)$ where $\widehat{z}_j = z_j/|z_j|$.

Denote the image coordinates by $z_k = x_k + iy_k$. Then define h_j by $[z_0 : \cdots : z_n] \xrightarrow{h_j} (x_0, y_0, \cdots, x_{j-1}, y_{j-1}, x_{j+1}, y_j$

Note that $|z_j| = 1 - \sum_{i \neq j} |z_i|^2$, so this is a one-to-one correspondence (i.e. we can recover the magnitude of z_j from the image point).

So what is f in these coordinates? We can write

$$\begin{aligned}
f \circ h_j^{-1}(x_0, y_0, \cdots, \widehat{x}_j, \widehat{y}_j, \cdots x_n, y_n) &= \sum i \neq j i(x_i^2 + y_i^2) + j|z_j|^2 \\
&= \sum_{i \neq j} i(x_i^2 + y_i^2) + j - \sum_{i \neq j} j(x_i^2 + y_i^2) \\
&= j + \sum_{i \neq j} (i - j)(x_i^2 + y_i^2) \\
&= j + (-j)(x_0^2 + y_0^2) + (-j+1)(x_1^2 + y_1^2) + \cdots + (n-j)(x_n^2 + y_n^2).
\end{aligned}$$

)

So what are the critical points? The derivative is zero iff $x_i = y_i = 0$ for some $i \neq j$. So there is only one critical point, $p_j = [0 : 0 : \cdots : \cdots 1_j : \cdots : 0]$. Thus there are $n+1$ critical points given by $\text{crit}(f) = \{p_0, \cdots, p_n\}$. Using the above equations, we can find that $\text{Ind}_f(p_j) = 2j$ (count positive and negative terms).

Note that we had the inequality $b_i(M) \leq |\{\text{critical points with index } i\}|$. Noting that $H^i(\mathbb{CP}^n; \mathbb{F}) = \mathbb{F}$ for $i = 0, 2, 4, \cdots, 2n$ and 0 otherwise, so we have exact equality here.

Note that there are $\mathbb{Q}HS$ where the inequality has to be strict, but equality can be obtained with $\mathbb{CP}^n, S^n$, etc.

7.2 Rearrangement

Fix a Morse function $f : W \longrightarrow [0,1]$, with $p, q \in \text{crit}(f)$ and $f(p) < f(q)$. Can we change f to a new Morse function g such that $\text{crit}(g) = \text{crit}(f)$ and $g(p) > g(q)$, where $g = f + \text{const.}$ in a neighborhood of p and a neighborhood of q?

Note that we obtain elementary cobordisms in each case:

Pick ξ a gradient-like vector field for f, which decomposes $W^*(p) = W^s(p) \bigcup W^u(p)$.

Lemma 7.1.
Let $f : W \longrightarrow I$ be a Morse function with 2 critical points p, q and ξ a gradient-like vector field for f such that $W^*(p) \bigcap W^*(q) = \emptyset$. Then for any two points $a, b \in I$, there exists a Morse function g such that:

1. ξ is gradient-like for g,
2. $\text{crit}(g) = \text{crit}(f)$, with $g(p) = a$ and $g(q) = b$.
3. $g = f$ near M_0 and M_1, and $g = f + \text{const.}$ in some neighborhood of p, and some neighborhood of q.

So this is stronger: we can modify our Morse function to take on any two real numbers.

Idea of proof: We want the two purple sections here, since we want to modify p and q separately:

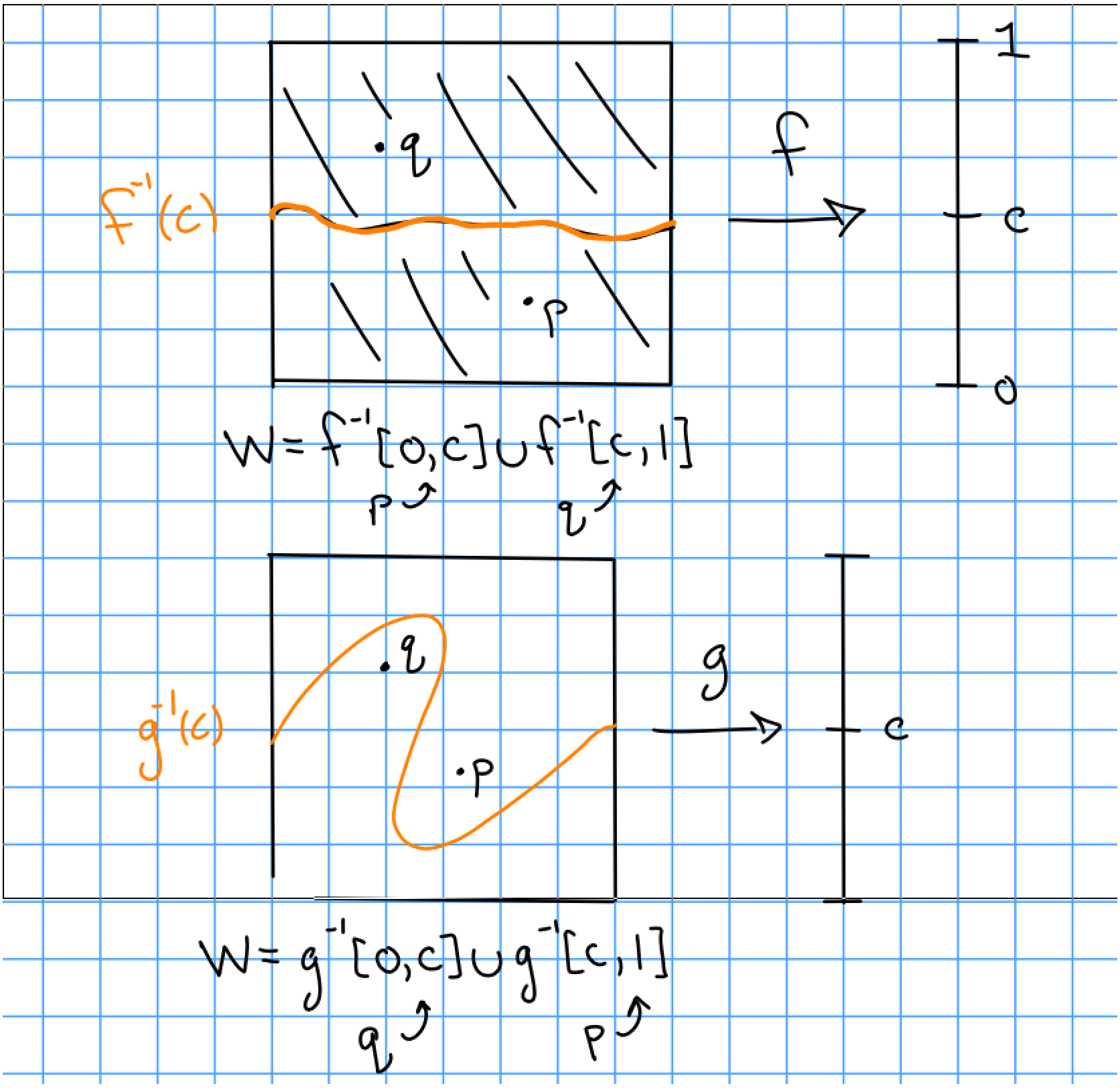


Figure 1: Image

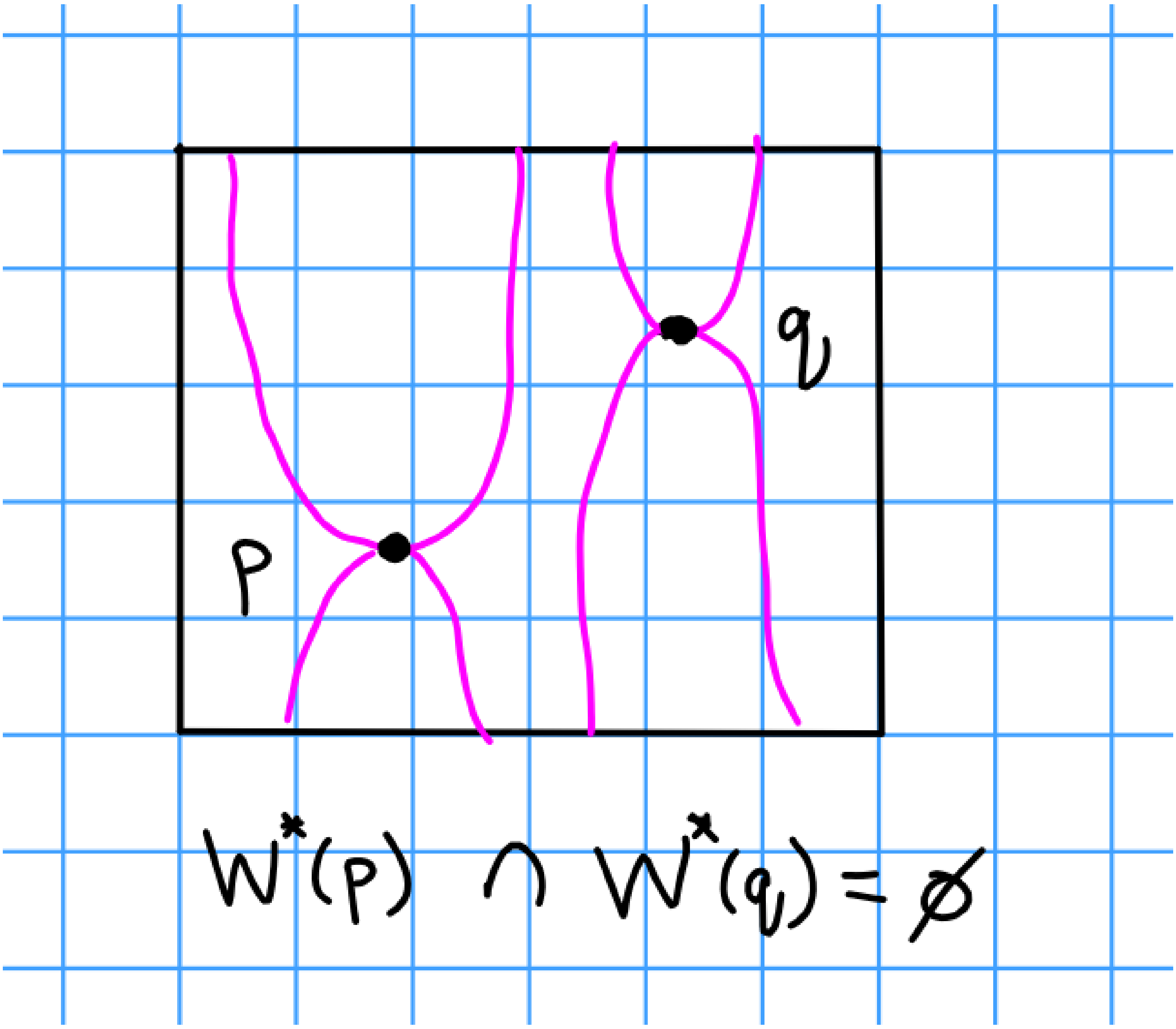
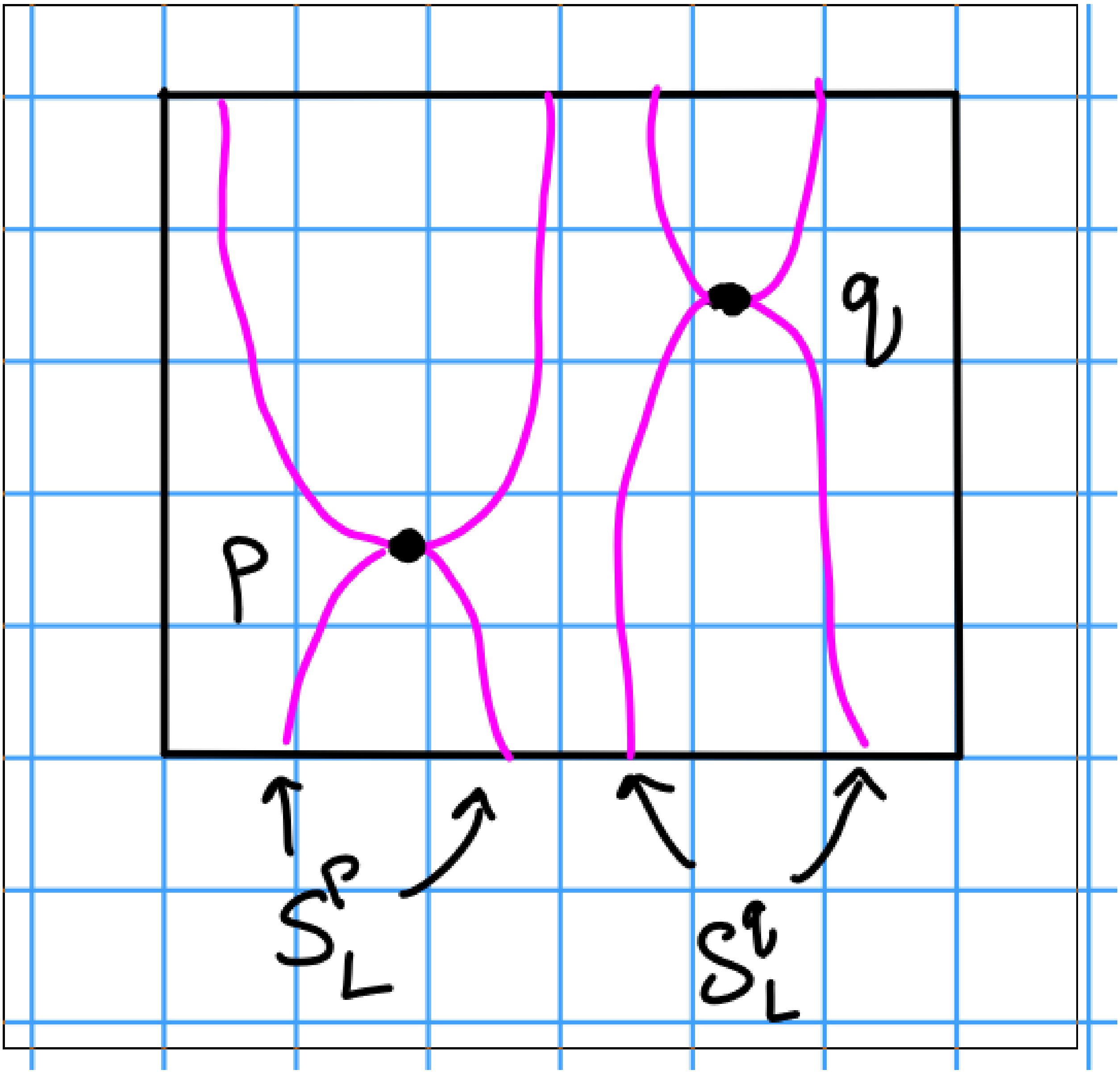


Figure 2: Image

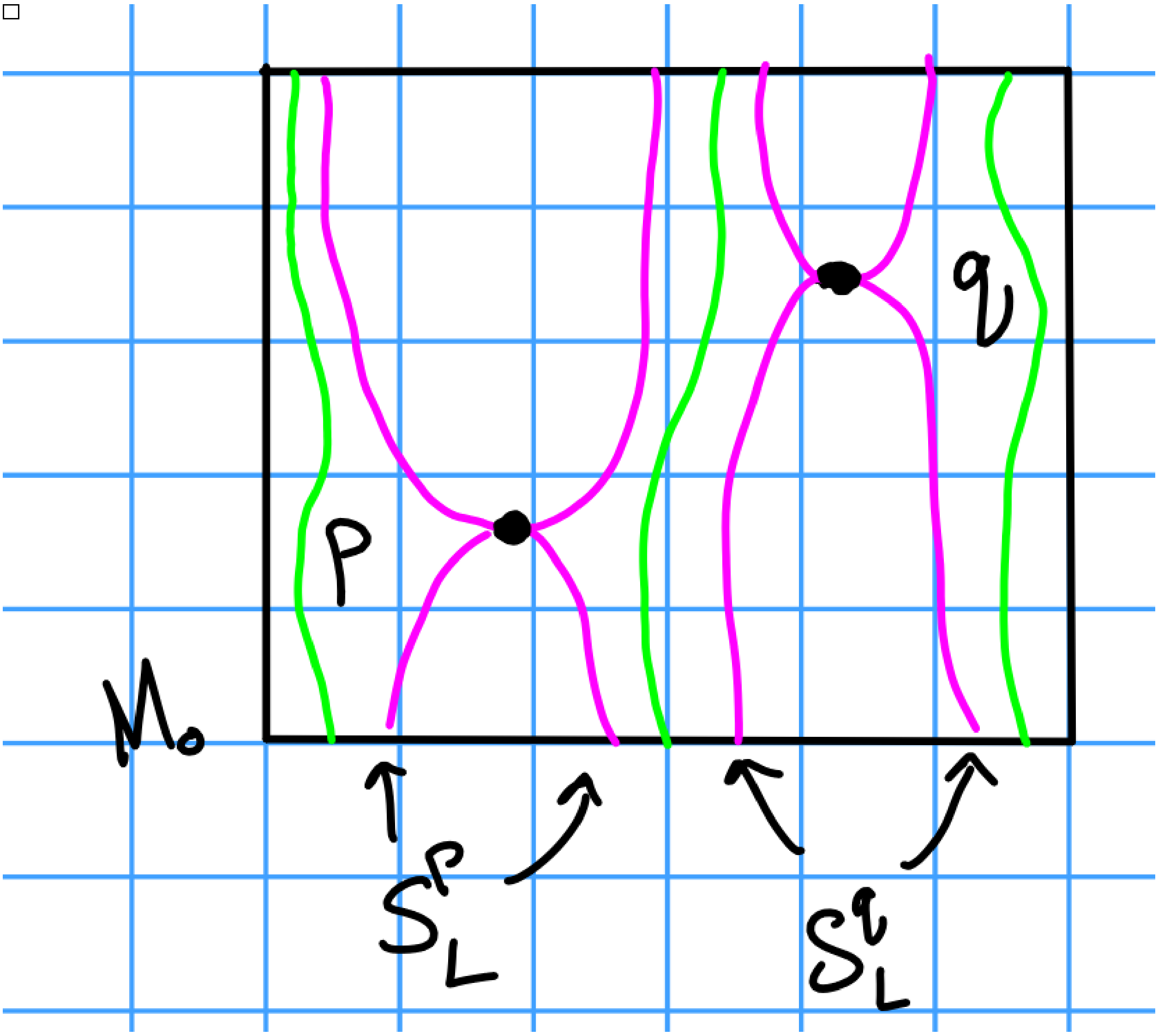


7.2.1 Proof of Lemma

We can find a $\mu : M_0 \longrightarrow I$ such that $\mu \equiv 0$ near S_L^P and $\mu \equiv 1$ near S_L^q. So extend μ to $\overline{\mu} : W \longrightarrow I$ such that $\overline{\mu}$ is constant over the integral curves of ξ and $\overline{\mu} \equiv 0$ near $W^*(p)$ and $\mu \equiv 1$ near $W^*(q)$.

Here the integral curves are green:

□

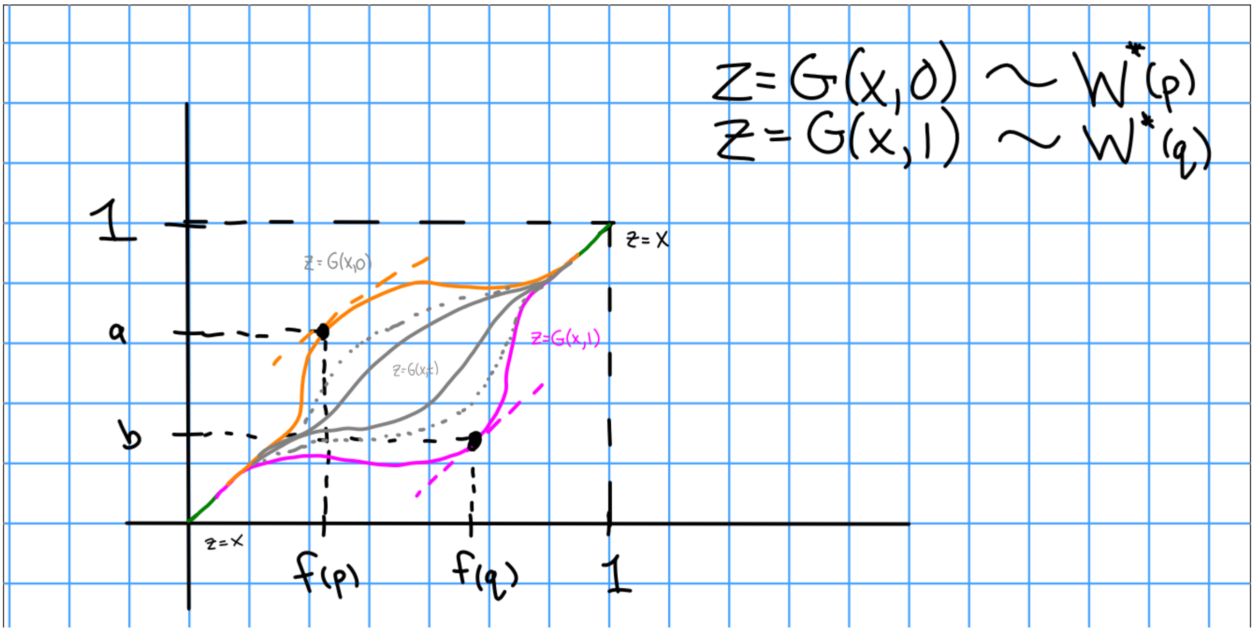


Let $g(z) = G(f(z), \overline{\mu}(z))$ where $G : I \times I \longrightarrow I$ will be defined as follows:

1. Fix a $y \in W$ to $\overline{\mu}$ is constant, then $G(\,\cdot\,, y) : I \longrightarrow I$ is increasing (since f is increasing) and surjective, i.e. $\dfrac{\partial G}{\partial x} > 0$ everywhere.
2. $G(x, y) = x$ whenever x is near 0 or 1.
3. $\dfrac{\partial G}{\partial x}(x, 0) = 1$ for x near $f(p)$ and $\dfrac{\partial g}{\partial x}(x, 1) = 1$ for x near $f(q)$.

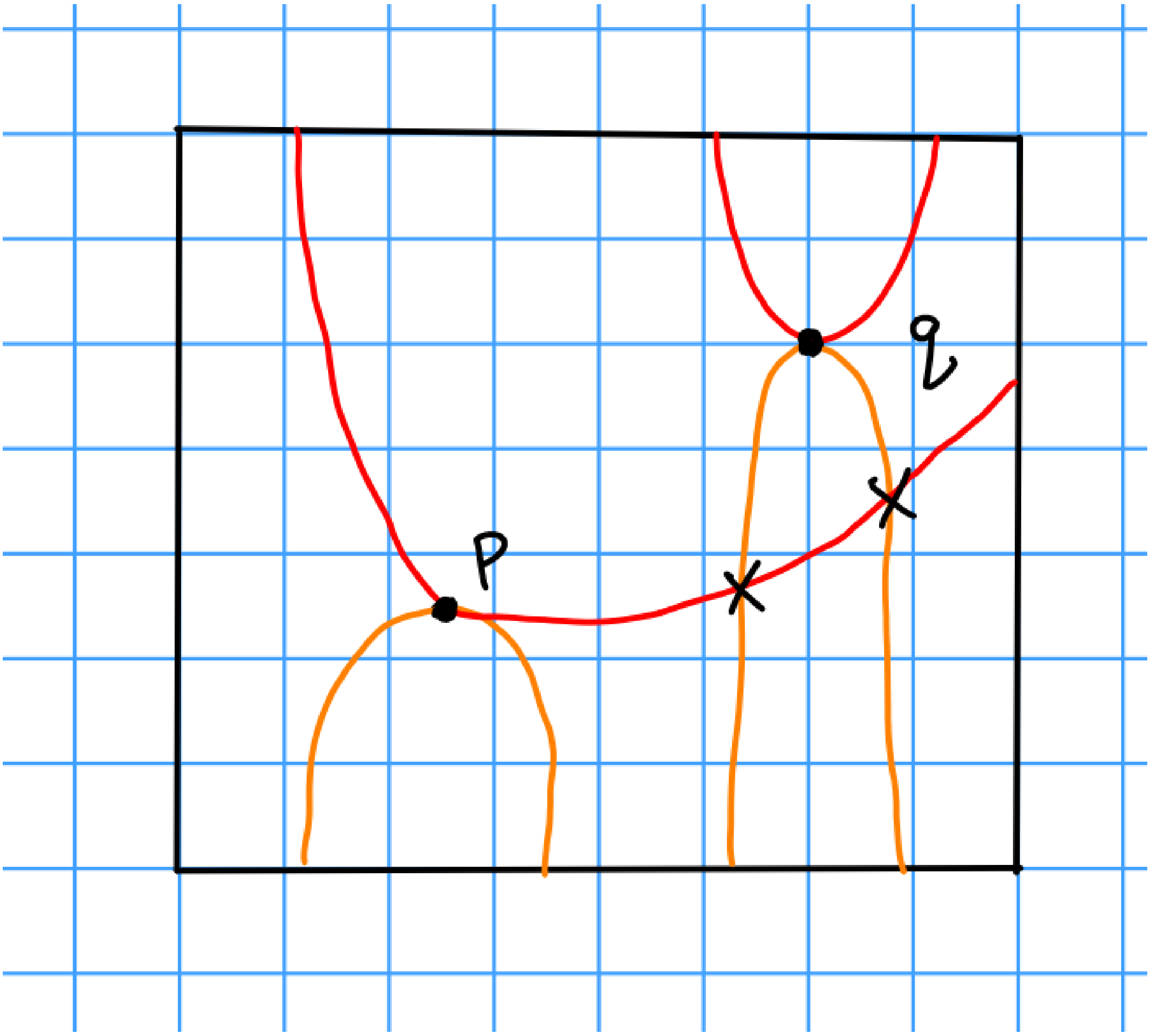
Note that $a = g(p)G(f(p), 0)$ and $b = g(q) = G(f(q), 1)$, and the slope should be constant near a, b.

We get something like the following graphs:



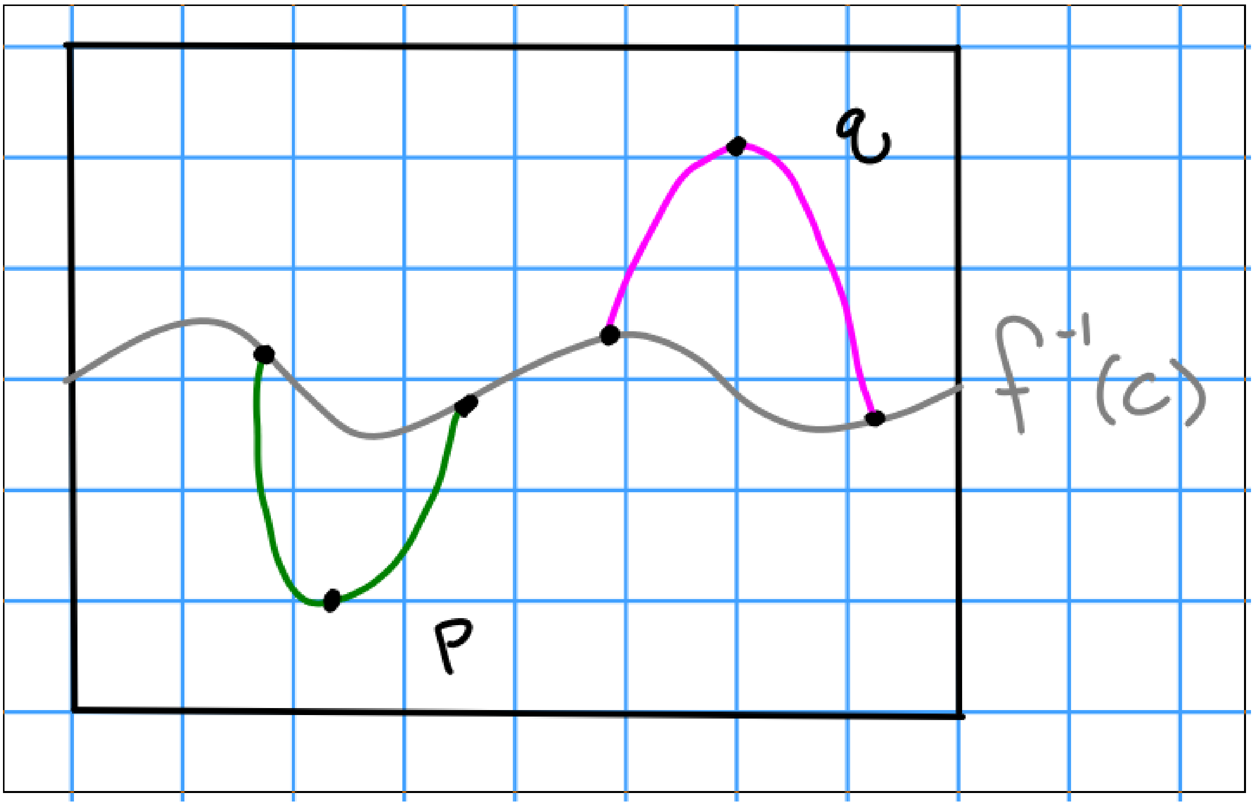
■

When can such a function exist? I.e. is this a relatively strong condition? If $f(p) < f(q)$, it is possible that $W^u(p) \bigcap W^s(q) \neq \emptyset$:



Theorem 7.2 ***(Modifying a Vector Field to Separate W's).***
If $f(p) < f(q)$ and $\text{Ind}(p) \geq \text{Ind}(q)$, then it is possible to change ξ in a neighborhood of $f^{-1}(x)$ for some $f(p) < c < f(q)$ such that $W_p^u \bigcap W_q^s = \emptyset$.

Main Idea:



Note that $W_p^u \bigcap W_q^s = \emptyset$ iff $S_R^c(p) \bigcap S_L^c(q) = \emptyset$, where $S_R^C(p) = W_p^u \bigcap f^{-1}(c)$ and $S_L^c(q) = W_q^s \bigcap f^{-1}(c)$.

We have the implication

$$\begin{cases} \dim W = n \\ \operatorname{Ind}(p) = \lambda \\ \operatorname{Ind}(q) = \lambda' \end{cases} \implies \begin{cases} \dim f^{-1}(c) = n-1 \\ \dim S_R^c(p) = n - \lambda - 1 \\ \dim S_L^c(q) = \lambda' - 1 \end{cases}.$$

and thus

$$\dim S_R^c(p) + \dim S_L^c(q) = n - \lambda - \lambda' 2 < n - 1 = \dim f^{-1}(c).$$

Lemma 7.3 ***(1).***
If $M^m, N^n \subset V^v$ are smooth submanifolds and $m + n < v$ then there exists a diffeomorphism $h : V \longrightarrow V$ which is isotopic to the identity such that $h(M) \bigcap N = \emptyset$.

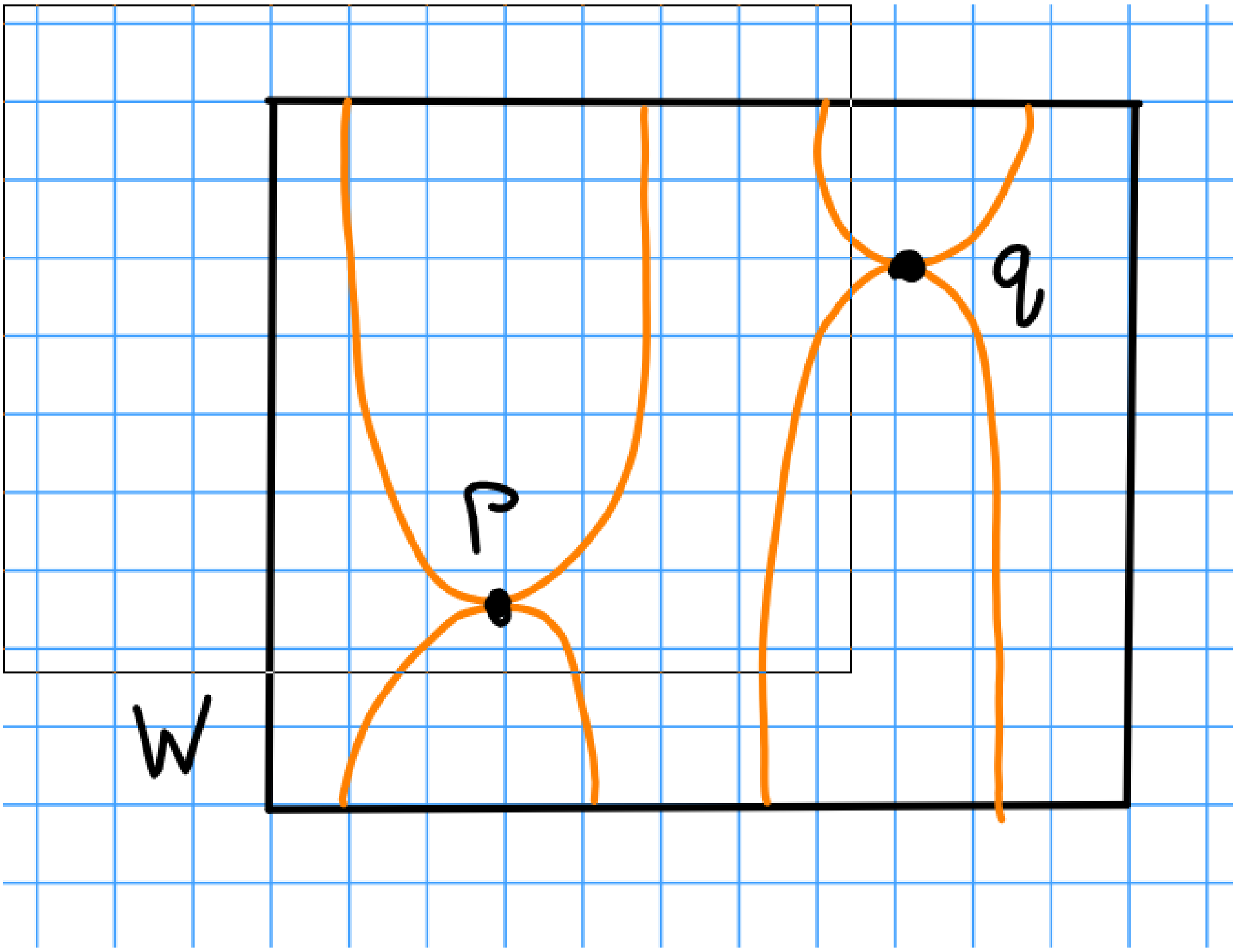
Idea: We'll use this new diffeomorphism to modify the vector field ξ to make things disjoint.

8 Tuesday February 4th

8.1 Modifying Vector Fields

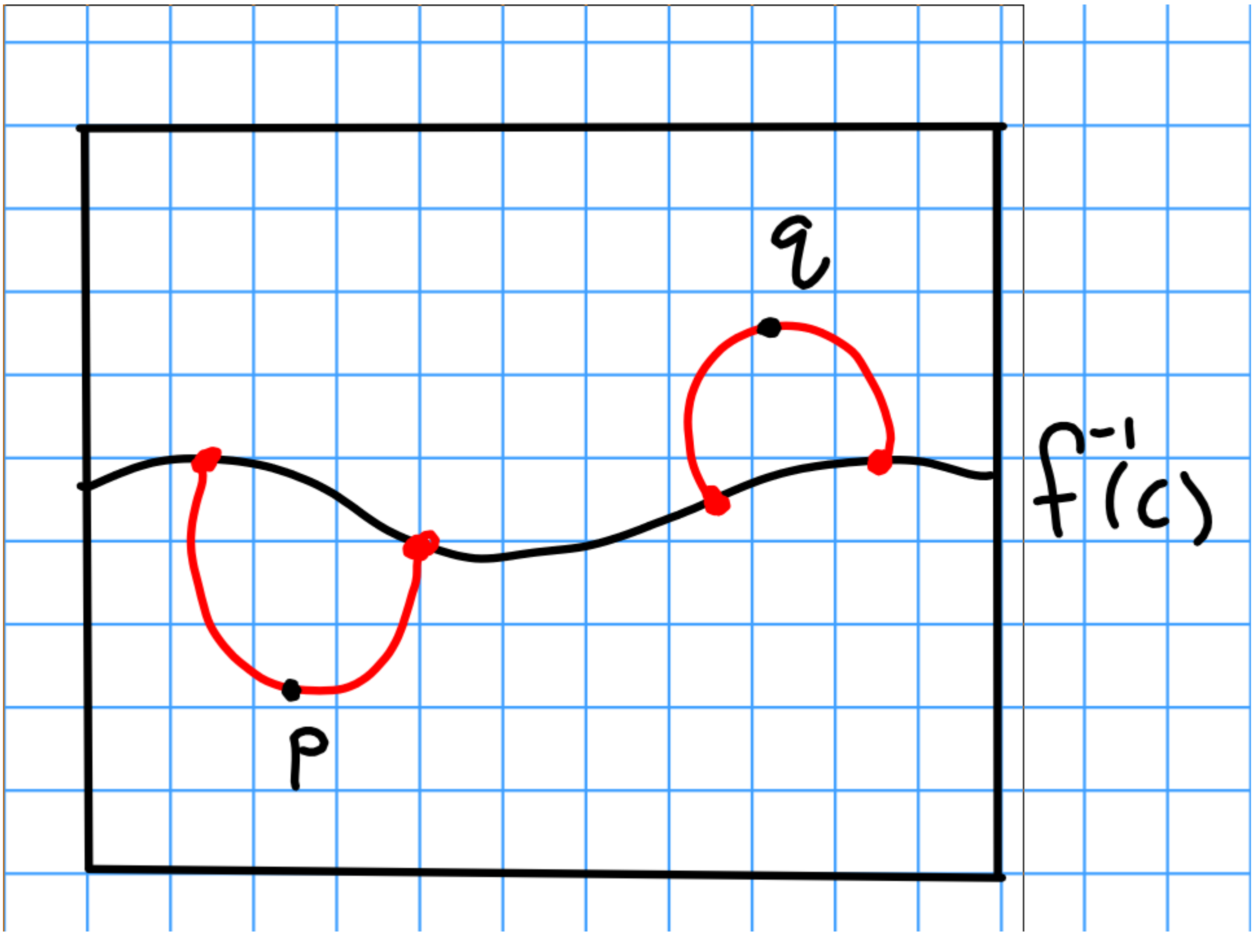
Recall: Let $f : W \longrightarrow I$ be Morse, $\text{crit}(F) = \{p, q\}$ where $f(p) < f(q)$, and ξ a gradient-like vector field for f.

Theorem 8.1 ***(When Critical Points of Morse Functions Can Take On Any Value).*** If $W^*(p) \bigcap W^*(q) = \emptyset$ then for any $a, b \in (0, 1)$ we can change f "nicely" to a new Morse function g such that $g(p) = a$ and $g(q) = b$.



Note that these are disjoint iff $W^u(p) \bigcap W^s(q) = \emptyset$ iff $S^c_R(p) \bigcap S^c_L(q) = \emptyset$. If $\in (p) \geq \in (q)$ then

$$\dim S^c_R(p) = \dim S^c_L(q) < n - 1 = \dim f^{-1}(c).$$



Lemma 8.2 ***(1, Small Submanifolds Are Disjoint Up to Isotopy).***
For $M^m, N^n \subset V^v$ submanifolds with $m + n < v$, there exists a diffeomorphism $h : V \longrightarrow V$ smoothly isotopic to id_V such that $h(M) \bigcap N = \emptyset$.

I.e. low enough dimension submanifolds can smoothly be made disjoint.

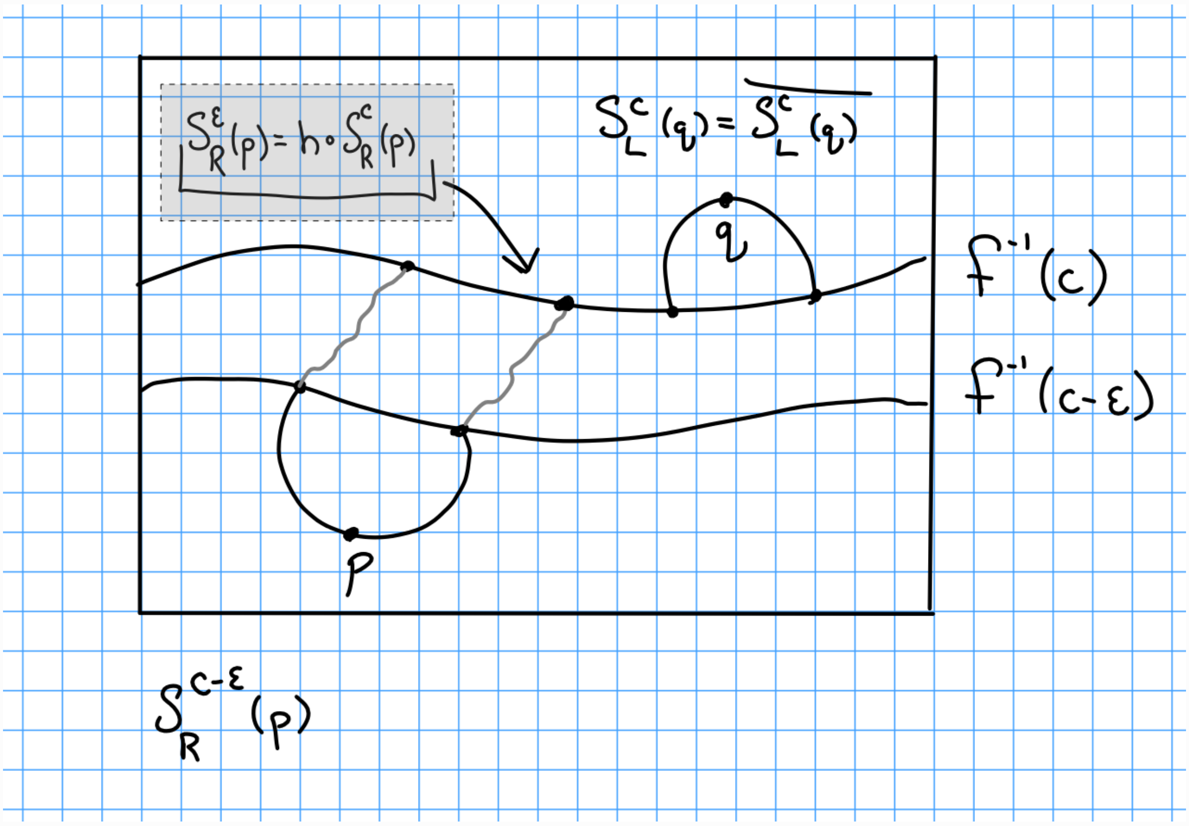
Lemma 8.3 ***(2, ??).***
Let $f : W \longrightarrow I$ be Morse with gradient-like vector field ξ and regular value $x \in (0, 1)$. Let $h : f^{-1}(c) \longrightarrow f^{-1}(c)$ be smoothly isotopic to the identity, and define $M := f^{-1}(c)$. Then we can change ξ over $f^{-1}[c - \varepsilon, c]\lambda$ to a new gradient-like vector field $\overline{\xi}$ such that if we let

$$\Phi : f^{-1}(c - \varepsilon) \longrightarrow M$$

be the flow induced by ξ and

$$\overline{\Phi} : f^{-1}(c - \varepsilon) \longrightarrow M$$

be induced by $\overline{\xi}$.



Note that the left/right spheres are defined in terms of gradient-like vector fields, so "bar" here refers to a new gradient-like vector field.

Then picking h from lemma 1, we can arrange so that $\overline{S}_L^c(q) \bigcap \overline{S}_R^c(p) = \emptyset$.
Then $\overline{\xi} = h \circ \Phi$.

8.1.1 Proof of Lemma 2

We have

$$[c-\varepsilon, c] \times M \xrightarrow{\phi} f^{-1}[c-\varepsilon, c] \xrightarrow{f} [c-\varepsilon, c],$$

which we can factor by projection onto the first component. This satisfies the following properties:

1. $\phi_*(\frac{\partial}{\partial t}) = \widehat{\xi} := \frac{1}{\xi(f)}\xi$
2. $\phi|_{\{x\}\times M} = \mathrm{id}$

Note: the product cobordism $[c-\varepsilon, c] \times M$ is easier to work with here, we can then push it to $f^{-1}[c-\varepsilon, c]$ via ϕ.

We now define h_t by the properties

- For t near $c-\varepsilon$, $h_t = \mathrm{id}$, and
- For t near x, $h_t = h$.

We use this to construct a diffeomorphism

$$\begin{aligned} H : [c-\varepsilon, c] \times M &\longrightarrow [c-\varepsilon, c] \times M \\ (t,x) &\mapsto (t, h_t(x)). \end{aligned}$$

Both the domain and codomain map via ϕ to $f^{-1}[c-\varepsilon, c]$, so we can consider

$$H_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial H}{\partial t}(t,x) = \left(1, \frac{\partial h_t}{\partial t}(x)\right) \quad \text{for } t \text{ near } c-\varepsilon \text{ and } c.$$

We then have $\xi' = (\phi \circ H \circ \phi^{-1})_* \widehat{\xi} = (\phi \circ H)_* \left(\frac{\partial}{\partial t}\right)$. Thus for t near $c-\varepsilon$ and c, we have $\xi' = \phi_*\left(\frac{\partial}{\partial t}\right) = \widehat{\xi}$. So define

$$\overline{\xi} = \begin{cases} \xi(f) \cdot \xi' & \text{on } f^{-1}[c-\varepsilon, c] \\ \xi & \text{everywhere else} \end{cases}.$$

On $[c-\varepsilon, c] \times M$, what are the integral curves of $H_*(\frac{\partial}{\partial t})$? Picking a $t \in [c-\varepsilon, c]$, we have $H(t,x) = (t, h_t(x))$ by definition, and thus the integral curves of $\widehat{\xi}$ are given by $\phi(t, h_t(x))$ for all $x \in M$. Then $\phi(c-\varepsilon, x) = \phi(c-\varepsilon, h_{c-\varepsilon}(x))$ for $t = c-\varepsilon$, which is just $\Phi^{-1}(x)$. Then for $t = c$ we get $\phi(c, h(x)) = h(x)$. Thus $\overline{\Phi} \circ \Phi^{-1}(x) = h(x)$, yielding $\overline{\Phi} = h \circ \Phi$.

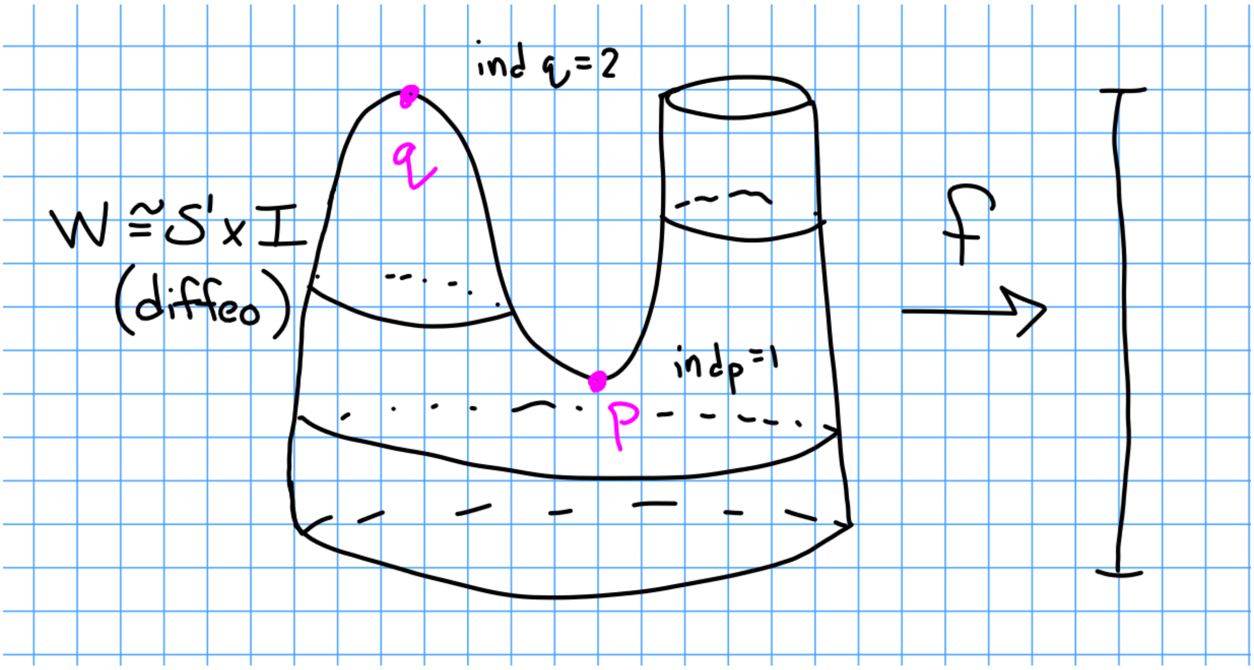
Corollary 8.4.
Given any Morse function f on an n-dimensional cobordism $(W^n; M_0, M_1)$ we can get a new Morse function g such that

- $\mathrm{crit}(g) = \mathrm{crit}(f)$,
- $g(p) = \mathrm{Ind}(p)$ (since we can make the critical points take on any values)
- $g^{-1}(-1/2) = M_0$ and $g^{-1}(n+1/2) = M_1$

Such a Morse function is called *self-indexing*.

8.2 Cancellation

Note that we may have "extraneous" critical points:



Here note that W is diffeomorphic to a product cobordism.

Let $f : W \longrightarrow [0, 1]$ be Morse, $\text{crit}(f) = \{p, q\}$, $\text{Ind}(p) = \lambda$ and $\in (q) = \lambda + 1$ with $f(p) < f(q)$.

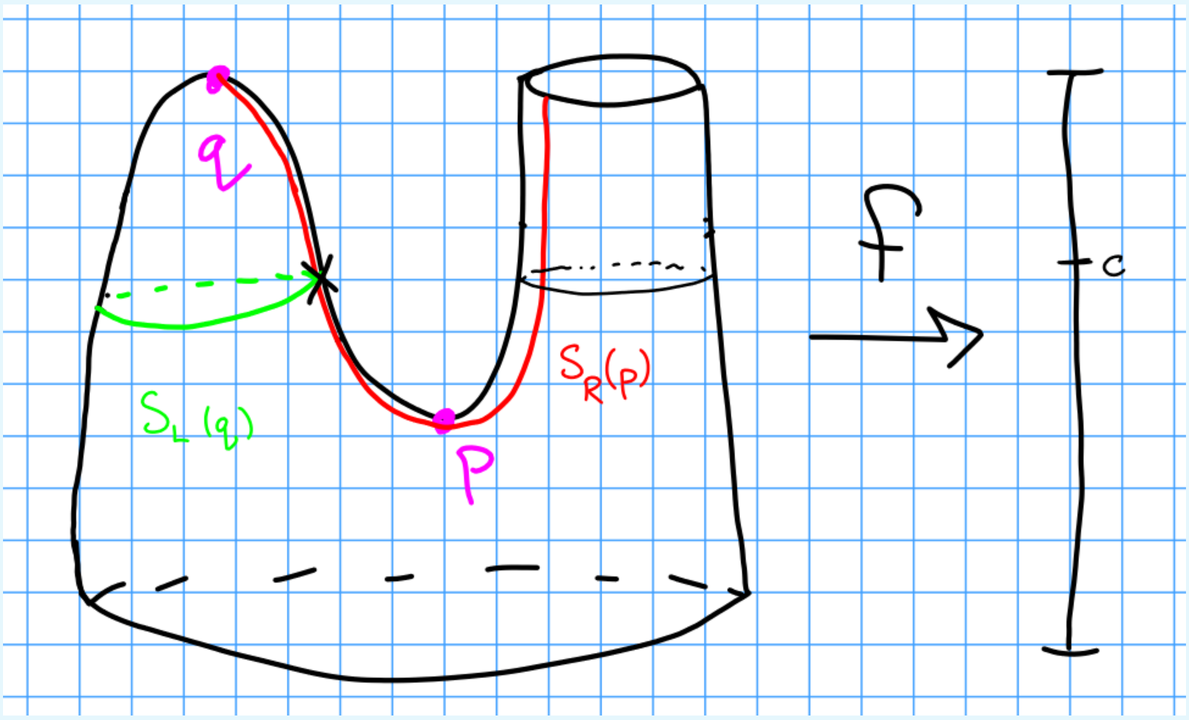
Pick $c \in [f(p), f(q)]$, then consider $S^c_R(p) \bigcap S^c_L(q)$. We have $\dim S^c_R(p) = n - \lambda - 1$ and $\dim S^c_L(q) = \lambda + 1 - 1$, and so the dimension of their intersection is $n - 1$, i.e. $\dim f^{-1}(c)$.

Definition 8.4.1 (Transverse Submanifolds)**.**
Submanifolds $M^m, N^n \subset V^v$ are called *transverse* if for any $p \in M \bigcap N$, $T_pV \subset \text{span}\{T_pM, T_pN\}$ and we write $M \pitchfork N$.

Example 8.1.
- If $m + n < v$, then $M \pitchfork N$ iff $M \bigcap N = \emptyset$.
- If $m + n = v$, then $M \pitchfork N$ iff $\dim M \bigcap N = 0$.
- In general, if $M \pitchfork N$ then $M \bigcap N$ is a smooth submanifold of dimension $m + n - v$.

Theorem 8.5 ***(Submanifolds are Transverse up to Isotopy).***
For submanifolds $M_m, N^n \subset V$, then there exist $h : V \xrightarrow{\text{diff}} V$ smoothly isotopic to id_V such that $h(M) \pitchfork N$.

Corollary 8.6.
We can change ξ in $f^{-1}[c - \varepsilon, c]$ such that $S^c_R(p) \pitchfork S^c_L(q)$, so their intersection consists of finitely many points.



Proposition 8.7 ***(First Cancellation).***
If $S_R^c(p)$intersects $S_L^c(q)$ in exactly one point, then W is a product cobordism.

Idea of proof:

1. Change ξ in a neighborhood of the integral curve from p to q such that the new vector field is nonvanishing.
2. Change f to g with no critical point such that the new vector field is gradient-like for ξ.

9 Thursday February 6th

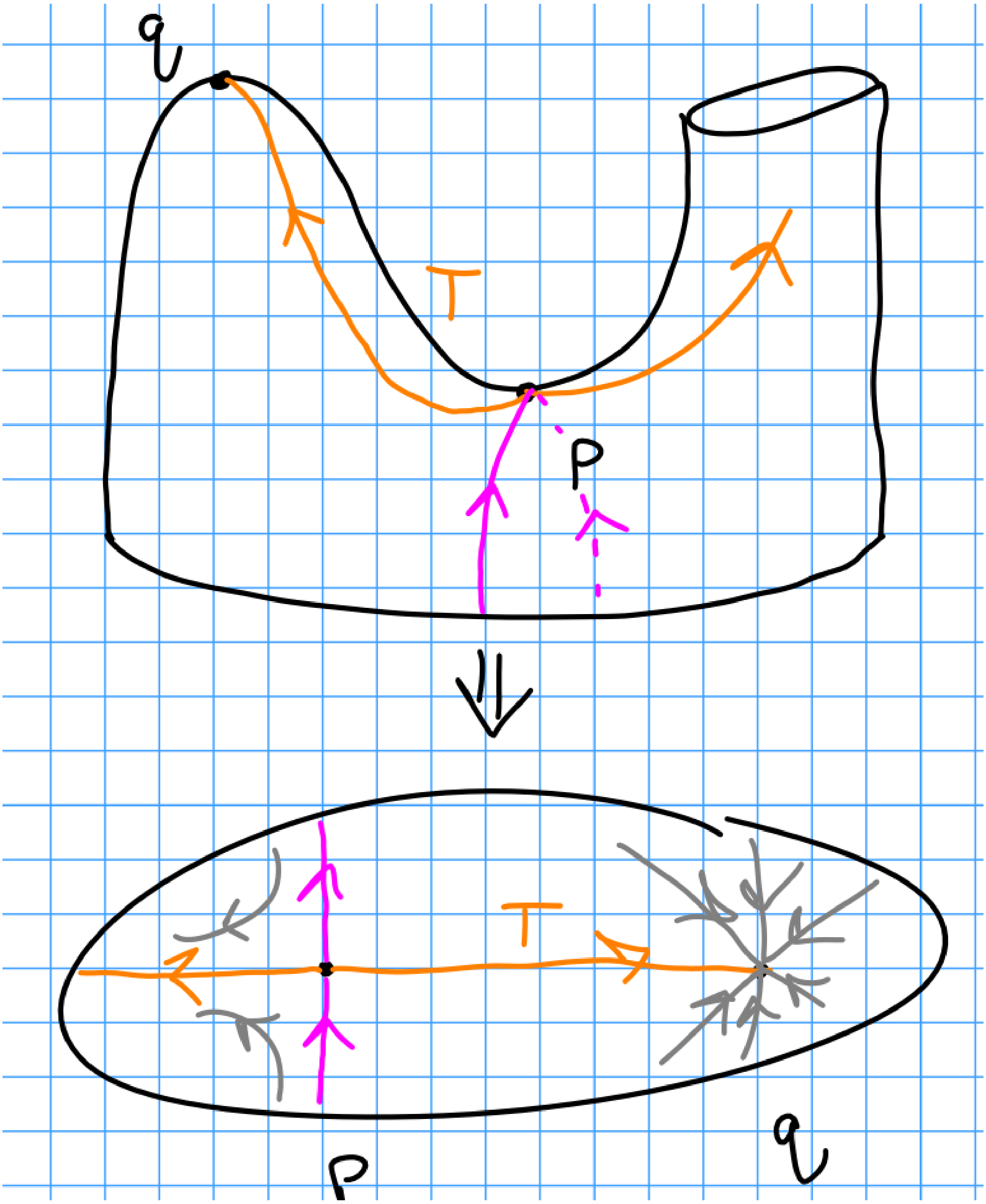
Cancellation: Let $f : W \longrightarrow I$ be Morse, $\mathrm{crit}(f) = \{p, q\}$ with $f(p) < f(q)$ and $\mathrm{Ind}(p) = \lambda$, $\mathrm{Ind}(q) = \lambda + 1$. Let ξ be its gradient-like vector field, then $S_R^c(p) \bigcap S_L^c(q) = \{\mathrm{pt}\}$, so there exists a unique integral curve T from p to q.

In this situation W is diffeomorphic to the product cobordism.

We will show

Theorem 9.1 ***(1, Modifying Vector Fields).***
We can change ξ in a compact neighborhood of T to get a nonvanishing vector field ξ' for which the integral curves originate at M_0 and end at M_1.

Example 9.1.



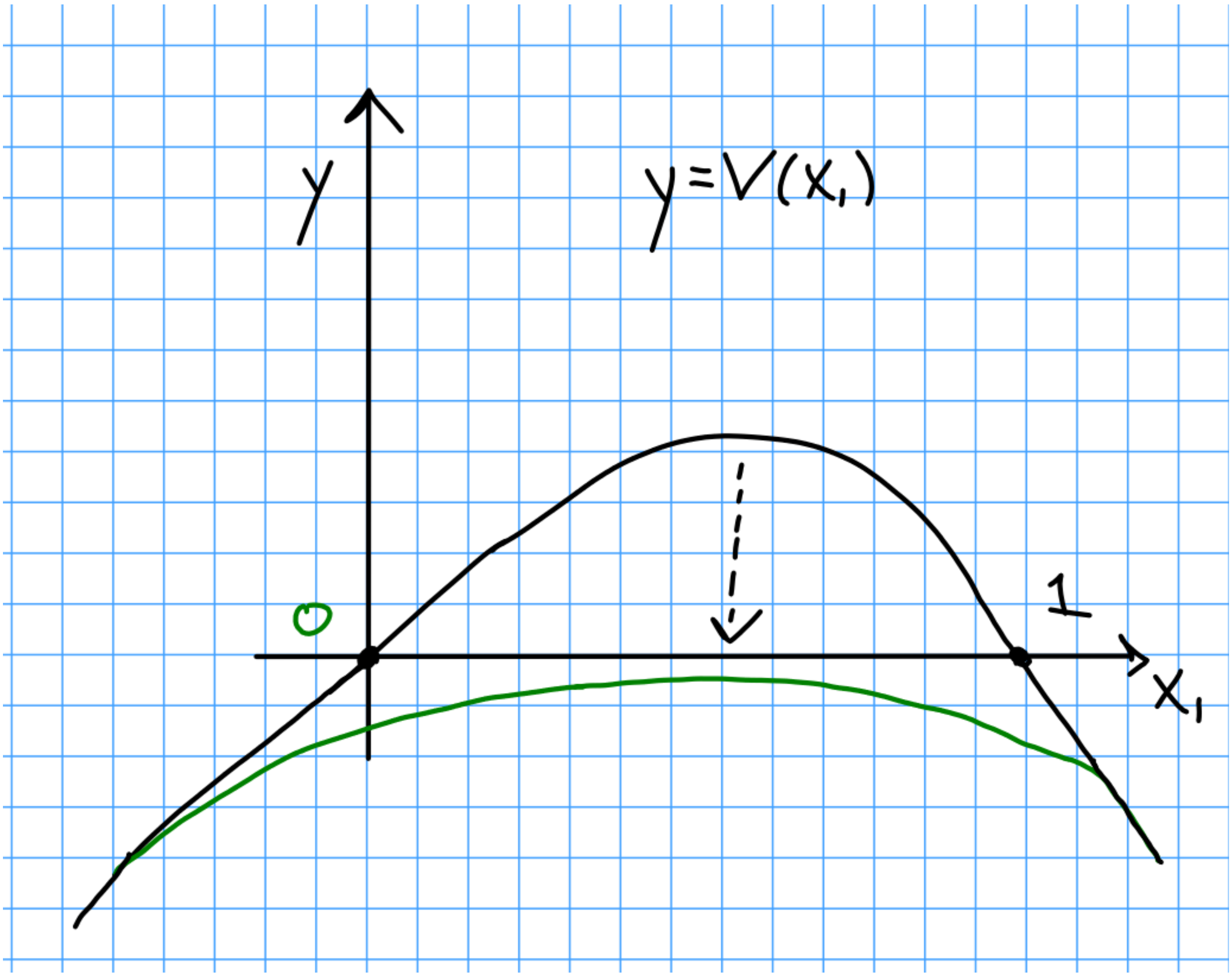
Moreover, it takes a particularly nice standard form, described in the following way:

Proposition 9.2.
There exists a neighborhood U_T and a coordinate chart $h : U_T \longrightarrow \mathbb{R}^n$ such that

1. $h(p) = (0, \cdots, 0)$ and $h(q) = (1, 0, \cdots, 0)$.
2. $h_*\xi = (V(x_1), -x_2, -x_3, \cdots, -x_{\lambda+1}, x_{\lambda+2}, \cdots, x_n)$.
3. $V(x)$ is smooth and positive over $(0,1)$with $V(0) = V(1) = 0$, and $V(x) < 0$ everywhere else.
4. (Minor) $|V'(0)| = |V'(1)| = 1$.

Thus we have

$$\begin{aligned} (x_1, -x_2, \cdots, -x_{\lambda+1}, x_{\lambda+2}, \cdots, x_n) & \quad \text{near } p \\ (-x_1, -x_2, \cdots, -x_{\lambda+1}, x_{\lambda+2}, x_n) & \quad \text{near } q. \end{aligned}$$



9.0.1 Proof of Theorem

Step 1:

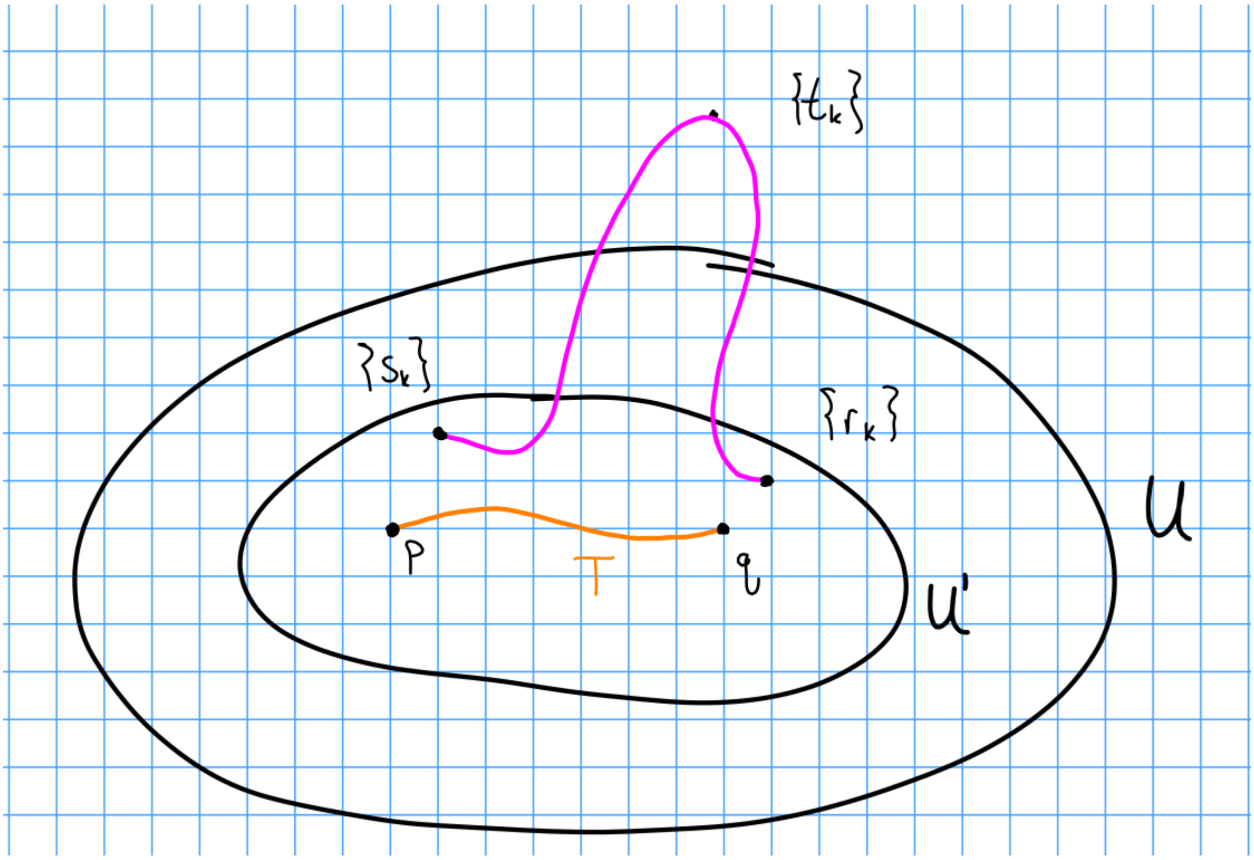
Consider $(U(x_1, \rho), -x_2, -x_3, \cdots, -x_{\lambda+1}, x_{\lambda+2}, \cdots, x_n)$ where $\rho(\mathbf{x}) = (x_2^2 + x_3^2 + \cdots x_n^2)^{1/2}$, which measures the distance between the two curves above. Some facts:

1. $U(x_1, \phi)$ is equal to $V(x_1)$ outside of a compact neighborhood of $h(T)$ in $h(U_T)$.
2. $U(x_1, 0) < 0$ for all x_1.

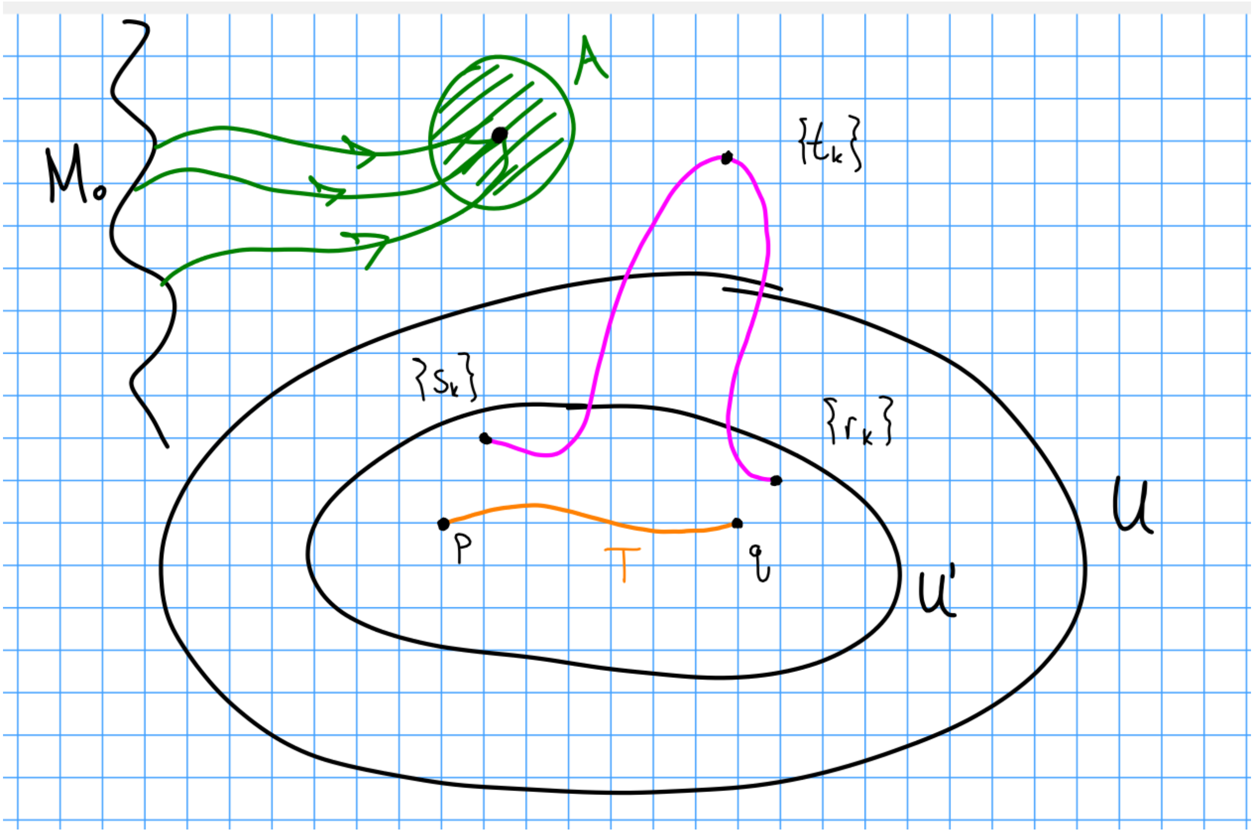
Then $\xi' = h_*(u, -x_2, \cdots, x_n)$ in U_T and $\xi' = \xi$ everywhere else. Thus ξ' is nowhere vanishing.

Step 2:

We want to pick U' such that $T \subset U' \subset \overline{U} \subset U_T$ where $\overline{U}$ is a compact set such that any trajectory of ξ that exits U never re-enters U'.



Suppose such a U' does not exist. Then there exist sequences of points $\{s_k\}, \{r_k\} \subset U$ and $\{t_k\} \subset W \setminus U$ all on the same integral curves γ_k such that $\{s_k\} \longrightarrow T$ and $\{r_k\} \longrightarrow T$. Since $W \setminus U$ is compact, $\{t_k\}$ has a limit point A. Then consider $\psi_A(t)$, which are integral curves that originate from M_0 and end on M_1.



Then there exists a neighborhood A such that for each $a \in A$, the integral curves (half trajectories) containing a originate on M_0. Moreover, for k large enough, all t_k are in A. The union of all of these half trajectories has a positive distance from T, so there is a small enough U disjoint from these trajectories, so $\{s_k\} \not\longrightarrow T$, a contradiction.

■

We now consider the flow lines of ξ' in $h(U_T)$. We have

$$\frac{\partial x_1}{\partial t} = u(x_1, \phi), \frac{\partial x_2}{\partial t} = -x_2, \cdots, \frac{\partial x_n}{\partial t} = x_n.$$

Thus $x_2 = x_2^0 e^{-t}, \cdots, x_n = x_n^0 e^t$.

So if $x_i \neq 0$ for some $\lambda + 2 \leq i \leq n$, the $|x_i|$ is increasing exponential and thus it will escape $h(U)$. The corresponding trajectory will then escape U, and so it will follow the integral curves of the original ξ and ends at M_1. If $x_{\lambda+2}^0 = \cdots = x_n^0 = 0$, then

$$\phi(\mathbf{x}) = (x_2^2 + \cdots + x_{\lambda+1}^2)^{1/2} = e^{-t}\left(\sum (x_i^0)^2\right)^{1/2}.$$

Thus $\phi(x)$ will decrease exponentially.

If it leaves U, we are in the previous case. Otherwise, if it doesn't leave U, then there exists an $\varepsilon > 0$ such that $u(x_1, \phi) < 0$ for all

$$N_\varepsilon = \left\{(x, p) \in h(U) \;\middle|\; \phi < \varepsilon\right\}.$$

Thus there exists a $-\alpha < 0$ such that $u < -\alpha$ on N_ε.

For t large enough,

$$\phi(\mathbf{x}(t)) \in N_\varepsilon \implies \frac{\partial x_1}{\partial t} = u(x_1, \phi) < -\alpha.$$

Thus $x_1(t) < -\alpha t + \text{const.}$ for large enough t, and as t increases $\mathbf{x}(t)$ will go out of U. By the previous argument, it must end at M_1.

Thus every integral curve of ξ starts at M_0 and ends at M_1.

■

Lemma 9.3.
ξ' gives a diffeomorphism from

$$W' = (M_0 \times I; M_0 \times 0, M_1 \times 1) \longrightarrow W = (W; M_0, M_1).$$

Proof .
Take $\pi : W \longrightarrow M_0$ and follow the integral curves backward. Then for all $x \in M_0$, there is a $\tau(x) \in \mathbb{R}^{\geq 0}$ such that $\psi_X(\tau(x)) \in M_1$.
So we get a

$$\widehat{\xi} = \tau(\pi(q))^{-1} \xi'_q$$

and we can define $\phi(x, t) = \widehat{\phi}_X(t)$.

■

Theorem 9.4 ***(2, Modifying a Vector Field Away From Critical Points).***
ξ' is a gradient-like vector field for some Morse function $g : W \longrightarrow I$ such that g has no critical points (since ξ' has no zeros) and $g = f$ near M_0 and M_1.

9.0.2 Proof of Theorem

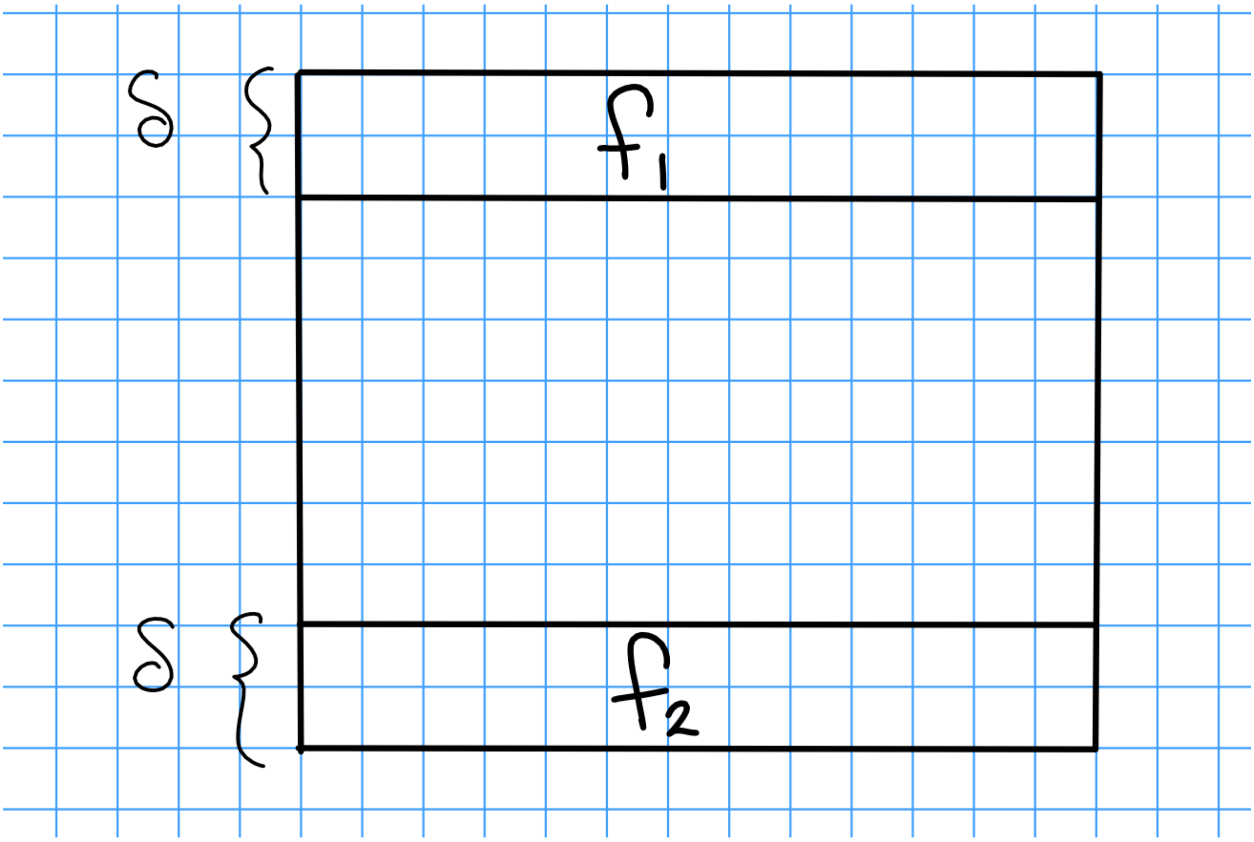
We want to build a $k : M_0 \times I \longrightarrow I$ such that the following diagram commutes:

$$\begin{tikzcd}
M_0 \times I \arrow[r, dashed, "k"] \arrow[d, "\phi"'] & I \\
W \arrow[ur, "g"] &
\end{tikzcd}$$

This needs to satisfy

1. k is equal to $f_1 := f \circ \phi$ near $M_0 \times 0$ and $M_0 \times 1$.
2. $\dfrac{\partial k}{\partial t} < 0$.

Since $\dfrac{\partial f_1}{\partial t} > 0$ near $M_0 \times 0$ and $M_0 \times 1$, take $\delta > 0$ such that $\dfrac{\partial f_1}{\partial t} > 0$ on $M_0 \times [0, \delta)$ and $M_1 \times (1 - \delta, 1]$.



So pick $\lambda : I \longrightarrow I$ such that $\lambda \equiv 1$ near $t = 0, 1$ and $\lambda \equiv 0$ on $[\delta, 1-\delta]$.

Then pick any positive $\overline{K} : M_0 \longrightarrow \mathbb{R}$, and then take

$$K(x,u) := \int_0^u \lambda(t) \frac{\partial f_1}{\partial t} + (1-\lambda(t))\ \overline{K}(x)\ dt.$$

Then

$$\frac{\partial K}{\partial u} = \lambda(u) \frac{\partial f_1}{\partial u} + (1-\lambda(u))\ \overline{K}(x) > 0$$

since the first term is positive near $M_0 \times 1$ or 0, and $\overline{K}$ is positive everywhere.

To see that it satisfies the first property, note that $\displaystyle\int_0^s \frac{\partial f_1}{\partial t}\ dt = f_1$ for s near 0.

To see that property 2, note

$$\begin{aligned} \int_0^1 \lambda(t) \frac{\partial f_1}{\partial t}\ dt + \overline{K} \int_0^1 (1-\lambda(t))\ dt = g(x,1) = f_1(x) \\ \implies \overline{K}(x) = \frac{f_1(x) - \displaystyle\int_0^1 \lambda(t) \frac{\partial f_1}{\partial t}\ dt}{\displaystyle\int_0^1 (1-\lambda(t))\ dt}. \end{aligned}$$

■

10 Tuesday February 11th

10.1 Cancellation

The setup: $f : W \longrightarrow [0,1]$ a morse function with $\mathrm{crit}(f) = \{p, q\}$ with $\mathrm{Ind}(p) = \lambda$ and $\mathrm{Ind}(q) = \lambda+1$, with a gradient-like vector field ξ such that there exists a *single* flow line T from p to q.

Lemma 10.1 ***(Modifying Gradient-Like Vector Fields).***
There exists a gradient-like vector field ξ' for f such that

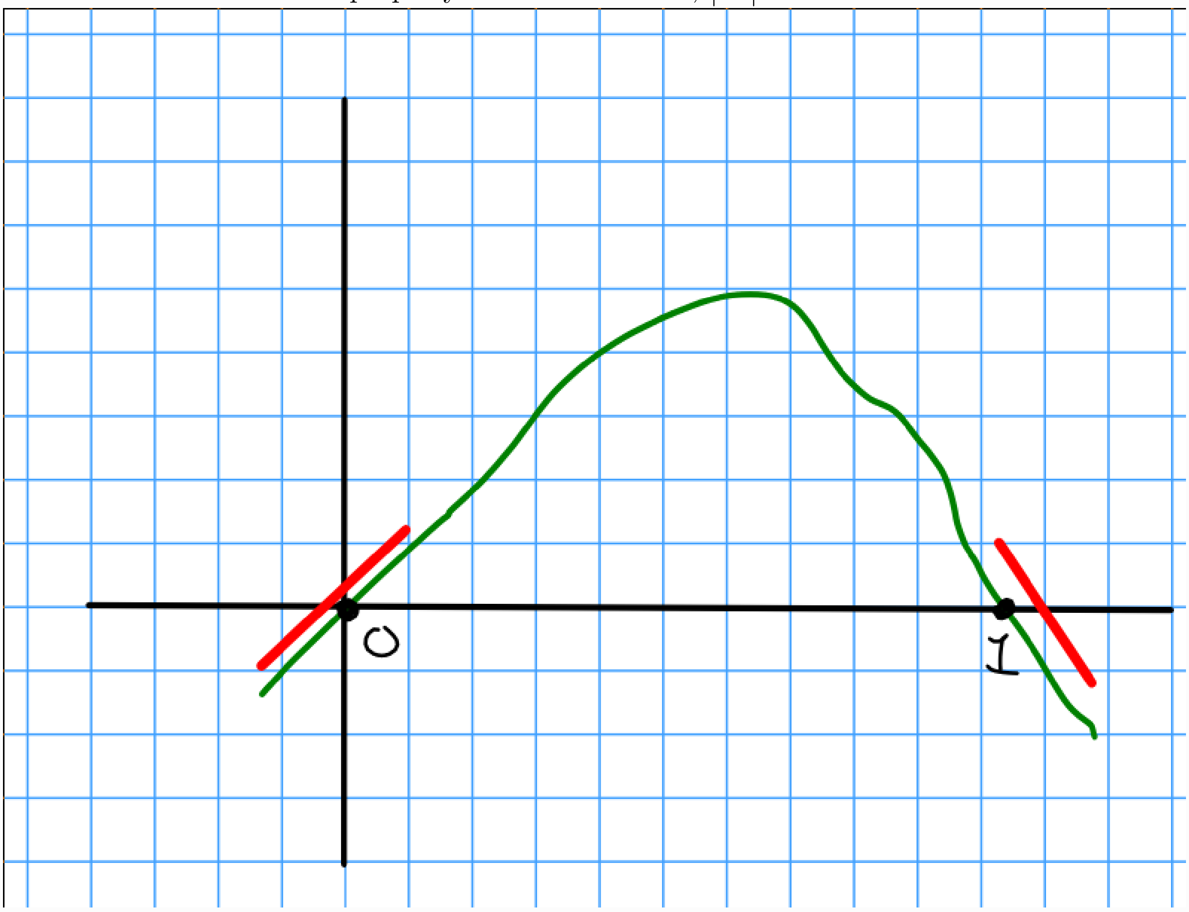
1. T is still the single flow line from p to q.
2. ξ' is *standard* in a neighborhood U_T of T, i.e. there exists $h : U_T \longrightarrow \mathbb{R}^n$ such that

$$h(p) = (0, 0, \cdots, 0)$$
$$h(q) = (1, 0, \cdots, 0).$$

$h(T)$ is contained in the x-axis, and

$$h_*\xi' = (V(x_1), -x_2, \cdots, -x_{\lambda+1}, x_{\lambda+2}, \cdots, x_n),$$

where V satisfies the property that near 0 and 1, $|V'| = 1$:



10.1.1 Proof

Let $\eta = V(x_1)$ from above. Define the following vector field:

$$F(\mathbf{x}) = f(p) + 2\int_0^{x_1} v(t)\ dt - x_2^2 - x_3^2 - \cdots x_{\lambda+1}^2 \cdots + x_n^2.$$

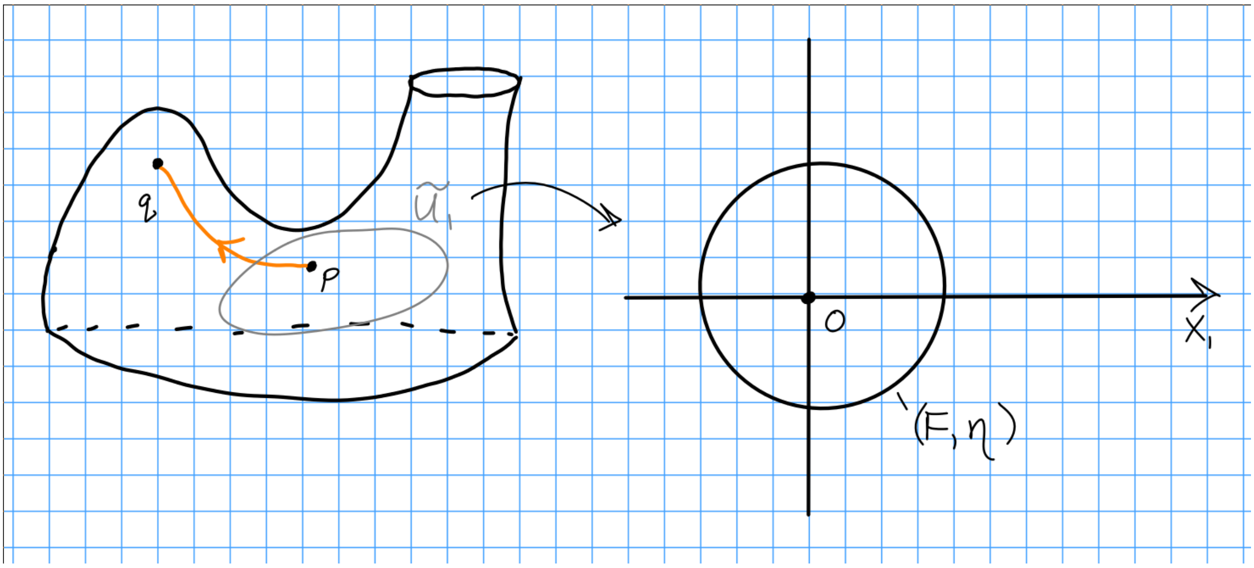
Then η is gradient-like for F, and we can pick $v(t)$ such that

$$F(1, 0, \cdots, 0) = f(p) + 2\int_0^1 v(t)\ dt = f(q)$$
$$\implies \int_0^1 v(t) = \frac{1}{2}(f(q) - f(p)).$$

We know that $v(t) = t$ near $(0, 0, \cdots, 0)$, and since $\int_0^1 t\ dt = 2\left(\frac{x_1^2}{2}\right)$, we have

$$F(\mathbf{x}) = f(p) + 2\int_0^{x_1} t\ dt + \cdots + x_n^2 = f(p) + x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + \cdots + x_n^2$$
$$\implies \eta(\mathbf{x}) = (x_1, -x_2, \cdots, -x_{\lambda+1}, x_{\lambda+2}, \cdots x_n).$$

Then there exists a neighborhood $\tilde{U}_1$ of p and $h_1 : \tilde{U}_1 \longrightarrow \mathbb{R}^n$ such that $\tilde{h}_1(p) = (0, 0, \cdots, 0)$ with $F \circ \tilde{h}_1 = f$ and $\tilde{h}_{1*} = \eta$.



Similarly, near $(1, 0, \cdots, 0)$ we have $v(t) = 1 - t$ and since $\int_0^1 v(t)\ dt = f(q) - f(p)$, we have

$$F(\mathbf{x}) = f(p) + 2\int_0^1 v(t)\ dt + 2\int_1^{x_1} (1 - t)\ dt - \cdots + x_n^2$$
$$= f(q) - (x_1 - 1)^2 - x_2^2 - \cdots + x_n^2,$$

and there exists a neighborhood $\tilde{U}_2$ of q and $\tilde{h}_2 : \tilde{U}_2 \longrightarrow \mathbb{R}^n$ such that $\tilde{h}_2(q) = (1, 0, \cdots, 0)$, $F \circ \tilde{h}_2 = f$, and $\tilde{h}_{2*}\xi = \eta$.

So pick $(\tilde{U}_1, \tilde{h}_1)$ and $(\tilde{U}_2, \tilde{h}_2)$ such that $\tilde{U}_1 \bigcap \tilde{U}_2 = \emptyset$ and $\tilde{h}_1(\tilde{U}_1) \bigcap \tilde{h}_2(\tilde{U}_2) = \emptyset$.

Pick

$$a_1 < f(p) < b_1 < b_2 < f(q) < a_2$$

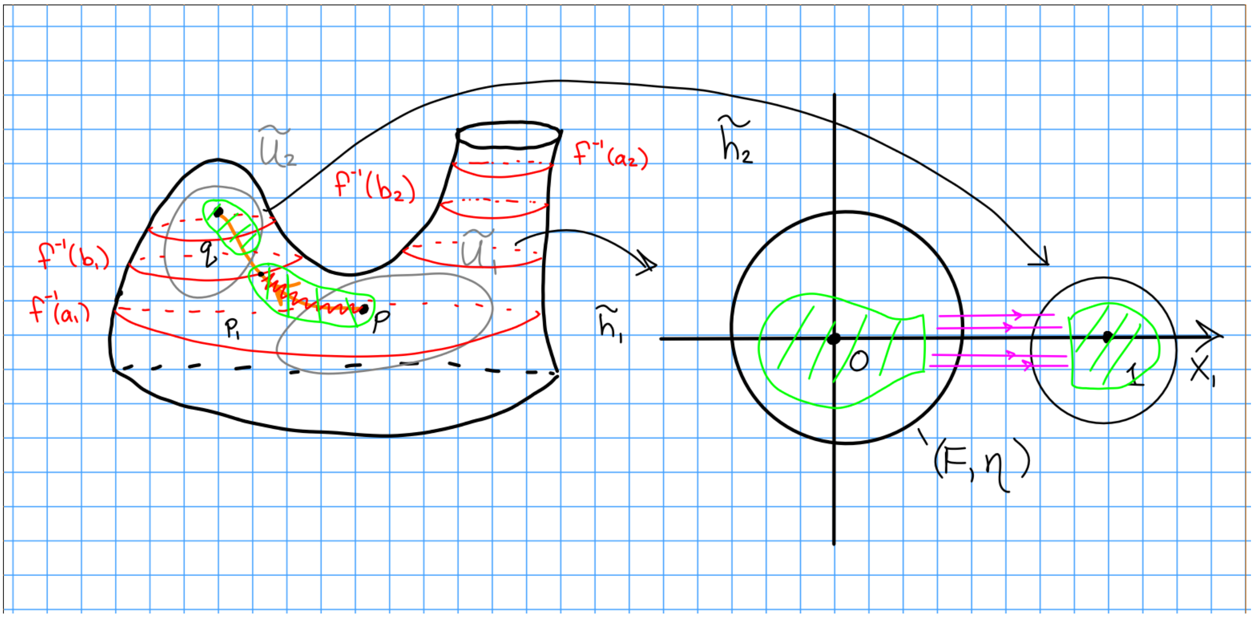
such that

$$f^{-1}[a_1, b_1] \bigcap T \subset \tilde{U}_1 \quad \text{and} \quad f^{-1}[b_2, a_2] \bigcap T \subset \tilde{U}_2$$

and set $p_i = f^{-1}(b_i) \bigcap T$.

Let U_1 and U_2 be closed neighborhood of the arc $p \longrightarrow p_1$ in $\tilde{U}_1 \bigcap f^{-1}[a_1, b_1]$ and $q \longrightarrow p_2$ in $\tilde{U}_2 \bigcap f^{-1}[b_2, a_2]$.

Let $h_i = \tilde{h}_i \Big|_{U_i}$. Then ξ yields a diffeomorphism $\psi : f^{-1}(b_1) \longrightarrow f^{-1}(b_2)$.



Fix a small neighborhood λ of $h_1(p_1)$ in $h_1(f^{-1}(b_1) \bigcap U_1)$, following the flow lines of η yields a diffeomorphism $\phi : V_1 \longrightarrow V_2$ where V_2 is a sufficiently small neighborhood of $h_2(p_2)$ in $h_2(f^{-1}(b_2) \bigcap U_2)$, the following diagram commutes:

$$\begin{tikzcd} f^{-1}(b_1) \arrow[r, "\psi"] & f^{-1}(b_2) \\ V_1 \arrow[u, "h_1^{-1}"] \arrow[r, "\phi"] & V_2 \arrow[u, "h_2^{-1}"'] \end{tikzcd}$$

If for V_1 small enough we have $h_2^{-1} \circ \phi \circ h_1$ restricted to $h_1^{-1}(V_1)$ is equal to ψ, then we can extend (h_1, h_2) to a diffeomorphism h from $U_1 \bigcap U_0 \bigcap U_2$, where U_0 is a small neighborhood of p_1p_2 such that it preserves the trajectories and level sets. We then obtain $h_*\eta = K\eta$, where K is some positive function.

We can extend K to a positive smooth function over W, which yields $\xi' \coloneqq \dfrac{1}{K}\xi$ and thus $h_*\xi' = \eta$. So ξ' is a gradient-like vector field for f.

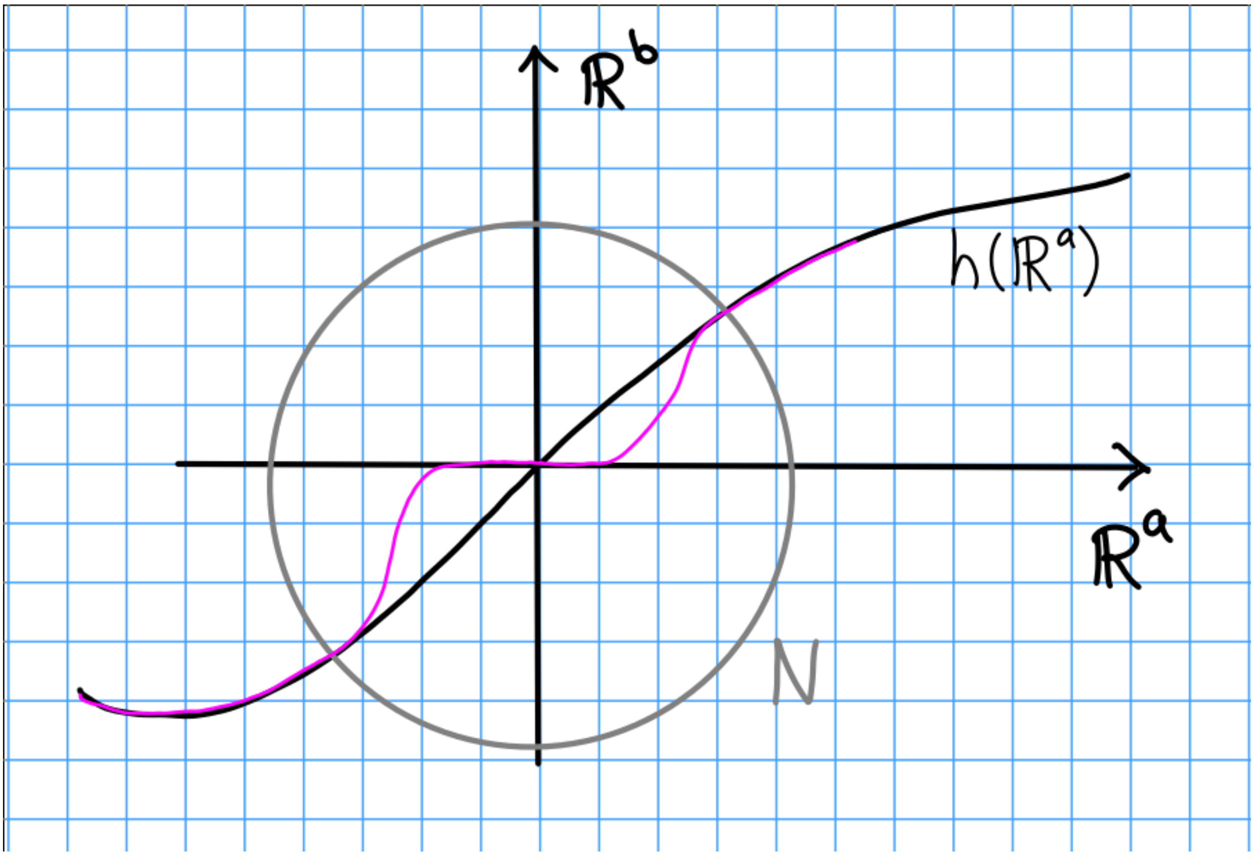
In case the above inequality does *not* hold, we can use an isotopy to change $\psi \longrightarrow \psi'$ and change $\xi \longrightarrow \xi'$ in $f^{-1}[a,b]$ such that the integral curves of ξ' induce ψ'. So find an isotopy such that ψ' is equal to $h_2^{-1}\phi h_1$ near p_1, and furthermore $\psi'(S_R^{b_1}(p))$ intersects $S_L^{b_2}(q)$ transversely in p_2, i.e. p_2 is the only intersection point.

We can do this last step locally. Let $\phi' = h_2^{-1}\phi h_1$, then $(\phi')^{-1}\psi : \tilde{V}_1 \hookrightarrow \tilde{V}_1$ for some small neighborhood $\tilde{V}_1 \subset V_1$ containing p_1. Note that if $S_R^{b_1}(p) \pitchfork S_L^{b_1}(q)$ at p_1, then $(\phi')^{-1}\psi S_R^{b_1}(p) \pitchfork (\phi')^{-1}\psi S_L^{b_1}(q)$ at p_1, and so $S_R^{b_1}(p) \pitchfork (\phi')^{-1}\psi S_L^{b_1}(q)$.

Theorem 10.2 *(?).*
Identify $\mathbb{R}^n = \mathbb{R}^a \oplus \mathbb{R}^b$ where $a+b=n$. Suppose that $h : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ is an orientation-preserving embedding such that $h(0)=0$ with $h(\mathbb{R}^a) \pitchfork \mathbb{R}^b$ with *intersection number* $+1$ at $\{\mathbf{0}\}$.
Then there exists a smooth isotopy $h_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

1. $h_0 = h$,
2. $h_t(0) = 0, h_t(x) = x$ for x outside of a neighborhood N of zero,
3. $h_1 = \mathrm{id}$ in a smaller neighborhood of zero in N
4. $h_1(\mathbb{R}^a) \bigcap \mathbb{R}^b = \{\mathbf{0}\}$.

Rough idea: modify $h(\mathbb{R}^1)$ in a neighborhood of 0:



Next time: a second cancellation theorem. Suppose W, M_0, M_1 are simply connected and $2 \leq \lambda < \lambda + 1 \leq n - 3$. If

$$S_R^c(p) \cdot S_L^c(q) = \pm 1,$$

then $W \cong M_0 \times [0,1]$ are diffeomorphic.

We'll briefly review the intersection number later. Also: homological intersection number.

11 Tuesday February 18th

11.1 Cancellation Theorems

Theorem 11.1 ***(Rearrangement).***
If a Morse function f has 2 critical points with $f(p) < c < f(q)$ and $\mathrm{Ind}(p) \geq \mathrm{Ind}(q)$, then ξ can be perturbed in a neighborhood of $f^{-1}(c)$ such that $W_p^s \bigcap W_q^u = \emptyset$.

Theorem 11.2 ***(First Cancellation).***
If $S_R \pitchfork S_L' = \{\{\mathrm{pt}\}\}$, then the cobordism is diffeomorphic to a product.

Theorem 11.3 ***(Second Cancellation).***
Suppose (W, V_0, V_1) is a cobordism and $f : W \longrightarrow \mathbb{R}$ has two critical points. If $S_R \cdot S_L' = \pm 1$,

then $W^n \cong V_0 \times [0,1]$.

Theorem 11.4 ***(Whitney's Trick).***
If $M, M' = M^m, M^n \subset V^{m+n}$ are closed submanifolds with $M \pitchfork M'$ such that $M, \nu(M')$ (the normal bundle) are oriented. Assume $m+n \geq 5$ and $n \geq 3$, and if $m = 1, 2$ then assume $\pi_1(V \setminus M') \hookrightarrow \pi_1(V)$. Let p, q be in the intersection, $\varepsilon(p) \cdot \varepsilon(q) = -1$ (local intersection numbers), and there exists a contractible loop $L \subset V$ such that

- $L = L_0 \bigcup L_1$
- L_0 is smooth in M, L_1 is smooth in M'.
- L_0, L_1 go from p to q.

And suppose each $L_i \bigcap (M \bigcap M' \setminus p, q) = \emptyset$.
Then

1. $h_0 = \mathrm{id}$,
2. $h_+ = \mathrm{id}$ near $M \bigcap M' \setminus \{p, q\}$
3. $h_1(M) \bigcap M' = M \bigcap M' \setminus \{p, q\}$.

Definition 11.4.1 (Homological Intersection Number)**.**
If $M, N \subset V$ are closed smooth submanifolds, then for $[M], [N] \subset H_*(V)$, then $[M] \cdot [N] = \sum_{p \in M \pitchfork N} \varepsilon(p)$ is the *homological intersection number.*

Sketch of proof:

1. Given L, find a $D^2 \hookrightarrow V$ that it bounds. Note that D^2 may have self-intersections.
2. Continuous maps can be approximated by smooth maps, and smooth intersections can be perturbed to be transverse. This lets the disc be perturbed, and since $2 + 2 \leq 5$, the self-intersection can be made zero.
3. Something else.

11.2 Facts From Differential Geometry

Let M be smooth, then there exists a Riemannian metric $\langle \cdot, \ \cdot \rangle$ on T_pM which is symmetric and positive definite.

Given $p, q \in M$ and a curve $c(t)$ from p to q, we want a parallel transport map $f_c : T_pM \longrightarrow T_qM$.

The exponential map: something that maps a neighborhood in T_pM to a neighborhood of p in M. Take geodesics starting at p and evaluate at $t = 1$.

Definition 11.4.2 (Geodesics)**.**
Geodesics are curves of global shortest length.

Definition 11.4.3 (Normal Bundle)**.**
For $M \subset V$, then $TM \subset TV|_M$ is a subbundle with a metric induced from the metric on V. The **normal bundle** is $TM \perp$.

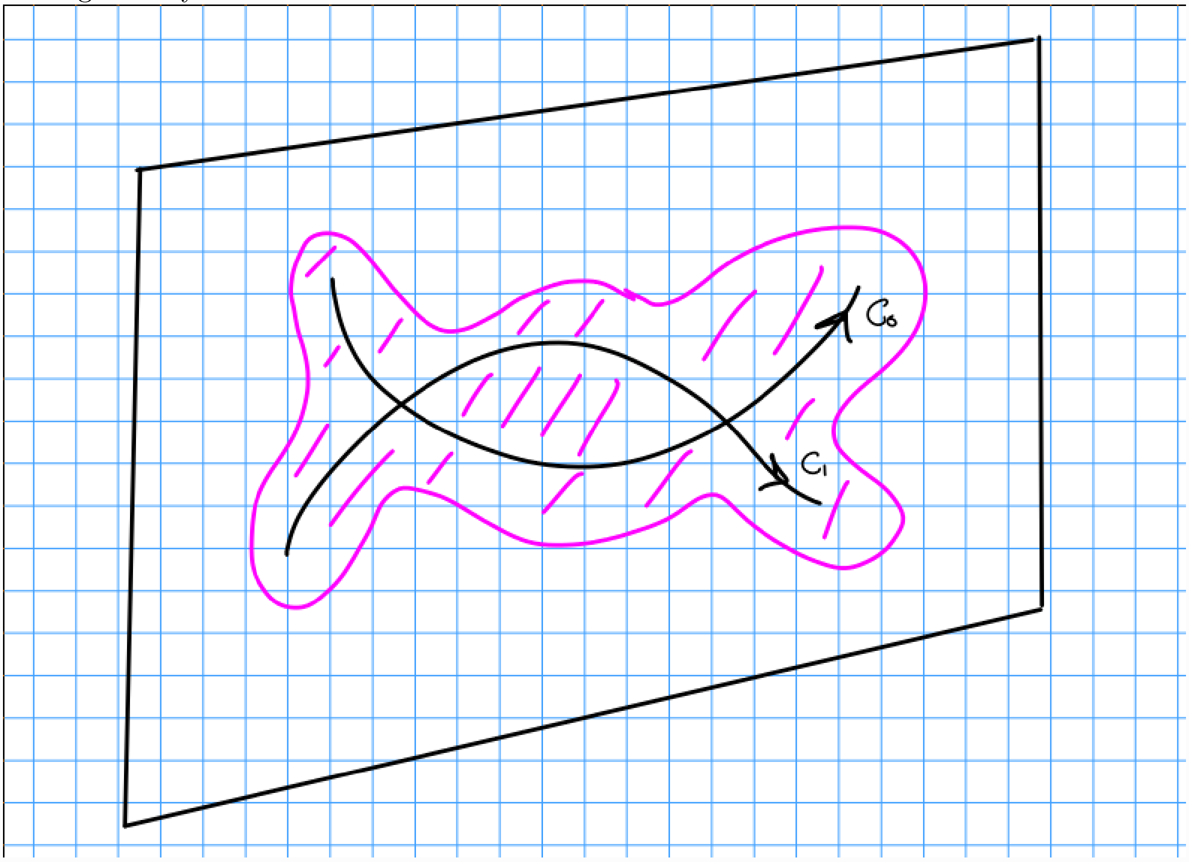
Definition 11.4.4 (Totally geodesic submanifold)**.**
If $M \subset V$ is a submanifold with $p \in M$ and $v \in T_pM$, then M is **totally geodesic** iff the entire geodesic starting at p with initial velocity v is entirely contained in M.

Fact (Existence of the Levi-Cevita Connection) Any Riemannian metric comes with a canonical **connection**: the Levi-Cevita connection.

Parallel transport along a curve in a totally geodesic submanifold (?).

11.3 Proving Whitney's Trick

Lemma 11.5.
Let L_0, L_1 be the image of $C_0, C_0' \subset \mathbb{R}^2$. Let U be a neighborhood of $C_0 \bigcup C_0'$ in $\mathbb{R}^2$, including the region they bound:



We can extend the maps embedding $U \bigcap \left(C_0 \bigcap C_0'\right)$ to $\phi_1 : U \longrightarrow V$ be the embedding, so $\phi_1|_{\partial D^2} = L$. We then get a map

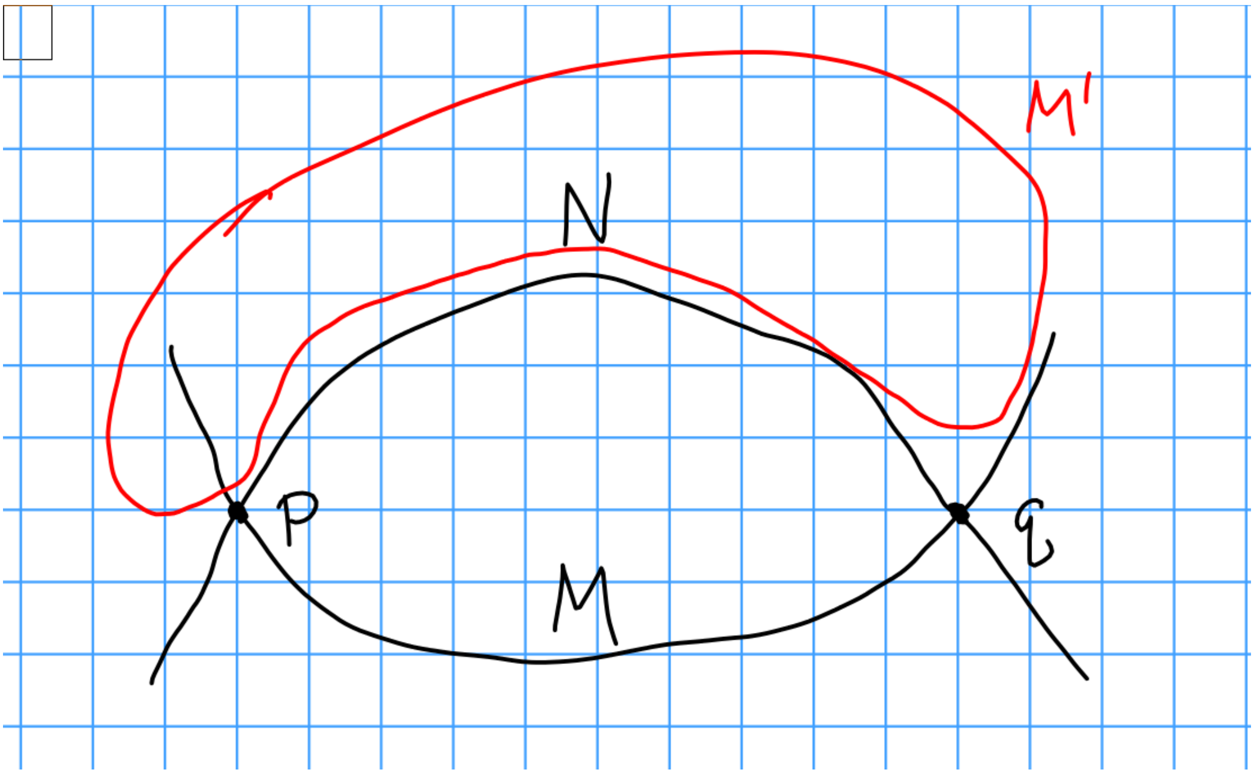
$$\phi : U \times \mathbb{R}^{m-1} \times \mathbb{R}^{n-1} \longrightarrow V$$
$$\phi^{-1}(M) = \left(U \bigcap C_0\right) \times \mathbb{R}^{m-1} \times \{0\}$$
$$\phi^{-1}(M') = \left(U \bigcap C_0'\right) \times \{0\} \times \mathbb{R}^{m-1}$$
.

12 Thursday February 27th

Setup: $M^m, N^n \subset V^{m+n}$ closed submanifolds, $M \pitchfork N$, M oriented (i.e. an orientation of TM) and N co-oriented (i.e. an orientation $\nu_N = TM/TV$).

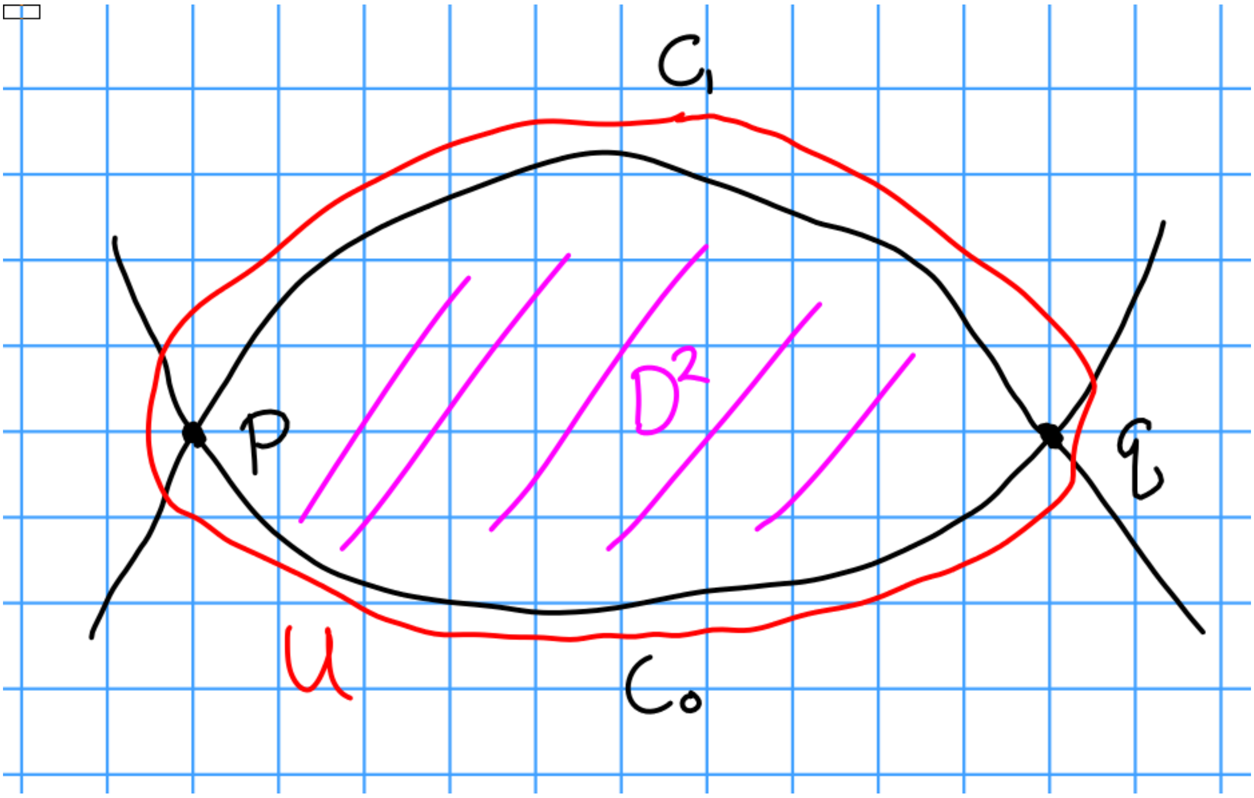
Each $p \in M \bigcap N$ has a sign $\varepsilon(p) \in \{-1, 1\}$. If $p, q \in M \bigcap N$ with $\varepsilon(p) = 1, \varepsilon(1) = -1$, we would like an isotopy $(h_t)_{0 \leq t \leq 1}$ of V such that $h_0 = \mathrm{id}$ and $h(M) \bigcap N = (M \bigcap N) \setminus \{p, q\}$.

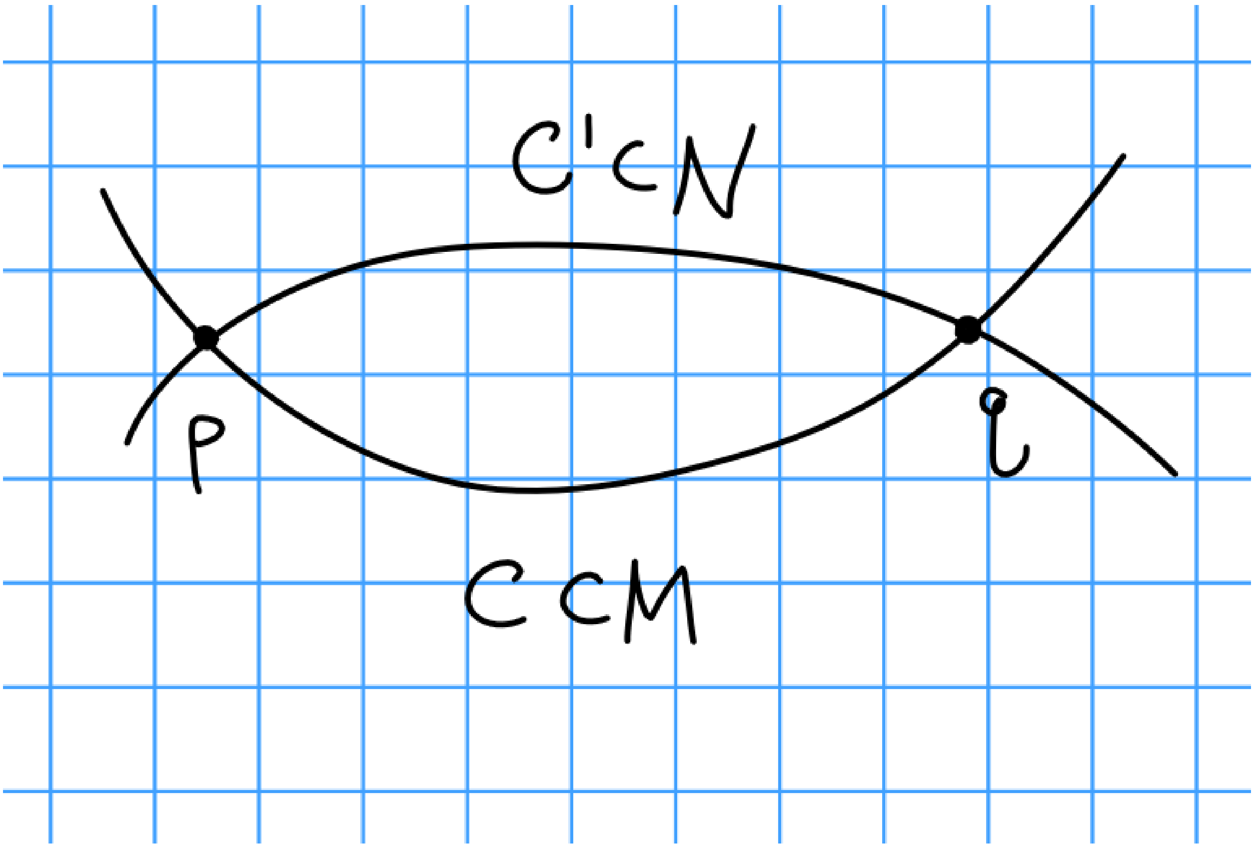
Idea: we want to push M off of N:



From last week, assume $\dim V \geq 5$ and $\dim N \geq 3$ and $\pi_1(V \setminus N) = \pi(V) = 0$.

Why? In the 2-dimensional model above, we want the disc in the middle to be contractible.

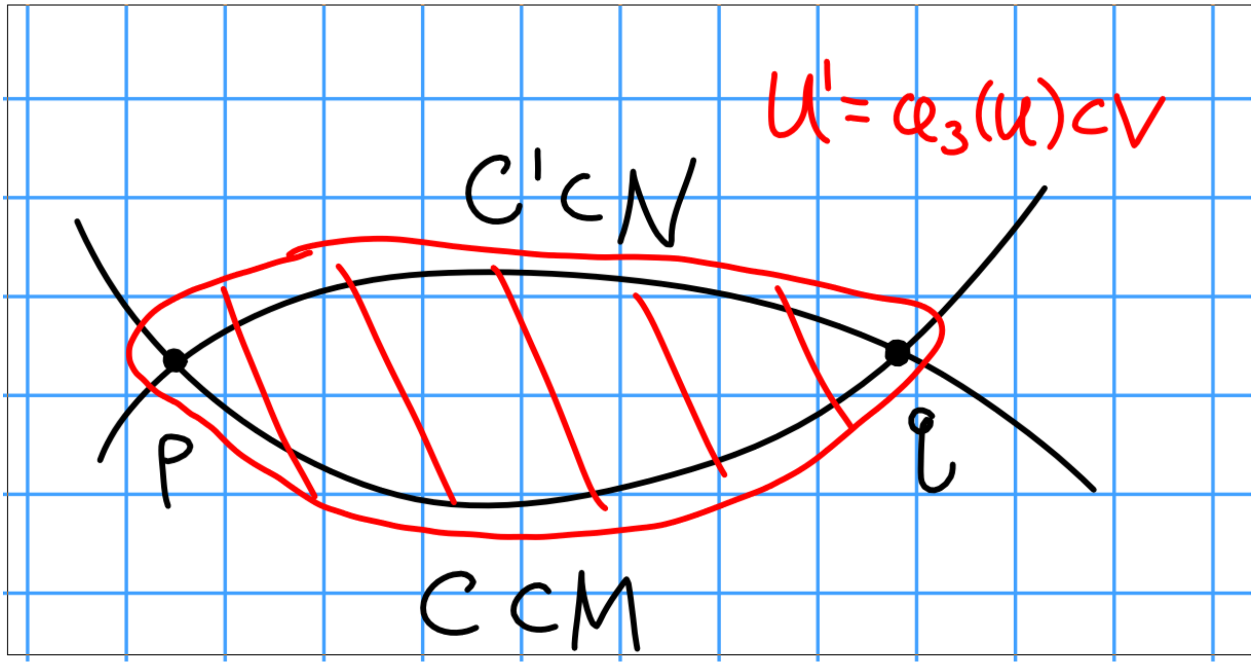

C_1
D^2
p
q
U
C_0


$C' \subset N$
p
q
$C \subset M$

Then there is a smooth embedding $\phi_3 : U \longrightarrow V$, 2-dimensional to $n+1$ dimensional, sending $U \bigcap C_i$ to C_i and $U(C_0 \bigcup C_0')$ to $V \setminus (M \bigcup N)$.

Goal for today: under the same hypotheses, ϕ_3 extends to an embedding $\phi : U \times \mathbb{R}^{m-1} \times \mathbb{R}^{n-1} \longrightarrow V$ such that $\phi^{-1}(M) = (U \bigcap C_0) \times \mathbb{R}^{m-1} \times \{0\}$ and $\phi^{-1}(N) = (U \bigcap C_0') \times \{0\} \times \mathbb{R}^{n-1}$.

Let $U' = \phi_3(U) \subset V$:



Lemma 12.1.
There exist vector fields along U', $\xi_1, \cdots, \xi_{m-1}, \eta_1, \cdots, \eta_{n-1}$, such that

1. These are orthonormal to each other and orthogonal to U'.

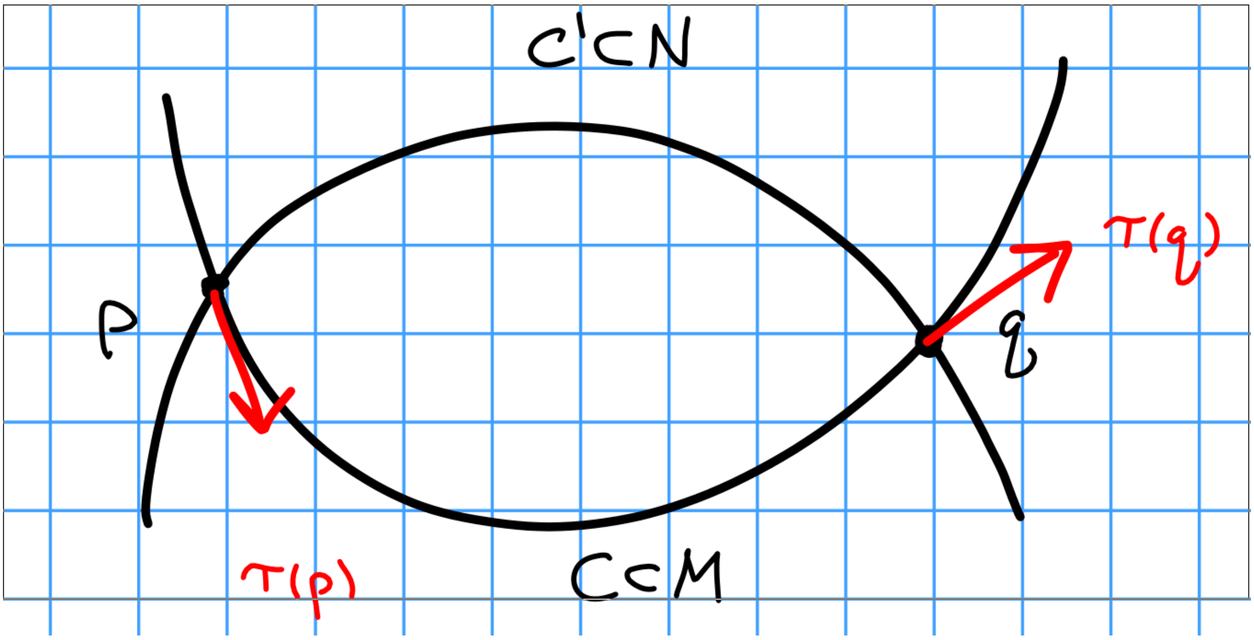
   Note that we'll need a Riemannian metric to make sense of this, and particularly one such that M, N are totally geodesic, and $T_pM \perp T_pN$ and $T_qM \perp T_qN$.

2. $\xi_1, \cdots, \xi_{m-1}$ are tangent to M along C
3. $\eta_1, \cdots, \eta_{n-1}$ are tangent to N along C_1'.

Given this, we have $\phi(u, \mathbf{x}, \mathbf{y}) = \exp_{\psi_3(u)} \left(\sum x_i \xi_i(\phi_3(u)) + \sum y_j \eta_j(\phi_r(u)) \right)$, where the exponential maps is evaluating a geodesic path at time 1.

12.0.1 Proof of Lemma

Let τ be the unit tangent vector field along C, oriented from p to q:



Let ν' be the unit vector field along C' normal to C_1' pointing toward the interior of U'. Thus $\nu'(p) = \tau(p)$ and $\nu'(q) = -\tau(q)$.

First, complete tangents to an orthonomal basis: choose $\xi_i(p)$ such that $\{\tau(p), \xi_1(p), \cdots\}$ is an oriented orthonormal basis for T_pM. Riemannian metrics induce a unique notion of parallel transport, extend ξ_i to all of C by parallel transport. This preserves inner products, and in particular we obtain an orthonormal basis for T_qM.

We can use this to obtain bases for the orthogonal complements, and thus for the normal bundles. Since $\varepsilon(p) = 1$, an orientation of T_pM yields an orientation of $(\nu_N)_p$. Thus $\{\nu'(p), \xi_i(p)\}$ is an oreinted basis, and similarly by flipping the sign of the first term, since $\nu'(q) = -\tau(q)$, $\{\nu'(q), \xi_i(q)\}$ is an oriented basis for $(\nu_N)_q$.

Consider the bundle over C' with fibers equal to orthonormal bases $\{w_1, \cdots, w_{n-1}\} \in (T_xV)^{n-1}$ with each w_i orthonormal and orthogonal to $\nu'(x)$. This has fiber $O(n-1)$, and since the base C' is contractible, this is a trivial bundle.

We have elements in the fiber over p and q inducing the same orientation, so are related by an element of $SO(n-1)$, which is connected and thus path-connected. This gives a path in the frame bundle connecting $\{\xi_1(p), \xi_{m-1}(p)\}$ to $\{\xi_1(q), \cdots, \xi_{m-1}(q)\}$.

So we extend the ξ' over all of C', remaining orthogonal to N and U'. So we have vector fields $\xi_1, \cdots, x_m$ along $C \bigcup C'$. We want to show that these can be extended over all of U_1' remaining orthogonal to U'.

Consider the bundle over U' whose fibers are $(m-1)$-tuples of vectors in T_xV that are orthonormal to T_xU'. Since U' is contractible, this bundle is trivial, and the fiber is orthonormal $(m-1)$-frames in an $(m+n-2)$-dimensional vector space, the orthogonal complement of T_xU'.

Since any orthonormal basis of size $m+n-2$ will send $m-1$ frames to other $m-1$ frames, with some redundancy if the upper-left block is the identity. Thus the fibers are isomorphic to $O(m+n-2)/O(n-1)$.

The construction of $\xi_1, \cdots, \xi_n$ over all of U' is now reduced to extending the loop on $O(m+n-2)/O(n-1)$ determined by ξ_i on $C \bigcup C'$ to a disk, i.e. U'.

In fact, π_1 of this space is 0, so this can be done. Once we have ξ_i, just take ν_i to be any orthonormal over U' such that ξ is orthogonal to TN along C'.

To see why this is, consider the fibrations $O(n-1) \longrightarrow O(n-m-2) \longrightarrow Q$ the quotient above and take the LES in homotopy, also consider $O(n) \longrightarrow O(n+1) \longrightarrow S^k$.

13 Thursday March 5th

Theorem 13.1.
Let (W, V, V') be a cobordism of dimension $n \geq 6$, f Morse with all critical points of indices in $[2, \cdots, n-2]$. Suppose $\pi_1 = 0$ for W, V, and V', and $H_*(W, V) = 0$; then this is homotopic (?) to the product cobordism.

13.0.1 Proof of Theorem

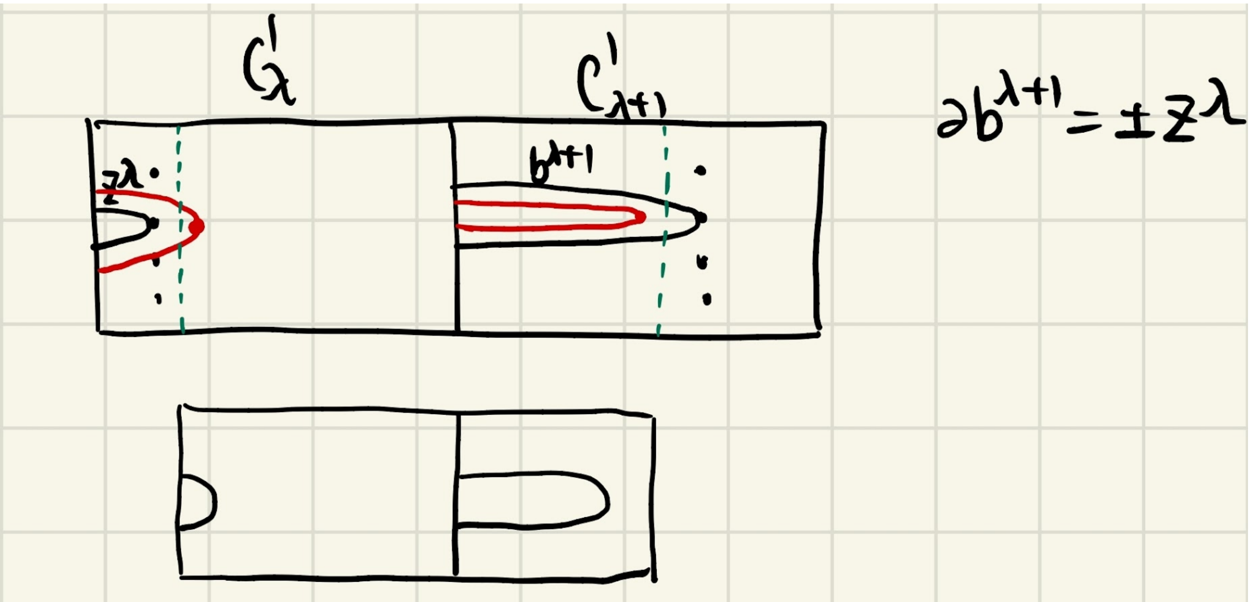
Factor $c = c_2 c_3 \cdots c_{n-2}$; then from $H_*(W, V) = 0$ we have

$$0 \longrightarrow C_{n-2} \xrightarrow{\partial} C_{n-3} \longrightarrow \cdots \longrightarrow C_2 \longrightarrow 0,$$

which has zero homology.

Thus for all λ, choose $\left\{z_1^{\lambda+1}, \cdots, z_{k_{\lambda+1}}^{\lambda+1}\right\}$ as basis of $\ker C_{\lambda+1} \longrightarrow C_\lambda$. Then choose $\left\{b_1^{\lambda+2}, \cdots, b_{\lambda+1}^{\lambda+2}\right\}$ such that $\partial b_i^{\lambda+2} = z_i^{\lambda+1}$. Then $\left\{b_i^\lambda, z_j^\lambda\right\}$ forms an integral basis of C_λ.

From the basis theorem, we can choose a pair (f', ξ') such that this basis is represented by D_L.



Claim 3.
$\partial b^{\lambda+1} = \pm z^{\lambda}$, so $S_R(q) \bigcap S_L(p) = \pm 1$. Using the 2nd Cancellation Theorem, the smaller cobordism is thus a product cobordism.

Proof .
Recall the following:

- If $X = \partial W$ with W oriented, then X is oriented by $\{\nu, \tau_1, \cdots, \tau_{n-1}\}$.
- There is a map

$$H_n(W, X) \longrightarrow H_{n-1}(X)$$
$$[0_W] \mapsto [0_X].$$

So choose an orientation of W (which we'll notate $\circ W$) and all D_L, and orient the normal bundle of D_R such that

- $(\circ(D_R), \circ(D_L)) = \circ W$
- $D_R(q_i) \bigcap D_L(q_i) = \pm 1$.

Then $\circ V D_R = \circ V S_R$ and $\circ D_R = \circ S_R$. The case for S_L and $V S_L$ are similar. ■

Lemma 13.2.
Given (W, V, V'), let $M \subset V'$ be a smooth submanifold and $[M] \in H_k(M)$ the fundamental class. Considering $h : H_k(M) \longrightarrow H_k(W, V)$, the image

$$h([M]) = \sum_{i=1}^{\ell} (S_R(q_i) \cdot_V [M]) \cdot D_L(q_i).$$

Corollary 13.3.
If $\partial_{\lambda+1} : C_{\lambda+1} \longrightarrow C_\lambda$, then

$$\partial(q_j') = \sum (S_R(q_j) \cdot S_L(q_j)) \cdot q_i.$$

This implies the claim.

■

Proof (of Lemma):).
Assume $\ell = 1$, then we have a diagram of the form ■

$$
\begin{tikzcd}
H_\lambda(M) \arrow[dr] \arrow[dd] \arrow[dddddd, r, bend left] & \\
& H_\lambda(V', V' \setminus S_R) \arrow[dd, "h_1"] \\
H_\lambda(V') \arrow[ur, "h_0"] \arrow[dd, "i_*"] & \\
& H_\lambda(V \bigcup D_L, V \bigcup (D_L \setminus \{q\})) \arrow[dd, "h_2"] \\
H_\lambda(W) \arrow[ddr] & \\
& H_\lambda(V', V' \setminus S_R) \arrow[d] \\
& H_\lambda(V', V' \setminus S_R)
\end{tikzcd}
$$