Discussion Notes for the Workshop on Morse Theory

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1 Monday, June 21st 2010

1.1 What is a gradient-like vector field?

Recall: (On a connected manifold) an analytic function is entirely determined by specifying the values on a set with an accumulation point, in particular, any non-empty open set. I.e. an analytic function is entirely determined by just one germ. Analytic manifolds are "rigid".
In smooth case, we have partitions of unity, i.e. a set of smooth functions, each with compact support, such that at each point we see only finitely many and they add up to 1. Partitions of unity allow us to do stuff on each coordinate chart first and then patch it together.
For example, take any smooth manifold, cover it by coordinate charts, each chart maps into Euclidean $\mathbb{R}^n$ which comes with a Riemannian metric (i.e. a positive-definite symmetric bilinear form on TM), patch these Riemannian metrics together using partitions of unity. We have just proven that any smooth manifold admits a compatible Riemannian structure. This is not necessarily of constant curvature. (Note: Ricci flow is a PDE which will evolve the metric to look more constant curvature. It will also run into singularities. Perelmann figured out what singularities can occur. If you do the right things with the singularities, you show Geometrization and the Poincare conjecture).
If you have a function $f : \mathbb{R}^n \to \mathbb{R}$, it is clear what the gradient is. But it doesn't behave like tangent vectors under coordinate transforms. In fact, $df \in T^*M$ is in the cotangent bundle.
But if we have a Riemannian metric $TM \times TM \to \mathbb{R}$ which is a non-degenerate quadratic form, we have a canonical isomorphism T^*M to TM: Hence a function f gives a section in the cotangent bundle T^*M and hence in the tangent bundle. So we get a vector-field. This is called "gradient-like" vector field.

An example for a function which is smooth but not analytic is $f(x) = 0$ if $x \leq 0$ and $f(x) = e^{-\frac{1}{x}}$ for $x > 0$.

1.2 Inverse Function Theorem

In the language of smooth manifolds, given a smooth map $M^n \to N^n$, it is a local diffeomorphism at p if rank $df(p) = n$.
Recall that an immersion is a map $f : M^k \to N^n$ that is locally a diffeomorphism onto its image, i.e. for every point p there is a neighborhood U such that $f_{|U} : U \to f(U)$ is a diffeomorphism. By the inverse function theorem this is equivalent to rank $df = k$ everywhere.

1.3 Implicit Function Theorem

In the language of smooth manifolds, it says that given a smooth map $f : M^n \to N^k$ such that rank $df = k$ everywhere, then $f^{-1}(p)$ is an $n - k$ dimensional submanifold of M^n.

Notice that submanifolds are always modeled on $\mathbb{R}^k \subset \mathbb{R}^n$, hence there are local coordinates $(u_1, \ldots, u_n)$ such that $f^{-1}(p)$ is the set where the last k coordinates vanish. Furthermore, if we pick coordinates $(v_1, \ldots, v_k)$ on N^k, then $(u_1, \ldots, u_{n-k}, v_1 \circ f, \ldots, v_k \circ f)$ are valid coordinates on M^n.
In particular if $f : U \subset \mathbb{R}^n \to \mathbb{R}$ has no critical point at 0, then near 0 we can change coordinates to make f look like $(x_1, \ldots, x_n) \to x_1$.

1.4 What do the indicies tell us?

If you count a critical point with even index as +1, and one with odd index as -1, the sum will be the Euler characteristic. E.g. the torus with the usual Morse function will have **1** critical point of index 0, **2** critical points of index 1 and **1** critical point of index 2, the alternating sum is $+1 - 2 + 1 = 0$ which is the Euler characteristic of the torus.
As explained this morning the Morse function induces a CW composition. A CW complex gives rise to a chain complex which gives cellular homology which turns out to be the same than singular homology. Each index k critical point gives a k-cell, hence the k-th group in the chain complex is $\mathbb{Z}^{i(k)}$ where $i(k)$ is number of critical points of index k. We haven't really talked about the attaching maps yet, and I think it is easier to see it with handle decompositions, and once know Kirby diagrams, we can read them off from there.
There is an equivalence which we see much better with handles: a k handle and $n-k$ handle are really the same bordism just turned upside down. If we flip the Morse function upside down, we just replace k handles by $n-k$ handles. So you see that we just turn around the chain complex. If we ignore torsion for a moment, we have just proven Poincare duality.
Poincare originally thought of triangulations and dualizing triangulations: here a k-simplex of a triangulation becomes a $n-k$-simplex of the dual triangulation.

1.5 Rob on the attaching maps

If the belt-sphere and attaching sphere (which correspond to the ascending and descending manifolds) are transverse, you count their intersection number, this gives you the maps in the chain complex.
He drew a picture of a 2-handle's attaching sphere intersecting a 1-handle's belt sphere twice and explained that this gives a $\times 2$ map in the chain complex. He dualized it to show that the torsion part shifts one dimension in Poincare duality.

1.6 Rob on 4-manifolds

Given an even integer less or equal than 2, there exists a unique orientable surface with that Euler characteristic.
The equivalent in dimension 4: given an even (odd) intersection form, there exists one (two) simply connected topological closed 4-manifolds with that intersection form. The two manifolds in the odd case can be distinguished by the Kirby-Siebenmann invariant which is in $\mathbb{Z}/2$.
He drew a picture of a Moebius band with a curve γ_1 representing a homology class. He pushed the curve off so that the resulting curve γ_2 was transverse to γ_1. Now γ_1 and γ_2 intersect each other once, so this gives the (self-)intersection number of $[\gamma_1]$. If we change γ_2 it might intersect γ_1 more than once, but keeping the orientations in mind and assigning crossings -1 and +1 accordingly, we will get 1 again.
He also explained that $p \times S^2$ and $S^2 \times p$ are the generators of $H_2(S^2 \times S^2)$ and they intersect at one point, hence the intersection form is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The determinant of this matrix has to be ± 1 because of Poincare duality. The parity of an intersection form is determined by the trace?

1.7 $f(x) = x^2$ is ...

1. A Morse function in the sense of the definition given this morning, i.e. under the trivial coordinates, it is of the form x^2.
2. A Morse function in the classical sense, i.e. the only critical point is non-degenerate, because $f''(0) = 2 \neq 0$.
3. Stable, i.e. for any smooth deformation f_t with $f_0 = f$, $\exists \epsilon : \forall t \in (-\epsilon, \epsilon) : f_t$ is locally equivalent to f.

The last one seems counterintuitive because it means that if you add any really crazy smooth function to f, you just need to scale the perturbation small enough, and f still looks like x^2 and has the same behavior with respect to crticial points.
To see the last statement, notice that f_t'' can be assured positive on some neighborhood around 0 for all small enough ϵ. Furthermore, as f' crossed the x-axis, a small perturbation will do so as well, giving a critical point of f_t near 0. By Morse Lemma, there exists coordinates such that f_t looks like x^2.

1.8 Morse functions are stable

The classical notion of Morse functions and the notions of stable functions are equivalent. You might need some extra conditions like critical values distinct and right definition of stable and locally stable.

1.9 $f(x) = x^3$ is not stable

Look at the family $f_t(x) = x^3 + tx$. For $t > 0$, this function is Morse because it has no critical points. For $t < 0$, this function is Morse and has two critical points. These cases give you different handle or CW decompositions (of an interval). So $f(x) = x^3$ is not stable because small perturbations give you different CW decompositons, whereas you want stable functions to give you CW decompositions independent of small perturbations.
So imagine the space of all smooth functions, then f_t is a path which is outside the dense open subspace of Morse/stable functions at exactly the time $t = 0$ and that is where the CW decomposition jumps. This is a 1-dimensional model of a handle cancelation.

1.10 Hessian under coordinate changes

Let's look at a function $f(x_1, x_2, \ldots, x_n) = +a_{11}x_1^2 + \ldots + a_{nn}x_n^2$. If we do a linear change of coordinates $(u_1, \ldots u_n)$ it means that $x_1, \ldots x_n$ are just linear combinations of $u_1, \ldots u_n$. So we can write down f in the new coordinates by just plugging in:

$$f(u_1, \ldots, u_n) = +a_{11}(\lambda_{11}u_1 + \lambda_{12}u_2 + \ldots \lambda_{1n}u_n)^2 \ldots + a_{nn}(\ldots)^2.$$

If we expand it, we get something like:

$$f(u_1, \ldots, u_n) = b_{11}u_1^2 + b_{12}u_1 \cdot u_2 + \ldots b_{nn}u_n^2.$$

The matrix of the b_{ij} and a_{ij} are related through the matrix M describing the coordinate change as follows:

$$A = M^T BM.$$

So the linear coordinate change is just the same than coordinate changes of quadratic forms. Each non-degenerate quadratic form can be brought into diagonal form. The index is an invariant: If we look at it as a function $\mathbb{R}^n \to \mathbb{R}$, the zero-set will be some cone of codimension 1 cutting some spaces out of the unit sphere. The dimensionality and topology of these spaces determine the index, so it is an intrinsic invariant not depending on the coordinates chosen.

For the analytic change, you have to add to f higher order terms and you also have to add to the equations for the coordinate changes ($u_i = \lambda_{i1}x_1 + \dots \lambda_{in}x_n$) higher order terms. For the smooth case, you have to add functions which have all derivatives vanishing at zero. Then you can expand the above equations.

But the second order terms are still just exactly the same terms as above, and you don't care about third order terms and higher when computing the index. So the index stays invariant.

If you have a function f with first order terms, coordinate transforms will potentially spill first order terms into the second order terms, so it will mess with the Hessian and the index is not an invariant. So the index of the Hessian only works at critical points.

The germ of smooth functions splits into the germ of analytic functions and functions which have all derivatives zero. The last part is really really small locally and can be kind of ignored for local approximations.

1.11 Problems

1.11.1 Problem 1

Let M be a compact manifold with boundary, let N be a connected component of ∂M. Show that there is an open neighborhood U of N diffeomorphic to $N \times [0, 1)$.

Sketch of solution: Use vector field transversal to N.

1.11.2 Problem 2

Let $f : M^m \to \mathbb{R}$ and $g : N^n \to \mathbb{R}$ be Morse functions. Find a Morse function on $M \times N$. What are the critical points of that Morse function in terms of critical points of f and g?

Solution: The function $f + g$ is a Morse function. To be more precise, $f \circ p_1 + g \circ p_2$ is Morse where p_1 and p_2 are the projections from $M \times N$ onto M and N. If $x_1, \dots x_m$ are coordinates on M and $y_1, \dots, y_m$, then $(x_1, \dots, x_m, y_1, \dots, y_m)$ are coordinates on $M \times N$. A critical point of $f + g$ has to have all derivatives $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_j}$ equal to zero so $(m, n) \in M \times N$ is critical if and only if m and n are critical. The critical points of $f + g$ are the product of critical points of f and g.
At these critical points, $f + g$ looks like $\pm x_1^2 + \dots \pm x_m^2 \pm y_1^2 \dots \pm y_n^2$, the number of minus sines just add up, so do the indices.
Compare this to the product of CW complexes. The product of an i-cell and a k-cell is a $i + k$-cell. In fact the Morse function $f + g$ gives the product of the CW decompositions corresponding to f and g.

1.11.3 Problem 3

Let X be a vector-field (which is a section of the tangent bundle TM) on a closed manifold M. A flow $\phi_t : M \to M$ generated by X is a one-dimensional family of smooth maps such that:

- for all $t, s \in \mathbb{R} : \phi_{t+s} = \phi_t \circ \phi_s$
- $X_p(f) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}$, in other words $df_p(X) = 1$

The image of a point p under ϕ_t is just a curve called flowline. In local coordinates, we can give such a curve just a velocity vector and check that this vector aggrees with the vector field. The above condition is coordinate independent however. Remember that a tangent vector in TM is a derivative and $X_p(f)$ denotes the resulte of f differentiated with respect to this derivative at the point p. The right hand side when just says that this value should be equal to the change happening to f when transported along the flow.

Solution: See John Milnor's "Morse Theory" Lemma 2.4.

1.11.4 Problem 4

Find a Morse function on $\mathbb{R}P^2$.

Solution 1: Take the ellipsoids $x_1^2/2 + x_2^2 + x_3^2 = 1$ and $x_1^2 + x_2^2 + 2x_3^2 = 1$. Both these ellipsoids can be thought of as functions $f, g : S^2 \to \mathbb{R}$ by assigning to a point $p \in S^2 \subset \mathbb{R}^3$ the distance from the origin to the intersection of the ellipsoid with the ray 0 to p. Look at $f + g : S^2 \to \mathbb{R}$. It will have index 0 critical points where S^2 intersects the x_3 coordinate axis, index 1 critical points where it intersects the x_2 coordinate axis and index 2 critical points where it intersects the x_1 coordinate axis. This Morse function is equivairant under the antipodal map, hence we get an induced Morse function on $\mathbb{R}P^2$. The handle decomposition of S^2 has a 2 0-handles and 2 1-handles forming a belt around the equator, and the 2-handles being attached above and below. $S^2 \to \mathbb{R}P^2$ will take the belt to a Moebius band and attach one 2-handle to it.

Solution 2: Take the saddle $x_3 = x_1^2 - x_2^2$ with $x_1^2 + x_2^2 \leq 1$. Smoothen it at the boundary so that it intersects the cylinder $x_1^2 + x_2^2 = 1$ perpendicular. Then identify points through the map $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$. This is $\mathbb{R}P^2$ (which is a disk attached to S^1 through a degree 2 map). x_3 will be a Morse function on it.

2 Tuesday, June 22nd 2010

2.1 Rob on the Whitney Trick

Rob drew a picture of an M^m and N^n manifold intersecting transversely in Q^{m+n} with $m + n \geq 5$. The manifolds intersect in points which carry $+$ and $-$ signs depending on orientation. With the right conditions, there is a Whitney circle spanned by a Whitney disk. Pushing either M^m or N^n along that Whitney disk will remove the two intersection points.

2.2 Exercises skipped

- Collar neigbhborhoods exits
- Given two manifolds M_1 and M_2 with boundary, collar neighborhoods around the boundary components $N_1 \subset \partial M_1$ and $N_2 \subset \partial M_2$ and an (orientation reversing) diffeomorphism $\phi : N_1 \to N_2$, we can glue M_1 and M_2 along ϕ and get a unique smooth structure. This is because a collar neighborhood is a diffeomorphism $\psi_i : [0, 1) \times N_i \to U_i \subset M_i$, and we know how to glue together two cylinders $[0, 1) \times N$ and $(-1, 0] \times N$.
- Prove that the smooth structure when we glue M_1 and M_2 this way does not depend on the collar neighborhoods. It is enough to show that if we have to collar neighborhoods ψ and $\psi' : [0, 1) \times N \to U \subset M$, then there is an automorphism on M fixing each boundary component and taking ψ to ψ'.
- Handle decompositions give us manifolds with "corners". How do you define these? Show there exists a way too "smoothen corners". Show that the resulting smooth manifold with boundary (and without corners) is unique.

2.3 Homological Algebra

Aaron defined what a chain complex is, how to define the homology of a chain complex and how the chain complex arises from a Delta-complex. I recommend Allen Hatcher's "Algebraic Topology" (freely available on his website).
The homology of a torus $S^1 \times S^1$ is $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^2, H_2 = \mathbb{Z}$.
The homology of $\mathbb{R}P^2$ with integer coefficients is $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/2, H_2 = 0$.

2.4 Handles in Dimension 0

Connected 0-manifolds are completely classified. Left as homework.

2.5 Handles in Dimension 1

I adopted the terminology from Robert Gompf and Andras Stipsicz's "4-manifolds and Kirby Calculus" where I also called the region around the belt sphere the "belt region".
Notice that the attaching region and attaching sphere (respectively belt region and belt sphere) are the same thing for handles in dimension 1. The attaching sphere corresponds to the descending manifold, and the belt sphere to the ascending manifold.
A 0-handle corresponds to a minimum of a Morse function. It can be drawn as an arc with the ends pointing upward. Notice that the attaching region is empty. Handles are really parts you glue to the belt region of lower handles along the attaching regions. 0-handles have a void attaching region, they don't need any manifold to be glued onto, so you can use this to bootstrap your building process.
A 1-handle looks the same upside down. Attaching region and belt region are reversed from 0-handles. You can use 1-handles to close up your manifold.

To build S^1 attach a 0-handle to 1-handle. There are other ways to build S^1 with several 0-handles and 1-handles. If you build the manifold from bottom to top, you can also look at the 0-dimensional boundary at each step. Attaching 0-handles and 1-handles to the 1-manifold can be regarded as surgery on the 0-dimensional boundary. A 0-handle adds two points to the boundary, and a 1-handle removes two points from the boundary.

The orientation of S^1 gives an orientation on the cores and cocores of the handles which are just intervals. The boundary of the core and cocore are just the attaching and belt sphere. They are pairs of points S^0 in this case, we just assign them $+1$ and -1.
In the chain complex for Morse homology, a handle is a generator, to get the boundary of a 1-handle follow its core, it will hit the cocore of two (or the same) 0-handle, multiply the above mentions $+1$ and -1 to get the image, i.e. the matrix coefficient where the column corresponds to the 0-handle and the row the 1-handle.

2.6 Handles in dimension 2

Each handle is just a ball, or a square. The square's boundary has 4 parts, two are parallel. Depending on how we divide these parallel parts, we get index 0, index 1 and index 2 handles.
We have the standard decomposition of a torus as shown in Gompf Stipsicz. It is easier to see when we draw the 0-handle plus first 1-handle as a bent cylinder or lower half of a torus. Attaching the second 1-handle to the lower half of the torus is the same than just removing a little disk at the top of a torus.
Notice that if we want orientation preserving, then how a 1-handle is attached is entirely determined by its end points of the core, i.e. its attaching sphere. If we allow non-orientable surfaces, we can twist the 1-handle. Here we need to specify how the attaching region is attached. In this case, that means we specify the image of the attaching sphere (two points) plus of direction. We ignore this for the moment, and come back to it when we talk about higher dimensions. (Aaron mentioned that we need a framing of the normal bundle of the image of the attaching sphere.)
For the torus, the chain complex is $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}$, and all the maps are zero. So we get the right homology.

The singular, cellular and Morse homology of a smooth manifold are all the same.

We can again regard attaching handles as surgery on the bounary. The boundary in this case is 1-dimensional so just circles. 0-handles introduce a new circle, 2-handles, kill a circle. 1-handles either split a circle into two (if its attaching sphere is in the same component) or joins two circles to one.

2.7 Problem 1

I have some handle decomposition, I see a 2-handle F where part of its attaching sphere is touching a 1-handle E_1, that connects to 0-handles V_0 and V_1. F corresponds to a generator of C_2 of the chain complex. The image of F is $\partial F = E_1$ plus some other stuff, hence $\partial\partial F = V_0 - V_1$ plus some other stuff which is non-zero. What did I do wrong that this is not a chain complex?

Solution: I only look at part of the handle decomposition. Let's try to fix the above thing to make it a valid handle decomposition. The two easiest ways: the boundary of V_0, V_1 with E_1 attached is a circle, use it to attach F. Now F's attaching sphere hits E_1 twice with opposite signs, so $\partial F = 0$ and we have a chain complex.
The other way: there is another 1-handle E_2 glued to V_0 and V_1 and $F's$ attaching sphere traverses E_1 and E_2. Now ∂F is $E_1 - E_2$ and since $\partial E_0 = \partial E_1 = V_1 - V_0$, $\partial\partial F = 0$. This exercise really just illustrates the statement from this morning that if there is an descending manifold of a k-index critical point hitting the ascending manifold of a $k-1$-index critical point and its descending manifold hits the ascending manifold of a $k-2$ handle, then this must happen again for a different (or possibly the same) $k-1$-index critical point such that things cancel in the chain complex.

2.8 Problem 2

Glue a twisted 1-handle to a 0-handle to get a Moebius band. Glue a 2-handle to get $\mathbb{R}P^2$. What is the homology?

Solution: We learned this morning to look at level sets, look at how the ascending and descending manifolds intersect in points and compute the resulting chain map using orientations. Because $\mathbb{R}P^2$ is non-orientable, just looking at a level set will fail because the level set won't inherent an orientation. The computation is somewhat harder to see, cellular homology seems to be more amenable. Also notice that $S^2 \to \mathbb{R}P^2$ is the orientation double cover, so $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, its abelianization is the homology $H_1 = \mathbb{Z}/2$.

2.9 Transversal intersections

Two curves in $\mathbb{R}^2$ intersect transversal if at the intersection points their tangent lines are not parallel. In general, two submanifolds M^m and N^n intersect transversal in Q^{m+n} if at their intersections points the tangent spaces of M^m and N^n span the tangent space of Q^{m+n}. So in $\mathbb{R}^3$ you should have a curve transversing a plane in mind. A self intersecting curve in $\mathbb{R}^3$ is not transverse, because at the intersection point you have only two tangent vectors. So you can disturb the curve a little bit two make it non-intersecting. With a transverse intersection, you can't disturb it a little to make it transverse, it is locked. That is why you need the Whitney trick Rob introduced earlier to prove Whitney's embedding theorem.
Let's assume M^m, N^n and Q^{m+n} are oriented and look at the intersection point again. Pick m tangent vectors $(t_1, \ldots, t_m)$ from M^m consistent and similarly for N^n, pick $(v_1, \ldots, v_n)$. Now $(t_1, \ldots, t_m, v_1, \ldots, v_n)$ gives an orientation on Q^{m+n} which might agree or might not agree with the orientation we already had fixed. If it does this intersection gets +1, otherwise -1. If there are multiple intersection points, we add them up and get the intersection number of M^m and N^n.

2.10 Morse homology revisited

Let's fix an orientation on M. Also for each handle fix an orientation for the core and cocore. Preferably make it compatible with the orientation on M. It does not really matter, since if you take the chain complex $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$, you get the same homology if you put in identity maps or minus identity maps in any order. This is the same than picking orientations on the descending D_p and ascending A_p manifolds of each critical point p.
To figure out the matrix coeffiecient in the boundary map, take the corresponding index k and $k-1$ critical point p and p' (or handle) and pick a level set ∂M_z in between these two points. Here M_z is

just $\{p \in M : f(p) \leq z\}$. The level set ∂M_z has an orientation induced from M_z. Similarly $D_p \cap \partial M_z$ and $A_p \cap \partial M_z$ have induced orientations. $D_p \cap \partial M_z$ and $A_p \cap \partial M_z$ intersect (in the generic case) transversely in ∂M_z. Their intersection number is what we have to write into the matrix for the boundary map.
In terms of handle decompositions: the intersection number of the attaching sphere of a k-handle and the belt sphere of a $k-1$-handle gives the matrix entry in the boundary map.

2.11 A transversal intersection with a point

We can have M a point and $N = Q$ where M and N as earlier when talking about transversal intersections. The point gives no tangent vector. Instead the orientation of a point is just a sign ± 1. If the orientations on N and Q agree, this sign is the intersection number.

2.12 The special case of 0-handles and n-handles

The intersection number calculation still works for 1-handles attached to 0-handles but look a little obscure. The cocore of a 0-handle is the entire 0-handle, assume it is oriented consistenly with the manifold. Orienting the core of the 1-handle just means assigning opposite signs to the two points in its attaching sphere S^0. So the boundary map of a 1-handle will be +1 times a 0-handle and -1 times a 0-handle, possibly the same 0-handle.

2.13 Homotopy type

If we just specify how the attaching spheres map into the belt regions, we can recover the homotopy type, because we have enough to construct the CW complex. In order to recover the diffeomorphism type, we need to specify how the attaching region maps into the belt regions.
This needs a framing. For the $\mathbb{R}P^2$ example, we have already seen an example where a framing was just a choice between two directions.

3 Wednesday, June 23rd 2010

The problem session is really about the images, which would take to much time to draw carefully and scan.

3.1 Overview

- Pictures of handles in dimension 3.
- How does a handle slide look like, examples for 1-handles in dimension 2 and 1-handles and 2-handles in dimension 3.
- We can always bring a Morse function into a form such that all handles of index k are above handles of index $k-1$.
- We can always bring a Morse function into a form such that there is only one 0-handle and one n-handle.
- Attaching 1-handle and 2-handle to a 3-manifold, corresponds to surgery on the boundary of the manifold increasing or decreasing the genus (if the attaching sphere is not separating). 1-handle means we remove to disks from the surface and connect the circle through a cylinder. 2-handle means we split the surface along the curve and cap it off with 2 disks.
- Heegard splitting: pick a value of the Morse function between all index 1 and index 2 critical points. The level surface is a genus g surface splitting the manifold into two handlebodies.

- We draw the attaching/belt spheres of the 2-handles/1-handles. This gives a Heegard diagram. (See Definition 2 on page 77 of "Knots, Links, Braids and 3-Manifolds" by V. Prasolov and A. Sossinsky.)
- Two Heegard diagrams of 3-manifolds are related through handle cancelations (their attaching and belt sphere have to intersect exactly once), introducing pairs of canceling handles and handle slides: the attaching sphere (really a circle) is sliding accross another attaching sphere.
- The genus of a 3-manifold is the minimal genus among all its Heegard splittings.
- Genus 0 is just S^3.
- Genus 1 are lens spaces.
- How to draw a 1-handle and 2-handle of a 4-manifold. See Chapter 5 of R. Gompf and A. Stipsicz "4-Manifold and Kirby Calculus".
- Every 3-manifold is boundary to a 4-manifold. Furthermore, we can simplify the 4-manifold to consist only of one 0-handle and 2-handles. The attaching spheres of the 2-handles are knots.
- We are in high enough dimension, so that the way the attaching regions is glued onto the boundary needs more information when just the image of the attaching sphere. We need a framing.
- Attaching a 2-handle to a 4-manifold is really a Dehn-surgery on the boundary of the 4-manifold: We need to remove a solid torus from the 3-manifold (Dehn drilling) and glue back in a solid torus (Dehn filling) possibly with a twist.
- Given a torus ($S^1 \times S^1$). To fill it with a solid torus, we glue in one more 2-cell (called the compressing disk) which will kill one $\mathbb{Z}$-subgroup in $H_1 = \pi_1$, then we glue in the 3-cell. The last step has no choice. The first step has a choice, to specify, we need to draw the boundary of the compression disk on $S^1 \times S^1$.
- This is displayed in "ribbon diagrams", or by giving integral or rational numbers to the components of a link.
- This gives "Kirby-diagrams".

The most beautiful reference is D. Rolfsen's "Knots and Links." A good reference for Kirby-Diagrams and Calculus is R. Gompf and A. Stipsicz "4-Manifolds and Kirby Calculus". Dehn filling has a role in hyperbolic manifolds, as explained in J. Ratcliffe's "Foundations of Hyperbolic Manifolds".

3.2 Problem 1

Give a lower bound on the genus of a 3-manifold.

Solution: The number of generators of the homology H_1 can be at most the number of generators of C_1 which is the genus of the Heegard splitting. So the genus is at least the number of generators of H_1.

3.3 Problem 2

Make a 3-manifold which has a torsion element H_1 respectively π_1.

3.4 Problem 3

Compute the homology of the picture on page 245 (Poincare manifold) of D. Rolfsen “Knots and Links”.

Solution: Pick one copy of the curves A and B, say $+A$ and $+B$. They are the belt spheres. The solid and dashed lines are the attaching spheres. Compute the intersection numbers of the attaching spheres and belt spheres. You get a matrix, the determinant is ± 1. Hence in the chain complex $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z}$, the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ is an isomorphism and hence $H_1 = H_2 = 0$. This is a homology sphere. π is non-trivial, it is related to the icosahedral group, the triangle group (2,3,5).

3.5 Problem 4

Can you do surgery on the trefoil knot to kill the homology?

Solution: See page 246 of D. Rolfsen's “Knots and Links”.

Remark: There exists several ways to specify the Poincare homology sphere (called “Poincare manifold” in Rolfsen or “Poincare dodecahedral” space in Ratcliffe):

- as a complex algebraic surface $z_1^2 + z_2^3 + z_3^5 = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$
- quotient of the universal cover of the Lie group $SU(2)$ by a lift of the triangle group (2,3,5)
- a 2/3/5-branched cover over the torus knot of type (3,5)/(2,5)/(2,3)
- Heegard splitting
- Surgery on the trefoil knot
- A gluing pattern on a (hyperbolic) dodecahedron.

References explaining these presentations are

- J. Milnor, “On the 3-dimensional Brieskorn Manifolds $M(p, q, r)$”
- D. Rolfsen, “Knots and Links”
- J. Ratcliffe, “Foundations of Hyperbolic Manifolds”