Title

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Problem 1.0.1 (Weibel 1.3.3)
Prove the 5-lemma. Suppose the following rows are exact:

$$
\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}
$$

Link to Diagram

a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
b. Show that if f_2, f_4 are epi and f_5 is monic, then f_3 is an epi.
c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):
Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with "$\hookrightarrow$" and the epis with "$\twoheadrightarrow$":

$$
\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[d, hook, "f_4"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4
\end{tikzcd}
$$

Link to Diagram

It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2.
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \to A_2 \to A_3$, so $x = 0$.

$$\begin{tikzcd}
y_5 \in A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] \arrow[dr, phantom, "3"] & y_2 \in A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] \arrow[dr, phantom, "2"] & x \in A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "1"] & y_1 \in A_4 \arrow[d, hook, "f_4"] \\
y_4 \in B_1 \arrow[r, "\partial_1^B"] & y_3 \in B_2 \arrow[r, "\partial_2^B"] & 0 \in B_3 \arrow[r, "\partial_3^B"] & 0 \in B_4
\end{tikzcd}$$

Link to Diagram

Solution (Part (b)):
Similarly, it suffices to consider the following portion of the diagram:

$$\begin{tikzcd}
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}$$

Link to Diagram

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4, pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \to A_5 \to B_5$.
- By injectivity of f_5, the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z - y \mapsto y_1 - y_1 = 0$ under $B_3 \to B_4$, using that $d(z - y) = d(z) - d(y)$ since these are R-module morphisms.

- By exactness of the bottom row, pull $z - y$ back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2, pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z - y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z - y$ under $A_3 \to B_3$.
- But then $y_3 - \zeta_3 \mapsto z - (z - y) = y$ under $A_3 \to B_3$ as desired.

$$\begin{tikzcd}
 & \zeta_3 \arrow[ddddd, dashed, bend right] & & \\
\zeta_2 \arrow[ur, dotted] \arrow[ddd, dotted, bend right] & y_3 \arrow[r, dotted] \arrow[dddd, dashed, bend left] & y_2 \arrow[r, dotted] \arrow[ddd, dotted, bend left] & 0 \arrow[ddd, dotted, bend left] \\
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] \arrow[dr, phantom, "3"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "2"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] \arrow[dr, phantom, "1"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5 \\
\zeta_1 \arrow[ddr, dotted] & y \arrow[r, dotted] & y_1 \arrow[r, dotted] & 0 \\
 & z & & \\
 & z - y & &
\end{tikzcd}$$

Link to Diagram

Solution (Part (c)):
Given the previous two result, if the outer maps are isomorphisms then f_3 is both monic and epi. Using a technical fact that monic epis are isomorphisms in a category $\mathcal{C}$ if and only if $\mathcal{C}$ is *balanced* and that all abelian categories are balanced, f_3 is isomorphism.

Problem 1.0.2 (Weibel 1.4.2)
Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$C_n \cong Z_n \oplus B_n'$$
$$Z_n = B_n \oplus H_n'.$$

Show that C is split exact if and only if $H_n' = 0$.

Solution:
For this problem, we'll use the fact that if d is an epimorphism, it satisfies the right-cancellation property: if $f \circ d = g \circ d$, then $f = g$. In particular, if $d_n = d_n s_{n-1} d_n$ with $d_n : C_n \to C_{n-1}$

surjective and $s_{n-1} : C_{n-1} \to C_n$, we can conclude $\mathbb{1}_{C_n} = d_n s_{n-1}$. We'll also use the fact that if we have a SES in any abelian category $\mathcal{A}$, then the following are equivalent:

- The sequence is split on the left.
- The sequence is split on the right.
- The middle term is isomorphic to the direct sum of the outer terms.

Fixing notation, we'll write $C \coloneqq (\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots)$, and we'll use concatenation fg to denote function composition $f \circ g$.
$\implies$:
Suppose C is split, so we have maps $\{s_n\}$ such that $\partial_n = \partial_n s_{n-1} \partial_n$.

Claim: The short exact sequence

$$0 \to Z_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

admits a right-splitting $f : B_{n-1} \to C_n$, and thus there is an isomorphism

$$C_n \cong Z_n \oplus B'_n \coloneqq Z_n \oplus B_{n-1}.$$

Proof (?).
That this sequence is exact follows from the fact that it can be written as

$$0 \to \ker \partial_n \hookrightarrow C_n \overset{\partial_n}{\twoheadrightarrow} \operatorname{im} \partial_n \to 0.$$

We proceed by constructing the splitting f. Noting that $s_{n-1} : C_{n-1} \to C_n$ and $B_{n-1} \leq C_{n-1}$, the claim is that its restriction $f \coloneqq s_{n-1}|_{B_{n-1}}$ works. It suffices to show that $(C_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{f} C_n)$ composes to the identity map $\mathbb{1}_{C_n}$. This follows from the splitting assumption, along with right-cancellability since ∂_n is surjective onto its image:

$$\partial_n = \partial_n s_{n-1} \partial_n \overset{\text{right-cancel } \partial_n}{\implies} \mathbb{1}_{C_n} = \partial_n s_{n-1} \coloneqq \partial_n f.$$

■

Claim: The SES

$$0 \to B_n \overset{\iota_{BZ}}{\hookrightarrow} Z_n \overset{\pi}{\twoheadrightarrow} \frac{Z_n}{B_n} \to 0$$

admits a left-splitting $f : Z_n \to B_n$, and thus there is an isomorphism

$$Z_n \cong B_n \oplus H'_n \coloneqq B_n \oplus H_n(C) \coloneqq B_n \oplus \frac{Z_n}{B_n}.$$

Proof (?).
We proceed by again constructing the splitting $f : Z_n \to B_n$. The situation is summarized in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[ddr, "\partial_{n+1}"'] & C_n \arrow[l, dashed, bend right, "s_n"'] \arrow[r] & \cdots \\
& & Z_n \arrow[u, hook, "\iota_Z"'] & \\
& & B_n \arrow[u, hook, "\iota_{BZ}"'] &
\end{tikzcd}$$

Link to Diagram

So a natural candidate for the map f is the composition

$$Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} B_n,$$

so $f \coloneqq \partial_{n+1} s_n \iota_Z$. We can simplify this slightly by regarding $Z_n \leq C_n$ as a submodule to suppress ι_Z, and identify s_n with its restriction to Z_n to write $f \coloneqq \partial_{n+1} s_n$. The claim is then that $f \iota_{BZ} = \mathbb{1}_{B_n}$. Anticipating using the fact that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$, we post-compose with ∂_{n+1} and compute:

$$\begin{aligned}
f \iota_{BZ} \partial_{n+1} &= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_n) \\
&= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{s_n|_{B_n}} C_{n+1} \xrightarrow{\partial_{n+1}}) \\
&= \partial_{n+1} s_n \partial_{n+1} \\
&= \partial_{n+1},
\end{aligned}$$

where in the last step we've used the splitting hypothesis and the fact that it remains true when everything is restricted to the submodule $B_n \leq C_n$. Using surjectivity of ∂_{n+1} onto B_n, we can now conclude as before:

$$f \iota_{BZ} \partial_{n+1} = \partial_{n+1} \overset{\text{right-cancel } \partial_n}{\implies} f \iota_{BZ} = \mathbb{1}_{B_n}.$$

■

Problem 1.0.3 (Weibel 1.4.3)
Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

Solution:
$\impliedby$:

C **is split**: Suppose $\mathbb{1}_C$ is nullhomotopic, so that there exist maps

$$s_n : C_n \to C_{n+1} \qquad\qquad \mathbb{1}_{C_n} = \partial_{n+1} s_n + s_{n-1}\partial_n.$$

We then have the following situation:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathbb{1}_{C_{n+1}}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathbb{1}_{C_n}"] \arrow[l, bend right, dashed, "s_n"'] \arrow[dl, "s_n"] & C_{n-1} \arrow[r] \arrow[d, "\mathbb{1}_{C_{n-1}}"] \arrow[l, bend right, dashed, "s_{n-1}"'] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here the nullhomotopy is outlined in red, and the map relevant to the splitting in green. Note that $s_n : C_n \to C_{n+1}$ is a candidate for a splitting, we just need to show that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$. We can proceed by post-composing the LHS with the identity $\mathbb{1}_C$, which allows us to substitute in the nullhomotopy:

$$\begin{aligned}
\partial_{n+1} &= \mathbb{1}_{C_n}\partial_{n+1} \\
&= \left(\partial_{n+1}s_n + s_{n-1}\partial_n\right)\partial_{n+1} \\
&= \partial_{n+1}s_n\partial_{n+1} + s_{n-1}\partial_n\partial_{n+1} \\
&= \partial_{n+1}s_n\partial_{n+1} + s_{n-1}0 && \text{since } \partial^2 = 0 \\
&= \partial_{n+1}s_n\partial_{n+1}.
\end{aligned}$$

C **is exact**: This follows from the fact that since $\mathbb{1}_C = \partial s + s\partial$ are equal as maps of chain complexes, the images $D_1 \coloneqq \mathbb{1}_C(C)$ and $D_2 \coloneqq (\partial s + s\partial)(C)$ are equal as chain complexes and have equal homology. Evidently $D_1 = C$, and on the other hand, each graded piece $(D_2)_n$ *only* consists of boundaries coming from various pieces of C, since $\partial s + s\partial$ necessarily lands in the images of the maps ∂_n. Thus $C_n(D_2) \subseteq B_n(D_2) = \varnothing$, i.e. there are no chains (or cycles) in D_2 which are *not* boundaries, and thus $H_n(D_2) \coloneqq Z_n(D_2)/B_n(D_2) = 0$ for all n. We can thus conclude that $C = D_2 \implies H(C) = H(D_2) = 0$, so C must be exact.

$\implies$: Suppose C is split. Then by exercise 1.4.2, we have R-module decompositions

$$\begin{aligned}
C_n &\cong Z_n \oplus B_{n-1} \\
Z_n &\cong B_n \oplus H_n
\end{aligned}$$

$$\implies C_n \cong B_n \oplus B_{n-1} \oplus H_n.$$

Supposing further that C is exact, we have $H_n = 0$, and thus $C_n \cong B_n \oplus B_{n-1}$. We first note that in this case, we can explicitly write the differential ∂_n. Letting $\mathbb{1}_n$ denote the identity on C_n, where by abuse of notation we also write this for its restriction to any submodules, we have:

$$
\begin{tikzcd}
C_n \arrow[r, "\partial_n"] \arrow[d, phantom, "\shortparallel"] & C_{n-1} \arrow[d, phantom, "\shortparallel"] \\
B_n \arrow[r, dashed, "0"] \arrow[ddr, dashed, "0"] \arrow[d, phantom, "\oplus"] & B_{n-1} \arrow[d, phantom, "\oplus"] \\
{} \arrow[d, phantom] & {} \\
B_{n-1} \arrow[uur, "\mathbb{1}_n"] \arrow[r, dashed, "0"] & B_{n-2}
\end{tikzcd}
$$

Link to Diagram

We can thus write ∂_n as the matrix

$$\partial_n = \begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

Similarly using this decomposition, we can construct a map $s_n : C_n \to C_{n+1}$:

$$
\begin{tikzcd}
C_n \arrow[r, "s_n"] \arrow[d, phantom, "\shortparallel"] & C_{n+1} \arrow[d, phantom, "\shortparallel"] \\
B_n \arrow[r, dashed, "0"] \arrow[ddr, "\mathbb{1}_n"] \arrow[d, phantom, "\oplus"] & B_{n+1} \arrow[d, phantom, "\oplus"] \\
{} & {} \\
B_{n-1} \arrow[uur, dashed, "0"] \arrow[r, dashed, "0"] & B_n
\end{tikzcd}
$$

Link to Diagram

We can write this as the following matrix:

$$s_n = \begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix}.$$

We can now verify that s_n is a nullhomotopy from a direct computation:

$$
\begin{aligned}
\partial_{n+1} s_n + s_{n-1} \partial_n &= \begin{bmatrix} 0 & \mathbb{1}_{n+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbb{1}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{1}_{B_n} & 0 \\ 0 & \mathbb{1}_{B_{n-1}} \end{bmatrix} \\
&= \mathbb{1}_{C_n},
\end{aligned}
$$

expressed as a map $B_n \oplus B_{n-1} \to B_n \oplus B_{n-1}$.

Problem 1.0.4 (Weibel 1.4.5)
Show that chain homotopy classes of maps form a quotient category K of $\mathrm{Ch}(R\text{-mod})$ and that the functors H_n factor through the quotient functor $\mathrm{Ch}(R\text{-mod}) \to K$ using the following steps:

1. Show that chain homotopy equivalence is an equivalence relation on $\left\{ f : C \to D \;\middle|\; f \text{ is a chain map} \right\}$. Define $\mathrm{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
2. Let $f \simeq g : C \to D$ be two chain homotopic maps. If $u : B \to C, v : D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $\mathrm{Ch}(R\text{-mod}) \to K$ is an additive functor.
4. Is K an abelian category? Explain.

Try at least two parts.

Problem 1.0.5 (Weibel 1.5.1)
Let $\mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C)$, so

$$\mathrm{cone}(C)_n = C_{n-1} \oplus C_n.$$

Show that $\mathrm{cone}(C)$ is split exact, with splitting map given by $(b, c) \mapsto (-c, 0)$.

Solution:
Fixing notation, let $\mathbb{1}_C$ be the identity chain map on C, $\mathbb{1}_n : C_n \to C_n$ its nth graded piece, and write $\widehat{C} \coloneqq \mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C)$, $\widehat{\mathbb{1}}$ for the identity on $\widehat{C}$, and $\widehat{\mathbb{1}}_n$ for its nth piece. The result will follow from a direct computation: from exercise 1.4.2, it suffices to show that $\widehat{\mathbb{1}}$ is nullhomotopic.

Problem 1.0.6 (Weibel 1.5.2)
Let $f : C \to D \in \mathrm{Mor}(\mathrm{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(s, f) : \mathrm{cone}(C) \to D.$$

Problem 1.0.7 (Extra)

a. Show that free implies projective.

b. Show that $\mathrm{Hom}_R(M, \cdot)$ is left-exact.

c. Show that P is projective if and only if $\mathrm{Hom}_R(P, \cdot)$ is exact.