Real Analysis Qual Prep Week 1: Preliminaries

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1 Week 1: Preliminaries

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- Normed spaces
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  - limsup and liminf,
  - Cauchy criteria for sums and integrals
- Basic inequalities (triangle, Cauchy-Schwarz)
- Tools from Calculus: MVT, Taylor's theorem & remainder
- Weierstrass approximation

1.2 Warmup

- Define what it means for a function to be **uniformly continuous**.
- Define what it means for a sequence of functions to converge **pointwise** and to converge **uniformly**.
- Give two different definitions for compactness in a metric space.
- Find an example of a metric space with a closed and bounded subspace that is not compact.
  - How can this be modified to obtain a necessary and sufficient condition?
- Show that if $\sum_{n\in\mathbb{N}} a_n < \infty$ converges, then

$$a_n \overset{n\to\infty}{\longrightarrow} 0$$

and the tail is small in the following sense:

$$\sum_{n>N} a_n \overset{N\to\infty}{\longrightarrow} 0$$

- Is it possible for a function $f : \mathbb{R} \to \mathbb{R}$ to be discontinuous precisely on the rationals $\mathbb{Q}$? If so, produce such a function, if not, why?
  - Can the set of discontinuities be precisely the irrationals $\mathbb{R} \setminus \mathbb{Q}$?
- Find a sequence of continuous functions that does *not* converge uniformly, but still has a pointwise limit that is continuous.

1.3 Exercises

- Prove the uniform limit theorem: a uniform limit of continuous function is continuous.
- Show that the uniform limit of bounded functions is uniformly bounded.
- Construct sequences of functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ which converge uniformly on some set E, and yet their product sequence $\{h_n\}_{n\in\mathbb{N}}$ with $h_n := f_n g_n$ does *not* converge uniformly.
  - Show that if f_n, g_n are additionally bounded, then h_n does converge uniformly.
- Show that if $f_n : [a, b] \to \mathbb{R}$ are continuously differentiable with derivatives f_n', the sequence of derivatives f_n' converges uniformly to some function g, and there exists *at least one* point x_0 such that $\lim_n f_n(x_0)$ exists, then $f_n \to f$ uniformly to some differentiable f, and $f' = g$.
  - Find a sequence of functions such that

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \frac{d}{dx} f_n(x)$$

  - Find a uniform limit of differentiable functions that is not differentiable.

2.4 Spring 2017 # 4 ✨

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

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2.5 Spring 2015 # 1 ✨

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

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2.1 Fall 2018 # 1 ✨

Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

- Let

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-test but that it nevertheless converges uniformly on $\mathbb{R}$.

- **4.** Let $f_n \colon [0, 1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1 + x^k}.$$

1. Prove that f_n converges to a function $f \colon [0, 1) \to \mathbb{R}$.

2. Prove that for every $0 < a < 1$ the convergence is uniform on $[0, a]$.

3. Prove that f is differentiable on $(0, 1)$.

- 3. (a) Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of all the rationals in $[0, 1]$ and define $f : [0, 1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q} \end{cases}.$$

Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0, 1]$ and conclude that set of all points at which f is discontinuous is precisely $[0, 1] \cap \mathbb{Q}$.

- 6. Let

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2 x}.$$

(a) Show that the series defining g does not converge uniformly on $(0, \infty)$, but none the less still defines a continuous function on $(0, \infty)$.

Hint for the first part: Show that if $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set X, then the sequence of functions $\{g_n\}$ must converge uniformly to 0 on X.

(b) Is g differentiable on $(0, \infty)$? If so, is the derivative function g' continuous on $(0, \infty)$?

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7. Let $h_n(x) = \dfrac{x}{(1+x)^{n+1}}$.
   (a) Prove that h_n converges uniformly to 0 on $[0, \infty)$.
   (b) i. Verify that
   $$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$
   ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0, \infty)$?
   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a, \infty)$ for any $a > 0$.

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1.4 Qual Questions

3.1 Spring 2020 # 1 ✨

Prove that if $f : [0, 1] \to \mathbb{R}$ is continuous then

$$\lim_{k\to\infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

3.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

a. Show that $f_n \to 0$ pointwise but not uniformly on $[0, 1]$.

b. Show that

$$\lim_{n\to\infty} \int_0^1 n(1-x)^n \sin x\, dx = 0$$

Hint for (a): Consider the maximum of f_n.

3.11 Fall 2020 # 1

Show that if x_n is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then

$$\lim_{n\to\infty} n x_n = 0.$$