

Title

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0.1 Big Theorems / Tools:

Proposition 0.1.1(*Fundamental Theorem of Calculus I*).

$$\frac{\partial}{\partial x} \int_a^x f(t) dt = f(x)$$

Proposition 0.1.2(*Generalized Fundamental Theorem of Calculus*).

$$\begin{aligned} \frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, t) dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt &= f(x, t) \cdot \frac{\partial}{\partial x}(t) \Big|_{t=a(x)}^{t=b(x)} \\ &= f(x, b(x)) \cdot b'(x) - f(x, a(x)) \cdot a'(x) \end{aligned}$$

If $f(x, t) = f(t)$ doesn't depend on x , then $\frac{\partial f}{\partial x} = 0$ and the second integral vanishes:

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(t) dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

Find examples.

Remark 0.1.1.

Note that you can recover the original FTC by taking

$$\begin{aligned} a(x) &= c \\ b(x) &= x \\ f(x, t) &= f(t). \end{aligned}$$

Corollary 0.1.1(?).

$$\frac{\partial}{\partial x} \int_1^x f(x, t) dt = \int_1^x \frac{\partial}{\partial x} f(x, t) dt + f(x, x)$$

Proposition 0.1.3(Extreme Value Theorem).

Todo

\todo[inline]{Todo}

Proposition 0.1.4(Mean Value Theorem).

$$\begin{aligned} f \in C^0(I) &\implies \exists p \in I : f(b) - f(a) = f'(p)(b - a) \\ &\implies \exists p \in I : \int_a^b f(x) dx = f(p)(b - a). \end{aligned}$$

Proposition 0.1.5(Rolle's Theorem).

todo

Proposition 0.1.6(L'Hopital's Rule).

If

- $f(x)$ and $g(x)$ are differentiable on $I - \{\text{pt}\}$, and

$$\lim_{x \rightarrow \{\text{pt}\}} f(x) = \lim_{x \rightarrow \{\text{pt}\}} g(x) \in \{0, \pm\infty\}, \quad \forall x \in I, g'(x) \neq 0, \quad \lim_{x \rightarrow \{\text{pt}\}} \frac{f'(x)}{g'(x)} \text{ exists,}$$

Then it is necessarily the case that

$$\lim_{x \rightarrow \{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \{\text{pt}\}} \frac{f'(x)}{g'(x)}.$$

Remark 0.1.2.

Note that this includes the following indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty, \quad \infty - \infty.$$

For $0 \cdot \infty$, can rewrite as $\frac{0}{1/\infty}$

For $1^\infty, \infty^0$, and 0^0 , set

$$L := \lim f^g \implies \ln L = \lim g \ln(f)$$

Proposition 0.1.7 (Taylor Expansion).

$$\begin{aligned}T(a, x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\&= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\&\quad + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{24}f^{(4)}(a)(x-a)^4 + \dots\end{aligned}$$

There is a bound on the error:

$$|f(x) - T_k(a, x)| \leq \left| \frac{f^{(k+1)}(a)}{(k+1)!} \right|$$

where $T_k(a, x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n$ is the k th truncation.

0.2 Differential

0.2.1 Limits

0.2.2 Tools for finding limits

How to find $\lim_{x \rightarrow a} f(x)$ in order of difficulty:

- Plug in: if f is continuous, $\lim_{x \rightarrow a} f(x) = f(a)$.
- Check for indeterminate forms and apply L'Hopital's Rule.
- Algebraic rules
- Squeeze theorem
- Expand in Taylor series at a
- Monotonic + bounded
- One-sided limits: $\lim_{x \rightarrow a^-} f(x) = \lim_{\varepsilon \rightarrow 0} f(a - \varepsilon)$
- Limits at zero or infinity:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{\frac{1}{x} \rightarrow 0} f\left(\frac{1}{x}\right) \text{ and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow \infty} f(1/x)$$

– Also useful:

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \deg p < \deg q \\ \infty & \deg p > \deg q \\ \frac{p_n}{q_n} & \deg p = \deg q \end{cases}$$

- Be careful: limits may not exist!!

- Example : $\lim_{x \rightarrow 0} \frac{1}{x} \neq 0$
- Asymptotes:
 - Vertical asymptotes: at values $x = p$ where $\lim_{x \rightarrow p} = \pm\infty$
 - Horizontal asymptotes: given by points $y = L$ where $L = \lim_{x \rightarrow \pm\infty} f(x) < \infty$
 - Oblique asymptotes: for rational functions, divide - terms without denominators yield equation of asymptote (i.e. look at the asymptotic order or “limiting behavior”).
 - ◇ Concretely: $f(x) = \frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{t(x)} \sim r(x)$
- Limit of a recurrence: $x_n = f(x_{n-1}, x_{n-2}, \dots)$
 - If the limit exists, it is a solution to $x = f(x)$

0.2.3 Derivatives

- Chain rule: $\frac{\partial}{\partial x} (f \circ g)(x) = f'(g(x))g'(x)$
- Product rule: $\frac{\partial}{\partial f} (x)g(x) = f'g + g'f$
 - Note for all rules: always prime the first thing!
- Quotient rule: $\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{f'g - g'f}{g^2}$
 - Mnemonic: Low d-high minus high d-low
- Inverse rule: $\frac{\partial f^{-1}}{\partial x} (f(x_0)) = \left(\frac{\partial f}{\partial x} \right)^{-1} (x_0) = 1/f'(x_0)$
- Implicit differentiation: $(f(x))' = f'(x) dx, (f(y))' = f'(y) dy$
 - Often able to solve for $\frac{\partial}{\partial y} [x]$ this way.
- Obtaining derivatives of inverse functions: if $y = f^{-1}(x)$ then write $f(y) = x$ and implicitly differentiate.
- Approximating change: $\Delta y \approx f'(x)\Delta x$

0.2.4 Related Rates

General series of steps: want to know some unknown rate y_t

- Lay out known relation that involves y
- Take derivative implicitly (say w.r.t t) to obtain a relation between y_t and other stuff.
- Isolate $y_t =$ known stuff
- Example: ladder sliding down wall
 - Setup: l, x_t and $x(t)$ are known for a given t , want y_t .
 - $x(t)^2 + y(t)^2 = l^2 \implies 2xx_t + 2yy_t = 2ll_t = 0$ (noting that l is constant)
 - So $y_t = -\frac{x(t)}{y(t)}x_t$
 - $x(t)$ is known, so obtain $y(t) = \sqrt{l^2 - x(t)^2}$ and solve.

0.3 Integral

- Average values:

$$f_{\text{avg}}(x) = \frac{1}{b-a} \int_a^b f(t) dt$$

- Proof: apply MVT to $F(x)$.
- Area Between Curves
 - Area in polar coordinates:

$$A = \int_{r_1}^{r_2} \frac{1}{2} r^2(\theta) d\theta$$

- Solids of Revolution
 - Disks: $A = \int \pi r(t)^2 dt$
 - Cylinders: $A = \int 2\pi r(t)h(t) dt$
- Arc lengths

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} & L &= \int ds \\ &= \int_{x_0}^{x_1} \sqrt{1 + \frac{\partial y}{\partial x}} dx \\ &= \int_{y_0}^{y_1} \sqrt{\frac{\partial x}{\partial y} + 1} dy \end{aligned}$$

- $SA = \int 2\pi r(x) ds$
- Center of Mass
 - Given a density $\rho(\mathbf{x})$ of an object R , the x_i coordinate is given by

$$x_i = \frac{\int_R x_i \rho(x) dx}{\int_R \rho(x) dx}$$

0.3.1 Big List of Integration Techniques

Given $f(x)$, we want to find an antiderivative $F(x) = \int f$ satisfying $\frac{\partial}{\partial x} F(x) = f(x)$

- Guess and check: look for a function that differentiates to f .
- u -substitution
 - More generally, any change of variables

$$x = g(u) \implies \int_a^b f(x) dx = \int_{g^{-1}(a)}^{g^{-1}(b)} (f \circ g)(x) g'(x) dx$$

- Integration by Parts:
 - The standard form:

$$\int u dv = uv - \int v du$$

- A more general form for repeated applications: let $v^{-1} = \int v$, $v^{-2} = \int \int v$, etc.

$$\begin{aligned}
 \int_a^b uv &= uv^{-1} \Big|_a^b - \int_a^b u^1 v^{-1} \\
 &= uv^{-1} - u^1 v^{-2} \Big|_a^b + \int_a^b u^2 v^{-2} \\
 &= uv^{-1} - u^1 v^{-2} + u^2 v^{-3} \Big|_a^b - \int_a^b u^3 v^{-3} \\
 &\vdots \\
 \Rightarrow \int_a^b uv &= \sum_{k=1}^n (-1)^k u^{k-1} v^{-k} \Big|_a^b + (-1)^n \int_a^b u^n v^{-n}
 \end{aligned}$$

- Generally useful when one term's n th derivative is a constant.
- Shoelace method:
- Note: you can choose u or v equal to 1! Useful if you know the derivative of the integrand.
- Differentiating under the integral

$$\begin{aligned}
 \frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, t) dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt &= f(x, \cdot) \frac{\partial}{\partial x} (\cdot) \Big|_{a(x)}^{b(x)} \\
 &= f(x, b(x)) b'(x) - f(x, a(x)) a'(x)
 \end{aligned}$$

- Proof: let $F(x)$ be an antiderivative and compute $F'(x)$ using the chain rule.
- #todo for constants, this should allow differentiating under the integral when f, f_x are “jointly continuous”
- LIPET: Log, Inverse trig, Polynomial, Exponential, Trig: generally let u be whichever one comes first.
- The ridiculous trig sub: for any integrand containing only trig terms
 - Transforms *any* such integrand into a rational function of x
 - Let $u = 2 \tan^{-1} x$, $du = \frac{2}{x^2 + 1}$, then

$$\int_a^b f(x) dx = \int_{\tan \frac{a}{2}}^{\tan \frac{b}{2}} f(u) du$$

◇ Example: $\int_0^{\pi/2} \frac{1}{\sin \theta} d\theta = 1/2$

Derivatives	Integrals	Signs	Result
u	v	NA	NA
u'	$\int v$	+	$u \int v$

Derivatives	Integrals	Signs	Result
u''	$\int \int v$	$-$	$-u' \int \int v$
\vdots	\vdots	\vdots	\vdots

Fill out until one column is zero (alternate signs). Get the result column by multiplying diagonally, then sum down the column.

- Trigonometric Substitution

$$\begin{array}{llll}
 \sqrt{a^2 - x^2} & \Rightarrow & x = a \sin(\theta) & dx = a \cos(\theta) d\theta \\
 \sqrt{a^2 + x^2} & \Rightarrow & x = a \tan(\theta) & dx = a \sec^2(\theta) d\theta \\
 \sqrt{x^2 - a^2} & \Rightarrow & x = a \sec(\theta) & dx = a \sec(\theta) \tan(\theta) d\theta
 \end{array}$$

- Partial Fractions
- Completing the Square #todo
- Trig Formulas
 - Double angle formulas:

$$\begin{array}{lll}
 \sin^2(x) & = & \frac{1}{2}(1 - \cos x) \\
 & = & \\
 & = & \\
 & = & \\
 & = &
 \end{array}$$

- Products of trig functions
 - Setup: $\int \sin^a(x) \cos^b(x) dx$
 - ◇ Both a, b even: $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$
 - ◇ a odd: $\sin^2 = 1 - \cos^2$, $u = \cos(x)$
 - ◇ b odd: $\cos^2 = 1 - \sin^2$, $u = \sin(x)$
 - Setup: $\int \tan^a(x) \sec^b(x) dx$
 - ◇ a odd: $\tan^2 = \sec^2 - 1$, $u = \sec(x)$
 - ◇ b even: $\sec^2 = \tan^2 + 1$, $u = \tan(x)$

Other small but useful facts:

$$\int_0^{2\pi} \sin \theta d\theta = \int_0^{2\pi} \cos \theta d\theta = 0.$$

0.3.2 Optimization

- Critical points: boundary points and wherever $f'(x) = 0$
- Second derivative test:

- $f''(p) > 0 \implies p$ is a min
- $f''(p) < 0 \implies p$ is a max

- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.
- Fundamental theorem of Calculus: If

$$\int f(x)dx = F(b) - F(a) \implies F'(x) = f(x).$$

- Min/maxing - either derivatives or Lagrange multipliers!
- Distance from origin to plane: equation of a plane

$$P : ax + by + cz = d.$$

- You can always just read off the normal vector $\mathbf{n} = (a, b, c)$. So we have $\mathbf{n}\mathbf{x} = d$.
- Since $\lambda\mathbf{n}$ is normal to P for all λ , solve $\mathbf{n}\lambda\mathbf{n} = d$, which is $\lambda = \frac{d}{\|\mathbf{n}\|^2}$
- A plane can be constructed from a point p and a normal n by the equation $np = 0$.
- In a sine wave $f(x) = \sin(\omega x)$, the period is given by $2\pi/\omega$. If $\omega > 1$, then the wave makes exactly ω full oscillations in the interval $[0, 2\pi]$.
- The directional derivative is the gradient dotted against a *unit vector* in the direction of interest
- Related rates problems can often be solved via implicit differentiation of some constraint function
- The second derivative of a parametric equation is not exactly what you'd intuitively think!
- For the love of god, remember the FTC!

$$\frac{\partial}{\partial x} \int_0^x f(y)dy = f(x)$$

- Technique for asymptotic inequalities: WTS $f < g$, so show $f(x_0) < g(x_0)$ at a point and then show $\forall x > x_0, f'(x) < g'(x)$. Good for big-O style problems too.
- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.

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