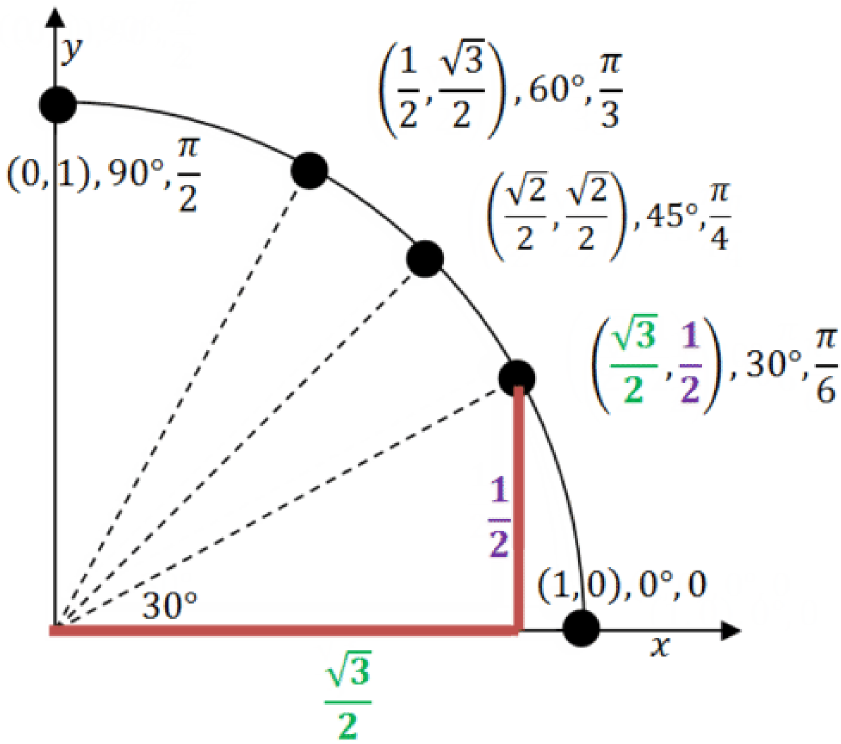

Notes: These are rough notes for the Math 1113 Precalculus course at the University of Georgia

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Precalculus

University of Georgia, Spring 2021

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

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1 Preface

2 Unit 1: Functions

Theorem 2.0.1 ***(The Pythagorean Theorem).***
If a, b are the legs of a right triangle with hypotenuse c, there is a relation

$$a^2 + b^2 = c^2.$$

Theorem 2.0.2 ***(The Distance Formula).***
If $p = (x_1, y_1)$ and $q = (x_2, y_2)$ are points in the Cartesian plane, then there is a **distance function**

$$\begin{aligned} d : \{\text{Pairs of points } (p,q)\} &\to \mathbb{R} \\ (p,q) &\mapsto d(p,q) \coloneqq \sqrt{(x_2 - x_1)^2 + (y_2 - y_q)^2}. \end{aligned}$$

Law of cosines

Definition 2.0.3 (Linear Functions)
A function $f : \mathbb{R} \to \mathbb{R}$ is **linear** if and only if f has a formula of the following form:

$$f(x) = \alpha x + \beta \qquad \alpha, \beta \in \mathbb{R}.$$

Definition 2.0.4 (Intercepts)
Given a function $f : \mathbb{R} \to \mathbb{R}$, an x**-intercept** of f is a point $(x_0, 0)$ on the graph of f, so $f(x_0) = 0$. Equivalently, it is a point on the intersection of the graph and the x-axis.

A y**-intercept** of f is a point $(0, y_0)$ on the graph of f, so $f(0) = y_0$. Equivalently, it is a point on the intersection of the graph and the y-axis.

Definition 2.0.5 (Relation)
A **relation** on two sets X and Y is a set of ordered pairs $(x, y) \in X \times Y$, so R can be described as a set:

$$R = \{(x_0, y_0), (x_1, y_2), \cdots\}.$$

The **domain** of the relation is the set of all $x \in X$ that occur in the first slot of these pairs, and the **range** is the set of all $y \in Y$ that occur in the second slot.

Definition 2.0.6 (Function)
A relation R is a **function** if it satisfies the following *deterministic property*: for every $x_0 \in$

> dom(R), there is exactly *one* pair of the form $(x_0, y_0) \in R$.

Remark 2.0.7: This says we can think of X as "inputs" and Y as "output", and a function is a way to unambiguously assign inputs to outputs. It can be useful to think of functions like programs: if I send in an x, what y should the program return to me? If I run this program today, tomorrow, and 100 years from now, sending in the same x every time, we might want it to give the same output every time, which is the *deterministic* property: I can *determine* a single unique output if I know what the input is. If my program tells me that $2 + 2 = 4$ today but $2 + 2 = 5$ tomorrow, who knows what it will return in 100 years! We can't "determine" it.

Slogan 2.0.8
For domains and ranges:

- Domains: the set of *meaningful* inputs that the function "knows" how to handle.
- Ranges: the set of *attainable* outputs that we can expect.

Remark 2.0.9: To determine a domain:

1. Naively hope it is *all* of $\mathbb{R}$.
2. Throw out "problematic" points.
3. Draw a number line and write out what you are left with in interval notation.

Example 2.0.10*(?)*: Define

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$

Then $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ and $\mathrm{range}(f) = \mathbb{R}$.

Example 2.0.11*(?)*: Define

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sqrt{x}. \end{aligned}$$

Then $\mathrm{dom}(f) = \mathbb{R} \setminus (-\infty, 0) = [0, \infty)$ and $\mathrm{range}(f) = [0, \infty)$.

3 Unit 2: Exponential and Logarithmic Functions

4 Unit 3: Trigonometric Functions

4.1 General Notes

- In this section, always draw a picture! Virtually 100% of the time.
  - In particular, a unit circle should almost always show up.
- Use exact ratios wherever possible.
- There are too many details and formulas to just memorize in this unit: focus on the **processes**.

4.2 Common Mistakes

Some facts to remember:

- $\sin^{-1}(\theta) \neq 1/\sin(\theta)$. Mnemonic: reciprocals of trigonometric functions already have a better name, here $\csc(\theta)$.

4.3 Basic Trigonometric Functions

Sin/cos/etc as ratios

4.4 Proportionality Relationships

Definition 4.4.1 (Radian)

What is a 1 radian?

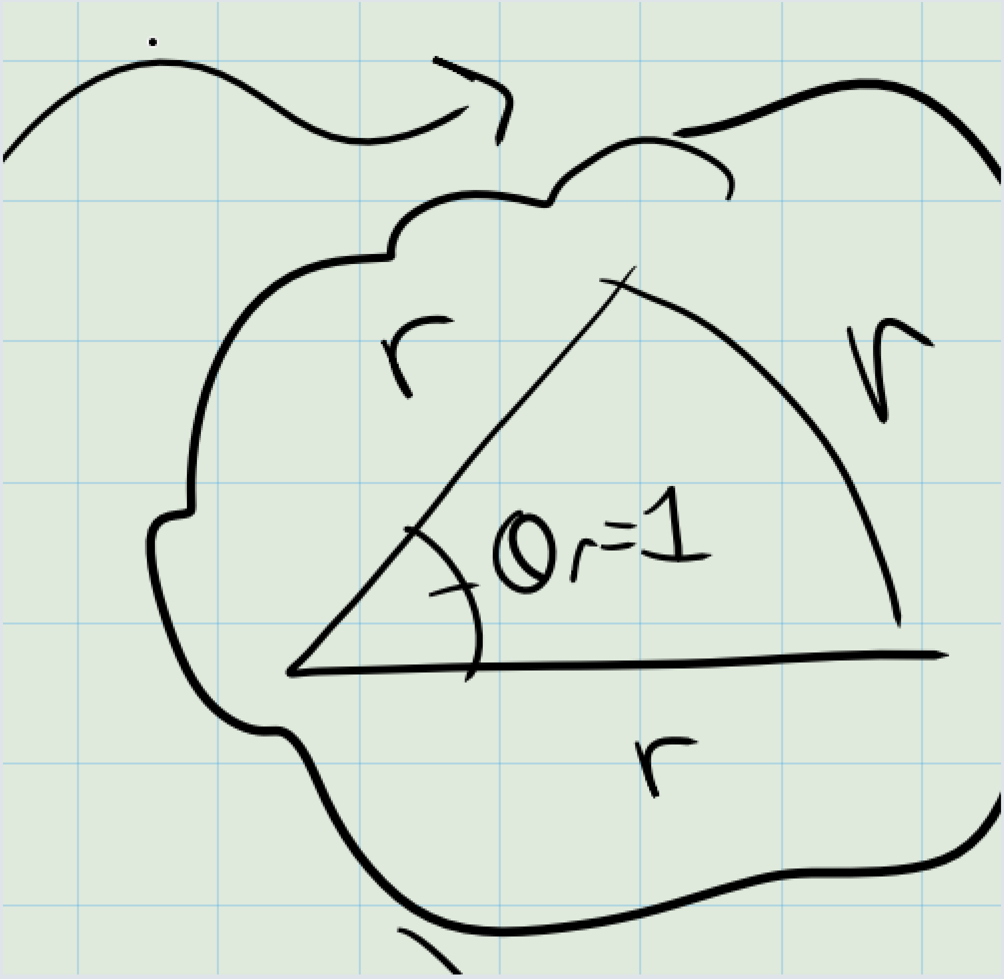


Figure 1: image_2021-04-18-21-51-59

Remark 4.4.2: In geometric terms, an angle in radians in the ratio of the arc length $s(\theta, R)$ to the radius R, so

$$\theta_R = \frac{s(\theta, R)}{R}.$$

Definition 4.4.3 (Coterminal Angles)
If θ is an abstract angle, we will say $\theta + k\,\mathrm{rev} \simeq \theta$ for any integer $k \in \mathbb{Z}$. Any such angle is said to be **coterminal** to θ.

Remark 4.4.4: In radians:

$$\theta_R \simeq \theta_R + k \cdot 2\pi \qquad k \in \mathbb{Z}.$$

In degrees:

$$\theta_D \simeq \theta_D + k \cdot 360^\circ \qquad k \in \mathbb{Z}.$$

Proposition 4.4.5 ***(Degrees are related to radians).***

todo

$$\frac{\theta}{1\,\text{rev}} = \frac{\theta_R}{2\pi\,\text{rad}} = \frac{\theta_D}{360^\circ}.$$

Proposition 4.4.6 ***(Arc length and sector area are related to radians).***

todo

$$\frac{\theta}{1\,\text{rev}} = \frac{s(R,\theta)}{2\pi R} = \frac{A(R,\theta)}{\pi R^2}.$$

This implies that

$$A(R,\theta) = \frac{R^2\theta}{2}$$
$$s(R,\theta) = R\theta.$$

4.5 Trigonometric Functions as Ratios

Definition 4.5.1 (?)
There are 6 trigonometric functions defined by the following ratios:

soh-cah-toa, cho-sha-cao

Function	Domain	Range
sin	$\mathbb{R}$	$[-1,1]$
cos	$\mathbb{R}$	$[-1,1]$
tan	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	?
csc	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?

sec	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	?
cot	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?

Proposition 4.5.2 ***(Domains of trigonometric functions).***

4.6 Polar Coordinates

Definition 4.6.1 (Unit Circle)
The **unit circle** is defined as

$$S^1 := \left\{\mathbf{p} = (x, y) \in \mathbb{R}^2 \;\middle|\; d(\mathbf{p}, \mathbf{0}) = 1\right\} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\},$$

the set of all points in the plane that are distance exactly 1 from the origin.

Theorem 4.6.2 ***(Polar Coordinates).***
If a vector $\mathbf{v}$ has at an angle of θ in radians and has length R, the corresponding point $\mathbf{p}$ at the end of $\mathbf{v}$ is given by

$$\mathbf{p} = [x, y] = [R\cos(\theta), R\sin(\theta)].$$

Conversely, if (x, y) are known, then the corresponding R and θ are given by

$$[R, \theta] = \left[\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right].$$

Corollary 4.6.3 ***(Polar Coordinates on S^1).***
If $R = 1$, so $\mathbf{v}$ is on the unit circle S^1, then

$$[x, y] = [\cos(\theta), \sin(\theta)].$$

Remark 4.6.4: This is a very important fact! The x, y coordinates on the unit circle *literally* corresponding to cosines and sines of subtended angles will be used frequently.

Slogan 4.6.5
Cosines are like x coordinates, sines are like y coordinates.

Example 4.6.6 ***(?):*** Given $\theta_R = 4\pi/3$, what is the corresponding point on the unit circle S^1?

⚠ **Warning 4.6.7**
Note that $\sin(\theta), \cos(\theta)$ work for any θ at all. However, $\cos(\theta) = 0$ sometimes, so $\tan(\theta) := \sin(\theta)/\cos(\theta)$ will on occasion be problematic. Similar story for the other functions.

4.7 Special Angles

For reference: the unit circle.

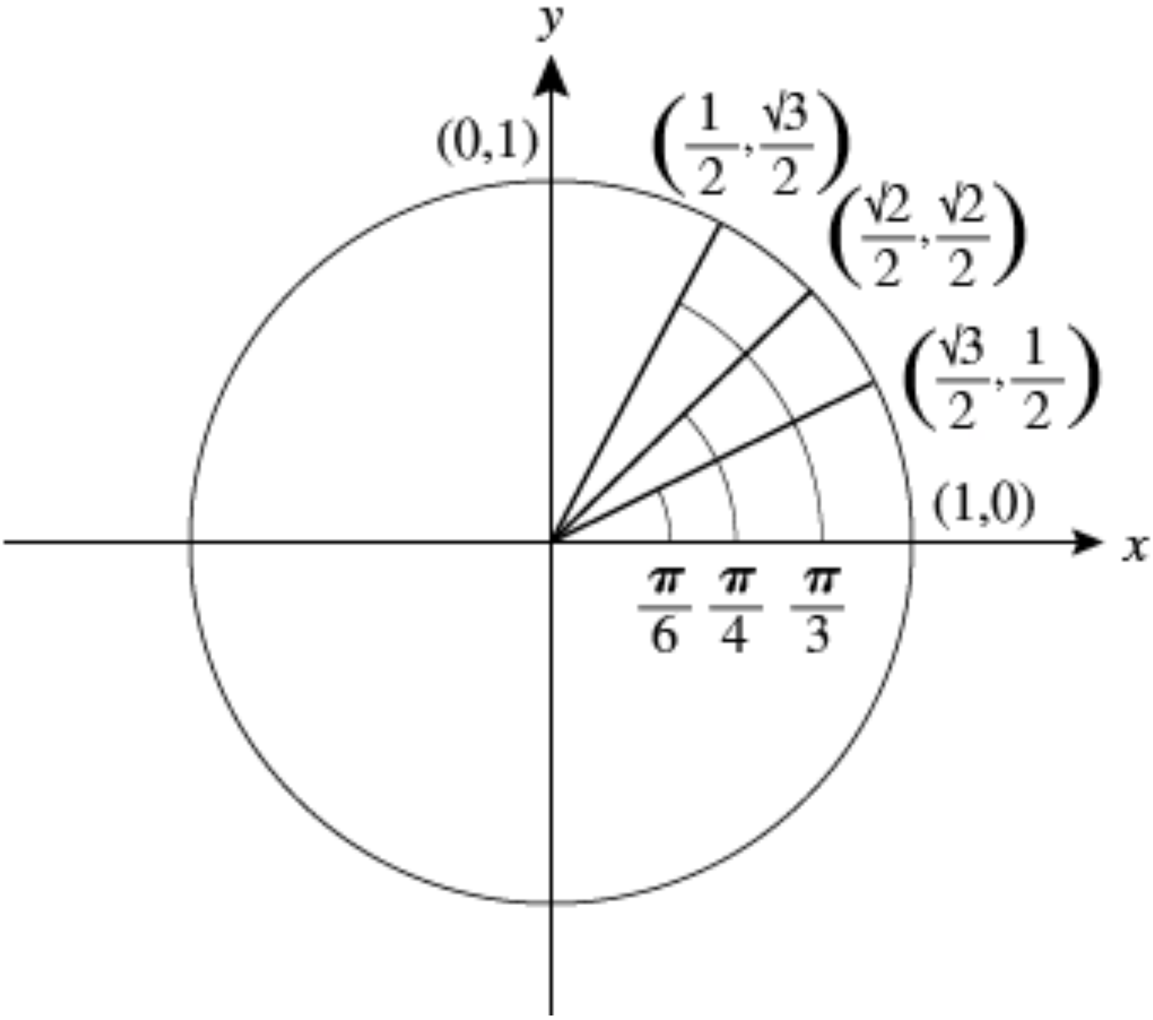


Figure 2: image_2021-04-18-21-06-45

Remark 4.7.1: Idea: we want to partition the circle simultaneously

- Into 8 pieces, so we increment by $2\pi/8 = \pi/4$
- Into 12 pieces, so we increment by $2\pi/12 = \pi/6$.

Proposition 4.7.2 ***(Trick to memorize special angles).***

Table of special angles, increasing/decreasing

4.8 Reference Angles and the Flipping Method

Definition 4.8.1 (Reference Angle)
Given a vector at of length R and angle θ, the **reference angle** θ_{Ref} is the acute angle in the triangle formed by dropping a perpendicular to the nearest horizontal axis.

Proposition 4.8.2 *(?).*
Reference angles for each quadrant:

$$\begin{aligned} \text{Quadrant II}: & \quad \theta + \theta_{\text{Ref}} = \pi \\ \text{Quadrant III}: & \quad \pi + \theta_{\text{Ref}} = \theta \\ \text{Quadrant IV}: & \quad \theta + \theta_{\text{Ref}} = 2\pi. \end{aligned}$$

Example 4.8.3 ***(?):*** Given $\sin(\theta) = 7/25$, what are the five remaining trigonometric functions of θ?

Method:

1. Draw a picture! Embed θ into a right triangle.
2. Find the missing side using the Pythagorean theorem.
3. Use definition of trigonometric functions are ratios.

Remark 4.8.4: Note that you can not necessarily find the angle θ here, but we didn't need it. If we *did* want θ, we would need an inverse function to free the argument:

$$\begin{aligned} \sin(\theta) &= 7/25 \\ \implies \arcsin(\sin(\theta)) &= \arcsin(7/25) \\ \implies \theta &= \arcsin(7/25) \end{aligned}$$

.

4.9 Identities Using Pythagoras

Proposition 4.9.1 *(?).*

$$\begin{aligned} (\sin(\theta))^2 + (\cos(\theta))^2 &= 1 \\ 1 + (\cot(\theta))^2 &= (\csc(\theta))^2 \\ (\tan(\theta))^2 + 1 &= (\sec(\theta))^2. \end{aligned}$$

Proof (?).
Derive first from Pythagorean theorem in S^1. Obtain the second by dividing through by $(\sin(\theta))^2$. Obtain the third by dividing through by $(\cos(\theta))^2$. ■

4.10 Even/Odd Properties

Question 4.10.1
Thinking of $\cos(\theta)$ as a function of θ, is it

- Even?
- Odd?
- Neither?

Remark 4.10.2: Why do we care? The Fundamental Theorem of Calculus.

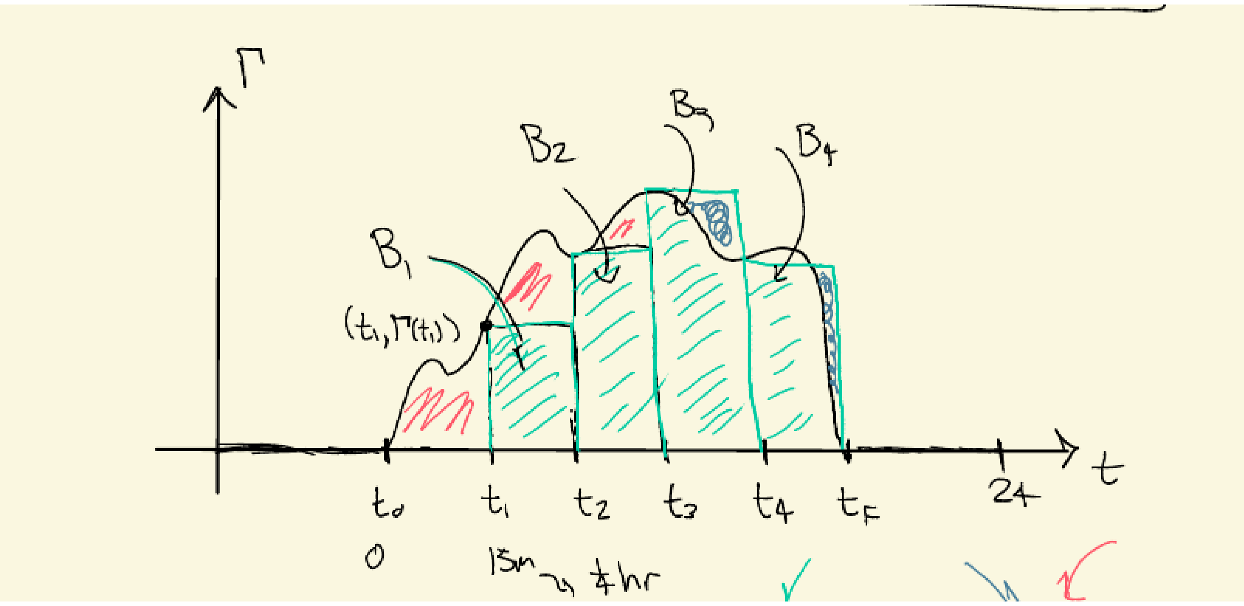


Figure 3: image_2021-04-18-22-39-08

Proposition 4.10.3 *(?).*

- $f(\theta) := \cos(\theta)$ is an even function.
- $g(\theta) := \sin(\theta)$ is an odd function.

Proof (?).
Plot vectors for $\theta, -\theta$ on S^1 and flip over the x-axis. ■

Corollary 4.10.4 *(?).*

- $\cos(t), \sec(t)$ are even.
- $\sin(t), \csc(t), \tan(t), \cot(t)$ are odd.

4.11 Wave Function

Remark 4.11.1: Motivation: let a vector run around the unit circle, where we think of θ as a time parameter. What are its x and y coordinates? What happens if we plot $x(t)$ in a new θ plane?

Definition 4.11.2 (Standard Form of a Wave Function)
The **standard form** of a wave function is given by

$$f(t) := A\cos(\omega(t-\varphi)) + \delta,$$

where

- A is the **amplitude**,
- ω is the **frequency**,
- φ is the **phase shift**, and
- δ is the **vertical shift**.
- $P := 2\pi/\omega$ is the **period**, so $f(t+kP) = f(t)$ for all $k \in \mathbb{Z}$.

Insert plot

Remark 4.11.3: Note that this is nothing more than a usual cosine wave, just translated/dilated in the x direction and the y direction.

Warning 4.11.4
Don't memorize equations like $y = \sin(Bt + C)$ and e.g. the phase shift if $\varphi = -C/B$. Instead, use a process: always put your equation in standard form, then you can just read off the parameters. For example:

$$\begin{aligned} f(t) &= \cos(Bt + C) \\ &= \cos(B(t + \frac{C}{B})) \\ &= \cos(\omega(t-\varphi)) \end{aligned}$$

$$\implies B = \omega, \varphi = -\frac{C}{B}.$$

Example 4.11.5 ***(?):*** Put the following wave in standard form:

$$f(t) \coloneqq 4\cos(3t+2).$$

Example 4.11.6 ***(?):*** Put the following wave in standard form:

$$f(t) \coloneqq \alpha\cos(\beta t+\gamma).$$

Proposition 4.11.7 ***(?).***
How to plot the graph of a wave equation:

1. Put in standard form.
2. Read off the parameters to build a rectangular box of width P and height $2|A|$ about the line $y=\delta$.
3. Break the box into 4 pieces using the key points $t=\varphi+\frac{k}{4}P$ for $k=0,1,2,3,4$.

Example 4.11.8 ***(Plotting):*** Plot the following function in the t plane:

$$f(t)=2\cos\left(5t-\frac{\pi}{2}\right)+7.$$

Example 4.11.9 ***(?):*** Plot the following:

$$f(t)=-2\sin(3t-7).$$

Proposition 4.11.10 ***(Determining the equation of a sine wave).***
Given a picture of a graph of a sine wave,

1. Draw a horizontal line cutting the wave in half. This will be δ.
2. Measure the distance from this midline to a peak. This will be $|A|$.
3. Restrict to one full period, starting either at a peak (if you want to match $\cos(t)$) or a zero (if you want to match $\sin(t)$). Pick the period starting as close as possible to the y-axis.
4. Measure the period P and reverse-engineer it to get ω: $P=2\pi/\omega \implies \omega=2\pi/P$.
5. Measure the distance from the starting point to the y-axis: this is φ.

Example 4.11.11 ***(?):*** Determine the equation of the following wave function:

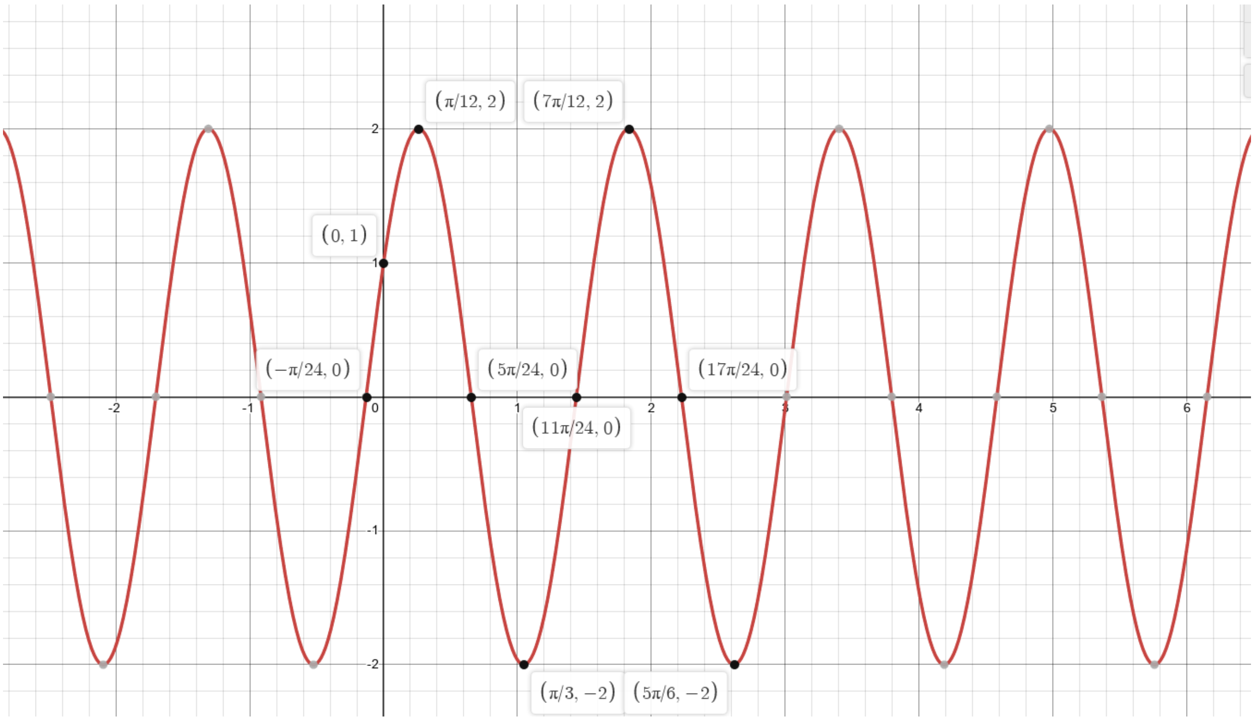


Figure 4: image_2021-04-18-20-51-34

Solution:

$$f(t) = 2\sin\left(4t + \frac{\pi}{6}\right).$$

Remark 4.11.12: Note that we can graph other trigonometric functions: they get pretty wild though.

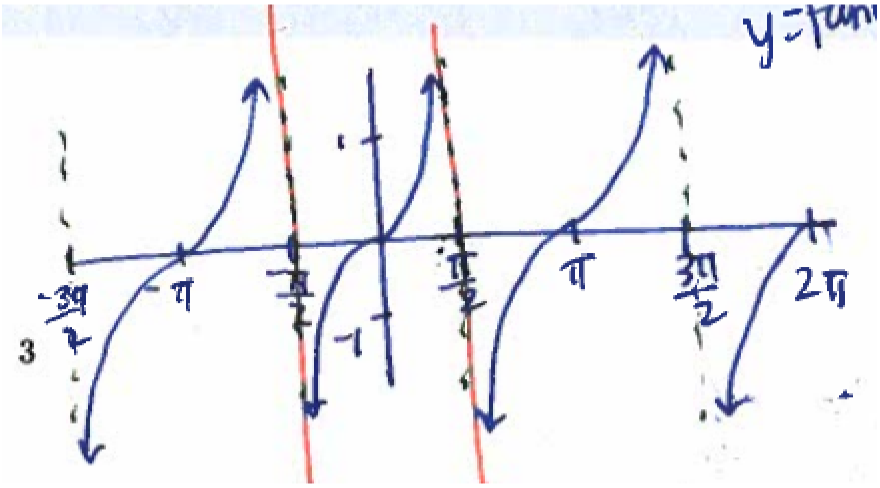
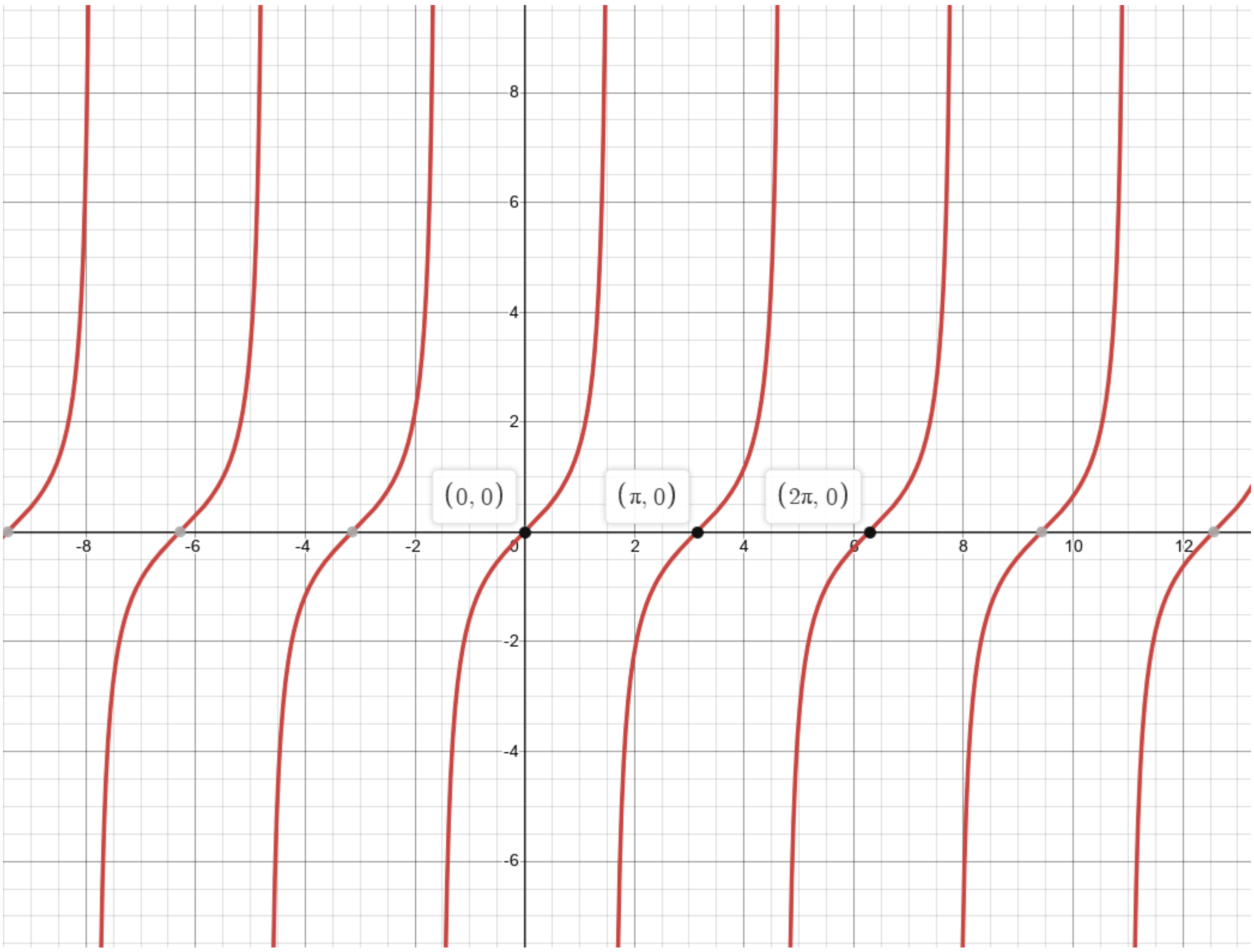


Figure 5: Tangent



4.12 Simplifying Identities

Remark 4.12.1: The goal: reduce a complicated mess of trigonometric functions to something as simple as possible. We'll use a **boxing-up method**.

Remark 4.12.2: On verifying identities: if you want to show $f(\theta) = g(\theta)$, start at one and arrive at the other:

$$\begin{aligned} f(\theta) &= \text{simplify } f \\ &= \cdots \\ &= \cdots \\ &= \cdots \\ &= g(\theta) \end{aligned}$$

.

⚠ **Warning 4.12.3**
If you end up with something like $1 = 1$ or $0 = 0$, this is hinting at a problem with your logic.

Exercise 4.12.4 (?)
Simplify the following:

$$F(\theta) := \left(\frac{\sin(\theta)\cos(\theta)}{\cot(\theta)} \right) \cos(\theta)\csc(\theta).$$

Solution:

$$F = s\left(\frac{s}{c}\right).$$

Remark 4.12.5: As an alternative, you can use the **transitivity of equality**: show that $f(\theta) = h(\theta)$ for some totally different function h, and then show $g(\theta) = h(\theta)$ as well.

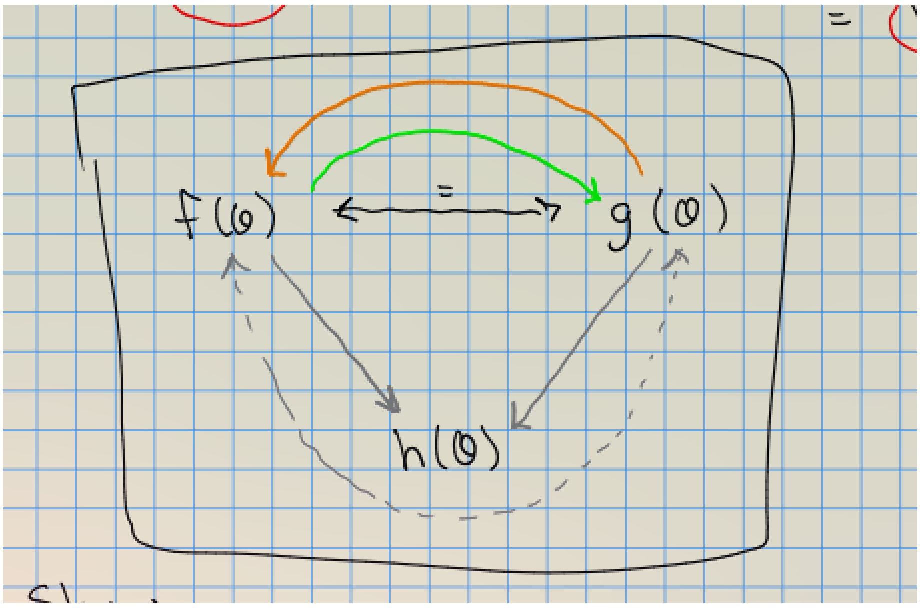


Figure 6: image_2021-04-18-21-58-52

Exercise 4.12.6 (Reducing both sides to a common expression)
Show the following identity:

$$\sin(-\theta) + \csc(\theta) = \cot(\theta)\cos(\theta)$$

by showing both sides are separately equal to $h(\theta) := \csc(\theta) - \sin(\theta)$.

4.13 Inverse Functions

4.13.1 Motivation

Remark 4.13.1: Motivation: we want a way to solve equations where the unknown θ is stuck in the argument of a trigonometric function. For example, for $\sin : \mathbb{R}_A \to \mathbb{R}_B$, this would be some function $f : \mathbb{R}_B \to \mathbb{R}_A$ such that

$$\begin{aligned} f(\sin(\theta)) &= \mathrm{id}(\theta) = \theta \\ \sin(f(y)) &= \mathrm{id}(y) = y. \end{aligned}$$

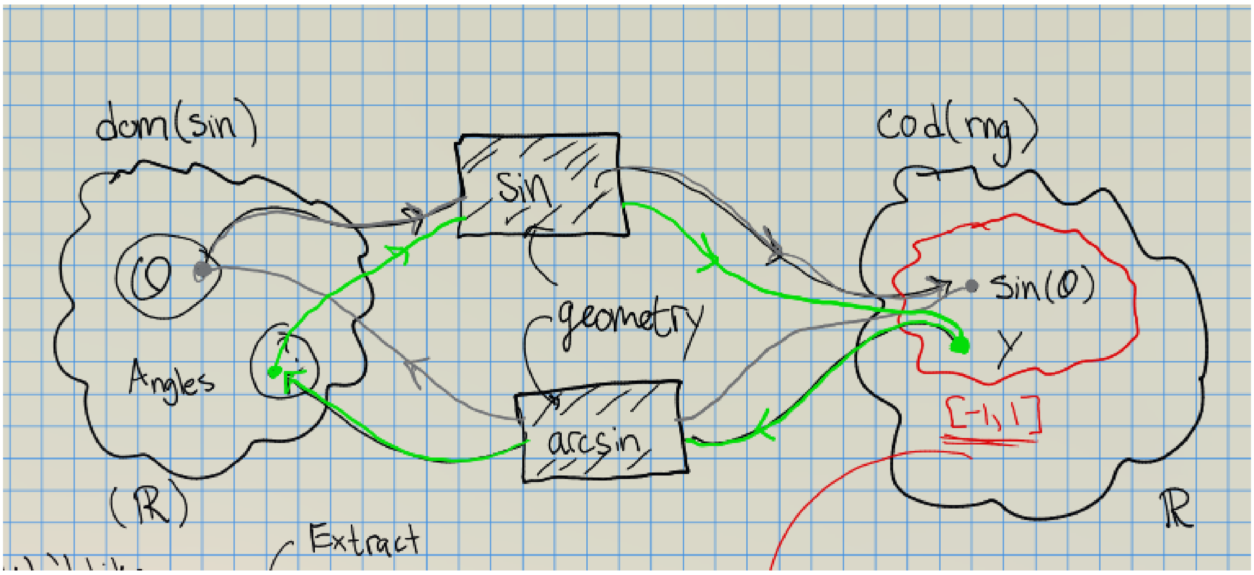
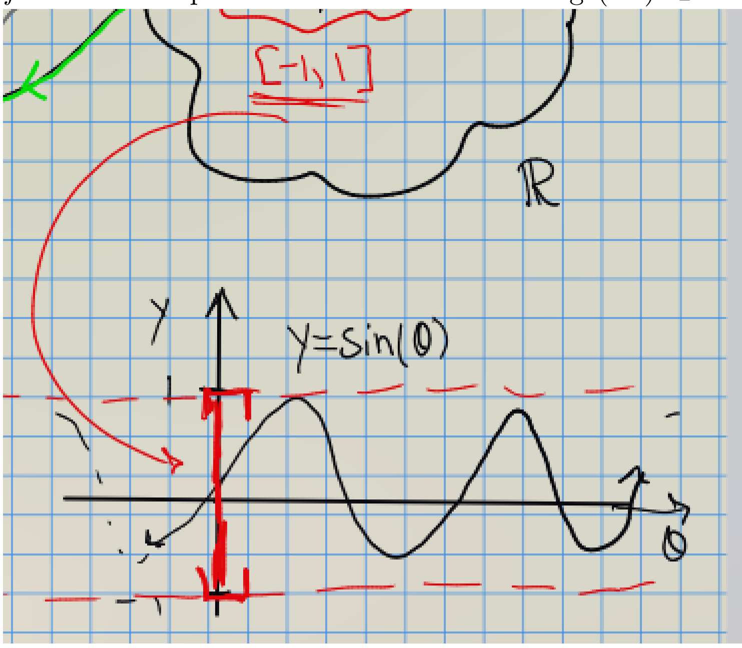


Figure 7: Input-Output perspective: important!

Note that we only ever have to define f on range(sin), since we're only ever sending outputs of f in as the inputs of sin. So we need range(sin) $\subset$ dom(f), noting that range(sin) $= [-1, 1]$:



Similarly, we need range(f) $\subset$ dom(sin).

4.13.2 Using Triangles

Remark 4.13.2: Optimistically imagine that we had some such inverse function. Then we could evaluate some expressions without even knowing anything else about it. The trick:

$$\begin{aligned} \theta &= \arccos(p/q) \\ \implies \cos(\theta) &= \cos(\arccos(p/q)) \\ \implies \cos(\theta) &= p/q. \end{aligned}$$

Now embed this in a triangle. We can't solve for θ, but we can solve for other trigonometric functions.

Exercise 4.13.3 (Using functional inverse property)

$$\cos\left(\arccos\left(\frac{\sqrt{5}}{5}\right)\right) = \frac{\sqrt{5}}{5}$$
$$\arccos\left(\cos\left(\frac{\sqrt{5}}{5}\right)\right) = \frac{\sqrt{5}}{5}$$
.

Exercise 4.13.4 (Using a triangle)

$$\tan\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{q^2 - p^2}}.$$

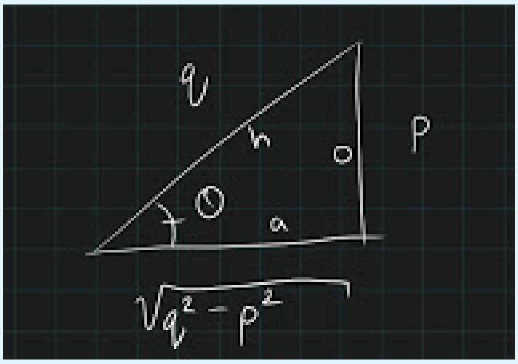


Figure 8: image_2021-04-22-22-14-13

Exercise 4.13.5 (Can't extract angles)
Compute $\arcsin(3/5)$.

⚠ **Warning 4.13.6**

This is equal to $\sin^{-1}(3/5)$, which is *not* equal to $\dfrac{1}{\sin(3/5)}$! One way to remember this is that we have another name for reciprocals, here $\csc(3/5)$.

Solution:

$$
\begin{aligned}
& \theta = \arcsin(3/5) \\
\implies & \sin(\theta) = (3/5) && \text{roughly by injectivity} \\
\implies & = \cdots ?.
\end{aligned}
$$

We are out of luck, since this isn't a special angle. So we can't find a numerical value of θ. We can find other trig functions of θ though:

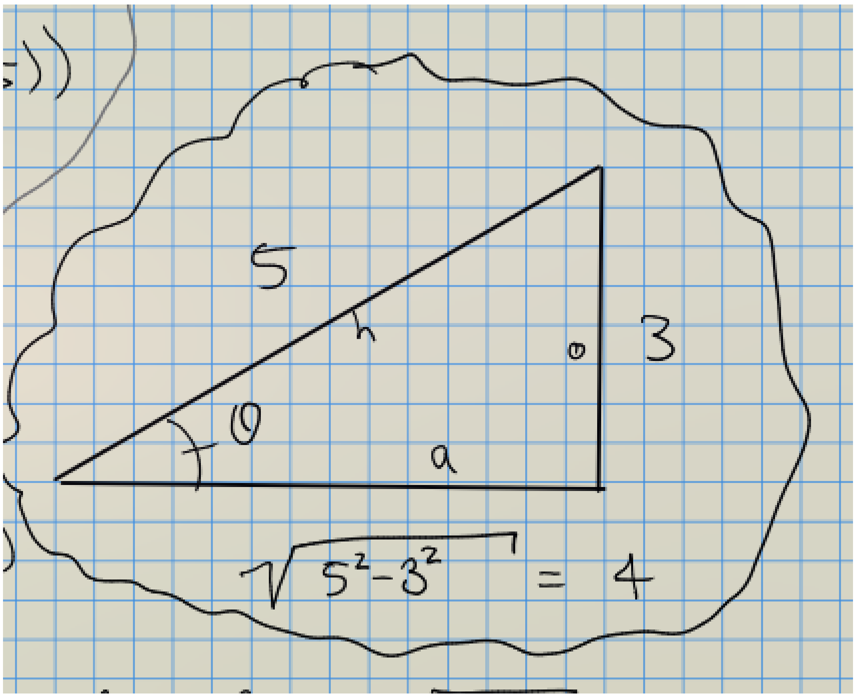


Figure 9: image_2021-04-18-22-30-09

So for example, $\cos(\arcsin(3/5)) = 4/5$.

Remark 4.13.7: Most inverse trigonometric functions can *not* be exactly solved! We'll have to approximate by calculator if we want the actual angle. If we just want *other* trigonometric functions though, we can always embed in a triangle.

Example 4.13.8 ***(Using triangles):*** Show the following:

- $\cos(\arcsin(24/26)) = 10/26$
  - Write $\theta = \arcsin(24/26)$, note θ is in $[-\pi/2, \pi/2] = \text{range}(\arcsin)$.
- $\tan(\arccos(-10/26)) = 10/26$
  - Write $\theta = \arccos(-10/26)$, note θ is in $[0, \pi] = \text{range}(\arccos)$

4.13.3 Defining Inverses

Remark 4.13.9: The setup: try swapping y and θ in the graph of $y = \sin(\theta)$:

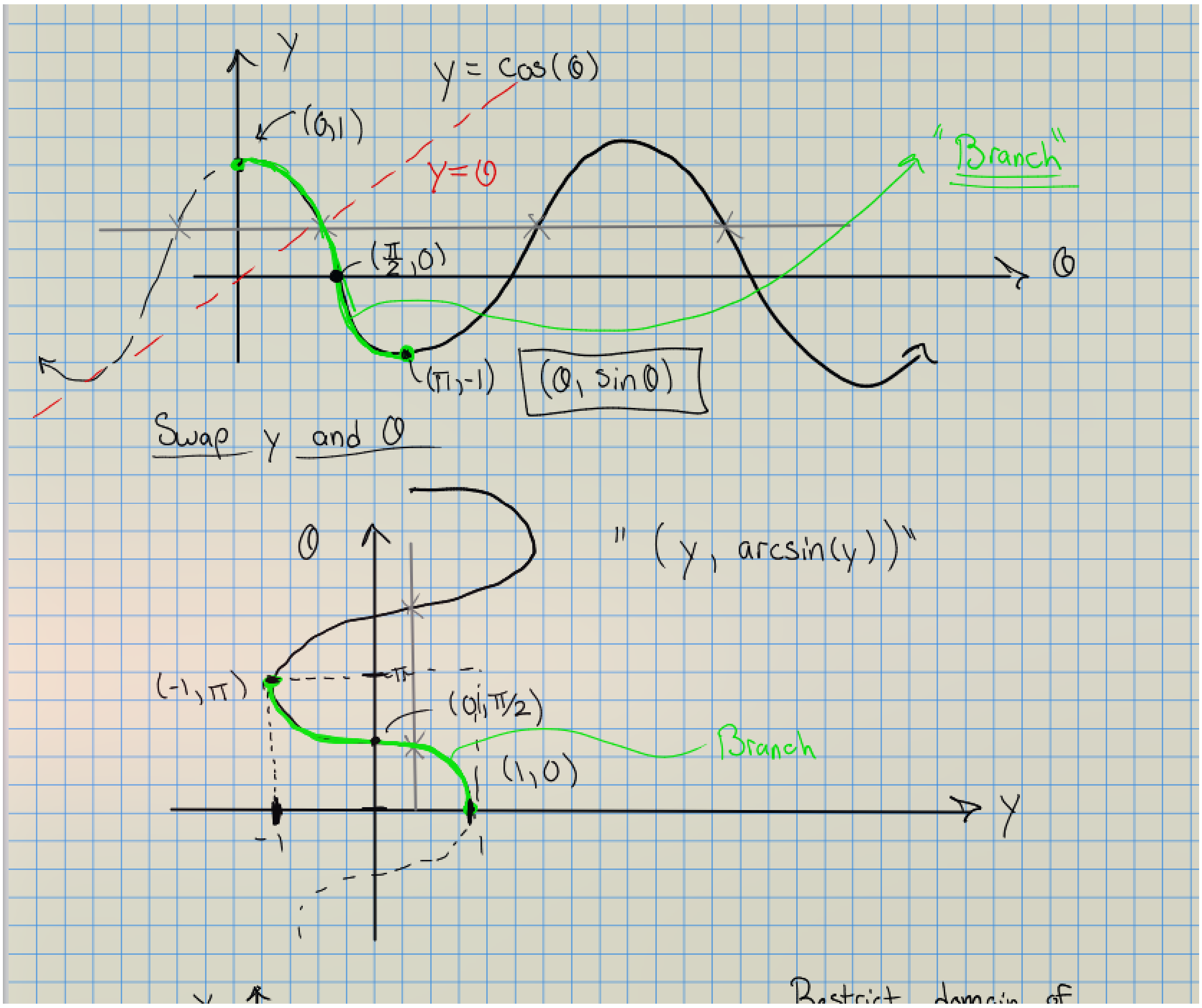


Figure 10: image_2021-04-18-22-32-36

Note that the latter is a function (vertical line test) iff the former is injective (horizontal line test). So we take the largest branch where the inverse is a function:

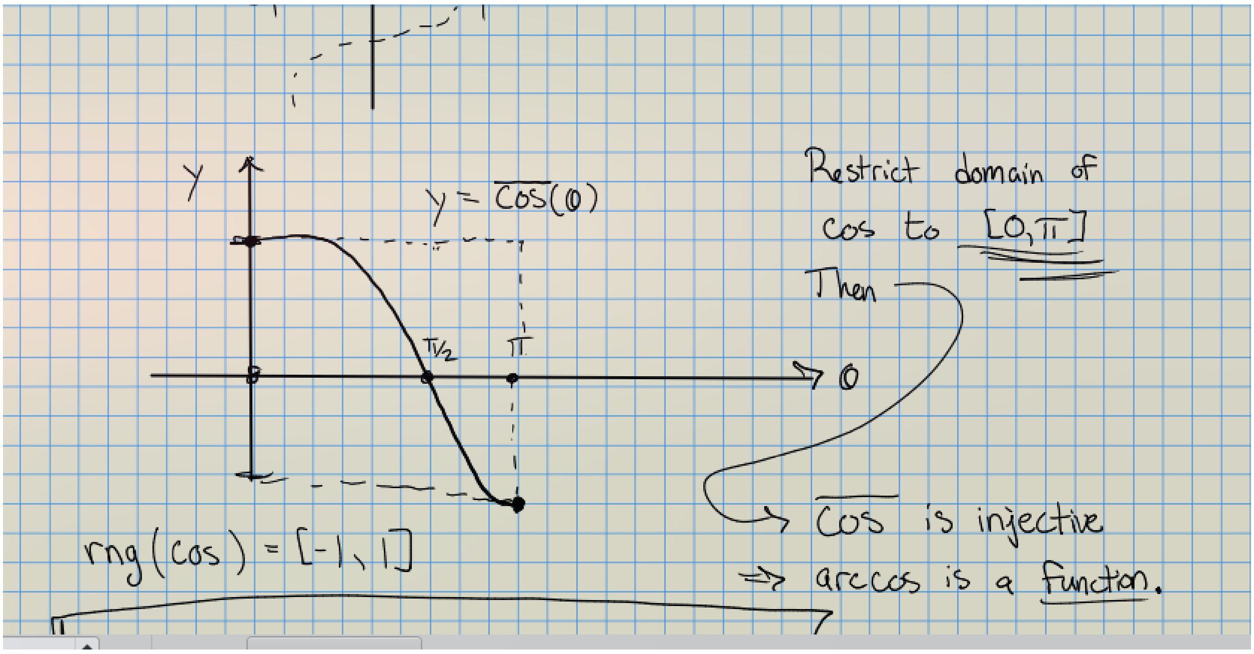


Figure 11: image_2021-04-18-22-33-27

Back on our original graph, this looks like the following:

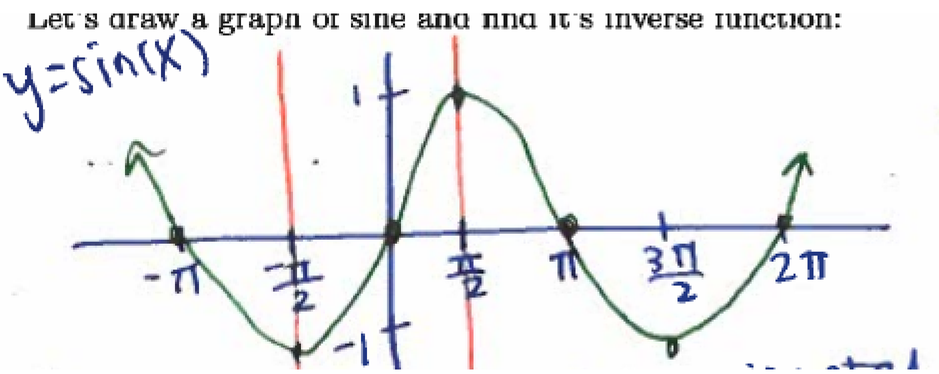


Figure 12: image_2021-04-18-20-53-25

Restricting, we get

- $\mathrm{dom}(\arccos) := \mathrm{range}(\cos) = [-1, 1]$.
- $\mathrm{range}(\arccos) := \mathrm{dom}(\cos) = [0, \pi]$.

Remark 4.13.10: A similar analysis works for $\sin(\theta)$:

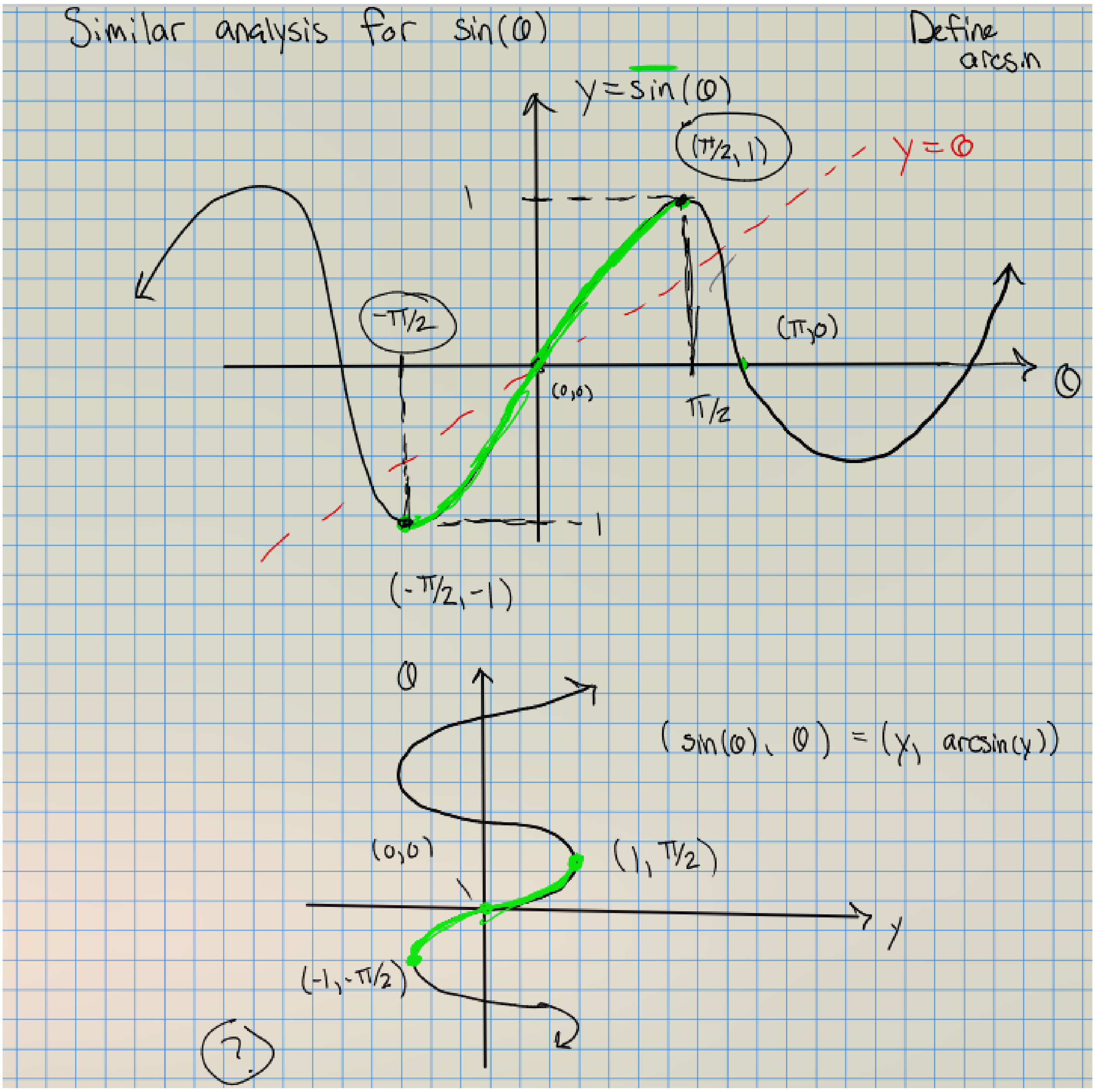


Figure 13: image_2021-04-18-22-34-21

Restricting, we get

- dom(arcsin) := range(sin) = $[-1, 1]$.
- range(arcsin) := dom(sin) = $[-\pi/2, \pi/2]$.

Remark 4.13.11: This gives us a new tool to solve equations:

$$\begin{aligned} \vdots &= \vdots \\ \implies \cos(x) &= b \\ \implies \arccos(\cos(x)) &= \arccos(b) \\ \implies x &= \arccos(b), \end{aligned}$$

but only if we know this makes sense based on domain/range issues.

Proposition 4.13.12 ***(Domains of inverse trigonometric functions).***
Restrict domains in the following ways:

- sin: $[-\pi/2, \pi/2]$
- $\cos : [0, \pi]$
- $\tan : [-\pi/2, \pi/2]$

Function	Domain	Range
arcsin	$[-1, 1]$	$[-\pi/2, \pi/2]$
arccos	$[-1, 1]$	$[0, \pi]$
arctan	$\mathbb{R}$	$(-\pi/2, \pi/2)$
arccsc	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?
arcsec	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	?
arccot	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?

Slogan 4.13.13
There is an easy way to remember this:

- Cosines are x-values, pick the upper (or lower) half of the circle to make them unique.
- Sines are y-values, pick the right (or left) half of the circle to make them unique.

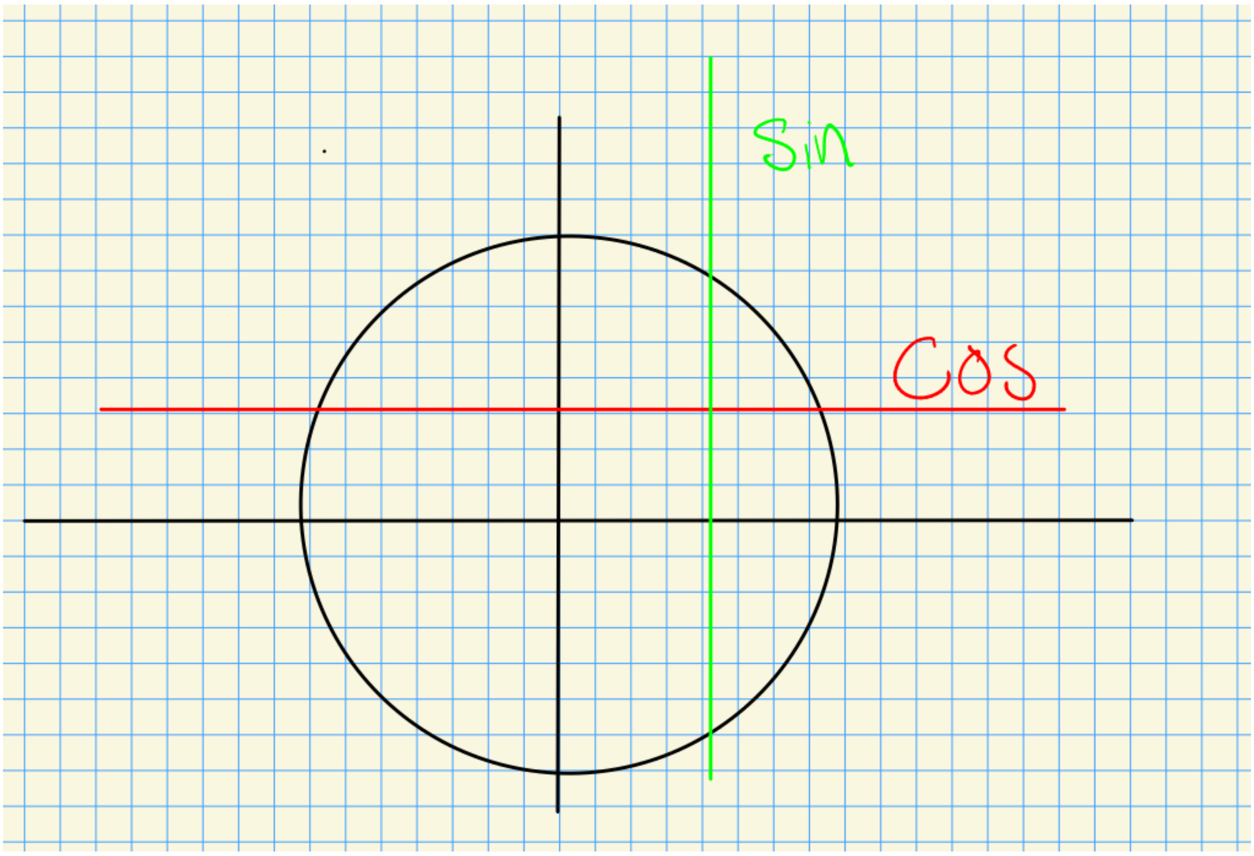


Figure 14: image_2021-04-22-22-00-04

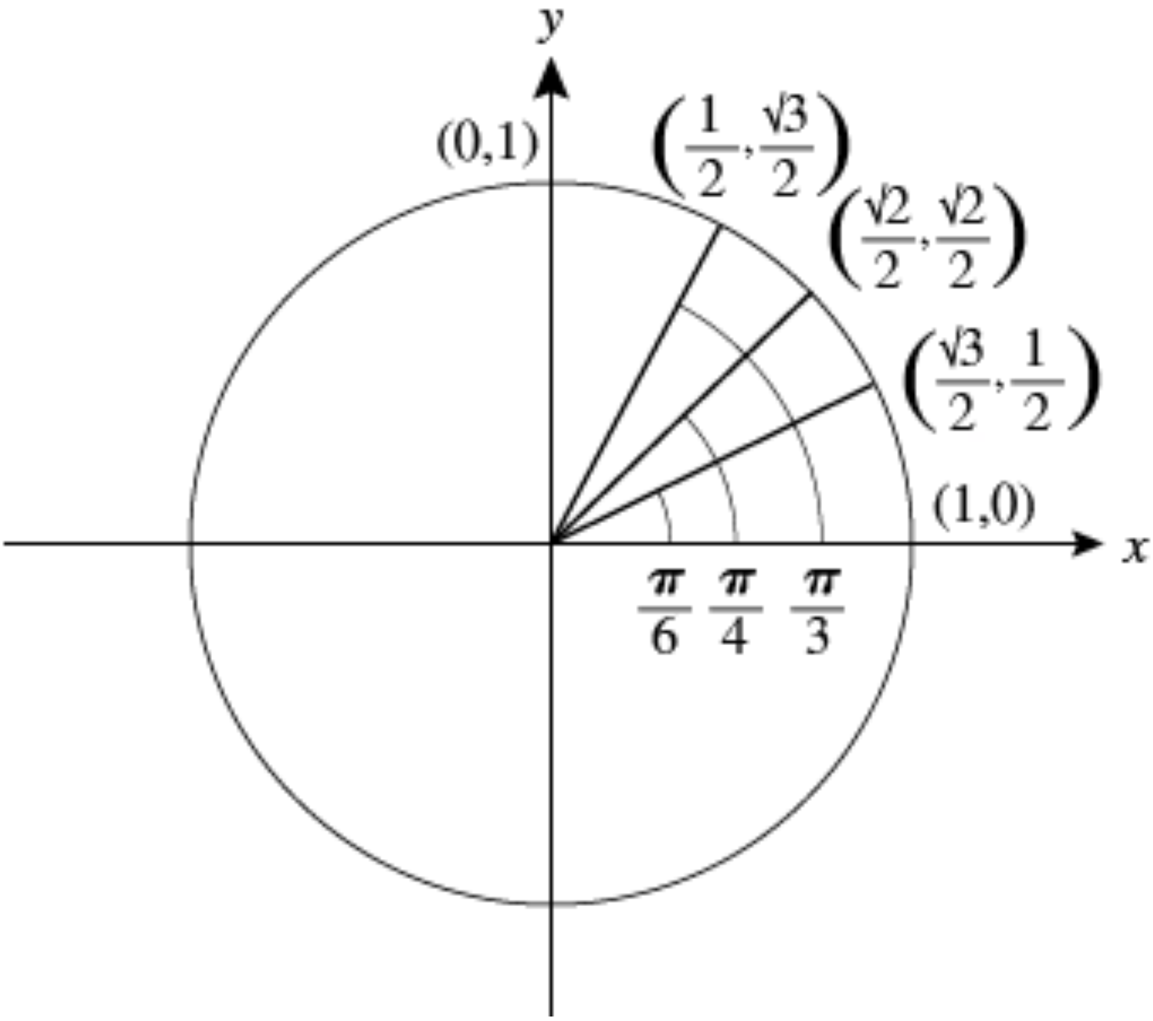


Figure 15: Unit Circle

Example 4.13.14*(Using special angles):* We have some exact values.

Sines should be in QI or QIV:

- $\arcsin(1/2) = \pi/6$
- $\arcsin(\sqrt{3}/2) = \pi/3$
- $\arcsin(-1/2) = -\pi/6$

Cosines should be in QI or QII:

- $\arccos(\sqrt{3}/2) = \pi/6$
- $\arccos(-\sqrt{2}/2) = 3\pi/4$
- $\arccos(1/2) = \pi/3$

Tangents should be in QI or QIV:

- $\arctan(\sqrt{3}/3) = \pi/6$
- $\arctan(0) = 0$
- $\arctan(1) = \pi/4$

Warning 4.13.15
Note that if f, g are an inverse pair, we have

$$f \circ g = \mathrm{id} \iff f(g(x)) = x, \quad g(f(x)) = x.$$

However, we have to be careful with domains for trigonometric functions:

- $\arcsin(\sin(x)) = x \iff x \in [-\pi/2, \pi/2]$ (restricted domain of sin)
- $\sin(\arcsin(x)) = x \iff x \in [-1, 1]$ (domain of arcsin)
- $\arccos(\cos(x)) = x \iff x \in [0, \pi]$ (restricted domain of cos)
- $\cos(\arccos(x)) = x \iff x \in [-1, 1]$ (domain of arccos)
- $\arctan(\tan(x)) = x \iff x \in [0]$ (restricted domain of tan)
- $\tan(\arctan(x)) = x \iff x \in \mathbb{R}$
  - Domain of arctan, then range is $[-\pi/2, \pi/2]$, which is in the domain of tan.

4.14 Double/Half-Angle Identities

Remark 4.14.1: Sometimes we are interested in **superposition** of waves. Mathematically this is modeled by multiplying two wave functions together. We can sometimes rewrite these as a *single* wave with a phase shift.

Proposition 4.14.2 *(?).*
Identities:

$$\begin{aligned}\sin(\theta + \psi) &= \sin(\theta)\cos(\psi) + \cos(\theta)\sin(\psi) \\ \cos(\theta + \psi) &= \cos(\theta)\cos(\psi) + \sin(\theta)\sin(\psi).\end{aligned}$$

Note that you can divide these to get

$$\tan(\theta + \psi) = \frac{\tan(\theta) + \tan(\psi)}{1 - \tan(\theta)\tan(\psi)},$$

and replace ψ with $-\psi$ and use even/odd properties to get formulas for $\sin(\theta - \psi), \cos(\theta - \psi)$

Slogan 4.14.3
Sines are friendly and cosines are clique-y!

Remark 4.14.4: The most interesting modifications of waves: superpositions and damped waves.

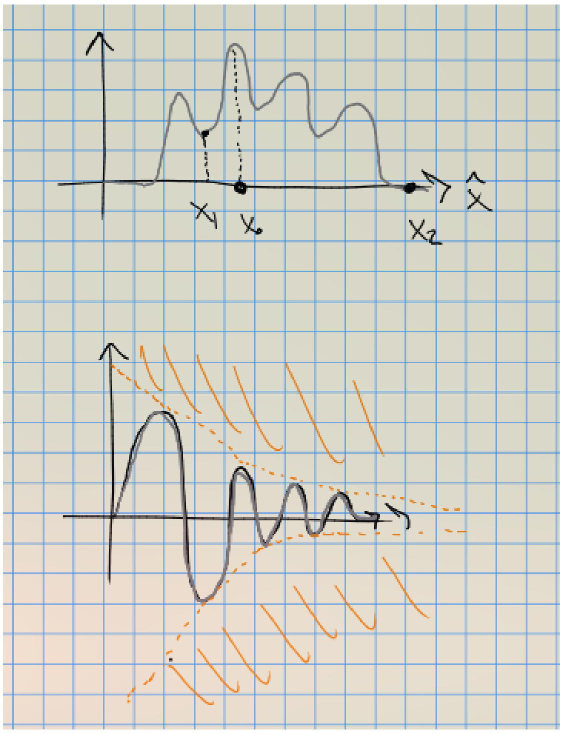


Figure 16: image_2021-04-18-22-06-08

Corollary 4.14.5 ***(Double angle identities).***
Taking $\theta = \psi$ is the above identities yields

$$\begin{aligned} \sin(2\theta) &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)\cos(\theta) \end{aligned}$$

$$\begin{aligned} \cos(2\theta) &= \cos(\theta)\cos(\theta) + \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta). \end{aligned}$$

⚠ **Warning 4.14.6**
The latter is not equal to 1! That would be $\cos^2(\theta) + \sin^2(\theta)$.

Remark 4.14.7: Why do we care? We had 16 special angles, this gives a lot more. For example,

$$\cos(\pi/12) = \cos(\pi/3 - \pi/4) = \cdots \text{ plug in.}$$

By allowing increments of $\pi/12$, we have 24 total angles.

Corollary 4.14.8 ***(?).***

Starting from the following:

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - \left(1 - \cos^2(\theta)\right) \\ &= 2\cos^2(\theta) - 1 && \text{using } s^2 + c^2 = 1, \end{aligned}$$

one can solve for

$$\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right).$$

Similarly

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \left(1 - \sin^2(\theta)\right) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta) && \text{using } s^2 + c^2 = 1, \end{aligned}$$

solving yields

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)).$$

4.15 Bonus: Complex Exponentials

Remark 4.15.1: Components of vectors: every $\mathbf{v} \in \mathbb{R}^2$ breaks up as the sum of two vectors, i.e. $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y$.

Remark 4.15.2: We've worked with the *Cartesian plane* all semester. One powerful tool is replacing this with the *complex* plane. We formally define a new symbol i such that $i^2 = -1$, and replace the $\widehat{\mathbf{y}}$ direction with the i direction – this amounts to replacing ordered pairs $(a, b) := a\widehat{\mathbf{x}} + b\widehat{\mathbf{y}}$ by a single number $x + iy$.

Proposition 4.15.3 ***(Euler's Identity).***

$$e^{i\pi} = -1.$$

Remark 4.15.4: The way you read this: $e^{i\theta} \in S^1$ is a complex number (identified with a vector!), and the θ tells you what direction it points in radians. π radians is directly to the left!

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