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Zeta
Functions

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The Weil Conjectures

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Zeta Functions

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}}_q \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

Definition (Zeta Function)

Let $J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}_{\mathbb{F}}^n$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^n \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n + 1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{l}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{l}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

- For a fixed variety X , we can consider its \mathbb{F} -points $X(\mathbb{F})$.
 - Note that $\#X(\mathbb{F}) < \infty$ is an integer
- For any L/\mathbb{F} , we can also consider $X(L)$
 - In particular, we can consider $X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
 - We again have $\#X(\mathbb{F}_{q^n}) < \infty$ and are integers for every such n .
- So we can consider the sequence

$$[N_1, N_2, \dots, N_n, \dots] := [\#X(\mathbb{F}), \#X(\mathbb{F}_{q^2}), \dots, \#X(\mathbb{F}_{q^n}), \dots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \dots.$$

Why Generating Functions?

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Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of F : we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n F(z) \Big|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} dz = N_n.$$

The latter form is very amenable to computer calculation.

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An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \rightarrow \mathbb{C}$ – in nice situations, we can hope for a closed-form representation.

A useful example: by integrating a geometric series we can derive

$$\begin{aligned}\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n && (= 1 + z + z^2 + \cdots) \\ \Rightarrow \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} \int z^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\ \Rightarrow -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} && \left(= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).\end{aligned}$$

For completeness, also recall that $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$

In particular, we have some choice in how to encode the sequence $[1, 1, \dots]$.

Note: a common other choice is associating an exponential generating function, $F(z) = \sum \frac{z^n}{n!}$.