Title

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Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of **??**; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$, then the data of a GL_n-torsor split by a τ-cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d] \\ X \end{tikzcd}$$

That U trivializes our torsor means that $\pi^* T = \pi^* G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\begin{tikzcd} \pi_1^* \pi^* T \arrow[r, "\sim"] \arrow[d] & \pi_2^* \pi^* T \arrow[d] \\ \pi_1^* \pi^* G \arrow[r, "\sim"] & \pi_2^* \pi^* G \end{tikzcd}$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n-torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{ét}}, \mathrm{GL}_n)$ is in bijection with n-dimensional vector bundles on X.

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Exercise 1.0.3 *(?)*: See if Hilbert 90 is true for groups other than GL_n.

1.1 Representability and Local Triviality

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).
You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:

$$\begin{tikzcd} T \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ T \arrow[r, "f"] \arrow[u, bend right, "\exists"'] & X \end{tikzcd}$$

The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that

$$\begin{tikzcd} & T \times_X T \arrow[r] \arrow[d] & T \arrow[d] \\ & T \arrow[r, "f"] \arrow[u, bend right, "\exists"'] & X \\ U \arrow[ur, hook, "\text{closed}"] \arrow[urr, bend right, "\exists \text{ét}"'] & & \end{tikzcd}$$

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Example 1.1.5 ***(non-smooth group schemes):***

- α_p, the kernel of Frobenius on $\mathbb{A}^1$ or $\mathbb{G}_a$,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on $\mathbb{G}_m$,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$, etc.