

# Homework 7

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## 1 Problem 1

### 1.1 Part 1

In order for  $IS$  to be a submodule of  $A$ , we need to show the following implication:

$$x \in IS, a \in A \implies xa, ax \in IS.$$

Suppose  $x \in IS$ . Then by definition,  $x = \sum_{i=1}^n r_i a_i$  for some  $r_i \in R, a_i \in A$ .

But then

$$\begin{aligned}
xa &= \left( \sum_{i=1}^n r_i a_i \right) a \\
&= \sum_{i=1}^n r_i a_i a \\
&:= \sum_{i=1}^n r_i a'_i,
\end{aligned}$$

where  $a'_i := a_i a$  for each  $i$ , which is still an element of  $A$  since  $A$  itself is a module and thus closed under multiplication.

But this expresses  $xa$  as an element of  $IS$ . Similarly, we have

$$\begin{aligned}
ax &= a \left( \sum_{i=1}^n r_i a_i \right) \\
&= \sum_{i=1}^n a r_i a_i a \\
&:= \sum_{i=1}^n r_i a a_i, \\
&:= \sum_{i=1}^n r_i a'_i,
\end{aligned}$$

and so  $ax \in IS$  as well.

## 1.2 Part 2

Letting  $R/I \curvearrowright A/IA$  be the action given by  $r + I \curvearrowright +IA := ra + IA$ , we need to show the following:

- $r.(x + y) = r.x + r.y$ ,
- $(r + r').x = r.x + r'.x$ ,
- $(rs).x = r.(s.x)$ , and
- $1.x = x$ .

Letting  $\oplus$  denote the addition defined on cosets, we have

$$\begin{aligned}
r \curvearrowright (x + IA \oplus y + IA) &:= r \curvearrowright x + y + IA \\
&:= r(x + y) + IA \\
&= rx + ry + IA \\
&:= rx + IA \oplus ry + IA \\
&:= (r \curvearrowright x + IA) \oplus (r \curvearrowright y + IA).
\end{aligned}$$

$$\begin{aligned}
(r + s) \curvearrowright x + IA &:= (r + s)x + IA \\
&:= rx + sx + IA \\
&:= rx + IA \oplus sx + IA \\
&:= (rs \curvearrowright IA) \oplus (sx \curvearrowright IA).
\end{aligned}$$

$$\begin{aligned}
(rs) \curvearrowright x + IA &:= rsx + IA \\
&= r(sx) + IA \\
&:= r \curvearrowright (sx + IA) \\
&= r \curvearrowright (s \curvearrowright x + IA).
\end{aligned}$$

$$1 \curvearrowright x + IA := 1x + IA = x + IA.$$

## 2 Problem 2

### 2.1 Part 1

We want to show that every simple  $R$ -module  $M$  is cyclic, i.e. if the only ideals of  $M$  are  $(0)$  and  $M$  itself, that  $M = \langle m \rangle$  for some element  $m \in M$ .

Towards a contradiction, let  $M$  be a simple  $R$ -module and suppose  $M$  is not cyclic, so  $M \neq \langle m \rangle$  for any  $m \in M$ . But then let  $a \in M$  be an arbitrary nontrivial element; then  $(a)$  is a non-empty ideal (since it contains  $a$ ), so  $(a) \neq 0$ . Since  $M$  is simple, we must have  $(a) = M$ , a contradiction.

### 2.2 Part 2

Let  $\phi : A \rightarrow A$  be a module endomorphism on a simple module  $A$ . Then  $\text{im } \phi := \phi(A)$  is a submodule of  $A$ . Since  $A$  is simple, we have either  $\text{im } \phi = 0$ , in which case  $\phi$  is the zero map, or  $\text{im } \phi = A$ , so  $\phi$  is surjective. In this case, we can also consider  $\ker \phi$ , which is a submodule of  $A$ . Since  $A$  is simple, we can again only have  $\ker \phi = A$ , which can not happen if  $\phi$  is not the zero map, or  $\ker \phi = 0$ , in which case  $\phi$  is both a surjective and an injective map and thus an isomorphism of modules.

## 3 Problem 3

### 3.1 Part 1

We want to show that if  $A, B$  are  $R$ -modules then  $X = (\text{hom}_{R\text{-mod}}(A, B), +)$  is an abelian group. Let  $f, g, h \in X$ , we then need to show the following:

- a. Closure:  $f + g \in X$
- b. Associativity:  $f + (g + h) = (f + g) + h$

- c. Identity:  $\text{id} \in X$
- d. Inverses:  $f^{-1} \in X$
- e. Commutativity:  $f + g = g + f$

Closure: This follows from the definition, because  $(f + g) \curvearrowright x := f(x) + g(x)$  pointwise, which is well-defined homomorphism  $A \rightarrow B$ .

Associativity: We have

$$\begin{aligned}
 f + (g + h) \curvearrowright x &:= f(x) + (g + h)(x) \\
 &:= f(x) + (g(x) + h(x)) \\
 &= (f(x) + g(x)) + h(x) \\
 &= (f + g) \curvearrowright x.
 \end{aligned}$$

Identity: We can define  $\mathbf{0} : A \rightarrow B$  by  $\mathbf{0}(x) = 0 \in B$ . Then

$$(f + \mathbf{0}) \curvearrowright x = f(x) + 0 = f(x) = 0 + f(x) = (\mathbf{0} + f) \curvearrowright x.$$

Inverses: Given  $f \in X$ , we can define  $-f : A \rightarrow B$  as  $-f(x) = -x$ . Then

$$\begin{aligned}
 (f + -f) \curvearrowright x &= f(x) + -f(x) = f(x) - f(x) = x - x = 0 = \mathbf{0} \curvearrowright x \\
 (-f + f) \curvearrowright x &= -f(x) + f(x) = -f(x) + f(x) = -x + x = 0 = \mathbf{0} \curvearrowright x.
 \end{aligned}$$

Commutativity: Since  $B$  is a module, by definition  $(B, +)$  is an abelian group. Thus

$$(f + g) \curvearrowright x = f(x) + g(x) = g(x) + f(x) = (g + f) \curvearrowright x.$$

### 3.2 Part 2

By part 1,  $(\text{hom}_{R\text{-mod}}(A, A), +)$  is an abelian group, We just need to check that  $(\text{hom}_R(A, A), \circ)$  is a monoid, i.e.:

- Associativity:  $f \circ (g \circ h) = (f \circ g) \circ h$
- Identity:  $\text{id} \circ f = f$
- Closure:  $f \circ g \in \text{hom}_{R\text{-mod}}(A, A)$

Associativity: We have

$$\begin{aligned}
 f \circ (g \circ h) \curvearrowright x &:= (f \circ (g \circ h))(x) \\
 &= f((g \circ h)(x)) \\
 &= f(g(h(x))) \\
 &= (f \circ g)(h(x)) \\
 &= ((f \circ g) \circ h)(x) \\
 &:= (f \circ g) \circ h \curvearrowright x.
 \end{aligned}$$

Identity: Take  $\text{id}_A : A \rightarrow A$  given by  $\text{id}_A(x) = x$ , then

$$f \circ \text{id}_A \curvearrowright x = f(\text{id}_A(x)) = f(x) = \text{id}_A(f(x)) = \text{id}_A \circ f \curvearrowright x.$$

Closure: If  $f : A \rightarrow A$  and  $g : A \rightarrow A$  are homomorphisms, then  $f \circ g : A \rightarrow A$  as a set map, and is an  $R$ -module homomorphism because

$$\begin{aligned} f \circ g \curvearrowright (r + s)(x + y) &= f(g((r + s)(x + y))) \\ &= f((r + s)(g(x) + g(y))) \\ &= (r + s)(f(g(x)) + f(g(y))) \\ &= (f \curvearrowright (r + s)(x + y)) \circ (g \curvearrowright (r + s)(x + y)). \end{aligned}$$

### 3.3 Part 3

For arbitrary  $x, y \in A$ , we need to check the following:

- a.  $f \curvearrowright (x + y) = f \curvearrowright x + f \curvearrowright y$
- b.  $(f + g) \curvearrowright x = f \curvearrowright x + g \curvearrowright x$
- c.  $f \circ g \curvearrowright x = f \curvearrowright (g \curvearrowright x)$
- d.  $\text{id}_A \curvearrowright x = x$

For (a):

$$\begin{aligned} f \curvearrowright (x + y) &:= f(x + y) \\ &= f(x) + f(y) \quad \text{since } f \text{ is a homomorphism} \\ &= f \curvearrowright x + f \curvearrowright y \end{aligned}$$

For (b):

$$\begin{aligned} (f + g) \curvearrowright x &= (f + g)(x) \\ &= f(x) + g(x) \\ &= f \curvearrowright x + g \curvearrowright x. \end{aligned}$$

For (c):

$$\begin{aligned} f \circ g \curvearrowright x &= (f \circ g)(x) \\ &= f(g(x)) \\ &= f \curvearrowright g(x) \\ &= f \curvearrowright (g \curvearrowright x). \end{aligned}$$

For (d):

$$\text{id}_A \curvearrowright x = \text{id}_A(x) = x.$$

## 4 Problem 4

**Injectivity:** We have the following situation:

$$\begin{array}{ccccccc}
 & a' & & a & & x & & 0 \\
 & & & & & & & \\
 A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \\
 \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow f & & \downarrow \alpha_4 \\
 B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 \\
 & & & & & & & \\
 & 0 & & \alpha_2(a) & & y = f(x) = 0 & & 0
 \end{array}$$

where we would like to show that  $f$  is a monomorphism, i.e. that  $\ker f = 0$ . So let  $x \in \ker f$ , so  $y := f(x) = 0 \in B_3$ .

We will show that  $x = 0 \in A_3$ :

- Since  $y = 0 \in B_3$ , applying  $B_3 \rightarrow B_4$  yields  $y \mapsto 0 \in B_4$  since these maps are homomorphisms and always map zero to zero.
- Pull back  $0 \in B_4$  to  $0 \in B_3$  along  $\alpha_4$ , which can be done since  $\alpha_4$  is injective, giving  $0 \in A_4$ .
- Since this is 0 in  $A_4$ , it is in the kernel of  $A_3 \rightarrow A_4$ , yielding some  $x \in A_3$ .
- By commutativity of the third square,  $x \mapsto f(x)$  under  $f : A_3 \rightarrow B_3$ .
- Since  $x \in \ker(A_3 \rightarrow A_4) = \text{im}(A_2 \rightarrow A_3)$  by exactness, there is some  $a \in A_2$  such that  $\alpha_2(a) = x \in A_3$ .
- By injectivity of  $\alpha_2$ ,  $a$  maps to a unique element  $\alpha_2(a) \in B_2$ .
- By commutativity of the middle square, since  $a \in A_2 \mapsto 0 \in B_3$ , we must have  $\alpha_2(a) \mapsto 0f(x)$  under  $B_2 \rightarrow B_3$ .
- Then  $\alpha_2(a) \in \ker(B_2 \rightarrow B_3) = \text{im}(B_1 \rightarrow B_2)$ , so it pulls back to some  $b \in B_1$ .
- By surjectivity of  $\alpha_1$ ,  $b$  pulls back to some  $a' \in A_1$ .
- By commutativity of square 1,  $a' \mapsto a$  under  $A_1 \rightarrow A_2$ .
- So  $a \mapsto x$  under  $A_1 \rightarrow A_3$ .
- But then  $a \in \text{im}(A_1 \rightarrow A_2) = \ker(A_2 \rightarrow A_3)$ , so  $a \mapsto 0$  under  $A_1 \rightarrow A_3$ .
- So  $x = 0$  as desired.

**Surjectivity:** We now have this situation:

$$\begin{array}{ccccccc}
 A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
 \downarrow \alpha_2 & & \downarrow f & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\
 B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
 \end{array}$$

Let  $y \in B_3$ ; we want to then show that there exists an  $x \in A_3$  such that  $f(x) = y$ .

- Apply  $B_3 \rightarrow B_4$  to  $y$  to obtain  $y_4 \in B_4$ .
- By surjectivity of  $\alpha_4$ , this pulls back to some  $a_4 \in A_4$ .
- Also by exactness of  $B_3 \rightarrow B_4 \rightarrow B_5$ ,  $y_4$  pushes forward to  $0 \in B_5$ .
- By injectivity of  $\alpha_5$ , this pulls back to  $0 \in A_5$ .
- By commutativity of the right square,  $y_4 \mapsto 0$  under  $A_4 \rightarrow A_5$ .
- Since  $a_4 \in \ker(A_4 \rightarrow A_5)$ , it pulls back to some  $x \in A_3$  by exactness of  $A_3 \rightarrow A_4 \rightarrow A_5$ .
- Then  $f(x) \in B_3$ , and it remains to show that  $f(x) = y$ .
- By commutativity of the middle square,  $f(x) \mapsto y_4$  under  $B_3 \rightarrow B_4$ .
- Since  $a \mapsto y_4$  as well, we have  $z := f(x) - y \in B_3$  maps to  $0 \in B_4$ .
- Since  $z \in \ker(B_3 \rightarrow B_4)$ , by exactness it pulls back to some  $b_2 \in B_2$ .
- By surjectivity of  $\alpha_2$ , this pulls back to some  $a_2 \in A_2$ .
- By commutativity of the first square,  $a_2 \mapsto z \in B_3$ .
- $a_2 \mapsto a_3 \in A_3$ , where  $a_3$  may not equal  $x$ , but  $f(a_3) = z := f(a) - y$ .
- Then  $f(a_3) = f(x) - y \implies y = f(x) - f(a_3) = f(x - a_3)$  since  $f$  is a homomorphism.
- This shows that  $x - a_3 \mapsto y$  under  $f$ , which is the element we wanted to produce.

## 5 Problem 5

### 5.1 Part (a)

We want to show that if  $(p) \trianglelefteq R$  is a prime ideal then  $R/(p)$  is a field, so we'll proceed by letting  $x + (p) \in R/(p)$  be arbitrary where  $x \notin (p)$  and producing a multiplicative inverse.

Since  $R$  is a principal ideal domain, prime ideals are maximal, so  $(p)$  is maximal. Then  $x \in R \setminus (p)$ , so define

$$I := \{p + rx \mid p \in (p), r \in R\} \trianglelefteq R,$$

which is an ideal in  $R$ .

In particular, since  $x \notin (p)$ , we have a strict containment  $(p) < I$ , but since  $(p)$  was maximal this forces  $I = R$ .

Then  $1 \in I$ , so there exists some  $p, r$  such that  $p + rx = 1$ , i.e.  $rx - 1 \in (p)$ .

But then

$$r + (p) \cdot x + (p) = rx + (p) = 1 + (p),$$

which says that  $(x + (p))^{-1} = r + (p)$  in  $R/(p)$ .

## 5.2 Part (b)

Images and kernels of module homomorphisms are always submodules, so define

$$\begin{aligned}\phi : A &\rightarrow A \\ x &\mapsto px.\end{aligned}$$

This is a module homomorphism, and

$$\begin{aligned}\text{im } \phi &:= \{px \mid x \in A\} := pA, \\ \ker \phi &:= \{a \in A \mid pa = 0\} := A[p].\end{aligned}$$

## 5.3 Part (c)

Since  $R/(p)$  is a field, we just need to show that  $A/pA \curvearrowright R/(p)$  defines a module.

$$r \cdot (x + y) = rx + ry:$$

$$\begin{aligned}r + (p) \curvearrowright x + pA \oplus y + pA &:= r + (p) \curvearrowright x + y + pA \\ &:= r(x + y) + pA \\ &= rx + ry + pA \\ &:= rx + pA \oplus ry + pA \\ &:= r \curvearrowright x + pA \oplus r \curvearrowright y + pA.\end{aligned}$$

$$(r + s) \cdot x = rx + sx:$$

$$\begin{aligned}r + (p) \oplus s + (p) \curvearrowright x + pA &:= r + s + (p) \curvearrowright x + pA \\ &:= (r + s)x + pA \\ &= rx + sx + pA \\ &:= rx + pA \oplus sx + pA \\ &:= r + (p) \curvearrowright x + pA \oplus s + (p) \curvearrowright x + pA.\end{aligned}$$

$$rs \cdot x = r \cdot (s \cdot x):$$

$$\begin{aligned}r + (p) \cdot s + (p) \curvearrowright x + pA &:= rs + (p) \curvearrowright x + pA \\ &= rsx + pA \\ &:= r + (p) \curvearrowright sx + pA \\ &:= r + (p) \curvearrowright s + (p) \curvearrowright x + pA.\end{aligned}$$



$$1 \cdot x = x:$$

$$1_R + (p) \curvearrowright x + pA = 1_R x + pA = x + pA.$$

## 5.4 Part (d)

Similarly, since  $R/(p)$  is a field, it suffices to show that  $R/(p) \curvearrowright A[p]$  defines a module.

$$r \cdot (x + y) = rx + ry:$$

$$\begin{aligned} r + (p) \curvearrowright (a + a') &:= r(a + a') \\ &= ra + ra' \\ &= r \curvearrowright a + r \curvearrowright a'. \end{aligned}$$

$$(r + s) \cdot x = rx + sx:$$

$$\begin{aligned} r + s + (p) \curvearrowright a &= (r + s)a \\ &= ra + sa \\ &= r \curvearrowright a + s \curvearrowright a. \end{aligned}$$

$$rs \cdot x = r \cdot (s \cdot x):$$

$$\begin{aligned} rs + (p) \curvearrowright a &= rsa \\ &= r \curvearrowright sa \\ &= r \curvearrowright s \curvearrowright a. \end{aligned}$$

$$1 \cdot x = x:$$

$$1_R + (p) \curvearrowright a = 1a = a.$$

## 6 Problem 6

Supposing that  $\dim V = n$ , let  $\mathcal{B} := \{\mathbf{b}_k \mid 1 \leq k \leq n\}$  be a basis for  $V$ , and define

$$\mathbf{e}_i := [0, 0, \dots, 1, \dots, 0] \in V^{\oplus m}$$

where the 1 occurs in the  $i$ th position. The claim is that  $\mathcal{B}^m := \{\mathbf{e}_i \mathbf{b}_k \mid 1 \leq i \leq n, 1 \leq k \leq m\}$  forms a basis for  $V^{\oplus m}$ .

Elements in  $\mathcal{B}^m$  are of the form

$$\begin{aligned} & [\mathbf{b}_1, 0, 0, \dots, 0] \\ & [\mathbf{b}_2, 0, 0, \dots, 0] \\ & \dots \\ & [0, \mathbf{b}_1, 0, \dots, 0] \\ & [0, \mathbf{b}_2, 0, \dots, 0] \\ & \dots, \end{aligned}$$

and by construction,  $|\mathcal{B}| = mn = m \dim V$ .

To see that this is a spanning set, let  $\mathbf{x} \in V^{\oplus m}$ , so  $\mathbf{x} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m]$  where each  $\mathbf{v}_i \in V$ .

Then each  $\mathbf{v}_i \in \mathcal{B}$ , so  $\mathbf{v}_i = \sum_{k=1}^n \alpha_{k,i} \mathbf{b}_k$ . But then

$$\mathbf{x} = \left[ \sum_{k=1}^n \alpha_{k,1} \mathbf{b}_k, \sum_{k=1}^n \alpha_{k,2} \mathbf{b}_k, \dots, \sum_{k=1}^n \alpha_{k,m} \mathbf{b}_k \right] := \sum_{i=1}^m \sum_{k=1}^n \alpha_{k,i} \mathbf{b}_k \mathbf{e}_i,$$

which exhibits  $\mathbf{x} \in \mathcal{B}^m$ .

To see that it is linearly independent, supposing that  $\mathbf{x} = \sum_i \sum_k \alpha_{k,i} \mathbf{b}_k \mathbf{e}_i = 0$ , this says that  $\mathbf{x} = [0, 0, \dots, 0]$ , which forces  $\sum_k \alpha_{k,i} \mathbf{b}_k$  to be zero for each  $i$ .

But for a fixed  $i$ , since  $\{\mathbf{b}_k\}$  was a basis for  $V$ , this means that  $\alpha_{k,i} = 0$  for all  $k$ . But then  $\alpha_{k,i} = 0$  for all pairs  $i, k$ .

## 7 Problem 7

Let  $F_1, F_2$  be free, so they have bases  $\mathcal{B}_1 = \{\mathbf{b}_{1,k}\}, \mathcal{B}_2 = \{\mathbf{b}_{2,k}\}$ . Supposing that they have the invariant dimension property, we can assume that  $\#\mathcal{B}_1 := \text{rank } F_1$  and similarly  $\#\mathcal{B}_2 := \text{rank } F_2$ .

The claim is that the set

$$\mathcal{B} = \{(v, 0) \mid v \in \mathcal{B}_1\} \cup \{(0, w) \mid w \in \mathcal{B}_2\}$$

is a basis for  $F_1 \oplus F_2$ , where  $\#\mathcal{B} = \#\mathcal{B}_1 + \#\mathcal{B}_2 = \text{rank } F_1 + \text{rank } F_2$ .

So see that  $\mathcal{B}$  spans  $F_1 \oplus F_2$ , let  $x \in F_1 \oplus F_2 = (f_1, f_2)$  be arbitrary. Since  $f_1 \in F_1$ , we have  $f_1 = \sum_i r_i \mathbf{b}_{1,i}$ , and similarly  $f_2 = \sum_j s_j \mathbf{b}_{2,j}$ .

We can then write

$$x = (f_1, f_2) = (f_1, 0) + (0, f_2) = \left( \sum_i r_i \mathbf{b}_{1,i}, 0 \right) + \left( 0, \sum_j s_j \mathbf{b}_{2,j} \right),$$

which exhibits  $x$  as a linear combination of elements in  $\mathcal{B}$ .

To see linear independence, we just note that

$$\begin{aligned}
x &= (0, 0) \\
&= \sum_i r_i(v_i, 0) + \sum_j s_j(0, w_j) \\
&= \sum_i (r_i v_i, 0) + \sum_j (0, s_j w_j) \\
&= (\sum_i r_i v_i, \sum_j s_j w_j) \\
&\implies \sum_i r_i v_i = 0 \quad \& \quad \sum_j s_j w_j = 0, .
\end{aligned}$$

but since the  $v_i$  were a basis of  $F_1$  and the  $w_j$  a basis of  $F_2$ , this forces  $r_i = 0, w_j = 0$  for all  $i$  and  $j$ .