

Floer Talk

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Wednesday 15th April, 2020

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1 Background and Notation

From the text:

- $(W, \omega \in \Omega_2(W))$ is a (compact?) symplectic manifold
- $C^\infty(A, B)$ is the space of smooth maps with the C^∞ topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C_{\text{loc}}^\infty(A, B)$ is the space with the C^∞ uniform convergence topology on compact subsets of A
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$ given by $H_t \in C^\infty(W; \mathbb{R})$ is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned}\mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) \, dt\end{aligned}$$

where $\mathcal{L}W$ is the contractible loop space of W , $u : \mathbb{D} \longrightarrow W$ is an extension of $x : S^1 \longrightarrow W$ to the disc with $u(\exp(2\pi it)) = x(t)$.

$$- \text{ Example: } W = \mathbb{R}^{2n} \implies \mathcal{A}_H(x) = \int_0^1 (H_t \, dt - p \, dq).$$

- Critical points of the action functional \mathcal{A}_H are given by orbits, i.e. contractible loops $x, y \in \mathcal{L}W$
- In general, x, y are two periodic orbits of H of period 1.

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- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u) = 0.$$

This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be J -holomorphic curves.

- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
 - They correspond to “embedded cylinders” with sides u and contractible caps x, y regarded as loops in W .
 - They also correspond to paths in $\mathcal{L}W$ from $x \rightarrow y$ (precisely: trajectories of the vector field $-\text{grad} \mathcal{A}_H$)
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Fig. 6.5

Here $u(s) \in \mathcal{L}W$ is a loop with value at time t given by $u(s, t)$, and $\lim_{s \rightarrow -\infty} u_s(t) = x$, $\lim_{s \rightarrow \infty} u_s(t) = y$.

- The energy of a solution is $E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds dt$.
- $\mathcal{M} = \{u \in C^\infty(\mathbb{R}; \mathcal{L}W) \mid E(u) < \infty\}$ (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x, y)$ is the space of solutions of the Floer equation connecting orbits x and y .
- $C_{\searrow}(x, y)$:

$$C_{\searrow}(x, y) := \{u \in C^\infty(\mathbb{R} \times S^1; W) \mid \lim_{s \rightarrow -\infty} u(s, t) = x(t), \lim_{s \rightarrow \infty} u(s, t) = y(t), \\ \left| \frac{\partial u}{\partial s}(s, t) \right| \leq K e^{-\delta|s|}, \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \leq K e^{-\delta|s|}\}$$

where $K, \delta > 0$ are constants depending on u . So

$$|\partial_s u(s, t)|, |\partial_t u(s, t) - X_H(u)| \sim e^{|s|}.$$

From the Appendices

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator: $\dim \ker - \dim \operatorname{coker}$.
- Fredholm operators: those for which the index makes sense, i.e. $\dim \ker < \infty, \dim \operatorname{coker} < \infty$.
- Elliptic operators: generalize the Laplacian Δ , coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset

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- Sobolev spaces: in dimension 1, define $\|u(t)\|_{s,p} = \sum_{i=0}^s \|\partial_t^i u(t)\|_{L^p}$ on $C^\infty(\bar{U})$, then take the completion and denote $W^{s,p}(\bar{U})$. Yields a distribution space, elements are functions with weak derivatives.
 - Distribution: $C_c^\infty(U)^\vee$, the dual of the space of smooth compactly supported functions on an open set $U \subset \mathbb{R}^n$.

2 Talk

Overview: Analyze the space $\mathcal{M}(x, y)$ of solutions to the Floer equation connecting two orbits x, y of H . Show $\mathcal{M}(x, y)$ is in fact a manifold of dimension $\mu(x) - \mu(y)$.

Strategy:

1. Describe $\mathcal{M}(x, y)$ as the zero set of a section of a vector bundle over the Banach manifold $\mathcal{P}(x, y)$.
2. Apply the Sard-Smale theorem: perturb H to make $\mathcal{M}(x, y)$ the inverse image of a regular value of some map.
3. Show that the tangent maps (?) are Fredholm operators of index $\mu(x) - \mu(y) = \dim \mathcal{M}(x, y)$.

Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of \mathcal{F} .

2.1 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Start by trying to construct a subspace $\mathcal{C}_\varepsilon^\infty(H) \subset C^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C^\infty_\varepsilon(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$ denote a perturbation of H .
- Fix $\varepsilon = \{\varepsilon_k \mid k \in \mathbb{Z}^{\geq 0}\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W, t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \mid |\alpha| = k \right\},$$

the maximum over all sets of multi-indices α of length k .

Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \dots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} |d^k h(x,t)|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

- Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \quad (?).$$

- Obtain the computable form

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \Psi_i^{-1})(z)|.$$

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \mid \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Proposition 2.1.

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma 2.2.

$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Proof (of Lemma, Sketch).

First prove for C^0 :

- **Idea:** reduce to polynomials in \mathbb{R}^m .
- Embed $W \times S^1 \hookrightarrow [-M, M]^m \cong I^m \subset \mathbb{R}^m$ for some large m , reduces to proving it for $C^\infty(I^m; \mathbb{R})$.
- Recall Stone-Weierstrass:

For $A \leq C^0(X; \mathbb{R})$ a subalgebra with X compact Hausdorff and A containing a nonzero constant function, A is dense iff it separates points (for all $a \neq b \in X$ there exists $f \in A$ such that $f(a) \neq f(b)$)
- Apply to $A = \mathbb{Q}[x_1, \dots, x_m]$ the subalgebra of polynomial functions, the nonzero constant function $c(x) = 1$, and show it separates points via $f(x) = x - a$, then $f(a) = 0$ and $f(b) = a - b \neq 0$ by assumption.

- Thus A is a countable dense subset.

Then prove for C^1 :

- **Idea:** Take polynomials convolved with a countable sequence of bump functions, which is still a countable dense subset.
- Choose a smooth bump function χ supported on $B(0, 1)$
- Define the sequence $\chi_k(x) := k^m \chi(kx)$.
- Prove that $(f * \chi_k) \xrightarrow{k \rightarrow \infty} f$ in the C_{loc}^0 sense (?)
- Show that for a fixed k , any other sequence $g_\ell \rightarrow f$ in C_{loc}^∞ , we have $g_\ell * \chi_k \rightarrow f * \chi_k$ in the C_{loc}^0 sense using

$$|g_\ell - f| \rightarrow 0 \implies \sup_K \left| \frac{\partial}{\partial x_i} (g_\ell - f) * \chi_k \right| \leq \sup_k |g_\ell - f| \cdot (\dots) \rightarrow 0 \quad \forall i$$

- Conclude $\lim_\ell \lim_k g_\ell * \chi_k = f$.
- Taking g_ℓ to be polynomial approximations, the following subset is countable and dense:

$$\bigcup_{k \in \mathbb{Z}^{\geq 0}} \{P * \chi_k \mid P \in \mathbb{Q}[x_1, \dots, x_m]\}$$

which are pushed through the charts Ψ_i to actually compute. ■

The second part of this proof generalizes to C^∞ .

Proof (of Proposition, Sketch).

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n , define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \mid k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable. ■

Why does this imply density? I don't know.

The next proposition establishes a version of this theorem with compact support:

Proposition 2.3.

For any $(\mathbf{x}, t) \in U \in W \times S^1$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the C^1 topology by functions in C_ε^∞ supported in U .

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \mid h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\text{supp}(h)$ is “far” from $\text{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

2.2 Review 8.2

What is \mathcal{F} ?

We started with the unadorned Floer map:

$$\begin{aligned} \mathcal{F} : \mathcal{C}^\infty(\mathbb{R} \times S^1; W) &\longrightarrow \mathcal{C}^\infty(\mathbb{R} \times S^1; TW) \\ u &\mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H_t) \end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u). \end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned} \mathcal{P}^{1,p}(x, y) & \text{ :? } \longrightarrow ? \\ (s, t) & \mapsto \exp_{w(s,t)} Y(s, t). \end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C_\infty^\infty(x, y)$.

2.3 8.4: Linearizing the Floer equation: The Differential of \mathcal{F}

Choose $m > n = \dim(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as Z_i , tangents to W along u or in a neighborhood B of u) with actual vectors in \mathbb{R}^m .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify

$$\text{im } \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

We’ve just replaced the codomain here.

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional \mathcal{A}_H ,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$ denotes a fixed solution to the Floer equation,
- $C_\infty(x, y)$ was the set of solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions.

Recall:

$$C_{\searrow}(x, y) := \{u \in C^\infty(\mathbb{R} \times S^1; W) \mid \lim_{s \rightarrow -\infty} u(s, t) = x(t), \lim_{s \rightarrow \infty} u(s, t) = y(t)\} \\ \left| \frac{\partial u}{\partial t}(s, t) \right| \quad \text{and} \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

Fix a solution

$$u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way: the loops x, y are contractible, so they bound discs. So we extend by pushing these discs out slightly::



From earlier in the book, we have

Assumption (6.22):

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle w^*TW , i.e. $\langle c_1(TW), \pi_2(W) \rangle = 0$ where c_1 denotes the first Chern class of the bundle TW .

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps $S^2 \longrightarrow W$ lift to $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$\begin{array}{ccc} \tilde{u}^*TW & \xrightarrow{d\tilde{u}} & TW \\ \downarrow & \lrcorner & \downarrow \\ S^2 & \xrightarrow{\tilde{u}} & W \end{array}$$

- Using the assumption, trivialize the pullback \tilde{u}^*TW to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

where

- The frame depends smoothly on $(s, t) \in S^2$,
- $\lim_{s \rightarrow \pm\infty} Z_i$ exists for each i .
-

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \partial t} \quad \curvearrowright \quad Z_i \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{for each } i$$

Claim: such trivializations exist, “using cylinders near the spherical caps in the figure”.

Recall what $\mathcal{P}^{1,p}(x, y)$, J , X_t are here.

- Use this frame to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

- Define and compute the differential of the composite map $\tilde{\mathcal{F}}$ defined as follows:

$$\begin{array}{ccc} & \tilde{\mathcal{F}} & \\ & \curvearrowright & \\ \mathcal{P}^{1,p}(x, y) & \xrightarrow{\mathcal{F}} L^p(\mathbb{R} \times S^1; TW) & \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\ & \tilde{\mathcal{F}} & \\ u & \xrightarrow{\quad\quad\quad} & \frac{\partial u}{\partial s} + J(u) \left(\frac{\partial u}{\partial t} - X_t(u) \right) \end{array}$$

- From now on, let \mathcal{F} denote $\tilde{\mathcal{F}}$.

- Take the vector

$$Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

- View Y as a vector in \mathbb{R}^m tangent to W , given by $Y = \sum_{i=1}^{2n} y_i Z_i$.

- Plug $u + Y$ into the equation for \mathcal{F} , directly yielding

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u) \\ \implies \mathcal{F}(u + Y) &= \frac{\partial(u + Y)}{\partial s} + J(u + Y) \frac{\partial(u + Y)}{\partial t} - J(u + Y) X_t(u + Y) \end{aligned}$$

- Extract the part that is linear in Y and collect terms:

$$\begin{aligned} (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) + \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \end{aligned}$$

- This is a sum of two differential operators:
 - * One of order 1, one of order 2 (Perspective 1)
 - * The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)
- Now compute in charts. Need a lemma:

Lemma 2.4 (Leibniz Rule).

For any source space X and any maps

$$\begin{aligned} J : X &\longrightarrow \text{End}(\mathbb{R}^m) \\ Y, v : X &\longrightarrow \mathbb{R}^m \end{aligned}$$

we have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

Proof .

Differentiate the map

$$\begin{aligned} J \cdot v : X &\longrightarrow \mathbb{R}^m \\ x &\mapsto J(x) \cdot v(x) \end{aligned}$$

to obtain

$$\begin{aligned} J(x+Y)v(x+y) &= (J(x) + (dJ)_x(Y)) \cdot (v(x) + (dv)_x(Y)) + \cdots \\ &= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x) + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots \end{aligned}$$

$$\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y).$$

■

- Using the chart i defined by $\{Z_i\}$ to write $Y = \sum_{i=1}^{2n} y_i Z_i$ and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where O_0 are order 0 terms (“they do not differentiate the y_i ”) and the O_1 are order 1 terms:

$$O_0 = \sum_{i=1}^{2n} \frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i$$

$$O_1 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right).$$

For every such smooth map $u : \mathbb{R} \times S^1 \rightarrow W$, $(d\mathcal{F})_u(Y) = O_1 + O_0$ where O_i are differential operators of order i , and in fact O_1 can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$ where $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^n)$ is linear of order 0, and in fact we have

Proposition 2.5.

If u solves Floer’s equation, then $(d\mathcal{F})_u = \bar{\partial} + S(s, t)$ where S is linear, tends to a symmetric operator as $s \rightarrow \pm\infty$, and $\lim \partial_t S = 0$ uniformly in t .

There is a very long computational proof.

Denote the order 0 part of $(d\mathcal{F})_u$ as $Y \mapsto S \cdot Y$ so $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^m)$ and define $S^\pm := \lim_{s \rightarrow \pm\infty} S(s, \cdot)$.

Proposition 2.6.

The equation $\partial_t Y = J_0 S^\pm Y$ linearizes Hamilton’s equation $\dot{z} = X_t(z)$ at $x = \lim_{s \rightarrow \pm\infty} u$ for S^+ and S^- respectively.

Proof: uses previous proposition.

Given a solution u , the product

$$\begin{aligned} u \cdot s &: ? \rightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and $\mathcal{F}(u \cdot s) = 0$ for all s .

Punchline:

Thus $\frac{\partial u}{\partial s}$ is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s} \mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left(\frac{\partial u}{\partial s} \right).$$

Along any nonconstant solution connecting x and y , $\dim \ker(d\mathcal{F})_u \geq 1$.