

# Title

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References: <https://www.daniellitt.com/etale-cohomology>

Prerequisites:

- Homological Algebra
  - Abelian Categories
  - Derived Functors
  - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
    - Etale morphism
    - Grothendieck topologies
    - The etale topology
    - Etale cohomology and the basis theorems
    - Etale cohomology of curves
    - Comparison theorems to singular cohomology
    - Focused on the case where coefficients are a constructible sheaf.
  - Prove the Weil Conjectures (more than one proof)
    - Proving the Riemann Hypothesis for varieties over finite fields
- One of the greatest pieces of 20th century mathematics!
- Topics
    - Weil 2 (Strengthening of RH, used in practice)
    - Formality of algebraic varieties (topological features unique to varieties)
    - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose  $X/\mathbb{C}$  is a quasiprojective variety: a finite type separated integral  $\mathbb{C}$ -scheme.

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If you take the complex points, it naturally has the structure of a complex analytic space  $X(\mathbb{C})^{\text{an}}$ : you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology  $H^i(X(\mathbb{C})^{\text{an}}, \mathbb{Z})$ , which satisfies several nice properties:

- Finitely generated  $\mathbb{Z}$ -modules
- Extra Hodge structure when tensored up to  $\mathbb{C}$  (same as  $\mathbb{C}$  coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general “nice” schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We’ll associate  $X$  a “nice scheme”  $\rightsquigarrow H^i(X_{\text{et}}, \mathbb{Z}/\ell^n \mathbb{Z})$ . Take the inverse limit over all  $n$  to obtain the  $\ell$ -adic cohomology  $H^i(X_{\text{et}}, \mathbb{Z}_\ell)$ . You can tensor with  $\mathbb{Q}$  to get something with  $\mathbb{Q}_\ell$  coefficients. And as in singular cohomology, you can a “twisted coefficient system”.

What are nice schemes:

- $X = \text{Spec } \mathcal{O}_k$ , the ring of integers over a number field.
- $X$  a variety over an algebraically closed field
  - Typical, most analogous to taking a variety over  $\mathbb{C}$ .
- $X$  a variety over a non-algebraically closed field

Some comparisons between the last two cases:

- For  $\mathbb{C}$ - variety,  $H_{\text{sing}}^i$  will vanish above  $i = 2d$ .
- Over a finite field,  $H^i$  will vanish for  $i > 2d + 1$  but generally not vanish for  $i = 2d + 1$ .

In good situations, these are finitely generated  $\mathbb{Z}/\ell^n \mathbb{Z}$ -modules, have Mayer-Vietoris and excision sequences, spectral sequences, etc.

Related invariants: for a scheme with a geometric point  $(X, \bar{x}) \rightsquigarrow \pi_1^{\text{étale}}(X, \bar{x})$ , which is a profinite topological group, which is a profinite topological group.

Note: a geometric point is a map from  $\text{Spec } X$  to an algebraically closed field.

More invariants beyond the scope of this course:

- Higher homotopy groups
- Homotopy type (equivalence class of spaces)

So we want homotopy-theoretic invariants for varieties.

**Remark 1.**

This cohomology theory is necessarily weird!

**Theorem 1.1 (Serre).**

There does not exist a cohomology theory for schemes over  $\bar{\mathbb{F}}_q$  with the following properties:

1. Functorial
2. Satisfies the Kunneth formula
3. For  $E$  an elliptic curve,  $H^1(E) = \mathbb{Q}^2$ .

Slogan: No cohomology theory with  $\mathbb{Q}$  coefficients.

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*Proof .*

Take  $E$  to be a supersingular elliptic curve. Then  $\text{End}(E) \otimes \mathbb{Q}$  is a quaternion algebra.

Fact: There are no algebra morphisms  $R \rightarrow \text{Mat}_{2 \times 2}(\mathbb{Q})$

**Exercise .**

Functoriality and Kunneth implies that  $\text{End}(E) \curvearrowright E$  yields an action on  $H^1(E)$ , which is precisely an algebra morphism  $\text{End}(E) \rightarrow \text{Mat}_{2 \times 2}(\mathbb{Q})$ , a contradiction.

The content: the sum of two endomorphisms act via their sum on  $H^1$ .

**Exercise .**

Prove the same thing for  $\mathbb{Q}_p$  coefficients, where  $p$  divides the characteristic of the ground field.

Proof the same, just need to know what quaternion algebras show up.

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This forces using some funky type of coefficients.

What are the Weil Conjectures?

Suppose  $X/\mathbb{F}_q$  is a variety, then

$$\zeta_X(t) = \exp \left( \sum_{n>0} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n \right).$$

Some comments:

- $\frac{\partial}{\partial t} \log \zeta_X(t)$  is an ordinary generating function for the number of rational points.
- Slogan: locations of zeros and poles of a meromorphic function control the growth rate of the coefficients of the Taylor series of the logarithmic derivative.

**Exercise 1.3.**

Make this slogan precise for rational functions, i.e. ratios of two polynomials.

The conjectures:

1.  $\zeta_X(t)$  is a rational function.
2. (Functional equation) For  $X$  smooth and proper

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t).$$

3. (RH) All roots and poles of  $\zeta_X(t)$  have absolute value  $q^{\frac{i}{2}}$  with  $i \in \mathbb{Z}$ , and these are equal to the  $i$ th Betti numbers if  $X$  lifts to characteristic zero.

Note: we'll generalize betti numbers so this makes sense in general.

All theorems! Proofs:

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1. Dwork, using  $p$ -adic methods. Proof here will follow from the fact that  $H_{\text{étale}}^i$  are finite-dimensional. Related to Lefschetz Trace Formula (how Grothendieck thought about it).
  2. Grothendieck, follows from some version of Poincaré duality.
  3. (and 4) Deligne.

Euler Product:

Let  $|X|$  denote the closed points of  $X$ , then there is an Euler product:

$$\begin{aligned}\zeta_X(q^{-n}t^{-1}) &= \pm q^{\frac{nE}{2}} t^E \zeta_X(t) = \prod_{x \in |X|} \exp \left( t^{\deg(x)} + \frac{t^{2\deg(x)}}{2} + \dots \right) \\ &= \prod_{x \in |X|} \exp \left( -\log(1 - t^{\deg(x)}) \right) \\ &= \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}.\end{aligned}$$

**Exercise 1.4.**

Check the first equality. If you have a point of  $\deg(x) = n$ , how many  $\mathbb{F}_{q^n}$  points does this contribute? I.e., how many maps are there  $\text{Spec}(\mathbb{F}_{q^n}) \rightarrow \text{Spec}(\mathbb{F}_q)$  over  $\mathbb{F}_q$ ?

There are exactly  $n$ : it's  $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ . But then division by  $n$  drops this contribution to one.

We can keep going by expanding and multiplying out the product:

$$\begin{aligned}\prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}} &= \prod_{x \in |X|} (1 + t^{\deg(x)} + t^{2\deg(x)} + \dots) \\ &= \sum_{n \geq 0} \left( \# \text{ of Galois-stable subset of } X(\bar{\mathbb{F}}_q) \text{ of size } n \right) t^n.\end{aligned}$$

Why? If you have a degree  $x$  point, it contributes a stable subset of size  $x$ : namely the  $\mathbb{F}_{q^n}$  points of  $\mathbb{F}_{q^n}$ . Taking Galois orbits like that correspond to multiplying this product.

But these are the points of some algebraic variety:

$$\dots = \sum_{n \geq 0} |\text{Sym}^n(X)(\mathbb{F}_q)| t^n,$$

where  $\text{Sym}^n(X) := X^n / \Sigma_n$ , the action of the symmetric group. Why is that? A  $\bar{\mathbb{F}}_q$  point of  $\text{Sym}^n(X)$  is an unordered  $n$ -tuple of  $\bar{\mathbb{F}}_q$  points without an ordering, and asking them to be Galois stable is the same as saying that they are an  $\mathbb{F}_q$  point.

**Theorem 1.2 (First Weil Conjecture for Curves).**

For  $X$  a smooth proper curve over  $\mathbb{F}_q$ ,  $\zeta_X(t)$  is rational.

*Proof .*

Claim: there is a set map

$$\begin{aligned}\mathrm{Sym}^n X &\longrightarrow \mathrm{Pic}^n X \\ D &\mapsto \mathcal{O}(D).\end{aligned}$$

Here the divisor is an  $n$ -tuple of points.

What are the fibers over a line bundle  $\mathcal{O}(D)$ ? A linear system, i.e. the projectivization of global sections  $\mathbb{P}\Gamma(X, \mathcal{O}(D))$ . What is the expected dimension? To compute the dimension of the space of line bundles on a curve, use Riemann-Roch:

$$\dim \mathbb{P}\Gamma(X, \mathcal{O}(D)) = \deg(D) + 1 - g + \dim H^1(X, \mathcal{O}(D)) - 1.$$

where the last  $-1$  comes from the fact that this is a projective space.

Claim: if  $\deg(D) = 2g - 2$ , then  $H^1(X, \mathcal{O}(D)) = 0$ .

This is because it's dual to  $H^0(X, \mathcal{O}(K - D))^\vee$ , but this has negative degree and a line bundle of negative degree can never have sections.

Note: should check to make sure you know why this is true!

Thus the fibers are isomorphic to  $\mathbb{P}^{n-g}$  for  $n > 2g - 2$ . Now make a reduction (exercise: justify why):

Wlog assume  $X(\mathbb{F}_q) \neq \emptyset$ . In this case,  $\mathrm{Pic}^n(X) \cong \mathrm{Pic}^{n+1}(X)$  for all  $n$ , since you can take  $\mathcal{O}(P)$  for  $P$  a point, a degree 1 line bundle, and tensor with it. It's an isomorphism because you can tensor with the dual bundle to go back.

Thus for all  $n > 2g - 2$ ,

$$|\mathrm{Sym}^n(X)(\mathbb{F}_q)| = |\mathbb{P}^{n-g}(\mathbb{F}_q)| \cdot |\mathrm{Pic}^n(X)(\mathbb{F}_q)| = |\mathbb{P}^{n-g}(\mathbb{F}_q)| \cdot |\mathrm{Pic}^0(X)(\mathbb{F}_q)|.$$

Thus  $\zeta_X(t)$  is a polynomial plus  $\sum_{n>2g-2} |\mathrm{Pic}^n(X)(\mathbb{F}_q)| (1 + q + q^2 + \cdots + q^{n-g}) t^n$ .

**Exercise .**

Show that this is a rational function using the formula for a geometric series.

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### Theorem 1.3 (Functional Equation).

The functional equation in the case of curves:

$$\zeta_X(q^{-1}t^{-1}) = \pm q^{\frac{2-2g}{2}} t^{2-2g} \zeta_X(t).$$

### Exercise 1.6 (Important).

Where it comes from in terms of  $\mathrm{Sym}^n$ : Serre duality.

We'll show the RH later.

### Theorem 1.4 (Dwork).

Suppose  $X/\mathbb{F}_q$  is any variety, then  $\zeta_X(t)$  is rational function.

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Roughly known to Weil, hinted at in original paper

*Proof (Grothendieck).*

Idea: take Frobenius (intentionally vague, arithmetic vs geometric vs ...)  $F : X \rightarrow X$ , then  $X(\mathbb{F}_q)$  are the fixed points of  $F$  acting on  $X_{\mathbb{F}_q}$ , and the  $\mathbb{F}_{q^n}$  points are the fixed points of  $F^n$ . Uses the Lefschetz fixed point formula, which will say for  $\ell \neq \text{char}(\mathbb{F}_q)$ ,

$$|X(\mathbb{F}_{q^n})| = \sum_{i=0}^{2 \dim(X)} (-1)^i \text{Tr}(F^n) H_c^i(X_{\mathbb{F}_q}, \mathbb{Q}_\ell).$$

Here  $H_c^i$  is compactly supported cohomology, we'll define this later in the course.

**Lemma 1.5.**

$$\exp \left( \sum \frac{\text{Tr}(F^n)}{n} t^n \right) \text{ is rational.}$$

This lemma implies the result, because if you plug the trace formula into the zeta function, you'll get an alternating product  $f \cdots \frac{1}{g} \cdot h \cdot \frac{1}{j} \cdots$  of functions of the form in the lemma, which is still rational.

*Proof (Of Lemma).*

It suffices to treat the case  $\dim(V) = 1$ , because otherwise you can just write this as a sum of powers of eigenvalues.

Then you have a scalar matrix, so you obtain

$$\exp \left( \sum \frac{\alpha^n}{n} t^n \right) = \exp(-\log(1 - \alpha t)) = \frac{1}{1 - \alpha t},$$

which is rational. ■

This proves rationality, contingent on

1. The Lefschetz fixed point formula
2. The finite dimensionality of étale cohomology

**Exercise 1.7.**

Try to figure out how Poincaré duality should give the functional equation.

Hint: try the lemma on a vector space where  $F$  scales a bilinear form.

**Theorem 1.6 (Serre, Kahler Analog).**

Suppose  $X/\mathbb{C}$  is a smooth projective variety