Interesting Topological Spaces in Algebraic Geometry

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- Calabi-Yau manifolds
  - Dimension 1: All elliptic curves (up to homeomorphism)
  - Dimension 2: $K3$ surfaces
  - Dimension 3 (threefolds): 500 million +, unknown if infinitely many
  - The bananafold
  - Hyperkähler
- Hurwitz schemes
- Topological galois groups, e.g. $G(\overline{F}/F)$ for $F = \mathbb{Q}, \mathbb{F}_p$.
- Spec (R) for R a DVR (a Sierpinski space)
- Quiver Grassmannians
- Rigid analytic spaces
- Affine line with two origins
- Moduli stack of elliptic curves $\mathcal{M}_{1,1}$.
- Abelian Surface
- Fano Varieties
- Curves: isomorphic to $\mathbb{P}^1$
- Surfaces: Del Pezzo surfaces
- Weighted projective space
- Toric Varieties
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> Due to Kunihiko Kodaira's classification of complex surfaces, we know that any compact hyperkähler 4-manifold is either a K3 surface or a compact torus T^{4}. (Every Calabi–Yau manifold in 4 (real) dimensions is a hyperkähler manifold, because SU(2) is isomorphic to Sp(1).)

> As was discovered by Beauville, the Hilbert scheme of k points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension 4k. This gives rise to two series of compact examples: Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

2 Analogies

Impossible goal: pick a category, understand all of the objects and all of the maps. Two main categories with a forgetful functor: **Diff** $\longrightarrow$ **Top**. Question:

- What's in the "image" of this functor? (Manifolds that admit a differentiable structure.)
- What is the "fiber" above a given topological manifold? (Distinct differentiable structures)

Differentiable Manifolds: classified by geometric structure in low dimensions (≤ 4), algebraic data/methods in high dimensions

2.1 Topological Category

Identify objects up to homeomorphism

- Initial object: empty set
- Dimension 0: The point (terminal object)
- 1-manifolds: $S^1, \mathbb{R}$

- 2-manifolds: $\left\langle \mathbb{S}, \mathbb{T}, \mathbb{RP} \;\middle|\; \mathbb{S} = 0, 3\mathbb{RP} = \mathbb{RP} + \mathbb{T} \right\rangle$. Classified by π_1 (orientability and "genus"). Riemann, Poincare, Klein.

2.2 Smooth Category

- 2-manifolds: Homeomorphic $\iff$ diffeomorphic.
  - Uniformization: admits a metric with one of three constant curvatures (negative, zero, positive).
  - Simply connected Riemann surfaces are conformally equivalent to one of $\mathbb{H}, \mathbb{D}^\circ, \mathbb{CP}^1$.
- 3-manifolds: Thurston's Geometrization
  - Oriented prime 3-manifolds can be decomposed into geometric "pieces" of 8 possible types
  - Geometric structure: a diffeo $M \cong \tilde{M}/\Gamma$ where Γ is a discrete Lie group acting freely/transitively on X
- 4-manifolds: classified in the topological category by surgery, but not in the smooth category
- n-manifolds, $n \geq 5$: classified by surgery