Title

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1.1 Frobenius Kernels

Let char $(k)p > 0$ and let G be an algebraic group scheme. We have a Frobenius map $F : G \to G$ given by $F((x_{ij})) = (x_{ij}^p)$, which we can iterate to get F^r for $r \in \mathbb{N}$. Setting $G_r = \ker F^r$ the rth Frobenius kernel, we get a normal series of group schemes

$$G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G.$$

There is an associated chain of finite dimensional Hopf algebras

$$\mathrm{Dist}(G_1) \leq \mathrm{Dist}(G_2) \leq \cdots \leq \mathrm{Dist}(G).$$

Then $k[G]^\vee = \mathrm{Dist}(G_r)$, and we get an equivalence of representations for G_r to representations for $\mathrm{Dist}(G_r)$.

A special case will be when G is a reductive algebraic group scheme. We'll start by finding a basis for $\mathrm{Dist}(G_r)$.

Recall the PBW theorem: we have a basis for $\mathfrak{g}$ given by

$$\begin{aligned}
\left\{ x_\alpha \;\middle|\; \alpha \in \Phi^+ \right\} &\text{ Positive root vectors} \\
\left\{ h_i \;\middle|\; i = 1, \cdots, n \right\} &\text{ A basis for } t \\
\left\{ x_\alpha \;\middle|\; \alpha \in \Phi^- \right\} &\text{ Negative root vectors}
\end{aligned}.$$

We can then obtain a basis for $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \left\langle \prod_{\alpha \in \Phi^+} x_\alpha^{n(\alpha)} \prod_{i=1}^{n} h_i^{k_i} \prod_{\alpha \in \Phi^+} x_{-\alpha}^{m(\alpha)} \right\rangle.$$

We can similarly obtain a basis for the distribution algebra

$$\mathrm{Dist}(G) = \left\langle \prod_{\alpha \in \Phi^+} \frac{x_\alpha^{n(\alpha)}}{n!} \prod_{i=1}^{n} \binom{h_i}{k_i} \prod_{\alpha \in \Phi^+} \frac{x_{-\alpha}^{n(\alpha)}}{n!} \right\rangle,$$

and we can similar get $\mathrm{Dist}(G_r)$ by restricting to $0 \leq n(\alpha), k_i, m(\alpha) \leq p^r - 1$. Above the k_i are allowed to be any integers. This yields a triangular decomposition

$$\mathrm{Dist}(G_r) = \mathrm{Dist}(U_r^+)\mathrm{Dist}(T_r)\mathrm{Dist}(U_r^-),$$

where we'll denote the first two terms $\mathrm{Dist}(B_r^+)$ and the last two as $\mathrm{Dist}(B_r)$.

1.2 Induced and Coinduced Modules

Goal: Classify simple G_r-modules. We know the classification of simple G-modules, so we'll follow similar reasoning. We started by realizing $L(\lambda) \hookrightarrow \mathrm{Ind}_B^G \lambda$ as a submodule (the socle) of some "universal" module.

Let M be a B_r-module, we can then define

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r},$$

where we're now taking the B_r-invariants. We get a decomposition as vector spaces,

$$k[G_r] = k[U_r^+] \otimes_k k[B_r]$$

and thus an isomorphism

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes M$$

since $k[B_r] \otimes M \cong \mathrm{Ind}_{B_r}^{B_r} M \cong M$.

We then define

$$\mathrm{Coind}_{B_r}^{G_r} = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \otimes M,$$

which is an analog of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M$.

We have $\mathrm{Dist}(U_r^+) \otimes \mathrm{Dist}(B_r) \cong \mathrm{Dist}(G_r)$, so

$$\mathrm{Coind}_{B_r}^{G_r} = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \otimes M \cong \mathrm{Dist}(U_r^+) \otimes_k \mathrm{Dist}(B_r) \otimes_{\mathrm{Dist}(B_r)} M \cong \mathrm{Dist}(U_r^+) \otimes_k M,$$

which we'll define as the **coinduced module**.

We can compute the dimension:

$$\dim \mathrm{Ind}_{B_r}^{G_r} M = \dim \mathrm{Coind}_{B_r}^{G_r} M = (\dim M) p^{r|\Phi^+|}.$$

Open: don't know how to compute composition factors.

Proposition 1.1 *(?).*

1. $$\mathrm{Coind}_{B_r}^{G_r} M \equiv \mathrm{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho,$$

   where the last term is a one-dimensional B_r-module and ρ is the *Weyl weight.*
2. $$\mathrm{Coind}_{B_r^+}^{G_r} M \cong \mathrm{Ind}_{B_r^+}^{G_r} M \otimes -2(p^r - 1)\rho$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} w_i.$$

Proof (Sketch for (1)).
Since the tensor product satisfies a universal property, we have a map

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, "B_r"] & & \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} M \arrow[dl, dotted, "\exists\psi"] \\
& N = M\,\mathrm{Ind}_{B_r}^{G_r} \otimes 2(p^r - 1)\rho &
\end{tikzcd}$$

1. We need to find a B_r morphism $f : M \to N$.
2. We need to show that f generates N as a G_r-module.

Note that if (1) and (2) hold, then ψ is surjective, but since $\dim \mathrm{Coind}_{B_r}^{G_r} M = \dim N$ this forces ψ to be an isomorphism.
We can write

$$\begin{aligned}
\mathrm{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho &= (k[G_r] \otimes M \otimes 2(p^r - 1)\rho)^{B_r} \\
&\cong \hom_{B_r} (\mathrm{Dist}(G_r), M \otimes 2(p^r - 1)\rho).
\end{aligned}$$

Let $g_m(x) := m \otimes 2(p^r - 1)\rho$ for any $x = \displaystyle\prod_{\alpha \in \Phi^+} \frac{x_\alpha^{p^r - 1}}{(p^r - 1)!}$, and $g_m(x) = 0$ for any other x.
Now define $f(m) = g_m$, and check that $\operatorname{im} f$ generates N. ∎

1.3 Verma Modules

Recall that $W(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda$ were the *Verma modules* for lie algebras.

Let $\lambda \in X(T)$, we have $T_r \leq T$ and restriction yields a map $X(T) \to X(T_r)$. Given a weight λ, we can write it p-adically as

$$\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots + \lambda_{r-1} + \cdots.$$

This yields an exact sequence

$$0 \to p^r X(T) \to X(T) \to X(T_r) \to 0,$$

and thus $X(T)/p^r X(T) \cong X(T_r)$.

Let $\lambda \in X(T_r)$, then λ becomes a B_r-module by letting U_r act trivially, since we have

$$\cdots U_r \to B_r \twoheadrightarrow T_r \to 0.$$

Set $Z(r) = \operatorname{Coind}_{B_r}^{G_r} \lambda$, and set $Z(r)' = \operatorname{Ind}_{B_r}^{G_r} \lambda$. Then $\dim Z_r(\lambda) = \dim Z_r'(\lambda) = p^{r\left|\Phi^+\right|}$. We'll then think of

- Coind $\twoheadrightarrow L_r(\lambda)$ being in the head,
- $L_r(\lambda) \hookrightarrow$ Ind being the socle.

Note that the dimensions aren't known, nor are the projective covers or injective hulls.

We have a form of translation invariance, namely

$$\begin{aligned} Z_r(\lambda + p^r \nu) &= Z_r(\lambda) \qquad \forall \nu \in X(T) \\ Z_r'(\lambda + p^r \nu) &= Z_r'(\lambda) \qquad \forall \nu \in X(T). \end{aligned}$$

Proposition 1.2 ***(?).***
Let $\lambda \in X(T)$.

1. $Z_r(\lambda)\downarrow_{B_r}$ is the projective cover of λ and the injective hull of $\lambda - 2(p^r - 1)\rho$.
2. $Z_r'(\lambda)\downarrow_{B_r^+}$ is the injective hull of λ and the projective hull of $\lambda - 2(p^r - 1)\rho$.