Full Notes

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1 Friday January 10

Recall that $\mathbb{C}$ is a field, where $z = x + iy \implies \overline{z} = x - iy$, and if $z \neq 0$ then $z^{-1} = \overline{z}/|z|^2$.

Lemma (Triangle Inequality: $|z + w| \leq |z| + |w|$

Proof:

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\overline{w}| - \Re z\overline{w}) \geq 0.$$

Lemma (Reverse Triangle Inequality): $||z| - |w|| \leq |z - w|$.

Proof:

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

Claim: $(\mathbb{C}, |\cdot|)$ is a normed space.

Definition: $\lim z_n = z \iff |z_n - z| \to 0 \in \mathbb{R}$.

Definition: A disc is defined as $D_r(z_0) \coloneqq \left\{ z \in \mathbb{C} \;\middle|\; |z - z_0| < r \right\}$, and a subset is open iff it contains a disc. By convention, D_r denotes a disc about $z_0 = 0$.

Definition: $\displaystyle\sum_k z_k$ converges iff $S_N \coloneqq \displaystyle\sum_{|k|<N} z_k$ converges.

Note that $z_n \to z$ and $z_n = x_n + iy_n$, and

$$|z_n - z| = \sqrt{(x_n - x)^2 - (y_n - y)^2} < \varepsilon \implies |x - x_n|, |y - y_n| < \varepsilon.$$

Since $\mathbb{R}$ is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

Note: This is useful precisely when you don't know the limiting term.

Note that $\sum_k z_k$ thus converges if $\left|\sum_{k=m}^{n} z_k\right| < \varepsilon$ for m, n large enough, so sums converges iff they have small tails.

Definition: $S_N = \sum^N z_k$ converges absolutely iff $\tilde{S} := \sum^N |z_k|$ converges.

Note that the partial sums $\sum^N |z_k|$ are monotone, so $\tilde{S}_N$ converges iff the partial sums are bounded above.

Definition: A sum of the form $\sum_{k=0}^{\infty} a_k z_k$ is a power series.

Examples:

$$\sum x^k = \frac{1}{1-x}$$
$$\sum (-x^2)^k = \frac{1}{1+x^2}.$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at $x = 1$ and the second as a pole at $x = i$.

2 Monday January 13th

Recall that $\sum z_k$ converges iff $s_n = \sum_{k=1}^{n} z_k$ converges.

Lemma: Absolute convergence implies convergence.

The most interesting series: $f(z) = \sum a_k z^k$, i.e. power series.

Divergence lemma: If $\sum z_k$ converges, then $\lim z_k = 0$.

Corollary: If $\sum z_k$ converges, $\{z_k\}$ is uniformly bounded by a constant $C > 0$, i.e. $|z_k| < C$ for all k.

Proposition: If $\sum a_k z_k$ converges at some point z_0, then it converges for all $|z| < |z|_0$.

The inequality is necessarily strict. For example, $\sum \frac{z^{n-1}}{n}$ converges at $z = -1$ (alternating harmonic series) but not at $z = 1$ (harmonic series).

Proof: Suppose $\sum a_k z_1^k$ converges. The terms are uniformly bounded, so $\left|a_k z_1^k\right| \leq C$ for all k. Then we have

$$|a_k| \leq C/|z_1|^k$$

, so if $|z| < |z_1|$ we have

$$\left|a_k z^k\right| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k.$$

So if $|z| < |z_1|$, the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting $r = |z|/|z_1|$, we have

$$\sum \left|a_k z^k\right| \leq \sum c r^k = \frac{c}{1-r},$$

and so we have absolute convergence.

■

Exercise (future problem set): Show that $\sum \frac{1}{k} z^{k-1}$ converges for all $|z| = 1$ except for $z = 1$. (Use summation by parts.)

Definition The radius of convergence is the real number R such that $f(z) = \sum a_k z^k$ converges precisely for $|z| < R$ and diverges for $|z| > R$. We denote a disc of radius R centered at zero by D_R.

If $R = \infty$, then f is said to be *entire*.

Proposition: Suppose that $\sum a_k z^k$ converges for all $|z| < R$. Then $f(z) = \sum a_k z^k$ is continuous on D_R, i.e. using the sequential definition of continuity, $\lim_{z \to z_0} f(z) = f(z_0)$ for all $z_0 \in D_R$.

Recall that $S_n(z) \to S(z)$ uniformly on Ω iff $\forall \varepsilon > 0$, there exists a $M \in \mathbb{N}$ such that $n > M \implies |S_n(z) - S(z)| < \varepsilon$ for all $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample: $f_n(x) = x^n$ on $[0, 1]$; then $f_n \to \delta(1)$. Note that it uniformly converges on $[0, 1 - \varepsilon]$ for any $\varepsilon > 0$.

Exercise: Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

Proof of proposition: Write $f(z) = \sum_{k=0}^{N} a_k z^k + \sum_{N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$. Note that if $|z| < R$, then there exists a T such that $|z| < T < R$ where $f(z)$ converges uniformly on D_T.

Check!

We need to show that $|R_N(z)|$ is uniformly small for $|z| < s < T$. Note that $\sum a_k z^k$ converges on D_T, so we can find a C such that $\left|a_k z^k\right| \leq C$ for all k. Then $|a_k| \leq C/T^k$ for all k, and so

$$\begin{aligned}\left|\sum_{k=N+1}^{\infty} a_k z^k\right| &\leq \sum_{k=N+1}^{\infty} |a_k||z|^k \\ &\leq \sum_{k=N+1}^{\infty} (c/T^k) s^k \\ &= c \sum |s/T|^k \\ &= c\frac{r^{N+!}}{1-r} & = C\varepsilon_n \to 0,\end{aligned}$$

which follows because $0 < r = s/T < 1$.

So $S_N(z) \to f(z)$ uniformly on $|z| < s$ and $S_N(z)$ are all continuous, so $f(z)$ is continuous.

There are two ways to compute the radius of convergence:

- Root test: $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$.
- Ratio test: $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$.

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function $f : \Omega \to \Omega$
2. Complex-differentiability: For $\Omega \subset \mathbb{C}$ open and $z_0 \in \Omega$, there exists $\varepsilon > 0$ such that $D_\varepsilon = \left\{z \;\middle|\; |z - z_0| < \varepsilon\right\} \subset \Omega$, and f is **holomorphic** (complex-differentiable) at z_0 iff

   $$\lim_{h\to 0} \frac{1}{h}(f(z_0 + h) - f(z_0))$$

   exists; if so we denote it by $f'(z_0)$.

Example: $f(z) = z$ is holomorphic, since $f(z + h) - f(z) = z + h - z = h$, so $f'(z_0) = \frac{h}{h} = 1$ for all z_0.

Example: Given $f(z) = \overline{z}$, we have $f(z+h) - f(z) = \overline{h}$, so the ratio is $\dfrac{\overline{h}}{h}$ and the limit doesn't exist (?).

We say f is holomorphic on an open set Ω iff it is holomorphic at every point, and is holomorphic on a closed set C iff there exists an open $\Omega \supset C$ such that f is holomorphic on Ω.

If f is holomorphic, writing $h = h_1 + ih_2$, then the following two limits exist and are equal:

$$\begin{aligned}
\lim_{h_1 \to 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\
\lim_{h_2 \to 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i}\frac{\partial f}{\partial y}(x_0, y_0) \\
\implies \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y}.
\end{aligned}$$

So if we write $f(z) = u(x,y) + iv(x,y)$, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Big|_{(x_0,y_0)},$$

and equating real and imaginary parts yields the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
\iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\end{aligned}$$

The usual rules of derivatives apply:

1. $(\sum f)' = \sum f'$

   Proof: Direct.

2. $(\prod f)' =$ product rule

   Proof: Consider $(f(z+h)g(z+h) - f(z)g(z))/h$ and use continuity of g at z.

3. Quotient rule

   Proof: Nice trick, write $q = \dfrac{f}{g}$ so $qg = f$, then $f' = q'g + qg'$and $q' = \dfrac{f'}{g} - \dfrac{fg'}{g^2}$.

4. Chain rule

Proof: Use the fact that if $f'(g(z)) = a$, then

$$f(z+h) - f(z) = ah + r(z,h), \quad |r(z,h)| = o(|h|) \to 0.$$

Write $b = g'(z)$, then

$$f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$$

by considering error terms, and so

$$\frac{1}{h}(f(g(z+h)) - f(g(z))) \to f'(g(z))g'(z)$$

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4 Appendix

Collection of facts used on problem sets

Standard forms of conic sections:

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x,y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Completing the square:

$$x^2 - bx = (x-s)^2 - s^2 \quad \text{where} s = \frac{b}{2}$$
$$x^2 + bx = (x+s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.$$

Properties of complex numbers

- $\Re(z) = \frac{1}{2}(z + \overline{z})$ and $\Im(z) = \frac{1}{2i}(z - \overline{z})$.
- $z\overline{z} = |z|^2$