Ch 5 # 1, 6, 16

1) $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$ $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}$

a) $\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$

b) $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$

c) $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$

6) a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix} = (1\ 2)(3\ 5\ 6)(4) = \alpha$

so $|\alpha| = \text{lcm}(2,3) = \boxed{6}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} = (1\ 7\ 5\ 3)(2\ 6\ 4) = \beta$

So $|\beta| = \text{lcm}(4,3) = \boxed{12}$

16) If α is even, prove α^{-1} is even. If α is odd, prove α^{-1} is odd.

Notice if $\sigma = (a_1\ a_2)$, $\sigma^{-1} = \sigma$. So the inverse of a transposition is itself. Assume α is even. Then $\alpha = \sigma_1 \sigma_2 \cdots \sigma_m$ where m is even and each σ_i is a transposition. So $\alpha^{-1} = \sigma_m^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}$
$\Rightarrow \alpha^{-1} = \sigma_m \cdots \sigma_2 \sigma_1$. Since m is even and α^{-1} is the composition an even # of transpositions, α^{-1} is even. The argument for α odd is the same except m is odd.

Ch5 # 2-5, 7, 10, 11, 15, 19, 20, 25 - 28, 30, 32, 34, 38, 45, 64, 65.

Team # 48, 61, 70

2) $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$

$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{bmatrix}$

a) $\alpha = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$

$\beta = (1)(2\ 3\ 8\ 4\ 7)(5\ 6)$

$\alpha\beta = (1\ 2\ 4\ 8\ 5\ 7\ 3\ 6)$

b) $\alpha = (1\ 5)(1\ 4)(1\ 3)(1\ 2)(6\ 8)(6\ 7)$

$\beta = (2\ 7)(2\ 4)(2\ 8)(2\ 3)(5\ 6)$

$\alpha\beta = (1\ 6)(1\ 3)(1\ 7)(1\ 5)(1\ 8)(1\ 4)(1\ 2)$

3) a) $(1\ 2\ 3\ 5)(4\ 1\ 3) = (1\ 5)(2\ 3\ 4)$

b) $(1\ 3\ 2\ 5\ 6)(2\ 3)(4\ 6\ 5\ 1\ 2) = (1\ 2\ 4)(3\ 5)$

c) $(1\ 2)(1\ 3)(2\ 3)(1\ 4\ 2) = (1\ 4\ 2\ 3)$

4) a) 2 b) 3 c) 5 d) k

5) a) 3 b) 12 c) 6 d) 6 e) 12 f) 3

7) 12

10) max order in A_{10} ← even perm on 10 elements

lengths	order
(9)	9
(7)(3)	21
(5)(5)	5
(5)(3)(1)	15
(3)(3)(3)	3

$\boxed{21}$

11) a) even

b) odd

c) even

d) $(12)(134)(152) = (15)(234)$ odd

e) $(1243)(3521) = (1)(2)(354)$ even

15) $n \in \mathbb{Z}_{>0}$

If n is odd, even. if n is even, odd.

19) $\alpha, \beta \in S_n$. Prove $\alpha\beta$ even iff α & β both even or both odd

(min # of 2-cycles to express $\alpha\beta$) = (min # of 2-cycles to express α) + (min # of 2-cycles to express β)

So $\alpha\beta$ even $\Leftrightarrow$ α & β even or α & β odd

20) even $\leftrightsquigarrow +1$ odd $\leftrightsquigarrow -1$

even · even is even $\longleftrightarrow$ $1 \cdot 1 = 1$

even · odd is odd $\longleftrightarrow$ $1 \cdot -1 = -1$

odd · odd is even $\longleftrightarrow$ $-1 \cdot -1 = 1$

25) odd permutations are not a subgrp b/c

(1) not closed

(2) does not contain ε

26) $\alpha, \beta \in S_n$ prove $\alpha^{-1}\beta^{-1}\alpha\beta$ is even.

Case 1: α & β even $\Rightarrow \alpha^{-1}$ & β^{-1} even so e·e·e·e = even

Case 2: α & β odd $\Rightarrow \alpha^{-1}$ & β^{-1} odd so o·o·o·o = e·e = even

Case 3: α even & β odd $\Rightarrow \alpha^{-1}$ even & β^{-1} odd so e·o·e·o = o·o = even.

Alt: Say α is m 2-cycles and β is n

then $\alpha^{-1}\beta^{-1}\alpha\beta$ has $m+n+m+n = 2(m+n)$ 2-cycles

27) a) $C_{\alpha_3} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
$(\alpha_3 = (13)(24))$

b) $C_{\alpha_{12}} = \{\alpha_1, \alpha_7, \alpha_{12}\}$
$(\alpha_{12} = (124))$

28) order 5 in S_7

can have length of 5 and this is the only way in S_7

So $\frac{7\cdot6\cdot5\cdot4\cdot3}{5} = 7\cdot6\cdot4\cdot3 = 24\cdot21 = \boxed{504}$

24×21: $24 + 480 = 504$

30) Prove $(1\,2\,3\,4)$ is not the product of 3-cycles.

$(1\,2\,3\,4)$ is odd.
3-cycles are even, and $\epsilon\cdot\epsilon = \epsilon$.

32) $\beta = (123)(145)$ $\quad \beta^{99} = ?$

$= (1\,4\,5\,2\,3) \Rightarrow |\beta| = 5$

So $\beta^{99} = \beta^{49} = \beta^4 = \beta^{-1} = \boxed{(1\,3\,2\,5\,4)}$

34) $(a_1 a_2 \cdots a_n)^{-1} = (a_n\, a_{n-1} \cdots a_2\, a_1)$

38) $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ ; } \beta(3) = 3\}$

- H subgrp : Clearly $\beta^{-1}(1) = 1$; $\beta^{-1}(3) = 3$ so $\beta^{-1} \in H$
  $(\alpha\beta)(1) = \alpha(1) = 1$ and $(\alpha\beta)(3) = \alpha(3) = 3$ so closed
- $|H| = 3\cdot2\cdot1 = 6$
- yes, subgrp w/ order $(n-2)!$
- If A_n instead, $\frac{(n-2)!}{2}$

45) S_n not Abelian for $n \geq 3$

$(1\,2), (2\,3) \in S_n \;\forall n \geq 3$

$(1\,2)(2\,3) = (1\,2\,3)$

$(2\,3)(1\,2) = (1\,3\,2)$ $\neq$

64) A_4 - see table 5.1

$|\alpha_1| = 1 \quad |\alpha_2| = 2 = |\alpha_3| = |\alpha_4|$

$|\alpha_5| = 3 = |\alpha_6| = |\alpha_7| = |\alpha_8| = |\alpha_9| = |\alpha_{10}| = |\alpha_{11}| = |\alpha_{12}|$

$|A_4| = 12$ and these divide 12

65) Show that everything in A_n, $n \geq 3$, can be expressed as a 3-cycle or product of 3-cycles

If $\alpha \in A_n$, it is even.

Each pair of 2-cycles either shares an element or is disjoint

If it shares one: $(a\,b)(a\,c) = (a\,c\,b)$

If disjoint: $(a\,b)(c\,d) = (c\,b\,d)(a\,c\,b)$