# Title

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# Prologue

## 0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

## 0.2 Notation

- If a property $P$ is said to hold **locally**, this means that for every point $p$ there is a neighborhood $U_p \ni p$ such that $P$ holds on $U_p$.

+————————————————-+————-+Notation Definition

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$k[\mathbf{x}] = k[x_1, \cdots, x_n]$ Polynomial ring in $n$ indeterminates. $k(\mathbf{x}) = k(x_1, \cdots, x_n)$ Rational function field in $n$ indeterminates $\mathcal{U} \rightrightarrows X$ An open cover $\mathcal{U} = \left\{ U_j \;\middle|\; j \in J \right\}$ $\Delta_X$ The diagonal $\left\{ (x,x) \;\middle|\; x \in X \right\} \subseteq X \times X$ $\mathbb{A}^n_{/k}$ Affine $n$-space
$\mathbb{A}^n_{/k} \coloneqq \left\{ \mathbf{a} = [a_1, \cdots, a_n] \;\middle|\; a_j \in k \right\}$ $\mathbb{P}^n_{/k}$ Projective $n$-space — $\mathbb{P}^n_{/k} \coloneqq (k^n \setminus \{0\}) / x \sim \lambda x$ — $=$ $\left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \;\middle|\; p, q, \in k[x_1, \cdots, x_n] \right\}$ $V(J), V_a(J)$ Variety associated to an ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ — $\coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \;\middle|\; f(\mathbf{x}) = 0, \forall f \in J \right\}$ $I(S), I_a(S)$ Ideal associated to a subset $S \subseteq \mathbb{A}^n_k$ — $\coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(\mathbf{x}\right.$ $A(X)$ Coordinate ring of a variety, $k[x_1, \cdots, x_n]/I(X)$ $V_p(J)$ Projective variety of an ideal — $\coloneqq$ $\left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \;\middle|\; f(\mathbf{x}) = 0, \forall f \in J \right\}$ $I_p(S)$ Projective ideal (?) — $\coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f \text{ is homogeneous and } f(x)\right.$ $S(X)$ Projective coordinate ring, $k[x_1, \cdots, x_n]/I_p(X)$ $f^h$ Homogenization, $x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$ $f^i$ Dehomogenization, $f(1, x_1, \cdots, x_n)$ $J^h$ Homogenization of an ideal, $\left\{ f^j \;\middle|\; f \in J \right\}$ $\overline{X}$ Projective closure of a subset — $\coloneqq V_p(J^h) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n \;\middle|\; f^h(\mathbf{x}) = 0 \forall f \in X \right\}$ $\mathcal{O}_X$ Structure sheaf $\left\{ f : U \to k \;\middle|\; f \in k(\mathbf{x}) \text{ locally}\right.$ $D(f)$ Distinguished open set, $D(f) = V(f)^c = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) \neq 0 \right\}$

## 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?

- What are the constructions $V(\,\cdot\,)$ and $I(\,\cdot\,)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety $X$ in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

## 0.4 Useful Examples

### 0.4.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x-p)$ a point.

### 0.4.2 Presheaves / Sheaves

- $C^\infty(\,\cdot\,, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\,\cdot\,, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\,\cdot\,)$, the sheaf of regular functions on $X$
- $\underline{\mathbb{R}}(\,\cdot\,)$, the constant sheaf associated to $\mathbb{R}$ (locally constant real-valued functions)
- $\text{Hol}(\,\cdot\,, \mathbb{C})$, a sheaf of holomorphic functions
- $K_p$ the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

## 0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

| Algebra | Geometry |
|---|---|
| $k[x_1, \cdots, x_n]$ | $\mathbb{A}^n_{/k}$ |
| Maximal ideals $\mathfrak{m} = x_1 - p_1, \cdots, x_n - p_n$ | Points $[a_1, \cdots, a_n]$ |
| Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$ | $V(J)$ the zero locus |
| Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \cdots, x_n])$ | Irreducible subsets |
| $I(S)$ the ideal of a set | $S \subseteq \mathbb{A}^n$ a subset |
| $I + J$ | $V(I) \cap V(J)$ |
| $\sqrt{I(V) + I(W)}$ | $V \cap W$ |
| $I \cap J, IJ$ | $V(I) \cup V(J)$ |
| $I(V) \cap I(W), \sqrt{I(V)I(W)}$ | $V \cup W$ |
| $I(V) : I(W)$ | $\overline{V \setminus W}$ |
| $k[x_1, \cdots, x_n]/I(X)$ | $A(X)$ (Functions on $X$) |
| $A(X)$ a domain | $X$ is irreducible |
| $A(X)$ indecomposable | $X$ is connected |
| Krull dimension $n$ (chaints of primes) | Topological dimension $n$ (chains of irreducibles) |