

# Final Exam

D. Zack Garza

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## 1 1

We prove a slightly stronger statement, namely:

**Theorem:**  $\mathbb{Z}$  is initial in the category of unital rings and ring homomorphisms.

This means that if we are given any such ring  $R$ , there is exactly one map  $\mathbb{Z} \rightarrow R$ .

Then, given an abelian group  $A$ , we can take  $R = \text{hom}_{\text{Ab}}(A, A)$ , the hom set of abelian group endomorphisms, which is itself a unital ring. This will imply that there is a unique map  $\mathbb{Z} \rightarrow \text{hom}_{\text{Ab}}(A, A)$ , and since all such maps induce  $\mathbb{Z}$ -module structures on  $A$ , the result will follow.

*Proof:* Let  $R$  be arbitrary and  $1_R$  be its multiplicative identity. We first show that there exists a ring homomorphism  $\mathbb{Z} \rightarrow R$ , namely

$$\begin{aligned}\phi : \mathbb{Z} &\rightarrow R \\ n &\mapsto \sum_{i=1}^n 1_R.\end{aligned}$$

Note that  $\phi(1) = 1_R$  and  $\phi(-1) = -1_R$ , and it is routine to check that  $\phi$  is a ring homomorphism.

Now toward a contradiction, suppose there were another such ring homomorphism  $\psi : \mathbb{Z} \rightarrow R$ . From the definition of a ring homomorphism,  $\psi$  must satisfy,

$$\begin{aligned}\psi(1) &= 1_R \\ \psi(-1) &= -1_R,\end{aligned}$$

and by  $\mathbb{Z}$ -linearity, we must have

$$\psi(n) = \psi\left(\sum_{i=1}^n 1\right) = \sum_{i=1}^n \psi(1) = \sum_{i=1}^n 1_R = \phi(n),$$

and so  $\psi(x) = \phi(x)$  for every  $x \in \mathbb{Z}$ . But this precisely means that  $\psi = \phi$  as ring homomorphisms. ■

## 2 2

### 2.1 a

Let  $\phi : \mathbb{Z}^4 \rightarrow \mathbb{Z}^3$  be a linear map which in the standard basis  $\mathcal{B}$  is represented by

$$T := [\phi]_{\mathcal{B}} = [f_1^t, f_2^t, f_3^t, f_4^t] = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -3 & 3 & 1 \\ -1 & 1 & 1 & 5 \end{bmatrix}.$$

Then  $\text{im } T = \text{span}_{\mathbb{Z}} \{f_1, f_2, f_3, f_4\} := N$  by construction.

We can then compute the echelon form

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 4 & 9 \end{pmatrix},$$

which has pivots in columns 1, 2, and 3, and thus

$$N = \text{span}_{\mathbb{Z}} \{f_1, f_2, f_3\}$$

## 2.2 b

Without loss of generality, we can consider the image of the reduced matrix

$$A' = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -3 & 3 \\ 1 & 1 & 1 \end{pmatrix},$$

since  $N = \text{im } A = \text{im } A'$ .

When computing the characteristic polynomial, we find that  $\chi_{A'}(x) = (x+3)(x+2)(x-2)$ , which means that  $A'$  has distinct eigenvalues. We can thus immediately write

$$JCF(A) = \left[ \begin{array}{c|c|c} 2 & 0 & 0 \\ \hline 0 & -2 & 0 \\ \hline 0 & 0 & -3 \end{array} \right].$$

From this, we can obtain the Smith normal form,

$$SNF(A') = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{bmatrix},$$

which allows us to read off

$$\text{im } A' \cong \mathbb{Z} \oplus \mathbb{Z} \oplus 12\mathbb{Z},$$

and thus

$$\mathbb{Z}^3/N \cong \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus \mathbb{Z} \oplus 12\mathbb{Z}} \cong \mathbb{Z}/12\mathbb{Z}..$$

### 3 3

The elementary divisors are given by:

$$\begin{array}{ccc} (x-1)^3 & (x^2+1)^4 & (x+2) \\ (x-1) & (x^2+1)^2 & \\ & (x^2+1)^2 & . \end{array}$$

The invariant factors are:

$$\begin{aligned} d_3 &= (x-1)^3(x^2+1)^4(x+2) \\ d_2 &= (x-1)(x^2+1)^2 \\ d_1 &= (x^2+1)^2. \end{aligned}$$

### 4 4

**Lemma:**  $(2, x) \trianglelefteq \mathbb{Z}[x]$  is not a principal ideal.

*Proof:* If this ideal were generated by a single element  $p(x)$ , then  $p \mid 2$  would force  $p \in \mathbb{Z}$ . But this means that the element  $x \notin (p)$ , a contradiction. ■

Suppose toward a contradiction that  $J = (2, x) \trianglelefteq \mathbb{Z}[x]$  is a direct sum of cyclic submodules of  $R := \mathbb{Z}[x]$ .

Then write

$$J = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

where each  $M_i = \alpha_i \mathbb{Z}[x]$  is a cyclic  $\mathbb{Z}[x]$ -module.

Note that by the lemma, we can not have  $n = 1$ , since this would mean  $J = \alpha_1 \mathbb{Z}[x] = (\alpha_1)$  where we can identify cyclic submodules with principal ideals.

On the other hand, we also can't have  $n \geq 2$ . Since the sum is direct, this forces (for example)  $M_1 \cap M_2 = \emptyset$ .

However, take the two generating elements  $\alpha_1, \alpha_2 \in \mathbb{Z}[x]$  and consider their product. Noting that  $\mathbb{Z}[x]$  is a commutative ring, we have

$$\alpha_1 \alpha_2 \in \alpha_1 \mathbb{Z}[x] = M_1 \text{ since } \alpha_2 \in \mathbb{Z}[x] \alpha_1 \alpha_2 = \alpha_2 \alpha_1 \in \alpha_2 \mathbb{Z}[x] = M_2 \text{ since } \alpha_1 \in \mathbb{Z}[x],$$

and so  $\alpha_1 \alpha_2 \in M_1 \cap M_2$ , a contradiction. So no such direct sum decomposition is possible. ■

## 5 5

**Irreducible:** Let  $a \in M$  be arbitrary; we can then consider the cyclic submodule  $aR \trianglelefteq M$ . Since  $M$  is irreducible, we must have  $aR = 0$  or  $aR = M$ . If  $aR = 0$  then  $a$  must be 0.

Otherwise,  $aR = M$  implies that  $M$  itself is a cyclic module with generator  $a$ . Since  $R$  is a PID, we can find an element  $p$  such that  $\text{Ann}_R(M) = (p) \trianglelefteq R$ , in which case  $M \cong R/(p)$ .

It is also necessarily the case that  $(p)$  is maximal, for if there were another ideal  $(p) \subsetneq J \trianglelefteq R$ , then  $J/(p) \trianglelefteq R/(p) \cong M$  is a submodule by the correspondence theorem for ideals. But this necessarily forces  $J/(p) = 0$  or  $M$  by irreducibility of  $M$ , so  $J = (p)$  or  $R$ .

Thus irreducible modules are exactly the cyclic modules, or equivalently those of the form  $R/(p)$  where  $(p)$  is a maximal ideal.

**Indecomposable:** We first note that by the structure theorem for modules over a PID, any module  $M$  has a primary decomposition  $M \cong \bigoplus_i R/(p_i^{k_i})$ .

This means that if  $M$  is indecomposable, we must have  $M \cong R/(p^n)$  (with a single summand) for some prime  $p \in R$ ; otherwise the primary decomposition would yield additional summands. Moreover, by the Chinese Remainder Theorem,  $M$  can not be decomposed further.

Thus all indecomposable module are of the form  $R/(p^n)$  for some  $n \geq 1$ .

## 6 6

Suppose  $T : V \rightarrow V$  is not invertible, then  $\dim \text{im } T < n$  and  $\dim \ker T > 0$  by the Rank-Nullity theorem. This means that there is a nontrivial  $\mathbf{v} \in \ker T$ , and a nontrivial vector  $\mathbf{w} \in \text{im } (T)$ , so let  $S$  be the matrix formed by the outer product  $\mathbf{v}\mathbf{w}^t$ .

We then consider how  $ST$  acts on vectors  $\mathbf{x}$ :

$$\begin{aligned} TS\mathbf{x} &= T\mathbf{v}\mathbf{w}^t\mathbf{x} \\ &= (T\mathbf{v})\mathbf{w}^t\mathbf{x} \\ &= \mathbf{0}\mathbf{w}^t\mathbf{x} \\ &= \mathbf{0}_n\mathbf{x} \\ &= \mathbf{0}, \end{aligned}$$

where  $\mathbf{0}_n$  is the  $n \times n$  matrix of all zeros.

Similarly,

$$\begin{aligned} ST\mathbf{x} &:= S\mathbf{y} \\ &= \mathbf{v}\mathbf{w}^t\mathbf{y} \\ &= \langle \mathbf{w}, \mathbf{y} \rangle \mathbf{v} \\ &= c_i \mathbf{v}_i, \end{aligned}$$

where  $\langle \mathbf{w}, \mathbf{y} \rangle := c_i \neq 0$  because  $\mathbf{y} \in \text{im } (T) = (\text{im } (T)^\perp)^\perp$ , so  $\mathbf{y}$  and  $\mathbf{w}$  can not be orthogonal.

■

## 7 7

### 7.1 a

Note that if  $A = 0$  or  $I$  then  $A$  is patently diagonal, so suppose otherwise. Since  $A^2 = A$ , we have  $A^2 - A = 0$  and thus  $A$  satisfies the polynomial  $p(x) = x^2 - 1 = x(x - 1)$ . Moreover, since  $A \neq 0, I$ , the minimal polynomial is at least degree 2 – since  $p$  is monic, it must in fact be the minimal polynomial.

We can immediately deduce that the size of the largest Jordan block corresponding to  $\lambda = 0$  is exactly 1, as is the size of the largest Jordan block corresponding to  $\lambda = 1$ . But this says that *all* Jordan blocks must be size 1, so  $JNF(A)$  has no off-diagonal entries and is thus diagonal.

### 7.2 b

If  $k$  is the multiplicity of  $\lambda = 0$  as an eigenvalue, we have

$$A \sim \left[ \begin{array}{ccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right],$$

which has a  $k \times k$  block of zeros and an  $(n - k) \times (n - k)$  block of 1s.

## 8 8

In both cases, we will need the characteristic polynomials  $\chi_A(x)$ , since  $RCF(A)$  will depend on the invariant factors of  $A$ . We will also use the fact that over the algebraic closure  $\mathbb{Q}$ , the minimal and characteristic polynomials must have the same roots.

### 8.1 a

Suppose  $m_A(x) = (x - 1)(x^2 + 1)^2$ , which is a degree 5 polynomial. Since  $\deg \chi_A$  must be 6 and  $m_A$  must divide  $\chi_A$  in  $\mathbb{Q}[x]$ , the only possibility in this case is that

$$\chi_A(x) = (x - 1)^2(x^2 + 2)^2.$$

To determine the possible invariant factors  $\{d_i\}$ , we can just note that  $\prod d_i = \chi_A(x)$  and  $d_n = m_A(x)$ . With these constraints, the only possibility is

$$\begin{aligned}d_1 &= (x - 1) \\d_2 &= (x - 1)(x^2 + 1)^2.,\end{aligned}$$

from which we can immediately obtain the elementary divisors:

$$(x - 1), (x - 1), (x^2 + 1)^2.$$

Then noting that

$$d_2 = d_2 = (x - 1)(x^2 + 1)^2 = x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4,$$

there is thus only one possible Rational Canonical form:

$$RCF(A) = \left[ \begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right].$$

## 8.2 b

The constraints  $m_A(x) = (x^2 + 1)^2(x^3 + 1)$  with  $\deg m_A(x) = 7$  and  $\deg \chi_A(x) = 10$  forces

$$\chi_A(x) = (x^2 + 1)^2(x^3 + 1)^2.$$

Furthermore, the invariant factors are similarly constrained, and so the only possibility is

$$\begin{aligned}d_1 &= (x^3 + 1) \\d_2 &= (x^2 + 1)^2(x^3 + 1)\end{aligned}$$

with corresponding elementary divisors

$$(x^3 + 1), (x^3 + 1), (x^2 + 1)^2.$$

Noting that

$$d_2 = (x^2 + 1)^2(x^3 + 1) = x^5 + x^3 + x^2 + 1,$$

we have

$$RCF(A) = \left[ \begin{array}{cc|cccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right].$$

## 9 9

The standard computation of  $\det(xI - A) = 0$  shows that  $\chi_A(x) = \det(xI - A) = (x - 1)^2(x + 1)^2$ , and so the eigenvalues of  $A$  are  $1, -1$ . We want the minimal polynomial of  $A$ , which is given by  $\prod (x - \lambda_i)^{\alpha_i}$  where  $\alpha_i = \dim E_{\lambda_i}$  is the geometric multiplicity of  $\lambda_i$ .

Another standard computation shows that

$$\lambda = 1 \implies \text{rank}(A - 1I) = 2 \implies \dim \ker(A - 1I) = 4 - 2 = 2$$

and similarly

$$\lambda = -1 \implies \text{rank}(A + I) = 3 \implies \dim \ker(A + I) = 4 - 3 = 1.$$

We thus have

$$\begin{aligned} p_A(x) &= (x - 1)(x + 1)^2 \\ \chi_A(x) &= (x - 1)^2(x + 1)^2. \end{aligned}$$

To compute  $JCF(A)$ , we use the following facts:

- For  $\lambda = 1$ ,
  - Since  $(x - 1)^1$  occurs in  $p_A(x)$ , the largest Jordan block for  $\lambda = 1$  is size 1.
  - Since  $(x - 1)^2$  occurs in  $\chi_A(x)$ , the sum of sizes of all such Jordan blocks is 2.
  - Since  $\dim E_1 = 2$ , there are 2 such Jordan blocks.
- For  $\lambda = -1$ ,
  - Since  $(x + 1)^2$  occurs in  $p_A(x)$ , the largest Jordan block for  $\lambda = -1$  is size 2.
  - Since  $(x + 1)^2$  occurs in  $\chi_A(x)$ , the sum of sizes of all such Jordan blocks is 2.
  - Since  $\dim E_{-1} = 1$ , there is 1 such Jordan block.

We can thus immediately write

$$JCF(A) = J_{-1}^2 \oplus 2J_1^1 = \left[ \begin{array}{cccc} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right].$$



By arguments similar to the previous two problems, the only possible invariant factor decomposition is given by

$$\begin{aligned}d_1 &= (x + 1) \\d_2 &= (x - 1)^2(x + 1)\end{aligned}$$

and thus

$$RCF(A) = C(d_1) \oplus C(d_2) = \left[ \begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

## 10 10

Suppose  $A^* = A$ . It is then a fact that  $A$  is self-adjoint, and so for every  $\mathbf{v} \in V$  we have

$$\langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, A^*\mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle.$$

### 10.1 a

Let  $(\lambda, \mathbf{v})$  be an eigenvalue of  $A$  with one of its corresponding eigenvectors, so  $A\mathbf{v} = \lambda\mathbf{v}$ .

On one hand,

$$\langle A\mathbf{v}, \mathbf{v} \rangle = \langle \lambda\mathbf{v}, \mathbf{v} \rangle = \lambda\langle \mathbf{v}, \mathbf{v} \rangle = \lambda\|\mathbf{v}\|^2,$$

while on the other hand,

$$\langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, A^*\mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle = \langle \mathbf{v}, \lambda\mathbf{v} \rangle = \bar{\lambda}\langle \mathbf{v}, \mathbf{v} \rangle = \bar{\lambda}\|\mathbf{v}\|^2.$$

Equating these expressions, we find that

$$\lambda = \bar{\lambda} \implies \lambda \in \mathbb{R}. \blacksquare$$

### 10.2 b

We can make use of the following fact:

**Theorem (Schur):** Every square matrix  $A \in M_n(\mathbb{C})$  is unitarily similar to an upper triangular matrix, i.e. there exists a unitary matrix  $U$  such that  $A = UTU^{-1}$  where  $T$  is upper-triangular.

Applying this theorem yields  $A = UTU^{-1}$  and thus  $T = U^{-1}AU$ . In particular,  $A \sim T$ .

Noting that if  $U$  is unitary then  $U^{-1} = U^*$ , we have

$$\begin{aligned}
T^* &= (U^{-1}AU)^* \\
&= U^*A^*(U^{-1})^* \\
&= U^*A^*U^{**} \\
&= U^{-1}A^*U \\
&= T,
\end{aligned}$$

and so  $T^* = T$ .

Since  $T$  is upper triangular, this forces  $T_{ij} = 0$  whenever  $i \neq j$ . But this makes  $T$  diagonal, so  $A$  is similar to a diagonal matrix. ■

*Proof of Schur's Theorem:* We'll proceed by induction on  $n = \dim_{\mathbb{C}}(V)$ , and showing that there is an orthonormal basis of  $V$  such that the matrix of  $A$  is upper triangular.

**Lemma:** If  $V$  is finite dimensional and  $\lambda$  is an eigenvalue of  $A$ , then  $\bar{\lambda}$  is an eigenvalue of  $A^*$ .

*Proof:*

$$\det(A - \lambda I) = 0 = \overline{\det(A^* - \bar{\lambda} I)}. \blacksquare$$

Since  $\mathbb{C}$  is algebraically closed, every matrix  $A \in M_n(\mathbb{C})$  will have an eigenvalue, since its characteristic polynomial will have a root by the Fundamental Theorem of Algebra.

So let  $\lambda_1, \mathbf{v}_1$  be an eigenvalue/eigenvector pair of the adjoint  $A^*$ .

Consider the space  $S = \text{span}_{\mathbb{C}}\{\mathbf{v}_1\}$ ; then  $V = S \oplus S^{\perp}$ . The claim is that the original  $A$  will restrict to an operator on  $S^{\perp}$ , which has dimension  $n - 1$ . The inductive hypothesis will then apply to  $A|_{S^{\perp}}$ .

Note that if this holds, there will be an orthonormal basis  $\mathcal{B}$  of  $S^{\perp}$  such that the matrix

$$\mathbf{A}' := [A|_{S^{\perp}}]_{\mathcal{B}}$$

will be upper triangular. We would then be able to obtain an orthonormal basis  $\mathcal{C} := \mathcal{B} \cup \{\mathbf{v}_1\}$  of  $S \oplus S^{\perp} = V$ .

Since we have a direct sum decomposition, the matrix of  $A$  with respect to  $\mathcal{C}$  can be written in block form as

$$[A]_{\mathcal{C}} = \begin{bmatrix} [A|_S]_{\mathcal{C}} & 0 \\ 0 & [A|_{S^{\perp}}]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} [A|_S]_{\{\mathbf{v}_1\}} & 0 \\ 0 & [A|_{S^{\perp}}]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \mathbf{A}' \end{bmatrix},$$

which is upper-triangular since  $\mathbf{A}'$  is upper-triangular.

To see that  $A$  does indeed restrict to an operator on  $S^{\perp}$ , we need to show that  $A(S^{\perp}) \subseteq S^{\perp}$ . So let  $\mathbf{s} \in S^{\perp}$ ; then  $\langle \mathbf{v}_1, \mathbf{s} \rangle = 0$  by definition. Then  $A\mathbf{s} \in S^{\perp}$  since

$$\begin{aligned}
\langle \mathbf{v}_1, A\mathbf{s} \rangle &= \langle A^* \mathbf{v}_1, \mathbf{s} \rangle \\
&= \langle \lambda_1 \mathbf{v}_1, \mathbf{s} \rangle \\
&= \lambda_1 \langle \mathbf{v}_1, \mathbf{s} \rangle \\
&= 0.
\end{aligned}$$

■