Lie Algebras

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Contents

1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5.

Here is a short overview of the topics we expect to cover:

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

Around chapter 3 or 4, we will describe the following series of correspondences:

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x,[y,z]]+[y,[x,z]]+[z,[x,y]]=\mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell+1,F)$
- $B_\ell \iff \mathfrak{so}(2\ell+1,F)$
- $C_\ell \iff \mathfrak{sp}(2\ell,F)$
- $D_\ell \iff \mathfrak{so}(2\ell,F)$

Exercise 1. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 1. An F**-algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition 2. An algebra is **associative** if $x(yz) = (xy)z$.

> Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 3. Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 4. We define $\mathrm{Der}(A) = \{\delta \ni \delta \text{ is a derivation }\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\,\cdot\,,\,\cdot\,]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x, [y, z]] = 0$.

Exercise 2. Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 3 (Turn in)**.** Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x, y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 1. Any F-vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\,\cdot\,,\,\cdot\,] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\,\cdot\,,\,\cdot\,]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 2. Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise 4. Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Compute the matrices of $\mathrm{ad}(e), \mathrm{ad}(h), \mathrm{ad}(g)$ with respect to this basis.

2.3 Ideals

Definition 5. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise 5. Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 6. A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 1. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \mathrm{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$\begin{aligned} 2e = [h, e] \in I &\implies e \in I, \\ 2f = [h, -f] \in I &\implies f \in I. \end{aligned}$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

$\square$

Note that there is a homework coming due next Monday, about 4 questions.