

Title

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Contents

1	Lecture 15: The L -Polynomial	3
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1 | Lecture 15: The L -Polynomial

Recall that we had $Z(t) + F(t) + G(t)$:

$$(q-1)F(t) = \sum_{0 \leq \deg C \leq 2g-2} q^{\ell(C)} t^{\deg(C)}$$

$$(q-1)G(t) = h \left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t} \right).$$

Note that $F(t)$ is a polynomial of degree at most $2g-2$, and clearing denominators in $G(t)$ yields a polynomial of degree at most $2g$

Definition 1.0.1 (The L -polynomial)

The L -polynomial is defined as

$$L(t) := (1-t)(1-qt)Z(t) = (1-t)(1-qt) \sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[t].$$

It turns out that the degree bound of $2g$ is sharp, and the coefficients closer to the middle are most interesting:

Theorem 1.0.2 (?).

Let K/\mathbb{F}_q be a function field of genus $g \geq 1$, then

- $\deg L = 2g$.
- $L(1) = h$
- $L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right)$.
- Writing $L(t) = \sum_{j=1}^{2g} a_j t^j$,
 - $a_0 = 1$ and $a_{2g} = q^g$.
 - For all $0 \leq j \leq g$, we have $a_{2g-j} = q^{g-j} a_j$.
 - $a_1 = |\Sigma(K/\mathbb{F}_q)| - (q+1)$, which notably does not depend on g .
 - Write $L(t) = \prod_{j=1}^{2g} (1 - \alpha_j t) \in \mathbb{C}[t]$ ^a
- The $\alpha_j \in \bar{\mathbb{Z}}$ ^b (which were *a priori* in \mathbb{C}) and can be ordered such that for all $1 \leq j \leq g$, we have $a_j a_{g+j} = q$. ^c

f. If $L_r(t) = (1-t)(1-q^r t)Z_r(t)$ then $L_r(t) = \prod_{j=1}^{2g} (1 - \alpha_j^r t)$, where K_r is the constant extension $K\mathbb{F}_{q^r}/\mathbb{F}_{q^r}$

^aThe polynomial isn't monic, but rather has a constant coefficient, so this expansion is somewhat more natural than (say) $\prod (t - \alpha)$.

^b $\bar{\mathbb{Z}}$ denotes the algebraic integers.

^cThis is the first hint at the Riemann hypothesis: if for example they all had the same complex modulus, this would force $|a_j| = \sqrt{q}$. Thus proving that they all have the same absolute value is 99% of the content!

Note that the α_i are reciprocal roots.

Proof (of a).

We saw from $Z(t) = F(t) + G(t)$ that $\deg L \leq 2g$. Equality will follow from the proof of (d) part 1, since this would imply that $a_{2g} = q^g \neq 0$. ■

Proof (of b).

Our formula $Z(t) = F(t) + G(t)$ and Schmidt's theorem (showing $\delta = 1$) gives

$$L(t) = (1-t)(1-qt)F(t) + \frac{h}{q-1} \left(q^g t^{2g-2} (1-t) - (1-qt) \right),$$

where we've expanded G but not F because it involves various $\ell(D)$ which are difficult to compute. It is some polynomial though, and we can evaluate L at 1 to get $L(1) = h$. Thus the class number is the sum of the coefficients! ■

Proof (of c).

This follows easily from the functional equation for $Z(t)$, which we already established using the Riemann-Roch theorem:

$$Z(t) = q^{g-1} t^{2g-2} Z\left(\frac{1}{qt}\right).$$

We can compute

$$\begin{aligned} q^g t^{2g} L\left(\frac{1}{qt}\right) &= q^g t^{2g} \left(1 - \frac{1}{qt}\right) \left(1 - \frac{1}{t}\right) Z\left(\frac{1}{qt}\right) \\ &= q^{g-1} t^{2g-2} (1-t)(1-qt) Z\left(\frac{1}{qt}\right) \\ &= (1-t)(1-qt) Z(t) \\ &:= L(t), \end{aligned}$$

where we've distributed one q and two t s in the first steps. ■

Proof (of d).

Using c , we can write

$$L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right) = \left(\frac{a_{2g}}{q^g}\right) + \left(\frac{a_{2g-1}}{q^{g-1}}\right) t + \cdots + (a_0 q^g) t^{2g}.$$

■