Moduli Spaces

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1 Thursday January 9th

1.1 Representability

Last time: Fix an S-scheme, i.e. a scheme over S.

Then there is a map

$$\begin{aligned}\mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{\mathrm{Sch}/S}(T, x).\end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ T \mapsto x &\longrightarrow \text{triangles} \end{aligned}$$

of the form

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd} .$$

Theorem 1.1 *(Yoneda)*.

$$\hom_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 1.2.

$$\hom_{\mathrm{Sch}/S}(x, y) \cong \hom_{\mathrm{Fun}}(h_x, h_y).$$

Definition 1.1.
A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 1.2.
A **moduli space** for that "something" appearing above is an $M \in \mathrm{Obj}(\mathrm{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \mathrm{Spec}\ (k)$.

h_m is the functor of points over M.

Remark (1) $h_m(\mathrm{Spec}\ (k)) = M(\mathrm{Spec}\ (k)) \cong$ "families over Spec k" $= F(\mathrm{Spec}\ k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M, and $\mathrm{id}_M \in \mathrm{Mor}_{\mathrm{Sch}/S}(M, M) = \xi_{Univ}$ is the universal family

Every family is uniquely the pullback of ξ_{Univ} This makes it much like a classifying space.

For $T \in \mathrm{Sch}/S$,

$$h_M \xrightarrow{\cong} F$$
$$f \in h_M(T) \xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\text{Univ}}).$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\begin{tikzcd}
\xi_M & \xi_{M'} \\
M \arrow[r, "f"] & M'
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[r, "g"] & M \\
\xi_{M'} & \xi_M
\end{tikzcd}$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.
A length 2 subscheme of $\mathbb{A}^1_k$ then $F(S) = \left\{ V(x^2 + bx + c) \right\} \subset \mathbb{A}'_5$ where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

Then F is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

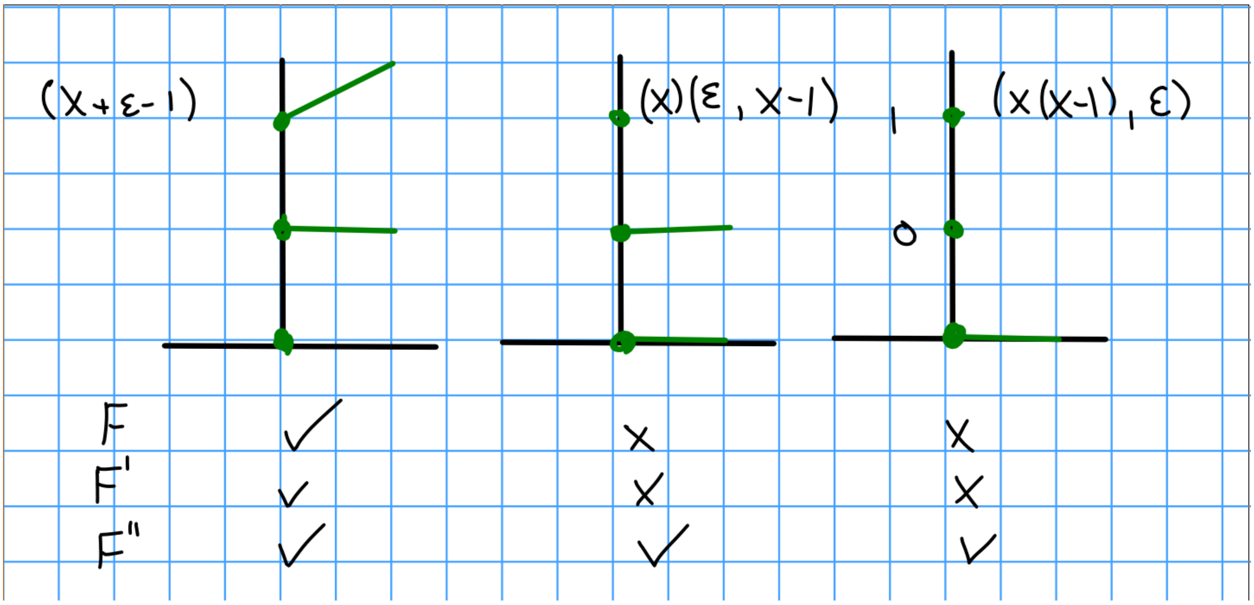
$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber.

$F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider Spec $k[\varepsilon]$.



$$\begin{tikzcd}
\text{Spec } k \arrow[r, hook, "i"] & \text{Spec } k[\varepsilon]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & {= F'(\text{Spec } k)} \arrow[dr] & \\
F(\text{Spec } k[\varepsilon]) \arrow[r, "F(i)"] & F(\text{Spec } k) \arrow[ur] & & {= F''(\text{Spec } k)} \\
T_pF^{','} \arrow[u, hook, "\subset"] & P = V(x(x-1)) \arrow[u, hook, "\in"] & &
\end{tikzcd}$$

We think of $T_pF^{','}$ as the tangent space at p.

If F is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\text{Spec } k[\varepsilon]) \arrow[r] & M(\text{Spec } k) \\
T_pM \arrow[u, hook, "\subset"] \arrow[r] & p \arrow[u, hook, "\subset"]
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \mathrm{Spec}\ k \arrow[dl, hook'] \arrow[dr, "?"] & \\
\mathrm{Spec}\ k[\varepsilon] \arrow[rr] & & \mathrm{Spec}\ \mathcal{O}_{M,p} \subset M
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
m_p & (\varepsilon) \\
m_p^2 & 0
\end{tikzcd}
$$

Moreover, $T_pM = (m_p/m_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$
\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda\varepsilon
\end{aligned}
$$

$$
\lambda^* : \mathrm{Spec}\ (k[\varepsilon]) \longrightarrow \mathrm{Spec}\ (k[\varepsilon])
$$

$$
\begin{aligned}
\lambda : M(\mathrm{Spec}\ (k[\varepsilon])) &\longrightarrow M(\mathrm{Spec}\ (k[\varepsilon])) \\
\cap ? \supset T_pM \qquad\qquad & \longrightarrow T_pM \subset .
\end{aligned}
$$

Conclusion: If F is representable, for each $p \in F(\mathrm{Spec}\ k)$ there exists a unique point of T_pF that are invariant under scaling.

1. If $F, F', G \in \mathrm{Fun}((\mathrm{Sch}/S)^{\mathrm{op}}, \mathrm{Set})$, there exists a fiber product

$$
\begin{tikzcd}
F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\
F \arrow[r] & G
\end{tikzcd}
$$

   where

$$
(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).
$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$
h_{X \times_T Y} = h_X \times_{h_t} h_Y.
$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}/T) &\longrightarrow \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

Claim: $\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(S) = \mathcal{O}_s^{n+1} &\longrightarrow L \longrightarrow 0/\sim . \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, "="] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $F(f)$ is given by pullbacks.

Remark $\mathbb{P}^n_S$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F_S(T) = \mathcal{O}_T^{n+1} &\longrightarrow L \longrightarrow 0/\sim . \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of claim).
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a

subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

■

Claim 2: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{tikzcd} F_i \arrow[r] & F \\ F_{ij} \arrow[u] \arrow[r] & F_j \arrow[u] \end{tikzcd}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example 1.2.
For $n = 1$, we can glue along open subschemes

$$\begin{tikzcd} & F_0 \\ F_{01} \arrow[ur] \arrow[dr] & \\ & F_1 \end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd} & & & F_0 \arrow[ddr] & \\ & & F_{01} \arrow[ur] \arrow[dr] & & \\ F_{012} \arrow[urr] \arrow[r] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\ & & F_{12} \arrow[ur] \arrow[dr] & & \\ & & & F_2 \arrow[uur] & \end{tikzcd}$$

This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S.

RHS to LHS comes from gluing.

Proof of (Claim 2)

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec $k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I \; F} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

2 Tuesday January 14th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, ... Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

Last time: Representability of functors, and specifically projective space $\mathbb{P}^n_{\mathbb{Z}}$ constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}^n_{\mathbb{Z}}} : \mathbb{P}^n_{\mathbb{Z}}\mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ s &\mapsto \mathbb{P}^n_{\mathbb{Z}}(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for L a line bundle, up to isomorphisms of diagrams:

$$
\begin{tikzcd}
\mathcal{O}_s^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}^n_{\mathbb{Z}}$ where $F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \;\middle|\; s_i \text{invertible} \right\}$ are representable and can be glued together, and projective space represents this functor.

Remark: Because projective space represents this functor, there is a universal object:

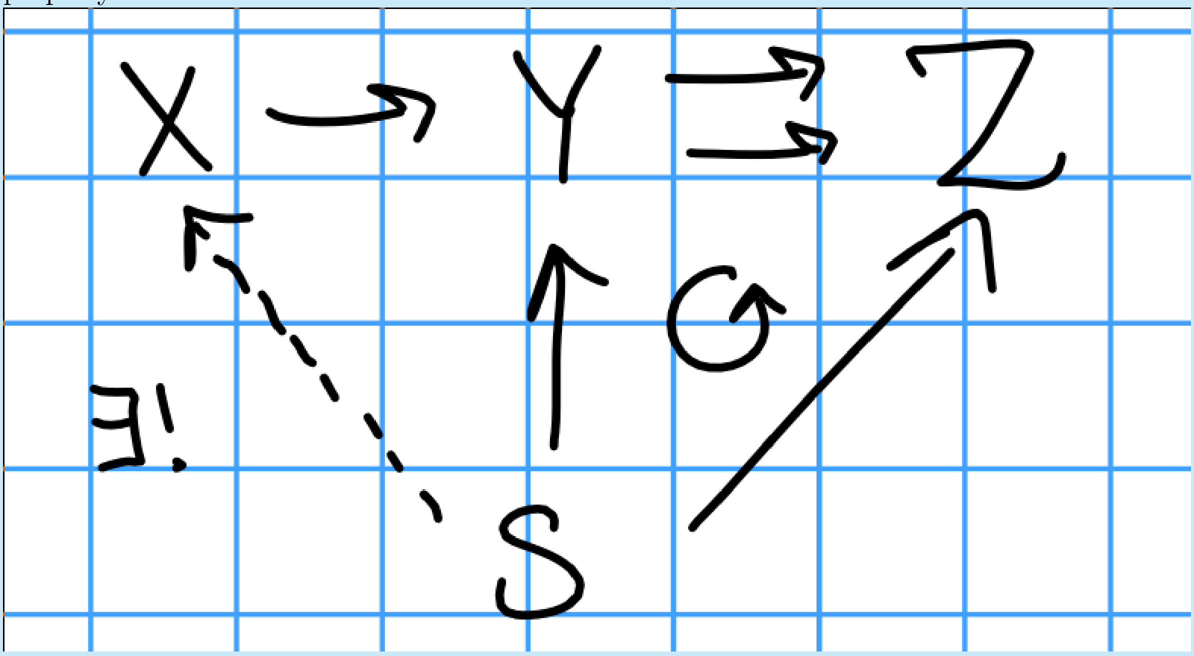
$$\begin{tikzcd} \mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{Z}}} \arrow[r] & L \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{O} \\ & \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) & \end{tikzcd}$$

and other functors are pullbacks of the universal one.

Example 2.1.
Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

Definition 2.1 (Equalizer)**.**
For a category C, we say a diagram $x \longrightarrow y \rightrightarrows z$ is an *equalizer* iff it is universal wrt the property:



(Here X is the universal object).

Example 2.2.
For sets, $X = \left\{ y \;\middle|\; f(y) = g(y) \right\}$ for $Y \xrightarrow{f,g} Z$.

Definition 2.2.
A coequalizer is the dual notion,

$$\begin{tikzcd}
 & S & \\
Z \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[u] \arrow[r] & X \arrow[ul, dashed, "\exists !"']
\end{tikzcd}$$

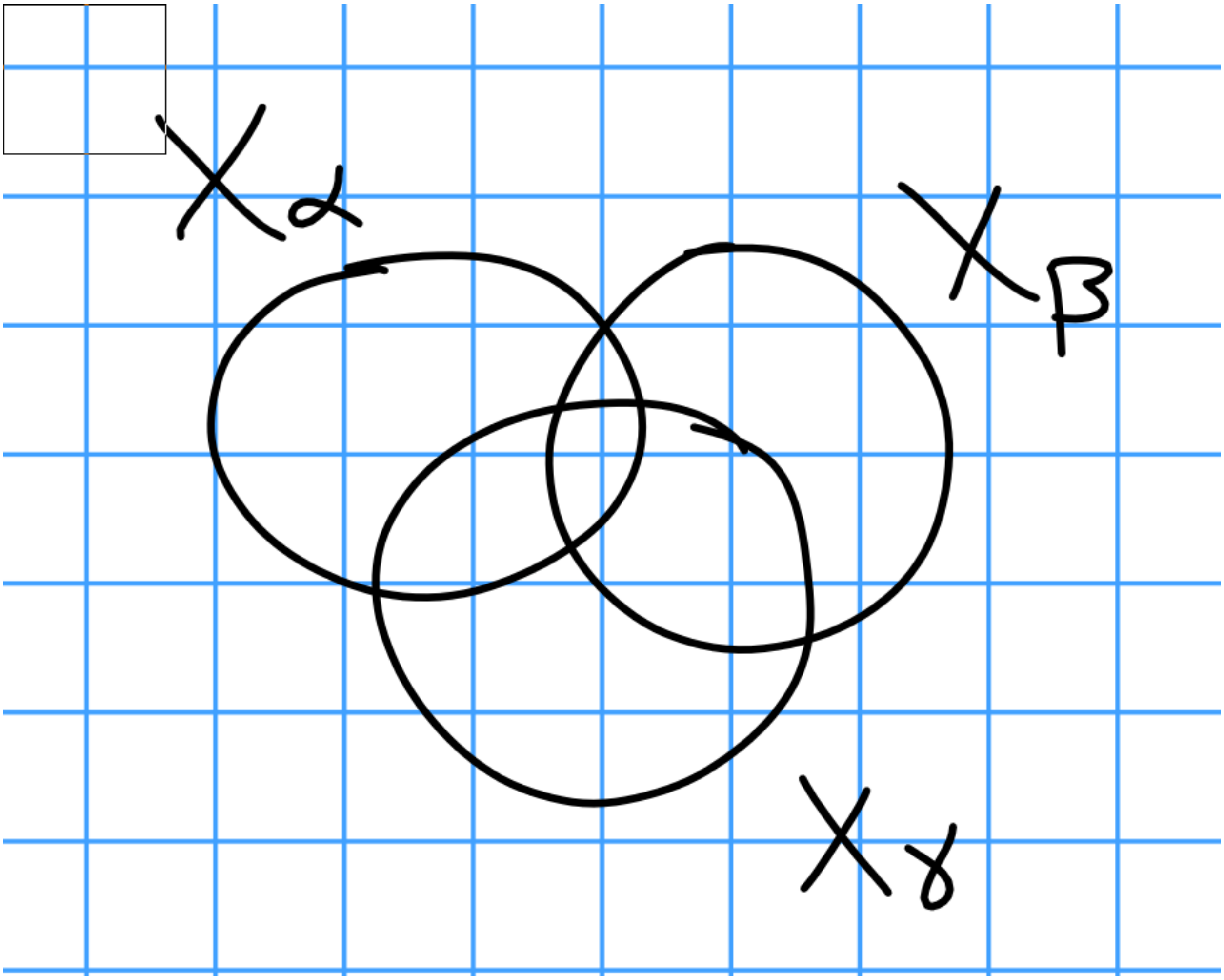
Example 2.3.
Take $C = Sch/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$\begin{tikzcd}
X_\alpha \arrow[r] & X & & X_\beta \arrow[r] & X \\
X_{\alpha\beta} \arrow[u] \arrow[r] & X_\beta \arrow[u] & & X_{\beta\alpha} \arrow[u] \arrow[r] & X_\alpha \arrow[u]
\end{tikzcd}$$

These are canonically isomorphic.

In Sch/S, we have $\coprod_{\alpha\beta} X_{\alpha\beta} \xrightarrow{f_{\alpha\beta}, g_{\alpha\beta}} \coprod X_\alpha \longrightarrow X$ where $f_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_\alpha$ and $g_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_\beta$; this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\phi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the cocycle condition:

1. $\phi_{\alpha\beta}^{-1}(X_{\beta\alpha} \bigcap X_{\beta\gamma}) = X_{\alpha\beta} \bigcap X_{\alpha\gamma}$, noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.
2. The following diagram commutes:

$$
\begin{tikzcd}
X_{\alpha\beta} \cap X_{\alpha\gamma} \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\alpha\beta}"'] & {} \arrow[d, phantom, "\circlearrowleft"] & X_{\gamma\alpha} \cap X_{\gamma\beta} \\
& X_{\beta\alpha} \cap X_{\beta\gamma} \arrow[ur, "\varphi_{\beta\gamma}"'] &
\end{tikzcd}
$$

Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \longrightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 2.3 (Zariski Sheaf)**.**
A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_α, the following is an equalizer:

$$F(T) \longrightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 2.4.
Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover $\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_\alpha T_\alpha \longrightarrow T$, since giving a local map to X that agrees on intersections if enough to specify a map from $T \longrightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 2.4 (Subfunctors, Open/Closed Functors)**.**
Suppose we have a morphism $F' \longrightarrow F$ in the category $\mathrm{Fun}(\mathrm{Sch}/S, \mathrm{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all T/S.
- ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over T, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes

$T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U.

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F', which maps to F (noting a section is the same as a map to the functor of points), and since $T \longrightarrow F$ and $F' \longrightarrow F$, we can form the fiber product $F' \times_F T$:

$$\begin{tikzcd}
F' \arrow[r] & F \\
F' \times_F T \arrow[u] \arrow[r] & T \arrow[u, "\xi"']
\end{tikzcd}$$

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \longrightarrow F$ is open/closed/locally closed iff

$$\begin{tikzcd} & F' \arrow[r] & F \\ F' \times_F T \arrow[r, phantom, ""] & T \arrow[u] \arrow[r, "g"] & T \arrow[u] \end{tikzcd}$$

$$F' \times_F T = \text{rep}$$

and g is open/closed/locally closed.

I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \longrightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

Definition 2.5 (coverings)**.**
A collection of open subfunctors $F_\alpha \subset F$ is an open cover iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \longrightarrow F$, the T_α in the following diagram are an open cover of T:

$$
\begin{tikzcd}
F_\alpha \arrow[r] & F \\
T_\alpha \arrow[r] \arrow[u] & T \arrow[u]
\end{tikzcd}
$$

Example 2.5.
Given $F(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}$ and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \longrightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

Proposition 2.1.
A Zariski sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ with a representable open cover is representable.

Proof .
Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \longrightarrow F_\alpha$ have the correct gluing maps. This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \longrightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x; we'll show that x represents F.
First produce a map $x \longrightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

Lemma 2.1.
If $E \longrightarrow F$ is a map of functors and E, F are zariski sheaves, where there are open covers $E_\alpha \longrightarrow E, F_\alpha \longrightarrow F$ with commutative diagrams

$$\begin{tikzcd}
E \arrow[r] & F \\
E_\alpha \arrow[u] \arrow[r, "\cong"'] & F_\alpha \arrow[u]
\end{tikzcd}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$
\begin{tikzcd}
X \arrow[r] & F \\
X_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u]
\end{tikzcd}
$$

Example 2.6.
For S and E a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} / \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E.

Then the restriction $F_i \longrightarrow \mathbb{P}E$ were $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i, $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 2.7.
For E locally free over S of rank n, take $r < n$ and consider the functor $\mathrm{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} / \sim$ (a Grassmannian) where Q is locally free of rank k.

Exercise

a. Show that this is representable

b. For the plucker embedding $\mathrm{Gr}(k, E) \longrightarrow \mathbb{P} \wedge^k E$, then a section over T is given by $f^* E \longrightarrow Q \longrightarrow 0$ corresponding to $\wedge^k f^* E \longrightarrow \wedge^k Q \longrightarrow 0$, noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \mathrm{Spec}\ k$, then E is a k-vector space V, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n.

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \longrightarrow Q \longrightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for $\mathbb{P}^n$ of all flat families of subschemes.