Weil Conjectures

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1 Notes from Daniel's Office Hours

0. Definition of Zeta functions
1. Statement of the conjectures
2. Easy examples: $\mathbb{P}^n_\daleth, \mathrm{Gr}_\daleth(k,n) = \mathrm{GL}(n, \daleth)/P$ the stabilizer of an $\daleth$-point in $\mathbb{C}^n, \mathbb{F}_{p^n}$.
3. Medium example: $E/\daleth$ an elliptic curve.
4. Work out a harder example as in Weil

1.1 Definition of Zeta Function

Fix q a prime and $\mathbb{F} \coloneqq \mathbb{F}_q$ the finite field with q elements, along with its unique degree n extensions

$$\mathbb{F}_n \coloneqq \mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_p \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\, n \in \mathbb{Z}^{\geq 2}$$

Definition 1.0.1.
Let

$$J = \langle f_1, \cdots, f_N \rangle \trianglelefteq k[x_0, \cdots, x_n]$$

be an ideal, then a *projective algebraic* variety $X \hookrightarrow \mathbb{P}^\infty_\mathbb{F}$ can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^\infty_\mathbb{F} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_N(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1,\cdots,i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Example: Take $f_1(x) = x \in \mathbb{F}[x]$, consider $V(\langle f_1 \rangle) \subset \mathbb{P}^1_{\mathbb{F}_n}$. This is given by the single point $x = \mathbf{0}$.

Fix $X/\mathbb{F}$ an N-dimensional projective algebraic variety. Note that it then has points in any finite extension L/K.

Definition 1.0.2.
Define the *local zeta function* (or *Hasse-Weil zeta function*) of X the following formal power series:

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$\zeta_X(0) = 1$$

$$z\left(\frac{\partial}{\partial z}\right) \log \zeta_X(z) = t\left(\frac{\zeta_X'(z)}{\zeta_X(z)}\right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \cdots,$$

which is an *ordinary generating function* for the sequence (α_n).

Thus if we define $G(x)$ to be the OGF for (α_n), we have $\zeta_X(t) = \exp$

Todo: why not an OGF.

Remark: Note that for an OGF $F(x) = \sum_{n=0}^{\infty} f_n x^n$, we can extract coefficients in the following way:

$$[x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

1.1.1 Why call it a Zeta function?

Suppose

$$\mathbb{A}^n_{\mathbb{Z}} \supseteq X = V(\langle f_1, \cdots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \cdots, x_{n-1}].$$

Then for every prime, we can reduce the equations mod p and consider

$$\mathbb{A}^n_{\mathbb{F}_p} \supseteq X_p := V(\langle f_1 \mod p, \cdots, f_d \mod p \rangle) \quad \text{where} \quad f_1 \mod p \in \mathbb{F}_p[x_0, \cdots, x_{n-1}]$$

Then define the "local at p" zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Note: the index set for the product may require some minor adjustment over $\mathbb{Q}$ in general. There are also potentially modifications needed to extend to schemes.

Taking $X = \text{Spec } \mathbb{Q}$ and $X_p = \text{Spec } \mathbb{F}_p$ (which is a single point since $\mathbb{F}_p$ is a field) and noting that

$$\begin{aligned}
\zeta_{X_p}(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^n\right) \\
&= \exp\left(-\log\left(1-z\right)\right) \\
&= \frac{1}{1-z},
\end{aligned}$$

we find that

$$\begin{aligned}
L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\
&= \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}}\right) \\
&= \zeta(s),
\end{aligned}$$

the classical Riemann Zeta function.

Example (Point): $X = \{x = 0\} / \mathbb{F}$ a single point over $\mathbb{F}$, then

$$\begin{aligned}
\#X(\mathbb{F}) &:= \alpha_1 = 1 \\
\#X(\mathbb{F}_2) &:= \alpha_2 = 1 \\
&\vdots \\
\#X(\mathbb{F}_n) &:= \alpha_n = 1 \\
&\vdots
\end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned}
\log(1+t) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} \\
\implies \log(1-t) &= -\sum_{n=1}^{\infty} \frac{t^n}{n} \\
\implies -\log(1-t) &= \sum_{n=1}^{\infty} \frac{t^n}{n} \\
&= 1 \cdot t + 1 \cdot t^2 + 1 \cdots t^3 + \cdots.
\end{aligned}$$

and so

$$\zeta_X(t) = \exp\left(-\log\left(1-t\right)\right) = \frac{1}{1-t}.$$

Example (Affine Line): $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then

$$\begin{aligned} X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n \end{aligned}.$$

where we just note that we can write $\mathbb{A}^1(\mathbb{F}_n) = \left\{(x_1) \;\middle|\; x_1 \in \mathbb{F}_n\right\}$.

Example (Projective Line): $X = \mathbb{P}^1/\mathbb{F}$ the projective line over $\mathbb{F}$, then

$$\begin{aligned} X(\mathbb{F}) &= q+1 \\ X(\mathbb{F}_2) &= q^2+1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n+1 \end{aligned}.$$

where we write $\mathbb{P}^1_{\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}} \coprod \{\infty\}$ is the affine line with a point at infinity. We can also count by coordinates:

$$\mathbb{P}^1(\mathbb{F}^n) = \left\{[x_1, x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}^n\right\} / \sim \qquad = \left\{[x_1, 1] \;\middle|\; x_1 \in \mathbb{F}^n\right\} \coprod \{[1,0]\}.$$

Example (Affine Space): Take $X = \mathbb{A}^n/\mathbb{F}$, then $\alpha_n = q^m + 1$ for a point at infinity, so

$$X(\mathbb{F}) = .$$

Thus

$$\zeta_X(t) = \frac{1}{(1-q^{-t})(1-q^{1-t})}.$$

Example (Projective Space): Take $X = \mathbb{P}^n_{\mathbb{F}}$, then $\alpha_n = 1 + q^m + (q^m)^2 + \cdots + (q^m)^n$, so

$$\zeta_X(t) = \left(\frac{1}{1-q^{-t}}\right)\left(\frac{1}{1-q^{1-t}}\right)\left(\frac{1}{1-q^{2-t}}\right)\cdots\left(\frac{1}{q^{n-t}}\right),$$

or equivalently, take your favorite curve $\gamma \in \mathbb{C}$ homotopic to $\mathbb{S}^1$.

Note: this is extremely amenable to numerical approximation if you have a closed form for F or even just a black-box numerical version of F! I.e. easy to throw at a computer.

Todo: how to manually count points in $\mathbb{P}^n$!

Example: Take $X = \mathrm{Gr}_{\mathbb{F}}(k, n)$, then ????? so

$$\zeta_X(t) = ?.$$

Questions about properties

- $\zeta_{X \coprod Y}(t) =_? \zeta_X(t)\zeta_Y(t)$?
- $\zeta_{X \times Y} = ?$

1.2 Statement of Weil Conjectures

1. (Rationality)

$$\zeta_X(t) = \frac{p_1(t)p_3(t)\cdots p_{2N-1}(t)}{p_0(t)p_2(t)\cdots p_N(t)} \in \mathbb{Z}(t), \quad \text{i.e.} \quad p_i(t) \in \mathbb{Z}[t]$$

$$\begin{aligned} P_0(t) &= 1 - t \\ P_{2n}(t) &= 1 - q^n t \\ P_i(t) &= \prod_j (1 - a_{ij}t), \quad a_{ij} \in \mathbb{C}. \end{aligned}$$

2. (Functional Equation and Poincare Duality)

$$\zeta_X(n - t) = \pm q^{\frac{1}{2}(nE) - Et} \zeta(x, t).$$

3. (Riemann Hypothesis)
4. (Betti Numbers)

1.3 Hard Example: An Elliptic Curve

Take $X = E/\mathbb{F}$, then $\alpha_n = q^n - (a^n + \bar{a}^n - 1)$ where $|a|_{\mathbb{C}} = |\bar{\alpha}|_{\mathbb{C}} = \sqrt{q}$. Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-s})}.$$