Complex Analysis Qual Prep Week 1: Things Named After Cauchy

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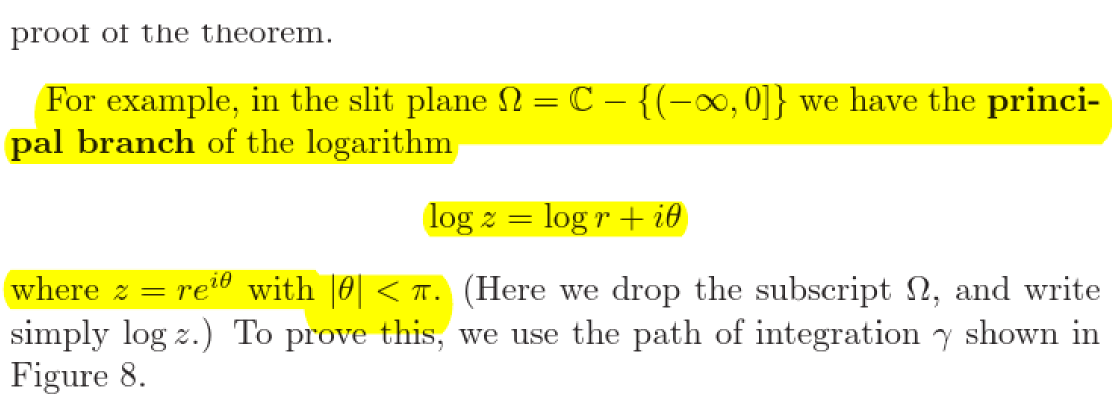
1.1 Review


proof of the theorem.

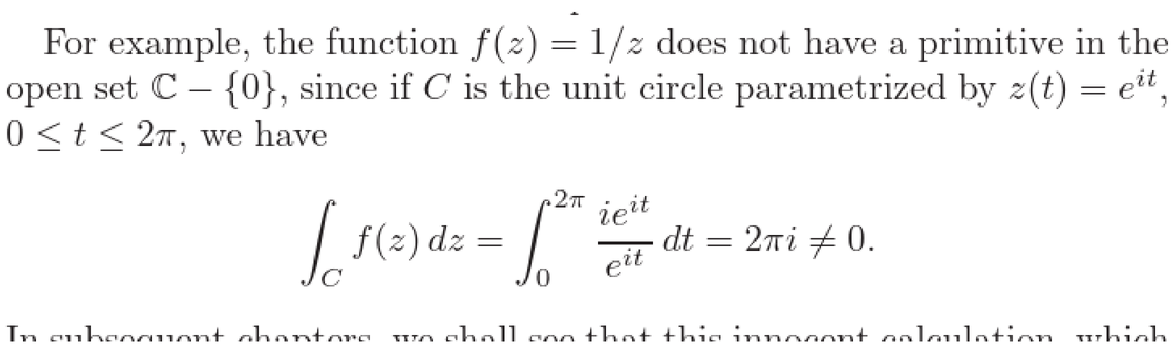
For example, in the slit plane $\Omega = \mathbb{C} - \{(-\infty, 0]\}$ we have the **principal branch** of the logarithm

$$\log z = \log r + i\theta$$

where $z = re^{i\theta}$ with $|\theta| < \pi$. (Here we drop the subscript Ω, and write simply $\log z$.) To prove this, we use the path of integration γ shown in Figure 8.

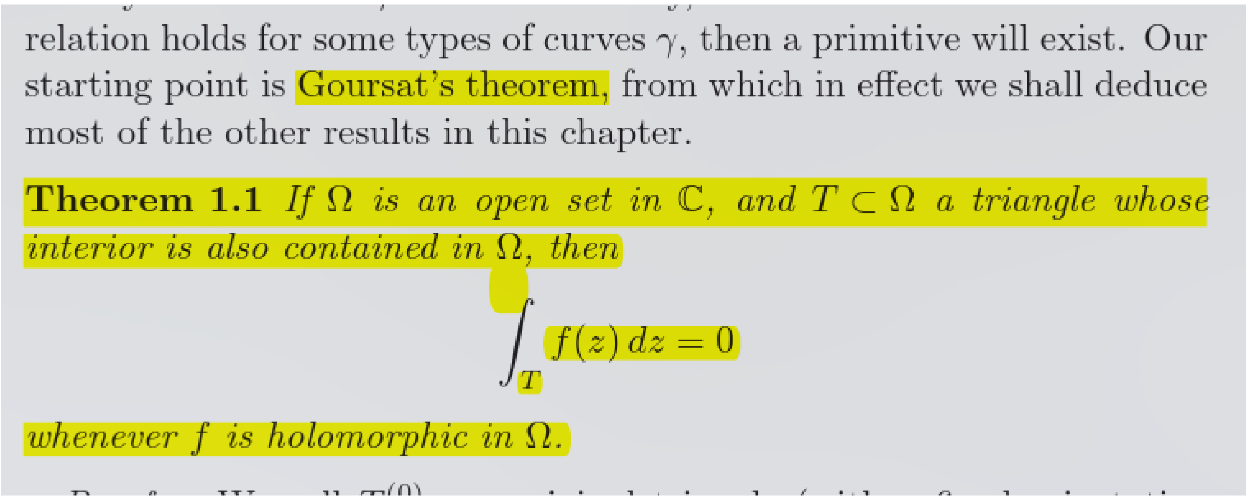
Figure 1: Complex log

1.1.1 Integrals and Residues


For example, the function $f(z) = 1/z$ does not have a primitive in the open set $\mathbb{C} - \{0\}$, since if C is the unit circle parametrized by $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, we have

$$\int_C f(z)\, dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\, dt = 2\pi i \neq 0.$$

Figure 2: Integrating 1/z manually



relation holds for some types of curves γ, then a primitive will exist. Our starting point is Goursat's theorem, from which in effect we shall deduce most of the other results in this chapter.

Theorem 1.1 *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\, dz = 0$$

whenever f is holomorphic in Ω.

Figure 3: Goursat

1.1.2 Residues

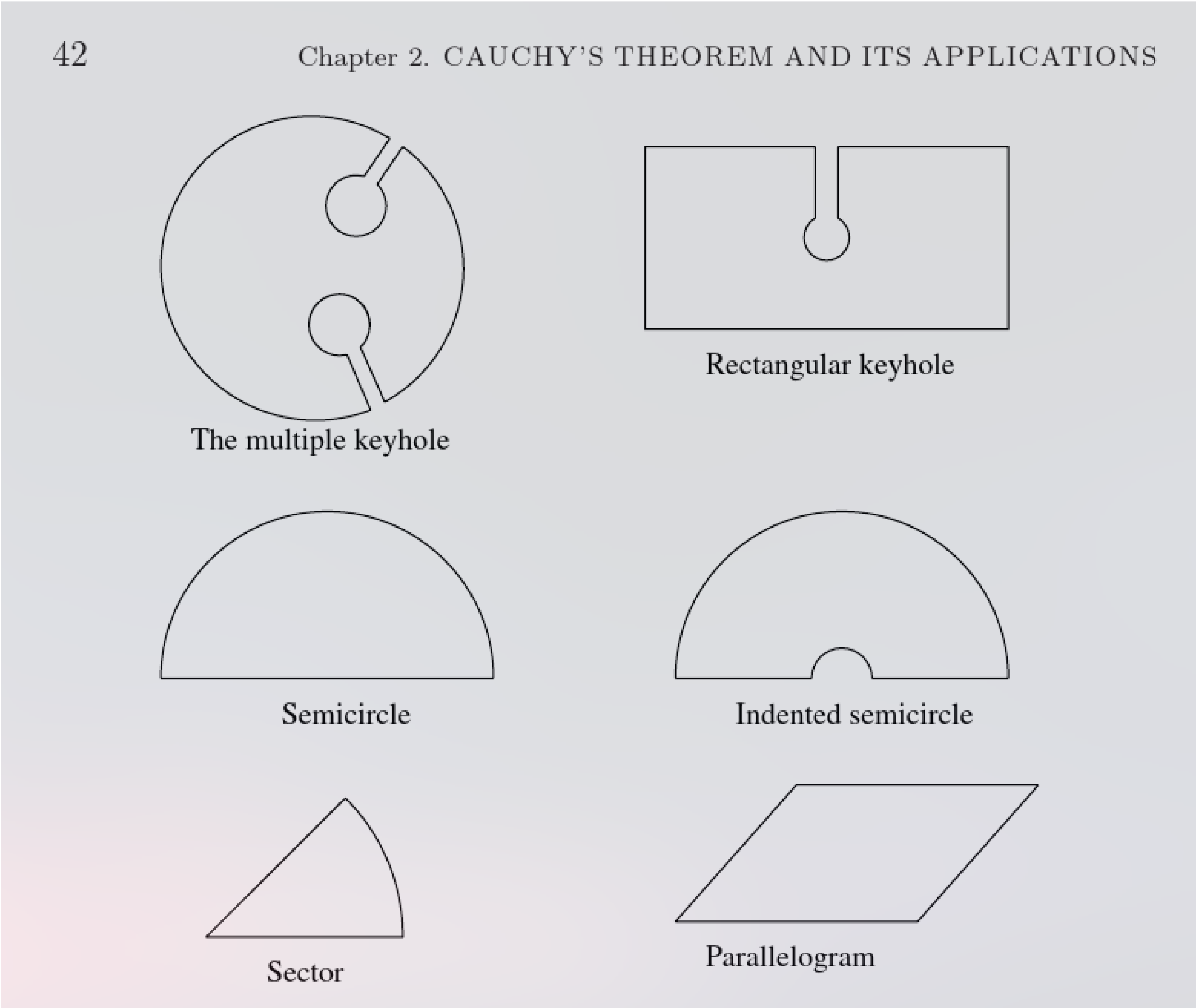


Figure 7. Examples of toy contours

Figure 4: Toy Contours

Corollary 2.2 *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, \ldots, z_N$ inside C. Then*

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

For the proof, consider a multiple keyhole which has a loop avoiding each one of the poles. Let the width of the corridors go to zero. In

Theorem 1.3 *If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)\,, \tag{1}$$

where G is a holomorphic function in a neighborhood of z_0.

Theorem 1.4 *If f has a pole of order n at z_0, then*

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-z_0)^n f(z).$$

The theorem is an immediate consequence of formula (1), which implies

$$(z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{n-1} +$$

$$+G(z)(z-z_0)^n.$$

The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

is called the **principal part** of f at the pole z_0, and the coefficient a_{-1} is the **residue** of f at that pole. We write $\operatorname{res}_{z_0} f = a_{-1}$. The importance of the residue comes from the fact that all the other terms in the principal

Simple poles

At a simple pole c, the residue of f is given by:

$$\operatorname{Res}(f,c)=\lim_{z\to c}(z-c)f(z).$$

Residue at infinity

In general, the residue at infinity is defined as:

$$\operatorname{Res}(f(z),\infty)=-\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right).$$

If the following condition is met:

$$\lim_{|z|\to\infty}f(z)=0,$$

then the residue at infinity can be computed using the following formula:

$$\operatorname{Res}(f,\infty)=-\lim_{|z|\to\infty}z\cdot f(z).$$

It may be that the function f can be expressed as a quotient of two functions, $\boxed{f(z) = \frac{g(z)}{h(z)}}$, where g and h are holomorphic functions in a neighbourhood of c, with $h(c) = 0$ and $h'(c) \neq 0$. In such a case, L'Hôpital's rule can be used to simplify the above formula to:

$$\begin{aligned}\boxed{\operatorname{Res}(f,c)} = \lim_{z\to c}(z-c)f(z) &= \lim_{z\to c}\frac{zg(z)-cg(z)}{h(z)}\\ &= \lim_{z\to c}\frac{g(z)+zg'(z)-cg'(z)}{h'(z)} \boxed{= \frac{g(c)}{h'(c)}}.\end{aligned}$$

Bounds

Consider a complex-valued, continuous function f, defined on a semicircu contour

$$C_R = \{Re^{i\theta} \mid \theta \in [0,\pi]\}$$

of positive radius R lying in the upper half-plane, centered at the origin. If function f is of the form

$$f(z) = e^{iaz}g(z), \quad z \in C_R,$$

with a positive parameter a, then Jordan's lemma states the following uppe bound for the contour integral:

Jordan's Lemma:

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi}{a}M_R \quad \text{where} \quad M_R := \max_{\theta\in[0,\pi]}\left|g\left(Re^{i\theta}\right)\right|.$$

(iii) *One has the inequality*

$$\left| \int_\gamma f(z)\,dz \right| \le \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$$

1.1.3 Blaschke Factors

7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left| \frac{w-z}{1-\overline{w}z} \right| < 1 \qquad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w-z}{1-\overline{w}z} \right| = 1 \qquad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \le (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.]

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

1.1.4 Cauchy's Integral Formula

toy contours.

The above ideas also lead us to a central result of this chapter, the Cauchy integral formula; this states that if f is holomorphic in an open set containing a circle C and its interior, then for all z inside C,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

as a consequence of the next theorem (see Exercises 11 and 12).

Theorem 4.1 *Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for any point } z \in D.$$

Corollary 4.2 *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

for all z in the interior of C.

From now on, we call the formulas of Theorem 4.1 and Corollary 4.2 the **Cauchy integral formulas**.

Corollary 4.3 (Cauchy inequalities) *If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$ denotes the supremum of $|f|$ on the boundary circle C.

1.1.5 Misc

Theorem 4.4 *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \textit{for all } n \geq 0.$$

expansion around 0, say $f(z) = \sum_{n=0}^{\infty} a_n z^n$, that converges in all of $\mathbb{C}$.

Corollary 4.5 (Liouville's theorem) *If f is entire and bounded, then f is constant.*

Proof. It suffices to prove that $f' = 0$, since $\mathbb{C}$ is connected, and we

2 Warmups

- Do any example from here
- ### 1.4 5 ✨

  Prove that there is no sequence of polynomials that uniformly converge to $f(z) = \frac{1}{z}$ on S^1.
- Anything from the homeworks
- Show that $f' = 0 \implies f$ is constant using integrals and *primitives* (i.e. antiderivatives).

  See S&S Corollary 3.4.
- **5.** Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

  $$\int_T f(z)\, dz = 0.$$

  This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

  [Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then

  $$\int_T F\, dx + G\, dy = \int_{\text{Interior of } T} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dxdy.$$
- Example 2. An integral that will play an important role in Chapter 6 is

  $$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin \pi a}, \qquad 0 < a < 1.$$

3 | Questions

- Can every continuous function on $\overline{\mathbb{D}}$ be uniformly approximated by polynomials in the variable z?

Hint: compare to Weierstrass for the real interval.

- Suppose f is analytic, defined on all of $\mathbb{C}$, and for each $z_0 \in \mathbb{C}$ there is at least one coefficient in the expansion $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ is zero. Prove that f is a polynomial.

Hint: use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.

11. Show that if $|\alpha| < r < |\beta|$, then

$$\int_\gamma \frac{1}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

where γ denotes the circle centered at the origin, of radius r, with positive orientation.

12. Assume f is continuous in the region: $x \geq x_0$, $0 \leq y \leq b$ and the limit

$$\lim_{x \to +\infty} f(x + iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x \to +\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.

9. Let $f(z)$ be analytic. Show that $\overline{f(\bar{z})}$ is also analytic.

4 | Qual Problems

2. Expand $\dfrac{1}{1-z^2} + \dfrac{1}{z-3}$ in a series of the form $\sum_{-\infty}^{\infty} a_n z^n$ so it converges for (a) $|z| < 1$, (b) $1 < |z| < 3$; and (c) $|z| > 3$.

Figure 5: Fall 2020 #2

3. Let $a \in \mathbb{R}$ with $0 < a < 3$. Evaluate $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^3} dx$.

Figure 6: Fall 2020 #3

1. Let z_1 and z_2 be two complex numbers.
   (a) Show that $|z_1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1-|z_1|^2)(1-|z_2|)$.
   (b) Show that if $|z_1| < 1$ and $|z_2| < 1$, then $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$.
   (c) Assume that $z_1 \neq z_2$. Show that $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$ if only if $|z_1| = 1$ or $|z_2| = 1$.

Figure 7: Spring 2021 #1

2. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\cosh(x)} dx$ where $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and ξ is real.
   Hint: Use an appropriate rectangular contour containing $[-R, R]$ as one side.

Figure 8: Spring 2021 #2

3. Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

Figure 9: Fall 2019 #3