

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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1 | Tuesday, January 12

1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
 - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic \mathbb{R}^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

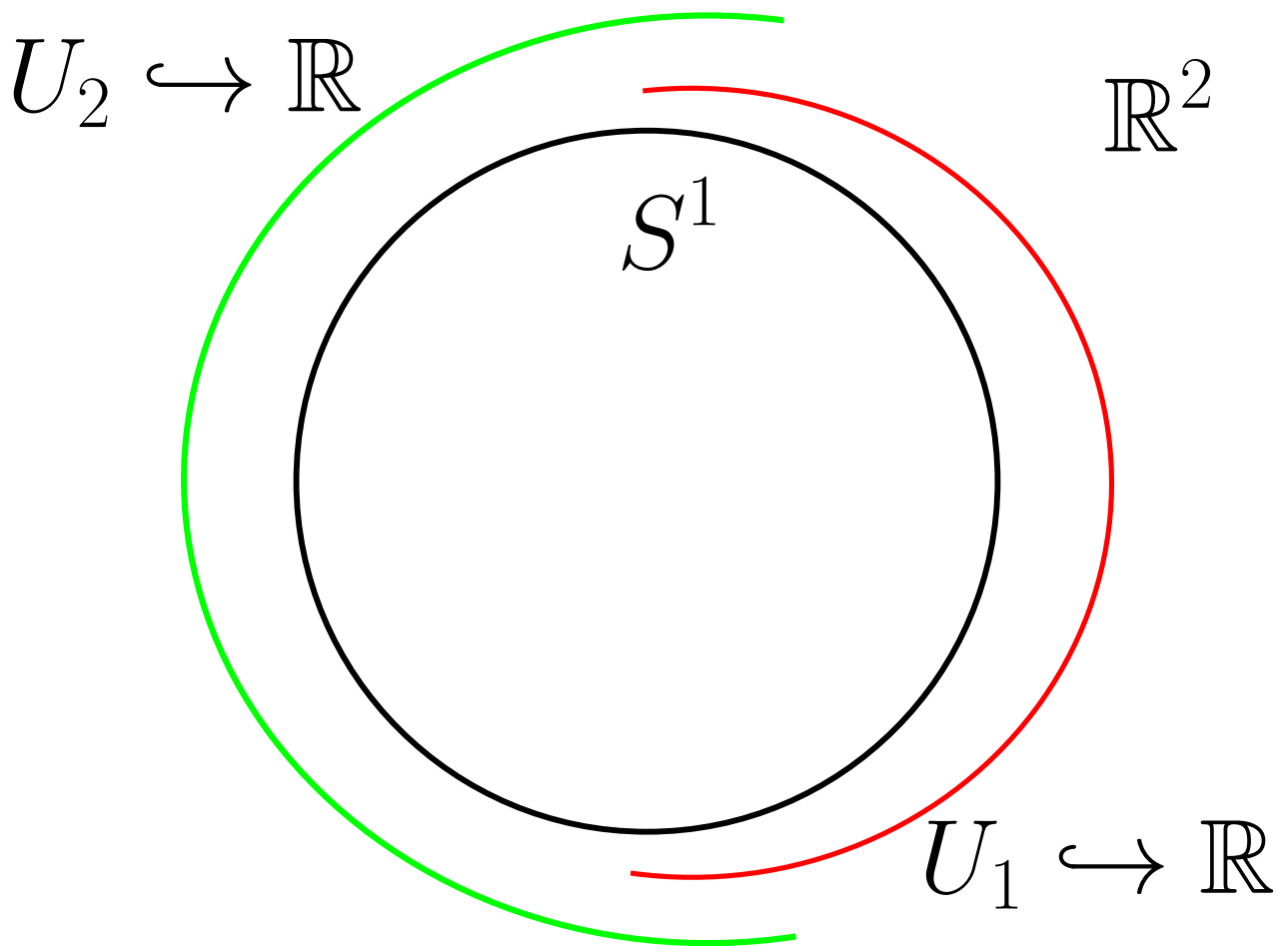
A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

Definition 1.2.1 (Topological Manifold)

Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to \mathbb{R}^n with a countable topological base, so we have charts $\varphi_u : U \rightarrow \mathbb{R}^n$ which are homeomorphisms from open sets covering X .

Example 1.2.2 (The circle): S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on \mathbb{CP}^N .

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6

Consider \mathbb{R}^n as a topological manifold: are any two smooth structures on \mathbb{R}^n diffeomorphic?

Remark 1.2.7: Fix a copy of \mathbb{R} and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\rightarrow \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on X , since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

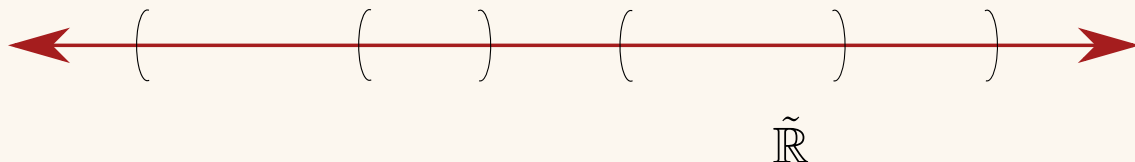
$$\begin{aligned} X &\rightarrow X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

Claim: \mathbb{R} admits a unique smooth structure.

Proof (sketch).

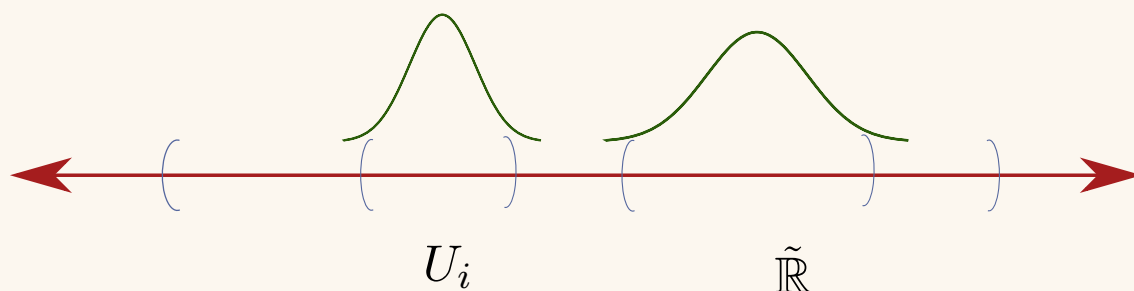
Let $\tilde{\mathbb{R}}$ be some exotic \mathbb{R} , i.e. a smooth manifold homeomorphic to \mathbb{R} . Cover this by coordinate charts to the standard \mathbb{R} :



Fact

There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \rightarrow \mathbb{R}$ with $f_i \geq 0$ and $\text{supp } f \subseteq U_i$ such that $\sum f_i = 1$ (i.e., *bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\rightarrow \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan

To understand smooth structures on X , we should try to solve differential equations on X .



Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2 .

Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.¹
2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $ks(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure.

¹Note that this doesn't start at C^0 , so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if $\text{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.
6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, \mathbb{R}^n for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes) \mathbb{R}^4 admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface Σ , the space of complex-analytic structures is a complex orbifold² of dimension $3g - 2$ where g is the genus of Σ , up to biholomorphism (i.e. *moduli*).

Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group.

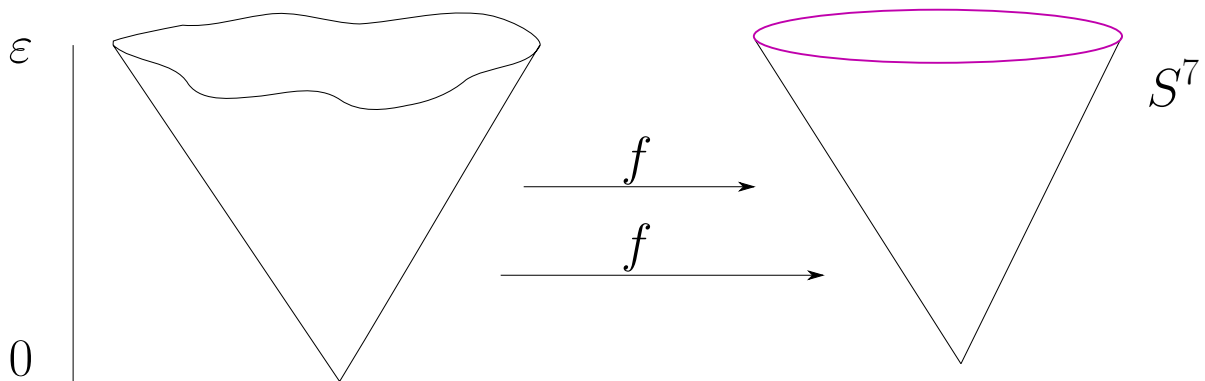
2 | Friday, January 15

Remark 2.0.1: Let

$$V := \{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\} \subseteq \mathbb{C}^5$$

$$S_\varepsilon := \{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \dots, 28$ yields the 28 smooth structures on S^7 . Note that V_k is the cone over $V_k \cap S_\varepsilon$.



? Admits a smooth structure, and $\bar{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure.

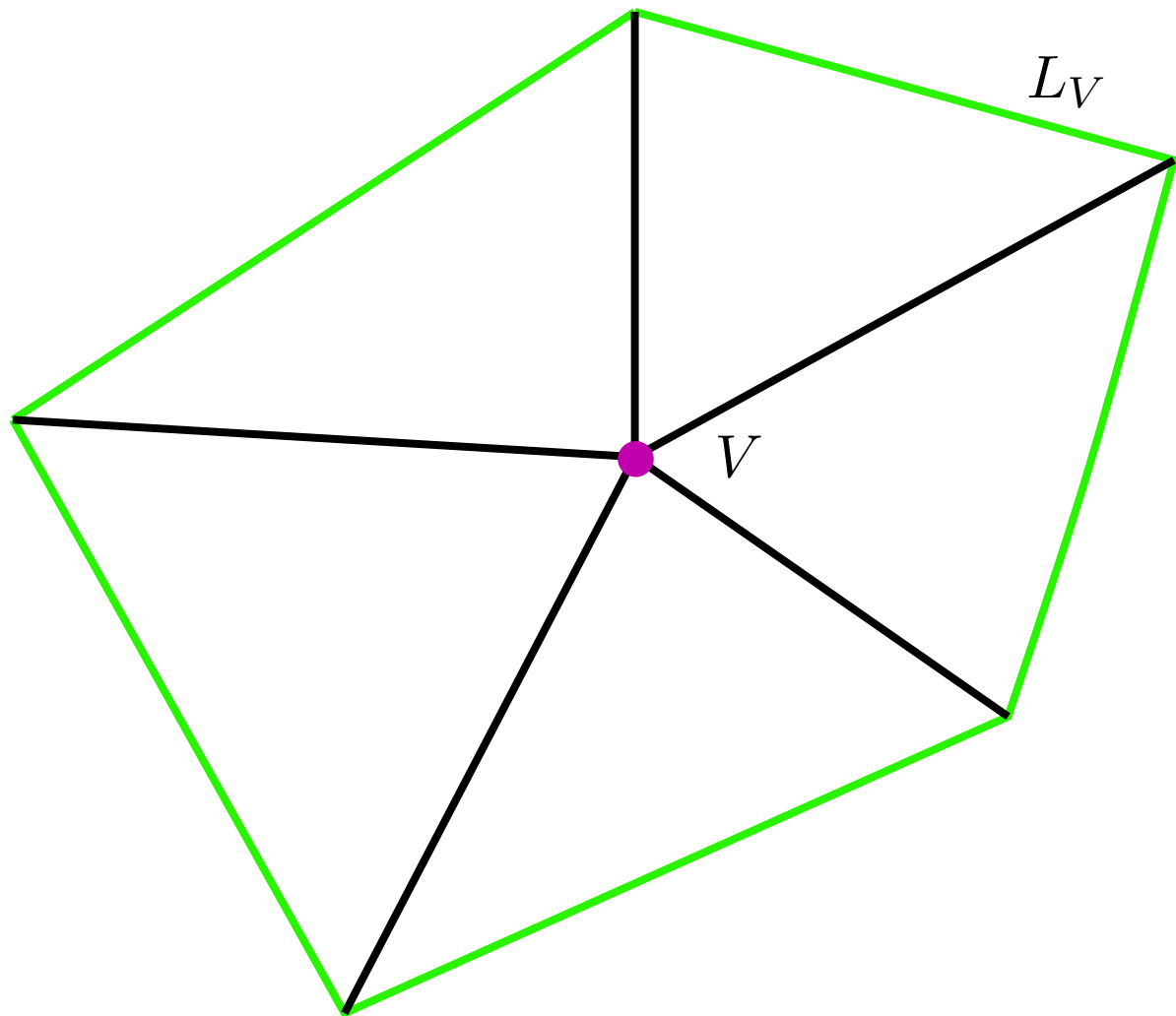
²Locally admits a chart to \mathbb{C}^n/Γ for Γ a finite group.

Question 2.0.2

Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

Answer 2.0.3

No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V :



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a “PL triangulation” where the links are spheres.

Remark 2.0.4: What’s special in dimension 4? Recall the **Kirby-Siebenmann** invariant $ks(x) \in H^4(X; \mathbb{Z}_2)$ for X a topological manifold where $ks(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)

Let X be a topological 4-manifold, then

$$\text{ks}(X) := \text{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2,$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\text{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures.} \end{cases}$$

Remark 2.0.7: $\text{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X; \mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha, \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 (Freedman).

If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\text{ks}(X) = \text{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha, \varphi\beta \rangle = \langle \alpha, \beta \rangle$.

Conversely, every **unimodular** bilinear form appears as $H^2(X; \mathbb{Z})$ for some X , i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\rightarrow H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for “being an element of the orthogonal group”, where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X; \mathbb{Z}) \xrightarrow{PD} H_2(X; \mathbb{Z})$ from Poincaré, where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1, \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \cap \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)

The **signature** of a topological 4-manifold is the signature of Q_X , where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X; \mathbb{R})$ and for some a, b

$$(H^2(X; \mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of +1s appearing in the matrix and b is the number of -1s. This is \mathbb{R}^{ab} where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a + 1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 (Rokhlin's Theorem).

Suppose $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X; \mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\text{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 (Donaldson).

Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X; \mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: \mathbb{R}^2 Has a Unique Smooth Structure

Remark 3.1.1: Last time we showed \mathbb{R}^1 had a unique smooth structure, so now we'll do this for \mathbb{R}^2 . The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)

A **Riemannian metric** $g \in \text{Sym}^2 T^*X$ for X a smooth manifold is a metric on every T_pX given by

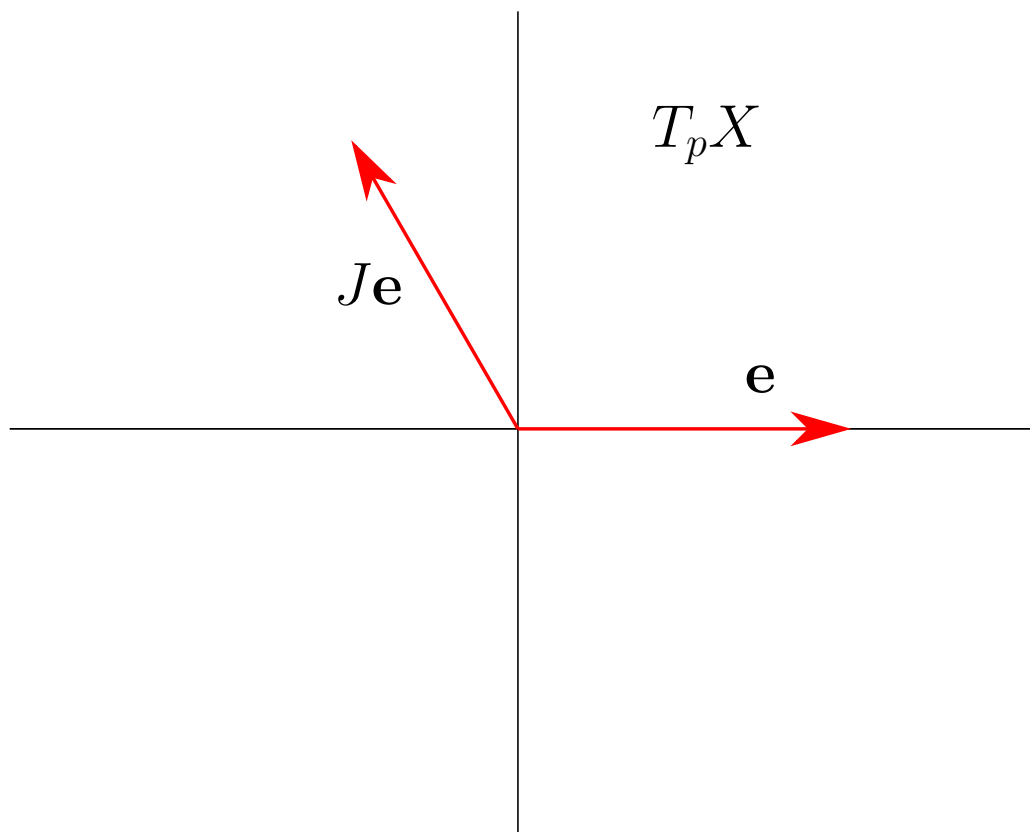
$$g_p : T_pX \times T_pX \rightarrow \mathbb{R}$$

$$g(v, v) \geq 0, g(v, v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)

An **almost complex structure** is a $J \in \text{End}(TX)$ such that $J^2 = -\text{id}$.

Remark 3.1.4: Let $e \in T_pX$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of T_pX .



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

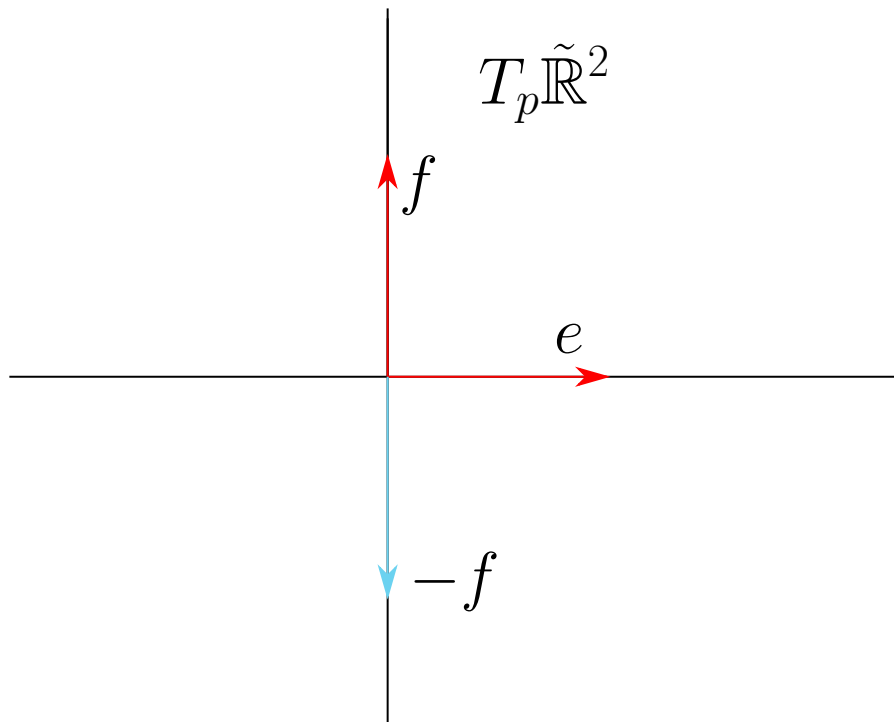
Let $\tilde{\mathbb{R}}^2$ be an exotic \mathbb{R}^2 .

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g := \sum f_I g_I$ with g_I metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e := f \in T_p \tilde{\mathbb{R}}^2$ such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$.
- $\{e, f\}$ is an oriented basis of $T_p \tilde{\mathbb{R}}^2$

This is because after choosing e , there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5(?).

Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \rightarrow \mathbb{C}$ such that J is multiplication by i .

So $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to \mathbb{D} or \mathbb{C} , both of which are diffeomorphic to \mathbb{R}^2 .

See the Newlander-Nirenberg theorem, a result in complex geometry.

4 | Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

4.1 Sheaves

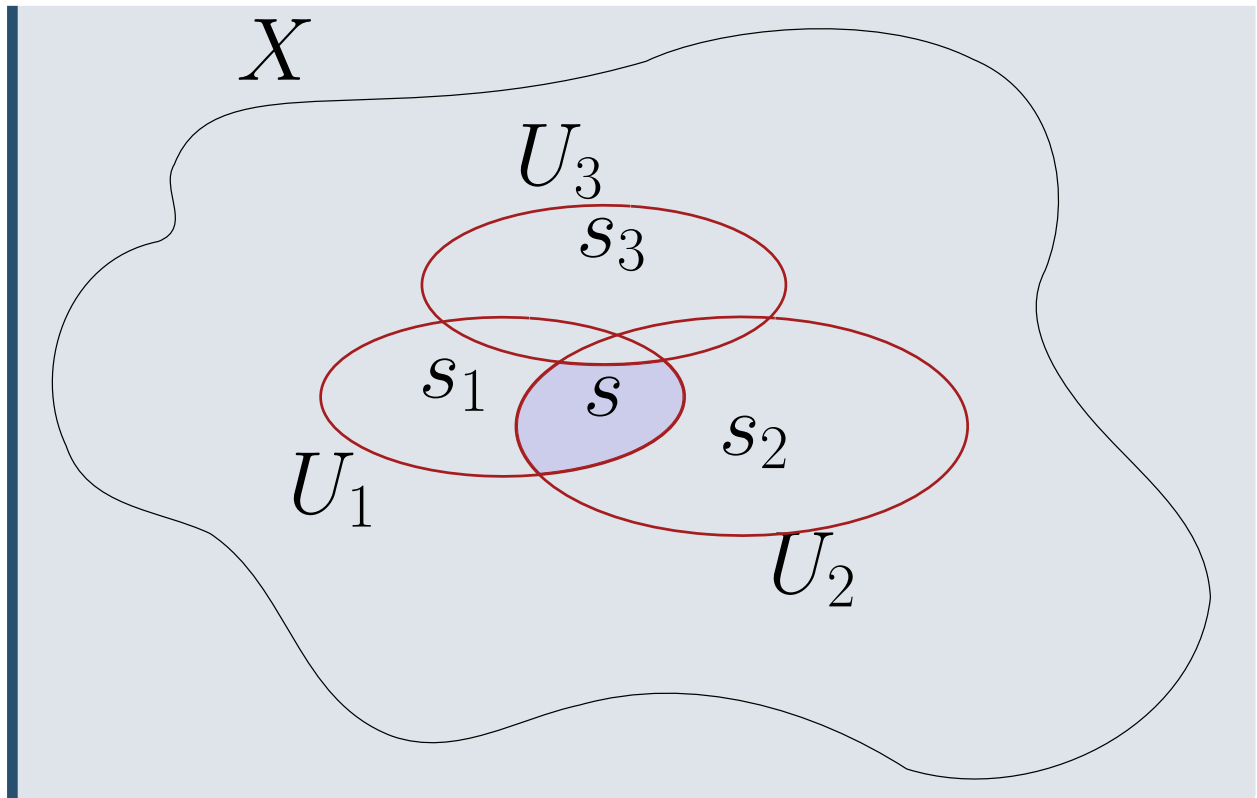
Definition 4.1.1 (Presheaves and Sheaves)

Recall that if X is a topological space, a **presheaf** of abelian groups \mathcal{F} is an assignment $U \rightarrow \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

- a. $\mathcal{F}(\emptyset) = 0$, the trivial abelian group.
- b. $\rho_{UU} : \mathcal{F}(U) \rightarrow \mathcal{F}(U) = \text{id}_{\mathcal{F}(U)}$
- c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say \mathcal{F} is a **sheaf** if additionally:

- d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2(?): Let X be a topological manifold, then $\mathcal{F} := C^0(\cdot, \mathbb{R})$ the set of continuous functionals form a sheaf. We have a diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\mathcal{F}} & C^0(U; \mathbb{R}) \\
 \uparrow & & \downarrow \text{restrict cts. functions} \\
 V & \xrightarrow{\mathcal{F}} & C^0(V; \mathbb{R})
 \end{array}$$

[Link to diagram](#)

Property (d) holds because given sections $s_i \in C^0(U_i; \mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in C^0(\bigcup_i U_i; \mathbb{R})$ such that $s|_{U_i} = s_i$ for all i – continuous functions glue.

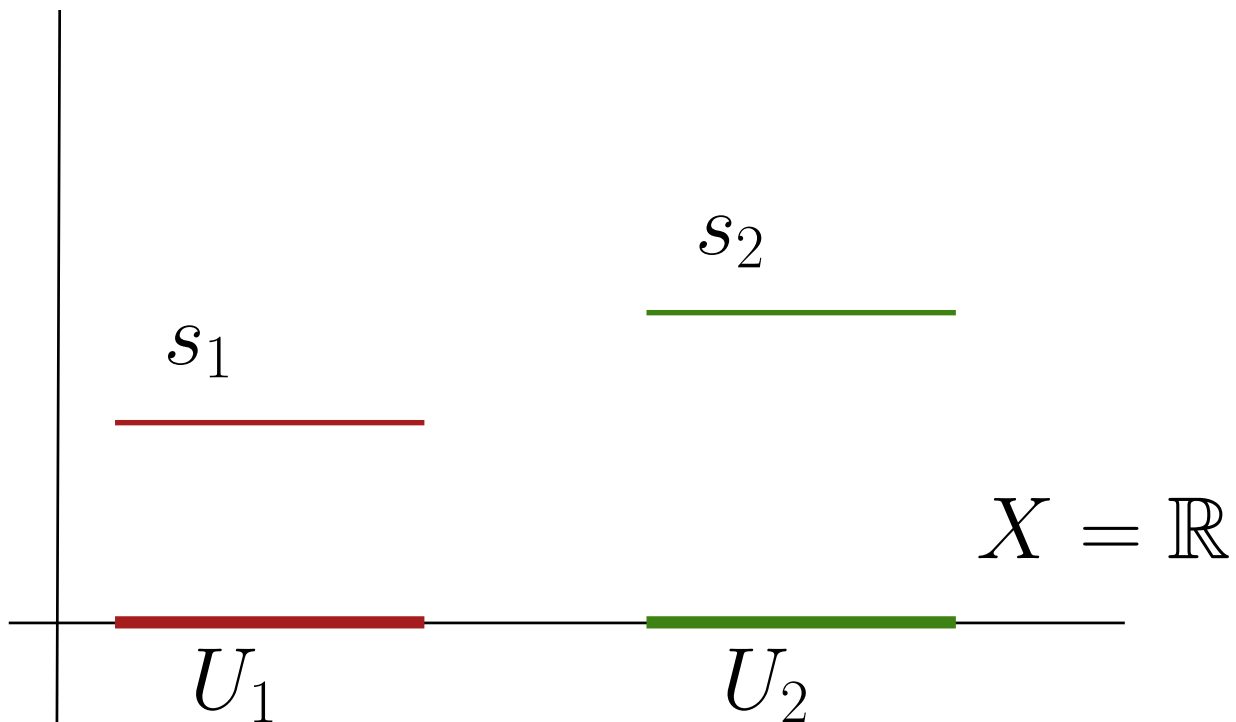
Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote \mathcal{O} . For example, $\mathcal{O} := C^0(\cdot; \mathbb{R})$ for topological manifolds, and $\mathcal{O} := C^\infty(\cdot; \mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set \mathcal{O} to be the sheaf of holomorphic functions.

Example 4.1.4(Locally Constant Sheaves): Let $A \in \mathbf{Ab}$ be an abelian group, then \underline{A} is the

sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \rightarrow A$. E.g. let $X \in \mathbf{Mfd}_{\mathbf{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant \implies globally constant in this case.

Warning 4.1.5

Note that the presheaf of constant functions doesn't satisfy (d)! Take \mathbb{R} and a function with two different values on disjoint intervals:



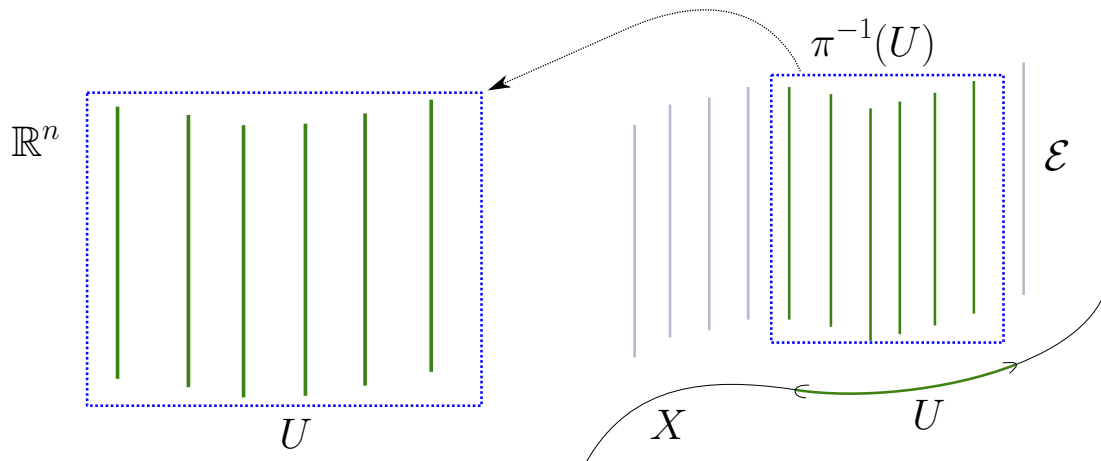
Note that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

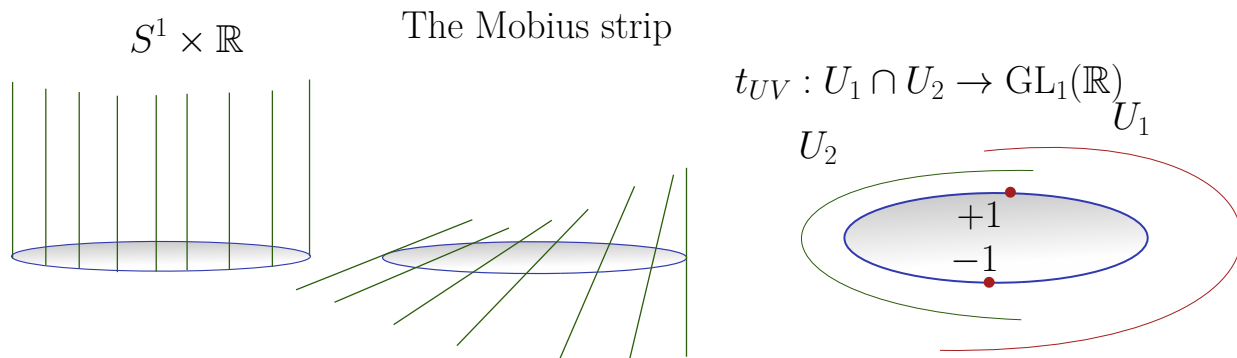
Remark 4.2.1: Let $\pi : \mathcal{E} \rightarrow X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \rightarrow \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 (Bundles over S^1): There are two \mathbb{R}^1 bundles over S^1 :



Note that the Möbius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: \mathcal{E} is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \rightarrow \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \text{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4(?): The trivial line bundle gives the sheaf $\mathcal{O} : \text{maps } U \xrightarrow{s} U \times Y \text{ for } Y = \mathbb{R}, \mathbb{C}$ such that $\pi \circ s = \text{id}$ are the same as maps $U \rightarrow Y$.

Definition 4.2.5 (\mathcal{O} -modules)

An \mathcal{O} -module is a sheaf \mathcal{F} such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6(?): If \mathcal{E} is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \curvearrowright s := fs$, i.e. just multiplying functions.

Example 4.2.7 (Non-example): The locally constant sheaf $\underline{\mathbb{R}}$ is not an \mathcal{O} -module: there isn't natural action since the sections of \mathcal{O} are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

Definition 4.2.8 (Morphisms of Sheaves)

A **morphism** of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:


$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Example 4.2.9 (*An \mathcal{O} -module that is not a vector bundle.*): Let $X = \mathbb{R}$ and define the skyscraper sheaf at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) := \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}$$

The $\mathcal{O}(U)$ -module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\rightarrow \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments. 

Example 4.2.10 (*of morphisms*): Let $X = \mathbb{R} \in \mathbf{Mfd}_{\text{Sm}}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x \cdot) : \mathcal{O} &\rightarrow \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$


for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.11 (?)

Check that $\ker \varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x \cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\text{coker}(x \cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to \mathbb{R}_0 . So $\mathcal{O} \rightarrow \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{(x \cdot)} \mathcal{O} \rightarrow \mathbb{R}_0 \rightarrow 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles. 

5 | Lecture 4 (Friday, January 22)

5.1 The Exponential Exact Sequence

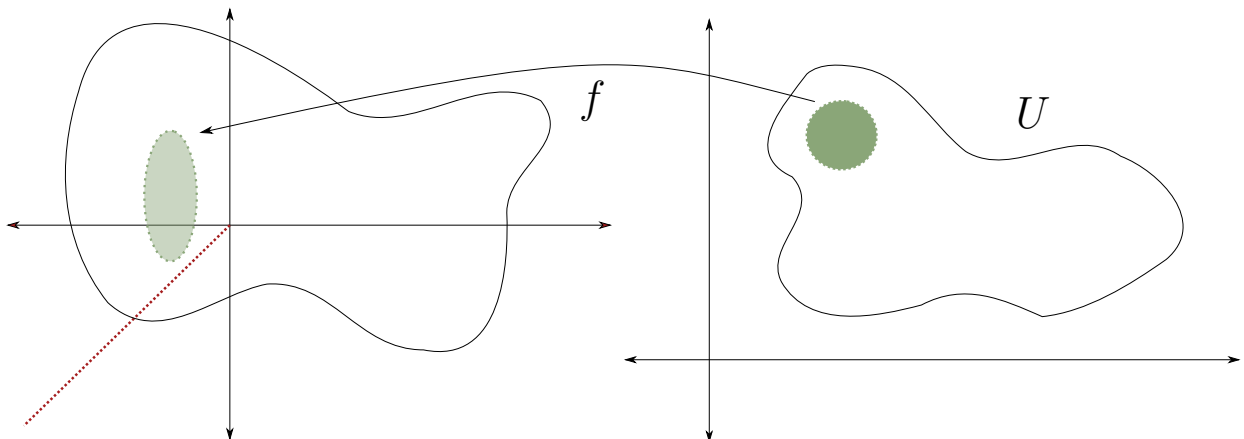
Let $X = \mathbb{C}$ and consider \mathcal{O} the sheaf of holomorphic functions and \mathcal{O}^\times the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \rightarrow \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \rightarrow \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\mathbb{Z}$, and we get an exact sequence

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \rightarrow \text{coker}(\exp) \rightarrow 0.$$

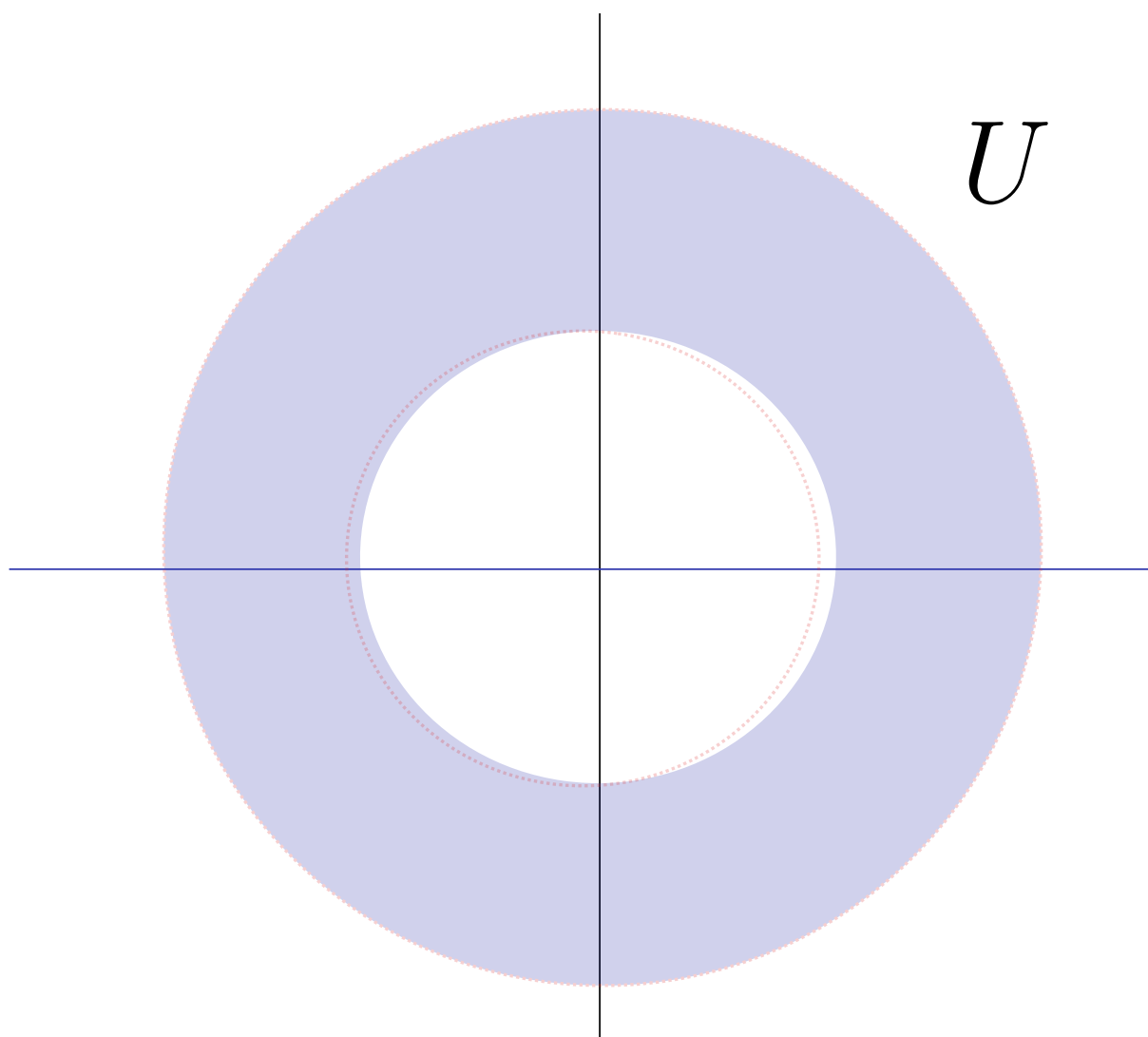
Question 5.1.1

What is the cokernel sheaf here?

Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U :



Thus on any sufficiently small open set, $\text{coker}(\exp) = 1$. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z , which is nontrivial.

Exercise 5.1.2 (?)

Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property $\text{Hom}_{\text{Presheaf}}(\mathcal{G}, \mathcal{F}^{\text{Presheaf}}) = \text{Hom}_{\text{Sheaf}}(\mathcal{G}, \mathcal{F}^{\text{Sh}})$ for any sheaf \mathcal{G} .

Definition 5.1.3 (Global Sections Sheaf)

The **global sections** sheaf of \mathcal{F} on X is given by $H^0(X; \mathcal{F}) = \mathcal{F}(X)$.

Example 5.1.4 (?):

- $C^\infty(X) = H^0(X, C^\infty)$ are the smooth functions on X
- $VF(X) = H^0(X; T)$ are the smooth vector fields on X for T the tangent bundle

- If X is a complex manifold then $\mathcal{O}(X) = H^0(X; \mathcal{O})$ are the globally holomorphic functions on X .
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

Remark 5.1.5: Given vector bundles V, W , we have constructions $V \oplus W, V \otimes W, V^\vee, \text{Hom}(V, W) = V^\vee \otimes W, \text{Sym}^n V, \bigwedge^p V$, and so on. Some of these work directly for sheaves:

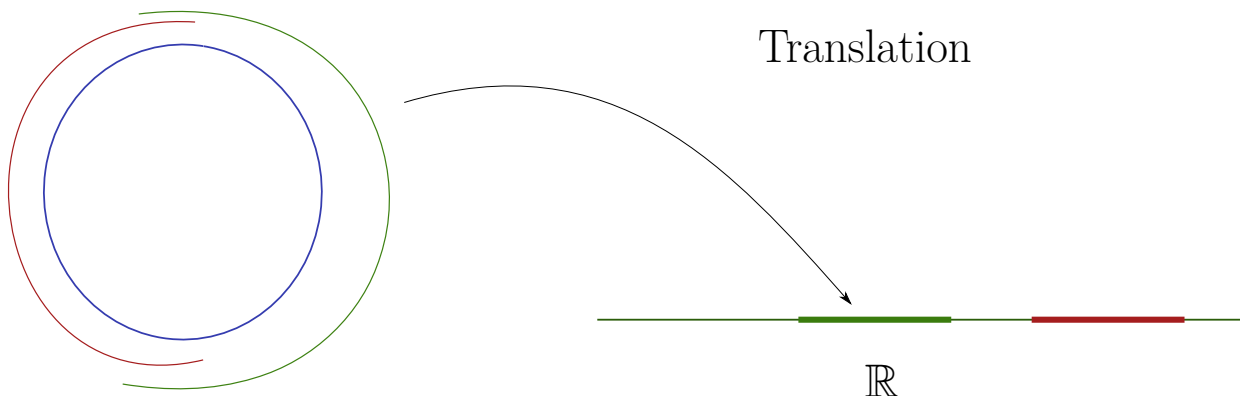
- $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathcal{H}\text{om}(V, W)$ we only get presheaves, so we need to sheafify.

⚠ Warning 5.1.6

$\text{Hom}(V, W)$ will denote the *global* homomorphisms $\mathcal{H}\text{om}(V, W)(X)$, which is a sheaf.

Example 5.1.7(?): Let $X^n \in \mathbf{Mfd}_{\text{sm}}$ and let Ω^p be the sheaf of smooth p -forms, i.e. $\bigwedge^p T^\vee$, i.e. $\Omega^p(U)$ are the smooth p forms on U , which are locally of the form $\sum f_{i_1, \dots, i_p}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_p}$ where the f_{i_1, \dots, i_p} are smooth functions.

Example 5.1.8(Sub-example): Take $X = S^1$, writing this as \mathbb{R}/\mathbb{Z} , we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x+c) = dx$ for c a constant:



Exercise 5.1.9(?)

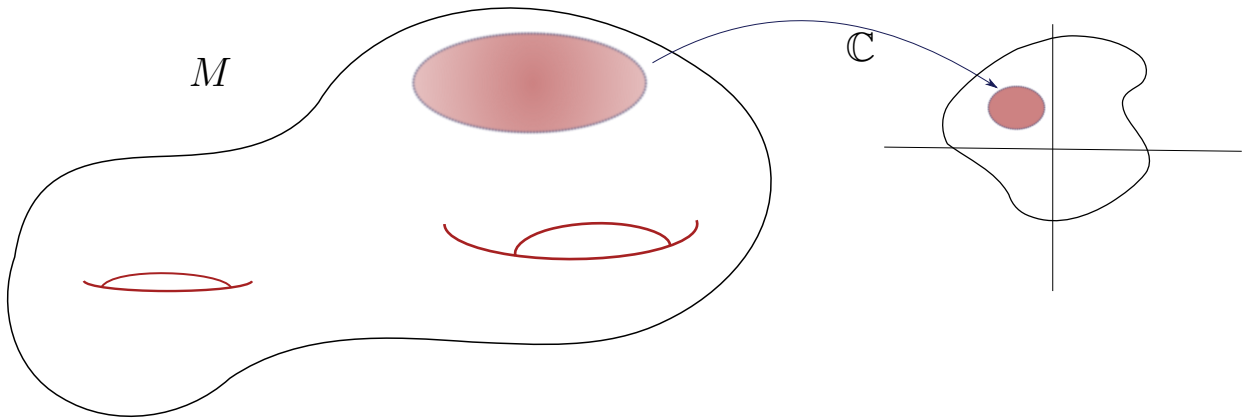
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.1.10: Note that there is a map $d : \Omega^p \rightarrow \Omega^{p+1}$ where $\omega \mapsto d\omega$.

⚠ Warning 5.1.11

d is **not** a map of \mathcal{O} -modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a \mathbb{C} -vector bundle.



Remark 5.1.12 (Subtlety 1): Note that Ω^p for complex manifolds is $\bigwedge^p T^\vee$, and so if we want to view $X \in \mathbf{Mfd}_\mathbb{R}$ we'll write $X_\mathbb{R}$. $TX_\mathbb{R}$ is then a real vector bundle of rank $2n$.

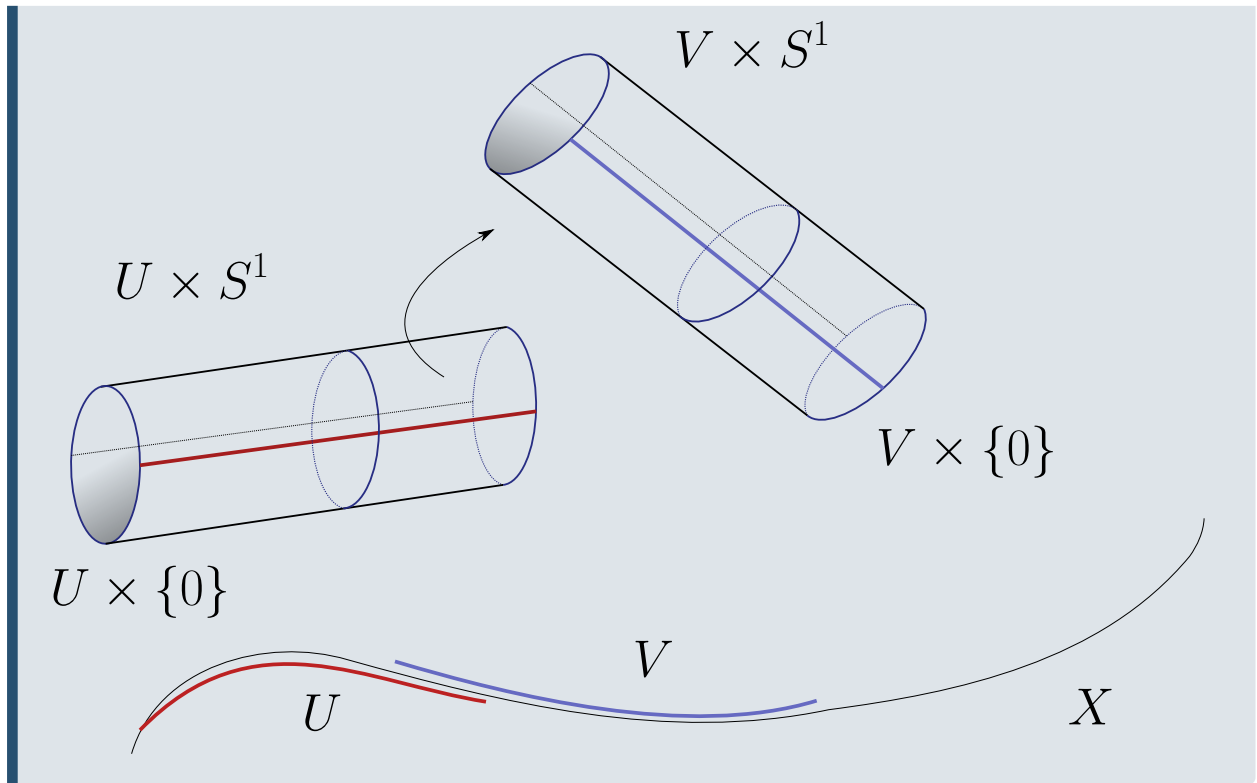
Remark 5.1.13 (Subtlety 2): Ω^p will denote *holomorphic* p -forms, i.e. local expressions $\sum f_I(z_1, \dots, z_n) \bigwedge dz_I$. For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$ -forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.1.14: Note that $T^\vee X_\mathbb{R} \otimes_\mathbb{C} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = f dz + g d\bar{z}$ where f, g are smooth. Then $\Omega^d X_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas Ω^1 is infinite-dimensional.

6 | Principal G -Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)

Let G be a (possibly disconnected) Lie group. Then a **principal G -bundle** $\pi : P \rightarrow X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \rightarrow G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \rightarrow G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \mathbf{Mfd}_{\text{Sm}}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \rightarrow \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)

Define $\text{Frame}_p(X) := \{\text{bases of } T_p X\}$, and $\text{Frame}X := \bigcup_{p \in X} \text{Frame}_p X$.

Remark 6.0.4: More generally, $\text{Frame}\mathcal{E}$ can be defined for any vector bundle \mathcal{E} , so $\text{Frame}X := \text{Frame}TX$. Note that $\text{Frame}X$ is a principal $\text{GL}_n(\mathbb{R})$ -bundle where $n := \text{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\text{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 (Important): A principal G -bundle admits a G -action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\rightarrow P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point:** the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)

The **orthogonal frame bundle** of a vector bundle \mathcal{E} equipped with a metric g is defined as $\text{OFrame}_p \mathcal{E} := \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r := \text{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \rightarrow \{x\}$ of a principal G -bundle are naturally **torsors** over G , i.e. a set with a free transitive G -action.

Definition 6.0.8 (?)

Let $\mathcal{E} \rightarrow X$ be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \rightarrow \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from \mathcal{E} .

Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

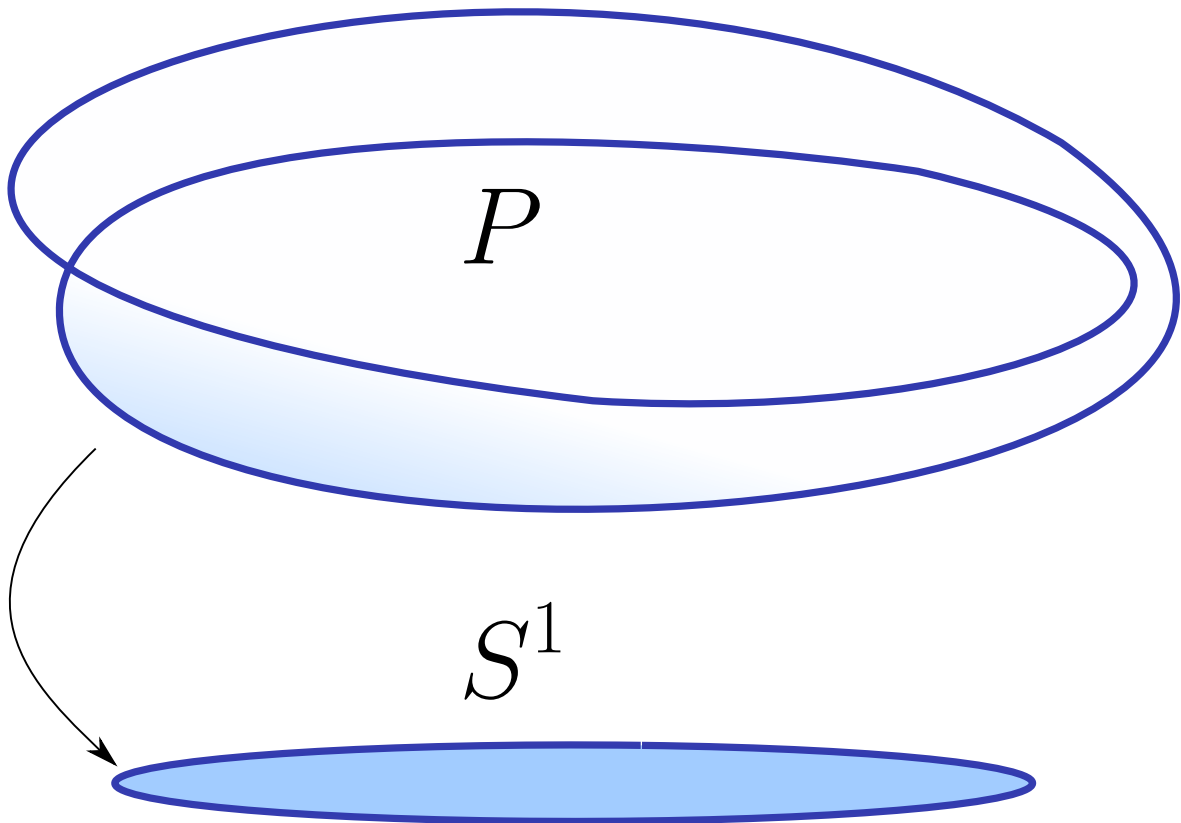
$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\mapsto (\bar{z}_1, \dots, \bar{z}_n). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)

We define the **unitary frame bundle** $\text{UFrame}(\mathcal{E}) := \bigcup_p \text{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of \mathcal{E}_p given by (e_1, \dots, e_n) where $h(e_i, \bar{e}_j) = \delta_{ij}$.

Remark 6.0.11: This is a principal G -bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\bar{A}^t = \text{id}$.

Example 6.0.12 (of more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G -bundle:



Example 6.0.13 (more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U , i.e.

$$P := \{(x, O) \mid x \in X, O \text{ is an orientation of } T_p X\}.$$

Note that P is an oriented manifold, $P \rightarrow X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \text{Frame}X / \text{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)

Let $P \rightarrow X$ be a principal G -bundle and let $G \rightarrow \text{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \{(p, v) \mid p \in P, v \in V\} / \sim \quad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 (?): Note that $\text{Frame}(\mathcal{E})$ is a $\text{GL}_r(\mathbb{R})$ -bundle and the map $\text{GL}_r(\mathbb{R}) \xrightarrow{\text{id}} \text{GL}(\mathbb{R}^r)$ is

a representation. At every fiber, we have $G \times_G V = (p, v) / \sim$ where there is a unique representative of this equivalence class given by (e, pv) . So $P \times_G V_p \rightarrow \{p\} \cong V_x$.

Exercise 6.0.16(?)

Show that $\text{Frame}(\mathcal{E}) \times_{\text{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \rightarrow G$, and so by the equivalence relation, $\text{im } t_{UV} \in \text{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Then $TM \rightarrow \text{Frame}(M)$ which is a principal $\text{SO}(3)$ -bundle. The universal cover is the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a \mathbb{C}^2 bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)

Let $\mathcal{E} \rightarrow X$ be a vector bundle, then a **connection** on \mathcal{E} is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\rightarrow \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge \omega + s \otimes d\omega. \end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \rightarrow \Omega^{p+1}$. The standard example is

$$\begin{aligned} d : \mathcal{O} &\rightarrow \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

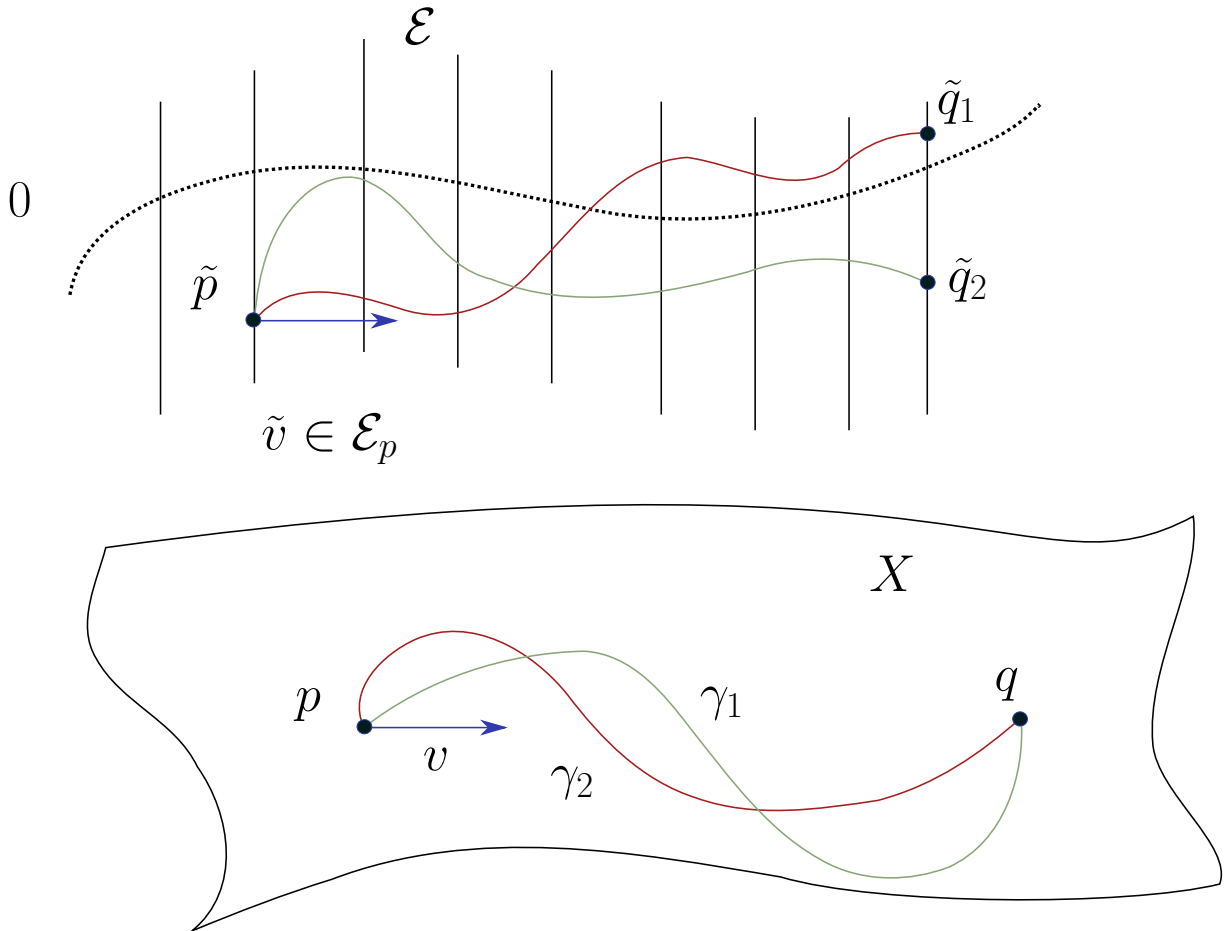
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇ , i.e. the map

$$F_{\nabla} := \nabla \circ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^2$$

is \mathcal{O} -linear, so $F_{\nabla}(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E} \otimes \Omega^{p+1}$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

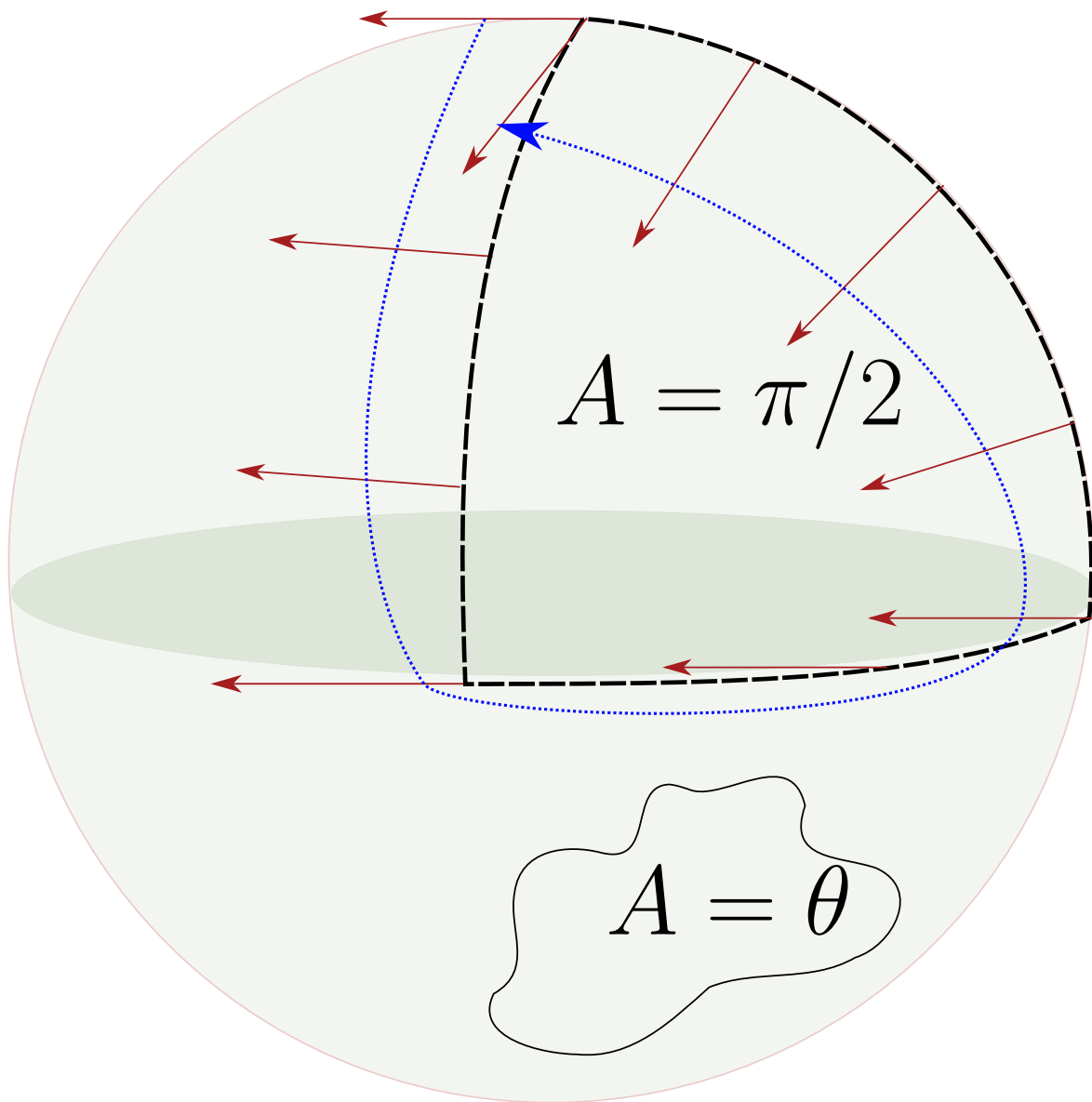


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1 , yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ .

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_{\nabla} = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ .

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 (A classic example): The Levi-Cevita connection ∇^{LC} on TX , which depends on a metric g . Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)

A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) := \left\{ s \in \mathcal{E}(U) \mid \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)

Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}}^d$. Furthermore $\text{rank}(L) = \text{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (?)

Let \mathcal{F} be a sheaf of abelian groups on a topological space X , and let $\mathfrak{U} := \{U_i\} \rightrightarrows X$ be an open cover of X . Let $U_{i_1, \dots, i_p} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1 < \dots < i_p} \mathcal{F}(U_{i_1, \dots, i_p})$$

with a differential

$$\begin{aligned} \partial^p : C_{\mathfrak{U}}^p(X, \mathcal{F}) &\rightarrow C_{\mathfrak{U}}^{p+1}(X, \mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \dots, i_p} := \prod_j (-1)^j \sigma_{i_0, \dots, \widehat{i_j}, \dots, i_p} \Big|_{U_{i_0, \dots, i_p}} \end{aligned}$$

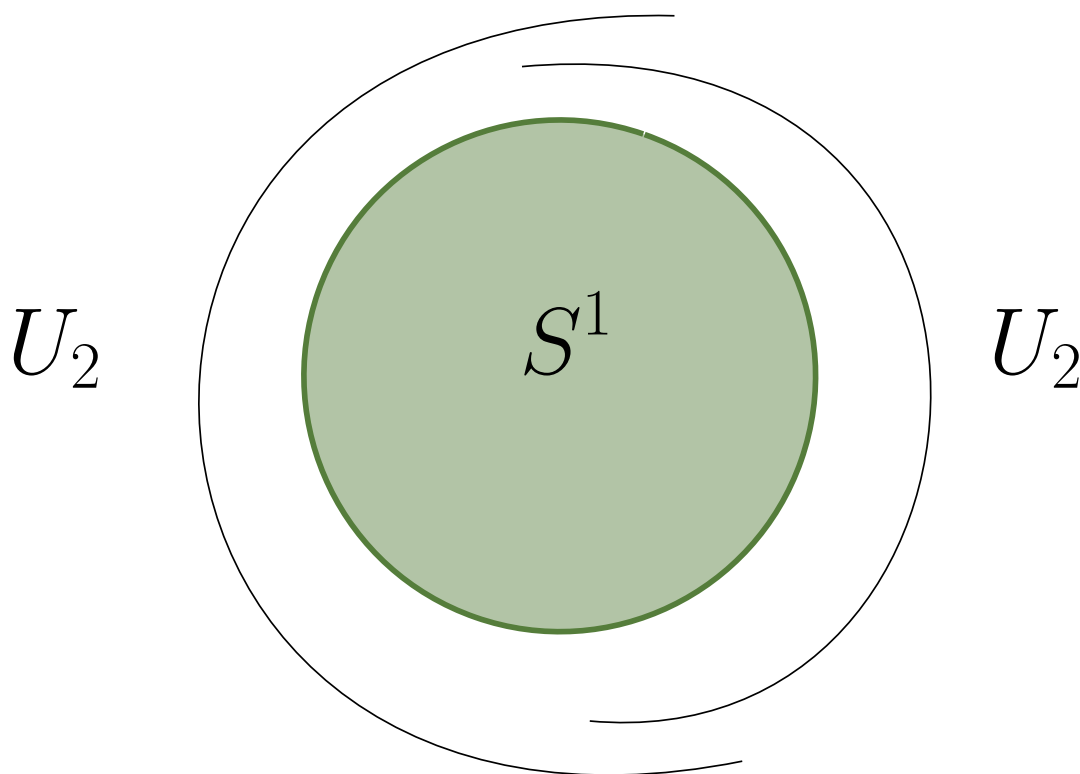
where we've defined this just on one given term in the product, i.e. a p -fold intersection.

Exercise 7.2.2 (?)

Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H_{\mathfrak{U}}^p(X, \mathcal{F})$ with respect to the cover \mathfrak{U} is defined as $\ker \partial^p / \text{im } \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4(?): Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



ere we have

$$C^0(S^1, \mathbb{Z}) = \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) = \mathbb{Z} \oplus \mathbb{Z},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\substack{\text{double} \\ \text{intersections}}} \mathbb{Z}(U_{ij})\mathbb{Z}(U_{12}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}.$$

We then get

$$\begin{aligned} C^0(S^1, \mathbb{Z}) &\xrightarrow{\partial} C^1(S^1, \mathbb{Z}) \\ \mathbb{Z} \oplus \mathbb{Z} &\rightarrow \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields $H^*(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex $C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1, \dots, i_p} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_p})$ for $\mathfrak{U} := \{U_i\}$ is an open cover of X and \mathcal{F} is a sheaf of abelian groups.

Fact 8.0.1

If \mathfrak{U} is a sufficiently fine cover then $H_{\mathfrak{U}}^p(X, \mathcal{F})$ is independent of \mathfrak{U} , and we call this $H^p(X; \mathcal{F})$.

Remark 8.0.2: Recall that we computed $H^p(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

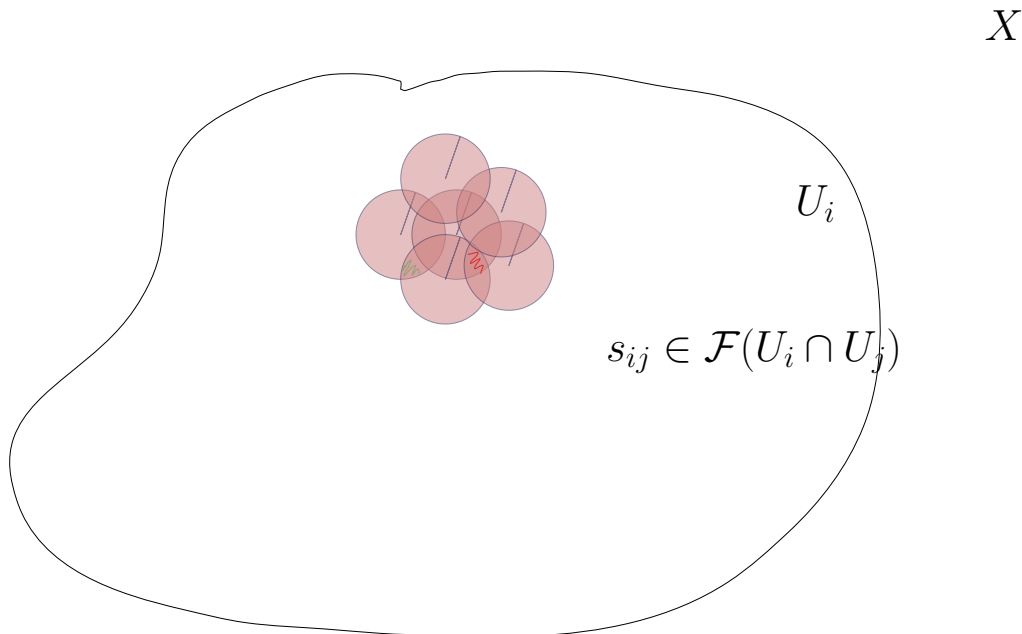
Theorem 8.0.3(?)

Let X be a paracompact and locally contractible topological space. Then $H^p(X, \mathbb{Z}) \cong H_{\text{Sing}}^p(X, \mathbb{Z})$. This will also hold more generally with \mathbb{Z} replaced by \underline{A} for any $A \in \mathbf{Ab}$.

Definition 8.0.4 (Acyclic Sheaves)

We say \mathcal{F} is *acyclic* on X if $H^{>0}(X; \mathcal{F}) = 0$.

Remark 8.0.5: How to visualize when $H^1(X; \mathcal{F}) = 0$:



On the intersections, we have $\text{im } \partial^0 = \{(s_i - s_j)_{ij} \mid s_i \in \mathcal{F}(U_i)\}$, which are *cocycles*. We have $C^1(X; \mathcal{F})$ are collections of sections of \mathcal{F} on every double overlap. We can check that $\ker \partial^1 = \{(s_{ij}) \mid s_{ij} - s_{ik} + s_{jk} = 0\}$, which is the cocycle condition. From the exercise from last class, $\partial^2 = 0$.

Theorem 8.0.6 (*Important!*).

Let X be a paracompact Hausdorff space and let

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a SES of sheaves of abelian groups, i.e. $\mathcal{F}_3 = \text{coker}(\varphi)$ and φ is injective. Then there is a LES in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \mathcal{F}_1) & \longrightarrow & H^0(X; \mathcal{F}_2) & \longrightarrow & H^0(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & H^1(X; \mathcal{F}_1) & \longrightarrow & H^1(X; \mathcal{F}_2) & \longrightarrow & H^1(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

Example 8.0.7(?): For X a manifold, we can define a map and its cokernel sheaf:

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

Corollary 8.0.8(of theorem).

Suppose $0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$ is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose I_n is acyclic for all $n \geq 0$. This is referred to as an **acyclic resolution**. Then the homology can be computed at $H^p(X; \mathcal{F}) = \ker(I_p(X) \rightarrow I_{p+1}(X)) / \text{im}(I_{p-1}(X) \rightarrow I_p(X))$.

Note that locally having kernels equal images is different than satisfying this globally!

Proof (of corollary).

This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{array}{ll} 0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} \operatorname{im}(d_0) \rightarrow 0 & \operatorname{im}(d_0) = \ker(d_1) \\ 0 \rightarrow \ker(d_1) \hookrightarrow I_1 \rightarrow I_1/\ker(d_1) = \operatorname{im}(d_1) & \operatorname{im}(d_1) = \ker(d_2) \\ & \cdot \end{array}$$

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving I_0 , we obtain:

$$\begin{array}{ccccc} & & & & \dots \\ & & & \nearrow & \\ H^{p-1}(\mathcal{F}) & \xleftarrow{\quad} & H^{p-1}(\mathcal{I}_i) = 0 & \xrightarrow{\quad} & H^{p-1}(\operatorname{im}(\lceil \cdot \rceil)) \\ & & \cong & \nwarrow & \\ H^p(\mathcal{F}) & \xleftarrow{\quad} & \dots = 0 & & \end{array}$$

The middle entries vanish since I_* was assumed acyclic, and so we obtain $H^p(\mathcal{F}) \cong H^{p-1}(\operatorname{im} d_0) \cong H^{p-1}(\ker d_1)$. Now taking the LES associated to I_1 , we get $H^{p-1}(\ker d_1) \cong H^{p-2}(\operatorname{im} d_1)$. Continuing this inductively, these are all isomorphic to $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$ after the p th step. ■

Corollary 8.0.9 (of the previous corollary).

Suppose $\mathfrak{U} \rightrightarrows X$, then if \mathcal{F} is acyclic on each U_{i_1, \dots, i_p} , then \mathfrak{U} is sufficiently fine to compute Čech cohomology, and $H_{\mathfrak{U}}^p(X; \mathcal{F}) \cong H^p(X; \mathcal{F})$.

Proof (?).

See notes. ■

Corollary 8.0.10 (of corollary).

Let $X \in \mathbf{Mfd}_{\setminus}$, then $H^p(X, \mathbb{R}) = H_{\mathrm{dR}}^p(X; \mathbb{R}R)$.

Proof (?).

Idea: construct an acyclic resolution of the sheaf \mathbb{R} on M . The following exact sequence works:

$$0 \rightarrow \underline{\mathbb{R}} \rightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \rightarrow \dots$$

So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that $\ker d^p = \operatorname{im} d^{p-1}$ on any contractible coordinate chart. In other words, we want to show that if $d\omega = 0$ for $\omega \in \Omega^p(\mathbb{R}^n)$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$. This is true by integration! Using the previous corollary, $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$. ■

Check Hartshorne to see how injective resolutions line up with derived functors!

9 | Monday, February 01

Remark 9.0.1: Last time $\underline{\mathbb{R}}$ on a manifold M has a resolution by vector bundles:

$$0 \rightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

This is an exact sequence of sheaves of any smooth manifold, since locally $d\omega = 0 \implies \omega = d\alpha$ (by the *Poincaré d-lemma*). We also want to know that Ω^k is an acyclic sheaf on a smooth manifold. ✍

Exercise 9.0.2 (?)

Let $X \in \mathbf{Top}$ and $\mathcal{F} \in \mathbf{Sh}(\mathbf{Ab})_X$. We say \mathcal{F} is **flasque** if and only if for all $U \supseteq V$ the map $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ is surjective. Show that \mathcal{F} is acyclic, i.e. $H^i(X; \mathcal{F}) = 0$. This can also be generalized with a POU.

Example 9.0.3(?): The function $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$, but doesn't extend to a continuous map on \mathbb{R} . So the restriction map is not surjective. ✍

Remark 9.0.4: Any vector bundle on a smooth manifold is acyclic. Using the fact that Ω^k is acyclic and the above resolution of $\underline{\mathbb{R}}$, we can write $H^k(X; \mathbb{R}) = \ker(d_k) / \operatorname{im} d_{k-1} := H_{dR}^k(X; \mathbb{R})$. ✍


Remark 9.0.5: Now letting $X \in \mathbf{Mfd}_{\mathbb{C}}$, recalling that Ω^p was the sheaf of holomorphic p -forms. Locally these are of the form $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$ where $f_I(\mathbf{z})$ is holomorphic. There is a resolution

$$0 \rightarrow \Omega^p \rightarrow A^{p,0},$$

where in $A^{p,0}$ we allowed also f_I are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth $(p, 0)$ form to check if it's holomorphic? ✍

Example 9.0.6 (?): For $p = 0$, we have

$$0 \rightarrow \mathcal{O} \rightarrow A^{0,0}.$$


where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations. 

Definition 9.0.7 (?)

Let $\omega \in A^{p,q}(X)$ where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J := \partial + \bar{\partial}$$

with $|I| = p, |J| = q$.


Example 9.0.8 (?): The function $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$ is smooth, and $df = \bar{z}dz + z d\bar{z}$. This can be checked by writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, and $\frac{\partial}{\partial \bar{z}} g = 0$ if and only if g is holomorphic. Here we get $\partial\omega \in A^{p+1,q}(X)$ and $\bar{\partial} \in A^{p,q+1}(X)$, and we can write $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$. 

Definition 9.0.9 (Cauchy-Riemann Equations)

Recall the Cauchy-Riemann equations: ω is a holomorphic $(p,0)$ -form on \mathbb{C}^n if and only if $\bar{\partial}\omega = 0$.

Remark 9.0.10: Thus to extend the previous resolution, we should take

$$0 \rightarrow \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \rightarrow \dots$$

The fact that this is exact is called the *Poincaré $\bar{\partial}$ -lemma*. 

Remark 9.0.11: There are no bump functions in the holomorphic world, and since Ω^p is a holomorphic bundle, it may not be acyclic. However, the $A^{p,q}$ are acyclic (since they are smooth vector bundles and thus admit POU's), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \text{im}(\bar{\partial}_{q-1}).$$

Note the similarity to H_{dR} , using $\bar{\partial}$ instead of d . This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers** $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Omega^p)$. These are analogies:

Smooth	Complex
\mathbb{R}	Ω^p
Ω^k	$A^{p,q}$
Betti numbers β_k	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

Theorem 9.0.12 (Properties of Singular Cohomology).

Let $X \in \mathbf{Top}$, then $H_{\text{Sing}}^i(X; \mathbb{Z})$ satisfies the following properties:

- Functoriality: given $f \in \text{Hom}_{\mathbf{Top}}(X, Y)$, there is a pullback $f^* : H^i(Y; \mathbb{Z}) \rightarrow H^i(X; \mathbb{Z})$.
- The cap product: a pairing

$$H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X; \mathbb{Z}) \rightarrow H_{j-i}(X; \mathbb{Z})$$

$$\varphi \otimes \sigma \mapsto \varphi \left(\sigma|_{\Delta_{0, \dots, j}} \right) \sigma|_{\Delta_{i, \dots, j}}.$$

This makes H_* a module over H^* .

- There is a ring structure induced by the cup product:

$$H^i(X; \mathbb{R}) \times H^j(X; \mathbb{R}) \rightarrow H^{i+j}(X; \mathbb{R}) \quad \alpha \cup \beta = (-1)^{ij} \beta \cup \alpha.$$

- Poincaré Duality: If X is an oriented manifold, there exists a fundamental class $[X] \in H_n(X; \mathbb{Z}) \cong \mathbb{Z}$ and $(\cdot) \cap X : H^i \rightarrow H_{n-i}$ is an isomorphism.

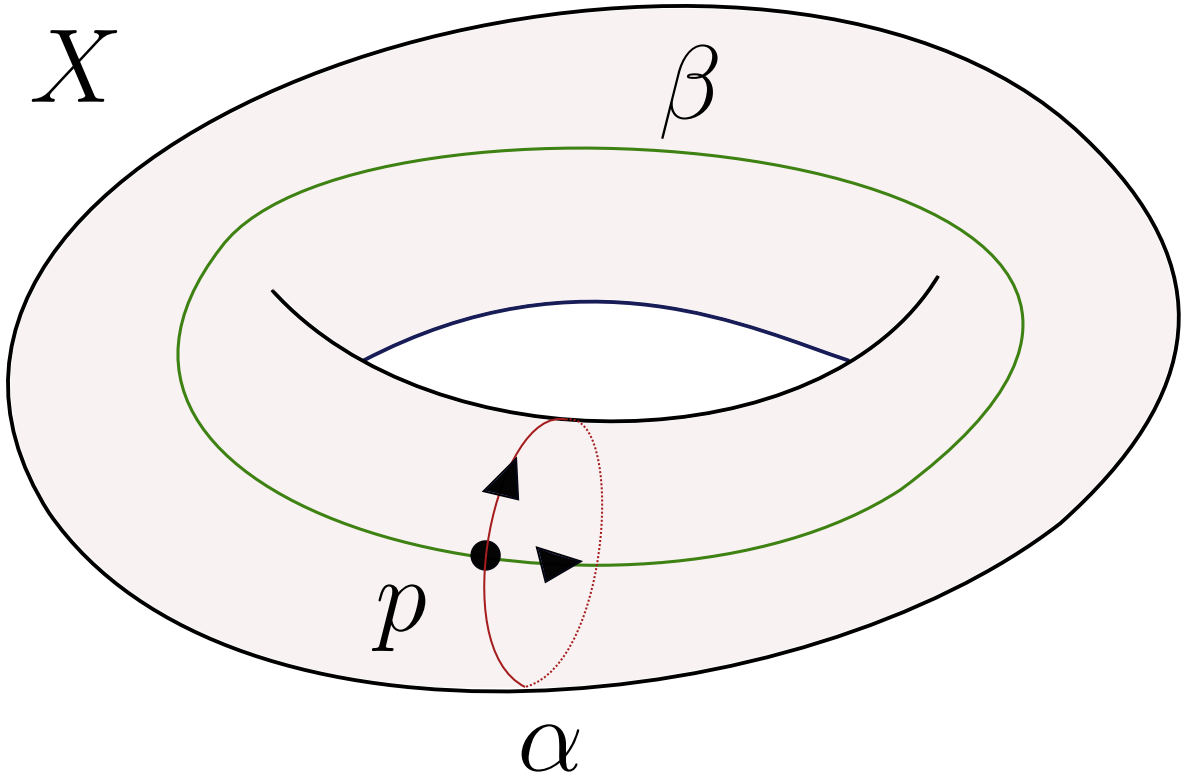
Remark 9.0.13: Let $M \subset X$ be a submanifold where X is a smooth oriented n -manifold. Then $M \hookrightarrow X$ induces a pushforward $H_n(M; \mathbb{Z}) \xrightarrow{\iota_*} H_n(X; \mathbb{Z})$ where $\sigma \mapsto \iota \circ \sigma$. Using Poincaré duality, we'll identify $H_{\dim M}(X; \mathbb{Z}) \rightarrow H^{\text{codim } M}(X; \mathbb{Z})$ and identify $[M] = PD(\iota_*([M]))$. In this case, if $M \pitchfork N$ then $[M] \cap [N] = [M \cap N]$, i.e. the cap product is given by intersecting submanifolds.

⚠ Warning 9.0.14

This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

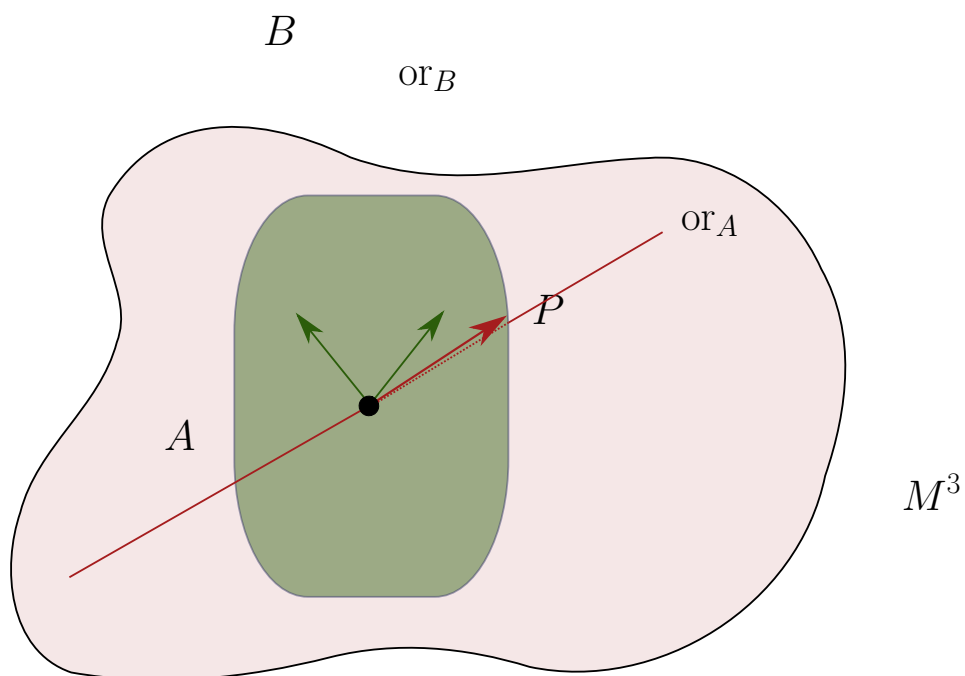


We can then take the fundamental classes of the submanifolds, say $[\alpha], [\beta] \in H^1(X; \mathbb{Z}) \xrightarrow{PD} H^1(X, \mathbb{Z})$. Here $T_p\alpha \oplus T_p\beta = T_pX$, since the intersections are transverse. Since α, β are oriented, let $\{e\}$ be a basis of $T_p\alpha$ (up to \mathbb{R}^+) and similarly $\{f\}$ a basis of $T_p\beta$. We can then ask if $\{e, f\}$ constitutes an *oriented* basis of T_pX . If so, we write $\alpha \cdot_p \beta := +1$ and otherwise $\alpha \cdot_p \beta = -1$. We thus have

$$[\alpha] \smile [\beta] \in H^2(X; \mathbb{Z}) \xrightarrow{PD} H_0(X; \mathbb{Z}) = \mathbb{Z}$$

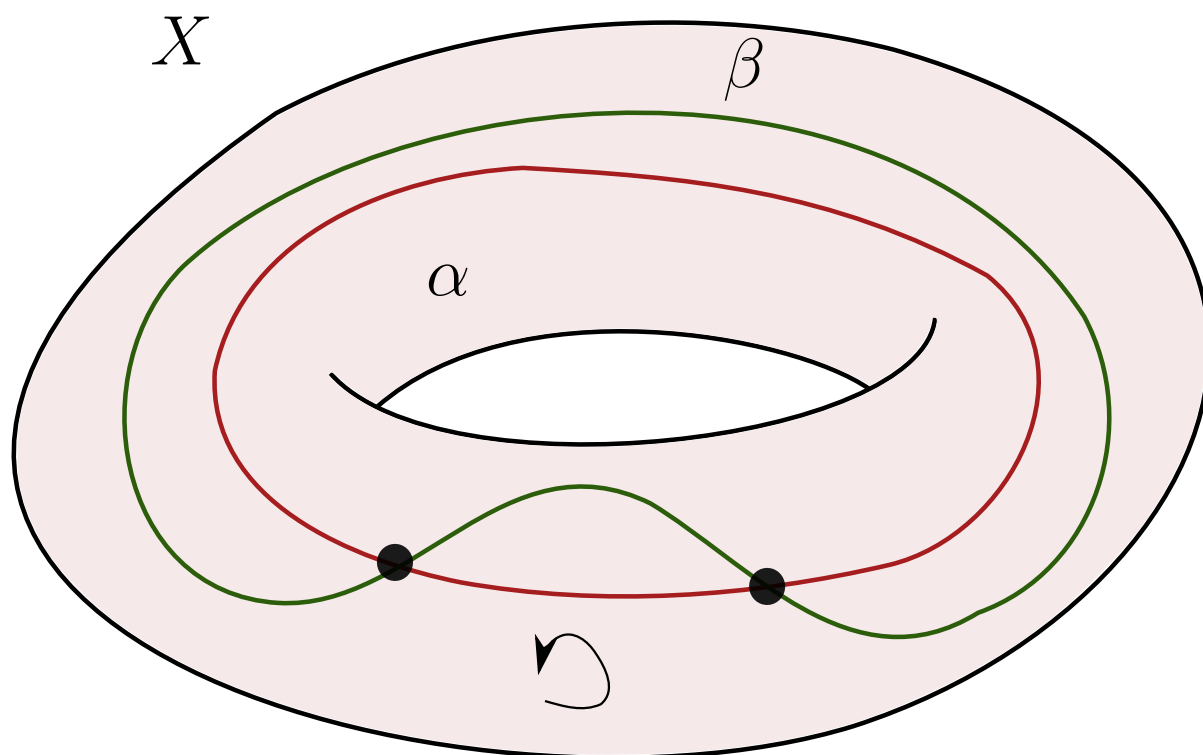
since X is connected. We can thus define the **intersection form** $\alpha \cdot \beta := [\alpha] \smile [\beta]$. In general if A, B are oriented transverse submanifolds of M which are themselves oriented, we'll have $[A] \smile [B] = [A \cap B]$. We need to be careful: how do we orient the intersection? This is given by comparing the orientations on A and B as before.

Example 10.0.1(?): If $\dim M = \dim A + \dim B$, then any $p \in A \cap B$ is oriented by comparing $\{\text{or}_A, \text{or}_B\}$ to or_M .



Here it suffices to check that $\{e, f_1, f_2\}$ is an oriented basis of $T_p M$.

Example 10.0.2(?): In this case, $[\alpha] \smile [\beta] = 0$ and so $\alpha \cdot \beta = 0$:



Remark 10.0.3: Note that cohomology with \mathbb{Z} coefficients can be defined for any topological space, and Poincaré duality still holds.

Remark 10.0.4: We'll be considering $M = M^4$, smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

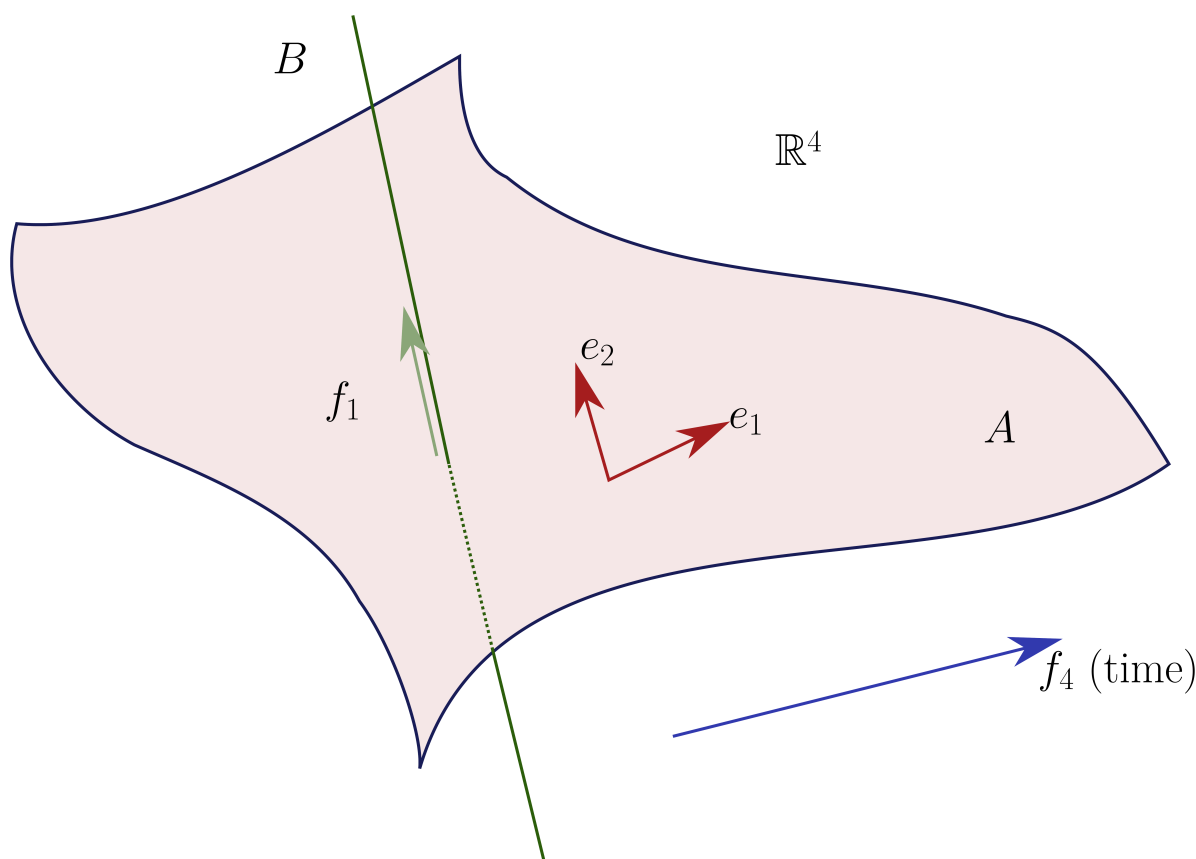


Figure 1: Picking one basis element in the time direction

Here α is oriented in the “forward time” direction, and this is a surface at time $t = 0$. Where $A \cdot B = +1$, since $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$ is a oriented basis for \mathbb{R}^4 . For α^2 , switching the order of α, β no longer yields an oriented basis, but in this case it is α and $A \cdot B = B \cdot A$. This is because

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

Remark 10.0.5: Let M^{2n} be an oriented manifold, then the cup product yields a bilinear map $H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ which is symmetric when n is odd and antisymmetric (or symplectic) when n is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$(H^n(M; \mathbb{Z})/\text{tors})^\vee \xrightarrow{\sim} H^n(M; \mathbb{Z})/\text{tors}$$

$$\alpha \smile \cdot \mapsto \alpha,$$

where the LHS are linear functionals on cohomology.

Remark 10.0.6: Recall the universal coefficients theorem:

$$H^i(X; \mathbb{Z})/\text{tors} \cong (H_i(X; \mathbb{Z})/\text{tors})^\vee.$$

The general theorem shows that $H^i(X; \mathbb{Z})_{\text{tors}} = H_{i-1}(X; \mathbb{Z})_{\text{tors}}$.

Remark 10.0.7: Note that if M is an oriented 4-manifold, then

	tors	torsionfree		tors	torsionfree	
H^0	0	\mathbb{Z}		H_0	0	\mathbb{Z}
H^1	0	\mathbb{Z}^{β_1}		H_1	A	\mathbb{Z}^{β_1}
H^2	A	\mathbb{Z}^{β_2}	\xrightarrow{PD}	H_2	A	\mathbb{Z}^{β_2}
H^3	A	\mathbb{Z}^{β_1}		H_3	0	\mathbb{Z}^{β_1}
H^4	0	\mathbb{Z}		H_4	0	\mathbb{Z}

In particular, if M is simply connected, then $H_1(M) = \mathbf{Ab}(\pi_1(M)) = 0$, which forces $A = 0$ and $\beta_1 = 0$.

Definition 10.0.8 (Lattice)

A **lattice** is a finite-dimensional free \mathbb{Z} -module L together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\rightarrow \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice (L, \cdot) is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\rightarrow L^\vee \\ \ell &\mapsto \ell \cdot (\cdot). \end{aligned}$$

Remark 10.0.9: How to determine if a lattice is unimodular: take a basis $\{e_1, \dots, e_n\}$ of L and form the *Gram matrix* $M_{ij} := (e_i \cdot e_j) \in \text{Mat}(n \times n, \mathbb{Z})^{\text{Sym}}$. Then (L, \cdot) is unimodular if and only if $\det(M) = \pm 1$ if and only if M^{-1} is integral. In this case, the rows of M^{-1} will form a basis of the dual basis.

Definition 10.0.10 (?)

The **index** of a lattice is $|\det M|$.

Exercise 10.0.11 (?)

Prove that $|\det M| = |L^\vee/L|$.

Remark 10.0.12: In general, for M^{4k} , the H^{2k}/tors is unimodular. For M^{4k+2} , the H^{2k+1}/tors is a unimodular *symplectic* lattice, which is obtained by replacing the word “symmetric” with “antisymmetric” everywhere above.

Example 10.0.13(?): For the torus, since the dimension is 2 (mod 4), you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check!

Definition 10.0.14 (?)

A lattice is **nondegenerate** if $\det M \neq 0$.

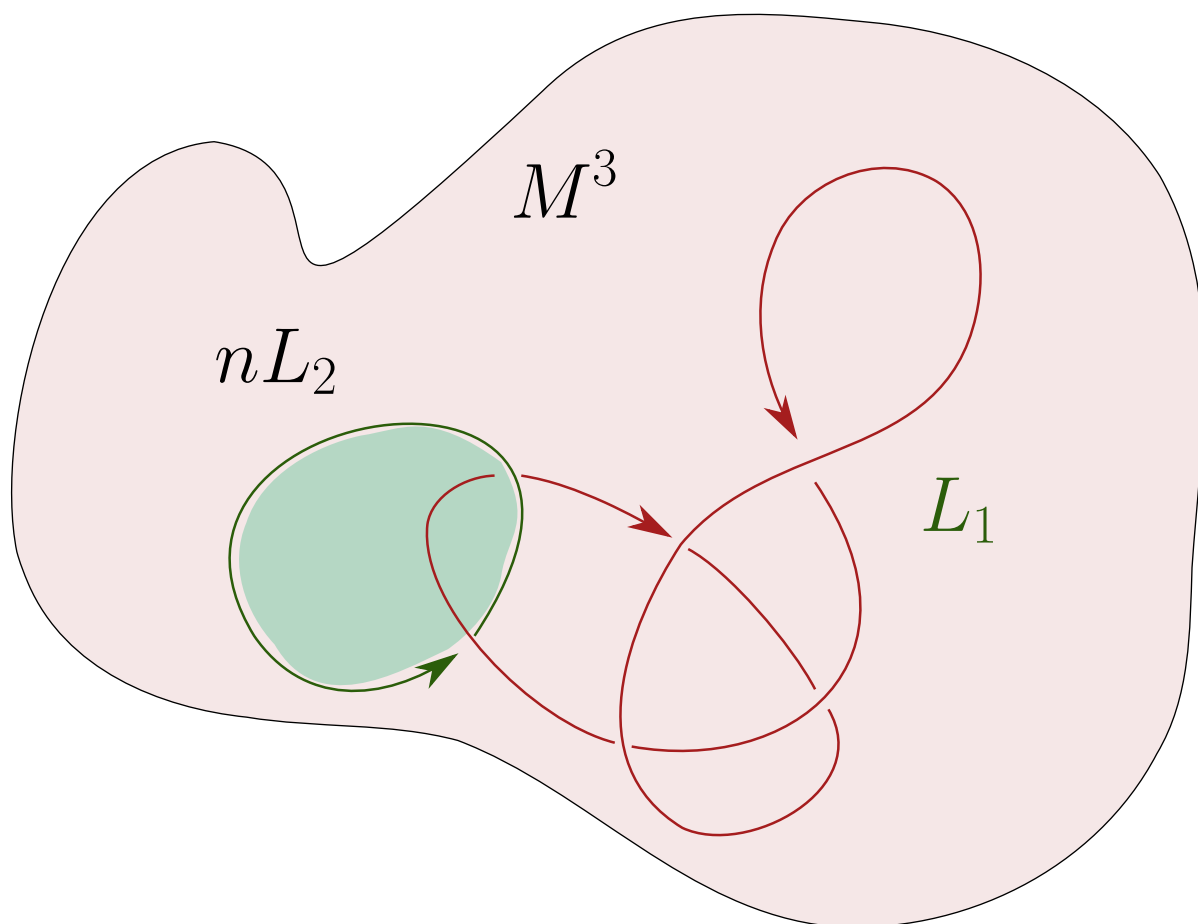
Definition 10.0.15 (?)

The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space with an \mathbb{R} -valued symmetric bilinear form. This allows extending the lattice from \mathbb{Z}^n to \mathbb{R}^n .

Remark 10.0.16: If (L, \cdot) is nondegenerate, then Gram-Schmidt will yield an orthonormal basis $\{v_i\}$. The number of positive norm vectors is an invariant, so we obtain $\mathbb{R}^{p,q}$ where p is the number of +1s in the Gram matrix and q is the number of -1s. The **signature** of (L, \cdot) is (p, q) , or by abuse of notation $p - q$. This is an invariant of the 4-manifold, as is the lattice itself $H^2(X; \mathbb{Z})/\text{tors}$ equipped with the intersection form.

Remark 10.0.17: There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$



Remark 10.0.18: $A \cdot B := \sum_{p \in A \cap B} \text{sgn}_p(A, B)$, where $A \pitchfork B$ and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \smile [\omega] = \int_A \omega.$$

11 | Friday, February 05

Remark 11.0.1: Recall that a lattice is **unimodular** if the map $L \rightarrow L^\vee := \text{Hom}(L, \mathbb{Z})$ is an isomorphism, where $\ell \mapsto \ell \cdot (\cdot)$. To check this, it suffices to check if the Gram matrix M of a basis $\{e_i\}$ satisfies $|\det M| = 1$.

Example 11.0.2 (Determinant 1 Integer Matrices): The matrices $[1]$ and $[-1]$ correspond to the lattice $\mathbb{Z}e$ where either $e^2 := e \cdot e = 1$ or $e^2 = -1$. If M_1, M_2 both have absolute determinant 1,

then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

So if L_1, L_2 are unimodular, then taking an orthogonal sum $L_1 \oplus L_2$ also yields a unimodular lattice. So this yields diagonal matrices with p copies of $+1$ and q copies of -1 . This is referred to as $rm1_{p,q}$, and is an *odd* unimodular lattice of signature (p, q) (after passing to \mathbb{R}). Here *odd* means that there exists a $v \in L$ such that v^2 is odd.

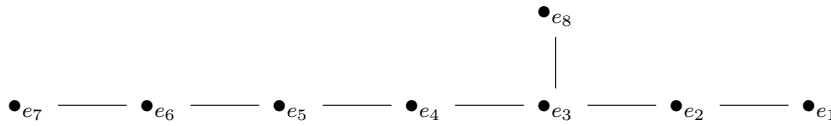
Example 11.0.3 (Even unimodular lattices): An even lattice must have no vectors of odd norm, so all of the diagonal elements are in $2\mathbb{Z}$. This is because $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i < j} 2n_i n_j e_i \cdot e_j$.

Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as H , sometimes referred to as the *hyperbolic cell* or *hyperbolic plane*.

Example 11.0.4 (A harder even unimodular lattice): This is built from the E_8 Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \rightarrow e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example, $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$. You can check that $\det(e_i \cdot e_j) = 1$, and this is referred to as the E_8 lattice. This is of signature $(0, 8)$, and it's negative definite if and only if $v^2 < 0$ for all $v \neq 0$. One can also negate the intersection form to define $-E_8$. Note that any simply-laced Dynkin diagram yields some lattice. For example, E_{10} is unimodular of signature $(1, 9)$, and it turns out that $E_{10} \cong E_8 \oplus H$.

Definition 11.0.5 (?)

Take

$$\Pi_{a,a+8b} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b E_8,$$

which is an even unimodular lattice since the diagonal entries are all -2 , and using the fact

that the signature is additive, is of signature $(a, a + 8b)$. Similarly,

$$\mathbf{II}_{a+8b,a} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b (-E_8),$$

which is again even and unimodular.

Remark 11.0.6: Thus

- $\mathbf{I}_{p,q}$ is odd, unimodular, of signature (p, q) .
- $\mathbf{II}_{p,q}$ is even, unimodular, of signature (p, q) only for $p \equiv q \pmod{8}$.

Theorem 11.0.7 (Serre).

Every unimodular lattice which is not positive or negative definite is isomorphic to either $\mathbf{I}_{p,q}$ or $\mathbf{II}_{p,q}$ with $8 \mid p - q$.

Remark 11.0.8: So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: E_8
16	2: $E_8^{\oplus 2}, D_{16}^+$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$> 8 \times 10^{16}!!!!$

Note that the signature of a definite lattice must be divisible by 8.

Remark 11.0.9: There is an isometry: $f : E_8 \rightarrow E_8$ where $f \in O(E_8)$, the linear maps preserving the intersection form (i.e. the Weyl group $W(E_8)$, given by $v \mapsto v + (v, e_i)e_i$. The Leech lattice also shows up in the sphere packing problems for dimensions 2, 4, 8, 24. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of L as a subset of \mathbb{R}^n , namely $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$ for example, and the map $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$ is an isometric embedding into \mathbb{R}^n with the standard form. Connection to classification of Lie groups: root lattices.

Remark 11.0.10: If M^4 is a compact oriented 4-manifold and if the intersection form on $H^2(M; \mathbb{Z})$ is indefinite, then the only invariants we can extract from that associated lattice are

- Whether it's even or odd, and
- Its signature

If the lattice is even, then the signature satisfies $8 \mid p - q$. So Poincaré duality forces unimodularity,

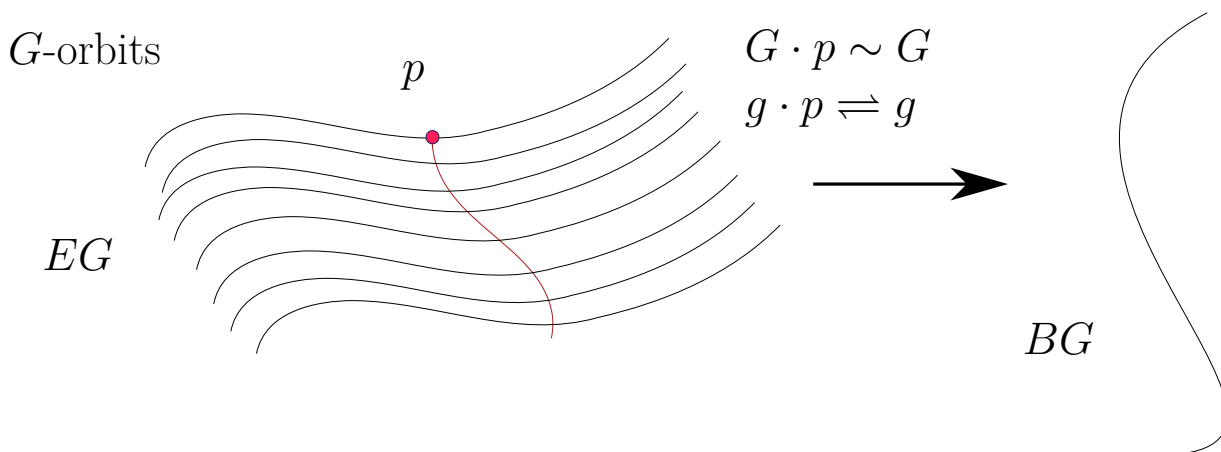
and then there are further number-theoretic restrictions. E.g. this prohibits $\beta_2 = 7$, since then the signature couldn't possibly be 8 if the intersection form is even.

11.1 Characteristic Classes

Definition 11.1.1 (?)

Let G be a topological group, then a **classifying space** EG is a contractible topological space admitting a free continuous G -action with a “nice” quotient.

Remark 11.1.2: Thus there is a map $EG \rightarrow BG := EG/G$ which has the structure of a principal G -bundle.



Here we use a point p depending on U in an orbit to identify orbits $g \cdot p$ with g , and we want to take transverse slices to get local trivializations of $U \in BG$. It suffices to know where $\pi^{-1}(U) \cong U \times G$, and it suffices to consider $U \times \{e\}$. Moreover, $EG \rightarrow BG$ is a universal principal G -bundle in the sense that if $P \rightarrow X$ is a universal G -bundle, there is an $f : X \rightarrow BG$.

$$\begin{array}{ccc} P & \xrightarrow{\quad\quad\quad} & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & BG \end{array}$$

[Link to Diagram](#)

Here bundles will be classified by homotopy classes of f , so

$$\{\text{Principal } G\text{-bundles}/_X\} \cong [X, BG].$$

⚠ Warning 11.1.3

This only works for paracompact Hausdorff spaces! The line \mathbb{R} with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

12 | Monday, February 08

Last time: BG and EG . See Milnor and Stasheff.

Example 12.0.1(?): Let $G := \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$, then we can take

$$EG = \mathbb{R}^\infty := \left\{ (a_1, a_2, \dots) \mid a_i \in \mathbb{R}, a_i \gg 0 = 0, a_i \text{ not all zero} \right\}.$$

Then \mathbb{R}^\times acts on EG by scaling, and we can take the quotient $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$, where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in \mathbb{R}^\times$. This yields \mathbb{RP}^∞ as the quotient. You can check that EG is contractible: it suffices to show that $S^\infty := \left\{ \sum |a_i| = 1 \right\}$ is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$. Similarly $BC^\times \cong \mathbb{CP}^\infty$ with $EC^\times := \mathbb{C}^\infty \setminus \{0\}$.

Example 12.0.2(?): Consider $G = \mathrm{GL}_n(\mathbb{R})$. It turns out that $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$, which is the set of linear subspaces of \mathbb{R}^∞ of dimension d . This is spanned by d vectors $\{e_i\}$ in some large enough $\mathbb{R}^N \subseteq \mathbb{R}^\infty$, since we can take N to be the largest nonvanishing coordinate and include all of the vectors into \mathbb{R}^∞ by setting $a_{>N} = 0$. For any $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$, since \mathbb{R}^d has a standard basis, there is a natural GL_d torsor: the set of ordered bases of linear subspaces. So define

$$EG := \{\text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty)\},$$

then any $A \in \mathrm{GL}_d(\mathbb{R})$ acts on EG by sending $(L, \{e_i\}) \mapsto (L, \{Le_i\})$. We can identify EG as d -tuples of linearly independent elements of \mathbb{R}^∞ , and there is a map

$$\begin{aligned} EG &\rightarrow BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over $B\mathrm{GL}_d$:

$$\begin{array}{ccc} \mathcal{E}_L := L & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & B\mathrm{GL}_d \end{array}$$

So $\mathcal{E} \subseteq B\mathrm{GL}_d \times \mathbb{R}^\infty$, where we can define $\mathcal{E} := \{(L, p) \mid p \in L\}$. In this case, $EG = \mathrm{Frame}(\mathcal{E})$ is the frame bundle of this universal bundle. The same setup applies for $G := \mathrm{GL}_d(\mathbb{C})$, except we take $\mathrm{Gr}_d(\mathbb{C}^\infty)$.

Example 12.0.3(?): Consider $G = O_d$, the set of orthogonal transformations of \mathbb{R}^d with the standard bilinear form, and U_d the set of unitary such transformations. To be explicit:

$$U_d := \left\{ A \in \text{Mat}(d \times d, \mathbb{C}) \mid \langle Av, Av \rangle = \langle v, v \rangle \right\},$$

where

$$\langle [v_1, \dots, v_n], [v_1, \dots, v_n] \rangle = \sum |v_i|^2.$$

Alternatively, $A^t A = I$ for O_d and $\overline{A}^t A = I$ for U_d . In this case, $BO_d = \text{Gr}_d(\mathbb{R}^\infty)$ and $BU_d = \text{Gr}_d(\mathbb{C}^\infty)$, but we'll make the fibers smaller: set the fiber over L to be

$$(EO_d)_L := \{\text{orthogonal frames of } L\}$$

and similarly $(EU_d)_L$ the unitary frames of L . That there are related comes from the fact that GL_d retracts onto O_d using the Gram-Schmidt procedure.

Remark 12.0.4: Recall that there is a bijective correspondence

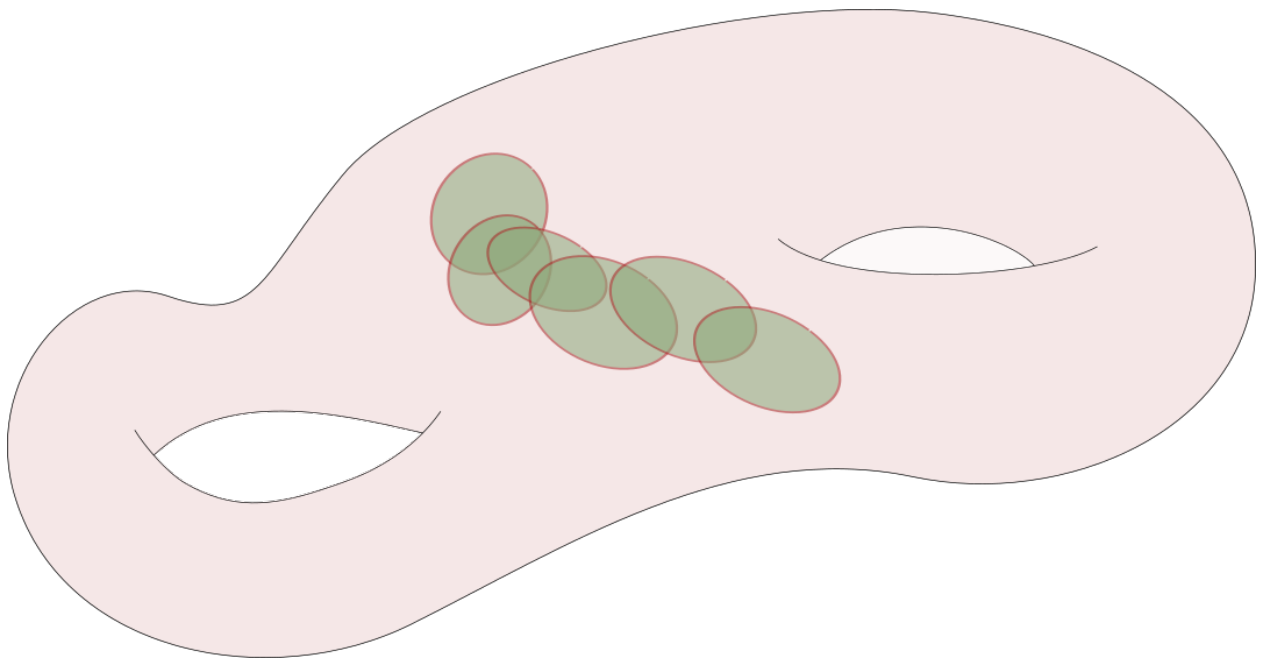
$$\left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \begin{array}{c} \text{Principal } \text{GL}_d\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Principal } O_d\text{-bundles} \\ \text{on } X \end{array} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles $\mathcal{E} \rightarrow X$ of rank d , and on the RHS we have bundles with a metric. In local trivializations $U \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, the metric is the standard one on \mathbb{R}^d . This is referred to as a **reduction of structure group**, i.e. a principal GL_d bundle admits possibly different trivializations for which the transition functions lie in the subgroup O_d .

Example 12.0.5(?): Given any trivial principal G -bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.



Remark 12.0.6: We want to compute $H^*(BU_d; \mathbb{Z})$. Why is this important? Given any complex vector bundle $\mathcal{E} \rightarrow X$ there is an associated principal U_d bundle by choosing a metric, so we get a homotopy class $[X, BU_d]$. Given any $f \in [X, BU_d]$ and any $\alpha \in H^k(BU_d; \mathbb{Z})$, we can take the pullback $f^*\alpha \in H^k(X; \mathbb{Z})$, which are **Chern classes**.

Exercise 12.0.7 (?)

Show that $H^*(BU_d; \mathbb{Z})$ stabilizes as $d \rightarrow \infty$ to an infinitely generated polynomial ring $\mathbb{Z}[c_1, c_2, \dots]$ with each c_i in cohomological degree $2i$, so $c_i \in H^{2i}(BU_d, \mathbb{Z})$.

Definition 12.0.8 (?)

There is a map $BU_{d-1} \rightarrow BU_d$, which we can identify as

$$\begin{aligned} \text{Gr}_{d-1}(C^\infty) &\rightarrow \text{Gr}_d(\mathbb{C}^\infty) \\ \{v_1, \dots, v_{d-1}\} &\mapsto \text{span}\{(1, 0, 0, \dots), sv_1, \dots, sv_{d-1}\}. \end{aligned}$$

This is defined by sending a basis where $s: \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$ is the map that shifts every coordinate to the right by one.

Question: does $\text{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?

This will yield a fiber sequence

$$S^{2d-1} \rightarrow BU_{d-1} \rightarrow BU_d$$

and using connectedness of the sphere and the LES in homotopy this will identify

$$H^*(BU_d) = H^*(BU_{d-1})[c_d] \quad \text{where } c_d \in H^{2d}(BU_d).$$

The **Chern class** of a vector bundle \mathcal{E} , denoted $c_k(\mathcal{E})$, will be defined as the pullback f^*c_k .

13 | Wednesday, February 10

Theorem 13.0.1 (?)

As $n \rightarrow \infty$, we have

$$H^*(BO_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \dots] \quad w_i \in H^i.$$

Definition 13.0.2 (?)

Given any principal O_n -bundle $P \rightarrow X$, there is an induced map $X \xrightarrow{f} BO_n$, so we can pull back the above generators to define the **Stiefel-Whitney classes** f^*w_i .

Remark 13.0.3: If $P := \text{OFrame}TX$, then f^*w_1 measures whether X has an orientation, i.e. $f^*w_1 = 0 \iff X$ can be oriented. We also have $f^*w_i(P) = w_i(\mathcal{E})$ where $P = \text{OFrame}(\mathcal{E})$. In general, we'll just write w_i for Stiefel-Whitney classes and c_i for Chern classes.

Definition 13.0.4 (Pontryagin Classes)

The **Pontryagin classes** of a real vector bundle \mathcal{E} are defined as

$$p_i(\mathcal{E}) = (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that the complexified bundle above is a complex vector bundle with the same transition functions as \mathcal{E} , but has a reduction of structure group from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{R})$.

Observation 13.0.5

\mathbb{RP}^∞ and \mathbb{CP}^∞ are examples of $K(\pi, n)$ spaces, which are the unique-up-to-homotopy spaces defined by

$$\pi_k K(\pi, n) = \begin{cases} \pi & k = n \\ 0 & \text{else.} \end{cases}$$

Theorem 13.0.6 (*Brown Representability*).

$$H^n(X; \pi) \cong [X, K(\pi, n)].$$

Example 13.0.7 (?):

$$[X, \mathbb{RP}^\infty] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$$

$$[X, \mathbb{CP}^\infty] \cong H^2(X; \mathbb{Z}).$$

Proposition 13.0.8 (?).

There is a correspondence

$$\{\text{Complex line bundles}\} \rightleftharpoons [X, \mathbb{CP}^\infty] = [X, BC^\times] \rightleftharpoons H^2(X; \mathbb{Z})$$

Importantly, note that for $X \in \mathbf{Mfd}_{\mathbb{C}}$, $H^2(X; \mathbb{Z})$ measures *smooth* complex line bundles and not holomorphic bundles.

Proof (?).

We'll take an alternate direct proof. Consider the exponential exact sequence on X :

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times.$$

Note that $\underline{\mathbb{Z}}$ consists of locally constant \mathbb{Z} -valued functions, \mathcal{O} consists of smooth functions, and \mathcal{O}^\times are ???.

Can't read screenshot! :(

This yields a LES in homology:

$$\begin{array}{ccccccc}
H^0(X; \mathbb{Z}) & \longrightarrow & H^0(X; \mathcal{O}) & \longrightarrow & H^0(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^1(X; \mathbb{Z}) & \longrightarrow & H^1(X; \mathcal{O}) & \longrightarrow & H^1(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^2(X; \mathbb{Z}) & \longrightarrow & H^2(X; \mathcal{O}) & \longrightarrow & H^2(X; \mathcal{O}^\times) & \longrightarrow &
\end{array}$$

[Link to Diagram](#)

Since \mathcal{O} admits a partition of unity, $H^{>0}(X; \mathcal{O}) = 0$ and all of the red terms vanish. For complex line bundles L , $H^1(X; \mathcal{O}^\times) \cong H^2(X; \mathbb{Z})$. Taking a local trivialization $L|_U \cong U \times \mathbb{C}$, we obtain transition functions

$$t_{UV} \in C^\infty(U \cap V, \mathrm{GL}_1(\mathbb{C}))$$

where we can identify $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. We then have

$$(t_{U_{ij}}) \in \prod_{i < j} \mathcal{O}^\times(U_i \cap U_j) = C^1(X; \mathcal{O}^\times).$$

Moreover,

$$(t_{U_{ij}} t_{U_{ik}}^{-1} t_{U_{jk}})_{i,j,k} = \partial(t_{U_{ij}})_{i,j} = 0,$$

since transition functions satisfy the cocycle condition. So in fact $(t_{U_{ij}}) \in Z^1(X; \mathcal{O}^\times) = \ker \partial^1$, and we can take its equivalence class $[(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times) = \ker \partial^1 / \mathrm{im} \partial^0$. Changing trivializations by some $s_i \in \prod_i \mathcal{O}^\times(U_i)$ yields a composition which is a different trivialization of the same bundle:

$$\begin{array}{ccccc}
L|_{U_i} & \xrightarrow{h_i} & U_i \times \mathbb{C} & \xrightarrow{\cdot s_i} & U_i \times \mathbb{C} \\
& & \searrow & & \nearrow \\
& & & &
\end{array}$$

[Link to Diagram](#)

So the $(t_{U_{ij}})$ change *exactly* by an $\partial^0(s_i)$. Thus the following map is well-defined:

$$L \mapsto [(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times).$$

There is another construction of the map

$$\begin{aligned}
\{L\} &\rightarrow H^2(X; \mathbb{Z}) \\
L &\mapsto c_1(L).
\end{aligned}$$

Take a smooth section of L and $s \in H^0(X; L)$ that intersects an \mathcal{O} -section of L transversely. Then

$$V(s) := \{x \in X \mid s(x) = 0\}$$

is a submanifold of real codimension 2 in X , and $c_1(L) = [V(s)] \in H^2(X; \mathbb{Z})$. ■

Theorem 13.0.9 (Splitting Principle for Complex Vector Bundles).

1. Suppose that $\mathcal{E} = \bigoplus_{i=1}^r L_i$ and let $c(\mathcal{E}) := \sum c_i(\mathcal{E})$. Then

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + c_i(L_i)).$$

2. Given any vector bundle $\mathcal{E} \rightarrow X$, there exists some Y and a map $Y \rightarrow X$ such that $f^* : H^k(X; \mathbb{Z}) \hookrightarrow H^k(Y; \mathbb{Z})$ is injective and $f^*\mathcal{E} = \bigoplus_{i=1}^r L_i$.

Slogan 13.0.10

To verify any identities on characteristic classes, it suffices to prove them in the case where \mathcal{E} splits into a direct sum of line bundles.

Example 13.0.11 (?) :


$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}).$$


To prove this, apply the splitting principle. Choose Y, Y' splitting $\mathcal{E}, \mathcal{F}'$ respectively, this produces a space Z and a map $f : Z \rightarrow X$ where both split. We can write

$$\begin{aligned} f^*\mathcal{E} &= \bigoplus L_i & c(f^*\mathcal{E}) &= \prod (1 + c_1(L_i)) \\ f^*\mathcal{F} &= \bigoplus M_j & c(f^*\mathcal{F}) &= \prod (1 + c_1(M_j)). \end{aligned}$$


We thus have

$$\begin{aligned} c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &= \prod (1 + c_1(L_i)) (1 + c_1(M_j)) \\ &= c(f^*\mathcal{E})c(f^*\mathcal{F}), \end{aligned}$$

and $f^*(c(\mathcal{E} \oplus \mathcal{F})) = f^*(c(\mathcal{E})c(\mathcal{F}))$. Since f^* is injective, this yields the desired identity. 

Example 13.0.12 (?) : We can compute $c(\text{Sym}^2 \mathcal{E})$, and really any tensorial combination involving \mathcal{E} , and it will always yield some formula in the $c_i(\mathcal{E})$. 

14 | Friday, February 12

Remark 14.0.1: Last time: the splitting principle. Suppose we have $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$ and let $x_i := c_1(L_i)$. Then $c_k(\mathcal{E})$ is the degree $2k$ part of $\prod_{i=1}^r (1 + x_i)$ where each x_i is in degree 2. This is equal to $e_k(x_1, \dots, x_r)$ where e_k is the k th elementary symmetric polynomial. 

Example 14.0.2(?): For example,

- $e_1 = x_1 + \cdots x_r$.
- $e_2 = x_1x_2 + x_1x_3 + \cdots = \sum_{i<j} x_ix_j$
- $e_3 = \sum_{i<j<k} x_ix_jx_k$, etc.

Remark 14.0.3: The theorem is that any symmetric polynomial is a polynomial in the e_i . For example, $p_2 = \sum x_i^2$ can be written as $e_1^2 - 2e_2$. Similarly, $p_3 = \sum x_i^3 = e_1^3 - 3e_1e_2 - 3e_3$. Note that the coefficients of these polynomials are important for representations of S_n , see *Schur polynomials*.

Remark 14.0.4: Due to the splitting principle, we can pretend that $x_i = c_i(L_i)$ exists even when \mathcal{E} doesn't split. If $\mathcal{E} \rightarrow X$, the individual symbols x_i don't exist, but we can write

$$x_1^3 + \cdots + x_r^3 = e_1^3 - 3e_1e_2 - 3e_3 := c_1(\mathcal{E})^3 + 3c_1(\mathcal{E})c_2(\mathcal{E}) + \cdots,$$

which is a well-defined element of $H^6(X; \mathbb{Z})$. So this polynomial defines a characteristic class of \mathcal{E} , and this can be done for any symmetric polynomial. We can change basis in the space of symmetric polynomials to now define different characteristic classes.

Definition 14.0.5 (Chern Character)

The **Chern character** is defined as

$$\begin{aligned} \text{ch}(\mathcal{E}) &:= \sum_{i=1}^r e^{x_i} \in H^*(X; \mathbb{Q}) \\ &:= \sum_{i=1}^r \sum_{k=0}^{\infty} \frac{x_i^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{p_k(x_1, \dots, x_r)}{k!} \\ &= \text{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 - c_2(\mathcal{E})}{2!} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) - 3c_3(\mathcal{E})}{3!} + \cdots \\ &\in H^0 + H^2 + H^4 + H^6 \\ &= \text{ch}_0(\mathcal{E}) + \text{ch}_1(\mathcal{E}) + \text{ch}_2(\mathcal{E}) + \cdots, \\ &\text{ch}_i(\mathcal{E}) \in H^{2i}(X; \mathbb{Q}). \end{aligned}$$

Definition 14.0.6 (Todd Class)

The **total Todd class**

$$\text{td}(\mathcal{E}) := \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}.$$

Note that

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \frac{x_i}{2} + \frac{x_i^2}{12} + \frac{x_i^4}{720} + \cdots = 1 + \frac{x_i}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x_i^{2i}.$$

where L'Hopital shows that the derivative at $x_i = 0$ exists, so it's analytic at zero and the expansion makes sense, and the B_i are Bernoulli numbers.

Remark 14.0.7 (Very important and useful!!): $\text{ch}(\mathcal{E} \oplus \mathcal{F}) = \text{ch}(\mathcal{E}) + \text{ch}(\mathcal{F})$ and $\text{ch}(\mathcal{E} \otimes \mathcal{F}) = \sum_{i,j} e^{x_i + y_j} = \text{ch}(\mathcal{E}) \text{ch}(\mathcal{F})$ using the fact that $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$. So ch is a “ring morphism” in the sense that it preserves multiplication \otimes and addition \oplus , making the Chern character even better than the total Chern class.

Definition 14.0.8 (Todd Class)

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$, then define the **Todd class** of X as $\text{td}_{\mathbb{C}}(X) := \text{td}(TX)$ where TX is viewed as a complex vector bundle. If $X \in \mathbf{Mfd}_{\mathbb{R}}$, define $\text{td}_{\mathbb{R}} = \text{td}(TX \otimes_{\mathbb{R}} \mathbb{C})$.

14.1 Section 5: Riemann-Roch and Generalizations

Remark 14.1.1: Let $X \in \mathbf{Top}$ and let \mathcal{F} be a sheaf of vector spaces. Suppose $h^i(X; \mathcal{F}) := \dim H^i(X; \mathcal{F}) < \infty$ for all i and is equal to 0 for $i \gg 0$.

Definition 14.1.2 (Euler Characteristic of a Sheaf)

The **Euler characteristic** of \mathcal{F} is defined as

$$\chi(X; \mathcal{F}) := \chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h_i(X; \mathcal{F}).$$

 **Warning 14.1.3**

This is not always well-defined!

Example 14.1.4(?): Let $X \in \mathbf{Mfd}_{\text{cpt}}$ and take $\mathcal{F} := \underline{\mathbb{R}}$, we then have

$$\chi(X; \underline{\mathbb{R}}) = h^0(X; \underline{\mathbb{R}}) - h^1(X; \underline{\mathbb{R}}) + \cdots = b_0 - b_1 + b_2 - \cdots := \chi_{\mathbf{Top}}(X).$$


Example 14.1.5(?): Let $X = \mathbb{C}$ and take $\mathcal{F} := \mathcal{O} := \mathcal{O}^{\text{holo}}$ the sheaf of holomorphic functions. We then have $h^{>0}(X; \mathcal{O}) = 0$, but $H^0(X; \mathcal{O})$ is the space of all holomorphic functions on \mathbb{C} , making $\dim_{\mathbb{C}} h^0(X; \mathcal{O})$ infinite.

Example 14.1.6(?): Take $X = \mathbb{P}^1$ with \mathcal{O} as above, $h^0(\mathbb{P}^1; \mathcal{O}) = 1$ since \mathbb{P}^1 is compact and the maximum modulus principle applies, so the only global holomorphic functions are constant. We can write $\mathbb{P}^1 = \mathbb{C}_1 \cup \mathbb{C}_2$ as a cover and $h^i(\mathbb{C}, \mathcal{O}) = 0$, so this is an acyclic cover and we can use it to compute $h^1(\mathbb{P}^1; \mathcal{O})$ using Čech cohomology. We have


- $C^0(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1) \oplus \mathcal{O}(\mathbb{C}_2)$
- $C^1(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1 \cap \mathbb{C}_2) = \mathcal{O}(\mathbb{C}^\times)$.
- The boundary map is given by

$$\begin{aligned} \partial_0 : C^0 &\rightarrow C^1 \\ (f(z), g(z)) &\mapsto g(1/z) - f(z) \end{aligned}$$

and there are no triple intersections.

Is every holomorphic function on \mathbb{C}^\times of the form $g(1/z) - f(z)$ with f, g holomorphic on \mathbb{C} . The answer is yes, by Laurent expansion, and thus $h^1 = 0$. We can thus compute $\chi(\mathbb{P}^1; \mathcal{O}) = 1 - 0 = 1$. 

15 | Monday, February 15


Remark 15.0.1: Last time: we saw that $\chi(\mathbb{P}^1, \mathcal{O}) = 1$, and we'd like to generalize to holomorphic line bundles on a Riemann surface. This will be the main ingredient for Riemann-Roch. 

Theorem 15.0.2(?).

Let $X \in \mathbf{Mfd}_{\mathbb{C}}$ be compact and let \mathcal{F} be a holomorphic vector bundle on X .^a Then χ is well-defined and

$$h^{>\dim_{\mathbb{C}} X}(X; \mathcal{F}) = 0.$$

^aOr more generally a finitely-generated \mathcal{O} -module, i.e. a coherent sheaf.

Remark 15.0.3: The locally constant sheaf $\underline{\mathbb{C}}$ is not an \mathcal{O} -module, i.e. $\underline{\mathbb{C}}(U) \notin \mathcal{O}(U)\text{-Mod}$. In fact, $h^{2i}(X, \underline{\mathbb{C}}) = \mathbb{C}$ for all i . 

Proof (?).

We'll resolve \mathcal{F} as a sheaf by first mapping to its smooth sections and continuing in the following way:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \rightarrow \dots,$$

where $\bar{\partial}f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$. Suppose we have a holomorphic trivialization of $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ and we have sections $(s_1, \dots, s_r) \in C^\infty \mathcal{F}(U)$, which are smooth functions on U . In local coordinates we have

$$\bar{\partial}s := (\bar{\partial}s_1, \dots, \bar{\partial}s_r),$$

but is this well-defined globally? Given a different trivialization over $V \subseteq X$, the s_i are related by transition functions, so the new sections are $t_{UV}(s_1, \dots, s_r)$ where $t_{UV} : U \cap V \rightarrow \mathrm{GL}_r(\mathbb{C})$.

Since t_{UV} are holomorphic, we have

$$\bar{\partial}(t_{UV}(s_1, \dots, s_r)) = t_{UV}\bar{\partial}(s_1, \dots, s_r).$$

This makes $\bar{\partial} : C^\infty \mathcal{F} \rightarrow F \otimes A^{0,1}$ a well-defined (but not \mathcal{O} -linear) map. We can thus continue this resolution using the Leibniz rule:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \xrightarrow{\bar{\partial}} \dots F \otimes A^{0,2} \xrightarrow{\bar{\partial}} \dots,$$

which is an exact sequence of sheaves since $(A^{0,\cdot}, \bar{\partial})$ is exact.

Why? Split into line bundles?

We can identify $C^\infty \mathcal{F} = \mathcal{F} \otimes A^{0,0}$, and $\mathcal{F} \otimes A^{0,q}$ is a smooth vector bundle on X . Using partitions of unity, we have that $\mathcal{F} \otimes A^{0,q}$ is acyclic, so its higher cohomology vanishes, and

$$H^i(X; \mathcal{F}) \cong \frac{\ker(\bar{\partial} : \mathcal{F} \otimes A^{0,i} \rightarrow \mathcal{F} \otimes A^{0,i+1})}{\operatorname{im}(\bar{\partial} : \mathcal{F} \otimes A^{0,i-1} \rightarrow \mathcal{F} \otimes A^{0,i})}.$$

However, we know that $A^{0,p} = 0$ for all $p > n := \dim_{\mathbb{C}} X$, since any wedge of $p > n$ forms necessarily vanishes since there are only n complex coordinates. ■

⚠ Warning 15.0.4

This only applies to holomorphic vector bundles or \mathcal{O} -modules! ✍

15.1 Riemann-Roch

Theorem 15.1.1 (*Riemann-Roch*).

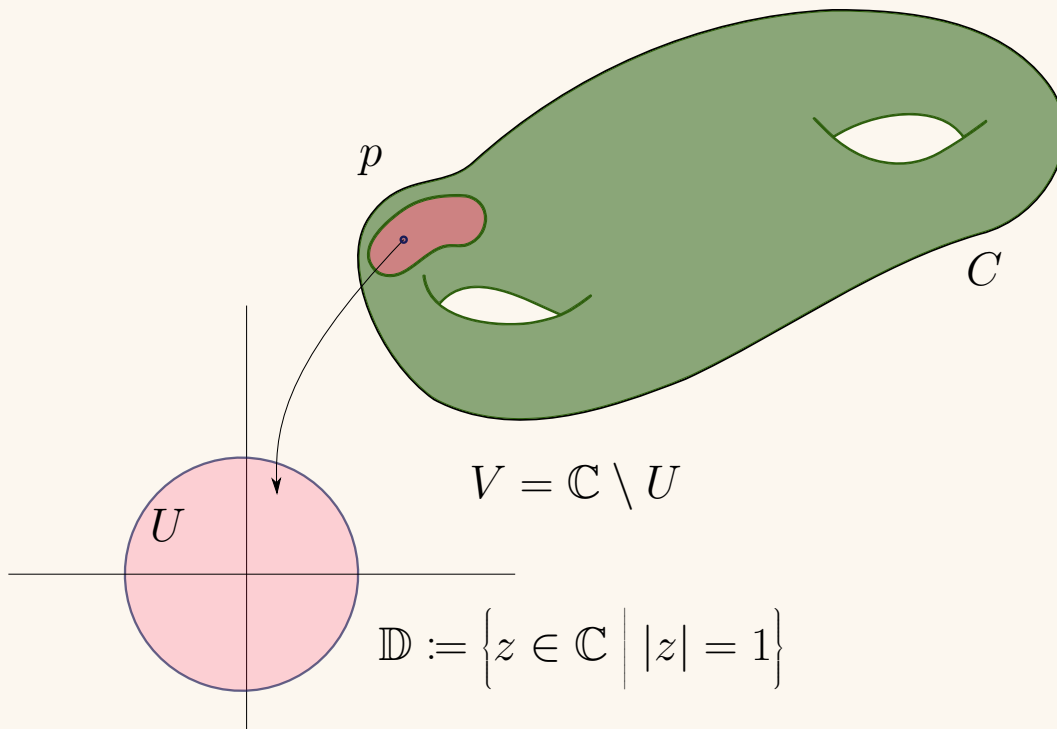
Let C be a compact connected Riemann surface, i.e. $X \in \mathbf{Mfd}_{\mathbb{C}}$ with $\dim_{\mathbb{C}}(X) = 1$, and let $\mathcal{L} \rightarrow C$ be a holomorphic line bundle. Then

$$\chi(C, \mathcal{L}) = \deg(L) + (1 - g) \quad \text{where } \int_C c_1(\mathcal{L})$$

and g is the genus of C .

Proof (?).

We'll introduce the notion of a “point bundle”, which are particularly nice line bundles, denoted $\mathcal{O}(p)$ for $p \in \mathbb{C}$.



Taking \mathbb{D} to be a disc of radius $1/2$ and V to be its complement, we have $t_{uv}(z) = z^{-1} \in \mathcal{O}^*(U \cap V)$. We can take a holomorphic section $s_p \in H^0(C, \mathcal{O}(p))$, where $s_p|_U = z$ and $s_p|_V = 1$. Then $t_{uv}(s_p|_U) = s_p|_V$ on the overlaps. We have a function which precisely vanishes to first order at p . Recall that $c_1(\mathcal{O}(p))$ is represented by $[V(s)] = [p]$, and moreover $\int_C c_1(\mathcal{O}(p)) = 1$.

We now want to generalize this to a **divisor**: a formal \mathbb{Z} -linear combination of points.

Example 15.1.2(?): Take $p, q, r \in C$, then a divisor can be defined as something like $D := 2[p] - [q] + 3[r]$.

Define $\mathcal{O}(D) := \bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}$ for any $D = \sum n_i [p_i]$. Here tensoring by negatives means taking duals, i.e. $\mathcal{O}(-[p]) := \mathcal{O}^{\otimes -1} := \mathcal{O}(p)^\vee$, the line bundle with inverted transition functions. $\mathcal{O}(D)$ has a meromorphic section given by

$$s_D := \prod s_{p_i}^{n_i} \in \text{Mero}(C, \mathcal{O}(D))$$

where we take the sections coming from point bundles. We can compute

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i := \deg(D).$$

Example 15.1.3(?):

$$\deg(2[p] - [q] + 3[r]) = 4.$$

Remark 15.1.4: Assume our line bundle L is $\mathcal{O}(D)$, we'll prove Riemann-Roch in this case by induction on $\sum |n_i|$. The base case is \mathcal{O} , which corresponds to taking an empty divisor. Then either

- Take $D = D_0 + [p]$ with $\deg(D_0) < \sum |n_i|$ (for which we need some positive coefficient),
or
- Take $D_0 = D + [p]$.

Claim: There is an exact sequence

$$0 \rightarrow \mathcal{O}(D_0) \rightarrow \mathcal{O}(D) \rightarrow \mathbb{C}_p \rightarrow 0$$

$$s \in \mathcal{O}(D_0)(U) \mapsto s \cdot s_p \in \mathcal{O}(D_0 + [p])(U),$$

where the last term is the skyscraper sheaf at p .

Proof (?).

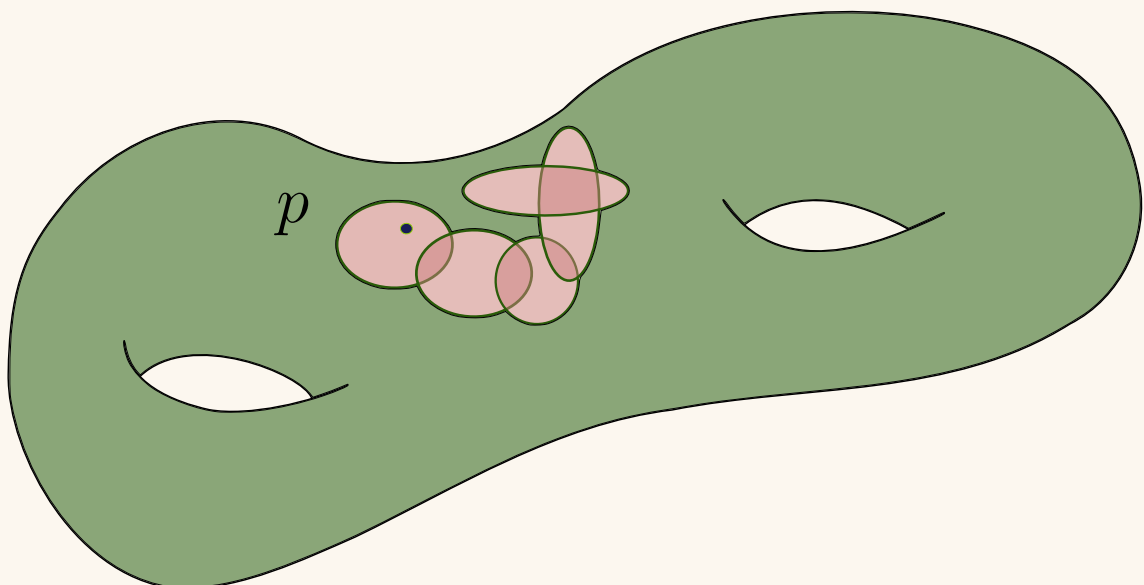
The given map is \mathcal{O} -linear and injective, since $s_p \neq 0$ and $ss_p = 0$ forces $s = 0$. Recall that we looked at $\mathcal{O} \xrightarrow{z} \mathcal{O}$ on \mathbb{C} , and this section only vanishes at p (and to first order). The same situation is happening here. ■

Thus there is a LES

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & & & \searrow & \\
 \rightarrow & H^0(\mathcal{O}(D_0)) & \longrightarrow & H^0(\mathcal{O}(D)) & \longrightarrow & H^0(\mathcal{O}(\mathbb{C}_p)) & \rightarrow \\
 & & & & & \searrow & \\
 \rightarrow & H^1(\mathcal{O}(D_0)) & \longrightarrow & H^1(\mathcal{O}(D)) & \longrightarrow & H^1(\mathcal{O}(\mathbb{C}_p)) = 0 & \rightarrow \\
 & & & & & \searrow & \\
 & & & & & \rightarrow & 0
 \end{array}$$

[Link to Diagram](#)

We also have $h^1(\mathbb{C}_p) = 0$ by taking a sufficiently fine open cover where p is only in one open set. So just checking Čech cocycles yields $C_U^1(C, \mathbb{C}_p) := \prod_{i < j} \mathbb{C}_p(U_i \cap U_j) = 0$ since p is in no intersection.



X

p

We obtain $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D_0)) + 1$, using that it is additive in SESs

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0 \implies \chi(\mathcal{E}_2) = \chi(\mathcal{E}_\infty) + \chi(\mathcal{E}_3)$$


and thus

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i = \deg(D) = \deg D_0 + 1.$$

The last step is to show that $\chi(C, \mathcal{O}) = 1 - g$, so just define g so that this is true! ■

Remark 15.1.5: Why is every $L \cong \mathcal{O}(D)$ for some D ? Easy to see if L has meromorphic sections: if s is a meromorphic section of L , then the following works:

$$D = \text{Div}(s) = \sum_p \text{Ord}_p(s)[p].$$

Then $\mathcal{O} \cong L \otimes \mathcal{O}(-D)$ has a meromorphic section ss_{-D} , a global nonvanishing section with $\text{Div}(ss_{-D}) = \emptyset$. Proving that every holomorphic line bundle has a meromorphic section is hard! 

16 | Friday, February 19

16.1 Applications of Riemann-Roch

Definition 16.1.1 (Curves)

A **curve** is a compact complex manifold of complex dimension 1.

Example 16.1.2(?): Let C be a curve, then Ω_C^1 is the sheaf of holomorphic 1-forms, and $\Omega_C^{>1} = 0$. We also have the sheaves $A^{1,0}, A^{0,1}, A^{1,1}$, the sheaves of smooth (p, q) -forms. Here the only nonzero combinations are $(0, 0), (0, 1), (1, 0), (1, 1)$ by dimensional considerations. Let L be a holomorphic line bundle on C , then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg(L) + 1 - g.$$

Remark 16.1.3: In general it can be hard to compute $h^1(L)$, since this is sheaf cohomology (sections over double overlaps, cocycle conditions, etc). On the other hand, h^0 is easy to understand, since $h^0(\Omega_C^1)$ is the dimension of the global holomorphic sections $H^0(C, L) = L(C)$. A key tool here is the following:

Proposition 16.1.4 (Serre Duality).

$$H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega_C^1)^\vee,$$

noting that these are both global sections of a line bundle.

Proof (?).

Recall that we had a resolution of the sheaf L given by smooth vector bundles:

$$0 \rightarrow L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar{\partial}} L \otimes A^{0,1} \xrightarrow{\bar{\partial}} 0.$$

So we know that $H^1(C, L) = H^0(L \otimes A^{0,1}) / \bar{\partial} H^0(L \otimes A^{0,0})$. Choose a Hermitian metric h on L , i.e. a map $h : L \otimes \bar{L} \rightarrow \mathcal{O}$. On fibers, we have $h_p : L_p \otimes \bar{L}_p \rightarrow \mathbb{C}$. We'll also choose a metric on C , say g . Since C is a Riemann surface, we have an associated volume form ν on C (essentially the determinant), so we can define a pairing between sections of $L \otimes A^{0,0}$:

$$\langle s, t \rangle := \int_C h(s, \bar{t}) d\nu.$$

Note that $\langle s, s \rangle = \int_C h(s, \bar{s}) d\nu \geq 0$ since $h(s, \bar{s})(p) = 0 \iff s_p = 0$, and moreover this integral is zero if and only if $s = 0$. So we have an inner product on $H^0(L \otimes A^{0,0})$. We can also define a pairing on sections of $L \otimes A^{0,1}$, say

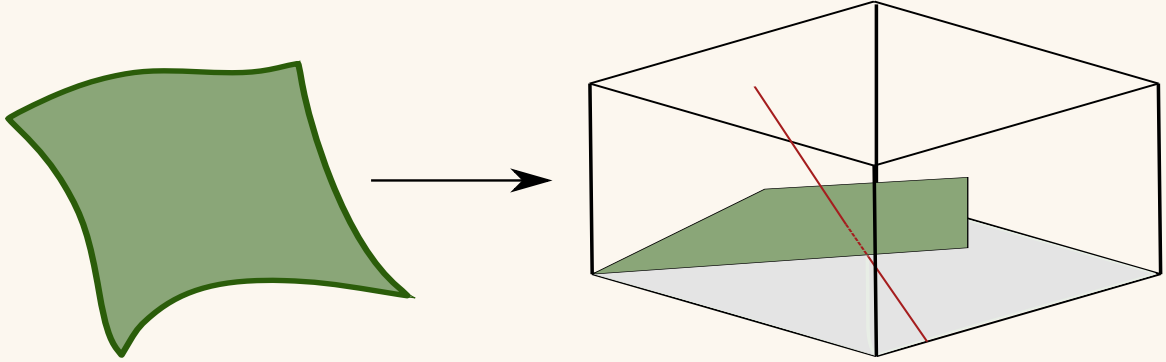
$$\langle s \otimes \alpha, t \otimes \beta \rangle = \int_C h(s, \bar{t}) \alpha \wedge \bar{\beta}.$$

Note that h is a smooth function and $\alpha \wedge \bar{\beta}$ is a $(1,1)$ -form. Moreover, this is positive and nondegenerate. We want to understand the cokernel of the linear map

$$H^0(L \otimes A^{0,0}) \xrightarrow{\bar{\partial}} H^0(L \otimes A^{0,1}).$$

To compute $\text{coker}(\bar{\partial})$, we can look at the kernel of the adjoint, and it suffices to find the orthogonal complement of $\text{im}(\bar{\partial})$, i.e.

$$\text{coker}(\bar{\partial}) = \left\{ t \in H^0(L \otimes A^{0,1}) \mid \langle \bar{\partial}s, t \rangle = 0 \forall s \right\}.$$



So we want to understand sections $t \in H^0(L \otimes A^{0,1})$ such that

$$\int_C (\bar{\partial}s) \bar{t} = 0 \quad \forall s \in H^0(L \otimes A^{0,0}),$$

where $\partial C = \emptyset$. We'll basically want to do integration by parts on this. Note that $h(s, t) = hst$ here where we view h as a certain section. Note that $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$, so we can replace ∂ with $d = \bar{\partial} + \partial$ and apply Stokes' theorem:

$$\begin{aligned} \int_C sd(h\bar{t}) &= 0 & \forall s \in H^0(L \otimes A^{0,0}) \\ 0 &= \int_C s\bar{\partial}(h\bar{t}) \\ &= \int_C s \frac{\bar{\partial}(h\bar{t})}{d\nu} d\nu \\ &= \left\langle s, \frac{\bar{\partial}(h\bar{t})}{d\nu} \right\rangle \end{aligned}$$

where $h \in C^\infty(L^{-1} \otimes \bar{L}^{-1})$ and $h\bar{t} \in C^\infty(L^{-1} \otimes A^{1,0})$. But the right-hand side is in $H^0(L \otimes A^{0,0})$ and by nondegeneracy we can conclude

$$\frac{\bar{\partial}(h\bar{t})}{d\nu} = 0 \iff \bar{\partial}(h\bar{t}) = 0.$$

We thus have $h\bar{t} \in H^0(L^{-1} \otimes A^{1,0})$ which is a holomorphic line bundle tensored with $A^{0,0}$. Thus $\text{coker}(\bar{\partial}) \cong_h H^0(L^{-1} \otimes \Omega^1)$. ■

Remark 16.1.5: We showed $\langle \bar{\partial}s, t \rangle = \langle s, Y(t) \rangle$ where Y is the adjoint given above. Then the kernel of Y wound up being where $\bar{\partial}$ vanishes, i.e. holomorphic sections of a separate bundle. Here we had

- $t \in H^0(L \otimes A^{0,1})$
- $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$
- $h \in H^0(L^{-1} \otimes \overline{L^{-1}})$



ToDos

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