Title

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1.1 Chapter 1

Let k be a field, not necessarily algebraically closed.

Definition 1.1.1 (Algebraic Function Field)**.**
An one variable **algebraic function field** F/K is a field extension F of K which factors as

$$\begin{tikzcd} F \arrow[dr, "\text{algebraic}"] & \\ & k(x) \arrow[dl, "\text{transcendental}"] \\ k & \end{tikzcd}$$

where $x \in \bar{k}$ is some element that is not algebraic over k.

Definition 1.1.2 (Field of Constants)**.**
The subfield

$$\tilde{k} := \left\{ z \in F \cap K^{\mathrm{alg}} \right\} \leq F,$$

consisting of elements that are algebraic over F is denoted the **field of constants**.

Definition 1.1.3 (Algebraically Closed)**.**
If $\tilde{k} = k$, we say that k is **algebraically closed** in F.

Definition 1.1.4 (Rational Function Field)**.**
An extension F/k is **rational** iff $F = k(y)$ for some $y \in k^{\mathrm{transc}}$ which is transcendental over k.

Definition 1.1.5 (Valuation Ring)**.**
A ring $\mathcal{O} \subseteq F$ is a **valuation ring** for F iff $k \subset \mathcal{O} \subseteq F$ and $z \in F \implies z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$.

Definition 1.1.6 (Discrete Valuation Ring).
A ring local R (thus with a unique maximal ideal) which is a PID but not a field is a **discrete valuation ring**.

Definition 1.1.7 (Place).
A **place** of a function field F/K is the maximal ideal of a valuation ring of F/K.

Definition 1.1.8 (Discrete Valuation).
A **discrete valuation** of F/k is a function

$$v : F \to \mathbb{Z} \cup \{\infty\}$$

that is

1. Nondegenerate: $v(x) = \infty$ iff $x = 0$.
2. Multiplicative: $v(xy) = v(x) + v(y)$.
3. Ultrametric triangle inequality: $v(x + y) \geq \min(v(x), v(y))$.
4. Fiber over one: there exist a $z \in F$ with $v(z) = 1$.
5. $v|_k = 0$.

Definition 1.1.9 (Rational Place).
A place of degree one is said to be a **rational place**.

Definition 1.1.10 (Valuation Ring of a Place).
The **valuation ring of a place** is defined by

$$\mathcal{O}_p := \left\{ z \in F \;\middle|\; z^{-1} \notin P \right\}.$$

Definition 1.1.11 (Degree of a Place).
The **degree** of a place P is defined by

$$\deg(P) := [F_p : k],$$

where $F_p = \mathcal{O}_P/P$.

Definition 1.1.12 (Discrete Valuation of a Place).
To any place P we associate the function

$$v_p : F \to \mathbb{Z} \cup \{\infty\}$$

defined by choosing any prime $t \in P$, writing any $x \in F$ as $x = t^n u$ with $u \in \mathcal{O}_P^\times$, and setting

$$v_p(x) = \begin{cases} n & \text{if } x = t^n u \\ 0 & x = \infty. \end{cases}$$

Note: from now on we assume $\tilde{K} = K$

Definition 1.1.13 (Divisor)**.**
The **divisor group** of F/K is the free abelian group on the set of places of F/K, i.e. a formal sum

$$D = \sum_{\text{Places}p} b_p P \qquad n_p \in \mathbb{Z}m$$

where cofinitely many n_p are zero.

Definition 1.1.14 (Degree of a Divisor)**.**

Definition 1.1.15 (Principle Divisors)**.**
The set of divisors

$$\mathrm{Princ}(F) \coloneqq \left\{ (x) \;\middle|\; 0 \neq x \in F \right\}.$$

Definition 1.1.16 (Divisor Class Group)**.**
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Definition 1.1.17 (Riemann-Roch Space)**.**
For a divisor $A \in \mathrm{Div}(F)$, the **Riemann-Roch** space is defined as

$$\mathcal{L}(A) \coloneqq \left\{ x \in F \;\middle|\; (x) \geq -A \right\} \cup \{0\}.$$