# Title

D. Zack Garza

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Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \,\forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety $X$ as $A(X) := k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on $X$.

Recall that a *topology* on $X$ can be defined as a collection of "closed" subsets of $X$ that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

**Definition 1.0.1** (Zariski Topology)**.**
Let $X$ be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in $X$ with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\displaystyle\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i\right)$.

**Remark 1.**
There are few closed sets, so this is a "weak" topology.

**Example 1.1.**
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A := \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But $A$ is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

**Example 1.2.**
Let $f : \mathbb{A}^1/k \longrightarrow \mathbb{A}^1/k$ be any injective map. Then $f$ is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

**Example 1.3.**
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\bigcup_{i=1}^{n} U_i \times V_i$ with $U_i, V_i$ open in $X, Y$ respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta := V(x-y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

**Example 1.4.**
Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 := V(x_1)$ and $X_2 := V(x_2)$. The Zariski topology can detect these decompositions.

**Definition 1.0.2** (Irreducibility and Connectedness)**.**
Let $X$ be a topological space.

a. $X$ is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, $X$ is said to be *irreducible*.
b. $X$ is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, $X$ is said to be *connected*.

**Example 1.5.**
$V(x_1x_2)$ is reducible but connected.

**Remark 2.**
$\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

**Proposition 1.1** ***(?).***
Let $X$ be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

*Proof .*
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1\emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \longrightarrow A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

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**Proposition 1.2*(?).***
An affine variety $X$ is irreducible $\iff$ $A(X)))$ is an integral domain.

*Proof .*
$\implies$ : By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\impliedby$ : Suppose $X$ is reducible with $X = X_1 \cup X_2$ with $X_i$ proper and closed. Define $J_i \coloneqq I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so $f_i$ vanishes on $X_i$. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain.

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