

Moduli Spaces

D. Zack Garza

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1 Thursday January 9th

1.1 Representability

Last time: Fix an S -scheme, i.e. a scheme over S .

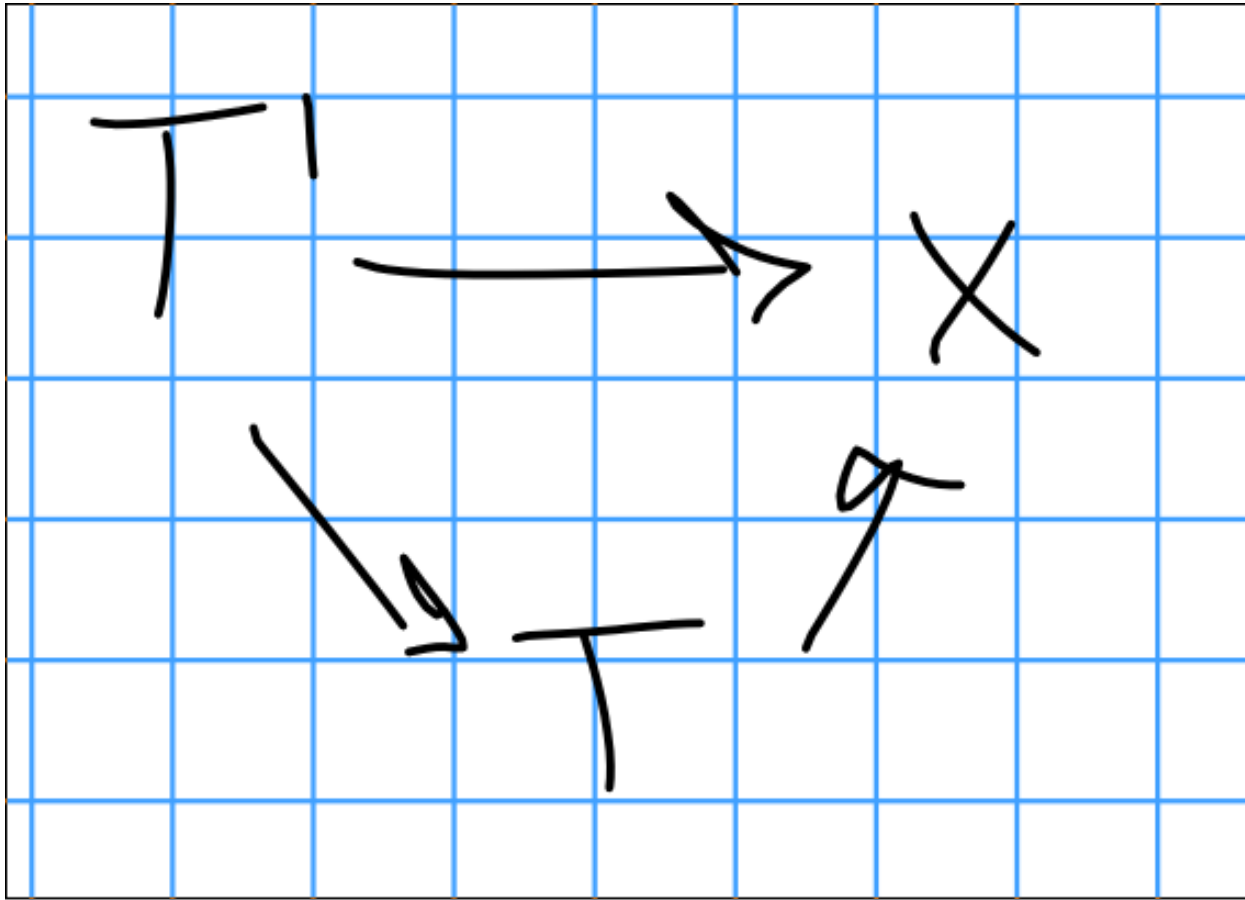
Then there is a map

$$\begin{aligned}\mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \mathrm{hom}_{\mathrm{Sch}/S}(T, x).\end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned}h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ T &\mapsto x \longrightarrow \text{triangles}\end{aligned}$$

of the form



Theorem 1.1 (Yoneda).

$$\mathrm{hom}_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 1.2.

$$\mathrm{hom}_{\mathrm{Sch}/S}(x, y) \cong \mathrm{hom}_{\mathrm{Fun}}(h_x, h_y).$$

Definition 1.1.

A **moduli functor** is a map

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

$$F(x) = \text{"Families of something over } x\text{"}$$

$$F(f) = \text{"Pullback"}.$$

Definition 1.2.

A **moduli space** for that “something” appearing above is an $M \in \text{Obj}(\text{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \text{Spec}(k)$.

h_m is the functor of points over M

Remark (1) $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{“families over Spec}k\text{”} = F(\text{Spec}k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M , and $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{\text{Univ}}$ is the universal family

Every family is uniquely the pullback of ξ_{Univ} . This makes it much like a classifying space.

For $T \in \text{Sch}/S$,

$$h_M \xrightarrow{\cong} F$$

$$f \in h_M(T) \xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\text{Univ}}).$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\text{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\xi_M \qquad \qquad \xi_{M'}$$

$$M \xrightarrow{f} M'$$

$$M' \xrightarrow{g} M$$

$$\xi_{M'} \qquad \qquad \xi_M$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.

A length 2 subscheme of \mathbb{A}_k^1 then $F(S) = \{V(x^2 + bx + c)\} \subset \mathbb{A}'_S$ where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

Then F is representable by $\mathbb{A}_k^2(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in \mathbb{A}'_S which are length 2 for every geometric fiber.

$F''(S)$ is the set of subschemes of \mathbb{A}_S^1 which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\text{Spec}k[\varepsilon]$.

F	✓	✗	✗
F'	✓	✗	✗
F''	✓	✓	✓

$$\begin{array}{ccc}
 \text{Spec}k & \xleftarrow{i} & \text{Spec}k[\varepsilon] \\
 & & \searrow \\
 F(\text{Spec}k[\varepsilon]) & \xrightarrow{F(i)} & F(\text{Spec}k) \xrightarrow{\quad} = F'(\text{Spec}k) \xrightarrow{\quad} = F''(\text{Spec}k)
 \end{array}$$

\subset
 \in

$$T_p F','' \quad P = V(x(x-1))$$

We think of $T_p F',''$ as the tangent space at p .

If F is representable, then it is actually the Zariski tangent space.

$$M(\text{Spec}k[\varepsilon]) \longrightarrow M(\text{Spec}k)$$

$$\subset \quad \subset$$

$$T_p M \longrightarrow p$$

$$\begin{array}{ccc}
 & \text{Spec} k & \\
 \swarrow & & \searrow \text{?} \\
 \text{Spec} k[\varepsilon] & \xrightarrow{\quad} & \text{Spec} \mathcal{O}_{M,p} \subset M
 \end{array}$$

$$\begin{array}{ccc}
 & & k \\
 & \nearrow & \uparrow \\
 \mathcal{O}_{M,p} & \xrightarrow{\quad} & k[\varepsilon]
 \end{array}$$

$$\begin{array}{ccc}
 m_p & & (\varepsilon) \\
 \\
 m_p^2 & & 0
 \end{array}$$

Moreover, $T_p M = (m_p/m_p^2)^\vee$, and in particular this is a k -vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
 \lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
 \varepsilon &\mapsto \lambda \varepsilon
 \end{aligned}$$

$$\lambda^* : \text{Spec}(k[\varepsilon]) \longrightarrow \text{Spec}(k[\varepsilon])$$

$$\begin{aligned}
 \lambda : M(\text{Spec}(k[\varepsilon])) &\longrightarrow M(\text{Spec}(k[\varepsilon])) \\
 &\supset T_p M \longrightarrow T_p M \subset .
 \end{aligned}$$

Conclusion: If F is representable, for each $p \in F(\text{Spec} k)$ there exists a unique point of $T_p F$ that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$\begin{array}{ccc}
 F \times_G F' & \xrightarrow{\quad} & F' \\
 \downarrow & & \downarrow \\
 F & \xrightarrow{\quad} & G
 \end{array}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\text{Sch}/T) &\longrightarrow \text{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}_{\mathbb{Z}}^n$, i.e. “rank 1 quotient of an $n + 1$ dimensional free module”.

Claim: $\mathbb{P}_{\mathbb{Z}}^n$ represents the following functor

$$\begin{aligned} F : \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ F(S) &= \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 / \sim. \end{aligned}$$

where \sim identifies diagrams of the following form:

$$\begin{array}{ccccc} \mathcal{O}_S^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow = & & \downarrow \cong & & \\ \mathcal{O}_S^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and $F(f)$ is given by pullbacks.

Remark \mathbb{P}_S^n represents the following functor:

$$\begin{aligned} F_S : (\text{Sch}/S)^{\text{op}} &\longrightarrow \text{Set} \\ F_S(T) &= \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of claim).

Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n + 1$ sections s_1, \dots, s_n of L , where surjectivity ensures that they are not the zero section.

So

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.

There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a

subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

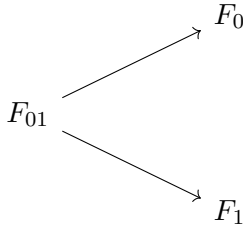
Claim 2: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \uparrow & & \uparrow \\ F_{ij} & \longrightarrow & F_j \end{array}$$

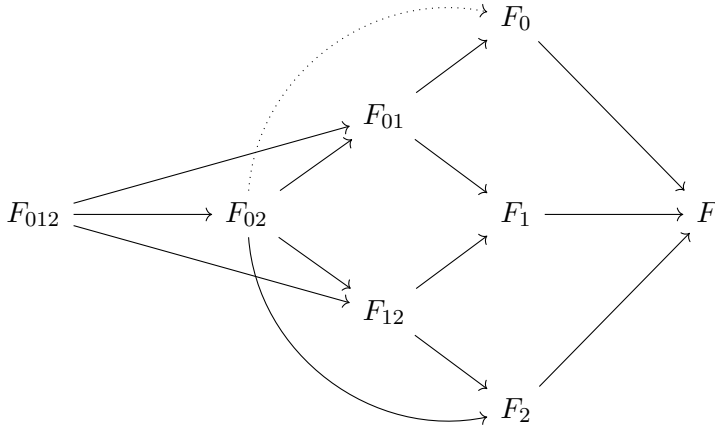
and each $F_{ij} \rightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example 1.2.

For $n = 1$, we can glue along open subschemes



For $n = 2$, we get overlaps of the following form:



This claim implies that we can glue together F_i to get a scheme M . We want to show that M represents F . $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S .

RHS to LHS comes from gluing.

Proof of (Claim 2)

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i .

So specifying $F_i(S)$ is equivalent to specifying $n - 1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by \mathbb{A}^n .

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots \widehat{f_j} \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. $\text{Spec} k[x_1, \dots, \widehat{x_i}, \dots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \dots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.