Spectral Sequence Review

Roughly speaking, a spectral sequence is a system for keeping tracking of collections of exact sequences with maps between them.

Recall the Snake Lemma: given A, B, C chain complexes fitting into a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

there is a canonical long exact sequence in homology

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to \cdots$$

where δ is the "connecting homomorphism".

Now specialize to the case where A_* is a chain complex, $B_* \subset A_*$ is a subcomplex, and consider the quotient A_*/B_*. We have a short exact sequence

$$0 \to B_* \xrightarrow{i} A_* \xrightarrow{p} A_*/B_* \to 0$$

Applying the snake lemma yields the long exact sequence in homology

$$\cdots \to H_n(B_*) \xrightarrow{i_*} H_n(A_*) \xrightarrow{p_*} H_n(A_*/B_*) \xrightarrow{\delta} H_{n-1}(B_*) \to \cdots$$

where δ is defined in the following way:

Given an arbitrary class $\alpha \in H_n(A_*/B_*)$, pick a representative $x \in A_*$ so that $\alpha = [x]$. Since
$\partial x \in B_*$, we can define
$\partial(\alpha) = \partial([x]) := [\partial x] \in H_{n-1}(B)$.

Supposing that the computation of the homologies for the subcomplex B_* and the quotient complex A_*/B_* are tractable, we can break this long exact sequence up into a collection of short exact sequences

$$0 \to \operatorname{coker} \delta \to H_i(A_*) \to \ker \delta \to 0$$

This yields the following procedure for computing $H_i(A_*)$:

1. Compute $H_i(B_*)$ and $H_i(A_*/B_*)$
2. Look at the two term chain complex $H_i(A_*/B_*) \xrightarrow{\delta} H_{i-1}(B_*)$
    1. Take its homology, yielding $G_1 H_i$ and $G_2 H_i$

3. Solve the extension problem for the short exact sequence $0 \to G_0 H_i \to H_i(A_*) \to G_1 H_i \to 0$

Filtrations

A *filtered R-module* is an R-module A with a sequence of submodules $\{A_i\}_{i \in \mathbb{Z}}$ such that $A_i \subset A_{i+1}$ and $\bigcup_{\mathbb{Z}} A_i = A$. Due to onerous index juggling, we write $A_i = F_i A$.

A good example of this is a CW-complex X, where $F_i X$ is the i-skeleton of X.

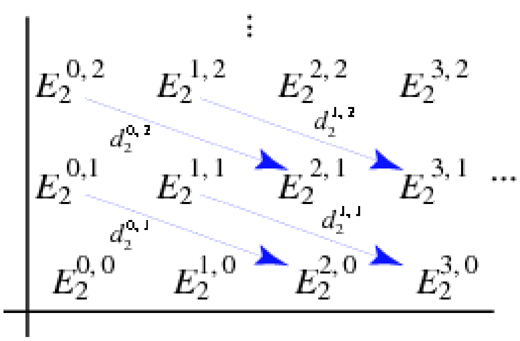
Given such a filtration, we can define an *associated graded module* B where $B_i = A_i / A_{i-1}$. This can yield a short exact sequence
$0 \to A_{i-1} \to A_i \to B_i \to 0$

A *filtered chain complex* is a chain complex (C_*, ∂) along with a filtration on each n-chain, $\{F_i C_n\}_{i \in \mathbb{Z}}$, such that $\partial(F_i C_n) \subseteq F_i C_{n-1}$ (i.e. the differential preserves the filtration).

Possible example: Compute Serre spectral sequences with $\mathbb{F}_p$ coefficients.

Example

The most basic example is a spectral sequence is $E^r_{p,q}$, where r denotes the page of the spectral sequence and the $E_{p,q}$ is a bigraded collection of abelian groups. Furthermore, we can take a "first quadrant" sequence, where only the $p > 0, q > 0$ terms are nontrivial. The differentials are then defined on any given page as a "shift map" that translates $p + r$ horizontal indices and $q - (r-1)$ vertical indices (direction depends on indexing vs. "coindexing"). Here is an example of an $r = 2$ page:



In this case, $\lim_{r \to \infty} E^r_{p,q}$ stabilizes for any given (p, q) term, so we define it as $E^\infty_{p,q}$.

Common Types

- Serre

    - Cohomology groups of spaces in a fibration
- Leray-Serre
    - “Cohomology” of complexes of sheaves
    - Special case of Grothendieck
- Grothendieck
    - The resulting derived functor from a composition of two known derived functors
- Adams
    - Higher homotopy groups of spheres