Problem Set 8

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1 Problem 1

1.1 Part a

Define a map

$$\begin{aligned} \phi_{\text{ev}} : \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) &\to A \\ (f : \mathbb{Z}_m \to A) &\mapsto f(1) \end{aligned}$$

Then noting that ϕ_{ev} is a homomorphism, forcing $f(\overline{0}) = 0_A$ (where $\overline{0} : \mathbb{Z}_m \to A$ is the zero map), we must have

$$0 = f(0) = f(m) = mf(1),$$

we must have $mf(1) = 0$ in A. So

$$\text{im } \phi_{\text{ev}} = \{a \in A \mid ma = 0\} := A[m].$$

It is also the case that

$$\ker \phi_{\text{ev}} = \{f \in \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \mid f(1) = 0\} = \left\{\overline{0}\right\},$$

which follows from the fact that $\mathbb{Z}_m = \langle 1 \mod m \rangle$ and $A = \langle 1_A \rangle$ as $\mathbb{Z}$-modules, so if $f(1 \mod m) = 0_A$ then

$$f(n \mod m) = nf(1 \mod m) = 0$$

and so f is necessarily the zero map. So $ker \phi = \overline{0}$.

We can then apply the first isomorphism theorem,

$$\frac{\hom_{\mathbb{Z}}(\mathbb{Z}_m, A)}{\ker \phi_{\text{ev}}} \cong \text{im } \phi_{\text{ev}} \implies \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m].$$

1.2 Part 2

The claim is that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{(m,n)}$, from which the result immediately follows by part 1.

Define a map

$$\begin{aligned} \phi : \mathbb{Z} &\to \mathbb{Z}_n[m] \\ x &\mapsto x \mod n \end{aligned}$$

Then ϕ is clearly surjective, since it is a quotient map, and

$$\begin{aligned} \ker \phi &= \{x \in \mathbb{Z} \ni x \equiv 0 \mod n \text{ and } mx = 0\} \\ &= \{x \in \mathbb{Z} \ni x \equiv 0 \mod m \text{ and } x \equiv 0 \mod n\} \\ &= \{x \in \mathbb{Z} \ni x \equiv 0 \mod \gcd(m,n)\} \\ &= \mathbb{Z}_{\gcd(m,n)}. \end{aligned}$$

By the first isomorphism theorem, we have

$$\frac{\mathbb{Z}}{\ker \phi} \cong \text{im } \phi \implies \mathbb{Z}_{\gcd(m,n)} \coloneqq \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}} \cong \mathbb{Z}_n[m].$$

1.3 Part 3

We identify

$$\mathbb{Z}^* = \hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$$