Title

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1.1 Pushforwards (Continued)

Last time: we saw the Leray spectral sequence, but no examples yet, so that's what we'll do now. We had $X \xrightarrow{f} Y \xrightarrow{g} Z$ to which we associated the spectral sequence $R^i f_* R^j f_*(\cdot) \Rightarrow R^{i+j}(g \circ f)_*(\cdot)$. To deduce existence we used that pushforwards preserve injectives, and we looked at some E_2 differentials.

Example 1.1.1 *(?)*: Let $X \xrightarrow{\pi} Z \coloneqq \operatorname{Spec} k$, where $k \neq \bar{k}$ necessarily. The spectral sequence for the functors π_*, Γ yields the Leray spectral sequence $H^i(k, R^j \pi_* \mathcal{F}) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathcal{F})$. The LHS is the étale cohomology of $\operatorname{Spec} k$, i.e. Galois cohomology. The Galois module corresponding to $R^j \pi_* \mathcal{F}$ is $H^j(X_{k^s}, \mathcal{F})$ by taking the $\bar{k}$ points of this functor So the Leray spectral sequence yields

$$H^i(k, H^j(X_{k^s, \text{ét}}, \mathcal{F})) \Rightarrow H^{i+j}(X_{\text{ét}}, \mathcal{F}).$$

Consider k a finite field and $X_{/k}$ a smooth projective variety. Then the Galois cohomology is given by

$$H^i(k, V) = \begin{cases} V^G & i = 0 & \text{the invariants} \\ V_G & i = 1 & \text{the coinvariants} \\ 0 & i > 1 & \end{cases}$$

This follows from computing the cohomology of $\widehat{\mathbb{Z}}$. Supposing we knew that the cohomological dimension of a smooth projective variety was $2n$ over $\bar{k}$ (e.g. taking $\mathcal{F} \coloneqq \mathbb{Z}/\ell\mathbb{Z}$ above), then the cohomological dimension of X would be $2n + 1$. This follows from E_2 vanishing for $i > 1$ in this case.

Remark 1.1.2: A general fact about the Leray spectral sequence for smooth proper morphisms: let $X \xrightarrow{\pi} Y$ such a morphism, then there is a spectral sequence

$$H^i(Y, R^j \pi_* \underline{\mathbb{Q}}) \Rightarrow H^{i+j}(X, \mathbb{Q}).$$

A fact due to Deligne is that this degenerates at E_2, which is proved with ℓ-adic cohomology (going through Weil II) using the theory of weights. Note that this is false for smooth proper morphisms between manifolds! Instead, for varieties, they behave more like products instead of "twisted" things.

We'll now be explicit about what these pushforwards are, so we'll give another description of them:

Proposition 1.1.3 *(?)*.
Let $X \xrightarrow{\pi} Y$, then $R^i \pi_* \mathcal{F}$ is the sheaf associated to the presheaf $U \to H^i(\pi^{-1}(U)_{\text{ét}}, \mathcal{F})$.

Proof (?).
Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\cdot$, then $\mathcal{H}^i(\pi_*\mathcal{I}^\cdot) \coloneqq R^i\pi_*\mathcal{F}$. Let's compute this pushforward in another way: we have

$$\begin{tikzcd}
\mathrm{Presh}(X_{\text{ét}}) \arrow[r] \arrow[d] & ? \arrow[d] \\
\mathrm{Sh}(X_{\text{ét}}) \arrow[r, "\pi_*"] & \mathrm{Sh}(Y_{\text{ét}})
\end{tikzcd}$$

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