Title

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Recall that the Riemann-Zeta function a product expansion

$$\zeta(s) = \sum n^{-s} = \prod_{p \in P} (1 - p^{-s})^{-1}$$

where the product is taken over all primes P.

Let $X = V(\{f_i\}) \coloneqq V(f)$ be the vanishing locus of a family of polynomials in $F = \mathbb{F}_q[x_1, \cdots, x_n]$ for some prime power q.

Let $N_m = \left|\left\{\mathbf{x} \in X(\mathbb{F}_q) \;\middle|\; f_i(\mathbf{x}) = 0\right\}\right| = |V(f)| \subset F$, the number of $\mathbb{F}_q$ points, or equivalently just the size of this variety.

Then the Hasse-Weil Zeta function is defined as

$$\zeta_X(t) = \exp \sum_{m \geq 1} \frac{N_m}{m} t^m$$

We immediately make a change of variables and send $t \to q^{-s}$ to obtain

$$\zeta_X(s) = \exp \sum_{m \geq 1} \frac{N_m}{m} (q^{-s})^m.$$

> Why? Turns the zeta function into a Dirichlet series in s. Yields $|t| = q^{-\Re(s)}$. Defined for $|t| < \frac{1}{q}$ in $\mathbb{C}$, extended to all of $\mathbb{C}$ as a rational function in x. Converts "All zeros of ζ_X have absolute value $\frac{1}{\sqrt{q}}$" to "All zeros of ζ_X have real part $\frac{1}{2}$".

Explanation of why exponential appears

Rough explanation: Take a bad first approximation and then correct. Let X be a fixed variety, for $p \in X$ define $\|p\|_X = q^n$ where n is the n occurring in the minimal field of definition of p, which is $\mathbb{F}_{q^n}$.

Attempt to define

$$\zeta_{X,q}(s) = \prod_{p \in X} \frac{1}{1 - \|p\|_X^{-s}}.$$

Note that $-\log(x+1) = \sum_{n \geq 1} \frac{x^n}{n}$.

Now fix one $p \in X$ and consider the factor it contributes, and take its logarithm:

$$\begin{aligned}
\log\left(\frac{1}{1 - \|p\|_X^{-s}}\right) &= -\log(1 - \|p\|_X^{-s}) \\
&= -\log(-\|p\|_X^{-s} + 1) \\
&= \sum_{j \geq 1} \frac{\|p\|_X^{-js}}{k} \\
&= \sum_{j \geq 1} \frac{q^{-nks}}{k} \\
&= \sum_{j \geq 1} \frac{n}{nk} (q^{-s})^{nk} \\
(m = nk) \qquad &= \sum_{j \geq 1} \frac{n}{m} (q^{-s})^m,
\end{aligned}$$

so we see this single point contributes n to N_m, when instead we'd like it to contribute exactly 1.

Fix: If p is minimally defined over $\mathbb{F}_{q^n}$, consider its Galois orbit (taking automorphisms of $\mathbb{F}_{q^n}$). There are exactly n points in the orbit of p – namely, the conjugates of p – so if we redefine

$$\zeta_{X,q}(s) = \prod_{\text{One } p \text{ in each Galois orbit}} \frac{1}{1 - \|p\|_X^{-s}}.$$

Then the above argument shows that each orbit now contributes n, and each orbit is of size n, so the contribution now accurately reflects the number of points.

1 Examples

1: $f(x) = x$ over $\mathbb{F}_q$.

Define $X_q = V(f)$, then this has exactly \$q points over $\mathbb{F}_q^n$ point for every n, so $N_n = 1$ and

$$\zeta_{X_q}(s) = \exp \sum_{n \geq 1} \frac{1}{n}(p^{-sn}) = e^{-\log(1-p^{-s})} = (1-p^{-s})^{-1}.$$

Note that the usual $\zeta_s = \prod_{p \text{ prime}} \zeta_{X_p}(s)$, i.e. Riemann Zeta is a product of Hasse-Weil zetas over all primes.

2. $V = \mathbb{CP}^1$ the projective line.

Here

$$\zeta_V(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}.$$

Corresponds to Riemann sphere, can check Betti numbers.

3. $V = \mathbb{CP}^n$:

$$\zeta_V(s) = \prod_{j=0}^{n} \frac{1}{1-q^{j-s}}.$$

4. An elliptic curve:

N_m is given by $1 - \alpha^m - \beta^m + q^m$ where $\alpha = \overline{\beta}$ are complex conjugates with absolute value $\sqrt{q}$.

$$\zeta(E,s) = \frac{(1-\alpha q^{-s})(1-\beta q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$