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1 | Wednesday, October 21

1.1 Strong Linkage

Let G be a semisimple algebraic group and $k = \overline{\mathbb{F}_p}$. We found that the *affine Weyl group* W_p played an important role here.

Theorem 1.1.1 ***(Strong Linkage I).***
Suppose we have a nonzero composition factor in the induced/Weyl module. Then

$$[H^0\lambda : L(\mu)] \neq 0] \implies \mu \uparrow \lambda.$$

In other words, there's a series of reflections sending μ to λ which doesn't increase it's value in the ordering.

Theorem 1.1.2 ***(Strong Linkage II).***
Let $\lambda \in X(T)$ with $\langle \lambda + \rho,\ \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$. Suppose $\mu \in X(T)_+$.

$$[H^i w \cdot \lambda : L(\mu)] \neq 0 \text{ for some } i \geq 0 \implies \mu \uparrow \lambda.$$

Remark 1.1.1.
Note that this is tells us slightly more than Bott-Borel-Weil.

Remark 1.1.2.
There is some history here:

1. Verma conjectured the first theorem in 1971.
2. Humphreys (1971) proved it for $Z_r(\lambda) = \operatorname{Ind}_{B_r}^{G_r} \lambda$.
3. Strong Linkage II proved by Andersen in 1980.
4. Jantzen proved strong linkage for Z_r, which implies strong linkage for $V(\lambda)$.
5. Doty (1987) proved strong linkage for $Z_r(\lambda)$ as a G_rT-modules, which implies strong linkage for $V(\lambda)$.

Remark 1.1.3.
One application is the following: let $\lambda, \mu \in X(T)_+$, then $\operatorname{Ext}_G^n(L(\lambda), L(\mu)) \neq 0$ for some $n \geq 0$. This implies that $\lambda \in W_p \cdot \mu$.

We can consider some cases

- If $n = 0$, we're reduced to previous situations.
- If $n = 1$, we can conclude that $L(\lambda)$ is in the second socle layer of $H^0\mu$, or vice-versa. In either case, $\lambda \in W_p \cdot \mu$.

We can compute this ext by considering an minimal injective resolution

$$0 \to L(\mu) \to I_0 = I(\mu) \to I_1 \to \cdots .$$

We can conclude that

$$[I(\mu) : H^0(\sigma)] = [H^0(\sigma) : L(\mu)] \neq 0.$$

by Brauer-Humphreys reciprocity, so $\sigma \in W_p \cdot \mu$. Similarly $[I(\mu) : L(\gamma)] \neq 0$ implies that $\gamma \in W_p \cdot \mu$, and continuing in this way we can write

$$I_1 = \bigoplus_{j=1}^{t} I(\gamma_j) \text{ with each } \gamma_j \in W_p \cdot \mu.$$

So all of these weights are strongly linked to μ.

But then we know $\mathrm{Ext}_G^n(L(\lambda), L(\mu)) \neq 0$ is a subquotient of $\hom_G(L(\lambda), I_n)$, which thus can not be zero. So $\lambda \in W_p \cdot \mu$

1.2 Translation Functors

Consider the case from category $\mathcal{O}$, e.g. by taking $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$:

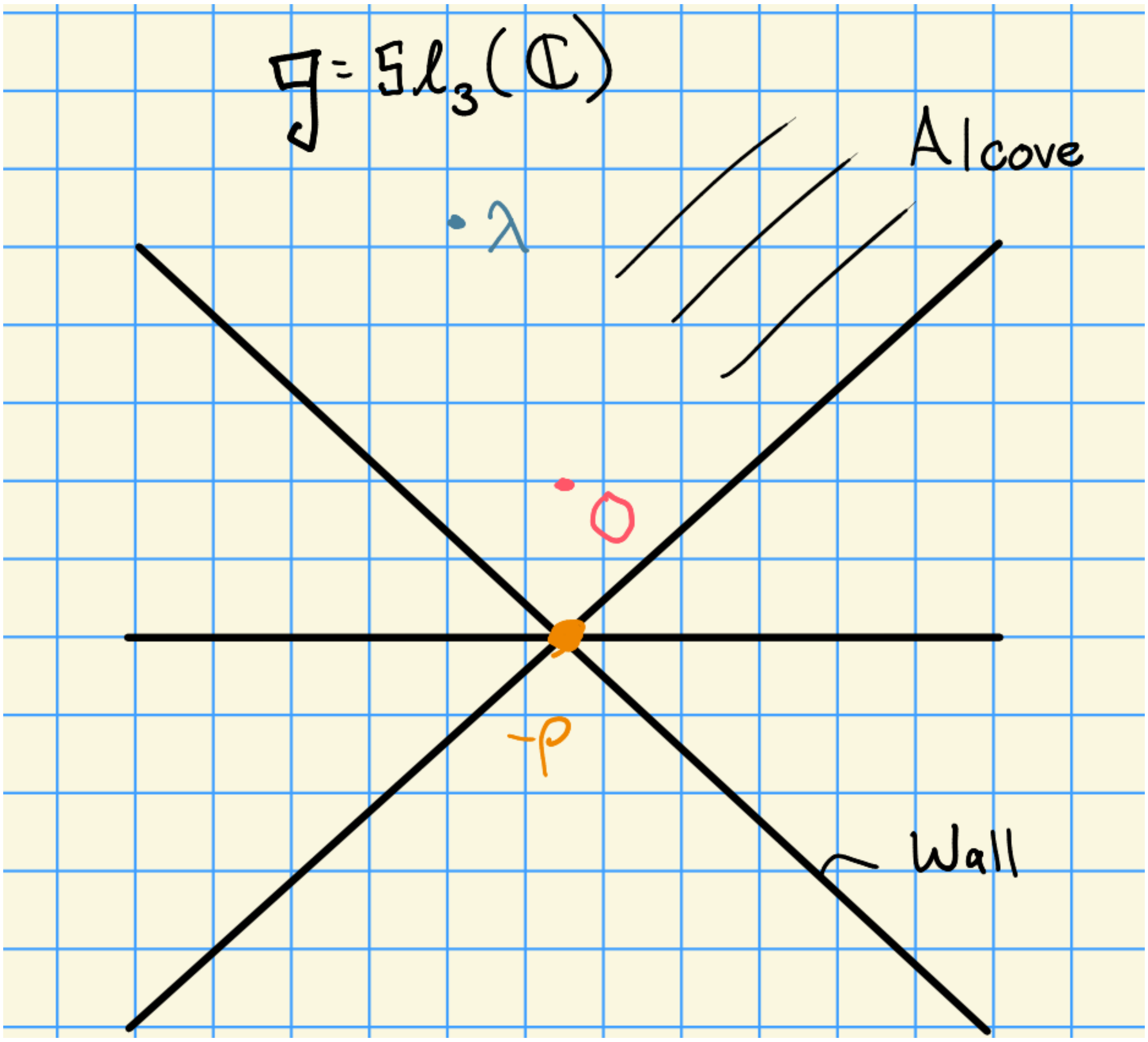

Figure 1: Image

For λ a regular weight, the principal block $\mathcal{B}_0$ is Morita-equivalent to $\mathcal{B}_\lambda$. If μ is a singular weight, then by Jantzen there are translation functors

$$T_\lambda^\mu : \mathcal{B}_\lambda \to \mathcal{B}_\mu$$
$$T_\mu^\lambda : \mathcal{B}_\mu \to \mathcal{B}_\lambda.$$

In the case where G is a semisimple algebraic group and $k = \overline{\mathbb{F}}_p$, we have the following picture instead:

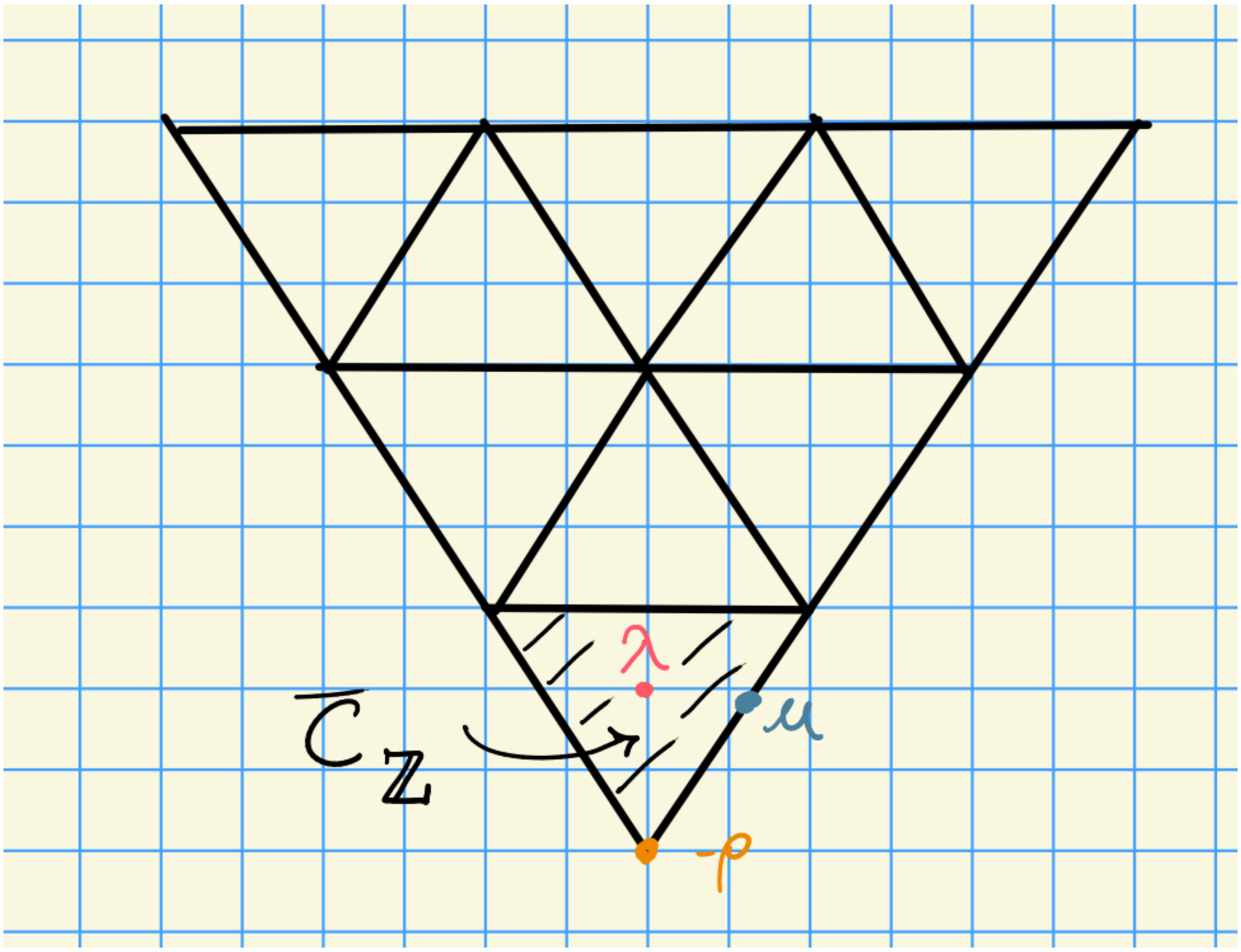

Figure 2: Image

1.2.1 Blocks

Two simple modules S, T are in the same *block* if we have a sequence $T_1, \cdots, T_n$ such that $S = T_1$ and $T_n = T$ where $\mathrm{Ext}^1(T_i, T_{i+1}) \neq 0$.

Lemma 1.1 ***(?).***
Let M, M' be H-modules and $\mathcal{B}(H)$ be the blocks of H. Then

1. $M = \bigoplus_{b \in \mathcal{B}(H)} M_b$ where $M_b = \sum_{M' \leq M} M'$ the sum of all submodules such that M has composition in the block b.
2. $$\mathrm{Ext}^i_H(, M') = \prod_{b \in \mathcal{B}(H)} \mathrm{Ext}^i_H(M_b, M'_b)$$

So the question becomes, what are the blocks of H? Let $\lambda \in X(T)_+$, so we can define $L(\lambda)$, and let $b(\lambda)$ be the G-block containing $L(\lambda)$.

We have $b(\lambda) \in \mathcal{B}(G)$ and $b(\lambda) \subseteq X(T)_+ \cap W_p \cdot \lambda$, i.e. we have strong linkage.

Here we refer to $b(\lambda)$ as both the block and the weights it contains.

Theorem 1.2.1 ***(Donkin).***
Let $\lambda \in X(T)_+$ be a dominant weight and let $r \in \mathbb{Z}$ be the largest integer such that $p^r \mid \langle \lambda + \rho, \alpha^\vee \rangle$ for all $\alpha \in \Phi$. Then

$$b(\lambda) = W_p^{(r)} \cdot \lambda \cap X(T)_+ \text{ where } W_p^{(r)} = W \rtimes p^r \mathbb{Z}\Phi.$$

Proposition 1.2.1 ***(?).***
Let B be a G-module and $\lambda \in X(T)$. Set $\mathrm{pr}_\lambda V$ to be the sum of all submodules of V with composition factors of the form $L(\mu)$ where $\mu \in W_p \cdot \lambda$. Then

- $V = \bigoplus_{\lambda \in Z} \mathrm{pr}_\lambda V$ where Z are representatives of the W_p orbits, i.e. one representative from each alcove in the weight lattice.
- $$\mathrm{Ext}_G^i(V, V') = \prod_{\lambda \in Z} \mathrm{Ext}_G^i(\mathrm{pr}_\lambda V, \mathrm{pr}_\lambda V')$$
- The projection functors $\mathrm{pr}_\lambda(\,\cdot\,)$ are exact.

Note that this still works for singular weights, not just regular weights.

Example 1.2.1.
We can compute

$$\mathrm{pr}_\lambda L(\mu) = \begin{cases} 0 & = \lambda \notin W_p \cdot \mu \\ L(\mu) & = \lambda \in W_p \cdot \mu \end{cases}.$$

Similarly, by strong linkage,

$$\mathrm{pr}_\lambda H^i(\mu) = \begin{cases} 0 & = \lambda \notin W_p \cdot \mu \\ H^i(\mu) & = \lambda \in W_p \cdot \mu \end{cases}.$$

Recall that

$$\overline{C}_\mathbb{Z} := \left\{ \lambda \in X(T) \;\middle|\; 0 \le \langle \lambda + \rho, \beta^\vee \rangle \le p \;\forall \beta \in \Phi^+ \right\}.$$

For every $\mu, \lambda \in \overline{C}_\mathbb{Z}$, consider $\mu - \lambda \in X(T)$. Then there is a way to conjugate it under the ordinary W action to land in the dominant region, i.e. some unique ν such that $\nu \in X(T)_+ \cap W(\mu - \lambda)$.

Definition 1.2.1 (Translation Functors)**.**
Define

$$T_\lambda^\mu V = \mathrm{pr}_\mu(L(\nu) \otimes \mathrm{pr}_\lambda V).$$

So project to λ, tensor with an irreducible representation, then project to μ. This is an exact functor

$$T_\lambda^\mu : G\text{-mod} \to G\text{-mod}.$$

Next time: we'll show that T_λ^μ and T_μ^λ form an adjoint pair. Note that if μ, λ are in the same block, these are the exact functor which product the categorical equivalence.