Chapter 9

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Important ideas:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

1 | Background from Chapter 8

- (M, ω) with $\omega \in \Omega^2(M)$ is a symplectic manifold
- $H \in C^\infty(M; \mathbb{R})$ a Hamiltonian with X_H the corresponding vector field.
  - Recall, could be time-dependent

- The Floer equation and its linearization:

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \text{grad}\,_u(H) = 0$$
$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}M$ is the *free loop space* of M, i.e. space of contractible loops on M, i.e. $C^\infty(S^1; M)$ with the C^∞ topology
  - Loops in $\mathcal{L}M$ can be viewed as maps $S^2 \to M$
- The action functional is given by

$$\mathcal{A}_H : \mathcal{L}W \to \mathbb{R}$$
$$x \mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt$$

  $u : \mathbb{D} \to W$ is an extension of $x : S^1 \to W$ to the disc with $u(\exp(()2\pi it)) = x(t)$.
  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq)$.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation.
- $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

2 | 9.1 and Review

- $\int_{S^2} u^*\omega = \sigma_1$ where $u \in C^\infty(S^2, W)$.
- $\langle c_1(TW),\ \pi_2(TW)\rangle = 0$?
- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k.
- x, y critical points of $\mathcal{A}_H$ with $\mathcal{M}(x, y)$ the moduli space of contractible solutions of finite energy connecting x, y.

2.1 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$, so we quotient to define $\mathcal{L}(x, y) =\coloneqq \mathcal{M}_{x,y}/\mathbb{R}$ with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \overset{n\to\infty}{\to} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n + s, \cdot\,) \overset{n\to\infty}{\to} u(s, \cdot).$$

Proposition 2.1 ***(?).***
$\mathcal{L}(x, y)$ is Hausdorff.

- Want to show $\mathcal{L}(x, y)$ is a compact 0-dimensional manifold.
- Have a differential

$$
\begin{aligned}
\partial : C_k(H) &\longrightarrow C_{k-1}(H) \\
\partial(x) &= \sum_{\mathrm{Ind}(y)=k-1} n(x, y) y.
\end{aligned}
$$

with $n(x, y)$ the number (mod 2) of trajectories of grad $\mathcal{A}_H$ connecting x, y, i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

  $$M := \overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

  and show that M is compact with ∂M equal to the last union.
- Last time: closure of space of trajectories connecting x, y contains "broken" trajectories.
- Last time: toward proving that M is compact

3 | 9.2

- Wanted to compactify $\mathcal{L}(x, y)$, needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

Theorem 3.1 ***(9.2.1).***
Let (H, J) be a regular pair with H nondegenerate.
Let x, z be two periodic trajectories of H such that $\mu(x) = \mu(z) + 2$.
Then $\overline{\mathcal{L}}(x, y)$ is a compact 1-manifold with boundary satisfying

$$\partial\overline{\mathcal{L}}(x, y) = \bigcup_{\mu(x)<\mu(y)<\mu(z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

As a corollary, $\partial^2 = 0$.

- Know $\overline{\mathcal{L}}(x, y)$ is compact and $\mathcal{L}(x, y)$ is a 1-manifold
- Now suffices to study in a neighborhood of boundary points ("gluing theorem")

3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function w_p which interpolates between u and v (not exactly a solution itself, but will be approximated by one later).

2. Constructing ψ a "true solution" from w_p using the Newton-Picard method. We'll have

$$\psi(p) = \exp_{w_p}(\gamma(p)) \qquad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x,z).$$

where $\mathcal{P} = ?$.

3. Get a lift $\widehat{\psi} = \pi \circ \psi$ where $\pi = ?$ satisfying

- $\widehat{\psi}(p) \overset{n\to\infty}{\to} (\widehat{u}, \widehat{v})$
- $\widehat{\varphi}$ is an embedding
- $\widehat{\psi}$ is unique in the following sense (the last point)

Theorem 3.2 ***(9.2.3 (Gluing Theorem)).***
Let x, y, z be critical points of the action functional $\mathcal{A}_H$ such that $\mu(x) = \mu(y) + 1 = \mu(z) + 2$. Let $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ be trajectories, inducing $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$.

- There exist a differentiable map $\psi : (\rho_0, \infty) \to \mathcal{M}(x, z)$ for some $\rho > 0$ such that
- $\pi \circ \psi : (\rho_0, \infty) \to \mathcal{L}(x, z)$ is an embedding
- $\widehat{\psi} \overset{\rho\to\infty}{\to} (\bar{u}, \bar{v}) \in \overline{\mathcal{L}(x, z)}$.
- If $\ell_n \in \mathcal{L}(x, z)$ with $\ell_n \overset{n\to\infty}{\to} (\bar{u}, \bar{v})$, then for $n \gg 1$ we have $\ell \in \Im(\widehat{\psi})$.

4 | 9.3: Pre-gluing

- Choose a bump function β on $\{0\}^c \subset \mathbb{R} \to [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^{\pm}$:

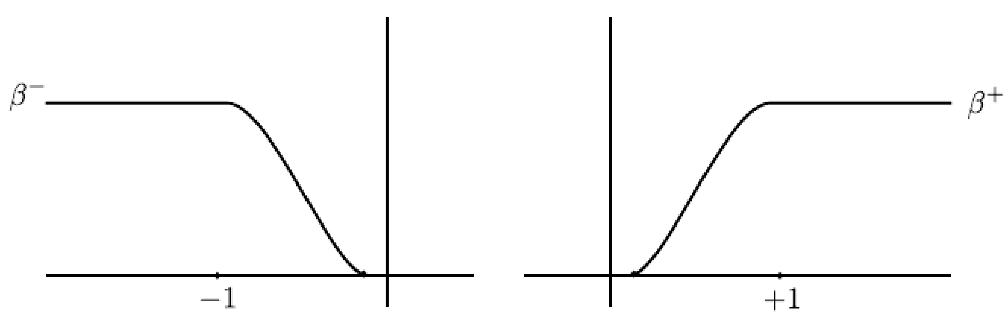

Figure 1: Figure 9.3

- Define the interpolation w_ρ from u to v in the following way:

$$w_\rho(s,t) = \begin{cases} u(s+\rho, t) & \text{if } s \leq -1 \\ \exp_{y(t)}\left(\beta^-(s)\exp^{-1}_{y(t)}(u(s+\rho,t)) + \beta^+(s)\exp^{-1}_{y(t)}(v(s-\rho,t))\right) & \text{if } s \in [-1, 1] \\ v(s-\rho, t) & \text{if } s \geq 1 \end{cases}$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \;\middle|\; \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\}.$$

5 | 9.4: Construction of ψ.

- Have constructed $w_\rho \in C^\infty_\searrow(x,z)C^\infty(x,z)$ for every $\rho \geq \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x,z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

  $$\mathcal{F} = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } Hx$$

  in the weak sense.
- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining $\mathcal{F}_\rho$.

$$\begin{aligned} W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\xrightarrow{\mathcal{F}_\rho} L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ (y_1, \ldots, y_{2n}) &\longmapsto \left[\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } H_t\right)\left(\exp_{w_\rho} \sum y_i Z_i^\rho\right)\right]_{Z_i} \end{aligned}$$

where $\mathcal{F}_\rho \coloneqq \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i. sd - Newton-Picard method, general idea

- Original method and variant: find the limit of a sequence

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

- Allows finding zeros of f given an approximate zero x_0.
- Linearize $\mathcal{F}_\rho$.