Title

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January 3, 2020

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1 Tuesday November 26th

1.1 Differentiation

Question: Let $f \in L^1([a,b])$ and $F(x) = \displaystyle\int_a^x f(y)\ dy$. Is F differentiable a.e. and $F' = f$?

If f is continuous, then absolutely yes.

Otherwise, we are considering

$$\frac{f(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(y)\ dy \to_? f(x)$$

so the more general question is

$$\lim_{\substack{m(I) \to 0 \\ x \in I}} \frac{1}{m(I)} \int_I f(y)\ dy =_? f(x) \text{ a.e.}$$

Note that if f is continuous, since $[a,b]$ is compact, we have uniform continuity and

$$\frac{1}{m(I)} \int_I f(y) - f(x)\ dy < \frac{1}{m(I)} \int_I \varepsilon \to 0.$$

1.2 Lebesgue Differentiation and Density Theorems

Theorem: If $f \in L^1(\mathbb{R}^n)$ then

$$\lim_{\substack{m(B) \to 0 \\ x \in B}} \int \frac{1}{m(B)} \int_B f(y)\ dy = f(x) \text{ a.e.}$$

Note: although it's not obvious at first glance, this really is a theorem about differentiation.

Corollary (Lebesgue Density Theorem): For any measurable set $E \subseteq \mathbb{R}^n$, we have

$$\lim_{r \to 0} \frac{m(E \bigcap B_r(x))}{m(B_r(x))} = 1 \ a.e.$$

Proof: Let $f = \chi_E$ in the theorem.

We want to show

$$Df(x) := \limsup_{\substack{m(B) \to 0 \\ x \in B}} \left| \frac{1}{m(B)} \int_B (f(y) - f(x)) \, dy \right| \to 0$$

Note that we can replace $\limsup \cdots$ with

$$\lim_{\varepsilon \to 0} \sup_{\substack{0 \le m(B) \le \varepsilon \\ x \in B}} \cdots ,$$

which is well defined as it is a decreasing sequence of numbers bounded below by zero.

Next we'll introduce that *Hardy-Littlewood Maximal Function*, given by

$$Mf(x) := \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| \, dy$$

Exercise: show that Mf is a measurable function. (Hint: it's easy to show that the appropriate preimage is open.)

Theorem (Hardy-Littlewood Maximal Function Theorem): Let $f \in L^1(\mathbb{R}^n)$, then

$$m(x \in \mathbb{R}^n \mid Mf(x) > \alpha) \le \frac{3^n}{\alpha} \|f\|_1.$$

Idea: if you look at all balls intersecting a given ball of radius α, the worst case is that the other ball doesn't intersect and is of the same radius. But then you can draw a ball of radius 3α and cover every such intersecting ball.

Exercise: As a corollary, $Mf(x) < \infty$ *a.e.*

This is called a *weak type* estimate, compared to a strong type $\|Mf\|_1 \le C\|f\|_1$. Note that by Chebyshev, a strong estimate would imply the weak one because

$$m(\left\{x \mid mf(x) > \alpha\right\}) \le \frac{1}{\alpha}\|Mf\|_1 \not\le \frac{C}{\alpha}\|f\|_1,$$

which is an inequality that doesn't hold (hence the theorem) because there is an L^1 function for which Mf is *not* L^1.

Proof of differentiation theorem: The goal is to show $Df(x) = 0$ a.e.

We will show that $m(\left\{x \mid Df(x) > \alpha\right\}) = 0$ for all $\alpha > 0$.

Some facts:

1. If g is continuous, then $Dg(x) = 0$ a.e. by uniform convergence.
2. $$D(f_1 + f_2)(x) \leq Df_1(x) + Df_2(x)$$
   by applying the triangle inequality and distributing the $\limsup$.
3. $$Df(x) \leq Mf(x) + |f(x)|$$

Fix an α and fix an ε. Choose a continuous g such that $\|f - g\|_1 < \varepsilon$.

Writing $f = f - g + g$, we have

$$\begin{aligned} Df(x) &\leq D(f-g)(x) + Dg(x) \\ &= D(f-g)(x) + 0 \\ &\leq M(f-g)(x) + |(f-g)(x)|., \end{aligned}$$

Then

$$Df(x) \geq \alpha \implies M(f-g)(x) \geq \frac{\alpha}{2}$$

or

$$|(f-g)(x)| \geq \frac{\alpha}{2}.$$

So we have

$$\left\{x \;\middle|\; Df(x) > \alpha\right\} \subseteq \left\{x \;\middle|\; M(f-g)(x) > \frac{\alpha}{2}\right\} \bigcup \left\{x \;\middle|\; |f(x) - g(x)| > \frac{\alpha}{2}\right\}.$$

Applying measures turns this into an inequality.

But then applying the maximal function theorem, we have

$$\begin{aligned} m(\left\{x \;\middle|\; Df(x) > \alpha\right\}) &\leq \frac{3^n}{\alpha/2}\|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \\ &\leq \varepsilon\left(\frac{2(3^n+1)}{\alpha}\right). \end{aligned}$$

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> Note that somehow proving the maximal function theorem here really paved the way, and allows some generalization. Here we computed an average over a solid ball, but there is a notion of surface measure, so we can consider averaging over the surface of spheres, which can include more exotic objects like spheres in $\mathbb{Z}^d$.

Proof of HL Maximal Function Theorem: Let

$$E_\alpha := \left\{ x \;\middle|\; Mf(x) > \alpha \right\}.$$

If $x \in E_\alpha$, then it follows that there is a B_x such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)| \, dy > \alpha \iff m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)| \, dy.$$

Note that if E_α were compact, there would only be finitely many such balls, so let $K \subseteq E_\alpha$ be a compact subset. We will be done if we can show that

$$m(K) < \frac{3^n}{\alpha} \|f\|_1,$$

since we can always find a compact K such that $m(E_\alpha \setminus K)$ is small.

There exists a finite collection $\{B_k\}^N$ such that each $B_k = B_x$ for some $x \in E_\alpha$, $K \subseteq \bigcup B_k$, and

$$m(B_k) \leq \frac{1}{\alpha} \int_{B_k} |f(y)| \, dy.$$

Supposing that the B_k were disjoint (which they are not!), then we would be done since

$$\begin{aligned} m(K) &\leq \sum m(B_k) \\ &\leq \frac{1}{\alpha} \sum \int_{B_k} |f(y)| \, dy \\ &\leq \frac{1}{\alpha} \int_{\mathbb{R}^n} |f(y)|. \end{aligned}$$

Lemma (The Vitali Covering Lemma): Given any collection of balls $B_1, \cdots, B_N$, there exists a sub-collection $A_1, \cdots, A_M$ which are disjoint with

$$m(\bigcup^N B_k) \leq 3^n \sum^M m(A_j).$$

> Note that this follows directly from picking the largest ball first, then picking further balls that avoid everything already picked and are chosen in decreasing order of size. The 3^n factor comes from the earlier fact that tripling the radius covers everything you didn't pick.

But now we can replace B_k with such a sub-collection A_k in the above set of inequalities, which proves the theorem. $\blacksquare$.