# 8005: Qual Problems

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## ① Classify all groups of order 14.

Let $n = 14 = \#G$. Noting that $n = 2 \cdot 7$, we have

$$r_7 = 1 \mod 7 \ \& \ r_7 \mid 2 \Rightarrow r_7 = 1,$$

So $G$ has a normal sylow 7-subgroup $S_7$, which is in fact cyclic since 7 is prime, and thus isomorphic to $\mathbb{Z}_7$. Similarly, the sylow 2-subgroup $S_2$ is isomorphic to $\mathbb{Z}_2$.

Since 7 & 2 are coprime, no nontrivial element can be in both subgroups – otherwise, it would generate a subgroup of both, which would have to have order dividing both 7 & 2 by Lagrange's theorem. Moreover, $|S_2 \cdot S_7| = \dfrac{|S_2| \cdot |S_7|}{|S_2 \cap S_7|} = \dfrac{2 \cdot 7}{1} = 14 = \#G$, so $G = S_2 S_7$.

So we have $S_2, S_7 \leq G$ where

- $S_7 \trianglelefteq G$
- $S_2 \cap S_7 = \{e\}$
- $S_2 \cdot S_7 = G$

and so $G \cong S_2 \rtimes_\psi S_7$, where $\begin{array}{l} S_2 \curvearrowright S_7 \\ a \mapsto \psi_a \in \mathrm{Aut}(S_7), \end{array}$

I.e., $G \cong \mathbb{Z}_2 \rtimes_\psi \mathbb{Z}_7$ for some $\psi \in \mathrm{Aut}(\mathbb{Z}_7)$, and since the map

$$\begin{aligned} \mathbb{Z}_2 &\to \mathrm{Aut}(\mathbb{Z}_7) \\ a &\mapsto \psi_a \end{aligned}$$

must be a homomorphism, $\psi$ must be order 2.

We have $\mathrm{Aut}(\mathbb{Z}_7) = \{ x \mapsto nx \mid 1 \leq n \leq 6 \}$, since any automorphism will map a generator to another generator, and any $n \neq 0$ in $\mathbb{Z}_7$ is a generator.

Then $\{\psi \in \mathrm{Aut}(\mathbb{Z}_7) \mid \text{order}(\psi) = 2\} = \{x \mapsto x, \ x \mapsto 6x\}$. We have $G \cong \langle a, b \mid a^2 = b^7 = e, \ aba^{-1} = \psi_a(b) \rangle$, so we obtain two groups:

1) $G \cong \langle a, b \mid a^2 = b^7 = e, \ aba^{-1} = b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_7 \cong \mathbb{Z}_{14}$,
2) $G \cong \langle a, b \mid a^2 = b^7 = e, \ aba^{-1} = b^6 \rangle \cong D_7$, the dihedral group. ■

② Show that $\#G = p^3 \Rightarrow G$ is abelian or $|Z(G)| = p$.

Since $Z(G) \leq G$, we must have $|Z(G)| \in \{1, p, p^2, p^3\}$.

- If $|Z(G)| = p^3$, then $G$ is abelian and we're done.
- $|Z(G)| \neq 1$ because $p$-groups have nontrivial centers. $*$
- If $|Z(G)| = p$, we're again done.
- If $|Z(G)| = p^2$, then $\left|\frac{G}{Z(G)}\right| = p^3/p^2 = p$, so $G/Z(G)$ is cyclic and (by a previous theorem) $G$ must be abelian. ∎

Proof of $*$:

If $\#G = p^n$ and $Z(G) \lneq G$ is proper, then by the class equation

$$|G| = |Z(G)| + \sum_{\substack{\text{One } g_i \text{ in each} \\ \text{conjugacy class}}} [G : C_G(g_i)]$$

where $g_i \notin Z(G)$. But each $C_G(g_i) \lneq G$ is then proper, so $|C_G(g_i)| = p^k$ for some $k \leq n-1$. So $p$ divides $[G : C_G(g_i)]$, and $p$ divides $|G|$, so $p$ must divide $|Z(G)|$ as well. ∎

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③ Let $p, q$ be distinct primes & $k$ be the smallest positive integer such that $p \mid q^k - 1$, and suppose $\#G = pq^k$. Then

$$n_p \mid q^k \Rightarrow n_p \in \{1, q, q^2, \cdots, q^k\}.$$

If $n_p = 1$, $G$ is not simple, so suppose $n_p = q^\ell$ for some $1 \leq \ell \leq k$. Then

$n_p \equiv 1 \bmod p \Rightarrow q^\ell - 1 \equiv 0 \bmod p \Rightarrow p \mid q^\ell - 1 \Rightarrow \ell = k$ by assumption.

So let $S_{p,i} \in Syl(p, G)$ be a sylow $p$-subgroup of $G$. Then $|S_{p,i}| = p$, so it is cyclic.

Since $S_{p,i} \cap S_{p,j} \leq S_{p,i}$ for example, these groups either coincide or intersect trivially. Thus

$$\left|\bigcup_{i=1}^{n_p} S_{p,i}\right| = n_p(p-1) = \boxed{q^k p - q^k.}$$

Now consider $Syl(q, G)$. If $n_q = 1$, $G$ is not simple, and since $n_q \mid p$, the only other possibility is $n_q = p$. Let $S_{q,i} \in Syl(q, G)$, so $|S_{q,i}| = q^k$. But since $n_q > 1$, we have

$\boxed{|\bigcup_i S_{q,i}| > q^k.}$ And since $p, q$ are coprime, $S_{q,i} \cap S_{p,j} = \{e\}$ $\forall i, j$, and so

$$\left(\bigcup_{i=1}^{n_q} S_{q,i}\right) \cup \left(\bigcup_{j=1}^{n_p} S_{p,j}\right) \subseteq G \Rightarrow \left|\bigcup_{i=1}^{n_q} S_{q,i}\right| + \left|\bigcup_{j=1}^{n_p} S_{p,j}\right| \leq |G| = pq^k$$

However, we've shown

$$\left.\begin{array}{l} \left|\bigcup_{i=1}^{n_q} S_{q,i}\right| > q^k \\ \left|\bigcup_{j=1}^{n_p} S_{p,j}\right| > q^k(p-1) \end{array}\right\} \Rightarrow \left|\bigcup_{i=1}^{n_q} S_{q,i}\right| + \left|\bigcup_{j=1}^{n_p} S_{p,j}\right| > q^k + q^k(p-1) = pq^k,$$

a contradiction. So we must have $n_p = 1$, and $G$ is not simple. ■

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4) <u>Show that $S_4$ is solvable and nonabelian.</u>

A group $G$ is solvable iff $G$ has a composition series in which each successive quotient is simple and abelian, so we can take

$$\begin{array}{lll} S_4 & & \\ \downarrow & \rightsquigarrow & S_4/A_4 \cong \mathbb{Z}_2, \text{ which is simple + abelian} \\ A_4 & & \\ \downarrow & \rightsquigarrow & A_4/\mathbb{Z}_2^2 \cong \mathbb{Z}_3, \quad \text{" " "} \\ H_1 \cong \mathbb{Z}_2^2 & & \\ \downarrow & \rightsquigarrow & \mathbb{Z}_2^2/\mathbb{Z}_2 \cong \mathbb{Z}_2, \quad \text{" " "} \\ H_2 \cong \mathbb{Z}_2 & & \\ \downarrow & \rightsquigarrow & \mathbb{Z}_2/0 \cong \mathbb{Z}_2, \quad \text{" " "} \\ 0 & & \end{array}$$

where $H_1 = \langle (12), (34) \rangle \trianglelefteq A_4 \trianglelefteq S_4$

$H_2 = \langle (34) \rangle \trianglelefteq A_4 \trianglelefteq S_4$. ■