Problem Set 7

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1 Regular Problems

1.1 Problem 1

Note that if either $p = 1$ or $q = 1$, G is a p-group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2 q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$\begin{aligned} n_p &\equiv 1 \mod p \qquad & n_q &\equiv 1 \mod q \\ n_p &\mid q & n_q &\mid p^2, \end{aligned}$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \left\{1, p, p^2\right\}.$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. This forces $n_p = q$, and we proceed by cases:

1.1.1 Case 1: $p = q$.

Then $\#G = p^3$ and G is a p-group. But every p-group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.1.2 Case 2: $p > q$.

Here, since $n_p \,\Big|\, q$, we must have $n_p < q$. But if $n_p < q < p$ and $n_p = 1 \mod p$, then $n_p = 1$.

1.1.3 Case 3: $q > p$.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q:

- $n_q = p$: If $n_q = p = 1 \mod q$ and $p < q$, this forces $p = 1$.
- $n_q = p^2$: We will reach a contradiction by showing that this forces

  $$\left| P \coloneqq \bigcup_{S_p \in \mathrm{Syl}(p,G)} S_p \setminus \{e\} \right| + \left| Q \coloneqq \bigcup_{S_q \in \mathrm{Syl}(q,G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

  We have

  $$\begin{aligned}
  |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\
  &= p^2(q-1) + q(p^2-1) + 1 \\
  &= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 \qquad \text{(since } q > 1\text{)} \\
  &= (p^2 q - p^2) + (p^2 - 1) + (q-1)(p^2-1) \\
  &= p^2 q + (q-1)(p^2-1) \\
  &\geq p^2 q + (2-1)(2^2-1) \qquad \text{(since } p, q \geq 2\text{)} \\
  &= p^2 q + 3 \\
  &> p^2 q = |G|,
  \end{aligned}$$

  which is a contradiction.

2 Qual Problems