3b.1 <u>Claim</u>: ω_f is well-defined

This amounts to showing that the sup and limit exist in

$$\omega_f(x) = \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

Let $x \in \mathbb{R}$ be arbitrary and δ fixed.

Since f is bounded, there is some M such that

$$\forall y \in \mathbb{R},\ |f(y)| < M \quad , \text{ and so}$$

$$y,z \in \mathbb{R} \Rightarrow |f(y) - f(z)| = |f(y) + (-f(z))| \leq |f(y)| + |-f(z)|$$

$$= |f(y)| + |f(z)| < 2M,$$

which holds for $y,z \in B_\delta(x) \subseteq \mathbb{R}$ as well.

And so $\{|f(y) - f(z)| \text{ s.t. } y,z \in B_\delta(x)\}$ is bounded above and thus has a <u>least</u> upper bound, and thus the following supremum exists.

$$S(\delta, x) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

To see that the $\lim_{\delta \to 0} S(\delta, x)$ exists, note that

$$\delta_1 \leq \delta_2 \Rightarrow B_{\delta_1}(x) \subseteq B_{\delta_2}(x)$$

and so for a fixed x, $S(\delta, x)$ is a monotonically decreasing function of δ that is bounded below by 0, which converges by the monotone convergence theorem. $\square$

<u>Claim</u>: f is continuous at x iff $\omega_f(x) = 0$.

($\Leftarrow$) Suppose $\omega_f(x) = 0$ and let $\varepsilon > 0$ be arbitrary; we will produce a δ to use in the definition of continuity.

Since $\omega_f(x) = \lim_{d \to 0^+} S(d,x) = 0$, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x)| < \varepsilon \quad , \qquad \text{which means}$$

$$d < \delta \Rightarrow \sup_{y,z \in B_d(x)} |f(y) - f(z)| < \varepsilon$$

So fix $z = x$ and let y vary, yielding

$$d < \delta \Rightarrow \sup_{y \in B_d(x)} |f(y) - f(x)| < \varepsilon$$

But now for an arbitrary $t \in B_\delta(x)$, we have $|x-t| < \delta$ and

$$|f(x) - f(t)| \le \sup_{y \in B_\delta(x)} |f(x) - f(y)| < \varepsilon,$$

which exactly says $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$. $\square$

($\Rightarrow$) Suppose f is continuous at x and let $\varepsilon > 0$ be arbitrary; we will show $\omega_f(x) < \varepsilon$.

Since f is continuous, choose δ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

We then have

$$y, z \in B_\delta(x) \Rightarrow |x-y| < \delta \quad \text{and} \quad |x-z| < \delta,$$

$$\Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x)-f(z)| < \frac{\varepsilon}{2}$$

$$\Rightarrow |f(y)-f(z)| \leq |f(y)-f(x)| + |f(x)-f(z)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and so

$$y, z \in B_\delta(x) \Rightarrow |f(y)-f(z)| < \varepsilon \quad \Rightarrow \sup_{y,z \in B_\delta(x)} |f(y)-f(z)| \leq \varepsilon$$

$$\Rightarrow S(\delta, x) \leq \varepsilon,$$

and since $S(d,x)$ is monotonically decreasing in d,

$$\omega_f(x) = \lim_{d \to 0^+} S(d,x) \leq S(\delta, x) \leq \varepsilon$$

as desired. $\blacksquare$

3b.2 We will show that

$$A_\varepsilon^c = \{ x \in \mathbb{R} \mid \omega_f(x) < \varepsilon \}$$

is open by showing every point is an interior point.

Fix $\varepsilon > 0$ and let $x \in A_\varepsilon^c$ be arbitrary. We want to produce a δ such that

$$B_\delta(x) \subsetneq A_\varepsilon^c \quad \text{or equivalently} \quad |y-x| < \delta \Rightarrow \omega_f(y) < \varepsilon.$$

Write $\omega_f(x) = \lim_{d \to 0^+} S(d,x)$; since $\omega_f(x) < \varepsilon$ and this limit exists, we can choose δ such that

$$d < \delta \Rightarrow |S(d,x) - 0| < \varepsilon \Rightarrow |S(d,x)| < \varepsilon.$$

Now suppose $y \in B_\delta(x)$, so $|y-x| < \delta$. Then there exists some δ' such that $B_{\delta'}(y) \subset B_\delta(x)$, and we claim that

$$S(\delta', y) \leq S(\delta, x)$$

Note that if this is true, then

S is monotonically decreasing in d

$$\omega_f(y) = \lim_{d \to 0} S(d,y) \leq S(\delta', y) \leq S(\delta, x) < \varepsilon.$$

To see why this is true, we just note that

$$a,b \in B_{\delta'}(y) \subset B_\delta(x) \Rightarrow a,b \in B_\delta(x)$$

$$\Rightarrow \sup_{a,b\in B_{\delta'}(y)} |f(y)-f(z)| \le \sup_{y,z\in B_\delta(x)} |f(y)-f(z)|,$$

since the supremum can only increase over a larger set.

So $\omega_f(y) < \varepsilon$ as desired. ■

Finally, note that if $D_f = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$,

then $$D_f = \{x \in \mathbb{R} \mid \omega_f(x) \neq 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} \mid \omega_f(x) \ge \tfrac{1}{n}\}$$

$$= \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$$

is a countable union of closed sets and thus F_σ. ■

④ <u>Claim</u>: f is increasing, i.e. $x \leq y \Rightarrow f(x) \leq f(y)$

Fix $x \in \mathbb{R}$, and define

$$A_x := \{ t \in X \mid x > t \}, \quad A_x^c := \{ t \in X \mid x \leq t \}.$$

(Note that $t \in A_x$ or $t \in A_x^c \Rightarrow t = x_n$ for some n, and $X = A_x \sqcup A_x^c$.)

Then noting that

$$x_n \in A_x \Rightarrow f_n(x) \equiv 1$$

and

$$x_n \in A_x^c \Rightarrow f_n(x) \equiv 0,$$

We can write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \cdot 1 + \sum_{\{n \mid x_n \in A_x^c\}} \frac{1}{n^2} \cdot 0$$

$$= \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2}.$$

Now if $y \geq x$, then $y \geq t$ for every $t \in A_x$, so $A_y \supseteq A_x$.

But then

$$f(x) = \sum_{\{n \mid x_n \in A_x\}} \frac{1}{n^2} \leq \sum_{\{n \mid x_n \in A_y\}} \frac{1}{n^2} = f(y),$$

where the inequality holds because

$$A_x \subseteq A_y \Rightarrow \{n \mid x_n \in A_x\} \subseteq \{n \mid x_n \in A_y\}$$
$$\Rightarrow |\{n \mid x_n \in A_x\}| \leq |\{n \mid x_n \in A_y\}|,$$

so the latter sum has <u>at least</u> as many terms and everything is positive. So $f(x) \leq f(y)$.

<u>Claim</u>: f is continuous on $\mathbb{R} \setminus X$ since $\sum f_n \xrightarrow{u} f$ and each f_n is continuous there.

Since $|f_n(x)| \leq 1$ by definition, and $|f_n(x)/n^2| \leq |1/n^2| := M_n$ where $\sum M_n < \infty$, $\sum f_n \xrightarrow{u} f$ by the M test.

Note that for a fixed n, $D_{f_n} = \{x_n\}$. This is because if we take a sequence $\{y_i\} \to x_n$ with each $y_i > x_n$, then $f(y_i) = 1$ for every i, and

$$\lim_{i\to\infty} f(y_i) = \lim_{i\to\infty} 1 = 1 \neq f(\lim_{i\to\infty} y_i) = f(x_n) = 0$$

So f_n is not continuous at $x = x_n$. Otherwise, either $x > x_n$ or $x < x_n$, in which case we can let ε be arbitrary and choose $\delta < |x - x_n|$ to get

$$y \in B_\delta(x) \Rightarrow \begin{cases} y > x_n \Rightarrow |f(y) - f(x)| = |0-0| < \varepsilon \\ y < x_n \Rightarrow |f(y) - f(x)| = |1-1| < \varepsilon. \end{cases}$$

Letting $F_N = \sum_{n=1}^{N} f_n$, we find that

$$F_N = f_1 + f_2 + \dots + f_N$$

discontinuous at: $\{x_1\} \cup \{x_2\} \cup \dots \cup \{x_N\}$

$\Bigg\}$ So F_N is continuous on $\mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$.

and since $\mathbb{R} \setminus X \subseteq \mathbb{R} \setminus \bigcup_{i=1}^{N} \{x_N\}$, F_N is continuous there too.

But then $f = \text{uniform limit}(F_N)$ is continuous on $\mathbb{R} \setminus X$. $\blacksquare$