Algebra

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Contents

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1 Lecture 1 (Thu 15 Aug 2019)

Definition: A *group* is an ordered pair $(G, \cdot : G \times G \to G)$ where G is a set and $\cdot$ is a binary operation, which satisfies the following axioms:

1. Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$
2. Identity: $\exists e \in G \ni ge = eg = g$
3. Inverses: $g \in G \implies \exists h \in G \ni gh = gh = e$.

Some examples of groups:

- $(\mathbb{Z}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Q}^\times, \times)$
- $(\mathbb{R}^\times, \times)$
- $(\mathrm{GL}(n, \mathbb{R}), \times) = \{A \in \mathrm{Mat}_n \ni \det(A) \neq 0\}$
- $(S_n, \circ)$

Definition: A subset $S \subseteq G$ is a *subgroup* of G iff

1. $s_1, s_2 \in S \implies s_1 s_2 \in S$
2. $e \in S$
3. $s \in S \implies s^{-1} \in S$

We denote such a subgroup $S \leq G$.

Examples:

- $(\mathbb{Z}, +) \leq (\mathbb{Q}, +)$
- $\mathrm{SL}(n, \mathbb{R}) \leq \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \ni \det(A) = 1\}$

1.1 Cyclic Groups

Definition: A group G is cyclic iff G is generated by a single element.

Exercise: Show $\langle g \rangle = \{g^n \ni n \in \mathbb{Z}\} \cong \bigcap\{H \leq G \ni g \in H\}$.

Theorem: Let G be a cyclic group, so $G\langle g \rangle$.

1. If $|G| = \infty$, then $G \cong \mathbb{Z}$.
2. If $|G| = n < \infty$, then $G \cong \mathbb{Z}_n$

Definition: Let $H \leq G$, and define a *right coset of* G by $aH = \{ah \ni H \in H\}$. A similar definition can be made for *left cosets*.

Then $aH = bH \iff b^{-1}a \in G$ and $Ha = Hb \iff ab^{-1} \in H$.

Some facts:

- Cosets partition H, i.e. $b \notin H \implies aH \bigcap bH = \{e\}$.
- $|H| = |aH| = |Ha|$ for all $a \in G$.

Theorem (Lagrange): If G is a finite group and $H \leq G$, then $|H| \mid |G|$.

Definition: $N \leq G$ is *normal* iff $gN = Ng$ for all $g \in G$, or equivalently $gNg^{-1} \subseteq N$. I denote this $N \trianglelefteq G$.

When $N \trianglelefteq G$, the set of left/right cosets of N themselves have a group structure. So we define $G/N = \{gN \ni g \in G\}$ where $(g_1N)(g_2N) = (g_1g_2)N$.

Given $H, K \leq G$, define $HK = \{hk \ni h \in H, k \in K\}$. We have a general formula,

$$|HK| = \frac{|H||K|}{|H \bigcap K|}.$$

1.2 Homomorphisms

Let G, G' be groups, then $\varphi : G \to G'$ is a *homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$.