CSE 166 HW #4

Wednesday, 22 November, 2017 06:10 AM

6.1)

a) Let $S = [\bar{s}_0\ \bar{s}_1\ \bar{s}_2] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, then $\det(S)=0$ iff $\langle s_i, s_j \rangle = 0$ for any $i,j = 1,2,3$.

but $\det S = 1(-1) - 1(2+1) + 1(-2) = -6 \neq 0$.

b) No, eg $\|s_1\| = \sqrt{2} \neq 1$. So take

$$\hat{S} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

c) $\hat{S}$ is this matrix

d) $\hat{S}\bar{f} = \left[\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right]\begin{bmatrix}3\\6\\-5\end{bmatrix}$

$$= \begin{bmatrix} \frac{3}{\sqrt{6}} + \frac{6}{\sqrt{2}} - \frac{5}{\sqrt{3}} \\ \frac{6}{\sqrt{6}} + 0 + \frac{5}{\sqrt{3}} \\ \frac{3}{\sqrt{6}} - \frac{6}{\sqrt{2}} - \frac{5}{\sqrt{3}} \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 3+6\sqrt{6}-5\sqrt{2} \\ 6+5\sqrt{2} \\ 3-6\sqrt{3}-5\sqrt{2} \end{bmatrix}.$$

e) $\underbrace{\hat{S}^{-1}\bar{f} = \hat{S}^T}_{\text{by orthogonality}}\bar{f} = \begin{bmatrix} \langle \frac{1}{\sqrt{6}}(1,2,1), (3,6,-5)\rangle \\ \langle \frac{1}{\sqrt{2}}(1,0,-1), (3,6,-5)\rangle \\ \langle \frac{1}{\sqrt{3}}(1,-1,1), (3,6,-5)\rangle \end{bmatrix}$ $\langle\ ,\ \rangle$ inner product

$$\underset{\text{bilinearity}}{=} \begin{bmatrix} \frac{1}{\sqrt{6}}\cdot 10 \\ \frac{1}{\sqrt{2}}\cdot 8 \\ \frac{1}{\sqrt{3}}\cdot -8 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 10 \\ 8\sqrt{3} \\ -8\sqrt{2} \end{bmatrix}$$

6.3)

This may not be a sufficient condition,

take $\mathcal{B} = \{\bar{v}_1, \bar{v}_2\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$, which is a basis for $\mathbb{R}^2$.

Note that the $\bar{v}_i$ are <u>real</u>, $\langle \bar{v}_1, \bar{v}_2 \rangle = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) + (\frac{-1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{-1}{2} = 0$, so we have orthogonality,

$\|\bar{v}_1\| = \|\bar{v}_2\| = 1$, but the matrix $B = [\bar{v}_1\ \bar{v}_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is <u>not</u>
(so normal & thus orthonormal)

symmetric, since $B^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \neq B$

In this case, we would have $r(x,u) = B[x,u]$ for $x = 1,2$ (the forward transform)
$s(x,u) = B^{-1}[x,u] = B^T[x,u]$ $u = 1,2$ (the reverse transform)

but $\underset{B[1,0]}{r(1,0)} = -1/\sqrt{2} \neq 1/\sqrt{2} = \underset{B^T[1,0]}{s(1,0)}$. (???)

(It is the case that $s(x,u) = r(u,x)$, though, since $B^{-1} = B^T$.)

6.17) Let $s(x,u)$ be a complex function defined for $\begin{cases} x = 0,1,\cdots,n \\ u = 0,1,\cdots,m. \end{cases}$

Define $\bar{s}_u(x) = [s(0,u), s(1,u), \cdots, s(n-1,u)]^T$ for $u = 0,1,\cdots,m$, then let $(S_{ij}) = s_i(j)$, so $S_m := \begin{bmatrix} s_1^T & \text{———} \\ s_2^T & \text{———} \\ \vdots & \\ s_m^T & \text{———} \end{bmatrix} = \begin{bmatrix} s_0(0) & s_0(1) & \cdots & s_0(n) \\ s_1(0) & s_1(1) & \cdots & s_1(n) \\ \vdots & \vdots & \ddots & \vdots \\ s_m(0) & s_m(1) & \cdots & s_m(n) \end{bmatrix}$ and $S_n := \begin{bmatrix} s_1^T & \text{———} \\ s_2^T & \text{———} \\ \vdots & \\ s_n^T & \text{———} \end{bmatrix}$

Then S is an <u>$m \times n$ rectangular matrix</u>.

Then if $F = (f(i,j))$ is an $n \times m$ 2D image to transform, we have

$$\boxed{\begin{array}{l} (1)\ \underset{(n\times n)}{S_n}\ \underset{(n\times m)}{F}\ \underset{(m\times m)}{S_m} := \underset{(n\times m)}{T} \\ \\ (2)\ \underset{(n\times n)}{S_n}\ \underset{(n\times m)}{T}\ \underset{(m\times m)}{S_m} = \underset{(n\times m)}{F}. \end{array}}$$

(1) the transformed image, and

(2) the reconstruction.

(630) $F = \sum_u \sum_v T(u,v) \cdot \underbrace{(\bar{s}_u \otimes \bar{s}_v)}_{\text{These are the basis images}}$

$S_{ij} = S(i,j) = \frac{1}{\sqrt{N}} h_j\left(\frac{i}{N}\right) \longrightarrow S = \begin{bmatrix} S(0,0) & S(0,1) \\ S(1,0) & S(1,1) \end{bmatrix} = \begin{bmatrix} (1/\sqrt{2})h_0\left(\frac{0}{2}\right) & (1/\sqrt{2})h_0\left(\frac{1}{2}\right) \\ (1/\sqrt{2})h_1\left(\frac{0}{2}\right) & (1/\sqrt{2})h_1\left(\frac{1}{2}\right) \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\longrightarrow \bar{h}_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\longrightarrow \bar{h}_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$h_0(x) = 1, \quad x \leq 1$

$h_1(x)$: $u = 1 = 2^0 + 0 \rightarrow p = q = 0$, so $2^{p/2} = 0$.

$q/2^p = 0/1 = 0$

$q+\frac{1}{2}/2^p = \frac{1}{2}/1 = \frac{1}{2}$

$q+1/2^p = 1/1 = 1 \longrightarrow h_1(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}) \\ -1, & x \in [\frac{1}{2}, 1) \end{cases}$.

$\longrightarrow h_1(0) = 1$

$h_1\left(\frac{1}{2}\right) = -1$

$$\boxed{\begin{array}{cc} \underset{h_0 \otimes h_0}{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}, & \underset{h_0 \otimes h_1}{\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}} \\ \\ \underset{h_1 \otimes h_0}{\begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}}, & \underset{h_1 \otimes h_1}{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}} \end{array}}$$

Basis images