# Problem Set 8

D. Zack Garza

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## Contents



# 1 Problem 1

## 1.1 Part a

It follows from the definition that $\|f\|_\infty = 0 \iff f = 0$ almost everywhere, and if $\|f\|_\infty$ is the best upper bound for $f$ almost everywhere, then $\|cf\|_\infty$ is the best upper bound for $cf$ almost everywhere.

So it remains to show the triangle inequality. Suppose that $|f(x)| \leq \|f\|_\infty$ a.e. and $|g(x)| \leq \|g\|_\infty$ a.e., then by the triangle inequality for the $|\cdot|_{\mathbb{R}}$ we have

$$\begin{aligned} |(f+g)(x)| &\leq |f(x)| + |g(x)| \quad a.e. \\ &\leq \|f\|_\infty + \|g\|_\infty \quad a.e., \end{aligned}$$

which means that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ as desired.

## 1.2 Part b

$\implies$ : Suppose $\|f_n - f\|_\infty \to 0$, then for every $\varepsilon$, $N_\varepsilon$ can be chosen large enough such that $|f_n(x) - f(x)| < \varepsilon$ a.e., which precisely means that there exist sets $E_\varepsilon$ such that $x \in E_\varepsilon \implies |f_n(x) - f(x)|$ and $m(E_\varepsilon^c) = 0$.

But then taking the sequence $\varepsilon_n := \frac{1}{n} \to 0$, we have $f_n \rightrightarrows f$ uniformly on $E := \bigcap_n E_n$ by definition, and $E^c = \bigcup_n E_n^c$ is still a null set.

$\impliedby$ : Suppose $f_n \rightrightarrows f$ uniformly on some set $E$ and $m(E^c) = 0$. Then for any $\varepsilon$, we can choose $N$ large enough such that $|f_n(x) - f(x)| < \varepsilon$ on $E$; but then $\varepsilon$ is an upper bound for $f_n - f$ almost everywhere, so $\|f_n - f\|_\infty < \varepsilon \to 0$.

## 1.3 Part c

To see that simple functions are dense in $L^\infty(X)$, we can use the fact that $f \in L^\infty(X) \iff$ there exists a $g$ that $f = g$ a.e. and $g$ is bounded.

Then there is a sequence $s_n$ of simple functions such that $\|s_n - g\|_\infty \to 0$, which follows from a proof in Folland:

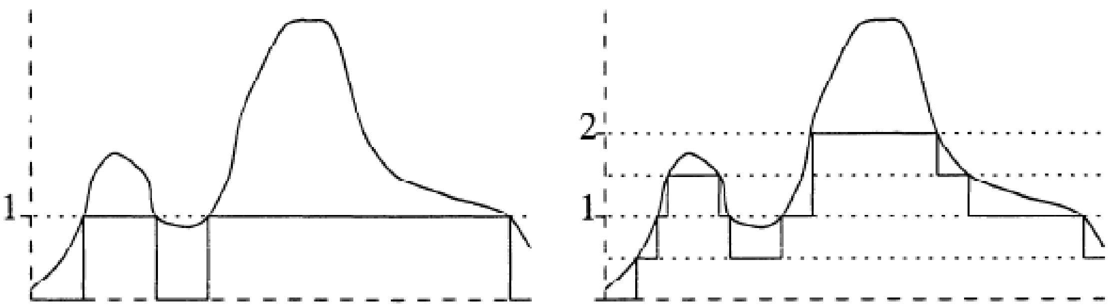
*Proof.* (a) For $n = 0, 1, 2, \ldots$ and $0 \le k \le 2^{2n} - 1$, let

$$E_n^k = f^{-1}\big((k2^{-n}, (k+1)2^{-n}]\big) \quad \text{and} \quad F_n = f^{-1}\big((2^n, \infty]\big),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n \chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that $\phi_n \le \phi_{n+1}$ for all $n$, and $0 \le f - \phi_n \le 2^{-n}$ on the set where $f \le 2^n$. The result therefore follows.




However, $C_c^0(X)$ is dense $L^\infty(X) \iff$ every $f \in L^\infty(X)$ can be approximated by a sequence $\{g_k\} \subset C_c^0(X)$ in the sense that $\|f - g_n\|_\infty \to 0$. To see why this can not be the case, let $f(x) = 1$, so $\|f\|_\infty = 1$ and let $g_n \to f$ be an arbitrary sequence of $C_c^0$ functions converging to $f$ pointwise.

Since every $g_n$ has compact support, say $E$, then $g_n|_{E^c} \equiv 0$ and $m(E^c) > 0$. In particular, this means that $\|f - g_n\|_\infty = 1$ for every $n$, so $g_n$ can not converge to $f$ in the infinity norm.

# 2 Problem 2

## 2.1 Part a

### 2.1.1 Part i

**Lemma:** $\|1\|_p = m(X)^{1/p}$

This follows from $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$ and taking $p$th roots.

By Holder with $p = q = 2$, we can now write

$$\begin{aligned} &\|f\|_1 = \|1 \cdot f\|_1 \leq \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\ \implies &\|f\|_1 \leq m(X)^{1/2} \|f\|_2. \end{aligned}$$

We also have

$$\begin{aligned} &\|f\|_2^2 = \int_X |f|^2 \leq \int_X |\|f\|_\infty|^2 = \|f\|_\infty \int_X 1 = \|f\|_\infty^2 m(X) \\ \implies &\|f\|_2 \leq m(X)^{1/2} \|f\|_\infty \\ \implies \quad &m(X)^{1/2} \|f\|_2 \leq m(X) \|f\|_\infty, \end{aligned}$$

and combining these yields

$$\|f\|_1 \leq m(X)^{1/2} \|f\|_2 \leq m(X) \|f\|_\infty.$$

, from which we can immediately conclude that if $m(X) < \infty$, we have $L^\infty(X) \subseteq L^2(X) \subseteq L^1(X)$.

To see that the inclusion is strict in general, we'll use the fact that $\sum_k k^{-2} < \infty$ and $\sum_k k^{-1} = \infty$: $\exists f \in L^1(X) \setminus L^2(X)$:

Let $E_k = X \bigcap B(t_k)$ where $B(t_k)$ is a ball centered at the origin (with radius depending on a parameter), $A_k = E_k \setminus E_{k-1}$ to be disjoint annuli where $t_k$ is chosen for each $k$ such that $m(A_k) = 1$.

(Note: since $m(X) < \infty$, $X \subseteq \bigcup A_k$.)

Then define $f : X \to \mathbb{R}$ by $f(x) = \sum_k s(k) \ \chi_{A_k}$, so we have

$$\begin{aligned} &\|f\|_1 = \int_X |f| = \sum_k \int_{A_k} s(k) = \sum_k s(k) \int_{A_k} 1 = \sum_k s(k) \\ &\|f\|_2 = \sum_k s(k)^2. \end{aligned}$$

Now setting $s(k) = \left(\frac{1}{k}\right)^{1/2}$ yields $\|f\|_1 < \infty$ but $\|f\|_2 = \infty$, so $f \notin L^2(X)$.

$\exists f \in L^2(X) \setminus L^\infty(X)$:

Let $X = (0, 1]$ and $f(x) = x^{-1/4}$. Then $\|f\|_2 = \int_0^1 \frac{1}{x^{1/4}} < \infty$ by the $p$-test, but $f$ is unbounded near 0. In particular, for any upper bound $M$, we have $m(\{x \ni f(x) > M\}) > 0$, so $\|f\|_\infty = \infty$ and $f \notin L^\infty(X)$.

### 2.1.2 Part ii

$\exists f \in L^2(X) \setminus L^1(X)$ when $m(X) = \infty$:

Take $X = [1, \infty)$ and let $f(x) = x^{-1}$. Then $\|f\|_2 < \infty$ but $\|f\|_1 = \infty$ by the $p$-test.

$\exists f \in L^\infty(X) \setminus L^2(X)$ when $m(X) = \infty$:

Take $X = \mathbb{R}$ and $f(x) = 1$. Then $\|f\|_\infty = 1 < \infty$ but $\|f\|_2 = \int_{\mathbb{R}} 1 = \infty$.

$L^2(X) \subseteq L^1(X) \implies m(X) < \infty$:

Let $f = \chi_X$, by assumption we can find a constant $M$ such that $\|\chi_X\|_2 \leq M\|\chi_X\|_1$.

Then pick a sequence of sets $E_k \nearrow X$ such that $m(E_k) < \infty$ for all $k$, $\chi_{E_k} \nearrow \chi_X$, and thus $\|\chi_{E_k}\|_p \leq M\|\chi_E\|_p$. By the lemma, $\|\chi_{E_k}\|_p = m(E)^{1/p}$, so we have

$$\begin{aligned}
\|\chi_{E_k}\|_2 \leq M\|\chi_{E_k}\|_1 &\implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \leq M \\
&\implies \frac{m(E_k)^{1/2}}{m(E_k)} \leq M \\
&\implies m(E_k)^{-1/2} \leq M \\
&\implies m(E_k) \leq M^2 < \infty.
\end{aligned}$$

and by continuity of measure, we have $\lim_K m(E_k) = m(X) \leq M^2 < \infty$. $\square$

## 2.2 Part b

# 3 Problem 3

# 4 Problem 4

# 5 Problem 5

# 6 Problem 6