Section 8.6: The Solutions of the Floer Equation are "Somewhere Injective".

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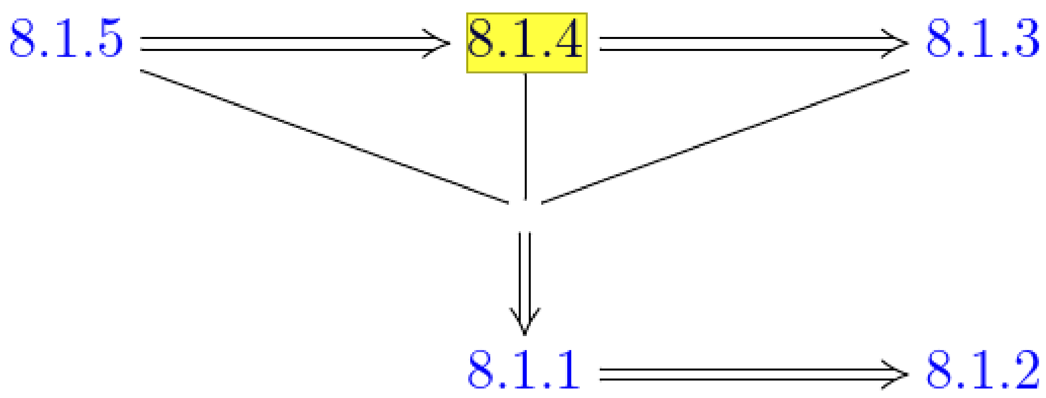
Contents

0.1 Outline

- Goal: Prove "injectivity" and continuation principle used in 8.5
- Prove Theorem 8.5.4
- Prove the continuation principle that was used in Proposition 8.1.4

Outline of statements:



- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- 8.1.4: $\Gamma : W^{1,p} \times C^\infty_\varepsilon \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x, y, J)$ is a Banach manifold
- 8.1.1: For $h \in \mathcal{H}_{\text{reg}}$, H_0+h is nondegenerate and $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(H_0+h, J)$.

- 8.1.2: For $h \in \mathcal{H}_{\text{reg}}$ and all contractible orbits x, y of H_0, $\mathcal{M}(x, y, H_0 + h)$ is a manifold of dimension $\mu(x) - \mu(y)$.

Set up notation:

- $z = s + it$
- u is a solution to an equation (appearing below)
- X is a vector field (time-dependent and periodic) on $\mathbb{R}^{2n}$
- X, J are smooth
- $C(u)$ the set of critical points u
- $R(u)$ the set of regular points of u

Goal: prove the following theorem

Theorem 0.1 ***(8.5.4).***
$C(u)$ is discrete and $R(u) \hookrightarrow \mathbb{R} \times S^1$ is open and dense.

Goal: prove a continuation principle:

Proposition 0.2 ***(8.6.6, Continuation Principle).***
Let Y be a solution to the perturbed CR equation on an open subset $U \subseteq \mathbb{R}^2$, then the set

$$C := \left\{ (s,t) \in U \;\middle|\; Y \text{ has an infinite order zero at } (s,t) \right\}$$

is clopen. In particular, if U is connected and $Y = 0$ on some nonempty $V \subset U$, then $Y \equiv 0$.

Proposition 0.3 ***(8.1.4,).***

Define

$$\mathcal{Z}(x, y, J) := \left\{ (u, H_0 + h) \,|\, h \in \mathcal{C}_\varepsilon^\infty (H_0) \text{ and } u \in \mathcal{M}(x, y, J, H) \right\}.$$

If $(u, H_0 + h) \in \mathcal{Z}(x, y)$ then the following map admits a continuous right-inverse and is surjective:

$$\begin{aligned} \Gamma : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \times \mathcal{C}_\varepsilon^\infty (H_0) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ (Y, h) &\longmapsto \left(d\mathcal{F}^{H_0+h}\right)_u (Y) + \operatorname{grad}_u h \end{aligned}$$

where $\mathcal{F}^{H_0+h}$ is the Floer operator corresponding to H_+h.

Used to show (via the implicit function theorem) that $\mathcal{Z}(x, y, J)$ is a Banach manifold when $x \neq y$.

Proposition 0.4 ***(8.6.1, Transform to CR-equation on R2).***
If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t, u)\left(\frac{\partial u}{\partial t} - X(t, u)\right) = 0.$$

Then there exist

- An almost complex structure J
- A diffeomorphism ϕ on W ?
- A map $v \in C^\infty(\mathbb{R}^2; W)$

satisfying

- $\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t}\right)v = 0$
- $v(s, t+1) = \varphi(v(s,t))$
- $C(u) = C(v)$, i.e. u, v have the same critical points
- $R(u) = R(v)$.

Proof: short, include.

Lemma 8.6.2: The set of critical points of v above is discrete. Precisely: There exists a constant $\delta > 0$ such that $(dv)_z \neq 0$ for any $0 << |z| < \delta$.

Proof: Postponed to p.264.

Definition: Multiple points

Proposition 8.6.3: Injectivity result. Let v be a smooth 1-periodic (in t) solution of the CR equation, i.e. $v(s, t+1) = \phi(v(s,t))$ for some smooth ϕ ? and $\frac{\partial v}{\partial s} \neq 0$. Then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.

0.2 Regular Points Are Open and Dense

Proof (BIG):

- Show $R(v)$ is open (easy)
- Show $R(v)$ is dense (delicate)

Long proof.

Lemma 8.6.4: For every $r > 0$ there exists a $\delta > 0$ such that

$$|t - t_0|,\ |s - s_0| < \delta \implies \exists s' \in B_r(s_j) \text{ s.t. } v(s,t) = v(s',t).$$

Proof: short.

Lemma 8.6.5: Let v_1, v_2 be two solutions of the CR-equation with $X_t \equiv 0$ on $B_\varepsilon(0)$, $v_1(0,0) = v_2(0,0)$ such that $(dv_1)_0, (dv_2)_0 \neq 0$. Also suppose

$$\forall \varepsilon\ \exists \delta \text{ s.t.}$$

$$\forall (s,t) \in B_\delta(0),\ \exists s' \in \mathbb{R} \begin{cases} (s',t) \in B_\varepsilon(0) \\ v_1(s,t) = v_2(s',t) \end{cases}.$$

Then

$$\forall z \in B_\varepsilon(0), \quad v_1(s,t) = v_2(s,t).$$

Take perturbed CR equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Fix $S \in C^\infty(\mathbb{R}^2; \mathrm{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$ be a solution to the perturbed CR equation and let $p > 2$. Then there exists $0 < \delta < \varepsilon$ and a map $A \in W^{1,p}(B_\delta, \mathrm{GL}(\mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_\delta \longrightarrow \mathbb{C}^n$ such that

$$\forall (s,t) \in B_\delta \quad Y(s,t) = A(s,t)\ \sigma(s+it) \quad \text{and} \quad J_0 A)s,t) = A(s,t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of $\bar\partial$) For every $p > 1$, the following operator is surjective and Fredholm:

$$\bar\partial : W^{1,p}\Big(S^2; \mathbb{C}^n\Big) \longrightarrow L^p(\Lambda^{0,1} T^* S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7