Cohomology groups of Lens spaces

Consider the scaling action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\} \simeq S^{2n+1}$, $n \geq 1$. By identifying $\mathbb{Z}/q$ with the q^{th} roots of unity in $\mathbb{C}^*$ we get an action of $\mathbb{Z}/q$ on S^{2n+1}. We call the quotient $L(n,q)$ a *Lens Space*. We allow $n = \infty$.

The action of $\mathbb{Z}/q$ on S^{2n+1} is clearly free, so the quotient map is a covering map with deck group $\mathbb{Z}/q$. S^{2n+1} is simply connected so it is the universal cover of $L(n,q)$. This tells us that $\pi_1 L(n,q) = \mathbb{Z}/q$ and all higher homotopy groups agree with those of the sphere. In particular $L(\infty,q) = K(\mathbb{Z}/q,1)$.

Covering maps are fibrations so we have a fibration

$$\begin{tikzcd} \mathbb{Z}/q\mathbb{Z} \arrow[r] & S^{2n+1} \arrow[d] \\ & L(n,q) \end{tikzcd}$$

At this point one is tempted to use the Serre spectral sequence and compute us some cohomology. Alas $L(n,q)$ is not simply connected. Instead we will write $L(n,q)$ as the *total space* of a fibration.

Note that even after modding out by $\mathbb{Z}/q$ we still have a "leftover" action of $S^1/(\mathbb{Z}/q) = S^1$ on $L(n,q)$. If we mod out by this action then we get $S^{2n+1}/S^1 = \mathbb{CP}^n$:

$$S^1 \longrightarrow L(n,q) \longrightarrow \mathbb{CP}^n$$

Now $S^{2n+1} \to \mathbb{CP}^n$ is a locally trivial fiber bundle with fiber S^1 - locally it looks like $S^1 \times U \to U$. But then $L(n,q) \to \mathbb{CP}^n$ locally looks like $S^1/(\mathbb{Z}/q) \times U \to U$. By Hurwitz's theorem, $L(n,q) \to \mathbb{CP}^n$ is a fibration with fiber S^1. This time the base space *is* simply connected so we get a spectral sequence

$$E_2^{p,q} = H^p(\mathbb{CP}^n, H^q(S^1,\mathbb{Z})) \to H^{p+q}(L(n,q),\mathbb{Z})$$

The E_2 page looks like

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & & 0 & \\ H^0(\mathbb{CP}^n, H^1(S^1,\mathbb{Z})) & 0 & H^2(\mathbb{CP}^n, H^1(S^1,\mathbb{Z})) & 0 & \dots & H^{2n}(\mathbb{CP}^n, H^1(S^1,\mathbb{Z})) & 0 \\ H^0(\mathbb{CP}^n, H^0(S^1,\mathbb{Z})) & 0 & H^2(\mathbb{CP}^n, H^0(S^1,\mathbb{Z})) & 0 & \dots & H^{2n}(\mathbb{CP}^n, H^0(S^1,\mathbb{Z})) & 0 \end{array}$$

where the point is that all other entries are zero. In other words it looks like

$$\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \dots & \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & \dots & \mathbb{Z} & 0 & 0 \end{array}$$

Because of the positioning of the nonzero entries, all differentials on pages 3 and higher are zero. So $E_3 = E_\infty$.

Now we claim that d_2 is multiplication by q. Recall that $H_1(L,\mathbb{Z}) = \mathbb{Z}/q$. By the universal coefficient theorem so is $H^2(L,\mathbb{Z})$. Let x be a generator of $\mathbb{Z} = H^0(\mathbb{CP}^n, H^1(S^1,\mathbb{Z}))$ and y be a generator of $\mathbb{Z} = H^2(\mathbb{CP}^n, H^0(S^1,\mathbb{Z}))$. Considering the second diagonal of the $E_\infty = E_3$ page we see that the image of the map $d_2 : E_2^{0,2} \to E_2^{1,0}$ is $q\mathbb{Z}$, so $d_2(x) = qy$ - multiplication by q. d_2 is a differential with respect to the cup product, so $d_2(xy) = d_2(x)y + xd_2(y) = qy^2$. Using the Kunneth formula we know the cup product structure on $H^*(\mathbb{CP}^n, H^0(S^1,\mathbb{Z}))$, and this tells us that y^2 is in fact the generator of $H^4(\mathbb{CP}^n, H^0(S^1,\mathbb{Z}))$. Continuing in this fashion we fnd that all the d_2 maps are multiplication by q, except for the last map $H^{2n}(\mathbb{CP}^n, H^1(S^1,\mathbb{Z})) \to H^{2n+2}(\mathbb{CP}^n, H^0(S^1,\mathbb{Z}))$ which is necessarily zero.

Therefore the E_3 page is concentrated in the bottom row except for one entry in the top (first) row. The extension problem for going from E_∞ to the cohomology of the total space is vacuous here, and we conclude

$$H^i(L(n,q),\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/q\mathbb{Z} & i=2j,\, j\leq n \\ \mathbb{Z} & i=2n+1 \\ 0 & otherwise \end{cases}$$

These formulas work for $n=\infty$ as well, interpreted in the obvious way.

Computation of $\pi_{n+1}(S^n)$

Let X be a CW complex and n be a natural number. By adding cells of dimension at least $n+2$ to X we can kill off all $\pi_i(X)$ for $i>n$, and by CW approximation this will have no effect on $\pi_i(X)$ for $i\leq n$. In this way we get a space Y_n with an inclusion $X \hookrightarrow Y_n$ inducing an isomorphism on π_i for $i\leq n$, and with $\pi_i(Y_n)=0$ for $i>n$. By adding even more cells of dimension at least $n+1$ we get a space Y_{n-1} satisfying similar conditions. Further, we can assume that the inclusion $Y_n \hookrightarrow Y_{n-1}$ is a fibration. By the long exact sequence of fundamental groups associated to a fibration, the fiber of this fibration is a $K(\pi_n(X),n)$. Summarizing the the above discussion, we get:

Lemma 1. *Let X be a CW complex. Then for any n there exists a fibration $\pi: Y_n \to Y_{n-1}$ of CW complexes with fiber $K(\pi_n(X),n)$, such that $\pi_i(Y_{n-1}) = \pi_i(X)$ for $i\leq n-1$ and $\pi_i(Y_{n-1})=0$ for $i\geq n$. Finally, Y_n (is homotopic to a CW complex that) differs from X only by cells of dimensino $\geq n+2$.*

The main result of this section is:

Theorem 1. $\pi_4(S^3) = \mathbb{Z}/2$

Proof. Apply the construction above in the case $n=4$, $X=S^3$, obtaining a fibration $Y_4 \to Y_3$ with fiber $F = K(\pi_4(S^3),4)$. Note that to get Y_3 we killed off all higher homotopy groups of S^3, so $Y_3 = K(\mathbb{Z},3)$. By the Hurewicz theorem,

$$H_q(Y_3,\mathbb{Z}) = \begin{cases} 0 & q=1,2 \\ \mathbb{Z} & q=3 \\ ? & q>3 \end{cases}$$

$$H_q(F,\mathbb{Z}) = \begin{cases} 0 & q=1,2,3 \\ \pi_4(S^3) & q=4 \\ ? & q>4 \end{cases}$$

Now consider the homology spectral sequence for the fibration $F \hookrightarrow Y_4 \to Y_3$. The E^2 page looks like

$$\begin{array}{ccccc}
\pi_4(S^3) & ? & ? & & \dots \\
0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & \dots \\
\mathbb{Z} & 0 & 0 & H_5(K(\mathbb{Z},3),\mathbb{Z}) & \dots
\end{array}$$

Note that
$$Y_4 = S^3 \cup (\textit{cells of dimension } \geq 6),$$
hence $H_4(Y_4) = 0 = H_5(Y_4)$. Thus all entries on the fourth and fifth diagonals of E^∞ are zero. The only differential that can affect $\pi_4(S^3)$ is $d^5 : H_5(K(\mathbb{Z},3),\mathbb{Z}) \to \pi_4(S^3)$, and by the previous remark, this map has to be an isomorphism. Hence
$$\pi_4(S^3) \cong H_5(K(\mathbb{Z},3),\mathbb{Z}).$$

By the cohomology spectral squence of the path fibration for $K(\mathbb{Z},3)$, one easily obtains

$$TorH^6(K(\mathbb{Z},3)) = \mathbb{Z}/2,\ FreeH^5(K(\mathbb{Z},3)) = 0,$$

hence $H_5(K(\mathbb{Z},3)) = \mathbb{Z}/2$.

$\square$

Corollary 1. $\pi_4(S^2) = \mathbb{Z}/2$

Proof. Apply the above calculation to the long exact sequence of homotopy groups for the Hopf fibration. $\square$

Theorem 2. *(Serre) For $n \geq 3$, $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$.*

Proof. Follows from $\pi_4(S^3) = \mathbb{Z}/2$ and the suspension theorem. $\square$

Whitehead Towers

Let X be a connected CW complex.

Definition. A Whitehead tower of X is a sequence of fibrations $\ldots \to X_n \to X_{n-1} \to \ldots \to X_0 = X$, such that

1. X_n is n-connected
2. $\pi_q(X_n) = \pi_q(X)$ for $q \geq n+1$
3. the fiber of $X_n \to X_{n-1}$ is a $K(\pi_n, n-1)$

Up to homotopy this may be viewed as a generalization of the universal cover construction: X_1 is a 1-connected space whose higher homtopy groups agree with those of X. The fiber of $X_1 \to X_0$ is a $K(\pi_1, 0)$, so it is (up to homotopy) a discrete space on $|\pi_1|$ many points.

Lemma 2. *For X a CW complex, Whitehead towers exist.*

Proof. We construct X_n inductively. Suppose that X_{n-1} has already been defined. Add cells to X_{n-1} to kill off $\pi_q(X_{n-1})$ for $q \geq n+1$. So we get a space Y which, by induction, is a $K(\pi_n, n)$. Now define the space

$$X_n := \{f : I \to Y, f(o) = *, f(1) \in X_{n-1}\}$$

consisting of of paths in Y beginning at a basepoint $* \in X_{n-1}$ and ending somewhere in X_{n-1}. Give it the compact-open topology. Then the map $\pi : X_n \to X_{n-1}$ defined by $\gamma \to \gamma(1)$ is a fibration.

Note that the fiber of π is just $\Omega Y = K(\pi_n, n-1)$. In particular it is a $K(\pi_n, n-1)$. Now consider the long exact sequence of homotopy groups associated to the fibration:

$$\ldots \pi_{i+1}(X_{n-1}) \to \pi_i(\Omega Y) \to \pi_i(X_n) \to \pi_i(X_{n-1}) \to \ldots$$

For $i < n-1$, and for $i > n$ get that $\pi_i(X_n) = \pi_i(X_{n-1}) = \pi_i$. The interesting part is

$$\pi_n(\Omega Y) \to \pi_n(X_n) \to \pi_n(X_{n-1}) \to \pi_{n-1}(\Omega Y) \to \pi_{n-1}(X_n) \to \pi_{n-1}(X_{n-1})$$

or,

$$0 \to \pi_n(X_n) \to \pi_n \to \pi_n \to \pi_{n-1}(X_n) \to 0$$

If we can show that the map $\pi_n \to \pi_n$ in the middle, i.e. the boundary map $\pi_n(X_{n-1}) \to \pi_{n-1}(\Omega Y)$, is an isomorphism, then we are done.

Note that we have an isomorphism $\pi_n(X_{n-1}) \to \pi_{n-1}(\Omega Y)$ by taking the map $[S^n, X_{n-1}] \to [S^n, Y]$ induced by the inclusion (which is an isomorphism by construction of Y) and following it with the natural isomorphism $[S^n, Y] \cong [S^{n-1}, \Omega Y]$. In fact the resulting map $\pi_n(X_{n-1}) \to \pi_{n-1}(\Omega Y)$ is precisely the boundary map from the long exact sequence above. Think about the definition of the boundary map. Recall that for $\alpha : (I^n, \partial I^n) \to (X_{n-1}, x_0)$ we use the lifting property of the fibration to get a map $\alpha' : (I^n, \partial I^n) \to (X_n, \Omega Y)$ and then restrict to get $S^{n-1} = \partial I^n \to \Omega Y$. In our case we can choose an explicit lift α'. Namely send $\vec{v} \in I^n$ to the path $t \to \alpha(t\vec{v})$. Restricted to ∂I^n this is just the map we get from the natural isomorphism.

□

Calculation of $\pi_4(S^3)$ and $\pi_5(S^3)$

Let us consider the Whitehead tower for $X = S^3$. S^3 is 2-connected so (in the notation from the definition of Whitehead towers) $X = X_1 = X_2$. Let $\pi_i := \pi_i(X)$. We have fibrations

$$\begin{tikzcd}
K(\pi_4, 3) \arrow[r] & X_4 \arrow[d] \\
K(\pi_3, 2) \arrow[r] & X_3 \arrow[d] \\
& S^3
\end{tikzcd}$$

Note that $\pi_3 = \mathbb{Z}$ so $K(\pi_3, 2) = \mathbb{CP}^\infty$. Moreover, we have by definition and Hurewicz that :

$$\pi_5(S^3) \cong \pi_5(X_4) \cong H_5(X_4)$$

$$\pi_4(S^3) \cong \pi_4(X_3) \cong H_4(X_3)$$

Now consider the cohomology spectral sequence for the bottom fibration,

$$E_2^{p,q} = H^p(S^3, H^q(\mathbb{CP}^\infty, \mathbb{Z})) \to H^{p+q}(X_3, \mathbb{Z})$$

The E_2 page looks like

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
H^0(S^3, H^4(\mathbb{CP}^\infty, \mathbb{Z}) & 0 & 0 & H^3(S^3, H^4(\mathbb{CP}^\infty, \mathbb{Z})) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
H^0(S^3, H^2(\mathbb{CP}^\infty, \mathbb{Z}) & 0 & 0 & H^3(S^3, H^2(\mathbb{CP}^\infty, \mathbb{Z})) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & \dots \\
H^0(S^3, H^0(\mathbb{CP}^\infty, \mathbb{Z}) & 0 & 0 & H^3(S^3, H^0(\mathbb{CP}^\infty, \mathbb{Z})) & 0 & \dots
\end{array}
$$

Thus $d_2 = 0$, so $E_2 = E_3$. In addition, for $r \geq 4$, $d_r = 0$. So $E_4 = E_\infty$. X_3 is 3-connected so (by Hurewicz) all entries on the 2nd and 3rd diagonals of E_4 are 0. In particular, $d_3 : H^0(S^3, H^2(\mathbb{CP}^\infty, \mathbb{Z}) \to H^3(S^3, H^0(\mathbb{CP}^\infty, \mathbb{Z}))$ must be an isomorphism. By the Kunneth formula, both of these groups are isomorphic to $\mathbb{Z}$. Let x be a generator of the former and u be a generator of the latter, so $d_3(x) = u$. From what we know of $\mathbb{CP}^\infty$, x^n generates $H^0(S^3, H^{2n}(\mathbb{CP}^\infty, \mathbb{Z})$. By the Leibnitz rule, $d_3 x^n = nx^{n-1}dx = nx^{n-1}u$. This tells us exactly what the E_4 page is like, and we get

$$
\begin{array}{rcl}
H^3(X_3, \mathbb{Z}) & = & 0 \\
H^4(X_3, \mathbb{Z}) & = & 0 \\
H^5(X_3, \mathbb{Z}) & = & \mathbb{Z}/2 \\
H^6(X_3, \mathbb{Z}) & = & 0 \\
H^7(X_3, \mathbb{Z}) & = & \mathbb{Z}/3 \\
H^8(X_3, \mathbb{Z}) & = & 0 \\
H^9(X_3, \mathbb{Z}) & = & \mathbb{Z}/4 \\
\vdots & & \vdots
\end{array}
$$

By the universal coefficient theorem,

$$
\begin{array}{rcl}
H_3(X_3, \mathbb{Z}) & = & 0 \\
H_4(X_3, \mathbb{Z}) & = & \mathbb{Z}/2 \\
H_5(X_3, \mathbb{Z}) & = & 0 \\
H_6(X_3, \mathbb{Z}) & = & \mathbb{Z}/3 \\
H_7(X_3, \mathbb{Z}) & = & 0 \\
H_8(X_3, \mathbb{Z}) & = & \mathbb{Z}/4 \\
\vdots & & \vdots
\end{array}
$$

In particular, $\pi_4 = H_4(X_3) = \mathbb{Z}/2$.

In order to get the next homotopy group, we use the *homology* spectral sequence for the top fibration. Note that (by Hurewicz) $H_i(K(\pi_4, 3), \mathbb{Z})$ is zero for $i < 3$ and $H_3(K(\pi_4, 3), \mathbb{Z}) = \pi_4 = \mathbb{Z}/2$. Thus the E^2

page of the homology spectral sequence looks like

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
H_0(X_3, H_5(K(\pi_4,3),\mathbb{Z})) & 0 & 0 & 0 & & & \\
? & 0 & 0 & 0 & & & \\
H_0(X_3, H_3(K(\pi_4,3),\mathbb{Z})) \cong \mathbb{Z}/2 & 0 & 0 & 0 & H_4(X_3, H^3(K(\pi_4,3),\mathbb{Z})) & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H_0(X_3,\mathbb{Z}) \cong \mathbb{Z} & 0 & 0 & 0 & H_4(X_3,\mathbb{Z}) \cong \mathbb{Z}/2 & 0 & \mathbb{Z}/3
\end{array}
$$

On the portion of the spectral sequence shown in the diagram above, $E_2 = E_4$. Further, we know that all entries on diagonals 3 and 4 at E^∞ are zero. Therefore the map $d^4 : H_4(X_3,\mathbb{Z}) \to H_0(X_3, H^3(K(\pi_4,3),\mathbb{Z}))$ must be an isomorphism. But the former group is $\mathbb{Z}/2$ and the latter group is $H^3(K(\pi_4,3),\mathbb{Z}) \cong \pi_4$. So we get back $\pi_4 = \mathbb{Z}/2$.

Moreover, by a spectral sequence argument on the path fibration of $K(\mathbb{Z}/2,3)$, we obtain: $H_5(K(\mathbb{Z}/2,3)) = \mathbb{Z}/2$. Note also that $E^2_{0,5} \cong \mathbb{Z}/2$, and this entry can only be affected by $d^6 : E^6_{6,0} \cong \mathbb{Z}/3 \to E^6_{0,5} = E^2_{0,5} \cong \mathbb{Z}/2$, which is the zero map, so $E^\infty_{0,5} = \mathbb{Z}/2$. Thus, on the fifth diagonal of E^∞, all entries are zero except $E^\infty_{0,5} = \mathbb{Z}/2$, which yields $H_5(X_4) = \mathbb{Z}/2$, i.e., $\pi_5(S^3) = \mathbb{Z}/2$.