Title

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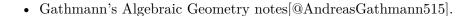# Contents

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

$$k[x_1, \cdots, x_n] \quad \text{The polynomial ring in } n \text{ indeterminates } \left\{ f(x_1, \cdots, x_n) = \sum \right\}$$

$$k(x_1, \cdots, x_n) V(J), V_a(J) \qquad \text{The variety associated to an ideal } J \trianglelefteq k[x_1, \cdots, x_n]$$

$$I(S), I_a(S) \qquad \text{The ideal associated to a subset } S \subseteq \mathbb{A}_k^n$$

$$A(X) \coloneqq k[x_1, \cdots, x_n]/I(X) \qquad \text{The coordinate ring of a variety}$$

$$\mathcal{O}_X \qquad \text{The structure sheaf } \left\{ f : U \to k \mid \ \right\}.$$

> Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:

    - The Zariski topology?

- Irreducibility?
- Connectedness?
- Dimension?

- What is the definition of a presheaf?

  - What are some examples and counterexamples?

- What is the definition of sheaf?

  - What are some examples?
  - What are some presheaves that are not sheaves?

- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?

  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?

- What is a morphism between two affine varieties?
- What is the definition of separatedness?

  - What are some examples of spaces that are and are not separated?

- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?

  - What are some examples and counterexamples of complete spaces?

0.4 Useful Algebra Facts

Fact 0.4.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f, then $\langle f \rangle$ is maximal

> **Proposition 0.4.2** *(Finitely generated polynomial rings are Noetherian).*
> A polynomial ring $k[x_1, \cdots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \cdots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \cdots, f_m \rangle$.

> *Proof (?).*
> A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.4.5), $k[x_1, \cdots, x_n]$ is thus Noetherian.
> ∎

> **Proposition 0.4.3** *(Properties and Definitions of Ideal Operations).*
>
> $$I + J := \left\{ f + g \mid f \in I, \, g \in J \right\}$$
>
> $$IJ := \left\{ \sum_{i=1}^{N} f_i g_i \mid f_i \in I, \, g_i \in J, N \in \mathbb{N} \right\}$$
>
> $I + J = \langle 1 \rangle \implies I \cap J = IJ$ (coprime or comaximal) $\langle a \rangle + \langle b \rangle = \langle a, b \rangle$.

> **Theorem 0.4.4** *(Noether Normalization).*
> Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

> **Theorem 0.4.5** *(Hilbert's Basis Theorem).*
> If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomi

$$I \mapsto V(I) := \left\{ a \;\middle|\; f(a) = 0 \forall f \in I \right\}$$

$$I(X) := \left\{ f \;\middle|\; f|_X = 0 \right\} \hookleftarrow X$$

Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets