

# Title

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Let  $k = \bar{k}$  and  $R$  a ring containing ideals  $I, J$ .

**Definition 1.0.1** (Radical).

Recall that the *radical* of  $I$  is defined as

$$\sqrt{I} = \left\{ r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

**Example 1.1.**

Let  $I = (x_1, x_2^2) \subset \mathbb{C}[x_1, x_2]$ , so  $I = \{f_1x_1 + f_2x_2 \mid f_1, f_2 \in \mathbb{C}[x_1, x_2]\}$ . Then  $\sqrt{I} = (x_1, x_2)$ , since  $x_2^2 \in I \implies x_2 \in \sqrt{I}$ .

Given  $f \in k[x_1, \dots, x_n]$ , take its value at  $a = (a_1, \dots, a_n)$  and denote it  $f(a)$ . Set  $\deg(f)$  to be the largest value of  $i_1 + \dots + i_n$  such that the coefficient of  $\prod x_j^{i_j}$  is nonzero.

**Example 1.2.**

$\deg(x_1 + x_2^2 + x_1x_2^3) = 4$

**Definition 1.0.2** (Affine Variety).

1. Affine  $n$ -space  $\mathbb{A}^n = \mathbb{A}_k^n$  is defined as  $\{(a_1, \dots, a_n) \mid a_i \in k\}$ .

Remark: not  $k^n$ , since we won't necessarily use the vector space structure (e.g. adding points).

2. Let  $S \subset k[x_1, \dots, x_n]$  to be a set of polynomials. Then define  $V(S) = \{x \in \mathbb{A}^n \mid f(x) = 0\} \subset \mathbb{A}^n$  to be an *affine variety*.

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**Example 1.3.**

- $\mathbb{A}^n = V(0)$ .
- For any point  $(a_1, \dots, a_n) \in \mathbb{A}^n$ , then  $V(x_1 - a_1, \dots, x_n - a_n) = \{a_1, \dots, a_n\}$  uniquely determines the point.
- For any finite set  $r_1, \dots, r_k \in \mathbb{A}^1$ , there exists a polynomial  $f(x)$  whose roots are  $r_i$ .

**Remark 1.**

We may as well assume  $S$  is an ideal by taking the ideal it generates,  $S \subseteq \langle S \rangle = \left\{ \sum g_i f_i \mid g_i \in k[x_1, \dots, x_n], f_i \in S \right\}$ .

Then  $V(\langle S \rangle) \subset V(S)$ .

Conversely, if  $f_1, f_2$  vanish at  $x \in \mathbb{A}^n$ , then  $f_1 + f_2, gf_1$  also vanish at  $x$  for all  $g \in k[x_1, \dots, x_n]$ . Thus  $V(S) \subset V(\langle S \rangle)$ .

**Lemma 1.1.**

1. If  $S_1 \subseteq S_2$  then  $V(S_1) \supseteq V(S_2)$ .
2.  $V(S_1 \cup S_2) = V(S_1 S_2) = V(S_1) \cap V(S_2)$ .

We thus have a map

$$V : \{\text{Ideals in } k[x_1, \dots, x_n]\} \longrightarrow \{\text{Affine varieties in } \mathbb{A}^n\}.$$

**Definition 1.1.1** (The Ideal of a Set).

Let  $X \subset \mathbb{A}^n$  be any set, then *the ideal of  $X$*  is defined as

$$I(X) := \left\{ f \in k[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in X \right\}.$$

**Example 1.4.**

Let  $X$  be the union of the  $x_1$  and  $x_2$  axes in  $\mathbb{A}^2$ , then  $I(X) = (x_1 x_2) = \{x_1 x_2 g \mid g \in k[x_1, x_2]\}$ .

Note that if  $X_1 \subset X_2$  then  $I(X_1) \supset I(X_2)$ .

**Proposition 1.2** (*The Image of  $V$  is Radical*).

$I(X)$  is a radical ideal, i.e.  $I(X) = \sqrt{I(X)}$ .

This is because  $f(x)^k = 0 \forall x \in X$  implies  $f(x) = 0$  for all  $x \in X$ , so  $f^k \in I(X)$  and thus  $f \in I(X)$ .

Our correspondence is thus

$$\begin{aligned} \{\text{Ideals in } k[x_1, \dots, x_n]\} &\xrightarrow{V} \{\text{Affine Varieties}\} \\ \{\text{Radical Ideals}\} &\xleftarrow{I} \{?\}. \end{aligned}$$

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**Proposition 1.3** (*Hilbert Nullstellensatz (Zero Locus Theorem)*).

- a. For any affine variety  $X$ ,  $V(I(X)) = X$ .
- b. For any ideal  $J$