8.8 Part 2, Computing the Index of L

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What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

- Define

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s,t)Y$$

  where

$$S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$$

$$S(s,t) \overset{s \to \pm\infty}{\Longrightarrow} S^\pm(t).$$

- 8.7: Shows L is Fredholm

- By the end of 8.8: replace L by L_1 with the same *index*

  - *(not the same kernel/cokernel)*

- Compute $\mathrm{Ind}\,L_1$: explicitly describe $\ker L_1, \mathrm{coker}\,L_1$.

- Replace in two steps:

  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s.
    * Replace $S(s,t)$ by a matrix

$$\tilde{S}(s,t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

    * Idea: approximate by cylinders at infinity.
    * Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    * Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\text{Ind}(L) = \mu\left(R^-(t)\right) - \mu\left(R^+(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

where $S : \mathbb{R} \longrightarrow \text{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\text{Ind}(R^\pm(t))$; then

$$\text{Ind}(L) = \text{Ind}(L_1) = \text{Ind}(R^-(t)) - \text{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing L_1 to $R^\pm$. Let $k^\pm := \text{Ind}(R^\pm)$; then $\text{Ind}(L_1) = k^- - k^+$.
- Lemma 8.8.4: $\text{Ind}(L_0) = \text{Ind}(L)$.
- Han's Talk:
    - Prop 8.8.3, using Lemma 8.8.5
- Me
    - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\text{Ind}(L_1) = K^- - k^+$.

Setup:

$$S(s) = \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) = \begin{cases} a_i^- & \text{if } s \le -s_0 \\ a_i^+ & \text{if } s \ge s_0 \end{cases}.$$

Statement: let $p > 2$ and define

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

> This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \text{Ker } F = 2 \cdot \#\left\{\ell \in \mathbf{Z} \,|\, a^- < 2\pi\ell < a^+\right\}$$
$$\dim \text{Ker } F^\star = 2 \cdot \#\left\{\ell \in \mathbf{Z} \,|\, a^+ < 2\pi\ell < a^-\right\}.$$

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2. Suppose $\sup\limits_{s\in\mathbb{R}}\|S(s)\| < 1$, then

$$\dim\operatorname{Ker} F = \#\left\{i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0\right\}$$

$$\dim\operatorname{Ker} F^\star = \#\left\{i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0\right\}.$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

0.3 Proof

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) = a_1(s) = a_2(s)$.

- Start with equation on $\mathbb{R}^2$,

$$Y(s,t) = (Y_1(s,t), Y_2(s,t))$$

- Replace with equation on $\mathbb{C}$:

$$Y(s,y) = Y_1(s,t) + iY_2(s,t)$$

.

- Rewrite the PDE $F(Y) = 0$ as $\bar{\partial}Y + S(s)Y = 0$, i.e.

$$\frac{\partial}{\partial s}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let $Y = B\tilde{Y}$ where $B \in \operatorname{GL}(1,\mathbb{C})$ satisfies $\left(\bar{\partial} + S\right)B = 0$ to obtain $\bar{\partial}\tilde{Y} = 0$.

    - Can choose $B = \begin{bmatrix} b(s)) 0 \\ 0 \quad b(s) \end{bmatrix}$ where $\dfrac{\partial b}{\partial s} = -a(s)b(s)$.
    - Explicitly, we can take the integral $b(s) = e^{\int_0^s -a(t)\,dt} = e^{-A(s)}$

- Remark: for some constants C_i, we have

$$A(s) = \begin{cases} C_1 + a^- s & s \le -\sigma_0 \\ C_2 + a^+ s & s \ge \sigma_0 \end{cases}.$$

- Remark: the new $\tilde{Y}$ satisfies CR. It is continuous and L^1_{loc} and thus by elliptic regularity C^∞. Its real/imaginary parts are harmonic.

Step 2: ?

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