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1 | Friday April 17th

Let X be locally compact and $\Gamma \curvearrowright X$ by homeomorphisms. Recall that Γ acts properly discontinuously at $x \in X$ iff there exists a neighborhood N_x such that $\left\{ \gamma \in \Gamma \;\middle|\; \gamma \cdot N_x \cdot N_x \neq \emptyset \right\}$ is finite. Equivalently, th e orbit $\Gamma \cdot x$ is discrete in X and the stabilizer Γ^x is finite. Moreover Γ acts freely iff Γ^x is trivial for all x.

Proposition For such X, Γ as above, the map $X \to \Gamma / X$ is a galois covering iff Γ acts freely and properly discontinuously.

Note: Galois covering here is just the usual notion involving deck transformations.

This requires Γ to be discrete, otherwise a sequence converging to the identity would have infinitely many neighborhoods converging on a neighborhood of the identity.

A *Fuchsian* group is a discrete subgroup of automorphisms of $\mathbb{H} \subset \mathbb{C}$, i.e. $\Gamma \subset \mathrm{Aut}_{\mathbb{C}}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$.

Fact Every Fuchsian group acts properly discontinuously on all of $\mathbb{H}$. Thus the discreteness which is necessary is also sufficient in this case. Then $\mathbb{H} \to \Gamma \backslash \mathbb{H}$ is a covering map and the action is free $\iff$ Γ has no elements of finite order.

Thus the complex structure (sheaf of holomorphic functions?) pushes forward to the quotient, giving it the structure of a Riemann surface.

Definition A *Kleinian* group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$.

Note that $\mathbb{H}$ carries a hyperbolic structure, and this group precisely preserves it.

Define the regular set as $\ell(\Gamma) = \left\{ x \in \mathbb{P}^1 \;\middle|\; x \text{ acts freely and PDC} \right\}$

Fact A Kleinian group is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a Fuschian group if it stabilizes a circle in $\mathbb{P}^1(\mathbb{C})$.

Exercise Let $\Gamma = \langle [q, 0; 0, 1] \rangle \subset \mathrm{PSL}(2, \mathbb{C})$ and $\Lambda(\Gamma) = \mathbb{P}^1 \setminus \{\mathbf{0}\}$. Show $\Lambda(\Gamma)$ is open but could be empty, has either 0,1,2, or infinitely many connected components.

Exercise Show that for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, $\Lambda(\Gamma) = \mathbb{H}^+ \cup \mathbb{H}^-$.

If $U(\Gamma) \neq \emptyset$, then $\Lambda(\Gamma) \to \Lambda(\Gamma)/\Gamma$ is a galois covering that endows the quotient with the structure of a Riemann surface.

For G a group and X a set with a map $M : G \times X \to X$, we say X is a G-set. We can make a category from this: given (X, μ_x) and (Y, μ_y), a G-map is given by a diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mu_x"] \arrow[d, "\mathrm{id} \times f"] & X \arrow[d, "f"] \\ G \times Y \arrow[r, "\mu_y"] & Y \end{tikzcd}$$

The set X is transitive if for all $x, y \in X$, there is a $g \in G$ such that $gx = y$. X is *simply transitive* if this g is unique.

Exercise $G \times G \xrightarrow{\mu} G$ given by group multiplication is simply transitive. Conversely, any simply transitive G-set X is isomorphic to G. Choosing $p_0 \in X$, consider the map $\varphi : X \to G$ where $p \mapsto p - p_0$.

This is a torsor, i.e. a principal homogeneous space?

Example: ODEs, the solutions for the inhomogeneous equation are torsors for the homogeneous solutions (?).

Let k be a field and G/k a commutative algebraic group. A torsor (or principal homogeneous space) under G is a nonempty k-variety X/A equipped with a morphism $\mu : G \times X \to X$ such that the following holds.

1st try: define $\mu(\bar{k}) : G(\bar{k}) \times X(\bar{k}) \to X(\bar{k})$ to be a simply transitive action. This is a good definition in characteristic zero. In general, we have an isomorphism

$$G \times X \to X(g, x) \qquad\qquad \mapsto (x, gx).$$

This is trivial iff $(X, \mu) \cong (G, \cdot)$ with its group structure.

Claim X is trivial $\iff X(k) \neq \emptyset$

Proof $\implies : X \cong_K G$ and $G(k) \neq \emptyset$, so $X(k) \neq \emptyset$.

$\impliedby$: Let $p_0 \in X(k)$, then the map

$$\mu(\,\cdot\,, p_0) : G \to X$$

is defined over k and is a bijection on $X(\bar{k})$, and thus an isomorphism in characteristic zero.

This implies that every X under G is a twisted form of G, i.e. $X/\bar{k} \cong G/\bar{k}$.

There's a group law on the torsors called the Baer sum, and a group structure yielding the Weil-Chevallet group. Every torsor will be given by a galois cohomological cocycle.