

# Linearization and Transversality

## Sections 8.3 and 8.4

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Linearization and  
Transversality

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Review 8.2

Section 8.3: The  
Space of  
Perturbations of  
 $H$

Section 8.4:  
Linearizing the  
Floer Equation:  
The Differential  
of  $F$

## Review 8.2

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## Section 8.3: The Space of Perturbations of $H$

# Goal

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**Goal:** Given a fixed Hamiltonian  $H \in C^\infty(W \times S^1; \mathbb{R})$ , perturb it (without modifying the periodic orbits) so that  $\mathcal{M}(x, y)$  are manifolds of the expected dimension.

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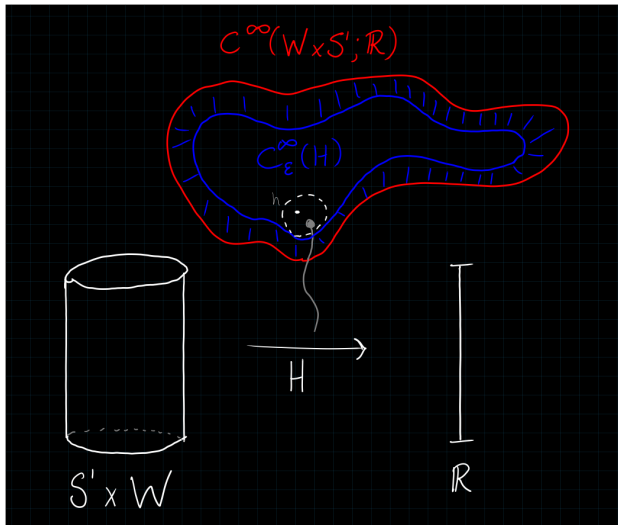
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Start by trying to construct a subspace  $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$ , the space of perturbations of  $H$  depending on a certain sequence  $\varepsilon = \{\varepsilon_k\}$ , and show it is a dense subspace.



# Define an Absolute Value

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Idea: similar to how you build  $L^2(\mathbb{R})$ , define a norm  $\|\cdot\|_\varepsilon$  on  $C_\varepsilon^\infty(H)$  and take the subspace of finite-norm elements.

- Let  $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$  denote a perturbation of  $H$ .
- Fix  $\varepsilon = \{\varepsilon_k \mid k \in \mathbb{Z}^{\geq 0}\} \subset \mathbb{R}^{>0}$  a sequence of real numbers, which we will choose carefully later.
- For a fixed  $\mathbf{x} \in W$ ,  $t \in \mathbb{R}$  and  $k \in \mathbb{Z}^{\geq 0}$ , define

$$|d^k h(\mathbf{x}, t)| = \max \left\{ d^\alpha h(\mathbf{x}, t) \mid |\alpha| = k \right\},$$

the maximum over all sets of multi-indices  $\alpha$  of length  $k$ .

*Note: I interpret this as*

$$d^{\alpha_1, \alpha_2, \dots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

*the partial derivatives wrt the corresponding variables.*

# Define a Norm

- Define a norm on  $C^\infty(W \times S^1; \mathbb{R})$ :

$$\|h\|_{\infty} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} |d^k h(x, t)|.$$

- Since  $W \times S^1$  is assumed compact (?), fix a finite covering  $\{B_i\}$  of  $W \times S^1$  such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

- Choose them in such a way we obtain charts

$$\psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \quad (?).$$

- Obtain the computable form

$$\|h\|_{\infty} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \psi_i^{-1})(z)|.$$

# Define a Banach Space

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- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \mid \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

which is a Banach space (normed and complete).

- Show that the sequence  $\{\varepsilon_k\}$  can be chosen so that  $C_\varepsilon^\infty$  is a *dense* subspace for the  $C^\infty$  topology, and in particular for the  $C^1$  topology.

## Theorem

*Such a sequence  $\{\varepsilon_k\}$  can be chosen.*

## Lemma

*$C^\infty(W \times S^1; \mathbb{R})$  with the  $C^1$  topology is separable as a topological space (contains a countable dense subset).*



# Sketch Proof of Theorem

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- By the lemma, produce a sequence  $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$  dense for the  $C^1$  topology.
- Using the norm on  $C^n(W \times S^1; \mathbb{R})$  for the  $f_n$ , define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \mid k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x, t)| \leq 2^{-n}$$

which is summable.

*Why does this imply density? I don't know.*

# Modified Theorem

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The next proposition establishes a version of this theorem with compact support:

## Theorem

*For any  $(\mathbf{x}, t) \in U \in W \times S^1$  there exists a  $V \subset U$  such that every  $h \in C^\infty(W \times S^1; \mathbb{R})$  can be approximated in the  $C^1$  topology by functions in  $C_\varepsilon^\infty$  supported in  $U$ .*

Then fix a time-dependent Hamiltonian  $H_0$  with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \mid h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then  $\text{supp}(h)$  is “far” from  $\text{Per}(H_0)$ , so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

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## Section 8.4: Linearizing the Floer Equation: The Differential of $F$

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Choose  $m > n = \dim(W)$  and embed  $TW \hookrightarrow \mathbb{R}^m$  to identify tangent vectors (such as  $Z_i$ , tangents to  $W$  along  $u$  or in a neighborhood  $B$  of  $u$ ) with actual vectors in  $\mathbb{R}^m$ .

*Why? Bypasses differentiating vector fields and the Levi-Cevita connection.*

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential  $d\mathcal{F}$ .

*We've just replaced the codomain here.*

# Definitions

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Recall that

- $x, y$  are contractible loops in  $W$  that are nondegenerate critical points of the action functional  $\mathcal{A}_H$ ,
- $u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^\infty$  denotes a fixed solution to the Floer equation,
- $C_{\searrow}(x, y) \subset \{u \in C^\infty(\mathbb{R} \times S^1; W)\}$  is the set of smooth solutions  $u : \mathbb{R} \times S^1 \rightarrow W$  satisfying some conditions:

$$\lim_{s \rightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \rightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left| \frac{\partial u}{\partial t}(s, t) \right|, \quad \left| \frac{\partial u}{\partial t}(s, t) - X_H(u) \right| \sim \exp(|s|)$$

# Compactify to Sphere

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Fix a solution

$$u \in \mathcal{M}(x, y) \subset C_{\text{loc}}^{\infty}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way:

The loops  $x, y$  are contractible, so they bound discs. So we extend by pushing these discs out slightly:

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$$u : R \times S^1 \longrightarrow W \quad \mapsto \quad \tilde{u} : S^2 \longrightarrow W$$

