8.8 Part 2, Computing the Index of L

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Contents

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$
\begin{aligned}
L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\
Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y
\end{aligned}
$$

  where

$$
\begin{aligned}
S : \mathbb{R} \times S^1 &\longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\
S(s,t) &\overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t).
\end{aligned}
$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
  - *(not the same kernel/cokernel)*
- Compute $\operatorname{Ind} L_1$: explicitly describe $\ker L_1, \operatorname{coker} L_1$.
- Replace in two steps:
  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s.
    * Replace $S(s,t)$ by a matrix

$$
\tilde{S}(s,t) = \begin{cases} S^{-}(t) & s \leq -\sigma_0 \\ S^{+}(t) & s \geq \sigma_0 \end{cases}.
$$

    * Idea: approximate by cylinders at infinity.
    * Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    * Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\operatorname{Ind}(L) = \mu\left(R^-(t)\right) - \mu\left(R^+(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

  where $S : \mathbb{R} \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\operatorname{Ind}(R^{\pm}(t))$; then

$$\operatorname{Ind}(L) = \operatorname{Ind}(L_1) = \operatorname{Ind}(R^-(t)) - \operatorname{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing L_1 to $R^{\pm}$. Let $k^{\pm} := \operatorname{Ind}(R^{\pm})$; then $\operatorname{Ind}(L_1) = k^- - k^+$.
- Lemma 8.8.4: $\operatorname{Ind}(L_0) = \operatorname{Ind}(L)$.
- Han's Talk:
  - Prop 8.8.3, using Lemma 8.8.5
- Me
  - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\operatorname{Ind}(L_1) = K^- - k^+$.

Setup:

$$S(s) = \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \qquad \text{with } a_i(s) = \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$

Statement: let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^{\pm} := a_1^{\pm} = a_2^{\pm}$. Then

$$\begin{aligned} \dim \operatorname{Ker} F &= 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^- < 2\pi\ell < a^+\right\} \\ \dim \operatorname{Ker} F^{\star} &= 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^+ < 2\pi\ell < a^-\right\}. \end{aligned}$$

2. Suppose $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$, then

$$\begin{aligned} \dim \operatorname{Ker} F &= \#\left\{ i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0 \right\} \\ \dim \operatorname{Ker} F^\star &= \#\left\{ i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0 \right\}. \end{aligned}$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

0.3 Proof

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) = a_1(s) = a_2(s)$.
- Start with equation on $\mathbb{R}^2$,

$$Y(s,t) = (Y_1(s,t), Y_2(s,t))$$

- Replace with equation on $\mathbb{C}$:

$$Y(s,y) = Y_1(s,t) + iY_2(s,t)$$

.

- Rewrite the PDE $F(Y) = 0$ as $\bar\partial Y + S(s)Y = 0$, i.e.

$$\frac{\partial}{\partial s}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let $Y = B\tilde{Y}$ where $B \in \mathrm{GL}(1,\mathbb{C})$ satisfies $\left(\bar\partial + S\right)B = 0$ to obtain $\bar\partial\tilde{Y} = 0$.
  - Can choose $B = \begin{bmatrix} b(s))0 & \\ 0 & b(s) \end{bmatrix}$ where $\frac{\partial b}{\partial s} = -a(s)b(s)$.
  - Explicitly, we can take the integral $b(s) = e^{\int_0^s -a(t)\ dt} = e^{-A(s)}$
- Remark: for some constants C_i, we have

$$A(s) = \begin{cases} C_1 + a^- s & s \le -\sigma_0 \\ C_2 + a^+ s & s \ge \sigma_0 \end{cases}.$$

- Remark: the new $\tilde{Y}$ satisfies CR. It is continuous and L^1_{loc} and thus by elliptic regularity C^∞. Its real/imaginary parts are C^∞ and harmonic.

Step 2: ?

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$

- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of $\tilde{Y}$ in z.
- Deduce that the solutions of the system are given by

$$\tilde{Y}(s+it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $c_\ell \in \mathbb{C}$ and this sequence converges for all s, t.

- Write in real coordinates as

$$\tilde{Y}(s,t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell} \left(\alpha_\ell \begin{pmatrix} \cos 2\pi\ell t \\ \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -\sin 2\pi\ell t \\ \cos 2\pi\ell t \end{pmatrix} \right).$$

- Return to $Y = B\tilde{Y}$:

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell} \left(\alpha_\ell \begin{pmatrix} e^{-A(s)} \cos 2\pi\ell t \\ e^{-A(s)} \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -e^{-A(s)} \sin 2\pi\ell t \\ e^{-A(s)} \cos 2\pi\ell t \end{pmatrix} \right).$$

- For $s \geq s_0$, for some constants K_i we can write

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{\left(2\pi\ell - a^-\right)s + K} \left(\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t\right) \\ e^{\left(2\pi\ell - a^-\right)s + K'} \left(\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t\right) \end{pmatrix}.$$

and for $s \geq s_0$

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{\left(2\pi\ell - a^+\right)s + C} \left(\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t\right) \\ e^{\left(2\pi\ell - a^+\right)s + C'} \left(\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t\right) \end{pmatrix}.$$

- Then $Y \in L^p \iff$ the exponential terms die at infinity. Forces the conditions:
  - $\ell \neq 0 \implies \alpha_\ell = \beta_\ell = 0 \; or \; 2\pi\ell < a^+$.
  - $\ell = 0 \implies \left(a_0 = 0 \text{ or } a^+ > 0\right) \; and \; \left(\beta_0 = 0 \text{ or } a^+ > .0\right)$.

This further forces

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } a^- < 2\pi\ell < a^+ & \ell \neq 0 \\ \left(\alpha_0 = 0 \text{ or } a^- < 0 < a^+\right) \text{ and } \left(\beta_0 = 0 \text{ or } a^- < 0 < a^+\right) & \ell = 0 \end{cases}.$$

- Finitely many such ℓ that satisfy these conditions
- Sufficient conditions for $Y(s,t) \in W^{1,p}$.

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

I.e. $F = \dfrac{\partial}{\partial s} + J_0 \dfrac{\partial}{\partial t} + S(s)$.

- Compute dimension of space of solutions:

$$\begin{aligned}\dim \operatorname{Ker} F &= 2\#\left\{\ell \in \mathbf{Z}^{\star} | a^{-} < 2\pi\ell < a^{+}\right\} && \left(+2\text{if } a^{-} < 0 < a^{+}\right) \\ &= 2\#\left\{\ell \in \mathbf{Z} | a^{-} < 2\pi\ell < a^{+}\right\}.\end{aligned}$$

Use this to deduce $\dim \ker F^{*}$:

- $Y \in \ker F^{*} \iff Z(s,t) \coloneqq Y(-s,t)$ is in the kernel of the operator

$$\begin{aligned}\tilde{F} : W^{1,q}\left(\mathbb{R} \times S^{1}; \mathbb{R}^{2}\right) &\longrightarrow L^{p}\left(\mathbb{R} \times S^{1}; \mathbb{R}^{2}\right) \\ Z &\mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y.\end{aligned}$$

- Obtain $\ker F^{*} \cong \ker \tilde{F}$