# Title

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Last time: we started discussing smoothness.

**Definition 1.0.1** (Tangent Space)
The **tangent space** $T_pX$ of a variety $X$ at a point $p \in X$ is defined as $V(\left\{ f_1 \mid f \in I(U_i), U_i \ni p = 0 \text{ affine} \right\})$ where $f_1$ denotes the degree 1 part.

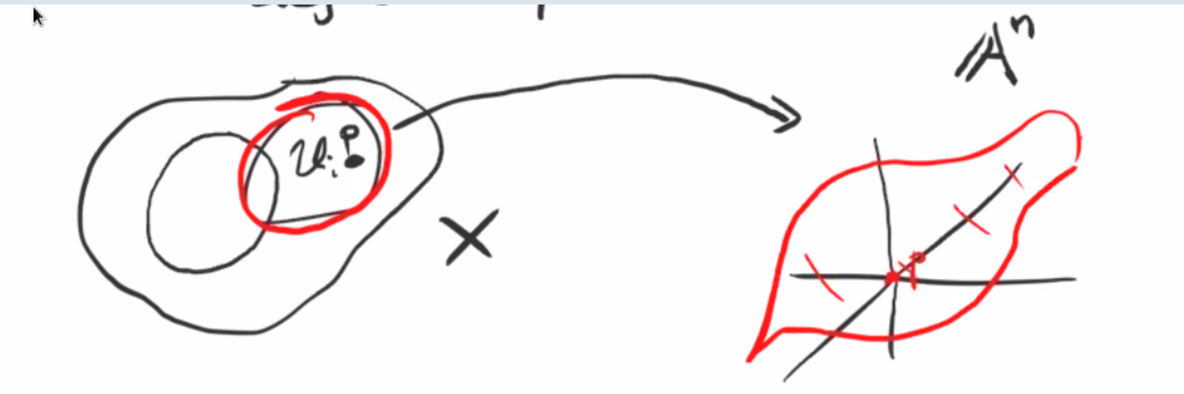


Figure 1: Image

**Remark 1.0.2:** We've really only defined it for affine varieties and $p = 0$, but this is a local definition. Note that this is also not a canonical definition, since it depends on the affine chart $U_i$.

**Example 1.0.3** *(?):* Consider $T_0V(xy) = V(f_1 \mid f \in \langle xy \rangle) = V(0) = \mathbb{A}^2$, since every polynomial in this ideal has degree at least 2. Letting $X = V(xy)$, note that we could embed $X \hookrightarrow \mathbb{A}^3$ as $X \cong V(xy, z)$. In this case we have $T_0X = V(f_1 \mid f \in \langle xy, z \rangle) = V(z) \cong \mathbb{A}^2$. So we get a vector space of a different dimension from this different affine embedding, but $\dim T_0X$ is the same.

**Example 1.0.4** *(?):* Let $X = V_p(xy - z^2) \subset \mathbb{P}^2$, which is a projective curve. What is $T_pX$ for $p = [0 : 1 : 0]$? Take an affine chart $\{y \neq 0\} \cap X$, noting that $\{y \neq 0\} \cong \mathbb{A}^2$. We could dehomogenize the ideal $\left\langle xy - z^2 \right\rangle\Big|_{y=1} = \left\langle x - z^2 \right\rangle$. Thus $X \cap D(y) = V(x - z^2) \subset \mathbb{A}^2$ and the point $[0 : 1 : 0] \in X$ gives $(0, 0)$ in this affine chart. Then $T_pX = V(f_1 \mid f \in \left\langle x - z^2 \right\rangle) = V(x)$. Then $f = (x - z^2)g$ implies that $f_1 = (xg)_1 = g_0x$, the constant term of $g$ multiplied by $x$, since $z^2$ kills any degree 1 part of $g$. So $T_pX$ is a line.

**Example 1.0.5** *(?):* Take $X$ to be the union of the coordinate axes in $\mathbb{A}^3$.