

Lie Algebras

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1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
 - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

1.1.2 Chapter 3-4

We will describe the following series of correspondences:



1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:





1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some important topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
 - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat- \mathcal{O} [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** \mathfrak{g} over F is an F -vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\cdot, \cdot]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0.$$

Exercise 1. Show that $[x, y] = -[y, x]$.

Definition 2. Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\text{End}(V) = \{f : V \rightarrow V \mid V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \mid \text{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group* $\text{SL}(n, V)$).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & I_n \\ \hline & -I_n & 0 \end{array} \right) & n = 2\ell + 1 \text{ odd,} \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

Proposition 1. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2 , and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and



zero elsewhere.

- For type A_ℓ , we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$.
- For type C_ℓ , we have $\dim \mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2 \left(\frac{\ell(\ell+1)}{2} \right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

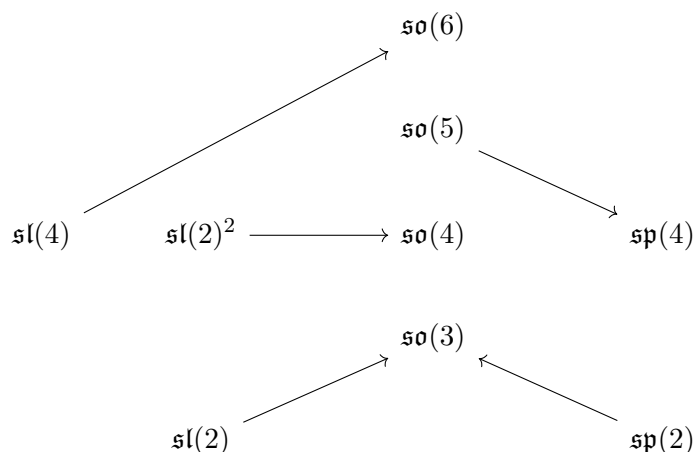
$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type B_ℓ , we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.



Theorem 1. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

Exercise 4. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 7. An F -algebra A is an F -vector space endowed with a bilinear map $A^2 \rightarrow A$, $(x, y) \mapsto xy$.

Definition 8. An algebra is **associative** if $x(yz) = (xy)z$.

Modern interest: simple Lie algebras, which have a good representation theory. Take a look at Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 9. Any map $\delta : A^2 \rightarrow A$ that satisfies the Leibniz rule is called a **derivation** of A , where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 10. We define $\text{Der}(A) = \{\delta \mid \delta \text{ is a derivation}\}$.

Any Lie algebra \mathfrak{g} is an F -algebra, since $[\cdot, \cdot]$ is bilinear. Moreover, \mathfrak{g} is associative iff $[x, [y, z]] = 0$.

Exercise 5. Show that $\text{Derg} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 6 (Turn in). Define the adjoint by $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$, $y \mapsto [x, y]$. Show that $\text{ad}_x \in \text{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 1. Any F -vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\cdot, \cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
 - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
 - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 2. Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise 7. Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\text{ad}(e)$, $\text{ad}(h)$, $\text{ad}(g)$ with respect to this basis.

2.3 Ideals

Definition 11. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise 8. Check that the following are all ideals of \mathfrak{g} :

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \mid x_i, y_i \in \mathfrak{g}\}$.
– Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \mid x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \cap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \mid x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 12. A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when \mathfrak{g} is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 2. Suppose that $\text{char } F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \text{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since $\text{char } F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I ; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since $\text{char } F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I,$$

$$2f = [h, -f] \in I \implies f \in I.$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

□

Note that there is a homework coming due next Monday, about 4 questions.

3 Lecture 3

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient \mathfrak{g}/I also yields a Lie algebra with the bracket given by $[x + I, y + I] = [x, y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x, y] + I = [x', y'] + I$.

3.1 Homomorphisms and Representations

Definition 13. A linear map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x, y] = [\phi(x), \phi(y)]$.

Remark. $\ker \phi \trianglelefteq \mathfrak{g}_1$ and $\text{im } \phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\text{hom } \phi : \mathfrak{g} \rightarrow \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I + J)/J \cong I/(I \cap J)$.

Definition: A *representation* of a Lie algebra \mathfrak{g} is a Lie algebra homomorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V .

We call V a \mathfrak{g} -module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

Corollary 1. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since \mathfrak{g} is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) &= \left\{ x \in \mathfrak{g} \mid \text{ad}_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \right\} \\ &= \ker \text{ad}_x. \end{aligned}$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad} \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

3.2 Automorphisms

Definition: An automorphism of \mathfrak{g} is an isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}$, and we define

$$\text{Aut}(\mathfrak{g}) = \{ \phi : \mathfrak{g} \rightarrow \mathfrak{g} \mid \phi \text{ is an isomorphism} \}.$$

Proposition: If $\delta \in \text{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) := \sum \frac{\delta^n}{n!} \in \text{Aut}(\mathfrak{g}).$$

This is well-defined because δ is nilpotent, and a binomial formula holds:

$$\frac{\delta^n([x, y])}{n!} = \sum_{i=0}^n \left[\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!} \right].$$

and for $n = 1$, $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\text{ad}_e) \exp(\text{ad}_{-f}) \exp(\text{ad}_e) \in \text{Aut} \mathfrak{g}.$$

where e, f are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of s is 2 and $W = \{1, s\}$.

4 Lecture 4

4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for \mathfrak{g} is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\vdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra \mathfrak{g} is *solvable* if there is some n for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let \mathfrak{g} be a Lie algebra, then

1. If \mathfrak{g} is solvable, then all subalgebras and all homomorphic images of \mathfrak{g} are also solvable.
2. If $I \trianglelefteq \mathfrak{g}$ and both I and \mathfrak{g}/I are solvable, then so is \mathfrak{g} .
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\text{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\text{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any \mathfrak{g} , the quotient $\mathfrak{g}/\text{Rad}(\mathfrak{g})$ is semisimple.

4.2 Nilpotency

$$\begin{aligned}\mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\ &\vdots \\ \mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i].\end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If \mathfrak{g} is nilpotent, then all subalgebras and all homomorphic images of \mathfrak{g} are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is \mathfrak{g} .
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Claim: If \mathfrak{g} is nilpotent, then $\text{ad}_x \in \text{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof: This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \dots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \dots [x_n, y]]] = 0$, and so $\text{ad}_{x_1} \circ \text{ad}_{x_2} \circ \dots \text{ad}_{x_n} = 0$ which implies that $\text{ad}_x^n = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If ad_x is nilpotent for all $x \in \mathfrak{g}$, then \mathfrak{g} is nilpotent.

Remark: This can be confusing if \mathfrak{g} is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}x$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while ad_x is nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V . If x is nilpotent as an endomorphism on V for all $x \in \mathfrak{g}$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases: - Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$. - Case $\dim \mathfrak{g} = 1$, left as an exercise.

Inductive step: Let A be a maximal proper subalgebra and define $\phi : A \rightarrow \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that ϕ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that $\text{im } \phi \leq \mathfrak{gl}(\mathfrak{g}/A)$ is a Lie subalgebra, and $\dim \text{im } \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \text{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\text{im } \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that A is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1y, a_2 + c_2y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1c_2[y, y]$. The last term is zero, the middle two terms are in A , and because A is closed under the bracket, the first term is in A as well.

But then $A \oplus F_y$ is a larger subalgebra than A , which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in A , $A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in A .

Proof to be continued on Friday!

5 Lecture 5

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of V , constructing $\text{im } \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus F_y$. Now consider

$$W = \{v \in V \mid Av = 0\},$$

which is \mathfrak{g} -invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But V is killed by any element in A , and both of these terms are in A . In particular, the y appearing in Fy also satisfies $y \in W$. Consider $y|_W \in \text{End}(W)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \text{End}(W)$, which is true exactly because y is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \text{End}(\mathfrak{g}/A)$ is nilpotent. Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by a , $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since m_ℓ, m_r commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly ad_a . But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if $\text{ad}_x \in \text{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then \mathfrak{g} is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to $\text{ad}_g \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that $\text{ad}_g v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension. Since ad_x is nilpotent, so is $\text{ad}_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space. \square

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of F , then the matrices of \mathfrak{g} with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on n , where the base case is easy. For the inductive step, we use the previous theorem to get a v_1 such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\bar{V} = F^n/Fv_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\bar{V})$ where $x \mapsto (\bar{y} \mapsto \overline{y(x)})$.

Then $\text{im } \phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \text{End}(F^{n-1})$ is nilpotent, since x was nilpotent on the larger space. But (see notes) then x can be written as a strictly upper-triangular matrix.

5.1 Chapter 2: Semisimple Lie Algebras

We now assume $\text{char } F = 0$ and $\bar{F} = F$.

Theorem: If \mathfrak{g} is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V , then V contains a common eigenvector for a $x \in \mathfrak{g}$, i.e. a $\lambda : \mathfrak{g} \rightarrow F, x \mapsto \lambda(x)$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$.

Proof: We will use induction on the dimension of \mathfrak{g} . For the inductive step:

Claim 1: There is an ideal $A \trianglelefteq \mathfrak{g}$ such that $\mathfrak{g} = A \oplus Fy$ for some $y \neq 0$, so A is a subalgebra of a solvable Lie algebra \mathfrak{g} and thus solvable itself. By hypothesis, we can produce a $w \in V \setminus \{0\}$, and thus a functional $\lambda : A \rightarrow F$ such that $aw = \lambda(a)w$ for all $a \in A$. So we define

$$V_\lambda = \{v \in V \mid av = \lambda(a)v \forall a \in A\}$$

where $w \in V_\lambda$.

Claim 2: $y(V_\lambda) \subseteq V_\lambda$, or $y|_{V_\lambda} \in \text{End}(V_\lambda)$.

Thus $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$ is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a $v \in V_\lambda$ and some $\mu \in F$ such that $y(v) = \mu v$. An arbitrary element $x \in \mathfrak{g}$ can be written as $x = a + cy$ for some $a \in A, c \in F$ and it acts by $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c\mu)v \in V_\lambda$.

6 Lecture n+1

Todo

7 Lecture n+2

Definition (Jordan Decomposition)

Let $X \in \text{End}(V)$ for V finite dimensional. Then,

- (a) There exists a unique $X_s, X_n \in \text{End}(V)$ such that $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.
- (b) There exists a $p(t), q(t) \in t\mathbb{F}[t]$ such that $X_s = p(X), X_n = q(X)$.

(Polynomials with no constant term.)

Proof of (a): Assume $X_s = X_s + X_n = X'_s + X'_n$, so both have bracket zero. Assuming that (b) holds, we have $X_s = p(X)$, and so

$$[X, X_s] = [X_s + X'_n, X'_s] = [X'_s, X'_s] + [X'_s, X'_n] = 0 \implies [p(X), X'_s] = 0 = [X_s, X'_s]$$

Using fact (c) from last time, then X_s, X'_s can be diagonalized simultaneously, and so $X_s - X'_s$ is semisimple.

On the other hand, if X'_n, X_n are nilpotent, and since these commute, $X_n - X'_n$ is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X'_s) + (X_n + X'_n)$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$ be the minimal polynomial of X , where each $m_i \geq 1$ and the λ_i are distinct. Then the primary composition of V is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V)^{m_i} \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomial $p \in \mathbb{F}[t]$ such that

$$\begin{aligned} p &= \lambda \pmod{(t - \lambda_i)^{m_i}} \quad \forall i, \\ p &= 0 \pmod{t}. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is $p(x) \curvearrowright V_i$? This acts by scalar multiplication by λ_i for all i . (Check). Because of the restrictive conditions, $p(x)$ has no constant term.

So $p(X) = X_s$ is the semisimple part we want. Now just set $q(t) = t - p(t)$, then $X_n := q(X) = X - X_s$ is nilpotent.

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have $X = X_s + X_n$, then $\text{ad}_X \in \text{End}(\text{End}(V))$. It can be shown that $(\text{ad}_X)_s + (\text{ad}_X)_n = \text{ad}(X_s) + \text{ad}(X_n)$.

$$p(x) \sim \begin{pmatrix} \boxed{\lambda_1 I_{v_1}} & & & \\ & \boxed{\lambda_2 I_{v_2}} & & \\ & & \ddots & \\ & & & \boxed{\lambda_r I_{v_r}} \end{pmatrix}$$

Figure 1: ???

Let e_{ij} be the elementary matrix with a 1 in the i, j position. You can write ad_X as a 4×4 matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$= \begin{pmatrix} e_{11} & 0 & 0 & 1 & 0 \\ e_{12} & -1 & 0 & 0 & 1 \\ e_{21} & 0 & 0 & 0 & 0 \\ e_{22} & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{array} \right)$$

$$= \begin{matrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \begin{pmatrix} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{pmatrix}$$

You can check that $(\text{ad}_X)_S = 0$, $\text{ad}(X_s) = 0$, and $(\text{ad}_X)_n$ is the Jordan form given above.

Lemma:

- (a) $x \in \text{End}(V) \implies \text{ad}(x)_s = \text{ad}(x_s)$ and $\text{ad}(x)_n = \text{ad}(x_n)$.
- (b) If A is a finite dimensional \mathbb{F} -algebra, then $\delta \in \text{Der}(A) \implies \delta_s, \delta_n \in \text{Der}(A)$ as well.

Proof of (a):

Check that $\text{ad}(x) = \text{ad}(x_s) + \text{ad}(x_n)$. Then for $y \in \text{End}(V)$, we have

$$\begin{aligned} (\text{ad}(x))(y) &= [x, y] \\ &= [x_s + x_n, y] \\ &= [x_s, y] + [x_n, y] \\ &= (\text{ad}(x_s))(y) + (\text{ad}(x_n))(y). \end{aligned}$$

Using theorem 3.3, x_n nilpotent $\implies \text{ad}(x_n)$ is also nilpotent. So write $x_s = \sum \lambda_i e_{ii}$ with the eigenvalues on the diagonal. Then $\text{ad}_{x_s}(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$ for all i, j .