# Title

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## Contents



### 0.1 Exercises

**Problem 1.**

Let $C$ denote the Cantor set.

1. Show that $C$ contains point that is not an endpoint of one of the removed intervals.
2. Show that $C$ is nowhere dense, meager, and has measure zero.
3. Show that $C$ is uncountable.

**Solution 1.**

1. First we will characterize the endpoints of the removed intervals. Let $C_n$ be the $n$Th stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:
$$C_n = [0, \frac{1}{3^n}] \bigcup [\frac{2}{2^n}, \frac{3}{3^n}] \bigcup \cdots \bigcup [\frac{3^n - 1}{n}, 1],$$
and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in $C_n$ is never removed in any later step, and so all endpoints remaining in $C$ are of this form where we allow $0 \leq n < \infty$.

   Thus, our goal is to produce a number $x \in [0, 1]$ such that $x \neq \frac{k}{3^n}$ for any $k$ or $n$, but also satisfies $x \in C$. So we will need a general characterization of all of the points in $C$.

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Lemma: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.
$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

Proof: By induction on the index $k$ in $a_k$, first consider note that if $x \in C$ then $x \in C_1 = [0,1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. So if $x \in C_1$, then $x \notin (\frac{1}{3}, \frac{2}{3})$. But note that $a_1$ is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \leq x < \frac{1}{3}, \\ 1 & \frac{1}{3} \leq x < \frac{2}{3}, \\ 2 & \frac{2}{3} \leq x < 1. \end{cases}$$

Since the interval $(\frac{1}{3}, \frac{2}{3})$ is deleted in $C_1$, we find that $a_1 = 1 \iff x = \frac{1}{3}$. In this case, however, we claim that we can find a ternary expansion of $x$ that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that $x = x'$, which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left( \sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left( \sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 (\frac{1}{1 - \frac{1}{3}}) - 2 - 2(3^{-1}) \\ &= 2 \left( \frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have $\frac{1}{3} = (0.1)_3 = (0.222\cdots)_3$ as ternary expansions, and a similar proof shows that such an expansion without 1s can be found for any endpoint.

For the inductive step, consider $a_n$: the claim is that if $a_n = 1$, then $x \notin C_{n+1}$ – that is, it is contained in one of the intervals deleted at the $n+1$st stage. Writing the deleted interval at this stage as $(a, b)$, we find that $a_n = 1$ if and only if $x \in [a, b)$. Since $x \in C$, the only way $a_n$ can be 1 is if $x$ was in fact the endpoint $a$ (since no previous digit was a 1, by hypothesis). However, as shown above, every such endpoint has a ternary expansion containing no 1s. $\square$

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Therefore, if we can produce an $x$ that satisfies $x \neq \frac{k}{3^n}$ for any $k, n$ **and** $x$ has no 1s in its ternary expansion, we will have an $x \in C$ that is not an endpoint.

So take

$$x = (0.\overline{02})_3 = (0.020202\cdots)_3.$$

This evidently has no 1s in its ternary expansion, and if we sum the corresponding geometric series, we find $x = \frac{1}{4}$. This is not of the form $\frac{k}{3^n}$ for any $k, n$, and thus fulfills both conditions.

2. We first show that $C$ is nowhere dense by showing that the interior of its closure is empty, i.e. $(\overline{C})^\circ = \emptyset$.

   To do so, we note that $C$ is itself closed and so $C = \overline{C}$. To see why this is, consider $C^c$; we'll show that it is open. By construction, $C_1^c$ is the open interval $(\frac{1}{3}, \frac{2}{3})$ that is deleted, and similarly $C_n^c$ is the finite union of the open intervals that are deleted at the $n$th stage. But then

   $$C^c = \left(\bigcap C_n\right)^c = \bigcup C_n^c$$

   is an infinite union of open sets, which is also open. So $C$ is closed.

   It is also the case that $C$ has empty interior, so $C^\circ = \emptyset$. Towards a contradiction, suppose $x \in C$ is an interior point; then there is some neighborhood $N_\varepsilon(x) \subset C$. Since we are on the real line, we can write this as an interval $(x-\varepsilon, x+\varepsilon)$, which has length $2\varepsilon > 0$. Moreover, we have the containment

   $$(x-\varepsilon, x+\varepsilon) \subset C \subset C_n$$

   for every $n$.

   Claim: The length of $C_n$ is $(\frac{2}{3})^n$ where we define $C_0 = [0,1]$. Letting $L_n$ be the length of $C_n$, one easy way to see that this is the case is to note that $L_n$ satisfies the recurrence relation

   $$L_{n+1} = \frac{2}{3}L_n,$$

   since an interval of length $\frac{1}{3}L_n$ is removed at each stage. With the initial conditions $L_0 = 1$, it can be checked that $L_n = \left(\frac{2}{3}\right)^n$ solves this relation.

   Now, since $x \in C = \bigcap C_n$, it is in every $C_n$. So we can choose $n$ large enough such that

   $$\left(\frac{2}{3}\right)^n \leq 2\varepsilon.$$

   Letting $\mu(X)$ denote the length of an interval, we always have $C \subseteq C_n$ and so $\mu(C) \leq \mu(C_n)$.

   Using the subadditivity of measures, we now have

   $$\begin{aligned} &(x-\varepsilon, x+\varepsilon) \subset C \subset C_n \\ \implies &\mu(x-\varepsilon, x+\varepsilon) \leq \mu(C) \leq \mu(C_n) \\ \implies &2\varepsilon \leq \left(\frac{2}{3}\right)^n, \end{aligned}$$

   a contradiction. So $C$ has no interior points.

   But this means that

   $$(\overline{C})^\circ = C^\circ = \emptyset,$$

and so $C$ is nowhere dense.

To see that $\mu(C) = 0$, we can use the fact that for any sets, measures are additive over disjoint sets and we have

$$\mu(A) + \mu(X \setminus A) = \mu(X) \implies \mu(X \setminus A) = \mu(X) - \mu(A).$$

Here we will take $X = [0, 1]$, so $\mu(X) = 1$, and $A = C$ the Cantor set.

By tracing through the construction of the Cantor set, letting $B_n$ be the length of the interval that is removed at each stage, we can deduce

$$\begin{aligned}
B_1 &= \frac{1}{3} \\
B_2 &= \frac{2}{9} \\
\cdots \\
B_n &= \frac{2^n}{3^{n+1}}.
\end{aligned}$$

We can identify $B_n = \mu(C_n^c)$, and using the fact that $C_n^c \bigcap C_{>n}^c = \emptyset$ and the fact that measures are additive over disjoint sets, we can compute

$$\begin{aligned}
\mu(C) &= 1 - \mu(C^c) \\
&= 1 - \mu((\bigcap_{n=0}^{\infty} C_n)^c) \\
&= 1 - \mu(\coprod_{n=0}^{\infty} C_n^c) \\
&= 1 - \sum_{n=0}^{\infty} \mu(C_n^c) \\
&= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{-n}} \\
&= 1 - \frac{1}{3} \sum_{n=0} \left(\frac{2}{3}\right)^n \\
&= 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) \\
&= 1 - \frac{1}{3}(3) = 0,
\end{aligned}$$

which is what we wanted to show. $\square$

3. Let $y \in [0, 1]$ be arbitrary, we will construct an element $x \in C$ such that $y = f(x)$. We first note that every number has a binary expansion, and we can write

$$y = \sum_{k=1}^{\infty} y_k 2^{-k} \quad \text{where } y_k \in \{0, 1\}.$$

Now we construct

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k = 2y_k \implies a_k \in \{0, 2\}.$$

By the characterization given in part (1), we see that $x \in C$ because it has no 1s in its ternary expansion. Moreover, under $f$, we have $a_k \mapsto \frac{1}{2}a_k = \frac{1}{2}(2a_k) = a_k$, and so $f(x) = y$ by construction.

This shows that $C$ surjects onto $[0,1]$, and in particular, $\#C \geq \#[0,1]$ holds for the cardinalities of these sets. Since $[0,1]$ is uncountable (say, by Cantor's diagonalization argument), this shows that $C$ is uncountable.

**Problem 2.**

1. Show that $X$ is $G_\delta$ iff $X^c$ is $F_\sigma$.
2. Show that $X$ closed $\implies$ $X$ is $G_\delta$ and $X$ open $\implies$ $X$ is $F_\sigma$.
3. Give an example of an $F_\sigma$ set that is not $G_\delta$, and a set that is neither.

**Solution 2.**

1. To show the forward direction, suppose $X$ is a $F_\sigma$, so $X = \bigcup_{i \in \mathbb{N}} A_i$ with each $A_i$ an closed set. By definition, each $A_i^c$ is open, and we have

   $$X^c = \left( \bigcup_{i \in \mathbb{N}} A_i \right)^c = \bigcap_{i \in \mathbb{N}} A_i^c,$$

   which exhibits $X^c$ as a countable intersection of closed sets, making it an $G_\delta$.

   The reverse direction proceeds analogously: supposing $X^c$ is $G_\delta$, we can write $X^c = \bigcap_{i \in \mathbb{N}} B_i$ with each $B_i$ open, where $B_i^c$ is closed by definition, and

   $$X = (X^c)^c = (\bigcap B_i)^c = \bigcup B_i^c$$

   which exhibits $X$ as a union of closed sets, and thus an $F_\sigma$.

2. Suppose $X$ is closed, we want to then write $X$ as a countable intersection of open sets. For every $x \in X$ and every $n \in \mathbb{N}$, define

   $$\begin{aligned} B_n(x) &= \left\{ y \in \mathbb{R}^n \ni |x - y| \leq \frac{1}{n} \right\}, \\ V_n &= \bigcup_{x \in X} B_n(x), \\ W &= \bigcap_{n \in \mathbb{N}} V_n. \end{aligned}$$

Explicitly, we have

$$W = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in X} B_n(x),$$

and the claim is that $W$ is a $G_\delta$ and $W = X$.

To see that the $V_n$ are open, note that $n$ is fixed and each $B_n(x)$ is an open ball around a point $x$. Any union of open sets is open, and thus so is $V_n$. By construction, $W$ is then a countable intersection of open sets, and thus $W$ is a $G_\delta$ by definition.

We show $W = X$ in two parts. To see that $X \subseteq W$, note that if $x \in X$, then $x \in B_n(x)$ for every $n$ and thus $x \in V_n$ for every $n$ as well. But this means that $x \in \bigcap_n V_n$, and so $x \in W$.

To see that $W \subseteq X$, let $w \in W$ be arbitrary. If $w \in X$, there is nothing to check, so suppose $w \notin X$ towards a contradiction.

Since $w \in \bigcap_n V_n$, it is in $V_n$ for every $n$. But this means that there is some particular $x_0$ such that $w \in B_n(x_0)$ for every $n$ as well, and moreover since we assumed $w \notin X$, we have $w \neq x_0$.

Then, letting $N_\varepsilon(w)$ be an arbitrary neighborhood of $w$, we can find an $n$ large enough such that $B_n(x) \subset N_\varepsilon(w)$. This means that $x_0 \neq w$ can be found in every neighborhood of $w$, which makes $w$ a limit point of $X$. However, since we assumed $X$ was closed, it contains all of its limit points, which would force $w \in X$, a contradiction. $\square$

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Now suppose $X$ is an open set, we want to show it is an $F_\sigma$ and can thus be written as a countable union of closed sets. We can use the fact that $X^c$ is closed, and by the previous result, $X^c$ is thus a $G_\delta$. But by an earlier result, $X^c$ is a $G_\delta \iff (X^c)^c = X$ is an $F_\sigma$, and we are done.

3. We want to construct a set that can be written as a countable union of closed sets, but not as a countable intersection of open sets. Note that in $\mathbb{R}$ with the usual topology, singletons are closed, and so $\{p\}^c$ is an open set for any point $p$.

   With this motivation, consider $X = \mathbb{Q}$ and $X^c = \mathbb{R} \setminus \mathbb{Q}$. We can write

   $$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\},$$

   which exhibits $X$ as a countable union of closed sets because $\mathbb{Q}$ itself is countable. So $\mathbb{Q}$ is an $F_\sigma$ set. Suppose towards a contradiction that $\mathbb{Q}$ is also $G_\delta$, so we have $\mathbb{Q} = \bigcap_{i \in \mathbb{N}} O_i$ with each $O_i$ open. So each $O_i$ covers $\mathbb{Q}$, i.e. $\mathbb{Q} \subseteq O_i$, which (importantly!) forces each $O_i$ to be dense in $\mathbb{R}$.

   But now note that we can also write

   $$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\},$$

   where we can note that $\mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$ for each $q$. We can appeal to the Baire category theorem twice, which tells us that any countable intersection of *open* dense sets will also be dense. This first tells us that the above intersection, and thus $\mathbb{R} \setminus \mathbb{Q}$, is dense in $\mathbb{R}$. Then, writing

   $$\left( \bigcap_{i \in \mathbb{N}} O_i \right) \bigcap \left( \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\} \right) = \mathbb{Q} \bigcap \mathbb{R} \setminus \mathbb{Q} = \emptyset,$$

we produce what is still just a countable intersection of open dense sets, and by Baire, the result would need to be dense as well. Since the empty set is *not* dense in $\mathbb{R}$, so we arrive at a contradiction.

**Problem 3.**

1. Let $r_n$ be an enumeration of the rationals, define $f(r_n) = \frac{1}{n}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that $\lim_{x \to c} f(x) = 0$ for every $c \in I$, and $D_f = \mathbb{Q} \bigcap I$.
2. Supposing $f$ is bounded, show that $\omega_f$ is (in general) well-defined, and that $f$ is continuous at $x \iff \omega_f(x) = 0$.
3. Show that for every $\varepsilon > 0$, the set $A(\varepsilon) = \{x \in \mathbb{R} \ni \omega_f(x) > \varepsilon\}$ is closed, and thus $D_f$ is an $F_\sigma$ set.

**Solution 3.**

1. We need to show that

$$\forall c \in I,\ \forall \varepsilon > 0,\ \exists \delta \ni \quad |x - c| \leq \delta \implies |f(x) - 0| \leq \varepsilon.$$

To that end, let $\{r_n\}$ be an arbitrary enumeration of $\mathbb{Q} \bigcap I$, let $\varepsilon$ be fixed, and let $c \in I$ be arbitrary. If $c \in I \setminus \mathbb{Q}$, then $f(c) = 0 < \varepsilon$ and there's nothing to prove. Otherwise, $c \in \mathbb{Q}$, so $c = r_n$ for some $n$, and $f(c) = \frac{1}{n}$. Let $S = \left\{ r_i \ni i \in \mathbb{N},\ \frac{1}{i} > \varepsilon \right\} \subset \mathbb{Q}$, and note that $S$ is finite by the archimedean property of $\mathbb{R}$. So choose

$$\delta < \min \{|c - s| \ni s \in S\},$$

so that $S \bigcap B_\delta(c) = \emptyset$.

This means that if $x \in B_\delta(c) \bigcap \mathbb{Q}$, then $x = r_m$ where $\frac{1}{m} < \varepsilon$ by construction. But then $|f(x)| = \frac{1}{m} < \varepsilon$, and we are done.

By the sequential definition of continuity, $f$ is continuous iff $\lim_{x \to c} f(x) = f(c)$. As we have shown, if $c \in I \setminus \mathbb{Q}$, then $\lim_{x \to c} f(x) = 0 = f(c)$, and so $f$ is continuous there. However, for $c \in I \bigcap \mathbb{Q}$, since $\lim_{x \to r_n} f(x) = 0 \neq \frac{1}{n}$, $f$ fails to be continuous there. Taken together, this says that $D_f = I \setminus \mathbb{Q}$ as desired.

2. To show that this is well-defined, we need to prove that the limit exists. By definition, since $f$ is bounded, there exists some $M$ that is independent of $x$ such that $x \in \mathbb{R} \implies |f(x)| \leq M$. In particular, for any fixed $\delta$, it is certainly the case that $B_\delta(x) \subset \mathbb{R}$, and so $x \in B_\delta(x) \implies |f(x)| \leq M$ as well.

We can then say that if $y, z \in B_\delta(x)$, then

$$|f(y) - f(z)| \leq |f(y)| + |f(z)| \leq 2M,$$

and thus the set $\{|f(y) - f(z)| \ni y, z \in B_\delta(x)\}$ is bounded above and thus has a least upper bound (since $\mathbb{R}$ has the least upper bound property). Thus the following supremum exists:

$$S(x, \delta) = \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|.$$

We now just need to show that $\lim_{\delta \to 0^+} S(x, \delta)$ exists. To this end, we can note that if $\delta_1 < \delta_2$, then $B_{\delta_1} \subset B_{\delta_2}$, and so $S$ is a monotonically decreasing function of $\delta$ that is bounded below by 0 (since $B_0(x) = \{x\} \implies y = z = x$ are the only choices), and is thus convergent by the monotone convergence theorem. So $\omega_f$ is well-defined.

To see that $f$ continuous at $x \implies \omega_f(x) = 0$, let $\varepsilon$ be arbitrary; we will show that $\omega_f(x) < \varepsilon$. Since $f$ is continuous, we can pick a $\delta$ such that $y, z \in B_\delta(x) \implies f(y), f(z) \in B_\varepsilon(f(x))$. Thus we have

$$
\begin{aligned}
|y - x| < \delta &\implies |f(y) - f(x)| < \varepsilon \\
|z - x| < \delta &\implies |f(z) - f(x)| < \varepsilon
\end{aligned}
$$
$$.$$

Moreover, we can write

$$|f(y) - f(z)| = |f(y) - f(x) + f(x) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| \leq 2\varepsilon,$$

and thus we also have

$$\sup_{y,z \in B_\delta(x)} |f(y) - f(z)| < 2\varepsilon.$$

We now want to take the limit as $\delta \to 0^+$; again since $\delta_1 \leq \delta_2 \implies B_{\delta_1} \subseteq B_{\delta_2}$, this can only make the left-hand-side of the above inequality smaller, and thus $\omega_f(x) \leq 2\varepsilon$. Taking $\varepsilon \to 0$ completes the proof.

To see that $\omega_f(x) = 0 \implies f$ is continuous at $x$, let $\varepsilon > 0$ be arbitrary; we want to produce a $\delta$ to use in the definition of continuity. Since $\omega_f(x) = 0$, we can find a $\delta$ such that