Problem Set 3

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Exercise 0.1 (Gathmann 2.33)**.**
Define

$$X \coloneqq \left\{ M \in \mathrm{Mat}(2 \times 3, k) \;\middle|\; \mathrm{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that X is an irreducible variety, and find its dimension.

Solution:
We'll use the following fact from linear algebra:

Definition ***(Matrix Minor).***
For an $m \times n$ matrix, a *minor of order* ℓ is the determinant of a $\ell \times \ell$ submatrix obtained by deleting any $m - \ell$ rows and any $n - \ell$ columns.

Theorem 0.1 ***(Rank is a Function of Minors).***
If $A \in \mathrm{Mat}(m \times n, k)$ is a matrix, then the rank of A is equal to the order of largest nonzero minor.

Thus

$$M_{ij} = 0 \text{ for all } \ell \times \ell \text{ minors } M_{ij} \iff \mathrm{rank}(M) < \ell,$$

following from the fact that if one takes $\ell = \min(m, n)$ and all $\ell \times \ell$ minors vanish, then the largest nonzero minor must be of size $j \times j$ for $j \leq \ell - 1$. But $\det M_{ij}$ is a polynomial f_{ij} in its entries, which means that X can be written as

$$X = V(\{f_{ij}\}),$$

which exhibits X as a variety. Thus

$$M = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \implies X = V(\langle xb - ya, yc - zb, xc - za \rangle) \subset \mathbb{A}^6.$$

Claim: The ideal above is prime, and so the coordinate ring $A(X)$ is a domain and thus X is irreducible.

Claim: $\dim(X) = 4$.

Heuristic: there are three degrees of freedom in choosing the first row x, y, z. To enforce the rank 1 condition, the second row must be a scalar multiple of the first, yielding one degree of freedom for the scalar.

Note: I looked at this for a couple of hours, but I don't know how to prove either of these statements with the tools we have so far!

Exercise 0.2 (Gathmann 2.34)**.**
Let X be a topological space, and show

a. If $\{U_i\}_{i\in I} \rightrightarrows X$, then $\dim X = \sup_{i\in I} \dim U_i$.

b. If X is an irreducible affine variety and $U \subset X$ is a nonempty subset, then $\dim X = \dim U$. Does this hold for any irreducible topological space?

Solution:

Strictly for notational convenience, we'll treat $\{U_i\}$ is if it were a countable open cover.

We first note that if $U \subseteq V$, then $\dim U \leq \dim V$. If this were not the case, one could find a chain $\{I_j\}$ of closed irreducible subsets of V of length $n > \dim U$. But then $I'_j := I_j \cap U$ would again be a closed irreducible set, yielding a chain of length n in U. Thus $\dim X \geq \dim U_i$, and it remains true that $\dim X \geq \sup \dim U_i$, so it suffices to show that $\dim X \leq \sup \dim U_i$.

Set $s := \sup_i \dim U_i$ and $n := \dim X$, we want to show that $s \geq n$. Let $\{I_j\}_{j\leq n}$ be a maximal chain of length n of closed irreducible subsets of X, so we have

$$\emptyset \subsetneq I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n \subseteq X.$$

Since $I_0 \subset X$ and $\{U_i\}$ covers X, we can find some $U_0 \in \{U_i\}$ such that $I_0 \cap U_0$ is nonempty, since otherwise there would be a point in $I_0 \cap (X \setminus \cup_{i\in J} U_i) = \emptyset$. We can do this for every I_j, so define $A_j := I_j \cap U_0$.

Each A_j is now closed in U_0, and must remain irreducible, since any decomposition of A_j would lift to a decomposition of I_0. To see that $A_0 \subsetneq A_1$, i.e. that the inclusions are still proper, we can just note that

$$x \in A_{i+1} \setminus A_i \iff x \in (I_{i+1} \cap U_0) \setminus (I_i \cap U_0) = (I_1 \setminus I_2) \cap U_0 = \emptyset.$$

But this exhibits a length n chain in U_0, so $\dim U_0 \geq n$. Taking suprema, we have

$$n \leq \dim U_0 \leq \sup_{i\in J} \dim U_i = s.$$

Exercise 0.3 (Gathmann 2.36)**.**
Prove the following:

a. Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.

b. A complex affine variety of dimension at least 1 is never compact in the classical topology.

Exercise 0.4 (Gathmann 2.40)**.**

Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1 x_4 - x_2 x_3 \rangle$$

and show the following:

a. R is an integral domain of dimension 3.

b. $x_1, \cdots, x_4$ are irreducible but not prime in R, and thus R is not a UFD.

c. $x_1 x_4$ and $x_2 x_3$ are two decompositions of the same element in R which are nonassociate.

d. $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

Exercise 0.5 (Problem 5)**.**
Consider a set U in the complement of $(0, 0) \in \mathbb{A}^2$. Prove that any regular function on U extends to a regular function on all of $\mathbb{A}^2$.