

Title

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Friday 21st August, 2020

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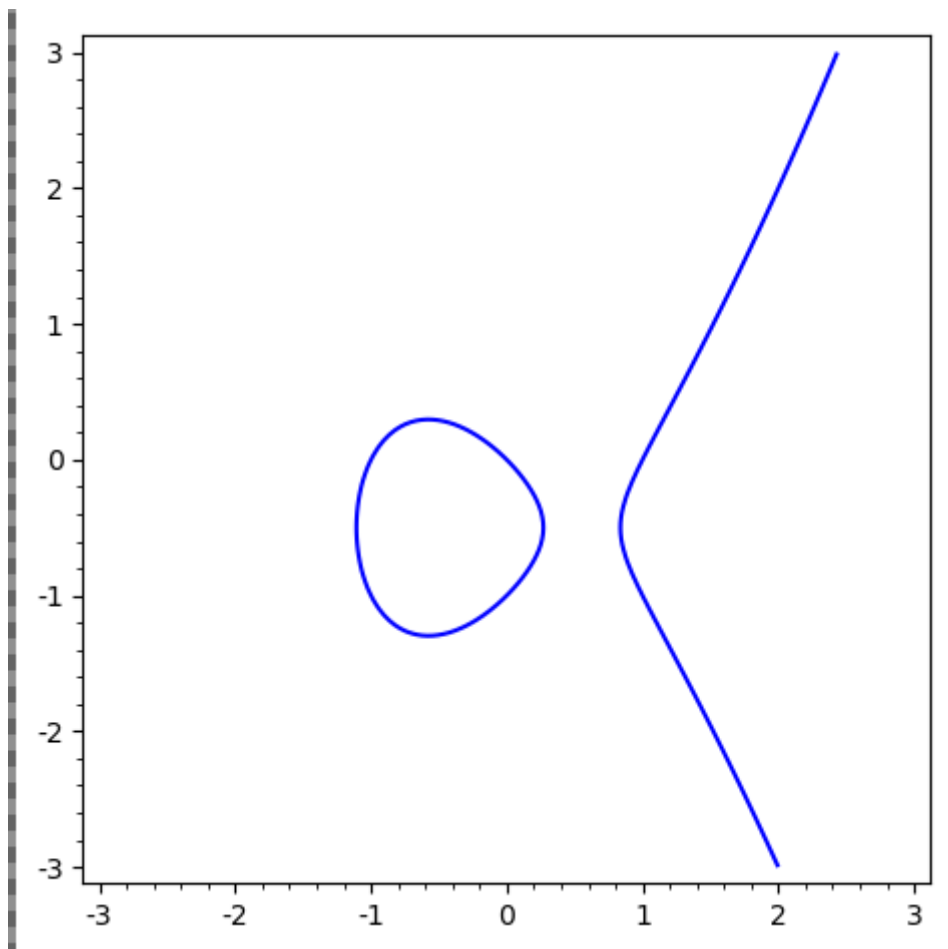
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Reference:
<https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf>

General idea: functions a coordinate ring $R[x_1, \dots, x_n]/I$ will correspond to the geometry of the variety cut out by I .

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over \mathbb{R}
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n = 1$: does it even contain a \mathbb{Q} -point? (Fermat's Last Theorem)
- The variety $\langle x^2 + 1 \rangle$, which has no \mathbb{R} -points.
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Theorem 1.1 (Harnack Curve Theorem).

If $f \in \mathbb{R}[x, y]$ is of degree d , then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.

Take the curve

$$X = \{ \mathbf{x} = (t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C} \}.$$

Then X is cut out by three equations:

- $y^2 = xz$

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- $x^2 = yz$
 - $z^2 = x^2y$