Mapping Class Groups

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Contents

1 | Setup

- All manifolds:
  - Connected
  - Oriented
  - 2nd countable (countable basis)
  - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \hom_{\mathrm{Top}}(X, Y) \coloneqq \left\{ f : X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
  - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology $\mathcal{O}_{\mathrm{CO}}$:

$$U \in \mathcal{O}_{\mathrm{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

1.1.1 Mapping Spaces

- So define

$$\mathrm{Map}(X, Y) \coloneqq (\hom_{\mathrm{Top}}(X, Y), \mathcal{O}_{\mathrm{CO}}) \qquad \text{where } \mathcal{O}_{\mathrm{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
  - $\mathrm{Homeo}(X, Y)$ the subspace of homeomorphisms
  - $\mathrm{Imm}(X, Y)$, the subspace of immersions (injective map on tangent spaces)
  - $\mathrm{Emb}(X, Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
  - $C^k(X, Y)$, the subspace of $k\times$ differentiable maps
  - $C^\infty(X, Y)$ the subspace of smooth maps
  - $\mathrm{Diffeo}(X, Y)$ the subspace of diffeomorphisms
  - $C^\omega(X, Y)$ the subspace of analytic maps
  - $\mathrm{Isom}(X, Y)$ the subspace of isometric maps (for Riemannian metrics)
  - $[X, Y]$ homotopy classes of maps

1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} \coloneqq \sup \left\{ d(f_n(x), f(x)) \,\middle|\, x \in K \right\} \overset{n\to\infty}{\to} 0 \quad \forall K \subseteq X \text{ compact.}$$

  - In words: $f_n \to f$ uniformly on every compact set.

- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{f_\alpha\} \subset \hom_{\mathrm{Top}}(X, Y)$$

  form a compact subset of $\mathrm{Map}(X, Y)$?

- Essentially answered by:

Theorem 1.1 ***(Ascoli).***
If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \hom_{\mathrm{Top}}(X, Y)$ has compact closure $\iff$ $\mathcal{F}$ is equicontinuous and $F_x \coloneqq \left\{ f(x) \,\middle|\, f \in \mathcal{F} \right\} \subset Y$ has compact closure.

Corollary 1.2 ***(Arzela).***
If $\{f_n\} \subset \hom_{\mathrm{Top}}(X, Y)$ is an equicontinuous sequence and $F_x \coloneqq \{f_n(x)\}$ is bounded for every X, it contains a uniformly convergent subsequence.

1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
  - Can take G to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} \coloneqq \hom_{\mathrm{TopGrp}}(G, S^1)$$

    where we consider $S^1 \subset \mathbb{C}$.

- Can integrate with respect to the Haar measure μ, define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) \coloneqq \int_G f(x) \overline{\chi(x)} d\mu(x).$$

2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\mathrm{Map}(\,\cdot\,, Y)$:

$$\begin{aligned} X = \{\mathrm{pt}\} \rightsquigarrow &\quad \mathrm{Map}(\{\mathrm{pt}\}, Y) \cong Y \\ X = I \rightsquigarrow &\quad \mathcal{P}Y \coloneqq \{f : I \to Y\} = Y^I \\ X = S^1 \rightsquigarrow &\quad \mathcal{L}Y \coloneqq \left\{f : S^1 \to Y\right\} = Y^{S^1}. \end{aligned}$$

- Adjoint property: there is a homeomorphism

$$\begin{aligned} \mathrm{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \mathrm{Map}(Z, Y^X) \\ H : X \times Z \to Y &\iff \tilde{H} : Z \to \mathrm{Map}(X, Y) \\ (x, z) \mapsto H(x, z) &\iff z \mapsto H(\,\cdot\,, z). \end{aligned}$$

  - Categorically, $\hom(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top.
  - A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space PY, the based loop space ΩY.
- Importance in homotopy theory: the path space fibration

$$\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y$$

  - Plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \mathrm{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},$$

  i.e. two maps f, g are homotopic $\iff$ they are connected by a path in $\mathrm{Map}(X, Y)$.

  Picture!

2.1.1 Proof

$$\mathcal{P}\mathrm{Map}(X, Y) = \mathrm{Map}(I, Y^X) \cong \mathrm{Map}(X \times I, Y),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \to Y$, the LHS are paths in the space of maps.

2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I, \cdot)$:

$$\begin{aligned}
\mathcal{P}\mathrm{Map}(X,Y) &\coloneqq \mathrm{Map}(I, Y^X) \\
\mathcal{P}^2\mathrm{Map}(X,Y) &\coloneqq \mathcal{P}\mathrm{Map}(I, Y^X) = \mathrm{Map}(I, (Y^X)^I) = \mathrm{Map}(I, Y^{XI}) \\
&\vdots \\
\mathcal{P}^n\mathrm{Map}(X,Y) &\coloneqq \mathcal{P}^{n-1}\mathrm{Map}(I, Y^{XI}) = \mathrm{Map}(X, Y^{XI^n}).
\end{aligned}$$

- Can interpret

$$\mathcal{P}^2\mathrm{Map}(X,Y) = \mathcal{P}\mathrm{Map}(X \times I, Y).$$

  as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

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3 Defining the Mapping Class Group

3.1 Isotopy

- Define a homotopy between $f, g : X \to Y$ as a map $F : X \times I \to Y$ restricting to f, g on the ends.
  - Equivalently: a *path*, an element of $\mathrm{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \to Y$ to be homeomorphisms for every t.
  - Equivalently: a path in the subspace of homeomorphisms, an element of $\mathrm{Map}(I, \mathrm{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\mathrm{Map}(X, Y)$, subspace of homeomorphisms.

3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
  - In a general category $\mathcal{C}$, we can always define the group $\mathrm{Aut}_{\mathcal{C}}(X)$.
    * If the group has a topology, we can consider $\pi_0 \mathrm{Aut}_{\mathcal{C}}(X)$, the set of path components.
    * Since groups have identities, we can consider $\mathrm{Aut}^0_{\mathcal{C}}(X)$, the path component containing the identity.
  - So we make a general definition, the *extended mapping class group*:

$$\mathrm{MCG}^{\pm}_{\mathcal{C}}(X) \coloneqq \mathrm{Aut}_{\mathcal{C}}(X)/\mathrm{Aut}^0_{\mathcal{C}}(X).$$

  - Here the $\pm$ indicates that we take both orientation preserving and non-preserving automorphisms.
  - Has an index 2 subgroup of orientation-preserving automorphisms, $\mathrm{MCG}^{+}(X)$.

Picture: quotienting out by identity component

3.3 Definitions in Several Categories

- Now restrict attention to

$$\mathrm{Homeo}(X) \coloneqq \mathrm{Aut}_{\mathrm{Top}}(X) = \left\{ f \in \mathrm{Map}(X, X) \;\middle|\; f \text{ is an isomorphism} \right\}$$
$$\text{equipped with } \mathcal{O}_{\mathrm{CO}}.$$

  - Taking $\mathrm{MCG}^{\pm}_{\mathrm{Top}}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\mathrm{Diffeo}(X) \coloneqq \mathrm{Aut}_{C^\infty}(X).$$

  - Taking $\mathrm{MCG}_{C^\infty}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\mathrm{ho}(X) \coloneqq \{[f] \in [X, Y]\}.$$

  - Taking $\mathrm{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences.*

3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
  - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi\mathrm{Aut}(X)$.
- For surfaces: For Σ a genus g surface, MCG(S) acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \to \mathrm{MCG}(\Sigma) \to T(\Sigma) \to \mathcal{M}_g \to 0$$

  where the last term is the moduli space of Riemann surfaces homeomorphic to X.
- $T(S)$ is the moduli space of complex structures on S, up to the action of homeomorphisms that are isotopic to the identity:
  - Points are isomorphism classes of marked Riemann surfaces
  - Equivalently the space of hyperbolic metrics
- Used in the Neilsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
  - Periodic,
  - Reducible (preserves some simple closed curves), or
  - Pseudo-Anosov (has directions of expansion/contraction)

Picture: $\mathcal{M}_g$.

4 Examples of MCG

4.1 The Plane: Straight Lines

- $\mathrm{MCG}_{\mathrm{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \to \mathbb{R}^2$, take the straight-line homotopy:

$$\begin{aligned} F &: \mathbb{R}^2 \times I \to \mathbb{R}^2 \\ F(x,t) &= tf(x) + (1-t)x. \end{aligned}$$

Picture: parameterize line between x and $f(x)$ and flow along it over time.

4.2 The Disc: The Alexander Trick

- $\mathrm{MCG}_{\mathrm{Top}}(\overline{\mathbb{D}}^2) = 1$: for any $f : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ such that $f|_{\partial \overline{\mathbb{D}}^2} = \mathrm{id}$, take

$$\begin{aligned} F &: \overline{\mathbb{D}}^2 \times I \to \overline{\mathbb{D}}^2 \\ F(x,t) &\coloneqq \begin{cases} tf\left(\dfrac{x}{t}\right) & \|x\| \in [0,t) \\ x & \|x\| \in [1-t, 1] \end{cases}. \end{aligned}$$

  - This is an isotopy from f to the identity.
  - Interpretation: "cone off" your homeomorphism over time:

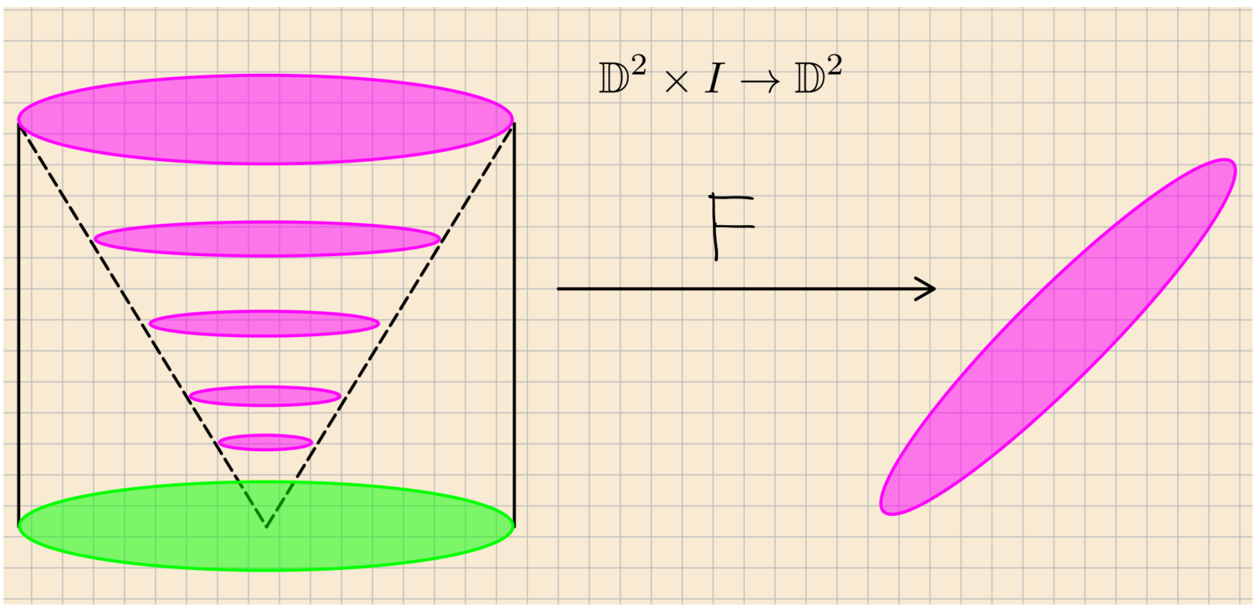

Figure 1: Image

  - Note that this won't work in the smooth category: singularity at origin

4.3 Overview of Big Results

- The word problem in $\mathrm{MCG}(\Sigma_g)$ is solvable
- For $g \geq 3$, the center of $\mathrm{MCG}(\Sigma_g)$ is trivial and $H_1(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = 1$
  - Why care: same as abelianization of the group. :::{.theorem title="Dehn-Neilsen-Baer"}

$$\mathrm{MCG}^{\pm}(\Sigma_g) \cong \mathrm{Out}(\pi_1(\Sigma_g)).$$

- $\mathrm{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation
- For $g \geq 4$, $H_2(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$
  - Why care: used to understand surface bundles

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

  - Find the classifying space $B\mathrm{Diffeo}$
  - Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\mathrm{Diffeo}(\Sigma_g)] \cong [S^{n-1}, \mathrm{Diffeo}(\Sigma_g)]$$

  - Theorem (Earle-Ells): For $g \geq 2$, $\mathrm{Diffeo}_0(\Sigma_g)$ is contractible. As a consequence, $\mathrm{Diffeo}(\Sigma_g) \twoheadrightarrow \mathrm{Mod}(\Sigma_g)$ is a homotopy equivalence, and there is a correspondence:

$$\left\{ \substack{\text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B} \right\} \Big/ \text{Bundle isomorphism} \iff \left\{ \substack{\text{Monodromy Representations} \\ \rho : \pi_1(B) \to \mathrm{MCG}(\Sigma_g)} \right\} \Big/ \text{Conjugacy}.$$

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5 | Dehn Twists