

Assignment 6: The Fourier Transform

D. Zack Garza

November 4, 2019

Contents

| | | |
|----------|--------------------|----------|
| 1 | Problem 1 | 1 |
| 2 | Problem 2 | 2 |
| 2.1 | Part (a) | 2 |
| 2.2 | Part (b) | 3 |
| 2.2.1 | (i) | 3 |
| 2.2.2 | (ii) | 3 |
| 3 | Problem 3 | 3 |
| 3.1 | (a) | 3 |
| 3.1.1 | (i) | 3 |
| 3.1.2 | (ii) | 4 |
| 3.2 | (b) | 4 |
| 4 | Problem 4 | 4 |
| 4.1 | (a) | 4 |
| 4.1.1 | (i) | 4 |
| 4.1.2 | (ii) | 5 |
| 4.2 | (b) | 5 |
| 5 | Problem 5 | 6 |
| 5.1 | (a) | 6 |
| 5.2 | (b) | 7 |
| 5.2.1 | (i) | 7 |
| 5.2.2 | (ii) | 7 |
| 5.3 | (c) | 7 |
| 6 | Problem 6 | 8 |

1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = \lim_{\xi' \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx$$

But as an immediate consequence, this yields

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \, dx \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\ &\rightarrow 0, \end{aligned}$$

which follows from continuity in L^1 since $f(x - \xi') \rightarrow f(x)$ as $\xi' \rightarrow 0$.

It thus only remains to show that the hint holds, and that $\xi' \rightarrow 0$ as $\xi \rightarrow \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned} \widehat{(f * g)}(\xi) &:= \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dy \, dx \\ &\stackrel{?}{=} \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dx \, dy \\ &= \int \int f(t) \exp(-2\pi i (x - y) \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dx \, dy \\ &\quad (t = x - y, \, dt = \, dx) \\ &= \int \int f(t) \exp(-2\pi i t \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dt \, dy \\ &= \int f(t) \exp(-2\pi i t \cdot \xi) \left(\int g(y) \exp(-2\pi i y \cdot \xi) \, dy \right) \, dt \\ &= \int f(t) \exp(-2\pi i t \cdot \xi) \hat{g}(\xi) \, dt \\ &= \hat{g}(\xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\ &= \hat{g}(\xi) \hat{f}(\xi). \end{aligned}$$

It thus remains to show that this swap is justified.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f} \hat{g} = \hat{g} \hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$(\widehat{f * g}) * h = \widehat{f * g} \hat{h} = \hat{f} \hat{g} \hat{h} = \hat{f} \widehat{g * h} = f * (g * h).$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f} \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi) \hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then $\lim_{|\xi| \rightarrow \infty} \hat{I}(\xi) \neq 0$, which by Problem 1 shows that I can not be in L^1 , a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x - y)$. We then have

$$\begin{aligned} \hat{g}(\xi) &:= \int g(x) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \exp(-2\pi i y \cdot \xi) \, dx \\ &= \exp(-2\pi i y \cdot \xi) \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \, dx \\ &\quad (t = x - y, dt = dx) \\ &= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\ &= \exp(-2\pi i y \cdot \xi) \hat{f}(\xi). \end{aligned}$$

3.1.2 (ii)

Let $h(x) = \exp(2\pi i x \cdot y) f(x)$. We then have

$$\begin{aligned}\hat{h}(\xi) &:= \int e^{2\pi i x \cdot y} f(x) e^{-2\pi i x \cdot \xi} dx \\ &= \int \exp(2\pi i x \cdot y - 2\pi i x \cdot \xi) f(x) dx \\ &= \int f(\xi - y) \exp(-2\pi i x \cdot (\xi - y)) dx \\ &= \hat{f}(\xi - y).\end{aligned}$$

3.2 (b)

We'll use the fact that if $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$\begin{aligned}\frac{1}{|\det T|} \hat{f}(T^{-T} \xi) &= \frac{1}{|\det T|} \int f(x) e^{-2\pi i x \cdot T^{-T} \xi} dx \\ &\quad x \mapsto Tx, \quad dx \mapsto |\det T| dx \\ &= \frac{1}{|\det T|} \int f(Tx) e^{-2\pi i Tx \cdot T^{-T} \xi} |\det T| dx \\ &= \int f(Tx) e^{-2\pi i x \cdot \xi} dx \\ &\quad \text{since } Tx \cdot T^{-T} \xi = T^{-1} Tx \cdot \xi = x \cdot \xi \\ &= \widehat{(f \circ T)}(\xi).\end{aligned}$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = x f(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial \xi} \hat{f}(\xi) &:= \frac{\partial}{\partial \xi} \int f(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int f(x) \frac{\partial}{\partial \xi} e^{-2\pi i x \cdot \xi} dx \\
&= \int f(x) 2\pi i x e^{-2\pi i x \cdot \xi} dx \\
&= 2\pi i \int x f(x) e^{-2\pi i x \cdot \xi} dx \\
&:= 2\pi i \hat{g}(\xi).
\end{aligned}$$

To see that the interchange is justified, we just note that we can apply the dominated convergence theorem, since $\int |f(x) e^{-2\pi i x \cdot \xi}| \leq \int |f| < \infty$, where we assumed $f \in L^1$.

4.1.2 (ii)

We have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) e^{-2\pi i x \cdot \xi} dx \\
&= f(x) e^{-2\pi i x \cdot \xi} \Big|_{x=-\infty}^{x=\infty} - \int f(x) (2\pi i \xi) e^{-2\pi i x \cdot \xi} dx \\
&\quad \text{(integrating by parts)} \\
&= - \int f(x) (-2\pi i \xi) e^{-2\pi i x \cdot \xi} dx \\
&\quad \text{(since } f(\infty) = f(-\infty) = 0\text{)} \\
&= 2\pi i \xi \int f(x) e^{-2\pi i x \cdot \xi} dx \\
&:= 2\pi i \xi \hat{f}(\xi).
\end{aligned}$$

4.2 (b)

Let $G(x) = e^{-\pi x^2}$ and ∂_ξ be the operator that differentiates with respect to ξ .

Then

$$\partial_\xi \left(\frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi)}{G(\xi)^2},$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi) = 0.$$

A direct computation shows that

$$\partial_\xi G(\xi) = -2\pi\xi G(\xi), \quad (1)$$

and we claim that $\partial_\xi \hat{G}(\xi) = -2\pi\xi \hat{G}(\xi)$ as well, which follows from the following computation:

$$\begin{aligned} \partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) e^{-2\pi i x \cdot \xi} dx \\ &= \int G(x) \partial_\xi e^{-2\pi i x \cdot \xi} dx \\ &= \int G(x) (-2\pi i x) e^{-2\pi i x \cdot \xi} dx \\ &= \int G(x) (-2\pi i x) e^{-2\pi i x \cdot \xi} dx \\ &= i \int 2\pi x G(x) e^{-2\pi i x \cdot \xi} dx \\ &= i \int \partial_x G(x) e^{-2\pi i x \cdot \xi} dx \quad \text{by (1)} \\ &:= i \widehat{\partial_x G(x)}(\xi) \\ &= i (2\pi i \xi \hat{G}(\xi)) \quad \text{by part (i)} \\ &= -2\pi \xi \hat{G}(\xi). \end{aligned}$$

We can thus write

$$G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi) = G(\xi) (-2\pi \xi \hat{G}(\xi)) - \hat{G}(\xi) (-2\pi \xi G(\xi)),$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0 , from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x) dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$\begin{aligned}
\hat{D}(\xi) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-2\pi i x \xi} dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) + i \sin(-2\pi x \xi) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) dx \\
&\quad (\text{since } \sin \text{ is odd and the domain is symmetric about } 0) \\
&= 2 \int_0^{\frac{1}{2}} \cos(-2\pi x \xi) dx \\
&\quad (\text{since } \cos \text{ is even and the domain is symmetric about } 0) \\
&= 2 \left(\frac{1}{2\pi \xi} \sin(-2\pi x \xi) \Big|_{x=0}^{x=\frac{1}{2}} \right) \\
&= \frac{\sin(\pi \xi)}{\pi \xi}.
\end{aligned}$$

5.2 (b)

5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left(\frac{\sin(\pi \xi)}{\pi \xi} \right)^2$.

5.2.2 (ii)

Letting \mathcal{F} denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if f is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting F be the box function, F can be seen to be even from its definition. Since $f := \mathcal{F}(F)$ by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that $\hat{f}(x) = F(x)$, the original box function.

5.3 (c)

By a direct computation of the integral in question, we have

$$\begin{aligned}
I(x) &:= \int e^{-2\pi|\xi|} e^{2\pi i x \xi} d\xi \\
&= \int_{-\infty}^0 e^{-2\pi(-\xi)} e^{-2\pi i x \xi} d\xi + \int_0^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&= \int_0^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} d\xi + \int_0^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&\quad \text{by the change of variables } \xi \mapsto -\xi, d\xi \mapsto -d\xi \text{ and swapping integration bounds} \\
&= \int_0^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} + e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&= \frac{1}{2\pi} \int_0^{\infty} e^{-u} e^{-i x u} + e^{-u} e^{i x u} du \\
&= \frac{1}{2\pi} \int_0^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)} du \\
&= \frac{1}{2\pi} \left(\left. \frac{-e^{-u(1+ix)}}{1+ix} \right|_{u=0}^{u=\infty} + \left. \frac{-e^{-u(1-ix)}}{1-ix} \right|_{u=0}^{u=\infty} \right) \\
&= \frac{1}{2\pi} \left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) \\
&= \frac{1}{2\pi} \frac{2}{1+x^2} \\
&= \frac{1}{\pi} \frac{1}{1+x^2},
\end{aligned}$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$\begin{aligned}
I(x) &= P(x) = \int \hat{P}(\xi) e^{-2\pi i x \xi} dx \\
&\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} = \int \hat{P}(\xi) e^{-2\pi i x \xi} dx \\
&\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} - \hat{P}(\xi) e^{-2\pi i x \xi} dx = 0 \\
&\implies \int \left(e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} dx = 0 \\
&\implies \left(e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} =_{a.e.} 0 \\
&\implies e^{-2\pi|\xi|} =_{a.e.} \hat{P}(\xi),
\end{aligned}$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

6 Problem 6

We first note that if $G_t(x) := t^{-n} e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2|\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt \right| dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt dx \\
&\quad \text{since the integrand and thus integral is positive.} \\
&= \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dx dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) dx \right) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} (1) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt,
\end{aligned}$$

which we claim is finite, so $f \in L^1$.

To see that the norm is finite, we note that

$$t \in [0, 1] \implies e^{-\pi t^2} < 1$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$t \in [1, \infty) \implies t^{2\varepsilon-1} \leq 1.$$

Thus

$$\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon-1} dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt \\
&\leq \int_0^1 t^{2\varepsilon-1} dt + \int_1^\infty e^{-\pi t^2} dt \\
&\leq \int_0^1 t^{2\varepsilon-1} dt + \int_0^\infty e^{-\pi t^2} dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

Justifying the interchange: we note that the integrand $G_t(x)e^{-\pi t^2}t^{2\varepsilon-1}$ is non-negative, and we've just showed that one of the iterated integrals is absolutely convergent, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on \mathbb{R}^n and the remaining terms are continuous on \mathbb{R} , so they are all measurable on \mathbb{R}^n and \mathbb{R} respectively. But then taking cylinders on everything in sight yields measurable functions, and the product of measurable functions is measurable.

If another interchange of integrals is justified, we can compute

$$\begin{aligned}
\hat{f}(\xi) &:= \int_{\mathbb{R}^n} \left(\int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt \right) e^{-2\pi i x \cdot \xi} dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} dt dx \\
&= ? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} dx dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} dx \right) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \hat{G}_t(\xi) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} e^{-\pi t^2 |\xi|^2} dt \\
&= \int_0^\infty e^{-\pi t^2 (1+|\xi|^2)} t^{2\varepsilon-1} dt \\
&= \int_0^\infty e^{-\pi (t\sqrt{1+|\xi|^2})^2} t^{2\varepsilon-1} dt \\
&\quad s = t\sqrt{1+|\xi|^2}, \quad ds = \sqrt{1+|\xi|^2} dt \\
&= \int_0^\infty e^{-\pi s^2} \left(\frac{s}{\sqrt{1+|\xi|^2}} \right)^{2\varepsilon-1} \frac{1}{\sqrt{1+|\xi|^2}} ds \\
&= (1+|\xi|^2)^{-\frac{2\varepsilon-1}{2}} (1+|\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon-1} ds \\
&= (1+|\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon-1} dt \\
&:= F(\xi) \|f\|_1.
\end{aligned}$$

To see that the interchange is justified, note that

$$\int_{\mathbb{R}^n} \int_0^\infty |G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi}| dt dx = \int_{\mathbb{R}^n} \int_0^\infty |G_t(x) e^{-\pi t^2} t^{2\varepsilon-1}| dt dx,$$

since $|e^{2\pi i x \cdot \xi}| = 1$. The integrand appearing is precisely what we showed was measurable when computed $\|f\|_1$ above, so Tonelli applies.

Thus $F(\xi)$ is the Fourier transform of the function $g(x) := f(x)/\|f\|_1$. \square