Category $\mathcal{O}$, Problem Set 4

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1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Show that the induced $U(\mathfrak{g})$-module structure $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

> Hence $M \not\in \mathcal{O}$.

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^{\vee} \cong M^{\vee} \otimes L^{\vee}$$

Reference for Dual of Sum

2.1 Solution

By theorem 3.2d, we have

$$M, N \in \mathcal{O} \implies (M \oplus N)^{\vee} \cong M^{\vee} \oplus N^{\vee}$$

and by definition, $M^{\vee} := \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} M_{\lambda}^{\vee}$ is the direct sum of the duals of various weight spaces.

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ.

For a concrete example, take Φ of type B_2 with a short simple root α and a long simple root β. If $\lambda := \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in Φ.

3.1 Solution

We would like to show the following two propositions:

1. $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. In general, the associated simple system $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \bigcap \Delta$.

3.1.1 Proof of Proposition 1

3.1.2 Proof of Proposition 2

Concretely, we can realize Φ and Δ as subsets of $\mathbb{R}^2$ in the following way:

$$\Phi = P_1 \coprod P_2 := \{[1,0], [0,1], [-1,0], [0,-1]\} \coprod \{[1,1], [-1,1], [1,-1], [-1,-1]\}$$
$$\Delta := \{\alpha, \beta\} := \{[1,0], [-1,1]\},$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\Phi_{[\lambda]} := \left\{ \gamma \in \Phi \mid \langle \lambda, \gamma^{\vee} \rangle \in \mathbb{Z} \right\}, \qquad \gamma^{\vee} := \frac{2}{\|\gamma\|^2} \gamma,$$
$$\Delta_{[\lambda]} := \left\{ \gamma \in \Delta \mid \langle \lambda, \gamma^{\vee} \rangle \in \mathbb{Z} \right\}.$$

Now choosing $\lambda := \dfrac{\alpha}{2} = \left[\dfrac{1}{2}, 0\right]$, we now consider the inner products $\langle \lambda, \ \gamma^\vee \rangle$ for $\gamma \in \Phi$:

Thus

$$\gamma_1 \in P_1 \implies \left\langle \left[\frac{1}{2}, 0\right], \ 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right)\langle [1,0], \ \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z}$$

$$\gamma_2 \in P_2 \implies \langle \lambda, \ \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right], \ \frac{2}{\left(\sqrt{2}\right)^2}[\pm 1, \pm 1] \right\rangle = \pm\frac{1}{2} \notin \mathbb{Z}$$

where $(\gamma_1)_1$ denotes the first component of γ_1.

We thus find that

$$\Phi_{[\lambda]} = P_1 \qquad\qquad\qquad\qquad\qquad \text{the short roots}$$
$$\Delta_{[\lambda]} = \{\alpha\} \qquad\qquad\qquad \text{the single short simple root.}$$

Choosing the following green hyperplane not containing any root, we can choose a positive system

$$\Phi^+ = \{[1,0], [0,1], [1,1], [-1,-1]\} = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$$

where we can note that $\Phi^+ \bigcap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi_{[\lambda]}^+ = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$. But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \ {\color{red}\neq}\ \{\alpha\} = \{\alpha, \alpha + \beta\} \bigcap \{\alpha, \beta\} = \Phi_{[\lambda]}^+ \bigcap \Delta,$$

which is what we wanted to show.

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that $\operatorname{coker} \phi$ has a standard filtration.