Title

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Wednesday 30$^{\text{th}}$ September, 2020

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Recall that we had a dominant weight $\lambda \in X(T)_+$ with

$$\begin{tikzcd}
 & V(\lambda) \arrow[dl, "\twoheadrightarrow"'] \arrow[dr, "\hookrightarrow"] & \\
L(\lambda) \arrow[dr, "\hookrightarrow"] & & T(\lambda) \arrow[dl, "\twoheadrightarrow"] \\
 & H^0(\lambda) &
\end{tikzcd}$$

where we have a module with both a *good* and a *Weyl* filtration.

If $B \subseteq P \subseteq G$ with P parabolic and $M \in \mathrm{Mod}(G)$, we have a "transfer theorem": maps

$$H^n(G; M) \xrightarrow{\text{Res}} H^n(P; M) \xrightarrow{\text{Res}} H^n(B; M)$$

induced by restrictions which are isomorphisms.

Proposition 1.1 ***(?).***
Let $M \in \mathrm{Mod}(P)$ with $P \supseteq B$.

a. If $\dim M < \infty$ then $\dim H^n(P; M) < \infty$.
b. If $H^j(P; M) \neq 0$ then there exists a weight λ of M such that $-\lambda \in \mathbb{N}\Phi^+$ and $\mathrm{ht}(-\lambda) \geq j$.

Part (a) is proved in the book, we won't show it here.

Proof (of part b).
Suppose $H^j(P; M) \neq 0$, then we have an injective resolution I_* for k. Tensoring with M yields an injective resolution for M,

$$0 \to M \to I_0 \otimes M \to I_1 \otimes M \to \cdots.$$

Since $H^j(B;M) \neq 0$, we know that the cocycles $\hom_B(k, I_j \otimes M) \neq 0$ and thus $\hom_T(k, I_j \otimes M) \neq 0$.
So there exists a weight $-\lambda$ of I_j with $\mathrm{ht}(-\lambda) \geq j$, and we know λ is a weight of M applying the previous lemma: namely we know that λ is invariant under the torus action, so there is a weight $-\lambda$ such that $-\lambda + \lambda = 0$.

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? Why the last part?

Theorem 1.2 ***(?)*.**
Let $\lambda, \mu \in X(T)_+$, then

1. The cohomology in the tensor product is zero, except in one special case:

$$H^i(G, H^0(\lambda) \otimes H^0(\mu)) = \begin{cases} 0 & i > 0 \\ k & i = 0, \lambda = -w_0\mu \end{cases}.$$

2. There are only extensions in one specific situation:

$$\mathrm{Ext}^i_G(V(\mu), H^0(\lambda)) = \begin{cases} 0 & i > 0 \\ k & i = 0, \lambda = \mu \end{cases}.$$

The following is an important calculation!

Proof .
Step 1: We'll use Frobenius reciprocity twice. We can write the term of interest in two ways:

$$H^i(G, H^0(\lambda) \otimes H^0(\mu)) = H^i(B, H^0(\lambda) \otimes \mu)$$

$$H^i(G, H^0(\lambda) \otimes H^0(\mu)) = H^i(G, \lambda \otimes H^0(\mu)).$$

Thus there exists a weight ν of $H^0(\lambda)$ and ν' of $H^0(\mu)$ such that

$$\mu + \nu, \lambda + \nu' \in -\mathbb{N}\Phi^+ \quad \mathrm{ht}(\mu + \nu), \mathrm{ht}(\lambda + \nu') \leq -i.$$

Since $w_0\lambda$ (resp. $w_0\mu$) is the lowest of weight of $H_0(\lambda)$ (resp. $H_0(\mu)$), it follows that

$$\mu + w_0\lambda, \lambda + w_0\mu \in -\mathbb{N}\Phi^+.$$

Since $w_0^2 = \mathrm{id}$, we can write $\lambda + w_0\mu = w_0(\mu + w_0\lambda)$. We know that the LHS is in $-\mathbb{N}\Phi^+$, and the term in parentheses on the RHS is also in $-\mathbb{N}\Phi^+$. Applying w_0 interchanges $\Phi^\pm$, so the RHS is in $\mathbb{N}\Phi^+$. But $\mathbb{N}\Phi^+ \cap -\mathbb{N}\Phi^+ = \{0\}$, forcing $\lambda + w_0\mu = 0$ and thus $\lambda = -w_0\mu$.
Since the height of zero is zero, we have

$$0 = \mathrm{ht}(\lambda + w_0\mu) \leq \mathrm{ht}(\lambda + \nu') \leq -i \implies i = 0.$$

This shows cohomological vanishing for $i > 0$, the first case in the theorem statement.
For the remaining case, we can check that $H^0(\lambda)^U = H^0(\lambda)_{w_0\lambda}$, and so

$$\left(H^0(\lambda) \otimes -w_0\lambda\right)^{U^+} = k.$$

This shows that $H^0(B; H^0(\lambda) \otimes -w_0\lambda) \cong k$, since

$$\left(H^0(\lambda) \otimes -w_0\lambda\right)^B = \left(\left(H^0(\lambda) \otimes -w_0\lambda\right)^U\right)^T.$$

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Proposition 1.3 ***(?).***
Let $\lambda, \mu \in X(T)_+$ with $\lambda \not> \mu$. Then we can calculate the ith ext by computing the $i-1$st: for $i > 0$,

$$\mathrm{Ext}^i_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^{i-1}_G(L(\lambda), H^0(\mu)/\mathrm{Soc}_G(H^0(\mu))).$$

Remark 1.
We showed this in a special case. Let $i = 1$ with $\lambda \not> \mu$, then

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(L(\lambda), H^0(\mu)/\mathrm{Soc}_G(H^0(\mu))).$$

Thus it suffices to understand only the previous layer:

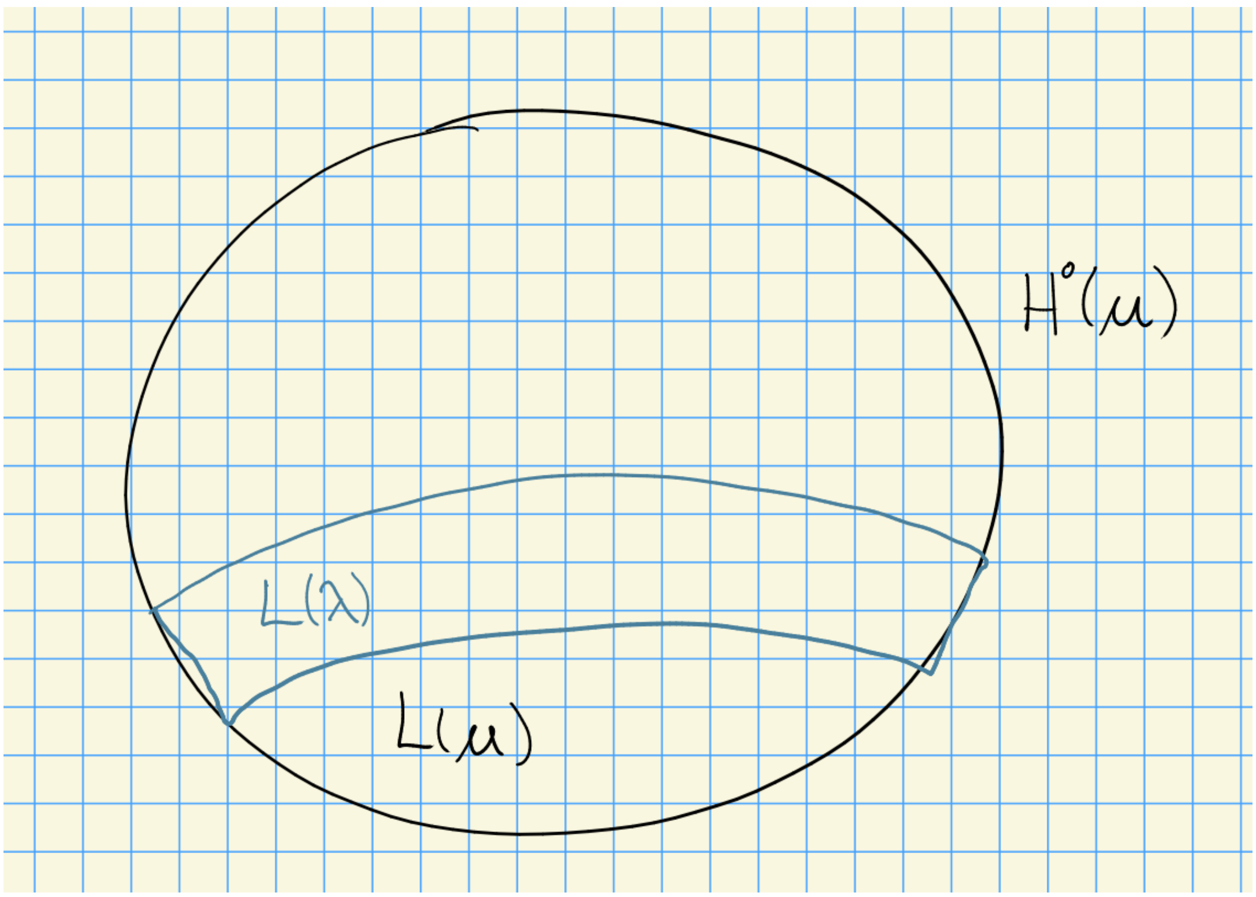

Figure 1: Image

Proof .
Consider the SES

$$0 \to L(\mu) \to H^0(\mu) \to H^0(\mu)/\operatorname{Soc}_G(H^0(\mu)) \to 0$$

which yields a LES in homology by applying $\hom_G(L(\lambda), \cdot)$. To obtain the statement, it suffices to show $\operatorname{Ext}^1_G(L(\lambda), H^0(\mu)) = 0$ for $i > 0$, since this is the middle column in the LES. We can write

$$\begin{aligned} \operatorname{Ext}^i_G(L(\lambda), H^0(\mu)) &= H^i(G, L(\lambda)^\vee \otimes H^0(\mu)) \quad \text{taking duals} \\ &= H^i(B, L(\lambda)^\vee \otimes \mu) \quad \text{by Frobenius reciprocity,} \end{aligned}$$

so we can obtain a weight σ of $L(\lambda)^\vee \otimes \mu$ such that $\sigma \in -\mathbb{N}\Phi^+$ and $\operatorname{ht}(-\sigma) \geq i > 0$ by applying the previous lemma. So $\sigma = \nu + \mu$ for ν some weight of $L(\lambda)^\vee$.
By rearranging, we find that $\sigma \in \mathbb{N}\Phi^-$. Letting λ be the lowest weight of $L(\lambda)^\vee$, we find $\sigma \geq -\lambda + \mu$ (since this can only lower the weight).
But then $-\lambda + \mu \in \mathbb{N}\Phi^-$, implying $-\mu + \lambda \in \mathbb{N}\Phi^-$, and the LHS here is equal to $\lambda - \mu$. This precisely says $\lambda > \mu$, which contradicts the assumption that λ did not dominate μ. It may also be the case that $\lambda = \mu$, which is handled separately.

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We now want criteria for when we can find the following types of lifts:

$$\begin{tikzcd} & V \\ L(\lambda) \arrow[ur, hook] \arrow[r, hook] & H^0(\lambda) \arrow[u, hook, dotted] \end{tikzcd}$$

Lemma 1.4*(Important!)*.
Let V be a G-module with $0 \neq \hom_G(L(\lambda), V)$. If

- $\hom(L(\mu), V) = 0$,
- $\operatorname{Ext}^1_G(V(\mu), V) = 0$ for all $\mu \in X(T)_+$ with $\mu < \lambda$,

then V contains a submodule isomorphic to $H^0(\lambda)$ and such a lift/extension exists.

Remark 2.
The ext criterion will be the most important. The idea is to quotient and continue applying it.

Proof .
Consider the SES

$$0 \to L(\lambda) \hookrightarrow V \to V/L(\lambda) \to 0$$

as well as

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0.$$

Now want to applying the LES in cohomology by applying $\hom_G(\,\cdot\,, V)$, we get a LES of homs over G:

$$\begin{aligned} 0 &\to \operatorname{Hom}(H^0(\lambda)/L(\lambda), V) \to \operatorname{Hom}(H^0(\lambda), V) \to \operatorname{Hom}(L(\lambda), V) \\ &\to \operatorname{Ext}^1(H^0(\lambda)/L(\lambda), V) \to \cdots . \end{aligned}$$

Thus it suffices to show this Ext^1 is zero.
Strategy: show all of the composition factors of $H^0(\lambda)/L(\lambda)$ are zero These are all of the form $L(\mu)$ for $\mu < \lambda$, so it now suffices to just show that $\operatorname{Ext}^1_G(L(\mu), V) = 0$ when $\mu < \lambda$.
Observe that we have

$$0 \to N \to V(\mu) \to L(\mu) \to 0$$

where N are $L(\sigma)$ composition factors for $\sigma < \mu$. So apply $\hom(\,\cdot\,, V)$:

$$\begin{aligned} 0 &\to \operatorname{Hom}(L(\mu), V) \to \operatorname{Hom}(V(\mu), V) \to \operatorname{Hom}(N, V) \\ &\to \operatorname{Ext}^(L(\mu), V) \to \operatorname{Ext}(V(\mu), V) \to \cdots . \end{aligned}$$

But we have $\operatorname{Hom}(N, V) = 0$ and $\operatorname{Ext}^1(V(\mu), V) = 0$, which *squeezes* and forces $\operatorname{Ext}^1(L(\mu), V) = 0$.

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Next time: state and prove a cohomological criterion (Donkin, Scott, proved independently) for a G-module to admit a good filtration. More about when tensor products of induced modules have good filtrations.