Problem Set 7

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1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence. We then have $\|x^j - x^k\|_{\ell^2} \to 0$, and we want to produce some $\mathbf{x} := \lim_{n \to \infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i := \lim_{n \to \infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left| x_i^j - x_i^k \right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left| x_i^j - x_i^k \right|^2$ is a Cauchy sequence of real numbers which necessarily converges by thethe completeness of $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \|\lim_{k \to \infty} x^k - x^j\|_{\ell^2} = \lim_{k \to \infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$
\begin{aligned}
\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\
&\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\
&\to M < \infty,
\end{aligned}
$$

where $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$ by the previous argument, and the second term is bounded because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \ \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on H takes values in $\mathbb{C}$, we have

$$\Re(\langle x, \ iy \rangle) = \Re(-i\langle x, \ y \rangle) = \Im(\langle x, \ y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\|x + iy\|^2 = \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ iy\rangle\right)$$
$$= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)$$

$$\|x - iy\|^2 = \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ iy\rangle\right)$$
$$= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)$$

and summing these all

$$\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x + iy\| = 4\,\Re\left(\langle x,\ y\rangle\right) + 4i\,\Im\left(\langle x,\ y\rangle\right)$$
$$= 4\langle x,\ y\rangle.$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 := \langle x,\ x\rangle = \langle Ux,\ Ux\rangle := \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\langle Ux,\ Uy\rangle = \frac{1}{4}\left(\|Ux + Uy\|^2 + \|Ux - Uy\|^2 + i\|Ux + Uy\|^2 - i\|Ux + Uy\|^2\right)$$
$$= \frac{1}{4}\left(\|U(x + y)\|^2 + \|U(x - y)\|^2 + i\|U(x + y)\|^2 - i\|U(x + y)\|^2\right)$$
$$= \frac{1}{4}\left(\|x + y\|^2 + \|x - y\|^2 + i\|x + y\|^2 - i\|x + y\|^2\right)$$
$$= \langle x,\ y\rangle.$$

$\square$

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| = |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle|$$
$$= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle|$$
$$\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|$$
$$\to 0 \cdot M + C \cdot 0 < \infty,$$

where $\|y_n\| \to M$ since $y_n \to y$ implies that $\|y\|$ is finite.

2.1 Part a:

We want to show that sequences in $E^\perp$ converge to elements of $E^\perp$. Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$\langle e,\ y\rangle = \left\langle \lim_n e_n,\ y\right\rangle = \lim_n \langle e_n,\ y\rangle = \lim_n 0 = 0,$$

so $e \in E^\perp$.

2.2 Part b:

Let $S := \mathrm{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n \to x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$y \in E^\perp \implies \langle x_n,\ y\rangle = \left\langle \sum_i a_i e_i,\ y\right\rangle = \sum_i a_i \langle e_i,\ y\rangle = 0 \implies x_n \in (E^\perp)^\perp.$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$y \in E^\perp \implies \langle x,\ y\rangle = \left\langle \lim_n x_n,\ y\right\rangle = \lim_n \langle x_n,\ y\rangle = 0 \implies x \in (E^\perp)^\perp,$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notation convenience, we'll just write S for $\overline{S}$. Let $x \in E^{\perp\perp}$. Noting that S is closed, we can define P, the operator projecting elements onto S, and write

$$x = Px + (x - Px) \in S \oplus S^\perp$$

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But since $\langle x, \ x - Px \rangle = 0$ because $x - Px \in E^\perp$ and $x \in (E^\perp)^\perp$, we can rewrite the first term in this inner product to obtain

$$0 = \langle x, \ x - Px \rangle = \langle Px + (x - Px), \ x - Px \rangle = \langle Px, \ x - Px \rangle + \langle x - Px, \ x - Px \rangle,$$

where we can note that the first term is zero because $Px \in S$ and $x - Px \in S^\perp$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\|e_0\|^2 = \int_0^1 1^2 \ dx = 1$$

$$\|e_1\|^2 = \int_0^1 3(2x - 1)^2 = \frac{1}{2}(2x - 1)^2 \Big|_0^1 = 1$$

$$\langle e_0, \ e_1 \rangle = \int_0^1 \sqrt{3}(2x - 1) \ dx = \frac{\sqrt{3}}{4}(2x - 1) \Big|_0^1 = 0.$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0, 1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$f(x) = \sum_i \left\langle x^3, \ e_i \right\rangle e_i$$

$$= \left\langle x^3, \ 1 \right\rangle 1 + \left\langle x^3, \ \sqrt{3}(2x-1) \right\rangle \sqrt{3}(2x-1)$$

$$= \int_0^1 x^2 \ dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1) \ dx$$

$$= \frac{1}{3} + \sqrt{3}(2x-1)\frac{\sqrt{3}}{6}$$

$$= x - \frac{1}{6}.$$

We can also compute

$$\|f - g\|_2^2 = \int_0^1 (x^2 - x + \frac{1}{6})^2 \ dx$$

$$= \frac{1}{180}$$

$$\implies \|f - g\|_2 = \frac{1}{\sqrt{180}}.$$

4 Problem 4

4.1 Part a

4.1.1 i

We can first note that $\left\langle 1/\sqrt{2}, \ \cos(2\pi nx) \right\rangle = \left\langle 1/\sqrt{2}, \ \sin(2\pi mx) \right\rangle = 0$ for any n or m, since this involves integrating either sine or cosine over an integer multiple of its period.

Letting $m, n \in \mathbb{Z}$, we can then compute

$$\langle \cos(2\pi nx), \ \sin(2\pi mx) \rangle = \int_0^1 \cos(2\pi nx)\sin(2\pi mx) \ dx$$

$$= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x) \ dx$$

$$= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2} \int_0^1 \sin(2\pi(n-m)x) \ dx$$

$$= 0,$$

which again follows from integration of sine over a multiple of its period (where we use the fact that $m + n, m - n \in \mathbb{Z}$).

Similarly,

$$\langle \cos(2\pi nx),\ \cos(2\pi mx) \rangle = \int_0^1 \cos(2\pi nx)\cos(2\pi mx)\ dx$$

$$= \frac{1}{2}\int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x)\ dx$$

$$= \begin{cases} \frac{1}{2}\int_0^1 \cos(4\pi nx) + 1\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.$$

$$\langle \sin(2\pi nx),\ \sin(2\pi mx) \rangle = \int_0^1 \sin(2\pi nx)\sin(2\pi mx)\ dx$$

$$= \frac{1}{2}\int_0^1 \cos(2\pi(m-n)x) + \cos(2\pi(m+n)x)\ dx$$

$$= \begin{cases} \frac{1}{2}\int_0^1 1 + \cos(4\pi nx)\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

4.1.2 ii

We have

$$\left\langle e^{2\pi kx},\ e^{-2\pi i\ell x} \right\rangle = \int_0^1 e^{2\pi ikx}\overline{e^{2\pi i\ell x}}\ dx$$

$$= \int_0^1 e^{2\pi ikx}e^{-2\pi i\ell x}\ dx$$

$$= \int_0^1 e^{2\pi i(k-\ell)x}\ dx$$

$$\left(= \int_0^1 1\ dx = 1 \quad \text{if } k = \ell, \text{ otherwise:}\right)$$

$$= \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)}\Big|_0^1$$

$$= \frac{e^{2\pi i(k-\ell)} - 1}{2\pi i(k-\ell)}$$

$$= 0,$$

since $e^{2\pi ik} = 1$ for every $k \in Z$, and $k - \ell \in \mathbb{Z}$. Thus this set is orthonormal.

4.2 Part b

4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials $P_n(x)$ such that $\|f - P_n\|_\infty \to 0$, i.e. the P_n uniformly approximate f on $[0,1]$.

Letting $\varepsilon > 0$, we can thus choose a P such that $\|f - P\|_\infty < \varepsilon$, which necessarily implies that $\|f - P\|_{L^1} < \varepsilon$ since we have

$$\int_0^1 |f(x) - P(x)| \ dx \le \int_0^1 \varepsilon \ dx = \varepsilon.$$

Thus we can write

$$f(x) = P(x) + (f(x) - P(x))$$

where $h(x) := f(x) - P(x)$ satisfies $\|h\|_{L^1} < \varepsilon$. It only remains to show that $P \in L^2([0,1])$, but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say $|P(x)| \le M < \infty$ for all $x \in [0,1]$, and thus

$$\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2 \ dx \le \int_0^1 M^2 \ dx = M^2 < \infty.$$

It follows that we can let $g = P$ and $h = f - P$ to obtain the desired result.

4.2.2 ii

By part (i), the claim is that it suffices to show this is true for $f \in L^2$. In this case, we can identify

$$\int_0^1 f(x) \cos(2\pi k x) \ dx := \Re(\hat{f}(k))$$
$$\int_0^1 f(x) \sin(2\pi k x) \ dx := \Im(\hat{f}(k)),$$

the real and imaginary parts of the kth Fourier coefficient of f respectively.

By Bessel's inequality, we know that $\left\{\hat{f}(k)\right\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$, and so $\sum_k \left|\hat{f}(k)\right| < \infty$.

But this is a convergent sequence of real numbers, which necessarily implies that $\left|\hat{f}(k)\right| \to 0$. In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have $f \in L^1$, write $f = g + h$ where $g \in L^2$ and $\|h\|_{L^1} \to 0$. Then

$$\begin{aligned}
\left|\int_0^1 f(x) \cos(2\pi k x) \ dx\right| &= \left|\int_0^1 (g(x) + h(x)) \cos(2\pi k x) \ dx\right| \\
&\le \left|\int_0^1 g(x) \cos(2\pi k x) \ dx\right| + \left|\int_0^1 h(x) \cos(2\pi k x) \ dx\right| \\
&\le \left|\int_0^1 g(x) \cos(2\pi k x) \ dx\right| + \int_0^1 |h(x)||\cos(2\pi k x)| \ dx \\
&= |\hat{g}(k)| + \varepsilon \\
&\to 0,
\end{aligned}$$

with a similar computation for $\int f(x) \sin(2\pi k x)$. $\quad\square$

5 Problem 5

5.1 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e}_1 = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i,\ \hat{e}_i \rangle \hat{e}_i$

The result set $\{\hat{e}_i\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$
\begin{aligned}
e_2 &= x - \langle x,\ \hat{e}_1 \rangle \hat{e}_1 \\
&= x - \langle x,\ 1 \rangle 1 \\
&= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x\ dx \\
&= x - \int \text{odd function} \\
&= x,
\end{aligned}
$$

and so $e_2 = x$. We can then check that

$$
\|e_2\| = \left(\int_{-1}^{1} x^2\ dx \right)^{1/2} = \sqrt{\frac{2}{3}},
$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}} x$.

We continue to compute

$$
\begin{aligned}
e_3 &= x^2 - \left\langle x^2,\ \hat{e}_1 \right\rangle \hat{e}_1 - \left\langle x^2,\ \hat{e}_2 \right\rangle \hat{e}_2 \\
&= x^2 - \frac{1}{2} \int_{-1}^{1} x^2\ dx - \frac{3}{2} x \int_{-1}^{1} x^3\ dx \\
&= x^2 - \left(\frac{1}{6} x^3 \right) \Big|_{-1}^{1} + \frac{3}{2} x \int_{-1}^{1} \text{odd function} \\
&= x^2 - \frac{1}{3}.
\end{aligned}
$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$\hat{e}_3 = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$$

$$= \frac{1}{2}\sqrt{\frac{45}{2}}\frac{1}{3}(3x^2 - 1)$$

$$= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right).$$

In summary, this yields

$$\hat{e}_1 = \frac{1}{\sqrt{2}}$$

$$\hat{e}_2 = x$$

$$\hat{e}_3 = \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right),$$

which are scalar multiples of the first three Legendre polynomials.

5.2 Part b

Let $p(x) = a + bx + cx^2$, we are then looking for p such that $\|x^3 - p(x)\|_2^2$ is minimized. Noting that

$$p(x) \in \text{span}\left\{1, x, x^2\right\} = \text{span}\left\{P_0(x), P_1(x), P_2(x)\right\} \coloneqq S,$$

we can conclude that $p(x)$ will be the projection of x^3 onto S. Thus $p(x) = \sum_{i=0}^{2}\langle x^3, \hat{e}_i\rangle\hat{e}_i$.

Proceeding to compute the terms in this expansion, we can note that $\langle x^3, f\rangle$ for any f that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$\langle x^3, x\rangle x = x\int_{-1}^{1} x^4 \, dx = \frac{2}{5}x$$

.

And thus $p(x) = \frac{2}{5}x$ is the minimizer.

5.3 Part c

The first three conditions necessitate $g \in S^{\perp}$ and $\|g\| = 1$. Since S is a closed subspace, we can write $x^3 = p(x) + (x^3 - p(x)) \in S \oplus S^{\perp}$, and so $x^3 - p(x) \in S^{\perp}$.

The claim is that $g(x) \coloneqq x^3 - p(x)$ is a scalar multiple of the desired maximizer. This follows from the fact that

$$\left|\langle x^3 - p, g\rangle\right| \leq \|x^3 - p\|\|g\|$$

by Cauchy-Schwarz, with equality precisely when $g = \lambda(x^3 - p)$ for some scalar λ. However, the restriction $\|g\| = 1$ forces $\lambda = \|x^3 - p\|^{-1}$.

A computation shows that

$$\|x^3 - p\|^2 = \int_0^1 (x^3 - \frac{2}{5}x)^2 \, dx = \frac{19}{525},$$

and so we can take

$$g(x) := \frac{25}{\sqrt{19}} \left(x^3 - \frac{2}{5}x \right).$$

6 Problem 6

6.1 Part a

To see that $g \in \mathcal{C}$, we can compute

$$\langle g, \, 1 \rangle = \int_0^1 18x^2 - 5 \, dx = 6 - 5 = 1$$
$$\langle g, \, x \rangle = \int_0^1 18x^3 - 5x \, dx = \frac{18}{4} - \frac{5}{2} = 2.$$

To see that $\mathcal{C} = g + S^\perp$, let $f \in \mathcal{C}$, so $\langle f, \, 1 \rangle = 1$ and $\langle f, \, x \rangle = 2$. We can then conclude that $f - g \in S^\perp$, since we have

$$\langle f - g, \, 1 \rangle = \langle f, \, 1 \rangle - \langle g, \, 1 \rangle = 1 - 1 = 0$$
$$\langle f - g, \, x \rangle = \langle f, \, x \rangle - \langle g, \, x \rangle = 2 - 2 = 0.$$

6.2 Part b

Note that this equivalent to finding an $f_0 \in \mathcal{C}$ such that $\|f_0\|$ is minimized.

Letting $f_0 \in \mathcal{C}$, be arbitrary and noting that by part (a) we have $f_0 = g + s$ where $s \in S^\perp$, we can compute

$$\begin{aligned} \|f_0\|^2 &= \langle f_0, \, f_0 \rangle \\ &= \langle g + s, \, g + s \rangle \\ &= \|g\|^2 + 2\Re\langle g, \, s \rangle + \|s\|^2, \end{aligned}$$

which can be minimized by taking $s = 0$, which forces $\|s\|^2 = 0$ and $\langle g, \, s \rangle = 0$. But this imposes the condition $f_0 = g + 0 = g$. $\quad \square$