Title

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Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p.

Notation	Definition
$k[\mathbf{x}] = k[x_1, \cdots, x_n]$ ———————————+	**0.3 Polynomial ring in n indeterminates**
$k(\mathbf{x}) = k(x_1, \cdots, x_n)$	Rational function field in n indeterminates

+———————————+———————————————————| | $\mathcal{U} \rightrightarrows X$ |
An open cover $\mathcal{U} = \left\{ U_j \,\middle|\, j \in J \right\}$ | +———————————+———————————
———————————| | Δ_X | The diagonal $\left\{ (x, x) \,\middle|\, x \in X \right\} \subseteq X \times X$ | +———————————
———+———————————————————| | $\mathbb{A}^n_{/k}$ | Affine n-space
$\mathbb{A}^n_{/k} := \left\{ \mathbf{a} = [a_1, \cdots, a_n] \,\middle|\, a_j \in k \right\}$ | +———————————+———————————
| | $\mathbb{P}^n_{/k}$ | Projective n-space | | — | $\mathbb{P}^n_{/k} := (k^n \setminus \{0\}) / x \sim \lambda x$ | | — | $= \left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \,\middle|\, p, q, \in k[x_1, \cdots, x_n] \right\}$
| +———————————+———————————| | $V(J), V_a(J)$ | Variety associated to an ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ | +———————————+———————————| |
— | $:= \left\{ \mathbf{x} \in \mathbb{A}^n \,\middle|\, f(\mathbf{x}) = 0, \, \forall f \in J \right\}$ | | $I(S), I_a(S)$ | Ideal associated to a subset $S \subseteq \mathbb{A}^n_k$ | | — | $:= \left\{ f \in k[x_1, \cdots, x_n] \,\middle|\, f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in X \right\}$ | | $A(X)$ | Coordinate ring of a variety, $k[x_1, \cdots, x_n]/I(X)$ | | $V_p(J)$ | Projective variety of an ideal | | — | $:= \left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \,\middle|\, f(\mathbf{x}) = 0, \, \forall f \in J \right\}$ | | $I_p(S)$ | Projective ideal (?) | | — | $:= \left\{ f \in k[x_1, \cdots, x_n] \,\middle|\, f \text{ is homogeneous and } f(x) = 0 \, \forall x \in S \right\}$ | | $S(X)$ | Projective coordinate ring, $k[x_1, \cdots, x_n]/I_p(X)$ | | f^h | Homogenization, $x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)$ | | f^i | Dehomogenization, $f(1, x_1, \cdots, x_n)$ | | J^h | Homogenization of an ideal, $\left\{ f^j \,\middle|\, f \in J \right\}$ | | $\overline{X}$ | Projective closure of a subset | | — | $:= V_p(J^h) := \left\{ \mathbf{x} \in \mathbb{P}^n \,\middle|\, f^h(\mathbf{x}) = 0 \, \forall f \in X \right\}$ | | $\mathcal{O}_X$ | Structure sheaf $\left\{ f : U \to k \,\middle|\, f \in k(\mathbf{x}) \text{ locally} \right\}$ | | $D(f)$ | Distinguished open set, $D(f) = V(f)^c =$

$\left\{x \in \mathbb{A}^n \;\middle|\; f(x) \neq 0\right\}$ | +——————————+——————————————
——————+

Fruit	Price	Advantages
Bananas	\$1.34	• built-in wrapper • bright color
Oranges	\$2.10	• cures scurvy • tasty

0.4 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\,\cdot\,)$ and $I(\,\cdot\,)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

0.5 Useful Examples

0.5.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x-p)$ a point.

0.5.2 Presheaves / Sheaves

- $C^\infty(\,\cdot\,, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\,\cdot\,, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\,\cdot\,)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\,\cdot\,)$, the constant sheaf associated to $\mathbb{R}$ (locally constant real-valued functions)
- $\mathrm{Hol}(\,\cdot\,, \mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.6 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

Algebra	Geometry
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n_{/k}$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \cdots, x_n - p_n$	Points $[a_1, \cdots, a_n]$
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
Prime ideals $\mathfrak{p} \in \mathrm{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
$k[x_1, \cdots, x_n]/I(X)$	$A(X)$ (Functions on X)
$A(X)$ a domain	X is irreducible
$A(X)$ indecomposable	X is connected
Krull dimension n (chaints of primes)	Topological dimension n (chains of irreducibles)