Title

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Recall the that *Hasse-Weil zeta function* of a one-variable function field $K/\mathbb{F}_q$ over a finite ground field is defined in the following way: let $A_n = A_n(K)$ be the number of effective divisors of degree n. We have proved that A_n is finite, and for $n > 2g-2$ we have a formula

$$Z(t) = \sum_{n=0}^{\infty} A_n t^n = \sum_{D \in \mathrm{Div}^+(K)} t^{\deg(D)} \in \mathbb{Z}[[t]],$$

which is a formal power series with integer coefficients.

Remark 1.0.1 : Recall that we have proved that it is a rational function of t, and in particular when $g = 0, \delta = 1$ [1] we get

$$Z(t) = \frac{1}{(1-qt)(1-t)}.$$

We got another expression which isn't fantastic: it involves this δ, which we'll work toward proving is equal to 1. When $g > 1$, we broke the zeta function into two pieces $Z(t) = F(t) + G(t)$. For divisors of sufficiently high degree, Riemann-Roch tells you what the dimension of the Riemann-Roch space is, and $G(t)$ explains the part coming from divisors of large degree. We obtained a formula previously for $F(t)$ and $G(t)$, and once we show $\delta = 1$ the formula for G will simplify. For $F(t)$, we specifically had

$$F(t) = \frac{1}{q-1} \sum_{0 \leq \deg(c) \leq 2g-2} q^{\ell(c) t^{\deg(c)}},$$

where the sum is over divisor classes and ℓ is the dimension of linear system corresponding to a divisor. But this isn't a great formula: what are these classes, dhow many are in each degree, and what is the dimension of the Riemann-Roch space?

Remark 1.0.2 : This is analogous to the Dedekind zeta function of a number field K, in which case

$$\zeta_K(s) = \dot{\sum_{T \in \ell(\mathbb{Z}_k)}} |\mathbb{Z}_k / I|^{-s},$$

which will be covered in a separate lecture on *Serre zeta functions*.

> **Theorem 1.0.1** ***(F.K. Schmidt).***
> For all $K/\mathbb{F}_q$, we have $\delta = I(K) = 1$ where I is the index.

This will follow from the associated, but it much weaker. However, this is one of the facts we'd like to establish to use to *prove* the Riemann hypothesis.

[1] The *index* of the function field, least positive degree of a divisor.

Remark 1.0.3 : Pete studied this in 2004 and found that every $I \in \mathbb{Z}^+$ arises as the index of a genus one function field $K/\mathbb{Q}$.

Notation: for $n \in \mathbb{Z}^+$, let μ_n denote the nth roots of unity in $\mathbb{C}$.

Lemma 1.1 ***(?).***
For $m, r \in \mathbb{Z}^+$, set $d \coloneqq \gcd(m, r)$. Then

$$\left(1 - t^{mr/d}\right)^d = \prod_{\xi \in \mu_r} 1 - (\xi t)^m.$$

Proof (?).
In $\mathbb{C}[x]$, we have

$$(X^{r/1} - 1)^d = \prod_{\xi \in \mu_r} (X - \xi^m),$$

where both sides are monic polynomials whose roots include the (r/d)th roots of unity, each with multiplicity d. On the LHS, the distinct roots are the r/dth roots of unity, then raising to the dth power gives them multiplicity d. On the RHS, this is an exercise in cyclic groups: consider the nth power map on $\mathbb{Z}/r\mathbb{Z}$ and compute its image and kernel. As ξ ranges over rth roots of unity, ξ^m ranges over all r/dth roots of unity, each occurring with multiplicity d. Substituting $X = t^{-m}$ and multiplying both sides by t^r yields the original result.

Special case: set $m = r$, so $d = r$, then the RHS is r copies of 1.

■

Next up, we want to compare the zeta function $Z(t)$ for a function field over $\mathbb{F}_q$ to the zeta function obtained when extending scalars to $\mathbb{Q}^r$.

Proposition 1.0.1 ***(Factorization identity for the zeta function).***
Let $K/\mathbb{F}_q$ be a function field, $r \in \mathbb{Z}^+$, and take the compositum K_r of K and $\mathbb{F}_q^r$ viewed as a function field over $\mathbb{F}_q^r$. Let $Z(t)$ be the zeta function of $K/\mathbb{F}_q$ and $Z_r(t)$ the zeta function of $K_r/\mathbb{F}_q^r$. Then

$$Z_r(t^r) = \prod_{\xi \in \mu_r} Z(\xi t).$$

Proof (?).
We have an Euler product formula

$$Z(t) = \prod_{p \in \Sigma(K/\mathbb{F}_q)} (1 - t^{\deg(p)})^{-1}.$$

where the sum is over places of the function field.

Proving this Euler product formula might show up in a separate lecture, but it is not any more difficult than proving it for the Riemann zeta function.

Exercise 1.0.1 (?): Why is this product expansion true? Write as a geometric series with ratio $t^{\deg(p)}$. Here just expand each summand to get

$$Z(t) = \prod_p \sum_{j=1}^{\infty} t^{j \deg(p)}.$$

Multiplying this out and collecting terms is in effect multiplying out the prime divisors to get effective divisors.
We now use the result about splitting that was stated (but not proved):

Claim: If $p \in \Sigma_m(K/\mathbb{F}_q)$ is a degree n place and $r \in \mathbb{Z}^+$, then there exist precisely $d := \gcd(m, r)$places p^r of K_r lying over p. Moreover, each place p^r has degree m/d.
In order to compare $Z_r(t)$ to $Z(t)$, we collect the p' into ones that have the same fiber. We then can range over all p first, then over all p' in the fiber above p, yielding

$$Z_r(t^r) = \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} \prod_{p'/p} \frac{1}{1 - t^{r \deg(p')}}.$$

Using the Euler product identity, we have for $p \in \Sigma_m(K_{/\mathbb{F}_q})$ and $d := \gcd(m, r)$ we can express the innermost product as

$$\prod_{p'/p} \frac{1}{1 - t^{r \deg(p')}} = (1 - t^{rm/d})^{-d} = \prod_{\xi \in \mu_r} (1 - (\xi t)^m)^{-1},$$

where we've used the fact that we know there are exactly d places and each contributes the same degree in the first expression. By using $-d$ in the previous lemma, we get the last term. Combining all of this yields

$$Z_r(t^r) = \prod_{\xi \in \mu_r} \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} (1 - (\xi t)^{\deg p})^{-1} = \prod_{\xi \in \mu_r} Z(\xi t).$$

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Remark 1.0.4 : Similar to taking an abelian extension of number fields and noting that the Dedekind zeta function factors into a finite product: the original zeta function, and in general, Hecke L functions. If you do this for an abelian number field over $\mathbb{Q}$, then the Dedekind zeta function of the upstairs number field will be a finite product where one of the terms in the Riemann zeta function and the others are Dirichlet L functions associated to certain Dirichlet characters. So this is some (perhaps simpler) version of that.

We can finally prove Schmidt's theorem that $\delta = 1$.

Proof ($\delta = 1$).
Take a δth root of unity $\xi \in \mu_\delta$. Then for all places $p \in \Sigma(K_{/\mathbb{F}_q})$, δ divides $\deg p$ by definition

since it is a gcd, and so we have

$$Z(\xi t) = \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} (q - (\xi t)^{\deg p})^{-1} = \prod_{p \in \Sigma_{K_{\mathbb{F}_q}}} \frac{1}{1 - t^{\deg p}} = Z(t),$$

using the fact that $\xi^{\deg p} = 1$.
We're now in a situation where we can apply the previous proposition, which gives the following identity for the zeta function over the degree δ extension:

$$Z_\delta(t^\delta) = \prod_{\xi \in \mu_\delta} Z(\xi t) = Z(t)^\delta.$$

Our previous formulas show that any zeta function for a 1-variable function field over a finite field has a simple pole at $t = 1$, and since $\mathrm{Ord}_{t-1}(t^\delta) = 0$, we get

$$-1 = \mathrm{Ord}_{t-1} Z_\delta(t^\delta) = \mathrm{Ord}_{t-1} Z(t)^\delta) = -\delta,$$

where for the first equality we're using the fact that the $(t-1)$-adic valuation of $Z_\delta(t^\delta)$ is one, and for the RHS, the ordinary zeta function has a simple pole at $t = 1$ and since we have a valuation, raising something to the δth power is just δ times the original valuation.

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There is some modest representation theory (character theory) that shows up when looking at zeta functions of abelian extensions.

Remark 1.0.5 : We can also conclude that every genus zero function field $K_{/\mathbb{F}_q}$ is isomorphic to $\mathbb{F}_q(t)$ and thus rational, since such a function field rational iff it has index one. Why? By Riemann-Roch, index one implies existence of a divisor of degree one, and taking a genus zero curve says that every divisor of nonnegative degree is linearly equivalent to an effective divisor. Thus if you have a divisor of degree one, you have an effective divisor of degree one, which makes the function field a degree one extension of a rational function field.

Exercise 1.0.2 (?): Let $K = \mathbb{F}_q(t)$, then show that $g = 0, \delta = 1$, and

$$Z(t) = \frac{1}{(1 - qt)(1 - t)}.$$

Hint: go back to complicated formulas and substitute $\delta = 1$ to simplify things.

Thus for rationality of the zeta function, we can get rid of the δ cluttering up formulas. Going back to the plan, we wanted to show

1.Rationality: $Z(t) \in \mathbb{Q}(t)$ and thus $Z(t) = P(t)/Q(t)$,

2. Understand the degrees of P and Q in terms of the genus, and

3. Ask about the roots of $P(t)$ to understand the analog of the Riemann Hypothesis for Dedekind zeta functions

We'll want to establish a functional equation, as is the usual yoga for zeta functions, since it helps establish a meromorphic continuation to $\mathbb{C}$.

The algebraic significance of the functional equation is that it aids in understand several equivalent packets of data:

- The number of effective divisors of a given degree,
- The number of places of a given degree,
- The number of rational points over each finite degree extension of the base field.

1.1 The Functional Equation

Theorem 1.1.1 ***(Functional Equation).***
Let $K_{/\mathbb{F}_q}$ be a function field of genus g, then

$$Z(t) = q^{g-1}t^{2g-2}Z\left(\frac{1}{qt}\right).$$

Proof (?).
For $g = 0$, we know that

$$Z(t) = \frac{1}{(1-t)(1-qt)},$$

and plugging in $\dfrac{1}{qt}$ is a straightforward calculation. So assume $g \geq 1$.
The idea was that we wrote $Z(t) = F(t) + G(t)$. The $F(t)$ piece came from summing over divisor classes of degree between 0 and $2g-2$ and recording the dimension of the associated linear system. The tricky piece $G(t)$ came from summing an infinite geometric series to get a more innocuous closed-form expression of a rational function. So the strategy here is to separately establish the functional equation for each of F and G separately. How to do this: for $g = 0$, there was no $F(t)$ piece. If we have a closed form it's just a computational check. For $F(t)$, we'll use our greatest weapon and dearest ally, the Riemann-Roch theorem. This will provide the extra symmetry we need.
We essentially already applied Riemann-Roch to $G(t)$ to get the closed-form expression, but we haven't applied it to the small degree divisors. This doesn't tell you what the dimension is, but rather gives you a duality result: ti gives the dimension in terms of the dimension of a complementary divisor.
Take a canonical divisor $\mathcal{K} \in \operatorname{Div}(K)$, so $\deg \mathcal{K} = 2g-2$. As C runs through all divisor classes of $\mathcal{K}$ of degree d with $0 \leq d \leq 2g-2$, so does the complementary divisor $\mathcal{K} - C$.
We can thus write

$$
\begin{aligned}
(q-1)F(t) &= \sum_{0 \le \deg C \le 2g-2} q^{\ell(C)} t^{\deg(C)} \\
(q-1)G(t) &= h\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t}\right).
\end{aligned}
$$

We can thus compute

$$
(q-1)F\left(\frac{1}{qt}\right) = \sum_{0 \le \deg C \le 2g-2} q^{\ell(C)} \left(\frac{1}{qt}\right)^{\deg C} = \sum_{0 \le \deg C \le 2g-2} q^{\ell(\mathcal{K}-C)} \left(\frac{1}{qt}\right)^{2g-2-\deg C},
$$

where in the second step we've exchanged C for $\mathcal{K}-C$ and noted that $\deg(\mathcal{K}-C) = 2g-2-\deg(C)$. We now do the calculation another way,

$$
\begin{aligned}
(q-1)F(t) = &\sum_{0 \le \deg C \le 2g-2} q^{\ell(C)} t^{\deg C} \\
&= q^{g-1} t^{2g-1} \sum_{0 \le \deg C \le 2g-2} q^{\deg(C)-(2g-2)+\ell(\mathcal{K}-C)} t^{\deg(C)-(2g-2)} \quad \text{by Riemann-Roch} \\
&= q^{g-1} t^{2g-2} \sum_{0 \le \deg C \le 2g-2} q^{\ell(\mathcal{K}-C)} \left(\frac{1}{qt}\right)^{\deg(\mathcal{K}-C)} \\
&= q^{g-1} t^{2g-2} (q-1) F\left(\frac{1}{qt}\right).
\end{aligned}
$$

where we've used Riemann-Roch to find that $\ell(C) = \ell(\mathcal{K}-C) + \deg(C) - g + 1$. Cancelling the common factor of $(q-1)$ establishes the functional equation for $F(T)$.
Now using the fact that $\delta = 1$, we have

$$
(q-1)G(t) = h\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t}\right),
$$

and thus

$$
\begin{aligned}
(q-1)q^{g-1}t^{2g-2} G\left(\frac{1}{qt}\right) &= h q^{g-1} t^{2g-2} \left(q^g \left(\frac{1}{qt}\right)^{2g-1} - \frac{1}{1-q\left(\frac{1}{qt}\right)} - \frac{1}{1-\frac{1}{qt}} \right) \\
&= h\left(\frac{-1}{1-t} + \frac{q^g t^{2g-1}}{1-qt}\right) \\
&= (q-1)G(t),
\end{aligned}
$$

which establishes the functional equation for $G(t)$.

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1.2 The L Polynomial

Definition 1.2.1 (The L Polynomial)**.**

$$L(t) \coloneqq (1-t)(1-qt)Z(t) \in \mathbb{Z}[t].$$

This essentially clears the denominators in $Z(t)$, so this is now a polynomial of degree at most $2g$.