

# Problem Set 1

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## 1 Problem 1

We'll use the following definition of a smooth map between manifolds:

Definition: Let  $M, N$  be smooth manifolds of dimensions  $m, n$  respectively and  $f : M \rightarrow N$  a continuous map. Then  $f$  is *smooth* iff for every  $p \in M$ , there exists a chart  $(U, \phi)$  with  $p \in U$  and a chart  $(V, \psi)$  with  $f(p) \in V$  such that

- $f(U) \subseteq V$ , and
- The induced map

$$\begin{aligned}\bar{f} : \phi(U) &\rightarrow \psi(V) \\ \bar{f} &= \psi \circ f \circ \phi^{-1}\end{aligned}$$

is smooth as a map from  $\mathbb{R}^m \rightarrow \mathbb{R}^n$ .

We will thus show that both  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  and  $f^{-1} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  are both smooth bijections, from which we can conclude that  $f$  is a diffeomorphism.

So identify  $0 = [0, 1]$  and  $\infty = [1, 0]$  in  $\mathbb{CP}^1$  and choose the following charts on  $\mathbb{CP}^1$  in terms of homogeneous coordinates:

$$\begin{aligned} (U, \phi) &:= \\ U = \mathbb{CP}^1 \setminus \{\infty\} &= \{[x, y] \mid x, y \in \mathbb{C}, y \neq 0\} \\ \phi : \mathbb{CP}^1 &\rightarrow \mathbb{C} \\ [x, y] &\mapsto x/y. \end{aligned}$$

$$\begin{aligned} \phi^{-1} : \mathbb{C} &\rightarrow \mathbb{CP}^1 \\ z &\mapsto [z, 1]. \end{aligned}$$

and

$$\begin{aligned} (V, \psi) &:= \\ V = \mathbb{CP}^1 \setminus \{0\} &= \{[x, y] \mid x, y \in \mathbb{C}, x \neq 0\} \\ \psi : \mathbb{CP}^1 &\rightarrow \mathbb{C} \\ [x, y] &\mapsto y/x. \end{aligned}$$

$$\begin{aligned} \psi^{-1} : \mathbb{C} &\rightarrow \mathbb{CP}^1 \\ z &\mapsto [1, z]. \end{aligned}$$

Now define

$$\begin{aligned} \tilde{f} : \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1 \\ p &\mapsto \begin{cases} p, & p = \infty \\ p + c & \text{otherwise} \end{cases} \end{aligned}$$

We then need to determine a formula for  $\tilde{f}$  in homogeneous coordinates. We compute

$$\begin{aligned} p \in U &\implies p = [a, b], \quad a, b \in \mathbb{C}, \quad b \neq 0 \\ &\implies \tilde{f}([a, b]) \Big|_U = (\phi^{-1} \circ f \circ \phi)([a, b]) \\ &= (\phi^{-1} \circ f)\left(\frac{a}{b}\right) \\ &= \phi^{-1}\left(\frac{a}{b} + c\right) \\ &= \left[\frac{a}{b} + c, 1\right] \\ &= [a + bc, b] \end{aligned}$$

and

$$\begin{aligned}
p \in V &\implies p = [a, b], \quad a, b \in \mathbb{C}, \quad a \neq 0 \\
&\implies \tilde{f}([a, b]) \Big|_V = (\psi^{-1} \circ f \circ \psi)([a, b]) \\
&= (\psi^{-1} \circ f)\left(\frac{b}{a}\right) \\
&= (\psi^{-1})\left(\frac{b}{a} + c\right) \\
&= \left[1, \frac{b}{a} + c\right] \\
&= [a, b + ac]
\end{aligned}$$

Since  $\mathbb{CP}^1 = U \cup V$ , we can note that if  $p \in M$  then either  $p \in U$  or  $p \in V$ . Moreover,  $p \in U \implies \tilde{f}(p) \in \tilde{f}(U) = U$ , since  $p = [a, b]$  with  $b \neq 0 \implies f(p) = [a + bc, b]$  where  $b \neq 0$  as well, so  $\tilde{f}(p) \in U$  and  $\tilde{f}(U) \subseteq U$ . Similarly,  $\tilde{f}(V) \subseteq V$ . So it only remains to check that the following two compositions are smooth:

- $f_U : \mathbb{C} \rightarrow \mathbb{C}, f_U := \phi \circ \tilde{f} \circ \phi^{-1}$ , and
- $f_V : \mathbb{C} \rightarrow \mathbb{C}, f_V := \psi \circ \tilde{f} \circ \psi^{-1}$ .

We can compute

$$\begin{aligned}
f_U(z) &:= (\phi \circ \tilde{f} \circ \phi^{-1})(z) \\
&= (\phi \circ \tilde{f})([z, 1]) \\
&= \phi([z + c, 1]) \\
&= z + c
\end{aligned}$$

$$\begin{aligned}
f_V(z) &:= (\psi \circ \tilde{f} \circ \psi^{-1})(z) \\
&= (\psi \circ \tilde{f})([1, z]) \\
&= \psi([1, z + c]) \\
&= z + c
\end{aligned}$$

## 2 Problem 3

### 2.1 Part 1

Note: Throughout this question, we will identify  $\{f : C^\infty(M) \rightarrow \mathbb{R}\} \cong C^\infty(M)^\vee$  as vector spaces.

Let  $M, N$  be smooth manifolds and  $f : M \rightarrow N$  be a fixed smooth map, and define a map

$$\begin{aligned}
\phi : C^\infty(N) \times TM &\rightarrow \mathbb{R} \\
(h, v) &\mapsto v(h \circ f)
\end{aligned}$$

## 2.2 Part 1

Using the derivation definition, we can identify this assignment as a map

$$\begin{aligned}\phi : C^\infty(N) \times C^\infty(M)^\vee &\rightarrow \mathbb{R} \\ (h, v) &\mapsto v(h \circ f)\end{aligned}$$

We'd like to show that this yields a well-defined element of  $T_p M = C^\infty(M)$ . So for some fixed  $v \in T_p M$ , define a map

$$\begin{aligned}\phi_v : C^\infty(N) &\rightarrow \mathbb{R} \\ h &\mapsto v(h \circ f),\end{aligned}$$

which will be an element of  $TM$  if it is a derivation. For  $x \in N$ , we have

$$\begin{aligned}\phi_v(h_1 \cdot h_2)(x) &:= v((h_1 h_2) \circ f)(x) \\ &= v((h_1 \circ f)(h_2 \circ f))(x) \\ &= v(h_1 \circ f)(x) \cdot h_2(x) + h_1(x) \cdot v(h_2 \circ f)(x) \quad \text{since } v \text{ is a derivation} \\ &= \phi_v(h_1)(x) \cdot h_2(x) + h_1(x) \cdot \phi_v(h_2)(x).\end{aligned}$$

## 2.3 Part 2

Using the integral curve definition,

## 3 Problem 4

### 3.1 Part 1

Let  $V = \mathbb{R}^n$  as a vector space, let  $g$  be a nonsingular matrix, and define a map

$$\begin{aligned}\phi : V &\rightarrow V^\vee \\ v &\mapsto (\phi_v : w \mapsto \langle v, gw \rangle)\end{aligned}$$

The claim is that  $\phi$  is a natural isomorphism. It is clearly linear (following from the linearity of the inner product and matrix multiplication), so it remains to check that it is a bijection.

To see that  $\ker \phi = 0$ , so that only the zero gets sent to the zero map, we can suppose that  $x \in \ker \phi$ . Then  $\phi_x : w \mapsto \langle x, gw \rangle$  is the zero map. But the inner product is nondegenerate by definition, i.e.  $\langle x, y \rangle = 0 \forall y \implies x = 0$ . So  $x$  could only have been the zero vector to begin with.

But  $\dim V = \dim V^\vee$ , so any injective linear map will necessarily be surjective as well.

### 3.2 Part 2

Let  $g : TM \otimes TM \rightarrow \mathbb{R}$  be a metric, and consider the tangent space  $TM$ . By definition, the cotangent space  $T_p^* M = (T_p M)^\vee$

## 4 Problem 5

### 4.1 Part 1

Let  $A \in \text{Mat}(n, n)$  be a positive definite  $n \times n$  matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and  $B \in \text{Mat}(n, n)$  be positive semi-definite, so

$$\langle v, Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v, (A + B)v \rangle &= \langle v, Av \rangle + \langle v, Bv \rangle \\ &> \langle v, Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

### 4.2 Part 2

Let  $M$  be a smooth manifold with tangent bundle  $TM$  and a maximal smooth atlas  $\mathcal{A}$ . Choose a covering of  $M$  by charts  $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$  such that  $M \subseteq \bigcup_{i \in I} U_i$ . Then choose a partition of unity  $\{f_i\}_{i \in I}$  subordinate to  $\mathcal{C}$ , so for each  $i$  we have

$$\begin{aligned} f_i &: M \rightarrow \mathbb{R} \\ \forall p \in M, \quad \sum_{i \in I} f_i(p) &= 1 \end{aligned}$$

In each copy of  $\phi_i(U_i) \cong \mathbb{R}^n$ , let  $g^i$  be the Euclidean metric given by the identity matrix, i.e.  $g^i_{jk} := \delta_{jk}$ . We then have

$$\begin{aligned} g^i &: T\phi_i(U_i) \otimes T\phi_i(U_i) \rightarrow \mathbb{R} \\ (\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is defined for pairs of vectors in  $T\phi_i(U_i) \cong T\mathbb{R}^n = \text{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$  on basis vectors as the Kronecker delta and extended linearly.

Note that each coordinate function  $\phi_i : U_i \rightarrow \mathbb{R}^n$  induces a map  $\tilde{\phi}_i : TU_i \rightarrow T\mathbb{R}^n$ .

Let  $G^i$  be the pullback of  $g^i$  along these induced maps  $\tilde{\phi}_i$ , so

$$\begin{aligned} G^i &: TU_i \otimes TU_i \rightarrow \mathbb{R} \\ G^i(x, y) &:= \left( (\tilde{\phi}_i)^* g^i \right) (x, y) := g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)) \end{aligned}$$

Then, for a point  $p \in M$ , define the following map:

$$\begin{aligned} g_p &: T_p M \otimes T_p M \rightarrow \mathbb{R} \\ (x, y) &\mapsto \sum_{i \in I} f_i(p) G^i(x, y). \end{aligned}$$

The claim is that  $g_p$  defines a metric on  $M$ , and thus the family  $\{g_p \mid p \in M\}$  yields a tensor field and thus a Riemannian metric on  $M$ . If we define the map

$$\begin{aligned} g &: M \rightarrow (TM \otimes TM)^\vee \\ p &\mapsto g_p \end{aligned}$$

then  $g$  can be expressed as

$$g = \sum_{i \in I} f_i G^i.$$

We can check that this is positive definite by considering  $x \in T_p M$  and computing

$$\begin{aligned} g(x, x) &:= g_p(x, x) \\ &= \sum_{i \in I} f_i(p) G^i(v, v) \\ &= \sum_{i \in I} f_i(p) g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)), \end{aligned}$$

where each term is positive semi-definite, and *at least one term* is positive definite because  $\sum_i f_i(p)$  must equal 1. By part 1, this means that the entire expression is positive definite, so  $g$  is a metric.  $\square$

## 5 Problem 6

### 5.1 Part 1

Let  $M = S^2$  as a smooth manifold, and consider a vector field on  $M$ ,

$$X : M \rightarrow TM$$

We want to show that there is a point  $p \in M$  such that  $X(p) = 0$ .

Every vector field on a compact manifold without boundary is complete, and since  $S^2$  is compact with  $\partial S^2 = \emptyset$ ,  $X$  is necessarily a complete vector field.

Thus every integral curve of  $X$  exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point  $p \in M$ .

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular,  $\phi_0 = \text{id}_M$ , and  $\phi_1 \in \text{Diff}(M, M)$ . Moreover  $\phi_0$  is homotopic to  $\phi_1$  via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to  $\phi_0$  and  $\phi_1$ . For an arbitrary map  $f : M \rightarrow M$ , we have

$$\Lambda(f) = \sum_k \text{Tr} \left( f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where  $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of  $M$ .

Since homotopic maps induce equal maps on homology, we also have  $\Lambda(\phi_1) = \chi(M)$ .

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have  $\chi(S^2) = 2 \neq 0$ , and thus  $\phi_1$  has a fixed point  $p_0$ , thus

$$\left. \frac{\partial}{\partial t} \phi_t(p_0) \right|_{t=1} \text{ so}$$

$$\begin{aligned} &\phi_t(p) = p \\ \implies &\frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies &\left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies &X(\phi_1(p_0)) := \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 && \text{by definition of } \phi_1 \end{aligned}$$

so  $X(\phi_1(p_0)) = 0$ , which shows that  $p_0$  is a zero of  $X$ . So  $X$  has at least one zero, as desired.  $\square$

## 5.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\rightarrow S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\rightarrow (\mathbf{x}, [1, 1]) \end{aligned}$$

which is the identity on the  $S^2$  component and assigns the constant vector  $[1, 1]$  to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & TS^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

can *not* have a nowhere vanishing section.  $\square$