Title

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We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for K/k of transcendence degree 1, all valuations on K which are trivial on k are discrete. We'll now want to go farther and describe the places $\Sigma(K/k)$, which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is K. Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if A is a Dedekind domain such that $\mathrm{ff}(A) = K$, then for all $\mathfrak{p} \in \mathrm{mSpec}(A)$ there exists a discrete valuation v_p on K. I.e., every maximal ideal induces a discrete valuation that is A-regular, so the valuation ring will contain A. How is this obtained? Take a nonzero $x \in K^\times$, and take the corresponding principal fractional ideal $\langle x \rangle := Ax$, which we can factor in a Dedekind domain as $Ax = \prod_{\mathfrak{p} \in \mathrm{mSpec}(A)} \mathfrak{p}^{\alpha_\mathfrak{p}}$ with $\alpha_\mathfrak{p} \in \mathbb{Z}$. This looks like an infinite product, but for any fixed x, only finitely many α are nonzero. Note that these α are exactly what we're looking for: the $\mathfrak{p}$-adic evaluation of x is given precisely by $v_\mathfrak{p}(x) := \alpha_\mathfrak{p}$, where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$\begin{aligned} v_\cdot : \mathrm{mSpec}(A) &\to \Sigma(K/A) \\ \mathfrak{p} &\mapsto v_\mathfrak{p}. \end{aligned}$$

So this sends a maximal ideal to a place that is A-regular, and it turns out to be a bijection.

Proposition 1.0.1 *(?)*.
The map v is a bijection, and thus we may write

$$\Sigma(K/A) \cong \mathrm{mSpec}(A).$$

Proof (?).

Claim: v is injective.
If $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{mSpec}(A)$ are two different maximal ideals. Then there exists an element $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$, and so $x^{-1} \in A_{\mathfrak{p}_2} \setminus A_{\mathfrak{p}_1}$. This follows since if x is not in $\mathfrak{p}_2$, its $\mathfrak{p}_2$-adic valuation is zero, and thus the $\mathfrak{p}_2$-adic valuation of x^{-1} is $-0 = 0$ as well. On the other hand, since $x \in \mathfrak{p}_1$, its $\mathfrak{p}_1$-adic valuation is positive and therefore $v_{\mathfrak{p}_1}(x^{-1}) < 0$ and x^{-1} is not in $A_{\mathfrak{p}_1}$.

Claim: v is surjective.
Let $v \in \Sigma(K/A)$, so $A \subset R_v$, i.e. take a valuation whose valuation ring contains A. Note that we're not assuming the valuation is discrete, this can be a general Krull valuation, but we're trying to show it's equal to a certain p-adic valuation. As always with a subring of a valuation ring, we can pull back the maximal ideal and consider $\mathfrak{m}_v \cap A \in \mathrm{Spec}(A)$. We're hoping that this is a maximal ideal, since maximals correspond to valuations. Since we're in a Dedekind

domain, the only prime ideal we *don't* want this to be is the zero ideal of A, so suppose it were. Then $A^\bullet \subset R_v^\times$, and so $K^\times \subset R_v^\times$. This is because the only element of the maximal ideal that lies in A is zero, so every nonzero element of A is not in this maximal ideal and is thus a unit. But for any unit, its inverse is also a unit, yielding the inclusion $K^\times \subset R_v^\times$. The only way this could possibly happen is if $R_v = K$, which yields the trivial valuation ring. However, by definition, in $\Sigma(K/A)$ we've excluded the trivial valuation, so this ideal can not be zero.

So we can conclude that the pullback $\mathfrak{m}_v \cap A \in \operatorname{mSpec}(A)$, and so $A_\mathfrak{p} \subset R_v$. This is from viewing elements in $A_\mathfrak{p}$ as quotients of elements in A whose denominator have $\mathfrak{p}$-adic valuation zero. Recall that we want to show that $R_v = A_\mathfrak{p}$. We know $R_v \subset K$ is a proper containment, and we can use the fact that a *discrete* valuation ring is maximal among all proper subrings of its fraction field. In other words, for R a DVR, there is no ring R' such that $R \subset R' \subset \operatorname{ff}(R)$. How do you prove this? This is similar to an early exercise in commutative algebra, where we looked at all rings between $\mathbb{Z}$ and $\mathbb{Q}$, which generalized to looking at all rings between a PID and its fraction field, and a DVR is a local PID. So proving this statement is actually easier.

This is enough to show that $A_\mathfrak{p} = R_v$, and this $v \sim v_\mathfrak{p}$.

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Remark 1.0.2: What the idea? For a general one variable function field K/k, we'll produce affine Dedekind domains R with $k \subset R \subset K$ and $\operatorname{ff}(R) = K$. This will give is subrings of this full ring of places that are mSpec of Dedekind domains. How many such domains will we need for their union to be the entire set of places? Just one won't work, since $\Sigma(K/k)$ is like a complete or projective object, and a projective variety of dimension 1 can't be covered by a single affine variety. However, it turns out that you can always cover it with 2. In fact, if you take any Dedekind domain between k and $\operatorname{ff}(K)$, the set of missing places (the ones that aren't regular for any of these domains) will be a nonempty finite set of places. So you can always cover it by finitely many, and two suffices: as a consequence of the Riemann-Roch theorem, after removing any nonempty finite set of places, you'll have the mSpec of a canonically associated Dedekind domain. We'll prove this by starting with the case of $K = k(t)$.

Claim:

$$|\Sigma(k(t)/k) \setminus \operatorname{mSpec} k[t]| = 1.$$

Note that $k \subset k[t] \subset k(t)$ and $k[t]$ is a Dedekind domain, so this fits into the above framework, and moreover we know the maximal ideals of polynomial rings: irreducible monic polynomials. Taking all of these misses exactly one place. How do we describe this missing place?

Suppose $v \in \Sigma(k(t)/k) \setminus \Sigma(k(t)/k[t])$, so the valuation ring of v contains k but does not contain $k[t]$. Then the valuation ring can not contain t, and thus $v(t) < 0$ and $v(1/t) = -v(t) > 0$. Since $k[1/t]$ is a PID, so if the valuation wasn't $tdash$regular, it's $1/t$-regular by definition. So $v \in \Sigma(k(t)/k[1/t])$. Note that $k[1/t] \cong k[t]$ as rings. How many valuations on this polynomial ring give positive valuation to $1/t$? Exactly one, since this corresponds to a prime ideal, namely $\langle 1/t \rangle$, so this unique valuation is $v = v_{\frac{1}{t}}$, the $1/t$-adic valuation.

That is, if we write $f \in k(t)$ as $(1/t)^n a(1/t)/b(1/t)$ with $a, b \in k[t]$ polynomials with nonzero constant terms, then $v_{\frac{1}{t}}(f) = n$. Note that this process is the same as the one used to compute the t-adic valuation v_t.

Recall that a valuation on a domain can be uniquely extended to its fraction field by setting $v(x/y) = v(x) - v(y)$.

Exercise 1.0.3*(?)*: Define $v_\infty : k(t)^\times \to \mathbb{Z}$ by $p(t)/q(t) \mapsto \deg q - \deg p$.

a. Show $v_\infty \in \Sigma(k(t)/k[1/t])$.

b. Show $v_\infty \sim v_{\frac{1}{t}}$ by showing they have the same valuation ring.

c. Show that $v_\infty = v_{\frac{1}{t}}$.

Note that $1/t$ is a uniformizer for v_∞

Theorem 1.0.4*(Complete description of places).*

$$\Sigma(k(t)/k) = \operatorname{mSpec} k[t] \coprod \{v_\infty\}.$$

Note that we know the maximal ideals – the irreducible monic polynomials – but it takes some effort to write them down. If k is algebraically closed, however, every such polynomial is linear of the form $t - \alpha$ for $\alpha \in k$. In this case, $\operatorname{mSpec} k(t) \cong k$, and so $\sigma(\bar{k}(t)/\bar{k}) = \bar{k} \coprod \{\infty\} = \mathbb{P}^1(\bar{k})$. More generally, the set of places on a rational function field will yield the scheme-theoretic set of closed points on the projective line over k, which is more complicated if $k \neq \bar{k}$ since not all closed points are k-rational. Another way to say this is that if you have a valuation, there is a residue field, and for any place on a one variable function field the residue field will be a finite degree extension of k. The degree 1 points will be the k-rational points, and so $\Sigma(k(t)/k)$ will always contain a copy of k but may have closed points of larger degree, making things slightly more complicated. This complication is handled well in both the scheme-theoretic and this valuation-theoretic approach.

The next theorem is a fact from commutative algebra:

Theorem 1.0.5*(?).*
Let A be a domain with $\operatorname{ff}(A) = K$. Suppose A is a finitely generated k-algebra, let L/K be a finite degree field extension, and let B be the integral closure of A in L. Then

a. B is finitely generated as an A-module.[a]

b. B is an integrally closed domain with $\operatorname{ff}(B) = L$ which is finitely generated as a k-algebra.

c. $\dim A = \dim B$[b]

d. If A is Dedekind, so is B.

[a] See CA notes, "Second Normalization Theorem", where normalization is a more geometric synonym for integral closure.

[b] Krull dimension, i.e. the supremum of lengths of chains of prime ideals.

Proof (?).
See Pete's CA notes sections 18 and 14. ∎

Remark 1.0.6: On why these should be true: we have a NTI square

$$\begin{tikzcd}
B \arrow[r, hook, "\subset"] & L \\
A \arrow[u, hook] \arrow[r, hook, "\subset"] & K \arrow[u, hook] \\
k \arrow[u, hook]
\end{tikzcd}$$

We have a domain A with a fraction field K, we take a finite degree extension L/K, and to complete the square we let B be the integral closure of A in L: the collection of elements in L satisfying monic polynomials with coefficients in A.

In our case, we're additionally assuming that A/k is finitely generated as a k-algebra.

Remark 1.0.7:
On (b): B being finitely generated as a k-algebra follows from assuming A is, and additionally that B is finitely generated as an A-module, and finite generation as a module provides finite generation as an algebra. The result follows from transitivity of finite generation of algebras.

On (c): This is just a property of integral extensions.

On (d): Use the characterization of being Noetherian, integrally closed, and Krull dimension 1. The only thing to check is that B is Noetherian, which follows from B being finitely generated as a k-algebra and applying the Hilbert basis theorem.

Remark 1.0.8: Note that we are not assuming that L/K is separable, which is an assumption that would simplify things. By the Krull-Akuzuki theorem, B will always be a Dedekind domain, but it need not be finitely generated over A. So the "stem" to k is grounding the situation: it's not just a Dedekind domain, but rather an *affine* domain: a domain that is finitely generated over a field. Note that this is much better than an arbitrary Dedekind domain!

Proposition 1.0.9 *(?)*.
Suppose that instead of $K = \mathrm{ff}(A)$, we instead have $A \subset K$ an arbitrary subring, and L/K a finite extension. Taking the integral closure B yields another NTI square:

$$\begin{tikzcd} B \arrow[r, hook, "\subset"] & L \\ A \arrow[u, hook] \arrow[r, hook, "\text{subring}"] & K \arrow[u, hook] \end{tikzcd}$$

Suppose we have an upstairs valuation v on L. Then it makes sense to restrict valuations to subfields, so

$$v \in \Sigma(L/B) \iff v\big|_K \in \Sigma(K/A).$$

So the original valuation is B-regular iff the restricted valuation is A-regular.

Proof (?).
$\Longleftarrow$: Since $A \subseteq B$, being B-regular implies being A-regular.

$\Longrightarrow$: Suppose $A \subset R_v$ and $x \in B$, and choose $a_0, \cdots, a_{n-1} \in A$ such that

$$p(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

So we have an integral relation for x, and we want to show $v(x) < 0$ and derive some contradiction from the fact that $v(a_i) \geq 0$. Note that we aren't grounded to the base field here, so this valuation may not be discrete and is rather some arbitrary Krull valuation.

If $x \notin R_v$, then $v(x) < 0$, and we can thus write

$$v(x^n) < \min\left\{ v(a_j x^j) \;\middle|\; 0 \leq j \leq n-1 \right\} \leq v(p(x)).$$

$\blacksquare$