

# Title

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## 1 | Chapter 9

## 2 | 9.1 and Review

- $(M, \omega)$  a symplectic manifold,  $H \in ?$  a Hamiltonian,  $X_H$  its ?
- $\int_{S^2} u^* \omega = \sigma_1$  where  $u \in C^\infty(S^2, W)$ .
- $\langle c_1(TW), \pi_2(TW) \rangle = 0?$
- $C_k(H) := \mathbb{Z}/2\mathbb{Z}[S]$  where  $S$  is the set of periodic orbits of  $X_H$  of Maslov index  $k$ .
- $x, y$  critical points of  $\mathcal{A}_H$  with  $\mathcal{M}(x, y)$  the moduli space of contractible solutions of finite energy connecting  $x, y$ .

### 2.1 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$ , so we quotient to define  $\mathcal{L}(x, y) := \mathcal{M}_{x,y}/\mathbb{R}$  with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

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**Proposition 2.1(?)**

$\mathcal{L}(x, y)$  is Hausdorff.

- Want to show  $\mathcal{L}(x, y)$  is a compact 0-dimensional manifold.
- Have a differential

$$\partial : C_k(H) \longrightarrow C_{k-1}(H)$$

$$\partial(x) = \sum_{\text{Ind}(y)=k-1} n(x, y)y.$$

with  $n(x, y)$  the number (mod 2) of trajectories of grad  $\mathcal{A}_H$  connecting  $x, y$ , i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

$$M := \overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

and show that  $M$  is compact with  $\partial M$  equal to the last union.

- Last time: closure of space of trajectories connecting  $x, y$  contains “broken” trajectories.
- Last time: toward proving that  $M$  is compact

## 3 | 9.2

- Wanted to compactify  $\mathcal{L}(x, y)$ , needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

**Theorem 3.1(9.2.1).**

Let  $(H, J)$  be a regular pair with  $H$  nondegenerate.

Let  $x, z$  be two periodic trajectories of  $H$  such that  $\mu(x) = \mu(z) + 2$ .

Then  $\overline{\mathcal{L}}(x, y)$  is a compact 1-manifold with boundary satisfying

$$\partial \overline{\mathcal{L}}(x, y) = \bigcup_{\mu(x) < \mu(y) < \mu(z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

As a corollary,  $\partial^2 = 0$ .

- Know  $\overline{\mathcal{L}}(x, y)$  is compact and  $\mathcal{L}(x, y)$  is a 1-manifold
- Now suffices to study in a neighborhood of boundary points (“gluing theorem”)

### 3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function  $w_p$  which interpolates between  $u$  and  $v$  (not exactly a solution itself, but will be approximated by one later).

2. Constructing  $\psi$  a “true solution” from  $w_p$  using the Newton-Picard method. We’ll have

$$\psi(p) = \exp_{w_p}(\gamma(p)) \quad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p} \mathcal{P}(x, z).$$

where  $\mathcal{P} = ?$ .

3. Get a lift  $\hat{\psi} = \pi \circ \psi$  where  $\pi = ?$  satisfying

- $\hat{\psi}(p) \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v})$
- $\hat{\varphi}$  is an embedding
- $\hat{\psi}$  is unique in the following sense (the last point)

**Theorem 3.2(9.2.3 (Gluing Theorem)).**

Let  $x, y, z$  be critical points of the action functional  $\mathcal{A}_H$  such that  $\mu(x) = \mu(y) + 1 = \mu(z) + 2$ . Let  $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$  be trajectories, inducing  $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$ .

- There exist a differentiable map  $\psi : (\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$  for some  $\rho > 0$  such that
- $\pi \circ \psi : (\rho_0, \infty) \rightarrow \mathcal{L}(x, z)$  is an embedding
- $\hat{\psi} \xrightarrow{\rho \rightarrow \infty} (\bar{u}, \bar{v}) \in \overline{\mathcal{L}(x, z)}$ .
- If  $\ell_n \in \mathcal{L}(x, z)$  with  $\ell_n \xrightarrow{n \rightarrow \infty} (\bar{u}, \bar{v})$ , then for  $n \gg 1$  we have  $\ell \in \mathfrak{F}(\hat{\psi})$ .

## 4 | 9.3: Pre-gluing

- Choose a bump function  $\beta$  on  $\{0\}^c \subset \mathbb{R} \rightarrow [0, 1]$  which is 1 on  $|x| \geq 1$  and 0 on  $|x| < \varepsilon$
- Split into positive and negative parts  $\beta^\pm$ :

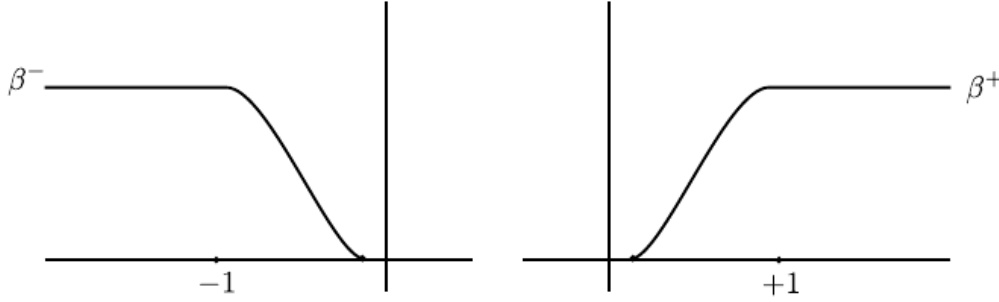


Figure 1: Figure 9.3

- Define the interpolation  $w_\rho$  from  $u$  to  $v$  in the following way:

$$w_\rho(s, t) = \begin{cases} u(s + \rho, t) & \text{if } s \leq -1 \\ \exp_{y(t)} \left( \beta^-(s) \exp_{y(t)}^{-1}(u(s + \rho, t)) + \beta^+(s) \exp_{y(t)}^{-1}(v(s - \rho, t)) \right) & \text{if } s \in [-1, 1] \\ v(s - \rho, t) & \text{if } s \geq 1 \end{cases}$$

- 
- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\}.$$

## 5 | 9.4: Construction of $\psi$ .

- Have constructed  $w_\rho \in C^\infty_\times(x, z)C^\infty(x, z)$  for every  $\rho \geq \rho_0$ , since there is exponential decay.
- Yields  $\psi_\rho \in \mathcal{M}(x, z)$  a true solution (to be defined).
- Need to check that  $\mathcal{F}(\psi_\rho) = 0$  where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- $\psi_\rho$  already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining  $\mathcal{F}_\rho$ .

$$\begin{aligned} & W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \xrightarrow{\mathcal{F}_\rho} L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \\ (y_1, \dots, y_{2n}) & \mapsto \left[ \left( \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } H_t \right) \left( \exp_{w_\rho} \sum y_i Z_i^\rho \right) \right]_{Z_i}. \end{aligned}$$