

Homotopy Groups of Spheres

Graduate Student Seminar

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Summary

Outline

Homotopy
Groups of
Spheres

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Summary

Examples

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

Definitions

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- For $f, g \in \text{hom}_{\text{Top}}(X, Y)$, we write $f \sim g \iff$ there exists an $F \in \text{hom}_{\text{Top}}(X \times I, Y)$ such that $F(0) = f, F(1) = g$.

Classification

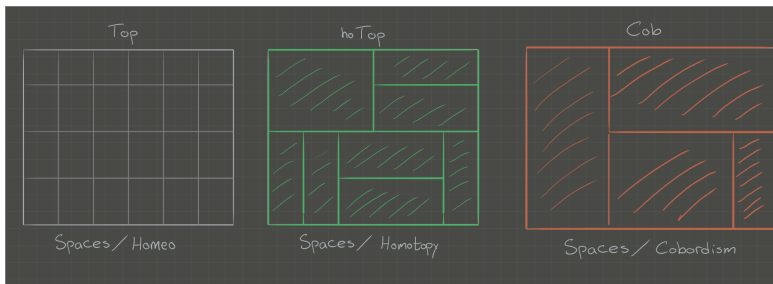
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- Holy grail: understand the topological category completely
 - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category $DTop$, its interplay with $hoTop$ is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

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Proposition: Let B be a CW complex; then isomorphism classes of \mathbb{R}^1 -bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G , the functor

$$h_G(\cdot) : \text{hoTop}^{\text{op}} \longrightarrow \text{Set}$$

$$X \mapsto G\text{-bundles over } X$$

is representable by a space called BG (Brown's representability theorem).

- Letting $I(G, X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism $I(G, X) \cong [X, BG]$. In general, identify $G = \text{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n , take $G = GL(n, \mathbb{R})$.

Note that for a poset of spaces (M_i, \hookrightarrow) , the space $M^\infty := \varinjlim M_i$. This are infinite dimensional “Hilbert manifolds”.

Proof:

$$I(\mathbb{R}^1, X) = [X, B(GL(1, \mathbb{R}))]$$

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Point 2

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Sphere 1

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