Title

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1.1 Strong Linkage

We have two categories:

- G_rT, with a notion of *strong linkage*, and
- G_r, which instead only has *linkage*.

We'll restate a few theorems.

Theorem 1.1.1 ***(?).***
Let $\lambda, \mu \in X(T)$.

1. If $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_rT} \neq 0$, then $\mu \uparrow \lambda$ are strongly linked.
2. If $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$, then $\mu \in W_p \cdot \lambda + p^r X(T)$.

Example 1.1.1(?): In the case of $\Phi = A_2$, we'll consider the two different categories.

We have the following picture for $\widehat{Z}$:

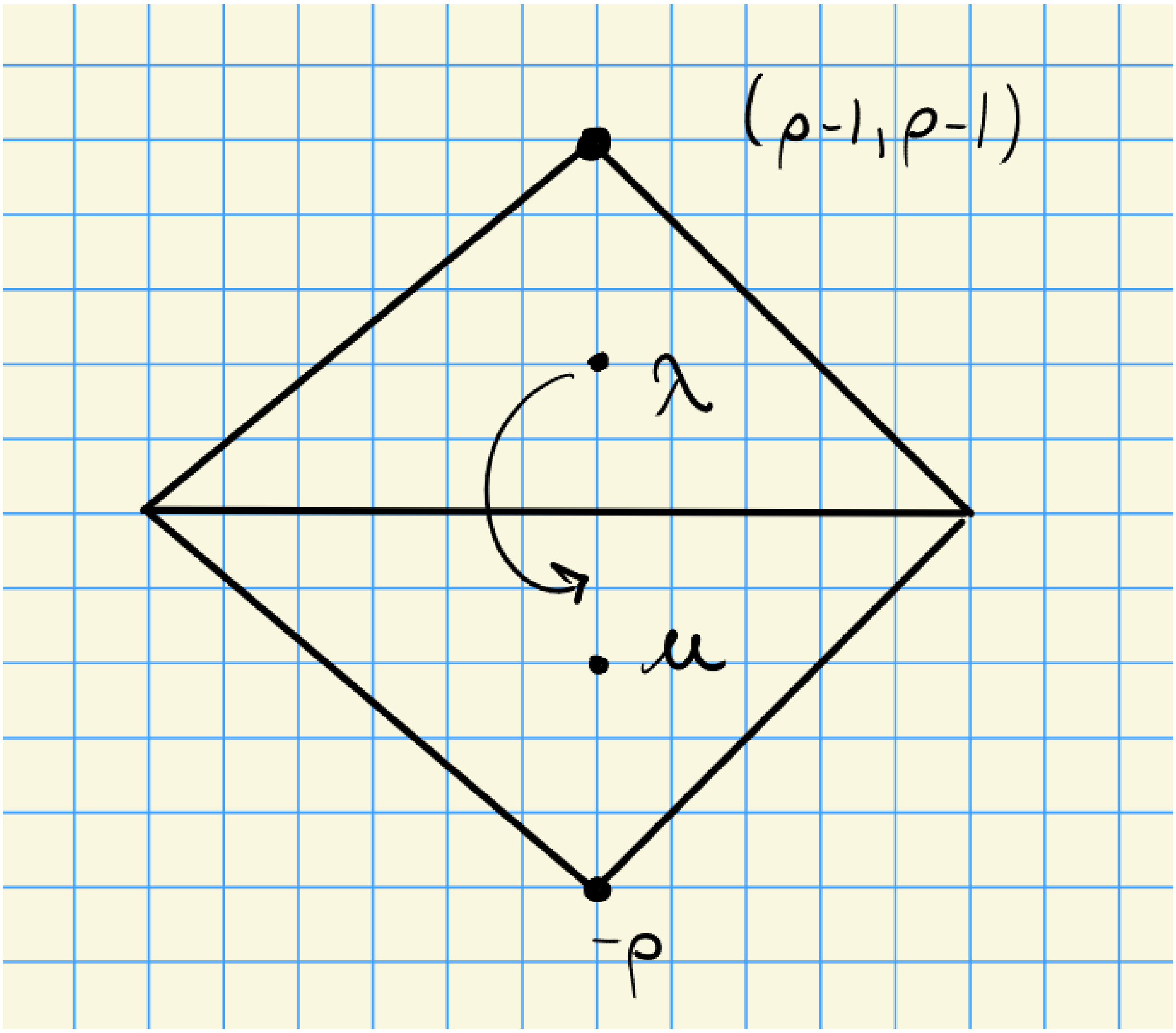

Figure 1: Image

Considering $X_1(T)$ and $[\widehat{Z}_1(\lambda) : \widehat{L}_1(\mu)] \neq 0$, and $\widehat{Z}_1(\lambda)$ has 6 composition factors as G_1T-modules.

On the other hand, for Z, we have the following:

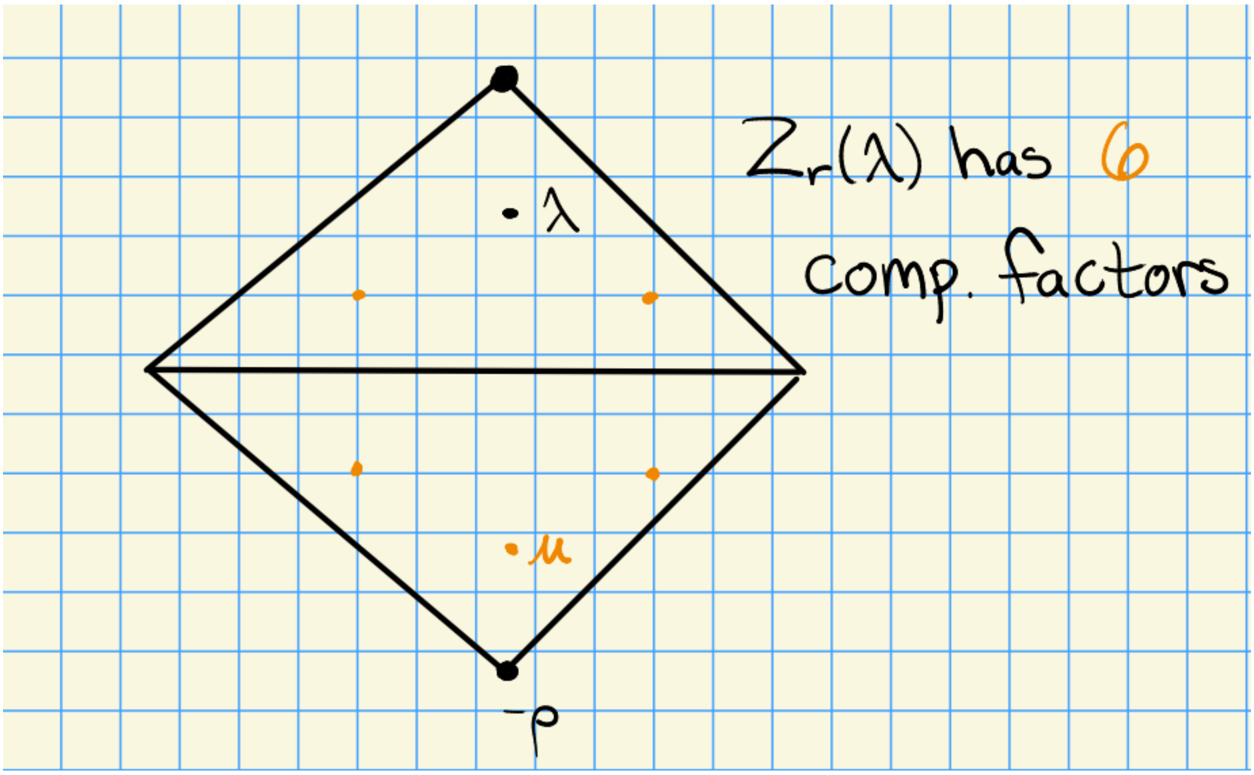

Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

1.2 Extensions

Let $\lambda, \mu \in X(T)$. We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\mathrm{Ext}^j_{G_rT}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) = \mathrm{Ext}^j_{G_r}\left(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)\right) \qquad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\mathrm{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right) = \bigoplus_{\gamma \in X(T)} \mathrm{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right)_\gamma$$

where we are taking the fixed points under the torus T action on the first factor (for which T_r acts trivially). We can write this as

$$\begin{aligned}
\cdots &= \bigoplus_{\gamma \in X(T)} \operatorname{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu) \otimes \gamma\right) \\
&= \bigoplus_{\gamma \in X(T)} \operatorname{Ext}^j_{G_r T}\left(L_r(\lambda), L_r(\mu) \otimes p^r v\right) \\
&= \bigoplus_{v \in X(T)} \operatorname{Ext}^j_{G_r T}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)\right).
\end{aligned}$$

So if we know extensions in the G_r category, we know them in the $G_r T$ category.

There is an isomorphism

$$\operatorname{Ext}^1_{G_r T}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) \cong \operatorname{Hom}_{G_R T}\left(\operatorname{rad}_{G_r T} \widehat{Z}_r(\lambda), \widehat{L}_r(\mu)\right).$$

Finally, for $\lambda, \mu \in X(T)$, if the above Ext^1 vanishes, then $\lambda \in W_p \cdot \mu$ (i.e. λ and μ are linked).