Lie Algebras

D. Zack Garza

November 4, 2019

Contents

1 Monday August 12

> The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

1.1.2 Chapter 3-4

We will describe the following series of correspondences:

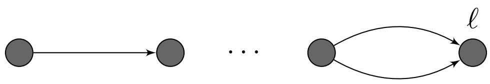
$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\oplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:

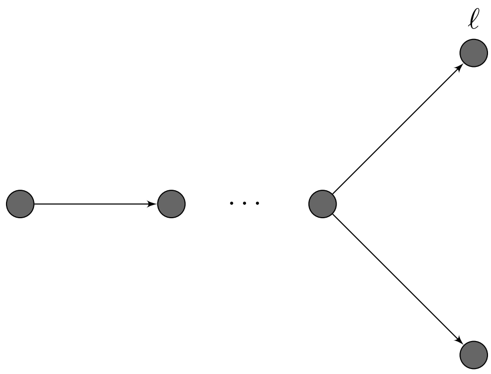
A_ℓ :

B_ℓ :

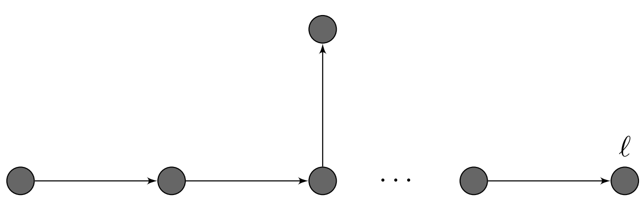


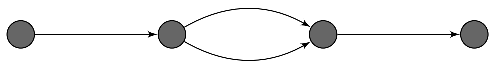
C_ℓ :

D_ℓ :



E_6, E_7, E_8 :



F_4: 

1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned}[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y].\end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear

- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Exercise 1. Show that $[x, y] = -[y, x]$.

Definition 2. Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\mathrm{End}(V) = \{f : V \to V \ni V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \ni \mathrm{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n, V)$.).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

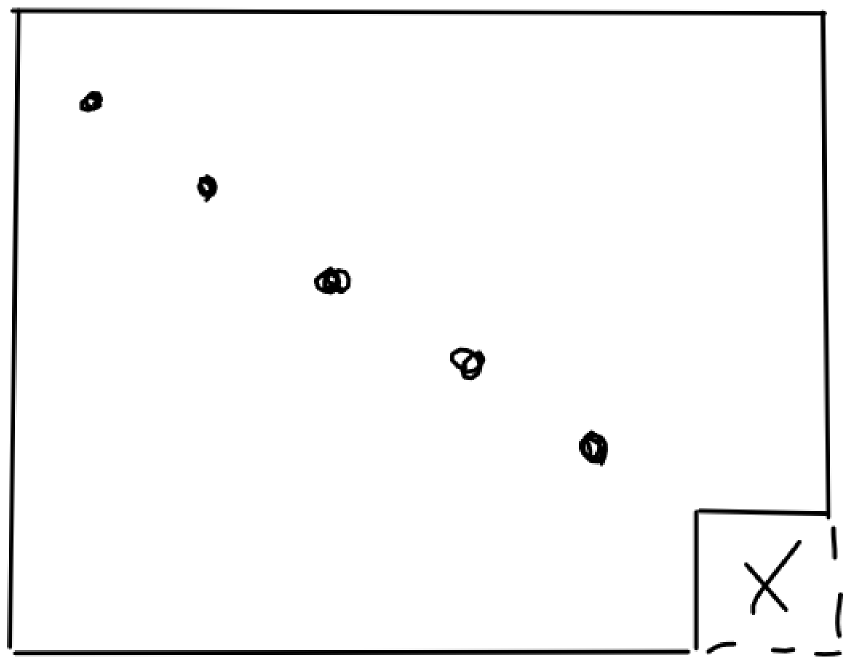
$$\mathfrak{sp}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where}$$
$$M = \begin{cases} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \end{array}\right) & n = 2\ell + 1 \text{ odd}, \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) & \text{else.} \end{cases}$$

Proposition 7. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n,\mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i,j position and



zero elsewhere.

- For type A_ℓ, we have $\dim \mathfrak{sl}(n,\mathbb{F}) = (\ell+1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n,\mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell,\mathbb{F}) = \dim\left\{\begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix}\right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type B_ℓ, we have $\dim \mathfrak{so}(2\ell,\mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array}\right).$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}$$

Theorem 8. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Wednesday August 14

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell+1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell+1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

Exercise 4. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 9. An F**-algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition 10. An algebra is **associative** if $x(yz) = (xy)z$.

> Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 11. Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 12. We define $\mathrm{Der}(A) = \{\delta \ni \delta \text{ is a derivation }\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\cdot, \cdot]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x, [y, z]] = 0$.

Exercise 5. Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 6 (Turn in)**.** Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x, y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 13. Any F-vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\cdot, \cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 14. Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise 7. Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\mathrm{ad}\,(e), \mathrm{ad}\,(h), \mathrm{ad}\,(g)$ with respect to this basis.

2.3 Ideals

Definition 15. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise 8. Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x,z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g},\mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n,F), \mathfrak{gl}(n,F)] = \mathfrak{sl}(n,F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I,J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 16. A Lie algebra is **simple** if $[\mathfrak{g},\mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Theorem 17. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2,F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2,F)$ given by $B = \{e,h,f\}$ where

- $[e,f] = h$,
- $[h,e] = 2e$,
- $[h,f] = -2f$.

So think of $[h,e] = \text{ad}\ _h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2,F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e,x] = 0 - 2be + ch$, and $[e,[e,x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2,F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$\begin{aligned} 2e = [h,e] \in I &\implies e \in I, \\ 2f = [h,-f] \in I &\implies f \in I. \end{aligned}$$

which implies that $I = \mathfrak{sl}(2,F)$ and thus it is simple.

$\square$

Note that there is a homework coming due next Monday, about 4 questions.

3 Friday August 16

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g},\mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x+I, y+I] = [x,y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x,y] + I = [x',y'] + I$.

3.1 Homomorphisms and Representations

Definition 18. A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x,y] = [\phi(x), \phi()]$.

Remark. $\ker \phi \trianglelefteq \mathfrak{g}_1$ and im $\phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I+J)/J \cong I/(I \bigcap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V.

We call V a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad } &: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \\ &x \mapsto [x, \cdot]. \end{aligned}$$

Corollary 19. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) = \Big\{ x \in \mathfrak{g} \ni \text{ad }_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \Big\} \\ = \ker \text{ad }_x. \end{aligned}$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad } \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

3.2 Automorphisms

Definition: An automorphism of $\mathfrak{g}$ is an isomorphism $\mathfrak{g}\circlearrowleft$, and we define

$$\text{Aut}(\mathfrak{g}) = \{\phi : \mathfrak{g}\circlearrowleft \ni \phi \text{ is an isomorphism } \}.$$

Proposition: If $\delta \in \text{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) := \sum \frac{\delta^n}{n!} \in \text{Aut}(\mathfrak{g}).$$

This is well-defined because δ is nilpotent, and a binomial formula holds:

$$\frac{\delta^{n([x,y])}}{n!} = \sum_{i=0}^{n} [\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!}].$$

and for $n = 1, \delta([x,y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x,y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\mathrm{ad}_{\ e}) \exp(\mathrm{ad}_{\ -f}) \exp(\mathrm{ad}_{\ e}) \in \mathrm{Aut}\mathfrak{g}.$$

where e, f are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of s is 2 and $W = \{1, s\}$.

4 Monday August 19

4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\cdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ is *solvable* if there is some n for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let $\mathfrak{g}$ be a Lie algebra, then

1. If $\mathfrak{g}$ is solvable, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also solvable.

2. If $I \trianglelefteq \mathfrak{g}$ and both I and $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\mathrm{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

4.2 Nilpotency

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\ &\cdots \\ \mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i]. \end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Claim: If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof: This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \cdots [x_n, y]]] = 0$, and so $\mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} \circ \cdots \mathrm{ad}_{x_n} = 0$ which implies that $\mathrm{ad}_x^n = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If ad_x is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while ad_x *is* nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V. If x is nilpotent as an endomorphism on V for all $x \in V$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases: - Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$. - Case $\dim g = 1$, left as an exercise.

Inductive step: Let A be a maximal proper subalgebra and define $\phi : A \to \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that ϕ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that im $\phi \leq \mathfrak{gl}(\mathfrak{g}/a)$ is a Lie subalgebra, and $\dim \mathrm{im}\ \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \mathrm{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\mathrm{im}\ \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that A is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2[y, y]$. The last term is zero, the middle two terms are in A, and because A is closed under the bracket, the first term is in A as well.

But then $A \oplus F_y$ is a larger subalgebra than A, which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in $A, A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in A.

Proof to be continued on Friday!

5 Wednesday August 21

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of V, constructing im $\phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus Fy$. Now consider

$$W = \{v \in V \ni Av = 0\},$$

which is $\mathfrak{g}$-invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But V is killed by any element in A, and both of these terms are in A. In particular, the y appearing in Fy also satisfies $y \in W$. Consider $y|_W \in \mathrm{End}(w)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \mathrm{End}(V)$, which is true exactly because y is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \mathrm{End}(\mathfrak{g}/A)$ is nilpotent. Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by a, $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since m_ℓ, m_r commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly $\mathrm{ad}\ _a$. But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if $\mathrm{ad}\ _x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to $\mathrm{ad}\ g \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that $\mathrm{ad}\ \mathfrak{g}v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension.

Since ad $_x$ is nilpotent, so is ad $_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space. $\square$

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of F, then the matrices of $\mathfrak{g}$ with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on n, where the base case is easy. For the inductive step, we use the previous theorem to get a v_1 such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\overline{V} = F^n/Fv_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \to \mathfrak{gl}(\overline{V})$ where $x \mapsto (\overline{y} \mapsto \overline{y(x)})$.

Then im $\phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \mathrm{End}(F^{n-1})$ is nilpotent, since x was nilpotent on the larger space. But (see notes) then x can be written as a strictly upper-triangular matrix.

5.1 Chapter 2: Semisimple Lie Algebras

We now assume char $F = 0$ and $\overline{F} = F$.

Theorem: If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V, then V contains a common eigenvector for a $x \in \mathfrak{g}$, i.e. a $\lambda : \mathfrak{g} \to F, x \mapsto \lambda(x)$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$.

Proof: We will use induction on the dimension of $\mathfrak{g}$. For the inductive step:

Claim 1: There is an ideal $A \trianglelefteq \mathfrak{g}$ such that $\mathfrak{g} = A \oplus Fy$ for some $y \neq 0$, so A is a subalgebra of a solvable Lie algebra $\mathfrak{g}$ and thus solvable itself. By hypothesis, we can produce a $w \in V \setminus \{0\}$, and thus a functional $\lambda : A \to F$ such that $aw = \lambda(a)w$ for all $a \in A$. So we define

$$V_\lambda = \{v \in V \ni av = \lambda(a)v \forall a \in A\}$$

where $w \in V_\lambda$.

Claim 2: $y(V_\lambda) \subseteq V_\lambda$, or $y|_{V_\lambda} \in \mathrm{End}(V_\lambda)$.

Thus $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$ is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a $v \in V_\lambda$ and some $\mu \in F$ such that $y(v) = \mu v$. An arbitrary element $x \in \mathfrak{g}$ can be written as $x = a + cy$ for some $a \in A, c \in F$ and it acts by $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c)v \in V_\lambda$.

6 Friday August 23

Chapter 3: Theorems of Lie and Cartan

6.1 4.1: Lie's Theorem

Theorem: Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is finite-dimensional. If $V \neq 0$, then V contains a common eigenvector for all of the endomorphisms in L.

Proof: Use induction on $\dim L$. The case $\dim L = 0$ is trivial. We'll attempt to mimic the proof of Theorem 3.3. The idea is to

1. Locate an ideal of K of codimension 1,

2. Show by induction that common eigenvectors exist for k,
3. Verify that L stabilizes a space consisting of such eigenvectors,
4. Find in that space an eigenvector for a single $z \in L$ satisfying $L = K + Fz$.

Step (1): Since L is solvable and of positive dimension, then $L \lneq [L, L]$. Otherwise, if $L = [L, L]$, then $L^{(1)} = L \implies L^{(n)} = L$, which would contradict L being solvable.

Since $[L, L]$ is abelian, any subspace is automatically an ideal. So take a subspace of codimension one, then its inverse image $K \trianglelefteq L$ is an ideal satisfying $[L, L] \subseteq K$.

Step (2): Use induction to find a common eigenvector $v \in V$ for K. (K is solvable; if $K = 0$ then L is abelian of dimension 1 and any eigenvector for a basis vector of L finishes the proof.)

This means that $x \in K \implies x \curvearrowright v = \lambda(x)v$ for some $\lambda : K \to F$ a linear functional. Fix this λ, and let $W = \{w \in V \ni x \curvearrowright w = \lambda(x)w \forall x \in K\}$; note that $W \neq 0$.

Step (3): This will involve showing that L leaves W invariant. Assume for the moment that this is done, and proceed to step (4).

Step (4):

Write $L = K + Fz$. Since F is algebraically closed, we can find an eigenvector $v_0 \in W$ of z for some eigenvalue of z. Then v_0 is a common eigenvector for L, and λ can be extended to a linear function on L satisfying $x \curvearrowright v_0 = \lambda(x)v_0$ where $x \in L$.

It remains to show that L stabilizes W. Let $w \in W, x \in L$. To test whether or not $x \curvearrowright w \in W$, we take an arbitrary $y \in K$ and examine

$$yx \curvearrowright w = xy \curvearrowright w - [x, y] \curvearrowright w = \lambda(y)x \curvearrowright w - \lambda([x, y])w.$$

> Note: the above equality is an important trick.

Thus we need to show that $\lambda([x, y]) = 0$. To this end, fix $w \in W, x \in L$. Let $n > 0$ be the smallest integer for which $w, x \curvearrowright w, \cdots x^n \curvearrowright w$ are all linearly *independent*. Let $W_i = \operatorname{span}(\{w, x \curvearrowright w, \cdots x^{i-1} \curvearrowright w\})$ and set $W_0 = 0$. Then $\dim W_n = n$, and $W_{n+i} = W_n$ for all $i \geq 0$. Moreover, x maps W_n into itself. It is easy to check that each $y \in K$ is represented by an upper-triangular matrix with diagonal entries equal to $\lambda(y)$. This follows immediately from the congruence

$$yx^i \curvearrowright w = \lambda(y)x^i \curvearrowright w \mod W_i,$$

which can by proved by induction on i. The case $i = 0$ is trivial. For the inductive step, write

$$yx^i \curvearrowright i = yxx^{i-1} \curvearrowright w = xyx^{i-1} \curvearrowright w = [x, y]x^{i-1} \curvearrowright w$$

By induction,

$$yx^{i-1} = \lambda(y)x^{i-1} \curvearrowright w + w',$$

where $w' \in W_{i-1}$. Since x maps W_{i-1} into W_i by construction, the congruence holds for all i.

According to our description of the action of $y \in K$ on W_n, we have $\operatorname{Tr}_{W_n}(y) = n\lambda(y)$. In particular, this is true for elements k of f of the special form $[x, y]$ where x is as it was above and y is in K.

But both x and y stabilize W_n, so $[x, y]$ acts on W_n as the commutator of two endomorphisms of W_n, and the trace is therefore zero.

We conclude that $n\lambda([x, y]) = 0$. Since $\text{char}F = 0$, this forces $\lambda([x, y]) = 0$ as required. $\square$

Corollary A (Lie's Theorem): Let $L \leq \mathfrak{gl}(V)$ be a solvable subalgebra where $\dim V = n < \infty$. Then L stabilizes some flag in V, i.e. the matrices of L relative to a suitable basis of V are upper triangular.

Proof: Use the above theorem, along with induction on $\dim V$. This is similar to the proof of corollary 3.3.

6.2 4.2: Jordan-Chevalley Decomposition

Fact 1:

The Jordan Canonical Form of a single endomorphism x over F algebraically closed is an expression of x in matrix form as a sum of blocks: $\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & 1 \\ 0 & & & a \end{pmatrix}$

Fact 2:

Call $x \in \text{End}V$ *semisimple* if the roots of its minimal polynomial over F are all distinct. Equivalently, if F is algebraically closed, then x is semisimple iff x is diagonalizable.

Fact 3:

Two commuting semisimple endomorphisms can be simultaneously diagonalized. Therefore, their sum or difference is again semisimple.

Proposition: Let V be a finite dimensional vector space over F and $x \in \text{End}V$. Then

a. There exist unique $x_s, x_n \in \text{End}V$ satisfying the conditions $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and x_s, x_n commute.

b. There exists polynomials $p(t), g(t)$ such that $x_s = p(x)$ and $x_n = g(x)$. In particular, x_s, x_n commute with any endomorphism commuting with x.

c. If $A < B < V$ are subspaces and x maps B into A, then x_s, x_n also map B into A.

The decomposition $x = x_s + x_n$ is called the (additive) **Jordan-Chevalley decomposition** of x, or just the Jordan decomposition. x_s, x_n are respectively called the **semisimple part** and the **nilpotent part** of x.

Example:

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies x_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $x_s x_n = x_n = x_n x_s$, $x_s = 2x - x^2$, and $x_n = x^2 - x$. We thus have $p(t) = 2t - t^2$ and $q(t) = t^2 - t$.

7 Monday August 26

Definition (Jordan Decomposition)

Let $X \in \mathrm{End}(V)$ for V finite dimensional. Then,

(a) There exists a unique $X_s, X_n \in \mathrm{End}(V)$ such that $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.

(b) There exists a $p(t), q(t) \in t\mathbb{F}[t]$ such that $X_s = p(X), X_n = q(X)$.

(Polynomials with no constant term.)

Proof of (a): Assume $X_s = X_s + X_n = X_s' + X_n'$, so both have bracket zero. Assuming that (b) holds, we have $X_s = p(X)$, and so

$$[X, X_s] = [X_s + X_n', X_s'] = [X_s', X_s'] + [X_s', X_n'] = 0 \implies [p(X), X_S'] = 0 = [X_s, X_s']$$

Using fact (c) from last time, then X_s, X_s' can be diagonalized simultaneously, and so $X_s - X_s'$ is semisimple.

On the other hand, if X_n', X_n are nilpotent, and since these commute, $X_n - X_n'$ is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X_s') + (X_n + X_n')$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$ be the minimal polynomial of X, where each $m_i \geq 1$ and the λ_i are distinct. Then the primary composition of V is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V) \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomail $p \in F[t]$ such that

$$\begin{aligned} p &= \lambda \mod (t - \lambda_i)^{m_i} \quad \forall i, \\ p &= 0 \mod t. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is $p(x) \curvearrowright V_i$? This acts by scalar multiplication by λ_i for all i. (Check). Because of the restrictive conditions, $p(x)$ has no constant term.

So $p(X) = X_s$ is the semisimple part we want. Now just set $q(t) = t - p(t)$, then $X_n := q(X) = X - X_s$ is nilpotent.

$$p(X) \sim \begin{pmatrix} \boxed{\lambda_1 I_{V_1}} & & & 0 \\ & \boxed{\lambda_2 I_{V_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_r I_{V_r}} \end{pmatrix}$$

Figure 1: ???

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have $X = X_s + X_n$, then ad $_X \in \text{End}(\text{End}(V))$. It can be shown that $(\text{ad }_X)_s + (\text{ad }_X)_n =$ ad $(X_s) +$ ad (X_n).

Let e_{ii} be the elementary matrix with a 1 in the i, j position. You can write ad $_X$ as a 4×4 matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X \quad = X_s \quad + \quad X_n$$

$$= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)$$

$$
= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array}
\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow[\text{JNF}]{} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)
$$

You can check that $(\text{ad }_X)_S = 0, \text{ad }(X_s) = 0$, and $(\text{ad } X)_n$ is the Jordan form given above.

Lemma:

(a) $x \in \text{End}(V) \implies \text{ad }(x)_s = \text{ad }(x_s)$ and $\text{ad }(x)_n = \text{ad }(x_n)$.

(b) If A is a finite dimensional $\mathbb{F}$-algebra, then $\delta \in \text{Der}(A) \implies \delta_s, \delta_n \in \text{Der}(A)$ as well.

Proof of (a):

Check that $\text{ad }(x) = \text{ad }(x_s) + \text{ad }(x_n)$. Then for $y \in \text{End}(V)$, we have

$$
\begin{aligned}
(\text{ad }(x))(y) &= [x, y] \\
&= [x_s + x_n, y] \\
&= [x_s, y] + [x_n, y] \\
&- (\text{ad }(x_s))(y) + (\text{ad }(x_n))(y).
\end{aligned}
$$

Using theorem 3.3, x_n nilpotent $\implies$ ad (x_n) is also nilpotent. So write $x_s = \sum \lambda_i e_{ii}$ with the eigenvalues on the diagonal. Then ad $x_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$ for all i, j. But then ad x_s is given

$$(\delta - (\lambda+\mu)I)^n([x,y])$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left[(\delta-\lambda I)^i(x), (\delta-\mu I)^{n-i}(y) \right]$$

Figure 2: Image

by a matrix with $\lambda_i - \lambda_j$ in the i, j position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

Proof of (b):

Since $\delta \in \mathrm{Der}(A)$, the primary decomposition with respect to δ is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \left\{ a \in A \ni (\delta - \lambda I)^k a = 0 \text{ for some } k >> 0 \right\}.$$

So $\delta_s \curvearrowright A_\lambda$ by scalar multiplication (by λ). Then for $\lambda, \mu \in \mathbb{F}$, we have

So $[A_x, A_y] \subseteq A_{\lambda+\mu}$ for all $x, y \in A$. But then

and so $\delta_s \in \mathrm{Der}(A)$, and $\delta_n = \delta - \delta_s \in \mathrm{Der}(A)$ as well.

8 Wednesday August 28

Todo

9 Friday August 30

Review of bilinear forms: let $V = \mathbb{F}^n$.

Definition: A bilinear form $\beta : V^2 \to \mathbb{F}$ can be represented by a matrix B with respect to a basis $\{\mathbf{v}_i\}$ such that

$$\beta\beta(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i) = (a_1 \ a_2 \ \cdots) B (b_1 \ b_2 \ \cdots)$$

- β is *symmetric* iff $\beta(a,b) = \beta(b,a)$.
- β is *symplectic* iff $\beta(a,b) = -\beta(b,a)$.
- β is *isotropic* iff $\beta(a,a) = 0$.

$$\delta_s([x,y])$$

$$\|$$

$$(\lambda+\mu)[x,y] = [\lambda x, y] + [x, \mu y]$$

$$\|$$

$$[\delta_s(x), y] + [x, \delta_s(y)]$$

Figure 3: Image

For a subspace $U \leq V$, define

$$U^\perp := \{\mathbf{v} \in V \ni \beta(\mathbf{u}, \mathbf{v}) = \mathbf{0} \ \forall u \in U\}.$$

> Note: in general, left/right orthogonality are distinguished, but these will be identical when β is symmetric/symplectic.

The form β is said to be *non-degenerate* iff $V^\perp = 0$ iff $\det B \neq 0$.

Assume F is an algebraically closed field, so $\overline{F} = F$, and $\mathrm{char} F \neq 2$, then

- If β is non-degenerate and symmetric, then $B \sim I_n$
- If β is non-degenerate and symplectic, then $B \sim [0, I_{n/2}; I_{n/2}, 0]$.

Remark:

$\mathfrak{so}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, F) \ni \beta(x(u), v) = -\beta(u, x(v))\}$, where B has the matrix $[0, I; I, 0]$ if n is odd, or this matrix with a 1 in the top-left corner if n is odd.

Similarly, $\mathfrak{sp}(2m, \mathbb{F})$ can be described this way with the matrix $[0, -I_m; -I_m, 0]$.

Overview: The killing form is defined as $\kappa : \mathfrak{g}^2 \to \mathbb{F}$ where $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$.

Then we have **Cartan's Criteria**:

- $\mathfrak{g}$ solvable $\iff \kappa(x, y) = 0 \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$.
- $\mathfrak{g}$ semisimple $\iff \kappa$ is non-degenerate.

Note that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \bigoplus_i I_i$ with each $I_i \trianglelefteq \mathfrak{g}$ and simple.

9.1 Cartan's Criteria

Some facts:

1. κ is symmetric
2. If $\mathfrak{g}$ is finite dimensional, then κ is associative, i.e $\kappa([x,y],z) = \kappa(x,[y,z])$.

Exercise: Show that if $I \trianglelefteq \mathfrak{g}$, then $I^\perp \leq \mathfrak{g}$ is an ideal.

Proof of (2): In section 4.3, it was shown that $\mathrm{tr}([a,b] \circ c) = \mathrm{tr}(a \circ [b,c])$ for all $a,b,c \in \mathrm{End}(V)$ (provided V is finite dimensional).

So

$$\begin{aligned}
\kappa([x,y],z) &= \mathrm{tr}(\mathrm{ad}_{\,[x,y]} \circ \mathrm{ad}_{\,z}) \\
&= \mathrm{tr}([\mathrm{ad}_{\,x}, \mathrm{ad}_{\,y}] \circ \mathrm{ad}_{\,z}) \\
&= \mathrm{tr}(\mathrm{ad}_{\,x} \circ [\mathrm{ad}_{\,y}, \mathrm{ad}_{\,z}]) \\
&= \mathrm{tr}(\mathrm{ad}_{\,x} \circ \mathrm{ad}_{\,[y,z]}) \\
&= \mathrm{tr}(x,[y,z])..
\end{aligned}$$

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Proof: $\implies$: We want to show that $\mathfrak{g}^\perp = 0$. Note that $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}$, and so for all $x \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$ and for any $y \in \mathfrak{g}^\perp$, we have

$$\kappa(x,y) = \mathrm{tr}(\mathrm{ad}_{\,x} \circ \mathrm{ad}_{\,y}) = 0$$

by the const(?) of $\mathfrak{g}^\perp$. This implies $\mathfrak{g}^\perp$ is solvable.

Using fact (2), we have $\mathfrak{g}^\perp \trianglelefteq \mathfrak{g}$ and thus $\mathfrak{g}^\perp \subseteq \mathrm{rad}(\mathfrak{g})$, which is 0 since because $\mathfrak{g}$ is semisimple. So either $\mathfrak{g}^\perp = 0$ or κ is nondegenerate.

Used the fact that the radical was a maximal solvable ideal.

$\impliedby$: We want to show that for all $I \trianglelefteq \mathfrak{g}$ where $[I,I] = 0$, we have $I^\perp \subseteq \mathfrak{g}^\perp$.

For $x \in I, y \in \mathfrak{g}$, we have

$$(\mathrm{ad}_{\,x} \circ \mathrm{ad}_{\,y})^2 = \mathfrak{g} \xrightarrow{\mathrm{ad}_{\,y}} \mathfrak{g} \xrightarrow{\mathrm{ad}_{\,x}} I \xrightarrow{\mathrm{ad}_{\,y}} I \xrightarrow{\mathrm{ad}_{\,x}} 0$$

And thus $\mathrm{tr}(\mathrm{ad}_{\,x} \circ \mathrm{ad}_{\,y}) = 0$ and $I \subseteq \mathfrak{g}^\perp$.

Suppose that $\mathfrak{g}$ is *not* semisimple. Then there exists a solvable ideal $J \neq 0$ such that the last term J^i in the derived series is an ideal $I \trianglelefteq \mathfrak{g}$ such that $[I,I] = 0$, forcing $J^i \subset \mathfrak{g}^\perp = 0$, which is a contradiction.

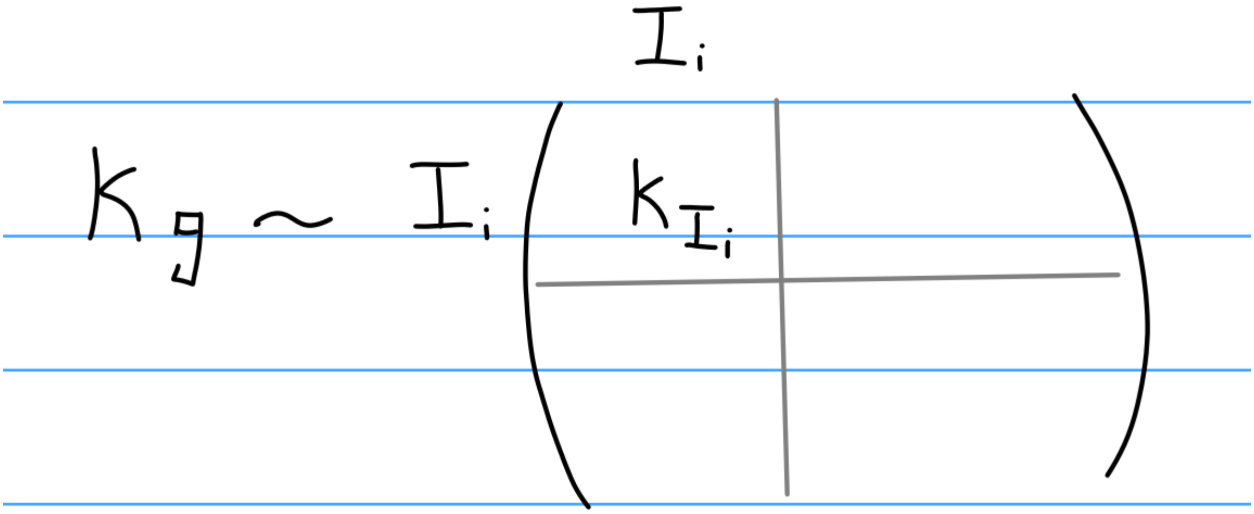


Figure 4: Image

9.2 Section 5.2

Theorem: If $\mathfrak{g}$ is semisimple, then

a. There exist ideals $I_i \trianglelefteq \mathfrak{g}$ which are simple Lie algebras satisfying $\mathfrak{g} = \bigoplus I_i$. Note that $[I_i, I_j] \subseteq I_i \bigcap I_j = 0$, since direct summands intersect only trivially.

b. Every simple $I \trianglelefteq \mathfrak{g}$ is one of these I_i.

c. $\kappa_{I_i} = \kappa_{\mathfrak{g}}\big|_{I_i \times I_i}$, so

Remark: $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g} = \bigoplus_i I_i$ for some simple Lie algebras I_i.

$\Longleftarrow$: For all $i, S \coloneqq \mathrm{rad}\mathfrak{g}, I_i \trianglelefteq I_i$ is a solvable ideal. This implies that it is 0, since I_i is simple.

By definition, simple Lie algebras are not abelian.

Supposing that $S = I_i$, we would then have $[S.S] \neq 0$ since $[I_i, I_i] \neq 0$ by definition. But $[S, S] \neq S$ because S is solvable, which says that S is not simple (a contradiction).

Note that $[\mathrm{rad}\mathfrak{g}, \mathfrak{g}] \subseteq \bigoplus [\mathrm{rad}\mathfrak{g}, I_i] = 0$, which forces $\mathrm{rad}\mathfrak{g} \subseteq Z(\mathfrak{g})$. Since I_i is simple, $Z(I_i) = 0$ for all i. But $Z(\mathfrak{g}) = \bigoplus Z(I_i) = 0$, and this forces $\mathrm{rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g}) \implies \mathrm{rad}\mathfrak{g} = 0$. So $\mathfrak{g}$ is semisimple.

Next time – starting the representation theory with $\mathfrak{sl}(2, \mathbb{F})$.

10 Monday September 2

Recall the killing form:

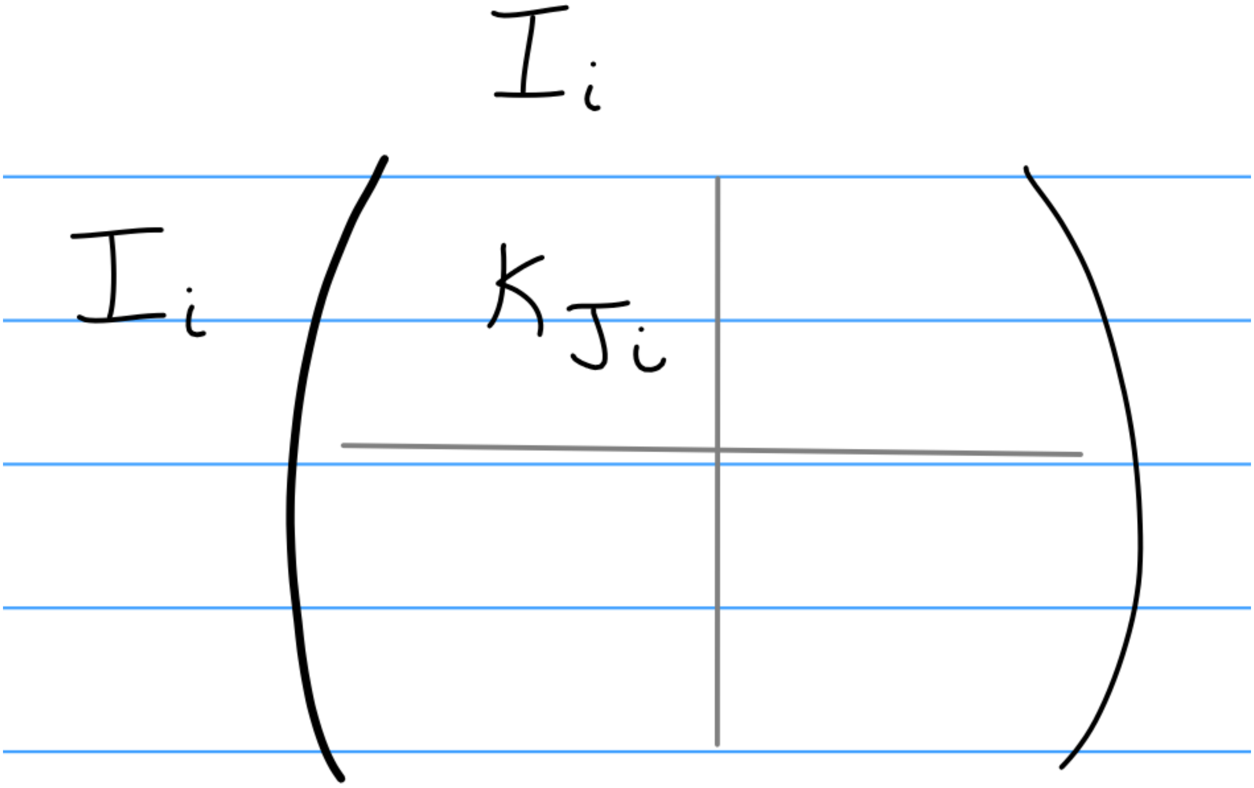


Figure 5: Image

$$\kappa : lieg^2 \to \mathbb{F}$$
$$(x, y) \mapsto \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y).$$

and Cartan's criteria:

1. $\mathfrak{g}$ is solvable $\iff \kappa(x, y) = 0\ \forall x \in \mathfrak{g}, \mathfrak{g}],\ y \in \mathfrak{g}$.
2. $\mathfrak{g}$ is semisimple $\iff \kappa$ is non-degenerate.

Theorem: If $\mathfrak{g}$ is semisimple, then

a. $\mathfrak{g} = \bigoplus_{i=1}^n I_i$ for some $I_i \trianglelefteq \mathfrak{g}$ which are all simple.
b. Every simple ideal $I \trianglelefteq \mathfrak{g}$ is one of the I_i.
c. $\kappa_{I_i} = \kappa_{\mathfrak{g}} \mid_{I_i \times I_i}$.

Proof of (a): Use induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ has no nonzero proper ideals, then $\mathfrak{g}$ is simple and we're done.

Otherwise, let I_1 be a minimal nonzero ideal of $\mathfrak{g}$. Then $I_1^\perp \trianglelefteq \mathfrak{g}$ is also an ideal, and thus $I := I_1 \bigcap I_1^\perp \trianglelefteq \mathfrak{g}$ is as well. Then for all $x \in [I, I]$, we must have $\kappa(x, y) = 0$ for any $y \in I \subseteq I_1^\perp$. So I is solvable, and thus $I = 0$. So $\mathfrak{g} = I_1 \oplus I_1^\perp$.

$$\operatorname{ad} x \sim \left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right) \qquad \kappa_{\mathfrak{g}}(x,y) = \operatorname{tr}\left(\left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right) \right)$$

$$\operatorname{ad} y \sim \left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right) \qquad = \operatorname{tr}\left(\begin{array}{c|c} A_x A_y & B_x B_y \\ \hline 0 & 0 \end{array}\right)$$

$$= \operatorname{tr}(A_x A_y)$$

$$= \kappa_{I_i}(x,y)$$

Figure 6: Image

Note that any ideal of $I_1^\perp$ is also an ideal of $\mathfrak{g}$, which implies that $\operatorname{rad}(I_1^\perp) \subseteq \operatorname{rad}(\mathfrak{g})$, which is zero since $\mathfrak{g}$ is semisimple, and thus $I_1^\perp$ is semisimple as well.

By the inductive hypothesis, $I_1^\perp = I_2 \oplus \cdots \oplus I_n$ where each $I_j \trianglelefteq I_i^\perp$ is simple. Then $I_j \trianglelefteq \mathfrak{g} \implies [I_1, I_j] \subset I_1 \bigcap I_j$, since I_1 has no contribution. But this is a subset of $I_1 \bigcap I_1^\perp = 0$. $\square$

Proof of (b): If $I \trianglelefteq \mathfrak{g}$, then $[I, \mathfrak{g}] \trianglelefteq I$ because $[[I, \mathfrak{g}], I] \subseteq [I, I] \subseteq [I, \mathfrak{g}]$.

Since $\mathfrak{g}$ is semisimple, $0 = \operatorname{rad}(\mathfrak{g}) \supseteq Z(\mathfrak{g})$. So $[I, \mathfrak{g}] \neq 0$, and thus $[I, \mathfrak{g}] = I$ since I is simple. But then $[I, \mathfrak{g}] = \bigoplus [I, I_i]$ is simple as well. So only one direct summand can survive, since otherwise this would produce at least 2 nontrivial ideals, and $[I, \mathfrak{g}] = [I, I_i]$ for some i.

So for all $j \neq i$, we must have $I_j \bigcap I = I_j \bigcap [I, I_i] = 0$, and so $I \subseteq I_i$. But then $I = I_i$ since I_i itself is simple, and we're done.

Proof of (c):

(Without using the simplicity of I_i)

For $x, y \in I_i$, we have

10.1 Inner Derivations

Recall that $\operatorname{ad} \mathfrak{g} \subseteq \operatorname{Der} \mathfrak{g}$, and in fact (lemma) this is an ideal.

Theorem: If $\mathfrak{g}$ is semisimple, then $\operatorname{ad} \mathfrak{g} = \operatorname{Der} \mathfrak{g}$.

Proof of lemma:

For all $\delta \in \mathrm{Der}\mathfrak{g}$ and all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}[\delta, \mathrm{ad}_{\,x}](y) &= \delta([x,y]) - [x, \delta(y)] \\ &= [\delta(x), y] \\ &= [\mathrm{ad}_{\,\delta(x)}](y),\end{aligned}$$

and so $[\delta, \mathrm{ad}\ x] \subseteq \mathrm{ad}\ \mathfrak{g}$. $\square$

Proof of theorem:

If $\mathfrak{g}$ is semisimple, then $0 = \mathrm{rad}\mathfrak{g} \supseteq Z(\mathfrak{g}) = \ker \mathrm{ad}$. Thus $\mathrm{ad}\ \mathfrak{g} \cong g/\ker \mathrm{ad}\ \cong \mathfrak{g}$ is also semisimple.

This means that $\kappa_{\mathrm{ad}\ \mathfrak{g}}$ is non-degenerate, and thus $\mathrm{ad}\ \mathfrak{g} \bigcap (\mathrm{ad}\ \mathfrak{g})^\perp = 0$, where $(\mathrm{ad}\ \mathfrak{g})^\perp \trianglelefteq \mathrm{Der}(\mathfrak{g})$.

(Note that the non-degeneracy of κ already forces $(\mathrm{ad}\ \mathfrak{g})^\perp = 0$.)

Then $[(\mathrm{ad}\ \mathfrak{g})^\perp, \mathrm{ad}\ \mathfrak{g}] = 0$, and so for all $\delta \in (\mathrm{ad}\ \mathfrak{g})^\perp$, we have $\delta(x) = [\delta, \mathrm{ad}\ x]$ by the lemma, but we've shown that this is zero.

But then δ must be zero because ad is an isomorphism, and in particular it is injective. This means that $(\mathrm{ad}\ \mathfrak{g})^\perp = 0$, and thus $\mathrm{ad}\ \mathfrak{g} = \mathfrak{g}$. $\square$

We can use this to define an abstract Jordan decomposition by pulling back decompositions on adjoints:

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11.1 4.3: Cartan's Criterion

Lemma: Let $A \subset B$ be two subspaces of $\mathfrak{gl}(V)$ where $\dim V < \infty$. Set $M = \{x \in \mathfrak{gl}(V) \ni [x, B] \subset A\}$. Suppose that $x \in M$ satisfies $\mathrm{Tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

Proof: Let $x = s + n$ (where $s = x_s$ and $n = x_n$) be be the Jordan decomposition of x. Fix a basis $v_1 \cdots v_m$ of V relative to which s has matrix $\mathrm{diag}(a_1 \cdots a_m)$. Let E be the vector subspace of F over the prime field Q spanned by the eigenvalues $a_1 \cdots a_m$. We have to show that $s = 0$, or equivalently that $E = 0$, since E has finite Q-dimension by construction. It will suffice to show that the dual space $E\vee$ is 0, i.e. that any linear functional $f : E \to Q$ is zero.

Given f, let y be the element of $\mathfrak{gl}(V)$ whose matrix is given by $\mathrm{diag}(\mathrm{f(a_1)}, \cdots \mathrm{f(a_m)})$. If $\{e_{ij}\}$ is a basis of $\mathfrak{gl}(V)$, then $\mathrm{ad}_{\,s}(e_{ij}) = (a_i - a_j)e_{ij}$ and $\mathrm{ad}_{\,y}(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$.

Now let $r(t) \in F[t]$ be a polynomial with no constant term, satisfying $r(a_i - a_j) = f(a_i) - f(a_j)$ for all pairs i, j. The existence of such $r(t)$ follows from Lagrange interpolation, and the fact that if $a_i = a_j$ then $0 = r(a_j) - r(a_i) = r(a_i - a_j) = r(0)$, so r has no constant term. Thus there is no ambiguity in the assigned values, since $a_i - a_j = a_j - a_l$ would imply (by linearity of f) that $f(a_i) - f(a_j) = f(a_k) - f(a_l)$. Thus $\mathrm{ad}_{\,y} = r(\mathrm{ad}_{\,s})$.

> Note that Lagrange Interpolation is a special case of the Chinese Remainder Theorem for polynomials. If all x_is are distinct, then $p_i(x) = x - x_i$ are all pairwise coprime. Then dividing $\frac{p(x)}{p_i(x)} = p(x_i)$. So letting $A_1 \cdots A_k$ be constants in k, there is a unique polynomial of degree less than k such that $p(x_i) = A_i$. Thus there is a polynomial $p(x)$ such that $p(x) = A_i \mod p_i(x)$, and $p(x_i) = A_i$.

$\mathfrak{g} \subseteq \mathrm{End}(V)$

$$\begin{array}{ccc} x & \xrightarrow{\quad\mathrm{ad}\quad} & \mathrm{ad}\,x \\ \Big\| \ \mathrm{JD} & & \Big\| \ \mathrm{JD} \\ x_s & \longmapsto & \mathrm{ad}\,x_s = (\mathrm{ad}\,x)_s \\ + & & \\ x_n & \longmapsto & \mathrm{ad}\,x_n = (\mathrm{ad}\,x)_n \end{array}$$

$\longleftarrow$

Can recover some x_s and x_n from the adjoints

Figure 7: Image

Now ad_s is the semisimple part of ad_x. By lemma A of 4.2, ad_s can be written as a polynomial in ad_x without a constant term. Therefore ad_y is also a polynomial in ad_x without constant term. By hypothesis, ad_x maps B into A, so we have $\text{ad}_y(B) \subset A$, and so $y \in M$. Using the hypothesis of the lemma, $\text{Tr}(xy) = 0$, and so $\sum a_i f(a_i) = 0$. The left side is a Q-linear combination of elements of E. Applying f, we obtain $\sum f(a_i)^2 = 0$. But the numbers $f(a_i)$ are rational, so this forces all of them to be zero. Finally, f must be identically 0 because the a_i span E. $\square$

> Note that $\text{Tr}([x,y]z) = \text{Tr}(x[y,z])$. To verify this, write $[x,y]z = xyz - yxz$ and $x[y,z] = xyz - xzy$, then use the fact that $\text{Tr}(y(xz)) = \text{Tr}((xz)y)$.

Theorem (Cartan's Criterion): Let $L \leq \mathfrak{gl}(V)$ be a subalgebra with V finite dimensional. Suppose $\text{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. Then L is solvable.

Proof: It suffices to show that $[L,L]$ is nilpotent, or just that all $x \in [L,L]$ are nilpotent endomorphisms. We apply the above lemma, with V as given, $A = [L,L]$, and $B = L$, so $M = \{x \in \mathfrak{gl}(V) \ni [x,L] \subset [L,L]\}$. We have $L \subset M$. Our hypothesis is that $\text{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. To use the lemma to reach the desired conclusion, we need a stronger result: that $\text{Tr}(xy) = 0$ for $x \in [L,L]$ and $y \in M$.

If $[x,y]$ is a generator of $[L,L]$ and $z \in M$, then $\text{Tr}([x,y]z) = \text{Tr}(x[y,z]) = \text{Tr}([y,z]x)$. By definition of M, $[y,z] \in [L,L]$, so the right side is 0 by hypothesis.

Corollary: Let L be a Lie algebraic such that $\text{Tr}(\text{ad}_x \circ \text{ad}_y) = 0$ for all $x \in [L,L], y \in L$. Then L is solvable.

Proof: Apply the theorem to the adjoint representation of L. We then get $\text{ad}\, L$ is solvable. Since $\ker \text{ad} = Z(L)$ is also solvable, L itself is solvable.

11.2 Killing Form

11.2.1 Criterion for Semisimplicity

Let L be any lie algebra. If $x, y \in L$, then define $\kappa(x,y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y)$. Then k is a symmetric bilinear form on L, called the **killing form**.

Theorem: $\mathfrak{g}$ is solvable $\iff$ $\kappa(x,y) = 0$ for all $x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$.

Proof: $\Longleftarrow$: By Cartan's Criterion.

$\Longrightarrow$: Exercise.

Example: The killing form of $\mathfrak{sl}(2,F)$.

We have

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then ad $_h = \mathrm{diag}(2, 0, -2)$, and

$$\text{ad }_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\text{ad }_y == \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

and thus k has the matrix

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

where $k_{ij} = \kappa(x_i, x_j)$ where x_i is a basis of L.

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Theorem: If L is semisimple and $x \in L$, there exists a unique x_s, x_n in L such that $x = x_s + x_n$, $[x_n, x_s] = 0$, ad x_s is semisimple, and ad x_n is nilpotent.

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Todo

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Let $S = \exp(\text{ad } e) \circ \exp(\text{ad } -f) \circ \exp(\text{ad } ei)$, which has the following matrix:

Where $\exp(\text{ad } e) = 1 + \text{ad } e + \frac{1}{2}(\text{ad } e)^2$, which would have the form

Theorem: If $\mathfrak{g}$ is semisimple, then any finite dimensional $\mathfrak{g}$-module V is completely reducible, i.e. it splits into a direct sum of simple modules.

14.1 Proof of Weyl's(?) Theorem

If V itself is simple, then we're done, so suppose it is not.

Assume there exists a nonzero submodule $U \subsetneq V$. It suffices to show that $V = U \oplus U'$ for some U'.

$$\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

Figure 8: Image

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \cdot & 2 & \\ & \cdot & -1 \\ & & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & 1 \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}$$

Figure 9: Image

14.1.1 Step 1:

If $\dim V = 2$ and $\dim U = 1$.

Then $U, V/U$ are both trivial modules. So $g \curvearrowright u = 0$ for all $u \in U$. But then $g \curvearrowright (v + U) = U$ for all $v \in V$, since $g \curvearrowright v \in U$.

So for all $x, y \in lieg$ and all $v \in V$, we have $[x, y] \curvearrowright v = x \curvearrowright (y \curvearrowright v) - y \curvearrowright (x \curvearrowright v)$. But both of the terms in parenthesis are in U, and all elements in $\mathfrak{g}$ kill elements in U, so this is zero. So $[\mathfrak{g}, \mathfrak{g}] \curvearrowright V$ trivially.

Exercise: If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

So $\mathfrak{g} \curvearrowright V$ trivially. Thus any U' that is a complementary subspace of U will be a submodule of V.

14.1.2 Step 2:

Suppose U is simple and $\dim U > 1$, so $\dim V/U = 1$.

Let Ω be the Casimir element on U (faithful representation?). Then $\Omega u = \check{c}$ for some $c \in \mathbb{F}$, and so $\Omega(U) \subseteq U$.

Since $\Omega : V \circlearrowleft$ is a homomorphism, $\ker \Omega \subseteq V$ is a $\mathfrak{g}$-submodule. Then $\dim V/U = 1 \implies V/U$ is a trivial module. So $\mathfrak{g} \curvearrowright V/U = 0$, i.e. $\mathfrak{g} \curvearrowright V \subseteq U$.

Then $\Omega(v) = \sum_i x_i \curvearrowright (y_i \curvearrowright v) \in U$ for all $v \in V$. What is the matrix of Ω?

In particular, $\mathrm{Tr}(\Omega \mid_{V/U}) = 0$. So $\mathrm{Tr}(\Omega) = \mathrm{Tr}(\Omega \mid_U)$. From 6.2, we know that $\mathrm{Tr}(\Omega) \neq 0 \implies c \neq 0$, where c is the scalar appearing above. So $\ker \Omega$ is 1-dimensional, and $\ker \Omega \bigcap U = \{0\}$.

So take $U' = \ker \Omega$.

14.1.3 Step 3:

Suppose U is *not* simple, but $\dim V/U = 1$.

We will induct on the dimension of U. Pick a proper nonzero submodule $\overline{U} \subsetneq U$, so that $\dim U/\overline{U} < \dim U$. Now $V/U \cong (V/\overline{U})/(U/\overline{U})$ by an isomorphism theorem. So $U/\overline{U}$ is a submodule of $V/\overline{U}$ of codimension 1. Applying the inductive hypothesis, we obtain $V/\overline{U} = U/\overline{U} \oplus \overline{V}/\overline{U}$ for some $\overline{V}$ such that $U \subseteq \overline{V} \subseteq V$.

In particular, since $U \subseteq \overline{V}$ has codimension 1, $\dim \overline{U} < \dim U$. So apply the inductive hypothesis again: $\overline{V} = \overline{U} \oplus U'$ for some U', and $V = U \oplus U'$.

14.1.4 Step 4: The general case

Recall that $\hom(V, U)$ is a $\mathfrak{g}$-module where

$$(g \curvearrowright \phi)(v) = g \curvearrowright \phi(v) - \phi(g \curvearrowright v).$$

$$\begin{array}{cc} & \begin{array}{cc} \quad U \quad & \quad V/U \end{array} \\ \begin{array}{c} U \\ V/U \end{array} & \left(\begin{array}{c|c} cI & * \\ \hline 0 \cdots\cdots 0 & 0 \end{array}\right) \end{array}$$

Figure 10: Image

Define

$$S = \{\phi \in \hom(V, U) \ni \phi\mid_U \in F 1_U\}.$$

Then $S \leq \hom(V, U)$ as a submodule. Define $T = \{\phi \in S \ni \phi\mid_U = 0\}$. Then $T \leq S$ as a submodule, and $\mathfrak{g}(S) \subseteq T$.

Now each $\phi \in S$is determined ($\mod T$) by the scalar $\phi\mid_U$. Note that $\dim(S/T) = 1$. By steps 1-3, we know that $S = T \oplus T'$ for some $T' \subseteq S$ of dimension 1. Then $T' = \operatorname{span}_{\mathbb{F}}(f)$ for some nonzero map $f : V \to U$ such that $f(u) = cu$ for some $c \neq 0$.

Then $\mathfrak{g}(T \oplus T') = \mathfrak{g}(S) \subseteq T \implies \mathfrak{g}(T') = 0$. So for all $g \in \mathfrak{g}$, we have $0 = (g \curvearrowright f)(v) = f \curvearrowright f(v) - f(g \curvearrowright v)$. Then $f : V \to U$ is a lie algebra homomorphism, $\ker f = U'$, and thus $V = U \oplus U'$. $\square$

Some consequences of Weyl's theorem:

14.2 Preservation of Jordan Decomposition

Recall that when $\mathfrak{g} \in \mathfrak{gl}(V)$ is a linear lie algebra, then for $x \in \mathfrak{g}$ we have:

Jordan Decomposition: $x = x_s + x_n$ where $x_s, x_n \in \operatorname{End}(V)$.

Abstract Jordan Decomposition:

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\text{ad}} \text{ad }(\mathfrak{g}) \\ x &\mapsto \text{ad } x \\ x_s &\leftarrow (\text{ad } x)_s \\ x_n &\leftarrow (\text{ad } x)_n. \end{aligned}$$

and so $x = x'_s + x'_n$ for some x'. The theorem will be that these recover the usual Jordan decomposition.

Theorem: If $\mathfrak{g} \in \mathfrak{gl}(V)$ is semisimple and V is finite dimensional, then $x_s, x_n \in \mathfrak{g}$, and $x_s = x'_s, x'_n$.

Corollary: If $\mathfrak{g}$ is semisimple and finite dimensional and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation, then if $x = x_s + x_n$ is the abstract Jordan decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the Jordan decomposition in $\mathfrak{gl}(V)$.

Example: If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is semisimple and finite dimensional, and h is diagonal, then by JD $h = h + 0, \phi(h) = \phi(h) + 0$. Then $h \curvearrowright V$ semisimply, or $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, where $V_\lambda = \{v \in V \ni h \curvearrowright v = \lambda v\}$ are the eigenspaces.

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Last time: The abstract Jordan Decomposition coincides with the actual Jordan Decomposition.

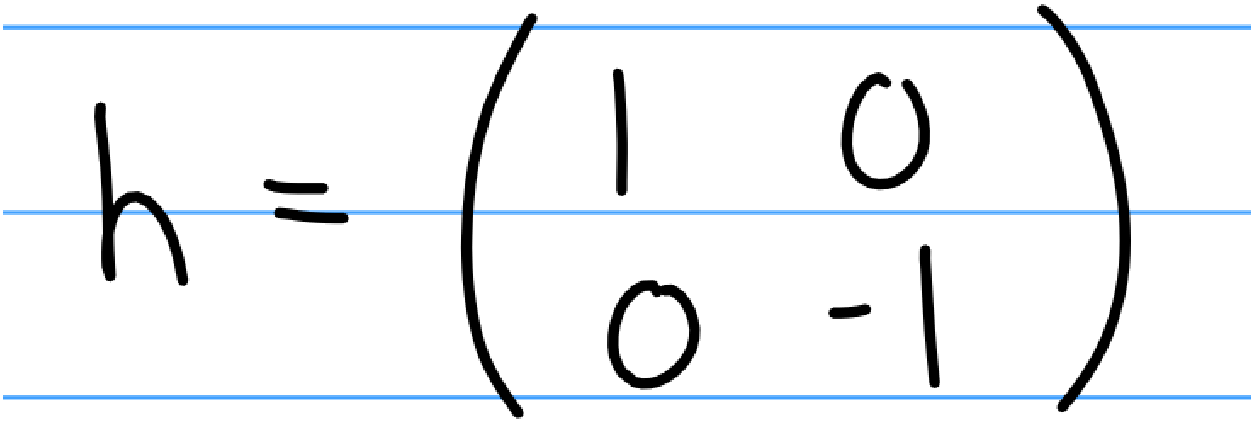


Figure 11: Image

$$
\begin{aligned}
\phi : \mathfrak{g} &\to \mathfrak{gl}(V) \\
x \mapsto \phi(x) = \phi(x)_s + \phi(x)_n &= \phi(x_n) + \phi(x_s) \\
x_s + x_n &\mapsto \phi(x_s) + \phi(x_n).
\end{aligned}
$$

Therefore $x_s \curvearrowright V$ semisimply. The example we saw last time was $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with a matrix $h = [1, 0; 0, -1]$ and $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$.

15.1 Finite Dimensional Representations of $\mathfrak{sl}(2, \mathbb{C})$

15.1.1 Weights and Maximal Vectors

Definition: If $V_\lambda \neq 0$, then V_λ is a *weight space* of V and $\lambda \in \mathbb{C}$ is a *weight* of h in V. We then define $W_t(V) = \{\text{weights in } V\}$.

Lemma: If $v \in V_\lambda$ then $e \curvearrowright v \in V_{\lambda+2}$ and $f \curvearrowright v \in V_{\lambda-2}$.

Proof:

$$
\begin{aligned}
h \curvearrowright (e \curvearrowright v) = [h, e] \curvearrowright v + e \curvearrowright (h \curvearrowright v) \\
= 2e \curvearrowright v + \lambda e \curvearrowright v \\
= (\lambda + 2) e \curvearrowright v.
\end{aligned}
$$

and

$$\begin{aligned} h \curvearrowright (f \curvearrowright v) &= [h,f] \curvearrowright v + f \curvearrowright (h \curvearrowright v) \\ &= -2f \curvearrowright v + \lambda f \curvearrowright v \\ &= (\lambda - 2) f \curvearrowright v. \end{aligned}$$

So if V is a finite-dimensional $\mathfrak{g}$-module, then there exists a $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. Any nonzero $v \in V_\lambda$ is called a *maximal vector*.

> Note: in category $\mathcal{O}$, these always exist?

Some computations:

- $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ Then $V = \mathbb{C}$ is the trivial module, and $g \curvearrowright V = 0$. So $W_t(V) = \{0\}$, and $V = V_0$.

If $V = \mathbb{C}^2$, then take the natural representation $\text{span}_{\mathbb{C}}\{v_1 = [1,0], v_2 = [0,1]\}$. Then $g \curvearrowright V$ by matrix multiplication, and if $h = [1,0;0,-1]$ then $h \curvearrowright v_1 = v_1$ and $h \curvearrowright v_2 = -v_2$ by just doing the matrix-vector multiplication. Then $\mathbb{C}([1,0]) = V_1, \mathbb{C}([0,1]) = V_{-1}$, so $W_t(V) = \{\pm 1\}$.

Taking $V = \mathbb{C}^3 = \text{ad}\ \mathfrak{g} = \text{span})_{\mathbb{C}}\{e, f, h\}$, then

$$\begin{aligned} h \curvearrowright f &= [h,f] = -2f \\ h \curvearrowright h &= [h,h] = 0h \\ h \curvearrowright e &= [h,e] = 2e. \end{aligned}$$

So $W_t(V) = \{2, 0, -2\}$ and $V_2 = \mathbb{C}e, V_0 = \mathbb{C}h, V_{-2} = \mathbb{C}f$.

> Note the pattern: some largest value, then jumping by 2 to lower values, ending at negative the largest value. In some sense, the rest of the theory will reduce to the case of $\mathfrak{sl}(2, \mathbb{C})$.

Lemma: Let V be a finite dimensional simple $\mathfrak{sl}(2, \mathbb{C})$-module, and $V_0 \in V_\lambda$ a maximal vector.

Set $V_{-1} = 0, V_i = f^{(i)} \curvearrowright v_0$ (where $f^{(i)} = \frac{f^i}{i!}$). Then for all $i \geq 0$, we have

a. $h \curvearrowright v_i = (\lambda - 2i) v_i$
b. $f \curvearrowright v_i = (i+1) v_{i+1}$
c. $e \curvearrowright v_i = (\lambda - i + 1) v_{i-1}$

Proof of (a): By lemma 7.1, we have $f \curvearrowright v_0 \in V_{\lambda-2}$, and so inductively $f^{(i)} \curvearrowright v_0 \in V_{\lambda - 2i}$

Proof of (b): By definition.

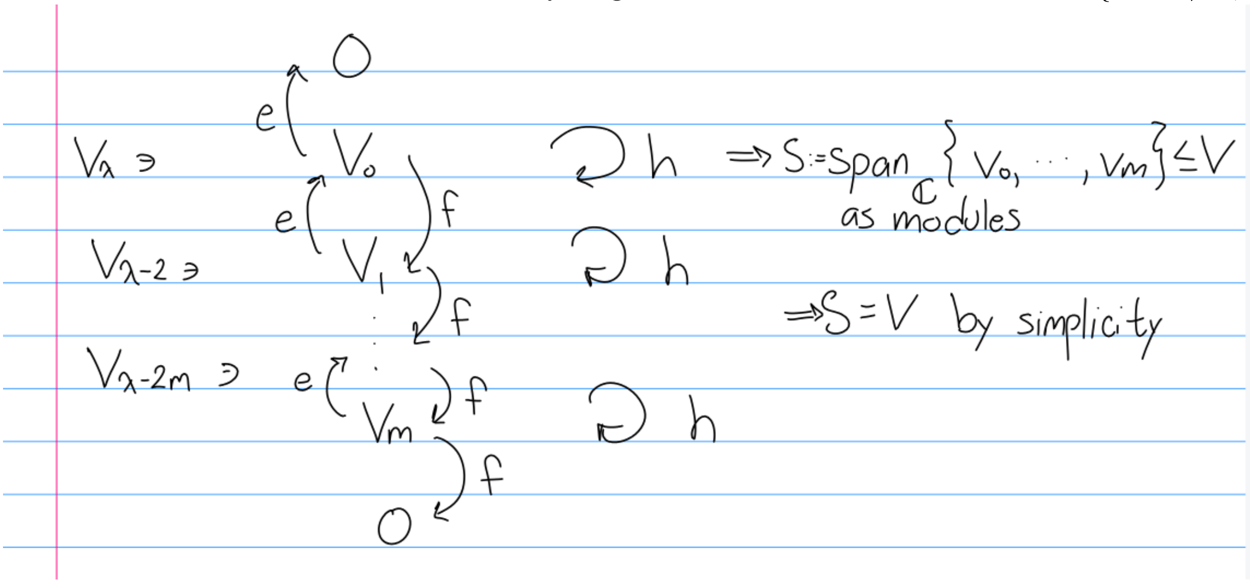
Proof of (c):

$$
\begin{aligned}
ie \curvearrowright v_i = ie \curvearrowright \frac{f^i \curvearrowright v_0}{i!} \\
= e \curvearrowright (f \curvearrowright v_{i-1}) \\
= [e, f] \curvearrowright v_{i-1} + f \curvearrowright (e \curvearrowright v_{i-1}) \\
= h \curvearrowright v_{i-1} + f \curvearrowright ((\lambda - i + 2) v_{i-2}) \text{ind} \\
= (\lambda - 2i + 2) v_{i-2} + (\lambda + i - 2)(i - 1 v_{i-1}) \\
= i (\text{RHS}).
\end{aligned}
$$

Theorem: If V is a finite dimensional and simple, then $V \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$ where $L(m) = \text{span}_{\mathbb{C}} \{v_0, v)1, \cdots v_m\}$ where each v_i is of weight $m - 2i$.

Thus $L(m) = L(m)_m \oplus L(m)_{m-2} \oplus \cdots \oplus L(m)_{-m}$ where $\dim L(m)_\mu = 1$ for all μ and $\dim L(m = m + 1.)$

Proof: Pick a maximal vector $v_0 \in V_\lambda$ for any weight λ. Define v_i as usual. Let $m = \min \{i \ni V_i \neq 0,\ V_{i+1} = 0\}$



Definition: A module V is a *highest weight module* of weight λ if $V = \mathfrak{g} \curvearrowright v_0$ for some maximal vector $v_0 \in V_\lambda$.

Then λ is referred to as the *highest weight*, and v_0 is the *highest weight vector.*

Corollary: If V is finite-dimensional, then

a. $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$
b. The number of summands $= \dim V_0 + \dim V_1$.

Proof of (a): By Weyl's theorem, we know $V = \oplus W_i$ for some simple W_i. By theorem 7.2, this is equal to $\oplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\mu_m}$

Proof of (b): $\dim V_0 = \# \{\text{summands where } m \text{ is even}\}$ $\dim V_1 = \# \{\text{summands where } m \text{ is odd}\}$

Remark: Let $V_d = \{f \in \mathbb{C}[x, y] \ni f \text{ is homogeneous of total degree } d\} = \text{span}_{\mathbb{C}} \left\{x^d, x^{d-1} y, \cdots, y^d\right\}$.

Then $\mathfrak{sl}(2, \mathbb{C}) \curvearrowright V_d$ by

$$e \mapsto x\frac{\partial}{\partial y}$$
$$f \mapsto y\frac{\partial}{\partial x}$$
$$h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Fact: For $L(m), \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(L(m))$, define

$$s = (\exp \phi(e)) \circ (\exp \phi(-f)) \circ (\exp \phi(e))$$

Then $s(v_i) = -v_{m-i}$.

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Last time: Construction of simple finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ module.

Today: Root space decomposition for semisimple finite-dimensional $\mathfrak{g}$.

16.1 Root Space Decomposition

Let $\mathfrak{g}$ be semisimple and finite dimensional, and let $\mathbb{F} = \mathbb{C}$.

16.1.1 Maximal Toral subalgebra and roots

Definition: A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is *toral* if $\mathfrak{h} \neq 0$ and it consists of only semisimple elements (i.e. $x_n = 0 \forall x \in \mathfrak{h}$)

Lemma:

a. There exists a toral subalgebra of $\mathfrak{g}$, which is a nontrivial maximal toral subalgebra.
b. Any toral subalgebra is abelian.

Proof of (a): Want to show that there exists an $x \in \mathfrak{g}$ such that $x_s \neq 0$, which will imply that $\mathfrak{h} = \mathbb{C}x_s$ is toral.

Suppose $x_s = 0$ for all $x \in \mathfrak{g}$, then ad $x =$ ad x_n is nilpotent. By Engel's theorem, this means $\mathfrak{g}$ must be nilpotent. But this contradicts $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (since $\mathfrak{g}$ is semisimple) so the derived series can never reach zero.

Proof of (b): Fix $x \in \mathfrak{h}$, want to show that $[x, h] = 0 \forall h \in \mathfrak{h}$. Then $x = x_s$, and so ad $x : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. It suffices to show that ad $x|_{\mathfrak{h}} = 0$ for all $\mathfrak{h}$.

Suppose that $[x, h] = ah$ for some vector h where $a \neq 0$. Decompose $\mathfrak{h}$ into eigenspaces, so $\mathfrak{h} = \bigoplus_\lambda \mathfrak{h}_\lambda$ where $\mathfrak{h}_\lambda = \{y \in \mathfrak{h} \ni [h, y] = \lambda y\}$. But then $[h, x] \in \mathfrak{h}_0$, since $[h, [h, x]] = [h, -ah] = 0$.

So write $x = \sum_\lambda c_\lambda x_\lambda$, where $c_\lambda \in \mathbb{C}$ and $x_\lambda \in \mathfrak{h}_\lambda$. Then

$$\begin{aligned}[h,x] &= \sum_\lambda c_\lambda [h, x_\lambda] \\ &= \sum_\lambda c_\lambda \lambda x_\lambda \in \mathfrak{h}_0,\end{aligned}$$

so $\lambda c\lambda = 0 \forall \lambda \neq 0$, which means $c_\lambda = 0 \forall \lambda \neq 0$, and thus $x \in \mathfrak{h}_0$ and $[h,x] = 0$. But this contradicts $[x,h] = ah$.

Now $\forall x, h \in \mathfrak{h}, g \in \mathfrak{g}$, we have $[h,[x,y]] = [x,[h,y]] + [y,[x,h]] = [x,[h,y]]$. Thus $\text{ad } h \circ \text{ad } x = \text{ad } x \circ \text{ad } h$ as elements of $\text{End}(\mathfrak{g})$.

So $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ni [h,x] = \alpha(h) x \forall h \in \mathfrak{h}\}$.

Note that $\mathfrak{g}_0 = \{x \in \mathfrak{g} \ni [h,x] = 0 \forall h \in \mathfrak{h}\} = C_{\mathfrak{g}}(\mathfrak{h}) \supseteq \mathfrak{h}$, i.e. the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Definition: Fix a toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then a *root* is a nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. $\mathfrak{g}_\alpha$ is referred to as the *root space*.

We write $\Phi = \{\text{roots}\}$ and $\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$.

Example: $\mathfrak{sl}(3, \mathbb{C})$.

TODO: Insert image from phone.

Then $\Phi = \{\alpha : \mathfrak{h} \to \mathbb{C}, h_1 \mapsto \alpha(h_1) \in \{\pm 1, \pm 2\}\}$. So

- $\mathfrak{g}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$
- $\mathfrak{g}_1 = \mathbb{C}f_2 \oplus \mathbb{C}e_3$
- $\mathfrak{g}_2 = \mathbb{C}e_1$
- $\mathfrak{g}_{-1} \mathbb{C}f_3 \oplus \mathbb{C}e_2$
- $\mathfrak{g}_{-2} = \mathbb{C}f_1$.

TODO: Insert second and third image from phone

From these computations, we collect the eigenvalues as ordered pairs. If we choose a larger toral subalgebra, we get a finer decomposition. And if we take a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$ and all $\dim \mathfrak{g}_\alpha = 1$.

Proposition (a): $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$.

Proposition (b): If $x \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$ then ad x is nilpotent.

Proposition (c): If $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta = 0$, then $\kappa(x,y) = 0 \forall x \in \mathfrak{g}_\alpha, y = \mathfrak{g}_\beta$.

Proof of (a): Easy exercise:

Proof of (b): For all $y \in \mathfrak{g}$, $y \in \mathfrak{g}_\mu$ for some $\mu \in \mathfrak{h}^*$. We have $\mathfrak{g}_u \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+\alpha} \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+2\alpha} \to \cdots$ by $y \mapsto [x,y] \mapsto \cdots$. Since $\mathfrak{g}$ is finite dimensional, this must terminate, so $(\text{ad } x)^n(y) = 0$ for some n.

Proof of (c): If $\alpha + \beta = 0$, then there exists an $h \in \mathfrak{h}$ such that $\alpha(h) + \beta(h) \neq 0$. Since the killing form is associative, we have

Corollary: $\kappa\mid_{\mathfrak{g}_0}$ is nondegenerate.

Proof: We want to show $\kappa(h,y) = 0 \forall y \in \mathfrak{g}_0 \implies h = 0$ holds for any choice of $y \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$.

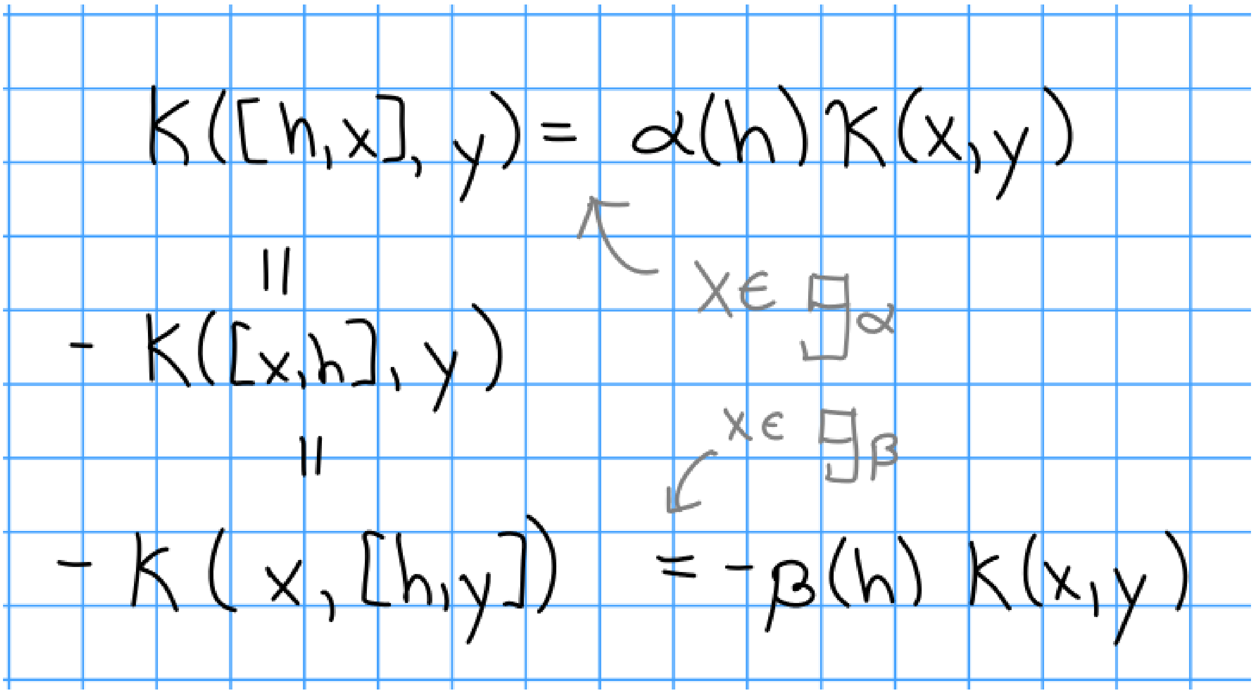

Figure 12: Image

By proposition (c), we have $\kappa(h, y) = 0$. Note that we have $\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha)$. This implies that $\kappa(h, y) = 0 \forall y \in \mathfrak{g}$. But then $h = 0$ because κ is nondegenerate and $\mathfrak{g}$ is semisimple.

17 Monday September 23

Last time: $\mathfrak{h}$ is a *toral* subalgebra if it contains only semisimple elements, and implies that there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ni [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ and $\Phi = \{\alpha : \mathfrak{h} \to \mathbb{C} \ni \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}$ and $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$.

Take larger $\mathfrak{h}$ yields finer decompositions, and a maximal $\mathfrak{h}$ gives $\dim \mathfrak{g}_\alpha = 1 \forall \alpha \in Phi$.

Corollary: $\kappa|_{\mathfrak{g}_0}$ is nondegenerate.

17.1 The Centralizer of $\mathfrak{h}$

If $x, y \in \mathrm{End}(V)$ where V is finite dimensional, $xy = yx$, and y is nilpotent, then xy is nilpotent and $\mathrm{Tr}(xy) = 0$.

Proposition: If $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$.

Proof:

Step 1: If $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$.

If $x \in \mathfrak{g}_0$, then $\mathrm{ad}\ x(\mathfrak{h}) \subseteq 0$. By proposition 4.2, $\mathrm{ad}\ x_s(\mathfrak{h}) \subseteq 0, \mathrm{ad}\ x_n(\mathfrak{h})$, and so $x_s, x_n \in \mathfrak{g}_0$.

Step 2: $\{x_s \ni x \in \mathfrak{g}_0\} \subseteq \mathfrak{h}$.

If $x \in \mathfrak{g}_0$, then by step 1 we have $x_s \in \mathfrak{g}_0$ and so $\mathfrak{h} + \mathbb{C}x_s$ is toral, and thus $x_s \in \mathfrak{h}$.

Step 3: $\kappa|_{\mathfrak{h}}$ is non-degenerate.

We want to show that $\kappa(h, x) = 0\ \forall x \in \mathfrak{g} \implies h = 0$. By the corollary, it suffices to show that $\kappa(h, x) = 0\ \forall x \in \mathfrak{g}_0$. By step 2, it suffices to check this only for $x \in \mathfrak{g}_0$ such that $x = x_n$.

If $x = x_n$, then $\mathrm{ad}\ x_n$ is nilpotent and $\mathrm{ad}\ h$ commutes with $\mathrm{ad}\ x$ because $[h, x] = 0$ (since $x \in \mathfrak{g}_0$). By the lemma, $\mathrm{Tr}(\mathrm{ad}\ h \circ \mathrm{ad}\ x) = 0$, since $\mathrm{ad}\ h = \kappa(h, x)$.

Step 4: $\mathfrak{g}_0$ is nilpotent.

Pick $x \in \mathfrak{g}_0$. Then by step 2, $x_s \in \mathfrak{h}$, so $\mathrm{ad}\ x_s : \mathfrak{g}_0 \circlearrowleft$ is a zero map and thus nilpotent.

So $\mathrm{ad}\ x_n$ is nilpotent, meaning that $\mathrm{ad}\ x$ is nilpotent. By Engel's theorem, this implies that $\mathfrak{g}_0$ itself is nilpotent.

Step 5: $\mathfrak{g}_0$ is abelian.

Suppose that $I := [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$. We have $I \trianglelefteq \mathfrak{g}_0$, and I is not nilpotent whereas $\mathfrak{g}_0$ is.

By Lemma 3.3, we have $I \bigcap Z(\mathfrak{g}_0) \neq 0$, so pick x in the intersection. Note that $\kappa(\mathfrak{h}, I) = \kappa(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0])$, which by associativity equals $\kappa([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = 0$.

By step 3, we have $\mathfrak{h} \bigcap I = 0$. By step 2, $x \neq x_s$, and thus $x_n \neq 0$. But we also have $x \in Z(\mathfrak{g}_0)$, so $[x, \mathfrak{g}_0] = 0$ and $\mathrm{ad}\ x(\mathfrak{g}_0) \subseteq 0$. By Proposition 4.2, this holds for x_s, x_n as well, which are both in the center. So for all $y \in \mathfrak{g}_0$, $\mathrm{ad}\ y$ commutes with $\mathrm{ad}\ x_n$, which is nilpotent.

By the lemma, this implies that $0 = \mathrm{Tr}(\mathrm{ad}\ y \circ \mathrm{ad}\ x_n) = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$. So $x_n = 0$.

Step 6: Suppose $\mathfrak{g}_0 \not\subset \mathfrak{h}$. By step 2, there exists an $x \in \mathfrak{g}_0$ such that $x \notin \mathfrak{h}$, where $x_n \neq 0$. By step 5, $[x_n, y] = 0$ for all $y \in \mathfrak{g}_0$. Then $\mathrm{ad}\ x$ (which is nilpotent) commutes with $\mathrm{ad}\ y$. By the lemma, $0 = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$, and thus $x_n = 0$. $\square$

> Main idea: Choose a maximal toral subalgebra to get a nice root space decomposition, and so it coincides with $\mathfrak{g}_0$.

Corollary: $\kappa|_{\mathfrak{g}}$ is nondegenerate.

Thus for all $\alpha \in \mathfrak{h}^*$, there exists a unique $t_\alpha \in \mathfrak{h}$ such that $\alpha = \kappa(t_\alpha, \cdot) : \mathfrak{h} \to \mathbb{C}$.

In other words, there is an identification

$$\begin{aligned} \mathfrak{h} &\xrightarrow{1-1} \mathfrak{h}^* \\ h &\mapsto \kappa(h, \cdot) \\ t_\alpha &\leftarrow \alpha. \end{aligned}$$

Definition: A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \ni [x, h] \subseteq \mathfrak{h}\}.$$

Note that $N_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.

Remark: If $\mathfrak{g}$ is semisimple and finite dimensional with $\mathrm{char}(F) = 0$, we will have a correspondence:

$$\{\text{CSAs of } \mathfrak{g}\} \iff \{\text{maximal toral subalgebras of } \mathfrak{g}\}.$$

Maximal toral subalgebras advantages over Cartan subalgebra definition:

- Yields the finest root space decomposition
- $\mathfrak{h}^* = \mathfrak{h}$, Weyl group?
- Existence is easy compared to CSAs

On the other hand, CSA advantages:

- All CSAs are conjugate under G (some group to be defined)
- The dimensions of all CSAs are the same, giving a well-defined notion of dimension ($\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$).

17.2 8.3: Orthogonality Properties

From now on, $\mathfrak{h}$ will be a maximal toral subalgebra.

Proposition: Let $\alpha \in \Phi$. Then

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$
c. $\forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \kappa(x, y)t_\alpha$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$ (let the nonzero scalar be λ)
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
f. For any nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha := \frac{\lambda}{\kappa(t_\alpha,t_\alpha)} t_\alpha$. Moreover, $\langle e_\alpha, f_\alpha, h_\alpha \rangle = \mathfrak{sl}(2, \mathbb{C})$.

18 Wednesday September 25

Today: Properties of the root space when the toral subalgebra is maximal.

Last time: We have $\mathfrak{g} = \mathfrak{g} \oplus (\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha)$ where $\kappa \mid_{\mathfrak{h}}$ is nondegenerate. We also have a correspondence

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}^* h \mapsto \kappa(\mathfrak{h}, \cdot\,) \\ t_\alpha &\leftarrow \alpha := \kappa(t_\alpha, \cdot\,). \end{aligned}$$

18.1 Orthogonality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$

c. $[x,y] = \kappa(x,y)t_\alpha$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$
f. For each nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha \coloneqq \frac{2}{\kappa(t_\alpha,t_\alpha)}t_\alpha$. Moreover, $\langle e_\alpha, t_\alpha, h_\alpha \rangle \cong \mathfrak{sl}(2,\mathbb{C})$.

Proof of (a): We want to show that $h \in \mathfrak{h}$ implies that if $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $h = 0$.

Take $x \in \mathfrak{g}_\alpha$. Then $[h,x] = \alpha(h)x = 0$. So $[\mathfrak{h}, \mathfrak{g}] = 0$ because $\mathfrak{h}$ is abelian. But then $[h, \mathfrak{g}] = 0$, or $h \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

Proof of (b): By Proposition 8.1c, we have $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$.

If $-\alpha \notin \Phi$, then $\mathfrak{g}_{-\alpha} = 0$. So $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$ by the non-degeneracy of κ.

Proof of (c): For all $h \in \mathfrak{h}$, we have

$$\begin{aligned}\kappa(h,[x,y]) &= \kappa([h,x],y) \\ &= \kappa(\alpha(h)x,y) \\ &= \kappa(t_\alpha,h)\kappa(x,y) \\ &= \kappa(\kappa(x,y)t_\alpha,h) \\ &= \kappa(h,\kappa(x,y)t_\alpha).\end{aligned}$$

which implies that $\kappa(h,[x,y] - \kappa(x,y)t_\alpha) = 0$, which forces the second argument to be zero by non-degeneracy.

Proof of (d): We will show that (d) implies (c), i.e. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}t_\alpha$.

We want to show $\kappa(x,y)$ is not always zero.

Pick any nonzero $x \in \mathfrak{g}_\alpha$. Then $\kappa(x, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$. If $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$, then $\kappa(x, \mathfrak{g}) = 0$. By non-degeneracy, this forces $x = 0$.

Proof of (e): We will skip this for now, and revisit with methods from later sections that make this proof simpler.

Proof of (f): Let $e_\alpha \neq 0$ in $\mathfrak{g}_\alpha$. Then there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, y) \neq 0$. Set $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(t_\alpha,t_\alpha)}$.

By (c), we have

$$\begin{aligned}[e_\alpha, f_\alpha] &= \kappa(e_\alpha, t_\alpha)t_\alpha \\ &= \frac{2}{\kappa(t_\alpha,t_\alpha)}t_\alpha \\ &= h_\alpha.\end{aligned}$$

and

$$\begin{aligned}[h_\alpha, e_\alpha] &= \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] \\ &= \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha)e_\alpha \\ &= 2e_\alpha.\end{aligned}$$

and similarly $[h_\alpha, f_\alpha] = -2f_\alpha$.

Definition:

Let $\mathfrak{sl}(2, \alpha) = \langle e_\alpha, f_\alpha, h_\alpha \rangle$ as in (f). A priori, this depends on a choice of $e_\alpha \neq 0$. We will show that this only depends on α.

18.2 Orthogonality/Integrality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. $\dim \mathfrak{g}_\alpha = 1$. (Note that in general, $\dim \mathfrak{g}_0 = \dim \mathfrak{h} \geq 1$)
b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
c. If $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
d. If $\beta \neq -\alpha \in \Phi$, then let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $\beta + q\alpha$ are both in Φ. Then $\beta + i\alpha \in \Phi$ for every $-p \leq i \leq q$, and

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = p - q \in \mathbb{Z}.$$

Proof of (a) and (b):

Let $M = \mathfrak{h} \oplus \left(\bigoplus_{c\neq 0} \mathfrak{g}_{c\alpha}\right) \leq \mathfrak{g}$ as a subspace. By a routine check, M is an $\mathfrak{sl}(2, \alpha)$ submodule of $\mathfrak{g}$. Recall that $M = \bigoplus_{\lambda\in\mathbb{Z}}$ as a direct sum of vector spaces. Applying Weyl's theorem, we also have $M = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} L(m)^{\oplus \mu_m}$ as a direct sum of (irreducible?) modules.

For $\mathfrak{h}$, if we have $[h_\alpha, h] = 0$ for all $h \in \mathfrak{h}$, then $h \in M_0$. For $\mathfrak{g}_{c\alpha}$, $[h_\alpha, x] = c\alpha(h_\alpha)x$ for all $x \in \mathfrak{g}_{c\alpha}$. But this equals $2cx$. So this implies that $\mathfrak{g}_{c\alpha} \subseteq M_{2c}$.

Thus $2c \in \mathbb{Z}$, and thus $c \in \frac{1}{2}\mathbb{Z}$, and $M_0 = \mathfrak{h}$.

We then have $\dim M_0 = \sum_{m\in 2\mathbb{Z}} \mu_m$. So write $h = \mathbb{C}t_\alpha \oplus \ker \alpha$ as vector spaces. Consider the action $\mathfrak{sl}(2, \alpha) \curvearrowright \ker \alpha$, which is trivial since $h \in \ker \alpha$. We $[h_\alpha, h] = 0$, $[e_\alpha, h] = -\alpha(h)e_\alpha = 0$ since $h \in \ker \alpha$, and similarly $[f_\alpha, h] = 0$.

Thus $\ker \alpha = L(0)^{\oplus \dim \mathfrak{h} - 1}$. Moreover, $\mathfrak{sl}(2, \alpha) = L(2) = \operatorname{span}(e_\alpha, t_\alpha, f_\alpha)^T$. But this forces the case that there is no other summand of the form $L(k)$ for k even in M.

Then $\mathfrak{g}_{2\alpha} \subseteq M_4$, which must be zero. So $2\alpha \notin \Phi$, so 2α is not a root. ("Twice a root is never a root")

So $\frac{1}{2}\alpha \notin \Phi$, otherwise we could apply this argument to conclude that α is not a root and reach a contradiction. Thus $M_1 = 0$, since $c \neq \frac{1}{2}$ implies that there is not summand of the form $L(k)$ for k odd in M. But this forces $M = \mathfrak{h} \oplus \mathfrak{sl}(2, \alpha)$.

Motto: reduce the complexity by using the $\mathfrak{sl}(2)$ module structure and its representation theory!

19 Friday September 27

Last time, we saw $\Phi \subseteq \mathfrak{h}^* = \{\alpha : \mathfrak{h} \to \mathbb{C}\}$.

Suppose $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a maximal toral subalgebra and take $F = \mathbb{C}$.

We have the following propositions:

a. $\dim \mathfrak{g}_\alpha = 1 \ \forall \alpha \in \Phi$
b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$, and $2\alpha \notin \Phi$ where $c\alpha : \mathfrak{h} \to \mathbb{C}$, $h \mapsto c \curvearrowright \alpha(h)$. Moreover, $M = \mathfrak{h} \oplus \left(\bigoplus_{c \neq 0} \mathfrak{g}_{c\alpha}\right)$
c. If $\alpha, \beta \in \Phi$ and $\beta \neq -\alpha$ Let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $b + q\alpha$ are in Φ. Moreover, $\beta(h_\alpha) = \kappa(t_\beta, t_\alpha) = p - q \in \mathbb{Z}$.

Proof of (c):

Set $M = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$, which is an $\mathfrak{sl}(2, \alpha)$ module. By (a), we have $\dim \mathfrak{g}_{\beta+i\alpha} = 1 \iff \beta + i\alpha \in \Phi$. But for all $x \in \mathfrak{g}_{\beta+i\alpha}$, we have $[h, x] = (\beta + i\alpha)(h)x$ for all $h \in \mathfrak{h}$. But then $[h_\alpha, x] = (\beta(h_\alpha) + i\alpha(h_\alpha))x = (\beta(h_\alpha) + 2i)x$

Then $\mathfrak{g}_{\beta+i\alpha} \subseteq M_{\beta(h_\alpha)+2i}$, so $\beta(h_\alpha) \in \mathbb{Z}$.

Moreover, $\mathrm{Wt}(M) = 2\mathbb{Z}$ or $2\mathbb{Z} + 1$, and in particular $\dim M_0 + \dim M_1 = 1$.

Thus M is irreducible, and $M \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$. Moreover, $\mathrm{Wt}(M) = \{m, m-2, \cdots - m\}$, and$simg\mathfrak{g}_{\beta+i\alpha} = 1$ for all $i \in [-p, q]$. Thus $\beta + i\alpha \in \Phi$.

Proof of 8.3(e): $\alpha(t_\alpha) \neq 0$. The claim is that for all $\beta \in Phi$, there exists an $r \in \mathbb{Q}$ such that $\beta(h) = r\alpha(h)$ for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

There are two cases: if $\beta = -\alpha$, then we're done by the previous argument.

Otherwise, $\beta \neq -\alpha$. Take $M = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$.

Then,

$$\begin{aligned}
\mathrm{Tr}_M(\mathrm{ad}\ h) = \sum_i \mathrm{Tr}_M((\mathrm{ad}\ e_i \circ \mathrm{ad}\ f_i) - (\mathrm{ad}\ f_i \circ \mathrm{ad}\ e_i)) = \sum_i \mathrm{Tr}_{\mathfrak{g}_{\beta+i\alpha}}(\mathrm{ad}\ h) \\
= \sum (\beta + i\alpha)(h) \dim \mathfrak{g}_{\beta+i\alpha} \\
= \sum_i \dim \mathfrak{g}_{\beta+i\alpha} \beta(h) + \sum_i i \dim(\mathfrak{g}_{\beta+i\alpha}) \\
\implies \beta(h) = \frac{-\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}{\sum_i \dim \mathfrak{g}_{\beta+i\alpha}} \alpha(h).
\end{aligned}$$

Now consider the killing form $\kappa(t_\beta, t_\alpha) = \beta(t_\alpha) = r\alpha(t_\alpha)$, where the last equality is what we are claiming.

Suppose that $\alpha(t_\alpha) = 0$. Then $\kappa(t_\beta, t_\alpha) = 0$ for all $\beta \in Phi$. By the non-degeneracy of κ, we have $t_\alpha = 0$ and thus $\alpha = 0$.

19.1 Summary

We have $\mathfrak{g}$ semisimple, finite dimensional, and $\mathfrak{h}$ a maximal toral subalgebra (i.e. the Cartan subalgebra). This implies that κ is nondegenerate, and we have a correspondence

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}\vee \\ h &\mapsto \kappa(h, \cdot) \\ t_\alpha &\leftarrow \alpha. \end{aligned}$$

This gives a symmetric bilinear form $(\cdot, \cdot) : \mathfrak{h}\vee \to \mathfrak{h}\vee$.

For $\alpha \in Phi$, define its *coroot* $\alpha\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

Note that $(\cdot)\vee$ is not linear: note that

$$(2\alpha)\vee = \frac{2}{(2\alpha, 2\alpha)}2\alpha = \frac{\alpha}{(\alpha, \alpha)} = \frac{\alpha\vee}{2}.$$

Assume that $\Phi = \{\alpha_i\}$. Define $E_{\mathbb{Q}} = \bigoplus_{i=1}^{\ell} \mathbb{Q}_{\alpha_i}$, and $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$.

Lemma: If $\alpha, \beta \in \Phi$, then

a. $(\beta, \alpha) \in \mathbb{Q}$,
b. $(\cdot, \cdot)$ on $E_{\mathbb{Q}}$ is positive definite, i.e. $x \neq 0 \implies (x, x) > 0$.

An immediate consequence of (b) is that $(\cdot, wait)$ on E is an inner product.

Proof: For all $\lambda, \mu \in \mathfrak{h}\vee$, we have

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\ t_\lambda \circ \mathrm{ad}\ t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(\ldots) + \sum_{\alpha\in\Phi} \mathrm{Tr}_{\mathfrak{g}_\alpha}(\ldots) = 0 + \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{[} \alpha \in \Phi] = \kappa(t_\alpha, t_\lambda)\kappa($$

So pick $\lambda = \mu = \alpha \in \Phi$. Then $(\alpha, \alpha) = \sum_{\beta\in\Phi}(\beta, \alpha)^2$.

Then

$$\frac{1}{(\alpha, \alpha)} = \sum_{\beta\in\Phi} \left(\frac{(\beta, \alpha\vee)}{2}\right)^2.$$

where $(\beta, \alpha\vee) = \cdots = \beta(h_\alpha) \in \mathbb{Z}$.

This means that $(\alpha, \alpha) \in \mathbb{Q}_{>0}$.

Summary of properties proved:

Let $\alpha, \beta \in \Phi$. Then

1. $0 \notin \Phi$ and Φ spans E
2. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\beta - (\beta, \alpha\vee)\alpha \in \Phi$
4. $(\beta, \alpha\vee) \in \mathbb{Z}$

Thus the assignment $(\mathfrak{g}, \mathfrak{h}) \mapsto (\Phi, E)$ defines a **root system**. This only works when $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is maximal toral.

Proof of (3):

We computed $(\beta, \alpha\vee) = p - q$. Then $-p \leq -(\beta, \alpha\vee) = q - p \leq q$. So this must be something on the root stream.

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Last time: Let $\mathfrak{g}$ be finite dimensional and $\mathfrak{h}$ a maximal toral subalgebra.

Then (Φ, E) is a *root system*, and we obtain a bilinear product

$$\begin{aligned} \langle\,\cdot\,,\ \cdot\,\rangle : E \times E &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \kappa(t_\alpha, t_\beta). \end{aligned}$$

Examples: $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ where

Todo: Insert clip image h1, h2

$$\begin{aligned} \alpha_1 : \mathfrak{h} &\to \mathbb{C} \\ h_1 \mapsto 2h_2 &\mapsto -1. \end{aligned}$$

$$\begin{aligned} \alpha_2 : \mathfrak{h} &\to \mathbb{C} \\ h_1 \mapsto -1h_2 &\mapsto 2. \end{aligned}$$

To find t_{α_i}, we need to look at $\kappa\mid_{\mathfrak{h}}$.

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Since we only need the trace, this suffice, and we find

$$\left.\begin{array}{c|cc} h_1 & h_2 & \\ h_1 & 12 & -6 \\ h_2 & -6 & 12 \end{array}\right].$$

We then get $t_{\alpha_1} = \frac{h_1}{6}$ and $t_{\alpha_2} = \frac{h_2}{6}$. Moreover

$$
\begin{aligned}
\langle \alpha_1,\ \alpha_1 \rangle = \kappa(t_{\alpha_1}, t_{\alpha_1}) = \frac{1}{3} \in \mathbb{Q} \\
\langle \alpha_1,\ \alpha_1 \rangle = \frac{1}{3} \\
\langle \alpha_1,\ \alpha_2 \rangle = -\frac{1}{6} \\
\langle \alpha_1,\ \alpha_2 \vee \rangle = \frac{2\langle \alpha_1,\ \alpha_2 \rangle}{\langle \alpha_2,\ \alpha_2 \rangle} = -1 \in \mathbb{Z} \langle \alpha_i,\ \alpha_i \vee \rangle = \frac{2\langle \alpha_i,\ \alpha_i \rangle}{\langle \alpha_i,\ \alpha_i \rangle} = 2 \in \mathbb{Z}.
\end{aligned}
$$

This leads to a nice fact: the matrix $\langle \alpha_i,\ \alpha_j \vee \rangle$ has $\mathbb{Z}$ entries, and this is called the *Cartan matrix*.

20.1 Ch.3: Root Systems

20.1.1 Axiomatics: Reflections

Fix a Euclidean space E.

Definition: A *hyperplane* in E is a subspace of codimension 1. A *reflection* in E is an element $s \in \mathfrak{gl}(E)$ such that

$$
\{E^s \coloneqq \{x \in E \ni sx = s\} \text{ is a hyperplane } H \text{ and } s(x) = -x \quad \forall x \in E \ni (x, H) = 0\}
$$

For nonzero $\alpha \in E$, its reflection is

$$
\begin{aligned}
S_\alpha : E \to E \\
\beta \mapsto \beta - \langle \beta,\ \alpha \vee \rangle \alpha.
\end{aligned}
$$

with respect to $H_\alpha = \{x \in E \ni \langle x,\ \alpha \rangle = 0\}$, where $\alpha \vee = \frac{2\alpha}{\langle \alpha,\ \alpha \rangle}$.

Lemma: Let $\Phi \subseteq E$ be finite such that $S_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

Suppose that $S \in \mathfrak{gl}(E)$ satisfies

1. $S(\Phi) = \Phi$,
2. $S(h) = h$ for all $h \in H$, and
3. $S(\alpha) = -\alpha$ for some $\alpha \in \Phi$,

then $S = S_\alpha$, i.e. this uniquely characterizes S

Proof:

Let $\tau = S \circ S_\alpha$. Then $\tau(\Phi) = \Phi$ and $\tau(\alpha) = \alpha$. This $\tau \curvearrowright \mathbb{R}\alpha$ by 1, and similarly $\tau \curvearrowright E/\mathbb{R}\alpha$ by 1 by picking a representative in H. Moreover, all eigenvalues of τ are 1. So the minimal polynomial of τ divides $(t-1)^{\dim E}$.

We want to show that $\tau \mathrel{\Big|} (t-1)^N$ for some large N, which forces $\tau \mathrel{\Big|} \gcd((t-1)^{\dim E}, t^N - 1) = 1$. For any $\beta \in \Phi$ and $k > |\Phi|$, not all vectors $\beta, \tau(\beta), \cdots \tau^k(\beta)$. So $\beta = \tau^{k_\beta}(\beta)$ for some k_β depending on β (noting that τ is invertible.)

Multiplying all of these k_βs together, we can get some k_Φ that is larger than $|\Phi|$, and so $\beta = \tau^{k_\Phi}$ for *all* $\beta \in \Phi$. But then $\tau^{k_\Phi} = 1$ in $\mathfrak{gl}(E)$.

20.1.2 Root Systems

Definition: A subset Φ of E a Euclidean space is called a *root system* iff

1. $|\Phi| < \infty, 0 \notin \Phi$, and $E = \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$
2. $\alpha \in \phi \implies \mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\alpha \in \Phi \implies S_\alpha(\Phi) = \Phi$
4. $\alpha, \beta \in \Phi \implies \langle \beta,\ \alpha \vee \rangle \in \mathbb{Z}$.

Definition: The *rank* of a root system is the dimension on E.

Definition: The *Weyl Group* of Φ is defined as

$$W = \langle S_\alpha \mid \alpha \in \Phi \rangle \subseteq \mathfrak{gl}(E)$$

Note that $W \hookrightarrow \Sigma_{|\Phi|}$, a permutation group of size $|\Phi|$.

Lemma: If $g \in \mathfrak{gl}(E)$ and $g(\Phi) = \Phi$, then for all $\alpha, \beta \in \Phi$, we have

$$\begin{aligned} g s_\alpha g^{-1} &= s_{g(\alpha)}, \\ \langle \beta,\ \alpha \vee \rangle &= \langle g(\beta),\ g(\alpha) \vee \rangle, \\ \langle \beta,\ \alpha \vee \rangle &= \langle w(\beta),\ w(\alpha) \vee \rangle \quad \forall w \in W. \end{aligned}$$

Proof: Check 1-3 in Lemma 9.1.

Proof of 1: We have

$$g s_\alpha g^{-1}(g(\beta)) = g s_\alpha(\beta) \in g(\Phi) = \Phi \quad \forall \beta \in \Phi,$$
$$.$$

Proof of 2: We have

$$\{g(\beta) \ni \beta \in \Phi\} = \Phi \implies g s_\alpha g^{-1}(\Phi) = \Phi \quad \forall h \in g H_\alpha$$
$$.$$

and so $g s_\alpha g^{-1}(h) = g g^{-1}(h) = h$, so h is a fixed point of this map.

Proof of 3: We have $g s_\alpha g^{-1}(g(\alpha) = g s_\alpha(\alpha) = -g(\alpha)$, and so $g s_\alpha g^{-1} = s_{g(\alpha)})$ by Lemma 9.1.

Finally, we have

$$\begin{aligned} g s_\alpha g^{-1}(g(\beta)) = g(s_\alpha(\beta)) = g(\beta - \langle \beta,\ \alpha \vee \rangle \alpha) = g(\beta) - \langle \beta,\ \alpha \vee \rangle g(\alpha) \\ = \\ s_{g(\alpha)} = g(\beta) - \langle g(\beta),\ g(\alpha) \vee \rangle g(\alpha). \end{aligned}$$

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Recall from last time:

1. $|\Phi| < \infty$ and Φ spans E, where $0 \notin \Phi$
2. If $\alpha \in \Phi$, then $C\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\alpha \in \Phi$, then $S_\alpha(\Phi) = \Phi$.
4. If $\alpha, \beta \in \Phi$, then $\langle \beta,\ \alpha\vee \rangle \in \mathbb{Z}$ where $(E, \langle \cdot,\ \cdot \rangle)$is Euclidean and

$$\begin{aligned} S_\alpha : E \to E \\ \beta \mapsto \beta - \langle \beta,\ \alpha\vee \rangle \alpha, \quad \alpha\vee = \frac{2}{\langle \alpha,\ \alpha \rangle}\alpha. \end{aligned}$$

Examples:

In Rank 1:

1. Prop 2 implies $\Phi = \{\pm\alpha\}$
2. Prop 1 implies $E = \mathbb{R}\alpha$
3. Prop 3: $S_\alpha(\alpha) = -\alpha$
4. Prop 4 implies $\langle \pm\alpha,\ \pm\alpha \rangle = \pm\frac{2\langle \alpha,\ \alpha \rangle}{\langle \alpha,\ \alpha \rangle} = \pm 2$

Rank 1 Diagram: Todo: Insert phone image

In Rank 2: Todo: Insert phone image

Exercise:

- Show that $\mathrm{ord}(S_\alpha, S_\beta) = 2, 3, 4, 6$ for types $A_1 \times A_1, B_2, G_2$.
- Show that $W(A_2) \cong \mathbb{Z}_3$ and $W(B_2) \cong D_8$.

21.1 Pairs of Roots

Lemma: Let $\alpha, \beta \in \Phi$ where $\beta \neq \pm\alpha$, then

1. $\langle \alpha,\ \beta\vee \rangle \langle \beta,\ \alpha\vee \rangle \in \{0, 1, 2, 3\}$ Moreover, assuming $|\beta| \geq |\alpha|$, we have the following table Todo: Insert table

2. If $\langle \alpha,\ \beta \rangle > 0$, then $\alpha - \beta \in \Phi$. Similarly, if $\langle \alpha,\ \beta \rangle > 0$, then $\alpha + \beta \in \Phi$.

3. Any root string is unbroken and has length greater than 4.

Proof of (1):

By the Law of Cosines, we can write $x \coloneqq \langle \beta,\ \alpha\vee \rangle \langle \alpha,\ \beta\vee \rangle = 4\cos^2(\theta) \in \mathbb{Z}$. This restricts the possibilities to $x \leq 4$. But $x = 4 \iff \alpha = c\beta$, i.e. $\theta = 0$, but we are assuming that $\alpha \neq \pm\beta$, so this can not happen.

Proof of (2):

Since $\langle \alpha,\ \beta \rangle > 0$ and $|\beta| \geq |\alpha|$, then $\langle \alpha,\ \beta\vee \rangle = 1$. But then $S_\beta(\alpha) = \alpha - \langle \alpha, \beta\vee,\ \beta \rangle \in \Phi$ by Prop 3. So this is equal to $\alpha - \beta$.

A similar argument works for $|\beta| \leq |\alpha|$.

Proof of (3): Let p, q be the largest integers such that $b-p\alpha, b+q\alpha \in \Phi$ respectively. Suppose that the root stream between these two is broken somewhere, say $\beta + s\alpha \in \Phi$ and $\beta + (s+1)\alpha \notin \Phi$ by counting up from $\beta - p\alpha$. Similarly, there is some t counting down from $b + q\alpha$ then $\beta + t\alpha \in \Phi$ but $\beta + (t-1)\alpha \notin \Phi$. In particular, $s < t$. From (2), we have $\langle \alpha,\ \beta + s\alpha \rangle \geq 0$, $\langle \alpha,\ \beta + t\alpha \rangle \leq 0$.

We have

$$\langle \alpha,\ \beta \rangle + t\langle \alpha,\ \alpha \rangle = \langle \alpha,\ \beta + t\alpha \rangle \leq 0 \leq \langle \alpha,\ beta + s\alpha \rangle = \langle \alpha,\ \beta \rangle + s\langle \alpha,\ \alpha \rangle$$

where we know that $\langle \alpha,\ \alpha \rangle > 0$.

Since $S_\alpha(\Phi) = \Phi$ and these $S_\alpha(\beta + i\alpha) = \beta - \mathbb{Z}\alpha$, we find that reflections permute the root string. We then find that $p = \langle \beta,\ \alpha\vee \rangle + q$, and so $\langle \beta,\ \alpha\vee \rangle = p - q \in [-3, 3]$.

21.2 Chapter 10: Simple Roots and Weyl Groups

Definition: A *base* of a root system Φ is a subset $\Pi \subseteq \Phi$ such that

1. Π is a basis for the underlying vector space E, and
2. Each $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Pi} \kappa_\alpha^\beta \alpha$ where all of the coefficients κ_α^β all have the same sign.

The roots in Π are called *simple*. A root β is *positive* (resp. *negative*) if the $\kappa_\alpha^\beta \geq 0$ for all $\beta in \Phi^+$ (resp ≤ 0 in Φ^-). The *height* of a β is the sum of the coefficients. Π defines a partial order on E where $\mu \leq \lambda \iff \lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha$.

Note that this is defined on the roots themselves, and can then be extended to all of E.

Todo: Insert phone image

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Last time:

Lemma 10.2

a. ?
b. $\alpha \in \Pi \implies S_\alpha \curvearrowright \Phi^+ \setminus \{\alpha\}$ by permutation
c. $\alpha_i \in \Pi$ and $S_{\alpha_1}, \cdots, S_{\alpha_{j-1}}(\alpha_j) \in \Phi^-$ then $S_{\alpha_1} \cdots S_{\alpha_j} = S_{\alpha_1} \cdots S_{alpha_{t-1}} \cdots S_{\alpha_{j-1}}$ for some t, where the former has j terms and the latter has $j-2$ terms.

Proof of (a): ?

Proof of (b):

Suppose towards a contradiction that $w(\alpha_j) \in \Phi^+$. Then consider $WS(\alpha_j) = -W(\alpha_j) \in \Phi^-$.

By Lemma 10.2(c), we have $W = S_{\alpha_1} \cdots S_{alpha_{t-1}} S_{\alpha_{t+1}} \cdots S_{\alpha_{j-1}} S_{\alpha_j}$, where this is $j-1$ terms. So $w = S_\alpha \cdots S_{\alpha_j}$ is not reduced.

22.1 Weyl Groups

Recall that the *chambers* are given by the connected component of $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$.

Theorem: Fix Π of Φ. Then

a. $W \curvearrowright \{\text{chambers}\}$ transitively
b. $W \curvearrowright \{\text{bases}\}$ transitively
c. $\forall \alpha\Phi,\ \exists w \in W \ni w(\alpha) \in Pi$
d. $W := \{S_\alpha \ni \alpha \in \Phi\} = \langle S_\alpha \mid \alpha \in \Pi\rangle := W_0$
e. $W \curvearrowright \{\text{bases}\}$ simply transitively, i.e. $w(\Pi) = \Pi \implies w = e$.

I.e. we can describe the Weyl group using only simple reflections

Proof: We will prove (a) – (c)$ for $W_0#.

Proof of (a): Recall the fundamental chamber, $C(\Pi) = \{x \in E \ni (x, a) > 0\ \forall \alpha \in \Pi\}$. We want to show that any chamber C is equal to $wC(\Pi)$.

Pick $\gamma \in C$ and $g \in W_0$ such that $(g(\gamma), \rho) = \max\{(w(\gamma), \rho) \ni w \in W_0\}$, which exists because W_0 is a finite group.

For all $\alpha \in \Pi, S_\alpha g \in W_0$ and so by maximality we have

$$\begin{aligned}(g(\gamma), \rho) &\geq (s_\alpha g(\gamma)\rho)\\ &= (g(\gamma), S_\alpha(\rho))\\ &= (g(\gamma), \rho - \alpha)\\ &= (g(\gamma), \rho) - (g(\gamma), \alpha).\end{aligned}$$

and so $(g(\gamma), 0) \geq 0$, because this can never be an equality since $\gamma \in C$. Thus $g(\gamma) \in C(\Pi)$.

Proof of (b):

This holds because there a correspondence between $\{C(\Pi)\} \iff \{\text{bases}\Pi\}$.

Proof of (c):

It suffices to show that $\alpha \in \Phi$ lies in some base $\Pi' = W(\Pi)$. Note that $\beta \neq \alpha \implies H_\beta \neq H_\alpha$, and so we can pick a $\gamma \in H_\alpha \bigcap H_\beta^c$ for every $\beta \in \Phi \setminus \pm\alpha$. Since $\langle \gamma,\ \alpha\rangle = 0$ but $\langle \gamma,\ \beta\rangle \neq 0$ for all $\beta \neq \pm\alpha$, we can choose some $\varepsilon > 0$ such that $|\langle \gamma',\ \beta\rangle| > \varepsilon$ for every $\beta \neq \pm\alpha$. Then $\gamma' \in C(\Pi')$ and thus $\alpha \in \Pi'$.

Proof of (d):

By definition, $W_0 \subseteq W$, so we need to show the reverse containment. For all $\alpha \in Phi$, we want to show $S_\alpha \in W_0$. By (c), there exists a $w \in W_0$ such that $w(\alpha) := \beta \in \Pi$, Then $S_\beta = S_{w(\alpha)} = ws_\alpha w^{-1}$. So $S_\alpha = w^{-1}S_\beta w$, where each term is in W_0, so the whole thing is in W_0 as well.

Proof of (e):

Suppose $W(\Pi) = \Pi$. Let $W = S_{\alpha_1} \cdots S_{\alpha_\ell}$ be a reduced expression, which exists by (d). By corollary 10.2b, we have $W(\alpha_\ell \in \Phi^-)$. But this forces $w = e$. $\square$

Remarks:

By (d), there is a well-defined notion of *length* for $w \in W$. We will now show that $\ell(w) = n(w) := \# N_w := \# \{\alpha \in \Phi^+ \ni W(\alpha) \in \Phi^-\}$, i.e. the number of roots that get sent to a negative root.

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Last time:

We have the Weyl group $W := \{S_\alpha \ni \alpha \in \Phi\} = \{S_\alpha \ni S_\alpha \in \Pi\}$. If $W \ni w = \prod i = 1^\ell W_{\alpha_i}$ is a product of simple reflections, then W is said to be *reduced* if ℓ is the smallest among all such products. Call $\ell(w)$ the length of W and let $n(W) = \# N_W$. By Corollary 10.2b, $N_W = \{\alpha \in \Phi^+ \ni W(\alpha) \in \Phi^-\}$, and if $W = \prod S_{\alpha_i}$ is reduced, then $w(\alpha_j) \in \Phi^-$.

Lemma: $\ell(w) = n(w)$.

Proof: Done in class, but see Humphrey's.

23.1 Classification

23.1.1 Cartan Matrix

Fix a base $\Pi \subset \Phi$ of rank ℓ.

Definition: Fix an order $(\alpha_1, \alpha_2, \cdots \alpha_\ell$ of Π. Then the *Cartan matrix* is given by $A_{ij} = \langle \alpha_i,\ \alpha_j \vee \rangle \in \mathrm{Mat}(\ell \times \ell, \mathbb{Z})$.

Examples:

Facts:

a. A depends on the chosen ordering of Π.
b. A is independent of the choice of Π.
c. A is invertible.
d. A uniquely determines the root system (up to isomorphism). I.e., if $A(\Phi) = A(\Phi')$ then there is an isomorphism $E \xrightarrow{\phi} E$ on the underlying Euclidean space such that $phi(\Phi) = \Phi'$ and $\langle \alpha,\ \beta \vee \rangle = \langle \phi(\alpha),\ \phi(\beta) \vee \rangle$ for all $\alpha, \beta \in \Phi$.

23.1.2 Dynkin Diagrams

Recall from Lemma 9.4 that $a_{ij} a_{ji} \in \{0, 1, 2, 3\}$.

Definition: Given a Cartan matrix A, its Coxeter diagram is an undirected multigraph $\Gamma = (I, E)$ where I is a vertex set and the edge set is given by edges between vertices corresponding to i, j (where $i \neq j$) with weight $a_{ij} a_{ji}$.

Examples:

Note that these diagrams don't encode which roots are longer, so we can decorate these diagrams with arrows to indicate this and obtain a partially-directed multigraph.

Definition: A *Dynkin diagram* is the partially-directed multigraph obtained from the Coxeter diagram by adding arrows on the double or triple edges between i, j precisely when $|a_i| > |a_j|$. (Note that this also occurs when $|a|_{ij} < |a|_{ji}$)

Definition: A non-empty root system is *irreducible* if $\Phi \neq \Phi_1 \oplus \Phi_2$ for some nonempty root system Φ_2 where $\alpha \in \Phi_1, \beta \in \Phi_2 \implies \langle \alpha,\ \beta \rangle = 0$.

For example: $\Phi(A_1 \times A_1)$ can be written as $\Phi(A_1) \oplus \phi(A_1)$ since the off-diagonal entries were zero, so it is reducible.

Facts:

a. Φ is irreducible iff the Dynkin diagram is connected
b. Φ can be uniquely written as the union of irreducible root systems (where the multiplicity of each system appearing is well-defined)

Thus to classify root systems, it suffices to classify connected Dynkin diagrams.

Examples of Dynkin diagrams:

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Recall from last time the Dynkin diagrams. If Φ is irreducible, then its diagram is one of the following:

Definition: A subset $A = \{v_1, \cdots, v_n\} \subseteq E$ is *admissible* iff

1. A is linearly independent.
2. $\langle v_i,\ v_i \rangle = 1$ for all i, and $\langle v_i,\ v_j \rangle \leq 0$ if $i \neq j$.
3. $s_{ij} = 4\langle v_i,\ v_j \rangle^2 \in \{0, 1, 2, 3\}$ if $i \neq j$.

Define a graph $\Gamma_A = (V_A, E_A)$ where $V_A = A$ and $E_A = \{s_{ij} \ni i \neq j\}$. If $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subseteq \Phi$ is a base, then $A := \left\{ v_i = \frac{\alpha_i}{\sqrt{\langle \alpha_i,\ \alpha_i \rangle}} \right\}$.

Lemma:

a. If A is admissible, then $\# \left\{ (v_i, v_j) \in E_A \ni 4\langle v_i,\ v_j \rangle^2 \neq 0 \right\} \leq |A| - 1$, and Γ_A contains no graph cycles.
b. $\deg V_i \leq 3$ for all i.
c. If Γ_A contains a path $p_1 \to \cdots \to p_t$, then $A' := \{p\} \bigcup A \setminus \{p_1, \cdots, p_t\}$ where $p := \sum p_i$. Moreover, $\Gamma_{A'}$ is obtained from Γ_A by contracting this path onto p.

Proof of theorem:

Assume the lemma holds. Let Γ be the Coxeter diagram of Φ; then Φ is connected.

Case 1: Γ has a triple edge. But then both vertices on this edge have degree 3, so this is the maximal number of edges between them. But since Γ must be connected, this is everything.

Case 2: Γ has no triple edges but some double edge. We will first show that Γ has only one double edge.

Suppose otherwise; then Γ has at least two double edges occurring. Without loss of generality (e.g. by taking a subgraph), these are connected by a path of single edges. By the lemma, we can contract this path to get an admissible subset. But then there is a vertex of degree 4, contracting $\deg V_i \leq 3$ for all i.

Now we'll show that Γ has no branching point, i.e. a vertex of degree exactly 3. If this occurs, then a double edge is connected to such a vertex by a path. Contracting this path yields a vertex of degree 4, again a contradiction.

By these two statements, Γ has the general form:

$$\Gamma = v_1 \to \circ \to \cdots \to v_p \to\to w_q \to \circ \to \cdots \to w_1.$$

Let $v = \sum iv_i$ and $w = \sum iw_i$, then $\langle v,\ v\rangle = \frac{1}{2}p(p+1)$, and $\langle w,\ w\rangle = \frac{1}{2}q(q+1)$. Note that $\langle v_i,\ w_j\rangle = -1/\sqrt{2}$ if $i = p$ and $j = q$, and 0 otherwise.

Thus $\langle v,\ w\rangle = \cdots = \frac{1}{2}p^2q^2$. By Cauchy-Schwarz, this is strictly less than $\langle v,\ v\rangle\langle w,\ w\rangle = \frac{1}{4}p(p+1)q(q+1)$. We then obtain $(p-1)(q-1) < 2$. Supposing wlog that $p \geq q$, we have either $p = q = 2$, in which case we get $\circ \to \circ \to\to \circ \to \circ$. Otherwise $q = 1$, and we get $\circ \to \cdots \to \circ \to\to \circ$.

Case 3: Γ has only single edges. We want to show Γ has only one branching point, i.e. a vertex of degree 3. If it has 2, we can contract the intermediate path to get a vertex of degree 4. So we have the following situation:

Define $x = \sum ix_i, y = \sum iy_i, w = \sum iw_i$, and $\hat{w}, \hat{x}, \hat{y}$ to be their normalization. Then $B = \{b_i\} := \{\hat{w}, \hat{y}, \hat{y}, z\}$ is orthonormal and linearly independent, so we can apply Gram-Schmidt. This yields a $z' \neq 0$ such that

$$z = \sum \left\langle z,\ \hat{b}_i \right\rangle \hat{b}_i$$

In particular, $\langle z,\ z'\rangle z' \neq 0z'$, otherwise z is a linear combination of the x_i, y_i, w_i. Thus $\langle z,\ \hat{w}\rangle^2 + \langle z,\ \hat{x}\rangle^2 + \langle z,\ \hat{y}\rangle^2 > 1$. We can compute $\langle z,\ \hat{w}\rangle = \frac{-q/2}{\sqrt{\frac{1}{2}q(q+1)}}$, and so $\langle z,\ \hat{w}\rangle^2 = \frac{q}{2(q+1)}$.

From this, we can obtain $\frac{1}{q+1} + \frac{1}{r+1} + \frac{1}{p+1} > 1$. We can assume $p \geq q \geq r \geq 1$, since these correspond to the lengths of paths in the above image. This allows us to do some case-by-case analysis.

Using this, we find $\frac{3}{r+1} > 1$, and so $r = 1$ must hold. Similarly, $\frac{2}{q+1} > \frac{1}{2}$, which forces $q =\in \{1, 2\}$.

Supposing $r = q = 1$, then we get type D_ℓ because p can be anything. Supposing otherwise that $r = 1, q = 2, p \in \{2, 3, 4\}$, we get type E.

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Last time:

Theorem: If Φ is irreducible, then the Dynkin diagram is given by $A - G$.

Definition: A subset $A = \{v_1, \cdots, v_n\}$ is *admissible* if

1. A is a linearly independent set,
2. $\langle v_i,\ v_i\rangle = 1$ for all i, and $\langle v_i,\ v_j\rangle \leq 0$ if $n \neq 0$.
3. $4\langle v_i,\ v_j\rangle^2 \in \{0, 1, 2, 3\}$ if $i \neq j$.

Thus the graph $\Gamma_A = (V_A, E_A)$ is given by $V_A = A$ and $E_A = \left\{ v_i \xrightarrow{4\langle v_i,\ v_j\rangle^2} v_j \mid i \neq j \right\}$.

Lemma:

a. If A is admissible, then the number of edges such that $4\langle v_i,\ v_j\rangle \neq 0$ is at most $|A| - 1$.
b. For every i, we have $\deg v_i \leq 3$.
c. If Γ_A contains a straight path of length t, then the graph Γ' obtained by contracting this path is also admissible.

Let p be the point obtained by contracting such a path.

Proof of (a): If $\{p_1, \cdots, p_t\}$ are linearly independent, then $p \neq 0$. Thus by positive-definiteness, we have $0 <_{pd} \langle p,\ p\rangle =_{\#2} t + \sum_{i<j} 2\langle p_i,\ p_j\rangle$. Then $t > \sum_{i<j}(-2)\langle p_i,\ p_j\rangle = \sum_{i<j}\sqrt{4\langle p_i,\ p_j\rangle^2}$, where the quantity in the square root is the number of edges, which is thus greater than or equal to the number of pairs connected.

Proof of (b): Fix i. Let $u_1 \cdots u_k$ be the vertices in A that are connected to v_i by a single edge. Then by (a), we have $\langle u_i,\ u_j\rangle = 0$ for all $i \neq j$.

Then the set $\{u_1, \cdots, u_k\}$ is an orthonormal basis for their span. Applying Gram-Schmidt, we can write each $v_i = \sum_{j=0}^{k} \langle v_i,\ u_j\rangle u_j$, where we pick u_0 such that the new set $\{u_0\} \bigcup \{u_1, \cdots, u_k\}$. Then $\langle v_i,\ u_0\rangle \neq 0$ for all i; otherwise we would have $\{u_1, \cdots, u_k, v_i\}$ would be linearly dependent, since $v_i = \sum c_i u_i$ from above, which contradicts our initial axiom/assumption. Then $1 = \langle v_i,\ v_i\rangle$ by A2, which equals $\sum_{j=0}^{k} \langle v_i,\ u_j\rangle^2 = \langle v_i,\ u_0\rangle^2 + \sum_{j=1}^{k} \langle v_i,\ u_j\rangle^2$, where the first term is strictly positive.

But then $1 > \sum_{j=1}^{k} \langle v_i,\ v_j\rangle^2 \geq \frac{k}{4}$ by A3, which then forces $k = \deg v_i \leq 3$.

Proof of (c): The conditions of A1 are satisfied. For A2, we have

$$\langle p_i,\ p_j\rangle = \begin{cases} -\frac{1}{2} & |i-j| = 1 \\ 0 & |i-j| > 1 \\ 1 & i = j. \end{cases}$$

We then have $\langle p,\ p\rangle = t + 2\sum i < j\langle p_i,\ p_j\rangle = t + 2\sum_{i=1}^{t-1} \langle p_i,\ p_{i+1}\rangle = 1$. Thus $\langle p,\ v_i\rangle = \sum_{j=1}^{t} \langle p_j,\ v_i\rangle \leq 0$.

For A3, fix $v_i \in A'$. Then v_i is connected (by a single edge) to at most one point p_j, otherwise there would be a cycle. Thus

$$\langle v_i,\ p\rangle = \begin{cases} \langle v_i,\ p_j\rangle & \text{if } v_i \text{ is connected to } p_j \\ 0 & \text{else.} \end{cases}$$

We thus have $4\langle v_i,\ p\rangle^2 = 4\langle v_i,\ p_j\rangle \in \{0, 1, 2, 3\}\, 1 v_i \sim p_j$.

25.1 Construction of Root Systems and Automorphisms

We'll start with the construction of types $A - G$.

Theorem: For Dynkin diagrams of type $A - G$, there exists an irreducible root system having the given diagram.

Proof: By explicit construction. Fix an orthonormal basis $\{\varepsilon_i\}$.

Type A_ℓ: Let

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq \ell + 1\}$$

Then $|\Phi| = \ell^2 + \ell$, and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell\}$. We then find that $\dim \mathfrak{g} = \ell^2 + 2\ell$.

Note that we don't know anything about $\mathfrak{g}$ yet, but already know its dimension.

Example: A_2. We have $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_3 - \varepsilon_2\}$. Then $A = (a_{ij})$ with $a_{ij} = \langle a_i,\ a_j \vee \rangle$, and $\alpha_1 \vee = \frac{2\alpha_1}{\langle \alpha_1,\ \alpha_1 \rangle} = \frac{2(\varepsilon_1 - \varepsilon_2)}{\langle \varepsilon_1 - \varepsilon_2,\ \varepsilon_1 - \varepsilon_2 \rangle} = \varepsilon_1 - \varepsilon_2 = \alpha_1$. Doing the computations, it turns out that $\langle \alpha_1,\ \alpha_2 \vee \rangle = -1$, $\langle \alpha_2,\ \alpha_1 \vee \rangle = -1$, and $\langle \alpha_i,\ \alpha_i \vee \rangle = 2$.

Thus $A = [2, -1; -1, 2]$, which has Dynkin diagram given by:

Type B_ℓ: Recall that these have one "short root":

Then $\Phi = \{\pm\varepsilon_j, \pm\varepsilon_j \mid 1 \leq i \neq j \leq \ell\} \bigcup \{\pm\varepsilon_i \mid 1 \leq i \leq \ell\}$, and we have $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i-1} \mid 1 \leq i \leq \ell - 1\} \bigcup \{\alpha$

After carrying out the computation, we have the following Cartan matrix:

And $\dim \mathfrak{g} = 2\ell^2 + \ell$, since $|\Phi| = 2\ell(\ell - 1) + 2\ell = 2\ell^2$.

Type D_ℓ:

We obtain $\Phi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell - 1\} \bigcup \{\alpha_\ell \coloneqq \varepsilon_{\ell-1} + \varepsilon_\ell\}$. We then find $\langle \alpha_{\ell-1},\ \alpha_\ell \vee \rangle = 0$ and $\langle \alpha_{\ell-2},\ \alpha_\ell \vee \rangle = -1$.

Type E_ℓ: We have $\Pi(E_\ell) = \Pi(D_{\ell-1}) \bigcup \left\{ \alpha_\ell \coloneqq -\frac{1}{2} \sum_{i=1}^{8} \varepsilon_i \right\}$.

This yields $|\Phi| = 72, 126, 240$ and $\dim \mathfrak{g} = 78, 133, 248$, corresponding to $\ell = 6, 7, 8$.

More results on exceptional Lie Algebras:

26 Wednesday October 16

Todo

27 Friday October 18

Todo

28 Monday October 21

28.1 Chapter 5: Existence Theorem

28.1.1 Universal Enveloping Algebra (UAE)

Some applications/motivations for UAEs:

1. Groups G are to group algebras $F[G]$ as Lie algebras $\mathfrak{g}$ are to UAE $U(\mathfrak{g})$. Any $\mathfrak{g}$-module then becomes a module over a ring, so the general theory applies.

2. PBW theorem: this yields a concrete F-basis of $U(\mathfrak{g})$. There is a triangular decomposition $U(\mathfrak{g}) = U(h) \otimes U(f) \otimes U(n)$. This allows constructing the Vermo module (and hence irreducible modules) for $\mathfrak{g}$, allowing for a description of BGG Category $\mathcal{O}$.
3. Harish-Chandra theorem: $Z)U(\mathfrak{g})) = S(\mathfrak{g})^W$. This characterizes central characters $\chi : Z(U(\mathfrak{g})) \to F$, which further allows describing the blocks of $\mathcal{O}$, i.e. when two irreducible modules have non-trivial extensions.
4. $U(\mathfrak{g})$ deforms to a quantum group $U_q(\mathfrak{g})$.

28.1.2 Tensor and Symmetric Algebras

Definition: For V a f.d. vector space, the *tensor algebra* over V is given by $T(V) = \bigoplus_{n\in\mathbb{N}} T^n(V)$ where $T^n(V) = \otimes_{i=1}^n V$ with an associative multiplication $T^a \times T^b \to T^{a+b}$ given by $(\otimes_{i=1}^a v_i, \otimes_{i=1}^b w_i) \mapsto \otimes_{i=1}^a v_i \otimes \otimes_{i=1}^b w_i$.

The tensor algebra satisfies a universal property: given any F-linear map $\phi : V \to A$. (See phone image)

Definition: The symmetric algebra on V is given by $S(V) = T(V)/I$ where $I = \langle x \otimes y - y \otimes x \rangle \trianglelefteq T(\mathfrak{g})$.

Some facts:

a. There is a natural grading $S(V) = \bigoplus_{n\in\mathbb{N}} S^n(V)$ where $S^0(V) = F, S^1(V) = V, S^n(V) = T^n(V)/(I \bigcap T^n V)$,
b. If $\{x_i\}^n$ is a basis of V, then $S(V) \cong F[x_1, \cdots, x_n]$.

28.1.3 Construction of UEA

Definition: For $\mathfrak{g}$ a lie algebra, define $U(\mathfrak{g}) = T(\mathfrak{g})/J$ where $J = \langle x \otimes y - y \otimes x - [x, y] \rangle \trianglelefteq T(\mathfrak{g})$.

Thus we have the following type of equation that holds in $U(\mathfrak{g})$:

$$\begin{aligned} v_1 \otimes \cdots \otimes v_a \otimes (x \otimes y) \otimes w_1 \otimes \cdots \otimes w_b = \\ v_1 \otimes \cdots \otimes v_a \otimes (y \otimes x) \otimes w_1 \otimes \cdots \otimes w_b + v_1 \otimes \cdots \otimes v_a \otimes ([x, y]) \otimes w_1 \otimes \cdots \otimes w_b. \end{aligned}$$

Proposition: $U(\mathfrak{g})$ has a universal property: given a lie algebra hom $\theta : \mathfrak{g} \to \mathcal{A}$ where $\mathcal{A}$ is any unital associative F-algebra with a lie bracket, there exists a unique $\psi : U(\mathfrak{g}) \to \mathcal{A}$ making the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\theta"'] & U(\mathfrak{g}) \arrow[d, dotted, "\exists\psi"] \\ & \mathcal{A} \end{tikzcd}$$

where $\iota : \mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ is given by $x \mapsto x + J$.

The upshot: There is a 1 to 1 correspondence

$$\left\{ \begin{array}{c} \text{Lie algebra} \\ \text{representations} \\ \mathfrak{g} \to \mathfrak{gl}(V) \end{array} \right\} \to \left\{ \begin{array}{c} \text{Algebras from} \\ U(\mathfrak{g}) \to \mathrm{End}(V) \end{array} \right\}$$

$$\theta \mapsto \psi$$

$$\theta = \psi \circ \iota \hookleftarrow \psi$$

Proof (existence):

$\theta : \mathfrak{g} \to \mathcal{A}$ extends to an algebra homomorphism $\tilde{\theta} : T(\mathfrak{g}) \to \mathcal{A}$ given by $\otimes_{i=1}^{n} x_i \mapsto \prod \theta(x_i)$. Note that $\tilde{\theta}(x \otimes y - y \otimes x - [x, y]) = \theta(x)\theta(y) - \theta(y)\theta(x) - \theta([x, y]) = 0$, and thus $J \trianglelefteq \ker \tilde{\theta}$ and $\phi : T(\mathfrak{g})/J \to \mathcal{A}$ is well-defined.

Uniqueness: Suppose that $\psi' : U(\mathfrak{g}) \to \mathcal{A}$ is another hom ψ' such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\theta"'] & U(\mathfrak{g}) \arrow[d, bend right, "\psi'"'] \arrow[d, bend left, "\psi"] \\ & \mathcal{A} \end{tikzcd}$$

Since $T(\mathfrak{g})$ is generated by $T^1(\mathfrak{g})$, $U(\mathfrak{g})$ is generated by $\iota(\mathfrak{g}) \in U(\mathfrak{g})$. Thus for all $x \in \mathfrak{g}$, $\psi \circ \iota(x) = \theta(x) = \psi' \circ \iota(x)$ by the commuting of each triangle. We then have $\psi = \psi'$ on $\iota(\mathfrak{g})$, and hence on $U(\mathfrak{g})$.

28.1.4 PBW Theorem

PBW: Poincaré-Birkhoff-Witt

Theorem: If $\mathfrak{g}$ has a basis $\{x_i\}_{i \in I}$ where $\leq$ is a total order on I, then let $y_i := \iota(x_i) \in U(\mathfrak{g})$. Then $U(\mathfrak{g})$ has an F-basis called a *PBW basis* which is given by

$$\left\{ y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \mid n \in \mathbb{N}, r_i \in \mathbb{N}, i_1 \leq \cdots \leq i_n \right\}.$$

We refer to each term appearing as a *PBW monomial*.

Examples:

Type A, $\mathfrak{g} = \mathfrak{sl}(2, F) = \langle f, h, e \rangle$. Pick an order $x_1 = f, x_2 = h, x_3 = e$, so $f < h < e$.

Then $U(\mathfrak{g})$ has a basis

$$B = \{1\} \bigcup \left\{ f^{r^1} \right\} \left\{ f^{r^1} h^{r_2} \right\} \bigcup \{ f^{r_1} h^{r_2} e^{r_3} \} \bigcup \{ h^{r_1} \} \bigcup \{ f^{r_1} e^{r_2} \} \bigcup \{ e^{r_1} \} \bigcup \{ h^{r_1} e^{r_2} \}.$$

i.e. $B = \left\{ f^a h^b e^c \ni a, b, c \in \mathbb{N} \right\}$.

If you pick a different order, say $f < e < h$, then we obtain $B = \left\{ f^a e^b h^c \ni a, b, c \in \mathbb{N} \right\}$.

29 Wednesday October 23

Recall from last time:

For $\mathfrak{g}$ a lie algebra, we define $T(\mathfrak{g})$ the tensor algebra, and the universal enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$ where $x \otimes y - y \otimes x \sim [x, y]$.

We also described the *PBW Theorem*, which provides a basis for $U(\mathfrak{g})$.

Proof of PBW Theorem:

We have $T(\mathfrak{g}) = \operatorname{span}\{x_{j_1} \otimes \cdots \otimes x_{j_k} \ni j_1, \cdots, j_k \in I\}$, where we note that there are not required to be ordered. Thus $U(\mathfrak{g}) = \operatorname{span}\{y_{j_1} \otimes \cdots \otimes y_{j_k} \ni j_1, \cdots, j_k \in I\}$, where which are again not required to be ordered. We would thus like to express every term here as some linear combination of monomials in the y_{i_j} with increasing indices. We proceed by inducting on k, the number of tensor factors occurring. The base case is clear.

For $k > 1$, supposing that the element is *not* a PBW monomial, then there is some inversion in the indices $(j_1, \cdots, j_k)$, i.e. there is at least one i such that $j_{i+1} < j_i$. Now for any two indices $a, b \in I$, we have

$$\iota(x_b \otimes x_a) = \iota(x_a \otimes x_b + [x_b, x_a]) \implies y_b y_a = y_a y_b + \iota([x_b, x_a])$$

Since $[x_b, x_a] = \sum_t F x_t$ and $\iota[x_b, x_a] = \sum_t F y_t$.

But then $y_{j_1} \cdots y_{j_k} = y_{i_1} y_{i_2} \cdots y_{j_k} +$ lower degree terms where $i_1 \leq i_2 \cdots i_k$ is a non-decreasing rearrangement of the j_i. By the inductive hypothesis, the lower degree terms are spanned by PBW monomials, so we're done.

Proof of linear independence:

Claim: let $\mathbf{x} := x_{j_1} \otimes \cdots \otimes x_{j_n}$ for an arbitrary indexing sequence, and $\mathbf{x}_{(k)}$ be this tensor with the j_k and j_{k+1} terms swapped, and $\mathbf{x}_{[k]}$ be this tensor with $x_{j_k}, x_{j_{k+1}}$ replaced by their bracket.

Then there exists a linear map

$$\begin{aligned} f : T(\mathfrak{g}) \to R := F[\{z_i\}_{i \in I}] \\ f(x_{i_1} \otimes \cdots \otimes x_{i_n}) = z_{i_1} \cdots z_{i_n} \\ f(\mathbf{x} - \mathbf{x}_{(k)}) = f(\mathbf{x}_{[k]}). \end{aligned}$$

By collecting terms, we can write

$$\mathbf{x} - \mathbf{x}_{(k)} - \mathbf{x}_{[k]} = x_{j_1} \otimes \cdots \otimes x_{j_{k-1}} \otimes \left((x_{j_k} \otimes x_{j_{k+1}}) - (x_{j_{k+1}} \otimes x_{j_k}) - [x_{j_k}, x_{j_{k+}}] \right) \otimes \cdots$$

So we can take J to be the ideal generated by all elements of this form, and we find that $J \subset \ker f$, and thus f descends to a map $\overline{f}$ on $U(\mathfrak{g})$. We then know that if $\overline{f}$ applied to any PBW monomial is $z_{i_1}^{r_1} \cdots z_{i_n}^{r_n}$, which are linearly independent in R, then any PBW monomial will be linearly independent in $U(\mathfrak{g})$.

Proof of claim:

For each $\mathbf{x}$, define an *index*

$$\lambda(\mathbf{x}) = \#\left\{(a,b) \in \{1,\cdots,n\}^2 \ \ni a < b, j_a < j_b\right\}.$$

Then

$$\{\mathbf{x} \ni \lambda(\mathbf{x}) = 0\} = \{x_{i_1} \otimes \cdots \otimes x_{i_n} \ \ni i_1 \leq \cdots \leq i_n\}.$$

So set $T^{n,k} = \{\mathbf{x} \in T^n(\mathfrak{g}) \ni \lambda(\mathbf{x}) \leq k\}$; we then have a filtration $T^{n,0} \hookrightarrow T^{n,1} \hookrightarrow \cdots \hookrightarrow T^n(\mathfrak{g})$.

Step 1: We'll construct f by induction on n.

For $n > 0$, set $f(\mathbf{x}) = z_{j_1} \cdots z_{j_n}$ if $\lambda(\mathbf{x}) = 0$. We now induct on the index k at a fixed power $n > 0$. The base case is clear.

For $k > 0$, there exists an inversion $(\ell, \ell+1)$, i.e. some indices $i_\ell >_{\ell+1}$. Set $f(\mathbf{x}) = f(\mathbf{x}_{(\ell)}) - f(\mathbf{x}_{[\ell]})$, where the LHS is in $T^{n,k}$ and the RHS terms are in $T^{n,k-1}$ and $T^{n-1}(\mathfrak{g}$ respectively.

Step 2: We'll check that f is well-defined.

In the above definition, note that $f(\mathbf{x})$ can be defined using different inversions of the indices, we'd like to show that these yield the same map.

Let $(\ell, \ell+1)$ and $(\ell', \ell'+1)$ be two distinct inversions. Then set

$$\begin{aligned} a &= x_{j_\ell} \\ b &= x_{j_{\ell+1}} \\ c &= x_{j'_\ell} \\ d &= x_{j_{\ell'+1}} \\ &. \end{aligned}$$

Then we have several cases:

Case 1: $\ell + 1 < \ell'$.

Then

$$\begin{aligned} f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\cdots b \otimes a \cdots c \otimes d \cdots) \\ +f(\cdots \otimes [a,b] \otimes \cdots c \otimes d \cdots) \\ &= f(\cdots b \otimes a \cdots d \otimes c \cdots) + f(\cdots b \otimes a \cdots [c,d] \cdots) + f(\cdots \otimes [a,b] \otimes \cdots d \otimes c \cdots \\ &= f(\mathbf{x}_{(\ell']}) + f(\mathbf{x}_{[\ell']}). \end{aligned}$$

Case 2: $\ell + 1 = \ell'$

Then

$$
\begin{aligned}
f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\cdots b \otimes a \otimes x) + f(\cdots [a,b] \otimes c) \\
&= f(b \otimes c \otimes a) + f(c \otimes [a,b]) + f(b \otimes [a,c]) + f([[a,b],c]) \\
&= f(c \otimes b \otimes a) + f(c \otimes [a,b]) + f(b \otimes [a,c]) + f([[a,b],c]) + f(b \otimes [a,c]) + f(a \otimes [b,c]) + f([[b, \\
&= f(\mathbf{x}_{(\ell')}) + f(\mathbf{x}_{[\ell']}).
\end{aligned}
$$

where the last equality is found by expanding the expression backwards.

30 Friday October 25

Theorem (PBW): The universal enveloping algebra $U(\mathfrak{g})$ has a basis consisting of the PBW monomials. If we fix a basis $\{x_i \ni i \in I\}$ of $\mathfrak{g}$ with a total order, then $\left\{y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \ni n \in \mathbb{N} > 0,\ i_j \in I,\ r_i \geq 1\right\}$.

We will construct a map

$$
\begin{aligned}
\iota : \mathfrak{g} &\to U(\mathfrak{g}) \\
x_i &\mapsto x_i + J \coloneqq y_i,
\end{aligned}
$$

where we can recall that $U(\mathfrak{g}) \coloneqq T(\mathfrak{g})/J$ where J was an ideal of specific relations.

Corollary:

a. The map ι is injective.
b. The map ι has no *zero divisors.*

We will use property (b) to study properties of Verma modules

Proof of (a): If $\sum c_i x_i \in \ker(\iota)$, then

$$
\begin{aligned}
0 = \iota(\textstyle\sum c_i x_i) &= \textstyle\sum c_i y_i \\
&\implies c_i = 0 \quad \forall i \text{ since } \{y_i\} \subsetneq \{\text{ PBW monomials }\} \\
&\implies \ker(\iota) = 0.
\end{aligned}
$$

Proof of (b): An arbitrary element in $U(\mathfrak{g})$ is of the form

$$
\begin{aligned}
a = \sum c_{\mathbf{i},\mathbf{r}}^a y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} &\text{ for some } c \in F \\
\coloneqq f_a(\mathbf{y}) + \text{ terms with smaller total degree }.
\end{aligned}
$$

where f is defined by picking out only those terms of highest total degree, e.g. $f(2y_1 + y_1y_2y_3 + y_2^2) = y_1y_2y_3$, which is of total degree 3.

We want to show that $a \neq 0$ and $b \neq 0$ then $ab \neq 0$, i.e. $(f_a(\mathbf{y}) + \cdots)(f_b(\mathbf{y}) + \cdots) \neq 0$.

Recall that $y_a y_b = y_b y_a + \sum_{a,b\in I}$ degree 1 monomials . Thus $f_a(\mathbf{y})(f_b(\mathbf{y})) \coloneqq f_a f_b(\mathbf{y}) + \sum$ terms of smaller total

Here we define $f_a(\mathbf{y})f_b(\mathbf{y})$ by e.g. if $b = y_2$, then $f_b(\mathbf{y}) = y_2$, and $f_a(\mathbf{y})f_b(\mathbf{y}) = y_1 y_2 y_3 y_2 = y_1 y_2^2 y_3 + y_1 y_2 [y_3, y_2]$. Note that the leading term is of total degree 4, and the remaining term is a sum of lower degree terms.

30.1 Free Lie Algebra

Let $X \coloneqq \{x_i \ni i \in I\}$ be a set. Define the *free associative algebra* $\mathcal{F}(X)$ as $\left\{\sum_k c_{\mathbf{i}} X_{\mathbf{i}} \mid \mathbf{i} = (i_1, \cdots, i_k) \in I^k,\ c_{\mathbf{i}} \in\right.$
Then the associated *free lie algebra* $\mathcal{FL}(x) = \bigcap_{\mathfrak{g}} \mathfrak{g}$ where $X \subseteq \mathfrak{g} \subseteq \mathcal{F}(X)$ is a containment of lie algebras.

Let $\iota : X \hookrightarrow \mathcal{FL}(X)$.

Proposition:

a. $\mathcal{FL}(X)$ satisfies a universal property – for any map $\theta : X \to \mathfrak{g}$ a lie algebra, there exists a unique ψ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\theta"'] & \mathcal{FL}(X) \arrow[d, dotted, "\exists!\psi"] \\ & \mathfrak{g} \end{tikzcd}$$

b. $U(\mathcal{FL}(X)) = \mathcal{F}(X)$.

> Upshot: we can define a Lie algebra $\mathfrak{g}$ using generators and relations, and define $\mathfrak{g} \coloneqq \mathcal{FL}(X)/(R)$ for some set of relations R.

30.2 Generators and Relations

Recall that we have a correspondence

$$\begin{aligned}
&\{\mathfrak{g} \ni \mathfrak{g} \text{ is a semisimple Lie Algebra }\} \\
&\qquad \iff \{\Phi,\ \text{root systems }\} \\
&\qquad \iff \{\text{Dynkin diagrams (Cartan Matrices)}\} \\
&(\mathfrak{g}, \mathfrak{h}) \to \Phi, \quad \{a_i\} \subseteq \{a\} \coloneqq \Pi \subseteq \Phi \qquad \mapsto A_{i,j} = \langle \alpha_i,\ \alpha_j \vee \rangle \\
&\mathfrak{g}(A) < -_{?}\Phi \qquad < -A.
\end{aligned}$$

We had an explicit construction to go from Dynkin diagrams to root systems, and an existence theorem of Serre's will take root systems Φ and produce semisimple Lie algebras from them. The question will be whether or not there is a one-to-one correspondence here, and that's what we'll spend the rest of the semester showing.

30.3 Cartan/Serre Relations

Recall from (8.3): For all $\alpha \in \Phi$, we have $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, then there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha \coloneqq \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$, where $t_\alpha \coloneqq \alpha = \kappa(t_\alpha, \cdot)$.

Fix $\Pi = \{\alpha_i \mid i \in I\}$, and write $h_i \coloneqq h_{\alpha_i}$, $e_i = e_{\alpha_i}$ for each i. Then $\alpha_i(h_j) = a_{ij}$. Now fix $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha}$ such that $[e_i, f_i] = h_i$ for every $i \in I$.

Proposition: $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i \mid i \in I\}$.

We have the Cartan relations for each $i, j \in I$:

$$\begin{aligned} [h_i, h_j] &= 0, & [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &a_{ji} e_j & [h_i, f_j] &= -a_{ji} f_j. \end{aligned}$$

as well as Serre relations for each $i \neq j$:

$$(\text{ad } e_i)^{1-a_{ji}}(e_j) = 0 \quad (ad f_i)^{1-a_{ji}}(f_j) = 0.$$

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \langle e_1 \coloneqq e, f_1 \coloneqq f, h_1 \coloneqq h \rangle$ satisfies $[h, e] = 2e$ and $[h, f] = -2f$, and since there are no higher order relation, there are no Serre relations. So we get $A = (2)$ as a matrix.

Example: $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ is of type C_2, and is generated by $\langle e_1, e_2, f_1, f_2, h_1, h_2 \rangle$ satisfying

- $[h_1, h_2] = 0$
- $[h_1, e_1] = 2e_1$
- $[h_1, e_2] = -2e_2$
- $\cdots$

Then e.g. we have $(\text{ad } e_1)^{1-a_{ij}}(e_2) = (\text{ad } e_1)^3(e_2) = 0$.

31 Monday October 28

31.1 Algebra Generated by a Cartan Matrix

Last time: The claim was that for a Cartan matrix A, there is a lie algebra $\mathfrak{g}(A)$ that is semisimple with CSA $\mathfrak{h}$ and a root system Φ that defines that Cartan matrix A.

The algebra $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i \mid i \in I = \{1, 2, \cdots \ell\}\}$, with relations

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ji} e_j \\ [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, f_j] &= -a_{ji} f_j, \end{aligned}$$

along with the Serre relations (which only appear in higher degrees):

$$
\begin{aligned}
s_{ij}^{+} &:= \text{ad }(e_i)^{1-a_{ji}}(e_j) = 0 \quad \text{if } i \neq j \\
s_{ij}^{-} &:= \text{ad }(f_i)^{1-a_{ji}}(f_j) = 0 \quad \text{if } i \neq j
\end{aligned}
$$

.

Proof:

1. Show that $\{e_i, f_i, h_i\}$ generates $\mathfrak{g}$.

The subalgebra $\mathfrak{h}$ is spanned by $\{t_{\alpha_i} \mid i \in I\}$ and hence spanned by $\{h_i \mid i \in I\}$. So it suffices to show that $\mathfrak{g}_\alpha \subseteq \langle e_i \rangle$ for all $\alpha - in\Phi^+$.

Write $\alpha = \alpha_i + \beta$ for each $i \in I, \beta \in \Phi^+$. Then $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta] = \mathfrak{g}_\alpha = \mathbb{C}e_\alpha$, so $e_\alpha = [e_i, e_\beta]$ for some nonzero $e_\beta \in \mathfrak{g}_\beta$.

By repeating this argument, we find that $e_\alpha = [[\cdots[e_{i_1}, e_{i_2}], e_{i_3}]\cdots], \cdots e_{i_k}]$.

2. Verify the relations

We need to check that $s_{ij}^{+} = 0$. The α_i root string through α_j is given by

$$
\alpha_j + p\alpha_i \to \cdots \to \alpha_j + q\alpha_i
$$

where $p \neq 0$ because $\alpha_j - \alpha_i \notin \Phi$ for any i, so the smallest root must be $\alpha_j \in \Phi$. By prop 8.4d, this means that $-q = \alpha_j(h_i) = \alpha_{ji}$.

Thus ad $(e_i)^{1-\alpha_{ji}}(e_j) = \text{ad }(e_i)^{1+q} \in \mathfrak{g}_{\alpha_j + (q+1)\alpha_i} = \{0\}$.

31.2 The Lie Algebra $\tilde{\mathfrak{g}}(A)$

Fix a Cartan matrix $A = (a_ij)_{i,j \in I}$where $I = \{1, \cdots, \ell\}$. Let $\tilde{J} \trianglelefteq \mathcal{FL}(\{e_i, f_i, h_i \mid i \in I\})$ generated by

- $[h_i, h_j]$,
- $[h_i, e_j] - a_{ji}e_j$,
- $[e_i, f_j] - \delta_{ij}h_i$
- $[h_i, f_j] + a_{ji}f_j$.

Then let J be the same ideal with the additional relations s^+, s^-, and set

- $\tilde{\mathfrak{g}}(A) = \mathcal{FL}(\{e_i, f_i, h_i\})/\tilde{J}$,
- $\mathfrak{g}(A) = \mathcal{FL}(\{e_i, f_i, h_i\})/J$.

Proposition:

a. Let $V = \mathcal{F}(\{f_1, \cdots, f_\ell\})$. Then $\pi : \tilde{\mathfrak{g}} \to \mathfrak{gl}(V)$ is a *representation* with

- $f_j : f_{i_1} \cdots f_{i_r} \mapsto f_j f_{i_1}$
- $h_j : f_{i_1} \cdots f_{i_r} \mapsto (\alpha_{ji_1} + \cdots)f_{i_1} \cdots f_{i_r}$
- $e_j : f_{i_1} \cdots f_{i_r} \mapsto (\sum \delta \sum a)(\alpha_{ji_1} + \cdots)f_{i_1} \cdots f_{i_r}$

b. $\{h_1, \cdots h_\ell\}$ is linearly independent set in $\tilde{\mathfrak{g}}$.

For (a), it suffices to check $[\pi(h_i), \pi(h_j)] = 0$, $[\pi(h_i), \pi(e_j)] = a_{ji}\pi(e_j)$, etc. For (b), it suffices to show that $\{\pi(h_i) \mid i \in I\}$ is linearly independent.

Suppose $\sum c_i\pi(h_i) = 0$ in $\mathfrak{gl}(V)$. Then,

$$\begin{aligned} 0 = \left(\sum_c c_i\pi(h_i)\right)(f_j) &= -\left(\sum_i c_i\alpha_{ji}\right) f_j \\ &\implies \sum c_i\alpha_{ji} = 0 \quad \forall\ j \\ &\implies c_i = 0 \quad \forall\ i, . \end{aligned}$$

since A is invertible.

Thus $\tilde{\mathfrak{h}} \coloneqq \operatorname{span}_{\mathbb{C}}\{h_i\}$ is a lie subalgebra of $\tilde{\mathfrak{g}}$..

Theorem:

a. $\tilde{\mathfrak{g}} = \bigoplus_{u\in\tilde{\mathfrak{h}}^*} \tilde{\mathfrak{g}}_\mu$ as vector spaces, where

$$\tilde{\mathfrak{g}}_\mu \coloneqq \left\{x \in \tilde{\mathfrak{g}} \ni [h, x] = \mu(h)x \quad \forall h \in \tilde{\mathfrak{h}}\right\}.$$

b. $\tilde{\mathfrak{g}} = \tilde{n}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{n}$ as vector spaces, where $\tilde{n}^- \coloneqq \langle f_i\rangle$ and $\tilde{n} \coloneqq \langle e_i\rangle$.

Proof of (a):

It's easy to check that $[\tilde{\mathfrak{g}}_\lambda, \tilde{\mathfrak{g}}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ for all $\lambda, \mu \in \tilde{\mathfrak{h}}^*$. Define $\alpha_i \in \tilde{\mathfrak{h}}^*$ by $h_j \mapsto a_{ij}$. Then

- $e_i \in \tilde{\mathfrak{g}}_{\alpha_i}, f_i \in \tilde{\mathfrak{g}}_{-\alpha_i}, h_i \in \tilde{\mathfrak{g}}_0$ for all i.
- Any $x \in \tilde{\mathfrak{g}}$ lies in $\tilde{\mathfrak{g}}_\mu$ for *some* μ.
- $\tilde{\mathfrak{g}} = \sum_\mu \tilde{\mathfrak{g}}_\mu$.

We just need to show that the last sum is in fact a direct sum.

Suppose that $\exists x \neq 0$ such that $x \in \tilde{\mathfrak{g}}_\mu, x = \sum_\nu x_\nu$ where $x_\nu \in \tilde{\mathfrak{g}}_\nu - \{0\}$ and ν runs over a finite set of weights that are not equal to μ.

Then $[h, x] = \mu(h)x$, and so $(\text{ad } h - \mu(h))(x) = 0$. On the other hand, $\prod_\nu(\text{ad } h - \nu(h))(x_\nu) = 0$. So pick some $h \in \tilde{\mathfrak{h}}$ such that $\mu(h) \neq \nu(h)$ for all ν. Then the polynomials $t - \mu(h), \prod_\nu(t - \nu(h))$ are coprime, and so there exist a, b such that

$$a(t - \mu(h) + b\prod_\nu(t - \nu(h)) = 1,$$

Then evaluating at $t = \text{ad } h$, we get

$$x = 1(x) = a(\text{ad } h)(\text{ad } h - \mu(h))(x) + b(\text{ad } h)(\prod_\nu \text{ad } h - \nu(h))(x) = 0,$$

and so $\tilde{\mathfrak{g}} = \oplus_\nu \tilde{\mathfrak{g}}_\mu$.

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Last time:

$$W \curvearrowright \mathfrak{h}^*, \lambda \mapsto w(\lambda), W \curvearrowright \mathfrak{h}, \mathfrak{h} \mapsto w \cdot \mathfrak{h}$$

such that $\lambda(w \cdot h) = (w^{-1}\lambda)(h) \forall \lambda \in \mathfrak{h}^*$.

We then get compatible squares:

<!--&   &   &   &

Proposition:

a. $\Theta_i := \exp(\text{ad } e_i) \circ \exp(\text{ad } (-f_i)) \circ \exp(\text{ad } e_i)$,
b. $\Theta_i(\mathfrak{h}) = \mathfrak{h}$, so it fixes Cartan subalgebra.
c. $\Theta_i \mid_{\mathfrak{h}} = s_i$ where s_i is the Weyl group action

Proof of (a):

We want to show that $\exp(\text{ad } e_i)$ is well-defined as an automorphism of $\mathfrak{g}$. It suffices to check that ad e_i is *locally nilpotent*, i.e. for all $x \in \mathfrak{g}$, there exists some $n_x > 0$ such that ad $(e_i)^n = 0$. We will also need to check that $\exp \text{ad } e_i$ is a derivation.

To see the local nilpotency, we can check

$$(\text{ad } e_i)^n([x, y]) = \sum_{t=0}^{n} \binom{n}{t} \left[(\text{ad } e_i)^t, (\text{ad } e_i)^{n-t} \right]$$

for all $x, y \in \mathfrak{g}$.

If x, y are locally nilpotent, then $[x, y]$ is as well.

It thus suffices to check that ad e_i acts on generators in a nilpotent way.

A direct computation shows ad $e_i = [e_i, e_i] = 0$, and $(\text{ad } e_i)^{1-a_{ji}}(e_j) = 0$ by the Serre relations.

We also find that ad $e_i(h_j) = [e_i, h_j] = -[h_j, e_i] = -a_{ij}e_i$, and applying it again yields $(\text{ad } e_i)^2(h_j) = -a_{ij}[e_i, e_i] = 0$.

We have ad $e_i(h_j) = 0$, and applying ad $e_i h_i$ multiple times yields $h_i, [e_i, h_i], 0$, so ad ${}^3 e_i(h_i) = 0$.

Proof of (b):

By a direct computation, we have $\Theta_i(h_j) = h_j - a_{ij}h_i \in \mathfrak{h}$. (See CJ's notes for full computation.)

Proof of (c):

Consider computing $s_i \cdot h_j$. This is the unique element satisfying $\lambda(s_i \cdot h_j) = (s_i^{-1}\lambda)(h_j)$, but we can compute

$$(s_i^{-1}\lambda)(h_j) = h_j - a_{ij}h_i = \Theta_i(h_j).$$

Theorem (Serre): Fix $\Phi \supseteq \Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ and $I = \{1, \cdots, \ell\}$. Define A by $a_{ij} = (\alpha_j, \alpha_i \vee)$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the algebra generated by these elements.

Then

a. $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n$ as vector spaces, where $n^- \cong \tilde{n}^-/s^-$, $\mathfrak{h} \cong \tilde{\mathfrak{h}}$, and $n \cong \tilde{n}/s^+$.
b. $\mathfrak{g} = \oplus_{\mu \in \mathfrak{h}^*} \mathfrak{g}_\mu$ as vector spaces, where $\mathfrak{g}_\mu = \{x \in \mathfrak{g} \ni [h, x] = \mu(h)x \forall h \in \mathfrak{h}\}$
c. $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$ if $\lambda \in W_\mu$,
d. $\dim \mathfrak{g} = \ell + |\Phi|$,
e. $\mathfrak{g}$ is semisimple,
f. $\mathfrak{h}$ is a Cartan subalgebra with root system Φ.

Proofs:

a. Follows from Theorem 18.2b and Lemma b.
b. Similar to Theorem 18.2a.

Proof of (c):

We may assume that $\lambda = s_i\mu$. Pick $x \in \mathfrak{g}_\lambda$. Then for all $h \in \mathfrak{h}$, we have

$$\begin{aligned} [\Theta_i(h), \Theta_i(x)] &= \Theta_i(h, x) \\ &= \lambda(h)\Theta_i(x) \\ &= \lambda(\Theta_i^{-1}(h))\Theta_i(x) \\ &= \lambda(s_i^{-1} \cdot h)\Theta_i(x) \\ &= (s_i\lambda)\Theta_i(x), \end{aligned}$$

so $\Theta_i(x) \in \mathfrak{g}_{s_i\lambda}$, and thus $\Theta_i(\mathfrak{g}_\lambda) \subseteq \mathfrak{g}_{s_i\lambda}$.

Replacing Θ_i with Θ_i^{-1} and λ by $s_i\lambda$, we find $\Theta_i^{-1}(\mathfrak{g}_{s_i\lambda}) \subseteq \mathfrak{g}_{s_is_i\lambda} = \mathfrak{g}_\lambda$, and so $\mathfrak{g}_\lambda \cong \mathfrak{g}_{s_i\lambda}$, i.e. $\mathfrak{g}_{s_i\lambda} \subseteq \Theta_i(\mathfrak{g})$.

Proof of (d):

By Corollary 18.2b, we have

$$\dim \mathfrak{g}_{k\alpha_{ii}} = \begin{cases} 1, & k = \pm 1 \\ 0, & k \notin \{0, \pm 1\} \\ \ell, & k = 0 \end{cases}.$$

Thus $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{h}}$.

Since s_{ij}^+ is of height $1 + a_{ji} \geq 2$, we have $\dim \mathfrak{g}_{\alpha_i} = \dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$ for all $i \in I$. Thus for any $\alpha \in \Phi$, we have $\alpha = w\alpha_i$ for some element of the Weyl group $w \in W$.

By parts (a) and (c), we have $\dim \mathfrak{g}_\alpha = 1$, so $\dim \mathfrak{g}_{k\alpha}$ satisfies the same cases as $\dim \mathfrak{g}_{k\alpha_{ii}}$ above.

It remains to show that there are no other root spaces, i.e. $\mathfrak{g}_\mu = 0$ if $\mu \notin \mathbb{Z}\alpha$ for all $\alpha \in \Phi$.

We can show this by considering reflections about hyperplanes again, i.e. that $\alpha \in \Phi \implies H_\mu \neq H_\alpha$.

If this is the case, it implies that there exists an $h \in \mathfrak{h}$ such that $h \in H_\mu \setminus H_\alpha$ for all $\alpha \in \Phi$. But then $\mu(h) = 0$ when $h \notin H_\alpha$ for all $\alpha \in \Phi$, so pick $w \in W$ such that $w^{-1} a_i(h) \in C(\Pi)$, the fundamental chamber. Thus $w^{-1}\alpha_i(h) > 0$ for all i, and is equal to $\alpha_i(w \cdot h)$, and

$$0 = \mu(h) = \kappa(t_\mu, h) = \cdots = (w\mu)(w \cdot h)$$

Writing $w_\mu = \sum_{i=1}^{\ell} m_i \alpha_i$, we have $0 = \sum_{i=1}^{\ell} m_i \alpha_i (w \cdot h)$, we find that note all m_i have the same sign, which is a contradiction. $\square$

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Upcoming