

# Chapter 9

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## 1 | Background, Notation, Setup

Important Ideas for This Chapter:

- Compactness of  $\mathcal{L}(x, y)$ .
- $\partial^2 = 0$ .
- Using broken trajectories to compactify
- Gluing

Goals

- Construct Floer homology and prove the Arnold Conjecture (“Symplectic Morse Inequalities?”):

$$\# \{1\text{-Periodic trajectories of } X_H\} \geq \sum_{k \in \mathbb{Z}} HM_k(W; \mathbb{Z}/2\mathbb{Z}).$$

Here  $HM_*(W)$  is the Morse homology.

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**Strategy:**

1. Define the action functional  $\mathcal{A}_H$ .
2. Construct the chain complex (graded vector space)  $CF_*$ .
3. Define  $X_H$ , which will be used to define  $\partial$  later.
4. Count trajectories.
5. Show finite-energy trajectories connect critical points of  $\mathcal{A}_H$ .
6. Show compactness property for space of trajectories of finite energy.
7. Define  $\partial$  (uses a compactness property in 9.1c)
8. Show space of trajectories is a manifold (plus genericity, “Smale property”)
9. **Show that  $\partial^2 = 0$ .**
10. Show that  $HF_*$  doesn’t depend on  $\mathcal{A}_H$  or  $X_H$
11. Show  $HF_* \cong HM_*$ , and compare dimensions of the vector spaces  $CM_*$  and  $CF_*$ .

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**Ingredients:**

- $(W, \omega)$  with  $\omega \in \Omega^2(W)$  is a symplectic manifold with an almost complex structure  $J$ .
- $H \in C^\infty(W; \mathbb{R})$  a Hamiltonian with  $X_H$  the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in  $T_x M$ :

$$\omega_x(\cdot, X_H(x)) = (dH)_x(\cdot).$$

- Zeros of vector field  $X_H$  correspond to critical points of  $H$ :

$$X_H(x) = 0 \iff (dH)_x = 0.$$

- Take the associated flow  $\psi^t : W \rightarrow W$ , assumed 1-periodic so  $\psi^1(x) = x$ : critical points of  $H$  are periodic trajectories.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$  is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y \end{aligned}$$

$$Y \in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- 
- $\mathcal{L}W$  is the *free loop space* on  $W$ , i.e. space of contractible loops on  $W$ , i.e.  $C^\infty(S^1; W)$  with the  $C^\infty$  topology
    - Elements  $x \in \mathcal{L}W$  can be viewed as maps  $S^1 \rightarrow W$ .
    - Can extend to maps from a closed disc,  $u : \bar{\mathbb{D}}^2 \rightarrow M$ .
    - Loops in  $\mathcal{L}W$  can be viewed as maps  $S^2 \rightarrow W$ , since they're maps  $I \times S^1 \rightarrow W$  with the boundaries pinched:



Figure 1: Loops in  $\mathcal{L}W$

- The action functional is given by

$$\mathcal{A}_H : \mathcal{L}W \rightarrow \mathbb{R}$$

$$x \mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) dt$$

- Example:  $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t dt - p dq)$ .
- Correspondence between trajectories of the gradient of  $\mathcal{A}_H$  and solutions to Floer equations.
- $x, y$  periodic orbits of  $H$  (nondegenerate, contractible), equivalently critical points of  $\mathcal{A}_H$ .

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- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every  $u \in C^\infty(S^2, W)$ ,

$$\int_{S^2} u^* \omega = 0 \quad \text{or equivalently} \quad \langle \omega, \pi_2 W \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every  $u \in C^\infty(S^2; M)$  there exists a symplectic trivialization of the fiber bundle  $u^*TM$ , equivalently

$$\langle c_1 TW, \pi_2 W \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- Maslov index: used the fact that
  - Every path in  $\gamma : I \rightarrow \text{Sp}(2n, \mathbb{R})$  can be assigned an integer coming from a map  $\tilde{\gamma} : I \rightarrow S^1$  and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$ , the moduli space of contractible finite-energy solutions to the Floer equation connecting  $x, y$ .
  - After perturbing  $H$  to get transversality, get a manifold of dimension  $\mu(x) - \mu(y)$ .
  - How we did it:
    - \* Describe as zeros of a section of a vector bundle over  $\mathcal{P}^{1,p}(x, y)$  (Banach manifold modeled on the Sobolev spaces  $W^{1,p}$ ),
    - \* Apply Sard-Smale to show  $\mathcal{M}(x, y)$  is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality.
    - \* Followed from showing  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .

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## 2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$$

$$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x, y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting  $x$  to  $y$ :

$$\mathcal{L}(x, y) := \mathcal{M}(x, y)/\mathbb{R}.$$

- Use the quotient topology, define sequentially:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

- When  $|\mu(x) - \mu(y)| = 1$ , get a compact 0-manifold, so the number of trajectories

$$n(x, y) := \#\mathcal{L}(x, y)$$

is well-defined.

- $C_k(H) := \mathbb{Z}/2\mathbb{Z}[S]$  where  $S$  is the set of periodic orbits of  $X_H$  of Maslov index  $k$ .
  - Finitely many since they are nondegeneracy implies they are isolated.

### Remark 1.

Some notation:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M}(x, z) \\ & & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

Hats will generally denote maps induced on quotient.

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- Defined a differential

$$\begin{aligned}\partial : C_k(H) &\rightarrow C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y)y\end{aligned}$$

$$\begin{aligned}n(x, y) &:= \# \{ \text{Trajectories of } \text{grad } \mathcal{A}_H \text{ connecting } x, y \} \pmod{2} \\ &= \# \mathcal{L}(x, y) \pmod{2}.\end{aligned}$$

- Examined  $\partial^2$ :

$$\begin{aligned}\partial^2 : C_k(H) &\rightarrow C_{k-2}(H) \\ x &\mapsto \partial(\partial(x)) \\ &= \partial \left( \sum_{\mu(y)=\mu(x)-1} n(x, y)y \right) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x, y)\partial(y) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x, y) \left( \sum_{\mu(z)=\mu(y)-1} n(y, z)z \right) \\ &= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x, y)n(y, z)z \\ &= \sum_{\mu(z)=\mu(y)-1} \left( \sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) \right) z \quad (\text{finite sums, swap order}),\end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When  $\mu(z) = \mu(x) - 2$ ,  $\mathcal{L}(x, z)$  is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get  $\bar{\mathcal{L}}(x, y)$ .

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- We'll then have

$$\bar{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial\bar{\mathcal{L}}(x, z), \quad \partial\bar{\mathcal{L}}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z),$$

which “space-ifies” the equation we want.

- We'll show  $\partial\bar{\mathcal{L}}(x, z)$  is a 1-manifold, which must have an even number of points, and thus

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = \#(\partial\bar{\mathcal{L}}(x, z)) \equiv 0 \pmod{2}.$$



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## 3 | Three Important Theorems

### 3.1 First Theorem: Convergence to Broken Trajectories

- Recall: *broken trajectories* are unions of intermediate trajectories connecting intermediate critical points.
- Shown last time: a sequence of trajectories can converge to a broken trajectory, i.e. there are broken trajectories in the closure of  $\mathcal{L}(x, z)$ .
- This theorem describes their behavior:

**Theorem 3.1 (9.1.7: Convergence to Broken Trajectories).**

Let  $\{u_n\}$  be a sequence in  $\mathcal{M}(x, z)$ , then there exist

- A subsequence  $\{u_{n_j}\}$
- Critical points  $\{x_0, x_1, \dots, x_{\ell+1}\}$  with  $x_0 = x$  and  $x_{\ell+1} = z$
- Sequences  $\{s_n^1\}, \{s_n^2\}, \dots, \{s_n^\ell\}$ .
- Elements  $u^k \in \mathcal{M}(x_k, x_{k+1})$  such that for every  $0 \leq k \leq \ell$ ,

$$u_{n_j} \cdot s_n^k \xrightarrow{n \rightarrow \infty} u^k.$$

- Upshots:
  - Every sequence upstairs has a subsequence which (after reparameterizing) converges
  - This descends to actual convergence after quotienting by  $\mathbb{R}$ ?
  - Yields uniqueness of limits in  $\mathcal{L}(x, z)$ , thus a separated topology
  - Sequentially compact  $\iff$  compact since  $\mathcal{L}(x, z)$  is a metric space?

**Corollary 3.2 (Compactness).**

$\bar{\mathcal{L}}(x, z)$  is compact.

### 3.2 Second Theorem: Compactness of $\bar{\mathcal{L}}(x, z)$

**Definition 3.2.1** (Regular Pair).

For an almost complex structure  $J$  and a Hamiltonian  $H$ , the pair  $(H, J)$  is **regular** if the Floer map  $\mathcal{F}$  is transverse to the zero section in the following vector bundle:

$$E_u := \{\text{Vector fields tangent to } M \text{ along } u\} \longrightarrow C^\infty(\mathbb{R} \times S^1; TM)$$

$$\begin{array}{ccc} & \begin{array}{c} \nearrow \\ \mathcal{F} \\ \searrow \end{array} & \\ & \downarrow & \\ & \begin{array}{c} \nwarrow \\ \mathbf{0} \\ \nearrow \end{array} & \\ & C^\infty(\mathbb{R} \times S^1; M) & \end{array}$$

Most of chapter 9 is spent proving this theorem:

**Theorem 3.3(9.2.1).**

Let  $(H, J)$  be a regular pair with  $H$  nondegenerate and  $x, z$  be two periodic trajectories of  $H$  such that

$$\mu(x) = \mu(z) + 2.$$

Then  $\bar{\mathcal{L}}(x, z)$  is a compact 1-manifold with boundary with

$$\partial \bar{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x, z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \quad \text{where} \quad \mathcal{I}(x, z) = \left\{ y \mid \mu(x) < \mu(y) < \mu(z) \right\}.$$

Note: possibly a typo in the book? Has  $x, y$  on the LHS.

**Corollary 3.4.**

$$\partial^2 = 0.$$

### 3.3 Third Theorem: Gluing

- We already know that  $\bar{\mathcal{L}}(x, z)$  is compact and  $\mathcal{L}(x, z)$  is a 1-manifold, so we look at neighborhoods of boundary points.

**Theorem 3.5 (9.2.3: Gluing).**

Let  $x, y, z$  be three critical points of  $\mathcal{A}_H$  with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a  $\rho_0 > 0$  and a differentiable map

$$\psi : [\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$$

such that  $\hat{\psi}$ , the induced map on the quotient

$$\begin{array}{ccc} [\rho_0, \infty) & \xrightarrow{\psi} & \mathcal{M}(x, z) \\ & \searrow \hat{\psi} & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

is an embedding that satisfies

$$\hat{\psi}(\rho) \xrightarrow{\rho \rightarrow \infty} (\hat{u}, \hat{v}) \in \bar{\mathcal{L}}(x, z).$$

2. (“Uniqueness”) For any sequence  $\{\ell_n\} \subseteq \mathcal{L}(x, z)$ ,

$$\ell_n \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v}) \quad \implies \quad \ell_n \in \text{im}(\hat{\psi}) \text{ for } n \gg 0.$$

## 4 | Gluing Theorem

Broken into three steps:

### 1. Pre-gluing:

- Get a function  $w_\rho$  which interpolates between  $u$  and  $v$  in the parameter  $\rho$ .

- 
- Not exactly a solution itself, just an “approximation”.

## 2. Newton’s Method:

- Apply the Newton-Picard method to  $w_p$  to construct a true solution

$$\begin{aligned}\psi : [-\rho, \infty) &\rightarrow \mathcal{M}(x, z) \\ \rho &\mapsto \exp_{w_p}(\gamma(p))\end{aligned}$$

$$\text{for some } \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p} \mathcal{P}(x, z)$$

where  $\mathcal{P} = ?$ .

- General idea: guess, intersect  $x$ -axis with tangent of graph, use that as a new guess and iterate.
- GIF of Newton’s Method

## 3. Project and Verify Properties:

- Check that the projection  $\hat{\psi} = \pi \circ \psi$  satisfies the conditions from the theorem.

# 5 | 9.3: Pre-gluing, Construction of $w_\rho$

- Choose (once and for all) a bump function  $\beta$  on  $B_\varepsilon(0)^c \subset \mathbb{R} \rightarrow [0, 1]$  which is 1 on  $|x| \geq 1$  and 0 on  $|x| < \varepsilon$
- Split into positive and negative parts  $\beta^\pm(s)$ :

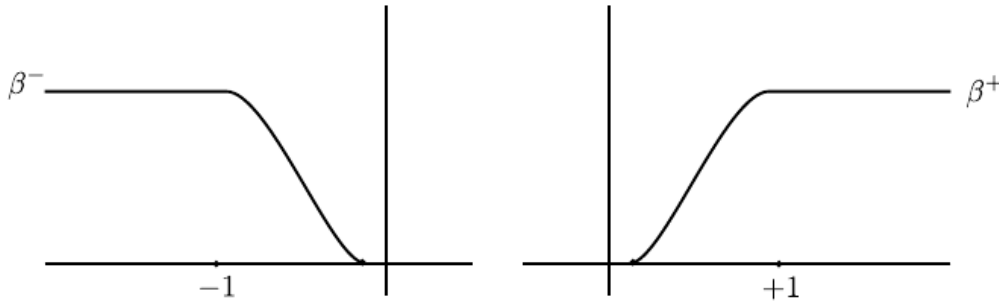


Figure 2: Bump away from zero

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- Define an interpolation  $w_\rho$  from  $u$  to  $v$  in the following way: let

- $\exp[\cdot] := \exp_{y(t)}(\cdot)$  and
- $\ln(\cdot) := \exp_{y(t)}^{-1}(\cdot)$ ,

then

$$w_\rho : x \rightarrow z$$

$$w_\rho(s, t) := \begin{cases} u(s + \rho, t) & s \in (-\infty, -1] \\ \exp[\beta^-(s) \ln(u(s + \rho, t)) + \beta^+(s) \ln(u(s - \rho, t))] & s \in [-1, 1] \\ u(s - \rho, t) & s \in [1, \infty) \end{cases}.$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\} \subseteq \text{im } \exp_{y(t)}(\cdot),$$

so we can apply  $\exp_{y(t)}^{-1}(\cdot)$ .

- Can make  $|s| \leq 1$  for large  $\rho$ , since

$$\begin{aligned} u(s, t) &\xrightarrow{s \rightarrow \infty} y(t) \\ v(s, t) &\xrightarrow{s \rightarrow -\infty} y(t). \end{aligned}$$

- So pick a  $\rho_0$  such that this holds for  $\rho > \rho_0$ .
- Might have to increase  $\rho_0$  later in the proof, so  $\rho > \rho_0$  just means  $\rho \gg 0$ .

- Some properties:

- $w_\rho \in C^\infty(x, z)$  and is differentiable in  $\rho$ .
- $s \in [-\varepsilon, \varepsilon] \implies w_\rho(s, t) = y(t)$ .

$$w_\rho(s - \rho, t) \xrightarrow{\rho \rightarrow \infty} u(s, t) \quad \text{in } C_{\text{loc}}^\infty$$

$$w_\rho(s, t) \xrightarrow{\rho \rightarrow \infty} y(t) \quad \text{in } C_{\text{loc}}^\infty.$$

- Now carry out the linearized version on tangent vectors

- 
- Let  $Y \in T_u\mathcal{P}(x, y)$
  - Let  $Z \in T_v\mathcal{P}(x, y)$
  - Replace  $w_\rho$  with the interpolation

$$Y \#_\rho Z \in T_{w_\rho}\mathcal{P}(x, y) = W^{1,p}(w_\rho^*TW).$$

defined by

$$(Y \#_\rho Z)(s, t) = \begin{cases} Y(s + \rho, t) & s \in (-\infty, -1] \\ \exp_T \left[ \beta^-(s) \ln_T(Y(s + \rho, t)) + \beta^+(s) \ln_T(Z(s - \rho, t)) \right] & s \in [-1, 1] \\ Z(s - \rho, t) & s \in [1, \infty) \end{cases},$$

where the subscript  $T$  indicates taking tangents of the exponential maps at appropriate points.

## 6 | 9.4: Construction of $\psi$ .

### 6.1 Summary

- $\mathcal{F}_\rho$  will be  $F \circ \exp w_\rho$  expanded in the bases  $Z_i$  coming from a trivialization.
- $L_\rho = (d\mathcal{F}_\rho)_0$  will be the linearization of the Floer operator at zero.
- Newton-Picard method, general idea:
  - Allows finding zeros of  $f$  given an approximate zero  $x_0$ , using the extra information of the 1st derivative  $f'$ .
  - Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(\mathbf{x}_0)}.$$

- Second variant more useful: only need to know the derivative at one point.

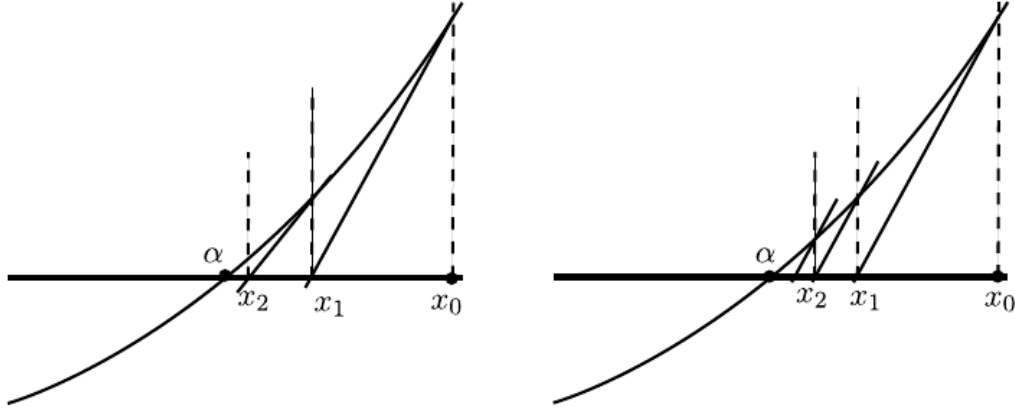


fig. 9.6

Figure 3: Newton Method Variants

- Adapting Newton-Picard to operators:
  - $\frac{1}{f'(x_0)}$  will correspond to  $L_\rho^{-1}$ , but it won't be invertible on entire space.
  - Decompose

$$T_{w_\rho} \mathcal{P}(x, z) = W^{1,p}(w_\rho^* TW) = W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) = \ker(L_\rho) \oplus W_\rho^\perp,$$

where  $L_\rho$  will have a right inverse on  $W_\rho^\perp \subseteq T_{w_\rho} \mathcal{P}(x, z)$ .

- Find that  $x_0 = 0$  is “approximately” a solution and run Newton-Picard in  $W_\rho^\perp$
- Where does  $W_\rho^\perp$  come from? Essentially the kernel of some linear functional given by an integral:

$$W_\rho^\perp := \left\{ Y \in W^{1,p} \mid \int_{\mathbb{R} \times S^1} \langle Y, \dots \rangle ds dt = 0, \text{ plus conditions} \right\}.$$

- Will obtain for every  $\rho \geq \rho_0$  an element  $\gamma(\rho) \in W_\rho^\perp$  with

$$\mathcal{F}_\rho(\gamma(\rho)) = 0.$$

- Where does  $\gamma$  come from? Intersection-theoretic interpretation on page

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$$\begin{aligned} (\exp_{w_\rho})^{-1} \mathcal{M}(x, z) \cap W_\rho^\perp &\subseteq T_{w_\rho} \mathcal{P}(x, z) && \rightsquigarrow \gamma \\ \mathcal{M}(x, z) \cap \left\{ \exp_{w_\rho} W_\rho^\perp \mid \rho \geq \rho_0 \right\} &\subseteq \mathcal{P}(x, z) && \rightsquigarrow \psi(\rho), \end{aligned}$$

which we get by exponentiating.

- \* This gives a codimension 1 subspace in  $\mathcal{M}(x, z)$ , which we take to be  $\psi(\rho)$ :

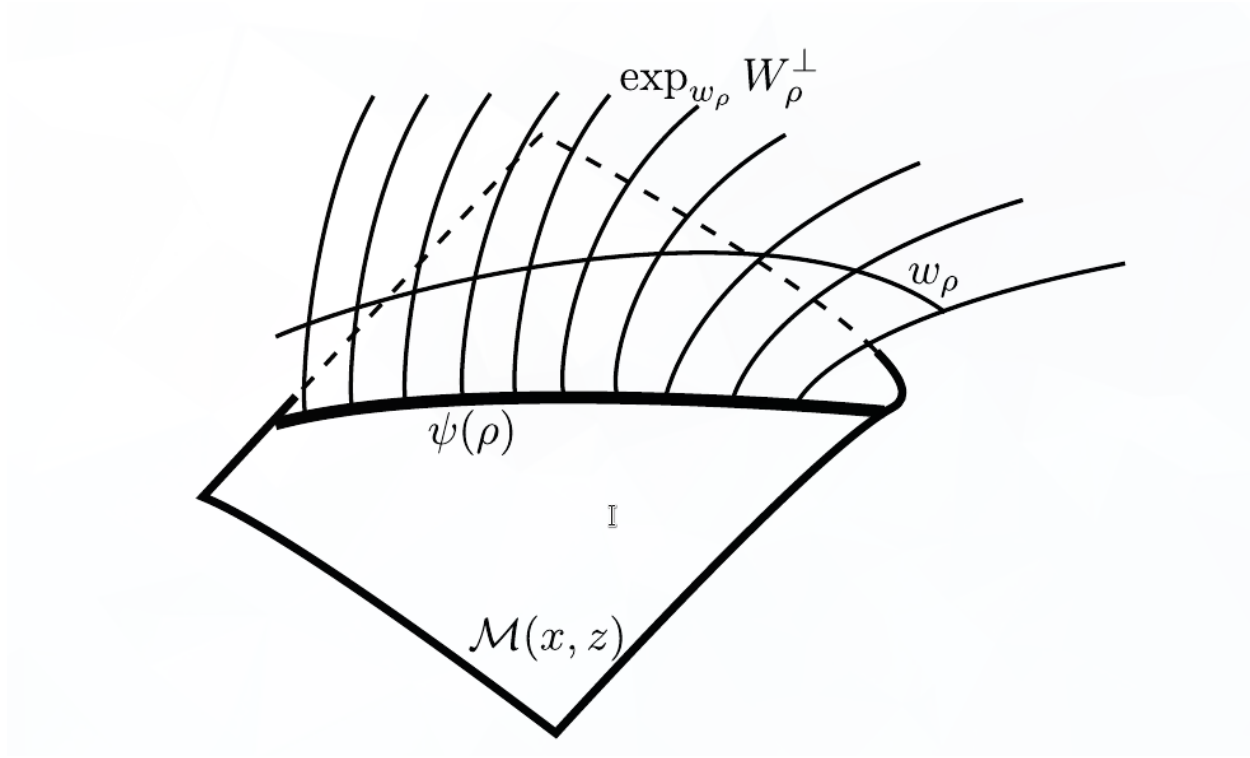


Figure 4: Intersection interpretation

- Apply the implicit function theorem to show differentiability of  $\gamma$  in  $\rho$ .
- Use a trivialization  $Z_i^\rho$  to get a vector field along  $w_\rho$  (also called  $\gamma(\rho)$ )
  - This is exactly an element of  $T_{w_\rho} \mathcal{P}(x, z)$
- Exponentiate to get an element of  $\mathcal{M}(x, z)$ :

$$\psi(\rho) := \exp_{w_\rho}(\gamma(\rho)).$$



- Project onto  $\mathcal{L}(x, z)$  to get  $\hat{\psi}$ .
- Prove  $\hat{\psi}$  is a proper injective immersion and thus an embedding.
- Show that the broken trajectory  $(\hat{u}, \hat{v})$  is the endpoint of an embedded interval in  $\bar{\mathcal{L}}(x, z)$ .
  - Then show that any other sequence converging to  $(\hat{u}, \hat{v})$  must approach via this interval, otherwise could have cuspidal points:

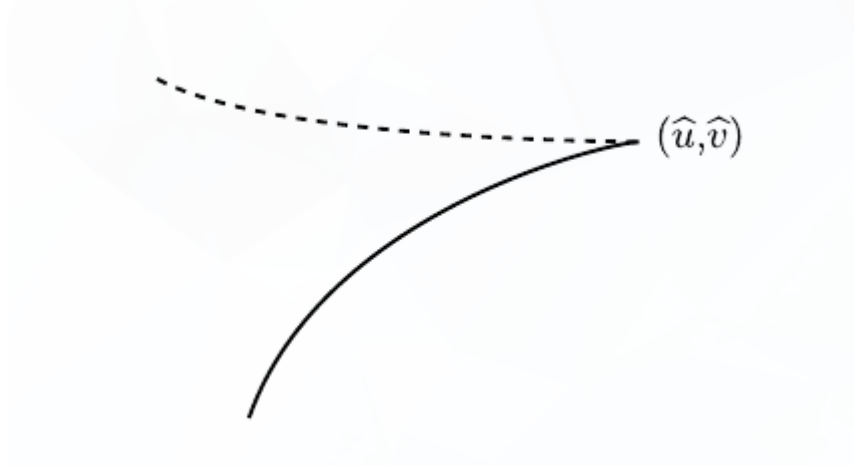


Figure 5: Cuspidal Point on Boundary

## 6.2 Section 9.4

- Have constructed  $w_\rho \in C^\infty_{\searrow}(x, z)C^\infty(x, z)$  for every  $\rho \geq \rho_0$ , since there is exponential decay.
- Yields  $\psi_\rho \in \mathcal{M}(x, z)$  a true solution (to be defined).
- Need to check that  $\mathcal{F}(\psi_\rho) = 0$  where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- $\psi_\rho$  already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining  $\mathcal{F}_\rho$ .

$$\begin{aligned}
W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n}) &\xrightarrow{\mathcal{F}_\rho} L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \\
(y_1, \dots, y_{2n}) &\mapsto \left[ \left( \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } H_t \right) (\exp_{w_\rho} \sum y_i Z_i^\rho) \right]_{Z_i}
\end{aligned}$$

where  $\mathcal{F}_\rho := \mathcal{F} \circ \exp_{w_\rho}$  written in the bases  $Z_i$ . sd

- Linearize  $\mathcal{F}_\rho$ .