Problem Set 1

D. Zack Garza

November 9, 2019

Contents

1	Problem 5															1																
	1.1	Part 1																										 				1
	1.2	Part 2									•					•	•															1
2	Problem 6															2	2															
	2.1	Part 1																										 			4	2
	2.2	Part 2		_			_						_	_				_	_	_	_	 			_	_		 			:	3

1 Problem 5

1.1 Part 1

Let $A \in \operatorname{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in Math(n, n)$ be positive semi-definite, so

$$\langle v, Bv \rangle \ge 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A+B)v \rangle \ge 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{split} \langle v,\ (A+B)v\rangle &= \langle v,\ Av\rangle + \langle v,\ Bv\rangle \\ &> \langle v,\ Av\rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{split}$$

1.2 Part 2

Let M be a smooth manifold with a maximal smooth atlas \mathcal{A} , and choose a covering of M by charts $(U_i, \phi_i) \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_i U_i$.

2 Problem 6

2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X:M\to TM$$

We want to show that there is a point $p \in M$ such that X(p) = 0.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi: M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} = X(\phi_t(p)),$$
$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t: M \to M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H: M \times I \to M$$

 $(p,t) \mapsto \phi_t(p).$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f: M \to M$, we have

$$\Lambda(f) = \sum_{k} \operatorname{Tr} \left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_*: H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f$$
 has at least one fixed point.

In particular, we have

$$\begin{split} \Lambda(\mathrm{id}_M) &= \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})}) \\ &= \sum_k \dim H_k(X;\mathbb{Q}) \\ &= \chi(M), \end{split}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2\\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2)=2\neq 0$, and thus ϕ_1 has a fixed point p_0 , thus $\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\phi_t(p) = p$$

$$\Rightarrow \frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 \qquad \text{by differentiating wrt } t$$

$$\Rightarrow \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \Big|_{t=0} = 0 \qquad \text{by evaluating at } t = 0$$

$$\Rightarrow X(\phi_1(p_0)) := \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \qquad \text{by definition of } \phi_1$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. \square

2.2 Part 2

The trivial bundle

$$\mathbb{R}^2 \longrightarrow S^2 \times \mathbb{R}^2$$

$$\downarrow^{r}$$

$$\downarrow^{s}$$

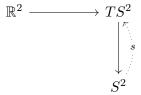
$$S^2$$

has a nowhere vanishing section, namely

$$s: S^2 \to S^2 \times \mathbb{R}^2$$

 $\mathbf{x} \to (\mathbf{x}, [1, 1])$

which is the identity on the S^2 component and assigns the constant vector [1, 1] to every point. However, as part 1 shows, the bundle



can *not* have a nowhere vanishing section. \Box