

Problem Set 2

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1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ can not lie in \mathcal{O} .

Useful facts:

- For any $\lambda \in \mathfrak{h}^\vee$, \mathbb{C}_λ is a 1-dimensional \mathfrak{b} -bimodule with a trivial \mathfrak{n} -action.
- $M(\lambda) = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ as a left $U(\mathfrak{g})$ -module.
- $M(\lambda) = U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$ -module.
- $M(\lambda)$ is generated as a $U(\mathfrak{g})$ -module by the maximal vector $v^+ = 1 \otimes 1$.
- The set of weights of $M(\lambda)$ is $\lambda - \Gamma$ where Γ is the semigroup in Λ_r generated by Φ^+ .
- $M(\lambda)$ has weights $\lambda, \lambda - 2, \lambda - 4, \dots$ each with multiplicity 1.

Questions

- What is the tensor product over? Guess: $\otimes_{\mathbb{C}}$.
- MSE: the product is no longer finitely generated.
 - Consider dimensions of weight spaces – eventually constant.
 - If $\text{wt } v = \lambda$ and $\text{wt}(u) = \mu$, then $\text{wt}(u \otimes v) = \lambda + \mu$.
 - Consider a weight space N_γ of M . This must have the form $\bigoplus_{a+b=\gamma} M_a \otimes_{\mathbb{C}} M_b$.
 - Example: consider $\lambda = \mu = 0$. Then $M = M(0) \otimes M(0)$ and N_{-2m} has dimension $m + 1$ for every $m \in \mathbb{Z}^+$.

2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h})$ be the Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ .

Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Useful facts:

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