

# Title

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## Contents

<b>1</b>	<b>Sunday, September 13</b>	<b>1</b>
1.1	General Notes . . . . .	1
1.2	1.a . . . . .	2
1.3	3.a . . . . .	2

# 1 | Sunday, September 13

## 1.1 General Notes

- If flipping logic and not using a direct proof (contradiction, contrapositive, etc), then signpost/announce it near the beginning of the proof.
- Say what you're assuming at the start of the proof.
- Put any important equations (i.e. major steps of the proof) on their own lines or in `displaymath` environments.
- Use some whitespace to separate parts of the proof and increase readability.
- Remember that limits of sequences need not exist, but  $\liminf$ s/ $\limsup$ s always do (just may be  $\pm\infty$ ).
- Try to avoid abbreviating the names of major theorems (example: "AP" can stand for many results, not just the Archimedean property!)
- It's not generally true that  $a \leq M \implies |a| \leq M$ , e.g. take  $a = -1$ . This only holds  $a \geq 0$ .
- A generic set may not contain its  $\inf$  or  $\sup$ . Example:  $\inf \left\{ \frac{1}{n} \right\} = 0$  and  $0 \notin \left\{ \frac{1}{n} \right\}$ , or  $\sup \left\{ 1 - \frac{1}{n} \right\} = 1$  with  $1 \notin \left\{ 1 - \frac{1}{n} \right\}$ .
- If there exists some element with a

**1.2 1.a***Proof* ( $A \implies B$ ).

- Suppose  $\{a_n\}$  is not bounded above.
- Then any  $k \in \mathbb{N}$  is not an upper bound for  $\{a_n\}$ .
- So choose a subsequence  $a_{n_k} > k$ , then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k \rightarrow \infty} a_{n_k} > \liminf_{k \rightarrow \infty} k = \infty.$$

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*Proof* ( $\neg A \implies \neg B$ ).

- Suppose  $\{a_n\}$  is bounded by  $M$ , so  $a_n < M < \infty$  for all  $n \in \mathbb{N}$ .
- Then if  $\{a_{n_k}\}$  is a subsequence, we have  $a_{n_k} \in \{a_n\}$ , so  $a_{n_k} < M$  for all  $k \in \mathbb{N}$ .
- But then

$$a_{n_k} < M \implies \limsup_{k \rightarrow \infty} a_{n_k} \leq M,$$

- Now note that if  $\lim_{k \rightarrow \infty} a_{n_k}$  exists,

$$\lim_{k \rightarrow \infty} a_{n_k} < \limsup_{k \rightarrow \infty} a_{n_k} \leq M < \infty,$$

so every subsequence is bounded and thus can not converge to  $\infty$ .

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**1.3 3.a***Proof* (Using definition (i)).

- Suppose  $x_n \leq M$  for all  $n$ , we will show that every subsequential limit is also bounded by  $M$ .
- Let

$$S := \left\{ x \in \mathbb{R} \mid x \text{ is a subsequential limit of } \{x_n\} \right\}$$

be the set of subsequential limits.

– Note that  $\inf S := \liminf_{n \rightarrow \infty} x_n$  by definition (i).

- Let  $\{x_{n_k}\} \in S$  be an arbitrary convergent subsequence (since we are only concerned about subsequences with well-defined limits).
- Then for every  $k$  we have  $x_{n_k} \in \{x_n\}$ , so

$$|x_{n_k}| \leq M.$$

- By order limit laws,

$$|x_{n_k}| \leq M \implies \lim_{k \rightarrow \infty} |x_{n_k}| \leq M,$$

- Since the map  $x \mapsto |x|$  is continuous, using the sequential definition of continuity we can pass the limit through the absolute value to obtain

$$\left| \lim_{k \rightarrow \infty} x_{n_k} \right| \leq M.$$

- Since the subsequence was arbitrary, we find that  $M$  is an upper bound for  $S$  and so  $\sup S \leq M$ .
- But

$$\inf S \leq \sup S \leq M \implies \inf S \leq M.$$

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*Proof (Using definition (ii)).*

- Suppose  $|x_n| \leq M$  for every  $n$ , we will directly show that  $\left| \liminf_{n \rightarrow \infty} x_n \right| \leq M$ .
- Let  $\{x_{n_k}\}$  be an arbitrary subsequence, then since  $x_{n_k} \in \{x_n\}$  for all  $k$ ,  $|x_{n_k}| \leq M$  for all  $k$ .
- By order-limit laws, for every fixed  $n$  we have

$$|x_{n_k}| \leq M \iff -M \leq x_{n_k} \leq M \implies -M \leq \inf_{k > n} x_{n_k} \leq M.$$

- Again applying order-limit laws,

$$-M \leq \inf_{k > n} x_{n_k} \leq M \implies -M \leq \lim_{n \rightarrow \infty} \inf_{k > n} x_{n_k} \leq M \iff \left| \lim_{n \rightarrow \infty} \inf_{k \geq n} x_{n_k} \right| \leq M.$$

- But by definition (i), this precisely says that  $\left| \liminf_{n \rightarrow \infty} x_n \right| \leq M$ .

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