

# Homotopy Groups of Spheres

## Graduate Student Seminar

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# Introduction

# Outline

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Examples

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

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## Homotopies of paths:



- Regard paths  $\gamma$  in  $X$  and homotopies of paths  $H$  as morphisms

$$\gamma \in \mathbf{hom}_{\mathbf{Top}}(I, X)$$

$$H \in \mathbf{hom}_{\mathbf{Top}}(I \times I, X).$$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write  $[\gamma]$  to denote a homotopy class of paths.

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- Why care about path homotopies? Historically: contour integrals in  $\mathbb{C}$



- By the residue theorem, for a meromorphic function  $f$  with simple poles  $P = \{p_i\}$  we know that

$$\oint_{\gamma} f(z) dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

# Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \text{hom}_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \text{hom}_{\text{Top}}(X \times I, Y)$$

such that  $F(0) = f, F(1) = g$ .

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] := \text{hom}_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \text{hom}(X, Y) \\ g \in \text{hom}(Y, X) \end{cases} \quad \text{such that } \begin{cases} f \circ g \sim \text{id}_Y \\ g \circ f \sim \text{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \text{Top} \mid Y \sim X \right\}.$$

# The Fundamental Group

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- $\pi_1(X)$  is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$



# Higher Homotopy Groups

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- Can now generalize to define

$$\pi_k(X) := [S^k, X]$$



*Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look  $\mathbb{A}^1$  or  $\mathbb{P}^1$ .*



# Classification

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- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



*Also have the derived category  $D\text{Top}$ , its interplay with  $\text{hoTop}$  is the subject of e.g. the Poincare conjecture(s).*

- Any representative from a green box: a *homotopy type*.

# Example: Homotopy Equivalence is Useful

**Proposition:** Let  $B$  be a CW complex; then isomorphism classes of  $\mathbb{R}^1$ -bundles over  $B$  are given by  $H^1(X, \mathbb{Z}/2\mathbb{Z})$ .

- Use the fact that for any fixed group  $G$ , the functor

$$\begin{aligned} h_G(\cdot) : \text{hoTop}^{\text{op}} &\longrightarrow \text{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

is representable by a space called  $BG$  (Brown's representability theorem).

- I.e., let  $\text{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$ , there is an isomorphism

$$\text{Bun}_G(X) \cong [X, BG]$$

- In general, identify  $G = \text{Aut}(F)$  the automorphism group of the fibers – for vector bundles of rank  $n$ , take  $G = GL(n, \mathbb{R})$ .

# Example: Homotopy Equivalence is Useful

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Proof:

$$\begin{aligned}\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, \mathrm{BGL}(1, \mathbb{R})] \\ &= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\ &= [X, \mathbb{RP}^\infty] \\ &= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\ &= H^1(X; \mathbb{Z}/2\mathbb{Z})\end{aligned}$$

*Note that for a poset of spaces  $(M_i, \hookrightarrow)$ , the space  $M^\infty := \varinjlim M_i$ . These are infinite dimensional “Hilbert manifolds”.*

The work being swept under the rug here is in identifying the homotopy type of the target representing space.

# Example: Homotopy Equivalence is Useful

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Corollary: There are 2 distinct line bundles over  $X = S^1$  (the cylinder and the mobius strip), since  $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ .

Corollary: A Riemann surface  $\Sigma_g$  satisfies  $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$  and thus there are  $2^{2g}$  distinct real line bundles over any such surface.

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# Point 2

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# Examples

# Sphere 1

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