Zack Garza

1a) Note that if $x \in C$ is an endpoint of a removed interval, then $x = k/3^n$ for some integers $n \geq 1$ and $0 \leq k \leq 3^n$. So we just need a real number $x \in (0,1)$ satisfying

a) x has <u>some</u> ternary expansion
$$x = \sum_{i=1}^\infty a_i 3^{-i} \text{ where } a_i \neq 1 \text{ for any } i, \text{ and}$$

b) $x \neq k/3^n$ for any $k, n \in \mathbb{N}^{>0}$,

then we will have $x \in C$ by (a) and x not an endpoint by (b).

<u>Claim</u>: $x = (0.\overline{02})_3 = (0.020202\cdots)_3$ works.

Base 3

<u>Pf</u>: By construction, x satisfies

(a) $$x = \sum_{i=0}^\infty a_i 3^{-i}, \quad a_i \in \{0,2\}$$

So no $a_i = 1$ and thus $x \in C$.

(b) To see that x satisfies (b), we can compute

$$x = (0.020202\cdots)_3$$
$$= 0\cdot 3^{-1} + 2\cdot 3^{-2} + 0\cdot 3^{-3} + 2\cdot 3^{-4} + \ldots$$
$$= \sum_{i=1}^{\infty} 2\cdot 3^{-2i} = 2\sum_{i=1}^{\infty} 3^{-2i} = 2\sum_{i=1}^{\infty}\left(\tfrac{1}{9}\right)^i$$
$$= 2\left(-1 + \sum_{i=0}^{\infty}\left(\tfrac{1}{9}\right)^i\right)$$
$$= 2\left(-1 + \frac{1}{1-\frac{1}{9}}\right) \boxed{= 1/4,}$$

where $4 \neq 3^n$ for any integer n. $\blacksquare$

(1b) If a set X is <u>nowhere dense</u> in a topological space, it equivalently satisfies

$$(\overline{X})^\circ = \emptyset$$

(i.e., the interior of the closure is empty.)

It then suffices to show that

a) C is closed, so $\overline{C} = C$, and

b) C has no interior points, so $C^\circ = \varnothing$.

(a) To see that C is closed, we will show $C^c := [0,1] \setminus C$ is open. An arbitrary union of open sets is open, so the claim is that $C^c = \bigcup_{j \in J} A_j$ for some collection of open sets $\{A_j\}_{j \in J}$.

Consider C_n, the n^{th} stage of the process used to construct the Cantor set, so $C = \bigcap_{i=1}^{\infty} C_n$. But by induction, C_n^c is a union of open sets. In particular, $C_1^c = (\frac{1}{3}, \frac{2}{3})$, and

$$C_n^c = \underbrace{\left(\bigcup_{i=1}^{n-1} C_i^c \right)}_{\text{Open by hypothesis}} \cup \underbrace{\left(\begin{array}{c} \text{Exactly } n \text{ open intervals} \\ \text{that were deleted} \end{array} \right)}_{\text{open by construction}},$$

So C_n^c is open for each n. But then

$$C^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c = \bigcup_{i=1}^{\infty} C_n^c$$

is a union of open sets and thus open. So C is closed.

(b) To see that $C^\circ = \emptyset$, suppose towards a contradiction that $x \in C^\circ$, so there exists some $\varepsilon > 0$ such that $N_\varepsilon(x) := (x-\varepsilon, x+\varepsilon) \subsetneq C$. Letting $\mu(I)$ denote the length of an interval, we have $\mu(N_\varepsilon(x)) = 2\varepsilon > 0$.

<u>Claim</u>: Let $L_n := \mu(C_n)$, then $L_n = \left(\frac{2}{3}\right)^n$.

This follows immediately by noting that L_n satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3} L_n, \qquad L_0 = 1$$

Since an interval of length $\frac{1}{3} L_{n-1}$ is removed at the n^{th} stage, which has the unique claimed solution.

But if $I_1 \subseteq I_2$ are real intervals, we must have $\underbrace{\mu(I_1) \le \mu(I_2)}$, whereas if we choose n large

Uses subadditivity of measure

enough such that $\left(\frac{2}{3}\right)^n < 2\varepsilon$, we have

$$(x-\varepsilon, x+\varepsilon) \subsetneq C = \bigcap_{i=1}^{\infty} C_i \Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq C_n, \text{ but}$$

$$\mu((x-\varepsilon, x+\varepsilon)) = 2\varepsilon > \left(\tfrac{2}{3}\right)^n = \mu(C_n), \text{ a contradiction.}$$

So such an $x \in C^\circ$ can't exist, and $C^\circ = \emptyset$.

Thus $(\overline{C})^\circ = C^\circ = \emptyset$, and C is nowhere dense,

and since a meager set is a countable union of nowhere dense sets, C is meager. $\square$

<u>Claim</u>: C is measure zero.

Measures are additive over disjoint sets, i.e.

$$A \cap B = \emptyset \Rightarrow \mu(A \sqcup B) = \mu(A) + \mu(B),$$

And if $A \subseteq B$, we have

$$\mu(B) = \mu(B \sqcup (B \setminus A)) = \mu(B) + \mu(B \setminus A)$$
$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A).$$

Now let B_n be the union of the intervals that are deleted at the n^{th} step. We have

$$\mu(B_0) = 0$$

$$\mu(B_1) = 1/3$$

$$\mu(B_2) = 2\left(\tfrac{1}{9}\right) = 2/9$$

$$\mu(B_3) = 4\left(\tfrac{1}{27}\right) = 4/27$$

$$\vdots$$

$$\mu(B_n) = 2^{n-1}/3^n$$

Moreover, if $i \neq j$, then $B_i \cap B_j = \emptyset$, and

$$C^c := [0,1] - C = \bigsqcup_{i=1}^{\infty} B_i .$$

We thus have

$$\begin{aligned} \mu(C) &= \mu([0,1]) - \mu(C^c) \\ &= 1 - \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) \\ &= 1 - \sum_{n=1}^{\infty} \mu(B_n) \\ &= 1 - \sum_{n=1}^{\infty} 2^{n-1}/3^n \end{aligned}$$

$$= 1 - (1/3) \sum_{n=0}^{\infty} \left(\tfrac{2}{3}\right)^n$$

$$= 1 - (1/3)\left(1/_{1-2/3}\right)$$

$$= 0. \qquad \blacksquare$$

1c Let $y \in [0,1]$ be arbitrary, we will produce an $x \in C$ such that $f(x) = c$.

Write $y = (a_1\, a_2 \cdots)_2 = \sum_{i=1}^{\infty} a_i 2^{-i}$ where $a_i \in \{0,1\}$

Now define

$$x = (2a_1\, 2a_2 \cdots)_3 = \sum_{i=1}^{\infty} (2a_i) 3^{-i} := \sum_{i=1}^{\infty} b_i 3^{-i}$$

Since $a_i \in \{0,1\}$, $b_i = 2a_i \in \{0,2\}$, meaning x has no 1^s in its ternary expansion and so $x \in C$.

Moreover, under f we have

$$\left.\begin{array}{ccc} b_i & \mapsto & \frac{1}{2} b_i \\ \parallel & & \parallel \\ 2a_i & \mapsto & \frac{1}{2}(2a_i) = a_i \end{array}\right\} \text{ So } b_i \mapsto a_i \text{ and thus } f(x) = y.$$

So $C \twoheadrightarrow [0,1]$, which is uncountable, thus so is C. $\blacksquare$

(2a) ($\Rightarrow$) Suppose X is G_δ, so $X = \bigcup_{n=1}^{\infty} A_i$ with each A_i closed. Then A_i^c is open by definition, and so

$$X^c = \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

is a countable intersection of open sets, and thus F_σ.

($\Leftarrow$) Suppose X^c is an F_σ, so $X^c = \bigcap_{i=1}^{\infty} B_i$ with each B_i open. Then each B_i^c is closed by definition, and

$$X = (X^c)^c = \left(\bigcap_{i=1}^{\infty} B_i\right)^c = \bigcup_{i=1}^{\infty} B_i^c$$

is a countable union of closed sets, and thus G_δ.

(2b) Suppose X is closed, we will show $X = \bigcap_{n=1}^{\infty} C_n$ with each C_n open. For each $x \in X$ and $n \in \mathbb{N}$, define

- $B_n(x) = \{ y \in \mathbb{R}^n \mid d(x,y) < \frac{1}{n} \}$
- $C_n = \bigcup_{x \in X} B_n(x)$
- $W = \bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \bigcup_{x \in X} B_n(x)$

Since each $B_n(x)$ is open by construction and C_n is a union of opens, each C_n is open.

Claim: $W = X$.

$X \subseteq W$: If $x \in X$, then $x \in B_n(x) \subseteq C_n$ for all n, and so

$$x \in \bigcap_{n=1}^{\infty} C_n = W.$$

$W \subseteq X$: Suppose there is some $w \in W \setminus X$ (so $w \neq x$ for any $x \in X$) towards a contradiction.

Since $w \in \bigcap_{i=1}^{n} C_n$, $w \in C_n$ for every n. So $w \in \bigcup_{x \in X} B_n(x)$ for every n. But then there is some particular $x_0 \in X$ such that $w \in B_n(x_0)$ for every n (otherwise we could take N large enough so that $w \notin B_N(x)$ for any $x \in X$, so $x \notin \bigcup_{x \in X} B_N(x)$) where $w \neq x_0$.

But then if $N_\varepsilon(x)$ is an arbitrary neighborhood of x, we can take $\frac{1}{n} < \varepsilon$ to obtain $w \in B_n(x) \subsetneq N_\varepsilon(x)$, which makes w a limit point of X. But since X is closed, it contains its limit points, forcing the contradiction $w \in X$.

So X is a countable intersection of open sets, and thus a G_δ set. $\blacksquare$

Now suppose X is open. Then X^c is closed, and thus a G_δ set. But then $(X^c)^c = X$ is an F_σ set by problem (2a). $\blacksquare$

(2c) Using the fact that singletons are closed in metric spaces, we can write $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ as a countable union of closed sets, so $\mathbb{Q}$ is an F_δ set. Suppose $\mathbb{Q}$ was also a G_δ set, so $\mathbb{Q} = \bigcap_{i=1}^{\infty} A_i$ with each A_i open. Then for any fixed n, $\mathbb{Q} \subseteq A_i$, so A_i is dense in $\mathbb{R}$ for every i.

However, it is also true that $\{q\}^c := \mathbb{R} \setminus \{q\}$ is an open, dense subset of $\mathbb{R}$, and we can write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$$

as in intersection of open dense sets; since $\mathbb{R}$ is a Baire space, countable intersections of open dense sets are dense.

But then $\left(\bigcap_{i=1}^{\infty} A_i\right) \cap \left(\bigcap_{q \in \mathbb{Q}} \{q\}^c\right) = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$

must be dense in $\mathbb{R}$, which is absurd.

Note that this argument also works when $\mathbb{R}$ is replaced with any open interval I and $\mathbb{Q}$ is replaced with $\mathbb{Q} \cap I$.

For a set that is neither G_δ nor F_δ, consider

$A = \mathbb{Q} \cap (0, \infty)$, positive rationals

$B = (\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, 0)$, negative irrationals

A is F_σ but not G_δ , using above argument, and

dually B is G_δ but not F_σ.

<u>Claim</u>: $X = A \cup B$ is neither G_δ nor F_σ.

Suppose X is G_δ. Then $X \cap \overbrace{(0,\infty)}^{\text{open}} = A$ is G_δ as well. $\#$

Suppose X is F_σ. Then X^c is G_δ, but

$$X^c = (A \cup B)^c = A^c \cap B^c = (\mathbb{Q} \cap (-\infty, 0)) \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap (0, \infty))$$

and thus $X^c \cap \overbrace{(-\infty, 0)}^{\text{open}} = A$ is G_δ. $\#$

So X is neither G_δ or F_σ.

∎

3a) <u>Claim</u>: $c \in [0,1] \Rightarrow \lim_{x \to c} f(x) = 0$.

This holds iff $\forall c \in I$, $\forall \varepsilon$, $\exists \delta$ s.t. $|x-c| < \delta \Rightarrow |f(x)| < \varepsilon$,

so let $\varepsilon > 0$ be arbitrary. Consider the set

$S = \{ n \in \mathbb{N} \mid \frac{1}{n} \geq \varepsilon \}$, which is a <u>finite</u> set, and so

$S_q = \{ r_n \in \mathbb{Q} \mid \frac{1}{n} \geq \varepsilon \}$ is <u>finite</u> as well.

So choose $\delta < \min_{r_n \in S_q} d(c, r_n)$ so $N_\delta(c) \cap S_q = \emptyset$

Then $|x-c| < \delta \Rightarrow \begin{cases} \cdot\ f(x) = 0 \text{ if } x \in I \setminus \mathbb{Q}, \text{ or} \\ \cdot\ x = r_m \in (\mathbb{Q} \setminus S_q) \cap I \text{ for some } m \text{ such that} \\ \quad 1/m < \varepsilon \text{ by construction.} \end{cases}$

But then $|f(x)| = |1/m| < \varepsilon$ as desired. $\square$

So
- $c \in I \setminus \mathbb{Q} \Rightarrow f(c) = 0 = \lim_{x \to c} f(x)$,
- $c = r_n \in I \cap \mathbb{Q} \Rightarrow f(c) = \frac{1}{n} \neq 0 = \lim_{x \to c} f(x)$

and f is discontinuous on $I \cap \mathbb{Q}$. $\blacksquare$