

Title

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Problem 1.0.1 (Weibel 1.3.3)

Prove the 5-lemma. Suppose the following rows are exact:

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{\partial_1^A} & A_2 & \xrightarrow{\partial_2^A} & A_3 & \xrightarrow{\partial_3^A} & A_4 & \xrightarrow{\partial_4^A} & A_5 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 B_1 & \xrightarrow{\partial_1^B} & B_2 & \xrightarrow{\partial_2^B} & B_3 & \xrightarrow{\partial_3^B} & B_4 & \xrightarrow{\partial_4^B} & B_5
 \end{array}$$

[Link to Diagram](#)

- Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
- Show that if f_2, f_4 are epi and f_5 is monic, then f_3 is an epi.
- Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_3 is an isomorphism.

Solution (Part (a)):

Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with “ \hookrightarrow ” and the epis with “ \twoheadrightarrow ”:

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{\partial_1^A} & A_2 & \xrightarrow{\partial_2^A} & A_3 & \xrightarrow{\partial_3^A} & A_4 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
 B_1 & \xrightarrow{\partial_1^B} & B_2 & \xrightarrow{\partial_2^B} & B_3 & \xrightarrow{\partial_3^B} & B_4
 \end{array}$$

[Link to Diagram](#)

It suffices to show that f_3 is an injection, and since these can be thought of as R -module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f_3(x) = 0 \in B_3$.
- Then under $A_3 \rightarrow B_3 \rightarrow B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \rightarrow A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \rightarrow B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2 .
- Commutativity of square 2 forces $y_3 \rightarrow 0$ under $B_2 \rightarrow B_3$.
- Exactness of the bottom row allows pulling this back to some $y_4 \in B_1$.
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \rightarrow A_2$ and $y_5 \mapsto x$ under $A_1 \rightarrow A_2 \rightarrow A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \rightarrow A_2 \rightarrow A_3$, so $x = 0$.

$$\begin{array}{ccccccc}
 y_5 \in A_1 & \xrightarrow{\partial_1^A} & y_2 \in A_2 & \xrightarrow{\partial_2^A} & x \in A_3 & \xrightarrow{\partial_3^A} & y_1 \in A_4 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
 y_4 \in B_1 & \xrightarrow{\partial_1^B} & y_3 \in B_2 & \xrightarrow{\partial_2^B} & 0 \in B_3 & \xrightarrow{\partial_3^B} & 0 \in B_4
 \end{array}$$

3
2
1

[Link to Diagram](#)

Solution (Part (b)):

Similarly, it suffices to consider the following portion of the diagram:

$$\begin{array}{ccccccc}
 A_2 & \xrightarrow{\partial_2^A} & A_3 & \xrightarrow{\partial_3^A} & A_4 & \xrightarrow{\partial_4^A} & A_5 \\
 \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 B_2 & \xrightarrow{\partial_2^B} & B_3 & \xrightarrow{\partial_3^B} & B_4 & \xrightarrow{\partial_4^B} & B_5
 \end{array}$$

[Link to Diagram](#)

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

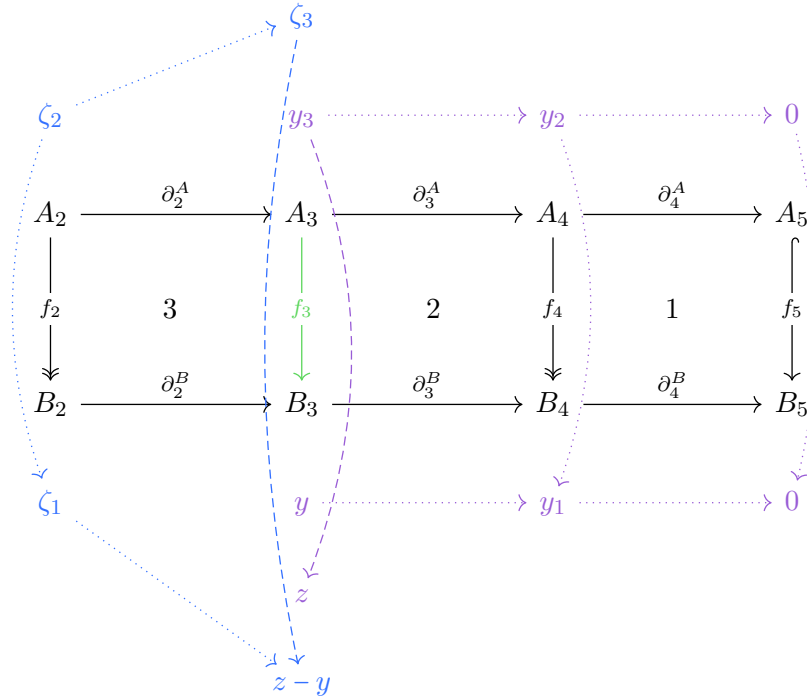
Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \rightarrow B_4$.
- By exactness of the bottom row, under $B_4 \rightarrow B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4 , pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \rightarrow A_5 \rightarrow B_5$.
- By injectivity of f_5 , the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \rightarrow A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \rightarrow B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \rightarrow B_4$
- Thus $z - y \mapsto y_1 - y_1 = 0$ under $B_3 \rightarrow B_4$, using that $d(z - y) = d(z) - d(y)$ since these are R -module morphisms.

- By exactness of the bottom row, pull $z - y$ back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2 , pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z - y$ under $A_2 \rightarrow B_2 \rightarrow B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \rightarrow A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z - y$ under $A_3 \rightarrow B_3$.
- But then $y_3 - \zeta_3 \mapsto z - (z - y) = y$ under $A_3 \rightarrow B_3$ as desired.



[Link to Diagram](#)

Solution (Part (c)):

Given the previous two results, if the outer maps are isomorphisms then f_3 is both monic and epi. Using a technical fact that monic epis are isomorphisms in a category \mathcal{C} if and only if \mathcal{C} is *balanced* and that all abelian categories are balanced, f_3 is an isomorphism.

Problem 1.0.2 (Weibel 1.4.2)

Let C be a chain complex. Show that C is split if and only if there are R -module decompositions

$$\begin{aligned} C_n &\cong Z_n \oplus B'_n \\ Z_n &= B_n \oplus H'_n. \end{aligned}$$

Show that C is split exact if and only if $H'_n = 0$.

Solution:

For this problem, we'll use the fact that if d is an epimorphism, it satisfies the right-cancellation property: if $f \circ d = g \circ d$, then $f = g$. In particular, if $d_n = d_n s_{n-1} d_n$ with $d_n : C_n \rightarrow C_{n-1}$

surjective and $s_{n-1} : C_{n-1} \rightarrow C_n$, we can conclude $\mathbb{1}_{C_n} = d_n s_{n-1}$. We'll also use the fact that if we have a SES in any abelian category \mathcal{A} , then the following are equivalent:

- The sequence is split on the left.
- The sequence is split on the right.
- The middle term is isomorphic to the direct sum of the outer terms.

Fixing notation, we'll write $C := (\cdots \rightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots)$, and we'll use concatenation fg to denote function composition $f \circ g$.

$\implies :$

Suppose C is split, so we have maps $\{s_n\}$ such that $\partial_n = \partial_n s_{n-1} \partial_n$.

Claim: The short exact sequence

$$0 \rightarrow Z_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0$$

admits a right-splitting $f : B_{n-1} \rightarrow C_n$, and thus there is an isomorphism

$$C_n \cong Z_n \oplus B'_n := Z_n \oplus B_{n-1}.$$

Proof (?).

That this sequence is exact follows from the fact that it can be written as

$$0 \rightarrow \ker \partial_n \hookrightarrow C_n \xrightarrow{\partial_n} \text{im } \partial_n \rightarrow 0.$$

We proceed by constructing the splitting f . Noting that $s_{n-1} : C_{n-1} \rightarrow C_n$ and $B_{n-1} \leq C_{n-1}$, the claim is that its restriction $f := s_{n-1}|_{B_{n-1}}$ works. It suffices to show that $(C_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{f} C_n)$ composes to the identity map $\mathbb{1}_{C_n}$. This follows from the splitting assumption, along with right-cancellability since ∂_n is surjective onto its image:

$$\partial_n = \partial_n s_{n-1} \partial_n \xrightarrow{\text{right-cancel } \partial_n} \mathbb{1}_{C_n} = \partial_n s_{n-1} := \partial_n f.$$

■

Claim: The SES

$$0 \rightarrow B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\pi} \frac{Z_n}{B_n} \rightarrow 0$$

admits a left-splitting $f : Z_n \rightarrow B_n$, and thus there is an isomorphism

$$Z_n \cong B_n \oplus H'_n := B_n \oplus H_n(C) := B_n \oplus \frac{Z_n}{B_n}.$$

Proof (?).

We proceed by again constructing the splitting $f : Z_n \rightarrow B_n$. The situation is summarized in the following diagram:

$$\begin{array}{ccccccc}
 & & & \xleftarrow{s_n} & & & \\
 & & & \partial_{n+1} & & & \\
 \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \longrightarrow & \cdots \\
 & & \searrow \partial_{n+1} & & \uparrow \iota_Z & & \\
 & & & & Z_n & & \\
 & & & & \uparrow \iota_{BZ} & & \\
 & & & & B_n & &
 \end{array}$$

[Link to Diagram](#)

So a natural candidate for the map f is the composition

$$Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} B_n,$$

so $f := \partial_{n+1} s_n \iota_Z$. We can simplify this slightly by regarding $Z_n \leq C_n$ as a submodule to suppress ι_Z , and identify s_n with its restriction to Z_n to write $f := \partial_{n+1} s_n$. The claim is then that $f \iota_{BZ} = \mathbb{1}_{B_n}$. Anticipating using the fact that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$, we post-compose with ∂_{n+1} and compute:

$$\begin{aligned}
 f \iota_{BZ} \partial_{n+1} &= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_n) \\
 &= (C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{s_n|_{B_n}} C_{n+1} \xrightarrow{\partial_{n+1}}) \\
 &= \partial_{n+1} s_n \partial_{n+1} \\
 &= \partial_{n+1},
 \end{aligned}$$

where in the last step we've used the splitting hypothesis and the fact that it remains true when everything is restricted to the submodule $B_n \leq C_n$. Using surjectivity of ∂_{n+1} onto B_n , we can now conclude as before:

$$f \iota_{BZ} \partial_{n+1} = \partial_{n+1} \xrightarrow{\text{right-cancel } \partial_n} f \iota_{BZ} = \mathbb{1}_{B_n}.$$

■

Problem 1.0.3 (Weibel 1.4.3)

Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

Problem 1.0.4 (Weibel 1.4.5)

Show that chain homotopy classes of maps form a quotient category K of $\text{Ch}(R\text{-mod})$ and that the functors H_n factor through the quotient functor $\text{Ch}(R\text{-mod}) \rightarrow K$ using the following

steps:

1. Show that chain homotopy equivalence is an equivalence relation on $\{f : C \rightarrow D \mid f \text{ is a chain map}\}$. Define $\text{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
2. Let $f \simeq g : C \rightarrow D$ be two chain homotopic maps. If $u : B \rightarrow C, v : D \rightarrow E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
3. Let $f_0, f_1, g_0, g_1 : C \rightarrow D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $\text{Ch}(R\text{-mod}) \rightarrow K$ is an additive functor.
4. Is K an abelian category? Explain.

Try at least two parts.

Problem 1.0.5 (Weibel 1.5.1)

Let $\text{cone}(C) := \text{cone}(\mathbb{1}_C)$, so

$$\text{cone}(C)_n = C_{n-1} \oplus C_n.$$

Show that $\text{cone}(C)$ is split exact, with splitting map given by $(b, c) \mapsto (-c, 0)$.

Problem 1.0.6 (Weibel 1.5.2)

Let $f : C \rightarrow D \in \text{Mor}(\text{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(s, f) : \text{cone}(C) \rightarrow D.$$

Problem 1.0.7 (Extra)

- a. Show that free implies projective.
- b. Show that $\text{Hom}_R(M, \cdot)$ is left-exact.
- c. Show that P is projective if and only if $\text{Hom}_R(P, \cdot)$ is exact.