Problem Set 7

D. Zack Garza

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1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence, so $\|x^j - x^k\|_{\ell^2} \to 0$. We want to produce some $\mathbf{x} := \lim\limits_{n\to\infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i := \lim_{n\to\infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left|x_i^j - x_i^k\right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left|x_i^j - x_i^k\right|^2$ is a Cauchy sequence of real numbers which necessarily converges in $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\|\lim_{k\to\infty} x^k - x^j\right\|_{\ell^2} = \lim_{k\to\infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$
\begin{aligned}
\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\
&\le \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\
&\to M < \infty,
\end{aligned}
$$

where $\|\mathbf{x}_i - x^j\|_{\ell^2} \to 0$ and the second sum is finite because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \ \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, we have

$$\Re(\langle x, \ iy \rangle) = \Re(-i\langle x, \ y \rangle) = \Im(\langle x, \ y \rangle).$$

We can compute the following:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2 \ \Re\left(\langle x, \ y \rangle\right)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2 \ \Re\left(\langle x, \ y \rangle\right)$$

$$
\begin{aligned}
\|x+iy\|^2 &= \|x\|^2 + \|y\|^2 + 2 \ \Re\left(\langle x, \ iy \rangle\right) \\
&= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle)
\end{aligned}
$$

$$
\begin{aligned}
\|x-iy\|^2 &= \|x\|^2 + \|y\|^2 - 2 \ \Re\left(\langle x, \ iy \rangle\right) \\
&= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle)
\end{aligned}
$$

and summing these all

$$
\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\| &= 4 \ \Re\left(\langle x, \ y \rangle\right) + 4i \ \Im\left(\langle x, \ y \rangle\right) \\
&= 4\langle x, \ y \rangle.
\end{aligned}
$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 := \langle x, \ x \rangle = \langle Ux, \ Ux \rangle := \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$
\begin{aligned}
\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2+\|Ux-Uy\|^2+i\|Ux+Uy\|^2-i\|Ux+Uy\|^2\right)\\
&= \frac{1}{4}\left(\|U(x+y)\|^2+\|U(x-y)\|^2+i\|U(x+y)\|^2-i\|U(x+y)\|^2\right)\\
&= \frac{1}{4}\left(\|x+y\|^2+\|x-y\|^2+i\|x+y\|^2-i\|x+y\|^2\right)\\
&= \langle x,\ y\rangle.
\end{aligned}
$$

□

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H\times H\to\mathbb{R}$ is continuous.

Proof:

Let $x_n\to x$ and $y_n\to y$, then

$$
\begin{aligned}
|\langle x_n,\ y_n\rangle-\langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle-\langle x,\ y_n\rangle+\langle x,\ y_n\rangle-\langle x,\ y\rangle|\\
&= |\langle x_n-x,\ y_n\rangle+\langle x,\ y_n-y\rangle|\\
&\le \|x_n-x\|\|y_n\|+\|x\|\|y_n-y\|\\
&\to 0\cdot M+C\cdot 0<\infty,
\end{aligned}
$$

where $\|y_n\|\to M$ since $y_n\to y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y\in E\implies\langle e_n,\ y\rangle=0$. Since H is complete, $e_n\to e\in H$; we can show that $e\in E^\perp$ by letting $y\in E$ be arbitrary and computing

$$
\langle e,\ y\rangle=\left\langle\lim_n e_n,\ y\right\rangle=\lim_n\langle e_n,\ y\rangle=\lim_n 0=0,
$$

so $e\in E^\perp$.

2.2 Part b:

Let $S:=\operatorname{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp}=\overline{S}$.

$\overline{S}\subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n\to x\in\overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$y \in E^{\perp} \implies \langle x_n,\ y \rangle = \left\langle \sum_i a_i e_i,\ y \right\rangle = \sum_i a_i \langle e_i,\ y \rangle = 0 \implies x_n \in (E^{\perp})^{\perp}.$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$y \in E^{\perp} \implies \langle x,\ y \rangle = \left\langle \lim_n x_n,\ y \right\rangle = \lim_n \langle x_n,\ y \rangle = 0 \implies x \in (E^{\perp})^{\perp},$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$: