Title

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Contents

1 Thursday January 9th

Recall: For M^n a closed smooth manifold, consider a smooth map $f : M^n \to \mathbb{R}$.

Definition: A critical point p of f is *non-degenerate* iff $\det(H \coloneqq \dfrac{\partial^i f}{\partial x_i \partial x_j}(p)) \neq 0$ in some coordinate system U.

Lemma (The Morse Lemma): For any non-degenerate critical point p there exists a coordinate system around p such that

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

λ is called the *index of f at p*.

Lemma: λ is equal to the number of *negative* eigenvalues of $H(p)$.

Proof: A change of coordinates sends $H(p) \to A^t H(p) A$, which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that A is invertible and not necessarily orthogonal.

This means that f can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Proof of Morse Lemma:

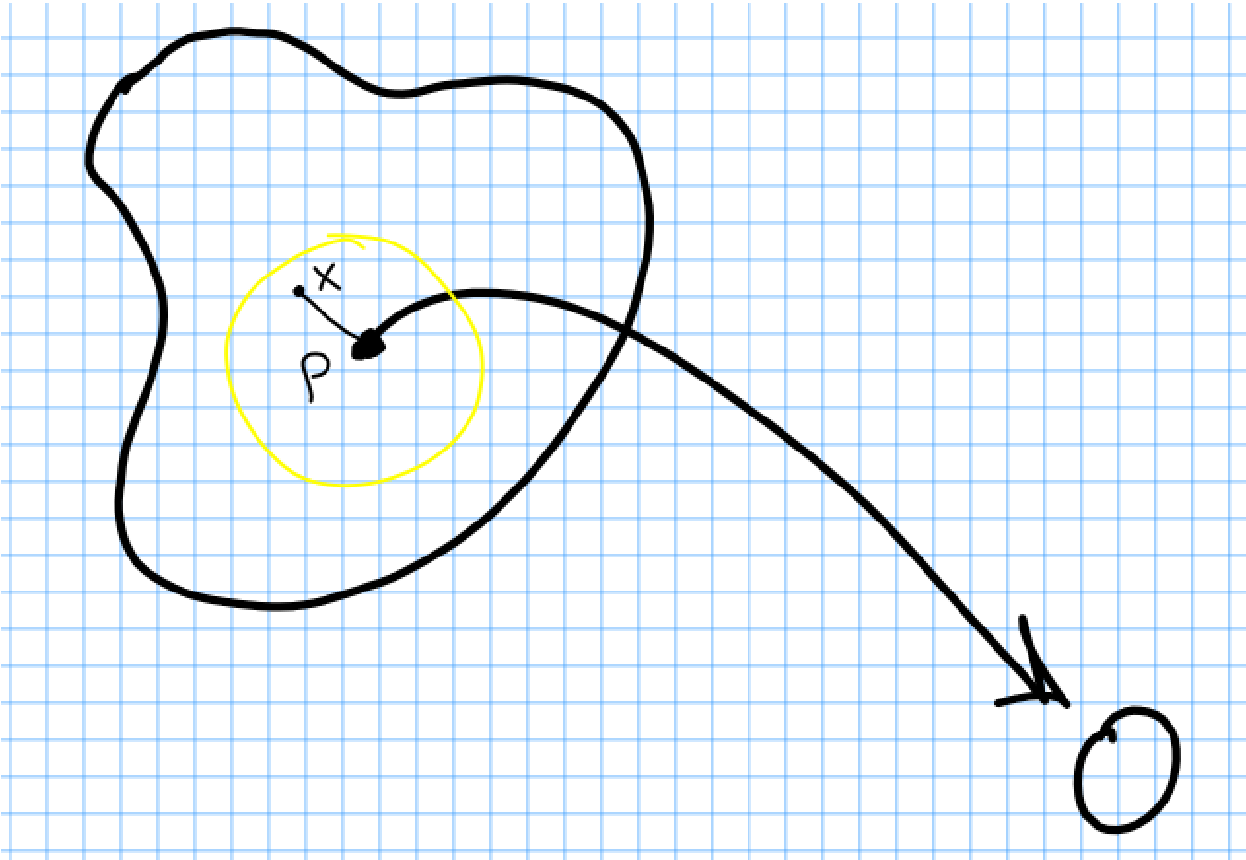
Suppose that we have a coordinate chart U around p such that $p \mapsto 0 \in U$ and $f(p) = 0$.

Step 1 – Claim: There exists a coordinate system around p such that

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

where $h_{ij}(x) = h_{ji}(x)$.

Proof: Pick a convex neighborhood V of $0 \in \mathbb{R}^n$.



Restrict f to a path between x and 0, and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \cdots, tx_n)}{dt}\, dt = f(x_1, \cdots, x_n) - f(0) = f(x_1, \cdots, x_n).$$

since $f(0) = 0$.

We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots \frac{\partial f}{\partial x_n} x_n\, dt \implies \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}\, dt = f(x).$$

We thus have $f(x) = \sum_{i=1}^{n} x_i g_i(x)$ where $\frac{\partial f}{\partial x_i}(0) = 0$, and $\frac{\partial f}{\partial x_i} = x_1 \frac{\partial g_1}{\partial x_i} + \cdots + g_i + x_i \frac{\partial g_i}{\partial x_i} + \cdots + x_n \frac{\partial g_n}{\partial x_i}$. When we plug $x = 0$ into this expression, the only term that doesn't vanish is g_i, and thus $\frac{\partial f}{\partial x_i}(0) = g_i(0)$ and $g_i(0) = 0$.

Applying the same result to g_i, we obtain $g_i(x) = \sum_{j=1}^{n} x_j h_{ij}(x)$, and thus $f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x)$.

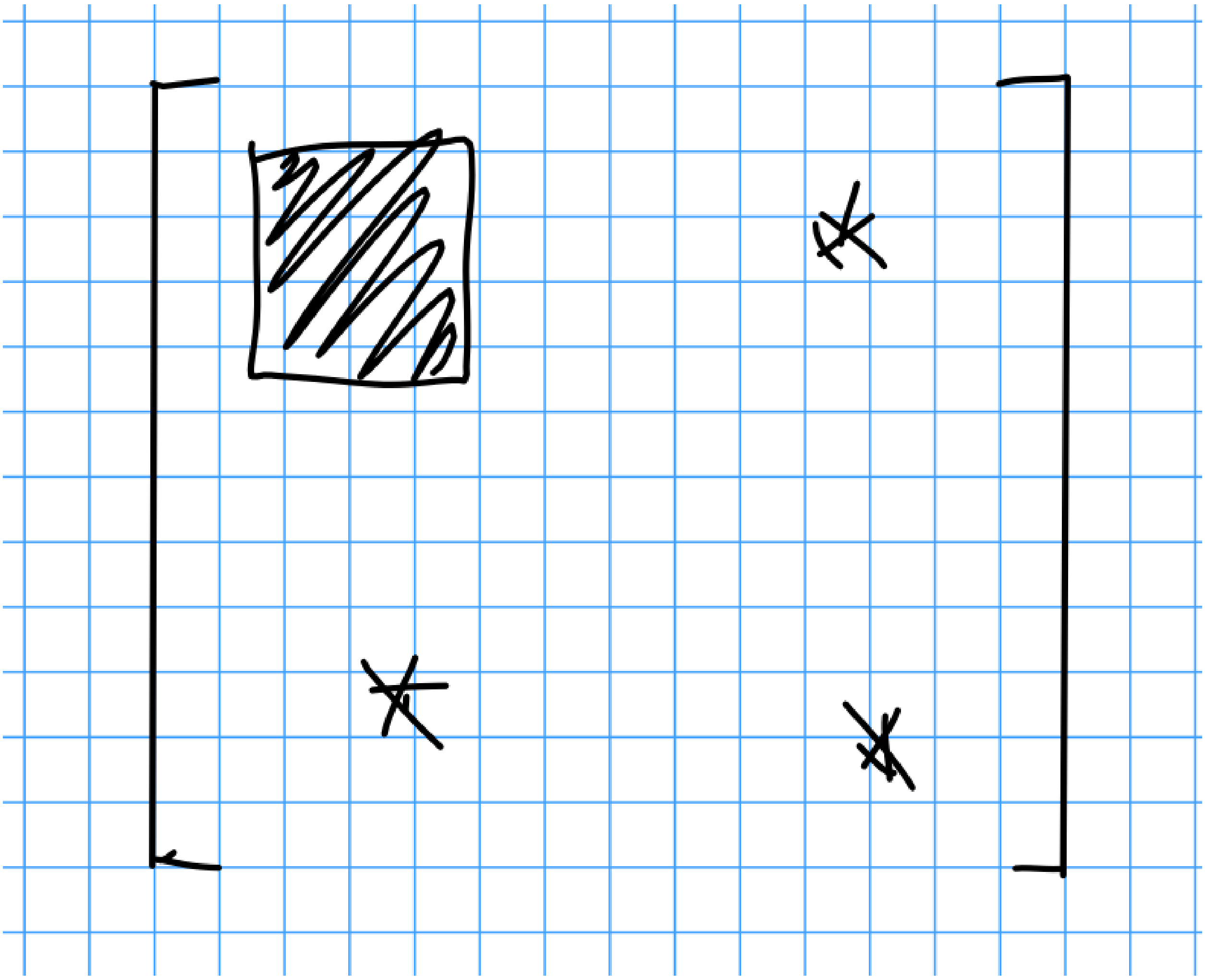
We still need to show h is symmetric. For every pair i, j, there is a term of the form $x_i x_j h_{ij} + x_j x_i h_{ji}$. So let $H_{ij}(x) = \frac{h_{ij}(x) + h_{ji}(x)}{2}$ (i.e. symmetrize/average h), then $f(x) = \sum_{i,j=1}^{n} x_i x_j H_{ij}(x)$ and this shows claim 1.

$\blacksquare$

Step 2 – Induction: Assume that in some coordinate system U_0,

$$f(y_1, \cdots, y_n) = \pm y_1^2 \pm y_2^2 \pm \cdots \pm y_{r-1}^2 + \sum_{i,j \geq r} y_i y_j H_{ij}(y_1, \cdots, y_n).$$

Note that $H_{rr}(0)$ is given by the top-left block of $H_{ij}(0)$, which is thus looks like



Note that this block is symmetric.

Claim 1: There exists a linear change of coordinates such that $H_{rr}(0) \neq 0$.

We can use the fact that $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(0) = H_{ij}(0) + H_{ji}(0) = 2H_{ij}(0)$, and thus $H_{ij}(0) = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x_i \partial x_j}\right)$.

Since $H(0)$ is non-singular, we can find A such that $A^t H(0) A$ has nonzero rr entry, namely by letting the first column of A be an eigenvector of $H(0)$, then $A = [\mathbf{v}, \cdots]$ and thus $H(0)A = [\lambda \mathbf{v}, \cdots]$ and $A^t[\lambda \mathbf{v}] = [\lambda \|\mathbf{v}\|^2, \cdots]$.

So