

# Assignment 6: The Fourier Transform

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## 1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = \lim_{\xi' \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx$$

But as an immediate consequence, this yields

$$\begin{aligned}
|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx \right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \, dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\
&\rightarrow 0,
\end{aligned}$$

which follows from continuity in  $L^1$  since  $f(x - \xi') \rightarrow f(x)$  as  $\xi' \rightarrow 0$ .

It thus only remains to show that the hint holds, and that  $\xi' \rightarrow 0$  as  $\xi \rightarrow \infty$ .

## 2 Problem 2

### 2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned}
\widehat{(f * g)}(\xi) &:= \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dy \, dx \\
&= \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dx \, dy \\
&= \int \int f(t) \exp(-2\pi i (x - y) \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dx \, dy \\
&\quad (t = x - y, \, dt = dx) \\
&= \int \int f(t) \exp(-2\pi i t \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dt \, dy \\
&= \int f(t) \exp(-2\pi i t \cdot \xi) \left( \int g(y) \exp(-2\pi i y \cdot \xi) \, dy \right) \, dt \\
&= \int f(t) \exp(-2\pi i t \cdot \xi) \hat{g}(\xi) \, dt \\
&= \hat{g}(\xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\
&= \hat{g}(\xi) \hat{f}(\xi).
\end{aligned}$$

It thus remains to show that this swap is justified.

### 2.2 Part (b)

We'll use the following lemma: if  $\hat{f} = \hat{g}$ , then  $f = g$  almost everywhere.

#### 2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f} \hat{g} = \hat{g} \hat{f} = \widehat{g * f},$$

and so by the lemma,  $f * g = g * f$ .

Similarly, we have

$$(\widehat{f * g}) * h = \widehat{f * g} \hat{h} = \hat{f} \hat{g} \hat{h} = \hat{f} \widehat{g * h} = f * (g * h).$$

### 2.2.2 (ii)

Suppose that there exists some  $I \in L^1$  such that  $f * I = f$ . Then  $\widehat{f * I} = \hat{f}$  by the lemma, so  $\hat{f} \hat{I} = \hat{f}$  by the above result.

But this says that  $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$  almost everywhere, and thus  $\hat{I}(\xi) = 1$  almost everywhere. Then  $\lim_{|\xi| \rightarrow \infty} \hat{I}(\xi) \neq 0$ , which by Problem 1 shows that  $I$  can not be in  $L^1$ , a contradiction.

## 3 Problem 3

### 3.1 (a)

#### 3.1.1 (i)

Let  $g(x) = f(x - y)$ . We then have

$$\begin{aligned} \hat{g}(\xi) &:= \int g(x) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \exp(-2\pi i y \cdot \xi) \, dx \\ &= \exp(-2\pi i y \cdot \xi) \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \, dx \\ &\quad (t = x - y, dt = dx) \\ &= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\ &= \exp(-2\pi i y \cdot \xi) \hat{f}(\xi). \end{aligned}$$

#### 3.1.2 (ii)

Let  $h(x) = \exp(2\pi i x \cdot y) f(x)$ . We then have

$$\begin{aligned} \hat{h}(\xi) &:= \int \exp(2\pi i x \cdot y) f(x) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int \exp(2\pi i x \cdot y - 2\pi i x \cdot \xi) f(x) \, dx \\ &= \int f(\xi - y) \exp(-2\pi i x \cdot (\xi - y)) \, dx \\ &= \hat{f}(\xi - y). \end{aligned}$$

### 3.2 (b)

We'll use the fact that if  $\langle \cdot, \cdot \rangle$  is an inner product on a vector space  $V$  and  $A$  is an invertible linear transformation, then for all  $\mathbf{x}, \mathbf{y} \in V$  we have

$$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle$$

where  $A^{-T}$  denotes the transpose of the inverse of  $A$  (or  $(A^{-1})^*$  if  $V$  is complex).

We then have

$$\begin{aligned} \frac{1}{|\det T|} \hat{f}(T^{-T} \xi) &= \frac{1}{|\det T|} \int f(x) \exp(-2\pi i x \cdot T^{-T} \xi) dx \\ &\quad x \mapsto Tx, \quad dx \mapsto |\det T| dx \\ &= \frac{1}{|\det T|} \int f(Tx) \exp(-2\pi i Tx \cdot T^{-T} \xi) |\det T| dx \\ &= \int f(Tx) \exp(-2\pi i x \cdot \xi) dx \\ &\quad \text{since } Tx \cdot T^{-T} \xi = T^{-1} Tx \cdot \xi = x \cdot \xi \\ &= \widehat{(f \circ T)}(\xi). \end{aligned}$$

## 4 Problem 4

### 4.1 (a)

#### 4.1.1 (i)

Let  $g(x) = xf(x)$ . Then if an interchange of the derivative and the integral is justified, we have

$$\begin{aligned} \frac{\partial}{\partial \xi} \hat{f}(\xi) &:= \frac{\partial}{\partial \xi} \int f(x) \exp(-2\pi i x \cdot \xi) dx \\ &=? \int f(x) \frac{\partial}{\partial \xi} \exp(-2\pi i x \cdot \xi) dx \\ &= \int f(x) 2\pi i x \exp(-2\pi i x \cdot \xi) dx \\ &= 2\pi i \int x f(x) \exp(-2\pi i x \cdot \xi) dx \\ &:= 2\pi i \hat{g}(\xi). \end{aligned}$$

It thus remains to show that this interchange is justified. TODO

#### 4.1.2 (ii)

We have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= f(x) \exp(-2\pi i x \cdot \xi) \Big|_{x=-\infty}^{x=\infty} - \int f(x) (2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx \\
&\quad \text{(integrating by parts)} \\
&= - \int f(x) (-2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx \\
&\quad \text{(since } f(\infty) = f(-\infty) = 0) \\
&= 2\pi i \xi \int f(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&:= 2\pi i \xi \hat{f}(\xi).
\end{aligned}$$

## 4.2 (b)

Let  $G(x) = \exp(-\pi x^2)$ .

Then

$$\partial_\xi \left( \frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi) \partial_\xi \hat{G}(\xi) - G(\xi) \partial_\xi \hat{G}(\xi)}{G(\xi)^2}$$

## 5 Problem 5

## 6 Problem 6