Lie Algebras

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Contents

1 Lecture 1

> The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory

- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

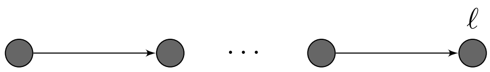
1.1.2 Chapter 3-4

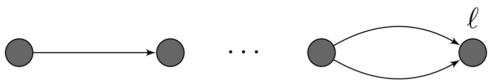
We will describe the following series of correspondences:

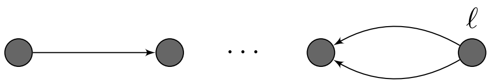
$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\bigoplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

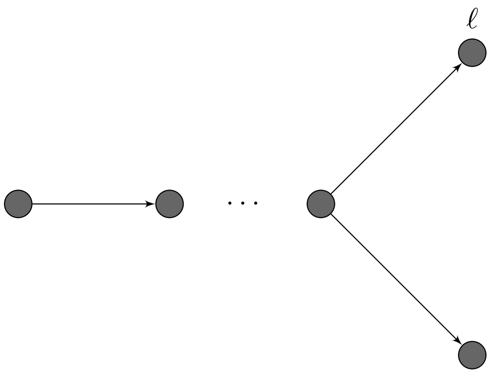
1.2 Classification

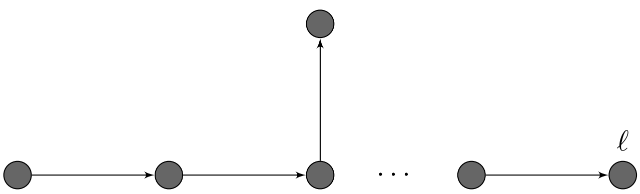
The classical Lie algebras can be essentially classified by certain classes of diagrams:

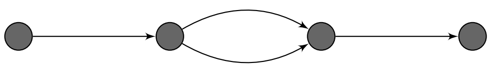
A_ℓ : 

B_ℓ : 

C_ℓ : 

D_ℓ : 

E_6, E_7, E_8 : 

F_4: 

1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
    - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned}[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y].\end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x,[y,z]]+[y,[x,z]]+[z,[x,y]]=\mathbf{0}.$$

Exercise 1. Show that $[x,y]=-[y,x]$.

Definition 2. Two Lie algebras $\mathfrak{g},\mathfrak{g}'$ are said to be isomorphic if $\varphi([x,y])=[\varphi(x),\varphi(y)]$.

1.5 Linear Lie Algebras

Let $V=\mathbb{F}^n$, and define $\mathrm{End}(V)=\{f:V\to V \ni V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n,V)$ by setting $[x,y]=(x\circ y)-(y\circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n,V)=\{f\in\mathfrak{gl}(n,V)\ni \mathrm{Tr}(f)=0\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n,V)$.).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

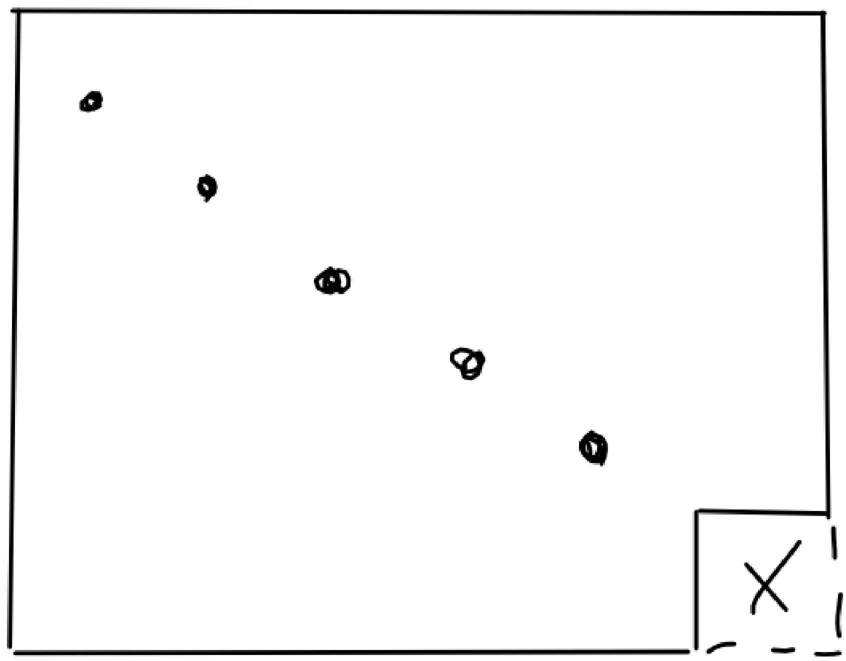
$$\mathfrak{sp}(2\ell,F)=\left\{A\in\mathfrak{gl}(2\ell,F)\ni MA-A^TM=0\right\} \text{ where } M=\left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0\end{array}\right).$$

Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell,F)=\left\{A\in\mathfrak{gl}(2\ell,F)\ni MA-A^TM=0\right\} \text{ where}$$
$$M=\begin{cases}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0\end{array}\end{array}\right) & n=2\ell+1 \text{ odd,}\\[2ex] \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0\end{array}\right) & \text{else.}\end{cases}$$

Proposition 1. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and



zero elsewhere.
- For type A_ℓ, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell+1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.
- For type B_ℓ, we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array}\right).$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}$$

Theorem 1. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

Exercise 4. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 7. An F-**algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition 8. An algebra is **associative** if $x(yz) = (xy)z$.

> Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 9. Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 10. We define $\mathrm{Der}(A) = \{\delta \ni \delta \text{ is a derivation }\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\cdot\,,\,\cdot]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x,[y,z]] = 0$.

Exercise 5. Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 6 (Turn in)**.** Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x,y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 1. Any F-vector space can be made into a Lie algebra by setting $[x,y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x,x] = 0 \implies [\cdot\,,\,\cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x,y]$. Some cases:
  - $[x,y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x,y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x,y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\cdot\,,\,\cdot]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 2. Let $V = \mathbb{R}^3$, and define $[a,b] = a \times b$ to be the usual cross product.

Exercise 7. Look at notes for basis elements of $\mathfrak{sl}(2,F)$,

$$\begin{aligned} e &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ f &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Compute the matrices of $\mathrm{ad}(e), \mathrm{ad}(h), \mathrm{ad}(g)$ with respect to this basis.

2.3 Ideals

Definition 11. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x,y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x,y] = [-y,x]$.

Exercise 8. Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x,z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n,F), \mathfrak{gl}(n,F)] = \mathfrak{sl}(n,F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 12. A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 2. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \mathrm{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$
\begin{aligned}
2e = [h, e] \in I &\implies e \in I, \\
2f = [h, -f] \in I &\implies f \in I.
\end{aligned}
$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

□

Note that there is a homework coming due next Monday, about 4 questions.

3 Lecture 3

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x + I, y + I] = [x, y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x, y] + I = [x', y'] + I$.

3.1 Homomorphisms and Representations

Definition 13. A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x, y] = [\phi(x), \phi()]$.

Remark. $\ker \phi \trianglelefteq \mathfrak{g}_1$ and im $\phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I + J)/J \cong I/(I \bigcap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V.

We call V a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

Corollary 1. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) = \Big\{ x \in \mathfrak{g} \ni \text{ad}_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \Big\} \\ = \ker \text{ad}_x. \end{aligned}$$

<!–> Idea: Define a semisimple Lie algebra–>

<!–> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while ad_x *is* nilpotent, which contradicts the above theorem.–>