Moduli Spaces

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Contents

List of Definitions

List of Theorems

1 Thursday January 9th

1.1 Representability

Last time: Fix an S-scheme, i.e. a scheme over S.

Then there is a map

$$\operatorname{Sch}/S \longrightarrow \operatorname{Fun}(\operatorname{Sch}/S^{\mathrm{op}}, \operatorname{Set})$$
$$x \mapsto h_x(T) = \hom_{\operatorname{Sch}/S}(T, x).$$

where $T' \xrightarrow{f} T$ is given by

$$h_x(f) : h_x(T) \longrightarrow h_x(T')$$
$$T \mapsto x \longrightarrow \text{triangles}$$

of the form

$$
\begin{array}{ccc}
T' & \longrightarrow & X \\
& \searrow \quad \nearrow & \\
& T &
\end{array}
$$
.

Theorem 1.1 *(Yoneda).*

$$\hom_{\text{Fun}}(h_x, F) = F(x).$$

Corollary 1.2.

$$\hom_{\text{Sch}/S}(x, y) \cong \hom_{\text{Fun}}(h_x, h_y).$$

Definition 1.1.
A **moduli functor** is a map

$$F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set}$$
$$F(x) = \text{"Families of something over } x\text{"}$$
$$F(f) = \text{"Pullback".}$$

Definition 1.2.
A **moduli space** for that "something" appearing above is an $M \in \text{Obj}(\text{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \text{Spec}(k)$.

h_m is the functor of points over M.

Remark (1) $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong$ "families over $\text{Spec } k$" $= F(\text{Spec } k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M, and $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{Univ}$ is the universal family

Every family is uniquely the pullback of ξ_{Univ} This makes it much like a classifying space.

For $T \in \text{Sch}/S$,

$$h_M \xrightarrow{\cong} F$$
$$f \in h_M(T) \xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\text{Univ}}).$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\text{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

ξ_M $\qquad\qquad$ $\xi_{M'}$

$M \xrightarrow{\quad f \quad} M'$

$M' \xrightarrow{\quad g \quad} M$

$\xi_{M'}$ $\qquad\qquad$ ξ_M

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.
A length 2 subscheme of $\mathbb{A}^1_k$ then $F(S) = \left\{ V(x^2 + bx + c) \right\} \subset \mathbb{A}'_5$ where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

Then F is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber.

$F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\text{Spec } k[\varepsilon]$.
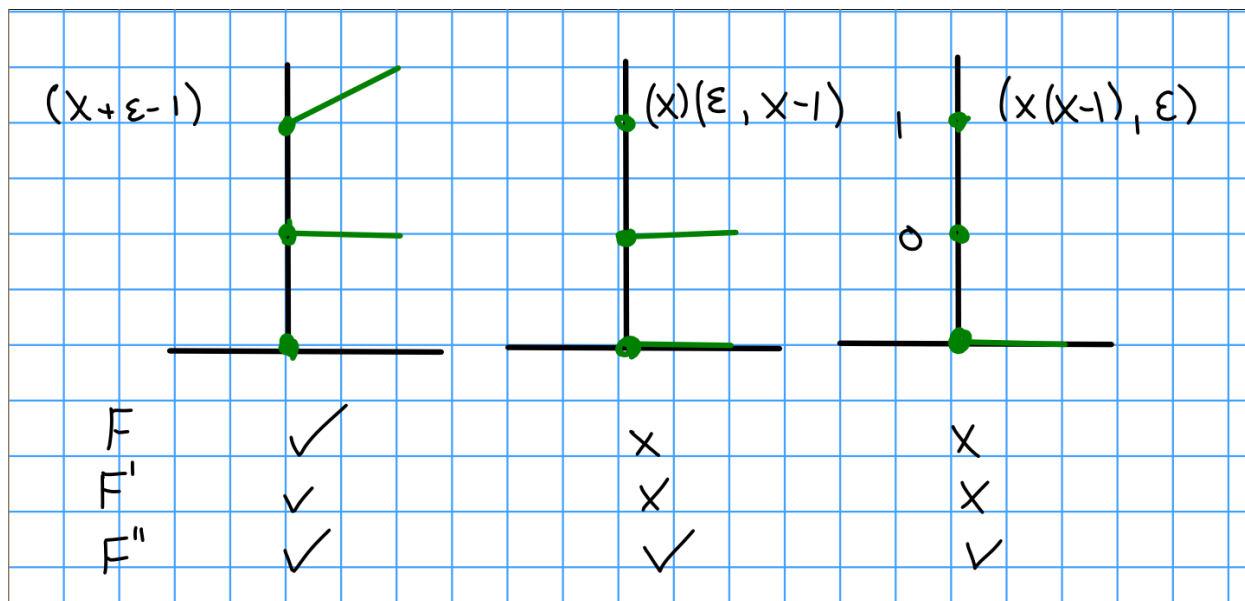
$$\text{Spec } k \xleftarrow{\quad i \quad} \text{Spec } k[\varepsilon] \qquad\qquad = F'(\text{Spec } k)$$

$$F(\text{Spec } k[\varepsilon]) \xrightarrow{\quad F(i) \quad} F(\text{Spec } k) \qquad\qquad = F''(\text{Spec } k)$$

$$\cup \Big\uparrow \qquad\qquad\qquad \in$$

$$T_p F^{',''} \qquad\qquad P = V(x(x-1))$$

We think of $T_p F^{',''}$ as the tangent space at p.

If F is representable, then it is actually the Zariski tangent space.

$$M(\text{Spec } k[\varepsilon]) \xrightarrow{\qquad\qquad} M(\text{Spec } k)$$

$$\subset \qquad\qquad\qquad \subset$$

$$T_p M \xrightarrow{\qquad\qquad} p$$

$$\text{Spec } k$$

$$\text{Spec } k[\varepsilon] \longrightarrow \text{Spec } \mathcal{O}_{M,p} \subset M$$

with label $?$ on the right arrow.

$$k$$

$$\mathcal{O}_{M,p} \longrightarrow k[\varepsilon]$$

$$m_p \qquad\qquad (\varepsilon)$$

$$m_p^2 \qquad\qquad 0$$

Moreover, $T_p M = (m_p/m_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$\lambda : k[\varepsilon] \longrightarrow k[\varepsilon]$$
$$\varepsilon \mapsto \lambda\varepsilon$$

$$\lambda^* : \text{Spec } (k[\varepsilon]) \longrightarrow \text{Spec } (k[\varepsilon])$$

$$\lambda : M(\text{Spec } (k[\varepsilon])) \longrightarrow M(\text{Spec } (k[\varepsilon]))$$
$$\supset T_p M \longrightarrow T_p M \subset .$$

Conclusion: If F is representable, for each $p \in F(\text{Spec } k)$ there exists a unique point of $T_p F$ that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$
\begin{array}{ccc}
F \times_G F' & \dashrightarrow & F' \\
\downarrow & & \downarrow \\
F & \longrightarrow & G
\end{array}
$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.

4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$F_T : (\text{Sch}/T) \longrightarrow \text{Set}$$
$$x \mapsto F(x).$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n + 1$ dimensional free module".

Claim: $\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$F : \text{Sch}^{\text{op}} \longrightarrow \text{Set}$$
$$F(S) = \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0/\sim .$$

where $\sim$ identifies diagrams of the following form:

$$
\begin{array}{ccccc}
\mathcal{O}_s^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & \\
\mathcal{O}_s^{n+1} & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

and $F(f)$ is given by pullbacks.

Remark $\mathbb{P}^n_S$ represents the following functor:

$$F_S : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set}$$
$$F_S(T) = \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0/\sim .$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of claim).
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n + 1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a

> subfunctor.
>> $F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.
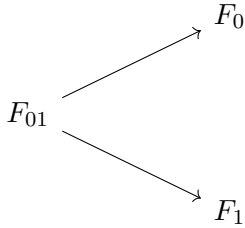>
> ∎

Claim 2: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$
\begin{array}{ccc}
F_i & \longrightarrow & F \\
\uparrow & & \uparrow \\
F_{ij} & \longrightarrow & F_j
\end{array}
$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example 1.2.
For $n = 1$, we can glue along open subschemes

For $n = 2$, we get overlaps of the following form:
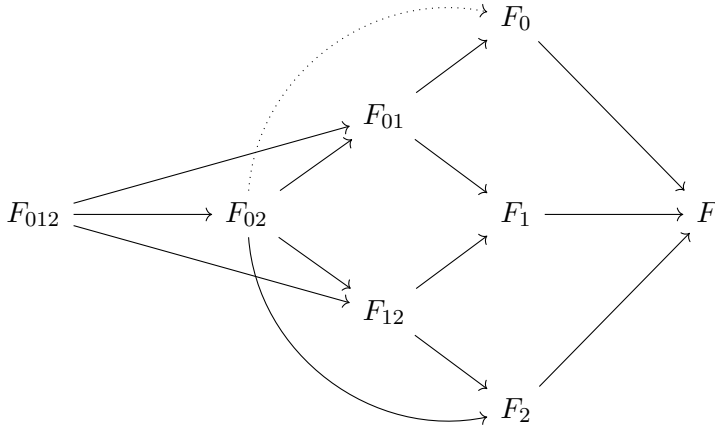
This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S.

RHS to LHS comes from gluing.

Proof of (Claim 2)

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i.

So specifying $F_i(S)$ is equivalent to specifying $n - 1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec $k[x_1, \cdots, \widehat{x_i}, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \underset{i \in I}{\underset{F}{\times}} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

2 Tuesday January 14th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
    - Slightly more detailed: Nitsure, ... Hilbert schemes, Fundamentals of Algebraic Geometry
    - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

Last time: Representability of functors, and specifically projective space $\mathbb{P}^n_{\mathbb{Z}}$ constructed via a functor of points, i.e.

$$
\begin{aligned}
h_{\mathbb{P}^n_{\mathbb{Z}}} : \mathbb{P}^n_{\mathbb{Z}} \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\
s &\mapsto \mathbb{P}^n_{\mathbb{Z}}(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}.
\end{aligned}
$$

for L a line bundle, up to isomorphisms of diagrams:

$$\mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow \bigcirc$$

$$\| \quad\quad\quad\quad \|\simeq$$

$$\mathcal{O}_S^{n+1} \longrightarrow M \longrightarrow \bigcirc$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}^n_{\mathbb{Z}}$ where $F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \;\middle|\; s_i \text{invertible} \right\}$ are representable and can be glued together, and projective space represents this functor.

Remark: Because projective space represents this functor, there is a universal object:

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \hookrightarrow L \twoheadrightarrow \mathcal{O}$$

$$\| $$

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$$

and other functors are pullbacks of the universal one.

Example 2.1.

Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over Spec $\mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$
\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spec} R & \longrightarrow & \operatorname{Spec} \mathbb{Z}
\end{array}
$$

Definition 2.1 (Equalizer)**.**
For a category C, we say a diagram $x \longrightarrow y \rightrightarrows z$ is an *equalizer* iff it is universal wrt the property:

$$X \longrightarrow Y \rightrightarrows Z$$

$$\exists! \qquad \circlearrowright$$

$$S$$

(Here X is the universal object).

Example 2.2.
For sets, $X = \left\{ y \mid f(y) = g(y) \right\}$ for $Y \xrightarrow{f,g} Z$.

Definition 2.2.
A coequalizer is the dual notion,

Example 2.3.
Take $C = Sch/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

These are canonically isomorphic.

In Sch/S, we have $\coprod_{\alpha\beta} X_{\alpha\beta} \xrightarrow{f_{\alpha\beta}, g_{\alpha\beta}} \coprod X_\alpha \longrightarrow X$ where $f_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_\alpha$ and $g_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_\beta$; this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:

Then $\phi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the cocycle condition:

1. $\phi_{\alpha\beta}^{-1}(X_{\beta\alpha} \bigcap X_{\beta\gamma}) = X_{\alpha\beta} \bigcap X_{\alpha\gamma}$, noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \longrightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

> **Definition 2.3** (Zariski Sheaf).
> A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_α, the following is an equalizer:
>
> $$F(T) \longrightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$
>
> where the maps are given by restrictions.

Example 2.4.
Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover $\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_\alpha T_\alpha \longrightarrow T$, since giving a local map to X that agrees on intersections if enough to specify a map from $T \longrightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

> **Definition 2.4** (Subfunctors, Open/Closed Functors).
> Suppose we have a morphism $F' \longrightarrow F$ in the category $\mathrm{Fun}(\mathrm{Sch}/S, \mathrm{Set})$.
> - This is a **subfunctor** if $\iota(T)$ is injective for all T/S.
> - ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over T, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F', which maps to F (noting a section is the same as a map to the functor of points), and since $T \longrightarrow F$ and $F' \longrightarrow F$, we can form the fiber product $F' \times_F T$:

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \longrightarrow F$ is open/closed/locally closed iff

$$
\begin{array}{ccc}
F' & \longrightarrow & F \\
\uparrow & & \uparrow \\
F' \times_F T & \xrightarrow{\ g\ } & T
\end{array}
$$

$$
F' \times_F T \ \| \ \text{rep}
$$

and g is open/closed/locally closed.

> I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \longrightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme

2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

> **Definition 2.5** (coverings).
> A collection of open subfunctors $F_\alpha \subset F$ is an open cover iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \longrightarrow F$, the T_α in the following diagram are an open cover of T:

$$\begin{array}{ccc} F_\alpha & \longrightarrow & F \\ \uparrow & & \uparrow \\ T_\alpha & \longrightarrow & T \end{array}$$

Example 2.5.
Given $F(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}$ and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \longrightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

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Proposition 2.1.
A Zariski sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ with a representable open cover is representable.

</div>

Proof .
Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \longrightarrow F_\alpha$ have the correct gluing maps. This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \longrightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x; we'll show that x represents F.
First produce a map $x \longrightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$
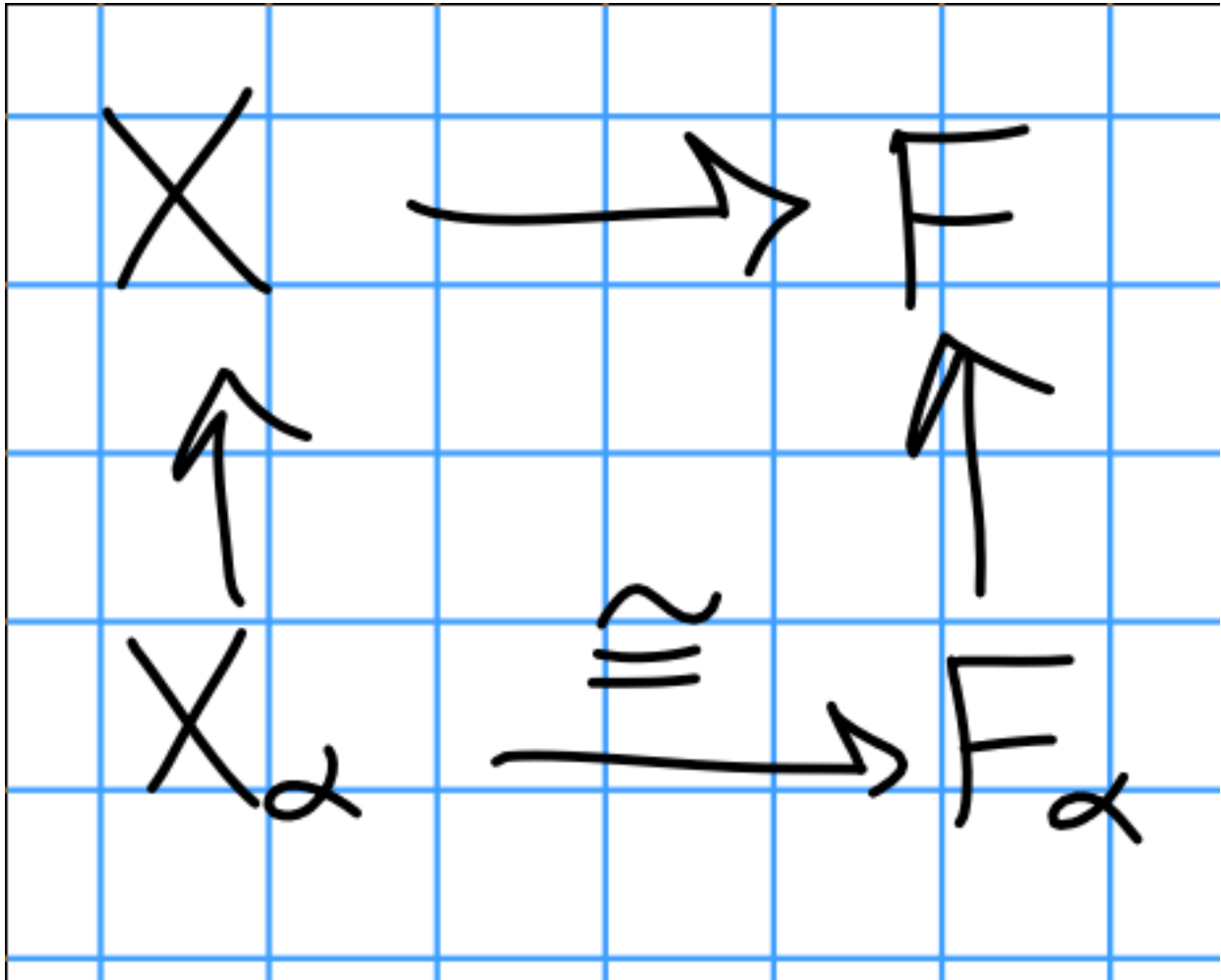
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Lemma 2.1.
If $E \longrightarrow F$ is a map of functors and E, F are zariski sheaves, where there are open covers $E_\alpha \longrightarrow E, F_\alpha \longrightarrow F$ with commutative diagrams

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$
\begin{array}{ccc}
X & \longrightarrow & F \\
\uparrow & & \uparrow \\
X_\alpha & \xrightarrow{\ \cong\ } & F_\alpha
\end{array}
$$

Example 2.6.

For S and E a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\}\,/\sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E.

Then the restriction $F_i \longrightarrow \mathbb{P}E$ were $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i, $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 2.7.

For E locally free over S of rank n, take $r < n$ and consider the functor $\mathrm{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\}\,/\sim$ (a Grassmannian) where Q is locally free of rank k.

Exercise

    a. Show that this is representable

    b. For the plucker embedding $\mathrm{Gr}(k, E) \longrightarrow \mathbb{P} \wedge^k E$, then a section over T is given by $f^*E \longrightarrow Q \longrightarrow 0$ corresponding to $\wedge^k f^*E \longrightarrow \wedge^k Q \longrightarrow 0$, noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \mathrm{Spec}\, k$, then E is a k-vector space V, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n.

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \longrightarrow Q \longrightarrow 0\} \ \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

> Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for $\mathbb{P}^n$ of all flat families of subschemes.