Title

D. Zack Garza

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Contents

1 Wednesday January 8

Course text: `http://math.uga.edu/~pete/integral2015.pdf`

Summary: The study of commutative rings, ideals, and modules over them.

The chapters we'll cover:

- 1 (Intro),
- 2 (Modules, partial),
- 3 (Ideals, CRT),
- 7 (Localization),
- 8 (Noetherian Rings),
- 11 (Nullstellensatz),
- 12 (Hilbert-Jacobson rings),
- 13 (Spectrum),
- 14 (Integral extensions),
- 17 (Valuation rings),
- 18 (Normalization),
- 19 (Picard groups),
- 20 (Dedekind domains),
- 22 (1-dim Noetherian domains)

In number theory, arises in the study of $\mathbb{Z}_k$, the ring of integers over a number field k, along with *localizations* and *orders* (both preserve the fraction field?).

In algebraic geometry, consider $R = k[t_1, \cdots, t_n]/I$ where k is a field and I is an ideal.

Some preliminary results:

1. In $\mathbb{Z}_k$, ideals factor uniquely into primes (i.e. it is a Dedekind domain).
2. $\mathbb{Z}_k$ has an integral basis (i.e. as a $\mathbb{Z}$-modules, $\mathbb{Z}_k \cong \mathbb{Z}^{[k:\mathbb{Q}]}$).

3. The Nullstellansatz: there is a bijective correspondence

$$\left\{\text{Irreducible Zariski closed subsets of } \mathbb{C}^n\right\} \iff \left\{\text{Prime ideals in } \mathbb{C}[t_1,\cdots,t_n]\right\}.$$

4. Noether normalization (a structure theorem for rings of the form R above).

All of these results concern particularly "nice" rings, e.g. $\mathbb{Z}_k, \mathbb{C}[t_1, \cdots, t_n]$. These rings are

- Domains
- Noetherian
- Finitely generated over other rings
- Finite Krull dimension (supremum of length of chains of prime ideals)
  - In particular, $\dim \mathbb{Z}_k = 1$ since nonzero prime ideals are maximal in a Dedekind domain
- Regular (nonsingularity condition, can be interpreted in scheme-theoretic language)

> Note: schemes will have "local charts" given by commutative rings, analogous to building a manifold from Euclidean n-space. General philosophy (Grothendieck): Every commutative ring is the ring of functions on some space, so we should study the category of commutative rings as a whole (i.e. let the rings be arbitrary). This does not hold for non-commutative rings! I.e. we can't necessarily associate a geometric space to every non-commutative ring. A common interesting example: $k[G]$, the group ring of an arbitrary group. Good references: Lam, 'Lectures on Modules and Rings'.

Example: Let X be a topological space and $C(X)$ be the continuous functions $f : X \to \mathbb{R}$. This is a ring under pointwise addition/multiplication. (This generally holds for the hom set into any commutative ring.)

Example: Take $X = [0, 1]$ and $C(X)$ as a ring.

Exercise:

1. Show that $C(X)$ is a not a domain. > Hint: find two nonzero functions whose product is identically zero, e.g. bump functions. > Note that they are not analytic/holomorphic.

2. Show that it is not noetherian (i.e. there is an ideal that is *not* finitely generated).

3. Fix a point $x \in [0, 1]$ and show that the ideal $\mathfrak{m}_x = \left\{ f \;\middle|\; f(x) = 0 \right\}$ is maximal.

4. Are all maximal ideals of this form? > Hint: See textbook chapter 5, or Gilman and Jerison 'Rings of Continuous Functions'.

Theorem of Swan: A theorem about topological vector bundles over $C([0, 1])$, see textbook. There is a categorical equivalence between vector bundles on a compact space and f.g. projective modules over this ring. (So commutative algebra has something to say about other branches of Mathematics!)

Definition: A topological space is called *boolean* (or a *Stone space*) iff it is compact, hausdorff, and totally disconnected.

Example: A projective variety over p-adics with $\mathbb{Q}_p$ points plugged in.

Definition: A ring is boolean if every element is idempotent, i.e. $x \in R \implies x^2 = x$.

Exercise: If R is a boolean domain, then it is isomorphic to the field with 2 elements.

Lemma: There is a categorical equivalence between Boolean spaces, Boolean rings, and so-called "Boolean algebras".