Category O

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1 Wednesday January 8

> Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$".

> Course Website: `https://faculty.franklin.uga.edu/brian/math-8030-spring-2020`

1.1 Chapter Zero: Review

> Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase mathfrak characters!

In this course, we'll take $k = \mathbb{C}$.

Recall that a Lie Algebra is a vector space $\mathfrak{g}$ with a bracket $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

- $[xx] = 0$ for all $x \in \mathfrak{g}$
  - Exercise: this implies $[xy] = -[yx]$.
    > Hint: Consider $[x+y, x+y]$. Note that the converse holds iff char $k \neq 2$.

    > Exercise: This implies Lie Algebras never have an identity.
- $[x[yz]] = [[xy]z] + [y[xz]]$ (The Jacobi identity)
  - This says x acts as a derivation.

Definition: $\mathfrak{g}$ is *abelian* iff $[xy] = 0$ for all $x, y \in \mathfrak{g}$.

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
  - A vector subspace that is closed under brackets.
- Homomorphisms
  - I.e. a linear transformation ϕ that commutes with the bracket, i.e. $\phi([xy]) = [\phi(x)\phi(y)]$.
- Ideals

Exercise: Given a vector space (possibly infinite-dimensional) over k, then (exercise) $\mathfrak{gl}(V) := \mathrm{End}_k(V)$ is a Lie algebra when equipped with $[fg] = f \circ g - g \circ f$.

Definition: A *representation* of $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ for some V.

Example: The adjoint representation is ad $: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where ad $(x)(y) := [xy]$.

Representations give $\mathfrak{g}$ the structure of a module over V, where $x \cdot v := \phi(x)(v)$. All of the usual module axioms hold, where now $[xy] \cdot v = x \cdot (\cdot v) - y \cdot (x \cdot v)$.

Example: The trivial representation $V = k$ where $x \cdot a = 0$.

Definition: V is *irreducible* (or *simple*) iff V as exactly two $\mathfrak{g}$-invariant subspaces, namely $0, V$.

Definition: V is *completely reducible* iff V is a direct sum of simple modules, and *indecomposable* iff V can not be written as $V = M \oplus N$, a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragradient/dual* $V^\vee := \hom_k(V, k)$ where $(x \cdot f) = -f(x \cdot v)$ for $f \in V^\vee, x \in \mathfrak{g}, v \in V$.
- The direct sum $V \oplus W$ where $x \cdot (v, w) = (x \cdot v, x \cdot w)$ and $x \cdot (v + w) = x \cdot v_x \cdot w$.
- The tensor product where $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.
- $\hom_k(V, W)$ where $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$.
  - Note that if we take $W = k$ then the first term vanishes and this recovers the dual.

1.2 Semisimple Lie Algebras

Definition: The derived ideal is given by $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \operatorname{span}_k\left(\left\{[xy] \;\middle|\; x, y \in \mathfrak{g}\right\}\right)$.

This is the analog of the commutator subgroup.

Lemma: $\mathfrak{g}$ is abelian iff $\mathfrak{g}^{(1)} = \{0\}$, and 1-dimensional algebras are always abelian.

This follows because if $[xy] := xy = yx$ then $[xy] = 0 \iff xy = yx$.

Definition: A lie group $\mathfrak{g}$ is *simple* iff the only ideals of $\mathfrak{g}$ are $0, \mathfrak{g}$ and $\mathfrak{g}^{(1)} \neq \{0\}$.

Note that thus rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

Definition: The derived series is defined by $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$, continuing inductively. $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

Lemma: Abelian implies solvable.

Review definition of nilpotent algebras.

Definition: $\mathfrak{g}$ is semisimple (s.s.) iff $\mathfrak{g}$ has no nonzero solvable ideals.

Exercise: Simple implies semisimple.

Some remarks:

1. Semisimple algebras $\mathfrak{g}$ will usually have solvable subalgebras.
2. $\mathfrak{g}$ is semisimple iff $\mathfrak{g}$ has no nonzero abelian ideals.

Definition: The Killing form is given by $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ where $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$, which is a symmetric bilinear form.

Lemma: $\kappa([xy], z) = \kappa(x, [yz])$.

Recall that if $\beta : V^{\otimes 2} \to k$ is any symmetric bilinear form, then its radical is defined by

$$\mathrm{rad}\beta = \left\{ v \in V \;\middle|\; \beta(v, w) = 0 \ \forall w \in V \right\}.$$

Definition: A bilinear form β is nondegenerate iff $\mathrm{rad}\beta = 0$.

Lemma: $\mathrm{rad}\kappa \trianglelefteq \mathfrak{g}$ is an ideal, which follows by the above associative property.

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Example: The standard example of a semisimple lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \;\middle|\; \mathrm{Tr}(x) = 0 \right\}$.

> Note: from now on, $\mathfrak{g}$ will denote a semisimple lie algebra over $\mathbb{C}$.

Theorem (Weyl): Every finite dimensional representation of a semisimple $\mathfrak{g}$ is completely reducible.

> I.e., the category of finite-dimensional representations is relatively uninteresting – there are no extensions, everything is a direct sum, so once you classify the simple algebras (which isn't terribly difficult) then you have complete information.

2 Friday January 10th

Let $\mathfrak{g}$ be a finite dimensional semisimple lie algebra over $\mathbb{C}$.

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the trace zero $n \times n$ matrices.

Let $\mathfrak{h}$ be a maximal toral subalgebra, where $x \in \mathfrak{g}$ is *toral* if x is semisimple, i.e. ad x is semisimple (i.e. diagonalizable).

Example: $\mathfrak{h}$ is the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$.

Fact: $\mathfrak{h}$ is abelian, so ad $\mathfrak{h}$ consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$ where $\alpha \in \mathfrak{h}^\vee$ is a linear functional.

Here $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, so $[hx] = 0$ corresponds to zero eigenvalues, and (fact) it turns out that $\mathfrak{h}$ is its own centralizer.

We then obtain a set of roots of $\mathfrak{h}, \mathfrak{g}$ given by $\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \;\middle|\; \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\} \right\}$.

Example: $\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$ for some $i \neq j$, the matrix with a 1 in the i, j position and zero elsewhere.

Fact: The restriction $\kappa|_{\mathfrak{h}}$ is nondegenerate, so we can identify $\mathfrak{h}, \mathfrak{h}^\vee$ via κ (can always do this with vector spaces with a nondegenerate bilinear form), where κ maps to another bilinear form $(\cdot, \cdot)$.

$$\mathfrak{h}^\vee \ni \lambda \iff t_\lambda \in \mathfrak{h}$$
$$\lambda(h) = \kappa(t_\lambda, h) \quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

2.1 Facts About Φ and Root Spaces

Let $\alpha, \beta \in \Phi$ be roots.

1. Φ spans $\mathfrak{h}^\vee$ and does not contain zero.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other scalar multiple of α is in Φ.

Aside:

- $\dim \mathfrak{g}_\alpha = 1$.
- If $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ then there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$ spans a 3-dimensional subalgebra in $\mathfrak{sl}_2$, given by $x_\alpha = [0, 1; 0, 0], y_\alpha = [0, 0; 1, 0], h_\alpha = [1, 0; 0, -1]$.
- Under the correspondence $\mathfrak{h} \iff \mathfrak{h}^\vee$ induced by κ, $h_\alpha \iff \alpha^\vee := \dfrac{2\alpha}{(\alpha, \alpha)}$. Thus for all $\lambda \in \mathfrak{h}^\vee$,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

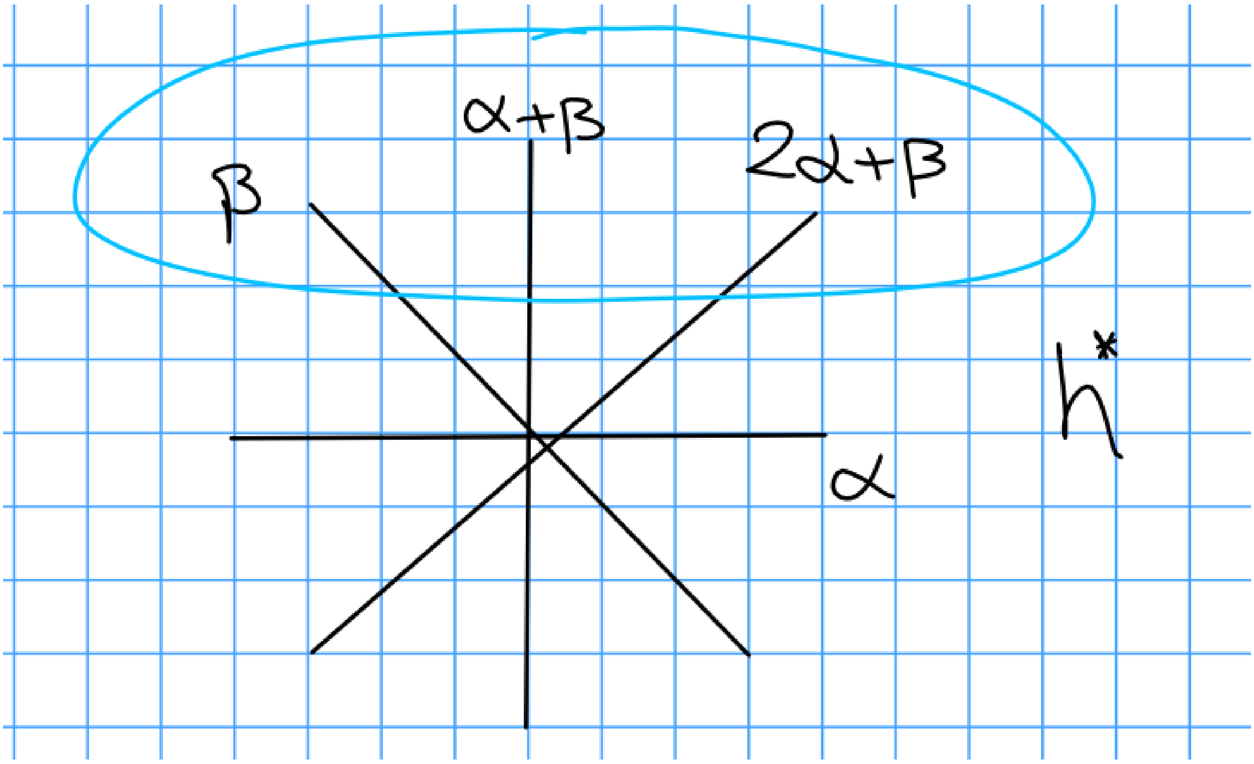
- If $\alpha + \beta \neq 0$, then $\kappa(g_\alpha, g_\beta) = 0$.

3. $(\beta, \alpha^\vee) \in \mathbb{Z}$
4. $S_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in \Phi$.

If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Example: If $\alpha = E_{ij}, \beta = E_{jk}$ where $k \neq i$, then $[E_{ij}, E_{jk}] = E_{ik}$.

- $\mathfrak{g}$ is generated as an algebra by the root spaces $\mathfrak{g}_\alpha$
- Root strings: If $\beta \neq \pm\alpha$, then the roots of the form $\alpha + k\beta$ for $k \in \mathbb{Z}$ form an unbroken string $\alpha - r\beta, \cdots, \alpha - \beta, \alpha, \alpha + \beta, \cdots, \alpha + \ell\beta$ consisting of at most 4 roots where $r - s = (\alpha, \beta^\vee)$.

Example: The circled roots below form the root string for β:



In general, a subset Φ of a real euclidean space E satisfying conditions (1) through (4) is an *(abstract) root system.*

When Φ comes from a $\mathfrak{g}$, $E := \mathbb{R}\Phi$.

2.1.1 The Root System

There exists a subset $\Delta \subseteq \Phi$ such that

- Δ is a $\mathbb{C}$-basis for $\mathfrak{g}^\vee$
- $\beta \in \Phi$ implies that $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either
  - All $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$ or $\beta < 0$.
  - All $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$ or $\beta > 0$.

Δ is called a *simple system.* If $\Delta = \{a_1, \cdots, a_\ell\}$ then $\Phi^+$are the *positive roots*, and $\Phi^+ \ni \beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$, then the *height* of β is defined as $\sum c_\alpha \in \mathbb{Z}_{>0}$.

Note that $\mathbb{Z}\Phi := \Lambda_r$ is a lattice, and is referred to as the *root lattice*, and $\Lambda_r \subset E = \mathbb{R}\Phi$. We also have $\Phi^+ = \left\{ \beta^\vee \;\middle|\; \beta \in \Phi \right\}$, the *dual root system*, is a root system with simple system $\Delta^\vee$.

Important subalgebras of $\mathfrak{g}$:

- Upper triangular with zero diagonal $\mathfrak{n} = \mathfrak{n}^+ = \sum_{\beta > 0} \mathfrak{g}_\beta$
- Lower triangular with zero diagonal $\mathfrak{n}^- = \sum_{\beta > 0} \mathfrak{g}_{-\beta}$

- Upper triangular, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra
- Lower triangular, $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

There is thus a triangular (Cartan) decomposition, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Fact: If $\beta \in \Phi^+ \setminus \Delta$, and if $\alpha \in \Delta$ such that $(\beta, \alpha^\vee) > 0$, then $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$ has height strictly less than the height of β.

> By root strings, $\beta - \alpha \in \Phi^+$ is positive root of height one less than β, yielding a way to induct on heights (useful technique).

2.1.2 Weyl Groups

For $\alpha \in \Phi$, define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\to \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in E perpendicular to α:

> Note that $S_\alpha^2 = \mathrm{id}$.

Define W as the subgroup of $\mathrm{gl}(E)$ generated by all s_α for $\alpha \in \Phi$, this is the *Weyl group* of $\mathfrak{g}$ or Φ, which is finite and $W = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$ is generated by simple reflections.

By (4), W leaves Φ invariant. In fact W is a finite Coxeter group with generators $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$ and defining relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ for $\alpha, \beta \in \Delta$ where $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$ and $m(\alpha, \alpha) = 1$.

> Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group.*

3 Monday January 13th

3.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over $\mathbb{C}$. We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group $W = \left\langle s_\beta \;\middle|\; \beta \in \Phi \right\rangle = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$ where $\Delta = \{a_i\}$ are the simple roots. For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) := n$ where $\ell(1) := 0$.

Facts:

1. $\ell(w)$ is the size of the set $\left\{ \beta \in \Phi^+ \;\middle|\; w\beta < 0 \right\}$

- The above set is equal to $\Phi^+ \bigcap w^{-1}\Phi^{-1}$.

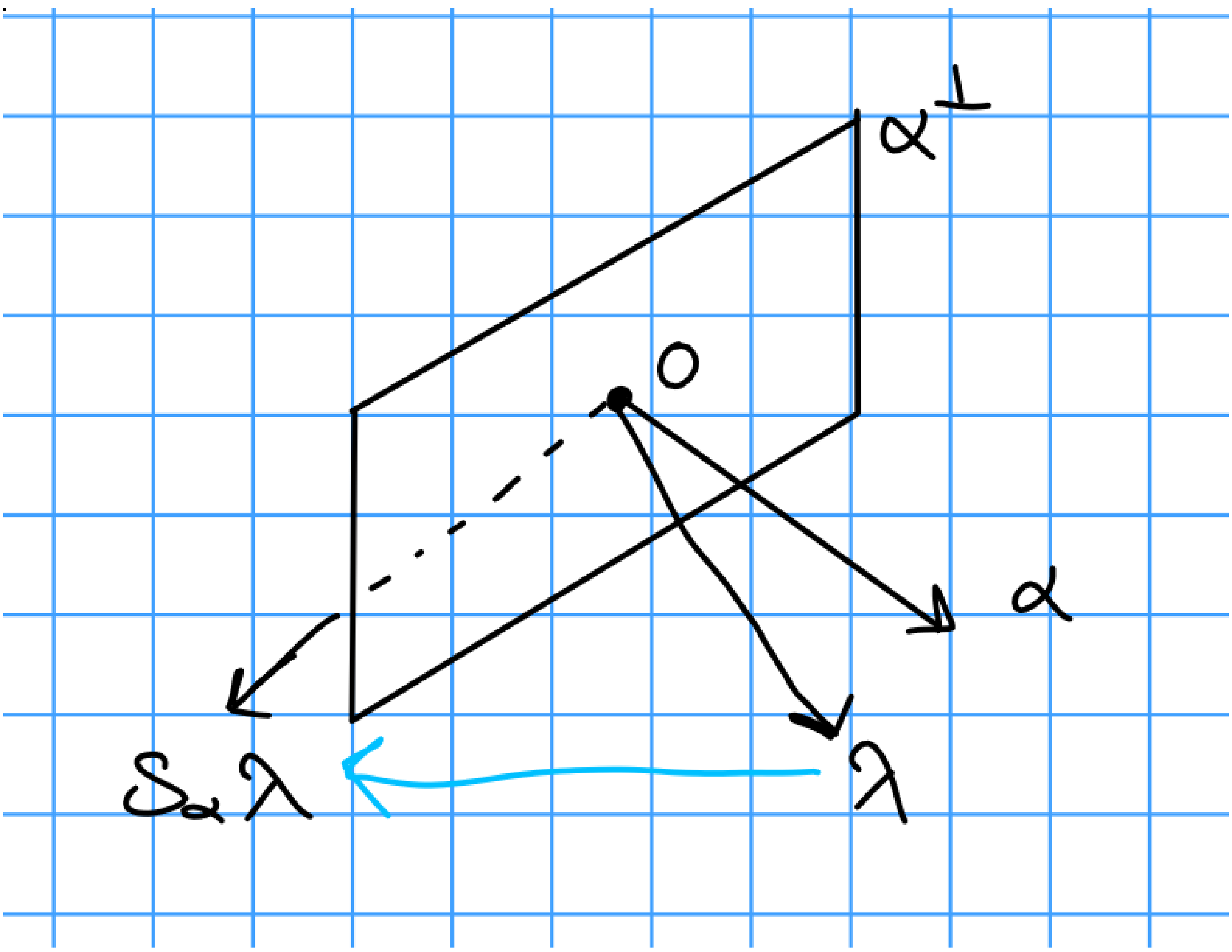


Figure 1: Image

- In particular, for $\beta \in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
- Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.

2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \bigcap w\Phi$ (replacing w^{-1} with w)

3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = \left|\Phi^+\right|$ and $w_0(\Phi^+) = \Phi^-$.

- Also $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$
\begin{aligned}
\ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\
\ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0
\end{aligned}
$$

.

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

3.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$. Then define

$$
T := \bigcup_{w \in W} wSw^{-1} = \left\{ s_\beta \;\middle|\; \beta \in \Phi^+ \right\}.
$$

This is the set of *all* reflections in W through hyperplanes in E.

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w, and refer to this as "the covering relation". So $w' \to w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that $w' = w_0 \to w_1 \to \cdots \to w_n = w$. This is called the **Bruhat-Chevalley order** on W.

Corollary: $w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order*. In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

Recall that lie algebras yield finite crystallographic coxeter groups.

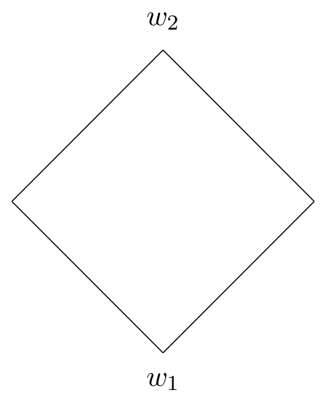
Properties: For (W, S) a coxeter group,

a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \cdots s_n$ for $w, where a subexpression is any subcollection of s_i in the same order.

Note that this implies that 1 is not only a minimal element in this order, but an infimum.

b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.

c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\left\{ w \in W \;\middle|\; w_1 < w < w_2 \right\}$ is either 0 or 2.



3.3 Properties of Universal Enveloping Algebras

Let $\mathfrak{g}$ be any lie algebra, and $\phi : \mathfrak{g} \to A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, dotted, "\tilde{\phi}"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[r, "\phi"'] & A \end{tikzcd}$$

Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_1$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, where $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \left\langle [x,y] - x \otimes y - y \otimes x \;\middle|\; x, y \in \mathfrak{g} \right\rangle.$$

Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\mathrm{ad}\ x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Big Theorem (Poincaré-Birkhoff-Witt, i.e. PBW): If $\{x_1, \cdots x_n\}$ is a basis for $\mathfrak{g}$, then $\left\{ x_1^{t_1}, \cdots, x_n^{t_n} \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$ (noting that $x^n = x \otimes x \otimes \cdots x$ and $\mathbb{Z}^+$ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If $\mathfrak{g}$ is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for $\mathfrak{g}$, then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

Notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

3.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the $\mathbb{R}$-span of the roots. We also have the **integral weight lattice**

$$\Lambda = \left\{ \lambda \in E \;\middle|\; (\lambda, \alpha^\vee) \in \mathbb{Z} \; \forall \alpha \in \Phi (\text{or } \Phi^+ \text{ or } \Delta) \right\}.$$

There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and $\mathfrak{h}^\vee$. We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+\Delta = \mathbb{Z}^+\Phi^+$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a $\mathbb{Z}$-basis for Λ. Then Λ is a free abelian group of rank ℓ, and $\Lambda^+ = \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

> Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.

4 Wednesday January 15th

4.1 Review

The Weyl vector is given by $\rho = \overline{\omega}_1 + \cdots + \overline{\omega}_\ell = \dfrac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \Lambda^+$.

- If $\alpha \in \Delta$ then $(\rho, \alpha^\vee) = 1$
- $s_\alpha(\rho) = \rho - \alpha$.

Let $\lambda \in \Lambda^+$; a few facts:

1. The size of $\left\{ \mu \in \Lambda^+ \;\middle|\; \mu \leq \lambda \right\}$ (with the partial order from last time) is finite.
2. $w\lambda < \lambda$ for all $w \in W$.

The Weyl chamber (for a fixed root, $E =$ Euclidean space) is $C = \left\{ \lambda \in E \;\middle|\; (\lambda, \alpha) > 0 \; \forall \alpha \in \Delta \right\}$ (Note that the hyperplane splits E into connected components, we mark this component as distinguished.)

- A connected component of the union of hyperplanes is orthogonal to roots
- They're in bijection with Δ
- They're permuted simply transitively by W.

And $\overline{C}$ denotes the fundamental domain.

4.2 Weight Representations

For $\lambda \in \mathfrak{h}^\vee$, we let $M_\lambda = \left\{ v \in M \;\middle|\; h \cdot v = \lambda(h)v \;\forall h \in \mathfrak{h} \right\}$ denote a *weight space* of M if $M_\lambda \neq 0$. In this case, λ is a *weight* of M. The dimension of M_λ is the *multiplicity* of λ in M, and we define the set of weights as $\Pi(M) = \left\{ \lambda \in \mathfrak{h}^\vee \;\middle|\; M_\lambda \neq 0 \right\}$.

Example if $M = \mathfrak{g}$ under the adjoint action, then $\Pi(M) = \Phi \bigcup \{0\}$.

Remark: The weight vectors for distinct weights are linearly independent. Thus there is a $\mathfrak{g}$-submodule given by $\sum\limits_\lambda M_\lambda$, which is in fact a direct sum.

> Note: It may not be the case that $\sum\limits_\lambda M_\lambda = M$, and can in fact be zero, although this is an odd situation. See Humphreys #1, #20.2, p. 110.

In our case, we'll have a *weight module* $M = \bigoplus\limits_\lambda M_\lambda$, so $\mathfrak{h} \curvearrowright M$ semisimply.

4.3 Finite-rimensional Modules

Recall Weyl's complete reducibility theorem, which implies that any finite dimensional $\mathfrak{g}$-module is a weight module. In fact, $\mathfrak{n}, \mathfrak{n}^- \curvearrowright M$ nilpotently.

Some facts:

- $\Pi(M) \subset \Lambda$ is a subset of the integral lattice.
- $\Pi(M)$ is W-invariant.
- $\dim M_\lambda = \dim M_{W\lambda}$ for any $\lambda \in \Pi(M), w \in W$.

4.4 Simple Finite Dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules

Fix the standard basis $\{x, h, y\}$ of $\mathfrak{sl}(2, \mathbb{C})$ with $[hx] = 2x, [hy] = -2y, [xy] = h$. Since $\dim \mathfrak{h} = 1$, there is a bijection $\mathfrak{h}^\vee \leftrightarrow \mathbb{C}$, $\Lambda \leftrightarrow \mathbb{Z}$, and $\Lambda_r \leftrightarrow 2\mathbb{Z}$ with $\alpha \to 2$ and $\rho \to 1$.

There is a correspondence between weights and simple modules:

$$\begin{aligned} \left\{\text{Isomorphism classes of simple modules}\right\} &\iff \Lambda^+ = \{0, 1, 2, 3, \cdots\} \\ L(\lambda) &\iff \lambda. \end{aligned}$$

Moreover, $L(\lambda)$ has a 1-dimensional weight spaces with weights $\lambda, \lambda-2, \cdots, -\lambda$ and thus $\dim L(\lambda) = \lambda + 1$.

Examples:

- $L(0) = \mathbb{C}$, the trivial representation,
- $L(1) = \mathbb{C}^2$, the natural representation where $\mathfrak{sl}(2, \mathbb{C})$ acts by matrix multiplication,
- $L(2) = \mathfrak{g}$, the adjoint representation.

Choose a basis $\{v_1, \cdots, v_\lambda\}$ for $L(\lambda)$ so that $\mathbb{C}v_0 = M_\lambda$, $\mathbb{C}v_1 = M_{\lambda-2}$, $\cdots \mathbb{C}v_\lambda M_{-\lambda}$. Then

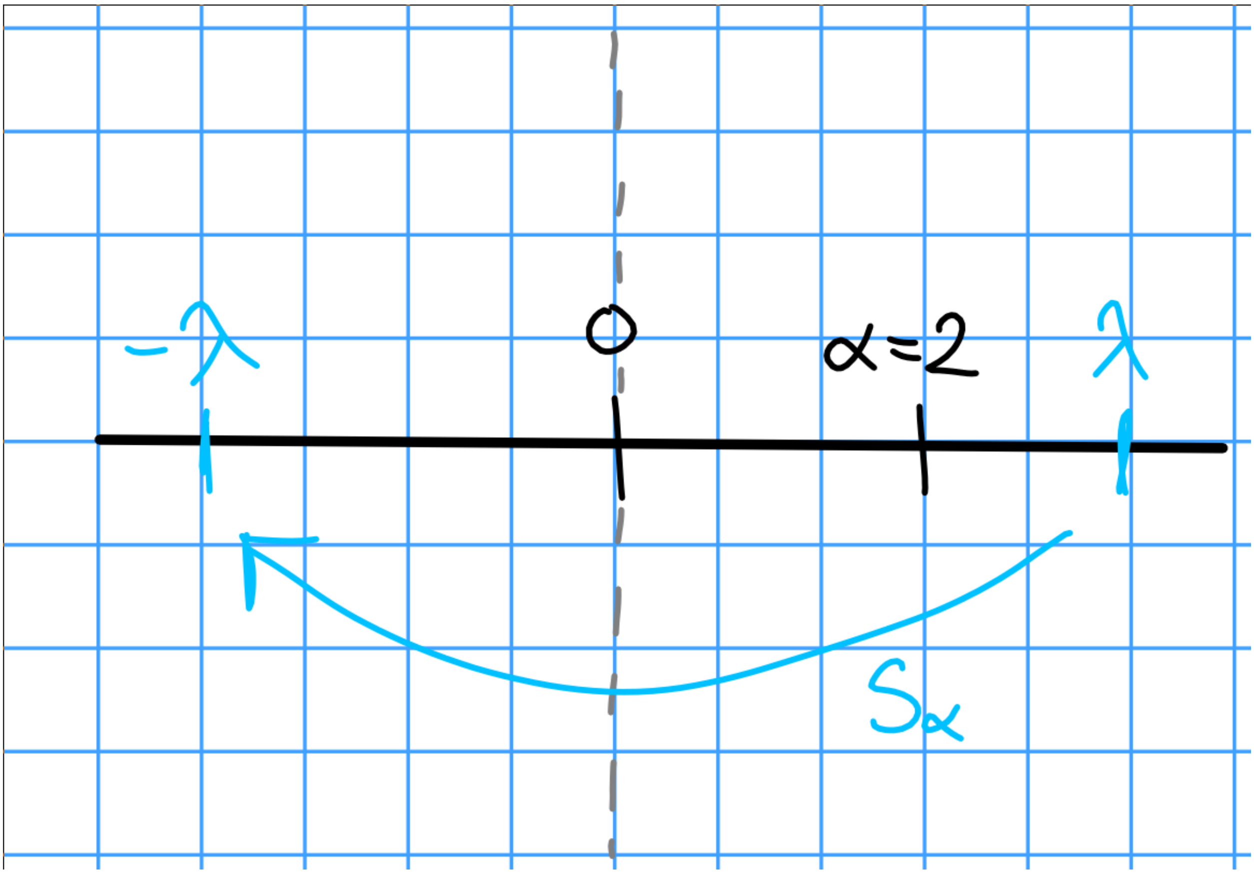


Figure 2: Image

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$, where $v_{-1} := 0$
- $y \cdot v_i = (i+1)v_{i+1}$ where $v_{\lambda+1} := 0$.

We then say $L(\lambda)$ is a highest weight module, since it is generated by a highest weight vector λ. Then $W = \{1, s_\alpha\}$, where s_α is reflection through 0 by the identification $\alpha = 2$.

5 Chapter 1: Category $\mathcal{O}$ Basics

The category of $U(\mathfrak{g})$-modules is too big. Motivated by work of Verma in 60s, started by Bernstein-Gelfand-Gelfand in the 1970s. Used to solve the Kazhdan-Lusztig conjecture.

5.1 Axioms and Consequences

Definition: $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$ modules consisting of M such that

1. M is finitely generated as a $U(\mathfrak{g})$-module.
2. M is $\mathfrak{h}$-semisimple, i.e. M is a weight module $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$.
3. M is locally n-finite, i.e. the dimension of $U(\mathfrak{n})v < \infty$ for all $v \in M$.

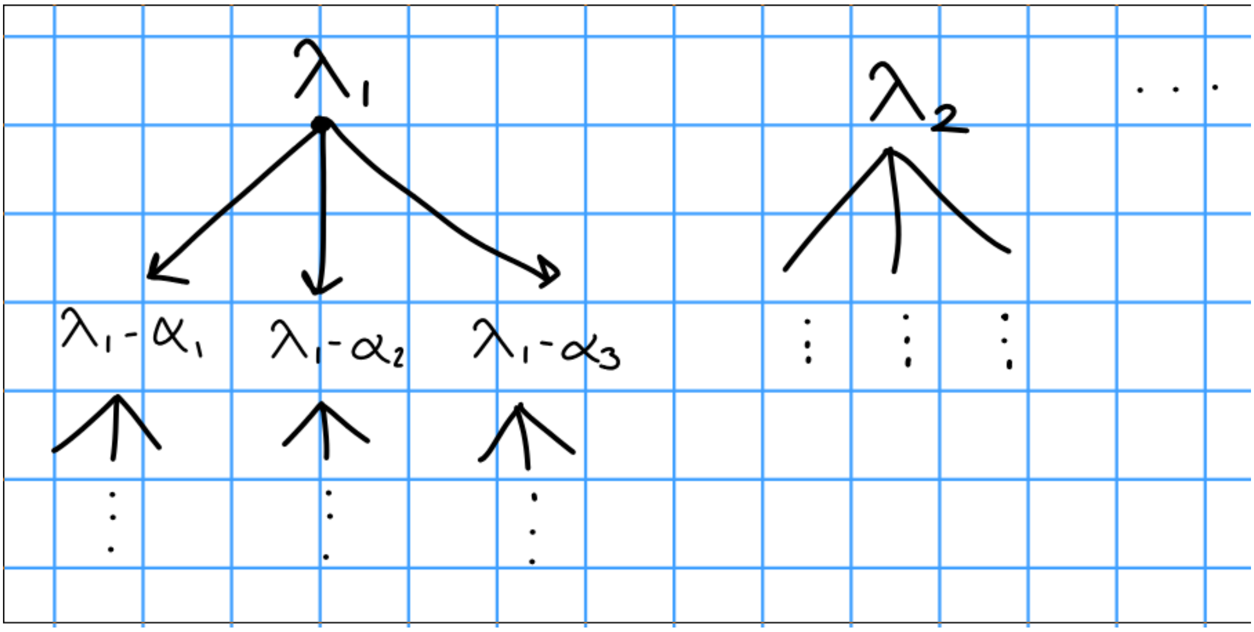


Figure 3: Image

Example: If $\dim M < \infty$, then M is $\mathfrak{h}$-semisimple, and axioms 1, 3 are obvious.

Lemma: Let $M \in \mathcal{O}$, then

4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$.
5. There exist $\lambda_1, \cdots \lambda_r \in \mathfrak{h}^\vee$ such that $\Pi(M) \subset \bigcup_{i=1}^{\lambda} (\lambda_i - \mathbb{Z}^+\Phi^+)$.

Proof: By axiom 2, every $v \in M$ is a finite sum of weight vectors in M. We can thus assume that our finite generating set consists of weight vectors. We can then reduce to the case where M is generated by a single weight vector v. So consider $U(\mathfrak{g}) \cdot v$. By the PBW theorem, there is a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$.

By axiom 3, $U(\mathfrak{n}) \cdot v$ is finite dimensional, so there are finitely many weights of finite multiplicity in the image. Then $U(\mathfrak{h})$ acts by scalar multiplication, and $U(\mathfrak{n}^-)$ produces the "cones" that result in the tree structure:

A weight of the form $\mu = \lambda_i - \sum n_i\alpha_i$ can arise from $y_{n_1}^{n_1} \cdots$.

6 Friday January 17th

Let M

1. be finitely generated,
2. semisimple $M = \oplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$,
3. locally finite
4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$,
5. satisfy the forest condition for weights.

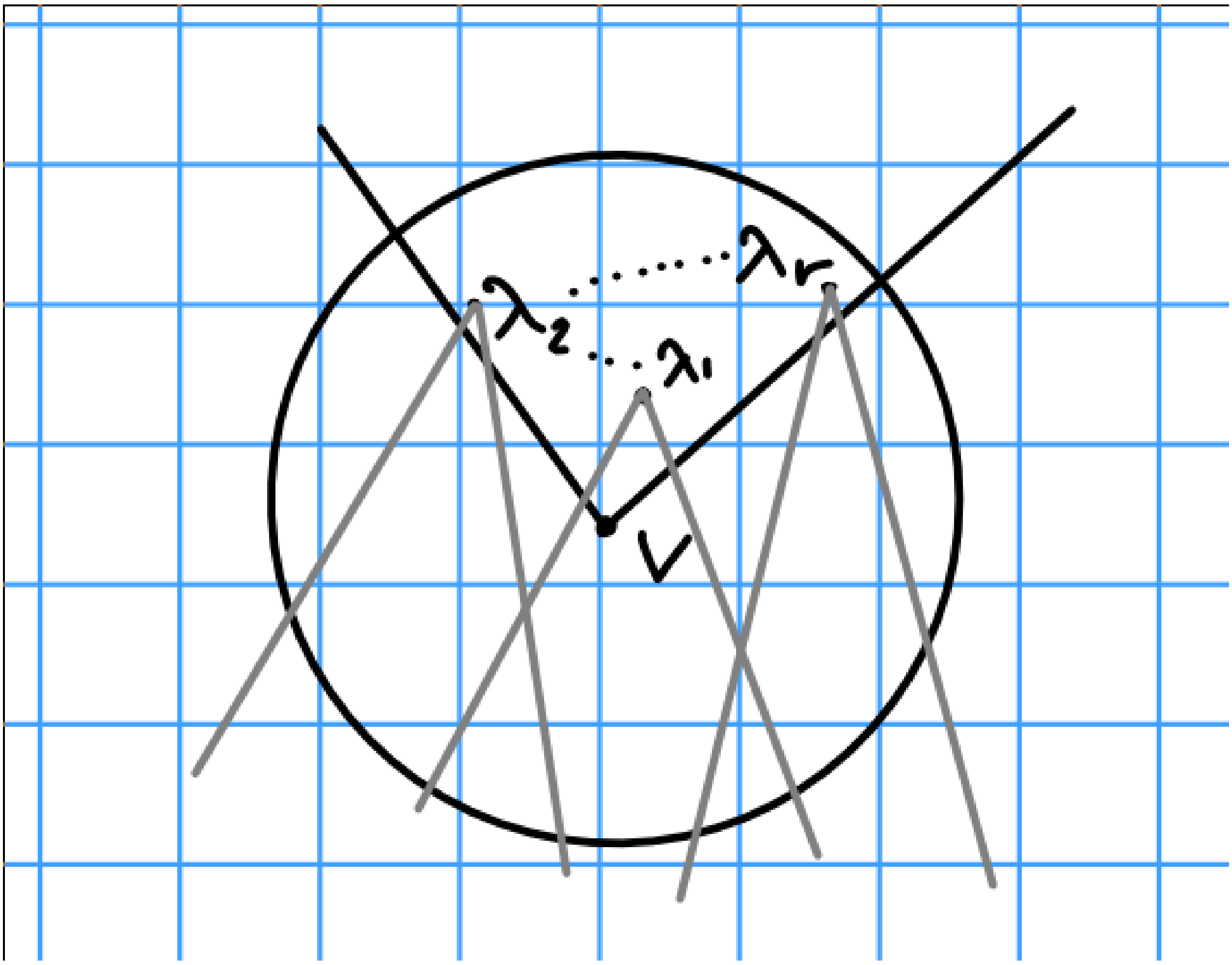


Figure 4: Image

Theorem:

a. $\mathcal{O}$ is Noetherian (ascending chain condition on submodules, i.e. no infinite filtrations by submodules)
b. $\mathcal{O}$ is closed under quotients, submodules, finite direct sums
c. $\mathcal{O}$ is abelian (similar to a category of modules)
d. If $M \in \mathcal{O}$, $\dim L < \infty$, then $L \otimes M \in \mathcal{O}$ and the endofunctor $M \mapsto L \otimes M$ is exact
e. If $M \in \mathcal{O}$ is locally $Z(\mathfrak{g})$-finite (recall: this is the center of $U(\mathfrak{g})$), i.e. $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ for all $v \in M$.
f. $M \in \mathcal{O}$ is finitely generated module. (?)

Proofs of a and b: See BA II, page 103.

Proof of c: Implied by (b), BA II Page 330.

Proof of d: Can check that $L \otimes M$ satisfies 2 and 3 above. Need to check first condition. Take a basis $\{v_i\}$ for L and $\{w_j\}$ a finite set of generators for M. The claim is that $B = \{v_i \otimes w_j\}$ generates $L \otimes M$. Let N be the submodule generated by B.

For any $v \in V$, $v \otimes w_j \in N$. For arbitrary $x \in \mathfrak{g}$, compute $x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + x \otimes (v \cdot w_j)$. Since the LHS is in N and the first term on the RHS is in N, the entire RHS is in N. By iterating, we find that $v \otimes (u \cdot w_j) \in N$ for all PBW monomials u. So $L \otimes M \in \mathcal{O}$.

Proof of e: Since $v \in M$is a sum of weight vectors, wlog we can assume $v \in M_\lambda$ is a weight vector (where $\lambda \in \mathfrak{h}^\vee$). For any central element $z \in Z(\mathfrak{g})$, we can compute $h \cdot (z \cdot v) = z \cdot (h \cdot v) = z \cdot \lambda(h) v = \lambda(h) z \cdot v$. Thus $z \cdot v \in M_\lambda$. By (4), we know that $\dim M_\lambda < \infty$, so $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ as well.

Proof of f: By 5, M is generated by a finite dimensional $U(\mathfrak{b})$ submodule N. Since we have a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-) U(\mathfrak{b})$, there is a basis of weight vectors for N that generates M as a $U(\mathfrak{n}^-)$ module.

6.1 Highest Weight Modules

Definition: A maximal vector $v^+ \in M \in \mathcal{O}$ is a nonzero vector such that $\mathfrak{n} \cdot v^+ = 0$.

Note: By 2 and 3, every nonzero $M \in \mathcal{O}$ has a maximal vector.

Definition: A highest weight module M of highest weight λ is a module generated by a maximal vector of weight λ, i.e. $M = U(\mathfrak{g}) v^+ = U(\mathfrak{n}^-) U(\mathfrak{h}) U(\mathfrak{n}) v^+ = U(\mathfrak{n}^-) v^+$.

Theorem: Let $M = U(\mathfrak{n}^-) v^+$ be a highest weight module, where $v^+ \in M_\lambda$. Fix $\Phi^+ = \{\beta_1, \cdots, \beta_n\}$ with root vectors $y_i \in \mathfrak{g}_{\beta_i}$.

a. M is the $\mathbb{C}$-span of PBW monomials $\left\langle y_1^{t_1} \cdots y_m^{t_m} \right\rangle$ of weight $\lambda - \sum t_i \beta_i$. Thus M is a module.
b. All weights μ of M satisfy $\mu \leq \lambda$
c. $\dim M_\mu < \infty$ for all $\mu \in T(M)$, and $\dim M_\lambda = 1$. In particular, property (3) holds and $M \in \mathcal{O}$.
d. Every nonzero quotient of M is a highest-weight module of highest weight λ.
e. Every submodule of M is a weight module, and any submodule generated by a maximal vector with $\mu < \lambda$ is proper. If M is semisimple, then the set of maximal weight vectors equals $\mathbb{C}^\times v^+$.
f. M has a unique maximal submodule N and a unique simple quotient L, thus M is indecomposable.

g. All simple highest weight modules of highest weight λ are isomorphic. For such M, $\dim \mathrm{End}(M) = 1$. (Category $\mathcal{O}$ version of Schur's Lemma, generalizes to infinite dimensional case)

Proofs of a to e: Either obvious or follows from previous results. First few imply M is in $\mathcal{O}$, and we know the latter hold for such modules.

Proof of f: N is a sum of submodules, so $N = \sum M_i$, proper submodules of M. So take $L = M/N$. To see indecomposability, there exists a better proof in section 1.3.

Proof of g: Let $M_1 = U(\mathfrak{n}^-)v_1^+$ and M_2 be define similarly, where the $v_i \in (M_i)_\lambda$ have the same weight. Then $M_0 \coloneqq M_1 \oplus M_2$ implies that $v^+ \coloneqq (v_1^+, v_2^+)$ is a maximal vector for M_0. So $N \coloneqq U(\mathfrak{n}^-)v^+$ is a highest weight module of highest weight λ.

We have the following diagram:

$$\begin{tikzcd}
 & & N \arrow[d, hook] \arrow[lldd, tail] \arrow[rrdd, tail] & & \\
 & & M_1 \oplus M_2 \arrow[lld] \arrow[rrd] & & \\
M_1 & & & & M_2
\end{tikzcd}$$

and since e.g. $N \to M_1$ is not the zero map, it is a surjection.

By (f), N is a unique simple quotient, so this forces $M_1 \cong M_2$. Since M is simple, any nonzero $\mathfrak{g}$-endomorphism ϕ must be an isomorphism, and so we take $v^+ \mapsto cv^+$ for some $c \neq 0$. Note that since ϕ is also a $\mathfrak{h}$-morphism, we have $\dim M_\lambda = 1$.

Since v^+ generates M and $\phi(u \cdot v^+) = u\phi(v^+) = cu \cdot v^+$, thus ϕ is multiplication by a constant.

7 Wednesday January 22nd

> Note: Try problems 1.1 and 1.3*.

Recall: In category $\mathcal{O}$, we have finite dimensional, semisimple modules over $\mathbb{C}$ with triangular decompositions.

If M is any $U(\mathfrak{g})$ module than a $v^+ \in M_\lambda$ a weight vector (so $\lambda \in \mathfrak{h}^\vee$) is primitive iff $\mathfrak{n} \cdot v^+ = 0$. Note: it doesn't have to be of maximal weight. M is a highest weight module of highest weight λ iff it's generated over $U(\mathfrak{g})$ as an associative algebra by a maximal vector v^+ of weight λ. Then $M = U(\mathfrak{g}) \cdot v^+$.

See structure of highest weight modules, and irreducibility.

Corollary: If $0 \neq M \in \mathcal{O}$, then M has a finite filtration with quotients highest weight modules, i.e. $M_0 \subset M_1 \subset \cdots \subset M_n = M$ with M_i/M_{i-1} highest weight modules. Note that the quotients are not necessarily simple, so this isn't a composition series, although we'll show such a series exists later.

Theorem: Let V be the $\mathfrak{n}$ submodule of M generated by a finite set of weight vectors which generate M as a $U(\mathfrak{g})$ module. I.e. take the finite set of weight vectors and act on them by $U(\mathfrak{n})$.

Then $\dim_{\mathbb{C}} V < \infty$ since M is locally $\mathfrak{n}$-finite.

Proof: Induction on $n = \dim V$. If $n = 1$, M itself is a highest weight module.

> Note that $\mathfrak{n}$ increases weights.

For $n > 1$, choose a weight vector $v_1 \in V$ of weight λ which is maximal among all weights of V. Set $M_1 \coloneqq U(\mathfrak{g})v_1$; this is a highest weight submodule of M of highest weight λ. ($\mathfrak{n}$ has to kill v_1, otherwise it increases weight and v_1 wouldn't be maximal.)

Let $\overline{M} = M/M_1 \in \mathcal{O}$, this is generated by the image of $\overline{V}$ of V and thus $\dim \overline{V} < \dim V$. By the IH, $\overline{M}$ has the desired filtration, say $0 \subset \overline{M}_2 \subset \overline{M}_{n-1} \subset \overline{M}_n = \overline{M}$. Let $\pi : M \to M/M_1$, then just take the preimages $\pi^{-1}(\overline{M}_i)$ to be the filtration on M.

> Note: by isomorphism theorems, the quotients in the series for M are isomorphic to the quotients for $\overline{M}$.

7.1 Verma and Simple Modules

Constructing *universal* highest weight modules using "algebraic induction". Start with a nice subalgebra of $\mathfrak{g}$ then "induce" via $\otimes$ to a module for $\mathfrak{g}$.

Recall $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, here $\mathfrak{h} \oplus \mathfrak{n}$ is the Borel subalgebra $\mathfrak{b}$, and $\mathfrak{n}$ corresponds to a fixed choice of positive roots in Φ^+ with basis Δ. Then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$. Given any $\lambda \in \mathfrak{h}^\vee$, let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{h}$-module (i.e. 1-dimensional $\mathbb{C}$-vector space)on which $\mathfrak{h}$ acts by λ.

Let $\{1\}$ be the basis for $\mathbb{C}$, so $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. Then there is a map $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$, so make C_λ a $\mathfrak{b}$-module via this map. This "inflate" C_λ into a 1-dimensional $\mathfrak{b}$-module.

Note that $\mathfrak{h}$ is solvable, and by Lie's Theorem, every finite dimensional irreducible $\mathfrak{b}$-module is of the form $\mathbb{C}_\lambda$ for some $\lambda \in \mathfrak{h}^\vee$.

Definition: $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \coloneqq \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ is the *Verma module of highest weight* λ. This process is called algebraic/tensor induction. This is a $U(\mathfrak{g})$ module via left multiplication, i.e. acting on the first tensor factor.

Since $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{h})$, we have $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$, but at what level of structure?

- As a vector space (clear)
- As a $\mathfrak{n}^-$-module via left multiplication
- As a $\mathfrak{h}^-$-module via the $\otimes$ action.

In particular, $M(\lambda)$ is a *free* $U(\mathfrak{n}^-)$-module of rank 1.

> Note: this always happens when tensoring with a vector space?

Consider $v^+ \coloneqq 1 \otimes 1 \in M(\lambda)$ (note that $U(\mathfrak{n}^-)$ is not homogeneous, so not graded, but does have a filtration). Then v^+ is nonzero, and freely generates $M(\lambda)$ as a $U(\mathfrak{n}^-)$-module. Moreover $\mathfrak{n} \cdot v^+ = 0$ since for $x \in \mathfrak{g}_\beta$ for $\beta \in \Phi^+$, we have

$$
\begin{aligned}
x(1 \otimes 1) &= x \otimes 1 \\
&= 1 \otimes x \cdot 1 \quad \text{since } x \in \mathfrak{b} \\
&= 1 \otimes 0 \implies x \in \mathfrak{n} \\
&= 0,
\end{aligned}
$$

and for $h \in \mathfrak{h}$,

$$
\begin{aligned}
h(1 \otimes 1) &= h1 \otimes 1 \\
&= 1 \otimes h1 \\
&= 1 \otimes \lambda(h)1 \\
&= \lambda(h)v^+.
\end{aligned}
$$

So $M(\lambda)$ is a highest weight module of highest weight λ, and thus $M(\lambda) \in \mathcal{O}$.

Observation: Any weight $\lambda \in \mathfrak{h}^\vee$ is the highest weight of some $M \in \mathcal{O}$. Let $\Pi(M)$ denote the set of weights of a module, then $\Pi(M(\lambda)) = \lambda - \mathbb{Z}^+\Phi^+$.

By PBW, we can obtain a basis for $M(\lambda)$ as $\left\{ y_1^{t_1} \cdots y_m^{t_m} v^+ \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$. Taking a fixed ordering $\{\beta_1, \cdots, \beta_m\} = \Phi^+$, then $0 \neq y_i \in \mathfrak{g}_{-\beta_i}$. Then every weight of this form is a weight of some $M(\lambda)$, and every weight of $M(\lambda)$ is of this form: $\lambda - \sum t_i\beta_i$.

Remark: The functor $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\,\cdot\,) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \,\cdot\,$ from the category of finite-dimensional $\mathfrak{g}$-semisimple $\mathfrak{b}$-modules to $\mathcal{O}$ is an exact functor, since it is naturally isomorphic to $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \,\cdot\,$ (which is clearly exact?)

Alternate construction of $M(\lambda)$: Let I by a left ideal of $U(\mathfrak{g})$ which annihilates v^+, so $I = \left\langle \mathfrak{n}, h - \lambda(h) \cdot 1 \;\middle|\; h \in \mathfrak{h} \right\rangle$. Since v^+ generates $M(\lambda)$ as a $U(\mathfrak{g})$-module, then (by a standard ring theory result) $M(\lambda) = U(\mathfrak{g})/I$, since I is the annihilator of $M(\lambda)$.

Theorem (Universal property of $M(\lambda)$): Let M be any highest weight module of highest weight λ generated by v. Then $I \cdot v = 0$, so I is the annihilator of v and thus M is a quotient of $M(\lambda)$. Thus $M(\lambda)$ is universal in the sense that every other highest weight module arises as a quotient of $M(\lambda)$.

By theorem 1.2, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ (nonstandard notation) and a unique simple quotient $L(\lambda)$ (standard notation).

Theorem: Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$ and is determined uniquely up to isomorphism by its highest weight. Moreover, there is an analog of Schur's lemma: $\dim \hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$, i.e. it's 1 iff $\lambda = \mu$ and 0 otherwise.

> Note: up to isomorphism, we've found all of the simple modules in $\mathcal{O}$, and most are finite-dimensional.

Proof: Next class.

8 Friday January 24th

A standard theorem about classifying simple modules in category $\mathcal{O}$:

Theorem: Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$, and is determined uniquely up to isomorphism by its highest weight. Moreover, $\dim \hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}$.

Proof: Let $L \in \mathcal{O}$ be irreducible. As observed in 1.2, L has a maximal vector v^+ of some weight λ.

> Recall: can increase weights and reach a maximal in a finite number of steps.

Since L is irreducible, L is generated by that weight vector, i.e. $LU(\mathfrak{g}) \cdot v^+$, so L must be a highest weight module.

> Standard argument: use triangular decomposition.

By the universal property, L is a quotient of $M(\lambda)$. But this means $L \cong L(\lambda)$, the unique irreducible quotient of $M(\lambda)$.

By Theorem 1.2 part g (see last Friday), $\dim \mathrm{End}_{\mathcal{O}}(L(\lambda)) = 1$ and $\hom_{\mathcal{O}}(L(\mu), L(\lambda)) = 0$ since both entries are irreducible.

■

Proof of Theorem 1.2 f:

Statement: A highest weight module M is indecomposable, i.e. can't be written as a direct sum of two nontrivial proper submodules.

Suppose $M = M_1 \oplus M_2$ where M is a highest weight module of highest weight λ. Category $\mathcal{O}$ is closed under submodules, so M_i are weight modules and have weight-space decompositions. But M_λ is 1-dimensional (triangular decomposition, only $\mathbb{C}$ acts), and thus $M_\lambda \subset M_1$. Since M_λ is a highest weight module, it generates the entire module, so $M \subset M_1$. The reverse holds as well, so $M = M_1$ and this forces $M_2 = 0$.

■

8.1 1.4: Maximal Vectors in Verma Modules

> 1.5: Examples in the case $\mathfrak{sl}(2)$, over $\mathbb{C}$ as usual.

First, some review from lie algebras.

Let $\mathfrak{g}$ be any lie algebra, and take $u, v \in U(\mathfrak{g})$. Recall that we have the formula $uv = [uv] + vu$, where we use the definition $[uv] = uv - vu$.

Let x, y_1, y_2 be in $\mathfrak{g}$, what is $[x, y_1 y_2]$? Use the fact that ad $x(y_1, y_2)$ acts as a derivation, and so $[x, y_1 y_2] = [xy_1]y_2 + y_1[xy_2]$, which is a bracket entirely in the Lie algebra. This extends to an action on $U(\mathfrak{g})$ by the product rule.

Recall that $\mathfrak{sl}(2)$ is spanned by $y = [0, 0; 1, 0], h = [1, 0; 0, -1], x = [0, 1; 0, 0]$, where each basis vector spans $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}$ respectively. Then $[xy] = h, [hx] = 2x, [hy] = -2y$, so $E_{ij}E_{kl} = \delta_{jk}E_{il}$ (should be able to compute easily!).

Then $\mathfrak{h} = \mathbb{C}$, so $\mathfrak{h}^\vee \cong \mathbb{C}$ where $\lambda \mapsto \lambda(h)$. So we identify λ with a complex number, this is kind of like a bundle of Verma modules over $\mathbb{C}$.

Consider $M(1)$, then $\lambda = 1$ will denote $\lambda(h) = 1$. As in any Verma module, $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$. We can think of v^+ as $1 \otimes 1$, with the action $yv^+ = y1 \otimes 1$. Note that y has weight -2.

Weight	Basis
1	v^+
-1	yv^+
-3	y^2v^+
-5	y^3v^+

Consider how $x \curvearrowright y^2v^+$. Note that x has weight $+2$. We have

$$\begin{aligned}
x \cdot y^2v^+ &= x \cdot y^2 \otimes 1_\lambda \\
&= ([xy^2] + y^2x) \otimes 1 \\
&= ([xy]y + y[xy]) \otimes 1 + y^2 \otimes x \cdot 1 \quad \text{moving } x \text{ across the tensor because ?} \\
&= ([xy]y + y[xy]) \otimes 1 + 0 \quad \text{since } x \text{ is maximal} \\
&= (hy + yh) \otimes 1 \\
&= hy \otimes 1 + y \otimes h \cdot 1 \\
&= hy \otimes 1 + \lambda(h)1 \\
&= hy \otimes 1 + 1 \\
&= ([xy] + yh) \otimes 1 + y \otimes 1 \\
&= -2y \otimes 1 + y \otimes 1 + y \otimes 1 \\
&= 0.
\end{aligned}$$

So y moves us downward through the table, and x moves upward, except when going from $-3 \to -1$ in which case the result is zero!

Thus there exists a morphism $\phi : M(-3) \to M(1)$, with image $U(\mathfrak{g})y^2v^+ = U(\mathfrak{n}^-)y^2v^+$. So the image of ϕ is everything spanned by the bases in the rows $-3, -5, \cdots$, which is exactly $M(-3)$. So $M(-3) \hookrightarrow M(1)$ as a submodule.

Motivation for next section: we want to find Verma modules which are themselves submodules of Verma modules.

It turns out that im $\phi \cong N(1)$. We should have $M(1)/N(1) \cong L(1)$. What is the simple module of weight 1 for $\mathfrak{sl}(2)$? The weights of $L(n)$ are $n, n-2, n-4, \cdots, -n$, so the representations are parameterized by $n \in \mathbb{Z}^+$. These are the Verma modules for $\mathfrak{sl}(2)$. What happens is that $y \curvearrowright -n \to -n-2$ gives a maximal vector, so the calculation above roughly goes through the same way. So we'll have a similar picture with $L(n)$ at the top.

8.2 Back to 1.4

Question 1: What are the submodules of $M(\lambda)$?

Question 2: What are the Verma submodules $M(\mu) \subset M(\lambda)$? Equivalently, when do maximal vectors of weight $\mu < \lambda$ (the interesting case) lie in $M(\lambda)$?

Question 3: As a special case, when do maximal vectors of weight $\lambda - k\alpha$ for $\alpha \in \Delta$ lie in $M(\lambda)$ for $k \in \mathbb{Z}^+$?

Fix a Chevalley basis for $\mathfrak{g}$ (see section 0.1) $h_1, \cdots, h_\ell \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi^+$. Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ and let $x_i = x_{\alpha_i}, y_i = y_{\alpha_i}$ be chosen such that $[x_i y_i] = h_i$.

Lemma: For $k \geq 0$ and $1 \leq i, j \leq \ell$, then

a. $[x_j, y_i^{k+1}] = 0$ if $j \neq i$

b. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$.

c. $[x_i, y_i^{k+1}] = -(k+1)y_i(k \cdot 1 - h_i)$.

Proof (sketch):

Both easy to prove by induction since $[x_j, y_i] \to \alpha_j - \alpha_i \notin \Phi$ is a difference of simple roots.

For $k = 0$, all identities are easy. For $k > 0$, an inductive formula that uses the derivation property, which we'll do next class.

9 Monday January 27th

9.1 Section 1.4

Fix $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$, $x_i \in g_{\alpha_i}$ and $y_i \in g_{-\alpha_i}$ with $h_i = [x_i y_i]$.

Lemma: For $k \geq 0$ and $1 \leq i, j \leq \ell$,

a. $[x_j y_i^{k+1}] = 0$ if $j \neq i$
b. $[h_j y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$
c. $[x_i y_i^{k+1}] = (k+1)y_i^k(k \cdot 1 - h_i)$.

Sketch of proof for (c):

By induction, where $k = 0$ is clear.

$$\begin{aligned}
[x + iy_i^{k+1}] &= [x_i y_i]y_i^k + y_i[x_i y_i^k] \\
&= h_i y_i^k + y_i(-ky_i^{k-1}((k-1)1 - h_i)) \quad \text{by I.H.} \\
&= (k+1)y_i^k h_i - (k^2 - k + 2k)y_i^k \\
&= -(k+1)y_i^k(k \cdot 1 - h_i)
\end{aligned}$$

.

■

Proposition: Suppose $\lambda \in \mathfrak{h}^\vee, \alpha \in \Delta$, and $n := (\lambda, \alpha^\vee) \in \mathbb{Z}^+$. Then in $M(\lambda)$, $y_\alpha^{n+1}v^+$ is a maximal weight vector of weight $\mu := \lambda - (n+1)\alpha < \lambda$.

Note this is free as an $U(\mathfrak{n}^-)$-module, so $v^+ \neq 0$. Note that $n = \lambda(h_\alpha)$.

By the universal property, there is a nonzero homomorphism $M(\mu) \to M(\lambda)$ with image contained in $N(\lambda)$, the unique maximal proper submodules of $M(\lambda)$.

Proof: Say $\alpha = \alpha_i$. Fix $j \neq i$.

$$\begin{aligned} x_i y_\alpha^{n+1} \otimes 1 &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes x_j \cdot 1 \\ &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes 0 \quad \text{by a} \\ &= 0. \end{aligned}$$

$$\begin{aligned} x_i y_i^{n+1} \otimes 1 &= [x_i y_i^{n+1} \otimes 1] \\ &= -(n+1) y_i^n (n \cdot 1 - h_i) \otimes 1 \\ &= -(n+1)(n - \lambda(h_i)) 1 \otimes 1 \\ &\coloneqq -(n+1)(\lambda(h_i) - \lambda(h_i)) 1 \otimes 1 \\ &= 0. \end{aligned}$$

Since g_{α_j} generate $\mathfrak{n}$ as a Lie algebra, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. This shows that $\mathfrak{n} \cdot y_i^{n+1} v^+ = 0$, and the weight of $y_i^{n+1} v^+$ is $\lambda - (n+1)\alpha_i$. So y_i^{n+1} is a maximal vector of weight μ. The universal property implies there is a nonzero map $M(\mu) \to M(\lambda)$ sending highest weight vectors to highest weight vectors and preserving weights. The image is proper since all weights of M_μ are less than or equal to $\mu < \lambda$.

■

Consider $\mathfrak{sl}(2)$, then $M(1) \supset M(-3)$. Note that reflecting through 0 doesn't send 1 to -3, but shifting the origin to -1 and reflecting about that with $s_\alpha \cdot$ fixes this problem. Note that $L(1)$ is the quotient.

For $\lambda \in \mathfrak{h}^\vee$ and $\alpha \in \Delta$, we can compute $s_\alpha \cdot \lambda \coloneqq s_\alpha(\lambda + \rho) - \rho$ where $\rho = \sum_{j=1}^{\ell} e_i$. Then $(w_j, \alpha_i^\vee) = \delta_{ij}$ and $(\rho, \alpha_i^\vee) = 1$.

$$\begin{aligned} s\alpha \cdot \lambda &= s_\alpha(\lambda + \rho) - \rho \\ &= (\lambda + \rho) - (\lambda + \rho, \alpha^\vee)\alpha - \rho \\ &= \lambda + \rho - ((\lambda < \alpha^\vee) + 1)\alpha - \rho \\ &= \lambda - (n+1)\alpha \\ &= \mu. \end{aligned}$$

So this gives a well-defined, nonzero map $M(s_\alpha \cdot \lambda) \to M(\lambda)$ for $s_\alpha \cdot \lambda < \lambda$.

Corollary: Let λ, α, n be as in the above proposition. Let $\overline{v}^+$ now be a maximal vector of weight λ in $L(\lambda)$. Then $y_\alpha^{n+1} \overline{v}^+ = 0$.

Proof: If not, then this would be a maximal vector, since it's the image of the vector $y_i^{n+1} v^+ \in M(\lambda)$ under the map $M(\lambda) \to L(\lambda)$ of weight $\mu < \lambda$. Then it would generate a proper submodules of $L(\lambda)$, but this is a contradiction since $L(\lambda)$ is irreducible.

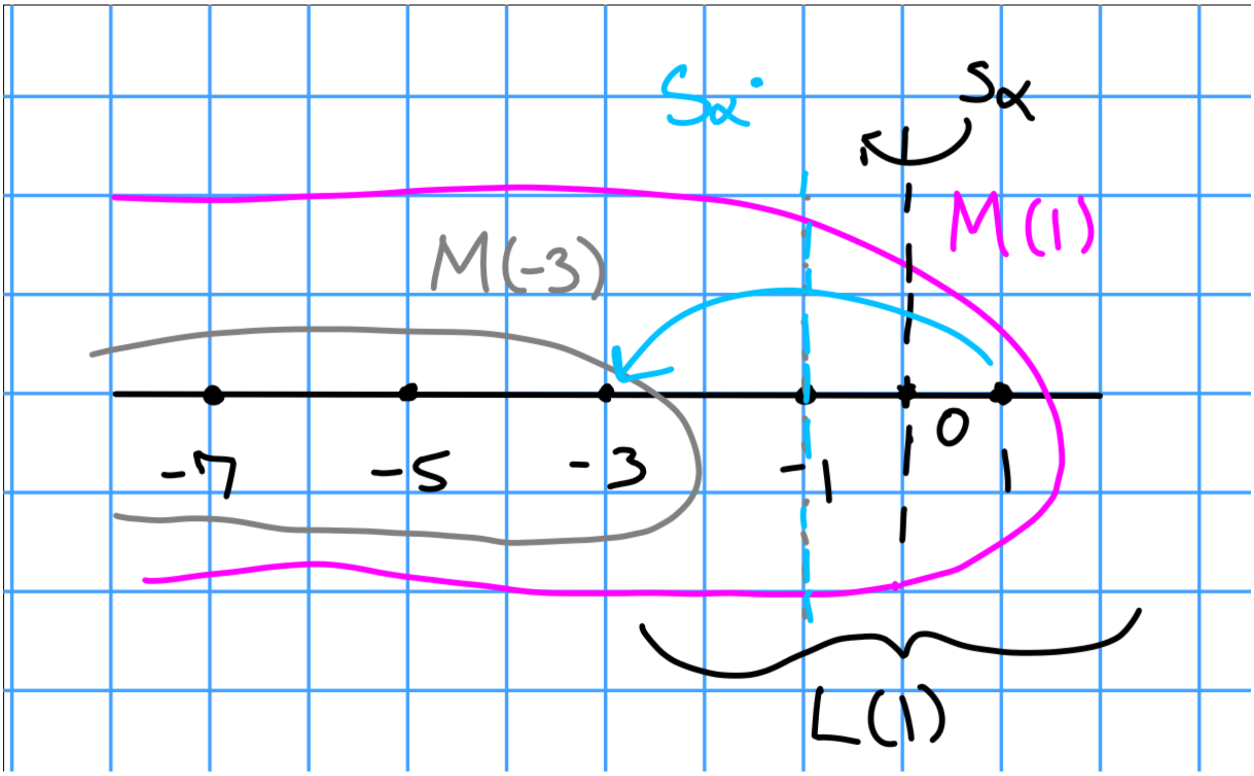


Figure 5: Image

9.2 Section 1.5

Example: $\mathfrak{sl}(2)$. What do Verma modules $M(\lambda)$ and their simple quotients $L(\lambda)$ look like?

Fix a Chevalley basis $\{y, h, x\}$ and let $\lambda \in \mathfrak{h}^\vee \cong \mathbb{C}$.

Fact 1: For $v^+ = 1 \otimes 1_\lambda$, we have $M(\lambda) = U(\mathfrak{n}^-)v^+ = \mathbb{C}\left\langle y^i v^+ \;\middle|\; i \in \mathbb{Z}^+ \right\rangle$ is a basis for $M(\lambda)$ with weights $\lambda - 2i$ where α corresponds to 2. So the weights of $M(\lambda)$ are $\lambda, \lambda - 2, \lambda - 4, \cdots$ each with multiplicity 1.

Letting $v_i = \dfrac{1}{i!} y^i v^+$ for $i \in \mathbb{Z}^+$; this is a basis for $M(\lambda)$. Using the lemma, we have

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i \\ y \cdot v_i &= (i+1)v_{i+1} \\ x \cdot v_i &= (\lambda - i + 1)v_{i-1}. \end{aligned}$$

Note that these are the same for *finite-dimensional* $\mathfrak{sl}(2)$-modules, see section 0.9.

Fact 2: We know from the proposition that if $\lambda \in \mathbb{Z}^+$, i.e. $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$, then $M(\lambda)$ has a maximal vector of weight $\lambda - (n+1)\alpha = \lambda - (\lambda+1)2 = -\lambda - 2 = s_\alpha \cdot \lambda$.

Exercise: Check that this maximal vector generates the maximal proper submodule $N(\lambda) = M(-\lambda - 2)$.

So the quotient $L(\lambda) = M(\lambda)/N(\lambda) = M(\lambda)/M(-\lambda-2)$ has weights $\lambda, \lambda-2, \cdots, -\lambda+2, -\lambda$. So when $\lambda \in \mathbb{Z}^+$, $L(\lambda)$ is the familiar simple $\mathfrak{sl}(2)$-module of highest weight λ.

Fact 3: When $\lambda \notin \mathbb{Z}^+$,

- $N(\lambda) = \{0\}$,
- $M(\lambda) = L(\lambda)$,
- $M(\lambda)$ is irreducible
- $L(\lambda)$ is infinite dimensional.

Proof: Argue by contradiction. If not, $M(\lambda) \supset M \neq 0$ is a proper submodule. So $M \in \mathcal{O}$, and thus M has a maximal weight vector w^+, and by the restriction of weights for modules in $\mathcal{O}$, we know w^+ has height $\lambda - 2m$ for some $m \in \mathbb{Z}^+$. Then $w^+ = cv_i$ where $0 \neq c \in \mathbb{C}$, and taking $v_{-1} := 0$ and $x \cdot v_i = (\lambda - i + 1)v_{i-1}$ for $i \geq 1$, so $\lambda = i - 1 \implies \lambda \in \mathbb{Z}^+$.

10 Friday January 31st

(3) A useful formula: $L(\lambda)^\vee \cong L(-w_0)$

Proof: $L(\lambda)^\vee$ is a finite dimensional module, and $(x \cdot f)(v) = -f(x \cdot v)$, so $L(\lambda)^\vee \cong L(\nu)$ for some $\nu \in \Lambda^+$. The weights of $L(\lambda)^\vee$ are the negatives of the weight of $L(\lambda)$. Thus the lowest weight of $L(\lambda)$ is $w_0\lambda$, since w_o reverses the partial order on $\mathfrak{h}^\vee$, i.e $w_0\Phi^+ = \Phi^-$

Then $\mu \in \Pi(L(\mu)) \implies w_0\mu \in \Pi(L(\lambda)) \implies w_0\mu \leq \lambda$. This shows that the lowest weight of $L(\lambda)$ is $w_0\lambda$, thus the highest weight $L(\lambda)^\vee$ is $-w_0\lambda$ by reversing this inequality.

> The inner product is W invariant and is its own inverse, so we can move it to the other side.

10.1 1.7: Action of $Z(\mathfrak{g})$

Next big goal: Every module in $\mathcal{O}$ has a *finite* composition series (Jordan-Holder series, the quotients are simple).

> Leads to Kazhdan-Lustzig conjectures from 1979/1980, which were solved, but are still open in characteristic p case.

The technique we'll use the Harish-Chandra homomorphism, which identifies $\mathcal{Z}(\mathfrak{g})$ explicitly.

> It's commutative, subalgebra of a Noetherian algebra, no zero divisors – could be a quotient, but then it'd have zero divisors, so this seems to force it to be a polynomial algebra on some unknown. Also note that $\mathcal{Z}(\mathfrak{g}) := Z(U(\mathfrak{g}))$.

Recall: $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on any $M \in \mathcal{O}$ – this is by theorem 1.1e, i.e. $v \in M_\mu$ and $z \in \mathcal{Z}(\mathfrak{g})$ implies that $zv \in M_\mu$. (The calculation just follows by computing the weight and commuting things through.)

Let $\lambda \in \mathfrak{h}^\vee$ and $M = U(\mathfrak{g})v^+$ a highest weight module of highest weight λ. Then for $z \in \mathcal{Z}(\mathfrak{g})$, $z \cdot v^+ \in M_\lambda$ which is 1-dimensional. Thus z acts by scalar multiplication here, and $z \cdot v^+ = \chi_\lambda(z)v^+$. Now if $u \in U(\mathfrak{n}^-)$, we have $z \cdot (u \cdot v^+) = u \cdot (z \cdot v^+) = u(\chi_\lambda(z)v^+) = \chi_\lambda(z)u \cdot v^+$. Thus z acts on *all* of M by the scalar $\chi_\lambda(z)$.

Exercise: Show that χ_λ is a nonzero additive and multiplicative function, so $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is linear and thus a morphism of algebras. Conclude that $\ker \chi_\lambda$ is a maximal ideal of $\mathcal{Z}(\mathfrak{g})$.

Note: this is called the infinitesimal character.

Note that χ_λ doesn't depend on which highest weight module M_λ was chosen, since they're all quotients of $M(\lambda)$. In fact, every submodule and subquotient of $M(\lambda)$ is the same infinitesimal character.

Definition: χ_λ is called the *central (or infinitesimal) character*, and $\widehat{\mathcal{Z}}(\mathfrak{g})$ denotes the set of all central characters. More generally, any algebra morphism $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is referred to as a central character. Central characters are in one-to-one correspondence with maximal ideals of $\mathcal{Z}(\mathfrak{g})$, where $\chi \iff \ker \chi$ and $\mathbb{C}[x_1, \cdots, x_n] \iff \langle x_1 - a_1, \cdots, x_n - a_n \rangle$ where $(\mathbf{a_i}) \in \mathbb{C}^n$.

Next goal: Describe $\chi_\lambda(z)$ more explicitly.

Using PBW, we can write $z \in \mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. Some observations:

1. Any PBW monomial in z ending with a factor in $\mathfrak{n}$ will kill v^+, and hence can not contribute to $\chi_\lambda(z)$.
2. Any PBW monomial in z beginning with a factor in $\mathfrak{n}^-$ will send v^+ to a lower weight space, so it also can't contribute.

So we only need to see what happens in the $\mathfrak{h}$ part. A relevant decomposition here is $U(\mathfrak{g}) = U(\mathfrak{g}) \oplus \left(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+\right)$.

Exercise: why is this sum direct?

Let $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection onto the first factor. Then $\chi_\lambda(z) = \lambda(\mathrm{pr} z)$ for all $z \in \mathcal{Z}(\mathfrak{g})$. Then if $\mathrm{pr}(z) = h_1^{m_1} \cdots h_\ell^{m_\ell}$, we can extend the action on $\mathfrak{h}$ to all polynomials in elements of $\mathfrak{h}$ (which is in fact evaluation on these monomials), and thus $\chi_\lambda(z) = \lambda(h_1)^{m_1} \cdots \lambda(h_\ell)^{m_\ell}$.

Note that for $\lambda \in \mathfrak{h}^\vee$, we've extended this to the "evaluation map" $\lambda : U(\mathfrak{g}) \cong S(\mathfrak{h})$, the symmetric algebra on $\mathfrak{h}$. Why is this the correct identification? The RHS is $T(\mathfrak{h}) / \langle x \otimes y - y \otimes x - [xy] \rangle$, but notice that the bracket vanishes since $\mathfrak{h}$ is abelian, and this is the exact definition of the symmetric algebra.

Thus $\chi_\lambda = \lambda \circ \mathrm{pr}$.

Observation:

$$\begin{aligned}
\lambda(\mathrm{pr}(z_1 z_2)) &= \chi_\lambda(z_1 z_1) \\
&= \chi_\lambda(z_1)\chi_\lambda(z_2) \\
&= \cdots \\
&= \lambda(\mathrm{pr}(z_1)\mathrm{pr}(z_2)).
\end{aligned}$$

Exercise: Show $\displaystyle\bigcap_{\lambda \in \mathfrak{h}^\vee} \ker \lambda = \{0\}$.

Definition: Let $\xi = \mathrm{pr}|_{\mathcal{Z}(\mathfrak{g})} : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h})$.

Lemma/Definition: ξ is an algebra morphism, and is referred to as the *Harish-Chandra homomorphism.*

See page 23 for interpretation of ξ without reference to representations.

Questions:

1. Is ξ injective?
2. What is im $\xi \subset U(\mathfrak{h})$?

When does $\chi_\lambda = \chi_\mu$? Proved last time: we introduced the $\cdot$ action and proved that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ where $\alpha \in \Delta$. It'll turn out that the statement holds for all $\lambda \in W$.

Wednesday: Section 1.8.

11 Wednesday February 5th

Recall the Harish-Chandra morphism ξ:

$$\begin{tikzcd} \mathcal{Z}(\mathfrak{g}) \arrow[r, hook] \arrow[dr, dashed, "\xi"] & U(\mathfrak{g}) = U(\mathfrak{g} \oplus \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \arrow[d, "\mathfrak{p}"] \\ & U(\mathfrak{h}) \end{tikzcd}$$

If M is a highest weight module of highest weight λ then $z \in \mathcal{Z}(\mathfrak{g})$ acts on M by scalar multiplication. Note that if we have $\chi_\lambda(z)$ where $z \cdot v = \chi_\lambda(z) v$ for all $v \in M$, we can identify $\lambda(\mathfrak{p}(z)) = \lambda(\xi(z))$.

11.1 Central Characters and Linkage

The χ_λ are not all distinct – for example, if $M(\mu) \subset M(\lambda)$, then $\chi_\mu = \chi_\lambda$. More generally, if $L(\mu)$ is a subquotient of $M(\lambda)$ then $\chi_\mu = \chi_\lambda$. So when do we have equality $\chi_\mu = \chi_\lambda$?

Given $\mathfrak{g} \supset \mathfrak{h}$ with $\Phi \supset \Phi^+ \supset \Delta$, then define $\rho = \dfrac{1}{2} \displaystyle\sum_{\beta \in \Phi^+} \beta \in \mathfrak{h}^\vee$. Note that $\alpha \in \Delta \implies s_\alpha \rho = \rho - \alpha$.

Definition: The *dot action* of W on $\mathfrak{h}^\vee$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, which implies $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Delta$. Then $\rho = \displaystyle\sum_{i=1}^{\ell} w$.

Exercise: Check that this gives a well-defined group action.

Definition: μ is *linked* to λ iff $\mu = w \cdot \lambda$ for some $w \in W$. Note that this is an equivalence relation, with equivalence classes/orbits where the orbit of λ is $\left\{ w \cdot \lambda \mathrel{\Big|} w \in W \right\}$ is called the *linkage class* of λ.

Note that this is a finite subset, since W is finite.

Note that orbit-stabilizer applies here, so bigger stabilizers yield smaller orbits and vice-versa.

Example: $w \cdot (-\rho) = w(-\rho + \rho) - \rho = -\rho$, so $-\rho$ is in its own linkage class.

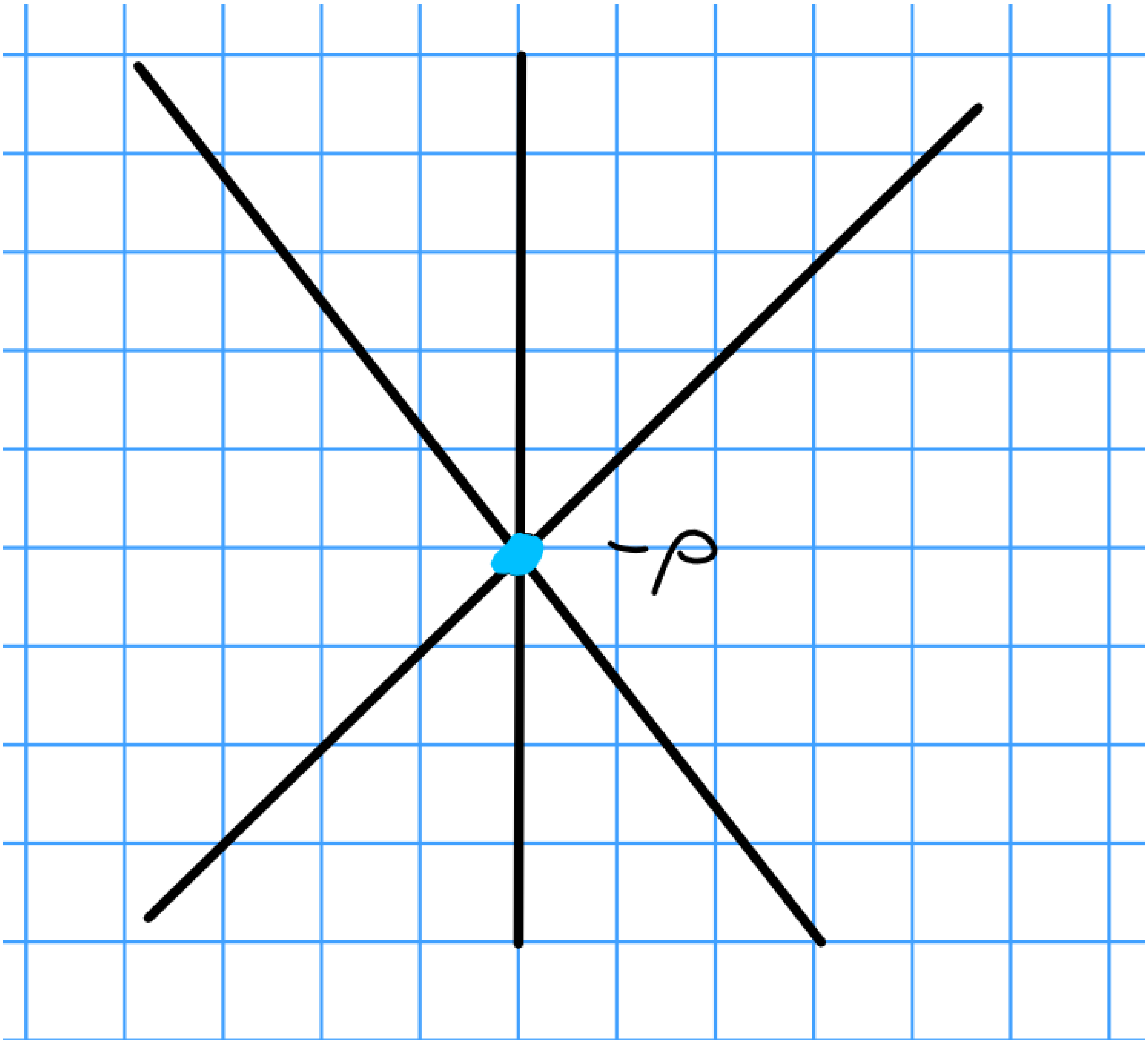
Definition: $\lambda \in \mathfrak{h}^\vee$ is *dot-regular* iff $|W \cdot \lambda| = |W|$, or equivalently if $(\lambda + \rho, \beta^\vee) \neq 0$ for all $\beta \in \Phi$.

To think about: does this hold if Φ is replaced by Δ?

We also say λ is *dot-singular* if λ is not dot-regular, or equivalently $\mathrm{Stab}_W . \lambda \neq \{1\}$.

I.e. lying on root hyperplanes.

Exercise: If $0 \in \mathfrak{h}^\vee$ is regular, then $-\rho$ is singular.



Proposition: If $\lambda \in \Lambda$ and $\mu \in W \cdot \lambda$, then $\chi_\mu = \chi_\lambda$.

Proof: Start with $\alpha \in \Delta$ and consider $\mu = s_\alpha \cdot \lambda$. Since $\lambda \in \Lambda$, we have $n := (\lambda, \alpha^\vee) \in \mathbb{Z}$ by definition. There are three cases:

1. $n \in \mathbb{Z}^+$, then $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. By Proposition 1.4, we have $\chi_\mu = \chi_\lambda$.
2. For $n = -1$, $\mu = s_\alpha \cdot \lambda = \lambda + \rho - (\lambda + \rho, \alpha^\vee)\alpha - \rho = \lambda + n + 1 = \lambda + 0$. So $\mu = \lambda$ and thus $M_\mu = M_\lambda$.
3. For $n \leq -2$,

$$
\begin{aligned}
(\mu, \alpha^\vee) &= (s_\alpha \cdot \lambda, \alpha^\vee) \\
&= (\lambda i(n+1)\alpha, \alpha^\vee) \\
&= n - 2(n+1) \\
&= -n - 2 \\
&\geq 0,
\end{aligned}
$$

so $\chi_\mu = \chi_{s_\alpha \cdot \mu} = \chi_{s\alpha \cdot (s_\alpha \cdot \lambda)} = \chi_\lambda$. Since W is generated by simple reflections and the linkage property is transitive, the result follows by induction on $\ell(w)$.

■

Exercise 1.8 (do but don't turn in): See book, show that certain properties of the dot action hold (namely nonlinearity).

11.2 1.9: Extending the Harish-Chandra Morphism

We want to extend the previous proposition from $\lambda \in \Lambda$ to $\lambda \in \mathfrak{h}^\vee$. We'll use a density argument from affine algebraic geometry, and switch to the Zariski topology on $\mathfrak{h}^\vee \subset \mathbb{C}^n$.

Fix a basis $\Delta = \{a_1, \cdots, a_\ell\}$ and use the Killing form to identify these with a basis for $\mathfrak{h} = \{h_1, \cdots, h_\ell\}$. Similarly, take $\{w_1, \cdots, w_\ell\}$ as a basis for $\mathfrak{h}^\vee$, and we'll use the identification

$$
\begin{aligned}
\mathfrak{h}^\vee &\iff \mathring{A}^\ell \\
\lambda &\iff (\lambda(h_1), \cdots, \lambda(h_\ell)).
\end{aligned}
$$

We identify $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[h_1, \cdots, h_\ell]$ with $P(\mathfrak{h}^\vee)$ which are polynomial functions on $\mathfrak{h}^\vee$. Fix $\lambda \in \mathfrak{h}^\vee$, extended λ to be a multiplicative function on polynomials. For $f \in \mathbb{C}[h_1, \cdots, h_\ell]$, we defined $\lambda(f)$. Under the identification, we send this to $\tilde{f}$ where $\tilde{f}(\lambda) = \lambda(f)$.

Note: we'll identify f and $\tilde{f}$ notationally going forward and drop the tilde everywhere.

Then W acts on $P(\mathfrak{h}^\vee)$ by the dot action: $(w \cdot \tilde{f})(\lambda) = \tilde{f}(w^{-1} \cdot \lambda)$.

Exercise: Check that this is a well-defined action.

Under this identification, we have

$$
\begin{aligned}
\mathfrak{h}^\vee &\iff \mathring{A}^\ell \\
\Lambda &\iff \mathbb{Z}^\ell.
\end{aligned}
$$

Note that Λ is discrete in the analytic topology, but is *dense* in the Zariski topology.

Proposition: A polynomial f on $\mathring{A}^\ell$ vanishing on $\mathbb{Z}^\ell$ must be identically zero.

Proof: For $\ell = 1$: A nonzero polynomial in one variable has only finitely many zeros, but if f vanishes on $\mathbb{Z}$ it has infinitely many zeros.

For $\ell > 1$: View $f \in \mathbb{C}[h_1, \cdots, h_{\ell-1}][h_\ell]$. Substituting any fixed integers for the h_i for $i \leq \ell - 1$ yields a polynomial in one variable which vanishes on $\mathbb{Z}$. By the first case, $f \equiv 0$, so the coefficients must all be zero and the coefficient polynomials in $\mathbb{C}[h_1, \cdots, h_{\ell-1}]$ vanish on $\mathbb{Z}^{\ell-1}$. By induction, these coefficient polynomials are identically zero.

■

Corollary: The only Zariski-closed subset of $\mathring{A}^\ell$ containing $\mathbb{Z}^\ell$ is $\mathring{A}^\ell$ itself, so the Zariski closure $\overline{\mathbb{Z}^\ell} = \mathring{A}^\ell$ and $\mathbb{Z}^\ell$ is dense in $\mathring{A}^\ell$.

12 Friday February 7th

So far, we have $\chi_\lambda = \chi_{w \cdot \cdot \lambda}$ if $\lambda \in \Lambda$ and $w \in W$. We have $\mathfrak{h}^\vee \supset \Lambda$ which is topologically equivalent to $\mathring{A}^\ell \supset \mathbb{Z}^\ell$, where $\mathbb{Z}^\ell$ is dense in the Zariski topology.

For $z \in \mathcal{Z}(\mathfrak{g})$, we have $\chi_\lambda(z) = \chi_{W \cdot \lambda}(z)$ and so $\lambda(\xi(z)) = (w \cdot \lambda)(\xi(z))$ where $\xi : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h}) \cong P(\mathfrak{h}^\vee)$ where we send $\lambda(f)$ to $f(\lambda)$.

Then $\xi(z)(\lambda) = \xi(z)(w \cdot \lambda)$ for all $\lambda \in \Lambda$, and so $\xi(z) = w^{-1}\xi(z)$ on Λ. But both sides here are polynomials and thus continuous, and $\Lambda \subset \mathfrak{h}^\vee$ is dense, so $\xi(z) = w^{-1}\xi(z)$ on all of $\mathfrak{h}^\vee$. I.e., $\chi_\lambda = \chi_{w \cdot} \lambda$ for all $\lambda \in \mathfrak{h}^\vee$.

This in fact shows that the image of $\mathcal{Z}(\mathfrak{g})$ under ξ consists of W-invariant polynomials.

It's customary to state this in terms of the natural action of W on polynomials without the row-shift. We do this by letting $\tau_\rho : S(\mathfrak{h}) \xrightarrow{\cong} S(\mathfrak{h})$ be the algebra automorphism induced by $f(\lambda) \mapsto f(\lambda - \rho)$. This is clearly invertible by $f(\lambda) \mapsto f(\lambda + \rho)$. We then define $\psi : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\xi} S(\mathfrak{h}) \xrightarrow{\tau_\rho} S(\mathfrak{h})$ as this composition; this is referred to as the Harish-Chandra (HC) homomorphism.

Exercise: Show $\chi_\lambda(z) = (\lambda + \rho)(\psi(z))$ and $\chi_{w \cdot \lambda}(w(\lambda + \rho))(\psi(z))$, where $w(\,\cdot\,)$ is the usual w-action. Replacing λ by $\lambda + \rho$ and w by w^{-1}, we get

$$w\psi(z) = \psi(z)$$

for all $z \in \mathcal{Z}(\mathfrak{g})$ and all $w \in W$ where $(w\psi(z))(\lambda) = \psi(z)(w^{-1}\lambda)$.

We've proved that

Theorem 1.9:

a. If $\lambda, \mu \in \mathfrak{h}^\vee$ that are linked, then $\chi_\lambda = \chi_\mu$.

b. The image of the twisted HC homomorphism $\psi : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h})$ lies in the subalgebra $S(\mathfrak{h})^W$.

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$. Recall from finite-dimensional representations there is a canonical element $c \in \mathcal{Z}(\mathfrak{g})$ called the Casimir element. For $\mathcal{O}$, we need information about the full center $\mathcal{Z}(\mathfrak{g})$ (hence introducing infinitesimal characters).

Expressing c in the PBW basis yields $c = h^2 + 2h + 4yx$, where $h^2 + 2h \in U(\mathfrak{h})$ and $4yx \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$.

Enveloping algebra convention: xs, hs, ys

Then $\xi(c) = \mathfrak{p}(c) = h^2 + 2h$, and under the identification $\mathfrak{h}^\vee \iff \mathbb{C}$ where $\lambda \iff \lambda(h)$, we can identify $\rho \iff \rho(h) = 1$. The row shift is given by $\psi(c) = (h-1)^2 + 2(h-1) = h^2 - 1$. This is W-invariant, since $s_{\alpha_h} = -h$. But $W = \langle s_\alpha, 1\rangle$, so s_α generates W.

We also have $\chi_\lambda(c) = (\lambda + \rho)(\psi(c)) = (\lambda+1)^2 - 1$. Then

$$\chi_\lambda(c) = \chi_\mu(c) \iff (\lambda+1)^2 - 1 = (\mu+1)^2 \iff \mu = \lambda \text{ or } \mu = -\lambda - 2$$

But $\lambda = 1 \cdot \lambda$ and $-\lambda - 2 = s_\alpha \cdot \lambda$, so $\mathcal{Z}(\mathfrak{g}) = \langle c \rangle := \mathbb{C}[c]$ as an algebra. So these characters are equal iff $\mu = w \cdot \lambda$ for $w \in W$.

13 Section 1.10: Harish-Chandra's Theorem

Goal: prove the converse of the previous theorem.

Theorem (HC): Let $\psi : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h})$ be the twisted HC homomorphism. Then

a. ψ is an *isomorphism* of $\mathcal{Z}(\mathfrak{g})$ onto $S(\mathfrak{h})^W$.

b. For all $\lambda, \mu \in \mathfrak{h}^\vee$, $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.

c. Every central character $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is a χ_λ.

Sketch of proof:

Part (a): Relies heavily on the *Chevalley Restriction Theorem* (which we won't prove here).

Initially we have a restriction map on polynomial functions $\theta : P(\mathfrak{g}) \to P(\mathfrak{h})$. We identified $P(\mathfrak{g}) = S(\mathfrak{g}^\vee)$, the formal polynomials on $\mathfrak{g}^\vee$. However, for $\mathfrak{g}$ semisimple, we can identify $S(\mathfrak{g}^\vee) \cong S(\mathfrak{g})$ via the Killing form.

By the Chinese Remainder Theorem, $\theta : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$ is an isomorphism, where the subgroup $G \leq \mathrm{Aut}(\mathfrak{g})$ is the *adjoint group* generated by $\left\{ \exp \mathrm{ad}_{\,x} \;\middle|\; x \text{ is nilpotent} \right\}$.

It turns out that $S(\mathfrak{g})^G$ is very close to $\mathcal{Z}(\mathfrak{g})$ – it is the associated graded of a natural filtration on $\mathcal{Z}(\mathfrak{g})$. This is enough to show that ψ is a bijection.

■

Part (b): We'll prove the contrapositive of the converse.

Suppose $W \cdot \lambda \bigcap W \cdot \mu = \emptyset$ and both are in $\mathfrak{h}^\vee$. Since these are finite sets, Lagrange interpolation yields a polynomial that is 1 on $W \cdot \lambda$ and 0 on $W \cdot \mu$. Let $g = \dfrac{1}{|W|} \displaystyle\sum_{w \in W} w \cdot f$.

Note: definitely the dot action here, may be a typo in the book.

Then g is a $W\cdot$ invariant polynomial with the same properties. By part (a), we can get rid of the row shift to obtain an isomorphism $\xi : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h})^(W\cdot)$, the $W\cdot$ invariant polynomials. Choose $z \in \mathcal{Z}(\mathfrak{g})$ such that $\xi(z) = g$, then $\chi_\lambda(z) = \lambda(\xi(z)) = \lambda(g) = g(\lambda) = 1$. So $\chi_\mu(z) = 0$ similarly, and $\chi_\lambda = \chi_\mu$.

■

Part (c): This follows from some commutative algebra, we won't say much here. Look at maximal ideals in $\mathbb{C}[x, y, \cdots]$ which correspond to evaluating on points in $\mathbb{C}^\ell$.

$\blacksquare$

Remark: Chevalley actually proved that $S(\mathfrak{h})^W \cong \mathbb{C}(p_1, \cdots, p_\ell)$ where the p_i are homogeneous polynomials of degrees $d_1 \leq \cdots \leq d_\ell$. These numbers satisfy some remarkable properties: $\prod d_i = |W|$ and $d_1 = 2$ (these are called the *degrees of* W)

14 Section 1.11

Theorem: Category $\mathcal{O}$ is *artinian*, i.e. every $M \in \mathcal{O}$ is Artinian (DCC) and $\dim \hom_{\mathfrak{g}}(M, N) < \infty$ for every M, N.

> Recall that $\mathcal{O}$ is known to be Noetherian from an earlier theorem. This will imply that every M has a composition/Jordan-Holder series, so we can take composition factors and multiplicities. Most interesting question: what are the factors/multiplicities of the simple modules and Verma modules?