Manifolds

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Tuesday 6^{th} October, 2020

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These are notes live-tex'd from a graduate course in Smooth Manifolds taught by David Gay at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Tuesday 6^{th} October, 2020
13:42

1 | Thursday, August 20

Exercise 1.1.
Show that $\left\{(\mathbb{R}^1, \mathrm{id}), (\mathbb{R}^1, x \mapsto x^3)\right\}$ is *not* a smooth atlas.

Exercise 1.2.
Let $S^1 := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\}$ with charts given by stereographic projection from $(0, 1)$ and $(0, -1)$ on $U = S^1 \setminus \{(0, 1)\} \to \mathbb{R}$ and $V = S^1 \setminus \{(0, -1)\} \to \mathbb{R}$.

Show that this gives a smooth atlas.

Exercise 1.3.
Write down a smooth atlas on the unit square.

2 | Tuesday, August 25

2.1 Submanifolds

Exercise 2.1.
Prove that charts on a manifold are smooth maps.

Hint: use the identity smooth structure on $\mathbb{R}^n$.

Exercise 2.2.
Show that open subsets of manifolds are again manifolds in a canonical way.

Exercise 2.3.
Show that S^1 is a manifold.

Example 2.1.
Prove that a submanifold is again a manifold.

3 | Thursday, September 24

Exercise 3.1 (?)**.**
Write down an explicit diffeomorphism between $\mathbb{CP}^1$ and S^2.

Exercise 3.2 (?)**.**
Show that the map

$$\begin{aligned} \mathbb{RP}^n &\to \mathbb{CP}^n \\ [x_0 : \cdots : x_n] &\mapsto [x_0 + 0i : \cdots : x_n + 0i] \end{aligned}$$

is an *embedding*, i.e. a differentiable map whose image is a submanifold, which is a diffeomorphism onto its image.

Exercise 3.3 (?)**.**
Define a vector field $V = -x_1 \partial_{x_1} + x_2 \partial_{x_2}$ on $M = (-1, 1)^2$. Find the best possible $\varepsilon : M \to (0, \infty]$, i.e. for each p, $\sup \left\{ t > 0 \,\middle|\, \Phi(t, p) \text{ is defined} \right\}$.

4 | Tuesday, September 29

Questions to look at for next Tuesday:

Exercise 4.1 (?)**.**
Show that the 3 natural coordinate charts on $\mathbb{CP}^2$ given by e.g. $\varphi_{U_0}([z_0 : z_1 : z_2]) = \left[\frac{z_1}{z_0}, \frac{z_2}{z_0} \right]$ yield a smooth atlas.

Exercise 4.2 (?)**.**

Consider the map

$$\pi : \mathbb{CP}^2 \to \mathbb{R}^2$$
$$[z_0 : z_1 : z_2] \mapsto \left[\frac{|z|_1^2}{|z|_0^2 + |z|_1^2 + |z|_2^2}, \frac{|z|_2^2}{|z|_0^2 + |z|_1^2 + |z|_2^2} \right].$$

- Show that $\operatorname{im}(\pi) = \{p_1, p_2 \geq 0, p_1 + p_2 \leq 1\}$.
- Show that π is smooth

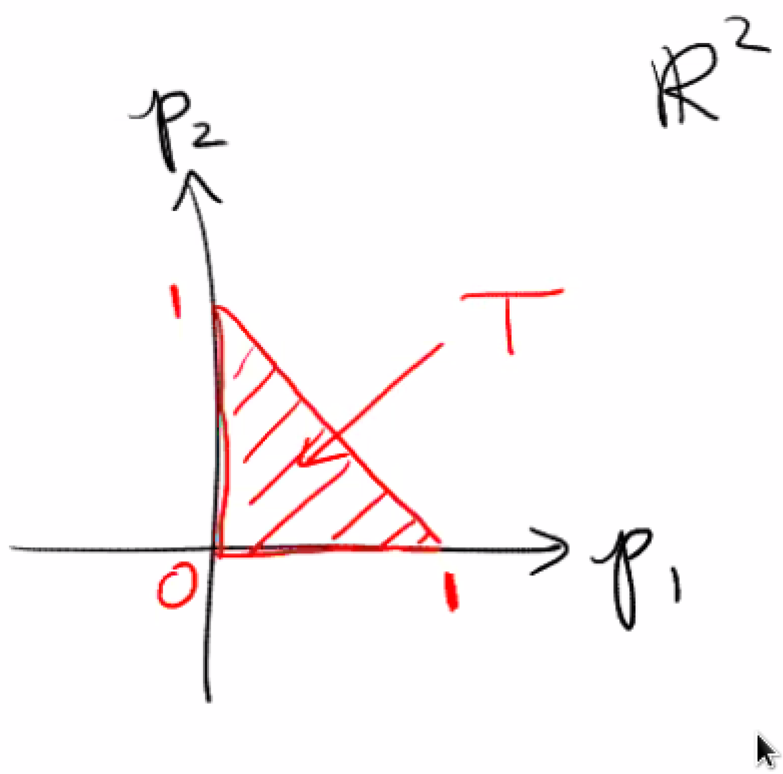

Figure 1: O

- If $[p_1, p_2] \in T^\circ$ is in the interior of the above triangle, then $\pi^{-1}(p_1, p_2) \cong S^1 \times S^1$ is diffeomorphic to a torus.
- If the point is on an edge, the fiber is diffeomorphic to S^1,
- If the point is on a vertex, the fiber is a single point.

Exercise 4.3 (?)**.**
Find a vector field V on some maximal subset of $\mathbb{CP}^2$ such that $D\pi(V) = p_1 \partial_{p_1} + p_2 \partial_{p_2}$ (the radial vector field).

I.e., for all $q \in \mathbb{CP}^2$, we have a map

$$D_1\pi : T_1\mathbb{CP}^2 \to T_{\pi(q)}\mathbb{R}^2$$

and $V(q) \in T_q\mathbb{CP}^2$, so we want $D_q\pi(V(q)) = p_1\partial_{p_1} + p_2\partial_{p_2}$.

Note that there will be a problem defining V on the fiber over the hypotenuse of T.

Theorem 4.1 ***(Collar Neighborhood).***
For all manifolds with boundary X, there exists an open neighborhood N of ∂X which is diffeomorphic to $(-\varepsilon, 0] \times \partial X$.

Proof strategy: construct a vector field pointing outward and flow it backward. Construct by forming local vector fields on open sets, then patch together using a partition of unity.

Definition 4.1.1 (Partition of Unity)**.**
A collection $\left\{\varphi_i : M \to \mathbb{R} \;\middle|\; i \in I\right\}$ such that

1. $\{\mathrm{supp}\varphi_i\}$ is locally finite, i.e. for all p, we have $\left|\left\{i \;\middle|\; p \in \mathrm{supp}(\varphi_i)\right\}\right| < \infty$.
2. $\varphi(p) \geq 0$ for all $p \in X$
3. For all $p \in X$, the sum $\sum_{i \in I} \varphi_i(p) = 1$.