Problem Set 10

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1 Problem 1

Let ϕ be an n-form. If suffices to show these statements for $n = 2$.

$\implies$: Suppose ϕ is alternating, then $\phi(b, b) = 0$ for all $b \in B$.

Letting $a, b \in B$ be arbitrary, we then have

$$\begin{aligned}
\phi(a+b, a+b) &= \phi(a, a+b) + \phi(b, a+b) \\
&= \phi(a, a) + \phi(a, b) + \phi(b, a) + \phi(b, b) \\
&= \phi(a, b) + \phi(b, a) \\
&\implies \phi(a, b) = -\phi(b, a),
\end{aligned}$$

which shows that ϕ is skew-symmetric.

$\impliedby$ Suppose ϕ is skew-symmetric, so $\phi(a, b) = -\phi(b, a)$ for all $a, b \in B$. Then $\phi(b, b) = -\phi(b, b)$ by transposing the terms, which says that $\phi(b, b) = 0$ for all $b \in B$ and thus ϕ is alternating.

2 Problem 2

Let $f(x) = \det(P + xQ) \in R[x]$, then f is a polynomial in x which is not identically zero.

To see that $f \not\equiv 0$, we can use that fact that P is invertible to evaluate $f(0) = \det(P) \neq 0$.

We can now note that f has finite degree, and thus finitely many zeroes in R.

3 Problem 3

Letting $k[x] \curvearrowright_{\phi} E$ to yield a $k[x]$-module structure on E and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \mid q_2 \mid \cdots \mid q_t$$

where $E_i = k[x]/(q_i)$. Then $q_t = q$, the minimal polynomial of E.

In particular, E_t is a ϕ-invariant subspace of E, and if $\deg q_t = m$, then E_t is in fact an m-dimensional cyclic module with basis $\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \cdots, \phi^{m-1}(\mathbf{v})\}$ for some $\mathbf{v} \in E_t$.

But since $E_t \leq E$ is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

4 Problem 4

$\implies$: Suppose $A \sim D$ where D is diagonal.