

Mapping Class Groups

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1 | Setup

- All manifolds:
 - Connected
 - Oriented
 - 2nd countable (countable basis)
 - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.

- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \{f : X \rightarrow Y \mid f \text{ is continuous}\}.$$

1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
 - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology \mathcal{O}_{CO} :

$$U \in \mathcal{O}_{\text{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

1.1.1 Mapping Spaces

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{\text{CO}}) \quad \text{where } \mathcal{O}_{\text{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
 - $\text{Homeo}(X, Y)$ the subspace of homeomorphisms
 - $\text{Imm}(X, Y)$, the subspace of immersions (injective map on tangent spaces)
 - $\text{Emb}(X, Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
 - $C^k(X, Y)$, the subspace of $k \times$ differentiable maps
 - $C^\infty(X, Y)$ the subspace of smooth maps
 - $\text{Diffeo}(X, Y)$ the subspace of diffeomorphisms
 - $C^\omega(X, Y)$ the subspace of analytic maps
 - $\text{Isom}(X, Y)$ the subspace of isometric maps (for Riemannian metrics)
 - $[X, Y]$ homotopy classes of maps

1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \{d(f_n(x), f(x)) \mid x \in K\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact.}$$

- In words: $f_n \rightarrow f$ uniformly on every compact set.
- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{f_\alpha\} \subset \text{hom}_{\text{Top}}(X, Y)$$

form a compact subset of $\text{Map}(X, Y)$?

- Essentially answered by:

Theorem 1.1 (Ascoli).

If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \text{hom}_{\text{Top}}(X, Y)$ has compact closure $\iff \mathcal{F}$ is equicontinuous and $F_x := \{f(x) \mid f \in \mathcal{F}\} \subset Y$ has compact closure.

Corollary 1.2 (Arzela).

If $\{f_n\} \subset \text{hom}_{\text{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every x , it contains a uniformly convergent subsequence.

1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
 - Can take G to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} := \text{hom}_{\text{TopGrp}}(G, S^1)$$

where we consider $S^1 \subset \mathbb{C}$.

- Can integrate with respect to the Haar measure μ , define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\text{Map}(\cdot, Y)$:

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\ X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\ X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}. \end{aligned}$$

- Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, Y^X) \\ H : X \times Z &\rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\ (x, z) &\mapsto H(x, z) \iff z \mapsto H(\cdot, z). \end{aligned}$$

- Categorically, $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top .
- A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space PY , the based loop space ΩY .
- Importance in homotopy theory: the path space fibration

$$\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y$$

- Plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences.
- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps f, g are homotopic \iff they are connected by a path in $\text{Map}(X, Y)$.

Picture!

2.1.1 Proof

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, Y^X) \cong \text{Map}(X \times I, Y),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \rightarrow Y$, the LHS are paths in the space of maps.

2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\text{Map}(I, \cdot)$:

$$\begin{aligned}
 \mathcal{P}\text{Map}(X, Y) &:= \text{Map}(I, Y^X) \\
 \mathcal{P}^2\text{Map}(X, Y) &:= \mathcal{P}\text{Map}(I, Y^X) = \text{Map}(I, (Y^X)^I) = \text{Map}(I, Y^{XI}) \\
 &\vdots \\
 \mathcal{P}^n\text{Map}(X, Y) &:= \mathcal{P}^{n-1}\text{Map}(I, Y^{XI}) = \text{Map}(X, Y^{XI^n}).
 \end{aligned}$$

- Can interpret

$$\mathcal{P}^2\text{Map}(X, Y) = \mathcal{P}\text{Map}(X \times I, Y).$$

as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

3 | Defining the Mapping Class Group

3.1 Isotopy

- Define a homotopy between $f, g : X \rightarrow Y$ as a map $F : X \times I \rightarrow Y$ restricting to f, g on the ends.
 - Equivalently: a *path*, an element of $\text{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \rightarrow Y$ to be homeomorphisms for every t .
 - Equivalently: a path in the subspace of homeomorphisms, an element of $\text{Map}(I, \text{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\text{Map}(X, Y)$, subspace of homeomorphisms.

3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
 - In a general category \mathcal{C} , we can always define the group $\text{Aut}_{\mathcal{C}}(X)$.
 - * If the group has a topology, we can consider $\pi_0\text{Aut}_{\mathcal{C}}(X)$, the set of path components.
 - * Since groups have identities, we can consider $\text{Aut}_{\mathcal{C}}^0(X)$, the path component containing the identity.
 - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X)/\text{Aut}_{\mathcal{C}}^0(X).$$

- Here the \pm indicates that we take both orientation preserving and non-preserving automorphisms.
- Has an index 2 subgroup of orientation-preserving automorphisms, $\text{MCG}^+(X)$.

Picture: quotienting out by identity component

3.3 Definitions in Several Categories

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with \mathcal{O}_{CO} .

- Taking $\text{MCG}_{\text{Top}}^{\pm}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking $\text{MCG}_{C^{\infty}}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f] \in [X, Y]\}.$$

- Taking $\text{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences*.

3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
 - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi\text{Aut}(X)$.
- For surfaces: For Σ a genus g surface, $\text{MCG}(S)$ acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \rightarrow \text{MCG}(\Sigma) \rightarrow T(\Sigma) \rightarrow \mathcal{M}_g \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to X .

- $T(S)$ is the moduli space of complex structures on S , up to the action of homeomorphisms that are isotopic to the identity:
 - Points are isomorphism classes of marked Riemann surfaces
 - Equivalently the space of hyperbolic metrics
- Used in the Nielsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
 - Periodic,
 - Reducible (preserves some simple closed curves), or
 - Pseudo-Anosov (has directions of expansion/contraction)

Picture: \mathcal{M}_g .

3.5 Easy Results

- $\text{MCG}_{\text{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, take the straight-line homotopy:

$$F : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$$

$$F(x, t) = tf(x) + (1 - t)x.$$

Picture: parameterize line between x and $f(x)$ and flow along it over time.

3.6 The Alexander Trick

- $\text{MCG}_{\text{Top}}(\bar{\mathbb{D}}^2) = 1$: for any $f : \bar{\mathbb{D}}^2 \rightarrow \bar{\mathbb{D}}^2$ such that $f|_{\partial\bar{\mathbb{D}}^2} = \text{id}$, take

$$F : \bar{\mathbb{D}}^2 \times I \rightarrow \bar{\mathbb{D}}^2$$

$$F(x, t) := \begin{cases} tf\left(\frac{x}{t}\right) & \|x\| \in [0, t) \\ x & \|x\| \in [1 - t, 1] \end{cases}.$$

- This is an isotopy from f to the identity.
- Interpretation: “cone off” your homeomorphism over time:

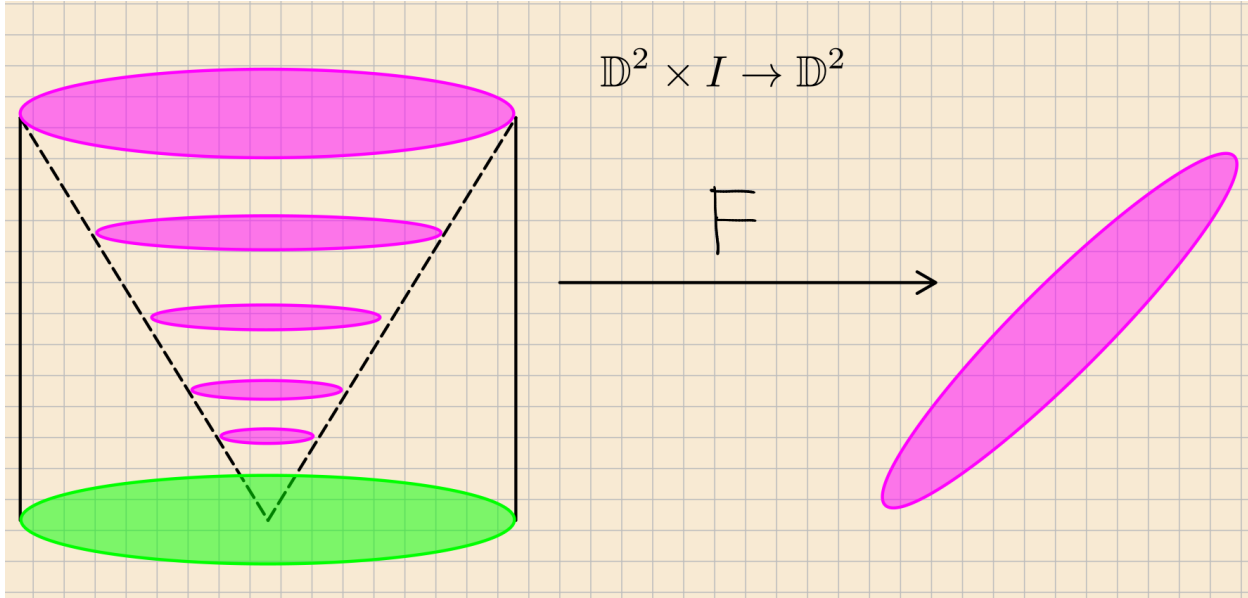


Figure 1: Image

- Note that this won't work in the smooth category: singularity at origin

3.7 Overview of Big Results

- The word problem in $\text{MCG}(\Sigma_g)$ is solvable
- For $g \geq 3$, the center of $\text{MCG}(\Sigma_g)$ is trivial and $H_1(\text{MCG}(\Sigma_g); \mathbb{Z}) = 1$

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- Why care: same as abelianization of the group.
 - $\text{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation
 - For $g \geq 4$, $H_2(\text{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$
 - Why care: used to understand surface bundles

$$\begin{array}{ccc} \Sigma_g & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

using

4 | Dehn Twists