

Title

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1 List of Topics

Chapters 1-9 of Dummit and Foote

- Left and right cosets
- Lagrange's theorem
- Isomorphism theorems
- Group generated by a subset
- Structure of cyclic groups

- Composite groups
 - HK is a subgroup iff $HK = KH$
- Normalizer
 - $HK \leq H$ if $H \leq N_G(K)$
- Symmetric groups
 - Conjugacy classes are determined by cycle types
- Group actions
 - Actions of G on X are equivalent to homomorphisms from G into $\text{Sym}(X)$
- Cayley's theorem
- Orbits of an action
- Orbit stabilizer theorem
- Orbits act on left cosets of subgroups
- Subgroups of index p , the smallest prime dividing $|G|$, are normal
- Action of G on itself by conjugation
- Class equation
- p -groups
 - Have non trivial center
- p^2 groups are abelian
- Automorphisms, the automorphism group
 - Inner automorphisms
 - $\text{Inn}(G) \cong Z/Z(G)$
 - $\text{Aut}(S_n) = \text{Inn}(S_n)$ unless $n = 6$
 - $\text{Aut}(G)$ for cyclic groups
 - $G \cong Z_p^n$, then $\text{Aut}(G) \cong GL_n(Z_p)$
- Proof of Sylow theorems
- A_n is simple for $n \geq 5$
- Recognition of internal direct product
- Recognition of semi-direct product
- Classifications:
 - pq
- Free group & presentations
- Commutator subgroup
- Solvable groups
 - S_n is solvable for $n \leq 4$
- Derived series
 - Solvable iff derived series reaches e
- Nilpotent groups
 - Nilpotent iff all sylow- p subgroups are normal
 - Nilpotent iff all maximal subgroups are normal
- Upper central series
 - Nilpotent iff series reaches G
- Lower central series
 - Nilpotent iff series reaches e
- Frattini's argument
- Rings
 - I maximal iff R/I is a field
 - Zorn's lemma
 - Chinese remainder theorem

- Localization of a domain
- Field of fractions
- Factorization in domains
- Euclidean algorithm
- Gaussian integers
- Primes and irreducibles
- Domains
 - * Primes are irreducible
- UFDs
 - * Have GCDs
 - * Sometimes PIDs
- PIDs
 - * Noetherian
 - * Irreducibles are prime
 - * Are UFDs
 - * Have GCDs
- Euclidean domains
 - * Are PIDs
- Factorization in $Z[i]$
- Polynomial rings
- Gauss' lemma
- Remainder and factor theorem
- Polynomials
- Reducibility
- Rational root test
- Eisenstein's criterion

2 Groups

2.1 Definitions

2.1.1 Subgroup Generated by a set A

- $\langle A \rangle = \{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_n^{\pm 1} : a_i \in A, n \in \mathbb{N}\}$
- Equivalently, the intersection of all H such that $A \subseteq H \leq G$

2.1.2 Free Group on a set X

- Equivalently, words over the alphabet X made into a group via concatenation

2.1.3 Centralizer of an element or a subgroup

- $C_G(a) = \{g \in G : ga = ag\}$
-

$$C_G(H) = \{g \in G : \forall h \in H, gh = hg\} = \bigcap_{h \in H} C_G(h)$$

- Note - requires the same g on both sides!

- Facts:
 - $C_G(H) \leq G$
 - $C_G(H) \trianglelefteq N_G(H)$
 - $C_G(G) = Z(G)$
 - $C_H(a) = H \cap C_G(a)$

2.1.4 Center of a group

- $Z(G) = \{g \in G : \forall x \in G, gx = xg\}$
- Facts
 -

$$Z(G) = \bigcap_{a \in G} C_G(a)$$

2.1.5 Normalizer of a subgroup

- - $N_G(H) = \{g \in G : gHg^{-1} = H\}$
- Equivalently, $\bigcup \{K : H \trianglelefteq K \leq G\}$ (the largest $K \leq G$ for which $H \trianglelefteq K$)
- Equivalently, the stabilizer of H under G acting on its subgroups via conjugation
- Differs from centralizer; can have $gh = h'g$
- Facts:
 - $C_G(H) \subseteq N_G(H) \leq G$
 - $N_G(H)/C_G(H) \cong A \leq \text{Aut}(H)$
 - Given $H \subseteq G$, let

$$S(H) = \bigcup_{g \in G} gHg^{-1}$$

, so $|S(H)|$ is the number of conjugates to H . Then

$$|S(H)| = [G : N_G(H)]$$

* i.e. the number of subgroups conjugate to H equals the index of the normalizer of H .

2.1.6 Normal Core of a subgroup

- - $H_G = \bigcap_{g \in G} gHg^{-1}$
- Equivalently, $H_G = \langle N : N \trianglelefteq G \text{ \& } N \leq H \rangle$
 - Largest normal subgroup that contains H
- Equivalently, $H_G = \ker \psi$ where $\psi : G \rightarrow \text{Sym}(G/H)$; $g \sim (xH) = (gx)H$
- Facts:
 - $H_G \trianglelefteq G$ and is an idempotent operation

2.1.7 Normal Closure of a subgroup

- $H^G = \{gHg^{-1} : g \in G\}$
- Equivalently,

$$H^G = \bigcap \{N : H \leq N \trianglelefteq G\}$$

- (The smallest normal subgroup of G containing H)

2.1.8 Group Action of a group on a set

- Given as a function

$$\phi : G \times X \rightarrow X, (g, x) \mapsto g \cdot x$$

which gives rise to a function

$$\phi_g : X \rightarrow X, x \mapsto g \cdot x$$

(which is a bijection) where \sim denotes a group element acting on a set element, and $\forall x \in X$,

- $e \cdot x = x$
- $(gh) \cdot x = g \cdot (h \cdot x)$

- Equivalently, a function

$$\psi : G \rightarrow \text{Sym}(X), g \mapsto \phi_g$$

–

$$\ker \psi = \bigcap_{x \in X} G_x$$

(intersection of all stabilizers)

- Interesting actions:
 - Left multiplication of G on G :

$$\phi : G \rightarrow G \rightarrow G \quad g \mapsto \phi_g : G \rightarrow G \quad h \mapsto gh$$

* $\mathcal{O}_x = G$ (transitive)

* $G_x = e$

- G acting via conjugation on itself:

$$\phi : G \rightarrow G \rightarrow G \quad g \mapsto \psi_g : G \rightarrow G \quad h \mapsto ghg^{-1}$$

* A common notation is $x^g = g^{-1}xg$ which obeys $(x^g)^h = x^{gh}$

* $\mathcal{O}_x = [x]$ (Conjugacy classes, so not generally transitive)

* $G_x = \{g \in G : gxg^{-1} = x\} = C_G(x)$

- G acting on $S = \{H : H \leq G\}$ via conjugation:

*

$$\phi : G \rightarrow S \rightarrow S \quad g \mapsto \psi_g : S \rightarrow S \quad H \mapsto gHg^{-1}$$

* $\mathcal{O}_H = [H] = \{gHg^{-1} : g \in G\}$, conjugate subgroups of H

* $G_x = N_G(H) = \{g \in G : gHg^{-1} = H\}$

2.1.9 Transitive group actions

- $\forall x, y \in X, \exists g \in G : g \cdot x = y$
- Equivalent, actions with a single orbit

2.1.10 Orbit of a set element

$$\mathcal{O}_x = \{g \sim x : x \in X\} = \bigcup_{g \in G} \{g \sim x\}$$

- The set of all orbits is denoted X/G or $X_G = \{\mathcal{O}_x : x \in X\}$
- Partitions X according to the equivalence relation $x \cong y \iff \exists g \in G : g \sim x = y$
- G acts transitively on X when restricted to any single orbit

2.1.11 Stabilizer of a set element

- $G_x = \{g \in G : g \sim x = x\}$
- Facts:
 - $G_x \leq G$, not usually normal
 - $x, y \in \mathcal{O}_x \Rightarrow G_x$ is conjugate to G_y

2.1.12 Automorphisms of a group

- $Aut(G) = \{\phi : G \rightarrow G : \phi \text{ is an isomorphism}\}$

2.1.13 Inner Automorphisms of a group

- $Inn(G) = \{\phi_g \in Aut(G) : \phi_g(x) = gxg^{-1}\}$
- Also consider the map

$$\psi : G \rightarrow Aut(G) : g \mapsto (\lambda : x \mapsto gxg^{-1})$$

Then $\text{im}\psi = Inn(G)$, $\ker\psi = Z(G)$

- Facts:
 - $Inn(G) \trianglelefteq Aut(G)$
 - $Inn(G) \cong G/Z(G)$

2.1.14 Outer Automorphisms of a group

- $Out(G) = Aut(G)/Inn(G)$

2.1.15 Conjugacy Class of an element

-

$$[a] = \{gag^{-1} : g \in G\} = \bigcup_{g \in G} \{gag^{-1}\}$$

- Equivalently, $[a] = \mathcal{O}_a$ under G acting on itself via conjugation
- Facts:
 - Equivalence relation, partitions the group
 - $|[a]|$ divides $|G|$
 - $a \in Z(G) \Rightarrow [a] = \{a\}$

2.1.16 Characteristic subgroup

- $H \text{ char } G \iff \forall \phi \in \text{Aut}(G), \phi(H) = H$
 - i.e., H is fixed by all automorphisms of G .

2.1.17 Simple group

- G is simple $\iff H \trianglelefteq G \Rightarrow H = e \text{ or } G$
 - No non-trivial normal subgroups

2.1.18 Commutator of an element, or of subgroups

- $[g, h] = ghg^{-1}h^{-1}$
- $[G, H] = \langle [g, h] : g \in G, h \in H \rangle$ (Subgroup generated by commutators)

2.2 Structural Results

- Cyclic \Rightarrow abelian
- $G/Z(G)$ cyclic $\Rightarrow G$ is abelian
- Intersections of subgroups are also subgroups

2.2.1 Isomorphisms Theorems

First Isomorphism Theorem

- Conditions:
 - $\phi : G \rightarrow G'$ is a homomorphism.
- Result:
 - $\ker \phi \trianglelefteq G$
 - $\text{im } \phi \leq G'$
 - $G/\ker \phi \cong \text{im } \phi$.
- Corollaries:
 - $\ker \phi = e \Rightarrow G \cong G'$

Second Isomorphism Theorem

- Conditions:
 - $N \trianglelefteq G, H \leq G$
- Results:
 - $HN \leq G$
 - $N \cap H \trianglelefteq H$
 -

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

- Corollaries:
 - (Weaker) Relaxing $N \trianglelefteq G$ to $H \subseteq N(N)$ yields
 - * $N \cap H \subseteq G$ (Not normal)
 - * $N \cap H \trianglelefteq H$

Third Isomorphism Theorem

- Conditions:
 - $N \trianglelefteq G, N \leq A \leq G$
- Results:
 - $A/N \leq G/N$
 - * Every subgroup of G/N is of this form for *some* such A
 - $$\frac{G/N}{A/N} \cong \frac{G}{A}$$
 - * Cancel the N !
- Corrolaries:
 - $A \trianglelefteq G \Rightarrow A/N \trianglelefteq G/N$
 - * All normal subgroups of G/N are of this form for some A .