Title

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Example (Hypersurfaces of contact type): The level sets of a Hamiltonian on $\mathbb{R}^{2n} = \operatorname{span}_{\mathbb{R}} \{\mathbf{p}, \mathbf{q}\}$ given by $H = K + U$ where $K = \frac{1}{2}\|\mathbf{p}\|^2$ and $U = U(\mathbf{q})$ is a function of only $\mathbf{q}$. (Usually kinetic + potential energy.)

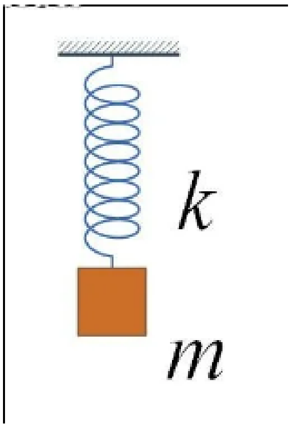
> Remark: all hypersurfaces of contact type (X, ω) look locally like $X \hookrightarrow \operatorname{Sp}(X)$, i.e. X embedded into its symplectification.

Basic Questions:

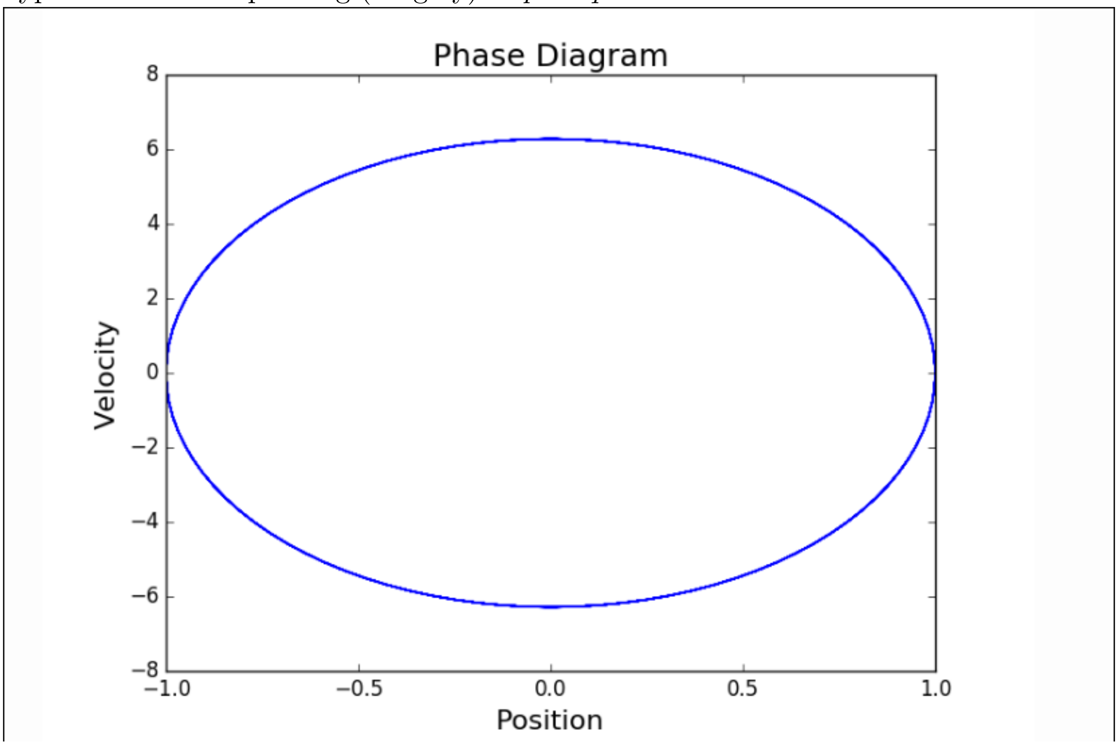
- Basic question: when does the flow of a vector field admit a *periodic orbit*?
- Does every/any vector field on a smooth manifold M admit a closed orbit?
  - Corollary: does every/any vector field on M admit a fixed point?
  - Note that if $\chi(M) \neq 0$, the Poincare-Hopf index theorem forces every vector field to have a fixed point.
- Does every vector field on S^3 admit a closed orbit?
  - Answer: no, very difficult to show, but turns out to hold for all 3-manifolds.

> *Remark*: The orbit of a Hamiltonian flow is contained in a single level set.

Example: Simple Harmonic Oscillator.



- $K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ where $p = mv$ is the momentum, given by $F = ma$
- $U = \frac{1}{2}kx^2$, given by Hooke's law
- $H(x,p) = U + K = \frac{1}{2}mv^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2 \sim p^2 + x^2$
- Has "phase space" $\Phi = \mathbb{R}^2 = \text{span}_{\mathbb{R}}\{x,p\}$, i.e. a position and momentum completely characterize the system at any fixed time.
- Conservation of energy shows that the time evolution of the system is governed by $\frac{\partial x}{\partial t} = -\frac{\partial H}{\partial p}$ and $\frac{\partial p}{\partial t} = \frac{\partial H}{\partial x}$
  - Corresponds to a path $\gamma : \mathbb{R} \to \Phi$ along which H is constant, i.e. a constant energy hypersurface corresponding (roughly) to $p^2 + q^2 = \text{const}$



    * If the Hamiltonian evolved over time, this region would travel around phases space, with the *volume* of this region invariant.

Definition (Reeb flow):

Definition (Reeb vector field):