Problem Set 2

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1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ may not lie in $\mathcal{O}$.

Proof:

Let $M(\lambda), M(\mu)$ be arbitrary Verma modules with highest weight vectors $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$ respectively. We can then consider the weight of $v \otimes w$ in $N \coloneqq M(\lambda) \otimes_{\mathbb{C}} M(\mu)$:

$$\begin{aligned} h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\ &= \lambda(h) v \otimes w + v \otimes \mu(h) w \\ &= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\ &= (\lambda(h) + \mu(h))(v \otimes w). \end{aligned}$$

Letting $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, so $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$, the claim is that it is possible for N to *not* be finitely-generated as a $U(\mathfrak{g})$-module.

Let $\{y, h, x\}$ be the usual basis for $\mathfrak{g}$, for which $U(\mathfrak{g})$ has the usual associated PBW basis. We can use the fact that $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$, so if we pick $\mu, \lambda \in \mathbb{Z}^{\leq 0}$ we have weight space decompositions

$$\begin{aligned} M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda - 2i} \coloneqq \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i \coloneqq \lambda - 2i}} M(\lambda)_{\lambda_i} \\ M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu - 2j} \coloneqq \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_i \coloneqq \mu - 2j}} M(\mu)_{\mu_j} \end{aligned}$$

where we can explicitly identify bases $M(\lambda)_{\lambda_i} = \operatorname{span}_{\mathbb{C}} \left\{ y^i\ v \right\}$ and $M(\mu)_{\mu_i} = \operatorname{span}_{\mathbb{C}} \left\{ y^i w \right\}$.

By the initial observation, this yields a weight space decomposition for N given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left(\bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) \coloneqq \bigoplus_{\nu \in \mathbb{Z}^+} N_\nu.$$

Since each weight space $N_\nu = \operatorname{span}_{\mathbb{C}} \left\{ y^i v \otimes y^j w \;\middle|\; i + j = \nu \right\}$ and there are infinitely many such weight spaces, no finite number of PBW monomials can generate N.

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2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the twisted Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ.

Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Proof:

Choosing a PBW basis $\{h_i\}_{i=1}^{\ell}$ for $U(\mathfrak{h})$, we can write

$$\begin{aligned}
\mathcal{Z}(\mathfrak{g}) \xrightarrow{\xi} U(\mathfrak{h}) \quad & \to S(\mathfrak{h}) = \mathbb{C}[\{h_i\}] = P(\mathfrak{h}^*) & \xrightarrow{\tau_\rho} \mathbb{C}[\{h_i\}] \\
z \mapsto z = \prod_{i=1}^{\ell} h_i^{t_i} \quad & \mapsto \left(\lambda \mapsto \prod_{i=1}^{\ell} \lambda(h_i)^{t_i} \right) & \mapsto \prod_{i=1}^{\ell} (\lambda - \rho)(h_i)^{t_i}.
\end{aligned}$$