Title

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1 | **Lecture 09**

Last time:

- The Čech-to-derived spectral sequence,
- The Mayer Vietoris LES,

  - Computes the étale cohomology of a scheme using a Zariski open cover.

- Étale cohomology of quasicoherent sheaves,

  - Agrees with Zariski cohomology, first legitimate computation!
  - Use this to compute:

- Étale cohomology of $\underline{\mathbb{F}_p}$ in characteristic p.

Last time we had a scheme $X_{/\mathbb{F}_p}$ and the *Artin-Schreier* exact sequence of sheaves of $X_{\text{ét}}$:

$$0 \to \underline{\mathbb{F}_p} \to \mathcal{O}_X^{\text{ét}} \xrightarrow{t \mapsto t^p - t} \mathcal{O}_X^{\text{ét}} \to 0.$$

The map appearing here is referred to as the *Artin-Schreier* map f This works over arbitrary fields of characteristic p, with a modified definition replacing t^p.

Exercise 1.0.1*(?):* Check that this is an additive homomorphism of abelian sheaves. This follows from the fact that Frobenius itself is.

Remark 1.0.2: From here onward, H^i will denote $H_{\text{ét}}^i$.

Recall that we had a theorem last time showing that the étale cohomology of quasicoherent sheaves is equivalent to the usual Zariski cohomology. From this we got a long exact sequence:

$$H^i(X_{\text{ét}}, \underline{\mathbb{F}_p}) \xrightarrow{\quad} H^i(X, \mathcal{O}_X) \xrightarrow{\ f\ } H^i(X, \mathcal{O}_X)$$

$$\xdownarrow{\delta}$$

$$\cdots \xrightarrow{\quad} H^{i-1}(X, \mathcal{O}_X)$$

We don't know how to compute $H^i(X_{\text{ét}}, \mathbb{F}_p)$ generally, but the affine case is easy. For X affine, $H^{>0}(X, \mathcal{O}_X) = 0$, which in facts holds for any quasicoherent sheaf replacing $\mathcal{O}_X$, and $H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|}$ where the exponent is the number of connected components of X. So we get an exact sequence

$$H^1(X, \mathcal{O}_X) \xrightarrow{\hspace{3cm}} 0$$

$$H^0(X, \mathbb{F}_p) = (\mathbb{F}_p)^{|\pi_0 X|} \xrightarrow{\hspace{1cm}} \mathcal{O}_X(X) \xrightarrow{f} \mathcal{O}_X(X)$$

$$0$$

Remark 1.0.3: $H^1(X, \mathcal{O}_X)$ is not finitely generated in general, e.g. take $X := \mathbb{A}^1$, then $\operatorname{coker}(t \mapsto t^p - t)$ as a map $k[t] \to k[t]$ is generally finite dimensional as a k-vector space. So in characteristic p, cohomology with $\mathbb{F}_p$ coefficients is ill-behaved: a nice cohomology theory would assign to every scheme a complex of finite dimensional vector spaces.

Remark 1.0.4: An aside: $\mathbb{G}_a$ is the representing object for $\mathcal{O}_X^{\text{ét}}$.

Remark 1.0.5: If X is proper, $H^i(X_{\text{ét}}, \mathbb{F}_p)$ is finite dimensional. Why? It follows from the exact sequence: by proper pushforward for coherent cohomology, the terms we're interested in are sandwiched between finite dimensional objects.

Example 1.0.6 *(?)*: However, these groups still won't have the expected dimension. For $X := E_{/k}$ where $k = \bar{k}, \operatorname{ch}(k) = p$, we have

$$H^1(E, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \text{if } E \text{ is ordinary} \\ 0 & \text{if } E \text{ is supersingular.} \end{cases}$$

This follows from the LES, since supersingularity is in terms of how Frobenius acts on the groups appearing. This is not what you'd expect: E is a torus, so you'd expect $\dim H^1 = 2$.

Remark 1.0.7: So this cohomology don't form a "good" cohomology theory in the sense that they won't prove the Weil conjectures or behave like the usual cohomology in characteristic zero, but can still be interesting and useful. This data is closely related to e.g. crystalline cohomology.

Example 1.0.8 *(?)*: We'll try to compute $H((\operatorname{Spec} k)_{\text{ét}}, \mathcal{F})$, the cohomology of the étale site of a field, using Čech cohomology. We had an equivalence of categories

$$\operatorname{Sh}^{\text{Ab}}(\operatorname{Spec} k)_{\text{ét}} \underset{\pi}{\overset{\iota}{\rightleftharpoons}} \{\text{Discrete } G\text{-modules}\},$$

where $G = \operatorname{Gal}(\bar{k}/k)$ is the absolute Galois group of k. What were the functors? Given a sheaf, you want to evaluate it on k^s (the separable closure), but this doesn't make sense since it's not an object on the étale site due to not being finitely presented. So you choose a separable closure, look at all intermediate extensions, and take the direct limit of evaluating the sheaf on those extensions. Going the other way, you can say what the value of a discrete G-module is on a finite extension L/k by taking its Galois fixed points: the fixed points of $\operatorname{Gal}(\bar{L}/L)$.

> **Corollary 1.0.9 *(?)*.**

$$H^i \left((\operatorname{Spec} k)_{\text{ét}}, \mathcal{F} \right) \xrightarrow{\sim} H^i(G, \iota\mathcal{F}).$$

Recall that this is because derived functor cohomology only depends on the equivalence class of the ambient category. Comparing this to Čech cohomology, suppose we have a cover $U := \operatorname{Spec} K$ where $L_{/k}$ is a separable field extension. Take the Čech complex

$$\check{C}(U_{/\operatorname{Spec} k}, \mathcal{F}) := \left(\mathcal{F}(U) \to \mathcal{F}(U \times U) \to \cdots \right).$$

Assume $L_{/k}$ is Galois with Galois group $G(L_{/k})$. We can rewrite this complex by identifying $U \times U = G(L_{/k}) \times \operatorname{Spec} L$, yielding

$$\check{C}(U/\operatorname{Spec} k, \mathcal{F}) := \left(\mathcal{F}(U) \to \mathcal{F}(G(L_{/k}) \times U) \to \mathcal{F}\left(G(L_{/k})^2 \times U \right) \to \cdots\cdots \right).$$

Exercise 1.0.10(*?*)**:** Show that this complex is the standard complex computing Galois cohomology $H^i(G(L_{/k}), \mathcal{F}(U))$. The terms are the same, so just identify the differentials. One can also take this as the definition of Galois cohomology.

As a corollary, this complex is quasi-isomorphic to the usual complex computing Galois cohomology, since that complex is the direct limit $\check{C}(U_{\operatorname{Spec} k}, \mathcal{F})$.

Question 1.0.11: When can étale cohomology can be computed as some kind of group cohomology.

Answer 1.0.12: This is true when $X = K(\pi, 1)$: it's connected and all of its homotopy groups vanish above degree 1, i.e. it's a classifying space for a discrete group. E.g. $S^1 = K(\mathbb{Z}, 1)$, or a compact orientable surface Σ_g of genus $g \geq 1$ has a contractible universal cover, and thus $\Sigma_g = K(\pi_1\Sigma_g, 1)$. In these cases, singular cohomology is the group cohomology of π_1. For G a finite group, BG will be an example, although e.g. this will not be true for GL_n. Another example will be affine curves.[1]

Goal for the next few classes: compute the étale cohomology of smooth (not necessarily projective) curves over $k = \bar{k}$, i.e. $H^i(C_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ where $\ell \neq \operatorname{ch}(k)$. We've seen what this is when $\ell = k$, and the answer will resemble the singular cohomology of a Riemann surface in terms of dimensions. This will be hard for $i > 2$, but we'll try to get to $i = 0, 1$. We can compute $i = 0$, since we're just asking for global sections to a sheaf:

$$H^i(C_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \mathbb{Z}/\ell^n\mathbb{Z},$$

since the definition of this sheaf was maps into $\mathbb{Z}/\ell^n\mathbb{Z}$, which is disconnected and so any map in is constant. For $i = 1$, we'll use an interpretation in terms of torsors, which are supposed to generalize principle homogeneous spaces. Typical example: given a space X, a Galois covering space will be a torsor for the Galois group: G acts on X and simply transitively on every fiber.

> **Definition 1.0.13** (*G*-Torsors)
> Idea: for $G \in \operatorname{Sh}^{\operatorname{Grp}}(X_{\text{ét}})$ a sheaf of (not necessarily abelian) groups, a *G***-torsor** is a sheaf

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[1] Here a curve will be a smooth separated scheme of finite type of dimension 1 over an algebraically closed field. We won't assume properness, and we'll generalize to singular curves.

$\mathcal{F} \in \mathrm{Sh}^{\mathrm{Set}}(X_{\text{ét}})$ with a G-action such that G acts on fibers simply and transitively.

Actual definition: a **torsor** is a sheaf $T \in \mathrm{Sh}^{\mathrm{Set}}(X)$ with an action $G \times T \xrightarrow{a} T$ (so $G(U)$ acts on $T(U)$ for every $U \in X_{\text{ét}}$) such that the following map is an isomorphism:

$$G \times T \xrightarrow{(a,\pi_2)} T \times T,$$

given by cross the action with the projection.