

CRAG

The Weil Conjectures

D. Zack Garza

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Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}}_q \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}_{\mathbb{F}}^n$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^n \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = 0 \right\}$$

where J is generated by *homogeneous* polynomials in $n + 1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{l}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{l}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

Problem: count points of a (smooth?) projective variety X/\mathbb{F} in all degree n extensions of \mathbb{F} . For reasons that will hopefully become clear in examples, define:

Definition

The *local zeta function* of X is the following formal power series:

$$Z_X(z) = \exp \left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$Z_X(0) = 1$$

$$z \left(\frac{\partial}{\partial z} \right) \log Z_X(z) = z \left(\frac{Z'_X(z)}{Z_X(z)} \right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \cdots,$$

which is an *ordinary generating function* for the sequence (α_n) .