Algebra Qual Prep Week 2: Finite Group Theory

D. Zack Garza

Table of Contents

Contents

1 Week 2: Finite Groups

See the Presentation Schedule

1.1 Topics

- Recognition of direct products and semidirect products
- Amalgam size lemma: $\#HK = \#H\#K/\#(H \cap K)$
- Group actions
  - Orbit-stabilizer
  - The class equation,
  - Burnside's formula
  - Important actions
    - Self-action by left translation (*the left-regular action*)
      - The assignment $g \mapsto \psi_g \in \mathrm{Sym}(G)$ where $\psi_g(x) := gx$ is sometimes referred to as the *Cayley representation* in qual questions, or sometimes a *permutation representation* since $\mathrm{Sym}(G) \cong S_n$ as sets where $n := \#G$
      - See the Strong Cayley Theorem
    - Self-action by conjugation
    - Action on subgroup lattice by left-translation
    - Action on cosets of a fixed G/H by left-translation
- Transitive subgroups
  - How these are related to Galois groups
- FTFGAG: The Fundamental Theorem of Finitely Generated Abelian Groups
  - Invariant factors
  - Elementary divisors
- Simple groups
- Automorphisms
  - Inner automorphisms
  - Outer automorphisms (not often tested directly)
  - Characteristic subgroups (not often tested directly)
- Series of groups (not often tested)
  - Normal series
  - Central series
  - The Jordan-Holder theorem
    - Composition series
  - Solvable groups

    - ♢ Derived series
  - Nilpotent groups
    - ♢ Lower central series
    - ♢ Upper central series

A remark: automorphisms and series of groups aren't often directly tested on the qual, but are useful practice. Simple/solvable groups do *come up often.*

1.2 Exercises

1.2.1 Warmup

- Show that if $H, K \leq G$ are subgroups and $H \in N_G(H)$, then HK is a subgroup.
  - Find a counterexample where $H \leq G$, K is only a subset and not a subgroup, and HK fails to be a subgroup?
- Prove the "Recognizing direct products" theorem: if H, K are normal in G with $H \cap K = \emptyset$ and $HK = G$, then $G \cong H \times K$.
  - Hint: write down a map $H \times K \to G$ and follow your nose!
  - How can you generalize this to 3 or more subgroups?
- State definitions of the following:
  - Group action
  - Orbit
  - Stabilizer
  - Fixed points
- State the orbit-stabilizer theorem
- State the class equation. Can you derive this from orbit-stabilizer?
- Show that the center of a p-group is nontrivial
- **Important**: Pick your favorite composite number $m = \prod p_i^{e_i}$ and classify all abelian groups of that order.
  - Write their invariant factor decompositions *and* their elementary divisor decompositions. Come up with an algorithm for converting back and forth between these.
- Prove that if $H \leq G$ is a proper subgroup, then G can not be written as a union of conjugates of H. - Use this to prove that if $G = \text{Sym}(X)$ is the group of permutations on a finite set X with $\#X = n$, then there exists a $g \in G$ with no fixed points in X.
- Define what a composition series is, and state what it means for a group to be simple, solvable, or nilpotent.
  - How are the derived and lower/upper central series defined? What type(s) of the groups above does each series correspond to?

1.2.2 Group Actions

- For each of the following group actions, identify what the orbits, stabilizers, and fixed points are. If possible, describe the kernel of each action, and its image in $\mathrm{Sym}(X)$.

  - G acting on $X = G$ by left-translation:
  $$g \cdot x := gx$$
  .
  - G acting on $X = G$ by conjugation:
  $$g \cdot x := gxg^{-1}$$
  - G acting on its set of subgroups $X := \left\{ H \;\middle|\; H \leq G \right\}$ by conjugation:
  $$g \cdot H := gHg^{-1}$$
  - For a fixed subgroup $H \leq G$, G acting on the set of cosets $X := G/H$ by left-translation:
  $$g \cdot xH := (gx)H$$

- Suppose X is a G-set, so there is a permutation action of G on X. Let $x_1, x_2 \in X$, and show that the stabilizer subgroups $\mathrm{Stab}_G(x_1), \mathrm{Stab}_G(x_2) \leq G$ are conjugate in G.

- Let $[G : H] = p$ be the smallest prime dividing the order of G. Show that H must be normal in G.

- Show that if G is an infinite simple group, then G can not have a subgroup of finite index.

  Hint: use the left-regular action on cosets.

- Show that every subgroup of order 5 in S_5 is a transitive subgroup.

1.2.3 Automorphisms

- How do you compute the totient $\varphi(p)$ for p prime? Or $\varphi(n)$ for n composite?

- What is the order of $\mathrm{GL}_n(\mathbb{F}_p)$?

- Identify $\mathrm{Aut}(\mathbb{Z}/p)$ and $\mathrm{Aut}(\prod_{i=1}^{n} \mathbb{Z}/p)$ for p a prime.

  - Identify $\mathrm{Aut}(\mathbb{Z}/n)$ for n composite.

- How many elements in $\mathrm{Aut}(\mathbb{Z}/20)$ have order 4?

- Find two groups $G \not\cong H$ where $\mathrm{Aut}G \cong \mathrm{Aut}H$.
- Let $H, K \leq G$ be subgroups with $H \cong K$. Is it true that $G/H \cong G/K$?

Hint: consider a group with distinct subgroups of order 2 whose quotients have order 4.

- Show that inner automorphisms send conjugate subgroups to conjugate subgroups.
- Show that for $n \neq 6$, $\mathrm{Aut}(S_n) = \mathrm{Inn}(S^n)$.

1.2.4 Series of Groups

- Determine all pairs $n, p \in \mathbb{Z}^{\geq 1}$ such that $\mathrm{SL}_n(\mathbb{F}_p)$ is solvable.
- If $\#G = pq$, is G necessarily nilpotent?

Hint: consider $Z(S_3)$.

- Show that if G is solvable, then G contains a nontrivial normal subroup.
  - What does this mean on the Galois theory side?

Hint: consider the derived series.

2 | Qual Problems

2.1 Fall 2019 #1 ✨

Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G. Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

Relevant concepts omitted.

5.5 Spring 2018 #1 ✨

a. Use the Class Equation (equivalently, the conjugation action of a group on itself) to prove that any p-group (a group whose order is a positive power of a prime integer p) has a nontrivial center.

b. Prove that any group of order p^2 (where p is prime) is abelian.

c. Prove that any group of order $5^2 \cdot 7^2$ is abelian.

d. Write down exactly one representative in each isomorphism class of groups of order $5^2 \cdot 7^2$.

3.3 Spring 2016 #5 🚩

Let G be a finite group acting on a set X. For $x \in X$, let G_x be the stabilizer of x and $G \cdot x$ be the orbit of x.

a. Prove that there is a bijection between the left cosets G/G_x and $G \cdot x$.

b. Prove that the center of every finite p-group G is nontrivial by considering that action of G on $X = G$ by conjugation.

3.5 Fall 2018 #2 ✨

a. Suppose the group G acts on the set X . Show that the stabilizers of elements in the same orbit are conjugate.

b. Let G be a finite group and let H be a proper subgroup. Show that the union of the conjugates of H is strictly smaller than G, i.e.

$$\bigcup_{g \in G} gHg^{-1} \subsetneq G$$

c. Suppose G is a finite group acting transitively on a set S with at least 2 elements. Show that there is an element of G with no fixed points in S.

3.4 Fall 2017 #1 🚩

Suppose the group G acts on the set A. Assume this action is faithful (recall that this means that the kernel of the homomorphism from G to $\mathrm{Sym}(A)$ which gives the action is trivial) and transitive (for all a, b in A, there exists g in G such that $g \cdot a = b$.)

a. For $a \in A$, let G_a denote the stabilizer of a in G. Prove that for any $a \in A$,

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \{1\}.$$

b. Suppose that G is abelian. Prove that $|G| = |A|$. Deduce that every abelian transitive subgroup of S_n has order n.

Needs some Sylow theory:

4.11 Fall 2019 #2 ✨

Let G be a group of order 105 and let P, Q, R be Sylow 3, 5, 7 subgroups respectively.

a. Prove that at least one of Q and R is normal in G.

b. Prove that G has a cyclic subgroup of order 35.

c. Prove that both Q and R are normal in G.

d. Prove that if P is normal in G then G is cyclic.