Theorems
Real Analysis

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1 Chapter 1

1.1 Topology

Theorem 1.1. *The closure of a set is a closed set.*

Theorem 1.2. *A set is closed if and only if it contains all of its limit points.*

Theorem 1.3 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Theorem 1.4. *In a metric space, sequential compactness is equivalent to compactness.*

1.2 Rectangles in $\mathbb{R}^d$

Theorem 1.5. *If a rectangle is the almost disjoint union of finitely many other rectangles, then the volume is the sum of the volumes. Symbolically, if $R = \bigcup_{k=1}^N R_k$, then*

$$|R| = \sum_{k=1}^N |R_k|$$

Theorem 1.6. *If a rectangle R is contained in a union of rectangles, then the volume of R does not exceed the sum of the volumes. Symbolically, if $R, R_1, \dots R_N$ are rectangles such that $R \subset \bigcup_{k=1}^N R_k$ then*

$$|R| \leq \sum_{k=1}^N |R_k|$$

Theorem 1.7. *Any collection of disjoint open intervals in $\mathbb{R}$ is countable.*

Proof. Let $\{I_\alpha\}_{\alpha \in A}$ be a collection of disjoint open intervals. Each I_α is nontrivial, so there is a rational $q_\alpha \in I_\alpha$. Thus we have q_α distinct rational numbers, since the I_α's are disjoint. There cannot be more than a countable number of rationals, so A is countable. $\square$

Theorem 1.8. *Every open subset $\mathcal{O} \subset \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Theorem 1.9. *Every open set $\mathcal{O} \subset \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.*

Theorem 1.10. *The Cantor middle-thirds set is compact, totally disconnected, and perfect.*

1.3 Exterior Lebesgue Measure

Theorem 1.11. *The exterior measure of a rectangle is equal to is volume.*

Theorem 1.12. *The exterior measure of $\mathbb{R}^d$ is infinite.*

Theorem 1.13. *The exterior measure of the Cantor (middle-thirds) set is zero.*

Theorem 1.14. *Let $E \subset \mathbb{R}^d$. For $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ such that*

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon$$

Theorem 1.15. *The exterior measure of a subset does not exceed the exterior measure of the containing set. Symbolically,*

$$E_1 \subset E_2 \implies m_*(E_1) \leq m_*(E_2)$$

Theorem 1.16. *Exterior measure is countably sub-additive. Symbolically,*

$$E = \bigcup_{j=1}^{\infty} E_j \implies m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$$

Theorem 1.17. *The exterior measure of E is equal to the infimum over the exterior measures of all open sets containing E. Symbolically,*

$$m_*(E) = \inf\{m_*(\mathcal{O}) : E \subset \mathcal{O} \textit{ and } \mathcal{O} \textit{ is open}\}$$

Theorem 1.18. *If two sets have positive distance from each other, then the exterior measure of the union is the sum of the exterior measures. Symbolically,*

$$d(E_1, E_2) < 0 \implies m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$$

Theorem 1.19. *The exterior measure of a countable union of almost disjoint cubes is equal to the sum of the measures of the cubes. Symbolically, if $\{Q_j\}_{j=1}^{\infty}$ is a collection of almost disjoint cubes, then*

$$m_*\left(\bigcup_{j=1}^{\infty} Q_j\right) = \sum_{j=1}^{\infty} |Q_j|$$

1.4 Lebesgue Measurable Sets

Theorem 1.20. *Open and closed sets in $\mathbb{R}^d$ are measurable.*

Theorem 1.21. *Any set with exterior measure zero is measurable, and has measure zero. More generally, any subset of a set of exterior measure zero is measurable and has measure zero. Symbolically,*

$$m_*(E) = 0 \text{ and } F \subset E \implies m(F) = 0$$

In other words, Lebesgue measure is complete. (See chapter 6 for definition of complete.)

Theorem 1.22. *The collection of measurable subsets of $\mathbb{R}^d$ forms a σ-algebra. That is, countable unions and intersections of measurable sets are measurable and the complement of a measurable set is measurable.*

Theorem 1.23. *The distance between a disjoint pair of a closed and a compact set is positive. Symbolically, if F is closed, K is compact, and $F \cap K = \emptyset$, then $d(F, K) > 0$.*

Theorem 1.24. *Lebesgue measure is σ-additive. That is, the measure of a countable union of disjoint measurable sets is the sum of the measures. Symbolically, if $\{E_n\}_{n=1}^{\infty}$ is a collection of disjoint measurable sets, then*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

Theorem 1.25. *The measure of the limit of an increasing sequence of measurable sets is the limit of the measures of the sets. Symbolically,*

$$E_n \nearrow E \implies m(E) = \lim_{n\to\infty} m(E_n)$$

where each E_n is assumed to be measurable.

Theorem 1.26. *The measure of the limit of a decreasing sequence of measurable sets is the limit of the measures of the sets, provided that the limit sets eventually have finite measure. Symbolically,*

$$E_n \searrow E \text{ and } \exists k \text{ such that } m(E_k) < \infty \implies m(E) = \lim_{n\to\infty} m(E_n)$$

where each E_n is assumed to be measurable. (Note that if there is some k such that $m(E_k) < \infty$, then every E_{k+j} also has finite measure.)

Theorem 1.27 (Borel-Cantelli Lemma). *Let $\{E_k\}_{k=1}^{\infty}$ be a countable family of measurable subsets of $\mathbb{R}^d$ such that $\sum_k m(E_k) < \infty$, and let $E = \limsup_{k\to\infty} E_k = \bigcap_n \bigcup_{k\geq n} E_k$. Then $m(E) = 0$.*

Theorem 1.28 (Theorem 3.4 and Exercise 26). *Let $E \subset \mathbb{R}^d$. The following are equivalent:*

1. *E is measurable.*

2. *For every $\epsilon > 0$, there exists an open set $\mathcal{O}$ such that $E \subset \mathcal{O}$ and $m(\mathcal{O} \setminus E) < \epsilon$.*

3. *For every $\epsilon > 0$, there exists a closed set F such that $F \subset E$ and $m(E \setminus F) < \epsilon$.*

Theorem 1.29. *Let $E \subset \mathbb{R}^d$ be measurable with $m(E) < \infty$. Then for $\epsilon > 0$,*

1. *There exists a compact set K with $K \subset E$ and $m(E \setminus K) < \epsilon$.*

2. *There exists a finite union $F = \bigcup_{j=1}^N Q_j$ of closed cubes such that $m(E \triangle F) < \epsilon$.*

Theorem 1.30 (Invariance Properties of Lebesgue Measure)**.** *Lebesgue measure is translation invariant, relatively dilation invariant, and reflection invariant. Symbolically, for $E \subset \mathbb{R}^d, h \in \mathbb{R}^d, \delta > 0$,*

$$\begin{aligned} m(E+h) &= m(E) \\ m(\delta E) &= \delta^d m(E) \\ m(-E) &= m(E) \end{aligned}$$

More generally, if $\delta = (\delta_1, \ldots, \delta_d)$ is a d-tuple of positive real numbers then

$$m(\delta E) = (\delta_1 \ldots \delta_d) m(E)$$

Theorem 1.31. *Let $E \subset \mathbb{R}^d$ be measureable and $L : \mathbb{R}^d \to \mathbb{R}^d$ a linear transformation. Then $L(E)$ is measurable.*

Theorem 1.32. *Let B be a ball in $\mathbb{R}^d$ with radius r. Then $m(B) = v_d r^d$ where v_d is the measure of the unit ball centered at the origin.*

Theorem 1.33. *G_δ sets and F_σ sets are Borel sets.*

Theorem 1.34. *Let $E \subset \mathbb{R}^d$. The following are equivalent:*

1. *E is measurable.*

2. *There exists $G \in G_\delta$ such that $m(E \setminus G) = 0$.*

3. *There exists $F \in F_\sigma$ such that $m(E \setminus F) = 0$.*

Theorem 1.35. *Let A, B, E be subset of $\mathbb{R}^d$ such that $A \subset E \subset B$, the sets A and B are measurable, and $m(A) = m(B)$. Then E is measurable, and thus $m(E) = m(A) = m(B)$.*

Theorem 1.36. *Let $E \subset \mathbb{R}$ where $m_*(E) > 0$. For each $\alpha \in (0, 1)$, there exists an open interval I so that $m_*(E \cap I) \geq \alpha m_*(I)$.*

Theorem 1.37. *There exists a non-measurable subset of $\mathbb{R}$.*

Theorem 1.38. *Every subset of $\mathbb{R}^d$ with strictly positive outer measure contains a non-measurable subset.*

Theorem 1.39. *The axiom of choice and the well-ordering principle are equivalent.*

1.5 Measurable Functions

Theorem 1.40. *If f is measurable, then $-f$ is measurable.*

Theorem 1.41. *Let $f : E \to \mathbb{R}$. The following are equivalent:*

1. *f is measurable.*
2. *$f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$.*
3. *$f^{-1}(F)$ is measurable for every closed set F.*

Theorem 1.42. *Continuous functions are measurable.*

Theorem 1.43. *The composition of a measurable and finite-valued function with a continuous function on the right is measurable. That is, if f is measurable and finite-valued and ϕ is continuous, then $\phi \circ f$ is measurable.*

Theorem 1.44. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Then*

$$\sup_n f_n \qquad \inf_n f_n \qquad \limsup_{n\to\infty} f_n \qquad \liminf_{n\to\infty} f_n$$

are also measurable functions.

Theorem 1.45. *If f is the limit of a sequence of measurable functions, then f is measurable. Symbolically,*

$$f(x) = \lim_{n\to\infty} f_n(x) \implies f \text{ is measurable}$$

Theorem 1.46. *The sum or pointwise multiplication of finite-valued measurable functions is measurable. Symbolically, if f, g are measurable and finite-valued, then $f + g$ and fg are measurable.*

Theorem 1.47. *Let f be a measurable function and suppose g is a function such that $f(x) = g(x)$ almost everywhere. Then g is measurable.*

Theorem 1.48. *Suppose f is a non-negative measurable function on $\mathbb{R}^d$. Then there exists an increasing sequence of non-negative simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges pointwise to f, that is,*

$$\phi_k(x) \le \phi_{k+1}(x) \qquad \lim_{k\to\infty} \phi_k(x) = f(x)$$

for all x.

Theorem 1.49. *Suppoet f is measurable on $\mathbb{R}^d$. Then there is a sequence of simple functions ϕ_k such that*

$$|\phi_k(x)| \le |\phi_{k+1}(x)| \qquad \lim_{k\to\infty} \phi_k(x) = f(x) \qquad |\phi_k(x)| \le |f(x)|$$

for all x. Note that this generalizes the above result.

Theorem 1.50. *Let f be measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions ψ_k that converges pointwise to $f(x)$ for almost every x. That is,*

$$\lim_{k\to\infty} \psi_k(x) = f(x) \quad a.e.\ x$$

Theorem 1.51. *Let f be measurable on $\mathbb{R}^d$. Then there exists a sequence f_k of continuous functions such that $f_k \to f$ pointwise for a.e. x.*

Littlewood's Three Principles

1. Every measurable set is nearly a finite union of intervals.
2. Every measurable function is nearly continuous. (see Lusin's Theorem)
3. Every convergent sequence of measurable functions is nearly uniformly continuous. (see Egorov's Theorem)

Theorem 1.52 (Egorov's Theorem)**.** *Suppose f_k is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, such that $f_k \to f$ a.e. on E. Then for every $\epsilon > 0$, there is a closed set $A_\epsilon \subset E$ such that $m(E \setminus A_\epsilon) < \epsilon$ and $f_k \to f$ uniformly on A_ϵ.*

Theorem 1.53 (Lusin's Theorem)**.** *Suppose f is measurable and finite-valued on E with $m(E) < \infty$. Then for every $\epsilon > 0$ there exists a closed set F_ϵ such that $F_\epsilon \subset E$ and $m(E \setminus F_\epsilon)$ such that $f|_{F_\epsilon}$ is continuous.*

Theorem 1.54 (Brunn-Minkowski Inequality)**.** *Let A, B be measurable sets in $\mathbb{R}^d$ so that $A + B$ is measurable. Then*

$$m(A+B)^{1/d} \geq m(A)^{1/d} + m(B)^{1/d}$$

2 Chapter 2

2.1 The Lebesuge Integral

Theorem 2.1 (Bounded Convergence Theorem)**.** *Suppose f_n is a sequence of measurable functions that are all bounded by M and supported on a set E of finite measure and $f_n(x) \to f(x)$ a.e. as $n \to \infty$. Then f is measurable, bounded, supported on E, and*

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

As a result,

$$\lim_{n\to\infty} \int f_n = \int f$$

Theorem 2.2. *If $f \geq 0$ and $\int f = 0$, then $f = 0$ almost everywhere.*

Theorem 2.3. *If f is integrable, then $f(x) < \infty$ almost everywhere.*

Theorem 2.4 (Agreement with Riemann Integral)**.** *If f is Riemann integrable on $[a,b]$, then f is measurable and the Riemann integral $\int_a^b f$ is equal to the Lebesgue integral $\int_{[a,b]} f$.*

Theorem 2.5. *Define the functions*

$$f_a(x) = \begin{cases} |x|^{-a} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$
$$F_a(x) = \frac{1}{1+|x|^a}$$

Then f_a is integrable if and only if $a < d$. F_a is integrable if an only if $a > d$.

Theorem 2.6 (Properties of Lebesgue Integral)**.** *Let f, g be integrable functions. Then*

$$a, b \in \mathbb{R} \implies \int (af + bg) = a \int f + b \int g$$
$$E \cap F = \emptyset \implies \int_{E \cup F} f = \int_E f + \int_F f$$
$$f \leq g \implies \int f \leq \int g$$
$$\left| \int f \right| \leq \int |f|$$

Theorem 2.7 (Fatou's Lemma)**.** *Let f_n be a sequence of nonnegative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then*

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

Theorem 2.8 (Corollary to Fatou's Lemma)**.** *Let f be a nonnegative measurable function and f_n a sequence of nonnegative measurable functions with $f_n \leq f$ and $f_n \to f$ for a.e. x. Then*

$$\lim_{n\to\infty} \int f_n = \int f$$

Theorem 2.9 (Monotone Convergence Theorem)**.** *Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$ (that is, $f_n \leq f_{n+1}$ a.e. and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e.). Then*

$$\lim_{n\to\infty} \int f_n = \int f$$

Theorem 2.10. *Let $\sum_{k=1}^{\infty} a_k(x)$ be a series where each a_k is a nonnegative measurable function. Then*

$$\int \sum_{k=1}^{\infty} a_k(x) dx = \sum_{k=1}^{\infty} \int a_k(x) dx$$

Consequently, if $\sum_k \int a_k(x)dx$ is finite, the series $\sum_k a_k(x)$ converges for a.e. x.

Theorem 2.11. *Let f be integrable on $\mathbb{R}^d$. For every $\epsilon > 0$, there exists a set B of finite measure such that*

$$\int_{\mathbb{R}^d \setminus B} |f| < \epsilon$$

Theorem 2.12. *Let f be integrable on $\mathbb{R}^d$. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$m(E) < \delta \implies \int_E |f| < \epsilon$$

Theorem 2.13 (Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. and there exists an integrable function g such that $|f_n(x)| \leq g(x)$. Then*

$$\lim_{n\to\infty} \int |f_n - f| = 0$$
$$\lim_{n\to\infty} \int f_n = \int f$$

2.2 The Banach Space of Integrable Functions

Theorem 2.14 (Properties of L^1)**.** *Let $f, g \in L^1$ and $a \in \mathbb{R}$. Then*

$$\|af\| = |a| \, \|f\|$$
$$\|f + g\| \leq \|f\| + \|g\|$$
$$\|f\| = 0 \iff f = 0 \text{ a.e.}$$

That is, the map $f \mapsto \int |f|$ is a norm on L^1. Additionally, $d(f, g) = \|f - g\|$ defines a metric on L^1.

Theorem 2.15 (Riesz-Fischer Theorem, for $p = 1$)**.** *The vector space L^1 is complete in its metric.*

Theorem 2.16. *L^1 is a Banach space.*

Theorem 2.17. *If f_n is a sequence of L^1 functions that converges to f in the L^1 norm, then there is a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ a.e.*

Theorem 2.18. *The following families of functions are dense in L^1: simple functions, step functions, and continuous functions of compact support.*

Theorem 2.19 (Transformation Invariance Properties of the Integral)**.** *Let $f \in L^1$. Then for $h \in \mathbb{R}^d$ and $\delta > 0$ we have*

$$\int_{\mathbb{R}^d} f(x - h)dx = \int_{\mathbb{R}^d} f(x)dx$$
$$\int_{\mathbb{R}^d} f(\delta x)dx = \delta^{-d} \int_{\mathbb{R}^d} f(x)dx$$
$$\int_{\mathbb{R}^d} f(-x)dx = \int_{\mathbb{R}^d} f(x)dx$$

Theorem 2.20. *Let $f \in L^1$ and $h \in \mathbb{R}^d$. Then $\|f_h - f\| \to 0$ as $h \to 0$. Analogously, for $\delta > 0$, $\|f(\delta x) - f(x)\| \to 0$ as $\delta \to 1$.*

2.3 Fubini's Theorem and Consequences

Theorem 2.21 (Fubini's Theorem)**.** *Let $f(x,y)$ be integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$,*

1. *The slice f^y is integrable on $\mathbb{R}^{d_1}$.*
2. *The function $g : \mathbb{R}^{d_2} \to \mathbb{R}$ define by $g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$.*
3. *Integrating g gives the integral of f, that is,*

$$\int_{\mathbb{R}^{d_2}} g(y)dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^{d_1+d_2}} f$$

Consequently, we can interchange the order of integration as follows:

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)dy \right) dx$$

Theorem 2.22 (Tonelli's Theorem, AKA Fubini's Theorem Part Two)**.** *Let f be a non-neagative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$,*

1. *The slice f^y is integrable on $\mathbb{R}^{d_1}$.*
2. *The function $g : \mathbb{R}^{d_2} \to \mathbb{R}$ define by $g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$.*
3. *Integrating g gives the integral of f, that is,*

$$\int_{\mathbb{R}^{d_2}} g(y)dy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^{d_1+d_2}} f$$

Consequently, we can interchange the order of integration as follows:

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)dy \right) dx$$

Theorem 2.23. *Let E be a measurable subset of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$, the slice*

$$E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$$

is a measurable subset of $\mathbb{R}^{d_1}$. Moreover, $m(E^y)$ is a measurable function of y and

$$m(E) = \int_{\mathbb{R}^{d_1}} m(E^y)dy$$

A symmetric result holds for x-slices of $\mathbb{R}^{d_2}$.

Theorem 2.24. *If $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}_d$, and $m_*(E_2) > 0$, then E_1 is measurable.*

Theorem 2.25. *For $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$, we have $m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2)$. (Note that for this inequality, we interpret the product of zero and infinity to be zero.)*

Theorem 2.26. *Let $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$ be measurable sets. Then $E_1 \times E_2$ is measurable in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $m(E) = m(E_1)m(E_2)$. (we interpret zero times infinity to be zero.)*

Theorem 2.27. *Let $f : \mathbb{R}^{d_1} \to [-\infty, \infty]]$ be a measurable function. Then the function $\widetilde{f} : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to [-\infty, \infty]$ defined by $\widetilde{f}(x, y) = f(x)$ is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.*

Theorem 2.28 (Area Under a Curve)**.** *Let $f : \mathbb{R}^d \to [0, \infty]$ be a non-negative measurable function. Let*

$$A = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$$

Then f is measurable on $\mathbb{R}^d$ if and only if A is measurable in $\mathbb{R}^{d+1}$, and if f is measurable, then

$$\int_{\mathbb{R}^d} f(x)dx = m(A)$$

This says that the measure of the area under an integrable function is equal to the integral of that function.

Theorem 2.29. *If f is a measurable function on $\mathbb{R}^d$, then the function $\widetilde{f}(x, y) = f(x - y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$.*

Theorem 2.30. *Let f be integrable on $\mathbb{R}$. Then $F(x) = \int_{-\infty}^{x} f(t)dt$ is uniformly continuous.*

Theorem 2.31 (Tchebychev Inequality)**.** *Let $f \geq 0$ and f be integrable. For $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, we have*

$$m(E_\alpha) \leq \frac{1}{\alpha} \int_{E_\alpha} f$$

3 Chapter 3

Theorem 3.1 (Hardy-Littlewood Maximal Function)**.** *Let $f \in L^1(\mathbb{R}^d)$. Then f^* is measurable, $f^*(x) < \infty$ for a.e. x, and for all $\alpha > 0$*

$$m(\{x \in \mathbb{R}^d : f^*(x) > \alpha\} \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)|dy$$

Compare this to the Tchebychev inequality, which says

$$m(\{x \in \mathbb{R}^d : f(x) > \alpha\} \leq \frac{1}{\alpha} \int_{\mathbb{R}^d} |f(y)|dy$$

Theorem 3.2 (Vitality Covering Lemma)**.** *Let $\{B_1, \ldots, B_N\}$ be a finite collection of open balls in $\mathbb{R}^d$. There exists a disjoint subcollection $B_{i_1}, \ldots, B_{i_k}$ such that*

$$\bigcup_{n=1}^{N} B_n \subset \bigcup_{j=1}^{k} 3B_{i_j}$$

Thus

$$m\left(\bigcup_{n=1}^{N} B_n\right) \leq 3^d \sum_{j=1}^{k} m(B_{i_j})$$

Theorem 3.3 (Lebesgue Differentiation Theorem)**.** *If $f \in L^1(\mathbb{R}^d)$, then*

$$\lim_{\substack{m(B)\to 0\\ x\in B}} \frac{1}{m(B)} \int_B f(y)dy = f(x)$$

for almost every x. In fact, the result holds if we only assume that f is locally integrable.

Theorem 3.4. *Let E be a measurable subset or $\mathbb{R}^d$ and let A be the set of Lebesgue density points of E. Then almost every $x \in E$ is in A and almost every $x \in E^c$ is in A^c. Equivalently,*

$$m(E \setminus A) = 0 \qquad m(A \setminus E) = 0 \qquad m(E) = m(A) = m(E \cap A)$$

3.1 Bounded Variation and Absolute Continuity

Theorem 3.5. *If F is real-valued, monotonic, and bounded, then F is of bounded variation.*

Theorem 3.6. *If F if differentiable everywhere and F' is bounded, then F is of bounded variation. Furthermore, F is absolutely continuous.*

Theorem 3.7. *Every* BV *function can be written as a difference of two increasing functions.*

Theorem 3.8. *Every* BV *function is differentiable almost everywhere.*

Theorem 3.9 (Rising Sun Lemma)**.** *Let G be real-valued and continuous on $\mathbb{R}$, and let*

$$E = \{x : G(x+h) > G(x) \textit{ for some } h > 0\}$$

If E is nonempty, then it is open. In this case, E can be written as a countable disjoint union of open intervals $E = \bigcup(a_k, b_k)$ such that

$$G(b_k) = G(a_k)$$

Theorem 3.10. *If F is increasing and continuous, then F' exists almost everywhere. Additionally, F' is measurable and nonnegative and*

$$\int_a^b F'(x)dx \leq F(b) - F(a)$$

Note: To get equality, we need stronger conditions on F. Specifically, we need absolute continuity.

Relevant "counter-example" to the obvious stronger version of the previous theorem: Let F be the Cantor-Lebesgue function. Then $F'(x) = 0$ a.e., so $\int_a^b F'(x)dx = 0$, but $F(1) = 1$ and $F(0) = 0$.

Theorem 3.11. *Absolutely continuous functions are uniformly continuous.*

Theorem 3.12. *Absolutely continuous functions are of bounded variation.*

Theorem 3.13. *If F is absolutely continuous on $[a, b]$, then T_F is absolutely continuous on $[a, b]$.*

Theorem 3.14. *If f is integrable and $F(x) = \int_a^x f(y)dy$, then F is absolutely continuous.*

Theorem 3.15. *If F is absolutely continuous on $[a, b]$, then $F'(x)$ exists almost everywhere. If $F'(x) = 0$ for a.e. x, then F is constant.*

Theorem 3.16. *Suppose E is a set of finite measure and $\mathcal{B}$ is a Vitali covering of E. Then for any $\delta > 0$ there is a finite, disjoint, collection of balls $B_1, \ldots, B_N$ in $\mathcal{B}$ such that*

$$\sum_{i=1}^{N} m(B_i) \geq m(E) - \delta$$

That is, we can "approximate" the E with coverings of balls whose total measure only barely exceeds that of E.

Theorem 3.17. *Suppose E is a set of finite measure and $\mathcal{B}$ is a Vitali covering of E. Then for any $\delta > 0$ there is a finite, disjoint, collection of balls $B_1, \ldots, B_N$ in $\mathcal{B}$ such that*

$$m\left(E \setminus \bigcup_{i=1}^{N} B_i\right) < 2\delta$$

Theorem 3.18. *Suppose F is absolutely continuous on $[a, b]$. Then F' exists almost everywhere and is integrable. Moreover,*

$$\int_a^x F'(y)dy = F(x) - F(a)$$

for all $a \leq x \leq b$. In particular, we can choose $x = b$ to get

$$\int_a^b F'(y)dy = F(b) - F(a)$$

Conversely, if f is integrable on $[a, b]$ then if we define $F(x) = \int_a^x f(y)dy$, then $F'(x) = f(x)$ almost everywhere.

Theorem 3.19. *A bounded increasing function on $[a, b]$ has at most countably many jump discontinuities.*

Theorem 3.20. *Let F be increasing and bounded on $[a, b]$. Then $J_F(x)$ is discontinuous exactly at the points $\{x_n\}$ and has a jump at x_n equal that of F. Furthermore, the function $F(x) - J_F(x)$ is increasing and continuous.*

Theorem 3.21. *Let F be increasing and bounded on $[a, b]$ and let $J_F(x)$ be its jump function. Then $J'(x)$ exists a.e. and $J'(x) = 0$ a.e.*

Theorem 3.22. *If $F \in \mathrm{BV}[a, b]$, then*

$$\int_a^b |F'(x)|dx \leq T_F(b)$$

Equality holds if and only if F is absolutely continuous.

Theorem 3.23. *If $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then f maps sets of measure zero to sets of measure zero, and f maps measurable sets to measurable sets.*

Theorem 3.24 (Change of Variable Formula)**.** *Let F be absolutely continuous and increasing on $[a, b]$ and set $A = F(a)$ and $B = F(b)$. Let f be a measurable function on $[A, B]$. Then $f(F(x))F'(x)$ is measurable on $[a, b]$, and if f is integrable on $[A, B]$ then*

$$\int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx$$

4 Chapter 6

4.1 Abstract Measure Spaces

Theorem 4.1. *Let m_* denote the Lebesgue outer measure. Then m_* is an outer measure.*

Theorem 4.2. *Let m_* denote the Lebesgue outer measure. Then a set $E \subset \mathbb{R}^d$ is Carathéodory measurable with respect to m_* if and only if E is Lebesgue measurable.*

Theorem 4.3. *Let X be a set and μ_* be an outer measure. Then the collection $\mathcal{M}$ of Carathéodory measurable sets forms a σ-algebra, and $\mu_*|_{\mathcal{M}}$ is a measure.*

Theorem 4.4. *If μ_* is a metric exterior measure on a metric space X, then the Borel sets in X are measurable. Therefore, $\mu_*|_{B_X}$ is a measure.*

Theorem 4.5. *Let (X, d) be a measure set and μ is a Borel measure on X such that for a ball B of finite radius, $\mu(B)$ is finite. Then μ is a regular measure.*

Theorem 4.6. *If μ_0 is a premeasure on an algebra $\mathcal{A}$, define μ_* on any subset E of X by*

$$\mu_*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(E_j) : E \subset \bigcup_{j=1}^{\infty} E_j,\ E_j \in \mathcal{A}\right\}$$

Then μ_* is an exterior measure on X that satisfies $\mu_*(E) = \mu_0(E)$ for $E \in \mathcal{A}$, and all sets in $\mathcal{A}$ are Carathéodory measurable.

Theorem 4.7. *Let $\mathcal{A}$ be an algebra of sets in X and μ_0 a premeasure on $\mathcal{A}$ and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. Then there is a measure μ on $\mathcal{M}$ that extends μ_0.*

4.2 Integration in Abstract Measure Spaces

All of the following definitions, concepts, and theorems are easily generalized from the development of Lebesgue measure and Lebesgue integration on $\mathbb{R}^d$ to a general σ-finite measure space.

1. Almost everywhere
2. Measurable functions, simple functions

3. Every non-negative measurable function can be approximated by an increasing sequence of simple functions.

4. Every measurable function can be approximated by a sequence of simple functions.

5. Egorov's Theorem

6. Integrable functions

7. Fatou's Lemma, Monotone Convergence Theorem, Dominated Convergence Theorem

8. The space $L^1(X, \mu)$ of integrable functions is a Banach space.

9. Fubini and Tonelli Theorems

Theorem 4.8. *Let F be an increasing and normalized function on $\mathbb{R}$. Then there is a unique measure μ (also denoted dF) on the Borel sets of $\mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$ for $a < b$. Conversely, if μ is a measure on the Borel sets of $\mathbb{R}$ that is finite on bounded intervals, then F defined by*

$$F(x) = \begin{cases} -\mu((-x, 0]) & x < 0 \\ 0 & x = 0 \\ \mu((0, x]) & x > 0 \end{cases}$$

is increasing and normalized.

Theorem 4.9. *Two increasing functions F and G give the same measure if $F - G$ is constant.*

Theorem 4.10. *If F is absolutely continuous on $[a, b]$, then*

$$\int_a^b f(x) dF(x) = \int_a^b f(x) F'(x) dx$$

for every Borel measurable function f that is integrable with respect to $d\mu$.

Theorem 4.11. *Let ν be a signed measure. Then the total variation of ν, denoted $|\nu|$, is a positive measure, and satisfies $\nu \leq |\nu|$.*

Theorem 4.12. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu(E) = 0$ for all E.*

Theorem 4.13. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mu)$. Then ν defined by*

$$\nu(E) = \int_E f d\mu$$

is a signed measure on X. Furthermore, $\nu \ll \mu$.

Theorem 4.14 (Radon-Nikodym Theorem)**.** *Let μ be a σ-finite positive measure on the measure space $(X, \mathcal{M})$ and let ν be a σ-finite signed measure on $\mathcal{M}$. Then there exist unique signed measure ν_a and ν_s so that $\nu_a \ll \mu$ and $\nu_s \perp \mu$ and $\nu = \nu_a + \nu_s$. In addition, the measure ν_a is of the form $d\nu_a = f d\mu$, that is,*

$$\nu_a(E) = \int_E f(x) d\mu$$

for some extended μ-integrable function f.

Theorem 4.15. *Let $C([a, b])$ denote the vector space of continuous functions on the compact interval $[a, b]$. If μ is a Borel measure on $[a, b]$ with $\mu([a, b]) < \infty$, then $\ell : C([a, b]) \to [\infty, \infty]$ given by*

$$\ell(f) = \int_a^b f(x) d\mu$$

is a linear functional. It is positive ($f \geq 0 \implies \ell(f) \geq 0$). Conversely, if ℓ is a positive linear functional on $C([a, b])$, then there is a unique Borel measure μ so that $\ell(f) = \int_a^b f d\mu$ for $f \in C([a, b])$.