Problem Set 2

D. Zack Garza

February 9, 2020

Contents

1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ can not lie in $\mathcal{O}$.

Useful facts:

- For any $\lambda \in \mathfrak{h}^\vee$, $\mathbb{C}_\lambda$ is a 1-dimensional $\mathfrak{b}$-bimodule with a trivial $\mathfrak{n}$-action.
- $M(\lambda) = \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ as a left $U(\mathfrak{g})$-module.
- $M(\lambda) = U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$-module.
- $M(\lambda)$ is generated as a $U(\mathfrak{g})$-module by the maximal vector $v^+ = 1 \otimes 1$.
- The set of weights of $M(\lambda)$ is $\lambda - \Gamma$ where Γ is the semigroup in Λ_r generated by Φ^+.
- $M(\lambda)$ has weights $\lambda, \lambda - 2, \lambda - 4, \cdots$ each with multiplicity 1.

Questions

- What is the tensor product over? Guess: $\otimes_{\mathbb{C}}$.
- MSE: the product is no longer finitely generated.
  - Consider dimensions of weight spaces – eventually constant.
  - If $\mathrm{wt} v = \lambda$ and $\mathrm{wt}(u) = \mu$, then $\mathrm{wt}(u \otimes v) = \lambda + \mu$.
  - Consider a weight space N_γ of M. This must have the form $\displaystyle\bigoplus_{a+b=\gamma} M_a \otimes_{\mathbb{C}} M_b$.
  - Example: consider $\lambda = \mu = 0$. Then $M = M(0) \otimes M(0)$ and N_{-2m} has dimension $m + 1$ for every $m \in \mathbb{Z}^+$.

Solution:

Let $M(\lambda), M(\mu)$ be arbitrary Verma modules for with highest weight vectors v, w respectively. We can then consider the weight of $v \otimes w$ in $N := M(\lambda) \otimes_{U(\mathfrak{g})} M(\mu)$:

$$
\begin{aligned}
h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\
&= \lambda(h) v \otimes w + v \otimes \mu(h) w \\
&= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\
&= (\lambda(h) + \mu(h))(v \otimes w).
\end{aligned}
$$

2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ.

Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Useful facts:

-