

Problem Set 3

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Exercise 0.1 (Gathmann 2.33).

Define

$$X := \left\{ M \in \text{Mat}(2 \times 3, k) \mid \text{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that X is an irreducible variety, and find its dimension.

Exercise 0.2 (Gathmann 2.34).

Let X be a topological space, and show

- a. If $\{U_i\} \rightrightarrows X$, then $\dim X = \sup_{i \in I} \dim U_i$.
- b. If X is an irreducible affine variety and $U \subset X$ is a nonempty subset, then $\dim X = \dim U$. Does this hold for any irreducible topological space?

Exercise 0.3 (Gathmann 2.36).

Prove the following:

- a. Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.
- b. A complex affine variety of dimension at least 1 is never compact in the classical topology.

Exercise 0.4 (Gathmann 2.40).

Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1x_4 - x_2x_3 \rangle$$

and show the following:

- a. R is an integral domain of dimension 3.
- b. x_1, \dots, x_4 are irreducible but not prime in R , and thus R is not a UFD.
- c. x_1x_4 and x_2x_3 are two decompositions of the same element in R which are nonassociate.
- d. $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

Exercise 0.5 (Problem 5).

Consider a set U in the complement of $(0, 0) \in \mathbb{A}^2$. Prove that any regular function on U extends to a regular function on all of \mathbb{A}^2 .