

Problem Set 2

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1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ can not lie in \mathcal{O} .

Useful facts:

- For any $\lambda \in \mathfrak{h}^\vee$, \mathbb{C}_λ is a 1-dimensional \mathfrak{b} -bimodule with a trivial \mathfrak{n} -action.
- $M(\lambda) = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ as a left $U(\mathfrak{g})$ -module.
- $M(\lambda) = U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$ -module.
- $M(\lambda)$ is generated as a $U(\mathfrak{g})$ -module by the maximal vector $v^+ = 1 \otimes 1$.
- The set of weights of $M(\lambda)$ is $\lambda - \Gamma$ where Γ is the semigroup in Λ_r generated by Φ^+ .
- $M(\lambda)$ has weights $\lambda, \lambda - 2, \lambda - 4, \dots$ each with multiplicity 1.

Questions

- What is the tensor product over? Guess: $\otimes_{\mathbb{C}}$.
- MSE: the product is no longer finitely generated.
 - Consider dimensions of weight spaces – eventually constant.
 - If $\text{wt}(v) = \lambda$ and $\text{wt}(u) = \mu$, then $\text{wt}(u \otimes v) = \lambda + \mu$.
 - Consider a weight space N_γ of M . This must have the form $\bigoplus_{a+b=\gamma} M_a \otimes_{\mathbb{C}} M_b$.
 - Example: consider $\lambda = \mu = 0$. Then $M = M(0) \otimes M(0)$ and N_{-2m} has dimension $m + 1$ for every $m \in \mathbb{Z}^+$.

Solution:

Let $M(\lambda), M(\mu)$ be arbitrary Verma modules with highest weight vectors $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$ respectively. We can then consider the weight of $v \otimes w$ in $N := M(\lambda) \otimes_{\mathbb{C}} M(\mu)$:

$$\begin{aligned}
h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\
&= \lambda(h)v \otimes w + v \otimes \mu(h)w \\
&= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\
&= (\lambda(h) + \mu(h))(v \otimes w).
\end{aligned}$$

Letting $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, so $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$, the claim is that it is possible for N to *not* be finitely-generated as a $U(\mathfrak{g})$ -module.

Let $\{y, h, x\}$ be the usual basis. We can use the fact that $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$, so if we pick $\mu, \lambda \in \mathbb{Z}^{\leq 0}$ we have weight space decompositions

$$\begin{aligned}
M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda-2i} := \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i := \lambda-2i}} M(\lambda)_{\lambda_i} \\
M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu-2j} := \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_j := \mu-2j}} M(\mu)_{\mu_j}
\end{aligned}$$

where we can explicitly identify bases $M(\lambda)_{\lambda_i} = \text{span}_{\mathbb{C}} \{y^i v\}$ and $M(\mu)_{\mu_i} = \text{span}_{\mathbb{C}} \{y^i w\}$.

By the initial observation, this yields a weight space decomposition for N given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left(\bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_{\nu}.$$

Since each weight space $N_{\nu} \supseteq \text{span}_{\mathbb{C}} \{y^i v \otimes y^j w \mid i + j = \nu\}$ and there are infinitely many such weight spaces, extending to a PBW basis for $U(\mathfrak{g})$ shows that no finite number of elements in $U(\mathfrak{g})$ can generate N . ■

2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h})$ be the Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ .

Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Useful facts:

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