Title

Contents

1 | Elementary Algebra

- Looking at real roots:
  - Let p be number of sign changes in $f(x)$;
  - Let q be number of sign changes in $f(-x)$;
  - Let n be the degree of f.
  - Then p gives the maximum number of positive real roots, q gives the maximum number of negative real roots, and $n-p-q$ gives the *minimum* number of complex roots.
  - Rational Roots Theorem: If $p(x) = ax^n + \cdots + c$ and $r = \dfrac{p}{q}$ where $p(r) = 0$, then $p \,\Big|\, c$ and $q \,\Big|\, a$.

2 | Abstract Algebra

- Order p: One, Z_p
- Order p^2: Two abelian groups, Z_{p^2}, Z_p^2
- Order p^3:
  - 3 abelian $Z_{p^3}, Z_p \times Z_{p^2}.Z_p^3$,
  - 2 others $Z_p \rtimes Z_{p^2}$.
    - ♢ The other is the quaternion group for p = 2 and a group of exponent p for p > 2.
- Order pq:
  - $p \,\Big|\, q-1$: Two groups, Z_{pq} and $Z_q \rtimes Z_p$
  - Else cyclic, Z_{pq}

- Every element in a permutation group is a product of disjoint cycles, and the order is the lcm of the order of the cycles.
- The product ideal IJ is *not* just elements of the form ij, it is all sums of elements of this form! The product alone isn't enough.
- The intersection of any number of ideals is also an ideal

3 | Complex Numbers

- $\lim_{z \to z_0} f(z) = x_0 + iy_0$ iff the component functions limit to x_0 and y_0 respectively. Moreover, both ways are equal!

4 | Analysis

- f injective implies f has a nonzero derivative (in neighborhoods)
- In $\mathbb{R}$, singletons are closed. This means any finite subset is closed, as a finite union of singleton sets! Good for counterexamples to continuity.
- Definition of topology: arbitrary unions and finite intersections of open sets are open. Equivalently, arbitrary intersections and finite unions of closed sets are closed.
- The best source of examples and counterexamples is the open/closed unit interval in $\mathbb{R}$. Always test against these first!
- Every Cauchy sequence converges in a complete metric space

5 | Combinatorics and Probability

- Counting non-isomorphic things: Pick a systematic way. Can descend my maximum vertex degree, or ascend by adding nodes/leaves.

6 | Linear Algebra

- An $m \times n$ matrix is a map from n-dimensional space to m-dimensional space. Number of *rows* tell you the dimension of the codomain, the number of *columns* tell you the dimension of the *domain.*
- The column space of A (i.e. linear combinations of the columns) are a basis for the *image* of A.
- The row space is a basis for the *coimage*, which is nullspace perp.
- Not enough pivots implies columns don't span the entire target domain
- The determinant of an RREF matrix is the product of the diagonals

- An $n \times n$ matrix P is diagonalizable iff its eigenspace is all of $\mathbb{R}^n$ (i.e. there are n linearly independent eigenvectors, so they span the space.) Equivalently, if there is a basis of eigenvectors for the range of P
- Projections decompose the range into the into the direct sum of its nullspace and nullspace perp.
- The space of matrices is not an integral domain!
- The transition matrix from a given basis $\mathcal{B} = b_i$ to the standard basis is given by just creating a matrix with each b_i being a column.
  - The transition matrix from the standard basis to $\mathcal{B}$ is just the inverse of the above!
- Inverting matrices quickly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{where } ad - bc \neq 0$$

The pattern?

1. Always divide by determinant
2. Swap the diagonals
3. Hadamard product with checkerboard [\begin{bmatrix}

- & - \
- & + \end{bmatrix}]

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} ei - fh & -(bi - ch) & bf - ce \\ -(di - fg) & ai - cg & -(af - cd) \\ dh - eg & -(ah - bg) & ae - bd \end{bmatrix}.$$

The pattern:

1. Divide by determinant
2. Each entry is determinant of submatrix of A with corresponding col/row deleted
3. Hadamard product with checkerboard

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

4. Transpose at the end!!

7 Calculus

- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)

- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.
- Fundamental theorem of Calculus: If $\int f(x)dx = F(b) - F(a)$, then $F'(x) = f(x)$.
- Min/maxing - either derivatives of Lagranage multipliers!
- Distance from origin to plane: equation of a plane P is given by $ax+by+cz=d$, and you can always just read off the normal vector $\mathbf{n}=(a,b,c)$. So we have $\mathbf{n}\mathbf{x}=d$. Since $t\mathbf{n}$ is normal to P for all t, solve $\mathbf{n}t\mathbf{n}=d$, which is $t = \frac{d}{\left|n\right|^2}$
- A plane can be constructed from a point p and a normal n by the equation $np = 0$.
- In a sine wave $f(x) = \sin(\omega x)$, the period is given by $2\pi/\omega$. If $\omega > 1$, then the wave makes exactly ω full oscillations in the interval $[0, 2\pi]$.
- The directional derivative is the gradient dotted against a *unit vector* in the direction of interest
- Related rates problems can often be solved via implicit differentiation of some constraint function
- The second derivative of a parametric equation is not exactly what you'd intuitively think!
- For the love of god, remember the FTC! $\frac{\partial}{\partial x}\int_0^x f(y)dy = f(x)$.
- Technique for asymptotic inequalities: WTS $f < g$, so show $f(x_0) < g(x_0)$ at a point and then show $\forall x > x_0, f'(x) < g'(x)$. Good for big-O style problems too.