Real Analysis Qual Prep Week 1: Preliminaries

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1.2 Background / Warmup / Review

- Derive the reverse triangle inequality from the triangle inequality.
- Let $E \subseteq \mathbb{R}$. Define $\sup E$ and $\inf E$.
- What is the **Archimedean** property?

1.2.1 Metric Spaces / Topology

- What does it mean for a metric space to be **complete**?
- Give two or more equivalently definitions for **compactness** in a complete metric space.
- What is an interior point? An isolated point? A limit point?
- What does it mean for a set to be open? Closed?
- What is the **closure** of a subspace $E \subseteq X$?
- What does it mean for $E \subseteq X$ to be a **dense** subspace?
- What does it mean for a family of sets to form a **basis** for a topology?
  - What is a basis for the standard topology on $\mathbb{R}^d$?
- Let X be a subset of $\mathbb{R}^d$. Prove the Heine-Borel theorem:
  - Show that X compact $\implies X$ is closed
  - Show that X compact $\implies X$ is bounded
  - Show that a closed subset of a compact set must be bounded.
  - Show that if X closed and bounded $\implies X$ is compact.
- Find an example of a metric space with a closed and bounded subspace that is not compact.
  - How can this be modified to obtain a necessary and sufficient condition?
- Determine if the following subsets of $\mathbb{R}$ are opened, closed, both, or neither:
  - $\mathbb{Q}$
  - $\mathbb{Z}$
  - $\{1\}$
  - $\left\{p \in \mathbb{Z}^{\geq 0} \mathrel{\Big|} p \text{ is prime}\right\}$
  - $\left\{\frac{1}{n} \mathrel{\Big|} n \in \mathbb{Z}^{\geq 0}\right\}$
  - $\left\{\frac{1}{n} \mathrel{\Big|} n \in \mathbb{Z}^{\geq 0}\right\} \cup \{0\}$

1.2.2 Sequences

- Can a convergent sequence of real numbers have a subsequence converging to a different limit?
- What does it mean for a sequence of functions to converge **pointwise** and to converge **uniformly**?
  - Give an example of a sequence that converges pointwise but not uniformly.
- Prove that every sequence admits a monotone subsequence.
- Prove the monotone convergence theorem for sequences.
- Prove the Bolzano-Weierstrass Theorem.

1.2.3 Series

– What does it mean for a series to converge? How can you check this? - What does it mean for a series to converge *uniformly*? What do you have to show to prove it does *not* converge uniformly? - Show that if $\sum_{n\in\mathbb{N}} a_n < \infty$ converges, then

$$a_n \overset{n\to\infty}{\longrightarrow} 0$$

. - Show that convergent sequences *have small tails* in the following sense:

$$\sum_{n>N} a_n \overset{N\to\infty}{\longrightarrow} 0$$

. - Is this a necessary and sufficient condition for convergence? - State the ratio, root, integral, and alternating series tests. - Prove that the harmonic series diverges - Derive a formula for the sum of a geometric series. - State and prove the p-test. - What does it mean for a series to converge absolutely? - Find a sequence that converges but not absolutely.

1.2.4 Continuity and Discontinuity

- What does it mean for a function to be **uniformly continuous** on a set?
- Is it possible for a function $f : \mathbb{R} \to \mathbb{R}$ to be discontinuous precisely on the rationals $\mathbb{Q}$? If so, produce such a function, if not, why?
  - Can the set of discontinuities be precisely the irrationals $\mathbb{R} \setminus \mathbb{Q}$?
- Find a sequence of continuous functions that does *not* converge uniformly, but still has a pointwise limit that is continuous.

1.3 Exercises

- Find a function that is differentiable but not continuously differentiable.
- Prove the **uniform limit theorem**: a uniform limit of continuous function is continuous.
- Show that the uniform limit of bounded functions is uniformly bounded.
- Construct sequences of functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ which converge uniformly on some set E, and yet their product sequence $\{h_n\}_{n\in\mathbb{N}}$ with $h_n := f_n g_n$ does *not* converge uniformly.
  - Show that if f_n, g_n are additionally bounded, then h_n does converge uniformly.

- Find a sequence of functions such that

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\frac{d}{dx} f_n(x)$$

- Find a uniform limit of differentiable functions that is not differentiable.

- Prove that the Cantor set is a Borel set.

- Show the Cantor ternary set is totally disconnected; that is show it contains no nonempty open interval.

- II.5 (a) Show the set of irrational numbers is a G_δ set but is not an F_σ set. **Hint:** Show $\mathbb{Q}$ is not a G_δ, for otherwise you could obtain a decreasing sequence G_n of dense open sets that have empty intersection. Then use the decomposition of each G_n into a disjoint countable union of open intervals.

  (b) Using the fact that the set of rational numbers in any closed interval $a \leq x \leq b$ where $a < b$ is not a G_δ set, give an example of a Borel subset of $\mathbb{R}$ which is neither an F_σ or a G_δ set.

  (c) Let f be any function from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set of points of discontinuity of f is of type F_σ.

  (d) Can a function from $\mathbb{R}$ to $\mathbb{R}$ be continuous on the rationals and discontinuous on the irrationals? What if the roles of the rationals and irrationals are interchanged?

- I.7 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Prove that the following are equivalent.

  (a) $\lim_{n\to\infty} x_n = a$.

  (b) Every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to a.

1.4 Qual Questions

a.

- I.8 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

is continuous at $(x = 1, y = 0)$ and $f(1, 0) = 0$. Explain carefully.

I.14 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that, given x and h, $f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi)$ for some ξ.

(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

2.4 Spring 2017 # 4 ✨

Let $f(x, y)$ on $[-1, 1]^2$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Determine if f is integrable.

2.5 Spring 2015 # 1 ✨

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $x_0 \in X$.

Prove that the following statements are equivalent:

1. For every $\varepsilon > 0 \quad \exists \delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta$.
2. The sequence $\{f(x_n)\}_{n=1}^{\infty} \to f(x_0)$ for every sequence $\{x_n\} \to x_0$ in X.

2.1 Fall 2018 # 1 ✨

Let $f(x) = \dfrac{1}{x}$. Show that f is uniformly continuous on $(1, \infty)$ but not on $(0, \infty)$.

> Let
>
> $$f_n(x) = \begin{cases} \frac{1}{n} & x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] \\ 0 & \text{otherwise.} \end{cases}$$
>
> Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-test but that it nevertheless converges uniformly on $\mathbb{R}$.

4. Let $f_n\colon [0,1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1+x^k}.$$

1. Prove that f_n converges to a function $f\colon [0,1) \to \mathbb{R}$.

2. Prove that for every $0 < a < 1$ the convergence is uniform on $[0,a]$.

3. Prove that f is differentiable on $(0,1)$.

- 3. (a) Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of all the rationals in $[0,1]$ and define $f : [0,1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}.$$

Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0,1]$ and conclude that set of all points at which f is discontinuous is precisely $[0,1] \cap \mathbb{Q}$.

- 6. Let

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2 x}.$$

(a) Show that the series defining g does not converge uniformly on $(0,\infty)$, but none the less still defines a continuous function on $(0,\infty)$.
Hint for the first part: Show that if $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on a set X, then the sequence of functions $\{g_n\}$ must converge uniformly to 0 on X.

(b) Is g differentiable on $(0,\infty)$? If so, is the derivative function g' continuous on $(0,\infty)$?

- 7. Let $h_n(x) = \dfrac{x}{(1+x)^{n+1}}$.

(a) Prove that h_n converges uniformly to 0 on $[0,\infty)$.

(b) i. Verify that

$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$

ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0,\infty)$?

(c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a,\infty)$ for any $a > 0$.

-

exists.

I.19 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0 & \text{, if } x \leq 0 \end{cases}$$

(a) Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1, 2, 3, \cdots$. Show details.

(b) Does f have a power series expansion at 0?

(c) Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0, 1)$.

- interval $[a, b]$ is absolutely continuous.

IV.9 A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in I is called a *Lipschitz function*.

(a) Show that a Lipschitz function is absolutely continuous.

(b) Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is essentially bounded.

- If f is nonnegative and integrable on $[0, 1]$, then $\lim_{n\to\infty} \int_0^1 \sqrt[n]{f} = m\{x | f(x) > 0\}$

- My Solution:

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots)$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
(b) Construct a sequence $\{s_n\}$ which does not converges, although $\lim \sigma_n = 0$.
(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]
(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

- If $m < n$, then

  – Note: outline omitted!

3.1 Spring 2020 # 1 ✨

Prove that if $f : [0,1] \to \mathbb{R}$ is continuous then

$$\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x)\, dx = f(1).$$

3.4 Fall 2017 # 4 ✨

Let

$$f_n(x) = nx(1-x)^n, \quad n \in \mathbb{N}.$$

a. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$.

b. Show that

$$\lim_{n \to \infty} \int_0^1 n(1-x)^n \sin x\, dx = 0$$

Hint for (a): Consider the maximum of f_n.

3.11 Fall 2020 # 1

Show that if x_n is a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, then

$$\lim_{n \to \infty} nx_n = 0.$$