

# Title

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## 1 Friday February 21st

Question: how do we define  $h_{V,D}$ ?

Answer: write  $D = D_1 - D_2$  which are (very) ample divisors and basepoint free. We then obtain embeddings

$$\begin{aligned}\varphi_1 : V &\hookrightarrow \mathbb{P}_K^{n_1} \\ \varphi_2 : V &\hookrightarrow \mathbb{P}_K^{n_2}.\end{aligned}$$

So write

$$h_{V,D}(p) = h(\varphi_1(p)) - h(\varphi_2(p)) + O(1)$$

### Example 1.1.

For  $E/K$  an elliptic curve,

- $2[0]$  is an ample divisor
- $3[0]$  is a very ample divisor.

Let  $K$  be a local field (i.e.  $\mathbb{C}, \mathbb{R}$ , a  $p$ -adic field, or  $\mathbb{F}_q((t))$  formal Laurent series) and  $A/K$  be an abelian variety; we want to understand  $A(K)$ . We know this has the structure of compact abelian  $K$ -analytic Lie group.

- Question 1: What does Lie theory say?
- Question 2: What extra information comes from  $A/K$  being a  $g$ -dimensional abelian variety?

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If  $K = \mathbb{C}$ , then  $A(K) \cong (\mathbb{R}/\mathbb{Z})^{2g}$ . If  $K = \mathbb{R}$ , then  $A(K) \cong (\mathbb{R}/\mathbb{Z})^g \oplus \prod_{i=1}^d \mathbb{Z}/2\mathbb{Z}$  where  $0 \leq d \leq g$ .

Fix  $d$ , then

- Let  $E_1/\mathbb{R}$  with  $\Delta > 0$  (and thus 3 real roots), then  $E_1(\mathbb{R})[2] = (\mathbb{Z}/2\mathbb{Z})^2$ .
- Let  $E_2/\mathbb{R}$  with  $\Delta < 0$  (and 1 real root), then  $E_2(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z}$ .

By taking products of  $E_1$  and  $E_2$ , i.e.  $A = (E_1)^d \times (E_2)^{g-d}$ .

Todo: find reference in Silverman?

**Fact**  $A(K)$  is totally disconnected and homeomorphic to a Cantor set.

**Fact (From Lie Theory, Serre p.116)** If  $G$  is an abelian compact  $K$ -analytic Lie group, then there exists a filtration by open finite index subgroups

$$G = G^0 \supset G^1 \supset \dots \supset G^m \supset \dots$$

such that

1. The successive quotients are finite, and each  $G^i$  is *standard*, i.e. obtained by evaluating a formal group law on  $(\mathfrak{m}^i)^g$ .
2.  $\bigcap_i G^i = (0)$ .
3.  $G^i/G^{i+1}$  has exponent  $p$ , i.e. it is a finite dimensional  $\mathbb{Z}/p\mathbb{Z}$ -vector space.
4.  $G'[\text{tors}] = G'[p^\infty]$ , all of the prime-to- $p$  torsion is  $p$ -primary.

Todo: define  $p$ -primary torsion,  $\mathbb{Q}_p$ .

What structure theorem does this give?

**Theorem 1.1 (C-Lacy).**

Let  $G$  be a compact, second countable,  $K$ -analytic abelian Lie group of dimension  $g \geq 1$ . Then

- a. If  $\text{char } K = 0$  and  $d = [K : \mathbb{Q}_p]$ , then

$$G \cong_{\text{TopGrp}} \mathbb{Z}_p^{dg} \oplus G[\text{tors}]$$

where  $G[\text{tors}]$  is finite.

- b. If  $\text{char } K = p$ , i.e.  $K = \mathbb{F}_q((t))$ , and if it is true that  $G[\text{tors}]$  is finite iff  $G[p]$  finite, then

$$G \cong_{\text{TopGrp}} \prod_{i=1}^{\infty} \mathbb{Z}_p \oplus G[\text{tors}]$$

For any  $g \geq 1$ ,  $(T, +)$  a finite discrete abelian group  $(R, +) \cong (\mathbb{Z}_p^d, +)$  and  $R^g \oplus T$  is a compact abelian  $K$ -analytic Lie group isomorphic to  $\mathbb{Z}_p^{dg} \oplus T$  (?).

Question: what does this mean for  $G = S^1$ ? Ask Pete!

**Theorem 1.2 (Cartan).**

Let  $K$  be a local field,  $\mathbb{Q} \hookrightarrow K$  dense (so  $K = \mathbb{R}, \mathbb{Q}_p$ ). Then if  $G_1, G_2$  are  $K$ -analytic, and  $\varphi \in \text{hom}_{\text{TopGrp}}(G_1, G_2)$ , then  $\varphi \in \text{hom}_{k\text{-analytic}}(G_1, G_2)$ .

**Example 1.2.**

For  $R = \mathbb{F}_q[[t]]$ ,  $(R, +)^g[p] = (R, +)^g$ .

**Example 1.3.**

Take  $G = \mathbb{G}_a^g(K)$  the additive group or  $A/K$  a  $g$ -dimensional abelian variety (i.e.  $G = A(K)$ ) then  $G[p] \subsetneq (\mathbb{Z}/p\mathbb{Z})^{2g}$  and is finite.

**1.1 Proof of Cartan's Theorem****1.1.1 Step 1**

We want to show that  $G[p] < \infty$ , then  $G[\text{tors}] < \infty$ . We'll use the filtration in Serre's result; then for  $i \gg 1$ , we'll have  $G^i[p] = 0$ . Thus for  $i \gg 1$ , we'll have  $G^i[p^\infty] = 0$ ; but this is the only torsion that can appear. We'll then obtain an injection  $G[\text{tors}] \hookrightarrow G/G^i < \infty$ .

**Lemma 1.3.**

Let  $H$  be an abelian torsionfree pro- $p$  group (e.g.  $\prod \mathbb{Z}_p$ ). Then  $H \cong \prod_{i \in I} \mathbb{Z}_p$ , and if  $H$  is second-countable, then  $I$  is countable.

*Proof .*

Omitted. Idea: use Pontryagin duality  $G^\vee := \text{hom}_{\text{Top}}(G, \mathbb{R}/\mathbb{Z})$  for locally compact abelian groups.

Use the fact that this is naturally an exact contravariant functor, this lets you trade in profinite groups for discrete torsion abelian groups. ■

Note: look up pro- $p$  groups. Is the Pontryagin dual a cohomotopy group?