

# Math 200A Homework Question Compendium

D. Zack Garza

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## 1 One

1. Given:  $\forall x \in G, x^2 = e$   
Show:  $G \in \mathbf{Ab}$
2. Given:  $|G| < \infty, |G| \equiv 0 \pmod{2}$   
Show:  $\exists g \in G \ni o(g) = 2$
3. Given:  $G \in \mathbf{Ab}$   
Show:  $T(G) \leq G$  (where  $T(G) = \{g \in G : |g| < \infty\}$ )
4. Show: Every finite group is finitely generated.
  - Show:  $\mathbb{Z}$  is finitely generated
  - Show:  $H \leq (\mathbb{Q}, +) \implies H$  is cyclic
  - Show:  $\mathbb{Q}$  is not finitely generated
5. Show:  $\mathbb{Q}/\mathbb{Z}$  has, for each coset, exactly one representative in  $[0, 1) \cap \mathbb{Q}$ 
  - Show: Every element of  $\mathbb{Q}/\mathbb{Z}$  has finite order.

- Show: There are elements in  $\mathbb{Q}/\mathbb{Z}$  of arbitrarily large order.
  - Show:  $\mathbb{Q}/\mathbb{Z} = T(\mathbb{R}/\mathbb{Z})$
  - Show:  $\mathbb{Q}/\mathbb{Z} \cong \mathbb{C}^x$
6. Given:  $G/Z(G)$  is cyclic  
Show:  $G$  is abelian
7. Given:  $H \trianglelefteq G, K \trianglelefteq G, H \cap K = e$   
Show:  $\forall h \in H, \forall k \in K, hk = kh$
8. Given:  $|G| < \infty, H \leq G, N \trianglelefteq G, (|H|, [G : N]) = 1$   
Show:  $H \leq N$
9. Given:  $|G| < \infty, N \trianglelefteq G, (|N|, [G : N]) = 1$   
Show:  $N$  is the unique subgroup of order  $|N|$

## 2 Two

1. Given: For every triplet in  $G$ , two elements commute  
Show:  $G$  is abelian
2. Given:  $H_1, H_2, H_3 \leq G, G = H_1 \cup H_2$   
Show:  $G = H_1 \vee G = H_2$
3. Given:  $G = H_1 \cup H_2 \cup H_3, G$  finite  
Show:  $G = H_i \vee \forall i, [G : H_i] = 2$
4. Show: TFAE;  $\text{clos}(H)$  satisfies:
- It is the smallest normal subgroup of  $G$  containing  $H$ .
  - It is the subgroup generated by all conjugates of  $H$ .
  - $\text{clos}(H) = \bigcap_{H \leq N \trianglelefteq G} N$
  - If  $\phi : G \rightarrow X$  and  $\phi(H) = e$ , then  $\phi$  factors through  $G/\text{clos}(H)$
5. Given:  $H, K \trianglelefteq HK \leq G$   
Show:

$$\frac{HK}{H \cap K} \cong \frac{HK}{H} \times \frac{HK}{K}$$

6. Given:  $H \leq G, N \trianglelefteq G, H \in \text{Hall}(G)$   
Show:

$$H \cap N \in \text{Hall}(N) \text{ and } \frac{HN}{N} \in \text{Hall}\left(\frac{G}{N}\right)$$

7. Given:  $|G| = n, G$  cyclic,  $\sigma_i : G \rightarrow G \ni x \mapsto x^i$

- Show  $\sigma_i \in \text{End}(G)$
- Show  $\sigma_i \in \text{Aut}(G)$  iff  $(i, n) = 1$
- $\sigma_i = \sigma_j$  iff  $i = j \pmod n$
- $\tau \in \text{Aut}(G) \implies \exists i \ni \tau = \sigma_i$
- $\sigma_i \circ \sigma_j = \sigma_{ij}$
- The map

$$\begin{aligned} \psi : \mathbb{Z}_n^\times &\rightarrow \text{Aut}(G) \\ i &\mapsto \sigma_i \end{aligned}$$

is an isomorphism.

8. Given:  $G$  is cyclic

Show:  $\text{Aut}(G)$  is abelian of order  $\phi(n)$

9. Show:  $D_\infty \cong \langle a, b \mid b^2 = e, ba = a^{-1}b \rangle$
10. Show:  $Q_8 \cong \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1} \rangle$
11. Show:  $\langle x, y \mid xy^2 = y^3, yx^2 = x^3y \rangle = \langle e \rangle$

### 3 Three

1. Given:  $G \curvearrowright X$  transitively,  $H \trianglelefteq G$

- Show:  $H \curvearrowright X$ , but possibly not transitively
- Show:  $G$  acts transitively on  $\{\mathcal{O}_h : h \in H\}$
- Show:  $\forall i, j, |\mathcal{O}_{h_i}| = |\mathcal{O}_{h_j}|$
- Given:  $x \in \mathcal{O}_h$  Show:  $|\mathcal{O}_h| = |H : H \cap G_x|$
- Show:  $|\{\mathcal{O}_h\}_{h \in H}| = [G : HG_x]$

2. Given:  $\mathcal{K}$  a conjugacy class in  $S_n$ ,  $\{\mathcal{O}_s : s \in S_n\}$  orbits of an  $A_n$ -action on  $S_n$

Show:  $\mathcal{K} = \mathcal{O}_s \vee \mathcal{K} = \mathcal{O}_{s_i} \cup \mathcal{O}_{s_j}$

Show: Case 2 occurs iff  $\{k_i\}$ , the cycle lengths in disjoint cycle form, are odd and distinct

3. i:  $|G| < \infty, H < G$

- Show:  $\{gHg^{-1} : g \in G\} = [G : N_G(H)]$
- Show:

$$G \neq \bigcup_{g \in G} gHg^{-1}$$

4. Prove Cauchy's Theorem. Given  $p \mid o(G) < \infty$

$$X = \left\{ (a_i)_{i=1}^p \in G^p \ni \prod_{i=1}^p a_i = e \right\}$$

- Show:  $(a_1 a_2 \cdots a_p) = e \implies (a_2 a_3 \cdots a_p a_1) = e$
  - Show:  $(\mathbb{Z}_p, +) \curvearrowright X$  and  $\bar{1} \curvearrowright (a_1 a_2 \cdots a_p) = (a_2 a_3 \cdots a_p a_1)$
  - Show:  $|X| = |G|^{p-1}$
  - Show:  $\{\mathcal{O}_x : |\mathcal{O}_x| = 1\} > 1$  and  $\exists a \in G \ni a^p = e$
5. Given:  $G \curvearrowright X$ ,  $|G| < \infty$ ,  $1 < |X| < \infty$
- Show:  $\exists g \in G$  such that  $\forall x \in X, g \curvearrowright x \neq x$
  - Show: This holds if  $|G| = \infty$ , but not if  $|X| = \infty$  as well.
6. Given:  $H \leq G$ . Show:  $\text{core}(H)$  is
- The largest  $N \trianglelefteq G, N \subseteq H$
  - Generated by all normal subgroups contained in  $H$
  - Given by  $\bigcap_{g \in G} gHg^{-1}$
  - The kernel of  $G \curvearrowright \frac{G}{H} \ni x \curvearrowright gH = (xg)H$
7. Given:  $[H : G] = n < \infty$
- Show:  $[\text{core}(H) : G]$  divides  $n!$
  - Show:  $G$  simple  $\implies o(G) \mid n!$  and  $|G| < \infty$
8. Given:  $A_n$  is simple for  $n \geq 5$
- Show:  $\nexists H \in A_n \ni [H : A_n] < n$
- Show:  $\exists H [H : A_n] = n$
9. Given:  $r$  beads of  $n$  colors
- Show: How many distinct circular bracelets can be made.

## 4 Four

1. Given:  $H \text{ char } G$   
Show:  $H \trianglelefteq G$
2. Given:  $H \text{ char } K \trianglelefteq G$   
Show:  $H \trianglelefteq G$
3. Given:  $K = \langle k \rangle \trianglelefteq G$   
Show:  $H \leq K \implies H \trianglelefteq G$
4. Show  $H \trianglelefteq K \trianglelefteq G \not\implies H \trianglelefteq G$
5. Given:  $P \leq H \leq K \leq G < \infty, P \in \text{Syl}_p(G)$   
Show:  $P, H \trianglelefteq K \implies P \trianglelefteq K$
6. Show:  $N_G(N_G(P)) = N_G(P)$
7. Given:  $\sigma \in \text{Aut}(G)$   
Show:  $\sigma \text{Inn}(G) \sigma^{-1} = \text{Inn}(G)$  iff  $\forall g \in G, g^{-1} \sigma(g) \in \mathbb{Z}(G)$

8. Show:  $\text{Inn}(G) \text{ char } \text{Aut}(G)$
9. Given:  $H \subseteq G, P \in \text{Syl}_p(G)$   
 Show:  $\exists g \in G \ni gPg^{-1} \in \text{Syl}_p(H)$   
 Given:  $H \trianglelefteq G$   
 Show:  $P \cap H \in \text{Syl}_p(H)$   
 Given:  $P \trianglelefteq G$   
 Show:  $P \cap H \in \text{Syl}_p(H)$  and  $|\text{Syl}_p(H)| = 1$
10. Given:  $|G| = pqr, p < q < r$   
 Show:  $\exists P_i \in \text{Syl}_i(G) \trianglelefteq G$
11. Given:  $|G| = 595$   
 Show: All sylow subgroups are normal
12. Given:  $|G| = p(p+1)$   
 Show:  $\exists N \trianglelefteq G$  where  $|N| = p$  or  $p+1$

## 5 Five

1. Given:  $G = H \rtimes_{\psi} K$

$$\begin{aligned}\psi : K &\rightarrow \text{Aut}(H) \\ k &\mapsto \psi(k)\end{aligned}$$

$$\theta \in \text{Aut}(H) \quad \rho : K \rightarrow K$$

$$\begin{aligned}\phi_{\theta} : \text{Aut}(H) &\rightarrow \text{Aut}(H) \\ \rho &\mapsto \theta \circ \rho \circ \theta^{-1}\end{aligned}$$

$$\begin{aligned}\psi_2 : K &\rightarrow \text{Aut}(H) \\ k &\mapsto (\phi_{\theta} \circ \psi)(k)\end{aligned}$$

$$\begin{aligned}\psi_3 : K &\rightarrow \text{Aut}(H) \\ k &\mapsto (\psi \circ \rho)(k)\end{aligned}$$

$$\text{Show: } H \rtimes_{\psi} K \cong H \rtimes_{\psi_2} K \cong H \rtimes_{\psi_3} K$$

2. Classify groups of order  $pq, p < q, p \mid q-1$
3. Classify groups of order 20.
4. Classify groups of order 75.
5. Show:  $|G| < 60 \implies G$  is not simple.

6. Show:  $|G| < 60 \implies G$  is solvable
7. Given:  $|G| < \infty$ ,  $H \leq G$  maximal  $\implies [G : H] = p$ , a prime.
  - Show:  $|G|$  is solvable
    - Given:  $P \in \text{Syl}_p(G)$  and  $\exists H \ni N_G(P) \leq H \leq G$  Show:  $[G : H] = 1 \pmod p$
    - Given:  $p \mid o(G)$ , the largest such prime Show:  $\exists P \trianglelefteq G \in \text{Syl}_p(G)$ ,
8.  $|G| < \infty$ 
  - Given:  $G$  is characteristically simple Show:  $\exists H$  (simple)  $\ni G \cong H^n$ . Show: Whether or not the converse holds
  - Given:  $N \trianglelefteq G$  minimal Show:  $N$  is characteristically simple,  $N \cong H^n$

## 6 Six

1. Given:  $G$  is nilpotent
 

Show:  $H \leq G \implies H, G/H$  are nilpotent
2. Show:  $G/Z(G)$  is nilpotent  $\implies G$  is nilpotent
3. Given:  $|G| < \infty$ 

Show:  $|G|$  is nilpotent iff  $a, b \in G, (a, b) = 1 \implies ab = ba$
4. Show:  $D_{2n}$  is nilpotent iff  $n = 2^i$
5. Given:  $|G| < \infty$ 
  - Show  $\Phi(G)$  char  $G$
  - Show  $\Phi(G)$  is nilpotent
  - Given:  $|P| = p^e$ 
    - Show:  $P/\Phi(P)$  is an elementary abelian  $p$ -group
    - Show:  $N \trianglelefteq P, P/N$  is elementary abelian  $\implies \Phi(P) \subseteq N$
6. Given:  $R$  a commutative ring,  $x, y \in R$  nilpotent
  - Show:  $x + y$  is nilpotent
  - Show:  $\{x \in R : x \text{ is nilpotent}\} \trianglelefteq R$ 
    - Given:  $u \in R^\times, x \in R$  nilpotent
    - Show:  $u + x \in R^\times$
    - Show: An counterexample to 1 when  $R$  is noncommutative.
7. Given:  $R$  a commutative ring,  $R[[x]]$  its formal power series
  - Show:  $\sum_{i=0}^{\infty} a_i x^i \in R[[x]]^\times \iff a_0 \in R^\times$
  - Show:  $R$  a domain  $\implies R[[x]]$  a domain
  - Given:  $R$  a field
    - Show:  $I = \{r \in R[[x]] : r_0 = 0\}$  is a maximal ideal of  $R[[x]]$
    - Show:  $I$  is the unique maximal ideal
8. Given:  $R$  a commutative ring,  $G$  a finite group,  $RG$  a group ring.
  - Given:  $\mathcal{K} = \{k_1, k_2, \dots, k_m\}$  a conjugacy class in  $G$

– Show:

$$K = \sum_{i=1}^m k_i \in RG \implies K \in Z(RG)$$

- Given:  $\mathcal{K}_1 \cdots \mathcal{K}_r$  distinct conjugacy classes in  $G$ ,  $K_i = \sum_j k_j \ni k_j \in \mathcal{K}_i$ 
  - Show:  $Z(RG) = \{\sum a_l K_l : \forall 1 \leq l \leq r, a_l \in R\}$  (All  $R$ -linear combinations of the  $\mathcal{K}_i$ )

9. Given:  $R$  a ring,  $M_n(R)$  its matrix ring

- Given:  $I \trianglelefteq R$  (two-sided)
  - Show:  $M_n(I) \trianglelefteq M_n(R)$
  - Show:

$$\frac{M_n(R)}{M_n(I)} \cong M_n\left(\frac{R}{I}\right)$$

- Show:  $\forall I_M \trianglelefteq M_n(R), I$  is of the form  $M_n(I)$  for some  $I \trianglelefteq R$ 
  - Show:  $R$  a division ring  $\implies M_n(R)$  is a simple ring.

## 7 Seven

## 8 Eight