Title

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1 Week 1

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1.1 Exercise 1.3H: Right Exactness of Tensoring

Show that the following endofunctor

$$\begin{aligned} F : R\text{-mod} &\longrightarrow R\text{-mod} \\ X &\mapsto X \otimes_R N \\ (X \xrightarrow{f} Y) &\mapsto (X \otimes_R N \xrightarrow{f \otimes \mathrm{id}_N} Y \otimes_R N) \end{aligned}$$

is exact.

Solution:

> Note: to make sense of the functor, we may need to show that there is an isomorphism
>
> $$\hom_{R\text{-mod}}(X, Y) \otimes_R \hom_{R\text{-mod}}(A, B) \longrightarrow \hom_{R\text{-mod}}(X \otimes_R A, Y \otimes_R B).$$
>
> This is what makes taking $f : X \longrightarrow Y$ and $g : A \longrightarrow B$ and forming $f \otimes g : X \otimes A \longrightarrow Y \otimes B$ well-defined?

Let $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence, so

- $\operatorname{im} f = \ker g$ by exactness at B
- $\operatorname{im} g = C$ by exactness at C.

Applying the above F yields

$$A \otimes_R N \xrightarrow{f \otimes \mathrm{id}_N} B \otimes_R N \xrightarrow{g \otimes \mathrm{id}_N} C \otimes_R N \longrightarrow 0.$$

We thus need to show

1. Exactness as $C \otimes_R N$: im $(g \otimes \mathrm{id}_N) = C \otimes_R N$, i.e. this is surjective.
2. Exactness at $B \otimes_R N$: im $(f \otimes \mathrm{id}_N) = \ker(g \otimes id_N)$.

We'll use the fact that every element in a tensor product is a finite sum of elementary tensors.

- Claim: im $(g \otimes \mathrm{id}_N) \subseteq C \otimes_R N$.
  - Let $b \otimes n \in B \otimes_R N$ be an elementary tensor
  - Then $(g \otimes \mathrm{id}_N)(b \otimes n) \coloneqq g(b) \otimes \mathrm{id}_N(n) = g(b) \otimes n$
  - Since im $(g) = C$, there exists a $c \in C$ such that $g(b) = c$, so $g(b) \otimes n = c \otimes n \in C \otimes_R N$
  - Extend by linearity:

$$(g \otimes_R \mathrm{id}_N)\left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i\right) = \sum_{i=1}^m (g \otimes \mathrm{id}_N)(r_i \cdot b_i \otimes n_i) \coloneqq \sum_{i=1}^m g(r_i \cdot b_i) \otimes \mathrm{id}_N(n_i) =_H \sum_{i=1}^m r_i \cdot c_i \otimes n_i \in C \otimes_R N$$

where we've used bilinearity for the first equality, and the equality marked with H uses above the proof for elementary tensors, and noted that we can pull ring scalars $r_i \in R$ through R-mod morphisms. - Claim: $C \otimes_R N \subseteq$ im $(g \otimes \mathrm{id}_N)$. - Let $c \otimes n \in C \otimes_R N$ be an elementary tensor. - Then $c \in C =$ im (g) implies $c = g(b)$ for some $b \in B$. - So $c \otimes n = g(b) \otimes n = (g \otimes \mathrm{id}_N)(b \otimes n) \in B \otimes_R N$. - Extend by linearity:

$$\sum_{i=1}^m r_i \cdot c_i \otimes n_i =_H \sum_{i=1}^m g(r_i \cdot b_i) \otimes n_i = \sum_{i=1}^m (g \otimes \mathrm{id}_N)(r_i \cdot b_i \otimes n_i) = (g \otimes \mathrm{id}_N)\left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i\right).$$

This proves (1).

- Claim: im $(f \otimes \mathrm{id}_N) \subseteq \ker(g \otimes \mathrm{id}_N)$.
  - Let $b \otimes n \in$ im $(f \otimes \mathrm{id}_N)$, we want to show $(g \otimes \mathrm{id}_N)(b \otimes n) = 0 \in C \otimes_R N$.
  - Then $b \otimes n = f(a) \otimes n$ for some $a \in A$.
  - By exactness of the original sequence, im $f \subseteq \ker g$, so $g(f(a)) = 0 \in C$
  - Then

    $$(g \otimes \mathrm{id}_N)(b \otimes n) = (g \otimes \mathrm{id}_N)(f(a) \otimes n) \coloneqq g(f(a)) \otimes n = 0 \otimes n = 0 \in C \otimes_R N$$

    where we've used the fact that $0 \otimes x = 0$ in any tensor product.
  - Extend by linearity.

- Claim (**nontrivial part**): $\ker(g \otimes \mathrm{id}_N) \subseteq$ im $(f \otimes \mathrm{id}_N)$.

  Note: the problem is that

  $$x \in \ker(g \otimes \mathrm{id}_N) \implies x = \sum_{i=1}^m r_i \cdot b_i \otimes n_i \implies (g \otimes \mathrm{id}_N)\left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i\right) = \sum_{i=1}^m r_i \cdot g(b_i) \otimes n_i = 0 \in C \otimes_R N$$

  but this does not imply that $g(b_i) = 0 \in C$ for all i, which is what you would need to use im $f = \ker g$ to write $g(b_i) = 0 \implies \exists a_i, f(a_i) = b_i$ and pull everything back to $A \otimes_R N$.

- Strategy: use the first claim and the first isomorphism theorem to obtain this situation:

$$\begin{tikzcd} \dfrac{B \otimes_R N}{\mathrm{im}\,(f \otimes_R \mathrm{id}_N)} \arrow[r, hook, "i"] \arrow[rrr, dotted, bend left, "\alpha"] & \dfrac{B \otimes_R N}{\ker(g \otimes_R \mathrm{id}_N)} \arrow[r, "\cong"] & \mathrm{im}\,(g \otimes_R \mathrm{id}_N) \arrow[r, equal] & C \otimes_R N \end{tikzcd}$$

- The first injection i will exist because $\mathrm{im}\,(g \otimes_R \mathrm{id}_N) \subseteq \ker(g \otimes_R \mathrm{id}_N)$ by the first claim.
- The middle isomorphism is the first isomorphism theorem.
- The RHS equality follows from surjectivity of $g \otimes_R \mathrm{id}_N$
- We then apply a strengthened version of the 1st isomorphism theorem for modules:

> Hungerford Ch.4 Thm 1.7: If $f : A \longrightarrow B$ is a R-module morphism and $C \leq \ker f$ then there is a unique map $\tilde{f} : A/C \longrightarrow B$ which is an isomorphism iff f is an epimorphism and $C = \ker f$.
> Following Hungerford Ch.4 Prop. 5.4, p.210.

- Since $\mathrm{im}\,(g \otimes_R \mathrm{id}_N) \subseteq \ker(g \otimes_R \mathrm{id}_N)$, by the theorem the map α exists and satisfies the same formula, i.e. $\alpha = \tilde{g} \otimes \tilde{\mathrm{id}}_N$ where the tilde denotes the induced map on quotients, so $\alpha([b \otimes n]) = g(b) \otimes n$.
  * We will show it is an isomorphism, which forces $\mathrm{im}\,(g \otimes_R \mathrm{id}_N) \cong \ker(g \otimes_R \mathrm{id}_N)$ by the above theorem.
- Constructing the inverse map: define

$$\tilde{\alpha}^{-1} : C \times N \longrightarrow \frac{B \otimes_R N}{\mathrm{im}\,(g \otimes_R \mathrm{id}_N)}$$
$$(c, n) \mapsto (b \otimes n) \mod \mathrm{im}\,(g \otimes_R \mathrm{id}_N) \quad \text{where} \quad b \in g^{-1}(c),$$

  which we will show well-defined (i.e. independent of choice of b) and R-linear, lifting to a map α^{-1} out of the tensor product by the universal property which is a two-sided inverse for α.
- Well-defined:
  * $g^{-1}(b)$ exists because g is surjective.
  * If $b \neq b'$ and $g(b') = 0$, then $0 = g(b) - g(b') = g(b - b')$ so $b - b' \in \ker g$.
  * By the original exactness, $b - b' \in \mathrm{im}\ f$ so $b - b' = f(a)$ for some $a \in A$.
  * Then $f(a) \otimes n \in \mathrm{im}\,(f \otimes \mathrm{id})$ implies $f(a) \otimes n \equiv 0 \mod \mathrm{im}\,(f \otimes \mathrm{id})$.
  * Then noting that $b - b' = f(a) \implies b = f(a) + b'$, working mod $\mathrm{im}\,(g \otimes_R \mathrm{id}_N)$ we have

$$b \otimes n \equiv (f(a) + b') \otimes n \equiv (f(a) \otimes n) + (b' \otimes n) \equiv b' \otimes n.$$

- R-linear:
  * ?
- Two-sided identity:
  * $(\alpha \circ \alpha^{-1})(c \otimes n) = \alpha(b \otimes n) = g(b) \otimes n = c \otimes n$, so $\alpha \circ \alpha^{-1} = \mathrm{id}$.
  * $(\alpha^{-1} \circ \alpha)([b \otimes n]) = \alpha^{-1}(g(b) \otimes n) = [b' \otimes n]$ where $b' \in g^{-1}(g(b))$ implies $b' = b$, so $\alpha \circ \alpha^{-1} = \mathrm{id}$.

$\blacksquare$

2 More Exercises

2.1 1.3.K

Note: I think this is an exercise about base change.

Part a: For M an A-module and $\varphi : A \longrightarrow B$ a morphism of rings, give $B \otimes_A M$ the structure of a B-module and show that it describes a functor B-Mod $\longrightarrow$ A-Mod.

Solution

- $B \otimes_A M$ makes sense: B is a (B, A)-bimodule with the usual multiplication on the left and the right action

$$\begin{aligned} A &\longrightarrow \mathrm{End}(B) \\ a &\mapsto (b \mapsto b \cdot \varphi(a)). \end{aligned}$$

- $B \otimes_A M$ is a left B-module via the following action:

$$\begin{aligned} B &\longrightarrow \mathrm{End}(B \otimes_A M) \\ b_0 &\mapsto (b \otimes m \mapsto b_0 b \otimes m). \end{aligned}$$

- This describes a functor:

$$\begin{aligned} F : A\text{-Mod} &\longrightarrow B\text{-Mod} \\ X &\mapsto B \otimes_A X \\ (X \xrightarrow{f} Y) &\mapsto (B \otimes_A X \xrightarrow{\mathrm{id}_B \otimes f} B \otimes_A Y). \end{aligned}$$

  - Need to check:
    - Preserves identity morphism, i.e. $X \in A$-Mod implies $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ in B-Mod.
    - Preserves composition: $F(f \circ g) = F(f) \circ F(g)$.

- Preserving identity morphisms:
  - By construction $X \circlearrowleft_{\mathrm{id}_X}$ maps to $B \otimes_A X \xrightarrow{\mathrm{id}_B \otimes \mathrm{id}_X} B \otimes_A X$, can argue that this is the identity map for B-modules.

- Preserving composition:

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (B \otimes_A X \xrightarrow{\mathrm{id}_B \otimes f} B \otimes_A Y \xrightarrow{\mathrm{id}_B \otimes g} B \otimes_A Z) = (B \otimes_A X \xrightarrow{\mathrm{id}_B \otimes (g \circ f)} B \otimes_A Z).$$

Note: not sure if there's anything to show here.

Part b: If $\psi : A \longrightarrow C$ is another ring morphism, show that $B \otimes_A C$ has a ring structure.

Solution:

- Note $B \otimes_A C$ makes sense, since C is a left A-module via $a \mapsto (c \mapsto \psi(a)c)$.

- Need to define $(B \otimes_A C, P, M)$ such that it's an abelian group under P (plus), a monoid under M (multiplication), and left/right distributivity.
- Start by defining on cartesian products:

$$\begin{gathered} P : (B \otimes_A C)^{\times 2} \longrightarrow B \otimes_A C \\ P((b_1 \otimes c_1), (b_2 \otimes c_2)) = (b_1 +_B b_2) \otimes (c_1 +_C c_2), \end{gathered}$$

$$\begin{gathered} M : (B \otimes_A C)^{\times 2} \longrightarrow B \otimes_A C \\ M((b_1 \otimes c_1), (b_2 \otimes c_2)) = (b_1 \cdot_B b_2) \otimes (c_1 \cdot_C c_2). \end{gathered}$$

- Check A-bilinearity:

$$\begin{aligned} P(a \cdot (b_1 \otimes c_1),\, (b_2 \otimes c_2)) &\coloneqq (a \cdot (b_1 + b_2)) \otimes (c_1 + c_2) \\ &= ((b_1 + b_2)) \otimes a \cdot (c_1 + c_2) \quad \text{since } C \text{ is a left } A\text{-module} \\ &\coloneqq P((b_1 \otimes c_1),\, a \cdot (b_2 \otimes c_2)). \end{aligned}$$

$$\begin{aligned} M(a \cdot (b_1 \otimes c_1),\, (b_2 \otimes c_2)) &\coloneqq (a \cdot (b_1 \cdot b_2)) \otimes (c_1 \cdot c_2) \\ &= (b_1 \cdot b_2) \otimes (a \cdot (c_1 \cdot c_2)) \quad \text{since } C \text{ is a left } A\text{-module} \\ &\coloneqq M((b_1 \otimes c_1),\, a \cdot (b_2 \otimes c_2)). \end{aligned}$$

- So these lift to maps out of $(B \otimes_A C)^{\otimes 2}$.
- P forms an abelian group: clear because $+_B, +_C$ do, and commuting is just done within each factor.
- M forms a monoid: clear for some reason.
- Checking distributivity, claim: it suffices to check on elementary tensors and extend by linearity?

$$\begin{aligned} (b_0 \otimes c_0) \cdot ((b_1 \otimes c_1) + (b_2 \otimes c_2)) &= (b_0 \otimes c_0) \cdot ((b_1 + b_2) \otimes (c_1 + c_2)) \\ &= (b_0(b_1 + b_2)) \otimes (c_0(c_1 + c_2)) \\ &= (b_0 b_1 + b_0 b_2) \otimes (c_0 c_1 + c_0 c_2) \\ &= \cdots . \end{aligned}$$

2.2 1.3.L

If $S \subseteq A$ is multiplicative and $M \in A$-Mod, describe a natural isomorphism

$$\eta : (S^{-1} A) \otimes_A M \longrightarrow (S^{-1} M)$$

as both $S^{-1}A$-modules and A-modules.

Solution

- Recall the definition

$$S^{-1}A := \left\{ \frac{a}{s} \;\middle|\; a \in A, s \in S \right\} / \sim$$
$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \iff \exists s \in S \text{ such that } s(s_2 a_1 - s_1 a_2) = 0_A.$$

- Similarly $S^{-1}M = \left\{ \frac{m}{s} \right\} / \sim$.

The universal property: in A-Mod, $M \longrightarrow S^{-1}M$ is initial among all morphisms $\alpha : M \longrightarrow N$ such that $\alpha(S) \subseteq N^{\times}$:

$$\begin{tikzcd} & S^{-1}M \arrow[d, dotted, "\exists!\tilde{\alpha}"] \\ M \arrow[ur, "S^{-1}\cdot"] \arrow[r, "\alpha"] & N \end{tikzcd}$$

Strategy: define a map $M \longrightarrow S^{-1}A \otimes_A M$ such that S is invertible in the image to obtain a map? Show they satisfy the same universal property?

- Since $M \in A$-Mod, we have an action $a \cdot m$, so define

$$\eta : (S^{-1}A) \times M \longrightarrow (S^{-1}M)$$
$$\left(\frac{a}{s}, m \right) \mapsto \frac{a \cdot m}{s}.$$

- The tensor product $S^{-1}A \otimes_A M$ makes sense.
  - $S^{-1}A$ is a right A-module by $a_0 \mapsto \left(\frac{a}{s} \mapsto \frac{a_0 a}{s} \right)$.
  - $S^{-1}M$ is a left A-module by $a_0 \mapsto (m \mapsto a_0 \cdot m)$ where the action comes from the A-module structure of M.
- The map makes sense as an A-module morphism
  - $S^{-1}A \otimes_A M$ is a left A-module by $a_0 \mapsto \left(\frac{a}{s} \otimes m \mapsto \frac{a_0 a}{s} \otimes m \right)$
  - $S^{-1}M$ is a left A-module by $a_0 \mapsto \left(\frac{m}{s} \mapsto \frac{a_0 \cdot m}{s} \right)$ using the A-module structure on M.
- The map makes sense as an $S^{-1}A$-module morphism
  - $S^{-1}A \otimes_A M$ is a left $S^{-1}A$-module by $\frac{a_0}{s_0} \mapsto \left(\frac{a}{s} \otimes m \mapsto \frac{a_0 a}{s_0 s} \otimes m \right)$
  - $S^{-1}M$ is a left $S^{-1}A$-module by $\frac{a_0}{s_0} \mapsto \left(\frac{m}{s} \mapsto \frac{a_0 \cdot m}{s_0 s} \right)$ by the A-module structure on M.
- Well-defined: ?

- A-bilinear: let $r \in A$, then

$$\begin{aligned}
\eta\left(r \cdot \frac{a}{s}, m\right) &:= \eta\left(\frac{r \cdot a}{s}, m\right) \\
&:= \frac{\psi(r \cdot a)(m)}{s} \\
&= \frac{r \cdot \psi(a)(m)}{s} && \text{since } \psi \text{ is a ring morphism} \\
&= \frac{\psi(a)(r \cdot m)}{s} && \text{since } \psi(a) \text{ is a ring morphism} \\
&:= \eta\left(\frac{a}{s}, r \cdot m\right).
\end{aligned}$$

  So this lifts to a map out of the tensor product.
- $S^{-1}A$-bilinear?

2.3 1.3.P

Show that the fiber product over the terminal object is the cartesian product.

Solution:

- Recall definition: T is terminal iff every object X admits a morphism $X \longrightarrow T$.
- Strategy: use both universal products to produce an isomorphism
- Let $\mathfrak{p}_X, \mathfrak{p}_Y$ by the cartesian product projections, and $\mathfrak{p}_X^T, \mathfrak{p}_Y^T$ be the fiber product projections
- Let T_X, T_Y be the maps $X \longrightarrow T, Y \longrightarrow T$.
- Since $X \times Y$ is an object in this category, it admits one unique map to T

$$\begin{tikzcd}
X \times Y \arrow[r, "\mathfrak{p}_Y"] \arrow[d, "\mathfrak{p}_X"'] \arrow[dr, "T_{X \times Y}"] & Y \arrow[d, "T_Y"] \\
X \arrow[r, "T_X"'] & T
\end{tikzcd}$$

- But now $T_Y \circ \mathfrak{p}_Y : X \times Y \longrightarrow T$ is another such map, so it must equal $T_{X \times Y}$.
- Similarly $T_X \circ \mathfrak{p}_X$ is equal to $T_{X \times Y}$.
- Thus $T_Y \circ \mathfrak{p}_Y = T_X \circ \mathfrak{p}_Y$, which is part of the universal property for $X \times_T Y$.
- By the universal property of $X \times Y$, for every W admitting maps to X, Y we get the following h_0:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists! h_0"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r, "\mathfrak{p}_Y"] \arrow[d, "\mathfrak{p}_X"] & Y \arrow[d, "T_Y"] \\
& X \arrow[r, "T_X"] & T
\end{tikzcd}$$

  Note that T doesn't matter in this particular diagram.
- This gives us the LHS diagram, the RHS comes from the universal property of $X \times Y$:

$$
\begin{tikzcd}
X \times_T Y \arrow[drr, bend left, "\mathfrak{p}_Y^T"] \arrow[dr, dotted, "\exists! h_0"] \arrow[ddr, bend right, "\mathfrak{p}_X^T"'] & & \\
& X \times Y \arrow[r, "\mathfrak{p}_Y"] \arrow[d, "\mathfrak{p}_X"] & Y \arrow[d, "T_Y"] \\
& X \arrow[r, "T_X"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[drr, bend left, "\mathfrak{p}_Y"] \arrow[dr, dotted, "\exists! h_1"] \arrow[ddr, bend right, "\mathfrak{p}_X"'] & & \\
& X \times_T Y \arrow[r, "\mathfrak{p}_Y^T"] \arrow[d, "\mathfrak{p}_X^T"] & Y \arrow[d, "T_Y"] \\
& X \arrow[r, "T_X"] & T
\end{tikzcd}
$$

- By commutativity, $h_0 \circ h_1 = \mathrm{id}_{X \times Y}$ and vice-versa?

2.4 1.3.Q

Show that if the two squares in this diagram are cartesian, then then outer square is also cartesian:

$$
\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}
$$

Solution:

- Need to show that given two maps $R \longrightarrow V$ and $R \longrightarrow Y$ such that $(V \longrightarrow Z) \circ (U \longrightarrow V) = (Y \longrightarrow Z) \circ (R \longrightarrow Y)$, then there is a unique map $R \longrightarrow U$ giving a commuting diagram:

$$
\begin{tikzcd}
U \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R \arrow[ul, dotted, "\exists!\,?"'] \arrow[r] \arrow[d] & V \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}
$$

- Applying the bottom square:
  - Need to produce maps $R \longrightarrow X$ and $R \longrightarrow Y$
  - We're given a map $R \longrightarrow Y$ by assumption.
  - We can build a map $R \longrightarrow X$ by taking $(V \longrightarrow X) \circ (R \longrightarrow V)$.
  - We then get a map $R \longrightarrow W$:

$$
\begin{tikzcd}
W \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R \arrow[ul, dotted, "\exists!"'] \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}
$$

- Applying the top square:
  - We have a map $R \longrightarrow V$ by assumption
  - We have a map $R \longrightarrow W$ from step 1
  - We have maps $V \longrightarrow X$ and $W \longrightarrow X$ from the top square
  - We thus obtain

$$\begin{tikzcd}
U \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R \arrow[ul, dotted, "\exists!"'] \arrow[r] \arrow[d] & V \arrow[d] \\
& W \arrow[r] & X
\end{tikzcd}$$