# Real Analysis

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August 15, 2019

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# 1 Lecture 1 (Thu 15 Aug 2019 11:04)

See Folland's Real Analysis, definitely a recommEnded reference.

Possible first day question: how can we "measure" a subset of $\mathbb{R}$? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of $\mathbb{R}$. We'll come back to this in a few lectures.

## 1.1 Notions of "smallness" in $\mathbb{R}$

**Definition 1.** Let $E$ be a set, then $E$ is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes $\emptyset, \mathbb{N}$.

**Definition 2.** A set $E$ is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

**Exercise 1.** Show that any finite subset of $\mathbb{R}$ is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in $Y$ iff the closure of $X$ is all of $Y$. So we'll make the following definition:

**Definition 3.** A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval $I$ contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. For example, $\mathbb{Q}$ is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently,

- $A^c$ contains a dense, open set.
- The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than $\varepsilon$.

**Definition 4.** Definition: $E$ is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

**Exercise 2.** Show that a countable union of null sets is null.

We have several relationships

- Countable $\implies$ Meager, but not the converse.
- Countable $\implies$ Null, but not the converse.

**Exercise 3.** Show that the "middle third" Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

**Theorem 1.** Every $E \subseteq \mathbb{R}$ can be written as $E = A \coprod B$ where $A$ is null and $B$ is meager.

This gives some information about how nullity and meagerness interact – in particular, $\mathbb{R}$ itself is neither meager nor null. Idea: if meager $\implies$ null, this theorem allows you to write $\mathbb{R}$ as the union of two null sets. This is bad!

*Proof.* We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Around each $q_j$, put an interval around it of size $1/2^{j+k}$ where we'll allow $k$ to vary, yielding multiple intervals around $q_j$. To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that $A$ is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick $k$ such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that $A^c$ :
$B$ is meager. Note that $G_k$ covers the rationals, and is a countable union of open sets, so it is dense. So $G_k$ is an open and dense set. By one of the equivalent formulations of meagerness, this means that $G_k^c$ is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager. $\square$

## 1.2 $\mathbb{R}$ is not small

**Theorem 2.**

- A (Cantor): $\mathbb{R}$ is not countable.
- B (Baire): $\mathbb{R}$ is not meager. (Baire Category Theorem)
- C (Borel): $\mathbb{R}$ is not null.

Note that theorems B and C imply theorem A. You can also replace $\mathbb{R}$ with any nonempty interval $I = [a, b]$ where $a < b$. This is a strictly stronger statement – if any subset of $\mathbb{R}$ is not countable, then certainly $\mathbb{R}$ isn't, and so on.

*Proof of (A).* Begin by thinking of $I = [0, 1]$, then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry. $\square$

*A second proof of (A).* Take an interval $I$, and suppose it is countable so $I = \{x_i\}$. Choose $I_1 \subseteq I$ that avoids $x_1$, so $x_1 \notin I_1$. Choose $I_2 \subseteq I_1$ avoiding $x_2$ and so on to produce a nested sequence of closed intervals. Since $\mathbb{R}$ is complete, the intersection $\bigcap_{n=1}^{\infty} I_n$ is nonempty, so say it contains $x$. But then $x \in I_1 \in I$, for example, but $x \neq x_i$ for any $i$, so $x \notin I$, a contradiction. $\square$ $\square$

*Proof of (B).* Suppose $I = \bigcup_{i=1}^{\infty} A_n$ where each $A_n$ is nowhere dense. We'll again construct a nested sequence of closed sets. Let $I_1 \subseteq I$ be a subinterval that misses all of $A_1$, so $A_1 \bigcap I_1 = \emptyset$ using the fact that $A_1$ is nowhere dense. Repeat the same process, let $I_2 \subset I_1 \setminus A_2$. By the nested interval property, there is some $x \in \bigcap A_i$. $\square$

> Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of $\mathbb{R}$ is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

Consider the Thomae function:

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It is continuous on $\mathbb{Q}$, but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Can this be switched to get some function $f$ that is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$? The answer is no. The set of discontinuities of a function is *always* an $F_\sigma$ set, and $\mathbb{R} \setminus \mathbb{Q}$ is not one. Equivalently, the rationals are not a $G_\delta$ set.

Some facts:

- The pointwise limit of continuous functions has a meager set of discontinuities.
- If $f$ is integrable, the set of discontinuities is null.
- If $f$ is monotone, they are countable.
- There is a continuous nowhere differentiable function: let $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$, and in fact *most* functions are like this.
- If $f$ is continuous and monotone, the discontinuities are null.

**Theorem 3.** Let $I = [a, b]$. If $I \subseteq \bigcup_{i=1}^{\infty} I_i$, then $|I| \leq \sum_{i=1}^{\infty} |I_i|$.

*Proof.* The proof is by induction. Assume $I \subseteq \bigcup_n^{N+1} I_n$, where wlog we can assume that $a < a_{N+1} < b \leq b_{N+1}$, then $[a, a_{N+1}] \subset \bigcup_{n=1}^{N} I_n$ so the inductive hypothesis applies. But then $b - a \leq b_{N+1} - a = (b_{N+1} - a_{N+1}) + (a_{N+1} - a) \leq \sum_{n=1}^{N+1} |I_n|$. $\square$

> Note that this proves that the reals are uncountable!

# 2 Lecture 2