Title

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Contents

1 | Monday, October 12

1.1 Proof of Bott-Borel-Weil

Recall the Bott-Borel-Weil theorem: in characteristic zero, we're looking at the closure of the region containing the fundamental region $C_{\mathbb{Z}}$:

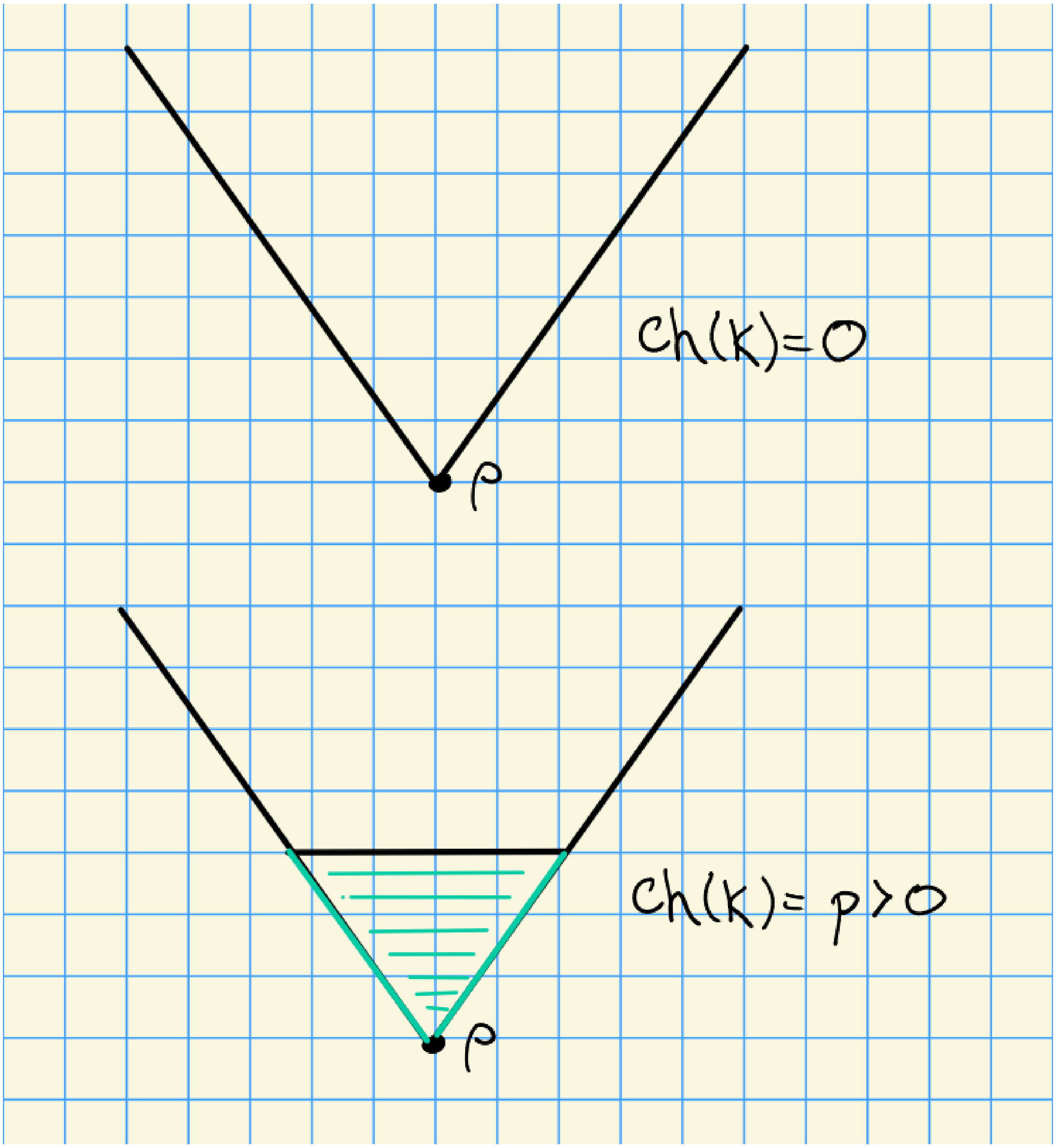

Figure 1: Image

Theorem 1.1.1 ***(due to Aandersen).*** a. If $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\lambda \notin X(T)_+$ then $H^0(w \circ \lambda) = 0$.
b. If $\lambda \in \overline{C}_{\mathbb{Z}} \cap X(T)_+$ then for all $w \in W$, we have

$$H^i(w \cdot \lambda) = \begin{cases} H^0(\lambda) & i = \ell(w) \\ 0 & \text{otherwise} \end{cases}.$$

Proof (of a).
For (a): we use induction on $\ell(w)$. For $\ell(w) = 0$, we have $w = \mathrm{id}$. Let $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\lambda \notin X(T)_+$. Then

$$\begin{aligned} 0 &\leq \langle \lambda + \rho,\ \alpha^\vee \rangle \\ &= \langle \lambda,\ \alpha^\vee \rangle + 1 \\ \implies \langle \lambda,\ \alpha^\vee \rangle &= -1. \end{aligned}$$

Applying the previous proposition, we get $H^0(\lambda) = 0$.

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Proof (of b).
For the base case $w = \mathrm{id}$, this follows from Kempf vanishing. Assuming the result holds for any word of length $l < \ell(w)$, if $\ell(w) > 0$, there exists some simple reflection s_α for $\alpha \in \Delta$ such that $\ell(s_\alpha w) = \ell(w) - 1$. Moreover, $w^{-1}(\alpha) \in -\Phi^+$, so set $\beta = -w^{-1}(\alpha) \in \Phi^+$. We can the make the following computation:

$$\begin{aligned} \langle (s_\alpha w) \cdot \lambda,\ \alpha^\vee \rangle &= \langle (s_\alpha w)(\lambda + \rho) - \rho,\ \alpha^\vee \rangle \\ &= \langle (s_\alpha w)(\lambda + \rho),\ \alpha^\vee \rangle - 1 \\ &= \langle w(\lambda + \rho),\ s_\alpha \alpha^\vee \rangle - 1 \\ &= -\langle w(\lambda + \rho),\ \alpha^\vee \rangle - 1 \\ &= \left\langle \lambda + \rho,\ -w^{-1} \alpha^\vee \right\rangle - 1 \\ &= \langle \lambda + \rho,\ \beta^\vee \rangle - 1 \\ &\geq -1 \end{aligned}$$

and $\langle (s_\alpha w) \cdot \lambda,\ \alpha^\vee \rangle < \rho$ since $\lambda \in \overline{C}_{\mathbb{Z}}$. Note that we've used the fact that the inner product is W-invariant.

Now if $\langle (s_\alpha w) \cdot \lambda,\ \alpha^\vee \rangle \geq 0$, we can apply the prior proposition part (d). Here we use the fact that $\mathrm{Ind}_B^{P_\alpha} (s_\alpha w)\lambda$ is simple. Applying the inductive hypothesis yields

$$H^i(s_\alpha - \lambda) = H^{i+1}(w \cdot \lambda).$$

Now if $\langle s_\alpha w \cdot \lambda,\ \alpha^\vee \rangle = -1$, then

$$\begin{aligned} -1 &= \langle \lambda + \rho,\ \beta^\vee \rangle - 1 \\ \implies \langle \lambda + \rho,\ \beta^\vee \rangle &= 0 \\ \implies \langle \lambda,\ \beta^\vee \rangle &= 0 \\ &\cdots. \end{aligned}$$

Missing computation

Then applying (a) yields $H^1(w \cdot \lambda) = 0$.

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1.2 Serre Duality and Grothendieck Vanishing

Let P be a parabolic subgroup, i.e. $P_J = P \coloneqq L_J \rtimes U_J$ for some $J \subseteq \Delta$. Set $n(P) = \left|\Phi^+\right| - \left|\Phi_J^+\right|$.

Example 1.2.1.
Let $\Phi = A_4$, which has ten simple roots:

- $\alpha_i, 1 \leq i \leq 4$
- $\alpha_i + \alpha_{i+1}$, $i = 1, 2, 3$.
- $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_2 + \alpha_3 + \alpha_4$
- $\sum_{i=1}^{4} \alpha_i$.

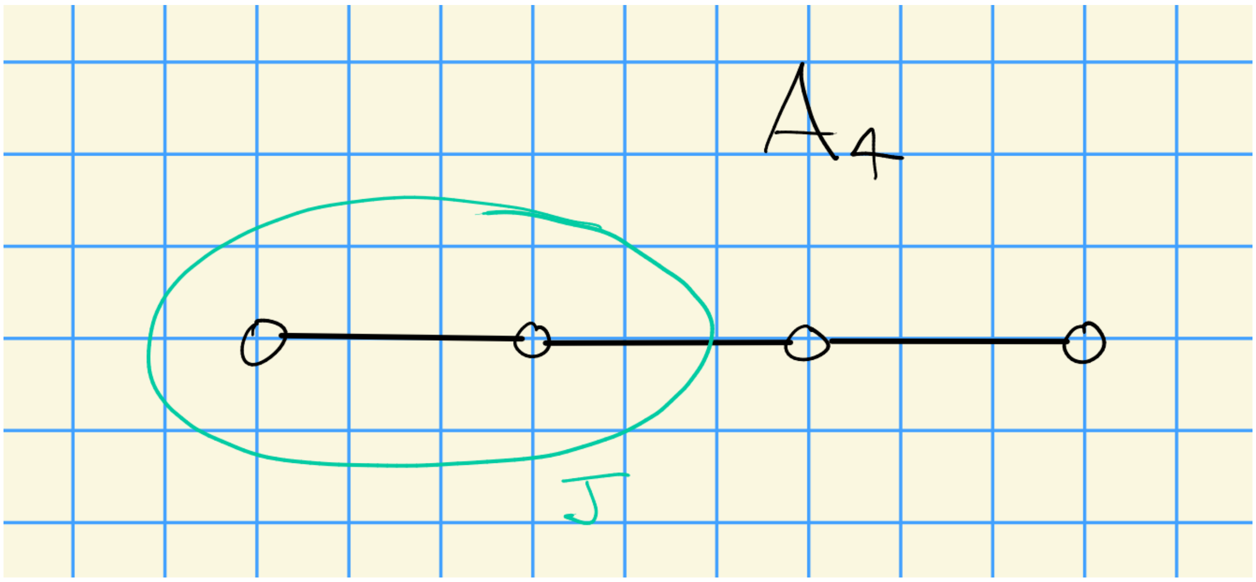

Figure 2: Image

Then $n(P) = 10 - 3 = 7$.

Theorem 1.2.1 ***(Grothendieck Vanishing).***

$$R^i \operatorname{Ind}_P^G M = 0 \qquad \text{for } i > n(P).$$

Theorem 1.2.2 ***(Serre Duality).***

$$\left(R^i \operatorname{Ind}_B^G M\right)^\vee \cong R^{n(P)-i} \operatorname{Ind}_P^G M^\vee \otimes (-2\rho_P).$$

where

$$\rho_p := \frac{1}{2} \sum_{\beta \in \Phi^+ \setminus \Phi_J} \beta$$

Example 1.2.2.
Take $B = P$ and $M = \lambda$. Then $\lambda^\vee = -\lambda$, so

$$\left(R^i \operatorname{Ind}_B^G \lambda\right)^\vee \cong R^{\left|\Phi^+\right| - i} \operatorname{Ind}_P^G (-\lambda)^\vee \otimes (-2\rho).$$

From this we can conclude

$$H^i(\lambda) = H^{n-i}(-\lambda - 2\rho)^\vee,$$

where $n = \left|\Phi^+\right|$.

Corollary 1.2.1 ***(?)*.**
Let $\lambda \in X(T)_+ \cap \overline{C}_{\mathbb{Z}}$ be a dominant weight. Then

a. The irreducible representations are given by $L(\lambda) = H^0(\lambda)$.
b. $\operatorname{Ext}_G^1(L(\lambda), L(\mu)) = 0$ for all λ, μ in $\overline{C}_{\mathbb{Z}}$.
c. If char $(k) = 0$, so $X(T)_+ \subset \overline{C}_{\mathbb{Z}}$, then all G-modules are completely reducible.

Proof (of a).
Note that the longest element takes positive roots to negative roots, so $w_0 \rho = -\rho$, and moreover $-w_0(\overline{C}_{\mathbb{Z}}) = \overline{C}_{\mathbb{Z}}$. We also have

$$\begin{aligned} w_0 \cdot (w_0 \lambda) &= w_0(-w_0 \lambda + \rho) - \rho \\ &= -\lambda + w_0 \rho - \rho \\ &= -\lambda - 2\rho. \end{aligned}$$

By Serre duality, if we take the Weyl module we obtain

$$\begin{aligned} V(-w_0 \lambda) &:= H^0(\lambda)^\vee \\ &= H^n(-\lambda - 2\rho) \\ &= H^n(w_0 \cdot (-w_0 \lambda)) \\ &= H^n(-w_0 \lambda) \qquad \text{by Bott-Borel-Weil,} \end{aligned}$$

where we've used that $\ell(w_0) = \left|\Phi^+\right|$. We know that $L(-w_0\lambda) \subseteq \operatorname{Soc} H^0(-w_0\lambda) = V(-w_0\lambda) \twoheadrightarrow L(-w_0\lambda)$, where the last term is contained in the head. But this means that this splits, so by indecomposability we must have $L(-w_0\lambda) = H^0(-w_0\lambda) = V(-w_0\lambda)$. So we can conclude

$$L(\mu) = H^0(\mu) = V(\mu) \qquad \forall \mu \in X(T)_+ \cap \overline{C}_{\mathbb{Z}}.$$

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Proof (of b and c).
Suppose $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, then $L(\lambda)$ is in $H^0(\mu)/\mathrm{Soc}\,_G H^0(\mu) = 0$ and $L(\mu)$ is in $H^0(\lambda)/\mathrm{Soc}\,_G H^0(\lambda) = 0$, but this forces $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$.

Part (c) follows from part (b).

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1.3 Weyl's Character Formula

Problem: Determine char $H^0\lambda$ for $\lambda \in X(T)_+$.

Solution: Let $A(\lambda) = \sum_{w \in W} \mathrm{sgn}(w) e^{w\lambda} \in \mathbb{Z}[X(T)]$, where we sum over the usual Weyl group and not the affine Weyl groups, taken as a formal sum in the group algebra on the weight lattice. We can then state Weyl's character formula:

$$\text{char } H^0(\lambda) = \frac{A(\lambda + \rho)}{A(\rho)} \qquad \text{for } \lambda \in X(T)_+.$$

This is a formal sum, so it's surprising that the bottom term even divides the top. But there is a great deal of cancellation, we'll see this in examples such as GL_3.

1.3.1 Formal Characters

Let M be a T-module, then define the *character*

$$\text{char } M \coloneqq \sum_{\mu \in X(T)} (\dim M_\mu) e^\mu \quad \in \mathbb{Z}[X(T)].$$

We then define the *Euler characteristic*

$$\chi(M) \coloneqq \sum_{i \geq 0} (-1)^i \text{char } H^i(M).$$

Note that by Grothendieck vanishing, $H^i(M) = 0$ for $i > \left|\Phi^+\right| = \dim(G/B)$, so this is a finite sum. In fact, if M is a G-module, then this is W-invariant and thus in fact $\chi(M) \in \mathbb{Z}[X(T)]^W$.