

# Problem Set 7

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October 27, 2019

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## 1 Regular Problems

### 1.1 Problem 1

Note that if either  $p = 1$  or  $q = 1$ ,  $G$  is a  $p$ -group, which is a nontrivial center that is always normal. So assume  $p \neq 1$  and  $q \neq 1$ .

We want to show that  $G$  has a non-trivial normal subgroup. Noting that  $\#G = p^2q$ , we will proceed by showing that either  $n_p$  or  $n_q$  must be 1.

We immediately note that

$$\begin{array}{ll} n_p \equiv 1 \pmod{p} & n_q \equiv 1 \pmod{q} \\ n_p \mid q & n_q \mid p^2, \end{array}$$

which forces

$$n_p \in \{1, q\}, \quad n_q \in \{1, p, p^2\}.$$

If either  $n_p = 1$  or  $n_q = 1$ , we are done, so suppose  $n_p \neq 1$  and  $n_q \neq 1$ . This forces  $n_p = q$ , and we proceed by cases:

### 1.1.1 Case 1: $p = q$ .

Then  $\#G = p^3$  and  $G$  is a  $p$ -group. But every  $p$ -group has a non-trivial center  $Z(G) \leq G$ , and the center is always a normal subgroup.

### 1.1.2 Case 2: $p > q$ .

Here, since  $n_p \mid q$ , we must have  $n_p < q$ . But if  $n_p < q < p$  and  $n_p \equiv 1 \pmod{p}$ , then  $n_p = 1$ .

### 1.1.3 Case 3: $q > p$ .

Since  $n_p \neq 1$  by assumption, we must have  $n_p = q$ . Now consider sub-cases for  $n_q$ :

- $n_q = p$ : If  $n_q = p \equiv 1 \pmod{q}$  and  $p < q$ , this forces  $p = 1$ .
- $n_q = p^2$ : We will reach a contradiction by showing that this forces

$$\left| P := \bigcup_{S_p \in \text{Syl}(p, G)} S_p \setminus \{e\} \right| + \left| Q := \bigcup_{S_q \in \text{Syl}(q, G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

We have

$$\begin{aligned} |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\ &= p^2(q-1) + q(p^2-1) + 1 \\ &= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 \quad (\text{since } q > 1) \\ &= (p^2q - p^2) + (p^2 - 1) + (q-1)(p^2-1) + 1 \\ &= p^2q + (q-1)(p^2-1) \\ &\geq p^2q + (2-1)(2^2-1) \quad (\text{since } p, q \geq 2) \\ &= p^2q + 3 \\ &> p^2q = |G|, \end{aligned}$$

which is a contradiction.  $\square$

## 1.2 Problem 2

We'll use the fact that  $H \trianglelefteq N(H)$  for any subgroup  $H$  (following directly from the closure axioms for a subgroup), and thus

$$P \trianglelefteq N(P) \quad \text{and} \quad N(P) \trianglelefteq N^2(P).$$

Since it is then clear that  $N(P) \subseteq N^2(P)$ , it remains to show that  $N^2(P) \subseteq N(P)$ .

So if we let  $x \in N^2(P)$ , so  $x$  normalizes  $N(P)$ , we need to show that  $x$  normalizes  $P$  as well, i.e.  $xPx^{-1} = P$ .

However, supposing that  $|G| = p^k m$  where  $(p, m) = 1$ , we have

$$P \leq N(P) \leq G \implies p^k \mid |N(P)| \mid p^k m,$$

so in fact  $P \in \text{Syl}(p, N(P))$  since it is a maximal  $p$ -subgroup.

Then  $P' := xPx^{-1} \in \text{Syl}(p, N(P))$  as well, since all conjugates of Sylow  $p$ -subgroups are also Sylow  $p$ -subgroups.

But since  $P \leq N(P)$ , there is only *one* Sylow  $p$ -subgroup of  $N(P)$ , namely  $P$ . This forces  $P = P'$ , i.e.  $P = xPx^{-1}$ , which says that  $x \in N(P)$  as desired.  $\square$

### 1.3 Problem 3

By definition,  $G$  is simple iff it has no non-trivial subgroups, so we will show that if  $|G| = 148$  then it must contain a normal subgroup.

Noting that  $248 = p^2 q$  where  $p = 2, q = 37$ , we find that (for example)  $n_2 \mid 37$  but  $n \equiv 1 \pmod{2}$ ; but the only odd divisor of 7 is 1, forcing  $n_2 = 1$ . So  $G$  has a normal Sylow 2-subgroup and we are done.

### 1.4 Problem 4

Let  $\tau := (t_1, t_2)$  denote the transposition and  $\sigma = (s_1, s_2, \dots, s_p)$  denote the  $p$ -cycle, and let  $S = \langle \sigma, \tau \rangle$ . We would like to show that  $S = S_p$ , and since  $S \subseteq S_p$  is clear, we just need to show that  $S_p \subseteq S$ .

We first note that because  $p$  is prime,  $\sigma^k$  is a  $p$ -cycle for every  $1 \leq k \leq p$ , and  $\langle \sigma \rangle = \langle \sigma^k \rangle$  for any such  $k$ .

Then note that  $t_1 = s_i$  for some  $i$  and  $t_2 = s_j$  for some  $j$ , so we can take  $k = j - i$  to get a cycle  $\sigma^k$  that sends  $t_1$  to  $t_2$ . So without loss of generality, we can replace  $\sigma$  with

$$\sigma = (t_1, t_2, \dots)$$

But now, we can relabel all of the elements of  $S_p$  simultaneously (i.e. replace  $\langle \sigma, \tau \rangle$  with another subgroup in the same conjugacy class) in such a way that  $t_1$  becomes 1 and  $t_2$  becomes 2. We can then assume wlog that

$$\tau = (1, 2), \quad \sigma = (1, 2, \dots, p)$$

We can then get all adjacent transpositions: noting that

$$\begin{aligned} \sigma^{-1} \tau \sigma &= (2, 3) \\ \sigma^{-2} \tau \sigma^2 &= (3, 4) \\ &\dots \\ \sigma^{-k} \tau \sigma^k &= (k+1 \pmod{p}, k+2 \pmod{p}) \quad \forall 1 \leq k \leq p, \end{aligned}$$

where we use the fact that for any  $\gamma \in S_p$ , we have  $\gamma\tau\gamma = (\gamma(1), \gamma(2))$ .

But this also gives us all transpositions of the form  $(1, j)$  for each  $2 \leq j \leq p$ :

$$\begin{aligned}(2, 3)^{-1}(1, 2)(2, 3) &= (1, 3) \\ (3, 4)^{-1}(1, 3)(3, 4) &= (1, 4) \\ &\dots \\ (j-1, j)^{-1}(1, j-1)(j-1, j) &= (1, j) \quad \forall 1 \leq j \leq p.\end{aligned}$$

Thus we have  $\langle \{(1, j) \mid 2 \leq j \leq p\} \rangle \subseteq S$ .

But now if  $\gamma = (g_1, g_2, \dots, g_k) \in S_p$  is an arbitrary cycle, we can write

$$\gamma = (g_1, g_2, \dots, g_k) = (1, g_1)(1, g_2), \dots (1, g_k).$$

Then writing any arbitrary permutation as a product of disjoint cycles, we find that  $S_p \in \langle (1, j) \mid 2 \leq j \leq p \rangle \subseteq S$ , and so  $S_p \subseteq S$  as desired.

## 2 Qual Problems