Space X	$\pi_0(X)$	$\pi_1(X)$	$H_0(X)$	$H_1(X)$	$H_2(X)$	$H_3(X)$
$\mathbb{R}^n$		0	$\mathbb{Z}$	0		
$\mathbb{R}^n - k \text{ pts}$						
B^n		0		0		
S^0		0	0	0	0	0
S^1		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
S^2		0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
S^3		0	$\mathbb{Z}$	0	0	?
$S^n, n \geq 4$		0	$\mathbb{Z}$	0	0	0
$S^n - k \text{ pts}$						
$T^2 = S^1 \times S^1$		$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$	$Z \oplus Z$		
$\prod_n S^1$		$F_n \cong \prod_n \mathbb{Z}$	$\mathbb{Z}$	$F_n^{(ab)} = \bigoplus_n \mathbb{Z}$		
$\prod_n S^1 - k \text{ pts}$						
$\bigvee_n S^1$	0	$*_n \mathbb{Z}$	$\mathbb{Z}$			
$\mathbb{RP}^1$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$		0
$\mathbb{RP}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$\mathbb{RP}^3$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\mathbb{RP}^n, n \geq 4$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
$\mathbb{CP}^1$	0	0		0		
$\mathbb{CP}^n, n \geq 2$	0	0		0		
Mobius Band						
Klein Bottle				$\mathbb{Z} \times \mathbb{Z}_2$		
$Gr(n,k)$						
$S^3 -$ Hopf Link						
n-fold dunce cap						

Notation

- $A \times B$ and $\prod X_i$ are direct products of groups, $A \oplus B$ and $\bigoplus X_i$ are direct sums. These are equivalent when there are finitely many terms involved; the latter is a subspace of the former when there are infinitely many terms.
- $A * B$ is a free product, $*_n \mathbb{Z}$ is the **free group** on n generators
- $\bigoplus_n \mathbb{Z}$ is the **free abelian group** on n generators
    - Every free abelian group is $\bigoplus_{i \in I} \mathbb{Z}$ for some set I.
    - $(*_n \mathbb{Z})^{\text{ab}} = \bigoplus_n \mathbb{Z}$.
    - $x \in \langle a_1, \cdots, a_n \rangle \implies x = \sum_n c_i a_i$ for some $c_i \in \mathbb{Z}$ (integer linear combination of generators)
- $\mathbb{RP}^n = S^n / S^0 = S^n / \mathbb{Z}_2$

- $\mathbb{CP}^n = S^{2n+1}/S^1$
- $G(n,k)$ where $G(n,1) = \mathbb{RP}^n$ is the real n-dimensional grassman manifold, also the set of k dimensional subspaces of $\mathbb{R}^n$.

Spheres

- $\pi_i(S^n) = 0$ for $i < n$, $\pi_n(S^n) = \mathbb{Z}$
  - Not necessarily true that $\pi_i(S^n) = 0$ when $i > n$!!!
    - E.g. $\pi_3(S^2) = \mathbb{Z}$ by Hopf fibration
- $H_i(S^n) = \mathbf{1}[i \in 0, n]$
- $H_n(\bigvee_i X_i) \cong \bigoplus_i H_n(X_i)$ for "good pairs"
  - Corollary: $H_n(\bigvee_k S^n) = \mathbb{Z}^k$
- $S^n/S^k \simeq S^n \vee \Sigma S^k$
  - $\Sigma S^n = S^{n+1}$
- S^n has the CW complex structure of 2 k-cells for each $0 \leq k \leq n$.

Torii

- $\pi_k(\times_n S^1) = 0$ for $k \geq 2$.
- $H_k(\times_n S^1) = \bigoplus_{\binom{n}{k}} \mathbb{Z}$

Projective Spaces

- $\mathbb{RP}^n = S^n/\mathbb{Z}_2$, an antipodal action.
- $\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$ for $k \geq 2$ $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for $n > 1$.
- $\pi_k(\mathbb{CP}^n) = \pi_k(S^{2n+1})$ for $k \geq 3$
- $H_i(\mathbb{RP}^n) = \mathbb{Z} \cdot 1[i = 0] + \mathbb{Z} \cdot 1[i = n, n \text{ odd})] + \mathbb{Z}_2 \cdot 1[1 \leq i < n, i \text{ odd}]$

Homotopy groups of real projective spaces

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
RP^1	Z	0	0	0	0	0	0	0	0	0	0	0
RP^2	Z_2	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^3	Z_2	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
RP^4	Z_2	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2

Homotopy groups of complex projective spaces												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
CP^1	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
CP^2	0	Z	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}
CP^3	0	Z	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0
CP^4	0	Z	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}

Theorems

- Techniques:
  - Fundamental group:
    - Van Kampen
    - Covering space actions?
      - $\pi_1(X/\Gamma) = \Gamma$ when $\pi_1(X) = 0$ and Γ acts freely
  - Homotopy Groups
    - Hurewicz
  - Homology Groups
    - Mayer-Vietoris
      $\cdots \to H_i(A \bigcap B) \to H_i(A) \oplus H_i(B) \to H_i(A \bigcup B) \to H_{i-1}(A \bigcap B) \to \cdots$
    - Excision?
- $\pi_k(X)$ for $k \geq 2$ is always abelian.
- Rank π_0 / H_0 = number of connected components.
- $\pi_1(X) = \mathbb{Z}$ iff X is simply connected.
- $H_1(X) = \mathbf{Ab}(\pi_1(X))$ (the abelianization of the fundamental group.)
- $\pi_k(\prod X_i) \cong \prod \pi_k(X_i)$ (homotopy groups commute with products)
- $\pi_1(\bigvee_n X_i) = *_n \pi_1(X_i)$ (fundamental groups of wedges split into free products)
- X simply connected implies $\pi_k(X) \cong H_k(X)$ for first nonvanishing H_k
- X an $n-1$ connected space implies $\pi_k(X) \cong H_k(X)$ for all $2 < k \leq n$. ($k = 1$ case is abelianization)
- For $n \geq k+2$, $\pi_{n+k}(S^n)$ does not depend on n. i.e., Homotopy groups stabilize. Diagonals show where diagonals become constant:

Homotopy groups of spheres												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
S^1	Z	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\boxed{Z}$	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^3	0	0	Z	$\boxed{Z_2}$	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	$Z_2 \times Z_2$
S^4	0	0	0	Z	Z_2	$\boxed{Z_2}$	$Z \times Z_{12}$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	$Z_{24} \times Z_3$	Z_{15}	Z_2
S^5	0	0	0	0	Z	Z_2	Z_2	$\boxed{Z_{24}}$	Z_2	Z_2	Z_2	Z_{30}
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	$\boxed{0}$	Z	Z_2
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	$\boxed{0}$
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Matrix Groups

- General list:
  - $O_n, U_n, SO_n, SU_n, Sp_n$
- $\pi_k(U_n) = \mathbb{Z} \cdot \mathbf{1}[k \text{ odd}]$ $\pi_1(U_n) = 1$
- $\pi_k(SU_n) = \mathbb{Z} \cdot \mathbf{1}[k \text{ odd}]$ $\pi_1(SU_n) = 0$
- $\pi_k(U_n) = \mathbb{Z}_2 \cdot \mathbf{1}[k = 0, 1/8] + \mathbb{Z} \cdot 1[k = 3, 7/8]$
- $\pi_k(SP_n) = \mathbb{Z}_2 \cdot \mathbf{1}[k = 4, 5/8] + \mathbb{Z} \cdot 1[k = 3, 7/8]$
-

A1.1.3.4 Exceptional groups

Homotopy groups of exceptional groups												
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
G_2	0	0	Z	0	0	Z_3	0	Z_2	Z_6	0	$Z \times Z_2$	0
F_4	0	0	Z	0	0	0	0	Z_2	Z_2	0	$Z \times Z_2$	0
E_6	0	0	Z	0	0	0	0	0	Z	0	Z	Z_{12}
E_7	0	0	Z	0	0	0	0	0	0	0	Z	Z_2
E_8	0	0	Z	0	0	0	0	0	0	0	0	0