

# Title

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# 1 | Lecture 07

Last time: stalks, sheafification, and  $\mathrm{Sh}(X_{\text{ét}})$  is abelian. Next up, we're aiming to define sheaf cohomology for  $\mathrm{Sh}(X_{\text{ét}})$ .

**Remark 1.0.1 (Esoteric!):** Related to a question asked by a viewer: there is not in fact a morphism from  $X_{\text{fppf}} \rightarrow X_{\text{ét}}$ , since locally finitely-presented need not be finitely presented (part of the condition for fppf). There is instead a morphism  $X_{\text{fppf}} \rightarrow X_{\text{ét,fp}}$  to a corresponding finitely presented site. There is also a map  $X_{\text{ét}} \rightarrow X_{\text{ét,fp}}$  inducing an equivalence on the category of sheaves via pushforward. 

**Theorem 1.0.2 (Enough injectives).**

$\mathrm{Sh}(X_{\text{ét}})$  has enough injectives.

*Proof (?)*.

Given  $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$  we want an injective sheaf  $\mathcal{I}$  and an injection  $\mathcal{F} \hookrightarrow \mathcal{I}$ . For each  $x \in X$ , choose a geometric point  $\bar{x}$  over  $x$ , and let  $I(\bar{x})$  be an injective  $\mathbb{Z}$ -module with a map  $\mathcal{F}_{\bar{x}} \rightarrow I(\bar{x})$ . These exist because the category of  $\mathbb{Z}$ -modules has enough injectives. The injectives in this category are **divisible** abelian groups.

**Claim:** The following object works:

$$\mathcal{I} := \prod_{\bar{x}} (\iota_{\bar{x}})_* I(\bar{x}).$$

We need to check

1. There is a map  $\mathcal{F} \rightarrow \mathcal{I}$ : The RHS is a product, so we map into the components.  $\mathcal{F}_{\bar{x}}$  maps into its own associated skyscraper sheaf where the map is sending sections to their germs. Then the skyscraper sheaf for  $\mathcal{F}_{\bar{x}}$  maps into the skyscraper sheaf for  $I(\bar{x})$  by pushforward.
2. This is a monomorphism: check on stalks.
3.  $\mathcal{I}$  is injective: check the lifting property directly.

■

## 1.1 What Else We Get From Sheafification

**Remark 1.1.1:** We now know that  $\mathrm{Sh}(X_{\text{ét}})$  is abelian with enough injectives. This is true for  $\mathrm{Sh}(\tau)$  for any site  $\tau$ , but this is substantially harder to show.

### 1.1.1 Inverse Images

For  $f : X \rightarrow Y$ , we have a map on presheaves

$$f^{-1} : \text{Presh}(Y_{\text{ét}}) \rightarrow \text{Presh}(X_{\text{ét}})$$

$$\mathcal{F}(V \xrightarrow{\text{ét}} X) \mapsto \varinjlim \mathcal{F}(U \rightarrow X),$$

where the limit is over diagrams of the form

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow \text{ét} & & \downarrow \text{ét} \\ X & \longrightarrow & Y \end{array}$$

**Fact 1.1.2:**  $f^{-1}$  is left adjoint to pushforward as functors on presheaves.

**Exercise 1.1.3(?):** Check this.

**Definition 1.1.4** (Inverse Image Sheaf)

$$f^* \mathcal{F} := (f^{-1} \mathcal{F})^a.$$

**Theorem 1.1.5(?).**

$f^*$  is left adjoint to  $f_*$ .

*Proof* (?).

Sheafification is a left adjoint. ■

**Example 1.1.6(?):**

- For  $\bar{x} \xrightarrow{\iota} X$  a geometric point, we have  $\iota^* \mathcal{F} = \mathcal{F}_{\bar{x}}$ .
- For  $Y \xrightarrow{f} X$ , we have  $f^* \underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$ .
- More generally, for  $Y \xrightarrow{f} X$  and any representable functor  $\mathcal{F} := \underline{\text{hom}}_X(\cdot, Z)$ , we have  $f^* \mathcal{F} = \underline{\text{hom}}_Y(\cdot, Y \times_X Z)$ .

## 1.2 Étale Cohomology

See ?? for the definition of étale cohomology. How do we compute  $H^i(X_{\text{ét}}, \mathcal{F})$ ? Choose an injective resolution

$$\mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$$

with the  $\mathcal{I}^j$  injectives. From the general theory of derived functors, we obtain

$$H^i(X_{\text{ét}}, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)),$$

where the RHS is a complex of abelian groups. Injective resolutions are difficult to find in general. Suppose  $\pi : X_{\text{ét}} \rightarrow Y_{\text{ét}}$  comes from a map of schemes, then we can compute derived functors of other functors such as the pushforward,

$$(R^i \pi_*) \mathcal{F} = H^i(\pi_* \mathcal{I}^\bullet),$$

where the RHS are sheaves on  $Y_{\text{ét}}$ . Implicit here is the claim that  $\pi_*$  is left-exact. You can also find  $(L^{>0} \pi^*) \mathcal{G} = 0$ .

**Exercise 1.2.1 (?)**: Check that pullback is exact.

**Proposition 1.2.2 (Properties of étale cohomology).**

1.  $H^0(X_{\text{ét}}, \mathcal{F}) = \mathcal{F}(X)$ , aka the global sections  $\Gamma(X, \mathcal{F})$ .
2.  $H^{>0}(\mathcal{I}) = 0$  for  $\mathcal{I}$  injective.
3. Given a SES