

Category \mathcal{O}

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1 Wednesday January 8

Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category \mathcal{O} ".

Course Website: <https://faculty.franklin.uga.edu/brian/math-8030-spring-2020>

1.1 Chapter Zero: Review

Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase \mathfrak{g} characters!

In this course, we'll take $k = \mathbb{C}$.

Recall that a Lie Algebra is a vector space \mathfrak{g} with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

- $[xx] = 0$ for all $x \in \mathfrak{g}$
 - Exercise: this implies $[xy] = -[yx]$.
- $[x[yz]] = [[xy]z] + [y[xz]]$ (The Jacobi identity)
 - This says x acts as a derivation.

Hint: Consider $[x + y, x + y]$. Note that the converse holds iff $\text{char } k \neq 2$.

Exercise: This implies Lie Algebras never have an identity.

Definition: \mathfrak{g} is *abelian* iff $[xy] = 0$ for all $x, y \in \mathfrak{g}$.

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
 - A vector subspace that is closed under brackets.
- Homomorphisms
 - I.e. a linear transformation ϕ that commutes with the bracket, i.e. $\phi([xy]) = [\phi(x)\phi(y)]$.
- Ideals

Exercise: Given a vector space (possibly infinite-dimensional) over k , then (exercise) $\mathfrak{gl}(V) := \text{End}_k(V)$ is a Lie algebra when equipped with $[fg] = f \circ g - g \circ f$.

Definition: A *representation* of \mathfrak{g} is a homomorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ for some V .

Example: The adjoint representation is $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, where $\text{ad}(x)(y) := [xy]$.

Representations give \mathfrak{g} the structure of a module over V , where $x \cdot v := \phi(x)(v)$. All of the usual module axioms hold, where now $[xy] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

Example: The trivial representation $V = k$ where $x \cdot a = 0$.

Definition: V is *irreducible* (or *simple*) iff V has exactly two \mathfrak{g} -invariant subspaces, namely $0, V$.

Definition: V is *completely reducible* iff V is a direct sum of simple modules, and *indecomposable* iff V can not be written as $V = M \oplus N$, a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragredient/dual* $V^\vee := \text{hom}_k(V, k)$ where $(x \cdot f) = -f(x \cdot v)$ for $f \in V^\vee, x \in \mathfrak{g}, v \in V$.
- The direct sum $V \oplus W$ where $x \cdot (v, w) = (x \cdot v, x \cdot w)$ and $x \cdot (v + w) = x \cdot v + x \cdot w$.
- The tensor product where $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.
- $\text{hom}_k(V, W)$ where $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$.
– Note that if we take $W = k$ then the first term vanishes and this recovers the dual.

1.2 Semisimple Lie Algebras

Definition: The derived ideal is given by $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \text{span}_k(\{[xy] \mid x, y \in \mathfrak{g}\})$.

This is the analog of the commutator subgroup.

Lemma: \mathfrak{g} is abelian iff $\mathfrak{g}^{(1)} = \{0\}$, and 1-dimensional algebras are always abelian.

This follows because if $[xy] := xy - yx$ then $[xy] = 0 \iff xy = yx$.

Definition: A lie group \mathfrak{g} is *simple* iff the only ideals of \mathfrak{g} are $0, \mathfrak{g}$ and $\mathfrak{g}^{(1)} \neq \{0\}$.

Note that this rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

Definition: The derived series is defined by $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$, continuing inductively. \mathfrak{g} is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some n .

Lemma: Abelian implies solvable.

Review definition of nilpotent algebras.

Definition: \mathfrak{g} is semisimple (s.s.) iff \mathfrak{g} has no nonzero solvable ideals.

Exercise: Simple implies semisimple.

Some remarks:

1. Semisimple algebras \mathfrak{g} will usually have solvable subalgebras.
2. \mathfrak{g} is semisimple iff \mathfrak{g} has no nonzero abelian ideals.

Definition: The Killing form is given by $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow k$ where $\kappa(x, y) = \text{Tr}(\text{ad } x \text{ ad } y)$, which is a symmetric bilinear form.

Lemma: $\kappa([xy], z) = \kappa(x, [yz])$.

Recall that if $\beta : V^{\otimes 2} \rightarrow k$ is any symmetric bilinear form, then its radical is defined by

$$\text{rad}\beta = \left\{ v \in V \mid \beta(v, w) = 0 \ \forall w \in V \right\}.$$

Definition: A bilinear form β is nondegenerate iff $\text{rad}\beta = 0$.

Lemma: $\text{rad}\kappa \trianglelefteq \mathfrak{g}$ is an ideal, which follows by the above associative property.

Theorem: \mathfrak{g} is semisimple iff κ is nondegenerate.

Example: The standard example of a semisimple lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{Tr}(x) = 0 \right\}$.

Note: from now on, \mathfrak{g} will denote a semisimple lie algebra over \mathbb{C} .

Theorem (Weyl): Every finite dimensional representation of a semisimple \mathfrak{g} is completely reducible.

I.e., the category of finite-dimensional representations is relatively uninteresting – there are no extensions, everything is a direct sum, so once you classify the simple algebras (which isn't terribly difficult) then you have complete information.

2 Friday January 10th

Let \mathfrak{g} be a finite dimensional semisimple lie algebra over \mathbb{C} .

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the trace zero $n \times n$ matrices.

Let \mathfrak{h} be a maximal toral subalgebra, where $x \in \mathfrak{g}$ is *toral* if x is semisimple, i.e. $\text{ad } x$ is semisimple (i.e. diagonalizable).

Example: \mathfrak{h} is the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$.

Fact: \mathfrak{h} is abelian, so $\text{ad } \mathfrak{h}$ consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

where $\mathfrak{g}_{\alpha} = \left\{ x \in \mathfrak{g} \mid [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$ where $\alpha \in \mathfrak{h}^{\vee}$ is a linear functional.

Here $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, so $[hx] = 0$ corresponds to zero eigenvalues, and (fact) it turns out that \mathfrak{h} is its own centralizer.

We then obtain a set of roots of $\mathfrak{h}, \mathfrak{g}$ given by $\Phi = \left\{ \alpha \in \mathfrak{h}^{\vee} \mid \alpha \neq 0, \mathfrak{g}_{\alpha} \neq \{0\} \right\}$.

Example: $\mathfrak{g}_{\alpha} = \mathbb{C}E_{ij}$ for some $i \neq j$, the matrix with a 1 in the i, j position and zero elsewhere.

Fact: The restriction $\kappa|_{\mathfrak{h}}$ is nondegenerate, so we can identify $\mathfrak{h}, \mathfrak{h}^\vee$ via κ (can always do this with vector spaces with a nondegenerate bilinear form), where κ maps to another bilinear form (\cdot, \cdot) .

$$\mathfrak{h}^\vee \ni \lambda \iff t_\lambda \in \mathfrak{h}$$

$$\lambda(h) = \kappa(t_\lambda, h) \quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

2.1 Facts About Φ and Root Spaces

Let $\alpha, \beta \in \Phi$ be roots.

1. Φ spans \mathfrak{h}^\vee and does not contain zero.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other scalar multiple of α is in Φ .

Aside:

- $\dim \mathfrak{g}_\alpha = 1$.
- If $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ then there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$ spans a 3-dimensional subalgebra in \mathfrak{sl}_2 , given by $x_\alpha = [0, 1; 0, 0], y_\alpha = [0, 0; 1, 0], h_\alpha = [1, 0; 0, -1]$.
- Under the correspondence $\mathfrak{h} \iff \mathfrak{h}^\vee$ induced by κ , $h_\alpha \iff \alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$. Thus for all $\lambda \in \mathfrak{h}^\vee$,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

- If $\alpha + \beta \neq 0$, then $\kappa(g_\alpha, g_\beta) = 0$.

3. $(\beta, \alpha^\vee) \in \mathbb{Z}$
4. $S_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in \Phi$.

If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Example: If $\alpha = E_{ij}, \beta = E_{jk}$ where $k \neq i$, then $[E_{ij}, E_{jk}] = E_{ik}$.

- \mathfrak{g} is generated as an algebra by the root spaces \mathfrak{g}_α
- Root strings: If $\beta \neq \pm\alpha$, then the roots of the form $\alpha + k\beta$ for $k \in \mathbb{Z}$ form an unbroken string $\alpha - r\beta, \dots, \alpha - \beta, \alpha, \alpha + \beta, \dots, \alpha + \ell\beta$ consisting of at most 4 roots where $r - s = (\alpha, \beta^\vee)$.

Example: The circled roots below form the root string for β :



In general, a subset Φ of a real euclidean space E satisfying conditions (1) through (4) is an (*abstract*) *root system*.

When Φ comes from a \mathfrak{g} , $E := \mathbb{R}\Phi$.

2.1.1 The Root System

There exists a subset $\Delta \subseteq \Phi$ such that

- Δ is a \mathbb{C} -basis for \mathfrak{g}^\vee
- $\beta \in \Phi$ implies that $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either
 - All $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$ or $\beta < 0$.
 - All $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$ or $\beta > 0$.

Δ is called a *simple system*. If $\Delta = \{a_1, \dots, a_\ell\}$ then Φ^+ are the *positive roots*, and $\Phi^+ \ni \beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$,

then the *height* of β is defined as $\sum c_\alpha \in \mathbb{Z}_{>0}$.

Note that $\mathbb{Z}\Phi := \Lambda_r$ is a lattice, and is referred to as the *root lattice*, and $\Lambda_r \subset E = \mathbb{R}\Phi$. We also have $\Phi^+ = \{\beta^\vee \mid \beta \in \Phi\}$, the *dual root system*, is a root system with simple system Δ^\vee .

Important subalgebras of \mathfrak{g} :

- Upper triangular with zero diagonal $\mathfrak{n} = \mathfrak{n}^+ = \sum_{\beta > 0} \mathfrak{g}_\beta$
- Lower triangular with zero diagonal $\mathfrak{n}^- = \sum_{\beta > 0} \mathfrak{g}_{-\beta}$

- Upper triangular, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra
- Lower triangular, $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

There is thus a triangular (Cartan) decomposition, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Fact: If $\beta \in \Phi^+ \setminus \Delta$, and if $\alpha \in \Delta$ such that $(\beta, \alpha^\vee) > 0$, then $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$ has height strictly less than the height of β .

By root strings, $\beta - \alpha \in \Phi^+$ is positive root of height one less than β , yielding a way to induct on heights (useful technique).

2.1.2 Weyl Groups

For $\alpha \in \Phi$, define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\rightarrow \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in E perpendicular to α :

Note that $S_\alpha^2 = \text{id}$.

Define W as the subgroup of $\text{gl}(E)$ generated by all s_α for $\alpha \in \Phi$, this is the *Weyl group* of \mathfrak{g} or Φ , which is finite and $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$ is generated by simple reflections.

By (4), W leaves Φ invariant. In fact W is a finite Coxeter group with generators $S = \{s_\alpha \mid \alpha \in \Delta\}$ and defining relations $(s_\alpha s_\beta)^{m(\alpha, \beta)} = 1$ for $\alpha, \beta \in \Delta$ where $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$ and $m(\alpha, \alpha) = 1$.

Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group*.

3 Monday January 13th

3.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over \mathbb{C} . We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group $W = \langle s_\beta \mid \beta \in \Phi \rangle = \langle s_\alpha \mid \alpha \in \Delta \rangle$ where $\Delta = \{a_i\}$ are the simple roots. For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) := n$ where $\ell(1) := 0$.

Facts:

1. $\ell(w)$ is the size of the set $\{\beta \in \Phi^+ \mid w\beta < 0\}$
 - The above set is equal to $\Phi^+ \cap w^{-1}\Phi^-$.



Figure 1: Image

- In particular, for $\beta \in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
 - Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.
2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \cap w\Phi$ (replacing w^{-1} with w)
 3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = |\Phi^+|$ and $w_0(\Phi^+) = \Phi^-$.
- Also $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$\begin{aligned}\ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0\end{aligned}$$

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

3.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \{s_\alpha \mid \alpha \in \Delta\}$. Then define

$$T := \bigcup_{w \in W} wSw^{-1} = \{s_\beta \mid \beta \in \Phi^+\}.$$

This is the set of *all* reflections in W through hyperplanes in E .

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w , and refer to this as “the covering relation”. So $w' \rightarrow w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that $w' = w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_n = w$. This is called the **Bruhat-Chevalley order** on W .

Corollary: $w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order*. In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

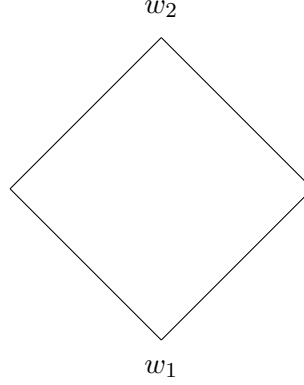
Recall that lie algebras yield finite crystallographic coxeter groups.

Properties: For (W, S) a coxeter group,

- a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \dots s_n$ for w , where a subexpression is any subcollection of s_i in the same order.

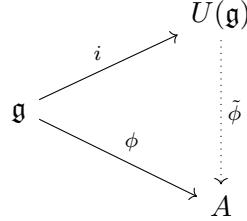
Note that this implies that 1 is not only a minimal element in this order, but an infimum.

- b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.
- c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\{w \in W \mid w_1 < w < w_2\}$ is either 0 or 2.



3.3 Properties of Universal Enveloping Algebras

Let \mathfrak{g} be any lie algebra, and $\phi : \mathfrak{g} \rightarrow A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:



Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \rightarrow U(\mathfrak{g}_2)$, where \mathfrak{g} generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle [x, y] - x \otimes y - y \otimes x \mid x, y \in \mathfrak{g} \rangle.$$

Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\text{ad } x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Big Theorem (Poincaré-Birkhoff-Witt, i.e. PBW): If $\{x_1, \dots, x_n\}$ is a basis for \mathfrak{g} , then $\{x_1^{t_1}, \dots, x_n^{t_n} \mid t_i \in \mathbb{Z}^+\}$ (noting that $x^n = x \otimes x \otimes \dots \otimes x$ and \mathbb{Z}^+ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If \mathfrak{g} is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for \mathfrak{g} , then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \rightarrow \text{gl}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

Notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

3.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the \mathbb{R} -span of the roots. We also have the **integral weight lattice**

$$\Lambda = \left\{ \lambda \in E \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in \Phi (\text{or } \Phi^+ \text{ or } \Delta) \right\}.$$

There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and \mathfrak{h}^\vee . We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+ \Delta = \mathbb{Z}^+ \Phi^+$. For a basis $\Delta = \{\alpha_1, \dots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a \mathbb{Z} -basis for Λ . Then Λ is a free abelian group of rank ℓ , and $\Lambda^+ = \mathbb{Z}^+ w_1 + \dots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.

4 Wednesday January 15th

4.1 Review

The Weyl vector is given by $\rho = \bar{w}_1 + \dots + \bar{w}_\ell = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \Lambda^+$.

- If $\alpha \in \Delta$ then $(\rho, \alpha^\vee) = 1$
- $s_\alpha(\rho) = \rho - \alpha$.

Let $\lambda \in \Lambda^+$; a few facts:

1. The size of $\left\{ \mu \in \Lambda^+ \mid \mu \leq \lambda \right\}$ (with the partial order from last time) is finite.
2. $w\lambda < \lambda$ for all $w \in W$.

The Weyl chamber (for a fixed root, $E = \text{Euclidean space}$) is $C = \left\{ \lambda \in E \mid (\lambda, \alpha) > 0 \forall \alpha \in \Delta \right\}$ (Note that the hyperplane splits E into connected components, we mark this component as distinguished.)

- A connected component of the union of hyperplanes is orthogonal to roots
- They're in bijection with Δ
- They're permuted simply transitively by W .

And \bar{C} denotes the fundamental domain.

4.2 Weight Representations

For $\lambda \in \mathfrak{h}^\vee$, we let $M_\lambda = \{v \in M \mid h \cdot v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$ denote a *weight space* of M if $M_\lambda \neq 0$. In this case, λ is a *weight* of M . The dimension of M_λ is the *multiplicity* of λ in M , and we define the set of weights as $\Pi(M) = \{\lambda \in \mathfrak{h}^\vee \mid M_\lambda \neq 0\}$.

Example if $M = \mathfrak{g}$ under the adjoint action, then $\Pi(M) = \Phi \cup \{0\}$.

Remark: The weight vectors for distinct weights are linearly independent. Thus there is a \mathfrak{g} -submodule given by $\sum_\lambda M_\lambda$, which is in fact a direct sum.

Note: It may not be the case that $\sum_\lambda M_\lambda = M$, and can in fact be zero, although this is an odd situation. See Humphreys #1, #20.2, p. 110.

In our case, we'll have a *weight module* $M = \bigoplus_\lambda M_\lambda$, so $\mathfrak{h} \curvearrowright M$ semisimply.

4.3 Finite-dimensional Modules

Recall Weyl's complete reducibility theorem, which implies that any finite dimensional \mathfrak{g} -module is a weight module. In fact, $\mathfrak{n}, \mathfrak{n}^- \curvearrowright M$ nilpotently.

Some facts:

- $\Pi(M) \subset \Lambda$ is a subset of the integral lattice.
- $\Pi(M)$ is W -invariant.
- $\dim M_\lambda = \dim M_{w\lambda}$ for any $\lambda \in \Pi(M), w \in W$.

4.4 Simple Finite Dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules

Fix the standard basis $\{x, h, y\}$ of $\mathfrak{sl}(2, \mathbb{C})$ with $[hx] = 2x, [hy] = -2y, [xy] = h$. Since $\dim \mathfrak{h} = 1$, there is a bijection $\mathfrak{h}^\vee \leftrightarrow \mathbb{C}$, $\Lambda \leftrightarrow \mathbb{Z}$, and $\Lambda_r \leftrightarrow 2\mathbb{Z}$ with $\alpha \rightarrow 2$ and $\rho \rightarrow 1$.

There is a correspondence between weights and simple modules:

$$\begin{aligned} \{\text{Isomorphism classes of simple modules}\} &\iff \Lambda^+ = \{0, 1, 2, 3, \dots\} \\ L(\lambda) &\iff \lambda. \end{aligned}$$

Moreover, $L(\lambda)$ has a 1-dimensional weight spaces with weights $\lambda, \lambda - 2, \dots, -\lambda$ and thus $\dim L(\lambda) = \lambda + 1$.

Examples:

- $L(0) = \mathbb{C}$, the trivial representation,
- $L(1) = \mathbb{C}^2$, the natural representation where $\mathfrak{sl}(2, \mathbb{C})$ acts by matrix multiplication,
- $L(2) = \mathfrak{g}$, the adjoint representation.

Choose a basis $\{v_1, \dots, v_\lambda\}$ for $L(\lambda)$ so that $\mathbb{C}v_0 = M_\lambda, \mathbb{C}v_1 = M_{\lambda-2}, \dots, \mathbb{C}v_\lambda M_{-\lambda}$. Then



Figure 2: Image

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$, where $v_{-1} := 0$
- $y \cdot v_i = (i + 1)v_{i+1}$ where $v_{\lambda+1} := 0$.

We then say $L(\lambda)$ is a highest weight module, since it is generated by a highest weight vector λ . Then $W = \{1, s_\alpha\}$, where s_α is reflection through 0 by the identification $\alpha = 2$.

5 Chapter 1: Category \mathcal{O} Basics

The category of $U(\mathfrak{g})$ -modules is too big. Motivated by work of Verma in 60s, started by Bernstein-Gelfand-Gelfand in the 1970s. Used to solve the Kazhdan-Lusztig conjecture.

5.1 Axioms and Consequences

Definition: \mathcal{O} is the full subcategory of $U(\mathfrak{g})$ modules consisting of M such that

1. M is finitely generated as a $U(\mathfrak{g})$ -module.
2. M is \mathfrak{h} -semisimple, i.e. M is a weight module $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$.
3. M is locally n -finite, i.e. the dimension of $U(\mathfrak{n})v < \infty$ for all $v \in M$.



Figure 3: Image

Example: If $\dim M < \infty$, then M is \mathfrak{h} -semisimple, and axioms 1, 3 are obvious.

Lemma: Let $M \in \mathcal{O}$, then

4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$.
5. There exist $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^\vee$ such that $\Pi(M) \subset \bigcup_{i=1}^r (\lambda_i - \mathbb{Z}^+ \Phi^+)$.

Proof: By axiom 2, every $v \in M$ is a finite sum of weight vectors in M . We can thus assume that our finite generating set consists of weight vectors. We can then reduce to the case where M is generated by a single weight vector v . So consider $U(\mathfrak{g}) \cdot v$. By the PBW theorem, there is a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$.

By axiom 3, $U(\mathfrak{n}) \cdot v$ is finite dimensional, so there are finitely many weights of finite multiplicity in the image. Then $U(\mathfrak{h})$ acts by scalar multiplication, and $U(\mathfrak{n}^-)$ produces the “cones” that result in the tree structure:

A weight of the form $\mu = \lambda_i - \sum n_i \alpha_i$ can arise from $y_{n_1}^{n_1} \dots$.

6 Friday January 17th

Let M

1. be finitely generated,
2. semisimple $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$,
3. locally finite
4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$,
5. satisfy the forest condition for weights.



Figure 4: Image

Theorem:

- a. \mathcal{O} is Noetherian (ascending chain condition on submodules, i.e. no infinite filtrations by submodules)
- b. \mathcal{O} is closed under quotients, submodules, finite direct sums
- c. \mathcal{O} is abelian (similar to a category of modules)
- d. If $M \in \mathcal{O}$, $\dim L < \infty$, then $L \otimes M \in \mathcal{O}$ and the endofunctor $M \mapsto L \otimes M$ is exact
- e. If $M \in \mathcal{O}$ is locally $Z(\mathfrak{g})$ -finite (recall: this is the center of $U(\mathfrak{g})$), i.e. $\dim \text{span} Z(\mathfrak{g}) \cdot v < \infty$ for all $v \in M$.
- f. $M \in \mathcal{O}$ is finitely generated module. (?)

Proofs of a and b: See BA II, page 103.

Proof of c: Implied by (b), BA II Page 330.

Proof of d: Can check that $L \otimes M$ satisfies 2 and 3 above. Need to check first condition. Take a basis $\{v_i\}$ for L and $\{w_j\}$ a finite set of generators for M . The claim is that $B = \{v_i \otimes w_j\}$ generates $L \otimes M$. Let N be the submodule generated by B .

For any $v \in V$, $v \otimes w_j \in N$. For arbitrary $x \in \mathfrak{g}$, compute $x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + x \otimes (v \cdot w_j)$. Since the LHS is in N and the first term on the RHS is in N , the entire RHS is in N . By iterating, we find that $v \otimes (u \cdot w_j) \in N$ for all PBW monomials u . So $L \otimes M \in \mathcal{O}$.

Proof of e: Since $v \in M$ is a sum of weight vectors, wlog we can assume $v \in M_\lambda$ is a weight vector (where $\lambda \in \mathfrak{h}^\vee$). For any central element $z \in Z(\mathfrak{g})$, we can compute $h \cdot (z \cdot v) = z \cdot (h \cdot v) = z \cdot \lambda(h)v = \lambda(h)z \cdot v$. Thus $z \cdot v \in M_\lambda$. By (4), we know that $\dim M_\lambda < \infty$, so $\dim \text{span} Z(\mathfrak{g}) \cdot v < \infty$ as well.

Proof of f: By 5, M is generated by a finite dimensional $U(\mathfrak{b})$ submodule N . Since we have a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{b})$, there is a basis of weight vectors for N that generates M as a $U(\mathfrak{n}^-)$ module.

6.1 Highest Weight Modules

Definition: A maximal vector $v^+ \in M \in \mathcal{O}$ is a nonzero vector such that $\mathfrak{n} \cdot v^+ = 0$.

Note: By 2 and 3, every nonzero $M \in \mathcal{O}$ has a maximal vector.

Definition: A highest weight module M of highest weight λ is a module generated by a maximal vector of weight λ , i.e. $M = U(\mathfrak{g})v^+ = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})v^+ = U(\mathfrak{n}^-)v^+$.

Theorem: Let $M = U(\mathfrak{n}^-)v^+$ be a highest weight module, where $v^+ \in M_\lambda$. Fix $\Phi^+ = \{\beta_1, \dots, \beta_n\}$ with root vectors $y_i \in \mathfrak{g}_{\beta_i}$.

- a. M is the \mathbb{C} -span of PBW monomials $\langle y_1^{t_1} \cdots y_m^{t_m} \rangle$ of weight $\lambda - \sum t_i \beta_i$. Thus M is a module.
- b. All weights μ of M satisfy $\mu \leq \lambda$
- c. $\dim M_\mu < \infty$ for all $\mu \in T(M)$, and $\dim M_\lambda = 1$. In particular, property (3) holds and $M \in \mathcal{O}$.
- d. Every nonzero quotient of M is a highest-weight module of highest weight λ .
- e. Every submodule of M is a weight module, and any submodule generated by a maximal vector with $\mu < \lambda$ is proper. If M is semisimple, then the set of maximal weight vectors equals $\mathbb{C}^\times v^+$.
- f. M has a unique maximal submodule N and a unique simple quotient L , thus M is indecomposable.

- g. All simple highest weight modules of highest weight λ are isomorphic. For such M , $\dim \text{End}(M) = 1$. (Category \mathcal{O} version of Schur's Lemma, generalizes to infinite dimensional case)

Proofs of a to e: Either obvious or follows from previous results. First few imply M is in \mathcal{O} , and we know the latter hold for such modules.

Proof of f: N is a sum of submodules, so $N = \sum M_i$, proper submodules of M . So take $L = M/N$. To see indecomposability, there exists a better proof in section 1.3.

Proof of g: Let $M_1 = U(\mathfrak{n}^-)v_1^+$ and M_2 be define similarly, where the $v_i \in (M_i)_\lambda$ have the same weight. Then $M_0 := M_1 \oplus M_2$ implies that $v^+ := (v_1^+, v_2^+)$ is a maximal vector for M_0 . So $N := U(\mathfrak{n}^-)v^+$ is a highest weight module of highest weight λ .

We have the following diagram:



and since e.g. $N \rightarrow M_1$ is not the zero map, it is a surjection.

By (f), N is a unique simple quotient, so this forces $M_1 \cong M_2$. Since M is simple, any nonzero \mathfrak{g} -endomorphism ϕ must be an isomorphism, and so we take $v^+ \mapsto cv^+$ for some $c \neq 0$. Note that since ϕ is also a \mathfrak{h} -morphism, we have $\dim M_\lambda = 1$.

Since v^+ generates M and $\phi(u \cdot v^+) = u\phi(v^+) = cu \cdot v^+$, thus ϕ is multiplication by a constant.

7 Wednesday January 22nd

Note: Try problems 1.1 and 1.3*.

Recall: In category \mathcal{O} , we have finite dimensional, semisimple modules over \mathbb{C} with triangular decompositions.

If M is any $U(\mathfrak{g})$ module than a $v^+ \in M_\lambda$ a weight vector (so $\lambda \in \mathfrak{h}^\vee$) is primitive iff $\mathfrak{n} \cdot v^+ = 0$. Note: it doesn't have to be of maximal weight. M is a highest weight module of highest weight λ iff it's generated over $U(\mathfrak{g})$ as an associative algebra by a maximal vector v^+ of weight λ . Then $M = U(\mathfrak{g}) \cdot v^+$.

See structure of highest weight modules, and irreducibility.

Corollary: If $0 \neq M \in \mathcal{O}$, then M has a finite filtration with quotients highest weight modules, i.e. $M_0 \subset M_1 \subset \dots \subset M_n = M$ with M_i/M_{i-1} highest weight modules. Note that the quotients are not necessarily simple, so this isn't a composition series, although we'll show such a series exists later.

Theorem: Let V be the \mathfrak{n} submodule of M generated by a finite set of weight vectors which generate M as a $U(\mathfrak{g})$ module. I.e. take the finite set of weight vectors and act on them by $U(\mathfrak{n})$.

Then $\dim_{\mathbb{C}} V < \infty$ since M is locally \mathfrak{n} -finite.

Proof: Induction on $n = \dim V$. If $n = 1$, M itself is a highest weight module.

Note that \mathfrak{n} increases weights.

For $n > 1$, choose a weight vector $v_1 \in V$ of weight λ which is maximal among all weights of V . Set $M_1 := U(\mathfrak{g})v_1$; this is a highest weight submodule of M of highest weight λ . (\mathfrak{n} has to kill v_1 , otherwise it increases weight and v_1 wouldn't be maximal.)

Let $\overline{M} = M/M_1 \in \mathcal{O}$, this is generated by the image of \overline{V} of V and thus $\dim \overline{V} < \dim V$. By the IH, \overline{M} has the desired filtration, say $0 \subset \overline{M}_2 \subset \overline{M}_{n-1} \subset \overline{M}_n = \overline{M}$. Let $\pi : M \rightarrow M/M_1$, then just take the preimages $\pi^{-1}(\overline{M}_i)$ to be the filtration on M .

Note: by isomorphism theorems, the quotients in the series for M are isomorphic to the quotients for \overline{M} .

7.1 Verma and Simple Modules

Constructing *universal* highest weight modules using “algebraic induction”. Start with a nice subalgebra of \mathfrak{g} then “induce” via \otimes to a module for \mathfrak{g} .

Recall $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, here $\mathfrak{h} \oplus \mathfrak{n}$ is the Borel subalgebra \mathfrak{b} , and \mathfrak{n} corresponds to a fixed choice of positive roots in Φ^+ with basis Δ . Then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$. Given any $\lambda \in \mathfrak{h}^\vee$, let \mathbb{C}_λ be the 1-dimensional \mathfrak{h} -module (i.e. 1-dimensional \mathbb{C} -vector space) on which \mathfrak{h} acts by λ .

Let $\{1\}$ be the basis for \mathbb{C} , so $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. Then there is a map $\mathfrak{b} \rightarrow \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$, so make \mathbb{C}_λ a \mathfrak{b} -module via this map. This “inflate” \mathbb{C}_λ into a 1-dimensional \mathfrak{b} -module.

Note that \mathfrak{h} is solvable, and by Lie’s Theorem, every finite dimensional irreducible \mathfrak{b} -module is of the form \mathbb{C}_λ for some $\lambda \in \mathfrak{h}^\vee$.

Definition: $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda := \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ is the *Verma module of highest weight λ* . This process is called algebraic/tensor induction. This is a $U(\mathfrak{g})$ module via left multiplication, i.e. acting on the first tensor factor.

Since $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$, we have $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$, but at what level of structure?

- As a vector space (clear)
- As a \mathfrak{n}^- -module via left multiplication
- As a \mathfrak{h}^- -module via the \otimes action.

In particular, $M(\lambda)$ is a *free* $U(\mathfrak{n}^-)$ -module of rank 1.

Note: this always happens when tensoring with a vector space?

Consider $v^+ := 1 \otimes 1 \in M(\lambda)$ (note that $U(\mathfrak{n}^-)$ is not homogeneous, so not graded, but does have a filtration). Then v^+ is nonzero, and freely generates $M(\lambda)$ as a $U(\mathfrak{n}^-)$ -module. Moreover $\mathfrak{n} \cdot v^+ = 0$ since for $x \in \mathfrak{g}_\beta$ for $\beta \in \Phi^+$, we have

$$\begin{aligned}
x(1 \otimes 1) &= x \otimes 1 \\
&= 1 \otimes x \cdot 1 \quad \text{since } x \in \mathfrak{b} \\
&= 1 \otimes 0 \implies x \in \mathfrak{n} \\
&= 0,
\end{aligned}$$

and for $h \in \mathfrak{h}$,

$$\begin{aligned}
h(1 \otimes 1) &= h1 \otimes 1 \\
&= 1 \otimes h1 \\
&= 1 \otimes \lambda(h)1 \\
&= \lambda(h)v^+.
\end{aligned}$$

So $M(\lambda)$ is a highest weight module of highest weight λ , and thus $M(\lambda) \in \mathcal{O}$.

Observation: Any weight $\lambda \in \mathfrak{h}^\vee$ is the highest weight of some $M \in \mathcal{O}$. Let $\Pi(M)$ denote the set of weights of a module, then $\Pi(M(\lambda)) = \lambda - \mathbb{Z}^+ \Phi^+$.

By PBW, we can obtain a basis for $M(\lambda)$ as $\{y_1^{t_1} \cdots y_m^{t_m} v^+ \mid t_i \in \mathbb{Z}^+\}$. Taking a fixed ordering $\{\beta_1, \dots, \beta_m\} = \Phi^+$, then $0 \neq y_i \in \mathfrak{g}_{-\beta_i}$. Then every weight of this form is a weight of some $M(\lambda)$, and every weight of $M(\lambda)$ is of this form: $\lambda - \sum t_i \beta_i$.

Remark: The functor $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \cdot$ from the category of finite-dimensional \mathfrak{g} -semisimple \mathfrak{b} -modules to \mathcal{O} is an exact functor, since it is naturally isomorphic to $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \cdot$ (which is clearly exact?)

Alternate construction of $M(\lambda)$: Let I be a left ideal of $U(\mathfrak{g})$ which annihilates v^+ , so $I = \langle \mathfrak{n}, h - \lambda(h) \cdot 1 \mid h \in \mathfrak{h} \rangle$. Since v^+ generates $M(\lambda)$ as a $U(\mathfrak{g})$ -module, then (by a standard ring theory result) $M(\lambda) = U(\mathfrak{g})/I$, since I is the annihilator of $M(\lambda)$.

Theorem (Universal property of $M(\lambda)$): Let M be any highest weight module of highest weight λ generated by v . Then $I \cdot v = 0$, so I is the annihilator of v and thus M is a quotient of $M(\lambda)$. Thus $M(\lambda)$ is universal in the sense that every other highest weight module arises as a quotient of $M(\lambda)$.

By theorem 1.2, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ (nonstandard notation) and a unique simple quotient $L(\lambda)$ (standard notation).

Theorem: Every simple module in \mathcal{O} is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$ and is determined uniquely up to isomorphism by its highest weight. Moreover, there is an analog of Schur's lemma: $\dim \text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$, i.e. it's 1 iff $\lambda = \mu$ and 0 otherwise.

Note: up to isomorphism, we've found all of the simple modules in \mathcal{O} , and most are finite-dimensional.

Proof: Next class.

8 Friday January 24th

A standard theorem about classifying simple modules in category \mathcal{O} :

Theorem: Every simple module in \mathcal{O} is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$, and is determined uniquely up to isomorphism by its highest weight. Moreover, $\dim \text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}$.

Proof: Let $L \in \mathcal{O}$ be irreducible. As observed in 1.2, L has a maximal vector v^+ of some weight λ .

Recall: can increase weights and reach a maximal in a finite number of steps.

Since L is irreducible, L is generated by that weight vector, i.e. $LU(\mathfrak{g}) \cdot v^+$, so L must be a highest weight module.

Standard argument: use triangular decomposition.

By the universal property, L is a quotient of $M(\lambda)$. But this means $L \cong L(\lambda)$, the unique irreducible quotient of $M(\lambda)$.

By Theorem 1.2 part g (see last Friday), $\dim \text{End}_{\mathcal{O}}(L(\lambda)) = 1$ and $\text{hom}_{\mathcal{O}}(L(\mu), L(\lambda)) = 0$ since both entries are irreducible. ■

Proof of Theorem 1.2 f:

Statement: A highest weight module M is indecomposable, i.e. can't be written as a direct sum of two nontrivial proper submodules.

Suppose $M = M_1 \oplus M_2$ where M is a highest weight module of highest weight λ . Category \mathcal{O} is closed under submodules, so M_i are weight modules and have weight-space decompositions. But M_λ is 1-dimensional (triangular decomposition, only \mathbb{C} acts), and thus $M_\lambda \subset M_1$. Since M_λ is a highest weight module, it generates the entire module, so $M \subset M_1$. The reverse holds as well, so $M = M_1$ and this forces $M_2 = 0$. ■

8.1 1.4: Maximal Vectors in Verma Modules

1.5: Examples in the case $\mathfrak{sl}(2)$, over \mathbb{C} as usual.

First, some review from lie algebras.

Let \mathfrak{g} be any lie algebra, and take $u, v \in U(\mathfrak{g})$. Recall that we have the formula $uv = [uv] + vu$, where we use the definition $[uv] = uv - vu$.

Let x, y_1, y_2 be in \mathfrak{g} , what is $[x, y_1 y_2]$? Use the fact that $\text{ad } x(y_1, y_2)$ acts as a derivation, and so $[x, y_1 y_2] = [x y_1] y_2 + y_1 [x y_2]$, which is a bracket entirely in the Lie algebra. This extends to an action on $U(\mathfrak{g})$ by the product rule.

Recall that $\mathfrak{sl}(2)$ is spanned by $y = [0, 0; 1, 0], h = [1, 0; 0, -1], x = [0, 1; 0, 0]$, where each basis vector spans $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}$ respectively. Then $[xy] = h, [hx] = 2x, [hy] = -2y$, so $E_{ij}E_{kl} = \delta_{jk}E_{il}$ (should be able to compute easily!).

Then $\mathfrak{h} = \mathbb{C}$, so $\mathfrak{h}^\vee \cong \mathbb{C}$ where $\lambda \mapsto \lambda(h)$. So we identify λ with a complex number, this is kind of like a bundle of Verma modules over \mathbb{C} .

Consider $M(1)$, then $\lambda = 1$ will denote $\lambda(h) = 1$. As in any Verma module, $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda}$. We can think of v^+ as $1 \otimes 1$, with the action $yv^+ = y1 \otimes 1$. Note that y has weight -2 .

Weight	Basis
1	v^+
-1	yv^+
-3	y^2v^+
-5	y^3v^+

Consider how $x \curvearrowright y^2v^+$. Note that x has weight $+2$. We have

$$\begin{aligned}
x \cdot y^2v^+ &= x \cdot y^2 \otimes 1_{\lambda} \\
&= ([xy^2] + y^2x) \otimes 1 \\
&= ([xy]y + y[xy]) \otimes 1 + y^2 \otimes x \cdot 1 \quad \text{moving } x \text{ across the tensor because ?} \\
&= ([xy]y + y[xy]) \otimes 1 + 0 \quad \text{since } x \text{ is maximal} \\
&= (hy + yh) \otimes 1 \\
&= hy \otimes 1 + y \otimes h \cdot 1 \\
&= hy \otimes 1 + \lambda(h)1 \\
&= hy \otimes 1 + 1 \\
&= ([xy] + yh) \otimes 1 + y \otimes 1 \\
&= -2y \otimes 1 + y \otimes 1 + y \otimes 1 \\
&= 0.
\end{aligned}$$

So y moves us downward through the table, and x moves upward, except when going from $-3 \rightarrow -1$ in which case the result is zero!

Thus there exists a morphism $\phi : M(-3) \rightarrow M(1)$, with image $U(\mathfrak{g})y^2v^+ = U(\mathfrak{n}^-)y^2v^+$. So the image of ϕ is everything spanned by the bases in the rows $-3, -5, \dots$, which is exactly $M(-3)$. So $M(-3) \hookrightarrow M(1)$ as a submodule.

Motivation for next section: we want to find Verma modules which are themselves submodules of Verma modules.

It turns out that $\text{im } \phi \cong N(1)$. We should have $M(1)/N(1) \cong L(1)$. What is the simple module of weight 1 for $\mathfrak{sl}(2)$? The weights of $L(n)$ are $n, n-2, n-4, \dots, -n$, so the representations are parameterized by $n \in \mathbb{Z}^+$. These are the Verma modules for $\mathfrak{sl}(2)$. What happens is that $y \curvearrowright -n \rightarrow -n-2$ gives a maximal vector, so the calculation above roughly goes through the same way. So we'll have a similar picture with $L(n)$ at the top.

8.2 Back to 1.4

Question 1: What are the submodules of $M(\lambda)$?

Question 2: What are the Verma submodules $M(\mu) \subset M(\lambda)$? Equivalently, when do maximal vectors of weight $\mu < \lambda$ (the interesting case) lie in $M(\lambda)$?

Question 3: As a special case, when do maximal vectors of weight $\lambda - k\alpha$ for $\alpha \in \Delta$ lie in $M(\lambda)$ for $k \in \mathbb{Z}^+$?

Fix a Chevalley basis for \mathfrak{g} (see section 0.1) $h_1, \dots, h_\ell \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi^+$. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ and let $x_i = x_{\alpha_i}, y_i = y_{\alpha_i}$ be chosen such that $[x_i y_i] = h_i$.

Lemma: For $k \geq 0$ and $1 \leq i, j \leq \ell$, then

- a. $[x_j, y_i^{k+1}] = 0$ if $j \neq i$
- b. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$.
- c. $[x_i, y_i^{k+1}] = -(k+1)y_i(k \cdot 1 - h_i)$.

Proof (sketch):

Both easy to prove by induction since $[x_j, y_i] \rightarrow \alpha_j - \alpha_i \notin \Phi$ is a difference of simple roots.

For $k = 0$, all identities are easy. For $k > 0$, an inductive formula that uses the derivation property, which we'll do next class.

9 Monday January 27th

9.1 Section 1.4

Fix $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, $x_i \in \mathfrak{g}_{\alpha_i}$ and $y_i \in \mathfrak{g}_{-\alpha_i}$ with $h_i = [x_i y_i]$.

Lemma: For $k \geq 0$ and $1 \leq i, j \leq \ell$,

- a. $[x_j y_i^{k+1}] = 0$ if $j \neq i$
- b. $[h_j y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$
- c. $[x_i y_i^{k+1}] = (k+1)y_i^k(k \cdot 1 - h_i)$.

Sketch of proof for (c):

By induction, where $k = 0$ is clear.

$$\begin{aligned}
 [x + i y_i^{k+1}] &= [x_i y_i] y_i^k + y_i [x_i y_i^k] \\
 &= h_i y_i^k + y_i (-k y_i^{k-1} ((k-1)1 - h_i)) \quad \text{by I.H.} \\
 &= (k+1) y_i^k h_i - (k^2 - k + 2k) y_i^k \\
 &= -(k+1) y_i^k (k \cdot 1 - h_i)
 \end{aligned}$$

■

Proposition: Suppose $\lambda \in \mathfrak{h}^\vee$, $\alpha \in \Delta$, and $n := (\lambda, \alpha^\vee) \in \mathbb{Z}^+$. Then in $M(\lambda)$, $y_\alpha^{n+1} v^+$ is a maximal weight vector of weight $\mu := \lambda - (n+1)\alpha < \lambda$.

Note this is free as an $U(\mathfrak{n}^-)$ -module, so $v^+ \neq 0$. Note that $n = \lambda(h_\alpha)$.

By the universal property, there is a nonzero homomorphism $M(\mu) \rightarrow M(\lambda)$ with image contained in $N(\lambda)$, the unique maximal proper submodules of $M(\lambda)$.

Proof: Say $\alpha = \alpha_i$. Fix $j \neq i$.

$$\begin{aligned} x_i y_\alpha^{n+1} \otimes 1 &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes x_j \cdot 1 \\ &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes 0 \quad \text{by a} \\ &= 0. \end{aligned}$$

$$\begin{aligned} x_i y_i^{n+1} \otimes 1 &= [x_i y_i^{n+1} \otimes 1] \\ &= -(n+1) y_i^n (n \cdot 1 - h_i) \otimes 1 \\ &= -(n+1)(n - \lambda(h_i)) 1 \otimes 1 \\ &:= -(n+1)(\lambda(h_i) - \lambda(h_i)) 1 \otimes 1 \\ &= 0. \end{aligned}$$

Since g_{α_j} generate \mathfrak{n} as a Lie algebra, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. This shows that $\mathfrak{n} \cdot y_i^{n+1} v^+ = 0$, and the weight of $y_i^{n+1} v^+$ is $\lambda - (n+1)\alpha_i$. So y_i^{n+1} is a maximal vector of weight μ . The universal property implies there is a nonzero map $M(\mu) \rightarrow M(\lambda)$ sending highest weight vectors to highest weight vectors and preserving weights. The image is proper since all weights of M_μ are less than or equal to $\mu < \lambda$. ■

Consider $\mathfrak{sl}(2)$, then $M(1) \supset M(-3)$. Note that reflecting through 0 doesn't send 1 to -3, but shifting the origin to -1 and reflecting about that with $s_\alpha \cdot$ fixes this problem. Note that $L(1)$ is the quotient.

For $\lambda \in \mathfrak{h}^\vee$ and $\alpha \in \Delta$, we can compute $s_\alpha \cdot \lambda := s_\alpha(\lambda + \rho) - \rho$ where $\rho = \sum_{j=1}^{\ell} e_i$. Then $(w_j, \alpha_i^\vee) = \delta_{ij}$ and $(\rho, \alpha_i^\vee) = 1$.

$$\begin{aligned} s_\alpha \cdot \lambda &= s_\alpha(\lambda + \rho) - \rho \\ &= (\lambda + \rho) - (\lambda + \rho, \alpha^\vee) \alpha - \rho \\ &= \lambda + \rho - ((\lambda < \alpha^\vee) + 1) \alpha - \rho \\ &= \lambda - (n+1) \alpha \\ &= \mu. \end{aligned}$$

So this gives a well-defined, nonzero map $M(s_\alpha \cdot \lambda) \rightarrow M(\lambda)$ for $s_\alpha \cdot \lambda < \lambda$.

Corollary: Let λ, α, n be as in the above proposition. Let \bar{v}^+ now be a maximal vector of weight λ in $L(\lambda)$. Then $y_\alpha^{n+1} \bar{v}^+ = 0$.

Proof: If not, then this would be a maximal vector, since it's the image of the vector $y_i^{n+1} v^+ \in M(\lambda)$ under the map $M(\lambda) \rightarrow L(\lambda)$ of weight $\mu < \lambda$. Then it would generate a proper submodules of $L(\lambda)$, but this is a contradiction since $L(\lambda)$ is irreducible.



Figure 5: Image

9.2 Section 1.5

Example: $\mathfrak{sl}(2)$. What do Verma modules $M(\lambda)$ and their simple quotients $L(\lambda)$ look like?

Fix a Chevalley basis $\{y, h, x\}$ and let $\lambda \in \mathfrak{h}^\vee \cong \mathbb{C}$.

Fact 1: For $v^+ = 1 \otimes 1_\lambda$, we have $M(\lambda) = U(\mathfrak{n}^-)v^+ = \mathbb{C} \langle y^i v^+ \mid i \in \mathbb{Z}^+ \rangle$ is a basis for $M(\lambda)$ with weights $\lambda - 2i$ where α corresponds to 2. So the weights of $M(\lambda)$ are $\lambda, \lambda - 2, \lambda - 4, \dots$ each with multiplicity 1.

Letting $v_i = \frac{1}{i!} y^i v^+$ for $i \in \mathbb{Z}^+$; this is a basis for $M(\lambda)$. Using the lemma, we have

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i \\ y \cdot v_i &= (i + 1)v_{i+1} \\ x \cdot v_i &= (\lambda - i + 1)v_{i-1}. \end{aligned}$$

Note that these are the same for *finite-dimensional* $\mathfrak{sl}(2)$ -modules, see section 0.9.

Fact 2: We know from the proposition that if $\lambda \in \mathbb{Z}^+$, i.e. $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$, then $M(\lambda)$ has a maximal vector of weight $\lambda - (n + 1)\alpha = \lambda - (\lambda + 1)2 = -\lambda - 2 = s_\alpha \cdot \lambda$.

Exercise: Check that this maximal vector generates the maximal proper submodule $N(\lambda) = M(-\lambda - 2)$.

So the quotient $L(\lambda) = M(\lambda)/N(\lambda) = M(\lambda)/M(-\lambda - 2)$ has weights $\lambda, \lambda - 2, \dots, -\lambda + 2, -\lambda$. So when $\lambda \in \mathbb{Z}^+$, $L(\lambda)$ is the familiar simple $\mathfrak{sl}(2)$ -module of highest weight λ .

Fact 3: When $\lambda \notin \mathbb{Z}^+$,

- $N(\lambda) = \{0\}$,
- $M(\lambda) = L(\lambda)$,
- $M(\lambda)$ is irreducible
- $L(\lambda)$ is infinite dimensional.

Proof: Argue by contradiction. If not, $M(\lambda) \supset M \neq 0$ is a proper submodule. So $M \in \mathcal{O}$, and thus M has a maximal weight vector w^+ , and by the restriction of weights for modules in \mathcal{O} , we know w^+ has height $\lambda - 2m$ for some $m \in \mathbb{Z}^+$. Then $w^+ = cv_i$ where $0 \neq c \in \mathbb{C}$, and taking $v_{-1} := 0$ and $x \cdot v_i = (\lambda - i + 1)v_{i-1}$ for $i \geq 1$, so $\lambda = i - 1 \implies \lambda \in \mathbb{Z}^+$.

10 Friday January 31st

(3) A useful formula: $L(\lambda)^\vee \cong L(-w_0)$

Proof: $L(\lambda)^\vee$ is a finite dimensional module, and $(x \cdot f)(v) = -f(x \cdot v)$, so $L(\lambda)^\vee \cong L(\nu)$ for some $\nu \in \Lambda^+$. The weights of $L(\lambda)^\vee$ are the negatives of the weight of $L(\lambda)$. Thus the lowest weight of $L(\lambda)$ is $w_0\lambda$, since w_0 reverses the partial order on \mathfrak{h}^\vee , i.e. $w_0\Phi^+ = \Phi^-$.

Then $\mu \in \Pi(L(\mu)) \implies w_0\mu \in \Pi(L(\lambda)) \implies w_0\mu \leq \lambda$. This shows that the lowest weight of $L(\lambda)$ is $w_0\lambda$, thus the highest weight $L(\lambda)^\vee$ is $-w_0\lambda$ by reversing this inequality.

The inner product is W invariant and is its own inverse, so we can move it to the other side.

■

10.1 1.7: Action of $Z(\mathfrak{g})$

Next big goal: Every module in \mathcal{O} has a *finite* composition series (Jordan-Holder series, the quotients are simple). Leads to Kazhdan-Lusztig conjectures from 1979/1980, which were solved, but are still open in characteristic p case.

The technique we'll use the Harish-Chandra homomorphism, which identifies $\mathcal{Z}(\mathfrak{g})$ explicitly.

It's commutative, subalgebra of a Noetherian algebra, no zero divisors – could be a quotient, but then it'd have zero divisors, so this seems to force it to be a polynomial algebra on some unknown. Also note that $\mathcal{Z}(\mathfrak{g}) := Z(U(\mathfrak{g}))$.

Recall: $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on any $M \in \mathcal{O}$ – this is by theorem 1.1e, i.e. $v \in M_\mu$ and $z \in \mathcal{Z}(\mathfrak{g})$ implies that $zv \in M_\mu$. (The calculation just follows by computing the weight and commuting things through.)

Let $\lambda \in \mathfrak{h}^\vee$ and $M = U(\mathfrak{g})v^+$ a highest weight module of highest weight λ . Then for $z \in \mathcal{Z}(\mathfrak{g})$, $z \cdot v^+ \in M_\lambda$ which is 1-dimensional. Thus z acts by scalar multiplication here, and $z \cdot v^+ = \chi_\lambda(z)v^+$. Now if $u \in U(\mathfrak{u}^-)$, we have $z \cdot (u \cdot v^+) = u \cdot (z \cdot v^+) = u(\chi_\lambda(z)v^+) = \chi_\lambda(z)u \cdot v^+$. Thus z acts on *all* of M by the scalar $\chi_\lambda(z)$.

Exercise: Show that χ_λ is a nonzero additive and multiplicative function, so $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is linear and thus a morphism of algebras. Conclude that $\ker \chi_\lambda$ is a maximal ideal of $\mathcal{Z}(\mathfrak{g})$.

Note: this is called the infinitesimal character.

Note that χ_λ doesn't depend on which highest weight module M_λ was chosen, since they're all quotients of $M(\lambda)$. In fact, every submodule and subquotient of $M(\lambda)$ is the same infinitesimal character.

Definition: χ_λ is called the *central (or infinitesimal) character*, and $\widehat{\mathcal{Z}}(\mathfrak{g})$ denotes the set of all central characters. More generally, any algebra morphism $\chi : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is referred to as a central character. Central characters are in one-to-one correspondence with maximal ideals of $\mathcal{Z}(\mathfrak{g})$, where $\chi \iff \ker \chi$ and $\mathbb{C}[x_1, \dots, x_n] \iff \langle x_1 - a_1, \dots, x_n - a_n \rangle$ where $(\mathbf{a}_i) \in \mathbb{C}^n$.

Next goal: Describe $\chi_\lambda(z)$ more explicitly.

Using PBW, we can write $z \in \mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. Some observations:

1. Any PBW monomial in z ending with a factor in \mathfrak{n} will kill v^+ , and hence can not contribute to $\chi_\lambda(z)$.
2. Any PBW monomial in z beginning with a factor in \mathfrak{n}^- will send v^+ to a lower weight space, so it also can't contribute.

So we only need to see what happens in the \mathfrak{h} part. A relevant decomposition here is

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^+).$$

Exercise: Why is this sum direct?

Let $\text{pr} : U(\mathfrak{g}) \rightarrow U(\mathfrak{h})$ be the projection onto the first factor. Then $\chi_\lambda(z) = \lambda(\text{pr}z)$ for all $z \in \mathcal{Z}(\mathfrak{g})$. Then if $\text{pr}(z) = h_1^{m_1} \dots h_\ell^{m_\ell}$, we can extend the action on \mathfrak{h} to all polynomials in elements of \mathfrak{h} (which is in fact evaluation on these monomials), and thus $\chi_\lambda(z) = \lambda(h_1)^{m_1} \dots \lambda(h_\ell)^{m_\ell}$.

Note that for $\lambda \in \mathfrak{h}^\vee$, we've extended this to the "evaluation map" $\lambda : U(\mathfrak{g}) \cong S(\mathfrak{h})$, the symmetric algebra on \mathfrak{h} . Why is this the correct identification? The RHS is $T(\mathfrak{h}) / \langle x \otimes y - y \otimes x - [xy] \rangle$, but notice that the bracket vanishes since \mathfrak{h} is abelian, and this is the exact definition of the symmetric algebra.

Thus $\chi_\lambda = \lambda \circ \text{pr}$.

Observation:

$$\begin{aligned} \lambda(\text{pr}(z_1 z_2)) &= \chi_\lambda(z_1 z_2) \\ &= \chi_\lambda(z_1) \chi_\lambda(z_2) \\ &= \dots \\ &= \lambda(\text{pr}(z_1) \text{pr}(z_2)). \end{aligned}$$

Exercise: Show $\bigcap_{\lambda \in \mathfrak{h}^\vee} \ker \lambda = \{0\}$.

Definition: Let $\xi = \text{pr}|_{\mathcal{Z}(\mathfrak{g})} : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h})$.

Lemma/Definition: ξ is an algebra morphism, and is referred to as the *Harish-Chandra homomorphism*.

See page 23 for interpretation of ξ without reference to representations.

Questions:

1. Is ξ injective?
2. What is $\text{im } \xi \subset U(\mathfrak{h})$?

When does $\chi_\lambda = \chi_\mu$? Proved last time: we introduced the \cdot action and proved that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ where $\alpha \in \Delta$. It'll turn out that the statement holds for all $\lambda \in W$.

Wednesday: Section 1.8.

11 Wednesday February 5th

Recall the Harish-Chandra morphism ξ :

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\quad} & U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}) \\ & \searrow \xi & \downarrow \text{pr} \\ & & U(\mathfrak{h}) \end{array}$$

If M is a highest weight module of highest weight λ then $z \in \mathcal{Z}(\mathfrak{g})$ acts on M by scalar multiplication. Note that if we have $\chi_\lambda(z)$ where $z \cdot v = \chi_\lambda(z)v$ for all $v \in M$, we can identify $\lambda(\text{pr}(z)) = \lambda(\xi(z))$.

11.1 Central Characters and Linkage

The χ_λ are not all distinct – for example, if $M(\mu) \subset M(\lambda)$, then $\chi_\mu = \chi_\lambda$. More generally, if $L(\mu)$ is a subquotient of $M(\lambda)$ then $\chi_\mu = \chi_\lambda$. So when do we have equality $\chi_\mu = \chi_\lambda$?

Given $\mathfrak{g} \supset \mathfrak{h}$ with $\Phi \supset \Phi^+ \supset \Delta$, then define $\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathfrak{h}^\vee$. Note that $\alpha \in \Delta \implies s_\alpha \rho = \rho - \alpha$.

Definition: The *dot action* of W on \mathfrak{h}^\vee is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, which implies $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Delta$. Then $\rho = \sum_{i=1}^{\ell} w_i$.

Exercise: Check that this gives a well-defined group action.

Definition: μ is *linked* to λ iff $\mu = w \cdot \lambda$ for some $w \in W$. Note that this is an equivalence relation, with equivalence classes/orbits where the orbit of λ is $\{w \cdot \lambda \mid w \in W\}$ is called the *linkage class* of λ .

Note that this is a finite subset, since W is finite.

Note that orbit-stabilizer applies here, so bigger stabilizers yield smaller orbits and vice-versa.

Example: $w \cdot (-\rho) = w(-\rho + \rho) - \rho = -\rho$, so $-\rho$ is in its own linkage class.

Definition: $\lambda \in \mathfrak{h}^\vee$ is *dot-regular* iff $|W \cdot \lambda| = |W|$, or equivalently if $(\lambda + \rho, \beta^\vee) \neq 0$ for all $\beta \in \Phi$.

To think about: does this hold if Φ is replaced by Δ ?

We also say λ is *dot-singular* if λ is not dot-regular, or equivalently $\text{Stab}_W \cdot \lambda \neq \{1\}$.

I.e. lying on root hyperplanes.

Exercise: If $0 \in \mathfrak{h}^\vee$ is regular, then $-\rho$ is singular.



Proposition: If $\lambda \in \Lambda$ and $\mu \in W \cdot \lambda$, then $\chi_\mu = \chi_\lambda$.

Proof: Start with $\alpha \in \Delta$ and consider $\mu = s_\alpha \cdot \lambda$. Since $\lambda \in \Lambda$, we have $n := (\lambda, \alpha^\vee) \in \mathbb{Z}$ by definition. There are three cases:

1. $n \in \mathbb{Z}^+$, then $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. By Proposition 1.4, we have $\chi_\mu = \chi_\lambda$.
2. For $n = -1$, $\mu = s_\alpha \cdot \lambda = \lambda + \rho - (\lambda + \rho, \alpha^\vee)\alpha - \rho = \lambda + n + 1 = \lambda + 0$. So $\mu = \lambda$ and thus $M_\mu = M_\lambda$.
3. For $n \leq -2$,

$$\begin{aligned}
(\mu, \alpha^\vee) &= (s_\alpha \cdot \lambda, \alpha^\vee) \\
&= (\lambda i(n+1)\alpha, \alpha^\vee) \\
&= n - 2(n+1) \\
&= -n - 2 \\
&\geq 0,
\end{aligned}$$

so $\chi_\mu = \chi_{s_\alpha \cdot \mu} = \chi_{s_\alpha \cdot (s_\alpha \cdot \lambda)} = \chi_\lambda$. Since W is generated by simple reflections and the linkage property is transitive, the result follows by induction on $\ell(w)$. ■

Exercise 1.8 (do but don't turn in): See book, show that certain properties of the dot action hold (namely nonlinearity).

11.2 1.9: Extending the Harish-Chandra Morphism

We want to extend the previous proposition from $\lambda \in \Lambda$ to $\lambda \in \mathfrak{h}^\vee$. We'll use a density argument from affine algebraic geometry, and switch to the Zariski topology on $\mathfrak{h}^\vee \subset \mathbb{C}^n$.

Fix a basis $\Delta = \{a_1, \dots, a_\ell\}$ and use the Killing form to identify these with a basis for $\mathfrak{h} = \{h_1, \dots, h_\ell\}$. Similarly, take $\{w_1, \dots, w_\ell\}$ as a basis for \mathfrak{h}^\vee , and we'll use the identification

$$\begin{aligned}
\mathfrak{h}^\vee &\Longleftrightarrow \mathbb{A}^\ell \\
\lambda &\Longleftrightarrow (\lambda(h_1), \dots, \lambda(h_\ell)).
\end{aligned}$$

We identify $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[h_1, \dots, h_\ell]$ with $P(\mathfrak{h}^\vee)$ which are polynomial functions on \mathfrak{h}^\vee . Fix $\lambda \in \mathfrak{h}^\vee$, extended λ to be a multiplicative function on polynomials. For $f \in \mathbb{C}[h_1, \dots, h_\ell]$, we defined $\lambda(f)$. Under the identification, we send this to \tilde{f} where $\tilde{f}(\lambda) = \lambda(f)$.

Note: we'll identify f and \tilde{f} notationally going forward and drop the tilde everywhere.

Then W acts on $P(\mathfrak{h}^\vee)$ by the dot action: $(w \cdot \tilde{f})(\lambda) = \tilde{f}(w^{-1} \cdot \lambda)$.

Exercise: Check that this is a well-defined action.

Under this identification, we have

$$\begin{aligned}
\mathfrak{h}^\vee &\Longleftrightarrow \mathbb{A}^\ell \\
\Lambda &\Longleftrightarrow \mathbb{Z}^\ell.
\end{aligned}$$

Note that Λ is discrete in the analytic topology, but is *dense* in the Zariski topology.

Proposition: A polynomial f on \mathbb{A}^ℓ vanishing on \mathbb{Z}^ℓ must be identically zero.

Proof: For $\ell = 1$: A nonzero polynomial in one variable has only finitely many zeros, but if f vanishes on \mathbb{Z} it has infinitely many zeros.

For $\ell > 1$: View $f \in \mathbb{C}[h_1, \dots, h_{\ell-1}][h_\ell]$. Substituting any fixed integers for the h_i for $i \leq \ell - 1$ yields a polynomial in one variable which vanishes on \mathbb{Z} . By the first case, $f \equiv 0$, so the coefficients must all be zero and the coefficient polynomials in $\mathbb{C}[h_1, \dots, h_{\ell-1}]$ vanish on $\mathbb{Z}^{\ell-1}$. By induction, these coefficient polynomials are identically zero.

■

Corollary: The only Zariski-closed subset of \mathbb{A}^ℓ containing \mathbb{Z}^ℓ is \mathbb{A}^ℓ itself, so the Zariski closure $\overline{\mathbb{Z}^\ell} = \mathbb{A}^\ell$ and \mathbb{Z}^ℓ is dense in \mathbb{A}^ℓ .

12 Friday February 7th

So far, we have $\chi_\lambda = \chi_{w \cdot \lambda}$ if $\lambda \in \Lambda$ and $w \in W$. We have $\mathfrak{h}^\vee \supset \Lambda$ which is topologically equivalent to $\mathbb{A}^\ell \supset \mathbb{Z}^\ell$, where \mathbb{Z}^ℓ is dense in the Zariski topology.

For $z \in \mathcal{Z}(\mathfrak{g})$, we have $\chi_\lambda(z) = \chi_{w \cdot \lambda}(z)$ and so $\lambda(\xi(z)) = (w \cdot \lambda)(\xi(z))$ where $\xi : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h}) \cong P(\mathfrak{h}^\vee)$ where we send $\lambda(f)$ to $f(\lambda)$.

Then $\xi(z)(\lambda) = \xi(z)(w \cdot \lambda)$ for all $\lambda \in \Lambda$, and so $\xi(z) = w^{-1}\xi(z)$ on Λ . But both sides here are polynomials and thus continuous, and $\Lambda \subset \mathfrak{h}^\vee$ is dense, so $\xi(z) = w^{-1}\xi(z)$ on all of \mathfrak{h}^\vee . I.e., $\chi_\lambda = \chi_{w \cdot \lambda}$ for all $\lambda \in \mathfrak{h}^\vee$.

This in fact shows that the image of $\mathcal{Z}(\mathfrak{g})$ under ξ consists of W -invariant polynomials.

It's customary to state this in terms of the natural action of W on polynomials without the row-shift. We do this by letting $\tau_\rho : S(\mathfrak{h}) \xrightarrow{\cong} S(\mathfrak{h})$ be the algebra automorphism induced by $f(\lambda) \mapsto f(\lambda - \rho)$. This is clearly invertible by $f(\lambda) \mapsto f(\lambda + \rho)$. We then define $\psi : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\xi} S(\mathfrak{h}) \xrightarrow{\tau_\rho} S(\mathfrak{h})$ as this composition; this is referred to as the Harish-Chandra (HC) homomorphism.

Exercise: Show $\chi_\lambda(z) = (\lambda + \rho)(\psi(z))$ and $\chi_{w \cdot \lambda}(w(\lambda + \rho))(\psi(z))$, where $w(\cdot)$ is the usual w -action. Replacing λ by $\lambda + \rho$ and w by w^{-1} , we get

$$w\psi(z) = \psi(z)$$

for all $z \in \mathcal{Z}(\mathfrak{g})$ and all $w \in W$ where $(w\psi(z))(\lambda) = \psi(z)(w^{-1}\lambda)$.

We've proved that

Theorem 1.9:

- a. If $\lambda, \mu \in \mathfrak{h}^\vee$ that are linked, then $\chi_\lambda = \chi_\mu$.
- b. The image of the twisted HC homomorphism $\psi : \mathcal{Z}(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ lies in the subalgebra $S(\mathfrak{h})^W$.

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$. Recall from finite-dimensional representations there is a canonical element $c \in \mathcal{Z}(\mathfrak{g})$ called the Casimir element. For \mathcal{O} , we need information about the full center $\mathcal{Z}(\mathfrak{g})$ (hence introducing infinitesimal characters).

Expressing c in the PBW basis yields $c = h^2 + 2h + 4yx$, where $h^2 + 2h \in U(\mathfrak{h})$ and $4yx \in \mathfrak{n}^-U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$.

Enveloping algebra convention: xs, hs, ys

Then $\xi(c) = \mathfrak{p}(c) = h^2 + 2h$, and under the identification $\mathfrak{h}^\vee \iff \mathbb{C}$ where $\lambda \iff \lambda(h)$, we can identify $\rho \iff \rho(h) = 1$. The row shift is given by $\psi(c) = (h-1)^2 + 2(h-1) = h^2 - 1$. This is W -invariant, since $s_{\alpha_h} = -h$. But $W = \langle s_\alpha, 1 \rangle$, so s_α generates W .

We also have $\chi_\lambda(c) = (\lambda + \rho)(\psi(c)) = (\lambda + 1)^2 - 1$. Then

$$\chi_\lambda(c) = \chi_\mu(c) \iff (\lambda + 1)^2 - 1 = (\mu + 1)^2 \iff \mu = \lambda \text{ or } \mu = -\lambda - 2$$

But $\lambda = 1 \cdot \lambda$ and $-\lambda - 2 = s_\alpha \cdot \lambda$, so $\mathcal{Z}(\mathfrak{g}) = \langle c \rangle := \mathbb{C}[c]$ as an algebra. So these characters are equal iff $\mu = w \cdot \lambda$ for $w \in W$.

13 Section 1.10: Harish-Chandra's Theorem

Goal: prove the converse of the previous theorem.

Theorem (Harish-Chandra): Let $\psi : \mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})$ be the twisted HC homomorphism. Then

- a. ψ is an *isomorphism* of $\mathcal{Z}(\mathfrak{g})$ onto $S(\mathfrak{h})^W$.
- b. For all $\lambda, \mu \in \mathfrak{h}^\vee$, $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.
- c. Every central character $\chi : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is a χ_λ .

Sketch of proof:

Part (a): Relies heavily on the *Chevalley Restriction Theorem* (which we won't prove here).

Initially we have a restriction map on polynomial functions $\theta : P(\mathfrak{g}) \rightarrow P(\mathfrak{h})$. We identified $P(\mathfrak{g}) = S(\mathfrak{g}^\vee)$, the formal polynomials on \mathfrak{g}^\vee . However, for \mathfrak{g} semisimple, we can identify $S(\mathfrak{g}^\vee) \cong S(\mathfrak{g})$ via the Killing form.

By the Chinese Remainder Theorem, $\theta : S(\mathfrak{g})^G \rightarrow S(\mathfrak{h})^W$ is an isomorphism, where the subgroup $G \leq \text{Aut}(\mathfrak{g})$ is the *adjoint group* generated by $\left\{ \exp \text{ad}_x \mid x \text{ is nilpotent} \right\}$.

It turns out that $S(\mathfrak{g})^G$ is very close to $\mathcal{Z}(\mathfrak{g})$ – it is the associated graded of a natural filtration on $\mathcal{Z}(\mathfrak{g})$. This is enough to show that ψ is a bijection. ■

Part (b): We'll prove the contrapositive of the converse.

Suppose $W \cdot \lambda \cap W \cdot \mu = \emptyset$ and both are in \mathfrak{h}^\vee . Since these are finite sets, Lagrange interpolation yields a polynomial that is 1 on $W \cdot \lambda$ and 0 on $W \cdot \mu$. Let $g = \frac{1}{|W|} \sum_{w \in W} w \cdot f$.

Note: definitely the dot action here, may be a typo in the book.

Then g is a $W \cdot$ invariant polynomial with the same properties. By part (a), we can get rid of the row shift to obtain an isomorphism $\xi : \mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})^{(W \cdot)}$, the $W \cdot$ invariant polynomials. Choose $z \in \mathcal{Z}(\mathfrak{g})$ such that $\xi(z) = g$, then $\chi_\lambda(z) = \lambda(\xi(z)) = \lambda(g) = g(\lambda) = 1$. So $\chi_\mu(z) = 0$ similarly, and $\chi_\lambda = \chi_\mu$. ■

Part (c): This follows from some commutative algebra, we won't say much here. Look at maximal ideals in $\mathbb{C}[x, y, \dots]$ which correspond to evaluating on points in \mathbb{C}^ℓ .

■

Remark: Chevalley actually proved that $S(\mathfrak{h})^W \cong \mathbb{C}(p_1, \dots, p_\ell)$ where the p_i are homogeneous polynomials of degrees $d_1 \leq \dots \leq d_\ell$. These numbers satisfy some remarkable properties: $\prod d_i = |W|$ and $d_1 = 2$ (these are called the *degrees of W*)

14 Section 1.11

Theorem: Category \mathcal{O} is *artinian*, i.e. every $M \in \mathcal{O}$ is Artinian (DCC) and $\dim \text{hom}_{\mathfrak{g}}(M, N) < \infty$ for every M, N .

Recall that \mathcal{O} is known to be Noetherian from an earlier theorem. This will imply that every M has a composition/Jordan-Holder series, so we can take composition factors and multiplicities. Most interesting question: what are the factors/multiplicities of the simple modules and Verma modules?