Full Notes

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Contents

1 Thursday, Week 1

Motivation: Gauss' Unicursal Problem. How many distinct curves $\alpha : [0,1] \to \mathbb{R}^2$ are there with no triple crossings?

Note that if we compactify the plane to the Riemann sphere (and possibly take the curve to be piecewise linear) then we obtain a *tiling* of the sphere. We can also take a *dual tiling* by taking the barycenters of each polygon and connecting them by an edge iff their corresponding polygons share an edge.

Definition: A *translation surface* is the 2-dimensional topological manifold obtained by taking any set of polygons in $\mathbb{R}^2$ and gluing their edges by translations.

Example: Any elliptic curve (topologically a torus) is a translation surface.

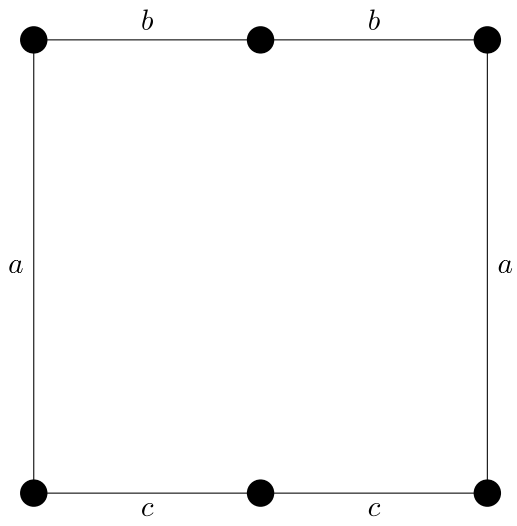
We take equivalence up to cutting, pasting, and rearranging.

Lemma: Two elliptic curves (of genus 2) are isomorphic iff the translation surfaces differ by a homothety, i.e. a rotation and scaling.

> Note: we will eventually see that the data of a translation surface is equivalently a holomorphic 1-form.

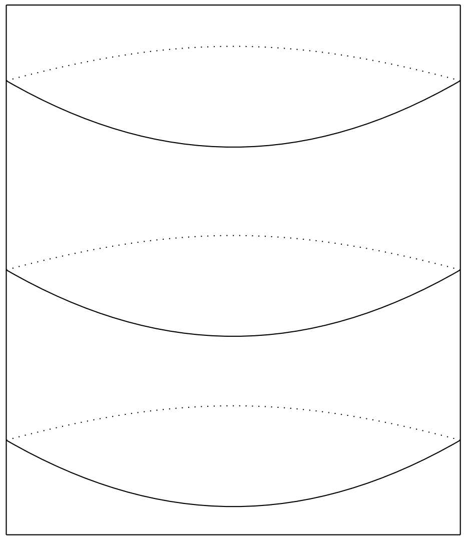
Definition: A *half-translation surface* is a translation surface where we now additionally allow gluing by rotations of π radians.

Example:



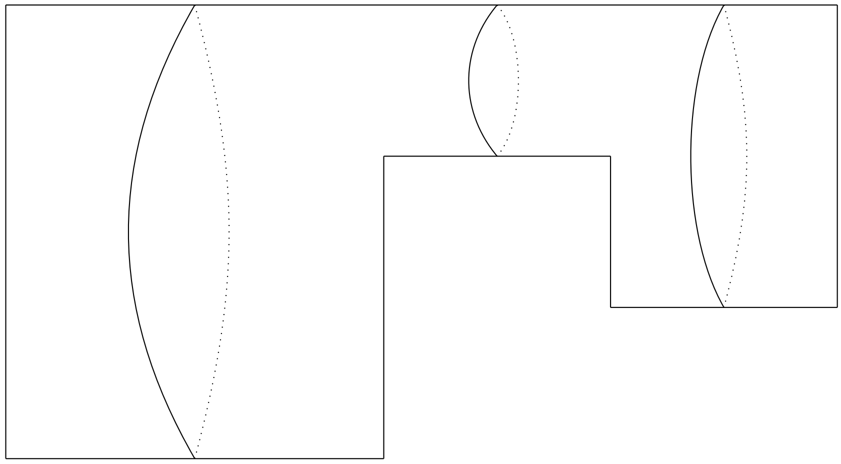
Note that the gluing b now requires a rotation.

By gluing edges with matching letters, we get a “hot pocket” surface:

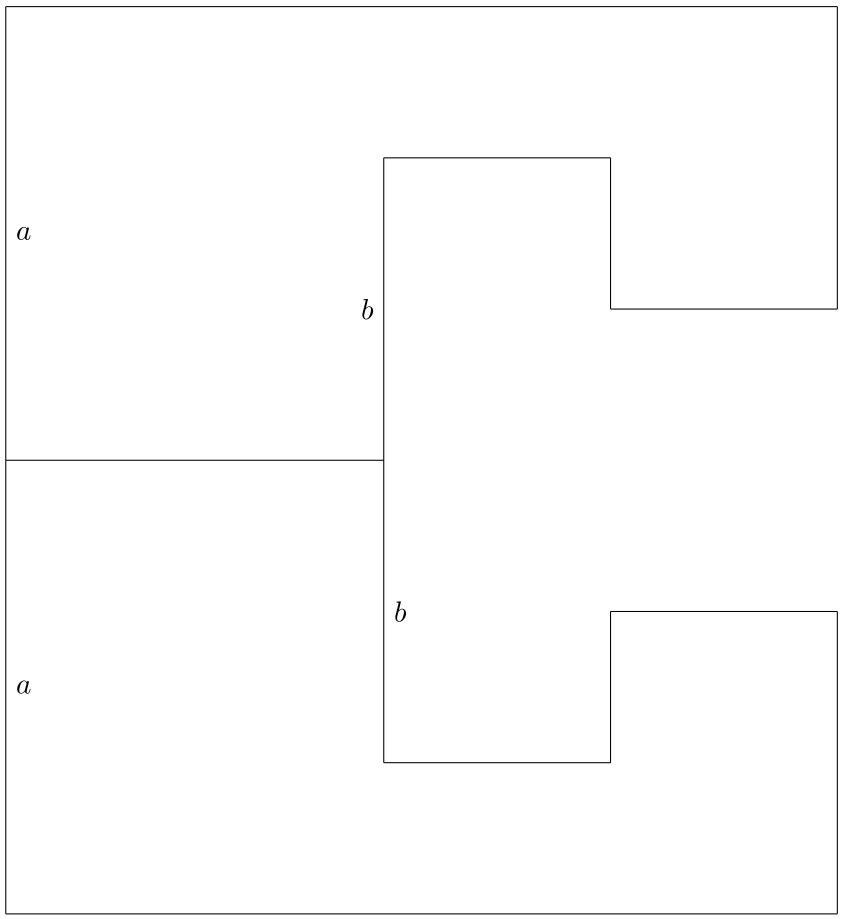


Lemma: Any game of *rectangular billiards* yields a half-translation surface.

Given any domain for rectangular billiards, we inflate it to a surface:



We can then cut along everything but the bottom edge to “unwrap” it into a half-translation surface:



Some questions related to rectangular billiards:

Question 1: Given a random starting point and direction, what proportion of the total region is traversed? Will the trajectory entire a given region? How long does the billiard spend in any given region?

Theorem: The percentage of time spent in a given region is equal to the proportion of its area to the total area.

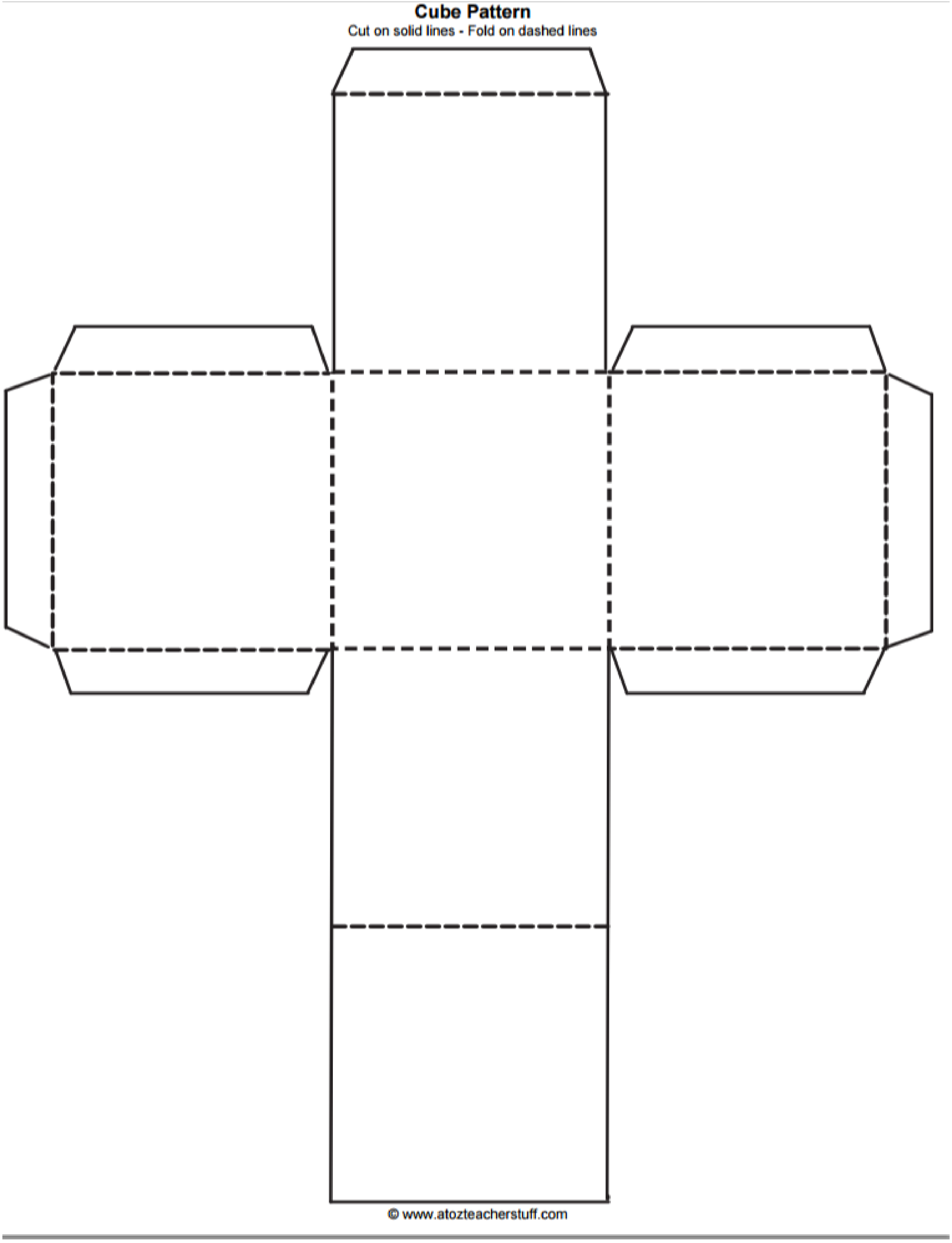
This requires some ergodic theory.

Question 2: If you shine a laser from a given spot, is the entire region illuminated?

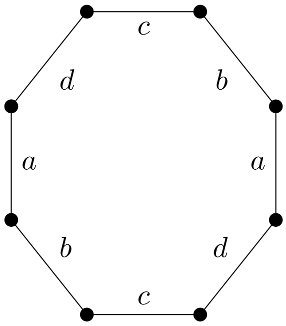
Theorem: No! There is a counterexample with 18 sides. Moreover, no positive-area region can be avoided, but certain finite sets can.

Definition: A *flat surface* is a generalization of translation surfaces that now allows gluing by any isometry of $\mathbb{R}^2$.

Example: A cube in $\mathbb{R}^3$ is a flat surface, noting that the planar gluing diagram for it now requires rotations of $\pi/2$ radians:



Note that we can form higher genus surfaces using polygon gluing:



Exercise: Check that there is only one vertex in this polygon.

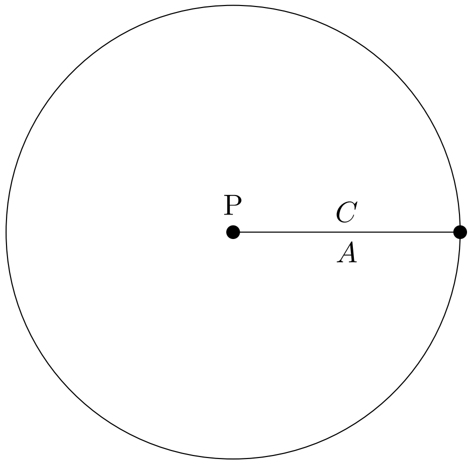
In general, the vertices may have total angle greater than $2\pi n$. We refer to these as *cone points*, and the total angle as the *cone angle*.

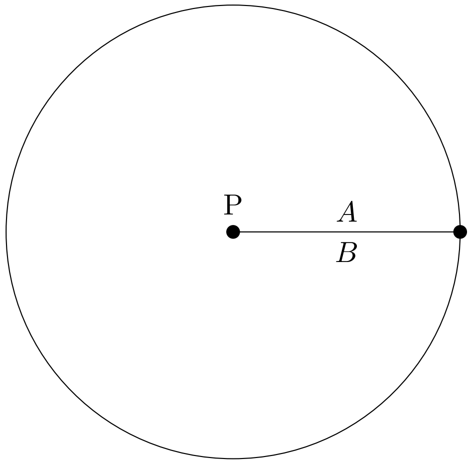
Exercise: Check that the cone angle in the above example is 6π by taking a loop around the cone point.

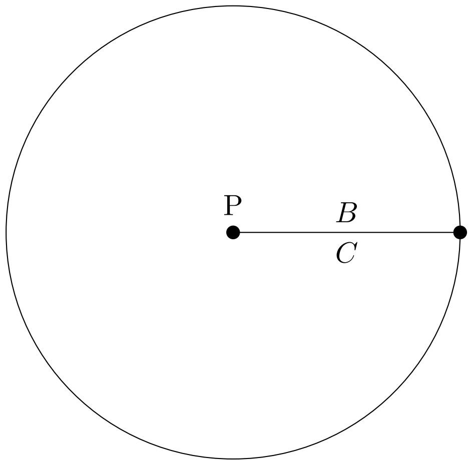
Note on a weird phenomenon: it seems difficult to find $\mathbb{Z}^2$ or $\mathbb{Q}^2$ points on the sloped edges of a regular polygon, based on computer drawings.

Remark: This flat surface admits charts to $\mathbb{C}$ with the following transition functions. Let P denote the single cone point.

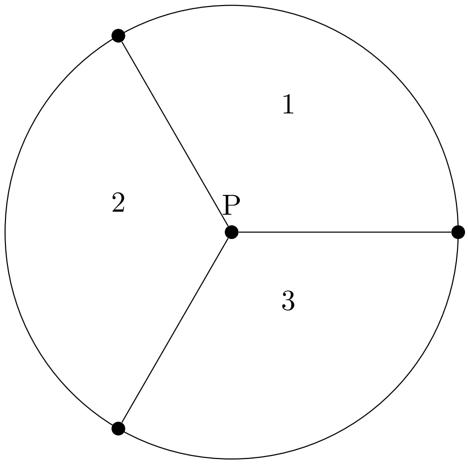
Then there is a 3-fold cover give by the following space, thought of as a singular helical





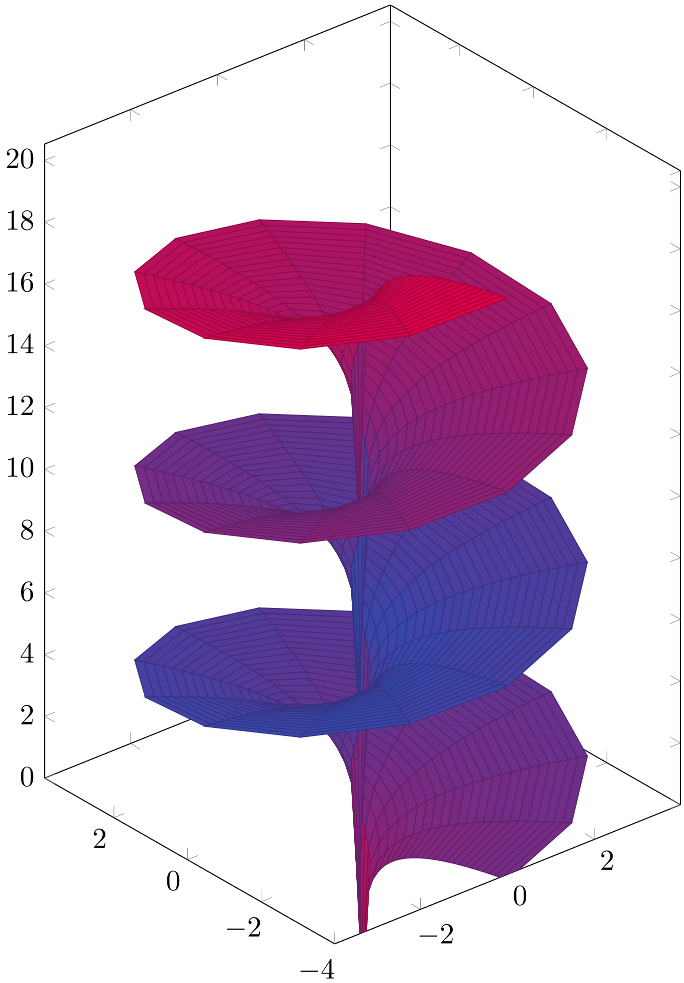


which maps onto the unit disc in $\mathbb{C}$:



The covering map from the former to the latter is given by $z \mapsto z^{\frac{1}{3}}$, which coincides with the fact that the cone angle at P is $3(2\pi) = 6\pi$.

One can also imagine this space in $\mathbb{R}^3$ with a projection onto the plane:



Here the helicoid goes through three full twists, where the top and bottom pieces are identified.

Proposition: In a neighborhood of a cone point P with cone angle $2\pi n$, the map $z \mapsto z^{\frac{1}{n}}$ will be a

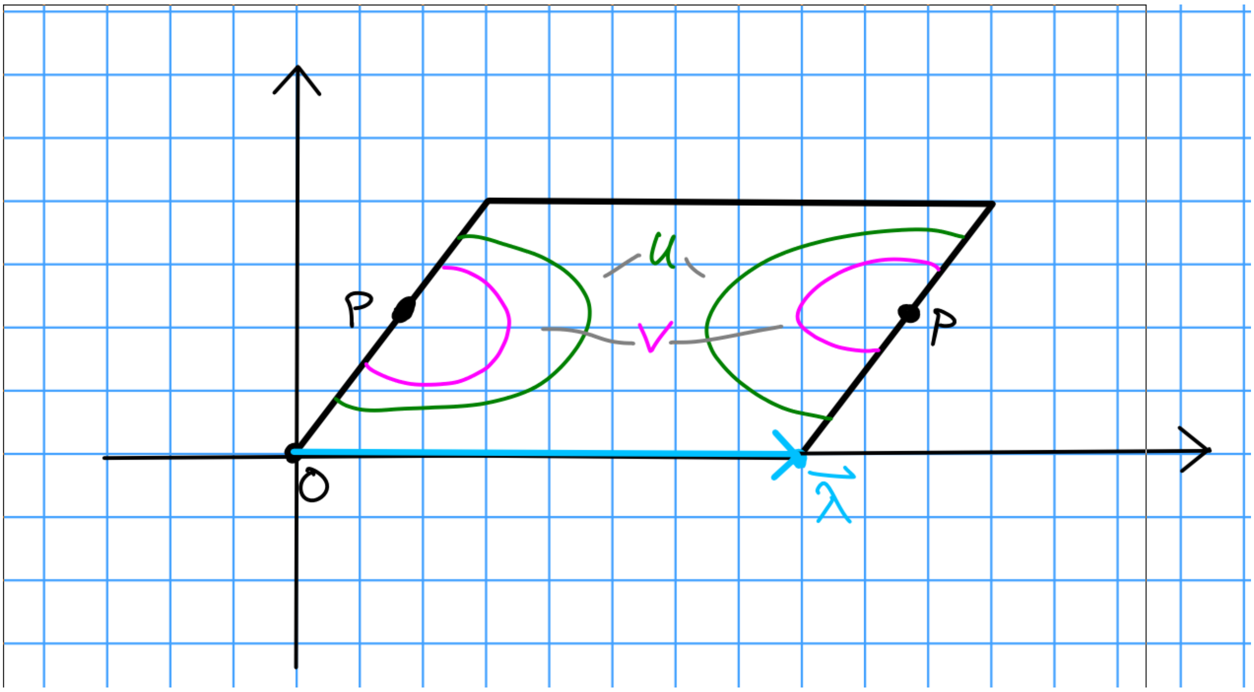


Figure 1: Image

local chart for any z in a neighborhood of P.

This gives the resulting surface the structure of a *Riemann surface*, i.e. a surface admitting charts to $\mathbb{C}$ with holomorphic transition functions.

Let X be a Riemann surface, we can look at the *canonical bundle* over X with sections that are compatible collections of $f_u(z_u)\ dz_u$ for each chart z_u, and for each such chart a holomorphic function $f_u(z_u)$ on $z_u(U)$ where on overlaps

$$\begin{aligned} f_u(z_u)\ dz_u &= f_v(z_v)\ dz_v \\ &= f_v(z_v \circ z_u) z_v'(z_u)\ dz_u. \end{aligned}$$

Updated Definition: A translation surface is a Riemann surface with a section of its canonical bundle.

Example: C/Λ for Λ any rank-2 lattice:

Note that the translation here is given by λ.

Definition: A holomorphic 1-form is a section of the canonical bundle.

Example: For the above surface, we have $z_v = z_u + \lambda$ and thus $dz_u = dz_v$ is nonvanishing.

Note that dz is a holomorphic 1-form on the complement of the vertex/vertices of the polygon.

Proposition: dz extends holomorphically to charts containing cone points with cone angle $2\pi n$ and has a zero of order $n-1$ at such cone points.

Example: For the chart $z = w^3$ where w is the local coordinate, we have $dz = 3w^2$, yielding a zero of order 2.

2 Random Notes

Holomorphic Forms: A holomorphic p-form on X is a section of $\Lambda^p T^\vee X$, the pth exterior power of the holomorphic cotangent bundle of X.

For $n = \dim_{\mathbb{C}} X$, the n-forms are an important special case. Any such form w is given in local coordinates $(z_1, \cdots, z_n)$ by

$$w = w(z_1, \cdots, z_n)dz_1 \wedge \cdots \wedge dz_n$$

for some holomorphic function $w : \mathbb{C}^n \to ?$.

Canonical Bundle: Given a complex manifold M, we can define the tangent bundle $\mathbb{C}^n \to TM \to M$ and the cotangent bundle $C^n \to T^\vee M \to M$, which we'll just denote $T^\vee M$. Then the canonical bundle is the bundle $\mathbb{C} \to \Lambda^n T^\vee M \to M?$, denoted by ω, obtained by taking the nth exterior power. It is a theorem that the fibers are in fact lines. For vector bundles, this is referred to as the *determinant bundle.* If M is a smooth manifold, then ω has a global section.

Note: a holomorphic n-form is exactly the same as a section of the canonical bundle.

Interesting aside: a Calabi-Yau is a manifold with a nowhere vanishing holomorphic n-form, which implies that the canonical bundle admits a map to a trivial line bundle that is an isomorphism, i.e. the canonical bundle is trivial.

Exercise: For Σ_g a compact Riemann surface of genus g, the space of holomorphic sections of the canonical is a vector space (over $\mathbb{C}$?) of dimension g.

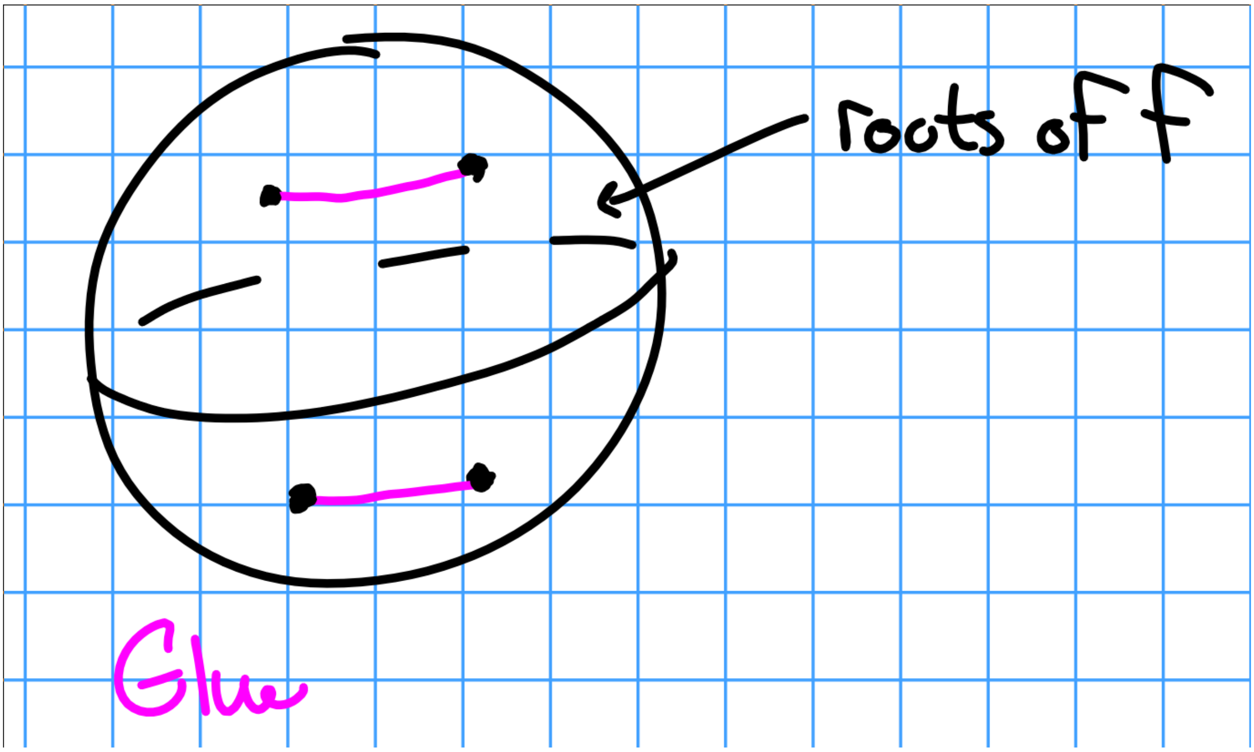
3 Thursday January 16th

3.1 Correspondence

Recall: Start with a translation surface with cone points with angles $2\pi ni$. This yields a Riemann surface Σ and a holomorphic 1-form ω with zeros of order $n_i - 1$.

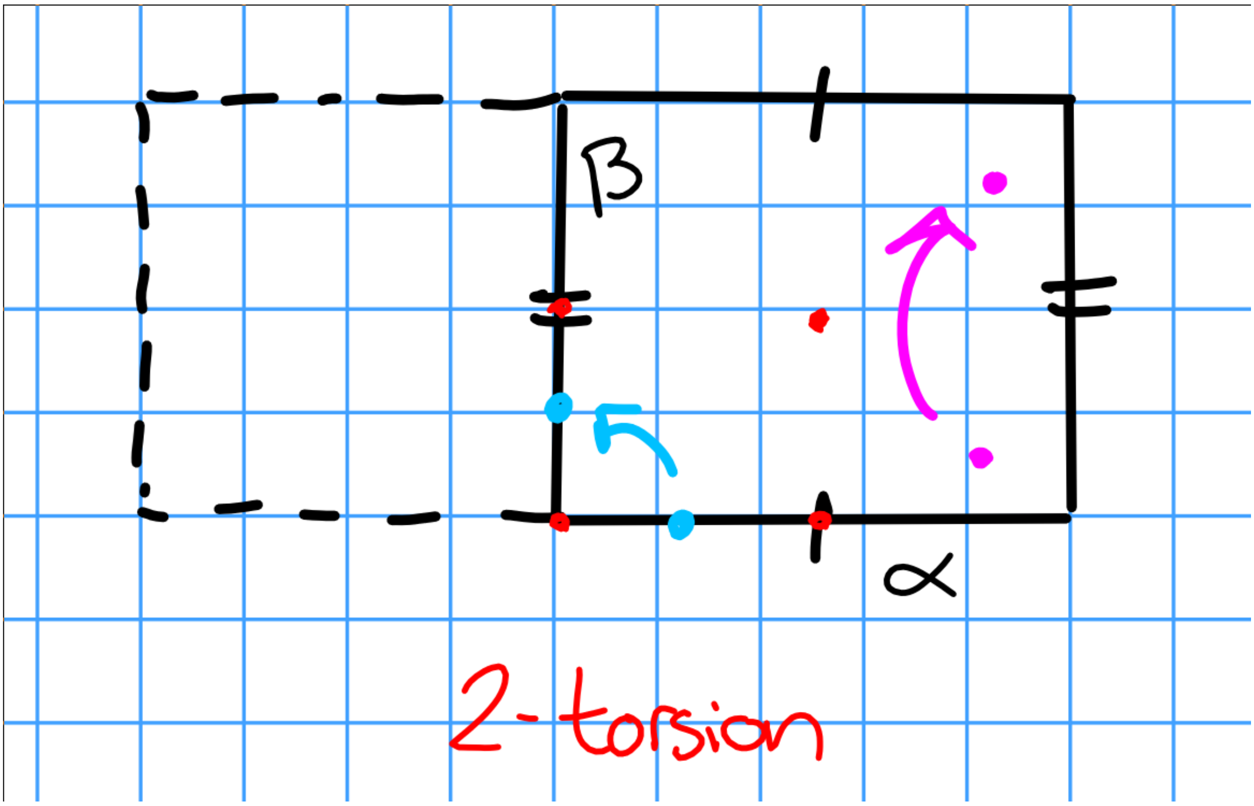
Given a square fundamental domain, there is an order 4 automorphism given by rotating 90 degrees. In charts, this is multiplication by i and possibly a translation, which is a holomorphic map $f : \Sigma \circlearrowleft$. We then have $f^*(dz) = d(f(z)) = idz$, so dz is an eigenvector for f^*.

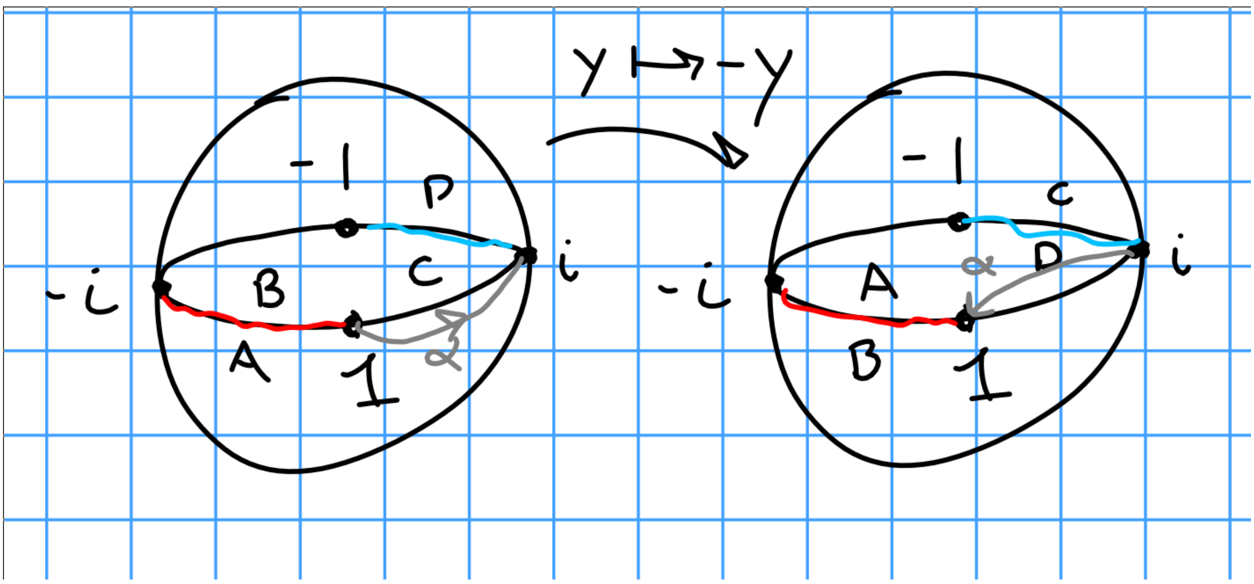
An elliptic curve can be specified by $y^2 = f_4(x)$ for a degree 4 polynomial, so we can obtain it as a double branched cover of S^2. (i.e. glue along the slits joining pairs of roots)



Take $E : y^2 = x^4 - 1$, this is the only elliptic curve with an order 4 automorphism. In coordinates, this is generated by $(x, y) \mapsto (-ix, y)$. So $\omega = c\dfrac{dx}{y}$, i.e. dx/y up to scaling, and $f^*(dx/y) = i\dfrac{dx}{y}$. What is the constant c?

Take closed cycles on E given by α, β (see diagram), then by FTC $\int_\alpha \omega = z\Big|_0^1 = 1$. Negation is 4 fixed points on the elliptic curve, the 2-torsion points.





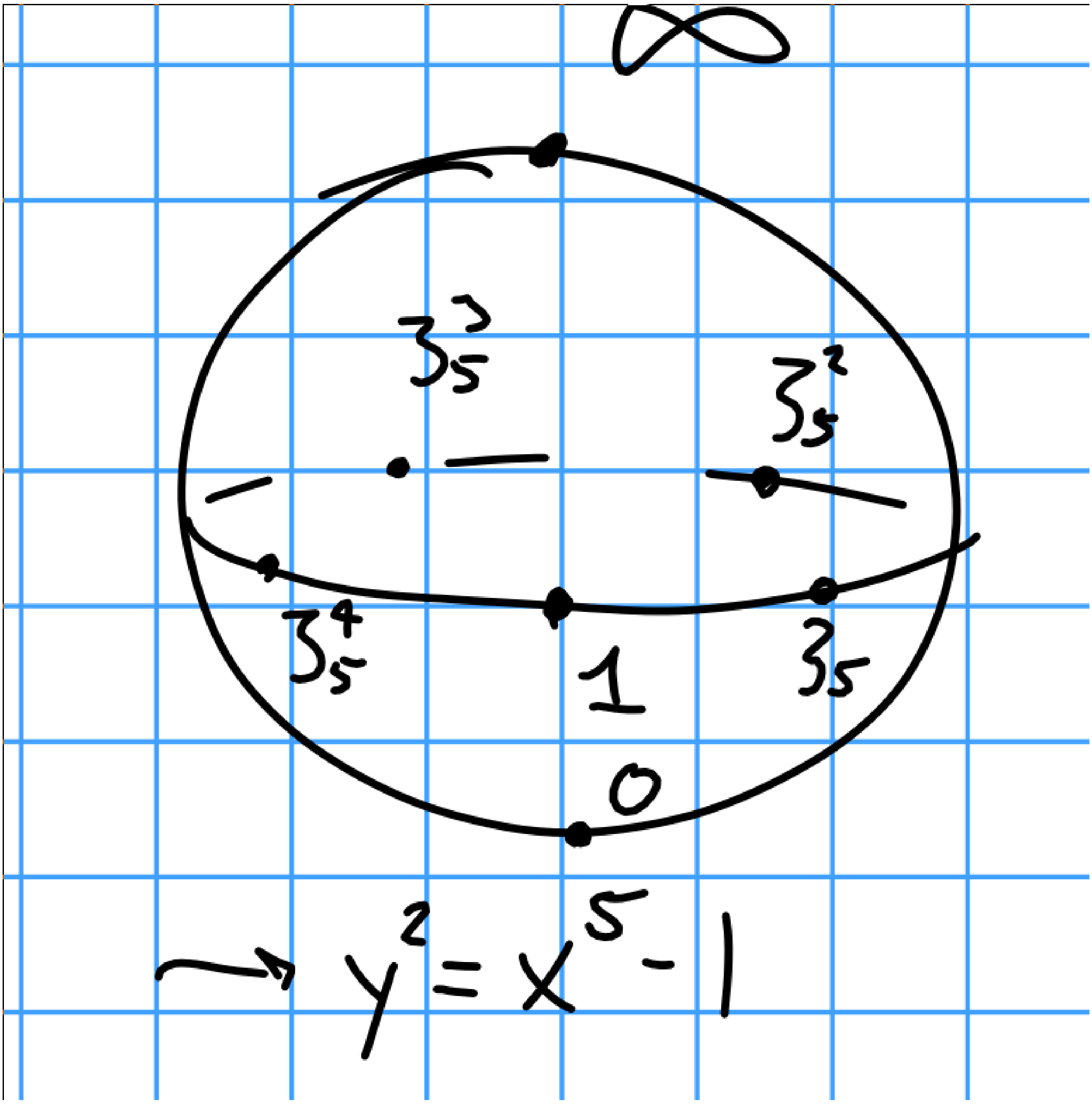
We can then compute

$$I = \int_\alpha \frac{dx}{y} = \int_1^2 \frac{dx}{\sqrt{x^4 - 1}}.$$

and since $2cI = 1$, this uniquely determines c.

> Note that this can be numerically evaluated, but this is an elliptic integral with (possibly) no elementary antiderivative.

Consider the decagon with sides identified. We get a complex structure (Σ, ω) with cone angles 4π and dz has 2 zeros of order 1 (namely the two cone points).



What is the genus? The degree of the canonical bundle is

$$g(\Sigma) = \deg K_\Sigma = 2g - 2 = \sum_{p \in \Sigma} \mathrm{Ord}_p \omega$$

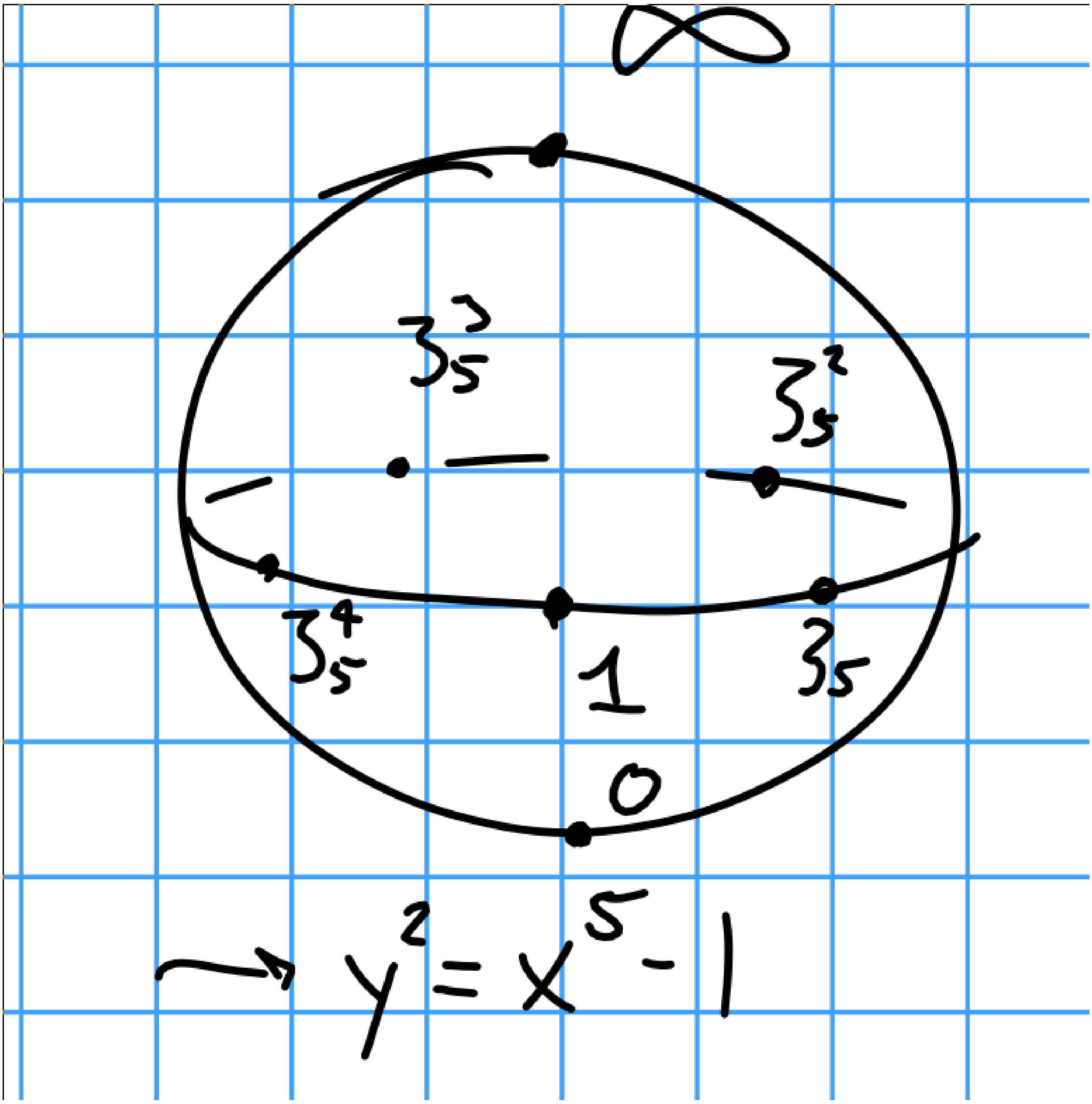
and thus $g = 2$.

Fact: Every genus 2 curve is a double branched cover of $\mathbb{P}^1$ branched over 6 points.

Use Riemann-Roch.

Consider automorphisms that preserve the decagon. Rotation by $\pi/10$ swaps the two cone points, to take rotation by $-2\pi/5$ (inserting the negative to account for pullbacks). Then $f^*\omega = \zeta_5\omega$, where again we just write locally $z \xrightarrow{f} \zeta_5^{-1} + c \implies f^*dz = \zeta_5 dz$.

Consider points of order 5 on $\mathbb{P}^1$, we can take ζ_5^k and ∞.



This corresponds to the curves $y^2 = x^5 - 1$, with an automorphism $(x, y) \mapsto (\zeta_5 x, y)$. This is the only genus 2 curve with an order 5 automorphism.

Fact: The space of sections of the canonical $H^0(K_\mathcal{E}) = \mathbb{C}^g$.

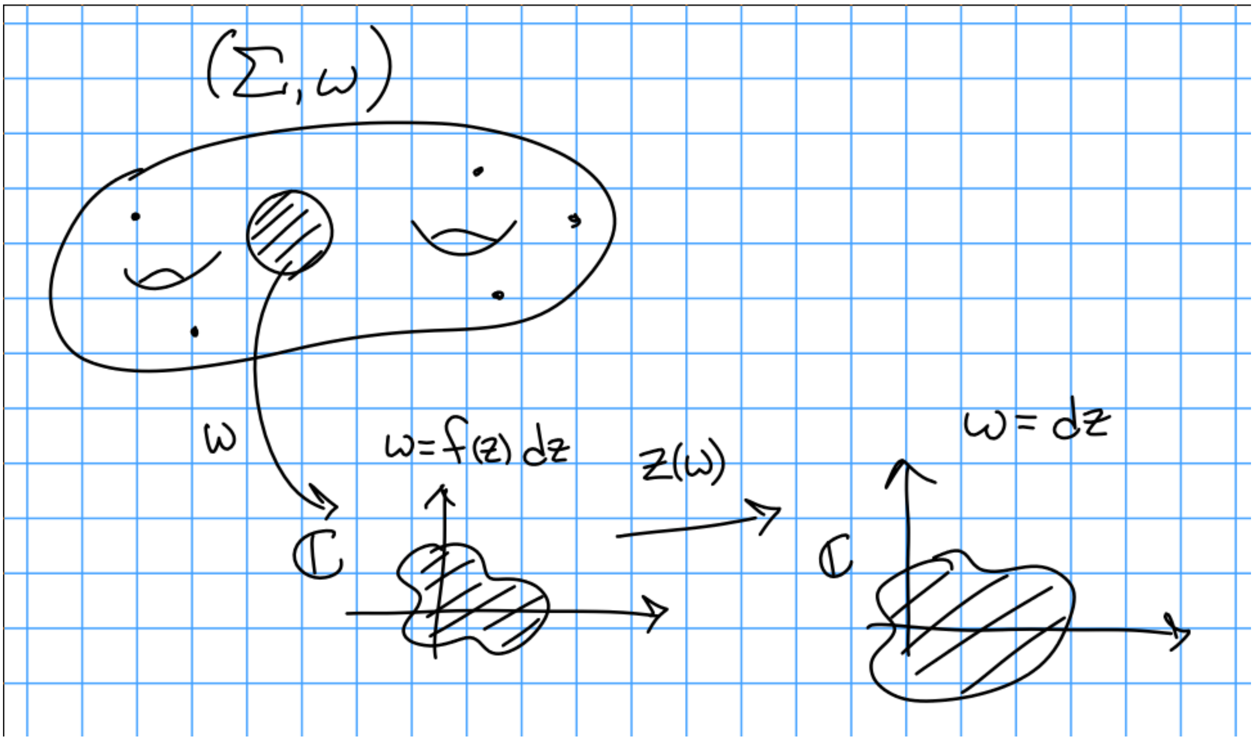
We can write a basis for the space of 1-forms: $\dfrac{dx}{y}$ and $(1-x)\dfrac{dx}{y}$. Alternatively, $\omega = (a+bx)\dfrac{dx}{y}$ where $a, b \in \mathbb{C}$. What are the zeros of ω? $V(\omega) = V(a+bx)$ where if $b = 0$ it's ∞. Because this has to preserve the order 5 symmetric and map cone points to themselves, this forces $\omega = bx\dfrac{dx}{y}$, which has exactly two zeros: $(x, y) = (0, \pm i)$.

We can also consider the doubled pentagon, which has only one point. This has an automorphism given by rotating each pentagon by 1/5, it has cone angle 6π, and ω has a double zero at the cone point. Since there is only *one* genus 2 curve with order 5 automorphisms, this yields the previous Riemann surface but a distinct translation surface and a distinct form.

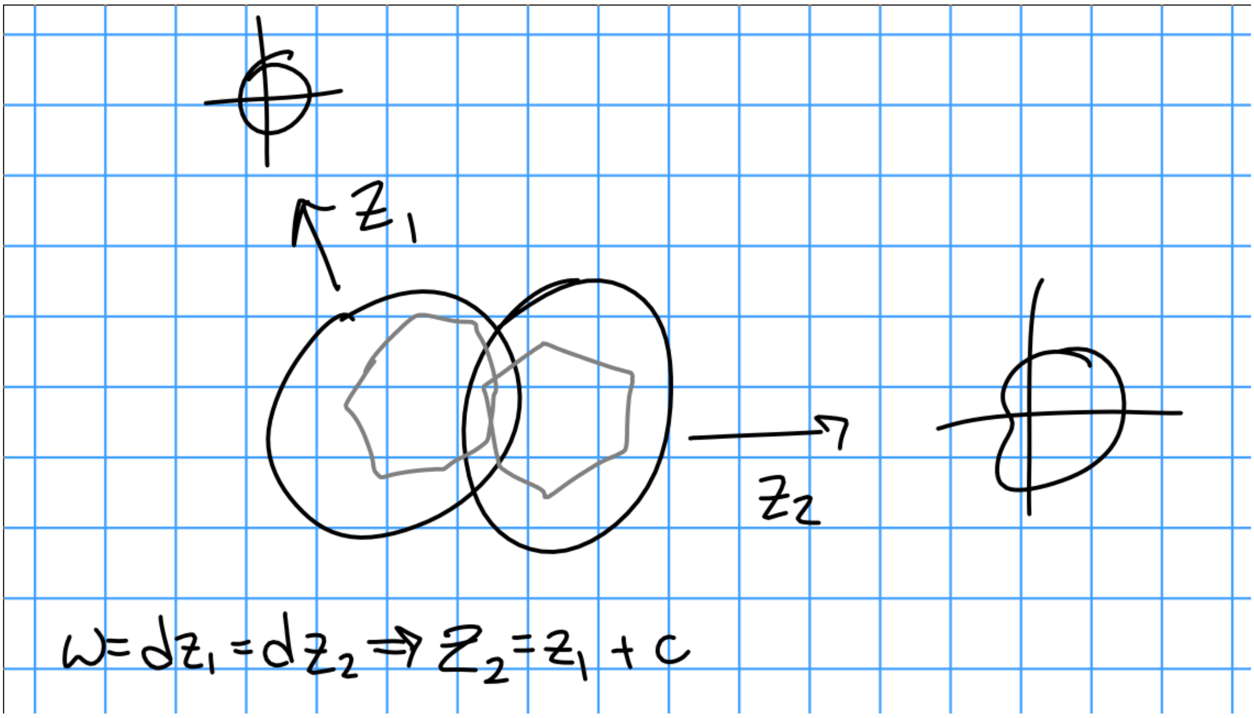
We can write $\alpha = a\dfrac{dx}{y}$.

Proposition: One Riemann surface has many translation structures, and the space of such structures is the space of 1-forms.

Proof: Pick a chart $w : U \to \mathbb{C}$ avoiding the zeros of ω. Then $\omega = f(z)\,dz$ in this chart, and we want to compose with a biholomorphism x to obtain a new chart z such that $\omega = dz$ in this chart. We can solve $dz = e = f(w)\,dw$ and $\dfrac{dz}{dw} = f(w)$ and by integrating we get $z(w) = \displaystyle\int_{p_0}^{w} f(w_0)dw_0$. In this chart, $\omega = dz$, and we can correct near zeros by finding charts such that $\omega = z^n dz$ where n is the order of the zero.



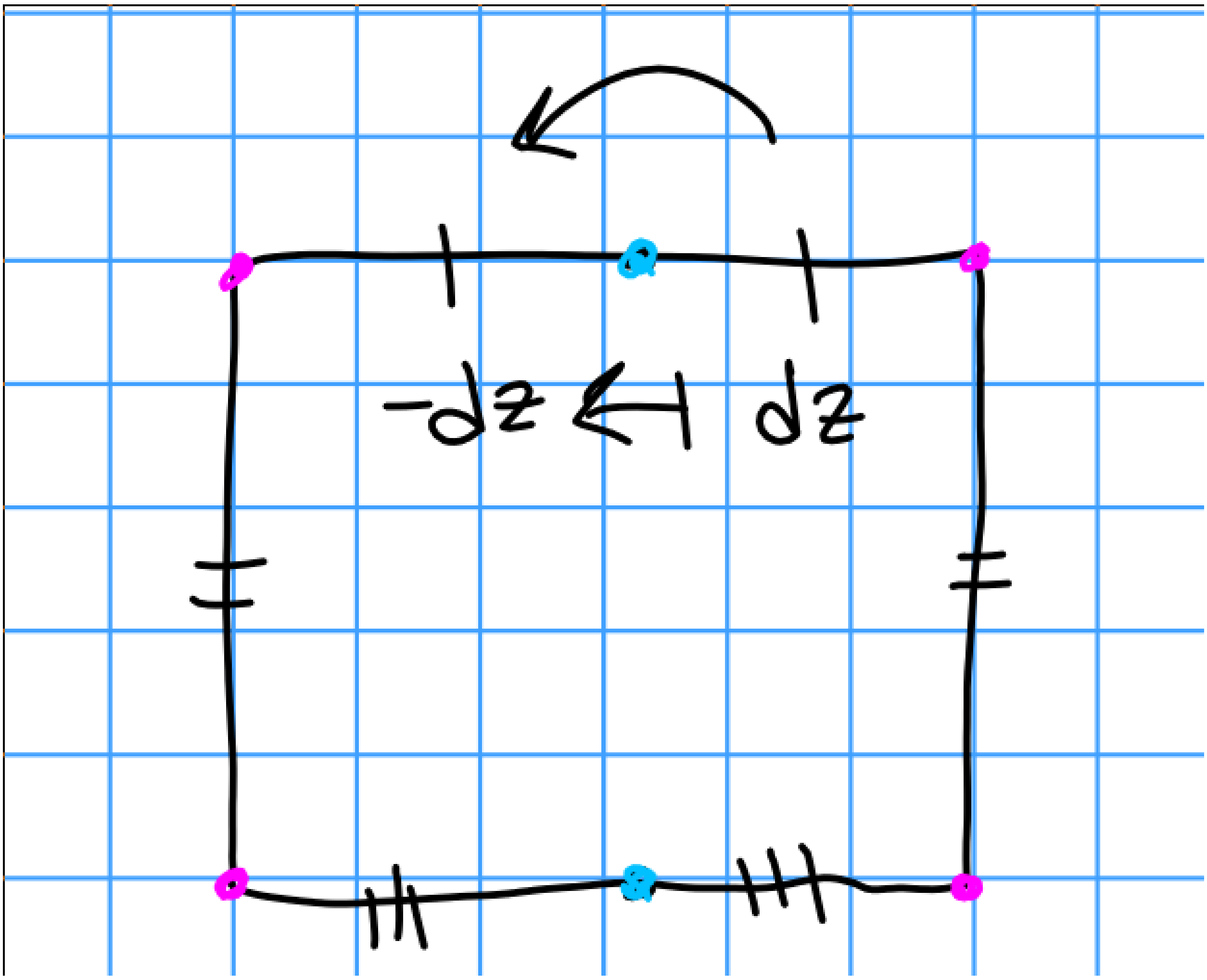
What does this buy us? We get a translation structure by considering transitions, and $\omega = dz_1 = dz_2 \implies z_1 = z_2 + c$, which is exactly a translation structure. Thus every (Σ, ω) has a translation structure for which $\omega = dz$ in local polygonal charts.



Theorem: There is a bijection

$$\left\{\text{Translation surfaces with cone angles } 2\pi n i\ \right\} \iff \left\{\begin{array}{c}(\Sigma,\omega) \text{ a Riemann surface with holomorphic} \\ \text{1-forms with zeros of order } n_i - 1\end{array}\right\}.$$

What do half-translations correspond to? Note that $dz \mapsto -dz$, so we don't get a well-defined holomorphic 1-form.



The fix? $(dz)^2$ is some well-defined object. What is it?

The set $\left\{f(z)(dz)^2 = g(w)(dw)^2\right\}$ corresponds with line bundles with transition functions given by $(dz/dw)^2$.

Thus the correspondence is

$$\left\{\text{Half translation surfaces wit cone angle } \pi n_i\right\} \iff \left\{\text{Riemann surfaces with } q \text{ a section of } K_\Sigma^{\otimes 2}\right\}.$$

A defining property of $q = (dz)^2$ for half-translation charts z: we can measure the order of the zero by going to charts, finding a chart to $\mathbb{C}$ (see image) e.g. $w = z^{2/3}$, and then defining

$$\omega = dz = d(w^{3/2}) = 3/2w^{1/2}dw$$

Then $q = w^2 = (dz)^2$ is well-defined and equal to $\dfrac{9}{4}w(dw)^2$ in the local chart w.

In this case, we get points that are *poles* of order 1 for the quadratic differential (sections of $K^{\otimes 2}$).

Note: 1-forms are referred to as "abelian differentials" in the literature.

We know that $K_{\mathbb{P}^1} = \mathcal{O}(-2)$ and $K_{\mathbb{P}^1}^{\otimes 2} = \mathcal{O}(-4)$.

3.2 Moduli Spaces

Definition: $\mathcal{H}(k_1, \cdots, k_n) = \left\{ \Sigma \text{with abelian differential} \div \omega = \sum k p_i \right\}$ where the k_i record the orders of zeros of ω.

Second condition on divisor records zeros.

Similarly, define $Q(k_1, \cdots, k_n) = \left\{ \cdots \;\middle|\; \text{quadratic differential} \right\}$ where now $k_i \geq -1, \neq 0$.

These moduli spaces are called *strata of abelian/quadratic differentials.*

Consider M_g, the moduli space of genus g surfaces. There is a vector bundle (the Hodge bundle) over M_g where the fiber over $[c]$ is $H^0(K_C)$ with strata given by Q.

This is a bundle of rank g, and by Riemann-Roch, ? $3g - 3$ and is the (co?)tangent bundle of something.

There is an $\mathrm{SL}(2, \mathbb{R})$ action on any stratum. How to define – use isomorphism with translation structures (see image) by just applying any such form γ to the entire plane and gluing polygons in the same way. This gives a non-holomorphic action on any stratum.

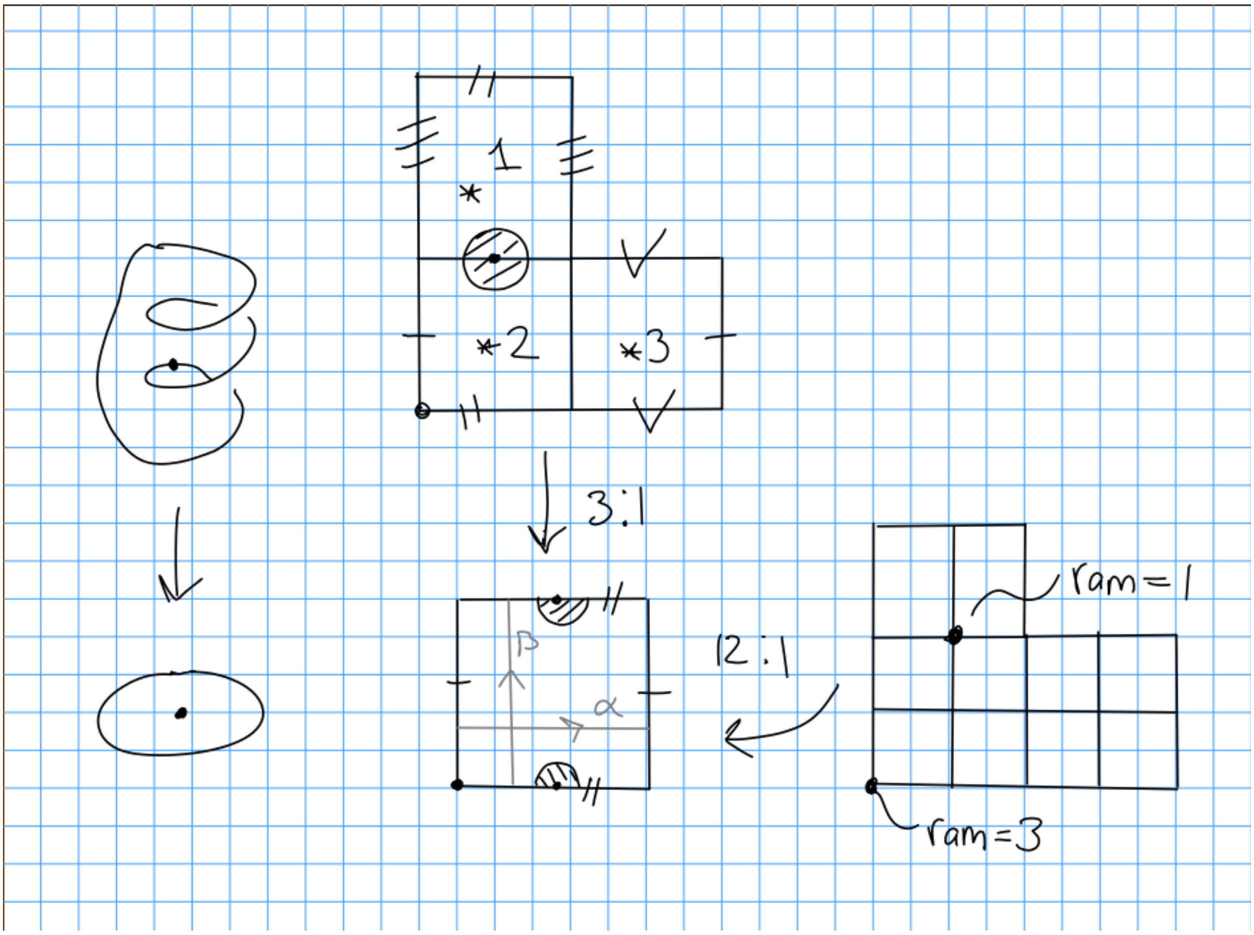
Next time: to special case of square tiled surfaces.

References:

- Esk. Ok 2001 Branched Covers of Torus, and 2005 for half-translation surfaces
- Elliptic orbifolds, E. 2017
- 1-cylinder surfaces

Square tiled surfaces in $\mathcal{H}(k)$ with d squares correspond to degree d branched covers of the identification square, branched over the origin, with profile (?).

To count square-tiled surfaces: label squares, look at inverse images of $*$ by $\{1, \cdots, d\}$. Consider the monodromy representation $\rho : \pi_1(\mathbb{T} \setminus \{0\}, *) \to S_d$ where $\sigma = \rho(\alpha) = (1)(23)$ and $\tau = \rho(\beta) = (12)(3)$. We compute ramification orders by considering the commutators $[\alpha, \beta]$. Then $\rho([\alpha, \beta])$ has cycle type $(1, 1, \cdots 1, 1 + k, \cdots, 1 + k_n)$. Note that $[(23), (12)]$ is a 3 cycle.



Conclusion: The number of square-tiled surfaces in $\mathcal{H}(k)$ with d squares is exactly $\frac{1}{d!}\left|\sigma, \tau \in S_d \;\middle|\; [\sigma, \tau] \in C_{1,\cdots,1,1}\right.$

Note that the division is due to the artificial labeling of squares.

Presentation next week: Eskin-Okounov, Branched Covers of Torus.

Main theorem: The generating function $f_\kappa(q) := \sum_{d \geq 1} \#\left\{\text{Square tiled, } d \text{ squares in } \mathcal{H}(k)\right\} q^d$ is a modular form. Follows from taking $q = e^{2\pi i \tau}$ which is holomorphic on $\mathbb{H}$ the upper half-plane, satisfying a transformation rule with respect to $\tau \mapsto -1/\tau$, which is a finite-dimensional space.

Actual: quasimodular mixed form.

The weights are bounded by $|\kappa| + \ell(\kappa)$.

Concretely, $f_\kappa \in \mathbb{Q}[E_2, E_4, E_8]$ where $E_k(q) = \text{const} + \sum_{d \geq 1} \sigma_{k-1}(d) q^d$, where $\sigma_{k-1}(d) = \sum_{e \mid d} e^{k-1}$.

This is the ring of quasimodular forms.

- 1 is weight 0,
- E_2 is weight 2,
- E_2^2, E_4 are weight 4,
- E_2^3, E_2E_4, E_6 are weight 6, etc

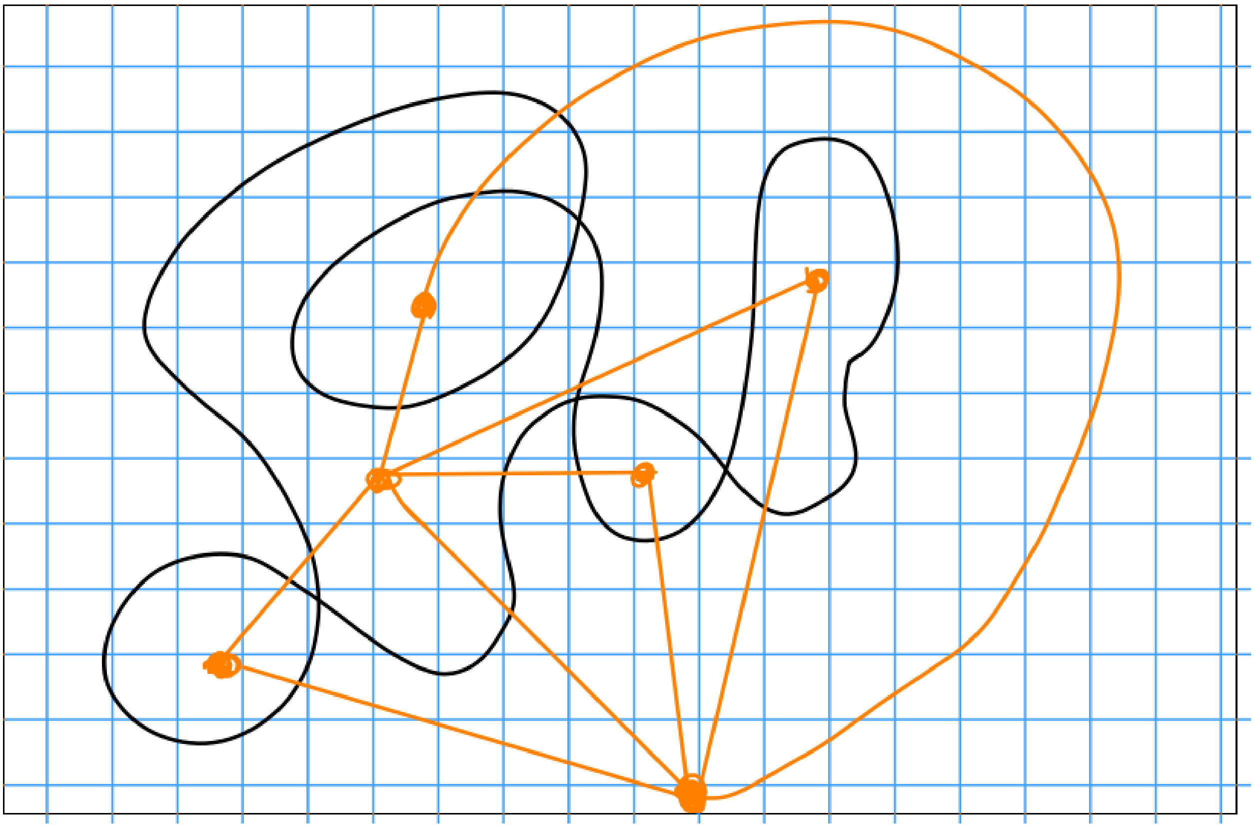


Figure 2: Image

Example: Take $\kappa = \{2\} \iff \mathcal{H}(2)$, then $|\kappa| + \ell(\kappa) = 3$ and $f_{\{z\}}(q) = c_1 + c_2 E_2(q)$. In this case $[q^1] = [q^2] = 0$.

Note that surfaces in $\mathcal{H}(2)$ have 1 vertex of cone angle 6π and all others of angle 2π, corresponding to an abelian differential with a single zero of order 2.

A special type of square-tiled surfaces: 1 cylinder surfaces, where $\rho(\alpha)$ is a full length cycle.

This is in $\mathcal{H}(3, 1)$, and corresponding surface (Σ, ω), which is a holomorphic 1-form with a triple zero and a single zero. By Riemann-Hurwitz, $2g - 2 = \deg \omega = 3 + 1 \implies g = 3$.

Note: difficult to compute otherwise!

Another paper: "One Cylinder Surfaces", Delecroix, Goujard, Zovrch, Zograf. See Phil for appendix.

Result: 1-cylinder surfaces have roughly a $1/d$ proportion in all square tiled surfaces, where $d = \dim \mathcal{H}(\kappa)$.

Recall that we can get a square tiled surface from any unicursal curve

Note that these aren't always translation surfaces:

This has transition maps that looks like $z \mapsto i^k z + z_0 = w$ and thus $dz = i^k dw$, so dz^k is the well-defined object here.

Recall

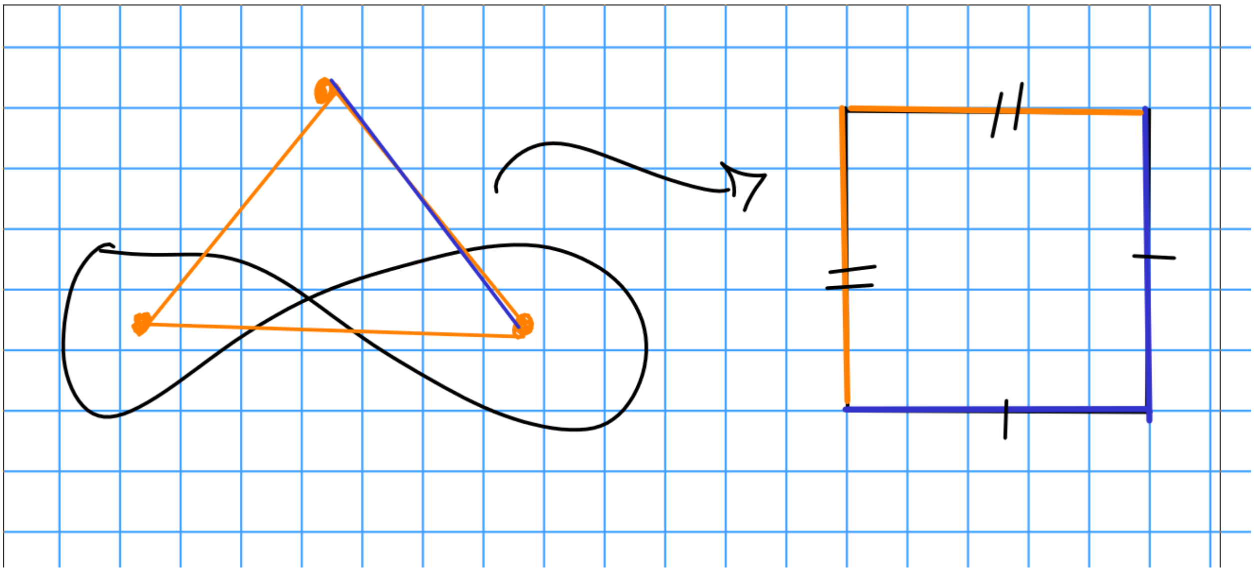
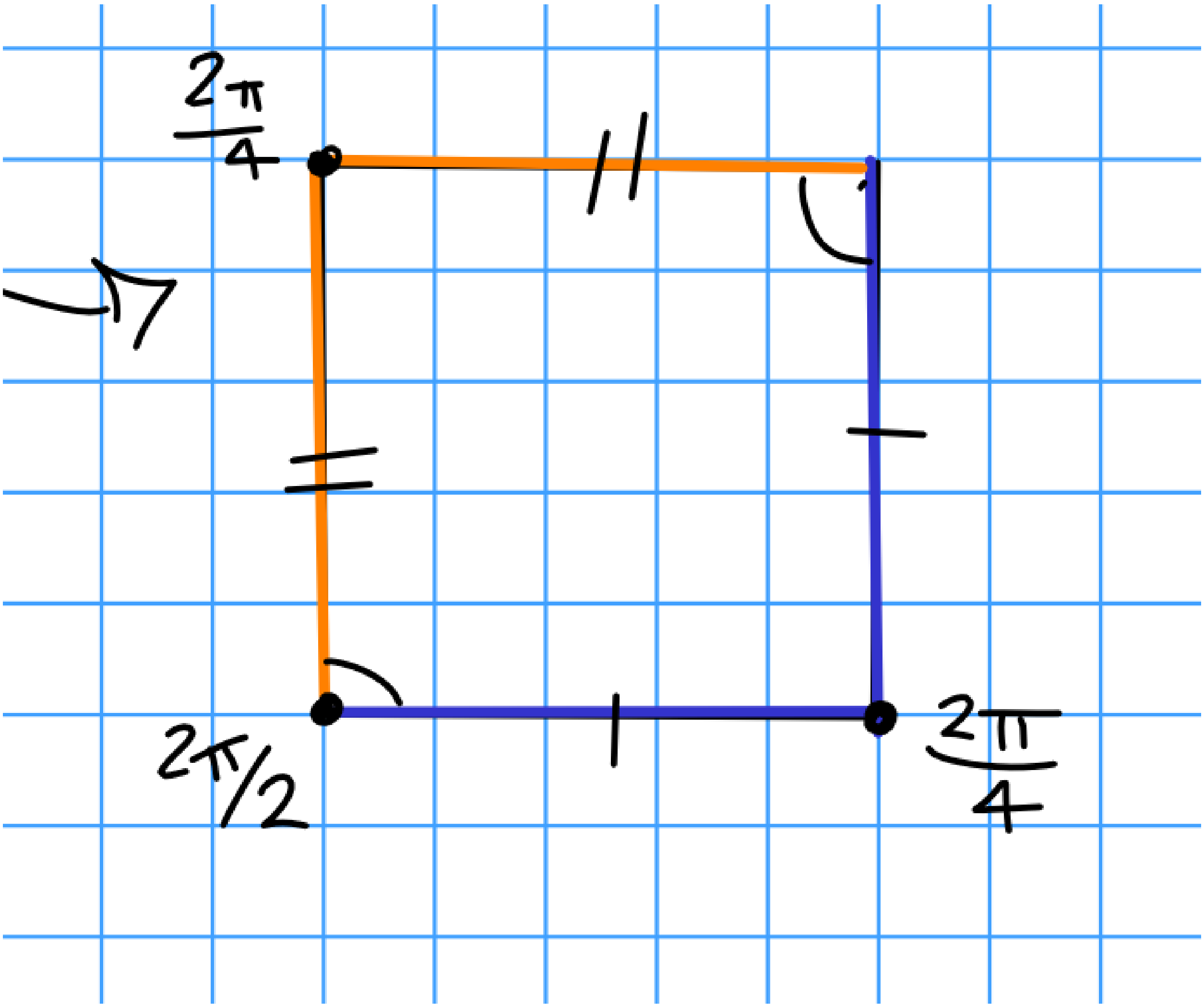


Figure 3: Image

- $(\Sigma, dz) \iff$ translation surfaces
- $(\Sigma, (dz)^2) \iff \frac{1}{2}$-translation surfaces
- $(\Sigma, (dz)^4) \iff \frac{1}{4}$-translation surfaces
  - I.e. a Riemann surface with a section of $K_{\varepsilon}^{\otimes 4}$.

Can consider a tricursal curve instead (lift the pen 3 times), taking dual complex yields a cube. This has charts $w = z^{4/3}$ and thus $(dz)^4 = w^{-1}(dw)^4$. Let $\mathcal{H}(\kappa)$ be the quartic differentials with $dv\omega = \sigma\kappa_i p_i$. Then the cube is in $\mathcal{H}_4(-1, \cdots -1)$ with 8 copies of -1.

This gives cone angles $n\dfrac{2\pi}{4}$ and the order of the zero/pole is $n - 4$.



This example is in $\mathcal{H}_4(-3,-3,-2)$.

Proposition: The generating functions for square-tiled surfaces $\mathcal{H}_4(\kappa)$ is now a quasimodular form for $\Gamma_1(4)$.

Question (can find numerical evidence?): How can we count these in terms of the symmetric group? Analogous result to proportion result earlier? Can try to lift square example, but admits no map from a torus – instead, quotient square by $\mathbb{Z}/4\mathbb{Z}$ and take fundamental domain. What kind of branching do these covers have?

Every center of a square is branched of order 4. Every center of an edge is branched of order 2. The ramification order of a vertex is its valence, divided by the number of squares meeting at that vertex. The degree of the covering map is $4n$ where n is the number of squares.

Identify the fundamental domain with $\mathbb{P}^1$, We get a monodromy representaiton

$$\begin{aligned}\rho : \pi_1(\mathbb{P}^1 \setminus (0,1,\infty), *) &\to S_{4d}\\ \gamma_0 &\mapsto \rho(\gamma_0).\end{aligned}$$

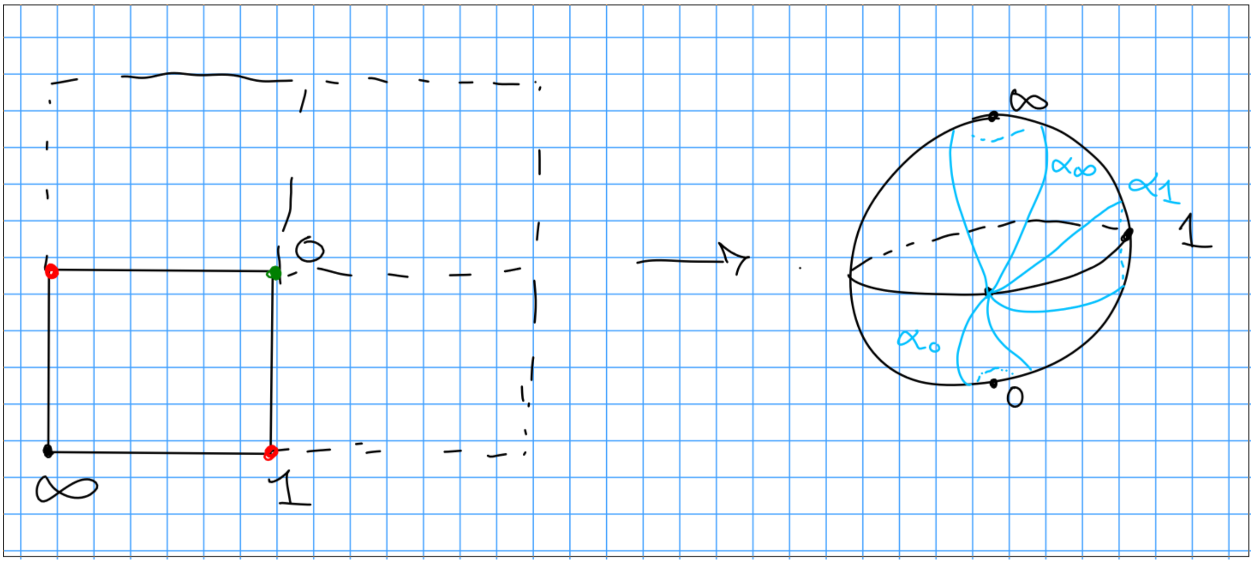


Figure 4: Image

Note that $\gamma_0\gamma_1\gamma_\infty = 1$.

It then follows that this has cycle type $(4, \cdots, 4)$.

So the number of square tiled surfaces in $\mathcal{H}_4(\kappa)$ is given by

$$\frac{1}{(4d)!} = \#\left\{(\sigma_0, \sigma_1, \sigma_\infty) \;\middle|\; \sigma_0 \in C_{4,\cdots,4}(d), \sigma_1 \in C_{2,\cdots,2}(2d), \sigma_\infty \in C_{4,\cdots,4,4+k,\cdots,4+k_n}, \sigma_0\sigma_1\sigma_\infty = 1\right\}.$$

Would be nice to figure out what the proportionality constant here is.