Problem Set One

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January 27, 2020

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1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee / \Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$.

Proposition: $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M

Proof: It suffices to check that $\mathfrak{g} \curvearrowright M^{[\lambda]} \subseteq M^[\lambda]$, i.e. this module is closed under the action of $U(\mathfrak{g})$. Let $g \in U(\mathfrak{g})$ and $m \in M^{[\lambda]}$ be arbitrary. Choose a ordered basis $\{e_i\}$ for $\mathfrak{g}$, then this can be extended to a PBW basis for $U(\mathfrak{g})$ given by $\left\{ \prod_i e_i^{t_i} \mathrel{\Big|} t_i \in \mathbb{Z} \right\}$. Then take a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. We can then write $u = \prod_i a_i^{t_i} \prod_j h_j^{t_j} \prod_k b_k^{t_k}$ and consider how each component acts.

First considering how the b_k act, we compute their weights; we want to show that if $\mu \in M_\mu$ for some $\mu \in [\lambda]$, then $b_k \curvearrowright \mu \in M_{u'}$ for some $m' \in [\lambda]$.

We know $h \curvearrowright m = \mu(h)m$ for each $m \in M_\mu$. Noting that $b_k \in g_\alpha$ for some positive root α, we have $[hg] = \alpha(h)g$, and so

$$\begin{aligned}
h \curvearrowright (b_k \curvearrowright m) &= b_k \curvearrowright (h \curvearrowright m) + [hb_k] \curvearrowright m \\
&= b_k \curvearrowright (\mu(h)m) + [hb_k] \curvearrowright m \\
&= b_k(\mu(h)m) + \alpha(h)b_k m \\
&= (\mu(h) + \alpha(h))b_k m \\
&\in M_{\mu+\alpha}.
\end{aligned}$$

But then $\mu + \alpha - \mu = \alpha \in \mathbb{Z}\Phi = \Lambda_r$, so μ and $\mu + \alpha$ are in the same coset $[\lambda]$. The same argument shows that $h \curvearrowright (b_k^t \curvearrowright m))$ is in the weight space $M_{\mu+t\alpha}$, which still only differs by an integral number of roots.

But this shows that $U(\mathfrak{n})$ and $U(\mathfrak{n}^-)$ leave this space invariant, and $U(\mathfrak{h})$ acts by scaling, which preserves subspaces. So $M^{[\lambda]}$ is closed under the action of $\mathfrak{g}$.

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Proposition: M is the direct sum of finitely many submodules of the form $M^{[\lambda]}$.

Proof:

By axiom 1 for Category $\mathcal{O}$, M is finitely generated, say by $\{m_j\}$, This category is closed under subobjects, so if we write $M = \oplus_{[\lambda]} M^{[\lambda]}$ as a union over all cosets, each $M^{[\lambda]}$ is finitely generated as well. Since m_1 is in this direct sum, it is in *finitely* many summands by definition of the direct sum, so for each j, $m_j \in \bigoplus_{k=1}^{R_j} M^[\lambda_{jk}]$ for some finite constant R_j and some coset depending on j and k. But then $M = \bigoplus_j \bigoplus_k M^{[\lambda_{jk}]}$ is still a finite direct sum, which is what we wanted to show.

Proposition: If M is indecomposable, then all weights of M lie in a single coset.

Proof: By (a), we can write $M = \bigoplus_{[\lambda_i]} M^[\lambda_i]$ for some finite set of λ_is. If M is indecomposable, then there can only be one summand, and so $M = M^[\lambda]$ for exactly 1 λ. We can then write $M = \sum_{\mu \in [\lambda]} M_\mu$, which decomposes M as a sum of weight spaces. But then if any $\sigma \in \Pi(M)$ is a weight, it must be one of the μ occurring above. So every weight of M is in the coset $[\lambda]$, and in particular they are all in the same coset.

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1.2 b

Proposition: The weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee / \Lambda_r$.

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Proposition: For any $M \in \mathcal{O}$, $M(\lambda)$ satisfies the following property:

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right).$$

Proof:

Noting that

- $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$,
- $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$ is an identification of the $\mathfrak{g}$-module M has a $\mathfrak{b}$- module by restricting the action of $\mathfrak{g}$,

consider the following two maps:

$$
\begin{aligned}
F : \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) &\to \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) \\
\phi &\mapsto (F\phi : z \mapsto \phi(1 \otimes z)),
\end{aligned}
$$

and using the action of $\mathfrak{g}$ on M,

$$
\begin{aligned}
G : \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) &\to \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) \\
\psi &\mapsto (G\psi : g \otimes v \mapsto g \curvearrowright \psi(v)).
\end{aligned}
$$

Note that the maps $G\psi$ are defined on ordered pairs, but are clearly bilinear and thus uniquely extend to maps on the tensor product.

It suffices to show that these maps are well-defined and mutually inverse.

To see that F is well-defined, let $\phi : U(\mathfrak{g}) \otimes C_\lambda \to M$ be fixed; we will show that the set map $F\phi : \mathbb{C}_\lambda \to M$ is $U(\mathfrak{b})$-linear. Let $b \in U(\mathfrak{b})$, then

$$
\begin{aligned}
b \curvearrowright F\phi(v) &:= b \curvearrowright (z \mapsto \phi(1 \otimes z))(v) \\
&:= b \curvearrowright \phi(1 \otimes v) \\
&= \phi(b \curvearrowright (1 \otimes v)) \quad \text{since } \phi \text{ is } U(\mathfrak{g})\text{-linear and } b \in U(\mathfrak{g}) \\
&= \phi((b \curvearrowright 1) \otimes v) \quad \text{by the definition/construction of } M(\lambda) \text{ as a } U(\mathfrak{g})\text{-module.} \\
&= \phi(1 \otimes (b \curvearrowright v)) \quad \text{since } \mathbb{C}_\lambda \text{ is a } \mathfrak{b}\text{-module and the tensor is over } U(\mathfrak{b}) \\
&:= (z \mapsto \phi(1 \otimes z))(b \curvearrowright v) \\
&:= F\phi(b \curvearrowright v).
\end{aligned}
$$

To see that G is well-defined, let $\psi : C_\lambda \to M$ be fixed; we will show that the set map $G\psi : U(\mathfrak{g}) \otimes C_\lambda \to M$ is $U(\mathfrak{g})$-linear. Let $u \in U(\mathfrak{g})$, then

$$
\begin{aligned}
u \curvearrowright G\psi(g \otimes v) &:= u \curvearrowright (g \otimes v \mapsto g \curvearrowright \psi(v))(g \otimes v) \\
&:= u \curvearrowright (g \curvearrowright \psi(v)) \\
&= (ug) \curvearrowright \psi(v) \quad \text{since } M \text{ is a } \mathfrak{g}\text{-module with a well-defined action.} \\
&:= (g \otimes v \mapsto g \curvearrowright \psi(v))(ug \otimes v) \\
&:= G\psi(ug \otimes v).
\end{aligned}
$$

To see that FG is the identity, let ϕ be defined as above and fix $g_0 \otimes v_0 \in U(\mathfrak{g}) \otimes \mathbb{C}_\lambda$. Then

$$
\begin{aligned}
GF\phi(g_0 \otimes v_0) &= G(v \mapsto \phi(1 \otimes v))(g_0 \otimes v_0) \\
&\coloneqq G(f) \quad \text{for notational convenience} \\
&\coloneqq G(g \otimes v \mapsto g \curvearrowright f(v))(g_0 \otimes v_0) \\
&= g_0 \curvearrowright f(v_0) \\
&= g_0 \curvearrowright \phi(1 \otimes v_0) \\
&= \phi(g \curvearrowright (1 \otimes v_0)) \quad \text{since } g_0 \in \mathfrak{g} \text{ and } \phi \text{ thus commutes with the } \mathfrak{g}\text{-action by definition} \\
&= \phi(g_0 \curvearrowright 1 \otimes v_0) \quad \text{by definition of the action on } U(\mathfrak{g}) \otimes C_\lambda \text{ as a } U(\mathfrak{g}) \text{ module} \qquad \coloneqq \phi(g_0 \\
&.
\end{aligned}
$$

To see that $GF \coloneqq G \circ F$ is the identity, let ψ be defined as above and fix $z_0 \in \mathbb{C}_\lambda$. Then

$$
\begin{aligned}
FG\psi(z_0) &= F(g \otimes v \to g \curvearrowright \psi(v))(z_0) \\
&\coloneqq F(\lambda)(z_0) \quad \text{for notational convenience} \\
&= (v \mapsto \lambda(1 \otimes v))(z_0) \\
&= \lambda(1 \otimes z_0) \\
&\coloneqq 1 \curvearrowright \psi(z_0) \\
&= \psi(z_0).
\end{aligned}
$$

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