Algebra Qual Prep Week 1: Groups Warmup

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1 | Week 1: Finite Groups

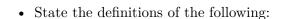
See the Presentation Schedule

1.1 Week 1 Topics



- Subgroups
 - The one-step subgroup test
 - Cosets
 - The index of a subgroup
 - Normal subgroups
 - Quotients
 - The normalizer of a subgroup
 - Maximal and proper subgroups
 - Characteristic subgroup
- Cauchy's theorem
- Lagrange's theorem
- Definitions and properties of common special families of groups:
 - Cyclic groups C_n
 - Symmetric groups S_n
 - Alternating groups A_n
 - Dihedral groups D_n
 - The quaternion group Q_8
 - Matrix groups $GL_n(k)$, $O_n(k)$, $SL_n(k)$, $SO_n(k)$
 - p-groups
 - Free groups F_n (and presentations/relations)
- The 4 fundamental isomorphism theorems
- Finite groups of order $\#G \le 20$
- Structure:
 - Cyclic -> Abelian -> Nilpotent -> Solvable -> All Groups

1.2 Review Exercises



- Group morphism (aka group homomorphism)
- Centralizer
- Normalizer

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- Conjugacy class
- Center
- Inner automorphism
- Commutator
- p-group
- Write definitions or presentations for all of the special families of groups appearing above.
- State what it means for a cycle to be even or odd.
- Find a counterexample for the converse of Lagrange's theorem.
- State the 4 fundamental isomorphism theorems

1.3 Unsorted Questions

For everything that follows, assume G is a finite group.

- $H \leq G$ denotes that H is a subgroup of G.
- #G denotes the order of G.
- e or e_G denotes the identity element of G.
- Multiplicative notation is generally used everywhere to denote the (possibly noncommutative) binary operation

1.3.1 Orders

- Prove Lagrange's theorem.
- Prove Cauchy's theorem.
- Prove that if #G is prime, then G is cyclic
- Prove that for every $g \in G$, the order of g divides the order of G.
- Prove that if #G = n, then $g^n = e$ for every $g \in G$

1.3.2 Cosets

- Let $H \leq G$ be a subgroup (not necessarily normal), and let G/H denote the set of left cosets of G by H. Prove that any two cosets $xH, yH \in G/H$ have the same cardinality.
- Prove the fundamental theorem of cosets: for $xH, yH \in G/H$,

$$xH = yH \iff x^{-1}y \in H \iff y^{-1}x \in H$$

• Suppose #G = pq with $p, q \ge 2$ prime, and let $H \le G$ be a proper subgroup. Prove that H must be cyclic.

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1.3 Unsorted Questions

1.3.3 Normal Subgroups

- Let $s \in G$, and state the definition of the **centralizer** of $C_G(s)$ of s in G.
 - Show that $C(s) \leq G$ is a subgroup.
 - Let $\langle s \rangle \subseteq C_G(s)$, where $\langle s \rangle$ is the subgroup of G generated by s.
 - Prove that $\langle s \rangle \subseteq G$ is in fact a **normal** subgroup.
- Let $H \leq G$ be a subgroup and $N \subseteq G$ be a normal subgroup. Show that $NH \leq G$ is a subgroup.
- Let G_1, G_2 be groups and $H_2 \leq G_2$ a subgroup. Suppose $\varphi : G_1 \to G_2$ is a group morphism.
- Suppose $\varphi: G_1 \to G_2$ is a group morphism.
 - Show that the image $\varphi(G_1) \leq G_2$ is a subgroup of G_2
 - Show that the preimage $\varphi^{-1}(H_2) \leq G_1$ is a subgroup of G_1 ,
 - Show that the kernel $\ker \varphi \subseteq G_1$ is a normal subgroup of G_1 .
 - Prove that group morphisms preserve coset structure in the following sense:

$$xH_1 = yH_1 \iff \varphi(x)H_2 = \varphi(y)H_2.$$

- Prove the first isomorphism theorem: φ is injective \iff ker $\varphi = \{e_{G_1}\}.$

1.3.4 Symmetric Groups

- Let $\sigma = (421)(6132) \in S_6$ in cycle notation.
 - Write σ as a product of disjoint cycles.
 - Compute the order of σ . What is the general theorem about the order of cycles?
 - Determine if σ is even or odd. What is the general theorem?
- Suppose $\varphi: S_n \to G$ with n even and #G = m odd.
 - Prove that if $\tau \in S_n$ is a transposition, then $\tau \in \ker \varphi$.
 - Prove that in fact every $\sigma \in S_n$ satisfies $\sigma \in \ker \varphi$, so φ is the trivial morphism.
 - Does this hold if n is odd?

1.3.5 Matrix Groups

- Let \mathbb{F}_p be the finite field with p elements, where p is a prime. Show that the centers of $GL_n(\mathbb{F}_p)$ and $SL_n(\mathbb{F}_p)$ consist only of scalar matrices.
 - Show that the scalars ζ that appear in scalar matrices $Z(\mathrm{SL}_n(\mathbb{F}_p))$ are roots of unity in \mathbb{F}_p , i.e. $\zeta^p = 1$.

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• Determine the orders $\#\operatorname{GL}_n(\mathbb{F}_p)$ and $\#\operatorname{SL}_n(\mathbb{F}_p)$.

1.3 Unsorted Questions

1.4 Warmup Problems



- (Important) Prove that if G/Z(G) is cyclic then G is abelian.
- (Important) Classify all groups of order p^2 .
- (Important) Show that if $H \leq G$ and [G:H] = 2 then H is normal.
 - Suppose that the same result holds with 2 replaced by p defined as the smallest prime factor of #G
- Prove that if $H \leq G$ is a proper subgroup, then G can not be written as a union of conjugates of H.
 - Use this to prove that if $G = \operatorname{Sym}(X)$ is the group of permutations on a finite set X with #X = n, then there exists a $g \in G$ with no fixed points in X.
- Let $G \leq H$ where H is a finite p-group, and suppose $\varphi : G \to H/[H,H]$ be defined by composing the inclusion $G \hookrightarrow H$ with the natural quotient map $H \to H/[H,H]$.

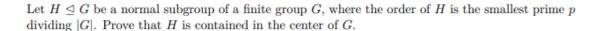
Prove that G = H by induction on #H in the following way:

- Letting $N \leq H$ be any nontrivial normal subgroup of H, use the inductive hypothesis to show that H = GN.
- Let Z = Z(H) be the center of H. Using that GZ = H by (1), show that $G \cap Z \neq \emptyset$. Set $N := G \cap Z$ and apply (1) to conclude.
- Determine all pairs $n, p \in \mathbb{Z}^{\geq 1}$ such that $\mathrm{SL}_n(\mathbb{F}_p)$ is solvable.

1.5 Qual Problems



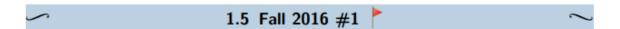
1.13 Spring 2021 #2



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Let G be a finite group with n distinct conjugacy classes. Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G. Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

Relevant concepts omitted.



Let G be a finite group and $s, t \in G$ be two distinct elements of order 2. Show that subgroup of G generated by s and t is a dihedral group.

Recall that the dihedral groups of order 2m for $m\geq 2$ are of the form

$$D_{2m} = \left\langle \sigma, \tau \mid \sigma^m = 1 = \tau^2, \tau \sigma = \sigma^{-1} \tau \right\rangle.$$

\sim 1.6 Fall 2015 #1 $\stackrel{ extstyle \sim}{ extstyle \sim}$

Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H which is of finite index.

$$\sim$$
 1.7 Spring 2015 #1 $\stackrel{ extstyle }{\sim}$

For a prime p, let G be a finite p-group and let N be a normal subgroup of G of order p. Prove that N is contained in the center of G.

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1.12 Fall 2019 Midterm #5

Let G be a nonabelian group of order p^3 for p prime. Show that Z(G) = [G, G].

\sim 1.13 Spring 2021 #2 $\stackrel{ extstyle }{\sim}$

Let $H \subseteq G$ be a normal subgroup of a finite group G, where the order of H is the smallest prime p dividing |G|. Prove that H is contained in the center of G.

1.5 Qual Problems