Problem Set 2

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February 9, 2020

Contents

1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ can not lie in $\mathcal{O}$.

Useful facts:

- For any $\lambda \in \mathfrak{h}^\vee$, $\mathbb{C}_\lambda$ is a 1-dimensional $\mathfrak{b}$-bimodule with a trivial $\mathfrak{n}$-action.
- $M(\lambda) = \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ as a left $U(\mathfrak{g})$-module.
- $M(\lambda) = U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$ as a left $U(\mathfrak{n}^-)$-module.
- $M(\lambda)$ is generated as a $U(\mathfrak{g})$-module by the maximal vector $v^+ = 1 \otimes 1$.
- The set of weights of $M(\lambda)$ is $\lambda - \Gamma$ where Γ is the semigroup in Λ_r generated by Φ^+.
- $M(\lambda)$ has weights $\lambda, \lambda - 2, \lambda - 4, \cdots$ each with multiplicity 1.

Questions

- What is the tensor product over? Guess: $\otimes_{\mathbb{C}}$.
- MSE: the product is no longer finitely generated.
  - Consider dimensions of weight spaces – eventually constant.
  - If $\mathrm{wt} v = \lambda$ and $\mathrm{wt}(u) = \mu$, then $\mathrm{wt}(u \otimes v) = \lambda + \mu$.
  - Consider a weight space N_γ of M. This must have the form $\bigoplus_{a+b=\gamma} M_a \otimes_{\mathbb{C}} M_b$.
  - Example: consider $\lambda = \mu = 0$. Then $M = M(0) \otimes M(0)$ and N_{-2m} has dimension $m + 1$ for every $m \in \mathbb{Z}^+$.

Solution:

Let $M(\lambda), M(\mu)$ be arbitrary Verma modules with highest weight vectors $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$ respectively. We can then consider the weight of $v \otimes w$ in $N := M(\lambda) \otimes_{\mathbb{C}} M(\mu)$:

$$\begin{aligned} h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\ &= \lambda(h) v \otimes w + v \otimes \mu(h) w \\ &= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\ &= (\lambda(h) + \mu(h))(v \otimes w). \end{aligned}$$

Letting $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, so $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$, the claim is that it is possible for N to *not* be finitely-generated as a $U(\mathfrak{g})$-module.

Let $\{y, h, x\}$ be the usual basis for $\mathfrak{g}$, for which $U(\mathfrak{g})$ has the usual associated PBW basis. We can use the fact that $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$, so if we pick $\mu, \lambda \in \mathbb{Z}^{\leq 0}$ we have weight space decompositions

$$\begin{aligned} M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda - 2i} := \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i := \lambda - 2i}} M(\lambda)_{\lambda_i} \\ M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu - 2j} := \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_i := \mu - 2j}} M(\mu)_{\mu_j} \end{aligned}$$

where we can explicitly identify bases $M(\lambda)_{\lambda_i} = \operatorname{span}_{\mathbb{C}} \left\{ y^i \ v \right\}$ and $M(\mu)_{\mu_i} = \operatorname{span}_{\mathbb{C}} \left\{ y^i w \right\}$.

By the initial observation, this yields a weight space decomposition for N given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left(\bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_\nu.$$

Since each weight space $N_\nu = \operatorname{span}_{\mathbb{C}} \left\{ y^i v \otimes y^j w \;\middle|\; i + j = \nu \right\}$ and there are infinitely many such weight spaces, no finite number of elements in the PBW basis for $U(\mathfrak{g})$ can generate N.

$\blacksquare$

2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ.

> Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Useful facts:

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