Title

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1 Chapter 5: Submanifolds

1.1 When Submanifolds are Embedded

The most important type of manifolds: embedded submanifolds. Most often described as the *level set* of a smooth map, but needs extra conditions. The level sets of constant rank maps are always embedded submanifolds.

More general: immersed submanifolds. Locally embedded, but may have global topology different than the subspace topology.

Definition 1.0.1 (Embedded Submanifolds)**.**
For $S \subseteq M$ in the subspace topology, with a smooth structure such that the inclusion $S \hookrightarrow M$ is smooth. If $S \hookrightarrow M$ is a proper map, then S is **properly embedded**.

Definition 1.0.2 (Embedded Hypersurface)**.**
An embedded submanifold of codimension 1.

Proposition 1.1 ***(Embedded Codimension 0 Subsets are Open Submanifolds).***
A subset $S \subseteq M$ of codimension zero is an embedded submanifold iff S is an open submanifold.

A way to produce submanifolds: :::{.proposition} If $F : N \longrightarrow M$, then $F(N)$ is a submanifold of M with the subspace topology and a unique smooth structure making F a diffeomorphism onto its image and $F(N) \hookrightarrow M$ and embedding. :::

Thus every embedded submanifold is the image of an embedding, namely its inclusion.

Embedded submanifolds are exactly the images of smooth embeddings: :::{.proposition} The slices $M \times \{p\}$ for $p \in N$ are embedded submanifolds of $M \times N$ diffeomorphic to M. :::

Proposition 1.2.
For $f : U \longrightarrow N$ with $U \subseteq M$,

$$\Gamma(f) \coloneqq \left\{(x, f(x)) \in M \times N \;\middle|\; x \in U,\right\} \hookrightarrow M \times N$$

is an embedded submanifold.

Note: any manifold that is locally the graph of a smooth function is an embedded submanifold.

Proposition 1.3.
$S \hookrightarrow M$ is a properly embedded submanifolds $\iff$ S is a closed subset of M. Thus every compact embedded submanifold is properly embedded.

1.2 The Slice Condition

Embedded submanifolds are locally modeled on the standard embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ where $\mathbf{x} \mapsto [\mathbf{x}, \mathbf{0}]$.

Proposition 1.4 ***(Local k-slice Condition).***
$S \subseteq M$ satisfies the **local k-slice condition** iff each $s \in S$ is in the domain of a smooth chart (U, φ) such that $S \bigcap U$ is a single k-slice in U.

Proposition 1.5 ***(Local Slice Criterion for Embeddings).***
$S \hookrightarrow M$ is an embedded k-dimensional submanifold $\iff$ S satisfies the local k-slice condition. Moreover, there is a unique smooth structure on S for which this holds.

For manifolds with boundary, $\partial M \hookrightarrow M$ is a proper embedding. Every such manifold can be embedded in a larger manifold $\tilde{M}$ without boundary.

1.3 Level Sets

Definition 1.5.1 (Level Sets).
For $\varphi : M \longrightarrow N$ and $c \in N$, $\varphi^{-1}(c)$ is a *level set* of φ.

Examples:

- $f(x, y) = x^2 - y$, then $V(f) \hookrightarrow \mathbb{R}^2$ is an embedding since it is the graph of the smooth function $x \mapsto x^2$.
- $f(x, y) = x^2 - y^2$ is not an embedded submanifold.
- $f(x, y) = x^2 - y^3$ is not an embedded submanifold.

Every closed $S \subset M$ is the zero set of some smooth function $M \longrightarrow \mathbb{R}$.

Theorem 1.6 ***(Constant Rank Level Set Theorem).***
For $\varphi : M \longrightarrow N$ with constant rank r, each level set of φ is a properly embedded codimension r submanifold.

Corollary 1.7 ***(Submersion Level Set Theorem).***
If $\varphi : M \longrightarrow N$ is a smooth submersion, then the level sets are properly embedded of codimension $\dim N$.

Proof .
Every smooth submersion has constant rank equal to the dimension of the codomain. ■

Analogy: for $L : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ a surjective linear map, $\ker L \leq \mathbb{R}^m$ has codimension r by rank-nullity. Surjective linear maps are analogous to smooth submersions.

Definition 1.7.1 (Regular and Critical Points)**.**
If $\varphi : M \longrightarrow N$ is smooth, $p \in M$ is a **regular point** if $d\varphi$ is surjective and a **critical point** otherwise. A point $c \in N$ is a **regular value** if every point in $\varphi^{-1}(c)$ is a regular point, and a **critical value** otherwise. A set $\varphi^{-1}(c)$ is a **regular level set** iff c is a regular value.

Note that if every point of M is critical then $\dim M < \dim N$, and every point is regular $\iff$ F is a submersion. The set of regular points is always open.

Theorem 1.8 ***(Regular Level Set Theorem).***
Every regular level set of a smooth map $\varphi : M \longrightarrow N$ is a properly embedded submanifold of codimension $\dim N$.

Definition 1.8.1 (Defining Map for an Embedding)**.**
If $S \hookrightarrow M$ is an embedded submanifold, a **defining map** for S is the smooth map $\varphi : M \longrightarrow N$ such that S is a regular level set of φ, if such a map exists.

Example: $f(\mathbf{x}) = \|\mathbf{x}\|^2$ is the defining map for S^n.

Not every embedded submanifold is the level set of a smooth submersion globally, but this does hold locally. I.e., every embedded submanifold admits a local defining map: :::{.proposition} $S_k \hookrightarrow M_m$ is an embedded k-dimensional submanifold $\iff$ every $s \in S$ admits a neighborhood U such that $U \bigcap S$ is the level set of a smooth submersion $U \longrightarrow \mathbb{R}^{m-k}$. :::

1.4 Immersed Submanifolds

Immersed submanifolds: more general than embedded submanifolds. Encountered when studying Lie subgroups, where subsets will be the images of injective immersions but not necessarily embeddings (example: figure eight curve).

Definition 1.8.2 (Immersed Submanifold)**.**
A subset $S \subseteq M$ equipped with some topology for which the inclusion $S \hookrightarrow M$ is a smooth immersion is said to be an **immersed submanifold**.

Convention: smooth submanifolds always denote immersions, whereas embeddings are a special case.