MATH 8320 HOMEWORK

PETE L. CLARK

0. Problem Set 0

Exercise 0.1. *There three notions of finite generation in play for a field extension l/k: (i) l is finitely generated as a k-module (equivalently, finite-dimensional as a k-vector space) – we also say that l/k has finite degree – (ii) l is finitely generated as a k-algebra: there are $x_1, \ldots, x_n \in l$ such that $l = k[x_1, \ldots, x_n]$: every element of l can be expressed as a polynomial in $x_1, \ldots, x_n$ with coefficients in k. (iii) l/k is finitely generated as a field extension.*

a) *Show: l/k finitely generated as a module implies l/k finitely generated as a k-algebra implies l/k finitely generated as a field extension.*
b) *Let $k(t)$ be the rational function field over k – the fraction field of the polynomial ring $k[t]$. Show: $k(t)/k$ is finitely generated as a field extension but is not finitely generated as a k-algebra.*
c) *Show: $k[t]/k$ is finitely generated as a k-algebra but not as a k-module. (However $k[t]$ is not a field!)*
d) *Can you exhibit a field extension l/k such that l is finitely generated as a k-algebra but not as a k-vector space?*
(Hint: no, you can't – this is a famous result of commutative algebra!)
e) *Suppose l/k is algebraic and finitely generated as a field extension. Show that l/k has finite degree.*

Exercise 0.2. *Show that every finitely generated field extension $K = k(x_1, \ldots, x_n)$ is the fraction field of a quotient of $k[t_1, \ldots, t_n]$ by a (not necessarily principal) prime ideal.*

Exercise 0.3. *Let k be a field, and let $k(a, b)$ be a field extension of k of transcendence degree 1.*

a) *Let $k[x, y]$ be the polynomial ring in two variables. Let $f : k[x, y] \to k(a, b)$ be the unique k-algebra homomorphism such that $f(x) = a$ and $f(y) = b$. Show that the kernel $\mathfrak{p}$ of f is a prime ideal, and let K be the fraction field of $k[x, y]/\mathfrak{p}$. Show that f induces a k-algebra isomorphism $K \xrightarrow{\sim} k(a, b)$.*
b) *Show: $\mathfrak{p}$ is generated by an irreducible polynomial, and deduce that there is an irreducible polynomial $f \in k[x, y]$, unique up to scaling by an element of $k^\times$, such that $f(a, b) = 0$ and $k(a, b)$ is the fraction field of $k[x, y]/(f)$.*
(Suggestion: by [CA, Cor. 12.17]*, the prime ideal $\mathfrak{p}$ has height 0, 1 or 2. Rule out the possibilities of height 0 and height 2, and then find and use a fact about height one prime ideals in a UFD.)*
c) *Show that if K/k is a separable one variable function field, then $K = k(a, b)$ for some a and b.*
(Remark: In the third lecture I mention that in this case we can actually take the polynomial f to be geometrically irreducible.)

Exercise 0.4. *Let k be a field, let G be a finite group of order n, and let $G \hookrightarrow S_n$ be the Cayley embedding. Permutation of variables gives a natural action of S_n and hence also G on $k(t_1, \ldots, t_n)$. Put $l := k(t_1, \ldots, t_n)^G$, so $k(t_1, \ldots, t_n)/l$ is a finite Galois extension with automorphism group G. Notice that this is an instance of the Lüroth problem.*

a) *Let $k = \mathbb{Q}$. Show: if $l/\mathbb{Q}$ is purely transcendental, then G occurs as a Galois group over $\mathbb{Q}$. Thus: an affirmative answer to the Lüroth problem yields an affirmative answer to the Inverse*

Galois Problem over $\mathbb{Q}$.
(Suggestion: This holds whenever k is a Hilbertian field.)

b) *Alas, $l/\mathbb{Q}$ need not be purely transcendental. Explore the literature on this – the first example was due to Swan, where G is cyclic of order 47.*

Exercise 0.5. *Let R_1 and R_2 be two k-algebras that are also domains, with fraction fields K_1 and K_2. Show that $R_1 \otimes_k R_2$ is a domain iff $K_1 \otimes_k K_2$ is a domain.*

Exercise 0.6. a) *Let l/k be an algebraic field extension. Show: $l \otimes_k l$ is a domain iff $l = k$.*
b) *Let l/k be any field extension. Show: $k(t) \otimes_k l$ is always a domain with fraction field $l(t)$. It is already a field iff l/k is algebraic.*

Exercise 0.7. *Describe the $\mathbb{R}$-algebra $\mathbb{C}(t) \otimes_{\mathbb{R}} \mathbb{C}$.*

Exercise 0.8. a) *Show: $k(t)/k$ is regular.*
b) *Show: every purely transcendental extension is regular.*
c) *Show: every extension K/k is regular iff k is algebraically closed.*
d) *Show: K/k is regular iff every finitely generated s subextension is regular.*

Exercise 0.9. *Let k be a field, let $d \geq 2$ be such that $4 \nmid d$, and let $p(x) \in k[x]$ be a polynomial of positive degree. In $\overline{k}[t]$ we factor p as $(x-a_1)^{e_1} \cdots (x-a_r)^{e_r}$ with $a_1, \ldots, a_r$ distinct elements of $\overline{k}$ and $e_1, \ldots, e_r \in \mathbb{Z}^+$. Suppose that there is some $1 \leq i \leq r$ such that $d \nmid e_i$. Show that the*

$$f(x,y) = y^d - p(x) \in k[x,y]$$

is geometrically irreducible and thus the fraction field of $k[x,y]/(y^d - p(x))$ is a regular one variable function field over k.
(Suggestion: use [**?**, Thm. 9.21]*.)*

Exercise 0.10. *Let k be a field of characteristic different from 2.*
a) *Show that the function field K_f attached to $f(x,y) = x^2 - y^2 - 1$ is rational: i.e., there is $z \in K$ such that $K_f = k(z)$.*
b) *Show that the function field K_f attached to $f(x,y) = x^2 + y^2 - 1$ is rational.*
c) *If $k = \mathbb{C}$, show that the function field K_f attached to $f(x,y,) = x^2 + y^2 + 1$ is rational.*
d) *If $k = \mathbb{R}$, is the function field attached to $f(x,y) = x^2 + y^2 + 1$ rational?*
(Answer: it is not, but at the moment we have precisely no tools to show that a regular function field is not rational, so I don't know how you could prove this. But keep it in mind – as we develop more theory, it will become possible, then easy, then clear.)

Exercise 0.11. *Give a purely algebraic proof of the Lüroth Theorem: for any field k, if K is a field such that $k \subsetneq K \subset k(t)$, then $K = k(f)$ for some $f \in K$.*

Exercise 0.12. *Fix $n \in \mathbb{Z}^+$. Exhibit a finite degree field extension l/k such that needs $n+1$ generators: that is, $l \neq k(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n \in l$.*

I do not know how to do the following exercise:

Exercise 0.13. a) *For each $n \in \mathbb{Z}^+$, find a one variable function field K/k that needs $n+1$ generators or show that no such exists.*
(Idea: As in Exercise 0.12, there is a finite degree field extension l/k that needs $n+1$ generators. It seems likely that $l(t)/k$ also needs $n+1$ generators!)
b) *Prove or disprove: every one variable function field K/k with $\kappa(K) = k$ is 2-generated.*

References

[CA] P.L. Clark, *Commutative Algebra.* `http://math.uga.edu/~pete/integral2015.pdf`
[FT] P.L. Clark, *Field Theory.* `http://math.uga.edu/~pete/FieldTheory.pdf`