

Title

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1 Appendix

An alternative characterization of **uniform continuity**:

$$\|\tau_y f - f\|_u \rightarrow 0 \text{ as } y \rightarrow 0$$

Lemma: Measurability is not preserved by homeomorphisms.

Counterexample: there is a homeomorphism that takes that Cantor set (measure zero) to a fat Cantor set

1.1 Undergraduate Analysis Review

- Some inclusions on the real line:

Differentiable with a bounded derivative \subset Lipschitz continuous \subset absolutely continuous
 \subset uniformly continuous \subset continuous
Proofs: Mean Value Theorem, Triangle inequality, Definition of absolute continuity
specialized to one interval, Definition of uniform continuity

- **Bolzano-Weierstrass**: Every bounded sequence has a convergent subsequence.
- **Heine-Borel**:

$$X \subseteq \mathbb{R}^n \text{ is compact} \iff X \text{ is closed and bounded.}$$

- **Baire Category Theorem**: If X is a complete metric space, then

- For any sequence $\{U_k\}$ of open, dense sets, $\bigcap_k U_k$ is also dense.
- X is *not* a countable union of nowhere-dense sets
- **Nested Interval Characterization of Completeness:** \mathbb{R} being complete \implies for any sequence of intervals $\{I_n\}$ such that $I_{n+1} \subseteq I_n$, $\bigcap I_n \neq \emptyset$.
- **Convergence Characterization of Completeness:** \mathbb{R} being complete is equivalent to “absolutely convergent implies convergent” for sums of real numbers.
- Compact subsets $K \subseteq \mathbb{R}^n$ are also *sequentially compact*, i.e. every sequence in K has a convergent subsequence.
- **Urysohn’s Lemma:** For any two sets A, B in a metric space or compact Hausdorff space X , there is a function $f : X \rightarrow I$ such that $f(A) = 0$ and $f(B) = 1$.
- Continuous compactly supported functions are
 - Bounded almost everywhere
 - Uniformly bounded
 - Uniformly continuous

Proof:

- Uniform convergence allows commuting sums with integrals
- Closed subsets of compact sets are compact.
- Every compact subset of a Hausdorff space is closed
- Showing that a series converges: (*Todo*)

1.2 Big Counterexamples

1.2.1 For Limits

- Differentiability \implies continuity but not the converse:
 - The Weierstrass function is continuous but nowhere differentiable.
- f continuous does not imply f' is continuous: $f(x) = x^2 \sin(1/x)$.
- Limit of derivatives may not equal derivative of limit:

$$f(x) = \frac{\sin(nx)}{n^c} \text{ where } 0 < c < 1.$$

- Also shows that a sum of differentiable functions may not be differentiable.
- Limit of integrals may not equal integral of limit:

$$\sum \mathbb{1}[x = q_n \in \mathbb{Q}].$$

- A sequence of continuous functions converging to a discontinuous function:

$$f(x) = x^n \text{ on } [0, 1].$$

- The Thomae function (*todo*)

1.2.2 For Convergence

- Notions of convergence:
 1. Uniform
 2. Pointwise
 3. Almost everywhere
 4. In norm

Uniform \implies pointwise \implies almost everywhere.

See Section 17.3.

Almost everywhere convergence does not imply L^p convergence for any $1 \leq p \leq \infty$

See notes section 1

Sequences $f_k \xrightarrow{a.e.} f$ but $f_k \not\xrightarrow{L^p} f$:

- For $1 \leq p < \infty$: The skateboard to infinity, $f_k = \chi_{[k, k+1]}$.

Then $f_k \xrightarrow{a.e.} 0$ but $\|f_k\|_p = 1$ for all k .

Converges pointwise and a.e., but not uniformly and not in norm.

- For $p = \infty$: The sliding boxes $f_k = k \cdot \chi_{[0, \frac{1}{k}]}$.

Then similarly $f_k \xrightarrow{a.e.} 0$, but $\|f_k\|_p = 1$ and $\|f_k\|_\infty = k \rightarrow \infty$

Converges a.e., but not uniformly, not pointwise, and not in norm.

The Converse to the DCT does not hold

L^p boundedness does not imply a.e. boundedness.

I.e. it is not true that $\lim \int f_k = \int f$ implies that $\exists g \in L^p$ such that $f_k < g$ a.e. for every k .

Take

- $b_k = \sum_{j=1}^k \frac{1}{j} \rightarrow \infty$
- $f_k = \chi_{[b_k, b_{k+1}]}$

Then

- $f_k \xrightarrow{a.e.} f = 0$,
- $\int f_k = \frac{1}{k} \rightarrow 0 \implies \|f_k\|_p \rightarrow 0$,

- $0 = \int f = \lim \int f_k = 0$
- But $g > f_k \implies g > \|f_k\|_\infty = 1$ a.e. $\implies g \notin L^p(\mathbb{R})$.

1.3 Errata

- **Equicontinuity:** If $\mathcal{F} \subset C(X)$ is a family of continuous functions on X , then \mathcal{F} *equicontinuous* at x iff

$$\forall \varepsilon > 0 \exists U \ni x \text{ such that } y \in U \implies |f(y) - f(x)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

- **Arzela - Ascoli 1:** If \mathcal{F} is pointwise bounded and equicontinuous, then \mathcal{F} is totally bounded in the uniform metric and its closure $\overline{\mathcal{F}} \in C(X)$ in the space of continuous functions is compact.
- **Arzela - Ascoli 2:** If $\{f_k\}$ is pointwise bounded and equicontinuous, then there exists a continuous f such that $f_k \xrightarrow{u} f$ on every compact set.

Example: Using Fatou to compute the limit of a sequence of integrals:

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{n^2}{1+n^2x^2} e^{-\frac{x^2}{n^3}} dx \stackrel{\text{Fatou}}{\geq} \int_0^\infty \lim_{n \rightarrow \infty} \frac{n^2}{1+n^2x^2} e^{-\frac{x^2}{n^3}} dx \rightarrow \int \infty.$$

Note that MCT might work, but showing that this is non-decreasing in n is difficult.

Lemma:

$$f_k \xrightarrow{a.e.} f, \quad \|f_k\|_p \leq M \implies f \in L^p \text{ and } \|f\|_p \leq M.$$

Proof: Apply Fatou to $|f|^p$:

$$\int |f|^p = \int \liminf |f_k|^p \leq \liminf \int |f_k|^p = M.$$

Lemma: If f is uniformly continuous, then

$$\|\tau_h f - f\|_p \xrightarrow{L^p} 0 \quad \text{for all } p.$$

Lemma: $\|\tau_h f - f\|_p \rightarrow 0$ for every p .

- i.e. “Continuity in L^1 ” holds for all L^p .
- i.e. Translation operators are continuous.

Proof: Take $g_k \in C_c^0 \rightarrow f$, then g is uniformly continuous, so

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p \rightarrow 0.$$

Lemma: For $f \in L^p, g \in L^q$, $f * g$ is uniformly continuous.

Proof: Use Young's inequality

$$\|\tau_h(f * g) - f * g\|_\infty = \|(\tau_h f - f) * g\|_\infty \leq \|\tau_h f - f\|_p \|g\|_q \rightarrow 0.$$

Lemma: If $\int f \phi = 0$ for every $\phi \in C_c^0$, then $f = 0$ almost everywhere.

Proof: Let A be an interval, choose $\phi_k \rightarrow \chi_A$, then $\int f \chi_A = 0$ for all intervals. So this holds for any Borel set A . Then just take $A_1 = \{f > 0\}$ and $A_2 = \{f < 0\}$, then $\int_{\mathbb{R}} f = \int_{A_1} f + \int_{A_2} f = 0$.

1.4 The Fourier Transform

Some Useful Properties:

$$\begin{aligned}\widehat{f * g}(\xi) &= \hat{f}(\xi) \cdot \hat{g}(\xi) \\ \widehat{\tau_h f}(\xi) &= e^{2\pi i \xi \cdot h} \hat{f}(\xi) \\ e^{2\pi i \xi \cdot h} \widehat{f}(\xi) &= \widehat{\tau_{-h} f}(\xi) \\ \widehat{f \circ T}(\xi) &= |\det T|^{-1} (\hat{f} \circ T^{-t})(\xi) \\ \frac{\partial}{\partial \xi} \hat{f}(\xi) &= -2\pi i \cdot \xi \hat{f}(\xi) \\ \widehat{\frac{\partial}{\partial \xi} f}(\xi) &= 2\pi i \xi \cdot \hat{f}(\xi).\end{aligned}$$

Some Useful Transform Pairs:

$$\text{Dirichlet:} \quad \chi_{\{-\frac{1}{2} \leq x \leq \frac{1}{2}\}} \iff \text{sinc}(\xi)$$