

Title

D. Zack Garza

Contents

1 Lecture 14: Monday, November 02

3

1 | Lecture 14: Monday, November 02

Recall that the *Hasse-Weil zeta function* of a one-variable function field K/\mathbb{F}_q over a finite ground field is defined in the following way: let $A_n = A_n(K)$ be the number of effective divisors of degree n . We have proved that A_n is finite, and for $n > 2g - 2$ we have a formula

$$Z(t) = \sum_{n=0}^{\infty} A_n t^n = \sum_{D \in \text{Div}^+(K)} t^{\deg(D)} \in \mathbb{Z}[[t]],$$

which is a formal power series with integer coefficients.

Remark 1.0.1 : Recall that we have proved that it is a rational function of t , and in particular when $g = 0, \delta = 1$ ¹ we get

$$Z(t) = \frac{1}{(1 - qt)(1 - t)}.$$

We got another expression which isn't fantastic: it involves this δ , which we'll work toward proving is equal to 1. When $g > 1$, we broke the zeta function into two pieces $Z(t) = F(t) + G(t)$. For divisors of sufficiently high degree, Riemann-Roch tells you what the dimension of the Riemann-Roch space is, and $G(t)$ explains the part coming from divisors of large degree. We obtained a formula previously for $F(t)$ and $G(t)$, and once we show $\delta = 1$ the formula for G will simplify. For $F(t)$, we specifically had

$$F(t) = \frac{1}{q-1} \sum_{0 \leq \deg(c) \leq 2g-2} q^{\ell(c)t^{\deg(c)}},$$

where the sum is over divisor classes and ℓ is the dimension of linear system corresponding to a divisor. But this isn't a great formula: what are these classes, how many are in each degree, and what is the dimension of the Riemann-Roch space?

Remark 1.0.2 : This is analogous to the Dedekind zeta function of a number field K , in which case

$$\zeta_K(s) = \sum_{T \in \ell(\mathbb{Z}_K)} |\mathbb{Z}_K/I|^{-s},$$

which will be covered in a separate lecture on *Serre zeta functions*.

Theorem 1.0.1 (F.K. Schmidt).

For all K/\mathbb{F}_q , we have $\delta = I(K) = 1$ where I is the index.

This will follow from the associated, but it much weaker. However, this is one of the facts we'd like to establish to use to *prove* the Riemann hypothesis.

¹The *index* of the function field, least positive degree of a divisor.

Remark 1.0.3 : Pete studied this in 2004 and found that every $I \in \mathbb{Z}^+$ arises as the index of a genus one function field K/\mathbb{Q} .

Notation: for $n \in \mathbb{Z}^+$, let μ_n denote the n th roots of unity in \mathbb{C} .

Lemma 1.1 (?).

For $m, r \in \mathbb{Z}^+$, set $d := \gcd(m, r)$. Then

$$\left(1 - t^{mr/d}\right)^d = \prod_{\xi \in \mu_r} 1 - (\xi t)^m.$$

Proof (?).

In $\mathbb{C}[x]$, we have

$$(X^{r/d} - 1)^d = \prod_{\xi \in \mu_r} (X - \xi^m),$$

where both sides are monic polynomials whose roots include the (r/d) th roots of unity, each with multiplicity d . On the LHS, the distinct roots are the r/d th roots of unity, then raising to the d th power gives them multiplicity d . On the RHS, this is an exercise in cyclic groups: consider the n th power map on $\mathbb{Z}/r\mathbb{Z}$ and compute its image and kernel. As ξ ranges over r th roots of unity, ξ^m ranges over all r/d th roots of unity, each occurring with multiplicity d . Substituting $X = t^{-m}$ yields the original result.

Special case: set $m = r$, so $d = r$, then the RHS is r copies of 1. ■

Next up, we want to compare the zeta function for a function field over \mathbb{F}_q to the zeta function obtained when extending scalars to \mathbb{Q}^r .

Proposition 1.0.1 (?).

Let K/\mathbb{F}_q be a function field, $r \in \mathbb{Z}^+$, and take the compositum of K and \mathbb{F}_q^r viewed as a function field over \mathbb{F}_q^r . Let $Z(t)$ be the zeta function of K/\mathbb{F}_q and $Z_r(t)$ the zeta function of K_r/\mathbb{F}_q^r . Then

$$Z_r(t^r) = \prod_{q \in \mu_r} Z(qt).$$

Proof (?).

We have an Euler product formula

$$Z(t) = \prod_{p \in \Sigma(K/\mathbb{F}_q)} (1 - t^{\deg(p)})^{-1}.$$

where the sum is over places of the function field.

Exercise 1.0.1 (?): Why is this true? Write as a geometric series with ratio $t^{\deg(p)}$. Here just

expand each summand to get

$$Z(t) = \prod_p \sum_{j=1}^{\infty} t^{j \deg(p)}.$$

Multiplying this out and collecting terms is in effect multiplying out the prime divisors to get effective divisors.

We use the result that was stated (but not proved): If $p \in \Sigma_m(K/\mathbb{F}_q)$ is a degree n place and $r \in \mathbb{Z}^+$, then there exist precisely $d := \gcd(m, r)$ places p^r of K_r lying over p . Moreover, each place p^r has degree m/d . ■