

# Homological Algebra Problem Sets

## Problem Set 3

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# 1 | Wednesday, February 17

**Problem 1.0.1** (Prove Corollary 2.3.2)

For  $R$  a PID, show that an  $R$ -module  $A$  is divisible if and only if  $A$  is injective.

*Recall that a module is divisible if and only if for every  $r \neq 0 \in R$  and every  $a \in A$ , we have  $a = br$  for some  $b \in A$ .*

**Solution:**

Note: we'll assume  $R$  is commutative, and since  $R$  is a domain, it has no nonzero zero divisors and thus all elements  $r \in R$  are left-cancelable.

$\Rightarrow$  : Suppose  $A$  is divisible, we then want to show every  $R$ -module morphism of the following form lifts, where we regard the ideal  $J$  and the ring  $R$  as  $R$ -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since  $R$  is a PID, we have  $J = jR$  for some  $j \in R$ , so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider  $f(j) \in A$ . Since  $A$  is divisible, we have  $A = jA$ , so we can write  $f(j) = j\mathbf{a}'$  for some  $\mathbf{a}' \in A$ . Using  $R$ -linearity and the fact that  $j$  is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending  $R$ -linearly yields a well-defined  $R$ -module morphism. Moreover, the diagram commutes by construction, since  $\iota(1_R) = 1_R$ .

$\Leftarrow$  : Suppose  $A \in R\text{-Mod}$  is injective, where by Baer's criterion we equivalently have a lift of the following form for every  $J \leq R$ :

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let  $j \in R$  be a nonzero element that is not a zero-divisor, we then want to show that  $A = jA$ , i.e. that for every  $\mathbf{a} \in A$ , there is a  $\mathbf{a}' \in A$  such that  $\mathbf{a} = j\mathbf{a}'$ . Fixing  $\mathbf{a} \in A$ , define a map  $f_a : J \rightarrow A$  in the following way: for  $x \in J$ , use the fact that  $\langle j \rangle := jR$  to first write  $x = jr$  for some  $r \in R$ , and then set  $f_a(x) = f_a(jr) := r\mathbf{a}$ . To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion  $jR \hookrightarrow R$  and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an  $r$  on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting  $\mathbf{a}$  range over all elements of  $A$  yields the desired result.

**Problem 1.0.2** (Calculating Ext Groups)

Calculate  $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$  for distinct primes  $p, q$ .

The following are several claims that are later used in the actual solution:

**Claim 1:** For any  $m \in \mathbb{Z}$ ,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

*Proof (?)*.

Note that there is an injection

$$1 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \rightarrow \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \rightarrow 1$$

where  $\pi_n$  is the canonical quotient morphism, and applying the left-exact contravariant functor  $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$  yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any  $\mathbb{Z}$ -module morphism  $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$ ,

1. Since  $\mathbb{Z}$  is a free  $\mathbb{Z}$ -module with generator 1,  $f$  is entirely determined by  $f(1)$ , and
2.  $f$  descends to a map  $\tilde{f} : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$  if and only if  $f(n) \in \mathbb{Z}$ , i.e.  $f(n) = [0]$  is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule  $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$  as all of those morphism  $\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  which descend to make the following diagram commute.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z} \\ \pi_n \downarrow & \nearrow \exists \tilde{f} & \\ \mathbb{Z}/n & & \end{array}$$

[Link to Diagram](#)

To characterize these, it suffices to determine all of the possible images  $f(1)$ . Moreover, we can restrict our attention to coset representatives in the interval  $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$ , where we want to find all  $q := f(1) \in [0, 1)$  such that  $nq = 1$ . A complete list of  $n$  such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}.$$

Setting  $f_i(1) := \left[ \frac{i}{n} \right]$  (where we take the equivalence class mod  $\mathbb{Z}$ ) yields  $n$  distinct morphisms  $f_i : \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  that descend to  $\tilde{f}_i : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$ . We can define a map

$$\begin{aligned} \Psi : \mathbb{Z} &\rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i, \end{aligned}$$

and using the fact that if  $i = i' \pmod{n}$ , write  $i' = i + kn$  for some  $k \in \mathbb{Z}$ , then

$$f_{i'}(1) = f_{i+kn}(1) = \left\lfloor \frac{i+kn}{n} \right\rfloor = \left\lfloor \frac{i}{n} + k \right\rfloor = \left\lfloor \frac{i}{n} \right\rfloor = f_i(1),$$

since  $k \in \mathbb{Z}$ , so by the first isomorphism theorem  $\Psi$  descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

**Claim 2:**  $\mathbb{Q}/\mathbb{Z}$  is an injective object in  $\mathbb{Z}$ -modules.

*Proof (?)*.

By the previous exercise, it suffices to show that  $\mathbb{Q}/\mathbb{Z}$  is divisible. More generally, if any group  $G$  is divisible and  $N \trianglelefteq G$  is a normal subgroup, then  $G/N$  will be divisible. This follows from the fact that if  $\bar{a}, \bar{b} \in G/N$  and  $n \in \mathbb{Z}$ , we can write  $\bar{a} = a + N$  and  $\bar{b} = b + N$  for some coset representatives, use divisibility to write  $a = nb$ , and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That  $\mathbb{Q}$  is divisible is a straightforward check: let  $n \in \mathbb{Z}$  and  $a \in \mathbb{Q}$ , we then want a  $b \in \mathbb{Q}$  such that  $a = nb$ , and  $b := \frac{a}{n} \in \mathbb{Q}$  works. Since  $\mathbb{Q}$  is an abelian group,  $\mathbb{Z}$  is automatically normal, and the result follows.

■

**Claim:**

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \quad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

*Proof (?)*.

Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-Mod},$$

and taking

- $M := \mathbb{Z}/m$ ,
- $A := \mathbb{Z}$ ,
- $I := n\mathbb{Z}$ ,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n(\mathbb{Z}/m)} \in \mathbb{Z}\text{-Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\rightarrow \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
 \ker \varphi &= \{x \in \mathbb{Z} \mid x(1 \otimes 1) = 0\} \\
 &= \{x \in \mathbb{Z} \mid n \mid x \text{ or } m \mid x\} \\
 &= \langle n, m \rangle \\
 &= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
 &:= \langle d \rangle.
 \end{aligned}$$

Now applying the first isomorphism theorem yields the result. ■

### Solution:

We'll follow the procedure outlined in Weibel:

- Define the contravariant functor  $F(\cdot) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \cdot)$ , then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution  $I$  of  $\mathbb{Z}/q$ .
- Write  $F(I)$  as a new (not necessarily exact) chain complex.
- Compute  $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$ .

We can first take the following injective resolution:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbb{Z}/q & \xrightarrow{d^{-1}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & \mathbb{Q}/\mathbb{Z} \xrightarrow{d^1} 1 \\
 & & & & & & \\
 & & [1]_q & \longrightarrow & \begin{bmatrix} 1 \\ q \end{bmatrix} & & 
 \end{array}$$

$$[x] \longrightarrow [qx]$$

[Link to Diagram](#)

This is a chain complex by construction, since  $d^2([1]_q) = \left[ q \left( \frac{1}{q} \right) \right] = [1] = [0]$ . We now delete the augmentation and apply  $F(\cdot)$ :

$$\begin{array}{ccccccc}
1 & \longrightarrow & I^0 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & I^1 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^1} & 1 \\
& & \downarrow F(\cdot) & & & & \\
1 & \longrightarrow & F(I^0) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^0 := F(d^0)} & F(I^1) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^1 := F(d^1)} & 1 \\
\parallel & & \uparrow \Psi \cong & & \uparrow \Psi \cong & & \parallel \\
1 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^0} & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^1} & 1
\end{array}$$

[Link to Diagram](#)

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that  $d^0(x) := qx$  was multiplication by  $q$ , we have  $\partial^0(f) = d^0 \circ f$  is post-composition by the multiplication by  $q$  map, and  $\tilde{\partial}^0$  similarly becomes multiplication by  $q$ .

We now take homology:

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\text{im } \partial^0} = \frac{\mathbb{Z}/p}{q(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where  $d := \gcd(p, q) = 1$  if  $p, q$  are coprime.

**Problem 1.0.3** (Weibel 2.3.2)

Let  $A \in \mathbf{Ab}$ , and show that the following map is injective:

$$\begin{aligned}
\varepsilon_A : A &\rightarrow I(A) := \prod_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\
a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},
\end{aligned}$$

i.e. when looking at the image  $\varepsilon_A(a)$  in the product, the component indexed by  $f$  is an element of  $\mathbb{Q}/\mathbb{Z}$  obtained by evaluating  $f(a)$ .

*Hint: if  $a \in A$ , find a map  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $f(a) \neq 0$  and extend this to a map  $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$ .*

**Solution:**

By contrapositive, we'll suppose  $a \neq 0$  and show  $\varepsilon_A(a) \neq 0$ . Following the hint, we first consider the cyclic subgroup  $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$  and define a map

$$\begin{aligned}
f_a : a\mathbb{Z} &\rightarrow \mathbb{Z} \\
an &\mapsto n.
\end{aligned}$$

We now pick  $\ell > 1 \in \mathbb{Z}$  to be any integer, and define a composition  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ :



$$\begin{array}{ccccccc}
 a\mathbb{Z} & \xrightarrow{f_a} & \mathbb{Z} & \xleftarrow{\iota} & \mathbb{Q} & \xrightarrow{x \mapsto \frac{x}{\ell}} & \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \\
 & \nearrow f & & & & & \nearrow f \\
 & & & & & & \\
 an & \longmapsto & n & \longmapsto & n & \longmapsto & \frac{n}{\ell} \longmapsto \left[ \frac{n}{\ell} \right] \\
 & & & & & & \\
 a & \xrightarrow{\quad\quad\quad} & & & & & \left[ \frac{1}{\ell} \right]
 \end{array}$$

[Link to Diagram](#)

By choice of  $\ell$ , this map satisfies  $f(a) = [1/\ell] \neq 0$ , so the map is nonzero. Since  $\mathbb{Q}/\mathbb{Z}$  is injective, the universal property provides a lift  $\tilde{f}$ :

$$\begin{array}{ccc}
 A & & \\
 \uparrow & \searrow \exists \tilde{f} & \\
 a\mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z}
 \end{array}$$

[Link to Diagram](#)

Since  $\tilde{f}$  lifts  $f$ , it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) := f(a) \neq 0,$$

so the  $f$  component of the image of  $a$  is nonzero and thus  $\mathbf{a} := \varepsilon_A(a) \neq 0$  in the product.

**Problem 1.0.4** (Weibel 2.4.2)

If  $U : \mathcal{B} \rightarrow \mathcal{C}$  is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

**Solution:**

We'll show that  $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$  for every object  $X$ . Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let  $P \rightarrow X$  be a projective resolution of  $X$ . Fixing labeling, we have the following situation:

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X \xrightarrow{0} 0 \\
 & & & & & \downarrow F(\cdot) & & & \\
 \cdots & \longrightarrow & FP_2 & \xrightarrow{F(\partial_2)} & FP_1 & \xrightarrow{F(\partial_1)} & FP_0 & \xrightarrow{0} & 0
 \end{array}$$

[Link to Diagram](#)

We now have by definition

$$L_i F(X) := \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U(L_i F(X)) := U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution  $P \rightarrow X$ , and apply a similar process:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X & \xrightarrow{0} & 0 \\ & & & & & \Downarrow (U \circ F)(\cdot) & & & & & \\ \cdots & \longrightarrow & UFP_2 & \xrightarrow{(UF)(\partial_2)} & UFP_1 & \xrightarrow{(UF)(\partial_1)} & UFP_0 & \xrightarrow{0} & 0 & & \end{array}$$

[Link to Diagram](#)

Again, by definition,

$$(L_i(UF))(X) := \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor  $U$  to the following SES to produce a new SES, from which we'll produce the desired isomorphism  $f$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{im} F(\partial_{i+1}) & \xrightarrow{\iota_i} & \ker F(\partial_i) & \xrightarrow{\pi_i} & \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \longrightarrow 0 \\ & & & & \Downarrow U & & \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{U(\pi_i)} & U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow f \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{\tilde{\pi}_i} & \frac{U(\ker F(\partial_i))}{U(\operatorname{im} F(\partial_{i+1}))} \longrightarrow 0 \end{array}$$

[Link to Diagram](#)

Here  $\tilde{\pi}_i$  is the natural quotient map, whose image is  $\operatorname{coker} U(\iota_i)$ . Finally, the map  $f$  exists in any abelian category, using that whenever  $0 \rightarrow A \xrightarrow{g_1} B \xrightarrow{g_2} C \rightarrow 0$  is exact, there is an isomorphism  $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$ .

*Problem 1.0.5 (Weibel 2.4.3)*

- If  $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$  is exact with  $P$  projective or  $F$ -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

- Show that if

$$0 \rightarrow M_m \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

is exact with  $P_i$  projective or  $F$ -acyclic, then

$$L_i F(A) \cong L_{i-m-1} F(M_m) \quad i \geq m+2.$$

– Moreover show that  $L_{m+1} F(A)$  is the kernel of  $F(M_m) \rightarrow F(P_m)$ .

- Conclude that if  $P \rightarrow A$  is an  $F$ -acyclic resolution of  $A$ , then  $L_i F(A) = H_i(F(P))$ .

**Solution:**

**Claim:**

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

*Proof (of claim).*

Following the proof of Weibel Theorem 2.4.6, let  $P_M \rightarrow M$  and  $P_A \rightarrow A$  be projective resolutions of  $M$  and  $A$  respectively. Then applying the Horseshoe Lemma, there is a projective resolution  $P_P \rightarrow P$  of  $P$  such that the following is a short exact sequence of chain complexes:

$$0 \rightarrow P_M \rightarrow P_P \rightarrow P_A \rightarrow 0,$$

where in fact in each degree  $n$  piece, this induces a *split* exact sequence. Using that  $F$  is additive and additive functors preserve split exact sequences, the following is a SES for every  $n$ :

$$0 \rightarrow FP_M^n \rightarrow FP_P^n \rightarrow FP_A^n \rightarrow 0,$$

which implies that there is a SES of chain complexes

$$0 \rightarrow FP_M \rightarrow FP_P \rightarrow FP_A \rightarrow 0.$$

Thus there is an associated LES of derived functors:

$$\begin{array}{ccccccc} & & & & \dots & & \\ & & & & \nearrow & & \\ & & & \partial_{i+1} & & & \\ \hookrightarrow & L_i FM & \longrightarrow & L_i FP & \longrightarrow & L_i FA & \longrightarrow \dots \\ & & & \searrow & & & \\ & & & \partial_i & & & \\ \hookrightarrow & L_{i-1} FM & \longrightarrow & L_{i-1} FP & \longrightarrow & \dots & \end{array}$$

[Link to Diagram](#)

Using that  $P$  is  $F$ -acyclic, the middle terms  $L_i FP = 0$  for all  $i > 0$ , and thus this splits into a collection of SESs:

$$\begin{array}{l} 0 \rightarrow L_2 FA \xrightarrow{\partial_2} L_1 FM \rightarrow 0 \\ 0 \rightarrow L_3 FA \xrightarrow{\partial_3} L_2 FM \rightarrow 0 \\ \vdots \\ 0 \rightarrow L_i FA \xrightarrow{\partial_i} L_{i-1} FM \rightarrow 0. \end{array}$$

This makes every  $\partial_i$  for  $i \geq 2$  an isomorphism. ■

**Claim:**

$$L_i FA \cong \ker(FM \rightarrow FP).$$

*Proof (?)*.

Using the same argument as above, consider the lower order terms of the associated LES:

$$\begin{array}{ccccccc}
 & & & & \dots & & \\
 & & & & \nearrow \partial_2 & & \\
 L_1 FM & \xrightarrow{\quad} & L_1 FP = 0 & \xrightarrow{\quad} & L_1 FA & & \\
 & & \nwarrow \partial_1 & & & & \\
 L_0 FM & \xrightarrow{\quad} & L_0 FP & \xrightarrow{\quad} & L_0 FA & \longrightarrow & 0
 \end{array}$$

[Link to Diagram](#)

Noting that  $L_1 FP = 0$  by  $F$ -acyclicity, the highlighted portion forms a four term exact sequence. We can form another exact sequence and compare the two:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & L_1 FA & \xrightarrow{\partial_1} & L_0 FM & \longrightarrow & L_0 FP & \longrightarrow & L_0 FA & \longrightarrow & 0 \\
 & & \downarrow \text{dotted} & & \parallel & & \parallel & & \parallel & & \\
 0 & \longrightarrow & \ker(L_0 FM \rightarrow L_0 FP) & \longrightarrow & L_0 FM & \longrightarrow & L_0 FP & \longrightarrow & L_0 FA & \longrightarrow & 0
 \end{array}$$

[Link to Diagram](#)

That the map indicated by the dotted line exists and is an isomorphism holds in any abelian category, using that fact that whenever  $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$  is a SES we have  $A \cong \ker f$ . ■

**Claim:** If  $P \rightarrow A$  is an  $F$ -acyclic resolution of  $A$ , then there is an isomorphism

$$L_i FA \cong H_i(FP).$$

*Problem 1.0.6 (Weibel 2.5.2)*

Show that the following are equivalent:

- $A$  is a projective  $R$ -module.
- $\text{Hom}_R(A, \cdot)$  is an exact functor.
- $\text{Ext}_R^{i \geq 1}(A, B) = 0$  and for all  $B$ , i.e.  $A$  is  $\text{Hom}_R(\cdot, B)$ -acyclic for all  $B$ .
- $\text{Ext}_R^1(A, B)$  vanishes for all  $B$ .

**Solution:**

We'll show

- $a \iff b$
- $b \implies c$
- $c \iff d$ :
- $d \implies b$

*Proof* ( $a \iff b$ ).Let  $\xi$  be the following SES:

$$\xi : 0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$$

and define the functor  $F(\cdot) := \text{Hom}_R(A, \cdot)$ . This is a covariant left-exact functor, and so applying it to the above sequence yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & F(\cdot) := \text{Hom}_R(A, \cdot) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & FM_1 & \xrightarrow{F(f): \lambda \mapsto f \circ \lambda} & FM_2 & \xrightarrow{F(g): \lambda \mapsto g \circ \lambda} & FM_3 \end{array}$$

[Link to Diagram](#) $\implies$  :

For  $F$  to be exact, it suffices to show it is right-exact, i.e. that  $F(g)$  is surjective. This amounts to asking that every  $\varphi \in FM_3 := \text{Hom}_R(A, M_3)$  lifts to a preimage  $\tilde{\varphi} \in FM_2 := \text{Hom}_R(A, M_2)$  satisfying  $F(g)(\tilde{\varphi}) = \varphi$ . Unwinding definitions, this requires that  $g \circ \tilde{\varphi} = \varphi$ , which is precisely the lift required for the universal property of projective objects:

$$\begin{array}{ccc} & A & \\ & \downarrow \varphi & \\ M_2 & \xrightarrow{g} & M_3 \longrightarrow 0 \end{array}$$

$\exists \tilde{\varphi}$  (dashed arrow from  $M_2$  to  $A$ )

[Link to Diagram](#)

If  $A$  is projective, this lift always exists, so  $\text{Hom}_R(A, \cdot)$  is an exact functor. Conversely, if  $\text{Hom}_R(A, \cdot)$  is exact, this lift always exists, so  $A$  satisfies the universal property of a projective object. ■

*Proof* ( $b \implies c$ ).

Suppose  $F(\cdot) := \text{Hom}_R(A, \cdot)$  is exact, then since  $F$  is left-exact covariant it has right-derived functors  $\text{Ext}_R^i(A, B) := R^i F(B)$  which are computed in the following way

1. Taking an *injective* resolution of

$$B \rightarrow I := (I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \dots).$$

2. Applying  $\text{Hom}_R(A, \cdot)$  to get the complex

$$FI := (0 \rightarrow \text{Hom}_R(A, I_0) \xrightarrow{F(\partial_0)} \text{Hom}_R(A, I_1) \xrightarrow{F(\partial_1)} \dots).$$

3. Defining

$$R^i F(B) := \ker F(\partial_i) / \text{im } F(\partial_{i-1}).$$

Note that in step (2), if  $\text{Hom}_R(A, \cdot)$  is an exact functor, then since  $I$  is an acyclic complex,  $FI$  is again acyclic and so  $\ker F(\partial_i) = \text{im } F(\partial_{i-1}) = 0$  for  $i \geq 1$ . So

$$\text{Ext}_R^{\geq 1}(A, B) := R^{\geq 1} F(B) = 0.$$

■

*Proof* ( $c \iff d$ ).

$\implies$  : This direction is clear, since if  $\text{Ext}_R^i(A, B) = 0$  for all  $B$ , then taking  $i = 1$  is the statement of (d).

$\impliedby$  : This follows from the dimension-shifting isomorphism in a previous exercise. Let  $F(\cdot) := \text{Hom}_R(A, \cdot)$  and suppose  $\text{Ext}_R^1(A, B) := L_1 F(B) = 0$  for all  $B$ . Let  $B'$  be arbitrary, it then suffices to show that  $\text{Ext}_R^i(A, B') := L_i(B') = 0$  for all  $i > 1$ , since we can take  $B'$  as one such  $B$  in the assumption for the  $i = 1$  case.

The dimension shifting results states that if  $P_i$  are  $F$ -acyclic, then for every exact sequence

$$0 \rightarrow M_m \rightarrow P_m \rightarrow \dots \rightarrow P_0 \rightarrow B' \rightarrow 0$$

we obtain an isomorphism

$$L_i F(B') \cong L_{i-m-1} F(M_m) \iff L_i F(M_m) \cong L_{i+m+1} F(B').$$

So take any  $F$ -acyclic resolution of  $P$ , say

$$B' \xrightarrow{\partial_{-1}} I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \dots,$$

then consider truncating it at the  $m$ th stage:

$$0 \rightarrow B' \xrightarrow{\partial_{i-1}} I_0 \rightarrow I_1 \rightarrow \dots \xrightarrow{\partial_{m-1}} I_m \rightarrow M_m := \text{coker } \partial_{m-1} \rightarrow 0$$

By assumption, we have  $L_1 F(M_m) = 0$  for every  $m$ , and thus

$$\begin{aligned}
 0 &= L_1 F(B') \quad \text{by assumption} \\
 0 &= L_1 F(M_0) \cong L_2 F(B') \\
 0 &= L_1 F(M_1) \cong L_3 F(B') \\
 0 &= L_1 F(M_2) \cong L_4 F(B') \\
 &\vdots \\
 0 &= L_1 F(M_m) \cong L_{m+2} F(B') \quad \forall m \geq 0.
 \end{aligned}$$

and so  $L_i(B') = 0$  for all  $i \geq 1$ . ■

*Proof* ( $d \implies b$ ).

Take an arbitrary SES

$$\xi : 0 \rightarrow B' \xrightarrow{f} B \xrightarrow{g} B'' \rightarrow 0$$

and consider applying the left-exact covariant functor  $F(\cdot) := \text{Hom}_R(A, \cdot)$  and taking the associated LES:

$$\begin{array}{ccccccc}
 0 & & & & & & \\
 \downarrow & & & & & & \\
 \text{Hom}_R(A, B') & \xrightarrow{f^*} & \text{Hom}_R(A, B) & \xrightarrow{g^*} & \text{Hom}_R(A, B'') & \rightarrow & 0 \\
 & \searrow \delta_0 & & & & & \\
 & \text{Ext}_R^1(A, B') & \longrightarrow & \text{Ext}_R^1(A, B) & \longrightarrow & \text{Ext}_R^1(A, B'') & \longrightarrow 0 \\
 & \searrow \delta_1 & & & & & \\
 & \dots & & & & & 
 \end{array}$$

[Link to Diagram](#)

By assumption, all of the higher Ext terms vanish, and in particular the red term  $\text{Ext}_R^1(A, B') = 0$ . This implies that  $g^*$  is surjective, making the following sequence exact:

$$0 \rightarrow \text{Hom}_R(A, B') \xrightarrow{f^*} \text{Hom}_R(A, B) \xrightarrow{g^*} \text{Hom}_R(A, B'') \rightarrow 0,$$

making  $\text{Hom}_R(A, \cdot)$  an exact functor. ■

*Problem 1.0.7* (Weibel 2.6.4)

Show that  $\text{colim}$  is left adjoint to  $\Delta$ , and conclude that  $\text{colim}$  is right-exact when  $\mathcal{A}$  is



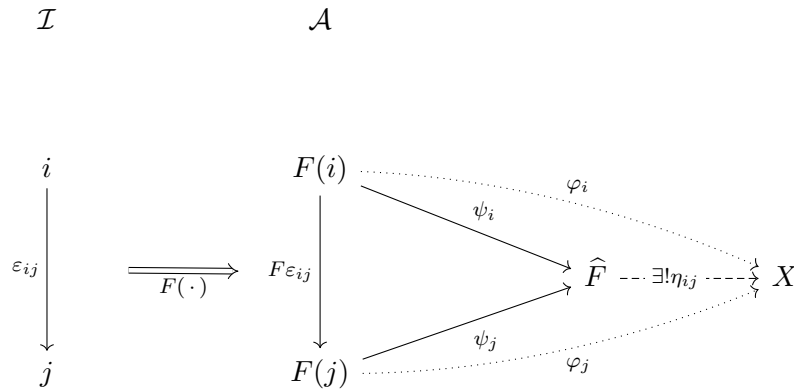
abelian and colim exists. Show that the pushout, i.e.  $\bullet \leftarrow \bullet \rightarrow \bullet$ , is not an exact functor on **Ab**.

**Solution:**

Fixing some index category  $\mathcal{I}$  and a functor  $F : \mathcal{I} \rightarrow \mathcal{A}$ , so  $F \in \mathcal{A}^{\mathcal{I}}$ , write  $\hat{A} := \text{colim}_{i \in \mathcal{I}} F(i)$ . We want to show that  $\mathcal{A}^{\mathcal{I}} \xrightleftharpoons[\Delta]{\text{colim}} \mathcal{A}$  defines an adjoint pair, so that colim is a left-adjoint and  $\Delta$  is a right-adjoint. By definition, this is equivalent to showing the existence of natural bijections of sets

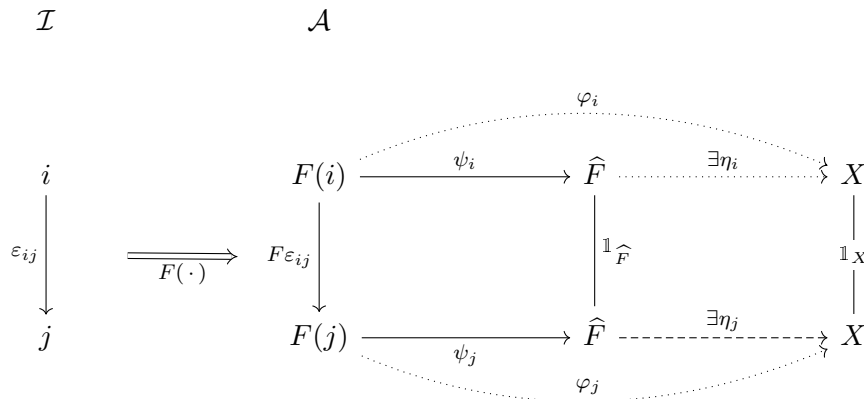
$$\tau_{FX} : \text{Hom}_{\mathcal{A}}(\hat{F}, X) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \quad \forall X \in \mathcal{A}, F \in \mathcal{A}^{\mathcal{I}}.$$

We first note that the data of  $\hat{F}$  is equivalent to the following universal property:



[Link to Diagram](#)

That is,  $\hat{F} \in \text{Ob}(\mathcal{A})$  is an object equipped with structure maps  $\psi_i$  for every object  $F(i)$  in the image of  $F$  such that the solid triangle commutes, and for any object  $X$  with maps  $\varphi_i : F(i) \rightarrow X, \varphi_j : F(j) \rightarrow X$  making the outer triangle commute, there is a unique map  $\eta_{ij} : \hat{F} \rightarrow X$  making the entire diagram commute. We can rewrite this condition in a more suggestive way:



[Link to Diagram](#)

Applying the  $\Delta$  functor, we can view this as a simpler universal property in  $\mathcal{A}^{\mathcal{I}}$ , since the above data is precisely the data of a natural transformation:

$$\begin{array}{ccc}
 & & \Delta F \\
 & \nearrow \Psi & \uparrow \exists! \eta \\
 F & \xrightarrow{\Phi} & \Delta X
 \end{array}$$

[Link to Diagram](#)

That is, the functor  $\Delta\hat{F}$  is equipped with structure maps  $\Phi : F \rightarrow \Delta\hat{F}$  which assemble into a natural transformation (i.e. a morphism in  $\mathcal{A}^{\mathcal{I}}$ ) such that any other natural transformation from  $F$  to a diagonal object  $\Delta X$  produces a unique natural transformation  $\eta : \Delta X \rightarrow \Delta F$ . This provides exactly the data needed to specify  $\tau$ :

$$\begin{aligned}
 \tau_{FX} : \text{Hom}_{\mathcal{A}}(\hat{F}, X) &\xrightarrow{\sim} \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \\
 \left( \hat{F} \xrightarrow{f} X \right) &\mapsto \left( F \xrightarrow{\Psi} \Delta\hat{F} \xrightarrow{\Delta(f)} \Delta X \right),
 \end{aligned}$$

i.e. we take the image  $\Delta(f)$  and pre-compose with the structure morphism  $\Psi$ .

**Claim:** This is a bijection of sets.

*Proof (?)*.

This is surjective by the universal property: any morphism  $F \xrightarrow{g} \Delta X$  in  $\mathcal{A}^{\mathcal{I}}$  factors through  $\Delta\hat{F}$ , and so all such morphisms are of this form.

That this is injective: ???

■

**Claim:** This is a natural isomorphism, i.e. for all  $X \xrightarrow{f} Y \in \text{Mor}(\mathcal{A})$  and all  $F \xrightarrow{\eta} G \in \text{Mor}(\mathcal{A}^{\mathcal{I}})$ , there is a commuting diagram

$$\begin{array}{ccccc}
 \text{Hom}(\hat{G}, X) & \xrightarrow{\hat{\eta}_*} & \text{Hom}(\hat{F}, X) & \xrightarrow{f_*} & \text{Hom}(\hat{F}, Y) \\
 \downarrow \tau_{GX} & & \downarrow \tau_{FX} & & \downarrow \tau_{FY} \\
 \text{Hom}(G, \Delta X) & \xrightarrow{\eta_*} & \text{Hom}(F, \Delta X) & \xrightarrow{(\Delta f)_*} & \text{Hom}(F, \Delta Y)
 \end{array}$$

[Link to Diagram](#)

*Proof (?)*.

Todo.

■

**Claim:** If  $\mathcal{A}$  is abelian and  $\mathcal{I}$  is an index category such that  $\text{colim}_{i \in \mathcal{I}} F(i)$  exists for all  $F \in \mathcal{A}^{\mathcal{I}}$ , then the functor  $\text{colim} : \mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$  is right-exact.

*Proof (Sketch).*

A sketch of the proof proceeds by showing every right adjoint is left-exact:

- Since  $\text{Hom}(LA, \cdot)$  is left-exact, we can apply it to a SES  $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ .
- Applying the natural isomorphisms coming from the adjunction, this is isomorphic to a sequence involving terms  $\text{Hom}(\cdot, RB)$ .
- This sequence is exact, so applying Yoneda yields an exact sequence

$$0 \rightarrow RB' \rightarrow RB \rightarrow RB'',$$

making  $R$  left-exact.

Finally, if  $L$  is a left adjoint out of  $\mathcal{A}$ , then  $L^{\text{op}}$  is a right adjoint out of  $\mathcal{A}^{\text{op}}$ . Thus  $L^{\text{op}}$  is left-exact by the above argument, making  $L$  right-exact. ■

**Claim:** Let  $\mathcal{I} := (\bullet \leftarrow \bullet \rightarrow \bullet)$  and define the pushout as  $\text{colim} : \mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$ . Then taking  $\mathcal{A} := \mathbf{Ab}$ , the pushout does not define an exact functor  $\mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$ .