Analysis Review Notes

D. Zack Garza

January 21, 2020

Contents

1 Inequalities and Equalities

AM-GM Inequality:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Reverse Triangle Inequality

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|.$$

Chebyshev's Inequality

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha} \right)^p.$$

Holder's Inequality:

$$\frac{1}{p} + \frac{1}{q} = 1 \implies \|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Application: For finite measure spaces,

$$1 \leq p < q \leq \infty \implies L^q \subset L^p \quad (\text{ and } \ell^p \subset \ell^q)$$

Proof: Fix p, q, let $r = \dfrac{q}{p}$ and $s = \dfrac{r}{r-1}$ so $r^{-1} + s^{-1} = 1$. Then let $h = |f|^p$:

$$\|f\|_p^p = \|h \cdot 1\|_1 \leq \|1\|_s \|h\|_r = \mu(X)^{\frac{1}{s}} \|f\|_q^{\frac{q}{r}} \implies \|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

Note: doesn't work for ℓ spaces, but just note that $\sum |x_n| < \infty \implies x_n < 1$ for large enough n, and thus $p < q \implies |x_n|^q \leq |x_n|^q$.

Cauchy-Schwarz:

$$|\langle f,\ g\rangle| = \|fg\|_1 \leq \|f\|_2 \|g\|_2,$$

with equality $\iff f = \lambda g$.

Relates inner product to norm, and only happens to relate norms in L^1.

Minkowski's Inequality:

$$1 \leq p < \infty \implies \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Note: does not handle $p = \infty$ case. Use to prove L^p is a normed space.

Young's Inequality:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \implies \|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Useful specific cases:

$$\begin{aligned}
\|f * g\|_1 &\leq \|f\|_1 \|g\|_1 \\
\|f * g\|_p &\leq \|f\|_1 \|g\|_p, \\
\|f * g\|_\infty &\leq \|f\|_2 \|g\|_2 \\
\|f * g\|_\infty &\leq \|f\|_p \|g\|_q.
\end{aligned}$$

Bessel's Inequality:

For $x \in H$ a Hilbert space and $\{e_k\}$ an orthonormal sequence,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

Note: this does not need to be a basis.

Parseval's identity:

Equality in Bessel's inequality, attained when $\{e_k\}$ is a *basis*, i.e. it is complete, i.e. the span of its closure is all of H.

2 Basics

Useful Technique: $\lim f_n = \limsup f_n = \liminf f_n$ iff the limit exists, so $\limsup f_n \leq g \leq \liminf f_n$ implies that $g = \lim f$. Similarly, a limit does not exist iff $\liminf f_n > \limsup f_n$.

Lemma: $\sum a_n < \infty \implies a_n \to 0$ and $\sum_{k=N}^{\infty} \overset{N \to \infty}{\to} 0$, i.e. the terms/tails of convergent sums go to zero.

Lemma (Heine-Borel): A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Lemma (Geometric Series):

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \iff |x| < 1.$$

> *Corollary:* $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1$.

Definition: A set S is **nowhere dense** iff the closure of S has empty interior iff every interval contains a subinterval that does not intersect S.

Definition: A set is **meager** if it is a *countable* union of nowhere dense sets.

> Note that a *finite* union of nowhere dense is still nowhere dense.

Lemma: The Cantor set is closed with empty interior.

> Proof: Its complement is a union of open intervals, and can't contain an interval since intervals have positive measure and $m(C_n)$ tends to zero.

> **Corollary:** The Cantor set is nowhere dense.

Definition: An F_σ set is a union of closed sets, and a G_δ set is an intersection of opens.

> Mnemonic: "F" stands for *ferme*, which is "closed" in French, and σ corresponds to a "sum", i.e. a union.

Lemma: Singleton sets in $\mathbb{R}$ are closed, and thus $\mathbb{Q}$ is an F_σ set.

Theorem (Baire): $\mathbb{R}$ is a Baire space, i.e. countable intersections of open, dense sets are still dense. Thus $\mathbb{R}$ can not be written as a countable union of nowhere dense sets.

Lemma: There is a function discontinuous precisely on $\mathbb{Q}$.

> *Proof:* $f(x) = \frac{1}{n}$ if $x = r_n \in \mathbb{Q}$ is an enumeration of the rationals, and zero otherwise. The limit at every point is 0.

Lemma: There *do not* exist functions that are discontinuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

> *Proof:* D_f is always an F_σ set, which follows by considering the oscillation ω_f. $\omega_f(x) = 0 \iff f$ is continuous at x, and $D_f = \bigcup_n A_{\frac{1}{n}}$ where $A_\varepsilon = \{\omega_f \geq \varepsilon\}$ is closed.

Lemma: Any nonempty set which is bounded from above (resp. below) has a well-defined supremum (resp. infimum).

3 Uniform Convergence

Theorem (Egorov):

Let $E \subseteq \mathbb{R}^n$ be measurable with $m(E) > 0$ and $\{f_k : E \to \mathbb{R}\}$ be measurable functions such that $f(x) \coloneqq \lim_{k\to\infty} f_k(x) < \infty$ exists almost everywhere.

Then $f_k \to f$ *almost uniformly*, i.e.

$$\forall \varepsilon > 0,\ \exists F \subseteq E \text{ closed such that } m(E \setminus F) < \varepsilon \quad\text{and}\quad f_k \xrightarrow{u} f \text{ on } F.$$

Theorem (Important Example): The space $X = C([0,1])$, continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$, is a **complete** metric space.

Proof:
Step 0: Let $\{f_k\}$ be Cauchy in X.
Step 1: Define a candidate limit using pointwise convergence:
Fix an x; since

$$|f_k(x) - f_j(x)| \leq \|f_k - f_k\| \to 0,$$

the sequence $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. So define $f(x) \coloneqq \lim_k f_k(x)$.
Step 2: Show that $\|f_k - f\| \to 0$:

$$|f_k(x) - f_j(x)| < \varepsilon\ \forall x \implies \lim_j |f_k(x) - f_j(x)| < \varepsilon\ \forall x$$

Alternatively, $\|f_k - f\| \leq \|f_k - f_N\| + \|f_N - f_j\|$, where N, j can be chosen large enough to bound each term by $\varepsilon/2$.
Step 3: Show that $f \in X$:
The uniform limit of continuous functions is continuous. (Note: in other cases, you may need to show the limit is bounded, or has bounded derivative, or whatever other conditions define X.) ■

Lemma: Metric spaces are compact iff they are sequentially compact, (i.e. every sequence has a convergent subsequence).

Corollary: The unit ball in $C([0,1])$ with the sup norm is not compact.

Proof: Take $f_k(x) = x^n$, which converges to a dirac delta at 1. The limit is not continuous, so no subsequence can converge.

Lemma: A uniform limit of continuous functions is continuous.

Lemma (Testing Uniform Convergence): $f_n \to f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \to 0$.

Negating: find an x which depends on n for which the norm is bounded below.

Useful Technique: If f_n has a global maximum (computed using f_n' and the first derivative test) $M_n \to 0$, then $f_n \to 0$ uniformly.

Lemma (Baby Commuting Limits with Integrals): If $f_n \to f$ uniformly, then $\displaystyle\int f_n = \int f$.

Lemma (Uniform Convergence and Derivatives) If $f_n' \to g$ uniformly for some g and $f_n \to f$ pointwise (or at least at one point), then $g = f'$.

Lemma (Uniform Convergence of Series): If $f_n(x) \leq M_n$ **for a fixed** x where $\sum M_n < \infty$, then the series $f(x) = \sum f_n(x)$ converges pointwise.

Lemma: If $\sum f_n$ converges then $f_n \to 0$ uniformly.

Useful Technique: For a fixed x, if $f = \sum f_n$ converges *uniformly* on some $B_r(x)$ and each f_n is continuous at x, then f is also continuous at x .

Lemma (M-test for Series): If $|f_n(x)| \leq M_n$ which does not depend on x, then $\sum f_n$ converges uniformly.

Lemma (p-tests): Let n be a fixed dimension and set $B = \left\{x \in \mathbb{R}^n \;\middle|\; \|x\| \leq 1\right\}$.

$$\begin{aligned}
\sum \frac{1}{n^p} &< \infty \iff p > 1 \\
\int_\varepsilon^\infty \frac{1}{x^p} &< \infty \iff p > 1 \\
\int_0^1 \frac{1}{x^p} &< \infty \iff p < 1 \\
\int_B \frac{1}{|x|^p} &< \infty \iff p < n \\
\int_{B^c} \frac{1}{|x|^p} &< \infty \iff p > n
\end{aligned}$$

.

4 Measure Theory

Useful Technique: $s = \inf\{x \in X\} \implies$ for every ε there is an $x \in X$ such that $x \leq s + \varepsilon$.

Useful Techniques: Always consider bounded sets, and if E is unbounded write $E = \bigcup_n B_n(0) \bigcap E$ and use countable subadditivity or continuity of measure.

Lemma: Every open subset of $\mathbb{R}$ (resp $\mathbb{R}^n$) can be written as a unique countable union of disjoint (resp. almost disjoint) intervals (resp. cubes).

Definition: The outer measure of a set is given by

$$m_*(E) = \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

Lemma (Properties of [Outer] Measure):

- Montonicity: $E \subseteq F \implies m_*(E) \leq m_*(F)$.
- Countable Subadditivity: $m_*(\bigcup E_i) \leq \sum m_*(E_i)$.
- Approximation: For all E there exists a $G \supseteq E$ such that $m_*(G) \leq m_*(E) + \varepsilon$.
- Disjoint* Additivity: $m_*(A \coprod B) = m_*(A) + m_*(B)$.

  Note: this holds for outer measure **iff** $\operatorname{dist}(A, B) > 0$.

Lemma (Subtraction of Measure): $m(A) = m(B) + m(C)$ and $m(C) < \infty$ implies that $m(A) - m(C) = m(B)$.

Lemma (Continuity of Measure):

$$\begin{aligned} E_i \nearrow E &\implies m(E_i) \to m(E) \\ m(E_1) < \infty \text{ and } E_i \searrow E &\implies m(E_i) \to m(E). \end{aligned}$$

Proof: 1. Break into disjoint annuli $A_2 = E_2 \setminus E_1$, etc then apply countable disjoint additivity to $E = \coprod A_i$.

2. Use $E_1 = (\coprod E_j \setminus E_{j+1}) \coprod (\bigcap E_j)$, taking measures yields a telescoping sum,and use countable disjoint additivity.

Lemma: Lebesgue measure is translation and dilation invariant.

Proof: Obvious for cubes; if $Q_i \rightrightarrows E$ then $Q_i + k \rightrightarrows E + k$, etc.

Theorem (Non-Measurable Sets): There is a non-measurable set.

Proof:

- Use AOC to choose one representative from every coset of $\mathbb{R}/\mathbb{Q}$ on $[0, 1)$, which is countable, and assemble them into a set N
- Enumerate the rationals in $[0, 1]$ as q_j, and define $N_j = N + q_j$. These intersect trivially.
- Define $M := \coprod N_j$, then $[0, 1) \subseteq M \subseteq [-1, 2)$, so the measure must be between 1 and 3. By translation invariance, $m(N_j) = m(N)$, and disjoint additivity forces $m(M) = 0$, a contradiction.

Lemma (Borel Characterization of Measurable Sets)

If E is Lebesgue measurable, then $E = H \coprod N$ where $H \in F_\sigma$ and N is null.

Useful technique: F_σ sets are Borel, so establish something for Borel sets and use this to extend it to Lebesgue.

Proof: For every $\frac{1}{n}$ there exists a closed set $K_n \subset E$ such that $m(E \setminus K_n) \le \frac{1}{n}$. Take $K = \bigcup K_n$, wlog $K_n \nearrow K$ so $m(K) = \lim m(K_n) = m(E)$. Take $N := E \setminus K$, then $m(N) = 0$.

Lemma:

$$\begin{aligned} \limsup_n A_n &= \bigcap_n \bigcup_{j \ge n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for inf. many } n \right\} \\ \liminf_n A_n &= \bigcup_n \bigcap_{j \ge n} A_j = \left\{ x \;\middle|\; x \in A_n \text{ for all except fin. many } n \right\} \end{aligned}$$

.

Lemma: If A_n are all measurable, $\limsup A_n$ and $\liminf A_n$ are measurable.

Proof: Measurable sets form a sigma algebra, and these are expressed as countable unions/intersections of measurable sets.

Lemma (Borel-Cantelli):

Let $\{E_k\}$ be a countable collection of measurable sets. Then

$$\sum_k m(E_k) < \infty \implies \text{almost every } x \in \mathbb{R} \text{ is in at most finitely many } E_k.$$

Application:

$$m\left(\left\{x \text{ such that } \exists \text{ inf. many } \frac{p}{q} \text{ with } \left|x - \frac{p}{q}\right| \leq \frac{1}{q^3}\right\}\right) = 0.$$

Proof: Write E_j to be the above set with p, q replaced by p_j, q_j where $r_j = \frac{p_j}{q_j}$ is an enumeration of $\mathbb{Q}$, then $m(E_j) \leq \frac{2}{q^3}$ and $\sum \frac{1}{q^3} < \infty$.

Lemma:

- Characteristic functions are measurable
- If f_n are measurable, so are $|f_n|, \limsup f_n, \liminf f_n, \lim f_n$,
- Sums and differences of measurable functions are measurable,
- Cones $F(x, y) = f(x)$ are measurable,
- Compositions $f \circ T$ for T a linear transformation are measurable,
- "Convolution-ish" transformations $(x, y) \mapsto f(x - y)$ are measurable

Proof (Convolution): Take the cone on f to get $F(x, y) = f(x)$, then compose F with the linear transformation $T = [1, -1; 1, 0]$.

5 Integration

Definition: $f \in L^+$ iff f is measurable and non-negative.

Useful techniques:

- Break integration domain up into disjoint annuli.
- Break real integrals up into $x < 1$ and $x > 1$.

Definition: A measurable function is integrable iff $\|f\|_1 < \infty$.

Useful facts about C_c functions:

- Bounded almost everywhere
- Uniformly continuous

5.1 Convergence Theorems

Monotone Convergence Theorem (MCT):

If $f_n \in L^+$ and $f_n \nearrow f$ a.e., then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \quad \int f_n \to \int f.$$

Needs to be positive and increasing.

Dominated Convergence Theorem (DCT):

If $f_n \in L^1$ and $f_n \to f$ a.e. with $|f_n| \leq g$ for some $g \in L^1$, then

$$\lim \int f_n = \int \lim f_n = \int f \quad \text{i.e.} \int f_n \to \int f,$$

and more generally,

$$\int |f_n - f| \to 0.$$

> Positivity *not* needed.

> Generalized DCT: can relax $|f_n| < g$ to $|f_n| < g_n \to g \in L^1$.

Lemma: If $f \in L^1$, then

$$\int |f_n - f| \to 0 \iff \int |f_n| \to |f|.$$

> *Proof:* Let $g_n = |f_n| - |f_n - f|$, then $g_n \to |f|$ and
>
> $$|g_n| = ||f_n| - |f_n - f|| \leq |f_n - (f_n - f)| = |f| \in L^1,$$
>
> so the DCT applies to g_n and
>
> $$\begin{aligned}\|f_n - f\|_1 = \int |f_n - f| + |f_n| - |f_n| &= \int |f_n| - g_n \\ &\to_{DCT} \lim \int |f_n| - \int |f|.\end{aligned}$$

Fatou's Lemma:

If $f_n \in L^+$, then

$$\begin{gathered}\int \liminf_n f_n \leq \liminf_n \int f_n \\ \limsup_n \int f_n \leq \int \limsup_n f_n.\end{gathered}$$

> Only need positivity.

Theorem (Tonelli): For $f(x, y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \displaystyle\int f(x, y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x \coloneqq \left\{ y \;\middle|\; (x, y) \in E \right\}$ are measurable.
- $\displaystyle\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem (Fubini): For $f(x, y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) = \int f(x, y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x := \left\{ y \,\middle|\, (x, y) \in E \right\}$ are measurable.
- $\int f = \int \int f(x, y)$, i.e. any iterated integral is equal to the original

Theorem (Fubini/Tonelli): If any iterated integral is **absolutely integrable**, i.e. $\int \int |f(x, y)| < \infty$, then f is integrable and $\int f$ equals any iterated integral.

Differentiating under the integral:

If $\left| \frac{\partial}{\partial t} f(x, t) \right| \leq g(x) \in L^1$, then letting $F(t) = \int f(x, t)\, dt$,

$$\begin{aligned}\frac{\partial}{\partial t} F(t) &:= \lim_{h \to 0} \int \frac{f(x, t+h) - f(x, t)}{h} dx \\ &\overset{\text{DCT}}{=} \int \frac{\partial}{\partial t} f(x, t)\, dx.\end{aligned}$$

To justify passing the limit, let $h_k \to 0$ be any sequence and define

$$f_k(x, t) = \frac{f(x, t+h_k) - f(x, t)}{h_k},$$

so $f_k \overset{\text{pointwise}}{\to} \frac{\partial}{\partial t} f$.

Apply the MVT to f_k to get $f_k(x, t) = f_k(\xi, t)$ for some $\xi \in [0, h_k]$, and show that $f_k(\xi, t) \in L_1$.

Lemma (Swapping Sum and Integral) If f_n are non-negative and $\sum \int |f|_n < \infty$, then $\sum \int f_n = \int \sum f_n$.

> *Proof: MCT.* Let $F_N = \sum^N f_n$ be a finite partial sum; then there are simple functions $\phi_n \nearrow f_n$ and so $\sum^N \phi_n \nearrow F_N$, so apply MCT.

Lemma: If $f_k \in L^1$ and $\sum \|f_k\|_1 < \infty$ then $\sum f_k$ converges almost everywhere and in L^1.

> *Proof:* Define $F_N = \sum^N f_k$ and $F = \lim_N F_N$, then $\|F_N\|_1 \leq \sum^N \|f_k\| < \infty$ so $F \in L^1$ and $\|F_N - F\|_1 \to 0$ so the sum converges in L^1. Almost everywhere convergence: ?

5.2 L^1 Facts

Lemma (Translation Invariance): The Lebesgue integral is translation invariant, i.e. $\int f(x)\, dx = \int f(x+h)\, dx$ for any h.

Proof:

- For characteristic functions, $\displaystyle\int_E f(x+h) = \int_{E+h} f(x) = m(E+h) = m(E) = \int_E f$ by translation invariance of measure.
- So this also holds for simple functions by linearity
- For $f \in L^+$, choose $\phi_n \nearrow f$ so $\displaystyle\int \phi_n \to \int f$.
- Similarly, $\tau_h \phi_n \nearrow \tau_h f$ so $\displaystyle\int \tau_h f \to \int f$
- Finally $\displaystyle\left\{\int \tau_h \phi\right\} = \left\{\int \phi\right\}$ by step 1, and the suprema are equal by uniqueness of limits.

Lemma (Integrals Distribute Over Disjoint Sets):

If $X \subseteq A \bigcup B$, then $\displaystyle\int_X f \le \int_A f + \int_{A^c} f$ with equality iff $X = A \coprod B$.

Lemma (L^1 functions may Decay Rapidly):

If $f \in L^1$ and f is uniformly continuous, then $f(x) \overset{|x|\to\infty}{\to} 0$.

Doesn't hold for general L^1 functions, take any train of triangles with height 1 and summable areas.

Lemma (L^1 functions have Small Tails):

If $f \in L^1$, then for every ε there exists a radius R such that if $A = B_R(0)^c$, then $\displaystyle\int_A |f| < \varepsilon$.

Proof: Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$ with $g \in C_c$, then choose N large enough so that $g = 0$ on $E := B_N(0)^c$, then $\displaystyle\int_E |f| \le \int_E |f-g| + \int_E |g|$.

Lemma (L^1 functions have absolutely continuity):

$m(E) \to 0 \implies \displaystyle\int_E f \to 0$.

Proof: Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$, then $g \le M$ so $\displaystyle\int_E f \le \int_E f - g + \int_E g \to 0 + M \cdot m(E) \to 0$.

Lemma (L^1 functions are finite almost everywhere):

If $f \in L^1$, then $m(\{f(x) = \infty\}) = 0$.

Proof (Split up domain2): Let $A = \{f(x) = \infty\}$, then $\displaystyle\infty > \int f = \int_A f + \int_{A^c} f = \infty \cdot m(A) + \int_{A^c} f \implies m(X) = 0$.

Lemma (Continuity in L^1): $\|\tau_h f - f\|_1 \to 0$ as $h \to 0$.

Proof: Approximate with compactly supported functions. Take $g \overset{L_1}{\to} f$ with $g \in C_c$.

$$\begin{aligned}
\int f(x+h) - f(x) \leq \\
\int f(x+h) - g(x+h) + \int g(x+h) - g(x) + \int g(x) - f(x) \\
\to 2\varepsilon + \int g(x+h) - g(x) \\
= \int_K g(x+h) - g(x) + \int_{K^c} g(x+h) - g(x) \to 0,
\end{aligned}$$

which follows because we can enlarge the support of g to K where the integrand is zero on K^c, then apply uniform continuity on K.

Theorem (Integration by Parts, Special Case):

$$\begin{aligned}
F(x) := \int_0^x f(y)dy \quad \text{and} \quad G(x) := \int_0^x g(y)dy \\
\implies \int_0^1 F(x)g(x)dx = F(1)G(1) - \int_0^1 f(x)G(x)dx.
\end{aligned}$$

Proof: Fubini-Tonelli, and sketch region to change integration bounds.

Theorem (Lebesgue Density):

$$A_h(f)(x) := \frac{1}{2h}\int_{x-h}^{x+h} f(y)dy \implies \|A_h(f) - f\| \overset{h\to 0}{\to} 0.$$

Proof: Fubini-Tonelli, and sketch region to change integration bounds, and continuity in L^1.

5.3 L^p Spaces

Lemma: The following are dense subspaces of $L^2([0,1])$:

- Simple functions
- Step functions
- $C_0([0,1])$
- Smoothly differentiable functions $C_0^\infty([0,1])$
- Smooth compactly supported functions C_c^∞

Dual Spaces: In general, $(L^p)^\vee \cong L^q$

- For qual, supposed to know the $p=1$ case, i.e. $(L^1)^\vee \cong L^\infty$
  - For the analogous $p=\infty$ case: $L^1 \subset (L^\infty)^\vee$, since the isometric mapping is always injective, but *never* surjective. So this containment is always proper (requires Hahn-Banach Theorem).
- The $p=2$ case: Easy by the Riesz Representation for Hilbert spaces.

6 Fourier Series and Convolution

Definition (Convolution)

$$f * g(x) = \int f(x-y)g(y)dy.$$

Definition (Dilation)

$$\phi_t(x) = t^{-n}\phi\left(t^{-1}x\right).$$

Definition (The Fourier Transform):

$$\hat{f}(\xi) = \int f(x)\ e^{2\pi i x\cdot\xi}\ dx.$$

Lemma: $\hat{f} = \hat{g} \implies f = g$ almost everywhere.

Lemma (Riemann-Lebesgue)

$$f \in L^1 \implies \hat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

Motto: Fourier transforms decay.

Lemma: If $f \in L^1$, then $\hat{f}$ is continuous and bounded.

Proof: $\left|\hat{f}\right| \leq \int |f| \cdot |e^{\cdots}| \leq \|f\|_1$, and the DCT shows that $\left|\hat{f}(\xi_n) - \hat{f}(\xi)\right| \to 0.$

Todo: search qual alerts.