

Title

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- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \left\{ f : X \rightarrow Y \mid f \text{ is continuous} \right\}.$$

- Topologize with the *compact-open* topology: $U \in \text{hom}_T(X, X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
 - * If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact}.$$

In words: $f_n \rightarrow f$ uniformly on every compact set.

- So define $\text{Map}(X, Y) = \text{hom}_{\text{Top}}(X, Y)$ equipped with the compact-open topology.
 - Can immediately consider a lot of interesting spaces:

$$\begin{aligned} X = I &:= [0, 1] \rightsquigarrow P(Y; x_0) := \left\{ f : I \rightarrow Y \mid f(0) = x_0 \right\} = Y^I \\ X = S^1 &\rightsquigarrow \Omega(Y; x_0) = \mathcal{L}(Y; x_0) := \left\{ f : S^1 \rightarrow Y \right\} = Y^{S^1}. \end{aligned}$$

- Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property:

$$\begin{aligned} \text{Map}(X \times Z, Y) &\rightarrow \text{Map}(Z, \text{Map}(X, Y)) \\ H(x, z) &\mapsto \tilde{H}(z)(x). \end{aligned}$$

- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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