

Title

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Saturday 26th September, 2020

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1.1 Compact-Open Topology

- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \{f : X \rightarrow Y \mid f \text{ is continuous}\}.$$

- General idea: it's nice to *cartesian closed* categories, which require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it (i.e. here we'd like the homs to again be spaces).
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
 - * Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
 - * Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Topologize with the *compact-open* topology \mathcal{O}_{CO} :

$$U \in \mathcal{O}_{\text{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

- * If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \{d(f_n(x), f(x)) \mid x \in K\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact}.$$

In words: $f_n \rightarrow f$ uniformly on every compact set.

- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.
- Useful in analysis: when is a family of functions $\mathcal{F} = \{f_\alpha\} \subset \text{hom}_{\text{Top}}(X, Y)$ compact? Essentially answered by Arzela-Ascoli

Theorem 1.1 (Ascoli).

If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \text{hom}_{\text{Top}}(X, Y)$ has compact closure $\iff \mathcal{F}$ is equicontinuous and $F_x := \{f(x) \mid f \in \mathcal{F}\} \subset Y$ has compact closure.

Corollary 1.2 (Arzela).

If $\{f_n\} \subset \text{hom}_{\text{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every x , it contains a uniformly convergent subsequence.

- Useful in Number Theory / Rep Theory / Fourier Series: can take G to be a locally compact abelian topological group and define its Pontryagin dual $\hat{G} := \text{hom}_{\text{TopGrp}}(G, S^1)$ where we consider $S^1 \subset \mathbb{C}$.
 - * Can integrate with respect to the Haar measure μ , define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\hat{f} \in L^p(\hat{G})$.

$$\hat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{\text{CO}}) \quad \text{where } \mathcal{O}_{\text{CO}} \text{ is the compact-open topology.}$$

$\text{Map}(X, Y) = \text{hom}_{\text{Top}}(X, Y)$ equipped with the compact-open topology.

- Can immediately consider some interesting spaces via the functor $\text{Map}(\cdot, Y)$:

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\ X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\ X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}. \end{aligned}$$

Note: take basepoints to obtain the base path space PY , the based loop space ΩY .

- Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, \text{Map}(X, Y)) \\ H : X \times Z &\rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\ (x, z) &\mapsto H(x, z) \iff z \mapsto H(\cdot, z). \end{aligned}$$

Categorically, $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top .

A form of this adjunction holds in any cartesian closed category.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps f, g are homotopic \iff they are connected by a path in $\text{Map}(X, Y)$.

* Proof:

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, \text{Map}(X, Y)) \cong \text{Map}(Y \times I, X),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- * Note that we can interpret the RHS as the space of paths
- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\text{Map}(I, \cdot)$: define

$$\text{Map}_I^1(X, Y) := \text{Map}(I, \text{Map}(X, Y)) = \mathcal{P}\text{Map}(X, Y)$$

and then

$$\begin{aligned} \text{Map}_I^2(X, Y) &:= \text{Map}(I, \text{Map}_I^1(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(I, \text{Map}(X, Y))) = \mathcal{P}(\mathcal{P}(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(Y \times I, X)) \\ &:= \mathcal{P}\text{Map}(Y \times I, X). \end{aligned}$$

Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

1.2 Self-Homeomorphisms

- Now restrict attention to

$$\text{Homeo}(X) := \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}.$$

- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.