

# Title

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## Contents

1 Friday, September 25

1

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- For  $X, Y$  topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \{f : X \rightarrow Y \mid f \text{ is continuous}\}.$$

- Topologize with the *compact-open* topology:  $U \in \text{hom}_T(X, X)$  open iff for every  $f \in U$ ,  $f(K)$  is open for every compact  $K \subseteq X$ .

- \* If  $Y = (Y, d)$  is a metric space, this is the topology of “uniform convergence on compact sets”: for  $f_n \rightarrow f$  in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \{d(f_n(x), f(x)) \mid x \in K\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact}.$$

In words:  $f_n \rightarrow f$  uniformly on every compact set.

- If  $X$  itself is compact and  $Y$  is a metric space,  $C(X, Y)$  can be promoted to a metric space with  $d(f, g) = \sup_{x \in X} (f(x), g(x))$ .
- So define  $\text{Map}(X, Y) = \text{hom}_{\text{Top}}(X, Y)$  equipped with the compact-open topology.
  - Can immediately consider a lot of interesting spaces by considering  $\text{Map}(\cdot, Y)$ :

$$X = I := [0, 1] \rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I$$

$$X = S^1 \rightsquigarrow \Omega Y := \{f : S^1 \rightarrow Y\} = Y^{S^1} = \mathcal{L}(Y; x_0).$$

- Importance in homotopy theory: the path space fibration  $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$  (plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property: there is a homeomorphism

$$\text{Map}(X \times Z, Y) \leftrightarrow \cong \text{Map}(Z, \text{Map}(X, Y))$$

$$H : X \times Z \rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y)$$

$$(x, z) \mapsto H(x, z) \iff z \mapsto H(\cdot, z).$$

Categorically,  $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$  form an adjoint pair in  $\text{Top}$ .

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- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps  $f, g$  are homotopic  $\iff$  they are connected by a path in  $\text{Map}(X, Y)$ .

\* Proof:

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, \text{Map}(X, Y)) \cong \text{Map}(Y \times I, X),$$

and just check that  $\gamma(0) = f \iff H(x, 0) = f$  and  $\gamma(1) = g \iff H(x, 1) = g$ .

\* Note that we can interpret the RHS as the space of paths

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor*  $\text{Map}(I, \cdot)$ : define

$$\text{Map}_I^1(X, Y) := \text{Map}(I, \text{Map}(X, Y)) = \mathcal{P}\text{Map}(X, Y)$$

and then

$$\begin{aligned} \text{Map}_I^2(X, Y) &:= \text{Map}(I, \text{Map}_I^1(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(I, \text{Map}(X, Y))) = \mathcal{P}(\mathcal{P}(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(Y \times I, X)) \\ &:= \mathcal{P}\text{Map}(Y \times I, X). \end{aligned}$$

Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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