Title

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Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of **??**.

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$, then the data of a GL_n-torsor split by a τ-cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$
\begin{tikzcd}
U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\
U \arrow[d] \\
X
\end{tikzcd}
$$

That U trivializes our torsor means that $\pi^* T = \pi^* G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$
\begin{tikzcd}
\pi_1^* \pi^* T \arrow[r, "\sim"] \arrow[d] & \pi_2^* \pi^* T \arrow[d] \\
\pi_1^* \pi^* G \arrow[r, "\sim"] & \pi_2^* \pi^* G
\end{tikzcd}
$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n-torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{ét}}, \mathrm{GL}_n)$ is in bijection with n-dimensional vector bundles on X.

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