Title

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- For X, Y topological spaces, consider

$$Y^X = C(X,Y) = \hom_{\text{Top}}(X,Y) \coloneqq \left\{ f : X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

  - Topologize with the *compact-open* topology: $U \in \hom_T(X,X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
    - If $Y = (Y,d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty,K} \coloneqq \sup\left\{ d(f_n(x), f(x)) \;\middle|\; x \in K \right\} \overset{n\to\infty}{\to} 0 \quad \forall K \subseteq X \text{ compact}.$$

      In words: $f_n \to f$ uniformly on every compact set.
  - If X itself is compact and Y is a metric space, $C(X,Y)$ can be promoted to a metric space with $d(f,g) = \sup\limits_{x\in X}(f(x), g(x))$.
- So define $\mathrm{Map}(X,Y) = \hom_{\text{Top}}(X,Y)$ equipped with the compact-open topology.
  - Can immediately consider a lot of interesting spaces by considering $\mathrm{Map}(\,\cdot\,, Y)$:

$$\begin{aligned} X = I \coloneqq [0,1] \rightsquigarrow &\quad P(Y;x_0) \coloneqq \left\{ f : I \to Y \;\middle|\; f(0) = x_0 \right\} = Y^I \\ X = S^1 \rightsquigarrow &\quad \Omega(Y;x_0) = \mathcal{L}(Y;x_0) \coloneqq \left\{ f : S^1 \to Y \right\} = Y^{S^1}. \end{aligned}$$

  - Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma\mapsto\gamma(1)} Y$ (plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences).
  - Adjoint property: there is a homeomorphism

$$\begin{aligned} \mathrm{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \mathrm{Map}(Z, \mathrm{Map}(X,Y)) \\ H : X \times Z \to Y &\iff \tilde{H} : Z \to \mathrm{Map}(X,Y) \\ (x,z) \mapsto H(x,z) &\iff z \mapsto H(\,\cdot\,, z). \end{aligned}$$

Categorically, $\hom(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \mathrm{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},$$

  i.e. two maps f, g are homotopic $\iff$ there is a path in $Map(X, Y)$ connecting them.
  * Proof:

$$\mathrm{Map}(I, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(Y \times I, X),$$

    and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.
- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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