

# Weil Conjectures

D. Zack Garza

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## 1 Notes from Daniel's Office Hours

0. Definition of Zeta functions
1. Statement of the conjectures
2. Easy examples:  $\mathbb{P}^n$ ,  $\text{Gr}_\mathbb{A}(k, n) = \text{GL}(n, \mathbb{A})/P$  the stabilizer of an  $\mathbb{A}$ -point in  $\mathbb{C}^n, \mathbb{F}_{p^n}$ .
3. Medium example:  $E/\mathbb{A}$  an elliptic curve.
4. Work out a harder example as in Weil

### References

- [http://www-personal.umich.edu/~mmustata/zeta\\_book.pdf](http://www-personal.umich.edu/~mmustata/zeta_book.pdf)
- <https://youtu.be/wEz7fCvK6sM?t=293>
- Explanation of exponential appearing
- <https://arxiv.org/pdf/1807.10812.pdf>
- [http://www.math.canterbury.ac.nz/~j.boomer/expos/weil\\_conjectures.pdf](http://www.math.canterbury.ac.nz/~j.boomer/expos/weil_conjectures.pdf)
- Weil's Paper

### 1.1 Definition of Zeta Function

Fix  $q$  a prime and  $\mathbb{F} := \mathbb{F}_q$  the finite field with  $q$  elements, along with its unique degree  $n$  extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}_p} \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

**Definition 1.0.1.**

Let

$$J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$$

be an ideal, then a *projective algebraic* variety  $X \subset \mathbb{P}_{\mathbb{F}}^N$  can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^{\infty} \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in  $n + 1$  variables, i.e. there is some fixed  $d \in \mathbb{Z}^{\geq 1}$  such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field  $\mathbb{F}_p$ . We will be thinking of  $K$ -valued points for  $K/\mathbb{F}_p$  algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into  $\mathbb{P}^{\infty}$ ?

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Fix  $X/\mathbb{F} \subset \mathbb{P}$  an  $N$ -dimensional projective algebraic variety, and say it's cut out by the equations  $f_1, \dots, f_M \in \mathbb{F}[x_0, \dots, x_n]$ . Note that it then has points in any finite extension  $L/K$ .

**Definition 1.0.2.**

Define the *local zeta function* of  $X$  the following formal power series:

$$Z_X(z) = \exp \left( \sum_{n=1}^{\infty} \alpha_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Concretely, for  $X \subset \mathbb{P}^M$  a variety cut out by  $\{f_i\} \subset \mathbb{F}[x_0, \dots, x_M]$  we are measuring the sizes of the sets

$$\alpha_n := \# \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}_{q^n}}^M \mid f_i(\mathbf{x}) = \mathbf{0} \, \forall i \right\}.$$

Note the following two properties:

$$Z_X(0) = 1$$

$$z \left( \frac{\partial}{\partial z} \right) \log Z_X(z) = t \left( \frac{Z'_X(z)}{Z_X(z)} \right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \dots,$$

which is an *ordinary generating function* for the sequence  $(\alpha_n)$ .

Todo: why not an OGF.

Remark: Note that for an OGF  $F(x) = \sum_{n=0}^{\infty} f_n x^n$ , we can extract coefficients in the following way:

$$f_n := [x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

Note: this is extremely amenable to numerical approximation if you have a closed form for  $F$  or even just a black-box numerical version of  $F$ ! I.e. easy to throw at a computer.

### 1.1.1 Simple but Useful Example: A Point

Take  $X = \{x = 0\} / \mathbb{F}$  a single point over  $\mathbb{F}$ , then

$$\begin{aligned} \#X(\mathbb{F}) &:= \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &:= \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &:= \alpha_n = 1 \\ &\vdots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n &&= 1 + z + z^2 + \dots \\ \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n &&= \sum_{n=0}^{\infty} \int z^n = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots \\ \implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n}. \end{aligned}$$

and so

$$\begin{aligned} Z_{\{\text{pt}\}}(z) &= \exp \left( 1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \dots \right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}. \end{aligned}$$

## 1.2 Statement of Weil Conjectures

(Weil 1949)

Let  $X$  be a smooth projective variety of dimension  $N$  over  $\mathbb{F}_q$  for  $q$  a prime, let  $Z_X(z)$  be its zeta function, and define  $\zeta_X(s) = Z_X(q^{-s})$ .

### 1. (Rationality)

$Z_X(z)$  is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$\begin{aligned} P_0(z) &= 1 - z \\ P_{2N}(z) &= 1 - q^N z \\ P_j(z) &= \prod_{i=1}^{\beta_j} (1 - a_{j,i} z) \quad \text{for some reciprocal roots } a_{j,i} \in \mathbb{C} \end{aligned}$$

where we've factored each  $P_i$  using its reciprocal roots  $a_{ij}$ .

In particular, this implies the existence of a meromorphic continuation of the associated function  $\zeta_X(s)$ , which a priori only converges for  $\Re(s) \gg 0$ .

### 2. (Functional Equation and Poincare Duality)

Let  $\chi(X)$  be the Euler characteristic of  $X$ , i.e. the self-intersection number of the diagonal embedding  $\Delta \hookrightarrow X \times X$ ; then  $Z_X(z)$  satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm \left(q^{\frac{N}{2}} z\right)^{\chi(X)} Z_X(z).$$

Equivalently,

$$\zeta_X(N - s) = \pm \left(q^{\frac{N}{2} - s}\right)^{\chi(X)} \zeta_X(s)$$

Note that when  $N = 1$ , e.g. for a curve, this relates  $\zeta_X(s)$  to  $\zeta_X(1 - s)$ .

Equivalently, there is an involutive map on the (reciprocal) roots

$$\begin{aligned} z &\longleftrightarrow \frac{q^N}{z} \\ \alpha_{j,k} &\longleftrightarrow \alpha_{2N-j,k} \end{aligned}$$

which sends roots of  $p_j$  to roots of  $p_{2N-j}$ .

### 3. (Riemann Hypothesis)

The reciprocal roots  $a_{j,k}$  are *algebraic* integers (roots of some monic  $p \in \mathbb{Z}[x]$ ) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}} \quad \forall 1 \leq j \leq 2N-1, \forall k.$$

4. (Betti Numbers) If  $X$  is a “good reduction mod  $q$ ” of a nonsingular projective variety  $\tilde{X}$  in characteristic zero, then the  $\beta_i = \deg p_i(z)$  are the Betti numbers of the topological space  $\tilde{X}(\mathbb{C})$ .

Why is (3) called the “Riemann Hypothesis”?

We can use the facts that

- a.  $|\exp(z)| = \exp(\Re(z))$  and
- b.  $a^z := \exp(z \operatorname{Log}(a))$ ,

to replace the polynomials  $P_i$  with

$$L_j(s) := \zeta_X(q^{-s}) = \prod_{k=1}^{\beta_j} (1 - \alpha_{j,k} q^{-s}).$$

Now consider the roots of  $L_j(s)$ : we have

$$\begin{aligned} L_j(s_0) &= 0 \\ \iff q^{-s_0} &= \frac{1}{\alpha_{j,k}} \quad \text{for some } k \\ \implies |q^{-s_0}| &= \left| \frac{1}{\alpha_{j,k}} \right| \quad \text{by assumption} \quad q^{-\frac{j}{2}} \\ \implies q^{-\frac{j}{2}} &\stackrel{(a)}{=} \exp\left(-\frac{j}{2} \cdot \operatorname{Log}(q)\right) = |\exp(-s_0 \cdot \operatorname{Log}(q))| \\ &\stackrel{(b)}{=} |\exp(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \operatorname{Log}(q))| \\ &\stackrel{(a)}{=} \exp(-(\Re(s_0)) \cdot \operatorname{Log}(q)) \\ \implies -\frac{j}{2} \cdot \operatorname{Log}(q) &= -\Re(s_0) \cdot \operatorname{Log}(q) \quad \text{by injectivity} \\ \implies \Re(s_0) &= \frac{j}{2}. \end{aligned}$$

Roughly speaking, realizing that we would need to apply a logarithm (a conformal map) to send the  $\alpha_{j,k}$  to zeros of the  $L_j$ , this says that the zeros all must lie on the “critical lines”  $\frac{j}{2}$ .



In particular, the zeros of  $L_1$  have real part  $\frac{1}{2}$ , analogous to the classical Riemann hypothesis.

Moral: the Diophantine properties of a variety's zeta function are governed by its (algebraic) topology. Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information. Plug for Langland's: it similarly asks for every  $L$  function arising from an automorphic representation that (essentially) satisfy Weil 2 and 3.

Historical note

- Desire for a “cohomology theory of varieties” drove 25 years of progress in AG

Remarks:

- Resolved for varieties over  $\mathbb{F}_q$
- On  $L_X$ :
  - Conjectured for smooth varieties over  $\mathbb{Q}$  (rationality  $\sim$  analytically continues to a meromorphic function, some functional equation), little is known.
  - Resolved for elliptic curves (Taylor-Wiles c/o the Taniyama-Shimura conjecture), implies  $L_X$  is an  $L$  function coming from a modular form.

### 1.2.1 Aside: Why call it a Zeta function?

Knowing the zeta function of a point, we can now make a precise analogy.

Suppose we have an algebraic variety cut out by equations:

$$\mathbb{A}_{\mathbb{Z}}^n \supseteq X = V(\langle f_1, \dots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \dots, x_{n-1}].$$

Then for every prime  $q$ , we can reduce the equations mod  $p$  and consider

$$\mathbb{A}_{\mathbb{F}_q}^n \supseteq X_q := V(\langle f_1 \bmod q, \dots, f_d \bmod q \rangle) \quad \text{where} \quad f_i \bmod q \in \mathbb{F}_q[x_0, \dots, x_{n-1}]$$

Then define the *Hasse-Weil* zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Take  $X = \text{Spec } \mathbb{Q}$  and  $X_p = \text{Spec } \mathbb{F}_p$ , which is a single point since  $\mathbb{F}_p$  is a field. The previous example shows that

$$\zeta_{X_p}(z) = \frac{1}{1-z},$$

We then find that

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left( \frac{1}{1-p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

which is the Euler product expansion of the classical Riemann Zeta function.

Moreover, it is a theorem (difficult, not proved here!) that for any variety  $X/\mathbb{F}_p$ , we have

$$\zeta_X(t) = \prod_{x \in X_{\text{cl}}} \left( \frac{1}{1-t^{\deg(x)}} \right) \xrightarrow{t=p^{-s}} \zeta_X(s) = \prod_{x \in X_{\text{cl}}} \left( \frac{1}{1-(p^{\deg(x)})^{-s}} \right),$$

which we can think of as attaching a “weight” to each closed point,  $|x| := p^{\deg(x)}$ , and the usual Riemann Zeta corresponds to assigning a weight of 1 to each point.

Note that this immediately implies that  $\zeta_X(t) \in \mathbb{Z}[[t]]$  is a *rational* function.

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

After modifying  $\zeta$  to make it symmetric about  $\Re(s) = \frac{1}{2}$  and eliminate the trivial zeros at  $-2\mathbb{Z}$  to obtain  $\widehat{\zeta}(s)$ , there are three relevant properties

- “Rationality”:  $\widehat{\zeta}(s)$  has a meromorphic continuation to  $\mathbb{C}$  with simple poles at  $s = 0, 1$ .
- “Functional equation”:  $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
- “Riemann Hypothesis”: The only zeros of  $\widehat{\zeta}$  have  $\Re(s) = \frac{1}{2}$ .

### 1.2.2 More Examples

**Example (Affine Line):**  $X = \mathbb{A}^1/\mathbb{F}$  the affine line over  $\mathbb{F}$ , then Note that we can write

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1] \mid x_1 \in \mathbb{F}_n \right\}$$

as the set of one-component vectors with entries in  $\mathbb{F}_n$ , so

$$\begin{aligned} X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp \left( \sum_{n=1}^{\infty} \frac{q^n}{n} z^n \right) = \frac{1}{1 - qz}.$$

**Example (Affine Space):** Set  $X = \mathbb{A}^m/\mathbb{F}$ , affine  $m$ -space over  $\mathbb{F}$ , so we can just repeat with now  $m$  coordinates

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1, \dots, x_m] \mid x_i \in \mathbb{F}_n \right\}$$

Counting yields

$$\begin{aligned} X(\mathbb{F}) &= q^m \\ X(\mathbb{F}_2) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_n) &= (q^n)^m. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp \left( \sum_{n=1}^{\infty} \frac{q^{nm}}{n} z^n \right) = \frac{1}{1 - q^m z}.$$

**Example (Projective Line):**  $X = \mathbb{P}^1/\mathbb{F}$  the projective line over  $\mathbb{F}$ , then we can write use some geometry to write

$$\mathbb{P}_{\mathbb{F}}^1 = \mathbb{A}_{\mathbb{F}}^1 \coprod \{\infty\}$$

as the affine line with a point added at infinity.

We can then count by enumerating coordinates:

$$\begin{aligned} \mathbb{P}^1(\mathbb{F}_n) &= \left\{ [x_1, x_2] \mid x_1, x_2 \neq 0 \in \mathbb{F}_n \right\} / \sim \\ &= \left\{ [x_1, 1] \mid x_1 \in \mathbb{F}_n \right\} \coprod \{[1, 0]\}. \end{aligned}$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &\cdot \end{aligned}$$

Thus

$$\zeta_X(z) = \frac{1}{(1 - z)(1 - qz)}.$$



**Example (Projective Space):** Take  $X = \mathbb{P}_{\mathbb{F}}^n$ ,



Example image of  $\mathbb{P}_{\mathbb{F}(3)}^2$ :

Note that we can identify  $X = \text{Gr}_{\mathbb{F}}(1, n)$  as the space of lines in  $\mathbb{A}_{\mathbb{F}}^n$ .

**Proposition 1.1.**

The number of  $k$ -dimensional subspaces of  $\mathbb{A}_{\mathbb{F}}^m$  is the  $q$ -binomial coefficient:

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

*Proof.*

To choose a  $k$ -dimensional subspace,

- Choose a nonzero vector  $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$  in

$$q^m - 1$$

ways.

- Identify  $\#\text{span}\{\mathbf{v}_1\} = \#\{\lambda \mathbf{v}_1 \mid \lambda \in \mathbb{F}\} = \#\mathbb{F} = q$ .

- Choose a nonzero vector  $\mathbf{v}_2$  *not* in the span of  $\mathbf{v}_1$  in

$$q^m - q$$

ways.

- Identify  $\#\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \#\{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mid \lambda_i \in \mathbb{F}\} = q \cdot q = q^2$ .

- Choose a nonzero vector  $\mathbf{v}_3$  not in the span of  $\mathbf{v}_1, \mathbf{v}_2$  in

$$q^m - q^2$$

ways.

- ... until  $\mathbf{v}_k$  is chosen in

$$(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1})$$

ways.

- This yields a  $k$ -tuple of linearly independent vectors spanning a  $k$ -dimensional subspace  $V_k$
- This overcounts because many linearly independent sets span  $V_k$ , we need to divide out by the number of choose a basis inside of  $V_k$ .
- By the same argument, this is given by

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

Thus

$$\begin{aligned} \# \text{subspaces} &= \frac{(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})} \\ &= \frac{q^m - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{m-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{m-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{m-(k-1)} - 1}{q^{k-(k-1)-1}}. \end{aligned}$$

■

We obtain a nice simplification for the number of lines corresponding to setting  $k = 1$ :

$$\begin{bmatrix} m \\ 1 \end{bmatrix}_q = \frac{q^m - 1}{q - 1} = q^{m-1} + q^{m-2} + \cdots + q + 1 = \sum_{j=0}^{m-1} q^j.$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= \sum_{j=0}^{m-1} q^j \\ X(\mathbb{F}_2) &= \sum_{j=0}^{m-1} (q^2)^j \\ &\vdots \\ X(\mathbb{F}_n) &= \sum_{j=0}^{m-1} (q^n)^j. \end{aligned}$$

So

$$\zeta_X(z) = \left(\frac{1}{1-z}\right) \left(\frac{1}{1-qz}\right) \left(\frac{1}{1-q^2z}\right) \cdots \left(\frac{1}{1-q^mz}\right),$$

Note that geometry can help us here: we have a “cell decomposition”  $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$ , and so inductively

$$\mathbb{P}^n = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^n,$$

### 1.3 Hard Example: An Elliptic Curve

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and it's straightforward to prove that

$$\zeta_{X \coprod Y}(z) = \zeta_X(z) \cdot \zeta_Y(z)$$

and recalling that  $\zeta_{\mathbb{A}^j}(z) = \frac{1}{1 - q^j z}$  we have

$$\zeta_{\mathbb{P}^m}(z) = \prod_{j=0}^m \zeta_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1 - q^j z}.$$

Example: Take  $X = \text{Gr}_{\mathbb{F}}(k, n)$ , then ????? so

$$\zeta_X(t) = ?.$$

### 1.3 Hard Example: An Elliptic Curve

The Weyl conjectures take on a particularly nice form for curves. Let  $X/\mathbb{F}$  be a smooth projective curve of genus  $g$ , then

1. (Rationality)

$$\zeta_X(z) = \frac{p(z)}{(1-z)(1-qz)}$$

2. (Functional Equation)

$$\zeta_X\left(\frac{1}{qz}\right) = q^{1-g} z^{2-2g} \zeta_X(z)$$

3. (Riemann Hypothesis)

$$p(t) = \prod_{i=1}^{2g} (q - a_i z) \quad \text{where} \quad |a_i| = \frac{1}{\sqrt{q}}$$

Take  $X = E/\mathbb{F}$ .

Consider the curve E defined by the following equation:

$$E : y^2 + y = x^3 - x^2$$

This is a cubic, whose graph is presented in Figure 1.



Figure 1: Implicit plot of E

Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-t})}.$$

The betti numbers are  $[1, 2, 1, 0, \dots]$ .

The number of points are

$$X(\mathbb{F}_n) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

Rough outline of proof:

- ??

The (complex?) dimension of  $X$  is  $N = 1$ , The WC say we should be able to write this as

$$\frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1 - \alpha_{1,1}z)(1 - \alpha_{1,2}z)}{(1-z)(1-qz)}.$$

Since we know the number of points, we can compute

$$\begin{aligned}
\zeta_X(z) &= \exp \sum_{n=1}^{\infty} \#X(\mathbb{F}_n) \frac{z^n}{n} \\
&= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
&= \exp \left( \sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
&= \exp(-\log(1 - qz)) \exp(-\log(1 - z)) \exp(\log(1 - \alpha z)) \exp(\log(1 - \bar{\alpha} z)) \\
&= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),
\end{aligned}$$

which is indeed a rational function.

Originally conjectured for curves by Artin Proved by Weil in 1949, proposed generalization to projective varieties Proof had work contributed by Dwork (rationality using p-adic analysis), Artin, Grothendieck (etale cohomology), with completion by Deligne in 1970s (RH)

## 1.4 Very Hard Example: A Diagonal Hypersurface

### Reference

- Set  $q$  to be a prime power and consider  $X/\mathbb{F}_q$  defined by

$$X = V(a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}) \subset \mathbb{F}_q^{r+1}.$$

- We want to compute  $N = \#X$ .
- Set  $d_i = \gcd(n_i, q - 1)$ .
- Define the character

$$\begin{aligned}
\psi_q : \mathbb{F}_q &\longrightarrow \mathbb{C}^\times \\
a &\mapsto \exp \left( \frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p} \right).
\end{aligned}$$

- By Artin's theorem for linear independence of characters,  $\psi_q \not\equiv 1$  and every additive character of  $\mathbb{F}_q$  is of the form  $a \mapsto \psi_q(ca)$  for some  $c \in \mathbb{F}_q$ .

- Fix an injective multiplicative map

$$\psi : \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times.$$

- Define

$$\begin{aligned}
\chi_{\alpha, n} : \bar{\mathbb{F}}_q^\times &\longrightarrow \mathbb{C}^\times \\
x &\mapsto \phi(x)^{\alpha(q^n - 1)}
\end{aligned}$$

$$\text{for } \alpha \in \mathbb{Q}/\mathbb{Z}, n \in \mathbb{Z}, \quad \alpha(q^n - 1) \equiv 0 \pmod{1}.$$

- Extend this to  $\mathbb{F}_{q^n}$  by

$$\begin{cases} 1 & \alpha \equiv 0 \pmod{1} \\ 0 & \text{else} \end{cases}.$$

- Set  $\chi_\alpha = \chi_{\alpha,1}$ .

- Shorthand notation: say  $a \sim 0 \iff a \equiv 0 \pmod{1}$ .

- Proposition:

$$\alpha(q-1) \equiv 0 \pmod{1} \implies \chi_{\alpha,n}(x) = \chi_\alpha(\text{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))$$

- Proposition:

$$d := \gcd(n, q-1), u \in \mathbb{F}_q \implies \#\{x \in \mathbb{F}_1 \mid x^n = u\} = \sum_{d\alpha \sim 0} \chi_\alpha(u)$$

- This implies

$$\begin{aligned} N &= \sum_{\substack{\alpha=[\alpha_0, \dots, \alpha_r] \\ d_i \alpha_i \sim 0}} \sum_{\substack{\mathbf{u}=[u_0, \dots, u_r] \\ \sum a_i u_i = 0}} \prod_{j=0}^r \chi_{\alpha_j}(u_j) \\ &= q^r + \sum_{\substack{\alpha, \alpha_i \in (0,1) \\ d_i \alpha_i \sim 0}} \left( \prod_{j=0}^r \chi_{\alpha_j}(a_j^{-1}) \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) \right). \end{aligned}$$

since the inner sum is zero if some *but not all* of the  $\alpha_i \sim 0$ .

- Evaluate the innermost sum by restricting to  $u_0 \neq 0$  and setting  $u_i = u_0 v_i$  and  $v_0 := 1$ :

$$\begin{aligned} \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) &= \sum_{u_0 \neq 0} \chi_{\sum \alpha_i}(u_0) \sum_{\sum v_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(v_j) \\ &= \begin{cases} (q-1) \sum_{\sum v_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(v_j) & \text{if } \sum \alpha_i \sim 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

- Define the *Jacobi sum* for  $\alpha$  where  $\sum \alpha_i \sim 0$ :

$$J(\alpha) := \left( \frac{1}{q-1} \right) \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) = \sum_{\sum v_i = 0} \prod_{j=1}^r \chi_{\alpha_j}(v_j)$$

- Express  $N$  in terms of Jacobi sums as

$$N = q^r + (q-1) \sum_{\substack{\sum \alpha_i \sim 0 \\ d_i \alpha_i \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^r \chi_{\alpha_j}(a_j^{-1}) J(\alpha).$$

- Evaluate  $J(\alpha)$  using Gauss sums: for  $\chi : \mathbb{F}_q \rightarrow \mathbb{C}$  a multiplicative character, define

$$G(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \psi_q(x).$$

- Proposition: for any  $\chi \neq \chi_0$ ,

- $|G(\chi)| = q^{\frac{1}{2}}$
- $G(\chi)G(\bar{\chi}) = q\chi(-1)$
- $G(\chi_0) = 0$

$$\chi(t) = \frac{G(\chi)}{q} \sum_{x \in \mathbb{F}_q} \bar{\chi}(x) \psi_q(tx).$$

- Proposition: if  $\sum \alpha_i \sim 0$ , then  $J(\alpha) = \frac{1}{q} \prod_{k=1}^r G(\chi_{\alpha_k})$  and  $|J(\alpha)| = q^{\frac{r-1}{2}}$ .
- We thus obtain

$$N = q^r + \left( \frac{q-1}{q} \right) \sum_{\substack{\sum \alpha_i \sim 0 \\ d_i \alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \prod_{j=0}^r \chi_{\alpha_j}(a_j^{-1}) G(\chi_{\alpha_j}).$$

- We now ask for number of points in  $\mathbb{F}_{q^\nu}$
- Theorem (Davenport, Hasse)  $(q-1)\alpha \sim 0 \implies -G(\chi_{\alpha,\nu}) = (-G(\chi_\alpha))^\nu$ .

- 
- Now restrict to  $n_0 = \dots = n_r = n$  a constant, and we consider a point count

$$\bar{N}_\nu = \# \left\{ [x_0 : \dots : x_r] \in \mathbb{P}_{\mathbb{F}_q}^r \mid \sum_{i=0}^r a_i x_i^n = 0 \right\}.$$

- We have a relation  $(q^\nu - 1)\bar{N}_\nu = N_\nu$ .
- This lets us write

$$\bar{N}_\nu = \sum_{j=0}^{r-1} q^{j\nu} + \sum_{\substack{\sum \alpha_i \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \prod_{j=0}^r \bar{\chi}_{\alpha_j,\nu}(a_i) J_\nu(\alpha).$$

- Plugging into the zeta function  $Z$  and setting  $\mu(\alpha) = \min \left\{ \mu \mid (q^\mu - 1)\alpha \sim 0 \right\}$  yields

$$\exp \left( \sum_{\nu=1}^{\infty} \bar{N}_\nu \frac{T^\nu}{\nu} \right).$$