Title

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Contents

1 | Wednesday, October 07

1.1 Schur Algebras

Let $G = \mathrm{GL}(n, k)$, then polynomial representations of G are equivalent to $S(n, d)$ modules for all $d \geq 0$, where we can note that $S(n, d) = \mathrm{End}_{\Sigma_d}(V^{\otimes d})$. We'll have a correspondence

$$\left\{ {\scriptstyle L(\lambda) \text{ simple modules for } S(n,d)} \right\} \iff \Lambda^+(n, d), \text{ partitions of } d \text{ with at most } n \text{ parts,}$$

Example 1.1.1.

Good example, can see all filtrations at work, tilting modules, etc.

Consider $S(3, 3)$ for $p = 3$, we then have the partitions $\Lambda^+(3, 3) = \{(3), (2, 1), (1, 1, 1)\}$. We can think of these in the ε basis as $(3) = (3, 0, 0), (2, 1) = (2, 1, 0)$. Since $\mathrm{SL}(3, k) \subset \mathrm{GL}(3, k)$, we can find the $SL(3, k)$ weights by taking successive differences to yield $(3, 0), (1, 1), (0, 0)$ with the corresponding picture

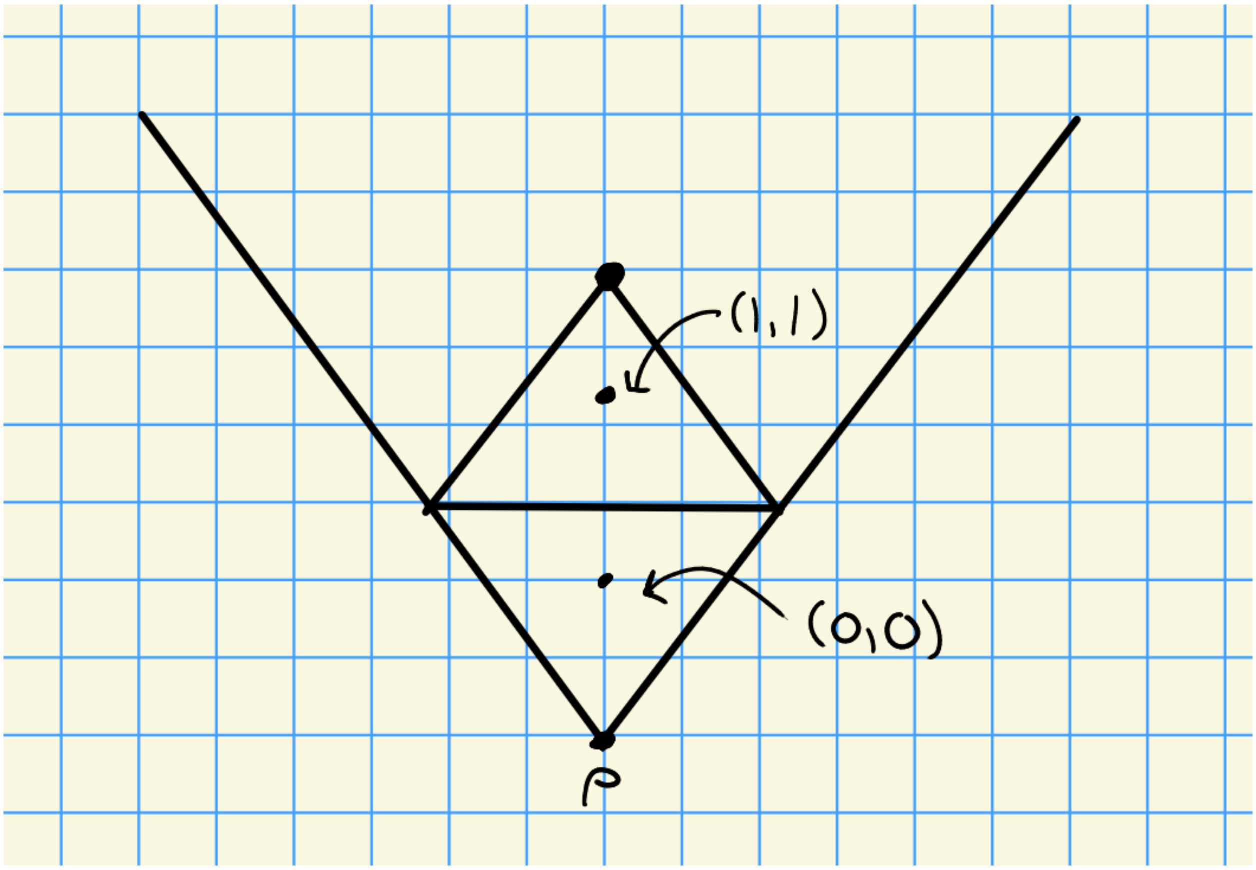

Figure 1: Image

We can compute

- $L(1,1,1) = H^0(1,1,1)$
- $L(2,1) = H^0(2,1)$
- $L(3) = H^0(3)$

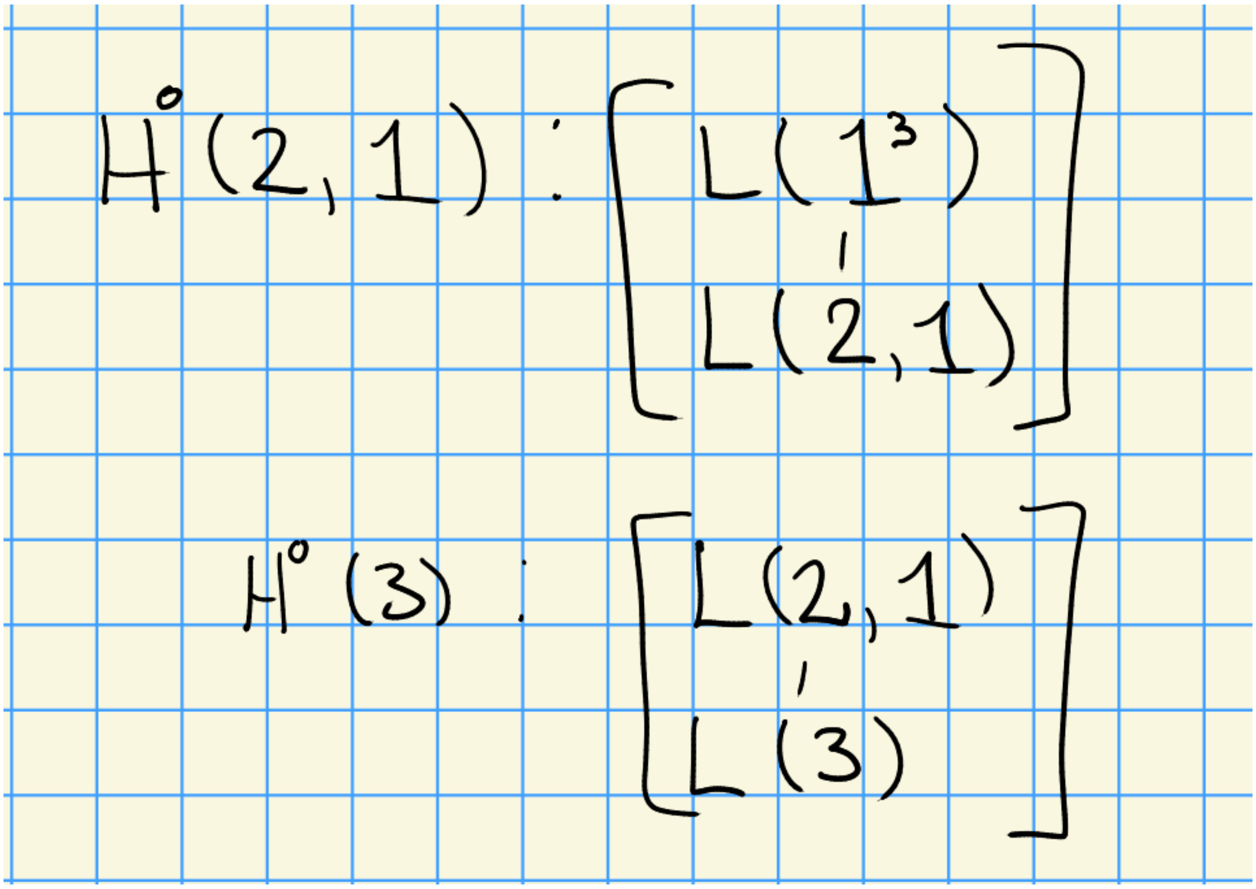

Figure 2: Image

We have a form of Brauer reciprocity:

$$[I(\lambda) : H^0(\mu)] = [H^0(\mu) : L(\lambda)].$$

We can now compute the injective hulls:

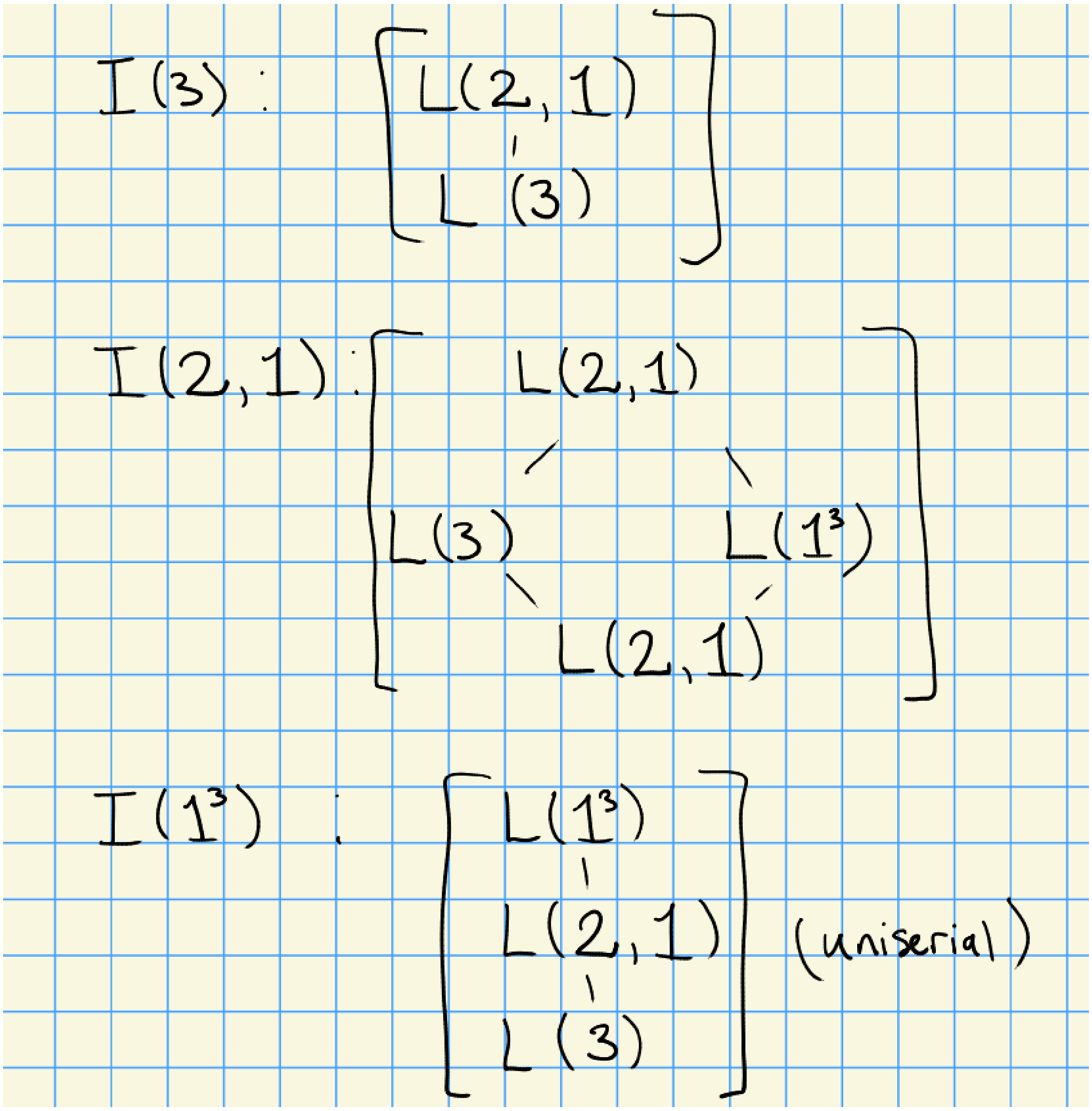

Figure 3: Image

What are the tilting modules? We can use the fact that $L(1^3) = V(1^3)$. It has a good filtration and a Weyl filtration and thus must be the tilting module for $L(1^3)$.

Using the following fact:

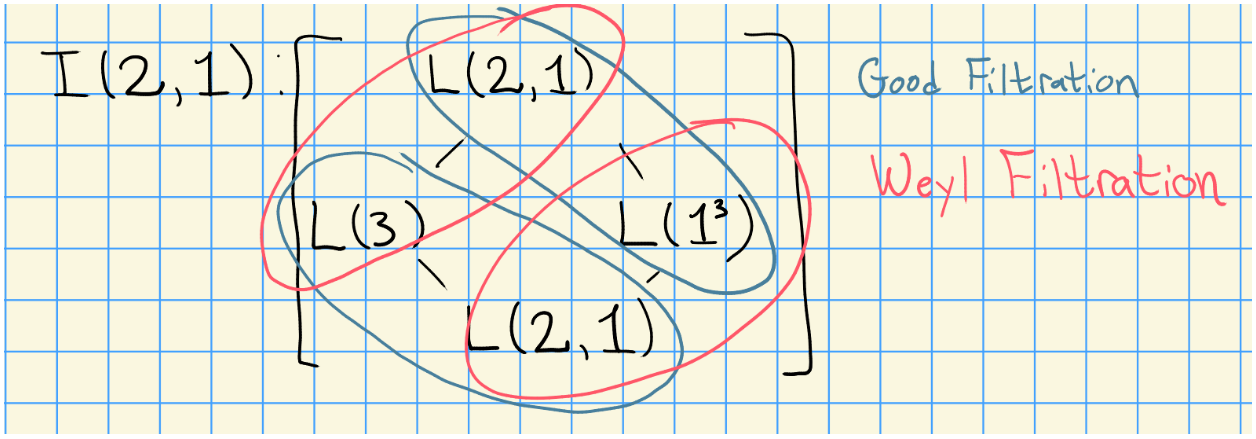

Figure 4: Image

We can compute the following:

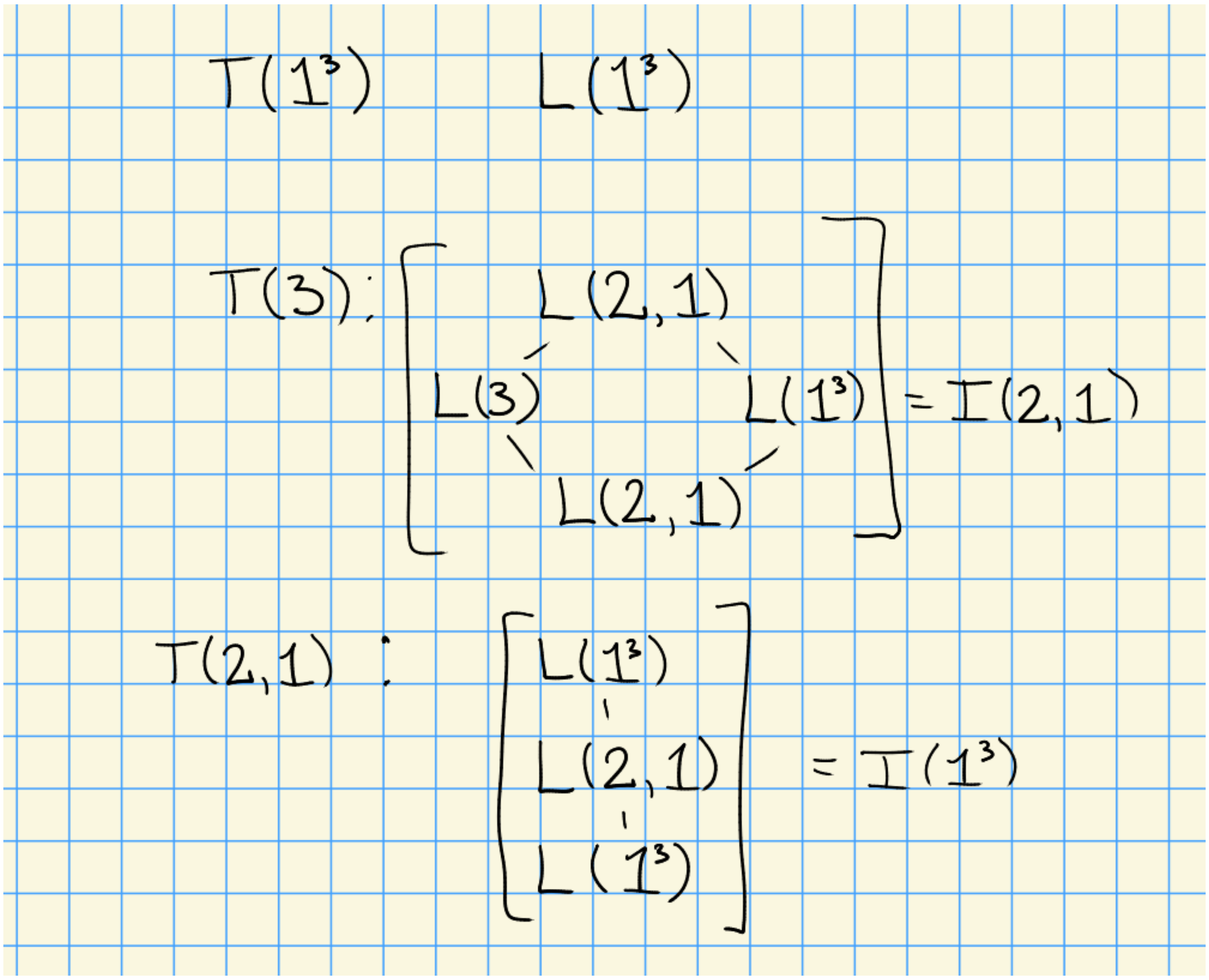

Figure 5: Image

1.2 Simplicity of $H^0(\lambda)$

1. $k = \mathbb{C}$ implies $L(\lambda) = H^0(\lambda)$ for all $\lambda \in X(T)_+$
2. $k = \overline{\mathbb{F}}_p$ implies $L(\lambda) = H^0(\lambda)$ if $\langle \lambda,\ \alpha_0^\vee \rangle \leq 1$ where α_0 is the highest short root.

Such λ are referred to as *minuscule weights.*

Example 1.2.1.

For type A_n, we have $\alpha_0 = \sum_{i=1}^{n} \alpha_i$. For type G_2, we have $\alpha_0^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee$.

Example 1.2.2.

In type A_n, set $\lambda = \sum_{j=1}^{n} c_j w_j$ where $c_j \geq 0$. Then $\langle \lambda,\ \alpha_0^\vee \rangle = \sum c_j \leq 1$, so λ is minuscule iff $\lambda = 0$

or $\lambda = w_j$ for some j.

Remark 1.2.1.
Quick timeline:

- 2015, Cantrell lectures by Dick Gross at UGA
- Fall 2015: email to Dan Nakano from Skip Garibaldi, conjecture from Gross without a proof

Proposition 1.2.1 ***(Gross).***
The simple module is equal to the induced module, so $L(\lambda) = H^0(\lambda)$, for all p iff λ is minuscule, or if $L(\lambda) = \mathfrak{g}$ for $\Phi = E_8$.

- Proved by Garibaldi-Nakano-Guralnick, appeared in Journal of Algebra

1.3 Bott-Borel-Weil Theorem

We can consider the higher right-derived functors of λ, given by $H^i(\lambda) = R^i \operatorname{Ind}_B^G \lambda$ for $\lambda \in X(T)$. You can think of this as the higher sheaf cohomology of the flag variety, $\mathcal{H}^i(G/B, \mathcal{L}(\lambda))$.

We have **Kempf Vanishing**: $H^i(\lambda) = 0$ for all $i > 0$ when $\lambda \in X(T)_+$ is dominant (although other things may happen for non-dominant weights). There is a correspondence $(G, T) \iff (W, \Phi)$, and since W is generated by simple reflections, we can write any $w \in W$ as $w = \prod s_{\alpha_i}$. A *reduced expression* is one in which the length can not be shortened, and any two reduced expressions necessarily have the same length (number of simple reflections).

Example 1.3.1.
For $\Phi = A_2$, we have $w_0 = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$.

1.3.1 Dot Action on Weights

We can let W act on $X(T)$ by reflections by the formula $s_\alpha \lambda = \lambda - \langle \lambda,\ \alpha^\vee \rangle \alpha$. We then shift the action by setting $s_\alpha \cdot \lambda = w(\lambda + \rho) - \rho$ where $\rho = \dfrac{1}{2} \displaystyle\sum_{\alpha \in \Phi^+} \alpha = \sum_{j=1}^n w_j$.

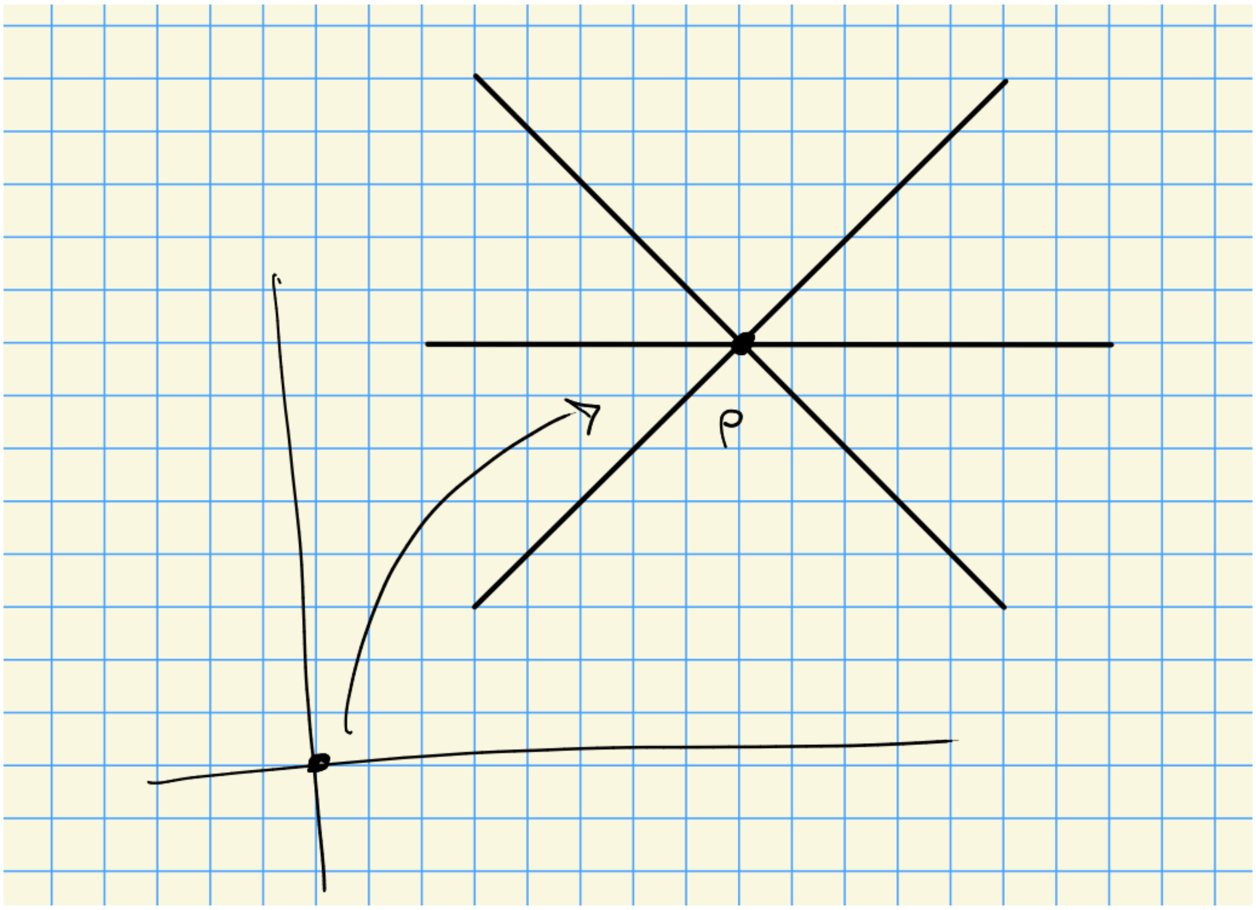

Figure 6: Image

Theorem 1.3.1 ***(Bott-Borel-Weil).***
Let G be a reductive algebraic group and $k = \mathbb{C}$. For $\lambda \in X(T)_+$, we can describe the sheaf cohomology:

$$\mathcal{H}^i(w \cdot \lambda) = \begin{cases} H^0(\lambda) & i = \ell(w) \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, if $\lambda \notin X(T)_+$ and $\langle \lambda + \rho,\ \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, then $\mathcal{H}^i(w \cdot \lambda) = 0$ for all $w \in W$.

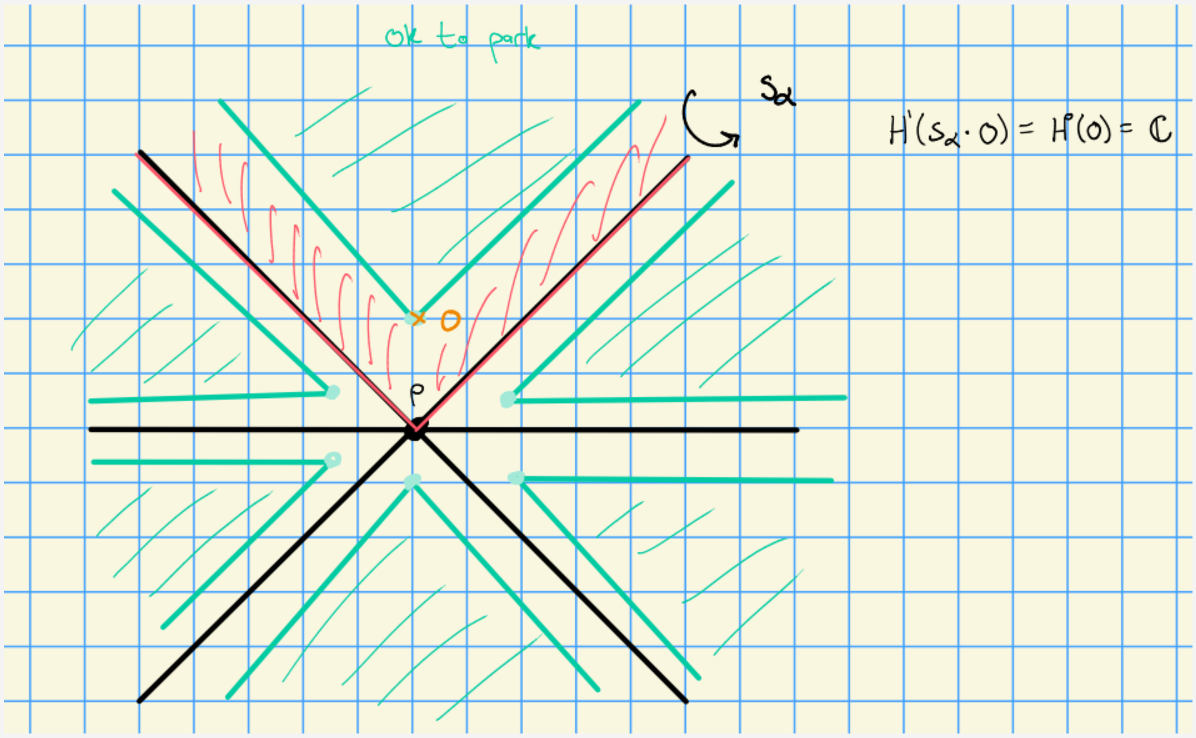

Figure 7: Image

Wide open in characteristic p, can say some things. We'll prove this in characteristic zero.

Recall that $k = \mathbb{C}$ and $H^0(\lambda) = L(\lambda)$. We'll want to reduce to $\mathrm{SL}(2, \mathbb{C})$ parabolics. For $\alpha \in \Delta$, let P_α be the associated parabolic $P_\alpha = L_\alpha \rtimes U_\alpha$, which is parabolic of type A_1.

Idea: α generates an SL_2 subgroup (the Levi factor), like the Borel but sticks out in one dimension:

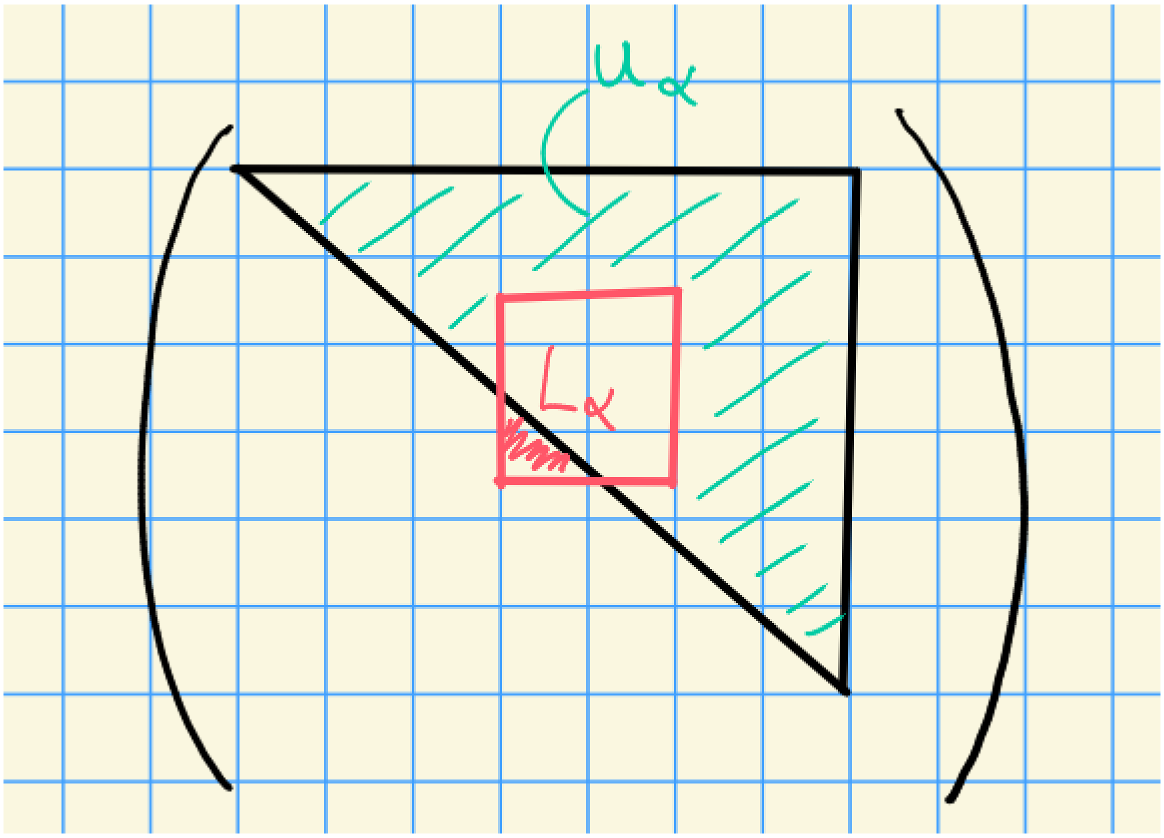

Figure 8: Image

Then

$$\begin{aligned} s_\alpha \cdot \lambda &= s_\alpha(\lambda+\rho) - \rho \\ &= \lambda + \rho - \langle \lambda+\rho,\ \alpha^\vee\rangle\alpha - \rho \\ &= \lambda - \langle \lambda+\rho,\ \alpha^\vee\rangle\alpha. \end{aligned}$$

Next time: proof of Bott-Borel-Weil and its generalization to $k = \bar{\mathbb{F}}_p$. For $B \subset P_\alpha \subset G$, we'll have a spectral sequence

$$E_2^{i,j} = R^i \operatorname{Ind}_{P_\alpha}^G R^j \operatorname{Ind}_B^{P_\alpha} \Rightarrow R^{i+j} \operatorname{Ind}_B^G \lambda = H^{i+j}(\lambda).$$