Problem Set One

D. Zack Garza

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1 Humphreys 1.1

1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee / \Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$.

Proposition: $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M

Proof:

Proposition: M is the direct sum of finitely many submodules of the form $M^{[\lambda]}$.

Proof:

1.2 b

Proposition: The weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee / \Lambda_r$.

2 Humphreys 1.3*

Proposition: For any $M \in \mathcal{O}$, $M(\lambda)$ satisfies the following property:

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right).$$

Proof:

Noting that

- $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$,
- $\mathfrak{g}$-morphisms can always be lifted to $U(\mathfrak{g})$-morphisms,
- $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$ is an identification of the $\mathfrak{g}$-module M has a $\mathfrak{b}$- module by restricting the action of $\mathfrak{g}$,

consider the following two maps:

$$\begin{aligned} F : \hom_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) &\to \hom_{\mathfrak{b}}(\mathbb{C}_\lambda, M) \\ \phi &\mapsto (F\phi : v \mapsto \phi(1 \otimes v)), \end{aligned}$$

and

$$\begin{aligned} G : \hom_{\mathfrak{b}}(\mathbb{C}_\lambda, M) &\to \hom_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) \\ \psi &\mapsto (G\psi : g \otimes v \mapsto g \cdot \psi(v)). \end{aligned}$$

It suffices to show that these maps are well-defined and mutually inverse.

To see that F is well-defined, let $\phi : U(\mathfrak{g}) \otimes C_\lambda \to M$ be fixed; we will show that the set map $F\phi : \mathbb{C}_\lambda \to M$ is $\mathfrak{b}$-linear.

- $F\phi(v + w) = \phi(1 \otimes (v + w)) = \phi((1 \otimes v) + (1 \otimes w)) = \phi(1 \otimes v) + \phi(1 \otimes w) = F\phi(v) + F\phi(w)$.
- $(F\phi_1 + F\phi_2)(v) = (\phi_1 + \phi_2)(1 \otimes v) = \phi_1(1 \otimes v) + \phi_2(1 \otimes v) = F\phi_1(v) + F\phi_2(v)$.