Title

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1.1 End of Chapter 4

Recall the proposition: morphisms between affine varieties are in bijection with k-algebra morphisms between their coordinate rings. As a result, we'll redefine an affine variety to be a ringed space isomorphic to an affine variety.

This allows you to say that affine varieties embedded in different ways are the same.

Example 1.1.1.
$\mathbb{A}^2$ vs $V(x) \subset \mathbb{A}^n$. In fact, the map

$$f : \mathbb{A}^2 \to \mathbb{A}^3(y, z) \qquad \mapsto (0, y, z).$$

This is continuous and the pullback of regular functions are again regular.

Remark 1.1.1.
With the new definition, there is a bijection between affine varieties up to isomorphisms and finitely generated k-algebras up to algebra isomorphism.

Proposition 1.1.1*(?)*.
Let $D(f) \subset X$ be a distinguished open, then $D(f)$ is a ringed space since $(X, \mathcal{O}_X)$ is and we can restrict the structure sheaf.

Proof .

Set

$$Y \coloneqq \left\{ (x,t) \in X \times \mathbb{A}^1 \;\middle|\; tf(x) = 1 \right\} \subset X \times \mathbb{A}^1.$$

This is an affine variety, since $Y = V(I + \langle ft - 1 \rangle)$. This is isomorphic to $D(f)$ by the map

$$Y \to D(f)(x,t) \qquad \mapsto x.$$

with inverse $x \mapsto (x, \dfrac{1}{f(x)})$.

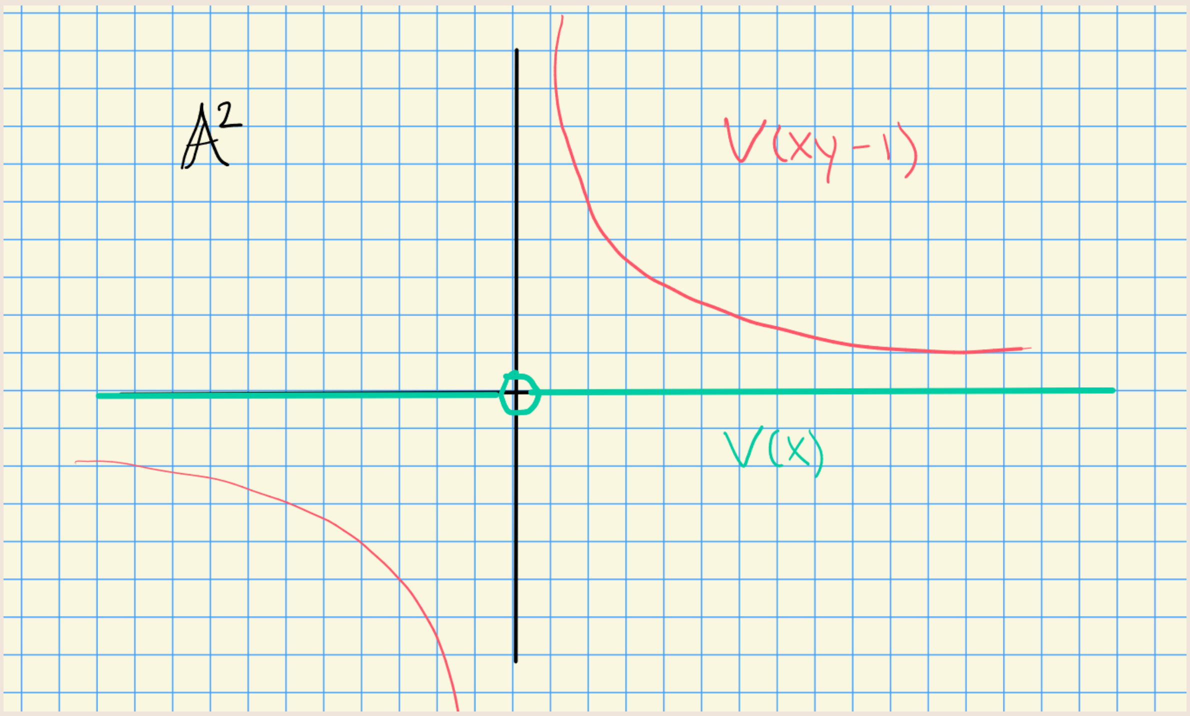

Figure 1: Image

Note that $\pi : X \times \mathbb{A}^1 \to X$ is regular, using prop 3.8: if the coordinates of a map are regular functions, then the entire map is a morphism of ringed spaces. We can then note that $\frac{1}{f(x)}$ is regular on $D(f)$, since $f \neq 0$ there.

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Example 1.1.2.
$\mathbb{A}^2 \setminus \{0\}$ is not an affine variety. Note that this is also not a distinguished open.

We showed on a HW problem that the regular functions on $\mathbb{A}^2 \setminus \{0\}$ are $k[x,y]$, which are also the regular functions on $\mathbb{A}^2$. So there is a map inducing a pullback

$$\begin{gathered} \iota : \mathbb{A}^2 \setminus \{0\} \to \mathbb{A}^2 \\ \iota^* k[x,y] \xrightarrow{\sim} k[x,y]. \end{gathered}$$

Note that ι^* is an isomorphism on the space of regular functions, but ι itself is not an isomorphism of topological spaces. Why? i^{-1} is not defined at zero.

1.2 Chapter 5

Definition 1.2.1 (Prevariety).
A *prevariety* is a ringed spaced X with a finite open cover by affine varieties. This is a topological space X with an open cover $\{U_i\}_{i=1}^n \rightrightarrows X$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine variety. We'll call $\mathcal{O}_X$ the sheaf of *regular functions* and $U_i \subset X$ *affine open sets.*

One way to construct prevarieties from affine varieties is by *gluing*:

Definition 1.2.2 (Glued Spaces).
let X_1, X_2 be prevarieties which are themselves actual varieties. Let $U_{12} \subset X_1, U_{21} \subset X_2$ be opens and $f : U_{12} \to U_{21}$ an isomorphism of ringed spaces.

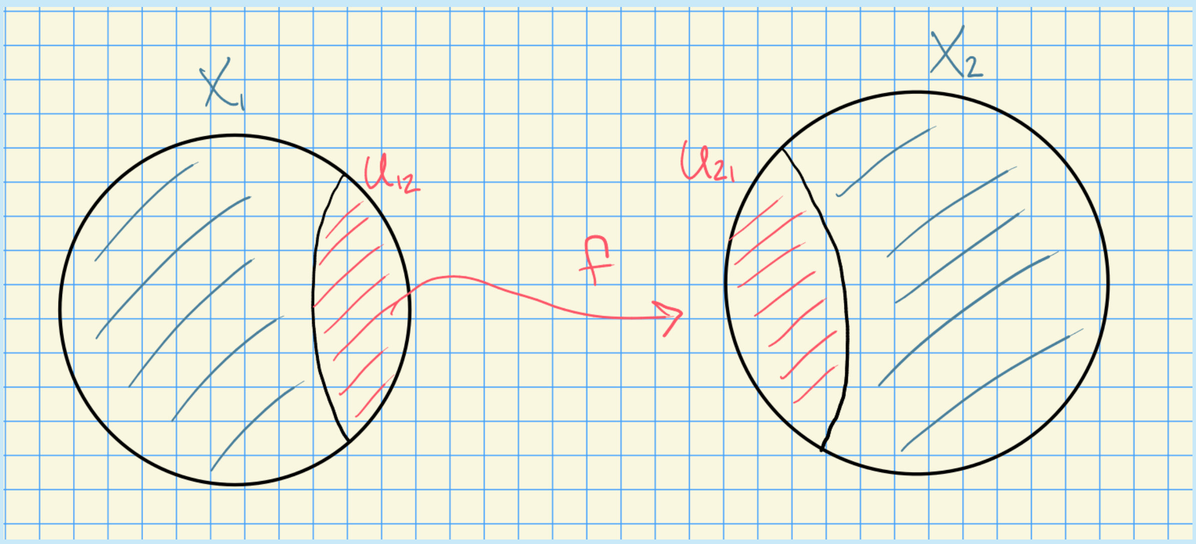

Figure 2: Image

As a set, take $X = X_1 \coprod X_2 / \sim$ where $a \sim f(a)$ for all $a \in U_{12}$. As a topological space, $U \subset X$ is open iff $U_i := U \cap X_i$ are open in X_i. As a ringed space, we take $\mathcal{O}_X(U) := \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U_i} \in \mathcal{O}_{X_i} \right\}$.

Example 1.2.1.
The prototypical example is $\mathbb{P}^1/k$ constructed from two copies of $\mathbb{A}^1/k$. Set $X_1 = \mathbb{A}^1, X_2 = \mathbb{A}^2$, with $U_{12} := D(x) \subset X_1$ and $U_{21} := D(y) \subset X_2$. Then let

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$

This defines a regular function on U_{12} so defines a morphism $U_{12} \xrightarrow{\sim} \mathbb{A}^1$.

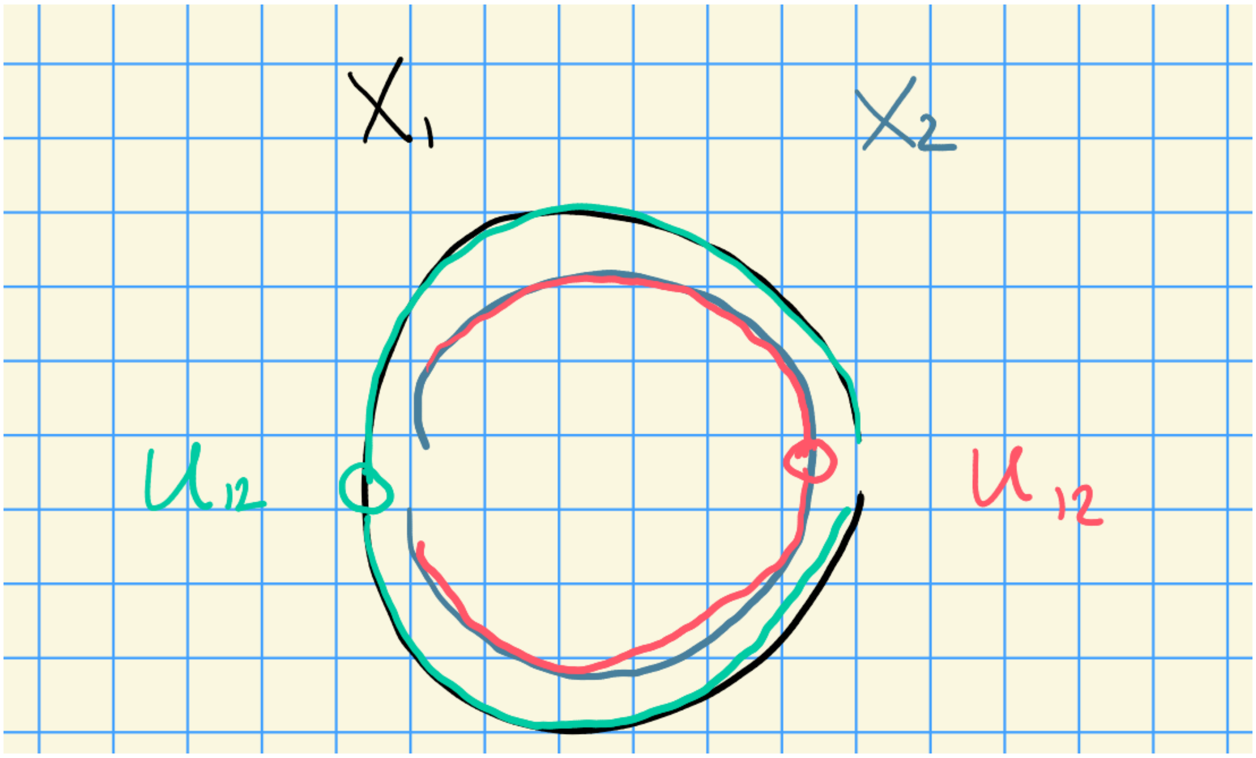

Figure 3: Image

Over $\mathbb{C}$, topologically this yields a sphere

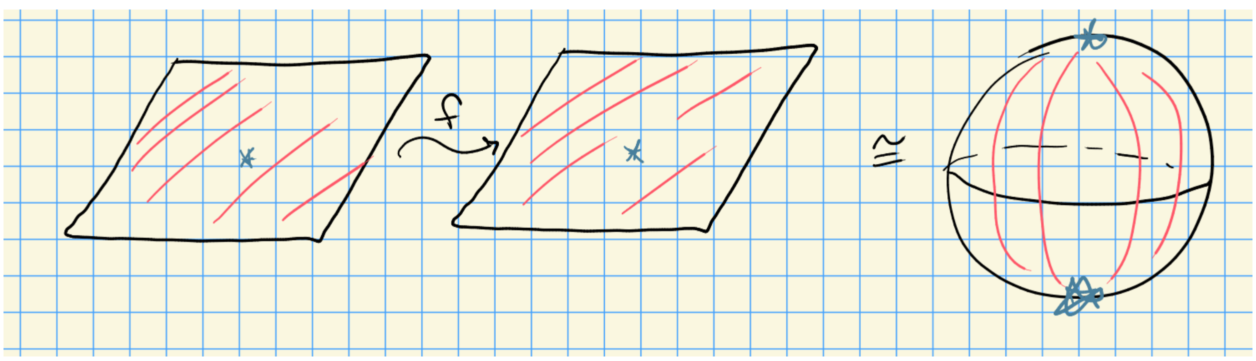

Figure 4: Image

Given a ringed space $X = X_1 \cup X_2$ with a structure sheaf $\mathcal{O}_X$, what is $\mathcal{O}_X(X)$? By definition, it's

$$\mathcal{O}_X(X) \coloneqq \left\{ \varphi : X \to k \;\middle|\; \varphi|_{X_1}, \varphi|_{X_2} \text{ are regular} \right\}.$$

Then if $\varphi|_{X_1} = f(x)$ and $\varphi|_{X_2} = g(y)$, we have $y = 1/x$ on the overlap and so $f(x)|_{D(x)} = g(1/x)|_{D(x)}$. Since f, g are rational functions agreeing on an infinite set, $f(x) = g(1/x)$ both being polynomial forces $f = g = c$ for some constant $c \in k$. Thus $\mathcal{O}_X(X) = k$.

What about $\mathcal{O}_X(X_1)$? This is just $k[x]$, and similarly $\mathcal{O}_X(X_2) = k[y]$. We can also consider $\mathcal{O}_X(X_1 \cap X_2) = D(x) \subset X$, so this yields $k[x, 1/x]$. We thus have a diagram

$$\begin{tikzcd}
 & \mathcal{O}_X(X_1) = k[x] \arrow[dr, "x \mapsto x"] & \\
\mathcal{O}_X(X) \arrow[ur] \arrow[dr] & & \mathcal{O}_X(X_1 \cap X_2) = k[x, 1/x] \\
 & \mathcal{O}_X(X_2) = k[y] \arrow[ur, "y \mapsto 1/x"] &
\end{tikzcd}$$