# Title

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# 1 Lecture 3

Today we'll be wrapping up the last of the preliminaries. Upcoming: one-variable function fields and their valuation rings.

## 1.1 Polynomials Defining Regular Function Fields

**Question 1.1.1:** Where's the curve in all of this?

**Answer 1.1.2:** This will come from an equation like $f(x, y) = 0$.

**Exercise 1.1.3:** Let $R_1, R_2$ be $k$-algebras that are also domains with fraction fields $K_i$. Show $R_1 \otimes_k R_2$ is a domain $\iff$ $K_1 \otimes_k K_2$ is a domain.[1]

**Definition 1.1.4** (Geometrically Irreducible)
A polynomial of positive degree $f \in k[t_1, \cdots, t_n]$ is **geometrically irreducible** if $f \in \bar{k}[t_1, \cdots, t_n]$ is irreducible as a polynomial.

**Remark 1.1.5:** If $n = 1$ then $f$ is geometrically irreducible $\iff$ $f$ is linear, i.e. of degree 1. Let $f$ be irreducible, then since polynomial rings are UFDs then $\langle f \rangle$ is a prime ideal (irreducibles generate principal ideals) and $k[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. Let $K_f$ be the fraction field.

**Exercise 1.1.6** ***(an easy one):***

a. Above for $1 \leq i \leq n$ let $x_i$ be the image of $t_i$ in $K_f$. Show that $K_f = k(x_1, \cdots, x_n)$.

b. Show that if $K/k$ is generated by $x_1, \cdots, x_n$, then it is the fraction field of $k[t_1, \cdots, t_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ (equivalently, a height 1 ideal).

**Proposition 1.1.7** ***(?).***
Suppose that $f$ is geometrically irreducible.

a. The function field $K/k$ is regular.

b. For all $\ell/k$, $f \in \ell[t_1, \cdots, t_n]$ is irreducible.

In this case we say $f$ is *absolutely irreducible* as a synonym for geometrically irreducible.

*Proof .*
By definition of geometric irreducibility, $\bar{k}[t_1, \cdots, t_n]/\langle f \rangle = k[t_1, \cdots, t_n]/\langle f \rangle \otimes_k \bar{k}$ is a domain. The exercise shows that $K_f \otimes_k k$ is a domain, so $K_f$ is regular. It follows that for all $\ell/k$,

[1] Hint: use a denominator clearing argument.

$K_f \otimes_k \ell$ is a domain, so $\ell[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. ■

Moral: geometrically irreducible polynomials are good sources of regular function fields.

**Exercise 1.1.8:** Let $k$ be a field, $d \in \mathbb{Z}^+$ such that $4 \nmid d$ and $p(x) \in k[x]$ be positive degree. Factor $p(x) = \prod_{i=1}^{r} (x - a_i)^{\ell_i}$ in $\bar{k}[x]$.

a. Suppose that for some $i$, $d \nmid \ell_i$. Show that $f(x, y) \coloneqq y^d - p(x) \in k[x, y]$ is geometrically irreducible. Conclude that $K_f \coloneqq ff\left(k[x, y]/\left\langle y^d - p(x)\right\rangle\right)$ is a regular one-variable function field over $k$, and thus elliptic curves yield regular function fields.[2]

b. What happens when $4 \,\big|\, d$?

**Exercise 1.1.9** ***(Nice, Recommended):*** Assume $k$ is a field, if necessary assuming $\mathrm{ch}(k) \neq 2$.

a. Let $f(x, y) = x^2 - y^2 - 1$ and show $K_f$ is is rational: $K_f = k(z)$.

b. Let $f(x, y) = x^2 + y^2 - 1$. Show that $K_f$ is again rational.

c. Let $k = \mathbb{C}$ and $f(x, y) = x^2 + y^2 + 1$, $K_f$ is rational.

d. Let $k = \mathbb{R}$. For $f(x, y) = x^2 + y^2 + 1$, is $K_f$ rational?[3]

**Question 1.1.10:** Can we always construct regular function fields using geometrically irreducible polynomials?

**Answer 1.1.11:** In several variables, no, since not every variety is birational to a hypersurface. In one variable, yes, as the following theorem shows:

**Theorem 1.1.12** ***(Regular Function Fields in One Variable are Geometrically Irreducible).***
Let $K/k$ be a one variable function field (finitely generated, transcendence degree one). Then

a. If $K/k$ is separable, then $K = k(x, y)$ for some $x, y \in K$.

b. If $K/k$ is regular (separable + constant subfield is $k$, so stronger) then $K \cong K_f$ for a geometrically irreducible $f \in k[x, y]$.

Recall separable implies there exists a separating transcendence basis.

[2] Referred to as *hyperelliptic* or *superelliptic* function fields. Hint: use FT 9.21 or Lang's Algebra.
[3] This is an example of a non-rational genus zero function field.

*Proof (of a).*
This means there exists a primitive element $x \in K$ such that $K/k(x)$ is finite and separable. By the Primitive Element Corollary (FT 7.2), there exist a $y \in K$ such that $K = k(x, y)$. ∎

*Proof (of b).*
Omitted for now, slightly technical. ∎

Importance of last result: a regular function field on one variable corresponds to a nice geometrically irreducible polynomial $f$.

**Remark 1.1.13:** Note: the plane curve module may not be smooth, and in fact usually is not possible. I.e. $k[x, y]/\langle f \rangle$ is a one-dimensional noetherian domain, which need not be integrally closed.

**Question 1.1.14:** Can every one variable function field be 2-generated?

**Answer 1.1.15:** Yes, as long as the ground field is perfect. In positive characteristic, the suspicion is no: there exists finite inseparable extensions $\ell/k$ that need arbitrarily many generators. However, what if $K/k$ has constant field $k$ but is not separable? Riemann-Roch may have something to say about this.

**Example 1.1.16:** Example from earlier lecture:

$$ax^p + b - y^b$$

**Remark 1.1.17:** We can find examples of nice function fields by taking irreducible polynomials in two variables. This will define a one-variable function field. If the polynomial is geometrical reducible, this produces regular function fields.

Next: One variable function fields and their valuations.