Title

Contents

$$\frac{\partial}{\partial x}\int_1^x f(x,t)dt = \int_1^x \frac{\partial}{\partial x} f(x,t)dt + f(x,x)$$

0.1 Big Theorems / Tools:

- The Fundamental Theorem of Calculus:
$$\frac{\partial}{\partial x}\int_a^x f(t)dt = f(x)$$
$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} g(t)dt = g(b(x))b'(x) - g(a(x))a'(x)$$
- The generalized Fundamental Theorem of Calculus
$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt = f(x,\cdot)\frac{\partial}{\partial x}(\cdot)\Big|_{a(x)}^{b(x)}$$
$$= f(x,b(x))\ b'(x) - f(x,a(x))\ a'(x)$$
  - Recover FTC by taking $a(x) = c, b(x) = x, f(x,t) = f(t)$.
    - ♢ Note that if $f(x,t) = f(t)$ alone, then $\frac{\partial x}{\partial f}(t) = 0$ and the second integral vanishes
- Extreme Value Theorem
- Involving the Derivative:
  - Mean Value Theorem:
$$f \in C^0(I) \implies \exists p \in I : f(b) - f(a) = f'(p)(b-a).$$
  - Useful variant for integrals and average value:
$$f \in C^0(I) \implies \exists p \in I : \int_a^b f(x)\ dx = f(p)(b-a)$$
  - Rolle's Theorem
  - L'Hopital's Rule: If
  - $f(x), g(x)$ differentiable on $I - \{\text{pt}\}$
  - $\lim_{x\to pt} f(x) = \lim_{x\to\{\text{pt}\}} g(x) \in \{0, \pm\infty\}$
  - $\forall x \in I, g'(x) \neq 0$
  - $\lim_{x\to\{\text{pt}\}} \frac{f'(x)}{g'(x)}$ exists
$$\implies \lim_{x\to\{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x\to\{\text{pt}\}} \frac{f'(x)}{g'(x)}$$

– Taylor Expansions:

$$\begin{aligned} T(a,x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &\quad + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{24}f^{(4)}(a)(x-a)^4 + \cdots \end{aligned}$$

Bounded error: $|f(x) - T_k(a,x)| < \left|\frac{1}{(k+1)!}f^{(k+1)}(a)\right|$ where $T_k(a,x)$ is the Taylor series truncated up to and including the x^k term.

0.2 Differential

0.2.1 Limits

- Tools for finding $\lim_{x\to a} f(x)$, in order of difficulty:
  - Plug in: equal to $f(a)$ if continuous
  - L'Hopital's Rule (only for indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$)
    - ♢ For $\lim f(x)^{g(x)} = 1^\infty, \infty^0, 0^0$, let $L = \lim f^g \implies \ln L = \lim g \ln f$
  - Algebraic rules
  - Squeeze theorem
  - Expand in Taylor series at a
  - Monotonic + bounded
- One-sided limits: $\lim_{x\to a^-} f(x) = \lim_{\varepsilon\to 0} f(a-\varepsilon)$
- Limits at zero or infinity:

$$\lim_{x\to\infty} f(x) = \lim_{\frac{1}{x}\to 0} f(\frac{1}{x}) \text{ and } \lim_{x\to 0} f(x) = \lim_{x\to\infty} f(1/x)$$