

# Problem Set 5

D. Zack Garza

October 22, 2019

## Contents

<b>1</b>	<b>Problem 1</b>	<b>1</b>
<b>2</b>	<b>Problem 2</b>	<b>2</b>
<b>3</b>	<b>Problem 3</b>	<b>3</b>

## 1 Problem 1

We first make the following claim (TODO):

$$S := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$$
$$T := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = \sup \left\{ \sum_{(j,k) \in C} a_{kj} \mid C \subset \mathbb{N}^2, |C| < \infty \right\}.$$

We will show that  $S = T$  by showing that  $S \leq T$  and  $T \leq S$ .

Let  $B \subset \mathbb{N}^2$  be finite, so  $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$ .

Now letting  $R > \max(I, J)$ , we can define  $C = [0, R]^2$ , which satisfies  $B \subseteq C \subset \mathbb{N}^2$  and  $|C| < \infty$ .

Moreover, since  $a_{jk} \geq 0$  for all pairs  $(j, k)$ , we have the following inequality:

$$\sum_{(j,k) \in B} a_{jk} < \sum_{(k,j) \in C} a_{jk} \leq \sum_{(k,j) \in C} a_{kj} \leq T,$$

since  $T$  is a supremum over *all* such sets  $C$ , and the terms of any finite sum can be rearranged.

But since this holds for every  $B$ , we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_B \sum_{(k,j) \in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that  $T \leq S$ , yielding the desired equality.  $\square$

## 2 Problem 2

We want to show the following equality:

$$\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx.$$

To that end, we can rewrite this using the integral definition of  $g(x)$ :

$$\int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx = \int_0^1 f(x) \, dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned} \int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx &= \int_0^1 \int_0^t \frac{f(t)}{t} \, dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} \int_0^t dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} (t - 0) \, dt \\ &= \int_0^1 f(t) \, dt, \end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this switch of integrals is justified.

To this end, define

$$\begin{aligned} F : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x, t) &\mapsto \frac{\chi_A(x, t) \hat{f}(x, t)}{t}. \end{aligned}$$

where  $A = \{(x, t) \in \mathbb{R}^2 \mid 0 \leq x \leq t \leq 1\}$  and  $\hat{f}(x, t) := f(t)$  is the cylinder on  $f$ .

This defines a measurable function on  $\mathbb{R}^2$ , since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular,  $|F|$  is measurable and non-negative, and so we can apply Tonelli to  $|F|$ . This allows us to write

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right| dx dt \\
&= \int_0^1 \int_0^t \frac{|f(t)|}{t} dx dt \quad \text{since } t > 0 \\
&= \int_0^1 \frac{|f(t)|}{t} \int_0^t dx dt \\
&= \int_0^1 |f(t)| < \infty,
\end{aligned}$$

where the switch is justified by Tonelli and the last inequality holds because  $f$  was assumed to be measurable.

Since this shows that  $F \in L^1(\mathbb{R}^2)$ , and we can thus apply Fubini to  $F$  to justify the initial switch.  $\square$

### 3 Problem 3

Let  $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$ , and define

$$F(x, y) = \chi_A(x, y) \frac{x^{1/3}}{(1 + xy)^{3/2}}.$$

Then, if all iterated integrals exist and a switch of integration order is justified, we would have

$$\begin{aligned}
\int_{\mathbb{R}^2} F &\stackrel{?}{=} \int_0^\infty \int_y^\infty f(x, y) dx dy \\
&\stackrel{?}{=} \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1 + xy)^{3/2}} dy dx \\
&= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3} \sqrt{1 + x^2}} dx \\
&= 2 \int_0^1 \frac{1}{x^{2/3} \sqrt{1 + x^2}} dx + 2 \int_1^\infty \frac{1}{x^{2/3} \sqrt{1 + x^2}} dx \\
&\leq \int_0^1 x^{-2/3} dx + \int_0^\infty x^{-5/3} \\
&= 2(3) + 2 \left( \frac{3}{2} \right) < \infty,
\end{aligned}$$

where the first term in the split integral is bounded by using the fact that  $\sqrt{1 + x^2} \geq \sqrt{x^2} = x$ , and the second term from  $x > 1 \implies x > 0 \implies \sqrt{1 + x^2} \geq \sqrt{1}$ .

Since  $F$  is non-negative, we have  $|F| = F$ , and so the above computation would thus imply that  $F \in L^1(\mathbb{R}^2)$ . It thus remains to show that  $\int F$  is equal to its iterated integrals, and that the switch of integration order is justified

Toward that end, we can note for  $(x, y) \in A$ , we have  $x, y \in [0, \infty)$ , and so  $F(x, y)$  is a non-negative function on  $\mathbb{R}^2$ . In particular,  $|F(x, y)| = F(x, y)$ . But this means that Tonelli applies, and  $\int F$  is equal to any iterated integral. Since we computed one such iterated integral and found it to be finite, this means that  $\int |F| = \int F < \infty$ .