

# Lie Algebras

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## 1 Monday August 12

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

### 1.1 Overview

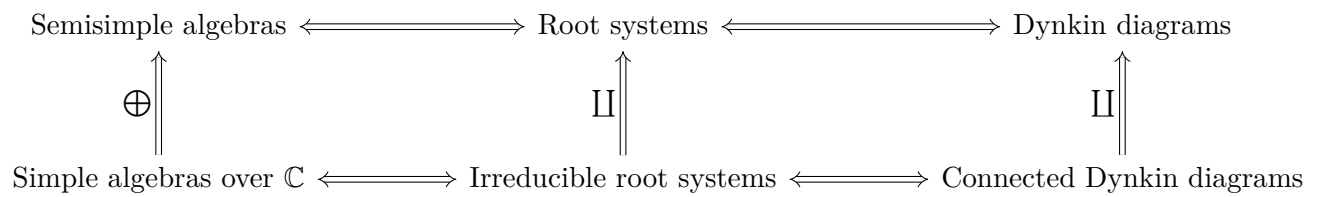
Here is a short overview of the topics we expect to cover:

#### 1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

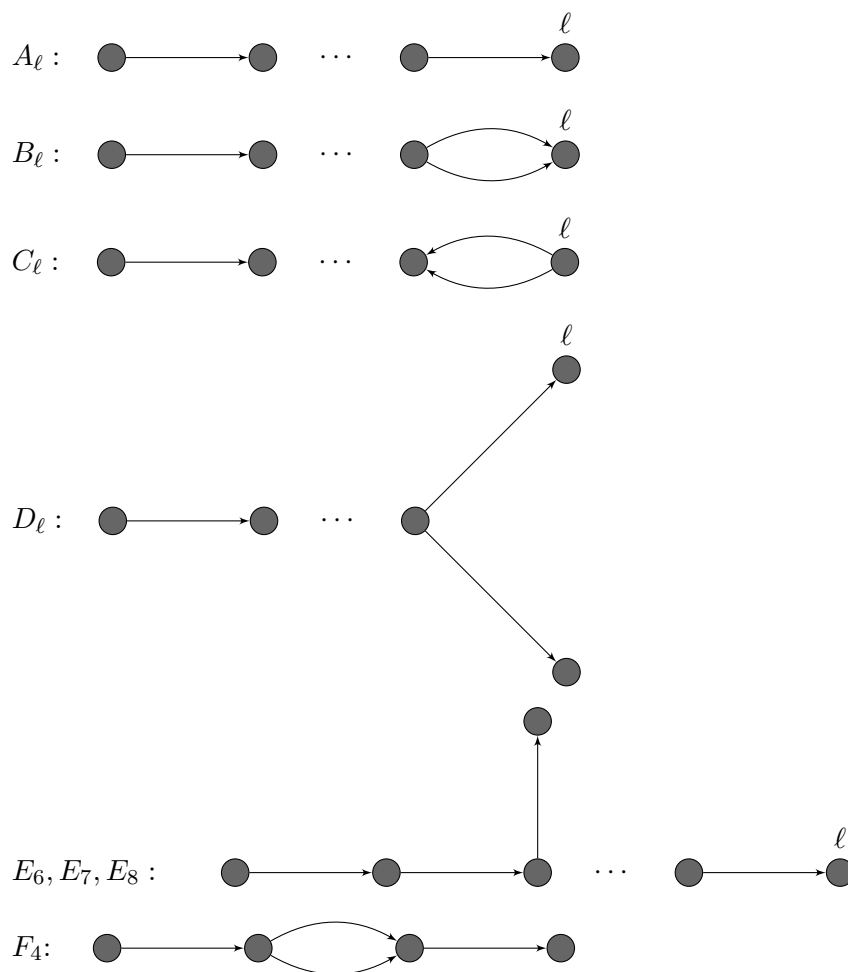
### 1.1.2 Chapter 3-4

We will describe the following series of correspondences:



### 1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:



### 1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

### 1.3.1 Chapter 6

Some important topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type  $A_\ell$  algebra used for the first time; however, it differs from the other types in several important/significant ways.

### 1.3.2 Chapter 7

Skip!

### 1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat- $\mathcal{O}$  [Humphrey 08]

## 1.4 Content

Fix  $F$  a field of characteristic zero – note that prime characteristic is closer to a research topic.

**Definition 1.** A **Lie Algebra**  $\mathfrak{g}$  over  $F$  is an  $F$ -vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\cdot, \cdot]$  is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0.$$

**Exercise 1.** Show that  $[x, y] = -[y, x]$ .

**Definition 2.** Two Lie algebras  $\mathfrak{g}, \mathfrak{g}'$  are said to be isomorphic if  $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ .

## 1.5 Linear Lie Algebras

Let  $V = \mathbb{F}^n$ , and define  $\text{End}(V) = \{f : V \rightarrow V \mid f \text{ is linear}\}$ . We can then define  $\mathfrak{gl}(n, V)$  by setting  $[x, y] = (x \circ y) - (y \circ x)$ .

**Exercise 2.** Verify that  $V$  is a Lie algebra.

**Definition 3.** Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \mid \text{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group*  $\text{SL}(n, V)$ ).

**Definition 4.** A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

**Definition 5.** The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where } M = \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

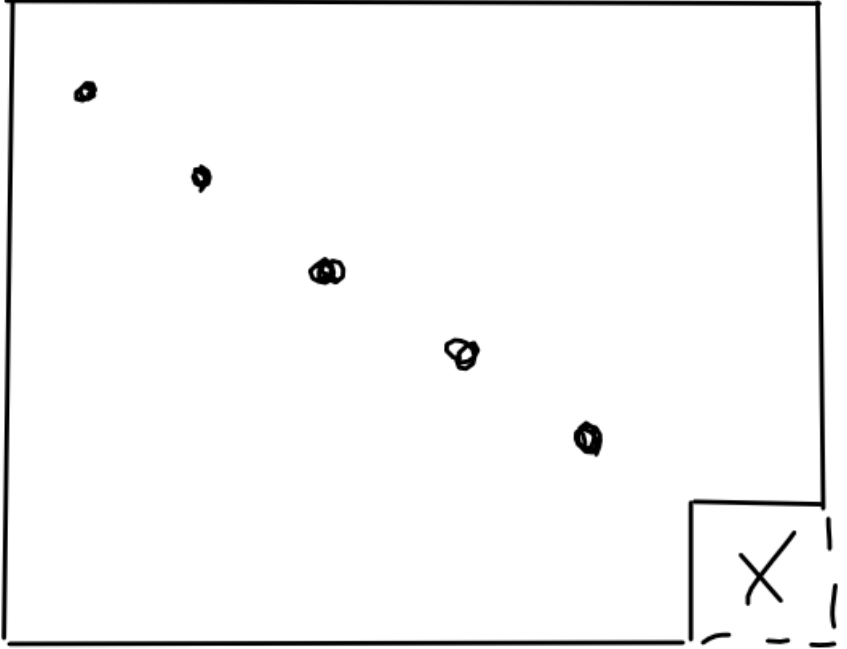
**Definition 6.** The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left( \begin{array}{c|c|c} 1 & 0 & \\ \hline 0 & 0 & I_n \\ \hline & -I_n & 0 \end{array} \right) & n = 2\ell + 1 \text{ odd,} \\ \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

**Proposition 7.** The dimensions of these algebras can be computed;

- The dimension of  $\mathfrak{gl}(n, \mathbb{F})$  is  $n^2$ , and has basis  $\{e_{i,j}\}$  the matrices if a 1 in the  $i, j$  position and



zero elsewhere.

- For type  $A_\ell$ , we have  $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$ .
- For type  $C_\ell$ , we have  $\dim \mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2 \left( \frac{\ell(\ell+1)}{2} \right)$ , and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type  $D_\ell$  we have

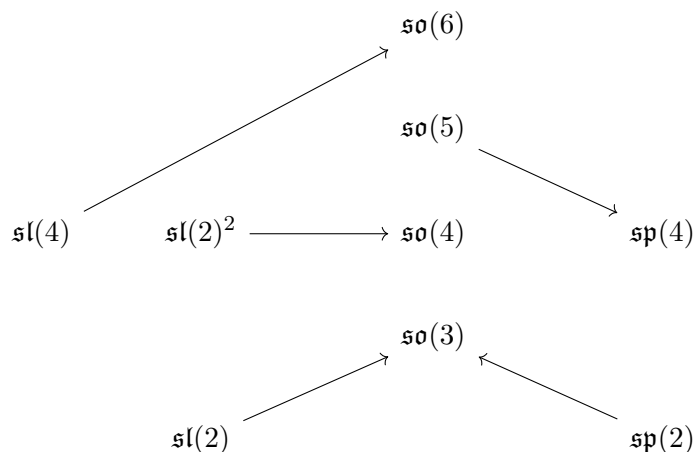
$$\dim \mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be  $2\ell^2 - \ell$ .

- For type  $B_\ell$ , we have  $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$ , with elements of the form

$$\left( \begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

**Exercise 3.** Use the relation  $MA = A^{tM}$  to reduce restrictions on the blocks.



**Theorem 8.** These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

## 2 Wednesday August 14

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

**Exercise 4.** Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

### 2.1 Lie Algebras of Derivations

**Definition 9.** An  $F$ -algebra  $A$  is an  $F$ -vector space endowed with a bilinear map  $A^2 \rightarrow A$ ,  $(x, y) \mapsto xy$ .

**Definition 10.** An algebra is **associative** if  $x(yz) = (xy)z$ .

Modern interest: simple Lie algebras, which have a good representation theory. Take a look at Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

**Definition 11.** Any map  $\delta : A^2 \rightarrow A$  that satisfies the Leibniz rule is called a **derivation** of  $A$ , where the rule is given by  $\delta(xy) = \delta(x)y + x\delta(y)$ .

**Definition 12.** We define  $\text{Der}(A) = \{\delta \mid \delta \text{ is a derivation}\}$ .



Any Lie algebra  $\mathfrak{g}$  is an  $F$ -algebra, since  $[\cdot, \cdot]$  is bilinear. Moreover,  $\mathfrak{g}$  is associative iff  $[x, [y, z]] = 0$ .

**Exercise 5.** Show that  $\text{Derg} \leq \mathfrak{gl}(\mathfrak{g})$  is a Lie subalgebra. One needs to check that  $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$ .

**Exercise 6** (Turn in). Define the adjoint by  $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$ . Show that  $\text{ad}_x \in \text{Der}(\mathfrak{g})$ .

## 2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of  $\mathfrak{gl}(V)$ . Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

**Example 13.** Any  $F$ -vector space can be made into a Lie algebra by setting  $[x, y] = 0$ ; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write  $\mathfrak{g} = Fx$ , and so  $[x, x] = 0 \implies [\cdot, \cdot] = 0$ . So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write  $\mathfrak{g} = Fx \oplus Fy$ , the only nontrivial bracket here is  $[x, y]$ . Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$  is abelian.
  - $[x, y] = ax + by \neq 0$ . Assume  $a \neq 0$  and set  $x' = ax + by, y' = \frac{y}{a}$ . Now compute  $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$ . Punchline:  $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$ .

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	$x'$	$y'$
$x'$	0	$x'$
$y'$	$-x'$	0

**Example 14.** Let  $V = \mathbb{R}^3$ , and define  $[a, b] = a \times b$  to be the usual cross product.

**Exercise 7.** Look at notes for basis elements of  $\mathfrak{sl}(2, F)$ ,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of  $\text{ad}(e), \text{ad}(h), \text{ad}(f)$  with respect to this basis.

## 2.3 Ideals

**Definition 15.** A subspace  $I \subseteq \mathfrak{g}$  is called an **ideal**, and we write  $I \trianglelefteq \mathfrak{g}$ , if  $x, y \in I \implies [x, y] \in I$ .

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using  $[x, y] = [-y, x]$ .

**Exercise 8.** Check that the following are all ideals of  $\mathfrak{g}$ :

- $\{0\}, \mathfrak{g}$ .
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra  $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \mid x_i, y_i \in \mathfrak{g}\}$ .  
– Moreover,  $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$ .

Fact: If  $I, J \trianglelefteq \mathfrak{g}$ , then

- $I + J = \{x + y \mid x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \cap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \mid x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

**Definition 16.** A Lie algebra is **simple** if  $[\mathfrak{g}, \mathfrak{g}] \neq 0$  (i.e. when  $\mathfrak{g}$  is not abelian) and has no non-trivial ideals. Note that this implies that  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .

**Theorem 17.** Suppose that  $\text{char } F \neq 2$ , then  $\mathfrak{sl}(2, F)$  is not simple.

*Proof.* Recall that we have a basis of  $\mathfrak{sl}(2, F)$  given by  $B = \{e, h, f\}$  where

- $[e, f] = h$ ,
- $[h, e] = 2e$ ,
- $[h, f] = -2f$ .

So think of  $[h, e] = \text{ad } h$ , so  $h$  is an eigenvector of this map with eigenvalues  $\{0, \pm 2\}$ . Since  $\text{char } F \neq 2$ , these are all distinct. Suppose  $\mathfrak{sl}(2, F)$  has a nontrivial ideal  $I$ ; then pick  $x = ae + bh + cf \in I$ . Then  $[e, x] = 0 - 2be + ch$ , and  $[e, [e, x]] = 0 - 0 + 2ce$ . Again since  $\text{char } F \neq 2$ , then if  $c \neq 0$  then  $e \in I$ . Now you can show that  $h \in I$  and  $f \in I$ , but then  $I = \mathfrak{sl}(2, F)$ , a contradiction. So  $c = 0$ .

Then  $x = bh \neq 0$ , so  $h \in I$ , and we can compute

$$\begin{aligned} 2e &= [h, e] \in I \implies e \in I, \\ 2f &= [h, -f] \in I \implies f \in I. \end{aligned}$$

which implies that  $I = \mathfrak{sl}(2, F)$  and thus it is simple. □

Note that there is a homework coming due next Monday, about 4 questions.

### 3 Friday August 16

Last time, we looked at ideals such as  $0, \mathfrak{g}, Z(\mathfrak{g})$ , and  $[\mathfrak{g}, \mathfrak{g}]$ .

**Definition:** If  $I \trianglelefteq \mathfrak{g}$  is an ideal, then the quotient  $\mathfrak{g}/I$  also yields a Lie algebra with the bracket given by  $[x + I, y + I] = [x, y] + I$ .

**Exercise:** Check that this is well-defined, so that if  $x + I = x' + I$  and  $y + I = y' + I$  then  $[x, y] + I = [x', y'] + I$ .

### 3.1 Homomorphisms and Representations

**Definition 18.** A linear map  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a *Lie homomorphism* if  $\phi[x, y] = [\phi(x), \phi(y)]$ .

**Remark.**  $\ker \phi \trianglelefteq \mathfrak{g}_1$  and  $\text{im } \phi \leq \mathfrak{g}_2$  is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence  $\{I \trianglelefteq \mathfrak{g}\} \iff \{\text{hom } \phi : \mathfrak{g} \rightarrow \mathfrak{g}'\}$  where  $I \mapsto (x \mapsto x + I)$  and the inverse is given by  $\phi \mapsto \ker \phi$ .

Theorem (Isomorphism theorem for Lie algebras):

- If  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a Lie algebra homomorphism, then  $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If  $I, J \trianglelefteq \mathfrak{g}$  are ideals and  $I \subset J$  then  $J/I \trianglelefteq \mathfrak{g}/I$  and  $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$ .
- If  $I, J \trianglelefteq \mathfrak{g}$  then  $(I + J)/J \cong I/(I \cap J)$ .

Definition: A *representation* of a Lie algebra  $\mathfrak{g}$  is a Lie algebra homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  into a linear Lie algebra for some vector space  $V$ .

We call  $V$  a  $\mathfrak{g}$ -module with action  $g \cdot v = \phi(g)(v)$ .

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

**Corollary 19.** Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since  $\mathfrak{g}$  is simple, the center  $Z(\mathfrak{g}) = 0$ . We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) &= \{x \in \mathfrak{g} \mid \text{ad } x(y) = 0 \quad \forall y \in \mathfrak{g}\} \\ &= \ker \text{ad } x. \end{aligned}$$

Using the first isomorphism theorem, we have  $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im } \text{ad} \subseteq \mathfrak{gl}(\mathfrak{g})$ . But  $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$  here, so we are done.

### 3.2 Automorphisms

Definition: An automorphism of  $\mathfrak{g}$  is an isomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}$ , and we define

$$\text{Aut}(\mathfrak{g}) = \{\phi : \mathfrak{g} \rightarrow \mathfrak{g} \mid \phi \text{ is an isomorphism}\}.$$

Proposition: If  $\delta \in \text{Der}(\mathfrak{g})$  is nilpotent, then

$$\exp(\delta) := \sum \frac{\delta^n}{n!} \in \text{Aut}(\mathfrak{g}).$$

This is well-defined because  $\delta$  is nilpotent, and a binomial formula holds:

$$\frac{\delta^n([x, y])}{n!} = \sum_{i=0}^n \left[ \frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!} \right].$$

and for  $n = 1$ ,  $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ .

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!}.$$

Example: Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$  and define

$$s = \exp(\operatorname{ad} e) \exp(\operatorname{ad} -f) \exp(\operatorname{ad} e) \in \operatorname{Aut} \mathfrak{g}.$$

where  $e, f$  are defined as (todo, see written notes).

Then define the Weyl group  $W = \langle s \rangle$ .

Exercise: Check that  $s(e) = -f, s(f) = -e, s(h) = -h$ , and so the order of  $s$  is 2 and  $W = \{1, s\}$ .

## 4 Monday August 19

### 4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for  $\mathfrak{g}$  is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\vdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra  $\mathfrak{g}$  is *solvable* if there is some  $n$  for which  $\mathfrak{g}^{(n)} = 0$ .

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in  $\mathfrak{gl}(n, \mathbb{F})$ .

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let  $\mathfrak{g}$  be a Lie algebra, then

1. If  $\mathfrak{g}$  is solvable, then all subalgebras and all homomorphic images of  $\mathfrak{g}$  are also solvable.

2. If  $I \trianglelefteq \mathfrak{g}$  and both  $I$  and  $\mathfrak{g}/I$  are solvable, then so is  $\mathfrak{g}$ .
3. If  $I, J \trianglelefteq \mathfrak{g}$  are solvable, then so is  $I + J$ .

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical*  $\text{Rad}(\mathfrak{g})$ .

Definition: A Lie algebra is semisimple if  $\text{Rad}(\mathfrak{g}) = 0$ .

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any  $\mathfrak{g}$ , the quotient  $\mathfrak{g}/\text{Rad}(\mathfrak{g})$  is semisimple.

## 4.2 Nilpotency

$$\begin{aligned}\mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\ &\vdots \\ \mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i].\end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If  $\mathfrak{g}$  is nilpotent, then all subalgebras and all homomorphic images of  $\mathfrak{g}$  are also nilpotent.
2. If  $\mathfrak{g}/Z(\mathfrak{g})$  is nilpotent, then so is  $\mathfrak{g}$ .
3. If  $\mathfrak{g} \neq 0$  is nilpotent, then  $Z(\mathfrak{g}) \neq 0$ .

Claim: If  $\mathfrak{g}$  is nilpotent, then  $\text{ad}_x \in \text{End}(\mathfrak{g})$  is nilpotent for all  $x \in \mathfrak{g}$ .

Proof: This is because  $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \dots]]] = 0$ , and so for every  $x_i, y \in \mathfrak{g}$  we have  $[x_1, [x_2, \dots [x_n, y]]] = 0$ , and so  $\text{ad}_{x_1} \circ \text{ad}_{x_2} \circ \dots \circ \text{ad}_{x_n} = 0$  which implies that  $\text{ad}_x^n = 0$  for all  $x \in \mathfrak{g}$ .

Theorem [Engel]: If  $\text{ad}_x$  is nilpotent for all  $x \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent.

Remark: This can be confusing if  $\mathfrak{g}$  is a linear algebra, we can consider elements  $x \in \mathfrak{g}$  and ask if it is the case  $x$  being nilpotent (as an endomorphism) iff  $\mathfrak{g}\mathfrak{g}$  is nilpotent? False, a counterexample is  $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ , where there exists an  $x$  which is *not* nilpotent while  $\text{ad}_x$  is nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  be a Lie subalgebra for some finite dimensional vector space  $V$ . If  $x$  is nilpotent as an endomorphism on  $V$  for all  $x \in \mathfrak{g}$ , then there exists a nonzero vector  $v \in V$  such that  $\mathfrak{g}v = 0$ , so  $x \in \mathfrak{g} \implies x(v) = 0$ .

Proof of lemma Use induction on  $\dim \mathfrak{g}$ , splitting into two separate base cases: - Case  $\dim \mathfrak{g} = 0$ , then  $\mathfrak{g} = \{0\}$ . - Case  $\dim \mathfrak{g} = 1$ , left as an exercise.

Inductive step: Let  $A$  be a maximal proper subalgebra and define  $\phi : A \rightarrow \mathfrak{gl}(\mathfrak{g}/A)$  where  $a \mapsto (x + A \mapsto [a, x] + A)$ . We need to check that  $\phi$  is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that  $\text{im } \phi \leq \mathfrak{gl}(\mathfrak{g}/A)$  is a Lie subalgebra, and  $\dim \text{im } \phi < \dim \mathfrak{g}$ . The claim is that  $\phi(a) \in \text{End}(\mathfrak{g}/A)$  is nilpotent for all  $a \in A$ . By the inductive hypothesis, there is a nonzero coset  $y + A \in \mathfrak{g}/A$  such that  $(\text{im } \phi) \cdot (y + A) = A$ . Since  $y \notin A$ , then  $\phi(a)(y + A) = A$  for all  $a \in A$ , and so  $[a, y] \in A$ .

We want to show that  $A$  is a subalgebra of codimension 1, and  $A \oplus F_y \leq \mathfrak{g}$  is a Lie subalgebra. This is because  $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2[y, y]$ . The last term is zero, the middle two terms are in  $A$ , and because  $A$  is closed under the bracket, the first term is in  $A$  as well.

But then  $A \oplus F_y$  is a larger subalgebra than  $A$ , which was maximal, so it must be everything. So  $A \oplus F_y = \mathfrak{g}$ . So  $A \trianglelefteq \mathfrak{g}$  because  $[a_1, a_2 + cy]$  is in  $A$ ,  $A \oplus F_y = \mathfrak{g}$  respectively, and this equals  $[a_1, a_2] + c[a_1, y]$ , where both terms are in  $A$ .

Proof to be continued on Friday!

## 5 Wednesday August 21

Last time: we had a theorem that said that if  $\mathfrak{g} \in \mathfrak{gl}(V)$  and every  $x \in \mathfrak{g}$  is nilpotent, then there exists a nonzero  $v \in V$  such that  $\mathfrak{g}v = 0$ .

We proceeded by induction on the dimension of  $V$ , constructing  $\text{im } \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$ , and showed that  $\mathfrak{g} = A \oplus F_y$ . Now consider

$$W = \{v \in V \mid Av = 0\},$$

which is  $\mathfrak{g}$ -invariant, so  $\mathfrak{g}(W) \subseteq W$ , or for all  $a \in A, x \in \mathfrak{g}, v \in W$ , we have  $a \curvearrowright x(v) = 0$ . This is true because  $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$ . But  $V$  is killed by any element in  $A$ , and both of these terms are in  $A$ . In particular, the  $y$  appearing in  $F_y$  also satisfies  $y \in W$ . Consider  $y|_W \in \text{End}(W)$ , and we want to apply the inductive hypothesis to  $F_y|_W \subseteq \mathfrak{gl}(W)$ .

We need to check that  $y|_W \in \text{End}(W)$ , which is true exactly because  $y$  is nilpotent. So we can construct a nonzero  $v \in W \subset V$  such that  $y(v) = 0$ , and so  $\mathfrak{g}v = 0$ .

Claim:  $\phi(a) \in \text{End}(\mathfrak{g}/A)$  is nilpotent. Each  $a \in A \subset \mathfrak{g}$  is nilpotent by assumption. Define the maps for left multiplication by  $a$ ,  $m_\ell : x \mapsto ax$ , and the right multiplication  $m_r : x \mapsto xa$ . These are nilpotent, and since  $m_\ell, m_r$  commute, the difference  $m_\ell - m_r$  is nilpotent, and this is exactly  $\text{ad } a$ . But then  $\phi(a)$  is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if  $\text{ad } x \in \text{End}(\mathfrak{g})$  is nilpotent for every  $x \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to  $\text{ad } \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ . So we can produce the nonzero  $v \in \mathfrak{g}$  such that  $\text{ad } \mathfrak{g}v = 0$ . Then  $[x, v] = 0$  for all  $x \in \mathfrak{g}$ , so either  $v \in Z(\mathfrak{g})$  or  $Z(\mathfrak{g}) \neq 0$ . In either case,  $\mathfrak{g}/Z(\mathfrak{g})$  has smaller dimension.

Since  $\text{ad } x$  is nilpotent, so is  $\text{ad } x + Z(\mathfrak{g})$ , and so  $\mathfrak{g}/Z(\mathfrak{g})$  is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space.  $\square$

Let  $\mathfrak{N}(F)$  be the subalgebra of  $\mathfrak{gl}(F)$  consisting of strictly upper triangular matrices. We have a corollary: if  $\mathfrak{g} \subset \mathfrak{gl}(n, F)$  is a Lie subalgebra such every  $x \in \mathfrak{g}$  is nilpotent as an endomorphism of  $F$ , then the matrices of  $\mathfrak{g}$  with respect to some bases of in  $\mathfrak{N}(n, F)$ .

The proof is by induction on  $n$ , where the base case is easy. For the inductive step, we use the previous theorem to get a  $v_1$  such that  $x(v_1) = 0$  for all  $x \in \mathfrak{g}$ . Let  $\bar{V} = F^n/Fv_1 \cong F^{n-1}$ , and define  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\bar{V})$  where  $x \mapsto (\bar{y} \mapsto \overline{y(x)})$ .

Then  $\text{im } \phi \leq \mathfrak{gl}(n-1, F)$  as a subalgebra, and every  $\phi(x) \in \text{End}(F^{n-1})$  is nilpotent, since  $x$  was nilpotent on the larger space. But (see notes) then  $x$  can be written as a strictly upper-triangular matrix.

## 5.1 Chapter 2: Semisimple Lie Algebras

We now assume  $\text{char } F = 0$  and  $\bar{F} = F$ .

Theorem: If  $\mathfrak{g}$  is a solvable Lie subalgebra of  $\mathfrak{gl}(V)$  for some finite dimensional  $V$ , then  $V$  contains a common eigenvector for a  $x \in \mathfrak{g}$ , i.e. a  $\lambda : \mathfrak{g} \rightarrow F, x \mapsto \lambda(x)$  such that  $x(v) = \lambda(x)v$  for all  $x \in \mathfrak{g}$ .

Proof: We will use induction on the dimension of  $\mathfrak{g}$ . For the inductive step:

Claim 1: There is an ideal  $A \trianglelefteq \mathfrak{g}$  such that  $\mathfrak{g} = A \oplus Fy$  for some  $y \neq 0$ , so  $A$  is a subalgebra of a solvable Lie algebra  $\mathfrak{g}$  and thus solvable itself. By hypothesis, we can produce a  $w \in V \setminus \{0\}$ , and thus a functional  $\lambda : A \rightarrow F$  such that  $aw = \lambda(a)w$  for all  $a \in A$ . So we define

$$V_\lambda = \{v \in V \mid av = \lambda(a)v \forall a \in A\}$$

where  $w \in V_\lambda$ .

Claim 2:  $y(V_\lambda) \subseteq V_\lambda$ , or  $y|_{V_\lambda} \in \text{End}(V_\lambda)$ .

Thus  $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$  is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a  $v \in V_\lambda$  and some  $\mu \in F$  such that  $y(v) = \mu v$ . An arbitrary element  $x \in \mathfrak{g}$  can be written as  $x = a + cy$  for some  $a \in A, c \in F$  and it acts by  $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c\mu)v \in V_\lambda$ .

## 6 Friday August 23

Chapter 3: Theorems of Lie and Cartan

### 6.1 4.1: Lie's Theorem

Theorem: Let  $L$  be a solvable subalgebra of  $\mathfrak{gl}(V)$ , where  $V$  is finite-dimensional. If  $V \neq 0$ , then  $V$  contains a common eigenvector for all of the endomorphisms in  $L$ .

Proof: Use induction on  $\dim L$ . The case  $\dim L = 0$  is trivial. We'll attempt to mimic the proof of Theorem 3.3. The idea is to

1. Locate an ideal of  $K$  of codimension 1,

2. Show by induction that common eigenvectors exist for  $k$ ,
3. Verify that  $L$  stabilizes a space consisting of such eigenvectors,
4. Find in that space an eigenvector for a single  $z \in L$  satisfying  $L = K + Fz$ .

Step (1): Since  $L$  is solvable and of positive dimension, then  $L \subsetneq [L, L]$ . Otherwise, if  $L = [L, L]$ , then  $L^{(1)} = L \implies L^{(n)} = L$ , which would contradict  $L$  being solvable.

Since  $[L, L]$  is abelian, any subspace is automatically an ideal. So take a subspace of codimension one, then its inverse image  $K \trianglelefteq L$  is an ideal satisfying  $[L, L] \subseteq K$ .

Step (2): Use induction to find a common eigenvector  $v \in V$  for  $K$ . ( $K$  is solvable; if  $K = 0$  then  $L$  is abelian of dimension 1 and any eigenvector for a basis vector of  $L$  finishes the proof.)

This means that  $x \in K \implies x \curvearrowright v = \lambda(x)v$  for some  $\lambda : K \rightarrow F$  a linear functional. Fix this  $\lambda$ , and let  $W = \{w \in V \mid x \curvearrowright w = \lambda(x)w \forall x \in K\}$ ; note that  $W \neq 0$ .

Step (3): This will involve showing that  $L$  leaves  $W$  invariant. Assume for the moment that this is done, and proceed to step (4).

Step (4):

Write  $L = K + Fz$ . Since  $F$  is algebraically closed, we can find an eigenvector  $v_0 \in W$  of  $z$  for some eigenvalue of  $z$ . Then  $v_0$  is a common eigenvector for  $L$ , and  $\lambda$  can be extended to a linear function on  $L$  satisfying  $x \curvearrowright v_0 = \lambda(x)v_0$  where  $x \in L$ .

It remains to show that  $L$  stabilizes  $W$ . Let  $w \in W, x \in L$ . To test whether or not  $x \curvearrowright w \in W$ , we take an arbitrary  $y \in K$  and examine

$$yx \curvearrowright w = xy \curvearrowright w - [x, y] \curvearrowright w = \lambda(y)x \curvearrowright w - \lambda([x, y])w.$$

Note: the above equality is an important trick.

Thus we need to show that  $\lambda([x, y]) = 0$ . To this end, fix  $w \in W, x \in L$ . Let  $n > 0$  be the smallest integer for which  $w, x \curvearrowright w, \dots, x^n \curvearrowright w$  are all linearly *independent*. Let  $W_i = \text{span}(\{w, x \curvearrowright w, \dots, x^{i-1} \curvearrowright w\})$  and set  $W_0 = 0$ . Then  $\dim W_n = n$ , and  $W_{n+i} = W_n$  for all  $i \geq 0$ . Moreover,  $x$  maps  $W_n$  into itself. It is easy to check that each  $y \in K$  is represented by an upper-triangular matrix with diagonal entries equal to  $\lambda(y)$ . This follows immediately from the congruence

$$yx^i \curvearrowright w = \lambda(y)x^i \curvearrowright w \pmod{W_i},$$

which can be proved by induction on  $i$ . The case  $i = 0$  is trivial. For the inductive step, write

$$yx^i \curvearrowright w = yx^{i-1} \curvearrowright w = xyx^{i-1} \curvearrowright w = [x, y]x^{i-1} \curvearrowright w$$

By induction,

$$yx^{i-1} \curvearrowright w = \lambda(y)x^{i-1} \curvearrowright w + w',$$

where  $w' \in W_{i-1}$ . Since  $x$  maps  $W_{i-1}$  into  $W_i$  by construction, the congruence holds for all  $i$ .

According to our description of the action of  $y \in K$  on  $W_n$ , we have  $\text{Tr}_{W_n}(y) = n\lambda(y)$ . In particular, this is true for elements  $k$  of  $f$  of the special form  $[x, y]$  where  $x$  is as it was above and  $y$  is in  $K$ .



**But both  $x$  and  $y$  stabilize  $W_n$ , so  $[x, y]$  acts on  $W_n$  as the commutator of two endomorphisms of  $W_n$ , and the trace is therefore zero.**

We conclude that  $n\lambda([x, y]) = 0$ . Since  $\text{char} F = 0$ , this forces  $\lambda([x, y]) = 0$  as required.  $\square$

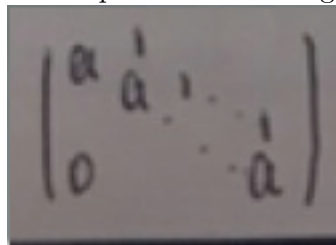
**Corollary A (Lie's Theorem):** Let  $L \leq \mathfrak{gl}(V)$  be a solvable subalgebra where  $\dim V = n < \infty$ . Then  $L$  stabilizes some flag in  $V$ , i.e. the matrices of  $L$  relative to a suitable basis of  $V$  are upper triangular.

**Proof:** Use the above theorem, along with induction on  $\dim V$ . This is similar to the proof of corollary 3.3.

## 6.2 4.2: Jordan-Chevalley Decomposition

**Fact 1:**

The Jordan Canonical Form of a single endomorphism  $x$  over  $F$  algebraically closed is an expression



of  $x$  in matrix form as a sum of blocks:

**Fact 2:**

Call  $x \in \text{End} V$  *semisimple* if the roots of its minimal polynomial over  $F$  are all distinct. Equivalently, if  $F$  is algebraically closed, then  $x$  is semisimple iff  $x$  is diagonalizable.

**Fact 3:**

Two commuting semisimple endomorphisms can be simultaneously diagonalized. Therefore, their sum or difference is again semisimple.

**Proposition:** Let  $V$  be a finite dimensional vector space over  $F$  and  $x \in \text{End} V$ . Then

- There exist unique  $x_s, x_n \in \text{End} V$  satisfying the conditions  $x = x_s + x_n$ ,  $x_s$  is semisimple,  $x_n$  is nilpotent, and  $x_s, x_n$  commute.
- There exists polynomials  $p(t), g(t)$  such that  $x_s = p(x)$  and  $x_n = g(x)$ . In particular,  $x_s, x_n$  commute with any endomorphism commuting with  $x$ .
- If  $A < B < V$  are subspaces and  $x$  maps  $B$  into  $A$ , then  $x_s, x_n$  also map  $B$  into  $A$ .

The decomposition  $x = x_s + x_n$  is called the (additive) **Jordan-Chevalley decomposition** of  $x$ , or just the Jordan decomposition.  $x_s, x_n$  are respectively called the **semisimple part** and the **nilpotent part** of  $x$ .

**Example:**

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies x_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that  $x_s x_n = x_n = x_n x_s$ ,  $x_s = 2x - x^2$ , and  $x_n = x^2 - x$ . We thus have  $p(t) = 2t - t^2$  and  $q(t) = t^2 - t$ .

## 7 Monday August 26

Definition (Jordan Decomposition)

Let  $X \in \text{End}(V)$  for  $V$  finite dimensional. Then,

- (a) There exists a unique  $X_s, X_n \in \text{End}(V)$  such that  $X = X_s + X_n$  where  $X_s$  is semisimple,  $X_n$  is nilpotent, and  $[X_s, X_n] = 0$ .
- (b) There exists a  $p(t), q(t) \in t\mathbb{F}[t]$  such that  $X_s = p(X), X_n = q(X)$ .

(Polynomials with no constant term.)

Proof of (a): Assume  $X_s = X_s + X_n = X'_s + X'_n$ , so both have bracket zero. Assuming that (b) holds, we have  $X_s = p(X)$ , and so

$$[X, X_s] = [X_s + X'_n, X'_s] = [X'_s, X'_s] + [X'_s, X'_n] = 0 \implies [p(X), X'_s] = 0 = [X_s, X'_s]$$

Using fact (c) from last time, then  $X_s, X'_s$  can be diagonalized simultaneously, and so  $X_s - X'_s$  is semisimple.

On the other hand, if  $X'_n, X_n$  are nilpotent, and since these commute,  $X_n - X'_n$  is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X'_s) + (X_n + X'_n)$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let  $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$  be the minimal polynomial of  $X$ , where each  $m_i \geq 1$  and the  $\lambda_i$  are distinct. Then the primary composition of  $V$  is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V)^{m_i} \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomial  $p \in F[t]$  such that

$$\begin{aligned} p &= \lambda \pmod{(t - \lambda_i)^{m_i}} \quad \forall i, \\ p &= 0 \pmod{t}. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is  $p(x) \curvearrowright V_i$ ? This acts by scalar multiplication by  $\lambda_i$  for all  $i$ . (Check). Because of the restrictive conditions,  $p(x)$  has no constant term.

So  $p(X) = X_s$  is the semisimple part we want. Now just set  $q(t) = t - p(t)$ , then  $X_n := q(X) = X - X_s$  is nilpotent.

$$p(x) \sim \begin{pmatrix} \boxed{\lambda_1 I_{v_1}} & & & \\ & \boxed{\lambda_2 I_{v_2}} & & \\ & & \ddots & \\ & & & \boxed{\lambda_r I_{v_r}} \end{pmatrix}$$

Figure 1: ???

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have  $X = X_s + X_n$ , then  $\text{ad } X \in \text{End}(\text{End}(V))$ . It can be shown that  $(\text{ad } X)_s + (\text{ad } X)_n = \text{ad } (X_s) + \text{ad } (X_n)$ .

Let  $e_{ij}$  be the elementary matrix with a 1 in the  $i, j$  position. You can write  $\text{ad } X$  as a  $4 \times 4$  matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$= \begin{pmatrix} e_{11} & 0 & 0 & 1 & 0 \\ e_{12} & -1 & 0 & 0 & 1 \\ e_{21} & 0 & 0 & 0 & 0 \\ e_{22} & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \left( \begin{array}{c|ccc} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{array} \right)$$

$$= \begin{matrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \begin{pmatrix} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{pmatrix}$$

You can check that  $(\text{ad } X)_S = 0$ ,  $\text{ad } (X_S) = 0$ , and  $(\text{ad } X)_n$  is the Jordan form given above.

Lemma:

- (a)  $x \in \text{End}(V) \implies \text{ad } (x)_S = \text{ad } (x_S)$  and  $\text{ad } (x)_n = \text{ad } (x_n)$ .
- (b) If  $A$  is a finite dimensional  $\mathbb{F}$ -algebra, then  $\delta \in \text{Der}(A) \implies \delta_S, \delta_n \in \text{Der}(A)$  as well.

Proof of (a):

Check that  $\text{ad } (x) = \text{ad } (x_S) + \text{ad } (x_n)$ . Then for  $y \in \text{End}(V)$ , we have

$$\begin{aligned} (\text{ad } (x))(y) &= [x, y] \\ &= [x_S + x_n, y] \\ &= [x_S, y] + [x_n, y] \\ &= (\text{ad } (x_S))(y) + (\text{ad } (x_n))(y). \end{aligned}$$

Using theorem 3.3,  $x_n$  nilpotent  $\implies \text{ad } (x_n)$  is also nilpotent. So write  $x_S = \sum \lambda_i e_{ii}$  with the eigenvalues on the diagonal. Then  $\text{ad } x_S(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$  for all  $i, j$ . But then  $\text{ad } x_S$  is given

$$\begin{aligned}
 & (\delta - (\lambda + \mu)I)^n([x, y]) \\
 &= \sum_{i=0}^n \binom{n}{i} \left[ (\delta - \lambda I)^i(x), (\delta - \mu I)^{n-i}(y) \right]
 \end{aligned}$$

Figure 2: Image

by a matrix with  $\lambda_i - \lambda_j$  in the  $i, j$  position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

Proof of (b):

Since  $\delta \in \text{Der}(A)$ , the primary decomposition with respect to  $\delta$  is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \{a \in A \mid (\delta - \lambda I)^k a = 0 \text{ for some } k \gg 0\}.$$

So  $\delta_s \sim A_\lambda$  by scalar multiplication (by  $\lambda$ ). Then for  $\lambda, \mu \in F$ , we have

So  $[A_x, A_y] \subseteq A_{\lambda+\mu}$  for all  $x, y \in A$ . But then

and so  $\delta_s \in \text{Der}(A)$ , and  $\delta_n = \delta - \delta_s \in \text{Der}(A)$  as well.

## 8 Wednesday August 28

Todo

## 9 Friday August 30

Review of bilinear forms: let  $V = \mathbb{F}^n$ .

Definition: A bilinear form  $\beta : V^2 \rightarrow \mathbb{F}$  can be represented by a matrix  $B$  with respect to a basis  $\{\mathbf{v}_i\}$  such that

$$\beta(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i) = (a_1 \ a_2 \ \cdots) B (b_1 \ b_2 \ \cdots)$$

- $\beta$  is *symmetric* iff  $\beta(a, b) = \beta(b, a)$ .
- $\beta$  is *symplectic* iff  $\beta(a, b) = -\beta(b, a)$ .
- $\beta$  is *isotropic* iff  $\beta(a, a) = 0$ .

$$S_S([x, y])$$

||

$$(\lambda + \mu)[x, y] = [\lambda x, y] + [x, \mu y]$$

||

$$[S_S(x), y] + [x, S_S(y)]$$

Figure 3: Image

For a subspace  $U \leq V$ , define

$$U^\perp := \{v \in V \mid \beta(u, v) = 0 \forall u \in U\}.$$

Note: in general, left/right orthogonality are distinguished, but these will be identical when  $\beta$  is symmetric/symplectic.

The form  $\beta$  is said to be *non-degenerate* iff  $V^\perp = 0$  iff  $\det B \neq 0$ .

Assume  $F$  is an algebraically closed field, so  $\bar{F} = F$ , and  $\text{char} F \neq 2$ , then

- If  $\beta$  is non-degenerate and symmetric, then  $B \sim I_n$
- If  $\beta$  is non-degenerate and symplectic, then  $B \sim [0, I_{n/2}; I_{n/2}, 0]$ .

Remark:

$\mathfrak{so}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, F) \mid \beta(x(u), v) = -\beta(u, x(v))\}$ , where  $B$  has the matrix  $[0, I; I, 0]$  if  $n$  is odd, or this matrix with a 1 in the top-left corner if  $n$  is even.

Similarly,  $\mathfrak{sp}(2m, \mathbb{F})$  can be described this way with the matrix  $[0, -I_m; -I_m, 0]$ .

Overview: The Killing form is defined as  $\kappa : \mathfrak{g}^2 \rightarrow \mathbb{F}$  where  $\kappa(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y)$ .

Then we have **Cartan's Criteria**:

- $\mathfrak{g}$  solvable  $\iff \kappa(x, y) = 0 \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$ .
- $\mathfrak{g}$  semisimple  $\iff \kappa$  is non-degenerate.

Note that if  $\mathfrak{g}$  is semisimple, then  $\mathfrak{g} = \bigoplus_i I_i$  with each  $I_i \trianglelefteq \mathfrak{g}$  and simple.

## 9.1 Cartan's Criteria

Some facts:

1.  $\kappa$  is symmetric
2. If  $\mathfrak{g}$  is finite dimensional, then  $\kappa$  is associative, i.e  $\kappa([x, y], z) = \kappa(x, [y, z])$ .

Exercise: Show that if  $I \trianglelefteq \mathfrak{g}$ , then  $I^\perp \leq \mathfrak{g}$  is an ideal.

Proof of (2): In section 4.3, it was shown that  $\text{tr}([a, b] \circ c) = \text{tr}(a \circ [b, c])$  for all  $a, b, c \in \text{End}(V)$  (provided  $V$  is finite dimensional).

So

$$\begin{aligned} \kappa([x, y], z) &= \text{tr}(\text{ad}_{[x, y]} \circ \text{ad}_z) \\ &= \text{tr}([\text{ad}_x, \text{ad}_y] \circ \text{ad}_z) \\ &= \text{tr}(\text{ad}_x \circ [\text{ad}_y, \text{ad}_z]) \\ &= \text{tr}(\text{ad}_x \circ \text{ad}_{[y, z]}) \\ &= \text{tr}(x, [y, z]).. \end{aligned}$$

Theorem:  $\mathfrak{g}$  is semisimple iff  $\kappa$  is nondegenerate.

Proof:  $\implies$  : We want to show that  $\mathfrak{g}^\perp = 0$ . Note that  $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}$ , and so for all  $x \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$  and for any  $y \in \mathfrak{g}^\perp$ , we have

$$\kappa(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y) = 0$$

by the const(?) of  $\mathfrak{g}^\perp$ . This implies  $\mathfrak{g}^\perp$  is solvable.

Using fact (2), we have  $\mathfrak{g}^\perp \trianglelefteq \mathfrak{g}$  and thus  $\mathfrak{g}^\perp \subseteq \text{rad}(\mathfrak{g})$ , which is 0 since because  $\mathfrak{g}$  is semisimple. So either  $\mathfrak{g}^\perp = 0$  or  $\kappa$  is nondegenerate.

Used the fact that the radical was a maximal solvable ideal.

$\impliedby$  : We want to show that for all  $I \trianglelefteq \mathfrak{g}$  where  $[I, I] = 0$ , we have  $I^\perp \subseteq \mathfrak{g}^\perp$ .

For  $x \in I, y \in \mathfrak{g}$ , we have

$$(\text{ad}_x \circ \text{ad}_y)^2 = \mathfrak{g} \xrightarrow{\text{ad}_y} \mathfrak{g} \xrightarrow{\text{ad}_x} I \xrightarrow{\text{ad}_y} I \xrightarrow{\text{ad}_x} 0$$

And thus  $\text{tr}(\text{ad}_x \circ \text{ad}_y) = 0$  and  $I \subseteq \mathfrak{g}^\perp$ .

Suppose that  $\mathfrak{g}$  is *not* semisimple. Then there exists a solvable ideal  $J \neq 0$  such that the last term  $J^i$  in the derived series is an ideal  $I \trianglelefteq \mathfrak{g}$  such that  $[I, I] = 0$ , forcing  $J^i \subset \mathfrak{g}^\perp = 0$ , which is a contradiction.



$$\kappa_{\mathfrak{g}} \sim I_i \begin{pmatrix} \kappa_{I_i} & \\ & \end{pmatrix}$$

Figure 4: Image

## 9.2 Section 5.2

Theorem: If  $\mathfrak{g}$  is semisimple, then

- There exist ideals  $I_i \trianglelefteq \mathfrak{g}$  which are simple Lie algebras satisfying  $\mathfrak{g} = \bigoplus I_i$ . Note that  $[I_i, I_j] \subseteq I_i \cap I_j = 0$ , since direct summands intersect only trivially.
- Every simple  $I \trianglelefteq \mathfrak{g}$  is one of these  $I_i$ .
- $\kappa_{I_i} = \kappa_{\mathfrak{g}}|_{I_i \times I_i}$ , so

Remark:  $\mathfrak{g}$  is semisimple  $\iff \mathfrak{g} = \bigoplus_i I_i$  for some simple Lie algebras  $I_i$ .

$\Leftarrow$  : For all  $i$ ,  $S := \text{rad } \mathfrak{g}$ ,  $I_i \trianglelefteq I_i$  is a solvable ideal. This implies that it is 0, since  $I_i$  is simple.

By definition, simple Lie algebras are not abelian.

Supposing that  $S = I_i$ , we would then have  $[S, S] \neq 0$  since  $[I_i, I_i] \neq 0$  by definition. But  $[S, S] \neq S$  because  $S$  is solvable, which says that  $S$  is not simple (a contradiction).

Note that  $[\text{rad } \mathfrak{g}, \mathfrak{g}] \subseteq \bigoplus [\text{rad } \mathfrak{g}, I_i] = 0$ , which forces  $\text{rad } \mathfrak{g} \subseteq Z(\mathfrak{g})$ . Since  $I_i$  is simple,  $Z(I_i) = 0$  for all  $i$ . But  $Z(\mathfrak{g}) = \bigoplus Z(I_i) = 0$ , and this forces  $\text{rad } \mathfrak{g} \subseteq Z(\mathfrak{g}) \implies \text{rad } \mathfrak{g} = 0$ . So  $\mathfrak{g}$  is semisimple.

Next time – starting the representation theory with  $\mathfrak{sl}(2, \mathbb{F})$ .

## 10 Monday September 2

Recall the killing form:



Figure 5: Image

$$\begin{aligned} \kappa : \mathfrak{lieg}^2 &\rightarrow \mathbb{F} \\ (x, y) &\mapsto \text{tr}(\text{ad}_x \circ \text{ad}_y). \end{aligned}$$

and Cartan's criteria:

1.  $\mathfrak{g}$  is solvable  $\iff \kappa(x, y) = 0 \ \forall x \in \mathfrak{g}, \mathfrak{g}], \ y \in \mathfrak{g}$ .
2.  $\mathfrak{g}$  is semisimple  $\iff \kappa$  is non-degenerate.

Theorem: If  $\mathfrak{g}$  is semisimple, then

- a.  $\mathfrak{g} = \bigoplus_{i=1}^n I_i$  for some  $I_i \trianglelefteq \mathfrak{g}$  which are all simple.
- b. Every simple ideal  $I \trianglelefteq \mathfrak{g}$  is one of the  $I_i$ .
- c.  $\kappa_{I_i} = \kappa_{\mathfrak{g}}|_{I_i \times I_i}$ .

Proof of (a): Use induction on  $\dim \mathfrak{g}$ . If  $\mathfrak{g}$  has no nonzero proper ideals, then  $\mathfrak{g}$  is simple and we're done.

Otherwise, let  $I_1$  be a minimal nonzero ideal of  $\mathfrak{g}$ . Then  $I_1^\perp \trianglelefteq \mathfrak{g}$  is also an ideal, and thus  $I := I_1 \cap I_1^\perp \trianglelefteq \mathfrak{g}$  is as well. Then for all  $x \in [I, I]$ , we must have  $\kappa(x, y) = 0$  for any  $y \in I \subseteq I_1^\perp$ . So  $I$  is solvable, and thus  $I = 0$ . So  $\mathfrak{g} = I_1 \oplus I_1^\perp$ .

$$\begin{aligned}
 \text{ad } x &\sim \left( \begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array} \right) & \kappa_{\mathfrak{g}}(x, y) &= \text{tr} \left( \left( \begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array} \right) \left( \begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array} \right) \right) \\
 \text{ad } y &\sim \left( \begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array} \right) & &= \text{tr} \left( \begin{array}{c|c} A_x A_y & B_x B_y \\ \hline 0 & 0 \end{array} \right) \\
 & & &= \text{tr}(A_x A_y) \\
 & & &= \chi_{\mathcal{I}_i}(x, y)
 \end{aligned}$$

Figure 6: Image

Note that any ideal of  $I_1^\perp$  is also an ideal of  $\mathfrak{g}$ , which implies that  $\text{rad}(I_1^\perp) \subseteq \text{rad}(\mathfrak{g})$ , which is zero since  $\mathfrak{g}$  is semisimple, and thus  $I_1^\perp$  is semisimple as well.

By the inductive hypothesis,  $I_1^\perp = I_2 \oplus \cdots \oplus I_n$  where each  $I_j \trianglelefteq I_i^\perp$  is simple. Then  $I_j \trianglelefteq \mathfrak{g} \implies [I_1, I_j] \subset I_1 \cap I_j$ , since  $I_1$  has no contribution. But this is a subset of  $I_1 \cap I_1^\perp = 0$ .  $\square$

Proof of (b): If  $I \trianglelefteq \mathfrak{g}$ , then  $[I, \mathfrak{g}] \trianglelefteq I$  because  $[[I, \mathfrak{g}], I] \subseteq [I, I] \subseteq [I, \mathfrak{g}]$ .

Since  $\mathfrak{g}$  is semisimple,  $0 = \text{rad}(\mathfrak{g}) \supseteq Z(\mathfrak{g})$ . So  $[I, \mathfrak{g}] \neq 0$ , and thus  $[I, \mathfrak{g}] = I$  since  $I$  is simple. But then  $[I, \mathfrak{g}] = \bigoplus [I, I_i]$  is simple as well. So only one direct summand can survive, since otherwise this would produce at least 2 nontrivial ideals, and  $[I, \mathfrak{g}] = [I, I_i]$  for some  $i$ .

So for all  $j \neq i$ , we must have  $I_j \cap I = I_j \cap [I, I_i] = 0$ , and so  $I \subseteq I_i$ . But then  $I = I_i$  since  $I_i$  itself is simple, and we're done.

Proof of (c):

(Without using the simplicity of  $I_i$ )

For  $x, y \in I_i$ , we have

## 10.1 Inner Derivations

Recall that  $\text{ad } \mathfrak{g} \subseteq \text{Derg}$ , and in fact (lemma) this is an ideal.

Theorem: If  $\mathfrak{g}$  is semisimple, then  $\text{ad } \mathfrak{g} = \text{Derg}$ .

Proof of lemma:

For all  $\delta \in \text{Der } \mathfrak{g}$  and all  $x, y \in \mathfrak{g}$ , we have

$$\begin{aligned} [\delta, \text{ad } x](y) &= \delta([x, y]) - [x, \delta(y)] \\ &= [\delta(x), y] \\ &= [\text{ad } \delta(x)](y), \end{aligned}$$

and so  $[\delta, \text{ad } x] \subseteq \text{ad } \mathfrak{g}$ .  $\square$

Proof of theorem:

If  $\mathfrak{g}$  is semisimple, then  $0 = \text{rad } \mathfrak{g} \supseteq Z(\mathfrak{g}) = \ker \text{ad}$ . Thus  $\text{ad } \mathfrak{g} \cong \mathfrak{g} / \ker \text{ad} \cong \mathfrak{g}$  is also semisimple.

This means that  $\kappa_{\text{ad } \mathfrak{g}}$  is non-degenerate, and thus  $\text{ad } \mathfrak{g} \cap (\text{ad } \mathfrak{g})^\perp = 0$ , where  $(\text{ad } \mathfrak{g})^\perp \leq \text{Der}(\mathfrak{g})$ .

(Note that the non-degeneracy of  $\kappa$  already forces  $(\text{ad } \mathfrak{g})^\perp = 0$ .)

Then  $[(\text{ad } \mathfrak{g})^\perp, \text{ad } \mathfrak{g}] = 0$ , and so for all  $\delta \in (\text{ad } \mathfrak{g})^\perp$ , we have  $\delta(x) = [\delta, \text{ad } x]$  by the lemma, but we've shown that this is zero.

But then  $\delta$  must be zero because  $\text{ad}$  is an isomorphism, and in particular it is injective. This means that  $(\text{ad } \mathfrak{g})^\perp = 0$ , and thus  $\text{ad } \mathfrak{g} = \mathfrak{g}$ .  $\square$

We can use this to define an abstract Jordan decomposition by pulling back decompositions on adjoints:

## 11 Wednesday September 4

### 11.1 4.3: Cartan's Criterion

Lemma: Let  $A \subset B$  be two subspaces of  $\mathfrak{gl}(V)$  where  $\dim V < \infty$ . Set  $M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subset A\}$ . Suppose that  $x \in M$  satisfies  $\text{Tr}(xy) = 0$  for all  $y \in M$ . Then  $x$  is nilpotent.

Proof: Let  $x = s + n$  (where  $s = x_s$  and  $n = x_n$ ) be the Jordan decomposition of  $x$ . Fix a basis  $v_1 \cdots v_m$  of  $V$  relative to which  $s$  has matrix  $\text{diag}(a_1 \cdots a_m)$ . Let  $E$  be the vector subspace of  $F$  over the prime field  $Q$  spanned by the eigenvalues  $a_1 \cdots a_m$ . We have to show that  $s = 0$ , or equivalently that  $E = 0$ , since  $E$  has finite  $Q$ -dimension by construction. It will suffice to show that the dual space  $E^*$  is 0, i.e. that any linear functional  $f : E \rightarrow Q$  is zero.

Given  $f$ , let  $y$  be the element of  $\mathfrak{gl}(V)$  whose matrix is given by  $\text{diag}(f(a_1), \dots, f(a_m))$ . If  $\{e_{ij}\}$  is a basis of  $\mathfrak{gl}(V)$ , then  $\text{ad } s(e_{ij}) = (a_i - a_j)e_{ij}$  and  $\text{ad } y(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$ .

Now let  $r(t) \in F[t]$  be a polynomial with no constant term, satisfying  $r(a_i - a_j) = f(a_i) - f(a_j)$  for all pairs  $i, j$ . The existence of such  $r(t)$  follows from Lagrange interpolation, and the fact that if  $a_i = a_j$  then  $0 = r(a_j) - r(a_i) = r(a_i - a_j) = r(0)$ , so  $r$  has no constant term. Thus there is no ambiguity in the assigned values, since  $a_i - a_j = a_j - a_l$  would imply (by linearity of  $f$ ) that  $f(a_i) - f(a_j) = f(a_k) - f(a_l)$ . Thus  $\text{ad } y = r(\text{ad } s)$ .

Note that Lagrange Interpolation is a special case of the Chinese Remainder Theorem for polynomials. If all  $x_i$ s are distinct, then  $p_i(x) = x - x_i$  are all pairwise coprime. Then dividing  $\frac{p(x)}{p_i(x)} = p(x_i)$ . So letting  $A_1 \cdots A_k$  be constants in  $k$ , there is a unique polynomial of degree less than  $k$  such that  $p(x_i) = A_i$ . Thus there is a polynomial  $p(x)$  such that  $p(x) \equiv A_i \pmod{p_i(x)}$ , and  $p(x_i) = A_i$ .

$$\mathfrak{g} \subseteq \text{End}(V)$$

$$x \xrightarrow{\text{ad}} \text{ad } x$$

$$\parallel \text{JD}$$

$$\parallel \text{JD}$$

$$x_s \mapsto \text{ad } x_s = (\text{ad } x)_s$$

+

$$x_n \mapsto \text{ad } x_n = (\text{ad } x)_n$$



Can recover some  $x_s$  and  $x_n$  from the adjoints

Figure 7: Image

Now  $\text{ad}_s$  is the semisimple part of  $\text{ad}_x$ . By lemma A of 4.2,  $\text{ad}_s$  can be written as a polynomial in  $\text{ad}_x$  without a constant term. Therefore  $\text{ad}_y$  is also a polynomial in  $\text{ad}_x$  without constant term. By hypothesis,  $\text{ad}_x$  maps  $B$  into  $A$ , so we have  $\text{ad}_y(B) \subset A$ , and so  $y \in M$ . Using the hypothesis of the lemma,  $\text{Tr}(xy) = 0$ , and so  $\sum a_i f(a_i) = 0$ . The left side is a  $Q$ -linear combination of elements of  $E$ . Applying  $f$ , we obtain  $\sum f(a_i)^2 = 0$ . But the numbers  $f(a_i)$  are rational, so this forces all of them to be zero. Finally,  $f$  must be identically 0 because the  $a_i$  span  $E$ .  $\square$

Note that  $\text{Tr}([x, y]z) = \text{Tr}(x[y, z])$ . To verify this, write  $[x, y]z = xyz - yxz$  and  $x[y, z] = xyz - xzy$ , then use the fact that  $\text{Tr}(y(xz)) = \text{Tr}((xz)y)$ .

**Theorem (Cartan's Criterion):** Let  $L \leq \mathfrak{gl}(V)$  be a subalgebra with  $V$  finite dimensional. Suppose  $\text{Tr}(xy) = 0$  for all  $x \in [L, L]$  and  $y \in L$ . Then  $L$  is solvable.

**Proof:** It suffices to show that  $[L, L]$  is nilpotent, or just that all  $x \in [L, L]$  are nilpotent endomorphisms. We apply the above lemma, with  $V$  as given,  $A = [L, L]$ , and  $B = L$ , so  $M = \{x \in \mathfrak{gl}(V) \mid [x, L] \subset [L, L]\}$ . We have  $L \subset M$ . Our hypothesis is that  $\text{Tr}(xy) = 0$  for all  $x \in [L, L]$  and  $y \in L$ . To use the lemma to reach the desired conclusion, we need a stronger result: that  $\text{Tr}(xy) = 0$  for  $x \in [L, L]$  and  $y \in M$ .

If  $[x, y]$  is a generator of  $[L, L]$  and  $z \in M$ , then  $\text{Tr}([x, y]z) = \text{Tr}(x[y, z]) = \text{Tr}([y, z]x)$ . By definition of  $M$ ,  $[y, z] \in [L, L]$ , so the right side is 0 by hypothesis.

**Corollary:** Let  $L$  be a Lie algebraic such that  $\text{Tr}(\text{ad}_x \circ \text{ad}_y) = 0$  for all  $x \in [L, L], y \in L$ . Then  $L$  is solvable.

**Proof:** Apply the theorem to the adjoint representation of  $L$ . We then get  $\text{ad } L$  is solvable. Since  $\ker \text{ad} = Z(L)$  is also solvable,  $L$  itself is solvable.

## 11.2 Killing Form

### 11.2.1 Criterion for Semisimplicity

Let  $L$  be any lie algebra. If  $x, y \in L$ , then define  $\kappa(x, y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y)$ . Then  $\kappa$  is a symmetric bilinear form on  $L$ , called the **killing form**.

**Theorem:**  $\mathfrak{g}$  is solvable  $\iff \kappa(x, y) = 0$  for all  $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$ .

**Proof:**  $\Leftarrow$  : By Cartan's Criterion.

$\Rightarrow$  : Exercise.

**Example:** The killing form of  $\mathfrak{sl}(2, F)$ .

We have

$$\begin{aligned} x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Then  $\text{ad } h = \text{diag}(2, 0, -2)$ , and

$$\begin{aligned}\text{ad } x &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \text{ad } y &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.\end{aligned}$$

and thus  $k$  has the matrix

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

where  $k_{ij} = \kappa(x_i, x_j)$  where  $x_i$  is a basis of  $L$ .

## 12 Wednesday September 11

Theorem: If  $L$  is semisimple and  $x \in L$ , there exists a unique  $x_s, x_n$  in  $L$  such that  $x = x_s + x_n$ ,  $[x_n, x_s] = 0$ ,  $\text{ad } x_s$  is semisimple, and  $\text{ad } x_n$  is nilpotent.

## 13 Friday September 13

Todo

## 14 Monday September 16

Let  $S = \exp(\text{ad } e) \circ \exp(\text{ad } -f) \circ \exp(\text{ad } ei)$ , which has the following matrix:

Where  $\exp(\text{ad } e) = 1 + \text{ad } e + \frac{1}{2}(\text{ad } e)^2$ , which would have the form

Theorem: If  $\mathfrak{g}$  is semisimple, then any finite dimensional  $\mathfrak{g}$ -module  $V$  is completely reducible, i.e. it splits into a direct sum of simple modules.

### 14.1 Proof of Weyl's(?) Theorem

If  $V$  itself is simple, then we're done, so suppose it is not.

Assume there exists a nonzero submodule  $U \subsetneq V$ . It suffices to show that  $V = U \oplus U'$  for some  $U'$ .

$$\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

Figure 8: Image

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \cdot & 2 & \\ & \cdot & -1 \\ & & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & 1 \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}$$

Figure 9: Image



### 14.1.1 Step 1:

If  $\dim V = 2$  and  $\dim U = 1$ .

Then  $U, V/U$  are both trivial modules. So  $g \curvearrowright u = 0$  for all  $u \in U$ . But then  $g \curvearrowright (v + U) = U$  for all  $v \in V$ , since  $g \curvearrowright v \in U$ .

So for all  $x, y \in \text{lieg}$  and all  $v \in V$ , we have  $[x, y] \curvearrowright v = x \curvearrowright (y \curvearrowright v) - y \curvearrowright (x \curvearrowright v)$ . But both of the terms in parenthesis are in  $U$ , and all elements in  $\mathfrak{g}$  kill elements in  $U$ , so this is zero. So  $[\mathfrak{g}, \mathfrak{g}] \curvearrowright V$  trivially.

Exercise: If  $\mathfrak{g}$  is semisimple, then  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .

So  $\mathfrak{g} \curvearrowright V$  trivially. Thus any  $U'$  that is a complementary subspace of  $U$  will be a submodule of  $V$ .

### 14.1.2 Step 2:

Suppose  $U$  is simple and  $\dim U > 1$ , so  $\dim V/U = 1$ .

Let  $\Omega$  be the Casimir element on  $U$  (faithful representation?). Then  $\Omega u = c u$  for some  $c \in \mathbb{F}$ , and so  $\Omega(U) \subseteq U$ .

Since  $\Omega : V \rightarrow V$  is a homomorphism,  $\ker \Omega \subseteq V$  is a  $\mathfrak{g}$ -submodule. Then  $\dim V/U = 1 \implies V/U$  is a trivial module. So  $\mathfrak{g} \curvearrowright V/U = 0$ , i.e.  $\mathfrak{g} \curvearrowright V \subseteq U$ .

Then  $\Omega(v) = \sum_i x_i \curvearrowright (y_i \curvearrowright v) \in U$  for all  $v \in V$ . What is the matrix of  $\Omega$ ?

In particular,  $\text{Tr}(\Omega|_{V/U}) = 0$ . So  $\text{Tr}(\Omega) = \text{Tr}(\Omega|_U)$ . From 6.2, we know that  $\text{Tr}(\Omega) \neq 0 \implies c \neq 0$ , where  $c$  is the scalar appearing above. So  $\ker \Omega$  is 1-dimensional, and  $\ker \Omega \cap U = \{0\}$ .

So take  $U' = \ker \Omega$ .

### 14.1.3 Step 3:

Suppose  $U$  is *not* simple, but  $\dim V/U = 1$ .

We will induct on the dimension of  $U$ . Pick a proper nonzero submodule  $\bar{U} \subsetneq U$ , so that  $\dim U/\bar{U} < \dim U$ . Now  $V/U \cong (V/\bar{U})/(U/\bar{U})$  by an isomorphism theorem. So  $U/\bar{U}$  is a submodule of  $V/\bar{U}$  of codimension 1. Applying the inductive hypothesis, we obtain  $V/\bar{U} = U/\bar{U} \oplus \bar{V}/\bar{U}$  for some  $\bar{V}$  such that  $U \subseteq \bar{V} \subseteq V$ .

In particular, since  $U \subseteq \bar{V}$  has codimension 1,  $\dim \bar{U} < \dim U$ . So apply the inductive hypothesis again:  $\bar{V} = \bar{U} \oplus U'$  for some  $U'$ , and  $V = U \oplus U'$ .

### 14.1.4 Step 4: The general case

Recall that  $\text{hom}(V, U)$  is a  $\mathfrak{g}$ -module where

$$(g \curvearrowright \phi)(v) = g \curvearrowright \phi(v) - \phi(g \curvearrowright v).$$

$$\begin{array}{c}
 u \qquad \qquad \qquad v/u \\
 \left( \begin{array}{c|c}
 u & * \\
 \hline
 v/u & 0 \dots 0
 \end{array} \right)
 \end{array}$$

Handwritten diagram of a block matrix. The matrix is partitioned into four blocks by a vertical line. The top-left block is labeled  $u$ , the top-right block is labeled  $*$ , the bottom-left block is labeled  $v/u$ , and the bottom-right block is labeled  $0 \dots 0$ . The matrix is enclosed in large parentheses.

Figure 10: Image

Define

$$S = \{\phi \in \text{hom}(V, U) \mid \phi|_U \in F1_U\}.$$

Then  $S \leq \text{hom}(V, U)$  as a submodule. Define  $T = \{\phi \in S \mid \phi|_U = 0\}$ . Then  $T \leq S$  as a submodule, and  $\mathfrak{g}(S) \subseteq T$ .

Now each  $\phi \in S$  is determined (mod  $T$ ) by the scalar  $\phi|_U$ . Note that  $\dim(S/T) = 1$ . By steps 1-3, we know that  $S = T \oplus T'$  for some  $T' \subseteq S$  of dimension 1. Then  $T' = \text{span}_{\mathbb{F}}(f)$  for some nonzero map  $f : V \rightarrow U$  such that  $f(u) = cu$  for some  $c \neq 0$ .

Then  $\mathfrak{g}(T \oplus T') = \mathfrak{g}(S) \subseteq T \implies \mathfrak{g}(T') = 0$ . So for all  $g \in \mathfrak{g}$ , we have  $0 = (g \curvearrowright f)(v) = f \curvearrowright f(v) - f(g \curvearrowright v)$ . Then  $f : V \rightarrow U$  is a lie algebra homomorphism,  $\ker f = U'$ , and thus  $V = U \oplus U'$ .  $\square$

Some consequences of Weyl's theorem:

## 14.2 Preservation of Jordan Decomposition

Recall that when  $\mathfrak{g} \in \mathfrak{gl}(V)$  is a linear lie algebra, then for  $x \in \mathfrak{g}$  we have:

Jordan Decomposition:  $x = x_s + x_n$  where  $x_s, x_n \in \text{End}(V)$ .

Abstract Jordan Decomposition:

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\text{ad}} \text{ad}(\mathfrak{g}) \\ x &\mapsto \text{ad } x \\ x_s &\leftarrow (\text{ad } x)_s \\ x_n &\leftarrow (\text{ad } x)_n. \end{aligned}$$

and so  $x = x'_s + x'_n$  for some  $x'$ . The theorem will be that these recover the usual Jordan decomposition.

Theorem: If  $\mathfrak{g} \in \mathfrak{gl}(V)$  is semisimple and  $V$  is finite dimensional, then  $x_s, x_n \in \mathfrak{g}$ , and  $x_s = x'_s, x'_n$ .

Corollary: If  $\mathfrak{g}$  is semisimple and finite dimensional and  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is a finite dimensional representation, then if  $x = x_s + x_n$  is the abstract Jordan decomposition, then  $\phi(x) = \phi(x_s) + \phi(x_n)$  is the Jordan decomposition in  $\mathfrak{gl}(V)$ .

Example: If  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  is semisimple and finite dimensional, and  $h$  is diagonal, then by JD  $h = h + 0$ ,  $\phi(h) = \phi(h) + 0$ . Then  $h \curvearrowright V$  semisimply, or  $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ , where  $V_\lambda = \{v \in V \mid h \curvearrowright v = \lambda v\}$  are the eigenspaces.

## 15 Wednesday September 18

Last time: The abstract Jordan Decomposition coincides with the actual Jordan Decomposition.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Figure 11: Image

$$\begin{aligned} \phi : \mathfrak{g} &\rightarrow \mathfrak{gl}(V) \\ x &\mapsto \phi(x) = \phi(x)_s + \phi(x)_n = \phi(x_n) + \phi(x_s) \\ x_s + x_n &\mapsto \phi(x_s) + \phi(x_n). \end{aligned}$$

Therefore  $x_s \curvearrowright V$  semisimply. The example we saw last time was  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ , with a matrix  $h = [1, 0; 0, -1]$  and  $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ .

## 15.1 Finite Dimensional Representations of $\mathfrak{sl}(2, \mathbb{C})$

### 15.1.1 Weights and Maximal Vectors

Definition: If  $V_\lambda \neq 0$ , then  $V_\lambda$  is a *weight space* of  $V$  and  $\lambda \in \mathbb{C}$  is a *weight* of  $h$  in  $V$ . We then define  $W_t(V) = \{\text{weights in } V\}$ .

Lemma: If  $v \in V_\lambda$  then  $e \curvearrowright v \in V_{\lambda+2}$  and  $f \curvearrowright v \in V_{\lambda-2}$ .

Proof:

$$\begin{aligned} h \curvearrowright (e \curvearrowright v) &= [h, e] \curvearrowright v + e \curvearrowright (h \curvearrowright v) \\ &= 2e \curvearrowright v + \lambda e \curvearrowright v \\ &= (\lambda + 2)e \curvearrowright v. \end{aligned}$$

and

$$\begin{aligned}
h \curvearrowright (f \curvearrowright v) &= [h, f] \curvearrowright v + f \curvearrowright (h \curvearrowright v) \\
&= -2f \curvearrowright v + \lambda f \curvearrowright v \\
&= (\lambda - 2)f \curvearrowright v.
\end{aligned}$$

So if  $V$  is a finite-dimensional  $\mathfrak{g}$ -module, then there exists a  $V_\lambda \neq 0$  such that  $V_{\lambda+2} = 0$ . Any nonzero  $v \in V_\lambda$  is called a *maximal vector*.

Note: in category  $\mathcal{O}$ , these always exist?

Some computations:

- $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$  Then  $V = \mathbb{C}$  is the trivial module, and  $g \curvearrowright V = 0$ . So  $W_t(V) = \{0\}$ , and  $V = V_0$ .

If  $V = \mathbb{C}^2$ , then take the natural representation  $\text{span}_{\mathbb{C}} \{v_1 = [1, 0], v_2 = [0, 1]\}$ . Then  $g \curvearrowright V$  by matrix multiplication, and if  $h = [1, 0; 0, -1]$  then  $h \curvearrowright v_1 = v_1$  and  $h \curvearrowright v_2 = -v_2$  by just doing the matrix-vector multiplication. Then  $\mathbb{C}([1, 0]) = V_1, \mathbb{C}([0, 1]) = V_{-1}$ , so  $W_t(V) = \{\pm 1\}$ .

Taking  $V = \mathbb{C}^3 = \text{ad } \mathfrak{g} = \text{span}_{\mathbb{C}} \{e, f, h\}$ , then

$$\begin{aligned}
h \curvearrowright f &= [h, f] = -2f \\
h \curvearrowright h &= [h, h] = 0h \\
h \curvearrowright e &= [h, e] = 2e.
\end{aligned}$$

So  $W_t(V) = \{2, 0, -2\}$  and  $V_2 = \mathbb{C}e, V_0 = \mathbb{C}h, V_{-2} = \mathbb{C}f$ .

Note the pattern: some largest value, then jumping by 2 to lower values, ending at negative the largest value. In some sense, the rest of the theory will reduce to the case of  $\mathfrak{sl}(2, \mathbb{C})$ .

Lemma: Let  $V$  be a finite dimensional simple  $\mathfrak{sl}(2, \mathbb{C})$ -module, and  $V_0 \in V_\lambda$  a maximal vector.

Set  $V_{-1} = 0, V_i = f^{(i)} \curvearrowright v_0$  (where  $f^{(i)} = \frac{f^i}{i!}$ ). Then for all  $i \geq 0$ , we have

- $h \curvearrowright v_i = (\lambda - 2i)v_i$
- $f \curvearrowright v_i = (i + 1)v_{i+1}$
- $e \curvearrowright v_i = (\lambda - i + 1)v_{i-1}$

Proof of (a): By lemma 7.1, we have  $f \curvearrowright v_0 \in V_{\lambda-2}$ , and so inductively  $f^{(i)} \curvearrowright v_0 \in V_{\lambda-2i}$

Proof of (b): By definition.

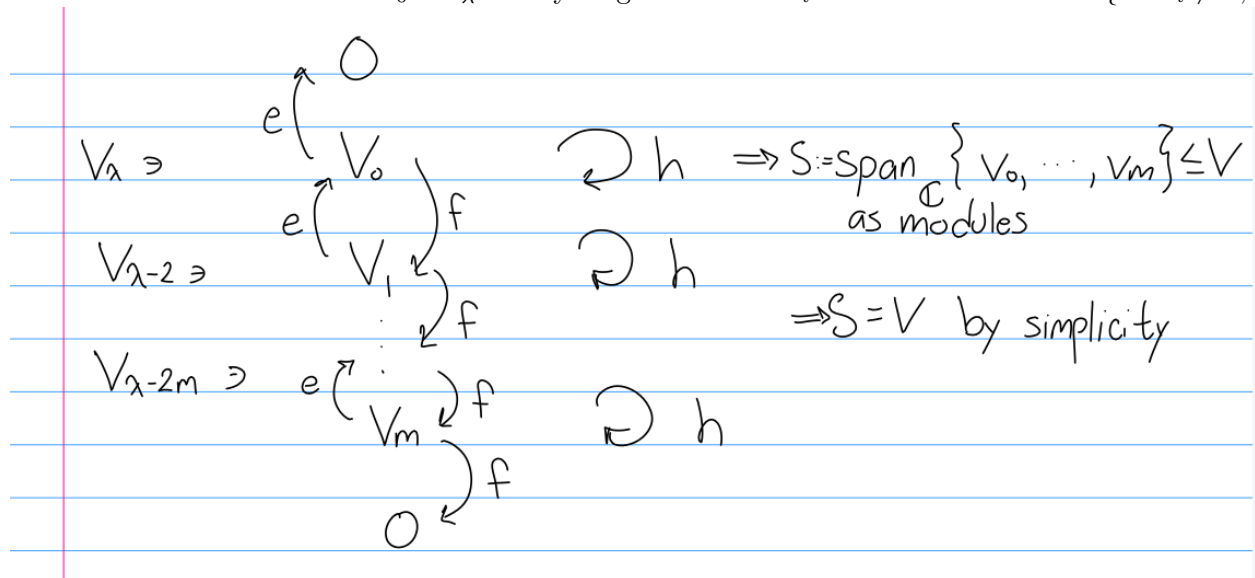
Proof of (c):

$$\begin{aligned}
ie \curvearrowright v_i &= ie \curvearrowright \frac{f^i \curvearrowright v_0}{i!} \\
&= e \curvearrowright (f \curvearrowright v_{i-1}) \\
&= [e, f] \curvearrowright v_{i-1} + f \curvearrowright (e \curvearrowright v_{i-1}) \\
&= h \curvearrowright v_{i-1} + f \curvearrowright ((\lambda - i + 2)v_{i-2}) \text{ind} \\
&= (\lambda - 2i + 2)v_{i-2} + (\lambda + i - 2)(i - 1)v_{i-1} \\
&= i(\text{RHS}).
\end{aligned}$$

Theorem: If  $V$  is a finite dimensional and simple, then  $V \cong L(m)$  for some  $m \in \mathbb{Z}_{\geq 0}$  where  $L(m) = \text{span}_{\mathbb{C}} \{v_0, v_1, \dots, v_m\}$  where each  $v_i$  is of weight  $m - 2i$ .

Thus  $L(m) = L(m)_m \oplus L(m)_{m-2} \oplus \dots \oplus L(m)_{-m}$  where  $\dim L(m)_\mu = 1$  for all  $\mu$  and  $\dim L(m) = m + 1$ .)

Proof: Pick a maximal vector  $v_0 \in V_\lambda$  for any weight  $\lambda$ . Define  $v_i$  as usual. Let  $m = \min \{i \ni V_i \neq 0, V_{i+1} = 0\}$



Definition: A module  $V$  is a *highest weight module* of weight  $\lambda$  if  $V = \mathfrak{g} \curvearrowright v_0$  for some maximal vector  $v_0 \in V_\lambda$ .

Then  $\lambda$  is referred to as the *highest weight*, and  $v_0$  is the *highest weight vector*.

Corollary: If  $V$  is finite-dimensional, then

- $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$
- The number of summands =  $\dim V_0 + \dim V_1$ .

Proof of (a): By Weyl's theorem, we know  $V = \bigoplus W_i$  for some simple  $W_i$ . By theorem 7.2, this is equal to  $\bigoplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\mu_m}$

Proof of (b):  $\dim V_0 = \# \{\text{summands where } m \text{ is even}\}$   $\dim V_1 = \# \{\text{summands where } m \text{ is odd}\}$

Remark: Let  $V_d = \{f \in \mathbb{C}[x, y] \ni f \text{ is homogeneous of total degree } d\} = \text{span}_{\mathbb{C}} \{x^d, x^{d-1}y, \dots, y^d\}$ .

Then  $\mathfrak{sl}(2, \mathbb{C}) \curvearrowright V_d$  by

$$\begin{aligned}
e &\mapsto x \frac{\partial}{\partial y} \\
f &\mapsto y \frac{\partial}{\partial x} \\
h &\mapsto x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.
\end{aligned}$$

Fact: For  $L(m), \phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(L(m))$ , define

$$s = (\exp \phi(e)) \circ (\exp \phi(-f)) \circ (\exp \phi(e))$$

Then  $s(v_i) = -v_{m-i}$ .

## 16 Friday September 20

Last time: Construction of simple finite-dimensional  $\mathfrak{sl}(2, \mathbb{C})$  module.

Today: Root space decomposition for semisimple finite-dimensional  $\mathfrak{g}$ .

### 16.1 Root Space Decomposition

Let  $\mathfrak{g}$  be semisimple and finite dimensional, and let  $\mathbb{F} = \mathbb{C}$ .

#### 16.1.1 Maximal Toral subalgebra and roots

Definition: A subalgebra  $\mathfrak{h} \leq \mathfrak{g}$  is *toral* if  $\mathfrak{h} \neq 0$  and it consists of only semisimple elements (i.e.  $x_n = 0 \forall x \in \mathfrak{h}$ )

Lemma:

- There exists a toral subalgebra of  $\mathfrak{g}$ , which is a nontrivial maximal toral subalgebra.
- Any toral subalgebra is abelian.

Proof of (a): Want to show that there exists an  $x \in \mathfrak{g}$  such that  $x_s \neq 0$ , which will imply that  $\mathfrak{h} = \mathbb{C}x_s$  is toral.

Suppose  $x_s = 0$  for all  $x \in \mathfrak{g}$ , then  $\text{ad } x = \text{ad } x_n$  is nilpotent. By Engel's theorem, this means  $\mathfrak{g}$  must be nilpotent. But this contradicts  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  (since  $\mathfrak{g}$  is semisimple) so the derived series can never reach zero.

Proof of (b): Fix  $x \in \mathfrak{h}$ , want to show that  $[x, h] = 0 \forall h \in \mathfrak{h}$ . Then  $x = x_s$ , and so  $\text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g}$  is diagonalizable. It suffices to show that  $\text{ad } x|_{\mathfrak{h}} = 0$  for all  $\mathfrak{h}$ .

Suppose that  $[x, h] = ah$  for some vector  $h$  where  $a \neq 0$ . Decompose  $\mathfrak{h}$  into eigenspaces, so  $\mathfrak{h} = \bigoplus_{\lambda} \mathfrak{h}_{\lambda}$  where  $\mathfrak{h}_{\lambda} = \{y \in \mathfrak{h} \mid [h, y] = \lambda y\}$ . But then  $[h, x] \in \mathfrak{h}_0$ , since  $[h, [h, x]] = [h, -ah] = 0$ .

So write  $x = \sum_{\lambda} c_{\lambda} x_{\lambda}$ , where  $c_{\lambda} \in \mathbb{C}$  and  $x_{\lambda} \in \mathfrak{h}_{\lambda}$ . Then

$$\begin{aligned} [h, x] &= \sum_{\lambda} c_{\lambda} [h, x_{\lambda}] \\ &= \sum_{\lambda} c_{\lambda} \lambda x_{\lambda} \in \mathfrak{h}_0, \end{aligned}$$

so  $\lambda c_{\lambda} = 0 \forall \lambda \neq 0$ , which means  $c_{\lambda} = 0 \forall \lambda \neq 0$ , and thus  $x \in \mathfrak{h}_0$  and  $[h, x] = 0$ . But this contradicts  $[x, h] = ah$ .

Now  $\forall x, h \in \mathfrak{h}, g \in \mathfrak{g}$ , we have  $[h, [x, y]] = [x, [h, y]] + [y, [x, h]] = [x, [h, y]]$ . Thus  $\text{ad } h \circ \text{ad } x = \text{ad } x \circ \text{ad } h$  as elements of  $\text{End}(\mathfrak{g})$ .

So  $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{\alpha}$ , where  $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ .

Note that  $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [h, x] = 0 \forall h \in \mathfrak{h}\} = C_{\mathfrak{g}}(\mathfrak{h}) \supseteq \mathfrak{h}$ , i.e. the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$ .

Definition: Fix a toral subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ , then a *root* is a nonzero  $\alpha \in \mathfrak{h}^*$  such that  $\mathfrak{g}_{\alpha} \neq 0$ .  $\mathfrak{g}_{\alpha}$  is referred to as the *root space*.

We write  $\Phi = \{\text{roots}\}$  and  $\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha})$ .

Example:  $\mathfrak{sl}(3, \mathbb{C})$ .

TODO: Insert image from phone.

Then  $\Phi = \{\alpha : \mathfrak{h} \rightarrow \mathbb{C}, h_1 \mapsto \alpha(h_1) \in \{\pm 1, \pm 2\}\}$ . So

- $\mathfrak{g}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$
- $\mathfrak{g}_1 = \mathbb{C}f_2 \oplus \mathbb{C}e_3$
- $\mathfrak{g}_2 = \mathbb{C}e_1$
- $\mathfrak{g}_{-1} = \mathbb{C}f_3 \oplus \mathbb{C}e_2$
- $\mathfrak{g}_{-2} = \mathbb{C}f_1$ .

TODO: Insert second and third image from phone

From these computations, we collect the eigenvalues as ordered pairs. If we choose a larger toral subalgebra, we get a finer decomposition. And if we take a maximal toral subalgebra, then  $\mathfrak{h} = \mathfrak{g}_0$  and all  $\dim \mathfrak{g}_{\alpha} = 1$ .

Proposition (a):  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$  for all  $\alpha, \beta \in \mathfrak{h}^*$ .

Proposition (b): If  $x \in \mathfrak{g}_{\alpha}$  and  $\alpha \neq 0$  then  $\text{ad } x$  is nilpotent.

Proposition (c): If  $\alpha, \beta \in \mathfrak{h}^*$  and  $\alpha + \beta = 0$ , then  $\kappa(x, y) = 0 \forall x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{\beta}$ .

Proof of (a): Easy exercise:

Proof of (b): For all  $y \in \mathfrak{g}$ ,  $y \in \mathfrak{g}_{\mu}$  for some  $\mu \in \mathfrak{h}^*$ . We have  $\mathfrak{g}_u \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+\alpha} \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+2\alpha} \rightarrow \dots$  by  $y \mapsto [x, y] \mapsto \dots$ . Since  $\mathfrak{g}$  is finite dimensional, this must terminate, so  $(\text{ad } x)^n(y) = 0$  for some  $n$ .

Proof of (c): If  $\alpha + \beta = 0$ , then there exists an  $h \in \mathfrak{h}$  such that  $\alpha(h) + \beta(h) \neq 0$ . Since the killing form is associative, we have

Corollary:  $\kappa|_{\mathfrak{g}_0}$  is nondegenerate.

Proof: We want to show  $\kappa(h, y) = 0 \forall y \in \mathfrak{g}_0 \implies h = 0$  holds for any choice of  $y \in \mathfrak{g}_{\alpha}$  with  $\alpha \neq 0$ .



$$\begin{array}{c}
 K([h, x], y) = \alpha(h) K(x, y) \\
 \parallel \\
 - K([x, h], y) \\
 \parallel \\
 - K(x, [h, y]) = -\beta(h) K(x, y)
 \end{array}
 \begin{array}{c}
 \nwarrow x \in \mathfrak{g}_\alpha \\
 \swarrow x \in \mathfrak{g}_\beta
 \end{array}$$

Figure 12: Image

By proposition (c), we have  $\kappa(h, y) = 0$ . Note that we have  $\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha)$ . This implies that  $\kappa(h, y) = 0 \forall y \in \mathfrak{g}$ . But then  $h = 0$  because  $\kappa$  is nondegenerate and  $\mathfrak{g}$  is semisimple.

## 17 Monday September 23

Last time:  $\mathfrak{h}$  is a *toral* subalgebra if it contains only semisimple elements, and implies that there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$  and  $\Phi = \{\alpha : \mathfrak{h} \rightarrow \mathbb{C} \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}$  and  $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ .

Take larger  $\mathfrak{h}$  yields finer decompositions, and a maximal  $\mathfrak{h}$  gives  $\dim \mathfrak{g}_\alpha = 1 \forall \alpha \in \Phi$ .

Corollary:  $\kappa|_{\mathfrak{g}_0}$  is nondegenerate.

### 17.1 The Centralizer of $\mathfrak{h}$

If  $x, y \in \text{End}(V)$  where  $V$  is finite dimensional,  $xy = yx$ , and  $y$  is nilpotent, then  $xy$  is nilpotent and  $\text{Tr}(xy) = 0$ .

Proposition: If  $\mathfrak{h} \subseteq \mathfrak{g}$  is a maximal toral subalgebra, then  $\mathfrak{h} = \mathfrak{g}_0$ .

Proof:

Step 1: If  $x \in \mathfrak{g}_0$ , then  $x_s, x_n \in \mathfrak{g}_0$ .

If  $x \in \mathfrak{g}_0$ , then  $\text{ad } x(\mathfrak{h}) \subseteq 0$ . By proposition 4.2,  $\text{ad } x_s(\mathfrak{h}) \subseteq 0$ ,  $\text{ad } x_n(\mathfrak{h})$ , and so  $x_s, x_n \in \mathfrak{g}_0$ .

*Step 2:*  $\{x_s \ni x \in \mathfrak{g}_0\} \subseteq \mathfrak{h}$ .

If  $x \in \mathfrak{g}_0$ , then by step 1 we have  $x_s \in \mathfrak{g}_0$  and so  $\mathfrak{h} + \mathbb{C}x_s$  is toral, and thus  $x_s \in \mathfrak{h}$ .

*Step 3:*  $\kappa|_{\mathfrak{h}}$  is non-degenerate.

We want to show that  $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g} \implies h = 0$ . By the corollary, it suffices to show that  $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g}_0$ . By step 2, it suffices to check this only for  $x \in \mathfrak{g}_0$  such that  $x = x_n$ .

If  $x = x_n$ , then  $\text{ad } x_n$  is nilpotent and  $\text{ad } h$  commutes with  $\text{ad } x$  because  $[h, x] = 0$  (since  $x \in \mathfrak{g}_0$ ). By the lemma,  $\text{Tr}(\text{ad } h \circ \text{ad } x) = 0$ , since  $\text{ad } h = \kappa(h, x)$ .

*Step 4:*  $\mathfrak{g}_0$  is nilpotent.

Pick  $x \in \mathfrak{g}_0$ . Then by step 2,  $x_s \in \mathfrak{h}$ , so  $\text{ad } x_s : \mathfrak{g}_0 \odot$  is a zero map and thus nilpotent.

So  $\text{ad } x_n$  is nilpotent, meaning that  $\text{ad } x$  is nilpotent. By Engel's theorem, this implies that  $\mathfrak{g}_0$  itself is nilpotent.

*Step 5:*  $\mathfrak{g}_0$  is abelian.

Suppose that  $I := [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$ . We have  $I \trianglelefteq \mathfrak{g}_0$ , and  $I$  is not nilpotent whereas  $\mathfrak{g}_0$  is.

By Lemma 3.3, we have  $I \cap Z(\mathfrak{g}_0) \neq 0$ , so pick  $x$  in the intersection. Note that  $\kappa(\mathfrak{h}, I) = \kappa(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0])$ , which by associativity equals  $\kappa([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = 0$ .

By step 3, we have  $\mathfrak{h} \cap I = 0$ . By step 2,  $x \neq x_s$ , and thus  $x_n \neq 0$ . But we also have  $x \in Z(\mathfrak{g}_0)$ , so  $[x, \mathfrak{g}_0] = 0$  and  $\text{ad } x(\mathfrak{g}_0) \subseteq 0$ . By Proposition 4.2, this holds for  $x_s, x_n$  as well, which are both in the center. So for all  $y \in \mathfrak{g}_0$ ,  $\text{ad } y$  commutes with  $\text{ad } x_n$ , which is nilpotent.

By the lemma, this implies that  $0 = \text{Tr}(\text{ad } y \circ \text{ad } x_n) = \kappa(x_n, y)$  for all  $y \in \mathfrak{g}_0$ . So  $x_n = 0$ .

*Step 6:* Suppose  $\mathfrak{g}_0 \not\subseteq \mathfrak{h}$ . By step 2, there exists an  $x \in \mathfrak{g}_0$  such that  $x \notin \mathfrak{h}$ , where  $x_n \neq 0$ . By step 5,  $[x_n, y] = 0$  for all  $y \in \mathfrak{g}_0$ . Then  $\text{ad } x$  (which is nilpotent) commutes with  $\text{ad } y$ . By the lemma,  $0 = \kappa(x_n, y)$  for all  $y \in \mathfrak{g}_0$ , and thus  $x_n = 0$ .  $\square$

Main idea: Choose a maximal toral subalgebra to get a nice root space decomposition, and so it coincides with  $\mathfrak{g}_0$ .

Corollary:  $\kappa|_{\mathfrak{g}}$  is nondegenerate.

Thus for all  $\alpha \in \mathfrak{h}^*$ , there exists a unique  $t_\alpha \in \mathfrak{h}$  such that  $\alpha = \kappa(t_\alpha, \cdot) : \mathfrak{h} \rightarrow \mathbb{C}$ .

In other words, there is an identification

$$\begin{aligned} \mathfrak{h} &\xrightarrow{1-1} \mathfrak{h}^* \\ h &\mapsto \kappa(h, \cdot) \\ t_\alpha &\leftarrow \alpha. \end{aligned}$$

Definition: A subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  is a *Cartan subalgebra* if  $\mathfrak{h}$  is nilpotent and

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \ni [x, h] \subseteq \mathfrak{h}\}.$$

Note that  $N_{\mathfrak{g}}(\mathfrak{h})$  is the largest subalgebra of  $\mathfrak{g}$  in which  $\mathfrak{h}$  is an ideal.

Remark: If  $\mathfrak{g}$  is semisimple and finite dimensional with  $\text{char}(F) = 0$ , we will have a correspondence:

$$\{\text{CSAs of } \mathfrak{g}\} \iff \{\text{maximal toral subalgebras of } \mathfrak{g}\}.$$

Maximal toral subalgebras advantages over Cartan subalgebra definition:

- Yields the finest root space decomposition
- $\mathfrak{h}^* = \mathfrak{h}$ , Weyl group?
- Existence is easy compared to CSAs

On the other hand, CSA advantages:

- All CSAs are conjugate under  $G$  (some group to be defined)
- The dimensions of all CSAs are the same, giving a well-defined notion of dimension ( $\text{rank } \mathfrak{g} = \dim \mathfrak{h}$ ).

## 17.2 8.3: Orthogonality Properties

From now on,  $\mathfrak{h}$  will be a maximal toral subalgebra.

Proposition: Let  $\alpha \in \Phi$ . Then

- a.  $\Phi$  spans  $\mathfrak{h}^*$
- b.  $-\alpha \in \Phi$
- c.  $\forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ , we have  $[x, y] = \kappa(x, y)t_\alpha$
- d.  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$  (let the nonzero scalar be  $\lambda$ )
- e.  $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$ .
- f. For any nonzero  $e_\alpha \in \mathfrak{g}_\alpha$ , there exists a unique  $f_\alpha \in \mathfrak{g}_{-\alpha}$  such that  $[e_\alpha, f_\alpha] = h_\alpha := \frac{\lambda}{\kappa(t_\alpha, t_\alpha)}t_\alpha$ .  
Moreover,  $\langle e_\alpha, f_\alpha, h_\alpha \rangle = \mathfrak{sl}(2, \mathbb{C})$ .

## 18 Wednesday September 25

Today: Properties of the root space when the toral subalgebra is maximal.

Last time: We have  $\mathfrak{g} = \mathfrak{g} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$  where  $\kappa|_{\mathfrak{h}}$  is nondegenerate. We also have a correspondence

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}^* h \mapsto \kappa(\mathfrak{h}, \cdot) \\ t_\alpha &\leftarrow \alpha := \kappa(t_\alpha, \cdot). \end{aligned}$$

### 18.1 Orthogonality Properties

Proposition: Let  $\alpha \in \Phi$ . Then:

- a.  $\Phi$  spans  $\mathfrak{h}^*$
- b.  $-\alpha \in \Phi$

- c.  $[x, y] = \kappa(x, y)t_\alpha$  for all  $x \in \mathfrak{g}_\alpha$  and  $y \in \mathfrak{g}_{-\alpha}$
- d.  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$
- e.  $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$
- f. For each nonzero  $e_\alpha \in \mathfrak{g}_\alpha$ , there exists a unique  $f_\alpha \in \mathfrak{g}_{-\alpha}$  such that  $[e_\alpha, f_\alpha] = h_\alpha := \frac{2}{\kappa(t_\alpha, t_\alpha)}t_\alpha$ .  
Moreover,  $\langle e_\alpha, t_\alpha, h_\alpha \rangle \cong \mathfrak{sl}(2, \mathbb{C})$ .

Proof of (a): We want to show that  $h \in \mathfrak{h}$  implies that if  $\alpha(h) = 0$  for all  $\alpha \in \Phi$ , then  $h = 0$ .

Take  $x \in \mathfrak{g}_\alpha$ . Then  $[h, x] = \alpha(h)x = 0$ . So  $[\mathfrak{h}, \mathfrak{g}] = 0$  because  $\mathfrak{h}$  is abelian. But then  $[h, \mathfrak{g}] = 0$ , or  $h \in Z(\mathfrak{g}) = 0$  since  $\mathfrak{g}$  is semisimple.

Proof of (b): By Proposition 8.1c, we have  $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$  for all  $\beta \neq -\alpha$ .

If  $-\alpha \notin \Phi$ , then  $\mathfrak{g}_{-\alpha} = 0$ . So  $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$  by the non-degeneracy of  $\kappa$ .

Proof of (c): For all  $h \in \mathfrak{h}$ , we have

$$\begin{aligned}
 \kappa(h, [x, y]) &= \kappa([h, x], y) \\
 &= \kappa(\alpha(h)x, y) \\
 &= \kappa(t_\alpha, h)\kappa(x, y) \\
 &= \kappa(\kappa(x, y)t_\alpha, h) \\
 &= \kappa(h, \kappa(x, y)t_\alpha).
 \end{aligned}$$

which implies that  $\kappa(h, [x, y]) - \kappa(x, y)t_\alpha = 0$ , which forces the second argument to be zero by non-degeneracy.

Proof of (d): We will show that (d) implies (c), i.e.  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}t_\alpha$ .

We want to show  $\kappa(x, y)$  is not always zero.

Pick any nonzero  $x \in \mathfrak{g}_\alpha$ . Then  $\kappa(x, \mathfrak{g}_\beta) = 0$  for all  $\beta \neq -\alpha$ . If  $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$ , then  $\kappa(x, \mathfrak{g}) = 0$ . By non-degeneracy, this forces  $x = 0$ .

Proof of (e): We will skip this for now, and revisit with methods from later sections that make this proof simpler.

Proof of (f): Let  $e_\alpha \neq 0$  in  $\mathfrak{g}_\alpha$ . Then there exists a  $y \in \mathfrak{g}_{-\alpha}$  such that  $\kappa(e_\alpha, y) \neq 0$ . Set  $f_\alpha \in \mathfrak{g}_{-\alpha}$  such that  $\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$ .

By (c), we have

$$\begin{aligned}
 [e_\alpha, f_\alpha] &= \kappa(e_\alpha, t_\alpha)t_\alpha \\
 &= \frac{2}{\kappa(t_\alpha, t_\alpha)}t_\alpha \\
 &= h_\alpha.
 \end{aligned}$$

and

$$\begin{aligned}
[h_\alpha, e_\alpha] &= \frac{2}{\kappa(t_\alpha, t_\alpha)} [t_\alpha, e_\alpha] \\
&= \frac{2}{\kappa(t_\alpha, t_\alpha)} \alpha(t_\alpha) e_\alpha \\
&= 2e_\alpha.
\end{aligned}$$

and similarly  $[h_\alpha, f_\alpha] = -2f_\alpha$ .

Definition:

Let  $\mathfrak{sl}(2, \alpha) = \langle e_\alpha, f_\alpha, h_\alpha \rangle$  as in (f). A priori, this depends on a choice of  $e_\alpha \neq 0$ . We will show that this only depends on  $\alpha$ .

## 18.2 Orthogonality/Integrality Properties

Proposition: Let  $\alpha \in \Phi$ . Then:

- a.  $\dim \mathfrak{g}_\alpha = 1$ . (Note that in general,  $\dim \mathfrak{g}_0 = \dim \mathfrak{h} \geq 1$ )
- b.  $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$
- c. If  $\beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ , then  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ .
- d. If  $\beta \neq -\alpha \in \Phi$ , then let  $p, q \in \mathbb{Z}$  be the largest such that  $\beta - p\alpha$  and  $\beta + q\alpha$  are both in  $\Phi$ . Then  $\beta + i\alpha \in \Phi$  for every  $-p \leq i \leq q$ , and

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = p - q \in \mathbb{Z}.$$

Proof of (a) and (b):

Let  $M = \mathfrak{h} \oplus \left( \bigoplus_{c \neq 0} \mathfrak{g}_{c\alpha} \right) \leq \mathfrak{g}$  as a subspace. By a routine check,  $M$  is an  $\mathfrak{sl}(2, \alpha)$  submodule of  $\mathfrak{g}$ . Recall that  $M = \bigoplus_{\lambda \in \mathbb{Z}} L(m)^{\oplus \mu_m}$  as a direct sum of vector spaces. Applying Weyl's theorem, we also have  $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\oplus \mu_m}$  as a direct sum of (irreducible?) modules.

For  $\mathfrak{h}$ , if we have  $[h_\alpha, h] = 0$  for all  $h \in \mathfrak{h}$ , then  $h \in M_0$ . For  $\mathfrak{g}_{c\alpha}$ ,  $[h_\alpha, x] = c\alpha(h_\alpha)x$  for all  $x \in \mathfrak{g}_{c\alpha}$ . But this equals  $2cx$ . So this implies that  $\mathfrak{g}_{c\alpha} \subseteq M_{2c}$ .

Thus  $2c \in \mathbb{Z}$ , and thus  $c \in \frac{1}{2}\mathbb{Z}$ , and  $M_0 = \mathfrak{h}$ .

We then have  $\dim M_0 = \sum_{m \in 2\mathbb{Z}} \mu_m$ . So write  $h = \mathbb{C}t_\alpha \oplus \ker \alpha$  as vector spaces. Consider the action  $\mathfrak{sl}(2, \alpha) \curvearrowright \ker \alpha$ , which is trivial since  $h \in \ker \alpha$ . We  $[h_\alpha, h] = 0$ ,  $[e_\alpha, h] = -\alpha(h)e_\alpha = 0$  since  $h \in \ker \alpha$ , and similarly  $[f_\alpha, h] = 0$ .

Thus  $\ker \alpha = L(0)^{\oplus \dim \mathfrak{h} - 1}$ . Moreover,  $\mathfrak{sl}(2, \alpha) = L(2) = \text{span}(e_\alpha, t_\alpha, f_\alpha)^T$ . But this forces the case that there is no other summand of the form  $L(k)$  for  $k$  even in  $M$ .

Then  $\mathfrak{g}_{2\alpha} \subseteq M_4$ , which must be zero. So  $2\alpha \notin \Phi$ , so  $2\alpha$  is not a root. ("Twice a root is never a root")

So  $\frac{1}{2}\alpha \notin \Phi$ , otherwise we could apply this argument to conclude that  $\alpha$  is not a root and reach a contradiction. Thus  $M_1 = 0$ , since  $c \neq \frac{1}{2}$  implies that there is not summand of the form  $L(k)$  for  $k$  odd in  $M$ . But this forces  $M = \mathfrak{h} \oplus \mathfrak{sl}(2, \alpha)$ .

Motto: reduce the complexity by using the  $\mathfrak{sl}(2)$  module structure and its representation theory!

## 19 Friday September 27

Last time, we saw  $\Phi \subseteq \mathfrak{h}^* = \{\alpha : \mathfrak{h} \rightarrow \mathbb{C}\}$ .

Suppose  $\mathfrak{g}$  is semisimple and  $\mathfrak{h}$  is a maximal toral subalgebra and take  $F = \mathbb{C}$ .

We have the following propositions:

- a.  $\dim \mathfrak{g}_\alpha = 1 \ \forall \alpha \in \Phi$
- b.  $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$ , and  $2\alpha \notin \Phi$  where  $c\alpha : \mathfrak{h} \rightarrow \mathbb{C}, h \mapsto c \curvearrowright \alpha(h)$ . Moreover,  $M = \mathfrak{h} \oplus \left( \bigoplus_{c \neq 0} \mathfrak{g}_{c\alpha} \right)$
- c. If  $\alpha, \beta \in \Phi$  and  $\beta \neq -\alpha$  Let  $p, q \in \mathbb{Z}$  be the largest such that  $\beta - p\alpha$  and  $\beta + q\alpha$  are in  $\Phi$ . Moreover,  $\beta(h_\alpha) = \kappa(t_\beta, t_\alpha) = p - q \in \mathbb{Z}$ .

Proof of (c):

Set  $M = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$ , which is an  $\mathfrak{sl}(2, \alpha)$  module. By (a), we have  $\dim \mathfrak{g}_{\beta+i\alpha} = 1 \iff \beta+i\alpha \in \Phi$ . But for all  $x \in \mathfrak{g}_{\beta+i\alpha}$ , we have  $[h, x] = (\beta+i\alpha)(h)x$  for all  $h \in \mathfrak{h}$ . But then  $[h_\alpha, x] = (\beta(h_\alpha) + i\alpha(h_\alpha))x = (\beta(h_\alpha) + 2i)x$

Then  $\mathfrak{g}_{\beta+i\alpha} \subseteq M_{\beta(h_\alpha)+2i}$ , so  $\beta(h_\alpha) \in \mathbb{Z}$ .

Moreover,  $\text{Wt}(M) = 2\mathbb{Z}$  or  $2\mathbb{Z} + 1$ , and in particular  $\dim M_0 + \dim M_1 = 1$ .

Thus  $M$  is irreducible, and  $M \cong L(m)$  for some  $m \in \mathbb{Z}_{\geq 0}$ . Moreover,  $\text{Wt}(M) = \{m, m-2, \dots, -m\}$ , and  $\dim \mathfrak{g}_{\beta+i\alpha} = 1$  for all  $i \in [-p, q]$ . Thus  $\beta + i\alpha \in \Phi$ .

Proof of 8.3(e):  $\alpha(t_\alpha) \neq 0$ . The claim is that for all  $\beta \in \Phi$ , there exists an  $r \in \mathbb{Q}$  such that  $\beta(h) = r\alpha(h)$  for all  $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ .

There are two cases: if  $\beta = -\alpha$ , then we're done by the previous argument.

Otherwise,  $\beta \neq -\alpha$ . Take  $M = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$ .

Then,

$$\begin{aligned} \text{Tr}_M(\text{ad } h) &= \sum_i \text{Tr}_M((\text{ad } e_i \circ \text{ad } f_i) - (\text{ad } f_i \circ \text{ad } e_i)) = \sum_i \text{Tr}_{\mathfrak{g}_{\beta+i\alpha}}(\text{ad } h) \\ &= \sum_i (\beta + i\alpha)(h) \dim \mathfrak{g}_{\beta+i\alpha} \\ &= \sum_i \dim \mathfrak{g}_{\beta+i\alpha} \beta(h) + \sum_i i \dim(\mathfrak{g}_{\beta+i\alpha}) \\ &\implies \beta(h) = \frac{-\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}{\sum_i \dim \mathfrak{g}_{\beta+i\alpha}} \alpha(h). \end{aligned}$$

Now consider the killing form  $\kappa(t_\beta, t_\alpha) = \beta(t_\alpha) = r\alpha(t_\alpha)$ , where the last equality is what we are claiming.

Suppose that  $\alpha(t_\alpha) = 0$ . Then  $\kappa(t_\beta, t_\alpha) = 0$  for all  $\beta \in \Phi$ . By the non-degeneracy of  $\kappa$ , we have  $t_\alpha = 0$  and thus  $\alpha = 0$ .

## 19.1 Summary

We have  $\mathfrak{g}$  semisimple, finite dimensional, and  $\mathfrak{h}$  a maximal toral subalgebra (i.e. the Cartan subalgebra). This implies that  $\kappa$  is nondegenerate, and we have a correspondence

$$\begin{aligned}\mathfrak{h} &\Longleftrightarrow \mathfrak{h}^\vee \\ h &\mapsto \kappa(h, \cdot) \\ t_\alpha &\leftarrow \alpha.\end{aligned}$$

This gives a symmetric bilinear form  $(\cdot, \cdot) : \mathfrak{h}^\vee \rightarrow \mathfrak{h}^\vee$ .

For  $\alpha \in \Phi$ , define its *coroot*  $\alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha$ .

Note that  $(\cdot)^\vee$  is not linear: note that

$$(2\alpha)^\vee = \frac{2}{(2\alpha, 2\alpha)}2\alpha = \frac{\alpha}{(\alpha, \alpha)} = \frac{\alpha^\vee}{2}.$$

Assume that  $\Phi = \{\alpha_i\}$ . Define  $E_{\mathbb{Q}} = \bigoplus_{i=1}^{\ell} \mathbb{Q}_{\alpha_i}$ , and  $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$ .

Lemma: If  $\alpha, \beta \in \Phi$ , then

- a.  $(\beta, \alpha) \in \mathbb{Q}$ ,
- b.  $(\cdot, \cdot)$  on  $E_{\mathbb{Q}}$  is positive definite, i.e.  $x \neq 0 \implies (x, x) > 0$ .

An immediate consequence of (b) is that  $(\cdot, \cdot)$  on  $E$  is an inner product.

Proof: For all  $\lambda, \mu \in \mathfrak{h}^\vee$ , we have

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \text{Tr}_{\mathfrak{g}}(\text{ad } t_\lambda \circ \text{ad } t_\mu) = \text{Tr}_{\mathfrak{g}}(\dots) + \sum_{\alpha \in \Phi} \text{Tr}_{\mathfrak{g}_\alpha}(\dots) = 0 + \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) = \kappa(t_\lambda, t_\mu) = \kappa(t_\mu, t_\lambda) = (\mu, \lambda)$$

So pick  $\lambda = \mu = \alpha \in \Phi$ . Then  $(\alpha, \alpha) = \sum_{\beta \in \Phi} (\beta, \alpha)^2$ .

Then

$$\frac{1}{(\alpha, \alpha)} = \sum_{\beta \in \Phi} \left( \frac{(\beta, \alpha^\vee)}{2} \right)^2.$$

where  $(\beta, \alpha^\vee) = \dots = \beta(h_\alpha) \in \mathbb{Z}$ .

This means that  $(\alpha, \alpha) \in \mathbb{Q}_{>0}$ .

Summary of properties proved:

Let  $\alpha, \beta \in \Phi$ . Then

1.  $0 \notin \Phi$  and  $\Phi$  spans  $E$
2.  $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$
3.  $\beta - (\beta, \alpha^\vee)\alpha \in \Phi$
4.  $(\beta, \alpha^\vee) \in \mathbb{Z}$

Thus the assignment  $(\mathfrak{g}, \mathfrak{h}) \mapsto (\Phi, E)$  defines a **root system**. This only works when  $\mathfrak{g}$  is semisimple and  $\mathfrak{h}$  is maximal toral.

Proof of (3):

We computed  $(\beta, \alpha^\vee) = p - q$ . Then  $-p \leq -(\beta, \alpha^\vee) = q - p \leq q$ . So this must be something on the root stream.

## 20 Monday September 30

Last time: Let  $\mathfrak{g}$  be finite dimensional and  $\mathfrak{h}$  a maximal toral subalgebra.

Then  $(\Phi, E)$  is a *root system*, and we obtain a bilinear product

$$\begin{aligned} \langle \cdot, \cdot \rangle : E \times E &\rightarrow \mathbb{R} \\ (\alpha, \beta) &\mapsto \kappa(t_\alpha, t_\beta). \end{aligned}$$

Examples:  $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$  and  $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$  where

Todo: Insert clip image h1, h2

$$\begin{aligned} \alpha_1 : \mathfrak{h} &\rightarrow \mathbb{C} \\ h_1 &\mapsto 2h_2 \mapsto -1. \end{aligned}$$

$$\begin{aligned} \alpha_2 : \mathfrak{h} &\rightarrow \mathbb{C} \\ h_1 &\mapsto -1h_2 \mapsto 2. \end{aligned}$$

To find  $t_{\alpha_i}$ , we need to look at  $\kappa|_{\mathfrak{h}}$ .

Todo: Insert phone image

Since we only need the trace, this suffice, and we find

$$\begin{bmatrix} h_1 & h_2 \\ h_1 & 12 & -6 \\ h_2 & -6 & 12 \end{bmatrix}.$$

We then get  $t_{\alpha_1} = \frac{h_1}{6}$  and  $t_{\alpha_2} = \frac{h_2}{6}$ . Moreover



$$\begin{aligned}
\langle \alpha_1, \alpha_1 \rangle &= \kappa(t_{\alpha_1}, t_{\alpha_1}) = \frac{1}{3} \in \mathbb{Q} \\
\langle \alpha_1, \alpha_1 \rangle &= \frac{1}{3} \\
\langle \alpha_1, \alpha_2 \rangle &= -\frac{1}{6} \\
\langle \alpha_1, \alpha_2 \vee \rangle &= \frac{2\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_2, \alpha_2 \rangle} = -1 \in \mathbb{Z} \quad \langle \alpha_i, \alpha_i \vee \rangle = \frac{2\langle \alpha_i, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2 \in \mathbb{Z}.
\end{aligned}$$

This leads to a nice fact: the matrix  $\langle \alpha_i, \alpha_j \vee \rangle$  has  $\mathbb{Z}$  entries, and this is called the *Cartan matrix*.

## 20.1 Ch.3: Root Systems

### 20.1.1 Axiomatics: Reflections

Fix a Euclidean space  $E$ .

Definition: A *hyperplane* in  $E$  is a subspace of codimension 1. A *reflection* in  $E$  is an element  $s \in \mathfrak{gl}(E)$  such that

$$\{E^s := \{x \in E \mid sx = x\}\} \text{ is a hyperplane } H \text{ and } s(x) = -x \quad \forall x \in E \mid (x, H) = 0\}$$

For nonzero  $\alpha \in E$ , its reflection is

$$\begin{aligned}
S_\alpha : E &\rightarrow E \\
\beta &\mapsto \beta - \langle \beta, \alpha \vee \rangle \alpha.
\end{aligned}$$

with respect to  $H_\alpha = \{x \in E \mid \langle x, \alpha \rangle = 0\}$ , where  $\alpha \vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ .

Lemma: Let  $\Phi \subseteq E$  be finite such that  $S_\alpha(\Phi) = \Phi$  for all  $\alpha \in \Phi$ .

Suppose that  $S \in \mathfrak{gl}(E)$  satisfies

1.  $S(\Phi) = \Phi$ ,
2.  $S(h) = h$  for all  $h \in H$ , and
3.  $S(\alpha) = -\alpha$  for some  $\alpha \in \Phi$ ,

then  $S = S_\alpha$ , i.e. this uniquely characterizes  $S$

Proof:

Let  $\tau = S \circ S_\alpha$ . Then  $\tau(\Phi) = \Phi$  and  $\tau(\alpha) = \alpha$ . This  $\tau \curvearrowright \mathbb{R}\alpha$  by 1, and similarly  $\tau \curvearrowright E/\mathbb{R}\alpha$  by 1 by picking a representative in  $H$ . Moreover, all eigenvalues of  $\tau$  are 1. So the minimal polynomial of  $\tau$  divides  $(t - 1)^{\dim E}$ .

We want to show that  $\tau \mid (t - 1)^N$  for some large  $N$ , which forces  $\tau \mid \gcd((t - 1)^{\dim E}, t^N - 1) = 1$ . For any  $\beta \in \Phi$  and  $k > |\Phi|$ , not all vectors  $\beta, \tau(\beta), \dots, \tau^k(\beta)$ . So  $\beta = \tau^{k_\beta}(\beta)$  for some  $k_\beta$  depending on  $\beta$  (noting that  $\tau$  is invertible.)

Multiplying all of these  $k_\beta$ s together, we can get some  $k_\Phi$  that is larger than  $|\Phi|$ , and so  $\beta = \tau^{k_\Phi}$  for all  $\beta \in \Phi$ . But then  $\tau^{k_\Phi} = 1$  in  $\mathfrak{gl}(E)$ .

### 20.1.2 Root Systems

Definition: A subset  $\Phi$  of  $E$  a Euclidean space is called a *root system* iff

1.  $|\Phi| < \infty, 0 \notin \Phi$ , and  $E = \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$
2.  $\alpha \in \Phi \implies \mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$
3.  $\alpha \in \Phi \implies S_\alpha(\Phi) = \Phi$
4.  $\alpha, \beta \in \Phi \implies \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ .

Definition: The *rank* of a root system is the dimension on  $E$ .

Definition: The *Weyl Group* of  $\Phi$  is defined as

$$W = \langle S_\alpha \mid \alpha \in \Phi \rangle \subseteq \mathfrak{gl}(E)$$

Note that  $W \hookrightarrow \Sigma_{|\Phi|}$ , a permutation group of size  $|\Phi|$ .

Lemma: If  $g \in \mathfrak{gl}(E)$  and  $g(\Phi) = \Phi$ , then for all  $\alpha, \beta \in \Phi$ , we have

$$\begin{aligned} g s_\alpha g^{-1} &= s_{g(\alpha)}, \\ \langle \beta, \alpha^\vee \rangle &= \langle g(\beta), g(\alpha)^\vee \rangle, \\ \langle \beta, \alpha^\vee \rangle &= \langle w(\beta), w(\alpha)^\vee \rangle \quad \forall w \in W. \end{aligned}$$

Proof: Check 1-3 in Lemma 9.1.

Proof of 1: We have

$$g s_\alpha g^{-1}(g(\beta)) = g s_\alpha(\beta) \in g(\Phi) = \Phi \quad \forall \beta \in \Phi,$$

Proof of 2: We have

$$\{g(\beta) \mid \beta \in \Phi\} = \Phi \implies g s_\alpha g^{-1}(\Phi) = \Phi \quad \forall h \in g H_\alpha$$

and so  $g s_\alpha g^{-1}(h) = g g^{-1}(h) = h$ , so  $h$  is a fixed point of this map.

Proof of 3: We have  $g s_\alpha g^{-1}(g(\alpha)) = g s_\alpha(\alpha) = -g(\alpha)$ , and so  $g s_\alpha g^{-1} = s_{g(\alpha)}$  by Lemma 9.1.

Finally, we have

$$\begin{aligned} g s_\alpha g^{-1}(g(\beta)) &= g(s_\alpha(\beta)) = g(\beta - \langle \beta, \alpha^\vee \rangle \alpha) = g(\beta) - \langle \beta, \alpha^\vee \rangle g(\alpha) \\ &= \\ s_{g(\alpha)} &= g(\beta) - \langle g(\beta), g(\alpha)^\vee \rangle g(\alpha). \end{aligned}$$

## 21 Wednesday October 2

Recall from last time:

1.  $|\Phi| < \infty$  and  $\Phi$  spans  $E$ , where  $0 \notin \Phi$
2. If  $\alpha \in \Phi$ , then  $C\alpha \cap \Phi = \{\pm\alpha\}$
3.  $\alpha \in \Phi$ , then  $S_\alpha(\Phi) = \Phi$ .
4. If  $\alpha, \beta \in \Phi$ , then  $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$  where  $(E, \langle \cdot, \cdot \rangle)$  is Euclidean and

$$S_\alpha : E \rightarrow E$$

$$\beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha, \quad \alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$

Examples:

In Rank 1:

1. Prop 2 implies  $\Phi = \{\pm\alpha\}$
2. Prop 1 implies  $E = \mathbb{R}\alpha$
3. Prop 3:  $S_\alpha(\alpha) = -\alpha$
4. Prop 4 implies  $\langle \pm\alpha, \pm\alpha \rangle = \pm \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \pm 2$

Rank 1 Diagram: Todo: Insert phone image

In Rank 2: Todo: Insert phone image

Exercise:

- Show that  $\text{ord}(S_\alpha, S_\beta) = 2, 3, 4, 6$  for types  $A_1 \times A_1, B_2, G_2$ .
- Show that  $W(A_2) \cong \mathbb{Z}_3$  and  $W(B_2) \cong D_8$ .

### 21.1 Pairs of Roots

Lemma: Let  $\alpha, \beta \in \Phi$  where  $\beta \neq \pm\alpha$ , then

1.  $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$  Moreover, assuming  $|\beta| \geq |\alpha|$ , we have the following table Todo:  
Insert table
2. If  $\langle \alpha, \beta \rangle > 0$ , then  $\alpha - \beta \in \Phi$ . Similarly, if  $\langle \alpha, \beta \rangle < 0$ , then  $\alpha + \beta \in \Phi$ .
3. Any root string is unbroken and has length greater than 4.

Proof of (1):

By the Law of Cosines, we can write  $x := \langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle = 4 \cos^2(\theta) \in \mathbb{Z}$ . This restricts the possibilities to  $x \leq 4$ . But  $x = 4 \iff \alpha = c\beta$ , i.e.  $\theta = 0$ , but we are assuming that  $\alpha \neq \pm\beta$ , so this can not happen.

Proof of (2):

Since  $\langle \alpha, \beta \rangle > 0$  and  $|\beta| \geq |\alpha|$ , then  $\langle \alpha, \beta^\vee \rangle = 1$ . But then  $S_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta \in \Phi$  by Prop 3. So this is equal to  $\alpha - \beta$ .

A similar argument works for  $|\beta| \leq |\alpha|$ .

Proof of (3): Let  $p, q$  be the largest integers such that  $b - p\alpha, b + q\alpha \in \Phi$  respectively. Suppose that the root stream between these two is broken somewhere, say  $\beta + s\alpha \in \Phi$  and  $\beta + (s + 1)\alpha \notin \Phi$  by counting up from  $\beta - p\alpha$ . Similarly, there is some  $t$  counting down from  $b + q\alpha$  then  $\beta + t\alpha \in \Phi$  but  $\beta + (t - 1)\alpha \notin \Phi$ . In particular,  $s < t$ . From (2), we have  $\langle \alpha, \beta + s\alpha \rangle \geq 0, \langle \alpha, \beta + t\alpha \rangle \leq 0$ .

We have

$$\langle \alpha, \beta \rangle + t\langle \alpha, \alpha \rangle = \langle \alpha, \beta + t\alpha \rangle \leq 0 \leq \langle \alpha, \beta + s\alpha \rangle = \langle \alpha, \beta \rangle + s\langle \alpha, \alpha \rangle$$

where we know that  $\langle \alpha, \alpha \rangle > 0$ .

Since  $S_\alpha(\Phi) = \Phi$  and these  $S_\alpha(\beta + i\alpha) = \beta - \mathbb{Z}\alpha$ , we find that reflections permute the root string. We then find that  $p = \langle \beta, \alpha^\vee \rangle + q$ , and so  $\langle \beta, \alpha^\vee \rangle = p - q \in [-3, 3]$ .

## 21.2 Chapter 10: Simple Roots and Weyl Groups

Definition: A *base* of a root system  $\Phi$  is a subset  $\Pi \subseteq \Phi$  such that

1.  $\Pi$  is a basis for the underlying vector space  $E$ , and
2. Each  $\beta \in \Phi$  can be written as  $\beta = \sum_{\alpha \in \Pi} \kappa_\alpha^\beta \alpha$  where all of the coefficients  $\kappa_\alpha^\beta$  all have the same sign.

The roots in  $\Pi$  are called *simple*. A root  $\beta$  is *positive* (resp. *negative*) if the  $\kappa_\alpha^\beta \geq 0$  for all  $\beta \in \Phi^+$  (resp  $\leq 0$  in  $\Phi^-$ ). The *height* of a  $\beta$  is the sum of the coefficients.  $\Pi$  defines a partial order on  $E$  where  $\mu \leq \lambda \iff \lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$ .

Note that this is defined on the roots themselves, and can then be extended to all of  $E$ .

Todo: Insert phone image

## 22 Monday October 7

Last time:

Lemma 10.2

- a. ?
- b.  $\alpha \in \Pi \implies S_\alpha \curvearrowright \Phi^+ \setminus \{\alpha\}$  by permutation
- c.  $\alpha_i \in \Pi$  and  $S_{\alpha_1}, \dots, S_{\alpha_{j-1}}(\alpha_j) \in \Phi^-$  then  $S_{\alpha_1} \cdots S_{\alpha_j} = S_{\alpha_1} \cdots S_{\alpha_{j-1}}$  for some  $t$ , where the former has  $j$  terms and the latter has  $j - 2$  terms.

Proof of (a): ?

Proof of (b):

Suppose towards a contradiction that  $w(\alpha_j) \in \Phi^+$ . Then consider  $WS(\alpha_j) = -W(\alpha_j) \in \Phi^-$ .

By Lemma 10.2(c), we have  $W = S_{\alpha_1} \cdots S_{\alpha_{j-1}} S_{\alpha_{j+1}} \cdots S_{\alpha_{j-1}} S_{\alpha_j}$ , where this is  $j - 1$  terms. So  $w = S_{\alpha_1} \cdots S_{\alpha_j}$  is not reduced.

## 22.1 Weyl Groups

Recall that the *chambers* are given by the connected component of  $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ .

Theorem: Fix  $\Pi$  of  $\Phi$ . Then

- a.  $W \curvearrowright \{\text{chambers}\}$  transitively
- b.  $W \curvearrowright \{\text{bases}\}$  transitively
- c.  $\forall \alpha \in \Phi, \exists w \in W \ni w(\alpha) \in \Pi$
- d.  $W := \{S_\alpha \mid \alpha \in \Phi\} = \langle S_\alpha \mid \alpha \in \Pi \rangle := W_0$
- e.  $W \curvearrowright \{\text{bases}\}$  simply transitively, i.e.  $w(\Pi) = \Pi \implies w = e$ .

I.e. we can describe the Weyl group using only simple reflections

Proof: We will prove (a) – (c) for  $W_0$ .

Proof of (a): Recall the fundamental chamber,  $C(\Pi) = \{x \in E \mid (x, \alpha) > 0 \ \forall \alpha \in \Pi\}$ . We want to show that any chamber  $C$  is equal to  $wC(\Pi)$ .

Pick  $\gamma \in C$  and  $g \in W_0$  such that  $(g(\gamma), \rho) = \max \{(w(\gamma), \rho) \mid w \in W_0\}$ , which exists because  $W_0$  is a finite group.

For all  $\alpha \in \Pi$ ,  $S_\alpha g \in W_0$  and so by maximality we have

$$\begin{aligned} (g(\gamma), \rho) &\geq (S_\alpha g(\gamma), \rho) \\ &= (g(\gamma), S_\alpha(\rho)) \\ &= (g(\gamma), \rho - \alpha) \\ &= (g(\gamma), \rho) - (g(\gamma), \alpha). \end{aligned}$$

and so  $(g(\gamma), \alpha) \geq 0$ , because this can never be an equality since  $\gamma \in C$ . Thus  $g(\gamma) \in C(\Pi)$ .

Proof of (b):

This holds because there is a correspondence between  $\{C(\Pi)\} \iff \{\text{bases}\}$ .

Proof of (c):

It suffices to show that  $\alpha \in \Phi$  lies in some base  $\Pi' = W(\Pi)$ . Note that  $\beta \neq \alpha \implies H_\beta \neq H_\alpha$ , and so we can pick a  $\gamma \in H_\alpha \cap H_\beta^c$  for every  $\beta \in \Phi \setminus \pm\alpha$ . Since  $\langle \gamma, \alpha \rangle = 0$  but  $\langle \gamma, \beta \rangle \neq 0$  for all  $\beta \neq \pm\alpha$ , we can choose some  $\varepsilon > 0$  such that  $|\langle \gamma', \beta \rangle| > \varepsilon$  for every  $\beta \neq \pm\alpha$ . Then  $\gamma' \in C(\Pi')$  and thus  $\alpha \in \Pi'$ .

Proof of (d):

By definition,  $W_0 \subseteq W$ , so we need to show the reverse containment. For all  $\alpha \in \Phi$ , we want to show  $S_\alpha \in W_0$ . By (c), there exists a  $w \in W_0$  such that  $w(\alpha) := \beta \in \Pi$ . Then  $S_\beta = S_{w(\alpha)} = w S_\alpha w^{-1}$ . So  $S_\alpha = w^{-1} S_\beta w$ , where each term is in  $W_0$ , so the whole thing is in  $W_0$  as well.

Proof of (e):

Suppose  $W(\Pi) = \Pi$ . Let  $W = S_{\alpha_1} \cdots S_{\alpha_\ell}$  be a reduced expression, which exists by (d). By corollary 10.2b, we have  $W(\alpha_\ell \in \Phi^-)$ . But this forces  $w = e$ .  $\square$

Remarks:

By (d), there is a well-defined notion of *length* for  $w \in W$ . We will now show that  $\ell(w) = n(w) := \#N_w := \#\{\alpha \in \Phi^+ \ni W(\alpha) \in \Phi^-\}$ , i.e. the number of roots that get sent to a negative root.

## 23 Wednesday October 9

Last time:

We have the Weyl group  $W := \{S_\alpha \ni \alpha \in \Phi\} = \{S_\alpha \ni S_\alpha \in \Pi\}$ . If  $W \ni w = \prod i = 1^\ell W_{\alpha_i}$  is a product of simple reflections, then  $W$  is said to be *reduced* if  $\ell$  is the smallest among all such products. Call  $\ell(w)$  the length of  $W$  and let  $n(W) = \#N_W$ . By Corollary 10.2b,  $N_W = \{\alpha \in \Phi^+ \ni W(\alpha) \in \Phi^-\}$ , and if  $W = \prod S_{\alpha_i}$  is reduced, then  $w(\alpha_j) \in \Phi^-$ .

Lemma:  $\ell(w) = n(w)$ .

Proof: Done in class, but see Humphrey's.

### 23.1 Classification

#### 23.1.1 Cartan Matrix

Fix a base  $\Pi \subset \Phi$  of rank  $\ell$ .

Definition: Fix an order  $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$  of  $\Pi$ . Then the *Cartan matrix* is given by  $A_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \in \text{Mat}(\ell \times \ell, \mathbb{Z})$ .

Examples:

Facts:

- $A$  depends on the chosen ordering of  $\Pi$ .
- $A$  is independent of the choice of  $\Pi$ .
- $A$  is invertible.
- $A$  uniquely determines the root system (up to isomorphism). I.e., if  $A(\Phi) = A(\Phi')$  then there is an isomorphism  $E \xrightarrow{\phi} E$  on the underlying Euclidean space such that  $\phi(\Phi) = \Phi'$  and  $\langle \alpha, \beta^\vee \rangle = \langle \phi(\alpha), \phi(\beta)^\vee \rangle$  for all  $\alpha, \beta \in \Phi$ .

#### 23.1.2 Dynkin Diagrams

Recall from Lemma 9.4 that  $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$ .

Definition: Given a Cartan matrix  $A$ , its Coxeter diagram is an undirected multigraph  $\Gamma = (I, E)$  where  $I$  is a vertex set and the edge set is given by edges between vertices corresponding to  $i, j$  (where  $i \neq j$ ) with weight  $a_{ij}a_{ji}$ .

Examples:

Note that these diagrams don't encode which roots are longer, so we can decorate these diagrams with arrows to indicate this and obtain a partially-directed multigraph.

Definition: A *Dynkin diagram* is the partially-directed multigraph obtained from the Coxeter diagram by adding arrows on the double or triple edges between  $i, j$  precisely when  $|a_i| > |a_j|$ . (Note that this also occurs when  $|a_{ij}| < |a_{ji}|$ )

Definition: A non-empty root system is *irreducible* if  $\Phi \neq \Phi_1 \oplus \Phi_2$  for some nonempty root system  $\Phi_2$  where  $\alpha \in \Phi_1, \beta \in \Phi_2 \implies \langle \alpha, \beta \rangle = 0$ .

For example:  $\Phi(A_1 \times A_1)$  can be written as  $\Phi(A_1) \oplus \phi(A_1)$  since the off-diagonal entries were zero, so it is reducible.

Facts:

- a.  $\Phi$  is irreducible iff the Dynkin diagram is connected
- b.  $\Phi$  can be uniquely written as the union of irreducible root systems (where the multiplicity of each system appearing is well-defined)

Thus to classify root systems, it suffices to classify connected Dynkin diagrams.

Examples of Dynkin diagrams:

## 24 Friday October 11

Recall from last time the Dynkin diagrams. If  $\Phi$  is irreducible, then its diagram is one of the following:

Definition: A subset  $A = \{v_1, \dots, v_n\} \subseteq E$  is *admissible* iff

1.  $A$  is linearly independent.
2.  $\langle v_i, v_i \rangle = 1$  for all  $i$ , and  $\langle v_i, v_j \rangle \leq 0$  if  $i \neq j$ .
3.  $s_{ij} = 4\langle v_i, v_j \rangle^2 \in \{0, 1, 2, 3\}$  if  $i \neq j$ .

Define a graph  $\Gamma_A = (V_A, E_A)$  where  $V_A = A$  and  $E_A = \{s_{ij} \mid i \neq j\}$ . If  $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subseteq \Phi$  is a base, then  $A := \left\{ v_i = \frac{\alpha_i}{\sqrt{\langle \alpha_i, \alpha_i \rangle}} \right\}$ .

Lemma:

- a. If  $A$  is admissible, then  $\# \left\{ (v_i, v_j) \in E_A \mid 4\langle v_i, v_j \rangle^2 \neq 0 \right\} \leq |A| - 1$ , and  $\Gamma_A$  contains no graph cycles.
- b.  $\deg V_i \leq 3$  for all  $i$ .
- c. If  $\Gamma_A$  contains a path  $p_1 \rightarrow \dots \rightarrow p_t$ , then  $A' := \{p\} \cup A \setminus \{p_1, \dots, p_t\}$  where  $p := \sum p_i$ . Moreover,  $\Gamma_{A'}$  is obtained from  $\Gamma_A$  by contracting this path onto  $p$ .

Proof of theorem:

Assume the lemma holds. Let  $\Gamma$  be the Coxeter diagram of  $\Phi$ ; then  $\Phi$  is connected.

Case 1:  $\Gamma$  has a triple edge. But then both vertices on this edge have degree 3, so this is the maximal number of edges between them. But since  $\Gamma$  must be connected, this is everything.

Case 2:  $\Gamma$  has no triple edges but some double edge. We will first show that  $\Gamma$  has only one double edge.

Suppose otherwise; then  $\Gamma$  has at least two double edges occurring. Without loss of generality (e.g. by taking a subgraph), these are connected by a path of single edges. By the lemma, we can contract this path to get an admissible subset. But then there is a vertex of degree 4, contradicting  $\deg V_i \leq 3$  for all  $i$ .

Now we'll show that  $\Gamma$  has no branching point, i.e. a vertex of degree exactly 3. If this occurs, then a double edge is connected to such a vertex by a path. Contracting this path yields a vertex of degree 4, again a contradiction.

By these two statements,  $\Gamma$  has the general form:

$$\Gamma = v_1 \rightarrow \circ \rightarrow \cdots \rightarrow v_p \rightarrow \rightarrow w_q \rightarrow \circ \rightarrow \cdots \rightarrow w_1.$$

Let  $v = \sum iv_i$  and  $w = \sum iw_i$ , then  $\langle v, v \rangle = \frac{1}{2}p(p+1)$ , and  $\langle w, w \rangle = \frac{1}{2}q(q+1)$ . Note that  $\langle v_i, w_j \rangle = -1/\sqrt{2}$  if  $i = p$  and  $j = q$ , and 0 otherwise.

Thus  $\langle v, w \rangle = \cdots = \frac{1}{2}p^2q^2$ . By Cauchy-Schwarz, this is strictly less than  $\langle v, v \rangle \langle w, w \rangle = \frac{1}{4}p(p+1)q(q+1)$ . We then obtain  $(p-1)(q-1) < 2$ . Supposing wlog that  $p \geq q$ , we have either  $p = q = 2$ , in which case we get  $\circ \rightarrow \circ \rightarrow \rightarrow \circ \rightarrow \circ$ . Otherwise  $q = 1$ , and we get  $\circ \rightarrow \cdots \rightarrow \circ \rightarrow \rightarrow \circ$ .

Case 3:  $\Gamma$  has only single edges. We want to show  $\Gamma$  has only one branching point, i.e. a vertex of degree 3. If it has 2, we can contract the intermediate path to get a vertex of degree 4. So we have the following situation:

Define  $x = \sum ix_i, y = \sum iy_i, w = \sum iw_i$ , and  $\hat{w}, \hat{x}, \hat{y}$  to be their normalization. Then  $B = \{b_i\} := \{\hat{w}, \hat{x}, \hat{y}, z\}$  is orthonormal and linearly independent, so we can apply Gram-Schmidt. This yields a  $z' \neq 0$  such that

$$z = \sum \langle z, \hat{b}_i \rangle \hat{b}_i$$

In particular,  $\langle z, z' \rangle z' \neq 0z'$ , otherwise  $z$  is a linear combination of the  $x_i, y_i, w_i$ . Thus  $\langle z, \hat{w} \rangle^2 + \langle z, \hat{x} \rangle^2 + \langle z, \hat{y} \rangle^2 > 1$ . We can compute  $\langle z, \hat{w} \rangle = \frac{-q/2}{\sqrt{\frac{1}{2}q(q+1)}}$ , and so  $\langle z, \hat{w} \rangle^2 = \frac{q}{2(q+1)}$ .

From this, we can obtain  $\frac{1}{q+1} + \frac{1}{r+1} + \frac{1}{p+1} > 1$ . We can assume  $p \geq q \geq r \geq 1$ , since these correspond to the lengths of paths in the above image. This allows us to do some case-by-case analysis.

Using this, we find  $\frac{3}{r+1} > 1$ , and so  $r = 1$  must hold. Similarly,  $\frac{2}{q+1} > \frac{1}{2}$ , which forces  $q \in \{1, 2\}$ .

Supposing  $r = q = 1$ , then we get type  $D_\ell$  because  $p$  can be anything. Supposing otherwise that  $r = 1, q = 2, p \in \{2, 3, 4\}$ , we get type  $E$ .

## 25 Monday October 14

Last time:

Theorem: If  $\Phi$  is irreducible, then the Dynkin diagram is given by  $A - G$ .

Definition: A subset  $A = \{v_1, \dots, v_n\}$  is *admissible* if

1.  $A$  is a linearly independent set,
2.  $\langle v_i, v_i \rangle = 1$  for all  $i$ , and  $\langle v_i, v_j \rangle \leq 0$  if  $i \neq j$ .
3.  $4\langle v_i, v_j \rangle^2 \in \{0, 1, 2, 3\}$  if  $i \neq j$ .



Thus the graph  $\Gamma_A = (V_A, E_A)$  is given by  $V_A = A$  and  $E_A = \left\{ v_i \xrightarrow{4\langle v_i, v_j \rangle^2} v_j \mid i \neq j \right\}$ .

Lemma:

- a. If  $A$  is admissible, then the number of edges such that  $4\langle v_i, v_j \rangle \neq 0$  is at most  $|A| - 1$ .
- b. For every  $i$ , we have  $\deg v_i \leq 3$ .
- c. If  $\Gamma_A$  contains a straight path of length  $t$ , then the graph  $\Gamma'$  obtained by contracting this path is also admissible.

Let  $p$  be the point obtained by contracting such a path.

Proof of (a): If  $\{p_1, \dots, p_t\}$  are linearly independent, then  $p \neq 0$ . Thus by positive-definiteness, we have  $0 <_{pd} \langle p, p \rangle = \sum_{i=1}^t 2\langle p_i, p_i \rangle + \sum_{i < j} 2\langle p_i, p_j \rangle$ . Then  $t > \sum_{i < j} (-2)\langle p_i, p_j \rangle = \sum_{i < j} \sqrt{4\langle p_i, p_j \rangle^2}$ , where the quantity in the square root is the number of edges, which is thus greater than or equal to the number of pairs connected.

Proof of (b): Fix  $i$ . Let  $u_1 \dots u_k$  be the vertices in  $A$  that are connected to  $v_i$  by a single edge. Then by (a), we have  $\langle u_i, u_j \rangle = 0$  for all  $i \neq j$ .

Then the set  $\{u_1, \dots, u_k\}$  is an orthonormal basis for their span. Applying Gram-Schmidt, we can write each  $v_i = \sum_{j=0}^k \langle v_i, u_j \rangle u_j$ , where we pick  $u_0$  such that the new set  $\{u_0\} \cup \{u_1, \dots, u_k\}$ . Then  $\langle v_i, u_0 \rangle \neq 0$  for all  $i$ ; otherwise we would have  $\{u_1, \dots, u_k, v_i\}$  would be linearly dependent, since  $v_i = \sum c_i u_i$  from above, which contradicts our initial axiom/assumption. Then  $1 = \langle v_i, v_i \rangle$  by A2, which equals  $\sum_{j=0}^k \langle v_i, u_j \rangle^2 = \langle v_i, u_0 \rangle^2 + \sum_{j=1}^k \langle v_i, u_j \rangle^2$ , where the first term is strictly positive.

But then  $1 > \sum_{j=1}^k \langle v_i, u_j \rangle^2 \geq \frac{k}{4}$  by A3, which then forces  $k = \deg v_i \leq 3$ .

Proof of (c): The conditions of A1 are satisfied. For A2, we have

$$\langle p_i, p_j \rangle = \begin{cases} -\frac{1}{2} & |i - j| = 1 \\ 0 & |i - j| > 1 \\ 1 & i = j. \end{cases}$$

We then have  $\langle p, p \rangle = t + 2 \sum_{i < j} \langle p_i, p_j \rangle = t + 2 \sum_{i=1}^{t-1} \langle p_i, p_{i+1} \rangle = 1$ . Thus  $\langle p, v_i \rangle = \sum_{j=1}^t \langle p_j, v_i \rangle \leq 0$ .

For A3, fix  $v_i \in A'$ . Then  $v_i$  is connected (by a single edge) to at most one point  $p_j$ , otherwise there would be a cycle. Thus

$$\langle v_i, p \rangle = \begin{cases} \langle v_i, p_j \rangle & \text{if } v_i \text{ is connected to } p_j \\ 0 & \text{else.} \end{cases}$$

We thus have  $4\langle v_i, p \rangle^2 = 4\langle v_i, p_j \rangle \in \{0, 1, 2, 3\}$   $1v_i \sim p_j$ .

## 25.1 Construction of Root Systems and Automorphisms

We'll start with the construction of types  $A - G$ .

**Theorem:** For Dynkin diagrams of type  $A - G$ , there exists an irreducible root system having the given diagram.

Proof: By explicit construction. Fix an orthonormal basis  $\{\varepsilon_i\}$ .

**Type  $A_\ell$ :** Let

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq \ell + 1\}$$

Then  $|\Phi| = \ell^2 + \ell$ , and  $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell\}$ . We then find that  $\dim \mathfrak{g} = \ell^2 + 2\ell$ .

Note that we don't know anything about  $\mathfrak{g}$  yet, but already know its dimension.

Example:  $A_2$ . We have  $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_3 - \varepsilon_2\}$ . Then  $A = (a_{ij})$  with  $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ , and  $\alpha_1^\vee = \frac{2\alpha_1}{\langle \alpha_1, \alpha_1 \rangle} = \frac{2(\varepsilon_1 - \varepsilon_2)}{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle} = \varepsilon_1 - \varepsilon_2 = \alpha_1$ . Doing the computations, it turns out that  $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ ,  $\langle \alpha_2, \alpha_1^\vee \rangle = -1$ , and  $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ .

Thus  $A = [2, -1; -1, 2]$ , which has Dynkin diagram given by:

**Type  $B_\ell$ :** Recall that these have one “short root”:

Then  $\Phi = \{\pm \varepsilon_j, \pm \varepsilon_j \mid 1 \leq i \neq j \leq \ell\} \cup \{\pm \varepsilon_i \mid 1 \leq i \leq \ell\}$ , and we have  $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i-1} \mid 1 \leq i \leq \ell - 1\} \cup \{\alpha_\ell = \varepsilon_\ell\}$ .

After carrying out the computation, we have the following Cartan matrix:

And  $\dim \mathfrak{g} = 2\ell^2 + \ell$ , since  $|\Phi| = 2\ell(\ell - 1) + 2\ell = 2\ell^2$ .

**Type  $D_\ell$ :**

We obtain  $\Phi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq \ell - 1\} \cup \{\alpha_\ell := \varepsilon_{\ell-1} + \varepsilon_\ell\}$ . We then find  $\langle \alpha_{\ell-1}, \alpha_\ell^\vee \rangle = 0$  and  $\langle \alpha_{\ell-2}, \alpha_\ell^\vee \rangle = -1$ .

**Type  $E_\ell$ :** We have  $\Pi(E_\ell) = \Pi(D_{\ell-1}) \cup \{\alpha_\ell := -\frac{1}{2} \sum_{i=1}^{\ell-1} \varepsilon_i\}$ .

This yields  $|\Phi| = 72, 126, 240$  and  $\dim \mathfrak{g} = 78, 133, 248$ , corresponding to  $\ell = 6, 7, 8$ .

More results on exceptional Lie Algebras:

## 26 Wednesday October 16

Todo

## 27 Friday October 18

Todo

## 28 Monday October 21

### 28.1 Chapter 5: Existence Theorem

#### 28.1.1 Universal Enveloping Algebra (UAE)

Some applications/motivations for UAEs:

1. Groups  $G$  are to group algebras  $F[G]$  as Lie algebras  $\mathfrak{g}$  are to UAE  $U(\mathfrak{g})$ . Any  $\mathfrak{g}$ -module then becomes a module over a ring, so the general theory applies.

2. PBW theorem: this yields a concrete  $F$ -basis of  $U(\mathfrak{g})$ . There is a triangular decomposition  $U(\mathfrak{g}) = U(\mathfrak{h}) \otimes U(\mathfrak{f}) \otimes U(\mathfrak{n})$ . This allows constructing the Verma module (and hence irreducible modules) for  $\mathfrak{g}$ , allowing for a description of BGG Category  $\mathcal{O}$ .
3. Harish-Chandra theorem:  $Z(U(\mathfrak{g})) = S(\mathfrak{g})^W$ . This characterizes central characters  $\chi : Z(U(\mathfrak{g})) \rightarrow F$ , which further allows describing the blocks of  $\mathcal{O}$ , i.e. when two irreducible modules have non-trivial extensions.
4.  $U(\mathfrak{g})$  deforms to a quantum group  $U_q(\mathfrak{g})$ .

### 28.1.2 Tensor and Symmetric Algebras

Definition: For  $V$  a f.d. vector space, the *tensor algebra* over  $V$  is given by  $T(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)$  where  $T^n(V) = \bigotimes_{i=1}^n V$  with an associative multiplication  $T^a \times T^b \rightarrow T^{a+b}$  given by  $(\bigotimes_{i=1}^a v_i, \bigotimes_{i=1}^b w_i) \mapsto \bigotimes_{i=1}^a v_i \otimes \bigotimes_{i=1}^b w_i$ .

The tensor algebra satisfies a universal property: given any  $F$ -linear map  $\phi : V \rightarrow A$ . (See phone image)

Definition: The symmetric algebra on  $V$  is given by  $S(V) = T(V)/I$  where  $I = \langle x \otimes y - y \otimes x \rangle \trianglelefteq T(\mathfrak{g})$ .

Some facts:

- a. There is a natural grading  $S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V)$  where  $S^0(V) = F, S^1(V) = V, S^n(V) = T^n(V)/(I \cap T^n V)$ ,
- b. If  $\{x_i\}^n$  is a basis of  $V$ , then  $S(V) \cong F[x_1, \dots, x_n]$ .

### 28.1.3 Construction of UEA

Definition: For  $\mathfrak{g}$  a lie algebra, define  $U(\mathfrak{g}) = T(\mathfrak{g})/J$  where  $J = \langle x \otimes y - y \otimes x - [x, y] \rangle \trianglelefteq T(\mathfrak{g})$ .

Thus we have the following type of equation that holds in  $U(\mathfrak{g})$ :

$$v_1 \otimes \dots \otimes v_a \otimes (x \otimes y) \otimes w_1 \otimes \dots \otimes w_b = v_1 \otimes \dots \otimes v_a \otimes (y \otimes x) \otimes w_1 \otimes \dots \otimes w_b + v_1 \otimes \dots \otimes v_a \otimes ([x, y]) \otimes w_1 \otimes \dots \otimes w_b.$$

Proposition:  $U(\mathfrak{g})$  has a universal property: given a lie algebra hom  $\theta : \mathfrak{g} \rightarrow \mathcal{A}$  where  $\mathcal{A}$  is any unital associative  $F$ -algebra with a lie bracket, there exists a unique  $\psi : U(\mathfrak{g}) \rightarrow \mathcal{A}$  making the following diagram commute:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\iota} & U(\mathfrak{g}) \\ & \searrow \theta & \vdots \exists \psi \\ & & \mathcal{A} \end{array}$$

where  $\iota : \mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$  is given by  $x \mapsto x + J$ .

The upshot: There is a 1 to 1 correspondence

$$\left\{ \begin{array}{c} \text{Lie algebra} \\ \text{representations} \\ \mathfrak{g} \rightarrow \mathfrak{gl}(V) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Algebras from} \\ U(\mathfrak{g}) \rightarrow \text{End}(V) \end{array} \right\}$$

$$\theta \mapsto \psi$$

$$\theta = \psi \circ \iota \leftarrow \psi$$

Proof (existence):

$\theta : \mathfrak{g} \rightarrow \mathcal{A}$  extends to an algebra homomorphism  $\tilde{\theta} : T(\mathfrak{g}) \rightarrow \mathcal{A}$  given by  $\otimes_{i=1}^n x_i \mapsto \prod \theta(x_i)$ . Note that  $\tilde{\theta}(x \otimes y - y \otimes x - [x, y]) = \theta(x)\theta(y) - \theta(y)\theta(x) - \theta([x, y]) = 0$ , and thus  $J \subseteq \ker \tilde{\theta}$  and  $\phi : T(\mathfrak{g})/J \rightarrow \mathcal{A}$  is well-defined.

Uniqueness: Suppose that  $\psi' : U(\mathfrak{g}) \rightarrow \mathcal{A}$  is another hom  $\psi'$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\iota} & U(\mathfrak{g}) \\ & \searrow \theta & \downarrow \psi' \\ & & \mathcal{A} \end{array}$$

Since  $T(\mathfrak{g})$  is generated by  $T^1(\mathfrak{g})$ ,  $U(\mathfrak{g})$  is generated by  $\iota(\mathfrak{g}) \in U(\mathfrak{g})$ . Thus for all  $x \in \mathfrak{g}$ ,  $\psi \circ \iota(x) = \theta(x) = \psi' \circ \iota(x)$  by the commuting of each triangle. We then have  $\psi = \psi'$  on  $\iota(\mathfrak{g})$ , and hence on  $U(\mathfrak{g})$ .

#### 28.1.4 PBW Theorem

PBW: Poincaré-Birkhoff-Witt

Theorem: If  $\mathfrak{g}$  has a basis  $\{x_i\}_{i \in I}$  where  $\leq$  is a total order on  $I$ , then let  $y_i := \iota(x_i) \in U(\mathfrak{g})$ . Then  $U(\mathfrak{g})$  has an  $F$ -basis called a *PBW basis* which is given by

$$\left\{ y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \mid n \in \mathbb{N}, r_i \in \mathbb{N}, i_1 \leq \cdots \leq i_n \right\}.$$

We refer to each term appearing as a *PBW monomial*.

Examples:

Type A,  $\mathfrak{g} = \mathfrak{sl}(2, F) = \langle f, h, e \rangle$ . Pick an order  $x_1 = f, x_2 = h, x_3 = e$ , so  $f < h < e$ .

Then  $U(\mathfrak{g})$  has a basis

$$B = \{1\} \cup \{f^{r_1}\} \cup \{f^{r_1}h^{r_2}\} \cup \{f^{r_1}h^{r_2}e^{r_3}\} \cup \{h^{r_1}\} \cup \{f^{r_1}e^{r_2}\} \cup \{e^{r_1}\} \cup \{h^{r_1}e^{r_2}\}.$$

$$\text{i.e. } B = \{f^a h^b e^c \mid a, b, c \in \mathbb{N}\}.$$

If you pick a different order, say  $f < e < h$ , then we obtain  $B = \{f^a e^b h^c \mid a, b, c \in \mathbb{N}\}.$

## 29 Wednesday October 23

Recall from last time:

For  $\mathfrak{g}$  a lie algebra, we define  $T(\mathfrak{g})$  the tensor algebra, and the universal enveloping algebra  $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$  where  $x \otimes y - y \otimes x \sim [x, y]$ .

We also described the *PBW Theorem*, which provides a basis for  $U(\mathfrak{g})$ .

Proof of PBW Theorem:

We have  $T(\mathfrak{g}) = \text{span}\{x_{j_1} \otimes \cdots \otimes x_{j_k} \mid j_1, \dots, j_k \in I\}$ , where we note that there are not required to be ordered. Thus  $U(\mathfrak{g}) = \text{span}\{y_{j_1} \otimes \cdots \otimes y_{j_k} \mid j_1, \dots, j_k \in I\}$ , where which are again not required to be ordered. We would thus like to express every term here as some linear combination of monomials in the  $y_{i_j}$  with increasing indices. We proceed by inducting on  $k$ , the number of tensor factors occurring. The base case is clear.

For  $k > 1$ , supposing that the element is *not* a PBW monomial, then there is some inversion in the indices  $(j_1, \dots, j_k)$ , i.e. there is at least one  $i$  such that  $j_{i+1} < j_i$ . Now for any two indices  $a, b \in I$ , we have

$$\iota(x_b \otimes x_a) = \iota(x_a \otimes x_b + [x_b, x_a]) \implies y_b y_a = y_a y_b + \iota([x_b, x_a])$$

Since  $[x_b, x_a] = \sum_t F x_t$  and  $\iota[x_b, x_a] = \sum_t F y_t$ .

But then  $y_{j_1} \cdots y_{j_k} = y_{i_1} y_{i_2} \cdots y_{j_k} + \text{lower degree terms}$  where  $i_1 \leq i_2 \cdots i_k$  is a non-decreasing rearrangement of the  $j_i$ . By the inductive hypothesis, the lower degree terms are spanned by PBW monomials, so we're done.

Proof of linear independence:

Claim: let  $\mathbf{x} := x_{j_1} \otimes \cdots \otimes x_{j_n}$  for an arbitrary indexing sequence, and  $\mathbf{x}_{(k)}$  be this tensor with the  $j_k$  and  $j_{k+1}$  terms swapped, and  $\mathbf{x}_{[k]}$  be this tensor with  $x_{j_k}, x_{j_{k+1}}$  replaced by their bracket.

Then there exists a linear map

$$\begin{aligned} f : T(\mathfrak{g}) &\rightarrow R := F[\{z_i\}_{i \in I}] \\ f(x_{i_1} \otimes \cdots \otimes x_{i_n}) &= z_{i_1} \cdots z_{i_n} \\ f(\mathbf{x} - \mathbf{x}_{(k)}) &= f(\mathbf{x}_{[k]}). \end{aligned}$$

By collecting terms, we can write

$$\mathbf{x} - \mathbf{x}_{(k)} - \mathbf{x}_{[k]} = x_{j_1} \otimes \cdots \otimes x_{j_{k-1}} \otimes ((x_{j_k} \otimes x_{j_{k+1}}) - (x_{j_{k+1}} \otimes x_{j_k}) - [x_{j_k}, x_{j_{k+1}}]) \otimes \cdots$$

So we can take  $J$  to be the ideal generated by all elements of this form, and we find that  $J \subset \ker f$ , and thus  $f$  descends to a map  $\bar{f}$  on  $U(\mathfrak{g})$ . We then know that if  $\bar{f}$  applied to any PBW monomial is  $z_{i_1}^{r_1} \cdots z_{i_n}^{r_n}$ , which are linearly independent in  $R$ , then any PBW monomial will be linearly independent in  $U(\mathfrak{g})$ .

Proof of claim:

For each  $\mathbf{x}$ , define an *index*

$$\lambda(\mathbf{x}) = \# \left\{ (a, b) \in \{1, \dots, n\}^2 \ni a < b, j_a < j_b \right\}.$$

Then

$$\{\mathbf{x} \ni \lambda(\mathbf{x}) = 0\} = \{x_{i_1} \otimes \dots \otimes x_{i_n} \ni i_1 \leq \dots \leq i_n\}.$$

So set  $T^{n,k} = \{\mathbf{x} \in T^n(\mathfrak{g}) \ni \lambda(\mathbf{x}) \leq k\}$ ; we then have a filtration  $T^{n,0} \hookrightarrow T^{n,1} \hookrightarrow \dots \hookrightarrow T^n(\mathfrak{g})$ .

Step 1: We'll construct  $f$  by induction on  $n$ .

For  $n > 0$ , set  $f(\mathbf{x}) = z_{j_1} \dots z_{j_n}$  if  $\lambda(\mathbf{x}) = 0$ . We now induct on the index  $k$  at a fixed power  $n > 0$ . The base case is clear.

For  $k > 0$ , there exists an inversion  $(\ell, \ell + 1)$ , i.e. some indices  $i_\ell > i_{\ell+1}$ . Set  $f(\mathbf{x}) = f(\mathbf{x}_{(\ell)}) - f(\mathbf{x}_{[\ell]})$ , where the LHS is in  $T^{n,k}$  and the RHS terms are in  $T^{n,k-1}$  and  $T^{n-1}(\mathfrak{g})$  respectively.

Step 2: We'll check that  $f$  is well-defined.

In the above definition, note that  $f(\mathbf{x})$  can be defined using different inversions of the indices, we'd like to show that these yield the same map.

Let  $(\ell, \ell + 1)$  and  $(\ell', \ell' + 1)$  be two distinct inversions. Then set

$$\begin{aligned} a &= x_{j_\ell} \\ b &= x_{j_{\ell+1}} \\ c &= x_{j'_\ell} \\ d &= x_{j_{\ell'+1}} \\ &\dots \end{aligned}$$

Then we have several cases:

Case 1:  $\ell + 1 < \ell'$ .

Then

$$\begin{aligned} f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\dots b \otimes a \dots c \otimes d \dots) \\ &+ f(\dots \otimes [a, b] \otimes \dots c \otimes d \dots) \\ &= f(\dots b \otimes a \dots d \otimes c \dots) + f(\dots b \otimes a \dots [c, d] \dots) + f(\dots \otimes [a, b] \otimes \dots d \otimes c \dots) \\ &= f(\mathbf{x}_{(\ell')}) + f(\mathbf{x}_{[\ell']}). \end{aligned}$$

Case 2:  $\ell + 1 = \ell'$

Then

$$\begin{aligned}
f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\cdots b \otimes a \otimes x) + f(\cdots [a, b] \otimes c) \\
&= f(b \otimes c \otimes a) + f(c \otimes [a, b]) + f(b \otimes [a, c]) + f([a, b], c) \\
&= f(c \otimes b \otimes a) + f(c \otimes [a, b]) + f(b \otimes [a, c]) + f([a, b], c) + f(b \otimes [a, c]) + f(a \otimes [b, c]) + f([b, c], a) \\
&= f(\mathbf{x}_{(\ell')}) + f(\mathbf{x}_{[\ell']}).
\end{aligned}$$

where the last equality is found by expanding the expression backwards.