Title

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1 Thursday, September 10

Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined $\dim X$ to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking $R = A(X)$, i.e. $\dim \mathbb{A}^n/k = n$.

Proposition 1.1 ***(Dimensions).***
Let $k = \bar{k}$.

a. The dimension of $k[x_1, \cdots, x_n]$ is n.
b. All maximal chains of prime ideals have length n.

1.1 Proof of Dimension Proposition

The case for $n = 0$ is trivial, just take $P_0 = \langle 0 \rangle$. For $n = 1$, easy to see since the only prime ideals in $k[x]$ are $\langle 0 \rangle$ and $\langle x - a \rangle$, since any polynomial factors into linear factors.

Let $P_0 \subsetneq \cdots \subsetneq P_m$ be a maximal chain of prime ideals in $k[x_1, \cdots, x_n]$; we then want to show that $m = n$. Assume $P_0 = \langle 0 \rangle$, since we can always extend our chain to make this true (using maximality). Then P_1 is a minimal prime and P_m is a maximal ideal (and maximals are prime).

Claim: P_1 is principle, i.e. $P_1 = \langle f \rangle$ for some irreducible f.

1.1.1 Proof That P_1 is Principle

Claim: $k[x_1, \cdots, x_n]$ is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD $\implies R[x]$ is a UFD for any R.

See Gauss' lemma.

Claim: In a UFD, minimal primes are principal. Let $r \in P$, and write $r = u \prod p_i^{n_i}$ with p_i irreducible and u a unit. So some $p_i \in P$, and p_i irreducible implies $\langle p_i \rangle$ is prime. Since $0 \subsetneq \langle p_i \rangle \subset P$, but P was prime and assumed minimal, so $\langle p_i \rangle = P$.

The idea is to now transfer the chain $P_0 \subsetneq \cdots \subsetneq P_m$ to a maximal chain in $k[x_1, \cdots, x_{n-1}]$. The first step is to make a linear change of coordinates so that f is monic in the variable x_n.

Example 1.1.
Take $f = x_1 x_2 + x_3^2 x_4$ and map $x_3 \mapsto x_3 + x_4$.

So write

$$f(x_1, \cdots, x_n) = x_n^d + f_1(x_1, \cdots, x_{n-1}) x_n^{d-1} + \cdots + f_d(x_1, \cdots, x_{n-1}).$$

We can then descend to $k[x_1, \cdots, x_n]$ to $k[x_1, \cdots, x_n] / \langle f \rangle$:

$$\begin{tikzcd}
P_0 \arrow[r] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m \arrow[d] \\
& P_1/P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m/P_1 \arrow[d] \\
& P_1/P_1 \cap k[x_1, \cdots, x_{n-1}] \arrow[r] & \cdots \arrow[r] & (P_m/P_1) \cap k[x_1, \cdots, x_{n-1}]
\end{tikzcd}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

Definition 1.1.1 (Integral Extension)**.**
An *integral* ring extension $R \hookrightarrow R'$ of R is one such that all $r' \in R'$ satisfying a monic polynomial with coefficients in R, where R' is finitely generated.

In this case, also implies that R' is a finitely-generated R module.

In this case, $k[x_1, \cdots, x_{n-1}] \hookrightarrow k[x_1, \cdots, x_n] / \langle f \rangle$ is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

Lemma 1.2.
Suppose $P', Q' \subset R'$ are distinct prime ideals with $R \hookrightarrow R'$ an integral extension. Then if $P' \cap R = Q' \cap R$, neither contains the other, i.e. $P' \not\subset Q'$ and $Q' \not\subset P'$.

Proof .
Toward a contradiction, suppose $P' \subset Q'$, we then want to show that $Q' \supset P'$. Let $a \in Q' \setminus P'$

(again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.
Then $\bar{a} \neq 0$ in R'/P', and there exists a monic polynomial of minimal degree that $\bar{a}$ satisfies, $p(x) = x^n + \sum_{i=2}^{n} \bar{c}_i x^{n-i}$. This implies $\bar{c}_n \in Q'/P'$ (which will contradict $c_n \in P'$), since if $\bar{c}_n = 0$ then factoring out x yields a lower degree polynomial that $\bar{a}$ satisfies.
But then $\bar{a}_n \in Q' \cap R$, so ???

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Question: Given $R \hookrightarrow R'$ is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the "Going Up" Theorem: given $P \subset R$ prime, there exists $P' \subset R'$ prime such that $P' \cap R = P$. Furthermore, we can lift $P_1 \subset P_2$ to $P_1' \subset P_2'$, as well as "lifting sandwiches":

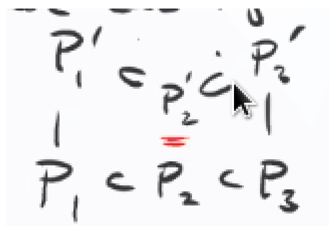

Figure 1: Image

In this process, the length of the chain decreased since $\langle 0 \rangle$ was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies. ■

1.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if $R \hookrightarrow R'$ is integral then $\dim R = \dim R'$.

Claim: Any affine variety has finite dimension.

Proof .
We have $\dim X = \dim A(X)$, where $A(X) := k[x_1, \cdots, x_n]I$ for some $I(X) = \sqrt{I(X)}$.
The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated k-algebra is an integral extension of some polynomial ring $k[y_1, \cdots, y_d]$. I.e., the following extension is integral:

$$k[y_1, \cdots, y_d] \hookrightarrow k[x_1, \cdots, x_n]/I.$$

We can conclude that $\dim A(X) = d < \infty$.

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Proposition 1.3 *(?)*.
Let X, Y be irreducible affine varieties. Then

a. $\dim X \times Y = \dim X + \dim Y$.
b. $Y \subset X \implies \dim X = \dim Y + \operatorname{codim}_X Y$.
c. If $f \in A(X)$ is nonzero, then any component of $V(f)$ has codimension 1.

Proof .

Remark .
Why is $X \times Y$ again an affine variety? If $X \subset \mathbb{A}^n/k$, $Y \subset \mathbb{A}^m/k$ with $X = V(I), Y = V(J)$, then $X \times Y \subset \mathbb{A}^n/k \times \mathbb{A}^m/k = \mathbb{A}^{n+m}/k$ can be given by taking $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$ using the natural inclusions of $k[x_1, \cdots, x_\ell]$.
Note that we can write

$$k[x_1, \cdots, x_n, y_1, \cdots, y_m] = k[x_1, \cdots, x_n] \otimes_k k[y_1, \cdots, y_n]$$

where we think of $x_i = x_i \otimes 1, y_j = 1 \otimes y_j$. We thus map I, J to $I \otimes 1 + 1 \otimes J$ and obtain $V(I \otimes 1 + 1 \otimes J) = X \times Y$ and $A(X \times Y) = A(X) \otimes_k A(Y)$.
In general, for k-algebras R, S,

$$R/I \otimes_k S/J \cong R \otimes_k S/\left\langle I \otimes 1 + 1 \otimes J \right\rangle.$$

Remark .
For R, S finitely generated k-algebras, $\dim R \otimes_k S = \dim R + \dim S$.

Part (a) is proved by the above remarks.
For part (b), translating into ring theory yields the equivalent statement that $P \subset A(X)$ with $I(Y) \subset P$ is a member of some maximal chain, along with the statement that all maximal chains have the same length.
Part (c) follows from Krull's principal ideal theorem, which says that minimal prime ideals are principal for finitely generated k-algebras.

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