

# Title

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# Prologue

## 0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

## 0.2 Notation

- If a property  $P$  is said to hold **locally**, this means that for every point  $p$  there is a neighborhood  $U_p \ni p$  such that  $P$  holds on  $U_p$ .

Notation	Definition
$k[\mathbf{x}] = k[x_1, \dots, x_n]$	Polynomial ring in $n$ indeterminates
$k(\mathbf{x}) = k(x_1, \dots, x_n)$	Rational function field in $n$ indeterminates
$\mathcal{U} \rightrightarrows X$	An open cover $\mathcal{U} = \{U_j \mid j \in J\}, X = \bigcup_{j \in J} U_j$
$\Delta_X$	The diagonal $\Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X$
$\mathbb{A}_{/k}^n$	Affine $n$ -space $\mathbb{A}_{/k}^n := \{\mathbf{a} = [a_1, \dots, a_n] \mid a_j \in k\}$
$\mathbb{P}_{/k}^n$	Projective $n$ -space $\mathbb{P}_{/k}^n := (k^n \setminus \{0\}) / x \sim \lambda x$ $\mathbb{P}_{/k}^n = \{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \mid p, q, \in k[x_1, \dots, x_n]\}$
$V(J), V_a(J)$	Variety associated to an ideal $J \trianglelefteq k[x_1, \dots, x_n]$ $V_a(J) := \{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$
$I(S), I_a(S)$	Ideal associated to a subset $S \subseteq \mathbb{A}_k^n$ $I_a(S) := \{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in S\}$
$A(X)$	Coordinate ring of a variety $A(X) := k[x_1, \dots, x_n]/I(X)$

Notation	Definition
$V_p(J)$	Projective variety of an ideal $V_p(J) := \left\{ \mathbf{x} \in \mathbb{P}_k^n \mid f(\mathbf{x}) = 0, \forall f \in J \right\}$
$I_p(S)$	Projective ideal (?) $I_p(S) := \left\{ f \in k[x_1, \dots, x_n] \mid f \text{ is homogeneous and } f(x) = 0 \forall x \in S \right\}$
$S(X)$	Projective coordinate ring $S(X) := k[x_1, \dots, x_n]/I_p(X)$
$f^h$	Homogenization $f^h := x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$
$f^i$	Dehomogenization $f^i := f(1, x_1, \dots, x_n)$
$J^h$	Homogenization of an ideal $J^h := \left\{ f^j \mid f \in J \right\}$
$\bar{X}$	Projective closure of a subset $\bar{X} := V_p(J^h) := \left\{ \mathbf{x} \in \mathbb{P}^n \mid f^h(\mathbf{x}) = 0 \forall f \in X \right\}$
$\mathcal{O}_X$	Structure sheaf $\mathcal{O}_X(U) := \left\{ f : U \rightarrow k \mid f \in k(\mathbf{x}) \text{ locally} \right\}$
$D(f)$	Distinguished open set, $D(f) := V(f)^c = \left\{ x \in \mathbb{A}^n \mid f(x) \neq 0 \right\}$

### 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?

- What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for  $X = D(f)$  a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety  $X$  in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

## 0.4 Useful Examples

### 0.4.1 Varieties

- $V(xy - 1) \subseteq \mathbb{A}^2$  a hyperbola
- $V(x)$  a coordinate axis
- $V(x - p)$  a point.

### 0.4.2 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$ , a sheaf of smooth functions
- $C^0(\cdot, \mathbb{R})$ , a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$ , the sheaf of regular functions on  $X$
- $\underline{\mathbb{R}}(\cdot)$ , the constant sheaf associated to  $\mathbb{R}$  (locally constant real-valued functions)
- $\text{Hol}(\cdot, \mathbb{C})$ , a sheaf of holomorphic functions
- $K_p$  the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

## 0.5 The Algebra-Geometry Dictionary

Let  $k = \bar{k}$ , we're setting up correspondences

Algebra	Geometry
$k[x_1, \dots, x_n]$	$\mathbb{A}_{/k}^n$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \dots, x_n - p_n$	Points $[a_1, \dots, a_n]$
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \dots, x_n]$	$V(J)$ the zero locus
Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$	Irreducible subsets
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
$k[x_1, \dots, x_n]/I(X)$	$A(X)$ (Functions on $X$ )
$A(X)$ a domain	$X$ is irreducible
$A(X)$ indecomposable	$X$ is connected
Krull dimension $n$ (chains of primes)	Topological dimension $n$ (chains of irreducibles)