

## Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

# Summary/Outline

# Outline

What we're trying to prove:

- 8.1.5:  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .

What we have so far:

- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$
$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

# Outline

- Took  $R^\pm : I \longrightarrow \text{Sp}(2n; \mathbb{R})$ : symplectic paths associated to  $S^\pm$
- These paths defined  $\mu(x), \mu(y)$
- Section 8.7:

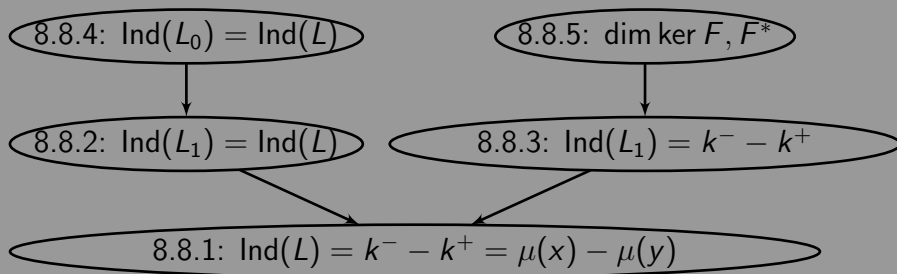
$$R^\pm \in \mathcal{S} := \left\{ R(t) \mid R(0) = \text{id}, \det(R(1) - \text{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\text{Ind}(L) \stackrel{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

# From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know  $\text{Ind}(L) = \text{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing  $\text{Ind}(L_1)$  by computing kernels.



## 8.8.5: $\dim \ker F, F^*$

# Recall

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^* : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^q(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here  $\frac{1}{p} + \frac{1}{q} = 1$  are conjugate exponents.

# Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since  $\text{coker } L_1 \cong \ker L_1^*$ , it suffices to compute  $\ker L_1^*$ .
- We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \text{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the  $n = 1$  case.



# Setup

$L_1$  used a path of diagonal matrices constant near  $\infty$ :

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



# Statement of Later Lemma (8.8.5)

Let  $p > 2$  and define

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

*Note:  $F$  is  $L_1$  for  $n = 1$ :*

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

# Statement of Lemma

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose  $a_i^\pm \notin 2\pi\mathbb{Z}$ .

- ① Suppose  $a_1(s) = a_2(s)$  and set  $a^\pm := a_1^\pm = a_2^\pm$ . Then

$$\dim \text{Ker } F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \subset \mathbb{R} \right\}$$

$$\dim \text{Ker } F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

- ② Suppose  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ , then

$$\dim \text{Ker } F = \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$

$$\dim \text{Ker } F^* = \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

# Statement of Lemma

In words:

- 1 If  $S(s)$  is a scalar matrix, set  $a^\pm = a_1^\pm = a_2^\pm$  to the limiting scalars and count the integer multiples of  $2\pi$  between  $a^-$  and  $a^+$ .
- 2 Otherwise, if  $S$  is uniformly bounded by 1, count the number of entries the flip from positive to negative as  $s$  goes from  $-\infty \rightarrow \infty$ .

$$S_K := \begin{bmatrix} a_1^- & 0 \\ 0 & a_2^- \end{bmatrix} \quad S(s) = \begin{bmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{bmatrix} \quad S_K^* = \begin{bmatrix} a_1^+ & 0 \\ 0 & a_2^+ \end{bmatrix}$$

The diagram below shows a horizontal axis labeled  $s$  with endpoints  $-\infty$  and  $\infty$ . Two vertical red lines mark the points  $-S_0$  and  $S_0$  on the axis. A horizontal bracket is drawn above the axis, spanning from  $-S_0$  to  $S_0$ .

# Proof of Assertion 1

- ① Suppose  $a_1(s) = a_2(s)$  and set  $a^\pm := a_1^\pm = a_2^\pm$ . Then

$$\begin{aligned}\dim \ker F &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \subset \mathbb{R} \right\} \\ \dim \ker F^* &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.\end{aligned}$$

## Step 1: Transform to Cauchy-Riemann Equations

- Write  $a(s) := a_1(s) = a_2(s)$ .
- Start with equation on  $\mathbb{R}^2$ ,

$$\mathbf{Y}(s, t) = [Y_1(s, t), Y_2(s, t)].$$

- Replace with equation on  $\mathbb{C}$ :

$$\mathbf{Y}(s, t) = Y_1(s, t) + iY_2(s, t).$$

# Assertion 1, Step 1: Reduce to CR

- Expand definition of the PDE

$$F(\mathbf{Y}) = 0 \rightsquigarrow \bar{\partial}\mathbf{Y} + S\mathbf{Y} = 0$$

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\mathbf{Y} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \mathbf{Y} = 0.$$

- Change of variables: want to reduce to  $\bar{\partial}\tilde{Y} = 0$
- Choose  $B \in \text{GL}(1, \mathbb{C})$  such that  $\bar{\partial}B + SB = 0$
- Set  $Y = B\tilde{Y}$ , which (?) reduces the previous equation to

$$\bar{\partial}\tilde{Y} = 0.$$

# Assertion 1, Step 1: Reduce to CR

Can choose (and then solve)

$$B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix} \quad \text{where} \quad \frac{\partial b}{\partial s} = -a(s)b(s)$$

$$\implies b(s) = \exp \left( \int_0^s -a(\sigma) d\sigma \right) := \exp(-A(s)).$$

Remarks:

- For some constants  $C_i$ , we have

$$A(s) = \begin{cases} C_1 + a^- s, & s \leq -\sigma_0 \\ C_2 + a^+ s, & s \geq \sigma_0 \end{cases}.$$

- The new  $\tilde{Y}$  satisfies CR, is continuous and  $L^1_{\text{loc}}$ , so elliptic regularity  $\implies C^\infty$ .
- The real/imaginary parts of  $\tilde{Y}$  are  $C^\infty$  and harmonic.

# Assertion 1, Step 2: Solve CR

- Identify  $s + it \in \mathbb{R} \times S^1$  with  $u = e^{2\pi z}$
- Apply Laurent's theorem to  $\tilde{Y}(u)$  on  $\mathbb{C} \setminus \{0\}$  to obtain an expansion of  $\tilde{Y}$  in  $z$ .
- Deduce that the solutions of the system are given by

$$\tilde{Y}(u) = \sum_{\ell \in \mathbb{Z}} c_\ell u^\ell \implies \tilde{Y}(s + it) = \sum_{\ell \in \mathbb{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where  $\{c_\ell\}_{\ell \in \mathbb{Z}} \subset \mathbb{C}$  converges for all  $s, t$ .



# Assertion 1, Step 2: Solve CR

Use  $e^{s+it} = e^s(\cos(t) + i \sin(t))$  to write in real coordinates:

$$\tilde{Y}(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s \ell} \begin{bmatrix} \cos(2\pi \ell t) & -\sin(2\pi \ell t) \\ \sin(2\pi \ell t) & \cos(2\pi \ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

Use

$$Y = B\tilde{Y} = \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \tilde{Y}$$

to write

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s \ell} \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \begin{bmatrix} \cos(2\pi \ell t) & -\sin(2\pi \ell t) \\ \sin(2\pi \ell t) & \cos(2\pi \ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

For  $s \leq s_0$  this yields for some constants  $K, K'$ :

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi \ell - a^-} \begin{bmatrix} e^K (\alpha_\ell \cos(2\pi \ell t) - \beta_\ell \sin(2\pi \ell t)) \\ e^{K'} (\alpha_\ell \sin(2\pi \ell t) + \beta_\ell \cos(2\pi \ell t)) \end{bmatrix}.$$

# Condition on $L^p$ Solutions

For  $s \leq s_0$  we had

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^K (\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'} (\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

and similarly for  $s \geq s_0$ , for some constants  $C, C'$  we have:

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^+)s} \begin{bmatrix} e^C (\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'} (\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

Then

$$Y \in L^p \iff \text{exponential terms} \xrightarrow{\ell \rightarrow \infty} 0.$$

# Condition on $L^p$ Solutions: Small Tails

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^K(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

- $\ell \neq 0$ : Need  $\alpha_\ell = \beta_\ell = 0$  **or**  $2\pi\ell > a^-$
- $\ell = 0$ : Need both
  - $\alpha_0 = 0$  or  $a^- < 0$  and
  - $\beta_0 = 0$  or  $a^- < 0$ .

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^+)s} \begin{bmatrix} e^C(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

- $\ell \neq 0$ : Need  $\alpha_\ell = \beta_\ell = 0$  **or**  $2\pi\ell < a^+$
- $\ell = 0$ : Need both
  - $\alpha_0 = 0$  or  $a^+ > 0$  and
  - $\beta_0 = 0$  or  $a^+ > 0$ .

# Counting Solutions

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } 2\pi\ell \in (a^-, a^+) & \ell \neq 0 \\ (\alpha_0 = 0 \text{ or } 0 \in (a^-, a^+)) \text{ and } (\beta_0 = 0 \text{ or } 0 \in (a^-, a^+)) & \ell = 0 \end{cases}.$$

- Finitely many such  $\ell$  that satisfy these conditions
- Sufficient conditions for  $Y(s, t) \in W^{1,p}$ .

Compute dimension of space of solutions:

$$\begin{aligned} \dim \text{Ker } F &= 2 \cdot \# \left\{ \ell \in \mathbb{Z}^* \mid 2\pi\ell \in (a^-, a^+) \right\} + 2 \cdot \mathbb{1} [0 \in (a^-, a^+)] \\ &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \right\}. \end{aligned}$$

*Note: not sure what  $\mathbb{Z}^*$  is: most likely  $\mathbb{Z} \setminus \{0\}$ .*

# Counting Solutions

Use this to deduce  $\dim \ker F^*$ :

- $Y \in \ker F^* \iff Z(s, t) := Y(-s, t)$  is in the kernel of the operator

$$\begin{aligned}\tilde{F} : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Z &\mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y.\end{aligned}$$

- Obtain  $\ker F^* \cong \ker \tilde{F}$ .
- Formula for  $\dim \ker \tilde{F}$  almost identical to previous formula, just swapping  $a^-$  and  $a^+$ .

# Assertion 2

**Assertion 2:** Suppose  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ , then

$$\begin{aligned} \dim \ker F &= \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 < a_i^+ \right\} \\ \dim \ker F^* &= \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 < a_i^- \right\}. \end{aligned}$$

We use the following:

- Lemma 8.8.7:

$$\sup_{s \in \mathbb{R}} \|S(s)\| < 1 \implies \text{the elements in } \ker F, \ker F^* \text{ are independent of } t.$$

- Proof: in subsection 10.4.a.

# Proof of Assertion 2

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

- Given as a fact:

$$\mathbf{Y} \in \ker F \implies \frac{\partial}{\partial s} \mathbf{Y} = \mathbf{a}(s) \mathbf{Y} \quad := \begin{bmatrix} -a_1(s) & 0 \\ 0 & -a_2(s) \end{bmatrix} \mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c}_0 \exp(-\mathbf{A}(s)) \quad \text{where} \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma) d\sigma.$$

# Proof of Assertion 2

- Explicitly in components:

$$\begin{cases} \frac{\partial Y_1}{\partial s} = -a_1(s)Y_1 \\ \frac{\partial Y_2}{\partial s} = -a_2(s)Y_2 \end{cases} \implies Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma) d\sigma.$$

- As before, for some constants  $C_{j,i}$ ,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

- Thus

$$Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+),$$

establishing

$$\dim \ker F = \# \left\{ i \in \{1, 2\} \mid 0 \in (a_i^-, a_i^+) \right\}.$$



$$8.8.3: \text{Ind}(L_1) = k^- - k^+$$

# Statement and Outline

Statement: let  $k^\pm := \text{Ind}(R^\pm)$ ; then  $\text{Ind}(L_1) = k^- - k^+$ .

- Consider four cases, depending on parity of  $k^\pm - n$
- Show all 4 lead to  $\text{Ind}(L_1) = k^- - k^+$

- 1  $k^- \equiv k^+ \equiv n \pmod{2}$ .
- 2  $k^- \equiv n, k^+ \equiv n - 1 \pmod{2}$
- 3  $k^- \equiv n - 1, k^+ \equiv n \pmod{2}$ .
- 4  $k^- \equiv k^+ \equiv n - 1 \pmod{2}$

$k^-$	$k^+$	$n$
✓	✓	✓
✓		✓
	✓	✓
✓	✓	

# Case 1: $k^+ \equiv k^- \equiv n \pmod{2}$

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^+)\pi & \\ & & & & & & (n-1-k^+)\pi \end{bmatrix}.$$

# Case 1: $k^- \equiv k^+ \equiv n \pmod{2}$

- Take  $a_1(s) = a_2(s)$  so  $a_1^\pm = a^\pm$
- Apply the proved lemma to obtain

$$\begin{aligned} \dim \ker L_1 &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\ell \in (n-1-k^-, n-1-k^+) \right\} \\ &= \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$\begin{aligned} \dim \ker L_1^* &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\ell \in (k^- - n + 1, k^+ - n + 1) \right\} \\ &= \begin{cases} k^+ - k^- & k^+ > k^- \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\implies \text{Ind}(L_1) = k^- - k^+.$$

# Case 2: $k^+ \not\equiv k^- \equiv n \pmod{2}$

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\varepsilon\pi & & & \\ & & & & -\varepsilon\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & \varepsilon & & & \\ & & & & -\varepsilon & & \\ & & & & & (n-2-k^+)\pi & \\ & & & & & & (n-2-k^+)\pi \end{bmatrix}.$$

## Case 2: $k^+ \not\equiv k^- \equiv n \pmod{2}$

- Take  $a_1(s) = a_2(s)$  everywhere except the  $n - 1$ st block, where we can assume  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ .
- Assertion 2 applies and we get

$$\begin{aligned} \dim \ker L_1 &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\ell \in (n - 1 - k^-, n - 2 - k^+) \right\} + 1 \\ &= \begin{cases} (k^- - k^+ - 1) + 1 & k^- > k^+ \\ 1 & \text{otherwise} \end{cases} \implies \text{Ind}(L_1) \end{aligned}$$