Title

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Lecture 10

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Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of ??; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{\'et}}, X_{\text{fppf}}$, then the data of a GL_n -torsor split by a τ -cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$U \times_X U$$

$$\pi_1 \downarrow \downarrow \pi_2$$

$$U$$

$$\downarrow$$

$$\downarrow$$

That U trivializes our torsor means that $\pi^*T = \pi^*G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\pi_1^*\pi^*T \xrightarrow{\sim} \pi_2^*\pi^*T$$

$$\downarrow \qquad \qquad \downarrow$$

$$\pi_1^*\pi^*G \xrightarrow{\sim} \pi_2^*\pi^*G$$

Both of the bottom objects are isomorphic to $G|_{U\times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n -torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{\'et}}, GL_n)$ is in bijection with n-dimensional vector bundles on X.

Exercise 1.0.3(?): See if Hilbert 90 is true for groups other than GL_n .

1.1 Representability and Local Triviality

Lecture 10 3

1 Lecture 10

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

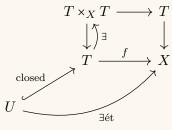
Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).

You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:

$$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow \mathring{\bigcap} \exists & & \downarrow \\ T & \stackrel{f}{\longrightarrow} & X \end{array}$$

The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that



Example 1.1.5 (non-smooth group schemes):

- α_p , the kernel of Frobenius on \mathbb{A}^1 or \mathbb{G}_a ,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on \mathbb{G}_m ,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times GL_n$, etc.