

Floer Talk

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Sunday 12th April, 2020

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Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of \mathcal{F} .

What is \mathcal{F} ?

We started with the unadorned Floer map:

$$\begin{aligned}\mathcal{F} : \mathcal{C}^\infty(\mathbf{R} \times S^1; W) &\longrightarrow \mathcal{C}^\infty(\mathbf{R} \times S^1; TW) \\ u &\longmapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H_t)\end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u).\end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned}\mathcal{P}^{1,p}(x, y) &:\rightarrow ? \\ (s, t) &\mapsto \exp_{w(s,t)} Y(s, t).\end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C_\infty^\infty(x, y)$.

1 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian H , perturb (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the right dimension.

Start by construction $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty$, the space of perturbations of H . Idea: define a norm $\|\cdot\|_\varepsilon$ and take the subspace of finite-norm elements.

$$\begin{aligned}\|h\|_\varepsilon &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} |d^k h(x,t)| \\ &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \Psi_i^{-1})(z)|.\end{aligned}$$

Where $\{\varepsilon_k\} \subset \mathbb{R}$ is chosen such that $\mathcal{C}_\varepsilon^\infty \hookrightarrow \mathcal{C}^\infty(W \times S^1)$ is dense for the C^∞ topology, and the $\Psi_i : B_i \rightarrow \overline{B(0,1)}$ is a fixed finite sequence of diffeomorphisms where $\bigcup_i B_i^\circ = W \times S^1$.

Note that we'll only use density for the C^1 topology in our case.

Proposition 1.1.

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Proof.

Show that $C^\infty(W \times S^1)$ is separable, yielding a sequence $(f_n) \subset C^\infty(W \times S^1)$ that is dense in the C^1 topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k \leq n} \|f_k\|_{C^n(W \times S^1)}}$$

where the diffeomorphisms Ψ_i are used to compute these norms. ■

Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\text{Per}(H_0 + h) = \text{Per}(H_0)$ and are nondegenerate.

1.1 8.4: Linearizing the Floer equation: The Differential of \mathcal{F}

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as Z_i , tangents to W along u or in a neighborhood B of u) with actual vectors in \mathbb{R}^m .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify $\text{im } \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R} \times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that x, y are contractible loops in W that are nondegenerate critical points of the action functional \mathcal{A}_H , and $C_{\searrow}(x, y)$ was the set of maps $u : \mathbb{R} \times S^1 \rightarrow W$ satisfying some conditions.

We lift each map to $\tilde{u} : S^2 \rightarrow W$ in the following way: the loops x, y are contractible, so they bound discs.