

# Weil Conjectures

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## 1 Notes from Daniel's Office Hours

0. Definition of Zeta functions
1. Statement of the conjectures
2. Easy examples:  $\mathbb{P}^n_{\mathbb{F}_p}, \text{Gr}_{\mathbb{F}_p}(k, n) = \text{GL}(n, \mathbb{F}_p)/P$  the stabilizer of an  $\mathbb{F}_p$ -point in  $\mathbb{C}^n, \mathbb{F}_{p^n}$ .
3. Medium example:  $E/\mathbb{F}_p$  an elliptic curve.
4. Work out a harder example as in Weil

### References

- [http://www-personal.umich.edu/~mmustata/zeta\\_book.pdf](http://www-personal.umich.edu/~mmustata/zeta_book.pdf)

### 1.1 Definition of Zeta Function

Fix  $q$  a prime and  $\mathbb{F} := \mathbb{F}_q$  the finite field with  $q$  elements, along with its unique degree  $n$  extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}_p} \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

#### Definition 1.0.1.

Let

$$J = \langle f_1, \dots, f_N \rangle \trianglelefteq k[x_0, \dots, x_n]$$

be an ideal, then a *projective algebraic* variety  $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}}$  can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \mid f_1(\mathbf{x}) = \dots = f_N(\mathbf{x}) = 0 \right\}$$

where an ideal generated by *homogeneous* polynomials in  $n + 1$  variables, i.e. there is some fixed  $d \in \mathbb{Z}^{\geq 1}$  such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field  $\mathbb{F}_p$ . We will be thinking of  $K$ -valued points for  $K/\mathbb{F}_p$  algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into  $\mathbb{P}^\infty$ ?

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Example: Take  $f_1(x) = x \in \mathbb{F}[x]$ , consider  $V(\langle f_1 \rangle) \subset \mathbb{P}_{\mathbb{F}_n}^1$ . This is given by the single point  $x = \mathbf{0}$ .

Fix  $X/\mathbb{F}$  an  $N$ -dimensional projective algebraic variety. Note that it then has points in any finite extension  $L/K$ .

**Definition 1.0.2.**

Define the *local zeta function* (or *Hasse-Weil zeta function*) of  $X$  the following formal power series:

$$\zeta_X(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$\begin{aligned} \zeta_X(0) &= 1 \\ z \left( \frac{\partial}{\partial z} \right) \log \zeta_X(z) &= t \left( \frac{\zeta'_X(z)}{\zeta_X(z)} \right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \cdots, \end{aligned}$$

which is an *ordinary generating function* for the sequence  $(\alpha_n)$ .

Thus if we define  $G(x)$  to be the OGF for  $(\alpha_n)$ , we have  $\zeta_X(t) = \exp$

Todo: why not an OGF.

Remark: Note that for an OGF  $F(x) = \sum_{n=0}^{\infty} f_n x^n$ , we can extract coefficients in the following way:

$$[x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

### 1.1.1 Why call it a Zeta function?

Suppose

$$\mathbb{A}_{\mathbb{Z}}^n \supseteq X = V(\langle f_1, \dots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \dots, x_{n-1}].$$

Then for every prime, we can reduce the equations mod  $p$  and consider

$$\mathbb{A}_{\mathbb{F}_p}^n \supseteq X_p := V(\langle f_1 \bmod p, \dots, f_d \bmod p \rangle) \quad \text{where} \quad f_i \bmod p \in \mathbb{F}_p[x_0, \dots, x_{n-1}]$$

Then define the “local at  $p$ ” zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Note: the index set for the product may require some minor adjustment over  $\mathbb{Q}$  in general. There are also potentially modifications needed to extend to schemes.

Taking  $X = \text{Spec } \mathbb{Q}$  and  $X_p = \text{Spec } \mathbb{F}_p$  (which is a single point since  $\mathbb{F}_p$  is a field) and noting that

$$\begin{aligned} \zeta_{X_p}(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^n\right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}, \end{aligned}$$

we find that

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left( \frac{1}{1-p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

the classical Riemann Zeta function.

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Example (Point):  $X = \{x = 0\} / \mathbb{F}$  a single point over  $\mathbb{F}$ , then

$$\begin{aligned} \#X(\mathbb{F}) &:= \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &:= \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &:= \alpha_n = 1 \\ &\vdots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned}\log(1+t) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} \\ \implies \log(1-t) &= - \sum_{n=1}^{\infty} \frac{t^n}{n} \\ \implies -\log(1-t) &= \sum_{n=1}^{\infty} \frac{t^n}{n} \\ &= 1 \cdot t + 1 \cdot t^2 + 1 \cdots t^3 + \cdots\end{aligned}$$

and so

$$\zeta_X(t) = \exp(-\log(1-t)) = \frac{1}{1-t}.$$

Example (Affine Line):  $X = \mathbb{A}^1/\mathbb{F}$  the affine line over  $\mathbb{F}$ , then

$$\begin{aligned}X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n \\ &\cdot\end{aligned}$$

where we just note that we can write  $\mathbb{A}^1(\mathbb{F}_n) = \{(x_1) \mid x_1 \in \mathbb{F}_n\}$ .

Example (Projective Line):  $X = \mathbb{P}^1/\mathbb{F}$  the projective line over  $\mathbb{F}$ , then

$$\begin{aligned}X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &\cdot\end{aligned}$$

where we write  $\mathbb{P}_{\mathbb{F}}^1 = \mathbb{A}_{\mathbb{F}}^1 \coprod \{\infty\}$  is the affine line with a point at infinity. We can also count by coordinates:

$$\mathbb{P}^1(\mathbb{F}^n) = \{[x_1, x_2] \mid x_1, x_2 \neq 0 \in \mathbb{F}^n\} / \sim = \{[x_1, 1] \mid x_1 \in \mathbb{F}^n\} \coprod \{[1, 0]\}.$$

Example (Affine Space): Take  $X = \mathbb{A}^n/\mathbb{F}$ , then  $\alpha_n = q^n + 1$  for a point at infinity, so

$$X(\mathbb{F}) = \cdot$$

Thus

$$\zeta_X(t) = \frac{1}{(1 - q^{-t})(1 - q^{1-t})}$$

Example (Projective Space): Take  $X = \mathbb{P}_{\mathbb{F}}^n$ , then  $\alpha_n = 1 + q^m + (q^m)^2 + \cdots + (q^m)^n$ , so

$$\zeta_X(t) = \left( \frac{1}{1 - q^{-t}} \right) \left( \frac{1}{1 - q^{1-t}} \right) \left( \frac{1}{1 - q^{2-t}} \right) \cdots \left( \frac{1}{1 - q^{n-t}} \right)$$

or equivalently, take your favorite curve  $\gamma \in \mathbb{C}$  homotopic to  $\mathbb{S}^1$ .

Note: this is extremely amenable to numerical approximation if you have a closed form for  $F$  or even just a black-box numerical version of  $F$ ! I.e. easy to throw at a computer.

Todo: how to manually count points in  $\mathbb{P}^n$ !

Example: Take  $X = \text{Gr}_{\mathbb{F}}(k, n)$ , then ????? so

$$\zeta_X(t) = ?.$$

Questions about properties

- $\zeta_{X \amalg Y}(t) = ? \zeta_X(t) \zeta_Y(t)$ ?
- $\zeta_{X \times Y} = ?$

## 1.2 Statement of Weil Conjectures

### 1. (Rationality)

$$\zeta_X(t) = \frac{p_1(t)p_3(t) \cdots p_{2N-1}(t)}{p_0(t)p_2(t) \cdots p_N(t)} \in \mathbb{Z}(t), \quad \text{i.e.} \quad p_i(t) \in \mathbb{Z}[t]$$

$$\begin{aligned} P_0(t) &= 1 - t \\ P_{2n}(t) &= 1 - q^n t \\ P_i(t) &= \prod_j (1 - a_{ij}t), \quad a_{ij} \in \mathbb{C}. \end{aligned}$$

### 2. (Functional Equation and Poincare Duality)

$$\zeta_X(n - t) = \pm q^{\frac{1}{2}(nE) - Et} \zeta(x, t).$$

3. (Riemann Hypothesis)
4. (Betti Numbers)

### 1.3 Hard Example: An Elliptic Curve

Take  $X = E/\mathbb{F}$ , then  $\alpha_n = q^n - (a^n + \bar{a}^n - 1)$  where  $|a|_{\mathbb{C}} = |\bar{a}|_{\mathbb{C}} = \sqrt{q}$ . Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-s})}.$$