Problem Set 1

D. Zack Garza

November 9, 2019

Contents

1	Problem 5															1											
	1.1	Part 1]
	1.2	Part 2																									2
2	Problem 6																	2									
	2.1	Part 1																									2
	2.2	Part 2																									

1 Problem 5

1.1 Part 1

Let $A \in \operatorname{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in Math(n, n)$ be positive semi-definite, so

$$\langle v, Bv \rangle \ge 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A+B)v \rangle \ge 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\langle v, (A+B)v \rangle = \langle v, Av \rangle + \langle v, Bv \rangle$$

> $\langle v, Av \rangle + 0$
 $\geq 0 + 0$
= 0.

1.2 Part 2

Let M be a smooth manifold with tangent bundle TM and a maximal smooth atlas \mathcal{A} . Choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$. Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to \mathcal{C} .

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} := \delta_{jk}$. We then have

$$g^{i}: T\phi_{i}(U_{i}) \times T\phi_{i}(U_{i}) \to \mathbb{R}$$

$$(\partial x_{i}, \partial x_{j}) \mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n \cong \mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Let G^i be the pullback of g^i along the coordinate functions ϕ_i , so

$$G^{i}: TU_{i} \times TU_{i} \to \mathbb{R}$$

$$G^{i}(p,q) := \left((\phi_{i})^{*} g^{i} \right)(p,q) := g^{i}(\phi_{i}(p), \phi_{i}(q))$$

Then define the following inner product on M which is defined by

$$g: TM \times TM \to \mathbb{R}$$

 $(p,q) \mapsto \sum_{i \in I} f_i G^i(p,q)$

2 Problem 6

2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X: M \to TM$$

We want to show that there is a point $p \in M$ such that X(p) = 0.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi: M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} = X(\phi_t(p)),$$
$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t: M \to M \in \mathrm{Diff}(M,M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H: M \times I \to M$$

 $(p,t) \mapsto \phi_t(p).$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f: M \to M$, we have

$$\Lambda(f) = \sum_{k} \operatorname{Tr} \left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_*: H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the induced map on homology, and

 $\Lambda(f) \neq 0 \iff f$ has at least one fixed point.

In particular, we have

$$\Lambda(\mathrm{id}_M) = \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})})$$
$$= \sum_k \dim H_k(X;\mathbb{Q})$$
$$= \chi(M),$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2\\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0 , thus

$$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$$
 so

$$\phi_t(p) = p$$

$$\Rightarrow \frac{\partial}{\partial t}\phi_t(p) = \frac{\partial}{\partial t}p = 0 \qquad \text{by differentiating wrt } t$$

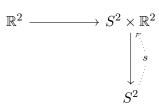
$$\Rightarrow \frac{\partial}{\partial t}\phi_t(p) \Big|_{t=1} = 0 \Big|_{t=0} = 0 \qquad \text{by evaluating at } t = 0$$

$$\Rightarrow X(\phi_1(p_0)) := \frac{\partial}{\partial t}\phi_t(p) \Big|_{t=1} = 0 \qquad \text{by definition of } \phi_1$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. \square

2.2 Part 2

The trivial bundle

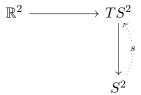


has a nowhere vanishing section, namely

$$s: S^2 \to S^2 \times \mathbb{R}^2$$

 $\mathbf{x} \to (\mathbf{x}, [1, 1])$

which is the identity on the S^2 component and assigns the constant vector [1, 1] to every point. However, as part 1 shows, the bundle



can not have a nowhere vanishing section.