Title

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Contents

1 List of Topics

Chapters 1-9 of Dummit and Foote

- Left and right cosets
- Lagrange's theorem
- Isomorphism theorems
- Group generated by a subset
- Structure of cyclic groups
- Composite groups
  - HK is a subgroup iff $HK = KH$
- Normalizer
  - $HK \leq H$ if $H \leq N_G(K)$
- Symmetric groups
  - Conjugacy classes are determined by cycle types
- Group actions
  - Actions of G on X are equivalent to homomorphisms from G into Sym(X)
- Cayley's theorem
- Orbits of an action
- Orbit stabilizer theorem
- Orbits act on left cosets of subgroups
- Subgroups of index p, the smallest prime dividing $|G|$, are normal
- Action of G on itself by conjugation
- Class equation

- p-groups
  - Have non trivial center
- p^2 groups are abelian
- Automorphisms, the automorphism group
  - Inner automorphisms
  - $Inn(G) \cong Z/Z(G)$
  - $Aut(S_n) = Inn(S_n)$ unless $n = 6$
  - $Aut(G)$ for cyclic groups
  - $G \cong Z_p^n$, then $Aut(G) \cong GL_n(Z_p)$
- Proof of Sylow theorems
- A_n is simple for $n \geq 5$
- Recognition of internal direct product
- Recognition of semi-direct product
- Classifications:
  - pq
- Free group & presentations
- Commutator subgroup
- Solvable groups
  - S_n is solvable for $n \leq 4$
- Derived series
  - Solvable iff derived series reaches e
- Nilpotent groups
  - Nilpotent iff all sylow-p subgroups are normal
  - Nilpotent iff all maximal subgroups are normal
- Upper central series
  - Nilpotent iff series reaches G
- Lower central series
  - Nilpotent iff series reaches e
- Fratini's argument
- Rings
  - I maximal iff R/I is a field
  - Zorn's lemma
  - Chinese remainer theorem
  - Localization of a domain
  - Field of fractions
  - Factorization in domains
  - Euclidean algorithm
  - Gaussian integers
  - Primes and irreducibles
  - Domains
    - Primes are irreducible
  - UFDs
    - Have GCDs
    - Sometimes PIDs
  - PIDs
    - Noetherian
    - Irreducibles are prime
    - Are UFDs

    * Have GCDs
  - Euclidean domains
    * Are PIDs
  - Factorization in $Z[i]$
  - Polynomial rings
  - Gauss' lemma
  - Remainder and factor theorem
  - Polynomials
  - Reducibility
  - Rational root test
  - Eisenstein's criterion

2 Groups

2.1 Definitions

2.1.1 Subgroup Generated by a set A

- $< A >= \{a_1^{\pm 1}, a_2^{\pm 1}, \cdots a_2^{\pm 1} : a_i \in A, n \in \mathbb{N}\}$
- Equivalently, the intersection of all H such that $A \subseteq H \leq G$

2.1.2 Free Group on a set X

- Equivalently, words over the alphabet X made into a group via concatenation

2.1.3 Centralizer of an element or a subgroup

- $C_G(a) = \{g \in G : ga = ag\}$
-
$$C_G(H) = \{g \in G : \forall h \in H, gh = hg\} = \bigcap_{h \in H} C_G(h)$$
  - Note - requires the same g on both sides!
- Facts:
  - $C_G(H) \leq G$
  - $C_G(H) \trianglelefteq N_G(H)$
  - $C_G(G) = Z(G)$
  - $C_H(a) = H \bigcap C_G(a)$

2.1.4 Center of a group

- $Z(G) = \{g \in G : \forall x \in G, gx = xg\}$
- Facts
  -
$$Z(G) = \bigcap_{a \in G} C_G(a)$$

2.1.5 Normalizer of a subgroup

- $$N_G(H) = \{g \in G : gHg^{-1} = H\}$$
- Equivalently, $\bigcup\{K : H \trianglelefteq K \leq G\}$ (the largest $K \leq G$ for which $H \trianglelefteq K$)
- Equivalently, the stabilizer of H under G acting on its subgroups via conjugation
- Differs from centralizer; can have $gh = h'g$
- Facts:
  - $C_G(H) \subseteq N_G(H) \leq G$
  - $N_G(H)/C_G(H) \cong A \leq Aut(H)$
  - Given $H \subseteq G$, let
    $$S(H) = \bigcup_{g \in G} gHg^{-1}$$
    , so $|S(H)|$ is the number of conjugates to H. Then
    $$|S(H)| = [G : N_G(H)]$$
    * i.e. the number of subgroups conjugate to H equals the index of the normalizer of H.