Linearization Continued

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Linearization Continued Section 8.4 Follow-Up

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The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

– We fixed a solution and lifted it to a sphere:

$$u \in C^{\infty}(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^{\infty}(S^2; W)$$

- We use the assumption: For every $w \in C^{\infty}(S^2, W)$ there exists a symplectic trivialization of the fiber bundle w^*TW , i.e. $\langle c_1(TW), \pi_2(W) \rangle =$ 0 where c_1 denotes the first Chern class of the bundle TW.
- We use this trivialize the pullback \tilde{u}^*TW to obtain an orthonormal unitary frame

$${Z_i}_{i=1}^{2n} \subset T_{u(s,t)}W$$

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– We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x,y)$ given by

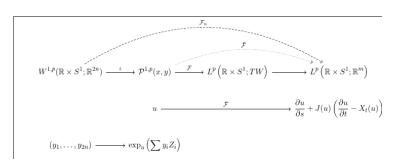
$$\iota: W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow \mathcal{P}^{1,p}(x,y)$$
$$Y = (y_1, \dots, y_{2n}) \longmapsto \exp_u\left(\sum y_i Z_i\right).$$

– We regard Y(s,t) as a tangent vector to W in some Euclidean embedding.

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- We seek to compute the composite map in charts:



Add a Tangent

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$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - J(u)X_t(u)$$

$$\mathcal{F}(u+Y) = \frac{\partial (u+Y)}{\partial s} + J(u+Y)\frac{\partial (u+Y)}{\partial t} - J(u+Y)X_t(u+Y)$$

Extract the part that is linear in *Y* and collect terms:

$$(d\mathcal{F})_{u}(Y)$$

$$= \frac{\partial Y}{\partial s} + (dJ)_{u}(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_{u}(Y)X_{t} - J(u)(dX_{t})_{u}(Y)$$

$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right)$$

$$+ \left((dJ)_{u}(Y)\frac{\partial u}{\partial t} - (dJ)_{u}(Y)X_{t} - J(u)(dX_{t})_{u}(Y)\right)$$

.

Linearization Continued Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$(d\mathcal{F})_{u}(Y) = \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right)$$

$$+ \left((dJ)_{u}(Y)\frac{\partial u}{\partial t} - (dJ)_{u}(Y)X_{t} - J(u)(dX_{t})_{u}(Y)\right)$$

$$= \sum_{i=1}^{2n} \left(\frac{\partial y_{i}}{\partial s}Z_{i} + \frac{\partial y_{i}}{\partial t}J(u)Z_{i}\right)$$

$$+ \sum_{i=1}^{2n} y_{i} \left(\frac{\partial Z_{i}}{\partial s} + J(u)\frac{\partial Z_{i}}{\partial t} + (dJ)_{u}(Z_{i})\frac{\partial u}{\partial t} - J(u)(dX_{t})_{u}Z_{i} - (dJ)_{u}(Z_{i})X_{t}\right).$$

Use the fact that this is $O_1 + O_0$ in Y.

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Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial} (y_1, \dots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}=\mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \overline{\partial} Y + SY$$

where $S \in C^{\infty}(\mathbb{R} \times S^1; \operatorname{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

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Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s,t)$$

where

- S is linear
- S tends to a symmetric operator as $s \longrightarrow \pm \infty$, and

$$\frac{\partial S}{\partial s}(s,t) \stackrel{s \longrightarrow \pm \infty}{\longrightarrow} 0$$
 uniformly in t

Collect terms in the order zero part:

$$O_{0} = \sum_{i=1}^{2n} y_{i} \left[\frac{\partial Z_{i}}{\partial s} + J(u) \frac{\partial Z_{i}}{\partial t} + (dJ)_{u}(Z_{i}) \frac{\partial u}{\partial t} - J(u)(dX_{t})_{u}Z_{i} - (dJ)_{u}(Z_{i})X_{t} \right]$$

$$= \sum_{i=1}^{2n} y_{i} \left[\frac{\partial Z_{i}}{\partial s} + (dJ)_{u}(Z_{i}) \left(\frac{\partial u}{\partial t} - (Z_{i})X_{t} \right) + J(u) \frac{\partial Z_{i}}{\partial t} - J(u)(dX_{t})_{u}Z_{i} \right].$$

- Claim: the terms in blue and orange vanish in the limit $s \longrightarrow \pm \infty$, so it suffices to prove that the remaining yields a symmetric operator.

Proof

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- The term in blue vanishes as $s \longrightarrow \pm \infty$
 - Using the fact that u is a solution
 - Uses $\frac{\partial u}{\partial s}$ 0 uniformly (as do its derivatives?)
- Suffices to show the remaining part is symmetric in the limit