Title

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1 | Monday, November 09

1.1 Strong Linkage

We have two categories:

- $G_r T$, with a notion of *strong linkage*, and
- G_r, which instead only has *linkage*.

We'll restate a few theorems.

> **Theorem 1.1.1***(?).*
> Let $\lambda, \mu \in X(T)$.
>
> 1. If $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_r T} \neq 0$, then $\mu \uparrow \lambda$ are strongly linked.
>
> 2. If $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$, then $\mu \in W_p \cdot \lambda + p^r X(T)$.

Example 1.1.1(?): In the case of $\Phi = A_2$, we'll consider the two different categories.
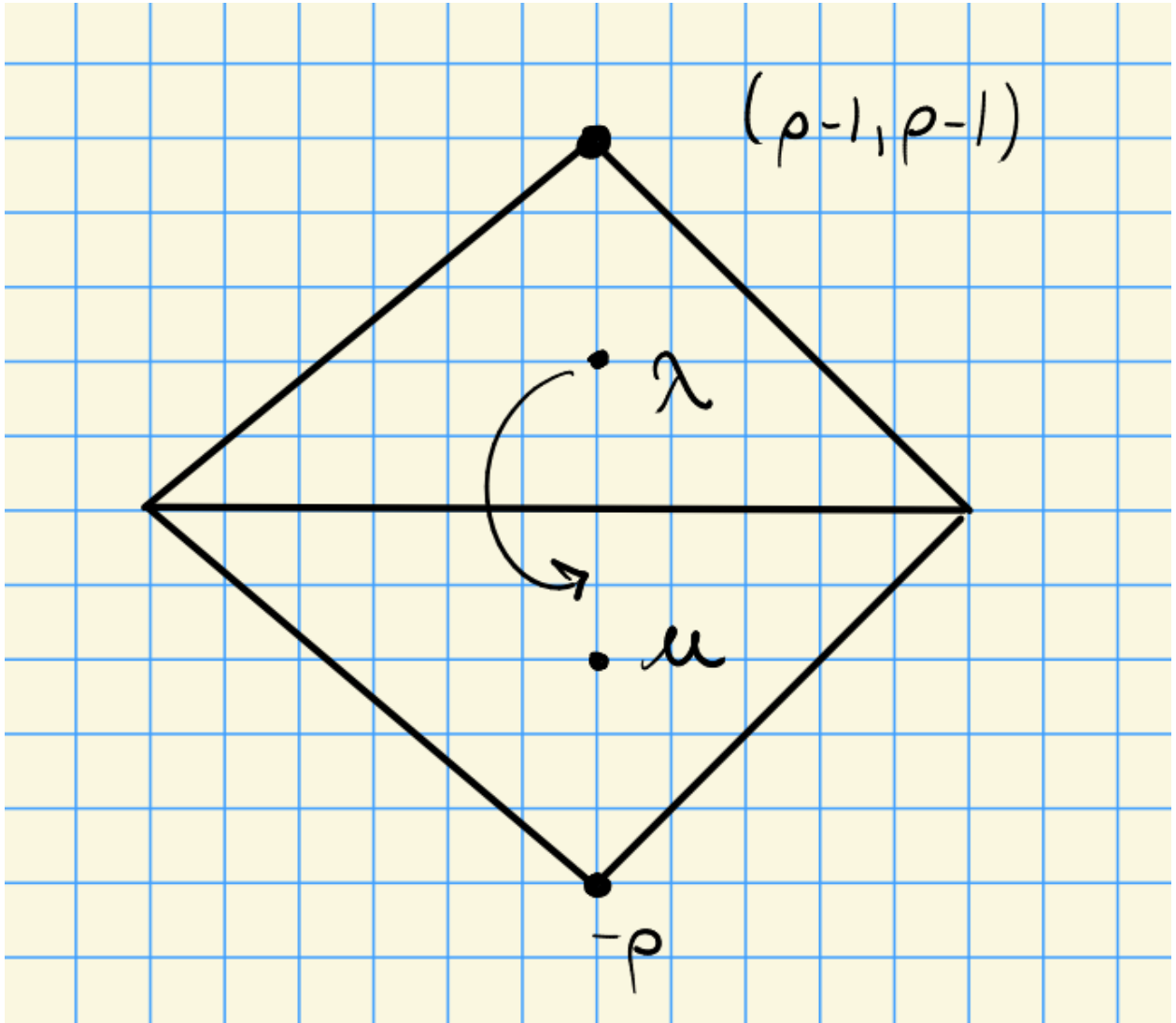
We have the following picture for $\widehat{Z}$:

Figure 1: Image

Considering $X_1(T)$ and $[\widehat{Z}_1(\lambda) : \widehat{L}_1(\mu)] \neq 0$, and $\widehat{Z}_1(\lambda)$ has 6 composition factors as G_1T-modules.

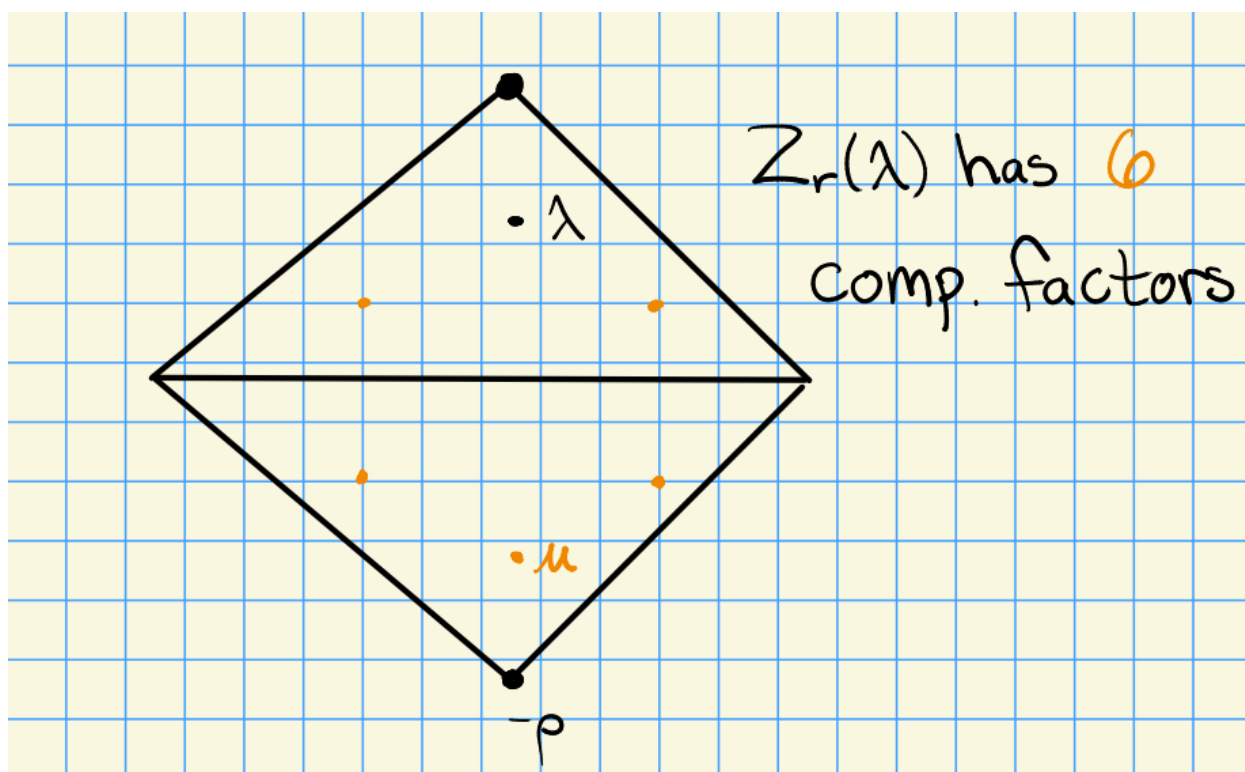On the other hand, for Z, we have the following:

Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

1.2 Extensions

Let $\lambda, \mu \in X(T)$. We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\text{Ext}^j_{G_r T}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) = \text{Ext}^j_{G_r}\left(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)\right) \qquad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\text{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right) = \bigoplus_{\gamma \in X(T)} \text{Ext}^j_{G_r}\left(L_r(\lambda), L_r(\mu)\right)_\gamma$$

where we are taking the fixed points under the torus T action on the first factor (for which T_r acts

trivially). We can write this as

$$\cdots = \bigoplus_{\gamma \in X(T)} \mathrm{Ext}^{j}_{G_r}\left(L_r(\lambda), L_r(\mu) \otimes \gamma\right)$$

$$= \bigoplus_{\gamma \in X(T)} \mathrm{Ext}^{j}_{G_r T}\left(L_r(\lambda), L_r(\mu) \otimes p^r v\right)$$

$$= \bigoplus_{v \in X(T)} \mathrm{Ext}^{j}_{G_r T}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)\right).$$

So if we know extensions in the G_r category, we know them in the $G_r T$ category.

There is an isomorphism

$$\mathrm{Ext}^{1}_{G_r T}\left(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)\right) \cong \mathrm{Hom}_{G_R T}\left(\mathrm{rad}_{G_r T}\widehat{Z}_r(\lambda), \widehat{L}_r(\mu)\right).$$

Finally, for $\lambda, \mu \in X(T)$, if the above Ext^1 vanishes, then $\lambda \in W_p \cdot \mu$ (i.e. λ and μ are linked).

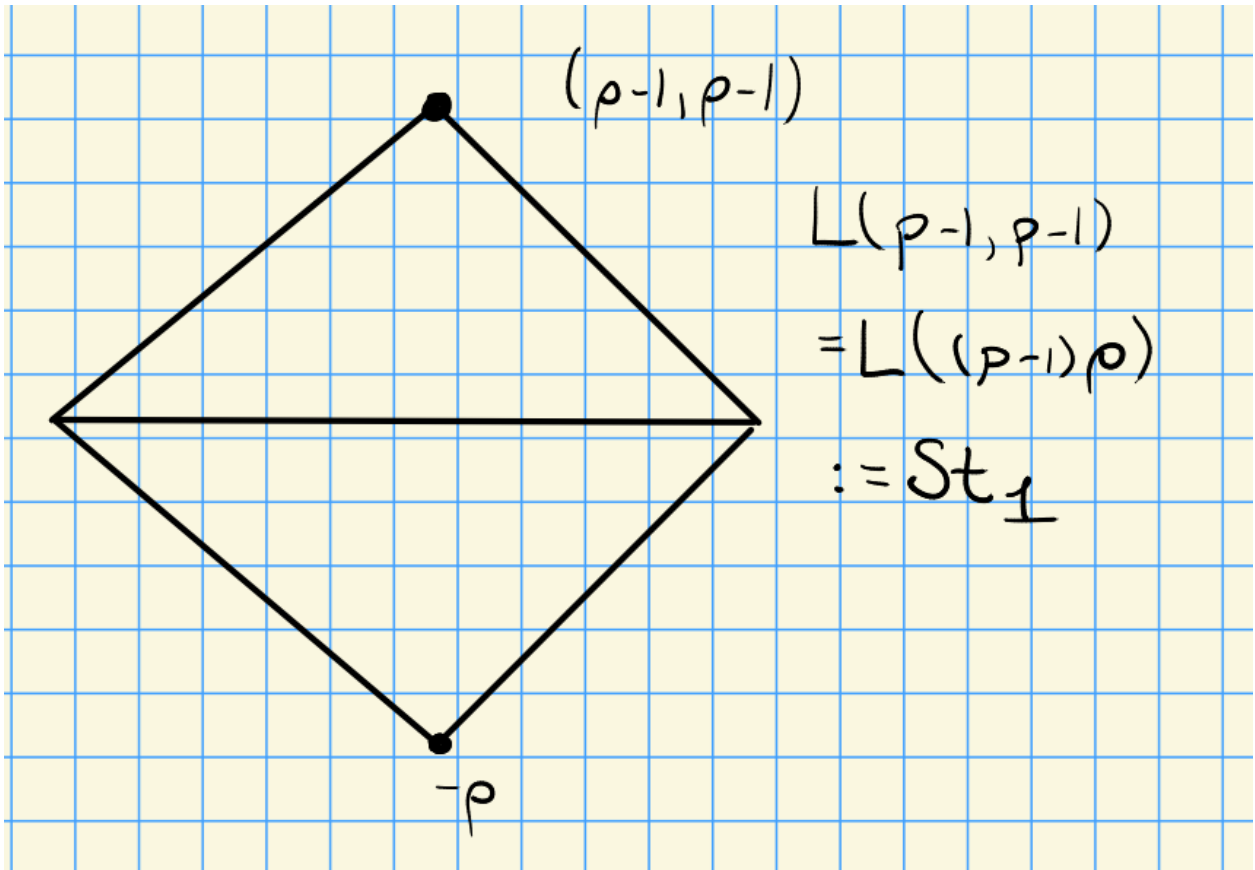## 1.3 The Steinberg Modules

Example 1.3.1(Steinberg): Consider A_2:

Figure 3: Image

Taking the representation corresponding to $(p-1, p-1)$ yields the "first Steinberg module"

$$L(p-1, p-1) = L((p-1)\rho) := \mathrm{St}_1.$$

In this case, we have an equality of many modules:

$$H^0((p-1)\rho) = L((p-1)\rho) = V((p-1)\rho) = T((p-1)\rho).$$

> **Definition 1.3.1** (Steinberg Modules)**.**
> The rth **Steinberg module** is defined to be $L((p^r - 1)\rho)$.

Remark 1.3.1: In general, we have

$$L((p^r - 1)\rho) = H^0((p^r - 1)\rho) = V((p^r - 1)\rho).$$

We also have

$$\widehat{Z}_r((p^r - 1)\rho) \cong L((p^r - 1)\rho) \downarrow_{G_r T} .$$

Theorem 1.3.1*(?).*
The Steinberg module is both injective and projective as