Title

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Recall the killing form:

$$\begin{aligned}\kappa : \mathfrak{g}^2 &\to \mathbb{F} \\ (x, y) &\mapsto \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y).\end{aligned}$$

and Cartan's criteria:

1. $\mathfrak{g}$ is solvable $\iff \kappa(x, y) = 0 \; \forall x \in \mathfrak{g}, \mathfrak{g}], \; y \in \mathfrak{g}$.
2. $\mathfrak{g}$ is semisimple $\iff \kappa$ is non-degenerate.

Theorem: If $\mathfrak{g}$ is semisimple, then

a. $\mathfrak{g} = \bigoplus_{i=1}^n I_i$ for some $I_i \trianglelefteq \mathfrak{g}$ which are all simple.
b. Every simple ideal $I \trianglelefteq \mathfrak{g}$ is one of the I_i.
c. $\kappa_{I_i} = \kappa_{\mathfrak{g}} \mid_{I_i \times I_i}$.

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Proof of (a): Use induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ has no nonzero proper ideals, then $\mathfrak{g}$ is simple and we're done.

Otherwise, let I_1 be a minimal nonzero ideal of $\mathfrak{g}$. Then $I_1^\perp \trianglelefteq \mathfrak{g}$ is also an ideal, and thus $I := I_1 \bigcap I_1^\perp \trianglelefteq \mathfrak{g}$ is as well. Then for all $x \in [I, I]$, we must have $\kappa(x, y) = 0$ for any $y \in I \subseteq I_1^\perp$. So I is solvable, and thus $I = 0$. So $\mathfrak{g} = I_1 \oplus I_1^\perp$.

Note that any ideal of $I_1^\perp$ is also an ideal of $\mathfrak{g}$, which implies that $\operatorname{rad}(I_1^\perp) \subseteq \operatorname{rad}(\mathfrak{g})$, which is zero since $\mathfrak{g}$ is semisimple, and thus $I_1^\perp$ is semisimple as well.

By the inductive hypothesis, $I_1^\perp = I_2 \oplus \cdots \oplus I_n$ where each $I_j \trianglelefteq I_i^\perp$ is simple. Then $I_j \trianglelefteq \mathfrak{g} \implies [I_1, I_j] \subset I_1 \bigcap I_j$, since I_1 has no contribution. But this is a subset of $I_1 \bigcap I_1^\perp = 0$. $\square$

Proof of (b): If $I \trianglelefteq \mathfrak{g}$, then $[I, \mathfrak{g}] \trianglelefteq I$ because $[[I, \mathfrak{g}], I] \subseteq [I, I] \subseteq [I, \mathfrak{g}]$.

Since $\mathfrak{g}$ is semisimple, $0 = \mathrm{rad}(\mathfrak{g}) \supseteq Z(\mathfrak{g})$. So $[I, \mathfrak{g}] \neq 0$, and thus $[I, \mathfrak{g}] = I$ since I is simple. But then $[I, \mathfrak{g}] = \bigoplus [I, I_i]$ is simple as well. So only one direct summand can survive, since otherwise this would produce at least 2 nontrivial ideals, and $[I, \mathfrak{g}] = [I, I_i]$ for some i.

So for all $j \neq i$, we must have $I_j \bigcap I = I_j \bigcap [I, I_i] = 0$, and so $I \subseteq I_i$. But then $I = I_i$ since I_i itself is simple, and we're done.

Proof of (c):

(Without using the simplicity of I_i)

For $x, y \in I_i$, we have

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1.1 Inner Derivations

Recall that ad $\mathfrak{g} \subseteq \mathrm{Der}\mathfrak{g}$, and in fact (lemma) this is an ideal.

Theorem: If $\mathfrak{g}$ is semisimple, then ad $\mathfrak{g} = \mathrm{Der}\mathfrak{g}$.

Proof of lemma:

For all $\delta \in \mathrm{Der}\mathfrak{g}$ and all $x, y \in \mathfrak{g}$, we have

$$
\begin{aligned}
[\delta, \mathrm{ad}\,_x](y) &= \delta([x, y]) - [x, \delta(y)] \\
&= [\delta(x), y] \\
&= [\mathrm{ad}\,_{\delta(x)}](y),
\end{aligned}
$$

and so $[\delta, \mathrm{ad}\ x] \subseteq \mathrm{ad}\ \mathfrak{g}$. $\square$

Proof of theorem:

If $\mathfrak{g}$ is semisimple, then $0 = \mathrm{rad}\mathfrak{g} \supseteq Z(\mathfrak{g}) = \ker \mathrm{ad}\,$. Thus ad $\mathfrak{g} \cong g / \ker \mathrm{ad}\, \cong \mathfrak{g}$ is also semisimple.

This means that $\kappa_{\mathrm{ad}\ \mathfrak{g}}$ is non-degenerate, and thus ad $\mathfrak{g} \bigcap (\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, where $(\mathrm{ad}\ \mathfrak{g})^{\perp} \trianglelefteq \mathrm{Der}(\mathfrak{g})$.

(Note that the non-degeneracy of κ already forces $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$.)

Then $[(\mathrm{ad}\ \mathfrak{g})^{\perp}, \mathrm{ad}\ \mathfrak{g}] = 0$, and so for all $\delta \in (\mathrm{ad}\ \mathfrak{g})^{\perp}$, we have $\delta(x) = [\delta, \mathrm{ad}\ x]$ by the lemma, but we've shown that this is zero.

But then δ must be zero because ad is an isomorphism, and in particular it is injective. This means that $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, and thus ad $\mathfrak{g} = \mathfrak{g}$. $\square$

We can use this to define an abstract Jordan decomposition by pulling back decompositions on adjoints:

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