Homework 6

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1 Homework Problems

1.1 Problem 1

Todo

1.2 Problem 2

We can note that since f has 4 roots, the Galois group G of its splitting field will be a subgroup of S_4. Moreover, G must be a *transitive subgroup* of S_4, i.e. the action of G on the roots of f should be transitive. This reduces the possibilities to $G \cong S^4, A^4, D^4, \mathbb{Z}_4, \mathbb{Z}_2^2$.

Since f has exactly 2 real roots and thus a pair of roots that are complex conjugates, the automorphism given by complex conjugation is an element of G. But this corresponds to a 2-cycle $\tau = (ab)$, and we can then make the following conclusions:

- Not A_4: A_4 contains only even cycles, and τ is odd.
- Not Z_4: This subgroup is generated by a single 4-cycle σ, which up to conjugacy is (1234), and σ^n is not a 2-cycle for any n.

- Not $\mathbb{Z}_2^2$: In order to be transitive, this subgroup must be $\{e, (12)(34), (13)(24), (14)(23)\}$, which does not contain τ.

The only remaining possibilities are S^4 and D^4. $\square$

1.3 Problem 3

1.3.1 Part 1

To see that $\phi(n)$ is even for all $n > 2$, we can take a prime factorization of n and write

$$\phi(n) = \phi\left(\prod_{i=1}^{m} p_i^{k_i}\right) = \prod_{i=1}^{m} \phi(p_i^{k_i}) = \prod_{i=1}^{m} p^{k_i - 1}(p-1) = \prod_{i=1}^{m} p^{k_i-1} \prod_{i=1}^{m} (p-1)$$

where each $k_i \geq 1 \implies k_i - 1 \geq 0$. But every prime power is odd, and a product of odd numbers is odd, so the first product is odd. It is also true that $p-1$ is even for every prime p, and the second term is a product of even terms and thus even. So $\phi(n)$ is the product of an even and an odd number, which is always even.

1.3.2 Part 2

Suppose $\phi(n) = 2$. Take a prime factorization of n, so we have

$$2 = \phi(n) = \prod_{i=1}^{m} \phi(p_i^{k_i})$$

Since the only factors of 2 are 1 and 2, we must have $\phi(p_i^{k_i}) = 2$ for exactly one i, and the rest must be equal to 1.

Consider the term that equals 2. We have $\phi(p_i^{k_i}) = p^{k_i-1}(p-1) = 2$, so we must have either

- Case 1: $p - 1 = 2$ and $p^{k_i - 1} = 1$, so $p = 3$ and $k_i = 1$. So $3 \,\Big|\, n$, but 3^ℓ does *not* divide n for any $\ell > 1$.
- Case 2: $p^{k_i - 1} = 2$ and $(p-1) = 1$, so $p = 2$ and $k_i = 2$. Thus 2^2 divides n but 2^ℓ does not for any $\ell > 2$.

In either case, it remains to check are whether the other factors where $\phi(p_j^{k_j}) = 1$ can contribute any other distinct divisors to n. We can note that $\phi(p_j^{k_j}$ iff $p^{k_j-1}(p-1) = 1$, so this forces $p = 2$ and $k_j = 1$. So n may or may not contain a single factor of 2, but by uniqueness of prime factorization, this can only happen in case 1. Note that this also forces $2 \,\Big|\, n$ but 2^2 does not divide n.

In summary, we've found that $\phi(n) = 2$ implies that

- $3 \,\Big|\, n, 9 \not\Big|\, n$, and
  - $2 \,\Big|\, n, 4 \not\Big|\, n$
  - $2 \not\Big|\, n$
- $2^2 \,\Big|\, n, 2^3 \not\Big|\, n$.

This reduces the possibilities to the finite set $n \in \{6, 3, 4\}$, and $\phi(6) = \phi(3) = \phi(4) = 2$. $\square$

1.4 Problem 4

1.5 Problem 5

1.6 Problem 6

2 Qual Problems

2.1 Problem 1

2.2 Problem 2

2.3 Problem 3