Title

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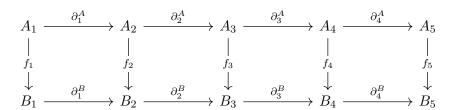
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Problem 1.0.1 (Weibel 1.3.3)

Prove the 5-lemma. Suppose the following rows are exact:



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- a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
- b. Show that if f_2 , f_4 are epi and f_5 is monic, then f_3 is an epi.
- c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):

Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with " \rightarrow " and the epis with " \rightarrow ":

$$A_{1} \xrightarrow{\partial_{1}^{A}} A_{2} \xrightarrow{\partial_{2}^{A}} A_{3} \xrightarrow{\partial_{3}^{A}} A_{4}$$

$$\downarrow \qquad \downarrow$$

$$\downarrow \qquad \qquad \qquad \qquad \qquad \downarrow$$

$$B_{1} \xrightarrow{\partial_{1}^{B}} B_{2} \xrightarrow{\partial_{2}^{B}} B_{3} \xrightarrow{\partial_{3}^{B}} B_{4}$$

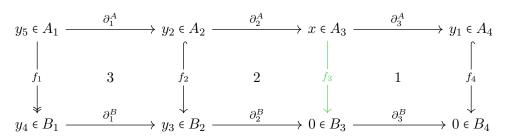
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It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2 .
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

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- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \rightarrow A_2 \rightarrow A_3$, so x = 0.



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Solution (Part (b)):

Similarly, it suffices to consider the following portion of the diagram:

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We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

Step 1 (the right-hand side approximation):

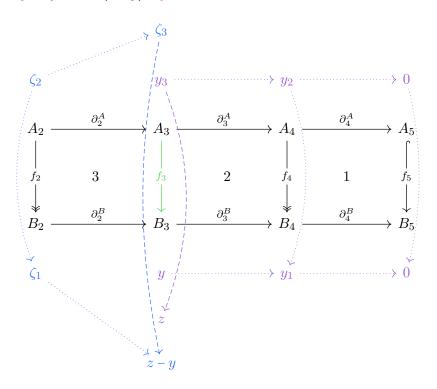
- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4 , pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \rightarrow A_5 \rightarrow B_5$.
- By injectivity of f_5 , the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z y \mapsto y_1 y_1 = 0$ under $B_3 \to B_4$, using that d(z y) = d(z) d(y) since these are R-module morphisms.

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- By exactness of the bottom row, pull z y back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2 , pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z y$ under $A_3 \to B_3$.
- But then $y_3 \zeta_3 \mapsto z (z y) = y$ under $A_3 \to B_3$ as desired.



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Problem 1.0.2 (Weibel 1.4.2)

Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$C_n \cong Z_n \oplus B'_n$$

 $Z_n = B_n \oplus H'_n$.

Show that C is split exact if and only if $H'_n = 0$.

Problem 1.0.3 (Weibel 1.4.3)

Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

Problem 1.0.4 (Weibel 1.4.5)

Show that chain homotopy classes of maps form a quotient category K of Ch(R-mod) and that the functors H_n factor through the quotient functor $Ch(R\text{-mod}) \to K$ using the following steps:

1. Show that chain homotopy equivalence is an equivalence relation on

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 $\{f: C \to D \mid f \text{ is a chain map}\}$. Define $\operatorname{Hom}_K(C,D)$ to be the equivalence classes of such maps and show that it is an abelian group.

- 2. Let $f \simeq g: C \to D$ be two chain homotopic maps. If $u: B \to C, v: D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
- 3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $Ch(R\text{-mod}) \to K$ is an additive functor.
- 4. Is K an abelian category? Explain.

Try at least two parts.

Problem 1.0.5 (Weibel 1.5.1) Let $cone(C) := cone(\mathbb{1}_C)$, so

 $cone(C)_n = C_{n-1} \oplus C_n.$

Show that cone(C) is split exact, with splitting map given by $(b, c) \mapsto (-c, 0)$.

Problem 1.0.6 (Weibel 1.5.2)

Let $f: C \to D \in \text{Mor}(\text{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(s, f) : \operatorname{cone}(C) \to D.$$

Problem 1.0.7 (Extra)

- a. Show that free implies projective.
- b. Show that $\operatorname{Hom}_R(M, \cdot)$ is left-exact.
- c. Show that P is projective if and only if $\operatorname{Hom}_R(P,\cdot)$ is exact.

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