Problem Set 1

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Contents

1	Pro	blem 5																										1
	1.1	Part 1																										1
	1.2	Part 2														•												1
2	2 Pro	Problem 6																1										
	2.1	Part 1																										1
		Part 2																										_

1 Problem 5

1.1 Part 1

Let $A \in \operatorname{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in Math(n, n)$ be non-negative definite, so

$$\langle v, Bv \rangle \ge 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, ABv \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

1.2 Part 2

2 Problem 6

2.1 Part 1

Let $M=S^2$ as a smooth manifold, and consider a vector field on M,

$$X:M\to TM$$

We want to show that there is a point $p \in M$ such that X(p) = 0.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi: M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} = X(\phi_t(p)),$$

$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t: M \to M \in \mathrm{Diff}(M,M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H: M \times I \to M$$

 $(p,t) \mapsto \phi_t(p).$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f: M \to M$, we have

$$\Lambda(f) = \sum_{k} \operatorname{Tr} \left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_*: H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f$$
 has at least one fixed point.

In particular, we have

$$\Lambda(\mathrm{id}_M) = \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})})$$
$$= \sum_k \dim H_k(X;\mathbb{Q})$$
$$= \chi(M),$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2\\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0 , thus $\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{split} \phi_t(p) = p \\ \Longrightarrow \frac{\partial}{\partial t} \phi_t(p) = & \frac{\partial}{\partial t} p = 0 \\ \Longrightarrow \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \Big|_{t=0} = 0 \end{split} \qquad \text{by differentiating wrt } t \\ \Longrightarrow \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \Big|_{t=0} = 0$$
 by evaluating at $t=0$
$$\Longrightarrow X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0$$
 by definition of ϕ_1

so $X(\phi_1(p_0))=0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. \square

2.2 Part 2

The trivial bundle

$$\mathbb{R}^2 \longrightarrow S^2 imes \mathbb{R}^2$$

$$\downarrow^{r}$$

$$S^2$$

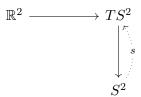
$$S^2$$

has a nowhere vanishing section, namely

$$s: S^2 \to S^2 \times \mathbb{R}^2$$

 $\mathbf{x} \to (\mathbf{x}, [1, 1])$

which is the identity on the S^2 component and assigns the constant vector [1, 1] to every point. However, as part 1 shows, the bundle



can *not* have a nowhere vanishing section. \Box