

Problem Set 2

D. Zack Garza

Wednesday 16th September, 2020

1 | Exercises

Exercise 1.1 (Gathmann 2.17).

Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

Solution:

Since $x = yz$ for all points in X , we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V(x - yz, y(z^2 - y)) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &:= X_1 \cup X_2. \end{aligned}$$

Claim: These two subvarieties are irreducible.

It suffices to show that the $A(X_i)$ are integral domains. We have

$$A(X_1) := \mathbb{C}[x, y, z] / \langle x - yz, y \rangle \cong \mathbb{C}[y, z] / \langle y \rangle \cong \mathbb{C}[z],$$

which is an integral domain since \mathbb{C} is a field and thus an integral domain, and

$$A(X_2) := \mathbb{C}[x, y, z] / \langle x - yz, z^2 - y \rangle \cong \mathbb{C}[y, z] / \langle z^2 - y \rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

Exercise 1.2 (Gathmann 2.18).

Let $X \subset \mathbb{A}^n$ be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

Solution:

Identify \bar{X} as the intersection of all closed sets containing X .

Claim: If X is closed (so $\bar{X} = X$), we have $V(I(X)) = X$.

$\bar{X} \subseteq$:

We have $X \subseteq V(I(X))$ and since $V(J)$ is closed in the Zariski topology for any ideal $J \subseteq k[x_1, \dots, x_n]$ by definition, $V(I(X))$ is closed. Thus $\bar{X} \subseteq V(I(X))$ as well.

$V(I(X)) \subseteq \bar{X}$:

Using the order-preserving property of $V(I(\cdot))$ (noting that $V(\cdot), I(\cdot)$ are individually order-reversing), we have

$$X \subseteq \bar{X} \implies V(I(X)) \subseteq V(I(\bar{X})) = \bar{X},$$

where we've used the above claim in the last equality.

Exercise 1.3 (Gathmann 2.21).

Let $\{U_i\}_{i \in I} \rightrightarrows X$ be an open cover of a topological space with $U_i \cap U_j \neq \emptyset$ for every i, j .

- Show that if U_i is connected for every i then X is connected.
- Show that if U_i is irreducible for every i then X is irreducible.

Solution:

?

Exercise 1.4 (Gathmann 2.22).

Let $f : X \rightarrow Y$ be a continuous map of topological spaces.

- Show that if X is connected then $f(X)$ is connected.
- Show that if X is irreducible then $f(X)$ is irreducible.

Solution:

?

Definition 1.0.1 (Ideal Quotient).

For two ideals $J_1, J_2 \subseteq R$, the *ideal quotient* is defined by

$$J_1 : J_2 := \left\{ f \in R \mid f J_2 \subseteq J_1 \right\}.$$

Solution:

?

Exercise 1.5 (Gathmann 2.23).

Let X be an affine variety.

- a. Show that if $Y_1, Y_2 \subset X$ are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

- b. If $J_1, J_2 \subseteq A(X)$ are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

Solution:

?

Exercise 1.6 (Gathmann 2.24).

Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible affine varieties, and show that $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible.

Solution:

?