Title

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1.1 Singularities

Recall that there are three types of singularities:

- Removable
- Poles
- Essential

Recall that a function g is holomorphic at z_0 iff

$$\lim_{z \longrightarrow z_0} (z - z_0) g(z) = 0$$

Theorem 1.1 *(3.2).*
An isolated singularity z_0 of f is a pole $\iff \lim_{z \longrightarrow z_0} f(z) = \infty$.

Theorem 1.2 *(3.3, Casorati-Weierstrass).*
If f is holomorphic in $D_r(z_0) \setminus \{z_0\}$ and has an essential singularity z_0, then there exists a radius r such that $f(D_r(\{z_0\}) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof .
Proceed by contradiction. Suppose there exists a $w \in \mathbb{C}$ and a $\delta > 0$ such that

$$D_\delta(w) \bigcap f(D_r(\{z_0\}) \setminus \{z_0\}) = \emptyset.$$

If $z \in D_r(w) \setminus z_0$, then $|f(z) - w| > \delta$. Define $g(z) = \dfrac{1}{f(z) - w}$ on $D_r(z_0) \setminus \{z_0\}$; then $|g(z)| < \dfrac{1}{\delta}$.

Note that this implies that $g(z)$ is holomorphic on $D_r(z_0) \setminus \{z_0\}$. $g(z)$ being holomorphic here follows from f being holomorphic here.

Then $g(z)$ has a removable singularity at $z = z_0$ by theorem 3.1.
If $g(z_0) \neq 0$, then $f(z) - w$ is holmorphic at z_0, contradicting the fact that z_0 is an essential singularity.
If instead $g(z_0) = 0$, then z_0 is a pole, again a contradiction.

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Note: revisit why this is a contradiction.

1.2 Singularities at Infinity

The point $z = \infty$ can be one of three types of singularities:

1. *Removable* $\iff f(z) = \sum_{k=-1}^{\infty} c_k \frac{1}{z^k}$.
   - I.e. only one positive exponent.
2. *Pole* $\iff f(z) = \sum_{k=-\infty}^{n} c_k z^k$
   - I.e. there are finitely many positive exponents.
3. *Essential* $\iff f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$
   - There are infinitely many positive exponents.

Definition 1.2.1 (Meromorphic)**.**
A function f is **meromorphic** on Ω iff there exists a sequence $\{z_i\} \subset \Omega$ with no limit point in Ω such that

1. f is holomorphic on $\Omega \setminus \{z_i\}$, and
2. f has poles at each z_i.

Theorem 1.3 ***(3.4, Meromorphic Functions are Rational).***
f is meromorphic on $\mathbb{CP}^1$ iff f is a rational function.

Proof .

$\implies$: By part 1 of the definition above, the point $z = 0$ is either a pole or a removable singularity of the function $F(z) = f\left(\frac{1}{z}\right)$. By part 2, F has finitely many poles $\{z_k\}_{k=1}^{N}$. So for each k, write

$$f(z) = f_k(z) + g_k(z)$$

where f_k is the principal part and g_k is holomorphic in a neighborhood of z_k. Then $f_k(z)$ is a polynomial in $\left(\dfrac{1}{z-z_k}\right)$, say of degree m_k. But then

$$F(z) \coloneqq f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

where $\tilde{f}_\infty(z)$ is a polynomial in z, and $\tilde{g}_\infty(z)$ is holomorphic near zero. Thus $\tilde{f}_\infty\left(\dfrac{1}{z}\right)$ is a polynomial in $\dfrac{1}{z}$.

Define $f_\infty(z) = \tilde{f}_\infty\left(\dfrac{1}{z}\right)$ and

$$H(z) = f(z) - f_\infty(z) - \sum_k f_k(z).$$

Then H is entire and bounded and thus constant, and since $\lim\limits_{z\longrightarrow\infty} H(z) = 0$, H is identically zero. Thus

$$f(z) = f_\infty(z) + \sum_k f_k(z)$$

$\Longleftarrow$: To be continued, uses the argument principle, Rouche's theorem, and Jordan's lemma.

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