

Problem Sets

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1 1.1

1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee / \Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$. Prove that $M^{[\lambda]}$ is a $U(\mathfrak{g})$ -submodule of M and that M is the direct sum of finitely many such submodules.

1.2 b

Deduce that the weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee / \Lambda_r$.

2 1.3*

Show that $M(\lambda)$ has the following property: for any $M \in \mathcal{O}$,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M) \cong \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M),$$

where $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}}$ is the restriction functor.

Hint: use the universal mapping property of tensor products.

3 Relevant information (?):

3.1 1

- $\mathfrak{h} \leq \mathfrak{g}$ is the Cartan subalgebra.
 - In finite-dimensional setting: maximal toral
 - Nilpotent subalgebra, i.e. LCS terminates, i.e. $\text{ad}_h = [h, \cdot]$ is a nilpotent operator so $\text{ad}_h^n = 0$ for some n .
 - Self-normalizing, so for a fixed y , $[h, y] \in \mathfrak{h} \ \forall h \in \mathfrak{h} \implies y \in \mathfrak{h}$.
- $\lambda \in \mathfrak{h}^\vee$ is a linear functional $\lambda : \mathfrak{h} \rightarrow \mathbb{C}$
 - λ is a root relative to \mathfrak{h} if $\lambda \neq 0$ and there is some $g \in \mathfrak{g}$ such that $[hg] = \lambda(h)g$ for all $h \in \mathfrak{h}$.
- $\Phi \subset \mathfrak{h}^\vee$ is the root system of \mathfrak{g} relative to \mathfrak{h} .
 - Each $\lambda \in \Phi$ is a root
 - Each root λ has a corresponding root space $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [hx] = \lambda(h)x \ \forall h \in \mathfrak{h}\}$.
- $\Lambda_r := \text{span}_{\mathbb{Z}} \{\lambda \in \Phi\} \subset \mathbb{C}^n$ is the root lattice.
- $M_\mu := \{v \in M \mid h \cdot v = \mu(h)v \ \forall h \in \mathfrak{h}\}$ is the weight space for μ .
- $M^{[\lambda]} = \sum_{\mu \in [\lambda]} M_\mu$

$M \in \mathcal{O} \implies$

- M is finitely generated as a $U(\mathfrak{g})$ -module.
- M is a weight module, so $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$
- For every $v \in M$, $U(\mathfrak{n}) \cdot v$ is finite-dimensional

3.2 2

$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ where $\mathfrak{b} \leq \mathfrak{g}$ is a fixed Borel subalgebra corresponding to a choice of positive roots, and \mathbb{C}_λ is the 1-dimensional \mathfrak{b} -module defined for any $\lambda \in \mathfrak{h}^\vee$ by the fact that $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ and thus $\mathfrak{n} \curvearrowright \mathfrak{h}$ can be taken to be a trivial action. The induction functor is given by $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})}(\cdot)$.

The restriction functor is given by $\text{Res}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = ?$

Frobenius Reciprocity for groups looks like

$$\begin{aligned} \text{hom}_{k[G]}(k[G] \otimes_{k[H]} V, W) &\rightarrow \text{hom}_{k[H]}(V, W) \\ \lambda &\mapsto 1 \otimes (\cdot) = (v \mapsto \lambda(1 \otimes v)) \\ (g \otimes v \mapsto g \cdot f(v)) &\leftarrow f. \end{aligned}$$