Homotopy Groups of Spheres

Graduate Student Seminar

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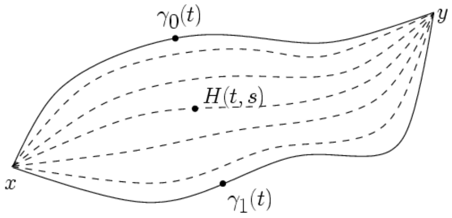
Introduction

Outline

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

Intuition

Homotopies of paths:



- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \hom_{\mathrm{Top}}(I, X)$$
$$H \in \hom_{\mathrm{Top}}(I \times I, X).$$

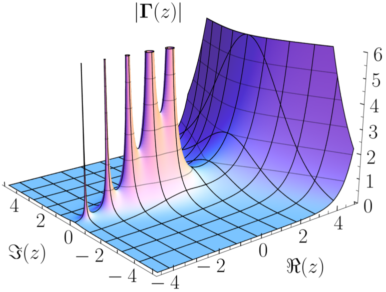
- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

Intuition

- Why care about path homotopies? Historically: contour integrals in $\mathbb{C}$



- By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_\gamma f(z)\ dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \hom_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \hom_{\text{Top}}(X \times I, Y)$$

such that $F(0) = f, F(1) = g$.

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] \coloneqq \hom_{\text{Top}}(X, Y)/\sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \hom(X, Y) \\ g \in \hom(Y, X) \end{cases} \text{ such that } \begin{cases} f \circ g \sim \mathrm{id}_Y \\ g \circ f \sim \mathrm{id}_X \end{cases}$$

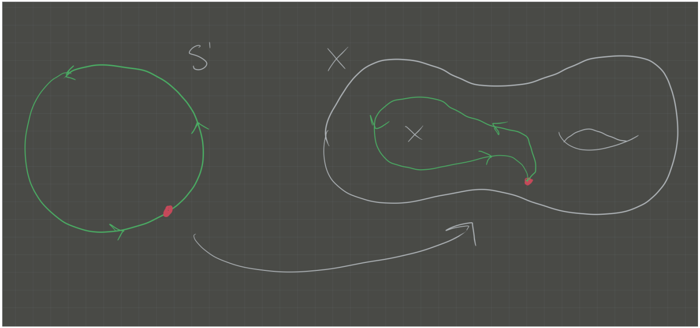
- Similarly write

$$[X] = \left\{ Y \in \text{Top} \;\middle|\; Y \sim X \right\}.$$

The Fundamental Group

- $\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

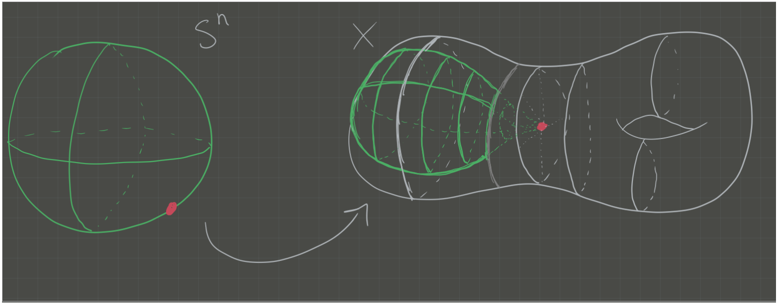
$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

- Can now generalize to define

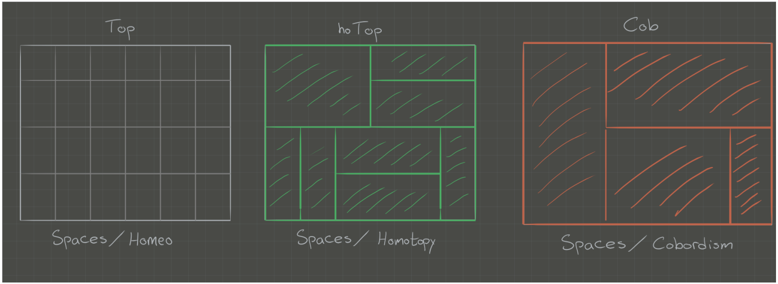
$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look $\mathbb{A}^1$ or $\mathbb{P}^1$.

Classification

- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category DTop, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G, the functor

$$\begin{aligned} h_G(\,\cdot\,) : \text{hoTop}^{\text{op}} &\longrightarrow \text{Set} \\ X &\mapsto G\text{-bundles over } X \end{aligned}$$

  is representable by a space called BG (Brown's representability theorem).
- Letting $I(G, X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism $I(G, X) \cong [X, BG]$. In general, identify $G = \mathrm{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

  Note that for a poset of spaces $(M_i, \hookrightarrow)$, the space $M^\infty := \varinjlim M_i$. This are infinite dimensional "Hilbert manifolds".

Proof:

$$I(\mathbb{R}^1, X) = [X, B(\mathrm{GL}(1, \mathbb{R}))] \tag{1}$$

Point 1

Point 2

Examples

Sphere 1