Problem 5. Let $f_n \in L^2[0,1]$ for $n \in \mathbb{N}$. Assume that

(a) $\|f_n\|_2 \le n^{-51/100}$, for all $n \in \mathbb{N}$, and

(b) $\hat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, that is

$\hat{f}_n(k) = \int_0^1 f_n(x)\, e^{-2\pi i k x}\, dx = 0$, for $k \notin [2^n, 2^{n+1}]$.

Prove that $\boxed{\sum_{n=1}^\infty f_n}$ converges in the Hilbert space $\boxed{L^2([0,1])}$.

(Hint: Plancherel's identity may be helpful.)

$n^{-\frac{51}{100}} \le n^{-\frac{1}{2}}$

Defs

- $L_2([0,1]) := \{ f : \mathbb{R} \to \mathbb{R} \mid \|f\|_{L^2([0,1])} < \infty \}$

where

$$\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p dx \right)^{1/p} \quad \text{for } 1 \le p < \infty$$

- A Hilbert space is
  1) A vector space H, (usually over $\mathbb{C}$)
     possibly $\dim X = \infty$
  2) With an inner product $\langle \cdot, \cdot \rangle : H \to \mathbb{C}$

a+a'+b= Hermitian form

a) Linearity: $\langle \alpha x + y, z \rangle = \alpha \langle x+y, z\rangle = \alpha(\langle x,z\rangle + \langle y,z\rangle)$

a') Sesquilinearity: $\langle x, \alpha z\rangle = \bar{\alpha}\langle x, z\rangle$

b) Conjugate symmetry: $\langle x,y\rangle = \overline{\langle y,x\rangle}$

c) Positive-definite: $\langle x,x\rangle > 0$ when $\vec{x} \neq \vec{0}_H$

c') Non-degeneracy: $\langle x,x\rangle = 0 \Leftrightarrow x = 0_H$

Induces a norm: $\|x\|_H := \sqrt{\langle x,x\rangle_H}$

3) $(H, \|\cdot\|_H)$ is complete, i.e. every Cauchy sequence in H converges to a vector in H.

$\forall \{\vec{x}_j\}_{j\in\mathbb{N}}$, $\|\vec{x}_j - \vec{x}_k\|_H \xrightarrow{j,k\to\infty} 0 \Rightarrow \exists\, \vec{x} \in H$ s.t.

$\|\vec{x}_j - \vec{x}\|_H \xrightarrow{j\to\infty} 0$ (by def, $\vec{x}_j \xrightarrow{j\to\infty} \vec{x}$)

Mnemonic

- Banach space = complete normed vector space
- Hilbert space = inner product space inducing a Banach space.

Ex: Define $\langle f, g\rangle_H := \int_H f(x)\overline{g(x)}\,dx$

$$\Rightarrow \|f\|_H = \left(\int_H |f(x)|^2 dx\right)^{\frac{1}{2}} = \|f\|_{L^2(H)}$$

$\Rightarrow L^2([0,1]), L^2(\mathbb{R})$ are Hilbert spaces.

Fourier Transform

$$\hat{f}(\vec{\xi}) := \int_H f(\vec{x})\, e^{2\pi i \vec{x}\cdot\vec{\xi}}\, d\vec{x}$$

Plancherel

? One of the following

- $\int_H |f(x)|^2 dx = \int_H |\hat{f}(\xi)|^2 d\xi$
- $\|f\|_H = \|\hat{f}\|_H$ (Fourier transform is unitary?)
- $\int_H f(x)\overline{g(x)}\,dx = \int_H \hat{f}(x)\overline{\hat{g}(x)}\,dx$.
- $\langle f, g\rangle_H = \langle \hat{f}, \hat{g}\rangle_H$ (Fourier transform is an isometry?)

abs value

Bessel

$\vec{x} \in H$, $\{\vec{e}_k\}_{k\in\mathbb{Z}}$ ON seq

$\sum_{k\in\mathbb{Z}} |\langle \vec{x}, \vec{e}_k\rangle_H| \le \|\vec{x}\|_H$

Parseval if $\{e_k\}$ a basis — equality

Complete ON ($\mathrm{Cl}_H(\mathrm{Span}\{e_k\}) = H$)

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← Cauchy seq?

WTS: Let $S_N := \sum_{n=1}^N f_n$, then S_N converges iff Cauchy (by completeness), so we want

$$\|S_N - S_M\|_H \longrightarrow 0$$

$\Rightarrow \|\sum_{n=M+1}^N f_n\|_H < \varepsilon_{M,N}\ \forall \varepsilon > 0$] Can maybe take tail? $\|\sum_{n=M}^\infty f_n\|_H \to 0$?

OK reduction? $\|\sum_{n=M+1}^N f_n\|_H^2 < \varepsilon$ $\quad \|\cdot\|^2 = (\sqrt{\langle \cdot,\cdot\rangle})^2$?

$\Rightarrow \int_0^1 |\sum_{n=M+1}^N f_n(x)|^2 dx \xrightarrow[M\to\infty]{N\to\infty} 0 \rightsquigarrow \|f_n\|_{L^2}$

$\le \sum_{n=M+1}^N \int_0^1 |f_n(x)|\,dx$

$= \sum_{n=M+1}^N \|f\|_{L^2}$ $\qquad \|f_n\|_{L^2}$

$= \int_0^1 |\sum_{n=M+1}^N f_n(x)| \cdot |\sum_{n=M+1}^N f_n(x)|\,dx$

$\le \int_0^1 \left(\sum_{n=M+1}^N |f_n(x)|\right)\cdot\left(\sum_{n=M+1}^N |f_n(x)|\right) dx$ (Δ-ineq)

$= \int_0^1 \sum_{n=M+1}^N \sum_{m=?}^{?} |f_m(x)|\cdot|f_{n-m}(x)|\,dx$] probably won't work...

$\vdots \quad ? \quad \vdots$

$= \sum_{n=M+1}^N \int_0^1 |f_n(x)|\,dx$ (finite sum)

$= \sum_{n=M+1}^N \|f_n\|_H^2$

$= \sum_{n=M+1}^N \|\hat{f}_n\|_H^2$

$:= \sum_{n=M+1}^N \int_0^1 |\hat{f}_n(\xi)|\,d\xi$

$:= \sum_{n=M+1}^N \int_{2^n}^{2^{n+1}} |\hat{f}_n(\xi)|\,d\xi$

$= \sum_{n=M+1}^N \int_{2^n}^{2^{n+1}} \left[\int_0^1 |f_n(x)\, e^{2\pi i x\cdot\xi}|\,dx \right] d\xi$

$= \sum_{n=M+1}^N \int_{2^n}^{2^{n+1}} \int_0^1 |f_n(x)\, e^{2\pi i x\cdot\xi}|\,dx\,d\xi$

$\le \sum_{n=M+1}^N \int_{2^n}^{2^{n+1}} \int_0^1 |f_n(x)|\,dx\,d\xi$

$\le \sum_{n=M+1}^N \int_{2^n}^{2^{n+1}} n^{-51/100}\,d\xi$

$\le \sum_{n=M+1}^N n^{-51/100} \underbrace{\int_{2^n}^{2^{n+1}} d\xi}$ $\qquad 2^{n+1} - 2^n = 2^n(2-1) = 2^n$

$\le \sum_{n=M+1}^N n^{-51/100}\, 2^n$

$= \sum_{n=M+1}^N 2^n / n^{51/100}$

$< \infty$ $\qquad$ Didn't use $n^{51/100}$ .....

p-test: $\sum_{n=1}^\infty 1/n^p < \infty \Leftrightarrow p > 1$

$$\left\| \sum_{n=M+1}^{N} f_n \right\|_H^2 = \int_0^1 \left| \sum_{n=M+1}^{N} f_n(x) \right|^2 dx$$

$$= \int_0^1 \left| \sum_{n=M+1}^{N} f_n(x) \right| \cdot \left| \sum_{n=M+1}^{N} f_n(x) \right| \, dx$$

$$= \int_0^1 \sum_{n=M+1}^{N} |f_n(x)| \cdot \sum_{n=M+1}^{N} |f_n(x)| \, dx$$

$$= \sum_m \sum_n \int_0^1 |f_n(x)| \cdot \overline{|f_m(x)|} \, dx$$

$$= \sum_m \sum_n \left| \langle f_n, f_m \rangle_H \right|$$