

# Title

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# 1 | Lecture 5A

We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for  $K/k$  of transcendence degree 1, all valuations on  $K$  which are trivial on  $k$  are discrete. We'll now want to go farther and describe the places  $\Sigma(K/k)$ , which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is  $K$ . Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if  $A$  is a Dedekind domain such that  $\text{ff}(A) = K$ , then for all  $\mathfrak{p} \in \text{mSpec}(A)$  there exists a discrete valuation  $v_{\mathfrak{p}}$  on  $K$ . I.e., every maximal ideal induces a discrete valuation that is  $A$ -regular, so the valuation ring will contain  $A$ . How is this obtained? Take a nonzero  $x \in K^{\times}$ , and take the corresponding principal fractional ideal  $\langle x \rangle := Ax$ , which we can factor in a Dedekind domain as  $Ax = \prod_{\mathfrak{p} \in \text{mSpec}(A)} \mathfrak{p}^{\alpha_{\mathfrak{p}}}$  with  $\alpha_{\mathfrak{p}} \in \mathbb{Z}$ . This looks like an infinite product, but for any fixed  $x$ , only finitely many  $\alpha$  are nonzero. Note that these  $\alpha$  are exactly what we're looking for: the  $\mathfrak{p}$ -adic evaluation of  $x$  is given precisely by  $v_{\mathfrak{p}}(x) := \alpha_{\mathfrak{p}}$ , where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$\begin{aligned} v. : \text{mSpec}(A) &\rightarrow \Sigma(K/A) \\ \mathfrak{p} &\mapsto v_{\mathfrak{p}}. \end{aligned}$$

So this sends a maximal ideal to a place that is  $A$ -regular, and it turns out to be a bijection.

## Proposition 1.0.1(?).

The map  $v$  is a bijection, and thus we may write

$$\Sigma(K/A) \cong \text{mSpec}(A).$$

*Proof* (?).

**Claim:**  $v$  is injective.

If  $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{mSpec}(A)$  are two different maximal ideals. Then there exists an element  $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ , and so  $x^{-1} \in A_{\mathfrak{p}_2} \setminus A_{\mathfrak{p}_1}$ . This follows since if  $x$  is not in  $\mathfrak{p}_2$ , its  $\mathfrak{p}_2$ -adic valuation is zero, and thus the  $\mathfrak{p}_2$ -adic valuation of  $x^{-1}$  is  $-0 = 0$  as well. On the other hand, since  $x \in \mathfrak{p}_1$ , its  $\mathfrak{p}_1$ -adic valuation is positive and therefore  $v_{\mathfrak{p}_1}(x^{-1}) < 0$  and  $x^{-1}$  is not in  $A_{\mathfrak{p}_1}$ .

**Claim:**  $v$  is surjective.

Let  $v \in \Sigma(K/A)$ , so  $A \subset R_v$ , i.e. take a valuation whose valuation ring contains  $A$ . Note that we're not assuming the valuation is discrete, this can be a general Krull valuation, but we're trying to show it's equal to a certain  $p$ -adic valuation. As always with a subring of a valuation ring, we can pull back the maximal ideal and consider  $\mathfrak{m}_v \cap A \in \text{Spec}(A)$ . We're hoping that this is a maximal ideal, since maximals correspond to valuations. Since we're in a Dedekind

domain, the only prime ideal we *don't* want this to be is the zero ideal of  $A$ , so suppose it were. Then  $A^\bullet \subset R_v^\times$ , and so  $K^\times \subset R_v^\times$ . This is because the only element of the maximal ideal that lies in  $A$  is zero, so every nonzero element of  $A$  is not in this maximal ideal and is thus a unit. But for any unit, its inverse is also a unit, ■