Linearization and Transversality

Sections 8.3 and 8.4

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Review 8.2

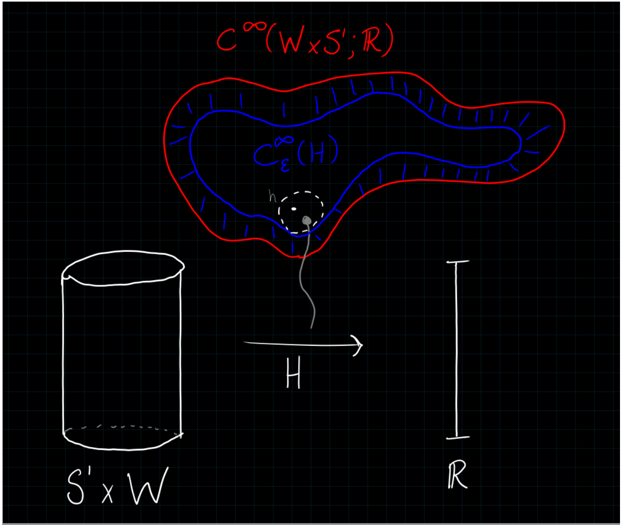
Section 8.3: The Space of Perturbations of H

Goal

Goal: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Goal

Start by trying to construct a subspace $\mathcal{C}^{\infty}_{\varepsilon}(H) \subset \mathcal{C}^{\infty}(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Define an Absolute Value

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C_\varepsilon^\infty(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$ denote a perturbation of H.
- Fix $\varepsilon = \left\{ \varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0} \right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k \right\},$$

  the maximum over all sets of multi-indices α of length k.
  Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1}\, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

  the partial derivatives wrt the corresponding variables.

Define a Norm

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left| d^k h(x,t) \right|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

- Obtain the computable form

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} \left| d^k (h \circ \Psi_i^{-1})(z) \right|.$$

Define a Banach Space

- Define
$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \;\middle|\; \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$
which is a Banach space (normed and complete).
- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Theorem

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma

$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Sketch Proof of Theorem

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n, define

$$\frac{1}{\varepsilon_n} = 2^n \max\left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x,t)| \leq 2^{-n}$$

which is summable.

Why does this imply density? I don't know.

Modified Theorem

The next proposition establishes a version of this theorem with compact support:

Theorem

For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ *there exists a* $V \subset U$ *such that every* $h \in C^\infty(W \times S^1; \mathbb{R})$ *can be approximated in the* C^1 *topology by functions in* C_ε^∞ *supported in* U*.*

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \;\middle|\; h(x,t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\operatorname{supp}(h)$ is "far" from $\mathrm{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$$

and are both nondegenerate.

Section 8.4: Linearizing the Floer Equation: The Differential of F

Goal

Choose $m > n = \dim(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

> *Why? Bypasses differentiating vector fields and the Levi-Cevita connection.*

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

> *We've just replaced the codomain here.*

Definitions

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$,
- $u \in \mathcal{M}(x, y) \subset C^\infty_{\text{loc}}$ denotes a fixed solution to the Floer equation,
- $C_\searrow(x, y) \subset \left\{u \in C^\infty(R \times S^1; W)\right\}$ is the set of smooth solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions:

$$\lim_{s \longrightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left|\frac{\partial u}{\partial t}(s, t)\right|, \quad \left|\frac{\partial u}{\partial t}(s, t) - X_H(u)\right| \sim \exp(|s|)$$