Problem Set 1

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Contents

Source: Section 1 of Gathmann

Exercise 0.1 (1.19)**.**
Prove that every affine variety $X \subset \mathbb{A}^n/k$ consisting of only finitely many points can be written as the zero locus of n polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in X have different x_1-coordinates.

Exercise 0.2 (1.21)**.**
Determine $\sqrt{I}$ for

$$I \coloneqq \left\langle x_1^3 - x_2^6,\, x_1 x_2 - x_2^3 \right\rangle \trianglelefteq \mathbb{C}[x_1, x_2].$$

Solution:
For notational purposes, let $\mathcal{I}, \mathcal{V}$ denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$, the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 & = 0 \\ xy - y^3 & = 0. \end{cases}$$

In the second equation, we have $(x - y^2)y = 0$, and since $\mathbb{C}[x, y]$ is an integral domain, one term must be zero.

1. If $y = 0$, then $x^3 = 0 \implies x = 0$, and thus the

   $$(0, 0) \in \mathcal{V}(I),$$

   i.e. the origin is contained in this vanishing locus.

2. Otherwise, if $x - y^2 = 0$, then $x = y^2$, with no further conditions coming from the first equation. So

$$P := \left\{ (t^2, t) \;\middle|\; t \in \mathbb{C} \right\} \subset \mathcal{V}(I).$$

Since the origin is in the latter set, this simplifies to $P = \mathcal{V}(I)$, and so taking the ideal generated by P yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \left\langle y - x^2 \right\rangle \in \mathbb{C}[x, y]$$

and thus $\sqrt{I} = \left\langle y - x^2 \right\rangle$.

Exercise 0.3 (1.22)**.**
Let $X \subset \mathbb{A}^3/k$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$ and show that it can not be generated by fewer than 3 elements.

Exercise 0.4 (1.23: Relative Nullstellensatz)**.**
Let $Y \subset \mathbb{A}^n/k$ be an affine variety and define $A(Y)$ by the quotient

$$\pi : k[x_1, \cdots, x_n] \longrightarrow A(Y) := k[x_1, \cdots, x_n]/I(Y).$$

a. Show that $V_Y(J) = V(\pi^{-1}(J)$ for every $J \trianglelefteq A(Y)$.

b. Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every affine subvariety $X \subseteq Y$.

c. Using the fact that $I(V(J)) \subset \sqrt{J}$ for every $J \trianglelefteq k[x_1, \cdots, x_n]$, deduce that $I_Y(V_Y(J)) \subset \sqrt{J}$ for every $J \trianglelefteq A(Y)$.

Conclude that there is an inclusion-reversing bijection

$$\left\{ \substack{\text{Affine subvarieties} \\ \text{of } Y} \right\} \iff \left\{ \substack{\text{Radical ideals} \\ \text{in } A(Y)} \right\}.$$

Exercise 0.5 (Extra)**.**
Let $J \trianglelefteq k[x_1, \cdots, x_n]$ be an ideal, and find a counterexample to $I(V(J)) = \sqrt{J}$ when k is not algebraically closed.