Title

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1 Wednesday November 20

Last time:

$$\begin{aligned}
\mathbb{Z}\Lambda &\iff \{\mathfrak{h}^* \to \mathbb{Z}_{\geq 0} \mid \sim\} \\
& e(\mu) \mapsto e_\mu \\
e(\lambda)e(\mu) = e(\lambda+\mu) \mapsto f \star g(\lambda) &= \sum_{a+b=\lambda} f(a)g(b)
\end{aligned}$$

and $\mathrm{ch}L(\lambda) = \sum_{\mu \in \Lambda} \dim L(\lambda)_\mu e(\mu)$.

We have the Kostant function $p(\lambda) = \#\{(k_\alpha)_\alpha \mid -\lambda = \sum_{\alpha \in \Phi^+} k_\alpha \alpha\}$ and the Weyl function $q = e_\rho \star \prod_{\alpha \in \Phi^+}(1 - e_{-\alpha}) = \prod_{\alpha \in \Phi^+}(e_{\alpha/2} - e_{-\alpha/2})$.

Lemma: $p \star e_\lambda = \mathrm{ch}M(\lambda)$, so $q \star \mathrm{ch}M(\lambda) = e_{\lambda+\rho}$ and $q \star p = e_\rho$.

1.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of W is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, i.e. a reflection for hyperplanes passing through $-\rho$.

E.g. for type $A2$, where $W(0) = 0$, we have:

Type A2

And for the dot action, we have

Image

where $W \cdot 0 = 0$ and $s(\alpha_1) = -\alpha_1$.

Theorem (Harish-Chandra): If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$ for $\mu \leq \lambda$.

Proof: Postponed.

ch are characters, $L(\lambda)$ is a Verma module.

Remark: if we sum over $\mu \leq \lambda$, we obtain

$$\begin{aligned}
\mathrm{ch} M(\lambda) &= \sum_{\mu \in W \cdot \lambda} a_{\lambda\mu} \mathrm{ch} L(\mu) \\
\mathrm{ch} L(\lambda) &= \sum_{\mu \in W \cdot \lambda} b_{\lambda\mu} \mathrm{ch} M(\mu) \\
&= \sum_{W \cdot \lambda \in \Lambda} c_{\lambda W} \mathrm{ch} M(w \cdot \lambda).
\end{aligned}$$

Theorem (Weyl's Character Formula): If $\lambda \in \Lambda^+$, then

$$\mathrm{ch} L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)}$$

Proof:

We have $\mathrm{ch} L(\lambda) = \sum_w c_{\lambda w} \mathrm{ch} M(w \cdot \lambda)$, and so by the lemma,

$$q * \mathrm{ch} L(\lambda) = \sum c_{\lambda w} q \star \mathrm{ch} M(W(\lambda + \rho) - \rho) = \sum_w c_{\lambda w} e_{W(\lambda + p)}$$

Thus for all $\alpha \in \Phi^+$, we have

$$s_\alpha(q \star \mathrm{ch} L(\lambda)) = \sum_w c_{\lambda, s_\alpha w} e_{w(\lambda+\rho)}$$