

# Title

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Friday 21<sup>st</sup> August, 2020

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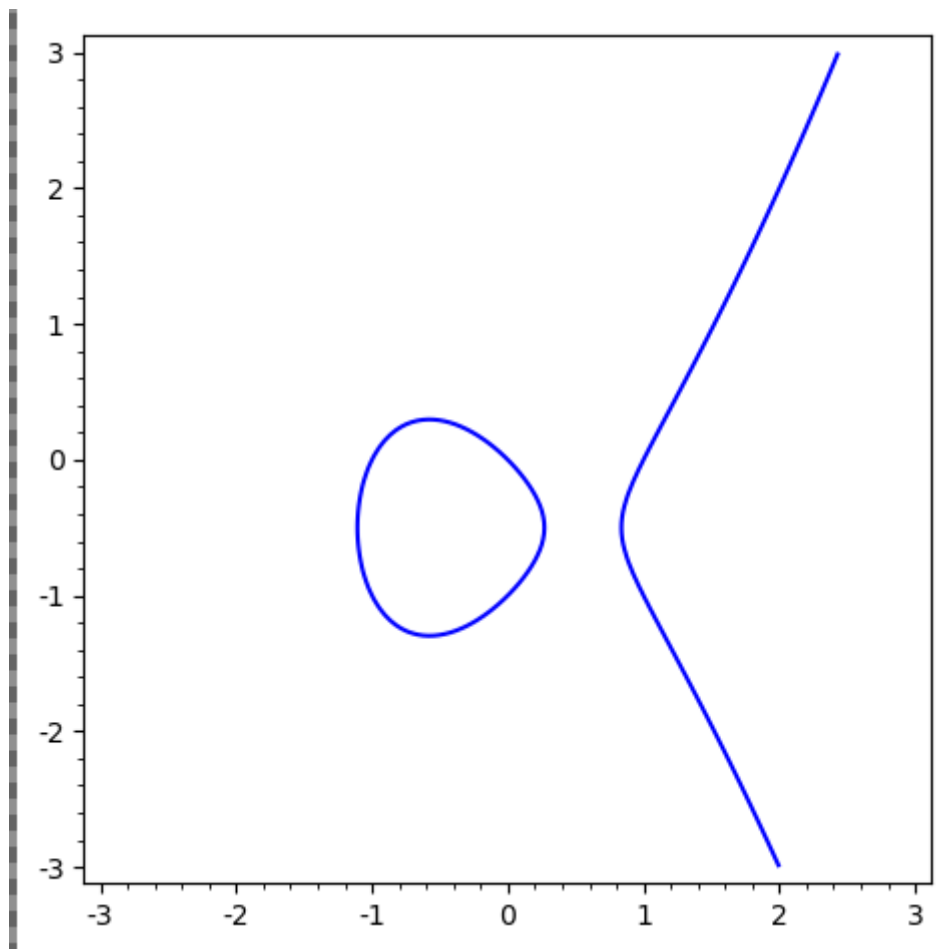
## 1 Friday, August 21

Reference:  
<https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf>

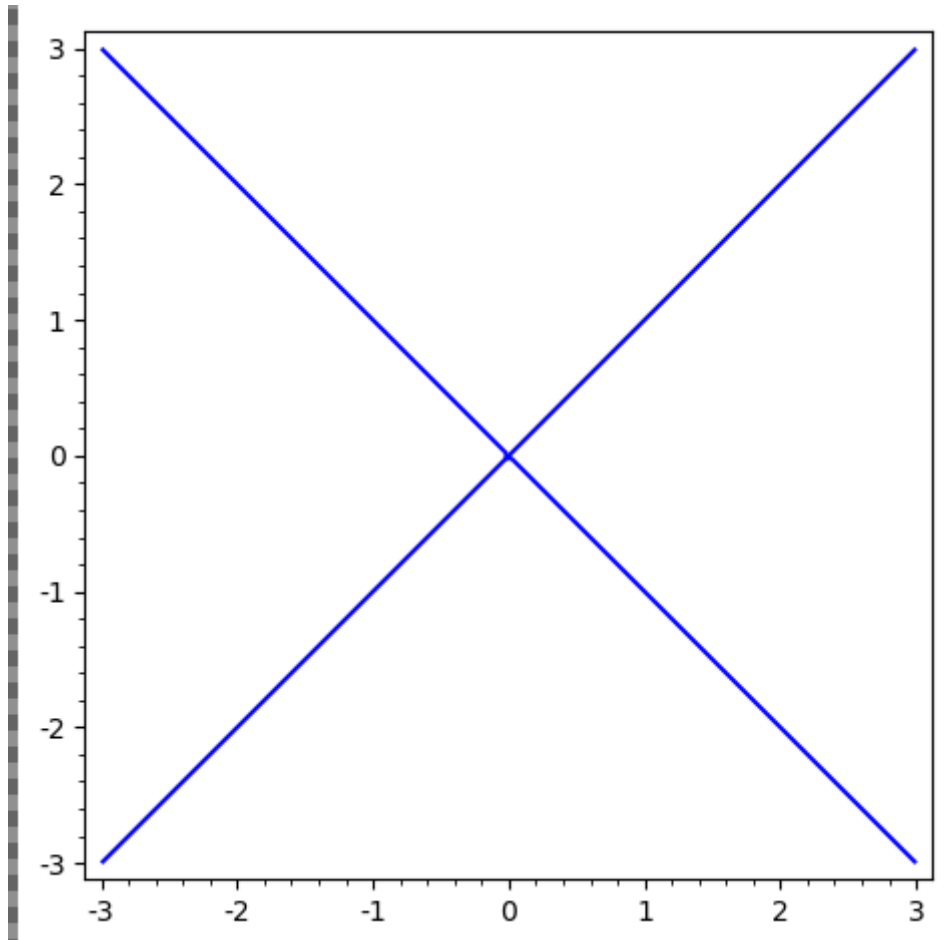
General idea: functions a coordinate ring  $R[x_1, \dots, x_n]/I$  will correspond to the geometry of the variety cut out by  $I$ .

### Example 1.1.

- $x^2 + y^2 - 1$  defines a circle, say, over  $\mathbb{R}$
- $y^2 = x^3 - x$  gives an elliptic curve:



- $x^n + y^n = 1$ : does it even contain a  $\mathbb{Q}$ -point? (Fermat's Last Theorem)
- $x^2 + 1$ , which has no  $\mathbb{R}$ -points.
- $x^2 + y^2 + 1/\mathbb{R}$  has vanishes nowhere, so ring of functions is not  $\mathbb{R}[x, y]/\langle x^2 + y^2 + 1 \rangle$  (problem:  $\mathbb{R}$  is not algebraically closed)
- $x^2 - y^2 = 0$  over  $\mathbb{C}$  is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$ , which has 3 points over  $\mathbb{F}_3^2$ , but  $f(x, y) = (x^3 - x)(y^3 - y)$  vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in  $\mathbb{C}$ ?)
  - $V(f) = \mathbb{F}_3^2$ , so the coordinate ring is zero instead of  $\mathbb{F}_3[x, y]/\langle f \rangle$  (addressed by scheme theory)

**Theorem 1.1 (Harnack Curve Theorem).**

If  $f \in \mathbb{R}[x, y]$  is of degree  $d$ , then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

**Example 1.2.**

Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C} \right\}.$$

Then  $X$  is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2y$

**Exercise 1.1.**

Show that the vanishing locus of the first two equations above is  $X \cup L$  for  $L$  a line.

Compare to linear algebra: codimension  $d$  iff cut out by exactly  $d$  equations.

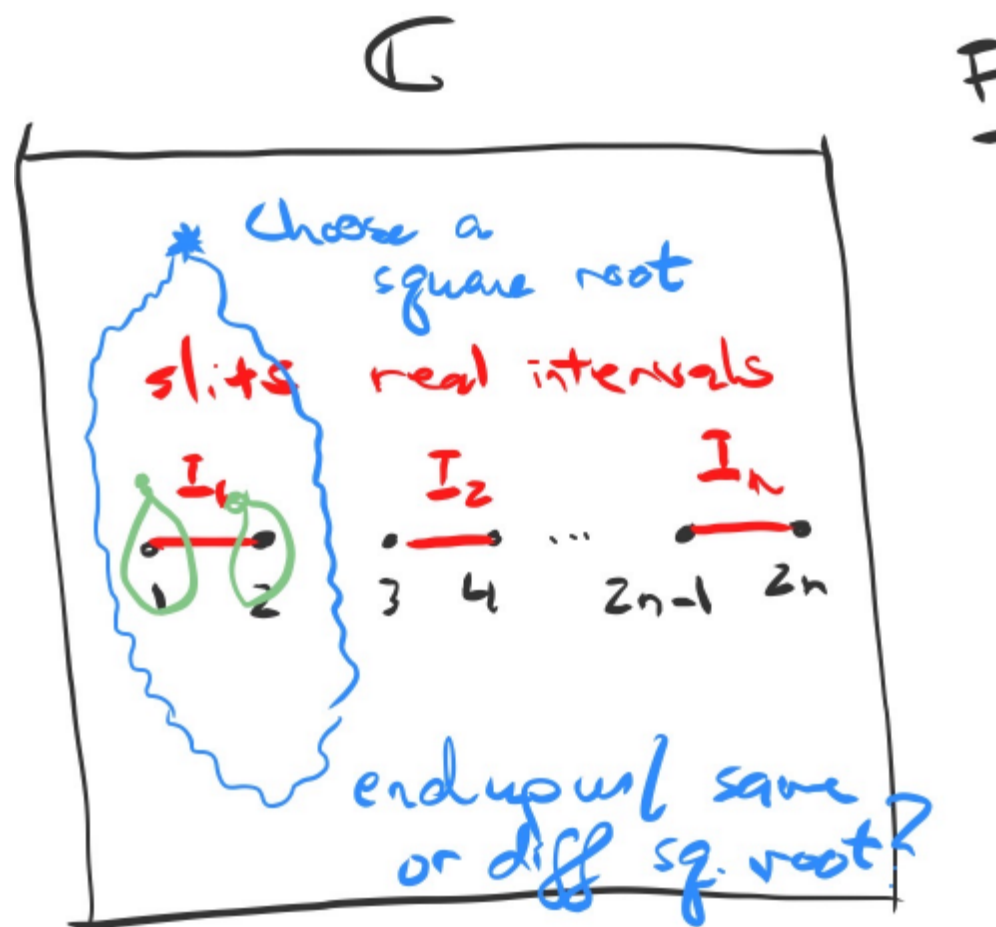
**Example 1.3.**

Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

how to visualize the solution set?

Fact: on  $\mathbb{C}$  with some slits, you can consistently choose a square root of the RHS.



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Away from  $x = 1, \dots, 2n$ , there are two solutions for  $y$  given  $x$ .