Problem Set 3

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1 Problem 31

1.1 a

Theorems used:

- Continuous bijection + open map (or closed map) $\implies$ homeomorphism.
- **Closed** subsets of compact sets are compact.
- The continuous image of a compact set is compact.
- Closed subsets of Hausdorff spaces are compact.

So we'll show that f is a closed map.

Let $U \in X$ be closed.

- Since X is compact, U is compact
- Since f is continuous, $f(U)$ is compact
- Since Y is Hausdorff, $f(U)$ is closed.

1.2 b

Note that any finite space is clearly compact.

Take $f : ([2], \tau_1) \to ([2], \tau_2)$ to be the identity map, where τ_1 is the discrete topology and τ_2 is the indiscrete topology. Any map into an indiscrete topology is continuous, and f is clearly a bijection.

Let g be the inverse map; then note that $1 \in \tau_1$ but $g^{-1}(1) = 1$ is not in τ_2, so g is not continuous.

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2 Problem 32

$\implies$:

- Let $p \in \Delta^c$.
- Then p is of the form (x, y) where $x \neq y$ and $x, y \in X$.
- Since X is Hausdorff, pick N_x, N_y in X such that $N_x \bigcap N_y = \emptyset$.
- Then $N_p := N_x \times N_y$ is an open set in X^2 containing p.
- Claim: $N_p \bigcap \Delta = \emptyset$.
  - If $q \in N_p \bigcap \Delta$, then $q = (z, z)$ where $z \in X$, and $q \in N_p \implies q \in N_x \bigcap N_y = \emptyset$.
- Then $\Delta^c = \bigcup_p N_p$ is open.

$\impliedby$:

- Let $x \neq y \in X$.
- Consider $(x, y) \in \Delta^c \subset X^2$, which is open.
- Thus $(x, y) \in B$ for some box in the product topology.
- $B = U \times V$ where U, V are open in X, and $B \subset \Delta^c$.
- So $x \in U, y \in V$.
- Claim: $U \bigcap V = \emptyset$.
  - Otherwise, $z \in U \bigcap V \implies (z, z) \in B$, but $B \bigcap \Delta^c = \emptyset$.

3 Problem 38

$\mathbb{R}$ is clearly Hausdorff, and $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology, and is thus non-Hausdorff. So take the quotient map $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$.

Direct proof that $\mathbb{R}/\mathbb{Q}$ isn't Hausdorff:

- Pick $[x] \subset U \neq [y] \subset V \in \mathbb{R}/\mathbb{Q}$ and suppose $U \cap V = \emptyset$.
- Pull back $U \to A, V \to B$ open disjoint sets in $\mathbb{R}$
- Both A, B contain intervals, so they contain rationals $p \in A, q \in B$
- Then $[p] = [q] \in U \bigcap V$.

4 Problem 42

Proof that $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology:

- Let $U \subset \mathbb{R}/\mathbb{Q}$ be open and nonempty, show $U = \mathbb{R}/\mathbb{Q}$.
- Let $[x] \in U$, then $x \in \pi^{-1}(U) := V \subset \mathbb{R}$ is open.
- Then V contains an interval (a, b)

- Every $y \in V$ satisfies $y + q \in V$ for all $q \in \mathbb{Q}$, since $y + q - y \in \mathbb{Q} \implies [y + q] = [y]$.
- So $(a - q, b + q) \in V$ for all $q \in \mathbb{Q}$.
- So $\bigcup_{q \in \mathbb{Q}} (a - q, b + q) \in V \implies \mathbb{R} \subset V$.
- So $\pi(V) = \mathbb{R}/\mathbb{Q} = U$, and thus the only open sets are the entire space and the empty set.

5 Problem 44

5.1 a

- Suppose X has a countable basis $B = \{B_i\}$.
- Choose an arbitrary $x_i \in B_i$ for each i. Define $Q = \{x_i\}$.
- Let $y \in N_y \subset X$.
- By definition of a basis, there exists some B_i such that $y \in B_i \subset N_y$.
- Since $x_i \in B_i$, $Q \bigcap N_y \neq \emptyset$.
- Thus Q is dense in X.

5.2 b

- Let $\{q_i\}$ be a countable dense subset.
- Define $B_{i,j} = B_{\frac{1}{i}}(q_j)$, which is still countable.
- Property 1: Every $x \in B_{i,j}$
  - Take $x \in N_{\frac{1}{2}}(x) \ni q_j$ by density.
  - Then $x \in B_{\frac{1}{2},j}$.
- Property 2: $x \in B_{i_1,j_1} \bigcap B_{i_2,j_2} \implies x \in B_{i_3,j_3} \subset B_{i_1,j_1} \cap B_{i_2,j_2}$:
  - Take $i < \min(i_1, i_2)$, then $N_i(x) \ni q_j$. for some j.
  - Thus $x \in B_{i,j}$.