

# Title

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Monday 31<sup>st</sup> August, 2020

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## 1 Sunday, August 30

Last of preliminaries. Upcoming: one-variable function fields and their valuation rings.

### 1.1 Polynomials Defining Regular Function Fields

Where's the curve:  $f(x, y) = 0$ .

#### Exercise 1.1.

Let  $R_1, R_2$  be  $k$ -algebras that are also domains with fraction fields  $K_i$ . Show  $R_1 \otimes_k R_2$  is a domain  $\iff K_1 \otimes_k K_2$  is a domain.

Denominator-clearing argument.

#### Definition 1.0.1 (Geometrically Irreducible).

A polynomial of positive degree  $f \in k[t_1, \dots, t_n]$  is *geometrically irreducible* if  $f \in \bar{k}[t_1, \dots, t_n]$  is irreducible as a polynomial.

If  $n = 1$  then  $f$  is geometrically irreducible  $\iff$  it's linear, i.e. of degree 1.

Let  $f$  be irreducible, then since polynomial rings are UFDs then  $\langle f \rangle$  is a prime ideal (irreducibles generate principal ideals) and  $k[t_1, \dots, t_n]/\langle f \rangle$  is a domain. Let  $K_f$  be the fraction field.

#### Exercise 1.2.

Easy:

- Above for  $1 \leq i \leq n$  let  $x_i$  be the image of  $t_i$  in  $K_f$ . Show that  $K_f = k(x_1, \dots, x_n)$ .
- Show that if  $K/k$  is generated by  $x_1, \dots, x_n$ , then it is the fraction field of  $k[t_1, \dots, t_n]/\mathfrak{p}$  for some prime ideal  $\mathfrak{p}$  (equivalently, a height 1 ideal).

**Proposition 1.1 (?)**.

Suppose that  $f$  is geometrically irreducible.

- a. The function field  $K/k$  is regular.
- b. For all  $\ell/k$ ,  $f \in \ell[t_1, \dots, t_n]$  is irreducible.

In this case we say  $f$  is *absolutely irreducible* as a synonym for geometrically irreducible.

*Proof.*

By definition of geometric irreducibility,  $\bar{k}[t_1, \dots, t_n]/\langle f \rangle = k[t_1, \dots, t_n]/\langle f \rangle \otimes_k \bar{k}$  is a domain.

The exercise shows that  $K_f \otimes_k k$  is a domain, so  $K_f$  is regular.

It follows that for all  $\ell/k$ ,  $K_f \otimes_k \ell$  is a domain, so  $\ell[t_1, \dots, t_n]/\langle f \rangle$  is a domain. ■

Moral: geometrically irreducible polynomials are good sources of regular function fields.

**Exercise 1.3.**

Let  $k$  be a field,  $d \in \mathbb{Z}^+$  such that  $4 \nmid d$  and  $p(x) \in k[x]$  be positive degree. Factor  $p(x) = \prod_{i=1}^r (x - a_i)^{\ell_i}$  in  $\bar{k}[x]$ .

- a. Suppose that for some  $i$ ,  $d \nmid \ell_i$ . Show that  $f(x, y) := y^d - p(x) \in k[x, y]$  is geometrically irreducible. Conclude that  $K_f := k[x, y]/\langle y^d - p(x) \rangle$  is a regular one-variable function field over  $k$ , and thus elliptic curves yield regular function fields.

Referred to as *hyperelliptic* or *superelliptic* function fields. Hint: use FT 9.21 or Lang's Algebra.

- b. What happens when  $4 \mid d$ ?

**Exercise 1.4** (Nice, Recommended).

Assume  $k$  is a field, if necessary assuming  $\text{char}(k) \neq 2$ .