

Chapter 9

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Important Theorems:

Important ideas:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

1 | Background from Chapter 8

- (M, ω) with $\omega \in \Omega^2(M)$ is a symplectic manifold with an almost complex structure J .
- $H \in C^\infty(M; \mathbb{R})$ a Hamiltonian with X_H the corresponding symplectic gradient.
 - Defined by how it acts on tangent vectors in $T_x M$:

$$\omega_x(\cdot, X_H(x)) = (dH)_x(\cdot).$$

- Zeros of vector field X_H correspond to critical points of H :

$$X_H(x) = 0 \iff (dH)_x = 0.$$

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- Take the associated flow $\psi^t : M \rightarrow M$, assumed 1-periodic so $\psi^1(x) = x$: critical points of H are periodic trajectories.
 - $u \in C^\infty(\mathbb{R} \times S^1; M)$ is a solution to the Floer equation.
 - The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y\end{aligned}$$

$$Y \in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).$$

- \mathcal{LM} is the *free loop space* of M , i.e. space of contractible loops on M , i.e. $C^\infty(S^1; M)$ with the C^∞ topology
 - Loops in \mathcal{LM} can be viewed as maps $S^2 \rightarrow M$, since they're maps $I \times S^1 \rightarrow M$ with the boundaries pinched:



Figure 1: Loops in \mathcal{LM}

- Elements $x \in \mathcal{LM}$ can be viewed as maps $S^1 \rightarrow M$.
- Can extend to maps from a closed disc, $u : \mathbb{D}^2 \rightarrow M$.
- The action functional is given by

$$\begin{aligned}\mathcal{A}_H : \mathcal{LW} &\rightarrow \mathbb{R} \\ x &\mapsto - \int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t)) \, dt\end{aligned}$$

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- Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t dt - p dq)$.
 - Correspondence between trajectories of the gradient of \mathcal{A}_H and solutions to Floer equations.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, M)$,

$$\int_{S^2} u^* \omega = 0 \quad \text{or equivalently} \quad \langle \omega, \pi_2 M \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle $u^* TM$, equivalently

$$\langle c_1 TM, \pi_2 M \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of \mathcal{A}_H .
- Maslov index: used the fact that
 - $\text{Sp}(2n, \mathbb{R})$ retracts onto $U(n)$: use a polar decomposition $S = PQ$ as a PSD times orthogonal, then homotope P to I .
 - $\pi_1 U_n = \mathbb{Z}$: use $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$ by the determinant, and $\pi_1 SU(n, \mathbb{C}) = 0$.
 - Thus every path in $\gamma : I \rightarrow \text{Sp}(2n, \mathbb{R})$ can be assigned an integer by getting a map $\tilde{\gamma} : I \rightarrow S^1$ and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y .
 - Showed that after perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
 - How did we do it: describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$ (Banach manifold modeled on the Sobolev spaces $W^{1,p}$), apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
 - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality. Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

Goals

- Construct Floer homology and prove the Arnold Conjecture (“Symplectic Morse Inequalities?”):

$$\# \{1\text{-Periodic trajectories of } X_H\} \geq \sum_{k \in \mathbb{Z}} HM_k(w; \mathbb{Z}/2\mathbb{Z}).$$

Steps

1. Define the action functional \mathcal{A}_H .
2. Construct the chain complex (graded vector space) CF_* .
3. Define X_H , which will be used to define ∂ later.

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4. Count trajectories.
 5. Show finite-energy trajectories connect critical points of \mathcal{A}_H .
 6. Show compactness property for space of trajectories of finite energy.
 7. Define ∂ (uses a compactness property in 9.1c)
 8. Show space of trajectories is a manifold (plus genericity, “Smale property”)
 9. Show that $\partial^2 = 0$.
 10. Show that HF_* doesn't depend on \mathcal{A}_H or X_H
 11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_* .

2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$$

$$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x, y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting x to y :

$$\mathcal{L}(x, y) := \mathcal{M}(x, y)/\mathbb{R}.$$

- Use the quotient topology, define sequentially:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

- When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

$$n(x, y) := \#\mathcal{L}(x, y)$$

is well-defined.

- $C_k(H) := \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k .
 - Finitely many since they are nondegeneracy implies they are isolated.
- Defined a differential

$$\begin{aligned} \partial : C_k(H) &\rightarrow C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x, y)y \end{aligned}$$

$$\begin{aligned} n(x, y) &:= \#\{\text{Trajectories of } \text{grad } \mathcal{A}_H \text{ connecting } x, y\} \pmod{2} \\ &= \#\mathcal{L}(x, y) \pmod{2}. \end{aligned}$$

- Examined ∂^2 :

$$\begin{aligned}
\partial^2 : C_k(H) &\rightarrow C_{k-2}(H) \\
x &\mapsto \partial(\partial(x)) \\
&= \partial \left(\sum_{\mu(y)=\mu(x)-1} n(x, y)y \right) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x, y)\partial(y) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x, y) \left(\sum_{\mu(z)=\mu(y)-1} n(y, z)z \right) \\
&= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x, y)n(y, z)z \\
&= \sum_{\mu(z)=\mu(y)-1} \left(\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) \right) z \quad (\text{finite sums, swap order}),
\end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x, z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\bar{\mathcal{L}}(x, y)$.
- We'll then have

$$\bar{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial\bar{\mathcal{L}}(x, z), \quad \partial\bar{\mathcal{L}}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

- Want to show $\mathcal{L}(x, y)$ is a compact 0-manifold, proceed by showing $\bar{\mathcal{L}}(x, y)$ is a compact 1-manifold with $\mathcal{L}(x, y)$

2.1 Three Important Theorems

- Shown last time: a sequence of trajectories can converge to a broken trajectory.

Theorem 2.1(9.1.7).

Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, y)$, then there exist

- A subsequence $\{u_{n_j}\}$

- Critical points $\{x_0, x_1, \dots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = y$
- For $0 \leq k \leq \ell$, sequences $\{s_n^k\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_n \cdot s_n^k \xrightarrow{n \rightarrow \infty} u^k.$$

Theorem 2.2(9.2.1).

Let (H, J) be a regular pair with H nondegenerate and x, z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\bar{\mathcal{L}}(x, z)$ is a compact 1-manifold with boundary with

$$\partial \bar{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x, z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z) \quad \text{where} \quad \mathcal{I}(x, z) = \{y \mid \mu(x) < \mu(y) < \mu(z)\}.$$

Note: possibly a typo in the book?

Remark 1.

- As a corollary, $\partial^2 = 0$.
- Most of chapter 9 is spent proving this theorem.

Remark 2.

Some notation:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M}(x, z) \\ & & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

Hats will generally denote maps induced on quotient.

Theorem 2.3(9.2.3: Gluing).

Let x, y, z be three critical points of \mathcal{A}_H with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

$$\psi : [\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$$

such that $\widehat{\psi}$, the induced map on the quotient

$$\begin{array}{ccc} [\rho_0, \infty) & \xrightarrow{\psi} & \mathcal{M}(x, z) \\ & \searrow \widehat{\psi} & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

is an embedding that satisfies

$$\widehat{\psi}(\rho) \xrightarrow{\rho \rightarrow \infty} (\widehat{u}, \widehat{v}) \in \overline{\mathcal{L}}(x, z).$$

2. For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x, z)$,

$$\ell_n \xrightarrow{n \rightarrow \infty} (\widehat{u}, \widehat{v}) \implies \ell_n \in \text{im}(\widehat{\psi}) \text{ for } n \gg 0.$$

2.2 Review Last Time

- $\mathbb{R} \curvearrowright \mathcal{M}_{x,y}$, so we quotient to define $\mathcal{L}(x, y) := \mathcal{M}_{x,y}/\mathbb{R}$ with the quotient topology.
- Topology defined by when sequences converge:

$$\tilde{u}_n \xrightarrow{n \rightarrow \infty} \tilde{u} \iff \exists \{s_n\} \subseteq \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \xrightarrow{n \rightarrow \infty} u(s, \cdot).$$

Proposition 2.4(?).

$\mathcal{L}(x, y)$ is Hausdorff.

- Want to show $\mathcal{L}(x, y)$ is a compact 0-dimensional manifold.
- Have a differential

$$\begin{aligned} \partial : C_k(H) &\longrightarrow C_{k-1}(H) \\ \partial(x) &= \sum_{\text{Ind}(y)=k-1} n(x, y)y. \end{aligned}$$

with $n(x, y)$ the number (mod 2) of trajectories of $\text{grad } \mathcal{A}_H$ connecting x, y , i.e solutions to the Floer equation.

- Want to prove that the following is a 1-dimensional manifold:

$$M := \overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup_{\mu(y)=\mu(x)+1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

and show that M is compact with ∂M equal to the last union.

- Last time: closure of space of trajectories connecting x, y contains “broken” trajectories.
- Last time: toward proving that M is compact

3 | 9.2

- Wanted to compactify $\mathcal{L}(x, y)$, needed to go to space of broken trajectories.
- Main theorem of chapter 9: 9.2.1.

Theorem 3.1(9.2.1).

Let (H, J) be a regular pair with H nondegenerate.

Let x, z be two periodic trajectories of H such that $\mu(x) = \mu(z) + 2$.

Then $\bar{\mathcal{L}}(x, y)$ is a compact 1-manifold with boundary satisfying

$$\partial \bar{\mathcal{L}}(x, y) = \bigcup_{\mu(x) < \mu(y) < \mu(z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

As a corollary, $\partial^2 = 0$.

- Know $\bar{\mathcal{L}}(x, y)$ is compact and $\mathcal{L}(x, y)$ is a 1-manifold
- Now suffices to study in a neighborhood of boundary points (“gluing theorem”)

3.1 Three steps to gluing theorem

1. Pre-gluing: Get a function w_p which interpolates between u and v (not exactly a solution itself, but will be approximated by one later).
2. Constructing ψ a “true solution” from w_p using the Newton-Picard method. We’ll have

$$\psi(p) = \exp_{w_p}(\gamma(p)) \quad \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p} \mathcal{P}(x, z).$$

where $\mathcal{P} = ?$.

3. Get a lift $\hat{\psi} = \pi \circ \psi$ where $\pi = ?$ satisfying
 - $\hat{\psi}(p) \xrightarrow{n \rightarrow \infty} (\hat{u}, \hat{v})$
 - $\hat{\varphi}$ is an embedding
 - $\hat{\psi}$ is unique in the following sense (the last point)

Theorem 3.2(9.2.3 (Gluing Theorem)).

Let x, y, z be critical points of the action functional \mathcal{A}_H such that $\mu(x) = \mu(y) + 1 = \mu(z) + 2$.

Let $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ be trajectories, inducing $(\bar{u}, \bar{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$.

- There exist a differentiable map $\psi : (\rho_0, \infty) \rightarrow \mathcal{M}(x, z)$ for some $\rho > 0$ such that
- $\pi \circ \psi : (\rho_0, \infty) \rightarrow \mathcal{L}(x, z)$ is an embedding
- $\hat{\psi} \xrightarrow{\rho \rightarrow \infty} (\bar{u}, \bar{v}) \in \bar{\mathcal{L}}(x, z)$.
- If $\ell_n \in \mathcal{L}(x, z)$ with $\ell_n \xrightarrow{n \rightarrow \infty} (\bar{u}, \bar{v})$, then for $n \gg 1$ we have $\ell \in \mathfrak{F}(\hat{\psi})$.

4 | 9.3: Pre-gluing

- Choose a bump function β on $\{0\}^c \subset \mathbb{R} \rightarrow [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts β^\pm :



Figure 2: Figure 9.3

- Define the interpolation w_ρ from u to v in the following way:

$$w_\rho(s, t) = \begin{cases} u(s + \rho, t) & \text{if } s \leq -1 \\ \exp_{y(t)} \left(\beta^-(s) \exp_{y(t)}^{-1}(u(s + \rho, t)) + \beta^+(s) \exp_{y(t)}^{-1}(v(s - \rho, t)) \right) & \text{if } s \in [-1, 1] \\ v(s - \rho, t) & \text{if } s \geq 1 \end{cases}$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \mid \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\}.$$

5 | 9.4: Construction of ψ .

- Have constructed $w_\rho \in C^\infty_\times(x, z)C^\infty(x, z)$ for every $\rho \geq \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x, z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining \mathcal{F}_ρ .

$$W^{1,p}(\mathbf{R} \times S^1; \mathbf{R}^{2n}) \xrightarrow{\mathcal{F}_\rho} L^p(\mathbf{R} \times S^1; \mathbf{R}^{2n})$$

$$(y_1, \dots, y_{2n}) \mapsto \left[\left(\frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } H_t \right) \left(\exp_{w_\rho} \sum y_i Z_i^\rho \right) \right]_{Z_i}$$

where $\mathcal{F}_\rho := \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i . sd - Newton-Picard method, general idea

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- Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(\textcolor{red}{x}_0)}.$$

- Allows finding zeros of f given an approximate zero x_0 .
- Linearize \mathcal{F}_ρ .