# Weil Conjectures

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3. Medium example: $E/\daleth$ an elliptic curve.
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References

- http://www-personal.umich.edu/~mmustata/zeta_book.pdf
- https://youtu.be/wEz7fCvK6sM?t=293
- Explanation of exponential appearing

## 1.1 Definition of Zeta Function

Fix $q$ a prime and $\mathbb{F} \coloneqq \mathbb{F}_q$ the finite field with $q$ elements, along with its unique degree $n$ extensions

$$\mathbb{F}_n \coloneqq \mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_p \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\, n \in \mathbb{Z}^{\geq 2}$$

**Definition 1.0.1.**
Let

$$J = \langle f_1, \cdots, f_N \rangle \trianglelefteq k[x_0, \cdots, x_n]$$

be an ideal, then a *projective algebraic* variety $X \hookrightarrow \mathbb{P}^\infty_{\mathbb{F}}$ can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^\infty_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_N(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1,\cdots,i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field $\mathbb{F}_p$. We will be thinking of $K$-valued points for $K/\mathbb{F}_p$ algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into $\mathbb{P}^\infty$?

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Example: Take $f_1(x) = x \in \mathbb{F}[x]$, consider $V(\langle f_1 \rangle) \subset \mathbb{P}^1_{\mathbb{F}_n}$. This is given by the single point $x = \mathbf{0}$.

Fix $X/\mathbb{F}$ an $N$-dimensional projective algebraic variety. Note that it then has points in any finite extension $L/K$.

**Definition 1.0.2.**
Define the *local zeta function* (or *Hasse-Weil zeta function*) of $X$ the following formal power series:

$$\zeta_X(z) = \exp\left( \sum_{n=1}^\infty \frac{\alpha_n}{n} z^n \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$\zeta_X(0) = 1$$
$$z\left(\frac{\partial}{\partial z}\right) \log \zeta_X(z) = t\left(\frac{\zeta_X'(z)}{\zeta_X(z)}\right) = \sum_{n=1}^\infty \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \cdots,$$

which is an *ordinary generating function* for the sequence $(\alpha_n)$.

Thus if we define $G(x)$ to be the OGF for $(\alpha_n)$, we have $\zeta_X(t) = \exp$

Todo: why not an OGF.

Remark: Note that for an OGF $F(x) = \sum_{n=0}^\infty f_n x^n$, we can extract coefficients in the following way:

$$[x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

### 1.1.1 Simple but Useful Example: A Point

Take $X = \{x = 0\} / \mathbb{F}$ a single point over $\mathbb{F}$, then

$$\begin{aligned} \#X(\mathbb{F}) &\coloneqq \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &\coloneqq \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &\coloneqq \alpha_n = 1 \\ &\vdots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned} \log(1+t) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} \\ \implies \log(1-t) &= -\sum_{n=1}^{\infty} \frac{t^n}{n} \\ \implies -\log(1-t) &= \sum_{n=1}^{\infty} \frac{t^n}{n} \\ &= 1 \cdot t + 1 \cdot t^2 + 1 \cdot t^3 + \cdots . \end{aligned}$$

and so

$$\zeta_{\{\mathrm{pt}\}}(t) = \exp\left(-\log\left(1-t\right)\right) = \frac{1}{1-t}.$$

### 1.1.2 Aside: Why call it a Zeta function?

Knowing the above calculation, we can now make a precise analogy.

Suppose

$$\mathbb{A}^n_{\mathbb{Z}} \supseteq X = V(\langle f_1, \cdots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \cdots, x_{n-1}].$$

Then for every prime, we can reduce the equations mod $p$ and consider

$$\mathbb{A}^n_{\mathbb{F}_p} \supseteq X_p \coloneqq V(\langle f_1 \mod p, \cdots, f_d \mod p \rangle) \quad \text{where} \quad f_1 \mod p \in \mathbb{F}_p[x_0, \cdots, x_{n-1}]$$

Then define the "local at $p$" zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

> Note: the index set for the product may require some minor adjustment over $\mathbb{Q}$ in general. There are also potentially modifications needed to extend to schemes.

Taking $X = \text{Spec } \mathbb{Q}$ and $X_p = \text{Spec } \mathbb{F}_p$ (which is a single point since $\mathbb{F}_p$ is a field) and noting that

$$\begin{aligned}\zeta_{X_p}(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^n\right) \\ &= \exp\left(-\log\left(1-z\right)\right) \\ &= \frac{1}{1-z},\end{aligned}$$

we find that

$$\begin{aligned}L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}}\right) \\ &= \zeta(s),\end{aligned}$$

the classical Riemann Zeta function.

Moreover, it is a theorem (difficult, not proved here!) that for any variety $X/\mathbb{F}_p$, we have

$$\zeta_X(t) = \prod_{x \in X_{\text{cl}}} \left(\frac{1}{1-t^{\deg(x)}}\right) \quad \overset{t=p^{-s}}{\Longrightarrow} \quad \zeta_X(s) = \prod_{x \in X_{\text{cl}}} \left(\frac{1}{1-\left(p^{\deg(x)}\right)^{-s}}\right),$$

which we can think of as attaching a "weight" to each closed point, $|x| := p^{\deg(x)}$, and the usual Riemann Zeta corresponds to assigning a weight of 1 to each point.

> Note that this immediately implies that $\zeta_X(t) \in \mathbb{Z}[[t]]$ is a *rational* function.

> Note for experts: $\zeta_X(z)$ an honest generating function for the 0-cycles on $X$ $(F(X_{\text{cl}}))$ where are effective (nonnegative coefficients).

### 1.1.3 More Examples

**Example (Affine Line):** $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then Note that we can write

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{\mathbf{x} = [x_1] \;\middle|\; x_1 \in \mathbb{F}_n\right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned}X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n.\end{aligned}$$

Thus

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{n} z^n\right) = \frac{1}{1-qz}.$$

**Example (Affine Space):** Set $X = \mathbb{A}^m/\mathbb{F}$, affine $m$-space over $\mathbb{F}$, so we can just repeat with now $m$ coordinates

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_n \right\}$$

Counting yields

$$\begin{aligned} X(\mathbb{F}) &= q^m \\ X(\mathbb{F}_2) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_n) &= (q^n)^m. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp\left( \sum_{n=1}^{\infty} \frac{q^{nm}}{n} z^n \right) = \frac{1}{1 - z^m t}.$$

**Example (Projective Line):** $X = \mathbb{P}^1/\mathbb{F}$ the projective line over $\mathbb{F}$, then we can write use some geometry to write

$$\mathbb{P}^1_{\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}} \coprod \{\infty\}$$

as the affine line with a point added at infinity.

We can then count by enumerating coordinates:

$$\begin{aligned} \mathbb{P}^1(\mathbb{F}_n) &= \left\{ [x_1, x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}_n \right\} / \sim \\ &= \left\{ [x_1, 1] \;\middle|\; x_1 \in \mathbb{F}_n \right\} \coprod \{[1, 0]\}. \end{aligned}$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &. \end{aligned}$$

Thus

$$\zeta_X(t) = \frac{1}{(1 - q^{-t})(1 - q^{1-t})}$$
.

**Example (Projective Space):** Take $X = \mathbb{P}^n_{\mathbb{F}}$, then $\alpha_n = 1 + q^m + (q^m)^2 + \cdots + (q^m)^n$, so

$$\zeta_X(t) = \left(\frac{1}{1-q^{-t}}\right)\left(\frac{1}{1-q^{1-t}}\right)\left(\frac{1}{1-q^{2-t}}\right)\cdots\left(\frac{1}{q^{n-t}}\right)$$

,

or equivalently, take your favorite curve $\gamma \in \mathbb{C}$ homotopic to $\mathbb{S}^1$.

Note: this is extremely amenable to numerical approximation if you have a closed form for $F$ or even just a black-box numerical version of $F$! I.e. easy to throw at a computer.

Todo: how to manually count points in $\mathbb{P}^n$!

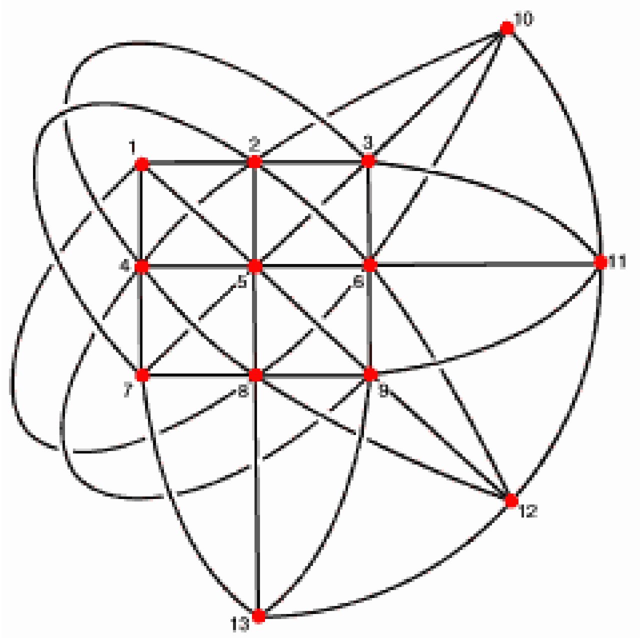


Image of $\mathbb{P}^2_{\mathbb{GF}(3)}$

Example: Take $X = \mathrm{Gr}_{\mathbb{F}}(k, n)$, then ????? so

$$\zeta_X(t) = ?.$$

Questions about properties

- $\zeta_{X \coprod Y}(t) =_? \zeta_X(t)\zeta_Y(t)$?
- $\zeta_{X \times Y} = ?$

## 1.2 Statement of Weil Conjectures

1. (Rationality)

$$\zeta_X(t) = \frac{p_1(t)p_3(t)\cdots p_{2N-1}(t)}{p_0(t)p_2(t)\cdots p_N(t)} \in \mathbb{Z}(t), \quad \text{i.e.} \quad p_i(t) \in \mathbb{Z}[t]$$

$$\begin{aligned} P_0(t) &= 1 - t \\ P_{2n}(t) &= 1 - q^n t \\ P_i(t) &= \prod_j (1 - a_{ij}t), \quad a_{ij} \in \mathbb{C}. \end{aligned}$$

2. (Functional Equation and Poincare Duality)

$$\zeta_X(n-t) = \pm q^{\frac{1}{2}(nE)-Et}\zeta(x,t).$$

3. (Riemann Hypothesis)
4. (Betti Numbers)

## 1.3 Hard Example: An Elliptic Curve

Take $X = E/\mathbb{F}$, then $\alpha_n = q^n - (a^n + \bar{a}^n - 1)$ where $|a|_{\mathbb{C}} = |\bar{\alpha}|_{\mathbb{C}} = \sqrt{q}$. Then

$$\zeta_X(t) = \frac{(1-aq^{-t})(1-\bar{a}q^{-t})}{(1-q^{-t})(1-q^{1-s})}.$$