Thursday, 26 January, 2017 10:45 PM

Intro/Logistics

- What are lightning talks?
- Why have them
- How do I sign up?

- SUMS Update
  - PMs registered club
  - General meetings near start of month
    - ↳ Give email & watch Slack!
  - Upcoming Events
    - ↳ Latex Workshop + refreshments
    - ↳ Lightning Talks 2
    - ↳ Pi Day ... ?

Review from last time:

- What is a category?
  - $Ob(\mathcal{C})$: class of objects

    Elts are $X, Y \in Ob(\mathcal{C})$
  - $Hom(\mathcal{C})$: set of morphisms/arrows

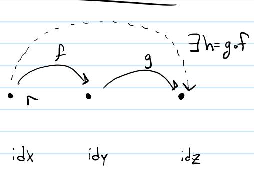
    $X \xrightarrow{f} Y$ then $f \in Hom_{\mathcal{C}}(X,Y) = Hom(X,Y)$
  - A binary op. on $Hom(\mathcal{C})$

    $\circ : Hom(\mathcal{C}) \times Hom(\mathcal{C}) \to Hom(\mathcal{C})$

    specifically $\circ : Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$

    $(f, g) \longmapsto h = g \circ f$

  - Where $\circ$ is associative and there exist unique two sided identities
- What's the picture?



- Special directed graphs
  - Nodes all have self-loops
  - Can "concatenate" paths & go directly to dest
  - Not every graph is a category though!

More review (in graph-theoretic terms)

- Duality: Reverse all arrows
- Functors: Graph isomorphisms
  - Covariant: $G \mapsto G$
  - Contravariant: $G \mapsto G^{op}$

Fun application:

Brouwer's Fixed Point Theorem

f cts fn. from a compact, convex set (into itself) $\to \exists x : f(x) = x$

- Take piece of paper, make a copy, crumple, place it on top
  - ~ Sitting in $\mathbb{R}^3$, P_1 is a 2d plane with $(x,y,0)$ coords

    P_2 is a surface in $\mathbb{R}_3$ with (x,y,z) coords w/ $z \geq 0$, $P_2 = f(P_1)$
  - ~ There is some point on P_2 that ends up above its original point in P_1 - same (x,y) coords!
  - ~ Could crumple the same way with a thumbtack through this point!

How does the proof work? Functors

- Initial objects - Sources
- Final objects - Sinks
- Injectivity - monomorphisms, cancel on left

  In sets, easy: $f(x) = f(y) \Rightarrow x = y$

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

and $(f \circ \varphi = f \circ \psi \Rightarrow \varphi = \psi)$

- Surjectivity - epimorphisms, cancel on right

  In sets, $\forall y \in Codom(f), \exists x \in Dom(f) : f(x) = y$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & Z \end{tikzcd}$$

and $(\alpha \circ f = \beta \circ f \Rightarrow \alpha = \beta)$

Surjectivity $\Rightarrow$ Right cancellable: $f : X \to Y$

$\forall \varphi, \psi : Y \to Z$

$\underbrace{\varphi \circ f = \psi \circ f}_{} \Rightarrow \underbrace{\varphi = \psi}_{WTS}$

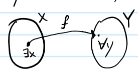
Sp. this $\overset{?}{}$ & f is surjective. Then $\forall y \in Codom(f) = Y, \exists x \in X : f(x) = y$

Surjectivity → Right cancellation: $f: X \to Y$

$\forall \varphi, \psi : Y \to Z$

$\underbrace{\varphi \circ f = \psi \circ f}_{?} \Rightarrow \underbrace{\varphi = \psi}_{\text{WTS}}$

Sp. this & f is surjective. Then $\forall y \in \mathrm{Codom}(f) = Y, \exists x \in X : f(x) = y$



Let y be arbitrary, then

$$\begin{aligned} \varphi(y) &= \varphi(f(x)) && \{\text{by surject. of } f\} \\ &= (\varphi \circ f)(x) \\ &= (\psi \circ f)(x) && \{\text{by hypoth.}\} \\ &= \psi(f(x)) \\ &= \psi(y) \end{aligned}$$

$\longrightarrow \forall y,\ \varphi(y) = \psi(y)$. ∎

· Isomorphism

If f is mono & epi. Situation:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[loop right, "id_B"] \arrow[l, bend left, "g"] \end{tikzcd}$$

mono:

$$\begin{tikzcd} A \arrow[r, bend left, "g \circ f"] \arrow[r, bend right, "id_A"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

$f \circ (g \circ f) = f \circ id_A \longrightarrow \underline{g \circ f = id_A}$

epi:

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "f \circ g"] \arrow[r, bend right, "id_B"'] & B \end{tikzcd}$$

$(f \circ g) \circ f = id_B \circ f \longrightarrow \underline{f \circ g = id_B}.$

A Survey of Mathematics

What can you do with a set?

1) Algebraic 2) Analytic

1- Algebraic

$\boxed{\text{Set}}$

Magma: + binary op
Semigroup: + assoc
Monoid: + an identity
Group: + inverses
Ab. Gp: + comm.

Ring: Group$(G,+) \otimes$ Monoid$(G,\cdot)$ $\{\otimes$ denotes fused structure on one set$\}$

Field: Group$(G,+) \otimes$ Group$(G,\cdot)$
~ Upgrade ring monoid to group

Module: $\underbrace{\text{Ab. Gp.}(G,\hat{+})}_{\text{(vectors)}} \wedge \underbrace{\text{Ring}(R,+,\cdot)}_{\text{(scalars)}}$
Combine two distinct structs

Vector Space: $\underbrace{\text{Ab. Gp}(G,\hat{+})}_{\text{(vectors)}} \wedge \underbrace{\text{Field}(\mathbb{F},+,\cdot)}_{\text{(scalars)}}$

Upgrade module ring to a field

Algebras: Ring$(G,\hat{+},\hat{\cdot}) \wedge$ Ring$(R,+,\cdot)$

A vector space with a bilinear product (like cross prod)
i.e. introduce multiplication of vectors

2) Analytic

$\boxed{\text{Set}}$ For notions of distance, area, angle

Topological Space - rough distance — Homeomorphism
Metric Space - pointwise Real # distance — Isometric Isomorphisms
Measure Space - volumes

Completions

Topological Vector Space - Vector Space $\otimes$ Topology
Normed Space - TVS & $\|\cdot\|$ or Metric Space $\otimes$ Vector Space → Banach

Inner product Space - TVS & $\langle \cdot, \cdot \rangle$ → Hilbert

Topological Vector Space - Vector Space & Topology

Normed Space - TVS & $\|\cdot\|$ $\longrightarrow$ Banach

or Metric Space $\otimes$ Vector space

Inner product Space - TVS & $\langle \cdot , \cdot \rangle$ $\longrightarrow$ Hilbert

Sets - Cardinality

- Objs = Sets A, B
- Morphs = Total set fns $f: A \to B$ (Total: defined $\forall a \in A$)
- Isos = Injective + Surjective set fns

or all f injective s.t. $\exists g: B \to A$ injective

$$\left.\begin{array}{l} f \text{ injective} \longrightarrow |A| \leq |B| \\ g \text{ injective} \longrightarrow |B| \leq |A| \end{array}\right\} |A| = |B|$$

Binary Relations

Objs - $(A, \sim)$, $(B, \propto)$ where $\sim \subseteq A \times A$ (ie ordered pairs, $a \sim b$ iff $(a,b) \subseteq \sim$)

Morphs - Rel-preserving set fns $f: (A, \sim) \to (B, \propto)$

$\forall a_1, a_2 \in A$, $a_1 \sim a_2 \Longrightarrow f(a_1) \propto f(a_2)$