8.8 Part 2, Computing the Index of L

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Contents

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

  where

$$\begin{aligned} S : \mathbb{R} \times S^1 &\longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\ S(s,t) &\overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t). \end{aligned}$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
  - *(not the same kernel/cokernel)*
- Compute $\operatorname{Ind} L_1$: explicitly describe $\ker L_1, \operatorname{coker} L_1$.
- Replace in two steps:
  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s.
    - Replace $S(s,t)$ by a matrix

$$\tilde{S}(s,t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

    - Idea: approximate by cylinders at infinity.
    - Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    - Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\mathrm{Ind}(L) = \mu\left(R^{-}(t)\right) - \mu\left(R^{+}(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

  where $S : \mathbb{R} \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\mathrm{Ind}(R^{\pm}(t))$; then

$$\mathrm{Ind}(L) = \mathrm{Ind}(L_1) = \mathrm{Ind}(R^{-}(t)) - \mathrm{Ind}(R^{+}(t)).$$

- Prop 8.8.3: Reducing L_1 to $R^{\pm}$. Let $k^{\pm} \coloneqq \mathrm{Ind}(R^{\pm})$; then $\mathrm{Ind}(L_1) = k^{-} - k^{+}$.
- Lemma 8.8.4: $\mathrm{Ind}(L_0) = \mathrm{Ind}(L)$.
- Han's Talk:
  - Prop 8.8.3, using Lemma 8.8.5
- Me
  - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\mathrm{Ind}(L_1) = K^{-} - k^{+}$.