# Linearization and Transversality

## Sections 8.3 and 8.4

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# Review 8.2

# Section 8.3: The Space of Perturbations of H

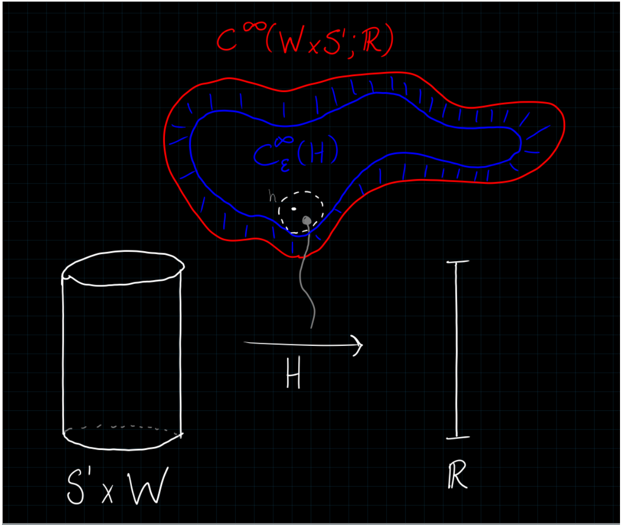
# Goal



**Goal**: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

# Goal

Start by trying to construct a subspace $\mathcal{C}_{\varepsilon}^{\infty}(H) \subset \mathcal{C}^{\infty}(W \times S^1; \mathbb{R})$, the space of perturbations of $H$ depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



# Define an Absolute Value

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C_\varepsilon^\infty(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$ denote a perturbation of $H$.
- Fix $\varepsilon = \left\{ \varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0} \right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k \right\},$$

  the maximum over all sets of multi-indices $\alpha$ of length $k$.
  *Note: I interpret this as*

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1}\, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

  *the partial derivatives wrt the corresponding variables.*

# Define a Norm



- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left| d^k h(x,t) \right| .$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

- Obtain the computable form

$$\|h\|_{,,} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} \left| d^k (h \circ \Psi_i^{-1})(z) \right|.$$

# Define a Banach Space



- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \;\middle|\; \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

  which is a Banach space (normed and complete).
- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that $C_\varepsilon^\infty$ is a *dense* subspace for the $C^\infty$ topology, and in particular for the $C^1$ topology.

**Theorem**

*Such a sequence $\{\varepsilon_k\}$ can be chosen.*

**Lemma**

*$C^\infty(W \times S^1; \mathbb{R})$ with the $C^1$ topology is separable as a topological space (contains a countable dense subset).*

# Sketch Proof of Theorem



- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the $C^1$ topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the $f_n$, define

$$\frac{1}{\varepsilon_n} = 2^n \max\left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x,t)| \leq 2^{-n}$$

which is summable.

*Why does this imply density? I don't know.*

# Modified Theorem

The next proposition establishes a version of this theorem with compact support:

**Theorem**

*For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the $C^1$ topology by functions in $C_\varepsilon^\infty$ supported in $U$.*

Then fix a time-dependent Hamiltonian $H_0$ with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \;\middle|\; h(x,t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\operatorname{supp}(h)$ is "far" from $\mathrm{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$$

and are both nondegenerate.

# Section 8.4: Linearizing the Floer Equation: The Differential of F

# Goal



Choose $m > n = \mathbf{dim}(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as $Z_i$, tangents to $W$ along $u$ or in a neighborhood $B$ of $u$) with actual vectors in $\mathbb{R}^m$.

*Why? Bypasses differentiating vector fields and the Levi-Cevita connection.*

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

*We've just replaced the codomain here.*

# Definitions

Recall that

- $x, y$ are contractible loops in $W$ that are nondegenerate critical points of the action functional $\mathcal{A}_H$,
- $u \in \mathcal{M}(x, y) \subset C^\infty_{\text{loc}}$ denotes a fixed solution to the Floer equation,
- $C_\searrow(x, y) \subset \left\{u \in C^\infty(R \times S^1; W)\right\}$ is the set of smooth solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions:

$$\lim_{s \longrightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left|\frac{\partial u}{\partial t}(s, t)\right|, \quad \left|\frac{\partial u}{\partial t}(s, t) - X_H(u)\right| \sim \exp(|s|)$$

# Compactify to Sphere



Fix a solution

$$u \in \mathcal{M}(x, y) \subset C^{\infty}_{\mathrm{loc}}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

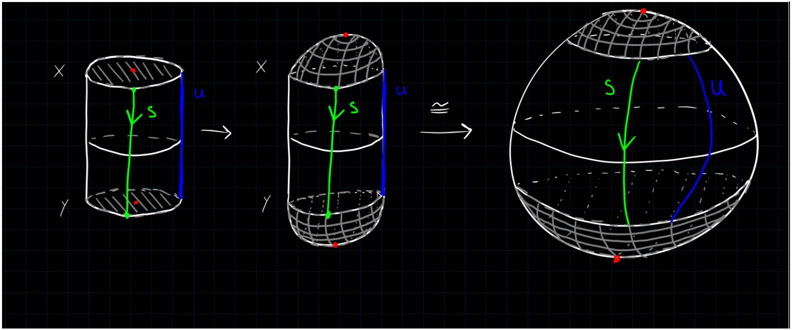
in the following way:

The loops $x, y$ are contractible, so they bound discs. So we extend by pushing these discs out slightly:

# Lift to 2-Sphere



$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$



# Trivial the Pullback



From earlier in the book, we have

**Assumption (6.22)**:

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where $c_1$ denotes the first Chern class of the bundle $TW$.

*Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:*

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W;\mathbb{Z})$$

*Assumption is satisfied when all maps* $S^2 \longrightarrow W$ *lift to* $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$\begin{tikzcd}
\tilde{u}^* TW \arrow[r, "d\tilde{u}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TW \arrow[d] \\
S^2 \arrow[r, "\tilde{u}"] & W
\end{tikzcd}$$

# Choose a Frame



- Using the assumption, trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

where

  - The frame depends smoothly on $(s, t) \in S^2$,
  - $\lim_{s \longrightarrow \infty} Z_i$ exists for each $i$.
  -

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \, \partial t} \quad \curvearrowright Z_i \xrightarrow{s \longrightarrow \pm\infty} 0 \quad \text{for each } i$$

Claim: such trivializations exist, "using cylinders near the spherical caps in the figure".

# Define "Banach Manifold Charts"

Recall we had $W^{1,p}(x,y)$ a completion of $C^\infty$

$$\mathcal{M}(x,y) \subset C^\infty_\searrow(x,y) \subset \mathcal{P}^{1,p}(x,y) \underset{\text{defn}}{\subset} \left\{ (s,t) \xrightarrow{\varphi} \exp_{w(s,t)} Y(s,t) \right\}.$$

where we restrict to

- $Y \in W^{1,p}(w^* TW)$,
- $w \in C^\infty_\searrow(x,y)$

Use the chosen frame $\{Z_i\}$ to define a chart centered at $u$ of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y) \\ \mathbf{y} = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

- Note that the derivative at zero is $\sum_{i=1}^{2n} y_i Z_i$.

# Define the Floer Map in Charts



Define and compute the differential of the composite map $\tilde{\mathcal{F}}$ defined as follows:

$$
\begin{tikzcd}
\mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, bend left, dotted, "\tilde{\mathcal{F}}"] & L^p\left(\mathbb{R}\times S^1; TW\right) \arrow[r] & L^p\left(\mathbb{R}\times S^1; \mathbb{R}^m\right)
\end{tikzcd}
$$

$$
u \xrightarrow{\quad\tilde{\mathcal{F}}\quad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)
$$

– From now on, let $\mathcal{F}$ denote $\tilde{\mathcal{F}}$.

# Add a Tangent

- Take the vector

$$Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

  - View $Y$ as a vector in $\mathbb{R}^m$ tangent to $W$, given by $Y = \sum_{i=1}^{2n} y_i Z_i$.
- Plug $u + Y$ into the equation for $\mathcal{F}$, directly yielding

# Add a Tangent



$$\begin{aligned}
\mathcal{F}(u) &= \frac{\partial u}{\partial s} && + J(u)\frac{\partial u}{\partial t} && - J(u)X_t(u) \\
\mathcal{F}(u+Y) &= \frac{\partial (u+Y)}{\partial s} && + J(u+Y)\frac{\partial (u+Y)}{\partial t} && - J(u+Y)X_t(u+Y)
\end{aligned}$$

# asd as

Extract the part that is linear in $Y$ and collect terms:

$$
\begin{aligned}
&(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y
\end{aligned}
$$

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