

# Title

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# Prologue

## 0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

## 0.2 Notation

- If a property  $P$  is said to hold **locally**, this means that for every point  $p$  there is a neighborhood  $U_p \ni p$  such that  $P$  holds on  $U_p$ .

$k[\mathbf{x}] := k[x_1, \dots, x_n]$	The polynomial ring in $n$ indeterminates
$k(\mathbf{x}) := k(x_1, \dots, x_n)$	The rational function field $\{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), p, q, \in k[x_1, \dots, x_n]\}$
$V(J), V_a(J)$	The variety associated to an ideal $J \leq k[x_1, \dots, x_n]$
	$V(J) := \{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$
$I(S), I_a(S)$	The ideal associated to a subset $S \subseteq \mathbb{A}_k^n$
	$I(S) := \{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in S\}$
$A(X) := k[x_1, \dots, x_n]/I(X)$	The coordinate ring of a variety
$\mathcal{O}_X$	The structure sheaf $\{f : U \rightarrow k \mid f \in k(\mathbf{x}) \text{ locally}\}$ .

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \leq k[x_1, \dots, x_n]$	$V(J)$ the zero locus
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$	Irreducible subsets

## 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for  $X = D(f)$  a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety  $X$  in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

## 0.4 Useful Examples

### 0.4.1 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$ , a sheaf of smooth functions
- $C^0$ , a sheaf of continuous functions

## 0.5 Useful Algebra Facts

### Fact 0.5.1:

- $\mathfrak{p} \trianglelefteq R$  is prime  $\iff R/\mathfrak{p}$  is a domain.
- $\mathfrak{p} \trianglelefteq R$  is maximal  $\iff R/\mathfrak{p}$  is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If  $R$  is a PID and  $\langle f \rangle \trianglelefteq R$  is generated by an irreducible element  $f$ , then  $\langle f \rangle$  is maximal

### Proposition 0.5.2 (*Finitely generated polynomial rings are Noetherian*).

A polynomial ring  $k[x_1, \dots, x_n]$  on finitely many generators is Noetherian. In particular, every ideal  $I \trianglelefteq k[x_1, \dots, x_n]$  has a finite set of generators and can be written as  $I = \langle f_1, \dots, f_m \rangle$ .

*Proof* (?).

A field  $k$  is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.5.5),  $k[x_1, \dots, x_n]$  is thus Noetherian. ■

### Proposition 0.5.3 (*Properties and Definitions of Ideal Operations*).

$$I + J := \{f + g \mid f \in I, g \in J\}$$

$$IJ := \left\{ \sum_{i=1}^N f_i g_i \mid f_i \in I, g_i \in J, N \in \mathbb{N} \right\}$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ$$

$$(\text{coprime or comaximal}) \langle a \rangle + \langle b \rangle = \langle a, b \rangle.$$

### Theorem 0.5.4 (*Noether Normalization*).

Any finitely-generated field extension  $k_1 \hookrightarrow k_2$  is a finite extension of a purely transcendental extension, i.e. there exist  $t_1, \dots, t_\ell$  such that  $k_2$  is finite over  $k_1(t_1, \dots, t_\ell)$ .

### Theorem 0.5.5 (*Hilbert's Basis Theorem*).

If  $R$  is a Noetherian ring, then  $R[x]$  is again Noetherian.

## 0.6 The Algebra-Geometry Dictionary

Let  $k = \bar{k}$ , we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \dots, x_n]$	$\mathbb{A}^n/k := \{[a_1, \dots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \dots, x_n - a_n \rangle$	Points $[a_1, \dots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \dots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k$ , vanishing locii of polynomials
	$I \mapsto V(I) := \{a \mid f(a) = 0 \forall f \in I\}$
	$I(X) := \{f \mid f _X = 0\} \triangleleft A(X)$
Radical ideals containing $I(X)$ , i.e. ideals in $A(X)$	closed subsets of $X$ , i.e. affine subvarieties
$A(X)$ is a domain	$X$ irreducible
$A(X)$ is not a direct sum	$X$ connected
Prime ideals in $A(X)$	Irreducible closed subsets of $X$
Krull dimension $n$ (longest chain of prime ideals)	$\dim X = n$ , (longest chain of irreducible closed subsets)