

Title

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Last time: we started discussing smoothness.

Definition 1.0.1 (Tangent Space)

The **tangent space** $T_p X$ of a variety X at a point $p \in X$ is defined as

$$V \left(\left\{ f_1 \mid f \in I(U_i), U_i \ni p = 0 \text{ affine} \right\} \right)$$

where f_1 denotes the degree 1 part.

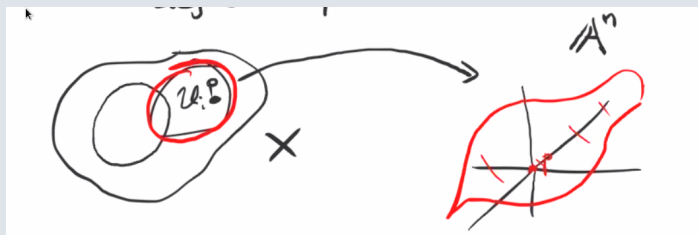


Figure 1: Image

Remark 1.0.2: We've really only defined it for affine varieties and $p = 0$, but this is a local definition. Note that this is also not a canonical definition, since it depends on the affine chart U_i .

Example 1.0.3(?): Consider $T_0 V(xy) = V(f_1 \mid f \in \langle xy \rangle) = V(0) = \mathbb{A}^2$, since every polynomial in this ideal has degree at least 2. Letting $X = V(xy)$, note that we could embed $X \hookrightarrow \mathbb{A}^3$ as $X \cong V(xy, z)$. In this case we have $T_0 X = V(f_1 \mid f \in \langle xy, z \rangle) = V(z) \cong \mathbb{A}^2$. So we get a vector space of a different dimension from this different affine embedding, but $\dim T_0 X$ is the same.

Example 1.0.4(?): Let $X = V_p(xy - z^2) \subset \mathbb{P}^2$, which is a projective curve. What is $T_p X$ for $p = [0 : 1 : 0]$? Take an affine chart $\{y \neq 0\} \cap X$, noting that $\{y \neq 0\} \cong \mathbb{A}^2$. We could dehomogenize the ideal $\langle xy - z^2 \rangle|_{y=1} = \langle x - z^2 \rangle$. Thus $X \cap D(y) = V(x - z^2) \subset \mathbb{A}^2$ and the point $[0 : 1 : 0] \in X$ gives $(0, 0)$ in this affine chart. Then $T_p X = V(f_1 \mid f \in \langle x - z^2 \rangle) = V(x)$. Then $f = (x - z^2)g$ implies that $f_1 = (xg)_1 = g_0 x$, the constant term of g multiplied by x , since z^2 kills any degree 1 part of g . So $T_p X$ is a line.

Example 1.0.5(?): Take X to be the union of the coordinate axes in \mathbb{A}^3 .

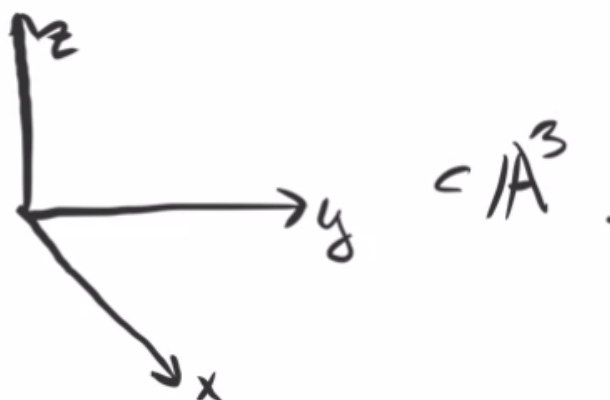


Figure 2: Image

Then $I(X) = \langle xy, yz, xz \rangle$ and $T_0X = V(f_1 \mid f \in I(X)) = V(0) = \mathbb{A}^3$, since the minimal degree of any such polynomial is 2. Note that $\dim X = 1$ but $\dim T_0X = 3$

Example 1.0.6(?): Take $Y = V(xy(x-y)) \subset \mathbb{A}^2$. Then $T_0X = V(0) = \mathbb{A}^2$:

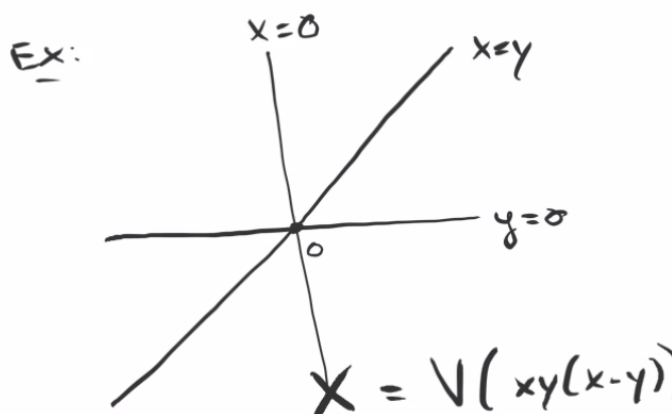


Figure 3: Image

Remark 1.0.7: Note that X and Y both consists of 3 copies of \mathbb{A}^1 intersecting at a single point.

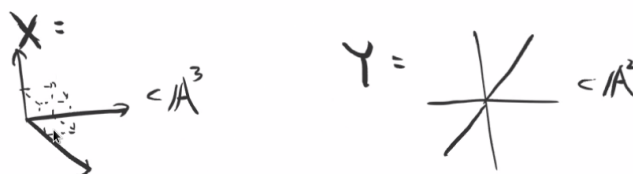


Figure 4: Image

Note that $\dim T_0 X = 3$ but $\dim T_0 Y = 3$, and interestingly $X \not\cong Y$ as affine varieties. There is a bijective morphism that is not invertible.

Remark 1.0.8: We will prove that $\dim T_p X$ is invariant under choice of affine embedding.

Example 1.0.9(?): How to compute $T_{(1,0,0)} V(xy, yz, xz)$: first move $(1, 0, 0)$ to the origin, yielding $T_{(0,0,0)} V((x+1)y, yz, (x+1)z)$. This is a different choice of affine embedding into \mathbb{A}^3 which sends $(1, 0, 0) \mapsto (0, 0, 0)$. Taking the vanishing locus of linear parts, it suffices to take the linear parts of the generators, which yields the x -axis $V(y, z)$, making the dimension of the tangent space 1.

Lemma 1.0.10(?).

Let $X \subset \mathbb{A}^n$ be an affine variety and let $0 = p \in X$. Then

$$T_0(X)^\vee := \text{hom}_k(T_0 X, k) \cong I_X(p)/I_X(p)^2$$

Remark 1.0.11: Note that the hom involves an affine embedding, but the quotient of ideals does not. We know that the category of affine varieties is equivalent to the category of reduced k -algebras, since the points of X biject with the maximal ideals of the coordinate ring $A(X)$. $I_X(p)$ is the maximal ideal in $A(X)$ of regular functions vanishing at p .

Proof (?).

Consider the map

$$\begin{aligned} \varphi : I_X(p) &\rightarrow T_0(X)^\vee \\ \bar{f} &\mapsto f_1|_{T_0(X)}. \end{aligned}$$

E.g. given $\bar{x}_1 - \bar{x}_2^2 \in A(X)$, we first lift to $x_1 - x_2^2 \in A(\mathbb{A}^n)$, restrict to the linear part x_1 , then restrict to $T_0(X)$. Note that $I_X(p) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \in k[x_1, \dots, x_n]/I(X)$, and we need to check that this is well-defined since there is ambiguity in choosing the above lift.

Claim: φ is well-defined.

Consider two lifts f, f' of $\bar{f} \in A(X) = k[x_1, \dots, x_n]/I(X)$. Then $f - f' \in I(X)$, so $(f - f')_1 = f_1 - f'_1$ is the linear part of some element in $I(X)$. The definition of $T_0(X)$ was the vanishing locus of linear parts of elements in $I(X)$, which contains $f_1 - f'_1$, and thus $(f_1 - f'_1)|_{T_0(X)} = 0$. So $f_1 = f'_1$ on $T_0(X)$.

Claim: $I_X(p)^2 \rightarrow 0$.

We know $I_X(p) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$, and so $I_X(p)^2 = \langle \bar{x}_i \bar{x}_j \rangle$. Given any $\bar{f} \in I_X(p)^2$, we can lift this to some $f \in \langle x_i x_j \rangle$, in which case $f_1 = 0$.

So φ descends to

$$\bar{\varphi} : I_X(p)/I_X(p)^2 \rightarrow T_0(X)^\vee$$

Claim: φ is injective and surjective.

That $\bar{\varphi}$ is surjective follows from the fact that if $\bar{x}_1, \dots, \bar{x}_n \in I_X(p)$, then the restrictions $x_1|_{T_0(X)}, \dots, x_n|_{T_0(X)}$ are in $\text{im } \bar{\varphi}$. These elements generate $T_0(X)^\vee$, since $T_0(X) \subset \mathbb{A}^n$. For injectivity, suppose $\bar{\varphi}(\bar{f}) = 0$, then $f_1|_{T_0(X)} = 0$, so f_1 is the linear part of some $f' \in I(X)$. Then $f' \in I(X)$ and f, f' have the same linear part f_1 , and $f - f'$ has no linear part. Thus $f - f' \in \langle x_i x_j \rangle$, which implies that $\bar{f} - \bar{f}' \in I_p(X)^2$ and $\bar{f} \equiv \bar{f}' \in I_p(X)/I_p(X)^2$. But $f' \equiv 0$ since $f' \in I(X)$. ■

Remark 1.0.12: So for X an affine variety, the cotangent space has a more intrinsic description, and we can recover the tangent space by dualizing:

$$T_p(X) := (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$$

where $\mathfrak{m}_p = I_X(p)$ is the maximal ideal of regular functions vanishing at p . So how can we get rid of the word affine? Given X any variety, we can define $T_p(X) := (\mathfrak{m}/\mathfrak{m}^2)^\vee$ where \mathfrak{m} is the maximal ideal of the local ring $\mathcal{O}_{X,p}$. This allows us to work on affine patches and localize. Moreover, this will be left invariant under the localization. ✍