Problem Set 7

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1 Problem 1

Note that if either $p = 1$ or $q = 1$, G is a p-group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2 q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$
\begin{aligned}
n_p &\equiv 1 \mod p & \qquad n_q &\equiv 1 \mod q \\
n_p &\,\Big|\, q & n_q &\,\Big|\, p^2,
\end{aligned}
$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \left\{1, p, p^2\right\}.$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. This forces $n_p = q$, and we proceed by cases:

1.1 Case 1: $p = q$.

Then $\#G = p^3$ and G is a p-group. But every p-group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.2 Case 2: $p > q$.

Here, since $n_p \,\Big|\, q$, we must have $n_p < q$. But if $n_p < q < p$ and $n_p = 1 \mod p$, then $n_p = 1$.

1.3 Case 3: $q > p$.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q:

- $n_q = p$: If $n_q = p = 1 \mod q$ and $p < q$, this forces $p = 1$.
- $n_q = p^2$: We will reach a contradiction by showing that this forces

  $$\left| P := \bigcup_{S_p \in \mathrm{Syl}(p,G)} S_p \setminus \{e\} \right| + \left| Q := \bigcup_{S_q \in \mathrm{Syl}(q,G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

  We have

  $$
  \begin{aligned}
  |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\
  &= p^2(q-1) + q(p^2-1) + 1 \\
  &= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 \qquad \text{(since } q > 1) \\
  &= (p^2 q - p^2) + (p^2-1) + (q-1)(p^2-1) + 1 \\
  &= p^2 q + (q-1)(p^2-1) \\
  &\geq p^2 q + (2-1)(2^2-1) \qquad \text{(since } p, q \geq 2) \\
  &= p^2 q + 3 \\
  &> p^2 q = |G|,
  \end{aligned}
  $$

  which is a contradiction. $\square$

2 Problem 2

We'll use the fact that $H \trianglelefteq N(H)$ for any subgroup H (following directly from the closure axioms for a subgroup), and thus

$$P \trianglelefteq N(P) \quad \text{and} \quad N(P) \trianglelefteq N^2(P).$$

Since it is then clear that $N(P) \subseteq N^2(P)$, it remains to show that $N^2(P) \subseteq N(P)$.

So if we let $x \in N^2(P)$, so x normalizes $N(P)$, we need to show that x normalizes P as well, i.e. $xPx^{-1} = P$.

However, supposing that $|G| = p^k m$ where $(p, m) = 1$, we have

$$P \leq N(P) \leq G \implies p^k \;\Big|\; |N(P)| \;\Big|\; p^k m,$$

so in fact $P \in \mathrm{Syl}(p, N(P))$ since it is a maximal p-subgroup.

Then $P' := xPx^{-1} \in \mathrm{Syl}(p, N(P))$ as well, since all conjugates of Sylow p-subgroups are also Sylow p-subgroups.

But since $P \trianglelefteq N(P)$, there is only *one* Sylow p- subgroup of $N(P)$, namely P. This forces $P = P'$, i.e. $P = xPx^{-1}$, which says that $x \in N(P)$ as desired. $\square$

3 Problem 3

By definition, G is simple iff it has no non-trivial subgroups, so we will show that if $|G| = 148$ then it must contain a normal subgroup.

Noting that $248 = p^2 q$ where $p = 2, q = 37$, we find that (for example) $n_2 \;\Big|\; 37$ but $n \equiv 1 \mod 2$; but the only odd divisor of 7 is 1, forcing $n_2 = 1$. So G has a normal Sylow 2-subgroup and we are done.

4 Problem 4

Let $\tau := (t_1, t_2)$ denote the transposition and $\sigma = (s_1, s_2 \cdots, s_p)$ denote the p-cycle, and let $S = \langle \sigma, \tau \rangle$. We would like to show that $S = S_p$, and since $S \subseteq S_p$ is clear, we just need to show that $S_p \subseteq S$.

We first note that because p is prime, σ^k is a p-cycle for every $1 \leq k \leq p$, and $\langle \sigma \rangle = \langle \sigma^k \rangle$ for any such k.

Then note that $t_1 = s_i$ for some i and $t_2 = s_j$ for some j, so we can take $k = j - i$ to get a cycle σ^k that sends t_1 to t_2. So without loss of generality, we can replace σ with

$$\sigma = (t_1, t_2, \cdots)$$

But now, we can relabel all of the elements of S_p simultaneously (i.e. replace $\langle \sigma, \tau \rangle$ with another subgroup in the same conjugacy class) in such a way that t_1 becomes 1 and t_2 becomes 2. We can then assume wlog that

$$\tau = (1,2), \quad \sigma = (1,2,\cdots,p)$$

We can then get all adjacent transpositions: noting that

$$\begin{aligned}
\sigma^{-1}\tau\sigma &= (2,3) \\
\sigma^{-2}\tau\sigma^{2} &= (3,4) \\
&\cdots \\
\sigma^{-k}\tau\sigma^{k} &= (k+1 \mod p,\ k+2 \mod p) \quad \forall 1 \leq k \leq p,
\end{aligned}$$

where we use the fact that for any $\gamma \in S_p$, we have $\gamma\tau\gamma = (\gamma(1),\ \gamma(2))$.

But this also gives us all transpositions of the form $(1,j)$ for each $2 \leq j \leq p$:

$$\begin{aligned}
(2,3)^{-1}(1,2)(2,3) &= (1,3) \\
(3,4)^{-1}(1,3)(3,4) &= (1,4) \\
&\cdots \\
(j-1,j)^{-1}(1,j-1)(j-1,j) &= (1,j) \quad \forall 1 \leq j \leq p.
\end{aligned}$$

Thus we have $J \coloneqq \langle \{(1,j) \mid 2 \leq j \leq p\} \rangle \subseteq S$.

But now if $\gamma = (g_1, g_2, \cdots, g_k) \in S_p$ is an arbitrary cycle, we can write

$$\gamma = (g_1, g_2, \cdots, g_k) = (1, g_1)(1, g_2), \cdots (1, g_k),$$

so $\gamma \in J$. Then writing any arbitrary permutation as a product of disjoint cycles, we find that $S_p \subseteq J \subseteq S$, and so $S_p \subseteq S$ as desired. $\square$

5 Problem 5

Since G is a p-group, it has a nontrivial center. Since p is prime and $Z(G)$ is a subgroup, this forces $\#Z(G) \in \{p, p^2\}$, where p^3 is ruled out because this would make G abelian.

Supposing that $\#Z(G) = p^2$,we would have $[G : Z(G)] = p$, and since $Z(G) \trianglelefteq G$, we can take the quotient and $\#(G/Z(G)) = p$. But this means $G/Z(G)$ is cyclic, which implies that G is abelian, a contradiction.

So we must have $\#Z(G) = p$, and $\#(G/Z(G)) = p^2$.

But any group of p^2 is abelian, and we can characterize $G' \coloneqq [G, G]$ in the following way:

$G' \leq G$ is the unique subgroup of G such that if $N \trianglelefteq G$ and G/N is abelian, then $N \leq G'$.

5.1 Case 2: $\#Z(G) = p$:

6 Problem 6

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9 Problem 9

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