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Garza

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## The Weil Conjectures

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## Background: Generating Functions

Fix  $q$  a prime and  $\mathbb{F} := \mathbb{F}_q$  the (unique) finite field with  $q$  elements, along with its (unique) degree  $n$  extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

## Definition (Projective Algebraic Varieties)

Let  $J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$  be an ideal, then a *projective algebraic variety*  $X \subset \mathbb{P}_{\mathbb{F}}^n$  can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^n \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = 0 \right\}$$

where  $J$  is generated by *homogeneous* polynomials in  $n+1$  variables, i.e. there is a fixed  $d = \deg f_i \in \mathbb{Z}^{\geq 1}$  such that

$$f(\mathbf{x}) = \sum_{\substack{I=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_I \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^\times.$$

# Point Counts

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- For a fixed variety  $X$ , we can consider its  $\mathbb{F}$ -points  $X(\mathbb{F})$ .
  - Note that  $\#X(\mathbb{F}) < \infty$  is an integer
- For any  $L/\mathbb{F}$ , we can also consider  $X(L)$ 
  - In particular, we can consider  $X(\mathbb{F}_{q^n})$  for any  $n \geq 2$ .
  - We again have  $\#X(\mathbb{F}_{q^n}) < \infty$  and are integers for every such  $n$ .
- So we can consider the sequence

$$[N_1, N_2, \dots, N_n, \dots] := [\#X(\mathbb{F}), \#X(\mathbb{F}_{q^2}), \dots, \#X(\mathbb{F}_{q^n}), \dots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \dots.$$

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Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of  $F$ : we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left( \frac{\partial}{\partial z} \right)^n F(z) \Big|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} dz = N_n.$$

*The latter form is very amenable to computer calculation.*

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An OGF is an infinite series, which we can interpret as an analytic function  $\mathbb{C} \rightarrow \mathbb{C}$  – in nice situations, we can hope for a closed-form representation.

A useful example: by integrating a geometric series we can derive

$$\begin{aligned}\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n && (= 1 + z + z^2 + \cdots) \\ \Rightarrow \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} \int z^n \quad \text{for } |z| < 1 \quad \text{by uniform convergence} \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\ \Rightarrow -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} && \left( = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \right).\end{aligned}$$

For completeness, also recall that

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

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# Zeta Functions

# Definition: Local Zeta Function

Problem: count points of a (smooth?) projective variety  $X/\mathbb{F}$  in all (finite) degree  $n$  extensions of  $\mathbb{F}$ .

## Definition (Local Zeta Function)

The *local zeta function* of an algebraic variety  $X$  is the following formal power series:

$$Z_X(z) = \exp \left( \sum_{n=1}^{\infty} N_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note that

$$\begin{aligned} z \left( \frac{\partial}{\partial z} \right) \log Z_X(z) &= z \frac{\partial}{\partial z} \left( N_1 z + N_2 \frac{z^2}{2} + N_3 \frac{z^3}{3} + \cdots \right) \\ &= z (N_1 + N_2 z + N_3 z^2 + \cdots) \quad (\text{unif. conv.}) \\ &= N_1 z + N_2 z^2 + \cdots = \sum_{n=1}^{\infty} N_n z^n, \end{aligned}$$

which is an *ordinary* generating function for the sequence  $(N_n)$ .



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## Examples

## Example: A Point

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Take  $X = \{\text{pt}\} = V(\{f(x) = 0\})/\mathbb{F}$  a single point over  $\mathbb{F}$ , then

$$\#X(\mathbb{F}_q) := N_1 = 1$$

$$\#X(\mathbb{F}_{q^2}) := N_2 = 1$$

$$\vdots$$

$$\#X(\mathbb{F}_{q^n}) := N_n = 1$$

$$\vdots$$

and so

$$\begin{aligned} Z_{\{\text{pt}\}}(z) &= \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \cdots\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}. \end{aligned}$$

*Notice:  $Z$  admits a closed form **and** is a rational function.*

# Example: The Affine Line

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Take  $X = \mathbb{A}^1/\mathbb{F}$  the affine line over  $\mathbb{F}$ , then We can write

$$\mathbb{A}^1(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1] \mid x_1 \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in  $\mathbb{F}_n$ , so

$$X(\mathbb{F}_q) = q$$

$$X(\mathbb{F}_{q^2}) = q^2$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = q^n.$$

Then

$$\begin{aligned} Z_X(z) &= \exp \left( \sum_{n=1}^{\infty} q^n \frac{z^n}{n} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{(qz)^n}{n} \right) \\ &= \exp(-\log(1 - qz)) \\ &= \frac{1}{1 - qz}. \end{aligned}$$

## Example: Affine m-space

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Take  $X = \mathbb{A}^m/\mathbb{F}$  the affine line over  $\mathbb{F}$ , then We can write

$$\mathbb{A}^m(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1, \dots, x_m] \mid x_i \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in  $\mathbb{F}_n$ , so

$$X(\mathbb{F}_q) = q^m$$

$$X(\mathbb{F}_{q^2}) = (q^2)^m$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = q^{nm}.$$

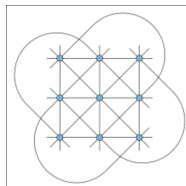


Figure:  $\mathbb{A}^2/\mathbb{F}_3$  ( $q = 3, m = 2, n = 1$ )

Then

$$\begin{aligned} Z_X(z) &= \exp \left( \sum_{n=1}^{\infty} q^{nm} \frac{z^n}{n} \right) = \exp \left( \sum_{n=1}^{\infty} \frac{(q^m z)^n}{n} \right) \\ &= \exp(-\log(1 - q^m z)) \\ &= \frac{1}{1 - q^m z}. \end{aligned}$$

# Example: Projective Line

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Take  $X = \mathbb{P}^1/\mathbb{F}$ , we can still count by enumerating coordinates:

$$\mathbb{P}^1(\mathbb{F}_{q^n}) = \left\{ [x_1 : x_2] \mid x_1, x_2 \neq 0 \in \mathbb{F}_{q^n} \right\} / \sim = \left\{ [x_1 : 1] \mid x_1 \in \mathbb{F}_{q^n} \right\} \coprod \{[1 : 0]\}.$$

Thus

$$X(\mathbb{F}_q) = q + 1$$

$$X(\mathbb{F}_{q^2}) = q^2 + 1$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = q^n + 1.$$

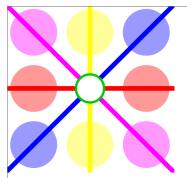


Figure:  $\mathbb{P}^1/\mathbb{F}_3$  ( $q = 3, n = 1$ )

Thus

$$\begin{aligned} Z_X(z) &= \exp \left( \sum_{n=1}^{\infty} (q^n + 1) \frac{z^n}{n} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} q^n \frac{z^n}{n} + \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \\ &= \frac{1}{(1 - qz)(1 - z)}. \end{aligned}$$

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# The Weil Conjectures

(Weil 1949)

Let  $X$  be a smooth projective variety of dimension  $N$  over  $\mathbb{F}_q$  for  $q$  a prime, let  $Z_X(z)$  be its zeta function, and define  $\zeta_X(s) = Z_X(q^{-s})$ .

## 1 (Rationality)

$Z_X(z)$  is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$

$$P_{2N}(z) = 1 - q^N z$$

$$P_j(z) = \prod_{i=1}^{\beta_j} (1 - a_{j,i} z) \quad \text{for some reciprocal roots } a_{j,i} \in \mathbb{C}$$

where we've factored each  $P_i$  using its reciprocal roots  $a_{ij}$ .

In particular, this implies the existence of a meromorphic continuation of the associated function  $\zeta_X(s)$ , which a priori only converges for  $\Re(s) \gg 0$ . This also implies that for  $n$  large enough,  $N_n$  satisfies a linear recurrence relation.

## 2 (Functional Equation and Poincare Duality)

Let  $\chi(X)$  be the Euler characteristic of  $X$ , i.e. the self-intersection number of the diagonal embedding  $\Delta \hookrightarrow X \times X$ ; then  $Z_X(z)$  satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm \left(q^{\frac{N}{2}} z\right)^{\chi(X)} Z_X(z).$$

Equivalently,

$$\zeta_X(N-s) = \pm \left(q^{\frac{N}{2}-s}\right)^{\chi(X)} \zeta_X(s)$$

Note that when  $N = 1$ , e.g. for a curve, this relates  $\zeta_X(s)$  to  $\zeta_X(1-s)$ .

Equivalently, there is an involutive map on the (reciprocal) roots

$$z \longleftrightarrow \frac{q^N}{z}$$

$$\alpha_{j,k} \longleftrightarrow \alpha_{2N-j,k}$$

which sends roots of  $p_j$  to roots of  $p_{2N-j}$ .



**3** (Riemann Hypothesis)

The reciprocal roots  $a_{j,k}$  are *algebraic* integers (roots of some monic  $p \in \mathbb{Z}[x]$ ) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}}, \quad 1 \leq j \leq 2N - 1, \quad \forall k.$$

**4** (Betti Numbers)

If  $X$  is a “good reduction mod  $q$ ” of a nonsingular projective variety  $\tilde{X}$  in characteristic zero, then the  $\beta_i = \deg p_i(z)$  are the Betti numbers of the topological space  $\tilde{X}(\mathbb{C})$ .

Moral:

- The Diophantine properties of a variety’s zeta function are governed by its (algebraic) topology.
- Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information.
- Langland’s: similarly asks for every  $L$  function arising from an automorphic representation to satisfy Weil 2 and 3.

# Why is (3) called the “Riemann Hypothesis”?

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Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

After modifying  $\zeta$  to make it symmetric about  $\Re(s) = \frac{1}{2}$  and eliminate the trivial zeros to obtain  $\hat{\zeta}(s)$ , there are three relevant properties

- “Rationality”:  $\hat{\zeta}(s)$  has a meromorphic continuation to  $\mathbb{C}$  with simple poles at  $s = 0, 1$ .
- “Functional equation”:  $\hat{\zeta}(1 - s) = \hat{\zeta}(s)$
- “Riemann Hypothesis”: The only zeros of  $\hat{\zeta}$  have  $\Re(s) = \frac{1}{2}$ .

# Why is (3) called the “Riemann Hypothesis”?

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Suppose it holds. We can use the facts that

a.  $|\exp(z)| = \exp(\Re(z))$  and

b.  $a^z := \exp(z \operatorname{Log}(a)),$

and to replace the polynomials  $P_i$  with

$$L_j(s) := P_j(q^{-s}) = \prod_{k=1}^{\beta_j} (1 - \alpha_{j,k} q^{-s}).$$

# Analogy to Riemann Hypothesis

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Now consider the roots of  $L_j(s)$ : we have

$$L_j(s_0) = 0$$

$$\iff q^{-s_0} = \frac{1}{\alpha_{j,k}} \quad \text{for some } k$$

$$\implies |q^{-s_0}| = \left| \frac{1}{\alpha_{j,k}} \right| \quad \text{by assumption } q^{-\frac{j}{2}}$$

$$\implies q^{-\frac{j}{2}} \stackrel{(a)}{=} \exp\left(-\frac{j}{2} \cdot \text{Log}(q)\right) = |\exp(-s_0 \cdot \text{Log}(q))|$$

$$\stackrel{(b)}{=} |\exp(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \text{Log}(q))|$$

$$\stackrel{(a)}{=} \exp(-(\Re(s_0)) \cdot \text{Log}(q))$$

$$\implies -\frac{j}{2} \cdot \text{Log}(q) = -\Re(s_0) \cdot \text{Log}(q) \quad \text{by injectivity}$$

$$\implies \Re(s_0) = \frac{j}{2}.$$

# Analogy with Riemann Hypothesis

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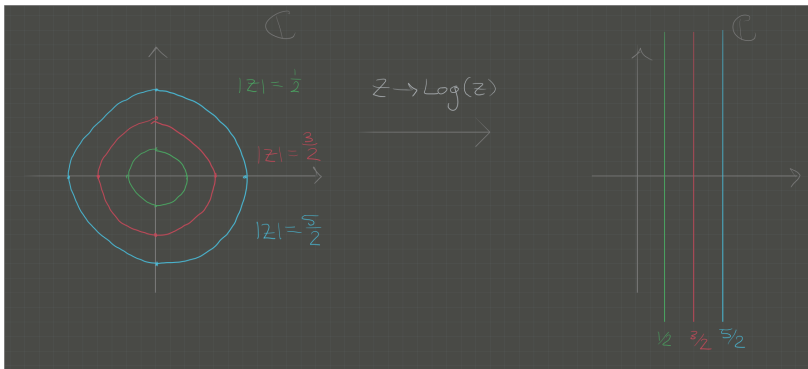
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Roughly speaking, realizing that we would need to apply a logarithm (a conformal map) to send the  $\alpha_{j,k}$  to zeros of the  $L_j$ , this says that the zeros all must lie on the “critical lines”  $\frac{j}{2}$ .



In particular, the zeros of  $L_1$  have real part  $\frac{1}{2}$ , analogous to the classical Riemann hypothesis.

# Precise Relation

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- Difficult to find in the literature! Idea: make a similar definition for schemes, then take  $X = \text{Spec } \mathbb{Z}$ .
- Define the “reductions mod  $q$ ”  $X_q$  for closed points  $q$ .
- Define the *local* zeta functions  $\zeta_{X_p}(s) = Z_{X_p}(q^{-s})$ .
- (Potentially incorrect) Evaluate to find  $Z_{X_p}(z) = \frac{1}{1-z}$ .
- Take a product over all closed points to define

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

which is the Euler product expansion of the classical Riemann Zeta function.

*If anyone knows a reference for this, let me know!*

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## Weil for Elliptic Curves

# Example: An Elliptic Curve

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The Weyl conjectures take on a particularly nice form for curves. Let  $X/\mathbb{F}$  be a smooth projective curve of genus  $g$ , then

1 (Rationality)

$$\zeta_X(z) = \frac{p(z)}{(1-z)(1-qz)}$$

2 (Functional Equation)

$$\zeta_X\left(\frac{1}{qz}\right) = q^{1-g} z^{2-2g} \zeta_X(z)$$

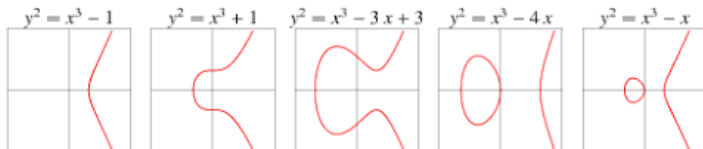
3 (Riemann Hypothesis)

$$p(t) = \prod_{i=1}^{2g} (q - a_i z) \quad \text{where} \quad |a_i| = \frac{1}{\sqrt{q}}$$

Take  $X = E/\mathbb{F}$ .



Figure: Some Elliptic Curves



- The number of points is given by

$$N_n := X(\mathbb{F}_{q^n}) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

- Proof: Unsure! Maybe someone can point me to a reference. Involves trace (or eigenvalues?) of Frobenius.
- The Poincare polynomial is given by  $P(x) = \sum \beta_i x^i = 1 + 2x + x^2$ .
- The dimension of  $X$  over  $\mathbb{C}$  is  $N = 1$ ,

The WC say we should be able to write as

$$Z_E(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1-\alpha_{1,1}z)(1-\alpha_{1,2}z)}{(1-z)(1-qz)}.$$

Since we know the number of points, we can compute

$$\begin{aligned}
 Z_E(z) &= \exp \sum_{n=1}^{\infty} \#E(\mathbb{F}_{q^n}) \frac{z^n}{n} \\
 &= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
 &= \exp \left( \sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
 &= \exp(-\log(1 - qz)) \exp(-\log(1 - z)) \exp(\log(1 - \alpha z)) \exp(\log(1 - \bar{\alpha} z)) \\
 &= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),
 \end{aligned}$$

which is indeed a rational function.

Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-t})}.$$

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## Weil for Projective m-space

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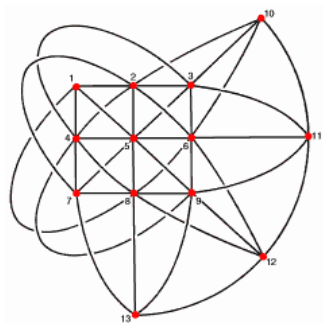
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Take  $X = \mathbb{P}^m/\mathbb{F}$  We can write

$$\mathbb{P}^m(\mathbb{F}_{q^n}) = \mathbb{A}^{m+1}(\mathbb{F}_{q^n}) \setminus \{\mathbf{0}\} / \sim = \left\{ \mathbf{x} = [x_0, \dots, x_m] \mid x_i \in \mathbb{F}_{q^n} \right\} / \sim$$

But how many points are actually in this space?

Figure: Points and Lines in  $\mathbb{P}^2/\mathbb{F}_3$



*A nontrivial combinatorial problem!*

# q-Analogs and Grassmannians

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To illustrate, this can be done combinatorially: identify  $\mathbb{P}_{\mathbb{F}}^m = \text{Gr}_{\mathbb{F}}(1, m+1)$  as the space of lines in  $\mathbb{A}_{\mathbb{F}}^{m+1}$ .

## Theorem

*The number of  $k$ -dimensional subspaces of  $\mathbb{A}_{\mathbb{F}_q}^N$  is the  $q$ -analog of the binomial coefficient:*

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

*Remark: Note  $\lim_{q \rightarrow 1} \begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k}$ , the usual binomial coefficient.*

**Proof:** To choose a  $k$ -dimensional subspace,

- Choose a nonzero vector  $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$  in  $q^N - 1$  ways.
  - For next step, note that  $\#\text{span}\{\mathbf{v}_1\} = \#\left\{\lambda \mathbf{v}_1 \mid \lambda \in \mathbb{F}_q\right\} = \#\mathbb{F}_q = q$ .
- Choose a nonzero vector  $\mathbf{v}_2$  *not* in the span of  $\mathbf{v}_1$  in  $q^N - q$  ways.
  - Now note  $\#\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \#\left\{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mid \lambda_i \in \mathbb{F}\right\} = q \cdot q = q^2$ .

# Proof continued

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- Choose a nonzero vector  $\mathbf{v}_3$  *not* in the span of  $\mathbf{v}_1, \mathbf{v}_2$  in  $q^N - q^2$  ways.
- $\dots$  until  $\mathbf{v}_k$  is chosen in

$$(q^N - 1)(q^N - q) \cdots (q^N - q^{k-1}) \quad \text{ways}$$

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- This yields a  $k$ -tuple of linearly independent vectors spanning a  $k$ -dimensional subspace  $V_k$
- This overcounts because many linearly independent sets span  $V_k$ , we need to divide out by the number of ways to choose a basis inside of  $V_k$ .
- By the same argument, this is given by

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

Thus

$$\begin{aligned} \# \text{subspaces} &= \frac{(q^N - 1)(q^N - q)(q^N - q^2) \cdots (q^N - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})} \\ &= \frac{q^N - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{N-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{N-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{N-(k-1)} - 1}{q^{k-(k-1)-1}} \\ &= \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}. \end{aligned}$$

# Counting Points

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Note that we've actually computed the number of points in any Grassmannian.

Identify  $\mathbb{P}_{\mathbb{F}}^m = \text{Gr}_{\mathbb{F}}(1, m+1)$  as the space of lines in  $\mathbb{A}_{\mathbb{F}}^{m+1}$ .

We obtain a nice simplification for the number of lines corresponding to setting  $k = 1$ :

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = \frac{q^{m+1} - 1}{q - 1} = q^m + q^{m-1} + \cdots + q + 1 = \sum_{j=0}^m q^j.$$

Thus

$$X(\mathbb{F}_q) = \sum_{j=0}^m q^j$$

$$X(\mathbb{F}_{q^2}) = \sum_{j=0}^m (q^2)^j$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = \sum_{j=0}^m (q^n)^j.$$

# Computing the Zeta Function

So

$$\begin{aligned}Z_X(z) &= \exp \left( \sum_{n=1}^{\infty} \sum_{j=0}^m (q^n)^j \frac{z^n}{n} \right) \\&= \exp \left( \sum_{n=1}^{\infty} \sum_{j=0}^m \frac{(q^j z)^n}{n} \right) \\&= \exp \left( \sum_{j=0}^m \sum_{n=1}^{\infty} \frac{(q^j z)^n}{n} \right) \\&= \exp \left( \sum_{j=0}^{m-1} -\log(1 - q^j z) \right) \\&= \prod_{j=0}^m (1 - q^j z)^{-1} \\&= \left( \frac{1}{1-z} \right) \left( \frac{1}{1-qz} \right) \left( \frac{1}{1-q^2 z} \right) \cdots \left( \frac{1}{1-q^m z} \right),\end{aligned}$$

*Miraculously, still a rational function!*

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# An Easier Proof

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Quick recap:

$$Z_{\{\text{pt}\}} = \frac{1}{1-z} \quad Z_{\mathbb{P}^1}(z) = \frac{1}{1-qz} \quad Z_{\mathbb{A}^1}(z) = \frac{1}{(1-z)(1-qz)}.$$

Note that  $\mathbb{P}^1 = \mathbb{A}^1 \amalg \{\infty\}$  and correspondingly  $Z_{\mathbb{P}^1}(z) = Z_{\mathbb{A}^1}(z) \cdot Z_{\{\text{pt}\}}(z)$ .  
This works in general:

## Lemma (Excision)

*If  $Y/\mathbb{F}_q \subset X/\mathbb{F}_q$  is a closed subvariety, for  $U = X \setminus Y$ ,  
 $Z_X(z) = Z_Y(z) \cdot Z_U(z)$ .*

**Proof:** Let  $N_n = \#Y(\mathbb{F}_{q^n})$  and  $M_n = \#U(\mathbb{F}_{q^n})$ , then

$$\begin{aligned} \zeta_X(z) &= \exp \left( \sum_{n=1}^{\infty} (N_n + M_n) \frac{z^n}{n} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n} + \sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n} \right) \cdot \exp \left( \sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n} \right) = \zeta_Y(z) \cdot \zeta_U(z). \end{aligned}$$

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Note that geometry can help us here: we have a stratification  $\mathbb{P}^n = \mathbb{P}^{n-1} \amalg \mathbb{A}^n$ , and so inductively

$$\mathbb{P}^m = \coprod_{j=0}^m \mathbb{A}^j = \mathbb{A}^0 \amalg \mathbb{A}^1 \amalg \cdots \amalg \mathbb{A}^m,$$

and recalling that

$$Z_{X \amalg Y}(z) = Z_X(z) \cdot Z_Y(z)$$

and  $Z_{\mathbb{A}^j}(z) = \frac{1}{1-q^j z}$  we have

$$Z_{\mathbb{P}^m}(z) = \prod_{j=0}^m Z_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1-q^j z}.$$

*Notice that the highest degree is exactly  $m$ , and there is exactly one factor for each  $j \leq m$ . Note that  $\mathbb{P}^m/\mathbb{F}_q$  can be thought of as a mod  $q$  reduction of  $\mathbb{R}P^m$  or  $\mathbb{C}P^m$ , and somehow  $Z$  “sees” its dimension.*

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## Grassmannian

Consider now  $X = \text{Gr}(k, m)/\mathbb{F}$  – by the previous computation, we know

$$X(\mathbb{F}_{q^n}) = \left[ \begin{matrix} m \\ k \end{matrix} \right]_{q^n} := \frac{(q^{nm} - 1)(q^{nm-1} - 1) \cdots (q^{nm-n(k-1)} - 1)}{(q^{nk} - 1)(q^{n(k-1)} - 1) \cdots (q^n - 1)}$$

but the corresponding Zeta function is much more complicated than the previous examples:

$$Z_X(z) = \exp \left( \sum_{n=1}^{\infty} \left[ \begin{matrix} m \\ k \end{matrix} \right]_{q^n} \frac{z^n}{n} \right) = \cdots?.$$

Note that  $\dim_{\mathbb{R}} \text{Gr}_{\mathbb{R}}(k, m) = k(m - k)$  as a real manifold, so