Title

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1 | 2021-04-09

1.1 18:12

Proof. Clear. ▲

Example 1.19 Not every quasi-isomorphism is a homotopy equivalence. Consider the complex

$$\cdots \to 0 \to \mathbb{Z}/\cdot 2 \to \mathbb{Z} \to 0 \to 0 \to \ldots$$

so $H^0 = \mathbb{Z}/2\mathbb{Z}$ and all cohomologies are 0. We have a quasi-isomorphism from the above complex to the complex

$$\cdots \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to \ldots$$

but no inverse can be defined (no map from $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$).

- **Proposition 1.20** *Additive functors preserve chain homotopies*

2 | 2021-04-08

2.1 18:11

limit us. We will apply the Riemann-Roch theorem to solve this problem.

Recall that a (Weil) divisor on a curve is a finite formal sum of closed points of the curve. If D is a divisor, then $\deg(D)$ is the sum of the coefficients of D. Functions and differentials define divisors via their orders at each point. Thus, if f is an element of the function field $K(C)$ of a curve C, then we write:

$$\mathrm{Div}(f) = \sum_{P \in C} \mathrm{ord}_P(f)P.$$

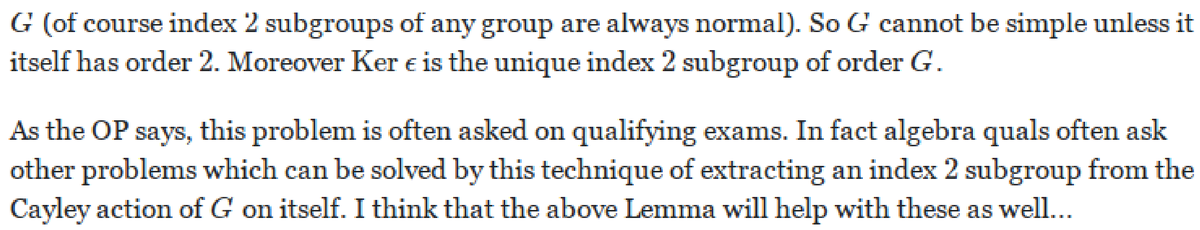
- Such a divisor is called **principal**. Similarly for differentials. Since curves are one dimensional and, for us, smooth, the sheaf of regular differentials $\Omega^1_{C/R}$ is a line bundle.

2.2 23:24

- Reading recommendations

3 | 2021-04-04

3.1 16:53



G (of course index 2 subgroups of any group are always normal). So G cannot be simple unless it itself has order 2. Moreover Ker ϵ is the unique index 2 subgroup of order G.

As the OP says, this problem is often asked on qualifying exams. In fact algebra quals often ask other problems which can be solved by this technique of extracting an index 2 subgroup from the Cayley action of G on itself. I think that the above Lemma will help with these as well...

Figure 1: image_2021-04-04-16-53-57

4 | 2021-04-03

4.1 16:28

The connections between FGLs and homotopy are key. Read Quillen's paper, J.F. Adams' blue book, Ravenel, etc. Ravenel has some slides on Quillen's work (good entry pt). If you really want to go in depth, I enjoyed learning from Hazewinkel's book "Formal Groups and Applications".

4.2 16:30

4.2.1 Setting Goals

- Don't set goals to be outcome-dependent. Set input goals! E.g. "Write a book that I'm proud of and I like."
- Don't set goals that depend on factors outside of your control – hitting some metric, getting a specific award, etc.
- Check in with your emotional state. It's not always healthy to push through not feeling like doing something. This can be useful if it's a matter of discipline, i.e. if pushing through is

actually serving you well. But it can also serve you poorly in the long run by exacerbating poor mental state or leading to burnout.

- It's okay to take a break. Check in to ask yourself if it's coming from a place of self-care or instead procrastination.
- Choose to be satisfied! The story you tell yourself after the fact does not change the reality of the past. Does it serve you well to say "I didn't do enough?" It's okay to choose to be satisfied.

4.2.2 Studying

- Have a mental model is the key priority, then go to rote memorization only when absolutely necessary. Remember because you understand the subject!
- Use active recall for the *learning* process – actively test yourself **as** you're reading.
- See book: "Make It Stick"
- The evidence is that popular techniques have low utility: rereading, highlighting, summarizing, taking notes.
- Summarizing and taking notes: low utility in general. This has some marginal utility if you're particularly skilled or trained in summarizing effectively. So not very effective, but do it if it brings you joy!
- Active recall and practice testing: high utility! Not very time-intensive, doesn't require special skills/training.
- Do practice testing **during each study session**. Studies show 10-15% improvement. Try to ask yourself inference questions.
- Studies show that those doing retrieval practice actually predict the smallest improvements and achieve the highest improvements. Very counterintuitive, trust the process!
- Reading once and practice testing can be more effective than rereading 4 times, *and* takes less time.
- Make first draft of notes with the book closed, no references.
- Spaced repetition: also works on a micro scale! E.g. within a single study session.
  - Recalling within a single session leads to almost a 30% increase in retention. Multiple successive recalls leads to no incremental gain, but multiple *spaced* recalls boosts by an additional 50%. Note that the last two are the same amount of work – letting yourself forget slightly forces retrieval.

- Practice a little bit each day over a long period of time.
- Extremely important: scope your subject. Know the broad outline of the course inside and out. Can scope based on actual exams.
- Start from topics you *don't* know as well. E.g. work backwards though lecture notes.
- Try interleaved practice: don't necessarily need to master a topic before moving on, realizing that you'll be reviewing it several times again before it's needed.
  - Getting through a large number of topics can be more useful than getting through a single topic in detail.
- Idea for tracking system: put course outline by topics into a spreadsheet, then record dates of review next to each topic. Color code cells based on quality of recall.
- Revision needs to be a fluid process, not worth timetabling down to each section due to different time requirements for each. Pick what makes your brain work the hardest.

4.3 19:04

4.3.1 Spectral Sequences

Link to PDF

are induced from the maps in the double complex C.

Example 6. Let us use the above theorem to prove the Snake Lemma, which says that the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "f"] & B \arrow[r] \arrow[u, "g"] & C \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

gives rise to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0.$$

To do this, we first use the second spectral sequence, with E_0-page as follows.

$$\begin{array}{|ccc}
0 & 0 & 0 \\
A & A' & 0 \\
B & B' & 0 \\
C & C' & 0 \\
0 & 0 & 0 \\
\hline
\end{array}$$

By exactness, this spectral sequence converges to zero. Let us now look at the first spectral sequence, with the following E_0-page.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & A' & B' & C' & 0 \\
0 & A & B & C & 0 \\
\hline
\end{array}$$

The E_1-page will look like

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \operatorname{coker} f & \operatorname{coker} g & \operatorname{coker} h & 0 \\
0 & \ker f & \ker g & \ker h & 0 \\
\hline
\end{array}$$

and the E_2-page is as follow.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \ker(\operatorname{coker} f \longrightarrow \operatorname{coker} g) & 0 & 0 & 0 \\
0 & 0 & 0 & (\ker h)/(\ker g) & 0 \\
\hline
\end{array}$$

- Snake Lemma

Since the spectral sequence converges to zero, we have an isomorphism

$$d : \ker(\operatorname{coker} f \longrightarrow \operatorname{coker} g) \longrightarrow (\ker h)/(\ker g).$$

Therefore $\delta := d^{-1}$ induces the connecting homomorphism in the Snake Lemma, while the other maps are the natural ones (which also corresponds to the differentials in the E_1-page above).

Example 8. Let $K(\mathbb{Z}, n)$ be the Eilenberg-Maclane space with $\pi_n($
and $\pi_i(K(\mathbb{Z}, n)) = 0$ otherwise. We now show by induction that th
ring

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \begin{cases} \mathbb{Q}[z], & \text{if } n \text{ is even,} \\ \mathbb{Q}[z]/z^2 & \text{if } n \text{ is odd,} \end{cases}$$

where $|z| = n$. (This is still true if we replace $\mathbb{Q}$ by a field of char
For $n = 1$ one has $K(\mathbb{Z}, 1) = S^1$, so this is clearly true. F
consider

$$\Omega K(\mathbb{Z}, n) \longrightarrow PK(\mathbb{Z}, n) \longrightarrow K(\mathbb{Z}, n),$$

where $\Omega K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n-1)$ is the loop space and $PK(\mathbb{Z}, n) \simeq$
space of $K(\mathbb{Z}, n)$. Note that $\pi_1(K(\mathbb{Z}, n)) = 1$, so by the Serre spec

$$E_2^{p,q} = H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) \Rightarrow H^{p+q}(*$$

This implies that the E_∞ page of the spectral sequence is $\mathbb{Z}$ at coo
and zero everywhere.
Let us first consider the case where $n \geq 2$ is even. Then $E_2^{0,n-1}$
by some fixed 2-torsion element $z \in H^{n-1}(K(\mathbb{Z}, n-1))$, and fo
converge to 0 along this diagonal we require $d_n(z) = x$ with d_n an
Thus $E_2^{n,n-1}$ is generated by xz, and by the Leibniz rule

$$d_n(xz) = d_n(x)z + (-1)^n x d_n(z) = x^2,$$

- Rational cohomology of $K(Z, n)$

since $d_n(x) = 0$. Inductively, we get the following diagram on the E_2-page.

$$\begin{array}{|cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & 0 & \dots & 0 & xz & 0 & \dots & 0 & x^2z & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & 0 & \dots & 0 & x & 0 & \dots & 0 & x^2 & 0 \\
\hline
\end{array}$$

This gives us what we want for $n \geq 2$ even. Next we consider the case where $n \geq 2$ is odd. Then $E_2^{0,k(n-1)}$ must be isomorphic to $E_2^{n,(k-1)(n-1)}$ for any positive integer n, and the rest of the E_2 page is zero except at $(0,0)$. Similar to before, $E_2^{0,n-1}$ is generated by $z \in H^{n-1}(K(\mathbb{Z}, n-1))$, and $d_n(z) = x$ with $d_n : E_2^{0,n-1} \longrightarrow E_2^{n,0}$ an isomorphism. Now

$$d_n(z^2) = d_n(z)z + (-1)^{n-1} z d_n(z) = 2xz,$$

and so $E_2^{n,n-1}$ is generated by xz (since $\mathbb{Q}$ has characteristic zero).

$$\begin{array}{|cccccc}
0 & 0 & \dots & 0 & 0 & 0 \; \dots \\
\frac{1}{2}z^2 & 0 & \dots & 0 & xz^2 & 0 \; \dots \\
0 & 0 & \dots & 0 & 0 & 0 \; \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 \; \dots \\
z & 0 & \dots & 0 & xz & 0 \; \dots \\
0 & 0 & \dots & 0 & 0 & 0 \; \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 \; \dots \\
1 & 0 & \dots & 0 & x & 0 \; \dots \\
\hline
\end{array}$$

Since the $2n^{th}$ column of the E_2 page is zero, we conclude that x must be 2-torsion, as desired.

Example 9. We show that

$$H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1}), \qquad |x| = 2.$$

To do this, we use the Hopf fibration

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n .$$

Explcitly, the map $S^{2n+1} \longrightarrow \mathbb{CP}^n$ is defined by the projection, where
both S^{2n+1} and $\mathbb{CP}^n$ as the usual subspace and subquotient of $\mathbb{C}^{n+1}$.

- Cohomology ring of $\mathbb{CP}^n$

Since $\mathbb{CP}^n$ is simply connected, the Serre spectral sequence in this case is

$$E_2^{p,q} = H^p(\mathbb{CP}^n, H^q(S^1)) \Rightarrow H^{p+q}(S^{2n+1}).$$

The E_2-page has nontrivial terms only on the first two rows.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & \dots \\
H^0(\mathbb{CP}^n) & H^1(\mathbb{CP}^n) & H^2(\mathbb{CP}^n) & H^3(\mathbb{CP}^n) & \dots \\
H^0(\mathbb{CP}^n) & H^1(\mathbb{CP}^n) & H^2(\mathbb{CP}^n) & H^3(\mathbb{CP}^n) & \dots \\
\hline
\end{array}$$

Since the spectral sequence converges after one page to $H^{p+q}(S^{2n+1})$, which equals $\mathbb{Z}$ when $p+q \in \{0, 2n+1\}$ and is 0 otherwise, it is easily seen that

$$H^i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \dots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

As the cup product structure is compatible with the product on E_2, the cohomology ring is as claimed.

sum of its filtration.

Example 12. Let C_n be the cyclic group of order n. It is well-known
Chapter 4.8 of [2]) that

$$H^*(C_n, Z) = \mathbb{Z}[z]/(nz)$$

where $|z| = 2$. We now demonstrate the lifting problem via the short
sequence

$$0 \longrightarrow C_2 \longrightarrow C_4 \longrightarrow C_2 \longrightarrow 0.,$$

- The Lifting Problem

which is a central extension. Using the Lyndon-Hoschschild-Serre spectral sequence the corner of the E_2-page is

$$\begin{array}{|ccccccc}
y^2 & 0 & y^2x & 0 & y^2x^2 & 0 & y^2x^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & yx & 0 & yx^2 & 0 & yx^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & x & 0 & x^2 & 0 & x^3 \\
\hline
\end{array}$$

It is clear that $E_2 = E_\infty$, and $Tot(E_\infty) = \mathbb{Z}[x,y]/(2x, 2y)$. However, this is not the cohomology of C_4.

Application. The homotopy group $\pi_i(S^n)$ is finite if n is odd and

Proof. For $n = 1$, covering space theory tells us that $\pi_i(S^1) = \pi_i($
$i > 1$. Now let $n > 1$ be odd. The sphere S^n is a finite CW
$H_i S^n \in \mathcal{C}_{FG}$ for all $i > 0$. Since S^n is 1-connected, by (1)(b), $\pi_i(S^n)$
all $i \geq 0$.

We now show that $\pi_i S^n$ is a torsion group for $i \neq n$. Let $f : S^n$
be the map inducing an isomorphism on π_n. Then $\pi_i(f)$ is an isomo
$i \leq n$ and an epimorphism for $i = n+1$. Applying (3) with the clas
see that

$$H_i(f) : H_i(S^n; \mathbb{Z}) \to H_i(K(\mathbb{Z}, n); \mathbb{Z})$$

is an isomorphism for $i \leq n$ and an epimorphism for $i = n+1$.The
of S^n and $K(\mathbb{Z}, n)$ with coefficients in $\mathbb{Q}$ vanish in degrees $> n$,[7] so

$$H_i(f) : H_i(S^n; \mathbb{Q}) \to H_i(K(\mathbb{Z}, n); \mathbb{Q})$$

is an isomorphism for all i. By example 18,

$$H_i(f) : H_i(S^n; \mathbb{Z}) \to H_i(K(\mathbb{Z}, n); \mathbb{Z})$$

is a $\mathcal{C}_{tors}$-isomorphism for all i. Applying (3) with the class $\mathcal{C}_{tors}$,
that

$$\pi_i(f) : \pi_i(S^n) \to \pi_i(K(\mathbb{Z}, n))$$

is a $\mathcal{C}_{tors}$-isomorphism for all i. Since $\pi_i(K(\mathbb{Z}, n)) = 0$ for $i \neq n$, $\pi_i($
for $i \neq n$.

We conclude that $\pi_i S^n$ is a finite group for $i \neq n$.

- Homotopy groups of spheres

5 2021-04-02

5.1 22:38

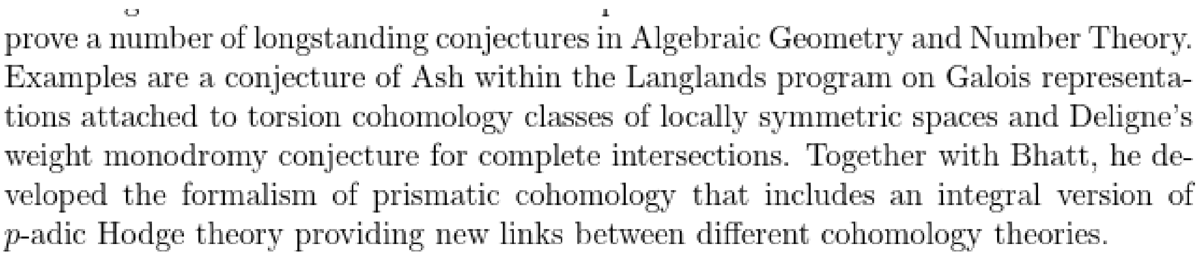

prove a number of longstanding conjectures in Algebraic Geometry and Number Theory. Examples are a conjecture of Ash within the Langlands program on Galois representations attached to torsion cohomology classes of locally symmetric spaces and Deligne's weight monodromy conjecture for complete intersections. Together with Bhatt, he developed the formalism of prismatic cohomology that includes an integral version of p-adic Hodge theory providing new links between different cohomology theories.

Figure 2: image_2021-04-02-22-38-21

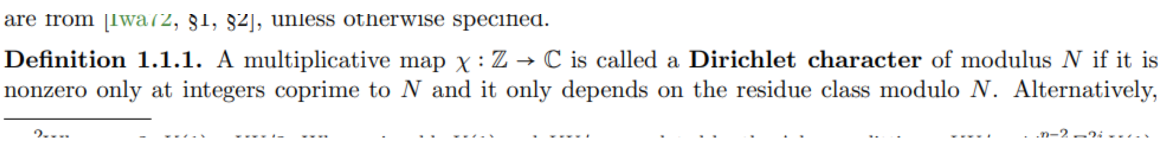
6 2021-03-28

6.1 23:18

What is the **Weight-Monodromy Conjecture**?

7 2021-03-26

7.1 20:00


are from [Iwa72, §1, §2], unless otherwise specified.

Definition 1.1.1. A multiplicative map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a **Dirichlet character** of modulus N if it is nonzero only at integers coprime to N and it only depends on the residue class modulo N. Alternatively,

Figure 3: image_2021-03-26-20-00-58

- A Dirichlet character is equivalent to a group homomorphism

$$\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times.$$

- Definition of a Dirichlet L-function:

$$L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} == \prod_p (1-\chi(p)p^{-s})^{-1}.$$

Definition 1.1.2. The ordinary Bernoulli numbers are defined to by

$$F(t) = \frac{te^t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Let χ be a Dirichlet character with conductor N. We define the generalized Bernoulli numbers associated to χ by setting

$$F_\chi(t) = \sum_{a=1}^{N} \frac{\chi(a)te^{at}}{e^{Nt}-1} = \sum_{n=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}. \tag{1.1.3}$$

Remark 1.1.4. Notice that the conductor of the trivial character χ^0 is 1. So we have $F_{\chi^0}(t) = F(t)$ and $B_{k,\chi^0} = B_k$.

Figure 4: image_2021-03-26-20-03-27

2.1. **The J-homomoprhism and the e-invariant.** The J-homomorphism is a group homomorphism $J_{k,n} : \pi_k(\mathrm{SO}(n)) \to \pi_{n+k}(S^n)$. This map passes to a stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$.

Definitions 2.1.1. The (unstable) J-homomoprhism is defined in the following ways:

(1) Loop spaces. An linear isometry of $\mathbb{R}^n$ restricts to a boundary preserving isometry of the unit ball D^n and thus induces a selfmap $S^n \to S^n$. From this, we get a continuous map $g_n : \mathrm{SO}(n) \to \Omega^n S^n$. We define

$$J_{k,n} := \pi_k(g_n) : \pi_k(\mathrm{SO}(n)) \longrightarrow \pi_k(\Omega^n S^n) \simeq \pi_{n+k}(S^n).$$

(2) Framed cobordism. Geometrically, the image of the J-homomorphism identifies the framed k-dimensional submanifolds of S^{n+k} whose underlying submanifolds are S^k. As the normal bundle of $S^k \hookrightarrow S^{n+k}$ is trivial, a framing of this embedding is equivalent a map $f : S^k \to O(n)$. One can further show two framings of the embedding $S^k \hookrightarrow S^{n+k}$ are equivalent iff the associated maps are homotopical. Thus we get a map $J_{k,n} : \pi_k(O(n)) \to \pi_{n+k}(S^n)$.

(3) Thom space. A map $f \in \pi_k(\mathrm{SO}(n)) \simeq \pi_{k+1}(B\mathrm{SO}(n))$ induces a n-dimensional oriented vector bundle ξ_f over S^{k+1}. The Thom space of ξ_f is a two-cell complex $\mathrm{Th}(\xi_f) = S^n \cup e^{n+k+1}$. Define $J_{k,n}(f)$ to be the gluing map of $\mathrm{Th}(\xi_f)$, i.e.

$$S^{n+k} = \partial e^{n+k+1} \xrightarrow{J_{k,n}(f)} S^n \longrightarrow \mathrm{Th}(\xi_f).$$

Proposition 2.1.2. *The definitions above are equivalent up to a sign.*

Figure 5: The J homomorphism

Definition 2.2.1. A cohomology theory E is called **complex oriented** if it is multiplicative and it satisfies the Thom isomorphism theorem for complex vector bundles. It is **even periodic** if E_* is concentrated in even degrees and there is a $\beta \in E^{-2}(\mathrm{pt})$ such that β is invertible in E_*.

Figure 6: image_2021-03-26-20-06-00

- Uniformizer π: generator of a maximal ideal.

8 | 2021-03-25

8.1 00:08

Definition 3.1. A **left Kan extension** of a functor $F : \mathcal{C} \to \mathcal{D}$ along a functor $G : \mathcal{C} \to \mathcal{E}$ is a pair of a functor $\mathrm{Lan}_G(F) : \mathcal{E} \to \mathcal{D}$ fitting into the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\ \mathcal{E} \arrow[ur, dotted, "\mathrm{Lan}_G(F)"'] & \end{tikzcd}$$

together with a natural transformation $\eta_L : F \to \mathrm{Lan}_G(F) \circ G$ that is *initial* among all such pairs.

A **right Kan extension** of F along G is a pair of a functor $\mathrm{Ran}_G(F) : \mathcal{E} \to \mathcal{D}$ fitting into the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\ \mathcal{E} \arrow[ur, dotted, "\mathrm{Ran}_G(F)"'] & \end{tikzcd}$$

together with a natural transformation $\eta_R : \mathrm{Ran}_G(F) \circ G \to F$ that is *final* among all such pairs.

When Kan extensions exists they are unique by their universal properties.

6

Figure 7: Kan extensions

8.2 00:09

Definition 3.2. Let $\mathcal{C}$ be a model category, $\mathcal{D}$ be any category, and $F : \mathcal{C} \to \mathcal{D}$ a functor. A **left derived functor** $\mathbb{L}'F$ of F is a *right* Kan extension of F along the $\mathcal{C}$-localization $\Gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$:

$$\mathbb{L}'F = \mathrm{Ran}_{\Gamma_{\mathcal{C}}}(F) : \mathrm{ho}\mathcal{C} \to \mathcal{D}.$$

A **right derived functor** $\mathbb{R}'F$ of F is a *left* Kan extension

$$\mathbb{R}'F = \mathrm{Lan}_{\Gamma_{\mathcal{C}}}(G) : \mathrm{ho}\mathcal{C} \to \mathcal{D}.$$

When both $\mathcal{C}$ and $\mathcal{D}$ are model categories, there is a notion of a *total* derived functor.

Definition 3.3. Let $\mathcal{C}, \mathcal{D}$ be model categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. A **total left derived functor** $\mathbb{L}F$ of F is a *right* Kan extension of the composition $\Gamma_{\mathcal{D}} \circ F$ along the $\mathcal{C}$-localization $\Gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$:

$$\mathbb{L}F = \mathrm{Ran}_{\Gamma_{\mathcal{C}}}(\Gamma_{\mathcal{D}} \circ F) : \mathrm{ho}\mathcal{C} \to \mathrm{ho}\mathcal{D}.$$

A **right derived functor** $\mathbb{R}F$ of F is a *left* Kan extension

$$\mathbb{R}F = \mathrm{Lan}_{\Gamma_{\mathcal{C}}}(\Gamma_{\mathcal{D}} \circ G) : \mathrm{ho}\mathcal{C} \to \mathrm{ho}\mathcal{D}.$$

Recall, we say a functor $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to a functor $U : \mathcal{D} \to \mathcal{C}$ if there exists a natural isomorphism of bifunctors $\mathcal{C} \times \mathcal{D} \to \mathsf{Set}$

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \to \mathrm{Hom}_{\mathcal{C}}(-, U(-)).$$

Thus, for each $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, U(Y)).$$

In symbols, we write $F \dashv G$.

Figure 8: Derived functors

Definition 3.4. An adjunction between model categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "U"] \end{tikzcd}$$

is a **Quillen adjunction** if F preserves cofibrations and acyclic cofibrations.

Figure 9: Quillen adjunctions

Definition 3.8. A **Quillen equivalence** is a Quillen pair $F \dashv U$ such that $\mathbb{L}F \dashv \mathbb{R}U$ is an equivalence of respective homotopy categories.

Theorem 3.9 (Quillen). *Suppose $F \dashv U$ is a Quillen pair. If, for every cofibrant object $X \in \mathcal{C}^{cof}$ and fibrant object $Y \in \mathcal{C}^{fib}$ one has*

$$(f : X \to UY) \in \mathcal{W}_{\mathcal{C}} \iff (\eta f : FX \to Y) \in \mathcal{W}_{\mathcal{D}}$$

then $\mathbb{L}F$ (and $\mathbb{R}G$) defines an equivalence of categories

$$\begin{tikzcd} \mathrm{ho}\mathcal{C} \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, phantom, "\cong"] & \mathrm{ho}\mathcal{D}. \arrow[l, bend left, "\mathbb{R}U"] \end{tikzcd}$$

Figure 10: Quillen Equivalence

Definition 2.1. An (∞, n)-**category** is a higher category in which all k-morphisms are invertible for $k > n$. We will simply refer to a $(\infty, 1)$-category as an ∞-**category**.

Figure 11: Infty n category

Using the motivating perspective of 2-categories as categories enriched in 1-categories, we arrive at the following model for an $(\infty, 1)$-category: it is a category enriched in $(\infty, 0)$-categories. That is, a *topological category*.

Definition 2.3. A **topological category** is a category enriched over the category of topological spaces Top. That is, a category $\mathcal{C}$ in which the set of morphisms between any two objects $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathsf{Top}$ is a topological space.

We will refer to the category of topological categories by $\mathsf{Cat}_{\mathsf{Top}}$.

Figure 12: Topological categories

which takes a topological space X to its path components $\pi_0 X$.

Definition 4.1. Let $\mathcal{C}$ be a topological category. Define its **homotopy category** ho $\mathcal{C}$ to be the (ordinary) category whose objects are the same as that of $\mathcal{C}$ and whose morphisms are

$$\mathrm{Hom}_{\mathrm{ho}\,\mathcal{C}}(X, Y) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Remark 4.2. The set $\pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is precisely the set of homotopy classes of maps $X \to Y$ and is sometimes written as $[X, Y]$.

Figure 13: Homotopy category

2. Transferring from $\mathsf{Vect}^{\mathsf{dg}}$

First of all, there is a natural model structure on the category of dg vector spaces given by the following:

- a map $f : V \to W$ in $\mathsf{Vect}_k^{\mathsf{dg}}$ is a fibration if an only if $f : V^n \to W^n$ is surjective for each n;
- a map $f : V \to W$ in $\mathsf{Vect}_k^{\mathsf{dg}}$ is a cofibration if and only if $f : V^n \to W^n$ is injective for each n;
- a map $f : V \to W$ in $\mathsf{Vect}_k^{\mathsf{dg}}$ is a weak equivalence if and only if f is a quasi-isomorphism.

Figure 14: Model structure on dg vector spaces

necessarily preserve these canonical maps.

An algebra A together with a map $\epsilon : A \to k$ is called *augmented*. Geometrically, augmented algebras correspond to pointed affine schemes

$$* = \mathrm{Spec}(k) \to \mathrm{Spec}(A).$$

Figure 15: Augmented algebras are pointed schemes

1.2. **Initial / final objects.** For an ordinary category, an object $0 \in \mathcal{C}$ is *initial* if $\mathrm{Hom}(0, X) = \{\star\}$ is the singleton set for any $X \in \mathcal{C}$. For the appropriate notion in ∞-categories it is then fairly clear what we must say: an object $0 \in \mathcal{C}$ is initial if the *space* of maps from 0 to any other object is *contractible*.

Definition 1.3. Let $\mathcal{C}$ be an ∞-category. An object $0 \in \mathcal{C}$ is **initial** if

$$\mathrm{Map}_{\mathcal{C}}(0, X)$$

is contractible for all $X \in \mathcal{C}$. An object $\star \in \mathcal{C}$ is **final** if

$$\mathrm{Map}_{\mathcal{C}}(X, \star)$$

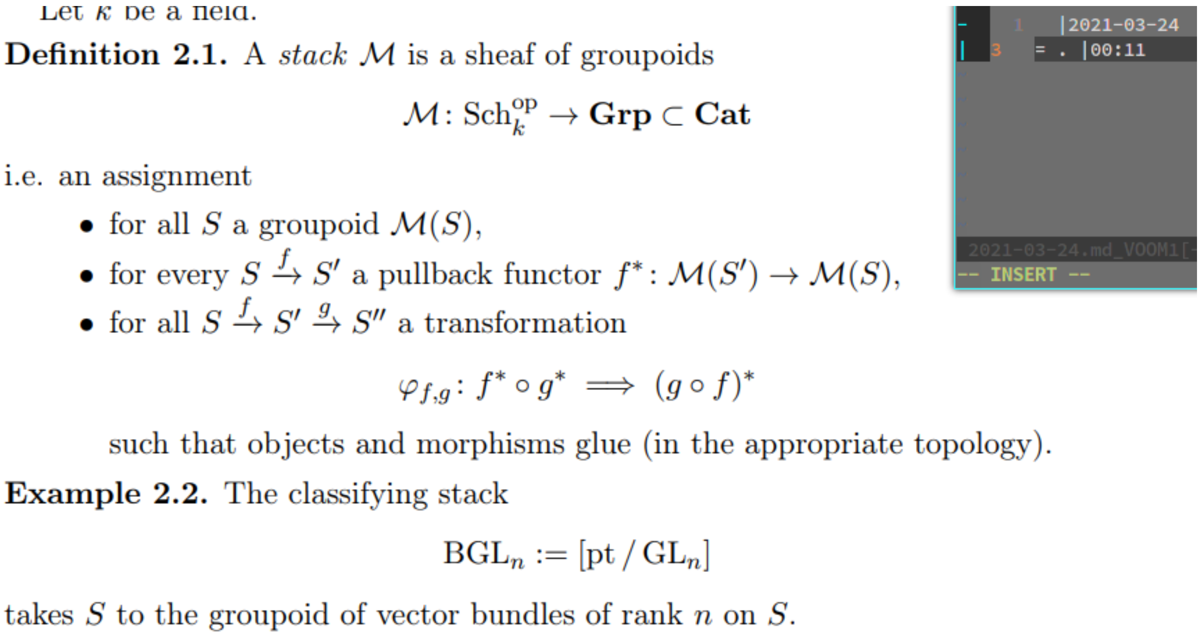
is contractible for all $X \in \mathcal{C}$.

2

Figure 16: Initial and final objects in infty-categories

9 | 2021-03-24

9.1 00:11



Let k be a field.

Definition 2.1. A *stack* $\mathcal{M}$ is a sheaf of groupoids

$$\mathcal{M}\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{Grp} \subset \mathbf{Cat}$$

i.e. an assignment

- for all S a groupoid $\mathcal{M}(S)$,
- for every $S \xrightarrow{f} S'$ a pullback functor $f^*\colon \mathcal{M}(S') \to \mathcal{M}(S)$,
- for all $S \xrightarrow{f} S' \xrightarrow{g} S''$ a transformation

$$\varphi_{f,g}\colon f^* \circ g^* \implies (g \circ f)^*$$

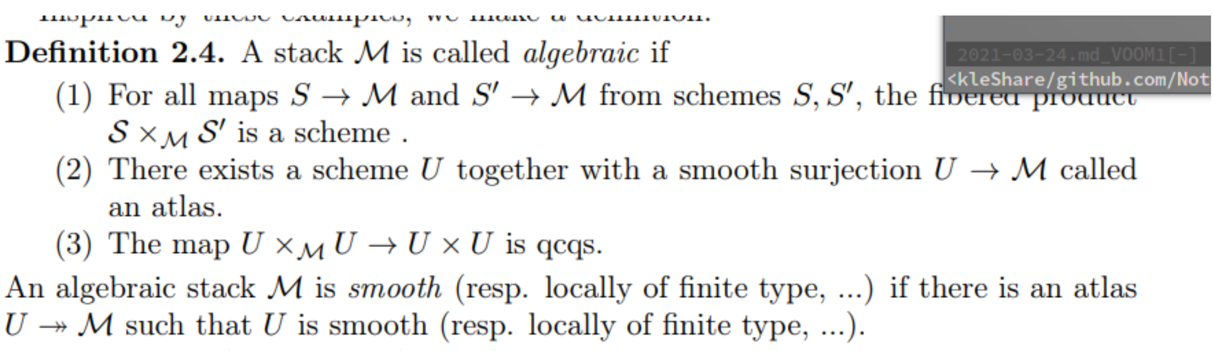
such that objects and morphisms glue (in the appropriate topology).

Example 2.2. The classifying stack

$$\mathrm{BGL}_n := [\mathrm{pt} \,/\, \mathrm{GL}_n]$$

takes S to the groupoid of vector bundles of rank n on S.

Figure 17: Definition of Stack



Inspired by these examples, we make a definition.

Definition 2.4. A stack $\mathcal{M}$ is called *algebraic* if

(1) For all maps $S \to \mathcal{M}$ and $S' \to \mathcal{M}$ from schemes S, S', the fibered product $S \times_{\mathcal{M}} S'$ is a scheme .
(2) There exists a scheme U together with a smooth surjection $U \to \mathcal{M}$ called an atlas.
(3) The map $U \times_{\mathcal{M}} U \to U \times U$ is qcqs.

An algebraic stack $\mathcal{M}$ is *smooth* (resp. locally of finite type, ...) if there is an atlas $U \twoheadrightarrow \mathcal{M}$ such that U is smooth (resp. locally of finite type, ...).

Figure 18: Algebraic and smooth stacks

- What are Quot schemes?

10 | 2021-03-17

10.1 17:00

- What is a local field?
- What is a global field?
  - Why are these generally more difficult than local fields?
- What is a field that is not local or global?
- What is a fibration of varieties?
- What is a del Pezzo surface?
- What is the Jacobian of a curve?
- What is the genus of a curve?
- What is a torsor?
- What is a complete intersection?
- What are some examples of p-adic fields?
- What is a Severi-Brauer variety?
- What is Hensel's Lemma?
- What is quadratic reciprocity?
  - Conics over global fields fail to have rational points at an even number of places?
- What is a ramified/unramified extension?
- Interpretation of Weil conjectures: has lots of points over big enough extensions?
- What is the Hasse Principle?
- What are adelic points?
  - Product of K_v points!
- What is the Brauer group?
- What is a central simple algebra?
- What are the main theorems of class field theory?
- Why is the following SES important:

$$0 \to \operatorname{Br} k \to \oplus_v \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z} \to 0.$$

- What is a model of a variety?
- What is a special fiber?

11 | 2021-01-21

11.1 13:37

- Local coefficient systems

11.2 14:27

- Separated
- Locally quasi-finite
- Quasi-affine morphisms

11.3 16:13

- Smooth
- Geometrically connected

12 | 2021-01-20

12.1 16:52

- Geometric point
- Hyperelliptic curve
- Jacobian
- Mordell-Weil Theorem

12.2 20:00

- Exact and closed forms.

13 | 2021-01-19

13.1 16:29

- Canonical bundle!
- $\mathcal{O}(1)$
- $\omega^{\otimes k}$.

14 | 2021-01-05

14.1 00:01

- Chow groups
- Picard group

15 | 2021-01-03

15.1 18:05

- Vojta's conjecture: Gives good height estimates, can be used to prove something is finite.

- Northcott's theorem: Used in arithmetic dynamics since bounded height and bounded degree implies finite.