

# Title

*D. Zack Garza*

# Contents

1	Tuesday, December 01	3
---	----------------------	---

# 1 | Tuesday, December 01

Last time: we started discussing smoothness.

## Definition 1.0.1 (Tangent Space)

The **tangent space**  $T_p X$  of a variety  $X$  at a point  $p \in X$  is defined as

$$V\left(\left\{f_1 \mid f \in I(U_i), U_i \ni p = 0 \text{ affine}\right\}\right)$$

where  $f_1$  denotes the degree 1 part.

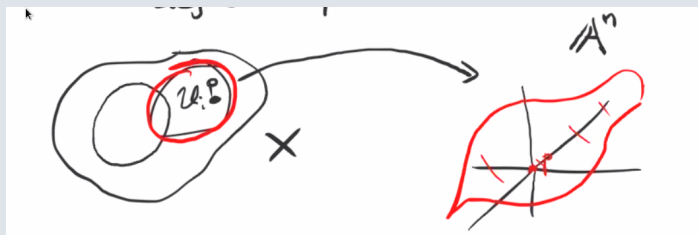


Figure 1: Image

**Remark 1.0.2:** We've really only defined it for affine varieties and  $p = 0$ , but this is a local definition. Note that this is also not a canonical definition, since it depends on the affine chart  $U_i$ .

**Example 1.0.3(?):** Consider  $T_0 V(xy) = V(f_1 \mid f \in \langle xy \rangle) = V(0) = \mathbb{A}^2$ , since every polynomial in this ideal has degree at least 2. Letting  $X = V(xy)$ , note that we could embed  $X \hookrightarrow \mathbb{A}^3$  as  $X \cong V(xy, z)$ . In this case we have  $T_0 X = V(f_1 \mid f \in \langle xy, z \rangle) = V(z) \cong \mathbb{A}^2$ . So we get a vector space of a different dimension from this different affine embedding, but  $\dim T_0 X$  is the same.

**Example 1.0.4(?):** Let  $X = V_p(xy - z^2) \subset \mathbb{P}^2$ , which is a projective curve. What is  $T_p X$  for  $p = [0 : 1 : 0]$ ? Take an affine chart  $\{y \neq 0\} \cap X$ , noting that  $\{y \neq 0\} \cong \mathbb{A}^2$ . We could dehomogenize the ideal  $\langle xy - z^2 \rangle|_{y=1} = \langle x - z^2 \rangle$ . Thus  $X \cap D(y) = V(x - z^2) \subset \mathbb{A}^2$  and the point  $[0 : 1 : 0] \in X$  gives  $(0, 0)$  in this affine chart. Then  $T_p X = V(f_1 \mid f \in \langle x - z^2 \rangle) = V(x)$ . Then  $f = (x - z^2)g$  implies that  $f_1 = (xg)_1 = g_0 x$ , the constant term of  $g$  multiplied by  $x$ , since  $z^2$  kills any degree 1 part of  $g$ . So  $T_p X$  is a line.

**Example 1.0.5(?):** Take  $X$  to be the union of the coordinate axes in  $\mathbb{A}^3$ .

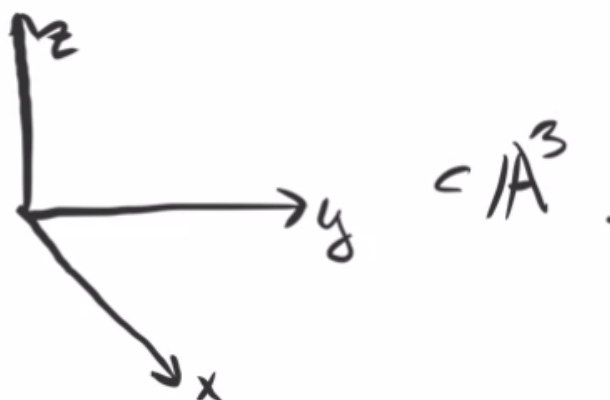


Figure 2: Image

Then  $I(X) = \langle xy, yz, xz \rangle$  and  $T_0X = V(f_1 \mid f \in I(X)) = V(0) = \mathbb{A}^3$ , since the minimal degree of any such polynomial is 2. Note that  $\dim X = 1$  but  $\dim T_0X = 3$

**Example 1.0.6(?)**: Take  $Y = V(xy(x-y)) \subset \mathbb{A}^2$ . Then  $T_0Y = V(0) = \mathbb{A}^2$ :

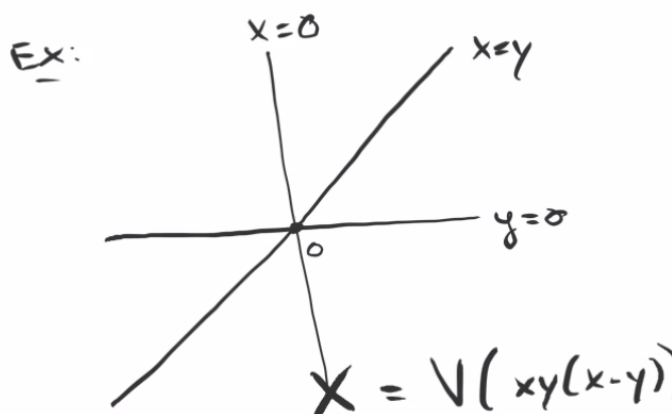


Figure 3: Image

**Remark 1.0.7**: Note that  $X$  and  $Y$  both consist of 3 copies of  $\mathbb{A}^1$  intersecting at a single point.

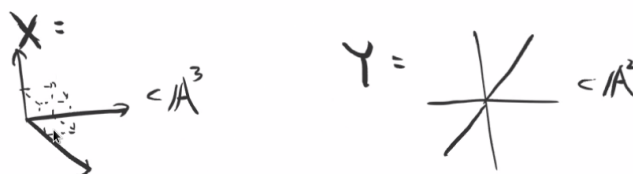


Figure 4: Image

Note that  $\dim T_0 X = 3$  but  $\dim T_0 Y = 3$ , and interestingly  $X \not\cong Y$  as affine varieties. There is a bijective morphism that is not invertible.

**Remark 1.0.8:** We will prove that  $\dim T_p X$  is invariant under choice of affine embedding.

**Example 1.0.9(?):** How to compute  $T_{(1,0,0)} V(xy, yz, xz)$ : first move  $(1, 0, 0)$  to the origin, yielding  $T_{(0,0,0)} V((x+1)y, yz, (x+1)z)$ . This is a different choice of affine embedding into  $\mathbb{A}^3$  which sends  $(1, 0, 0) \mapsto (0, 0, 0)$ . Taking the vanishing locus of linear parts, it suffices to take the linear parts of the generators, which yields the  $x$ -axis  $V(y, z)$ , making the dimension of the tangent space 1.

**Lemma 1.0.10(?).**

Let  $X \subset \mathbb{A}^n$  be an affine variety and let  $0 = p \in X$ . Then

$$T_0(X)^\vee := \text{hom}_k(T_0 X, k) \cong I_X(p)/I_X(p)^2$$

**Remark 1.0.11:** Note that the hom involves an affine embedding, but the quotient of ideals does not. We know that the category of affine varieties is equivalent to the category of reduced  $k$ -algebras, since the points of  $X$  biject with the maximal ideals of the coordinate ring  $A(X)$ .  $I_X(p)$  is the maximal ideal in  $A(X)$  of regular functions vanishing at  $p$ .

*Proof (?).*

Consider the map

$$\begin{aligned} \varphi : I_X(p) &\rightarrow T_0(X)^\vee \\ \bar{f} &\mapsto f_1|_{T_0(X)}. \end{aligned}$$

E.g. given  $\bar{x}_1 - \bar{x}_2^2 \in A(X)$ , we first lift to  $x_1 - x_2^2 \in A(\mathbb{A}^n)$ , restrict to the linear part  $x_1$ , then restrict to  $T_0(X)$ . Note that  $I_X(p) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \in k[x_1, \dots, x_n]/I(X)$ , and we need to check that this is well-defined since there is ambiguity in choosing the above lift.

**Claim:**  $\varphi$  is well-defined.

Consider two lifts  $f, f'$  of  $\bar{f} \in A(X) = k[x_1, \dots, x_n]/I(X)$ . Then  $f - f' \in I(X)$ , so  $(f - f')_1 = f_1 - f'_1$  is the linear part of some element in  $I(X)$ . The definition of  $T_0(X)$  was the vanishing locus of linear parts of elements in  $I(X)$ , which contains  $f_1 - f'_1$ , and thus  $(f_1 - f'_1)|_{T_0(X)} = 0$ . So  $f_1 = f'_1$  on  $T_0(X)$ .

**Claim:**  $I_X(p)^2 \rightarrow 0$ .

We know  $I_X(p) = \langle \bar{x}_1, \dots, \bar{x}_n \rangle$ , and so  $I_X(p)^2 = \langle \bar{x}_i \bar{x}_j \rangle$ . Given any  $\bar{f} \in I_X(p)^2$ , we can lift this to some  $f \in \langle x_i x_j \rangle$ , in which case  $f_1 = 0$ .

So  $\varphi$  descends to

$$\bar{\varphi} : I_X(p)/I_X(p)^2 \rightarrow T_0(X)^\vee$$

**Claim:**  $\varphi$  is injective and surjective.

That  $\bar{\varphi}$  is surjective follows from the fact that if  $\bar{x}_1, \dots, \bar{x}_n \in I_X(p)$ , then the restrictions  $x_1|_{T_0(X)}, \dots, x_n|_{T_0(X)}$  are in  $\text{im } \bar{\varphi}$ . These elements generate  $T_0(X)^\vee$ , since  $T_0(X) \subset \mathbb{A}^n$ . For injectivity, suppose  $\bar{\varphi}(\bar{f}) = 0$ , then  $f_1|_{T_0(X)} = 0$ , so  $f_1$  is the linear part of some  $f' \in I(X)$ . Then  $f' \in I(X)$  and  $f, f'$  have the same linear part  $f_1$ , and  $f - f'$  has no linear part. Thus  $f - f' \in \langle x_i x_j \rangle$ , which implies that  $\bar{f} - \bar{f}' \in I_p(X)^2$  and  $\bar{f} \equiv \bar{f}' \in I_p(X)/I_p(X)^2$ . But  $f' \equiv 0$  since  $f' \in I(X)$ . ■

**Remark 1.0.12:** So for  $X$  an affine variety, the cotangent space has a more intrinsic description, and we can recover the tangent space by dualizing:

$$T_p(X) := (\mathfrak{p}/\mathfrak{m}_p^2)^\vee$$

where  $\mathfrak{m}_p = I_X(p)$  is the maximal ideal of regular functions vanishing at  $p$ . So how can we get rid of the word affine? Given  $X$  any variety, we can define  $T_p(X) := \mathfrak{m}/\mathfrak{m}^2$  where  $\mathfrak{m}$  is the maximal ideal of the local ring  $\mathcal{O}_{X,p}$ . 