

# Title

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## 1 | Thursday, November 05

Today: projective spaces. We defined  $\mathbb{P}_k^n := k^{n+1} \setminus \{0\} / \sim$  where  $x \sim \lambda x$  for all  $x \in k^\times$ , which we identified with lines through the origin in  $k^{n+1}$ . We have homogeneous coordinates  $p = [x_0 : \cdots : x_n]$ .

We say an ideal is *homogeneous* iff for all  $f \in I$ , the homogeneous part  $f_d \in I$  for all  $d$ . In this case  $V_p(I) \subset \mathbb{P}_k^n$  defined as the vanishing locus of all homogeneous elements of  $I$  is well-defined. Think of this as the “projective version” of a vanishing locus.

Similarly we defined  $I_p(S)$  defined as the ideal generated by all homogeneous  $f \in k[x_1, \dots, x_n]$  such that  $f(x) = 0$  for all  $x \in S$ .

### Remark 1.0.1.

Observe that  $V_a(I)$  defined as the cone over  $V_p(I)$  is the set of points in  $\mathbb{A}^{n+1} \setminus \{0\} \cup \{0\}$  which map to  $V_p(I)$ .

We have an alternative definition of a cone in  $\mathbb{A}^{n+1}$ , characterized as a closed subset  $C$  which is closed under scaling, so  $kC \subseteq C$ .

### Proposition 1.0.1.

- If  $S \subset k[x_1, \dots, x_n]$  is a set of homogeneous polynomials, then  $V_a(S)$  is a cone since it is closed and closed under scaling. This follows from the fact that  $f(x) = 0 \iff f(\lambda x) = 0$  for  $\lambda \in k^\times$  when  $f$  is homogeneous.
- If  $C$  is a cone, then its affine ideal  $I_a(C)$  is homogeneous.

*Proof (?)*.

Let  $f \in I_a(C)$ , then  $f(x) = 0$  for all  $x \in C$ . Since  $C$  is closed under scaling,  $f(\lambda x) = 0$  for all  $x \in C$  and  $\lambda \in k^\times$ . Decompose  $f = \sum_d f_d$  into homogeneous pieces, then

$$x \in C \implies 0 = f(\lambda x) = \sum \lambda^d f_d(x).$$

Fixing  $x \in C$ , the quantities  $f_d(x)$  are constants, so the resulting polynomial in  $\lambda$  vanishes for all  $\lambda$ . But since  $k$  is infinite, this forces  $f_d(x) = 0$  for all  $d$ , which shows that  $f_d \in I_a(C)$ . ■

### Lemma 1.1 (?).

There is a bijective correspondence

$$\begin{aligned} \{\text{Cones}\} &\Longleftrightarrow \{\text{Projective Varieties}\} \\ \mathbb{A}^{n+1} \supset X &\mapsto \mathbb{P}X \subset \mathbb{P}^n \\ \mathbb{A}^{n+1} \supset CX &\mapsto X \subset \mathbb{P}^n \end{aligned}$$

*Proof (?)*.

$\mathbb{P}V_a(S) = V_p(S)$  for any set  $S$  of homogeneous polynomials, and  $C(V_p(S)) = V_a(S)$ , where  $V_p(S)$  is a cone by part (a) of the previous proposition. Conversely, every cone is the variety associated to some homogeneous ideal. ■

## 1.1 Projective Nullstellensatz

**Definition 1.1.1** (Irrelevant Ideal).

The homogeneous ideal  $I_0 := (x_0, \dots, x_n) \subset k[x_1, \dots, x_n]$  is denoted the **irrelevant ideal**.

**Proposition 1.1.1** (*Projective Nullstellensatz*).

- a. For all  $X \subseteq \mathbb{P}^n$ ,  $V_p(I_p(X)) = X$ .
- b. For all homogeneous ideal  $J \subset k[x_1, \dots, x_n]$  such that (importantly)  $\sqrt{J} \neq I_0$ ,  $I_p(V_p(J)) = \sqrt{J}$ .

*Proof (of a).*

⊃: If we let  $I$  denote the ideal of all homogeneous polynomials vanishing on  $X$ , then this certainly contains  $X$ .

⊂: This follows from part (b), since  $X = V_p(J)$  implies that  $(V_p I_p V_p)(J) = V_p(\sqrt{J}) = V_p(J) = X$ , since taking roots of homogeneous polynomials doesn't change the vanishing locus. ■

*Proof (of b).*

That  $I_p(V_p(J)) \supset \sqrt{J}$  is obvious, since  $f \in \sqrt{J}$  vanishes on  $V_p(J)$ .

Check

It remains to show  $\sqrt{J} \subset I_p(V_p(J))$ , but we can write  $I_p(V_p(J))$  as  $\langle f \in k[x_1, \dots, x_n] \mid \text{the set of homogeneous polynomials vanishing on } V_p(S), \text{ which is equal to those vanishing on } V_a(J) \setminus \{0\} \rangle$ .

But since  $I_p(\dots)$  is closed, this is equal to the  $f$  that vanish on  $\overline{V_a(J) \setminus \{0\}}$ , which is only equal to  $V_a(J)$  iff  $V_a(J) \neq \{0\}$ .

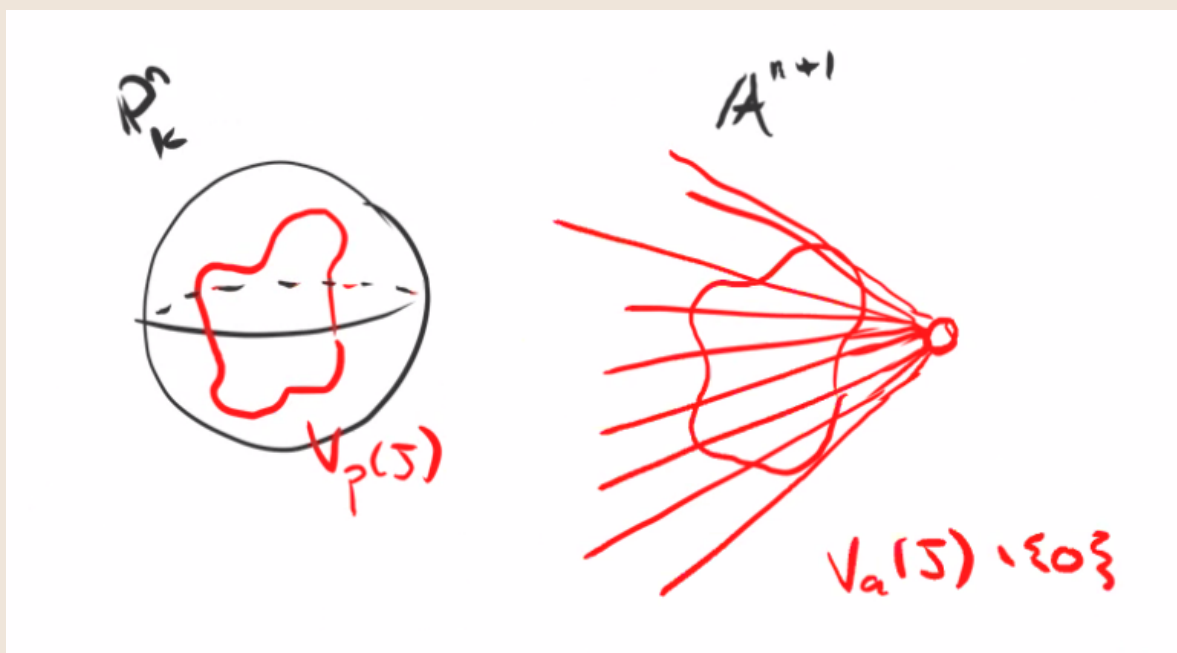


Figure 1: Image

By the affine Nullstellensatz,

$$V_a(J) = \{0\} \iff \sqrt{J} = I_0.$$

Thus  $I_p(V_p(J)) = \langle f \mid \text{homogeneous vanishing on } V_a(J) \rangle$ . Using the fact that  $V_a(J)$  is a cone, its ideal is homogeneous and thus generated by homogeneous polynomials by part (b) of the previous proposition. Thus

$$I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J},$$

where the last equality follows from the affine Nullstellensatz. ■

### Corollary 1.1.1(?).

There is an order-reversing bijection

$$\left\{ \begin{array}{c} \text{Projective varieties} \\ X \subset \mathbb{P}^n \end{array} \right\} \left\{ \begin{array}{c} \text{Homog non-irrelevant radical ideals} \\ \in k[x_1, \dots, x_n] \end{array} \right\}$$

$$X \mapsto I_p(X)$$

$$? \leftarrow ?.$$

**Remark 1.1.1.**

A better definition of a cone over  $X \subset \mathbb{P}_k^n$  is  $\overline{\pi^{-1}(X)} \subset \mathbb{A}_k^{n+1}$  where

$$\begin{aligned}\pi : \mathbb{A}^{n+1} \setminus \{0\} &\rightarrow \mathbb{P}^n \\ [x_0, \dots, x_n] &\mapsto [x_0 : \dots : x_n].\end{aligned}$$

**Definition 1.1.2** (Projective coordinate ring).

Given  $X \subset \mathbb{P}^n$  a projective variety, the **projective coordinate ring** of  $X$  is given by

$$S(X) := k[x_1, \dots, x_n]/I_p(X).$$

**Remark 1.1.2.**

This is a graded ring since  $I_p(X)$  is homogeneous. This follows since the quotient of a graded ring by a homogeneous ideal yields a grading on the quotient.

**Remark 1.1.3.**

We have relative versions of everything. Projective subvarieties of projective varieties are given by  $Y \subset X \subset \mathbb{P}^n$  where  $X$  is a projective variety. We have a topology on  $X$  where the closed subsets are projective subvarieties.

**Remark 1.1.4.**

Given  $J \subset S(X)$ , where  $S(X)$  is the projective coordinate ring of  $X$  and has a grading, we can take  $V_p(J) \subset X$ . Conversely, given a set  $Y \subset S(X)$ , we can take  $I_p(Y) \subset S(X)$  those homogeneous elements vanishing on  $Y$ . Thus there is an order-reversing bijection

$$\left\{ \begin{array}{c} \text{Projective subvarieties} \\ Y \subset X \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Homogeneous nonirrelevant radical ideals} \\ I \subset S(X) \end{array} \right\}.$$