

# Category $\mathcal{O}$ , Problem Set 3

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February 23, 2020

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## 1 Humphreys 1.10

Prove that the transpose map  $\tau$  fixes  $Z(\mathfrak{g})$  pointwise.

Check that  $\tau$  commutes with the Harish-Chandra morphism  $\xi$  and use the fact that  $\xi$  is injective.

### 1.1 Solution

We first note that after choosing a PBW basis for  $\mathfrak{g}$ ,  $\tau$  is defined on  $\mathfrak{g}$  in the following way:

$$\begin{aligned}\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha\end{aligned}$$

which lifts to an anti-involution  $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$  by extending linearly over PBW monomials. We can note that since  $\tau$  fixes  $\mathfrak{h}$  pointwise by definition, its lift also fixes  $U(\mathfrak{h})$  pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned}\xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}.\end{aligned}$$

**Proposition 1.1.**

The following diagram commutes

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \\ \downarrow \tau & & \downarrow \tau \\ Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \end{array}$$

*Proof.*

We will show that for all  $z \in Z(\mathfrak{g})$ ,  $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$ . Expand  $z$  in a PBW basis as  $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}$ . We then make the following computations:

$$\begin{aligned} (\xi \circ \tau)(z) &= (\xi \circ \tau) \left( \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j} \right) \\ &= \xi \left( \prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_j} \right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\tau \circ \xi)(z) &= \tau \left( \prod_j h_j^{s_j} \right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of  $\xi$ , we have

$$\begin{aligned} (\xi \circ \tau)z &= \xi(z) \\ \implies \tau(z) &= z. \end{aligned}$$

■

## 2 Humphreys 1.12

Fix a central character  $\chi$  and let  $\{V^{(\lambda)}\}$  be a collection of modules in  $\mathcal{O}$  indexed by the weights  $\lambda$  for which  $\chi = \chi_\lambda$  satisfying

1.  $\dim V^{(\lambda)} = 1$
2.  $\mu < \lambda$  for all weights  $\mu$  of  $V^{(\lambda)}$ .

Then the symbols  $[V^{(\lambda)}]$  form a  $\mathbb{Z}$ -basis for the Grothendieck group  $K(\mathcal{O}_x)$ .

For example take  $V^{(\lambda)} = M(\lambda)$  or  $L(\lambda)$ .

## 3 Humphreys 1.13

Suppose  $\lambda \neq \mu$ , so the linkage class  $W \cdot \lambda$  is the disjoint union of its nonempty intersections of various cosets of  $\Lambda_r \in \mathfrak{h}^\vee$ .

Prove that each  $M \in \mathcal{O}_{\chi_\lambda}$  has a corresponding direct sum decomposition  $M = \bigoplus M_i$  in which all weights of  $M_i$  lie in a single coset.

Recall exercise 1.1b.