2. The fundamental group

1 (Spring '15). Let S^1 denote the unit circle in $\mathbb{C}$, X be any topological space, $x_0 \in X$, and $\gamma_0, \gamma_1 : S^1 \to X$ two continuous maps such that $\gamma_0(1) = \gamma_1(1) = x_0$. Prove that γ_0 is homotopic to γ_1 if and only if the elements represented by γ_0 and γ_1 in $\pi_1(X, x_0)$ are conjugate.

2 (Spring '09/Spring '07/Fall '07/Fall '06).

(a) State van Kampen's theorem.
(b) Calculate the fundamental group of the space obtained by taking two copies of the torus $T = S^1 \times S^1$ and gluing them along a circle $S^1 \times \{p\}$ where p is a point in S^1.
(c) Calculate the fundamental group of the Klein bottle.
(d) Calculate the fundamental group of the one-point union of $S^1 \times S^1$ and S^1.
(e) Calculate the fundamental group of the one-point union of $S^1 \times S^1$ and $\mathbb{R}P^2$.

3 (Fall '18). Prove the following portion of van Kampen's theorem. If $X = A \cup B$ and A, B, and $A \cap B$ are nonempty and path connected with $* \in A \cap B$, then there is a surjection $\pi_1(A, *) * \pi_1(B, *) \to \pi_1(X, *)$.

4 (Spring '15). Let X denote the quotient space formed from the sphere S^2 by identifying two distinct points. Compute the fundamental group and the homology groups of X.

5 (Spring '06). Start with the unit disk D^2 and identify points on the boundary if their angles, thought of in polar coordinates, differ a multiple of $\pi/2$. Let X be the resulting space. Use van Kampen's theorem to compute $\pi_1(X, *)$.

6 (Spring '08). Let L be the union of the z-axis and the unit circle in the xy-plane. Compute $\pi_1(\mathbb{R}^3 \setminus L, *)$.

7 (Fall '16). Let A be the union of the unit sphere in $\mathbb{R}^3$ and the interval $\{(t, 0, 0) : -1 \leq t \leq 1\} \subset \mathbb{R}^3$. Compute $\pi_1(A)$ and give an explicit description of the universal cover of X.

8 (Spring '13). (a) Let S_1 and S_2 be disjoint surfaces. Give the definition of their connected sum $S_1 \# S_2$.

(b) Compute the fundamental group of the connected sum of the projective plane and the two-torus.

9 (Fall '15). Compute the fundamental group, using any technique you like, of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

10 (Fall '11) Let $V = D^2 \times S^1 = \{(z, e^{it}) | |z| \leq 1, 0 \leq t < 2\pi\}$ be the "solid torus" with boundary given by the torus $T = S^1 \times S^1$. For $n \in \mathbb{Z}$ define $\phi_n : T \to T$ by $\phi_n(e^{is}, e^{it}) = (e^{is}, e^{i(ns+t)})$. Find the fundamental group of the identification space

$$V_n = \frac{V \coprod V}{\sim_n}$$

where the equivalence relation $\sim_n$ identifies a point x on the boundary T of the first copy of V with the point $\phi_n(x)$ on the boundary of the second copy of V.

11 (Fall '16). Let S_k be the space obtained by removing k disjoint open disks from the sphere S^2. Form X_k by gluing k Möbius bands onto S_k, one for each circle boundary component of S_k (by identifying the boundary circle of a Möbius band homeomorphically with a given boundary component circle). Use van Kampen's theorem to calculate $\pi_1(X_k)$ for each $k > 0$ and identify X_k in terms of the classification of surfaces.

12 (Spring '13).

(i) Let A be a subspace of a topological space X. Define what it means for A to be a *deformation retract* of X.
(ii) Consider X_1 the "planar figure eight" and $X_2 = S^1 \cup (\{0\} \times [-1, 1])$ (the "theta space"). Show that X_1 and X_2 have isomorphic fundamental groups.
(iii) Prove that the fundamental group of X_2 is a free group on two generators.

3. Covering spaces

1 (Spring 11/Spring '14). (a) Give the definition of a covering space $\hat{X}$ (and covering map $p : \hat{X} \to X$) for a topological space X.

(b) State the homotopy lifting property of covering spaces. Use it to show that a covering map $p : \hat{X} \to X$ induces an injection $p_* : \pi_1(\hat{X}, \hat{x}) \to \pi_1(X, p(\hat{x}))$ on fundamental groups.

(c) Let $p : \hat{X} \to X$ be a covering map with Y and X path-connected. Suppose that the induced map p_* on π_1 is an isomorphism. Prove that p is a homeomorphism.

2 (Fall '06/Fall '09/Fall '15). (a) Give the definitions of *covering space* and *deck transformation* (or *covering transformation*).

(b) Describe the universal cover of the Klein bottle and its group of deck transformations.

(c) Explicitly give a collection of deck transformations on $\{(x, y)| - 1 \leq x \leq 1, -\infty < y < \infty\}$ such that the quotient is a Möbius band.

(d) Find the universal cover of $\mathbb{R}P^2 \times S^1$ and explicitly describe its group of deck transformations.

3 (Spring '06/Spring '07/Spring '12). (a) What is the definition of a *regular* (or *Galois*) covering space?

(b) State, without proof, a criterion in terms of the fundamental group for a covering map $p : \tilde{X} \to X$ to be regular.

(c) Let Θ be the topological space formed as the union of a circle and its diameter (so this space looks exactly like the letter Θ). Give an example of a covering space of Θ that is *not* regular.

4 (Spring '08). Let S be the closed orientable surface of genus 2 and let C be the commutator subgroup of $\pi_1(S, *)$. Let $\tilde{S}$ be the cover corresponding to C. Is the covering map $\tilde{S} \to S$ regular? (The term "normal" is sometimes used as a synonym for regular in this context.) What is the group of deck transformations? Give an example of a nontrivial element of $\pi_1(S, *)$ which lifts to a trivial deck transformation.

5 (Fall '04). Describe the 3-fold connected covering spaces of $S^1 \vee S^1$.

6 (Spring '17). Find all three-fold covers of the wedge of two copies of $\mathbb{R}P^2$. Justify your answer.

7 (Fall '17). Describe, as explicitly as you can, two different (non-homeomorphic) connected two-sheeted covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^3$, and prove that they are not homeomorphic.

8 (Spring '19). Is there a covering map from

$$X_3 = \{x^2 + y^2 = 1\} \cup \{(x - 2)^2 + y^2 = 1\} \cup \{(x + 2)^2 + y^2 = 1\} \subset \mathbb{R}^2$$

to the wedge of two S^1's? If there is, give an example; if not, give a proof.

9 (Spring '05). (a) Suppose Y is an n-fold connected covering space of the torus $S^1 \times S^1$. Up to homeomorphism, what is Y? Justify your answer.

(b) Let X be the topological space obtained by deleting a disk from a torus. Suppose Y is a 3-fold covering space of X. What surfaces could Y be? Justify your answer, but you need not exhibit the covering maps explicitly.

10 (Spring '07). Let S be a connected surface, and let U be a connected open subset of S. Let $p : \tilde{S} \to S$ be the universal cover of S. Show that $p^{-1}(U)$ is connected if and only if the homeomorphism $i_* : \pi_1(U) \to \pi_1(S)$ induced by the inclusion $i : U \to S$ is onto.

11 (Fall '10). Suppose that X has universal cover $p : \tilde{X} \to X$ and let $A \subset X$ be a subspace with $p(\tilde{a}) = a \in A$. Show that there is a group isomorphism $\ker(\pi_1(A, a) \to \pi_1(X, a)) \cong \pi_1(p^{-1}A, \tilde{a})$.

12 (Fall '14). Prove that every continuous map $f : \mathbb{R}P^2 \to S^1$ is homotopic to a constant. (Hint: think about covering spaces.)

13 (Spring '16). Prove that the free group on two generators contains a subgroup isomorphic to the free group on five generators by constructing an appropriate covering space of $S^1 \vee S^1$.

14 (Fall '12). Use covering space theory to show that $\mathbb{Z}_2 * \mathbb{Z}$ (that is, the free product of $\mathbb{Z}_2$ and $\mathbb{Z}$) has two subgroups of index 2 which are not isomorphic to each other.

15 (Spring '17). (a) Show that any finite index subgroup of a finitely generated free group is free. State clearly any facts you use about the fundamental groups of graphs.

(b) Prove that if N is a nontrivial normal subgroup of infinite index in a finitely generated free group F, then N is not finitely generated.

16 (Spring '19). Let $p : X \to Y$ be a covering space, where X is compact, path-connected, and locally path-connected. Prove that for each $x \in X$ the set $p^{-1}(\{p(x)\})$ is finite, and has cardinality equal to the index of $p_*(\pi_1(X, x))$ in $\pi_1(Y, p(x))$.