Title

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Contents

1 Monday January 13

1.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta\in\Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over $\mathbb{C}$. We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group $W = \left\langle s_\beta \;\middle|\; \beta \in \Phi \right\rangle = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$ where $\Delta = \{a_i\}$ are the simple roots. For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) \coloneqq n$ where $\ell(1) \coloneqq 0$.

Facts:

1. $\ell(w)$ is the size of the set $\left\{\beta \in \Phi^+ \;\middle|\; w\beta < 0\right\}$

- The above set is equal to $\Phi^+ \bigcap w^{-1}\Phi^{-1}$.
- In particular, for $\beta in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
- Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.

2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \bigcap w\Phi$ (replacing w^{-1} with w)
3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = \left|\Phi^+\right|$ and $w_0(\Phi^+) = \Phi^-$.

- Also $\ell(w_0 w) = \ell(w_0) - \ell(w) >$ Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$\begin{aligned}\ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0\\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0\end{aligned}$$
.

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

1.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$. Then define $T \coloneqq \bigcup_{w \in W} wSw^{-1} = \left\{ s_\beta \;\middle|\; \beta \in \Phi^+ \right\}$. This is the set of *all* reflections in W through hyperplanes in E.

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w, and refer to this as "the covering relation". So $w' \to w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that $w' = w_0 \to w_1 \to \cdots \to w_n = w$. This is called the **Bruhat-Chevalley order** on W.

Corollary: $w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

> It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order.* In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

> Recall that lie algebras yield finite crystallographic coxeter groups.

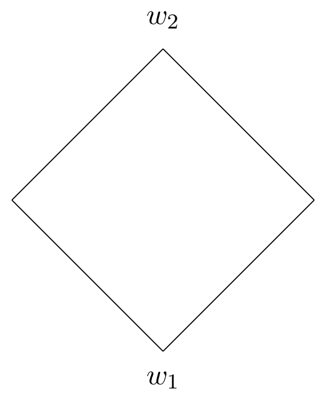
Properties: For (W, S) a coxeter group,

a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \cdots s_n$ for $w, where a subexpression is any subcollection of s_i in the same order.

> Note that this implies that 1 is not only a minimal element in this order, but an infimum.

b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.

c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\left\{ w \in W \;\middle|\; w_1 < w < w_2 \right\}$ is either 0 or 2.



1.3 Properties of Universal Enveloping Algebras

Let $\mathfrak{g}$ be any lie algebra, and $\phi : \mathfrak{g} \to A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, dotted, "\tilde{\phi}"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[r, "\phi"'] & A \end{tikzcd}$$

Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_1$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, where $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as $T(\mathfrak{g})/\left\langle [x, y] - x \otimes y - y \otimes x \;\middle|\; x, y \in \mathfrak{g} \right\rangle$. Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\text{ad } x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Big Theorem (Poincaré- Birkhoff - Witt, i.e. PBW): If $\{x_1, \cdots x_n\}$ is a basis for $\mathfrak{g}$, then $\left\{ x_1^{t_1}, \cdots, x_n^{t_n} \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$ (noting that $x^n = x \otimes x \otimes \cdots x$ and $\mathbb{Z}^+$ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If $\mathfrak{g}$ is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for $\mathfrak{g}$, then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \to \text{gl}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

Notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

1.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the $\mathbb{R}$-span of the roots. We also have the **integral weight lattice** $\Lambda = \left\{\lambda \in E \;\middle|\; (\lambda, \alpha^\vee) \in \mathbb{Z}\ \forall \alpha \in \Phi(\text{or } \Phi^+ \text{ or } \Delta)\right\}$. There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and $\mathfrak{h}^\vee$. We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+\Delta = \mathbb{Z}^+\Phi^+$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a $\mathbb{Z}$-basis for Λ. Then Λ is a free abelian group of rank ℓ, and $\Lambda^+ = \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.