Problem Set One

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1 Humphreys 1.1

1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee / \Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces M_μ for which $\mu \in [\lambda]$.

Proposition: $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of M

Proof:

Proposition: M is the direct sum of finitely many submodules of the form $M^{[\lambda]}$.

Proof:

1.2 b

Proposition: The weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee / \Lambda_r$.

2 Humphreys 1.3*

Proposition: For any $M \in \mathcal{O}$, $M(\lambda)$ satisfies the following property:

$$\operatorname{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \operatorname{Hom}_{U(\mathfrak{g})}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \operatorname{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right).$$

Proof:

Noting that

- $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$,
- $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{g}} M$ is an identification of the $\mathfrak{g}$-module M has a $\mathfrak{b}$- module by restricting the action of $\mathfrak{g}$,

consider the following two maps:

$$\begin{aligned} F : \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) &\to \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) \\ \phi &\mapsto (F\phi : z \mapsto \phi(1 \otimes z)), \end{aligned}$$

and using the action of $\mathfrak{g}$ on M,

$$\begin{aligned} G : \hom_{U(\mathfrak{b})}(\mathbb{C}_\lambda, M) &\to \hom_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, M) \\ \psi &\mapsto (G\psi : g \otimes v \mapsto g \curvearrowright \psi(v)). \end{aligned}$$

It suffices to show that these maps are well-defined and mutually inverse.

To see that F is well-defined, let $\phi : U(\mathfrak{g}) \otimes C_\lambda \to M$ be fixed; we will show that the set map $F\phi : \mathbb{C}_\lambda \to M$ is $U(\mathfrak{b})$-linear. Let $b \in U(\mathfrak{b})$, then

$$\begin{aligned} b \curvearrowright F\phi(v) &:= b \curvearrowright (z \mapsto \phi(1 \otimes z))(v) \\ &:= b \curvearrowright \phi(1 \otimes v) \\ &= \phi(b \curvearrowright (1 \otimes v)) \quad \text{since } \phi \text{ is } U(\mathfrak{g})\text{-linear and } b \in U(\mathfrak{g}) \\ &= \phi((b \curvearrowright 1) \otimes v) \quad \text{by the definition/construction of } M(\lambda) \text{ as a } U(\mathfrak{g})\text{-module.} \\ &= \phi(1 \otimes (b \curvearrowright v)) \quad \text{since } \mathbb{C}_\lambda \text{ is a } \mathfrak{b}\text{-module and the tensor is over } U(\mathfrak{b}) \\ &:= (z \mapsto \phi(1 \otimes z))(b \curvearrowright v) \\ &:= F\phi(b \curvearrowright v). \end{aligned}$$

To see that G is well-defined, let $\psi : C_\lambda \to M$ be fixed; we will show that the set map $G\psi : U(\mathfrak{g}) \otimes C_\lambda \to M$ is $U(\mathfrak{g})$-linear. Let $u \in U(\mathfrak{g})$, then

$$\begin{aligned} u \curvearrowright G\psi(g \otimes v) &:= u \curvearrowright (g \otimes v \mapsto g \curvearrowright \psi(v))(g \otimes v) \\ &:= u \curvearrowright (g \curvearrowright \psi(v)) \\ &= (ug) \curvearrowright \psi(v) \quad \text{since } M \text{ is a } \mathfrak{g}\text{-module with a well-defined action.} \\ &:= (g \otimes v \mapsto g \curvearrowright \psi(v))(ug \otimes v) \\ &:= G\psi(ug \otimes v). \end{aligned}$$

To see that $GF := G \circ F$ is the identity, let ψ be defined as above and fix $z_0 \in \mathbb{C}_\lambda$. Then

$$
\begin{aligned}
FG\psi(z_0) &= F(g \otimes v \to g \curvearrowright \psi(v))(z_0) \\
&:= F(\lambda)(z_0) \quad \text{for notational convenience} \\
&= (v \mapsto \lambda(1 \otimes v))(z_0) \\
&= \lambda(1 \otimes z_0) \\
&:= 1 \curvearrowright \psi(z_0) \\
&= \psi(z_0).
\end{aligned}
$$

Similarly, to see that FG is the identity, let ϕ be defined as above and fix $g_0 \otimes v_0 \in U(\mathfrak{g}) \otimes \mathbb{C}_\lambda$. Then

$$
\begin{aligned}
GF\phi(g_0 \otimes v_0) &= G(g \otimes v \mapsto g \curvearrowright \phi(v))(g_0 \otimes v_0) \\
&= g_0 \curvearrowright \phi(v_0).
\end{aligned}
$$