

# Problem Set 2

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## 1 | Exercises

**Exercise 1.1** (Gathmann 2.17).

Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

**Solution:**

Since  $x = yz$  for all points in  $X$ , we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V(x - yz, y(z^2 - y)) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &:= X_1 \cup X_2. \end{aligned}$$

**Claim:** These two subvarieties are irreducible.

It suffices to show that the  $A(X_i)$  are integral domains. We have

$$A(X_1) := \mathbb{C}[x, y, z] / \langle x - yz, y \rangle \cong \mathbb{C}[y, z] / \langle y \rangle \cong \mathbb{C}[z],$$

which is an integral domain since  $\mathbb{C}$  is a field and thus an integral domain, and

$$A(X_2) := \mathbb{C}[x, y, z] / \langle x - yz, z^2 - y \rangle \cong \mathbb{C}[y, z] / \langle z^2 - y \rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

**Exercise 1.2** (Gathmann 2.18).

Let  $X \subset \mathbb{A}^n$  be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

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**Solution:**

$\bar{X} \subseteq V(I(X))$ :

We have  $X \subseteq V(I(X))$  and since  $V(J)$  is closed in the Zariski topology for any ideal  $J \trianglelefteq k[x_1, \dots, x_n]$  by definition,  $V(I(X))$  is closed. Thus

$$X \subseteq V(I(X)) \text{ and } V(I(X)) \text{ closed} \implies \bar{X} \subseteq V(I(X)),$$

since  $\bar{X}$  is the intersection of all closed sets containing  $X$ .

$V(I(X)) \subseteq \bar{X}$ :

Noting that  $V(\cdot), I(\cdot)$  are individually order-reversing, we find that  $V(I(\cdot))$  is order-*preserving* and thus

$$X \subseteq \bar{X} \implies V(I(X)) \subseteq V(I(\bar{X})) = \bar{X},$$

where in the last equality we've used part (i) of the Nullstellensatz: if  $X$  is an affine variety, then  $V(I(X)) = X$ . This applies here because  $\bar{X}$  is always closed, and the closed sets in the Zariski topology are precisely the affine varieties.

**Exercise 1.3** (Gathmann 2.21).

Let  $\{U_i\}_{i \in I} \rightrightarrows X$  be an open cover of a topological space with  $U_i \cap U_j \neq \emptyset$  for every  $i, j$ .

- Show that if  $U_i$  is connected for every  $i$  then  $X$  is connected.
- Show that if  $U_i$  is irreducible for every  $i$  then  $X$  is irreducible.

**Solution:**

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**Exercise 1.4** (Gathmann 2.22).

Let  $f : X \rightarrow Y$  be a continuous map of topological spaces.

- Show that if  $X$  is connected then  $f(X)$  is connected.
- Show that if  $X$  is irreducible then  $f(X)$  is irreducible.

**Solution:**

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**Definition 1.0.1** (Ideal Quotient).

For two ideals  $J_1, J_2 \trianglelefteq R$ , the *ideal quotient* is defined by

$$J_1 : J_2 := \left\{ f \in R \mid f J_2 \subset J_1 \right\}.$$

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**Solution:**

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**Exercise 1.5** (Gathmann 2.23).

Let  $X$  be an affine variety.

- a. Show that if  $Y_1, Y_2 \subset X$  are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

- b. If  $J_1, J_2 \trianglelefteq A(X)$  are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

**Solution:**

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**Exercise 1.6** (Gathmann 2.24).

Let  $X \subset \mathbb{A}^n$ ,  $Y \subset \mathbb{A}^m$  be irreducible affine varieties, and show that  $X \times Y \subset \mathbb{A}^{n+m}$  is irreducible.

**Solution:**

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