

# Category $\mathcal{O}$ , Problem Set 3

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## 1 Humphreys 1.10

Prove that the transpose map  $\tau$  fixes  $Z(\mathfrak{g})$  pointwise.

Check that  $\tau$  commutes with the Harish-Chandra morphism  $\xi$  and use the fact that  $\xi$  is injective.

### 1.1 Solution

We first note that after choosing a PBW basis for  $\mathfrak{g}$ ,  $\tau$  is defined on  $\mathfrak{g}$  in the following way:

$$\begin{aligned}\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha\end{aligned}$$

which lifts to an anti-involution  $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$  by extending linearly over PBW monomials. We can note that since  $\tau$  fixes  $\mathfrak{h}$  pointwise by definition, its lift also fixes  $U(\mathfrak{h})$  pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned} \xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}. \end{aligned}$$

**Proposition 1.1.**

The following diagram commutes

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \\ \downarrow \tau & & \downarrow \tau \\ Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \end{array}$$

*Proof.*

We will show that for all  $z \in Z(\mathfrak{g})$ ,  $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$ . Expand  $z$  in a PBW basis as  $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k}$ . We then make the following computations:

$$\begin{aligned} (\xi \circ \tau)(z) &= (\xi \circ \tau) \left( \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} \right) \\ &= \xi \left( \prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_k} \right) \quad \text{since } \tau \text{ is an anti-homomorphism} \\ &= \prod_j h_j^{s_j} \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\tau \circ \xi)(z) &= \tau \left( \prod_j h_j^{s_j} \right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of  $\xi$ , we have

$$\begin{aligned} (\xi \circ \tau)(z) &= \xi(z) \\ \implies \tau(z) &= z. \end{aligned}$$

■

## 2 Humphreys 1.12

Fix a central character  $\chi$  and let  $\{V^{(\lambda)}\}$  be a collection of modules in  $\mathcal{O}$  indexed by the weights  $\lambda$  for which  $\chi = \chi_\lambda$  satisfying

1.  $\dim V^{(\lambda)} = 1$
2.  $\mu < \lambda$  for all weights  $\mu$  of  $V^{(\lambda)}$ .

Then the symbols  $[V^{(\lambda)}]$  form a  $\mathbb{Z}$ -basis for the Grothendieck group  $K(\mathcal{O}_\chi)$ .

For example take  $V^{(\lambda)} = M(\lambda)$  or  $L(\lambda)$ .

### 2.1 Solution

Following a similar proof outlined here.

Fix a  $\lambda_0$  such that  $\chi = \chi_{\lambda_0}$  by Harish-Chandra's theorem, fix some order on the Weyl group  $W = \{w_j \mid 1 \leq j \leq |W| < \infty\}$ , and note that  $\chi_{\lambda_0} = \chi_{w \cdot \lambda_0}$  for each  $w \in W$ .

#### Proposition 2.1.

The simple modules  $\{L(w \cdot \lambda_0) \mid w \in W\}$  form a  $\mathbb{Z}$ -basis for  $\mathcal{O}_\chi$ .

*Proof.*

Write  $\mathcal{L} = \text{span}_{\mathbb{Z}} \{[L(w_j \cdot \lambda_0)] \mid 1 \leq j \leq |W|\} \subset K(\mathcal{O}_\chi)$ .

**Spanning:** Let  $M \in \mathcal{O}_\chi$  be arbitrary, and consider  $[M] \in K(\mathcal{O}_\chi)$ . By Humphreys Theorem 1.11,  $M$  has a finite composition series

$$M = M_1 > M_2 > \cdots > M_n$$

with simple quotients  $M^{i+1}/M^i \cong L(\lambda_i)$  for some  $\lambda_i \in \mathfrak{h}^\vee$ . By collecting terms, we can write

$$[M] = \sum_{i=1}^n [L(\lambda_i)] = \sum_{i=1}^{n'} c_i [L(\lambda_i)] \in K(\mathcal{O}_\chi),$$

where each  $c_i$  is the multiplicity of  $L(\lambda_i)$  in the above composition series.

By definition,  $M \in \mathcal{O}_\chi \iff L(\lambda_i) \in \mathcal{O}_\chi$ , i.e.  $M$  is in this block precisely when all of its composition factors are. But this forces each  $L(\lambda_i) = L(w_j \cdot \lambda_0)$  for some  $j$ , and so we have

$$[M] = \sum_{i=j}^{n'} c_j [L(w_j \cdot \lambda_0)] \in \mathcal{L}.$$

**Linear Independence:** Define a family of maps

$$r_j : \mathcal{O}_\chi \longrightarrow \mathbb{Z}^{\geq 0}$$

$$M \mapsto \left| \left\{ M^{i+1}/M^i \mid M^{i+1}/M^i \cong L(w_j \cdot \lambda_0) \right\} \right|,$$

i.e. the map that counts the multiplicity of  $L(w_j \cdot \lambda_0)$  appearing in any composition series of  $M$  for a fixed  $j$ .

This lifts to a group morphism  $r_j : K(\mathcal{O}_\chi) \longrightarrow \mathbb{Z}^{\geq 0}$  which satisfies

$$r_j(L(w_i \cdot \lambda_0)) = \delta_{ij},$$

i.e. it takes the value 1 on the Verma modules in  $\mathcal{L}$  precisely when  $i = j$  and zero otherwise.

Now suppose  $\sum_{i=1}^n a_i [L(w_i \cdot \lambda_0)] = [0]$  in  $K(\mathcal{O}_\chi)$ . For each fixed  $j$ , we can then apply the above group morphism to obtain

$$\begin{aligned} r_j \left( \sum_{i=1}^n a_i [L(w_i \cdot \lambda_0)] \right) &= \sum_{i=1}^n a_i r_j([L(w_i \cdot \lambda_0)]) \\ &= \sum_{i=1}^n a_i r_j \delta_{ij} \\ &= a_j. \end{aligned}$$

Since group morphisms preserve equalities and  $r_j([0]) = 0 \in \mathbb{Z}$ , this forces  $a_j = 0$  for each  $j$ . ■

**Proposition 2.2.**

An arbitrary set of the stated form  $\mathcal{V} = \left\{ V^{(\lambda_i)} \mid 1 \leq i < N < \infty \right\}$  is also a  $\mathbb{Z}$ -basis of  $K(\mathcal{O}_\chi)$ .

*Proof.*

We first note that we can similarly write  $V^{(\lambda_i)} = V^{(w_j \cdot \lambda_0)}$  for some  $j$ , so wlog we reindex the  $\lambda_i$  to  $\lambda_j$ s. Similarly, fixing a  $V^{\lambda_j}$ , for  $\mu < \lambda_j$ , there is an  $i$  such that  $\mu = w_i \cdot \lambda_0$ , so we reindex all lower weights accordingly as well.

By the previous proposition, for each fixed  $V^{(\lambda_i)}$ , we can write

$$[V^{(\lambda_j)}] = [L(w_j \cdot \lambda_0) + \sum_{\mu_i < \lambda_j} a_{ij} [L(w_i \cdot \lambda_0)]].$$

The matrix  $A = (a_{ij})$  is then strictly upper-triangular with ones on the diagonal, and is thus invertible, and so expresses a change of basis matrix  $\mathcal{L} \rightarrow \mathcal{V}$ . ■

### 3 Humphreys 1.13

Suppose  $\lambda \notin \Lambda$ , so the linkage class  $W \cdot \lambda$  is the disjoint union of its nonempty intersections of various cosets of  $\Lambda_r \in \mathfrak{h}^\vee$ .

Prove that each  $M \in \mathcal{O}_{\chi_\lambda}$  has a corresponding direct sum decomposition  $M = \bigoplus M_i$  in which all weights of  $M_i$  lie in a single coset.

Recall exercise 1.1b.

#### 3.1 Solution

Fix a nonintegral  $\lambda \in \mathfrak{h}^\vee \setminus \Lambda$  and  $M \in \mathcal{O}_{\chi_\lambda}$ , and write

$$\mathfrak{h}^\vee / \Lambda = \left\{ \lambda_i + \Lambda \mid i \in I \right\} = \left\{ [\lambda_i] \mid i \in I \right\}$$

for some indexing set  $I$ . As in exercise 1.1, for each  $i$  we can define

$$M_i = M^{[\lambda_i]} := \sum_{\mu \in [\lambda_i]} M_\mu,$$

the sum of weight spaces  $M_\mu$  for which  $\mu \in [\lambda_i]$ . Note that by construction, all of the weights of  $M_i$  lie in the single coset  $[\lambda_i]$ .

By the result of that exercise,  $M$  decomposes as a finite direct sum of such modules.

Let  $W \cdot \lambda$  be the orbit of  $\lambda$  under the action of  $W$ , i.e. the linkage class of  $\lambda$ . Since  $\lambda \notin \Lambda$ , we can write the image of  $W \cdot \lambda$  in  $\mathfrak{h}^\vee / \Lambda$  as  $\{[\eta_1], \dots, [\eta_N]\}$  for some  $N \geq 2$ .

This yields

$$M = \bigoplus_{i=1}^N M^{[\eta_i]},$$

which satisfies the desired property. ■