Full Notes

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Contents

1 Thursday, Week 1

Motivation: Gauss' Unicursal Problem. How many distinct curves $\alpha : I \to \mathbb{R}^2$ with are there with no triple crossings?

Note that if we compactify the plane to the Riemann sphere (and possibly take the curve to be piecewise linear) then we obtain a *tiling* of the sphere. We can also take a *dual tiling* by taking the barycenters of each polygon and connecting them by an edge iff their corresponding polygons share an edge.

Definition: A *translation surface* is the 2-dimensional topological manifold obtained by taking any set of polygons in $\mathbb{R}^2$ and gluing their edges by translations.

Example: Any elliptic curve (topologically a torus).

We take equivalence up to cutting, pasting, and rearranging.

Lemma: Two elliptic curves (of genus 2) are isomorphic iff the translation surfaces differ by a homothety, i.e. a rotation and scaling.

> Note: we will eventually see that the data of a translation surface is equivalently a holomorphic 1-form.
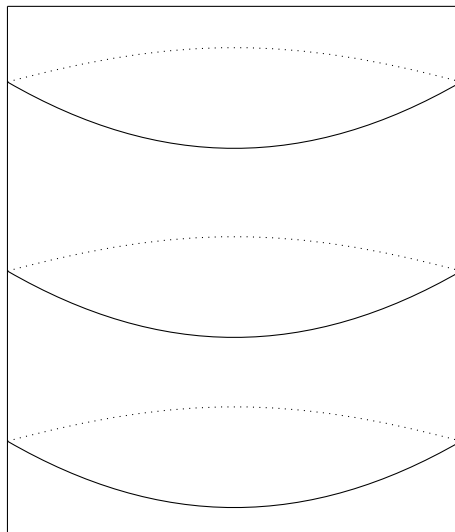
Definition: A *half-translation surface* is a translation surface where we now additionally allow gluing by rotations of π radians.
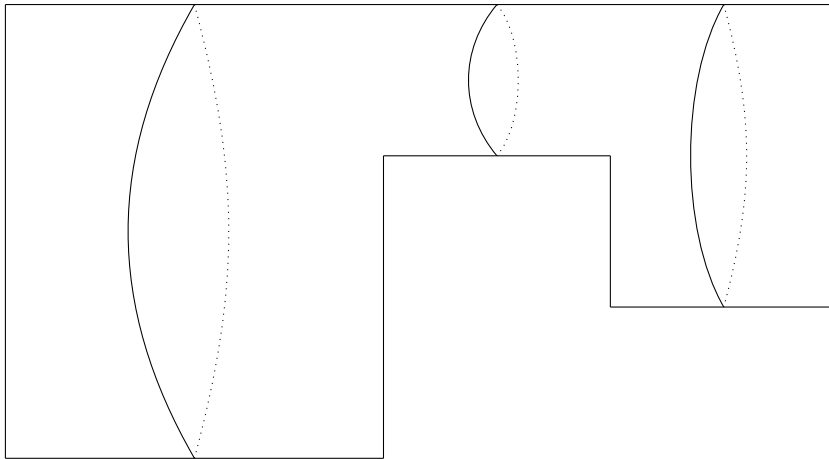
Example:
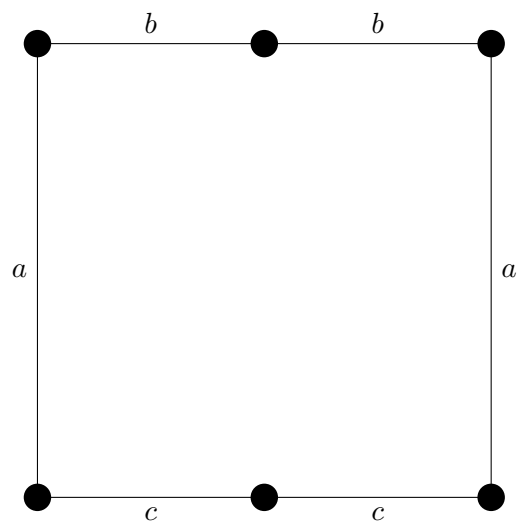
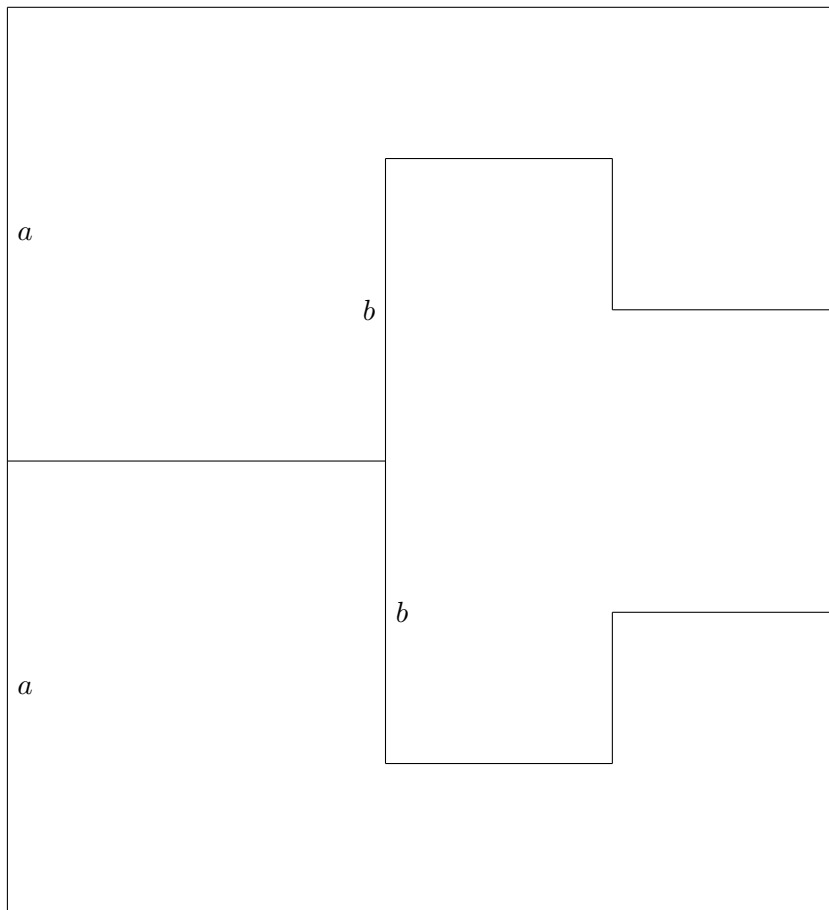This glues to a "hot pocket" surface:

Lemma: Any game of *rectangular billiards* yields a half-translation surface.

Given any domain for rectangular billiards, we inflate it to a surface:

We can then cut along everything but the bottom edge to "unwrap" it into a half-translation surface:

Some questions related to rectangular billiards:

1. Given a random starting point and direction, what proportion of the total region is traversed? Will the trajectory entire a given region? How long does the billiard spend in any given region?

Theorem: The percentage of time spent in a given area is equal to the proportion of its area to the total area.

> This requires some ergodic theory.

2. If you shine a laser from a given spot, is the entire region illuminated?

Theorem: No! There is a counterexample with 18 sides. Moreover, no positive-area region can be avoided, but certain finite sets can.

Definition: A *flat surface* is a generalization of translation surfaces that now allows gluing by any isometry of $\mathbb{R}^2$.

Example: A cube is a flat surface, noting that there is a planar gluing diagram for it that now requires rotations of $\pi/2$ radians:

Cube Pattern
Cut on solid lines - Fold on dashed lines

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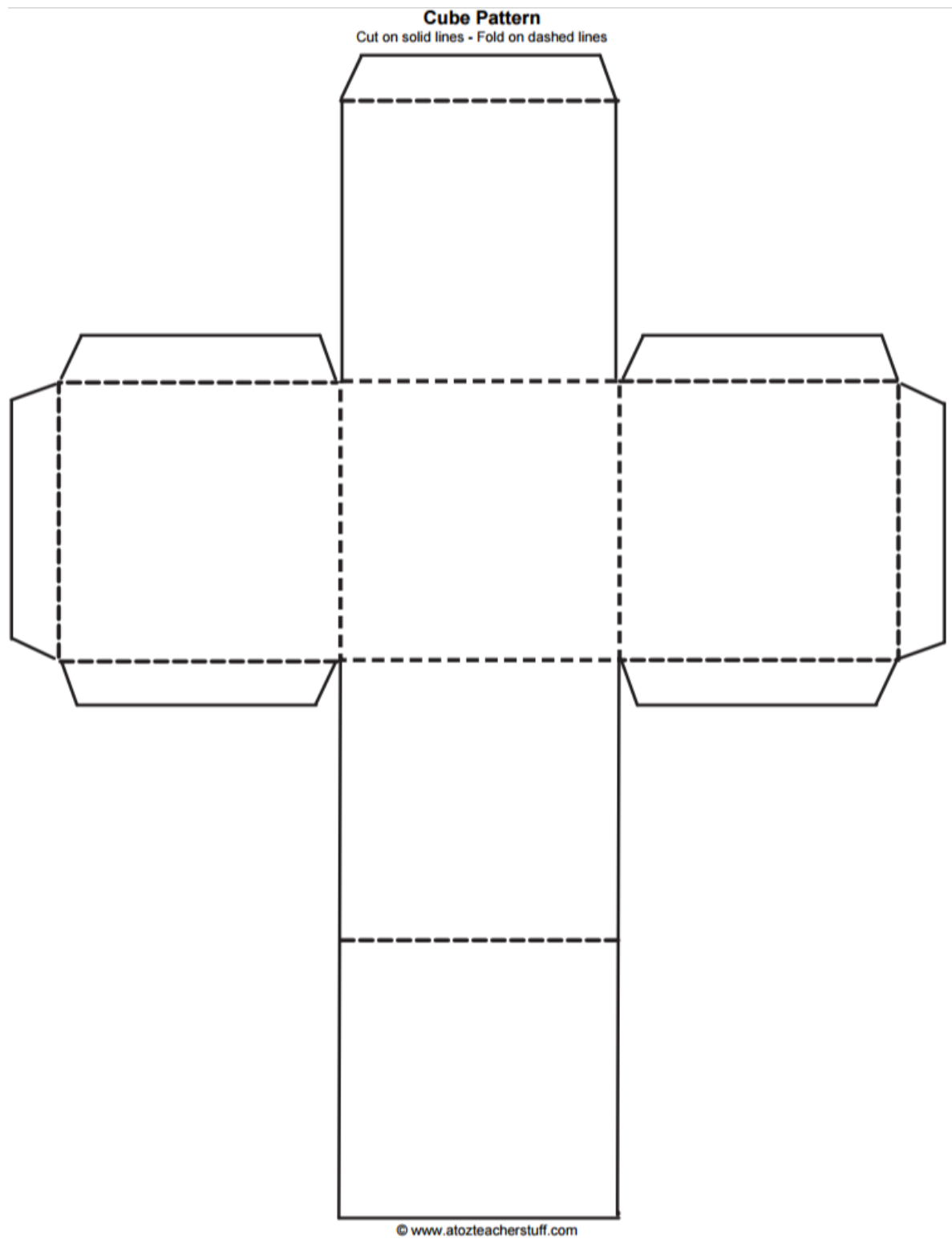
Note that we can form higher genus surfaces using polygon gluing:

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Exercise: Check that there is only one vertex in this polygon.

In general, the vertices may have total angle greater than $2\pi n$. We refer to these as *cone points*, and the total angle as the *cone angle*.

Exercise: Check that the cone angle in the above example is 6π by taking a loop around the cone point.

Note on a weird phenomenon: it seems difficult to find $\mathbb{Z}^2$ or $\mathbb{Q}^2$ points on the sloped edges of a regular polygon, based on computer drawings.

Remark: This flat surface admits charts to $\mathbb{C}$ with the following transition functions. Let P denote the single cone point.

Then there is a 3-fold cover give by the following space, thought of as a singular helical

which maps onto the unit disc in $\mathbb{C}$:
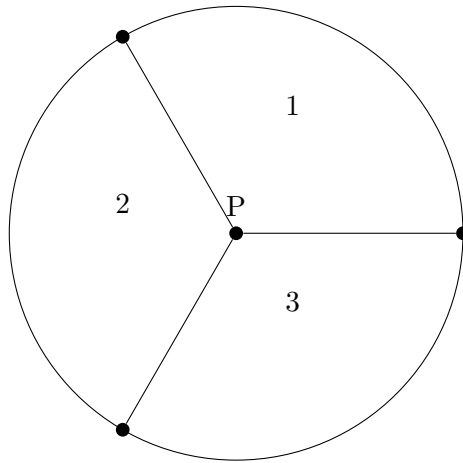
The covering map from the former to the latter is given by $z \mapsto z^{\frac{1}{3}}$, which coincides with the fact that the cone angle at P is $3(2\pi)$.

One can also imagine this space in $\mathbb{R}^3$ with a projection onto the plane:
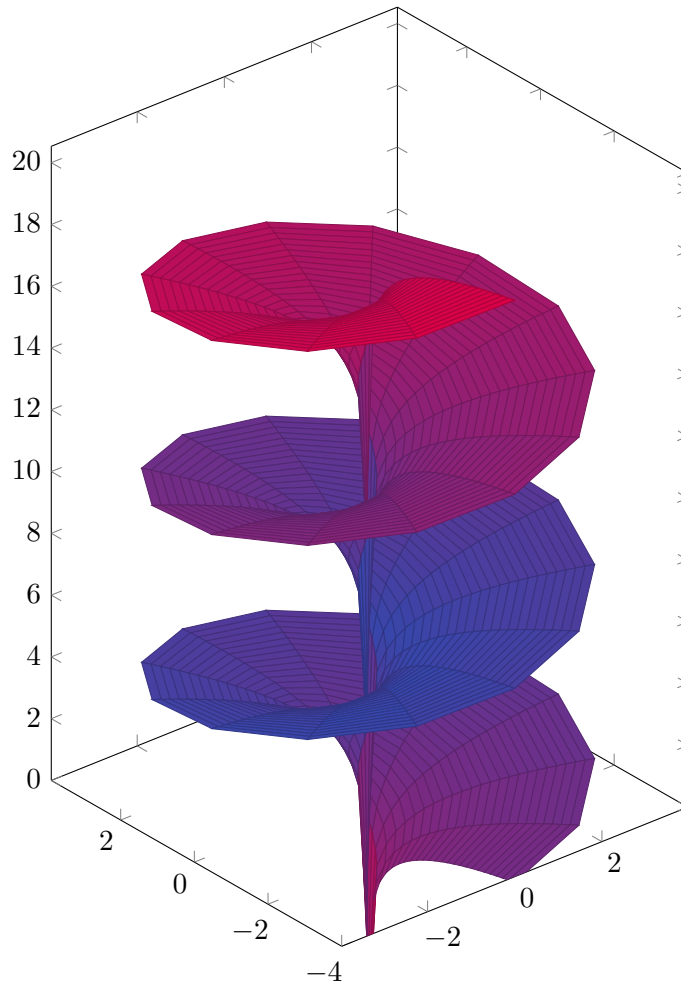
Proposition: In a neighborhood of a cone point P with cone angle $2\pi n$, the map $z \mapsto z^{\frac{1}{n}}$ will be a local chart for any z in a neighborhood of P.
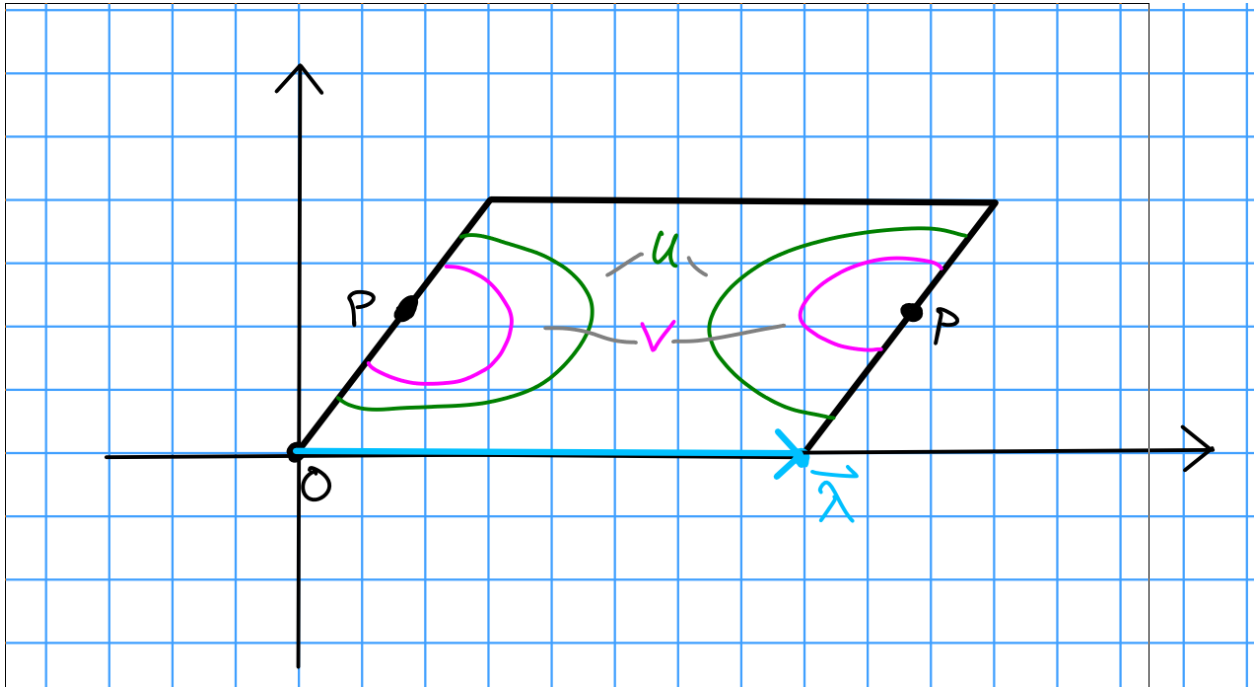
Figure 1: Image

This gives the resulting surface the structure of a *Riemann surface*, i.e. a surface admitting charts to $\mathbb{C}$ with holomorphic transition functions.

Let X be a Riemann surface, we can look at the *canonical bundle* over X with sections that are compatible collections of $f_u(z_u)\,dz_u$ for each chart z_u, and for each such chart a holomorphic function $f_u(z_u)$ on $z_u(U)$ where on overlaps

$$f_u(z_u)\,dz_u = f_v(z_v)\,dz_v$$
$$= f_v(z_v \circ z_u)z'_v(z_u)\,dz_u.$$

Updated Definition: A translation surface is a Riemann surface with a section of its canonical bundle.

Example: C/Λ for Λ any rank-2 lattice:

Note that the translation here is given by λ.

Definition: A holomorphic 1-form is a section of the canonical bundle.

Example: For the above surface, we have $z_v = z_u + \lambda$ and thus $dz_u = dz_v$ is nonvanishing.

Note that dz is a holomorphic 1-form on the complement of the vertex/vertices of the polygon.

Claim: dz extends holomorphically to charts containing cone points with cone angle $2\pi n$ and has a zero of order $n-1$ at such cone points.

Example: For the chart $z = w^3$ where w is the local coordinate, we have $dz = 3w^2$, yielding a zero of order 2.

2 Notes on Paper

2.1 Section 1

Flat surfaces are characterized as surfaces with a flat metric and (finitely many?) cone-like singularities. These surfaces appear to be isomorphic to moduli spaces of holomorphic 1-forms. It is profitable to study the orbit of the surface under the Teichmüller geodesic flow, as well as a GL_n action.

Some introductory surveys:

- A. Eskin [E],
- G. Forni [For2],
- P. Hubert and T. Schmidt [HuSdt5]
- H. Masur [Ma7],
- H. Masur and S. Tabachnikov [MaT]
- J. Smillie [S].

We usually associate

- Constant positive curvature $= S^2$
- Constant zero curvature $= S^1 \times S^1 := T^1 = \Sigma_1$
- Constant negative curvature $= \Sigma_g$ for $g \geq 2$, a surface of higher genus

Proposition: Any surface can be given a flat metric, possibly introducing singular points.

Idea: Push all of the curvature into a cone point.

Example: The standard cube embedded in $\mathbb{R}^3$.

This is a flat surface with 8 cone points located at the vertices. Note that the metric is non-degenerate on the edges, since any neighborhood of a point on an edge is still homeomorphic to $\mathbb{R}^2$:

Any neighborhood of a vertex is isometric to the vertex of the usual notion of a cone. The cone angle can be measured by cutting a cone from the base to the vertex, yielding a flat pattern that sits in $\mathbb{R}_2$, and measuring the "missing" angle in the resulting circle:

This shows that the cone angle of the cube is $3\pi/2$, which coincides with the fact that there are 3 square (and thus 3 right angles) adjacent to any cone point.

Definition: Geodesic (todo)

Definition: Ergodic (todo)

Here this means that a typical geodesic will visit any region in phase space and time spent in a region is proportional to volume.

Definition: Holonomy and Holonomy group (todo)

General (wildly open) problem:

Figure 2: Image

Figure 3: Image

- Describe the behavior of a generic geodesic on a surface
- Prove that the geodesic flow is ergodic on a typical flat surface
- Does almost every surface have a closed geodesic that does not pass through singular points?
  - If so, how many?
  - Find the (asymptotic) number of such closed geodesics of length shorter than L

This remains unsolved for S^2 with 3 singularities (equivalent to a certain billiards problem). It is not even known if any flat sphere admits a single closed geodesic.

Flat surfaces have nontrivial holonomy, which makes them resemble Riemannian manifolds more than flat torii.

If we take the surface and puncture the conical points, it is locally isometric to the punctured Euclidean plane. This allows a notion of parallel transport of tangent vectors.

Parallel transports along homotopically trivial loops are always the identity; otherwise, for homotopically nontrivial loops this rotates the vector by some angle. Parallel transport around a cone point rotates by exactly the cone angle. Nontrivial holonomy forces geodesics to self-intersect many times.

> Exercise: parallel transport a vector around the cone point of a cube.

The flat torus has trivial linear holonomy – all geodesics either close up, or never self-intersect and produce a dense winding path.

Definition: A translation surface is closed orientable surface with a flat metric, a finite number of conical singularities, and trivial linear holonomy.

Note that trivial linear holonomy implies that cone angles are all integer multiples of 2π.

> Convention: we assume all flat surfaces come with a distinguished direction.

Remark: Billiards gives rise to a flat surface with nontrivial linear holonomy.

Definition: A half-translation surface is a surface with a flat metric and holonomy group $\mathbb{Z}/(2)$.

> In this case, a vector $\mathbf{v}$ may come back as $-\mathbf{v}$ after parallel transport.

We'll often consider families of flat surface sharing the same genus and number/type of conical singularities. These will correspond to strata of moduli spaces of one-forms.

It will often be useful to let $\mathrm{SL}(2,\mathbb{R})$ act on these families, consider the orbits, and take its closure.

Central problem / conjecture: Taking the closure of an orbit under the action of $\mathrm{GL}^+(2,\mathbb{R})$ is a complex subvariety, so both the moduli space of holomorphic one-forms and the moduli space of quadratic differentials resemble homogeneous spaces under the action of a unipotent group.

The there is a projection from these orbits (Teichmüller discs) to the moduli space of complex structures (?), which will be denoted $\mathcal{M}_g$. It is well-known that moduli spaces are not homogeneous spaces, but the conjecture here is that they behave as if they were.

2.2 Section 2: Motivations

Open problems in rectangular billiards:

1. Describe the behavior of a billiard trajectory in a generic triangle, and prove that the billiard flow is ergodic.
2. Does (almost) any table have at least one regular periodic trajectory? Is it preserved under deformations?
3. Asymptotically in length, how many periodic trajectories are there?
4. Does any obtuse triangle have a single periodic trajectory?

Known example: Acute triangles have at least 1, see Fagnano trajectory

Fox-Kershner construction: Yields a way to go from billiard trajectories to geodesics on a flat surface. General idea: glue two copies of the billiard table along the edge to get a flat sphere; then paths lift to geodesics. Such surfaces are not "very flat", i.e. they have nontrivial linear holonomy.

3 Random Notes

Holomorphic Forms: A holomorphic p-form on X is a section of $\Lambda^p T^\vee X$, the pth exterior power of the holomorphic cotangent bundle of X.

For $n = \dim_\mathbb{C} X$, the n-forms are an important special case. Any such form w is given in local coordinates $(z_1, \cdots, z_n)$ by

$$w = w(z_1, \cdots, z_n) dz_1 \wedge \cdots \wedge dz_n$$

for some holomorphic function $w : \mathbb{C}^n \to ?$.

Canonical Bundle: Given a complex manifold M, we can define the tangent bundle $\mathbb{C}^n \to TM \to M$ and the cotangent bundle $C^n \to T^\vee M \to M$, which we'll just denote $T^\vee M$. Then the canonical bundle is the bundle $\mathbb{C} \to \Lambda^n T^\vee M \to M$?, denoted by ω, obtained by taking the nth exterior power. It is a theorem that the fibers are in fact lines. For vector bundles, this is referred to as the *determinant bundle*. If M is a smooth manifold, then ω has a global section.

> Note: a holomorphic n-form is exactly the same as a section of the canonical bundle.

Interesting aside: a Calabi-Yau is a manifold with a nowhere vanishing holomorphic n-form, which implies that the canonical bundle admits a map to a trivial line bundle that is an isomorphism, i.e. the canonical bundle is trivial.

Exercise: For Σ_g a compact Riemann surface of genus g, the space of holomorphic sections of the canonical is a vector space (over $\mathbb{C}$?) of dimension g.

4 Thursday January 16th

4.1 Correspondence

Recall: Start with a translation surface with cone points with angles $2\pi n i$. This yields a Riemann surface Σ and a holomorphic 1-form ω with zeros of order $n_i - 1$.

Given a square fundamental domain, there is an order 4 automorphism given by rotating 90 degrees. In charts, this is multiplication by i and possibly a translation, which is a holomorphic map $f : \Sigma \circlearrowleft$. We then have $f^*(dz) = d(f(z)) = i dz$, so dz is an eigenvector for f^*.

An elliptic curve can be specified by $y^2 = f_4(x)$ for a degree 4 polynomial, so we can obtain it as a double branched cover of S^2. (i.e. glue along the slits joining pairs of roots)
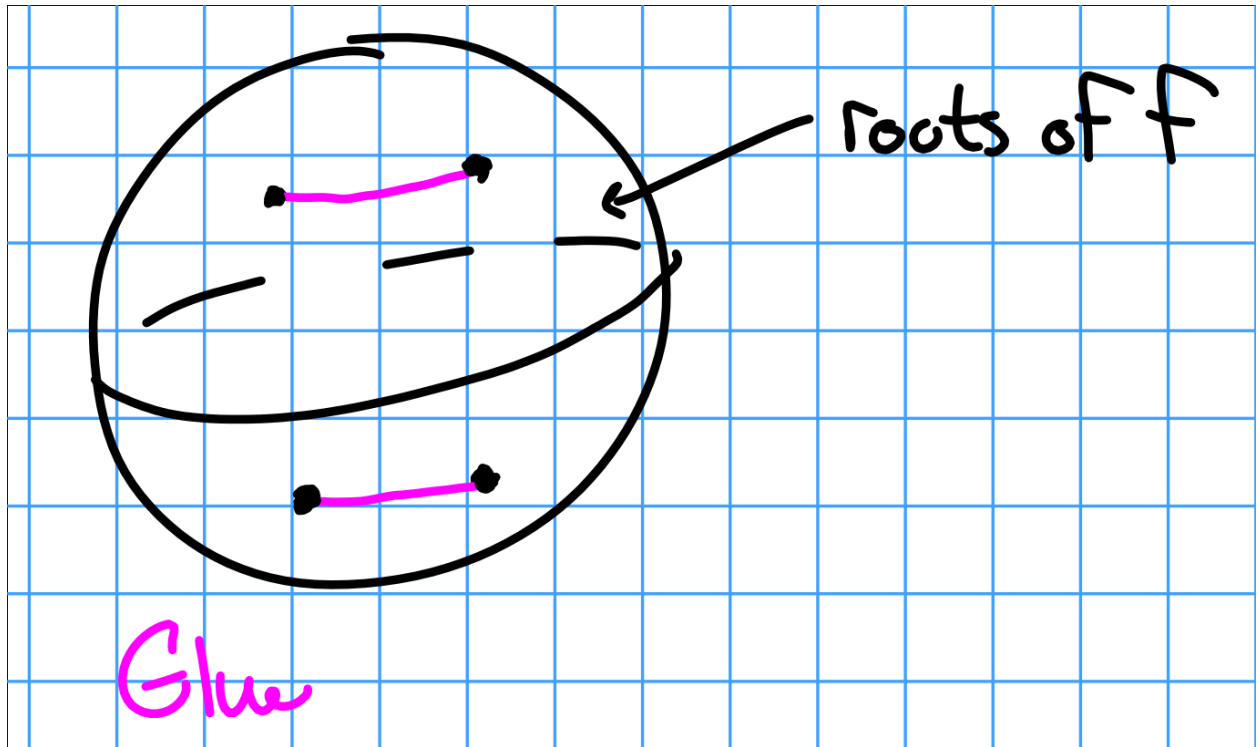
Take $E : y^2 = x^4 - 1$, this is the only elliptic curve with an order 4 automorphism. In coordinates, this is generated by $(x, y) \mapsto (-ix, y)$. So $\omega = c\dfrac{dx}{y}$, i.e. dx/y up to scaling, and $f^*(dx/y) = i\dfrac{dx}{y}$. What is the constant c?

Take closed cycles on E given by α, β (see diagram), then by FTC $\displaystyle\int_\alpha \omega = z \Big|_0^1 = 1$. Negation is 4 fixed points on the elliptic curve, the 2-torsion points.
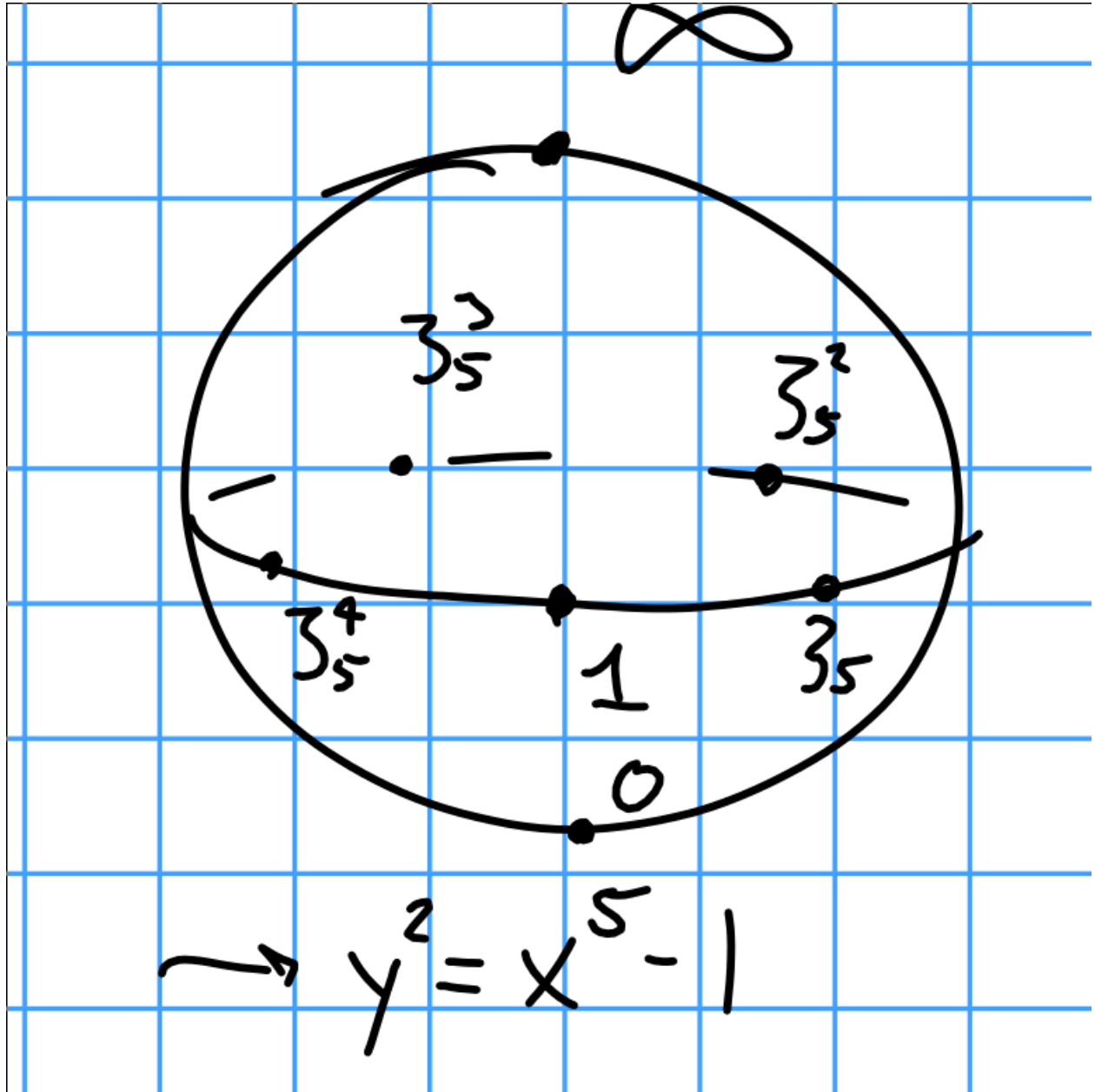
2-torsion

$$y \mapsto -y$$

We can then compute

$$I = \int_\alpha \frac{dx}{y} = \int_1^2 \frac{dx}{\sqrt{x^4 - 1}}.$$

and since $2cI = 1$, this uniquely determines c.

Consider the decagon with sides identified. We get a complex structure (Σ, ω) with cone angles 4π and dz has 2 zeros of order 1 (namely the two cone points).

What is the genus? The degree of the canonical bundle is

$$g(\Sigma) = \deg K_\Sigma = 2g - 2 = \sum_{p \in \Sigma} \mathrm{Ord}_p \omega$$
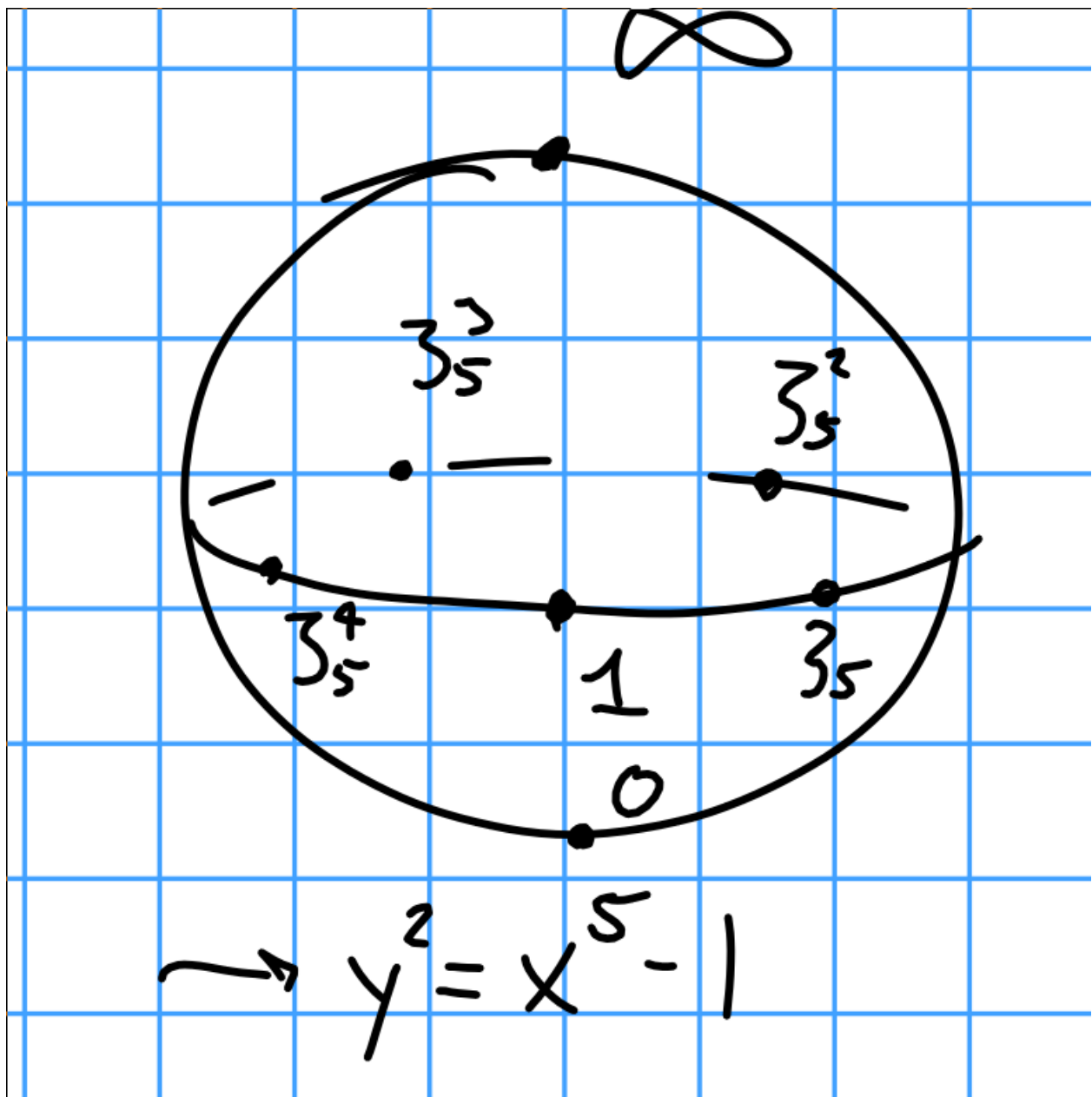
and thus $g = 2$.

Fact: Every genus 2 curve is a double branched cover of $\mathbb{P}^1$ branched over 6 points.

Consider automorphisms that preserve the decagon. Rotation by $\pi/10$ swaps the two cone points, to take rotation by $-2\pi/5$ (inserting the negative to account for pullbacks). Then $f^*\omega = \zeta_5\omega$, where again we just write locally $z \xrightarrow{f} \zeta_5^{-1} + c \implies f^*dz = \zeta_5 dz$.

Consider points of order 5 on $\mathbb{P}^1$, we can take ζ_5^k and ∞.

This corresponds to the curves $y^2 = x^5 - 1$, with an automorphism $(x, y) \mapsto (\zeta_5 x, y)$. This is the only genus 2 curve with an order 5 automorphism.

Fact: The space of sections of the canonical $H^0(K_{\mathcal{E}}) = \mathbb{C}^g$.

We can write a basis for the space of 1-forms: $\dfrac{dx}{y}$ and $(1-x)\dfrac{dx}{y}$. Alternatively, $\omega = (a+bx)\dfrac{dx}{y}$ where $a, b \in \mathbb{C}$. What are the zeros of ω? $V(\omega) = V(a+bx)$ where if $b=0$ it's ∞. Because this has to preserve the order 5 symmetric and map cone points to themselves, this forces $\omega = bx\dfrac{dx}{y}$, which has exactly two zeros: $(x, y) = (0, \pm i)$.

We can also consider the doubled pentagon, which has only one point. This has an automorphism given by rotating each pentagon by $1/5$, it has cone angle 6π, and ω has a double zero at the cone point. Since there is only *one* genus 2 curve with order 5 automorphisms, this yields the previous Riemann surface but a distinct translation surface and a distinct form.

We can write $\alpha = a\dfrac{dx}{y}$.

Proposition: One Riemann surface has many translation structures, and the space of such structures is the space of 1-forms.

Proof: Pick a chart $w : U \to \mathbb{C}$ avoiding the zeros of ω. Then $\omega = f(z)\, dz$ in this chart, and we want to compose with a biholomorphism x to obtain a new chart z such that $\omega = dz$ in this chart. We can solve $dz = e = f(w)\, dw$ and $\dfrac{dz}{dw} = f(w)$ and by integrating we get $z(w) = \displaystyle\int_{p_0}^{w} f(w_0)\, dw_0$. In this chart, $\omega = dz$, and we can correct near zeros by finding charts such that $\omega = z^n dz$ where n is the order of the zero.


What does this buy us? We get a translation structure by considering transitions, and $\omega = dz_1 = dz_2 \implies z_1 = z_2 + c$, which is exactly a translation structure. Thus every (Σ, ω) has a translation structure for which $\omega = dz$ in local polygonal charts.

$$\omega = dz_1 = dz_2 \Rightarrow z_2 = z_1 + c$$

Theorem: There is a bijection

$$\left\{ \text{Translation surfaces with cone angles } 2\pi n i \right\} \iff \left\{ \begin{array}{c} (\Sigma, \omega) \text{ a Riemann surface with holomorphic} \\ \text{1-forms with zeros of order } n_i - 1 \end{array} \right\}.$$

What do half-translations correspond to? Note that $dz \mapsto -dz$, so we don't get a well-defined holomorphic 1-form.
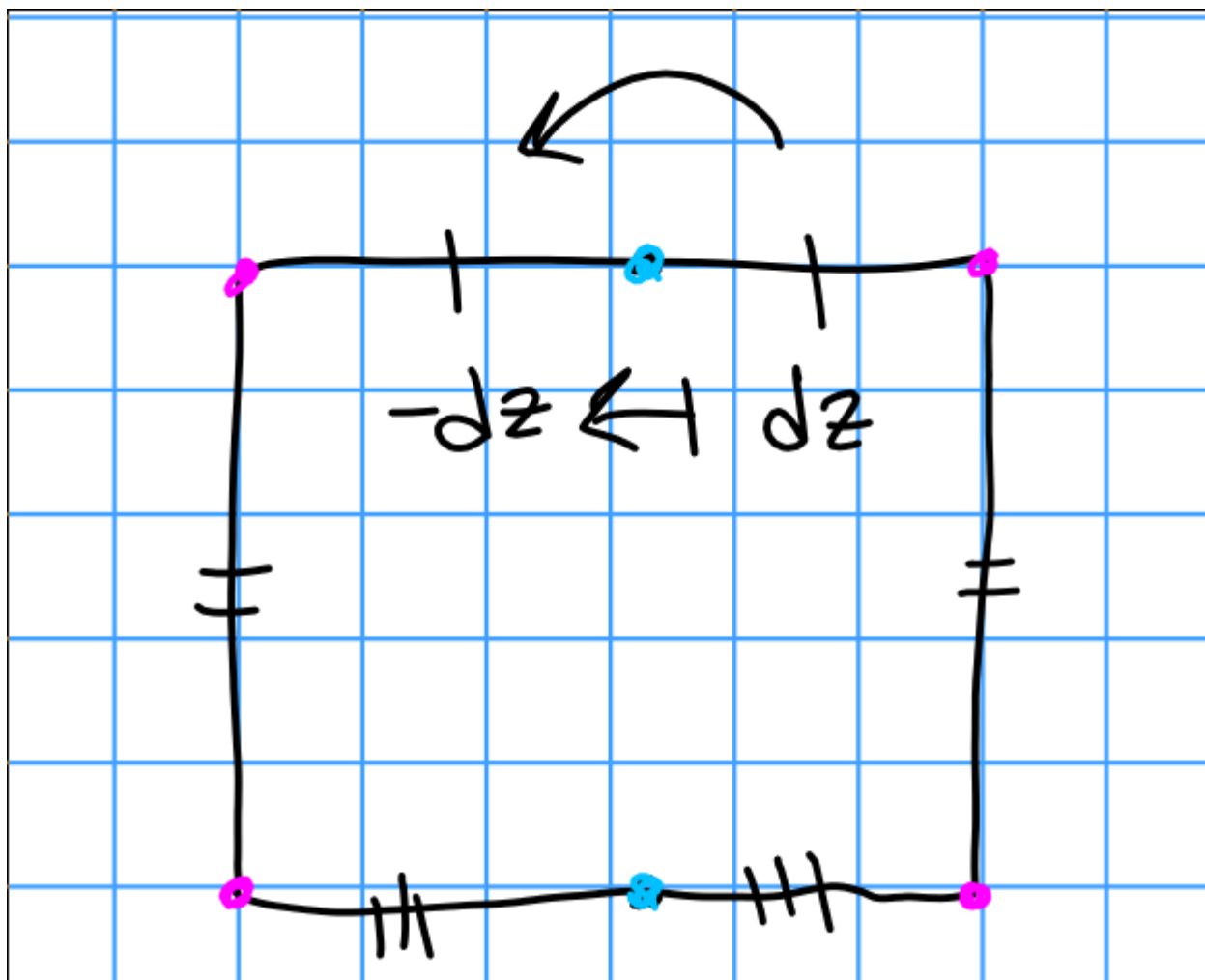
The fix? $(dz)^2$ is some well-defined object. What is it?

The set $\left\{ f(z)(dz)^2 = g(w)(dw)^2 \right\}$ corresponds with line bundles with transition functions given by $(dz/dw)^2$.

Thus the correspondence is

$$\left\{ \text{Half translation surfaces wit cone angle } \pi n_i \right\} \iff \left\{ \text{Riemann surfaces with } q \text{ a section of } K_\Sigma^{\otimes 2} \right\}.$$

A defining property of $q = (dz)^2$ for half-translation charts z: we can measure the order of the zero by going to charts, finding a chart to $\mathbb{C}$ (see image) e.g. $w = z^{2/3}$, and then defining $\omega = dz = d(w^{3/2}) = 3/2 w^{1/2} dw$; then $q = w^2 = (dz)^2$ is well-defined and equal to $9/4 w(dw)^2$ in the local chart w.

In this case, we get points that are *poles* of order 1 for the quadratic differential (sections of $K^{\otimes 2}$).

> Note: 1-forms are referred to as "abelian differentials" in the literature.

We know that $K_{\mathbb{P}^1} = \mathcal{O}(-2)$ and $K_{\mathbb{P}^1}^{\otimes 2} = \mathcal{O}(-4)$.

4.2 Moduli Spaces

Definition: $\mathcal{H}(k_1, \cdots, k_n) = \left\{ \Sigma \text{with abelian differential} \div \omega = \sum k p_i \right\}$ where the k_i record the orders of zeros of ω.

> Second condition on divisor records zeros.

Similarly, define $Q(k_1, \cdots, k_n) = \left\{ \cdots \mid \text{quadratic differential} \right\}$ where now $k_i \geq -1, \neq 0$.

These moduli spaces are called *strata of abelian/quadratic differentials.*

Consider M_g, the moduli space of genus g surfaces. There is a vector bundle (the Hodge bundle) over M_g where the fiber over $[c]$ is $H^0(K_C)$ with strata given by Q.

This is a bundle of rank g, and by Riemann-Roch, ? $3g - 3$ and is the (co?)tangent bundle of something.

There is an $SL(2, \mathbb{R})$ action on any stratum. How to define – use isomorphism with translation structures (see image) by just applying any such form γ to the entire plane and gluing polygons in the same way. This gives a non-holomorphic action on any stratum.

> Next time: to special case of square tiled surfaces.