Algebraic Curves

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List of Theorems

These are notes live-tex'd from a graduate course in Algebraic Curves taught by Pete Clark at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Friday 28th August, 2020
02:13

1 Monday, August 24

Review of lecture one:

Theorem 1.1 ***(Finitely Generated in Towers).***
See video

- Transcendence bases
- Luroth problem

2 Friday, August 28

2.1 Field Theory

See Chapter 11 of Field Theory notes.

2.1.1 Notion 1

Definition 2.0.1 (Finitely Generated Field Extension)**.**
A field extension ℓ/k is *finitely generated* if there exists a finite set $x_1, \cdots, x_n \in \ell$ such that $\ell = k[x_1, \cdots, x_n]$ and ℓ is the smallest field extension of k.
Concretely, every element of ℓ is a quotient of the form $\dfrac{p(x_1, \cdots, x_n)}{q(x_1, \cdots, x_n)}$ with $p, q \in k[x_1, \cdots, x_n]$.

There are three different notions of finite generation for fields, the above is the weakest.

2.1.2 Notion 2

The second is being finitely generated as an algebra:

Definition 2.0.2 (Finitely Generated Algebras)**.**
For $R \subset S$ finitely generated algebras, S is finitely generated over R is every element of S is a polynomials in $x_1, \cdots, x_n$, with coefficients in R, i.e. $S = R(x_1, \cdots, x_n)$.

Note that this implies the previous definition.

2.1.3 Notion 3

The final notion: ℓ/k is finite (finite degree) if ℓ is finitely generated as a k-module, i.e. a finite-dimensional k-vector space.

Definition 2.0.3 (Rational Function Field)**.**
A *rational function field* is $k(t_1, \cdots, t_n) \coloneqq ff k[t_1, \cdots, t_n]$.

Note that we can make a similar definition for infinitely many generators by taking a direct limit (here: union), and in fact every element will only involve finitely many generators.

Exercise 2.1.

a. Show $k(t)/k$ is finitely generated by notion (3) but not by (2).

b. Show that $k[t]/k$ is (2) but not (1).

   Note $k[t]$ is not a field.

c. Show that it is not possible for a **field** extension to satisfy (2) but not (1).

   Hint: Zariski's lemma.

d. Show that if ℓ/k is finitely generated by (3) and algebraic, then it satisfies (1).

Theorem 2.1 ***(Field Theory Notes 11.19).***
If $L/K/F$ are field extensions, then L/F is finitely generated $\iff$ K/F and L/K are finitely generated.

See Artin-Tate Lemma, this doesn't necessarily hold for general rings.

Definition 2.1.1 (Algebraically Independent)**.**
For ℓ/k, a subset $\{x_i\} \subset \ell$ is *algebraically independent* over k if no finite subset satisfies a nonzero polynomial with k coefficients.
In this case, $k[\{x_i\}]/k$ is *purely transcendental* as a rational function field.

Theorem 2.2 ***(?).***
For ℓ/k a field extension,

a. There exists a subset $\{x_i\} \subset \ell$ algebraically independent over k such that $\ell/k(\{x_i\})$ is algebraic.
b. If $\{y_t\}$ is another set of algebraically independent elements such that $\ell/k(\{y_t\})$ is algebraic, then $|\{x_i\}| = |\{y_t\}|$.

Thus every field extension is algebraic over a purely transcendental extension. A subset as above is called a *transcendence basis*, and every 2 such bases have the same cardinality.

We have a notion of generation (similar to "spanning"), independence, and bases, so there are analogies to linear algebra (e.g. every vector space has a basis, any two have the same cardinality). There is a common generalization: matroids.

Definition 2.2.1 (Transcendence Degree)**.**
The *transcendence degree* of ℓ/k is the cardinality of any transcendence basis.

Analogy: dimension in linear algebra.

Theorem 2.3 ***(Transcendence Degree is Additive in Towers).***
If $L/K/F$ are fields then $\operatorname{trdeg}(L/F) = \operatorname{trdeg}(K/F) + \operatorname{trdeg}(L/K)$.

Theorem 2.4 ***(Bounds on Transcendence Degree).***
Let K/k be finitely degenerated, so $K = k(x_1, \cdots, x_n)$. Then $\operatorname{trdeg}(K/k) \leq n$, with equality iff K/k is purely transcendental.

Proof .
Suppose K is monogenic, i.e. generated by one element. Then $\operatorname{trdeg}(F(x)/F) = \mathbb{1}[x/F \text{ is transcendental}]$.
So the degree increases when a transcendental element is added, and doesn't change when x is algebraic.
By additivity in towers, we take $k \hookrightarrow k(x_1) \hookrightarrow k(x_1, x_2) \hookrightarrow \cdots \hookrightarrow k(x_1, \cdots, x)n) = K$ to obtain a chain of length n. The transcendence degree is thus the number of indices i such that x_i is transcendental over $k(x_1, \cdots, x_{i-1})$.

Similar to checking if a vector is in the span of a collection of previous vectors.

■

Definition 2.4.1 (Function Fields)**.**
For $d \in \mathbb{Z}^{\geq 0}$, an extension K/k is *a function field in d variables* (i.e. of dimension d) if K/k is finitely generated of transcendence degree d.

The study of such fields is birational geometry over the ground field k. $k = \mathbb{C}$ is of modern interest, things get more difficult in other fields.

The case of $d = 1$ is much easier: the function field will itself be the geometric object and everything will built from that.

Main tool: valuation theory, which will correspond to points on the curve.

2.2 Case Study: The Luroth Problem.

For which fields k and $d \in \mathbb{Z}^{\geq 0}$ is it true that if $k \subset \ell \subset k(t_1, \cdots, t_d)$ with $k(t_1, \cdots, t_d)/\ell$ finite then ℓ is purely transcendental?

Theorem 2.5 ***(Luroth).***
True for $d = 1$: For any $k \subset \ell \subset k(t)$, $\ell = k(x)$.

Theorem 2.6 ***(Castelnuovo).***
Also true for $d = 2, k = \mathbb{C}$.

Theorem 2.7 ***(Zariski).***
No if $d = 2$, $k = \bar{k}$, and k is positive characteristic. Also no if $d = 2, k \neq \bar{k}$ in characteristic zero.

Theorem 2.8 ***(Clemens-Griffiths).***
No if $d \geq 3$ and $k = \mathbb{C}$.

Unirational need not imply rational for varieties.

Exercise 2.2.
Let k be a field, G a finite group with $G \hookrightarrow S_n$ the Cayley embedding. Then S_n acts by permutation of variables on $k(t_1, \cdots, t_n)$, thus so does G. Set $\ell := k(t_1, \cdots, t_n)^G$ the fixed field, then by Artin's observation in Galois theory: if you have a finite field acting effectively by automorphisms on a field then taking the fixed field yields a galois extension with automorphism group G.

So $\mathrm{Aut}(k(t_1, \cdots, t_n)/\ell) = G$.A

a. Suppose $k = \mathbb{Q}$, and show that an affirmative answer to the Luroth problem implies an affirmative answer to the inverse galois problem for $\mathbb{Q}$.

Hint: works for any field for which Hilbert's Irreducibility Theorem holds.

b. $\ell/\mathbb{Q}$ need not be a rational function field, explore the literature on this: first example due to Swan with $|G| = 47$.

c. Can still give many positive examples using the Shepherd-Todd Theorem.

What's a global field?

2.3 Onto Business

Definition 2.8.1 (?).
For K/k a field extension, set $\kappa(K)$ to be the algebraic closure of k in K, i.e. special case of *integral closure*. If K/k is finitely generated, then $\kappa(K)/k$ is finite degree.
Here $\kappa(K)$ is called the *field of constants*, and K is also a function field over $\kappa(K)$.

In practice, we don't want $\kappa(K)$ to be a proper extension of k.

If this isn't the case, we replace considering K/k by $K/\kappa(K)$. If K/k is finitely generated, then

$$k \xhookrightarrow{\text{finite}} \kappa(K) \xhookrightarrow{\text{finitely generated}} K$$

Where we use the fact that from above, $\kappa(K)/k$ is finitely generated and algebraic and thus finite, and by a previous theorem, if K/k is transcendental then $K/\kappa(K)$ is as well, and thus finitely generated.

Thus if you have a function field over k, you can replaced k by $\kappa(K)$ and regard it as a function field over $\kappa(K)$.