

# Discussion Notes

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## 1 Discussion 1

If  $X$  is an  $F_\sigma$  set, then

$$X = \bigcup_{i=1}^{\infty} F_i \quad \text{with each } F_i \text{ closed.}$$

If  $X$  is a  $G_\delta$  set, then

$$X = \bigcap_{i=1}^{\infty} G_i \quad \text{with each } G_i \text{ open.}$$

A set  $A$  is *nowhere dense* iff  $(\overline{A})^\circ = \emptyset$  iff for any interval  $I$ , there exists a subinterval  $S$  such that  $S \cap A = \emptyset$ . This is a set that is not dense in any nonempty open set. If the closure of a subset of  $\mathbb{R}$  contains no open intervals, it will be nowhere dense.

A set  $A$  is *meager* or *first category* if it can be written as

$$A = \bigcup_{i \in \mathbb{N}} A_i \quad \text{with each } A_i \text{ nowhere dense}$$

A set  $A$  is *null* if for any  $\varepsilon$ , there exists a cover of  $A$  by countably many intervals of total length less than  $\varepsilon$ , i.e. there exists  $\{I_k\}_{k \in \mathbb{N}}$  such that  $A \subseteq \bigcup_{j \in \mathbb{N}} I_j$  and  $\sum_{j \in \mathbb{N}} \mu(I_j) < \varepsilon$ . If  $A$  is null, we say  $\mu(A) = 0$ .

Some facts:

- If  $f_n \rightarrow f$  and each  $f_n$  is continuous, then  $D_f$  is meager.

- If  $f \in \mathcal{R}(a, b)$  and  $f$  is bounded, then  $D_f$  is null.
- If  $f$  is monotone, then  $D_f$  is countable.
- If  $f$  is monotone and differentiable on  $(a, b)$ , then  $D_f$  is null.

We define the *oscillation of  $f$*  as

$$\omega_f(x) := \lim_{\delta \rightarrow 0^+} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|$$

## 1.1 Uniform Convergence

We say that  $f_n \rightarrow f$  *converges uniformly on  $A$*  if  $\|f_n - f\|_\infty = \sup_{x \in A} |f_n(x) - f(x)| \rightarrow 0$ . (Note that this defines a sequence of *numbers* in  $\mathbb{R}$ .)

This means that one can find an  $n$  large enough that for every  $x \in A$ , we have  $|f_n(x) - f(x)| \leq \varepsilon$  for any  $\varepsilon$ .

- Showing uniform convergence: find some  $M_n$ , independent of  $x$ , such that  $|f_n(x) - f(x)| \leq M_n$  where  $M_n \rightarrow 0$ .
- Negating: Fix  $\varepsilon$ , let  $n$  be arbitrary, and find a bad  $x$  (which can depend on  $n$ ) such that  $|f_n(x) - f(x)| \geq \varepsilon$ .

Example:  $\frac{1}{1+nx} \rightarrow 0$  pointwise on  $(0, \infty)$ , which can be seen by fixing  $x$  and taking  $n \rightarrow \infty$ . To see the convergence is not uniform, choose  $x = \frac{1}{n}$  and  $\varepsilon = \frac{1}{2}$ . Then

$$\sup_{x>0} \left| \frac{1}{1+nx} - 0 \right| \geq \frac{1}{2} \not\rightarrow 0.$$

Here, the problem is at small scales – note that the convergence *is* uniform on  $[a, \infty)$  for any  $a > 0$ . To see this, note that

$$x > a \implies \frac{1}{x} < \frac{1}{a} \implies \left| \frac{1}{1+nx} \right| \leq \left| \frac{1}{nx} \right| \leq \frac{1}{na} \rightarrow 0$$

since  $a$  is fixed.

## 1.2 Uniformly Cauchy

Let  $C^0([a, b], \|\cdot\|_\infty)$  be the metric space of continuous functions of  $[a, b]$ , endowed with the metric

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in [a, b]} |f(x) - g(x)|$$

This is a complete metric space, and

$$f_n \xrightarrow{U} f \iff \forall \varepsilon \exists N \ni m \geq n \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon \forall x \in X$$

$\implies$  : Use the triangle inequality.

$\impliedby$  : Find a candidate limit  $f$ : first fix an  $x$ , so that each  $f_n(x)$  is just a number. Now we can consider the sequence  $\{f_n(x)\}_{n \in \mathbb{N}}$ , which (by assumption) is a Cauchy sequence in  $\mathbb{R}$  and thus

converges. So define  $f(x) := \lim_n f_n(x)$ . Aside: we note that if  $a_n < \varepsilon$  for all  $n$  and  $a_n \rightarrow a$ , then  $a \leq \varepsilon$ .

So take  $m \rightarrow \infty$ , i.e.

$$|f_n(x) - f_m(x)| < \varepsilon \forall x \implies \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \leq \varepsilon \forall x \implies f_n \rightarrow^U f.$$

Note:  $f_n \rightarrow^U f$  does not imply that  $f'_n \rightarrow^U f'$ .

Counterexample: Let  $f_n(x) = \frac{1}{n} \sin(n^2 x)$ , which converges to 0 uniformly, but  $f'_n(x) = \cos(n^2 x)$  does not even converge pointwise.

To make this work, the theorem is that if  $f'_n \rightarrow^U g$  for some  $g$  and for at least 1 point  $x$  we have  $f_n(x) \rightarrow f(x)$ , then  $g = \lim f'_n$ .

Exercise: Let  $f(x) = \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ .

Does it converge at all, say on  $(0, \infty)$ ?

We can check pointwise convergence by fixing  $x$ , say  $x = 1$ , and noting that

$$x = 1 \implies \left| \frac{nx^2}{n^3+x^2} \right| \leq \left| \frac{n}{n^3+1} \right| \leq \frac{1}{n^2} := M_n,$$

where  $\sum M_n < \infty$ . To see why it does not converge uniformly, we can let  $x = n$ . Then,

$$x = n \implies \left| \frac{nx^2}{n^3+x^2} \right| = \frac{n^3}{2n^3} = \frac{1}{2} \not\rightarrow 0,$$

so there is a problem at large values of  $x$ .

However, if we restrict attention to  $(0, b)$  for some fixed  $b$ , we have  $x < b$  and so

$$\left| \frac{nx^2}{n^3+x^2} \right| \leq \frac{nb^2}{n^3+b^2} \leq b^2 \left( \frac{n}{n^3} \right) = b^2 \frac{1}{n^2} \rightarrow 0.$$

Note that this actually tells us that  $f$  is *continuous* on  $(0, \infty)$ , since if we want continuity at a specific point  $x$ , we can take  $b > x$ . Since each term is a continuous function of  $x$ , and we have uniform convergence, the limit function is the uniform limit of continuous functions on this interval and thus also continuous here. Checking  $x = 0$  separately, we find that  $f$  is in fact continuous on  $[0, \infty)$ .

### 1.3 Series of Functions

Let  $f_n$  be a function of  $x$ , then we say  $\sum_{n=1}^{\infty} f_n$  converges uniformly to  $S$  on  $A$  iff the partial sums  $s_n = f_1 + f_2 + \dots$  converges to  $S$  uniformly on  $A$ .

This equivalently requires that

$$\forall \varepsilon \exists N \ni n \geq m \geq N \implies |s_n - s_m| = \left| \sum_{k=m}^n f_k(x) \right| \leq \varepsilon \quad \forall x \in A.$$

Showing uniform convergence of a series: **Always use the M-test!!!** I.e. if  $|f_n(x)| \leq M_n$ , which doesn't depend on  $x$ , and  $\sum M_n < \infty$ , then  $\sum f_n$  converges uniformly.

Example: Let  $f(x) = \sum \frac{1}{x^2+n^2}$ .

Does it converge at all? Fix  $x \in \mathbb{R}$ , say  $x = 1$ , then  $\frac{1}{1+n^2} \leq \frac{1}{n^2}$  which is summable. So this converges pointwise. But since  $x^2 > 0$ , we generally have  $\frac{1}{x^2+n^2} \leq \frac{1}{n^2}$  for any  $x$ , so this actually converges uniformly.

### 1.3.1 Negating Uniform Convergence for Series

???

### 1.4 Misc

A useful inequality:

$$(1+x)^n = \sum_{k=1}^n \binom{n}{k} x^k = 1 + nx + n^2x \geq 1 + nx + nx^2 > 1 + nx$$

A summary of convergence results:

- Functions  $f_n \rightarrow^U f$ :
  - Showing:
    - \*  $M$  test. Produce a bound  $\|f_n - f\|_\infty < M_n$  which doesn't depend on  $n$ , where  $M_n \rightarrow 0$ .
  - Negating:
    - \* If  $f_n$  is continuous but  $f$  is not,
    - \* Let  $n$  be arbitrary, then find a bad  $x$  (which can depend on  $n$ ) and  $\varepsilon$  such that  $\sup |f_n(x) - f(x)| \geq \varepsilon$ .
- Series of function  $\sum f_n \rightarrow^U f$ :
  - Showing:
    - \*  $M$  test. Produce a bound  $\|f_n\|_\infty < M_n$  where  $\sum M_n < \infty$ .
  - Negating:
    - \* If each partial sum is continuous, but  $f$  is not.
    - \* If  $f_n \not\rightarrow^U 0$ .
    - \* Find a bad  $x$ ?