CRAG

The Weil Conjectures

D. Zack Garza

April 2020

Zeta Functions

Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\; n \in \mathbb{Z}^{\geq 2}$$

Definition (Zeta Function)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic* variety $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \mathbf{deg}\, f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I} = (i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

Point Counts

- For a fixed variety X, we can consider its $\mathbb{F}$-points $X(\mathbb{F})$.
  - Note that $\#X(\mathbb{F}) < \infty$ is an integer
- For any $L/\mathbb{F}$, we can also consider $X(L)$
  - In particular, we can consider $X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
  - We again have $\#X(\mathbb{F}_{q^n}) < \infty$ and are integers for every such n.
- So we can consider the sequence

$$[N_1, N_2, \cdots, N_n, \cdots] := [\#X(\mathbb{F}),\ \#X(\mathbb{F}_{q^2}), \cdots,\ \#X(\mathbb{F}_{q^n}), \cdots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \cdots.$$

Why Generating Functions?

Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of F: we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n F(z)\bigg|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}\, dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z}\, dz = N_n.$$

The latter form is very amenable to computer calculation.

Why Generating Functions?

An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \longrightarrow \mathbb{C}$ – in nice situations, we can hope for a closed-form representation.

A useful example: by integrating a geometric series we can derive

$$
\begin{aligned}
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n && \left(= 1 + z + z^2 + \cdots\right) \\
\implies \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \int z^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\
\implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} && \left(= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).
\end{aligned}
$$

For completeness, also recall that

$$
\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}
$$

In particular, we have some choice in how to encode the sequence $[1, \ 1, \cdots]$:

> *Note: a common other choice is associating an exponential generating function,* $F(z) = \sum \frac{z^n}{n!}$.