# Problem Sets

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January 24, 2020

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# 1 1.1

## 1.1 a

If $M \in \mathcal{O}$ and $[\lambda] = \lambda + \Lambda_r$ is any coset of $\mathfrak{h}^\vee/\Lambda_r$, let $M^{[\lambda]}$ be the sum of weight spaces $M_\mu$ for which $\mu \in [\lambda]$. Prove that $M^{[\lambda]}$ is a $U(\mathfrak{g})$-submodule of $M$ and that $M$ is the direct sum of finitely many such submodules.

## 1.2 b

Deduce that the weights of an indecomposable module $M \in \mathcal{O}$ lie in a single coset of $\mathfrak{h}^\vee/\Lambda_r$.

# 2 1.3*

Show that $M(\lambda)$ has the following property: for any $M \in \mathcal{O}$,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \mathrm{Hom}_{U(\mathfrak{g})}\left(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right) \cong \mathrm{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right),$$

where $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}}$ is the restriction functor.

Hint: use the universal mapping property of tensor products.

## 3 Relevant information (?):

$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ where $\mathfrak{b} \leq \mathfrak{g}$ is a fixed Borel subalgebra corresponding to a choice of positive roots, and $C_\lambda$ is the 1-dimensional $\mathfrak{b}$-module defined for any $\lambda \in \mathfrak{h}^\vee$ by the fact that $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$ and thus $\mathfrak{n} \curvearrowright \mathfrak{h}$ can be taken to be a trivial action.

The induction functor is given by $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\,\cdot\,) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\,\cdot\,)$.

The restriction functor is given by $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}}(\,\cdot\,) =?$