# **Homological Algebra Problem Sets**

Problem Set 3

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# 1 Wednesday, February 17

Problem 1.0.1 (Prove Corollary 2.3.2)

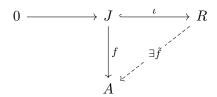
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible if and only if for every  $r \neq 0 \in R$  and every  $a \in A$ , we have a = br for some  $b \in A$ .

#### Solution:

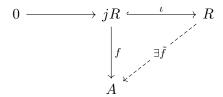
Note: we'll assume R is commutative.

 $\implies$ : Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:



#### Link to Diagram

Since R is a PID, we have J = jR for some  $j \in \overline{R}$ , so it suffices to produce lifts of the following form:



#### Link to Diagram

Consider  $f(j) \in A$ . Since A is divisible, we have A = jA, so we can write  $f(j) = j\mathbf{a}'$  for some  $\mathbf{a}' \in A$ . Using R-linearity and the fact that j can not be a zero divisor and is thus left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

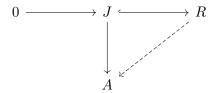
$$\tilde{f}: R \to A$$

$$1_R \mapsto \mathbf{a}'.$$

and extending by R-linearity yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since  $\iota(1_R) = 1_R$ .

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 $\Leftarrow$ : Suppose  $A \in R$ -Mod is injective, where by Baer's criterion we equivalently have a lift of the following form for every  $J \subseteq R$ :

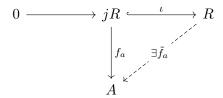


#### Link to Diagram

Let  $j \in R$  be a nonzero element that is not a zero-divisor, we then want to show that A = jA, i.e. that for every  $\mathbf{a} \in A$ , there is a  $\mathbf{a}' \in A$  such that  $\mathbf{a} = j\mathbf{a}'$ . Fixing  $\mathbf{a} \in A$ , define a map  $f_a : J \to A$  in the following way: for  $x \in J$ , use the fact that  $\langle j \rangle \coloneqq jR$  to first write x = jr for some  $r \in R$ , and then set  $f_a(x) = f_a(jr) \coloneqq r\mathbf{a}$ . To summarize, we have

$$f_a: J = jR \to R$$
  
 $x = jr \mapsto r\mathbf{a}.$ 

By injectivity, we can take the inclusion  $jR \hookrightarrow R$  and get a lift:



#### Link to Diagram

We can now use the fact that

$$r\mathbf{a} = f_a(jr)$$

$$= \tilde{f}_a(\iota(jr))$$

$$= \tilde{f}_a(jr)$$

$$= jr\tilde{f}_a(1_R) \qquad \text{using $R$-linearity and $j,r \in R$}$$

$$= rj\tilde{f}_a(1_R) \qquad \text{since $R$ is commutative}$$

$$\implies \mathbf{a} = j\tilde{f}_a(1_R) \in jA,$$

where in the last step we have canceled an r on the left, using the fact a PID is a domain and thus has no nonzero zero divisors, and in particular r is not a zero divisor and is thus left-cancelable. So in the definition of divisibility, we can take

$$\mathbf{a}' \coloneqq \tilde{f}_a(1_R),$$

and letting a range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups) Calculate  $\operatorname{Ext}_{\mathbb{Z}}^{i}(\mathbb{Z}/p,\mathbb{Z}/q)$  for distinct primes p,q.

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*Problem* 1.0.3 (Weibel 2.3.2)

For  $A \in \mathbf{Ab}$ , define  $I(A) := \bigoplus_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ , and let  $e_A : A \to I(A)$ . Show that  $e_A$  is

injective.

Hint: if  $a \in A$ , find a map  $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$  with  $f(a) \neq 0$  and extend this to a map  $f' : A \to \mathbb{Q}/\mathbb{Z}$ .

*Problem* 1.0.4 (Weibel 2.4.2)

If  $U: \mathcal{B} \to \mathcal{C}$  is an exact functor, show that

$$U(L_iF) \cong L_i(UF)$$
.

Problem 1.0.5 (Weibel 2.4.3)

If  $0 \to M \to P \to A \to 0$  is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} FM$$
  $i \ge 2.$ 

Show that  $L_{m+1}F(A)$  is the kernel of  $F(M_m) \to F(P_m)$ . Conclude that if  $P \to A$  is an F-acyclic resolution of A, then  $L_iF(A) = H_i(F(P))$ .

*Problem* 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

- a. A is a projective R-module.
- b.  $\operatorname{Hom}_R(\,\cdot\,,A)$  is an exact functor.
- c.  $\operatorname{Ext}_R^{i\neq 0}(A,B)=0$  and for all B, i.e. A is  $\operatorname{Hom}_R(\,\cdot\,,B)$ -acyclic for all B.
- d.  $\operatorname{Ext}_R^1(A,B)$  vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)

Show that colim is left adjoint to  $\Delta$ , and conclude that colim is right-exact when when  $\mathcal{A}$  is abelian and colim exists. Show that the pushout, i.e.  $\bullet \leftarrow \bullet \rightarrow \bullet$ , is not an exact functor on  $\mathbf{Ab}$ .