Representation Stability in the Braid Group (Dan Margalit)

The Braid group: Homological stability

$$H_i(B_n;\mathbb{Q}) = \begin{cases} \mathbb{Q}, & i=1 \\ 0, & \text{else} \end{cases}$$

Total winding number

The Pure Braid group

$$H_1(PB_n;\mathbb{Q}) = \mathbb{Q}^{\binom{n}{2}}$$

Pairwise winding number, "instability"

Church-Farb: Representation of S_n

$$1 \to PB_n \to S_n \to B_n \to 1$$

$\to H_1 \cong V_0 \oplus V_1 \oplus V_2$ irreducible reps

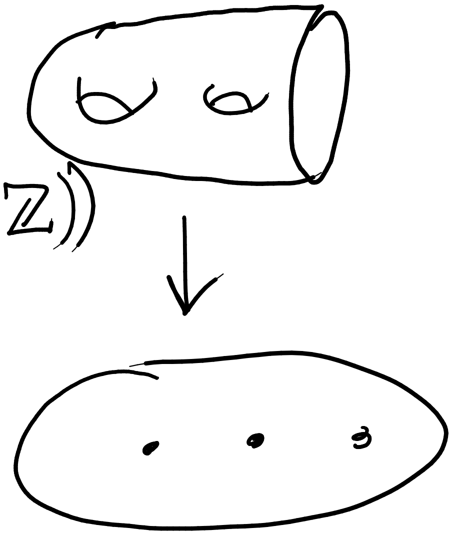
"Representation stability", big in 2010

Today: The Level 4 Braid group

$B_n = MCG(\text{Punctured disc})$

gen'd by d^2 for d = dehn twist

Burau rep: $B_n \to Gl_n(\mathbb{Z}[t]) \to Gl_n(\mathbb{Z})$



$\ker(B_n \to Gl_n(\mathbb{Z}) \to Gl_n(\mathbb{Z}/m\mathbb{Z}))$

$:= B[m]$

$B[1] = B_n$, $B[2] = PB_n$

$B[4] = PB_n^2$ (= ker x2 map)

$= \pi_1(\underbrace{X_n[4]}_{})$

moduli space of open hyperelliptic curves with level 4 structures

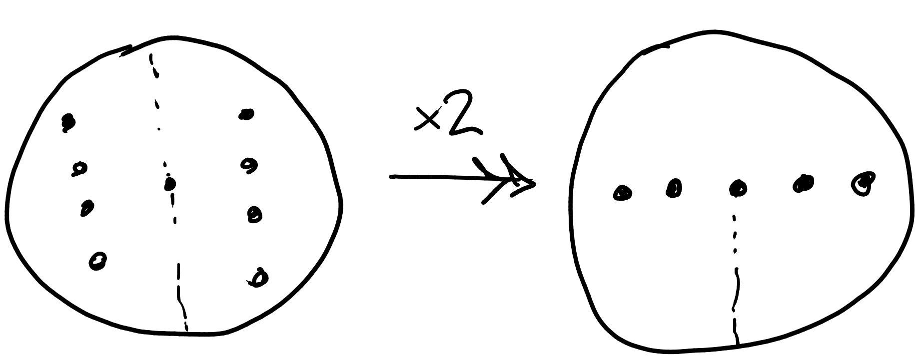
Also = a Kummer cover ⇝ algebraic variety with equations

Q: Does B[4] enjoy rep. stability?

Note $B[0] = \pi_1$ of a branch locus of a period mapping to a h.e. Torelli group.

Open Q: is it finitely generated?

The new idea: lift to covers



The tie to rep. stability

The group that acts is

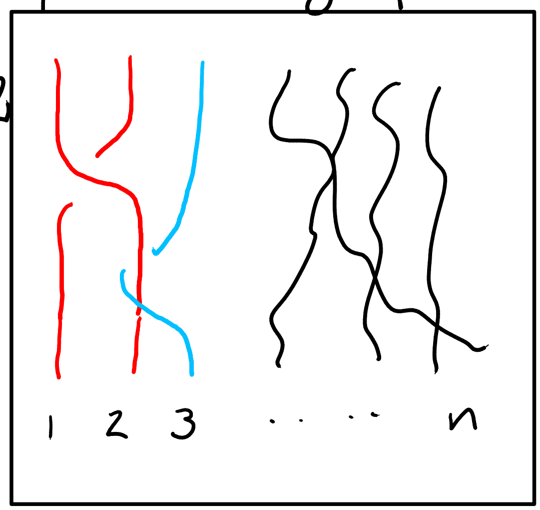
$$Z_n = B_n / B_n[4]$$

$$1 \to \mathbb{Z}/2\mathbb{Z}^{\binom{n}{2}} \to Z_n \to S_n \to 1$$

Paper develops rep. theory of this group.

Get a hom $\to \mathbb{Z}/2\mathbb{Z}$ by counting the winding of red/blue

Similar constructions yield irreducible reps.



<u>Theorem</u>: $H_1(B_n[4]; \mathbb{C}) = \bigoplus V_i$ which don't depend on n (uniform rep. stability)

See slides for applications and further open questions.

Atiyah-Kodaira bundles

$$\Sigma_g \to E \to B$$

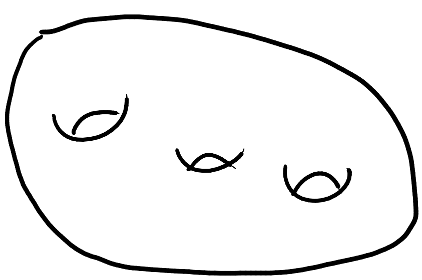
manifold

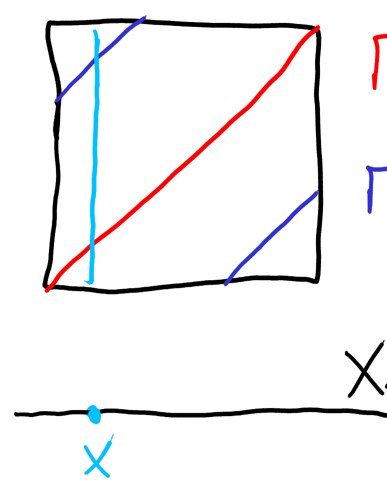
Structure gp. = Homeo Σ_g (or Diff)

Ex

Mapping Torus, glued by some homeo

A-K: Works for any X with a $\mathbb{Z}/m\mathbb{Z}$ action. Ex





4-mfd

$$\underbrace{\Sigma_6}_{\text{Branched cover}} \to E \to \underbrace{\Sigma_{129}}_{\text{comes from homology cover}}$$

Props

- Holomorphic
- $\kappa_1 \neq 0$ ($\mathrm{Sig}\, E \neq 0$)

2015
- Does not admit Riemannian metric that is non-positively curved (even if base/fibers do)
- Conjecturally not flat (Foliation of total space)

$$\begin{tikzcd} & \nexists & \mathrm{Diff}(\Sigma_g)^{\mathbb{Z}/m\mathbb{Z}} \arrow[d] \\ & \pi_1 B \arrow[ur, dashed] \arrow[r] & \mathrm{Mod}(\Sigma_g) \end{tikzcd}$$

(stabilizer)

- Multiple fiberings
  ~ generally the case with surface bundles over the circle

Monodromy - Arithmeticity

Earle - Eells: If $\chi(S) < 0$, for any B,

$$\left\{ \begin{tikzcd} S \arrow[r] & E \arrow[d, "f"] \\ & B \end{tikzcd} \right\} \leftrightarrow \left\{ \begin{array}{c} \pi_1 B \to \mathrm{Mod}(S) \\ \text{homeos} \end{array} \right\} \Big/ \text{conj}$$

$\longrightarrow$ monodromy

$\longleftarrow$ miracle!

Monodromy – Topology dictionary

Thurston: $S = \Sigma_g$, $B = S^1$ then there is exactly 1 fibering iff $H^1(\Sigma_g;\mathbb{Q})^f = 0$

Salter: if $B = B^2$, replace f with $\pi_1 B$ and you get just the converse

Chen: Can show AK admits exactly two fiberings iff $(H^1\Sigma_g)^{\pi_1 B} \cong H_1 X$

Deligne: $E \longrightarrow B$ quasiprojective

holomorphic

$\Rightarrow \Gamma_E \subseteq Sp(2g,\mathbb{Z})$ has Zariski closure G semisimple

monodromy gp

Q: Is $[\Gamma_E : G(\mathbb{Z})]$ $\underbrace{\text{finite}}_{\text{"arithmetic"}}$ or $\underbrace{\text{infinite}}_{\text{"thin"}}$?

Some known ranges, in general difficult to ascertain.

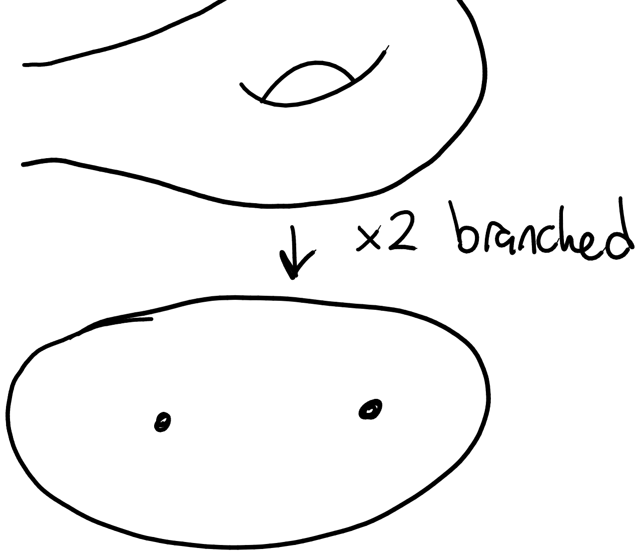
__Ex__

$$\begin{tikzcd} \Sigma_g \arrow[r] & E \arrow[d] \\ & \mathrm{Conf}(\mathbb{C}, k) \end{tikzcd}$$

Braid monodromies

See Deligne-Mostow

Monodromy $\quad \begin{array}{c} \pi_1 \mathrm{Conf}(\mathbb{C}, k) \\ \| \\ B_k \end{array} \longrightarrow Sp(2g, \mathbb{Z})$



__Theorem__: $m \geq 2$, $n \geq 5$ $\rightarrow$ The monodromy gp of the surface bundle

$$\begin{tikzcd} \Sigma_g \arrow[r] & E_{m,n} \arrow[d] \\ & B \end{tikzcd}$$

is arithmetic

with identifiable Zariski closure.

Lift pt-pushing maps to MCGs

Sketch of proof: 3 steps

1) The monodromy of the $\underbrace{\text{AK bundle}}$ factors

only on fin. index subgps

$$E_{h,m} \to B$$

$$p: \mathrm{Mod}(\Sigma_h, *) \dashrightarrow \mathrm{Mod}(\Sigma_g)^{\mathbb{Z}/m\mathbb{Z}} \to Sp(2g, \mathbb{Z})^{\mathbb{Z}/m\mathbb{Z}}$$

$$\triangledown$$

$$\pi_1 \Sigma_h = \mathrm{Ker}[\mathrm{Mod}(\Sigma_h, *) \to \mathrm{Mod}\,\Sigma_h]$$

Let $\Gamma_E = \mathrm{Im}\left(p\big|_{\pi_1 \Sigma_h}\right) < Sp(2g, \mathbb{Z})^{\mathbb{Z}/m\mathbb{Z}}$

2) Show Γ_E is arithmetic by showing it contains enough unipotents. hard part

(Generalizes previous results to branched covers

3) Margalis normal subgroups. ∎

Sometimes false!

Hyperbolicity in automorphisms of free groups (Caglar Uyanik)

Main Theorem: Let F_n be the free group, then

$H < Out(F_n)$ contains an <u>atoroidal</u> subgroup

or $\exists H_0 < H$, $1 \neq g \in F_n$ s.t. $H_0[g] = [g]$

Nielson-Thurston classification

$f \in Mod(S) = Homeo^+(S) / isotopy$ ($+$: orientation-preserving)

1) f is periodic,

2) reducible, or

3) Pseudo-Anosov

$M_f = S \times I / (x,1) \sim (f(x), 0)$

for some $f \in [S,S] \in Top$

M_f is hyperbolic iff f is pseudo-Anosov

These are generic; take a random walk on Mod(S) and you will land on one with prob. 1.

<u>Thm</u> (Ivano) $H < Mod(S)$ contains a pA elt <u>or</u> contains a fin index s.g. that fixes a curve $\gamma \subseteq S$

Why study Aut F_n via Mod(S)?

$$1 \to \mathrm{Inn}\, F_n \to \mathrm{Out}\, F_n \to \mathrm{Aut}\, F_n \to 1$$

$\psi \in \mathrm{Aut}\, F_n$ is atoroidal iff no power of ψ fixes a conjugacy class

$\varphi \in \mathrm{Out}\, F_n$ is fully irreducible iff no power of φ fixes a free factor

Thm: φ is both iff $\exists S$, $\pi_1 S = F_n$, ∂S has 1 component, and $\varphi = f_*$ where $f: S \circlearrowleft$ is pA.

Define for subgroups iff no fin index s.g. fixes free factors.

Thm H is fully irreducible $\Rightarrow$ H is atoroidal or H is geometric, i.e.

$$\exists S : \pi_1 S = F_n,\ H < i(\mathrm{Mod}\, S) < \mathrm{Out}\, F_n$$

induced by $f: S \circlearrowleft$

Theorem

"Mapping Torus" for $\mathrm{Aut}\,F_n$

$$M_\varphi = \langle x_1, \dots, x_n, t \mid t\varphi(x_i)t^{-1} = x_i \rangle$$

is δ-hyperbolic iff φ is atoroidal.

Well known: $1 \to F_n \to \mathrm{Aut}\,F_n \xrightarrow{q} \mathrm{Out}\,F_n \to 1$

$$\cup\!\!|$$

$$1 \to F_n \to q^{-1}(\Gamma) \to \Gamma \to 1$$

$G := E_\Gamma$, hyperbolic extensions of free groups

Q: When is E_Γ δ-hyperbolic?

- E_Γ hyperbolic $\Rightarrow$ Γ is purely atoroidal

↳ Conjecture: Converse holds

↳ Dowdall-Taylor: Add condition

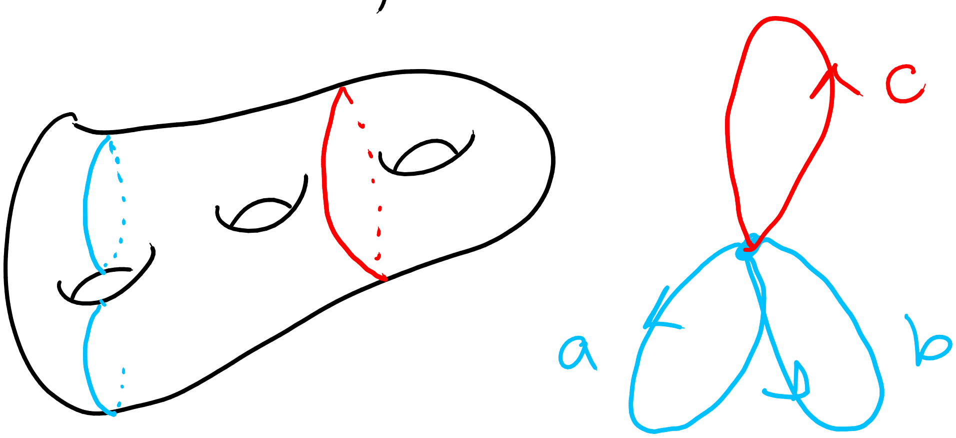
$\Gamma \hookrightarrow FF_n$ a q.i. embedding, then yes.

ie Γ is convex & cocompact

<u>Theorem</u> There are many hyp. extensions, eg free <u>non</u>convex cocompact.

Not pA → preserves a collection of curves

No reduction theory



Use the Aut

$a \mapsto aba$
$b \mapsto ba$
$c \mapsto cabba^2$

Then a,b are preserved but c mixes.

<u>Fact</u> There is no Out F_n graph whose loxodromic isometries yield exactly the atoroidal elts.

What goes into the proof?

1) Handel - Mosher decomposition

Generally need more than just hyperbolic geom.

Look at space of currents, $\mathrm{Curr}\, F_n$, whose projectivization is compact where scalar multiples of conjugacy classes are dense,

+ Do some ergodic theory.

Strong Tits' alternative:

$\varphi \in H$ p.A. $\Rightarrow$ H is virtually cyclic <u>or</u>

$\exists \Gamma < H$, $\Gamma \cong F_2$ purely p.A.

Can generalize to fully irreducible

(Atoroidal ~ fully irreducible/p.A.)

<u>Thm</u>: If $\varphi \in H$ is atoroidal then

$\exists \Gamma < H$, $\Gamma \simeq F_2$ purely atoroidal iff

$H|_{\mathcal{L}}$ to each minimal H-invariant free factor is <u>not</u> virtually cyclic. $\blacksquare$

Counting hyperbolic manifolds which bound geometrically (Michelle Chu)

An n-dim mfd M bounds an $n+1$-dim mfd N iff $\partial N \cong M \in Top$

Some classical results

- Every closed compact 3-mfd bounds
- Every closed hyperbolic 4-mfd bounds

  (generally, restriction on signature)

Def A fin vol hyperbolic n-mfd M bounds geometrically if $M \cong \partial N$ isom. with totally geodesic boundary.

Ratcliffe-Tschantz construct an explicit example of M a 3-mfd.

Q: Does every mfd geom. bound? A: No.

Obstructions

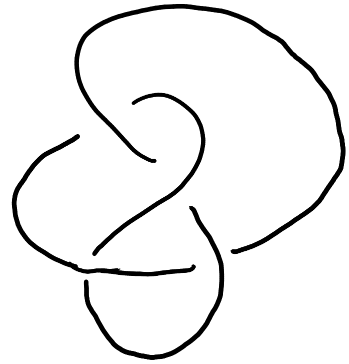
2000 · If M^3 bounds, $\eta(M^3) \in \mathbb{Z}$ (eta invar.)

'92 Thm: If M is a 1-cusped hyp 3-mfds, $\{\eta(M_{p/q})\} \subseteq \mathbb{R}$ is dense

So most do not bound.

'18 · Cusp field obstruction

1^{st} cusped

$S^3 -$  Figure 8 complement

don't bound

$S^3 -$  Twist knots

some twists

2001 <u>Thm</u>: There exists an M^n for every n that bounds

Arithmetic hyperbolic mfds of simple type

Let Q be a quadratic form over $\mathbb{Z}$ with signature $(n,1) \longrightarrow O^+(Q,\mathbb{R}) \cong \mathrm{Isom}\,\mathbb{H}^n$

Then $\Gamma := O^+(Q,\mathbb{Z})$ is a discrete subgroup

so $\mathbb{H}^n/\Gamma$ is a fin. vol. orbifold

(this works with other number fields)

<u>Def.</u> M is <u>arith of simplest type</u> if it is commensurable to $\mathbb{H}^n/O^+(Q,\mathcal{O}_k)$

ring of integers in some # field k

2010 <u>Thm</u> For $n \geq 3$,

$$\left|\{\text{arith. hyp } n\text{-mfds of vol} \leq x\}\right| \sim x^x$$

super exponential

Let $FV_n(X) = \#\{$ " which bound $\}$

$C_n(X) = \#\{$ " + compact $\}$

18-19 Thm: For $3 \le n \le 19$ & $n = 21$, $FN_n(X) \sim x^x$,

$3 \le n \le 8$ $\quad C_n(X) \sim X^X$.

Proof: look at hyperbolic Coxeter polytopes

These dims are where we have explicit examples.

Want to embed nonorientable mfds M

totally geodesically in some N s.t.

$\tilde{N}$ (the orientation cover) bounds geom.

Thm: M arith. of simplest type then M embeds

totally geodesically in some N^{n+1} or its 2x-cover.

Take Γ an arithmetic reflection group (coxeter)

$\longrightarrow \Gamma = \langle \{r_i\}_{i \in S} \rangle$ reflections w/ $|S| < \infty$

Want

- $S' \subseteq S$ with $\Gamma_{S'}$ virtually free
- $\gamma \in \Gamma$ orient. reversing, no letters in S'
- Γ' torsion free, $\gamma \in \Gamma'$, $[\Gamma:\Gamma_S] < \infty$

$$\Gamma \longrightarrow \Gamma_{S'}$$

$$r_i \mapsto \begin{cases} 1, & i \in S' \\ r_i, & \text{else} \end{cases}$$

$$\gamma \mapsto 1$$

(Virtually free $\longrightarrow$ lots of subgroups, can count by index)

Congruence subgroups: If $\Gamma < Gl_n(\mathbb{Z})$,

$$\Gamma \xrightarrow{p_m} Gl_n(\mathbb{Z}/m\mathbb{Z})$$

and $K = \ker \varphi_m$ satisfies $[\Gamma : K] < \infty$

and K is torsion free

Trick

For $\varphi_i : \Gamma \to G_i$ finite with torsion-free kernel, then if $\gamma \in \Gamma$ s.t. for $|\varphi_1(\gamma)|$, $|\varphi_2(\gamma)|$ and any prime appearing in both w/ different powers

Then $\varphi_1 \times \varphi_2^{-1}(\langle \varphi_1(\gamma), \varphi_2(\gamma) \rangle)$ is torsion free.

Cubical dimension of groups (Kasia Kankiewicz)

The geometric dim of G is the dim of a contractible CW complex upon which G acts freely

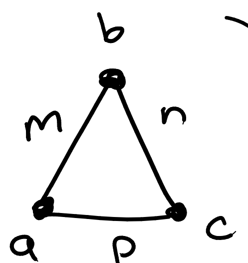
Can generalize to Cat(0) dim on Cat(0) complexes by semisimple isom.

Plus cubical dimension by free actions

We have

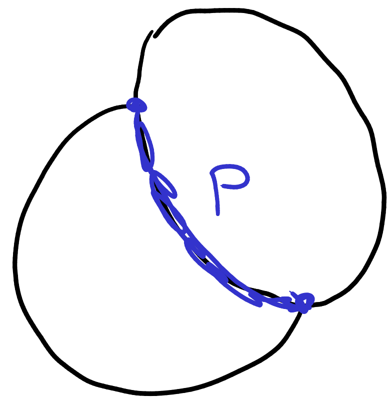
geom dim $\leq$ Cat(0) dim $\leq$ cubical dim

Certain Artin groups with dim $< \infty$

$A(m,n,p) = A\Bigg($  $\Bigg) = \langle a,b,c \mid (ab)^m = (ba)^m \rangle$

<u>Thm</u> For every n $\exists$ a fin. presented $C'(1/6)$ gp with cubical dim $> n$

$G = \langle S \mid R \rangle$ is $C'(1/6)$ if in the Cayley graph $Cay(G,S)$, $|P| \leq \frac{1}{6}|R|$ where P is the overlap



- $C'(1/6)$ groups are hyperbolic
- The 2-complex is aspherical $\Rightarrow$

  geom dim $G \leq 2$
- The Cayley complex can be folded into a Cat(-1) complex $\rightarrow$ Cat(0) dim ≤ 2
- They act properly and cocompactly on Cat(0) complexes

Ex There exists a group G s.t.

- geom dim $= 2$
- Cubical dim $= \infty$
- G acts freely on a locally finite Cat(0) complex

For $G = \langle S \mid R \rangle$, define

$$\omega(G, S) = \lim_{n \to \infty} \sqrt[n]{\underbrace{|B_S(G)|}_{\text{in Cayley graph}}}$$

Say growth is uniformly exponential if $\inf \omega(G, S) > 1$.

Thm: If G acts freely on a 2-dim Cat(0) complex then either

- G is virtually abelian
- G has uniformly exponential growth

Main lemma

Let a, b be isometries of X, a 2-dim Cat(0) cube complex then $\exists$ a pair u, v, one of length ≤ 10, that generate a free subgroup that stabilizes a flat?.

Can construct a $C'(1/6)$ group with cubical dim > 2: $G = \langle a, b \mid R_{u,v} \ \forall u, v \rangle$

where $R_{uv} = u^{\alpha_1} v^{\beta_1} u^{\alpha_2} v^{\beta_2} \cdots$

↳ No $\langle u, v \rangle$ is free

↳ Can choose α_i, β_i to get $G = C'(1/6)$

$\Rightarrow$ G is virtually abelian

$\Rightarrow$ G does not act freely on a 2d Cat(0) complex

Main Lemma

Let a, b be hyperbolic isometries of

an n-dim cubical Cat(0) complex.

- $\exists$ pair $\{u, v\}$, $|u|, |v| \leq C(\sim)$ that generate a Free ~~~
- $\exists$ pair that stabilize a hyperplane
- $\langle a^c, b^c \rangle$ stabilize a flat

Quasi-isometric rigidity of 2-dimensional Artin groups (Jingyin Huang)

Artin groups, 2-dim

$\Gamma = (\{v_i\}, \{e_i\})$

$A_\Gamma = \langle \{v_i\} \mid (v_i v_j)^{m_{ij}/2} = (v_j v_i)^{m_{ij}/2} \quad \forall\ e_\ell = (v_i, v_j) \rangle$

A f.g. gp is <u>strongly rigid</u> if any self quasi-isometry is uniformly close to an autom.

Goal: Find such 2-dim Artin gps

Right-angled groups ($m_{ij} = 2$) <u>not</u>

Artin (3,3,3) <u>is</u>. (commensurate to $S^2 - \{*\}^5$)

- Dehn twist flats
- Autos of curve complex induced by a mapping class

An n-dim quasi-flat is a map $\mathbb{E}^n \to X$,

For A_Γ,

- $n \geq 3 \Rightarrow$ no n-qfs
- Any quasi-isom preserves 2-qfs

Examples of qfs in A_Γ

- Any $\mathbb{Z}^2$ subgp
- Any $\underset{\ell_1 \in}{F_n} \times \underset{\ell_2 \in}{F_m}$ subgp, take $\ell_1 \times \ell_2$

<u>Thm</u>: $\exists$ finitely many subgps $\{H_i\}$ s.t.

- $H_i \cong F_n \times F_m \quad \forall i$
- $Q \subseteq \dot{\bigcup\limits_{i \in J}} H_i \quad , |J| < \infty$

Can define Dehn twists, curve complexes, show hyperbolicity, etc

An Artin group is rigid when .... isom wrt the word metric

Large type: $m_{ij} \geq 3$

Thm

A_Γ is rigid (if large type & triangle free) if

- $A_\Gamma \not\sim F_n \times \mathbb{Z}$ ($\sim$: commensurable)
- $|\mathrm{Out}\, A_\Gamma| < \infty$

$\boxed{\text{Open Q: Are 2-dim Artin gps Cat(0)?}}$

↳ There are 2d gps that can not act geometrically on any 2d Cat(0) space

1) Take presentation complex of A_Γ, then take universal cover. Metrize 2-cells as flat polygons.

⇝ Yields a tiling of a Cat(0) plane by triangles; complicated due to some irrational angles.