Full Notes

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1 Friday January 10

Recall that $\mathbb{C}$ is a field, where $z = x + iy \implies \overline{z} = x - iy$, and if $z \neq 0$ then $z^{-1} = \overline{z}/|z|^2$.

Lemma (Triangle Inequality: $|z + w| \leq |z| + |w|$

Proof:

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\overline{w}| - \Re z\overline{w}) \geq 0.$$

Lemma (Reverse Triangle Inequality): $||z| - |w|| \leq |z - w|$.

Proof:

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

Claim: $(\mathbb{C}, |\cdot|)$ is a normed space.

Definition: $\lim z_n = z \iff |z_n - z| \to 0 \in \mathbb{R}$.

Definition: A disc is defined as $D_r(z_0) := \left\{ z \in \mathbb{C} \,\middle|\, |z - z_0| < r \right\}$, and a subset is open iff it contains a disc. By convention, D_r denotes a disc about $z_0 = 0$.

Definition: $\sum\limits_k z_k$ converges iff $S_N := \sum\limits_{|k|<N} z_k$ converges.

Note that $z_n \to z$ and $z_n = x_n + iy_n$, and

$$|z_n - z| = \sqrt{(x_n - x)^2 - (y_n - y)^2} < \varepsilon \implies |x - x_n|, |y - y_n| < \varepsilon.$$

Since $\mathbb{R}$ is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

Note: This is useful precisely when you don't know the limiting term.

Note that $\sum\limits_k z_k$ thus converges if $\left| \sum\limits_{k=m}^{n} z_k \right| < \varepsilon$ for m, n large enough, so sums converges iff they have small tails.

Definition: $S_N = \sum\limits^{N} z_k$ converges absolutely iff $\tilde{S} := \sum\limits^{N} |z_k|$ converges.

Note that the partial sums $\sum\limits^{N} |z_k|$ are monotone, so $\tilde{S}_N$ converges iff the partial sums are bounded above.

Definition: A sum of the form $\sum\limits_{k=0}^{\infty} a_k z_k$ is a power series.

Examples:

$$\begin{aligned} \sum x^k &= \frac{1}{1-x} \\ \sum (-x^2)^k &= \frac{1}{1+x^2}. \end{aligned}$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at $x = 1$ and the second as a pole at $x = i$.

2 Monday January 13th

Recall that $\sum z_k$ converges iff $s_n = \sum\limits_{k=1}^{n} z_k$ converges.

Lemma: Absolute convergence implies convergence.

The most interesting series: $f(z) = \sum a_k z^k$, i.e. power series.

Divergence lemma: If $\sum z_k$ converges, then $\lim z_k = 0$.

Corollary: If $\sum z_k$ converges, $\{z_k\}$ is uniformly bounded by a constant $C > 0$, i.e. $|z_k| < C$ for all k.

Proposition: If $\sum a_k z_k$ converges at some point z_0, then it converges for all $|z| < |z|_0$.

The inequality is necessarily strict. For example, $\sum \frac{z^{n-1}}{n}$ converges at $z = -1$ (alternating harmonic series) but not at $z = 1$ (harmonic series).

Proof: Suppose $\sum a_k z_1^k$ converges. The terms are uniformly bounded, so $\left|a_k z_1^k\right| \leq C$ for all k. Then we have

$$|a_k| \leq C/|z_1|^k$$

, so if $|z| < |z_1|$ we have

$$\left|a_k z^k\right| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k.$$

So if $|z| < |z_1|$, the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting $r = |z|/|z_1|$, we have

$$\sum \left|a_k z^k\right| \leq \sum c r^k = \frac{c}{1-r},$$

and so we have absolute convergence.

■

Exercise (future problem set): Show that $\sum \frac{1}{k} z^{k-1}$ converges for all $|z| = 1$ except for $z = 1$. (Use summation by parts.)

Definition The radius of convergence is the real number R such that $f(z) = \sum a_k z^k$ converges precisely for $|z| < R$ and diverges for $|z| > R$. We denote a disc of radius R centered at zero by D_R.

If $R = \infty$, then f is said to be *entire*.

Proposition: Suppose that $\sum a_k z^k$ converges for all $|z| < R$. Then $f(z) = \sum a_k z^k$ is continuous on D_R, i.e. using the sequential definition of continuity, $\lim_{z \to z_0} f(z) = f(z_0)$ for all $z_0 \in D_R$.

Recall that $S_n(z) \to S(z)$ uniformly on Ω iff $\forall \varepsilon > 0$, there exists a $M \in \mathbb{N}$ such that $n > M \implies |S_n(z) - S(z)| < \varepsilon$ for all $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample: $f_n(x) = x^n$ on $[0,1]$; then $f_n \to \delta(1)$. Note that it uniformly converges on $[0, 1-\varepsilon]$ for any $\varepsilon > 0$.

Exercise: Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

Proof of proposition: Write $f(z) = \sum_{k=0}^{N} a_k z^k + \sum_{N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$. Note that if $|z| < R$, then there exists a T such that $|z| < T < R$ where $f(z)$ converges uniformly on D_T.

Check!

We need to show that $|R_N(z)|$ is uniformly small for $|z| < s < T$. Note that $\sum a_k z^k$ converges on D_T, so we can find a C such that $\left|a_k z^k\right| \leq C$ for all k. Then $|a_k| \leq C/T^k$ for all k, and so

$$\begin{aligned}
\left|\sum_{k=N+1}^{\infty} a_k z^k\right| &\leq \sum_{k=N+1}^{\infty} |a_k||z|^k \\
&\leq \sum_{k=N+1}^{\infty} (c/T^k) s^k \\
&= c\sum |s/T|^k \\
&= c\frac{r^{N+!}}{1-r} && = C\varepsilon_n \to 0,
\end{aligned}$$

which follows because $0 < r = s/T < 1$.

So $S_N(z) \to f(z)$ uniformly on $|z| < s$ and $S_N(z)$ are all continuous, so $f(z)$ is continuous.

There are two ways to compute the radius of convergence:

- Root test: $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$.
- Ratio test: $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$.

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function $f : \Omega \to \Omega$
2. Complex-differentiability: For $\Omega \subset \mathbb{C}$ open and $z_0 \in \Omega$, there exists $\varepsilon > 0$ such that $D_\varepsilon = \left\{z \;\middle|\; |z - z_0| < \varepsilon\right\} \subset \Omega$, and f is **holomorphic** (complex-differentiable) at z_0 iff

$$\lim_{h\to 0} \frac{1}{h}(f(z_0 + h) - f(z_0))$$

exists; if so we denote it by $f'(z_0)$.

Example: $f(z) = z$ is holomorphic, since $f(z+h) - f(z) = z + h - z = h$, so $f'(z_0) = \frac{h}{h} = 1$ for all z_0.

Example: Given $f(z) = \overline{z}$, we have $f(z+h) - f(z) = \overline{h}$, so the ratio is $\frac{\overline{h}}{h}$ and the limit doesn't exist. Note that if $h \in \mathbb{R}$, then $\overline{h} = h$ and the ratio is identically 1, while if h is purely imaginary, then $\overline{h} = -h$ and the limit is identically -1.

We say f is holomorphic on an open set Ω iff it is holomorphic at every point, and is holomorphic on a closed set C iff there exists an open $\Omega \supset C$ such that f is holomorphic on Ω.

If f is holomorphic, writing $h = h_1 + ih_2$, then the following two limits exist and are equal:

$$\begin{aligned}
\lim_{h_1 \to 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\
\lim_{h_2 \to 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i}\frac{\partial f}{\partial y}(x_0, y_0) \\
\implies \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y}.
\end{aligned}$$

So if we write $f(z) = u(x, y) + iv(x, y)$, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Big|_{(x_0,y_0)},$$

and equating real and imaginary parts yields the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
\iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\end{aligned}$$

The usual rules of derivatives apply:

1. $(\sum f)' = \sum f'$

    Proof: Direct.

2. $(\prod f)' =$ product rule

    Proof: Consider $(f(z+h)g(z+h) - f(z)g(z))/h$ and use continuity of g at z.

3. Quotient rule

    Proof: Nice trick, write $q = \frac{f}{g}$ so $qg = f$, then $f' = q'g + qg'$and $q' = \frac{f'}{g} - \frac{fg'}{g^2}$.

4. Chain rule

Proof: Use the fact that if $f'(g(z)) = a$, then

$$f(z+h) - f(z) = ah + r(z,h), \quad |r(z,h)| = o(|h|) \to 0.$$

Write $b = g'(z)$, then

$$f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$$

by considering error terms, and so

$$\frac{1}{h}(f(g(z+h)) - f(g(z))) \to f'(g(z))g'(z)$$

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4 Friday January 17th

Reference: See Lang's Complex Analysis, there are plenty of solution manuals.

Let $f; \Omega \to \mathbb{C}$ be a complex-valued function. Recall that f is *complex differentiable* iff the usual ratio/limit exists. Note that $h = x + iy$ and $h \to 0 \iff x, y \to 0$.

We can write $f'(z) = \dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$. This follows from Cauchy-Riemann since $u_x = v_y$ and $u_y = -v_x$.

Definition: We want to define $\partial, \overline{\partial}$ operators. We have the identities

$$x = \frac{z + \overline{z}}{z} \quad y = \frac{z - \overline{z}}{iz}.$$

We can then write

$$\begin{aligned} dz &= dx + idy \\ d\overline{z} &= dx - idy. \end{aligned}$$

We define the dual operators by $\left\langle \dfrac{\partial}{\partial z},\ dz \right\rangle = 1$ and similarly $\left\langle \dfrac{\partial}{\partial \overline{z}},\ d\overline{z} \right\rangle = 1$. By the chain rule, we can write

$$\begin{aligned} f_z &= f_x x_z + f_y y_z \\ &= \frac{1}{2} f_x + f_y \frac{1}{2i} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f, \end{aligned}$$

and similarly $f_{\overline{z}} = f_x x_{\overline{z}} + f_y z_{\overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{1}{2i}\dfrac{\partial}{\partial y} \right) f$.

We thus find $\partial_x = \partial_z + \partial_{\overline{z}}$ and $\partial_y = i(\partial_z - \partial_{\overline{z}})$, and define

$$\partial f = \frac{\partial f}{\partial z} dz$$
$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}$$
$$df = f_z dz + f_{\overline{z}} d\overline{z}.$$

Proposition: f is holomorphic iff $f_{\overline{z}} = 0$.

This means that f depends on z alone and not $\overline{z}$.

Proof: $\overline{\partial} f = 0$ iff $\frac{1}{2}(f_x + i f_y) = 0$, so $(u_x - v_y) + i(v_x + u_y) = 0$. ■

Application to PDEs: We can write $u_{xx} = v_{xy}, u_{yy} = v_{yx}$ and so $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. Thus $\Delta f = 0$, and f satisfies Laplace's equation and is said to be *harmonic*.

Corollary: If f is analytic, then u, v are both harmonic functions.

Theorem (Chain Rule): Let $w = f(z)$ and $g(w) = g(f(z))$. Then

$$h_z = g_w f_z + g_{\overline{w}} \overline{f}_z$$
$$h_{\overline{z}} = g_w f_{\overline{z}} + g_{\overline{w}} \overline{f}_{\overline{z}}.$$

If f, g are holomorphic, $f_{\overline{z}} = g_{\overline{w}} = 0$, so $h_{\overline{z}} = 0$ and h is holomorphic and $h_z = g_w f_z$.

Example: Given a power series $f = \sum a_n (z - z_0)^n$. Then

1. There exists a radius of convergence R such that f converges precisely on $D_R(z_0)$.
2. f is continuous on $D_R(z_0)^\circ$.
3. By the root test, $R = (\limsup |a_n|^{1/n})^{-1} = \liminf |a_n/a_{n+1}| = (\limsup |a_{k+1}/a_k|)^{-1}$.

Recall the ratio test: $\sum a_k$ converges absolutely iff $\limsup |a_{k+1}/a_k| < 1$

Theorem: If $f(z) = \sum_{n=0} a_n z^n$ is holomorphic on $|z| < R$ for $R > 0$ then $f'(z) = \sum_{n=1} a_n n z^{n-1}$.

Exercise: Show $\lim_n n^{\frac{1}{n}} = 1$. Also tricky: show $\lim \sin(n)$ doesn't exist, and $\sin(n)$ is dense in $[-1, 1]$.

Proof: Consider $\limsup |a_n n|^{\frac{1}{n}}$.

Remark: An analytic function is holomorphic in its domain of convergence, so analytic implies holomorphic. The converse requires Cauchy's integral formula.

Note: look for 13 equivalent statements, Springer GTM Lipman.

Proof: Given $|z| < R$, fix $r > 0$ such that $|z| < r < R$. Suppose that $|w - z| < r - |z|$, so $|w| < r$.

We want to show

$$|S| = \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1} a_n n z^{n-1} \right| \to 0 \quad \text{as } w \to z.$$

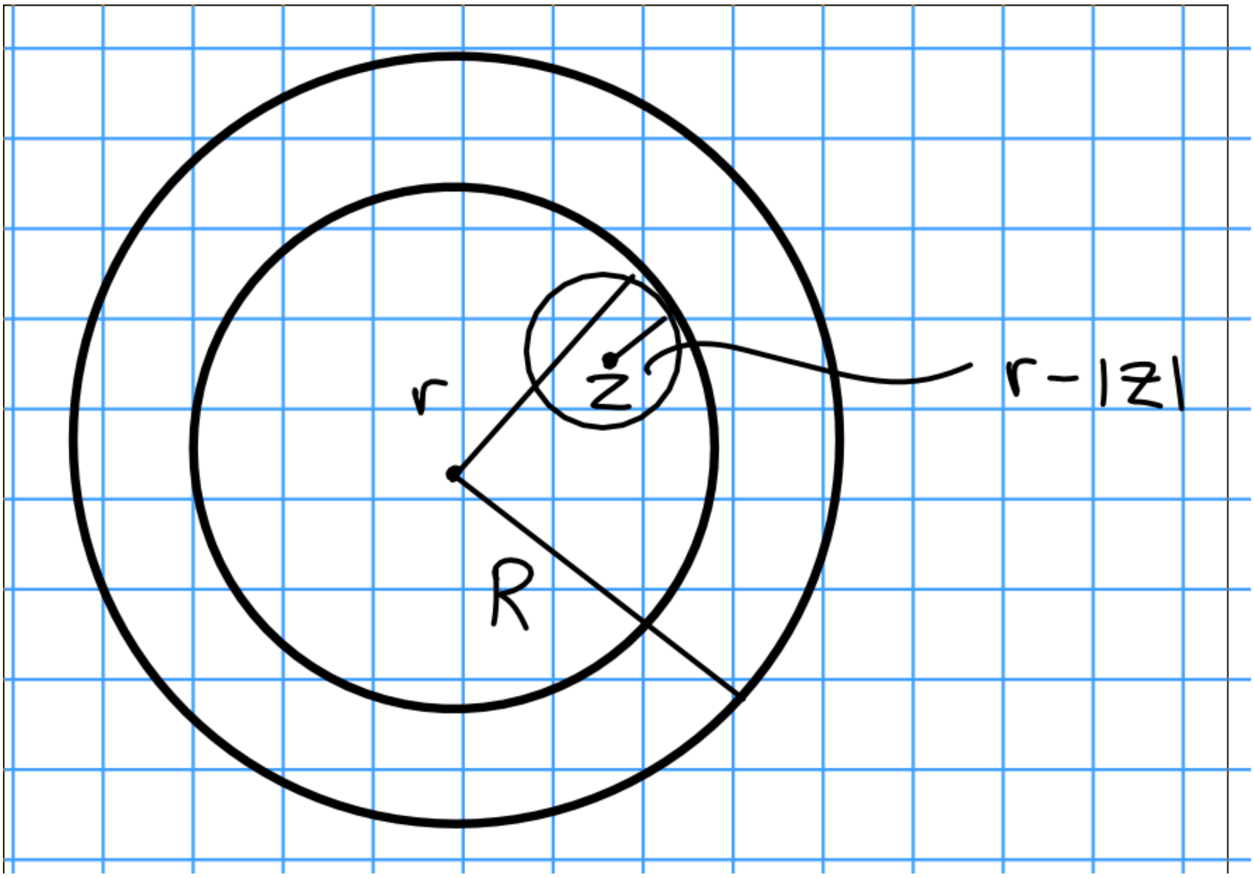

Figure 1: Image

Idea: write everything in terms of power series. Use the fact that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots)$, and so $\left|(w^k - z^k)/(w-z)\right| \leq kr^{k-1}$.

$$\begin{aligned}
S &= \sum_{n=1} a_n \left(\frac{w^n - z^n}{w - z} - nz^{n-1} \right) \\
&= \sum a_n \left(w^{n-1} + w^{n-2}z + \cdots + z^{n-1} + nz^{n-1} \right) \\
&= \sum a_n \left((w^{n-1} - z^{n-1}) + (w^{n-2} - z^{n-2})z + \cdots + (w-z)z^{n-2} \right) \quad = \sum a_n (w-z) \left(\cdots + z^{n-2} \right) \\
&\leq \sum_{n=2} |a_n| \frac{1}{2} n(n-1) r^{n-2} |z - w|.
\end{aligned}$$

■

Next time: trying to prove holomorphic functions are analytic.

5 Wednesday January 22nd

> Note: multiple complex variables, see Hormander or Steven Krantz

Recall from last time that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $z_0 \neq 0$ has radius of convergence $R = (\limsup |a_n|^{1/n})^{-1} > 0$, then f' exists and is obtained by differentiating term-by-term. We have f analytic implies f holomorphic (and smooth), we want to show the converse. For this, we need integration.

Definition: A parameterized curve is a function $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$.

Definition: The curve is said to be smooth iff z' exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for any t. At the boundary $\{a, b\}$, we define the derivative by taking one-sided limits.

Definition: A curve is said to be piecewise smooth iff $z(t)$ is continuous on $[a, b]$ and there are $a < a_1 < \cdots < a_n = b$ with z smooth on each $[a_k, a_{k+1}]$.

> Note: may fail to have tangent lines at a_i.

Definition: Two parameterizations $z : [a, b] \to \mathbb{C}, \tilde{z} : [c, d] \to \mathbb{C}$ are equivalent iff there exists a C^1 bijection $s : [c, d] \to [a, b]$ where $s \mapsto t(s)$ such that $s' > 0$ and $\tilde{z}(s) = z(s(t))$.

Note that $s' > 0$ preserves orientation and $s' < 0$ reverses orientation.

Definition:

$$\gamma : [a, b] \to \mathbb{C} \implies \gamma^- := [a, b] to \mathbb{C}, \; t \mapsto \gamma(a + b - t).$$

Definition: A curve is closed iff $z(a) = z(b)$, and is simple iff $z(t) \neq z_{t_1}$ for $t \neq t_1$.

Definition: For $C_r(z_0) := \left\{ z \;\middle|\; |z - z_0| = r \right\}$, the positive orientation is given by $z(t) = z_0 + re^{2\pi i t}$ for $t \in [0, 1]$.

Definition: The integral of f over γ is defined as

$$\int_\gamma f\ dz = \int_a^b f(z(t))z'(t)\ dt.$$

Note: This doesn't depend on parameterization, since if $t = t(s)$, then a change of variables yields

$$\int_\gamma f\ dz - \int_c^d f(z(t(s)))z'(t(s))t'(s)\ ds = \int_c^d f(\tilde{z}(s))\tilde{z}'(s)\ ds.$$

Definition: The length of γ is defined as $|\gamma| = \displaystyle\int |z'(t)|\ dt$.

Proposition:

1. We can extend this definition to piecewise smooth curves by
$$\int_\gamma f\ dz = \sum \int_{a_k}^{a_{k+1}} f\ dz$$
2. This integral is linear and $\displaystyle\int_\gamma f = -\int_{\gamma^-} f$.
3. We have an inequality
$$\left|\int_\gamma f\right| \leq \max_{a \leq t \leq b} |f(z(t))||\gamma|.$$

Definition: A function F is a primitive for f on Ω iff F is holomorphic on Ω and $F'(z) = f(z)$ on Ω.

Recall that in $\mathbb{R}$, we have $F(x) \displaystyle\int_a^x f(t)\ dt$ as an antiderivative with $F'(x) = f(x)$, and $\displaystyle\int f = F(b) - F(a)$.

Theorem: If f is continuous, has a primitive F in Ω, and γ is a curve beginning at w_0 and ending at w_1, then $\displaystyle\int_\gamma f = F(w_1) - F(w_0)$.

Proof: Use definitions, write $z(t)$ where $z(a) = w_1, z(b) = w_2$. Then

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(z(t))z'(t)\ dt \\
&= \int_a^b F'(z(t))z'(t)\ dt \\
&= \int_a^b F_t\ dt \\
&= F(z(b)) - F(z(a)) \quad \text{by FTC} \\
&= F(w_1) - F(w_2).
\end{aligned}$$

Note that if γ is piecewise smooth, the sum of the integrals telescopes to yield the same conclusion.

Corollary: If f is continuous and γ is a closed curve in Ω, and f has a primitive in Ω, then $\oint f = 0$.

6 Friday January 24th

Corollary: If γ is a closed curve on Ω an open set and f is continuous with a primitive in Ω (i.e. an F holomorphic in Ω with $F' = f$) then $\int_\gamma f\ dz = 0$.

Proof (easy):

$$\int_\gamma f\ dz = \int_\gamma F' = F'(z)z(t)\ dt = F(z(b)) - F(z(a)) = 0.$$

Corollary: If f is holomorphic with $f' = 0$ on Ω, then f is constant.

Proof (easy): Pick $w_0 \in \Omega$; we want to fix $w_0 \in \Omega$ and show $f(w) = f(w_0)$ for all $w \in \Omega$.

Take any path $\gamma : w_0 \to w$, then

$$0 = \int_\gamma f' = f(w) - f(w_0).$$

Example: Let $f(z) = e^{-z^2}$, this is holomorphic. Write $f(z) = \sum(-1)^n z^{2n}/n!$, so $\int f = \sum(-1)^n z^{2n+1}/(n!(2n+1))$. Since f is entire, $\int f$ is entire, and $(\int f)' = f$ so this function has a primitive. Thus $\int_\gamma f(z) = 0$ for *any* closed curve. So take γ a rectangle with vertices $\pm a, \pm a + ib$.

So

$$\int_\gamma f = \int_{-a}^{a} e^{-x^2}\ dx + \int e^{-(a+iy)^2} i\ dy - \int_{-a}^{a} e^{-(x+ib)^2}\ dx - \int_0^b e^{-(a+iy)^2} i dy = 0.$$

We can do some estimates,

$$\begin{aligned} e^{-(a+iy)^2} = e^{-(a^2+2iay-y^2)} = e^{-a^2+y^2} e^{2iay} \leq e^{-a^2+y^2} \leq e^{-a^2+b^2} \\ \left| \int_0^b e^{-(a+ib)^2} i\ dy \right| \leq e^{-a^2+b^2} \cdot b \\ \int_{-a}^{a} e^{-(x^2+2ibx)-b^2} = e^{b^2} \int_{-a}^{a} e^{-x^2}(\cos(2bx) - i\sin(2bx)) \overset{\text{odd fn}}{=} e^{b^2} \int_{-a}^{a} e^{-x^2} \cos(2bx)\ dx. \end{aligned}$$

Now take $a \to \infty$ to obtain

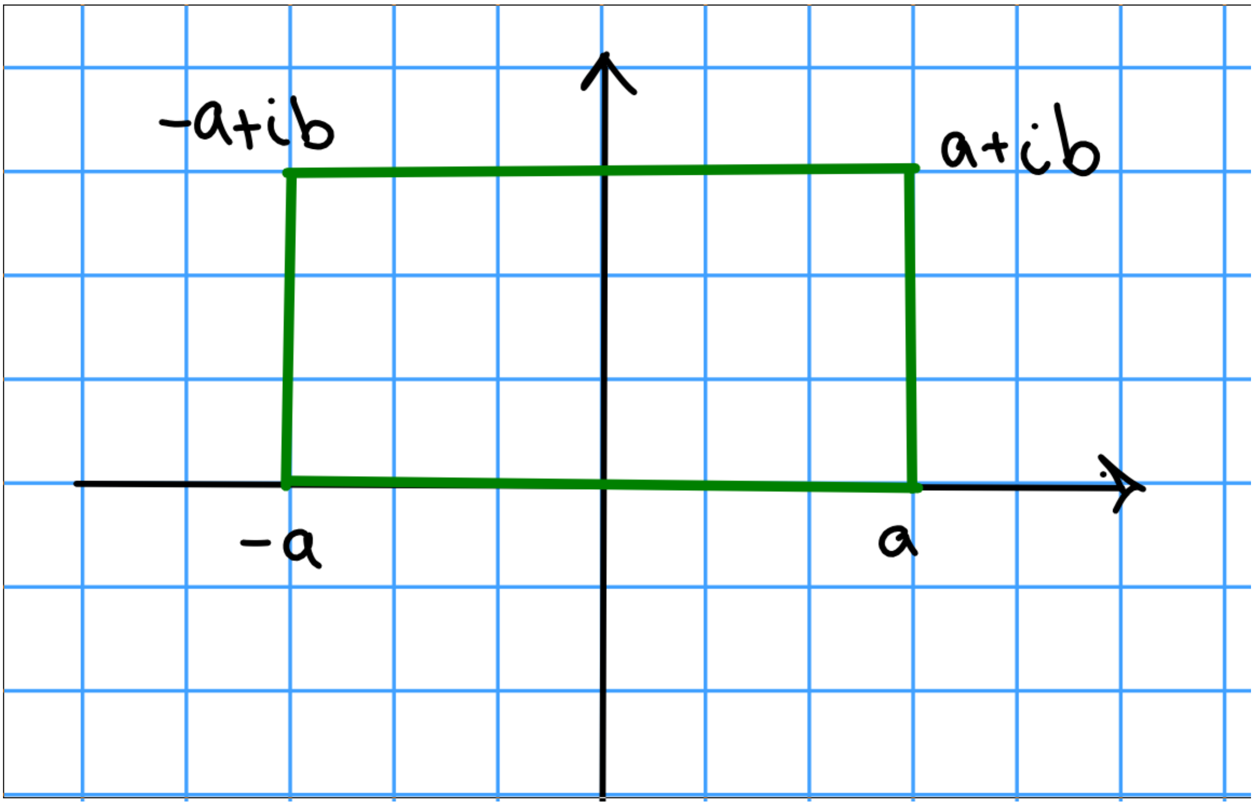

Figure 2: Image

$$\int_{\mathbb{R}} e^{-x^2}\ dx = e^{b^2} \int_{\mathbb{R}} e^{-x^2} \cos(2bx)\ dx.$$

We can compute

$$\int_{\mathbb{R}} e^{-x^2} = \left[\left(\int_{\mathbb{R}} e^{-x^2} \right)^2 \right]^{1/2} = \left(\int_0^{2\pi} \int_0^{\infty} e^{r^2} r\ dr\ d\theta \right) = \sqrt{\pi}.$$

and then conclude

$$\int_{\mathbb{R}} e^{-x^2} \cos(2bx) = \sqrt{\pi} e^{-b^2}.$$

Make a change of variables $2b = 2\pi\xi$, so $b = \pi\xi$, then

$$\int_{\mathbb{R}} e^{-x^2} \cos(2\pi\xi x)\ dx = \sqrt{\pi} e^{-\pi^2 \xi^2}.$$

Thus $\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\pi^2 \xi^2}$, allowing computation of the Fourier transform. Note that this can be used to prove the Fourier inversion formula.

Exercise: Show that this is an approximate identity and prove the Fourier inversion formula.

Exercise: Show $\mathcal{F}(e^{-ax^2}) = \sqrt{\pi/a} e^{-\pi^2/a \cdot \xi^2}$, and thus taking $a = \pi$ makes $e^{\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue 1.

Theorem: If f has a primitive on Ω then $F(z)$ is holomorphic and $\displaystyle\int_{\gamma} f = 0$. If f is holomorphic, then $\displaystyle\int_{\gamma} f = 0$.

Theorem (Green's): Take $\Omega \in \mathbb{R}^2$ bounded with $\partial\Omega$ piecewise smooth. If $f, g \in C^1\overline{\Omega}$, then

$$\int_{\partial\Omega} f\ dx + g\ dy = \iint_{\Omega} (g_x - f_y)\ dA.$$

Proof: Not given here!

Proof of Theorem: Write $\gamma = \partial\Gamma$, and noting that $f_z = f_x = \dfrac{1}{i} f_y$ implies that $\dfrac{\partial f}{\partial \overline{z}}$, so

$$
\begin{aligned}
\int_\gamma f\ dz &= \int_\gamma f(z)\ (dx + idy) \\
&= \int f(z)\ dx + if(z)\ dy \\
&= \iint_\Gamma (if_x - f_y)\ dA \\
&= i \iint_\Gamma \left(f_x - \frac{1}{i} f_y\right)\ dA \\
&= i \iint 0\ dA = 0.
\end{aligned}
$$

Next class: We'll prove that this integral over any triangle is zero by a limiting process.

7 Appendix

Collection of facts used on problem sets

Standard forms of conic sections:

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x, y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Completing the square:

$$
\begin{aligned}
x^2 - bx &= (x - s)^2 - s^2 \quad \text{where} s = \frac{b}{2} \\
x^2 + bx &= (x + s)^2 - s^2 \quad \text{where} s = \frac{b}{2}.
\end{aligned}
$$

Useful Properties

- $\Re(z) = \frac{1}{2}(z + \overline{z})$ and $\Im(z) = \frac{1}{2i}(z - \overline{z})$.
- $z\overline{z} = |z|^2$

- $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
- $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Useful Series

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Cauchy-Riemann Equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

7.1 Useful Techniques

Showing a function is constant: Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.

Deriving Polar Cauchy-Riemann: See walkthrough here. Take derivative along two paths, along a ray with constant angle θ_0 and along a circular arc of constant radius r_0. Then equate real and imaginary parts. See problem set 1.

Computing Arguments: $\mathrm{Arg}(z/w) = \mathrm{Arg}(z) - \mathrm{Arg}(w)$.

The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.