Title

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1.1 Group Schemes

> **Definition 1.0.1** (Representable Functors).
> Let $F :: k\text{-alg} \to \text{Set}$ be a functor, then F is **representable** iff $F(R)$ corresponds to "solutions to equations in R".

Example 1.1.
Let $F(\,\cdot\,) = \text{SL}(2, \,\cdot\,)$, then the corresponding equations are $\det(x_{ij}) = 1$.

If F is representable, there is a correspondence $F(R) \cong \hom_R(A, R)$. In the above example,

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}] / \langle x_{11}x_{22} - x_{12}x_{21} \rangle ,$$

which is exactly the coordinate algebra.

> **Definition 1.0.2** (Affine Group Scheme).
> An *affine group scheme* is a representable functor $F : k\text{-alg} \to \text{Groups}$.

Suppose G is an affine group scheme, and let $A = k[G]$ be the representing object. Then there is a correspondence

$$G\text{-modules} \iff k[G]^{\vee}\text{-modules}.$$

For G reductive, the RHS is equivalent to $\text{Dist}(G)$-modules.

> **Definition 1.0.3** (Finite Group Schemes)**.**
> G is a **finite** group scheme iff $k[G]$ is finite dimensional.

If G is finite, then $A^\vee \cong k[G]^\vee$ is a cocommutative Hopf algebras. Thus representations for *finite* group schemes are equivalent to representations for finite-dimensional cocommutative Hopf algebras.

> On group scheme side: see reduction, spectral sequences, conceptual arguments. On the algebra side: have bases, underlying vector space, can do concrete computations. Can take $\mathrm{Spec}\,(k[G])^\vee$ to recover a group scheme.

1.2 Hopf Algebras

For A a k-alg, we have a multiplication and a unit, which can be defined in terms of diagrams. To categorically reverse arrows, we can ask for a comultiplication and a counit.

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to k.$$

We'll want another map, an *antipode*

$$s : A \to A.$$

The comultiplication should satisfy

$$
\begin{array}{ccc}
A^{\otimes 3} & \xleftarrow{\ 1 \otimes A\ } & A^{\otimes 2} \\
{\scriptstyle \Delta \otimes 1}\big\uparrow & & \big\uparrow{\scriptstyle \Delta} \\
A^{\otimes 2} & \xleftarrow{\ \Delta\ } & A
\end{array}
$$

The counit should satisfy

$$
\begin{array}{ccc}
k \otimes A & \xleftarrow{\ \varepsilon \otimes 1\ } & A^{\otimes 2} \\
{\scriptstyle \cong}\big\downarrow & & \big\uparrow{\scriptstyle \Delta} \\
A & \xrightarrow{\ \cong\ } & A
\end{array}
$$

And the antipode should satisfy

$$
\begin{array}{ccc}
A & \xleftarrow{\ m(s \otimes 1)\ } & A \\
\big\uparrow & & \big\uparrow{\scriptstyle \Delta} \\
A & \xleftarrow{\ \varepsilon\ } & A
\end{array}
$$

1.2.1 Module Constructions

Let A be a Hopf algebra.

1. For A-modules M, N, we can form the A-module $M \otimes_k N$ with

$$\Delta(a) = \sum a_i \otimes a_j$$

$$a(m \otimes n) = \sum a_1 m \otimes a_2 n.$$

2. If M is finite-dimensional over A, then $M^{\vee} = \hom_k(M, k) \ni f$ is an A-module, and we can define $(af)(x) := f(s(a)x)$ for $a \in A, x \in M$.

Example 1.2.
$A = kG$ the group algebra on a finite group is a Hopf algebra:

$$\Delta :; A \to A^{\otimes 2}$$

$$g \mapsto g \otimes g.$$

The module action is diagonal, namely $g(m \otimes n) = gm \otimes gn$. The antipode is given by $s(g) = g^{-1}$, and the unit is $\varepsilon(g) = 1$ for all $g \in G$.

Example 1.3.
Let $A = U(\mathfrak{g})$, the universal enveloping algebra for $\mathfrak{g}$ a Lie algebra. Recall that $\mathfrak{g}$-modules are equivalent to $U(\mathfrak{g})$-modules (unitary representations, some big associative algebra). Then A is a Hopf algebra, with $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ for $\ell \in \mathfrak{g}$. The unit is $\varepsilon(\ell) = 0$, and the antipode is $s(\ell) = -\ell$.

Example 1.4.
Take the additive group $\mathbb{G}_a$, then $A = k[\mathbb{G}_a] \cong k[x]$ is a commutative Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0, s(x) = -x$.

Example 1.5.
For $\mathbb{G}_m$, we have $A = k[\mathbb{G}_m] \cong k[x, x^{-1}], \varepsilon(x) = 1, s(x) = x^{-1}$.

1.3 Frobenius Kernels

Let G be an algebraic group (scheme) over k, where char $(k) = p$. Let $F : G \to G$ be the Frobenius, where e.g.

$$F : \mathrm{GL}(n, \cdot) \to \mathrm{GL}(n, \cdot)$$

$$(x_{ij}) \mapsto (x_{ij}^p).$$

Then F is a map of group schemes.

> **Definition 1.0.4** (Frobenius Kernels)**.**
> $G_r := \ker F^r$, where $F^r := F \circ F \circ \cdots \circ F$ is the r-fold composition of the Frobenius.
> This yields a nesting $G_1 \trianglelefteq G_2 \trianglelefteq G_3 \cdots \leq G$.

Recall that

$$\mathrm{Dist}(G) = \left\langle \frac{x_\alpha^n}{n!} \right\rangle.$$