

Title

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1 | Ordinary Differential Equations

1.1 Techniques Overview

$$p(y)y' = q(x) \quad \text{separable}$$

$$y' + p(x)y = q(x) \quad \text{integrating factor}$$

$$y' = f(x, y), f(tx, ty) = f(x, y) \quad y = xV(x) \text{ COV reduces to separable}$$

$$y' + p(x)y = q(x)y^n \quad \text{Bernoulli, divide by } y^n \text{ and COV } u = y^{1-n}$$

$$M(x, y)dx + N(x, y)dy = 0 \quad M_y = N_x : \varphi(x, y) = c(\varphi_x = M, \varphi_y = N)$$

$$P(D)y = f(x, y) \quad x^k e^{rx} \text{ for each root}$$

Where e^{zx} yields $e^{ax} \cos bx, e^{ax} \sin bx$

1.2 Types of Equations

- Separable equations:

$$p(y) \frac{dy}{dx} - q(x) = 0 \implies \int p(y) dy = \int q(x) dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)} dy = \int f(x) dx + C$$

- Population growth:

$$\frac{dP}{dt} = kP \implies P = P_0 e^{kt}$$

- Logistic growth:

$$\diamond \text{ General form: } \frac{dP}{dt} = (B(t) - D(t))P(t)$$

- \diamond Assume birth rate is constant $B(t) = B_0$ and death rate is proportional to instantaneous population $D(t) = D_0 P(t)$. Then let $r = B_0, C = B_0/D_0$ be the *carrying capacity*:

$$\frac{dP}{dt} = r \left(1 - \frac{P}{C}\right) P \implies P(t) = \frac{P_0}{\frac{P_0}{C} + e^{-rt}(1 - \frac{P_0}{C})}$$

- First order linear:

$$\frac{dy}{dx} + p(x)y = q(x) \implies I(x) = e^{\int p(x) dx}, \quad y(x) = \frac{1}{I(x)} \left(\int q(x) I(x) dx + C \right)$$

- Exact:

$$- M(x, y)dx + N(x, y)dy = 0 \text{ is exact} \iff \exists \varphi : \frac{\partial \varphi}{\partial x} = M(x, y), \quad \frac{\partial \varphi}{\partial y} = N(x, y)$$

$$\iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- General solution:

$$\varphi(x, y) = \int^x M(s, y) ds + \int^y N(x, t) dt - \int^y \frac{\partial}{\partial t} \left(\int^x M(s, t) ds \right) dt$$

(where $\int^x f(t) dt$ means take the antiderivative of f and consider it a function of x)

- Cauchy Euler: #todo
- Bernoulli: todo

1.3 Linear Homogeneous

General form:

$$y^{(n)} + c_{n-1}y^{(n-1)} + \dots + c_2y'' + cy' + cy = 0$$

$$p(D)y = \prod (D - r_i)^{m_i} y = 0$$

where p is a polynomial in the differential operator D with roots r_i :

- Real roots: contribute m_i solutions of the form

$$e^{rx}, xe^{rx}, \dots, x^{m_i-1}e^{rx}$$

- Complex conjugate roots: for $r = a + bi$, contribute $2m_i$ solutions of the form

$$e^{(a \pm bi)x}, xe^{(a \pm bi)x}, \dots, x^{m_i-1}e^{(a \pm bi)x} \\ = e^{ax} \cos(bx), e^{ax} \sin(bx), xe^{ax} \cos(bx), xe^{ax} \sin(bx), \dots,$$

Example: by cases, second order equation of the form

$$ay'' + by' + cy = 0$$

- Two distinct roots: $c_1 e^{r_1 x} + c_2 e^{r_2 x}$ - One real root: $c_1 e^{rx} + c_2 x e^{rx}$ - Complex conjugates $\alpha \pm i\beta$: $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

1.4 Linear Inhomogeneous

General form:

$$y^{(n)} + c_{n-1}y^{(n-1)} + \dots + c_2y'' + c_1y' + cy = F(x) \\ p(D)y = \prod (D - r_i)^{m_i}y = 0$$

Then solutions are of the form $y_c + y_p$, where y_c is the solution to the associated homogeneous system and y_p is a particular solution.

Methods of obtaining particular solutions

1.4.1 Undetermined Coefficients

- Find an operator $p(D)$ the annihilates $F(x)$ (so $q(D)F = 0$)
- Find solution of $q(D)p(D) = 0$, subtract of known solutions from homogeneous part to obtain the form of the trial solution $A_0 f(x)$, where A_0 is the undetermined coefficient
- Substitute trial solution into original equation to determine A_0

Useful Annihilators:

$$\begin{aligned} F(x) &= p(x) : & D^{\deg(p)+1} \\ F(x) &= p(x)e^{ax} : & (D - a)^{\deg(p)+1} \\ F(x) &= \cos(ax) + \sin(ax) : & D^2 + a^2 \\ F(x) &= e^{ax}(a_0 \cos(bx) + b_0 \sin(bx)) : & (D - z)(D - \bar{z}) = D^2 - 2aD + a^2 + b^2 \\ F(x) &= p(x)e^{ax} \cos(bx) + p(x)e^{ax} \sin(bx) : & ((D - z)(D - \bar{z}))^{\max(\deg(p), \deg(q))+1} \end{aligned}$$

1.4.2 Variation of Parameters

todo

1.4.3 Reduction of Order

todo

1.5 Systems of Differential Equations

General form:

$$\frac{\partial \mathbf{x}(t)}{\partial t} = A\mathbf{x}(t) + \mathbf{b}(t) \iff \mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

General solution to homogeneous equation:

$$c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_n \mathbf{x}_n(t) = \mathbf{X}(t)\mathbf{c}$$

If A is a matrix of constants: $\mathbf{x}(t) = e^{\lambda_i t} \mathbf{v}_i$ is a solution for each eigenvalue/eigenvector pair $(\lambda_i, \mathbf{v}_i)$
 - If A is defective, you'll need generalized eigenvectors.

Inhomogeneous Equation: particular solutions given by

$$\mathbf{x}_p(t) = \mathbf{X}(t) \int^t \mathbf{X}^{-1}(s) \mathbf{b}(s) ds$$

1.6 Laplace Transforms

Definitions:

$$H_a(t) := \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

$$\delta(t) : \int_{\mathbb{R}} \delta(t-a) f(t) dt = f(a), \quad \int_{\mathbb{R}} \delta(t-a) dt = 1$$

$$(f * g)(t) = \int_0^t f(t-s) g(s) ds$$

$$L[f(t)] = L[f] = \int_0^\infty e^{-st} f(t) dt = F(s).$$

Useful property: for $a \leq b$, $H_a(t) - H_b(t) = \mathbb{1}[[a, b]]$.

$t^n, n \in \mathbb{N}$	\iff	$\frac{n!}{s^{n+1}}, \quad s > 0$
$t^{-\frac{1}{2}}$	\iff	$\sqrt{\pi} s^{-\frac{1}{2}}, \quad s > 0$
e^{at}	\iff	$\frac{1}{s-a}, \quad s > a$
$\cos(bt)$	\iff	$\frac{s}{s^2 + b^2}, \quad s > 0$
$\sin(bt)$	\iff	$\frac{b}{s^2 + b^2}, \quad s > 0$
$\cosh(bt)$	\iff	$\frac{s}{s^2 - b^2}, \quad ?$
$\sinh(bt)$	\iff	$\frac{b}{s^2 - b^2}, \quad ?$
$\delta(t-a)$	\iff	e^{-as}
$H_a(t)$	\iff	$s^{-1}e^{-as}$
$e^{at}f(t)$	\iff	$F(s-a)$
$H_a(t)f(t-a)$	\iff	$e^{-as}F(s)$
$f'(t)$	\iff	$sL(f) - f(0)$
$f''(t)$	\iff	$s^2L(f) - sf(0) - f'(0)$
$f^{(n)}(t)$	\iff	$s^nL(f) - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0)$
$f(t)g(t)$	\iff	$F(s) * G(s)$

- For f periodic with period T , $L(f) = \frac{1}{1 + e^{-sT}} \int_0^T e^{-st} f(t) dt$

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