

# Algebraic Topology 2: Smooth Manifolds

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The key point of this class will be a discussion of *smooth structures*. As you may recall, a sensational result of Milnor's exhibited exotic spheres with smooth structures – i.e., a differentiable manifold  $M$  which is homeomorphic but *not* diffeomorphic to a sphere.

Summary of this result: Look at bundles  $S^3 \rightarrow X \rightarrow S^4$ , then one can construct some  $X \cong S^7 \in \mathbf{Top}$  but  $X \not\cong S^7 \in \mathbf{Diff}^\infty$ . There are in fact 7 distinct choices for  $X$ .

It is not known if there are exotic smooth structures on  $S^4$ . The Smooth Poincaré conjecture is that these do not exist; this is believed to be false.

The other key point of this course is to show that  $X \in \mathbf{Diff}^\infty \implies X \hookrightarrow \mathbb{R}^n$  for some  $n$ , and is in fact a topological subspace.

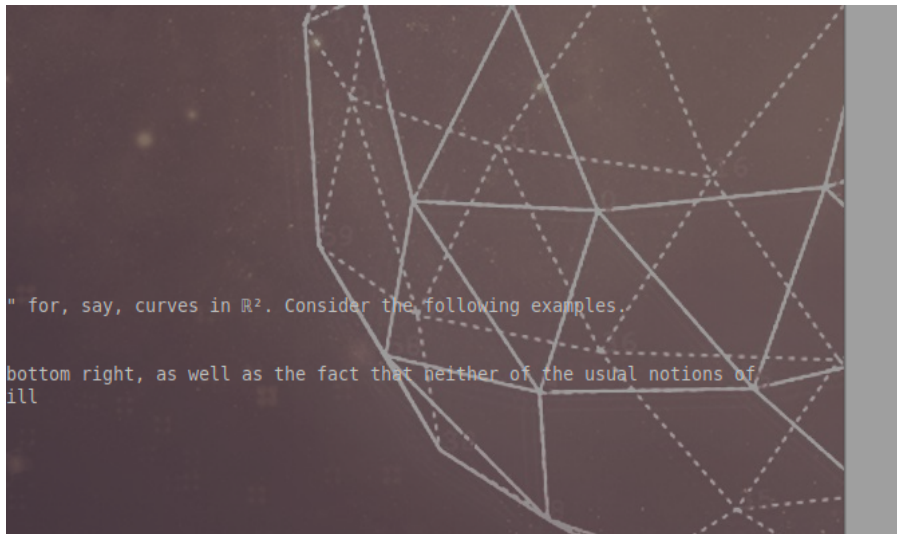
A short list of words/topics we hope to describe:

- Differentiable manifolds
- Local charts
- Submanifolds
- Projective spaces
- Lie groups
- Tangent spaces
- Vector fields
- Cotangent spaces
- Differentials of smooth maps
- Differential forms
- de Rham's theorem

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We'd like a notion of “convergence” for, say, curves in  $\mathbb{R}^2$ . Consider the following examples.

Note the problematic point on the bottom right, as well as the fact that neither of the usual notions of pointwise or uniform convergence will



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The key point of this class will be a result of Milnor's exhibited as a space which is homeomorphic but not diffeomorphic to  $S^7$ .

Summary of this result: Look at  $X \subset S^7 \in \mathbf{Diff}^\infty$ . There is a natural map  $X \rightarrow S^7$ .

It is not known if there is a manifold which is that these do not exist.

The other key point of this class is that a topological subspace of  $\mathbb{R}^n$  is not necessarily a topological manifold.

A short list of words/topics to be covered:

- Differentiable manifold
- Local charts
- Submanifolds