Problem Set 5

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1 Problem 1

We first make the following claim (TODO):

$$\begin{aligned}
S &:= \sum_{j=1}^\infty \sum_{k=1}^\infty a_{jk} = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \ni B \subset \mathbb{N}^2,\ |B| < \infty \right\} \\
T &:= \sum_{k=1}^\infty \sum_{j=1}^\infty a_{kj} = \sup \left\{ \sum_{(j,k) \in C} a_{kj} \ni C \subset \mathbb{N}^2,\ |B| < \infty \right\}.
\end{aligned}$$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs (j,k), we have the following inequality:

$$\sum_{(j,k)\in B} a_{jk} < \sum_{(k,j)\in C} a_{jk} \leq \sum_{(k,j)\in C} a_{jk} \leq T,$$

since T is a supremum over *all* such sets C, and the terms of any finite sum can be rearranged.

But since this holds for every B, we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S \coloneqq \sup_B \sum_{(k,j)\in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that $T \leq S$, yielding the desired equality. $\square$

2 Problem 2

We want to show the following equality:

$$\int_0^1 g(x)\ dx = \int_0^1 f(x)\ dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx = \int_0^1 f(x)\ dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned}
\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx &=_? \int_0^1 \int_0^t \frac{f(t)}{t}\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t} \int_0^t\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t}(t-0)\ dt \\
&= \int_0^1 f(t)\ dt,
\end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$\begin{aligned}
F : \mathbb{R}^2 &\to \mathbb{R} \\
(x,t) &\mapsto \frac{\chi_A(x,t)\hat{f}(x,t)}{t}.
\end{aligned}$$

where $A = \big\{(x,t) \subset \mathbb{R}^2 \ni 0 \le x \le t \le 1\big\}$ and $\hat{f}(x,t) := f(t)$ is the cylinder on f.

This defines a measurable function on $\mathbb{R}^2$, since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right| \, dx \; dt \\
&= \int_0^1 \int_0^t \frac{|f(t)|}{t} \, dx \; dt \quad \text{since } t > 0 \\
&= \int_0^1 \frac{|f(t)|}{t} \int_0^t \, dx \; dt \\
&= \int_0^1 |f(t)| < \infty,
\end{aligned}$$

where the switch is justified by Tonelli and the last inequality holds because f was assumed to be measurable.

Since this shows that $F \in L^1(\mathbb{R}^2)$, and we can thus apply Fubini to F to justify the initial switch. $\square$

3 Problem 3

Let $A = \{0 \le x \le y\} \subset \mathbb{R}^2$, and define

$$\begin{aligned}
f(x,y) &= \frac{x^{1/3}}{(1+xy)^{3/2}} \\
F(x,y) &= \chi_A(x,y) f(x,y).
\end{aligned}$$

Note that F Then, if all iterated integrals exist and a switch of integration order is justified, we would have

$$
\begin{aligned}
\int_{\mathbb{R}^2} F &=_? \int_0^\infty \int_y^\infty f(x,y)\ dx\ dy \\
&=_? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1+xy)^{3/2}}\ dy\ dx \\
&= 2\int_{\mathbb{R}} \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx \\
&= 2\int_0^1 \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx + 2\int_1^\infty \frac{1}{x^{2/3}\sqrt{1+x^2}}\ dx \\
&\leq \int_0^1 x^{-2/3}\ dx + \int_0^\infty x^{-5/3} \\
&= 2(3) + 2\left(\frac{3}{2}\right) < \infty,
\end{aligned}
$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1+x^2} \geq \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1+x^2} \geq \sqrt{1}$.

Since F is non-negative, we have $|F| = F$, and so the above computation would imply that $F \in L^1(\mathbb{R}^2)$. It thus remains to show that $\int F$ is equal to its iterated integrals, and that the switch of integration order is justified

Since F is non-negative, Tonelli can be applied directly if F is measurable in $\mathbb{R}^2$. But f is measurable on A, since it is continuous at almost every point in A, and χ_A is measurable, so F is a product of measurable functions and thus measurable.

4 Problem 4

4.1 Part (a)

For any $x \in \mathbb{R}^n$, let $A_x \coloneqq A \bigcap (x - B)$.

We can then write $A_t \coloneqq A \bigcap (t - B)$ and $A_s \coloneqq A \bigcap (s - B)$, and thus

$$
\begin{aligned}
g(t) - g(s) &= m(A_t) - m(A_s) \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x)\ dx - \int_{\mathbb{R}^n} \chi_{A_s}(x)\ dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_s}(x)\ dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_t}(t - s + x)\ dx \\
&\quad \text{(since } x \in s - B \iff s - x \in B \iff t - (s - x) \in t - B\text{)},
\end{aligned}
$$

and thus by continuity in L^1, we have

$$
|g(t) - g(s)| \leq \int_{\mathbb{R}^n} |\chi_{A_t}(x) - \chi_{A_t}(t - s + x)|\ dx \to 0 \quad \text{as} \quad t \to s
$$

which means g is continuous.

To see that $\int g = m(A)m(B)$, if an interchange of integrals is justified, we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} g(t)\ dt &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{A_t}(x)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\chi_{t-B}(x,t)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\chi_{t-B}(x,t)\ dx\ dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\ \chi_B(t-x)\ dx\ dt \\
&\qquad \text{(since } x \in t-B \iff t-x \in B\text{)} \\
&=_? \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x)\ \chi_B(t-x)\ \mathbf{dt\ dx} \\
&= \int_{\mathbb{R}^n} \chi_A(x) \int_{\mathbb{R}^n} \chi_B(t-x)\ dt\ dx \\
&= \int_{\mathbb{R}^n} \chi_A(x)\ m(B)\ dt \\
&\qquad \text{(by translation invariance of Lebesgue integral)} \\
&= m(B) \int_{\mathbb{R}^n} \chi_A\ dt \\
&= m(B)m(A).
\end{aligned}$$

4.1.1 Justification for integral switch

To see that this is justified, we note that that the map $F(x,t) = \chi_A(x)\ \chi_B(x-t)$ is non-negative, and we claim is measurable in $\mathbb{R}^{2n}$.

- The first component is $\chi_A(x)$, which is measurable on $\mathbb{R}^n$, and thus the cylinder over it will be measurable on $\mathbb{R}^{2n}$.
- The second component involves $\chi_B(t-x)$, which is $\chi_B(x)$ composed with a reflection (which is still measurable) followed by a translation (which is again still measurable).
- Thus, as a product of two measurable functions, the integrand is measurable.

So Tonelli applies to $|F|$, and thus $\int |F| = m(A)m(B) < \infty$ since A, B were assumed to be bounded. But then F is integrable by Fubini, and the claimed equality holds.

4.2 Part (b)

Supposing that $m(A), m(B) > 0$, we have $\int g(t)\ dt > 0$, using the fact that $\int g = 0$ a.e. $\iff g = 0$ a.e., we can conclude that if $T = \{t \ni g(t) \neq 0\}$, then $m(T) > 0$. So there is some $t \in \mathbb{R}^n$ such that $g(t) \neq 0$, and since g is continuous, there is in fact some open ball B_t containing t such that $t' \in B_t \implies g(t') \neq 0$. So we have

- $\forall t' \in B_t,\ A \bigcap t' - B \neq \emptyset \iff$
- $\forall t' \in B_t,\ \exists x \in A \bigcap t' - B \iff$
- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $x \in t' - B \iff$

- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $x = t' - B$ for some $b \in B \iff$
- $\forall t' \in B_t,\ \exists x$ such that $x \in A$ and $t' = x + B$ for some $b \in B \iff$
- $\forall t' \in B_t,\ \exists t'$ such that $t' \in A + B$

And thus $B_t \subseteq A + B$.

5 Problem 5

If the iterated integrals exist and are equal (so an interchange of integration order is justified), we have

$$\begin{aligned}
\int_0^1 F(x)g(x) &\coloneqq \int_0^1 \left(\int_0^x f(y)\ dy\right) g(x)\ dx \\
&= \int_0^1 \int_0^x f(y)g(x)\ dy\ dx \\
&=_? \int_0^1 \int_y^1 f(y)g(x)\ \mathbf{dx\ dy} \\
&= \int_0^1 f(y) \left(\int_y^1 g(x)\ dx\right)\ dy \\
&= \int_0^1 f(y)(G(1) - G(y))\ dy \\
&= G(1)\int_0^1 f(y)\ dy - \int_0^1 f(y)G(y)\ dy \\
&= G(1)(F(1) - F(0)) - \int_0^1 f(y)G(y)\ dy \\
&= G(1)F(1) - \int_0^1 f(y)G(y)\ dy \qquad \text{since } F(0) = 0,
\end{aligned}$$

which is what we want to show.

To see that this is justified, let $I = [0,1]$ and note that the integrand can be written as $H(x,y) = \hat{f}(x,y)\hat{g}(x,y)$ where $\hat{f}(x,y) = \chi_I f(y)$ and $\hat{g}(x,y) = \chi_I g(x)$ are cylinders over f and g respectively. Since f, g are in $L^1(I)$, their cylinders are measurable over $\mathbb{R} \times I$, and thus $\hat{f}, \hat{g}$ are measurable on $\mathbb{R}^2$ as products of measurable functions. Then H is a measurable function as a product of measurable functions as well.

But then $|H|$ is non-negative and measurable, so by Tonelli all iterated integrals will be equal. We want to show that $H \in L^1(\mathbb{R}^2)$ in order to apply Fubini, so we will show that $\int |H| < \infty$.

To that end, noting that $f, g \in L^1$, we have $\int_0^1 f \coloneqq C_f < \infty$ and $\int_0^1 g \coloneqq C_g < \infty$. Then,

$$
\begin{aligned}
\int_{\mathbb{R}^2} |H| &= \int_0^1 \int_0^1 |f(x)g(y)| \ dx \ dy \\
&= \int_0^1 \int_0^1 |f(x)| \ |g(y)| \ dx \ dy \\
&= \int_0^1 |g(y)| \left(\int_0^1 |f(x)| \ dx \right) \ dy \\
&= \int_0^1 |g(y)| C_f \ dy \\
&= C_f \int_0^1 |g(y)| \ dy \\
&= C_f C_g < \infty,
\end{aligned}
$$

and thus by Fubini, the original interchange of integrals was justified.

6 Problem 6

6.1 Part (a)

We have

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \ dy \right| \ dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \left| \int_{x-h}^{x+h} f(y) \ dy \right| \ dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \left(\int_{x-h}^{x+h} |f(y)| \ dy \right) \ dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \ dy \ dx \\
&=_? \ \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \ \mathbf{dx \ dy} \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \int_{y-h}^{y+h} \ dx \ dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \left((y+h) - (y-h) \right) \ dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} 2h |f(y)| \ dy \\
&= \int_{\mathbb{R}} |f(y)| \ dy < \infty
\end{aligned}
$$

.

since f was assumed to be in $L^1(\mathbb{R})$, where the changed bounds of integration are determined by considering the following diagram:

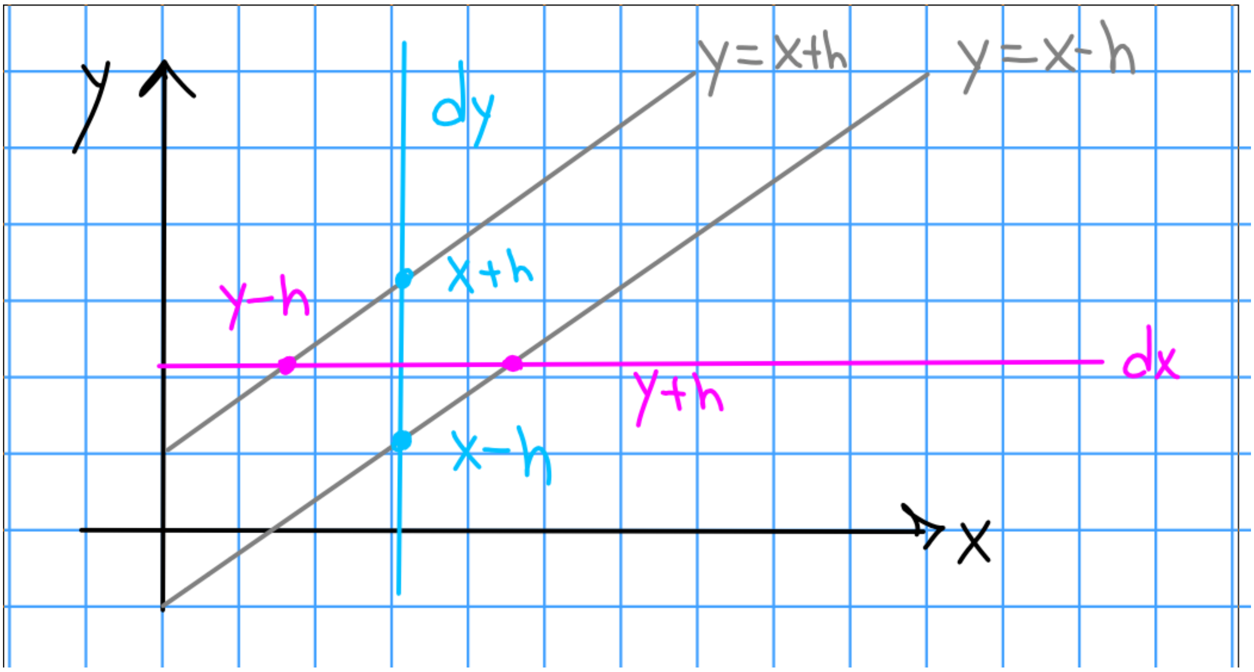

Figure 1: Changing the bounds of integration

To justify the change in the order of integration, consider the function $H(x,y) = \frac{1}{2h}\chi_A(x,y)f(y)$ where $A = \{(x,y) \in \mathbb{R}^2 \ni -\infty < x-h \leq x, y \leq x+h\}$. Since f is measurable, the constant function $(x,y) \mapsto \frac{1}{2h}$ is measurable, and characteristic functions are measurable, H is a product of measurable functions and thus measurable.

Thus it makes sense to write $\int |H|$ as an iterated integral by Tonelli, and since $\int_{\mathbb{R}^2} |H| = \int_{\mathbb{R}} |A_h(f)| < \infty$ by the above calculation, we have $H \in L^1(\mathbb{R}^2)$, and Fubini applies.

6.2 Part (b)

Let $\varepsilon > 0$; we then have

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \; dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - f(x) \right| \; dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \; dy \right| \; dx \\
& \quad \text{since } \frac{1}{2h} \int_{x-h}^{x+h} f(x) \; dy = \frac{1}{2h} f(x)((x+h) - (x-h)) = \frac{1}{2h} f(x) 2h = f(x) \\
&= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{B(h,x)} f(y) - f(x) \; dy \right| \; dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - f(x) \; dy| \; dx
\end{aligned}
$$

.

but since $h \to 0$ will force $y \to x$ in the integral, by continuity in L^1, we can choose an h' small enough such that $x, y \in B_h(x) \implies \int |f(y) - f(x)| < \varepsilon$.

Thus, for $h < h'$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \; dx &\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - f(x) \; dy| \; dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \varepsilon \; dx \\
&\leq \lim_{h \to 0} \frac{1}{2h} \varepsilon \int_{\mathbb{R}} \; dx
\end{aligned}
$$

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