

Title

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1 | Lecture 07

Last time: stalks, sheafification, and $\mathrm{Sh}(X_{\text{ét}})$ is abelian. Next up, we're aiming to define sheaf cohomology for $\mathrm{Sh}(X_{\text{ét}})$.

Remark 1.0.1 (Esoteric!): Related to a question asked by a viewer: there is not in fact a morphism from $X_{\text{fppf}} \rightarrow X_{\text{ét}}$, since locally finitely-presented need not be finitely presented (part of the condition for fppf). There is instead a morphism $X_{\text{fppf}} \rightarrow X_{\text{ét,fp}}$ to a corresponding finitely presented site. There is also a map $X_{\text{ét}} \rightarrow X_{\text{ét,fp}}$ inducing an equivalence on the category of sheaves via pushforward. 

Theorem 1.0.2 (Enough injectives).

$\mathrm{Sh}(X_{\text{ét}})$ has enough injectives.

Proof (?).

Given $\mathcal{F} \in \mathrm{Sh}(X_{\text{ét}})$ we want an injective sheaf \mathcal{I} and an injection $\mathcal{F} \hookrightarrow \mathcal{I}$. For each $x \in X$, choose a geometric point \bar{x} over x , and let $I(\bar{x})$ be an injective \mathbb{Z} -module with a map $\mathcal{F}_{\bar{x}} \rightarrow I(\bar{x})$. These exist because the category of \mathbb{Z} -modules has enough injectives. The injectives in this category are **divisible** abelian groups.

Claim: The following object works:

$$\mathcal{I} := \prod_{\bar{x}} (\iota_{\bar{x}})_* I(\bar{x}).$$

We need to check

1. There is a map $\mathcal{F} \rightarrow \mathcal{I}$: The RHS is a product, so we map into the components. $\mathcal{F}_{\bar{x}}$ maps into its own associated skyscraper sheaf where the map is sending sections to their germs. Then the skyscraper sheaf for $\mathcal{F}_{\bar{x}}$ maps into the skyscraper sheaf for $I(\bar{x})$ by pushforward.
2. This is a monomorphism: check on stalks.
3. \mathcal{I} is injective: check the lifting property directly.

■

1.1 What Else We Get From Sheafification

Remark 1.1.1: We now know that $\mathrm{Sh}(X_{\text{ét}})$ is abelian with enough injectives. This is true for $\mathrm{Sh}(\tau)$ for any site τ , but this is substantially harder to show.

1.1.1 Inverse Images

For $f : X \rightarrow Y$, we have a map on presheaves

$$f^{-1} : \text{Presh}(Y_{\text{ét}}) \rightarrow \text{Presh}(X_{\text{ét}})$$

$$\mathcal{F}(V \xrightarrow{\text{ét}} X) \mapsto \varinjlim \mathcal{F}(U \rightarrow X),$$

where the limit is over diagrams of the form

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow \text{ét} & & \downarrow \text{ét} \\ X & \longrightarrow & Y \end{array}$$

Fact 1.1.2: f^{-1} is left adjoint to pushforward as functors on presheaves.

Exercise 1.1.3(?): Check this.

Definition 1.1.4 (Inverse Image Sheaf)

$$f^* \mathcal{F} := (f^{-1} \mathcal{F})^a.$$

Theorem 1.1.5(?).

f^* is left adjoint to f_* .

Proof (?).

Sheafification is a left adjoint. ■

Example 1.1.6(?):

- For $\bar{x} \xrightarrow{\iota} X$ a geometric point, we have $\iota^* \mathcal{F} = \mathcal{F}_{\bar{x}}$.
- For $Y \xrightarrow{f} X$, we have $f^* \underline{\mathbb{Z}/\ell\mathbb{Z}} = \underline{\mathbb{Z}/\ell\mathbb{Z}}$.
- More generally, for $Y \xrightarrow{f} X$ and any representable functor $\mathcal{F} := \underline{\text{hom}}_X(\cdot, Z)$, we have $f^* \mathcal{F} = \underline{\text{hom}}_Y(\cdot, Y \times_X Z)$.

1.2 Étale Cohomology

See ?? for the definition of étale cohomology. How do we compute $H^i(X_{\text{ét}}, \mathcal{F})$? Choose an injective resolution

$$\mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$$

with the \mathcal{I}^j injectives. From the general theory of derived functors, we obtain

$$H^i(X_{\text{ét}}, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)),$$

where the RHS is a complex of abelian groups. Injective resolutions are difficult to find in general. Suppose $\pi : X_{\text{ét}} \rightarrow Y_{\text{ét}}$ comes from a map of schemes, then we can compute derived functors of other functors such as the pushforward,

$$(R^i \pi_*) \mathcal{F} = H^i(\pi_* \mathcal{I}^\bullet),$$

where the RHS are sheaves on $Y_{\text{ét}}$. Implicit here is the claim that π_* is left-exact. You can also find $(L^{>0} \pi^*) \mathcal{G} = 0$.

Exercise 1.2.1(?): Check that pullback is exact.

Proposition 1.2.2 (Properties of étale cohomology).

1. $H^0(X_{\text{ét}}, \mathcal{F}) = \mathcal{F}(X)$, aka the global sections $\Gamma(X, \mathcal{F})$.
2. $H^{>0}(\mathcal{I}) = 0$ for \mathcal{I} injective.
3. Given a SES of sheaves in $\text{Sh}(X_{\text{ét}})$

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

there is a LES

$$\dots \rightarrow H^{i+1}(X_{\text{ét}}, C) \xrightarrow{\delta} H^i(X_{\text{ét}}, A) \rightarrow \dots$$

Example 1.2.3(?): Suppose k is a field, not necessarily algebraically closed, and consider $\text{Sh}((\text{Spec } k)_{\text{ét}})$. Let $G := \text{Gal}(k^s/k)$ for a choice of separable closure k^s/k .

Claim: There is a functor from $\text{Sh}((\text{Spec } k)_{\text{ét}})$ to discrete G -modules¹ inducing an equivalence of categories.

Note that when thinking of Galois representations, \mathbb{Z}_ℓ is not an example of this, but a representation over a finite field works. E.g. the Tate module (the inverse limit of torsion) of an elliptic curve is not a discrete G -module since the Galois action is not continuous in the discrete topology (although it is in the ℓ -adic topology).

¹ G is a topological group in the inverse limit topology, so a discrete G -module is a module with the discrete topology where the G -action is continuous. In particular, the action on any element factors through a finite quotient of G .

To prove this claim, the map is given by

$$\iota : \mathrm{Sh}((\mathrm{Spec} k)_{\mathrm{\acute{e}t}}) \rightarrow \text{Discrete } G\text{-modules}$$

$$\mathcal{F} \mapsto \varprojlim_{k \subset L \subset k^s} \mathcal{F}(\mathrm{Spec} L).$$

The idea here: you want to evaluate \mathcal{F} on k^s , which doesn't make sense because k^s is not locally finitely-presented, so we take a limit instead. The claim is that the image is a discrete G -module and this is an equivalence. This follows because each term is, and taking limits preserves this property.

Corollary 1.2.4(?).

$H^i((\mathrm{Spec} K)_{\mathrm{\acute{e}t}}, \mathcal{F}) = H^i(G, \iota\mathcal{F})$, which is the Galois cohomology.

Why? Derived functors only depend on the ambient category, so it suffices to check H^0 .

Proof (of claim).

We get a G -module since G acts on the entire diagram and thus its limit.

Exercise 1.2.5(?): Check that this is a discrete G -module.

There is an inverse functor: given $V \rightarrow \mathrm{Spec} k$ an étale map, by the classification of étale k -algebras we have $V = \coprod_{k \subset K'} \mathrm{Spec} k'$ where K' is the set of all finite separable k'/k . Given a discrete G -module M , send it to the Galois fixed points $V \rightarrow \prod M^{G'_s}$ where $G'_s := \mathrm{Gal}(k^s/k')$.

Exercise 1.2.6(Check): Check that this is an inverse, it follows from Galois descent. ■

Proof (of corollary).

$\Gamma(\mathrm{Spec} k, \mathcal{F}) = (\iota\mathcal{F})^G$, taking the G -invariants. So $H^0 \xrightarrow{\iota}$ to taking invariants, and thus the higher derived functors agree, where the RHS are group cohomology. ■