Title

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Contents

1 | **Thursday, August 27**

1.1 Consequence of the Nullstellensatz

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{ \text{Radical ideals } k[x_1, \cdots, x_n] \right\} \xrightarrow{V(\,\cdot\,)I(\,\cdot\,)} \left\{ \text{Affine varieties in } \mathbb{A}^n \right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

> **Corollary 1.1.1*(?)*.**
> If $V(I)$ is empty, then $I = \langle 1 \rangle$.

Remark 1.1.2: This is because no common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Slogan 1.1.3: The only ideals that vanish nowhere are trivial.

> *Proof .*
> By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty. ∎

1.2 Proof of Remaining Part of Nullstellensatz

We now return to the remaining hard part of the proof of the Nullstellensatz:

$I(V(J)) \subseteq \sqrt{J}$ (hard part)"}

Let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal

$$\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$$

Observation 1.2.1: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$.

However, if $f = 0$, then $ft - 1 \neq 0$, so

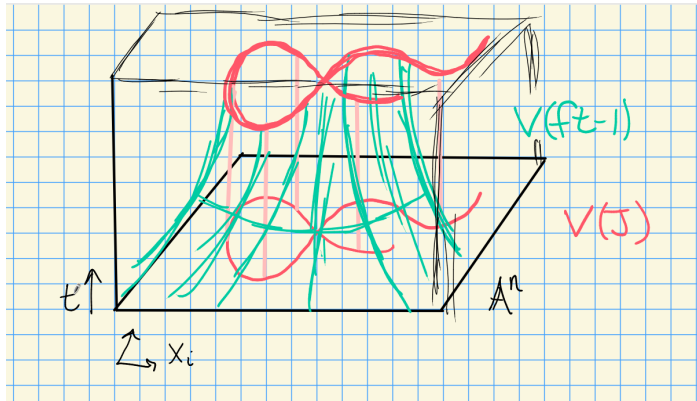$$V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$$

Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so

$$1 = \langle ft - 1 \rangle \, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$$

with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N \coloneqq (ft - 1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t. Now take $k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle.$$

:::{.proposition title="?"}{#prop:inclusion}

There is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t]/\langle ft - 1 \rangle.$$

:::

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

:::

Exercise 1.2.2*(?):* Why is this true?

Example 1.2.3: Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then

$$I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J,$$

so this loses information.

Example 1.2.4: Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\left\{ \text{Points of } \mathbb{A}^n \right\} \iff \left\{ \text{Maximal ideals of } k[x_1, \cdots, x_n] \right\}.$$

> **Theorem 1.2.5** *(Properties of I).*
>
> a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
>
> b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

> *Proof .*
> We proved (a) on the variety side.
> For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so
>
> $$\begin{aligned} I(X_1 \cap X_2) &= I\left(VI(X_1) \cap VI(X_2)\right) \\ &= IV\left(I(X_1) + I(X_2)\right) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$
>
> ∎

Example 1.2.6: Example of property (b):

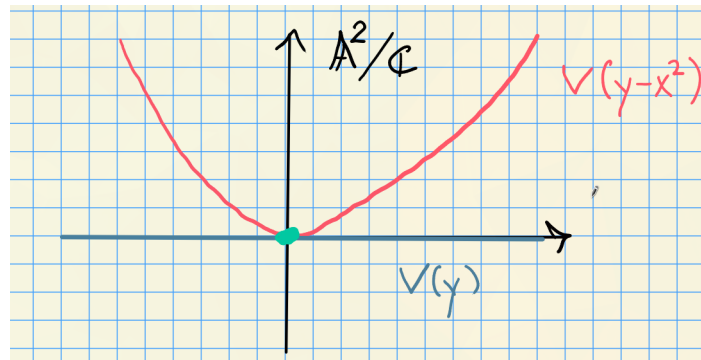Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

Figure 2: Intersecting $V(y - x^2)$ and $V(y)$

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but

$$I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$$

> **Proposition 1.2.7 *(?)*.**
> If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

> *Proof* .
> Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$.
> ∎

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

> **Definition 1.2.8** (Coordinate Ring)
> For an affine variety X, the *coordinate ring of X* is
>
> $$A(X) := k[x_1, \cdots, x_n]/I(X).$$
>
> Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation 1.2.9: The constructions $V(\,\cdot\,), I(\,\cdot\,)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \left\{ x \in Y \mid f(x) = 0 \; \forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \left\{ f \in A(Y) \mid f(x) = 0 \; \forall x \in X \right\} \subseteq A(Y).$$

Example 1.2.10: For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$A(\mathbb{A}^n) \xrightarrow{\;\cdot/I(Y)\;} A(Y) \xrightarrow{\;\cdot/I(X)\;} A(X)$$

with an arc labeled $\cdot/I(X)$ from $A(\mathbb{A}^n)$ to $A(X)$.

> **Theorem 1.2.11 *(?)*.**
> Let $X \subset Y$ be an affine subvariety, then
>
>   a. $A(X) = A(Y)/I_Y(X)$

b. There is a correspondence

$$\left\{\text{Affine subvarieties of } Y\right\} \iff \left\{\text{Radical ideals in } A(Y)\right\}$$
$$X \mapsto I_Y(X)$$
$$V_Y(J) \leftmapsto J.$$

Proof .

Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ∎

Example 1.2.12: Let $Y = V(y - x^2) \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x - 1, y - 1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \left\langle x - 1, y - 1 \right\rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$
\begin{array}{ccccc}
A(\mathbb{A}^n) & \longrightarrow & A(Y) & \longrightarrow & A(X) \\
\| & & \| & & \| \\
k[x,y] & \longrightarrow & k[x,y]/\left\langle y - x^2 \right\rangle & \longrightarrow & k[x,y]/\left\langle x - 1, y - 1 \right\rangle
\end{array}
$$