Homological Algebra Problem Sets

Problem Set 3

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Problem 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
$\Longleftarrow$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\
& A &
\end{tikzcd}$$

Link to Diagram

Since R is a PID, we can write $J = \langle j \rangle$ for some generator. Fixing a $\mathbf{a} \in A$, define a map $f_a : J \to A$ on generators as follows: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\to R \\ jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we get a lift:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f_a"] & R \arrow[dl, dashed, "\exists \tilde{f}_a"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

Problem 1.0.3 (Weibel 2.3.2)
For $A \in \mathbf{Ab}$, define $I(A) := \displaystyle\bigoplus_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \to I(A)$. Show that e_A is injective.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_iF) \cong L_i(UF).$$

Problem 1.0.5 (Weibel 2.4.3)
If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_iF(A) \cong L_{i-1}FM \qquad i \geq 2.$$

Show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

a. A is a projective R-module.

b. $\mathrm{Hom}_R(\,\cdot\,, A)$ is an exact functor.

c. $\mathrm{Ext}_R^{i\neq 0}(A, B) = 0$ and for all B, i.e. A is $\mathrm{Hom}_R(\,\cdot\,, B)$-acyclic for all B.

d. $\mathrm{Ext}_R^1(A, B)$ vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on **Ab**.