

Problem Set 7

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November 14, 2019

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1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence, so $\|x^j - x^k\|_{\ell^2} \rightarrow 0$. We want to produce some $\mathbf{x} := \lim_{n \rightarrow \infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i , define

$$\mathbf{x}_i := \lim_{n \rightarrow \infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i |x_i^j - x_i^k|^2 \rightarrow 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i , the sequence $|x_i^j - x_i^k|^2$ is a Cauchy sequence of real numbers which necessarily converges by completeness of \mathbb{R} .

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \rightarrow 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\| \lim_{k \rightarrow \infty} x^k - x^j \right\|_{\ell^2} = \lim_{k \rightarrow \infty} \|x^k - x^j\|_{\ell^2} \rightarrow 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \rightarrow \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$\begin{aligned} \|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\ &\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\ &\rightarrow M < \infty, \end{aligned}$$

where $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$ by the previous argument, and the second term is bounded because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$. \square

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z , we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on H takes values in \mathbb{C} , we have

$$\Re(\langle x, iy \rangle) = \Re(-i\langle x, y \rangle) = \Im(\langle x, y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, y \rangle)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, y \rangle)$$

$$\begin{aligned} \|x + iy\|^2 &= \|x\|^2 + \|y\|^2 + 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

$$\begin{aligned} \|x - iy\|^2 &= \|x\|^2 + \|y\|^2 - 2 \Re(\langle x, iy \rangle) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x, y \rangle) \end{aligned}$$

and summing these all

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 &= 4 \Re(\langle x, y \rangle) + 4i \Im(\langle x, y \rangle) \\ &= 4\langle x, y \rangle. \end{aligned}$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 := \langle x, x \rangle = \langle Ux, Ux \rangle := \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned} \langle Ux, Uy \rangle &= \frac{1}{4} \left(\|Ux + Uy\|^2 + \|Ux - Uy\|^2 + i\|Ux + iUy\|^2 - i\|Ux - iUy\|^2 \right) \\ &= \frac{1}{4} \left(\|U(x + y)\|^2 + \|U(x - y)\|^2 + i\|U(x + iy)\|^2 - i\|U(x - iy)\|^2 \right) \\ &= \frac{1}{4} \left(\|x + y\|^2 + \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right) \\ &= \langle x, y \rangle. \end{aligned}$$

□

2 Problem 2

Lemma: The map $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$ is continuous.

Proof:

Let $x_n \rightarrow x$ and $y_n \rightarrow y$, then

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
&= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\
&\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \\
&\rightarrow 0 \cdot M + C \cdot 0 < \infty,
\end{aligned}$$

where $\|y_n\| \rightarrow M$ since $y_n \rightarrow y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

We want to show that sequences in E^\perp converge to elements of E^\perp . Using the lemma, letting $\{e_n\}$ be a sequence in E^\perp , so $y \in E \implies \langle e_n, y \rangle = 0$. Since H is complete, $e_n \rightarrow e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$\langle e, y \rangle = \left\langle \lim_n e_n, y \right\rangle = \lim_n \langle e_n, y \rangle = \lim_n 0 = 0,$$

so $e \in E^\perp$.

2.2 Part b:

Let $S := \text{span}_H(E)$; then the smallest closed subspace containing E is \overline{S} , the closure of S . We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S , so $x_n \rightarrow x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$y \in E^\perp \implies \langle x_n, y \rangle = \left\langle \sum_i a_i e_i, y \right\rangle = \sum_i a_i \langle e_i, y \rangle = 0 \implies x_n \in (E^\perp)^\perp.$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$y \in E^\perp \implies \langle x, y \rangle = \left\langle \lim_n x_n, y \right\rangle = \lim_n \langle x_n, y \rangle = 0 \implies x \in (E^\perp)^\perp,$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notation convenience, we'll just write S for \overline{S} . Let $x \in E^{\perp\perp}$. Noting that S is closed, we can define P , the operator projecting elements onto S , and write

$$x = Px + (x - Px) \in S \oplus S^\perp$$

But since $\langle x, x - Px \rangle = 0$ because $x - Px \in E^\perp$ and $x \in (E^\perp)^\perp$, we can rewrite the first term in this inner product to obtain

$$0 = \langle x, x - Px \rangle = \langle Px + (x - Px), x - Px \rangle = \langle Px, x - Px \rangle + \langle x - Px, x - Px \rangle,$$

where we can note that the first term is zero because $Px \in S$ and $x - Px \in S^\perp$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\begin{aligned}\|e_0\|^2 &= \int_0^1 1^2 dx = 1 \\ \|e_1\|^2 &= \int_0^1 3(2x-1)^2 dx = \frac{1}{2}(2x-1)^2 \Big|_0^1 = 1 \\ \langle e_0, e_1 \rangle &= \int_0^1 \sqrt{3}(2x-1) dx = \frac{\sqrt{3}}{4}(2x-1) \Big|_0^1 = 0.\end{aligned}$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0, 1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$\begin{aligned}
f(x) &= \sum_i \langle x^3, e_i \rangle e_i \\
&= \langle x^3, 1 \rangle 1 + \langle x^3, \sqrt{3}(2x-1) \rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2 dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1) dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1) \frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}$$

We can also compute

$$\begin{aligned}
\|f - g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2 dx \\
&= \frac{1}{180} \\
\Rightarrow \|f - g\|_2 &= \frac{1}{\sqrt{180}}.
\end{aligned}$$

4 Problem 4

4.1 Part a

4.1.1 i

We can first note that $\langle 1/\sqrt{2}, \cos(2\pi nx) \rangle = \langle 1/\sqrt{2}, \sin(2\pi mx) \rangle = 0$ for any n or m , since this involves integrating either sine or cosine over an integer multiple of its period.

Letting $m, n \in \mathbb{Z}$, we can then compute

$$\begin{aligned}
\langle \cos(2\pi nx), \sin(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx) \sin(2\pi mx) dx \\
&= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x) dx \\
&= \frac{1}{2} \int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2} \int_0^1 \sin(2\pi(n-m)x) dx \\
&= 0,
\end{aligned}$$

which again follows from integration of sine over a multiple of its period (where we use the fact that $m+n, m-n \in \mathbb{Z}$).

Similarly,

$$\begin{aligned}
\langle \cos(2\pi nx), \cos(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx) \cos(2\pi mx) \, dx \\
&= \frac{1}{2} \int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x) \, dx \\
&= \begin{cases} \frac{1}{2} \int_0^1 \cos(4\pi nx) + 1 \, dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}$$

$$\begin{aligned}
\langle \sin(2\pi nx), \sin(2\pi mx) \rangle &= \int_0^1 \sin(2\pi nx) \sin(2\pi mx) \, dx \\
&= \frac{1}{2} \int_0^1 \cos(2\pi(m-n)x) - \cos(2\pi(m+n)x) \, dx \\
&= \begin{cases} \frac{1}{2} \int_0^1 1 - \cos(4\pi nx) \, dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

4.1.2 ii

We have

$$\begin{aligned}
\langle e^{2\pi kx}, e^{-2\pi i\ell x} \rangle &= \int_0^1 e^{2\pi ikx} \overline{e^{-2\pi i\ell x}} \, dx \\
&= \int_0^1 e^{2\pi ikx} e^{-2\pi i\ell x} \, dx \\
&= \int_0^1 e^{2\pi i(k-\ell)x} \, dx \\
&= \int_0^1 1 \, dx = 1 \quad \text{if } k = \ell, \text{ otherwise:} \\
&= \left. \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)} \right|_0^1 \\
&= \frac{e^{2\pi i(k-\ell)} - 1}{2\pi i(k-\ell)} \\
&= 0,
\end{aligned}$$

since $e^{2\pi ik} = 1$ for every $k \in \mathbb{Z}$, and $k - \ell \in \mathbb{Z}$. Thus this set is orthonormal.

4.2 Part b

4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials $P_n(x)$ such that $\|f - P_n\|_\infty \rightarrow 0$, i.e. the P_n uniformly approximate f on $[0, 1]$.

Letting $\varepsilon > 0$, we can thus choose a P such that $\|f - P\|_\infty < \varepsilon$, which necessarily implies that $\|f - P\|_{L^1} < \varepsilon$ since we have

$$\int_0^1 |f(x) - P(x)| \, dx \leq \int_0^1 \varepsilon \, dx = \varepsilon.$$

Thus we can write

$$f(x) = P(x) + (f(x) - P(x))$$

where $h(x) := f(x) - P(x)$ satisfies $\|h\|_{L^1} < \varepsilon$. It only remains to show that $P \in L^2([0, 1])$, but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say $|P(x)| \leq M < \infty$ for all $x \in [0, 1]$, and thus

$$\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2 \, dx \leq \int_0^1 M^2 \, dx = M^2 < \infty.$$

It follows that we can let $g = P$ and $h = f - P$ to obtain the desired result.

4.2.2 ii

By part (i), the claim is that it suffices to show this is true for $f \in L^2$. In this case, we can identify

$$\begin{aligned} \int_0^1 f(x) \cos(2\pi kx) \, dx &:= \Re(\hat{f}(k)) \\ \int_0^1 f(x) \sin(2\pi kx) \, dx &:= \Im(\hat{f}(k)), \end{aligned}$$

the real and imaginary parts of the k th Fourier coefficient of f respectively.

By Bessel's inequality, we know that $\{\hat{f}(k)\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$, and so $\sum_k |\hat{f}(k)| < \infty$.

But this is a convergent sequence of real numbers, which necessarily implies that $|\hat{f}(k)| \rightarrow 0$. In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have $f \in L^1$, write $f = g + h$ where $g \in L^2$ and $\|h\|_{L^1} \rightarrow 0$. Then

$$\begin{aligned} \left| \int_0^1 f(x) \cos(2\pi kx) \, dx \right| &= \left| \int_0^1 (g(x) + h(x)) \cos(2\pi kx) \, dx \right| \\ &\leq \left| \int_0^1 g(x) \cos(2\pi kx) \, dx \right| + \left| \int_0^1 h(x) \cos(2\pi kx) \, dx \right| \\ &\leq \left| \int_0^1 g(x) \cos(2\pi kx) \, dx \right| + \int_0^1 |h(x)| |\cos(2\pi kx)| \, dx \\ &= |\hat{g}(k)| + \varepsilon \\ &\rightarrow 0, \end{aligned}$$

with a similar computation for $\int f(x) \sin(2\pi kx)$. \square

5 Problem 5

5.1 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e}_1 = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i, \hat{e}_k \rangle \hat{e}_k$

The result set $\{\hat{e}_i\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$\begin{aligned} e_2 &= x - \langle x, \hat{e}_1 \rangle \hat{e}_1 \\ &= x - \langle x, 1 \rangle 1 \\ &= x - \int_{-1}^1 \frac{1}{\sqrt{2}} x \, dx \\ &= x - \int \text{odd function} \\ &= x, \end{aligned}$$

and so $e_2 = x$. We can then check that

$$\|e_2\| = \left(\int_{-1}^1 x^2 \, dx \right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}}x$.

We continue to compute

$$\begin{aligned} e_3 &= x^2 - \langle x^2, \hat{e}_1 \rangle \hat{e}_1 - \langle x^2, \hat{e}_2 \rangle \hat{e}_2 \\ &= x^2 - \frac{1}{2} \int_{-1}^1 x^2 \, dx - \frac{3}{2} x \int_{-1}^1 x^3 \, dx \\ &= x^2 - \left(\frac{1}{6} x^3 \right) \Big|_{-1}^1 + \frac{3}{2} x \int_{-1}^1 \text{odd function} \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$\begin{aligned}
\hat{e}_3 &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \\
&= \frac{1}{2} \sqrt{\frac{45}{2}} \frac{1}{3} (3x^2 - 1) \\
&= \frac{1}{3} \sqrt{\frac{45}{2}} \left(\frac{3x^2 - 1}{2} \right).
\end{aligned}$$

In summary, this yields

$$\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{2}} \\
\hat{e}_2 &= x \\
\hat{e}_3 &= \frac{1}{3} \sqrt{\frac{45}{2}} \left(\frac{3x^2 - 1}{2} \right),
\end{aligned}$$

which are scalar multiples of the first three Legendre polynomials.

5.2 Part b

Let $p(x) = a + bx + cx^2$, we are then looking for p such that $\|x^3 - p(x)\|_2^2$ is minimized. Noting that

$$p(x) \in \text{span} \{1, x, x^2\} = \text{span} \{P_0(x), P_1(x), P_2(x)\} := S,$$

we can conclude that $p(x)$ will be the projection of x^3 onto S . Thus $p(x) = \sum_{i=0}^2 \langle x^3, \hat{e}_i \rangle \hat{e}_i$.

Proceeding to compute the terms in this expansion, we can note that $\langle x^3, f \rangle$ for any f that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$\langle x^3, x \rangle x = x \int_{-1}^1 x^4 dx = \frac{2}{5}x$$

And thus $p(x) = \frac{2}{5}x$ is the minimizer.

5.3 Part c

The first three conditions necessitate $g \in S^\perp$ and $\|g\| = 1$. Since S is a closed subspace, we can write $x^3 = p(x) + (x^3 - p(x)) \in S \oplus S^\perp$, and so $x^3 - p(x) \in S^\perp$.

The claim is that $g(x) := x^3 - p(x)$ is a scalar multiple of the desired maximizer. This follows from the fact that

$$|\langle x^3 - p, g \rangle| \leq \|x^3 - p\| \|g\|$$

by Cauchy-Schwarz, with equality precisely when $g = \lambda(x^3 - p)$ for some scalar λ . However, the restriction $\|g\| = 1$ forces $\lambda = \|x^3 - p\|^{-1}$.

A computation shows that

$$\|x^3 - p\|^2 = \int_0^1 (x^3 - \frac{2}{5}x)^2 dx = \frac{19}{525},$$

and so we can take

$$g(x) := \frac{25}{\sqrt{19}} \left(x^3 - \frac{2}{5}x \right).$$

6 Problem 6

6.1 Part a

To see that $g \in \mathcal{C}$, we can compute

$$\begin{aligned} \langle g, 1 \rangle &= \int_0^1 18x^2 - 5 dx = 6 - 5 = 1 \\ \langle g, x \rangle &= \int_0^1 18x^3 - 5x dx = \frac{18}{4} - \frac{5}{2} = 2. \end{aligned}$$

To see that $\mathcal{C} = g + S^\perp$, let $f \in \mathcal{C}$, so $\langle f, 1 \rangle = 1$ and $\langle f, x \rangle = 2$. We can then conclude that $f - g \in S^\perp$, since we have

$$\begin{aligned} \langle f - g, 1 \rangle &= \langle f, 1 \rangle - \langle g, 1 \rangle = 1 - 1 = 0 \\ \langle f - g, x \rangle &= \langle f, x \rangle - \langle g, x \rangle = 2 - 2 = 0 \end{aligned}$$

6.2 Part b