Problem Set 7

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October 26, 2019

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1 Regular Problems

1.1 Problem 1

Note that if either $p = 1$ or $q = 1$, G is a p-group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2 q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$
\begin{aligned}
n_p &\equiv 1 \mod p \qquad & n_q &\equiv 1 \mod q \\
n_p &\,\Big|\, q & n_q &\,\Big|\, p^2,
\end{aligned}
$$

which forces

$$
n_p \in \{1, q\}, \quad n_1 \in \left\{1, p, p^2\right\}.
$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. Proceeding by cases:

1.1.1 Case 1: $p = q$.

Then $\#G = p^3$ and G is a p-group. But every p-group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.1.2 Case 2: $p > q$.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q:

- $n_q = p$: If $n_q = p = 1 \mod q$ and $p < q$, this forces $p = 1$.
- $n_1 = p^2$: We will reach a contradiction by showing that this forces

$$\left| S := \bigcup_{S_p \in \mathrm{Syl}(p,G)} S_p \right| + \left| P := \bigcup_{S_q \in \mathrm{Syl}(q,G)} S_q \right| > |G|.$$

Towards this end,

2 Qual Problems