

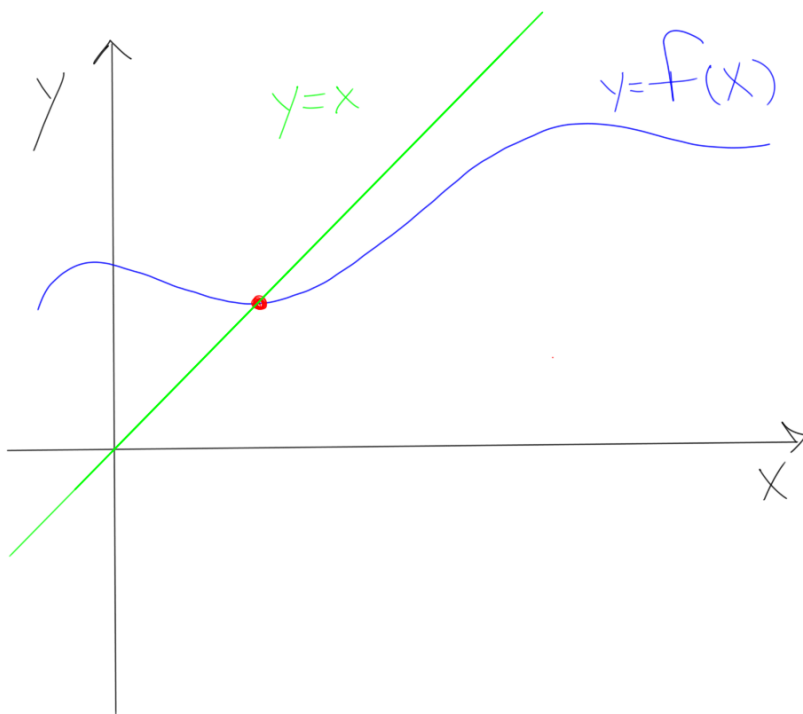
Major Questions:

- What is Topology?
- What is a fixed point?

What is a fixed point?

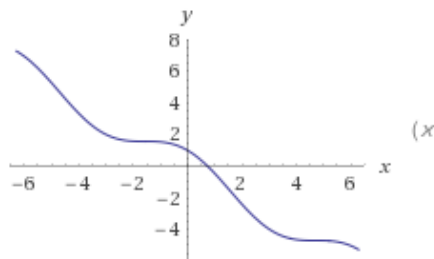
- Algebraically, for a given $A \ni f$, defined by $F = \{x : f(x) = x\}$.
 - That is, you plug in a number, you get the same number out. Think of $A = [0, 1]$ or \mathbb{R} .
- Geometrically, several equivalent ways to think about it
 - Draw graph of f in the plane, then find an x such that $f(x) = x$: this is exactly the intersection of the graphs of $y = f(x)$ and $y = x$.

Draw $f(x) = \cos(x)$



- Equivalently, draw graph of $g(x) := f(x) - x$ and find a zero. Draw $g(x) = \cos(x) - x$:

Plots:



- Okay, but what about $\mathbb{R}^n \circlearrowleft_f$? Now we need a different model, think of this as a *deformation* of the space:

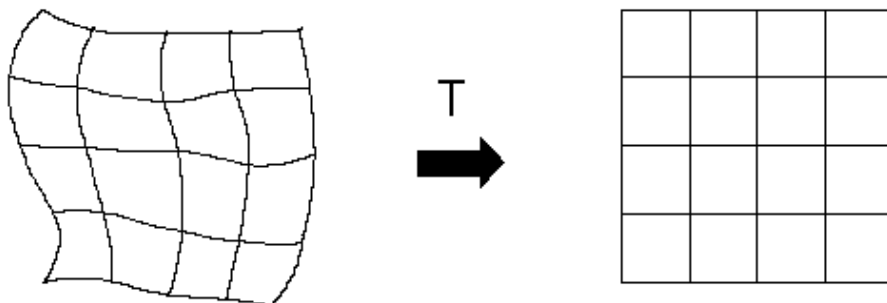


Figure 4.5 Geometric transform on a plane.

Imagine a movie from right pic to left - a fixed point remains stationary! ## Examples {#examples }

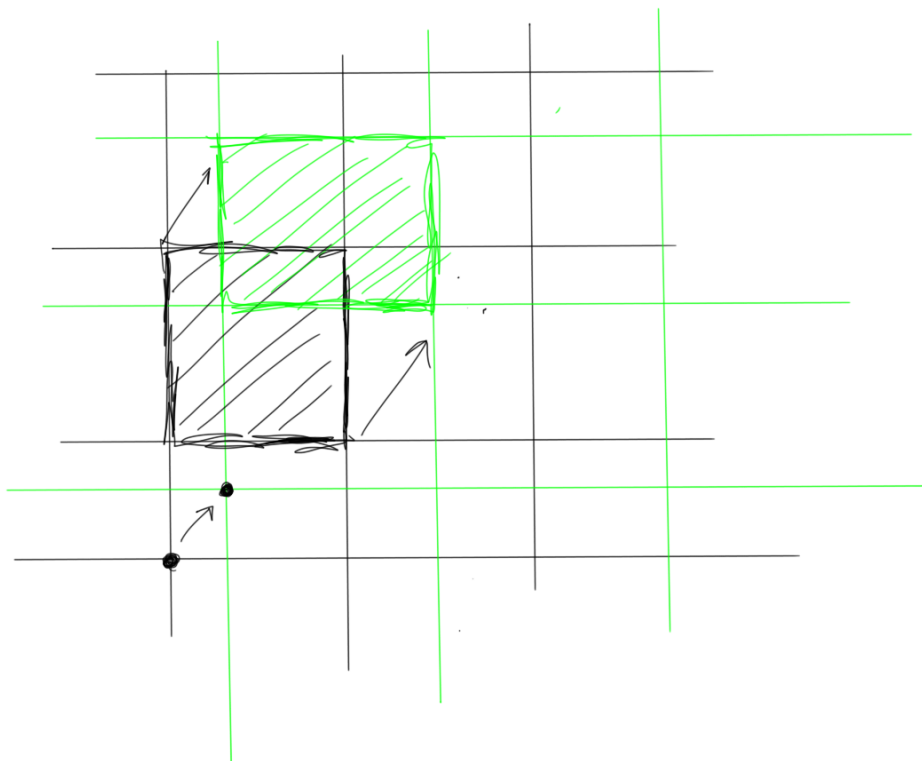
(People can guess, show algebraically solving $f(x) = x$)

- Dumb example: $\mathbb{R} \xrightarrow{x \mapsto x} \mathbb{R} \implies F = \mathbb{R}$
- Equally dumb example: $\mathbb{R} \xrightarrow{x \mapsto c} \mathbb{R} \implies F = \{c\}$
- Less dumb example: $\mathbb{R} \xrightarrow{x \mapsto x^2} \mathbb{R} \implies F = \{0\}$
- Less dumb example: $\mathbb{R} \xrightarrow{x \mapsto ax} \mathbb{R} \implies F = \{0\}$

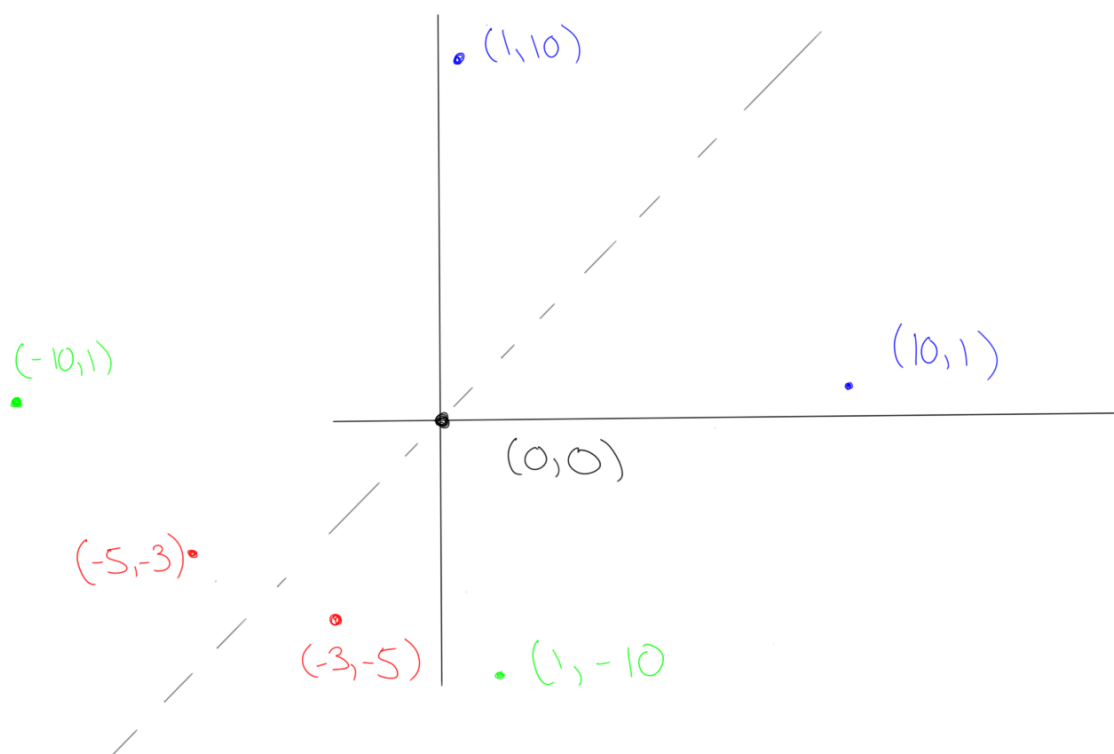
Show dilation of grid

- Less dumb example: $\mathbb{R} \xrightarrow{x \mapsto ax+b} \mathbb{R} \implies F = \emptyset$ (as long as $b \neq 0$)

How do we know there are no fixed points? Show dilation and translation of grid



- Moderate example: $\mathbb{R}^2 \xrightarrow{(x,y) \mapsto (y,x)} \Rightarrow F = \{y = x\}$ Just swapping coordinates, look at a few:



This is a reflection through the line $y = x$.

- Nontrivial example: $\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$ where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(Spoiler: $F = \{0\}$, this is a counter-clockwise rotation by $\pi/2$)

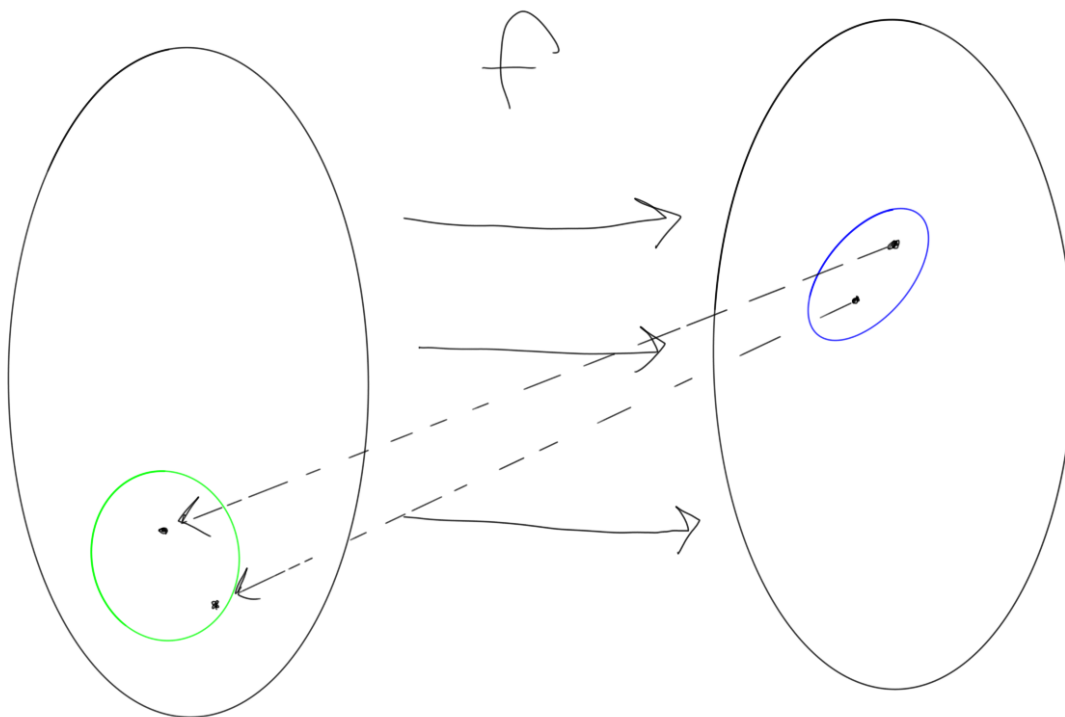
- Key example: $\mathbb{R} \xrightarrow{x \mapsto \cos(x)} \mathbb{R} \implies F = \{0.739085 \dots\}$.
- Moral: Fixed points can be non-obvious!

Why are fixed points important?

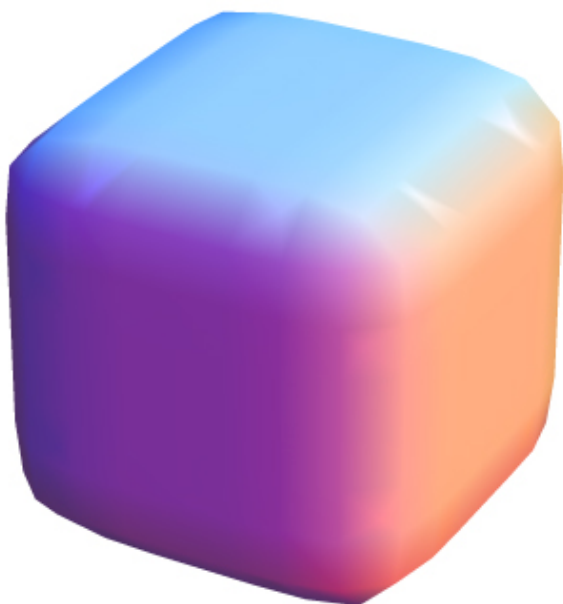
- Essentially any problem that asks for an x such that the equation $f(x) = 0$ holds can be equivalently stated as a fixed point of some different equation $g(x) = x$
 - Example: $f(x) = x^4 - x - 10$ goes to $g(x) = 10/(x^3 - 1)$ (move 10, divide x) or $(x + 10)^{1/4}$ (move $x - 10$, take root)
- The core of a huge swatch of numerical approximation techniques: Zeros of polynomials, ODEs, PDEs, Eigenvalues, etc.
- Important theoretically - finite group theory, lie groups, topology where groups “act” on a space, etc. Used as a concrete numerical computation, maps without *any* fixed point are kind of special (like rotations).

Topology

What is continuity?

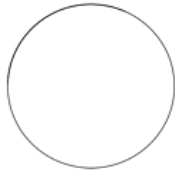
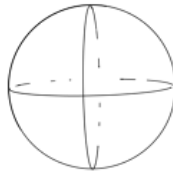


Visual indication of continuity



Here's an example of something halfway between a sphere and a cube

What kind of spaces we will be using:

S_0  S_1  S_2  D_0  D_1  D_2 

Spheres and Discs

Fundamental Group:

A functor from Top to Grp

Send identity maps to isomorphisms.

Facts: $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(D^1) = 0$

Brouwer's Fixed Point Theorem

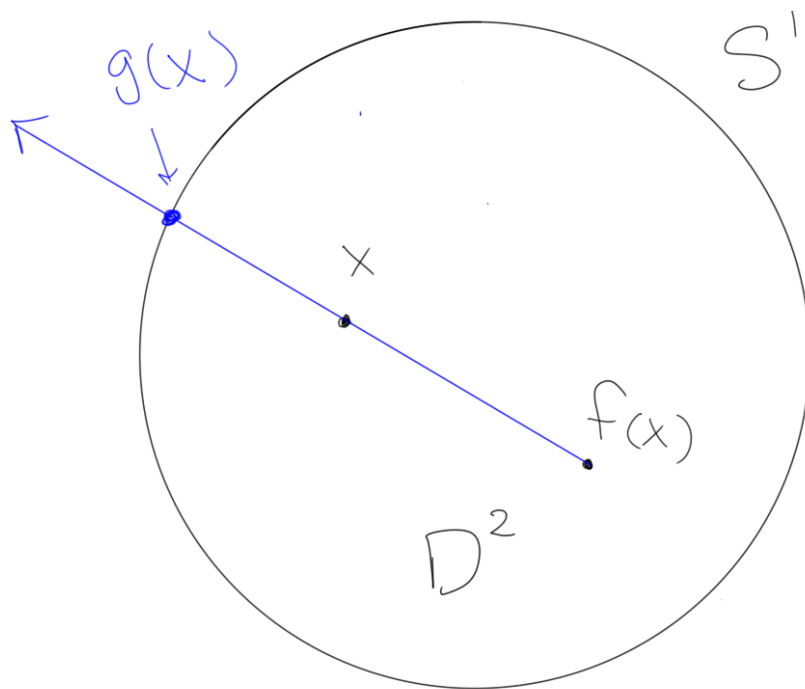
Every continuous function $D^n \rightarrow D^n$ has a fixed point. (Generally, compact convex subset of \mathbb{R}^n)

Examples

- Crumpled pieces of paper with grid $D^2 \rightarrow D^2$
- Map on a table $D^2 \rightarrow D^2$
- Stir drink $D^3 \rightarrow D^3$

Proof

Construct g ,



Definition of $g : D^2 \rightarrow S^1$

This fixes ∂D^2 . Then

$$S^1 \hookrightarrow D^2 \twoheadrightarrow^g S^1$$

induces

$$\begin{aligned}\pi_1(S^1) &\xrightarrow{i_*} \pi_1(D^2) \xrightarrow{f} \pi_1(S^1) \\ &= \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}\end{aligned}$$

Then $f \circ \iota = \text{id}_{S^1}$, but ι_* is the zero map, so this can't induce an isomorphism - contradiction.

Borsuk Ulask:

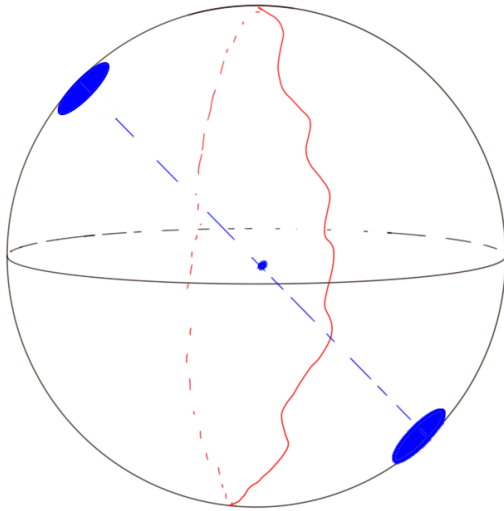
There exist a pair of antipodal points with same temperature and pressure. There are *infinitely* many antipodal points with the same temperature!

What is an antipode? Given any point A , you can find its antipode by drawing a line through the center of the sphere - the antipode is the other spot in which this line intersects the sphere.

Idea: pick a point A (say the bottom blue point in the diagram), then it has a unique antipode B (say the top blue circle). Place a thermometer at both locations, then pick any path from A to B over the surface of the Earth - thinking of this path as something traversed over time, we can normalize so that this it is traversed between $t = 0$ and $t = 1$.

This gives a continuous function $\gamma : [0, 1] \rightarrow S^2$ where $\gamma(0) = A$ and $\gamma(1) = B$.

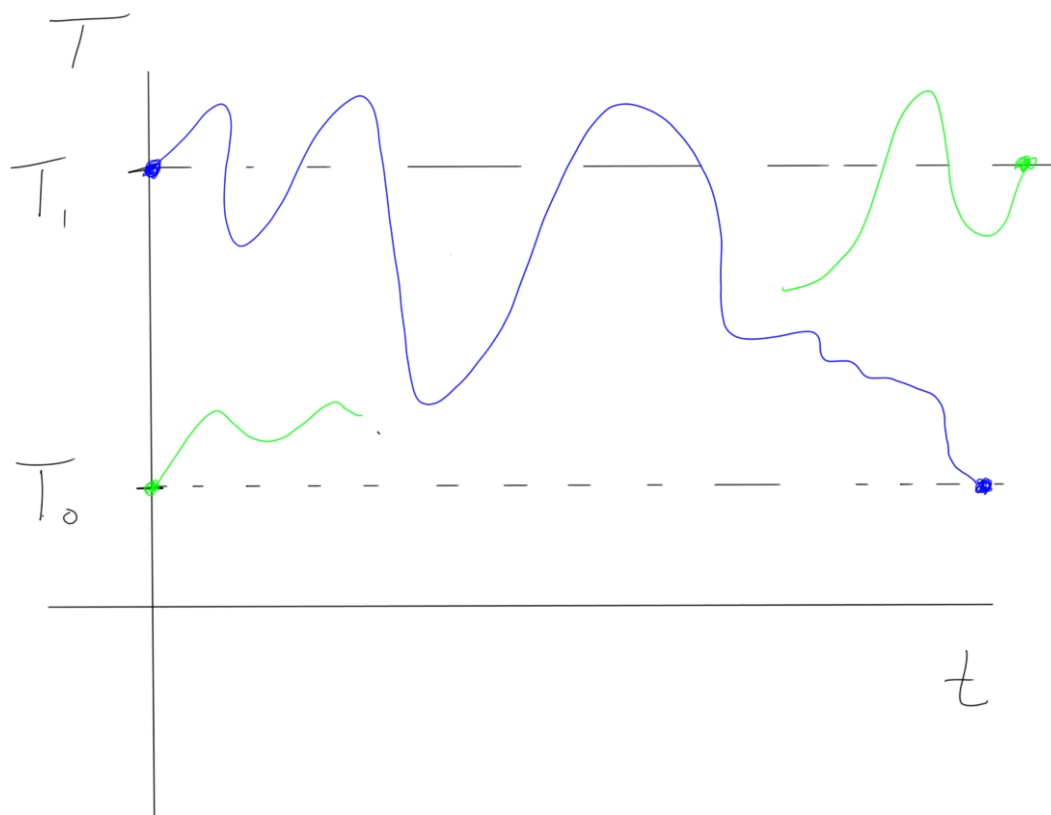
Then this induces a “mirror” path that B takes, if we always place B at the antipode of A , wherever it is along the path.



Band of continuous stuff

Then consider holding a thermometer, taking readings at A and B , and continuously taking temperatures as you traverse this path. Since A ends up at where B started, following this path just swaps the temperatures!

So let's look at what's happening on a temperature vs. time graph:



Path taken by temperature changes

Say blue is the path taken by A , and green by B . We know the temperatures measured by A and B at the start of their trip, and that the temperatures must have swapped by the end. We also know that the temperature varies on A 's journey looks something like the blue line - so this means the two lines have to cross at *some* point in this interval!

This is an intuitive argument, but can be made rigorous by looking at a function like

$$f : [0, 1] \rightarrow \mathbb{R}$$

$$t \mapsto T_1(\gamma(t)) - T_0(\gamma(t))$$

(All this means is that we evaluate the temperature at every physical point along the path.)

Without loss of generality, $T_1(\gamma(0)) > T_2(\gamma(0))$, so $f(0) > 0$. But because these switch by the end, we must have $T_1(\gamma(1)) < T_0(\gamma(1))$, and so $f(1) < 0$. Since f is continuous, we can apply the Intermediate Value Theorem to conclude that $f(t) = 0$ for some $t \in [0, 1]$.

But this works for *any* path, which means there must be a band homeomorphic to S^1 , which is mirrored antipodally, for which the antipodal temperatures are equal. This is the red curved line in the first figure. This

is the set of points for which there is an antipodal point of the same temperature. Note that this set must be a connected band - if it were not, you could find one path avoiding all of those points, yielding a path with no equal antipodal temperatures, contradicting our earlier findings.

So we can apply this argument again with some other real measurement - say pressure, to find two antipodal points on this S^1 that have the same value.