Title

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Tuesday 15th September, 2020

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1.1 Review

Let $k = \bar{k}$, we're setting up correspondences

$$
\begin{aligned}
\text{Ring Theory} &\quad \text{Geometry/Topology of Affine Varieties} \\
\text{Polynomial functions} &\quad \text{Affine space} \\
k[x_1, \cdots, x_n] &\quad \mathbb{A}^n/k \coloneqq \{[a_1, \cdots, a_n] \in k^n\} \\
\text{Maximal ideals } \langle x_1 - a_1, \cdots, x_n - a_n \rangle &\quad \text{Points } [a_1, \cdots, a_n] \in \mathbb{A}^n/k \\
\text{Radical ideals } I \trianglelefteq k[x_1, \cdots, x_n] &\quad \text{Affine varieties } X \subset \mathbb{A}^n/k, \text{ vanishing locii of polynomials} \\
& I \mapsto V(I) \coloneqq \left\{ a \;\middle|\; f(a) = 0 \forall f \in I \right\} \\
I(X) \coloneqq \left\{ f \;\middle|\; f|_X = 0 \right\} \mapsfrom X & \\
\text{Radical ideals containing } I(X) \text{, i.e. ideals in } A(X) &\quad \text{closed subsets of } X \text{, i.e. affine subvarieties} \\
A(X) \text{ is a domain} &\quad X \text{ irreducible} \\
A(X) \text{ is not a direct sum} &\quad X \text{ connected} \\
\text{Prime ideals in } A(X) &\quad \text{Irreducible closed subsets of } X \\
\text{Krull dimension } n \text{ (longest chain of prime ideals)} &\quad \dim X = n, \text{ (longest chain of irreducible closed subsets).}
\end{aligned}
$$

> Recall that we defined the coordinate ring $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, which contained no nilpotents.

We had some results about dimension

1. $\dim X < \infty$ and $\dim \mathbb{A}^n = n$.
2. $\dim Y + \operatorname{codim}_X Y = \dim X$ when $Y \subset X$ is irreducible.
3. Only over $\bar{k} = k$, $\operatorname{codim}_X V(f) = 1$.

Example 1.1.
Take $V(x^2 + y^2) \subset \mathbb{A}^2/\mathbb{R}$

Definition 1.0.1 (?).
An affine variety Y of

- $\dim Y = 1$ is a **curve**,
- $\dim Y = 2$ is a **surface**,
- $\mathrm{codim}_X Y = 1$ is a **hypersurface in** X

Question: Is every hypersurface the vanishing locus of a *single* polynomials $f \in A(X)$?

Answer: This is true iff $A(X)$ is a UFD.

Definition 1.0.2 (Codimension in a Ring).
$\mathrm{codim}_R \mathfrak{p}$ is the length of the longest chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}.$$

Recall that f is irreducible if $f = f_1 f_2 \implies f_i \in R^\times$ for one i, and f is prime iff $\langle f \rangle$ is a prime ideal, or equivalently $f \,\Big|\, ab \implies f \,\Big|\, a$ or $f \,\Big|\, b$.

Note that prime implies irreducible, since f divides itself.

Proposition 1.1 ***(?).***
Let R be a Noetherian domain, then TFAE

a. All prime ideals of codimension 1 are principal.
b. R is a UFD.

Proof .
$a \implies b$:
Let f be a nonzero non-unit, we'll show it admits a prime factorization. If f is not irreducible, then $f = f_1 f_1'$, both non-units. If f_1' is not irreducible, we can repeat this, to get a chain

$$\langle f \rangle \subsetneq \langle f_1' \rangle \subsetneq \langle f_2' \rangle \subsetneq \cdots,$$

which must terminate.
This yields a factorization $f = \prod f_i$ with f_i irreducible. To show that R is a UFD, it thus suffices to show that the f_i are prime. Choose a minimal prime ideal containing f. We'll use Krull's Principal Ideal Theorem: if you have a minimal prime ideal $\mathfrak{p}$ containing f, its codimension $\mathrm{codim}_R \mathfrak{p}$ is one. By assumption, this implies that $\mathfrak{p} = \langle g \rangle$ is principal. But $g \,\Big|\, f$ with f irreducible, so f, g differ by a unit, forcing $\mathfrak{p} = \langle f \rangle$. So $\langle f \rangle$ is a prime ideal.

$b \implies a$:
Let $\mathfrak{p}$ be a prime ideal of codimension 1. If $\mathfrak{p} = \langle 0 \rangle$, it is principal, so assume not. Then there exists some nonzero non-unit $f \in \mathfrak{p}$, which by assumption has a prime factorization since R is assumed a UFD. So $f = \prod f_i$.
Since $\mathfrak{p}$ is a prime ideal and $f \in \mathfrak{p}$, some $f_i \in \mathfrak{p}$. Then $\langle f_i \rangle \subset \mathfrak{p}$ and $\mathfrak{p}$ minimal implies $\langle f_i \rangle = \mathfrak{p}$,

so $\mathfrak{p}$ is principal. ■

Corollary 1.2 *(?).*
Every hypersurface $Y \subset X$ is cut out by a single polynomial, so $Y = V(f)$, iff $A(X)$ is a UFD.

Example 1.2.
Apply this to $R = A(X)$, we find that there is a bijection

$$\text{codim1 prime ideals} \iff \text{codim1 closed irreducible subsets } Y \subset X, \text{ i.e. hypersurfaces.}$$

Taking $A(X) = \mathbb{C}[x, y, z] / \left\langle x^2 + y^2 - z^2 \right\rangle$, whose real points form a cone:

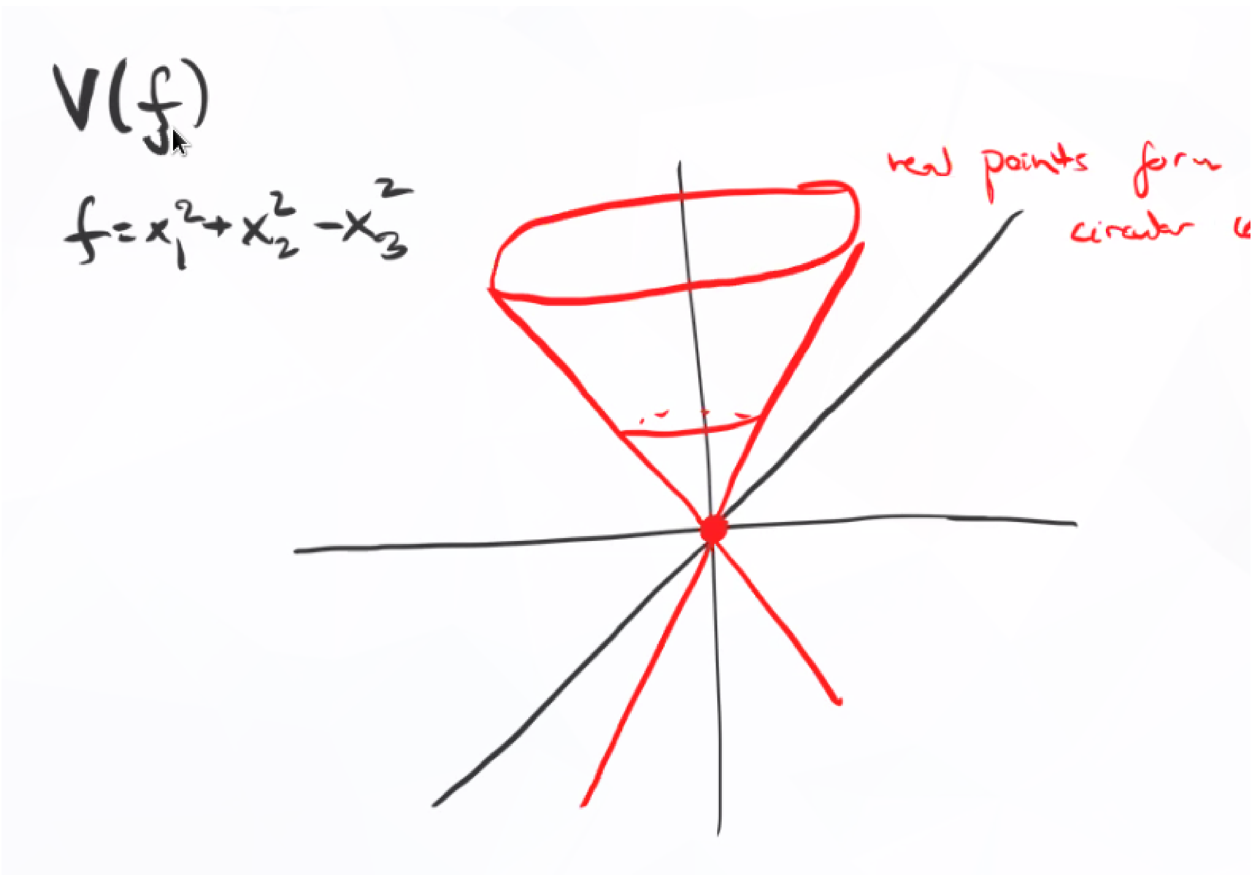

Figure 1: Image

Note that $x^2 + y^2 = (x - iy)(x + iy) = z^2$ in this quotient, so this is not a UFD.

Then taking a line through its surface is a codimension 1 subvariety not cut out by a single polynomial. Such a line might be given by $V(x + iy, z)$, which is 2 polynomials, so why not codimension 2?

Note that $V(z)$ is the union of the lines

- $z = 0, x + iy = 0$,
- $z = 0, x - iy = 0$.

Note that it suffices to show that this ring has an irreducible that is not prime. Supposing $z = f_1 f_2$, some f_i is a unit, then z is not prime because $z \mid xy$ but divides neither of x, y.

Example 1.3.
Note that $k[x_1, \cdots, x_n]$ is a UFD since k is a UFD. Applying the corollary, every hypersurface in $\mathbb{A}^n$ is cut out by a single irreducible polynomial.

Definition 1.2.1 (?).
An affine variety X is of **pure dimension** d iff every irreducible component X_i is of dimension d.

Note that X is a Noetherian space, so has a unique decomposition $X = \cup X_i$.

Given $X \subset \mathbb{A}^n/k$ of pure dimension $n-1$, $X = \cup X_i$ with X_i hypersurfaces with $I(X_j) = \langle f_j \rangle$, $I(X) = \langle f \rangle$ where $f = \prod f_i$.