Homological Algebra Problem Sets

Problem Set 3

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Table of Contents

Contents

1 | Wednesday, February 17

Problem 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.
$\implies$: Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R-linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\to A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending R-linearly yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

$\Longleftarrow$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\
& A &
\end{tikzcd}$$

Link to Diagram

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\to R \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f_a"] & R \arrow[dl, dashed, "\exists \tilde{f}_a"] \\
& A &
\end{tikzcd}$$

Link to Diagram

We can now use the fact that

$$\begin{aligned}
r\mathbf{a} &= f_a(jr) \\
&= \tilde{f}_a(\iota(jr)) \\
&= \tilde{f}_a(jr) \\
&= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\
&= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\
\implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA,
\end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting $\mathbf{a}$ range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).
Note that there is an injection

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \to \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \to 1$$

where π_m is the canonical quotient morphism, and applying the left-exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any $\mathbb{Z}$-module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "f"] \arrow[d, "\pi_m"'] & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/n \arrow[ur, dashed, "\exists \tilde{f}"'] &
\end{tikzcd}$$

Link to Diagram

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0, 1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n} \right\}.$$

Setting $f_i(1) := \left[\frac{i}{n} \right]$ (where we take the equivalence class mod $\mathbb{Z}$) yields n distinct morphisms $f_i : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\begin{aligned}
\Psi : \mathbb{Z} &\to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\
i &\mapsto f_i,
\end{aligned}$$

and using the fact that if $i = i' \pmod n$, write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left[\frac{i+kn}{n}\right] = \left[\frac{i}{n} + k\right] = \left[\frac{i}{n}\right] = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

Claim 2: $\mathbb{Q}/\mathbb{Z}$ is an injective object in $\mathbb{Z}$-modules.

Proof (?).
By the previous exercise, it suffices to show that $\mathbb{Q}/\mathbb{Z}$ is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N \coloneqq n(b + N) = n\bar{b}.$$

That $\mathbb{Q}$ is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b \coloneqq \frac{a}{n} \in \mathbb{Q}$ works. Since $\mathbb{Q}$ is an abelian group, $\mathbb{Z}$ is automatically normal, and the result follows.

■

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \qquad\qquad d \coloneqq \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).
Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M \coloneqq \mathbb{Z}/m$,
- $A \coloneqq \mathbb{Z}$,
- $I \coloneqq n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n\,(\mathbb{Z}/m)} \qquad\qquad \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\to \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
\ker \varphi &= \left\{ x \in \mathbb{Z} \;\middle|\; x(1 \otimes 1) = 0 \right\} \\
&= \left\{ x \in \mathbb{Z} \;\middle|\; n \;\middle|\; x \text{ or } m \;\middle|\; x \right\} \\
&= \langle n, m \rangle \\
&= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
&:= \langle d \rangle \,.
\end{aligned}$$

Now applying the first isomorphism theorem yields the result.

■

Solution:
We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\,\cdot\,) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \,\cdot\,)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of $\mathbb{Z}/q$.
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/q \arrow[r, "d^{-1}"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^0"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
& {[1]_q} \arrow[r] & \left[\frac{1}{q}\right] & & \\
& & {[x]} \arrow[r] & {[qx]} &
\end{tikzcd}$$

Link to Diagram

This is a chain complex by construction, since $d^2([1]_q) = \left[q\left(\frac{1}{q}\right)\right] = [1] = [0]$. We now delete the augmentation and apply $F(\,\cdot\,)$:

$$
\begin{tikzcd}
1 \arrow[r] & I^0 := \mathbb{Q}/\mathbb{Z} \arrow[rr, "d^0"] & & I^1 := \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
 & & \ \arrow[d, Rightarrow, "F(\,\cdot\,)"] & & \\
1 \arrow[r] \arrow[d, equal] & F(I^0) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[rr, "\partial^0 := F(d^0)"] & & F(I^1) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial^1 := F(d^1)"] & 1 \arrow[d, equal] \\
1 \arrow[r] & \mathbb{Z}/p \arrow[rr, "\tilde{\partial}^0"] \arrow[u, "\Psi \cong"] & & \mathbb{Z}/p \arrow[r, "\tilde{\partial}^1"] \arrow[u, "\Psi \cong"] & 1
\end{tikzcd}
$$

Link to Diagram

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q, we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q.
We now take homology:

$$
\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\operatorname{im} \partial^0} = \frac{\mathbb{Z}/p}{q\,(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,
$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)
Let $A \in \mathbf{Ab}$, and show that the following map is injective:

$$
\begin{aligned}
\varepsilon_A : A \to I(A) &:= \prod_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\
a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},
\end{aligned}
$$

i.e. when looking at the image $\varepsilon_A(a)$ in the product, the component indexed by f is an element of $\mathbb{Q}/\mathbb{Z}$ obtained by evaluating $f(a)$.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Solution:
By contrapositive, we'll suppose $a \neq 0$ and show $\varepsilon_A(a) \neq 0$. Following the hint, we first consider the cyclic subgroup $a\mathbb{Z} = \left\{ an \;\middle|\; n \in \mathbb{Z} \right\}$ and define a map

$$
\begin{aligned}
f_a : a\mathbb{Z} &\to \mathbb{Z} \\
an &\mapsto n.
\end{aligned}
$$

We now pick $\ell > 1 \in \mathbb{Z}$ to be any integer, and define a composition $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$:

$$\begin{tikzcd}
a\mathbb{Z} \arrow[r, "f_a"] \arrow[rrrr, bend left, dashed, "f"] & \mathbb{Z} \arrow[r, hook, "\iota"] & \mathbb{Q} \arrow[r, "x \mapsto \frac{x}{\ell}"] & \mathbb{Q} \arrow[r, "\pi"] & \mathbb{Q}/\mathbb{Z} \\
an \arrow[r, mapsto] & n \arrow[r, mapsto] & n \arrow[r, mapsto] & \frac{n}{\ell} \arrow[r, mapsto] & \left[\frac{n}{\ell}\right] \\
a \arrow[rrrr, dashed] & & & & \left[\frac{1}{\ell}\right]
\end{tikzcd}$$

Link to Diagram

By choice of ℓ, this map satisfies $f(a) = [1/\ell] \neq 0$, so the map is nonzero. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the universal property provides a lift $\tilde{f}$:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists \tilde{f}"] & \\
a\mathbb{Z} \arrow[u, hook] \arrow[r, "f"] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

Link to Diagram

Since $\tilde{f}$ lifts f, it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) \coloneqq f(a) \neq 0,$$

so the f component of the image of a is nonzero and thus $\mathbf{a} \coloneqq \varepsilon_A(a) \neq 0$ in the product.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Solution:
We'll show that $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$ for every object X. Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let $P \to X$ be a projective resolution of X. Fixing labeling, we have the following situation:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, Rightarrow, "F(\,\cdot\,)"] & P_0 \arrow[r, "\varepsilon"] & X \arrow[r, "0"] & 0 \\
\cdots \arrow[r] & FP_2 \arrow[r, "F(\partial_2)"'] & FP_1 \arrow[r, "F(\partial_1)"'] & FP_0 \arrow[r, "0"'] & 0 &
\end{tikzcd}$$

Link to Diagram

We now have by definition

$$L_iF(X) \coloneqq \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U\left(L_iF(X)\right) \coloneqq U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution $P \to X$, and apply a similar process:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, Rightarrow, "(U\circ F)(\,\cdot\,)"] & P_0 \arrow[r, "\varepsilon"] & X \arrow[r, "0"] & 0 \\
\cdots \arrow[r] & UFP_2 \arrow[r, "(UF)(\partial_2)"'] & UFP_1 \arrow[r, "(UF)(\partial_1)"'] & UFP_0 \arrow[r, "0"'] & 0
\end{tikzcd}$$

Link to Diagram

Again, by definition,

$$(L_i(UF))(X) \coloneqq \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor U to the following SES to produce a new SES, from which we'll produce the desired isomorphism f:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} F(\partial_{i+1}) \arrow[r, "\iota_i"] & \ker F(\partial_i) \arrow[r, "\pi_i"] \arrow[d, Rightarrow, "U"] & \dfrac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \arrow[r] & 0 \\
0 \arrow[r] & U\left(\operatorname{im} F(\partial_{i+1})\right) \arrow[r, "U(\iota_i)"] \arrow[d, equal] & U\left(\ker F(\partial_i)\right) \arrow[r, "U(\pi_i)"] \arrow[d, equal] & U\left(\dfrac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & U\left(\operatorname{im} F(\partial_{i+1})\right) \arrow[r, "U(\iota_i)"] & U\left(\ker F(\partial_i)\right) \arrow[r, "\tilde{\pi}_i"] & \dfrac{U(\ker F(\partial_i))}{U(\operatorname{im} F(\partial_{i+1}))} \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here $\tilde{\pi}_i$ is the natural quotient map, whose image is $\operatorname{coker} U(\iota_i)$. Finally, the map f exists an any abelian category, using that whenever $0 \to A \xrightarrow{g_1} B \xrightarrow{g_2} C \to 0$ is exact, there is an isomorphism $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$.

Problem 1.0.5 (Weibel 2.4.3)

- If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} FM \qquad i \geq 2.$$

- Show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

  is exact with P_i projective or F-acyclic, then

$$L_i F(A) \cong L_{i-m-1} F(M_m) \qquad i \geq m+2.$$

  - Moreover show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$.

- Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Solution:

Claim:

$$L_i F(A) \cong L_{i-1} FM \qquad i \geq 2.$$

Proof (of claim).
Following the proof of Weibel Theorem 2.4.6, let $P_M \to M$ and $P_A \to A$ be projective resolutions of M and A respectively. Then applying the Horseshoe Lemma, there is a projective resolution $P_P \to P$ of P such that the following is a short exact sequence of chain complexes:

$$0 \to P_M \to P_P \to P_A \to 0,$$

where in fact in each degree n piece, this is induces a *split* exact sequence. Using that F is additive and additive functors preserve split exact sequences, the following is a SES for every n:

$$0 \to FP_M^n \to FP_P^n \to FP_A^n \to 0,$$

which implies that there is a SES of chain complexes

$$0 \to FP_M \to FP_P \to FP_A \to 0.$$

Thus there is an associated LES of derived functors:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial_{i+1}"] \\
L_iFM \arrow[r] & L_iFP \arrow[r] & L_iFA \arrow[dll, "\partial_i"] \\
L_{i-1}FM \arrow[r] & L_{i-1}FP \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Using that P is F-acyclic, the middle terms $L_iFP = 0$ for all $i > 0$, and thus this splits into a collection of SESs:

$$0 \to L_2FA \xrightarrow{\partial_2} L_1FM \to 0$$
$$0 \to L_3FA \xrightarrow{\partial_3} L_2FM \to 0$$
$$\vdots$$
$$0 \to L_iFA \xrightarrow{\partial_3} L_{i-1}FM \to 0.$$

This makes every ∂_i for $i \geq 2$ an isomorphism. ■

Claim:

$$L_iFA \cong \ker(FM \to FP).$$

Proof (?).
Using the same argument as above, consider the lower order terms of the associated LES:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\partial_2"'] \\
L_1FM \arrow[r] & L_1FP = 0 \arrow[r] & L_1FA \arrow[dll, "\partial_1"'] \\
L_0FM \arrow[r] & L_0FP \arrow[r] & L_0FA \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Noting that $L_1FP = 0$ by F-acyclicity, the highlighted portion forms a four term exact sequence. We can form another exact sequence and compare the two:

$$\begin{tikzcd}
0 \arrow[r] & L_1FA \arrow[r, "\partial_1"] \arrow[d, dashed] & L_0FM \arrow[r] \arrow[d, equal] & L_0FP \arrow[r] \arrow[d, equal] & L_0FA \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(L_0FM \to L_0FP) \arrow[r] & L_0FM \arrow[r] & L_0FP \arrow[r] & L_0FA \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

That the map indicated by the dotted line exists and is an isomorphism holds in any abelian category, using that fact that whenever $0 \to A \to B \xrightarrow{f} C \to 0$ is a SES we have $A \cong \ker f$.

■

Claim: If $P \to A$ is an F-acyclic resolution of A, then there is an isomorphism

$$L_iFA \cong H_i(FP).$$

Proof (?).

Extend this to the exact sequence with m terms, and show that the last conclusion holds.

■

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

a. A is a projective R-module.

b. $\mathrm{Hom}_R(A, \cdot)$ is an exact functor.

c. $\mathrm{Ext}_R^{i\geq 1}(A, B) = 0$ and for all B, i.e. A is $\mathrm{Hom}_R(\cdot, B)$-acyclic for all B.

d. $\mathrm{Ext}_R^1(A, B)$ vanishes for all B.

Solution:
We'll show

- $a \iff b$
- $b \implies c$
- $c \iff d$:
- $d \implies b$

Proof ($a \iff b$).
Let ξ be the following SES:

$$\xi : 0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

and define the functor $F(\cdot) := \mathrm{Hom}_R(A, \cdot)$. This is a covariant left-exact functor, and so applying it to the above sequence yields

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] & M_2 \arrow[r, "g"] \arrow[d, Rightarrow, "F(\cdot):=\mathrm{Hom}_R(A,\cdot)"] & M_3 \arrow[r] & 0 \\
0 \arrow[r] & FM_1 \arrow[r, "F(f):\lambda\mapsto f\circ\lambda"'] & FM_2 \arrow[r, "F(g):\lambda\mapsto g\circ\lambda"'] & FM_3
\end{tikzcd}$$

[*Link to Diagram*]

$\implies$:
For F to be exact, it suffices to show it is right-exact, i.e. that $F(g)$ is surjective. This amounts to asking that every $\varphi \in FM_3 := \mathrm{Hom}_R(A, M_3)$ lifts to a preimage $\tilde{\varphi} \in FM_2 := \mathrm{Hom}_R(A, M_2)$ satisfying $F(g)(\tilde{\varphi}) = \varphi$. Unwinding definitions, this requires that $g \circ \tilde{\varphi} = \varphi$, which is precisely the lift required for the universal property of projective objects:

$$\begin{tikzcd}
 & A \arrow[dl, dashed, "\exists\tilde{\varphi}"'] \arrow[d, "\varphi"] & \\
M_2 \arrow[r, "g"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

[*Link to Diagram*]

If A is projective, this lift always exists, so $\mathrm{Hom}_R(A, \cdot)$ is an exact functor. Conversely, if $\mathrm{Hom}_R(A, \cdot)$ is exact, this lift always exists, so A satisfies the universal property of a projective object.

■

Proof (b $\implies$ c).
Suppose $F(\,\cdot\,) := \operatorname{Hom}_R(A, \,\cdot\,)$ is exact, then since F is left-exact covariant it has right-derived functors $\operatorname{Ext}_R^i(A, B) := R^i F(B)$ which are computed in the following way

1. Taking an *injective* resolution of
$$B \to I := (I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots).$$
2. Applying $\operatorname{Hom}_R(A, \,\cdot\,)$ to get the complex
$$FI := (0 \to \operatorname{Hom}_R(A, I_0) \xrightarrow{F(\partial_0)} \operatorname{Hom}_R(A, I_1) \xrightarrow{F(\partial_1)} \cdots).$$
3. Defining
$$R^i F(B) := \ker F(\partial_i) / \operatorname{im} F(\partial_{i-1}).$$

Note that in step (2), if $\operatorname{Hom}_R(A, \,\cdot\,)$ is an exact functor, then since I is an acyclic complex, FI is again acyclic and so $\ker F(\partial_i) = \operatorname{im} F(\partial_{i-1}) = 0$ for $i \geq 1$. So
$$\operatorname{Ext}_R^{\geq 1}(A, B) := R^{\geq 1} F(B) = 0.$$

■

Proof (c $\iff$ d).
$\implies$: This direction is clear, since if $\operatorname{Ext}_R^i(A, B) = 0$ for all B, then taking $i = 1$ is the statement of (d).

$\impliedby$: This follows from the dimension-shifting isomorphism in a previous exercise. Let $F(\,\cdot\,) := \operatorname{Hom}_R(A, \,\cdot\,)$ and suppose $\operatorname{Ext}_R^1(A, B) := L_1 F(B) = 0$ for all B. Let B' be arbitrary, it then suffices to show that $\operatorname{Ext}^i(A, B') := L_i(B') = 0$ for all $i > 1$, since we can take B' as one such B in the assumption for the $i = 1$ case.
The dimension shifting results states that if P_i are F-acyclic, then for every exact sequence
$$0 \to M_m \to P_m \to \cdots \to P_0 \to B' \to 0$$
we obtain an isomorphism
$$L_i F(B') \cong L_{i-m-1} F(M_m) \iff L_i F(M_m) \cong L_{i+m+1} F(B').$$
So take any F-acyclic resolution of P, say
$$B' \xrightarrow{\partial_{-1}} I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots,$$
then consider truncating it at the mth stage:
$$0 \to B' \xrightarrow{\partial_{i-1}} I_0 \to I_1 \to \cdots \xrightarrow{\partial_{m-1}} I_m \to M_m := \operatorname{coker} \partial_{m-1} \to 0$$

By assumption, we have $L_1F(M_m) = 0$ for every m, and thus

$$\begin{aligned}
0 &= L_1F(B') \quad \text{by assumption} \\
0 &= L_1F(M_0) \cong L_2F(B') \\
0 &= L_1F(M_1) \cong L_3F(B') \\
0 &= L_1F(M_2) \cong L_4F(B') \\
&\vdots \\
0 &= L_1(M_m) \cong L_{m+2}(B') \quad \forall m \geq 0.
\end{aligned}$$

and so $L_i(B') = 0$ for all $i \geq 1$.

■

Proof (d $\implies$ b).
Take an arbitrary SES

$$\xi : 0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$$

and consider applying the left-exact covariant functor $F(\,\cdot\,) \coloneqq \mathrm{Hom}_R(A,\,\cdot\,)$ and taking the associated LES:

$$\begin{tikzcd}
0 \arrow[d] & & \\
\mathrm{Hom}_R(A, B') \arrow[r, "f^*"] & \mathrm{Hom}_R(A, B) \arrow[r, "g^*"] & \mathrm{Hom}_R(A, B'') \arrow[dll, "\delta_0"] \\
\mathrm{Ext}_R^1(A, B') \arrow[r] & \mathrm{Ext}_R^1(A, B) \arrow[r] & \mathrm{Ext}_R^1(A, B'') \arrow[dll, "\delta_1"] \\
\cdots & &
\end{tikzcd}$$

Link to Diagram

By assumption, all of the higher Ext terms vanish, and in particular the red term $\mathrm{Ext}_R^1(A, B') = 0$. This implies that g^* is surjective, making the following sequence exact:

$$0 \to \mathrm{Hom}_R(A, B') \xrightarrow{f^*} \mathrm{Hom}_R(A, B) \xrightarrow{g^*} \mathrm{Hom}_R(A, B'') \to 0,$$

making $\mathrm{Hom}_R(A,\,\cdot\,)$ an exact functor.

■

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is

abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \to \bullet$, is not an exact functor on **Ab**.

Fixing some index category $\mathcal{I}$ and a functor $F : \mathcal{I} \to \mathcal{A}$, so $F \in \mathcal{A}^{\mathcal{I}}$, write $\widehat{A} := \operatorname{colim}_{i \in I} F(i)$. We want to show that $\mathcal{A}^{\mathcal{I}} \underset{\Delta}{\overset{\operatorname{colim}}{\underset{\perp}{\rightleftarrows}}} \mathcal{A}$ defines an adjoint pair, so that colim is a left-adjoint and Δ is a right-adjoint. By definition, this is equivalent to showing the existence of natural bijections of sets

$$\tau_{FX} : \operatorname{Hom}_{\mathcal{A}}(\widehat{F}, X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \quad \forall X \in \mathcal{A}, F \in \mathcal{A}^{\mathcal{I}}.$$

We first note that the data of $\widehat{F}$ is equivalent to the following universal property:

$$\begin{tikzcd}
\mathcal{I} & & \mathcal{A} & & \\
i \arrow[dd, "\varepsilon_{ij}"'] & & F(i) \arrow[dd, "F\varepsilon_{ij}"'] \arrow[dr, "\psi_i"] \arrow[drr, dotted, bend left, "\varphi_i"] & & \\
{} \arrow[rr, Rightarrow, "F(\,\cdot\,)"'] & & {} & \widehat{F} \arrow[r, dashed, "\exists!\eta_{ij}"] & X \\
j & & F(j) \arrow[ur, "\psi_j"'] \arrow[urr, dotted, bend right, "\varphi_j"'] & &
\end{tikzcd}$$

Link to Diagram

That is, $\widehat{F} \in \operatorname{Ob}(\mathcal{A})$ is an object equipped with structure maps ψ_i for every object $F(i)$ in the image of F such that the solid triangle commutes, and for any object X with maps $\varphi_i : F(i) \to X, \varphi_j : F(j) \to X$ making the outer triangle commute, there is a unique map $\eta_{ij} : \widehat{F} \to X$ making the entire diagram commute. We can rewrite this condition in a more suggestive way:

$$\begin{tikzcd}
\mathcal{I} & & \mathcal{A} & & \\
i \arrow[dd, "\varepsilon_{ij}"'] & & F(i) \arrow[dd, "F\varepsilon_{ij}"'] \arrow[r, "\psi_i"] \arrow[rr, dotted, bend left, "\varphi_i"] & \widehat{F} \arrow[r, dotted, "\exists\eta_i"] \arrow[dd, dash, "\mathbb{1}_{\widehat{F}}"] & X \arrow[dd, dash, "\mathbb{1}_X"] \\
{} \arrow[rr, Rightarrow, "F(\,\cdot\,)"'] & & {} & & \\
j & & F(j) \arrow[r, "\psi_j"'] \arrow[rr, dotted, bend right, "\varphi_j"'] & \widehat{F} \arrow[r, dashed, "\exists\eta_j"] & X
\end{tikzcd}$$

Link to Diagram

Applying the Δ functor, we can view this as a simpler universal property in $\mathcal{A}^{\mathcal{I}}$, since the above data is precisely the data of a natural transformation:

$$\begin{tikzcd}
 & \Delta F \\
F \arrow[ur, "\Psi"] \arrow[r, "\Phi"'] & \Delta X \arrow[u, dashed, "\exists!\eta"']
\end{tikzcd}$$

Link to Diagram

That is, the functor $\Delta\widehat{F}$ is equipped with structure maps $\Phi : F \to \Delta\widehat{F}$ which assemble into a natural transformation (i.e. a morphism in $\mathcal{A}^{\mathcal{I}}$) such that any other natural transformation from F to a diagonal object ΔX produces a unique natural transformation $\eta : \Delta X \to \Delta F$. This provides exactly the data needed to specify τ:

$$\tau_{FX} : \mathrm{Hom}_{\mathcal{A}}(\widehat{F}, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X)$$
$$\left(\widehat{F} \xrightarrow{f} X\right) \mapsto \left(F \xrightarrow{\Psi} \Delta\widehat{F} \xrightarrow{\Delta(f)} \Delta X\right),$$

i.e. we take the image $\Delta(f)$ and pre-compose with the structure morphism Ψ.

Claim: This is a bijection of sets.

Proof (?).
This is surjective by the universal property: any morphism $F \xrightarrow{g} \Delta X$ in $\mathcal{A}^{\mathcal{I}}$ factors through $\Delta\widehat{F}$, and so all such morphisms are of this form.

Todo: showing surjectivity or constructing inverse.

■

Claim: This is a natural isomorphism, i.e. for all $X \xrightarrow{f} Y \in \mathrm{Mor}(\mathcal{A})$ and all $F \xrightarrow{\eta} G \in \mathrm{Mor}(\mathcal{A}^{\mathcal{I}})$, there is a commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}(\widehat{G}, X) \arrow[r, "\widehat{\eta}_*"] \arrow[d, "\tau_{GX}"] & \mathrm{Hom}(\widehat{F}, X) \arrow[r, "f_*"] \arrow[d, "\tau_{FX}"] & \mathrm{Hom}(\widehat{F}, Y) \arrow[d, "\tau_{FY}"] \\
\mathrm{Hom}(G, \Delta X) \arrow[r, "\eta_*"] & \mathrm{Hom}(F, \Delta X) \arrow[r, "(\Delta f)_*"] & \mathrm{Hom}(F, \Delta Y)
\end{tikzcd}$$

Link to Diagram

Proof (?).

Show that the diagrams above always commute.

■

Claim: If $\mathcal{A}$ is abelian and $\mathcal{I}$ is an index category such that $\operatorname{colim}_{i\in\mathcal{I}} F(i)$ exists for all $F \in \mathcal{A}^{\mathcal{I}}$, then the functor $\operatorname{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ is right-exact.

Proof (Sketch).
A sketch of the proof proceeds by showing every right adjoint is left-exact:

- Since $\operatorname{Hom}(LA, \cdot\,)$ is left-exact, we can apply it to a SES $0 \to B' \to B \to B'' \to 0$.
- Applying the natural isomorphisms coming from the adjunction, this is isomorphic to a sequence involving terms $\operatorname{Hom}(\,\cdot\,, RB)$.
- This sequence is exact, so applying Yoneda yields an exact sequence

  $$0 \to RB' \to RB \to RB'',$$

  making R left-exact.

Finally, if L is a left adjoint out of $\mathcal{A}$, then L^{op} is a right adjoint out of $\mathcal{A}^{\mathrm{op}}$. Thus L^{op} is left-exact by the above argument, making L right-exact.

■

Claim: Let $\mathcal{I} \coloneqq (\bullet \leftarrow \bullet \to \bullet)$ and define the pushout as $\operatorname{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$. Then taking $\mathcal{A} \coloneqq \mathbf{Ab}$, the pushout does not define an exact functor $\mathcal{A}^{\mathcal{I}} \to \mathcal{A}$.

Proof (?).
We proceed by constructing a counterexample. Unwinding definitions, we first note that an exact sequence of objects in $\mathcal{A}^{\mathcal{I}}$ corresponds precisely to an exact sequence of diagrams. For pushouts, writing X_i for the pushout of $A_i \hookleftarrow P_i \hookrightarrow B_i$, this gives an exact sequence of diagrams. If pushout were exact, this would in turn correspond to an exact sequence of the pushout objects X_i shown on the right:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & & & 0 \arrow[dr] & & & & \\
0 \arrow[dr] & P_1 \arrow[r] \arrow[d] \arrow[dr] & B_1 \arrow[dr] & & & X_1 \arrow[dr] & & & \\
& A_1 \arrow[dr] & P_2 \arrow[r] \arrow[d] \arrow[dr] & B_2 \arrow[dr] & {} \arrow[r, Rightarrow] & {} & X_2 \arrow[dr] & & \\
& & A_2 \arrow[dr] & P_3 \arrow[r] \arrow[d] \arrow[dr] & B_3 \arrow[dr] & & & X_3 \arrow[dr] & \\
& & & A_3 \arrow[dr] & 0 & 0 & & & 0 \\
& & & & 0 & & & &
\end{tikzcd}
$$

Link to Diagram

If we let $f_i : P_i \to B_i$ be arbitrary maps between abelian groups and push out along $A_i = 0$, we recover to cokernels of the f_i:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & & & 0 \arrow[dr] & & & & \\
0 \arrow[dr] & P_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr] & B_1 \arrow[dr] & & & \operatorname{coker} f_1 \arrow[dr] & & & \\
& 0 \arrow[dr] & P_2 \arrow[r, "f_2"] \arrow[d] \arrow[dr] & B_2 \arrow[dr] & {} \arrow[r, Rightarrow] & {} & \operatorname{coker} f_2 \arrow[dr] & & \\
& & 0 \arrow[dr] & P_3 \arrow[r, "f_3"] \arrow[d] \arrow[dr] & B_3 \arrow[dr] & & & \operatorname{coker} f_3 \arrow[dr] & \\
& & & 0 \arrow[dr] & 0 & 0 & & & 0 \\
& & & & 0 & & & &
\end{tikzcd}
$$

Link to Diagram

However, the sequence of cokernels appearing on the right is not exact in general, since this precisely fits into the diagram and exact sequence shown in the snake lemma:

$$\begin{tikzcd}
0 \arrow[r] & \ker f_1 \arrow[r] & \ker f_2 \arrow[r] & \ker f_3 \arrow[dddll] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_1 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & P_1 \arrow[r] & P_1 \arrow[r] & P_1 \arrow[r] & 0 \\
 & \operatorname{coker} f_1 \arrow[r] & \operatorname{coker} f_2 \arrow[r] & \operatorname{coker} f_3 \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here we know that the map involved in the red terms $\operatorname{coker} f_1 \to \operatorname{coker} f_2$ is not injective in general, provided the green term $\ker f_3 \neq 0$. Thus an exact sequence of diagrams does not necessarily yield an exact sequence of their pushouts.

■