Problem Set 1

D. Zack Garza

Wednesday 2^{nd} September, 2020

Contents

Source: Section 1 of Gathmann

Exercise 0.1 (Gathmann 1.19)**.**
Prove that every affine variety $X \subset \mathbb{A}^n/k$ consisting of only finitely many points can be written as the zero locus of n polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in X have different x_1-coordinates.

Solution:
Let $X = \{p_1, \cdots, p_d\} = \{p_j\}_{j=1}^d$, where each p_j can be written in coordinates $p_j \coloneqq \left[p_j^1, p_j^2, \cdots, p_j^n\right]$.

Claim: Without loss of generality, we can assume all of the first components $\left\{p_j^1\right\}_{j=1}^d$ are distinct.

Todo: by some change of basis?

We will use the following fact

Theorem 0.1 ***(Lagrange)*****.**
Given a set of d points $\{(x_i, y_i)\}_{i=1}^d$ with all x_i distinct, there exists a unique polynomial of degree d in $f \in k[x]$ such that $\tilde{f}(x_i) = y_i$ for every i.
This can be explicitly given by

$$\tilde{f}(x) = \sum_{i=1}^d y_i \left(\prod_{\substack{0 \leq m \leq d \\ m \neq i}} \left(\frac{x - x_m}{x_i - x_m} \right) \right).$$

Equivalently, there is a polynomial f defined by $f(x_i) = \tilde{f}(x_i) - y_i$ of degree d whose roots are precisely the x_i.

Using this theorem, we define a system of n polynomials in the following way:

- Define $f_1 \in k[x_1] \subseteq k[x_1, \cdots, x_n]$ by
$$f_1(x) = \prod_{i=1}^{d} \left(x - p_i^1\right).$$
Then the roots of f_1 are precisely the first components of the points p.
- Define $f_2 \in k[x_1, x_2] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs
$$\left\{(x_1, x_2) = (p_j^1, p_j^2)\right\},$$
then taking the unique Lagrange interpolating polynomial satisfying $f_2(p_j^1) = p_j^2$ for all $1 \leq j \leq d$.
- Define $f_3 \in k[x_1, x_3] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs
$$\left\{(x_1, x_3) = (p_j^1, p_j^3)\right\},$$
then taking the unique Lagrange interpolating polynomial satisfying $f_2(p_j^1) = p_j^3$ for all $1 \leq j \leq d$.
- $\cdots$

Continuing in this way up to $f_n \in k[x_1, x_n]$ yields a system of n polynomials.
Claim: $V(f_1, \cdots, f_n) = X$.
$X \subset V(f_1, \cdots, f_n)$: This follows by construction; letting $p_j \in X$ be arbitrary, we find that
$$f(p_j) = \prod_{i=1}^{d} \left(p_j^1 - p_i^1\right) = \prod_{\substack{i \leq d \\ i \neq j}} \left(p_j^1 - p_i^1\right).$$

Exercise 0.2 (Gathmann 1.21)**.**
Determine $\sqrt{I}$ for
$$I \coloneqq \left\langle x_1^3 - x_2^6,\, x_1 x_2 - x_2^3 \right\rangle \trianglelefteq \mathbb{C}[x_1, x_2].$$

Solution:
For notational purposes, let $\mathcal{I}, \mathcal{V}$ denote the maps in Hilbert's Nullstellensatz, we then have
$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$
So we consider $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$, the vanishing locus of these two polynomials, which yields the system
$$\begin{cases} x^3 - y^6 & = 0 \\ xy - y^3 & = 0. \end{cases}$$
In the second equation, we have $(x - y^2)y = 0$, and since $\mathbb{C}[x, y]$ is an integral domain, one term must be zero.

1. If $y = 0$, then $x^3 = 0 \implies x = 0$, and thus $(0, 0) \in \mathcal{V}(I)$,i.e. the origin is contained in this vanishing locus.

2. Otherwise, if $x - y^2 = 0$, then $x = y^2$, with no further conditions coming from the first equation. So

$$P := \left\{ (t^2, t) \;\middle|\; t \in \mathbb{C} \right\} \subset \mathcal{V}(I).$$

The corresponding ideal
Since the origin is in the latter set, this simplifies to $P = \mathcal{V}(I)$, and so taking the ideal generated by P yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \left\langle y - x^2 \right\rangle \in \mathbb{C}[x, y]$$

and thus $\sqrt{I} = \left\langle y - x^2 \right\rangle$.

Exercise 0.3 (Gathmann 1.22)**.**
Let $X \subset \mathbb{A}^3/k$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$ and show that it can not be generated by fewer than 3 elements.

Solution:
Claim:

$$I(X) = \langle x_1x_2,\, x_1x_3,\, x_2x_3 \rangle \,.$$

Exercise 0.4 (Gathmann 1.23: Relative Nullstellensatz)**.**
Let $Y \subset \mathbb{A}^n/k$ be an affine variety and define $A(Y)$ by the quotient

$$\pi : k[x_1, \cdots, x_n] \longrightarrow A(Y) := k[x_1, \cdots, x_n]/I(Y).$$

a. Show that $V_Y(J) = V(\pi^{-1}(J)$ for every $J \trianglelefteq A(Y)$.

b. Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every affine subvariety $X \subseteq Y$.

c. Using the fact that $I(V(J)) \subset \sqrt{J}$ for every $J \trianglelefteq k[x_1, \cdots, x_n]$, deduce that $I_Y(V_Y(J)) \subset \sqrt{J}$ for every $J \trianglelefteq A(Y)$.

Conclude that there is an inclusion-reversing bijection

$$\left\{ \substack{\text{Affine subvarieties} \\ \text{of } Y} \right\} \iff \left\{ \substack{\text{Radical ideals} \\ \text{in } A(Y)} \right\}.$$

Exercise 0.5 (Extra)**.**
Let $J \trianglelefteq k[x_1, \cdots, x_n]$ be an ideal, and find a counterexample to $I(V(J)) = \sqrt{J}$ when k is not algebraically closed.