Title

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1.1 Group Schemes

Definition 1.0.1 (Representable Functors)**.**
Let $F :: k\text{-alg} \to \text{Set}$ be a functor, then F is **representable** iff $F(R)$ corresponds to "solutions to equations in R".

Example 1.1.
Let $F(\,\cdot\,) = \mathrm{SL}(2,\,\cdot\,)$, then the corresponding equations are $\det(x_{ij}) = 1$.

If F is representable, there is a correspondence $F(R) \cong \hom_R(A, R)$. In the above example,

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}] / \left\langle x_{11}x_{22} - x_{12}x_{21} \right\rangle,$$

which is exactly the coordinate algebra.

Definition 1.0.2 (Affine Group Scheme)**.**
An *affine group scheme* is a representable functor $F : k\text{-alg} \to \text{Groups}$.

Suppose G is an affine group scheme, and let $A = k[G]$ be the representing object. Then there is a correspondence

$$G\text{-modules} \iff k[G]^\vee\text{-modules}.$$

For G reductive, the RHS is equivalent to $\mathrm{Dist}(G)$-modules.

Definition 1.0.3 (Finite Group Schemes)**.**
G is a **finite** group scheme iff $k[G]$ is finite dimensional.

If G is finite, then $A^\vee \cong k[G]^\vee$ is a cocommutative Hopf algebras. Thus representations for *finite* group schemes are equivalent to representations for finite-dimensional cocommutative Hopf algebras.

> On group scheme side: see reduction, spectral sequences, conceptual arguments. On the algebra side: have bases, underlying vector space, can do concrete computations. Can take $\operatorname{Spec}(k[G])^\vee$ to recover a group scheme.

1.2 Hopf Algebras

For A a k-alg, we have a multiplication and a unit, which can be defined in terms of diagrams. To categorically reverse arrows, we can ask for a comultiplication and a counit.

$$\Delta : A \to A^{\otimes 2}$$

$$\epsilon : A \to k.$$

We'll want another map, an *antipode*

$$s : A \to A.$$

The comultiplication should satisfy

$$\begin{tikzcd}
A^{\otimes 3} & A^{\otimes 2} \arrow[l, "1\otimes A"'] \\
A^{\otimes 2} \arrow[u, "\Delta\otimes 1"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

The counit should satisfy

$$\begin{tikzcd}
k \otimes A \arrow[d, "\cong"] & A^{\otimes 2} \arrow[l, "\varepsilon\otimes 1"'] \\
A \arrow[r, "\cong"] & A \arrow[u, "\Delta"']
\end{tikzcd}$$

And the antipode should satisfy

$$\begin{tikzcd}
A & A \arrow[l, "m(s\otimes 1)"'] \\
A \arrow[u] & A \arrow[l, "\varepsilon"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

1.2.1 Module Constructions

Let A be a Hopf algebra.

1. For A-modules M, N, we can form the A-module $M \otimes_k N$ with

$$\Delta(a) = \sum a_i \otimes a_j$$

$$a(m \otimes n) = \sum a_1 m \otimes a_2 n.$$

2. If M is finite-dimensional over A, then $M^\vee = \hom_k(M, k) \ni f$ is an A-module, and we can define $(af)(x) \coloneqq f(s(a)x)$ for $a \in A, x \in M$.

Example 1.2.
$A = kG$ the group algebra on a group is a Hopf algebra:

$$\begin{aligned} \Delta : A &\to A^{\otimes 2} \\ g &\mapsto g \otimes g. \end{aligned}$$

The module action is diagonal, namely $g(m \otimes n) = gm \otimes gn$. The antipode is given by $s(g) = g^{-1}$, and the unit is $\varepsilon(g) = 1$ for all $g \in G$.

Example 1.3.
Let $A = U(\mathfrak{g})$, the universal enveloping algebra for $\mathfrak{g}$ a Lie algebra. Recall that $\mathfrak{g}$-modules are equivalent to $U(\mathfrak{g})$-modules (unitary representations, some big associative algebra). Then A is a Hopf algebra, with $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ for $\ell \in \mathfrak{g}$. The unit is $\varepsilon(\ell) = 0$, and the antipode is $s(\ell) = -\ell$.

Example 1.4.
Take the additive group $\mathbb{G}_a$, then $A = k[\mathbb{G}_a] \cong k[x]$ is a commutative Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0, s(x) = -x$.

Example 1.5.
For $\mathbb{G}_m$, we have $A = k[\mathbb{G}_m] \cong k[x, x^{-1}], \varepsilon(x) = 1, s(x) = x^{-1}$.

1.3 Frobenius Kernels

Let G be an algebraic group (scheme) over k, where char $(k) = p$. Let $F : G \to G$ be the Frobenius, where e.g.

$$\begin{aligned} F : \mathrm{GL}(n, \cdot\,) &\to \mathrm{GL}(n, \cdot\,) \\ (x_{ij}) &\mapsto (x_{ij}^p). \end{aligned}$$

Then F is a map of group schemes.

Definition 1.0.4 (Frobenius Kernels)**.**
$G_r \coloneqq \ker F^r$, where $F^r \coloneqq F \circ F \circ \cdots \circ F$ is the r-fold composition of the Frobenius.
This yields a nesting $G_1 \trianglelefteq G_2 \trianglelefteq G_3 \cdots \leq G$.

Recall that

$$\mathrm{Dist}(G) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \right\rangle.$$

We get a chain of finite dimensional algebras

$$\mathrm{Dist}(G_1) \leq \mathrm{Dist}(G_2) \leq \cdots \leq \mathrm{Dist}(G)$$

where

$$\mathrm{Dist}(G_1) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \;\middle|\; 0 \leq n, m, k \leq p-1 \right\rangle,$$

where in general $\mathrm{Dist}(G_\ell)$ goes up to $p^\ell - 1$. Recall that G_r representations were equivalent to $\mathrm{Dist}(G_r)$ representations.

Some basic questions (Curtis, Steinberg, 1960s):

1. What are the simple modules for Frobenius kernels? I.e., what are the irreducible representations for G_r?
2. How are the representations for G_r related to those for G?

> It turns out the representations for G_r will lift to representations to G. Use "twisted tensor product" (Steinberg).

Remark 1.
It turns out that G_1 is special.

$$\mathrm{Dist}(G_1) \cong u(\mathfrak{g}) \coloneqq U(\mathfrak{g}) / \left\langle x^p - x^{[p]} \right\rangle,$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ is a *restricted lie algebra* (N. Jacobson). Note that for $D \in \mathfrak{g}$ a derivation, we define $D^{[p]} \coloneqq D \circ \cdots \circ D$ is the p-fold composition.

G_1-modules are equivalent to $\mathfrak{g}$-modules which are *restricted* in the sense that

$$\begin{aligned} \rho : g &\to \mathfrak{gl}(V) \\ x^{[p]} &\mapsto \rho(x)^p. \end{aligned}$$