

Title

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Review: Regular functions. Given an affine variety X and $U \subseteq X$ open, a *regular function* $\varphi : U \rightarrow k$ is one locally (wrt the zariski topology) a fraction. We write the set of regular functions as \mathcal{O}_X .

Example 1.1.

$X = V(x_1x_4 - x_2x_3)$ on $U = V(x_2, x_4)^c$, the following function is regular:

$$\varphi : U \rightarrow k$$
$$x \mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0 \\ \frac{x_3}{x_4} & x_4 \neq 0 \end{cases}.$$

Note that this is not globally a fraction.

Definition 1.0.1 (Distinguished Open Sets).

A *distinguished open set* $D(f) \subseteq X$ for some $f \in A(X)$ is $V(f)^c := \{x \in X \mid f(x) \neq 0\}$.

These are useful because the $D(f)$ form a base for the zariski topology.

Proposition 1.1 (?).

For X an affine variety, $f \in A(X)$, we have

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \mid g \in A(X), n \in \mathbb{N} \right\}.$$

Proof .

The first reduction we made was that $\varphi \in \mathcal{O}_X(D(f))$ is expressible as $\frac{g_a}{f_a}$ on distinguished opens $D(f_a)$ covering $D(f)$. We also noted that

$$\frac{g_a}{f_a} = \frac{g_b}{f_b} \text{ on } D(f_a) \cap D(f_b) \implies f_b g_a = f_a g_b \text{ in } A(X).$$

The second step was writing $D(f) = \cup D(f_a)$, and so $V(f) = \cap_a V(f_a)$ implies that $f \in I(V(\{f_a \mid a \in U\}))$. By the Nullstellensatz, $f \in \sqrt{\langle f_a \mid a \in U \rangle}$, so $f^N = \sum k_a f_a$ for some N . So construct $g = \sum k_a g_a$, then compute

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b \sum k_a f_a = g_b f^N.$$

Thus $g/f^N = g_b/f_b$ for all b , and we can thus conclude

$$\varphi := \left\{ \frac{g_b}{f_b} \text{ on } D(f_b) \right\} = g/f^N.$$

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Corollary 1.2(?).

For X an affine variety, $\mathcal{O}_X(X) = A(X)$.

⚠ Warning: For k not algebraically closed, the proposition and corollary are both false. Take $X = \mathbb{A}^1/\mathbb{R}$, then $\frac{1}{x^2+1} \in \mathbb{R}(x)$, but $\mathcal{O}_X(X) \neq \mathbb{R}[x]$.