Homotopy
Groups of
Spheres

D. Zack Garza

Introduction

Spheres

Homotopy Groups of Spheres
Graduate Student Seminar

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Introduction

Outline

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- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

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Homotopies of paths:

- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \hom_{\mathsf{Top}}(I, X)$$
$$H \in \hom_{\mathsf{Top}}(I \times I, X).$$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

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– Why care about path homotopies? Historically: contour integrals in $\mathbb{C}$

$|\mathbf{\Gamma}(z)|$

– By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_\gamma f(z)\ dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

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- Generalize to a homotopy of *morphisms*:

$$f, g \in \mathsf{hom}_{\mathsf{Top}}(X, Y) \quad f \sim g \iff \exists F \in \mathsf{hom}_{\mathsf{Top}}(X \times I, Y)$$

such that $F(0) = f, F(1) = g$.

- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] := \mathsf{hom}_{\mathsf{Top}}(X, Y)/ \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \mathsf{hom}(X, Y) \\ g \in \mathsf{hom}(Y, X) \end{cases} \text{ such that } \begin{cases} f \circ g \sim \mathsf{id}_Y \\ g \circ f \sim \mathsf{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \mathsf{Top} \ \middle| \ Y \sim X \right\}.$$

The Fundamental Group

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– $\pi_1(X)$ is the group of homotopy classes of loops:

– Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$

Higher Homotopy Groups

– Can now generalize to define

$$\pi_k(X) := [S^k, X]$$

Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look $\mathbb{A}^1$ or $\mathbb{P}^1$.

Classification

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– Holy grail: understand the topological category completely
  – I.e. have a well-understood geometric model one space of each
    homeomorphism type

Also have the derived category DTop, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

– Any representative from a green box: a *homotopy type*.

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

– Use the fact that for any fixed group G, the functor

$$h_G(\,\cdot\,) : \mathrm{hoTop}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$
$$X \mapsto \{G\text{-bundles over } X\}$$

is representable by a space called BG (Brown's representability theorem).

– I.e., let $\mathrm{Bun}_G(X) = \{G\text{-bundles}/B\} / \sim$, there is an isomorphism

$$\mathrm{Bun}_G(X) \cong [X, BG]$$

– In general, identify $G = \mathrm{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

Note that for a poset of spaces $(M_i, \hookrightarrow)$, the space $M^\infty := \varinjlim M_i$. These are infinite dimensional "Hilbert manifolds".

Proof:

$$
\begin{aligned}
\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, B\mathrm{GL}(1, \mathbb{R})] \\
&= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\
&= [X, \mathbb{RP}^\infty] \\
&= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\
&= H^1(X; \mathbb{Z}/2\mathbb{Z})
\end{aligned}
$$

Work being swept under the rug: identifying the homotopy type of the representing object.

Corollary: There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.

Cylinder

Mobius strip

A

A

A

A

Example: Higher Homotopy Groups are Useful

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– Application: computing $\pi_1(SO(n, \mathbb{R})$ (rigid rotations in $\mathbb{R}^n$).

– The fibration

$$SO(n, \mathbb{R}) \longrightarrow SO(n+1, \mathbb{R}) \longrightarrow S^n$$

yields a LES in homotopy:

$$\cdots \longrightarrow \pi_2(SO(n, \mathbb{R})) \longrightarrow \pi_2(SO(n, \mathbb{R})) \longrightarrow \pi_2(S^n)$$

$$\pi_1(SO(n, \mathbb{R})) \longleftarrow \longrightarrow \pi_1(SO(n, \mathbb{R})) \longrightarrow \pi_1(S^n)$$

Knowing $\pi_k S^n$, this reduces to

$$\cdots 0 \longrightarrow \pi_2(SO(n, \mathbb{R})) \longrightarrow \pi_2(SO(n, \mathbb{R})) \longrightarrow 0$$

$$\pi_1(SO(n, \mathbb{R})) \longrightarrow \pi_1(SO(n, \mathbb{R})) \longrightarrow 0$$

- Thus $\pi_1(SO(3, \mathbb{R})) \cong \pi_1(SO(4, \mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(SO(3, \mathbb{R}))$ (stabilization)
- Use the fact that "accidental" homeomorphism in low dimension $SO(3, \mathbb{R}) \cong_{\text{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

  Can also use the fact that $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is a double cover from the universal cover.

Uses of Higher Homotopy

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- – Important consequence: $SO(3, \mathbb{R})$ is not simply connected!
- – See "plate trick": non-contractible loop of rotations that squares to the identity.
- – Robotics: paths in configuration spaces with singularities

- – Computer graphics: smoothly interpolating between quaternions for rotated camera views

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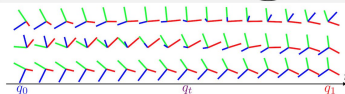
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Setup

– Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate: $X = S^n$

– Can consider the bigraded group $\pi_S := [S^k, S^n]$:

$\pi_k(S^n)$

But Wait!

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The corresponding picture in homology is very easy:

Slogan: "conservation/duality of complexity"

History

- 1895: Poincare, *Analysis situs* ("the analysis of position") in analogy to Euler *Geometria situs* in 1865 on the Kongisberg bridge problem
  - Studies spaces arising from gluing polygons, polyhedra, etc (surfaces!), first use of "algebraic invariant theory" for spaces by introducing π_1 and homology.
- 1920s: Rigorous proof of classification of surfaces (Klein, Möbius, Clifford, Dehn, Heegard)
  - Captured entirely by π_1 (equivalently, by genus and orientability).
- **1931: Hopf discovers a nontrivial (not homotopic to identity) map $S^3 \longrightarrow S^2$**

History

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– 1932/1935: Cech (resp Hurewicz) introduce higher homotopy groups, gives a map relating homotopy to homology, shows they are **abelian** groups for $n \geq 2$.
  – Withdrew his paper because of this theorem!
– 1951: Serre uses spectral sequences to show that **all groups $\pi_k S^n$ are torsion except**,
  – $k = n$, since $\pi_n S^n = \mathbb{Z}$, i -$k \equiv 3 \mod 4, n \equiv 0 \mod 2$, then equal to $\mathbb{Z} \oplus T$
  – Tight bounds on where p-torsion can occur.
– 1953: Whitehead shows the homotopy groups of spheres split into stable and unstable ranges.

Today: We know $\pi_{n+k} S^n$ for

– $k \leq 64$ when $n \geq k + 2$ (stable range)
– $k \leq 19$ when $n < k + 2$ (unstable range)
– We *only* have a complete list for S^0 and S^1, and know *no* patterns beyond this!

Sphere 1

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