

# Title

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# Prologue

## 0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

## 0.2 Notation

- If a property  $P$  is said to hold **locally**, this means that for every point  $p$  there is a neighborhood  $U_p \ni p$  such that  $P$  holds on  $U_p$ .

Notation	Definition
$k[\mathbf{x}] = k[x_1, \dots, x_n]$	Polynomial ring
$k(\mathbf{x}) = k(x_1, \dots, x_n)$	field in $n$
$\mathcal{U} \rightrightarrows X$	indeterminates.
$\Delta_X$	Rational function
$\mathbb{A}_{/k}^n$	field in $n$
$\mathbb{P}_{/k}^n$	indeterminates
$— V(J), V_a(J) — I(S), I_a(S) — A(X)$	An open cover
$V_p(J)$	$\mathcal{U} =$
$I_p(S) — S(X)$	$\{U_j \mid j \in J\}$
$f^h f^i J^h \bar{X} — \mathcal{O}_X$	The diagonal
$D(f)$	$\{(x, x) \mid x \in X\} \subseteq$
	$X \times X$ Affine
	$n$ -space
	$\mathbb{A}_{/k}^n :=$
	$\{\mathbf{a} = [a_1, \dots, a_n] \mid a_j \in k\}$
	Projective
	$n$ -space $\mathbb{P}_{/k}^n :=$
	$(k^n \setminus \{0\})/x \sim$
	$\lambda x =$
	$\{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \mid p, q, \in k[x_1, \dots, x_n]\}$
	Variety
	associated to an
	ideal
	$J \leq k[x_1, \dots, x_n]$
	$:=$
	$\{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$
	Ideal associated
	to a subset
	$S \subseteq \mathbb{A}_k^n :=$
	$\{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in X\}$
	Coordinate ring
	of a variety,
	$k[x_1, \dots, x_n]/I(X)$
	Projective variety
	of an ideal $:=$
	$\{\mathbf{x} \in \mathbb{P}_{/k}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$
	Projective ideal
	$(?) :=$
	$\{f \in k[x_1, \dots, x_n] \mid f \text{ is homogeneous of degree } d\}$
	Projective
	coordinate ring,
	$k[x_1, \dots, x_n]/I_p(X)$
	Homogenization,
	$x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$
	Dehomogenization,
	$f(1, x_1, \dots, x_n)$
	Homogenization
	of an ideal,
	$\{f^j \mid f \in J\}$
	Projective

## 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for  $X = D(f)$  a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety  $X$  in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

## 0.4 Useful Examples

### 0.4.1 Varieties

- $V(xy - 1) \subseteq \mathbb{A}^2$  a hyperbola
- $V(x)$  a coordinate axis
- $V(x - p)$  a point.

### 0.4.2 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$ , a sheaf of smooth functions
- $C^0(\cdot, \mathbb{R})$ , a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$ , the sheaf of regular functions on  $X$
- $\underline{\mathbb{R}}(\cdot)$ , the constant sheaf associated to  $\mathbb{R}$  (locally constant real-valued functions)
- $\text{Hol}(\cdot, \mathbb{C})$ , a sheaf of holomorphic functions
- $K_p$  the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

## 0.5 The Algebra-Geometry Dictionary

Let  $k = \bar{k}$ , we're setting up correspondences

Algebra	Geometry
$k[x_1, \dots, x_n]$	$\mathbb{A}_{/k}^n$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \dots, x_n - p_n$	Points $[a_1, \dots, a_n]$
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \dots, x_n]$	$V(J)$ the zero locus
Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$	Irreducible subsets
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
$k[x_1, \dots, x_n]/I(X)$	$A(X)$ (Functions on $X$ )
$A(X)$ a domain	$X$ is irreducible
$A(X)$ indecomposable	$X$ is connected
Krull dimension $n$ (chains of primes)	Topological dimension $n$ (chains of irreducibles)