Title

Contents

1 | Monday April 20th

Today: torsors.

Let k be a (perfect, separable) field and G/k a commutative algebraic group (a finite type reduced group ?).

Definition A variety X/k is a *torsor under* G is $\mu : G \times X \to X$ a group action such that the map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (\mu(gx), x) \end{aligned}$$

is an isomorphism.

For ℓ/k any field extension, the base change to X/ℓ induces μ_ℓ making X/ℓ a G/ℓ torsor. X is trivial iff it is isomorphic to

$$\begin{aligned} \mu : G \times G &\to G \\ (g, hg) &\mapsto gh. \end{aligned}$$

Claim X is trivial iff $X(k) \neq \emptyset$.

First "proof": for $p \in X(k)$, define $\mu(\,\cdot\,, p) : G \to X$. Want to get a map $G(k^{\text{sep}}) \to X(k^{\text{sep}})$, when does this happen? In characteristic zero, we have some map $G \to X$ (???) which is surjective with trivial kernel and thus an isogeny but has not k^{sep} points. But this doesn't work in positive characteristic.

Second proof: the map $G \times X \to X \times X$ being an isomorphism says that upon base change on $X \to \operatorname{Spec} k$, X becomes isomorphic to G. But then it also becomes isomorphic over base change for which X is intermediate. So if we have

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & \operatorname{Spec} k \\ & X \arrow[ur] & \end{tikzcd}$$

which factors through Y, if $p \in X(k)$ then $\operatorname{Spec} k \to X$ and thus $X/k \cong G/k$.

The form of the assumed isomorphic implies that the base change of the G-torsor X from $\operatorname{Spec} k$ to X is trivial as a $G \times X$ torsor over X.

∎

For k a field, G/k, an equivalent definition would be that a G torsor is X/k with a G action that becomes trivial over k^{sep}. Therefore A G torsor X is a k^{sep}/k twisted form of X where $X/k^{\text{sep}} \cong G/k^{\text{sep}}$.

Example: Let $G = E$ an elliptic curve, and X/k is a nice curve of genus 1, but $X(k)$ is likely empty. Conversely, given such a curve of genus 1, we can take the Picard variety $\underline{\mathrm{Pic}}^0 X$, i.e. the Jacobian. Then there is an isomorphism

$$\begin{aligned} X &\xrightarrow{\cong} \underline{\mathrm{Pic}}^1 X \\ p &\mapsto [p]. \end{aligned}$$

So every nice curve is a torsor for its Jacobian (?). Note that in higher dimensions, we'd need to take the albanese, and the same statement would work: every abelian variety is a torsor over its albanese.

For G/k commutative, we can make the set of torsors X for G/k modulo equivalence into a commutative group. We define the Weil-Chatelet group of G/k as $WC(k, G)$. For two torsors, we can define the *Baer sum* $X_1 \oplus X_2$ by first defining a map

$$\begin{aligned} \mu_\pm : G \times (X_1 \times X_2) &\to X_1 \times X_2 \\ (g, x_1, x_2) &\mapsto (\mu_1(g, x_1), \mu_2([-1]g, x_2)) \end{aligned}$$

and defining $X_1 \oplus X_2 = (X_1 \times X_2)/\mu_\pm$. Then the action $\mu_\pm$ on $X_1 \oplus X_2$ is a G torsor.

This makes $WC(k, G)$ into a commutative group where $\mu : G \times G \to G$ defines $[-1](X, \mu) := (X, \mu([-1] \cdot))$.

Exercise For C/k a nice genus one curve, $G = E = \underline{\mathrm{Pic}}^0 C$ and $C = \underline{\mathrm{Pic}}^1 C$. Show that $n[C]\underline{\mathrm{Pic}}^n C$.

Note that by adding divisor classes, there is a map $\underline{\mathrm{Pic}}^1 C \times \underline{\mathrm{Pic}}^1 C \to \underline{\mathrm{Pic}}^2 E$.

Corollary For E/k an elliptic curve, $WC(k, E)$ is a torsion abelian group iff for all genus 1 curves C, there exists an $n \in \mathbb{Z}^{\geq 0}$ such that $(\underline{\mathrm{Pic}}^n C)(k) \neq \emptyset$.

We can define the *period* of an elliptic curve as the least n for which the torsor becomes trivial, this is an interesting numerical invariant.

Next up: cocycles and descent.