Title

D. Zack Garza

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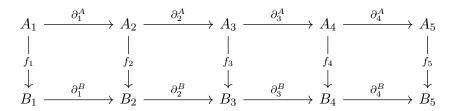
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Problem 1.0.1 (Weibel 1.3.3)

Prove the 5-lemma. Suppose the following rows are exact:



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- a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
- b. Show that if f_2 , f_4 are epi and f_5 is monic, then f_3 is an epi.
- c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):

Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with " \rightarrow " and the epis with " \rightarrow ":

$$A_{1} \xrightarrow{\partial_{1}^{A}} A_{2} \xrightarrow{\partial_{2}^{A}} A_{3} \xrightarrow{\partial_{3}^{A}} A_{4}$$

$$\downarrow \qquad \downarrow$$

$$\downarrow \qquad \qquad \qquad \qquad \qquad \downarrow$$

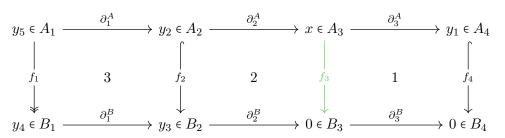
$$B_{1} \xrightarrow{\partial_{1}^{B}} B_{2} \xrightarrow{\partial_{2}^{B}} B_{3} \xrightarrow{\partial_{3}^{B}} B_{4}$$

Link to Diagram

It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2 .
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \rightarrow A_2 \rightarrow A_3$, so x = 0.



Link to Diagram

Solution (Part (b)):

Similarly, it suffices to consider the following portion of the diagram:

Link to Diagram

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

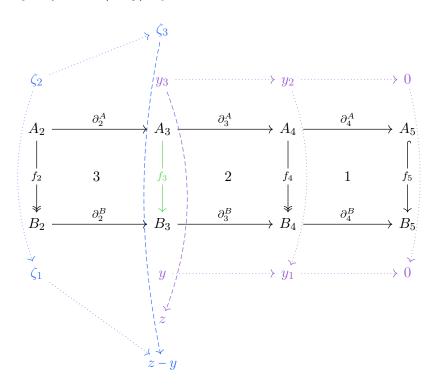
Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4 , pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \rightarrow A_5 \rightarrow B_5$.
- By injectivity of f_5 , the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z y \mapsto y_1 y_1 = 0$ under $B_3 \to B_4$, using that d(z y) = d(z) d(y) since these are R-module morphisms.

- By exactness of the bottom row, pull z y back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2 , pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z y$ under $A_3 \to B_3$.
- But then $y_3 \zeta_3 \mapsto z (z y) = y$ under $A_3 \to B_3$ as desired.



Link to Diagram

Solution (Part (c)):

Given the previous two result, if the outer maps are isomorphisms then f_3 is both monic and epi. Using a technical fact that monic epis are isomorphisms in a category \mathcal{C} if and only if \mathcal{C} is balanced and that all abelian categories are balanced, f_3 is isomorphism.

Problem 1.0.2 (Weibel 1.4.2)

Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$C_n \cong Z_n \oplus B'_n$$

 $Z_n = B_n \oplus H'_n$.

Show that C is split exact if and only if $H'_n = 0$.

Solution:

For this problem, we'll use the fact that if d is an epimorphism, it satisfies the right-cancellation property: if $f \circ d = g \circ d$, then f = g. In particular, if $d_n = d_n s_{n-1} d_n$ with $d_n : C_n \to C_{n-1}$

surjective and $s_{n-1}: C_{n-1} \to C_n$, we can conclude $\mathbb{1}_{C_n} = d_n s_{n-1}$. We'll also use the fact that if we have a SES in any abelian category \mathcal{A} , then the following are equivalent:

- The sequence is split on the left.
- The sequence is split on the right.
- The middle term is isomorphic to the direct sum of the outer terms.

Fixing notation, we'll write $C := (\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots)$, and we'll use concatenation fg to denote function composition $f \circ g$.

⇒ :

Suppose C is split, so we have maps $\{s_n\}$ such that $\partial_n = \partial_n s_{n-1} \partial_n$.

Claim: The short exact sequence

$$0 \to Z_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

admits a right-splitting $f: B_{n-1} \to C_n$, and thus there is an isomorphism

$$C_n \cong Z_n \oplus B'_n = Z_n \oplus B_{n-1}$$
.

Proof (?).

That this sequence is exact follows from the fact that it can be written as

$$0 \to \ker \partial_n \hookrightarrow C_n \stackrel{\partial_n}{\twoheadrightarrow} \operatorname{im} \partial_n \to 0.$$

We proceed by constructing the splitting f. Noting that $s_{n-1}: C_{n-1} \to C_n$ and $B_{n-1} \le C_{n-1}$, the claim is that its restriction $f := s_{n-1}|_{B_{n-1}}$ works. It suffices to show that $(C_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{f} C_n)$ composes to the identity map $\mathbbm{1}_{C_n}$. This follows from the splitting assumption, along with right-cancellability since ∂_n is surjective onto its image:

$$\partial_n = \partial_n s_{n-1} \partial_n \overset{\text{right-cancel } \partial_n}{\Longrightarrow} \mathbb{1}_{C_n} = \partial_n s_{n-1} \coloneqq \partial_n f.$$

Claim: The SES

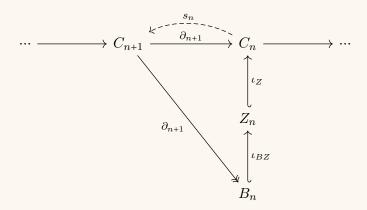
$$0 \to B_n \stackrel{\iota_{BZ}}{\hookrightarrow} Z_n \stackrel{\pi}{\twoheadrightarrow} \frac{Z_n}{B_n} \to 0$$

admits a left-splitting $f: \mathbb{Z}_n \to \mathbb{B}_n$, and thus there is an isomorphism

$$Z_n \cong B_n \oplus H'_n \coloneqq B_n \oplus H_n(C) \coloneqq B_n \oplus \frac{Z_n}{B_n}.$$

Proof (?).

We proceed by again constructing the splitting $f: \mathbb{Z}_n \to \mathbb{B}_n$. The situation is summarized in the following diagram:



Link to Diagram

So a natural candidate for the map f is the composition

$$Z_n \xrightarrow{\iota_Z} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} B_n$$

so $f \coloneqq \partial_{n+1} s_n \iota_Z$. We can simplify this slightly by regarding $Z_n \le C_n$ as a submodule to suppress ι_Z , and identify s_n with its restriction to Z_n to write $f \coloneqq \partial_{n+1} s_n$. The claim is then that $f\iota_{BZ} = \mathbbm{1}_{B_n}$. Anticipating using the fact that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$, we post-compose with ∂_{n+1} and compute:

$$f\iota_{BZ}\partial_{n+1} = \left(C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{\iota_{BZ}} Z_n \xrightarrow{\iota_{Z}} C_n \xrightarrow{s_n} C_{n+1} \xrightarrow{\partial_{n+1}} C_n\right)$$

$$= \left(C_{n+1} \xrightarrow{\partial_{n+1}} B_n \xrightarrow{s_n|_{B_n}} C_{n+1} \xrightarrow{\partial_{n+1}}\right)$$

$$= \partial_{n+1}s_n\partial_{n+1}$$

$$= \partial_{n+1},$$

where in the last step we've used the splitting hypothesis and the fact that it remains true when everything is restricted to the submodule $B_n \leq C_n$. Using surjectivity of ∂_{n+1} onto B_n , we can now conclude as before:

$$f\iota_{BZ}\partial_{n+1}=\partial_{n+1}\overset{\text{right-cancel }\partial_n}{\Longrightarrow}f\iota_{BZ}=\mathbbm{1}_{B_n}.$$

Problem 1.0.3 (Weibel 1.4.3)

Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

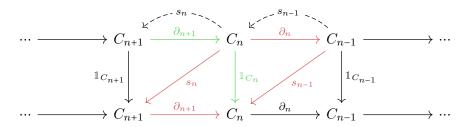
Solution:

⇐=:

C is split: Suppose $\mathbb{1}_C$ is nullhomotopic, so that there exist maps

$$s_n: C_n \to C_{n+1} \qquad \qquad \mathbb{1}_{C_n} = \partial_{n+1} s_n + s_{n-1} \partial_n.$$

We then have the following situation:



Link to Diagram

Here the nullhomotopy is outlined in red, and the map relevant to the splitting in green. Note that $s_n: C_n \to C_{n+1}$ is a candidate for a splitting, we just need to show that $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$. We can proceed by post-composing the LHS with the identity $\mathbb{1}_C$, which allows us to substitute in the nullhomotopy:

$$\partial_{n+1} = \mathbb{1}_{C_n} \partial_{n+1}$$

$$= (\partial_{n+1} s_n + s_{n-1} \partial_n) \partial_{n+1}$$

$$= \partial_{n+1} s_n \partial_{n+1} + s_{n-1} \partial_n \partial_{n+1}$$

$$= \partial_{n+1} s_n \partial_{n+1} + s_{n-1} \mathbf{0}$$

$$= \partial_{n+1} s_n \partial_{n+1}.$$
since $\partial^2 = 0$

$$= \partial_{n+1} s_n \partial_{n+1}.$$

C is exact: This follows from the fact that since $\mathbb{1}_C = \partial s + s\partial$ are equal as maps of chain complexes, the images $D_1 := \mathbb{1}_C(C)$ and $D_2 := (\partial s + s\partial)(C)$ are equal as chain complexes and have equal homology. Evidently $D_1 = C$, and on the other hand, each graded piece $(D_2)_n$ only consists of boundaries coming from various pieces of C, since $\partial s + s\partial$ necessarily lands in the images of the maps ∂_n . Thus $C_n(D_2) \subseteq B_n(D_2) = \emptyset$, i.e. there are no chains (or cycles) in D_2 which are not boundaries, and thus $H_n(D_2) := Z_n(D_2)/B_n(D_2) = 0$ for all n. We can thus conclude that $C = D_2 \implies H(C) = H(D_2) = 0$, so C must be exact.

 \implies : Suppose C is split. Then by exercise 1.4.2, we have R-module decompositions

$$C_n \cong Z_n \oplus B_{n-1}$$
$$Z_n \cong B_n \oplus H_n$$

$$\implies C_n \cong B_n \oplus B_{n-1} \oplus H_n.$$

Supposing further that C is exact, we have $H_n = 0$, and thus $C_n \cong B_n \oplus B_{n-1}$. We first note that in this case, we can explicitly write the differential ∂_n . Letting $\mathbb{1}_n$ denote the identity on C_n , where by abuse of notation we also write this for its restriction to any submodules, we have:

$$C_n \xrightarrow{\partial_n} C_{n-1}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$B_n \xrightarrow{-0} \xrightarrow{-0} B_{n-1}$$

$$\oplus \qquad \oplus$$

$$B_{n-1} \xrightarrow{-0} \xrightarrow{-0} B_{n-2}$$

Link to Diagram

We can thus write ∂_n as the matrix

$$\partial_n = \begin{bmatrix} 0 & \mathbb{1}_n \\ 0 & 0 \end{bmatrix}.$$

Similarly using this decomposition, we can construct a map $s_n: C_n \to C_{n+1}$:

$$C_n \xrightarrow{s_n} C_{n+1}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$B_n \xrightarrow{-0} \xrightarrow{-0} B_{n+1}$$

$$\oplus \qquad \oplus$$

$$B_{n-1} \xrightarrow{-0} \xrightarrow{-0} B_n$$

Link to Diagram

We can write this as the following matrix:

$$s_n = \begin{bmatrix} 0 & 0 \\ \mathbb{1}_n & 0 \end{bmatrix}.$$

We can now verify that s_n is a nullhomotopy from a direct computation:

$$\begin{split} \partial_{n+1} s_n + s_{n-1} \partial_n &= \begin{bmatrix} 0 & \mathbbm{1}_{n+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbbm{1}_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbbm{1}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbbm{1}_n \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbbm{1}_n & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbbm{1}_{n-1} \end{bmatrix} \\ &= \mathbbm{1}_{C_n}, \end{split}$$

expressed as a map $B_n \oplus B_{n-1} \to B_n \oplus B_{n-1}$.

Problem 1.0.4 (Weibel 1.4.5)

Show that chain homotopy classes of maps form a quotient category K of Ch(R-mod) and that the functors H_n factor through the quotient functor $Ch(R\text{-mod}) \to K$ using the following steps:

- 1. Show that chain homotopy equivalence is an equivalence relation on $\{f: C \to D \mid f \text{ is a chain map}\}$. Define $\operatorname{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
- 2. Let $f \simeq g: C \to D$ be two chain homotopic maps. If $u: B \to C, v: D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
- 3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $Ch(R\text{-mod}) \to K$ is an additive functor.
- 4. Is K an abelian category? Explain.

Try at least two parts.

Problem 1.0.5 (Weibel 1.5.1) Let cone(C) := cone($\mathbb{1}_C$), so

$$cone(C)_n = C_{n-1} \oplus C_n.$$

Show that cone(C) is split exact, with splitting map given by $(b,c) \mapsto (-c,0)$.

Solution:

Fixing notation, let $\mathbb{1}_C$ be the identity chain map on C, $\mathbb{1}_n : C_n \to C_n$ its nth graded piece, and write $\widehat{C} := \operatorname{cone}(C) := \operatorname{cone}(\mathbb{1}_C)$, $\widehat{\mathbb{1}}$ for the identity on \widehat{C} , and $\widehat{\mathbb{1}}_n$ for its nth piece. The result will follow from a direct computation: from exercise 1.4.2, it suffices to show that $\widehat{\mathbb{1}}$ is nullhomotopic.

Problem 1.0.6 (Weibel 1.5.2)

Let $f: C \to D \in \text{Mor}(\text{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(s, f) : \operatorname{cone}(C) \to D.$$

Problem 1.0.7 (Extra)

- a. Show that free implies projective.
- b. Show that $\operatorname{Hom}_R(M, \cdot)$ is left-exact.
- c. Show that P is projective if and only if $\operatorname{Hom}_{R}(P,\cdot)$ is exact.