Fiber Bundles

What is a fiber bundle? Generally speaking, it is similar to a fibration - we require the homotopy lifting property to hold, although it is not necessary that path lifting is unique.

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_0"] \arrow[d, "X\times\{0\}"'] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "f"'] \arrow[ur, dotted, "\tilde{f}"] & B \end{tikzcd}$$

However, it also satisfies more conditions - in particular, the condition of *local triviality*. This requires that the total space looks like a product locally, although there may some type of global monodromy. Thus with some mild conditions[1], fiber bundles will be instances of fibrations (or alternatively, fibrations are a generalization of fiber bundles, whichever you prefer!)

As with fibrations, we can interpret a fiber bundle as "a family of Bs indexed/parameterized by Fs", and the general shape data of a fiber bundle is similarly given by

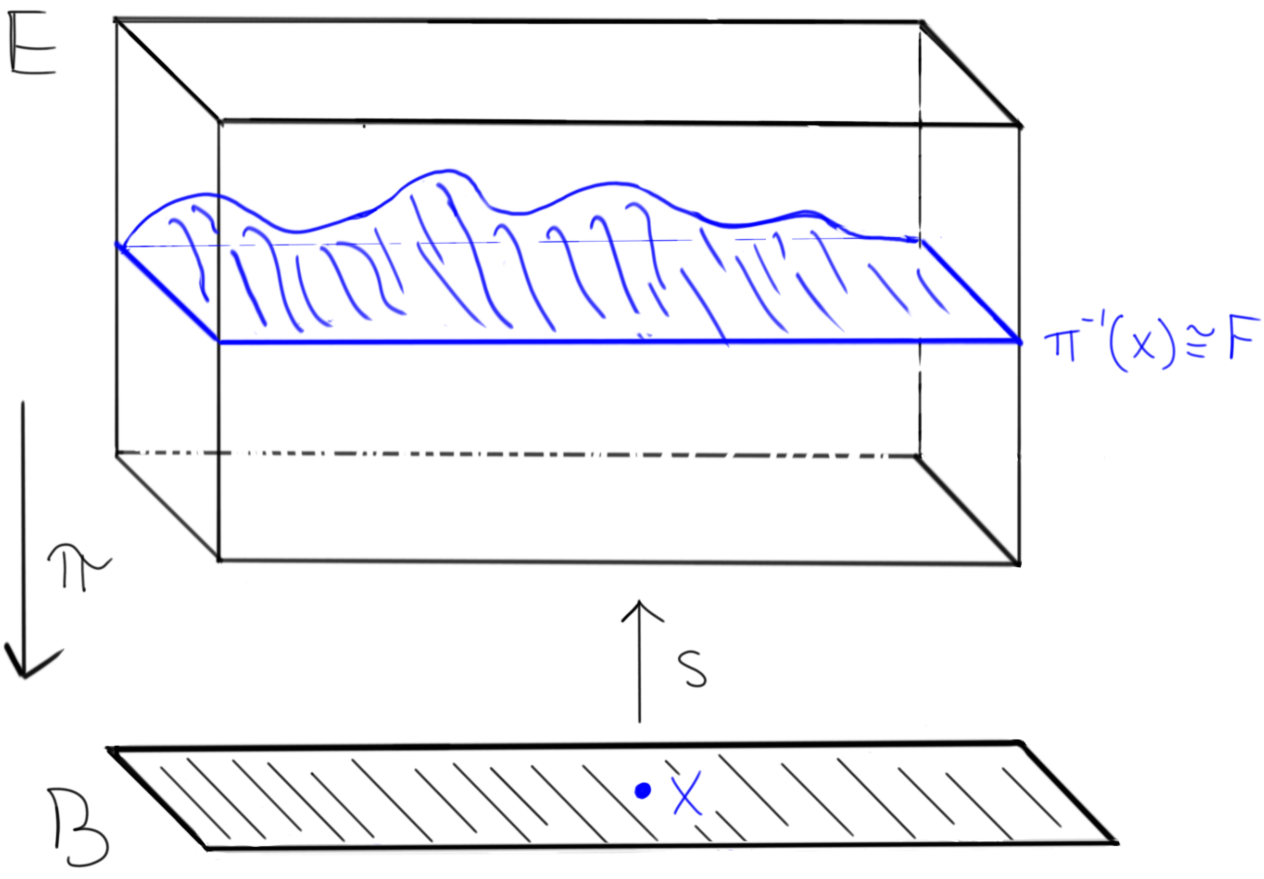
$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, two heads, "\pi"] \\ & B \arrow[u, dotted, bend left, "s"] \end{tikzcd}$$

where B is the base space, E is the total space, $\pi : E \to B$ is the projection map, and F is "the" fiber (in this case, unique up to homeomorphism). Fiber bundles are often described in shorthand by the data $E \xrightarrow{\pi} B$, or occasionally by tuples such as (E, π, B).

The local triviality condition is a requirement that the projection π locally factors through the product; that is, for each open set $U \in B$, there is a homeomorphism φ making this diagram commute:

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, dashed, "\varphi"] \arrow[d, two heads, "\pi"'] & U \times F \arrow[dl, "(a,b)\mapsto a"] \\
U &
\end{tikzcd}
$$

Fiber bundles may admit right-inverses to the projection map $s : B \to E$ satisfying $\pi \circ s = \mathrm{id}_B$, denoted *sections*. Equivalently, for each $b \in B$, a section is a choice of an element e in the preimage $\pi^{-1}(b) \simeq F$ (i.e. the fiber over b). Sections are sometimes referred to as *cross-sections* in older literature, due to the fact that a choice of section yields might be schematically represented as such:



Here, we imagine each fiber as a cross-section or “level set” of the total space, giving rise to a “foliation” of E by the fibers.[2]

For a given bundle, it is generally possible to choose sections locally, but there may or may not exist globally defined sections. Thus one key question is **when does a fiber bunde admit a global section?**

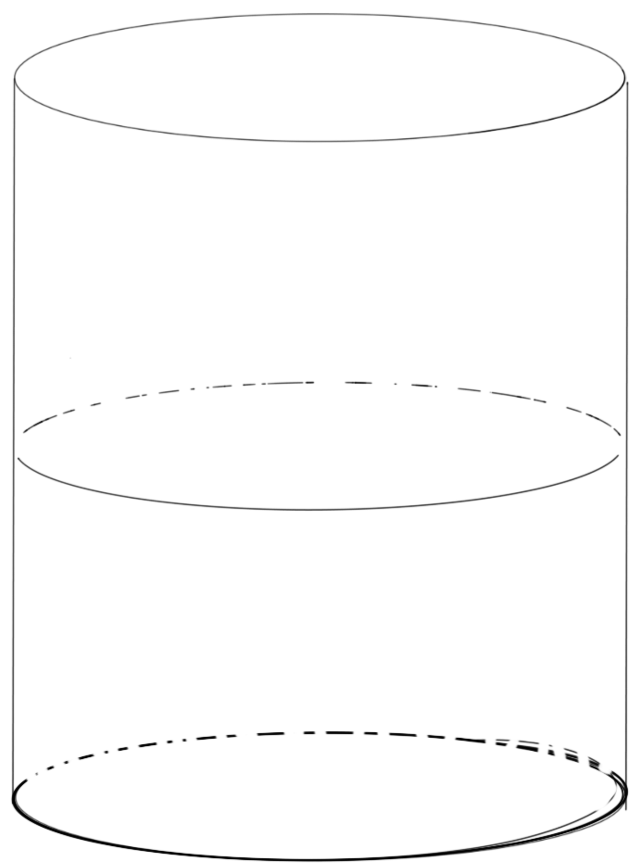
A bundle is said to be *trivial* if $E = F \times B$, and so another important question is **when is a fiber bundle trivial?**

Definition: A fiber bundle in which F is a k-vector space for some field k is referred to as a *rank n vector bundle.* When $k = \mathbb{R}, \mathbb{C}$, they are denoted real/complex vector bundles respectively. A vector bundle of rank 1 is often referred to as a *line bundle*.

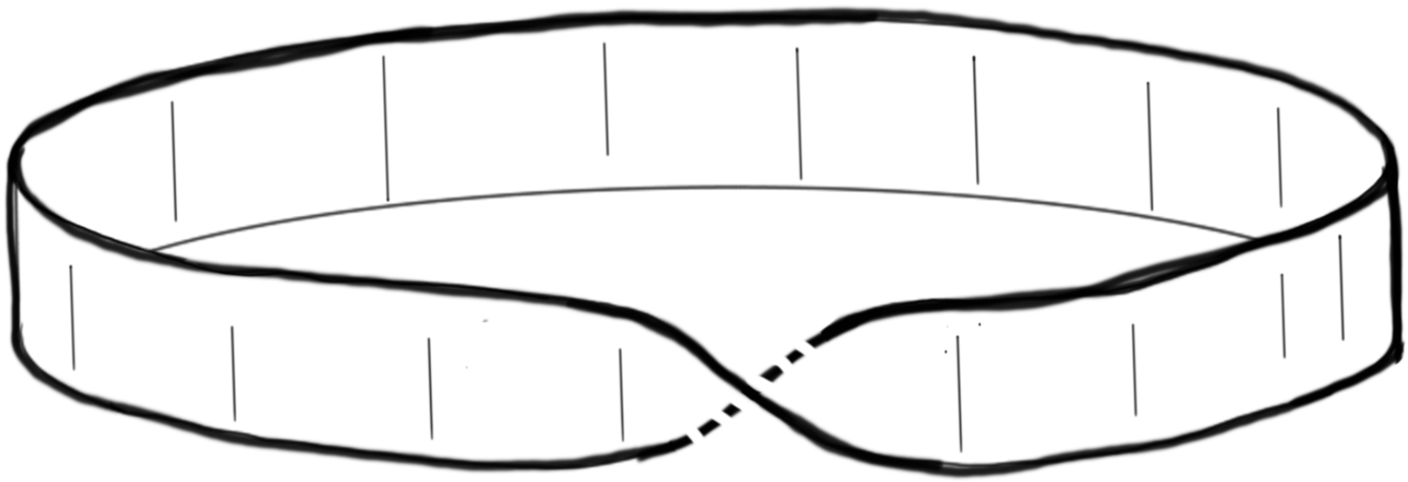
Example: There are in fact non-trivial fiber bundles. Consider the space E that can appear as the total space in a line bundle over the circle

$$\mathbb{R}^1 \to E \to S^1$$

That is, the total spaces that occur when a one-dimensional real vector space (i.e. a real line) is chosen at each point of S^1. One possibility is the trivial bundle $E \cong S^1 \times \mathbb{R} \cong S^1 \times I^\circ \in \mathrm{DiffTop}$, which is an "open cylinder":



But another possibility is $E \cong M^\circ \in \mathrm{DiffTop}$, an open Mobius band:



Here we can take the base space B to be the circle through the center of the band; then every open neighborhood U of a point $b \in B$ contains an arc of the center circle crossed with a vertical line segment that misses ∂M. Thus the local picture looks like $S^1 \times I^\circ$, while globally $M \ncong S^1 \times I^\circ \in \mathrm{Top}$.[3]

So in terms of fiber bundles, we have the following situation

$$\begin{array}{ccccc} \mathbb{R} & \to & M & \to & S^1 \\ \parallel & & \not\cong & & \parallel \\ \mathbb{R} & \to & S^1 \times I^\circ & \to & S^1 \end{array}$$

and thus M is associated to a nontrivial line bundle over the circle.

Remark: In fact, these are the only two line bundles over S^1. This leads us to a natural question, similar to the group extension question: **given a base B and fiber F, what are the isomorphism classes of fiber bundles over B with fiber F?** In general, we will find that these classes manifest themselves in homology or homotopy. As an example, we have the following result:

Notation: Let $I(F, B)$ denote isomorphism classes of fiber bundles of the form $F \to \cdot \to B$.

Proposition:

The set of isomorphism classes of smooth line bundles over a space B satisfies the following isomorphism of abelian groups:

$$I(\mathbb{R}^1, B) \cong H^1(B; \mathbb{Z}_2) \in \mathrm{Ab}$$

in which the RHS is generated by the first Stiefel-Whitney class $w_1(B)$.

Proof:

Lemma: The structure group of a vector bundle is a general linear group. (Or orthogonal group, by Gram-Schmidt)

Lemma: The classifying space of $\mathrm{GL}(n, \mathbb{R})$ is $\mathrm{Gr}(n, \mathbb{R}^\infty)$

Lemma: $\mathrm{Gr}(n, \mathbb{R}^\infty) = \mathbb{RP}^\infty \simeq K(\mathbb{Z}_2, 1)$

Lemma: For G an abelian group and X a CW complex, $[X, K(G, n)] \cong H^n(X; G)$

The structure group of a vector bundle can be taken to be either the general linear group or the orthogonal group, and the classifying space of both groups are homotopy-equivalent to an infinite real Grassmanian.

$$\begin{aligned} I(\mathbb{R}^1, B) &= [B, B((\mathrm{Sym}\,\mathbb{R})|_{\mathrm{Vect}})] \\ &= [B, B(\mathrm{GL}(1, \mathbb{R}))] \\ &= [B, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\ &= [B, \mathbb{RP}^\infty] \\ &= [B, K(\mathbb{Z}_2, 1)] \\ &= H^1(B; \mathbb{Z}_2) \end{aligned}$$

$\blacksquare$

This is the general sort of pattern we will find - isomorphism classes of bundles will be represented by homotopy classes of maps into classifying spaces, and for nice enough classifying spaces, these will represent elements in cohomology.

Corollary: There are two isomorphism classes of line bundles over S^1, generated by the Mobius strip, since $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$ (Note: this computation follows from the fact that $H_1(S^1) = \mathbb{Z}$ and an application of both universal coefficient theorems.)

Note: The Stiefel-Whitney class is only a complete invariant of *line* bundles over a space. It is generally an incomplete invariant; for higher dimensions or different types of fibers, other invariants (so-called *characteristic classes*) will be necessary.

Another important piece of data corresponding to a fiber bundle is the *structure group*, which is a subgroup of $\mathrm{Sym}(F) \in \mathrm{Set}$ and arises from imposing conditions on the structure of the transition functions between local trivial patches. A fiber bundle with structure group G is referred to as a G*-bundle*.

Vector Bundles

Definition: A *rank n vector bundle* is a fiber bundle in which the fibers F have the structure of a vector space k^n for some field k; the structure group of such a bundle is a subset of $\mathrm{GL}(n, k)$.

Note that a vector bundle always has one global section: namely, since every fiber is a vector space, you can canonically choose the 0 element to obtain a global zero section.

Proposition: A rank n vector bundle is trivial iff it admits k linearly independent global sections.

Example: The tangent bundle of a manifold is an $\mathbb{R}$-vector bundle. Let M^n be an n- dimensional manifold. For any point $x \in M$, the tangent space T_xM exists, and so we can define

$$TM = \coprod_{x \in M} T_xM = \{(x, t) \mid x \in M, t \in T_xM\}$$

Then TM is a manifold of dimension $2n$ and there is a corresponding fiber bundle

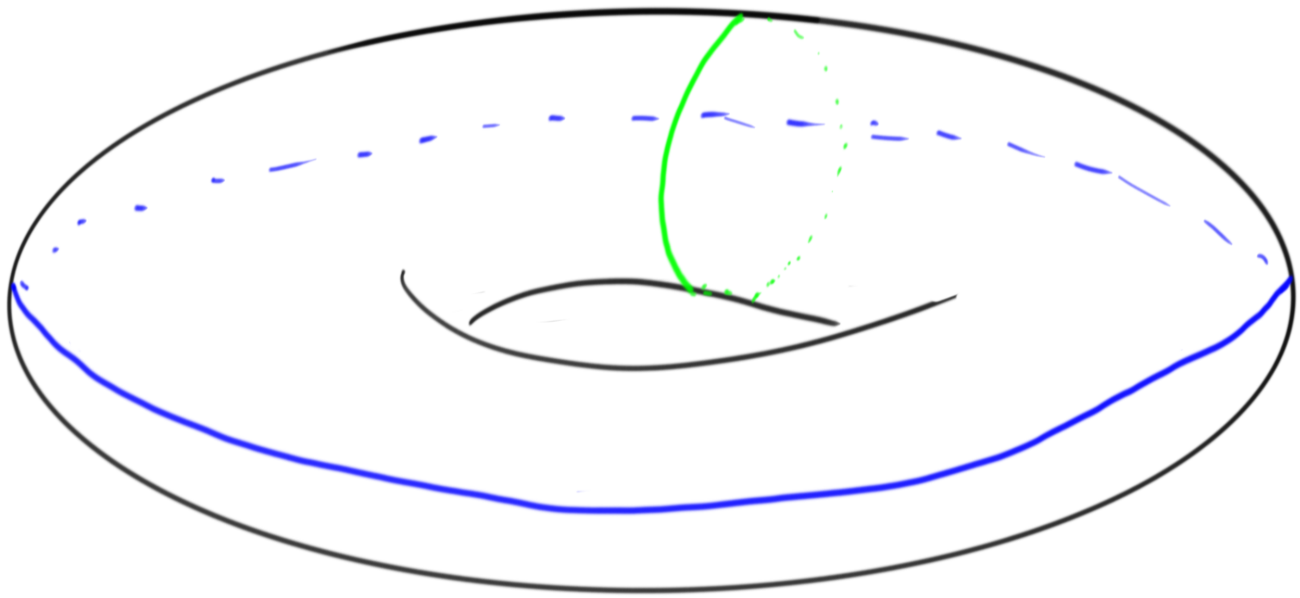
$$\mathbb{R}^n \to TM \xrightarrow{\pi} M$$

given by a natural projection $\pi : (x, t) \mapsto x$

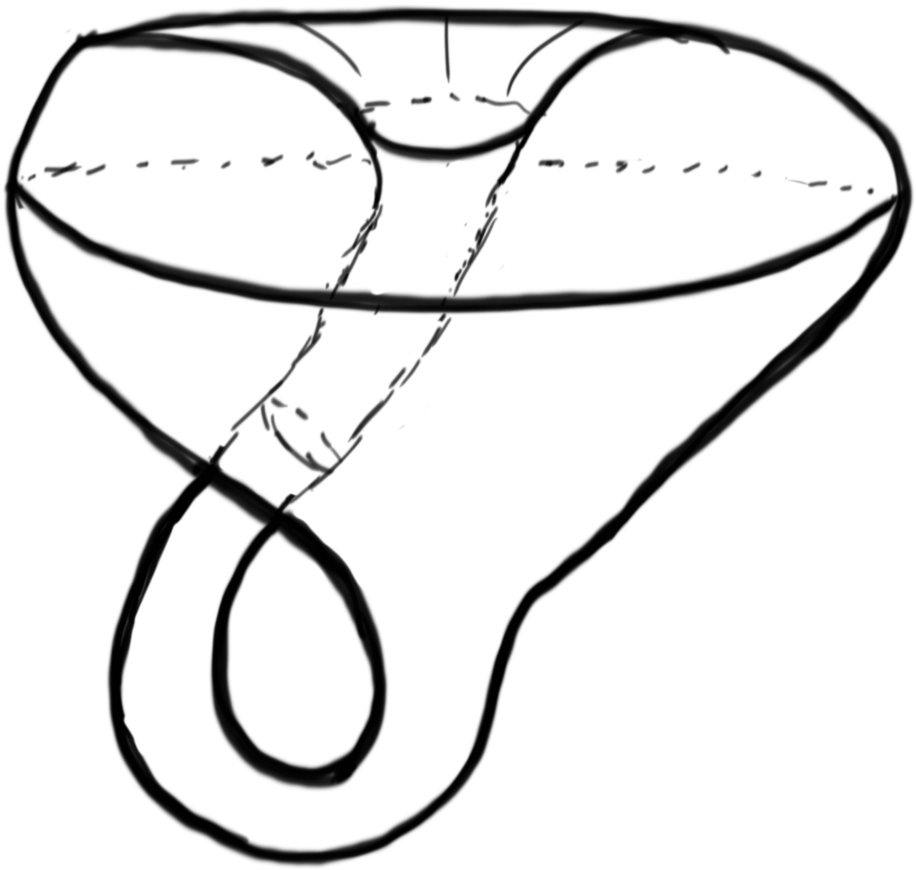
Example A circle bundle is a fiber bundle in which the fiber is isomorphic to S^1 as a topological group. Consider circle bundles over a circle, which are of the form

$$S^1 \to E \xrightarrow{\pi} S^1$$

There is a trivial bundle, when $E = S^1 \times S^1 = T^2$, the torus:



There is also a nontrivial bundle, $E = K$, the Klein bottle:



As in the earlier example involving the Mobius strip, since K is nonorientable, $T^2 \ncong K$ and there are thus at least two distinct bundles of this type.

Remark: A section of the tangent bundle TM is equivalent to a *vector field* on M.

Definition: If the tangent bundle of a manifold is trivial, the manifold is said to be *parallelizable.*

Proposition: The circle S^1 is parallelizable.

Proof Let $M = S^1$, then there is a rank 1 vector bundle

$$\mathbb{R} \to TM \to M$$

and since $TM = S^1 \times \mathbb{R}$ (why?), we find that S^1 is parallelizable.

■

Proposition: The sphere S^2 is not parallelizable.

Proof: Let $M = S^2$, which is associated to the rank 2 vector bundle

$$\mathbb{R}^2 \to TM \to M$$

Then TM is trivial iff there are 2 independent global sections. Since there is a zero section, a second independent section must be everywhere-nonzero - however, this would be a nowhere vanishing vector field on S^2, which by the Hairy Ball theorem does not exist.

Alternate proof: such a vector field would allow a homotopy between the identity and the antipodal map on S^2, contradiction by basic homotopy theory.

$\blacksquare$

Classifying Spaces

Definition: A *principal G- bundle* is a fiber bundle $F \to E \to B$ in which for each fiber $\pi^{-1}(b) := F_b$, satisfying the condition that G acts freely and transitively on F_b. In other words, there is a continuous group action $\curvearrowright : E \times G \to E$ such that for every $f \in F_b$ and $g \in G$, we have $g \curvearrowright f \in F_b$ and $g \curvearrowright f \neq f$.

Example: A covering space $\hat{X} \xrightarrow{p} X$ yields a principal $\pi_1(X)$- bundle.

Remark: A consequence of this is that each $F_b \cong G \in \mathrm{TopGrp}$ (which may also be taken as the definition). Furthermore, each F_b is then a *homogeneous space*, i.e. a space with a transitive group action $G \curvearrowright F_b$ making $F_b \cong G/G_x$.

Remark: Although each fiber F_b is isomorphic to G, there is no preferred identity element in F_b. Locally, one can form a local section by choosing some $e \in F_b$ to serve as the identity, but the fibers can only be given a global group structure iff the bundle is trivial. This property is expressed by saying F_b is a G-*torsor*.

Remark: Every fiber bundle $F \to E \to B$ is a principal $\mathrm{Aut}(F)$- fiber bundle. Also, in local trivializations, the transition functions are elements of G.

Proposition: A principal bundle is trivial iff it admits a global section. Thus all principal vector bundles are trivial, since the zero section always exists.

Definition: A principal bundle $F \to E \xrightarrow{\pi} B$ is *universal* iff E is weakly contractible, i.e. if E has the homotopy type of a point.

Definition: Given a topological group G, a *classifying space*, denoted BG, is the base space of a universal principal G- bundle

$$G \to EG \xrightarrow{\pi} BG$$

making BG a quotient of the contractible space EG by a G- action. We shall refer to this as *the classifying bundle*.

Classifying spaces satisfy the property that any other principal G- bundle over a space X is isomorphic to a pullback of the classifying bundle along a map $X \to BG$.

Let $I(G, X)$ denote the set of isomorphism classes of principal G- bundles over a base space X, then

$$I(G, X) \cong [X, BG]_{\text{hoTop}}$$

So in other words, isomorphism classes of principal G- bundles over a base X are equivalent to homotopy classes of maps from X into the classifying space of G.

Proposition: Grassmanians are classifying spaces for vector bundles. That is, there is a bijective correspondence:

$$[X, \mathrm{Gr}(n, \mathbb{R})] \cong \{\text{isomorphism classes of rank } n \ \mathbb{R}\text{-vector bundles over } X\}$$

\

It is also the case that every such vector bundle is a pullback of the principal bundle

$$\mathrm{GL}(n, \mathbb{R}) \to V_n(\mathbb{R}^\infty) \to \mathrm{Gr}(n, \mathbb{R})$$

---

1. A fiber bundle $E \to B$ is a fibration when B is paracompact. ↩
2. When E is in fact a product $F \times B$, this actually is a foliation in the technical sense. ↩
3. Due to the fact that, for example, M is nonorientable and orientability distinguishes topological spaces up to homeomorphism. ↩