History

Poincare, *Analysis Situs* papers in 1895. Coined “homeomorphism”, defined homology, gave rigorous definition of homotopy, established “method of invariants” and essentially kicked off algebraic topology.

Motivation

Generalized Topological Poincare Conjecture: When is a homotopy sphere also a topological sphere? i.e. when does $\pi_* X \cong_{Grp} \pi_* S^n \implies X \cong_{Top} S^n$?

- $n=1$: True. Trivial
- $n=2$: True. Proved by Poincare, classical
- $n=3$: True. Perelman (2006) using Ricci flow + surgery
- $n=4$: True. Freedman (1982), Fields medal!
- $n=5$: True. Zeeman (1961)
- $n=6$: True. Stalling (1962)
- $n\geq 7$: True. Smale (1961) using h-cobordism theorem, uses handle decomposition + Morse functions

Smooth Poincare Conjecture: When is a homotopy sphere a *smooth* sphere?

- $n=1$: True. Trivial
- $n=2$: True. Proved by Poincare, classical
- $n=3$: True. (Top = PL = Smooth)
- $n=4$: **Open**
- $n=5$: Zeeman (1961)
- $n=6$: Stalling (1962)
- $n\geq 7$: False in general (Milnor and Kervaire, 1963), Exotic S^n, 28 smooth structures on S^7

> It is unknown whether or not B^4 admits an exotic smooth structure. If not, the smooth 4-dimensional Poincare conjecture would have an affirmative answer.

> Current line of attack: Gluck twists on on S^4. Yield homeomorphic spheres, *suspected* not to be diffeomorphic, but no known invariants can distinguish smooth structures on S^4.

Relation to homotopy: Define a monoid G_n with

- Objects: smooth structures on the n sphere (identified as oriented smooth $n\dash$manifolds which are homeomorphic to S^n)
- Binary operation: Connect sum

For $n\neq 4$, this is a group. Turns out to be isomorphic to Θ_n, the group of $h\dash$cobordism classes of "homotopy S^ns"

Recently (almost) resolved question: what is Θ_n for all n?

> Application: what spheres admit unique smooth structures?

- Define $bP_{n+1} \leq \Theta_n$ the subgroup of spheres that bound *parallelizable* manifolds (define in a moment).
- The Kervaire invariant is an invariant of a framed manifold that measures whether the manifold could be surgically converted into a sphere. 0 if true, 1 otherwise.
- Hill/Hopkins/Ravenel (2016): = 0 for $n \geq 254$.
- Kervaire invariant = 1 only in 2, 6, 14, 30, 62. Open case: 126.
- Punchline: there is a map $\Theta_n/bP_{n+1} \to \pi_n^S/ J$, (to be defined) and the Kervaire invariant influences the size of bP_{n+1}. This reduces the differential topology problem of classifying smooth structures to (essentially) computing homotopy groups of spheres.
- Open question: is there a manifold of dimension 126 with Kervaire invariant 1?

> Parallelizable/framed: Trivial tangent bundle, i.e. the principal frame bundle has a smooth global section. Parallelizable spheres S^0, S^1, S^3, S^7 corresponding to $\RR, \CC, \mathbb{H}, \mathbb{O}$.

- Framed Bordism Classes of manifolds – $\Omega^{fr}_n \cong \pi_n^S$ > Note: bordism is one of the coarsest equivalence relations we can put on manifolds. Hope to understand completely!

Background

Definition (Homotopy) Given two paths $P_1, P_2: I \to X$ (where we identify the paths with their images under these maps), then a *homotopy* from P_1 to P_2 is a function $$ H: I \to (I \to X) \\ H(0, \wait) = x_0 \\ H(1, \wait) = x_1 \\ \\ H(\wait, 0) = P_1(\wait)\\ H(\wait, 1) = P_2(\wait)\\ $$

such that the associated "partially applied" function $H_t: I \to X$ is continuous.

Definition (Homotopic Maps) Given two maps $f, g: X \to Y$, we say f is *homotopic* to g and write $f \sim g$ if there is a homotopy $$ H: I \to (X \to Y) \\ H(0, \wait) = f(\wait)\\ H(1, \wait) = g(\wait)\\ $$

such that $H_t: X \to Y$ is continuous.

> Can think of this as a map from the cylinder on X into Y, or deformations through continuous functions.

> Note: This is an equivalence relation. If $f: X \to Y$ is a map, we write $[X, Y]$ to denote the homotopy classes of maps X to Y.

Definition (Fundamental Group) $$ \pi_1(X) \definedas [S^1, X]. $$

> Note that this actually measures homotopy classes of *loops* in X.

Example: $\pi_1 T^2 = \ZZ^{\ast 2}$, a *free* abelian group of rank 2.

Definition (Higher Homotopy Groups) $$ \pi_n(X) \definedas [S^n, X]. $$

Introduced by Cech in 1932, Alexandrov reportedly told him to withdraw because it couldn't possibly be the right generalization due to the following theorem:

Theorem: $$ n\geq 2 \implies [S^n, X] \in \mathrm{Ab}. $$ In words, higher homotopy groups are abelian. We have a complete classification of abelian groups, so we know $\pi_n(X) = F \oplus T$ for some free and torsion parts.

Theorem (Hopf, 1931): $$[S^3, S^2] = \ZZ \neq 0$$

> Recall that homology vanishes above the dimension of a given manifold!

This group is generated by the *Hopf fibration*, and provides infinitely many ways of "wrapping" a 3-sphere around a 2-sphere nontrivially! This was surprising and unexpected

Definition (CW Complex) A CW complex is any space built from the following inductive process:

> Denote X_n the $n\dash$skeleton.

- Let X_0 by a discrete set of points.
- Let X_{n+1} be obtained from X_n by taking a collection of $n\dash$ balls and glue them to X_n by maps $$\phi: \del B^n \to X_n$$.
- If infinitely many stages, let $X = \union X_n$ with the weak topology
(i.e. a set $A \subset X$ is open iff $A \intersect X_n$ is open for all n)

Example: Every graph is a 1-dimensional CW complex

Example: Identification polyhedra for surfaces

Example: $S_n = e_0 + e_n$ by gluing B^{n+1} to a point by a map $\phi: \del B^{n+1} \to \pt$, i.e. $B^{n+1} / B^n \cong S^n$. Can also attach two hemispheres at each $i\leq n$ to get $S^n = e_0 + e_1 + 2e_2 + \cdots + 2e_n$.

> Note: Cellular homology is very easy to compute!

> Note: Replacing ϕ with a homotopic map yields an equivalent CW complex. So understanding CW complexes boils down to understanding $[S^n, S^m]$ for $m < n$, i.e. higher homotopy groups of spheres.

Definition (Cellular Map) A map between $f:X \to Y$ between CW complex is *cellular* if $f(X_{(k)}) \subseteq Y_{(k)}$ for every k.

Theorem (Cellular Approximation): Any map $f: X \to Y$ is homotopic to a cellular map.

> Application: $\pi_k S^n = 0$ if $k < n$. Use $f\in \pi_kS^n \iff f\in [S_k, S_n]$, deform f to be cellular, then $f(S^k_{(k)}) \injects S^n_{(k)} = \pt$, so $f \homotopic c_0$, a constant map.

Definition (Homotopy Equivalence) Two spaces X, Y are said to be *homotopy equivalent* if there exists a maps $f: X \to Y$ and $f\inv: Y \to X$ such that $$ f\circ f\inv \homotopic \id_Y \\ f\inv \circ f \homotopic \id_X $$

Definition (Weak Equivalence) A continuous map $$ f: X \to Y $$ is called a *weak homotopy equivalence* if the induced map $$ f_*: \pi_*(X) \to \pi_*(Y) $$ is a graded isomorphism.

> Note that this is a strictly weaker notion than homotopy equivalence – we don't require an explicit inverse.

> Note that a weak homotopy equivalence also induces isomorphisms on all homology and cohomology.

Theorem (Whitehead): If $f: X\to Y$ is a weak equivalence between CW complexes, then it is a homotopy equivalence.

> Corollary (Relative Whitehead): If $f:X\to Y$ between CW complexes induces an isomorphism $H_* X \cong H_* Y$, then f is a weak equivalence.

Theorem (CW Approximation): For every topological space X, there exists a CW complex $\tilde X$ and a weak homotopy equivalence $f: X \to \tilde X$.

> Note: Weak equivalences = equivalences for CW complexes, which means we can essentially throw out the distinction!

> Note: This says that if we understand CW complexes, we essentially understand the category hoTop completely. Moreover, we only have to understand spaces up to *weak* equivalence, i.e. we just need to check induced maps on π_* instead of checking for inverse maps.

Definition (Connectedness): A space is said to be $n\dash$connected if $\pi_{\leq n} X = 0$.

> Recall that a space is *simply connected* iff $\pi_1 X = 0$.

Theorem (Hurewicz): Given a fixed space X, any map $f \in \pi_kX = [S^k, X]$ has the type $f: S^k \to X$. This induces a map $f_*: H_* S^k \to H_* X$. Since $H_k S^k \cong \ZZ \cong \generators{\mu}$, define a family of maps $$ h_k: \pi_k X \to H_k X \\ [f] \mapsto f_*(\mu) $$

If $n\geq 2$ and X is $n-1$ connected, then h_k is an isomorphism for all $k\leq n$.

> Note: If $k=1$, then h_1 is the abelianization of π_1.

Application

If X a simply connected, closed 3-manifold is a homology sphere, then it is a homotopy sphere.

- $H_0 X = \ZZ$ since X is path-connected
- $H_1 X = 0$ since X is simply-connected
- $H_3 X = \ZZ$ since X is orientable
- $H_2 X = H^1 X$ by **Poincare duality**. What group is this?
    - $0 \rightarrow \operatorname{Ext}_{\ZZ}^{1}\left(H_{0}(X ; \mathbb{Z}), \mathbb{Z}\right) \rightarrow H^{1}(X ; \mathbb{Z}) \rightarrow \operatorname{Hom}_{\mathbb{Z}}\left(H_{1}(X ; \mathbb{Z}), \mathbb{Z}\right) \rightarrow 0$ yields
    - $0 \rightarrow \operatorname{Ext}_{\ZZ}^{1}\left(\ZZ, \mathbb{Z}\right) \rightarrow H^{1}(X ; \mathbb{Z}) \rightarrow \operatorname{Hom}_{\mathbb{Z}}\left(0, \mathbb{Z}\right) \rightarrow 0$
    - Then $\operatorname{Ext}_{\ZZ}^{1}\left(\ZZ, \mathbb{Z}\right) = 0$ because $\ZZ$ is a projective $\ZZ\dash$module, so $H^1 X = 0$.
- So $H_*(X) = [\ZZ, 0, 0, \ZZ, 0, \cdots]$
- So $h_3: \pi_3 X \to H_3 X$ is an isomorphism by **Hurewicz**. Pick some $f\in \pi_3 X \cong \ZZ$. By partial application, this induces an isomorphism $H_* S^3 \to H_* X$.
- Taking **CW approximations** for S^3, X, we find that f is a homotopy equivalence.

Other Topics

Theorem (Freudenthal Suspension): If X is an $n\dash$connected CW complex, then there is a map $$ \Sigma: \pi_i X \to \pi_{i+1}\Sigma X $$

which is an isomorphism for $i \leq 2n$ and a surjection for $i=2n+1$.

> Note: $[S^k, X] \mapsto [\Sigma S^k, \Sigma X] = [S^{k+1}, \Sigma X]$

> Application: $\pi_2 S^2 = \pi_3 S^3 = \cdots$ since 2 is already in the stable range.

A consequence: $\pi_1 X \to \pi_2 \Sigma X \to \pi_3 \Sigma^2 X \to \cdots$ is eventually constant, we say the homotopy groups *stabilize*. So define the *stable homotopy groups $$ \pi_i^S \definedas \lim_{k\to\infty} \pi_{i+k} X $$

$X = S^n$ yields *stable homotopy groups of spheres*, ties back to initial motivation.

Noting that $\Sigma S^n = S^{n+1}$, we could alternatively define $\mathbb{S} \definedas \lim_k \Sigma^k S^0$, then it turns out that $\pi_n \mathbb{S} = \pi_n^S$.

This object is a *spectrum*, which vaguely resembles a chain complex with a differential: $$ X_0 \mapsvia{\Sigma} X_2 \mapsvia{\Sigma} X_3 \mapsvia{\Sigma} \cdots $$

Spectra *represent* invariant theories (like cohomology) in a precise way. For example, $$ HG \definedas \left(K(G, 1) \mapsvia{\Sigma} K(G, 2) \mapsvia{\Sigma} \cdots \right) $$

then $H^n(X; G) \cong [X, K(G, 1)]$, and we can similarly extract $H^*(X; G)$ from (roughly) $\pi_* HG \definedas [\mathbb{S}, HG \bigwedge X]$.

> Note: this glosses over some important details! Also, smash product basically just looks like the tensor product in the category of spectra.

A modern direction is cooking up spectra that represent *extraordinary* cohomology theories. There are Eilenberg–Steenrod axioms that uniquely characterize homology on spaces; if we drop $H^* \pt = 0$, we get generalized alternatives.

Other Topics

- The homotopy hypothesis
- Generalized Cohomology theories
- Stable Homotopy Theory
- Infinity Categories
- Higher Homotopy Groups of Spheres

- Eilenberg Mclane and Moore Spaces

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	${\mathbb{Z}_2}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

Image

- Below jagged line: Zero by cellular approximation, or stable by Freudenthal suspension.
- Above line: Unstable range. Need to throw everything in the book at these guys to compute!