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Recall that an affine variety is given by $X = V(I) \subset \mathbb{A}^n/k$, and we have sheaves of rings of regular functions $\mathcal{O}_X$ on X. A prevariety is a ringed space that is covered by finitely many affine spaces. A morphism of prevarieties $f : X \to Y$ is a continuous map such that the pullbacks of regular functions are regular, i.e. for all $\varphi \in \mathcal{O}_X(U)$ we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. We can form a category PreVar_k of prevarieties over k, where we have several important constructions

1. Gluing
2. Products: Given X, Y, there is a unique prevariety $X \times Y$ such that

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists! h"] \arrow[drr, bend left, "f_x"] \arrow[ddr, bend right, "f_y"'] & & \\
& X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

We had an analogue of being Hausdorff: the diagonal Δ_X is closed.

Example 1.0.1.
Glue $D(x) \subset \mathbb{A}^1$ to $D(y) \subset \mathbb{A}^1$ by the isomorphism

$$\begin{aligned} D(x) &\xrightarrow{\sim} D(y) \\ x &\mapsto y. \end{aligned}$$

This yields an affine line with two origins:

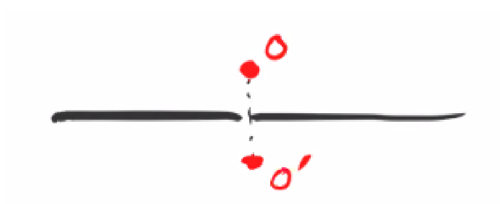

Figure 1: Line with two origins.

Consider the product:

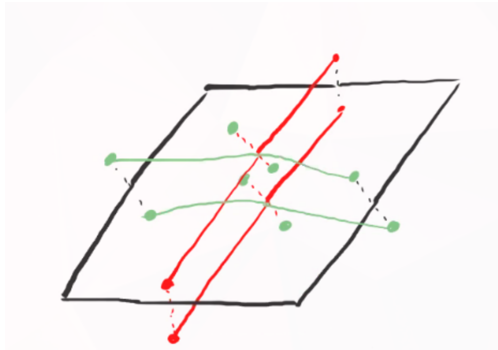

Figure 2: Product of lines with two origins

Since the diagonal is given by $\Delta_X = \left\{ (x,x) \;\middle|\; x \in X \right\}$, we have the following situation in blue:

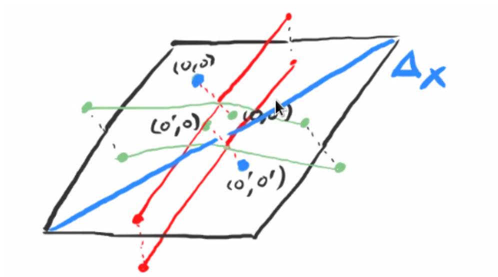

Figure 3: Image

We claim that Δ_X is not closed, and for example $(0, 0') \in \overline{\Delta}_X$. Consider $U \times U' \subset X \times X$ where U, U' are the two copies of $\mathbb{A}^1$ in X. This is an affine open set, since it's isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$.

If Δ_X were closed, then $S \coloneqq \Delta_X \cap (U \times U') = \left\{ (x,x) \;\middle|\; x \neq 0 \right\}$ would be closed in $U \times U'$.

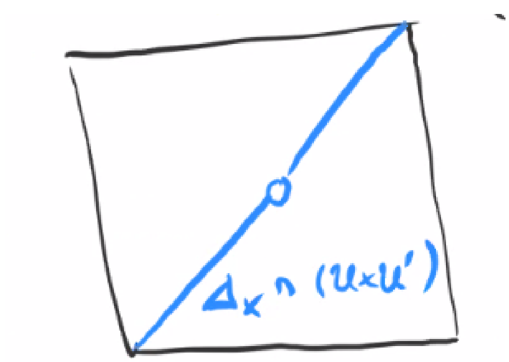

Figure 4: Open diagonal in a product.

This is because any polynomial vanishing on S must vanish at $(0, 0)$, so S is an affine variety. But then $V(I(S)) = \Delta_{\mathbb{A}^1}$.

Lemma 1.1 ***(?).***

a. Any affine variety is a variety.
b. Open and closed subprevarieties of a variety X are themselves varieties.

Thus it makes since to consider *open* and *closed subvarieties*.

Proof (of a).
We need to check that $\Delta_X \subset X^2$ is closed for any affine $X \subset \mathbb{A}^n$. Note that we can write.

$$\Delta_X = X^2 \cap V\left(\left\{x_j - y_j \;\middle|\; 1 \leq j \leq n\right\}\right) \subset \mathbb{A}^n \times \mathbb{A}^2$$

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Proof (of b).
Let $\iota : Y \to X$ be the inclusion of either an open or closed subset. Then we have a morphism $(\iota, \iota) : Y^2 \to X^2$ by the universal property. Then $\Delta_Y = (\iota, \iota)^{-1}(\Delta_X)$, so is closed by the continuity of (ι, ι) and the fact that Δ_X. Thus Y is a variety.

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1.1 Properties of Varieties

Proposition 1.1.1 ***(Properties of Varieties).***
Let $f, g : X \to Y$ be morphisms of prevarieties and assume Y is a variety.

a. The graph of f, given by $\Gamma_f := \left\{(x, f(x)) \;\middle|\; x \in X\right\}$, is closed in $X \times Y$.
b. The set $\left\{x \in X \;\middle|\; f(x) = g(x)\right\}$ is closed in X.

Proof (of a).
Consider the product morphism $(f, \mathrm{id}) : X \times Y \to Y^2$. Since Δ_Y is closed, $(f, \mathrm{id})^{-1}(\Delta_Y)$ is closed, and is the locus where $f(x) = y$, so this is Γ_f.

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Proof (of b).
Consider $(f, g) : X \to Y^2$. Since $\Delta_Y \subset Y^2$ is closed,

$$(f, g)^{-1}(\Delta_Y) = \left\{x \in X \;\middle|\; f(x) = g(x)\right\} \subset X$$

is closed.

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1.2 Chapter 6: Projective Varieties

Note that affine varieties of positive dimension over $\mathbb{C}$ are not compact in the classical topology, but *are* compact in the Zariski topology. Similarly, they are Hausdorff classically, but not in the Zariski topology. We want to find notions equivalent to Hausdorffness and compactness in the classical setting, which end up also applying to varieties: i.e. if P holds The fix in the latter case was considering “separatedness”. The fix for compactness will be the following:

Definition 1.2.1 (Complete)**.**
A variety X is **complete** iff for any variety Y the projection map $\pi_Y : X \times Y \to Y$ is a closed map, i.e. $\pi_Y(U)$ is closed whenever U is closed.

Example 1.2.1.
$\mathbb{A}^1$ is not complete. Let $Y = \mathbb{A}^1$ and $Z = V(xy - 1) \subset X \times Y$. Then $\pi_Y(Z) = D(y) \subset Y \subset \mathbb{A}^1$ is not closed.