

Algebra

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1 Major Theorems

Theorem 1 (Cauchy). For any prime p dividing the order of G , there is an element x of order p (and thus a subgroup $H = \langle x \rangle$).

Theorem 2 (Lagrange). If $H \leq G$ is a subgroup, then $|H| \mid |G|$.

Theorem 3 (Sylow 1). If $|G| = n = \prod p_i^{a_i}$ as a prime factorization, then G has subgroups of order $p_i^{a_i}$ for every i . Moreover, this holds for any $1 \leq r \leq a_i$.

Theorem 4 (Classification of finitely generated abelian groups). If G is a finitely generated abelian group, then $G \cong F \oplus T$, where F is free abelian and T is a torsion group. If $|T| = n$, then $T \cong \bigoplus \mathbb{Z}_{p_i^{\alpha_i}}$ where $n = \prod p_i^{\alpha_i}$ is some factorization of n with the p_i **not necessarily distinct**.

Theorem 5. Conjugacy classes partition G

$$|G| = |Z(G)| + \sum_{\text{One representative in each orbit}} |C_G(g_i)| = \sum_{\text{asdsa}} [G : C(g_i)].$$

Some nice lemmas:

- Every subgroup of a cyclic group is itself cyclic.

2 Lecture 1 (Thu 15 Aug 2019)

We'll be using Hungerford's Algebra text. Show that a finite abelian group that is not cyclic contains a subgroup which is isomorphic

2.1 Definitions

The following definitions will be useful to know by heart:

- The order of a group
- Cartesian product
- Relations
- Equivalence relation
- Partition
- Binary operation
- Group
- Isomorphism
- Abelian group
- Cyclic group
- Subgroup
- Greatest common divisor
- Least common multiple
- Permutation
- Transposition
- Orbit
- Cycle
- The symmetric group S^n
- The alternating group A_n
- Even and odd permutations
- Cosets
- Index
- The direct product of groups
- Homomorphism

- Image of a function
- Inverse image of a function
- Kernel
- Normal subgroup
- Factor group
- Simple group

Here is a rough outline of the course:

- Group Theory
 - Groups acting on sets
 - Sylow theorems and applications
 - Classification
 - Free and free abelian groups
 - Solvable and simple groups
 - Normal series
- Galois Theory
 - Field extensions
 - Splitting fields
 - Separability
 - Finite fields
 - Cyclotomic extensions
 - Galois groups
 - Solvability by radicals
- Module theory
 - Free modules
 - Homomorphisms
 - Projective and injective modules
 - Finitely generated modules over a PID
- Linear Algebra
 - Matrices and linear transformations
 - Rank and determinants
 - Canonical forms
 - Characteristic polynomials
 - Eigenvalues and eigenvectors

2.2 Preliminaries

Definition 1. A **group** is an ordered pair $(G, \cdot : G \times G \rightarrow G)$ where G is a set and \cdot is a binary operation, which satisfies the following axioms:

- Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$,
- Identity: $\exists e \in G \ni ge = eg = g$,
- Inverses: $g \in G \implies \exists h \in G \ni gh = gh = e$.

Example 1.

- $(\mathbb{Z}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Q}^\times, \times)$

- $(\mathbb{R}^\times, \times)$
- $(\text{GL}(n, \mathbb{R}), \times) = \{A \in \text{Mat}_n \ni \det(A) \neq 0\}$
- (S_n, \circ)

Definition 2. A subset $S \subseteq G$ is a **subgroup** of G iff

1. $s_1, s_2 \in S \implies s_1 s_2 \in S$
2. $e \in S$
3. $s \in S \implies s^{-1} \in S$

We denote such a subgroup $S \leq G$.

Examples of subgroups:

- $(\mathbb{Z}, +) \leq (\mathbb{Q}, +)$
- $\text{SL}(n, \mathbb{R}) \leq \text{GL}(n, \mathbb{R})$, where $\text{SL}(n, \mathbb{R}) = \{A \in \text{GL}(n, \mathbb{R}) \ni \det(A) = 1\}$

2.3 Cyclic Groups

Definition 3. A group G is **cyclic** iff G is generated by a single element.

Exercise 1. Show $\langle g \rangle = \{g^n \ni n \in \mathbb{Z}\} \cong \bigcap \{H \leq G \ni g \in H\}$.

Theorem 6. Let G be a cyclic group, so $G \cong \langle g \rangle$.

- If $|G| = \infty$, then $G \cong \mathbb{Z}$.
- If $|G| = n < \infty$, then $G \cong \mathbb{Z}_n$.

Definition 4. Let $H \leq G$, and define a **right coset** of G by $aH = \{ah \ni H \in H\}$. A similar definition can be made for **left cosets**.

Then $aH = bH \iff b^{-1}a \in H$ and $Ha = Hb \iff ab^{-1} \in H$.

Some facts:

- Cosets partition H , i.e. $b \notin H \implies aH \cap bH = \{e\}$.
- $|H| = |aH| = |Ha|$ for all $a \in G$.

Theorem 7 (Lagrange). If G is a finite group and $H \leq G$, then $|H| \mid |G|$.

Definition 5. A subgroup $N \leq G$ is **normal** iff $gN = Ng$ for all $g \in G$, or equivalently $gNg^{-1} \subseteq N$. I denote this $N \trianglelefteq G$.

When $N \trianglelefteq G$, the set of left/right cosets of N themselves have a group structure. So we define

$$G/N = \{gN \ni g \in G\} \text{ where } (g_1N)(g_2N) = (g_1g_2)N.$$

Given $H, K \leq G$, define $HK = \{hk \ni h \in H, k \in K\}$. We have a general formula,

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

2.4 Homomorphisms

Definition 6. Let G, G' be groups, then $\varphi : G \rightarrow G'$ is a **homomorphism** if $\varphi(ab) = \varphi(a)\varphi(b)$.

Example 2. • $\exp : (\mathbb{R}, +) \rightarrow (\mathbb{R}^{>0}, \cdot)$ where $\exp(a+b) = e^{a+b} = e^a e^b = \exp(a)\exp(b)$.

- $\det : (\text{GL}(n, \mathbb{R}), \times) \rightarrow (\mathbb{R}^\times, \times)$ where $\det(AB) = \det(A)\det(B)$.
- Let $N \trianglelefteq G$ and $\varphi G \rightarrow G/N$ given by $\varphi(g) = gN$.
- Let $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ where $\phi(g) = [g] = g \pmod n$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$

Definition 7. Let $\varphi : G \rightarrow G'$. Then φ is a **monomorphism** iff it is injective, an **epimorphism** iff it is surjective, and an **isomorphism** iff it is bijective.

2.5 Direct Products

Let G_1, G_2 be groups, then define

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\} \text{ where } (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2).$$

We have the formula $|G_1 \times G_2| = |G_1||G_2|$.

2.6 Finitely Generated Abelian Groups

Definition 8. We say a group is **abelian** if G is commutative, i.e. $g_1, g_2 \in G \implies g_1 g_2 = g_2 g_1$.

Definition 9. A group is **finitely generated** if there exist $\{g_1, g_2, \dots, g_n\} \subseteq G$ such that $G = \langle g_1, g_2, \dots, g_n \rangle$.

This generalizes the notion of a cyclic group, where we can simply intersect all of the subgroups that contain the g_i to define it.

We know what cyclic groups look like – they are all isomorphic to \mathbb{Z} or \mathbb{Z}_n . So now we'd like a structure theorem for abelian finitely generated groups.

Theorem 8. Let G be a finitely generated abelian group. Then

$$G \cong \mathbb{Z}^r \times \prod_{i=1}^s \mathbb{Z}_{p_i^{\alpha_i}}$$

for some finite $r, s \in \mathbb{N}$ and p_i are (not necessarily distinct) primes.

Example 3. Let G be a finite abelian group of order 4. Then $G \cong \mathbb{Z}_4$ or \mathbb{Z}_2^2 , which are not isomorphic because every element in \mathbb{Z}_2^2 has order 2 where \mathbb{Z}_4 contains an element of order 4.

2.7 Fundamental Homomorphism Theorem

Let $\varphi : G \rightarrow G'$ be a group homomorphism and define $\ker \varphi = \{g \in G \mid \varphi(g) = e'\}$.

2.7.1 The First Homomorphism Theorem

Theorem 9. There exists a map $\varphi' : G/\ker \varphi \rightarrow G'$ such that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{\quad \varphi \quad} & G' \\
 \eta \downarrow & \nearrow \varphi' & \\
 G/\ker \varphi & &
 \end{array}$$

That is, $\varphi = \varphi' \circ \eta$, and φ' is an isomorphism onto its image, so $G/\ker \varphi = \text{im } \varphi$. This map is given by $\varphi'(g(\ker \varphi)) = \varphi(g)$.

Exercise 2. Check that φ is well-defined.

2.7.2 The Second Theorem

Theorem 10. Let $K, N \leq G$ where $N \trianglelefteq G$. Then

$$\frac{K}{N \cap K} \cong \frac{NK}{N}$$

Proof. Define a map $K \xrightarrow{\varphi} NK/N$ by $\varphi(k) = kN$. You can show that φ is onto by looking at $\ker \varphi$; note that $kN = \varphi(k) = N \iff k \in N$, and so $\ker \varphi = N \cap K$. \square

3 Lecture 2

Last time: the fundamental homomorphism theorems.

Theorem 1: Let $\varphi : G \rightarrow G'$ be a homomorphism. Then there is a canonical homomorphism $\eta : G \rightarrow G/\ker \varphi$ such that the usual diagram commutes. Moreover, this map induces an isomorphism $G/\ker \varphi \cong \text{im } \varphi$.

Theorem 2: Let $K, N \leq G$ and suppose $N \trianglelefteq G$. Then there is an isomorphism

$$\frac{K}{K \cap N} \cong \frac{NK}{N}$$

(Show that $K \cap N \trianglelefteq K$, and NK is a subgroup exactly because N is normal).

Theorem 3: Let $H, K \trianglelefteq G$ such that $H \leq K$.

1. H/K is normal in G/K .
2. The quotient $(G/K)/(H/K) \cong G/H$.

Proof: We'll use the first theorem. First make a map

$$\begin{aligned}
 G/K &\rightarrow G/H \\
 \phi(gk) &= gH
 \end{aligned}$$

Exercise: Show that this map is onto, and that $\ker \phi \cong H/K$.

3.1 Permutation Groups

Let A be a set, then a *permutation* on A is a bijective map $A \rightarrow A$. This can be made into a group with a binary operation given by composition of functions. Denote S_A the set of permutations on A .

Theorem: S_A is in fact a group. Check associativity, inverses, identity, etc.

In the special case that $A = \{1, 2, \dots, n\}$, then $S_n := S_A$.

Recall two line notation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

Moreover, $|S_n| = n!$ by a combinatorial counting argument.

Example: S_3 is the symmetries of a triangle (see notes).

Example: The symmetries of a square are *not* given by S_4 , it is instead D_4 (see notes).

3.2 Orbits

Permutations S_A “acts” on A , and if $\sigma \in S_A$, then $\langle \sigma \rangle$ also acts on A .

Define $a \sim b$ iff there is some n such that $\sigma^n(a) = b$. This is an equivalence relation, and thus induces a partition of A . See notes for diagram. The equivalence classes under this relation are called the *orbits* under σ .

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (18)(2)(364)(57).$$

Definition: A permutation $\sigma \in S_n$ is a *cycle* iff it contains at most one orbit with more than one element. The *length* of a cycle is the number of elements in the largest orbit.

Recall cycle notation: $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$. Note that this is read right-to-left by convention!

Theorem: Every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Definition: A *transposition* is a cycle of length 2. Moreover, we have

and so every permutation is a product of transpositions. This is not a unique decomposition, however, as e.g. $\text{id} = (12)^2 = (34)^2$.

Theorem: Any $\sigma \in S_n$ can be written as **either** an even number of transpositions or an odd number of transpositions.

Define $A_n = \{\sigma \in S_n \mid \sigma \text{ is even}\}$. We claim that $A_n \trianglelefteq S_n$.

1. Closure: If τ_1, τ_2 are both even, then $\tau_1\tau_2$ also has an even number of transpositions.
2. The identity has an even number of transpositions, since zero is even.
3. Inverses: If $\sigma = \prod_{i=1}^s \tau_i$ where s is even, then $\sigma^{-1} = \prod_{i=1}^s \tau_{s-i}$. But each τ is order 2, so $\tau^{-1} = \tau$, so there are still an even number of transpositions.

So A_n is a subgroup. It is normal because it is index 2, or the kernel of a homomorphism, or by a direct computation.

3.3 Groups Acting on Sets

Think of this as a generalization of a G -module.

Definition: A group G is said to *act* on a set X if there exists a map $G \times X \rightarrow X$ such that

1. $e \curvearrowright x = x$

Examples:

1. $G = S_A \curvearrowright A$
2. $H \leq G$, then $G \curvearrowright X = G/H$ where $g \curvearrowright xH = (gx)H$.
3. $G \curvearrowright G$ by conjugation, i.e. $g \curvearrowright x = gxg^{-1}$.

Definition: Let $x \in X$, then define the *stabilizer subgroup*

$$G_x = \{g \in G \mid g \curvearrowright x = x\} \leq G$$

We can also look at the dual thing,

$$X_g = \{x \in X \mid g \curvearrowright x = x\}.$$

We then define the *orbit* of an element x as

$$Gx = \{g \curvearrowright x \mid g \in G\}$$

and we have a similar result where $x \sim y \iff x \in Gy$, and the orbits partition X .

Theorem: Let G act on X . We want to know the number of elements in an orbit, and it turns out that

Proof: Construct a map $Gx \xrightarrow{\psi} G/Gx$ where $\psi(g \curvearrowright x) = gGx$. Exercise: Show that this is well-defined, so if 2 elements are equal then they go to the same coset. Exercise: Show that this is surjective.

Injectivity: $\psi(g_1x) = \psi(g_2x)$, so $g_1Gx = g_2Gx$ and $(g_2^{-1}g_1)Gx = Gx$ so $g_2^{-1}g_1 \in Gx \iff g_2^{-1}g_1 \curvearrowright x = x \iff g_1x = g_2x$.

Next time: Burnside's theorem, proving the Sylow theorems.

4 Lecture 3 (Aug 22)

Last time: let G be a group and X be a set; we say G *acts* on X (or that X is a G -set) when there is a map $G \times X \rightarrow X$ such that $ex = x$ and $(gh) \curvearrowright x = g \curvearrowright (h \curvearrowright x)$. We then define the *stabilizer of x* as

$$G_x = \{g \in G \mid g \curvearrowright x = x\} \leq G,$$

and the *orbit*

$$G.x = \mathcal{O}_x = \{g \curvearrowright x \mid x \in X\} \subseteq X.$$

When G is finite, we have

$$\#G.x = \frac{\#G}{\#G_x}.$$

We can also consider the fixed points of X ,

$$X_g = \{x \in X \mid g \curvearrowright x = x \forall g \in G\} \subseteq X$$

4.1 Burnside's Theorem

Theorem (Burnside): Let X be a G -set and v be the number of orbits. Then

$$v\#G = \sum_{g \in G} \#X_g.$$

Proof:

Define $N = \{(g, x) \mid g \curvearrowright x = x\} \subseteq G \times X$, we then have

$$\begin{aligned} |N| &= \sum_{g \in G} |X_g| \\ &= \sum_{x \in X} |G_x| \\ &= \sum_{x \in X} \frac{|G|}{|G.x|} \\ &= |G| \left(\sum_{x \in X} \frac{1}{|G.x|} \right) \\ &= |G|v. \end{aligned}$$

Since the orbits partition X , say into $X = \bigcup_{i=1}^v \sigma_i$, let $\sigma = \{\sigma_i \mid 1 \leq i \leq v\}$ and abuse notation slightly by replacing each orbit in σ with a representative element $x_i \in \sigma_i \subset X$. We then have

$$\sum_{x \in \sigma} \frac{1}{|G.x|} = \frac{1}{|Gx|} |\sigma| = 1.$$

Application: Consider seating 10 people around a circular table. How many distinct seating arrangements are there?

Let X be the set of configurations, $G = S_{10}$, and let $G \curvearrowright X$ by permuting configurations. Then v , the number of orbits under this action, yields the number of distinct seating arrangements. By Burnside, we have

$$v = \frac{1}{|G|} \sum_{g \in G} |X_g| = \frac{1}{10} (10!) = 9!,$$

since $X_g = \{x \in X \mid gx = x\} = \emptyset$ unless $g = e$, and $X_e = X$.

4.2 Sylow Theory

Recall Lagrange's theorem: If $H \leq G$ and G is finite, then $\#H \mid \#G$.

Consider the converse: if $n \mid \#G$, does there exist a subgroup of size n ? The answer is no in general, and a counterexample is A_4 which has $4!/2 = 12$ elements but no subgroup of order 6.

4.2.1 Class Functions

Let X be a G -set, and choose orbit representatives $x_1 \cdots x_v$. Then

$$|X| = \sum_{i=1}^v |Gx_i|.$$

We can then separately count all orbits with exactly one element, which is exactly $X_G = \{x \in G \mid g \curvearrowright x = x \ \forall g\}$

We then have

$$|X| = |X_G| + \sum_{i=j}^v$$

for some j where $|Gx_i| > 1$ for all $i \geq j$.

Theorem: Let G be a group of order p^n for p a prime, then

$$|X| \equiv |X_G| \pmod{p}$$

Proof: We know that $|Gx_i| = [G : G_{x_i}]$ for $j \leq i \leq v$, and $|Gx_i| > 1$ implies that $G_{x_i} \neq G$ and thus $p \mid [G : G_{x_i}]$. The result follows.

Application: If $|G| = p^n$, then the center $Z(G)$ is nontrivial. Let $X = G$ act on itself by conjugation, so $g \curvearrowright x = gxg^{-1}$. Then

$$X_G = \{x \in G \mid gxg^{-1} = x\} = \{x \in G \mid gx = xg\} = Z(G)$$

But then, by the previous theorem, we have $|Z(G)| \equiv |X| \equiv |G| \pmod{p}$, but since $Z(G) \leq G$ we have $|Z(G)| \leq |G|$, and so in particular, $|Z(G)| \neq 0 \pmod{p}$, and so in particular, $Z(G) \neq \{e\}$.

Definition: A group G is a p -group iff every element in G has order p^k for some k . A subgroup is a p -group exactly when it is a p -group in its own right.

4.2.2 Cauchy's Theorem

Theorem (Cauchy): Let G be a finite group, where $p \mid |G|$ is a prime. Then G has an element (and thus a subgroup) of order p .

Proof: Consider $X = \{(g_1, g_2, \dots, g_p) \in G^{\oplus p} \mid g_1 g_2 \cdots g_p = e\}$. Given any $p-1$ elements, say $g_1 \cdots g_{p-1}$, the remaining element is completely determined by $g_p = (g_1 \cdots g_{p-1})^{-1}$. So $|X| = |G|^{p-1}$.

Since $p \mid |G|$, we have $p \mid |X|$. Now let $\sigma \in S_p$ the symmetric group act on X by index permutation, i.e. $\sigma \curvearrowright (g_1, g_2, \dots, g_p) = (g_{\sigma(1)}, g_{\sigma(2)}, \dots, g_{\sigma(p)})$.

Exercise: Check that this gives a well-defined group action.

Let $\sigma = (1 \ 2 \ \dots \ p) \in S_p$, and note $\langle \sigma \rangle \leq S_p$ also acts on X where $|\langle \sigma \rangle| = p$. Therefore we have

$$|X| = |X_{\langle \sigma \rangle}| \pmod{p}.$$

Since $p \mid |X|$, it follows that $|X_{\langle \sigma \rangle}| = 0 \pmod{p}$, and thus $p \mid |X_{\langle \sigma \rangle}|$.

If $\langle \sigma \rangle$ fixes (g_1, g_2, \dots, g_p) , then $g_1 = g_2 = \dots = g_p$.

Note that $(e, e, \dots) \in X_{\langle \sigma \rangle}$, as is (a, a, \dots, a) since $p \mid |X_{\langle \sigma \rangle}|$. So there is some $a \in G$ such that $a^p = 1$. Moreover, $\langle a \rangle \leq G$ is a subgroup of size p .

4.2.3 Normalizers

Let G be a group and $X = S$ be the set of subgroups of G . Let G act on X by $g \curvearrowright H = gHg^{-1}$. What is the stabilizer? $G_x = G_H = \{g \in G \mid gHg^{-1} = H\}$, making G_H the largest subgroup such that $H \trianglelefteq G_H$. So we define $N_G(H) = G_H$.

Lemma: Let H be a p -subgroup of G of order p^n . Then $[N_G(H) : H] = [G : H] \pmod{p}$.

Proof: That $S = G/H$ be the set of left H -cosets in G . Now let H act on S by $H \curvearrowright x + H = (hx) + H$.

By a previous theorem, $|G/H| = |S| = |S_H| \pmod{p}$, where $|G/H| = [G : H]$. What is S_H ? Thus is given by $S_H = \{x + H \in S \mid xHx^{-1} \in H \forall h \in H\}$. Therefore $x \in N_G(H)$.

Corollary: Let $H \leq G$ be a subgroup of order p^n . If $p \nmid [G : H]$ then $N_G(H) = H$. Proof: Exercise.