Title

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Contents

1 | Monday, September 21

Let G be a reductive algebraic group scheme, $k = \overline{\mathbb{F}}_p$ with $p > 0$, equipped with the Frobenius map $F : G \to G$ with F^r its r-fold composition. We defined *Frobenius kernels* $G_r \coloneqq \ker F^r$, which are in correspondence with the cocommutative Hopf algebras $\mathrm{Dist}(G_r)$.

Goal: We want to classify simple G_r-modules, and to do this we'll use socles.

We have a maximal torus $T \subseteq G$ and thus $T_r \subseteq G_r$ after acting by Frobenius. This yields a SES

$$0 \to p_r X(T) \to X(T) \to X(T)/p^r X(T) = X(T_r) \to 0.$$

How to think about this: take $\lambda \in X(T_r)$, then we can write $\lambda = \lambda + p^r \sigma$ in $X(T_r)$ for some other weight $\sigma \in X(T)$. We'll define the "baby Verma modules"

$$\begin{aligned} Z_r(\lambda) &\coloneqq \mathrm{Coind}_{B_r^+}^{G_r} \lambda \\ Z_r'(\lambda) &\coloneqq \mathrm{Ind}_{B_r^+}^{G_r} \lambda, \end{aligned}$$

and we have $\dim Z_r(\lambda) = \dim Z_r'(\lambda) = p^{r\left|\Phi^+\right|}$.

Proposition 1.1 ***(?).***
Let $\lambda \in X(T)$ be a weight.

1. $Z_r(\lambda) \downarrow_{B_r}$ is the *projective cover* of λ and the *injective hull* of $\lambda - 2(p^r - 1)\rho$.
2. $Z_r'(\lambda) \downarrow_{B_r^+}$ is the *injective hull* of λ and the *projective cover* of $\lambda - 2(p^r - 1)\rho$.

Note the latter are T_r-modules, so we let U^+ act trivially.

Proof (of 1).
What we need to do:

1. Show $Z_r(\lambda) \downarrow_{B_r}$ is projective.
2. Show $Z_r(\lambda)$ is the smallest projective module such that $Z_r(\lambda) \twoheadrightarrow \lambda$.

For (1), we can write

$$\mathrm{Dist}(G_r) = \mathrm{Dist}(U_r^+)\mathrm{Dist}(B_r) = \mathrm{Dist}(B_r^+)\mathrm{Dist}(U_r), ,$$

and so

$$\begin{aligned} Z_r(\lambda) &= \mathrm{Coind}_{B_r^+}^{G_r} \lambda \\ &= \Big(\mathrm{dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \lambda\Big) \downarrow_{B_r^+} \\ &= \mathrm{Dist}(U_r^+) \otimes \lambda \\ &= \mathrm{Dist}(B_r^+) \otimes_{\mathrm{Dist}(T_r)} \lambda \\ &= \mathrm{Coind}_{T_r}^{B_r^+} \lambda. \end{aligned}$$

Why is this projective? Look at cohomology, suffices to show that higher Exts vanish. So consider

$$\begin{aligned} \mathrm{Ext}^n_{B_r^+}(\mathrm{Coind}_{T_r}^{B_r^+}, M) &= \mathrm{Ext}^n_{T_r}(\lambda, M) \qquad \text{by Frobenius reciprocity} \\ &= 0 \qquad \text{for } n \geq 0, \end{aligned}$$

since representations for T_r are completely reducible, and we've used the fact that $\mathrm{Coind}_{T_r}^{B_r^+}(\,\cdot\,)$ is exact.

Note: general algebra fact that higher exts vanish for projective modules.

For (2), we can write

$$\begin{aligned} \hom_{B_r^+}(Z_r(\lambda), \mu) &= \hom_{B_r^+}(\mathrm{Coind}_{T_r}^{B_r^+} \lambda, \mu) \\ &= \hom_{T_r}(\lambda, \mu) \qquad \text{by Frobenius reciprocity} \\ &= \begin{cases} k \& \lambda = \mu \\ 0 \& \text{else.} \end{cases} \end{aligned}$$

Thus $Z_r(\lambda)/\mathrm{rad}\, Z_r(\lambda) \downarrow B_r^+ = \lambda$.
If we now write $A = \mathrm{Dist}(B_r^+)$ and $\mathfrak{g} = \mathfrak{n}^+ \oplus t \oplus \mathfrak{n}$ with $\mathfrak{b}^+ := \mathfrak{n}^+ \oplus t$,

$$\begin{aligned} &\sum_S (\dim P(S))(\dim(S)) \\ &= \sum_{\lambda \in X(T_r)} (\dim Z_r(\lambda))(\dim \lambda) \\ &= \sum_{\lambda \in X(T_r)} p^{r\left|\Phi^+\right|} \cdot 1 \\ &= |X(T_r)| p^{r\left|\Phi^+\right|} \\ &= p^{rn} p^{r\left|\Phi^+\right|} \qquad n = \dim t \\ &= p^{r \dim \mathfrak{b}^+} \\ &= \dim A \end{aligned}$$

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1.1 Simple G-modules

We know that after taking fixed points, $Z_r(\lambda)^{U_r}$ and $Z'_r(\lambda)^{U_r^+}$ are one-dimensional, and thus

$$Z_r(\lambda)/\text{rad } Z_r(\lambda) \cong L_r(\lambda) \qquad \text{Soc}_{G_r} Z'_r(\lambda) = L_r(\lambda)$$

following the same argument considering $H_0(\lambda)$.

For any $\lambda \in X(T_r)$ we have $0 \neq L_r = \text{Soc}_{G_r} Z'_r(\lambda)$. By the one-dimensionality above, we know

$$L_r(\mu) = L_r(\lambda) \iff \lambda = \mu \in X(T_r).$$

Letting N be a simple G_r-module, we can consider it as a B_r-module, and the simple B_r-modules are one dimensional and obtained from simple T_r-modules. We then know that for some $\lambda \in X(T_r)$,

$$\begin{aligned} 0 \neq \hom_{B_r}(N, \lambda) \\ &= \hom_{G_r}(N, \text{Ind}_{B_r}^{G_r} \lambda), \end{aligned}$$

which implies that $N \hookrightarrow \text{Ind}_{B_r}^{G_r} \lambda = Z'_r(\lambda)$ as a submodule, and thus $N = L_r(\lambda)$.

Theorem 1.2 ***(Main Theorem).***
Let Λ be a set of representatives of $XX(T)/p^r X(T) \cong X(T_r)$. Then there exists a one-to-one correspondence

$$\Lambda \iff \{L_r(\lambda)\lambda \in \Lambda\},$$

where the RHS are simple G_r-modules.

How to think about this: **restricted regions**. Choose dominant weights as representatives

$$\begin{aligned} X_r(T) &= \left\{ \lambda \in X(T)_+ \;\middle|\; 0 \leq \langle \lambda,\ \alpha^\vee \rangle < p^r \,\forall \alpha \in \Delta \right\} \\ &= \left\{ \lambda \in X(T)_+ \;\middle|\; \lambda = \sum_{i=1}^{\ell} n_i w_i,\, 0 \leq n_j \leq p^r - 1 \,\forall j \right\} \end{aligned}$$

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Pictures:

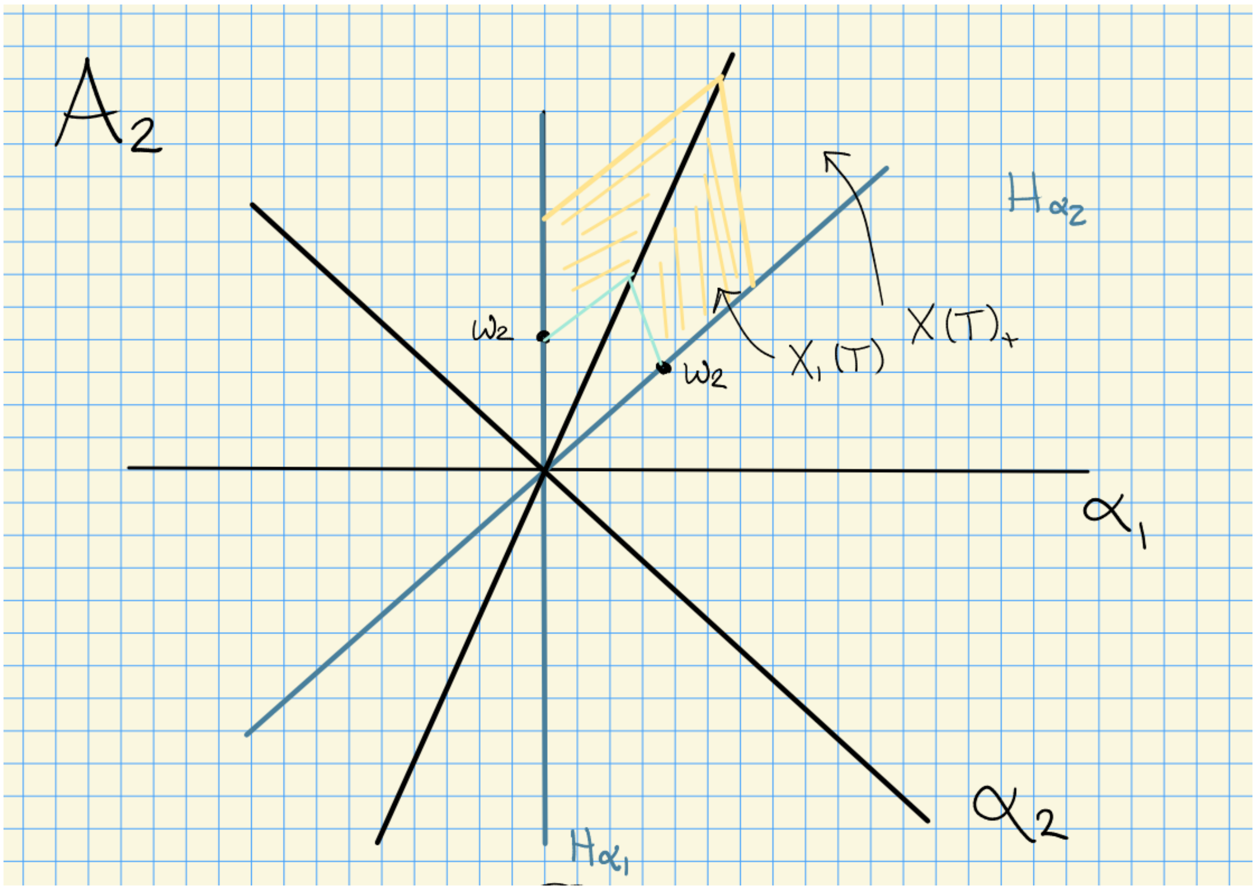

Figure 1: Root systems, chambers formed by dominant weights

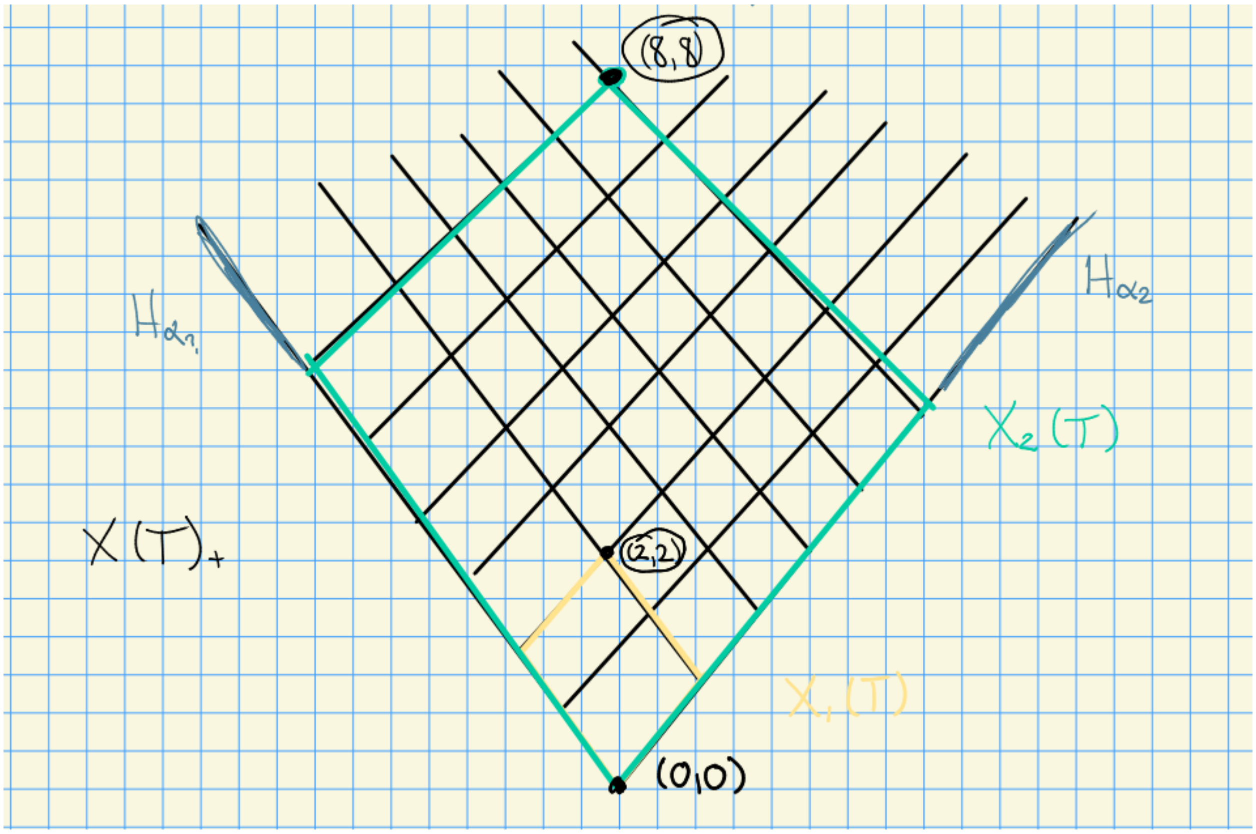

Figure 2: Restricted regions

Some facts:

If $\lambda \in X(T)_+$, then $L(\lambda)$ is a simple G-module.

Question 1: What happens when we restrict $L(\lambda) \downarrow_{G_r}$?

Answer: This remains irreducible over G_r iff $\lambda \in X_r(T)$, i.e. if $L(\lambda) \downarrow_G \cong L_r(\lambda)$ when $\lambda \in X_r(T)$.

Question 2: Given $L(\lambda)$ for $\lambda \in X(T)_+$, can we express $L(\lambda)$ in terms of simple G_r-modules?

Answer: Yes, can be formulated in terms of *Steinberg's twisted tensor product.*