Weil Conjectures

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1 Notes from Daniel's Office Hours

- 0. Definition of Zeta functions
- 1. Statement of the conjectures
- 2. Easy examples: \mathbb{P}^n_{\exists} , $\operatorname{Gr}_{\exists}(k,n) = \operatorname{GL}(n,\exists)/P$ the stabilizer of an \exists -point in \mathbb{C}^n , \mathbb{F}_{p^n} .
- 3. Medium example: E/\mathbb{k} an elliptic curve.
- 4. Work out a harder example as in Weil

References

• http://www-personal.umich.edu/~mmustata/zeta_book.pdf

1.1 Definition of Zeta Function

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the finite field with q elements, along with its unique degree n extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}}_p \mid x^{q^n} - x = 0 \right\} \quad \forall \ n \in \mathbb{Z}^{\geq 2}$$

Definition 1.0.1.

Let

$$J = \langle f_1, \cdots, f_N \rangle \le k[x_0, \cdots, x_n]$$

be an ideal, then a projective algebraic variety $X\hookrightarrow \mathbb{P}^\infty_{\mathbb{F}}$ can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^{\infty} \mid f_1(\mathbf{x}) = \dots = f_N(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by homogeneous polynomials in n+1 variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I} = (i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field \mathbb{F}_p . We will be thinking of K-valued points for K/\mathbb{F}_p algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into \mathbb{P}^{∞} ?

Examples:

• Dimension 1: Curves

• Dimension 2: Surfaces

• Codimension 1: Hypersurfaces

Example: Take $f_1(x) = x \in \mathbb{F}[x]$, consider $V(\langle f_1 \rangle) \subset \mathbb{P}^1_{\mathbb{F}_n}$. This is given by the single point $x = \mathbf{0}$.

Fix X/\mathbb{F} an N-dimensional projective algebraic variety. Note that it then has points in any finite extension L/K.

Definition 1.0.2.

Define the local zeta function (or Hasse-Weil zeta function) of X the following formal power series:

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^n\right) \in \mathbb{Q}[[z]] \text{ where } \alpha_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$\zeta_X(0) = 1$$

$$z\left(\frac{\partial}{\partial z}\right)\log\zeta_X(z) = t\left(\frac{\zeta_X'(z)}{\zeta_X(z)}\right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \cdots,$$

which is an ordinary generating function for the sequence (α_n) .

Thus if we define G(x) to be the OGF for (α_n) , we have $\zeta_X(t) = \exp$

Todo: why not an OGF.

Remark: Note that for an OGF $F(x) = \sum_{n=0}^{\infty} f_n x^n$, we can extract coefficients in the following way:

$$[x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!} \left(\frac{\partial}{\partial x}\right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

1.1.1 Simple but Useful Example: A Point

Take $X = \{x = 0\} / \mathbb{F}$ a single point over \mathbb{F} , then

$$\#X(\mathbb{F}) := \alpha_1 = 1$$

 $\#X(\mathbb{F}_2) := \alpha_2 = 1$
 \vdots
 $\#X(\mathbb{F}_n) := \alpha_n = 1$
 \vdots

Recall that by integrating a geometric series we can derive

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n}$$

$$\implies \log(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$$

$$\implies -\log(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}$$

$$= 1 \cdot t + 1 \cdot t^2 + 1 \cdot t^3 + \cdots$$

and so

$$\zeta_{\{\text{pt}\}}(t) = \exp\left(-\log\left(1 - t\right)\right) = \frac{1}{1 - t}.$$

1.1.2 Aside: Why call it a Zeta function?

Knowing the above calculation, we can now make a precise analogy.

Suppose

$$\mathbb{A}^n_{\mathbb{Z}} \supseteq X = V(\langle f_1, \cdots, f_d \rangle)$$
 where $f_i \in \mathbb{Z}[x_0, \cdots, x_{n-1}].$

Then for every prime, we can reduce the equations mod p and consider

$$\mathbb{A}^n_{\mathbb{F}_p} \supseteq X_p \coloneqq V(\langle f_1 \mod p, \cdots, f_d \mod p \rangle) \quad \text{where} \quad f_1 \mod p \in \mathbb{F}_p[x_0, \cdots, x_{n-1}]$$

Then define the "local at p" zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Note: the index set for the product may require some minor adjustment over \mathbb{Q} in general. There are also potentially modifications needed to extend to schemes.

Taking $X = \operatorname{Spec} \mathbb{Q}$ and $X_p = \operatorname{Spec} \mathbb{F}_p$ (which is a single point since \mathbb{F}_p is a field) and noting that

$$\zeta_{X_p}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^n\right)$$
$$= \exp\left(-\log\left(1 - z\right)\right)$$
$$= \frac{1}{1 - z},$$

we find that

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s})$$
$$= \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}}\right)$$
$$= \zeta(s),$$

the classical Riemann Zeta function.

Moreover, it is a theorem (difficult, not proved here!) that for any variety X/\mathbb{F}_p , we have

$$\zeta_X(t) = \prod_{x \in X_{cl}} \left(\frac{1}{1 - t^{\deg(x)}} \right) \quad \stackrel{t = p^{-s}}{\Longrightarrow} \quad \zeta_X(s) = \prod_{x \in X_{cl}} \left(\frac{1}{1 - p^{-s \deg(x)}} \right).$$

Note that this immediately implies that $\zeta_X(t) \in \mathbb{Z}[[t]]$ is a rational function.

Note for experts: $\zeta_X(z)$ an honest generating function for the 0-cycles on X ($F(X_{cl})$) where are effective (nonnegative coefficients).

1.1.3 More Examples

Example (Affine Line): $X = \mathbb{A}^1/\mathbb{F}$ the affine line over \mathbb{F} , then Note that we can write

$$\mathbb{A}^{1}(\mathbb{F}_{n}) = \left\{ \mathbf{x} = [x_{1}] \mid x_{1} \in \mathbb{F}_{n} \right\}$$

as the set of one-component vectors with entries in \mathbb{F}_n , so

$$X(\mathbb{F}) = q$$
$$X(\mathbb{F}_2) = q^2$$
$$\vdots$$
$$X(\mathbb{F}_n) = q^n.$$

Thus

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} q^n z^n\right).$$

Example (Affine Space): $X = \mathbb{A}^m/\mathbb{F}$, affine m-space over \mathbb{F} , then

$$\mathbb{A}^{1}(\mathbb{F}_{n}) = \left\{ \mathbf{x} = [x_{1}, \cdots, x_{m}] \mid x_{i} \in \mathbb{F}_{n} \right\}$$

$$X(\mathbb{F}) = q^{n}$$

$$X(\mathbb{F}_{2}) = (q^{2})^{n}$$

$$\vdots$$

$$X(\mathbb{F}_{n}) = (q^{m})^{n}.$$

Example (Projective Line): $X = \mathbb{P}^1/\mathbb{F}$ the projective line over \mathbb{F} , then

$$X(\mathbb{F}) = q + 1$$

$$X(\mathbb{F}_2) = q^2 + 1$$

$$\vdots$$

$$X(\mathbb{F}_n) = q^n + 1$$

where we write $\mathbb{P}^1_{\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}} \coprod \{\infty\}$ is the affine line with a point at infinity. We can also count by coordinates:

$$\mathbb{P}^{1}(\mathbb{F}^{n}) = \left\{ [x_{1}, x_{2}] \mid x_{1}, x_{2} \neq 0 \in \mathbb{F}^{n} \right\} / \sim = \left\{ [x_{1}, 1] \mid x_{1} \in \mathbb{F}^{n} \right\} \coprod \left\{ [1, 0] \right\}.$$

Example (Affine Space): Take $X = \mathbb{A}^n/\mathbb{F}$, then $\alpha_n = q^m + 1$ for a point at infinity, so

$$X(\mathbb{F}) = .$$

Thus

$$\zeta_X(t) = \frac{1}{(1 - q^{-t})(1 - q^{1-t})}$$

Example (Projective Space): Take $X = \mathbb{P}_{\mathbb{F}}^n$, then $\alpha_n = 1 + q^m + (q^m)^2 + \cdots + (q^m)^n$, so

$$\zeta_X(t) = \left(\frac{1}{1 - q^{-t}}\right) \left(\frac{1}{1 - q^{1-t}}\right) \left(\frac{1}{1 - q^{2-t}}\right) \cdots \left(\frac{1}{q^{n-t}}\right)$$

or equivalently, take your favorite curve $\gamma \in \mathbb{C}$ homotopic to \mathbb{S}^1 .

Note: this is extremely amenable to numerical approximation if you have a closed form for F or even just a black-box numerical version of F! I.e. easy to throw at a computer.

Todo: how to manually count points in \mathbb{P}^n !

Example: Take $X = Gr_{\mathbb{F}}(k, n)$, then ????? so

$$\zeta_X(t) = ?.$$

Questions about properties

- $\zeta_{X\coprod Y}(t) =_{?} \zeta_X(t)\zeta_Y(t)$? $\zeta_{X\times Y} =_{?}$

1.2 Statement of Weil Conjectures

1. (Rationality)

$$\zeta_X(t) = \frac{p_1(t)p_3(t)\cdots p_{2N-1}(t)}{p_0(t)p_2(t)\cdots p_N(t)} \in \mathbb{Z}(t), \quad \text{i.e.} \quad p_i(t) \in \mathbb{Z}[t]$$

$$P_0(t) = 1 - t$$

$$P_{2n}(t) = 1 - q^n t$$

$$P_i(t) = \prod_j (1 - a_{ij}t), \quad a_{ij} \in \mathbb{C}.$$

2. (Functional Equation and Poincare Duality)

$$\zeta_X(n-t) = \pm q^{\frac{1}{2}(nE)-Et}\zeta(x,t).$$

- 3. (Riemann Hypothesis)
- 4. (Betti Numbers)

1.3 Hard Example: An Elliptic Curve

Take $X = E/\mathbb{F}$, then $\alpha_n = q^n - (a^n + \bar{a}^n - 1)$ where $|a|_{\mathbb{C}} = |\bar{\alpha}|_{\mathbb{C}} = \sqrt{q}$. Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-s})}.$$