Title

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References: `https://www.daniellitt.com/tale-cohomology`

Prerequisites:

- Homological Algebra
  - Abelian Categories
  - Derived Functors
  - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
  - Etale morphism
  - Grothendieck topologies
  - The etale topology
  - Etale cohomology and the basis theorems
  - Etale cohomology of curves
  - Comparison theorems to singular cohomology
  - Focused on the case where coefficients are a constructible sheaf.
- Prove the Weil Conjectures (more than one proof)
  - Proving the Riemann Hypothesis for varieties over finite fields

  One of the greatest pieces of 20th century mathematics!
- Topics
  - Weil 2 (Strengthening of RH, used in practice)
  - Formality of algebraic varieties (topological features unique to varieties)
  - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose $X/\mathbb{C}$ is a quasiprojective variety: a finite type separated integral $\mathbb{C}$-scheme.

If you take the complex points, it naturally has the structure of a complex analytic space $X(\mathbb{C})^{\mathrm{an}}$: you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology $H^i(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Z})$, which satisfies several nice properties:

- Finitely generated $\mathbb{Z}$-modules
- Extra Hodge structure when tensored up to $\mathbb{C}$ (same as $\mathbb{C}$ coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general "nice" schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We'll associate X a "nice scheme" $\rightsquigarrow H^i(X_{\mathrm{et}}, \mathbb{Z}/\ell^n\mathbb{Z})$. Take the inverse limit over all n to obtain the ℓ-adic cohomology $H^i(X_{\mathrm{et}}, \mathbb{Z}_\ell)$. You can tensor with $\mathbb{Q}$ to get something with $\mathbb{Q}_\ell$ coefficients. And as in singular cohomology, you can a "twisted coefficient system".

What are nice schemes:

- $X = \mathrm{Spec}\ \mathcal{O}_k$, the ring of integers over a number field.
- X a variety over an algebraically closed field
  - Typical, most analogous to taking a variety over $\mathbb{C}$.
- X a variety over a non-algebraically closed field

Some comparisons between the last two cases:

- For $\mathbb{C}$- variety, H^i_{sing} will vanish above $i = 2d$.
- Over a finite field, H^i will vanish for $i > 2d + 1$ but generally not vanish for $i = 2d + 1$.

In good situations, these are finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, have Mayer-Vietoris and excision sequences, spectral sequences, etc.

Related invariants: for a scheme with a geometric point $(X, \bar{x}) \rightsquigarrow \pi_1^{\text{étale}}(X, \bar{x})$, which is a profinite topological group, which is a profinite topological group.

Note: a geometric point is a map from Spec X to an algebraically closed field.

More invariants beyond the scope of this course:

- Higher homotopy groups
- Homotopy type (equivalence class of spaces)

So we want homotopy-theoretic invariants for varieties.

Remark 1.
This cohomology theory is necessarily weird!

Theorem 1.1 ***(Serre).***
There does not exists a cohomology theory for schemes over $\bar{\mathbb{F}}_q$ with the following properties:

1. Functorial
2. Satisfies the Kunneth formula
3. For E an elliptic curve, $H^1(E) = \mathbb{Q}^2$.

Slogan: No cohomology theory with $\mathbb{Q}$ coefficients.

Proof .
Take E to be a supersingular elliptic curve. Then $\mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra.
Fact: There are no algebra morphisms $R \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$

Exercise .
Functoriality and Kunneth implies that $\mathrm{End}(E) \curvearrowright E$ yields an action on $H^1(E)$, which is precisely an algebra morphism $\mathrm{End}(E) \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$, a contradiction.

The content: the sum of two endomorphisms act via their sum on H^1.

Exercise .
Prove the same thing for $\mathbb{Q}_p$ coefficients, where p divides the characteristic of the ground field.

Proof the same, just need to know what quaternion algebras show up.

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