Title

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1 | Definitions

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n.$$

1.1 Set Theory

- Injectivity

$$\begin{aligned} f : X \to Y \text{ injective } &\iff \forall x_1, x_2 \in X, \quad f(x_1) = f(x_2) \implies x_1 = x_2 \\ &\iff \forall x_1, x_2 \in X, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2). \end{aligned}$$

- Surjectivity

$$f : X \to Y \text{ surjective } \iff \forall y \in Y,\ \exists x \in X : f(x) = y.$$

- Preimage

$$f : X \to Y, U \subseteq Y \implies f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

1.2 Calculus

- Limit

$$\lim_{x\to p} f(x) = L \iff \forall \varepsilon,\ \exists \delta : \\ d(x,p) < \delta \implies d(f(x), L) < \varepsilon$$

- Continuity
  - Epsilon-delta definition:

$$\begin{aligned} f : X \to Y \text{ continuous at } p &\iff \forall \varepsilon,\ \exists \delta : \\ d_X(x,p) < \delta &\implies d_Y(f(x), f(p)) < \varepsilon \end{aligned}$$

- Limit/Sequential definition:

$$f : X \to Y \text{ continuous at } p \iff \forall \{x_i\}_{i\in\mathbb{N}} \subseteq X : \{x_i\} \to p, \\ \lim_{i\to\infty} f(x_i) = f(\lim_{i\to\infty} x_i) = f(p)$$

- Topological Definition:

$$f : X \to Y \text{ continuous } \iff U \text{ open in } \operatorname{im}(f) \subseteq Y \implies f^{-1}(U) \text{ open in } X.$$

- Differentiability and the Derivative
  - For single variable functions:

$$f : \mathbb{R} \to \mathbb{R} \text{ differentiable at } p \iff \forall \{x_i\}_{i\in\mathbb{N}} \to p, \\ f'(p) \coloneqq \lim_{i\to\infty} \frac{f(x_i) - f(p)}{x_i - p} < \infty$$

  - For multivariable functions:

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m \text{ differentiable at } \mathbf{p} \iff \exists \text{ a linear map } \mathbf{J} : \mathbb{R}^n \to \mathbb{R}^m \text{ such that:} \\ \lim_{\mathbf{h}\to 0} \frac{\|\mathbf{f}(\mathbf{p}+\mathbf{h}) - \mathbf{f}(\mathbf{p}) - \mathbf{J}(\mathbf{h})\|_{\mathbb{R}^n}}{\|\mathbf{h}\|_{\mathbb{R}^m}} = 0$$

- Gradient

$$\nabla f = [f_x, f_y, f_z].$$

- Divergence
- Curl
- Taylor Series (at a point a)
  - Single Variable $\mathbb{R} \to \mathbb{R}$

$$T_a(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ \implies T_a(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  - Multivariable $\mathbb{R}^n \to \mathbb{R}$:

$$T_a(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} - \mathbf{a})^T \nabla f(\mathbf{a}).$$

  - Multivariable $\mathbb{R}^n \to \mathbb{R}^m$:

$$T_{(a,b)}(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\Big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{yy}(a,b) + (y-b)^2 f_{yx}(a,b)\Big) + \cdots.$$

$$T_a(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a})^{\mathrm{T}}\mathbf{J}(\mathbf{a}) + \frac{1}{2!}(\mathbf{x}-\mathbf{a})^{\mathrm{T}}\mathbf{H}(\mathbf{a})(\mathbf{x}-\mathbf{a}) + \cdots$$
$$\implies T_a(\mathbf{x}) = \sum_{|\alpha|\geq 0} \frac{(\mathbf{x}-\mathbf{a})^{\alpha}}{\alpha!}\left(\partial^{\alpha} f\right)(\mathbf{a})$$

1.3 Analysis

- Archimedean Property: $x \in \mathbb{R} \implies \exists n \in \mathbb{N}: \; x < n$ and $x > 0 \implies \exists n: \; \frac{1}{n} < x$
- Upper Bound (for S $\subseteq \mathbb{R}$)

$$\alpha \text{ is an upper bound for } S \iff s \in S \implies s < \alpha.$$

- Triangle Inequality
  - $|a+b| \leq |a| + |b|$
  - $|a-b| \leq |a| + |b|$
- Reverse Triangle Inequality
  - $||a| - |b|| \leq |a-b|$
- Least Upper Bound / Supremum (for S $\subseteq \mathbb{R}$)

$$\alpha \text{ is a LUB for } S \iff s \in S \implies s < \alpha \text{ and } \forall t : (s \in S \implies s < t), \; \alpha < t.$$

- Greatest Lower Bound / Infimum (for S $\subseteq \mathbb{R}$)

$$\alpha \text{ is a GLB for } S \iff s \in S \implies \alpha < s \text{ and } \forall t : (s \in S \implies t < s), \; t < \alpha.$$

- Open Set
- Closed Set
- Limit Point
- Interior Point
- Closure of a Set
- Boundary
- Metric
- Cauchy Sequence:

$$\{a_i\} \text{ is a cauchy sequence } \iff \forall \varepsilon \; \exists N \in \mathbb{N} : \quad m, n > N \implies d(x_m, x_n) < \varepsilon.$$

- Connected: S is connected $\iff \nexists U, V \subset S$ nonempty, open, disjoint such that $S = U \cup V$
- Compact: Every open cover has a finite subcover:

$$X \subseteq \cup_{j \in J} V_j \implies \exists I \subseteq J : |I| < \infty \text{ and } X \subseteq \cup_{i \in I} V_i.$$

- Sequential Compactness Every sequence has a convergent subsequence:

$$\{x_i\}_{i \in I} \subseteq X \implies \exists J \subseteq I,\ \exists p \in X : \quad \{x_j\}_{j \in J} \to p.$$

- Bounded (sequences, subsets, metric spaces)

$$U \subseteq X \text{ is bounded } \iff \exists x \in X, \exists M \in \mathbb{R} : \quad u \in U \implies d(x, u) < M.$$

- Totally Bounded

todo

- Pointwise Convergence

$$\text{For } \{f_n : X \to Y\}_{n \in \mathbb{N}},$$
$$f_n \to f \iff \forall \varepsilon > 0,\ \forall x \in X,\ \exists N(x, \varepsilon) \in \mathbb{N} : \quad n > N \implies d_Y\left(f_n(x), f(x)\right) < \varepsilon$$

- Uniform Convergence

$$\text{For } \{f_n : X \to Y\}_{n \in \mathbb{N}},$$
$$f_n \rightrightarrows f \iff \forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N} : \quad \forall x \in X,\ n > N \implies d_Y(f_n(x), f(x)) < \varepsilon$$

- Generalized Mean Value Theorem

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

1.4 Linear Algebra

Convention: always over a field k, and $T : k^n \to k^m$ is a generic linear map (or $m \times n$ matrix).

- Consistent

  A system of linear equations is *consistent* when it has at least one solution.

- Inconsistent

  A system of linear equations is *inconsistent* when it has no solutions.

- Rank

  The number of nonzero rows in RREF

- Elementary Matrix
- Row Equivalent
- Pivot

- Cofactor

$$\text{cofactor}(A)_{i,j} = (-1)^{i+j} M_{i,j}$$

where $M_{i,j}$ is the minor obtained by deleting the i-th row and j-th column of A.

- Adjugate

$$\text{adjugate}(A) = \text{cofactor}(A)^T = (-1)^{i+j} M_{j,i}.$$

- Vector Space Axioms
  - Let k be a field and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s, t \in k$. A vector space V over k satisfies:
    1. Closure under addition: $\mathbf{v} + \mathbf{w} \in V$
    2. Closure under scalar multiplication: $r\mathbf{v} \in V$
    3. Commutativity of addition: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
    4. Associativity of addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
    5. Existence of an additive zero $\mathbf{0}$ satisfying $\mathbf{v} + 0 = 0 + \mathbf{v} = \mathbf{v}$
    6. Existence of additive inverse $-\mathbf{v}$ satisfying $v + (-\mathbf{v}) = 0$
    7. Unit property: $1\mathbf{v} = \mathbf{v}$
    8. Associativity of scalar multiplication: $(rs)\mathbf{v} = r(s\mathbf{v})$
    9. Distribution of scalars multiplication over vector addition: $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$
    10. Distribution of scalar multiplication over scalar addition: $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$

- Subspace
  - A nonempty subset $W \subseteq V$ that is a vector space and satisfies

$$\left\{ \sum_i c_i \mathbf{x}_i \;\middle|\; c_i \in \mathbb{F},\ x_i \in W \right\} \subseteq W.$$

  - Quick counter-check: find $\mathbf{x}, \mathbf{y}$ such that $a\mathbf{x} + b\mathbf{y} \notin W$

- Span Given a set of n vectors $S = \{\mathbf{x}_i\}_{i=1}^n$, defined as

$$\text{Span}(S) = \left\{ \sum_{i=1}^n c_i \mathbf{x}_i \;\middle|\; c_i \in k \right\}.$$

- Row Space
  - The range of the linear map T.
  - Given $T = \begin{bmatrix} \mathbf{x}_1 \rightarrow \\ \mathbf{x}_2 \rightarrow \\ \vdots \\ \mathbf{x}_m \rightarrow \end{bmatrix}$, defined as

$$\text{Span}(\{\mathbf{x}_i\}_{i=1}^m) \subseteq k^m.$$

  - $\text{rowspace}(T)^\perp = \text{null}(T)$

  - $|\text{rowspace}(T)| = \text{Rank}(T)$
- Column Space
- Null Space
  - Defined as $\text{null}(T) = \left\{ \mathbf{x} \in k^n \;\middle|\; T(\mathbf{x}) = 0 \in k^m \right\}$
  - $\text{null}(T)^{\perp} = \text{rowspace}(T)$
- Eigenvalue
  - A value λ such that $Ax = \lambda x$
  - Invariant under similarity.
- Eigenspace
  - For a linear map T with eigenvalue λ, defined as $E_\lambda = \left\{ \mathbf{x} \in k^n \;\middle|\; T(\mathbf{x}) = \lambda \mathbf{x} \right\}$
- Dimension
  - The cardinality of a basis of V
- Basis
  - A linearly independent set of vectors $S = \{\mathbf{x}_i\} \subset V$ such that $\text{Span}(S) = V$
- Linear independence
  - A set of vectors $\{\mathbf{x}_i\}_{i=1}^n$ is linearly independent $\iff \displaystyle\sum_{i=1}^n c_i \mathbf{x}_i = 0 \implies c_i = 0$ for all i.
  - Can be detected by considering the matrix

$$T = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]^T.$$

    (linearly independent iff T is singular)
- Rank
  - Dimension of rowspace
- Rank-Nullity Theorem
  - $|\text{Nullspace}(A)| + |\text{Rank}(A)| = |\text{Codomain}(A)|$
- Nullspace
  - $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$
- Singular
  - A square $n \times n$ matrix T is singular iff $\text{Rank}(T) < n$
- Similarity
  - Two matrices A, B are similar iff there exists an invertible matrix S such that $B = SAS^{-1}$
- Diagonalizable
  - A matrix X is diagonalizable if it can be written $X = EDE^{-1}$ where D is diagonal.

  - If X is $n \times n$ and has n linearly independent eigenvectors λ_i, then $D_{ii} = \lambda_i$, and $E = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$

- Positive Definite
  - A matrix A is positive definite iff $\forall \mathbf{x} \in k^n$, we have the scalar inequality $\mathbf{x}^T A \mathbf{x} > 0$

- Projection
  - The projection of a vector $\mathbf{v}$ onto $\mathbf{u}$ is given by $P_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle \, \widehat{u}$
  - The projection of a vector $\mathbf{v}$ onto a space $U = \mathrm{Span}(\{\mathbf{u}_i\})$ is given by

$$P_U(\mathbf{v}) = \sum_i P_{\mathbf{u}_i}(\mathbf{v}) = \sum_i \langle \mathbf{u}_i, \mathbf{v} \rangle \, \widehat{u}_i.$$

- Orthogonal Complement
  - Given a subspace $U \subseteq V$, defined as $U^\perp = \left\{ \mathbf{v} \in V \;\middle|\; \forall \mathbf{u} \in U, \langle \mathbf{u}, \mathbf{v} \rangle = 0 \right\}$

- Determinant

$$\det(A) = \sum_{\tau \in S^n} \prod_{i=1}^n \sigma(\tau) a_{i,\tau(i)}.$$

- Trace

$$\mathrm{Tr}(A) = \sum_{i=1}^n A_{ii}.$$

- Characteristic Polynomial
  - $p_A(x) = \det(xI - A)$
  - Roots of p_A are eigenvalues of A

- Symmetric: $A = A^T$

- Skew-Symmetric: $A = -A^T$

- Inner Product
  - $\langle \mathbf{x},\ \mathbf{x} \rangle \geq 0$
  - $\langle \mathbf{x},\ \mathbf{x} \rangle = 0 \iff \mathbf{x} = \mathbf{0}$
  - $\langle \mathbf{x},\ \mathbf{y} \rangle = \overline{\langle \mathbf{y},\ \mathbf{x} \rangle}$
  - $\langle [,\ k \rangle \mathbf{x}]\mathbf{y} = k\langle \mathbf{x},\ \mathbf{y} \rangle = \langle \mathbf{x},\ k\mathbf{y} \rangle$
  - $\langle \mathbf{x} + \mathbf{y},\ \mathbf{z} \rangle = \langle \mathbf{x},\ \mathbf{z} \rangle + \langle [,\ y \rangle]\mathbf{z}$
  - $\langle [,\ a \rangle \mathbf{x}]b\mathbf{y} = \langle \mathbf{x},\ \mathbf{x} \rangle + \langle a\mathbf{x},\ y \rangle + \langle \mathbf{x},\ b\mathbf{y} \rangle + \langle \mathbf{y},\ \mathbf{y} \rangle$
  - Defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\ \mathbf{x} \rangle} \implies \|\mathbf{x}\|^2 = \langle \mathbf{x},\ \mathbf{x} \rangle$

- Cauchy-Schwarz Inequality: $|\langle \mathbf{x},\ \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$

- Orthogonality:
  - For vectors: $\mathbf{x}^\perp \mathbf{y} \iff \langle \mathbf{x},\ \mathbf{y} \rangle = 0$
  - For matrices: A is orthogonal $\iff A^{-1} = A^T$

- Orthogonal Projection of $\mathbf{x}$ onto $\mathbf{y}$:

$$P(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x},\ \mathbf{y} \rangle \widehat{y} = \langle \mathbf{x},\ \mathbf{y} \rangle \frac{\mathbf{y}}{\|\mathbf{y}\|^2}.$$

  - Note $\|P(\mathbf{x}, \mathbf{y})\| = \|\mathbf{x}\| \cos \theta_{x,y}$

- Defective: An $n \times n$ matrix A is defective $\iff$ the number of linearly independent eigenvectors of A is less than n.

1.5 Differential Equations

- Homogeneous

$$f(x, y) \text{ homogeneous of degree } n \iff \exists n \in \mathbb{N} : f(tx, ty) = t^n f(x, y)..$$

- Separable

$$p(y) \frac{dy}{dx} - q(x) = 0.$$

- Wronskian:

$$W\left[f_1, f_2, \ldots, f_k\right](x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_k(x) \\ f_1'(x) & f_2'(x) & \ldots & f_k'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \ldots & f_k^{(k-1)}(x) \end{vmatrix}$$

- Laplace Transform:

$$L_f(s) = \int_0^\infty e^{-st} f(t)\ dt.$$

1.6 Algebra

- Ring
- Group
- Subgroup
  - Two step subgroups test:
- Integral Domain
- Division Ring
- Principal Ideal Domain
- Tensor Product: #todo insert construction

1.7 Complex Analysis

- Analytic
- Harmonic
- Cauchy-Euler Equations

- Holomorphic
- The Complex Derivative
- Meromorphic
- The Gamma Function: Satisfies $\Gamma(p+1) + p\Gamma(p)$ and $\Gamma(1) = 1$, defined as

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \quad p > 0.$$