CRAG

The Weil Conjectures

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Zeta Functions

Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\ n \in \mathbb{Z}^{\geq 2}$$

Definition (Zeta Function)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic* variety $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \mathbf{deg}\, f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I} = (i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

Point Counts

- For a fixed variety X, we can consider its $\mathbb{F}$-points $X(\mathbb{F})$.
  - Note that $\#X(\mathbb{F}) < \infty$
- For any $L/\mathbb{F}$, we can also consider $X(L)$
  - In particular, we can consider $X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
  - We again have $\#X(\mathbb{F}_{q^n}) < \infty$ for every such n.
- So we can consider the sequence

$$[\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots] := [\#X(\mathbb{F}), \#X(\mathbb{F}_{q^2}), \cdots, \#X(\mathbb{F}_{q^n}), \cdots].$$