Title

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Contents

1 | Lecture 14: Monday, November 02

Recall the that *Hasse-Weil zeta function* of a one-variable function field $K/\mathbb{F}_q$ over a finite ground field is defined in the following way: let $A_n = A_n(K)$ be the number of effective divisors of degree n. We have proved that A_n is finite, and for $n > 2g-2$ we have a formula

$$Z(t) = \sum_{n=0}^{\infty} A_n t^n = \sum_{D \in \mathrm{Div}^+(K)} t^{\deg(D)} \in \mathbb{Z}[[t]],$$

which is a formal power series with integer coefficients.

Remark 1.0.1 : Recall that we have proved that it is a rational function of t, and in particular when $g = 0, \delta = 1$ [1] we get

$$Z(t) = \frac{1}{(1-qt)(1-t)}.$$

We got another expression which isn't fantastic: it involves this δ, which we'll work toward proving is equal to 1. When $g > 1$, we broke the zeta function into two pieces $Z(t) = F(t) + G(t)$. For divisors of sufficiently high degree, Riemann-Roch tells you what the dimension of the Riemann-Roch space is, and $G(t)$ explains the part coming from divisors of large degree. We obtained a formula previously for $F(t)$ and $G(t)$, and once we show $\delta = 1$ the formula for G will simplify. For $F(t)$, we specifically had

$$F(t) = \frac{1}{q-1} \sum_{0 \leq \deg(c) \leq 2g-2} q^{\ell(c) t^{\deg(c)}},$$

where the sum is over divisor classes and ℓ is the dimension of linear system corresponding to a divisor. But this isn't a great formula: what are these classes, dhow many are in each degree, and what is the dimension of the Riemann-Roch space?

Remark 1.0.2 : This is analogous to the Dedekind zeta function of a number field K, in which case

$$\zeta_K(s) = \dot{\sum_{T \in \ell(\mathbb{Z}_k)}} |\mathbb{Z}_k / I|^{-s},$$

which will be covered in a separate lecture on *Serre zeta functions*.

> **Theorem 1.0.1** ***(F.K. Schmidt).***
> For all $K/\mathbb{F}_q$, we have $\delta = I(K) = 1$ where I is the index.

This will follow from the associated, but it much weaker. However, this is one of the facts we'd like to establish to use to *prove* the Riemann hypothesis.

[1] The *index* of the function field, least positive degree of a divisor.

Remark 1.0.3 : Pete studied this in 2004 and found that every $I \in \mathbb{Z}^+$ arises as the index of a genus one function field $K/\mathbb{Q}$.

Notation: for $n \in \mathbb{Z}^+$, let μ_n denote the nth roots of unity in $\mathbb{C}$.

Lemma 1.1 ***(?).***
For $m, r \in \mathbb{Z}^+$, set $d := \gcd(m, r)$. Then

$$\left(1 - t^{mr/d}\right)^d = \prod_{\xi \in \mu_r} 1 - (\xi t)^m.$$

Proof (?).
In $\mathbb{C}[x]$, we have

$$(X^{r/1} - 1)^d = \prod_{\xi \in \mu_r} (X - \xi^m),$$

where both sides are monic polynomials whose roots include the (r/d)th roots of unity, each with multiplicity d. On the LHS, the distinct roots are the r/dth roots of unity, then raising to the dth power gives them multiplicity d. On the RHS, this is an exercise in cyclic groups: consider the nth power map on $\mathbb{Z}/r\mathbb{Z}$ and compute its image and kernel. As ξ ranges over rth roots of unity, ξ^m ranges over all r/dth roots of unity, each occurring with multiplicity d. Substituting $X = t^{-m}$ and multiplying both sides by t^r yields the original result.

Special case: set $m = r$, so $d = r$, then the RHS is r copies of 1.

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Next up, we want to compare the zeta function $Z(t)$ for a function field over $\mathbb{F}_q$ to the zeta function obtained when extending scalars to $\mathbb{Q}^r$.

Proposition 1.0.1 ***(?).***
Let $K/\mathbb{F}_q$ be a function field, $r \in \mathbb{Z}^+$, and take the compositum K_r of K and $\mathbb{F}_q^r$ viewed as a function field over $\mathbb{F}_q^r$. Let $Z(t)$ be the zeta function of $K/\mathbb{F}_q$ and $Z_r(t)$ the zeta function of $K_r/\mathbb{F}_q^r$. Then

$$Z_r(t^r) = \prod_{q \in \mu_r} Z(qt).$$

Proof (?).
We have an Euler product formula

$$Z(t) = \prod_{p \in \Sigma(K/\mathbb{F}_q)} (1 - t^{\deg(p)})^{-1}.$$

where the sum is over places of the function field.

Proving this Euler product formula might show up in a separate lecture, but it is not any more difficult than proving it for the Riemann zeta function.

Exercise 1.0.1 (?): Why is this product expansion true? Write as a geometric series with ratio $t^{\deg(p)}$. Here just expand each summand to get

$$Z(t) = \prod_{p} \sum_{j=1}^{\infty} t^{j \deg(p)}.$$

Multiplying this out and collecting terms is in effect multiplying out the prime divisors to get effective divisors.
We use the result that was stated (but not proved): If $p \in \Sigma_m(K/\mathbb{F}_q)$ is a degree n place and $r \in \mathbb{Z}^+$, then there exist precisely $d := \gcd(m, r)$places p^r of K_r lying over p. Moreover, each place p^r has degree m/d.

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