Problem Set 8

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Contents

1 Regular Problems

1.1 Problem 1

1.1.1 Part a

Define a map

$$\begin{aligned}\phi_{\text{ev}} : \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) &\to A \\ (f : \mathbb{Z}_m \to A) &\mapsto f(1)\end{aligned}$$

Then ϕ_{ev} is a $\mathbb{Z}$-module homomorphism, since

$$\begin{aligned}\phi_{\text{ev}}(nf+g) &= (nf+g)(1) \\ &= nf(1) + g(2) \\ &= n\phi_{\text{ev}}(f) + \phi_{\text{ev}}(g)\end{aligned}$$

But this forces $f(\overline{0}) = 0_A$ (where $\overline{0} : \mathbb{Z}_m \to A$ is the zero map), we have

$$0 = f(0) = f(m) = mf(1),$$

we must have $mf(1) = 0$ in A. So

$$\text{im } \phi_{\text{ev}} = \{a \in A \mid ma = 0\} \coloneqq A[m].$$

It is also the case that

$$\ker \phi_{\text{ev}} = \{f \in \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \mid f(1) = 0\} = \{\overline{0}\},$$

which follows from the fact that $\mathbb{Z}_m = \langle 1 \mod m \rangle$ and $A = \langle 1_A \rangle$ as $\mathbb{Z}$-modules, so if $f(1 \mod m) = 0_A$ then

$$f(n \mod m) = nf(1 \mod m) = 0$$

and so f is necessarily the zero map. So $\ker \phi = \overline{0}$.

We can then apply the first isomorphism theorem,

$$\frac{\hom_{\mathbb{Z}}(\mathbb{Z}_m, A)}{\ker \phi_{\text{ev}}} \cong \text{im } \phi_{\text{ev}} \implies \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m].$$

1.1.2 Part 2

Lemma: If $x \mid n$ and $x \mid m$ then $x \mid \gcd(m, n)$

Proof: We have $x \mid km + \ell n$ for any integers k, ℓ. So let $d = \gcd(m, n)$, then there exist integers a, b such that $am + bn = d$. But we can now just take $k = a$ and $\ell = b$. $\square$

We claim that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{(m,n)}$, from which the result immediately follows by part 1.

Define a map

$$\begin{aligned}\phi : \mathbb{Z} &\to \mathbb{Z}_n[m] \\ 1 &\mapsto [1]_{\mod n}.\end{aligned}$$

The claim is that this is an isomorphism.

Then ϕ is clearly surjective (since $\mathbb{Z} \to \mathbb{Z}_n$ is a quotient map and $\mathbb{Z}_n[m]$ is a subgroup of $\mathbb{Z}_n$) and if we let $d \coloneqq \gcd(m, n)$, we have

$$\begin{aligned}\ker \phi &= \{x \in \mathbb{Z}_n \ni mx = 0\} \\ &= \left\{x \in \mathbb{Z}_n \ni x \;\middle|\; m\right\} \\ &= \left\{x \in \mathbb{Z} \ni x \;\middle|\; n \text{ and } x \;\middle|\; m\right\} \\ &= \left\{x \in \mathbb{Z} \ni x \;\middle|\; d\right\} \quad \text{by the lemma} \\ &= d\mathbb{Z}\ .\end{aligned}$$

Then by the first isomorphism theorem, we have

$$\frac{\mathbb{Z}}{\ker \phi} \cong \text{im } \phi \implies \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}} \coloneqq \mathbb{Z}_{\gcd(m,n)} \cong \mathbb{Z}_n[m].$$

1.1.3 Part 3

Let $f \in \mathbb{Z}^* = \hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, so $f : \mathbb{Z}_m \to \mathbb{Z}$. These are both $\mathbb{Z}$-modules generated by their identity elements, so such a map is determined by where it send $[1]_{\mod m}$.

So let $f([1]_{\mod m}) = n \in \mathbb{Z}$. Since f is a module homomorphism, we have $f([0]_{\mod m}) = 0$, and in particular we have

$$\begin{aligned}0 &= f([0]_{\mod m}) \\ &= f([m]_{\mod m}) \\ &= f([1m]_{\mod m}) \\ &= mf([1]_{\mod m}),\end{aligned}$$

which forces $f([1]) \in \mathbb{Z}[m] = \{0\}$, so f must be the zero map and $\mathbb{Z}^* = 0$.

Note: $\mathbb{Z}[m] = 0$ because $\mathbb{Z}$ is an integral domain, so $mx = 0$ forces $m = 0$ or $x = 0$.

1.1.4 Part 4

To see that $\mathbb{Z}_m$ is a $\mathbb{Z}_{mk}$ module, we define an action

$$
\begin{aligned}
\mathbb{Z}_{mk} &\curvearrowright \mathbb{Z}_m \\
[x]_{mk} \curvearrowright [y]_m &\coloneqq [xy]_m
\end{aligned}
$$

This is a well-defined action:

If $[x_1]_{mk} = [x_2]_{mk}$ are two representatives of the same equivalence class, then

$$[x_1]_{mk} - [x_2]_{mk} = [x_1 - x_2]_{mk} = [0]_{mk} \implies m \;\Big|\; x_1 - x_2.$$

But then

$$
\begin{aligned}
([x_1]_{mk} \curvearrowright [y]_m) - ([x_2]_{mk} \curvearrowright [y]_m) &= [x_1 y]_m - [x_2 y]_m \\
&= [(x_1 - x_2)y]_m \\
&= [0]_m,
\end{aligned}
$$

which shows that their resulting actions on $\mathbb{Z}_m$ are equal.

This action yields a module structure:

- $r.(x+y) = r.x + r.y$:

$$[r]_{mk} \curvearrowright ([x]_m + [y]_m) = [r]_{mk} \curvearrowright [x+y]_m = [r(x+y)]_m = [rx]_m + [ry]_m.$$

- $(r+s).x = r.x + s.x$:

$$[r]_{mk} + [s]_{mk} \curvearrowright [x]_m = [r+s]_{mk} \curvearrowright [x]_m = [(r+s)x]_m = [rx]_m + [sx]_m.$$

- $(rs).x = r.s.x$:

$$
\begin{aligned}
[r]_{mk} \cdot [s]_{mk} \curvearrowright [x]_m &= [rs]_{mk} \curvearrowright [x]_m \\
&= [(rs)x]_m \\
&= [r]_{mk} \curvearrowright [sx]_m \\
&= [r]_{mk} \curvearrowright ([s]_{mk} \curvearrowright [x]_m).
\end{aligned}
$$

- $1.x = x$:

$$[1]_{mk} \curvearrowright [x]_m = [1x]_m = [x]_m.$$

$\mathbb{Z}_m^* \coloneqq \hom_{\mathbb{Z}_{mk}}(\mathbb{Z}_m, \mathbb{Z}_{mk}) \cong \mathbb{Z}_m$:

Define a map

$$
\begin{aligned}
\phi : \hom_{\mathbb{Z}_{mk}}(\mathbb{Z}_m, \mathbb{Z}_{mk}) &\to \mathbb{Z}_m \\
f &\mapsto [f([1]_m)]_m
\end{aligned}
$$

ϕ **is a homomorphism**, as

$$\begin{aligned}\phi(f+g) = [(f+g)([1]_m)]_m = [f([1]_m) + g([1]_m)]_m = [f([1]_m)]_m + [g([1]_m)]_m, \\ \phi([r]_{mk} \curvearrowright f) = [[r]_{mk} f([1]_m)]_m = [r]_m \cdot [f([1]_m)]_m = [r]_{mk} \curvearrowright \phi(f).\end{aligned}$$

ϕ **is injective**, as $[f([1]_m)]_m = [0]_m$, then for any $1 \leq \ell \leq m$, we have

$$[f([\ell]_m)]_m = [\ell f([1]_m)]_m = \ell[f([1]_m)]_m = \ell[0]_m = [0]_m,$$

so f must be the zero map.

ϕ **is surjective**, since if $[\ell]_m \in \mathbb{Z}_m$, we can define

$$\begin{aligned} f_\ell : \mathbb{Z}_m &\to \mathbb{Z}_{mk} \\ [1]_m &\mapsto [\ell]_{mk} \end{aligned}$$

which makes sense and is well-defined because $\mathbb{Z}_m \hookrightarrow \mathbb{Z}_{mk}$, and the map is defined on the generator.

So we have the desired bijection. $\square$

1.2 Problem 2

We have the map

$$\begin{aligned} \pi : \mathbb{Z} &\to \mathbb{Z}_2 \\ x &\mapsto [x]_2 \end{aligned}$$

which is a surjection and thus an epimorphism in the category $\mathbb{Z}$-Mod, and if we apply the functor $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \cdot\,)$ to π we obtain an induced map

$$\begin{aligned} \overline{\pi} : \hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) &\to \hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ f &\mapsto \pi \circ f. \end{aligned}$$

The claim is that $\overline{\pi}$ is *not* a surjection, and thus not an epimorphism (in the same category).

To see that this is the case, we can simply note that $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) = 0$ by part 3 of Problem 1, whereas $\hom_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \neq 0$.

For example, one can define $\mathrm{id}_{\mathbb{Z}_2} : \mathbb{Z}_2 \to \mathbb{Z}_2,\ [x]_2 \to [x]_2$, which is a nontrivial module homomorphisms.

So any such f appearing must be the zero map, and thus $\overline{\pi}$ is also the zero map. $\square$

1.3 Problem 3

Let $f : R \to R$ be an endomorphism of R in the category of rings. We can then check that for any $r \in R$, we have $f(r) = f(r1_R) = rf(1_R)$, which says that f is given by right-multiplication by some fixed element $x_f \coloneqq f(1_R)$, i.e.

$$\begin{aligned} f : R &\to R \\ r &\mapsto r \cdot x_f \end{aligned}$$

and so we can attempt to define

$$\begin{aligned} \phi_1 : \hom_R(R, R) &\to R \\ f &\mapsto x_f \coloneqq f(1_R) \end{aligned}$$

We can check that

$$(g \circ f(r)) = g(f(r)) = g(r \cdot x_f) = r \cdot x_f \cdot x_g,$$

which shows that in fact

$$\phi(g \circ f) = x_f \cdot x_g,$$

which reverses the multiplication. So the correct codomain is R^{op}, and we amend the definition:

$$\begin{aligned} \phi_2 : \hom_R(R, R) &\to R^{op} \\ f &\mapsto x_f \coloneqq f(1_R) \end{aligned}$$

By construction, ϕ_s **is a ring homomorphism**. If R is commutative, then $x_f \cdot x_g = x_g \cdot x_f$, which makes ϕ_1 a ring homomorphism as well. It remains to check that it is an isomorphism/

ϕ_1 **is in injective**: We can check that $\ker \phi_1 = 0$ as a ring. To that end, suppose $\phi_1(f) = x_f = 0$. Then $f(r) = r \cdot 0 = 0$, so f can only be the zero map.

ϕ_1 **is surjective**: Let $x \in R$ be arbitrary, then we can define $f : R \to R$ by $f(1_R) = x$, so $f(r) = r \cdot x$. This is an endomorphism of R, and thus an element of $\hom_R(R, R)$.

By the first isomorphism theorem for rings, we thus have $\hom_R(R, R) \cong R$. $\square$

1.4 Problem 4

We have maps

$$\begin{aligned} \theta_A : A &\to (A^\vee)^\vee \\ a &\mapsto (\mathrm{ev}_a : f \mapsto f(a)) \end{aligned}$$

$$\begin{aligned}\theta_B : B &\to (B^\vee)^\vee \\ b &\mapsto (\mathrm{ev}_b : g \mapsto g(b))\end{aligned}$$

$$\begin{aligned}f : A &\to B \\ a &\mapsto f(a)\end{aligned}$$

$$\begin{aligned}f^\vee : B^\vee &\to A^\vee \\ g &\mapsto g \circ f\end{aligned}$$

$$\begin{aligned}f^{\vee\vee} : A^{\vee\vee} &\to B^{\vee\vee} \\ h &\mapsto h \circ f^\vee\end{aligned}$$

We can now check that $f^{\vee\vee} \circ \theta_A = \theta_B \circ f$ as maps from A to $B^{\vee\vee}$. Letting $a \in A$, and $h \in B^{\vee\vee}$ (so $h : B^\vee \to R$), we will show that both maps act on h in the same way.

For notational convenience, write $\phi \curvearrowright h := h \circ \phi$. We then have

$$\begin{aligned}(f^{\vee\vee} \circ \theta_A)(a) \curvearrowright h &:= f^{\vee\vee}(\theta_A(a)) \curvearrowright h \\ &:= f^{\vee\vee}(\mathrm{ev}_a) \curvearrowright h \\ &= (\mathrm{ev}_a \circ f^\vee) \curvearrowright h \\ &:= h \circ (\mathrm{ev}_a \circ f) \\ &:= h(f(a)) \\ &= \mathrm{ev}_{f(a)} \curvearrowright h \\ &:= \theta_B(f(a)) \curvearrowright h \\ &:= (\theta_B \circ f)(a) \curvearrowright h,\end{aligned}$$

which shows that these actions agree, and thus the diagram commutes.

1.5 Problem 5

Let E be a free module over R an integral domain. Then E has a basis $\{\mathbf{e}_i\} \subseteq F$, so if $x \neq 0 \in E$, we have

$$x = \sum_i r_i \mathbf{e}_i$$

where each $r_i \in R$. Moreover, since $x \neq 0$, at least one $r_i \neq 0$, so let r_j denote one of the nonzero coefficients.

Now suppose x is a torsion element, so $mx = 0$ for some $m \neq 0 \in E$. We can then write

$$mx = m \sum_i r_i \mathbf{e}_i = \sum_i m r_i \mathbf{e}_i = 0$$

But by linear independence, this forces $mr_i = 0$ for all i. In particular, $mr_j = 0$ where $r_j \neq 0$. But this exhibits either m or r_j as a zero divisor, and since the only zero divisor in an integral domain is zero, we must have $m = 0$ or $r_j = 0$, a contradiction.

So x can not be a torsion element. But since $x \in E$ was arbitrary, E must be torsion-free.

For an example of a torsion-free module over an integral domain that is *not* free, consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $\mathbb{Q}$ is clearly torsion-free, since it is an integral domain and the same argument as above applies.

But $\mathbb{Q}$ is not free as $\mathbb{Z}$-module. Supposing that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots\} \subset \mathbb{Q}$ was a $\mathbb{Z}$-basis, consider $\mathbf{b}_1 = \frac{p_1}{q_1}$ and $\mathbf{b}_2 = \frac{p_2}{q_2}$. Then $\mathbf{b}_1, \mathbf{b}_2$ can not be linearly independent over $\mathbb{Z}$, which follows from the fact that

$$q_1 p_2 \mathbf{b}_1 + q_2 p_1 \mathbf{b}_2 = p_2 p_1 - p_1 p_2 = 0,$$

while $q_1 p_2,\ q_2 p_1 \neq 0 \in \mathbb{Z}$. $\square$

1.6 Problem 6

If A is a cyclic module over a commutative ring R, so we have $A = Ra$ for some $a \in A$. By Hungerford's definition, the submodule A has order $r \iff$ the element a has order $r \iff$ the order ideal $\mathcal{O}_a := \{x \in R \mid xa = 0\} = (r)$.

In particular, $ra = 0$.

1.6.1 Part 1

Since $(r, s) = (1)$, we can find $t_1, t_2 \in R$ such that

$$\begin{aligned}
t_1 r + t_2 s = 1 &\implies t_1 ra + t_2 sa = 1a \\
&\implies t_1(ra) + t_2 sa = a \\
&\implies t_2 sa = a && \text{since } ra = 0 \\
&\implies s(t_2 a) = a && \text{since } R \text{ is commutative,}
\end{aligned}$$

which implies that $a \in sA$ and thus $A \subseteq sA$. However, we always have $sA \subseteq A$ for modules, so this shows that $A = sA$.

To see that $A[s] = \{x \in A \mid sx = 0\} = 0$, let $x \in A[s]$; we will show $x = 0$. Since $x \in A = Ra$, we have $x = r_1 a$, and in particular

$$ra = 0 \implies rx = rr_1 a = r_1(ra) = 0.$$

So we now have $rx = 0$ and $sx = 0$, and we can write

$$
\begin{aligned}
x &= (t_1 r + t_2 s)x \\
&= t_1(rx) + t_2(sx) \\
&= t_1 0 + t_2 0 \\
&= 0.
\end{aligned}
$$

So $x = 0$ and thus $A[s] = 0$. $\square$

1.6.2 Part 2

Suppose $r = sk$. Toward an application of the first isomorphism theorem, define a map

$$
\begin{aligned}
\phi : R &\to sA = sRa \\
x &\mapsto sxa.
\end{aligned}
$$

ϕ is well-defined:

This follows from that fact that $a \in A \implies xA \in A$ for any $x \in R$, so the codomain is in fact sA.

ϕ is an R-module homomorphism:

We have

$$
\begin{aligned}
t \in R &\implies \phi(tx) = s(tx)a = t(sxa) = t\phi(x) \\
x, y \in R &\implies \phi(x+y) = s(x+y)a = sxa + sya + \phi(x) + \phi(y)
\end{aligned}
$$

$\ker \phi = (k)$:

Suppose $x \in \ker \phi$ so $sxa = 0_A$; we'd like to show $x \in (k)$.

By definition $sx \in \mathcal{O}_a$, and by assumption $\mathcal{O}_a = (r)$, so $sx = t_1 r$ for some $t_1 \in R$.

$$
\begin{aligned}
& sxa = 0_A \\
\implies & sx = t_1 r && \text{since } sx \in \mathcal{O}_a \\
\implies & sx = t_1(sk) && \text{since } r = sk \text{ by assumption} \\
\implies & sx = s(t_1 k) && \text{since elements in } R \text{ and } A \text{ commute} \\
\implies & x = t_1 k && \text{since } R \text{ is a domain, so } sm = sn, s \neq 0 \implies m = n,
\end{aligned}
$$

which exhibits $x = t_1 k \implies x \in (k)$ as desired.

ϕ is surjective:

Since $A = Ra$, we have $sA = sRA$ and thus $x \in sA \implies x = sra$ for some $r \in R$; but then $\phi(r) = sra = x$.

We thus have

$$R/\ker\phi \cong \text{im } \phi \implies R/(k) \cong sA.$$

Similarly, define a map

$$\begin{aligned}\psi : R &\to A[s] \\ x &\mapsto kxa\end{aligned}$$

ψ is well-defined:

It suffices to check that im $\psi \subseteq A[s]$ (since we will show surjectivity shortly), i.e. that s annihilates anything in the image. This follows from

$$s(kxa) = (sk)xa = rxa = x(ra) = 0,$$

since $ra = 0$ by assumption.

ψ is an R-module homomorphism:

We can check

$$\psi(tr_1 + r_2) = k(tr_1 + r_2)s = tkr_1s + kr_2s = t\psi(r_1) + \psi(r_2)$$

which follows because elements of R commute with those from A under multiplication.

$\ker\psi = (s)$:

Suppose $x \in \ker\psi$, so $kxa = 0$. Then $kx \in \mathcal{O}_a = (r)$, so $kx = rt_1$. Then

$$\begin{aligned}
&kxa = 0_A \\
&\implies kx = rt_1 && \text{since } kx \in \mathcal{O}_a \\
&\implies kx = (sk)t_1 && \text{since } r = sk \\
&\implies kx = k(st_1) && \text{since } R \text{ is commutative} \\
&\implies x = st_1 && \text{since } R \text{ is a domain,}
\end{aligned}$$

and so $x \in (s)$ as desired.

ψ is surjective:

Letting $y \in A[s]$ be arbitrary. We have

$$
\begin{aligned}
y \in A[s] &\implies x = t_1 a, \quad sx = 0 \\
&\implies s(t_1 a) = 0 \\
&\implies st_1 \in \mathcal{O}_a \implies \exists x \in R \ni st_1 = xr = x(sk) \\
&\implies st_1 = sxk \\
&\implies t_1 = xk \qquad\qquad \text{since } R \text{ is a domain} \\
&\implies y = t_1 a = (xk)a = kxa,
\end{aligned}
$$

so $\psi(x) = y$.

We can then apply the first isomorphism theorem

$$
R/\ker\psi \cong \operatorname{im}\psi \implies R/(s) \cong A[s].
$$

□

1.7 Problem 7

Lemma: If M is a cyclic module over a PID, then M has exactly 1 invariant factor.

Lemma: Let A be a cyclic module, so $A = Ra$. If the order of A is r, so $\mathcal{O}_a = (r)$, then $A \cong R/(r)$.

This means that we can write $A = R/(a)$ and $B = R/(b)$, and a, b are the invariant factors of A, B respectively, and $M := A \oplus B \cong R/(ab)$.

Since R is a PID, there is unique factorization, so we can write

$$
\begin{aligned}
r &= \prod_{i=1}^{n} p_i^{k_i} \\
s &= \prod_{i=1}^{n} p_i^{\ell_i} \\
\implies m := rs &= \prod_{i=1}^{n} p_i^{k_i + \ell_i},
\end{aligned}
$$

where we allow some $k_i, \ell_i = 0$ so that we can take the product over the same set of primes.

However, means that the elementary divisors of M are given by $\left\{ p_i^{k_i + \ell_i} \mid 1 \le i \le n \right\}$.

By definition, the first invariant factor is obtained from the elementary divisors as

$$
d_1 := \prod_{\{i \mid k_i, \ell_i \neq 0\}} p_i^{\max(k_i, \ell_i)}
$$

i.e., we collect all of the prime powers that divide m, and take the highest power of each prime occuring. However, this is exactly $\gcd(r, s)$.

Since $rs = \gcd(r,s) \cdot \operatorname{lcm}(r,s)$, the second invariant factor is obtained by performing the same process on $\frac{rs}{\gcd(r,s)} = \operatorname{lcm}(r,s)$. We can write

$$\frac{rs}{\gcd(r,s)} = \prod_{i=1}^{n} p_i^{k_i+\ell_i-\max(k_i,\ell_i)}$$

But we can note here that there is at least one p_i such that the exponent $k_i + \ell_i - \max(k_i, \ell_i) = 0$, i.e.,

2 Qual Problems

2.1 Problem 8

2.1.1 Part 1

The claim is that every element in $M := R^n/\text{im } A$ is torsion $\iff$ the matrix rank of A is exactly $n \iff$ the Smith normal form of A has exactly n nonzero invariant factors.

To see that this is the case, we can apply the structure theorem for finitely-generated modules over a PID. This gives us

$$M \cong F \oplus \bigoplus R/(r_i)$$

where F is free of finite rank, $R/(r_i)$ is cyclic torsion, and $r_i \;\Big|\; r_{i+1} \;\Big|\; \cdots$ are the invariant factors of M.

We thus have

$$M \cong R^n/\text{im } A \cong F \oplus \bigoplus R/(r_i),$$

which will be pure torsion if and only if $F = 0$.

But if we compute the smith normal for of A, we obtain

$$SNF(A) = \begin{bmatrix} d_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & d_2 & \cdot & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & d_n & \cdots & 0 \end{bmatrix}$$

where $d_1 \;\Big|\; d_2 \;\Big|\; \cdots \;\Big|\; d_n$, and thus

$$\begin{aligned} \text{im } A &\cong \text{im } SNF(A) \cong d_1R \oplus d_2R \oplus \cdots \oplus d_nR \\ \implies M = R^n/\text{im } A &\cong \frac{R^n}{d_1R \oplus d_2R \oplus \cdots d_nR} \\ &\cong R/(d_1) \oplus R/(d_2) \cdots \oplus R/(d_n) \end{aligned}$$

where $R/(d_i)$ is a cyclic torsion module precisely when $d_i \neq 0$. If instead some $d_i = 0$, we then have $R/(d_i) \cong R$, which is a free R-module, yielding non-torsion elements in M.

But $\det(A) = \det(SNF(A)) = \prod_{i=1}^{n} d_i$, and so if $d_i = 0$ for some i iff $\det A = 0$ iff $\operatorname{rank} A < n$.

2.1.2 Part 2

Identifying

$$R \times F = F[x] \oplus F \cong F[x] \oplus \frac{F[x]}{(f)}$$

where f is any degree 1 polynomial in $F[x]$, by the structure theorem we can pick a matrix $A \in M_2(F[x])$ with invariant factors $d_1 = 0, d_2 = f$. Then by the same argument given in part 1, we would have

$$(F[x])^2/\text{im } A \cong \frac{F[x]}{(d_1)} \oplus \frac{F[x]}{(d_2)} = F[x] \oplus \frac{F[x]}{(f)}$$

So we can choose $n = 2$, and say $f(x) = x + 1$, and then just pick a matrix that is already in Smith normal form:

$$A = \begin{bmatrix} x+1 & 0 \\ 0 & 0 \end{bmatrix}.$$

2.2 Problem 9

2.2.1 Part 1

Let M be a finitely generated module over R a PID.

Then

$$M \cong F \oplus \bigoplus_{i=1}^{n} R/(d_i)$$

where F is free of finite rank and $R/(d_i)$ are cyclic torsion modules (the *invariant factors*) satisfying $d_1 \mid d_2 \mid \cdots \mid d_n$.

Equivalently,

$$M \cong F \oplus \bigoplus_{i=1}^{n} R/(p_i^{s_i})$$

where F is free of finite rank, $p^i \in R$ are (not necessarily distinct) prime elements (the *elementary divisors*), and $s_i \in \mathbb{Z}^{\geq 1}$.

2.2.2 Part 2

Since $\mathbb{Z}^4$ is a finitely generated module over the PID $\mathbb{Z}$, the structure theorem applies, and we can write $M \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/(r_i)$ for some $k \leq 4$ and some collection r_i of invariant factors.

If we write $M \cong \mathbb{Z}^4/N$ where N is the submodule generated by the prescribed relations, then we can construct a homomorphism of $\mathbb{Z}$-modules $L : \mathbb{Z}^4 \to N$ which is given by the matrix

$$A_L = \begin{pmatrix} 3 & 12 & 3 & 6 \\ 0 & 6 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then im $A_L \cong N$, and we can compute the Smith normal form,

$$SNF(A_L) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which shows that the invariant factors are $3, 6, 6, 0$. We can thus write im $A_L \cong 3\mathbb{Z} \oplus 6\mathbb{Z} \oplus 6\mathbb{Z}$, and so

$$M \cong \frac{\mathbb{Z}^4}{3\mathbb{Z} \oplus 6\mathbb{Z} \oplus 6\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(6).$$

2.3 Problem 10

2.3.1 Part 1

An element $x \in M$ is *torsion* iff there exists some nonzero $r \in R$ such that $rm = 0$, or equivalently $\mathrm{Ann}(x) \neq 0$.

2.3.2 Part 2

Let $R = \mathbb{C}[x],\ M = \mathbb{C}^2$, and

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in M_2(\mathbb{C}).$$

Then $\mathbb{C}^2$ is a module over $\mathbb{C}[x]$ with action given by

$$p(x) \curvearrowright \mathbf{v} \coloneqq p(A)\mathbf{v}$$

Then M is cyclic as an R-module and generated by the basis vector $[1, 0]^2 \in \mathbb{C}^2$, since

$$
\begin{aligned}
(tA+s)\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}x\\y\end{bmatrix}\\
\implies \begin{bmatrix}t & 2t\\2t & t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}s\\0\end{bmatrix} &= \begin{bmatrix}x\\y\end{bmatrix}\\
\implies \begin{bmatrix}t+s\\2t\end{bmatrix} &= \begin{bmatrix}x\\y\end{bmatrix}\\
\implies \begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}t\\s\end{bmatrix} &= \begin{bmatrix}x\\y\end{bmatrix}\\
&,
\end{aligned}
$$

which is a linear system of equations represented by an invertible matrix, which always has a solution. So every $\mathbf{v} \in \mathbb{C}^2$ is the image of some polynomial in A.

It is then easy to see that $\mathbb{C}^2$ is torsion as a module over $\mathbb{C}[x]$, since by Cayley-Hamilton we have $\mathrm{Ann}(A) = (\mathrm{minpoly}(A)) = (x^2 - 2x - 3)$, and so letting $p(x) = x^2 - 2x - 3$, we find that

$$\forall\ \mathbf{v} \in \mathbb{C}^2 \quad p(A) \curvearrowright \mathbf{v} = 0 \curvearrowright \mathbf{v} = 0.$$

2.3.3 Part 3

Suppose R is a domain, M an R-module, and let

$$T(M) = \{m \in M \ni rm = 0 \text{ for some } r \neq 0 \in R\}.$$

Then $T(R)$ is a submodule iff for all $r \in R$ and all $m, n \in T(M)$ we have $rm + n \in T(M)$.

So pick annihilators $a_m, a_n \neq 0 \in R$ where $a_m m = 0$ and $a_n n = 0$.

Since $a_m \neq 0$ and $a_n \neq 0$, the product $a_m a_n \neq 0$ **because R is a domain**.

Since $0 \in T(M)$, we can suppose $rm + n \neq 0$ (otherwise this is in $T(M)$ trivially). Then

$$
\begin{aligned}
a_m a_n(rm+n) &= a_m a_n rm + a_m a_n n\\
&= ra_n(a_m m) + a_m(a_n n)\\
&= ra_n 0 + a_m 0\\
&= 0.
\end{aligned}
$$

where the commutativity of r, a_n, a_m follows from the fact that these are all elements of R, which is a domain, and in particular is commutative. $\square$