

# Title

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Recall Hilbert's Nullstellensatz:

- a. For any affine variety,  $V(I(X)) = X$ .
- b. For any ideal  $J \subseteq k[x_1, \dots, x_n]$ ,  $I(V(J)) = \sqrt{J}$ .

So there's an order-reversing bijection

$$\{\text{Radical ideals } k[x_1, \dots, x_n]\} \longrightarrow V(\cdot)I(\cdot) \{\text{Affine varieties in } \mathbb{A}^n\}.$$

In proving  $I(V(J)) \subseteq \sqrt{J}$ , we had an important lemma (Noether Normalization): the maximal ideals of  $k[x_1, \dots, x_n]$  are of the form  $\langle x - a_1, \dots, x - a_n \rangle$ .

#### Corollary 1.1(?).

If  $V(I)$  is empty, then  $I = \langle 1 \rangle$ .

Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus  $\implies$  trivial ideal, so there's a linear combination that equals 1.

*Proof.*

By contrapositive, suppose  $I \neq \langle 1 \rangle$ . By Zorn's Lemma, there exists a maximal ideal  $\mathfrak{m}$  such that  $I \subset \mathfrak{m}$ . By the order-reversing property of  $V(\cdot)$ ,  $V(\mathfrak{m}) \subseteq V(I)$ . By the classification of maximal ideals,  $\mathfrak{m} = \langle x - a_1, \dots, x - a_n \rangle$ , so  $V(\mathfrak{m}) = \{a_1, \dots, a_n\}$  is nonempty. ■

Returning to the proof that  $I(V(J)) \subseteq \sqrt{J}$ : let  $f \in I(V(J))$ , we want to show  $f \in \sqrt{J}$ . Consider the ideal  $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \dots, x_n, t]$ .

Observation:  $f = 0$  on all of  $V(J)$  by the definition of  $I(V(J))$ . But  $ft - 1 \neq 0$  if  $f = 0$ , so  $V(\tilde{J}) = V(J) \cap V(ft - 1) = \emptyset$ .

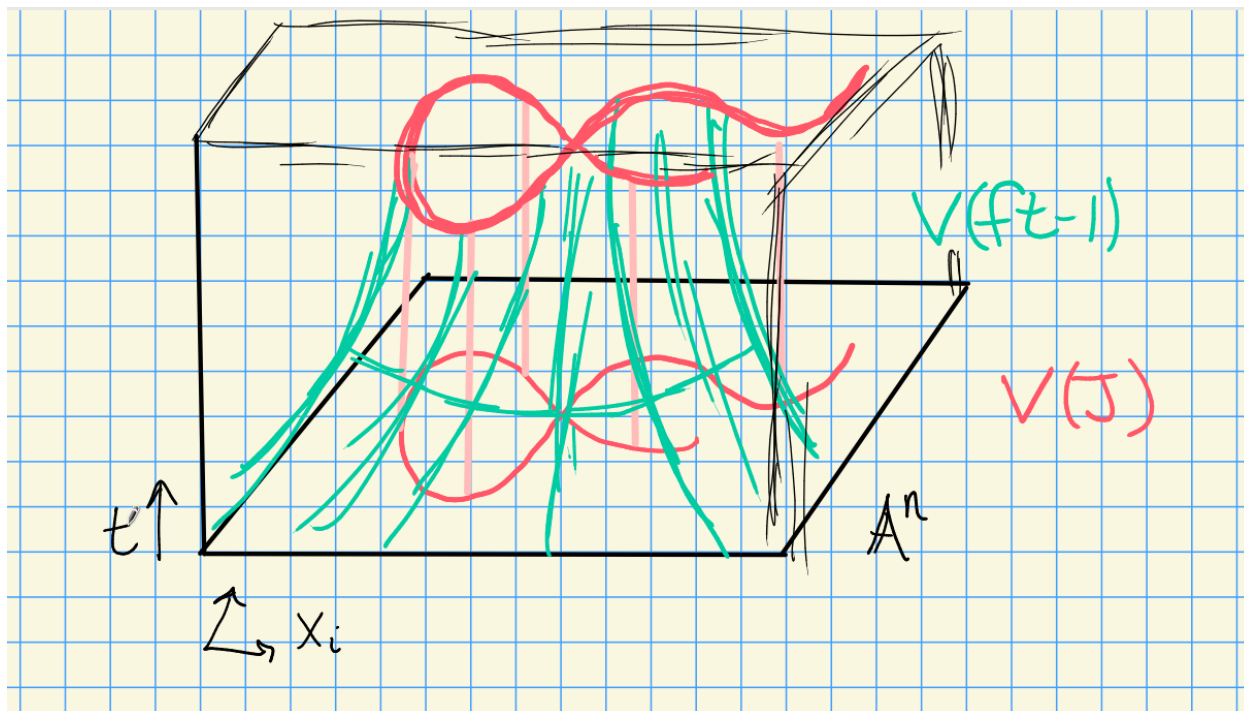


Figure 1: Effect, a hyperbolic tube around  $V(J)$ , so both can't vanish

Applying the corollary  $\tilde{J} = (1)$ , so  $1 = \langle ft - 1 \rangle g_0(x_1, \dots, x_n, t) + \sum f_i g_i(x_1, \dots, x_n, t)$  with  $f_i \in J$ . Let  $t^N$  be the largest power of  $t$  in any  $g_i$ . Thus for some polynomials  $G_i$ , we have

$$f^N := (ft - 1)G_0(x_1, \dots, x_n, ft) + \sum f_i G_i(x_1, \dots, x_n, ft)$$

noting that  $f$  does not depend on  $t$ .

Now take  $k[x_1, \dots, x_n, t] / \langle ft - 1 \rangle$ , so  $ft = 1$  in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \dots, x_n, 1) \in k[x_1, \dots, x_n, t] / \langle ft - 1 \rangle.$$

Observation: there is an inclusion

$$k[x_1, \dots, x_n] \hookrightarrow k[x_1, \dots, x_n, t] / \langle ft - 1 \rangle.$$

### Exercise 1.1.

Why is this true?

Since this is injective, this identity also holds in  $k[x_1, \dots, x_n]$ . But  $f_i \in J$ , so  $f \in \sqrt{J}$ .

### Example 1.1.

Consider  $k[x]$ . If  $J \subset k[x]$  is an ideal, it is principal, so  $J = \langle f \rangle$ . We can factor  $f(x) = \prod_{i=1}^k (x - a_i)^{n_i}$  and  $V(f) = \{a_1, \dots, a_k\}$ . Then  $I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J$ . Note that this loses information.

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**Example 1.2.**

Let  $J = \langle x - a_1, \dots, x - a_n \rangle$ , then  $I(V(J)) = \sqrt{J} = J$  with  $J$  maximal. Thus there is a correspondence

$$\{\text{Points of } \mathbb{A}^n\} \iff \{\text{Maximal ideals of } k[x_1, \dots, x_n]\}.$$

**Theorem 1.2 (Properties of  $I$ ).**

- a.  $I(X_1 \cup X_2) = I(X_1) \cap I(X_2).$
- b.  $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}.$

*Proof.*

We proved (a) at the level of  $V$ .

For (b), we have

