

# Notes on Lee's Manifolds

D. Zack Garza

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## 1 General Notes to Self

Interesting things to know:

- The Whitney embedding theorem
- The Jordan-Brouwer separation theorem
- The Poincare-Hopf theorem
- The Hopf degree theorem
- Generalized Stokes' Theorem
- Sard's Theorem
- The Frobenius Integrability Theorem

## 2 Preface: Point Set Review

### 2.1 Quotients

**Definition 2.0.1** (Saturated).

A subset  $A \subseteq X$  is *saturated* with respect to  $p : X \rightarrow Y$  if whenever  $p^{-1}(\{y\}) \cap A \neq \emptyset$ , then  $p^{-1}(\{y\}) \subseteq A$ .

Equivalently,  $A = p^{-1}(B)$  for some  $B \subseteq Y$ , i.e. it is a complete inverse image of some subset of  $Y$ , i.e.  $A$  is a union of fibers  $p^{-1}(b)$ .

**Definition 2.0.2** (Quotient Map).

A continuous surjective map  $p : X \rightarrow Y$  is a *quotient map* if  $U \subseteq Y$  is open **iff**  $p^{-1}(U) \subseteq X$  is open.

Note that  $\Rightarrow$  comes from the definition of continuity of  $p$ , but  $\Leftarrow$  is a stronger condition.

Equivalently,  $p$  maps saturated subsets of  $X$  to open subsets of  $Y$ .

**Definition 2.0.3** (Universal Property of Quotients).

For  $\pi : X \rightarrow Y$  a quotient map, if  $g : X \rightarrow Z$  is a map that is constant on each  $\pi^{-1}(\{y\})$ , then there is a unique map  $f$  making the following diagram commute:

$$\begin{array}{ccc} X & & \\ \downarrow \pi & \searrow g & \\ Y & \xrightarrow{f} & Z \end{array}$$

Fact: an injective quotient map is a homeomorphism.

Fact: a product of quotient maps need not be a quotient map.

### 2.2 Subspaces

**Definition 2.0.4** (The Subspace Topology).

$U \subseteq A$  is open iff  $U = V \cap A$  for some open  $V \subseteq X$ .

**Proposition 2.1** (*Universal Property of Subspaces*).

If  $X$  and  $\iota_S : S \hookrightarrow Y$  is a subspace, then every continuous map  $f : X \rightarrow S$  lifts to a continuous map  $\tilde{f} : X \rightarrow Y$  where  $\tilde{f} := \iota_S \circ f$ :

$$\begin{array}{ccc} & Y & \\ \exists! \tilde{f} \nearrow & \uparrow \iota_S & \\ X & \xrightarrow{f} & S \end{array}$$

Note that we can view  $\iota_S := \text{id}_Y|_S$ . The subspace topology is the unique topology for which this property holds.

Some properties of subspace:

- The inclusion  $\iota_S$  is a topological embedding.
- Restricting a continuous map to a subspace is still continuous.

- A basis for the subspace topology for  $A \subset X$  can be obtained by intersecting basis elements of  $X$  with  $A$ .
- If  $X$  is Hausdorff/first/second-countable, then so is  $A$ .

## 2.3 Products

**Definition 2.1.1** (The Product Topology).

The coarsest topology such that every projection map  $p_\alpha : \prod_{\beta} X_\beta \rightarrow X_\alpha$  is continuous, i.e. for every  $U_\alpha \subseteq X_\alpha$  open,  $p_\alpha^{-1}(U_\alpha) \in \prod X_\beta$  is open. For finite index sets, we can take the box topology: the collection of sets of the form  $\prod_{i=1}^N U_i$  with each  $U_i$  open in  $X_i$  forms a basis for the product topology on  $\prod_{i=1}^N X_i$ .

Why these differ: in  $\mathbb{R}^\infty$ , the set  $S = \prod(-1, 1)$  is open in the box topology but not the product topology, since  $\{0\}^\infty$  is not contained in any basic open neighborhood contained in  $S$ .

Some properties of products:

- Projections  $\pi_i$  are continuous by definition.
- A basis for the product topology can be obtained by taking the product of bases.
- A map  $f : X \rightarrow \prod Y_i$  into a product is continuous iff each component function  $F_i := \pi_i \circ f : X \rightarrow Y_i$  is continuous.
  - I.e. if we have continuous maps  $f_i : X \rightarrow Y_i$  then the composite map  $F = [f_1, f_2, \dots]$  is continuous.
- Separate continuity does not imply joint continuity: A map  $f : \prod X_i \rightarrow Y$  out of a product need not be continuous even if (defining  $\iota_j : X_j \hookrightarrow \prod X_i$ ) the map  $f \circ \iota_j : X_j \rightarrow Y$  is continuous for all arbitrary inclusions  $\iota_j$ .
- Any map of the form  $f_{\mathbf{a}_j} : X_j \rightarrow \prod_{i=1}^n X_i$  where  $x \mapsto (a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_n)$  is a topological embedding.
- If  $X_i$  are Hausdorff/first/second-countable, then so is  $\prod_{i=1}^n X_i$ .

## 2.4 Misc

**Definition 2.1.2** (Precompact).

A subset  $A \subseteq X$  is *precompact* iff its closure  $\text{cl}_X(A)$  is compact in  $X$ .

**Definition 2.1.3** (Locally Compact).

A space  $X$  is *locally compact* iff every  $x \in X$  has a neighborhood which is contained in some compact subset of  $X$ .

## 2.5 Analysis Review

**Definition 2.1.4** (Derivative, Real Valued).A function  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable at  $x$  iff there is a number  $y \in \mathbb{R}$  such that

$$\left( \frac{f(x+h) - f(x)}{h} - y \right) \xrightarrow{h \rightarrow 0} 0$$

where  $h \in \mathbb{R}$ .The number  $f'(x) := y$  is the *derivative* of  $f$  at  $x$ .

Note that this equivalently says

$$f(x+h) - f(x) = f'(x)h + r(h) \text{ where } \frac{r(h)}{h} \xrightarrow{h \rightarrow 0} 0.$$

**Definition 2.1.5** (Derivative, Vector Valued).For  $\mathbf{f} : (a, b) \rightarrow \mathbb{R}^n$ ,  $\mathbf{f}'(x)$  is the vector  $\mathbf{y} \in \mathbb{R}^n$  such that

$$\left( \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} - \mathbf{y} \right) \xrightarrow{h \rightarrow 0} \mathbf{0} \iff \frac{|\mathbf{f}(x+h) - \mathbf{f}(x) - h\mathbf{y}|}{|h|} \xrightarrow{h \rightarrow 0} 0$$

where  $h \in \mathbb{R}$ .The vector  $\nabla f := \mathbf{y}$  is the *derivative* (or *gradient*) of  $f$  at  $\mathbf{x}$ .

Note that this equivalently says

$$\mathbf{f}(x+h) - \mathbf{f}(x) = h\nabla f + \mathbf{r}(h) \text{ where } \frac{\mathbf{r}(h)}{h} \xrightarrow{h \rightarrow 0} \mathbf{0}.$$

**Definition 2.1.6** (Derivative, General Case).A function  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable iff there exists a linear transformation  $\mathbf{Y}$  such that

$$\frac{\|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{Y}\mathbf{h}\|_{\mathbb{R}^m}}{\|\mathbf{h}\|_{\mathbb{R}^n}} \xrightarrow{\mathbf{h} \rightarrow \mathbf{0}} 0.$$

The matrix  $D_f(\mathbf{x}) := \mathbf{Y}$  is the *total derivative* of  $f$  at  $\mathbf{x}$ .

Note that this equivalently says

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) = D_f \mathbf{h} + \mathbf{r}(\mathbf{h}) \text{ where } \frac{\|\mathbf{r}(\mathbf{h})\|}{\|\mathbf{h}\|} \xrightarrow{\mathbf{h} \rightarrow \mathbf{0}} 0.$$

Note that we can write  $(\nabla f)(\mathbf{x}) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathbf{e}_i$ .

**Theorem 2.2 (Chain Rule).**If  $E \subset \mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $E \xrightarrow{f} f(E) \xrightarrow{g} g(f(E))$  with  $f$  differentiable at  $\mathbf{x}_0$  and  $g$  differentiable at  $f(\mathbf{x}_0)$ , then the map  $F(\mathbf{x}) := g(f(\mathbf{x}))$  is differentiable at  $\mathbf{x}_0$  with derivative

$$D_F(\mathbf{x}_0) = D_g(f(\mathbf{x}_0)) \cdot D_f(\mathbf{x}_0).$$

**Definition 2.2.1** (Components of a Function).

If  $\mathcal{B}_n := \{\mathbf{e}_i\} \subset \mathbb{R}^n$  and  $\mathcal{B}_m := \{\mathbf{u}_i\} \subset \mathbb{R}^m$  are standard bases and  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , then the *components* of  $\mathbf{f}$  are the functions  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x})\mathbf{u}_i = [f_1(\mathbf{x}), \dots, f_m(\mathbf{x})]_{\mathcal{B}_m}.$$

**Definition 2.2.2** (Partial Derivative).

For  $\{\mathbf{e}_j\}$  the standard orthonormal basis of  $\mathbb{R}^n$ , define

$$\frac{\partial f_i}{\partial x_j} = (D_j f_i)(\mathbf{x}) = \lim_{t \rightarrow 0} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t}.$$

Warning:  $f$  continuous and existence of all  $\frac{\partial f_i}{\partial x_j}$  does not imply differentiability. If  $f$  is differentiable, however, then  $D_f$  is the Jacobian matrix.

**Theorem 2.3** (*Derivative Equals Jacobian*).

If  $f$  is differentiable at  $\mathbf{x}_0$ , then its derivative is an  $m \times n$  matrix, its partial derivatives exist, and

$$\begin{aligned} D_f(\mathbf{x})\mathbf{e}_j &= \sum_{i=1}^m \frac{\partial f_i}{\partial x_j} \mathbf{u}_i \\ &= \left[ \frac{\partial \mathbf{f}}{\partial x_1}, \dots, \frac{\partial \mathbf{f}}{\partial x_n} \right] = \begin{bmatrix} \nabla f_1 & \rightarrow \\ \nabla f_2 & \rightarrow \\ \vdots & \vdots \\ \nabla f_m & \rightarrow \end{bmatrix} = [\nabla f_1^t, \dots, \nabla f_m^t]^t = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}. \end{aligned}$$

**Remark** This implies that

$$D_f(\mathbf{x})\mathbf{h} = \sum_{i=1}^m \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} h_j \mathbf{u}_i.$$

**Theorem 2.4** (*Inverse Function Theorem*).

Suppose  $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$  and  $D_f(\mathbf{a}) \in \text{GL}(n, \mathbb{R})$  for some  $\mathbf{a}$  and  $\mathbf{b} = f(\mathbf{a})$ .

Then there exist  $U \ni \mathbf{a}$  and  $V \ni \mathbf{b}$  such that  $f(U) = V$  and  $f|_U$  is bijective with inverse  $g \in C^1(V)$ .

**Theorem 2.5.**

If  $f \in C^1(\mathbb{R}^n)$  and  $D_f(\mathbf{x}) \in \text{GL}(n, \mathbb{R})$  for all  $\mathbf{x} \in \mathbb{R}^n$ , then  $f$  is an open map (and thus *locally injective*)

**Theorem 2.6** (*Implicit Function Theorem*).

Let  $A : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and suppose  $A_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is invertible.

Then for every  $\mathbf{k} \in \mathbb{R}^m$  there exists a unique  $\mathbf{h} \in \mathbb{R}^n$  such that

$$A(\mathbf{h}, \mathbf{k}) = \mathbf{0} \quad \text{and} \quad \mathbf{h} = -A_x^{-1} A_y \mathbf{k}.$$

### 3 Chapter 1: Smooth Manifolds

**Definition 3.0.1** (Smooth Functions).

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by  $[f_1(\mathbf{x}^n), f_2(\mathbf{x}^n), \dots, f_m(\mathbf{x}^n)]$  (or any subsets thereof) is said to be  $C^\infty$  or **smooth** iff each  $f_i$  has continuous partial derivatives of all orders.

**Definition 3.0.2** (Diffeomorphism).

A smooth bijective map with a smooth inverse is a *diffeomorphism*.

**Remark** A diffeomorphism is necessarily a homeomorphism, but not conversely.

**Definition 3.0.3** (Transition Maps).

If  $(U, \varphi), (V, \psi)$  are two charts on  $M$  such that  $U \cap V \neq \emptyset$ , the composite map  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is a function  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  and is called the *transition map* from  $\varphi$  to  $\psi$ .



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Two charts are *smoothly compatible* iff  $U \cap V = \emptyset$  or  $\psi \circ \varphi^{-1}$  is a diffeomorphism.

**Definition 3.0.4.**

A collection of charts  $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}$  is an *atlas* for  $M$  iff  $\{U_\alpha\} \rightrightarrows M$ , and is a *smooth atlas* iff all of the charts it contains are pairwise smoothly compatible.

**Remark** To show an atlas is smooth, it suffices to show that an arbitrary  $\psi \circ \varphi^{-1}$  is smooth. This is because this immediately implies that its inverse is smooth, and these are diffeomorphisms. Alternatively, one can show that  $\psi \circ \varphi^{-1}$  is smooth, injective, and has nonsingular Jacobian at each point.

**Remark** Attempting to define a function  $f : M \rightarrow \mathbb{R}$  to be smooth iff  $f \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth for each  $\varphi$  may not work because many atlases give the “same” smooth structure in the sense that they all determine the same collection of smooth functions on  $M$ .

For example, take the following two atlases on  $\mathbb{R}^n$ :

$$\begin{aligned}\mathcal{A}_1 &= \{(\mathbb{R}^n, \text{Id}_{\mathbb{R}^n})\} \\ \mathcal{A}_2 &= \left\{ \left( \mathbb{D}_1(\mathbf{x}), \text{id}_{\mathbb{D}_1(\mathbf{x})} \right) \mid \mathbf{x} \in \mathbb{R}^n \right\} .\end{aligned}$$

Claim: a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth wrt either atlas iff it is smooth in the usual sense.

**Definition 3.0.5** (Maximal or Complete Atlas).

A smooth atlas on  $M$  is *maximal* iff it is not properly contained in any larger smooth atlas.

**Remark** Not every topological manifold admits a smooth structure. See Kervaire’s 10-dimensional manifold from 1960.

**Definition 3.0.6** (Smooth Structures and Smooth Manifolds).

If  $M$  is a topological manifold, a maximal smooth atlas  $\mathcal{A}$  is a *smooth structure* on  $M$ . The triple  $(M, \tau, \mathcal{A})$  where  $\mathcal{A}$  is a smooth structure is a *smooth manifold*.

**Remark** To show that two smooth structures are *distinct*, it suffices to show that they are not smoothly compatible, i.e. one of the transition functions  $\psi \circ \varphi^{-1}$  is not smooth. This is because any maximal atlas  $\mathcal{A}_1$  must contain  $\psi$  and likewise  $\mathcal{A}_2$  contains  $\varphi^{-1}$ , but no maximal atlas can contain  $\varphi$  and  $\psi$  because all charts in a maximal atlas are smoothly compatible by definition.

**Proposition 3.1.**

Let  $M$  be a topological manifold.

1. Every smooth atlas  $\mathcal{A}$  for  $M$  is contained in a unique maximal smooth atlas, called the *smooth structure determined by  $\mathcal{A}$* .
2. Two smooth atlases for  $M$  determine the same smooth structure  $\iff$  their union is a smooth atlas.

**Remark** That we can place many requirements on the functions  $\psi \circ \varphi^{-1}$  and get various other structures:  $C^k$ , real-analytic, complex-analytic, etc.  $C^0$  structures recover topological manifolds.

What does “determine the same collection of smooth functions” mean?

**Definition 3.1.1** (Smooth Charts, Maps, Domains).

If  $(M, \tau, \mathcal{A})$  is a smooth manifold, any chart  $(U, \varphi) \in \mathcal{A}$  is a *smooth chart*, where  $U$  is a *smooth coordinate domain* and  $\varphi$  is a *smooth coordinate map*. A *smooth coordinate ball* is a smooth coordinate domain  $U$  such that  $\varphi(U) = \mathbb{D}^n$ .

**Definition 3.1.2** (Regular Coordinate Ball).

A set  $B \subseteq M$  is a *regular coordinate ball* if there is a smooth coordinate ball  $B'$  such that  $\text{cl}_M(B) \subseteq B'$ , and a smooth coordinate map  $\varphi : B' \rightarrow \mathbb{R}^n$  such that for some positive numbers  $r < r'$ ,

- $\varphi(B) = \mathbb{D}_r(\mathbf{0})$ ,
- $\varphi(B') = \mathbb{D}_{r'}(\mathbf{0})$ , and
- $\varphi(\text{cl}_M(B)) = \text{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$ .

This says  $B$  “sits nicely” inside a larger coordinate ball.

**Remark**  $\text{cl}_M(B) \cong_{\text{Top}} \text{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$  which is closed and bounded and thus compact, so  $\text{cl}_M(B)$  is compact. Thus every regular coordinate ball in  $M$  is precompact.

**Proposition 3.2.**

Every smooth manifold has a countable basis of regular coordinate balls.

**Remark** There is only one 0-dimensional smooth manifold, up to equivalence of smooth structures.

**Definition 3.2.1** (Standard Smooth Structure on  $\mathbb{R}^n$ ).

Define the atlas  $\mathcal{A}_0 = \{(\mathbb{R}^n, \text{id}_{\mathbb{R}^n})\}$  and take the smooth structure it generates, this is the *standard smooth structure* on  $\mathbb{R}^n$ .

**Proposition 3.3.**

There are at least two distinct smooth structures on  $\mathbb{R}^n$ .

*Proof.*

Define  $\psi(x) = x^3$ ; then  $\mathcal{A}_1 := \{(\mathbb{R}^n, \varphi)\}$  defines a smooth structure.

Then  $\mathcal{A}_1 \neq \mathcal{A}_0$ , which follows because  $(\text{id}_{\mathbb{R}^n} \circ \varphi^{-1})(x) = x^{\frac{1}{3}}$ , which is not smooth at  $\mathbf{0}$ . ■

## 4 Chapter 1 Problems

### 4.1 Recommended Problems

Note: helpful theorem, two smooth structures induced by two smooth atlases  $\mathcal{A}_1, \mathcal{A}_2$  are equivalent iff  $\mathcal{A}_1 \cup \mathcal{A}_2$  is again a smooth atlas. So it suffices to check pairwise compatibility of charts.

**Exercise (Problem 1.6)** Show that if  $M^n \neq \emptyset$  is a topological manifold of dimension  $n \geq 1$  and  $M$  has a smooth structure, then it has uncountably many distinct ones.

Hint: show that for any  $s > 0$  that  $F_s(x) := |x|^{s-1}x$  defines a homeomorphism  $F_x : \mathbb{D}^n \rightarrow \mathbb{D}^n$  which is a diffeomorphism iff  $s = 1$ .

Recommended problem



Solution:

Define

$$F_s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\mathbf{x} \mapsto \|\mathbf{x}\|^{s-1} \mathbf{x}.$$

Claim:  $F_s$  restricted to  $\mathbb{D}^n$  is a continuous map  $\mathbb{D}^n \longrightarrow \mathbb{D}^n$ .

- Note that if  $\|\mathbf{x}\| \leq \varepsilon < 1$  then

$$\|F_s(\mathbf{x})\| = \|\|\mathbf{x}\|^s \hat{\mathbf{x}}\| = \|\mathbf{x}\|^s \leq \|\mathbf{x}\| \leq \varepsilon < 1,$$

so  $F_s(\mathbb{D}^n) \subseteq \mathbb{D}^n$  and moreover  $F_s(\mathbb{D}_\varepsilon^n) \subseteq \mathbb{D}_\varepsilon^n$ .

- We'll use the fact that  $F_s^{-1} = F_{\frac{1}{s}}$  is of the same form, and thus  $F_s^{-1}(\mathbb{D}^n) \subseteq \mathbb{D}^n$ , forcing  $F_s(\mathbb{D}^n) = \mathbb{D}^n$ .

- This is a continuous function on the punctured disc  $\mathbb{D}_0^n := \mathbb{D}^n \setminus \{\mathbf{0}\}$ , since it can be written as a composition of smooth functions:

$$\mathbb{D}_0^n \xrightarrow{\Delta} \mathbb{D}_0^n \times \mathbb{D}_0^n \xrightarrow{(\|\cdot\|, \text{id}_{\mathbb{D}_0^n})} \mathbb{D}_0^n \times \mathbb{D}_0^n \xrightarrow{((\cdot)^{s-1}, \text{id}_{\mathbb{D}_0^n})} \mathbb{D}_0^1 \times \mathbb{D}_0^n \xrightarrow{(a,b) \mapsto ab} \mathbb{D}_0^n$$

$$\mathbf{x} \longrightarrow (\mathbf{x}, \mathbf{x}) \longrightarrow (\|\mathbf{x}\|, \mathbf{x}) \longrightarrow (\|\mathbf{x}\|^{s-1}, \mathbf{x}) \longrightarrow \|\mathbf{x}\|^{s-1} \mathbf{x}$$

For any  $s \geq 0$ , continuity at zero follows from the fact that  $\|F_s(\mathbf{x})\| \leq \|\mathbf{x}\| \longrightarrow 0$ , so  $\lim_{\mathbf{x} \rightarrow \mathbf{0}} F_s(\mathbf{x}) = \mathbf{0}$  and the sequential definition of continuity applies. So  $F_s$  is continuous on  $\mathbb{D}^n$  for every  $s$ .

Here we are taking for granted the fact that taking norms, exponentiating, and multiplying are all smooth functions away from zero.

Claim:  $F_s$  is a bijection  $\mathbb{D}^n \setminus \mathbf{0} \circlearrowleft$  that extends to a bijection  $\mathbb{D}^n \circlearrowleft$ .

We can note that

$$F_s(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|^s \frac{\mathbf{x}}{\|\mathbf{x}\|} := \|\mathbf{x}\|^s \hat{\mathbf{x}} & \text{if } \|\mathbf{x}\| \neq 0 \\ \mathbf{0} & \text{if } \|\mathbf{x}\| = 0 \end{cases}$$

This follows because we can construct a two-sided inverse that composes to the identity, namely  $F_{\frac{1}{s}}$ , for  $\mathbf{x} \neq \mathbf{0}$ , and note that  $F_s(\mathbf{0}) = \mathbf{0}$ . Using the fact that  $\|t\mathbf{x}\| = t\|\mathbf{x}\|$  for any scalar  $t$ , we can check

that

$$\begin{aligned}
 (F_s \circ F_{\frac{1}{s}})(\mathbf{x}) &= F_s(\|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}}) \\
 &= \left\| \|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}} \right\|^s \cdot \widehat{\left\| \|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}} \right\|^s} \\
 &= \left( \|\mathbf{x}\|^{\frac{1}{s}} \right)^s \cdot \|\widehat{\mathbf{x}}\|^s \cdot \frac{\|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}}}{\left\| \|\mathbf{x}\|^{\frac{1}{s}} \widehat{\mathbf{x}} \right\|} \\
 &= \|\mathbf{x}\| \cdot 1^s \cdot \left( \frac{\|\mathbf{x}\|^{\frac{1}{s}}}{\|\mathbf{x}\|^{\frac{1}{s}}} \right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\
 &= \|\mathbf{x}\| \widehat{\mathbf{x}} \\
 &= \mathbf{x}.
 \end{aligned}$$

and similarly

$$\begin{aligned}
 (F_{\frac{1}{s}} \circ F_s)(\mathbf{x}) &= F_{\frac{1}{s}}(\|\mathbf{x}\|^s \widehat{\mathbf{x}}) \\
 &= \left\| \|\mathbf{x}\|^s \widehat{\mathbf{x}} \right\|^{\frac{1}{s}} \cdot \widehat{\left\| \|\mathbf{x}\|^s \widehat{\mathbf{x}} \right\|^{\frac{1}{s}}} \\
 &= \left( \|\mathbf{x}\|^s \right)^{\frac{1}{s}} \|\widehat{\mathbf{x}}\|^{\frac{1}{s}} \cdot \frac{\|\mathbf{x}\|^s \widehat{\mathbf{x}}}{\left\| \|\mathbf{x}\|^s \widehat{\mathbf{x}} \right\|} \\
 &= \|\mathbf{x}\| \cdot 1^{1-s} \cdot \left( \frac{\|\mathbf{x}\|^s}{\|\mathbf{x}\|^s} \right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\
 &= \|\mathbf{x}\| \widehat{\mathbf{x}} \\
 &= \mathbf{x}.
 \end{aligned}$$

Claim:  $F_s$  is a homeomorphism for all  $s$ .

This follows from the fact that the domain  $\mathbb{D}^n$  is compact and the codomain  $\mathbb{D}^n$  is Hausdorff, and a continuous bijection between such spaces is a homeomorphism.

Claim:  $F_s$  is a diffeomorphism iff  $s = 1$ .

If  $s = 1$ ,  $F_s = \text{id}_{\mathbb{D}^n}$  which is clearly a diffeomorphism.

Otherwise, we claim that  $F_s$  is not a diffeomorphism because either  $F_s$  or  $F_s^{-1}$  will fail to be smooth at  $\mathbf{x} = \mathbf{0}$ .

- If  $0 \leq s < 1$ , then  $F_s$  fails to be differentiable at zero.
- If  $1 < s < \infty$  then  $0 \leq \frac{1}{s} < 1$  and the same argument applies to  $F_s^{-1} := F_{\frac{1}{s}}$ .

Why? Should boil down to  $x \mapsto x^t$  for  $0 \leq t < 1$  failing to be differentiable at 0 in  $\mathbb{R}$

We now show that we can produce infinitely many distinct maximal atlases on  $M$ . Let  $\mathcal{A}$  be any smooth atlas on  $M$  and fix  $p_0 \in M$ .

Claim: We can modify  $\mathcal{A}$  to obtain an atlas  $\mathcal{A}'$  where  $p_0$  is in exactly one chart  $(V, \psi)$  with  $\psi(p_0) = \mathbf{0} \in \mathbb{R}^n$ .

- Pick a chart containing  $p_0$ , say  $(U, \varphi)$  where  $\varphi(p_0) := \mathbf{p}$
- Since  $\varphi(U) \subseteq \mathbb{R}^n$  is open, find a disc containing  $\mathbf{p}$ , say  $\mathbb{D}_R(\mathbf{p}) \subset \varphi(U)$ .
- Define  $V \subseteq M$  as  $V := \varphi^{-1}(\mathbb{D}_R(\mathbf{p}))$ .

- Define  $\psi : U \longrightarrow \mathbb{R}^n$  by

$$\begin{aligned}\psi : U &\longrightarrow \mathbb{R}^n \\ x &\mapsto \frac{\varphi(x) - \varphi(p_0)}{R}.\end{aligned}$$

- Note: this is constructed precisely so that  $\psi(V) = \mathbb{D}_1(\mathbf{0}) \in \mathbb{R}^n$  and  $\psi(p) = \mathbf{0}$ .
- This is a homeomorphism onto its image since we can write

$$\psi = \delta_{\frac{1}{R}} \circ \tau_{\mathbf{p}} \circ \varphi$$

is a composition of continuous functions, where  $\delta, \tau$  are dilations/translations in  $\mathbb{R}^n$  which are known to be continuous, and

$$\psi^{-1} = \varphi^{-1} \circ \tau_{-\mathbf{p}} \circ \delta_R$$

is again a composition of smooth (and in particular, continuous) functions.

- Define  $\mathcal{A}^1 := \mathcal{A} \cup \{(V, \psi|_V)\}$ 
  - This is a smooth atlas: any pair of charts coming from  $\mathcal{A}$  are smoothly compatible, so it suffices to check that an arbitrary chart from  $\mathcal{A}$  is smoothly compatible with the new chart.
  - Let  $(T, \xi)$  be any other chart, then if  $T \cap V \neq \emptyset$ , the transition function

$$\psi \circ \xi^{-1} = \delta_{\frac{1}{R}} \tau_{\mathbf{p}} \circ \varphi \circ \xi^{-1}$$

is a composition of smooth functions and thus smooth, and similarly for  $\xi \circ \psi^{-1}$ .

- Since the charts from  $\mathcal{A}$  cover  $M$ , so do the charts of  $\mathcal{A}^1$  since  $\mathcal{A} \subseteq \mathcal{A}^1$ .
- For every  $(U_\alpha, \varphi_\alpha) \in \mathcal{A}^1$ , define a new chart  $(U_\alpha \setminus \{p\}, \varphi_\alpha|_{U_\alpha \setminus \{p\}})$  and define this set of charts as  $\mathcal{A}^2$ .
  - This still covers  $M$ :  $p$  is in the chart  $(V, \psi|_V)$ , and if  $q \neq p$ , then  $q \in U_\alpha$  for some  $\alpha$  since  $\mathcal{A}$  was an atlas, and  $q \in U_\alpha \setminus \{p\}$ .
  - The coordinate maps are still homeomorphisms onto their images, because the restriction of a homeomorphism is again a homeomorphism.
  - The transition functions are still smooth because the restriction of a smooth function is again smooth.

Claim: We can define a new atlas  $\mathcal{A}_s$  from  $\mathcal{A}^2$  by only replacing the single chart  $(V, \psi)$  with  $(V, F_s \circ \psi)$ .

- $\mathcal{A}_s$  still covers  $M$ , since we haven't changed the coordinate domains
- All coordinate functions are still a homeomorphisms onto their images, since the only change is  $\psi$  is replaced with  $F_s \circ \psi$  and we've shown that  $F_s$  is a homeomorphism; a composition of homeomorphisms is again a homeomorphism.
- The chart  $(V, F_s \circ \psi)$  is still a valid chart, since  $F_s : \mathbb{D}_n \hookrightarrow \mathbb{R}^n$  and  $\psi(V) \cong \mathbb{D}^n$  by construction.
- All charts in  $\mathcal{A}_s$  are still smoothly compatible:
  - It suffices to check compatibility between an arbitrary  $(U_\alpha, \varphi_\alpha)$  and  $(V, F_s \circ \psi)$ , so we consider  $F_s \circ \psi \circ \varphi_\alpha^{-1}$
  - By construction,  $p \notin U_\alpha$ , and we know  $F_s$  is smooth away from  $\mathbf{0}$ , so this is a smooth function.



2. Show that  $\sigma$  is bijective and

$$\sigma^{-1}(\mathbf{u}) = \sigma^{-1}\left([u^1, \dots, u^n]\right) = \frac{1}{\|\mathbf{u}\|^2 + 1} [2u^1, \dots, 2u^n, \|\mathbf{u}\|^2 - 1].$$

3. Compute the transition map  $\tilde{\sigma} \circ \sigma^{-1}$  and verify that the atlas

$$\mathcal{A} := \{(S^n \setminus N, \sigma), (S^n \setminus S, \tilde{\sigma})\}$$

define a smooth structure on  $S^n$ .

4. Show that this smooth structure is equivalent to the standard smooth structure: Put graph coordinates on  $S^n$  as outlined in 5.2 to obtain  $\{(U_i^\pm, \varphi_i^\pm)\}$ .

For indices  $i < j$ , show that

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1} [u^1, \dots, u^n] = [u^1, \dots, \hat{u}^i, \dots, \pm \sqrt{1 - \|\mathbf{u}\|^2}, \dots, u^n]$$

where the square root appears in the  $j$ th position. Find a similar formula for  $i > j$ . Show that if  $i = j$ , then

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1} = \varphi_i^- \circ (\varphi_i^+)^{-1} = \text{id}_{\mathbb{D}^n}.$$

Show that these yield a smooth atlas.

Solution (1):

- Parameterize the line through  $\mathbf{x} \in S^n$  and  $\mathbf{N}$ :

$$\begin{aligned} \ell_{N,\mathbf{x}}(t) &= t\mathbf{x} + (1-t)\mathbf{N} \\ &= t[x^1, \dots, x^n, x^{n+1}] + (1-t)[0, \dots, 1] \\ &= [tx^1, \dots, x^n, tx^{n+1} + (1-t)] \\ &= [tx^1, \dots, x^n, 1 - t(1 - x^{n+1})] \end{aligned}$$

- Evaluate at  $t = \frac{1}{1 - x^{n+1}}$  to obtain  $\frac{1}{x^{n+1}} [x^1, \dots, x^n, 0] = [\sigma(\mathbf{x}), 0]$ .
- For  $\tilde{\sigma}(\mathbf{x})$ : Todo .

Todo

Solution (2):

- How to derive this formula: no clue.
  - Start with  $\mathbf{u} \in \mathbb{R}^n$ , parameterize the line  $\ell_{N,\mathbf{u}}(t)$ , solve for where  $\|\ell_{N,\mathbf{u}}(t)\| = 1$  and  $\mathbf{u} \neq \mathbf{N}$
  - Should yield  $t^2\|\mathbf{u}\|^2 + (1-t)^2 = 1$ , solve for nonzero  $t$ ; should get  $t = \frac{2}{\|\mathbf{u}\| + 1}$ , so
- Compute compositions  $\sigma \circ \sigma^{-1}$ : Todo.

Figure out how to invert.

Messy computations that didn't work out.

Solution (3):

- Computing the transition maps:

$$\begin{aligned}
 (\tilde{\sigma} \circ \sigma^{-1})(\mathbf{u}) &= -\sigma\left(\left(\frac{-1}{\|\mathbf{u}\|^2 + 1}\right)[2u^1, \dots, 2u^n, \|\mathbf{u}\|^2 - 1]\right) \\
 &= -1 \cdot \left[\frac{\frac{-2u^1}{\|\mathbf{u}\|^2 + 1}}{1 - \frac{1 - \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2}}, \dots, n\right] \\
 &= \left[\frac{2u^1}{\|\mathbf{u}\|^2 + 1} \cdot \frac{1 + \|\mathbf{u}\|^2}{1 + \|\mathbf{u}\|^2 - (1 - \|\mathbf{u}\|^2)}, \dots, n\right] \\
 &= \left[\frac{2u^1}{2\|\mathbf{u}\|^2}, \dots, n\right] \\
 &= \frac{\mathbf{u}}{\|\mathbf{u}\|^2} \\
 &:= \hat{\mathbf{u}},
 \end{aligned}$$

which is a smooth function on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ .

- Todo: computing  $(\sigma \circ \tilde{\sigma}^{-1})(\mathbf{u}) = \hat{\mathbf{u}}$
- Todo: argue that it suffices that these are smooth on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$

Computation.

Solution (4):

We want to argue that these define the same maximal smooth atlas, for which it suffices to the charts from each are pairwise smoothly compatible.

- Define  $\varphi_i([x^1, \dots, x^n]) = [x^1, \dots, \hat{x}^i, \dots, x^n]$  and  $\varphi_i^{-1}([x^1, \dots, x^{n-1}]) = [x^1, \dots, \sqrt{1 - \|\mathbf{x}\|^2}, \dots, x^{n-1}]$ .
- Compute  $(\varphi_i \circ \sigma^{-1})(\mathbf{u}) = \frac{1}{\|\mathbf{u}\| + 1}[2u^1, \dots, \hat{u}^i, \dots, 2u^n, \|\mathbf{u}\|^2 - 1]$ , which is (clearly) smooth?
- Compute  $(\sigma \circ \varphi_i^{-1})(\mathbf{u}) = \sigma([u^1, \dots, \sqrt{1 - \|\mathbf{u}\|^2}, \dots, u^n])$ , which is  $\frac{1}{1 - u^n}[u^1, \dots, \sqrt{1 - \|\mathbf{u}\|^2}, \dots, u^{n-1}]$ .
  - This is smooth if  $u^n \neq 1$ , but this corresponds to  $\mathbf{N}$  in  $S^2$ , in which case  $\varphi_i^{-1}(\mathbf{u})$  isn't in the domain of  $\sigma$  to begin with.

What are the actual domains and ranges of the transition functions? It seems like you pull back  $\mathbb{R}^n$  to  $S^n \setminus N$ , then push  $S^n \setminus \{N, S\}$  to  $\mathbb{R}^n \setminus 0$ , but this yields  $\mathbb{R}^n \rightarrow \mathbb{R}^n \setminus 0$  where we haven't deleted zero in the domain (problem: not smooth!).

**Exercise (Problem 1.8)** Define an *angle function* on  $U \subset S^1$  as any continuous function  $\theta : U \rightarrow \mathbb{R}$  such that  $e^{i\theta(z)} = z$  for all  $z \in U$ .

Show that  $U$  admits an angle function iff  $U \neq S^1$ , and for any such function  $\theta$ ,  $(U, \theta)$  is a smooth coordinate chart for  $S^1$  with its standard smooth structure.

Note that  $f : \mathbb{R} \rightarrow S^1$  given by  $f(x) = e^{ix}$  is a covering map (in fact the universal cover).

Some way to do this just with covering spaces?

$\Rightarrow$  :

- Suppose there exists an angle function  $\theta : U \rightarrow \mathbb{R}$ .
- Then  $f \circ \theta|_U = \text{id}_U$  by assumption, since  $u \xrightarrow{\theta|_U} \theta(u) \xrightarrow{f} e^{i\theta(u)} = u$ .
- So  $\theta$  has a left-inverse and is thus injective.
- Suppose  $U = S^1$ , which is compact.
- Then  $\theta$  is an injective continuous map on a compact set, so its image  $\theta(S^1) \subseteq \mathbb{R}$  is compact.

- Lemma: a continuous map from a compact space to a Hausdorff space is a closed map.
- Since  $\theta$  is injective and is surjective onto its image, since it is continuous it is a homeomorphism onto its image and  $S^1 \cong \theta(S^1)$ .
- Since  $S^1$  is connected,  $\theta(S^1)$  is connected, and the only connected subsets of  $\mathbb{R}$  are intervals.
- Since  $\theta(S^1)$  is compact, it must be a closed and bounded subset, so  $\theta(S^1) = [a, b] \subset \mathbb{R}$ .
- But this forces  $S^1 \cong [a, b]$  is a homeomorphism, which is a contradiction: removing one point from  $S^1$  yields one connected component, while removing  $\frac{1}{2}(b - a)$  from  $[a, b]$  produces a disconnected set.

$\Leftarrow$  :

- Suppose  $U \neq S^1$ , then there exists a point  $p \in S^1 \setminus U$ ; wlog suppose  $p = 1$ .
- Then  $U \subseteq S^1 \setminus \{1\}$
- Note that  $f^{-1}(\{1\}) = \{2k\pi \mid k \in \mathbb{Z}\}$ .
- Take the interval  $I = [0, 2\pi]$  and set  $\tilde{f} = f|_I$ .
- Since  $U \neq S^1$ ,  $\tilde{f}^{-1}(U) \subsetneq I$ .
- Then  $\tilde{f}$  restricted to  $\tilde{f}^{-1}(U)$  is injective, since  $\tilde{f}$  only fails injectivity at  $0, 2\pi$ .
- Then the restricted map  $\hat{f} := \tilde{f}|_{\tilde{f}^{-1}(U)} : \tilde{f}^{-1}(U) \rightarrow U$  is a continuous injection and surjects onto its image, thus a bijection
- Claim:  $\hat{f}$  is a homeomorphism
  - Define a candidate inverse  $\theta = \hat{f}^{-1} : S^1 \rightarrow \mathbb{R}$ .
  - Then  $f \circ \theta = \text{id}_{S^1}$  implies  $e^{i\theta(x)} = x$  for all  $x \in U$ .
  - Letting  $V \subseteq \tilde{f}^{-1}(U)$  be open, we have  $\theta^{-1}(V) = \hat{f}(V)$  which (claim?) is open since ???
  - So  $\theta$  is continuous.

Alternatively:

- Take  $I = (0, 2\pi)$ .
- Then  $\tilde{f}(I) = S^1 \setminus \{1\}$ , so  $U \subseteq \tilde{f}(I)$ .
- Claim:  $f : S^1 \setminus \{1\} \rightarrow I$  is a homeomorphism.
- Set  $\theta(x) = \tilde{f}|_I^{-1}U(x)$ ; the claim is that this works.
  - Taking a branch cut  $\{x + iy \mid x \in [0, \infty), y = 0\}$  for the complex logarithm defines an inverse.

How to prove?

$(U, \theta)$  is a smooth coordinate chart:

- Let  $\theta$  be arbitrary with  $e^{i\theta(z)} = z$  and  $\theta \subsetneq S^1$ .
- $U \subseteq S^1$  is open by assumption.
- We need to show that  $\theta : U \rightarrow \varphi(U)$  is a homeomorphism

**Exercise (Problem 1.9)** Show that  $\mathbb{CP}^n$  is a compact  $2n$ -dimensional topological manifold, and show how to equip it with a smooth structure, using the correspondence

Recommended problem

$$\mathbb{R}^{2n} \iff \mathbb{C}^n$$

$$[x^1, y^1, \dots, x^n, y^n] \iff [x^1 + iy^1, \dots, x^n + iy^n].$$

## 5 Chapter 1: Point-Set Properties of Topological Manifolds

Pages 1- 29.

## 5.1 Notes

**Definition 5.0.1** (Topological Manifold).

A topological space  $M$  that satisfies

1.  $M$  is Hausdorff, i.e. points can be separated by open sets
2.  $M$  is second-countable, i.e. has a countable basis
3.  $M$  is locally Euclidean, i.e. every point has a neighborhood homeomorphic to an open subset  $\hat{U}$  of  $\mathbb{R}^n$  for some fixed  $n$ .

The last property says  $p \in M \implies \exists U$  with  $p \in U \subseteq M$ ,  $\hat{U} \subseteq \mathbb{R}^n$ , and a homeomorphism  $\varphi : U \longrightarrow \hat{U}$ .

Note that second countability is primarily needed for existence of partitions of unity.

**Exercise** Show that in the last condition,  $\hat{U}$  can equivalently be required to be an open ball or  $\mathbb{R}^n$  itself.

**Theorem 5.1** (*Topological Invariance of Dimension*).

Two nonempty topological manifolds of different dimensions can not be homeomorphic.

**Exercise** Show that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique.

**Exercise** Show that subspaces and finite products of Hausdorff (resp. second countable) spaces are again Hausdorff (resp. second countable).

Thus any open subset of a topological manifold with the subspace topology is again a topological manifold.

**Exercise** Give an example of a connected, locally Euclidean Hausdorff space that is not second countable.

**Definition 5.1.1** (Charts).

A chart on  $M$  is a pair  $(U, \varphi)$  where  $U \subseteq M$  is open and  $\varphi : U \longrightarrow \hat{U}$  is a homeomorphism from  $U$  to  $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$ . If  $p \in M$  and  $\varphi(p) = 0 \in \bar{\hat{U}}$ , then the chart is said to be *centered* at  $p$ . Note that any chart about  $p$  can be modified to a chart  $(\varphi_1, \hat{U}_1)$  that is centered at  $p$  by defining  $\varphi_1(x) = x - \varphi(p)$ .





**Fig. 1.2** A coordinate chart

$U$  is the *coordinate domain* and  $\varphi$  is the *coordinate map*.

Note that we can write  $\varphi$  in components as  $\varphi(p) = [x^1(p), \dots, x^n(p)]$  where each  $x^i$  is a map  $x^i : U \rightarrow \mathbb{R}$ . The component functions  $x^i$  are the *local coordinates* on  $U$ .

Shorthand notation:  $[x^i] := [x^1, \dots, x^n]$ .

**Example 5.1** (Graphs of Continuous Functions).

Define

$$\Gamma(f) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U, y = f(x) \in \hat{U} \right\}.$$

This is a topological manifold since we can take  $\varphi : \Gamma(f) \rightarrow U$  by restricting  $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  to the subspace  $\Gamma(f)$ . Projections are continuous, restrictions of continuous functions are continuous.

This is a homeomorphism because the map  $g : x \mapsto (x, f(x))$  is continuous and  $g \circ \pi_1 = \text{id}_{\mathbb{R}^n}$  is continuous with  $\pi_1 \circ g = \text{id}_{\Gamma(f)}$ . Note that  $U \cong \Gamma(f)$ , and thus  $(U, \varphi) = (\Gamma(f), \varphi)$  is a single *global* coordinate chart, called the *graph coordinates* of  $f$ .

Thus graphs of continuous functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$  are locally Euclidean?

Note that this works in greater generality:: “The same observation applies to any subset of  $\mathbb{R}^{n+k}$  by setting *any*  $k$  of the coordinates equal to some continuous function of the other  $n$ .”

Coordinates as numbers vs functions?

**Example 5.2** (Spheres).

$S^n$  is a subspace of  $\mathbb{R}^{n+1}$  and is thus Hausdorff and second-countable by exercise 5.1.



**Fig. 1.3** Charts for  $\mathbb{S}^n$

To see that it's locally Euclidean, take

$$U_i^+ := \left\{ [x^1, \dots, x^n] \in \mathbb{R}^{n+1} \mid x^i > 0 \right\} \quad \text{for } 1 \leq i \leq n+1$$

$$U_i^- := \left\{ [x^1, \dots, x^n] \in \mathbb{R}^{n+1} \mid x^i < 0 \right\} \quad \text{for } 1 \leq i \leq n+1.$$

Define

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^{\geq 0}$$

$$\mathbf{x} \mapsto \sqrt{1 - \|\mathbf{x}\|^2}.$$

Note that we immediately need to restrict the domain to  $\mathbb{D}^n \subset \mathbb{R}^n$ , where  $\|x\|^2 \leq 1 \implies 1 - \|x\|^2 \geq 0$ , to have a well-defined real function  $f : \mathbb{D}^n \longrightarrow \mathbb{R}^{\geq 0}$ .

Then (claim)

$$\begin{aligned} U_i^+ \cap S^n & \text{ is the graph of } x^i = f(x^1, \dots, \hat{x}^i, \dots, x^{n+1}) \\ U_i^- \cap S^n & \text{ is the graph of } x^i = -f(x^1, \dots, \hat{x}^i, \dots, x^{n+1}). \end{aligned}$$

This is because

$$\begin{aligned} \Gamma(x^i) &:= \{(\mathbf{x}, f(\mathbf{x})) \subseteq \mathbb{R}^n \times \mathbb{R}\} \\ &= \left\{ [x_1, \dots, \hat{x}^i, \dots, x^{n+1}], f([x_1, \dots, \hat{x}^i, \dots, x^{n+1}]) \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \\ &= \left\{ [x_1, \dots, \hat{x}^i, \dots, x^{n+1}], \left( 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right)^{\frac{1}{2}} \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \end{aligned}$$

and any vector in this set has norm satisfying

$$\|(\mathbf{x}, y)\|^2 = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 + \left( 1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right) = 1$$

and is thus in  $S^n$ .

To see that any such point also has positive  $i$  coordinate and is thus in  $U_i^+$ , we can rearrange (?) coordinates to put the value of  $f$  in the  $i$ th coordinate to obtain

$$\Gamma(x_i) = \left\{ [x^1, \dots, f(x^1, \dots, \hat{x}^i, \dots, x^n), \dots, x^n] \right\}$$

Seems like  $f$  is always the \*last\* coordinate in the graph

and note that the square root only takes on positive values.

Thus each  $U_i^\pm \cap S^n$  is the graph of a continuous function and thus locally Euclidean, and we can define chart maps

$$\begin{aligned} \varphi_i^\pm : U_i^\pm \cap S^n &\longrightarrow \mathbb{D}^n \\ [x^1, \dots, x^n] &\mapsto [x^1, \dots, \hat{x}^i, \dots, x^{n+1}] \end{aligned}$$

yield  $2(n+1)$  charts that are graph coordinates for  $S^n$ .

### Example 5.3 (Projective Space).

Define  $\mathbb{RP}^n$  as the space of 1-dimensional subspaces of  $\mathbb{R}^{n+1}$  with the quotient topology determined by the map

$$\begin{aligned} \pi : \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{RP}^n \\ \mathbf{x} &\mapsto \text{span}_{\mathbb{R}} \{\mathbf{x}\}. \end{aligned}$$

How is this map a quotient map?

Notation: for  $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\}$  write  $[\mathbf{x}] := \pi(\mathbf{x})$ , the line spanned by  $\mathbf{x}$ .

Define charts:

$$\tilde{U}_i := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\} \mid x^i \neq 0 \right\}, \quad U_i = \pi(\tilde{U}_i) \subseteq \mathbb{RP}^n$$

and chart maps

$$\begin{aligned} \tilde{\varphi}_i : \tilde{U}_i &\longrightarrow \mathbb{R}^n \\ [x^1, \dots, x^{n+1}] &\mapsto \left[ \frac{x^1}{x^i}, \dots, \widehat{x^i}, \dots, \frac{x^{n+1}}{x^i} \right]. \end{aligned}$$

Then (claim) this descends to a continuous map  $\varphi_i : U_i \longrightarrow \mathbb{R}^n$  by the universal property of the quotient:

$$\begin{array}{ccc} \tilde{U}_i & & \\ \pi_U \downarrow & \searrow \tilde{\varphi}_i & \\ U_i & \xrightarrow{\varphi_i} & \mathbb{R}^n \end{array}$$

- The restriction  $\pi_U : \tilde{U}_i \longrightarrow U_i$  of  $\pi$  is still a quotient map because  $\tilde{U}_i = \pi_U^{-1}(U_i)$  where  $U_i \subseteq \mathbb{RP}^n$  is open in the quotient topology and thus  $\tilde{U}_i$  is saturated.

Thus  $\pi_U$  sends saturated sets to open sets and is thus a quotient map.

- $\tilde{\varphi}_i$  is constant on preimages under  $\pi_U$ : fix  $y \in U_i$ , then  $\pi_U^{-1}(\{y\}) = \{\lambda \mathbf{y} \mid \lambda \in \mathbb{R} \setminus \{0\}\}$ , i.e. the point  $y \in \mathbb{RP}^n$  pulls back to every nonzero point on the line spanned by  $\mathbf{y} \in \mathbb{R}^n$ .

But

$$\begin{aligned} \tilde{\varphi}_i(\lambda \mathbf{y}) &= \varphi_i([\lambda y^1, \dots, \lambda y^i, \dots, \lambda y^n]) \\ &= \left[ \frac{\lambda y^1}{\lambda y^i}, \dots, \widehat{\lambda y^i}, \dots, \frac{\lambda y^{n+1}}{\lambda y^i} \right] \\ &= \left[ \frac{y^1}{y^i}, \dots, \widehat{y^i}, \dots, \frac{y^{n+1}}{y^i} \right] \\ &= \tilde{\varphi}_i(\mathbf{y}). \end{aligned}$$

So this yields a continuous map

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n.$$

We can now verify that  $\varphi$  is a homeomorphism since it has a continuous inverse given by

$$\begin{aligned} \varphi_i^{-1} : \mathbb{R}^n &\longrightarrow U_i \subseteq \mathbb{RP}^n \\ \mathbf{u} := [u^1, \dots, u^n] &\mapsto [u^1, \dots, u^{i-1}, \mathbf{1}, u^{i+1}, \dots, u^n]. \end{aligned}$$

It remains to check:

Exercise

1. The  $n + 1$  sets  $U_1, \dots, U_{n+1}$  cover  $\mathbb{RP}^n$ .
2.  $\mathbb{RP}^n$  is Hausdorff
3.  $\mathbb{RP}^n$  is second-countable.

**Exercise (1.6)** Show that  $\mathbb{RP}^n$  is Hausdorff and second countable.

**Exercise (1.7)** Show that  $\mathbb{RP}^n$  is compact. (Hint: show that  $\pi$  restricted to  $S^n$  is surjective.)

**Definition 5.1.2** (Topological Embedding).

A continuous map  $f : X \rightarrow Y$  is a *topological embedding* iff it is injective and  $\tilde{f} : X \rightarrow f(X)$  is a homeomorphism.

**Example 5.4** (Product Manifolds).

Let  $M := M_1 \times \dots \times M_k$  be a product of manifolds of dimensions  $n_1, \dots, n_k$  respectively. A product of Hausdorff/second-countable spaces is still Hausdorff/second-countable, so just need to check that it's locally Euclidean.

- Let  $\mathbf{p} \in \prod_{i=1}^N M_i$ , so  $p_i \in M_i$
- Choose a chart  $(U_i, \varphi_i)$  with  $p_i \in U_i$  and assemble a product map:

$$\Phi := \prod \varphi_i : \prod U_i \rightarrow \prod \mathbb{R}^{n_i} \cong \mathbb{R}^{\sum n_i} := \mathbb{R}^N.$$

- Claim:  $\Phi$  is a homeomorphism onto its image in  $\mathbb{R}^N$ .
  - Each  $\varphi_i$  is a homeomorphism onto  $\varphi_i(U_i)$  (by the definition of a chart on  $M_i$ )
  - It suffices to show that  $\Phi^{-1}$  exists and is continuous, where

$$\Phi^{-1}(V) := \left( \prod \varphi_i \right)^{-1} \left( \prod V_i \right).$$

- $\Phi$  is a product of continuous functions and thus continuous.
- $\Phi^{-1} := \left( \prod \varphi_i \right)^{-1} = \prod \varphi_i^{-1}$ , which are all assumed continuous since  $\varphi_i$  were homeomorphisms.

**Example 5.5** (Torii).

$T^n := \prod_{i=1}^n S^1$  is a topological  $n$ -manifold.

**Definition 5.1.3** (Precompact).

A subset  $A \subseteq X$  is *precompact* iff its closure  $\text{cl}_X(A)$  is compact in  $X$ .

**Proposition 5.2.**

Every topological manifold has a countable basis of precompact coordinate balls.

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**Proposition 5.3.**

Let  $M$  be a topological manifold.

- $M$  is locally path-connected.
- $M$  is connected  $\iff M$  is path-connected
- The connected components and path components of  $M$  coincide.
- $\pi_0(M)$  is countable and each component is open and a connected topological manifold.

**Proposition 5.4.**

Every topological manifold  $M$  is locally compact.

*Proof .*

$M$  has a basis of precompact open sets. ■

**Theorem 5.5 (Manifolds are Paracompact).**

Given any open cover  $\mathcal{U} \rightrightarrows M$  of a topological manifold and any basis  $\mathcal{B}$  for the topology on  $M$ , there exists a countable locally finite open refinement of  $\mathcal{U}$  consisting of elements of  $\mathcal{B}$ .

**Proposition 5.6.**

$\pi_1(M)$  is countable.

## 6 Chapter 2

**Definition 6.0.1** (Smooth Functionals on Manifolds).

A function  $f : M^n \rightarrow \mathbb{R}^k$  is *smooth* iff for every  $p \in M$  there exists a smooth chart  $(U, \varphi)$  about  $p$  such that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$  is smooth as a real function.

Fact:  $C^\infty(M) := \{f : M \rightarrow \mathbb{R}\}$  is a vector space

**Definition 6.0.2** (Coordinate Representations of Functions).

Given a function  $f : M \rightarrow \mathbb{R}^k$ , the function  $\hat{f} : \varphi(U) \rightarrow \mathbb{R}^k$  where  $\hat{f}(x) = (f \circ \varphi^{-1})(x)$  is a *coordinate representation* of  $f$ .

Fact:  $f$  is smooth  $\iff f$  is smooth (in the above sense) in *some* smooth chart about each point.

**Definition 6.0.3** (Smooth Maps Between Manifolds).

$F : M \rightarrow N$  is *smooth* iff for every  $p \in M$  there exists charts  $p \in (U, \varphi)$  and  $F(p) \in (V, \psi)$  such that  $F(U) \subseteq V$  and  $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is smooth.

Fact: taking  $N = V = \mathbb{R}^k$  and  $\psi = \text{id}$  recovers the previous definition.

**Proposition 6.1.**

Every smooth map between manifolds is continuous.

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**Proposition 6.2 (Smoothness is Local).**

If  $F : M \rightarrow N$ , then

1. If every  $p \in M$  has a neighborhood  $U \ni p$  such that  $F$  restricted to  $U$  is smooth, then  $F$  is smooth.
2. If  $F$  is smooth, then its restriction to every open subset is smooth.

**Definition 6.2.1.**

For  $F : M \rightarrow N$  and  $(U, \varphi), (V, \psi)$  smooth charts in  $M, N$  respectively, then  $\hat{F} := \psi \circ F \circ \varphi^{-1}$  is the *coordinate representation* of  $F$ .

**Proposition 6.3.**

1. Constant maps  $c : M \rightarrow N, c(x) = n_0$  are smooth
2. The identity is smooth
3. Inclusion of open submanifolds  $U \hookrightarrow M$  is smooth
4.  $F : M \rightarrow N$  and  $G : N \rightarrow P$  smooth implies  $G \circ F$  is smooth.

**Proposition 6.4.**

A map  $F : N \rightarrow \prod_{i=1}^k M_i$  with at most one  $i$  such that  $\partial M_i \neq \emptyset$  is smooth iff each component map  $\pi_i \circ F : N \rightarrow M_i$  is smooth.

Proving a map between manifolds is smooth:

1. Write the map as a composition of known smooth functions.
2. Write in *smooth local coordinates* and recognize the component functions as compositions of smooth functions

Fact: projection maps from products are smooth

- Every closed subset  $A \subseteq M$  of a smooth manifold is the level set of some smooth nonnegative functional  $f : M \rightarrow \mathbb{R}$ , i.e.  $f^{-1}(0) = A$ .

## 7 Chapter 3

**Definition 7.0.1.**

For a fixed point  $\mathbf{a} \in \mathbb{R}^n$ , define the *geometric tangent space* at  $\mathbf{a}$  to be the set

$$\mathbb{R}_{\mathbf{a}}^n := \{\mathbf{a}\} \times \mathbb{R}^n = \{(\mathbf{a}, \mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^n\}.$$

Notation:  $\mathbf{v}_a$  denotes the tangent vector at  $\mathbf{v}$ , i.e. the pair  $(\mathbf{a}, \mathbf{v})$ . Think of this as a vector with its base at the point  $\mathbf{a}$ .

**Remark** There is a natural isomorphism  $\mathbb{R}_{\mathbf{a}}^n \cong \mathbb{R}^n$  given by  $(\mathbf{a}, \mathbf{v}) \mapsto \mathbf{v}$ .

This map is not explicitly stated.

**Proposition 7.1.**

$D_v|_a$  satisfies the product rule:

$$D_v|_a(fg) = f(a) \cdot D_v|_a g + D_v|_a f \cdot g(a).$$

Picking the standard basis for  $\mathbb{R}^n = \{\mathbf{e}_{i,a}\}_{i=1}^n$  and expanding  $\mathbf{v} = \sum_{i=1}^n v^i \mathbf{e}_{i,a}$ , we can explicitly write

$$D_v|_a f = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(a).$$

**Definition 7.1.1.**

Denote the space of all derivations of  $C^\infty(\mathbb{R}^n)$  at  $a$  as

$$T_a \mathbb{R}^n := \left\{ w \in \text{hom}_{\mathbb{R}\text{-mod}}(C^\infty(\mathbb{R}^n), \mathbb{R}) \mid w(fg) = f(a)w(g) + g(a)w(f) \right\},$$

i.e. a derivation  $w$  is an  $\mathbb{R}$ -linear map satisfying the Leibniz Rule (LR).

Facts:

1. If  $f$  is a constant function then  $v(f) = 0 \in \mathbb{R}$ .
2. If  $f(p) = g(p)$  for  $p \in M$  then  $v(fg) = 0 \in \mathbb{R}$ .

**Example 7.1.**

Claim: if  $f \in C^\infty(\mathbb{R}^n)$  is constant, say  $f(\mathbf{p}) = 1$  for all  $\mathbf{p} \in \mathbb{R}^n$ , then  $w(f) = 0$  for any derivation  $w$ .

Proof: WLOG suppose  $f(\mathbf{p}) = 1 \in \mathbb{R}$ . Note that  $f(\mathbf{p}) = f(\mathbf{p}) \cdot f(\mathbf{p})$ , so

$$w(f) = w(f \cdot f) \stackrel{LR}{=} f(\mathbf{p})w(f) + w(f)f(\mathbf{p}) = 2f(\mathbf{p})w(f) = 2w(f) \quad \text{since } f(\mathbf{p}) = 1,$$

and thus  $w(f) = 2w(f) \in \mathbb{R}$  forcing  $w(f) = 0$ .

**Remark** A geometric tangent vector provides a way of taking directional derivatives via the correspondence

$$\begin{aligned} \mathbb{R}^n &\longrightarrow C^\infty(\mathbb{R}^n)^\vee \\ \mathbf{v}_a &\mapsto D_{\mathbf{v}}|_a \end{aligned}$$

where

$$\begin{aligned} D_{\mathbf{v}}|_a : C^\infty(\mathbb{R}^n) &\longrightarrow \mathbb{R} \\ f &\mapsto D_{\mathbf{v}}f(\mathbf{a}) := \left. \frac{\partial}{\partial t} \right|_{t=0} f(\mathbf{a} + t\mathbf{v}). \end{aligned}$$

More precisely,

What does this equality mean? Is  $w(fg)$  a real number? Does  $wg = w(g)$ , so this is a number too?



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**Proposition 7.2 (Space of Derivations is Isomorphic to Geometric Tangent Space).**

For each geometric tangent vector  $\mathbf{v}_a \in \mathbb{R}^n$ , the map  $D_{\mathbf{v}}|_a$  is a derivation at  $a$ , and the map  $\mathbf{v}_a \mapsto D_{\mathbf{v}}|_a$  is an isomorphism.

**Theorem 7.3 (Basis of Tangent Space).**

For any  $\mathbf{p} \in \mathbb{R}^n$ , there is a basis for  $T_{\mathbf{p}}\mathbb{R}^n$  given by  $\left\{ \frac{\partial}{\partial x^i} \Big|_{\mathbf{p}} \mid 1 \leq i \leq n \right\} \subset C^\infty(\mathbb{R}^n, \mathbb{R})$  which are defined as

$$\begin{aligned} \frac{\partial}{\partial x^i} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x^i}(\mathbf{p}). \end{aligned}$$

**Definition 7.3.1 (Tangent Vector on a Manifold).**

Let  $M$  be smooth and  $p \in M$ , then

$$T_p M := \left\{ v : C^\infty(M) \longrightarrow \mathbb{R} \mid v(fg) = f(p)vg + g(p)vf \right\}.$$

**Definition 7.3.2 (Differential of a Map).**

For  $F : M \longrightarrow N$  a smooth map, for each  $p \in M$ , we define the *differential* of  $f$  at  $p$  as

$$\begin{aligned} dF_p : T_p M &\longrightarrow T_{F(p)} N \\ v &\mapsto (DF_p(v) : f \mapsto v(f \circ F)). \end{aligned}$$

Note that  $f \in C^\infty(N)$  implies that  $f \circ F \in C^\infty(M)$ , and since  $v \in T_p M$  is a functional in  $C^\infty(M)^\vee$ ,  $v$  can act on such objects.. Moreover,  $dF_p(v)$  is in fact a derivation at  $F(p)$ , since

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F) \cdot (g \circ F)) && \text{Why?} \\ &= (f \circ F)(p) \cdot v(g \circ F) + v(f \circ F) \cdot (g \circ F)(p) && \text{since } v \text{ is a derivation} \\ &:= (f \circ F)(p) dF_p(v)(g) + (g \circ F)(p) dF_p(v)(f) \\ &:= f(F(p)) dF_p(v)(g) + g(F(p)) dF_p(v)(f), \end{aligned}$$

which puts it in the form  $\partial(fg) = f(q)\partial(g) + \partial(f)g(q)$  where  $q = F(p)$ .

Facts:

- $dF_p$  is a linear map.
- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ .
- If  $F$  is a diffeomorphism, then  $dF_p$  is an isomorphism with  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ .

**Proposition 7.4 (Tangent Vectors Act Locally).**

If  $f, g \in C^\infty(M)$  agree on any neighborhood of  $p \in M$ , then  $v(f) = v(g)$ .

Warning: the action of a derivation depends only on the values of a function in arbitrarily small

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neighborhoods, and in particular, can only be applied to functions defined in a neighborhood of  $p$  (not necessarily on all of  $M$ ).

**Remark** The tangent space of an  $n$ -manifolds is  $n$ -dimensional, even on boundary point.

For a vector space  $V$ , there is a natural smooth structure (Example 1.24) and for any  $\mathbf{a}, \mathbf{v} \in V$  we can similarly define a map

$$D_{\mathbf{v}} \Big|_{\mathbf{a}} : C^\infty(V) \longrightarrow \mathbb{R}$$

$$f \mapsto D_{\mathbf{v}} f(\mathbf{a}) := \frac{\partial}{\partial t} \Big|_{t=0} f(\mathbf{a} + t\mathbf{v}).$$

**Proposition 7.5.**

If  $V$  is a vector space, for any  $\mathbf{a} \in V$ , the map  $\mathbf{a} \mapsto D_{\mathbf{v}} \Big|_{\mathbf{a}}$  yields an isomorphism  $V \cong T_{\mathbf{a}} V$ . Thus tangent vectors in  $V$  are routinely identified with elements of  $V$ .

**Definition 7.5.1** (Germ of a Function).

The *germ* of a function  $f$  at  $p$  is the equivalence class of ordered pairs  $(f, U)$  where  $U \subseteq M$  is open and  $f \in C^\infty(U, \mathbb{R})$ , where  $(f, U) \sim (g, V)$  iff there exists a neighborhood  $N \subset U \cap V$  containing  $p$  such that  $f|_N \equiv g|_N$ . The set of germs of functions at  $p$  is denoted  $C_p^\infty(M)$  and is an associative  $\mathbb{R}$ -algebra.

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