Floer Talk

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Sunday 12^{th} April, 2020

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Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of $\mathcal{F}$.

What is $\mathcal{F}$?

We started with the unadorned Floer map:

$$\begin{aligned}\mathcal{F} : \mathcal{C}^\infty\left(\mathbf{R} \times S^1; W\right) &\longrightarrow \mathcal{C}^\infty\left(\mathbf{R} \times S^1; TW\right) \\ u &\longmapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u\left(H_t\right)\end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u).\end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned}\mathcal{P}^{1,p}(x, y) :? &\longrightarrow ? \\ (s, t) &\mapsto \exp_{w(s,t)} Y(s, t).\end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C^\infty_\searrow(x, y)$.

1 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian H, perturb (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the right dimension.

Start by construction $\mathcal{C}^\infty_\varepsilon(H) \subset \mathcal{C}^\infty$, the space of perturbations of H. Idea: define a norm $\|\cdot\|_\varepsilon$ and take the subspace of finite-norm elements.

$$\begin{aligned}\|h\|_\varepsilon &= \sum_{k\geq 0} \varepsilon k \sup_{(x,t)\in W\times S^1} \left|d^k h(x,t)\right| \\ &= \sum_{k\geq 0} \varepsilon k \sup_{(x,t)\in W\times S^1} \sup_{i,z\in B(0,1)} \left|d^k(h\circ\Psi_i^{-1})(z)\right|.\end{aligned}$$

Where $\{\varepsilon_k\} \subset \mathbb{R}$ is chosen such that $\mathcal{C}^\infty_\varepsilon \hookrightarrow \mathcal{C}^\infty(W\times S^1)$ is dense for the C^∞ topology, and the $\Psi_i : B_i \longrightarrow \overline{B(0,1)}$ is a fixed finite sequence of diffeomorphisms where $\bigcup_i B_i^\circ = W \times S^1$.

Note that we'll only use density for the C^1 topology in our case.

Proposition 1.1.
Such a sequence $\{\varepsilon_k\}$ can be chosen.

Proof .
Show that $C^\infty(W\times S^1)$ is separable, yielding a sequence $(f_n) \subset C^\infty(W\times S^1)$ that is dense in the C^1 topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k\leq n} \|f_k\| C^n(W\times S^1)}$$

where the diffeomorphisms Ψ_i are used to compute these norms.

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Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$ and are nondegenerate.

1.1 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

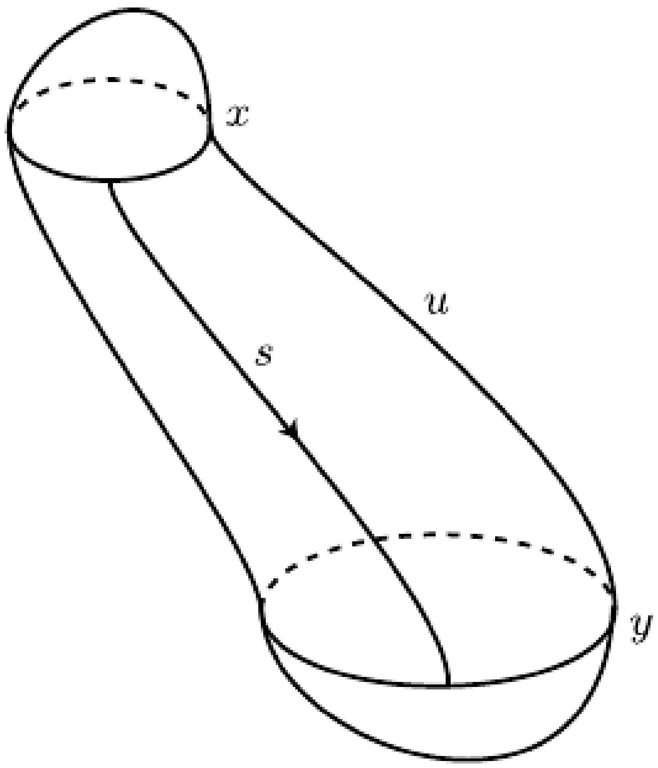
Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify im $\mathcal{F} = C^\infty(\mathbb{R}\times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R}\times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$ (i.e. solutions to the Floer equation), and $C_\searrow(x,y)$ was the set of maps $u : \mathbb{R}\times S^1 \longrightarrow W$ satisfying some conditions.

We lift each map to $\tilde{u} : S^2 \longrightarrow W$ in the following way: the loops x, y are contractible, so they bound discs. So we extend according to:



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