

Title

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Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

| | |
|-----------------------------------|--|
| $k[x_1, \dots, x_n]$ | The polynomial ring in n indeterminates |
| $k(x_1, \dots, x_n)$ | The rational function field $\{f(x) = p(x)/q(x), p, q, \in k[x_1, \dots, x_n]\}$ |
| $I(S), I_a(S)$ | The ideal associated to a subset $S \subseteq \mathbb{A}_k^n$ |
| $A(X) := k[x_1, \dots, x_n]/I(X)$ | The coordinate ring of a variety |
| \mathcal{O}_X | The structure sheaf $\{f : U \rightarrow k \mid \}$ |

Lots of notation to fill in.

| Algebra | Geometry |
|--|-------------------------------------|
| Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \dots, x_n]$ | $V(J)$ the zero locus |
| $I(S)$ the ideal of a set | $S \subseteq \mathbb{A}^n$ a subset |
| $I + J$ | $V(I) \cap V(J)$ |
| $\sqrt{I(V) + I(W)}$ | $V \cap W$ |
| $I \cap J, IJ$ | $V(I) \cup V(J)$ |
| $I(V) \cap I(W), \sqrt{I(V)I(W)}$ | $V \cup W$ |
| $I(V) : I(W)$ | $\overline{V \setminus W}$ |

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
 - The Zariski topology?
 - Irreducibility?

- Connectedness?
 - Dimension?
- What is the definition of a presheaf?
 - What are some examples and counterexamples?
- What is the definition of sheaf?
 - What are some examples?
 - What are some presheaves that are not sheaves?
- What is the definition of \mathcal{O}_X , the sheaf of regular functions?
 - How does one compute \mathcal{O}_X for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
 - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
 - What are some examples and counterexamples of complete spaces?

0.4 Useful Algebra Facts

Fact 0.4.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f , then $\langle f \rangle$ is maximal

Proposition 0.4.2 (Finitely generated polynomial rings are Noetherian).

A polynomial ring $k[x_1, \dots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \dots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \dots, f_m \rangle$.

Proof (?).

A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.4.5), $k[x_1, \dots, x_n]$ is thus Noetherian. ■

Proposition 0.4.3 (Properties and Definitions of Ideal Operations).

$$I + J := \{f + g \mid f \in I, g \in J\}$$

$$IJ := \left\{ \sum_{i=1}^N f_i g_i \mid f_i \in I, g_i \in J, N \in \mathbb{N} \right\}$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ \quad (\text{coprime or comaximal}) \quad \langle a \rangle + \langle b \rangle = \langle a, b \rangle.$$

Theorem 0.4.4 (Noether Normalization).

Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist t_1, \dots, t_ℓ such that k_2 is finite over $k_1(t_1, \dots, t_\ell)$.

Theorem 0.4.5 (Hilbert's Basis Theorem).

If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

| | | |
|--|--|--|
| | Ring Theory | Geometry/Topology of Affine Varieties |
| | Polynomial functions | Affine space |
| | $k[x_1, \dots, x_n]$ | $\mathbb{A}^n/k := \{[a_1, \dots, a_n] \in k^n\}$ |
| | Maximal ideals $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ | Points $[a_1, \dots, a_n] \in \mathbb{A}^n/k$ |
| | Radical ideals $I \subseteq k[x_1, \dots, x_n]$ | Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomials |
| | | $I \mapsto V(I) := \{a \mid f(a) = 0 \forall f \in I\}$ |
| | $I(X) := \{f \mid f _X = 0\} \leftarrow X$ | |
| Radical ideals containing $I(X)$, i.e. ideals in $A(X)$ | | closed subsets of X , i.e. affine subvarieties |
| | $A(X)$ is a domain | X irreducible |
| | $A(X)$ is not a direct sum | X connected |
| | Prime ideals in $A(X)$ | Irreducible closed subsets of X |
| Krull dimension n (longest chain of prime ideals) | | $\dim X = n$, (longest chain of irreducible closed subsets) |