Algebraic Topology 2: Smooth Manifolds

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1 Lecture 1

The key point of this class will be a discussion of *smooth structures*. As you may recall, a sensational result of Milnor's exhibited exotic spheres with smooth structures – i.e., a differentiable manifold M which is homeomorphic but *not* diffeomorphic to a sphere.

Summary of this result: Look at bundles $S^3 \to X \to S^4$, then one can construct some $X \cong S^7 \in \mathbf{Top}$ but $X \not\cong S^7 \in \mathbf{Diff}^\infty$. There are in fact 7 distinct choices for X.

> It is not known if there are exotic smooth structures on S^4. The Smooth Poincare' conjecture is that these do not exist; this is believed to be false.

The other key point of this course is to show that $X \in \mathbf{Diff}^\infty \implies X \hookrightarrow \mathbb{R}^n$ for some n, and is in fact a topological subspace.

A short list of words/topics we hope to describe:

- Differentiable manifolds
- Local charts
- Submanifolds
- Projective spaces
- Lie groups
- Tangent spaces
- Vector fields
- Cotangent spaces
- Differentials of smooth mapsG
- Differential forms
- de Rham's theorem

We'd like a notion of "convergence" for, say, curves in $\mathbb{R}^2$. Consider the following examples.

Note the problematic point on the bottom right, as well as the fact that neither of the usual notions of pointwise or uniform convergence will yield a point on the LHS that converges to the red point

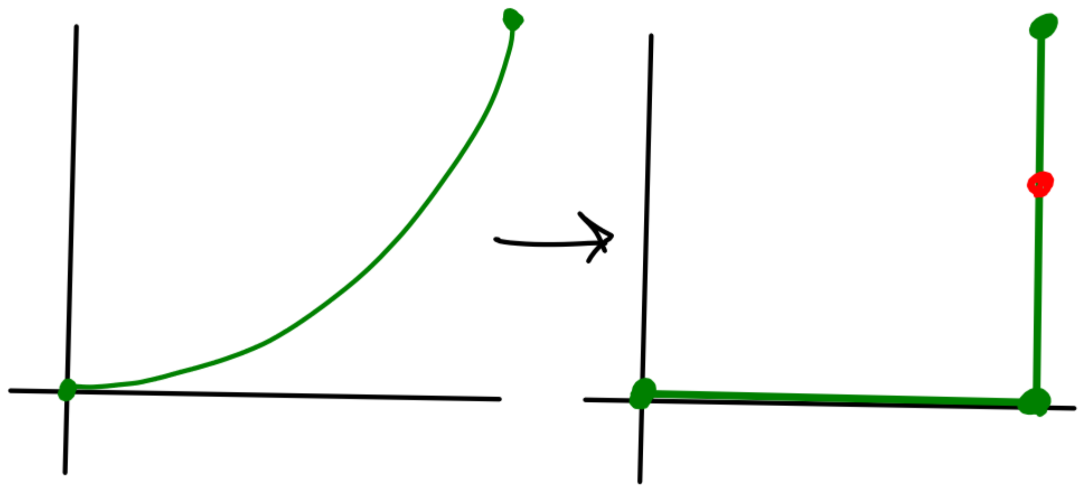

Figure 1: $y = x^n,\ n \to \infty$

on the RHS.

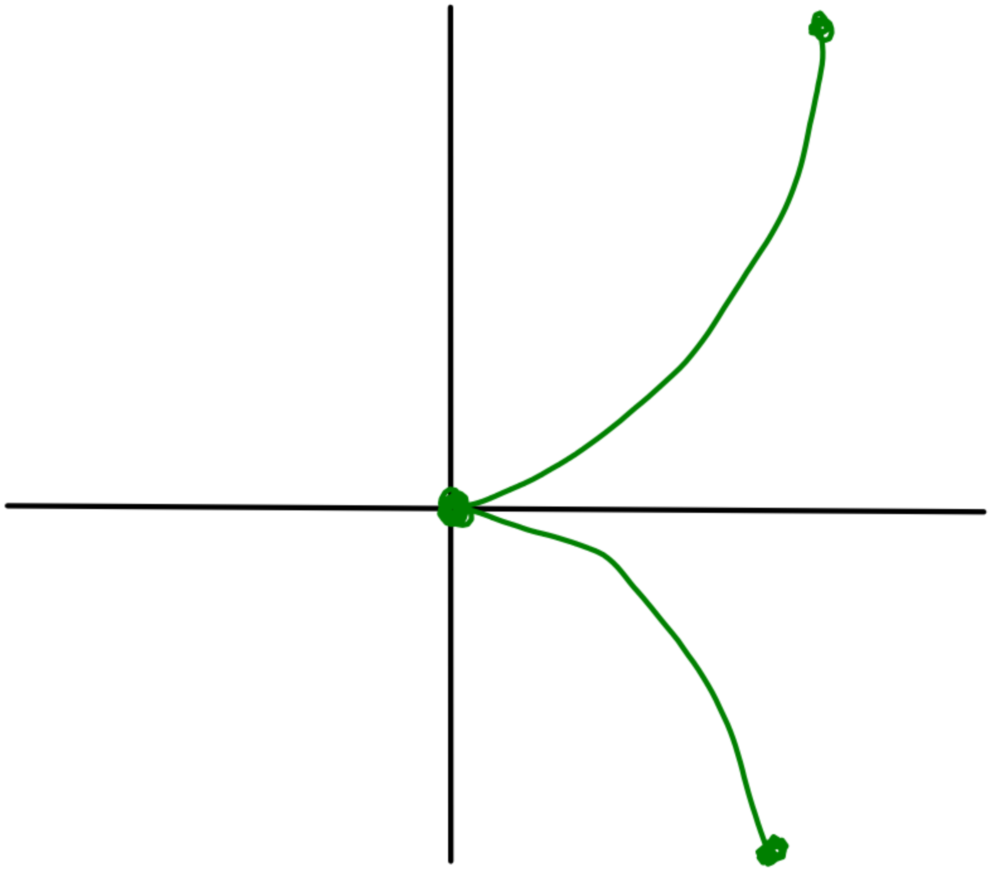

Figure 2: Image