# Title

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## Contents



# 1 List of Topics

Chapters 1-9 of Dummit and Foote

- Left and right cosets
- Lagrange's theorem
- Isomorphism theorems
- Group generated by a subset

- Structure of cyclic groups
- Composite groups
  - $HK$ is a subgroup iff $HK = KH$
- Normalizer
  - $HK \leq H$ if $H \leq N_G(K)$
- Symmetric groups
  - Conjugacy classes are determined by cycle types
- Group actions
  - Actions of G on X are equivalent to homomorphisms from G into Sym(X)
- Cayley's theorem
- Orbits of an action
- Orbit stabilizer theorem
- Orbits act on left cosets of subgroups
- Subgroups of index $p$, the smallest prime dividing $|G|$, are normal
- Action of $G$ on itself by conjugation
- Class equation
- $p$-groups
  - Have non trivial center
- $p^2$ groups are abelian
- Automorphisms, the automorphism group
  - Inner automorphisms
  - $Inn(G) \cong Z/Z(G)$
  - $Aut(S_n) = Inn(S_n)$ unless $n = 6$
  - $Aut(G)$ for cyclic groups
  - $G \cong Z_p^n$, then $Aut(G) \cong GL_n(Z_p)$
- Proof of Sylow theorems
- $A_n$ is simple for $n \geq 5$
- Recognition of internal direct product
- Recognition of semi-direct product
- Classifications:
  - $pq$
- Free group & presentations
- Commutator subgroup
- Solvable groups
  - $S_n$ is solvable for $n \leq 4$
- Derived series
  - Solvable iff derived series reaches $e$
- Nilpotent groups
  - Nilpotent iff all sylow-p subgroups are normal
  - Nilpotent iff all maximal subgroups are normal
- Upper central series
  - Nilpotent iff series reaches $G$
- Lower central series
  - Nilpotent iff series reaches $e$
- Fratini's argument
- Rings
  - $I$ maximal iff $R/I$ is a field
  - Zorn's lemma

- Chinese remainer theorem
- Localization of a domain
- Field of fractions
- Factorization in domains
- Euclidean algorithm
- Gaussian integers
- Primes and irreducibles
- Domains
  - Primes are irreducible
- UFDs
  - Have GCDs
  - Sometimes PIDs
- PIDs
  - Noetherian
  - Irreducibles are prime
  - Are UFDs
  - Have GCDs
- Euclidean domains
  - Are PIDs
- Factorization in $Z[i]$
- Polynomial rings
- Gauss' lemma
- Remainder and factor theorem
- Polynomials
- Reducibility
- Rational root test
- Eisenstein's criterion

# 2 Groups

## 2.1 Definitions

### 2.1.1 Subgroup Generated by a set $A$

- $< A >= \{a_1^{\pm 1}, a_2^{\pm 1}, \cdots a_2^{\pm 1} : a_i \in A, n \in \mathbb{N}\}$
- Equivalently, the intersection of all $H$ such that $A \subseteq H \leq G$

### 2.1.2 Free Group on a set $X$

- Equivalently, words over the alphabet $X$ made into a group via concatenation

### 2.1.3 Centralizer of an element or a subgroup

- $C_G(a) = \{g \in G : ga = ag\}$
- $$C_G(H) = \{g \in G : \forall h \in H, gh = hg\} = \bigcap_{h \in H} C_G(h)$$

  - Note - requires the same $g$ on both sides!
- Facts:
  - $C_G(H) \leq G$
  - $C_G(H) \trianglelefteq N_G(H)$
  - $C_G(G) = Z(G)$
  - $C_H(a) = H \bigcap C_G(a)$

### 2.1.4 Center of a group

- $Z(G) = \{g \in G : \forall x \in G, gx = xg\}$
- Facts
  -
$$Z(G) = \bigcap_{a \in G} C_G(a)$$

### 2.1.5 Normalizer of a subgroup

-
$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

- Equivalently, $\bigcup\{K : H \trianglelefteq K \leq G\}$ (the largest $K \leq G$ for which $H \trianglelefteq K$)
- Equivalently, the stabilizer of $H$ under $G$ acting on its subgroups via conjugation
- Differs from centralizer; can have $gh = h'g$
- Facts:
  - $C_G(H) \subseteq N_G(H) \leq G$
  - $N_G(H)/C_G(H) \cong A \leq Aut(H)$
  - Given $H \subseteq G$, let
$$S(H) = \bigcup_{g \in G} gHg^{-1}$$
    , so $|S(H)|$ is the number of conjugates to $H$. Then
$$|S(H)| = [G : N_G(H)]$$
    * i.e. the number of subgroups conjugate to $H$ equals the index of the normalizer of $H$.

### 2.1.6 Normal Core of a subgroup

-
$$H_G = \bigcap_{g \in G} gHg^{-1}$$
- Equivalently, $H_G =< N : N \trianglelefteq G \;\&\; N \leq H >$
  - Largest normal subgroup that contains $H$
- Equivalently, $H_G = \ker\psi$ where $\psi : G \to Sym(G/H);\; g \sim (xH) = (gx)H$
- Facts:
  - $H_G \trianglelefteq G$ and is an idempotent operation

### 2.1.7 Normal Closure of a subgroup

- $H^G = \{gHg^{-1} : g \in G\}$
- Equivalently,
$$H^G = \bigcap\{N : H \leq N \trianglelefteq G\}$$
  - (The smallest normal subgroup of $G$ containing $H$)

### 2.1.8 Group Action of a group on a set

- Given as a function
$$\phi : G \times X \to X(g, x) \mapsto g \sim x$$
  which gives rise to a function
$$\phi_g : X \to Xx \mapsto g \sim x$$
  (which is a bijection) where $\sim$ denotes a group element acting on a set element, and $\forall x \in X$,
  - $e \sim x = x$
  - $(gh) \sim x = g \sim (h \sim x)$
- Equivalently, a function
$$\psi : G \to Sym(X)g \mapsto \phi_g$$
  -
$$\ker\psi = \bigcap_{x\in X} G_x$$
    (intersection of all stabilizers)
- Interesting actions:
  - Left multiplication of $G$ on $G$:
$$\phi : G \to G \to G \qquad g \mapsto \phi_g : G \to G \qquad\qquad h \mapsto gh$$
    * $\mathcal{O}_x = G$ (transitive)
    * $G_x = e$
  - $G$ acting via conjugation on itself:
$$\phi : G \to G \to G \qquad g \mapsto \psi_g : G \to G \qquad\qquad h \mapsto ghg^{-1}$$
    * A common notation is $x^g = g^{-1}xg$ which obeys $(x^g)^h = x^{gh}$
    * $\mathcal{O}_x = [x]$ (Conjugacy classes, so not generally transitive)
    * $G_x = \{g \in G : gxg^{-1} = g\} = C_G(x)$
  - $G$ acting on $S = \{H : H \leq G\}$ via conjugation:
    *
$$\phi : G \to S \to S \qquad g \mapsto \psi_g : S \to S \qquad\qquad H \mapsto gHg^{-1}$$
    * $\mathcal{O}_H = [H] = \{gHg^{-1} : g \in G\}$, conjugate subgroups of $H$
    * $G_x = N_G(H) = \{g \in G : gHg^{-1} = H\}$

### 2.1.9 Transitive group actions

- $\forall x, y \in X, \exists g \in G : g \sim x = y$
- Equivalent, actions with a single orbit

### 2.1.10 Orbit of a set element

$$\mathcal{O}_x = \{g \sim x : x \in X\text{ă}\} = \bigcup_{g \in G} \{g \sim x\}$$

- The set of all orbits is denoted $X/G$ or $X_G = \{\mathcal{O}_x : x \in X\}$
- Partitions $X$ according to the equivalence relation $x \cong y \iff \exists g \in G : g \sim x = y$
- $G$ acts transitively on $X$ when restricted to any single orbit

### 2.1.11 Stabilizer of a set element

- $G_x = \{g \in G : g \sim x = x\}$
- Facts:
  - $G_x \leq G$, not usually normal
  - $x, y \in \mathcal{O}_x \Rightarrow G_x$ is conjugate to $G_y$

### 2.1.12 Automorphisms of a group

- $Aut(G) = \{\phi : G \to G : \phi \text{ is an isomorphism}\}$

### 2.1.13 Inner Automorphisms of a group

- $Inn(G) = \{\phi_g \in Aut(G) : \phi_g(x) = gxg^{-1}\}$
- Also consider the map

  $$\psi : G \to Aut(G) g \mapsto (\lambda : x \mapsto gxg^{-1})$$

  Then $\text{im}\psi = Inn(G), \text{ker}\psi = Z(G)$
- Facts:
  - $Inn(G) \trianglelefteq Aut(G)$
  - $Inn(G) \cong G/Z(G)$

### 2.1.14 Outer Automorphisms of a group

- $Out(G) = Aut(G)/Inn(G)$

### 2.1.15 Conjugacy Class of an element

- $$[a] = \{gag^{-1} : g \in G\} = \bigcup_{g \in G} \{gag^{-1}\}$$
- Equivalently, $[a] = \mathcal{O}_a$ under $G$ acting on itself via conjugation
- Facts:
  - Equivalence relation, partitions the group
  - $|[a]|$ divides $|G|$
  - $a \in Z(G) \Rightarrow [a] = \{a\}$

### 2.1.16 Characteristic subgroup

- $H$ char $G \iff \forall \phi \in Aut(G), \phi(H) = H$
  - i.e., $H$ is fixed by all automorphisms of $G$.

### 2.1.17 Simple group

- $G$ is simple $\iff H \trianglelefteq G \Rightarrow H = e$ or $G$
  - No non-trivial normal subgroups

### 2.1.18 Commutator of an element, or of subgroups

- $[g, h] = ghg^{-1}h^{-1}$
- $[G, H] =< [g, h] : g \in G, h \in H >$ (Subgroup generated by commutators)

## 2.2 Structural Results

- Cyclic $\Rightarrow$ abelian
- $G/Z(G)$ cyclic $\Rightarrow G$ is abelian
- Intersections of subgroups are also subgroups

### 2.2.1 Isomorphisms Theorems

**First Isomorphism Theorem**

- Conditions:
  - $\phi : G \to G'$ is a homomorphism.
- Result:
  - $\ker\phi \trianglelefteq G$
  - $\mathrm{im}\phi \leq G'$
  - $G/\ker\phi \cong \mathrm{im}\phi$.
- Corollaries:
  - $\ker\phi = e \Rightarrow G \cong G'$

**Second Isomorphism Theorem**

- Conditions:
  - $N \trianglelefteq G, H \leq G$
- Results:
  - $HN \leq G$
  - $N \bigcap H \trianglelefteq H$
  - $$\frac{H}{H \bigcap N} \cong \frac{HN}{N}$$
- Corrolaries:
  - (Weaker) Relaxing $N \trianglelefteq G$ to $H \subseteq N(N)$ yields
    - $N \bigcap H \subseteq G$ (Not normal)
    - $N \bigcap H \trianglelefteq H$

**Third Isomorphism Theorem**

- Conditions:
  - $N \trianglelefteq G, N \leq A \leq G$
- Results:
  - $A/N \leq G/N$
    - Every subgroup of $G/N$ is of this form for *some* such $A$
  - $$\frac{G/N}{A/N} \cong \frac{G}{A}$$
    - Cancel the $N$!
- Corrolaries:
  - $A \trianglelefteq G \Rightarrow A/N \trianglelefteq G/N$
    - All normal subgroups of $G/N$ are of this form for some $A$.

## 2.3 Misc Results

- $G/N$ is abelian $\iff [G, G] \leq N$
- $HK$ is not always a subgroup - see conditions in 2nd Isomorphism theorem'
  - Normal subgroups with trivial intersection commute
- $H$ char $G \Rightarrow H \trianglelefteq G$
  - Characteristic is a strictly stronger condition than normality
- $H$ char $K$ char $G \Rightarrow H$ char $G$
  - Characteristic is transitive
- $H \leq G$, $K \trianglelefteq G$, $H$ char $K \Rightarrow H \trianglelefteq G$
  - i.e., normality is **not** transitive, strengthening normality to char gives "weak transitivity"
- Recognizing (Internal) Direct Products:$H \leq G, K \leq G$
  - $H \bigcap K = e$
  - $\forall g \in G, \exists h \in H, k \in K : g = hk$
  - $H \trianglelefteq G, K \trianglelefteq G$
    - **OR** Every element in $H$ commutes with every element in $K$
- P Groups
  - $\bigcap P = O_P(G)$ char $G$. And $O_P(G) \trianglelefteq G$ as well.
  - $N \trianglelefteq G$ implies that $P_N \leq N$ are of the form $N \bigcap P_G$
  - $P \bigcap Q = e$