# CRAG
## The Weil Conjectures

D. Zack Garza

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– A big thanks to Daniel Litt for organizing this reading seminar, recommending papers, helping with questions!!

– Goals for this talk:
  – Understand the Weil conjectures,
  – Understand *why* the relevant objects should be interesting,
  – See elementary but concrete examples,
  – Count all of the things!

## Generating Functions

Fix $q$ a prime and $\mathbb{F}_q$ the (unique) finite field with $q$ elements, along with its (unique) degree $n$ extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \ \Big| \ x^{q^n} - x = 0 \right\} \quad \forall \ n \in \mathbb{Z}^{\geq 1}$$

### Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \cdots, f_M \rangle \ \trianglelefteq \ k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}_q} \ \Big| \ f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where $J$ is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{I = (i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_I \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

– For a fixed variety $X$, we can consider its $\mathbb{F}_q$-points $X(\mathbb{F}_q)$.

    – Note that $\#X(\mathbb{F}_q) < \infty$ is an integer

– For any $L/\mathbb{F}_q$, we can also consider $X(L)$

    – For $[L : \mathbb{F}_q]$ finite, $\#X(L) < \infty$ and are integers for every such $n$.

    – In particular, we can consider the finite counts $\#X(\mathbb{F}_{q^n})$ for any $n \geq 2$.

– So we can consider the sequence

$$[N_1, N_2, \cdots, N_n, \cdots] := [\#X(\mathbb{F}_q), \ \#X(\mathbb{F}_{q^2}), \cdots, \ \#X(\mathbb{F}_{q^n}), \cdots].$$

– Idea: associate some generating function (a formal power series) encoding the sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \cdots.$$

For *ordinary* generating functions, the coefficients are related to the real-analytic properties of $F$:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left( \frac{\partial}{\partial z} \right)^n F(z) \bigg|_{z=0} = N_n$$

and also to the complex analytic properties:

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} \, dz = N_n.$$

*Using the Residue theorem. The latter form is very amenable to computer calculation.*

# Why Generating Functions?

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– An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \longrightarrow \mathbb{C}$
– In nice situations, we can hope for a closed-form representation.
– A useful example: by integrating a geometric series we can derive

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \left( = 1 + z + z^2 + \cdots \right)$$

$$\implies \int \frac{1}{1-z} = \int \sum_{n=0}^{\infty} z^n$$

$$= \sum_{n=0}^{\infty} \int z^n \quad for |z| < 1 \quad \text{by uniform convergence}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1}$$

$$\implies -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \left( = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \right).$$

– For completeness, also recall that

$$\exp(z) \coloneqq \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

– We can regard $\exp, \log$ as elements in the ring of formal power series $\mathbb{Q}[[z]]$.

– In particular, for any $p(z) \in z \cdot \mathbb{Q}[[z]]$ we can consider $\exp(p(z)), \log(1 + p(z))$

– Since $\mathbb{Q} \hookrightarrow \mathbb{C}$, we can consider these as a complex-analytic functions, ask where they converge, etc.

## Zeta Functions

Problem: count points of a (smooth?) projective variety $X/\mathbb{F}$ in all (finite) degree $n$ extensions of $\mathbb{F}$.

## Definition (Local Zeta Function)

The *local zeta function* of an algebraic variety $X$ is the following formal power series:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n}\right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note that

$$z\left(\frac{\partial}{\partial z}\right) \log Z_X(z) = z\frac{\partial}{\partial z}\left(N_1 z + N_2 \frac{z^2}{2} + N_3 \frac{z^3}{3} + \cdots\right)$$

$$= z(N_1 + N_2 z + N_3 z^2 + \cdots) \qquad \text{(unif. conv.)}$$

$$= N_1 z + N_2 z^2 + \cdots = \sum_{n=1}^{\infty} N_n z^n,$$

which is an *ordinary* generating function for the sequence $(N_n)$.

# Examples

Take $X = \{\text{pt}\} = V(\{f(x) = x - c\})/\mathbb{F}$ for $c$ a fixed element of $\mathbb{F}$. This yields a single point over $\mathbb{F}$, then

$$\#X(\mathbb{F}_q) := N_1 = 1$$
$$\#X(\mathbb{F}_{q^2}) := N_2 = 1$$
$$\vdots$$
$$\#X(\mathbb{F}_{q^n}) := N_n = 1$$

and so

$$Z_{\{\text{pt}\}}(z) = \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \cdots\right) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right)$$
$$= \exp\left(-\log\left(1 - z\right)\right)$$
$$= \frac{1}{1 - z}.$$

*Notice: $Z$ admits a closed form **and** is a rational function.*

Take $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^1(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1] \ \middle| \ x_1 \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$X(\mathbb{F}_q) = q$$
$$X(\mathbb{F}_{q^2}) = q^2$$
$$\vdots$$
$$X(\mathbb{F}_{q^n}) = q^n.$$

Then

$$Z_X(z) = \exp\left( \sum_{n=1}^{\infty} q^n \frac{z^n}{n} \right)$$
$$= \exp\left( \sum_{n=1}^{\infty} \frac{(qz)^n}{n} \right)$$
$$= \exp(-\log(1 - qz))$$
$$= \frac{1}{1 - qz}.$$

Take $X = \mathbb{A}^m/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^m(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$X(\mathbb{F}_q) = q^m$$
$$X(\mathbb{F}_{q^2}) = (q^2)^m$$
$$\vdots$$
$$X(\mathbb{F}_{q^n}) = q^{nm}.$$



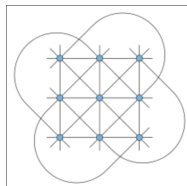Figure: $\mathbb{A}^2/\mathbb{F}_3$ ($q = 3, m = 2, n = 1$)

Then

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} q^{nm} \frac{z^n}{n}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(q^m z)^n}{n}\right)$$
$$= \exp(-\log(1 - q^m z))$$
$$= \frac{1}{1 - q^m z}.$$

Take $X = \mathbb{P}^1/\mathbb{F}$, we can still count by enumerating coordinates:

$$\mathbb{P}^1(\mathbb{F}_{q^n}) = \left\{ [x_1 : x_2] \ \middle| \ x_1, x_2 \neq 0 \in \mathbb{F}_{q^n} \right\} / \sim \ = \left\{ [x_1 : 1] \ \middle| \ x_1 \in \mathbb{F}_{q^n} \right\} \coprod \{[1:0]\}.$$

Thus

$$X(\mathbb{F}_q) = q + 1$$
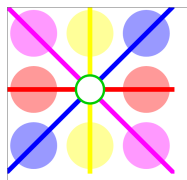$$X(\mathbb{F}_{q^2}) = q^2 + 1$$
$$\vdots$$
$$X(\mathbb{F}_{q^n}) = q^n + 1.$$



Figure: $\mathbb{P}^1/\mathbb{F}_3$ $(q = 3, n = 1)$

Thus

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} (q^n + 1)\frac{z^n}{n}\right)$$
$$= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} + \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n}\right)$$
$$= \frac{1}{(1 - qz)(1 - z)}.$$

## The Weil Conjectures

(Weil 1949)

Let $X$ be a smooth projective variety of dimension $N$ over $\mathbb{F}_q$ for $q$ a prime, let $Z_X(z)$ be its zeta function, and define $\zeta_X(s) = Z_X(q^{-s})$.

**1** (Rationality)

$Z_X(z)$ is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$
$$P_{2N}(z) = 1 - q^N z$$
$$P_j(z) = \prod_{j=1}^{\beta_i} (1 - a_{j,k} z) \quad \text{for some reciprocal roots} \quad a_{j,k} \in \mathbb{C}$$

where we've factored each $P_i$ using its reciprocal roots $a_{ij}$.

In particular, this implies the existence of a meromorphic continuation of the associated function $\zeta_X(s)$, which a priori only converges for $\Re(s) \gg 0$. This also implies that for $n$ large enough, $N_n$ satisfies a linear recurrence relation.

**2** (Functional Equation and Poincare Duality)
Let $\chi(X)$ be the Euler characteristic of $X$, i.e. the self-intersection number of the diagonal embedding $\Delta \hookrightarrow X \times X$; then $Z_X(z)$ satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm\left(q^{\frac{N}{2}}z\right)^{\chi(X)} Z_X(z).$$

Equivalently,

$$\zeta_X(N - s) = \pm\left(q^{\frac{N}{2}-s}\right)^{\chi(X)} \zeta_X(s).$$

*Note that when $N = 1$, e.g. for a curve, this relates $\zeta_X(s)$ to $\zeta_X(1 - s)$.*

Equivalently, there is an involutive map on the (reciprocal) roots

$$z \iff \frac{q^N}{z}$$

$$\alpha_{j,k} \iff \alpha_{2N-j,k}$$

which sends interchanges the coefficients in $p_j$ and $p_{2N-j}$.

**3** (Riemann Hypothesis)
The reciprocal roots $\alpha_{j,k}$ are *algebraic* integers (roots of some monic $p \in \mathbb{Z}[x]$) which satisfy

$$|\alpha_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}}, \qquad 1 \leq j \leq 2N - 1, \ \forall k.$$

**4** (Betti Numbers)
If $X$ is a "good reduction mod $q$" of a nonsingular projective variety $\tilde{X}$ in characteristic zero, then the $\beta_i = \deg p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$.

Moral:

- The Diophantine properties of a variety's zeta function are governed by its (algebraic) topology.
- Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information.

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

After modifying $\zeta$ to make it symmetric about $\Re(s) = \frac{1}{2}$ and eliminate the trivial zeros to obtain $\widehat{\zeta}(s)$, there are three relevant properties

1. "Rationality": $\widehat{\zeta}(s)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$.
2. "Functional equation": $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
3. "Riemann Hypothesis": The only zeros of $\widehat{\zeta}$ have $\Re(s) = \frac{1}{2}$.

Suppose it holds for some $X$. Use the facts:

- a. $|\exp(z)| = \exp(\Re(z))$ and
- b. $a^z := \exp(z \operatorname{Log}(a))$,

and to replace the polynomials $P_j$ with

$$L_j(s) := P_j(q^{-s}) = \prod_{k=1}^{\beta_j} \left(1 - \alpha_{j,k} q^{-s}\right).$$

Now consider the roots of $L_j(s)$: we have

$$L_j(s_0) = 0 \iff (1 - \alpha_{j,k} q^{-s}) = 0 \quad \text{for some} \quad k$$

$$\iff q^{-s_0} = \frac{1}{\alpha_{j,k}}$$

$$\iff \left| q^{-s_0} \right| = \left| \frac{1}{\alpha_{j,k}} \right| \quad \overset{\text{by assumption}}{=} \quad q^{-\frac{j}{2}}$$

$$\iff q^{-\frac{j}{2}} \overset{(a)}{=} \exp\left(-\frac{j}{2} \cdot \mathrm{Log}(q)\right) = \left| \exp\left(-s_0 \cdot \mathrm{Log}(q)\right) \right|$$

$$\overset{(b)}{=} \left| \exp\left(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \mathrm{Log}(q)\right) \right|$$

$$\overset{(a)}{=} \exp\left(-(\Re(s_0)) \cdot \mathrm{Log}(q)\right)$$

$$\iff -\frac{j}{2} \cdot \mathrm{Log}(q) = -\Re(s_0) \cdot \mathrm{Log}(q) \quad \text{by injectivity}$$

$$\iff \Re(s_0) = \frac{j}{2}.$$

Roughly speaking, we can apply $\mathrm{Log}$ (a conformal map) to send the $\alpha_{j,k}$ to zeros of the $L_j$, this says that the zeros all must lie on the "critical lines" $\frac{j}{2}$.
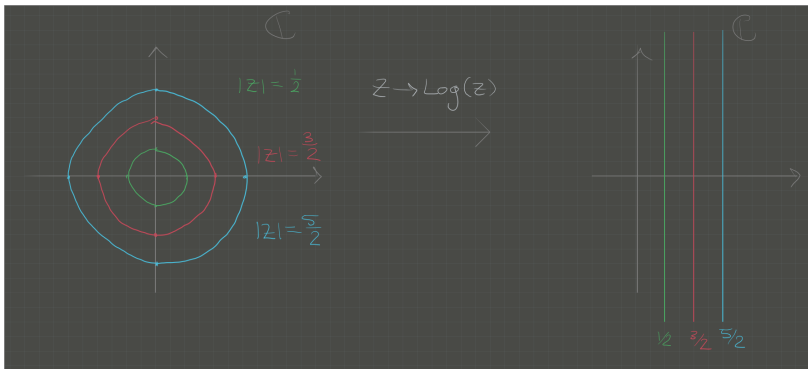


In particular, the zeros of $L_1$ have real part $\frac{1}{2}$ (analogy: classical Riemann hypothesis).

– Difficult to find in the literature!
– Idea: make a similar definition for schemes, then take $X = \operatorname{Spec} \mathbb{Z}$.
– Define the "reductions mod $p$" $X_p$ for closed points $p$.
– Define the *local* zeta functions $\zeta_{X_p}(s) = Z_{X_p}(p^{-s})$.
– (Potentially incorrect) Evaluate to find $Z_{X_p}(z) = \frac{1}{1-z}$.
– Take a product over all closed points to define

$$
\begin{aligned}
L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\
&= \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right) \\
&= \zeta(s),
\end{aligned}
$$

which is the Euler product expansion of the classical Riemann Zeta function.
*If anyone knows a reference for this, let me know!*

# Weil for Curves

The Weil conjectures take on a particularly nice form for curves. Let $X/\mathbb{F}_q$ be a smooth projective curve of genus $g$, then

1. (Rationality)

$$Z_X(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)}$$

2. (Functional Equation)

$$Z_X\left(\frac{1}{qz}\right) = (z\sqrt{q})^{2-2g} Z_X(z)$$

3. (Riemann Hypothesis)

$$p_1(z) = \prod_{i=1}^{\beta_1}(1 - a_i z) \quad \text{where} \quad |a_i| = \sqrt{q}$$

4. (Betti Numbers) $\mathcal{P}_{\Sigma_g}(x) = 1 + 2g \cdot x + x^2 \implies \deg p_1 = \beta_1 = 2g.$

$\mathcal{P}$ here is the Poincaré polynomial, the generating function for the Betti numbers. $\Sigma_g$ is the surface (real 2-dimensional smooth manifold) of genus $g$.

# The Projective Line

Recall $Z_{\mathbb{P}^1/\mathbb{F}_q}(z) = \frac{1}{(1-z)(1-qz)}$.

1. Rationality: Clear!
2. Functional Equation: $g = 0 \implies 2g - 2 = 2$

$$Z_{\mathbb{P}^1}\left(\frac{1}{qz}\right) = \frac{1}{(1-\frac{1}{qz})(1-\frac{q}{qz})} = \frac{qz^2}{(1-z)(1-qz)} = \frac{(\sqrt{q}z)^2}{(1-z)(1-qz)}.$$

3. Riemann Hypothesis: Nothing to check (no $p_1(z)$)
4. Betti numbers: Use the fact that $\mathcal{P}_{\mathbb{CP}^1} = 1 + 0 \cdot x + x^2$, and indeed $\deg p_0 = \deg p_2 = 1, \ \deg p_1 = 0$.

   *Note that even Betti numbers show up as degrees in the denominator, odd in the numerator. Allows us to immediately guess the zeta function for $\mathbb{P}^n/\mathbb{F}_q$ by knowing $H^*\mathbb{CP}^\infty$!*

Figure: Some Elliptic Curves
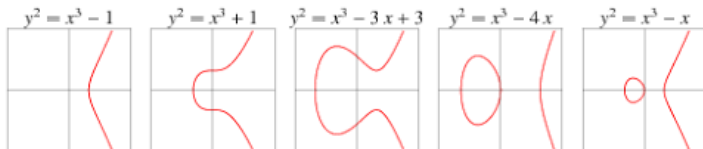


Consider $E/\mathbb{F}_q$.

– (Nontrivial!) The number of points is given by

$$N_n := E(\mathbb{F}_{q^n}) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

  – Proof: Involves trace (or eigenvalues?) of Frobenius, (could use references)

– $\dim_{\mathbb{C}} E/\mathbb{C} = N = 1$ and $g = 1$.

The Weil Conjectures say we should be able to write

$$Z_E(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1-a_1 z)(1-a_2 z)}{(1-z)(1-qz)}.$$

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**1** Rationality: using the point count, we can compute

$$Z_E(z) = \exp \sum_{n=1}^{\infty} \#E(\mathbb{F}_{q^n}) \frac{z^n}{n}$$

$$= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n}$$

$$= \exp \left( \sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right)$$

$$\exp \left( \sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right)$$

$$= \exp \left( -\log \left( 1 - qz \right) \right) \cdot \exp \left( -\log \left( 1 - z \right) \right)$$

$$\exp \left( \log \left( 1 - \alpha z \right) \right) \cdot \exp \left( \log \left( 1 - \bar{\alpha} z \right) \right)$$

$$= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),$$

which is a rational function of the expected form (Weil 1).

Note that the "expected" point counts show up in the denominator, along with the even Betti numbers, while the "correction" factor appears in the denominator and odd Betti numbers.

**2** Functional Equation: we use the equivalent formulation of "Poincaré duality":

$$\frac{(1-\alpha z)(1-\bar{\alpha}z)}{(1-z)(1-qz)} = \frac{p_1(z)}{p_0(z)p_2(z)} \implies \begin{cases} z & \Longleftrightarrow \frac{q}{z} \\ \alpha_{j,k} & \Longleftrightarrow \alpha_{2-j,k} \end{cases}$$

This amounts to checking that the coefficients of $p_0, p_2$ are interchanged, and that the two coefficients of $p_1$ are interchanged:

$$\mathrm{Coefs}(p_0) = \{1\} \xrightarrow{z \longrightarrow \frac{q}{z}} \left\{ \frac{1}{q} \right\} = \mathrm{Coefs}(p_2)$$

$$\mathrm{Coefs}(p_1) = \{\alpha, \bar{\alpha}\} \xrightarrow{z \longrightarrow \frac{q}{z}} \left\{ \frac{q}{\alpha}, \frac{q}{\bar{\alpha}} \right\} = \{\bar{\alpha}, \alpha\} \quad \text{using} \quad \alpha\bar{\alpha} = q.$$

**3** RH: Assumed as part of the point count ($|\alpha| = q^{\frac{1}{2}}$)

**4** Betti Numbers: $\mathcal{P}_{\Sigma_1}(x) = 1 + 2x + x^2$, and indeed
$\deg p_0 = \deg p_2 = 1, \ \deg p_1 = 2.$

# History

– 1801, Gauss: Point count and RH showed for specific elliptic curves

– 1924, Artin: Conjectured for algebraic curves ,

– 1934, Hasse: proved for elliptic curves.

– 1949, Weil: Proved for smooth projective curves over finite fields, proposed generalization to projective varieties

– 1960, Dwork: Rationality via $p$-adic analysis

– 1965, Grothendieck et al.: Rationality, functional equation, Betti numbers using étale cohomology
  – Trace of Frobenius on $\ell$-adic cohomology
  – Expected proof via *the standard conjectures*. Wide open!

– 1974, Deligne: Riemann Hypothesis using étale cohomology, circumvented the standard conjectures

– Recent: Hasse-Weil conjecture for arbitrary algebraic varieties over number fields
  – Similar requirements on $L$-functions: functional equation, meromorphic continuation
  – 2001: Full modularity theorem proved, extending Wiles, implies Hasse-Weil for elliptic curves
  – Inroad to Langlands: show every $L$ function coming from an algebraic variety also comes from an automorphic representation.

## Weil for Projective m-space

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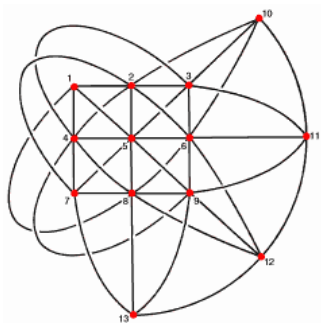Weil for
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## Setup

Take $X = \mathbb{P}^m / \mathbb{F}$ We can write

$$\mathbb{P}^m(\mathbb{F}_{q^n}) = \mathbb{A}^{m+1}(\mathbb{F}_{q^n}) \setminus \{\mathbf{0}\} / \sim \; = \left\{ \mathbf{x} = [x_0, \cdots, x_m] \; \middle| \; x_i \in \mathbb{F}_{q^n} \right\} / \sim$$

But how many points are actually in this space?

Figure: Points and Lines in $\mathbb{P}^2 / \mathbb{F}_3$



*A nontrivial combinatorial problem!*

To illustrate, this can be done combinatorially: identify $\mathbb{P}^m_{\mathbb{F}} = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}^{m+1}_{\mathbb{F}}$.

### Theorem

*The number of $k$-dimensional subspaces of $\mathbb{A}^N_{\mathbb{F}_q}$ is the q-analog of the binomial coefficient:*

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q^N - 1)(q^{N-1} - 1)\cdots(q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

*Remark: Note $\lim_{q \to 1} \begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k}$, the usual binomial coefficient.*

**Proof:** To choose a $k$-dimensional subspace,

   − Choose a nonzero vector $\mathbf{v}_1 \in \mathbb{A}^n_{\mathbb{F}}$ in $q^N - 1$ ways.
   
      − For next step, note that $\#\mathrm{span}\,\{v_1\} = \#\left\{ \lambda v_1 \;\middle|\; \lambda \in \mathbb{F}_q \right\} = \#\mathbb{F}_q = q$.
   
   − Choose a nonzero vector $\mathbf{v}_2$ *not* in the span of $\mathbf{v}_1$ in $q^N - q$ ways.
   
      − Now note $\#\mathrm{span}\,\{v_1, v_2\} = \#\left\{ \lambda_1 v_1 + \lambda_2 v_2 \;\middle|\; \lambda_i \in \mathbb{F} \right\} = q \cdot q = q^2$.

– Choose a nonzero vector $\mathbf{v}_3$ *not* in the span of $\mathbf{v}_1, \mathbf{v}_2$ in $q^N - q^2$ ways.
– $\cdots$ until $\mathbf{v}_k$ is chosen in
$$(q^N - 1)(q^N - q) \cdots (q^N - q^{k-1}) \quad \text{ways} \quad .$$
  – This yields a $k$-tuple of linearly independent vectors spanning a $k$-dimensional subspace $V_k$
– This overcounts because many linearly independent sets span $V_k$, we need to divide out by the number of ways to choose a basis inside of $V_k$.
– By the same argument, this is given by
$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

Thus

$$\#\text{subspaces} = \frac{(q^N - 1)(q^N - q)(q^N - q^2) \cdots (q^N - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})}$$

$$= \frac{q^N - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{N-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{N-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{N-(k-1)} - 1}{q^{k-(k-1)-1}}$$

$$= \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

■

# Counting Points

CRAG

D. Zack
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*Note that we've actually computed the number of points in any Grassmannian.*

Identify $\mathbb{P}^m_{\mathbb{F}} = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}^{m+1}_{\mathbb{F}}$.

We obtain a nice simplification for the number of lines corresponding to setting $k = 1$:

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = \frac{q^{m+1} - 1}{q - 1} = q^m + q^{m-1} + \cdots + q + 1 = \sum_{j=0}^{m} q^j.$$

Thus

$$X(\mathbb{F}_q) = \sum_{j=0}^{m} q^j$$

$$X(\mathbb{F}_{q^2}) = \sum_{j=0}^{m} \left(q^2\right)^j$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = \sum_{j=0}^{m} \left(q^n\right)^j.$$

So

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}(q^n)^j\frac{z^n}{n}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}\frac{\left(q^j z\right)^n}{n}\right)$$

$$= \exp\left(\sum_{j=0}^{m}\sum_{n=1}^{\infty}\frac{\left(q^j z\right)^n}{n}\right)$$

$$= \exp\left(\sum_{j=0}^{m-1}-\log(1-q^j z)\right)$$

$$= \prod_{j=0}^{m}\left(1-q^j z\right)^{-1}$$

$$= \left(\frac{1}{1-z}\right)\left(\frac{1}{1-qz}\right)\left(\frac{1}{1-q^2 z}\right)\cdots\left(\frac{1}{1-q^m z}\right),$$

*Miraculously, still a rational function!*

$$Z_X(z) = \prod_{j=0}^{m} \left( \frac{1}{1 - q^j z} \right).$$

**1** Rationality: Clear!

**2** Functional Equation: Less clear, but true:

$$
\begin{aligned}
Z_X\left(\frac{1}{q^m z}\right) &= \frac{1}{(1 - 1/q^m t)\,(1 - q/q^m t)\cdots(1 - q^m/q^m z)} \\
&= \frac{q^m z \cdot q^{m-1} z \ldots q z \cdot z}{(1 - z)(1 - qz)\ldots(1 - q^m z)} \\
&= q^{\frac{m(m+1)}{2}} z^{m+1} \cdot Z_X(z) \\
&= \left(q^{\frac{m}{2}} z\right)^{\chi(X)} \cdot Z_X(z)
\end{aligned}
$$

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3. Riemann Hypothesis: Reduces to the statement $\{\alpha_i\} = \left\{ \frac{q^m}{\alpha_j} \right\}$.

4. Betti Numbers: Use the fact that $\mathcal{P}_{\mathbb{CP}^m}(x) = 1 + x^2 + x^4 + \cdots + x^{2m}$.

Quick recap:

$$Z_{\{pt\}} = \frac{1}{1-z} \qquad Z_{\mathbb{P}^1}(z) = \frac{1}{1-qz} \qquad Z_{\mathbb{A}^1}(z) = \frac{1}{(1-z)(1-qz)}.$$

Note that $\mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$ and correspondingly $Z_{\mathbb{P}^1}(z) = Z_{\mathbb{A}^1}(z) \cdot Z_{\{pt\}}(z)$. This works in general:

### Lemma (Excision)

*If $Y/\mathbb{F}_q \subset X/\mathbb{F}_q$ is a closed subvariety, for $U = X \setminus Y$,*
*$Z_X(z) = Z_Y(z) \cdot Z_U(z)$.*

**Proof**: Let $N_n = \#Y(\mathbb{F}_{q^n})$ and $M_n = \#U(\mathbb{F}_{q^n})$, then

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} (N_n + M_n) \frac{z^n}{n}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n} + \sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right) = \zeta_Y(z) \cdot \zeta_U(z).$$

Note that geometry can help us here: we have a stratification $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$, and so inductively

$$\mathbb{P}^m = \coprod_{j=0}^{m} \mathbb{A}^j = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m,$$

and recalling that

$$Z_{X \coprod Y}(z) = Z_X(z) \cdot Z_Y(z)$$

and $Z_{\mathbb{A}^j}(z) = \frac{1}{1 - q^j z}$ we have

$$Z_{\mathbb{P}^m}(z) = \prod_{j=0}^{m} Z_{\mathbb{A}^j}(z) = \prod_{j=0}^{m} \frac{1}{1 - q^j z}.$$

# Grassmannians

Consider now $X = \mathrm{Gr}(k, m)/\mathbb{F}$ – by the previous computation, we know

$$X(\mathbb{F}_{q^n}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} := \frac{(q^{nm} - 1)(q^{nm-1} - 1) \cdots (q^{nm-n(k-1)} - 1)}{(q^{nk} - 1)(q^{n(k-1)} - 1) \cdots (q^n - 1)}$$

but the corresponding Zeta function is much more complicated than the previous examples:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} \frac{z^n}{n}\right) = \cdots?.$$

Since $\dim_{\mathbb{C}} \mathrm{Gr}_{\mathbb{C}}(k, m) = 2k(m - k)$, by Weil we should expect

$$Z_X(z) = \prod_{j=0}^{2k(m-k)} \frac{p_{2(j+1)}(z)}{p_{2j}(z)}$$

with $\deg p_j = \beta_j$.

It turns out that (proof omitted) one can show

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \sum_{j=0}^{k(m-k)} \lambda_{m,k}(j) q^j \implies Z_X(z) = \prod_{j=0}^{k(m-k)} \left( \frac{1}{1-q^j x} \right)^{\lambda_{m,k}(j)}$$

where $\lambda_{m,k}$ is the number of integer partitions of of $[i]$ into at most $m-k$ parts, each of size at most $k$.

- One proof idea: use combinatorial identities to write $q$-analog $\begin{bmatrix} m \\ k \end{bmatrix}_q$ as a *sum*
- Second proof idea: "pave by affines".

This lets us conclude that the Poincare polynomial of the complex Grassmannian is given by

$$\mathcal{P}_{\mathrm{Gr}_{\mathbb{C}}(m,k)}(x) = \sum_{n=1}^{k(m-k)} \lambda_{m,k}(n) x^{2n},$$

In particular, the cohomology vanishes in odd degree.

## Weil's Proof

Proof of rationality of $Z_X(T)$ for $X$ a diagonal hypersurface.

   – Set $q$ to be a prime power and consider $X/\mathbb{F}_q$ defined by

$$X = V(a_0 x_0^n + \cdots + a_r x_r^n) \subset \mathbb{F}_q^{r+1}.$$

   – We want to compute $N = \#X$.

   – Set $d_i = \gcd(n_i, q - 1)$.

   – Define the character

$$\psi_q : \mathbb{F}_q \longrightarrow \mathbb{C}^\times$$

$$a \mapsto \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}\right).$$

      – By Artin's theorem for linear independence of characters, $\psi_q \not\equiv 1$ and every additive character of $\mathbb{F}_q$ is of the form $a \mapsto \psi_q(ca)$ for some $c \in \mathbb{F}_q$.

   – Shorthand notation: say $a \sim 0 \iff a \equiv 0 \mod 1$.

– Fix an injective multiplicative map

$$\phi : \overline{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times.$$

– Define

$$\chi_{\alpha,n} : \mathbb{F}_{q^n}^\times \longrightarrow \mathbb{C}^\times$$
$$x \mapsto \phi(x)^{\alpha(q^n-1)}$$

$$\text{for} \quad \alpha \in \mathbb{Q}/\mathbb{Z}, n \in \mathbb{Z}, \quad \alpha(q^n - 1) \equiv 0 \mod 1.$$

– Extend this to $\mathbb{F}_{q^n}$ by

$$\begin{cases} 1 & \alpha \equiv 0 \mod 1 \\ 0 & \text{else} \end{cases}.$$

– Set $\chi_\alpha = \chi_{\alpha,1}$.

– Proposition:

$$\alpha(q - 1) \equiv 0 \mod 1 \implies \chi_{\alpha,n}(x) = \chi_\alpha(\mathrm{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))$$

– Proposition:

$$d := \gcd(n, q - 1), u \in \mathbb{F}_q \implies \#\left\{ x \in \mathbb{F}_q \;\middle|\; x^n = u \right\} = \sum_{d\alpha \sim 0} \chi_\alpha(u)$$

– This implies

$$N = \sum_{\substack{\alpha=[\alpha_0,\cdots,\alpha_r] \\ d_i\alpha_i \sim 0}} \sum_{\substack{u=[u_0,\cdots,u_r] \\ \Sigma\, a_i u_i = 0}} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j)$$

$$= q^r + \sum_{\substack{\alpha,\ \alpha_i \in (0,1) \\ d_i\alpha_i \sim 0}} \left( \prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) \sum_{\Sigma\, u_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) \right).$$

since the inner sum is zero if some *but not all* of the $\alpha_i \sim 0$.

– Evaluate the innermost sum by restricting to $u_0 \neq 0$ and setting $u_i = u_0 v_i$ and $v_0 := 1$:

$$\sum_{\Sigma\, u_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) = \sum_{u_0 \neq 0} \chi_{\Sigma\, \alpha_i}(u_0) \sum_{\Sigma\, v_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(v_j)$$

$$= \begin{cases} (q-1)\sum_{\Sigma\, v_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(v_j) & \text{if } \sum \alpha_i \sim 0 \\ 0 & \text{else} \end{cases}.$$

– Define the *Jacobi sum* for $\alpha$ where $\sum \alpha_i \sim 0$:

$$J(\alpha) := \left( \frac{1}{q-1} \right) \sum_{\Sigma \, u_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) = \sum_{\Sigma \, v_i = 0} \prod_{j=1}^{r} \chi_{\alpha_j}(v_j)$$

– Express $N$ in terms of Jacobi sums as

$$N = q^r + (q-1) \sum_{\substack{\Sigma \alpha_i \sim 0 \\ d_i \alpha_i \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) J(\alpha).$$

– Evaluate $J(\alpha)$ using Gauss sums: for $\chi : \mathbb{F}_q \longrightarrow \mathbb{C}$ a multiplicative character, define

$$G(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \psi_q(x).$$

– Proposition: for any $\chi \neq \chi_0$,
  – $|G(\chi)| = q^{\frac{1}{2}}$
  – $G(\chi)G(\bar{\chi}) = q\chi(-1)$
  – $G(\chi_0) = 0$

$$\chi(t) = \frac{G(\chi)}{q} \sum_{x \in \mathbb{F}_q} \bar{\chi}(x) \psi_q(tx).$$

# A Diagonal Hypersurface

– Proposition: if

$$\sum \alpha_i \sim 0 \implies J(\alpha) = \frac{1}{q} \prod_{k=1}^{r} G(\chi_{\alpha_k}) \quad \text{and} \quad |J(\alpha)| = q^{\frac{r-1}{2}}.$$

– We thus obtain

$$N = q^r + \left( \frac{q-1}{q} \right) \sum_{\substack{\sum \alpha_i \sim 0 \\ d_i \alpha_i \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) G(\chi_{\alpha_j}).$$

– We now ask for number of points in $\mathbb{F}_{q^\nu}$ and consider a point count

$$\overline{N}_\nu = \# \left\{ [x_0 : \cdots : x_r] \in \mathbb{P}^r_{\mathbb{F}^\nu_q} \;\middle|\; \sum_{i=0}^{r} a_i x_i^n = 0 \right\}.$$

– Theorem (Davenport, Hasse)

$$(q-1)\alpha \sim 0 \implies -G(\chi_{\alpha,\nu}) = (-G(\chi_\alpha))^\nu.$$

– We have a relation $(q^\nu - 1)\overline{N}_\nu = N_\nu$.

– This lets us write

$$\overline{N}_\nu = \sum_{j=0}^{r-1} q^{j\nu} + \sum_{\substack{\sum \alpha_i \, \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \prod_{j=0}^{r} \overline{\chi}_{\alpha_{j,\nu}}(a_i) J_\nu(\alpha).$$

– Set

$$\mu(\alpha) = \min \left\{ \mu \ \Big| \ (q^\mu - 1)\alpha \sim 0 \right\}$$

$$C(\alpha) = (-1)^{r+1} \prod_{j=1}^{r} \overline{\chi}_{\alpha_0, \mu(\alpha)}(a_j) \cdot J_{\mu(\alpha)}(\alpha).$$

– Plugging into the zeta function $Z$ yields

$$\exp \left( \sum_{\nu=1}^{\infty} \overline{N}_\nu \frac{T^\nu}{\nu} \right) = \prod_{j=0}^{r-1} \left( \frac{1}{1 - q^j T} \right) \prod_{\substack{\sum \alpha_i \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \left( 1 - C(\alpha) T^{\mu(\alpha)} \right)^{\frac{(-1)^r}{\mu(\alpha)}},$$

which is evidently a rational function.