Homological Algebra

Problem Set 6

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Problem 1.0.1 (5.7.1, Sentence One.)
In a Cartan-Eilenberg resolution, show that the following induced maps are projective resolutions in $\mathcal{A}$:

$$Z^p(\varepsilon) : Z_p(P, d^h) \to Z_p(A)$$
$$\varepsilon^p : P_{p,*} \to A_p.$$

Solution:
To show that these form projective resolutions, by definition we need to show that each object in the respective complexes is projective, and that each complex is exact, so kernels equal images. In what follows, fix a column p in $P_{*,*}$.

Claim: $Z_p(P, d^h)_q$ is a projective object for all $q \geq 0$.

Proof (?).
With p fixed, for every row q we have a SES

$$0 \to B_p(P, d^h)_q \to Z_p(P, d^h)_q \to H_p(P, d^h)_q \to 0.$$

Since by assumption $H_p(P, d^h)_*$ forms a projective resolution of A_p, each $H_p(P, d^h)_q$ is a projective object. So this sequence splits and we have

$$Z_p(P, d^h)_q \cong B_p(P, d^h)_q \oplus H_p(P, d^h)_q.$$

Working over R-modules, Z_p is projective if and only if it is a direct summand of a free module. By assumption, B_p, H_p are projective, and hence direct summands of free modules F_1, F_2. But then Z_p is a direct summand of $F_1 \oplus F_2$, which is still free, making Z_p projective.

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Claim: The complex $\left\{ Z_p(P, d^h)_q \;\middle|\; q \geq 0 \right\}$ is exact, making it a projective resolution of $Z_p(A)$.
We can assemble all of the above SESs into the following diagram:

$$\begin{tikzcd}
\vdots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \\
0 \arrow[r] & B_p(P, d^h)_q \arrow[r] \arrow[d] & Z_p(P, d^h)_q \arrow[r] \arrow[d] & H_p(P, d^h)_q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_p(P, d^h)_{q-1} \arrow[r] \arrow[d] & Z_p(P, d^h)_{q-1} \arrow[r] \arrow[d] & H_p(P, d^h)_{q-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & B_p(P, d^h)_1 \arrow[r] \arrow[d] & Z_p(P, d^h)_1 \arrow[r] \arrow[d] & H_p(P, d^h)_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_p(P, d^h)_0 \arrow[r] \arrow[d] & Z_p(P, d^h)_0 \arrow[r] \arrow[d] & H_p(P, d^h)_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_p(A) \arrow[r] & Z_p(A) \arrow[r] & H_p(A) \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

The vertical maps are all induced by the vertical maps in the original CE resolution. The blue portions are exact by assumption, since the H_p and B_p form projective resolutions of $H_p(A)$ and $B_p(A)$. Collapsing each vertical tower into a chain complex, we get a SES of complexes

$$0 \to B_p(P, d^h)_* \to Z_p(P, d^h)_* \to H_p(P, d^h)_* \to 0.$$

We thus get a long exact sequence in the homology of these complexes, where here we use $\mathcal{H}$ to distinguish this from the original homology and omit the asterisk for notational brevity:

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
\mathcal{H}_1 B_p(P, d^h) \arrow[r] & \mathcal{H}_1 Z_p(P, d^h) \arrow[r] & \mathcal{H}_1 H_p(P, d^h) \arrow[dll] \\
\mathcal{H}_0 B_p(P, d^h) \arrow[r] & \mathcal{H}_0 Z_p(P, d^h) \arrow[r] & \mathcal{H}_0 H_p(P, d^h) \arrow[d] \\
 & & 0
\end{tikzcd}$$

Link to Diagram

We can now use the fact that a complex is exact if and only if its homology vanishes in degree $d \geq 1$. Since B_p, H_p were exact, the edge terms in this LES are zero, yielding the following situation:

$$\begin{tikzcd}
& & \cdots \arrow[dll] \\
0 \arrow[r] & \mathcal{H}_2 Z_p(P, d^h) \arrow[r] & 0 \arrow[dll] \\
0 \arrow[r] & \mathcal{H}_1 Z_p(P, d^h) \arrow[r] & 0 \arrow[dll] \\
B_p(P, d^h) \arrow[r] & Z_p(P, d^h) \arrow[r] & H_p(P, d^h) \arrow[d] \\
& & 0
\end{tikzcd}$$

Link to Diagram

This forces $\mathcal{H}_d Z_p(P, d^h)_* = 0$ for $d \geq 1$, so the complex $Z_p(P, d^h)_*$ is exact as desired.

Claim: $P_{p,q}$ is projective object for all q and $Pp, *$ forms an exact complex, making it a projective resolution of A_p.

Proof (?).
Fixing q, we apply precisely the same argument as above to the SES

$$0 \to Z_p(P, d^h)_q \to P_{p,q} \to B_p(P, d^h)_q \to 0,$$

where we again use that B_p is projective to form the splitting which shows $P_{p,q}$ is projective, and that the complexes Z_p, B_p are exact to force exactness of the complex $P_{p,*}$.

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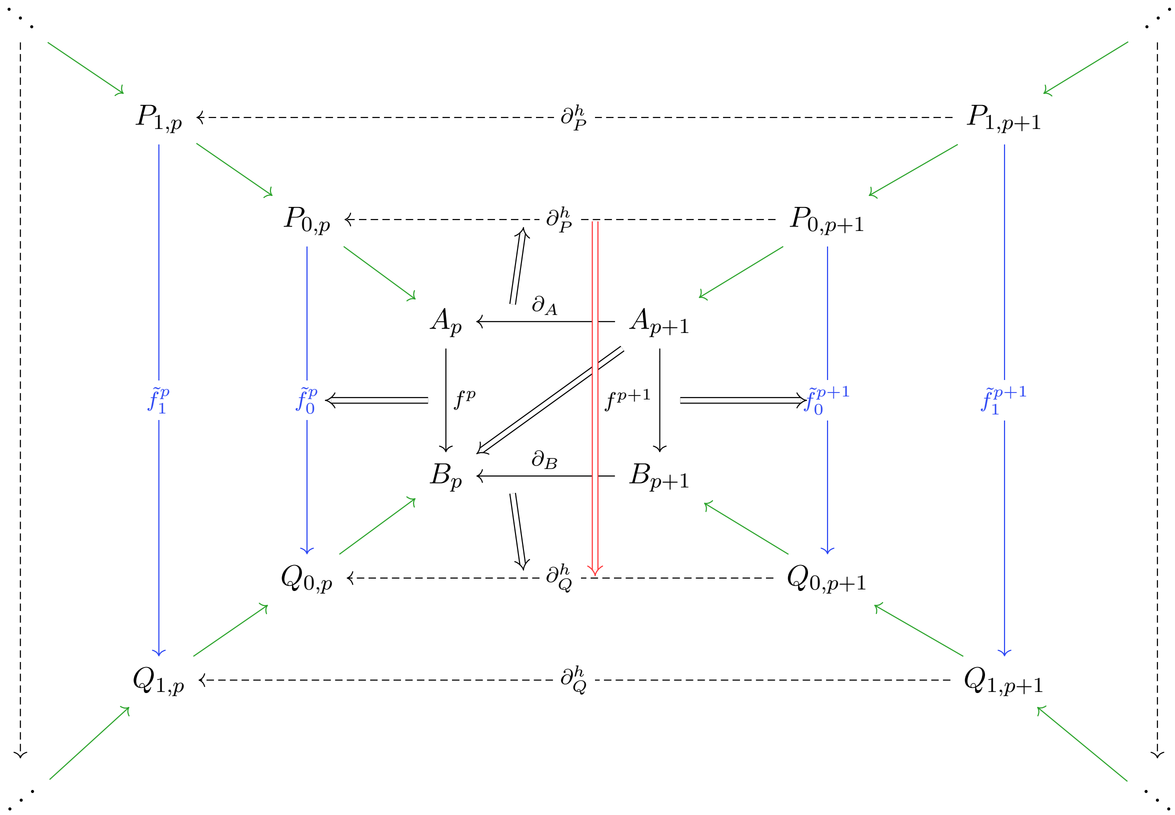
Problem 1.0.2 (5.7.2)
If $A \to B$ is a chain map and $P \to A, Q \to B$ are CE resolutions, show that there is a double complex map $\tilde{f} : P \to Q$ lifting f.

Solution:
Fixing p, let $P_{*,p} \to A_p$ be the column of the CE resolution above A_p, so it forms a projective resolution by the previous exercise, and let $Q_{*,p} \to B_p$ be the corresponding column above B_p. The claim is that there is a commutative tower of the following form:



Link to Diagram

- The green maps denote the vertical differentials in the respective CE resolutions, and the horizontal differentials are indicated with dotted lines.
- The blue maps $\tilde{f}_i^p$ and $\tilde{f}_i^{p+1}$ are supplied by the comparison theorem, which assemble to form chain maps $\tilde{f}^p : P_{*,p} \to Q_{*,p}$ that lift f^p for every p:

$$\begin{tikzcd}
P_{*,p} \arrow[r, dotted, "\exists \tilde{f}^p"] \arrow[d, two heads, "\varepsilon^A"] & Q_{*,p} \arrow[d, two heads, "\varepsilon^B"] \\
A_p \arrow[r, "f^p"] & B_p
\end{tikzcd}$$

Link to Diagram

- The double-arrows indicate squares that commute:
  - The square on the "ground floor" of the tower commutes because the original f was assumed to be a chain map.

- The squares corresponding to the North and South walls commute since they participate in the CE resolutions of A and B respectively.
- The squares corresponding to the East and West walls commute because the comparison theorem guarantees that the $\tilde{f}_i$ lift f and form a chain map.

Thus for these to assemble to form a map of double complexes $\tilde{f} : P_{*,*} \to Q_{*,*}$ that also lifts f, it only remains to show that the squares corresponding to the "ceiling" of each floor of the tower also commutes, as indicated with the red double arrow.

Not sure how to show this, doesn't seem guaranteed by the Comparison theorem. Hint from Weibel: modify the proof of theorem 2.4.6.

Problem 1.0.3 (5.7.3)

1. If $f, g : A \to B$ are homotopic maps of chain complexes and $\tilde{f}, \tilde{g} : P \to Q$ are maps of CE resolutions over them, show that $\tilde{f}$ is chain homotopic to $\tilde{g}$.

2. Show that any two CE resolutions P, Q of A are chain homotopy equivalent. Conclude that for any additive functor F, the chain complex $\mathrm{Tot}^{\oplus}(F(P))$ and $\mathrm{Tot}^{\oplus}(F(Q))$ are chain homotopy equivalent.

Solution:
It suffices to show that if $f : A \to B$ is a nullhomotopic map of chain complexes, then then induced map $\tilde{f} : P \to Q$ is a nullhomotopic map of double complexes, since $f \simeq g \iff f - g \simeq 0$. So suppose that $f : A \to B$ is nullhomotopic, which supplies us with a nullhomotopy $s : A \to B[1]$ such that $f = ds + sd$. By the previous exercise, the map of complexes $f : A \to B$ lifts to map of double complexes $\tilde{f} : P \to Q$, which in the pth component is a chain map $\tilde{f}^p : P_{p,*} \to Q_{p-1,*}$. Since f is nullhomotopic, we are supplied with a nullhomotopy $s : A \to B[1]$ (where $[1]$ denotes the shifted complex) satisfying $f = ds + sd$ in every component. By the same argument, this lifts to a double complex map $\tilde{s} : P \to Q[1, 0]$, and so fixing a p we have the following situation:

$$\begin{tikzcd}
P_{p-1,*} \arrow[d] \arrow[drr, "\tilde{s}_{p-1}"] & P_{p,*} \arrow[l] \arrow[d] \arrow[dr, "\tilde{f}^p"] \arrow[drr, "\tilde{s}_p"] & & \\
A_{p-1} \arrow[drr, "s_{p-1}"] & A_p \arrow[l] \arrow[dr, "f^p"] \arrow[drr, "s_p"] & Q_{p,*} \arrow[d] & Q_{p+1,*} \arrow[l] \arrow[d] \\
 & & B_p & B_{p+1} \arrow[l]
\end{tikzcd}$$

Link to Diagram

The base of this cube commutes by assumption, and $f^p = d^B s_p + s_{p-1} d^A$ since s was a nullhomotopy of f. The front and back sides commute by construction of the CE resolution, and the left and right sides commute since the $\tilde{s}_p$ lift the original s_p.

Claim: The map of double complexes $\tilde{s} : P \to Q[1,0]$ forms the horizontal component of a nullhomotopy $\mathbf{s} : P \to Q[1,1]$,
Remark 1.0.1: This will follow if we can construct a vertical homotopy s^v such that

$$\tilde{f}^p = \mathbf{s}\partial + \partial\mathbf{s} := \left(\tilde{s}_{p-1}\partial_P^h + \partial_Q^h \tilde{s}_p\right) + \left(s^v \partial_P^v + \partial_Q^v s^v\right)$$

as a map of complexes $P \to Q[1.1]$, where the ∂^h and ∂^v are the horizontal and vertical maps in the respective CE resolutions.

Claim: The vertical homotopy can be constructed to satisfy

$$\left(s^v \partial_P^v + \partial_Q^v s^v\right) = \tilde{f}^p - \left(\tilde{s}_{p-1}\partial_P^h + \partial_Q^h \tilde{s}_p\right).$$

Note sure where these vertical maps should come from.

Problem 1.0.4 (Extra)

1. Check the claim about B_p at the end of Lemma 5.7.2:

assume axiom (AB4) holds.

Lemma 5.7.2 *Every chain complex A_* has a Cartan-Eilenberg resolution $P_{**} \to A$.*

Proof For each p select projective resolutions P^B_{p*} of $B_p(A)$ and P^H_{p*} of $H_p(A)$. By the Horseshoe Lemma 2.2.8 there is a projective resolution P^Z_{p*} of $Z_p(A)$ so that

$$0 \to P^B_{p*} \to P^Z_{p*} \to P^H_{p*} \to 0$$

is an exact sequence of chain complexes lying over

$$0 \to B_p(A) \to Z_p(A) \to H_p(A) \to 0.$$

Applying the Horseshoe Lemma again, we find a projective resolution P^A_{p*} of A_p fitting into an exact sequence

$$0 \to P^Z_{p*} \to P^A_{p*} \to P^B_{p-1,*} \to 0.$$

We now define P_{**} to be the double complex whose p^{th} column is P^A_{p*} except that (using the Sign Trick 1.2.5) the vertical differential is multiplied by $(-1)^p$; the horizontal differential of P_{**} is the composite

$$P^A_{p+1,*} \to P^B_{p*} \hookrightarrow P^Z_{p*} \hookrightarrow P^A_{p*}.$$

The construction guarantees that the maps $\epsilon_p \colon P_{p0} \to A_p$ assemble to give a chain map ϵ, and that each $B_p(\epsilon)$ and $H_p(\epsilon)$ give projective resolutions (check this!). ◇

Definition 5.7.3 Let $f, g \colon D \to E$ be two maps *chain homotopy* from f to g consists of maps s^h_{pq} $D_{pq} \to E_{p,q+1}$ so that

$$g - f = (d^h s^h + s^h d^h) + (d^v s^v$$

2. Verify the remark following definition 5.7.3:

$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0.$$

This definition is set up so that $\{s^h + s^v \colon \mathrm{Tot}(D)_n \to \mathrm{Tot}(E)_{n+1}\}$ forms an ordinary chain homotopy between the maps $\mathrm{Tot}(f)$ and $\mathrm{Tot}(g)$ from $\mathrm{Tot}^{\oplus}(D)$ to $\mathrm{Tot}^{\oplus}(E)$.

Solution (Part 1):