Problem Set 5

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1 4.3

Proposition 1.1.
Suppose $\lambda+\rho \in \Lambda^{+}$. Then $M(w \cdot \lambda) \subset M(\lambda)$ for all $w \in W$. Thus all $[M(\lambda): L(w \cdot \lambda)]>0$.

More precisely, if $w=s_{n} \cdots s_{1}$ is a reduced expression for w in terms of simple reflections corresponding to roots α_{i}, then there is a sequence of embeddings:

$$M(w \cdot \lambda)=M\left(\lambda_{n}\right) \subset M\left(\lambda_{n-1}\right) \subset \cdots \subset M\left(\lambda_{0}\right)=M(\lambda)$$

Here

$$\begin{aligned} \lambda_{0}:=\lambda, \lambda_{k}:=s_{k} \cdot \lambda_{k-1}=\left(s_{k} \ldots s_{1}\right) \cdot \lambda \Longrightarrow \lambda_{n}=s_{n} \cdot \lambda_{n-1}=w \cdot \lambda \\ w \cdot \lambda=\lambda_{n} \leq \lambda_{n-1} \leq \cdots \leq \lambda_{0}=\lambda \text{with} \quad \left\langle\lambda_{k}+\rho, \alpha_{k+1}^{\vee}\right\rangle \in \mathbb{Z}^{+} \text{ for } k=0, \cdots, n-1. \end{aligned}$$

Assume $\lambda+\rho \in \Lambda^{+}$.

a. Prove that the unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_{\diamond} \cdot \lambda)$, where $w_{\diamond}$ is the longest element of W.

b. In case $\lambda \in \Lambda^{+}$, show that the inclusions obtained in the above proposition are all proper.

2 4.6

Theorem 2.1 ***(Verma).***
Let $\lambda \in \mathfrak{h}^\vee$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Work through the steps of Verma's Theorem in the special case discussed in the previous problem

3 4.11

In the case of $\mathfrak{sl}(3, \mathbb{C})$, what can be said at this point about Verma modules with a singular integral highest weight?

Aside from the trivial case $-\rho$, a typical linkage class has 3 elements. For example, if λ lies in the α hyperplane and is antidominant, the linked weights are $\lambda,\ s_\beta \cdot \lambda,\ s_\alpha s_\beta \cdot \lambda$.