Homological Algebra Problem Sets

Problem Set 1

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Last updated:

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Problem 1.0.1 (Weibel 1.1.2)
Show that a morphism $u : C \to D$ of chain complexes preserves boundaries and cycles respectively, hence inducing a map $H_n(C) \to H_n(D)$ for each n. Prove that $H_n : \mathrm{Ch}(R\text{-}\mathrm{mod}) \to R\text{-}\mathrm{mod}$ is a functor.

Solution:

Claim 1: The chain map u induces the following well-defined maps:

$$Z_n(u) : Z_n(C) \to Z_n(D)$$
$$B_n(u) : B_n(C) \to B_n(D).$$

Proof (of claim (1)).
We'll use the convention that $Z_n \coloneqq \ker d_n$ and $B_n \coloneqq \operatorname{im} d_{n+1}$ where we index chain complexes as $C = \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots\right)$. Unraveling definitions, we would like to show the existence of maps

$$\begin{aligned} Z_n(u) &: \ker d_n^C \to \ker d_n^D \\ B_n(u) &: \operatorname{im} d_{n+1}^C \to \operatorname{im} d_{n+1}^D. \end{aligned}$$

It suffices to show

a. $x \in \ker d_n^C \implies u_n(x) \in \ker d_n^D$, and
b. $y \in \operatorname{im} d_{n+1}^C \implies u_n(y) \in \operatorname{im} d_{n+1}^D$.

Since u is a morphism of chain complexes, we have a commuting ladder where $u_{n-1} \circ d_n^C = d_n^D \circ u_n$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n^C"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}^C"] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}^D"'] & D_n \arrow[r, "d_n^D"'] & D_{n-1} \arrow[r, "d_{n-1}^D"'] & \cdots
\end{tikzcd}$$

Link to Diagram

To see that (a) holds, we compute

$$\begin{aligned}
x \in \ker d_n^C & & \leq C_n & \\
\iff d_n^C(x) = 0_R & & \in C_{n-1} & \\
\iff (u_{n-1} \circ d_n^C)(x) = 0_R & & \in D_{n-1} & \quad \text{since } u_n \text{ is a ring morphism and sends } 0_R \to 0_R \\
\implies (d_n^D \circ u_n)(x) = 0_R & & \in D_{n-1} & \quad \text{using commutativity} \\
\implies x \in \ker(d_n^D \circ u_n) & & \leq D_{n-1} & \\
\iff u_n(x) \in \ker d_n^D & & \leq D_n. &
\end{aligned}$$

Similarly, for (b) we have

$$\begin{aligned}
y \in \operatorname{im} d_{n+1}^C &\iff \exists x \in C_{n+1} \text{ such that } d_{n+1}^C(x) = y \\
&\implies u_{n+1}(x) \in D_{n+1} \\
&\implies (d_{n+1}^D \circ u_{n+1})(x) \in \operatorname{im} d_{n+1}^D \leq D_n \\
&\implies (u_n \circ d_{n+1}^C)(x) \in \operatorname{im} d_{n+1}^D \leq D_n && \text{using commutativity} \\
&\iff u_n(y) \in \operatorname{im} d_{n+1}^D && \text{using } d_{n+1}^C(x) = y.
\end{aligned}$$

■

Now noting that $H_n(C) \coloneqq Z_n(C)/B_n(C)$, since u_n preserves Z_n there is a well-defined restriction of each $u_n : C_n \to D_n$ to $u_n : Z_n(C) \to Z_n(D)$. Composing with the projection $Z_n(D) \to Z_n(D)/B_n(D) \coloneqq H_n(D)$ yields maps $u_n : Z_n(C) \to H_n(D)$.

Problem 1.0.2 (Weibel 1.1.4)
Show that for every $A \in R\text{-}\mathrm{mod}$ and $C \in \mathrm{Ch}(R\text{-}\mathrm{mod})$ that $D. \coloneqq \mathrm{Hom}_{R\text{-}\mathrm{mod}}(A, C.)$ is a chain complex of abelian groups. Taking $A \coloneqq Z_n$, show that $H_n(D.) = 0 \implies H_n(C.) = 0$. Is the converse true?

Solution:
We first show that if $A \in R\text{-}\mathrm{mod}$ and $C \in \mathrm{Ch}(R\text{-}\mathrm{mod})$, then

$$D_n \coloneqq \mathrm{Hom}_{R\text{-}\mathrm{mod}}(A, C_n).$$

defines a chain complex of abelian groups. Fixing notation, we write

$$C \coloneqq (\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots).$$

1. D_n is an abelian group for all n: Define an operation

$$\begin{aligned} +_D : D_n \times D_n &\to D_n \\ (f, g) &\mapsto \left\{ \begin{matrix} A \xrightarrow{f+g} C_n \\ x \mapsto f(x) +_C g(x) \end{matrix} \right\}, \end{aligned}$$

   where $+_C$ is the addition on C_n provided by its structure as an R-module. We can then check that this operation is commutative:

$$\begin{aligned} (f +_D g)(x) &\coloneqq f(x) +_C g(x) \\ &= g(x) +_C f(x) && \text{since the addition on } C_n \text{ is commutative} \\ &= (g +_D f)(x), \end{aligned}$$

   The additive inverse of f is $-f$, there is an identity function $\mathrm{id}_{C_n}(x) \coloneqq x$, and the sum of two functions $A \to C_n$ is again a function $A \to C_n$, making D_n an abelian group for all n.

2. There exist differentials $D_n \xrightarrow{d_n^D} D_{n-1}$: Noting that we have differentials $C_n \xrightarrow{d_n^C} C_{n-1}$, we can define

$$\begin{aligned} d_n^D : D_n &\to D_{n-1} \\ (A \xrightarrow{f} C_n) &\mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1}), \end{aligned}$$

   i.e. we send $f \mapsto d_n^C \circ f$ be precomposing with the differential from C_*.

3. $(d^D)^2 = 0$: We can explicitly write

$$\begin{aligned} (d^D)^2 : D_n &\to D_{n-2} \\ (A \xrightarrow{f} C_n) &\mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1} \xrightarrow{d_{n-1}^C} C_{n-2}), \end{aligned}$$

and so $f \mapsto d^C_{n-1} \circ d^C_n \circ f$. The claim is that this is the zero map, which follows from writing this as $(d^C)^2 \circ f = 0 \circ f = 0$, using that C_* is a chain complex.

Thus

$$D \coloneqq (\cdots \to D_{n+1} \xrightarrow{d^D_{n+1}} D_n \xrightarrow{d^D_n} D_{n-1} \to \cdots) \in \mathrm{Ch}(\mathrm{Ab}).$$

Writing $Z_n \coloneqq Z_n(C) \coloneqq \ker d^C_n$, we now show the following:

$$H_n(\mathrm{Hom}_{R\text{-}\mathrm{mod}}(Z_n, C) = 0 \implies H_n(C) = 0.$$

It suffices to show that $\ker d^C_n \subseteq \operatorname{im} d^C_{n+1}$, so let $y \in \ker d^C_n$; we want to produce the following:

$$x \in C_{n+1}, \quad d^C_{n+1}(x) = y.$$

We can start with the inclusion map

$$\iota : \ker d^C_n \hookrightarrow C_n,$$

which by definition is an element of $D_n \coloneqq \hom(Z_n, C_n)$. By assumption, the following complex is exact at n since its homology vanishes at position n:

$$\begin{gathered}(\cdots \to D_{n+1} \to D_n \to D_{n-1} \to \cdots) \coloneqq \\ \cdots \to \mathrm{Hom}_R(Z_n, C_{n+1}) \xrightarrow{d^D_{n+1}} \mathrm{Hom}_R(Z_n, C_n) \xrightarrow{d^D_n} \mathrm{Hom}_R(Z_n, C_{n-1}) \to \cdots.\end{gathered}$$

Claim: $d^D_n(\iota) = 0$.

This can be seen by writing this out as the composition

$$d^D_n(\ker d^C_n \xrightarrow{\iota} C_n) = (\ker d^C_n \xrightarrow{\iota} C_n \xrightarrow{d^C_n} C_{n-1}).$$

We can now use the general fact that the $f(\ker f) = 0$ for any map f, i.e. the image of the kernel is necessarily zero. Taking $f = d^C_n$ shows that this composition is zero. By exactness, $\ker d^D_n = \operatorname{im} d^D_{n+1}$ and we can thus pull ι back to some $f \in D_{n+1} \coloneqq \mathrm{Hom}_R(Z_n, C_{n+1})$, and since our original $y \in \ker d^C_n \coloneqq Z_n$, it makes sense to consider $x \coloneqq f(y) \in C_{n+1}$ and to identity $y = \iota(y) \in C_n$:

$$\begin{tikzcd}
& & y \arrow[d, phantom, "\rotatebox[origin=c]{-90}{$\in$}"] & & \\
& & Z_n \arrow[dl, dashed, "\exists f"'] \arrow[d, hook, "\iota"'] & & \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"'] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & C_n \arrow[r, "d^C_n"'] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & C_{n-1} \arrow[r] & \cdots \\
& x \coloneqq f(y) & \iota(y) & &
\end{tikzcd}$$

Link to Diagram

Importantly, this f satisfies $\iota = d^D_{n+1}(f) \coloneqq d^C_{n+1} \circ f$, and so we can write

$$y = \iota(y) = (d^C_{n+1} \circ f)(y) \coloneqq d_{n+1}(x),$$

which is what we wanted to show.

Problem 1.0.3 (Weibel 1.1.6: Homology of a graph)
Let Γ be a finite graph with vertices $V \coloneqq \{v_1, \cdots, v_V\}$ and edge $E \coloneqq \{e_1, \cdots, e_E\}$. Define the **incidence matrix** of Γ to be the $V \times E$ matrix A where

$$A_{ij} = \begin{cases} 1 & e_j \text{ starts at } v_i, \\ -1 & e_j \text{ ends at } v_i, \\ 0 & \text{else.} \end{cases}$$

Define a chain complex by taking free R-modules:

$$C \coloneqq (\cdots \to 0 \to C_1 \to C_0 \to 0 \to \cdots) = (\cdots \to 0 \to R^E \xrightarrow{A} R^V \to 0 \to \cdots).$$

If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and $E - V + 1$ respectively.

Hint: choose a basis $\{v_1, v_2 - v_1, \cdots, v_V - v_1\}$ and use a path from $v_1 \rightsquigarrow v_i$ to produce an element $e \in C_1$ with $d(e) = v_i - v_1$.

Solution:
We first make the following two observations:

1. $H_0(C) = \operatorname{coker}(A) \cong R^V / \operatorname{im} A \implies \operatorname{rank} H_0(C) = V - \operatorname{rank} \operatorname{im} A$, and
2. $H_1(C) = \ker(A) \implies \operatorname{rank} H_1(C) = \operatorname{rank} \ker A$

Claim: $\operatorname{rank} \operatorname{im}(A) = V - 1$.
Given this claim, applying observation (1) we immediately obtain

$$\operatorname{rank} H_0(C) = V - (V - 1) = 1,$$

which is the first equality we want to show. For the second equality, we can use the first isomorphism theorem to get a SES of free R-modules

$$0 \to \ker(A) \hookrightarrow R^E \to \operatorname{im}(A) \to 0,$$

and since $\operatorname{im}(A)$ is free and thus projective, this sequence splits. So $R^E \cong \ker(A) \oplus \operatorname{im}(A)$, and taking free ranks yields

$$E = \operatorname{rank} \ker(A) + (V - 1) \implies \operatorname{rank} \ker(A) = E - V + 1,$$

and this yields the second equality by using observation (2) to identify the LHS with $\operatorname{rank} H_1(C)$.

Proof (of claim).
Using the fact that

$$\mathcal{B} \coloneqq \{v_1, \cdots, v_V\}$$

is a basis for R^V as a free R-module, we can make a change of basis to

$$\mathcal{B}' \coloneqq \{v_1, v_2 - v_1, \cdots, v_V - v_1\}.$$

That this is again a basis follows from the fact that the change-of-basis matrix M is upper-triangular with ones on the diagonal and thus satisfies $\det M = 1_R \in R^\times$, making it nonsingular. We can then observe that if e_i is an edge between two vertices $v_{i_1} \xrightarrow{e_i} v_{i_2}$, then $d(e_i) \coloneqq Ae_i = v_{i_2} - v_{i_2}$. By linearity, if $e_{i_1}, \cdots, e_{i_n}$ is a sequence of edges connecting v_1 to v_j for any $1 \leq j \leq V$, then

$$d(e_{i_1} + \cdots + e_{i_n}) = v_j - v_1.$$

Since Γ is connected, there always exists such a sequence of edges connecting each v_j to v_1, and thus $v_j - v_1$ is in $\operatorname{im}(A)$. We can conclude that

$$V - 1 \leq \operatorname{rank} \operatorname{im}(A) \leq V.$$

To see that $\operatorname{rank} \operatorname{im}(A) \neq V$, note that if e is any sequence of edges connecting v_1 to itself in a loop, then $d(e_1) = v_1 - v_1 = 0$. Any other path e' must necessarily start or end at some $v_j \neq v_1$ and satisfies $d(e') = v_j - v_1 \neq v_1$, and so $v_1 \notin \operatorname{im}(A)$. Thus

$$\operatorname{rank} \operatorname{im}(A) = V - 1.$$

■

Problem 1.0.4 (Weibel 1.1.7: Tetrahedra)
The **tetrahedron** T is a surface with 4 vertices, 6 edges, and 4 faces of dimension 2, and its homology is the homology of the complex

$$C. \coloneqq (\cdots \to 0 \to R^4 \to R^6 \to R^4 \to 0 \to \cdots).$$

Write down the matrices in this complex and computationally verify that

$$H_*(T) = [R, 0, R, 0, \cdots].$$

Problem 1.0.5 (Weibel 1.2.3)
Let $\mathcal{A}$ be the category $\operatorname{Ch}(R\text{-mod})$ and let f be a chain map. Show that the complex $\ker f$ is a (categorical) kernel of f and that $\operatorname{coker} f$ is a (categorical) cokernel of f.

Solution:
For a fixed map $f : A \to B$, the *kernel* of f is an object $\ker f$ satisfying the following universal property: for any object K with a morphism $K \xrightarrow{g} A$ making the following outer square

commute, there is a unique morphism $u : K \to \ker f$ making the entire diagram commute:

$$\begin{tikzcd}
K \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\exists! u"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f \arrow[r, "\iota^f"] \arrow[d, "0"'] & A \arrow[d, "f"] \\
& 0 \arrow[r, "0"'] & B
\end{tikzcd}$$

We'll use without proof that kernels exist in $\mathcal{A} = R$-mod and are given by $\ker f \coloneqq \left\{ a \in A \;\middle|\; f(a) = 0_B \right\}$ along with an inclusion map $\iota^f : \ker f \hookrightarrow A$.

Let $A, B \in \mathrm{Ch}(\mathcal{A})$ be chain complexes and $f : A \to B$ be a chain map. We will construct $\ker f$ as a chain complex and show it satisfies the correct universal property.

Claim 1: There are unique objects $\ker f_n \in R$-mod which can be assembled into a unique chain complex $(\ker f, \partial^f)$.

Proof (?).
Let $u : A \to B$ be a chain map, so that we have a commuting diagram of the following form:
% https://q.uiver.app/?q=WzAsMTAsWzIsMCwiQV97bisxfSJdLFs0LDAsIkFfbiJdLFs2LDAsIkFfe24tMX

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Appealing to the universal property of kernels in R-mod, we can produce unique objects $\ker f_n$ and morphisms $\iota^f_n : \ker f_n \to A_n$ satisfying $(\ker f_n \to A_n \to B_n) = 0$ for every n. We also claim that there are maps $\partial^f_n : \ker f_n \to \ker f_{n-1}$, yielding the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & \ker f_{n+1} \arrow[r, dotted, "\partial^f_{n+1}"] \arrow[d, "\iota^f_{n+1}"] \arrow[dr, phantom, "2"] & \ker f_n \arrow[r, dotted, "\partial^f_n"] \arrow[d, "\iota^f_n"] \arrow[dr, phantom, "3"] & \ker f_{n-1} \arrow[r, dotted] \arrow[d, "\iota^f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "u_{n+1}"] \arrow[dr, phantom, "1"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "u_n"] & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Why the ∂^f_n exist: this follows from the universal property of kernels in $\mathcal{A}$: Using the commutativity of square 1 we have

$$0 = (\ker f_{n+1} \to A_{n+1} \to B_{n+1} \to B_n) = (\ker f_{n+1} \to A_{n+1} \to A_n \to B_n),$$

where we've also used the fact that $(\ker f_{n+1} \to A_{n+1} \to B_{n+1} = 0)$ from the universal property of $\ker f_{n+1}$. So we can fit these into an appropriate diagram in $\mathcal{A}$, which supplies these differentials:

$$\begin{tikzcd}
\ker f_{n+1} \arrow[dr, dotted, "\exists! \partial^f_{n+1}"] \arrow[drr, bend left, "\partial^A_{n+1} \circ \iota^f_{n+1}"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f_n \arrow[r, "\iota^f"] \arrow[d, "0"'] & A_n \arrow[d, "f_n"] \\
& 0 \arrow[r, "0"'] & B_n
\end{tikzcd}$$

Why the $\iota^f : \ker f \to A$ assemble into a chain map: Note that everything here commutes, and we can break the northeast corner of this diagram up and rearrange things slightly to form the following diagram:
% https://q.uiver.app/?q=WzAsOCxbMCwwLCJcXGtlciBmX3tuKzF9Il0sWzIsMCwiQV97bisxfSJdLFsxLDEsIkFf

$$\begin{tikzcd}
\ker f_{n+1} \arrow[r, "\iota^f_{n+1}"] \arrow[d, dashed, "\exists! \partial^f_{n+1}"'] \arrow[dr, phantom, "2"] & A_{n+1} \arrow[d, "\partial^A_{n+1}"] \\
{} & {}
\end{tikzcd}$$

Claim 2: The complex $\ker f$ satisfies the universal property of kernels in $\mathrm{Ch}(\mathcal{A})$, i.e. if $g^K : K \to A$ is a chain map satisfying $K \to A \to B = 0$, there is a unique chain map $u : K \to \ker f$ making the appropriate diagram commute.

Proof (?).
Again using the universal property of kernels in R-mod, for each n we have a commutative diagram

$$\begin{tikzcd}
K_n \arrow[dr, dotted, "\exists ! u_n"] \arrow[drr, bend left, "g_n^K"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f_n \arrow[r, "\iota_n^f"] \arrow[d, "0"'] & A_n \arrow[d, "f"] \\
& 0 \arrow[r, "0"'] & B_n
\end{tikzcd}$$

This results in a diagram of the following form:

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n+1} \arrow[r, "\partial^K_{n+1}"] \arrow[d, dashed, "\exists u_{n+1}"] \arrow[dd, bend right, red, "g^K_{n+1}"'] \arrow[dr, phantom, "1"] & K_n \arrow[r, "\partial^K_{n+1}"] \arrow[d, dashed, "\exists u_n"] \arrow[dd, bend right, red, "g^K_n"'] & K_{n-1} \arrow[r] \arrow[d, dashed, "\exists u_{n-1}"] \arrow[dd, bend right, red, "g^K_{n-1}"'] & \cdots \\
\cdots \arrow[r] & \ker f_{n+1} \arrow[r, "\partial^f_{n+1}"] \arrow[d, "\iota^f_{n+1}"] \arrow[dr, phantom, "3"] & \ker f_n \arrow[r, "\partial^f_n"] \arrow[d, "\iota^f_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "\iota^f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "u_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "u_n"] & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

It only remains to check that the u_n assemble to a chain map $K \to \ker f$, which would follow from the commutativity of e.g. square (1). However, if (1) were *not* commutative, then the rectangle formed by (1) and (3) together would not be commutative – but g^K was assumed to be a chain map, so this rectangle commutes, yielding a contradiction. ■

Note: a proof of a similar flavor seems to work for the cokernel complex by reversing all of the arrows.

Problem 1.0.6 (?)
Verify exactness in the Snake Lemma in at least two other positions.

Solution:
This follows from the construction of the complex $\ker f$ above, specifically using the fact that the constructed differential ∂^f satisfies $(\partial^f)^2 = 0$.