

# Title

# Contents

0.1	Big Theorems / Tools: . . . . .	2
0.2	Differential . . . . .	3
0.2.1	Limits . . . . .	3
0.2.2	Derivatives . . . . .	4
0.2.3	Related Rates . . . . .	5
0.3	Integral . . . . .	5
0.3.1	Big List of Integration Techniques . . . . .	6
0.3.2	Optimization . . . . .	8

$$\frac{\partial}{\partial x} \int_1^x f(x, t) dt = \int_1^x \frac{\partial}{\partial x} f(x, t) dt + f(x, x)$$

## 0.1 Big Theorems / Tools:

**Proposition 0.1.1** (*Fundamental Theorem of Calculus I*).

$$\frac{\partial}{\partial x} \int_a^x f(t) dt = f(x)$$

**Proposition 0.1.2** (*Generalized Fundamental Theorem of Calculus*).

$$\begin{aligned} \frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, t) dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt &= f(x, \cdot) \frac{\partial}{\partial x} (\cdot) \Big|_{a(x)}^{b(x)} \\ &= f(x, b(x)) b'(x) - f(x, a(x)) a'(x) \end{aligned}$$

If  $f(x, t) = f(t)$  doesn't depend on  $x$ , then  $\frac{\partial f}{\partial x} = 0$  and the second integral vanishes:

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(t) dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

**Remark 0.1.1.**

Note that you can recover the original FTC by taking

$$\begin{aligned} a(x) &= c \\ b(x) &= x \\ f(x, t) &= f(t). \end{aligned}$$

- Extreme Value Theorem
- Involving the Derivative:
  - Mean Value Theorem:

$$f \in C^0(I) \implies \exists p \in I : f(b) - f(a) = f'(p)(b - a).$$

- Useful variant for integrals and average value:

$$f \in C^0(I) \implies \exists p \in I : \int_a^b f(x) \, dx = f(p)(b - a)$$

- Rolle's Theorem
- L'Hopital's Rule: If
- $f(x), g(x)$  differentiable on  $I - \{\text{pt}\}$
- $\lim_{x \rightarrow \text{pt}} f(x) = \lim_{x \rightarrow \{\text{pt}\}} g(x) \in \{0, \pm\infty\}$
- $\forall x \in I, g'(x) \neq 0$
- $\lim_{x \rightarrow \{\text{pt}\}} \frac{f'(x)}{g'(x)}$  exists

$$\implies \lim_{x \rightarrow \{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \{\text{pt}\}} \frac{f'(x)}{g'(x)}$$

- Taylor Expansions:

$$\begin{aligned} T(a, x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 \\ &\quad + \frac{1}{6}f'''(a)(x - a)^3 + \frac{1}{24}f^{(4)}(a)(x - a)^4 + \dots \end{aligned}$$

Bounded error:  $|f(x) - T_k(a, x)| < \left| \frac{1}{(k+1)!} f^{(k+1)}(a) \right|$  where  $T_k(a, x)$  is the Taylor series truncated up to and including the  $x^k$  term.

## 0.2 Differential

### 0.2.1 Limits

- Tools for finding  $\lim_{x \rightarrow a} f(x)$ , in order of difficulty:
  - Plug in: equal to  $f(a)$  if continuous
  - L'Hopital's Rule (only for indeterminate forms  $\frac{0}{0}, \frac{\infty}{\infty}$ )
    - ◇ For  $\lim f(x)^{g(x)} = 1^\infty, \infty^0, 0^0$ , let  $L = \lim f^g \implies \ln L = \lim g \ln f$
  - Algebraic rules
  - Squeeze theorem
  - Expand in Taylor series at  $a$
  - Monotonic + bounded

- One-sided limits:  $\lim_{x \rightarrow a^-} f(x) = \lim_{\varepsilon \rightarrow 0} f(a - \varepsilon)$
- Limits at zero or infinity:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{\frac{1}{x} \rightarrow 0} f\left(\frac{1}{x}\right) \text{ and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow \infty} f(1/x)$$

– Also useful:

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \deg p < \deg q \\ \infty & \deg p > \deg q \\ \frac{p_n}{q_n} & \deg p = \deg q \end{cases}$$

- Be careful: limits may not exist!!
  - Example :  $\lim_{x \rightarrow 0} \frac{1}{x} \neq 0$
- Asymptotes:
  - Vertical asymptotes: at values  $x = p$  where  $\lim_{x \rightarrow p} = \pm\infty$
  - Horizontal asymptotes: given by points  $y = L$  where  $\lim_{x \rightarrow \pm\infty} f(x) < \infty$
  - Oblique asymptotes: for rational functions, divide - terms without denominators yield equation of asymptote (i.e. look at the asymptotic order or “limiting behavior”).
    - ◇ Concretely:  $f(x) = \frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{t(x)} \sim r(x)$
- Limit of a recurrence:  $x_n = f(x_{n-1}, x_{n-2}, \dots)$ 
  - If the limit exists, it is a solution to  $x = f(x)$

### 0.2.2 Derivatives

- Chain rule:  $\frac{\partial}{\partial x} (f \circ g)(x) = f'(g(x))g'(x)$
- Product rule:  $\frac{\partial}{\partial f} (x)g(x) = f'g + g'f$ 
  - Note for all rules: always prime the first thing!
- Quotient rule:  $\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{f'g - g'f}{g^2}$ 
  - Mnemonic: Low d-high minus high d-low
- Inverse rule:  $\frac{\partial f^{-1}}{\partial x} (f(x_0)) = \left(\frac{\partial f}{\partial x}\right)^{-1} (x_0) = 1/f'(x_0)$
- Implicit differentiation:  $(f(x))' = f'(x) dx, (f(y))' = f'(y) dy$ 
  - Often able to solve for  $\frac{\partial}{\partial y} x$  this way.
- Obtaining derivatives of inverse functions: if  $y = f^{-1}(x)$  then write  $f(y) = x$  and implicitly differentiate.
- Approximating change:  $\Delta y \approx f'(x)\Delta x$

### 0.2.3 Related Rates

General series of steps: want to know some unknown rate  $y_t$

- Lay out known relation that involves  $y$
- Take derivative implicitly (say w.r.t  $t$ ) to obtain a relation between  $y_t$  and other stuff.
- Isolate  $y_t =$  known stuff
- Example: ladder sliding down wall
  - Setup:  $l, x_t$  and  $x(t)$  are known for a given  $t$ , want  $y_t$ .
  - $x(t)^2 + y(t)^2 = l^2 \implies 2xx_t + 2yy_t = 2ll_t = 0$  (noting that  $l$  is constant)
  - So  $y_t = -\frac{x(t)}{y(t)}x_t$
  - $x(t)$  is known, so obtain  $y(t) = \sqrt{l^2 - x(t)^2}$  and solve.

### 0.3 Integral

- Average values:

$$f_{\text{avg}}(x) = \frac{1}{b-a} \int_a^b f(t) dt$$

- Proof: apply MVT to  $F(x)$ .
- Area Between Curves
  - Area in polar coordinates:

$$A = \int_{r_1}^{r_2} \frac{1}{2} r^2(\theta) d\theta$$

- Solids of Revolution
  - Disks:  $A = \int \pi r(t)^2 dt$
  - Cylinders:  $A = \int 2\pi r(t)h(t) dt$
- Arc lengths

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} & L &= \int ds \\ &= \int_{x_0}^{x_1} \sqrt{1 + \frac{\partial y}{\partial x}} dx \\ &= \int_{y_0}^{y_1} \sqrt{\frac{\partial x}{\partial y} + 1} dy \end{aligned}$$

- $SA = \int 2\pi r(x) ds$
- Center of Mass
  - Given a density  $\rho(\mathbf{x})$  of an object  $R$ , the  $x_i$  coordinate is given by

$$x_i = \frac{\int_R x_i \rho(x) dx}{\int_R \rho(x) dx}$$

**0.3.1 Big List of Integration Techniques**

Given  $f(x)$ , we want to find an antiderivative  $F(x) = \int f$  satisfying  $\frac{\partial}{\partial x} F(x) = f(x)$

- Guess and check: look for a function that differentiates to  $f$ .
- $u$ - substitution
  - More generally, any change of variables

$$x = g(u) \implies \int_a^b f(x) dx = \int_{g^{-1}(a)}^{g^{-1}(b)} (f \circ g)(x) g'(x) dx$$

- Integration by Parts:
  - The standard form:

$$\int u dv = uv - \int v du$$

- A more general form for repeated applications: let  $v^{-1} = \int v$ ,  $v^{-2} = \int \int v$ , etc.

$$\begin{aligned} \int_a^b uv &= uv^{-1} \Big|_a^b - \int_a^b u^1 v^{-1} \\ &= uv^{-1} - u^1 v^{-2} \Big|_a^b + \int_a^b u^2 v^{-2} \\ &= uv^{-1} - u^1 v^{-2} + u^2 v^{-3} \Big|_a^b - \int_a^b u^3 v^{-3} \\ &\vdots \\ \implies \int_a^b uv &= \sum_{k=1}^n (-1)^k u^{k-1} v^{-k} \Big|_a^b + (-1)^n \int_a^b u^n v^{-n} \end{aligned}$$

- Generally useful when one term's  $n$ th derivative is a constant.
- Shoelace method:
- Note: you can choose  $u$  or  $v$  equal to 1! Useful if you know the derivative of the integrand.
- Differentiating under the integral

$$\begin{aligned} \frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, t) dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt &= f(x, \cdot) \frac{\partial}{\partial x} (\cdot) \Big|_{a(x)}^{b(x)} \\ &= f(x, b(x)) b'(x) - f(x, a(x)) a'(x) \end{aligned}$$

- Proof: let  $F(x)$  be an antiderivative and compute  $F'(x)$  using the chain rule.
- #todo for constants, this should allow differentiating under the integral when  $f, f_x$  are “jointly continuous”

- LIPET: Log, Inverse trig, Polynomial, Exponential, Trig: generally let  $u$  be whichever one comes first.
- The ridiculous trig sub: for any integrand containing only trig terms
  - Transforms *any* such integrand into a rational function of  $x$
  - Let  $u = 2 \tan^{-1} x$ ,  $du = \frac{2}{x^2 + 1}$ , then

$$\int_a^b f(x) dx = \int_{\tan \frac{a}{2}}^{\tan \frac{b}{2}} f(u) du$$

◇ Example:  $\int_0^{\pi/2} \frac{1}{\sin \theta} d\theta = 1/2$

Derivatives	Integrals	Signs	Result
$u$	$v$	NA	NA
$u'$	$\int v$	+	$u \int v$
$u''$	$\int \int v$	–	$-u' \int \int v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fill out until one column is zero (alternate signs). Get the result column by multiplying diagonally, then sum down the column.

- Trigonometric Substitution

$$\begin{array}{llll} \sqrt{a^2 - x^2} & \Rightarrow & x = a \sin(\theta) & dx = a \cos(\theta) d\theta \\ \sqrt{a^2 + x^2} & \Rightarrow & x = a \tan(\theta) & dx = a \sec^2(\theta) d\theta \\ \sqrt{x^2 - a^2} & \Rightarrow & x = a \sec(\theta) & dx = a \sec(\theta) \tan(\theta) d\theta \end{array}$$

- Partial Fractions
- Completing the Square #todo
- Trig Formulas
  - Double angle formulas:

$$\begin{array}{lll} \sin^2(x) & = & \frac{1}{2}(1 - \cos 2x) \\ & = & \\ & = & \\ & = & \\ & = & \end{array}$$

- Products of trig functions
  - Setup:  $\int \sin^a(x) \cos^b(x) dx$

- ◇ Both  $a, b$  even:  $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$
- ◇  $a$  odd:  $\sin^2 = 1 - \cos^2$ ,  $u = \cos(x)$
- ◇  $b$  odd:  $\cos^2 = 1 - \sin^2$ ,  $u = \sin(x)$
- Setup:  $\int \tan^a(x) \sec^b(x) dx$ 
  - ◇  $a$  odd:  $\tan^2 = \sec^2 - 1$ ,  $u = \sec(x)$
  - ◇  $b$  even:  $\sec^2 = \tan^2 + 1$ ,  $u = \tan(x)$

Other small but useful facts:

$$\int_0^{2\pi} \sin \theta \, d\theta = \int_0^{2\pi} \cos \theta \, d\theta = 0.$$

### 0.3.2 Optimization

- Critical points: boundary points and wherever  $f'(x) = 0$
- Second derivative test:
  - $f''(p) > 0 \implies p$  is a min
  - $f''(p) < 0 \implies p$  is a max
- Inflection points of  $h$  occur where the *tangent* of  $h'$  changes sign. (Note that this is where  $h'$  itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the  $x$  that satisfies this; it can be used in the original equation.
- Fundamental theorem of Calculus: If

$$\int_a^b f(x) dx = F(b) - F(a) \implies F'(x) = f(x).$$

- Min/maxing - either derivatives or Lagrange multipliers!
- Distance from origin to plane: equation of a plane

$$P : ax + by + cz = d.$$

- You can always just read off the normal vector  $\mathbf{n} = (a, b, c)$ . So we have  $\mathbf{n} \cdot \mathbf{x} = d$ .
- Since  $\lambda \mathbf{n}$  is normal to  $P$  for all  $\lambda$ , solve  $\mathbf{n} \cdot \lambda \mathbf{n} = d$ , which is  $\lambda = \frac{d}{\|\mathbf{n}\|^2}$
- A plane can be constructed from a point  $p$  and a normal  $n$  by the equation  $\mathbf{n} \cdot \mathbf{p} = 0$ .
- In a sine wave  $f(x) = \sin(\omega x)$ , the period is given by  $2\pi/\omega$ . If  $\omega > 1$ , then the wave makes exactly  $\omega$  full oscillations in the interval  $[0, 2\pi]$ .
- The directional derivative is the gradient dotted against a *unit vector* in the direction of interest



- Related rates problems can often be solved via implicit differentiation of some constraint function
- The second derivative of a parametric equation is not exactly what you'd intuitively think!
- For the love of god, remember the FTC!

$$\frac{\partial}{\partial x} \int_0^x f(y)dy = f(x)$$

- Technique for asymptotic inequalities: WTS  $f < g$ , so show  $f(x_0) < g(x_0)$  at a point and then show  $\forall x > x_0, f'(x) < g'(x)$ . Good for big-O style problems too.
- Inflection points of  $h$  occur where the *tangent* of  $h'$  changes sign. (Note that this is where  $h'$  itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the  $x$  that satisfies this; it can be used in the original equation.
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