Linearization Continued

Section 8.4 Follow-Up

D. Zack Garza

April 2020

– The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

– We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

– We use the assumption:
   For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivial-
   ization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW), \pi_2(W) \rangle =$
   0 where c_1 denotes the first Chern class of the bundle TW.

– We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an
   orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

– We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow \mathcal{P}^{1,p}(x, y)$$

$$Y = (y_1, \ldots, y_{2n}) \longmapsto \exp_u\left(\sum y_i Z_i\right).$$

– We regard $Y(s, t)$ as a tangent vector to W in some Euclidean embedding.

Linearization
Continued

D. Zack Garza

– We seek to compute the composite map in charts:

$$
\begin{array}{ccccccc}
W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) & \xrightarrow{\ \iota\ } & \mathcal{P}^{1,p}(x,y) & \xrightarrow{\ \mathcal{F}\ } & L^p\left(\mathbb{R} \times S^1; TW\right) & \longrightarrow & L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right)
\end{array}
$$

$$
u \xrightarrow{\qquad\qquad \mathcal{F} \qquad\qquad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)
$$

$$
(y_1, \ldots, y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)
$$

Add a Tangent

$$\mathcal{F}(u) = \qquad \frac{\partial u}{\partial s} \qquad +J(u)\frac{\partial u}{\partial t} \qquad -J(u)X_t(u)$$

$$\mathcal{F}(u+Y) = \quad \frac{\partial(u+Y)}{\partial s} \quad +J(u+Y)\frac{\partial(u+Y)}{\partial t} \quad -J(u+Y)X_t(u+Y)$$

Extract the part that is linear in Y and collect terms:

$$(d\mathcal{F})_u(Y)$$

$$= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y)$$

$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right)$$

$$\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y) \right)$$

.

Review

Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - J \, dv(Y)$$

$$\begin{aligned}
(d\mathcal{F})_u(Y) &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\
&+ \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \\
&= \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) \\
&+ \sum_{i=1}^{2n} y_i \Bigg(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \\
&\qquad\qquad - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \Bigg).
\end{aligned}$$

Use the fact that this is $O_1 + O_0$ in Y.

Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

1. *S is linear*
2. *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s, t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$O_0 = S(y_1, \cdots, y_{2n}) = \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right.$$

$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right]$$

$$= \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left(\frac{\partial u}{\partial t} - (Z_i) X_t \right) \right.$$

$$\left. + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right].$$

– Claim: the terms in blue and orange vanish in the limit
$s \longrightarrow \pm\infty$, so it suffices to prove that the red term limits to a
symmetric operator.

Proof: Blue Term Vanishes

$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since u is a solution and

$$\frac{\partial u}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

This seems to be the full argument for the blue term.

Linearization
Continued

D. Zack Garza

$$\frac{\partial Z_i}{\partial s} \overset{s \longrightarrow \pm \infty}{\longrightarrow} 0$$

Follows since the frame Z_i was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \, \partial t} \quad \curvearrowright Z_i \overset{s \longrightarrow \pm \infty}{\longrightarrow} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \overset{s \longrightarrow \pm \infty}{\longrightarrow} 0.$$

This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in t?

Proof: Symmetry

Write the remaining red term as

$$A := A(y_1, \ldots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u(Z_i) \right).$$

Extract the jth component:

$$A_j = \sum y_i \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u(Z_i), \quad Z_j \right\rangle.$$

We'll show that

$$\lim_{s \to \pm\infty} \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u(Z_i), Z_j \right\rangle$$

$$- \left\langle J(u) \frac{\partial Z_j}{\partial t} - J(u)(dX_t)_u Z_j, \quad Z_i \right\rangle = 0.$$

Proof: Symmetry

Use the fact that the frame $\{Z_i\}$ is unitary:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t} \langle J(u)Z_i, Z_j \rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u)\frac{\partial Z_i}{\partial t}, Z_j \right\rangle + \left\langle J(u)Z_i, \frac{\partial Z_j}{\partial t} \right\rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u)\frac{\partial Z_i}{\partial t}, Z_j \right\rangle - \left\langle Z_i, J(u)\frac{\partial Z_j}{\partial t} \right\rangle.
\end{aligned}
$$

Proof: Symmetry

Therefore it suffices to show

$$-\left\langle J(u)\left(dX_t\right)_u(Z_i),\quad Z_j\right\rangle$$
$$+\left\langle J(u)\left(dX_t\right)_u(Z_j),\quad Z_i\right\rangle$$
$$-\left\langle (dJ)_u\left(\frac{\partial u}{\partial t}\right)Z_i,\quad Z_j\right\rangle$$

$$\xrightarrow{s\longrightarrow\pm\infty} 0.$$

Proof: Symmetry

Using the fact that

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \xrightarrow{s \longrightarrow \pm \infty} 0$$

we can equivalently show

$$\begin{aligned}
&- \left\langle J(u)\, (dX_t)_u\, (Z_i)\,, \quad Z_j \right\rangle \\
&+ \left\langle J(u)\, (dX_t)_u\, (Z_j)\,, \quad Z_i \right\rangle \\
&- \left\langle (dJ)_u\, (X_t)\, Z_i, \quad Z_j \right\rangle
\end{aligned}$$

$$\xrightarrow{s \longrightarrow \pm \infty} 0$$

For a fixed (s, t), this expression only depends on Z_i at the point $u(s, t)$.

Lemma: For $p \in W$, $\{Z_i\}$ a unitary basis of $T_p W$,

$$
\begin{aligned}
&- \left\langle J(p) \left(dX_t \right)_p (Z_i), \quad Z_j \right\rangle \\
&+ \left\langle J(p) \left(dX_t \right)_p (Z_j), \quad Z_i \right\rangle \\
&- \left\langle (dJ)_p (X_t) Z_i, \quad\quad Z_j \right\rangle \\
&= 0.
\end{aligned}
$$

Claim: This lemma immediately concludes the previous proof?

Proof of Lemma

Extend $\{Z_i\}$ to a chart containing p and use the Leibniz rule to rewrite

$$-\left\langle J(p)\left(dX_t\right)_p(Z_i),Z_j\right\rangle + \left\langle J(p)\left(dX_t\right)_p(Z_j),Z_i\right\rangle - \left\langle (dJ)_p\left(X_t\right)Z_i,Z_j\right\rangle = 0$$

as

$$-\left\langle J\left(dX_t\right)(Z_i),Z_j\right\rangle + \left\langle J\left(dX_t\right)(Z_j),Z_i\right\rangle + \left\langle J\left(dZ_i\right)(X_t),Z_j\right\rangle - \left\langle d\left(JZ_i\right)(X_t),Z_j\right\rangle$$

$$= \left\langle J[X_t,Z_i],Z_j\right\rangle + \left\langle J\left(dX_t\right)(Z_j),Z_i\right\rangle - \left\langle d\left(JZ_i\right)(X_t),Z_j\right\rangle.$$

where we'll rewrite the red terms.

Proof of Lemma

Now use

$$X_t \langle JZ_i, \ Z_j \rangle = 0 \implies \langle d\left(JZ_i\right)\left(X_t\right), Z_j \rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right) \rangle = 0.$$

We now rewrite the RHS from before:

$$\langle J\left[X_t, Z_i\right], Z_j \rangle + \langle J\left(dX_t\right)\left(Z_j\right), Z_i \rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right) \rangle$$

$$= \langle J\left[X_t, Z_i\right], Z_j \rangle + \langle J\left(dX_t\right)\left(Z_j\right) - J\left(dZ_j\right)\left(X_t\right), Z_i \rangle$$
$$= \langle J\left[X_t, Z_i\right], Z_j \rangle - \langle J\left[X_t, Z_j\right], Z_i \rangle$$
$$= \omega\left(\left[X_t, Z_i\right], Z_j\right) - \omega\left(\left[X_t, Z_j\right], Z_i\right).$$

The symmetry follows from ω being closed and

$$0 = d\omega\left(X_t, Z_i, Z_j\right)$$
$$= X_t \cdot \omega\left(Z_i, Z_j\right) - Z_i \cdot \omega\left(X_t, Z_j\right) + Z_j \cdot \omega\left(X_t, Z_i\right)$$
$$\quad - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) - \omega\left(\left[Z_i, Z_j\right], X_t\right)$$

$$= -X_t \cdot \langle Z_i, JZ_j \rangle + Z_i \cdot \left(dH_t\right)\left(Z_j\right) - Z_j \cdot \left(dH_t\right)\left(Z_i\right)$$
$$\quad - \left(dH_t\right)\left(\left[Z_i, Z_j\right]\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right)$$

$$= d\left(dH_t\right)\left(Z_i, Z_j\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right)$$
$$= -\omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right). \qquad \blacksquare$$

Linearization of Hamilton's Equation

Recall

$$(d\mathcal{F})_u = \bar{\partial}Y + SY = (\bar{\partial} + S)Y$$

Now think of S as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

$$\partial_t Y = J_0 S^\pm Y$$

is a linearization of Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad at \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \longrightarrow \infty} u & \text{for } S^+ \end{cases} \quad respectively.$$

Proof

We first linearize Hamilton's equation at x:

$$\frac{\partial z}{\partial t} = X_t(z) \quad \overset{\text{linearized}}{\Longrightarrow} \quad \frac{\partial Y}{\partial t} = (dX_t)_x Y.$$

So write $Y = \sum y_i Z_i$ to obtain

$$\sum_i \frac{\partial y_i}{\partial t} Z_i = \sum_i y_i \left(-\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i) \right)$$

$$= \sum_i \sum_j y_i \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i), Z_j \right\rangle Z_j$$

$$= \sum_i \sum_j y_j \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle Z_i$$

$$\implies \frac{\partial y_i}{\partial t} = \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle y_j.$$

Proof

Thus we can rewrite the linearized equation as

$$\frac{\partial Y}{\partial t} = (dX_t)_x Y = B^- \cdot Y, \quad b_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x (Z_j), Z_i \right\rangle.$$

Recall

$$A := A(y_1, \ldots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i) \right).$$

Now take $s \longrightarrow -\infty$ and look at the order zero part of $(d\mathcal{F})_u$:

$$A\left(\sum y_i Z_i\right) = \sum_i \left(J(x) \frac{\partial Z_i}{\partial t} - J(x) (dX_t)_x (Z_i) \right)$$

$$= \sum_i \sum_j y_i \left\langle J \frac{\partial Z_i}{\partial t} - J (dX_t)(Z_i), Z_j \right\rangle Z_j$$

$$= \sum_i \sum_j y_j \left\langle J \frac{\partial Z_j}{\partial t} - J (dX_t)(Z_j), Z_i \right\rangle Z_i$$

$$= \sum_i \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), J Z_i \right\rangle_{y_j} Z_i.$$