

# Title

D. Zack Garza

Tuesday 1<sup>st</sup> September, 2020

## Contents

1 Tuesday, September 01

1

## 1 Tuesday, September 01

Last time:  $V(I) = \{x \in \mathbb{A}^n \mid f(x) = 0 \forall x \in I\}$  and  $I(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in X\}$ .

We proved the Hilbert Nullstellensatz  $I(V(J)) = \sqrt{J}$ , defined the coordinate ring of an affine variety  $X$  as  $A(X) := k[x_1, \dots, x_n]/I(X)$ , the ring of “regular” (polynomial) functions on  $X$ .

Recall that a *topology* on  $X$  can be defined as a collection of “closed” subsets of  $X$  that are closed under arbitrary intersections and finite unions. A subset  $Y \subset X$  inherits a subspace topology with closed sets of the form  $Z \cap Y$  for  $Z \subset X$  closed.

**Definition 1.0.1** (Zariski Topology).

Let  $X$  be an affine variety. The closed sets are affine subvarieties  $Y \subset X$ .

We have  $\emptyset, X$  closed, since

1.  $V_X(1) = \emptyset$ ,
2.  $V_X(0) = X$

Closure under finite unions: Let  $V_X(I), V_X(J)$  be closed in  $X$  with  $I, J \subset A(X)$  ideals. Then  $V_X(IJ) = V_X(I) \cup V_X(J)$ .

Closure under intersections: We have  $\bigcap_{i \in \sigma} V_X(J_i) = V_X\left(\sum_{i \in \sigma} J_i\right)$ .

**Remark 1.**

There are few closed sets, so this is a “weak” topology.

**Example 1.1.**

Compare the classical topology on  $\mathbb{A}^1/\mathbb{C}$  to the Zariski topology.

Consider the set  $A := \{x \in \mathbb{A}^1/\mathbb{C} \mid \|x\| \leq 1\}$ , which is closed in the classical topology.

---

But  $A$  is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

**Example 1.2.**

Let  $f : \mathbb{A}^1/k \rightarrow \mathbb{A}^1/k$  be any injective map. Then  $f$  is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

**Example 1.3.**

Consider  $X \times Y$  a product of affine varieties. Then there is a product topology where open sets are of the form  $\bigcup_{i=1}^n U_i \times V_i$  with  $U_i, V_i$  open in  $X, Y$  respectively.

This is the wrong topology! On  $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ , the diagonal  $\Delta := V(x - y)$  is closed in the Zariski topology on  $\mathbb{A}^2$  but not in the product topology.

**Example 1.4.**

Consider  $\mathbb{A}^2/\mathbb{C}$ , so the closed sets are curves and points. Observation:  $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$  decomposed into the union of the coordinate axes  $X_1 := V(x_1)$  and  $X_2 := V(x_2)$ . The Zariski topology can detect these decompositions.

**Definition 1.0.2** (Irreducibility and Connectedness).

Let  $X$  be a topological space.

- a.  $X$  is *reducible* iff there exist nonempty proper closed subsets  $X_1, X_2 \subset X$  such that  $X = X_1 \cup X_2$ . Otherwise,  $X$  is said to be *irreducible*.
- b.  $X$  is *disconnected* if there exist  $X_1, X_2 \subset X$  such that  $X = X_1 \amalg X_2$ . Otherwise,  $X$  is said to be *connected*.

**Example 1.5.**

$V(x_1x_2)$  is reducible but connected.

**Remark 2.**

$\mathbb{A}^1/\mathbb{C}$  is *not* irreducible, since we can write  $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$ .

**Proposition 1.1 (?)**

Let  $X$  be a disconnected affine variety with  $X = X_1 \amalg X_2$ . Then  $A(X) \cong A(X_1) \times A(X_2)$ .

*Proof .*

We have  $X_1 \cup X_2 = X$ , so  $I(X_1) \cap I(X_2) = I(X) = (0)$  in the coordinate ring  $A(X)$  (recalling that it is a quotient by  $I(X)$ .)

Since  $X_1 \cap X_2 = \emptyset$ , we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus  $I(X_1) + I(X_2) = \langle 1 \rangle$ , and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \longrightarrow A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely  $A(X_1) \times A(X_2)$ . ■

### Proposition 1.2(?).

An affine variety  $X$  is irreducible  $\iff A(X)$  is an integral domain.

*Proof .*

$\implies$  : By contrapositive, suppose  $f_1, f_2 \in A(X)$  are nonzero with  $f_1 f_2 = 0$ . Let  $X_i = V(f_i)$ , then  $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$  which are closed and proper since  $f_i \neq 0$ .

$\impliedby$  : Suppose  $X$  is reducible with  $X = X_1 \cup X_2$  with  $X_i$  proper and closed. Define  $J_i := I(X_i)$ , and note  $J_i \neq 0$  because  $V(J_i) = V(I(X_i)) = X_i$  by part (a) of the Nullstellensatz.

So there exists a nonzero  $f_i \in J_i = I(X_i)$ , so  $f_i$  vanishes on  $X_i$ . But then  $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$ , so  $X = V(f_1 f_2)$  and  $f_1 f_2 \in I(X) = \langle 0 \rangle$  and  $f_1 f_2 = 0$ . So  $A(X)$  is not a domain. ■

### Example 1.6.

Let  $X = \{p_1, \dots, p_d\}$  be a finite set in  $\mathbb{A}^n$ . The Zariski topology on  $X$  is the discrete topology, and  $X = \coprod \{p_i\}$ . So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^d A(\{p_i\}) = \prod_{i=1}^d k[x_1, \dots, x_n] / \langle x_j - a_j(p_i) \rangle_{j=1}^d .$$

### Example 1.7.

Set  $V(x_1 x_2) = X$ , then  $A(X) = k[x_1, x_2] / \langle x_1 x_2 \rangle$ . This not being a domain (since  $x_1 x_2 = 0$ ) corresponds to  $X = V(x_1) \cup V(x_2)$  not being irreducible.

### Example 1.8.

$\mathbb{A}^2/k$  is irreducible since  $k[x_1, \dots, x_n]$  is a domain.

### Example 1.9.

---

Let  $X_1$  be the  $xy$  plane and  $X_2$  be the line parallel to the  $y$ -axis through  $[0, 0, 1]$ , and let  $X = X_1 \coprod X_2$ . Then  $X_1 = V(z)$  and  $X_2 = V(x, z - 1)$ , and  $I(X) = \langle z \rangle \cdots \langle x, z - 1 \rangle = \langle xz, z^2 - z \rangle$ .

Then the coordinate ring is given by  $A(X) = \mathbb{C}[x, y, z] / \langle xz, z^2 - z \rangle$ .