Problem Set 1

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Sunday 6^{th} September, 2020

Source: Section 1 of Gathmann

1 Exercises

Exercise 1.1 (Gathmann 1.19)**.**
Prove that every affine variety $X \subset \mathbb{A}^n/k$ consisting of only finitely many points can be written as the zero locus of n polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in X have different x_1-coordinates.

Solution:
Let $X = \{\mathbf{p}_1, \cdots, \mathbf{p}_d\} = \{\mathbf{p}_j\}_{j=1}^d$, where each $\mathbf{p}_j \in \mathbb{A}^n$ can be written in coordinates

$$\mathbf{p}_j := \left[p_j^1, p_j^2, \cdots, p_j^n\right].$$

Remark.
Proof idea: for some fixed k with $2 \leq k \leq n$, consider the pairs $(p_j^1, p_j^k) \in \mathbb{A}^2$. Letting j range over $1 \leq j \leq d$ yields d points of the form $(x, y) \in \mathbb{A}^2$, so construct an interpolating polynomial such that $f(x) = y$ for each tuple. Then $f(x) - y$ vanishes at every such tuple.

Doing this for each k (keeping the first coordinate always of the form p_j^1 and letting the second coordinate vary) yields $n-1$ polynomials in $k[x_1, x_k] \subseteq k[x_1, \cdots, x_n]$, then adding in the polynomial $p(x) = \prod_j (x - p_j^1)$ yields a system the vanishes precisely on $\{\mathbf{p}_j\}$.

Claim: Without loss of generality, we can assume all of the first components $\left\{p_j^1\right\}_{j=1}^d$ are distinct.

Todo: follows from "rotation of axes"?

We will use the following fact:

Theorem 1.1 ***(Lagrange).***
Given a set of d points $\{(x_i, y_i)\}_{i=1}^{d}$ with all x_i distinct, there exists a unique polynomial of degree d in $f \in k[x]$ such that $\tilde{f}(x_i) = y_i$ for every i.
This can be explicitly given by

$$\tilde{f}(x) = \sum_{i=1}^{d} y_i \left(\prod_{\substack{0 \leq m \leq d \\ m \neq i}} \left(\frac{x - x_m}{x_i - x_m} \right) \right).$$

Equivalently, there is a polynomial f defined by $f(x_i) = \tilde{f}(x_i) - y_i$ of degree d whose roots are precisely the x_i.

Using this theorem, we define a system of n polynomials in the following way:

- Define $f_1 \in k[x_1] \subseteq k[x_1, \cdots, x_n]$ by

  $$f_1(x) = \prod_{i=1}^{d} \left(x - p_i^1 \right).$$

  Then the roots of f_1 are precisely the first components of the points p.

- Define $f_2 \in k[x_1, x_2] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs

  $$\left\{ (x_1, x_2) = (p_j^1, p_j^2) \right\},$$

  then taking the unique Lagrange interpolating polynomial $\tilde{f}_2$ satisfying $\tilde{f}_2(p_j^1) = p_j^2$ for all $1 \leq j \leq d$. Then set $f_2 := \tilde{f}_2(x_1) - x_2 \in k[x_1, x_2]$.

- Define $f_3 \in k[x_1, x_3] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs

  $$\left\{ (x_1, x_3) = (p_j^1, p_j^3) \right\},$$

  then taking the unique Lagrange interpolating polynomial $\tilde{f}_3$ satisfying $\tilde{f}_2(p_j^1) = p_j^3$ for all $1 \leq j \leq d$. Then set $f_3 := \tilde{f}_3(x_1) - x_3 \in k[x_1, x_3]$.
- $\cdots$

Continuing in this way up to $f_n \in k[x_1, x_n]$ yields a system of n polynomials.

Proposition 1.2.
$V(f_1, \cdots, f_n) = X$.

Proof .

Claim: $X \subseteq V(f_i)$:
This is essentially by construction. Letting $p_j \in X$ be arbitrary, we find that

$$f_1(p_j) = \prod_{i=1}^{d} \left(p_j^1 - p_i^1\right) = (p_j^1 - p_j^1) \prod_{\substack{i \le d \\ i \ne j}} \left(p_j^1 - p_i^1\right) = 0.$$

Similarly, for $2 \le k \le n$,

$$f_k(p_j) = \tilde{f}_k(p_j^1) - p_j^k = 0,$$

which follows from the fact that $\tilde{f}_k(p_j^1) = p_j^k$ for every k and every j by the construction of $\tilde{f}_k$.

Claim: $X^c \subseteq V(f_i)^c$:
This follows from the fact the polynomials f given by Lagrange interpolation are unique, and thus the roots of $\tilde{f}$ are unique. But if some other point was in $V(f_i)$, then one of its coordinates would be another root of some $\tilde{f}$.

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Exercise 1.2 (Gathmann 1.21)**.**
Determine $\sqrt{I}$ for

$$I := \left\langle x_1^3 - x_2^6,\, x_1 x_2 - x_2^3 \right\rangle \trianglelefteq \mathbb{C}[x_1, x_2].$$

Solution:
For notational purposes, let $\mathcal{I}, \mathcal{V}$ denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$, the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 & = 0 \\ xy - y^3 & = 0. \end{cases}$$

In the second equation, we have $(x - y^2)y = 0$, and since $\mathbb{C}[x, y]$ is an integral domain, one term must be zero.

1. If $y = 0$, then $x^3 = 0 \implies x = 0$, and thus $(0, 0) \in \mathcal{V}(I)$, i.e. the origin is contained in this vanishing locus.
2. Otherwise, if $x - y^2 = 0$, then $x = y^2$, with no further conditions coming from the first equation.

Combining these conditions,

$$P := \left\{ (t^2, t) \,\middle|\, t \in \mathbb{C} \right\} \subset \mathcal{V}(I).$$

where $I = \left\langle x^3 - y^6, xy - y^3 \right\rangle$.
We have $P = \mathcal{V}(I)$, and so taking the ideal generated by P yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \left\langle y - x^2 \right\rangle \in \mathbb{C}[x, y]$$

and thus $\sqrt{I} = \left\langle y - x^2 \right\rangle$.

Exercise 1.3 (Gathmann 1.22)**.**
Let $X \subset \mathbb{A}^3/k$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$ and show that it can not be generated by fewer than 3 elements.

Solution:
Claim:

$$I(X) = \langle x_2x_3,\, x_1x_3,\, x_1x_2 \rangle\,.$$

We can write $X = X_1 \cup X_2 \cup X_3$, where

- The x_1-axis is given by $X_1 := V(x_2x_3) \implies I(X_1) = \langle x_2x_3 \rangle$,
- The x_2-axis is given by $X_2 := V(x_1x_3) \implies I(X_2) = \langle x_1x_3 \rangle$,
- The x_3-axis is given by $X_3 := V(x_1x_2) \implies I(X_3) = \langle x_1x_2 \rangle$.

Here we've used, for example, that

$$I(V(x_2x_3)) = \sqrt{\langle x_2x_3 \rangle} = \langle x_2x_3 \rangle$$

by applying the Nullstellensatz and noting that $\langle x_2x_3 \rangle$ is radical since it is generated by a squarefree monomial.
We then have

$$\begin{aligned} I(X) &= I(X_1 \cup X_2 \cup X_3) \\ &= I(X_1) \cap I(X_2) \cap I(X_3) \\ &= \sqrt{I(X_1) + I(X_2) + I(X_3)} \\ &= \sqrt{\langle x_2, x_3 \rangle + \langle x_1x_3 \rangle + \langle x_1x_2 \rangle} \\ &= \sqrt{\langle x_2x_3,\, x_1x_3,\, x_1x_2 \rangle} && \text{since } \langle a \rangle + \langle b \rangle = \langle a, b \rangle \\ &= \langle x_2x_3,\, x_1x_3,\, x_1x_2 \rangle, \end{aligned}$$

where in the last equality we've again used the fact that an ideal generated by squarefree monomials is radical.

Claim: $I(X)$ can not be generated by 2 or fewer elements.

Let $J \coloneqq I(X)$ and $R \coloneqq k[x_1, x_2, x_3]$, and toward a contradiction, suppose $J = \langle r, s\rangle$. Define $\mathfrak{m} \coloneqq \langle x, y, z\rangle$ and a quotient map

$$\pi : J \longrightarrow J/\mathfrak{m}J$$

and consider the images $\pi(r), \pi(s)$.

Note that $J/\mathfrak{m}J$ is an $R/\mathfrak{m}$-module, and since $R/\mathfrak{m} \cong k$, $J/\mathfrak{m}J$ is in fact a k-vector space. Since $\pi(r), \pi(s)$ generate $J/\mathfrak{m}J$ as a k-module,

$$\dim_k J/\mathfrak{m}J \leq 2.$$

But this is a contradiction, since we can produce 3 k-linearly independent elements in $J/\mathfrak{m}J$: namely $\pi(x_1x_2), \pi(x_1x_3), \pi(x_2x_3)$. Suppose there exist α_i such that

$$\alpha_1\pi(x_1x_2) + \alpha_2\pi(x_1x_3) + \alpha_3\pi(x_2x_3) = 0 \in J/\mathfrak{m}J \iff \alpha_1x_1x_2 + \alpha_2x_1x_3 + \alpha_3x_2x_3 \in \mathfrak{m}J,$$

But now we can then note that

$$\mathfrak{m}J = \langle x_1, x_2.x_3\rangle \langle x_1x_2, x_1x_3, x_2x_3\rangle = \left\langle x_1^2x_2,\ x_1^2x_3,\ x_1x_2x_3, \cdots \right\rangle.$$

can't contain any elements of degree less than three.

Exercise 1.4 (Gathmann 1.23: Relative Nullstellensatz)**.**
Let $Y \subset \mathbb{A}^n/k$ be an affine variety and define $A(Y)$ by the quotient

$$\pi : k[x_1, \cdots, x_n] \longrightarrow A(Y) \coloneqq k[x_1, \cdots, x_n]/I(Y).$$

a. Show that $V_Y(J) = V(\pi^{-1}(J)$ for every $J \trianglelefteq A(Y)$.

b. Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every affine subvariety $X \subseteq Y$.

c. Using the fact that $I(V(J)) \subset \sqrt{J}$ for every $J \trianglelefteq k[x_1, \cdots, x_n]$, deduce that $I_Y(V_Y(J)) \subset \sqrt{J}$ for every $J \trianglelefteq A(Y)$.

Conclude that there is an inclusion-reversing bijection

$$\left\{\substack{\text{Affine subvarieties} \\ \text{of } Y}\right\} \iff \left\{\substack{\text{Radical ideals} \\ \text{in } A(Y)}\right\}.$$

Exercise 1.5 (Extra)**.**
Let $J \trianglelefteq k[x_1, \cdots, x_n]$ be an ideal, and find a counterexample to $I(V(J)) = \sqrt{J}$ when k is not algebraically closed.

Solution:
Take $J = \left\langle x^2 + 1 \right\rangle \trianglelefteq \mathbb{R}[x]$, noting that J is nontrivial and proper but $\mathbb{R}$ is not algebraically closed. Then $V(J) \subseteq \mathbb{R}$ is empty, and thus $I(V(J)) = I(\emptyset)$.

Claim: $I(V(J)) = \mathbb{R}[x]$.
Checking definitions, for any set $X \subset \mathbb{A}^n/k$ we have

$$I(X) = \left\{ f \in \mathbb{R}[x] \;\middle|\; \forall x \in X,\ f(x) = 0 \right\}$$

and so we vacuously have

$$I(\emptyset) = \left\{ f \in \mathbb{R}[x] \;\middle|\; \forall x \in \emptyset,\ f(x) = 0 \right\} = \{ f \in \mathbb{R}[x] \} = \mathbb{R}[x].$$

Claim: $\sqrt{J} \neq \mathbb{R}[x]$.
This follows from the fact that maximal ideals are radical, and $\mathbb{R}[x]/J \cong \mathbb{C}$ being a field implies that J is maximal. In this case $\sqrt{J} = J \neq \mathbb{R}[x]$.

That maximal ideals are radical follows from the fact that if $J \trianglelefteq R$ is maximal, we have $J \subset \sqrt{J} \subset R$ which forces $\sqrt{J} = J$ or $\sqrt{J} = R$.

But if $\sqrt{J} = R$, then

$$1 \in \sqrt{J} \implies 1^n \in J \text{ for some } n \implies 1 \in J \implies J = R,$$

contradicting the assumption that J is maximal and thus proper by definition.