

# Title

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## 1 | Elementary Algebra

- Looking at real roots:
  - Let  $p$  be number of sign changes in  $f(x)$ ;
  - Let  $q$  be number of sign changes in  $f(-x)$ ;
  - Let  $n$  be the degree of  $f$ .
  - Then  $p$  gives the maximum number of positive real roots,  $q$  gives the maximum number of negative real roots, and  $n - p - q$  gives the *minimum* number of complex roots.
  - Rational Roots Theorem: If  $p(x) = ax^n + \cdots + c$  and  $r = \frac{p}{q}$  where  $p(r) = 0$ , then  $p \mid c$  and  $q \mid a$ .

## 2 | Abstract Algebra

- Order  $p$ : One,  $Z_p$
- Order  $p^2$ : Two abelian groups,  $Z_{p^2}, Z_p^2$
- Order  $p^3$ :
  - 3 abelian  $Z_{p^3}, Z_p \times Z_{p^2}, Z_p^3$ ,
  - 2 others  $Z_p \rtimes Z_{p^2}$ .
  - ◊ The other is the quaternion group for  $p = 2$  and a group of exponent  $p$  for  $p > 2$ .
- Order  $pq$ :
  - $p \mid q - 1$ : Two groups,  $Z_{pq}$  and  $Z_q \rtimes Z_p$
  - Else cyclic,  $Z_{pq}$

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- Every element in a permutation group is a product of disjoint cycles, and the order is the lcm of the order of the cycles.
  - The product ideal  $IJ$  is *not* just elements of the form  $ij$ , it is all sums of elements of this form! The product alone isn't enough.
  - The intersection of any number of ideals is also an ideal

## 3 | Complex Numbers

- $\lim_{z \rightarrow z_0} f(z) = x_0 + iy_0$  iff the component functions limit to  $x_0$  and  $y_0$  respectively. Moreover, both ways are equal!

## 4 | Analysis

- $f$  injective implies  $f$  has a nonzero derivative (in neighborhoods)
- In  $\mathbb{R}$ , singletons are closed. This means any finite subset is closed, as a finite union of singleton sets! Good for counterexamples to continuity.
- Definition of topology: arbitrary unions and finite intersections of open sets are open. Equivalently, arbitrary intersections and finite unions of closed sets are closed.
- The best source of examples and counterexamples is the open/closed unit interval in  $\mathbb{R}$ . Always test against these first!
- Every Cauchy sequence converges in a complete metric space

## 5 | Combinatorics and Probability

- Counting non-isomorphic things: Pick a systematic way. Can descend my maximum vertex degree, or ascend by adding nodes/leaves.

## 6 | Linear Algebra

- An  $m \times n$  matrix is a map from  $n$ -dimensional space to  $m$ -dimensional space. Number of *rows* tell you the dimension of the codomain, the number of *columns* tell you the dimension of the *domain*.
- The column space of  $A$  (i.e. linear combinations of the columns) are a basis for the *image* of  $A$ .
- The row space is a basis for the *coimage*, which is nullspace perp.
- Not enough pivots implies columns don't span the entire target domain
- The determinant of an RREF matrix is the product of the diagonals

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- An  $n \times n$  matrix  $P$  is diagonalizable iff its eigenspace is all of  $\mathbb{R}^n$  (i.e. there are  $n$  linearly independent eigenvectors, so they span the space.) Equivalently, if there is a basis of eigenvectors for the range of  $P$
  - Projections decompose the range into the direct sum of its nullspace and nullspace perp.
  - The space of matrices is not an integral domain!
  - The transition matrix from a given basis  $\mathcal{B} = b_i$  to the standard basis is given by just creating a matrix with each  $b_i$  being a column.
    - The transition matrix from the standard basis to  $\mathcal{B}$  is just the inverse of the above!
  - Inverting matrices quickly:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{where } ad - bc \neq 0$$

The pattern?

1. Always divide by determinant
  2. Swap the diagonals
  3. Hadamard product with checkerboard  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $\& - \backslash$
  - $\& + \backslash \text{end}\{\text{bmatrix}\}$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} ei - fh & -(bi - ch) & bf - ce \\ -(di - fg) & ai - cg & -(af - cd) \\ dh - eg & -(ah - bg) & ae - bd \end{bmatrix}.$$

The pattern:

1. Divide by determinant
2. Each entry is determinant of submatrix of  $A$  with corresponding col/row deleted
3. Hadamard product with checkerboard

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

4. Transpose at the end!!

## 7 | Calculus

- Inflection points of  $h$  occur where the *tangent* of  $h'$  changes sign. (Note that this is where  $h'$  itself changes sign.)

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- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
  - If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the  $x$  that satisfies this; it can be used in the original equation.
  - Fundamental theorem of Calculus: If  $\int f(x)dx = F(b) - F(a)$ , then  $F'(x) = f(x)$ .
  - Min/maxing - either derivatives or Lagrange multipliers!
  - Distance from origin to plane: equation of a plane  $P$  is given by  $ax + by + cz = d$ , and you can always just read off the normal vector  $\mathbf{n} = (a, b, c)$ . So we have  $\mathbf{n}\mathbf{x} = d$ . Since  $t\mathbf{n}$  is normal to  $P$  for all  $t$ , solve  $\mathbf{n}t\mathbf{n} = d$ , which is  $t = \frac{d}{|\mathbf{n}|^2}$
  - A plane can be constructed from a point  $p$  and a normal  $n$  by the equation  $np = 0$ .
  - In a sine wave  $f(x) = \sin(\omega x)$ , the period is given by  $2\pi/\omega$ . If  $\omega > 1$ , then the wave makes exactly  $\omega$  full oscillations in the interval  $[0, 2\pi]$ .
  - The directional derivative is the gradient dotted against a *unit vector* in the direction of interest
  - Related rates problems can often be solved via implicit differentiation of some constraint function
  - The second derivative of a parametric equation is not exactly what you'd intuitively think!
  - For the love of god, remember the FTC!  $\frac{\partial}{\partial x} \int_0^x f(y)dy = f(x)$ .
  - Technique for asymptotic inequalities: WTS  $f < g$ , so show  $f(x_0) < g(x_0)$  at a point and then show  $\forall x > x_0, f'(x) < g'(x)$ . Good for big-O style problems too.