Assignment 6: The Fourier Transform

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \lim_{\xi'\to 0} \frac{1}{2} \int_{\mathbb{R}^n} (f()) - f(x - \xi'))e^{-2\pi i x \cdot \xi} \, dx$$

.

But as an immediate consequence, this yields

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\int_{\mathbb{R}^n} (f(x) - f(x - \xi'))e^{-2\pi i x \cdot \xi} \, dx\right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \left|e^{-2\pi i x \cdot \xi}\right| \, dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\
&\to 0,
\end{aligned}$$

which follows from continuity in L^1 since $f(x - \xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds, and that $\xi' \to 0$ as $\xi \to \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned}
\widehat{f * g}(\xi) &:= \int \int f(x - y)g(y) \, e^{-2\pi i x \cdot \xi} \, dy \, dx \\
&=_? \int \int f(x - y)g(y) \, e^{-2\pi i x \cdot \xi} \, dx \, dy \\
&= \int \int f(t)e^{-2\pi i (x - y) \cdot \xi} \, g(y) \, e^{-2\pi i y \cdot \xi} \, dx \, dy \\
&\qquad (t = x - y, \ dt = \ dx) \\
&= \int \int f(t)e^{-2\pi i t \cdot \xi} g(y)e^{-2\pi i y \cdot \xi} \, dt \, dy \\
&= \int f(t)e^{-2\pi i t \cdot \xi} \left(\int g(y) \, e^{-2\pi i y \cdot \xi} \, dy\right) \, dt \\
&= \int f(t)e^{-2\pi i t \cdot \xi} \, \hat{g}(\xi) \, dt \\
&= \hat{g}(\xi) \int f(t)e^{-2\pi i t \cdot \xi} \, dt \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}$$

To see that this swap is justified, we'll apply Fubini-Tonelli. Note that if $f, g \in L^1(\mathbb{R}^n)$, then the map $(x, y) \mapsto f(x - y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$. Since g is measurable as well, taking the cylinder on g is also measurable on $\mathbb{R}^n \times \mathbb{R}^n$. The exponential is continuous, and thus measurable on $\mathbb{R}^n$. Thus the integrand $F(x, y)$ is a product of measurable functions and thus measurable. In particular, $|F| = |fg|$ is measurable, and ths computation shows that one iterated integral is finite. From a previous homework question, we know that $f \in L^1 \implies \hat{f}$ is bounded, and thus $\hat{f}\hat{g}$ is bounded. Since $|F|$ is measurable and one iterated integrable was finite, Fubini-Tonelli applies.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$(\widehat{f * g}) * h = \widehat{f * g}\,\hat{h} = \hat{f}\,\hat{g}\,\hat{h} = \hat{f}\,\widehat{g * h} = f * (g * h).$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\,\hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then

$$\lim_{|\xi| \to \infty} \hat{I}(\xi) \neq 0,$$

which by Problem 1 shows that I can not be in L^1, a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x - y)$. We then have

$$\hat{g}(\xi) := \int g(x)e^{-2\pi i x \cdot \xi} \, dx$$

$$= \int f(x-y)e^{-2\pi i x \cdot \xi} \, dx$$

$$= \int f(x-y)e^{-2\pi i (x-y)\cdot \xi}e^{-2\pi i y \cdot \xi} \, dx$$

$$= e^{-2\pi i y \cdot \xi}\int f(x-y)e^{-2\pi i (x-y)\cdot \xi} \, dx$$

$$(t = x - y, dt = dx)$$

$$= e^{-2\pi i y \cdot \xi}\int f(t)e^{-2\pi i t \cdot \xi} \, dt$$

$$= e^{-2\pi i y \cdot \xi}\hat{f}(\xi).$$

3.1.2 (ii)

Let $h(x) = e^{2\pi i x \cdot y} f(x)$. We then have

$$\hat{h}(\xi) := \int e^{2\pi i x \cdot y} f(x)e^{-2\pi i x \cdot \xi} \, dx$$

$$= \int e^{2\pi i x \cdot y - 2\pi i x \cdot \xi)f(x} \, dx$$

$$= \int f(\xi - y)e^{-2\pi i x \cdot (\xi - y)} \, dx$$

$$= \hat{f}(\xi - y).$$

3.2 (b)

We'll use the fact that if $\langle \cdot , \ \cdot \rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle A\mathbf{x}, \ \mathbf{y} \rangle = \left\langle \mathbf{x}, \ A^T\mathbf{y} \right\rangle$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$\frac{1}{|\det T|}\hat{f}(T^{-T}\xi) = \frac{1}{|\det T|}\int f(x)e^{-2\pi i x \cdot T^{-T}\xi} \, dx$$

$$x \mapsto Tx, \ dx \mapsto |\det T| \, dx$$

$$= \frac{1}{|\det T|}\int f(Tx)e^{-2\pi i Tx \cdot T^{-T}\xi}|\det T| \, dx$$

$$= \int f(Tx)e^{-2\pi i x \cdot \xi} \, dx$$

$$\text{since } Tx \cdot T^{-T}\xi = T^{-1}Tx \cdot \xi = x \cdot \xi$$

$$= \widehat{(f \circ T)}(\xi).$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$
\begin{aligned}
\frac{\partial}{\partial \xi} \hat{f}(\xi) &:= \frac{\partial}{\partial \xi} \int f(x) e^{-2\pi i x \cdot \xi} \, dx \\
&=_? \int f(x) \frac{\partial}{\partial \xi} e^{-2\pi i x \cdot \xi} \, dx \\
&= \int f(x) 2\pi i x e^{-2\pi i x \cdot \xi} \, dx \\
&= 2\pi i \int x f(x) e^{-2\pi i x \cdot \xi} \, dx \\
&:= 2\pi i \hat{g}(\xi).
\end{aligned}
$$

> To see that the interchange is justified, we just note that we can apply the dominated convergence theorem, since $\int \left| f(x) e^{-2\pi i x \cdot \xi} \right| \leq \int |f| < \infty$, where we assumed $f \in L^1$.

4.1.2 (ii)

We have

$$
\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) e^{-2\pi i x \cdot \xi} \, dx \\
&= f(x) e^{-2\pi i x \cdot \xi} \Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i \xi) e^{-2\pi i x \cdot \xi} \, dx \\
&\quad \text{(integrating by parts)} \\
&= -\int f(x)(-2\pi i \xi) e^{-2\pi i x \cdot \xi} \, dx \\
&\quad \text{(since } f(\infty) = f(-\infty) = 0) \\
&= 2\pi i \xi \int f(x) e^{-2\pi i x \cdot \xi} \, dx \\
&:= 2\pi i \xi \hat{f}(\xi).
\end{aligned}
$$

4.2 (b)

Let $G(x) = e^{-\pi x^2}$ and ∂_ξ be the operator that differentiates with respect to ξ.

Then

$$
\partial_\xi \left(\frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi)}{G(\xi)^2},
$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = 0.$$

A direct computation shows that

$$\partial_\xi G(\xi) = -2\pi\xi G(\xi), \tag{1}$$

and we claim that $\partial_\xi\hat{G}(\xi) = -2\pi\xi\hat{G}(\xi)$ as well, which follows from the following computation:

$$
\begin{aligned}
\partial_\xi\hat{G}(\xi) &:= \partial_\xi \int G(x)e^{-2\pi ix\cdot\xi}\,dx \\
&= \int G(x)\partial_\xi e^{-2\pi ix\cdot\xi}\,dx \\
&= \int G(x)(-2\pi ix)e^{-2\pi ix\cdot\xi}\,dx \\
&= \int G(x)(-2\pi ix)e^{-2\pi ix\cdot\xi}\,dx \\
&= i\int 2\pi x G(x)e^{-2\pi ix\cdot\xi}\,dx \\
&= i\int \partial_x G(x)e^{-2\pi ix\cdot\xi}\,dx \qquad \text{by (1)} \\
&:= i\,\widehat{\partial_x G(x)}(\xi) \\
&= i\,(2\pi i\xi\hat{G}(\xi)) \qquad \text{by part (i)} \\
&= -2\pi\xi\hat{G}(\xi).
\end{aligned}
$$

We can thus write

$$G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x)\,dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$\hat{D}(\xi) := \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 e^{-2\pi i x \xi} \, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) + i \sin(-2\pi x \xi) \, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) \, dx$$

(since sin is odd and the domain is symmetric about 0)

$$= 2 \int_{0}^{\frac{1}{2}} \cos(-2\pi x \xi) \, dx$$

(since cos is even and the domain is symmetric about 0)

$$= 2 \left(\frac{1}{2\pi\xi} \sin(-2\pi x \xi) \Big|_{x=0}^{x=\frac{1}{2}} \right)$$

$$= \frac{\sin(\pi\xi)}{\pi\xi}.$$

5.2 (b)

5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi} \right)^2$.

5.2.2 (ii)

Letting $\mathcal{F}$ denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if f is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting F be the box function, F can be seen to be even from its definition. Since $f := \mathcal{F}(F)$ by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that $\hat{f}(x) = F(x)$, the original box function.

5.3 (c)

By a direct computation of the integral in question, we have

$$I(x) := \int e^{-2\pi|\xi|} e^{2\pi i x \xi} \, d\xi$$

$$= \int_{-\infty}^{0} e^{-2\pi(-\xi)} e^{-2\pi i x \xi} \, d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} \, d\xi$$

$$= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} \, d\xi + \int_{0}^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} \, d\xi$$

by the change of variables $\xi \mapsto -\xi$, $d\xi \mapsto -d\xi$ and swapping integration bounds

$$= \int_{0}^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} + e^{2\pi\xi} e^{2\pi i x \xi} \, d\xi$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-u} e^{-ixu} + e^{-u} e^{ixu} \, du$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)} \, du$$

$$= \frac{1}{2\pi} \left(\frac{-e^{-u(1+ix)}}{1 + ix} \Big|_{u=0}^{u=\infty} + \frac{-e^{-u(1-ix)}}{1 + ix} \Big|_{u=0}^{u=\infty} \right)$$

$$= \frac{1}{2\pi} \left(\frac{1}{1 + ix} + \frac{1}{1 - ix} \right)$$

$$= \frac{1}{2\pi} \frac{2}{1 + x^2}$$

$$= \frac{1}{\pi} \frac{1}{1 + x^2},$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$I(x) = P(x) = \int \hat{P}(\xi) e^{-2\pi i x \xi} \, dx$$

$$\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} = \int \hat{P}(\xi) e^{-2\pi i x \xi} \, dx$$

$$\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} - \hat{P}(\xi) e^{-2\pi i x \xi} \, dx = 0$$

$$\implies \int \left(e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} \, dx = 0$$

$$\implies \left(e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} =_{a.e.} 0$$

$$\implies e^{-2\pi|\xi|} =_{a.e} \hat{P}(\xi),$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

6 Problem 6

We first note that if $G_t(x) := t^{-n} e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2 |\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$
\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dt \right| \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dt \, dx \\
&\qquad \text{since the integrand and thus integral is positive.} \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} (1) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt,
\end{aligned}
$$

which we claim is finite, so $f \in L^1$.

To see that the norm is finite, we note that

$$
t \in [0,1] \implies e^{-\pi t^2} < 1
$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$
t \in [1, \infty) \implies t^{2\varepsilon-1} \leq 1.
$$

Thus

$$
\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon-1} \, dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon-1} \, dt \\
&\leq \int_0^1 t^{2\varepsilon-1} \, dt + \int_1^\infty e^{-\pi t^2} \, dt \\
&\leq \int_0^1 t^{2\varepsilon-1} \, dt + \int_0^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}
$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

> Justifying the interchange: we note that the integrand $G_t(x) e^{-\pi t^2} t^{2\varepsilon-1}$ is non-negative, and we've just showed that one of the iterated integrals is absolutely convergent, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on $\mathbb{R}^n$ and the remaining terms are continuous on $\mathbb{R}$, so they are all measurable on $\mathbb{R}^n$ and $\mathbb{R}$ respectively But then taking cylinders on everything in sight yields measurable functions, and the product of measurable functions is measurable.

If another interchange of integrals is justified, we can compute

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} \left(\int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \, dt \right) e^{-2\pi i x \cdot \xi} \, dx$$

$$= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \, dt \, dx$$

$$=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \, dx \, dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} \, dx \right) dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \hat{G}_t(\xi) \, dt$$

$$= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} e^{-\pi t^2 |\xi|^2} \, dt$$

$$= \int_0^\infty e^{-\pi t^2 (1+|\xi|^2)} t^{2\varepsilon-1} \, dt$$

$$= \int_0^\infty e^{-\pi (t\sqrt{1+|\xi|^2})^2} t^{2\varepsilon-1} \, dt$$

$$s = t\sqrt{1+|\xi|^2}, \ ds = \sqrt{1+|\xi|^2} dt$$

$$= \int_0^\infty e^{-\pi s^2} \left(\frac{s}{\sqrt{1-|\xi|^2}} \right)^{2\varepsilon-1} \frac{1}{\sqrt{1+|\xi|^2}} \, ds$$

$$= (1+|\xi|^2)^{-\frac{2\varepsilon-1}{2}} (1+|\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon-1} \, ds$$

$$= (1+|\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon-1} \, dt$$

$$:= F(\xi) \|f\|_1.$$

To see that the interchange is justified, note that

$$\int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} \right| \, dt \, dx = \int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \right| \, dt \, dx,$$

since $\left| e^{2\pi i x \cdot \xi} \right| = 1$. The integrand appearing is precisely what we showed was measurable when computed $\|f\|_1$ above, so Tonelli applies.

Thus $F(\xi)$ is the Fourier transform of the function $g(x) := f(x)/\|f\|_1$. □