Title

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Recall that good filtration is a chain $\{0\} \subseteq V_1 \subseteq \cdots \subseteq V$ satisfying $V = \bigcup V_i$ and $V_i/V_{i-1} \cong H^0(\lambda_i)$ for λ_i some weight of V.

Lemma 1.1(?).

Let V be a G-module and $\lambda \in X(T)_+$ with $\hom_G(L(\lambda), V)$. If $\hom_G(L(\mu), V) = 0$ for any $\mu < \lambda$ and $\operatorname{Ext}_G^1(V(\mu), V) = 0$ for all $\mu \in X(T)_+$, then V contains a submodule isomorphic to $H^0(\lambda)$.

That is, we have a lift of the following form:

$$L(\lambda) \stackrel{\exists}{\longleftrightarrow} V$$

$$\downarrow \qquad \qquad \downarrow$$

$$H^0(\lambda)$$

Theorem 1.0.1 (Cohomological Condition for Good Filtrations).

Let V be a G-module.

1. If V admits a good filtration, then the number of factors isomorphic to $H^0(\lambda)$, denoted $[V:H^0(\lambda)]$, is equal to dim $\hom_G(V(\lambda),V)$.

Analog of Jordan-Holder. Note that $H^0(\lambda)$ may not by irreducible, but changing the filtration can not change the number of composition factors.

- 2. Suppose $hom_G(V(\lambda), V) < \infty$, then TFAE:
- \bullet V admits a good filtration.
- $\operatorname{Ext}_G^i(V(\lambda), V) = 0$ for all $\lambda \in X(T)_+$ and all i > 0.
- $\operatorname{Ext}_G^1(V(\lambda), V) = 0$ for all $\lambda \in X(T)_+$.

Much like measuring projectivity: can check all exts, or just the first.

Proof (Part a).

Suppose V has a good filtration. Idea: induct on the filtration.

Suppose $V = H^0(\lambda_1)$, then

$$[V:H^0(\mu)] = \begin{cases} 0 & \mu \neq \lambda_1 \\ 1 & \mu = \lambda_1 \end{cases} = \dim \hom_G(V(\lambda_1), V),$$

since we know the dimensions of these hom spaces from a previous result. Suppose now that we have

$$0 \to H^0(\mu_1) \to VH^0(\mu_2) \to 0.$$

Applying $F := \text{hom}_G(V(\lambda), \cdot)$, we find that Ext_G^1 vanishes. So this leads a SES, and the dimensions are thus additive. The result follows since F is additive.

Proof (Part b).

1 \Longrightarrow 2: Use the fact that $\operatorname{Ext}_G^i(V(\lambda), H^0(\mu)) = 0$ for all i > 0 and all μ .

 $2 \implies 3$: Clear!

 $3 \implies 1$: Choose a total ordering of weights $\lambda_0, \lambda_1, \dots \in X(T)$ such that if $\lambda_i < \lambda_j$ then i < j. Since $V \neq 0$, there exists a dominant weight $\lambda \in X(T)_+$ such that $\hom_G(V(\lambda), V) \neq 0$, so choose i minimally in this order to produce such a λ_i . Idea: use this to start a filtration. Then $\hom(L(\lambda_i), V) \neq 0$, and we have

$$V(\lambda_i) \twoheadrightarrow L(\lambda_i) \hookrightarrow V$$
.

We know that

$$\begin{aligned} \hom_G(V(\mu),V) &= 0 \quad \forall \mu < \lambda_i \\ \hom_G(L(\mu),V) &= 0 \quad \forall \mu < \lambda_i \\ \mathrm{Ext}^1_G(L(\mu),V) &= 0 \quad \forall \mu \in X(T)_+ \text{ by assumption.} \end{aligned}$$

So the following map must be an injection, since there is no socle:

$$\begin{array}{ccc}
L(\lambda_i) & \longrightarrow V \\
\downarrow & & \downarrow \\
0 & \longrightarrow H^0(\lambda_i)
\end{array}$$

Set $V_1 = H^0(\lambda_i)$, so $V_1 \subseteq V$. We then have a SES

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

Applying $hom(V(\lambda), \cdot)$ we obtain

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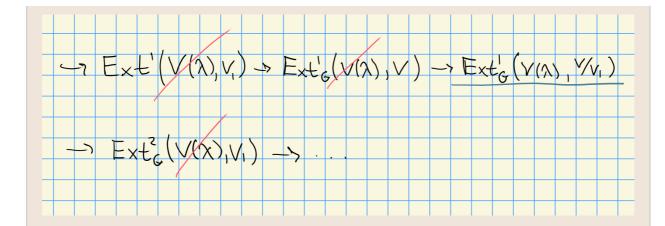


Figure 1: Cancellation in LES

Now iterate this process to obtain a chain $V_1 \subseteq V_2 \subseteq \cdots \subseteq V$, and set $V' := \bigcup_{i>0} V_i$. Then $\dim \hom_G(V(\lambda), V') = \dim \hom_G(V(\lambda), V)$ since $\dim \hom_G(V(\lambda), V) < \infty$. But then taking the SES

$$0 \to V' \to V \to V/V' \to 0$$

and applying $\operatorname{Hom}(V(\lambda), \cdot)$, we have $\operatorname{Hom}(V(\lambda), V/V') = 0$ and we get an isomorphism of homs. But then $\operatorname{hom}(V(\lambda), V/V') = 0$ for all $\lambda \in X(T)_+$, forcing V/V' = 0 and V = V'.

Corollary 1.0.1(?).

Let $0 \to V_1 \to V \to V_2 \to 0$ be a SES of G-modules with $\dim \operatorname{hom}_G(V(\lambda), V_2) < \infty$ for all $\lambda \in X(T)_+$. If V_1, V have good filtrations, then V_2 also has a good filtration.

Note: this is likely difficult to prove without cohomology! But here we can apply the ext criterion.

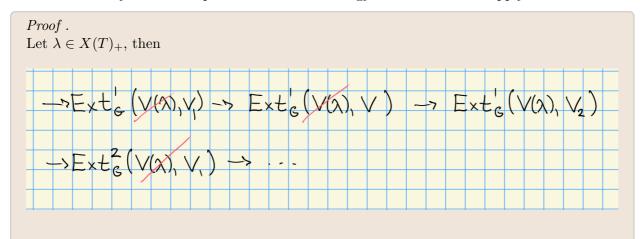


Figure 2: Image

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For $\lambda \in X(T)_+$, let $I(\lambda)$ be the injective hull of $L(\lambda)$, so we have

$$0 \to L(\lambda) \hookrightarrow I(\lambda)$$
.

Theorem 1.0.2(?).

Let $\lambda \in X(T)_+$ and $I(\lambda)$ be the injective hull of $L(\lambda)$.

- a. $I(\lambda)$ has a good filtration.
- b. The multiplicity $[I(\lambda): H^0(\mu)]$ is equal to $[H^0(\mu): L(\lambda)]$, the composition factor multi-

Brauer-Humphreys Reciprocity. Same idea as in category \mathcal{O} : multiplicity of Vermas equals multiplicity of irreducibles.

 $Proof\ (of\ a).$

How to check that it has a good filtration? The cohomological criterion! So consider $\operatorname{Ext}_G^1(V(\sigma),I(\lambda))$ for all $\sigma\in X(T)_+$. We want to show it's zero, but this follows because $I(\lambda)$ is injective.

 $Proof\ (of\ b).$

By the previous result, we have

$$\begin{split} [I(\lambda):H^0(\mu)] &= \dim \hom_G(V(\mu),I(\lambda)) \\ &= [V(\mu):L(\lambda)]. \end{split}$$

Why does this second equality hold? The functor $\hom_G(\cdot, I(\lambda))$ is exact, and $\hom_G(L(\mu), I(\lambda)) = \delta_{\lambda,\mu}$. If $\lambda = \mu$ there's only one morphism, since $L(\lambda) \hookrightarrow I(\lambda)$ and $\operatorname{Soc}_G I(\lambda) = L(\lambda)$. This means that they have the same character, char $H^0(\lambda) = \operatorname{char} V(\lambda)$, and this implies that they have the same composition factors.

Theorem 1.0.3 (Cohomological Criterion for Weyl Filtrations).

Let V be a G-module.

a. If V admits a Weyl filtration, then

$$[V:V(\lambda)] = \dim \hom_G(V, H^0(\lambda))$$

- b. Suppose that dim $\hom_G(V(\lambda), H^0(\lambda)) < \infty$ for all $\lambda \in X(T)_+$. Then TFAE
- V has a Weyl filtration.
- $\operatorname{Ext}^i_G(V, H^0(\lambda)) = 0$ for all $\lambda \in X(T)_+$ and i > 0. $\operatorname{Ext}^1_G(V, H^0(\lambda)) = 0$ for all $\lambda \in X(T)_+$.