

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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Table of Contents

Contents

Table of Contents	2
1 Tuesday, January 12	4
1.1 Background	4
1.2 Introduction	4
2 Friday, January 15	8
3 Main Theorems for the Course	10
3.1 Warm Up: \mathbb{R}^2 Has a Unique Smooth Structure	11
3.1.1 Sketch of Proof	12
4 Lecture 3 (Wednesday, January 20)	14
4.1 Sheaves	14
4.2 Bundles	17
5 Lecture 4 (Friday, January 22)	19
5.1 The Exponential Exact Sequence	19
6 Principal G-Bundles and Connections (Monday, January 25)	22
7 Wednesday, January 27	26
7.1 Bundles and Connections	26
7.2 Sheaf Cohomology	29
8 Sheaf Cohomology (Friday, January 29)	30
9 Monday, February 01	34
10 Wednesday, February 03	36
11 Friday, February 05	43
11.1 Characteristic Classes	46
12 Monday, February 08	47
ToDoS	49
Definitions	50
Theorems	51
Exercises	52

Figures	53
Bibliography	54

1 | Tuesday, January 12

1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
 - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic \mathbb{R}^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

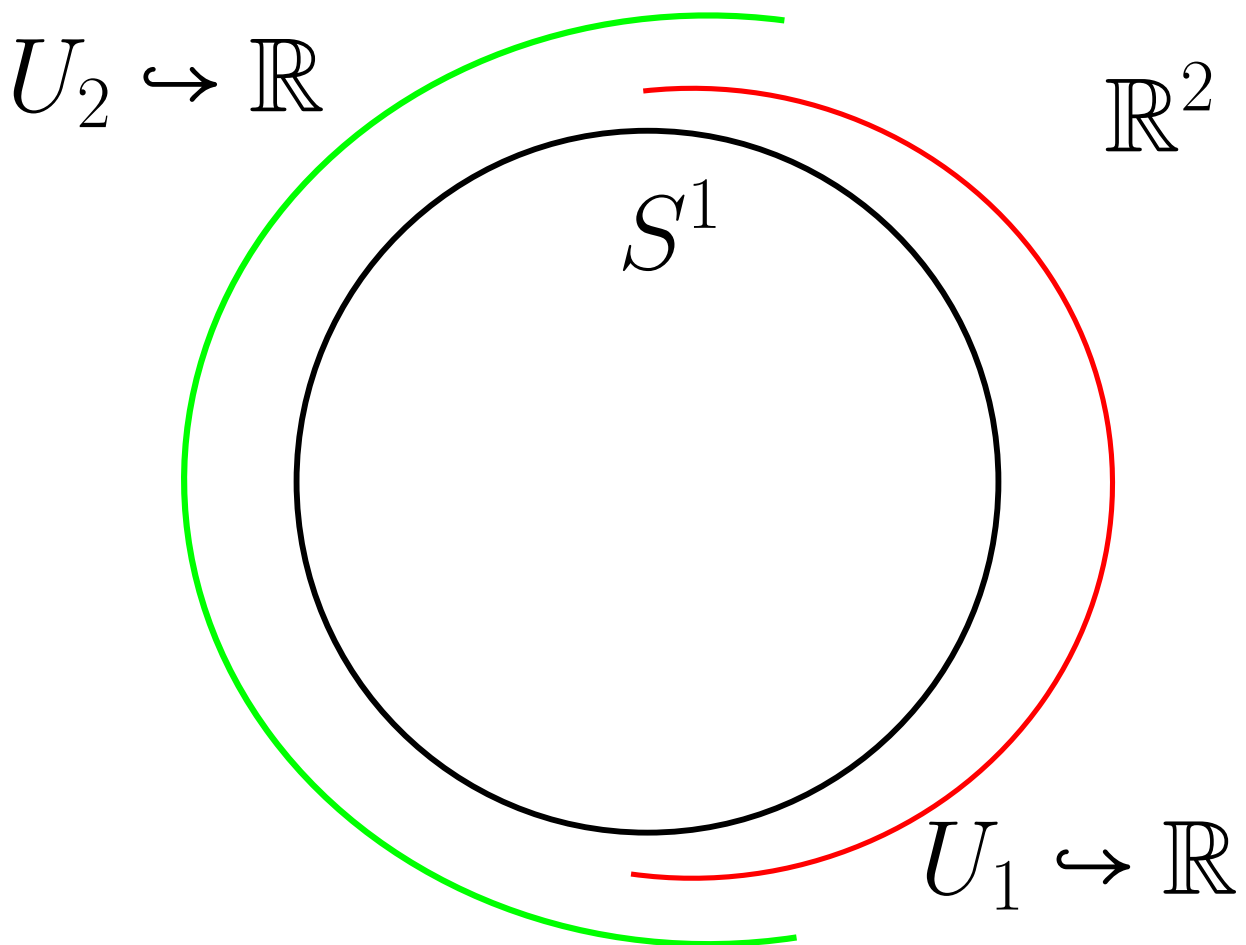
A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

Definition 1.2.1 (Topological Manifold)

Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to \mathbb{R}^n with a countable topological base, so we have charts $\varphi_u : U \rightarrow \mathbb{R}^n$ which are homeomorphisms from open sets covering X .

Example 1.2.2 (The circle): S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on \mathbb{CP}^N .

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6

Consider \mathbb{R}^n as a topological manifold: are any two smooth structures on \mathbb{R}^n diffeomorphic?

Remark 1.2.7: Fix a copy of \mathbb{R} and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\rightarrow \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on X , since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

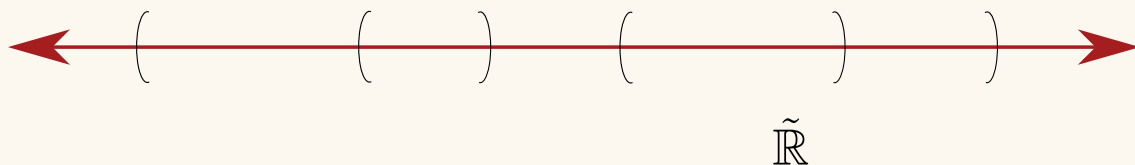
$$\begin{aligned} X &\rightarrow X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

Claim: \mathbb{R} admits a unique smooth structure.

Proof (sketch).

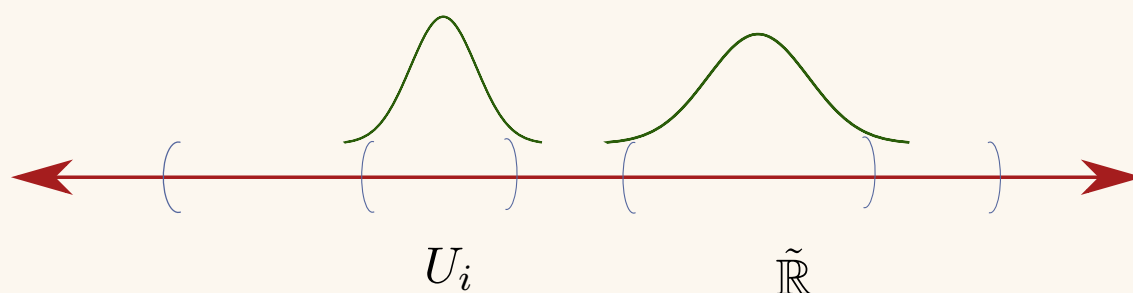
Let $\tilde{\mathbb{R}}$ be some exotic \mathbb{R} , i.e. a smooth manifold homeomorphic to \mathbb{R} . Cover this by coordinate charts to the standard \mathbb{R} :



Fact

There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \rightarrow \mathbb{R}$ with $f_i \geq 0$ and $\text{supp } f_i \subseteq U_i$ such that $\sum f_i = 1$ (i.e., *bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\rightarrow \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan

To understand smooth structures on X , we should try to solve differential equations on X .

■

Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2 .

Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?


Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.¹
2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $ks(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure.

¹Note that this doesn't start at C^0 , so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if $\text{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.
6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, \mathbb{R}^n for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes) \mathbb{R}^4 admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface Σ , the space of complex-analytic structures is a complex orbifold² of dimension $3g - 2$ where g is the genus of Σ , up to biholomorphism (i.e. *moduli*).

Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group. 

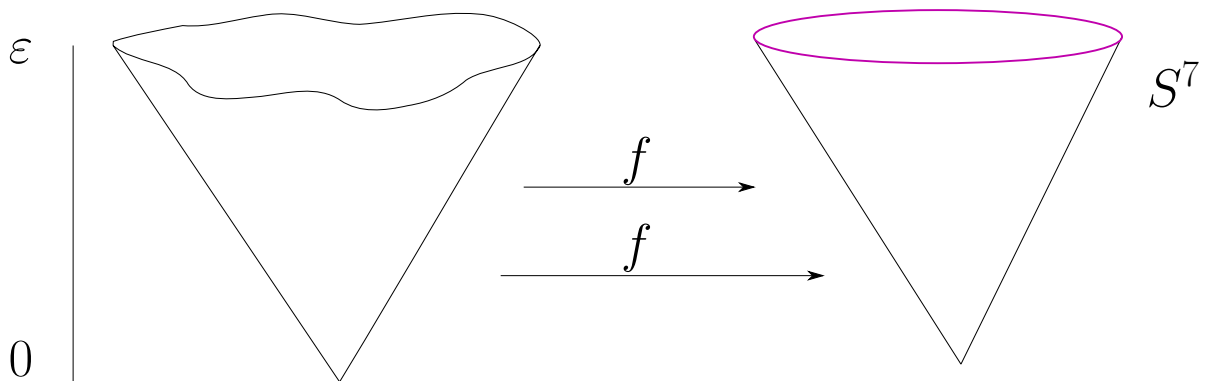
2 | Friday, January 15


Remark 2.0.1: Let

$$V := \{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\} \subseteq \mathbb{C}^5$$

$$S_\varepsilon := \{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \dots, 28$ yields the 28 smooth structures on S^7 . Note that V_k is the cone over $V_k \cap S_\varepsilon$.



? Admits a smooth structure, and $\bar{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure. 

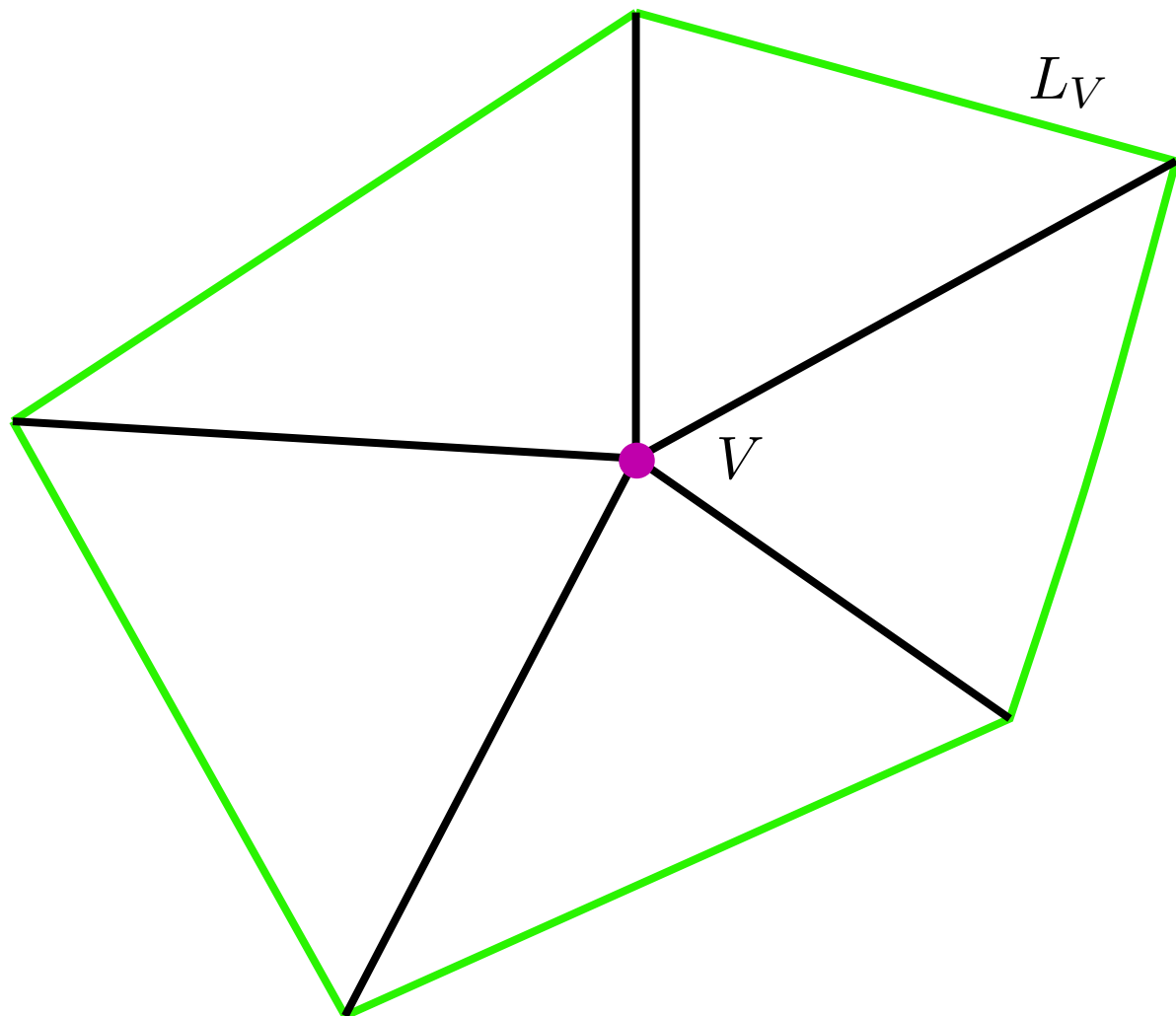
Question 2.0.2

Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

²Locally admits a chart to \mathbb{C}^n/Γ for Γ a finite group.

Answer 2.0.3

No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V :



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a “PL triangulation” where the links are spheres.

Remark 2.0.4: What’s special in dimension 4? Recall the **Kirby-Siebenmann** invariant $ks(x) \in H^4(X; \mathbb{Z}_2)$ for X a topological manifold where $ks(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)

Let X be a topological 4-manifold, then

$$ks(X) := ks(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\text{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} . \end{cases}$$

Remark 2.0.7: $\text{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X; \mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha, \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 (Freedman).

If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\text{ks}(X) = \text{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha, \varphi\beta \rangle = \langle \alpha, \beta \rangle$.

Conversely, every **unimodular** bilinear form appears as $H^2(X; \mathbb{Z})$ for some X , i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\rightarrow H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for “being an element of the orthogonal group”, where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X; \mathbb{Z}) \xrightarrow{PD} H_2(X; \mathbb{Z})$ from Poincaré, where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1, \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)

The **signature** of a topological 4-manifold is the signature of Q_X , where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X; \mathbb{R})$ and for some a, b

$$(H^2(X; \mathbb{R}), Q_X) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of +1s appearing in the matrix and b is the number of -1s. This is \mathbb{R}^{ab} where $e_i^2 = 1, i = 1 \dots a$ and $e_i^2 = -1, i = a + 1, \dots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 (Rokhlin's Theorem).

Suppose $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X; \mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\text{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 (Donaldson).

Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X; \mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: \mathbb{R}^2 Has a Unique Smooth Structure

Remark 3.1.1: Last time we showed \mathbb{R}^1 had a unique smooth structure, so now we'll do this for \mathbb{R}^2 . The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)

A **Riemannian metric** $g \in \text{Sym}^2 T^*X$ for X a smooth manifold is a metric on every T_pX given by

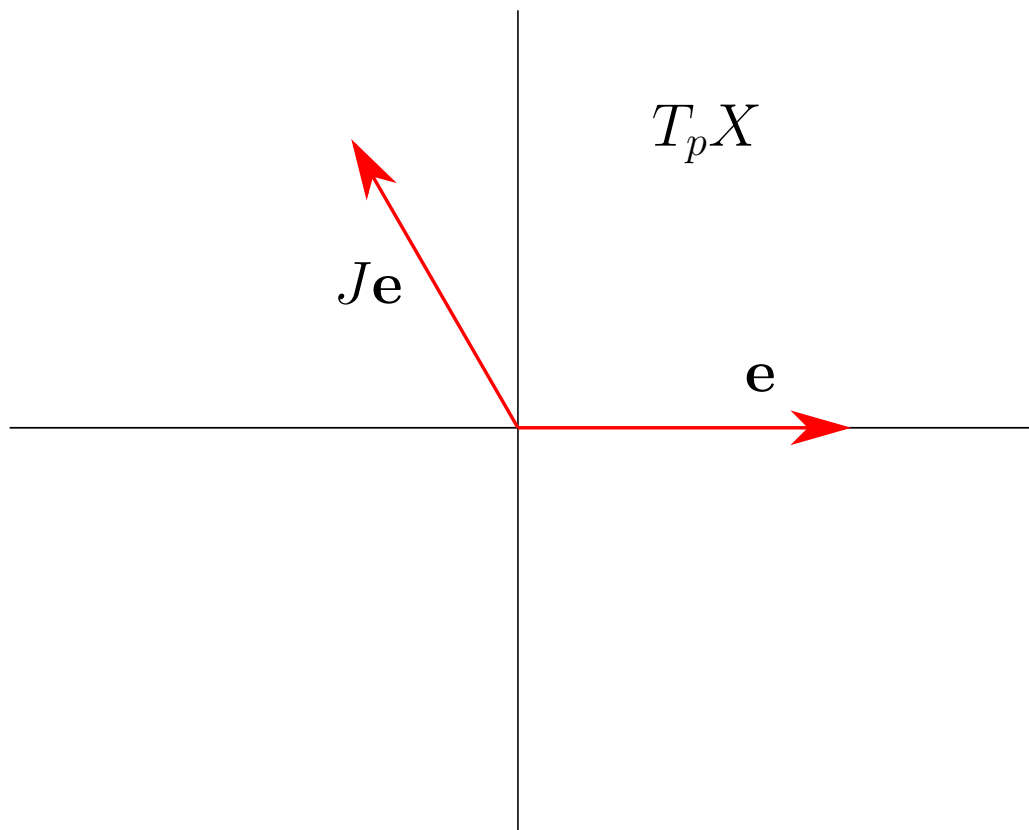
$$g_p : T_pX \times T_pX \rightarrow \mathbb{R}$$

$$g(v, v) \geq 0, g(v, v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)

An **almost complex structure** is a $J \in \text{End}(TX)$ such that $J^2 = -\text{id}$.

Remark 3.1.4: Let $e \in T_pX$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of T_pX .



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

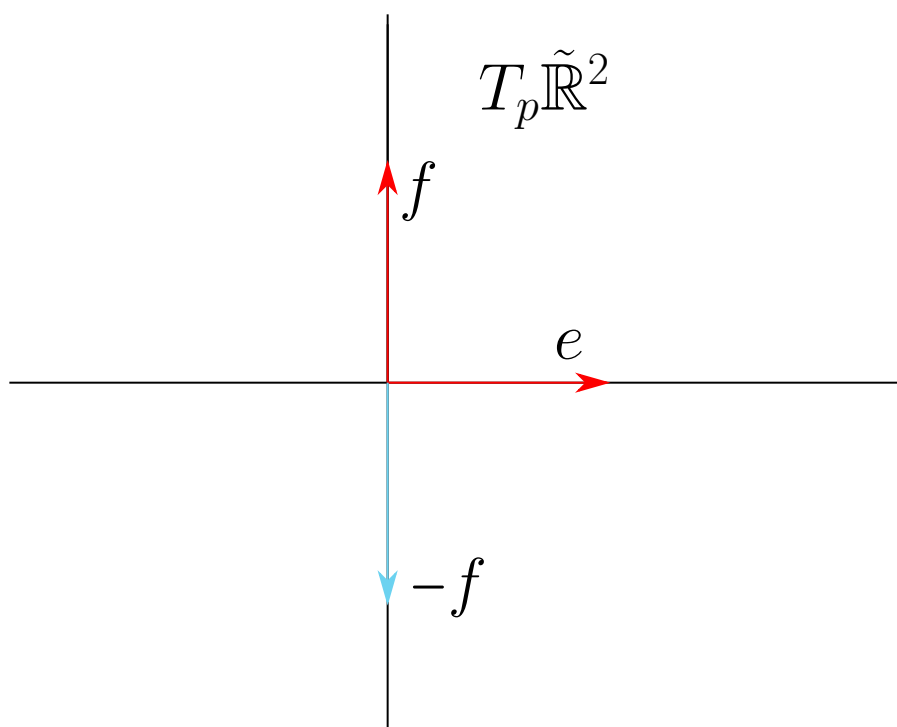
Let $\tilde{\mathbb{R}}^2$ be an exotic \mathbb{R}^2 .

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g := \sum f_I g_i$ with g_i metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e := f \in T_p \tilde{\mathbb{R}}^2$ such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$.
- $\{e, f\}$ is an oriented basis of $T_p \tilde{\mathbb{R}}^2$

This is because after choosing e , there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5(?).

Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \rightarrow \mathbb{C}$ such that J is multiplication by i .

So $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to \mathbb{D} or \mathbb{C} , both of which are diffeomorphic to \mathbb{R}^2 .

See the Newlander-Nirenberg theorem, a result in complex geometry.

4 | Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

4.1 Sheaves

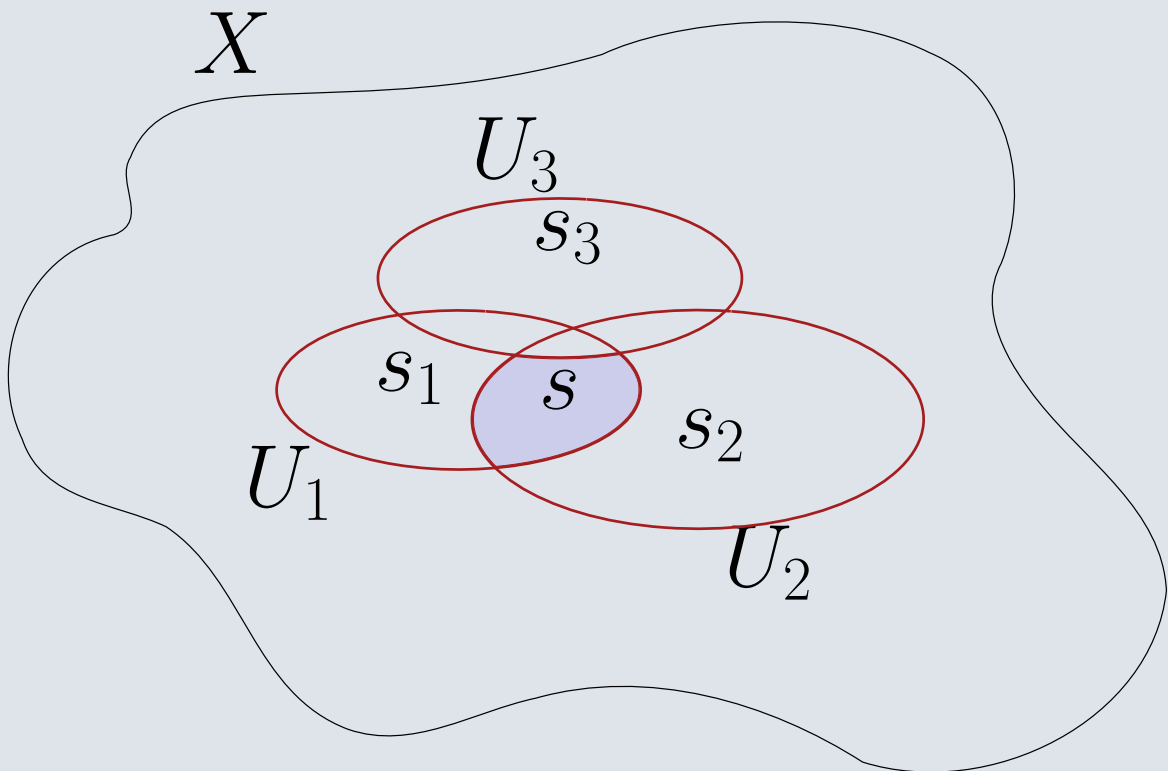
Definition 4.1.1 (Presheaves and Sheaves)

Recall that if X is a topological space, a **presheaf** of abelian groups \mathcal{F} is an assignment $U \rightarrow \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

- a. $\mathcal{F}(\emptyset) = 0$, the trivial abelian group.
- b. $\rho_{UU} : \mathcal{F}(U) \rightarrow \mathcal{F}(U) = \text{id}_{\mathcal{F}(U)}$
- c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say \mathcal{F} is a **sheaf** if additionally:

- d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2(?): Let X be a topological manifold, then $\mathcal{F} := C^0(\cdot, \mathbb{R})$ the set of continuous functions form a sheaf. We have a diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\mathcal{F}} & C^0(U; \mathbb{R}) \\
 \uparrow & & \downarrow \text{restrict cts. functions} \\
 V & \xrightarrow{\mathcal{F}} & C^0(V; \mathbb{R})
 \end{array}$$

[Link to diagram](#)

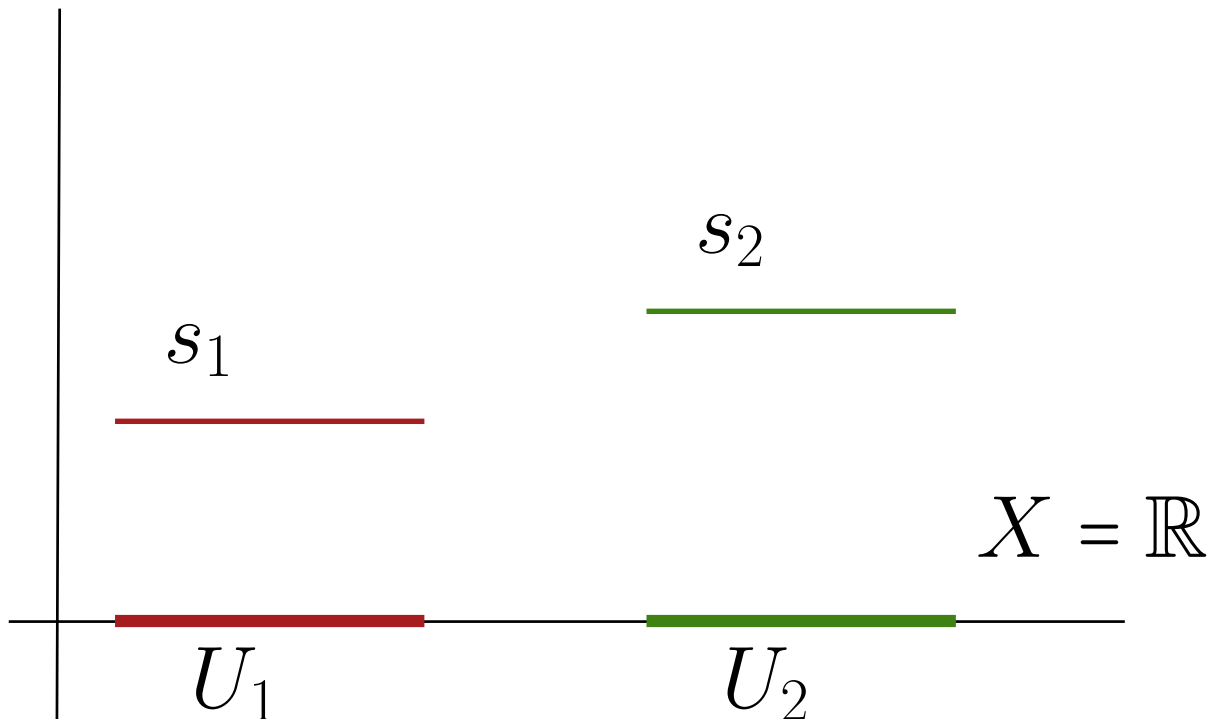
Property (d) holds because given sections $s_i \in C^0(U_i; \mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, there exists a unique $s \in C^0(\bigcup_i U_i; \mathbb{R})$ such that $s|_{U_i} = s_i$ for all i – continuous functions glue.

Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote \mathcal{O} . For example, $\mathcal{O} := C^0(\cdot; \mathbb{R})$ for topological manifolds, and $\mathcal{O} := C^\infty(\cdot; \mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set \mathcal{O} to be the sheaf of holomorphic functions.

Example 4.1.4 (Locally Constant Sheaves): Let $A \in \mathbf{Ab}$ be an abelian group, then \underline{A} is the sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \rightarrow A$. E.g. let $X \in \mathbf{Mfd}_{\text{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant \implies globally constant in this case.

Warning 4.1.5

Note that the presheaf of constant functions doesn't satisfy (d)! Take \mathbb{R} and a function with two different values on disjoint intervals:



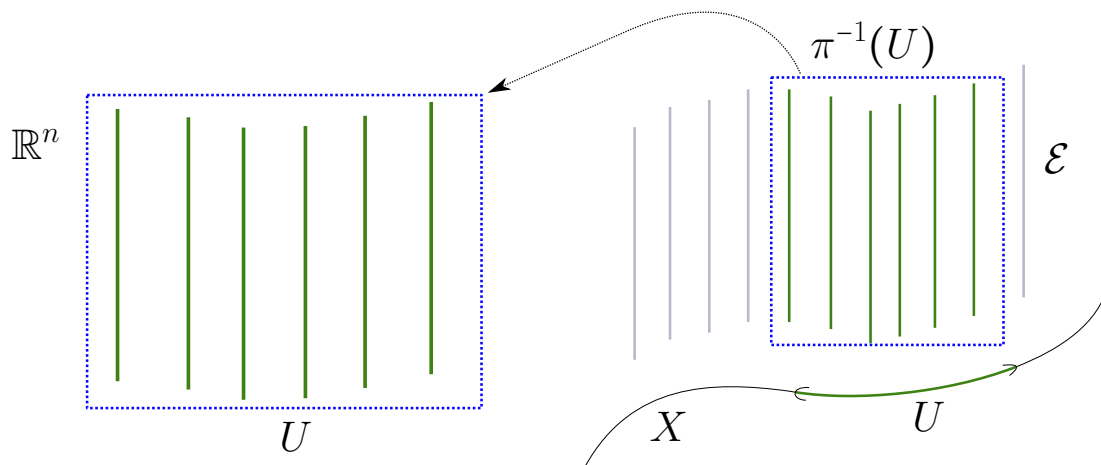
Note that $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

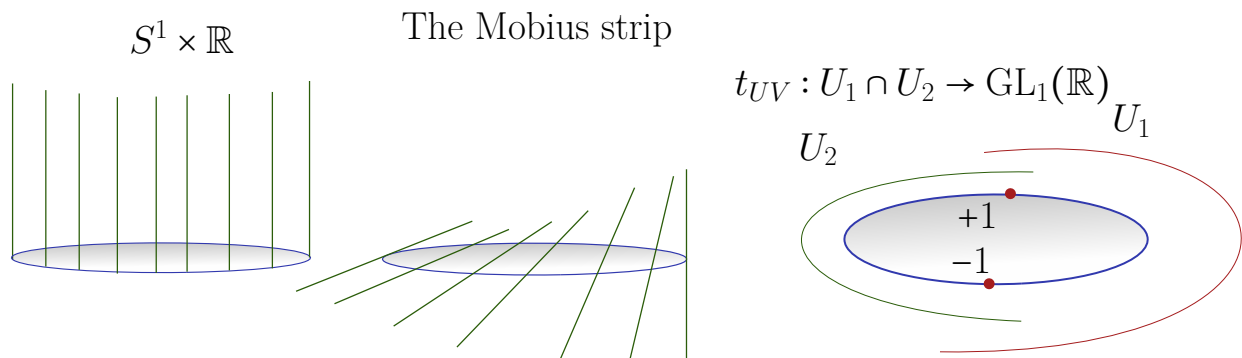
Remark 4.2.1: Let $\pi : \mathcal{E} \rightarrow X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \rightarrow \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 (Bundles over S^1): There are two \mathbb{R}^1 bundles over S^1 :



Note that the Möbius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: \mathcal{E} is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \rightarrow \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \mathrm{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4(?): The trivial line bundle gives the sheaf $\mathcal{O} : \text{maps } U \xrightarrow{s} U \times Y \text{ for } Y = \mathbb{R}, \mathbb{C} \text{ such that } \pi \circ s = \text{id}$ are the same as maps $U \rightarrow Y$.

Definition 4.2.5 (\mathcal{O} -modules)

An \mathcal{O} -module is a sheaf \mathcal{F} such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6(?): If \mathcal{E} is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \cdot s := fs$, i.e. just multiplying functions.

Example 4.2.7(Non-example): The locally constant sheaf \mathbb{R} is not an \mathcal{O} -module: there isn't natural action since the sections of \mathcal{O} are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

Definition 4.2.8 (Morphisms of Sheaves)

A **morphism** of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Example 4.2.9(An \mathcal{O} -module that is not a vector bundle.): Let $X = \mathbb{R}$ and define the **skyscraper sheaf** at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) := \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}$$

The $\mathcal{O}(U)$ -module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\rightarrow \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments.

Example 4.2.10(of morphisms): Let $X = \mathbb{R} \in \text{Mfd}_{\text{Sm}}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x \cdot) : \mathcal{O} &\rightarrow \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$

for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.11(?)

Check that $\ker \varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x \cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\text{coker}(x \cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to \mathbb{R}_0 . So $\mathcal{O} \rightarrow \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{(x \cdot)} \mathcal{O} \rightarrow \mathbb{R}_0 \rightarrow 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles.

5 | Lecture 4 (Friday, January 22)

5.1 The Exponential Exact Sequence

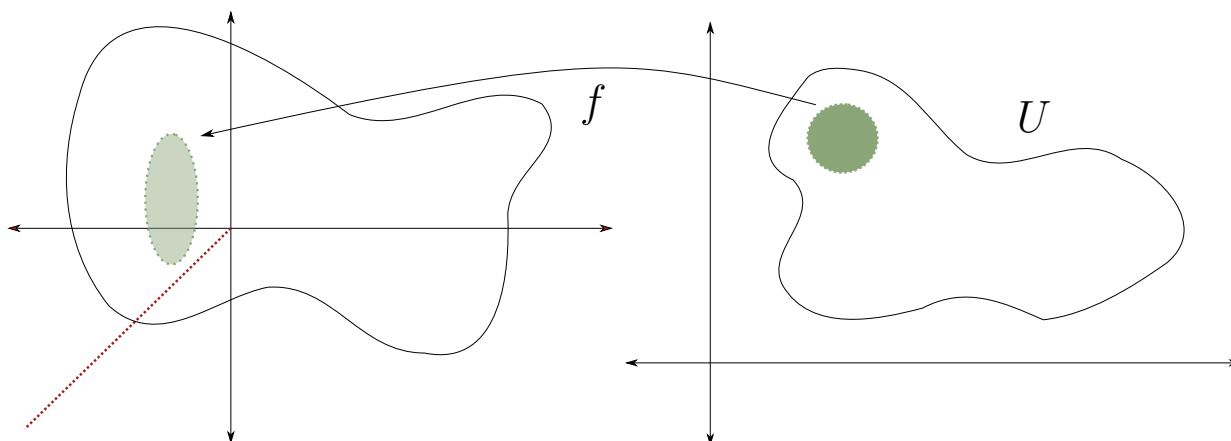
Let $X = \mathbb{C}$ and consider \mathcal{O} the sheaf of holomorphic functions and \mathcal{O}^\times the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \rightarrow \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \rightarrow \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\mathbb{Z}$, and we get an exact sequence

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \rightarrow \text{coker}(\exp) \rightarrow 0.$$

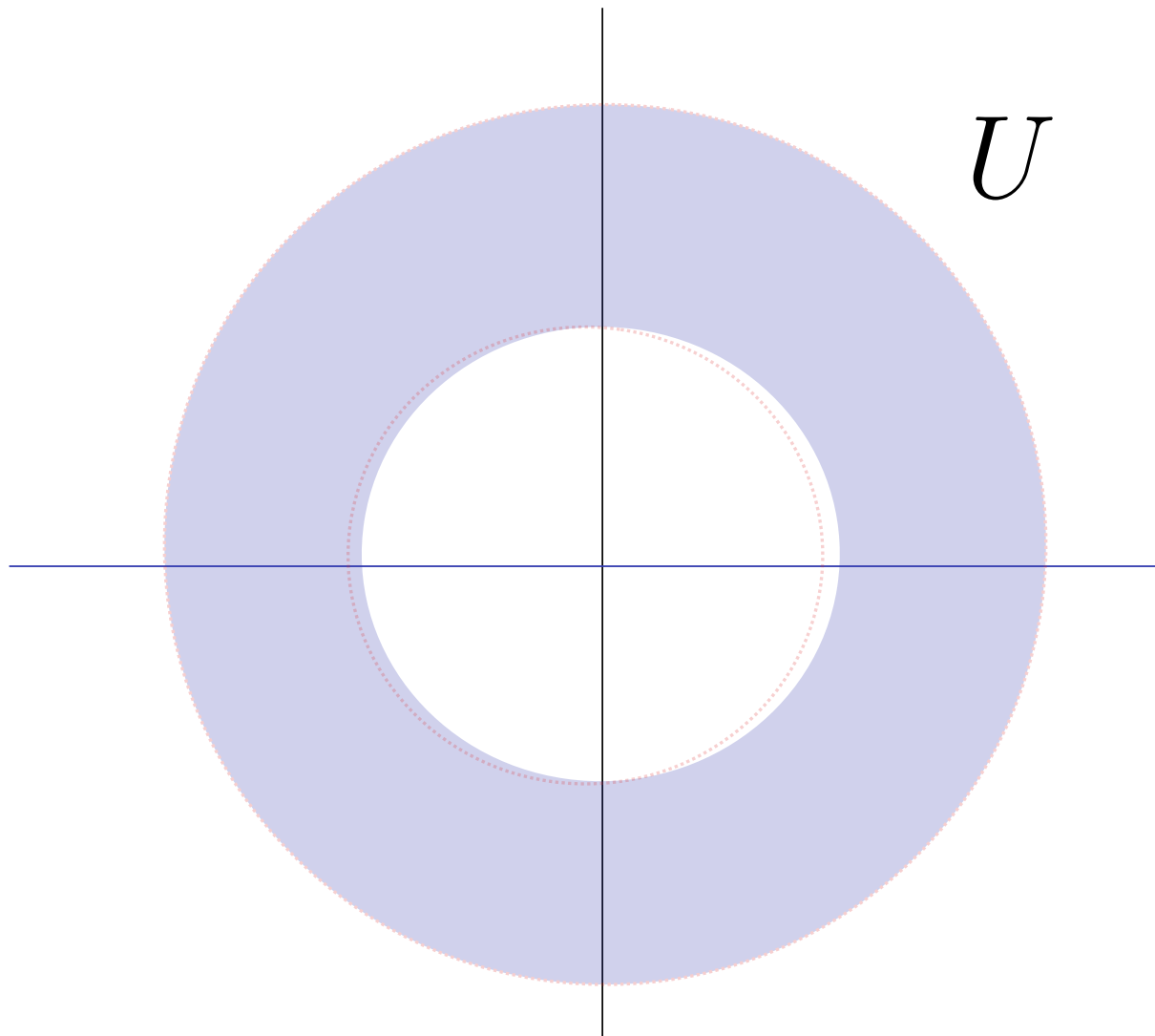
Question 5.1.1

What is the cokernel sheaf here?

Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}^\times(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U :



Thus on any sufficiently small open set, $\text{coker}(\exp) = 1$. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z , which is nontrivial.

Exercise 5.1.2 (?)

Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property $\text{Hom}_{\text{Presheaf}}(\mathcal{G}, \mathcal{F}^{\text{Presheaf}}) = \text{Hom}_{\text{Sheaf}}(\mathcal{G}, \mathcal{F}^{\text{Sh}})$ for any sheaf \mathcal{G} .

Definition 5.1.3 (Global Sections Sheaf)

The **global sections** sheaf of \mathcal{F} on X is given by $H^0(X; \mathcal{F}) = \mathcal{F}(X)$.

Example 5.1.4(?):

- $C^\infty(X) = H^0(X, C^\infty)$ are the smooth functions on X
- $VF(X) = H^0(X; T)$ are the smooth vector fields on X for T the tangent bundle
- If X is a complex manifold then $\mathcal{O}(X) = H^0(X; \mathcal{O})$ are the globally holomorphic functions on X .
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

Remark 5.1.5: Given vector bundles V, W , we have constructions $V \oplus W, V \otimes W, V^\vee, \text{Hom}(V, W) = V^\vee \otimes W, \text{Sym}^n V, \Lambda^p V$, and so on. Some of these work directly for sheaves:

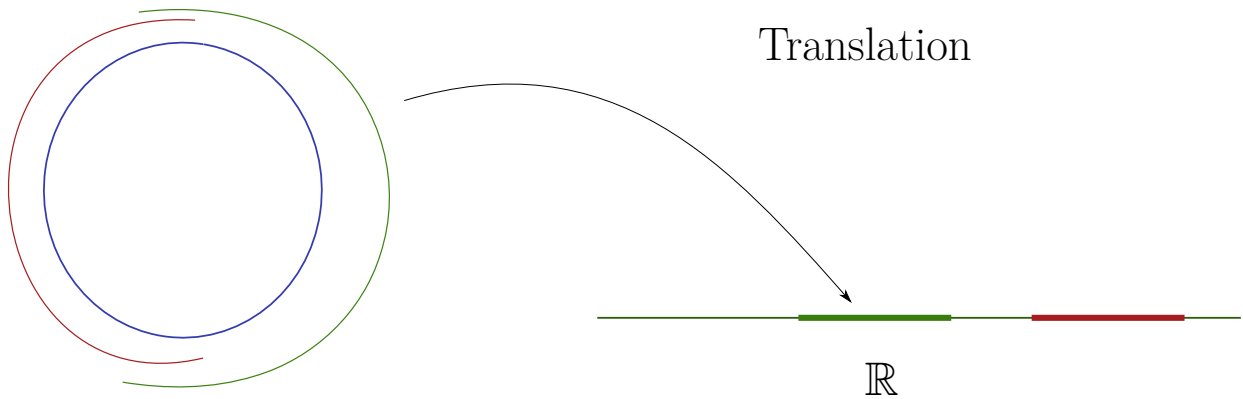
- $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathcal{H}\text{om}(V, W)$ we only get presheaves, so we need to sheafify.

⚠ Warning 5.1.6

$\text{Hom}(V, W)$ will denote the *global* homomorphisms $\mathcal{H}\text{om}(V, W)(X)$, which is a sheaf.

Example 5.1.7(?): Let $X^n \in \text{Mfd}_{\text{sm}}$ and let Ω^p be the sheaf of smooth p -forms, i.e. $\Lambda^p T^\vee$, i.e. $\Omega^p(U)$ are the smooth p forms on U , which are locally of the form $\sum f_{i_1, \dots, i_p}(x_1, \dots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$ where the f_{i_1, \dots, i_p} are smooth functions.

Example 5.1.8(Sub-example): Take $X = S^1$, writing this as \mathbb{R}/\mathbb{Z} , we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x+c) = dx$ for c a constant:

**Exercise 5.1.9(?)**

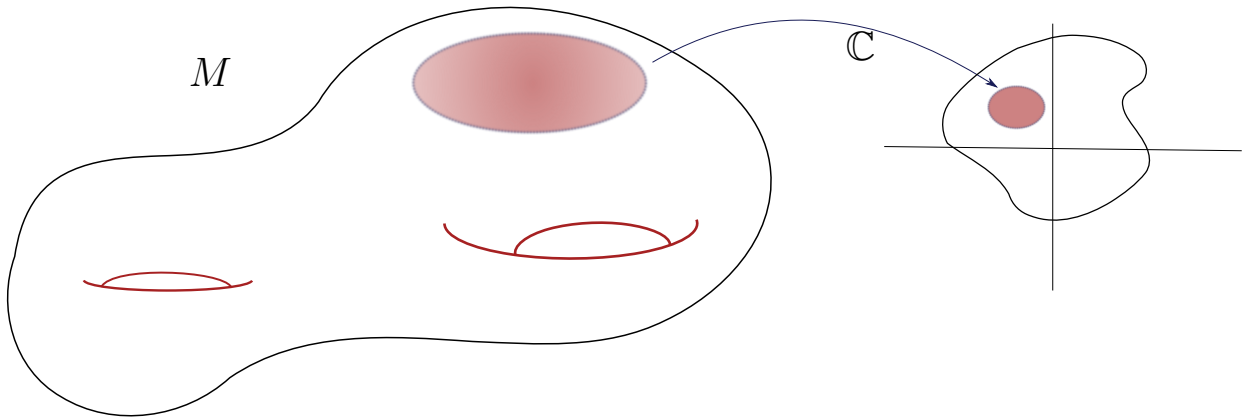
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.1.10: Note that there is a map $d: \Omega^p \rightarrow \Omega^{p+1}$ where $\omega \mapsto d\omega$.

⚠ Warning 5.1.11

d is **not** a map of \mathcal{O} -modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Let $X \in \text{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a \mathbb{C} -vector bundle.



Remark 5.1.12 (Subtlety 1): Note that Ω^p for complex manifolds is $\Lambda^p T^{\vee}$, and so if we want to view $X \in \text{Mfd}_{\mathbb{R}}$ we'll write $X_{\mathbb{R}}$. $TX_{\mathbb{R}}$ is then a real vector bundle of rank $2n$.

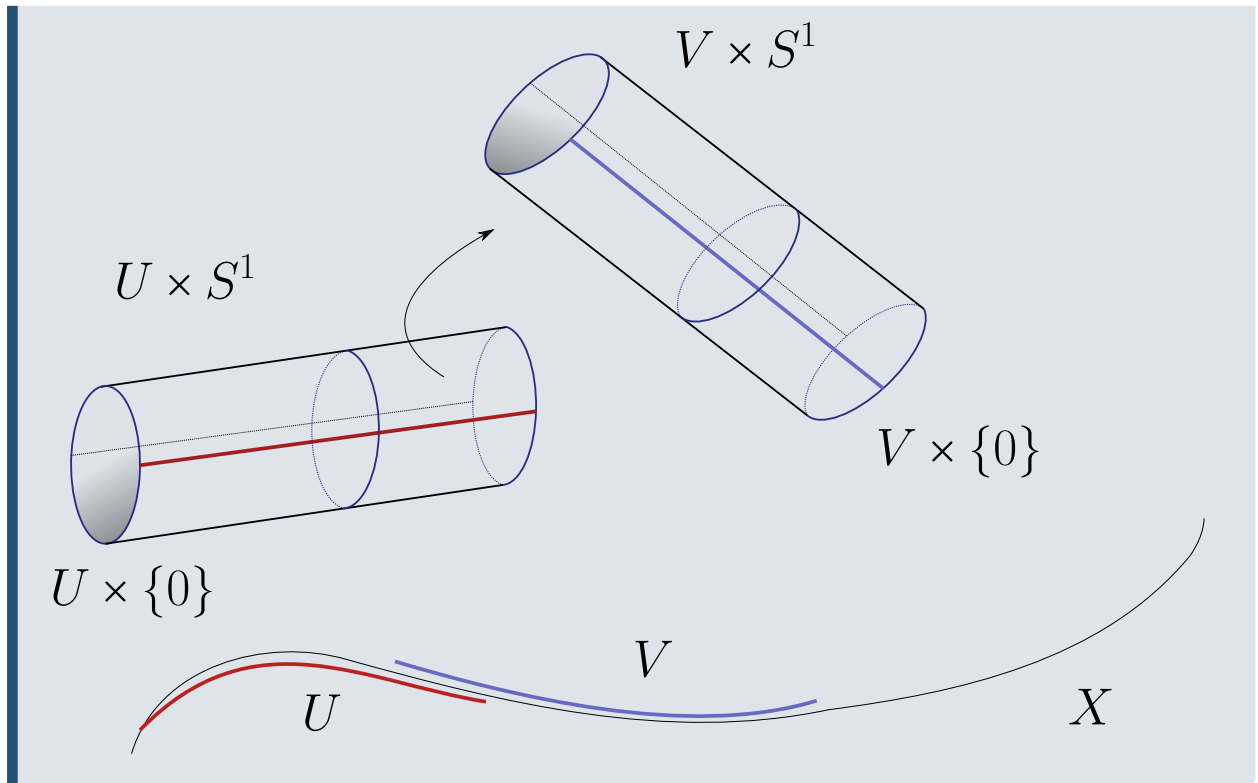
Remark 5.1.13 (Subtlety 2): Ω^p will denote *holomorphic* p -forms, i.e. local expressions $\sum f_I(z_1, \dots, z_n) \Lambda dz_I$. For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$ -forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.1.14: Note that $T^{\vee}X_{\mathbb{R}} \otimes_{\mathbb{C}} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = f dz + g d\bar{z}$ where f, g are smooth. Then $\Omega^d X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas Ω^1 is infinite-dimensional.

6 | Principal G -Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)

Let G be a (possibly disconnected) Lie group. Then a **principal G -bundle** $\pi : P \rightarrow X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \rightarrow G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \rightarrow G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \text{Mfd}_{\text{sm}}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \rightarrow \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)

Define $\text{Frame}_p(X) := \{\text{bases of } T_p X\}$, and $\text{Frame}X := \bigcup_{p \in X} \text{Frame}_p X$.

Remark 6.0.4: More generally, $\text{Frame}\mathcal{E}$ can be defined for any vector bundle \mathcal{E} , so $\text{Frame}X := \text{Frame}TX$. Note that $\text{Frame}X$ is a principal $\text{GL}_n(\mathbb{R})$ -bundle where $n := \text{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\text{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 (Important): A principal G -bundle admits a G -action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\rightarrow P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point:** the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)

The **orthogonal frame bundle** of a vector bundle \mathcal{E} equipped with a metric g is defined as $\text{OFrame}_p \mathcal{E} := \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r := \text{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \rightarrow \{x\}$ of a principal G -bundle are naturally **torsors** over G , i.e. a set with a free transitive G -action.

Definition 6.0.8 (?)

Let $\mathcal{E} \rightarrow X$ be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \rightarrow \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from \mathcal{E} .

Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

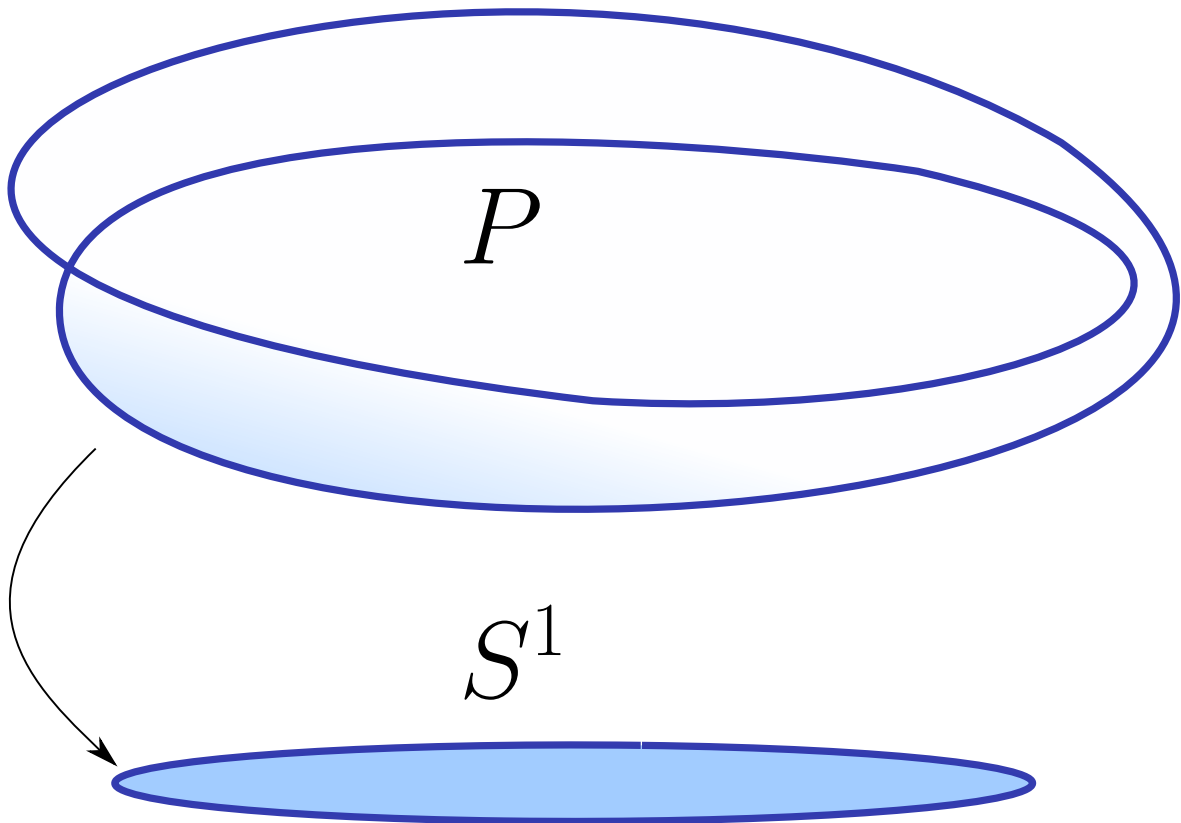
$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\mapsto (\bar{z}_1, \dots, \bar{z}_n). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)

We define the **unitary frame bundle** $\text{UFrame}(\mathcal{E}) := \bigcup_p \text{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of \mathcal{E}_p given by (e_1, \dots, e_n) where $h(e_i, \bar{e}_j) = \delta_{ij}$.

Remark 6.0.11: This is a principal G -bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\bar{A}^t = \text{id}$.

Example 6.0.12 (of more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G -bundle:



Example 6.0.13 (more principal bundles): For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U , i.e.

$$P := \{(x, O) \mid x \in X, O \text{ is an orientation of } T_p X\}.$$

Note that P is an oriented manifold, $P \rightarrow X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \text{Frame}X / \text{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)

Let $P \rightarrow X$ be a principal G -bundle and let $G \rightarrow \text{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \{(p, v) \mid p \in P, v \in V\} / \sim \quad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 (?): Note that $\text{Frame}(\mathcal{E})$ is a $\text{GL}_r(\mathbb{R})$ -bundle and the map $\text{GL}_r(\mathbb{R}) \xrightarrow{\text{id}} \text{GL}(\mathbb{R}^r)$ is

a representation. At every fiber, we have $G \times_G V = (p, v) / \sim$ where there is a unique representative of this equivalence class given by (e, pv) . So $P \times_G V_p \rightarrow \{p\} \cong V_x$.

Exercise 6.0.16 (?)

Show that $\text{Frame}(\mathcal{E}) \times_{\text{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \rightarrow G$, and so by the equivalence relation, $\text{im } t_{UV} \in \text{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Then $TM \rightarrow \text{Frame}(M)$ which is a principal $\text{SO}(3)$ -bundle. The universal cover is the double cover $\text{SU}(2) \rightarrow \text{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a \mathbb{C}^2 bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)

Let $\mathcal{E} \rightarrow X$ be a vector bundle, then a **connection** on \mathcal{E} is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\rightarrow \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge \omega + s \otimes d\omega. \end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \rightarrow \Omega^{p+1}$. The standard example is

$$\begin{aligned} d : \mathcal{O} &\rightarrow \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

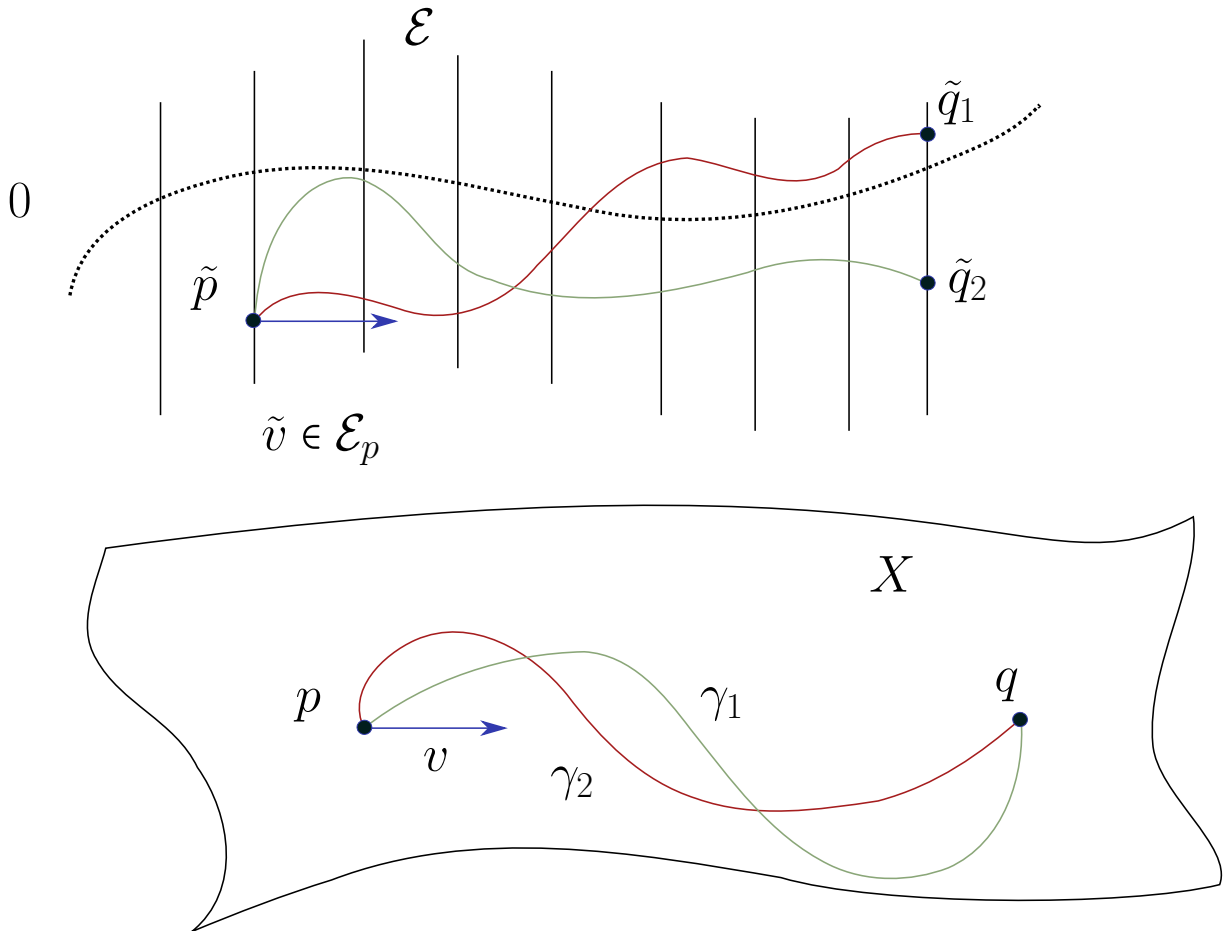
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇ , i.e. the map

$$F_{\nabla} := \nabla \circ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^2$$

is \mathcal{O} -linear, so $F_{\nabla}(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E} \otimes \Omega^{p+1}$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

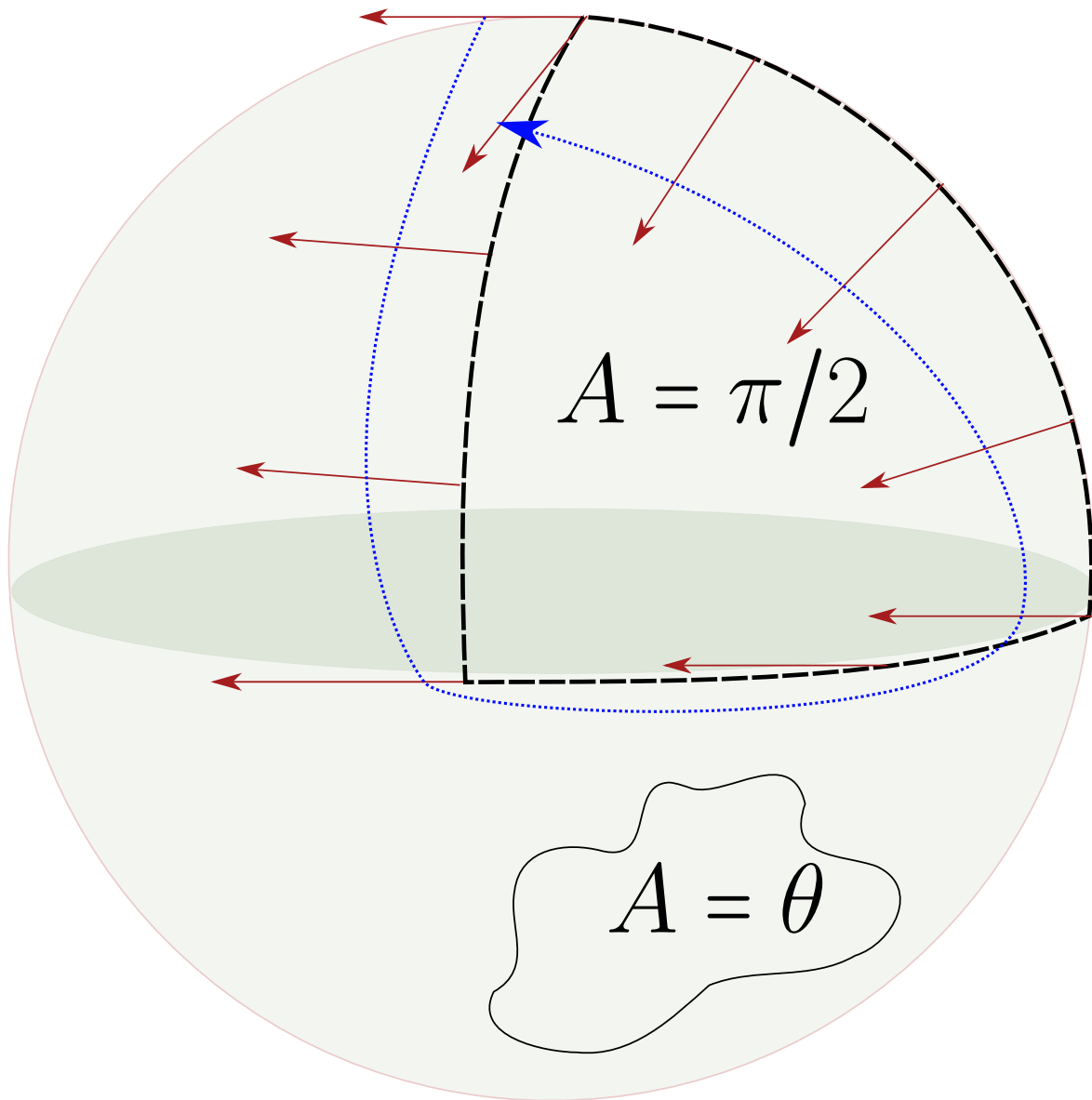


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1 , yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ .

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_{\nabla} = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ .

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 (A classic example): The Levi-Cevita connection ∇^{LC} on TX , which depends on a metric g . Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)

A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) := \left\{ s \in \mathcal{E}(U) \mid \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)

Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}}^d$. Furthermore $\text{rank}(L) = \text{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (?)

Let \mathcal{F} be a sheaf of abelian groups on a topological space X , and let $\mathfrak{U} := \{U_i\} \rightrightarrows X$ be an open cover of X . Let $U_{i_1, \dots, i_p} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1 < \dots < i_p} \mathcal{F}(U_{i_1, \dots, i_p})$$

with a differential

$$\begin{aligned} \partial^p : C_{\mathfrak{U}}^p(X, \mathcal{F}) &\rightarrow C_{\mathfrak{U}}^{p+1}(X, \mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \dots, i_p} := \prod_j (-1)^j \sigma_{i_0, \dots, \widehat{i_j}, \dots, i_p} \Big|_{U_{i_0, \dots, i_p}} \end{aligned}$$

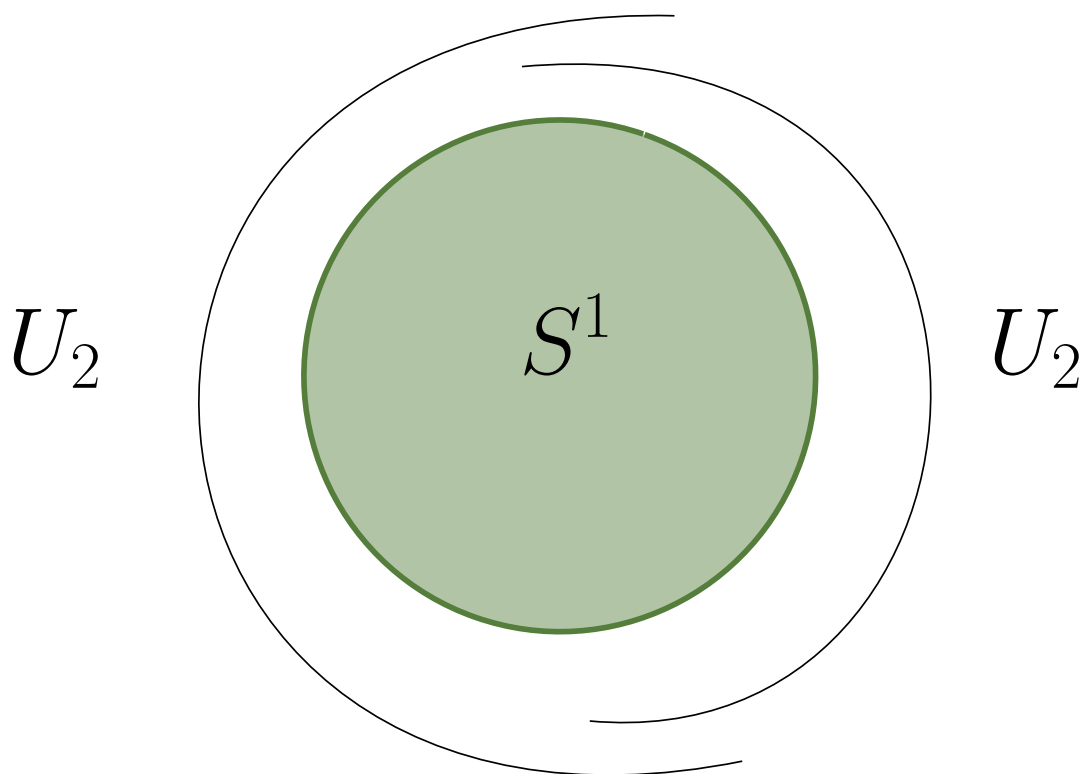
where we've defined this just on one given term in the product, i.e. a p -fold intersection.

Exercise 7.2.2 (?)

Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H_{\mathfrak{U}}^p(X, \mathcal{F})$ with respect to the cover \mathfrak{U} is defined as $\ker \partial^p / \text{im } \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4(?): Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



ere we have

$$C^0(S^1, \mathbb{Z}) = \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) = \mathbb{Z} \oplus \mathbb{Z},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\text{double intersections}} \mathbb{Z}(U_{ij})\mathbb{Z}(U_{12}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}.$$

We then get

$$\begin{aligned} C^0(S^1, \mathbb{Z}) &\xrightarrow{\partial} C^1(S^1, \mathbb{Z}) \\ \mathbb{Z} \oplus \mathbb{Z} &\rightarrow \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields $H^*(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex $C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1, \dots, i_p} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_p})$ for $\mathfrak{U} := \{U_i\}$ is an open cover of X and \mathcal{F} is a sheaf of abelian groups.

Fact 8.0.1

If \mathfrak{U} is a sufficiently fine cover then $H_{\mathfrak{U}}^p(X, \mathcal{F})$ is independent of \mathfrak{U} , and we call this $H^p(X; \mathcal{F})$.

Remark 8.0.2: Recall that we computed $H^p(S^1, \underline{\mathbb{Z}}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$.

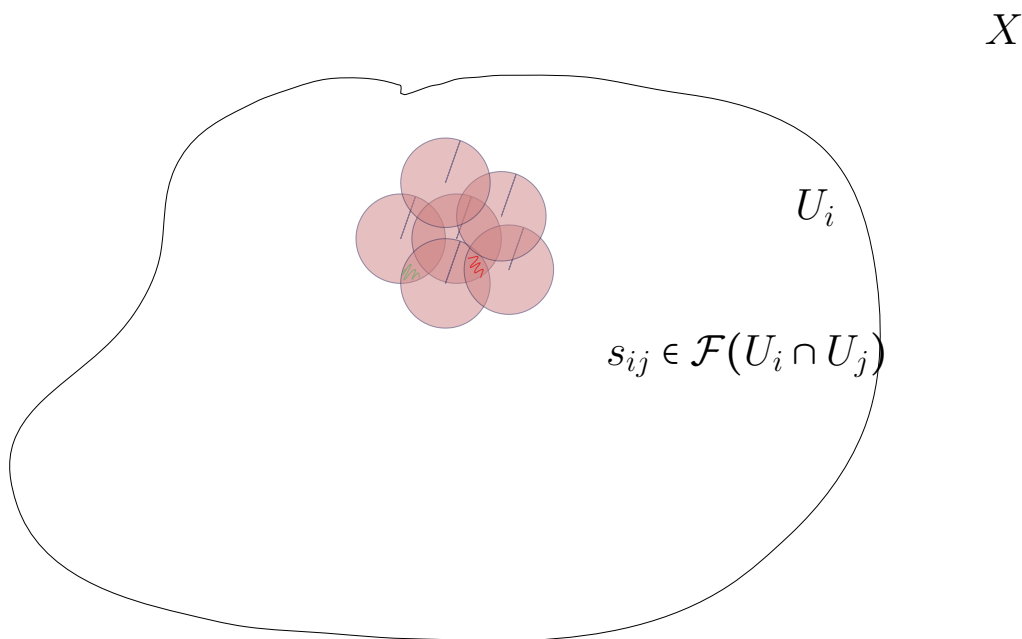
Theorem 8.0.3(?)

Let X be a paracompact and locally contractible topological space. Then $H^p(X, \underline{\mathbb{Z}}) \cong H_{\text{Sing}}^p(X, \mathbb{Z})$. This will also hold more generally with $\underline{\mathbb{Z}}$ replaced by \underline{A} for any $A \in \text{Ab}$.

Definition 8.0.4 (Acyclic Sheaves)

We say \mathcal{F} is *acyclic* on X if $H^{>0}(X; \mathcal{F}) = 0$.

Remark 8.0.5: How to visualize when $H^1(X; \mathcal{F}) = 0$:



On the intersections, we have $\text{im } \partial^0 = \{(s_i - s_j)_{ij} \mid s_i \in \mathcal{F}(U_i)\}$, which are *cocycles*. We have $C^1(X; \mathcal{F})$ are collections of sections of \mathcal{F} on every double overlap. We can check that $\ker \partial^1 = \{(s_{ij}) \mid s_{ij} - s_{ik} + s_{jk} = 0\}$, which is the cocycle condition. From the exercise from last class, $\partial^2 = 0$.

Theorem 8.0.6 (Important!).

Let X be a paracompact Hausdorff space and let

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a SES of sheaves of abelian groups, i.e. $\mathcal{F}_3 = \text{coker}(\varphi)$ and φ is injective. Then there is a LES in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \mathcal{F}_1) & \longrightarrow & H^0(X; \mathcal{F}_2) & \longrightarrow & H^0(X; \mathcal{F}_3) \\ & & & & \searrow & & \\ & & H^1(X; \mathcal{F}_1) & \longrightarrow & H^1(X; \mathcal{F}_2) & \longrightarrow & H^1(X; \mathcal{F}_3) \\ & & & & \searrow & & \\ & & \dots & & & & \end{array}$$

Example 8.0.7(?): For X a manifold, we can define a map and its cokernel sheaf:

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & \searrow & & \\ & & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & \searrow & & \\ & & \dots & & & & \end{array}$$

Corollary 8.0.8(of theorem).

Suppose $0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$ is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose I_n is acyclic for all $n \geq 0$. This is referred to as an **acyclic resolution**. Then the homology can be computed at $H^p(X; \mathcal{F}) = \ker(I_p(X) \rightarrow I_{p+1}(X)) / \text{im}(I_{p-1}(X) \rightarrow I_p(X))$.

Note that locally having kernels equal images is different than satisfying this globally!

Proof (of corollary).

This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{array}{ll} 0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} \text{im}(d_0) \rightarrow 0 & \text{im}(d_0) = \ker(d_1) \\ 0 \rightarrow \ker(d_1) \hookrightarrow I_1 \rightarrow I_1/\ker(d_1) = \text{im}(d_1) & \text{im}(d_1) = \ker(d_2) \end{array}$$

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving I_0 , we obtain:

$$\begin{array}{ccccc} & & & & \dots \\ & & & \nearrow & \\ H^{p-1}(\mathcal{F}) & \xleftarrow{\quad} & H^{p-1}(\mathcal{I}) = 0 & \xrightarrow{\quad} & H^{p-1}(\text{im}()) \\ & & \searrow \cong & & \\ H^p(\mathcal{F}) & \xleftarrow{\quad} & \dots = 0 & & \end{array}$$

The middle entries vanish since I_* was assumed acyclic, and so we obtain $H^p(\mathcal{F}) \cong H^{p-1}(\text{im } d_0) \cong H^{p-1}(\ker d_1)$. Now taking the LES associated to I_1 , we get $H^{p-1}(\ker d_1) \cong H^{p-2}(\text{im } d_1)$. Continuing this inductively, these are all isomorphic to $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$ after the p th step. ■

Corollary 8.0.9 (of the previous corollary).

Suppose $\mathfrak{U} \rightrightarrows X$, then if \mathcal{F} is acyclic on each U_{i_1, \dots, i_p} , then \mathfrak{U} is sufficiently fine to compute Čech cohomology, and $H_{\mathfrak{U}}^p(X; \mathcal{F}) \cong H^p(X; \mathcal{F})$.

Proof (?).

See notes. ■

Corollary 8.0.10 (of corollary).

Let $X \in \text{Mfd}_\downarrow$, then $H^p(X, \mathbb{R}) = H_{\text{dR}}^p(X; \mathbb{R})$.

Proof (?).

Idea: construct an acyclic resolution of the sheaf \mathbb{R} on M . The following exact sequence works:

$$0 \rightarrow \underline{\mathbb{R}} \rightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \rightarrow \dots$$


So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that $\ker d^p = \operatorname{im} d^{p-1}$ on any contractible coordinate chart. In other words, we want to show that if $d\omega = 0$ for $\omega \in \Omega^p(\mathbb{R}^n)$ then $\omega = d\alpha$ for some $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$. This is true by integration! Using the previous corollary, $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$. ■

Check Hartshorne to see how injective resolutions line up with derived functors!

9 | Monday, February 01


Remark 9.0.1: Last time $\underline{\mathbb{R}}$ on a manifold M has a resolution by vector bundles:


$$0 \rightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

This is an exact sequence of sheaves of any smooth manifold, since locally $d\omega = 0 \implies \omega = d\alpha$ (by the *Poincaré d-lemma*). We also want to know that Ω^k is an acyclic sheaf on a smooth manifold. 

Exercise 9.0.2 (?)


Let $X \in \mathbf{Top}$ and $\mathcal{F} \in \mathbf{Sh}(\mathbf{Ab})_X$. We say \mathcal{F} is **flasque** if and only if for all $U \supseteq V$ the map $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$ is surjective. Show that \mathcal{F} is acyclic, i.e. $H^i(X; \mathcal{F}) = 0$. This can also be generalized with a POU.

Example 9.0.3(?): The function $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$, but doesn't extend to a continuous map on \mathbb{R} . So the restriction map is not surjective. 

Remark 9.0.4: Any vector bundle on a smooth manifold is acyclic. Using the fact that Ω^k is acyclic and the above resolution of $\underline{\mathbb{R}}$, we can write $H^k(X; \underline{\mathbb{R}}) = \ker(d_k) / \operatorname{im} d_{k-1} := H_{dR}^k(X; \underline{\mathbb{R}})$. 


Remark 9.0.5: Now letting $X \in \mathbf{Mfd}_{\mathbb{C}}$, recalling that Ω^p was the sheaf of holomorphic p -forms. Locally these are of the form $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$ where $f_I(\mathbf{z})$ is holomorphic. There is a resolution

$$0 \rightarrow \Omega^p \rightarrow A^{p,0},$$

where in $A^{p,0}$ we allowed also f_I are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth $(p, 0)$ form to check if it's holomorphic? 

Example 9.0.6 (?): For $p = 0$, we have

$$0 \rightarrow \mathcal{O} \rightarrow A^{0,0}.$$


where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations. 

Definition 9.0.7 (?)

Let $\omega \in A^{p,q}(X)$ where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J := \partial + \bar{\partial}$$

with $|I| = p, |J| = q$.


Example 9.0.8 (?): The function $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$ is smooth, and $df = \bar{z}dz + z d\bar{z}$. This can be checked by writing $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, and $\frac{\partial}{\partial \bar{z}} g = 0$ if and only if g is holomorphic. Here we get $\partial\omega \in A^{p+1,q}(X)$ and $\bar{\partial} \in A^{p,q+1}(X)$, and we can write $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$. 

Definition 9.0.9 (Cauchy-Riemann Equations)

Recall the Cauchy-Riemann equations: ω is a holomorphic $(p,0)$ -form on \mathbb{C}^n if and only if $\bar{\partial}\omega = 0$.

Remark 9.0.10: Thus to extend the previous resolution, we should take

$$0 \rightarrow \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \rightarrow \dots$$

The fact that this is exact is called the *Poincaré $\bar{\partial}$ -lemma*. 

Remark 9.0.11: There are no bump functions in the holomorphic world, and since Ω^p is a holomorphic bundle, it may not be acyclic. However, the $A^{p,q}$ are acyclic (since they are smooth vector bundles and thus admit POUs), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \text{im}(\bar{\partial}_{q-1}).$$

Note the similarity to H_{dR} , using $\bar{\partial}$ instead of d . This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers** $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Omega^p)$. These are analogies:

Smooth	Complex
\mathbb{R}	Ω^p
Ω^k	$A^{p,q}$
Betti numbers β_k	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

Theorem 9.0.12 (Properties of Singular Cohomology).

Let $X \in \text{Top}$, then $H_{\text{Sing}}^i(X; \mathbb{Z})$ satisfies the following properties:

- Functoriality: given $f \in \text{Hom}_{\text{Top}}(X, Y)$, there is a pullback $f^* : H^i(Y; \mathbb{Z}) \rightarrow H^i(X; \mathbb{Z})$.
- The cap product: a pairing

$$H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X; \mathbb{Z}) \rightarrow H_{j-i}(X; \mathbb{Z})$$

$$\varphi \otimes \sigma \mapsto \varphi \left(\sigma|_{\Delta_{0, \dots, j}} \right) \sigma|_{\Delta_{i, \dots, j}}.$$

This makes H_* a module over H^* .

- There is a ring structure induced by the cup product:

$$H^i(X; \mathbb{R}) \times H^j(X; \mathbb{R}) \rightarrow H^{i+j}(X; \mathbb{R}) \quad \alpha \cup \beta = (-1)^{ij} \beta \cup \alpha.$$

- Poincaré Duality: If X is an oriented manifold, there exists a fundamental class $[X] \in H_n(X; \mathbb{Z}) \cong \mathbb{Z}$ and $(\cdot) \cap X : H^i \rightarrow H_{n-i}$ is an isomorphism.

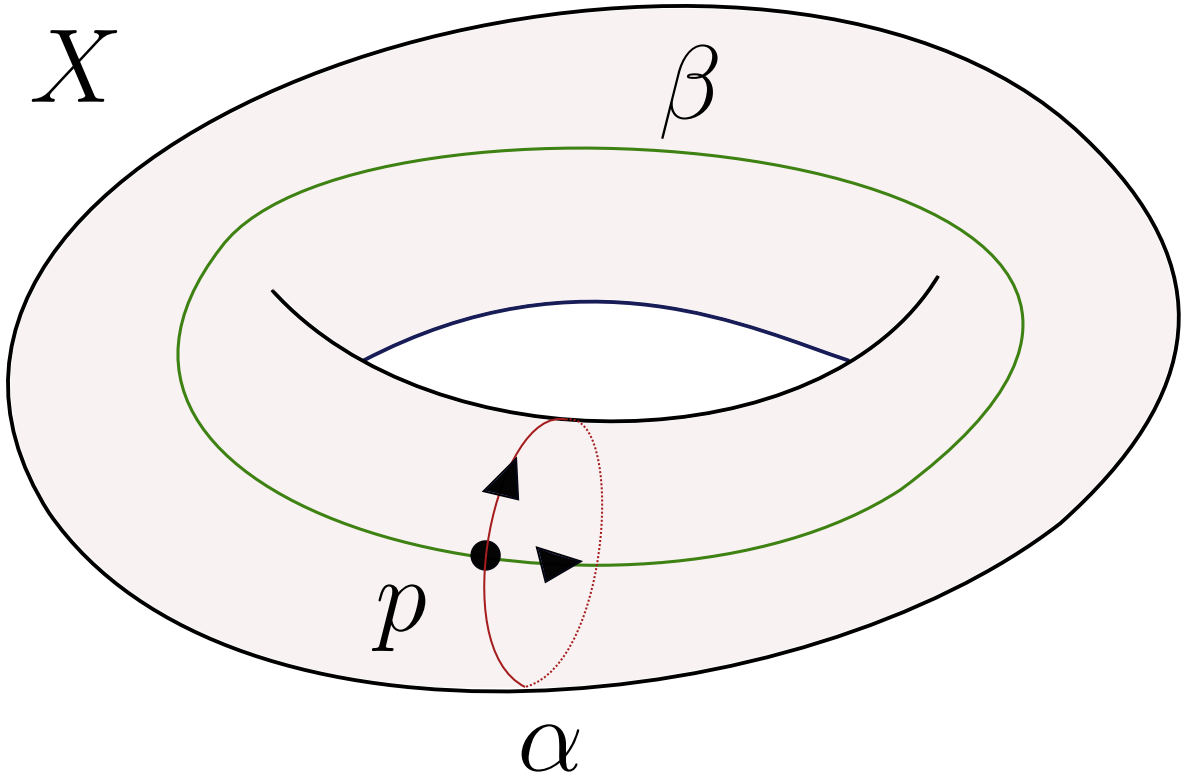
Remark 9.0.13: Let $M \subset X$ be a submanifold where X is a smooth oriented n -manifold. Then $M \hookrightarrow X$ induces a pushforward $H_n(M; \mathbb{Z}) \xrightarrow{\iota_*} H_n(X; \mathbb{Z})$ where $\sigma \mapsto \iota \circ \sigma$. Using Poincaré duality, we'll identify $H_{\dim M}(X; \mathbb{Z}) \rightarrow H^{\text{codim } M}(X; \mathbb{Z})$ and identify $[M] = PD(\iota_*([M]))$. In this case, if $M \pitchfork N$ then $[M] \cap [N] = [M \cap N]$, i.e. the cap product is given by intersecting submanifolds.

Warning 9.0.14

This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

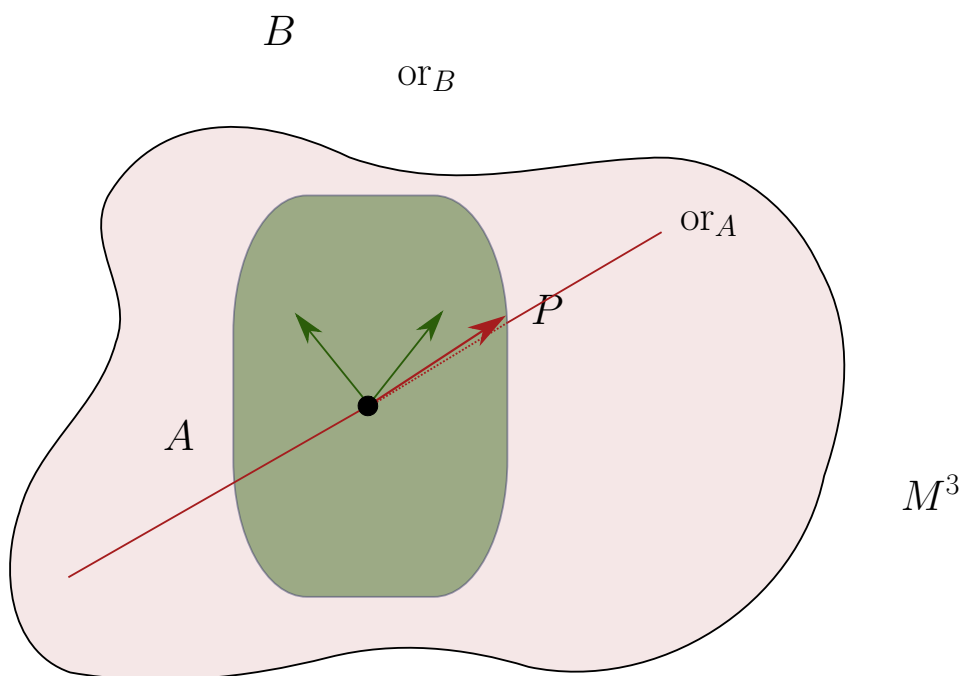


We can then take the fundamental classes of the submanifolds, say $[\alpha], [\beta] \in H^1(X; \mathbb{Z}) \xrightarrow{PD} H^1(X; \mathbb{Z})$. Here $T_p\alpha \oplus T_p\beta = T_pX$, since the intersections are transverse. Since α, β are oriented, let $\{e\}$ be a basis of $T_p\alpha$ (up to \mathbb{R}^+) and similarly $\{f\}$ a basis of $T_p\beta$. We can then ask if $\{e, f\}$ constitutes an *oriented* basis of T_pX . If so, we write $\alpha \cdot_p \beta := +1$ and otherwise $\alpha \cdot_p \beta = -1$. We thus have

$$[\alpha] \smile [\beta] \in H^2(X; \mathbb{Z}) \xrightarrow{PD} H_0(X; \mathbb{Z}) = \mathbb{Z}$$

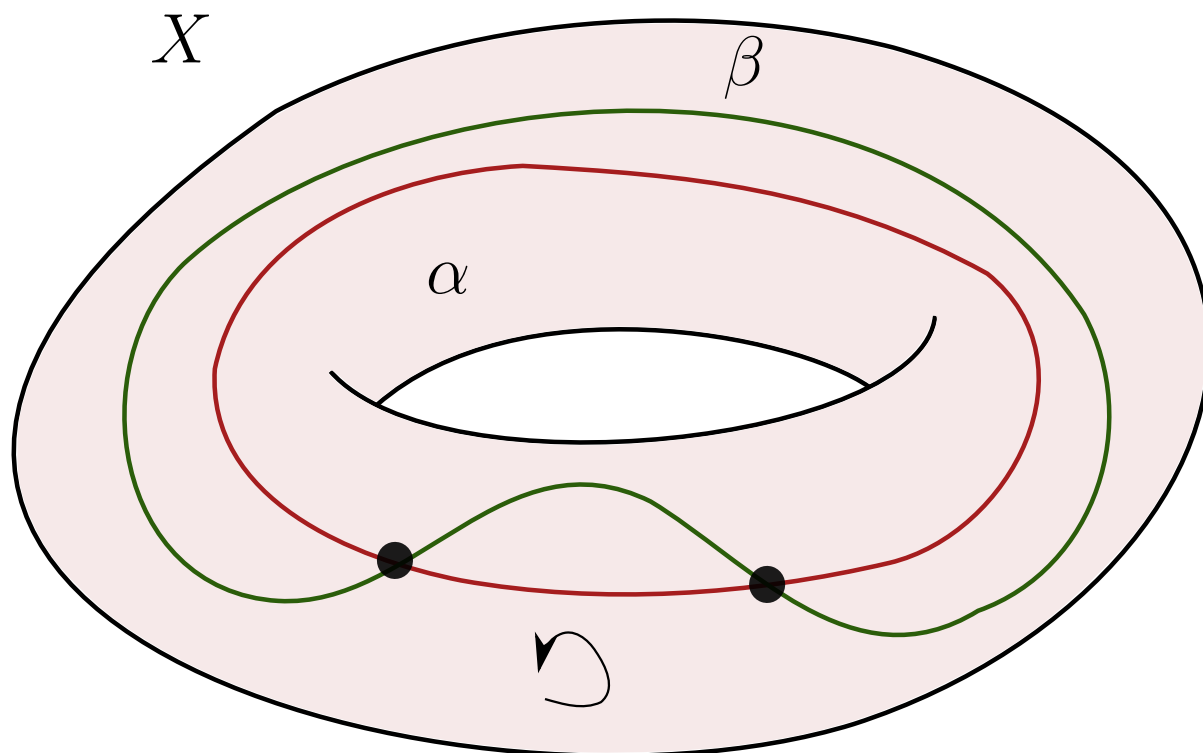
since X is connected. We can thus define the **intersection form** $\alpha \cdot \beta := [\alpha] \smile [\beta]$. In general if A, B are oriented transverse submanifolds of M which are themselves oriented, we'll have $[A] \smile [B] = [A \cap B]$. We need to be careful: how do we orient the intersection? This is given by comparing the orientations on A and B as before.

Example 10.0.1(?): If $\dim M = \dim A + \dim B$, then any $p \in A \cap B$ is oriented by comparing $\{\text{or}_A, \text{or}_B\}$ to or_M .



Here it suffices to check that $\{e, f_1, f_2\}$ is an oriented basis of $T_p M$.

Example 10.0.2(?): In this case, $[\alpha] \sim [\beta] = 0$ and so $\alpha \cdot \beta = 0$:



Remark 10.0.3: Note that cohomology with \mathbb{Z} coefficients can be defined for any topological space, and Poincaré duality still holds.

Remark 10.0.4: We'll be considering $M = M^4$, smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

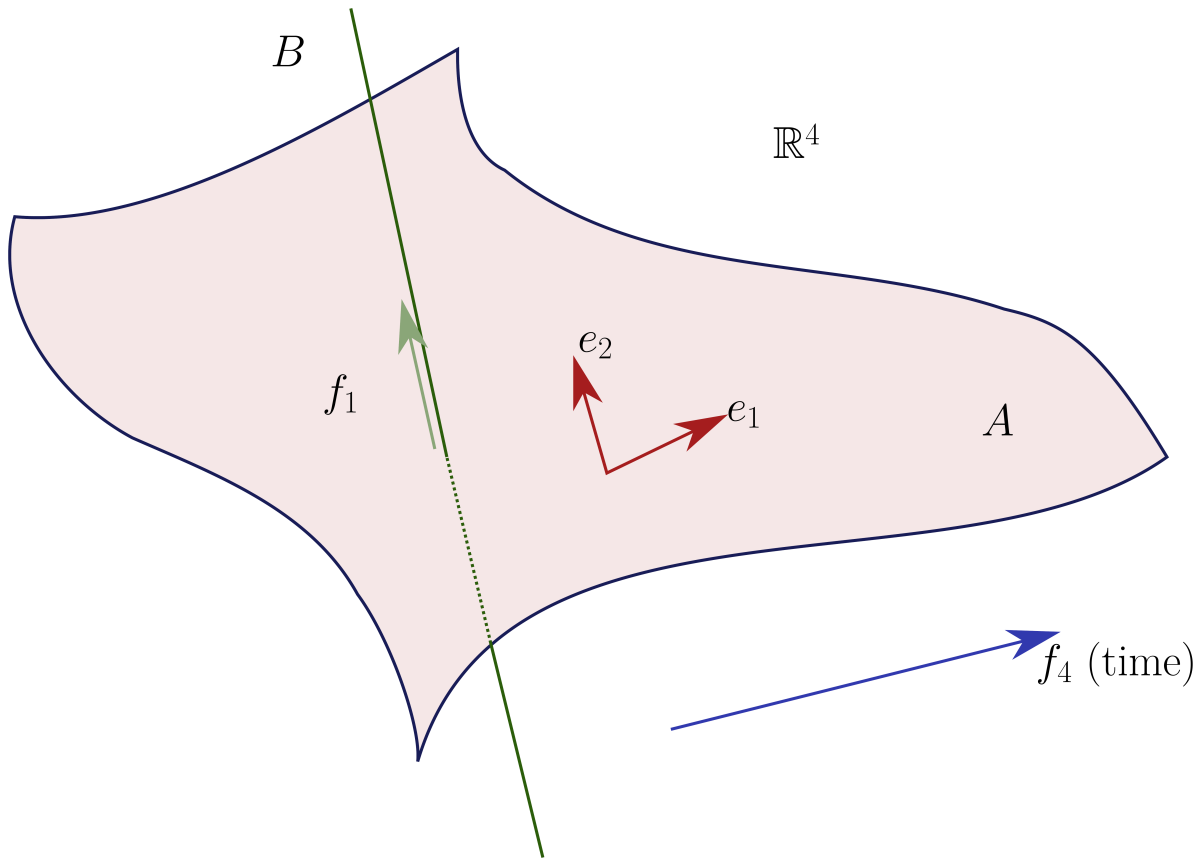


Figure 1: Picking one basis element in the time direction

Here f_4 is oriented in the “forward time” direction, and this is a surface at time $t = 0$. Where $A \cdot B = +1$, since $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$ is a oriented basis for \mathbb{R}^4 . For f_2 , switching the order of α, β no longer yields an oriented basis, but in this case it is f_2 and $A \cdot B = B \cdot A$. This is because

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

Remark 10.0.5: Let M^{2n} be an oriented manifold, then the cup product yields a bilinear map $H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ which is symmetric when n is odd and antisymmetric (or symplectic) when n is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$(H^n(M; \mathbb{Z})/\text{tors})^\vee \xrightarrow{\sim} H^n(M; \mathbb{Z})/\text{tors}$$

$$\alpha \smile \cdot \leftrightarrow \alpha,$$

where the LHS are linear functionals on cohomology.

Remark 10.0.6: Recall the universal coefficients theorem:

$$H^i(X; \mathbb{Z})/\text{tors} \cong (H_i(X; \mathbb{Z})/\text{tors})^\vee.$$

The general theorem shows that $H^i(X; \mathbb{Z})_{\text{tors}} = H_{i-1}(X; \mathbb{Z})_{\text{tors}}$.

Remark 10.0.7: Note that if M is an oriented 4-manifold, then

	tors	torsionfree		tors	torsionfree	
H^0	0	\mathbb{Z}		H_0	0	\mathbb{Z}
H^1	0	\mathbb{Z}^{β_1}		H_1	A	\mathbb{Z}^{β_1}
H^2	A	\mathbb{Z}^{β_2}	\xrightarrow{PD}	H_2	A	\mathbb{Z}^{β_2}
H^3	A	\mathbb{Z}^{β_1}		H_3	0	\mathbb{Z}^{β_1}
H^4	0	\mathbb{Z}		H_4	0	\mathbb{Z}

In particular, if M is simply connected, then $H_1(M) = \text{Ab}(\pi_1(M)) = 0$, which forces $A = 0$ and $\beta_1 = 0$.

Definition 10.0.8 (Lattice)

A **lattice** is a finite-dimensional free \mathbb{Z} -module L together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\rightarrow \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice (L, \cdot) is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\rightarrow L^\vee \\ \ell &\mapsto \ell \cdot (\cdot). \end{aligned}$$


Remark 10.0.9: How to determine if a lattice is unimodular: take a basis $\{e_1, \dots, e_n\}$ of L and form the *Gram matrix* $M_{ij} := (e_i \cdot e_j) \in \text{Mat}(n \times n, \mathbb{Z})^{\text{Sym}}$. Then (L, \cdot) is unimodular if and only if $\det(M) = \pm 1$ if and only if M^{-1} is integral. In this case, the rows of M^{-1} will form a basis of the dual basis.

Definition 10.0.10 (?)

The **index** of a lattice is $|\det M|$.

Exercise 10.0.11 (?)

Prove that $|\det M| = |L^\vee / L|$.

Remark 10.0.12: In general, for M^{4k} , the H^{2k}/tors is unimodular. For M^{4k+2} , the H^{2k+1}/tors is a unimodular *symplectic* lattice, which is obtained by replacing the word “symmetric” with “antisymmetric” everywhere above. 

Example 10.0.13 (?): For the torus, since the dimension is $2 \pmod{4}$, you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$


Check! 

Definition 10.0.14 (?)

A lattice is **nondegenerate** if $\det M \neq 0$.

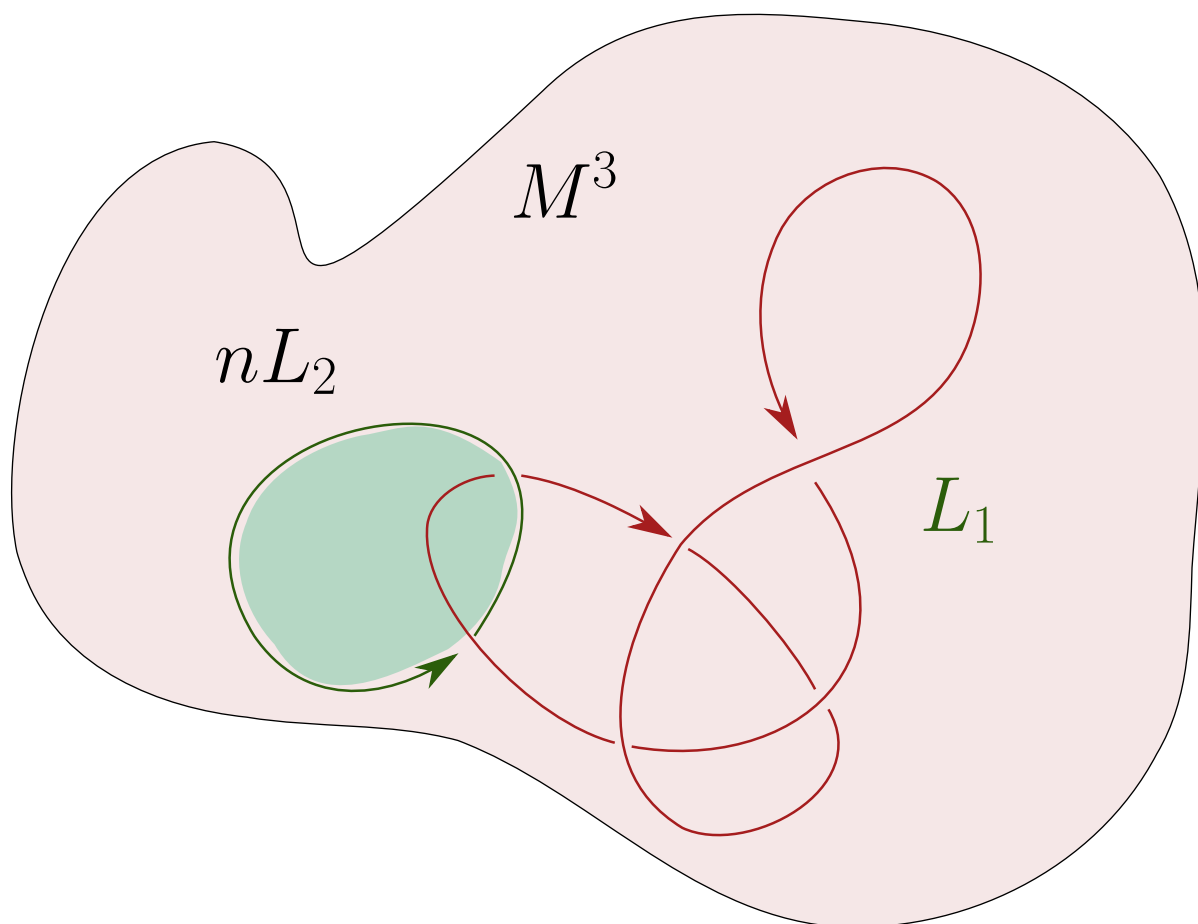
Definition 10.0.15 (?)

The tensor product $L \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space with an \mathbb{R} -valued symmetric bilinear form. This allows extending the lattice from \mathbb{Z}^n to \mathbb{R}^n .

Remark 10.0.16: If (L, \cdot) is nondegenerate, then Gram-Schmidt will yield an orthonormal basis $\{v_i\}$. The number of positive norm vectors is an invariant, so we obtain $\mathbb{R}^{p,q}$ where p is the number of +1s in the Gram matrix and q is the number of -1s. The **signature** of (L, \cdot) is (p, q) , or by abuse of notation $p - q$. This is an invariant of the 4-manifold, as is the lattice itself $H^2(X; \mathbb{Z})/\text{tors}$ equipped with the intersection form. 

Remark 10.0.17: There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$



Remark 10.0.18: $A \cdot B := \sum_{p \in A \cap B} \text{sgn}_p(A, B)$, where $A \pitchfork B$ and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \sim [\omega] = \int_A \omega.$$

11 | Friday, February 05

Remark 11.0.1: Recall that a lattice is **unimodular** if the map $L \rightarrow L^\vee := \text{Hom}(L, \mathbb{Z})$ is an isomorphism, where $\ell \mapsto \ell \cdot (\cdot)$. To check this, it suffices to check if the Gram matrix M of a basis $\{e_i\}$ satisfies $|\det M| = 1$.

Example 11.0.2 (Determinant 1 Integer Matrices): The matrices $[1]$ and $[-1]$ correspond to the lattice $\mathbb{Z}e$ where either $e^2 := e \cdot e = 1$ or $e^2 = -1$. If M_1, M_2 both have absolute determinant 1,

then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

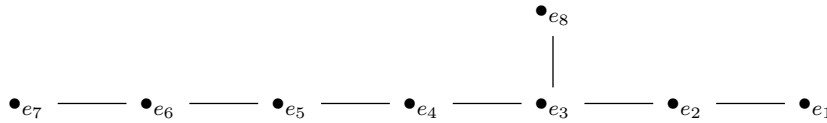
So if L_1, L_2 are unimodular, then taking an orthogonal sum $L_1 \oplus L_2$ also yields a unimodular lattice. So this yields diagonal matrices with p copies of $+1$ and q copies of -1 . This is referred to as $rm1_{p,q}$, and is an *odd* unimodular lattice of signature (p, q) (after passing to \mathbb{R}). Here *odd* means that there exists a $v \in L$ such that v^2 is odd.

Example 11.0.3 (Even unimodular lattices): An even lattice must have no vectors of odd norm, so all of the diagonal elements are in $2\mathbb{Z}$. This is because $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i < j} 2n_i n_j e_i \cdot e_j$. Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as H , sometimes referred to as the *hyperbolic cell* or *hyperbolic plane*.

Example 11.0.4 (A harder even unimodular lattice): This is built from the E_8 Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \rightarrow e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example, $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$. You can check that $\det(e_i \cdot e_j) = 1$, and this is referred to as the E_8 lattice. This is of signature $(0, 8)$, and it's negative definite if and only if $v^2 < 0$ for all $v \neq 0$. One can also negate the intersection form to define $-E_8$. Note that any simply-laced Dynkin diagram yields some lattice. For example, E_{10} is unimodular of signature $(1, 9)$, and it turns out that $E_{10} \cong E_8 \oplus H$.

Definition 11.0.5 (?)

Take

$$\text{ii}_{a,a+8b} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b E_8,$$

which is an even unimodular lattice since the diagonal entries are all -2 , and using the fact

that the signature is additive, is of signature $(a, a + 8b)$. Similarly,

$$\text{ii}_{a+8b,a} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b (-E_8),$$

which is again even and unimodular.

Remark 11.0.6: Thus

- is odd, unimodular, of signature (p, q) .
- is even, unimodular, of signature (p, q) only for $p \equiv q \pmod{8}$.

Theorem 11.0.7 (Serre).

Every unimodular lattice which is not positive or negative definite is isomorphic to either or with $8 \mid p - q$.

Remark 11.0.8: So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: E_8
16	2: $E_8^{\oplus 2}, D_{16}^+$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$> 8 \times 10^{16}!!!!$

Note that the signature of a definite lattice must be divisible by 8.

Remark 11.0.9: There is an isometry: $f : E_8 \rightarrow E_8$ where $f \in O(E_8)$, the linear maps preserving the intersection form (i.e. the Weyl group $W(E_8)$, given by $v \mapsto v + (v, e_i)e_i$. The Leech lattice also shows up in the sphere packing problems for dimensions 2, 4, 8, 24. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of L as a subset of \mathbb{R}^n , namely $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$ for example, and the map $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$ is an isometric embedding into \mathbb{R}^n with the standard form. Connection to classification of Lie groups: root lattices.

Remark 11.0.10: If M^4 is a compact oriented 4-manifold and if the intersection form on $H^2(M; \mathbb{Z})$ is indefinite, then the only invariants we can extract from that associated lattice are

- Whether it's even or odd, and
- Its signature

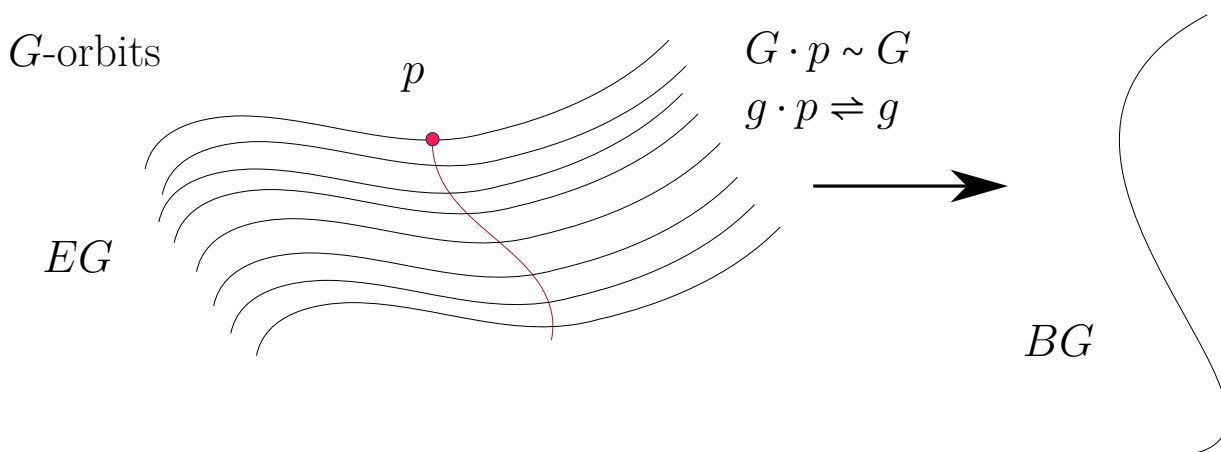
If the lattice is even, then the signature satisfies $8 \mid p - q$. So Poincaré duality forces unimodularity, and then there are further number-theoretic restrictions. E.g. this prohibits $\beta_2 = 7$, since then the signature couldn't possibly be 8 if the intersection form is even.

11.1 Characteristic Classes

Definition 11.1.1 (?)

Let G be a topological group, then a **classifying space** EG is a contractible topological space admitting a free continuous G -action with a “nice” quotient.

Remark 11.1.2: Thus there is a map $EG \rightarrow BG := EG/G$ which has the structure of a principal G -bundle.



Here we use a point p depending on U in an orbit to identify orbits $g \cdot p$ with g , and we want to take transverse slices to get local trivializations of $U \in BG$. It suffices to know where $\pi^{-1}(U) \cong U \times G$, and it suffices to consider $U \times \{e\}$. Moreover, $EG \rightarrow BG$ is a universal principal G -bundle in the sense that if $P \rightarrow X$ is a universal G -bundle, there is an $f : X \rightarrow BG$.

$$\begin{array}{ccc} P & \dashrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

[Link to Diagram](#)

Here bundles will be classified by homotopy classes of f , so $\{\text{Principal } G\text{-bundles}/X\} \cong [X, BG]$. This only works for paracompact Hausdorff spaces! The line \mathbb{R} with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

12 | Monday, February 08

Last time: BG and EG . See Milnor and Stasheff.

Example 12.0.1(?): Let $G := \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$, then we can take

$$EG = \mathbb{R}^\infty := \left\{ (a_1, a_2, \dots) \mid a_i \in \mathbb{R}, a_{i \gg 0} = 0, a_i \text{ not all zero} \right\}.$$

Then \mathbb{R}^\times acts on EG by scaling, and we can take the quotient $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$, where $\mathbf{a} \sim \lambda \mathbf{a}$ for all $\lambda \in \mathbb{R}^\times$. This yields \mathbb{RP}^∞ as the quotient. You can check that E_G is contractible: it suffices to show that $S^\infty := \left\{ \sum |a_i| = 1 \right\}$ is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$. Similarly $BC^\times \cong \mathbb{CP}^\infty$ with $EC^\times := C^\infty \setminus \{0\}$.

Example 12.0.2(?): Consider $G = \mathrm{GL}_n(\mathbb{R})$. It turns out that $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$, which is the set of linear subspaces of \mathbb{R}^∞ of dimension d . This is spanned by d vectors $\{e_i\}$ in some large enough $\mathbb{R}^N \subseteq \mathbb{R}^\infty$, since we can take N to be the largest nonvanishing coordinate and include all of the vectors into \mathbb{R}^∞ by setting $a_{>N} = 0$. For any $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$, since \mathbb{R}^d has a standard basis, there is a natural GL_d torsor: the set of ordered bases of linear subspaces. So define $EG := \{\text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty)\}$, then any $A \in \mathrm{GL}_d(\mathbb{R})$ acts on EG by sending $(L, \{e_i\}) \mapsto (L, \{Le_i\})$. We can identify EG as d -tuples of linearly independent elements of \mathbb{R}^∞ , and there is a map

$$\begin{aligned} EG &\rightarrow BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over BGL_d :

$$\begin{array}{ccc} \mathcal{E}_L := L & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & BGL_d \end{array}$$

So $\mathcal{E} \subseteq BGL_d \times \mathbb{R}^\infty$, where we can define $\mathcal{E} := \{(L, p) \mid p \in L\}$. In this case, $EG = \mathrm{Frame}(\mathcal{E})$ is the frame bundle of this universal bundle. The same setup applies for $G := \mathrm{GL}_d(\mathbb{C})$, except we take $\mathrm{Gr}_d(\mathbb{C}^\infty)$.

Example 12.0.3(?): Consider $G = O_d$, the set of orthogonal transformations of \mathbb{R}^d with the standard bilinear form, and U_d the set of unitary such transformations. To be explicit:

$$U_d := \left\{ A \in \mathrm{Mat}(d \times d, \mathbb{C}) \mid \langle Av, Av \rangle = \langle v, v \rangle \right\},$$

where $\langle [v_1, \dots, v_n], [v_1, \dots, v_n] \rangle = \sum |v_i|^2$. Alternatively, $A^t A = I$ for O_d and $\overline{A}^t A = I$ for U_d . In this case, $BO_d = \mathrm{Gr}_d(\mathbb{R}^\infty)$ and $BU_d = \mathrm{Gr}_d(\mathbb{C}^\infty)$, but we'll make the fibers smaller: set the fiber over L to be $(EO_d)_L := \{\text{orthogonal frames of } L\}$ and similarly $(EU_d)_L$ the unitary frames of L . That there are related comes from the fact that GL_d retracts onto O_d using the Gram-Schmidt procedure.

Remark 12.0.4: Recall that there is a bijective correspondence

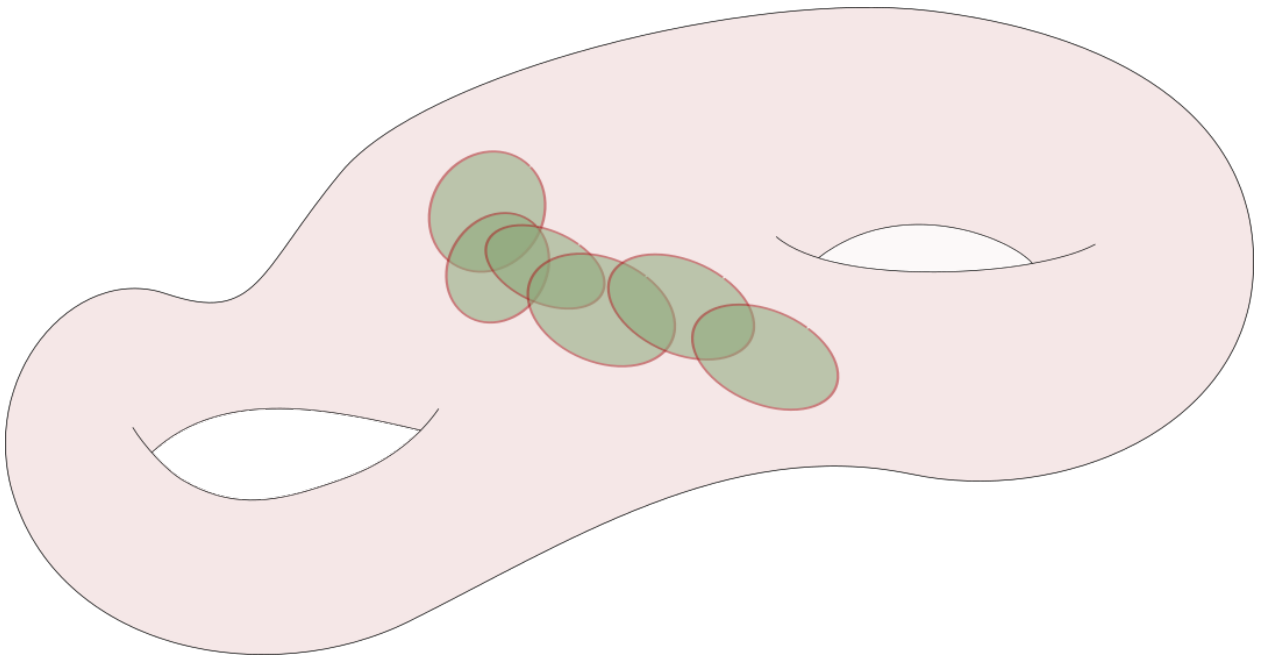
$$\left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \begin{array}{c} \text{Principal } GL_d\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Principal } \mathcal{O}_d\text{-bundles} \\ \text{on } X \end{array} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles $\mathcal{E} \rightarrow X$ of rank d , and on the RHS we have bundles with a metric. In local trivializations $U \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, the metric is the standard one on \mathbb{R}^d . This is referred to as a **reduction of structure group**, i.e. a principal GL_d bundle admits possibly different trivializations for which the transition functions lie in the subgroup O_d .

Example 12.0.5(?): Given any trivial principal G -bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.



Remark 12.0.6: We want to compute $H^*(BU_d; \mathbb{Z})$. Why is this important? Given any complex vector bundle $\mathcal{E} \rightarrow X$ there is an associated principal U_d bundle by choosing a metric, so we get a homotopy class $[X, BU_d]$. Given any $f \in [X, BU_d]$ and any $\alpha \in H^k(BU_d; \mathbb{Z})$, we can take the pullback $f^* \alpha \in H^k(X; \mathbb{Z})$, which are **Chern classes**.

Exercise 12.0.7 (?)

Show that $H^*(BU_d; \mathbb{Z})$ stabilizes as $d \rightarrow \infty$ to an infinitely generated polynomial ring $\mathbb{Z}[c_1, c_2, \dots]$ with each c_i in cohomological degree $2i$, so $c_i \in H^{2i}(BU_d, \mathbb{Z})$.

Definition 12.0.8 (?)

There is a map $BU_{d-1} \rightarrow BU_d$, which we can identify as $\text{Gr}_{d-1}(\mathbb{C}^\infty) \rightarrow \text{Gr}_d(\mathbb{C}^\infty)$. This is defined by sending a basis $\{v_1, \dots, v_{d-1}\} \mapsto \text{span}\{(1, 0, 0, \dots), sv_1, \dots, sv_{d-1}\}$ where $s: \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$

is the map that shifts every coordinate to the right by one.

Question: does $\text{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?

This will yield a fiber sequence $S^{2d-1} \rightarrow BU_{d-1} \rightarrow BU_d$, and using connectedness of the sphere and the LES in homotopy this will identify $H^*(BU_d) = H^*(BU_{d-1})[c_d]$ where $c_d \in H^{2d}(BU_d)$. The **Chern class** of a vector bundle \mathcal{E} , denoted $c_k(\mathcal{E})$, will be defined as the pullback f^*c_k .

ToDos

List of Todos

Check!	42
Revisit this last section, had to clarify a few things for myself!	46
Question: does $\text{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map?	49

Definitions

1.2.1	Definition – Topological Manifold	4
1.2.4	Definition – Restricted Structures on Manifolds	5
2.0.5	Definition – Kirby-Siebenmann Invariant of a 4-manifold	9
3.0.4	Definition – Signature	11
3.1.2	Definition – Riemannian Metrics	12
3.1.3	Definition – Almost complex structure	12
4.1.1	Definition – Presheaves and Sheaves	15
4.2.5	Definition – \mathcal{O} -modules	18
4.2.8	Definition – Morphisms of Sheaves	18
5.1.3	Definition – Global Sections Sheaf	20
6.0.1	Definition – Principal Bundles	22
6.0.3	Definition – The Frame Bundle	23
6.0.6	Definition – Orthogonal Frame Bundle	24
6.0.8	Definition – ?	24
6.0.10	Definition – Unitary Frame Bundle	24
6.0.14	Definition – Associated Bundles	25
7.1.1	Definition – Connections	26
7.1.6	Definition – Flat Connection and Flat Sections	29
7.2.1	Definition – ?	29
8.0.4	Definition – Acyclic Sheaves	31
9.0.7	Definition – ?	35
9.0.9	Definition – Cauchy-Riemann Equations	35
10.0.8	Definition – Lattice	41
10.0.10	Definition – ?	41
10.0.14	Definition – ?	42
10.0.15	Definition – ?	42
11.0.5	Definition – ?	44
11.1.1	Definition – ?	46
12.0.8	Definition – ?	48

Theorems

3.0.1	Theorem – Freedman	10
3.0.5	Theorem – Rokhlin's Theorem	11
3.0.7	Theorem – Donaldson	11
3.1.5	Theorem – ?	13
8.0.3	Theorem – ?	31
8.0.6	Theorem – (Important!)	31
9.0.12	Theorem – Properties of Singular Cohomology	36
11.0.7	Theorem – Serre	45

Exercises

4.2.11	Exercise – ?	19
5.1.2	Exercise – ?	20
5.1.9	Exercise – ?	21
6.0.16	Exercise – ?	26
7.1.3	Exercise – ?	26
7.1.7	Exercise – ?	29
7.2.2	Exercise – ?	29
9.0.2	Exercise – ?	34
10.0.11	Exercise – ?	41
12.0.7	Exercise – ?	48

Figures

List of Figures

1	Picking one basis element in the time direction	40
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