Problem Set 1

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Contents

1 Problem 1

We'll use the following definition of a smooth map between manifolds:

Definition: Let M, N be smooth manifolds of dimensions m, n respectively and $f : M \to N$ a continuous map. Then f is *smooth* iff for every $p \in M$, there exists a chart (U, ϕ) with $p \in U$ and a chart (V, ψ) with $f(p) \in V$ such that

- $f(U) \subseteq V$, and
- The induced map

$$\overline{f} : \phi(U) \to \psi(V)$$
$$\overline{f} = \psi \circ f \circ \phi^{-1}$$

is smooth as a map from $\mathbb{R}^m \to \mathbb{R}^n$.

We will thus show that both $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ and $f^{-1} : \mathbb{CP}^1 \to \mathbb{CP}^1$ are both smooth bijections, from which we can conclude that f is a diffeomorphism.

So choose the following charts on $\mathbb{CP}^1$:

$$(U_1, \phi) := \{[x, 1] \ni x \in \mathbb{C}\}, \quad U_1 \subseteq \mathbb{CP}^1 \setminus \{\infty\}$$
$$(U_2, \psi) := \{[1, x] \ni x \in \mathbb{C}\}, \quad U_2 \subseteq \mathbb{CP}^1 \setminus \{0\}$$

2 Problem 3

2.1 Part 1

Note: Throughout this question, we will identify $\{f : C^\infty(M) \to \mathbb{R}\} \cong C^\infty(M)^\vee$ as vector spaces.

Let M, N be smooth manifolds and $f : M \to N$ be a fixed smooth map, and define a map

$$\phi : C^\infty(N) \times TM \to \mathbb{R}$$
$$(h, v) \mapsto v(h \circ f)$$

2.2 Part 1

Using the derivation definition, we can identify this assignment as a map

$$\phi : C^\infty(N) \times C^\infty(M)^\vee \to \mathbb{R}$$
$$(h, v) \mapsto v(h \circ f)$$

We'd like to show that this yields a well-defined element of $T_pM = C^\infty(M)$. So for some fixed $v \in T_pM$, define a map

$$\phi_v : C^\infty(N) \to \mathbb{R}$$
$$h \mapsto v(h \circ f),$$

which will be an element of TM if it is a derivation. For $x \in N$, we have

$$
\begin{aligned}
\phi_v(h_1 \cdot h_2)(x) &:= v((h_1 h_2) \circ f)(x) \\
&= v((h_1 \circ f)(h_2 \circ f))(x) \\
&= v(h_1 \circ f)(x) \cdot h_2(x) + h_1(x) \cdot v(h_2 \circ f)(x) \quad \text{since } v \text{ is a derivation} \\
&= \phi_v(h_1)(x) \cdot h_2(x) + h_1(x) \cdot \phi_v(h_2)(x).
\end{aligned}
$$

2.3 Part 2

Using the integral curve definition,

3 Problem 4

3.1 Part 1

Let $V = \mathbb{R}^n$ as a vector space, let g be a nonsingular matrix, and define a map

$$\phi : V \to V^\vee$$
$$v \mapsto (\phi_v : w \mapsto \langle v,\ gw \rangle)$$

The claim is that ϕ is a natural isomorphism. It is clearly linear (following from the linearity of the inner product and matrix multiplication), so it remains to check that it is a bijection.

To see that $\ker \phi = 0$, so that only the zero gets sent to the zero map, we can suppose that $x \in \ker \phi$. Then $\phi_x : w \to \langle x,\ gw \rangle$ is the zero map. But the inner product is nondegenerate by definition, i.e. $\langle x,\ y \rangle = 0 \ \forall y \implies x = 0$. So x could only have been the zero vector to begin with.

But $\dim V = \dim V^\vee$, so any injective linear map will necessarily be surjective as well.

3.2 Part 2

Let $g : TM \otimes TM \to \mathbb{R}$ be a metric, and consider the tangent space TM. By definition, the cotangent space $T_p^* M = (T_p M)^\vee$

4 Problem 5

4.1 Part 1

Let $A \in \mathrm{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \mathrm{Math}(n, n)$ be positive semi-definite, so

$$\langle v,\ Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v,\ (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned}
\langle v,\ (A + B)v \rangle &= \langle v,\ Av \rangle + \langle v,\ Bv \rangle \\
&> \langle v,\ Av \rangle + 0 \\
&\geq 0 + 0 \\
&= 0.
\end{aligned}$$

4.2 Part 2

Let M be a smooth manifold with tangent bundle TM and a maximal smooth atlas $\mathcal{A}$. Choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$. Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to $\mathcal{C}$, so for each i we have

$$f_i : M \to I$$
$$\forall p \in M, \quad \sum_{i \in I} f_i(p) = 1$$

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} := \delta_{jk}$. We then have

$$g^i : T\phi_i(U_i) \otimes T\phi_i(U_i) \to \mathbb{R}$$
$$(\partial x_i, \partial x_j) \mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Note that each coordinate function $\phi_i : U_i \to \mathbb{R}^n$ induces a map $\tilde{\phi}_i : TU_i \to T\mathbb{R}^n$.

Let G^i be the pullback of g^i along these induced maps $\tilde{\phi}_i$, so

$$G^i : TU_i \otimes TU_i \to \mathbb{R}$$
$$G^i(x, y) := \left(\left(\tilde{\phi}_i \right)^* g^i \right)(x, y) := g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y))$$

Then, for a point $p \in M$, define the following map:

$$g_p : T_pM \otimes T_pM \to \mathbb{R}$$
$$(x, y) \mapsto \sum_{i \in I} f_i(p) G^i(x, y).$$

The claim is that g_p defines a metric on M, and thus the family $\{g_p \mid p \in M\}$ yields a tensor field and thus a Riemannian metric on M. If we define the map

$$g : M \to (TM \otimes TM)^{\vee}$$
$$p \mapsto g_p$$

then g can be expressed as

$$g = \sum_{i \in I} f_i G^i.$$

We can check that this is positive definite by considering $x \in T_pM$ and computing

$$g(x, x) := g_p(x, x)$$
$$= \sum_{i \in I} f_i(p) \, G^i(v, v)$$
$$= \sum_{i \in I} f_i(p) \, g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)),$$

where each term is positive semi-definite, and *at least one term* is positive definite because $\sum_i f_i(p)$ must equal 1. By part 1, this means that the entire expression is positive definite, so g is a metric. $\square$

5 Problem 6

5.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} = X(\phi_t(p)),$$
$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H : M \times I \to M$$
$$(p, t) \mapsto \phi_t(p).$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \mathrm{Tr} \left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_* : H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\Lambda(\mathrm{id}_M) = \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})})$$
$$= \sum_k \dim H_k(X;\mathbb{Q})$$
$$= \chi(M),$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus $\frac{\partial}{\partial t}\phi_t(p_0)\big|_{t=1}$ so

$$\phi_t(p) = p$$
$$\implies \frac{\partial}{\partial t}\phi_t(p) = \frac{\partial}{\partial t}p = 0 \qquad \text{by differentiating wrt } t$$
$$\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} = 0\Big|_{t=0} = 0 \qquad \text{by evaluating at } t = 0$$
$$\implies X(\phi_1(p_0)) := \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} = 0 \qquad \text{by definition of } \phi_1$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. $\qquad \square$

5.2 Part 2

The trivial bundle

$$\mathbb{R}^2 \longrightarrow S^2 \times \mathbb{R}^2$$
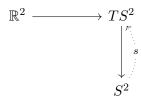$$\Big\downarrow {\scriptstyle s}$$
$$S^2$$

has a nowhere vanishing section, namely

$$s : S^2 \to S^2 \times \mathbb{R}^2$$
$$\mathbf{x} \to (\mathbf{x}, [1, 1])$$

which is the identity on the S^2 component and assigns the constant vector $[1, 1]$ to every point.

However, as part 1 shows, the bundle

$$\mathbb{R}^2 \longrightarrow TS^2$$
$$\downarrow s$$
$$S^2$$

can *not* have a nowhere vanishing section. $\square$