

Section 8.6: The Solutions of the Floer Equation are “Somewhere Injective”.

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0.1 Outline

Two Goals:

1. Prove critical points are discrete and regular points are open/dense.
2. Prove the continuation principle that was used in Proposition 8.1.4
3. Idea: A holomorphic function with all derivatives vanishing on a line is identically zero on \mathbb{C}

0.2 Outline of Statements



- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- 8.1.4: $\Gamma : W^{1,p} \times C_\varepsilon^\infty \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x, y, J)$ is a Banach manifold
- 8.1.1: For $h \in \mathcal{H}_{\text{reg}}$, $H_0 + h$ is nondegenerate and $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(H_0 + h, J)$.

- 8.1.2: For $h \in \mathcal{H}_{\text{reg}}$ and all contractible orbits x, y of H_0 , $\mathcal{M}(x, y, H_0 + h)$ is a manifold of dimension $\mu(x) - \mu(y)$.

0.3 Notation

- The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y \\ Y &\in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).\end{aligned}$$

- $z = s + it$
- X is a vector field (time-dependent and periodic) on \mathbb{R}^{2n} , J an almost complex structure
– X, J are smooth
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0$$

- $C(u)$ the set of critical points and $R(u)$ the set of regular points of u :

$$\begin{aligned}(s_0, t_0) \in C(u) \subseteq \mathbb{R} \times S^1 &\iff \frac{\partial u}{\partial s}(s_0, t_0) = 0 \\ (s_0, t_0) \in R(u) \subset \mathbb{R} \times S^1 &\iff (s_0, t_0) \notin C(u) \ \& \ s \neq s_0 \implies u(s_0, t_0) \neq u(s, t_0).\end{aligned}$$

0.4 Goal 1: Discrete Critical Points and Dense Regular Points

Goal 1: prove the following theorem

Theorem 0.1(8.5.4).

1. $C(u)$ is discrete and
2. $R(u) \hookrightarrow \mathbb{R} \times S^1$ is open and dense.

Outline of the proof:

- Prove 8.6.1 (direct, short) which transforms the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0 \quad \text{where } u \in C^\infty(\mathbb{R} \times S^1; W)$$

to a Cauchy-Riemann equation on \mathbb{R}^2 :

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0 \quad \text{where } v \in C^\infty(\mathbb{R}^2; W)$$

- Produces a map v which reduces Theorem 8.5.4 to two statements
 - 8.6.2: $C(v)$ (and thus $C(u)$) is discrete
 - * Proved later using similarity principle.

- 8.6.3 (Injectivity): If v is a smooth periodic solution of CR with $\frac{\partial v}{\partial s} \neq 0$ then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.
- Prove 8.6.3 (Injectivity)
 - Show open.
 - Show dense
- Prove 8.6.8 (similarity principle)
- Use similarity principle to prove 8.6.1, yields theorem.



0.5 8.6.1: Transform to Cauchy-Riemann

Proposition 0.2(8.6.1, Transform to CR-equation on \mathbb{R}^2).

If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0.$$

Then there exists

- An almost complex structure J_1
- A diffeomorphism φ on W ?
- A map $v \in C^\infty(\mathbb{R}^2; W)$

satisfying

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

where

1. $v(s, t+1) = \varphi(v(s, t))$
2. $C(u) = C(v)$, i.e. u, v have the same critical points
3. $R(u) = R(v)$.

Proof

- Since $W \times S^1$ is compact, the flow ψ_t of X_t is defined on all of W
 - We thus have a map $\psi_t : W \rightarrow W$ such that

$$\frac{\partial}{\partial t} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}$$

- Define the (important!) map

$$v(s, t) := (\psi_t^{-1} \circ u)(s, t)$$

- We can then compute

$$\begin{aligned} \frac{\partial u}{\partial s} &= (d\psi_t) \left(\frac{\partial v}{\partial s} \right) \\ \frac{\partial u}{\partial t} &= (d\psi_t) \left(\frac{\partial v}{\partial t} \right) + X_t(u). \end{aligned}$$

- Attempt at explanation: rearrange, use chain rule, and known derivative of ψ_t :

$$u(s, t) = (\psi_t \circ v)(s, t) \implies \frac{\partial u}{\partial s}(s, t) = \frac{\partial \psi_t}{\partial s}(v(s, t)) \cdot \frac{\partial v}{\partial s}(s, t)$$

$$? \implies \frac{\partial u}{\partial s} = (d\psi_t) \cdot \left(\frac{\partial v}{\partial s} \right)$$

and

$$\begin{aligned} \frac{\partial u}{\partial t}(s, t) &= \frac{\partial \psi_t}{\partial t}(v(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= (X_t \circ \psi_t)(v(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= (X_t \circ \psi_t \circ v)(s, t) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= X_t(u(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= X_t(u) \left(\frac{\partial v}{\partial t} \right) \dots???? \end{aligned}$$

- Continuing computations,

$$\begin{aligned}
 0 &= \frac{\partial u}{\partial s} + J \left(\frac{\partial u}{\partial t} - X_t(u) \right) && \text{since } u \text{ is a solution} \\
 &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} - JX_t(u) && \text{expanding terms} \\
 &= \left((d\psi_t) \left(\frac{\partial v}{\partial s} \right) \right) + J \left((d\psi_t) \left(\frac{\partial v}{\partial t} \right) + X_t(u) \right) - JX_t(u) && \text{by substitution} \\
 &= (d\psi_t) \left(\frac{\partial v}{\partial s} \right) + J(u) (d\psi_t) \left(\frac{\partial v}{\partial t} \right) && \text{cancelling} \\
 &= (d\psi_t) \left(\frac{\partial v}{\partial s} + (d\psi_t)^{-1} J(u) (d\psi_t) \left(\frac{\partial v}{\partial t} \right) \right) && \text{collecting terms} \\
 &:= (d\psi_t) \left(\frac{\partial v}{\partial s} + \psi_t^* J(v) \right) && \text{by definition.}
 \end{aligned}$$

- Conclude that v is a solution of

$$\frac{\partial v}{\partial s} + \psi_t^* J(v) \frac{\partial v}{\partial t} = 0.$$

- Set $\varphi := \psi_1$ and $J_1(v) := \psi_1^* J(v)$ to obtain

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

of which v is still a solution

- Attempt to check directly, using $\psi_{t+1} = \psi_t \circ \psi_1$:

$$\begin{aligned}
 v(s, t+1) &:= (\psi_t^{-1} \circ u)(s, t+1) \\
 &= (\psi_1 \circ \psi_t^{-1} \circ u)(s, t) \\
 &= \psi_1(v(s, t)) \\
 &:= \varphi(v(s, t)),
 \end{aligned}$$

which verifies property 1.

Note: just a guess from me!

- Recall definition of v :

$$v(s, t) := \psi_t^{-1}(u(s, t))$$

- Verifying that $C(v) = C(u)$: not spelled out. Property of flow?

– Need to check that

$$\frac{\partial u}{\partial s}(s_0, t_0) = 0 \implies \frac{\partial v}{\partial s}(s_0, t_0) = 0$$

where

$$\frac{\partial u}{\partial s} = (d\psi_t) \left(\frac{\partial v}{\partial s} \right)$$

- How: ?
- Verifying that $R(v) = R(u)$
 - Need to check that for $(s_0, t_0) \notin C(u)$ and $s \neq s_0$ we have

$$u(s_0, t_0) \neq u(s, t_0) \implies v(s_0, t_0) \neq v(s, t_0)$$

- Follows directly:

$$\begin{aligned} v(s_0, t_0) \neq v(s, t_0) &\iff \psi_t^{-1}(u(s_0, t_0)) \neq \psi_t^{-1}(u(s, t_0)) \quad \text{by definition} \\ &\iff (\psi_t \circ \psi_t^{-1})(u(s_0, t_0)) \neq (\psi_t \circ \psi_t^{-1})(u(s, t_0)) \quad \text{injectivity of } \psi_t \\ &\iff u(s_0, t_0) \neq u(s, t_0). \end{aligned}$$

■

0.6 Splitting the Main Theorem

- The main theorem is equivalent to two upcoming statements

Proposition 0.3 (8.6.2: Statement 1, Critical Points are Discrete).

Let $z = s + it$ where $(s, t) \in \mathbb{R}^1 \times S^1$ (?). There exists a constant $\delta > 0$ such that

$$0 < |z| < \delta \implies (dv)_z \neq 0.$$

Proof.

Postponed to p.264 because it relies on the 8.6.8 (Similarity Principle). ■

For the second statement, we set up some notation/definitions.

- $v \in C^\infty(\mathbb{R}^2; W)$ is a solution satisfying

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

$$v(s, t+1) = \varphi(v(s, t))$$

$$C(v) = C(u), R(v) = R(u).$$

- The **regular points** are given by

$$R(v) = \left\{ (s, t) \in \mathbb{R}^2 \mid \frac{\partial v}{\partial s}(s, t) \neq 0, \quad v(s, t) \neq x^\pm(t), \quad v(s, t) \notin v(\mathbb{R} \setminus \{s\}, t) \right\}.$$

Note: last condition should be equivalent to $s \neq s' \implies v(s, t) \neq v(s', t)$. For reference, also equivalent to $v(s, t) = v(s', t) \implies s = s'$.

- **Multiple points** are defined as follows:

- Set $\bar{\mathbb{R}} = \mathbb{R} \coprod \{\pm\infty\}$
- Extend $v : \mathbb{R}^2 \rightarrow W$ to

$$\begin{aligned} v : \bar{\mathbb{R}} \times \mathbb{R} &\rightarrow W \\ v(\pm\infty, t) &= x^\pm(t). \end{aligned}$$

- Define the set of *multiple points* as

$$M(s, t) := \left\{ (s', t) \in \mathbb{R}^2 \mid s \neq s' \in \bar{\mathbb{R}}, \quad v(s', t) = v(s, t) \right\}$$

Note that the same t is used throughout.

- Recast definition of $R(v)$ as “points in the complement of $C(v)$ that do not admit multiples”.
 - Potentially incorrect formulation:

$$R(v) = C(v)^c \bigcap_{(s, t) \in \bar{\mathbb{R}} \times \mathbb{R}} M(s, t)^c.$$

Proposition 0.4(8.6.3: Regular Points Open/Dense, "Injectivity").

Let v be a smooth solution of the Cauchy-Riemann equation, then

$$\left. \begin{array}{l} v(s, t+1) = \varphi(v(s, t)) \\ \frac{\partial v}{\partial s} \neq 0 \end{array} \right\} \implies R(v) \subseteq \mathbb{R}^2 \text{ is open and dense.}$$

Proof (Long).

Splits into two parts:

- Show $R(v)$ is open (easy)
- Show $R(v)$ is dense (delicate)

■

0.7 Regular Points Are Open

Proving the first part: $R(v)$ is open.

- Want to show $R(v)^c$ is closed.
- Toward a contradiction, suppose otherwise: $R(v)^c$ is *open*.
 - Use limit point definition: $R(v)^c$ is closed \iff it contains all of its limit points
 - So $R(v)^c$ does *not* contain one of its limit points
 - Produces a sequence

$$R(v)^c \supseteq \{(s_n, t_n)\}_{n \in \mathbb{N}} \xrightarrow{n \rightarrow \infty} (s, t) \in R(v)$$

- The first two conditions defining $R(v)$ are open conditions:
 - The two conditions:

$$\frac{\partial v}{\partial s}(s, t) \neq 0 \quad \text{Condition 1}$$

$$v(s, t) \neq x^\pm(t) \quad \text{Condition 2.}$$

- Thus for $N \gg 1$ we have

$$n \geq N \implies \frac{\partial v}{\partial s}(s_n, t_n) \neq 0, \quad v(s_n, t_n) \neq x^\pm(t)$$

- But $(s_n, t_n) \notin R(v)$ for such n , so they must fail the last condition: injectivity
 - Third condition:

$$s \neq s' \implies v(s, t) \neq v(s', t)$$

- Failing this conditions means:

$$\forall n > N, \exists s'_n \in \mathbb{R} \text{ s.t. } s'_n \neq s_n \text{ and } v(s_n, t_n) = v(s'_n, t_n).$$

- Produces a sequence $\{s'_n\}_{n \in \mathbb{N}}$, want to show it is bounded.

- Toward a contradiction, suppose not, then there is a subsequence

$$\{s_{n_k}\}_{n_k \in \mathbb{N}} \xrightarrow{n_k \rightarrow \infty} \pm\infty.$$

- This implies

$$\begin{aligned} v(s, t) &= \lim_{n_k \rightarrow \infty} v(s'_{n_k}, t'_{n_k}) \quad \text{using continuity of } v \\ &= v(\pm\infty, t) \\ &:= x^\pm(t). \end{aligned}$$

- Why? By definition, precisely because we extended v by setting $v(\pm\infty, t) = x^\pm(t)$.
- But condition 2 for points in $R(v)$ says $v(s, t) \neq x^\pm(t)$, so this contradicts $(s, t) \in R(v)$.
- Sequence is bounded, so apply Bolzano-Weierstrass to extract a convergent subsequence converging to some limit:

$$\{s'_{n_j}\}_{n_j \in \mathbb{N}} \xrightarrow{n_j \rightarrow \infty} s'.$$

- Use the fact that injectivity failed:

$$\begin{aligned} \forall n, s'_n \neq s_n \quad \text{and} \quad v(s_n, t_n) &= v(s'_n, t_n) \\ \implies \lim_{n_k \rightarrow \infty} v(s_n, t_n) &= \lim_{n_k \rightarrow \infty} v(s'_n, t'_n) \\ \iff v(s, t) &= v(s', t) \quad \text{using continuity.} \end{aligned}$$

- Use the fact that because $(s, t) \in R(v)$ we must have

$$s = s'.$$

- (*Minor technical point*) Now have $\{s'_{n_j}\}_{n_j \in \mathbb{N}} \rightarrow s'$ and $\{s_n\}_{n \in \mathbb{N}} \rightarrow s$
 - Since the latter sequence is convergent, every subsequence converges to the same limit, so $\{s_{n_j}\}_{n_j \in \mathbb{N}} \rightarrow s$.
- Again using failed injectivity, i.e.

$$\begin{aligned} v(s, t) &= v(s', t) \\ (\implies v(s, t) - v(s', t) &= 0). \end{aligned}$$

we have

$$\begin{aligned} s'_{n_j} \neq s_{n_j} \quad \text{and} \quad v(s_{n_j}, t_{n_j}) &= v(s'_{n_j}, t_{n_j}) \\ (\implies v(s_{n_j}, t_{n_j}) - v(s'_{n_j}, t_{n_j}) &= 0 \quad \forall n_j). \end{aligned}$$

- But by the last step, we do have equality in the limit, $s = s'$, thus

$$\frac{\partial v}{\partial s}(s, t) = \lim_{n_j \rightarrow \infty} \frac{v(s_{n_j}, t) - v(s'_{n_j}, t)}{s_{n_j} - s'_{n_j}} = 0.$$

Lemma 8.6.4: For every $r > 0$ there exists a $\delta > 0$ such that

$$|t - t_0|, |s - s_0| < \delta \implies \exists s' \in B_r(s_j) \text{ s.t. } v(s, t) = v(s', t).$$

Proof: short.

Lemma 8.6.5: Let v_1, v_2 be two solutions of the CR-equation with $X_t \equiv 0$ on $B_\varepsilon(0)$, $v_1(0, 0) = v_2(0, 0)$ such that $(dv_1)_0, (dv_2)_0 \neq 0$. Also suppose

$$\forall \varepsilon \exists \delta \text{ s.t. } \forall (s, t) \in B_\delta(0), \exists s' \in \mathbb{R} \begin{cases} (s', t) \in B_\varepsilon(0) \\ v_1(s, t) = v_2(s', t) \end{cases}.$$

Then

$$\forall z \in B_\varepsilon(0), \quad v_1(s, t) = v_2(s, t).$$

Take perturbed CR equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Fix $S \in C^\infty(\mathbb{R}^2; \text{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$ be a solution to the perturbed CR equation and let $p > 2$. Then there exists $0 < \delta < \varepsilon$ and a map $A \in W^{1,p}(B_\delta, \text{GL}(\mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_\delta \rightarrow \mathbb{C}^n$ such that

$$\forall (s, t) \in B_\delta \quad Y(s, t) = A(s, t) \sigma(s + it) \quad \text{and} \quad J_0 A(s, t) = A(s, t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of $\bar{\partial}$) For every $p > 1$, the following operator is surjective and Fredholm:

$$\bar{\partial} : W^{1,p}(S^2; \mathbb{C}^n) \longrightarrow L^p(\Lambda^{0,1} T^* S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7

1 Goal 2: Continuation Principle

Goal 2: prove a continuation principle:

Proposition 1.1 (8.6.6, Continuation Principle).

On an open $U \subset \mathbb{R}^2$, let Y be a solution to the perturbed CR equation

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0$$

where J_0 is the standard complex structure on \mathbb{R}^{2n} and $S \in C^\infty(\mathbb{R}^2, \text{End}(\mathbb{R}^{2n}))$. Say that f has an *infinite-order zero* at z_0 iff

$$\forall k \geq 0, \quad \sup_{|z-z_0| \leq t} \frac{|f(z)|}{r^k} \xrightarrow{r \rightarrow 0} 0.$$

For f smooth, equivalently $f^{(k)}(z_0) = 0$ for all k .

Then the set

$$C := \left\{ (s, t) \in U \mid Y \text{ has an infinite order zero at } (s, t) \right\}$$

is clopen. In particular, if U is connected and $Y = 0$ on some nonempty $V \subset U$, then $Y \equiv 0$.

Proposition 1.2(8.1.4).

Define

$$\mathcal{Z}(x, y, J) := \{ (u, H_0 + h) \mid h \in \mathcal{C}_\varepsilon^\infty(H_0) \text{ and } u \in \mathcal{M}(x, y, J, H) \}.$$

If $(u, H_0 + h) \in \mathcal{Z}(x, y)$ then the following map admits a continuous right-inverse and is surjective:

$$\begin{aligned} \Gamma : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \times \mathcal{C}_\varepsilon^\infty(H_0) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \\ (Y, h) &\longmapsto \left(d\mathcal{F}^{H_0+h} \right)_u(Y) + \text{grad}_u h \end{aligned}$$

where \mathcal{F}^{H_0+h} is the Floer operator corresponding to $H_0 + h$.

Used to show (via the implicit function theorem) that $\mathcal{Z}(x, y, J)$ is a Banach manifold when $x \neq y$.