Title

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Contents

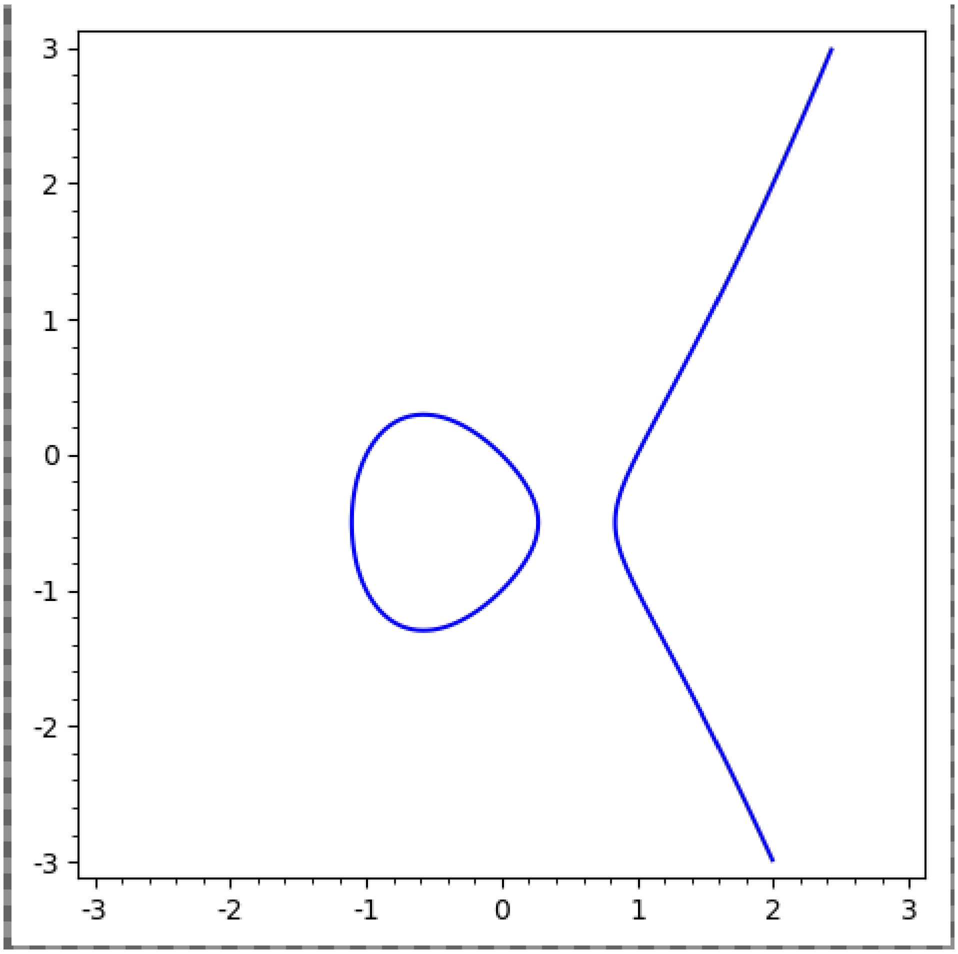
1 Friday, August 21

> Reference:
> `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)
- The variety $\left\langle x^2+1 \right\rangle$, which has no $\mathbb{R}$-points.
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Theorem 1.1 ***(Harnack Curve Theorem).***
If $f \in \mathbb{R}[x,y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.
Take the curve

$$X = \left\{ (x,y,z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$

- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.1.
Show that the vanishing locus of the first two equations above