

# Problem Set 2

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## 1 | Exercises

**Exercise 1.1** (Gathmann 2.17).

Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

**Solution:**

Since  $x = yz$  for all points in  $X$ , we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V(x - yz, y(z^2 - y)) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &:= X_1 \cup X_2. \end{aligned}$$

**Claim:** These two subvarieties are irreducible.

It suffices to show that the  $A(X_i)$  are integral domains. We have

$$A(X_1) := \mathbb{C}[x, y, z] / \langle x - yz, y \rangle \cong \mathbb{C}[y, z] / \langle y \rangle \cong \mathbb{C}[z],$$

which is an integral domain since  $\mathbb{C}$  is a field and thus an integral domain, and

$$A(X_2) := \mathbb{C}[x, y, z] / \langle x - yz, z^2 - y \rangle \cong \mathbb{C}[y, z] / \langle z^2 - y \rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

**Exercise 1.2** (Gathmann 2.18).

Let  $X \subset \mathbb{A}^n$  be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

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**Solution:**

$\bar{X} \subseteq V(I(X))$ :

We have  $X \subseteq V(I(X))$  and since  $V(J)$  is closed in the Zariski topology for any ideal  $J \trianglelefteq k[x_1, \dots, x_n]$  by definition,  $V(I(X))$  is closed. Thus

$$X \subseteq V(I(X)) \text{ and } V(I(X)) \text{ closed} \implies \bar{X} \subseteq V(I(X)),$$

since  $\bar{X}$  is the intersection of all closed sets containing  $X$ .

$V(I(X)) \subseteq \bar{X}$ :

Noting that  $V(\cdot), I(\cdot)$  are individually order-reversing, we find that  $V(I(\cdot))$  is order-*preserving* and thus

$$X \subseteq \bar{X} \implies V(I(X)) \subseteq V(I(\bar{X})) = \bar{X},$$

where in the last equality we've used part (i) of the Nullstellensatz: if  $X$  is an affine variety, then  $V(I(X)) = X$ . This applies here because  $\bar{X}$  is always closed, and the closed sets in the Zariski topology are precisely the affine varieties.

**Exercise 1.3** (Gathmann 2.21).

Let  $\{U_i\}_{i \in I} \rightrightarrows X$  be an open cover of a topological space with  $U_i \cap U_j \neq \emptyset$  for every  $i, j$ .

- Show that if  $U_i$  is connected for every  $i$  then  $X$  is connected.
- Show that if  $U_i$  is irreducible for every  $i$  then  $X$  is irreducible.

**Solution(a):**

Suppose toward a contradiction that  $X = X_1 \coprod X_2$  with  $X_i$  proper, disjoint, and open. Since  $\{U_i\} \rightrightarrows X$ , for each  $j \in I$  this would force one of  $U_j \subseteq X_1$  or  $U_j \subseteq X_2$ , since otherwise  $U_j \cap X_1 \cap X_2$  would be nonempty.

So without loss of generality (relabeling if necessary), assume  $U_j \subseteq X_1$  for some fixed  $j$ . But then for every  $i \neq j$ , we have  $U_i \cap U_j$  nonempty by assumption, and so in fact  $U_i \subseteq X_1$  for every  $i \in I$ . But then  $\cup_{i \in I} U_i \subseteq X_1$ , and since  $\{U_i\}$  was a cover, this forces  $X \subseteq X_1$  and thus  $X_2 = \emptyset$ .

**Solution(b):**

**Claim:**  $X$  is irreducible  $\iff$  any two open subsets intersect.

This follows because otherwise, if  $U, V \subset X$  are open and disjoint then  $X \setminus U, X \setminus V$  are proper and closed. But then we can write  $X = (X \setminus U) \coprod (X \setminus V)$  as a union of proper closed subsets, forcing  $X$  to not be irreducible.

So it suffices to show that if  $U, V \subset X$  then  $U \cap V$  is nonempty. Since  $\{U_i\} \rightrightarrows X$ , we can find a pair  $i, j$  such that there is at least one point in  $U \cap U_i$  and one point in  $V \cap U_j$ .

But by assumption  $U_i \cap U_j$  is nonempty, so both  $U \cap U_i$  and  $U_j \cap U_i$  are open nonempty subsets of  $U_i$ . Since  $U_i$  was assumed irreducible, they must intersect, so there exists a point

$$x_0 \in (U \cap U_i) \cap (U_j \cap U_i) = U \cap (U_i \cap U_j) := \tilde{U}.$$

We can now similarly note that  $\tilde{U} \cap V$  and  $U_j \cap V$  are nonempty open subsets of  $V$ , and thus intersect. So there is a point

$$\tilde{x}_0 \in (\tilde{U} \cap V) \cap (U_j \cap V) = \tilde{U} \cap V = U \cap V \cap (U_i \cap U_j),$$

and in particular  $\tilde{x}_0 \in U \cap V$  as desired.

**Exercise 1.4** (Gathmann 2.22).

Let  $f : X \rightarrow Y$  be a continuous map of topological spaces.

- Show that if  $X$  is connected then  $f(X)$  is connected.
- Show that if  $X$  is irreducible then  $f(X)$  is irreducible.

**Solution(a):**

Toward a contradiction, if  $f(X) = Y_1 \coprod Y_2$  with  $Y_1, Y_2$  nonempty and open in  $Y$ , then

$$f^{-1}(f(X)) \subseteq X$$

on one hand, and

$$f^{-1}(f(X)) = f^{-1}(Y_1) \coprod f^{-1}(Y_2)$$

on the other. If  $f$  is continuous, the preimages  $f^{-1}(Y_i)$  are open (and nonempty), so  $X$  contains a disconnected subset. However, every subset of a connected set must be connected, so this contradicts the connectedness of  $X$ .

**Solution(b):**

Suppose  $f(X) = Y_1 \cup Y_2$  with  $Y_i$  proper closed subsets of  $Y$ . Then  $f^{-1}(Y_1) \cup f^{-1}(Y_2) = (f^{-1} \circ f)(X) \subseteq X$  are closed in  $X$ , since  $f$  is continuous. Since  $X$  is irreducible, without loss of generality (by relabeling), this forces  $X_1 = \emptyset$ . But then  $f(X_1) = \emptyset$ , forcing  $f(X) = Y_2$ .

**Definition 1.0.1** (Ideal Quotient).

For two ideals  $J_1, J_2 \trianglelefteq R$ , the *ideal quotient* is defined by

$$J_1 : J_2 := \left\{ f \in R \mid f J_2 \subset J_1 \right\}.$$

**Exercise 1.5** (Gathmann 2.23).

Let  $X$  be an affine variety.

- Show that if  $Y_1, Y_2 \subset X$  are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

b. If  $J_1, J_2 \subseteq A(X)$  are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

**Solution:**

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**Exercise 1.6** (Gathmann 2.24).

Let  $X \subset \mathbb{A}^n$ ,  $Y \subset \mathbb{A}^m$  be irreducible affine varieties, and show that  $X \times Y \subset \mathbb{A}^{n+m}$  is irreducible.

**Solution:**

That  $X \times Y$  is again an affine variety follows from writing  $X = V(I)$ ,  $Y = V(J)$ , then  $X \times Y = V(I + J)$  where  $I + J \subseteq k[x_1, \dots, x_n, y_1, \dots, y_m]$ . So let

$$X \times Y = U \cup V$$

with  $U, V$  proper and closed, and let  $\pi_X, \pi_Y$  be the projections onto the factors.

**Claim:** For each  $x \in X$ ,  $\pi_X^{-1}(x) \cong Y$  is contained in only one of  $U$  or  $V$ .

Note that if this is true, we can write  $X = G_U \cup G_V$  where

$$G_U := \{x \in X \mid \pi_X^{-1}(x) \subseteq U\}$$

are the points for which the entire fiber lies in  $U$ , and similarly  $G_V$  are those for which the fiber lies in  $V$ . If we can then show that  $G_U, G_V$  are closed, by irreducibility of  $X$  this will force (wlog)  $G_V = \emptyset$  and  $X = G_U$ . But then

$$\pi_X^{-1}(X) = X \times Y \text{ and } \pi_X^{-1}(G_U) = U \implies X \times Y = U.$$

which shows that  $X \times Y$  is irreducible.

*Proof (Every fiber is contained in one irreducible component).*

For any fixed  $x$ , we can write

$$\pi_X^{-1}(x) = (\pi_X^{-1}(x) \cap U) \cup (\pi_X^{-1}(x) \cap V).$$

Since points are closed in the Zariski topology and  $\pi_X$  is continuous, each  $\pi_X^{-1}(x)$  is closed. and thus  $\pi_X^{-1}(x) \cap U$  is closed (and similarly for  $V$ ). Noting that  $\pi_X^{-1}(x) \cong \{x\} \times Y \cong Y$ , where we've assumed  $Y$  to be irreducible, we can conclude wlog that  $\pi_X^{-1}(x) \cap V = \emptyset$ . ■

*Proof ( $G_U, G_V$  are closed).*

Wlog consider  $G_U \subseteq X$ . Fixing any point  $y_0 \in Y$ , we have  $X \cong X \times \{y_0\} \subseteq X \times Y$ , so we can identify  $G_U$  ■