Title

D. Zack Garza

Monday 31st August, 2020

Contents

1 Sunday, August 30

1.1 Base Extension

Given some object A/k and $k \hookrightarrow \ell$ is a field extension, we would like some extended object A/ℓ.

Example 1.1.
An *affine variety* V/k is given by finitely many polynomials in $p_i \in k[t_1, \cdots, t_n]$, and base extension comes from the map $k[t_1, \cdots, t_n] \hookrightarrow \ell[t_1, \cdots, t_n]$.

More algebraically, we have the affine coordinate ring over k given by $k[V] = k[t_1, \cdots, t_n]/\left\langle p_i \right\rangle$, the ring of polynomial functions on the zero locus corresponding to this variety. We can similarly replace k be ℓ in this definition. Here we can observe that $\ell[V] \cong k[V] \otimes_k \ell$.

In general we have a map

$$\begin{aligned} & \cdot \otimes_k \ell \\ \{k\text{-vector space}\} &\longrightarrow \{\ell\text{-vector spaces}\} \\ \{k\text{-algebras}\} &\longrightarrow \{\ell\text{-algebras}\}. \end{aligned}$$

Note that this will be an exact functor on the category k-Vect, i.e. ℓ is a flat module. Here everything is free, and free $\implies$ flat, so things work out nicely.

What about for function fields?

Since k is a k-algebra, we can consider $k \otimes_k \ell$, however this need not be a field.

> Note: tensor products of fields come up very often, but don't seem to be explicitly covered in classes! We'll broach this subject here.

Exercise 1.1.
If ℓ/k is algebraic and $\ell \otimes_k \ell$ is a domain, the $\ell = k$.

I.e. this is rarely a domain. Hint: start with the monogenic case, and also reduce to the case where the extension is not just algebraic but finite.

Tensor products of field extensions are still interesting: if ℓ/k is finite, it is galois $\iff \ell \otimes_k \ell \cong \ell^{[\ell:k]}$. So its dimension as an ℓ-algebra is equal to the degree of ℓ/k, so it splits as a product of copies of ℓ.

Remark 1.
We'd like the tensor product of a field to be a field, or at least a domain where we can take the fraction field and get a field. This hints that we should not be tensoring algebraic extensions, but rather transcendental ones.

Exercise 1.2.
For ℓ/k a field extension,

a. Show $k(t) \otimes_k \ell$ is a domain with fraction field $\ell(t)$.

b. Show it is a field $\iff \ell/k$ is algebraic.

Proposition 1.1 ***(FT 12.7, 12.8).***
Let $k_1, k_2/k$ are field extensions, and suppose $k_1 \otimes_k k_2$ is a domain. Then this is a field $\iff$ at least one of k_1/k or k_2/k is algebraic.

Reminder: for ℓ/k and $\alpha \in \ell$ algebraic over k, then $k(\alpha) = k[\alpha]$.

So we'll concentrate on when $K \otimes_k \ell$ is a domain. What's the condition on a function field K/k that guarantees this, i.e. when extending scalars from k to ℓ still yields a domain? If this remains a domain, we'll take the fraction field and call it the *base change*.

Exercise 1.3.
If K/k is finitely generated (i.e. a function field) and $K \otimes_k \ell$ is a domain, then $ff(K \otimes_k \ell)/\ell$ is finitely generated.

The point: if taking a function field and extending scalars still results in a domain, we'll call the result a function field as well.

Most of all, we want to base change to the algebraic closure. We'll have issues if the constant field is not just k itself:

Lemma 1.2.
If $K \otimes_k \bar{k}$ is a domain, then the constant field $\kappa(K) = k$.

Proof .
Use the fact that $\cdot \otimes_k V$ is exact. We then get an injection

$$\begin{tikzcd}
\kappa(K) \otimes_k \kappa(K) \arrow[rr, hook] \arrow[dr, hook] & & K \otimes_k \bar{k} \\
& \kappa(K) \otimes_k \bar{k} \arrow[ur, hook] &
\end{tikzcd}$$

Here we use the injections $\kappa(K) \hookrightarrow \bar{k}$ and $\kappa(K) \hookrightarrow K$.
We now have an injection of k-algebras, and subrings of domains are domains. So apply the first exercise: the only way this can happen is if $\kappa(K) = k$.

■

Exercise 1.4.
The simplest possible case: describe $\mathbb{C}(t) \otimes_{\mathbb{R}} \mathbb{C}$, tensored as $\mathbb{R}$-algebras.

Won't be a domain by the lemma, some $\mathbb{C}(t)$-algebra of dimension 2.

In order to have a good base change for our function fields, we want to constant extension to be trivial, i.e. $\kappa(K) = k$. This requires that the ground field be algebraically closed.

In this case, you might expect that extending scalars to the algebraic closure would yield a field again. This is true in characteristic zero, but false in positive characteristic.

A more precise question: if $\kappa(K) = k$, must $K \otimes_K \bar{k}$ be a field? If that's true and we're in positive characteristic, recalling the for an algebraic extension this being a field is equivalent to it being a domain. But if that's a domain, the tensor product of every algebraic extension must be a domain, which is why this is an important case.

If so, then $K \otimes_k k^{\frac{1}{p}}$ is a field, where $k^{\frac{1}{p}} := k(\left\{ x^{\frac{1}{p}} \;\middle|\; x \in k \right\})$ is obtained by adjoining all pth roots of all elements. This is a purely inseparable extension. The latter condition (this tensor product being a field) is one of several equivalent conditions for a field to be separable.

Note that frobenius maps $k^{\frac{1}{p}} \twoheadrightarrow k$, so this is sort of like inverting this map.

Remember that K/k is transcendental, and there is an extended notion of separability for non-algebraic extensions.