Title

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1 Friday March 6th

Recall $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \ cdot \xi} \, dx$. Define $\mathcal{F}_a = ??$.

Definition 1.1 (Decay)**.**
$f \in \mathcal{F}_a$ iff 1. f is holomorphic in the strip $S_a = \left\{ z = x + iy \;\middle|\; \lvert y \rvert < a \right\}$. 2. There exists an $A > 0$ such that $\lvert f(x+iy) \rvert \frac{A}{1+x^2}$.

Examples:

- $e^{-z^2} \in \mathcal{F}_a$ for all a
- $\frac{1}{c^2 + z^2} \in \mathcal{F}_a$ for all $a > c$
- $\frac{1}{\cosh(\pi z)} \in \mathcal{F}_a$ for $a < \frac{1}{2}$.

Lemma 1.1.
If $f \in \mathcal{F}_a$, then $f^{(n)}(z) \in \mathcal{F}_b$ for all $b < a$.

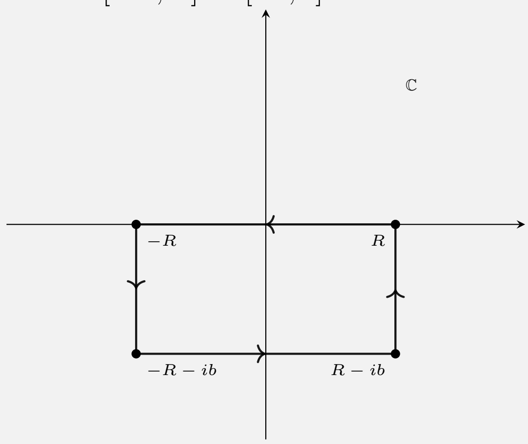
Theorem 1.2.
If $f \in \mathcal{F}_a$, then $\left\lvert \widehat{f}(\xi) \right\rvert \leq B e^{-2\pi b \lvert \xi \rvert}$ for some constants b, B.

Proof .

If $\xi = 0$,

$$\begin{aligned}\left|\widehat{f}(\xi)\right| &= \left|\int_{\mathbb{R}} f(x)e^{2\pi i x\cdot\xi}\right| \\ &\le \int_{\mathbb{R}} |f(x)|\ dx \\ &\le A\int_{\mathbb{R}} \frac{1}{1+x^2}\ dx \\ &= A\pi.\end{aligned}$$

For $\xi > 0$, integrate over the box $[-R, R]\times i[-b, 0]$:



Define $g(z) = f(z)e^{-2\pi i z\cdot\xi}$. The integral over the rectangle is zero, since g is holomorphic, so we can equate

$$\int_R^{R-ib} f(z)e^{-2\pi i z\cdot\xi}\ dz = \int_0^b f(R-it)e^{-2\pi i(R-it)\cdot\xi}(-i)\ dt$$

We can use the estimate in $\mathcal{F}_a$ to obtain

$$\begin{aligned}\int_0^b \cdots &\le \int_0^b \frac{A}{1+R^2}e^{-2\pi s\xi}\ ds \\ &\le O(R^{-2}).\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}\ d\xi &= \int_{-\infty-ib}^{\infty-ib} \cdots\ dz \\ &= \int_{\mathbb{R}} f(x-ib)e^{2\pi i(x-ib)\cdot\xi}\ dx \\ &\le \int_{\mathbb{R}} \frac{A}{1+x^2}e^{-2\pi b\xi}\ dx \\ &= A\pi e^{-2\pi b\xi},\end{aligned}$$

so we can take $B = A\pi$.
For $\xi > 0$, the same argument works with the rectangle above the axis.

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Theorem 1.3.
If $f \in \mathcal{F}_a$, then $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi$.

Proof .
Letting $L_1 = \{x - ib\}$ and $L_2 = \{x + ib\}$

$$
\begin{aligned}
I &= \int_0^\infty \widehat{f} \cdots + \int_{-\infty}^0 \widehat{f} \cdots \\
&= \int_0^\infty e^{2\pi i x \cdot \xi} \left(\int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} \, dz \right) d\xi + \int_\infty^0 e^{2\pi i x \cdot \xi} \left(\int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} \, dz \right) d\xi \\
&= \int_{L_1} \int_0^\infty e^{2\pi i x \xi - 2\pi i (s - ib)\xi} \, d\xi \, ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i x \cdot \xi - 2\pi i (s + ib)\xi} \, d\xi ds \\
&\qquad \text{by absolute convergence, where } z = s - ib \\
&= \int_{L_1} f(z) \int_0^\infty e^{2\pi i (x - s + ib)\xi} \, d\xi \, ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i (x - s + ib)\xi} \, d\xi \, ds \\
&= \int_{L_1} f(z) \frac{1}{2\pi i (x - i + ib)} \, ds + \int_{L_2} f(z) \frac{1}{2\pi i (x - s - ib)} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - x} \, dz \\
&= f(x),
\end{aligned}
$$

noting that

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\int_0^\infty e^{as} \, ds = \frac{1}{a} \quad \text{for } \Re(a) > 0.
$$

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Note the similar trick: for $\xi < 0$, move up, and $\xi > 0$ move down to form a rectangle. Use the fact that integration along the vertical edges is zero.