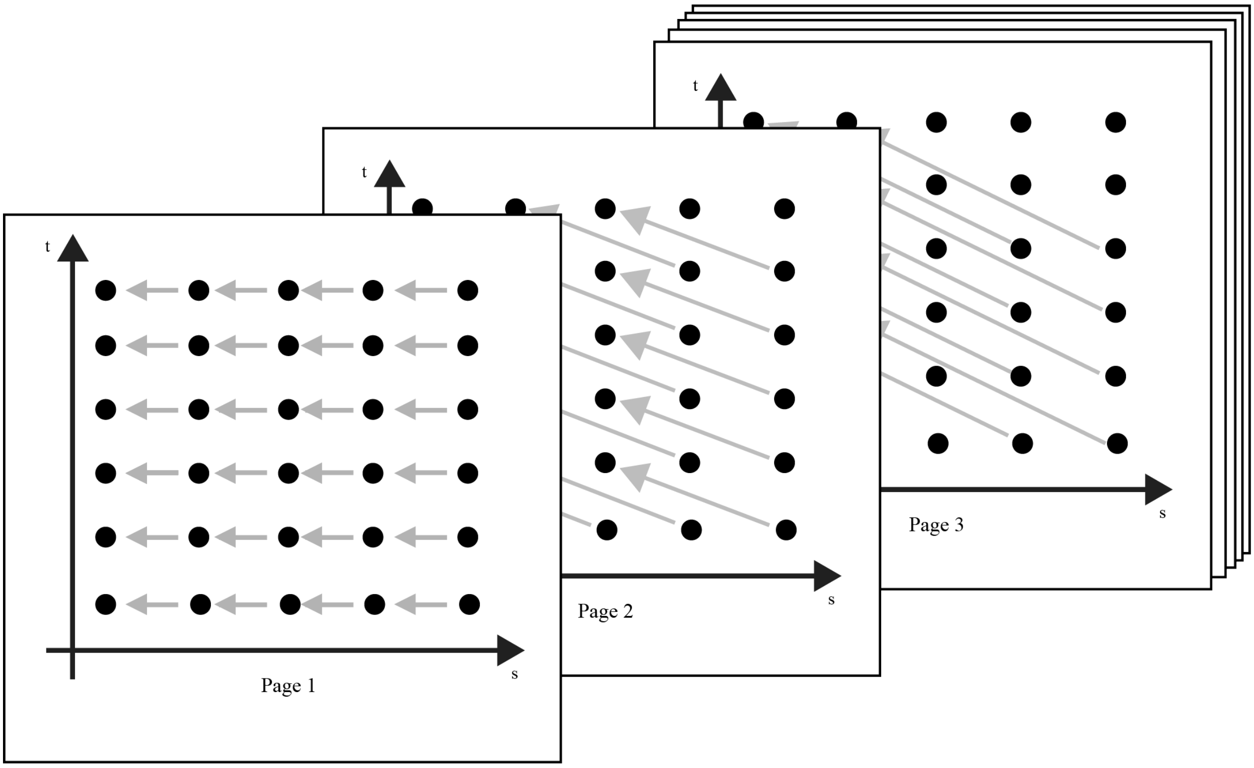

Notes: These are notes live-tex'd from a graduate course in Homological Algebra taught by Brian Boe at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Homological Algebra

Lectures by Brian Boe. University of Georgia, Spring 2021

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Table of Contents

Contents

1

Wednesday, January 13

Reference:

- The course text is Weibel [1].
- See the many corrections/errata: `http://www.math.rutgers.edu/~weibel/Hbook-corrections.html`
- Sections we'll cover:
    - 1.1-1.5,
    - 2.2-2.7,
    - 3.4,
    - 3.6,
    - 6.1,
    - 5.1-5.2,
    - 5.4-5.8,
    - 6.8,
    - 6.7,
    - 6.3,
    - 7.1-7.5,
    - 7.7-7.8,
    - Appendix A (when needed)
- Course Website: `https://uga.view.usg.edu/d2l/le/content/2218619/viewContent/33763436/View`

1.1 Overview

Definition 1.1.1 (Exact complexes)
A **complex** is given by

$$\cdots \xrightarrow{d_{i-1}} M_{i-1} \xrightarrow{d_i} M_i \xrightarrow{d_{i+1}} M_{i+1} \to \cdots .$$

where $M_i \in R$-**Mod** and $d_i \circ d_{i-1} = 0$, which happens if and only if $\operatorname{im} d_{i-1} \subseteq \ker d_i$. If $\operatorname{im} d_{i-1} = \ker d_i$, this complex is **exact**.

Example 1.1.2*(?)*: We can apply a functor such as $\otimes_R N$ to get a new complex

$$\cdots \xrightarrow{d_{i-1}\otimes 1_N} M_{i-1} \otimes_R N \xrightarrow{d_i \otimes 1} M_i \otimes N \to M_{i+1} \xrightarrow{d_{i+1}\otimes 1} \cdots .$$

Example 1.1.3 ***(?):*** Applying $\mathrm{Hom}(N, \cdot\,)$ similarly yields

$$\mathrm{Hom}_R(N, M_i) \xrightarrow{d_{i-1}^*} \mathrm{Hom}_R(N, M_{i+1}),$$

where $d_i^* = d_i \circ (\,\cdot\,)$ is given by composition.

Example 1.1.4 ***(?):*** Applying $\mathrm{Hom}(\,\cdot\,, N)$ yields

$$\mathrm{Hom}_R(M_i, N) \xrightarrow{d_i^*} \mathrm{Hom}_R(M_{i+1}, N)$$

where $d_i^* = (\,\cdot\,) \circ d_i$.

Remark 1.1.5: Note that we can also take complexes with arrows in the other direction. For F a functor, we can rewrite these examples as

$$d_i^* \circ d_{i-1}^* = F(d_i) \circ F(d_{i-1}) = F(d_i \circ d_{i-1}) = F(0) = 0,$$

provided F is nice enough and sends zero to zero. This follows from the fact that functors preserve composition. Even if the original complex is exact, the new one may not be, so we can define the following:

Definition 1.1.6 (Cohomology)

$$H^i(M^*) = \ker d_i^* / \operatorname{im} d_{i-1}^*.$$

Remark 1.1.7: These will lead to ***i*th derived functors**, and category theory will be useful here. See appendix in Weibel. For a category $\mathcal{C}$ we'll define

- $\mathrm{Obj}(\mathcal{C})$ as the objects
- $\mathrm{Hom}_{\mathcal{C}}(A, B)$ a set of morphisms between them, where a more modern notation might be $\mathrm{Mor}(A, B)$.
- Morphisms compose: $A \xrightarrow{f} B \xrightarrow{g} C$ means that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$
- Associativity
- Identity morphisms

See the appendix for diagrams defining zero objects and the zero map, which we'll need to make sense of exactness. We'll also needs notions of kernels and images, or potentially cokernels instead of images since they're closely related.

Remark 1.1.8: In the examples, we had $\ker d_i \subseteq M_i$, but this need not be true since the objects in the category may not be sets. Such an example is the category of complexes of R-modules: $\mathrm{Cx}(R\text{-}\mathbf{Mod})$. In this setting, kernels will be subcomplexes but not subsets.

Definition 1.1.9 (Functors)
Recall that **functors** are "functions" between categories $F : \mathcal{C} \to \mathcal{D}$ such that

- Objects are sent to objects,
- Morphisms are sent to morphisms, so $A \xrightarrow{f} B \rightsquigarrow F(A) \xrightarrow{F(f)} F(B)$,
- F respects composition and identities

Example 1.1.10 ***(Hom):*** $\operatorname{Hom}_R(N, \cdot\,) : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$, noting that the hom set may not have an R-module structure.

Remark 1.1.11: Taking cohomology yields the ith derived functors of F, for example $\operatorname{Ext}^i, \operatorname{Tor}_i$. Recall that functors can be *covariant* or contravariant. See section 1 for formulating simplicial and singular homology (from topology) in this language.

1.2 Chapter 1: Chain Complexes

1.2.1 Complexes of R-modules

Definition 1.2.1 (Exactness)
Let R be a ring with 1 and define R-**Mod** to be the category of *right* R-modules. $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if and only if $\ker g = \operatorname{im} f$, and in particular $g \circ f = 0$.

Definition 1.2.2 (Chain Complex)
A **chain complex** is

$$C_\cdot \coloneqq (C_\cdot, d_\cdot) \coloneqq \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \right)$$

for $n \in \mathbb{Z}$ such that $d_n \circ d_{n+1} = 0$. We drop the n from the notation and write $d^2 \coloneqq d \circ d = 0$.

Definition 1.2.3 (Cycles and boundaries)

- $Z_n = Z_n(C_\cdot) = \ker d_n$ are referred to as n-**cycles**.
- $B_n = B_n(C_\cdot) = \operatorname{im} d_{n+1}$ are the n-**boundaries**.

Definition 1.2.4 (Homology of a chain complex)
Note that if $d^2 = 0$ then $B_n \leq Z_n \leq C_n$. In this case, it makes sense to define the quotient module $H^n(C_\cdot) \coloneqq Z_n / B_n$, the n**th homology** of $C_\cdot$.

Definition 1.2.5 (Maps of chain complexes)
A map $u : C_\cdot \to D_\cdot$ of chain complexes is a sequence of maps $u_n : C_n \to D_n$ such that all of the following squares commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & C_n \arrow[r] \arrow[d, "u_n"] & C_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Remark 1.2.6: We can thus define a category $\mathrm{Ch}(R\text{-}\mathbf{Mod})$ where

- The objects are chain complexes,
- The morphisms are chain maps.

Exercise 1.2.7 (Weibel 1.1.2)
A chain complex map $u : C. \to D.$ restricts to

$$u_n : Z_n(C.) \to Z_n(D.)$$
$$u_n : B_n(D.) \to B_n(D.)$$

and thus induces a well-defined map $u_{n,*} : H_n(C.) \to H_n(D.)$.

Remark 1.2.8: Each H_n thus becomes a functor $\mathrm{Ch}(R\text{-}\mathbf{Mod}) \to R\text{-}\mathbf{Mod}$ where $H_n(u) := u_{*.n}$.

2 | Friday, January 15

2.1 Review

See assignment posted on ELC, due Wed Jan 27

Remark 2.1.1: Recall that a chain complex is $C.$ where $d^2 = 0$, and a map of chain complex is a ladder of commuting squares

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & \cdots \\
\cdots \arrow[r] & D_{n-1} \arrow[r, "d_{n-1}"] & D_n \arrow[r, "d_n"'] & D_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to diagram

Recall that $u_n : Z_n(C) \to Z_n(D)$ and $u_n : B_n(C) \to B_n(D)$ preserves these submodules, so there are induced maps $u_{\cdot,n} : H_n(D) \to H_n(D)$ where $H_n(C) \coloneqq Z_n(C)/B_n n - 1(C)$. Moreover, taking $H_n(\,\cdot\,)$ is a functor from $\mathrm{Ch}(R\text{-}\mathbf{Mod}) \to R\text{-}\mathbf{Mod}$ for any fixed n and on objects $C \mapsto H_n(C)$ and chain maps $u_n \to H_n(u) \coloneqq u_{*,n}$. Note the lower indices denote maps going down in degree.

2.2 Cohomology

Definition 2.2.1 (Quasi-isomorphism)
A chain map $u : C \to D$ is a **quasi-isomorphism** if and only if the induced map $u_{*,n} : H^n(C) \to H^n(D)$ is an isomorphism of R-modules.

Remark 2.2.2: Note that the usual notion of an isomorphism in the categorical sense might be too strong here.

Definition 2.2.3 (Cohomology)
A **cochain complex** is a complex of the form

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \cdots$$

where $d^n \circ d^{n-1} = 0$. We similarly write $Z^n(C) \coloneqq \ker d^n$ and $B^n(C) \coloneqq \operatorname{im} d^{n-1}$ and write the R-module $H^n(C) \coloneqq Z^n/B^n$ for the nth **cohomology** of C.

Remark 2.2.4: There is a way to go back and forth bw chain complexes and cochain complexes: set $C_n \coloneqq C^{-n}$ and $d_n \coloneqq d^{-n}$. This yields

$$C^{-n} \xrightarrow{d^{-n}} C^{-n+1} \iff C_n \xrightarrow{d^n} C_{n-1},$$

and the notions of $d^2 = 0$ coincide.

Definition 2.2.5 (Bounded complexes)
A cochain complex C is **bounded** if and only if there exists an $a \leq b \in \mathbb{Z}$ such that $C_n \neq 0 \iff a \leq n \leq b$. Similarly C^n is bounded above if there is just a b, and **bounded below** for just an a. All of the same definitions are made for cochain complexes.

Remark 2.2.6: See the book for classical applications:

- 1.1.3: Simplicial homology
- 1.1.5: Singular homology

2.3 Operations on Chain Complexes

Remark 2.3.1: Write Ch for Ch(R-**Mod**), then if $f, g : C \to D$ are chain maps then $f + g : C \to D$ can be defined as $(f + g)(x) = f(x) + g(x)$, since D has an addition coming from its R-module structure. Thus the hom sets $\mathrm{Hom}_{\mathrm{Ch}}(C, D)$ becomes an abelian group. There is a distinguished **zero object**[1] 0, defined as the chain complex with all zero objects and all zero maps. Note that we also have a zero map given by the composition $(C \to 0) \circ (0 \to D)$.

Definition 2.3.2 (Products and Coproducts)
If $\{A_\alpha\}$ is a family of complexes, we can form two new complexes:

- The **product** $\left(\prod_\alpha A_\alpha\right)_n := \prod_\alpha A_{\alpha,n}$ with the differential

$$\left(\prod d_\alpha\right)_n : \prod A_{\alpha,n} \xrightarrow{d_{\alpha,n}} \prod A_{\alpha,n-1}.$$

- The **coproduct** $\left(\coprod_\alpha A_\alpha\right)_n := \bigoplus_\alpha A_{\alpha,n}$, i.e. there are only finitely many nonzero entries, with exactly the same definition as above for the differential.

Remark 2.3.3: Note that if the index set is finite, these notions coincide. By convention, finite direct products are written as direct sums.

These structures make Ch into an **additive category**. See appendix for definition: the homs are abelian groups where composition distributes over addition, existence of a zero object, and existence of finite products. Note that here we have arbitrary products.

Definition 2.3.4 (Subcomplexes)
We say B is a **subcomplex** of C if and only if

- $B_n \leq C_n \in R$-**Mod** for all n,
- The differentials of B_n are the restrictions of the differentials of C_n.

Remark 2.3.5: This can be alternatively stated as saying the inclusion $i : B \to C$ given by $i_n : B_n \to C_n$ is a morphism of chain complexes. Recall that some squares need to commute, and this forces the condition on restrictions.

Definition 2.3.6 (Quotient Complexes)
When $B \leq C$, we can form the quotient complex C/B where

$$C_n/B_n \xrightarrow{\overline{d_n}} C_{n-1}/B_{n-1}.$$

Moreover there is a natural projection $\pi : C \to C/B$ which is a chain map.

---

[1] See appendix A 1.6 for initial and terminal objects. Note that $\emptyset$ is an initial but non-terminal object in Set, whereas zero objects are both.

Remark 2.3.7: Suppose $f : B \to C$ is a chain map, then there exist induced maps on the levelwise kernels and cokernels, so we can form the **kernel** and **cokernel** complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \ker f_n \arrow[r, dashed, "\exists d_n"] \arrow[d, "i_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "\pi_n"] & C_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{coker} f_n \arrow[r, dashed, "\exists d_n"] & \operatorname{coker} f_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here $\ker f \leq B$ is a subcomplex, and $\operatorname{coker} f$ is a quotient complex of C. The chain map $i : \ker f \to B$ is a categorical kernel of f in Ch, and π is similarly a cokernel. See appendix A 1.6. These constructions make Ch into an **abelian category**: roughly an additive category where every morphism has a kernel and a cokernel.

3 | 1.2 (Wednesday, January 20)

3.1 Taking Chain Complexes of Chain Complexes

See phone pic for missed first 10m

3.1.1 Double Complexes

Remark 3.1.1: Consider a double complex:

$$\begin{tikzcd}
 & & & C_{p,\cdot} : & & & \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \\
 & \cdots & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v_{p-1,q+1}"] & C_{p,q+1} \arrow[l, "d^h_{p,q+1}" description] \arrow[d, "d^v_{p,q+1}"] & C_{p+1,q+1} \arrow[l, "d^h_{p+1,q+1}" description] \arrow[d, "d^v_{p+1,q+1}"] & \cdots \arrow[l] & \\
C_{\cdot,q} : & \cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v_{p-1,q}"] & C_{p,q} \arrow[l, "d^h_{p,q}"'] \arrow[d, "d^v_{p,q}"] & C_{p+1,q} \arrow[l, "d^h_{p+1,q}"'] \arrow[d, "d^v_{p+1,q}"] & \cdots \arrow[l] & \\
 & \cdots & C_{p-1,q+1} \arrow[l] \arrow[d] & C_{p,q+1} \arrow[l, "d^h_{p,q+1}"'] \arrow[d] & C_{p+1,q+1} \arrow[l, "d^h_{p+1,q+1}"'] \arrow[d] & \cdots \arrow[l] & \\
 & & \vdots & \vdots & \vdots & &
\end{tikzcd}$$

Link to Diagram

All of the individual rows and columns are chain complexes, where $(d^h)^2 = 0$ and $(d^v)^2 = 0$, and the square anticommute: $d^v d^h + d^h d^v - 0$, so $d^v d^h = -d^h d^v$. This is almost a chain complex of chain complexes, i.e. an element of $\mathrm{Ch}(\mathrm{Ch}\, R\text{-}\mathbf{Mod}))$. It's useful here to consider lines parallel to the line $y = x$.

Definition 3.1.2 (Bounded Complexes)
A double complex $C_{\cdot,\cdot}$ is **bounded** if and only if there are only finitely many nonzero terms along each constant diagonal $p + q = n$.

Example 3.1.3 ***(?):*** A *first quadrant* double complex $\{C_{p,q}\}_{p,q \geq 0}$ is bounded: note that this can still have infinitely many terms, but each diagonal is finite because each will hit a coordinate axis.

Remark 3.1.4 ***(The sign trick):*** The squares anticommute, since the d^v are not chain maps between the horizontal chain complexes. This can be fixed by changing every one out of four signs, defining

$$f_{*,q} : C_{*,q} \to C_{*,q-1}$$
$$f_{p,q} \coloneqq (-1)^p d^v_{p,q} : C_{p,q} \to C_{p,q-1}.$$

This yields a new double complex where the signs of each column alternate:

$$\begin{tikzcd}
C_{0,q} \arrow[d, "d^v"] & C_{1,q} \arrow[l, "d^h"'] \arrow[d, "-d^v"] & C_{2,q} \arrow[l, "d^h"'] \arrow[d, "d^v"] \\
C_{0,q-1} & C_{1,q-1} \arrow[l, "d^h"'] & C_{2,q-1} \arrow[l, "d^h"']
\end{tikzcd}$$

Now the squares commute and $f_{\cdot,q}$ are chain maps, so this object is an element of $\mathrm{Ch}(\mathrm{Ch}\, R\text{-}\mathbf{Mod})$.

3.1.2 Total Complexes

Recall that products and coproducts of R-modules coincide when the indexing set is finite.

Definition 3.1.5 (Total Complexes)
Given a double complex $C_{\cdot,\cdot}$, there are two ordinary chain complexes associated to it referred to as **total complexes**:

$$(\mathrm{Tot}^{\prod} C)_n := \prod_{p+q=n} C_{p,q} (\mathrm{Tot}^{\oplus} C)_n := \bigoplus_{p+q=n} C_{p,q}.$$

Writing $\mathrm{Tot}(C)$ usually refers to the former. The differentials are given by

$$d_{p,q} = d^h + d^v : C_{p,q} \to C_{p-1,q} \oplus C_{p,q-1},$$

where $C_{p,q} \subseteq \mathrm{Tot}^{\oplus}(C)_n$ and $C_{p-1,q} \oplus C_{p,q-1} \subseteq \mathrm{Tot}^{\oplus}(C)_{n-1}$. Then you extend this to a differential on the entire diagonal by defining $d = \bigoplus_{p,q} d_{p,q}$.

Exercise 3.1.6 (?)
Check that $d^2 = 0$, using $d^v d^h + d^h d^v = 0$.

Remark 3.1.7: Some notes:

- $\mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\prod}(C)$ when C is bounded.
- The total complexes need not exist if C is unbounded: one needs infinite direct products and infinite coproducts to exist in $\mathcal{C}$. A category admitting these is called **complete** or **cocomplete**.[2]

---

[2] Recall that abelian categories are additive and only require *finite* products/coproducts. A counterexample: categories of *finite* abelian groups, where e.g. you can't take infinite sums and stay within the category.

3.1.3 More Operations

Definition 3.1.8 (Truncation below)
Fix $n \in \mathbb{Z}$, and define the **nth truncation** $\tau_{\geq n}(C)$ by

$$\tau_{\geq n}(C) = \begin{cases} 0 & i < n \\ Z_n & i = n \\ C_i & i > n. \end{cases}.$$

Pictorially:

$$\begin{tikzcd} \cdots & 0 \arrow[l] & Z_n \arrow[l, "d_n"'] & C_{n+1} \arrow[l, "d_{n+1}"'] & C_{n+2} \arrow[l, "d_{n+2}"'] & \cdots \arrow[l] \end{tikzcd}$$

Link to diagram

This is sometimes call the **good truncation of C below n**.

Remark 3.1.9: Note that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0 & i < n \\ H_i(C) & i \geq n. \end{cases}.$$

Definition 3.1.10 (Truncation above)
We define the quotient complex

$$\tau_{< n} C \coloneqq C / \tau_{\geq n} C.$$

which is C_i below n, C_n / Z_n at n. Thus is has homology

$$\begin{cases} H_i(C) & i < n. \\ 0 & i \geq n \end{cases}.$$

Definition 3.1.11 (Translation)
If C is a chain complex and $p \in \mathbb{Z}$, define a new complex $C[p]$ by

$$C[p]_n \coloneqq C_{n+p}.$$

$$\begin{tikzcd} \text{Degrees} & -p & & 0 & & p \\ C & C_{-p} & \cdots & C_0 \arrow[dll, dashed] & \cdots & C_p \arrow[dll, dashed] \\ C[p] & C_0 & \cdots & C_p & \cdots & C_{2p} \end{tikzcd}$$

Link to Diagram

Similarly, if C is a *cochain* complex, we set $C[p]^n := C^{n-p}$:

$$\begin{tikzcd}
\text{Degrees} & -p & & 0 & & p \\
C & C^{-p} \arrow[r] \arrow[rrd, dashed] & \cdots \arrow[r] & C^0 \arrow[r] \arrow[rrd, dashed] & \cdots \arrow[r] & C^p \\
C[p] & C^0 \arrow[r] & \cdots \arrow[r] & C^{-p} \arrow[r] & \cdots \arrow[r] & C^0
\end{tikzcd}$$

Link to Diagram

Mnemonic: Shift p positions in the same direction as the arrows.

In both cases, the differentials are given by the shifted differential $d[p] := (-1)^p d$. Note that these are not alternating: p is the fixed translation, so this is a constant that changes the signs of all differentials. Thus $H_n(C[p]) = H_{n+p}(C)$ and $H^n(C[p]) = H^{n-p}$.

Exercise 3.1.12
Check that if $C^n := C_{-n}$, then $C[p]^n = C[p]_{-n}$.

Remark 3.1.13: We can make translation into a functor $[p] : \mathrm{Ch} \to \mathrm{Ch}$: given $f : C \to D$, define $f[p] : C[p] \to D[p]$ by $f[p]_n := f_{n+p}$, and a similar definition for cochain complexes changing p to $-p$.

4 | Lecture 4 (Friday, January 22)

4.1 Long Exact Sequences

Remark 4.1.1: Some terminology: in an abelian category $\mathcal{A}$ an example of an **exact complex** in $\mathrm{Ch}(\mathcal{A})$ is

$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \cdots .$$

where *exactness* means ker = im at each position, i.e. $\ker f = 0, \operatorname{im} f = \ker g, \operatorname{im} g = C$. We say f is monic and g epic.

As a special case, if $0 \to A \to 0$ is exact then A must be zero, since the image of the incoming map must be 0. This also happens when every other term is zero. If $0 \to A \xrightarrow{f} B \to 0$, then $A \cong B$ since f is both injective and surjective (say for R-modules).

Theorem 4.1.2 ***(Long Exact Sequences).***
Suppose $0 \to A \to B \to C \to 0$ is a SES in $\mathrm{Ch}(\mathcal{A})$ (note: this is a sequence of *complexes*), then there are natural maps

$$\delta : H_n(C) \to H_{n-1}(A)$$

called **connecting morphisms** which decrease degree such that the following sequence is exact:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "f_*=H_n(f)"] & H_n(B) \arrow[r, "g_*=H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

This is referred to as the **long exact sequence in homology**. Similarly, replacing chain complexes by cochain complexes yields a similar connecting morphism that increases degree.

Note on notation: some books use ∂ for homology and δ for cohomology.

The proof that this sequence exists is a consequence of the *snake lemma*.

Lemma 4.1.3 ***(The Snake Lemma).***
The sequence highlighted in red in the following diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] & \ker(\alpha) \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & \ker(\gamma) \arrow[d] \arrow[dddll, dotted, "\exists\delta"] & & \\
 & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 & \\
 & 0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[r] \arrow[d] & 0 & \\
 & & \operatorname{coker}(\alpha) \arrow[r] & \operatorname{coker}(\beta) \arrow[r] & \operatorname{coker}(\gamma) \arrow[r] & \operatorname{coker}(g') \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proof (of the Snake Lemma: Existence).

- Start with $c \in \ker(\gamma) \leq C$, so $\gamma(c) = 0 \in C'$
- **Choose** $b \in B$ by surjectivity
    - We'll show it's independent of this choice.
- Then $b' \in B'$ goes to $0 \in C'$, so $b' \in \ker(B' \to C')$
- By exactness, $b' \in \ker(B' \to C') = \operatorname{im}(A' \to B')$, and now produce a unique $a' \in A'$ by injectivity
- Take the image $[a'] \in \operatorname{coker} \alpha$
- Define $\partial(c) \coloneqq [a']$.

■

Proof (of the Snake Lemma: Uniqueness).

- We chose b, suppose we chose a different $\tilde{b}$.
- Then $\tilde{b} - b \mapsto c - c = 0$, so the difference is in $\ker g = \operatorname{im} f$.
- Produce an $\tilde{a} \in A$ such that $\tilde{a} \mapsto \tilde{b} - b$
- Then $\bar{a} \coloneqq \alpha(\tilde{a})$, so apply f'.
- Define $\beta(\tilde{b}) = \tilde{b}' \in B$.
- Commutativity of the LHS square forces $\tilde{a}' \mapsto \tilde{b}' - b'$.
- Then $\bar{a} + a' \mapsto \tilde{b}' - b' + b' = \tilde{b}'$.
- So $\tilde{a}' + a'$ is the desired pullback of $\tilde{b}'$
- Then take $[\tilde{a}'] \in \operatorname{coker} \alpha$; are $a', \tilde{a}'$ in the same equivalence class?
- Use that fact that $\tilde{a} = a' + \bar{a}$, where $\bar{a} \in \operatorname{im} \alpha$, so $[\tilde{a}] = [a' + \bar{a}] = [a'] \in \operatorname{coker} \alpha \coloneqq A' / \operatorname{im} \alpha$.

■

A few changes in the middle, redo!

Proof (of the Snake Lemma: Exactness).

- Let's show $g : \ker \beta \to \ker \gamma$.
    - Let $b \in \ker \beta$, then consider $\gamma(g(\beta)) = g'(\beta(b)) = g'(0) = 0$ and so $g(b) \in \ker \gamma$.
- Now we'll show $\operatorname{im}(g|_{\ker \beta}) \subseteq \ker \delta$
    - Let $b \in \ker \beta, c = g(b)$, then how is $\delta(c)$ defined?
    - Use this b, then apply β to get $b' = \beta(b) = 0$ since $b \in \ker \beta$.
    - So the unique thing mapping to it a' is zero, and thus $[a'] = 0 = \delta(c)$.
- $\ker \delta \subseteq \operatorname{im}(g|_{\ker \beta})$
    - Let $c \in \ker \delta$, then $\delta(c) = 0 = [a'] \in \operatorname{coker} \alpha$ which implies that $a' \in \operatorname{im} \alpha$.
    - Write $a' = \alpha(a)$, then $\beta(b) = b' = f'(a') = f'(\alpha(a))$ by going one way around the LHS square, and is equal to $\beta(f(a))$ going the other way.
    - So $\tilde{b} \coloneqq b - f(a) \in \ker \beta$, since $\beta(b) = \beta(f(a))$ implies their difference is zero.

- Then $g(\tilde{b}) = g(b) - g(f(a)) = g(b) = c$, which puts $c \in g(\ker \beta)$ as desired.

■

Exercise 4.1.4 (?)
Show exactness at the remaining places – the most interesting place is at $\operatorname{coker} \alpha$. Also check that all of these maps make sense.

Remark 4.1.5: We assumed that $\mathcal{A} = R$-**Mod** here, so we could chase elements, but this happens to also be true in any abelian category $\mathcal{A}$ but by a different proof. The idea is to embed $\mathcal{A} \to R$-**Mod** for some ring R, do the construction there, and pull the results back – but this doesn't quite work! $\mathcal{A}$ can be too big. Instead, do this for the smallest subcategory $\mathcal{A}_0$ containing all of the modules and maps involved in the snake lemma. Then $\mathcal{A}_0$ is small enough to embed into R-**Mod** by the **Freyd-Mitchell Embedding Theorem**.

5 | Lecture 5 (Monday, January 25)

5.1 LES Associated to a SES

Theorem 5.1.1 *(?).*
For every SES of chain complexes, there is a long exact sequence in homology.

Proof (?).
Suppose we have a SES of chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

which means that for every n there is a SES of R-modules. Recall the diagram for the snake lemma, involving kernels across the top and cokernels across the bottom. Applying the snake lemma, by hypothesis $\operatorname{coker} g = 0$ and $\ker f = 0$. There is a SES

$$A_n/dA_{n+1} \to B_n/dB_{n+1} \to C_n/dC_{n+1} \to 0$$

Using the fact that $B_n \subseteq Z_n$, we can use the 1st and 2nd isomorphism theorems to produce

$$\begin{tikzcd}
 & H_n(A) \arrow[r, "f_*"] \arrow[d] & H_n(B) \arrow[r, "g_*"] \arrow[d] & H_n(C) \arrow[d] \arrow[dddll, dotted, "\exists\delta"] & \\
 & A_n/dA_{n+1} \arrow[r, "f"] \arrow[d, "d_n"'] & B/dB_{n+1} \arrow[r, "g"] \arrow[d, "d_n"'] & C/dC_{n+1} \arrow[r] \arrow[d, "d_n"'] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] \arrow[d] & Z_{n-1}(B) \arrow[r] \arrow[d] & Z_{n-1}(C) \arrow[d] & \\
 & \begin{array}{c}\operatorname{coker} d_n \\ =Z_{n-1}(A)/dA_n \\ =H_{n-1}(A)\end{array} \arrow[r, "f_*"'] & H_{n-1}(B) \arrow[r, "g_*"'] & H_{n-1}(C) &
\end{tikzcd}$$

Link to diagram

This yields an exact sequence relating H_n to H_{n-1}, and these can all be spliced together.

- $\ker(A_n/dA_{n-1} \to Z_{n-1}(A) = Z_n(A)/dA_{n+1} := H_n(A)$ using the 2nd isomorphism theorem

■

Remark 5.1.2: Note that d is *natural*, which means the following: there is a category $\mathcal{S}$ whose objects are SESs of chain complexes and whose maps are chain maps:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

There is another full subcategory $\mathcal{L}$ of Ch whose objects are LESs of objects in the original abelian category, i.e. exact chain complexes. The claim is that the LES construction in the theorem defines a functor $\mathcal{S} \to \mathcal{L}$. We've seen how this maps objects, so what is the map on morphisms? Given a morphism as in the above diagram, there is an induced morphism:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "H_n(u_A)"] & H_n(B) \arrow[r] \arrow[d, "H_n(u_B)"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "H_n(u_C)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(u_A)"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

The first two squares commute, and *naturality* means that the third square commutes as well.

Exercise 5.1.3 (?)
Check the details!

Remark 5.1.4: It is sometimes useful to explicitly know how to compute snake lemma boundary elements. See the book for a recipe for computing $\partial(\xi)$:

- Lift ξ to a cycle $c \in Z_n(C) \subseteq C_n$.
- Pull c back to a preimage $b \in B_n$ by surjectivity.
- Apply the differential to get $d(b) \in Z_{n-1}(B)$, using that images are contained in kernels.
- Since this is in kernel of the outgoing map, it's in the kernel of the incoming map and thus there exists an $a \in Z_{n-1}(A)$ such that $f(a) = db$
- So set $\delta(\xi) := [a] \in H_{n-1}(A)$.

Remark 5.1.5: Why is naturality useful? Suppose $H_n(B) = 0$, you get isomorphisms, and this allows inductive arguments up the LES. The LES in homology is sometimes abbreviated as an **exact triangle**:

$$\begin{tikzcd}
& H_*(A) \arrow[dr, "f"] & \\
H_*(C) \arrow[ur, rightsquigarrow, "\partial"] & & H_*(B) \arrow[ll, "g"]
\end{tikzcd}$$

Here $\partial : H_*(C) \to H_*(A)[1]$ shifts degrees. Note that this motivates the idea of **triangulated categories**, which is important in modern research. See Weibel Ch.10, and exercise 1.4.5 for how to construct these as quotients of Ch.

5.2 1.4: Chain Homotopies

Remark 5.2.1: Assume for now that we're in the situation of R-modules where R is a field, i.e. vector spaces. The main fact/advantage here that is not generally true for R-modules: every subspace has a complement. Since $B_n \subseteq Z_n \subseteq C_n$, we can write $C_n = Z_n \oplus B'_n$ for every n, and $Z_n = B_n \oplus H_n$. This notation is suggestive, since $H_n \cong Z_n/B_n$ as a quotient of vector spaces. Substituting, we get $C_n = B_n \oplus H_n \oplus B'_n$. Consider the projection $C_n \to B_n$ by projecting onto the first factor. Identifying $B_n := \operatorname{im}(C_{n+1} \to C_n) \cong C_{n+1}/Z_{n+1}$ by the 1st isomorphism theorem in the reverse direction. But this image is equal to B'_{n+1}, and we can embed this in C_{n+1}, so define $s_n : C_n \to C_{n+1}$ as the composition

$$s_n := (C_n \xrightarrow{\text{Proj}} B_n = \operatorname{im}(C_{n+1} \to C_n) \xrightarrow{d_{n+1}^{-1}} C_{n+1}/Z_{n+1} \xrightarrow{\cong} B'_{n+1} \hookrightarrow C_{n+1}.$$

Claim 1: $d_{n+1}s_n d_{n+1} = d_{n+1}$ are equal as maps.

Proof (?).

- Check on the first factor $B'_{n+1} \subseteq C_{n+1}$ directly to get $s_n d_{n+1}(x) = d_{n+1}(x)$ for $x \in B'_{n+1}$, and then applying d_{n+1} to both sides is the desired equality.
- On the second factor Z_{n+1}, both sides give zero since this is exactly the kernel.

■

Claim 2: $d_{n+1}s_n + s_{n-1}d_n = \mathrm{id}_{C_n}$ if and only if $H_n = 0$, i.e. the complex C is exact at C_n. This map is the sum of taking the two triangle paths in this diagram:

$$\begin{tikzcd}
 & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d_{n+1}"] & C_n &
\end{tikzcd}$$

Proof (?).
We again check this on both factors:

- Using the first claim, $s_n = 0$ on B'_n and thus $s_{n-1}d_n = \mathrm{id}_{B'_n}$.
- On H_n, $s_n = 0$ and $d_n = 0$, and so the LHS is $0 = \mathrm{id}_{H_n}$ *if and only if* $H_n = 0$.
- On B_n, and tracing through the definition of s_n yields $d_{n+1}s_n(x) = x$ and this yields id_{B_n}.

■

Next time: summary of decompositions, start general section on chain homotopies.

6 | Wednesday, January 27

See phone pic for missed first 10m.

6

Wednesday, January 27

6.1 1.4: Chain Homotopies

Definition 6.1.1 (Split Exact)
A complex is called **split** if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. In this case, the maps s_n are referred to as the **splitting maps**, and if C is additionally acyclic, we say C is **split exact**.

Remark 6.1.2: Note that when C is split exact, we have

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d"] & C_n &
\end{tikzcd}$$

Link to Diagram

Example 6.1.3 ***(Not all complexes split):*** Take

$$C = \left(0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{d} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0\right).$$

Then $\operatorname{im} d = \{0, 2\} = \ker d$, but this does not split since $\mathbb{Z}/2\mathbb{Z}^2 \not\cong \mathbb{Z}/4\mathbb{Z}$: one has an element of order 4 in the underlying additive group. Equivalently, there is no complement to the image. What might be familiar from algebra is $ds = \mathrm{id}$, but the more general notion is $dsd = d$.

Example 6.1.4 ***(?):*** The following complex is not split exact for the same reason:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \to \cdots .$$

Question 6.1.5
Given $f, g : C \to D$, when do we get equality $f_* = g_* : H_*(C) \to H_*(D)$?

Definition 6.1.6 (Homotopy Terminology for Chains)
A chain map $f : C \to D$ is **nullhomotopic** if and only if there exist maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$:

6 Wednesday, January 27

$$\begin{tikzcd}
& C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\text{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}$$

Link to Diagram

The map s is called a **chain contraction**. Two maps are **chain homotopic** (or initially: f is chain homotopic to g, since we don't yet know if this relation is symmetric) if and only if $f-g$ is nullhomotopic, i.e. $f-g=ds+sd$. The map s is called a **chain homotopy** from f to g. A map f is a **chain homotopy equivalence** if both fg and gf are chain homotopic to the identities on C and D respectively.

Lemma 6.1.7 ***(?).***
If map $f : C \to D$ is nullhomotopic then $f_* : H_*(C) \to H_*(D)$ is the zero map. Thus if f, g are chain homotopic, then they induce equal maps.

Proof (?).
An element in the quotient $H_n(C)$ is represented by an n-cycle $x \in Z_n(C)$. By a previous exercise, $f(x)$ is a well-defined element of $H_n(D)$, and using that $d(x)=0$ we have

$$f(x) = (ds+sd)(x) = d(s(x)),$$

and so $f[x] = [f(x)] = [0]$.

$$\begin{tikzcd}
& x & d(x)=0 \\
& C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\text{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n & \\
& d(s(x)) &
\end{tikzcd}$$

Link to Diagram

Now applying the first part to $f-g$ to get the second part. ■

See Weibel for topological motivations.

6.2 1.5 Mapping Cones

Remark 6.2.1: Note that we'll skip *mapping cylinders*, since they don't come up until the section on triangulated categories. The goal is to see how any two maps between homologies can be fit into a LES. This helps reduce questions about *quasi-isomorphisms* to questions about split exact complexes.

Definition 6.2.2 (Mapping Cones)
Suppose we have a chain map $f : B \to C$, then there is a chain complex $\operatorname{cone}(f)$, the **mapping cone of** f, defined by

$$\operatorname{cone}(f)_n = B_{n-1} \oplus C_n.$$

The maps are given by the following:

$$\begin{tikzcd}
B_{n-1} \arrow[r, "-d^B"] \arrow[ddr, "-f"'] & B_{n-2} \\
\oplus & \oplus \\
C_n \arrow[r, "d^C"] & C_{n-1}
\end{tikzcd}$$

Link to Diagram

We can write this down: $d(b,c) = (-d(b), -f(b) + d(c))$, or as a matrix

$$\begin{bmatrix} -d^B & 0 \\ -f & d^C \end{bmatrix}.$$

Exercise 6.2.3 (?)
Check that the differential on $\operatorname{cone}(f)$ squares to zero.

Exercise 6.2.4 (Weibel 1.5.1)
When $f = \mathrm{id} : C \to C$, we write $\operatorname{cone}(C)$ instead of $\operatorname{cone}(\mathrm{id})$. Show that $\operatorname{cone}(C)$ is split exact, with splitting map $s(b,c) = (-c, 0)$ for $b \in C_{n-1}, c \in C_n$.

Proposition 6.2.5 ***(?).***
Suppose $f : B \to C$ is a chain map, then the induced maps $f_* : H(B) \to H(C)$ fit into a LES. There is a SES of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & \operatorname{cone}(f) \arrow[r] & B[-1] \arrow[r] & 0 \\
 & c \arrow[r] & (0,c) & & \\
 & & (b,c) \arrow[r] & -b &
\end{tikzcd}$$

Link to Diagram

Exercise 6.2.6 *(?)*
Check that these are chain maps, i.e. they commute with the respective differentials d.

The corresponding LES is given by the following:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}\operatorname{cone}(f) \arrow[r, "\delta_*"] & H_{n+1}(B[-1]) = H_n(B) \arrow[dll, "\partial"] \\
H_n(C) \arrow[r] & H_n\operatorname{cone}(f) \arrow[r] & H_n(B[-1]) = H_{n-1}(B) \arrow[dll] \\
\cdots & &
\end{tikzcd}$$

Link to Diagram

Overflowing :(

Lemma 6.2.7 ***(?).***
The map $\partial = f_*$

Proof (?).
Letting $b \in B_n$ is an n-cycle.

1. Lift b to anything via δ, say $(-b, 0)$.
2. Apply the differential d to get $(db, fb) = (0, fb)$ since b was a cycle.
3. Pull back to C_n by the map $C \to \operatorname{cone}(f)$ to get fb.
4. Then the connecting morphism is given by $\partial[b] = [fb]$. But by definition of f_*, we have $[fb] = f_*[b]$.

■

7 | Friday, January 29

7.1 Mapping Cones

Remark 7.1.1: Given $f : B \to C$ we defined $\operatorname{cone}(f)_n \coloneqq B_{n-1} \oplus C_n$, which fits into a SES

$$0 \to C \to \operatorname{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

and thus yields a LES in cohomology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(\operatorname{cone}(f)) \arrow[r] & H_n(B) \arrow[dll, out=0, in=180, "\delta = f_*"] \\
H_n(C) \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r] & H_{n-1}(B) \arrow[dll, out=0, in=180, "\delta"] \\
\cdots & &
\end{tikzcd}$$

Link to Diagram

Corollary 7.1.2 *(?).*
$f : B \to C$ is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is exact.

Proof (?).
In the LES, all of the maps f_* are isomorphisms, which forces $H_n(\operatorname{cone}(f)) = 0$ for all n. ■

Remark 7.1.3: So we can convert statements about quasi-isomorphisms of complexes into exactness of a single complex.

We'll skip the rest, e.g. mapping cylinders which aren't used until the section on triangulated categories. We'll also skip the section on δ-functors, which is a slightly abstract language.

7.2 Ch. 2: Derived Functors

Remark 7.2.1: Setup: fix $M \in R$-**Mod**, where R is a ring with unit. Note that by an upcoming exercise, $\mathrm{Hom}_R(M, \cdot) : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is a *left-exact* functor, but not in general right-exact: given a SES $0 \to A \to B \to C \to 0 \in \mathrm{Ch}(\mathbf{Mod}\text{-}R)$, there is an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_*=f\circ(\cdot)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_*=g\circ(\cdot)"] & \mathrm{Hom}_R(M, A)
\end{tikzcd}$$

Link to Diagram

However, this is not generally surjective: not every $M \to C$ is given by composition with a morphism $M \to B$ (*lifting*). To create a LES here, one could use the cokernel construction, but we'd like to do this functorially by defining a sequence functors F^n that extend this on on the right to form a LES:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_*=f\circ(\cdot)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_*=g\circ(\cdot)"] & \mathrm{Hom}_R(M, A) \arrow[dll, out=0, in=180] \\
 & F^1(A) \arrow[r] & F^1(B) \arrow[r] & F^1(C) \arrow[dll, out=0, in=180] \\
 & F^2(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

It turns out such functors exist and are denoted $F^n(\cdot) \coloneqq \mathrm{Ext}_R^n(M, \cdot)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_*=f\circ(\cdot)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_*=g\circ(\cdot)"] & \mathrm{Hom}_R(M, A) \arrow[dll, out=0, in=180] \\
 & \mathrm{Ext}_R^1(A) \arrow[r] & \mathrm{Ext}_R^1(B) \arrow[r] & \mathrm{Ext}_R^1(C) \arrow[dll, out=0, in=180] \\
 & \mathrm{Ext}_R^2(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

By convention, we set $\operatorname{Ext}_R^0(\,\cdot\,) := \operatorname{Hom}_R(M, \,\cdot\,)$. This is an example of a general construction: **right-derived functors** of $\operatorname{Hom}_R(M, \,\cdot\,)$. More generally, if $\mathcal{A}$ is an abelian category (with a certain additional property) and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor (where $\mathcal{B}$ is another abelian category) then we can define right-derived functors $R^n F : \mathcal{A} \to \mathcal{B}$. These send SESs in $\mathcal{A}$ to LESs in $\mathcal{B}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
\\
0 \arrow[r] & FA \arrow[r] & FB \arrow[r] & FC \arrow[dll, out=0, in=180] & \\
 & R^1FA \arrow[r] & R^1FB \arrow[r] & R^1FC \arrow[dll, out=0, in=180] & \\
 & \cdots & & &
\end{tikzcd}$$

Link to Diagram

Similarly, if F is *right-exact* instead, there are left-derived functors $L^n F$ which form a LES ending with 0 at the right:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
 & & & \cdots & \\
 & LFA \arrow[urr, out=180, in=0] \arrow[r] & LFB \arrow[r] & LFC & \\
 & FA \arrow[urr, out=180, in=0] \arrow[r] & FB \arrow[r] & FC \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

7.3 2.2: Projective Resolutions

Definition 7.3.1 (Projective Modules)
Let $\mathcal{A} = R\text{-}\mathbf{Mod}$, then $P \in R\text{-}\mathbf{Mod}$ satisfies the following universal property:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Remark 7.3.2: Free modules are projective. Let $F = R^X$ be the free module on the set X. Then consider $\gamma(x) \in C$, by surjectivity these can be pulled back to some elements in B:

$$\begin{tikzcd}
 & X \arrow[ddl, dotted, "\exists\tilde{\beta}"'] \arrow[d, hook, "\iota_X"] & \\
 & F \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0 \\
\exists b \in g^{-1}(\gamma(x)) \coloneqq \beta(x) & \gamma(x) &
\end{tikzcd}$$

Link to Diagram

This follows from the universal property of free modules:

$$\begin{tikzcd}
 & \exists F(X) \arrow[d, dashed, "\exists! f' \in \mathrm{Hom}_R(F(X),X)"] & \\
X \arrow[ur, "\exists g \in \mathrm{Hom}_{\mathrm{Set}}(X,F(X))"] \arrow[r, "f \in \mathrm{Hom}_{\mathrm{Set}}(X,M)"] & M & \in R\text{-}\mathbf{Mod}
\end{tikzcd}$$

Link to Diagram

Proposition 7.3.3*(?).*
An R-module is projective if and only if it is a direct summand of a free module.

Exercise 7.3.4 (?)
Prove the $\impliedby$ direction!

Proof (?).
$\implies$: Assume P is projective, and let $F(P)$ be the free R-module on the underlying set of P. We can start with this diagram:

$$\begin{tikzcd}
& F(P) \arrow[d, dashed] \\
P \arrow[ur] \arrow[r, "\mathrm{id}_P"] & P
\end{tikzcd}$$

Link to Diagram

And rearranging, we get

$$\begin{tikzcd}
& & & P \arrow[dl, "\exists\iota"'] \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & \ker\pi \arrow[r, hook] & F(P) \arrow[r, two heads, "\pi"] & P \arrow[r] \arrow[l, bend left, dashed, "\iota"] & 0
\end{tikzcd}$$

Link to Diagram

Since $\pi \circ \iota$, the SES splits and this $F(P) \cong P \oplus \ker\pi$, making P a direct summand of a free module.

■

Example 7.3.5*(?):* Not every projective module is free. Let $R = R_1 \times R_2$ a direct product of unital rings. Then $P \coloneqq R_1 \times \{0\}$ and $P' \coloneqq \{0\} \times R_2$ are R-modules that are submodules of R. They're projective since R is free over itself as an R-module, and their direct sum is R. However they can not be free, since e.g. P has a nonzero annihilator: taking $(0,1) \in R$, we have $(0,1) \cdot P = \{(0,0)\} = 0_R$. No free module has a nonzero annihilator, since ix $0 \neq r \in R$ then $rR \neq 0$ since $r1_R \in rR$, which implies that $r\left(\bigoplus R\right) \neq 0$.

Example 7.3.6*(?):* Taking $R = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ admits projective R-modules which are not free.

Example 7.3.7*(?):* Let F be a field, define the ring $R \coloneqq \mathrm{Mat}(n \times n, F)$ with $n \geq 2$, and set $V = F^n$ thought of as column vectors. This is left R-module, and decomposes as $R = \bigoplus_{i=1}^{n} V$ corresponding to the columns of R, using that $AB = [Ab_1, \cdots, Ab_n]$. Then V is a projective R-module as a direct summand of a free module, but it is not free. We have vector spaces, so we can consider dimensions: $\dim_F R = n^2$ and $\dim_F V = n$, so V can't be a free R-module since this would force $\dim_F V = kn^2$ for some k.

Example 7.3.8 ***(?):*** How many projective modules are there in a given category? Let $\mathcal{C} := \mathbf{Ab}^{\text{fin}}$ be the category of *finite* abelian groups, where we take the full subcategory of the category of all abelian groups. This is an abelian category, although it is not closed under *infinite* direct sums or products, which has no projective objects.

Proof (?).
Over a PID, every submodule of a free module is free, and so we have free $\iff$ projective in this case. So equivalently, we can show there are no free $\mathbb{Z}$-modules, which is true because $\mathbb{Z}$ is infinite, and any such module would have to contain a copy of $\mathbb{Z}$. ■

Remark 7.3.9: The definition of projective objects extends to any abelian category, not just R-modules.

8 | Monday, February 01

Recall the universal of projective modules.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Definition 8.0.1 (Enough Projective)
If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has **enough projectives** if and only if for all $a \in \mathcal{A}$ there exists a projective object $P \in \mathcal{A}$ and a surjective morphism $P \twoheadrightarrow A$.

Example 8.0.2 ***(?):*** **Mod**-R has enough projectives: for all $A \in$ **Mod**-R, one can take $F(A) \twoheadrightarrow A$.

Example 8.0.3 ***(?):*** The category of finite abelian groups does *not* have enough projectives.

Why?

Lemma 8.0.4 ***(?).***
P is projective if and only if $\operatorname{Hom}_{\mathcal{A}}(P, \cdot\,)$ is an exact functor.

Exercise 8.0.5 (?)
Prove this!

Definition 8.0.6 ((Key))
Let $M \in \mathbf{Mod}\text{-}R$, then a **projective resolution** of M is an exact complex

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_1 \overset{\varepsilon}{\twoheadrightarrow} M \to 0.$$

We write $P_\cdot \overset{\epsilon}{\twoheadrightarrow} M$.

Lemma 8.0.7 ***((Key)).***
Every object $M \in \mathbf{Mod}\text{-}R$ has a projective resolution. This is true in any abelian category with enough projectives.

Proof (?).

- Since there are enough projectives, choose $P_0 \xrightarrow{\epsilon_0} M \to 0$.
- To extend this, set $M_0 := \ker \epsilon_0$, then find a projective cover $P_1 \xrightarrow{\epsilon_1} M_0$
- Use that $d_1 := \iota_0 \circ \epsilon_1$ and $\operatorname{im} d_1 = M_0 = \ker \epsilon_0$
- Then $d_2 := \iota_1 \circ \epsilon_2$ with $\operatorname{im} d_2 = M_1$, and $\ker d_1 = \ker \epsilon_1 = M_1$.
- Continuing in this fashion makes the complex exact at every stage.

$$\begin{tikzcd}
 & & 0 \arrow[dr, hook] & & 0 & & & & \\
 & & & M_1 \arrow[ur, two heads] \arrow[dr, hook] & & & & & \\
\cdots \arrow[rr, dashed] & & P_2 \arrow[ur, two heads] \arrow[rr, dashed, "\exists d_2"] & & P_1 \arrow[rr, dashed, "\exists d_1"] \arrow[dr, two heads, "\varepsilon_0"] & & P_0 \arrow[r, two heads, "\varepsilon_0"] & M & 0 \\
 & M_2 \arrow[ur, hook] & & & & M_0 \arrow[ur, hook, "\iota_0"] \arrow[dr, two heads] & & & \\
0 \arrow[ur, hook] & & & & 0 \arrow[ur, hook] & & 0 & &
\end{tikzcd}$$

Link to Diagram

■

8.1 Comparison Theorem

Theorem 8.1.1 ***(Comparison Theorem).***
Suppose $P_\cdot \xrightarrow{\epsilon} M$ is a projective resolution of an object in $\mathcal{A}$ and $(M \xrightarrow{f} N \in \operatorname{Mor}(\mathcal{A})$ and $Q_\cdot \xrightarrow{\eta} N$ a resolution of N. Then there exists a chain map $P \xrightarrow{f} Q$ lifting f which is unique

up to chain homotopy:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[d, dashed, "\exists f_2"] & P_1 \arrow[r, "d_1^P"] \arrow[d, dashed, "\exists f_1"] & P_0 \arrow[r, "\varepsilon = d_0^P"] \arrow[d, dashed, "\exists f_0"] & M \arrow[r] \arrow[d, "f_{-1} := f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2^Q"'] & Q_1 \arrow[r, "d_1^Q"'] & Q_0 \arrow[r, "\eta = d_0^Q"'] & N \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Remark 8.1.2: The proof will only use that $P \xrightarrow{\epsilon} M$ is a chain complex of projective objects, i.e. $d^2 = 0$, and that $\epsilon \circ d_1^p = 0$. To make the notation more consistent, we'll write $Z_{-1}(P) := M$ and $Z_{-1}(Q) := N$. Toward an induction, suppose that the f_i have been constructed for $i \leq n$, so $f_{i-1} \circ d = d \circ f_i$.

Proof (Existence).
A fact about chain maps is that they induce maps on the kernels of the outgoing maps, so there is a map $f_n' : Z_n(P) \to Z_n(Q)$. We get a diagram where the top row is not necessarily exact:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d"] \arrow[d, dashed, "\exists f_{n+1}"] & Z_n(P) \arrow[d, "f_{n'}"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0
\end{tikzcd}$$

Link to Diagram

Using the definition of projective, since P_{n+1} is projective, the map $f_{n+1} : P_{n+1} \to Q_{n+1}$ exists where $d \circ f_{n+1} = f_n' \circ d = f_n \circ d$, since $f_n = f_n'$ on $\operatorname{im} d \subseteq Z_n(P)$. This yields commutativity of the above square.

■

Proof (Uniqueness).
Suppose $g : P \to Q$ is another lift of f', the consider $h := f - g$. This is a chain map $P \to Q$ lifting of $f' - f' = 0$. We'll construct a chain contraction $\{s_n :; P_n \to Q_{n+1}\}$ by induction on n:
We have the following diagram:

$$\begin{tikzcd}
 & P_0 \arrow[r, "\varepsilon"] \arrow[d, "h_0 := f_0 - f_0'"'] & M \arrow[d, "f - f' = 0"] \\
Q_1 \arrow[r, "d"'] & Q_0 \arrow[r, "\eta"'] & N
\end{tikzcd}$$

Link to Diagram

Setting $P_{-1} \coloneqq 0$ and $s_{-1} : P_{-1} \to Q_0$ to be the zero map, we have $\eta \circ h_0 = \varepsilon(f' - f') = 0$. Using projectivity of P_0, there exists an s_0 as shown below which satisfies $h_0 = d \circ s_0 = ds_0 + s_{-1}d$ where $s_{-1}d = 0$:

$$\begin{tikzcd}
& P_0 \arrow[r, "d_0=0"] \arrow[dl, dashed, "\exists s_1"'] \arrow[d, "h_0"] & P_{-1} = 0 \arrow[dl, "s_{-1}=0"] \\
Q_1 \arrow[r, two heads] & d(Q_1) \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proceeding inductively, assume we have maps $s_i : P_i \to Q_{i+1}$ such that $h_{n-1} = ds_{n-1} + s_{n-2}d$, or equivalently $ds_{n-1} = h_{n-1} - s_{n-2}d$. We want to construct s_n in the following diagram:

$$\begin{tikzcd}
& P_n \arrow[r, "d"] \arrow[dl, dashed, "\exists s_n"] \arrow[d, "h_n"] & P_{n-1} \arrow[r, "d"] \arrow[dl, "s_{n-1}"] \arrow[d, "h_{n-1}"] & P_{n-2} \arrow[dl, "s_{n-2}"] \\
Q_{n+1} \arrow[r, "d"] & Q_n \arrow[r, "d"] & Q_{n-1} &
\end{tikzcd}$$

Link to Diagram

So consider $h_n - s_{n-1}d : P_n \to Q_n$, which we want to equal $d(s_n)$. We want exactness, so we need better control of the image! We have $d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d$. But this is equal to $dh_n - h_{n-1}d = 0$ since h is a chain map. Thus we get $h_n - s_{n-1}d : P_n \to Z_n(Q)$, and thus using projectivity one last time, we obtain the following:

$$\begin{tikzcd}
& P_n \arrow[dl, dashed, "\exists s_n"'] \arrow[d, "h_n - s_{n-1}d"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0
\end{tikzcd}$$

Link to Diagram

Since P_n is projective, there exists an $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$.

$\blacksquare$

9 | Wednesday, February 03

Remark 9.0.1: All rings have 1 in this course!

9.1 Horseshoe Lemma

Proposition 9.1.1 ***(Horseshoe Lemma).***
Suppose we have a diagram like the following, where the columns are exact and the rows are projective resolutions:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

Link to Diagram

Note that if the vertical sequence were split, one could sum together to two resolutions to get a resolution of the middle. This still works: there is a projective resolution of P of A given by

$$P_n \coloneqq P_n' \oplus P_n''$$

which lifts the vertical column in the above diagram to an exact sequence of complexes

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0,$$

where $\iota_n : P_n' \hookrightarrow P_n$ is the natural inclusion and $\pi_i : P_n \twoheadrightarrow P_n''$ the natural projection.

9.1.1 Proof of the Horseshoe Lemma

We can construct this inductively:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"'] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
\ker(\varepsilon) \arrow[r] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"'] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \\
\ker(\varepsilon'') \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"'] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Link to Diagram

- P_0'' projective and π_A surjective implies ε'' lifts to $\eta'' : P_0'' \to A$
- Composing yields $\eta' \coloneqq \iota_A \circ \eta' : P_0' \to A$
- Get $\varepsilon \coloneqq \eta' \oplus \eta'' : P_0 \coloneqq P_0' \oplus P_0'' \to A$.

Flipping the diagram, we can apply the snake lemma to the two columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"'] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"'] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[uurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"'] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Link to Diagram

We can now conclude that

- $\operatorname{coker} \varepsilon = 0$
- $\partial = 0$ since it lands on the zero moduli

So append a zero onto the far left column:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"'] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"'] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[d] \arrow[uuurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"'] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Link to Diagram

Thus the LHS column is a SES, and we have the first step of a resolution. Proceeding inductively, at the next step we have

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r, "d_1'"] & \ker(\varepsilon') \arrow[r] \arrow[d] & 0 \\
 & & \ker(\varepsilon) \arrow[d] & \\
\cdots \arrow[r] & P_1'' \arrow[r, "d_1''"] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

Link to Diagram

However, this is precisely the situation that appeared before, so the same procedure works.

Exercise 9.1.2 (?)
Check that the middle complex is exact! Follows by construction.

9.2 Injective Resolutions

Definition 9.2.1 (Injective Objects)
Let $\mathcal{A}$ be an abelian category, then $I \in \mathcal{A}$ is **injective** if and only if it satisfies the following universal property: A is projective if and only if for every monic $\alpha : A \to I$, any map $f : A \to B$ lifts to a map $B \to I$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \\
& I \arrow[ur, dashed, "\exists\beta"'] &
\end{tikzcd}$$

Link to Diagram

We say $\mathcal{A}$ **has enough injectives** if and only if for all A, there exists $A \hookrightarrow I$ where I is injective.

Slogan 9.2.2
Maps on subobjects extend.

Proposition 9.2.3 ***(Products of Injectives are Injective).***
If $\{I_\alpha\}$ is a family of injectives and $I \coloneqq \prod_\alpha I_\alpha \in A$, then I is again injective.

Proof (?).
Use the universal property of direct products. ■

9.3 Baer's Criterion

Proposition 9.3.1 ***(Baer's Criterion).***
An object $E \in R$-**Mod** is injective if and only if for every right ideal $J \trianglelefteq R$, every map $J \to E$ extends to a map $R \to E$. Note that J is a right R-submodule.

Proof (?).
$\implies$: This is essentially by definition. Instead of taking arbitrary submodules, we're just taking R itself and *its* submodules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\
 & E &
\end{tikzcd}$$

Link to Diagram

$\impliedby$: Suppose we have the following:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \\
 & E &
\end{tikzcd}$$

Link to Diagram

Let $\mathcal{E} := \left\{ \alpha' : A' \to E \;\middle|\; A \leq A' \leq B \right\}$, i.e. all of the intermediate extensions:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & A' \arrow[r] & B \\
 & E & &
\end{tikzcd}$$

Link to Diagram

Add a partial order to $\mathcal{E}$ where $\alpha' \leq \alpha''$ if and only if α'' extends α'. Applying Zorn's lemma (and abusing notation slightly), we can produce a maximal $\alpha' : A' \to E$. The claim is that $A' = B$. Supposing not, then A' is a proper submodule, so choose a $b \in B \setminus A'$. Then define the set $J := \left\{ r \in R \;\middle|\; br \in A' \right\}$, this is a right ideal of R since A' was a right R-module. Now applying the assumption of Baer's condition on E, we can produce a map $f : R \to E$:C

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, "b\cdot"'] & R \arrow[ddl, dashed, "\exists f"] \\
& A' \arrow[d, "\alpha'"'] & \\
& E &
\end{tikzcd}$$

Link to Diagram

Now let $A'' := A' + bR \leq B$, and provisionally define

$$\begin{aligned} \alpha'' : A'' &\to E \\ a + br &\mapsto \alpha'(a) + f(r). \end{aligned}$$

Remark 9.3.2: Is this well-defined? Consider overlapping terms, it's enough to consider elements of the form $br \in A'$. In this case, $r \in J$ by definition, and so $\alpha'(br) = f(r)$ by commutativity in the previous diagram, which shows that the two maps agree on anything in the intersection.

Note that α'' now extends α', but $A' \subsetneq A''$ since $b \in A'' \setminus A'$. But then A'' strictly contains A', contradicting its maximality from Zorn's lemma.

■

Remark 9.3.3: Big question: what *are* injective modules really? These are pretty nonintuitive objects.

10 | Friday, February 05

See missing first 10m Recall the definition of injectives.

Remark 10.0.1: Over a PID, divisible is equivalent (?) to injective as a module.

Example 10.0.2 ***(?):*** $\mathbb{Q}$ is divisible, and thus an injective $\mathbb{Z}$-module. Similarly $\mathbb{Q}/\mathbb{Z} \rightleftharpoons [0,1) \cap \mathbb{Q}$.

Example 10.0.3 ***(?):*** Let $p \in \mathbb{Z}$ be prime, then $\mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{Q}$ has elements of the form $\sum \frac{a_i}{p^{n_i}}$, and is not divisible. On the other hand, $\mathbb{Z}_{p^\infty} := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \rightleftharpoons \mathbb{Z}[\frac{1}{p}] \cap [0,1)$ is divisible since $p^n \left(\frac{a}{p^n}\right) = a \in \mathbb{Z}$, which equals zero in $\mathbb{Z}_{p^\infty}$. To solve $xr = a/p^n$ with $r, a \in \mathbb{Z}$ and $r \neq 0$, first assume $\gcd(r,p) = 1$ by just dividing through by any common powers of p. This amounts to solving $1 = srtp^n$ where

$s, t \in \mathbb{Z}$:

$$\begin{aligned}\frac{a}{p^n} &= sr\left(\frac{a}{p^n}\right) + tp^n\left(\frac{a}{p^n}\right) \\ &= \left(\frac{sa}{p^n}\right) r \\ &:= xr \in \mathbb{Z}_{p^\infty}.\end{aligned}$$

Fact 10.0.4
Every injective abelian group is isomorphic to a direct sum of copies of $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ for various primes p.

Example 10.0.5 *(?):* $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty}$. To prove this, do induction on the number of prime factors in the denominator.

Exercise 10.0.6 (2.3.2)
$\mathbf{Ab} = \mathbb{Z}\text{-}\mathbf{Mod}$ has enough injectives.

Remark 10.0.7: As a consequence, $\mathbf{Mod}\text{-}R$ has enough injectives for *any* ring R.

10.1 Transferring Injectives Between Categories

Next we'll use our background in projectives to deduce analogous facts for injectives.

Definition 10.1.1 (Opposite Category)
Let $\mathcal{A}$ be any category, then there is an opposite/dual category $\mathcal{A}^{\text{op}}$ defined in the following way:

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$
- $A \to B \in \text{Mor}(\mathcal{A}) \implies B \to A \in \text{Mor}(\mathcal{A}^{\text{op}}$, so

$$\text{Hom}_{\mathcal{A}}(A, B) \cong \text{Hom}_{\mathcal{A}^{\text{op}}}(B, A) f \rightleftharpoons f^{\text{op}}.$$

- We require that if $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, then $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$ where $C \xrightarrow{g^{\text{op}}} B \xrightarrow{f^{\text{op}}} A$.
- $\mathbb{1}_A^{\text{op}} = \mathbb{1}_A$ in $\mathcal{A}^{\text{op}}$.

⚠ **Warning 10.1.2**
Thinking of these as functions won't quite work! For $f : A \to B$, there may not be any map $B \to A$ – you'd need it to be onto to even define such a thing, and if it's not injective there are many choices. Note that initials and terminals are swapped, and since 0 is both. Counterintuitively, $A \to 0 \to B$ is 0, which maps to $B \to 0 \to A = 0^{\text{op}}$.

Remark 10.1.3: Note that $(\,\cdot\,)^{\mathrm{op}}$ switches

- Monics and epis,
- Initial and terminal objects,
- Kernels and cokernels.

Moreover, $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{\mathrm{op}}$ is abelian.

Definition 10.1.4 (Contravariant Functors)
A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

$$\begin{tikzcd}
C_1 \arrow[r, "f"] & C_2 & C_2 \arrow[r, "f^{\mathrm{op}}"] & C_1 & FC_2 \arrow[r, "F(f)"] & FC_1 \\
& \mathcal{C} \arrow[rr, squiggly] & & \mathcal{C}^{\mathrm{op}} \arrow[rr, squiggly] & & \mathcal{D}
\end{tikzcd}$$

In particular, $F(\mathbb{1}) = \mathbb{1}$ and $F(gf) = F(f)F(g)$

Link to Diagram

Example 10.1.5 *(?):* $\mathrm{Hom}_R(\,\cdot\,, A) : \textbf{Mod-}R \to \textbf{Ab}$ is a contravariant functor in the first slot.

Definition 10.1.6 (Left-Exact Functors)
A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is **left exact** if and only if the corresponding covariant functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$: That is, SESs in $\mathcal{A}$ get mapped to long left-exact sequences in $\mathcal{B}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, squiggly, "F(\,\cdot\,)"] & C \arrow[r] & 0 \\
0 \arrow[r] & FC \arrow[r] & FB \arrow[r] & FA \arrow[r] & \cdots
\end{tikzcd}$$

Lemma 10.1.7 *(?).*
If $\mathcal{A}$ is abelian and $A \in \mathcal{A}$, then the following are equivalent:

- A is injective in $\mathcal{A}$.
- A is projective in $\mathcal{A}^{\mathrm{op}}$.
- The contravariant functor $\mathrm{Hom}_{\mathcal{A}}(\,\cdot\,, A)$ is exact.

Lemma 10.1.8 *(?).*
If an abelian category $\mathcal{A}$ has enough injectives, then every $M \in \mathcal{A}$ has an injective resolution:

$$0 \to M \to I^0 \to I^1 \to \cdots .$$

which is an exact cochain complex with each I^n injective. There is a version of the comparison lemma that is proved in roughly the same way as for projective resolutions.

11 Friday, February 05

Next up: how to transport injective resolutions in $\mathbb{Z}$-**Mod** to R-**Mod**.

Observation 10.1.9
If $A \in \mathbf{Ab}$ and $N \in R\text{-}\mathbf{Mod}$ then $\operatorname{Hom}_{\mathbf{Ab}}(N, A) \in \mathbf{Mod}\text{-}R$ in the following way: taking $f : N \to A$ and $r \in R$, define a right action $(f \cdot r)(n) := f(rn)$.

Exercise 10.1.10 (?)
Check that this is a morphism of abelian groups, that this yields a module structure, along with other details. For noncommutative rings, it's crucial that the r is on the right in the action and on the left in the definition.

Lemma 10.1.11 ***(?).***
If $M \in \mathbf{Mod}\text{-}R$, then the following natural map τ is an isomorphism of abelian groups for each $A \in \mathbf{Ab}$:

$$\begin{aligned} \tau : \operatorname{Hom}_{\mathbf{Ab}}(?(M), A) &\to \operatorname{Hom}_{\mathbf{Mod}\text{-}R}(M, \operatorname{Hom}_{\mathbf{Ab}}(R, A)) \\ f &\mapsto \tau(f)(m)(r) := f(mr), \end{aligned}$$

where $m \in M$ and $r \in R$. Note that R is a left R-module, so the hom in the RHS is a right R-module and the hom makes sense.

Exercise 10.1.12 ***(?)***
Check the details here, particularly that the module structures all make sense.

There is a map μ going the other way: $\mu(g)(m) = g(m)(1_R)$ for $m \in M$.

Remark 10.1.13: A quick look at why these maps are inverses:

$$\begin{aligned} \mu(\tau(f)) &= (\tau f)(m)(1_R) \\ &= f(m \cdot 1) \\ &= f(m). \end{aligned}$$

Conversely,

$$\begin{aligned} \tau(\mu(g))(m)(r) &= (\mu g)(mr) \\ &= g(mr)(1) \\ &= g(m \cdot r) && \text{since } g \in \operatorname{Mor}_{R\text{-}\mathbf{Mod}} \\ &= g(m)(r \cdot 1) && \text{by observation earlier} \\ &= g(m)(r). \end{aligned}$$

Remark 10.1.14: The ? functor in the lemma will be the forgetful functor applied to M, yielding an adjoint pair.

11 | Monday, February 08

11.1 Transporting Injectives

Remark 11.1.1: Last time: we had a lemma that for any $M \in \mathbf{Mod}\text{-}R$ and $A \in \mathbf{Ab}$ there is an isomorphism

$$\operatorname{Hom}_{\mathbf{Ab}}(F(M), A) \cong \operatorname{Hom}_{\mathbf{Mod}\text{-}R}(M, \operatorname{Hom}_{\mathbf{Ab}}(R, A)),$$

where $F : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is the forgetful functor.

Definition 11.1.2 (Adjoin)
A pair of functors $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ are **adjoint** is there are natural bijections

$$\tau_{AB} : \operatorname{Hom}_{B}(L(A), B) \xrightarrow{\sim} \operatorname{Hom}_{A}(A, R(B)) \qquad \forall A \in A, B \in B,$$

where *natural* means that for all $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ there is a diagram

$$\begin{tikzcd}
\operatorname{Hom}_{B}(LA', B) \arrow[r, "(Lf)^*"] \arrow[d, "\tau"] & \operatorname{Hom}_{B}(LA, B) \arrow[r, "g_*"] \arrow[d, "\tau"] & \operatorname{Hom}_{B}(LA, B') \arrow[d, "\tau"] \\
\operatorname{Hom}_{A}(A', RB) \arrow[r, "f^*"] & \operatorname{Hom}_{A}(A, RB) \arrow[r, "(Rg)_*"] & \operatorname{Hom}_{A}(A, RB')
\end{tikzcd}$$

Link to Diagram

In this case we say L is **left adjoint** to R and R is **right adjoint** to L.

Remark 11.1.3: The lemma thus says that $\operatorname{Hom}_{\mathbf{Ab}}(R, \cdot\,) : \mathbf{Ab} \to \mathbf{Mod}\text{-}R$ (using that $R \in R\text{-}\mathbf{Mod}$ is a left R-module) is right adjoint to the forgetful functor $\mathbf{Mod}\text{-}R \to \ Ab$.

Remark 11.1.4: Recall that F is **additive** if $\operatorname{Hom}_{\mathcal{B}}(B', B) \to \operatorname{Hom}_{\mathcal{A}}(FB', FB)$ is a morphism of abelian groups for all $B, B' \in \mathcal{B}$.

Proposition 11.1.5 *(?).*
If $R : \mathcal{B} \to \mathcal{A}$ is an additive functor and right adjoint to an exact functor $L : \mathcal{A} \to \mathcal{B}$, then $I \in \mathcal{B}$ injective implies $R(I) \in \mathcal{A}$ is injective. Dually, if $\mathcal{L} : A \to B$ is additive and left adjoint to an exact functor $R : \mathcal{B} \to \mathcal{A}$, then $P \in \mathcal{A}$ projective implies $L(P) \in \mathcal{B}$ is projective.

Corollary 11.1.6 *(?).*
If $I \in \mathbf{Ab}$ is injective, then $\operatorname{Hom}_{\mathbf{Ab}}(R, I) \in \mathbf{Mod}\text{-}R$ is injective.

Proof (?).
This follows from the previous lemma: $\operatorname{Hom}_{\mathbf{Ab}}(R, \cdot\,)$ is right adjoint to the forgetful functor

R-**Mod** $\to$ **Ab** which is certainly exact. This follows from the fact that kernels and images don't change, since these are given in terms of set maps and equalities of sets.

■

Exercise 11.1.7 (2.3.5, 2.3.2)
Show that **Mod**-R has enough injectives, using that **Ab** has enough injectives.

Proof (of proposition).
It suffices to show that the contravariant functor $\operatorname{Hom}_{\mathcal{A}}(\,\cdot\,, RI)$ is exact. We know it's left exact, so we'll show surjectivity. Suppose we have a SES $0 \to A \xrightarrow{f} A'$ which is exact in $\mathcal{A}$. Then $0 \to LA \xrightarrow{Lf} LA'$ is exact, and we can apply hom to obtain the exact sequence

$$\operatorname{Hom}_{\mathcal{B}}(LA', I) \xrightarrow{(LF)^*} \operatorname{Hom}_{\mathcal{B}}(LA, I) \to 0.$$

Applying τ yields

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{B}}(LA', I) \arrow[r, "(Lf)^*"] \arrow[d, "\tau\sim" description] & \operatorname{Hom}_{\mathcal{B}}(LA, I) \arrow[r] \arrow[d, "\tau\sim" description] & 0 \arrow[d, dashed] \\
\operatorname{Hom}_{\mathcal{A}}(A', RI) \arrow[r, "f^*"] & \operatorname{Hom}_{\mathcal{A}}(A, RI) \arrow[r, dashed] & 0
\end{tikzcd}$$

Link to Diagram

- The top sequence is exact since I is injective in $\mathcal{B}$
- Therefore the bottom map is onto (diagram chase)

■

11.2 2.4: Left Derived Functors

Remark 11.2.1: Goal: define left derived functors of a right exact functor F, with applications the bifunctor $\cdot \otimes_R \cdot$, which is right exact and covariant in each variable. We're ultimately interested in Hom functors and Ext, but this is slightly more technical since it's covariant in one slot and contravariant in the other, so focusing on this functor makes the theory slightly easier to develop. This can be fixed by switching $\mathcal{C}$ with $\mathcal{C}^{\text{op}}$ once in a while. Everything for left derived functors will have a dual for right derived functors.

Remark 11.2.2: Let $\mathcal{A}, \mathcal{B}$ be abelian categories where $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor (which implicitly assumes F is additive). We want to define $L_i F : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$.

Definition 11.2.3 (Left Derived Functors)
For $A \in \mathcal{A}$, fix once and for all a projective resolution $P \xrightarrow{\varepsilon} A$, where $P_{<0} = 0$. Then define $FP = (\cdots \to F(P_1) \xrightarrow{Fd_1} F(P_0) \to 0$, noting that A no longer appears in this complex. We can write $H_0(FP) = FP_0/(Fd_1)(FP_1)$, and define

$$(L_iF)(A) \coloneqq H_i(FP).$$

Remark 11.2.4: Note that $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$ is exact, and since F is right exact, it can be shown that the following is a SES: $FP_1 \xrightarrow{Fd_1} FP_0 \xrightarrow{F\varepsilon} FA \to 0$. We can use this to compute the original homology, despite it not having any homology itself! From this, we can extra $L_0(A) \coloneqq FP_0/(Fd_1)(FP_1) = FP_0/\ker F(\varepsilon)$ using exactness at FP_0, and by the 1st isomorphism theorem this is isomorphic to the image FA using surjectivity. So $L_0F \cong F$.

Theorem 11.2.5 *(?).*
$L_iF : \mathcal{A} \to \mathcal{B}$ are additive functors.

Proof (?).
First, $\mathbb{1}_P : P \to P$ lifts $\mathbb{1}_A : A \to A$ since it yields a commuting ladder, and $F(\mathbb{1}) = \mathbb{1}$, so $(L_if)(\mathbb{1}) = \mathbb{1}$. Then in the following diagram, the outer rectangle commutes since the inner squares do:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\tilde{f}"] & A \arrow[d, "f"] \\
P' \arrow[r] \arrow[d, "\tilde{g}"] & A' \arrow[d, "g"] \\
P'' \arrow[r] & A''
\end{tikzcd}$$

Link to Diagram

So $\tilde{g} \circ \tilde{f}$ lifts $g \circ f$ and therefore $g_* f_* = (gf)_*$. Thus L_iF is a functor. That they are additive comes from checking the following diagram:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, shift right, "\tilde{g}"'] \arrow[d, shift left, "\tilde{f}"] & A \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \\
Q \arrow[r] & B
\end{tikzcd}$$

Link to Diagram

Then $\tilde{f} + \tilde{g}$ lifts $f + g$, and H_i is an additive functor: $(F\tilde{f} + F\tilde{g})_* = (F\tilde{f})_* + (F\tilde{g})_*$. Thus L_iF is additive.

■

12 | Wednesday, February 10

Remark 12.0.1: Setup: Let $\mathcal{A}, \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ has enough projectives. Let $P \xrightarrow{\varepsilon} A \in \mathcal{A}$ be a projective resolution, we want to define the left derived functors $L_i F(A) := H_i(FP)$.

Lemma 12.0.2 *(?).*
$L_i F(A)$ is well-defined up to natural isomorphism, i.e. if $Q \to A$ is a projective resolution, then there are canonical isomorphism $H_i(FP) \xrightarrow{\sim} H_i(FQ)$. In particular, changing projective resolutions yields a new functor $\widehat{L}_i F$ which are naturally isomorphic to F.

Proof (?).
We can set up the following situation

$$\begin{tikzcd}
P \arrow[r, "\varepsilon_P"] \arrow[d, "\exists f"] & A \arrow[r] \arrow[d, "\mathbb{1}_A"] & 0 \\
Q \arrow[r, "\varepsilon_Q"] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Here f exists by the comparison theorem, and thus there are induced maps $f_* : H_*(FP) \to H_*(FQ)$ by abuse of notation – really, this is more like $(f_*)_i = H_u(Ff)$. We're using that both F and H_i are both additive functors. Note that the lift f of $\mathbb{1}_A$ is not unique, but any other lift is chain homotopic to f, i.e. $f - f' = ds + sd$ where $s : P \to Q[1]$. So they induce the same maps on homology, i.e. $f'_* = f_*$. Thus the isomorphism is canonical in the sense that it doesn't depend on the choice of lift.
Similarly there exists a $g : Q \to P$ lifting $\mathbb{1}_A$, and so gf and $\mathbb{1}_P$ are both chain maps lifting $\mathbb{1}_A$, since it's the composition of two maps lifting $\mathbb{1}_A$. So they induce the same map on homology by the same reasoning above. We can write

$$g_* f_* = (gf)_* = (\mathbb{1}_{FP})_* = \mathbb{1}_{H_*(FP)},$$

and similarly $f_* g_* = \mathbb{1}_{H_*(FQ)}$, making f_* an isomorphism. ■

Corollary 12.0.3 *(?).*
If A is projective, then $L^{>0} FA = 0$.

Proof (?).
Use the projective resolution $\cdots \to 0 \to A \xrightarrow{\mathbb{1}_A} A \to 0 \to \cdots$. In this case $H_{>0}(FP) = 0$. ■

Remark 12.0.4: This is an interesting result, since it doesn't depend on the functor! Short aside on F-acyclic objects – we don't need something as strong as a *projective* resolution.

Definition 12.0.5 (F-acyclic objects)
An object $Q \in \mathcal{A}$ is **F-acyclic** if $L_{>0}FQ = 0$.

Remark 12.0.6: Note that projective implies F-acyclic for every F, but not conversely. For example, flat R-modules are acyclic for $\cdot \otimes_R \cdot$. In general, flat does not imply projective, although projective implies flat.

Definition 12.0.7 (F-acyclic resolutions)
An **F-acyclic resolution** of A is a left resolution $Q \to A$ for which every Q_i is F-acyclic.

Remark 12.0.8: One can compute $L_iF(A) \cong H_i(FQ)$ for any F-acyclic resolution. For the L_iF to be functors, we need to define them on maps!

Lemma 12.0.9 *(?).*
If $f : A \to A'$, there is a natural associated morphism $L_iF(f) : L_iF(A) \to L_iF(A')$.

Proof (?).
Again use the comparison theorem:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\exists \tilde{f}"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
P' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Then there is an induced map $\tilde{f}_* : H_*(FP) \to H_*(FP')$, noting that one first needs to apply F to the above diagram. As before, this is independent of the lift using the same argument as before, using the additivity of F and H_* and the chain homotopy is pushed through F appropriately. So set $(L_iF)(f) := (\tilde{f}_*)_i$.

■

We can now pick up the theorem from the end of last time:

Theorem 12.0.10 *(?).*
$L_iF : \mathcal{A} \to \mathcal{B}$ are additive functors.

Proof (?).
Done last time!

■

Theorem 12.0.11 *(?).*
Using the same assumptions as before, given a SES

$$0 \to A' \to A \to A'' \to 0$$

there are natural connecting maps δ yielding a LES

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\delta_{i+1}"'] \\
L_iF(A') \arrow[r] & L_iF(A) \arrow[r] & L_iF(A'') \arrow[dll, "\delta_i"'] \\
L_{i-1}F(A') \arrow[r, dashed] & \cdots \arrow[r, dashed] & \cdots \arrow[dll, "\delta_1"'] \\
FA' \arrow[r] & FA \arrow[r] & FA'' \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proof (?).
Using the Horseshoe lemma, we can obtain the following map:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
P' \arrow[r] \arrow[d] & A' \arrow[d] \\
P \arrow[r, dashed, "\exists"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
P'' \arrow[r] & A'' \arrow[d] \\
 & 0
\end{tikzcd}$$

Link to Diagram

So we get a SES of complexes over $\mathcal{A}$, $0 \to P' \to P \to P'' \to 0$. One can use that $P = P' \oplus P''$, or alternatively that each P''_n is a projective R-module, to see that there are splittings

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, "f"] & P \arrow[l, dashed, bend right, "f'"'] \arrow[r, "g"] & P'' \arrow[l, dashed, bend right, "g'"'] \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Note that this can be phrased in terms of $g'g = \mathbb{1}, f'f = \mathbb{1}$, or $g'g + f'f = \mathbb{1}$. Since F is additive, it preserves all of these relations, particularly the ones that define being split exact. So additive functors preserve split exact sequences. Thus $0 \to FP' \to FP \to FP'' \to 0$ is still split exact, even though F is only right exact. Now take homology and use the LES in homology to get the desired LES above, and δ is the connecting morphism that comes from the snake lemma.

Proving naturality: we start with the following setup.

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "g'"] & A \arrow[r] \arrow[d, "g"] & A'' \arrow[r] \arrow[d, "g''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

Naturality of δ will be showing that the following square commutes:

$$\begin{tikzcd}
L_{i+1}F(A'') \arrow[r, "\delta"] \arrow[d] & L_iF(A') \arrow[d] \\
L_{i+1}F(B'') \arrow[r, "\delta"] & L_iF(B')
\end{tikzcd}$$

We now apply the horseshoe lemma several times:

$$\begin{tikzcd}
0 & & P' \arrow[rr] \arrow[dr] \arrow[dddd, dotted, "\exists G'"] & & P \arrow[rr] \arrow[dr, dashed] \arrow[dddd, dashed, "\exists G"] & & P'' \arrow[dr] \arrow[dddd, dotted, "\exists G''"] & & 0 & \\
& 0 \arrow[rr] & & A' \arrow[rr] \arrow[d, "g'"] & & A \arrow[rr] \arrow[d, "g"] & & A'' \arrow[rr] \arrow[d, "g''"] & & 0 \\
& 0 \arrow[rr] & & B' \arrow[rr] & & B \arrow[rr] & & B'' \arrow[rr] & & 0 \\
& & & & & & & & & \\
0 & & Q' \arrow[rr] \arrow[uur] & & Q \arrow[rr] \arrow[uur, dashed] & & Q'' \arrow[uur] & & 0 &
\end{tikzcd}$$

It turns out (details omitted see Weibel p. 46) that G can be chosen such that we get a commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "G'"] & P \arrow[r] \arrow[d, "G"] & P'' \arrow[r] \arrow[d, "G''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

We proved naturality of the connecting maps ∂ in the corresponding LES in homology in general (see prop. 1.3.4). This translates to naturality of the maps $\delta_i : L_i(A'') \to L_{i-1}(A')$. ■

Remark 12.0.12: See exercise 2.4.3 for "dimension shifting". This is a useful tool for inductive arguments.

13 | Friday, February 12

Remark 13.0.1: Last time: right-exact functors have left-derived functors where a SES induces a LES. The functors are *natural* with respect to the connecting morphisms in the sense that certain squares commute. Weibel refers to $\{L_i F\}_{i\geq 0}$ as a **homological δ-functor**, i.e. anything that takes SESs to LESs which are natural with respect to connecting morphism.

13.1 Aside: Natural Transformations

Definition 13.1.1 (Natural Transformation)
Given functors $F, G, \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \implies G$ is the following data:

- For all $C \in \mathcal{C}$ there is a map $F(C) \xrightarrow{\eta_C} G(C) \in \operatorname{Mor}(\mathcal{D})$, sometimes referred to as $\eta(C)$.
- If $C \xrightarrow{f} C' \in \operatorname{Mor}(\mathcal{C})$, there is a diagram

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\eta_C"] & FC' \arrow[d, "\eta_{C'}"] \\
GC \arrow[r, "Gf"] & GC'
\end{tikzcd}$$

Link to Diagram

- η is a **natural isomorphism** if all of the η_C are isomorphisms, and we write $F \cong C$.

Definition 13.1.2 (Equivalence of Categories)
A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if and only if there exists a $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathbb{1}_{\mathcal{C}}$ and $FG \cong \mathbb{1}_{\mathcal{D}}$.

Example 13.1.3 *(?):* A category $\mathcal{C}$ is **small** if $\operatorname{Ob}(\mathcal{C})$ is a set, then take $\mathcal{C} := \textbf{Cat}$ whose objects are categories and morphisms are functors. Note that in all categories, all collections of morphisms should be sets, and the small condition guarantees it. In this case, natural transformations $\eta : F \to G$

is an additional structure yielding morphisms of morphisms. These are called **2-morphisms**, and in this entire structure is a **2-category**, and our previous notion is referred to as a **1-category**.

Theorem 13.1.4 *(?).*
Assume $\mathcal{A}, \mathcal{B}$ are abelian and $F : \mathcal{A} \to \mathcal{B}$ is a right-exact additive functor where $\mathcal{A}$ has enough projectives. Then the family $\{L_i F\}_{i \geq 0}$ is a *universal δ-functor* where $L_0 F \cong F$ is a natural isomorphism.

Remark 13.1.5: Here *universal* means that if $\{T_i\}_{i \geq 0}$ is also a δ-functor with a natural *transformation* (not necessarily an isomorphism) $\varphi_0 : T_0 \to F$, then there exist unique morphism of δ-functors $\{\varphi_i : T_i \to L_i F\}_{i \geq 0}$, i.e. a family of natural transformations that commute with the respective δ maps coming from both the T_i and the $L_i F$, which extend φ_0. This will be important later on when we try to show Ext and Tor are functors in either slot.

Proof (?).
Assume $\{T_i\}_{i \geq 0}$ and φ_0 are given, and assume inductively that $n > 0$ and we've defined $\varphi_i : T_i \to F$ for $0 \leq i < n$ which commute with the δ maps. Step 1: given $A \in \mathcal{A}$, fix a reference exact sequence: pick a projective mapping onto A and its kernel to obtain

$$0 \to K \to P \to A \to 0.$$

Applying the functors T_i and $L_i F$ yields

$$\begin{tikzcd}
 & x \arrow[r, dotted] & k \arrow[r, dotted] \arrow[ddd, dotted, bend right] & 0 \arrow[ddd, dotted, bend right] \\
 & T_n A \arrow[r, "\delta"] \arrow[d, dashed, "\exists \varphi_{n-1}(A)"] & T_{n-1} K \arrow[r] \arrow[d, "\varphi_{n-1}(K)"] & T_{n-1} P \arrow[d, "\varphi_{n-1}(P)"] \\
L_n FP = 0 \arrow[r] & L_n FA \arrow[r, hook, "\delta"'] & L_{n-1} FK \arrow[r] & L_{n-1} FP \\
 & \exists ! y \coloneqq \varphi_{n-1}(x) \arrow[r, dotted] & \ell \arrow[r, dotted] & 0
\end{tikzcd}$$

Link to Diagram

So define $\varphi_n(A)(x) \coloneqq y$, which makes the LHS square commute by construction. Note that $L_n FP$ vanishes (as do all its higher derived functors) since P is projective.

⚠ **Warning 13.1.6**
The map $\varphi_n(A)$ could depend on the choice of P!
We now want to show that φ_n is a natural transformation. Supposing $f : A' \to A$, we need to show φ_n commutes with f.

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, dashed, "\exists h"] & P' \arrow[r] \arrow[d, dashed, "\exists g"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Since P' is projective, we can lift f to $P' \to P$, and then define h to be the restriction of g to $K' \to K$.

$$\begin{tikzcd}
T_nA' \arrow[rrr, "T_nf"] \arrow[dr, "\delta'"] \arrow[ddd, "\varphi_n(A')"] & & & T_nA \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
& T_{n-1}K' \arrow[r, "T_{n-1}h"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}K \arrow[d, "\varphi_{n-1}"] & \\
& L_{n-1}FK' \arrow[r, "L_{n-1}Fh"] & L_{n-1}FK & \\
L_nFA' \arrow[ur, "\delta'"] \arrow[rrr, "L_nFf"] & & & L_nF(A) \arrow[ul, "\delta"']
\end{tikzcd}$$

Link to Diagram

Note that all of the quadrilaterals here commute. The middle top and bottom come from naturality of T_*, L_*F with respect to δ, the RHS/LHS due to the construction of the φ_n, and φ_{n-1} is natural by the inductive hypothesis. Now in order to traverse $T_nA' \to T_nA \to L_nF(A)$, we can pass the path through one commuting square at a time to make it equal to $T_nA' \to L_nFA' \to L_nFA$, so the outer square commutes. We have

$$\delta\varphi_n(A)T_nF = \delta L_nFf\varphi_n(A'),$$

and since δ is monic (using the previous vanishing due to projectivity), so we can cancel on the left and this yields the definition of naturality.

Corollary 13.1.7 *(?).*
The definition of $\varphi_n(A)$ does not depend on the choice of P. Taking $A' = A$ in the previous argument with $f = \mathbb{1}$, suppose $P' \neq P$. Then $T_nf = \mathbb{1} = L_nFf$ and setting $\varphi_n'(A)$ to be the map coming from P', we get $\varphi_n'(A) = \varphi_n(A)$ using the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, dashed, "\exists h"] & P' \arrow[r] \arrow[d, dashed, "\exists g"] & A \arrow[r] \arrow[d, "\mathbb{1}"] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

So the $\varphi_n(A)$ are uniquely defined. We now want to show that φ_n commutes with the δ_n coming from an *arbitrary* SES instead of a fixed reference SES.

$$\begin{tikzcd}
T_nA \arrow[rr, "\delta_{(2)}"'] \arrow[d, "="] \arrow[dr, phantom, "T_* \text{ a } \delta \text{ functor}"] & & T_{n-1}C \arrow[ddd, "\varphi_{n-1}"] \\
T_nA \arrow[r, "\delta_{(2)}"] \arrow[d, "\varphi_n"] \arrow[dr, phantom, "\text{reference}"] & T_{n-1}K \arrow[ur, "T_{n-1}h"] \arrow[d, "\varphi_{n-1}"'] \arrow[dr, phantom, "\varphi_{n-1}\text{natural}"] & \\
L_nFA \arrow[r, "\delta_{(1)}"] \arrow[dr, phantom, "L_*F \text{ a } \delta \text{ functor}"] & L_{n-1}FA \arrow[dr, "L_{n-1}Fh"] & \\
L_nFA \arrow[u, "="] \arrow[rr, "\delta_{(1)}"'] & & L_{n-1}FC
\end{tikzcd}$$

Link to Diagram

This diagram commutes for the reasons indicated, and commutativity of the outside square implies that φ_n commutes with the δ coming from any SES.

See section 2.4 in Weibel.

$\blacksquare$

14 | Monday, February 15

14.1 2.5: Right-Derived Functors

Remark 14.1.1: Today: right-derived functors of a left-exact functor. Luckily we can use some opposite category tricks which save us some work of re deriving everything.

Definition 14.1.2 (Right Derived Functors)
Let $F : \mathcal{A} \to \mathcal{B}$ be left-exact where $\mathcal{A}$ has enough injectives. Given $A \in \mathcal{A}$, fix an injective resolution $0 \to A \xrightarrow{\varepsilon} I$ and define

$$R^i\mathcal{F} := H^i(FA) \qquad\qquad i \geq 0.$$

Remark 14.1.3: Then

$$0 \to FA \xrightarrow{F\varepsilon} FI^0 \xrightarrow{Fd^0} FI^1$$

is exact, and

$$R^0FA = \ker F(d^0) / \langle 0 \rangle = \operatorname{im} F\varepsilon \cong FA,$$

and so there is naturally an isomorphism $R^0F \cong F$. Observe that F yields a right-exact functor $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$, where we note that $F^{\mathrm{op}}(f^{\mathrm{op}}) = F(f)^{\mathrm{op}}$. Note that taking the opposite category sends injectives to projectives and so $\mathcal{A}^{\mathrm{op}}$ has enough projectives. This means that L_iF^{op} are defined using the projective resolution I, so we have

$$R^iF(A) = (L_iF^{\mathrm{op}})^{\mathrm{op}}.$$

Thus all results about left-derived functors translate to right-derived functors:

- R_iF is independent of the choice of injective resolution, up to a natural isomorphism.
- If A is injective, then $R^{i>0}F(A) = 0$.
- The collection $\left\{R^iF\right\}_{i\geq 0}$ forms a universal cohomological δ-functor for F.
- An object $Q \in \mathcal{A}$ is F**-acyclic** if $R^{>0}F(Q) = 0$.
- R^iF can be computed using F-acyclic objects instead of injective resolutions.

Definition 14.1.4 (?)
Fix a right R-module $M \in$ **Mod**-R, then $F := \mathrm{Hom}_{\textbf{Mod-}R}(M, \cdot\,) : \textbf{Mod-}R \to \textbf{Ab}$ is a left-exact functor. Its right-derived functors are **ext functors** and denoted $\mathrm{Ext}^i_{\textbf{Mod-}R}(M, \cdot\,)$.

Example 14.1.5 ***(?):***

$$\mathrm{Ext}^i_{\textbf{Mod-}R}(M, A) = (R^iF)(A) = [R^i\mathrm{Hom}_{\textbf{Mod-}R}(M, \cdot\,)](A).$$

Remark 14.1.6: Exercises 2.5.1, 2.5.2 are important extensions of our existing characterizations of injectives and projectives in **Mod**-R. These upgrade the characterization involving Hom to one involving Ext. [3]

Remark 14.1.7: Fix $B \in$ **Mod**-R and consider $G := \mathrm{Hom}_{\textbf{Mod-}R}(\,\cdot\,, B) : \textbf{Mod-}R \to \textbf{Ab}$. Then G is still left-exact, but is now *contravariant.* We can regard it as a covariant functor left-exact functor $G : \textbf{Mod-}R^{\mathrm{op}} \to \textbf{Ab}$. So we define $R^iG(A)$ by an injective resolution of A in $\mathcal{A}^{\mathrm{op}}$, and this is the same as a projective resolution of A in $\mathcal{A}$. So apply G and take cohomology. It turns out that

$$R^i\mathrm{Hom}_{\textbf{Mod-}R}(\,\cdot\,, B) \cong R^i\mathrm{Hom}_{\textbf{Mod-}R}(A, \cdot\,)(B) := \mathrm{Ext}^i_{R\textbf{-Mod}}(A, B),$$

so we can use the same notation $\mathrm{Ext}^i_R(\,\cdot\,, B)$ for both cases.

---

[3] Note the typo in 2.5.1.3, it should say the following: "B is $\mathrm{Hom}_R(A, \cdot\,)$ is acyclic for all A."

14.2 2.6: Adjoint Functors and Left/Right Exactness

Slogan 14.2.1
$\cdot$ adjoints are $\cdot^{\text{op}}$ exact, since $\cdot$ adjoints have $\cdot$-derived functors.

Theorem 14.2.2 ***(?).***
Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "R"] \end{tikzcd}$$

be an adjoint pair of functors. Then there exists a natural isomorphism

$$\tau_{AB} : \operatorname{Hom}_{\mathcal{B}}(LA, B) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(A, RB) \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Moreover,

- L is right exact, and
- B is left exact.

Proposition 14.2.3 ***(1.6: Yoneda).***
A sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact in $\mathcal{A}$ if and only if for all $M \in \operatorname{Ob}(\mathcal{A})$, the sequence

$$\operatorname{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\alpha^* := \alpha \circ \cdot} \operatorname{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\beta^* := \beta \circ \cdot} \operatorname{Hom}_{\mathcal{A}}(M, C)$$

is exact.

Proof (?).

1. Take $M = A$, then $0 = \beta^* \alpha^* (\mathbb{1}_A) = \beta \alpha \mathbb{1} = \beta \alpha$. Thus $\operatorname{im} \alpha \subseteq \ker \beta$.
2. Take $M = \ker \beta$ and consider the inclusion $\iota : \ker M \hookrightarrow B$, then $\beta^*(\iota) = \beta \iota = 0$ and thus $\iota \in \ker \beta^* = \operatorname{im} \alpha^*$. So there exists $\sigma \in \operatorname{Hom}(\ker \beta, A)$ such that $\iota = \alpha^*(\sigma) := \alpha \sigma$, and thus $\ker \beta = \operatorname{im} \iota \subset \operatorname{im} \alpha$.

Thus $\ker \beta = \operatorname{im} \alpha$, yielding exactness of the bottom sequence. ■

Proof (of theorem).
We'll first prove that R is left-exact. Take a SES in B, say

$$0 \to B' \to B \to B'' \to 0.$$

Apply the left-exact covariant functor $\operatorname{Hom}_{\mathcal{B}}(LA, \cdot\,)$ followed by τ:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathcal{B}}(LA, B') \arrow[r] \arrow[d, squiggly, "\tau_{AB}" description] & \operatorname{Hom}_{\mathcal{B}}(LA, B) \arrow[r] \arrow[d, squiggly, "\tau_{AB}" description] & \operatorname{Hom}_{\mathcal{B}}(LA, B'') \arrow[d, squiggly, "\tau_{AB}" description] \\
0 \arrow[r, dashed] & \operatorname{Hom}_{\mathcal{B}}(A, RB') \arrow[r] & \operatorname{Hom}_{\mathcal{B}}(A, RB) \arrow[r] & \operatorname{Hom}_{\mathcal{B}}(A, RB'')
\end{tikzcd}$$

Link to Diagram

The bottom sequence is exact by naturality of τ. Now applying the Yoneda lemma, we obtain an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(A, RB') \to \operatorname{Hom}_{\mathcal{A}}(A, RB) \to \operatorname{Hom}_{\mathcal{A}}(A, RB'').$$

So R is left exact. Now $L^{\mathrm{op}} : \mathcal{A} \to \mathcal{B}$ is right adjoint to R^{op}, so L^{op} is left exact and thus L is right exact.

■

14.3 Tensor Product Functors and Tor

Remark 14.3.1: Let

- $R, S \in \mathbf{Ring}$,
- $B \in (R, S)\text{-}\mathbf{biMod}$,
- $C \in S\text{-}\mathbf{Mod}$.

Then $\operatorname{Hom}_S(B, C) \in \mathbf{Mod}\text{-}R$ in a natural way: given $f : B \to C$, define $(f \cdot r)(b) = f(rb)$.

Exercise 14.3.2 (?)
Check that this is a well-defined morphism of right S-modules.

Remark 14.3.3: We saw this structure earlier with $S = \mathbb{Z}$, see p.41.

Proposition 14.3.4 ***(?).***
Fix R, S and ${}_R B_S$ as above. Then for every $A \in \mathbf{Mod}\text{-}R$ and $C \in \text{--}\mathbf{Mod}S$ there is a natural isomorphism

$$\begin{aligned}
\tau : \operatorname{Hom}_S(A \otimes_R B, C) &\xrightarrow{\sim} \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C)) \\
f &\mapsto g(a)(b) = f(a \otimes b) \\
f(a \otimes b) = g(a)(b) &\mapsfrom g.
\end{aligned}$$

Note that the tensor product is a right S-module, and the hom on the right is a right R-module, so these expressions make sense. Here B is fixed, so A and C are variables and this is a statement about bifunctors

$$\cdot \otimes_R B : \mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}S,$$

which is left adjoint to

$$\mathrm{Hom}_S(B, \cdot\,) : \mathbf{Mod}\text{-}S \to \mathbf{Mod}\text{-}R.$$

So the former is a left adjoint and the latter is a right adjoint, so by the theorem, $\cdot \otimes_R B$ is right exact.

Remark 14.3.5: If B is only a left R-module, we can always take $S = \mathbb{Z}$, which makes this into a functor

$$\cdot \otimes_R B : \mathbf{Mod}\text{-}R \to \mathbf{Ab}.$$

Since this is a right exact functor from a category with enough injectives, we can define left-derived functors.

Definition 14.3.6 (?)
Let $B \in (R,S)\text{-}\mathbf{biMod}$ and let

$$T(\,\cdot\,) \coloneqq \cdot \otimes_R B : \mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}S.$$

Then define $\mathrm{Tor}_n^R(A,B) \coloneqq L_n T(A)$.

Remark 14.3.7: Note that these are easier to work with, since they're covariant in both variables.

15 | Friday, February 19

Remark 15.0.1: We looked at $B \in (R,S)\text{-}\mathbf{biMod}$ and showed $\cdot \otimes_R B : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ is left adjoint to hom, and has left-derived functors $\mathrm{Tor}_n^R(\,\cdot\,, B) \coloneqq L_n(\,\cdot \otimes_R B)$.

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[r, bend left, "\cdot \otimes_R B"] \arrow[r, phantom, "\perp"] & S\text{-}\mathbf{Mod}. \arrow[l, bend left, "\mathrm{Hom}_S(B, \cdot\,)"] \end{tikzcd}$$

Note that $\mathrm{Tor}_0^R(A,B) \cong A \otimes_R B$.

Remark 15.0.2: $A \otimes_R \cdot$ is also right exact, and it turns out that

$$L_n(A \otimes_R \cdot\,)(B) \cong L_n(\,\cdot \otimes_R B)(A).$$

So unambiguously denote either of this left derived functors as $\mathrm{Tor}_n(A,B)$.

15

Friday, February 19

15.1 Limits and Colimits

Definition 15.1.1 (Functor Category)
Given categories $\mathcal{I}, \mathcal{A}$, define a **functor category** $\mathcal{A}^{\mathcal{I}}$ by

- $\mathrm{Ob}(\mathcal{A}^{\mathcal{I}})$: functors $A : \mathcal{I} \to \mathcal{A}$.
- $\mathrm{Mor}(\mathcal{A}^{\mathcal{I}})$: natural transformations $\eta : A \to B$ between functors.

$\mathcal{I}$ is thought of as an index category, and we'll write $A_i := A(i) \in \mathcal{A}$ for $i \in \mathcal{I}$. If $\alpha : i \to j$ is a morphism in I, then denote $A(\alpha) := \alpha_*$, which is the morphism defined by the following:

$$\begin{tikzcd}
A_i \arrow[r, "\alpha_*"] \arrow[d, "\eta_i" description] & A_j \arrow[d, "\eta_j" description] \\
B_i \arrow[r, "\alpha_*"] & B_j
\end{tikzcd}$$

Link to Diagram

Composition is defined by $A \xrightarrow{\eta} B \xrightarrow{\zeta} C$ is given by $(\zeta_\eta)_i = \zeta_i \circ \eta_i$. We need the collection of morphisms to be sets, so we'll require $\mathcal{I}$ to be a *small category* (i.e. the class of objects forms a set).

Example 15.1.2*(Poset Category):* Take $(I, \leq)$ a poset (which is reflexive, antisymmetric, transitive, but not every two elements are comparable), define a category by

- $\mathrm{Ob}(\mathcal{I}) = I$
- $|\mathrm{Hom}_{\mathcal{I}}(i,j)| \leq 1$, and $i \to j \iff i \leq j$

Note that if $i \not\leq j$, then $\mathrm{Hom}_{\mathcal{I}}(i,j) = \emptyset$.

Remark 15.1.3: Both $\mathcal{A}, \mathcal{A}^{\mathcal{I}}$ are small, so we can consider functors between them.

Definition 15.1.4 (Diagonal Functor)
The **diagonal functor** is defined as $\Delta : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$ where for $B \in \mathcal{A}$ the functor $\Delta(B)$ is the constant functor, i.e. $\Delta(B)_i = B$ for all $i \in \mathcal{I}$. All morphism are sent to the identity, i.e. $i \xrightarrow{\alpha} j \xrightarrow{\Delta(B)} B \xrightarrow{\mathbb{1}_B} B$.

Work out how morphisms work here with respect to natural transformations.

Definition 15.1.5 (Colimit)
The **colimit** of a functor $A : \mathcal{I} \to \mathcal{A}$ is an object $C \in \mathcal{A}$ which we'll denote $\mathrm{colim}_{i \in \mathcal{I}} A_i$, along with a natural transformation $\eta : A \to \Delta(C)$ which is universal among natural transformations

of the form $\theta : A \to \Delta(B)$ for $B \in \mathcal{A}$. The unique map in the universal property is from $C \to B$, and we have the following situation:

$$\begin{tikzcd}
\mathcal{I} & & \\
i \arrow[d, "\alpha"] & A_i \arrow[r, "\eta_i"'] \arrow[d, "\alpha_*"] & C \arrow[d, equal] \\
j & A_j \arrow[r, "\eta_j"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
A_i \arrow[dr, "\eta_i"'] \arrow[drr, bend left, "\theta_i"] \arrow[dd, "\alpha_*"] & & \\
& C \arrow[r, dashed, "\exists!\gamma"] & B \\
A_j \arrow[ur, "\eta_j"] \arrow[urr, bend right, "\theta_j"'] & &
\end{tikzcd}$$

Link to Diagram

Example 15.1.6 ***(?):*** Let $(I, \leq)$ be a poset and take $\mathcal{I}$ its poset category. Then there are morphisms $i \to j \iff i \leq j$, and we have a diagram

$$\begin{tikzcd}
A_i \arrow[dr, "\eta_i"] \arrow[drr, bend left, "\theta_i"] & & \\
& C \arrow[r, dashed, "\exists!\gamma"] & D \\
A_j \arrow[ur, "\eta_j"'] \arrow[urr, bend right, "\theta_j"] & &
\end{tikzcd}$$

Link to Diagram

This is the **direct limit**. Note that for a poset of category of subsets, this ends up being the union.

Example 15.1.7 ***(?):*** Let $\text{Ob}(\mathcal{I}) = \{1, 2\}$, and take two maps, one of which we'll label by "0":

$$\begin{tikzcd}
1 \arrow[d, shift right] \arrow[d, shift left=2, "0"] \\
2
\end{tikzcd}$$

Link to Diagram

Suppose now that $\mathcal{A}$ is an abelian category, and suppose we're given a morphism $A_1 \xrightarrow{f} A_2$ in $\mathcal{A}$. Define $A \in \mathcal{A}^{\mathcal{I}}$, and define a functor

$$\begin{tikzcd}
1 \arrow[dd, shift right] \arrow[dd, shift left, "0"] & A_1 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "0"] \arrow[rrd, bend left, "\theta_1"] & & \\
& & & B \\
2 & A_2 \arrow[rru, bend right, "\theta_2"'] & &
\end{tikzcd}$$

Link to Diagram

By commutativity,

- $\theta_2 \circ 0 = \theta_1 \implies \theta_1 = 0$
- $\theta_2 \circ f = \theta_1 = 0.$

So suppose there was a colimit C, then it'd fit into this diagram as follows:

$$\begin{tikzcd}
1 \arrow[dd, shift right] \arrow[dd, shift left, "0"] & A_1 \arrow[dd, shift right, "f"'] \arrow[dd, shift left, "0"] \arrow[rd, "0"] \arrow[rrd, bend left, "\theta_1"] & & \\
& & C \arrow[r, dashed] & B \\
2 & A_2 \arrow[ru, "p"] \arrow[rru, bend right, "\theta_2"'] & &
\end{tikzcd}$$

Link to Diagram

Note that C is precisely the cokernel of f!

Remark 15.1.8: Think about this last diagram: what happens when you mod out by larger modules?

Exercise 15.1.9 (Colimits always exist)
Suppose I is a discrete category, i.e. $\mathrm{Hom}(i,j) = \emptyset$ unless $i = j$, in which case $\mathrm{Hom}(i,i) = \{\mathbb{1}_i\}$. Supposing that $A : I \to \mathcal{A}$, show that $\mathrm{colim}_{i\in\mathcal{I}} = \coprod_i A_i$.

Definition 15.1.10 (?)
A category $\mathcal{A}$ is **cocomplete** if every colimit $\mathrm{colim}_{i\in\mathcal{I}} A_i$ exists for every $A \in \mathcal{A}^{\mathcal{I}}$ and all small categories $\mathcal{I}$.

Exercise 15.1.11 (Taking colimits defines a functor for cocomplete categories)
Show that when $\mathcal{A}$ is cocomplete, $\mathrm{colim} : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$ defines a functor.

Exercise 15.1.12 (Weibel 2.6.4)
Show that the functor colim is left-adjoint to the diagonal functor Δ, so there is an adjunction

$$\begin{tikzcd}
\mathcal{A}^{\mathcal{I}} \arrow[r, shift left, "\mathrm{colim}"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "\Delta"]
\end{tikzcd}.$$

Thus when $\mathcal{A}$ is abelian and colim exists, it is right-exact (since left-adjoints are always right-exact). Note that it's not exact in general.

Proposition 15.1.13 ***(Cocomplete iff all coproducts exist).***
For any abelian category $\mathcal{A}$, the following are equivalent:

1. $\coprod A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$ (*set-indexed coproducts*).
2. $\mathcal{A}$ is cocomplete.

Remark 15.1.14: We'll prove this next time, note that $2 \implies 1$ since coproducts are special cases of limits.

16 | Monday, February 22

16.1 Colimits and Adjoint

Proposition 16.1.1 ***(?).***
Assume $\mathcal{A}$ is abelian so we have cokernels for maps. TFAE:

1. $\bigoplus A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$.
2. $\mathcal{A}$ is cocomplete, i.e. $\mathrm{colim}_{i\in I} A_i$ exists for every functor $\mathcal{I} \to \mathcal{A}$ with $\mathcal{I}$ small.

Proof (?).
Note that (1) is a special case of (2), so it suffices to show $1 \implies 2$. Given a functor $A : \mathcal{I} \to \mathcal{A}$ and let $f : \bigoplus\limits_{\alpha i \to j} A_i \to \bigoplus\limits_{i \in \mathcal{I}} A_i$ where $i, j \in \mathcal{I}$.

$$\begin{tikzcd}
i \arrow[r, "\alpha"] \arrow[d, squiggly, "A" description] & j \arrow[d, squiggly, "A" description] & & \mathcal{I} \arrow[d, squiggly, "A"] \\
A_i \arrow[r, "\alpha_*"] & A_j & & \mathcal{A}
\end{tikzcd}$$

Link to Diagram

Then the map $f(a_{i,\alpha}) = \alpha_*(a_i) - a_i \in A_j - A_i$, so this is $\alpha_* - \mathbb{1}$. Let $C := \operatorname{coker} f := \bigoplus\limits_{i \in I} A_i / \operatorname{im}(f)$, and we'll denote elements in this quotient with a bar.

Claim: $C = \operatorname{colim}_{i \in I} A_i$ with

$$\begin{aligned} \eta_i : A_i &\to C \\ a_i &\mapsto \overline{a_i}, \end{aligned}$$

where we first embed A_i into the direct sum and then take the quotient.

Exercise *(?)*
Use the universal property of cokernels in $\mathcal{A}$. Check that the following diagram commutes:

$$\begin{tikzcd}
A_i \arrow[dd, "\alpha_*"'] \arrow[dr, "\eta_i"] & \\
& C \\
A_j \arrow[ur, "\eta_j"'] &
\end{tikzcd}$$

This essentially follows from the fact that $\overline{\alpha_*(a_i)} = \overline{a_i}$.

■

Remark 16.1.3: **Mod-**R satisfies (1), since direct sums of R-modules still have an R-module structure. Thus **Mod-**R is cocomplete.

Definition 16.1.4 (Limits)
The **limit** of a functor $A : \mathcal{I} \to \mathcal{A}$ is the colimit of the dual functor $A^{\mathrm{op}} : I^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.

Remark 16.1.5: Note that this amounts to reversing arrows in the conditions of a colimit. Many of the results for colimits go through with arrows reversed. Examples: kernels, direct products. If I is a poset, then limits are referred to as **inverse limits**, using $\varprojlim\limits_{i \in I} A_i$.

Definition 16.1.6 (Complete Categories)
$\mathcal{A}$ is **complete** if and only if $\lim\limits_{i\in I} A_i$ exists whenever $\mathcal{I}$ is small and $A : \mathcal{I} \to \mathcal{A}$.

Theorem 16.1.7 ***(The Adjoint-Limit Theorem).***
Let $\mathcal{A} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$ be an adjoint pair, where now $\mathcal{A}, \mathcal{B}$ are now arbitrary categories (not necessarily abelian). Then

- The **left adjoint** L preserves **colimits** (direct sums, cokernels, etc). I.e. if $A : \mathcal{I} \to \mathcal{A}$ has a colimit, then so does $(L \circ A) : \mathcal{I} \to \mathcal{B}$, and $L(\operatorname{colim} A_i) = \operatorname{colim}(LA_i)$.
- The **right adjoint** R preserves **limits** (direct products, kernels, etc).

Proof (?).
Not given in the book! See MacLane's *Categories for the Working Mathematician.* ■

Remark 16.1.8: Recall left adjoints are right-exact and have left-derived functors.

Corollary 16.1.9 ***(?).***

If $\mathcal{A}$ is a cocomplete abelian category with enough projectives and $\mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{B}$. Then for every set-indexed collection of objects $\{A_i\}$,

$$(L_*F)\left(\bigoplus_{i\in I} A_i\right) = \bigoplus_{i\in I} L_*F(A_i)),$$

so left-derived functors commute with direct sums.

Proof (?).
Let P_i be the projective resolution of A_i, so $P_i \to A_i$, then $\bigoplus P_i \to \bigoplus A_i$ is a projective resolution, and by definition

$$\begin{aligned}
(L_*F)\left(\bigoplus A_i\right) &= H_*\left(F\left(\bigoplus P_i\right)\right) \\
&= H_*\left(\bigoplus FP_i\right) \quad \text{by the theorem} \\
&\cong \bigoplus H_*(FP_i) \text{homology commutes with } \oplus \in \operatorname{Ch}(\mathcal{A}) \\
&= \bigoplus_i L_*F(A_i).
\end{aligned}$$

■

Corollary 16.1.10 ***(?).***

For $A_i \in R\text{-}\mathbf{Mod}, B \in \mathbf{Mod}\text{-}R$,

$$\operatorname{Tor}_*^R\left(\bigoplus_{i\in I} A_i, B\right) \cong \bigoplus_{i\in I} \operatorname{Tor}_*^R(A_i, B).$$

Proof (of corollary).

$$\operatorname{Tor}_*^R(\,\cdot\,, B) = L_*F, \qquad F := (\cdot \otimes_R B),$$

and F is a left-adjoint by the tensor-hom adjunction. ■

Remark 16.1.11: One can also show directly from the definition that

$$\operatorname{Tor}_*^R(A, \bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I} \operatorname{Tor}_*^R(A, B_i).$$

This uses the fact that $P \otimes_R (\bigoplus_{i\in I} B_i) \cong \bigoplus_{i\in I}(P \otimes B_i)$.

Remark 16.1.12: We'll skip the rest of this section, we (hopefully) won't need filtered colimits.

16.2 Balancing Tor and Ext

Idea: their derived functors with either variable fixed will essentially be the same. We'll start by showing that the two left-derived functors of $\cdot \otimes_R \cdot$ give the same results, and similarly for the two right-derived functors $\operatorname{Hom}_R(\,\cdot\,,\,\cdot\,)$. We'll use double complexes!

16.2.1 Tensor Product Complexes

Suppose we have two chain complexes $(P)_R \in \operatorname{Ch}(\mathbf{Mod}\text{-}R), {}_R(Q) \in \operatorname{Ch}(R\text{-}\mathbf{Mod})$. Then there is a double complex where i, j indexes rows and columns: $P \otimes_R Q = \{P_i \otimes_R Q_j\}_{i,j}$, the **tensor product double complex** of P and Q. We use the sign trick from 1.2.5:

- $d^h := d^P \otimes \mathbb{1}$
- $d^v := (-1)^i 1 \otimes d^Q$

Taking the direct sum totalization $\operatorname{Tor}^{\oplus}(P \otimes_R Q)$ is the **total tensor product chain complex** of P and Q. Note that this has a single differential! The big theorem from this section:

Theorem 16.2.1 *(?).*

$$L_n(A \otimes_R \cdot)(B) \cong L_n(\cdot \otimes_R B)(A) := \operatorname{Tor}_n^R(A, B).$$

Remark 16.2.2: Note that this makes the right-hand side notation unambiguous.

Proof (?).
Choose projective resolutions $P \xrightarrow{\varepsilon} A \in \mathbf{Mod}\text{-}R$ and $Q \xrightarrow{\eta} B \in R\text{-}\mathbf{Mod}$. We'll form 3 tensor product double complexes.

- $P \otimes Q$: A first quadrant double complex, since the projective resolutions have nonnegative indices.
- $A \otimes Q$, embedding $A \hookrightarrow \operatorname{Ch}(\mathcal{A})$ as a complex concentrated in degree 0 (so one column)
- $P \otimes B$ (one row).

There are several maps of double complexes among these induced by ϵ, η:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A \otimes Q_2 & P_0 \otimes Q_2 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_1 \otimes Q_2 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_2 \otimes Q_2 \arrow[l, "d^P\otimes 1"'] \arrow[d, "1\otimes d^Q"] & \cdots \arrow[l] \\
A \otimes Q_1 & P_0 \otimes Q_1 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes d^Q"] & P_1 \otimes Q_1 \arrow[l, "d^P\otimes 1"] \arrow[d, "1\otimes d^Q"] & P_2 \otimes Q_1 \arrow[l, "d^P\otimes 1"] \arrow[d, "1\otimes d^Q"] & \cdots \arrow[l] \\
A \otimes Q_0 & P_0 \otimes Q_0 \arrow[l, "\epsilon\otimes 1"'] \arrow[d, "1\otimes\eta"] & P_1 \otimes Q_0 \arrow[l, "d^P\otimes 1"] \arrow[d, "1\otimes\eta"] & P_2 \otimes Q_0 \arrow[l, "d^P\otimes 1"] \arrow[d, "1\otimes\eta"] & \cdots \arrow[l] \\
 & P_0 \otimes B & P_1 \otimes B & P_2 \otimes B &
\end{tikzcd}$$

Link to Diagram

We'll show there are two maps:

$$A \otimes Q = \operatorname{Tot}(A \otimes Q) \xleftarrow{\varepsilon\otimes\mathbb{1}} \operatorname{Tor}(P \otimes Q) \xrightarrow{1\otimes\eta} \operatorname{Tor}(P \otimes B) = P \otimes B,$$

using that totalizing a one-row or one-column complex is summing along diagonals where each has one term, yielding actual equality of the first and last terms respectively above. Moreover, we'll show these are **quasi-isomorphisms**, and so

$$L_*(A \otimes \cdot) \xleftarrow{\varepsilon \otimes \mathbb{1}} H_*(\mathrm{Tor}(P \otimes Q)) \xrightarrow{\mathbb{1} \otimes \eta} L_*(\cdot \otimes B)(A).$$

We'll continue with the proof of this next time. ■

17 | Appendix: Extra Definitions

Definition 17.0.1 (Acyclic)
A chain complex C is **acyclic** if and only if $H_*(C) = 0$.

18 | Extra References

- `https://www.math.wisc.edu/~csimpson6/notes/2020_spring_homological_algebra/notes.pdf`

ToDos

List of Todos

Definitions

Theorems

Exercises

Figures

List of Figures

Bibliography

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