# Title

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# Contents

# 1 | Lecture 8: Riemann-Roch Spaces

Setting up for the single most important theorem in the course: the Riemann-Roch theorem. We start by motivating this by considering the following property of $K := k(t)$: for any degree 1[1] place $p \in \Sigma(K/k)$, there exists an $f \in K^\times$ such that $(f)_- = p$. In other words, $f$ is a rational function with a simple pole at the given place, and no other poles. Why? We just know precisely what all of the places are for this function field.

If $p = \infty$, we can just take $f(t) = t$, since any polynomial is regular away from $\infty$ and the valuation is $-\deg(f) = -1$ The other places $p$ correspond to $t - \alpha$ (the uniformizing element) for $\alpha \in k$, since they correspond to other points on $\mathbb{A}^1_{/k}$, and so we can take $f(t) = 1/(t - \alpha)$. This $f$ is regular at infinity since the degree of the numerator is larger than the degree of the denominator, and the denominator doesn't vanish at any other place.

**Remark 1.0.1:** With some thought, it can be found that this is a *characteristic* property of rational function fields: if $f \in K$, a one variable function field, and $\deg(d)_- = 1$[2] then the degree of the function is equal to the degree of the divisor of the zeros and the divisor of the poles, and thus the degree of the extension $[K : k(t)] = 1$ and thus $K = k(t)$ is rational. So having a rational with a simple pole at only one point *only* happens in you're in a rational function field.

On the other hand, we both wanted and used in our discussion of holomorphy rings the fact that given a nonempty finite subset $S \subset \Sigma(K/k)$, we want to find a rational function $f \in K^\times$ has poles at all of the points in $S$, so $\operatorname{supp}(f)_- = S$. Better yet, we'd like a bound on the degree of any such $f$, i.e. the orders of all of these poles. If $S$ is a single place, unless the function field is rational, we can't require the function to have a pole of degree 1 at that point. But can it admit a pole of degree at most 10, for example? This is what motivates the Riemann-Roch spaces and the Riemann-Roch theorem. If you're trying to give a quantitative bound on how high of an order of a pole you have to allow in order to have a rational function, this comes from a key invariant called the *genus* of the function field. The theorem that will tell us about the existence of rational functions with poles of prescribed degrees in terms of the genus is precisely the Riemann-Roch theorem, so that's where we are headed.

**Definition 1.0.2** (Riemann-Roch Space of $D$ (Key Definition))
For $D \in \operatorname{Div} K$, the **Riemann-Roch space** of $D$ is defined as

$$\mathcal{L}(D) := \left\{ f \in K^\times \;\middle|\; (t) \geq -D \right\} \cup \{0\}.$$

**Remark 1.0.3:** This will turn out to be a $k$-vector space, and is a sub $k$-vector space of $K$. One of the first things we'll prove is that it's always finite dimensional. This is only interesting when $D$ is linearly equivalent to an effective divisor, so we should think of $D$ as having a nonnegative degree, and in fact itself being an effective divisor. So this is the space of rational functions that have prescribes poles of a prescribed order.

[1] So the residue field of the corresponding DVR is $k$ itself rather than some proper finite degree extension.
[2] Recall that this is the divisor pole.

**Question 1.0.4:** Does $\mathcal{L}(D)$ contain any rational functions other than zero?

**Answer 1.0.5:** For any nonzero $f \in \mathcal{L}(D)^{\bullet}$, the divisor $D + (f)$ is effective, since $(f) \geq -D$, and also linearly equivalent to $D$. If $D$ is not linearly equivalent to an effective divisor, this is just the zero vector space.

**Exercise 1.0.6** ***(?):*** Let $K = k(t)$ and $n \in \mathbb{Z}^{\geq 0}$. Show that

$$L(n\infty) = \left\{ f \in k[t] \;\middle|\; \deg f \leq n \right\}$$

and in particular is a $k$-vector space of dimension $n + 1$.[3]

**Remark 1.0.7:** Note that $\infty$ is a degree 1 place, and multiplying it by $n$ yields an effective divisor. The Riemann-Roch space here is comprised of rational functions that regular away from $\infty$, which are polynomials, whose pole at $\infty$ has order at worst $n$. But the order of a pole at infinity is its degree as a polynomial, since the $\infty$-adic valuation is the negative degree, so this yields polynomials of degree at most $n$.

**Lemma 1.0.8** ***(?).***
For $D \in \operatorname{Div} K$,

$$\mathcal{L}(D) \neq \{0\} \iff 0 \text{ is equivalent to an effective divisor.}$$

*Proof (?).*
$\implies$ : If $f \in \mathcal{L}(D)^{\bullet}$, then $D + (f)$ is effective and linearly equivalent to zero.
$\impliedby$ : If $D' \geq 0$ and $D' \sim D$, then $D' = D + (f) \geq 0$. So $(f) \geq -D$ and thus $f \in \mathcal{L}(D)$. ■

**Example 1.0.9** ***(?):*** $\mathcal{L}(0) = \left\{ f \;\middle|\; (f) \geq 0 \right\} \cup \{0\}$, which consists of rational functions with no poles (so their divisor is the zero divisor), and thus $\mathcal{L}(0) = \kappa(K)$. I.e., these are the constants: they are regular everywhere and have no zeros or poles. We would like this space to have $k$-dimension 1, so we impose $\kappa(K) = k$.

**Exercise 1.0.10** ***(?):***

a. Show that for all $D$, $\mathcal{L}(D) \in \mathsf{Vect}_k$.
b.

$$D \sim D' \implies \mathcal{L}(D) \cong_{\mathsf{Vect}_k} \mathcal{L}(D').$$

**Remark 1.0.11:** You can frame the above as taking rational functions with poles of certain orders, and analyzing the orders of poles of their sums. If you take $D'$ and write it as $D + (f)$ for $f$ a rational function, then $f$ should produce this isomorphism. The moral: $\mathcal{L}(D)$ only depends on the linear equivalence class of $D$.

[3] Recall that $\infty$ is the $1/t$-adic place.

**Exercise 1.0.12*(?)*:** Let $D \in \operatorname{Div}^0 K$ be a degree zero divisor, then TFAE:

a. $\dim \mathcal{L}(D) \geq 1$
b. $\dim \mathcal{L}(D) = 1$,
c. $D$ is principal, i.e. the divisor of a rational function or linearly equivalent to zero.

**Slogan 1.0.13:** The only way a degree zero divisor can have a nontrivial Riemann-Roch space is if it's linearly equivalent to zero.

**Lemma 1.0.14*(?)*.**
Let $A \leq B$[a] in $\operatorname{Div} K$, then

a. $\mathcal{L}(A) \leq_{\mathrm{Vect}_k} \mathcal{L}(B)$ is a subspace,
b. $\dim \mathcal{L}(B)/\mathcal{L}(A) \leq \deg B - \deg A = \deg(B - A)$.

[a] These are formal linear combinations of places, so the coefficients in front of each place in $A$ should be less than the corresponding coefficient for $B$, or equivalently $B - A$ is effective.

**Remark 1.0.15:** Since $B \geq A$, you can think of this as starting with $A$ and adding an effective divisor to get $B$, namely $A + (B - A) = B$. How much does that decrease the dimension of the Riemann-Roch space? At most, by the degree of $B - A$ as a divisor.

**Corollary 1.0.16*(?)*.**
For $D \in \operatorname{Div} K$,

a. If $\deg D < 0$ then $\mathcal{L}(D) = 0$.
b. If $\deg(D) \geq 0$ then $\dim_k \mathcal{L}(D) \leq \deg(D) + 1 < \infty$.

**Remark 1.0.17:** This shows that Riemann-Roch spaces are always finite dimensional, and also gives a simple upper bound on that dimension.

*Proof (of corollary).*
For (a), a divisor of negative degree is not linearly equivalent to an effective divisor. ■