

# Problem Set 2

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## 1 Humphreys 1.5

**Proposition:** Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and  $M(\lambda), M(\mu)$  Verma modules. Then  $M(\lambda) \otimes M(\mu)$  may not lie in  $\mathcal{O}$ .

**Proof:**

Let  $M(\lambda), M(\mu)$  be arbitrary Verma modules with highest weight vectors  $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$  respectively. We can then consider the weight of  $v \otimes w$  in  $N := M(\lambda) \otimes_{\mathbb{C}} M(\mu)$ :

$$\begin{aligned} h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\ &= \lambda(h)v \otimes w + v \otimes \mu(h)w \\ &= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\ &= (\lambda(h) + \mu(h))(v \otimes w). \end{aligned}$$

Letting  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ , so  $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$ , the claim is that it is possible for  $N$  to *not* be finitely-generated as a  $U(\mathfrak{g})$ -module.

Let  $\{y, h, x\}$  be the usual basis for  $\mathfrak{g}$ , for which  $U(\mathfrak{g})$  has the usual associated PBW basis. We can use the fact that  $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$ , so if we pick  $\mu, \lambda \in \mathbb{Z}^{\leq 0}$  we have weight space decompositions

$$\begin{aligned} M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda-2i} := \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i := \lambda-2i}} M(\lambda)_{\lambda_i} \\ M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu-2j} := \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_j := \mu-2j}} M(\mu)_{\mu_j} \end{aligned}$$

where we can explicitly identify bases  $M(\lambda)_{\lambda_i} = \text{span}_{\mathbb{C}} \{y^i v\}$  and  $M(\mu)_{\mu_i} = \text{span}_{\mathbb{C}} \{y^i w\}$ .

By the initial observation, this yields a weight space decomposition for  $N$  given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left( \bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_{\nu}.$$

Since each weight space  $N_{\nu} = \text{span}_{\mathbb{C}} \{y^i v \otimes y^j w \mid i + j = \nu\}$  and there are infinitely many such weight spaces, no finite number of PBW monomials can generate  $N$ . ■

## 2 Humphreys 1.9

**Proposition:** Let  $\psi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h})$  be the twisted Harish-Chandra homomorphism. Then  $\psi$  is independent of the choice of a simple system in  $\Phi$ .

Hint: any simple system has the form  $w\Delta$  for some  $w \in W$ .

**Proof:**

Choosing a PBW basis  $\{h_i\}_{i=1}^{\ell}$  for  $U(\mathfrak{h})$ , we can write

$$\begin{array}{lll} Z(\mathfrak{g}) \xrightarrow{\xi} U(\mathfrak{h}) & \rightarrow S(\mathfrak{h}) = \mathbb{C}[\{h_i\}] = P(\mathfrak{h}^*) & \rightarrow \mathbb{C}[\{h_i\}] \\ z \mapsto z = \prod_{i=1}^{\ell} h_i^{t_i} & \mapsto \left( \lambda \mapsto \prod_{i=1}^{\ell} \lambda(h_i)^{t_i} \right) & \mapsto d. \end{array}$$