Qual Problems

D. Zack Garza

November 25, 2019

Contents

1 Problem 1

1.1 Part 1

Definition: An element $r \in R$ is *irreducible* if whenever $r = st$, then either s or t is a unit.

Definition: Two elements $r, s \in R$ are *associates* if $r = \ell s$ for some unit ℓ.

A ring R is a *unique factorization domain* iff for every $r \in R$, there exists a set $\{p_i \mid 1 \leq i \leq n\}$ such that $r = u \prod_{i=1}^n p_i$ where u is a unit and each p_i is irreducible.

Moreover, this factorization is unique in the sense that if $r = w \prod_{i=1}^n q_i$ for some w a unit and q_i irreducible elements, then each q_i is an associate of some p_i.

1.2 Part 2

A ring R is a *principal ideal domain* iff whenever $I \trianglelefteq R$ is an ideal of R, there is a single element $r_i \in R$ such that $I = \langle r_i \rangle$.

1.3 Part 3

An example of a UFD that is not a PID is given by $R = k[x, y]$ for k a field.

That R is a UFD follows from the fact that if k is a field, then k has no prime elements since every non-zero element is a unit. So the factorization condition holds vacuously for k, and k is a UFD. But then we can use the following theorem:

Theorem: If R is a UFD, then $R[x]$ is a UFD.

Since k is a UFD, the theorem implies that $k[x]$ is a UFD, from which it follows that $k[x][y] = [kx, y]$ is also a UFD.

To see that R is not a PID, consider the ideal $I = \langle x, y \rangle$, and suppose $I = \langle g \rangle$ for some single $g \in k[x, y]$.

Note that $I \neq R$, since I contains no degree zero polynomials. Moreover, since $\langle x \rangle \subset I = \langle g \rangle$ (and similarly for y), we have $g \mid x$ and $g \mid y$, which forces $\deg g = 0$. So in fact $g \in k$, but then $\langle g \rangle = g^{-1}\langle g \rangle = \langle 1 \rangle = k$, so this forces $I = k \trianglelefteq k[x, y]$. However, it is also the case that $x \notin k$ (nor y), which is a contradiction.