

# Problem Set 5

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## Contents

<b>1 Problem 1</b>	<b>1</b>
<b>2 Problem 2</b>	<b>2</b>
<b>3 Problem 3</b>	<b>3</b>
<b>4 Problem 4</b>	<b>4</b>
4.1 Part (a) . . . . .	4
4.2 Part (b) . . . . .	5
<b>5 Problem 5</b>	<b>6</b>
<b>6 Problem 6</b>	<b>7</b>
6.1 Part (a) . . . . .	7
6.2 Part (b) . . . . .	8

## 1 Problem 1

We first make the following claim:

$$S := \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$$
$$T := \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = \sup \left\{ \sum_{(j,k) \in C} a_{kj} \mid C \subset \mathbb{N}^2, |B| < \infty \right\}.$$

It suffices to show the first equality holds, as the other case will follow similarly. Let  $S = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}$  and  $S' = \sup \left\{ \sum_{(j,k) \in B} a_{jk} \mid B \subset \mathbb{N}^2, |B| < \infty \right\}$ .

Then consider any bounded set  $B \subset \mathbb{N}^2$ ; so  $B \subset \{1, \dots, n_1\} \times \{1, \dots, n_2\}$  for some  $n_1, n_2 \in \mathbb{N}$ . We then have

$$\sum_B a_{jk} \leq \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} a_{jk} \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}.$$

where the first equality holds  $a + jk \geq 0$  for all  $j, k$ , so the sum can only increase if we add more terms. But this holds for every  $B$  and thus holds if we take the supremum over all of them, so  $S' \leq S$ .

We will show that  $S = T$  by showing that  $S \leq T$  and  $T \leq S$ .

Let  $B \subset \mathbb{N}^2$  be finite, so  $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$ .

Now letting  $R > \max(I, J)$ , we can define  $C = [0, R]^2$ , which satisfies  $B \subseteq C \subset \mathbb{N}^2$  and  $|C| < \infty$ .

Moreover, since  $a_{jk} \geq 0$  for all pairs  $(j, k)$ , we have the following inequality:

$$\sum_{(j,k) \in B} a_{jk} < \sum_{(k,j) \in C} a_{jk} \leq \sum_{(k,j) \in C} a_{jk} \leq T,$$

since  $T$  is a supremum over *all* such sets  $C$ , and the terms of any finite sum can be rearranged.

But since this holds for every  $B$ , we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$S := \sup_B \sum_{(k,j) \in B} a_{jk} \leq T.$$

(Use epsilon-delta argument)

An identical argument shows that  $T \leq S$ , yielding the desired equality.  $\square$

## 2 Problem 2

We want to show the following equality:

$$\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx.$$

To that end, we can rewrite this using the integral definition of  $g(x)$ :

$$\int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx = \int_0^1 f(x) \, dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned} \int_0^1 \int_x^1 \frac{f(t)}{t} \, dt \, dx &= \int_0^1 \int_0^t \frac{f(t)}{t} \, dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} \int_0^t dx \, dt \\ &= \int_0^1 \frac{f(t)}{t} (t - 0) \, dt \\ &= \int_0^1 f(t) \, dt, \end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(x, t) \mapsto \frac{\chi_A(x, t) \hat{f}(x, t)}{t}.$$

where  $A = \{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq t \leq 1\}$  and  $\hat{f}(x, t) := f(t)$  is the cylinder on  $f$ .

This defines a measurable function on  $\mathbb{R}^2$ , since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular,  $|F|$  is measurable and non-negative, and so we can apply Tonelli to  $|F|$ . This allows us to write

$$\begin{aligned} \int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left| \frac{f(t)}{t} \right| dx dt \\ &= \int_0^1 \int_0^t \frac{|f(t)|}{t} dx dt \quad \text{since } t > 0 \\ &= \int_0^1 \frac{|f(t)|}{t} \int_0^t dx dt \\ &= \int_0^1 |f(t)| < \infty, \end{aligned}$$

where the switch is justified by Tonelli and the last inequality holds because  $f$  was assumed to be measurable.

Since this shows that  $F \in L^1(\mathbb{R}^2)$ , and we can thus apply Fubini to  $F$  to justify the initial switch.  $\square$

### 3 Problem 3

Let  $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$ , and define

$$f(x, y) = \frac{x^{1/3}}{(1 + xy)^{3/2}}$$

$$F(x, y) = \chi_A(x, y) f(x, y).$$

Note that  $F$  Then, if all iterated integrals exist and a switch of integration order is justified, we would have

$$\begin{aligned}
\int_{\mathbb{R}^2} F &= ? \int_0^\infty \int_y^\infty f(x, y) \, dx \, dy \\
&= ? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1+xy)^{3/2}} \, dy \, dx \\
&= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3} \sqrt{1+x^2}} \, dx \\
&= 2 \int_0^1 \frac{1}{x^{2/3} \sqrt{1+x^2}} \, dx + 2 \int_1^\infty \frac{1}{x^{2/3} \sqrt{1+x^2}} \, dx \\
&\leq \int_0^1 x^{-2/3} \, dx + \int_0^\infty x^{-5/3} \, dx \\
&= 2(3) + 2 \left( \frac{3}{2} \right) < \infty,
\end{aligned}$$

where the first term in the split integral is bounded by using the fact that  $\sqrt{1+x^2} \geq \sqrt{x^2} = x$ , and the second term from  $x > 1 \implies x > 0 \implies \sqrt{1+x^2} \geq \sqrt{1}$ .

Since  $F$  is non-negative, we have  $|F| = F$ , and so the above computation would imply that  $F \in L^1(\mathbb{R}^2)$ . It thus remains to show that  $\int F$  is equal to its iterated integrals, and that the switch of integration order is justified

Since  $F$  is non-negative, Tonelli can be applied directly if  $F$  is measurable in  $\mathbb{R}^2$ . But  $f$  is measurable on  $A$ , since it is continuous at almost every point in  $A$ , and  $\chi_A$  is measurable, so  $F$  is a product of measurable functions and thus measurable.

## 4 Problem 4

### 4.1 Part (a)

For any  $x \in \mathbb{R}^n$ , let  $A_x := A \cap (x - B)$ .

We can then write  $A_t := A \cap (t - B)$  and  $A_s := A \cap (s - B)$ , and thus

$$\begin{aligned}
g(t) - g(s) &= m(A_t) - m(A_s) \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) \, dx - \int_{\mathbb{R}^n} \chi_{A_s}(x) \, dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_s}(x) \, dx \\
&= \int_{\mathbb{R}^n} \chi_{A_t}(x) - \chi_{A_t}(t - s + x) \, dx \\
&\quad (\text{since } x \in s - B \iff s - x \in B \iff t - (s - x) \in t - B),
\end{aligned}$$

and thus by continuity in  $L^1$ , we have

$$|g(t) - g(s)| \leq \int_{\mathbb{R}^n} |\chi_{A_t}(x) - \chi_{A_t}(t - s + x)| \, dx \rightarrow 0 \quad \text{as } t \rightarrow s$$

which means  $g$  is continuous.

To see that  $\int g = m(A)m(B)$ , if an interchange of integrals is justified, we can write

$$\begin{aligned}
\int_{\mathbb{R}^n} g(t) dt &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{A_t}(x) dx dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x, t) dx dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_{t-B}(x, t) dx dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_B(t-x) dx dt \\
&\quad (\text{since } x \in t-B \iff t-x \in B) \\
&=? \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_A(x) \chi_B(t-x) dt dx \\
&= \int_{\mathbb{R}^n} \chi_A(x) \int_{\mathbb{R}^n} \chi_B(t-x) dt dx \\
&= \int_{\mathbb{R}^n} \chi_A(x) m(B) dt \\
&\quad (\text{by translation invariance of Lebesgue integral}) \\
&= m(B) \int_{\mathbb{R}^n} \chi_A dt \\
&= m(B)m(A).
\end{aligned}$$

To see that this is justified, we note that the map  $F(x, t) = \chi_A(x) \chi_B(x-t)$  is non-negative, and we claim it is measurable in  $\mathbb{R}^{2n}$ .

- The first component is  $\chi_A(x)$ , which is measurable on  $\mathbb{R}^n$ , and thus the cylinder over it will be measurable on  $\mathbb{R}^{2n}$ .
- The second component involves  $\chi_B(t-x)$ , which is  $\chi_B(x)$  composed with a reflection (which is still measurable) followed by a translation (which is again still measurable).
- Thus, as a product of two measurable functions, the integrand is measurable.

So Tonelli applies to  $|F|$ , and thus  $\int |F| = m(A)m(B) < \infty$  since  $A, B$  were assumed to be bounded. But then  $F$  is integrable by Fubini, and the claimed equality holds.

## 4.2 Part (b)

Supposing that  $m(A), m(B) > 0$ , we have  $\int g(t) dt > 0$ , using the fact that  $\int g = 0$  a.e.  $\iff g = 0$  a.e., we can conclude that if  $T = \{t \ni g(t) \neq 0\}$ , then  $m(T) > 0$ . So there is some  $t \in \mathbb{R}^n$  such that  $g(t) \neq 0$ , and since  $g$  is continuous, there is in fact some open ball  $B_t$  containing  $t$  such that  $t' \in B_t \implies g(t') \neq 0$ . So we have

- $\forall t' \in B_t, A \cap t' - B \neq \emptyset \iff$
- $\forall t' \in B_t, \exists x \in A \cap t' - B \iff$
- $\forall t' \in B_t, \exists x \text{ such that } x \in A \text{ and } x \in t' - B \iff$
- $\forall t' \in B_t, \exists x \text{ such that } x \in A \text{ and } x = t' - b \text{ for some } b \in B \iff$
- $\forall t' \in B_t, \exists x \text{ such that } x \in A \text{ and } t' = x + b \text{ for some } b \in B \iff$

- $\forall t' \in B_t, \exists t'$  such that  $t' \in A + B$

And thus  $B_t \subseteq A + B$ .

## 5 Problem 5

If the iterated integrals exist and are equal (so an interchange of integration order is justified), we have

$$\begin{aligned}
\int_0^1 F(x)g(x) &:= \int_0^1 \left( \int_0^x f(y) \, dy \right) g(x) \, dx \\
&= \int_0^1 \int_0^x f(y)g(x) \, dy \, dx \\
&\stackrel{?}{=} \int_0^1 \int_y^1 f(y)g(x) \, dx \, dy \\
&= \int_0^1 f(y) \left( \int_y^1 g(x) \, dx \right) \, dy \\
&= \int_0^1 f(y)(G(1) - G(y)) \, dy \\
&= G(1) \int_0^1 f(y) \, dy - \int_0^1 f(y)G(y) \, dy \\
&= G(1)(F(1) - F(0)) - \int_0^1 f(y)G(y) \, dy \\
&= G(1)F(1) - \int_0^1 f(y)G(y) \, dy \quad \text{since } F(0) = 0,
\end{aligned}$$

which is what we want to show.

To see that this is justified, let  $I = [0, 1]$  and note that the integrand can be written as  $H(x, y) = \hat{f}(x, y)\hat{g}(x, y)$  where  $\hat{f}(x, y) = \chi_I f(y)$  and  $\hat{g}(x, y) = \chi_I g(x)$  are cylinders over  $f$  and  $g$  respectively. Since  $f, g$  are in  $L^1(I)$ , their cylinders are measurable over  $\mathbb{R} \times I$ , and thus  $\hat{f}, \hat{g}$  are measurable on  $\mathbb{R}^2$  as products of measurable functions. Then  $H$  is a measurable function as a product of measurable functions as well.

But then  $|H|$  is non-negative and measurable, so by Tonelli all iterated integrals will be equal. We want to show that  $H \in L^1(\mathbb{R}^2)$  in order to apply Fubini, so we will show that  $\int |H| < \infty$ .

To that end, noting that  $f, g \in L^1$ , we have  $\int_0^1 f := C_f < \infty$  and  $\int_0^1 g := C_g < \infty$ . Then,

$$\begin{aligned}
\int_{\mathbb{R}^2} |H| &= \int_0^1 \int_0^1 |f(x)g(y)| \, dx \, dy \\
&= \int_0^1 \int_0^1 |f(x)| |g(y)| \, dx \, dy \\
&= \int_0^1 |g(y)| \left( \int_0^1 |f(x)| \, dx \right) \, dy \\
&= \int_0^1 |g(y)| C_f \, dy \\
&= C_f \int_0^1 |g(y)| \, dy \\
&= C_f C_g < \infty,
\end{aligned}$$

and thus by Fubini, the original interchange of integrals was justified.

## 6 Problem 6

### 6.1 Part (a)

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \, dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \left| \int_{x-h}^{x+h} f(y) \, dy \right| \, dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \left( \int_{x-h}^{x+h} |f(y)| \, dy \right) \, dx \\
&= \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \, dy \, dx \\
&\stackrel{=?}{=} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \, dx \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| \int_{y-h}^{y+h} dx \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} |f(y)| ((y+h) - (y-h)) \, dy \\
&= \frac{1}{2h} \int_{\mathbb{R}} 2h |f(y)| \, dy \\
&= \int_{\mathbb{R}} |f(y)| \, dy < \infty
\end{aligned}$$

since  $f$  was assumed to be in  $L^1(\mathbb{R})$ , where the changed bounds of integration are determined by considering the following diagram:

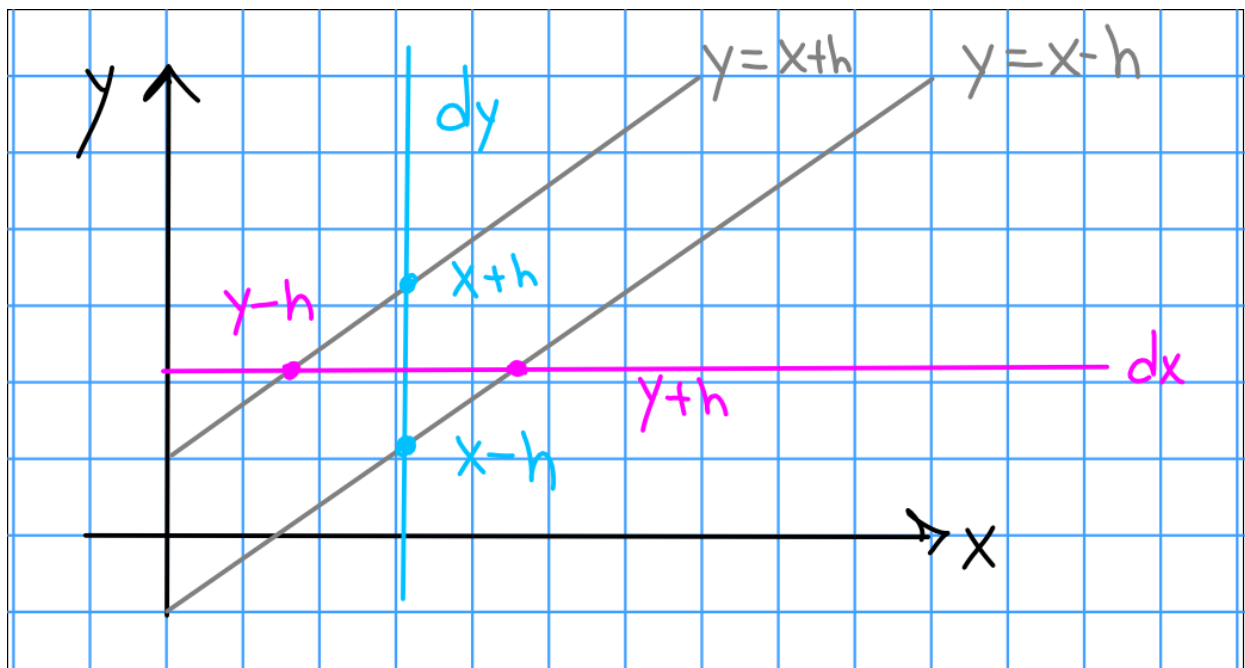


Figure 1: Changing the bounds of integration

To justify the change in the order of integration, consider the function  $H(x, y) = \frac{1}{2h} \chi_A(x, y) f(y)$  where  $A = \{(x, y) \in \mathbb{R}^2 : -\infty < x - h \leq x, y \leq x + h\}$ . Since  $f$  is measurable, the constant function  $(x, y) \mapsto \frac{1}{2h}$  is measurable, and characteristic functions are measurable,  $H$  is a product of measurable functions and thus measurable.

Thus it makes sense to write  $\int |H|$  as an iterated integral by Tonelli, and since  $\int_{\mathbb{R}^2} |H| = \int_{\mathbb{R}} |A_h(f)| < \infty$  by the above calculation, we have  $H \in L^1(\mathbb{R}^2)$ , and Fubini applies.

## 6.2 Part (b)

Let  $\varepsilon > 0$ ; we then have



$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \, dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - f(x) \right| \, dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \, dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \, dy \right| \, dx \\
&\quad \text{since } \frac{1}{2h} \int_{x-h}^{x+h} f(x) \, dy = \frac{1}{2h} f(x)((x+h) - (x-h)) = \frac{1}{2h} f(x)2h = f(x) \\
&= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{B(h,x)} f(y) - f(x) \, dy \right| \, dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - f(x)| \, dy \, dx \\
&\leq \int_{\mathbb{R}} \frac{1}{2h} \int_{-h}^h |f(y-x) - f(x)| \, dy \, dx
\end{aligned}$$

but since  $h \rightarrow 0$  will force  $y \rightarrow x$  in the integral, for a fixed  $x$  we can let  $\tau_x(y) = f(y-x)$  and we have  $\|\tau_x - f\|_1 \rightarrow 0$  by continuity in  $L^1$ . Thus  $\int_{-h}^h |f(y-x) - f(x)| \rightarrow 0$ , forcing  $\|A_h(f) - f\|_1 \rightarrow 0$  as  $h \rightarrow 0$ .  $\square$