

# Title

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## 1 | Friday, September 25

### 1.1 Compact-Open Topology

### 1.2 Isotopy

- Define a homotopy between  $f, g : X \rightarrow Y$  as a map  $F : S \times I \rightarrow S$  restricting to  $f, g$  on the ends.
  - Equivalently: a *path* in  $\text{Map}(X, Y)$ .
- Isotopy: require the partially-applied function  $F_t : S \rightarrow S$  to be homeomorphisms for every  $t$ .
  - Equivalently: a path in  $\text{Map}(X, Y)$

### 1.3 Self-Homeomorphisms

- In any category, the automorphisms form a group.
  - In a general category  $\mathcal{C}$ , we can always define the group  $\text{Aut}_{\mathcal{C}}(X)$ .
    - \* If the group has a topology, we can consider  $\pi_0 \text{Aut}_{\mathcal{C}}(X)$ , the set of path components.
    - \* Since groups have identities, we can consider  $\text{Aut}_{\mathcal{C}}^0(X)$ , the path component containing the identity.
  - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X) / \text{Aut}_{\mathcal{C}}^0(X).$$

- Here the  $\pm$  indicates that we take both orientation preserving and non-preserving automorphisms.

- Has an index 2 subgroup of orientation-preserving automorphisms,  $\text{MCG}^+(X)$ .

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with  $\mathcal{O}_{\text{CO}}$ .

- Taking  $\text{MCG}_{\text{Top}}^{\pm}(X)$  yields ??

- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking  $\text{MCG}_{C^{\infty}}(X)$  yields ??

- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f]\}.$$

- Taking  $\text{MCG}(X)$  here yields *homotopy classes of self-homotopy equivalences*.

- For topological manifolds: Isotopy classes of homeomorphisms

- In the compact-open topology, two maps are isotopic iff they are in the same component of  $\pi \text{Aut}(X)$ .

- For surfaces:  $\text{MCG}(S)$  on the Teichmuller space  $T(S)$ , yielding a SES

$$0 \rightarrow \text{MCG}(S) \rightarrow T(S) \rightarrow \widetilde{\mathcal{M}}_g(S) \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to  $X$ .

- $T(S)$  is the moduli space of complex structures on  $S$ , up to the action of homeomorphisms that are isotopic to the identity:

\* Points are isomorphism classes of marked Riemann surfaces

- Used in the Nielsen-Thurston Classification (for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of: periodic: reducible (preserves some simple closed curves), or pseudo-Anosov (has directions of expansion/contraction))

- Generated by Dehn twists: a self homeomorphism
- Any finite group is  $\text{MCG}(X)$  for some compact hyperbolic 3-manifold  $X$ .

**Theorem 1.1 (Dehn-Nielsen-Baer).**

$$\text{MCG}^{\pm}(\Sigma_g) \cong \text{Out}(\pi_1(\Sigma_g)).$$

## 1.4 Dehn Twists

**Claim:** Let  $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ , then  $\text{MCG}(A) \cong \mathbb{Z}$ , generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto \exp(2\pi i |z|) z. \end{aligned}$$