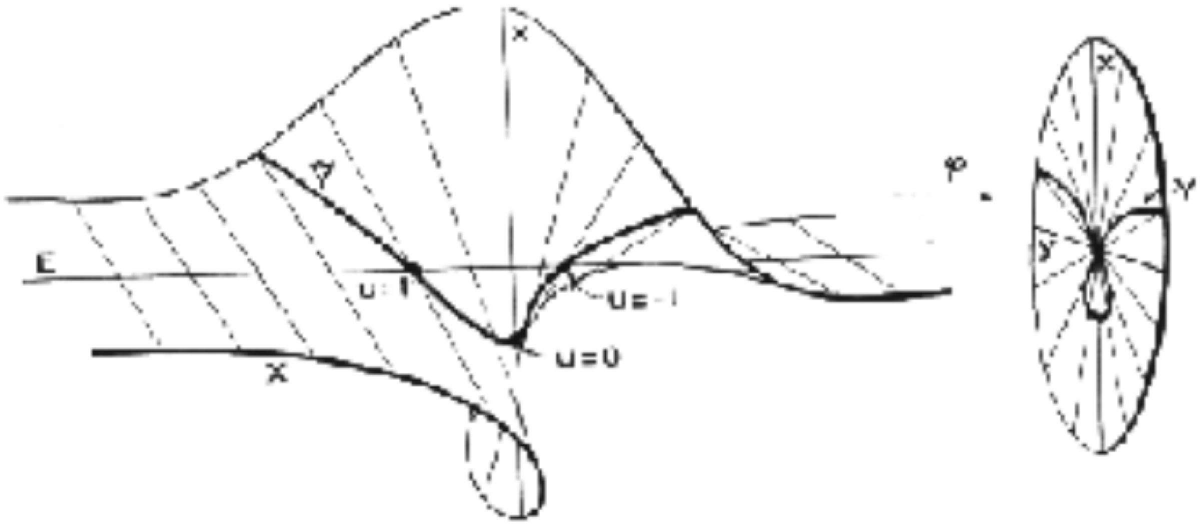

Notes: These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Philip Engel at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

Algebraic Geometry

University of Georgia, Fall 2020

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Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[1] `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

0.2 Notation

$V(I)$	The variety associated to an ideal $I \trianglelefteq k[x_1, \cdots, x_n]$.

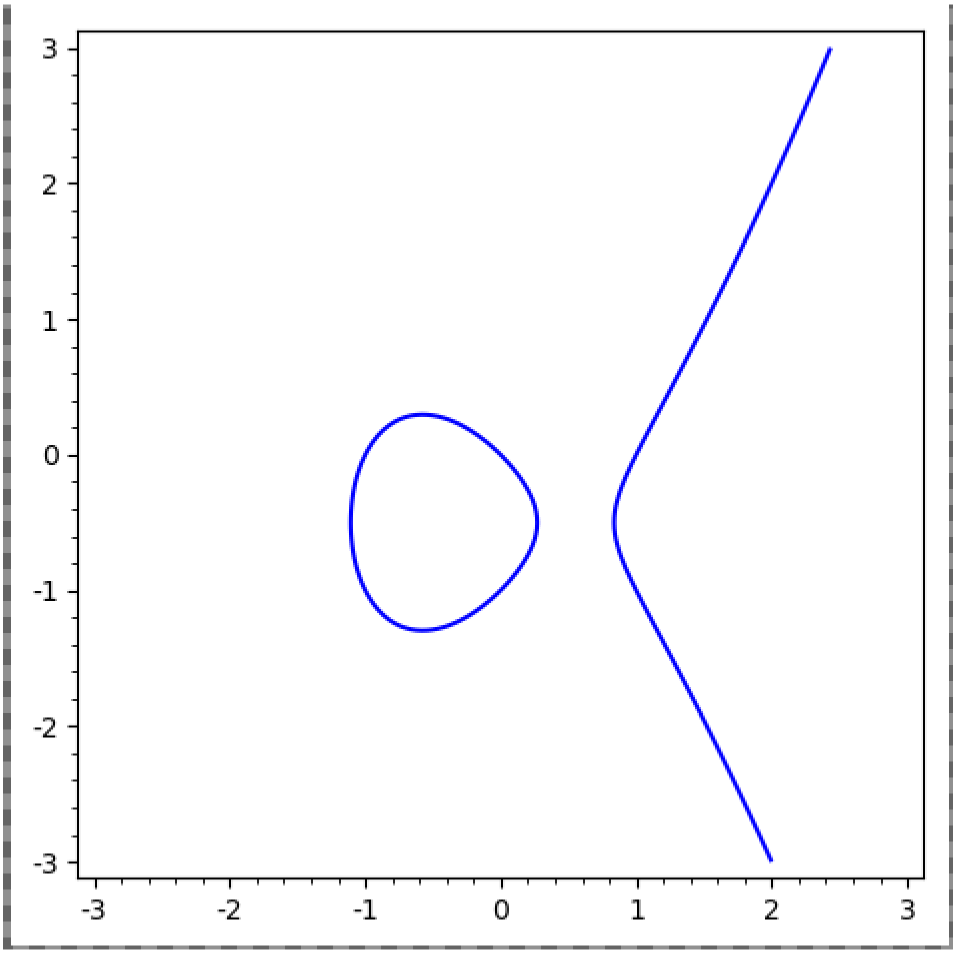
1 | Friday, August 21

Ref:
`https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.[1]

Example 1.0.1 :

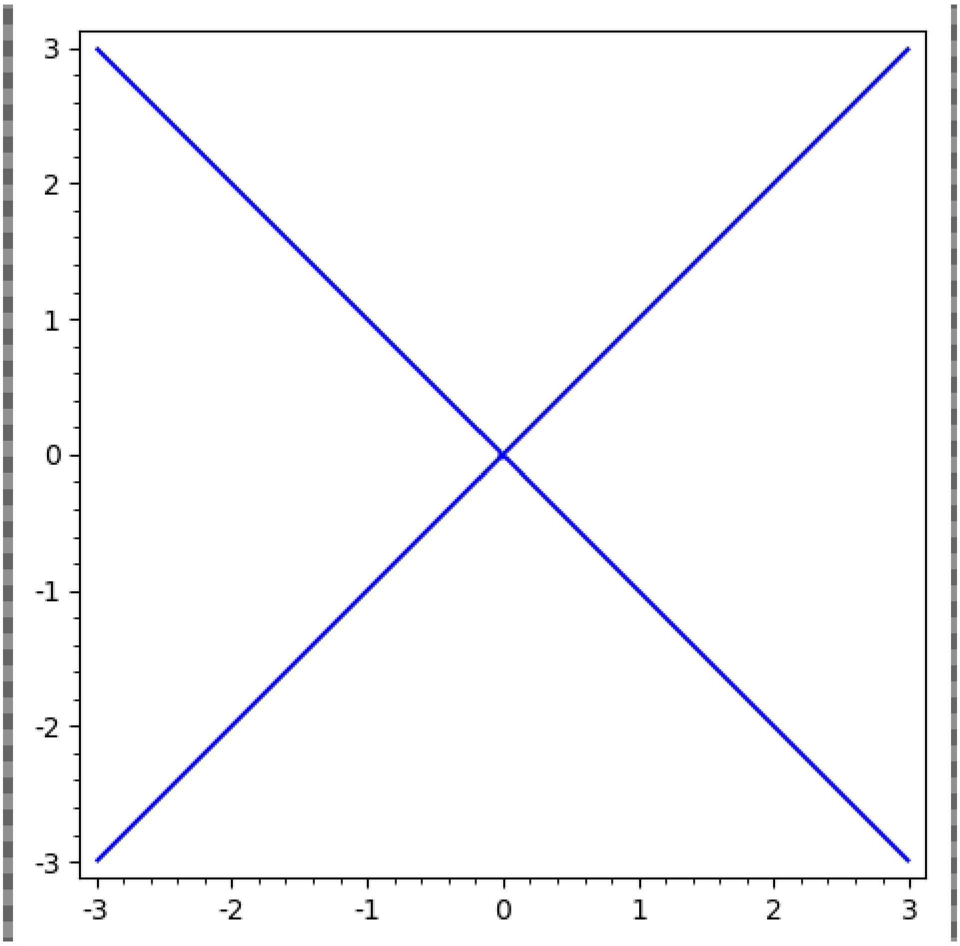
- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)

[1]Example footnote.

- $x^2 + 1$, which has no $\mathbb{R}$-points.
- $x^2 + y^2 + 1 / \mathbb{R}$ vanishes nowhere, so its ring of functions is not $\mathbb{R}[x, y] / \left\langle x^2 + y^2 + 1 \right\rangle$ (problem: $\mathbb{R}$ is not algebraically closed)
- $x^2 - y^2 = 0$ over $\mathbb{C}$ is not a manifold (no chart at the origin):



- $x + y + 1 / \mathbb{F}_3$, which has 3 points over $\mathbb{F}_3^2$, but $f(x, y) = (x^3 - x)(y^3 - y)$ vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in $\mathbb{C}$?)
  - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x, y] / \langle f \rangle$ (addressed by scheme theory)

Theorem 1.0.1 ***(Harnack Curve Theorem).***

If $f \in \mathbb{R}[x, y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.0.2 : Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.0.1 : Show that the vanishing locus of the first two equations above is $X \cup L$ for L a line.

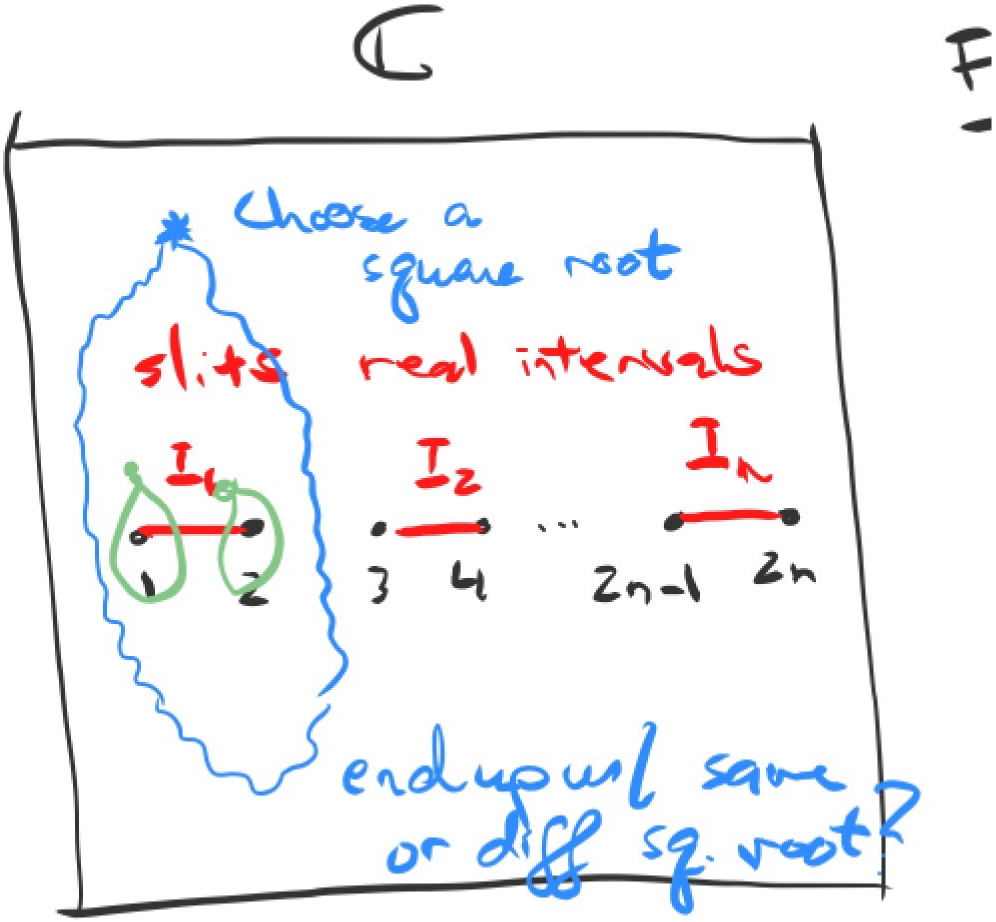
Compare to linear algebra: codimension d iff cut out by exactly d equations.

Example 1.0.3 : Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

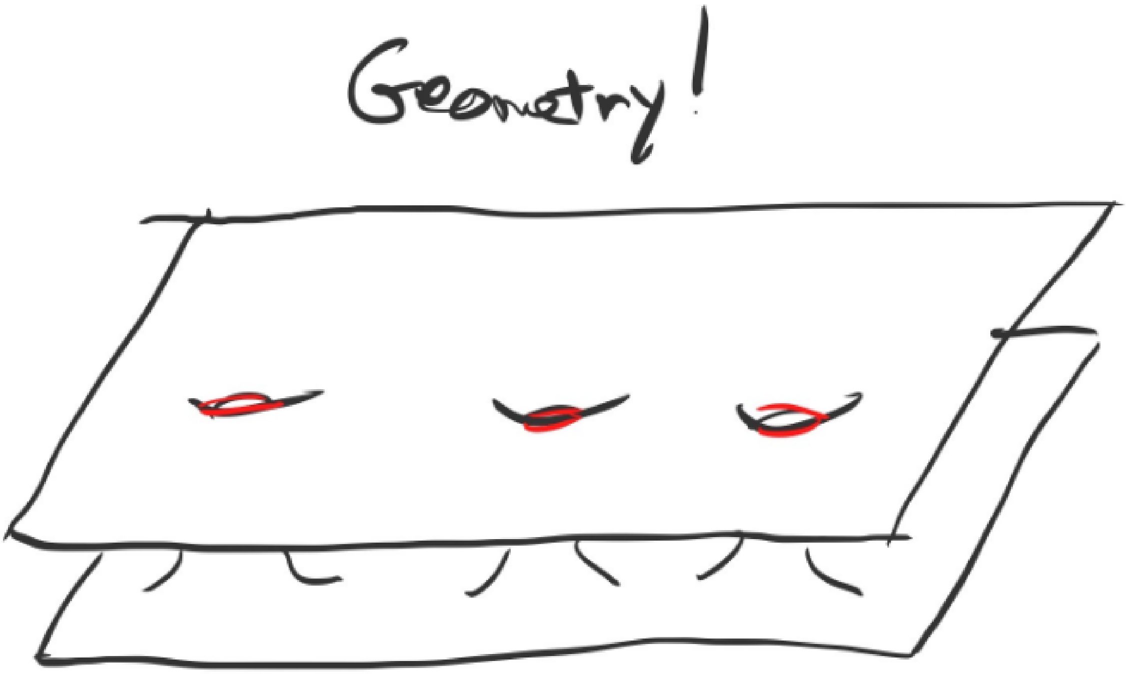
how to visualize the solution set?

Fact: on $\mathbb{C}$ with some slits, you can consistently choose a square root of the RHS.



Away from $x = 1, \cdots, 2n$, there are two solutions for y given x.

After gluing along strips, obtain:



2 | Tuesday, August 25

Let $k = \bar{k}$ and R a ring containing ideals I, J.

Definition 2.0.1 (Radical)**.**
Recall that the *radical* of I is defined as

$$\sqrt{I} = \left\{ r \in R \;\middle|\; r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

Example 2.0.1 : Let $I = (x_1, x_2^2) \subset \mathbb{C}[x_1, x_2]$, so $I = \left\{ f_1 x_1 + f_2 x_2 \;\middle|\; f_1, f_2 \in \mathbb{C}[x_1, x_2] \right\}$. Then $\sqrt{I} = (x_1, x_2)$, since $x_2^2 \in I \implies x_2 \in \sqrt{I}$.

Given $f \in k[x_1, \cdots, x_n]$, take its value at $a = (a_1, \cdots, a_n)$ and denote it $f(a)$. Set $\deg(f)$ to be the largest value of $i_1 + \cdots + i_n$ such that the coefficient of $\prod x_j^{i_j}$ is nonzero.

Example 2.0.2 : $\deg(x_1 + x_2^2 + x_1 x_2^3 = 4)$

Definition 2.0.2 (Affine Variety)**.**

1. Affine n-space $\mathbb{A}^n = \mathbb{A}^n_k$ is defined as $\left\{ (a_1, \cdots, a_n) \;\middle|\; a_i \in k \right\}$.

Remark: not k^n, since we won't necessarily use the vector space structure (e.g. adding points).

2. Let $S \subset k[x_1, \cdots, x_n]$ to be a set of polynomials. Then define $V(S) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \right\} \subset \mathbb{A}^n$ to be an *affine variety.*

Example 2.0.3 :

- $\mathbb{A}^n = V(0)$.
- For any point $(a_1, \cdots, a_n) \in \mathbb{A}^n$, then $V(x_1 - a_1, \cdots, x_n - a_n) = \{a_1, \cdots, a_n\}$ uniquely determines the point.
- For any finite set $r_1, \cdots, r_k \in \mathbb{A}^1$, there exists a polynomial $f(x)$ whose roots are r_i.

Remark 2.0.1 : We may as well assume S is an ideal by taking the ideal it generates, $S \subseteq \langle S \rangle = \left\{ \sum g_i f_i \;\middle|\; g_i \in k[x_1, \cdots, x_n],\, f_i \in S \right\}$. Then $V(\langle S \rangle) \subset V(S)$.

Conversely, if f_1, f_2 vanish at $x \in \mathbb{A}^n$, then $f_1 + f_2, gf_1$ also vanish at x for all $g \in k[x_1, \cdots, x_n]$. Thus $V(S) \subset V(\langle S \rangle)$.

Proposition 2.0.1 ***(Properties and Definitions of Ideal Operations).***

- $I + J := \left\{ f + g \;\middle|\; f \in I,\, g \in J \right\}$.
- $IJ := \left\{ \sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\, g_i \in J, N \in \mathbb{N} \right\}$.
- If $I + J = \langle 1 \rangle$ then $I \cap J = IJ$ (coprime or comaximal)

Note that if $I = \langle a \rangle$ and $J = \langle b \rangle$, then $I + J = \langle a \rangle + \langle b \rangle = \langle a, b \rangle$.

Proposition 2.0.2 ***(Properties of V).***

1. If $S_1 \subseteq S_2$ then $V(S_1) \supseteq V(S_2)$.
2. $V(S_1) \cup V(S_2) = V(S_1 S_2) = V(S_1 \cap S_2)$.
3. $\bigcap V(S_i) = V\left(\bigcup S_i \right)$.

We thus have a map

$$V : \{\text{Ideals in } k[x_1, \cdots, x_n]\} \to \{\text{Affine varieties in } \mathbb{A}^n\}.$$

Definition 2.0.3 (The Ideal of a Set)**.**
Let $X \subset \mathbb{A}^n$ be any set, then *the ideal of* X is defined as

$$I(X) := \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \, \forall x \in X \right\}.$$

Example 2.0.4 : Let X be the union of the x_1 and x_2 axes in $\mathbb{A}^2$, then $I(X) = (x_1 x_2) = \left\{ x_1 x_2 g \;\middle|\; g \in k[x_1, x_2] \right\}$.

Note that if $X_1 \subset X_2$ then $I(X_1) \subset I(X_2)$.

Proposition 2.0.3 ***(The Image of V is Radical).***
$I(X)$ is a radical ideal, i.e. $I(X) = \sqrt{I(X)}$.
This is because $f(x)^k = 0 \forall x \in X$ implies $f(x) = 0$ for all $x \in X$, so $f^k \in I(X)$ and thus $f \in I(X)$.

Our correspondence is thus

$$\left\{\text{Ideals in } k[x_1, \cdots, x_n]\right\} \xrightarrow{V} \left\{\text{Affine Varieties}\right\}$$
$$\left\{\text{Radical Ideals}\right\} \xleftarrow{I} \{?\}.$$

Proposition 2.0.4 ***(Hilbert Nullstellensatz (Zero Locus Theorem)).***

a. For any affine variety X, $V(I(X)) = X$.

b. For any ideal $J \subset k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

Thus there is a bijection between radical ideals and affine varieties.

2.1 Proof of Nullstellensatz

Remark 2.1.1 : Recall the Hilbert Basis Theorem: any ideal in a finitely generated polynomial ring over a field is again finitely generated.

We need to show 4 inclusions, 3 of which are easy.

a: $X \subset V(I(X))$:

- If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$.
- So $x \in V(I(X))$, since every $f \in I(X)$ vanishes at x.

b: $\sqrt{J} \subset I(V(J))$:

- If $f \in \sqrt{J}$ then $f^k \in J$ for some k.
- Then $f^k(x) = 0$ for all $x \in V(J)$.
- So $f(x) = 0$ for all $x \in V(J)$.
- Thus $f \in I(V(J))$.

c: $V(I(X)) \subset X$:

- Need to now use that X is an affine variety.
  - Counterexample: $X = \mathbb{Z}^2 \subset \mathbb{C}^2$, then $I(X) = 0$. But $V(I(X)) = \mathbb{C}^2$, but $\mathbb{C}^2 \not\subset \mathbb{Z}^2$.
- By (b), $I(V(J)) \supset \sqrt{J} \supset J$.
- Since $V(\,\cdot\,)$ is order-reversing, taking V of both sides reverses the containment.
- So $V(I(V(J))) \subset V(J)$, i.e. $V(I(X)) \subset X$.

d: $I(V(J)) \subset \sqrt{J}$ (hard direction)

Theorem 2.1.1 ***(1st Version of Nullstellensatz).***
Suppose k is algebraically closed and uncountable (still true in countable case by a different proof).
Then the maximal ideals in $k[x_1, \cdots, x_n]$ are of the form $(x_1 - a_1, \cdots, x_n - a_n)$.

Proof .
Let $\mathfrak{m}$ be a maximal ideal, then by the Hilbert Basis Theorem, $\mathfrak{m} = \langle f_1, \cdots, f_r \rangle$ is finitely generated.
Let $L = \mathbb{Q}[\{c_i\}]$ where the c_i are all of the coefficients of the f_i if $\mathrm{ch}(K) = 0$, **or** $\mathbb{F}_p[\{c_i\}]$ if $\mathrm{ch}(k) = p$. Then $L \subset k$.
Define $\mathfrak{m}_0 = \mathfrak{m} \cap L[x_1, \cdots, x_n]$. Note that by construction, $f_i \in \mathfrak{m}_0$ for all i, and we can write $\mathfrak{m} = \mathfrak{m}_0 \cdot k[x_1, \cdots, x_n]$.
Claim: $\mathfrak{m}_0$ is a maximal ideal.
If it were the case that

$$\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq L[x_1, \cdots, x_n],$$

then

$$\mathfrak{m}_0 \cdot k[x_1, \cdots, x_n] \subsetneq \mathfrak{m}_0' \cdot k[x_1, \cdots, x_n] \subsetneq k[x_1, \cdots, x_n].$$

So far: constructed a smaller polynomial ring and a maximal ideal in it.
Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is a field that is finitely generated over either $\mathbb{Q}$ or $\mathbb{F}_p$.

Theorem 2.1.2 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Note: this theorem is perhaps more important than the Nullstellensatz!

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is finite over some $\mathbb{Q}(t_1, \cdots, t_n)$, and since k is uncountable, there exists an embedding $\mathbb{Q}(t_1, \cdots, t_n) \hookrightarrow k$.

Use the fact that there are only countably many polynomials over a countable field.

This extends to an embedding of $\varphi : L[x_1, \cdots, x_n]/\mathfrak{m}_0 \hookrightarrow k$ since k is algebraically closed. Letting a_i be the image of x_i under φ, then $f(a_1, \cdots, a_n) = 0$ by construction, $f_i \in (x_i - a_i)$ implies that $\mathfrak{m} = (x_i - a_i)$ by maximality.

■

3 | Thursday, August 27

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{ \text{Radical ideals } k[x_1, \cdots, x_n] \right\} \to V(\,\cdot\,) I(\,\cdot\,) \left\{ \text{Affine varieties in } \mathbb{A}^n \right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 3.0.1 *(?)*.
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty.

■

Returning to the proof that $I(V(J)) \subseteq \sqrt{J}$: let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. But $ft - 1 \neq 0$ if $f = 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

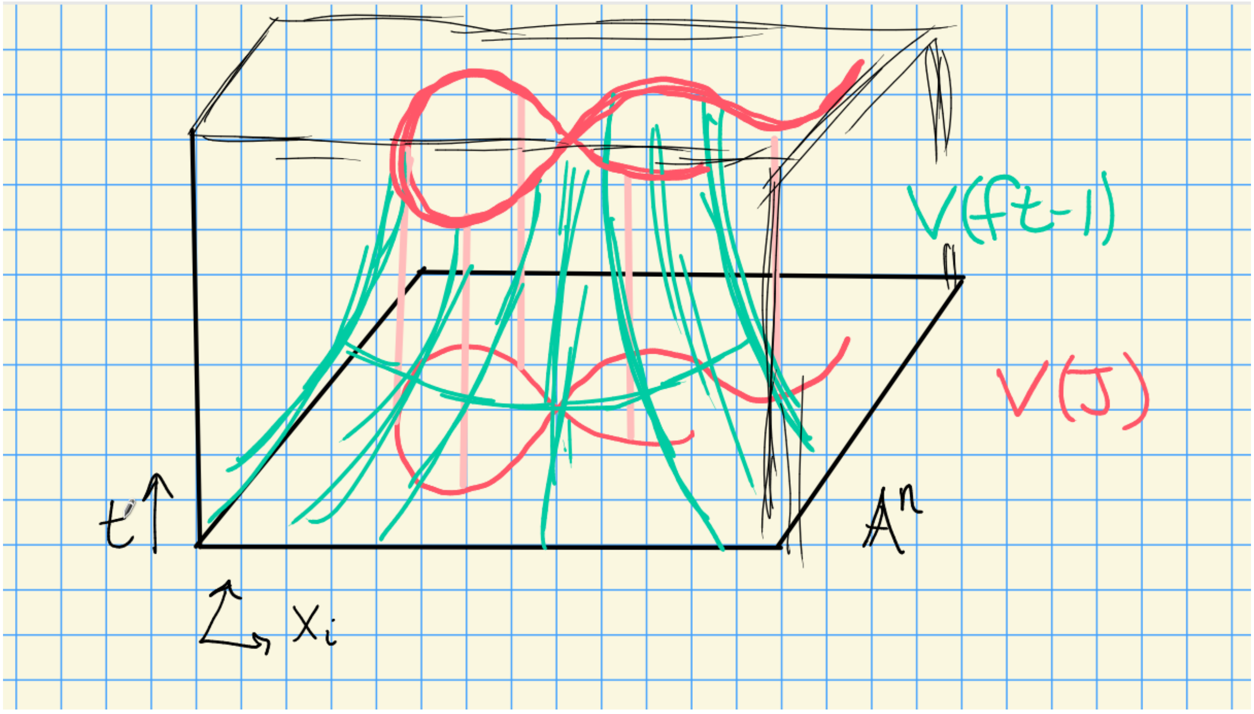


Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so $1 = \langle ft - 1 \rangle \, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$ with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N := (ft - 1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t.

Now take $k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Observation: there is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Exercise 3.0.1 : Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 3.0.1 : Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then $I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J$. Note that this loses information.

Example 3.0.2 : Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\{\text{Points of } \mathbb{A}^n\} \iff \{\text{Maximal ideals of } k[x_1, \cdots, x_n]\}.$$

Theorem 3.0.1 ***(Properties of*** I***).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) on the variety side.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

■

Example 3.0.3 : Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

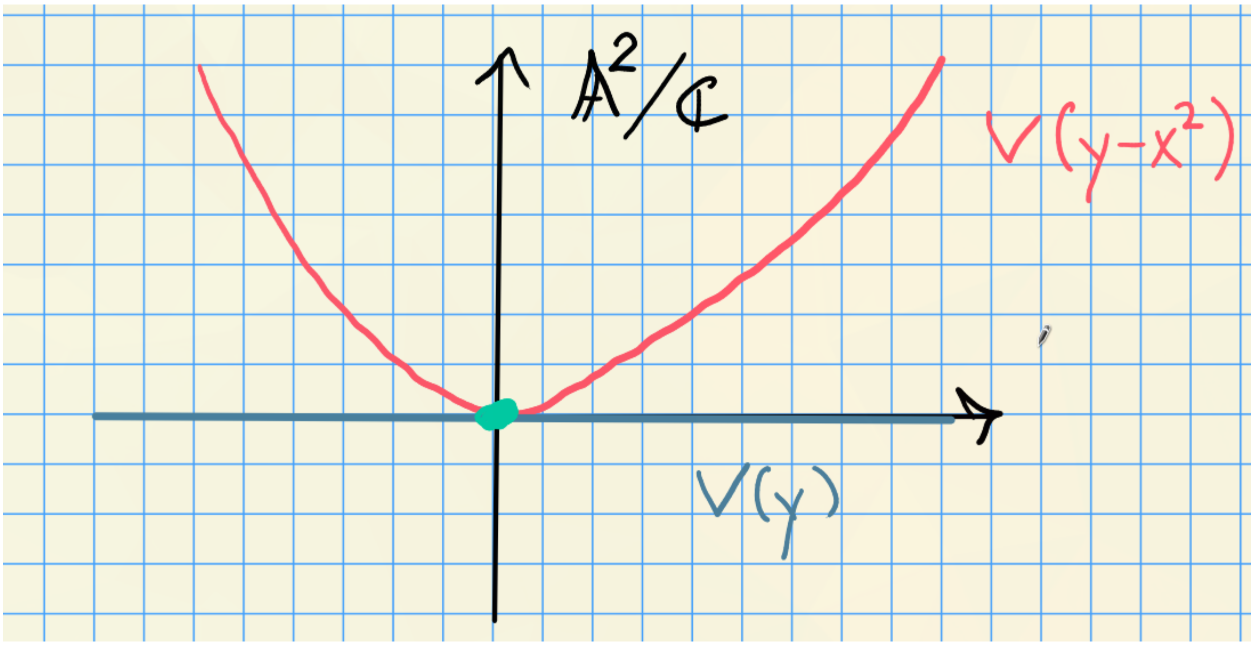


Figure 2: Image

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but $I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$.

Proposition 3.0.1 *(?)*.
If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof .
Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$. ∎

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

Definition 3.0.1 (Coordinate Ring).
For an affine variety X, the *coordinate ring of X* is

$$A(X) := k[x_1, \cdots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation: The constructions $V(\cdot), I(\cdot)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \left\{ x \in Y \;\middle|\; f(x) = 0 \; \forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \left\{ f \in A(Y) \;\middle|\; f(x) = 0 \; \forall x \in X \right\} \subseteq A(Y).$$

Example 3.0.4 : For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$\begin{tikzcd}
A(\mathbb{A}^n) \arrow[r, two heads, "\cdot / I(Y)"] \arrow[rr, bend left, two heads, "\cdot / I(X)"] & A(Y) \arrow[r, two heads, "\cdot / I(X)"] & A(X)
\end{tikzcd}$$

Theorem 3.0.2 ***(?).***
Let $X \subset Y$ be an affine subvariety, then

a. $A(X) = A(Y)/I_Y(X)$

b. There is a correspondence

$$\begin{aligned}
\left\{ \text{Affine subvarieties of } Y \right\} &\iff \left\{ \text{Radical ideals in } A(Y) \right\} \\
X &\mapsto I_Y(X) \\
V_Y(J) &\leftarrow\!\shortmid J.
\end{aligned}$$

Proof .
Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ■

Example 3.0.5 : Let $Y = V(y - x^2) \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x-1, y-1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \langle x-1, y-1 \rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$\begin{tikzcd}
A(\mathbb{A}^n) \arrow[r] \arrow[d, equal] & A(Y) \arrow[r] \arrow[d, equal] & A(X) \arrow[d, equal] \\
k[x,y] \arrow[r] & k[x,y]/\left\langle y - x^2 \right\rangle \arrow[r] & k[x,y]/\langle x-1, y-1 \rangle
\end{tikzcd}$$

4 | Tuesday, September 01

Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \, \forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \, \forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) := k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 4.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\displaystyle\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i\right)$.

Remark 4.0.1 : There are few closed sets, so this is a "weak" topology.

Example 4.0.1 : Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A := \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 4.0.2 : Let $f : \mathbb{A}^1/k \to \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 4.0.3 : Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\displaystyle\bigcup_{i=1}^{n} U_i \times V_i$ with U_i, V_i open in X, Y respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta := V(x-y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

Example 4.0.4 : Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 := V(x_1)$ and $X_2 := V(x_2)$. The Zariski topology can detect these decompositions.

Definition 4.0.2 (Irreducibility and Connectedness)**.**
Let X be a topological space.

a. X is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible*.

b. X is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, X is said to be *connected*.

Example 4.0.5 : $V(x_1x_2)$ is reducible but connected.

Remark 4.0.2 : $\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

Proposition 4.0.1 ***(?).***
Let X be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof .
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1 \emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \to A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

■

Proposition 4.0.2 ***(?).***
An affine variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof .
$\implies$: By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\Longleftarrow$: Suppose X is reducible with $X = X_1 \cup X_2$ with X_i proper and closed. Define $J_i := I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so f_i vanishes on X_i. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain. ■

Example 4.0.6 : Let $X = \{p_1, \cdots, p_d\}$ be a finite set in $\mathbb{A}^n$. The Zariski topology on X is the discrete topology, and $X = \coprod \{p_i\}$. So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^{d} A(\{p_i\}) = \prod_{i=1}^{d} k[x_1, \cdots, x_n] / \langle x_j - a_j(p_i) \rangle_{j=1}^{d}.$$

Example 4.0.7 : Set $V(x_1 x_2) = X$, then $A(X) = k[x_1, x_2] / \langle x_1 x_2 \rangle$. This not being a domain (since $x_1 x_2 = 0$) corresponds to $X = V(x_1) \cup V(x_2)$ not being irreducible.

Example 4.0.8 : $\mathbb{A}^2/k$ is irreducible since $k[x_1, \cdots x_n]$ is a domain.

Example 4.0.9 : Let X_1 be the xy plane and X_2 be the line parallel to the y-axis through $[0, 0, 1]$, and let $X = X_1 \coprod X_2$. Then $X_1 = V(z)$ and $X_2 = V(x, z-1)$, and $I(X) = \langle z \rangle \cdots \langle x, z-1 \rangle = \left\langle xz, z^2 - z \right\rangle$.

Then the coordinate ring is given by $A(X) = \mathbb{C}[x, y, z] / \left\langle xz, z^2 - z \right\rangle = \mathbb{C}[x, y, z] / \langle z \rangle \oplus \mathbb{C}[x, y, z] / \langle x, z-1 \rangle$.

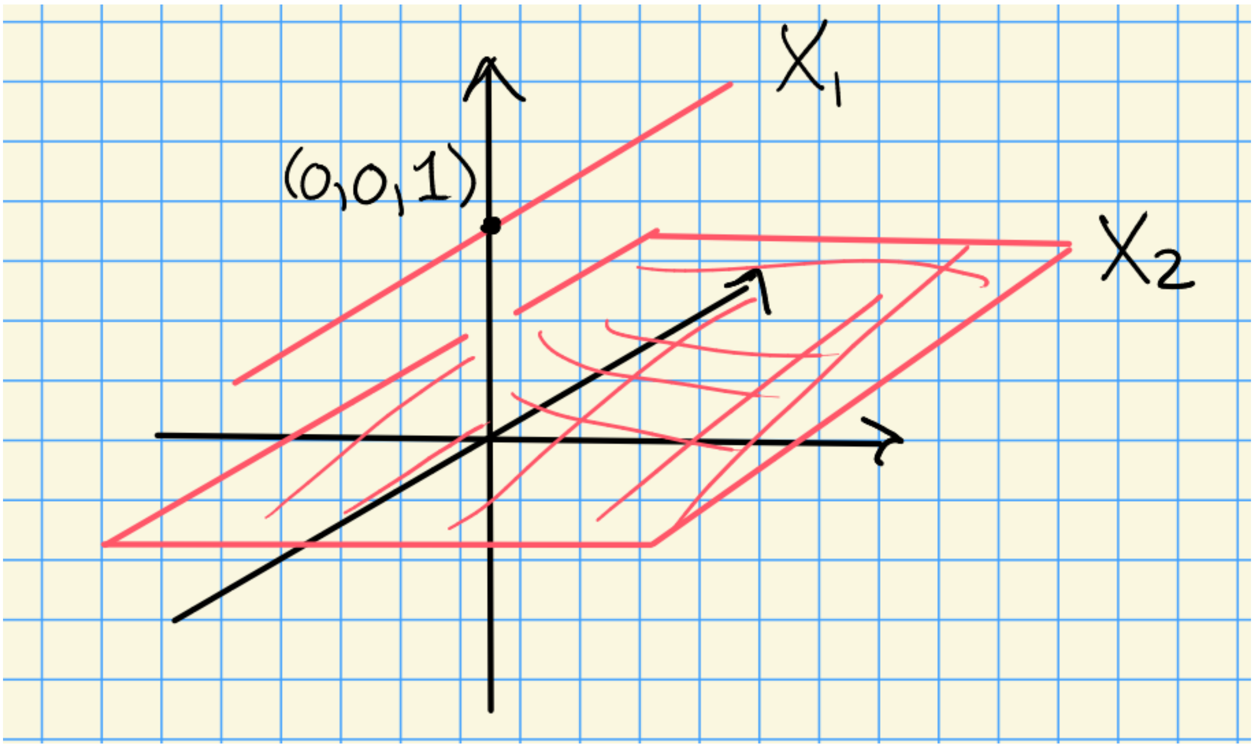


Figure 3: Image

5 | Thursday, September 03

Recall that the Zariski topology is defined on an affine variety $X = V(J)$ with $J \trianglelefteq k[x_1, \cdots, x_n]$ by describing the closed sets.

Proposition 5.0.1 *(?).*
X is irreducible if its coordinate ring $A(X)$ is a domain.

Proposition 5.0.2 *(?).*
There is a 1-to-1 correspondence

$$\left\{ \substack{\text{Irreducible subvarieties} \\ \text{of } X} \right\} \iff \left\{ \substack{\text{Prime ideals} \\ \text{in } A(X)} \right\}.$$

Proof .
Suppose $Y \subset X$ is an affine subvariety. Then

$$A(X)/I_X(Y) = A(Y).$$

By NSS, there is a bijection between subvarieties of X and radical ideals of $A(X)$ where $Y \mapsto I_X(Y)$. A quotient is a domain iff quotienting by a prime ideal, so $A(Y)$ is a domain iff $I_X(Y)$ is prime.

■

Recall that $\mathfrak{p} \trianglelefteq R$ is prime when $fg \in \mathfrak{p} \iff f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus $\bar{f}\bar{g} = 0$ in $R/\mathfrak{p}$ implies $\bar{f} = 0$ or $\bar{g} = 0$ in $R/\mathfrak{p}$, i.e. $R/\mathfrak{p}$ is a domain.

Finally note that prime ideals are radical (easy proof).

Example 5.0.1 : Consider $\mathbb{A}^2/\mathbb{C}$ and some subvarieties C_i:

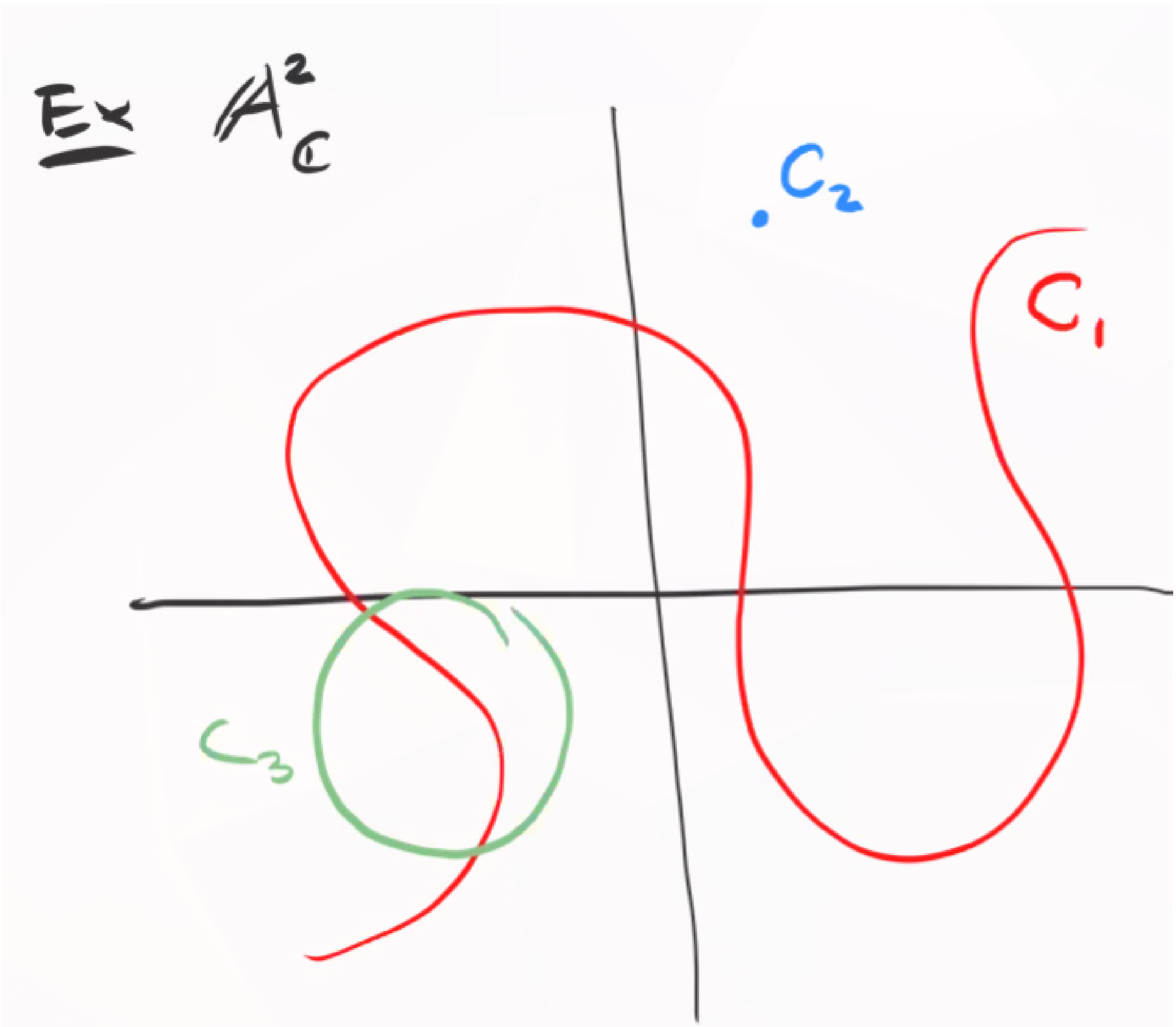


Figure 4: Subvarieties

Then irreducible subvarieties correspond to prime ideals in $\mathbb{C}[x,y]$. Here C_1, C_3 correspond to $V(f), V(g)$ for f, g irreducible polynomials, whereas C_2 corresponds to a maximal ideal, i.e. $V(x_1 - a_1, x_2 - a_2)$.

Note that $I(C_1 \cup C_2 \cup C_3)$ is not a prime ideal, since the variety is reducible as the union of 3 closed subsets.

Example 5.0.2 : A finite set is irreducible iff it contains only one point.

Example 5.0.3 : Any irreducible topological space is connected, since irreducible requires a union but connectedness requires a *disjoint* union.

Example 5.0.4 : $\mathbb{A}^n/k$ is irreducible: by prop 2.8, its irreducible iff the coordinate ring is a domain. However $A(\mathbb{A}^n) = k[x_1, \cdots, x_n]$, which is a domain.

Example 5.0.5 : $V(x_1 x_2)$ is not irreducible, since it's equal to $V(x_1) \cup V(x_2)$.

Definition 5.0.1 (Noetherian Space)**.**
A *Noetherian* topological space X is a space with no infinite strictly decreasing sequence of closed subsets.

Proposition 5.0.3*(?).*
An affine variety X with the zariski topology is a noetherian space.

Proof .
Let $X_0 \supsetneq X_1 \supsetneq \cdots$ be a decreasing sequence of closed subspaces. Then $I(X_0) \subsetneq I(X_1) \subsetneq$. Note that these containments are strict, otherwise we could use $V(I(X_1)) = X_1$ to get an equality in the original chain.
Recall that a ring R is Noetherian iff every ascending chain of ideals terminates. Thus it suffices to show that $A(X)$ is Noetherian.
We have $A(X) = k[x_1, \cdots, x_n]/I(X)$, and if this had an infinite chain $I_1 \subsetneq I_2 \subsetneq \cdots$ lifts to a chain in $k[x_1, \cdots, x_n]$, which is Noetherian. A useful fact: R noetherian implies that $R[x]$ is noetherian, and fields are always noetherian.

■

Remark 5.0.1 : Any subspace $A \subset X$ of a noetherian space is noetherian. To see why, suppose we have a chain of closed sets in the subspace topology,

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq \cdots .$$

Then $X_0 \supsetneq X_1 \supsetneq \cdots$ is a strictly decreasing chain of closed sets in X. Why strictly decreasing: $\cap^n X_i = \cap^{n+1} X_i \implies A \cap^n X_i = A \cap^{n+1} X_i$, a contradiction.

Proposition 5.0.4 ***(Important).***

Every noetherian space X is a finite union of irreducible closed subsets, i.e. $X = \bigcup_{i=1}^{k} X_i$. If we further assume $X_i \not\subset X_j$ for all i, j, then the X_i are unique up to permutation.

Remark 5.0.2 : The X_i are the **components** of X. In the previous example $C_1 \cup C_2 \cup C_3$ has three components.

Proof .
If X is irreducible, then $X = X$ and this holds.
Otherwise, write $X = X_1 \cup X_2$ with X_i proper closed subsets. If X_1 and X_1' are irreducible, we're done, so otherwise suppose wlog X_1' is not irreducible.
Then we can express $X = X_1 \cup (X_2 \cup X_2')$ with $X_2, X_2' \subset X_1'$ closed and proper.
Thus we can obtain a tree whose leaves are proper closed subsets:

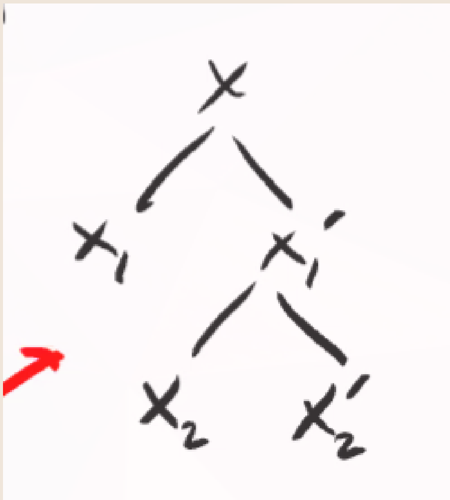


Figure 5: Image

This tree terminates because X is Noetherian: if it did not, this would generate an infinite decreasing chain of subspaces.
We now want to show that the decomposition is unique if no two components are contained in the other.
Suppose

$$X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=1}^{\ell} X_j'.$$

Note that $X_i \subset X$ implies that $X_i = \bigcup_{j=1}^{\ell} X_i \cap X_j'$. But X_i is irreducible and this would express X_i as a union of proper closed subsets, so some $X_i \cap X_j'$ is *not* a proper closed subset.
Thus $X_i = X_i \cap X_j'$ for some j, which forces $X_i \subset X_j'$. Applying the same argument to X_j' to obtain $X_j' \subset X_k$ for some k.

Then $X_i \subset X'_j \subset X_k$, but $X_i \not\subset X_j$ when $j \neq i$. Thus $X_i = X'_j = X_k$, forcing the X_i to be unique up to permutation.

■

Recall from ring theory: for $I \subset R$ and R noetherian, I has a *primary decomposition* $I = \bigcap_{i=1}^{k} Q_i$ with $\sqrt{Q_i}$ prime. Assuming the Q_i are minimal in the sense that $\sqrt{Q_i} \not\subset \sqrt{Q_j}$ for any i, j, this decomposition is unique.

Applying this to $I(X) \trianglelefteq k[x_1, \cdots, x_n] = R$ yields

$$I(X) = \bigcap_{i=1}^{k} Q_i \implies X = V(I(X)) = \bigcup_{i=1}^{k} V(Q_i).$$

Letting $P_i = \sqrt{Q_i}$, noting that the P_i are prime and thus radical, we have $V(Q_i) = V(P_i)$. Writing $X = \bigcup V(P_i)$, we have $I(V(P_i)) = P_i$ and thus $A(V(P_i)) = R/P_i$ is a domain, meaning $V(P_i)$ are irreducible affine varieties.

Conversely, if we express $X = \bigcup X_i$, we have $I = I\left(\bigcup X_i\right) = \bigcap I(X_i) = \bigcap P_i$ which are irreducible since they are prime.

Remark 5.0.3 : There is a correspondence

$$\left\{\begin{matrix}\text{Irreducible components} \\ \text{of } X\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Minimal prime ideals} \\ \text{in } A(X)\end{matrix}\right\},$$

where here *minimal* is the condition that no pair of ideals satisfies a subset containment.

Remark 5.0.4 : Let X be an irreducible topological space.

Proposition 5.0.5 ***(1).***
The intersection of nonempty two open sets is *never* empty.

Proof .
Let U, U' be open and $X \setminus U, X \setminus U'$ closed. Then $U \cap U' = \emptyset \iff (X \setminus U) \cup (X \setminus U') = X$, but this is not possible since X is irreducible.

■

Irreducible iff any two nonempty open sets intersect.

Proposition 5.0.6 ***(?).***
Any nonempty open set is dense, i.e. if $U \subset X$ is open then its closure $\text{cl}_X(U)$ is dense in X.

Proof .
Write $X = \text{cl}_X(U) \cup (X \setminus U)$. Since $X \setminus U \neq X$ and X is irreducible, we have $\text{cl}_X(U) = X$. ■

6 | Tuesday, September 08

Review: we discussed irreducible components. Recall that the *Zariski topology* on an affine variety X has affine subvarieties as closed sets, and a *noetherian space* has no infinitely decreasing chains of closed subspaces.

We showed that any noetherian space has a decomposition into irreducible components $X = \cup X_i$ with X_i closed, irreducible, and unique such that no two are subsets of each other. Applying this to affine varieties, a descending chain of subspaces $X_0 \supsetneq X_1 \cdots$ in X corresponds to an increasing chain of ideals $I(X_0) \subsetneq I(X_1) \cdots$ in $A(X)$. Since $k[x_1, \cdots, x_n]$ is a noetherian ring, this chain terminates, so affine varieties are noetherian.

6.1 Dimension

Definition 6.1.1 (Dimensions)**.**
Let X be a topological space.

1. The *dimension* $\dim X \in \mathbb{N} \cup \{\infty\}$ is either ∞ or the length n of the longest chain of **irreducible** closed subsets $\emptyset \neq Y_0 \subsetneq \cdots \subsetneq Y_n \subset X$ where Y_n need not be equal to X.

2. The *codimension* of Y in X, $\text{codim}_X(Y)$, for an irreducible subset $Y \subseteq X$ is the length of the longest chain $Y \subset Y_0 \subsetneq Y_1 \cdots \subset X$.

Example 6.1.1 : Consider $\mathbb{A}^1/k$, what are the closed subsets? The finite sets, the empty set, and the entire space.

What are the irreducible closed subsets? Every point is a closed subset, so sets with more than one point are reducible. So the only irreducible closed subsets are $\{a\}, \mathbb{A}^1/k$, since an affine variety is irreducible iff its coordinate ring is a domain and $A(\mathbb{A}^1/k) = k[x]$. We can check

$$\emptyset \subseteq Y_0 = \{a\} \subseteq Y_1 = \mathbb{A}^1/k,$$

which is of length 1, so $\dim(\mathbb{A}^1/k) = 1$.

Note that we count the number of nontrivial strict subset containments in this chain.

Example 6.1.2 : Consider $V(x_1x_2) \subset \mathbb{A}^2/k$, the union of the x_i axes. Then the closed subsets are $V(x_1), V(x_2)$, along with finite sets and their unions. What is the longest chain of irreducible closed subsets?

Note that $k[x_1, x_2]/\langle x_1 \rangle \cong k[x_2]$ is a domain, so $V(x_i)$ are irreducible. So we can have a chain

$$\emptyset \subsetneq \{a\} \subsetneq V(x_1) \subset X,$$

where a is any point on the x_2-axis, so $\dim(X) = 1$.

The only closed sets containing $V(x_1)$ are $V(x_1) \cup S$ for S some finite set, which can not be irreducible.

Remark 6.1.1 : You may be tempted to think that if X is noetherian then the dimension is finite. However, finite dimension requires a bounded length on descending/ascending chains, whereas noetherian only requires "termination", which may not happen in a bounded number of steps. So this is **false**!

Example 6.1.3 : Take $X = \mathbb{N}$ and define a topology by setting closed subsets be the sets $\{0, \cdots, n\}$ as n ranges over $\mathbb{N}$, along with $\mathbb{N}$ itself. Is X noetherian? Check descending chains of closed sets:

$$\mathbb{N} \supsetneq \{0, \cdots, N\} \supsetneq \{0, \cdots, N-1\} \cdots,$$

which has length at most N, so it terminates and X is noetherian.

But note that all of these closed subsets $X_N \coloneqq \{0, \cdots, N\}$ are irreducible. Why? If $X_n = X_i \cup X_j$ then one of i, j is equal to N, i.e $X_i, X_j = X_N$.

So for every N, there exists a chain of irreducible closed subsets of length N, implying that $\dim(\mathbb{N}) = \infty$.

Remark 6.1.2 : Let X be an affine variety. There is a correspondence

$$\left\{ \substack{\text{Chains of irreducible closed subsets} \\ Y_0 \subsetneq \cdots \subsetneq Y_n \text{ in } X} \right\} \left\{ \substack{\text{Chains of prime ideals} \\ P_0 \supsetneq \cdots \supsetneq P_n \text{ in } A(X)} \right\}.$$

Why? We have a correspondence between closed subsets and radical ideals. If we specialize to irreducible, we saw that these correspond to radical ideals $I \subset A(X)$ such that $A(Y) \coloneqq A(X)/I$ is a domain, which precisely correspond to prime ideal in $A(X)$.

We thus make the following definition:

Definition 6.1.2 (Krull Dimension)**.**
The *krull dimension* of a ring R is the length n of the longest chain of prime ideals

$$P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n.$$

Remark 6.1.3 : This uses the key fact from commutative algebra: a finitely generated k-algebra M satisfies

1. M has finite k-dimension
2. If M is a domain, every maximal chain has the same length.

Remark 6.1.4 : From scheme theory: for any ring R, there is an associated topological space $\operatorname{Spec} R$ given by the set of prime ideals in R, where the closed sets are given by

$$V(I) = \left\{ \text{Prime ideals } \mathfrak{p} \trianglelefteq R \;\middle|\; I \subseteq \mathfrak{p} \right\}.$$

If R is a noetherian ring, then $\operatorname{Spec}(R)$ is a noetherian space.

Example 6.1.4 : Using the fact above, let's compute $\dim \mathbb{A}^n/k$. We can take the following chain of prime ideals in $k[x_1, \cdots, x_n]$:

$$0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \cdots \subsetneq \langle x_1, \cdots, x_n \rangle .$$

By applying $V(\,\cdot\,)$ we obtain

$$\mathbb{A}^n/k \supsetneq \mathbb{A}^{n-1}/k \cdots \supsetneq \mathbb{A}^0/k = \{0\} \supsetneq \emptyset,$$

where we know each is irreducible and closed, and it's easy to check that these are maximal:

If there were an ideal $\langle x_1, x_2 \rangle \subset P \subset \langle x_1, x_2, x_3 \rangle$, then take $P \cap k[x_1, x_2, x_3]/\langle x_1, x_2 \rangle$ which would yield a polynomial ring in $k[x_1]$. But we know the only irreducible sets in $\mathbb{A}^1/k$ are a point and the entire space.

So this is a chain of maximal length, implying $\dim \mathbb{A}^n/k = n$.

7 | Thursday, September 10

Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined $\dim X$ to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking $R = A(X)$, i.e. $\dim \mathbb{A}^n/k = n$.

Proposition 7.0.1 *(Dimensions).*
Let $k = \bar{k}$.

a. The dimension of $k[x_1, \cdots, x_n]$ is n.
b. All maximal chains of prime ideals have length n.

7.1 Proof of Dimension Proposition

The case for $n = 0$ is trivial, just take $P_0 = \langle 0 \rangle$. For $n = 1$, easy to see since the only prime ideals in $k[x]$ are $\langle 0 \rangle$ and $\langle x - a \rangle$, since any polynomial factors into linear factors.

Let $P_0 \subsetneq \cdots \subsetneq P_m$ be a maximal chain of prime ideals in $k[x_1, \cdots, x_n]$; we then want to show that $m = n$. Assume $P_0 = \langle 0 \rangle$, since we can always extend our chain to make this true (using maximality). Then P_1 is a minimal prime and P_m is a maximal ideal (and maximals are prime).

Claim: P_1 is principle, i.e. $P_1 = \langle f \rangle$ for some irreducible f.

7.1.1 Proof That P_1 is Principle

Claim: $k[x_1, \cdots, x_n]$ is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD $\implies R[x]$ is a UFD for any R.

See Gauss' lemma.

Claim: In a UFD, minimal primes are principal. Let $r \in P$, and write $r = u \prod p_i^{n_i}$ with p_i irreducible and u a unit. So some $p_i \in P$, and p_i irreducible implies $\langle p_i \rangle$ is prime. Since $0 \subsetneq \langle p_i \rangle \subset P$, but P was prime and assumed minimal, so $\langle p_i \rangle = P$.

The idea is to now transfer the chain $P_0 \subsetneq \cdots \subsetneq P_m$ to a maximal chain in $k[x_1, \cdots, x_{n-1}]$. The first step is to make a linear change of coordinates so that f is monic in the variable x_n.

Example 7.1.1 : Take $f = x_1 x_2 + x_3^2 x_4$ and map $x_3 \mapsto x_3 + x_4$.

So write

$$f(x_1, \cdots, x_n) = x_n^d + f_1(x_1, \cdots, x_{n-1}) x_n^{d-1} + \cdots + f_d(x_1, \cdots, x_{n-1}).$$

We can then descend to $k[x_1, \cdots, x_n]$ to $k[x_1, \cdots, x_n] / \langle f \rangle$:

$$\begin{tikzcd}
P_0 \arrow[r] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m \arrow[d] \\
 & P_1/P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & P_m/P_1 \arrow[d] \\
 & P_1/P_1 \cap k[x_1, \cdots, x_{n-1}] \arrow[r] & \cdots \arrow[r] & (P_m/P_1) \cap k[x_1, \cdots, x_{n-1}]
\end{tikzcd}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

Definition 7.1.1 (Integral Extension)**.**
An *integral* ring extension $R \hookrightarrow R'$ of R is one such that all $r' \in R'$ satisfying a monic polynomial with coefficients in R, where R' is finitely generated.

In this case, also implies that R' is a finitely-generated R module.

In this case, $k[x_1, \cdots, x_{n-1}] \hookrightarrow k[x_1, \cdots, x_n]/\langle f \rangle$ is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

Lemma 7.1.
Suppose $P', Q' \subset R'$ are distinct prime ideals with $R \hookrightarrow R'$ an integral extension. Then if $P' \cap R = Q' \cap R$, neither contains the other, i.e. $P' \not\subset Q'$ and $Q' \not\subset P'$.

Proof .
Toward a contradiction, suppose $P' \subset Q'$, we then want to show that $Q' \supset P'$. Let $a \in Q' \setminus P'$ (again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.
Then $\bar{a} \neq 0$ in R'/P', and there exists a monic polynomial of minimal degree that $\bar{a}$ satisfies, $p(x) = x^n + \sum_{i=2}^{n} \bar{c}_i x^{n-i}$. This implies $\bar{c}_n \in Q'/P'$ (which will contradict $c_n \in P'$), since if $\bar{c}_n = 0$ then factoring out x yields a lower degree polynomial that $\bar{a}$ satisfies.
But then $\bar{a}_n \in Q' \cap R$, so ???

∎

Question: Given $R \hookrightarrow R'$ is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the "Going Up" Theorem: given $P \subset R$ prime, there exists $P' \subset R'$ prime such that $P' \cap R = P$. Furthermore, we can lift $P_1 \subset P_2$ to $P_1' \subset P_2'$, as well as "lifting sandwiches":

Figure 6: Image

In this process, the length of the chain decreased since $\langle 0 \rangle$ was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies. ■

7.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if $R \hookrightarrow R'$ is integral then $\dim R = \dim R'$.

Claim: Any affine variety has finite dimension.

Proof .
We have $\dim X = \dim A(X)$, where $A(X) := k[x_1, \cdots, x_n]I$ for some $I(X) = \sqrt{I(X)}$.
The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated k-algebra is an integral extension of some polynomial ring $k[y_1, \cdots, y_d]$. I.e., the following extension is integral:

$$k[y_1, \cdots, y_d] \hookrightarrow k[x_1, \cdots, x_n]/I.$$

We can conclude that $\dim A(X) = d < \infty$.

■

Proposition 7.2.1 *(?).*
Let X, Y be irreducible affine varieties. Then

a. $\dim X \times Y = \dim X + \dim Y$.
b. $Y \subset X \implies \dim X = \dim Y + \operatorname{codim}_X Y$.
c. If $f \in A(X)$ is nonzero, then any component of $V(f)$ has codimension 1.

Proof .

Remark 7.2.1 : Why is $X \times Y$ again an affine variety? If $X \subset \mathbb{A}^n/k$, $Y \subset \mathbb{A}^m/k$ with $X = V(I), Y = V(J)$, then $X \times Y \subset \mathbb{A}^n/k \times \mathbb{A}^m/k = \mathbb{A}^{n+m}/k$ can be given by taking $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$ using the natural inclusions of $k[x_1, \cdots, x_\ell]$.
Note that we can write

$$k[x_1, \cdots, x_n, y_1, \cdots, y_m] = k[x_1, \cdots, x_n] \otimes_k k[y_1, \cdots, y_n]$$

where we think of $x_i = x_i \otimes 1, y_j = 1 \otimes y_j$. We thus map I, J to $I \otimes 1 + 1 \otimes J$ and obtain $V(I \otimes 1 + 1 \otimes J) = X \times Y$ and $A(X \times Y) = A(X) \otimes_k A(Y)$.
In general, for k-algebras R, S,

$$R/I \otimes_k S/J \cong R \otimes_k S/\langle I \otimes 1 + 1 \otimes J \rangle .$$

Remark 7.2.2 : For R, S finitely generated k-algebras, $\dim R \otimes_k S = \dim R + \dim S$.

Part (a) is proved by the above remarks.
For part (b), the statement is equivalent to $P \subset A(X)$ with $I(Y) \subset P$ is a member of some maximal chain, along with the statement that all maximal chains are the same length. ■

8 | Tuesday, September 15

8.1 Review

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k \coloneqq \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k,$ vanishing locii of polynomi
	$I \mapsto V(I) \coloneqq \left\{a \;\middle\vert\; f(a) = 0 \forall f \in I\right\}$
$I(X) \coloneqq \left\{f \;\middle\vert\; f\|_X = 0\right\} \hookleftarrow X$	
Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets

> Recall that we defined the coordinate ring $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, which contained no nilpotents.

We had some results about dimension

1. $\dim X < \infty$ and $\dim \mathbb{A}^n = n$.
2. $\dim Y + \operatorname{codim}_X Y = \dim X$ when $Y \subset X$ is irreducible.
3. Only over $\bar{k} = k$, $\operatorname{codim}_X V(f) = 1$.

Example 8.1.1 : Take $V(x^2 + y^2) \subset \mathbb{A}^2/\mathbb{R}$

Definition 8.1.1 (?).
An affine variety Y of

- $\dim Y = 1$ is a **curve**,
- $\dim Y = 2$ is a **surface**,
- $\operatorname{codim}_X Y = 1$ is a **hypersurface in** X

Question: Is every hypersurface the vanishing locus of a *single* polynomials $f \in A(X)$?

Answer: This is true iff $A(X)$ is a UFD.

Definition 8.1.2 (Codimension in a Ring)**.**
$\operatorname{codim}_R \mathfrak{p}$ is the length of the longest chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}.$$

Recall that f is irreducible if $f = f_1 f_2 \implies f_i \in R^\times$ for one i, and f is prime iff $\langle f \rangle$ is a prime ideal, or equivalently $f \,\Big|\, ab \implies f \,\Big|\, a$ or $f \,\Big|\, b$.

Note that prime implies irreducible, since f divides itself.

Proposition 8.1.1 *(?)***.**
Let R be a Noetherian domain, then TFAE

a. All prime ideals of codimension 1 are principal.

b. R is a UFD.

Proof .
$a \implies b$:
Let f be a nonzero non-unit, we'll show it admits a prime factorization. If f is not irreducible, then $f = f_1 f_1'$, both non-units. If f_1' is not irreducible, we can repeat this, to get a chain

$$\langle f \rangle \subsetneq \langle f_1' \rangle \subsetneq \langle f_2' \rangle \subsetneq \cdots,$$

which must terminate.
This yields a factorization $f = \prod f_i$ with f_i irreducible. To show that R is a UFD, it thus suffices to show that the f_i are prime. Choose a minimal prime ideal containing f. We'll use Krull's Principal Ideal Theorem: if you have a minimal prime ideal $\mathfrak{p}$ containing f, its codimension $\operatorname{codim}_R \mathfrak{p}$ is one. By assumption, this implies that $\mathfrak{p} = \langle g \rangle$ is principal. But $g \,\Big|\, f$ with f irreducible, so f, g differ by a unit, forcing $\mathfrak{p} = \langle f \rangle$. So $\langle f \rangle$ is a prime ideal.

$b \implies a$:
Let $\mathfrak{p}$ be a prime ideal of codimension 1. If $\mathfrak{p} = \langle 0 \rangle$, it is principal, so assume not. Then there exists some nonzero non-unit $f \in \mathfrak{p}$, which by assumption has a prime factorization since R is assumed a UFD. So $f = \prod f_i$.
Since $\mathfrak{p}$ is a prime ideal and $f \in \mathfrak{p}$, some $f_i \in \mathfrak{p}$. Then $\langle f_i \rangle \subset \mathfrak{p}$ and $\mathfrak{p}$ minimal implies $\langle f_i \rangle = \mathfrak{p}$, so $\mathfrak{p}$ is principal.

■

Corollary 8.1.1 *(?)*.
Every hypersurface $Y \subset X$ is cut out by a single polynomial, so $Y = V(f)$, iff $A(X)$ is a UFD.

Example 8.1.2 : Apply this to $R = A(X)$, we find that there is a bijection

$$\text{codim}\,1 \text{ prime ideals} \iff \text{codim}\,1 \text{ closed irreducible subsets } Y \subset X, \text{ i.e. hypersurfaces.}$$

Taking $A(X) = \mathbb{C}[x, y, z]/\left\langle x^2 + y^2 - z^2 \right\rangle$, whose real points form a cone:

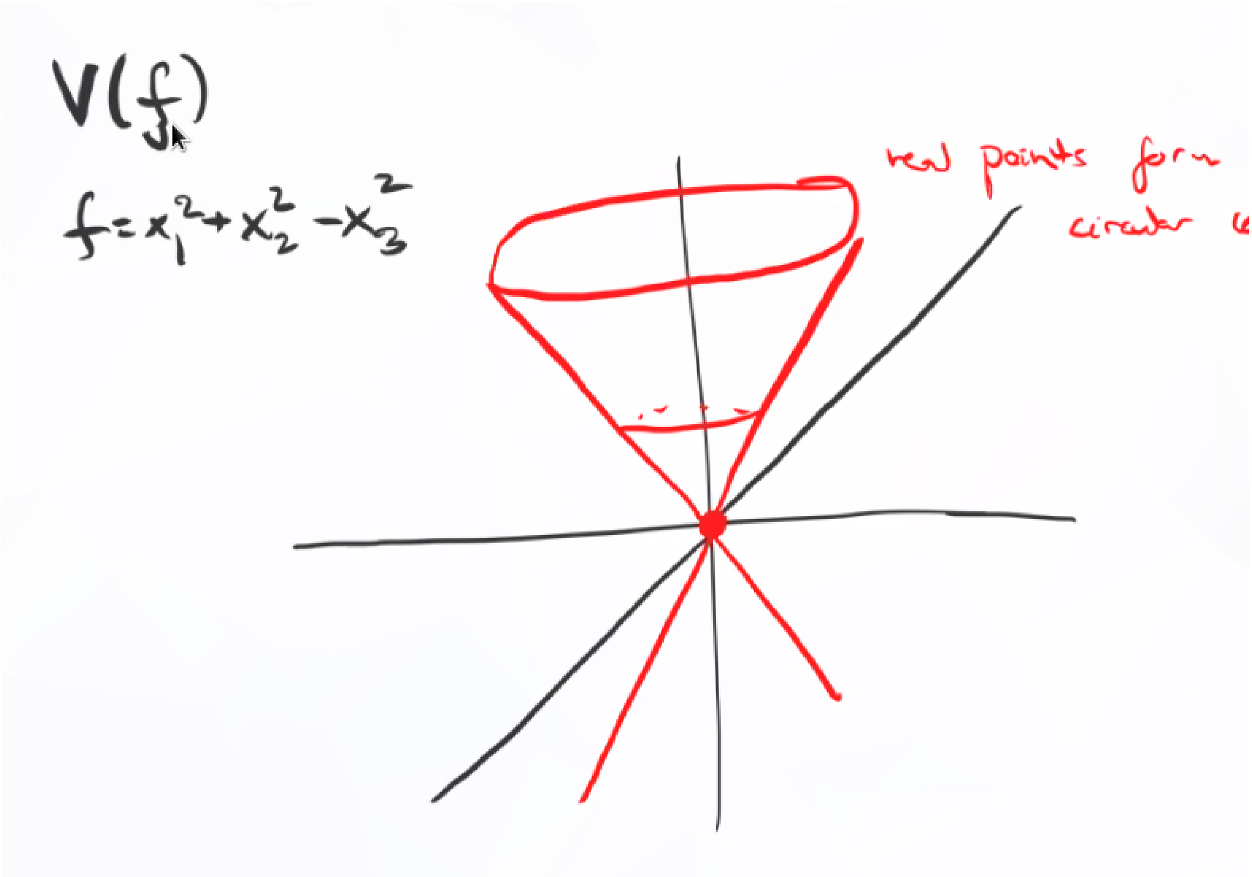


Figure 7: Image

Note that $x^2 + y^2 = (x - iy)(x + iy) = z^2$ in this quotient, so this is not a UFD.

Then taking a line through its surface is a codimension 1 subvariety not cut out by a single polynomial. Such a line might be given by $V(x + iy, z)$, which is 2 polynomials, so why not codimension 2?

Note that $V(z)$ is the union of the lines

- $z = 0, x + iy = 0$,
- $z = 0, x - iy = 0$.

Note that it suffices to show that this ring has an irreducible that is not prime. Supposing $z = f_1 f_2$, some f_i is a unit, then z is not prime because $z \mid xy$ but divides neither of x, y.

Example 8.1.3 : Note that $k[x_1, \cdots, x_n]$ is a UFD since k is a UFD. Applying the corollary, every hypersurface in $\mathbb{A}^n$ is cut out by a single irreducible polynomial.

Definition 8.1.3 (?).
An affine variety X is of **pure dimension** d iff every irreducible component X_i is of dimension d.

Note that X is a Noetherian space, so has a unique decomposition $X = \cup X_i$.

Given $X \subset \mathbb{A}^n/k$ of pure dimension $n-1$, $X = \cup X_i$ with X_i hypersurfaces with $I(X_j) = \langle f_j \rangle$, $I(X) = \langle f \rangle$ where $f = \prod f_i$.

Definition 8.1.4 (?).
Given such an X, define the **degree of a hypersurface** as the degree of f where $I(X) = \langle f \rangle$.

9 | Thursday, September 17

9.1 Regular Functions

See chapter 3 in the notes.

Some examples:

- X a manifold or an open set in $\mathbb{R}^n$ has a ring of C^∞ functions.
- $X \subset \mathbb{C}$ has a ring of holomorphic functions.
- $X \subset \mathbb{R}$ has a ring of real analytic functions

These all share a common feature: it suffices to check if a function is a member on an arbitrary open set about a point, i.e. they are *local.*

Definition 9.1.1 (?).
Let X be an affine variety and $U \subseteq X$ open. A **regular function** on U is a function $\varphi : U \to k$ such that φ is "locally a fraction", i.e. a ratio of polynomial functions.

More formally, for all $p \in U$ there exists a U_p with $p \in U_p \subseteq U$ such that $\varphi(x) = g(x)/f(x)$ for all $x \in U_p$ with $f, g \in A(X)$.

Example 9.1.1 : For X an affine variety and $f \in A(X)$, consider the open set $U := V(f)^c$. Then $\frac{1}{f}$ is a regular function on U, so for $p \in U$ we can take U_p to be all of U.

Example 9.1.2 : For $X = \mathbb{A}^1$, take $f = x - 1$. Then $\frac{x}{x-1}$ is a regular function on $\mathbb{A}^1 \setminus \{1\}$.

Example 9.1.3 : Let $X + V(x_1x_4 - x_2x_3)$ and

$$U := X \setminus V(x_2, x_4) = \left\{[x_1, x_2, x_3, x_4] \;\middle|\; x_1x_4 = x_2x_3, x_2 \neq 0 \text{ or } x_4 \neq 0\right\}.$$

Define

$$\begin{aligned} \varphi : U &\to K \\ [x_1, x_2, x_3, x_4] &\mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}. \end{aligned}$$

This is well-defined on $\{x_2 \neq 0\} \cap \{x_4 \neq 0\}$, since $\frac{x_1}{x_2} = \frac{x_3}{x_4}$. Note that this doesn't define an element of k at $[0, 0, 0, 1] \in U$. So this is not globally a fraction.

Notation: we'll let $\mathcal{O}_X(U)$ is the ring of regular function on U.

Proposition 9.1.1 *(?)*.
Let $U \subset X$ be an affine variety and $\varphi \in \mathcal{O}_X(U)$. Then $V(\varphi) := \left\{x \in U \;\middle|\; \varphi(x) = 0\right\}$ is closed in the subspace topology on U.

Proof .
For all $a \in U$ there exists $U_a \subset U$ such that $\varphi = g_a/f_a$ on U_a with $f_a, g_a \in A(X)$ with $f_a \neq 0$ on U_a.
Then

$$\left\{x \in U_a \;\middle|\; \varphi(x) \neq 0\right\} = U_a \setminus V(g_a) \cap U_a$$

is an open subset of U_a, so taking the union over a again yields an open set. But this is precisely $V(\varphi)^c$.

■

Proposition 9.1.2.
Let $U \subset V$ be open in X an *irreducible* affine variety. If $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ agree on U, then they are equal.

Proof .
$V(\varphi_1 - \varphi_2)$ contains U and is closed in V. It contains $\overline{U} \cap V$, by an earlier lemma, X irreducible implies that $\overline{U} = X$ and so $V(\varphi_1 - \varphi_2) = V$.

■

Compare and contrast: Let $U \subset V \subset \mathbb{R}^n$ be open. If $\varphi_1, \varphi_2 \in C^\infty(V)$ such that φ_1, φ_2 are equal when restricted $U \subset V$. Does this imply $\varphi_1 = \varphi_2$?

For $\mathbb{R}^n$, no, there exist smooth bump functions. You can make a bump function on $V \setminus U$ and extend by zero to U. For $\mathbb{C}$ and holomorphic functions, the answer is yes, by the uniqueness of analytic continuation.

Definition 9.1.2 ((Important) Distinguished Opens)**.**
A **distinguished open set** in an affine variety is one of the form

$$D(f) := X \setminus V(f) = \left\{ x \in X \;\middle|\; f(x) = 0 \right\}.$$

Proposition 9.1.3.
The distinguished open sets form a base of the zariski topology.

Proof .
Given $f, g \in A(X)$, we can check:

1. Closed under finite intersections: $D(f) \cap D(g) = D(fg)$.
2. $$U = X \setminus V(f_1, \cdots, f_k) = V \setminus \bigcap V(f_i) = \bigcup D(f_i),$$

   and any open set is a *finite* union of distinguished opens by the Hilbert basis theorem.

■

Proposition 9.1.4*(?)*.
The regular functions on $D(f)$ are given by

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N} \right\} = A(X)_{\langle f \rangle},$$

the localization of $A(X)$ at $\langle f \rangle$.

Note that if $f = 1$, then $\mathcal{O}_X(X) = A(X)$.

Proposition 9.1.5 *(?)*.
Note that $\frac{g}{f^n} \in \mathcal{O}_X(D(f))$ since $f^n \neq 0$ on $D(f)$. Let $\varphi : D(f) \to k$ be a regular function. By definition, for all $a \in D(f)$ there exists a local representation as a fraction $\varphi = g_a/f_a$ on $U_a \ni a$. Note that U_a can be covered by distinguished opens, one of which contains a. Shrink U_a if necessary to assume it is a distinguished open set $U_a = D(h_a)$.

Now replace

$$\varphi = \frac{g_a}{f_a} = \frac{g_a h_a}{f_a h_a},$$

which makes sense because $h_a \neq 0$ on U_a. We can assume wlog that $h_a = f_a$. Why? We have $\varphi = \frac{g_a}{f_a}$ on $D(f_a)$. Since f_a doesn't vanish on U_a, we have $V(f_a h_a) = V(h_a)$ since $V(f_a) \subset D(h_a)^c = V(h_a)$.
Consider $U_a = D(f_a)$ and $U_b = D(f_b)$, on which $\varphi = \frac{g_a}{f_a}$ and $\varphi = \frac{g_b}{f_b}$ respectively. On $U_a \cap U_b = D(f_a f_b)$, these are equal, i.e. $f_b g_a = f_a g_b$ in the coordinate ring $A(X)$.

Then $D(f) = \bigcup_a D(f_a)$, so take the component $V(f) = \cap V(f_a)$ by the Nullstellensatz $f \in I(V(f_a)) = I(V(g_a, a \in D_f)) = \sqrt{f_a \mid a \in D_f}$.
Then there exists an expression $f^n = \sum k_a f_a$ as a finite sum, so set $g - \sum g_a k_a$.
Claim: $\varphi = g/f^n$ on $D(f)$.

This follows because on $D(f_b)$, we have $\varphi = \frac{g_b}{f_b}$, and so $g f_b = \sum k_a g_a f_b$.

Finish next class

10 | Tuesday, September 22

10.1 Review: Regular Functions

Given an affine variety X and $U \subseteq X$ open, a *regular function* $\varphi : U \to k$ is one locally (wrt the zariski topology) a fraction. We write the set of regular functions as $\mathcal{O}_X$.

Example 10.1.1 : $X = V(x_1x_4 - x_2x_3)$ on $U = V(x_2, x_4)^c$, the following function is regular:

$$\varphi : U \to k$$
$$x \mapsto \begin{cases} \dfrac{x_1}{x_2} & x_2 \neq 0 \\ \\ \dfrac{x_3}{x_4} & x_4 \neq 0 \end{cases}.$$

Note that this is not globally a fraction.

Definition 10.1.1 (Distinguished Open Sets)**.**
A *distinguished open set* $D(f) \subseteq X$ for some $f \in A(X)$ is $V(f)^c := \left\{ x \in X \;\middle|\; f(x) \neq 0 \right\}$.

These are useful because the $D(f)$ form a base for the zariski topology.

Proposition 10.1.1 *(?)*.
For X an affine variety, $f \in A(X)$, we have

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N} \right\}.$$

Proof .
The first reduction we made was that $\varphi \in \mathcal{O}_X(D(f))$ is expressible as $\frac{g_a}{f_a}$ on distinguished opens $D(f_a)$ covering $D(f)$. We also noted that

$$\frac{g_a}{f_a} = \frac{g_b}{f_b} \text{ on } D(f_a) \cap D(f_b) \implies f_b g_a = f_a g_b \text{ in } A(X).$$

The second step was writing $D(f) = \cup D(f_a)$, and so $V(f) = \cap_a V(f_a)$ implies that $f \in I(V(\left\{ f_a \;\middle|\; a \in U \right\}))$. By the Nullstellensatz, $f \in \sqrt{\left\langle f_a \;\middle|\; a \in U \right\rangle}$, so $f^N = \sum k_a f_a$ for some N. So construct $g = \sum k_a g_a$, then compute

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b \sum k_a f_a = g_b f^N.$$

Thus $g/f^N = g_b/f_b$ for all b, and we can thus conclude

$$\varphi := \left\{ \frac{g_b}{f_b} \text{ on } D(f_b) \right\} = g/f^N.$$

■

Corollary 10.1.1 *(?)*.
For X an affine variety, $\mathcal{O}_X(X) = A(X)$.

Warning 10.1: For k not algebraically closed, the proposition and corollary are both false. Take $X = \mathbb{A}^1/\mathbb{R}$, then $\frac{1}{x^2+1} \in \mathbb{R}(x)$, but $\mathcal{O}_X(X) \neq A(X) = \mathbb{R}[x]$.

Definition 10.1.2 (Localization)**.**
Let R be a ring and S a set closed under multiplication, then the localization at S is defined by

$$R_S := \left\{ r/s \;\middle|\; r \in R, s \in S \right\} / \sim .$$

where $r_1/s_1 \sim r_2/s_2 \iff s_3(s_2 r_1 - s_1 r_2) = 0$ for some $s_3 \in S$.

Example 10.1.2 : Let $f \in R$ and take $S = \left\{ f^n \;\middle|\; n \geq 1 \right\}$, then $R_f := R_S$.

Corollary 10.1.2*(?).*
$\mathcal{O}_X(D(f)) = A(X)_f$ is the localization of the coordinate ring.

These requires some proof, since the LHS literally consists of functions on the topological space $D(f)$ while the RHS consists of formal symbols.

Proof .
Consider the map

$$\begin{aligned} A(X)_f &\to \mathcal{O}_X(D(f)) \\ \text{“}g/f^n\text{”} &\mapsto g/f^n : D(f) \to k. \end{aligned}$$

By definition, there exists a $k \geq 0$ such that

$$f^k(f^m g - f^n g') = 0 \implies f^k(f^m g - f^n g') = 0 \text{ as a function on } D(f).$$

Since $f^k \neq 0$ on $D(f)$, we have $f^m g = f^n g'$ as a function on $D(f)$, so $g/f^n = g'/g^m$ as functions on $D(f)$.

Surjectivity: By the proposition, we have surjectivity, i.e. any element of $|OO_x(D(f))$ can be represented by some g/f^n.

Injectivity: Suppose g/f^n defines the zero function on $D(f)$, then $g = 0$ on $D(f)$ implies that $fg = 0$ on X (i.e. $fg = 0 \in A(X)$), and we can write $f(g \cdot 1 - f^n \cdot 0) = 0$. Then $g/f^n \sim 0/1 \in A(X)_f$, which forces $g/f^n = 0 \in A(X)_f$.

■

10.2 Sheaves

Idea: spaces on functions on topological spaces.

Definition 10.2.1 (Presheaf)**.**
A *presheaf* (of rings) $\mathcal{F}$ on a topological space is

1. For every open set $U \subset X$ a ring $\mathcal{F}(U)$.
2. For any inclusion $U \subset V$ a restriction map $\mathrm{Res}_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying

a. $F(\emptyset) = 0$.
b. $\mathrm{Res}_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.
c. $\mathrm{Res}_{VW} \circ \mathrm{Res}_{UV} = \mathrm{Res}_{UW}$.

Example 10.2.1 : The smooth functions on $\mathbb{R}$ with the standard topology, $\mathcal{F} = C^\infty$ where $C^\infty(U)$ is the set of smooth functions $U \to \mathbb{R}$. It suffices to check the restriction condition, but the restriction of a smooth function is smooth: if f is smooth on U, it is smooth at every point in U, i.e. all derivatives exist at all points of U. So if $V \subset U$, all derivatives of f will exist at points $x \in V$, so f will be smooth on V.

Note that this also works with continuous functions.

Definition 10.2.2 (Sheaf)**.**
A *sheaf* is a presheaf satisfying an additional gluing property: given $\varphi_i \in \mathcal{F}(U_i)$ such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$, then there exists a unique $\varphi \in \mathcal{F}(\cup_i U_i)$ such that $\varphi|_{U_i} = \varphi_i$.

11 | Thursday, September 24

Recall that we defined the *regular functions* $\mathcal{O}_X(U)$ on an open set $U \subset X$ an affine variety as the set of functions $\varphi : U \to k$ such that φ is locally a fraction, i.e. for all $p \in U$ there exists a neighborhood of p, say $U_p \subset U$, such that φ restricted to U_p is given by $\frac{g_p}{f_p}$ for some $f_p, g_p \in A(X)$.

We proved that on a distinguished open set $D(f) = V(f)^c$, we have $\mathcal{O}_X(D(f)) = A(X)_f$. An important example was that $\mathcal{O}_X(X) = A(X)$.

Question: If X is a variety over $\mathbb{C}$, does $A(X) = \mathrm{Hol}(X)$? The answer is no, since taking $\mathbb{A}^1/\mathbb{C} \cong \mathbb{C} = X$ we obtain $A(X) = \mathbb{C}[x]$ but for example $e^z \in \mathrm{Hol}(X)$.

On the other hand, if you require that $f \in \mathrm{Hol}(X)$ is meromorphic at ∞, i.e. $f(\dfrac{1}{z})$ is meromorphic at zero, then you do get $\mathbb{C}[z]$. This is an example of GAGA!

Review: what is a category?

Review: what is a presheaf?

12 | Tuesday, September 29

Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on $\mathbb{R}^n$
- Holomorphic functions on $\mathbb{C}$

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on $\mathbb{R}$ a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \left\{ f : U \to \mathbb{R} \;\middle|\; f(x) = c \right\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$. Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0, 1)$ and $f_2 = 2$ on $(2, 3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf $\mathcal{G}$ on $\mathbb{R}$ given by *locally* constant function, i.e. $\left\{ f : U \to \mathbb{R} \;\middle|\; \forall p \in U, \exists U_p \ni p,\ f|_{U_p} \text{ is constant} \right\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 12.0.1 : Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open. Then define a sheaf by

$$\begin{aligned} \emptyset &\mapsto 0 \\ \{p\} &\mapsto R \\ \{q\} &\mapsto S \\ \implies \{p, q\} &\mapsto R \times S, \end{aligned}$$

where the sheaf condition forces the assignment of the whole space to be the product. Note that the first 3 assignments are automatically compatible, which means that we need a unique $f \in \mathcal{F}(X)$ restricting to R and S. In other words, $\mathcal{F}(X)$ needs to be unique and have maps to R, S, but this is exactly the universal property of the product.

Example 12.0.2 : Consider the presheaf on X given by $\mathcal{F}(X) = R \times S \times T$. Taking $T = \mathbb{Z}/2\mathbb{Z}$, we can force uniqueness to fail: by projecting to R, S, there are two elements in the fiber, namely $(r, s, 0) \mapsto r, s$ and $(r, s, 1) \mapsto r, s$.

Example 12.0.3 : Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Can check that it's closed under finite intersections and arbitrary unions, so this forms a topology. Now make the assignments

$$\begin{aligned} \{a\} &\mapsto A \\ \{b\} &\mapsto B \\ \{a, b\} &\mapsto C \\ X &\mapsto ?. \end{aligned}$$

We have a situation like this:

$$\begin{tikzcd}
& \mathcal{F}(X) \arrow[dl] \arrow[dr] & \\
B \arrow[dr] & & C \arrow[dl] \\
& A \arrow[d] & \\
& \emptyset &
\end{tikzcd}$$

Unique gluing says that given $r \in B, s \in C$ such that $\varphi_B(r) = \varphi_C(s)$, there should exist a unique

$t \in \mathcal{F}(X)$ such that $t|_{\{a,b\}} = r$ and $t|_{\{a,c\}} = s$. This recovers exactly the fiber product.

$$B \times_A C := \left\{ (r,s) \in B \times C \;\middle|\; \varphi_B(r) = \varphi_C(s) \in A \right\}.$$

Example 12.0.4 : Let X be an affine variety with the Zariski topology and let $\mathcal{F} := \mathcal{O}_X$ be the sheaf of regular functions:

$$\mathcal{O}_X(U) := \left\{ f : U \to k \;\middle|\; \forall p \in U, \exists U_p \ni p,\;\; f|_{U_p} = \frac{g_p}{h_p} \right\}.$$

Is this a presheaf? We can check that there are restriction maps:

$$\begin{aligned} \mathcal{O}_X(U) &\to \mathcal{O}_X(V) \\ \{f : U \to K\} &\mapsto \{ f|_V(x) := f(x) \text{ for } x \in V \}. \end{aligned}$$

This makes sense because if $V \subset U$, any $x \in V$ is in the domain of f. Given that f is locally a fraction, say $\rho = g_p/h_p$ on $U_p \ni p$, is $\varphi|_V$ locally a fraction? Yes: for all $p \in V \subset U$, $\varphi = g_p/f_p$ on U_p and this remains true on $U_p \cap V$.

To check that $\mathcal{O}_X$ is a sheaf, given a set of regular functions $\{\varphi_i : U_i \to k\}$ agreeing on intersections, define

$$\begin{aligned} \varphi : \cup U_i &\to k \\ \varphi(x) &:= \varphi_i(x) \text{ if } x \in U_i. \end{aligned}$$

This is well-defined, since if $x \in U_i \cap U_j$, $\varphi_i(x) = \varphi_j(x)$ since both restrict to the same function on $U_i \cap U_j$ by assumption.

Why is φ locally a fraction? We need to check that for all $p \in U := \cup U_i$ there exists a $U_p \ni p$ with $\varphi|_{U_p} = g_p/h_p$. But any $p \in \cup U_i$ implies $p \in U_i$ for some i. Then there exists an open set $U_{i,p} \ni p$ in U_i such that $\varphi|_{U_{i,p}} = g_p/h_p$ by definition of a regular function. So take $U_p = U_{i,p}$ and use the fact that $\varphi|_{U_i} = \varphi_i$ along with compatibility of restriction.

Remark 12.0.1 : General observation: any presheaf of functions is a sheaf when the functions are defined by a local property, i..e any property that can be checked at p by considering an open set $U_p \ni p$.

As in the examples of smooth or holomorphic functions, these were local properties. E.g. checking that a function is smooth involves checking on an open set around each point. On the other hand, being a constant function is not a local property.

Definition 12.0.1 (Restriction of a (Pre)sheaf)**.**
Given a sheaf $\mathcal{F}$ on X and an open set $U \subset X$, we can define a sheaf $\mathcal{F}|_U$ on U (with the subspace topology) by defining $\mathcal{F}|_U(V) := \mathcal{F}(V)$ for $U \subseteq V$.

Definition 12.0.2 (Stalks)**.**
Let $\mathcal{F}$ be a sheaf on X and $p \in X$ a point. The *stalk* of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$ for $p \in U$, is defined by

$$\mathcal{F}_p := \left\{ (U, \varphi) \,\middle|\, \varphi \in \mathcal{F}(U) \right\} / \sim$$

where $(U, \varphi) \sim (V, \varphi')$ iff there exists a $W \subset U \cap V$ and $p \in W$ such that $\varphi|_W = \varphi|_W{}'$.

Example 12.0.5 : What is the stalk of $\mathrm{Hol}(\mathbb{C})$ at $p = 0$?

Examples of equivalent elements in this stalk:

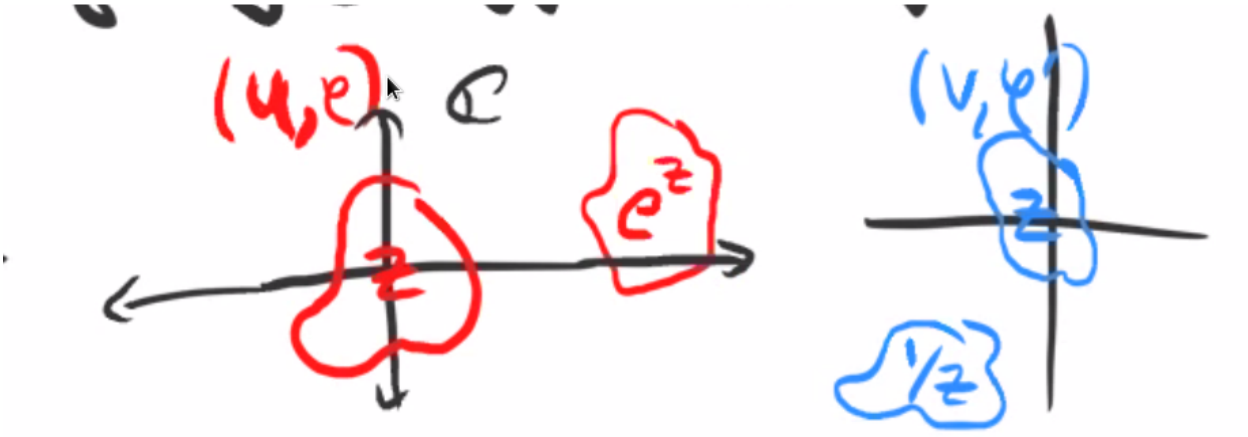


Figure 8: O

In this case

$$\mathrm{Hol}(\mathbb{C})_0 = \left\{ \varphi = \sum_{i>0} c_i z^i \,\middle|\, \varphi \text{ has a positive radius of convergence} \right\}.$$

Definition 12.0.3 (Sections)**.**
An element $f \in \mathcal{F}(U)$ is called a *section* over U, and elements of the stalk $f \in \mathcal{F}_p$ are called *germs* at p.

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13.1 Stalks and Localizations

Recall that a sheaf of rings on a topological space X is a ring $\mathcal{F}(U)$ for all open sets $U \subset X$ satisfying four properties:

1. The empty set is mapped to zeor.
2. The morphism $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity.
3. Given $W \subset V \subset U$ we have
4. Gluing: given sections $s_i \in \mathcal{F}(U_i)$ which agree on overlaps (restrict to the same function on $U_i \cap U_j$), there is a unique $s \in \mathcal{F}(\cup U_i)$.

Example 13.1.1 : If X is an affine variety with the zariski topology, $\mathcal{O}_X$ is a sheaf of regular functions, where we recall $\mathcal{O}_X(U)$ are the functions $\varphi : U \to k$ that are locally a fraction.

Recall that the *stalk* of a sheaf $\mathcal{F}$ at a point $p \in X$, is defined as

$$\mathcal{F}_p := \left\{ (U, \varphi) \;\middle|\; p \in U \text{ open }, \varphi \in \mathcal{F}(U) \right\} / \sim .$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists a $p \in W \subset U \cap U'$ such φ, φ' restricted to W are equal.

Recall that a *local ring* is a ring with a unique maximal ideal $\mathfrak{m}$. Given a prime ideal $\mathfrak{p} \in R$, so $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$, the complement $R \setminus P$ is closed under multiplication. So we can localize to obtain $R_\mathfrak{p} = \left\{ a/s \;\middle|\; s \in R \setminus P, a \in R \right\} / \sim$ where $a'/s' \sim a/s$ iff there exists a $t \in R \setminus P$ such that $t(a's - as') = 0$.

⚠ *Warning 13.1:* Note that R_f is localizing at the powers of f, whereas $R_\mathfrak{p}$ is localizing at the *complement* of $\mathfrak{p}$.

Since maximal ideals are prime, we can localize any ring R at a maximal ideal $R_\mathfrak{m}$, and this will be a local ring. Why? The ideals in $R_\mathfrak{m}$ biject with ideals in R contained in $\mathfrak{m}$. Thus all ideals in $R_\mathfrak{m}$ are contained in the maximal ideal generated by $\mathfrak{m}$, i.e. $\mathfrak{m}R_\mathfrak{m}$.

Lemma 13.1 *(?).*
Let X be an affine variety. The stalk of the sheaf of regular functions $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$ is isomorphic to the localization $A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p := I(\{p\})$.

Proof .
We can write

$$A(X)_{\mathfrak{m}_p} := \left\{ \frac{g}{f} \;\middle|\; g \in A(X),\ f \in A(X) \setminus \mathfrak{m}_p \right\} / \sim$$

where $g_1/f_1 \sim g_2/f_2 \iff \exists h(p) \neq 0$ where $0 = h(f_2 g_1 - f_1 g_2)$.

where the f are regular functions on X such that $f(p) = 0$.
We can also write

$$\mathcal{O}_{X,p} := \left\{ (U, \varphi) \;\middle|\; p \in U,\ \varphi \in \mathcal{O}_X(U) \right\} / \sim$$
$$\text{where } (U, \varphi) \sim (U', \varphi') \iff \exists p \in W \subset U \cap U' \text{ s.t. } \varphi|_W = \varphi'|_W.$$

So we can define a map

$$\begin{aligned} \Phi : A(X)_{\mathfrak{m}_p} &\to \mathcal{O}_{X,p} \\ \frac{g}{f} &\mapsto \left(D_f, \frac{g}{f} \right). \end{aligned}$$

Step 1: There are equivalence relations on both sides, so we need to check that things are well-defined.
We have

$$\begin{aligned} g/f \sim g'/f' &\iff \exists g \text{ such that } h(p) \neq 0,\ h(gf' - g'f) = 0 \in A(X) \\ &\iff \text{the functions } \frac{g}{f}, \frac{g'}{f'} \text{ agree on } W := D(f) \cap D(f') \cap D(h) \\ &\implies (D_f, g/f) \sim (D_{f'}, g'/f'), \end{aligned}$$

since there exists a $W \subset D_f \cap D_{f'}$ such that $g/f, g'/f'$ are equal.

Step 2: Surjectivity, since this is clearly a ring map with pointwise operations.
Any germ can be represented by (U, φ) with $\varphi \in \mathcal{O}_X(U)$. Since the sets D_f form a base for the topology, there exists a $D_f \subset U$ containing p. By definition, $(U, \varphi) = (D_f, \varphi|_{D_f})$ in $\mathcal{O}_{X,p}$. Using the proposition that $\mathcal{O}_X(D(f)) = A(X)_f$, this implies that $\varphi|_{D_f} = g/f^n$ for some n and $f(p) \neq 0$, so (U, φ) is in the image of Φ.

Step 3: Injectivity. We want to show that $g/f \mapsto 0$ implies that $g/f = 0 \in A(X)_{\mathfrak{m}_p}$.
Suppose that $(D_f, g/f) = 0 \in \mathcal{O}_{X,p}$ and $(U, \varphi) = 0 \in \mathcal{O}_{X,p}$, then there exists an open $W \subset D_f$ containing p such that after passing to some distinguished open $D_h \ni p$ such that $\varphi = 0$ on D_h. Wlog we can assume $\varphi = 0$ on U, since we could shrink U (staying in the same equivalence class) to make this true otherwise. Then $\varphi = g/f$ on D_h, using that $\mathcal{O}_X(D_f) = A(X)_f$, so $g/f = 0$ here. So there exists a k such that $f^k(g \cdot 1 - 0 \cdot f) = 0$ in $A(X)$, so $f^k g = 0 \in A(X)_{\mathfrak{m}_p}$. ■

Conclusion:

$$\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}.$$

Example 13.1.2 : Let $X = \{p, q\}$ with the discrete topology with the sheaf $\mathcal{F}$ given by $p \mapsto R, q \mapsto S, X \mapsto R \times S$.

Then $\mathcal{F}_p = R$, since if U is open and $p \in U$ then either $U = \{p\}$ or $U = X$. We can check that for (r, s) a section of $\mathcal{F}$, we have an equivalence of germs $(X, (r, s)) \sim (\{p\}, r)$ since $\{p\} \subset X \cap \{p\}$. Here X plays the role of U, $\{p\}$ of U', and the last $\{p\}$ the role of $W \subset U \cap U'$.

$$\begin{aligned} \mathcal{O}_{X,p} &\to A(X) \\ (\{p\}, r) &\mapsto r \\ \mathcal{F}_p &\cong R. \end{aligned}$$

Example 13.1.3 : Let M be a manifold and consider the sheaf C^∞ of smooth functions on M. Then the stalk C_p^∞ at p is defined as the set of smooth functions in a neighborhood of p modulo functions being equivalent if they agree on a small enough ball $B_\varepsilon(p)$. This contains a maximal ideal $\mathfrak{m}_p$, the smooth functions vanishing at p.

Then $\mathfrak{m}_p^2$ is again an ideal, equal to the set $\left\{ f \;\middle|\; \partial_i \partial_j f \Big|_p = 0, \forall i, j \right\}$. Thus $\mathfrak{m}_p / \mathfrak{m}_p^2 \cong \{\partial_v\}^\vee$, the dual of the set of directional derivatives.

13.2 What's the Point!

Problem: what should a map of affine varieties be? A bad definition would be just taking the continuous maps: for example, any bijection $\mathbb{A}^1_{\mathbb{C}}$ is a homeomorphism in the zariski topology. Why? This coincides with the cofinite topology, and the preimage of a cofinite set is cofinite.

How do we fix this?

1. $f : X \to Y$ is continuous, i.e. $f^{-1}(U)$ is open whenever U is open.

2. Given $U \subset Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the function $\varphi \circ f : f^{-1}(U) \to k$ is regular.

We'll take this to be the definition of a morphism $X \to Y$.

Example 13.2.1 : For smooth manifolds, we also require that there is a pullback that preserves smooth functions:

$$f^* : C^\infty(U) \to C^\infty(f^{-1}(U)).$$

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Note: the sheaf of locally constant functions valued in a set S is written $\underline{\mathbf{S}}$.

14.1 Gathmann Chapter 4

Definition 14.1.1 (Ringed Spaces)**.**
A **ringed space** is a topological space X together with a sheaf $\mathcal{O}_X$ of rings.

Example 14.1.1 :

1. X an affine variety and $\mathcal{O}_X$ its ring of regular functions.
2. X a manifold over $\mathbb{R}^n$ with $\mathcal{O}_X$ a ring of smooth or continuous functions on X.
3. $X = \{p, q\}$ with the discrete topology and $\mathcal{O}_X$ given by $p \mapsto R, q \mapsto S$.
4. Let $U \subset X$ an open subset of X an affine variety. Then declare $\mathcal{O}_U$ to be $OO_X|_U$.

Recall that the restriction of a sheaf $\mathcal{F}$ to an open subset $U \subset X$ is defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$.

Example 14.1.2 : Let X be a topological space and $p \in X$ a point. The *skyscraper sheaf at p* is defined by

$$K_p(U) := \begin{cases} K & p \in U \\ 0 & p \notin U \end{cases}.$$

Convention: we'll always assume that $\mathcal{O}_X$ is a sheaf of functions, so $\mathcal{O}_X(U)$ is a subring of all K-valued functions on U. Moreover, Res_{UV} is restriction of K-valued functions.

Definition 14.1.2 (Morphisms)**.**
A *morphism of ringed spaces*

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $X \to Y$ such that for all opens $U \subset Y$ and any $\varphi \in \mathcal{O}_Y(U)$, the pullback satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$, i.e. the pullback of a regular function is regular.

Note: need convention that $\mathcal{O}_X$ is a sheaf of K-valued functions in order to make sense of pullbacks. In general, for schemes, need some analog of $f^* : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Example 14.1.3 : If $(X, \mathcal{O}_X)$ is a ringed space associated to an affine variety, ?

Example 14.1.4 : Let $X = \mathbb{A}^1/K$ and $U = D(f)$ for $f(x) = x$, then $D(f) = \mathbb{A}^1 \setminus \{0\}$. Then $U \hookrightarrow X$ is continuous. Given an open set $D(f) \subset \mathbb{A}^1$, we have

$$\mathcal{O}_{\mathbb{A}^1}(D(f)) := \left\{ g/f^n \;\middle|\; g \in K[x] \right\}.$$

We want to show that $\iota : (U, \mathcal{O}_U) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism of ringed spaces where $\mathcal{O}_U(V) = \mathcal{O}_X(V)$. Does ι^* pull back regular functions to regular functions? Yes, since $\iota^{-1}(D(f)) = D(xf)$ and $g/f^n \in \mathcal{O}_U(\iota^{-1}(D(f)))$.

Example 14.1.5 : A non-example: take

$$h : \mathbb{A}^1 \to \mathbb{A}^1$$
$$x \mapsto \begin{cases} x & x \neq \pm 1 \\ -x & x = \pm 1 \end{cases}.$$

This is continuous because the zariski topology on $\mathbb{A}^1$ is the cofinite topology (since the closed sets are finite), so any injective map is continuous since inverse images of cofinite sets are again cofinite.

Question: Does h define a morphism of ringed spaces? I.e., is the pullback of a regular function on an open still regular? Take $U = \mathbb{A}^1$ and the regular function $x \in \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. Then $h^*x = x \circ h$, so

$$(x \circ h)(p) = \begin{cases} p & p \neq \pm 1 \\ -p & p = \pm 1 \end{cases} \notin K[x]$$

since this is clearly not a polynomial: if two polynomials agree on an infinite set of points, they are equal.

Example 14.1.6 : Consider $\iota : (\mathbb{R}^2, C^\infty) \hookrightarrow (\mathbb{R}^3, C^\infty)$ is the inclusion of a coordinate hyperplane. To say that this is a morphism of ringed spaces, we need that for all $U \subset \mathbb{R}^3$ open and $f : U \to \mathbb{R}$ a smooth function, we want $i^*f \in C^\infty(\iota^{-1}(U))$. But this is the same as $f \circ \iota \in C^\infty(\mathbb{R}^2 \cap U)$, which is true.

Proposition 14.1.1 ***(Properties of Morphisms of Ringed Spaces).***

1. They can be composed: if $\varphi \in \mathcal{O}_Z(U)$, then $g^*\varphi \in \mathcal{O}_Y(g^{-1}(U))$ and so $f^*g^*\varphi \in \mathcal{O}_X(f^{-1}g^{-1}(U))$.
2. The identity is a morphism.

Thus ringed spaces form a category, since composition is associative.

Lemma 14.1 ***(Gluing for Morphisms).***
Let $f : X \to Y$ be a continuous map between ringed spaces. Assume there exists an open cover $\{U_i\}_{i \in I} \rightrightarrows X$ such that $f|_{U_i}$ is a morphism, then f is a morphism.
Slogan: it suffices to check a morphism on an open cover.

Proof .
Part a: Need to check that f is continuous, can compute

$$f^{-1}(V) = \bigcup_{i \in I} U_i \cap f^{-1}(V) = \bigcup_{i \in I} f|_{U_i}{}^{-1}(V).$$

but the later is open as a union of open sets, where each constituent set is open by assumption.
Will finish proof next time.

■

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Proposition 15.0.1 ***(Gluing).***
Let $f : X \to Y$ be a map of ringed spaces such that there exists an open cover $U_i \rightrightarrows X$ such that $f|_{U_i}$ is a morphism of ringed spaces. Then f itself is a morphism is a morphism of ringed spaces.

Recall that we proved part (a).

Proof (part b).
We want to show that f^* sends sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$ (e.g. regular functions pullback). Let $V \subset Y$ be open and $\varphi \in \mathcal{O}_Y(V)$, then

$$f^*\varphi|_{U_i \cap f^{-1}(V)} \left(f^*\varphi|_{U_i \cap f^{-1}(V)} \right)^* \varphi \in \mathcal{O}_X(U_i f^{-1}(V)).$$

Since pullback commutes with restriction, $f^*\varphi$ is the unique k-valued function for which

$$f^*\varphi|_{U_i \cap f^{-1}V} = f|_{U_i \cap f^{-1}V}{}^* \varphi.$$

and all of the latter functions agree on overlaps $U_i \cap U_j$. This by unique gluing, $f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$.

■

Proposition 15.0.2 ***(?).***
Let $U \subset X$ be open in an affine variety and let $Y \subset \mathbb{A}^n$ be another affine variety. Then the morphisms $U \to Y$ of ringed spaces are the maps of the form $f = [f_1, \cdots, f_n] : U \to \mathbb{A}^n$ such that $f(U) \subset Y$ and $f_i \in \mathcal{O}_X(U)$ for all i.

Proof .
$\implies$: Assume that $f : U \to Y$ is a morphism. Then the coordinate functions $Y \xrightarrow{y_i} \mathbb{A}_1$ are regular functions, since they generate $\mathcal{O}_Y(Y) = k[y_1, \cdots, y_n]/I(Y)$. Then $f^* y_i$ is a regular function, so define $f_i := f^* y_i$. But then $f = [f_1, \cdots, f_n]$.
$\impliedby$: Conversely suppose $f := [f_1, \cdots, f_n] : U \to Y \subset \mathbb{A}^n$ is a map such that $f_i \in \mathcal{O}_U(U)$. We want to show that f is a morphism, i.e. that the pullback of every regular function is regular. We thus need to show

1. f is continuous, and

2. f^* pulls back regular functions.

For 1, suppose Z is closed, then it suffices to show $f^{-1}(Z)$ is closed. Then $Z = V(g_1, \cdots, g_n)$ for some $g_i \in A(Y)$. So we can write

$$f^{-1}(Z) = \left\{ x \in U \;\middle|\; g_i(f_1(x), \cdots, f_n(x)) = 0 \,\forall i \right\}.$$

The claim is that the functions g_i are regular, i.e. in $\mathcal{O}_U(U)$, because the g_i are polynomials in regular functions, which form a ring.
This is the common vanishing locus of m regular functions on U. By lemma 3.4, the vanishing locus of a regular function is closed, so $f^{-1}(Z)$ is closed.

For 2, let $\varphi \in \mathcal{O}_Y(W)$ be a regular function on $W \subset Y$ open. Then

$$\begin{aligned} f^*\varphi = \varphi \circ f : f^{-1}(W) &\to K \\ x &\mapsto \varphi(f_1(x), \cdots, f_n(x)). \end{aligned}$$

We want to show that this is a regular function. Since the f_i are regular functions, they are locally fractions, so for all $x \in f^{-1}(W)$ there is a neighborhood of $U_x \ni x$ such that (by intersecting finitely many neighborhoods) all of the f_i are fractions a_i/b_i.
Then at a point $p = [f_i(x)]$ in the image, there exists an open neighborhood W_p in W such that $\varphi = U/V$. But then $\varphi[a_i/b_i] = (U/V)([a_i/b_i])$, which is evaluation of a fraction of functions on fractions.

■

Example 15.0.1 : Let $Y = V(xy-1)$ and $U \subset \mathbb{A}^1$ be $D(x)$, so $U = \mathbb{A}^1 \setminus \{0\}$. Note that $A(Y) = k[x,y]/\langle xy-1 \rangle$ and $A(\mathbb{A}^1) = k[t]$, and $f_1 = t, f_2 = t^{-1} \in \mathcal{O}_U(U)$. Then

$$\begin{aligned} [f_1, f_2] : U &\to Y \subset \mathbb{A}^2 \\ p &\mapsto \left[p, \frac{1}{p}\right]. \end{aligned}$$

Thus the image lies in Y.

Conversely, there is a map

$$\begin{aligned} V(xy-1) &\to U = D(0) \subset \mathbb{A}^1 \\ [x,y] &\mapsto x. \end{aligned}$$

This a morphism from $V(xy-1)$ to $\mathbb{A}^1$, since the coordinates are regular functions. Since the image is contained in U, the definitions imply that this is in fact a morphism of ringed spaces. We thus have maps $U \xrightarrow{[t,t^{-1}]} V(xy-1)$ and $V(xy-1) \xrightarrow{x} U$ which are mutually inverse, so these are isomorphic as ringed spaces.

Thus maps of affine varieties (or their open subsets) are given by functions whose coordinates are regular.

Corollary 15.0.1 *(?).*
Let X, Y be affine varieties, then there is a correspondence

$$\left\{\text{Morphisms } X \to Y\right\} \iff \left\{k\text{-algebra morphisms } A(Y) \to A(X)\right\}$$
$$X \to Y \mapsto A(Y) \to A(X)$$
$$f \mapsto f^* \mathcal{O}_Y(Y) = \mathcal{O}_X(X).$$

Thus there is an equivalence of categories between reduced k-algebras and ???.

Proof .
We have a map in the forward direction. Conversely, given a k-algebra morphism $g : A(Y) \to A(X)$, we need to construct a morphism f such that $f^* = g$. Let $Y \subset \mathbb{A}^n$ with coordinate functions $y_1, \cdots, y_n$. Then $f_i = g(y_i) \in A(X) = \mathcal{O}_X(X)$. Set $f = [f_1, \cdots, f_n]$. Then by the proposition, f is a morphism to $\mathbb{A}^n$.
Let $h \in A(\mathbb{A}^n)$, then

$$\begin{aligned} (f^*h)(x) &= h(f(x)) \\ &= h([f_1(x), \cdots, f_n(x)]) \\ &= h(g(y_1), \cdots, g(y_n)) \\ &= g(h)(x) \qquad \text{since } g \text{ is an algebra morphism, } h \text{ is a polynomial} \end{aligned}$$

which follows since $f_i(x) = g(y_i)(x)$, where $g : A(Y) \to A(X)$. So $f^*(h) = g(h)$ for all $h \in A(\mathbb{A}^n)$, so the pullback of f is g. We now need to check that it's contained in the image. Let $h \in I(Y)$, then $f^*(h) = g(h) = 0$ since $h = 0 \in A(Y)$. So $\operatorname{im} f \subset Y$. Since the coordinate f_i are regular, this is a morphism, and we have $f^* = g$ as desired.

■

Example 15.0.2 : Isomorphisms are not necessarily bijective morphisms. Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$.

Then there is a morphism

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to X \\ t &\mapsto \left[t^2, t^3\right], \end{aligned}$$

since the coordinates t^2, t^3 are regular functions. Then φ is a bijection, since we can define a piecewise inverse

$$\begin{aligned} \varphi^{-1} : X &\to \mathbb{A}^1 \\ [x, y] &\mapsto \begin{cases} y/x & x \neq 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

However, φ^{-1} is not a morphism. For instance, pulling back the function t yields $\left(\varphi^{-1}\right)^* t \notin A(X)$, since it is equal to the map $[x, y] \mapsto y/x$ for $x \neq 0$ and 0 if $x = y = 0$, which is not a regular function.

Since φ is a morphism, we can consider the corresponding map of k-algebras

$$\begin{aligned}\varphi^* : A(X) &\to A(\mathbb{A}^1)\\ k[x,y]/\left\langle y^2 - x^3\right\rangle &\mapsto k[t]\\ x &\mapsto t^2\\ y &\mapsto t^3.\end{aligned}$$

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Last time: proved that if X, Y are affine varieties then there is a bijection

$$\left\{\substack{\text{Morphisms}\\ f:X\to Y}\right\} \iff \left\{\substack{k\text{-algebra morphisms}\\ A(Y)\to A(X)}\right\}$$
$$f \mapsto f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

Remark 16.0.1 : A morphism $f : X \to Y$ is by definition a morphism of ringed spaces where $\mathcal{O}_X, \mathcal{O}_Y$ are the sheaves of regular functions.

Remark 16.0.2 : This shows $X \cong Y$ as ringed spaces iff $A(X) \cong A(Y)$ as k-algebras.

Example 16.0.1 : Take

$$\begin{aligned}f : \mathbb{A}^1 &\to V(y^2 - x^3) \subset \mathbb{A}^2\\ t &\mapsto (t^2, t^3).\end{aligned}$$

This is a morphism by proposition 4.7.

We then get a map on algebras

$$\begin{aligned}f^* : A(V(y^2 - x^3)) = k[x,y]/\left\langle y^2 - x^3\right\rangle &\to k[t]\\ x &\mapsto t^2\\ y &\mapsto t^3,\end{aligned}$$

but even though f is a bijective morphism, it's not an isomorphism of ringed spaces. This can be seen from the fact that the image doesn't contain t.

Review of introductory category theory.

We'll define a category AffVar_k whose objects are affine varieties over k and morphisms in $\hom(X, Y)$ will be morphisms of ringed spaces. There is a contravariant functor A into reduced finitely generated k-algebras which sends X to $A(X)$ and sends morphisms $f : X \to Y$ to their pullbacks $f^* : A(Y) \to A(X)$, where "reduced" denotes the fact that there are no nilpotents.

Review of the universal property of the product.

Remark 16.0.3 : If we have X, Y affine varieties, we take $X \times Y$ to be the categorical product instead of the underlying product of topological spaces. We have

$$A(X \times Y) \cong A(X) \otimes_k A(Y) \cong k[x_1, \cdots, x_n, y_1, \cdots, y_m]/I(X) \otimes 1 + 1 \otimes I(Y).$$

This recovers the product, since if we have

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists H"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

where $H = (f, g)$.

Remark 16.0.4 : Products of spaces are sent to the tensor product of k-algebras, i.e. pullbacks are sent to pushouts.

Remark 16.0.5 : Note that the groupoid associated to a group does not have products: there can only be one element, but the outer triangles will not necessarily simultaneously commute.

17 | Thursday, October 15

17.1 End of Chapter 4

Recall the proposition: morphisms between affine varieties are in bijection with k-algebra morphisms between their coordinate rings. As a result, we'll redefine an affine variety to be a ringed space isomorphic to an affine variety.

This allows you to say that affine varieties embedded in different ways are the same.

Example 17.1.1 : $\mathbb{A}^2$ vs $V(x) \subset \mathbb{A}^n$. In fact, the map

$$f : \mathbb{A}^2 \to \mathbb{A}^3(y, z) \qquad \mapsto (0, y, z).$$

This is continuous and the pullback of regular functions are again regular.

Remark 17.1.1 : With the new definition, there is a bijection between affine varieties up to isomorphisms and finitely generated k-algebras up to algebra isomorphism.

Proposition 17.1.1 *(?).*
Let $D(f) \subset X$ be a distinguished open, then $D(f)$ is a ringed space since $(X, \mathcal{O}_X)$ is and we can restrict the structure sheaf.

Proof .
Set

$$Y := \left\{ (x, t) \in X \times \mathbb{A}^1 \;\middle|\; tf(x) = 1 \right\} \subset X \times \mathbb{A}^1.$$

This is an affine variety, since $Y = V(I + \langle ft - 1 \rangle)$. This is isomorphic to $D(f)$ by the map

$$Y \to D(f)(x, t) \qquad \mapsto x.$$

with inverse $x \mapsto (x, \dfrac{1}{f(x)})$.

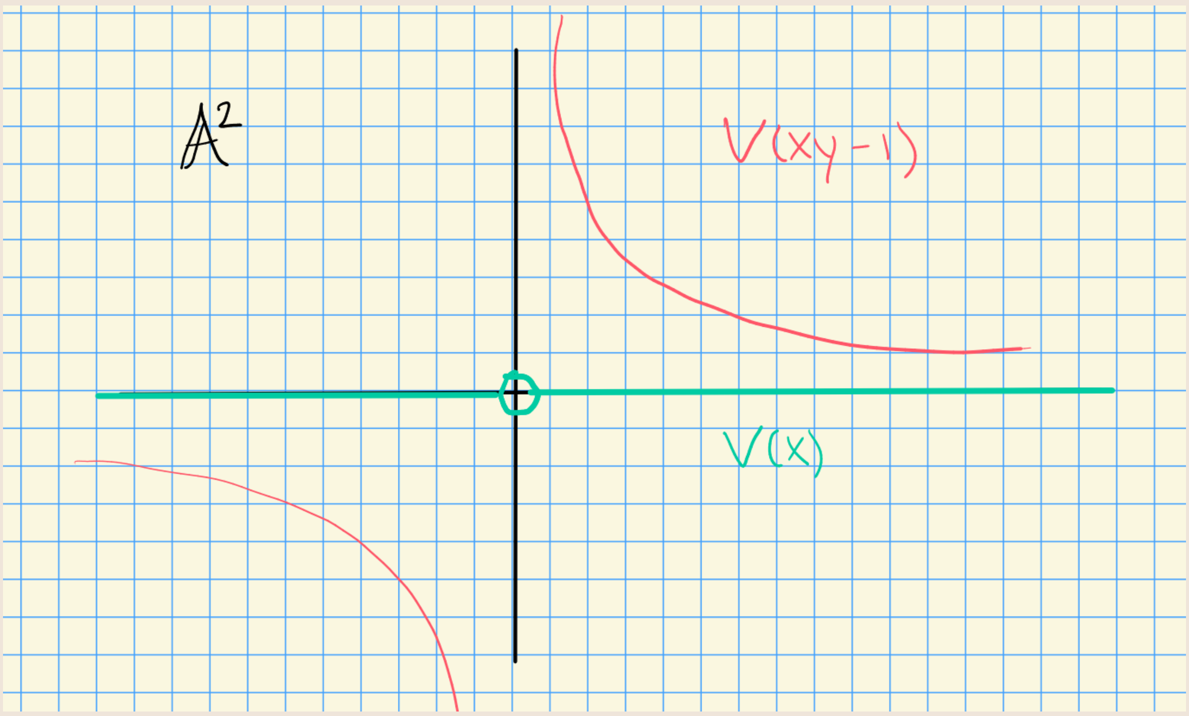

Figure 9: Image

Note that $\pi : X \times \mathbb{A}^1 \to X$ is regular, using prop 3.8: if the coordinates of a map are regular functions, then the entire map is a morphism of ringed spaces. We can then note that $\frac{1}{f(x)}$ is regular on $D(f)$, since $f \neq 0$ there.

■

Example 17.1.2 : $\mathbb{A}^2 \setminus \{0\}$ is not an affine variety. Note that this is also not a distinguished open.

We showed on a HW problem that the regular functions on $\mathbb{A}^2 \setminus \{0\}$ are $k[x, y]$, which are also the regular functions on $\mathbb{A}^2$. So there is a map inducing a pullback

$$\iota : \mathbb{A}^2 \setminus \{0\} \to \mathbb{A}^2$$
$$\iota^* k[x, y] \xrightarrow{\sim} k[x, y].$$

Note that ι^* is an isomorphism on the space of regular functions, but ι itself is not an isomorphism of topological spaces. Why? i^{-1} is not defined at zero.

17.2 Chapter 5

Definition 17.2.1 (Prevariety)**.**
A *prevariety* is a ringed spaced X with a finite open cover by affine varieties. This is a topological space X with an open cover $\{U_i\}_{i=1}^n \rightrightarrows X$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine variety. We'll call $\mathcal{O}_X$ the sheaf of *regular functions* and $U_i \subset X$ *affine open sets.*

One way to construct prevarieties from affine varieties is by *gluing*:

Definition 17.2.2 (Glued Spaces)**.**
let X_1, X_2 be prevarieties which are themselves actual varieties. Let $U_{12} \subset X_1, U_{21} \subset X_2$ be opens and $f : U_{12} \to U_{21}$ an isomorphism of ringed spaces.

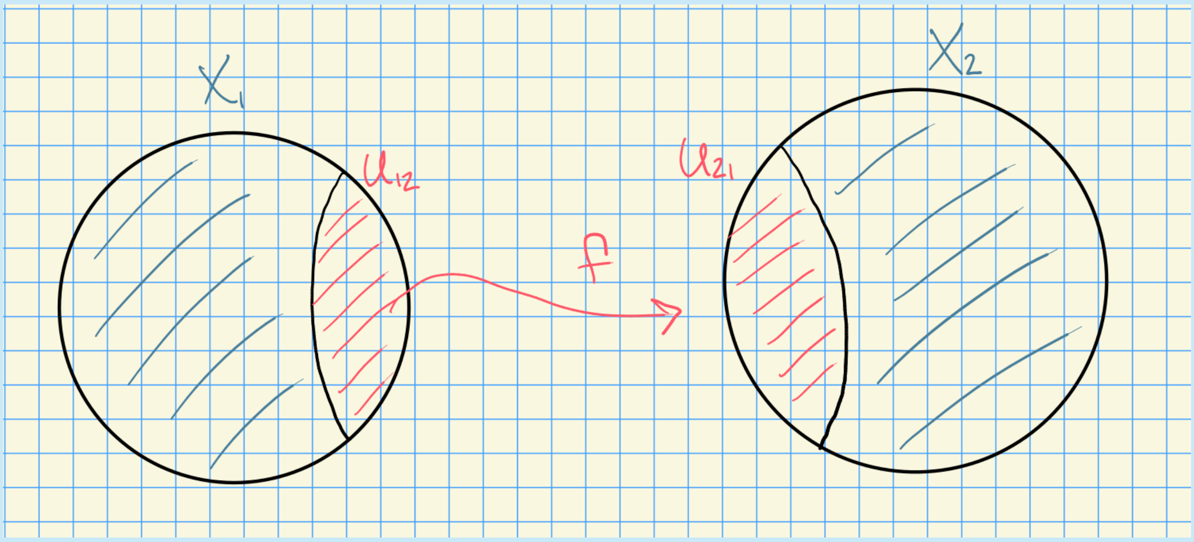

Figure 10: Image

As a set, take $X = X_1 \coprod X_2 / \sim$ where $a \sim f(a)$ for all $a \in U_{12}$. As a topological space, $U \subset X$ is open iff $U_i \coloneqq U \cap X_i$ are open in X_i. As a ringed space, we take $\mathcal{O}_X(U) \coloneqq \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U_i} \in \mathcal{O}_{X_i} \right\}$.

Example 17.2.1 : The prototypical example is $\mathbb{P}^1/k$ constructed from two copies of $\mathbb{A}^1/k$. Set $X_1 = \mathbb{A}^1, X_2 = \mathbb{A}^2$, with $U_{12} \coloneqq D(x) \subset X_1$ and $U_{21} \coloneqq D(y) \subset X_2$. Then let

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$

This defines a regular function on U_{12} so defines a morphism $U_{12} \xrightarrow{\sim} \mathbb{A}^1$.

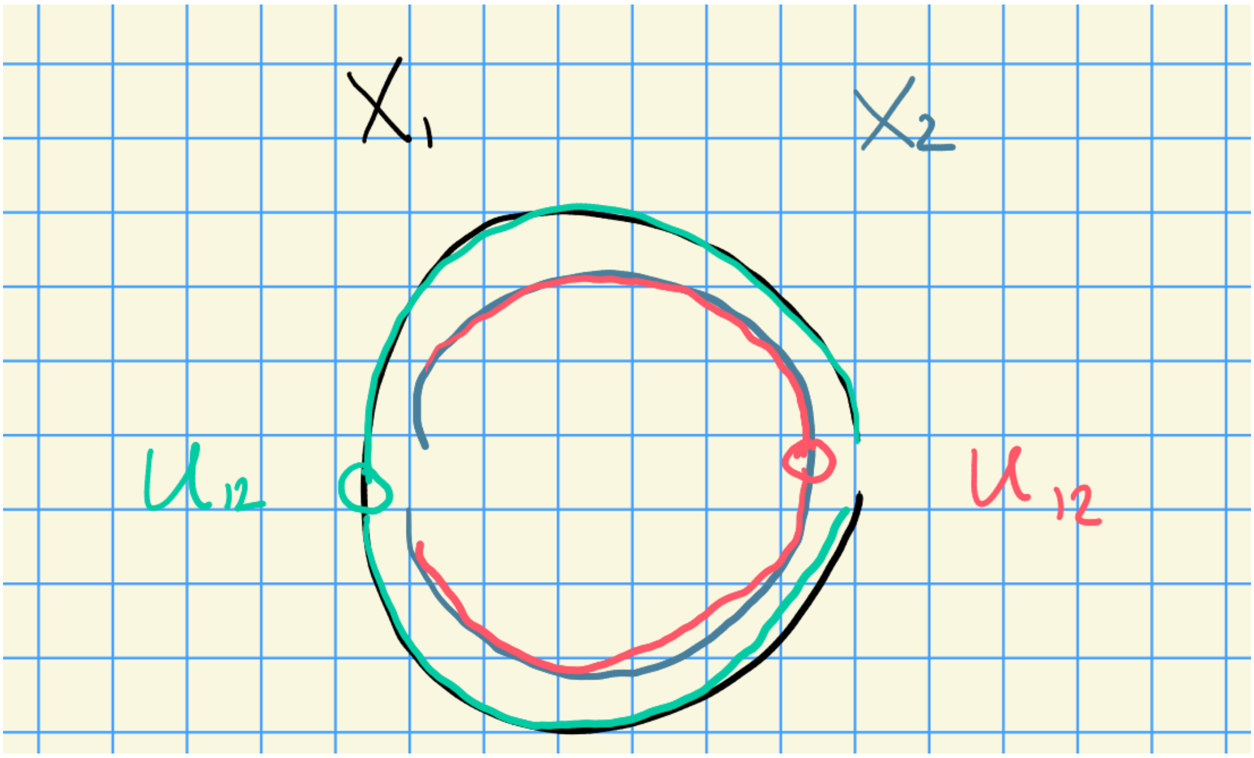

Figure 11: Image

Over $\mathbb{C}$, topologically this yields a sphere

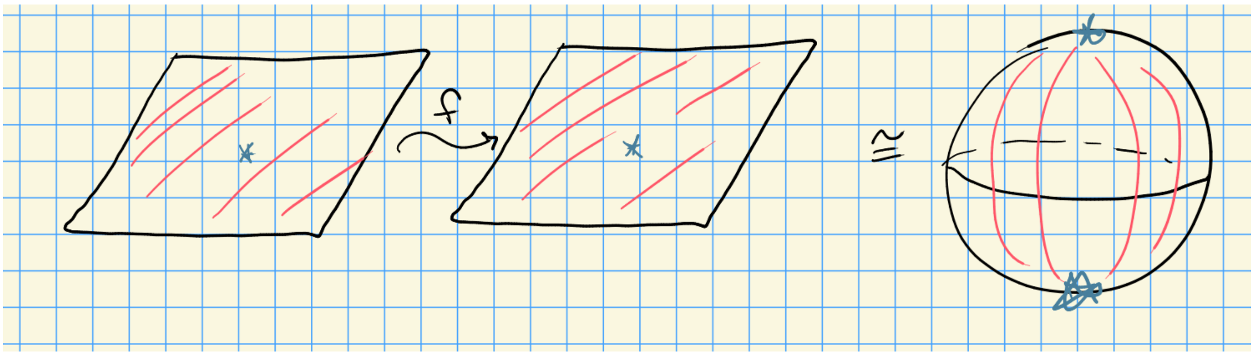

Figure 12: Image

Given a ringed space $X = X_1 \cup X_2$ with a structure sheaf $\mathcal{O}_X$, what is $\mathcal{O}_X(X)$? By definition, it's

$$\mathcal{O}_X(X) := \left\{ \varphi : X \to k \;\middle|\; \varphi|_{X_1}, \varphi|_{X_2} \text{ are regular} \right\}.$$

Then if $\varphi|_{X_1} = f(x)$ and $\varphi|_{X_2} = g(y)$, we have $y = 1/x$ on the overlap and so $f(x)|_{D(x)} = g(1/x)|_{D(x)}$. Since f, g are rational functions agreeing on an infinite set, $f(x) = g(1/x)$ both being polynomial forces $f = g = c$ for some constant $c \in k$. Thus $\mathcal{O}_X(X) = k$.

What about $\mathcal{O}_X(X_1)$? This is just $k[x]$, and similarly $\mathcal{O}_X(X_2) = k[y]$. We can also consider $\mathcal{O}_X(X_1 \cap X_2) = D(x) \subset X$, so this yields $k[x, 1/x]$. We thus have a diagram

$$\begin{tikzcd}
 & \mathcal{O}_X(X_1) = k[x] \arrow[dr, "x \mapsto x"] & \\
\mathcal{O}_X(X) \arrow[ur] \arrow[dr] & & \mathcal{O}_X(X_1 \cap X_2) = k[x, 1/x] \\
 & \mathcal{O}_X(X_2) = k[y] \arrow[ur, "y \mapsto 1/x"] &
\end{tikzcd}$$

18 | Tuesday, October 20

18.1 Gluing Two Opens

Recall that a *prevariety* is a ringed space that is locally isomorphic to an affine variety, where we recall that $(X, \mathcal{O}_X)$ is *locally isomorphic* to an affine variety iff there exists an open cover $U_i \rightrightarrows X$ such that $(U_i, \mathcal{O}_{U_i})$.

We found one way of producing these: the gluing construction. Given two ringed spaces $(X_1, \mathcal{O}_{X_1})$ and $(X_2, \mathcal{O}_{X_2})$ and open sets $U_{12} \in X_1$ and $U_{21} \in X_2$ and an isomorphism $(U_{12}, \mathcal{O}_{U_{12}}) \xrightarrow{f} (U_{21}, \mathcal{O}_{U_{21}})$, we defined

- The topological space as $X_1 \coprod_f X_2$
- The sheaf of rings as $\mathcal{O}_X = \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U \cap X_i} \text{ is regular for } i = 1, 2 \right\}$.

Example 18.1.1 : $\mathbb{P}^1/k = X_1 \cup X_2$ where $X_1 \cong \mathbb{A}^1, X_2 \cong \mathbb{A}^2$. Take $U_{12} = D(x)$ and $U_{21} = D(y)$ with

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto \frac{1}{x} = y. \end{aligned}$$

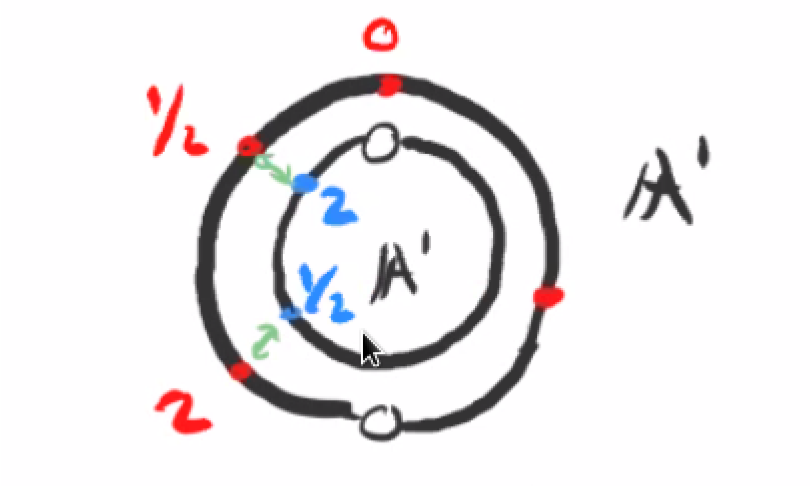

Figure 13: Supposing $\mathrm{ch}(k) \neq 2$. Note that for $\mathbb{C}$ this recovers S^2 in the classical topology.

Example 18.1.2 : Let $X_i = \mathbb{A}^1$ and $U_{12} = D(x), U_{21} = D(y)$ with

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto x = y. \end{aligned}$$

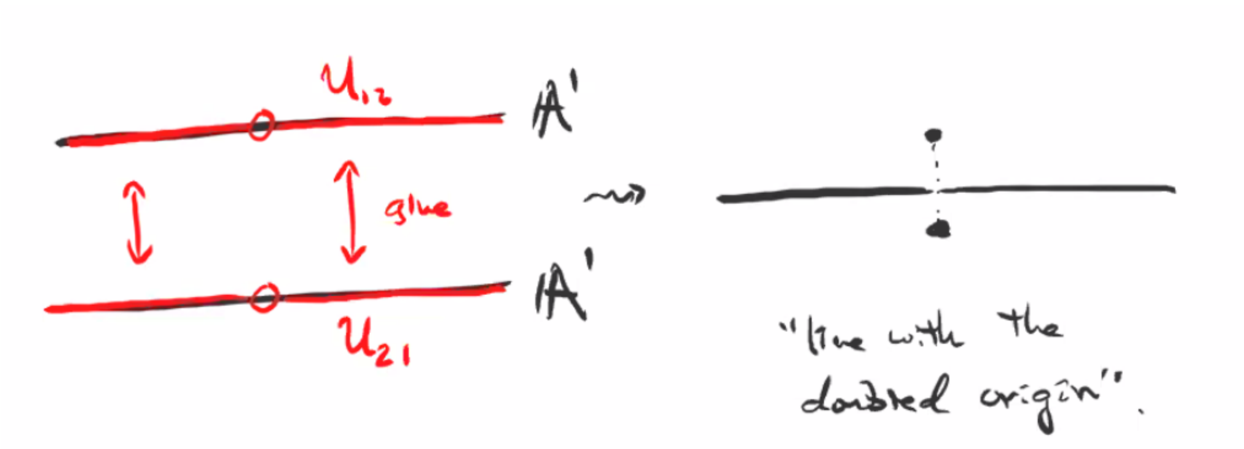

Figure 14: Line with the doubled origin.

Then $\mathcal{O}_X = \left\{ \varphi : X \to k \;\middle|\; \varphi|_{X_i} \text{ is regular} \right\} \cong k[x]$.

18.2 More General Gluing

Now we want to glue more than two open sets. Let I be an indexing set for prevarieties X_i. Suppose that for an ordered pair (i, j) we have open sets $U_{ij} \subset X_i$ and isomorphisms $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ such that

a. $f_{ji} = f_{ij}^{-1}$

b. $f_{jk} \circ f_{ij} = f_{ik}$ (cocycle condition)

Figure 15: Opens with isomorphisms.

Then the gluing construction is given by

1. $X \coloneqq \coprod X_i / \sim$ where $x \sim f_{ij}(x)$ for all i, j and all $x \in U_{ij}$.

2. $\mathcal{O}_x(U) \coloneqq \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U \cap X_i} \in \mathcal{O}_{X_i} \right\}$.

Every prevariety arises from the gluing construction applied to X_i affine varieties, since a prevariety $(X, \mathcal{O}_X)$ by definition has an open affine cover $X_i \rightrightarrows X$ and X is the result of gluing the X_is by the identity.

Example 18.2.1 : Let $X_1 = X_2 = X_3 = \mathbb{A}^2/k$. Glue by the following instructions:

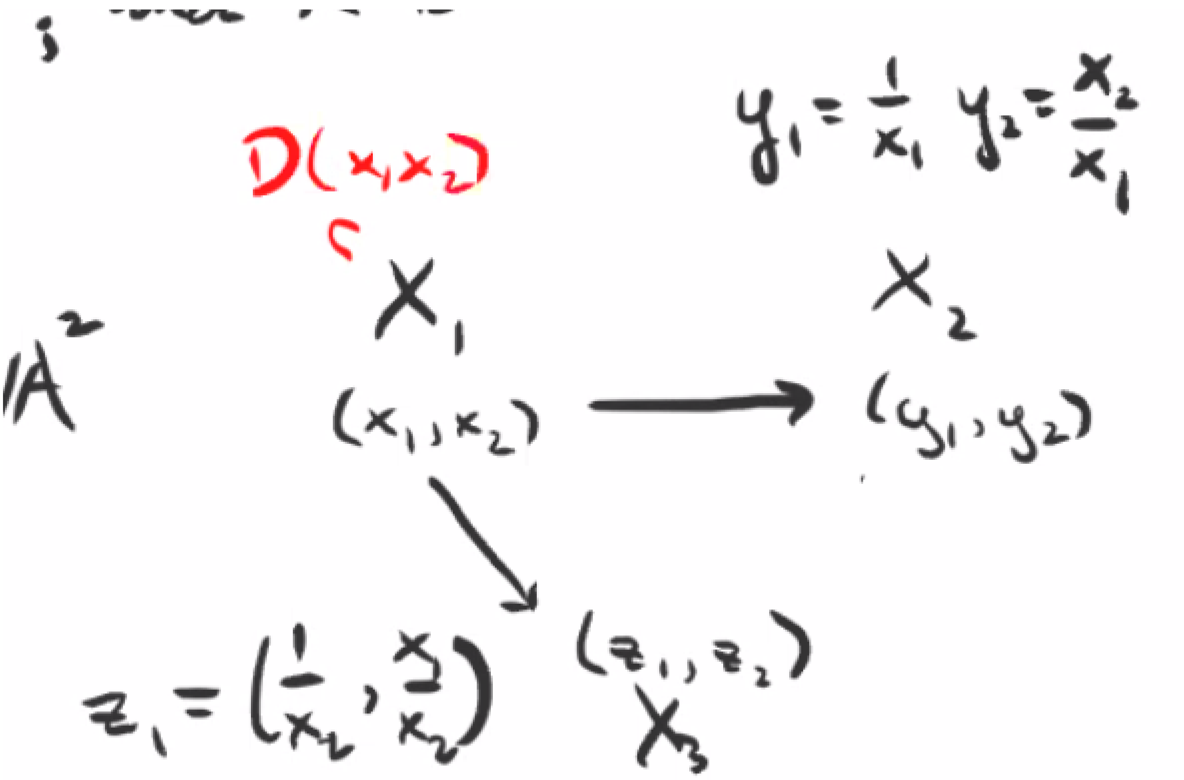

Figure 16: The map not shown is whatever formula is necessary to make the diagram commute.

Here

- $(y_1, y_2) = (1/x_1, x_2/x_1)$
- $(z_1, z_2) = (1/x_2, x_1/x_2)$
- $U_{12} = D(x_1)$
- $U_{21} = D(x_2)$.

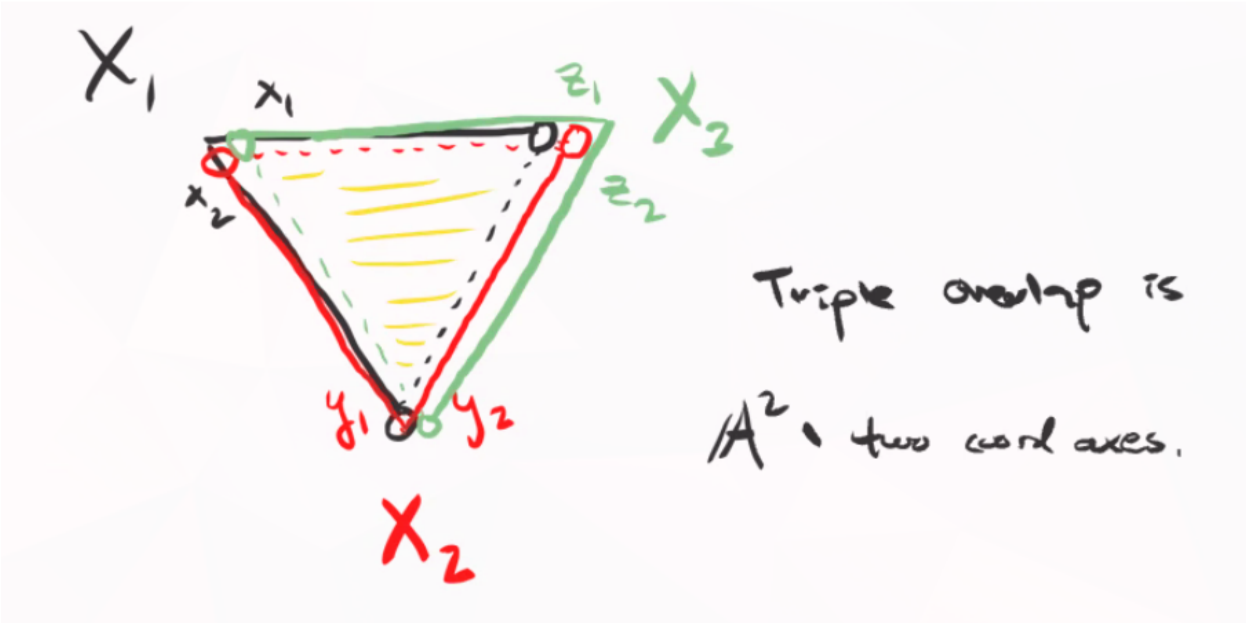

Figure 17: Yields $\mathbb{P}^2$

Here $X_1 = [1 : y/x : z/x]$, $X_2 = [x/y : 1 : z/y]$.

Example 18.2.2 : From Gathmann 5.10, open and closed subprevarieties. Let X be a prevariety and suppose $U \subset X$ is open. Then $(U, \mathcal{O}_U)$ is a prevariety where $\mathcal{O}_U = \mathcal{O}_X|_U$. How can we write U as (locally) an affine variety?

Since the U_i are covered by distinguished opens D_{ij} in X_i where $X = \cup X_i$ with X_i affine varieties, we can write $U = \bigcup_i U_i = \bigcup_{i,j} D_{ij}$.

Example 18.2.3 : Let $Y \subset X$ be a closed subset of a prevariety X. We need to define $\mathcal{O}_Y(U)$ for all $U \subset Y$ open, so we set

$$\mathcal{O}_Y(U) = \left\{ \varphi : U \to k \;\middle|\; \forall p \in U, \exists V_p \text{ with } p \in V_p \subset_{\text{open}} X \text{ and } \psi \in \mathcal{O}_X(V_p) \text{ s.t. } \psi|_{U \cap V} \varphi \right\}.$$

What's the picture?

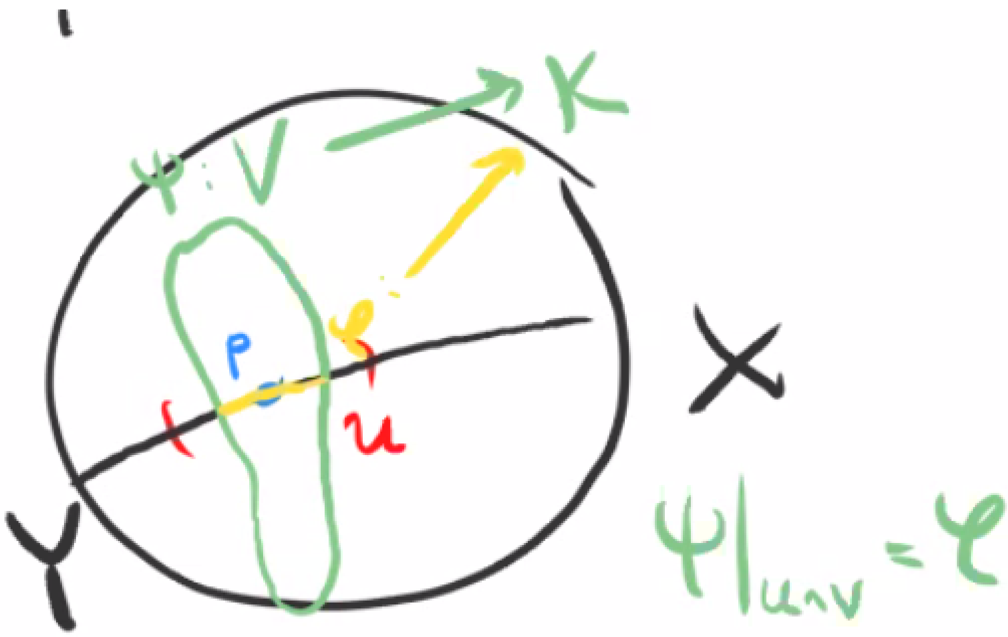

Figure 18: Sheaf for a closed subset.

It's an exercise to show that this is a prevariety.

Remark 18.2.1 : If $U \subset X$ is an open subprevariety or $Y \subset X$ is a closed subprevariety, then the inclusions are morphisms. We'd need to show that a pullback of a function is regular, but this is set up by definition.

Remark 18.2.2 : Define $\tilde{\mathcal{O}}_X(U)$ as the set of *all* functions $U \to k$. Then the inclusion $(X, \mathcal{O}_X) \hookrightarrow (X, \tilde{\mathcal{O}}_X)$ given by the identity on X is a morphism, but the identity in the reverse direction is not.

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Example 19.0.1 : Consider $\mathbb{A}^1$, whose polynomial functions are $k[x]$. Consider now $D(x) \subset \mathbb{A}^1$, which is equal to the affine variety $V(xy-1)$. Then the polynomial functions on $D(x)$ are $k[x, y]/\langle xy - 1\rangle \cong k[x, x^{-1}]$.

Recall that a *prevariety* is a ringed space $(X, \mathcal{O}_X)$ such that X has a finite open cover by affine varieties $(U_i, \mathcal{O}_X|_{U_i})$, and a *morphism* of prevarieties is a morphism of ringed spaces. We saw that one can construct prevarieties by gluing finite collections of prevarieties or affine varieties along open sets, and all prevarieties arise this way.

Similar to varieties, the product P of prevarieties X, Y will satisfy a universal property:

Tikz Link

$$\begin{tikzcd}
P \arrow[drr, bend left, "\pi_X"] \arrow[ddr, bend right, "\pi_Y"'] & & \\
& Z \arrow[ul, dashed, "\exists!"'] \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

Proposition 19.0.1 *(?).*
The product is unique up to unique isomorphism, i.e. there is a unique isomorphism between any two products.

Proof .
Standard! ■

Example 19.0.2 : Consider $\mathbb{A}^1 \times \mathbb{A}^1$, then the product is (and should be) $\mathbb{A}^2$, but $\mathbb{A}^2$ does not have the product topology. The open set $D(x-y)$ is not covered by products of open sets.

This happens because the Zariski topology is too weak.

Strategy to fix: use gluing. Let X, Y be prevarieties and $\{U_i\}, \{V_i\}$ be open affine covers of X and Y respectively. We can construct the product $U_i \times V_j \subset \mathbb{A}^{n+m}$, which is an affine variety and satisfies the universal property for products. We then glue two such products $U_{i_1} \times V_{j_1}$ and $U_{i_2} \times V_{j_2}$ along their common open subset in $(U_{i_1} \cap U_{i_2}) \cap (V_{j_1} \cap V_{j_2}) \subseteq X \times Y$.

Let $\tilde{U} \coloneqq U_{i_1} \cap U_{i_2} \times V_{j_1} \cap V_{j_2}$, we then need that

$$(\tilde{U}, \mathcal{O}_{U_{i_1} \times V_{j_1}}\Big|_{\tilde{U}}) \cong (\tilde{U}, \mathcal{O}_{U_{i_2} \times V_{j_2}}\Big|_{\tilde{U}}).$$

This follows from the universal property of products, since the open set $(U \times V, \mathcal{O}_{X \times Y}|_{U \times V})$ is a categorical product of ringed spaces, and the identity provides a unique isomorphism. By the gluing construction, this produces a ringed space $(X \times Y, \mathcal{O}_{X \times Y})$, we just need to check that this satisfies the universal property. We have projections π_X, π_Y set-theoretically, which restrict to morphisms on every $U_i \times V_j$. For any prevariety Z, we get a unique set map $h : Z \to X \times Y$ which commutes, so it suffices to check that h is a morphism of ringed spaces.

So consider $h^{-1}(U_i \times V_j) \subset Z$, which is an open subset of Z given by $f^{-1}(U) \times f^{-1}(V)$. Take an open cover and let W be an element in it. We can then restrict f and g to get $f|_W : W \to U_i$ and

$g|_W : W \to V_j$ and their product is a morphism of ringed spaces. So Z is covered by open sets for which h is a morphism of ringed spaces, making h itself a morphism.

What was the point of constructing the product? We want some notion analogous to being Hausdorff to distinguish spaces like $\mathbb{P}^1/k$ from the line with the doubled origin. The issue is that these spaces with the Zariski topology are never Hausdorff. So we make the following definition:

Definition 19.0.1 (Separated)**.**
A prevariety is **separated** iff $X \xhookrightarrow{\Delta_X} X \times X$ is a closed embedding, where $\Delta(x) = (x, x)$ is the diagonal morphism. i.e. $\mathrm{id}_X \times \mathrm{id}_X$.

Definition 19.0.2 (Variety)**.**
A **variety** is a separated prevariety.

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Recall that an affine variety is given by $X = V(I) \subset \mathbb{A}^n/k$, and we have sheaves of rings of regular functions $\mathcal{O}_X$ on X. A prevariety is a ringed space that is covered by finitely many affine spaces. A morphism of prevarieties $f : X \to Y$ is a continuous map such that the pullbacks of regular functions are regular, i.e. for all $\varphi \in \mathcal{O}_X(U)$ we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$. We can form a category PreVar_k of prevarieties over k, where we have several important constructions

1. Gluing

2. Products: Given X, Y, there is a unique prevariety $X \times Y$ such that

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists! h"] \arrow[drr, bend left, "f_x"] \arrow[ddr, bend right, "f_y"'] & & \\
& X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

We had an analogue of being Hausdorff: the diagonal Δ_X is closed.

Example 20.0.1 : Glue $D(x) \subset \mathbb{A}^1$ to $D(y) \subset \mathbb{A}^1$ by the isomorphism

$$\begin{aligned} D(x) &\xrightarrow{\sim} D(y) \\ x &\mapsto y. \end{aligned}$$

This yields an affine line with two origins:

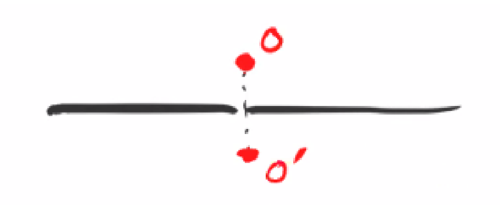

Figure 19: Line with two origins.

Consider the product:

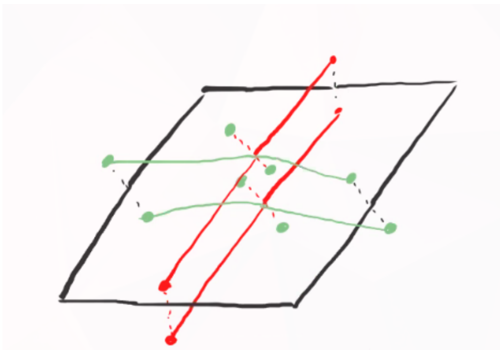

Figure 20: Product of lines with two origins

Since the diagonal is given by $\Delta_X = \left\{ (x, x) \mid x \in X \right\}$, we have the following situation in blue:

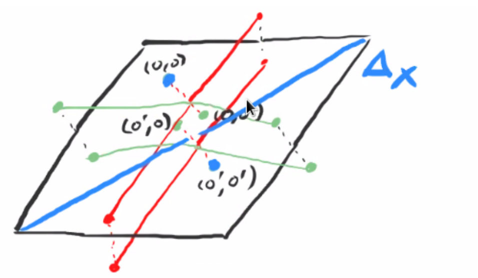

Figure 21: Image

We claim that Δ_X is not closed, and for example $(0,0') \in \overline{\Delta}_X$. Consider $U \times U' \subset X \times X$ where U, U' are the two copies of $\mathbb{A}^1$ in X. This is an affine open set, since it's isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$.

If Δ_X were closed, then $S \coloneqq \Delta_X \cap (U \times U') = \left\{(x,x) \;\middle|\; x \neq 0\right\}$ would be closed in $U \times U'$.

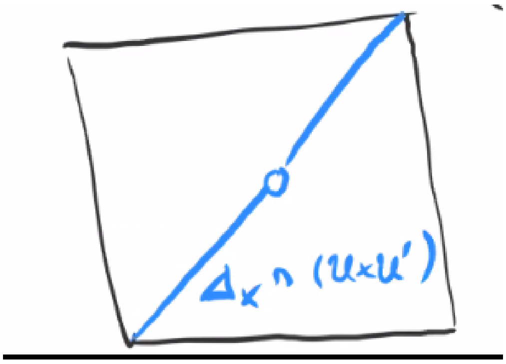

Figure 22: Open diagonal in a product.

This is because any polynomial vanishing on S must vanish at $(0,0)$, so S is an affine variety. But then $V(I(S)) = \Delta_{\mathbb{A}^1}$.

Lemma 20.1 *(?)*.

a. Any affine variety is a variety.

b. Open and closed subprevarieties of a variety X are themselves varieties.

Thus it makes since to consider *open* and *closed subvarieties*.

Proof (of a).
We need to check that $\Delta_X \subset X^2$ is closed for any affine $X \subset \mathbb{A}^n$. Note that we can write.

$$\Delta_X = X^2 \cap V\left(\left\{x_j - y_j \;\middle|\; 1 \leq j \leq n\right\}\right) \subset \mathbb{A}^n \times \mathbb{A}^2$$

■

Proof (of b).
Let $\iota : Y \to X$ be the inclusion of either an open or closed subset. Then we have a morphism $(\iota, \iota) : Y^2 \to X^2$ by the universal property. Then $\Delta_Y = (\iota, \iota)^{-1}(\Delta_X)$, so is closed by the continuity of (ι, ι) and the fact that Δ_X. Thus Y is a variety.

■

20.1 Properties of Varieties

Proposition 20.1.1 ***(Properties of Varieties).***
Let $f, g : X \to Y$ be morphisms of prevarieties and assume Y is a variety.

a. The graph of f, given by $\Gamma_f := \left\{ (x, f(x)) \;\middle|\; x \in X \right\}$, is closed in $X \times Y$.

b. The set $\left\{ x \in X \;\middle|\; f(x) = g(x) \right\}$ is closed in X.

Proof (of a).
Consider the product morphism $(f, \mathrm{id}) : X \times Y \to Y^2$. Since Δ_Y is closed, $(f, \mathrm{id})^{-1}(\Delta_Y)$ is closed, and is the locus where $f(x) = y$, so this is Γ_f.

■

Proof (of b).
Consider $(f, g) : X \to Y^2$. Since $\Delta_Y \subset Y^2$ is closed,

$$(f, g)^{-1}(\Delta_Y) = \left\{ x \in X \;\middle|\; f(x) = g(x) \right\} \subset X$$

is closed.

■

20.2 Chapter 6: Projective Varieties

Note that affine varieties of positive dimension over $\mathbb{C}$ are not compact in the classical topology, but *are* compact in the Zariski topology. Similarly, they are Hausdorff classically, but not in the Zariski topology. We want to find notions equivalent to Hausdorffness and compactness in the classical setting, which end up also applying to varieties. The fix in the latter case was considering "separatedness". The fix for compactness will be the following:

Definition 20.2.1 (Complete)**.**
A variety X is **complete** iff for any variety Y the projection map $\pi_Y : X \times Y \to Y$ is a closed map, i.e. $\pi_Y(U)$ is closed whenever U is closed.

Example 20.2.1 : $\mathbb{A}^1$ is not complete. Let $Y = \mathbb{A}^1$ and $Z = V(xy - 1) \subset X \times Y$. Then $\pi_Y(Z) = D(y) \subset Y \subset \mathbb{A}^1$ is not closed.

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21.1 Projective Space

Definition 21.1.1 (Projective Space)**.**
Let $n \in \mathbb{N}$, and define **projective n-space** over k by

$$\mathbb{P}^n/k = \left\{ \text{lines through the origin in } k^{n+1} \right\}.$$

Remark 21.1.1 : For notation, given $L \in \mathbb{P}^n/k$, it is spanned by any nonzero points $[x_0, \cdots, x_n] \in L$, and L is uniquely determined by this point up to scaling by elements in $k^\times$. In this case, we write $L = [x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$. We can then define $\mathbb{P}^n/k = k^{n+1} \setminus \{0\} / \sim$ where we mod out by scalar multiplication. We call $[x_1 : \cdots : x_n]$ the *homogeneous coordinates* on $\mathbb{P}^n/k$.

Remark 21.1.2 : Consider the map

$$\begin{aligned} \mathbb{A}^n &\to \mathbb{P}^n \\ [x_1, \cdots, x_n] &\mapsto [1 : x_1 : \cdots : x_n]. \end{aligned}$$

This is injective. Conversely, consider

$$\text{"}D(x_0)\text{"} \subset \mathbb{P}^n := \left\{ [x_0 : \cdots : x_n] \;\middle|\; x_0 \neq 0 \right\}.$$

This is a well-defined subset of $\mathbb{P}^n$, since it only depends on the equivalence class of a point. In this case, there is a unique $\lambda(x_0, \cdots, x_n)$, namely $\lambda = 1/x_0$, such that each point in this set is of the form $\left[1 : \dfrac{x_1}{x_0} : \cdots : \dfrac{x_n}{x_0}\right]$, yielding a copy of $\mathbb{A}^n \subset \mathbb{P}^n$ given by points $\left[\dfrac{x_1}{x_0}, \cdots, \dfrac{x_n}{x_0}\right]$. What is its complement?

It's given by $\{[0 : x_1 : \cdots : x_n]\} \subset \mathbb{P}^n$, which is equal (as a set) to a copy of $\mathbb{P}^{n-1}$ defined by the set of lines in k^n defined by $x_0 = 0$.

Example 21.1.1 (?): Note that $\mathbb{P}^1$ contains a copy of $\mathbb{A}^1$ where $x_0 \neq 0$ and a second copy where $x_1 \neq 0$, admitting maps

$$\begin{aligned} f_1 : \mathbb{A}^1 &\to \mathbb{P}^1 \\ [x_0 : x_1] &\mapsto \left[\frac{x_0}{x_1}\right]. \end{aligned}$$

and

$$f_2 : \mathbb{A}^1 \to \mathbb{P}^1$$
$$[x_0 : x_1] \mapsto \left[\frac{x_1}{x_0}\right],$$

since every line in $\mathbb{P}^1$ has either $x_0 \neq 0$ or $x_1 \neq 0$. These two copies cover $\mathbb{P}^1$, and the "transition map" is inversion.

Remark 21.1.3 : More generally, there are $n+1$ inclusions $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ given by dividing by the jth coordinate, and their union is the entire space. The gluing construction gives $\mathbb{P}^n$ the structure of a prevariety: we can consider $D(x_j) \subset \mathbb{P}^n$ where each has the structure of a ringed space $(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$. We have $D(x_i) \cap D(x_j) \subset D(x_i)$ is given coordinate x_k/x_i where $k \neq i$, and similarly this is a subset of $D(x_j)$ with coordinates x_k/x_j for $k \neq j$. Their intersection is $D(\dfrac{x_i}{x_j})$, which is a copy of $\mathbb{A}^{n-1}$.

Example 21.1.2 (?): Consider $\mathbb{P}^1$, then $D(x_0) \cong \mathbb{A}^1$ with which contains a copy of $\mathbb{A}^1$ with coordinate ring $k\left[\dfrac{x_1}{x_0}\right]$ and a subset $D\left(\dfrac{x_1}{x_0}\right)$ with coordinate ring $k[y, 1/y]$, and similarly, $D(x_1) \cong \mathbb{A}^1$ has coordinate ring $k\left[\dfrac{x_0}{x_1}\right]$ and contains $\supseteq D\left(\dfrac{x_0}{x_1}\right)$ with coordinate ring $\dfrac{k[z, 1}{z]}$. Consider their overlap $D(x_0) \cap D(x_1)$? When do y, z denote the same point in $\mathbb{P}^1$? When $y = 1/z$.

We can conclude that the $n+1$ copies $D(x_i) \subset \mathbb{P}^n$ are affine varieties isomorphic as ringed spaces on the overlaps, so the gluing construction makes $\mathbb{P}^n$ a prevariety.

Definition 21.1.2 (Homogeneous Polynomial)**.**
A polynomial f is homogeneous of degree f is every monomial in f has total degree d.

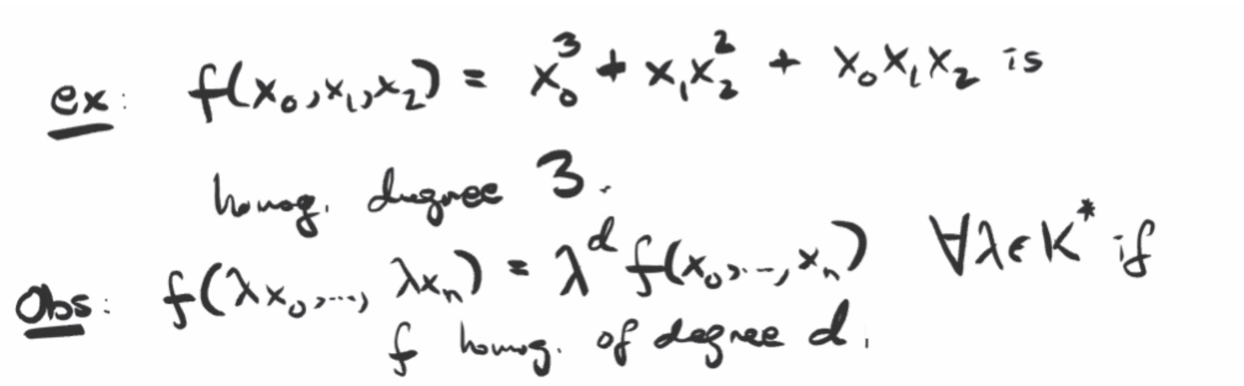

Figure 23: Image

Example 21.1.3 (?):

If f is homogeneous, $V(f) \subset \mathbb{P}^n$ is a well-defined subset, since $f(x_0, \cdots, x_n) = 0 \iff \lambda^d f(x_0, \cdots, x_n) = 0 \iff f(\lambda x_0, \cdots, \lambda x_n) = 0$.

Definition 21.1.3 (?).
A graded ring R is a ring R with abelian subgroups $R_d \subset R$ with $R = \bigoplus_{d \geq 0} R_d$ and for all $f \in R_d$ and $g \in R_{d'}$, we have $fg \in R_{d+d'}$ and $R_d + R_d \subset R_d$.

22 | Thursday, November 05

Today: projective spaces. We defined $\mathbb{P}^n_{/k} := k^{n+1} \setminus \{0\} / \sim$ where $x \sim \lambda x$ for all $x \in k^\times$, which we identified with lines through the origin in k^{n+1}. We have homogeneous coordinates $p = [x_0 : \cdots : x_n]$.

We say an ideal is *homogeneous* iff for all $f \in I$, the homogeneous part $f_d \in I$ for all d. In this case $V_p(I) \subset \mathbb{P}^n_{/k}$ defined as the vanishing locus of all homogeneous elements of I is well-defined. Think of this as the "projective version" of a vanishing locus.

Similarly we defined $I_p(S)$ defined as the ideal generated by all homogeneous $f \in k[x_1, \cdots, x_n]$ such that $f(x) = 0$ for all $x \in S$.

Remark 22.0.1 : Observe that $V_a(I)$ defined as the cone over $V_p(I)$ is the set of points in $\mathbb{A}^{n+1} \setminus \{0\} \cup \{0\}$ which map to $V_p(I)$.

We have an alternative definition of a cone in $\mathbb{A}^{n+1}$, characterized as a closed subset C which is closed under scaling, so $kC \subseteq C$.

Proposition 22.0.1.

- If $S \subset k[x_1, \cdots, x_n]$ is a set of homogeneous polynomials, then $V_a(S)$ is a cone since it is closed and closed under scaling. This follows from the fact that $f(x) = 0 \iff f(\lambda x) = 0$ for $\lambda \in k^\times$ when f is homogeneous.
- If C is a cone, then its affine ideal $I_a(C)$ is homogeneous.

Proof (?).
Let $f \in I_a(C)$, then $f(x) = 0$ for all $x \in C$. Since C is closed under scaling, $f(\lambda x) = 0$ for all $x \in C$ and $\lambda \in k^\times$. Decompose $f = \sum_d f_d$ into homogeneous pieces, then

$$x \in C \implies 0 = f(\lambda x) = \sum \lambda^d f_d(x).$$

Fixing $x \in C$, the quantities $f_d(x)$ are constants, so the resulting polynomial in λ vanishes for all λ. But since k is infinite, this forces $f_d(x) = 0$ for all d, which shows that $f_d \in I_a(C)$. ■

Lemma 22.1 *(?).*
There is a bijective correspondence

$$\begin{gathered}\{\text{Cones}\} \iff \{\text{Projective Varieties}\} \\ \mathbb{A}^{n+1} \supset X \mapsto \mathbb{P}X \subset \mathbb{P}^n \\ \mathbb{A}^{n+1} \supset CX \leftarrow\!\shortmid X \subset \mathbb{P}^n \\ .\end{gathered}$$

Proof (?).
$\mathbb{P}V_a(S) = V_p(S)$ for any set S of homogeneous polynomials, and $C(V_p(S)) = V_a(S)$, where $V_p(S)$ is a cone by part (a) of the previous proposition. Conversely, every cone is the variety associated to some homogeneous ideal. ■

22.1 Projective Nullstellensatz

Definition 22.1.1 (Irrelevant Ideal)**.**
The homogeneous ideal $I_0 \coloneqq (x_0, \cdots, x_n) \subset k[x_1, \cdots, x_n]$ is denoted the **irrelevant ideal**.

Proposition 22.1.1 *(Projective Nullstellensatz).*

a. For all $X \subseteq \mathbb{P}^n$, $V_p(I_p(X)) = X$.

b. For all homogeneous ideal $J \subset k[x_1, \cdots, x_n]$ such that (importantly) $\sqrt{J} \neq I_0$, $I_p(V_p(J)) = \sqrt{J}$.

Proof (of a).
$\supset$: If we let I denote the ideal of all homogeneous polynomials vanishing on X, then this certainly contains X.
$\subset$: This follows from part (b), since $X = V_p(J)$ implies that $(V_pI_pV_p)(J) = V_p(\sqrt{J}) = V_p(J) = X$, since taking roots of homogeneous polynomials doesn't change the vanishing locus. ■

Proof (of b).
That $I_p(V_p(J)) \supset \sqrt{J}$ is obvious, since $f \in \sqrt{J}$ vanishes on $V_p(J)$.
Check
It remains to show $\sqrt{J} \subset I_p(V_p(J))$, but we can write $I_p(V_p(J))$ as $\langle f \in k[x_1, \cdots, x_n]\rangle$ the set of homogeneous polynomials vanishing on $V_p(S)$, which is equal to those vanishing on $V_a(J) \setminus \{0\}$.

But since $I_p(\cdots)$ is closed, this is equal to the f that vanish on $\overline{V_a(J) \setminus \{0\}}$, which is only equal to $V_a(J)$ iff $V_a(J) \neq \{0\}$.

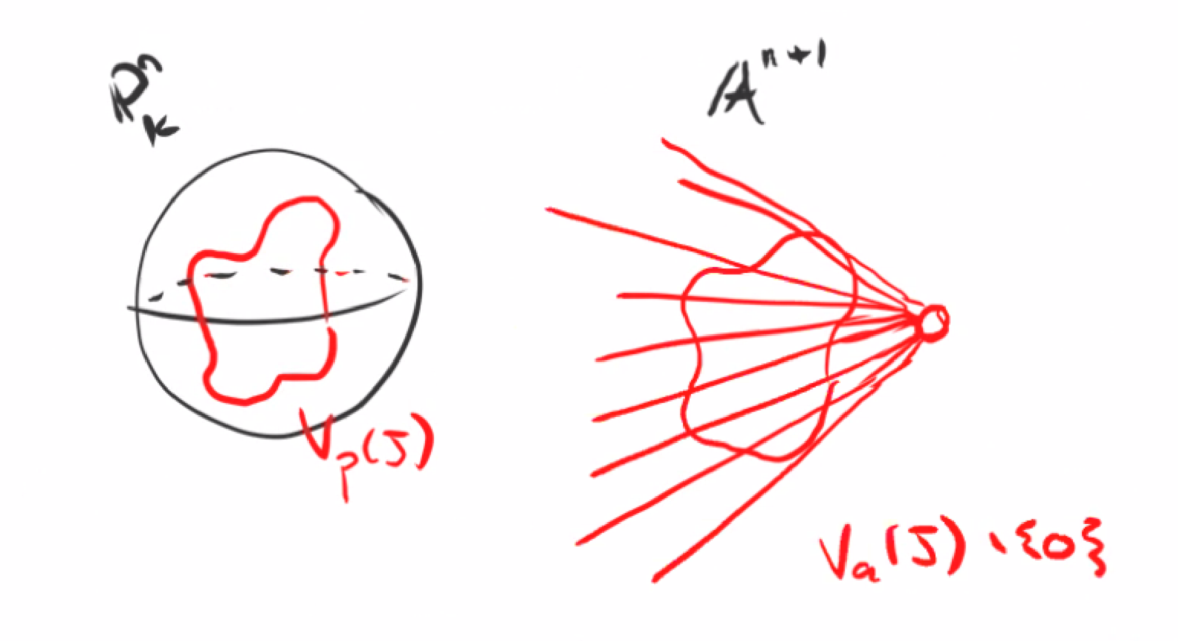

Figure 24: Image

By the affine Nullstellensatz,

$$V_a(J) = \{0\} \iff \sqrt{J} = I_0.$$

Thus $I_p(V_p(J)) = \left\langle f \;\middle|\; \text{homogeneous vanishing on } V_a(J) \right\rangle$. Using the fact that $V_a(J)$ is a cone, its ideal is homogeneous and thus generated by homogeneous polynomials by part (b) of the previous proposition. Thus

$$I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J},$$

where the last equality follows from the affine Nullstellensatz.

■

Corollary 22.1.1 *(?).*
There is an order-reversing bijection

$$\begin{aligned} \left\{ \substack{\text{Projective varieties} \\ X \subset \mathbb{P}^n} \right\} &\iff \left\{ \substack{\text{Homog non-irrelevant radical ideals} \\ \in k[x_1, \cdots, x_n]} \right\} \\ X &\mapsto I_p(X) \\ ? &\mapsfrom ?. \end{aligned}$$

Remark 22.1.1 : A better definition of a cone over $X \subset \mathbb{P}^n_{/k}$ is $\overline{\pi^{-1}(X)} \subset \mathbb{A}^{n+1}_{/k}$ where

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\to \mathbb{P}^n \\ [x_0, \cdots, x_n] &\mapsto [x_0 : \cdots : x_n]. \end{aligned}$$

Definition 22.1.2 (Projective coordinate ring)**.**
Given $X \subset \mathbb{P}^n$ a projective variety, the **projective coordinate ring** of X is given by

$$S(X) := k[x_1, \cdots, x_n]/I_p(X).$$

Remark 22.1.2 : This is a graded ring since $I_p(X)$ is homogeneous. This follows since the quotient of a graded ring by a homogeneous ideal yields a grading on the quotient.

Remark 22.1.3 : We have relative versions of everything. Projective subvarieties of projective varieties are given by $Y \subset X \subset \mathbb{P}^n$ where X is a projective variety. We have a topology on X where the closed subsets are projective subvarieties.

Remark 22.1.4 : Given $J \subset S(X)$, where $S(X)$ is the projective coordinate ring of X and has a grading, we can take $V_p(J) \subset X$. Conversely, given a set $Y \subset S(X)$, we can take $I_p(Y) \subset S(X)$ those homogeneous elements vanishing on Y. Thus there is an order-reversing bijection

$$\left\{ \substack{\text{Projective subvarieties} \\ Y \subset X} \right\} \iff \left\{ \substack{\text{Homogeneous nonirrelevant radical ideals} \\ I \subset S(X)} \right\}$$

and $S(X) = k[x_1, \cdots, x_n]/J \subset \overline{I_0}$.

Remark 22.1.5 : Every nontrivial homogeneous ideal J contained in I_0. Why? Suppose $f \in J \setminus I_0$ and $f_0 \neq 0$. Then $f_0 \in J$ but $f_0 \in k \subset k[x_1, \cdots, x_n]$, implying that $1 \in J$ and thus $J = \langle 1 \rangle$.

Remark 22.1.6 : It is sometimes useful to know that a projective variety is cut out by homogeneous polynomials all of *equal* degree, so $X = V(f_1, \cdots, f_m)$ with each f_i homogeneous of degree d_i. Then there is some maximum degree d. We can write

$$\begin{aligned} V(f_1) = V(x_0^k f_1, \cdots, x_n^k f_1) \qquad \forall k \geq 0 \\ X = \bigcap V(f_1) \cup V(x_i). \end{aligned}$$

This follows because V of a product is a union of the vanishing loci, but $\bigcap V(x_i) = \emptyset$. The equality follows because for all points $[x_0, \cdots, x_n] \in \mathbb{P}^n$, some x_i is nonzero.

23 | Tuesday, November 10

Last time: projective varieties $V(f_i) \subset \mathbb{P}^n_{/k}$ with f_i homogeneous. We proved the projective nullstellensatz: for any projective variety X, we have $V_p(I_p(X))$ and for any homogeneous ideal I with $\sqrt{I} \neq I_0$ the irrelevant ideal, $I_p(V_p(I)) = \sqrt{I}$. Recall that $I_0 = \langle x_0, \cdots, x_n \rangle$. We had a notion of a projective coordinate ring, $S(X) \coloneqq k[x_1, \cdots, x_n]/I_p(X)$, which is a graded ring since $I_p(X)$ is a homogeneous ideal.

Remark 23.0.1 : Note that $S(X)$ is not a ring of functions on X: e.g. for $X = \mathbb{P}^n$, $S(X) = k[x_1, \cdots, x_n]$ but x_0 is not a function on $\mathbb{P}^n$. This is because $f([x_0 : \cdots : x_n]) = f([\lambda x_0 : \cdots : \lambda x_n])$ but $x_0 \neq \lambda x_0$. It still makes sense to ask if f is zero, so $V_p(f)$ is a well-defined object.

Definition 23.0.1 (Dehomogenization of functions and ideals)**.**
Let $f \in k[x_1, \cdots, x_n]$ be a homogeneous polynomial, then we define its dehomogenization as

$$f^i \coloneqq f(1, x_1, \cdots, x_n) \in k[x_1, \cdots, x_n].$$

For a homogeneous ideal, we define

$$J^i \coloneqq \left\{ f^i \;\middle|\; f \in J \right\}.$$

Example 23.0.1 : This is usually not homogeneous. Take

$$\begin{aligned} f &= x_0^3 + x_0 x_1^2 + x_0 x_1 x_2 + x_0^2 + x_1 \\ \implies f^i &= 1 + x_1^2 + x_1 x_2 + x_1, \end{aligned}$$

where has terms of mixed degrees.

Remark 23.0.2 :

- $(fg)^i = f^i g^i$,
- $(f+g)^i = f^i + g^i$

In other words, evaluating at $x_0 = 1$ is a ring morphism.

Definition 23.0.2 (Homogenization of a function)**.**
Let $f \in k[x_1, \cdots, x_n]$, then the **homogenization** of f is defined by

$$f^h \coloneqq x_0^d f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

where $d \coloneqq \deg(f)$.

Example 23.0.2 (?): Let $f(x_1, x_2) = 1 + x_1^2 + x_1 x_2 + x_2^3$, then

$$f^h(x_0, x_1, x_2) = x_0^3 + x_0 x_1^2 + x_0 x_1 x_2 + x_2^3,$$

which is a homogeneous polynomial of degree 3. Note that $(f^h)^i = f$.

Example 23.0.3 (?): It need not be the case that $(f^i)^h = f$. Take $f = x_0^3 + x_0 x_1 x_2$, then $f^i = 1 + x_1 x_2$ and $(f^i)^h = x_0^2 + x_1 x_2$. Note that the total degree dropped, since everything was divisible by x_0.

Remark 23.0.3 :

$$(f^i)^h = f \iff x_0 \nmid f.$$

Definition 23.0.3 (Homogenization of an ideal)**.**
Given $J \subset k[x_1, \cdots, x_n]$, define its **homogenization** as

$$J^h := \left\{ f^h \;\middle|\; f \in J \right\}.$$

Example 23.0.4 : This is not a ring morphism, since $(f+g)^h \neq f^h + g^h$ in general. Taking $f = x_0^2 + x_1$ and $g = -x_0^2 + x_2$, we have $f^h + g^h = x_0 x_1 + x_0 x_2$ while $(f+g)^h = x_1 2 + x_2$.

Remark 23.0.4 : What is the geometric significance? Set

$$U_0 := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n_{/k} \;\middle|\; x_0 \neq 0 \right\} \cong \mathbb{A}^n_{/k}$$

with coordinates $\left[\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right]$.

Proposition 23.0.1 ***(?).***
The conclusion is thus that U_0 with the subspace topology is equal to $\mathbb{A}^n$ with the Zariski topology.

Proof (?).
If we define the Zariski topology on $\mathbb{P}^n$ as having closed sets $V_p(I)$, we would want to check that $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$ is closed in the subspace topology. This amounts to showing that $V_p(I) \cap U_0$ is closed in $\mathbb{A}^n \cong U_0$. We can check that

$$V_p(f, f \in I) = \left\{ [x_0 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0 \, \forall f \in I \right\}.$$

Intersecting with U_0 yields $\left\{ [x_1 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0, \, x_0 \neq 0 \right\}$. Equivalently, we can rewrite this set as

$$\left\{ [x_1 : \cdots : x_n] \;\middle|\; f\left(\left[1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right]\right) = 0, \, f \text{ homogeneous} \right\}$$

Since these are coordinates on $\mathbb{A}^1$, we have $V_p(I) \cap U_0 = V_a(I^i)$ which is closed.
Conversely, given a closed set $V(I)$, we can write this as $V(I) = U_0 \cap V_p(I^h)$. ∎

Corollary 23.0.1 ***(?).***
$\mathbb{P}^n$ is irreducible of dimension n, where the proof is that its covered by irreducible topological spaces of dimension n with nonempty intersection combined with a fact from the exercises.

Example 23.0.5 (?): Consider $f(x_1, x_2) = x_1^2 - x_2^2 - 1$ and consider $V(f) \subset \mathbb{A}^2_{/\mathbb{C}}$:

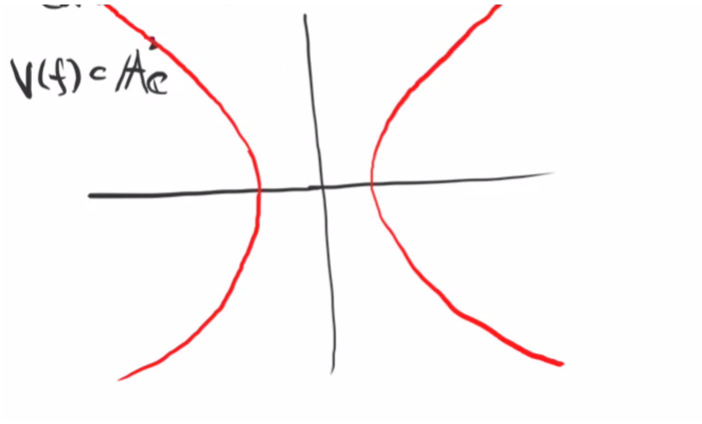

Figure 25: Image

Note that for real projective space, we can view this as a sphere with antipodal points identified. We can thus visualize this in the following way:

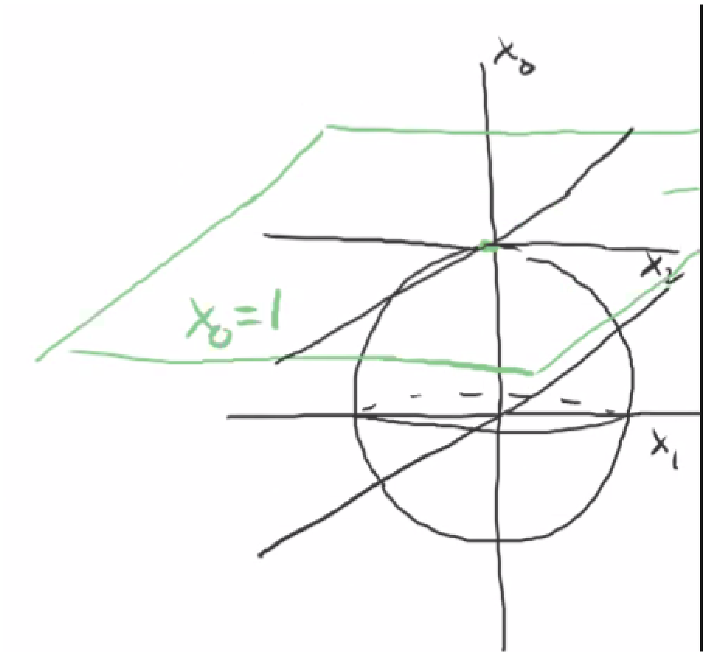

Figure 26: O

We can normalize the x_0 coordinate to one, hence the plane. We can also project $V(f)$ from the plane onto the sphere, mirroring to antipodal points:

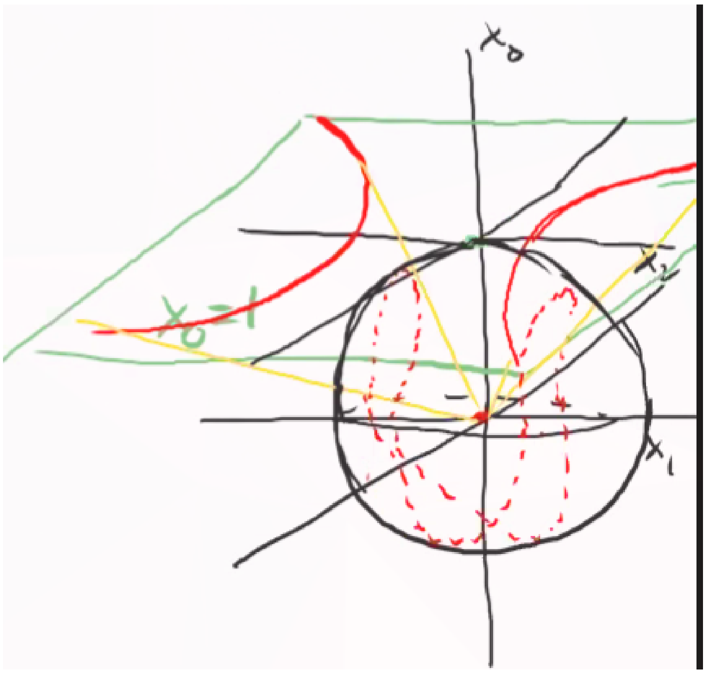

Figure 27: Image

This misses some points on the equator, since we aren't including points where $x_0 = 0$. Consider the homogenization $V(f^h) \subset \mathbb{P}^2_{/\mathbb{C}}$. It's given by $f^h = x_1^2 - x_2^2 - x_0^2$, then

$$V(f^h) \cap V(x_0) = \left\{ [0 : x_1 : x_2] \;\middle|\; f^h(0, x_1, x_2) = 0 \right\} = \{[0:1:1], [0:1:-1]\},$$

which can be seen in the picture as the points at infinity:

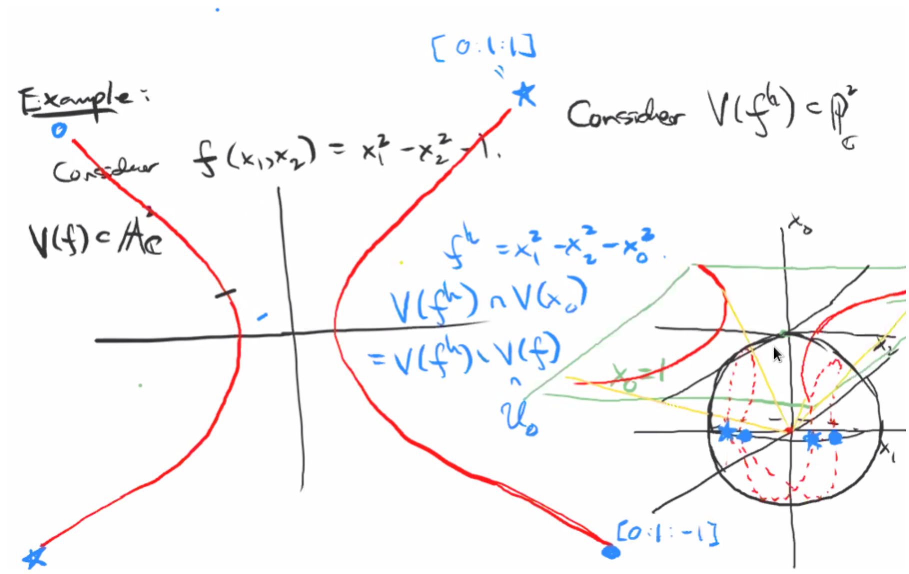

Figure 28: A

Note that the equator is $V(x_0) = \mathbb{P}^2_{/\mathbb{C}} \setminus U_0 \cong \mathbb{P}^2 \setminus \mathbb{A}^2$. So we get a circle of points at infinity, i.e. $V(x_0) = \mathbb{P}^1 = \{[0 : v_1 : v_2]\}$.

Example 23.0.6 (?): Consider $V(f)$ where f is a line in $\mathbb{A}^2_{/\mathbb{C}}$, say $f = ax_1 + bx_2 + c$. This yields $f^h = ax_1 + bx_2 + cx_0$ and we can consider $V(f^h) \cong \mathbb{P}^2_{\mathbb{C}}$. We know $\mathbb{P}^1_{\mathbb{C}}$ is topologically a sphere and $\mathbb{A}^1_{/\mathbb{C}}$ is a point:

Figure 29: $\mathbb{P}^1_{\mathbb{C}}$

The points at infinity correspond to

$$V(f^h) = V(f^h) \cap V(x_0) = \{[0 : -b : a]\},$$

which is a single point not depending on c.

Remark 23.0.5 : $\mathbb{P}^2_{/k}$ for any field k is a **projective plane**, which satisfies certain axioms:

1. There exists a unique line through any two distinct points,

2. Any two distinct lines intersect at a single point.

A famous example is the *Fano plane*:

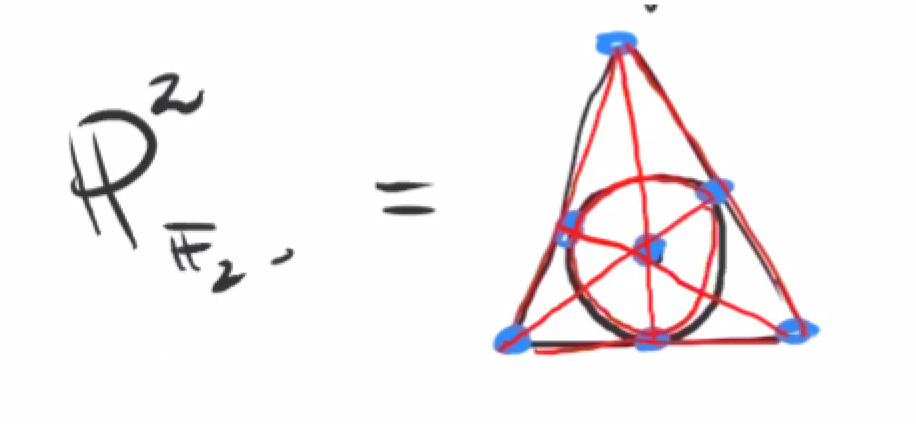

Figure 30: Fano Plane

Why is this true? $\mathbb{P}^2_{/k}$ is the set of lines in k^3, and the lines in $\mathbb{P}^2_{/k}$ are the vanishing loci of homogeneous polynomials and also planes in k^3, since any two lines determine a unique plane and any two planes intersect at the origin.

Proposition 23.0.2*(?).*
Let $J \subset k[x_1, \cdots, x_n]$ be an ideal. Let $X := V_a(J) \subset \mathbb{A}^n$ where we identify $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$. Then the closure $\overline{X} \subset \mathbb{P}^n$ is given by $\overline{X} = V_p(J^h)$. In particular, $V_a(J) = V_p(J^h)$.

Proof (?).
Note that it's clear that $V_p(J^h)$ is closed and contains $V_a(J)$. For the reverse containment, let $Y \supseteq X$ be closed; we want to show that $Y \supseteq V_p(J^h)$. Since Y is closed, $Y = V_p(J')$ where J' is some homogeneous ideal. Any element $f' \in J'$ satisfies $f' = x^d f$ for some maximal d where $x_0^d f$ vanishes on X. We also have $f = 0$ on X since $X \subset U_0$. We can compute

$$f \in I_a(X) = I_a(V_a(J)) = \sqrt{J},$$

so $f^m \in J$. Then $(f^h)^m \in J^h$ for some m, and this $f^h \in \sqrt{J^h}$.
We can conclude that $J' \subset \sqrt{J}$, which shows that $V_p(J') \supseteq V_p(J^h)$ as desired. ■

Definition 23.0.4 (Projective Closure)**.**
The **projective closure** of $X = V_a(J)$ is the smallest closed subset containing X and is given by

$$\overline{X} = V_p(J^h).$$

24 Misc Unsorted

algebra	$\leftrightarrow$	geometry
radical ideal $I = \sqrt{I}$	$\rightarrow$	$V(I)$ variety
$I(V)$ ideal of a set	$\leftarrow$	solution set V
sum of ideals $I + J$	$\rightarrow$	$V(I) \cap V(J)$ intersection of varieties
$\sqrt{I(V) + I(W)}$ radical of sum	$\leftarrow$	intersection of sets $V \cap W$
product of ideals IJ	$\rightarrow$	$V(I) \cup V(J)$ union of varieties
$\sqrt{I(V)I(W)}$ radical of product	$\leftarrow$	union of sets $V \cup W$
intersection of ideals $I \cap J$	$\rightarrow$	$V(I) \cup V(J)$ union of varieties
$I(V) \cap I(W)$	$\leftarrow$	union of sets $V \cup W$
quotient of ideals $I : J$	$\rightarrow$	$\overline{V(I) - V(J)}$ difference of varieties
$I(V) : I(W)$	$\leftarrow$	difference of sets $\overline{V - W}$
elimination $\sqrt{I \cap \mathbb{C}[x_{k+1}, \ldots, x_n]}$	$\leftrightarrow$	$\overline{\pi_k(V(I))}$ projection of varieties

Figure 31: Image

25 Indices

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26 Bibliography

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