Algebra

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1 Lecture 1 (Thu 15 Aug 2019)

Definition 1. A *group* is an ordered pair $(G, \cdot : G \times G \to G)$ where G is a set and $\cdot$ is a binary operation, which satisfies the following axioms:

1. Associativity: $(g_1 g_2)g_3 = g_1(g_2 g_3)$,
2. Identity: $\exists e \in G \ni ge = eg = g$,
3. Inverses: $g \in G \implies \exists h \in G \ni gh = gh = e$.

Some examples of groups:

- $(\mathbb{Z}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Q}^\times, \times)$
- $(\mathbb{R}^\times, \times)$
- $(\mathrm{GL}(n, \mathbb{R}), \times) = \{A \in \mathrm{Mat}_n \ni \det(A) \neq 0\}$
- $(S_n, \circ)$

Definition 2. A subset $S \subseteq G$ is a *subgroup* of G iff

1. $s_1, s_2 \in S \implies s_1 s_2 \in S$
2. $e \in S$
3. $s \in S \implies s^{-1} \in S$

We denote such a subgroup $S \leq G$.

Examples of subgroups:

- $(\mathbb{Z}, +) \leq (\mathbb{Q}, +)$
- $\mathrm{SL}(n, \mathbb{R}) \leq \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \ni \det(A) = 1\}$

1.1 Cyclic Groups

Definition 3. A group G is cyclic iff G is generated by a single element.

Exercise 1. Show $\langle g \rangle = \{g^n \ni n \in \mathbb{Z}\} \cong \bigcap \{H \leq G \ni g \in H\}$.

Theorem 1. Let G be a cyclic group, so $G\langle g \rangle$.

1. If $|G| = \infty$, then $G \cong \mathbb{Z}$.
2. If $|G| = n < \infty$, then $G \cong \mathbb{Z}_n$

Definition 4. Let $H \leq G$, and define a *right coset of G* by $aH = \{ah \ni H \in H\}$. A similar definition can be made for *left cosets*.

Then $aH = bH \iff b^{-1}a \in G$ and $Ha = Hb \iff ab^{-1} \in H$.

Some facts:

- Cosets partition H, i.e. $b \notin H \implies aH \bigcap bH = \{e\}$.
- $|H| = |aH| = |Ha|$ for all $a \in G$.

Theorem 2 (Lagrange)**.** If G is a finite group and $H \leq G$, then $|H| \mid |G|$.

Definition: $N \leq G$ is *normal* iff $gN = Ng$ for all $g \in G$, or equivalently $gNg^{-1} \subseteq N$. I denote this $N \trianglelefteq G$.

When $N \trianglelefteq G$, the set of left/right cosets of N themselves have a group structure. So we define $G/N = \{gN \ni g \in G\}$ where $(g_1N)(g_2N) = (g_1g_2)N$.

Given $H, K \leq G$, define $HK = \{hk \ni h \in H, k \in K\}$. We have a general formula,

$$|HK| = \frac{|H||K|}{|H \bigcap K|}.$$

1.2 Homomorphisms

Let G, G' be groups, then $\varphi : G \to G'$ is a *homomorphism* if $\varphi(ab) = \varphi(a)\varphi(b)$.

Examples:

- $\exp : (\mathbb{R}, +) \to (\mathbb{R}^{>0}, \cdot\,)$ where $\exp(a + b) = e^{a+b} = e^a e^b = \exp(a)\exp(b)$.
- $\det : (\mathrm{GL}(n, \mathbb{R}), \times) \to (\mathbb{R}^\times, \times)$ where $\det(AB) = \det(A)\det(B)$.
- Let $N \trianglelefteq G$ and $\varphi G \to G/N$ given by $\varphi(g) = gN$.
- Let $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ where $\phi(g) = [g] = g \mod n$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$

Definitions: Let $\varphi : G \to G'$. Then φ is a *monomorphism* iff it is injective, an *epimorphism* iff it is surjective, and an *isomorphism* iff it is bijective.

1.3 Direct Products

Let G_1, G_2 be groups, then define $G_1 \times G_2 = \{(g_1, g_2) \ni g_1 \in G, g_2 \in G_2\}$ where $(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2, h_2)$.

We have the formula $|G_1 \times G_2| = |G_1||G_2|$.

1.4 Finitely Generated Abelian Groups

We say a group is *abelian* if G is commutative, i.e. $g_1, g_2 \in G \implies g_1 g_2 = g_2 g_1$.

A group is *finitely generated* if there exist $\{g_1, g_2, \cdots g_n\} \subseteq G$ such that $G = \langle g_1, g_2, \cdots g_n \rangle$. This generalizes the notion of a cyclic group, where we can simply intersect all of the subgroups that contain the g_i to define it.

We know what cyclic groups look like – they are all isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$. So now we'd like a structure theorem for abelian f.g. groups.

Theorem: Let G be a f.g. abelian group. Then $G \cong \mathbb{Z}^r \times \prod_{i=1}^{s} \mathbb{Z}_{p_i^{\alpha_i}}$ for some finite $r, s \in \mathbb{N}$ and p_i are (not necessarily distinct) primes.

Example: Let G be a finite abelian group of order 4. Then $G \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$, which are not isomorphic because every element in $\mathbb{Z}_2^2$ has order 2 where $\mathbb{Z}_4$ contains an element of order 4.

1.5 Fundamental Homomorphism Theorem

Let $\varphi : G \to G'$ be a group homomorphism and define $\ker \varphi = \{g \in G \ni \varphi(g) = e'\}$.

1.5.1 The First Homomorphism Theorem

\begin{theorem theorem There exists a map $\varphi' : G/\ker \varphi \to G'$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, dotted, "\varphi"] \arrow[d, "\eta"'] & G' \\ G/\ker\varphi \arrow[ur, "\varphi'"] & \end{tikzcd}$$

That is, $\varphi = \varphi' \circ \eta$, and φ' is an isomorphism onto its image, so $G/\ker \varphi = \operatorname{im} \varphi$. This map is give by $\varphi'(g(\ker \varphi)) = \varphi(g)$, which can be checked to be well-defined.

1.5.2 The Second Theorem

Theorem 3. Let $K, N \leq G$ where $N \trianglelefteq G$. Then

$$\frac{K}{N \bigcap K} \cong \frac{NK}{N}$$

Proof. Proof: Define a map $K \xrightarrow{\varphi} NK/N$ by $\varphi(k) = kN$. You can show that φ is onto by looking at $\ker \varphi$; note that $kN = \varphi(k) = N \iff k \in N$, and so $\ker \varphi = N \bigcap K$. $\square$

2 Lecture 2