

# Qual Problems

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## 1 Problem 1

### 1.1 Part 1

*Definition:* An element  $r \in R$  is *irreducible* if whenever  $r = st$ , then either  $s$  or  $t$  is a unit.

*Definition:* Two elements  $r, s \in R$  are *associates* if  $r = \ell s$  for some unit  $\ell$ .

A ring  $R$  is a *unique factorization domain* iff for every  $r \in R$ , there exists a set  $\{p_i \mid 1 \leq i \leq n\}$  such that  $r = u \prod_{i=1}^n p_i$  where  $u$  is a unit and each  $p_i$  is irreducible.

Moreover, this factorization is unique in the sense that if  $r = w \prod_{i=1}^n q_i$  for some  $w$  a unit and  $q_i$  irreducible elements, then each  $q_i$  is an associate of some  $p_i$ .

### 1.2 Part 2

A ring  $R$  is a *principal ideal domain* iff whenever  $I \trianglelefteq R$  is an ideal of  $R$ , there is a single element  $r_i \in R$  such that  $I = (r_i)$ .

### 1.3 Part 3

An example of a UFD that is not a PID is given by  $R = k[x, y]$  for  $k$  a field.

That  $R$  is a UFD follows from the fact that if  $k$  is a field, then  $k$  has no prime elements since every non-zero element is a unit. So the factorization condition holds vacuously for  $k$ , and  $k$  is a UFD. But then we can use the following result:

**Theorem:** If  $R$  is a UFD, then  $R[x]$  is a UFD.

Since  $k$  is a UFD, the theorem implies that  $k[x]$  is a UFD, from which it follows that  $k[x][y] = k[x, y]$  is also a UFD.

To see that  $R$  is not a PID, consider the ideal  $I = (x, y)$ , and suppose  $I = (g)$  for some single  $g \in k[x, y]$ .

Note that  $I \neq R$ , since  $I$  contains no degree zero polynomials. Moreover, since  $(x) \subset I = (g)$  (and similarly for  $y$ ), we have  $g \mid x$  and  $g \mid y$ , which forces  $\deg g = 0$ .

So in fact  $g \in k$  and thus  $g$  is invertible, but then  $(g) = g^{-1}(g) = (1) = k$ , so this forces  $I = k \leq k[x, y]$ . However,  $x \notin k$  (nor  $y$ ), which is a contradiction.

## 2 Problem 2

**Lemma 1:** Then  $A$  has  $n$  distinct eigenvalues  $\iff m_A(x) = \chi_A(x)$ .

*Proof:*

We'll use the fact that every eigenvalue is always root of both  $m_A(x)$  and  $\chi_A(x)$  (potentially with differing multiplicities), so we can write

$$\begin{aligned} m_A(x) &= \prod_i (x - \lambda_i)^{p_i} \\ \chi_A(x) &= \prod_i (x - \lambda_i)^{q_i} \end{aligned}$$

where  $1 \leq p_i \leq q_i$  for every  $i$ .

$\implies$  : If  $A$  has  $n$  distinct eigenvalues, then  $\chi_A(x) = \prod_{i=1}^n (x - \lambda_i)$  in  $\bar{k}[x]$ . Noting that every exponent is 1, we have  $q_i = 1$  for all  $i$ , which forces  $p_i = 1$  and thus  $m_A(x) = \chi_A(x)$ .

$\impliedby$  : If  $m_A(x) = \chi_A(x)$ , then  $p_i = q_i$  for all  $i$ . If we then consider  $JCF(A)$ , we have

- The number of Jordan block  $J_{\lambda_i}$  is the dimension of the eigenspace  $E_{\lambda_i}$ ,
- $q_i$  = the sum of the sizes of all Jordan blocks  $J_{\lambda_i}$ , and
- $p_i$  = the size of the largest Jordan block  $J_{\lambda_i}$ .

Thus  $p_i = q_i$  for every  $i \iff$  there is one Jordan block for every  $\lambda_i \iff \dim E_{\lambda_i} = 1$  for every  $i$ .

But  $\dim E_{\lambda_i}$  is precisely the multiplicity of  $\lambda_i$  in  $\chi_A(x)$ , which means that  $\chi_A(x) = \prod_i (x - \lambda_i)$ . Since  $\chi_A(x)$  is a degree  $n$  polynomial, this says that  $\chi_A$  has  $n$  distinct linear factors, corresponding to  $n$  distinct eigenvalues of  $A$ .

□

**Lemma 2:** Let  $k[x] \curvearrowright V$  in the usual way with  $A$  to obtain an invariant factor decomposition

$$V = \frac{k[x]}{(f_1)} \oplus \frac{k[x]}{(f_2)} \oplus \cdots \oplus \frac{k[x]}{(f_k)}, \quad f_1 \mid f_2 \mid \cdots \mid f_k.$$

Then it is always the case that

- $m_A(x) = f_k(x)$ , i.e. the minimal polynomial is the invariant factor of largest degree,
- $\chi_A(x) = \prod_{i=1}^k f_i(x)$ , i.e. the characteristic polynomial is the product of all of the invariant factors.

□

Now to prove the main result:

(1)  $\implies$  (2):

Suppose

$$V = \text{span}_k \{ \mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \dots, A^{n-1}\mathbf{v} \} := \text{span}_k \mathcal{B}$$

where  $\dim_k V = n$ .

Then  $A^n\mathbf{v}$  is necessarily a linear combination of these basis elements, and in particular, there are coefficients  $c_i$  (not all zero) such that

$$A^n\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i\mathbf{v}.$$

The consider computing the matrix of  $A$  in  $\mathcal{B}$  by considering the images of all basis elements under  $A$ .

Letting  $\mathcal{B} = \{ \mathbf{w}_i := A^i\mathbf{v} \mid 0 \leq i \leq n-1 \}$ , we have

$$\begin{aligned} \mathbf{w}_0 &:= \mathbf{v} \mapsto A\mathbf{v} := \mathbf{w}_1 \\ \mathbf{w}_1 &:= A\mathbf{v} \mapsto A^2\mathbf{v} := \mathbf{w}_2 \\ \mathbf{w}_2 &:= A^2\mathbf{v} \mapsto A^3\mathbf{v} := \mathbf{w}_3 \\ &\vdots \\ \mathbf{w}_{n-2} &:= A^{n-2}\mathbf{v} \mapsto A^{n-1}\mathbf{v} := \mathbf{w}_{n-1} \\ \mathbf{w}_{n-1} &:= A^{n-1}\mathbf{v} \mapsto A^n\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i\mathbf{v}_i := \sum_{i=0}^{n-1} c_i \mathbf{w}_i. \end{aligned}$$

This means that with respect to the basis  $\mathcal{B}$ ,  $A$  has the following matrix representation:

$$[A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & c_{n-1} \end{bmatrix}$$

But this is the companion matrix for  $p(x) = \sum_{i=0}^{n-1} c_i x^i$ , which always satisfy the property that  $p(x)$  equals both their characteristic *and* their minimal polynomial.

Thus by lemma 1, the matrix  $[A]_{\mathcal{B}}$  has distinct eigenvalues.

(2)  $\implies$  (1):

Suppose  $A$  has distinct eigenvalues. By Lemma 1,  $\chi_A(x) = m_A(x)$ , and so we have

$$\chi_A(x) = f_k(x) = \prod_{i=1}^k f_i(x) = m_A(x),$$

which can only happen if  $f_1(x) = f_2(x) = \cdots = f_{n-1}(x) = 1$ , in which case there is only one nontrivial invariant factor.

So we have

$$V \cong \frac{k[x]}{(f_k)}, \quad \text{Ann}(V) = (f_k), \quad \deg f_k = n,$$

which exhibits  $V$  as a cyclic  $k[x]$ -module and thus we have  $V = k[x]\mathbf{v}$  for some  $\mathbf{v} \in V$ .

In particular, if we take the Rational Canonical Form of  $A$ , this says that  $RCF(A)$  has only a single block in a suitable ordered basis  $\mathcal{B} = \{\mathbf{w}_0, \cdots, \mathbf{w}_{n-1}\}$ .

Thus  $[A]_{\mathcal{B}}$  is the companion matrix to  $f_k(x) := \sum_{i=0}^n c_i x^i$  in this basis, which by construction satisfies

$$A\mathbf{w}_i = \begin{cases} \mathbf{w}_{i+1} & 0 \leq i < n-2 \\ \sum_{i=0}^{n-1} c_i \mathbf{w}_i & i = n-1 \end{cases}.$$

□

### 3 Problem 3

#### 3.1 Part 1

Let  $\mathbf{v} = [0, 1, 0]^t$ , We compute

$$M\mathbf{v} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(0) + 0(1) + x(0) \\ 0(0) + 1(1) + 0(0) \\ y(0) + 0(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which shows that  $\mathbf{v}$  is an eigenvector of  $M$  with eigenvalue  $\lambda = 1$ .

### 3.2 Part 2

Noting that the rank is the dimension of the column space, we find that

- $\text{rank}(M) \geq 1$ , since it is not the zero matrix,
- $\text{rank}(M) \geq 2$ , since neither  $[1, 0, y]^t$  or  $[x, 0, 1]^t$  can be in the span of  $[0, 1, 0]^t$ , and
- $\text{rank}(M) = 3 \iff \det(M) \neq 0$ .

So we compute

$$\det_M(x, y) = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{vmatrix} = 1(1 - 0) - 0(1 - xy) + x(-y) = 1 - xy,$$

and so  $\det_M(x, y) = 0 \iff xy = 1$ . Thus

$$\text{rank}(M) = \begin{cases} 3 & xy = 1 \\ 2 & \text{else.} \end{cases}$$

### 3.3 Part 3

Since  $M$  is diagonalizable  $\iff M$  is full rank, which in this case means  $\text{rank}(M) = 3$ , we have

$$S = \left\{ (x, y) \in \mathbb{R}^2 \mid M \text{ is diagonalizable} \right\} = \left\{ \left( x, \frac{1}{x} \right) \mid x \in \mathbb{R} \setminus \{0\} \right\} \subset \mathbb{R}^2.$$