Title

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Recall the that *Hasse-Weil zeta function* of a one-variable function field $K/\mathbb{F}_q$ over a finite ground field is defined in the following way: let $A_n = A_n(K)$ be the number of effective divisors of degree n. We have proved that A_n is finite, and for $n > 2g-2$ we have a formula

$$Z(t) = \sum_{n=0}^{\infty} A_n t^n = \sum_{D \in \mathrm{Div}^+(K)} t^{\deg(D)} \in \mathbb{Z}[[t]],$$

which is a formal power series with integer coefficients.

Remark 1.0.1 : Recall that we have proved that it is a rational function of t, and in particular when $g = 0, \delta = 1$ [1] we get

$$Z(t) = \frac{1}{(1-qt)(1-t)}.$$

We got another expression which isn't fantastic: it involves this δ, which we'll work toward proving is equal to 1. When $g > 1$, we broke the zeta function into two pieces $Z(t) = F(t) + G(t)$. For divisors of sufficiently high degree, Riemann-Roch tells you what the dimension of the Riemann-Roch space is, and $G(t)$ explains the part coming from divisors of large degree. We obtained a formula previously for $F(t)$ and $G(t)$, and once we show $\delta = 1$ the formula for G will simplify. For $F(t)$, we specifically had

$$F(t) = \frac{1}{q-1} \sum_{0 \leq \deg(c) \leq 2g-2} q^{\ell(c) t^{\deg(c)}},$$

where the sum is over divisor classes and ℓ is the dimension of linear system corresponding to a divisor. But this isn't a great formula: what are these classes, dhow many are in each degree, and what is the dimension of the Riemann-Roch space?

Remark 1.0.2 : This is analogous to the Dedekind zeta function of a number field K, in which case

$$\zeta_K(s) = \dot{\sum_{T \in \ell(\mathbb{Z}_k)}} |\mathbb{Z}_k / I|^{-s},$$

which will be covered in a separate lecture on *Serre zeta functions*.

> **Theorem 1.0.1** ***(F.K. Schmidt).***
> For all $K/\mathbb{F}_q$, we have $\delta = I(K) = 1$ where I is the index.

This will follow from the associated, but it much weaker. However, this is one of the facts we'd like to establish to use to *prove* the Riemann hypothesis.

[1] The *index* of the function field, least positive degree of a divisor.

Remark 1.0.3 : Pete studied this in 2004 and found that every $I \in \mathbb{Z}^+$ arises as the index of a genus one function field $K/\mathbb{Q}$.

Notation: for $n \in \mathbb{Z}^+$, let μ_n denote the nth roots of unity in $\mathbb{C}$.

Lemma 1.1 ***(?).***
For $m, r \in \mathbb{Z}^+$, set $d := \gcd(m, r)$. Then

$$\left(1 - t^{mr/d}\right)^d = \prod_{\xi \in \mu_r} 1 - (\xi t)^m.$$

Proof (?).
In $\mathbb{C}[x]$, we have

$$(X^{r/1} - 1)^d = \prod_{\xi \in \mu_r} (X - \xi^m),$$

where both sides are monic polynomials whose roots include the (r/d)th roots of unity, each with multiplicity d. On the LHS, the distinct roots are the r/dth roots of unity, then raising to the dth power gives them multiplicity d. On the RHS, this is an exercise in cyclic groups: consider the nth power map on $\mathbb{Z}/r\mathbb{Z}$ and compute its image and kernel. As ξ ranges over rth roots of unity, ξ^m ranges over all r/dth roots of unity, each occurring with multiplicity d. Substituting $X = t^{-m}$ and multiplying both sides by t^r yields the original result.

Special case: set $m = r$, so $d = r$, then the RHS is r copies of 1.

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Next up, we want to compare the zeta function $Z(t)$ for a function field over $\mathbb{F}_q$ to the zeta function obtained when extending scalars to $\mathbb{Q}^r$.

Proposition 1.0.1 ***(Factorization identity for the zeta function).***
Let $K/\mathbb{F}_q$ be a function field, $r \in \mathbb{Z}^+$, and take the compositum K_r of K and $\mathbb{F}_q^r$ viewed as a function field over $\mathbb{F}_q^r$. Let $Z(t)$ be the zeta function of $K/\mathbb{F}_q$ and $Z_r(t)$ the zeta function of $K_r/\mathbb{F}_q^r$. Then

$$Z_r(t^r) = \prod_{\xi \in \mu_r} Z(\xi t).$$

Proof (?).
We have an Euler product formula

$$Z(t) = \prod_{p \in \Sigma(K/\mathbb{F}_q)} (1 - t^{\deg(p)})^{-1}.$$

where the sum is over places of the function field.

Proving this Euler product formula might show up in a separate lecture, but it is not any more difficult than proving it for the Riemann zeta function.

Exercise 1.0.1 (?): Why is this product expansion true? Write as a geometric series with ratio $t^{\deg(p)}$. Here just expand each summand to get

$$Z(t) = \prod_p \sum_{j=1}^{\infty} t^{j \deg(p)}.$$

Multiplying this out and collecting terms is in effect multiplying out the prime divisors to get effective divisors.
We now use the result about splitting that was stated (but not proved):

Claim: If $p \in \Sigma_m(K/\mathbb{F}_q)$ is a degree n place and $r \in \mathbb{Z}^+$, then there exist precisely $d := \gcd(m, r)$places p^r of K_r lying over p. Moreover, each place p^r has degree m/d.
In order to compare $Z_r(t)$ to $Z(t)$, we collect the p' into ones that have the same fiber. We then can range over all p first, then over all p' in the fiber above p, yielding

$$Z_r(t^r) = \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} \prod_{p'/p} \frac{1}{1 - t^{r \deg(p')}}.$$

Using the Euler product identity, we have for $p \in \Sigma_m(K_{/\mathbb{F}_q})$ and $d := \gcd(m, r)$ we can express the innermost product as

$$\prod_{p'/p} \frac{1}{1 - t^{r \deg(p')}} = (1 - t^{rm/d})^{-d} = \prod_{\xi \in \mu_r} (1 - (\xi t)^m)^{-1},$$

where we've used the fact that we know there are exactly d places and each contributes the same degree in the first expression. By using $-d$ in the previous lemma, we get the last term. Combining all of this yields

$$Z_r(t^r) = \prod_{\xi \in \mu_r} \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} (1 - (\xi t)^{\deg p})^{-1} = \prod_{\xi \in \mu_r} Z(\xi t).$$

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Remark 1.0.4 : Similar to taking an abelian extension of number fields and noting that the Dedekind zeta function factors into a finite product: the original zeta function, and in general, Hecke L functions. If you do this for an abelian number field over $\mathbb{Q}$, then the Dedekind zeta function of the upstairs number field will be a finite product where one of the terms in the Riemann zeta function and the others are Dirichlet L functions associated to certain Dirichlet characters. So this is some (perhaps simpler) version of that.

We can finally prove Schmidt's theorem that $\delta = 1$.

Proof ($\delta = 1$).
Take a δth root of unity $\xi \in \mu_\delta$. Then for all places $p \in \Sigma(K_{/\mathbb{F}_q})$, δ divides $\deg p$ by definition

since it is a gcd, and so we have

$$Z(\xi t) = \prod_{p \in \Sigma(K_{/\mathbb{F}_q})} (q - (\xi t)^{\deg p})^{-1} = \prod_{p \in \Sigma_{K_{\mathbb{F}_q}}} \frac{1}{1 - t^{\deg p}} = Z(t),$$

using the fact that $\xi^{\deg p} = 1$.
We're now in a situation where we can apply the previous proposition, which gives the following identity for the zeta function over the degree δ extension:

$$Z_\delta(t^\delta) = \prod_{\xi \in \mu_\delta} Z(\xi t) = Z(t)^\delta.$$

Our previous formulas show that any zeta function for a 1-variable function field over a finite field has a simple pole at $t = 1$, and since $\mathrm{Ord}_{t-1}(t^\delta) = 0$, we get

$$1 = \mathrm{Ord}_{t-1} Z_\delta(t^\delta) = \mathrm{Ord}_{t-1} Z(t)^\delta) = \delta.$$

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