Title

D. Zack Garza

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0.1 Exercises

Problem 1.

Let C denote the Cantor set.

1. Show that C contains point that is not an endpoint of one of the removed intervals.
2. Show that C is nowhere dense, meager, and has measure zero.
3. Show that C is uncountable.

Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let C_n be the nth stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:
$$C_n = [0, \frac{1}{3^n}] \bigcup [\frac{2}{2^n}, \frac{3}{3^n}] \bigcup \cdots \bigcup [\frac{3^n - 1}{n}, 1],$$
and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in C_n is never removed in any later step, and so all endpoints remaining in C are of this form where we allow $0 \leq n < \infty$.

   Thus, our goal is to produce a number $x \in [0, 1]$ such that $x \neq \frac{k}{3^n}$ for any k or n, but also satisfies $x \in C$.

   Claim: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.
$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$
Proof: Towards a contradiction suppose that $x \in C$ and contains a 1 in its ternary expansion, so $a_k = 1$ for some k. Without loss of generality, we can consider the smallest k for which this happens.