

Weil Conjectures

D. Zack Garza

Tuesday 21st April, 2020

Contents

1	Notes from Daniel's Office Hours	1
1.1	Definition of Zeta Function	1
1.1.1	Simple but Useful Example: A Point	3
1.2	Statement of Weil Conjectures	4
1.2.1	Aside: Why call it a Zeta function?	6
1.2.2	More Examples	7
1.3	Hard Example: An Elliptic Curve	11
1.4	Very Hard Example: A Diagonal Hypersurface	13

1 Notes from Daniel's Office Hours

0. Definition of Zeta functions
1. Statement of the conjectures
2. Easy examples: \mathbb{P}^n , $\text{Gr}_\gamma(k, n) = \text{GL}(n, \mathbb{F})/P$ the stabilizer of an γ -point in $\mathbb{C}^n, \mathbb{F}_{p^n}$.
3. Medium example: E/\mathbb{F} an elliptic curve.
4. Work out a harder example as in Weil

References

- http://www-personal.umich.edu/~mmustata/zeta_book.pdf
- <https://youtu.be/wEz7fCvK6sM?t=293>
- Explanation of exponential appearing
- <https://arxiv.org/pdf/1807.10812.pdf>
- http://www.math.canterbury.ac.nz/~j.boomer/expos/weil_conjectures.pdf
- Weil's Paper

1.1 Definition of Zeta Function

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the finite field with q elements, along with its unique degree n extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}_q} \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

Definition 1.0.1.

Let

$$J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$$

be an ideal, then a *projective algebraic* variety $X \subset \mathbb{P}_{\mathbb{F}}^N$ can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^{\infty} \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in $n + 1$ variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field \mathbb{F}_p . We will be thinking of K -valued points for K/\mathbb{F}_p algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into \mathbb{P}^{∞} ?

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Fix $X/\mathbb{F} \subset \mathbb{P}$ an N -dimensional projective algebraic variety, and say it's cut out by the equations $f_1, \dots, f_M \in \mathbb{F}[x_0, \dots, x_n]$. Note that it then has points in any finite extension L/K .

Definition 1.0.2.

Define the *local zeta function* of X the following formal power series:

$$Z_X(z) = \exp \left(\sum_{n=1}^{\infty} \alpha_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Concretely, for $X \subset \mathbb{P}^M$ a variety cut out by $\{f_i\} \subset \mathbb{F}[x_0, \dots, x_M]$ we are measuring the sizes of the sets

$$\alpha_n := \# \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}_{q^n}}^M \mid f_i(\mathbf{x}) = \mathbf{0} \, \forall i \right\}.$$

Note the following two properties:

$$Z_X(0) = 1$$

$$z \left(\frac{\partial}{\partial z} \right) \log Z_X(z) = t \left(\frac{Z'_X(z)}{Z_X(z)} \right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \dots,$$

which is an *ordinary generating function* for the sequence (α_n) .

Todo: why not an OGF.

Remark: Note that for an OGF $F(x) = \sum_{n=0}^{\infty} f_n x^n$, we can extract coefficients in the following way:

$$f_n := [x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!} \left(\frac{\partial}{\partial x} \right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

Note: this is extremely amenable to numerical approximation if you have a closed form for F or even just a black-box numerical version of F ! I.e. easy to throw at a computer.

1.1.1 Simple but Useful Example: A Point

Take $X = \{x = 0\} / \mathbb{F}$ a single point over \mathbb{F} , then

$$\begin{aligned} \#X(\mathbb{F}) &:= \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &:= \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &:= \alpha_n = 1 \\ &\vdots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n &&= 1 + z + z^2 + \dots \\ \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n &&= \sum_{n=0}^{\infty} \int z^n = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots \\ \implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n}. \end{aligned}$$

and so

$$\begin{aligned} Z_{\{\text{pt}\}}(z) &= \exp \left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \dots \right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}. \end{aligned}$$

1.2 Statement of Weil Conjectures

(Weil 1949)

Let X be a smooth projective variety of dimension N over \mathbb{F}_q for q a prime, let $Z_X(z)$ be its zeta function, and define $\zeta_X(s) = Z_X(q^{-s})$.

1. (Rationality)

$Z_X(z)$ is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$\begin{aligned} P_0(z) &= 1 - z \\ P_{2N}(z) &= 1 - q^N z \\ P_j(z) &= \prod_{i=1}^{\beta_j} (1 - a_{j,i} z) \quad \text{for some reciprocal roots } a_{j,i} \in \mathbb{C} \end{aligned}$$

where we've factored each P_i using its reciprocal roots a_{ij} .

In particular, this implies the existence of a meromorphic continuation of the associated function $\zeta_X(s)$, which a priori only converges for $\Re(s) \gg 0$.

2. (Functional Equation and Poincare Duality)

Let $\chi(X)$ be the Euler characteristic of X , i.e. the self-intersection number of the diagonal embedding $\Delta \hookrightarrow X \times X$; then $Z_X(z)$ satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm \left(q^{\frac{N}{2}} z\right)^{\chi(X)} Z_X(z).$$

Equivalently,

$$\zeta_X(N - s) = \pm \left(q^{\frac{N}{2} - s}\right)^{\chi(X)} \zeta_X(s).$$

Note that when $N = 1$, e.g. for a curve, this relates $\zeta_X(s)$ to $\zeta_X(1 - s)$.

Equivalently, there is an involutive map on the (reciprocal) roots

$$\begin{aligned} z &\longleftrightarrow \frac{q^N}{z} \\ \alpha_{j,k} &\longleftrightarrow \alpha_{2N-j,k} \end{aligned}$$

which sends roots of p_j to roots of p_{2N-j} .

3. (Riemann Hypothesis)

The reciprocal roots $a_{j,k}$ are *algebraic* integers (roots of some monic $p \in \mathbb{Z}[x]$) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}} \quad \forall 1 \leq j \leq 2N-1, \forall k.$$

4. (Betti Numbers) If X is a “good reduction mod q ” of a nonsingular projective variety \tilde{X} in characteristic zero, then the $\beta_i = \deg p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$.

Why is (3) called the “Riemann Hypothesis”?

We can use the facts that

- a. $|\exp(z)| = \exp(\Re(z))$ and
- b. $a^z := \exp(z \operatorname{Log}(a))$,

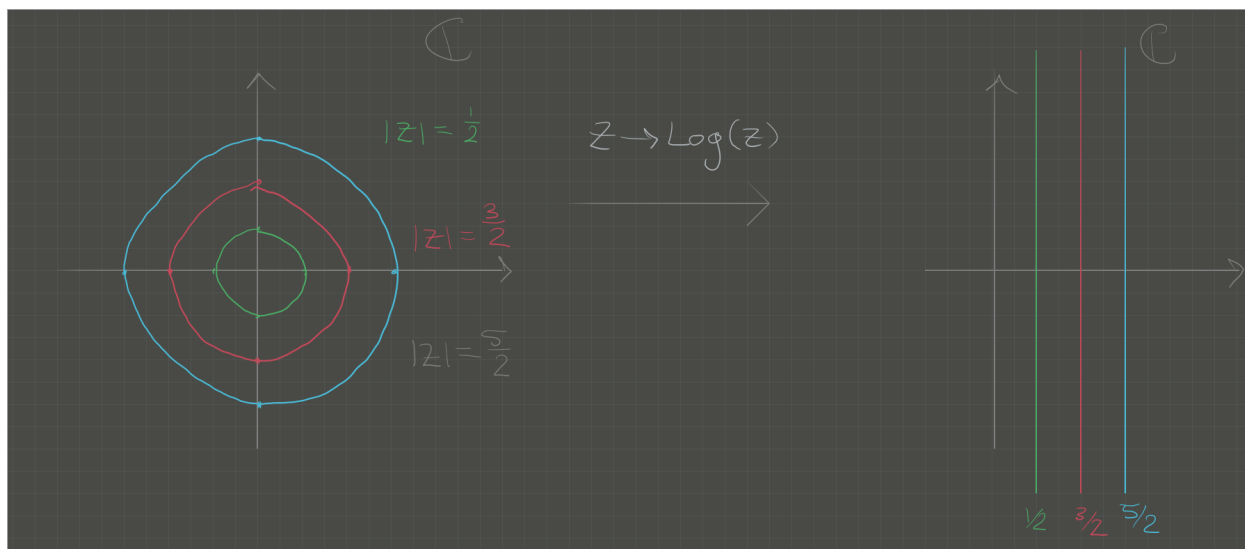
to replace the polynomials P_i with

$$L_j(s) := \zeta_X(q^{-s}) = \prod_{k=1}^{\beta_j} (1 - \alpha_{j,k} q^{-s}).$$

Now consider the roots of $L_j(s)$: we have

$$\begin{aligned} L_j(s_0) &= 0 \\ \iff q^{-s_0} &= \frac{1}{\alpha_{j,k}} \quad \text{for some } k \\ \implies |q^{-s_0}| &= \left| \frac{1}{\alpha_{j,k}} \right| \quad \text{by assumption} \quad q^{-\frac{j}{2}} \\ \implies q^{-\frac{j}{2}} &\stackrel{(a)}{=} \exp\left(-\frac{j}{2} \cdot \operatorname{Log}(q)\right) = |\exp(-s_0 \cdot \operatorname{Log}(q))| \\ &\stackrel{(b)}{=} |\exp(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \operatorname{Log}(q))| \\ &\stackrel{(a)}{=} \exp(-(\Re(s_0)) \cdot \operatorname{Log}(q)) \\ \implies -\frac{j}{2} \cdot \operatorname{Log}(q) &= -\Re(s_0) \cdot \operatorname{Log}(q) \quad \text{by injectivity} \\ \implies \Re(s_0) &= \frac{j}{2}. \end{aligned}$$

Roughly speaking, realizing that we would need to apply a logarithm (a conformal map) to send the $\alpha_{j,k}$ to zeros of the L_j , this says that the zeros all must lie on the “critical lines” $\frac{j}{2}$.



In particular, the zeros of L_1 have real part $\frac{1}{2}$, analogous to the classical Riemann hypothesis.

Moral: the Diophantine properties of a variety's zeta function are governed by its (algebraic) topology. Conversely, the analytic properties encode a lot of geometric/topological/algebraic information. Plug for Langland's: it similarly asks for every L function arising from an automorphic representation that (essentially) satisfy Weil 2 and 3.

Historical note

- Desire for a “cohomology theory of varieties” drove 25 years of progress in AG

Remarks:

- Resolved for varieties over \mathbb{F}_q
- On L_X :
 - Conjectured for smooth varieties over \mathbb{Q} (rationality \sim analytically continues to a meromorphic function, some functional equation), little is known.
 - Resolved for elliptic curves (Taylor-Wiles c/o the Taniyama-Shimura conjecture), implies L_X is an L function coming from a modular form.

1.2.1 Aside: Why call it a Zeta function?

Knowing the zeta function of a point, we can now make a precise analogy.

Suppose we have an algebraic variety cut out by equations:

$$\mathbb{A}_{\mathbb{Z}}^n \supseteq X = V(\langle f_1, \dots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \dots, x_{n-1}].$$

Then for every prime q , we can reduce the equations mod p and consider

$$\mathbb{A}_{\mathbb{F}_q}^n \supseteq X_q := V(\langle f_1 \bmod q, \dots, f_d \bmod q \rangle) \quad \text{where} \quad f_i \bmod q \in \mathbb{F}_q[x_0, \dots, x_{n-1}]$$

Then define the *Hasse-Weil* zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Take $X = \text{Spec } \mathbb{Q}$ and $X_p = \text{Spec } \mathbb{F}_p$, which is a single point since \mathbb{F}_p is a field. The previous example shows that

$$\zeta_{X_p}(z) = \frac{1}{1-z},$$

We then find that

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

which is the Euler product expansion of the classical Riemann Zeta function.

Moreover, it is a theorem (difficult, not proved here!) that for any variety X/\mathbb{F}_p , we have

$$\zeta_X(t) = \prod_{x \in X_{\text{cl}}} \left(\frac{1}{1-t^{\deg(x)}} \right) \xrightarrow{t=p^{-s}} \zeta_X(s) = \prod_{x \in X_{\text{cl}}} \left(\frac{1}{1-(p^{\deg(x)})^{-s}} \right),$$

which we can think of as attaching a “weight” to each closed point, $|x| := p^{\deg(x)}$, and the usual Riemann Zeta corresponds to assigning a weight of 1 to each point.

Note that this immediately implies that $\zeta_X(t) \in \mathbb{Z}[[t]]$ is a *rational* function.

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

After modifying ζ to make it symmetric about $\Re(s) = \frac{1}{2}$ and eliminate the trivial zeros at $-2\mathbb{Z}$ to obtain $\widehat{\zeta}(s)$, there are three relevant properties

- “Rationality”: $\widehat{\zeta}(s)$ has a meromorphic continuation to \mathbb{C} with simple poles at $s = 0, 1$.
- “Functional equation”: $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
- “Riemann Hypothesis”: The only zeros of $\widehat{\zeta}$ have $\Re(s) = \frac{1}{2}$.

1.2.2 More Examples

Example (Affine Line): $X = \mathbb{A}^1/\mathbb{F}$ the affine line over \mathbb{F} , then Note that we can write

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1] \mid x_1 \in \mathbb{F}_n \right\}$$

as the set of one-component vectors with entries in \mathbb{F}_n , so

$$\begin{aligned} X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp \left(\sum_{n=1}^{\infty} \frac{q^n}{n} z^n \right) = \frac{1}{1 - qz}.$$

Example (Affine Space): Set $X = \mathbb{A}^m/\mathbb{F}$, affine m -space over \mathbb{F} , so we can just repeat with now m coordinates

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1, \dots, x_m] \mid x_i \in \mathbb{F}_n \right\}$$

Counting yields

$$\begin{aligned} X(\mathbb{F}) &= q^m \\ X(\mathbb{F}_2) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_n) &= (q^n)^m. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp \left(\sum_{n=1}^{\infty} \frac{q^{nm}}{n} z^n \right) = \frac{1}{1 - q^m z}.$$

Example (Projective Line): $X = \mathbb{P}^1/\mathbb{F}$ the projective line over \mathbb{F} , then we can write use some geometry to write

$$\mathbb{P}_{\mathbb{F}}^1 = \mathbb{A}_{\mathbb{F}}^1 \amalg \{\infty\}$$

as the affine line with a point added at infinity.

We can then count by enumerating coordinates:

$$\begin{aligned} \mathbb{P}^1(\mathbb{F}_n) &= \left\{ [x_1, x_2] \mid x_1, x_2 \neq 0 \in \mathbb{F}_n \right\} / \sim \\ &= \left\{ [x_1, 1] \mid x_1 \in \mathbb{F}_n \right\} \amalg \{[1, 0]\}. \end{aligned}$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &\cdot \end{aligned}$$

Thus

$$\zeta_X(z) = \frac{1}{(1 - z)(1 - qz)}.$$

Example (Projective Space): Take $X = \mathbb{P}_{\mathbb{F}}^n$,



Example image of $\mathbb{P}_{\mathbb{F}(3)}^2$:

Note that we can identify $X = \text{Gr}_{\mathbb{F}}(1, n)$ as the space of lines in $\mathbb{A}_{\mathbb{F}}^n$.

Proposition 1.1.

The number of k -dimensional subspaces of $\mathbb{A}_{\mathbb{F}}^m$ is the q -binomial coefficient:

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

Proof.

To choose a k -dimensional subspace,

- Choose a nonzero vector $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$ in

$$q^m - 1$$

ways.

- Identify $\#\text{span}\{\mathbf{v}_1\} = \#\{\lambda \mathbf{v}_1 \mid \lambda \in \mathbb{F}\} = \#\mathbb{F} = q$.

- Choose a nonzero vector \mathbf{v}_2 *not* in the span of \mathbf{v}_1 in

$$q^m - q$$

ways.

- Identify $\#\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \#\{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mid \lambda_i \in \mathbb{F}\} = q \cdot q = q^2$.

- Choose a nonzero vector \mathbf{v}_3 not in the span of $\mathbf{v}_1, \mathbf{v}_2$ in

$$q^m - q^2$$

ways.

- ... until \mathbf{v}_k is chosen in

$$(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1})$$

ways.

- This yields a k -tuple of linearly independent vectors spanning a k -dimensional subspace V_k
- This overcounts because many linearly independent sets span V_k , we need to divide out by the number of choose a basis inside of V_k .
- By the same argument, this is given by

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

Thus

$$\begin{aligned} \# \text{subspaces} &= \frac{(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})} \\ &= \frac{q^m - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{m-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{m-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{m-(k-1)} - 1}{q^{k-(k-1)-1}}. \end{aligned}$$

■

We obtain a nice simplification for the number of lines corresponding to setting $k = 1$:

$$\begin{bmatrix} m \\ 1 \end{bmatrix}_q = \frac{q^m - 1}{q - 1} = q^{m-1} + q^{m-2} + \cdots + q + 1 = \sum_{j=0}^{m-1} q^j.$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= \sum_{j=0}^{m-1} q^j \\ X(\mathbb{F}_2) &= \sum_{j=0}^{m-1} (q^2)^j \\ &\vdots \\ X(\mathbb{F}_n) &= \sum_{j=0}^{m-1} (q^n)^j. \end{aligned}$$

So

$$\zeta_X(z) = \left(\frac{1}{1-z}\right) \left(\frac{1}{1-qz}\right) \left(\frac{1}{1-q^2z}\right) \cdots \left(\frac{1}{1-q^mz}\right),$$

Note that geometry can help us here: we have a “cell decomposition” $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$, and so inductively

$$\mathbb{P}^n = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^n,$$

1.3 Hard Example: An Elliptic Curve

and it's straightforward to prove that

$$\zeta_{X \coprod Y}(z) = \zeta_X(z) \cdot \zeta_Y(z)$$

and recalling that $\zeta_{\mathbb{A}^j}(z) = \frac{1}{1 - q^j z}$ we have

$$\zeta_{\mathbb{P}^m}(z) = \prod_{j=0}^m \zeta_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1 - q^j z}.$$

Example: Take $X = \text{Gr}_{\mathbb{F}}(k, n)$, then ????? so

$$\zeta_X(t) = ?.$$

1.3 Hard Example: An Elliptic Curve

The Weyl conjectures take on a particularly nice form for curves. Let X/\mathbb{F} be a smooth projective curve of genus g , then

1. (Rationality)

$$\zeta_X(z) = \frac{p(z)}{(1-z)(1-qz)}$$

2. (Functional Equation)

$$\zeta_X\left(\frac{1}{qz}\right) = q^{1-g} z^{2-2g} \zeta_X(z)$$

3. (Riemann Hypothesis)

$$p(t) = \prod_{i=1}^{2g} (q - a_i z) \quad \text{where} \quad |a_i| = \frac{1}{\sqrt{q}}$$

Take $X = E/\mathbb{F}$.

Consider the curve E defined by the following equation:

$$E : y^2 + y = x^3 - x^2$$

This is a cubic, whose graph is presented in Figure 1.



Figure 1: Implicit plot of E

Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-t})}.$$

The betti numbers are $[1, 2, 1, 0, \dots]$.

The number of points are

$$X(\mathbb{F}_n) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

Rough outline of proof:

- ??

The (complex?) dimension of X is $N = 1$, The WC say we should be able to write this as

$$\frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1 - \alpha_{1,1}z)(1 - \alpha_{1,2}z)}{(1-z)(1-qz)}.$$

Since we know the number of points, we can compute

$$\begin{aligned}
\zeta_X(z) &= \exp \sum_{n=1}^{\infty} \#X(\mathbb{F}_n) \frac{z^n}{n} \\
&= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
&= \exp \left(\sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
&= \exp(-\log(1 - qz)) \exp(-\log(1 - z)) \exp(\log(1 - \alpha z)) \exp(\log(1 - \bar{\alpha} z)) \\
&= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),
\end{aligned}$$

which is indeed a rational function.

Originally conjectured for curves by Artin Proved by Weil in 1949, proposed generalization to projective varieties Proof had work contributed by Dwork (rationality using p-adic analysis), Artin, Grothendieck (etale cohomology), with completion by Deligne in 1970s (RH)

1.4 Very Hard Example: A Diagonal Hypersurface

Reference

- Set q to be a prime power and consider X/\mathbb{F}_q defined by

$$X = V(a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}) \subset \mathbb{F}_q^{r+1}.$$

- We want to compute $N = \#X$.
- Set $d_i = \gcd(n_i, q - 1)$.
- Define the character

$$\begin{aligned}
\psi_q : \mathbb{F}_q &\longrightarrow \mathbb{C}^\times \\
a &\mapsto \exp \left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p} \right).
\end{aligned}$$

- By Artin's theorem for linear independence of characters, $\psi_q \not\equiv 1$ and every additive character of \mathbb{F}_q is of the form $a \mapsto \psi_q(ca)$ for some $c \in \mathbb{F}_q$.

- Fix an injective multiplicative map

$$\psi : \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times.$$

- Define

$$\begin{aligned}
\chi_{\alpha, n} : \bar{\mathbb{F}}_q^\times &\longrightarrow \mathbb{C}^\times \\
x &\mapsto \phi(x)^{\alpha(q^n - 1)}
\end{aligned}$$

$$\text{for } \alpha \in \mathbb{Q}/\mathbb{Z}, n \in \mathbb{Z}, \quad \alpha(q^n - 1) \equiv 0 \pmod{1}.$$

- Extend this to \mathbb{F}_{q^n} by

$$\begin{cases} 1 & \alpha \equiv 0 \pmod{1} \\ 0 & \text{else} \end{cases}.$$

- Set $\chi_\alpha = \chi_{\alpha,1}$.

- Shorthand notation: say $a \sim 0 \iff a \equiv 0 \pmod{1}$.

- Proposition:

$$\alpha(q-1) \equiv 0 \pmod{1} \implies \chi_{\alpha,n}(x) = \chi_\alpha(\text{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))$$

- Proposition:

$$d := \gcd(n, q-1), u \in \mathbb{F}_q \implies \#\{x \in \mathbb{F}_1 \mid x^n = u\} = \sum_{d\alpha \sim 0} \chi_\alpha(u)$$

- This implies

$$\begin{aligned} N &= \sum_{\substack{\alpha=[\alpha_0, \dots, \alpha_r] \\ d_i \alpha_i \sim 0}} \sum_{\substack{\mathbf{u}=[u_0, \dots, u_r] \\ \sum a_i u_i = 0}} \prod_{j=0}^r \chi_{\alpha_j}(u_j) \\ &= q^r + \sum_{\substack{\alpha, \alpha_i \in (0,1) \\ d_i \alpha_i \sim 0}} \left(\prod_{j=0}^r \chi_{\alpha_j}(a_j^{-1}) \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) \right). \end{aligned}$$

since the inner sum is zero if some *but not all* of the $\alpha_i \sim 0$.

- Evaluate the innermost sum by restricting to $u_0 \neq 0$ and setting $u_i = u_0 v_i$ and $v_0 := 1$:

$$\begin{aligned} \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) &= \sum_{u_0 \neq 0} \chi_{\sum \alpha_i}(u_0) \sum_{\sum v_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(v_j) \\ &= \begin{cases} (q-1) \sum_{\sum v_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(v_j) & \text{if } \sum \alpha_i \sim 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

- Define the *Jacobi sum* for α where $\sum \alpha_i \sim 0$:

$$J(\alpha) := \left(\frac{1}{q-1} \right) \sum_{\sum u_i = 0} \prod_{j=0}^r \chi_{\alpha_j}(u_j) = \sum_{\sum v_i = 0} \prod_{j=1}^r \chi_{\alpha_j}(v_j)$$

- Express N in terms of Jacobi sums as

$$N = q^r + (q-1) \sum_{\substack{\sum \alpha_i \sim 0 \\ d_0 \alpha_i \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^r \chi_{\alpha_j}(a_j^{-1}) J(\alpha).$$

- Evaluate $J(\alpha)$ using Gauss sums: for $\chi : \mathbb{F}_q \rightarrow \mathbb{C}$ a multiplicative character, define

$$G(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \psi_q(x).$$

- Proposition: for any $\chi \neq \chi_0$,

- $|G(\chi)| = q^{\frac{1}{2}}$
- $G(\chi)G(\bar{\chi}) = q\chi(-1)$
- $G(\chi_0) = 0$

$$\chi(t) = \frac{G(\chi)}{q} \sum_{x \in \mathbb{F}_q} \bar{\chi}(x) \psi_q(tx).$$

- Proposition: if $\sum \alpha_i \sim 0$, then $J(\alpha) = \frac{1}{q} \prod_{k=1}^r G(\chi_{\alpha_k})$ and $|J(\alpha)| = q^{\frac{r-1}{2}}$.