Problem Set 10

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December 1, 2019

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1 Problem 1

Let ϕ be an n-form. If suffices to show these statements for $n = 2$.

$\implies$: Suppose ϕ is alternating, then $\phi(b, b) = 0$ for all $b \in B$.

Letting $a, b \in B$ be arbitrary, we then have

$$\begin{aligned}
\phi(a+b, a+b) &= \phi(a, a+b) + \phi(b, a+b) \\
&= \phi(a,a) + \phi(a,b) + \phi(b,a) + \phi(b,b) \\
&= \phi(a,b) + \phi(b,a) \\
&\implies \phi(a,b) = -\phi(b,a),
\end{aligned}$$

which shows that ϕ is skew-symmetric.

$\impliedby$ Suppose ϕ is skew-symmetric, so $\phi(a, b) = -\phi(b, a)$ for all $a, b \in B$. Then $\phi(b, b) = -\phi(b, b)$ by transposing the terms, which says that $\phi(b, b) = 0$ for all $b \in B$ and thus ϕ is alternating.

2 Problem 2

Let $f(x) = \det(P + xQ) \in R[x]$, then f is a polynomial in x which is not identically zero.

To see that $f \not\equiv 0$, we can use that fact that P is invertible to evaluate $f(0) = \det(P) \neq 0$.

We can now note that f has finite degree, and thus finitely many zeroes in R.

3 Problem 3

Letting $k[x] \curvearrowright_\phi E$ to yield a $k[x]$-module structure on E and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \;\Big|\; q_2 \;\Big|\; \cdots \;\Big|\; q_t$$

where $E_i = k[x]/(q_i)$. Then $q_t = q$, the minimal polynomial of E.

In particular, E_t is a ϕ-invariant subspace of E, and if $\deg q_t = m$, then E_t is in fact an m-dimensional cyclic module with basis $\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \cdots, \phi^{m-1}(\mathbf{v})\}$ for some $\mathbf{v} \in E_t$.

But since $E_t \leq E$ is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

4 Problem 4

$\implies$: Suppose $A \sim D$ where D is diagonal. Then $JCF(A) = JCF(D) = D$, which means that every Jordan block of A has size exactly 1.

Since the elementary divisors of A are precisely the minimal polynomials of the Jordan blocks of A, and the minimal polynomial of any 1×1 matrix $[a_{ij}]$ is given by the linear polynomial $x - a_{ij}$, every elementary divisor of A must be linear.

$\impliedby$: Suppose all of the elementary divisors of A are linear. Every elementary divisor is the minimal polynomial of a Jordan block of A, and so if we write $JCF(A) = \bigoplus M_i$, then the minimal polynomial of each M_i is linear.

Supposing that M_i has minimal polynomial $p_i(x) = x - c$ for some scalar c, we have

$$p_i(M_i) = 0 \implies M_i - cI_n = 0 \implies M_i = cI_n,$$

which shows that M_i is a diagonal matrix with only c on its diagonal.

But if every Jordan block of A is diagonal,then $JCF(A) = D$ is diagonal and $A \sim D$.

5 Problem 5

5.1 Part 1

We'll use the fact that the minimal polynomial q is the invariant factor of highest degree, and so every other invariant factor must divide q.

Moreover, $RCF(A) = C_1 \oplus C_2 \oplus \cdots \oplus C_k$ where each C_i is the companion matrix of the ith invariant factor if we write $V \cong \bigoplus_{i=1}^k k[x]/(a_i)$. So it suffices to determine all of the possible distinct combinations of invariant factors.

We can restrict this list by noting that the characteristic polynomial satisfies $\chi_A(x) = \prod a_i$, and in particular, $\deg \chi_A(x) = 6$. Noting that $\deg q(x) = 3$, the degrees of the remaining invariant factors must sum to 3.

These are:

$$\begin{aligned}
R_1 : a_1 &= (x-2), & a_2 &= (x-2)^2, & a_3 &= q(x), \\
R_2 : a_1 &= (x-2), & a_2 &= (x-2)(x-3), & a_3 &= q(x), \\
R_3 : a_1 &= (x-3), & a_2 &= (x-2)(x-3), & a_3 &= q(x).
\end{aligned}$$

Noting that

$$\begin{aligned}
(x-2)^2 &= x^2 - 4x + 4 \\
(x-2)(x-3) &= x^2 - 5x + 6 \\
q(x) &= x^3 - 7x^2 + 16x - 12,
\end{aligned}$$

these choices correspond to the matrices

$$R_1 = \left[\begin{array}{c|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right], R_2 = \left[\begin{array}{c|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right], R_3 = \left[\begin{array}{c|cc|ccc} 3 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right].$$

Note: these are perhaps transposed from Hungerford's notation.

Since none of the associated polynomials were irreducible over $\mathbb{Q}$, $RCF(A)$ takes these forms over $\mathbb{C}$ as well.

5.2 Part 2

We'll first exhibit the possibilities over $\mathbb{C}$, then show what subset can be obtained over $\mathbb{Q}$.

Over CC, we have $x^2 + 1 = (x-i)(x+i)$. By the same argument used in Part 1, we know that $q(x)$ is the largest invariant factor, and since $\deg q = 3$, the degrees of the remaining factors must sum to 4 (since the degree χ_A will be 7, and it's the product of these factors).

The possibilities are thus

$$
\begin{array}{rrrr}
a_1 = (x-i) & a_2 = (x-i)(x-7) & & a_4 = q(x) \\
a_1 = (x+i) & a_2 = (x+i)(x-7) & a_3 = q(x) & \\
a_1 = (x-7) & a_2 = (x+i)(x-7) & a_3 = q(x) & \\
a_1 = (x-7) & a_2 = (x-i)(x-7) & a_3 = q(x) & \\
.
\end{array}
$$