Problem Set 2

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1 | Exercises

Exercise 1.1 (Gathmann 2.17)**.**
Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

Solution:
Since $x = yz$ for all points in X, we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V\Big(x - yz, y(z^2 - y)\Big) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &\coloneqq X_1 \cup X_2. \end{aligned}$$

Claim: These two subvarieties are irreducible.
It suffices to show that the $A(X_i)$ are integral domains. We have

$$A(X_1) \coloneqq \mathbb{C}[x, y, z]/\left\langle x - yz, y \right\rangle \cong \mathbb{C}[y, z]/\left\langle y \right\rangle \cong \mathbb{C}[z],$$

which is an integral domain since $\mathbb{C}$ is a field and thus an integral domain, and

$$A(X_2) \coloneqq \mathbb{C}[x, y, z]/\left\langle x - yz, z^2 - y \right\rangle \cong \mathbb{C}[y, z]/\left\langle z^2 - y \right\rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

Exercise 1.2 (Gathmann 2.18)**.**
Let $X \subset \mathbb{A}^n$ be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

Solution:

$\overline{X} \subseteq V(I(X))$:
We have $X \subseteq V(I(X))$ and since $V(J)$ is closed in the Zariski topology for any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ by definition, $V(I(X))$ is closed. Thus

$$X \subseteq V(I(X)) \text{ and } V(I(X)) \text{ closed } \implies \overline{X} \subseteq V(I(X)),$$

since $\overline{X}$ is the intersection of all closed sets containing X.

$V(I(X)) \subseteq \overline{X}$:
Noting that $V(\,\cdot\,), I(\,\cdot\,)$ are individually order-reversing, we find that $V(I(\,\cdot\,))$ is order-*preserving* and thus

$$X \subseteq \overline{X} \implies V(I(X)) \subseteq V(I(\overline{X})) = \overline{X},$$

where in the last equality we've used part (i) of the Nullstellensatz: if X is an affine variety, then $V(I(X)) = X$. This applies here because $\overline{X}$ is always closed, and the closed sets in the Zariski topology are precisely the affine varieties.

Exercise 1.3 (Gathmann 2.21)**.**
Let $\{U_i\}_{i \in I} \rightrightarrows X$ be an open cover of a topological space with $U_i \cap U_j \neq \emptyset$ for every i, j.

a. Show that if U_i is connected for every i then X is connected.

b. Show that if U_i is irreducible for every i then X is irreducible.

Solution *(a)*:
?

Solution *(b)*:

Claim: X is irreducible iff any two open subsets intersect.
This follows because otherwise, if $U, V \subset X$ are open and disjoint then $X \setminus U$, $X \setminus V$ are proper and closed. But then we can write $X = (X \setminus U) \coprod (X \setminus V)$ as a union of proper closed subsets, forcing X to not be irreducible.

So it suffices to show that if $U, V \subset X$ then $U \cap V$ is nonempty. Since $\{U_i\} \rightrightarrows X$, we can find a pair i, j such that there is at least one point in $U \cap U_i$ and one point in $V \cap U_j$.

But by assumption $U_i \cap U_j$ is nonempty, so both $U \cap U_i$ and $U_j \cap U_i$ are

Exercise 1.4 (Gathmann 2.22)**.**
Let $f : X \to Y$ be a continuous map of topological spaces.

a. Show that if X is connected then $f(X)$ is connected.

b. Show that if X is irreducible then $f(X)$ is irreducible.

Solution:
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Definition 1.0.1 (Ideal Quotient)**.**
For two ideals $J_1, J_2 \trianglelefteq R$, the *ideal quotient* is defined by

$$J_1 : J_2 \coloneqq \left\{ f \in R \;\middle|\; fJ_2 \subset J_1 \right\}.$$

Solution:
?

Exercise 1.5 (Gathmann 2.23)**.**
Let X be an affine variety.

a. Show that if $Y_1, Y_2 \subset X$ are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

b. If $J_1, J_2 \trianglelefteq A(X)$ are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

Solution:
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Exercise 1.6 (Gathmann 2.24)**.**
Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible affine varieties, and show that $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible.

Solution:
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