

Title

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Recall $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} dx$. Define $\mathcal{F}_a = ??$.

Definition 1.1 (Decay).

$f \in \mathcal{F}_a$ iff 1. f is holomorphic in the strip $S_a = \{z = x + iy \mid |y| < a\}$. 2. There exists an $A > 0$ such that $|f(x + iy)| \frac{A}{1 + x^2}$.

Examples:

- $e^{-z^2} \in \mathcal{F}_a$ for all a
- $\frac{1}{c^2 + z^2} \in \mathcal{F}_a$ for all $a > c$
- $\frac{1}{\cosh(\pi z)} \in \mathcal{F}_a$ for $a < \frac{1}{2}$.

Lemma 1.1.

If $f \in \mathcal{F}_a$, then $f^{(n)}(z) \in \mathcal{F}_b$ for all $b < a$.

Theorem 1.2.

If $f \in \mathcal{F}_a$, then $|\hat{f}(\xi)| \leq B e^{-2\pi b |\xi|}$ for some constants b, B .

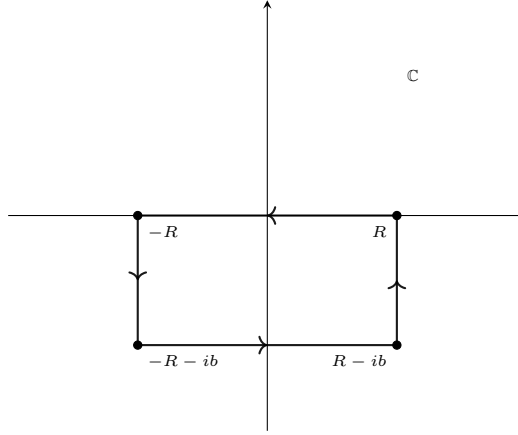
Proof.

If $\xi = 0$,

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$$\begin{aligned}
|\widehat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} \right| \\
&\leq \int_{\mathbb{R}} |f(x)| \, dx \\
&\leq A \int_{\mathbb{R}} \frac{1}{1+x^2} \, dx \\
&= A\pi.
\end{aligned}$$

For $\xi > 0$, integrate over the box $[-R, R] \times i[-b, 0]$.



Define $g(z) = f(z)e^{-2\pi i z \cdot \xi}$. The integral over the rectangle is zero, since g is holomorphic, so we can equate

$$\int_R^{R-ib} f(z) e^{-2\pi i z \cdot \xi} \, dz = \int_0^b f(R - it) e^{-2\pi i (R - it) \cdot \xi} (-i) \, dt$$

We can use the estimate in \mathcal{F}_a to obtain

$$\begin{aligned}
\int_0^b \dots &\leq \int_0^b \frac{A}{1+R^2} e^{-2\pi s \xi} \, ds \\
&\leq O(R^{-2}).
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} \, d\xi &= \int_{-\infty - ib}^{\infty - ib} \dots \, dz \\
&= \int_{\mathbb{R}} f(x - ib) e^{2\pi i (x - ib) \cdot \xi} \, dx \\
&\leq \int_{\mathbb{R}} \frac{A}{1+x^2} e^{-2\pi b \xi} \, dx \\
&= A\pi e^{-2\pi b \xi},
\end{aligned}$$

so we can take $B = A\pi$.

For $\xi > 0$, the same argument works with the rectangle above the axis.

Theorem 1.3.

If $f \in \mathcal{F}_a$, then $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$.

Proof.

Letting $L_1 = \{x - ib\}$ and $L_2 = \{x + ib\}$

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$$\begin{aligned}
I &= \int_0^\infty \widehat{f} \cdots + \int_{-\infty}^0 \widehat{f} \cdots \\
&= \int_0^\infty e^{2\pi i x \cdot \xi} \left(\int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi + \int_{-\infty}^0 e^{2\pi i x \cdot \xi} \left(\int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi \\
&= \int_{L_1} \int_0^\infty e^{2\pi i x \xi - 2\pi i (s-ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i x \cdot \xi - 2\pi i (s+ib)\xi} d\xi ds \\
&\quad \text{by absolute convergence, where } z = s - ib \\
&= \int_{L_1} f(z) \int_0^\infty e^{2\pi i (x-s+ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i (x-s+ib)\xi} d\xi ds \\
&= \int_{L_1} f(z) \frac{1}{2\pi i (x - i + ib)} ds + \int_{L_2} f(z) \frac{1}{2\pi i (x - s - ib)} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - x} dz \\
&= f(x).
\end{aligned}$$

Noting that

$$\int_0^\infty e^{as} ds = \frac{1}{a} \quad \text{for } \Re(a) > 0.$$

Note the similar trick: for $\xi < 0$, move up, and $\xi > 0$ move down to form a rectangle. Use the fact that integration along the vertical edges is zero.