

# Title

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# 1 | Lecture 8: Riemann-Roch Spaces

Setting up for the single most important theorem in the course: the Riemann-Roch theorem. We start by motivating this by considering the following property of  $K := k(t)$ : for any degree 1<sup>1</sup> place  $p \in \Sigma(K/k)$ , there exists an  $f \in K^\times$  such that  $(f)_- = p$ . In other words,  $f$  is a rational function with a simple pole at the given place, and no other poles. Why? We just know precisely what all of the places are for this function field.

If  $p = \infty$ , we can just take  $f(t) = t$ , since any polynomial is regular away from  $\infty$  and the valuation is  $-\deg(f) = -1$ . The other places  $p$  correspond to  $t - \alpha$  (the uniformizing element) for  $\alpha \in k$ , since they correspond to other points on  $\mathbb{A}_{/k}^1$ , and so we can take  $f(t) = 1/(t - \alpha)$ . This  $f$  is regular at infinity since the degree of the numerator is larger than the degree of the denominator, and the denominator doesn't vanish at any other place.

**Remark 1.0.1:** With some thought, it can be found that this is a *characteristic* property of rational function fields: if  $f \in K$ , a one variable function field, and  $\deg(f)_- = 1$ <sup>2</sup> then the degree of the function is equal to the degree of the divisor of the zeros and the divisor of the poles, and thus the degree of the extension  $[K : k(t)] = 1$  and thus  $K = k(t)$  is rational. So having a rational with a simple pole at only one point *only* happens in you're in a rational function field.

On the other hand, we both wanted and used in our discussion of holomorphy rings the fact that given a nonempty finite subset  $S \subset \Sigma(K/k)$ , we want to find a rational function  $f \in K^\times$  has poles at all of the points in  $S$ , so  $\text{supp}(f)_- = S$ . Better yet, we'd like a bound on the degree of any such  $f$ , i.e. the orders of all of these poles. If  $S$  is a single place, unless the function field is rational, we can't require the function to have a pole of degree 1 at that point. But can it admit a pole of degree at most 10, for example? This is what motivates the Riemann-Roch spaces and the Riemann-Roch theorem. If you're trying to give a quantitative bound on how high of an order of a pole you have to allow in order to have a rational function, this comes from a key invariant called the *genus* of the function field. The theorem that will tell us about the existence of rational functions with poles of prescribed degrees in terms of the genus is precisely the Riemann-Roch theorem, so that's where we are headed.

**Definition 1.0.2** (Riemann-Roch Space of  $D$  (Key Definition))

For  $D \in \text{Div } K$ , the **Riemann-Roch space** of  $D$  is defined as

$$\mathcal{L}(D) := \left\{ f \in K^\times \mid (f)_- \geq -D \right\} \cup \{0\}.$$

**Remark 1.0.3:** This will turn out to be a  $k$ -vector space, and is a sub  $k$ -vector space of  $K$ . One of the first things we'll prove is that it's always finite dimensional. This is only interesting when  $D$  is linearly equivalent to an effective divisor, so we should think of  $D$  as having a nonnegative degree, and in fact itself being an effective divisor. So this is the space of rational functions that have prescribes poles of a prescribed order.

<sup>1</sup>So the residue field of the corresponding DVR is  $k$  itself rather than some proper finite degree extension.

<sup>2</sup>Recall that this is the divisor pole.

**Question 1.0.4:** Does  $\mathcal{L}(D)$  contain any rational functions other than zero?

**Answer 1.0.5:** For any nonzero  $f \in \mathcal{L}(D)^\bullet$ , the divisor  $D + (f)$  is effective, since  $(f) \geq -D$ , and also linearly equivalent to  $D$ . If  $D$  is not linearly equivalent to an effective divisor, this is just the zero vector space.

**Exercise 1.0.6(?):** Let  $K = k(t)$  and  $n \in \mathbb{Z}^{\geq 0}$ . Show that

$$L(n\infty) = \{f \in k[t] \mid \deg f \leq n\}$$

and in particular is a  $k$ -vector space of dimension  $n + 1$ .<sup>3</sup>

**Remark 1.0.7:** Note that  $\infty$  is a degree 1 place, and multiplying it by  $n$  yields an effective divisor. The Riemann-Roch space here is comprised of rational functions that regular away from  $\infty$ , which are polynomials, whose pole at  $\infty$  has order at worst  $n$ . But the order of a pole at infinity is its degree as a polynomial, since the  $\infty$ -adic valuation is the negative degree, so this yields polynomials of degree at most  $n$ .

**Lemma 1.0.8(?).**

For  $D \in \text{Div } K$ ,

$$\mathcal{L}(D) \neq \{0\} \iff 0 \text{ is equivalent to an effective divisor.}$$

*Proof (?).*

$\implies$  : If  $f \in \mathcal{L}(D)^\bullet$ , then  $D + (f)$  is effective and linearly equivalent to zero.

$\impliedby$  : If  $D' \geq 0$  and  $D' \sim D$ , then  $D' = D + (f) \geq 0$ . So  $(f) \geq -D$  and thus  $f \in \mathcal{L}(D)$ . ■

**Example 1.0.9(?):**  $\mathcal{L}(0) = \{f \mid (f) \geq 0\} \cup \{0\}$ , which consists of rational functions with no poles (so their divisor is the zero divisor), and thus  $\mathcal{L}(0) = \kappa(K)$ . I.e., these are the constants: they are regular everywhere and have no zeros or poles. We would like this space to have  $k$ -dimension 1, so we impose  $\kappa(K) = k$ .

**Exercise 1.0.10(?):**

- Show that for all  $D$ ,  $\mathcal{L}(D) \in \text{Vect}_k$ .
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$$D \sim D' \implies \mathcal{L}(D) \cong_{\text{Vect}_k} \mathcal{L}(D').$$

**Remark 1.0.11:** You can frame the above as taking rational functions with poles of certain orders, and analyzing the orders of poles of their sums. If you take  $D'$  and write it as  $D + (f)$  for  $f$  a rational function, then  $f$  should produce this isomorphism. The moral:  $\mathcal{L}(D)$  only depends on the linear equivalence class of  $D$ .

<sup>3</sup>Recall that  $\infty$  is the  $1/t$ -adic place.

**Exercise 1.0.12(?):** Let  $D \in \operatorname{Div}^0 K$  be a degree zero divisor, then TFAE:

- a.  $\dim \mathcal{L}(D) \geq 1$
- b.  $\dim \mathcal{L}(D) = 1$ ,
- c.  $D$  is principal, i.e. the divisor of a rational function or linearly equivalent to zero.

**Slogan 1.0.13:** The only way a degree zero divisor can have a nontrivial Riemann-Roch space is if it's linearly equivalent to zero.

**Lemma 1.0.14(?).**

Let  $A \leq B^a$  in  $\operatorname{Div} K$ , then

- a.  $\mathcal{L}(A) \leq_{\operatorname{Vect}_k} \mathcal{L}(B)$  is a subspace,
- b.  $\dim \mathcal{L}(B)/\mathcal{L}(A) \leq \deg B - \deg A$ .

<sup>a</sup>These are formal linear combinations of places, so the coefficients in front of each place in  $A$  should be less than the corresponding coefficient for  $B$ , or equivalently  $B - A$  is effective.

**Remark 1.0.15:**