Reference List

- **Rational Homotopy Theory and Differential Forms** by Griffiths and Morgan
- **Differential Forms in Algebraic Topology** by Bott and Tu
- **Differential Topology** by Hirsch
- **Comprehensive Introduction to DIfferential Geometry** by Spivak
- **Topology from the Differentiable Viewpoint** by Milnor
- **Topology and Geometry** by Bredon
- **User's Guide to Spectral Sequences** by Mcleary
  - View Here
  - Apparently lots of technical details

General Notes

The standard Serre fibration: $\Omega X \to PX \xrightarrow{f} X$ where ΩX is the loop space, PX is the path space, and f is the "evaluation at the endpoint" map. Note that PX is contractible!

Consider a SES $0 \to A \to B \to C \to C$, then look at it as a 2-step filtration of B so $F^0 B = B, F^1 B = A, F^2 B = 0$. The graded pieces are $G_0 = C, G_1 = A$. Can use this to obtain LES from SS.

Homology in the ring-theoretic setting: If R is a Noetherian ring and $I \subseteq R$, then if I can be generated by n elements then $H_I^i(M) = 0$ for any R-module M and $i > n$. Thus to prove I can *not* be generated by n elements, it suffices to find a module M where $H_I^{n+1} \neq 0$.

Griffiths and Morgan

Overall purpose: want to relate C^∞ forms on a manifold to AT invariants. One significant result: given a manifold M, the singular cohomology $H^*(M, \mathbb{R})$ is isomorphic to the cohomology of the differential graded algebra of C^∞ forms, $H^*_{DR}(M)$.

Is this the de Rham cohomology..?

This DGA of smooth forms is actually enough to calculate all of the AT invariants, and can be used to build the Postnikov tower of M ($\otimes \mathbb{R}$)

One construction is the localization of a CW complex at $\mathbb{Q}$, this removes torsion and divisibility phenomena. The effect on the postnikov tower is just then tensoring with $\mathbb{Q}$.

Things that are homotopy equivalent to CW complexes:

- Manifolds
- Varieties
- Loop spaces of CW complexes
- Eilenberg-MacLane spaces? $K(\pi, n)$.

The Whitehead theorem holds for these: $X \xrightarrow{f} Y$ is an homotopy equivalence iff $\pi_*(X) \xrightarrow{f_*} \pi_*(Y)$ is an isomorphism.

Recall the weak topology for infinite CW complex: U is open in X iff $U \cap X^n$ is open for every n.

Theorem: Given any $X \xrightarrow{f} Y$, we can transform this into an inclusion up to homotopy equivalence. (Just replace Y my the mapping cylinder of f, denoted $M_f \simeq Y$).

A fibration is anything that satisfies the homotopy lifting property. Examples:

- Locally trivial fiber bundles
- Vector bundles
- Covering spaces

Path spaces are fibrations, loop spaces are contractible.

Homology can be defined with coefficients in any abelian group by tensoring the singular chain groups with G. That is, if we $H_*(X)$ obtained from

$\xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \cdots \xrightarrow{\partial_1} C_0(X)$

then we can define $H_*(X; G)$ via

$\xrightarrow{\partial_{n+1} \otimes 1} C_n(X) \otimes G \xrightarrow{\partial_n \otimes 1} C_{n-1}(X) \otimes G \cdots \xrightarrow{\partial_1 \otimes 1} C_0(X) \otimes G$

Note that homology has the structure of a graded group, while cohomology has the structure of a graded commutative ring.

Axioms of homology:

- $X \xrightarrow{f} Y$ always induces a map on homology $H_*(X) \xrightarrow{f_*} H_*(Y)$
- An orientation on S^n induces an isomorphism $H_n(S^n) \cong \mathbb{Z}$; reversing orientation induces the map $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z} : \phi(1) = -1$
- $Y \subseteq X$ yields the definition of relative homology $H_*(X, Y)$, and mayer vietoris holds; i.e. there is a long exact sequence $\cdots H_n(Y) \to H_n(X) \to H_n(X, Y) \to H_{n-1}(Y) \cdots$
- Excision: $U \subset Y \subset X$ and $\bar{U} \subset Y^\circ$ implies $H_*(X - U, Y - U) \cong H_*(X, Y)$.

Any homology theory satisfying these properties is equivalent to singular homology.

Use notation $[X, Y]$ for homotopy classes of maps $X \to Y$, then $\pi_1(X) = [S^1, X]$ and we can define $\pi_n(X) = [S^n, X]$. Homotopy groups fail excision.

Whitehead theorem: for CW complexes, if $X \xrightarrow{f} Y$ induces $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ and f_* is an isomorphism (and Y is connected), then f is a homotopy equivalence. For spaces that aren't CW complexes, this may fail, and we say f is a *weak homotopy equivalence* instead.

Hurewicz theorem: the bottom homology and homotopy groups are isomorphic, and homology below the bottom homology is zero.

General note: there are equivalent "relative versions" of most of these theorems.

Spectral Sequence: Page 45.

For any fibration $F \to E \xrightarrow{\pi} B$, we get a LES in homotopy $\pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F)$

Basic question: How are the cohomologies of F, E, B related? An easy case is when $E = F \times B$, but even then $\pi_n(F \times B) \neq \pi_n(F) \oplus \pi_n(B)$. Need the Kunneth theorem, formula is more complicated.

For CW complexes and a fibration, the relationship is nice - look at the total space of the fibration. It is filtered by increasing n-skeleta, and we use the LES. More general filtrations need a spectral sequence.

Note: use LES as trivial example of spectral sequence! Write out the pages, differentials, etc

The spectral sequence relates the cohomology of *successive pairs* in the filtration to the cohomology of the total space.

Theorem: If B is path-connected and $\pi_1(B, b_0)$ acts trivially on $H^*(F)$, then there are isomorphisms

$$\begin{aligned} H^n(E^p, E^{p-1}) &\cong \prod_{p\text{-cells in } B} H^n(\pi^{-1}e^p, \pi^{-1}\partial e^p) \\ &\cong C^p(B; H^{n-p}(F)) \end{aligned}$$

In other words, for any k, we can think of $H^*(E^p, E^{p-k})$ as a k-th approximation to $H^*(E^p)$.

(Should probably review results about polynomial and exterior algebras. And what does it mean for π_1 to act trivially on a fiber?)

EXAMPLES OF COMPUTATION: Page 54

- Cohomology of $\mathbb{CP}^n$ using $S^1 \to S^{2n+1} \to \mathbb{CP}^n$
  -
- Cohmology of the infinite Grassmanian $\lim_n G(k, n)$
  - Answer: $H^*(G(k)) = \mathbb{Z}[x_1, x_2, \cdots x_k]$

Postnikov Towers

A decomposition dual to cell decomposition, the atoms of the space are Eilenberg-Maclane spaces $K(\pi, n)$. (Note the spheres are atomic in homology, while the K are atomic in homotopy.)

Gives a way of going back and forth between X and $\pi_*(X)$: defined as a tower of spaces $X_0 \leftarrow X_1 \leftarrow \cdots$

- $X_{i-1} \leftarrow X_i$ is a fibration
- $\pi_k(X_n) = 1[k \leq n] \cdot \pi_k(X) + 1[k > n] \cdot 0$
  - So all lower homotopy groups agree at the n-th spot
- (Probably) $X_i \hookrightarrow X$

Unique up to homotopy, $X = \lim_n X_n$ (an inverse limit). Essentially constructs X out of $K(\pi_n(X), n)$.

Note: revisit and draw diagrams for Postnikov Towers

Homotopy and homology commute with direct limits.

$\cdot \otimes \mathbb{Q}$ is a right-exact functor, most results in this section are about how terms in exact sequences all become $\mathbb{Q}$-vector spaces. In particular, $H^*(X; \mathbb{Q}), H_*(X; \mathbb{Q})$ are.

Homotopy theory over $\mathbb{Q}$ is much easier than over $\mathbb{Z}$. Samples results:

$$\pi_i(S^{2n-1}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Then using the fact that $\pi_i(S^{2n-1})$ is always finitely generated, we can conclude

$$\pi_i(S^{2n-1}) = \begin{cases} \mathbb{Z} & i = 2n-1 \\ \text{a finite group } G & \text{otherwise} \end{cases}$$

This yields for even n:

$$\pi_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ \mathbb{Z} \oplus G & i = 2n-1 \\ H & \text{otherwise} \end{cases}$$

for some finite groups G, H!

Can also obtain Bott Periodicity this way.

Other Reading

Lots of good examples of computations here

Some fibrations

- Hopf: $S^1 \to S^3 \to S^2$
- $S^1 \to S^{2n+1} \to \mathbb{CP}^n$
- Path space: $\Omega S^n \to PS^n \to S^n$

Serre Spectral Sequence Example: For the fibration $S^1 \to S^3 \to S^2$, the E_2 page:

$$\begin{array}{c|ccc} 1 & H^0(S^2,\mathbb{Z}) & H^1(S^2,\mathbb{Z}) & H^2(S^2,\mathbb{Z}) \\ 0 & H^0(S^2,\mathbb{Z}) & H^1(S^2,\mathbb{Z}) & H^2(S^2,\mathbb{Z}) \\ \hline & 0 & 1 & 2 \end{array}$$

Which is equal to

$$\begin{array}{c|ccc} 1 & H^0(S^2,\mathbb{Z}) & 0 & H^2(S^2,\mathbb{Z}) \\ 0 & H^0(S^2,\mathbb{Z}) & 0 & H^2(S^2,\mathbb{Z}) \\ \hline & 0 & 1 & 2 \end{array}$$

And $E_3 = E_\infty$, so $d_2^{0,1}$ is an isomorphism.

Note: Probably a good starting point for basic calcuations? Fill out the missing details for this table.

Challenge: Prove $\pi_4(S^2) = \frac{\mathbb{Z}}{2\mathbb{Z}}$