Category O

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1 Definitions

- Indecomposable: doesn't decompose as $A \oplus B$. Weaker than irreducible.
- Irreducible: simple, i.e. no nontrivial proper submodules. Implies indecomposable.
- Completely reducible: Direct sum of irreducibles.
- Solvable: Derived series terminates.
- Borel: maximal solvable subalgebra.
- Radical: Largest solvable ideal.
- Semisimple: Direct sum of simple modules.
  - Acts in a diagonalizable way.
- Reductive: Radical equals center.
- Artinian: Descending chain condition on submodules.
- Antidominant weight: $\langle \lambda + \rho,\ \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$, equivalently $M(\lambda) = L(\lambda)$.
- Dominant weight: $\langle \lambda + \rho,\ \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$.
- Regular weight: λ is regular iff the isotropy/stabilizer group $\mathrm{Stab}_W(\lambda) := \left\{ w \in W \;\middle|\; w\lambda = w \right\} = 1$, equivalently $|W\lambda| = |W|$ so $\langle \lambda + \rho,\ \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$.

- Singular weight: Not regular.
- Linked: $\mu \sim \lambda \iff \mu \in W \cdot \lambda$, the orbit of λ under W, a.k.a. the linkage class of λ.
- Socle: Direct sum of all simple submodules.
- Radical: Intersection of all maximal submodules, smallest submodule such that quotient is semisimple.
- Head: $M/\mathrm{rad}(M)$.

2 List of Notation

- $M(\lambda)$: Verma Modules
- $L(\lambda)$: Unique simple *quotient* of $M(\lambda)$.
- $N(\lambda)$ the maximal *submodule* of $M(\lambda)$
- The root system

$$\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \;\middle|\; [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$$

  containing roots α

  - Abstractly: spans a Euclidean space, $\lambda\alpha \in \phi \implies \lambda = \pm 1$, and closed under reflections about orthogonal hyperplanes.
- Φ^+ the corresponding positive system (choose a hyperplane not containing any root), $\Phi \coloneqq \Phi^+ \coprod \Phi^-$.
- $$s_\alpha(\,\cdot\,) \coloneqq (\,\cdot\,) - 2\langle\,\cdot\,,\ \alpha\rangle \frac{\alpha}{\|\alpha\|^2}$$

  the corresponding reflection about the hyperplane H_α
- $\mathfrak{g}_\alpha \coloneqq \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$ the corresponding root space
- The triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{-\alpha} \coloneqq \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

- Δ the corresponding simple system of size ℓ, i.e $\alpha = \sum_{\delta_k \in \Delta} c_\delta \delta_k$ with $c_\delta \in \mathbb{Z}^{\geq 0}$.
- $\Lambda = \left\{ \lambda \in E \;\middle|\; \langle \lambda,\ \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi \right\}$ the integral weight lattice
- $\Lambda^+ = \mathbb{Z}^+\Omega$ the dominant integral weights
  - $\Omega \coloneqq \{\bar{\omega}_1, \cdots, \bar{\omega}_\ell\}$ the fundamental weights
- $[A : B]$ the composition factor multiplicity of B in a composition series for A.
- $(A : B)$ the composition factor multiplicity of B in a *standard filtration* for A.
- $\phi_{[\lambda]}$ the integral root system of λ

- $\Delta_{[\lambda]}$ the corresponding simple system
- $W_{[\lambda]}$ the integral Weyl group of λ
- $\mu \uparrow \lambda$: strong linkage of weights
- $\mathcal{O}_{\chi_\lambda}$: the block corresponding to λ.
- $\operatorname{char} M := \sum_{\lambda \in \mathfrak{h}^\vee} (\dim M_\lambda) e^\lambda$ the formal character.

3 Useful Facts

- λ dominant integral $\implies w\lambda \leq \lambda$ for all W.
- The dot action is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.
- For any filtration $0 \hookrightarrow M^n \hookrightarrow M^{n-1} \hookrightarrow \cdots \hookrightarrow M^1 \hookrightarrow M^0 = M$, we have

$$\operatorname{char} M = \sum_{i=1}^{n} \operatorname{char} \left(M^i / M^{i-1} \right),$$

  i.e. the character of M is the sum of the characters of its composition factors (with multiplicity).
- $\operatorname{Soc}(M(\lambda)) = M(w_0 \cdot \lambda)$.
- $\operatorname{Soc}(M(w \cdot \lambda)) = L(w_0 \cdot \lambda)$.

4 SL2 Theory

Definition 4.0.1.
The group and the algebra:

$$\begin{aligned} \mathfrak{sl}(n, \mathbb{C}) &= \left\{ M \in \mathrm{GL}(n, \mathbb{C}) \;\middle|\; \det(M) = 1 \right\} \\ \mathfrak{sl}(n, \mathbb{C}) &= \left\{ M \in \mathrm{GL}(n, \mathbb{C}) \;\middle|\; \operatorname{Tr}(M) = 0 \right\}. \end{aligned}$$

- The usual representation on $\mathbb{C}^2$: h has eigenvalues ± 1, yields $L(1)$.
- The adjoint representation on $\mathbb{C}^3$: $\operatorname{ad} h = \operatorname{diag}(2, 0, -2)$ with eigenvalues $0, \pm 2$, yields $L(2)$.

Generated by

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with relations

$$\begin{aligned} [hx] &= 2x \\ [hy] &= -2y \\ [xy] &= h \end{aligned}$$

.

Some identifications:

$$
\begin{aligned}
\Phi &= A_1 \\
\dim \mathfrak{h} &= 1 \\
\Lambda &\cong \mathbb{Z} \\
\Lambda_r &\cong \mathbb{Z}/2\mathbb{Z} \\
W &= \{1, s_0\} \quad \lambda - 2i \overset{s_0}{\iff} -(\lambda - 2i) \\
\chi_\lambda = \chi_\mu &\iff \mu = \lambda, -\lambda - 2 \quad \text{(linked)} \\
\Pi(M(\lambda)) &= \{\lambda, \lambda - 2, \cdots\}.
\end{aligned}
$$

For λ dominant integral

$$
\begin{aligned}
N(\lambda) &\cong L(-\lambda - 2) \\
\dim L(\lambda) &= \lambda + 1 \\
\Pi(L(\lambda)) &= \{\lambda, \lambda - 2, \cdots, -\lambda\} \\
\dim \left(L(\lambda)\right)_\mu &= 1 \qquad \forall \mu = \lambda - 2i.
\end{aligned}
$$

- Simple modules are parameterized by dominant integral weights:

$$M(\lambda) \text{ is simple } \iff \lambda \notin \mathbb{Z}^{\geq 0} = \Lambda^+ \iff \dim L(\lambda) = \infty$$

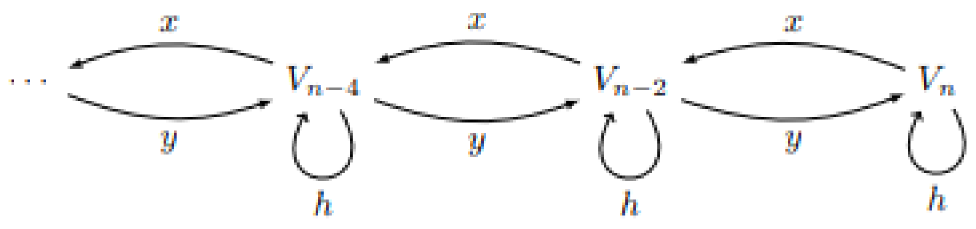


FIGURE 2.2. The action of x and y on the eigenspaces of an irreducible $\mathfrak{sl}_2$-module.

Finite-dimensional irreducible representations (i.e. simple modules) of $\mathfrak{sl}(2, \mathbb{C})$ are in bijection with dominant integral weights $n \in \Lambda$, i.e. $n \in \mathbb{Z}^{\geq 0}$, are denoted $M(n)$, and each admits a basis $\left\{ \mathbf{v}_i \;\middle|\; 0 \leq i \leq n \right\}$ where

$$
\begin{aligned}
h \cdot v_i &= (n - 2i) v_i \\
x \cdot v_i &= (n - i + 1) v_{i-1} \\
y \cdot v_i &= (i + 1) v_{i+1},
\end{aligned}
$$

setting $v_{-1} = v_{n+1} = 0$ and letting v_0 be the unique vector in $L(n)$ annihilated by x.

- rad $M(\lambda) = N(\lambda)$
- hd $M(\lambda) = L(\lambda)$.
- $M(\lambda)$ for $\lambda > 0$ not integral is simple, however $-\lambda - 2 \notin W \cdot \lambda$.
- $\lambda \geq 0 \implies \text{char } L(\lambda) = \text{char } M(\lambda) - \text{char } M(s_\alpha \cdot \lambda)$ where $s_\alpha \cdot \lambda = -\lambda - 2$.
- For $\lambda \geq 0$, $\dim L(\lambda) = \lambda + 1$ and so

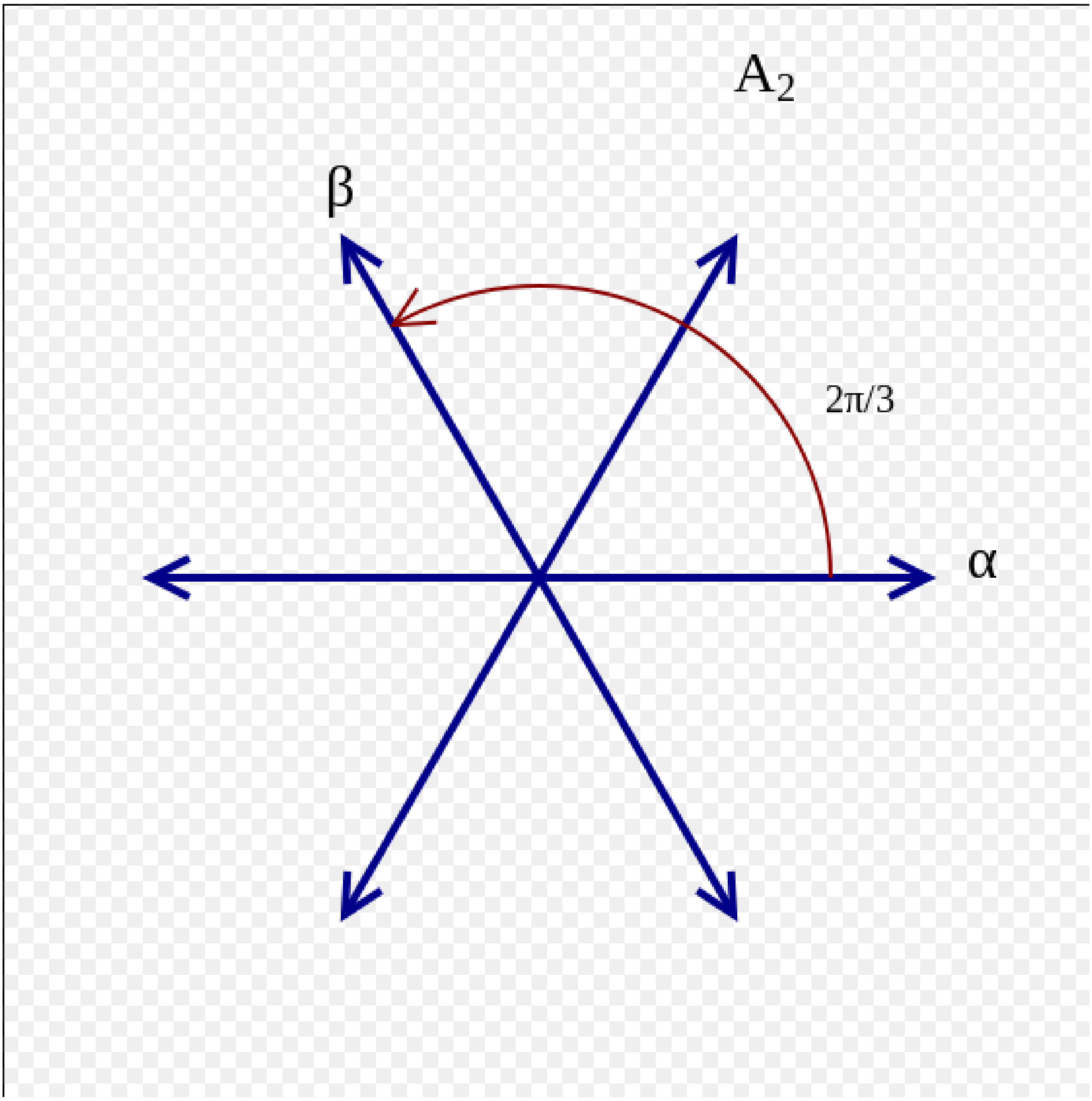
$$\text{char } L(\lambda) = e^\lambda + e^{\lambda-2} + \cdots + e^{-\lambda} = \frac{e^{\lambda+1} - e^{\lambda-1}}{e^1 - e^{-1}}.$$

- For $\lambda \neq \rho \in \mathbb{Z}$, the composition factors of $M(\lambda)$ are $M(\lambda), L(-\lambda-2)$.
- There is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & N(\lambda) \arrow[r] \arrow[d, equal] & M(\lambda) \arrow[r] \arrow[d, equal] & L(\lambda) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & L(-\lambda-2) \arrow[r] & M(\lambda) \arrow[r] & L(\lambda) \arrow[r] & 0
\end{tikzcd}$$

5 SL3

$\mathfrak{sl}(3, \mathbb{C})$ has root system A_2:



$$\Delta = \{\alpha, \beta\}$$
$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_0\}.$$

For λ regular, integral, and antidominant:

- $M(\lambda) = L(\lambda)$
- No other $M(w \cdot \lambda)$ is simple
- $\operatorname{Soc}(M(w \cdot \lambda)) = L(\lambda)$.
- $[M(w \cdot \lambda) : L(\lambda)] = [M(w \cdot \lambda) : L(w \cdot \lambda)] = 1$ for all w.
- $\operatorname{char} L(s_\alpha \cdot \lambda) = \operatorname{char} M(s_\alpha \cdot \lambda) - \operatorname{char} M(\lambda)$.

- char $M(s_\alpha \cdot \lambda) = \text{char } L(s_\alpha \cdot \lambda) + \text{char } L(\lambda)$.
- The Jantzen filtration when $w \in \{s_{\alpha\beta}, s_{\beta\alpha}, w_0\}$ is given by

$$
\begin{aligned}
M(w \cdot \lambda)^0 &= M(w \cdot \lambda) \\
M(w \cdot \lambda)^1 &= ? \\
M(w \cdot \lambda)^2 &= L(\lambda) \\
M(w \cdot \lambda)^{\geq 3} &= 0.
\end{aligned}
$$