

# Title

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## 1 | Monday, November 09

### 1.1 Strong Linkage

We have two categories:

- $G_r T$ , with a notion of *strong linkage*, and
- $G_r$ , which instead only has *linkage*.

We'll restate a few theorems.

**Theorem 1.1.1(?)**.

Let  $\lambda, \mu \in X(T)$ .

1. If  $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_r T} \neq 0$ , then  $\mu \uparrow \lambda$  are strongly linked.
2. If  $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$ , then  $\mu \in W_p \cdot \lambda + p^r X(T)$ .

*Example 1.1.1(?):* In the case of  $\Phi = A_2$ , we'll consider the two different categories.

We have the following picture for  $\widehat{Z}$ :

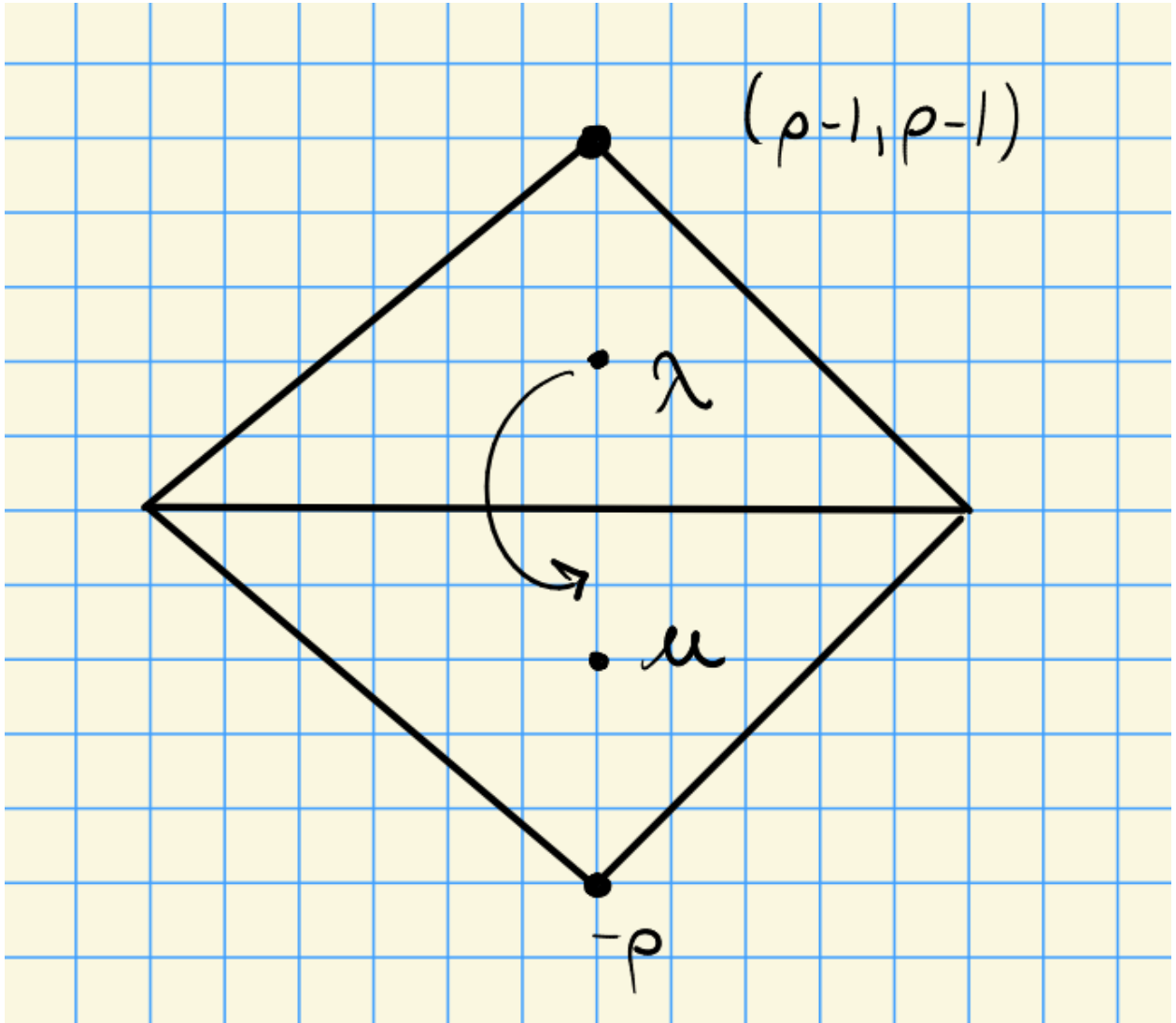


Figure 1: Image

Considering  $X_1(T)$  and  $[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] \neq 0$ , and  $\hat{Z}_1(\lambda)$  has 6 composition factors as  $G_1T$ -modules.

On the other hand, for  $Z$ , we have the following:

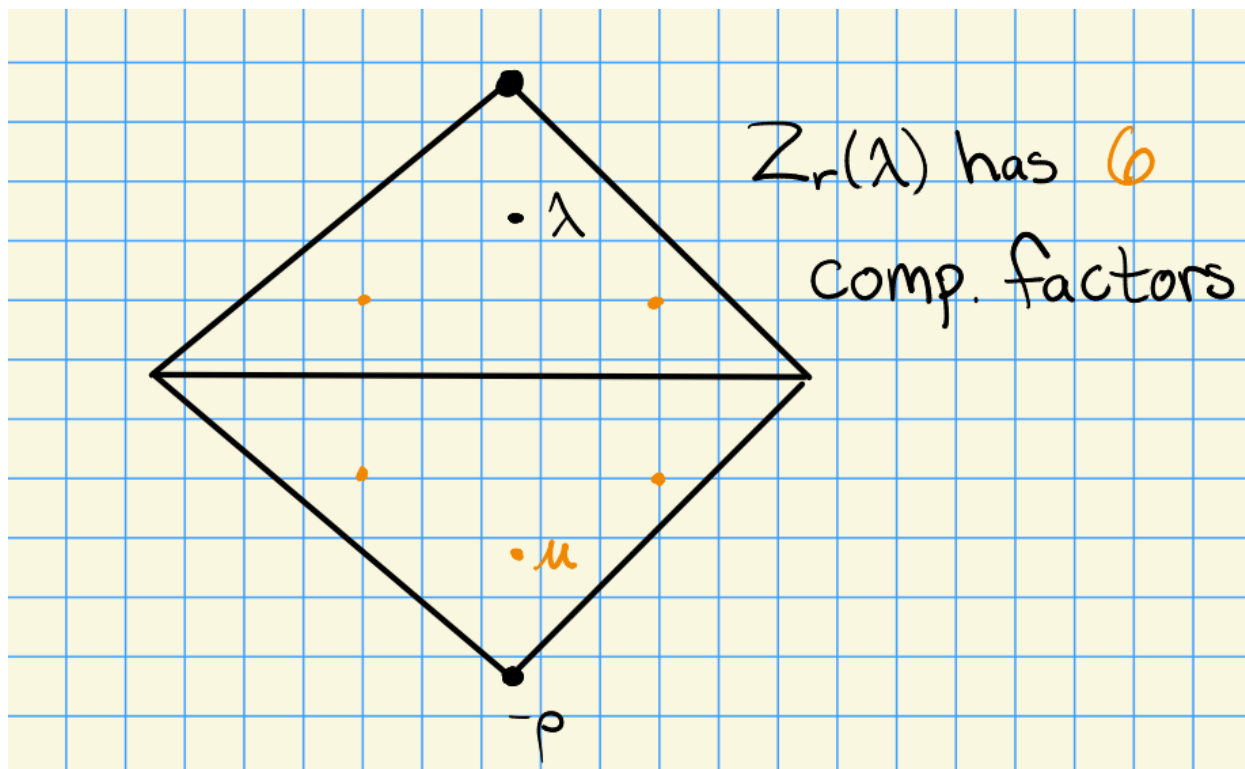


Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

## 1.2 Extensions

Let  $\lambda, \mu \in X(T)$ . We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\mathrm{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) = \mathrm{Ext}_{G_r}^j(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)) \quad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu)) = \bigoplus_{\gamma \in X(T)} \mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu))_{\gamma}$$

where we are taking the fixed points under the torus  $T$  action on the first factor (for which  $T_r$  acts

trivially). We can write this as

$$\begin{aligned}
 \cdots &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu) \otimes \gamma) \\
 &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r T}^j(L_r(\lambda), L_r(\mu) \otimes p^r v) \\
 &= \bigoplus_{v \in X(T)} \text{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)).
 \end{aligned}$$

So if we know extensions in the  $G_r$  category, we know them in the  $G_r T$  category.

There is an isomorphism

$$\text{Ext}_{G_r T}^1(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) \cong \text{Hom}_{G_r T}(\text{rad}_{G_r T} \widehat{Z}_r(\lambda), \widehat{L}_r(\mu)).$$

Finally, for  $\lambda, \mu \in X(T)$ , if the above  $\text{Ext}^1$  vanishes, then  $\lambda \in W_p \cdot \mu$  (i.e.  $\lambda$  and  $\mu$  are linked).

## 1.3 The Steinberg Modules

*Example 1.3.1(Steinberg):* Consider  $A_2$ :

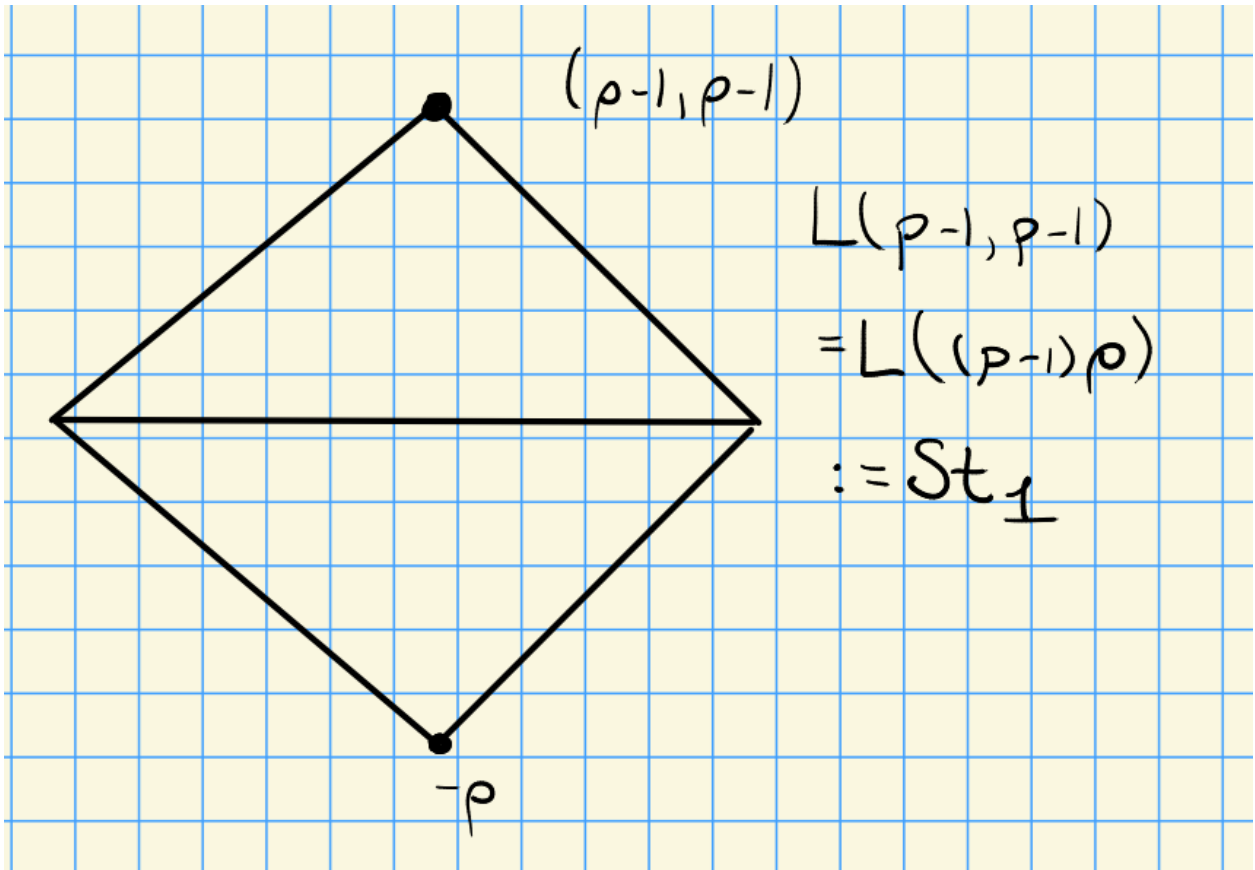


Figure 3: Image

Taking the representation corresponding to  $(p-1, p-1)$  yields the “first Steinberg module”

$$L(p-1, p-1) = L((p-1)\rho) := St_1.$$

In this case, we have an equality of many modules:

$$H^0((p-1)\rho) = L((p-1)\rho) = V((p-1)\rho) = T((p-1)\rho).$$

**Definition 1.3.1** (Steinberg Modules).

The  $r$ th **Steinberg module** is defined to be  $L((p^r - 1)\rho)$ .

*Remark 1.3.1:* In general, we have

$$L((p^r - 1)\rho) = H^0((p^r - 1)\rho) = V((p^r - 1)\rho).$$

We also have

$$\widehat{Z}_r((p^r - 1)\rho) \cong L((p^r - 1)\rho) \downarrow_{G_r T}.$$

**Theorem 1.3.1 (?).**

The Steinberg module is both injective and projective as both a  $G_r$ -module and a  $G_r T$ -module.

*Proof* (?).

It suffices to prove that  $\text{St}_r$  is projective over  $G_r T$ , then by a previous theorem, it will also be projective over  $G_r$ . Let  $\widehat{L}_r(\mu)$  be a simple  $G_r T$ -module, and consider

$$\text{Ext}_{G_r T}^1(\text{St}_r, \widehat{L}_r(\mu)) = \text{Ext}_{G_r T}^1(\widehat{L}_r((p^r - 1)\rho), \widehat{L}_r(\mu)).$$

If we show this is zero for every simple module, the result will follow.

Suppose  $(p^r - 1)\rho \not\prec \mu$ . In this case, the RHS above is zero.

Missed why: something to do with radical of the first term?

Otherwise, we have

$$\text{Ext}_{G_r T}^1(\widehat{L}_r(\mu), \text{St}_r) = \text{Hom}_{G_r T}(\text{rad}(\widehat{Z}_r(\mu)), \text{St}_r).$$

Suppose that the RHS is nonzero. Then  $\text{rad}(\widehat{Z}_r(\mu)) \rightarrow \text{St}_r$ , and thus

$$\dim \text{rad}(\widehat{Z}_r(\mu)) \geq \dim \text{St}_r = p^{r|\Phi^+|}$$

But we know that

$$\dim \text{rad}(\widehat{Z}_r(\mu)) < \dim \widehat{Z}_r(\mu) = p^{r|\Phi^+|},$$

so we've reached a contradiction and the hom must be zero. ■

**Proposition 1.3.1 (Conjecture, Donkin, MSRI 1990).**

Let  $G$  be a reductive group and  $M$  a finite-dimensional  $G$ -module. Then  $M$  has a good  $(p, r)$ -filtration iff  $\text{St}_r \otimes M$  has a good filtration.

*Remark 1.3.2:* See Nakano-Kildetoft (2015) and Bendel-Nakano-Pillen-