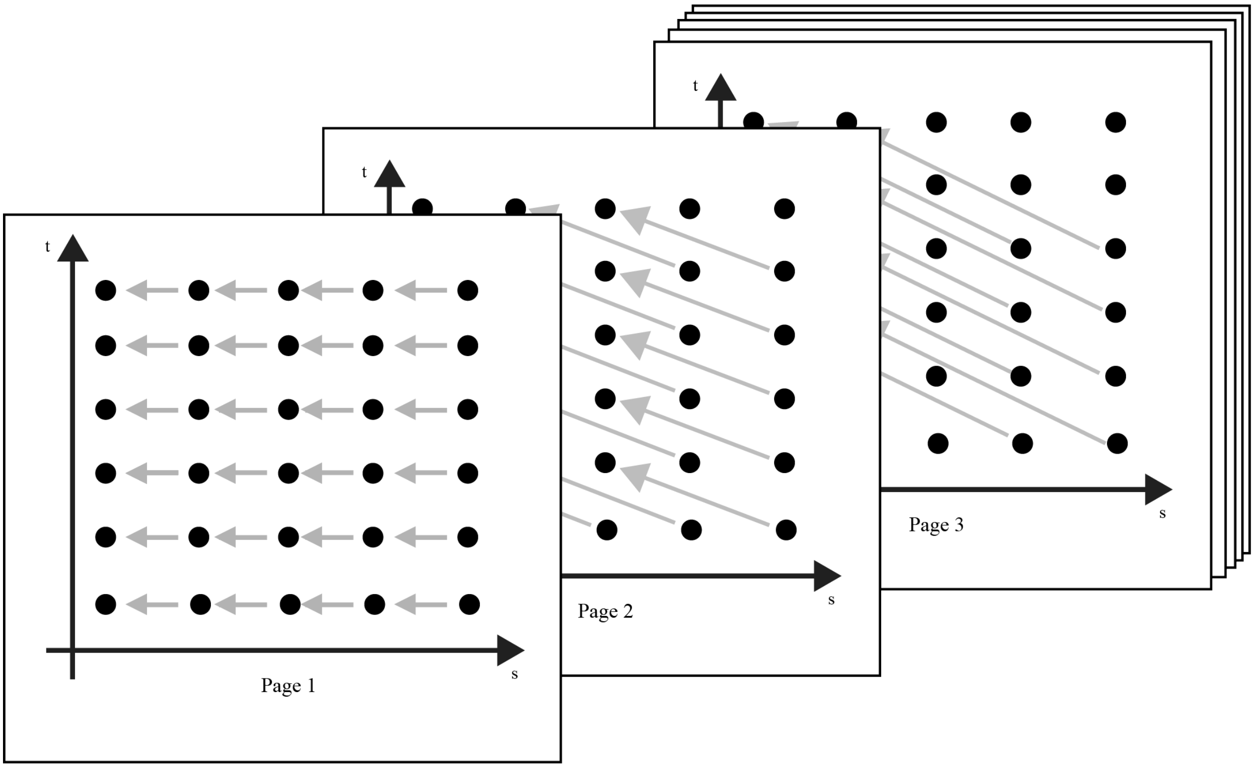

Notes: These are notes live-tex'd from a graduate course in Homological Algebra taught by Brian Boe at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Homological Algebra

Lectures by Brian Boe. University of Georgia, Spring 2021

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Reference:

- The course text is Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.
- See corrections: Many corrections to Weibel's book: http://www.math.rutgers.edu/~weibel/Hbook-corrections.html
- 1.1-1.5, 2.2-2.7, 3.4 3.6, 6.1, 5.1-5.2, 5.4-5.8, 6.8, 6.7, 6.3, 7.1-7.5, 7.7-7.8, along with most of Appendix A when needed.
- Course Website: https://uga.view.usg.edu/d2l/le/content/2218619/viewContent/33763436/View

1.1 Overview

Definition 1.1.1 (Exact complexes)
A **complex** is given by

$$\cdots \xrightarrow{d_{i-1}} M_{i-1} \xrightarrow{d_i} M_i \xrightarrow{d_{i+1}} M_{i+1} \to \cdots.$$

where $M_i \in R$-mod and $d_i \circ d_{i-1} = 0$, which happens if and only if $\operatorname{im} d_{i-1} \subseteq \ker d_i$. If $\operatorname{im} d_{i-1} = \ker d_i$, this complex is **exact**.

Example 1.1.2 *(?):* We can apply a functor such as $\otimes_R N$ to get a new complex

$$\cdots \xrightarrow{d_{i-1}\otimes 1_N} M_{i-1} \otimes_R N \xrightarrow{d_i \otimes 1} M_i \otimes N \to M_{i+1} \xrightarrow{d_{i+1}\otimes 1} \cdots.$$

Example 1.1.3 *(?):* Applying $\operatorname{Hom}(N, \cdot)$ similarly yields

$$\operatorname{Hom}_R(N, M_i) \xrightarrow{d_{i-1}^*} \operatorname{Hom}_R(N, M_{i+1}),$$

where $d_i^* = d_i \circ (\cdot)$ is given by composition.

Example 1.1.4 *(?):* Applying $\operatorname{Hom}(\cdot, N)$ yields

$$\operatorname{Hom}_R(M_i, N) \xrightarrow{d_i^*} \operatorname{Hom}_R(M_{i+1}, N)$$

where $d_i^* = (\cdot) \circ d_i$.

Remark 1.1.5: Note that we can also take complexes with arrows in the other direction. For F a functor, we can rewrite these examples as

$$d_i^* \circ d_{i-1}^* = F(d_i) \circ F(d_{i-1}) = F(d_i \circ d_{i-1}) = F(0) = 0,$$

provided F is nice enough and sends zero to zero. This follows from the fact that functors preserve composition. Even if the original complex is exact, the new one may not be, so we can define the following:

Definition 1.1.6 (Cohomology)

$$H^i(M^*) = \ker d_i^* / \operatorname{im} d_{i-1}^*.$$

Remark 1.1.7: These will lead to i**th derived functors**, and category theory will be useful here. See appendix in Weibel. For a category $\mathcal{C}$ we'll define

- $\mathrm{Obj}(\mathcal{C})$ as the objects
- $\mathrm{Hom}_{\mathcal{C}}(A, B)$ a set of morphisms between them, where a more modern notation might be $\mathrm{Mor}(A, B)$.
- Morphisms compose: $A \xrightarrow{f} B \xrightarrow{g} C$ means that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$
- Associativity
- Identity morphisms

See the appendix for diagrams defining zero objects and the zero map, which we'll need to make sense of exactness. We'll also needs notions of kernels and images, or potentially cokernels instead of images since they're closely related.

Remark 1.1.8: In the examples, we had $\ker d_i \subseteq M_i$, but this need not be true since the objects in the category may not be sets. Such an example is the category of complexes of R-modules: $\mathrm{Cx}(R\text{-}\mathrm{mod})$. In this setting, kernels will be subcomplexes but not subsets.

Definition 1.1.9 (Functors)
Recall that **functors** are "functions" between categories $F : \mathcal{C} \to \mathcal{D}$ such that

- Objects are sent to objects,
- Morphisms are sent to morphisms, so $A \xrightarrow{f} B \rightsquigarrow F(A) \xrightarrow{F(f)} F(B)$,
- F respects composition and identities

Example 1.1.10 ***(Hom):*** $\mathrm{Hom}_R(N, \cdot\,) : R\text{-}\mathrm{mod} \to \mathrm{Ab}$, noting that the hom set may not have an R-module structure.

Remark 1.1.11: Taking cohomology yields the ith derived functors of F, for example $\mathrm{Ext}^i, \mathrm{Tor}_i$. Recall that functors can be *covariant* or contravariant. See section 1 for formulating simplicial and singular homology (from topology) in this language.

1.2 Chapter 1: Chain Complexes

1.2.1 Complexes of R-modules

Definition 1.2.1 (Exactness)
Let R be a ring with 1 and define $R\text{-mod}$ to be the category of *right* R-modules. $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if and only if $\ker g = \operatorname{im} f$, and in particular $g \circ f = 0$.

Definition 1.2.2 (Chain Complex)
A **chain complex** is

$$C. \coloneqq (C., d.) \coloneqq \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \right)$$

for $n \in \mathbb{Z}$ such that $d_n \circ d_{n+1} = 0$. We drop the n from the notation and write $d^2 \coloneqq d \circ d = 0$.

Definition 1.2.3 (Cycles and boundaries)

- $Z_n = Z_n(C.) = \ker d_n$ are referred to as n**-cycles**.
- $B_n = B_n(C.) = \operatorname{im} d_{n+1}$ are the n**-boundaries**.

Definition 1.2.4 (Homology of a chain complex)
Note that if $d^2 = 0$ then $B_n \leq Z_n \leq C_n$. In this case, it makes sense to define the quotient module $H^n(C.) \coloneqq Z_n / B_n$, the n**th homology** of $C.$.

Definition 1.2.5 (Maps of chain complexes)
A map $u : C. \to D.$ of chain complexes is a sequence of maps $u_n : C_n \to D_n$ such that all of the following squares commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & C_n \arrow[r] \arrow[d, "u_n"] & C_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Remark 1.2.6: We can thus define a category $\mathrm{Ch}(R\text{-mod})$ where

- The objects are chain complexes,
- The morphisms are chain maps.

Exercise 1.2.7 (Weibel 1.1.2)
A chain complex map $u : C. \to D.$ restricts to

$$u_n : Z_n(C.) \to Z_n(D.)$$
$$u_n : B_n(D.) \to B_n(D.)$$

and thus induces a well-defined map $u_{n,*} : H_n(C.) \to H_n(D.)$.

Remark 1.2.8: Each H_n thus becomes a functor $\mathrm{Ch}(R\text{-mod}) \to R\text{-mod}$ where $H_n(u) \coloneqq u_{*.n}$.

2 | Friday, January 15

2.1 Review

See assignment posted on ELC, due Wed Jan 27

Remark 2.1.1: Recall that a chain complex is $C.$ where $d^2 = 0$, and a map of chain complex is a ladder of commuting squares

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & \cdots \\
\cdots \arrow[r] & D_{n-1} \arrow[r, "d_{n-1}"] & D_n \arrow[r, "d_n"'] & D_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to diagram Recall that $u_n : Z_n(C) \to Z_n(D)$ and $u_n : B_n(C) \to B_n(D)$ preserves these submodules, so there are induced maps $u_{.,n} : H_n(D) \to H_n(D)$ where $H_n(C) \coloneqq Z_n(C)/B_n n{-}1(C)$. Moreover, taking $H_n(\cdot)$ is a functor from $\mathrm{Ch}(R\text{-mod}) \to R\text{-mod}$ for any fixed n and on objects $C \mapsto H_n(C)$ and chain maps $u_n \to H_n(u) \coloneqq u_{*,n}$. Note the lower indices denote maps going down in degree.

2.2 Cohomology

Definition 2.2.1 (Quasi-isomorphism)
A chain map $u : C \to D$ is a **quasi-isomorphism** if and only if the induced map $u_{*,n} : H^n(C) \to H^n(D)$ is an isomorphism of R-modules.

Remark 2.2.2: Note that the usual notion of an isomorphism in the categorical sense might be too strong here.

Definition 2.2.3 (Cohomology)
A **cochain complex** is a complex of the form

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \cdots$$

where $d^n \circ d^{n-1} = 0$. We similarly write $Z^n(C) \coloneqq \ker d^n$ and $B^n(C) \coloneqq \operatorname{im} d^{n-1}$ and write the R-module $H^n(C) \coloneqq Z^n / B^n$ for the nth **cohomology** of C.

Remark 2.2.4: There is a way to go back and forth bw chain complexes and cochain complexes: set $C_n \coloneqq C^{-n}$ and $d_n \coloneqq d^{-n}$. This yields

$$C^{-n} \xrightarrow{d^{-n}} C^{-n+1} \iff C_n \xrightarrow{d^n} C_{n-1},$$

and the notions of $d^2 = 0$ coincide.

Definition 2.2.5 (Bounded complexes)
A cochain complex C is **bounded** if and only if there exists an $a \leq b \in \mathbb{Z}$ such that $C_n \neq 0 \iff a \leq n \leq b$. Similarly C^n is bounded above if there is just a b, and **bounded below** for just an a. All of the same definitions are made for cochain complexes.

Remark 2.2.6: See the book for classical applications:

- 1.1.3: Simplicial homology
- 1.1.5: Singular homology

2.3 Operations on Chain Complexes

Remark 2.3.1: Write Ch for $\mathrm{Ch}(R\text{-mod})$, then if $f, g : C \to D$ are chain maps then $f + g : C \to D$ can be defined as $(f+g)(x) = f(x) + g(x)$, since D has an addition coming from its R-module structure. Thus the hom sets $\mathrm{Hom}_{\mathrm{Ch}}(C, D)$ becomes an abelian group. There is a distinguished **zero object**[1] 0, defined as the chain complex with all zero objects and all zero maps. Note that we also have a zero map given by the composition $(C \to 0) \circ (0 \to D)$.

[1] See appendix A 1.6 for initial and terminal objects. Note that $\varnothing$ is an initial but non-terminal object in Set, whereas zero objects are both.

Definition 2.3.2 (Products and Coproducts)
If $\{A_\alpha\}$ is a family of complexes, we can form two new complexes:

- The **product** $\left(\prod_\alpha A_\alpha\right)_n \coloneqq \prod_\alpha A_{\alpha,n}$ with the differential

$$\left(\prod d_\alpha\right)_n : \prod A_{\alpha,n} \xrightarrow{d_{\alpha,n}} \prod A_{\alpha,n-1}.$$

- The **coproduct** $\left(\coprod_\alpha A_\alpha\right)_n \coloneqq \bigoplus_\alpha A_{\alpha,n}$, i.e. there are only finitely many nonzero entries, with exactly the same definition as above for the differential.

Remark 2.3.3: Note that if the index set is finite, these notions coincide. By convention, finite direct products are written as direct sums.

These structures make Ch into an **additive category**. See appendix for definition: the homs are abelian groups where composition distributes over addition, existence of a zero object, and existence of finite products. Note that here we have arbitrary products.

Definition 2.3.4 (?)
We say B is a **subcomplex** of C if and only if

- $B_n \leq C_n \in R\text{-mod}$ for all n,
- The differentials of B_n are the restrictions of the differentials of C_n.

Remark 2.3.5: This can be alternatively stated as saying the inclusion $i : B \to C$ given by $i_n : B_n \to C_n$ is a morphism of chain complexes. Recall that some squares need to commute, and this forces the condition on restrictions.

Definition 2.3.6 (Quotient Complex)
When $B \leq C$, we can form the quotient complex C/B where

$$C_n/B_n \xrightarrow{\overline{d_n}} C_{n-1}/B_{n-1}.$$

Moreover there is a natural projection $\pi : C \to C/B$ which is a chain map.

Suppose $f : B \to C$ is a chain map, then there exist induced maps on the levelwise kernels and cokernels, so we can form the **kernel** and **cokernel** complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \ker f_n \arrow[r, dashed, "\exists d_n"] \arrow[d, "i_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "\pi_n"] & C_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{coker} f_n \arrow[r, dashed, "\exists d_n"] & \operatorname{coker} f_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here $\ker f \leq B$ is a subcomplex, and $\operatorname{coker} f$ is a quotient complex of C. The chain map $i : \ker f \to B$ is a categorical kernel of f in Ch, and π is similarly a cokernel. See appendix A 1.6. These constructions make Ch into an **abelian category**: roughly an additive category where every morphism has a kernel and a cokernel.

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2 Definitions

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