Problem Set 2

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1 Humphreys 1.5

Proposition: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $M(\lambda), M(\mu)$ Verma modules. Then $M(\lambda) \otimes M(\mu)$ may not lie in $\mathcal{O}$.

Proof:

Let $M(\lambda), M(\mu)$ be arbitrary Verma modules with highest weight vectors $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$ respectively. We can then consider the weight of $v \otimes w$ in $N \coloneqq M(\lambda) \otimes_{\mathbb{C}} M(\mu)$:

$$\begin{aligned} h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\ &= \lambda(h) v \otimes w + v \otimes \mu(h) w \\ &= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\ &= (\lambda(h) + \mu(h))(v \otimes w). \end{aligned}$$

Letting $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, so $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$, the claim is that it is possible for N to *not* be finitely-generated as a $U(\mathfrak{g})$-module.

Let $\{y, h, x\}$ be the usual basis for $\mathfrak{g}$, for which $U(\mathfrak{g})$ has the usual associated PBW basis. We can use the fact that $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$, so if we pick $\mu, \lambda \in \mathbb{Z}^{\leq 0}$ we have weight space decompositions

$$\begin{aligned} M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda - 2i} \coloneqq \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i \coloneqq \lambda - 2i}} M(\lambda)_{\lambda_i} \\ M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu - 2j} \coloneqq \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_i \coloneqq \mu - 2j}} M(\mu)_{\mu_j} \end{aligned}$$

where we can explicitly identify $\mathbb{C}$-bases $M(\lambda)_{\lambda_i} = \text{span}_{\mathbb{C}}\left\{y^i\ v^+\right\}$ and $M(\mu)_{\mu_i} = \text{span}_{\mathbb{C}}\left\{y^i w^+\right\}$ where v^+, w^+ are maximal weight vectors for $M(\lambda), M(\mu)$ respectively.

By the initial observation, this yields a weight space decomposition for N given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left(\bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_\nu.$$

Since each weight space $N_\nu = \text{span}_{\mathbb{C}}\left\{y^i v^+ \otimes y^j w^+ \;\middle|\; i + j = \nu\right\}$ has dimension $p_2(\nu)$, the (combinatorial) number of partitions of ν into two parts. In particular, $p_2(\nu)$ takes on arbitrarily large values as ν ranges over $\mathbb{Z}^+$, and thus N has weight spaces of arbitrarily large dimension.

Now suppose toward a contradiction that N is finitely generated as a $U(\mathfrak{g})$-module, say by the n generators $\{m_1, \cdots, m_n\}$. Then the $\mathbb{C}$-vector spaces spanned by the m_i is of dimension no larger than n^2 – however, picking $\nu > n^2$ yields $p_2(\nu) > n^2$, and thus there is a $\mathbb{C}$-subspace of dimension greater than n^2 by the above argument – a contradiction.

■

2 Humphreys 1.9

Proposition: Let $\psi : Z(\mathfrak{g}) \to S(\mathfrak{h})$ be the twisted Harish-Chandra homomorphism. Then ψ is independent of the choice of a simple system in Φ.

Hint: any simple system has the form $w\Delta$ for some $w \in W$.

Proof:

For a given simple root system $\Delta_1 = \{\alpha_1, \cdots, \alpha_\ell\}$, we can choose a PBW basis $\left\{h_i^{t_j} \;\middle|\; 1 \leq i \leq \ell, j \in \mathbb{Z}^+\right\}$ for $U(\mathfrak{h})$. Then if $z \in \mathcal{Z}(\mathfrak{g})$, we can write $z = \sum_{i,j} c_{ij} h_i^{t_j}$ for some $c_{ij} \in \mathbb{C}$. We can then identify the (twisted) Harish-Chandra morphism as follows:

$$\begin{array}{lllll} \psi : \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\xi} U(\mathfrak{h}) & \to S(\mathfrak{h}) = \mathbb{C}[\{h_i\}] = P(\mathfrak{h}^*) & & \xrightarrow{\tau_\rho} \mathbb{C}[\{h_i\}] \\ \quad z & \mapsto z = \displaystyle\sum_{i=1}^{\ell} c_{ij} h_i^{t_j} & \mapsto \left(\lambda \mapsto \displaystyle\sum_{i=1}^{\ell} c_{ij} \lambda(h_i)^{t_j}\right) & & \mapsto \psi(z) = \displaystyle\prod_{i=1}^{\ell} (\lambda - \rho)(h_i)^{t_i}, \end{array}$$

where ξ is the Harish-Chandra morphism and τ_p is the twist sending $f(\lambda)$ to $f(\lambda - \rho)$. We thus find that ψ explicitly depends only on ρ and potentially the basis $\{h_i\}$

The claim is that if an alternative simple root system $\Delta_2 = \{\alpha_1', \cdots, \alpha_\ell'\}$ is chosen, $\psi(z)$ does not change. By the hint, there exists some uniform $w \in W$ such that $w\alpha_i = \alpha_i'$.

We can denote the positive root system induced by Δ_1 as Φ_1^+ and similarly Δ_2 induces Φ_2^+. From this, a priori we may have two distinct weyl vectors:

$$\rho_1 = \sum_{\beta \in \Phi_1^+} \beta$$
$$\rho_2 = \sum_{\beta' \in \Phi_2^+} \beta'$$
$$.$$

However, since W acts transitively on the Weyl chambers, it only permutes the elements in such a sum, and since $\Delta_1 = w\Delta_2$ we in fact obtain $\rho_1 = \rho_2 \coloneqq \rho$.