

Homological Algebra Problem Sets

Problem Set 3

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

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Problem 1.0.1 (Prove Corollary 2.3.2)

For R a PID, show that an R -module A is divisible if and only if A is injective.

Recall that a module is divisible if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.

Solution:

Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.

\Rightarrow : Suppose A is divisible, we then want to show every R -module morphism of the following form lifts, where we regard the ideal J and the ring R as R -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R -linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending R -linearly yields a well-defined R -module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

\Leftarrow : Suppose $A \in R\text{-Mod}$ is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \leq R$:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \rightarrow A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting \mathbf{a} range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)

Calculate $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q .

Solution:

We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\cdot) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \cdot)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of \mathbb{Z}/q .
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We'll need the following fact:

Claim: For any $m \in \mathbb{Z}$,

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/m.$$

Proof (?).

Note that there is an injection

$$1 \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \rightarrow \mathbb{Z} \xrightarrow{x \mapsto mx} \mathbb{Z} \xrightarrow{\pi_m} \mathbb{Z}/m \rightarrow 1$$

where π_m is the canonical quotient morphism, and applying the left-exact contravariant functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any \mathbb{Z} -module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since \mathbb{Z} is a free \mathbb{Z} -module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/m \rightarrow \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z} \\ \pi_m \downarrow & \nearrow \exists \tilde{f} & \\ \mathbb{Z}/m & & \end{array}$$

[Link to Diagram](#)

In the target, it suffices to restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$.

■

Problem 1.0.3 (Weibel 2.3.2)

For $A \in \mathbf{Ab}$, define $I(A) := \bigoplus_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \rightarrow I(A)$. Show that e_A is injective.

Hint: if $a \in A$, find a map $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)

If $U : \mathcal{B} \rightarrow \mathcal{C}$ is an exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Problem 1.0.5 (Weibel 2.4.3)

If $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$ is exact with P projective or F -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

Show that $L_{m+1} F(A)$ is the kernel of $F(M_m) \rightarrow F(P_m)$. Conclude that if $P \rightarrow A$ is an F -acyclic resolution of A , then $L_i F(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

- A is a projective R -module.
- $\text{Hom}_R(\cdot, A)$ is an exact functor.
- $\text{Ext}_R^{i \neq 0}(A, B) = 0$ and for all B , i.e. A is $\text{Hom}_R(\cdot, B)$ -acyclic for all B .
- $\text{Ext}_R^1(A, B)$ vanishes for all B .

Problem 1.0.7 (Weibel 2.6.4)

Show that colim is left adjoint to Δ , and conclude that colim is right-exact when \mathcal{A} is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on \mathbf{Ab} .