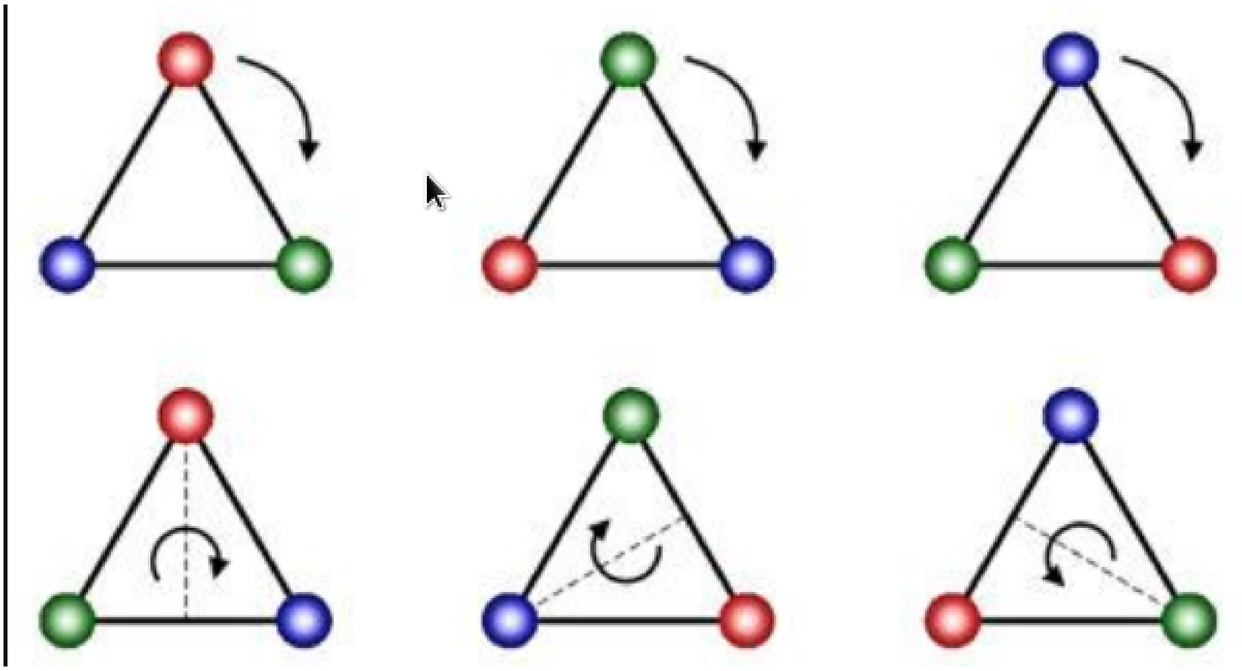


Notes: These are notes live-tex'd from a graduate course in Algebra taught by Dan Nakano at the University of Georgia in Fall 2019. As such, any errors or inaccuracies are almost certainly my own.

Algebra

University of Georgia, Fall 2019

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2020-10-22

Contents

1 | Summary

- Groups and rings, including Sylow theorems,
- Classifying small groups,
- Finitely generated abelian groups,
- Jordan-Holder theorem,
- Solvable groups,
- Simplicity of the alternating group,
- Euclidean domains,
- Principal ideal domains,
- Unique factorization domains,
- Noetherian rings,
- Hilbert basis theorem,
- Zorn's lemma, and
- Existence of maximal ideals and vector space bases.

Previous course web pages:

- Fall 2017, Asilata Bapat

2 | Thursday August 15th

We'll be using Hungerford's Algebra text.

2.1 Definitions

The following definitions will be useful to know by heart:

- The order of a group
- Cartesian product
- Relations
- Equivalence relation
- Partition
- Binary operation
- Group
- Isomorphism
- Abelian group
- Cyclic group
- Subgroup
- Greatest common divisor
- Least common multiple
- Permutation
- Transposition
- Orbit
- Cycle
- The symmetric group S_n
- The alternating group A_n
- Even and odd permutations
- Cosets
- Index
- The direct product of groups
- Homomorphism
- Image of a function
- Inverse image of a function
- Kernel
- Normal subgroup
- Factor group
- Simple group

Here is a rough outline of the course:

- Group Theory
  - Groups acting on sets
  - Sylow theorems and applications
  - Classification
  - Free and free abelian groups
  - Solvable and simple groups
  - Normal series
- Galois Theory
  - Field extensions
  - Splitting fields
  - Separability
  - Finite fields

  - Cyclotomic extensions
  - Galois groups
  - Solvability by radicals
- Module theory
  - Free modules
  - Homomorphisms
  - Projective and injective modules
  - Finitely generated modules over a PID
- Linear Algebra
  - Matrices and linear transformations
  - Rank and determinants
  - Canonical forms
  - Characteristic polynomials
  - Eigenvalues and eigenvectors

2.2 Preliminaries

Definition: A **group** is an ordered pair $(G, \cdot : G \times G \to G)$ where G is a set and $\cdot$ is a binary operation, which satisfies the following axioms:

1. **Associativity**: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$,
2. **Identity**: $\exists e \in G \;\middle|\; ge = eg = g$,
3. **Inverses**: $g \in G \implies \exists h \in G \;\middle|\; gh = gh = e$.

Examples of groups:

- $(\mathbb{Z}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Q}^\times, \times)$
- $(\mathbb{R}^\times, \times)$
- $(\mathrm{GL}(n, \mathbb{R}), \times) = \left\{ A \in \mathrm{Mat}_n \;\middle|\; \det(A) \neq 0 \right\}$
- $(S_n, \circ)$

Definition: A subset $S \subseteq G$ is a **subgroup** of G iff

1. **Closure**: $s_1, s_2 \in S \implies s_1 s_2 \in S$
2. **Identity**: $e \in S$
3. **Inverses**: $s \in S \implies s^{-1} \in S$

We denote such a subgroup $S \leq G$.

Examples of subgroups:

- $(\mathbb{Z}, +) \leq (\mathbb{Q}, +)$
- $\mathrm{SL}(n, \mathbb{R}) \leq \mathrm{GL}(n, \mathbb{R})$, where $\mathrm{SL}(n, \mathbb{R}) = \left\{ A \in \mathrm{GL}(n, \mathbb{R}) \;\middle|\; \det(A) = 1 \right\}$

2.3 Cyclic Groups

Definition: A group G is **cyclic** iff G is generated by a single element.

Exercise: Show

$$\langle g \rangle = \left\{ g^n \;\middle|\; n \in \mathbb{Z} \right\} \cong \cap_{g \in G} \left\{ H \;\middle|\; H \leq G \text{ and } g \in H \right\}.$$

Theorem: Let G be a cyclic group, so $G = \langle g \rangle$.

- If $|G| = \infty$, then $G \cong \mathbb{Z}$.
- If $|G| = n < \infty$, then $G \cong \mathbb{Z}_n$.

Definition: Let $H \leq G$, and define a **right coset of** G by $aH = \left\{ ah \;\middle|\; H \in H \right\}$.

A similar definition can be made for **left cosets**.

The "Fundamental Theorem of Cosets":

$$aH = bH \iff b^{-1}a \in H \text{ and } Ha = Hb \iff ab^{-1} \in H.$$

Some facts:

- Cosets partition H, i.e.

$$b \notin H \implies aH \cap bH = \{e\}.$$

- $|H| = |aH| = |Ha|$ for all $a \in G$.

Theorem (Lagrange): If G is a finite group and $H \leq G$, then $|H| \;\Big|\; |G|$.

Definition A subgroup $N \leq G$ is **normal** iff $gN = Ng$ for all $g \in G$, or equivalently $gNg^{-1} \subseteq N$. (I denote this $N \trianglelefteq G$.)

When $N \trianglelefteq G$, the set of left/right cosets of N themselves have a group structure. So we define

$$G/N = \left\{ gN \;\middle|\; g \in G \right\} \text{ where } (g_1 N) \cdot (g_2 N) \coloneqq (g_1 g_2) N.$$

Given $H, K \leq G$, define

$$HK = \left\{ hk \;\middle|\; h \in H,\ k \in K \right\}.$$

We have a general formula,

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

2.4 Homomorphisms

Definition: Let G, G' be groups, then $\varphi : G \to G'$ is a **homomorphism** if $\varphi(ab) = \varphi(a)\varphi(b)$.

Examples of homomorphisms:

- $\exp(:)(\mathbb{R}, +) \to (\mathbb{R}^{>0}, \cdot)$ since

$$\exp(()a+b) := e^{a+b} = e^a e^b := \exp(()a)\exp(()b).$$

- $\det : (\mathrm{GL}(n, \mathbb{R}), \times) \to (\mathbb{R}^\times, \times)$ since

$$\det(AB) = \det(A)\det(B).$$

- Let $N \trianglelefteq G$ and define

$$\begin{aligned} \varphi : G &\to G/N \\ g &\mapsto gN. \end{aligned}$$

- Let $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ where $\varphi(g) = [g] = g \mod n$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$

Definition: Let $\varphi : G \to G'$. Then φ is a **monomorphism** iff it is injective, an **epimorphism** iff it is surjective, and an **isomorphism** iff it is bijective.

2.5 Direct Products

Let G_1, G_2 be groups, then define

$$G_1 \times G_2 = \left\{ (g_1, g_2) \;\middle|\; g_1 \in G, g_2 \in G_2 \right\} \text{ where } (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2, h_2).$$

We have the formula $|G_1 \times G_2| = |G_1||G_2|$.

2.6 Finitely Generated Abelian Groups

Definition: We say a group is **abelian** if G is commutative, i.e. $g_1, g_2 \in G \implies g_1 g_2 = g_2 g_1$.

Definition: A group is **finitely generated** if there exist $\{g_1, g_2, \cdots g_n\} \subseteq G$ such that $G = \langle g_1, g_2, \cdots g_n \rangle$.

This generalizes the notion of a cyclic group, where we can simply intersect all of the subgroups that contain the g_i to define it.

We know what cyclic groups look like – they are all isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_n$. So now we'd like a structure theorem for abelian finitely generated groups.

Theorem: Let G be a finitely generated abelian group.

Then

$$G \cong \mathbb{Z}^r \times \prod_{i=1}^{s} \mathbb{Z}_{p_i^{\alpha_i}}$$

for some finite $r, s \in \mathbb{N}$ where the p_i are (not necessarily distinct) primes.

Example: Let G be a finite abelian group of order 4.

Then $G \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$, which are not isomorphic because every element in $\mathbb{Z}_2^2$ has order 2 where $\mathbb{Z}_4$ contains an element of order 4.

2.7 Fundamental Homomorphism Theorem

Let $\varphi : G \to G'$ be a group homomorphism and define

$$\ker \varphi \coloneqq \left\{ g \in G \;\middle|\; \varphi(g) = e' \right\}.$$

2.7.1 The First Homomorphism Theorem

Theorem: There exists a map $\varphi' : G/\ker\varphi \to G'$ such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, dotted, "\varphi"] \arrow[d, "\eta"'] & G' \\
G/\ker\varphi \arrow[ur, "\varphi'"] &
\end{tikzcd}$$

That is, $\varphi = \varphi' \circ \eta$, and φ' is an isomorphism onto its image, so $G/\ker\varphi = \operatorname{im}(\varphi)$.

This map is given by

$$\varphi'(g(\ker\varphi)) = \varphi(g).$$

Exercise: Check that φ is well-defined.

2.7.2 The Second Theorem

Theorem: Let $K, N \leq G$ where $N \trianglelefteq G$. Then

$$\frac{K}{N \cap K} \cong \frac{NK}{N}$$

Proof: Define a map

$$\begin{aligned} K &\xrightarrow{\varphi} NK/N \\ k &\mapsto kN. \end{aligned}$$

You can show that φ is onto, then look at $\ker\varphi$; note that

$$kN = \varphi(k) = N \iff k \in N,$$

and so $\ker\varphi = N \cap K$.

■

3 | Tuesday August 20th

3.1 The Fundamental Homomorphism Theorems

Theorem 1: Let $\varphi : G \to G'$ be a homomorphism. Then there is a canonical homomorphism $\eta : G \to G/\ker\varphi$ such that the usual diagram commutes.

Moreover, this map induces an isomorphism $G/\ker\varphi \cong \operatorname{im}(\varphi)$.

Theorem 2: Let $K, N \leq G$ and suppose $N \trianglelefteq G$. Then there is an isomorphism

$$\frac{K}{K \cap N} \cong \frac{NK}{N}$$

Proof Sketch: Show that $K \cap N \trianglelefteq G$, and NK is a subgroup exactly because N is normal.

Theorem 3: Let $H, K \trianglelefteq G$ such that $H \leq K$.

Then

1. H/K is normal in G/K.
2. The quotient $(G/K)/(H/K) \cong G/H$.

Proof: We'll use the first theorem.

Define a map

$$\begin{aligned} \varphi : G/K &\to G/H \\ gk &\mapsto gH. \end{aligned}$$

Exercise: Show that φ is surjective, and that $\ker\varphi \cong H/K$.

■

3.2 Permutation Groups

Let A be a set, then a *permutation* on A is a bijective map $A\circlearrowleft$. This can be made into a group with a binary operation given by composition of functions. Denote S_A the set of permutations on A.

Theorem: S_A is in fact a group.

Proof: Exercise. Follows from checking associativity, inverses, identity, etc.

■

In the special case that $A = \{1, 2, \cdots n\}$, then $S_n \coloneqq S_A$.

Recall two line notation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

Moreover, $|S_n| = n!$ by a combinatorial counting argument.

Example: S_3 is the symmetries of a triangle.

Example: The symmetries of a square are *not* given by S_4, it is instead D_4.

3.3 Orbits and the Symmetric Group

Permutations S_A *act* on A, and if $\sigma \in S_A$, then $\langle \sigma \rangle$ also acts on A.

Define $a \sim b$ iff there is some n such that $\sigma^n(a) = b$. This is an equivalence relation, and thus induces a partition of A. See notes for diagram. The equivalence classes under this relation are called the *orbits* under σ.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (18)(2)(364)(57).$$

Definition: A permutation $\sigma \in S_n$ is a *cycle* iff it contains at most one orbit with more than one element.

The *length* of a cycle is the number of elements in the largest orbit.

Recall cycle notation: $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$.

Note that this is read right-to-left by convention!

Theorem: Every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles.

Definition: A *transposition* is a cycle of length 2.

Proposition: Every permutation is a product of transpositions.

Proof:

$$(a_1 a_2 \cdots a_n) = (a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_2).$$

■

This is not a unique decomposition, however, as e.g. $\mathrm{id} = (12)^2 = (34)^2$.

Theorem: Any $\sigma \in S_n$ can be written as **either**

- An even number of transpositions, or
- An odd number of transpositions.

Proof:

Define

$$A_n = \left\{ \sigma \in S_n \;\middle|\; \sigma \text{ is even} \right\}.$$

We claim that $A_n \trianglelefteq S_n$.

1. Closure: If τ_1, τ_2 are both even, then $\tau_1 \tau_2$ also has an even number of transpositions.

2. The identity has an even number of transpositions, since zero is even.
3. Inverses: If $\sigma = \prod_{i=1}^{s} \tau_i$ where s is even, then $\sigma^{-1} = \prod_{i=1}^{s} \tau_{s-i}$. But each τ is order 2, so $\tau^{-1} = \tau$, so there are still an even number of transpositions.

So A_n is a subgroup.

It is normal because it is index 2, or the kernel of a homomorphism, or by a direct computation.

3.4 Groups Acting on Sets

Think of this as a generalization of a G-module.

Definition: A group G is said to *act* on a set X if there exists a map $G \times X \to X$ such that

1. $e \curvearrowright x = x$
2. $(g_1 g_2) \curvearrowright x = g_1 \curvearrowright (g_2 \curvearrowright x)$.

Examples:

1. $G = S_A \curvearrowright A$
2. $H \leq G$, then $G \curvearrowright X = G/H$ where $g \curvearrowright xH = (gx)H$.
3. $G \curvearrowright G$ by conjugation, i.e. $g \curvearrowright x = gxg^{-1}$.

Definition: Let $x \in X$, then define the **stabilizer subgroup**

$$G_x = \left\{ g \in G \;\middle|\; g \curvearrowright x = x \right\} \leq G$$

We can also look at the dual notion,

$$X_g = \left\{ x \in X \;\middle|\; g \curvearrowright x = x \right\}.$$

We then define the *orbit* of an element x as

$$Gx = \left\{ g \curvearrowright x \;\middle|\; g \in G \right\}$$

and we have a similar result where $x \sim y \iff x \in Gy$, and the orbits partition X.

Theorem: Let G act on X. We want to know the number of elements in an orbit, and it turns out that

$$|Gx| = [G : G_x]$$

Proof: Construct a map $Gx \xrightarrow{\psi} G/Gx$ where $\psi(g \curvearrowright x) = gGx$.

Exercise: Show that this is well-defined, so if 2 elements are equal then they go to the same coset.

Exercise: Show that this is surjective.

Injectivity: $\psi(g_1 x) = \psi(g_2 x)$, so $g_1 Gx = g_2 Gx$ and $(g_2^{-1} g_1)Gx = Gx$ so

$$g_2^{-1} g_1 \in Gx \iff g_2^{-1} g_1 \curvearrowright x = x \iff g_1 x = g_2 x.$$

■

Next time: Burnside's theorem, proving the Sylow theorems.

4 | Thursday August 22nd

4.1 Group Actions

Let G be a group and X be a set; we say G *acts* on X (or that X is a G- set) when there is a map $G \times X \to X$ such that $ex = x$ and

$$(gh) \curvearrowright x = g \curvearrowright (h \curvearrowright x).$$

We then define the **stabilizer** of x as

$$\mathrm{Stab}_G(x) = G_x := \left\{ g \in G \;\middle|\; g \curvearrowright x = x \right\} \leq G,$$

and the **orbit**

$$G.x = \mathcal{O}_x := \left\{ g \curvearrowright x \;\middle|\; x \in X \right\} \subseteq X.$$

When G is finite, we have

$$|G.x| = \frac{|G|}{|G_x|}.$$

We can also consider the **fixed points** of X,

$$X_g = \left\{ x \in X \;\middle|\; g \curvearrowright x = x \; \forall g \in G \right\} \subseteq X$$

4.2 Burnside's Theorem

Theorem (Burnside): Let X be a G-set and $v := |X/G|$ be the number of orbits. Then

$$v|G| = \sum_{g \in G} |X_g|.$$

Proof: Define

$$N = \left\{ (g, x) \;\middle|\; g \curvearrowright x = x \right\} \subseteq G \times X,$$

we then have

$$\begin{aligned}
|N| &= \sum_{g\in G} |X_g| \\
&= \sum_{x\in X} |G_x| \\
&= \sum_{x\in X} \frac{|G|}{|G.x|} \quad \text{by Orbit-Stabilizer} \\
&= |G| \left(\sum_{x\in X} \frac{1}{|G.x|} \right) \\
&= |G| \sum_{G.x \ \in \ X/G} \left(\sum_{y \ \in \ G.x} \frac{1}{|G.x|} \right) \\
&= |G| \sum_{G.x \ \in \ X/G} \left(|G.x| \frac{1}{|G.x|} \right) \\
&= |G| \sum_{G.x \ \in \ X/G} 1 \\
&= |G| v.
\end{aligned}$$

The last two equalities follow from the following fact: since the orbits partition X, say into $X = \coprod_{i=1}^{v} \sigma_i$, so let $\sigma = \left\{ \sigma_i \;\middle|\; 1 \leq i \leq v \right\}$.

By abuse of notation, replace each orbit in σ with a representative element $x_i \in \sigma_i \subset X$.

We then have

$$\sum_{x\in\sigma} \frac{1}{|G.x|} = \frac{1}{|G.x|} |\sigma| = 1.$$

■

Application: Consider seating 10 people around a circular table. How many distinct seating arrangements are there?

Let X be the set of configurations, $G = S_{10}$, and let $G \curvearrowright X$ by permuting configurations. Then v, the number of orbits under this action, yields the number of distinct seating arrangements.

By Burnside, we have

$$v = \frac{1}{|G|} \sum_{g\in G} |X_g| = \frac{1}{10}(10!) = 9!$$

since $X_g = \left\{ x \in X \;\middle|\; g \curvearrowright x = x \right\} = \emptyset$ unless $g = e$, and $X_e = X$.

4.3 Sylow Theory

Recall Lagrange's theorem:

If $H \leq G$ and G is finite, then $|H|$ divides $|G|$.

Consider the converse: if n divides $|G|$, does there exist a subgroup of size n?

The answer is **no** in general, and a counterexample is A_4 which has $4!/2 = 12$ elements but no subgroup of order 6.

4.3.1 Class Functions

Let X be a G-set, and choose orbit representatives $x_1 \cdots x_v$.

Then

$$|X| = \sum_{i=1}^{v} |G.x_i|.$$

We can then separately count all orbits with exactly one element, which is exactly

$$X_G = \left\{ x \in G \;\middle|\; g \curvearrowright x = x \;\forall g \in G \right\}$$

.

We then have

$$|X| = |X_G| + \sum_{i=j}^{v} |G.x_i|$$

for some j where $|G.x_i| > 1$ for all $i \geq j$.

Theorem: Let G be a group of order p^n for p a prime.

Then

$$|X| = |X_G| \mod p.$$

Proof: We know that

$$|G.x_i| = [G : G_{x_i}] \text{ for } j \leq i \leq v \text{ and } |Gx_i| > 1 \implies G.x_i \neq G,$$

and thus p divides $[G : Gx_i]$. The result follows.

■

Application: If $|G| = p^n$, then the center $Z(G)$ is nontrivial.

Let $X = G$ act on itself by conjugation, so $g \curvearrowright x = gxg^{-1}$. Then

$$X_G = \left\{ x \in G \;\middle|\; gxg^{-1} = x \right\} = \left\{ x \in G \;\middle|\; gx = xg \right\} = Z(G)$$

But then, by the previous theorem, we have

$$|Z(G)| \equiv |X| \equiv |G| \mod p,$$

but since $Z(G) \leq G$ we have $|Z(G)| \cong 0 \mod p$. So in particular, $Z(G) \neq \{e\}$.

Definition: A group G is a ***p*-group** iff every element in G has order p^k for some k. A subgroup is a p-group exactly when it is a p-group in its own right.

4.3.2 Cauchy's Theorem

Theorem (Cauchy): Let G be a finite group, where p is prime and divides $|G|$. Then G has an element (and thus a subgroup) of order p.

Proof: Consider

$$X = \left\{ (g_1, g_2, \cdots, g_p) \in G^{\oplus p} \;\middle|\; g_1 g_2 \cdots g_p = e \right\}.$$

Given any $p-1$ elements, say $g_1 \cdots g_{p-1}$, the remaining element is completely determined by $g_p = (g_1 \cdots g_{p-1})^{-1}$.

So $|X| = |G|^{p-1}$.and since $p \;\Big|\; |G|$, we have $p \;\Big|\; |X|$.

Now let $\sigma \in S_p$ the symmetric group act on X by index permutation, i.e.

$$\sigma \curvearrowright (g_1, g_2 \cdots g_p) = (g_{\sigma(1)}, g_{\sigma(2)}, \cdots, g_{\sigma(p)}).$$

Exercise: Check that this gives a well-defined group action.

Let $\sigma = (1\ 2\ \cdots\ p) \in S_p$, and note $\langle \sigma \rangle \leq S_p$ also acts on X where $|\langle \sigma \rangle| = p$. Therefore we have

$$|X| = \left| X_{\langle \sigma \rangle} \right| \mod p.$$

Since $p \;\Big|\; |X|$, it follows that $\left| X_{\langle \sigma \rangle} \right| = 0 \mod p$, and thus $p \;\Big|\; \left| X_{\langle \sigma \rangle} \right|$.

If $\langle \sigma \rangle$ fixes $(g_1, g_2, \cdots g_p)$, then $g_1 = g_2 = \cdots g_p$.

Note that $(e, e, \cdots) \in X_{\langle \sigma \rangle}$, as is $(a, a, \cdots a)$ since $p \;\Big|\; \left| X_{\langle \sigma \rangle} \right|$. So there is some $a \in G$ such that $a^p = 1$. Moreover, $\langle a \rangle \leq G$ is a subgroup of size p.

■

4.3.3 Normalizers

Let G be a group and $X = S$ be the set of subgroups of G. Let G act on X by $g \curvearrowright H = gHg^{-1}$. What is the stabilizer?

$$G_x = G_H = \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\},$$

making G_H the largest subgroup such that $H \trianglelefteq G_H$.

So we **define** $N_G(H) \coloneqq G_H$.

Lemma: Let H be a p-subgroup of G of order p^n. Then

$$[N_G(H) : H] = [G : H] \mod p.$$

Proof: Let $S = G/H$ be the set of left H-cosets in G. Now let H act on S by

$$H \curvearrowright x + H \coloneqq (hx) + H.$$

By a previous theorem, $|G/H| = |S| = |S_H| \mod p$, where $|G/H| = [G : H]$. What is S_H? This is given by

$$S_H = \left\{ x + H \in S \;\middle|\; xHx^{-1} \in H \forall h \in H \right\}.$$

Therefore $x \in N_G(H)$.

$\blacksquare$

Corollary: Let $H \leq G$ be a subgroup of order p^n. If $p \;\middle|\; [G : H]$ then $N_G(H) \neq H$.

Proof: Exercise. $\blacksquare$

Theorem: Let G be a finite group, then G is a p-group $\iff |G| = p^n$ for some $n \geq 1$.

Proof: Suppose $|G| = p^n$ and $a \in G$. Then $|\langle a \rangle| = p^\alpha$ for some α.

Conversely, suppose G is a p-group. Factor $|G|$ into primes and suppose $\exists q$ such that $q \;\middle|\; |G|$ but $q \neq p$.

By Cauchy, we can then get a subgroup $\langle c \rangle$ such that $|\langle c \rangle| \;\middle|\; q$, but then $|G| \neq p^n$.

$\blacksquare$

5 | Tuesday August 27th

Let G be a finite group and p a prime. TFAE:

- $|H| = p^n$ for some n
- Every element of H has order p^α for some α.

If either of these are true, we say H is a *p-group*.

Let H be a p-group, last time we proved that if $p \;\middle|\; [G : H]$ then $N_G(H) \neq H$.

5.1 Sylow Theorems

Let G be a finite group and suppose $|G| = p^n m$ where $(m, n) = 1$. Then

5.1.1 Sylow 1

> Idea: take a prime factorization of $|G|$, then there are subgroups of order p^i for *every* prime power appearing, up to the maximal power.

1. G contains a subgroup of order p^i for every $1 \leq i \leq n$.
2. Every subgroup H of order p^i where $i < n$ is a normal subgroup in a subgroup of order p^{i+1}.

Proof: By induction on i. For $i = 1$, we know this by Cauchy's theorem. If we show (2), that shows (1) as a consequence.

So suppose this holds for $i < n$. Let $H \leq G$ where $|H| = p^i$, we now want a subgroup of order p^{i+1}. Since $p \mid [G : H]$, by the previous theorem, $H < N_G(H)$ is a proper subgroup (?).

Now consider the canonical projection $N_G(H) \to N_G(H)/H$. Since

$$p \mid [N_G(H) : H] = |N_G(H)/H|,$$

by Cauchy there is a subgroup of order p in this quotient. Call it K. Then $\pi^{-1}(K) \leq N_G(H)$.

Exercise: Show that $\left|\varphi^{-1}(K)\right| = p^{i+1}$.

It now follows that $H \trianglelefteq \varphi^{-1}(K)$. ■

Definition: For G a finite group and $|G| = p^n m$ where p does not divide m.

Then a subgroup of order p^n is called a **Sylow p-subgroup**.

Note: by Sylow 1, these exist.

5.1.2 Sylow 2

If P_1, P_2 are Sylow p-subgroups of G, then P_1 and P_2 are conjugate.

Proof: Let $\mathcal{L}$ be the left cosets of P_1, i.e. $\mathcal{L} = G/P_1$.

Let P_2 act on $\mathcal{L}$ by

$$p_2 \curvearrowright (g + P_1) := (p_2 g) + P_1.$$

By a previous theorem about orbits and fixed points, we have

$$|\mathcal{L}_{P_2}| = |\mathcal{L}| \mod p.$$

Since p does not divide $|\mathcal{L}|$, we have p does not divide $|\mathcal{L}_{P_2}|$. So $\mathcal{L}_{P_2}$ is nonempty.

So there exists a coset xP_1 such that $xP_1 \in \mathcal{L}_{P_2}$, and thus

$$yxP_1 = xP_1 \text{ for all } y \in P_2.$$

Then $x^{-1}yxP_1 = P_1$ for all $y \in P_2$, and so $x^{-1}P_2x = P_1$. So P_1 and P_2 are conjugate.

■

5.1.3 Sylow 3

Let G be a finite group, and $p \mid |G|$. Let r_p be the number of Sylow p-subgroups of G.

Then

- $r_p \cong 1 \mod p$.
- $r_p \mid |G|$.

- $r_p = [G : N_G(P)]$

Proof:

Let $X = \mathcal{S}$ be the set of Sylow p-subgroups, and let $P \in X$ be a fixed Sylow p-subgroup.

Let $P \curvearrowright \mathcal{S}$ by conjugation, so for $\overline{P} \in \mathcal{S}$ let $x \curvearrowright \overline{P} = x\overline{P}x^{-1}$.

By a previous theorem, we have

$$|\mathcal{S}| = \mathcal{S}_P \mod p$$

What are the fixed points $\mathcal{S}_P$?

$$\mathcal{S}_P = \left\{ T \in \mathcal{S} \;\middle|\; xTx^{-1} = T \quad \forall x \in P \right\}.$$

Let $\mathcal{T} \in \mathcal{S}_P$, so $xTx^{-1} = T$ for all $x \in P$.

Then $P \leq N_G(T)$, so both P and T are Sylow p- subgroups in $N_G(H)$ as well as G.

So there exists a $f \in N_G(T)$ such that $T = gPg^{-1}$. But the point is that in the normalizer, there is only **one** Sylow p- subgroup.

But then T is the unique largest normal subgroup of $N_G(T)$, which forces $T = P$.

Then $\mathcal{S}_P = \{P\}$, and using the formula, we have $r_p \cong 1 \mod p$.

Now modify this slightly by letting G act on $\mathcal{S}$ (instead of just P) by conjugation.

Since all Sylows are conjugate, by Sylow (1) there is only one orbit, so $\mathcal{S} = GP$ for $P \in \mathcal{S}$. But then

$$r_p = |\mathcal{S}| = |GP| = [G : G_p] \;\Big|\; |G|.$$

Note that this gives a precise formula for r_p, although the theorem is just an upper bound of sorts, and $G_p = N_G(P)$.

5.2 Applications of Sylow Theorems

Of interest historically: classifying finite *simple* groups, where a group G is *simple* If $N \trianglelefteq G$ and $N \neq \{e\}$, then $N = G$.

Example: Let $G = \mathbb{Z}_p$, any subgroup would need to have order dividing p, so G must be simple.

Example: $G = A_n$ for $n \geq 5$ (see Galois theory)

One major application is proving that groups of a certain order are *not* simple.

Applications:

Proposition: Let $|G| = p^n q$ with $p > q$. Then G is not simple.

Proof:

> Strategy: Find a proper normal nontrivial subgroup using Sylow theory. Can either show $r_p = 1$, or produce normal subgroups by intersecting distinct Sylow p-subgroups.

Consider r_p, then $r_p = p^\alpha q^\beta$ for some α, β. But since $r_p \cong 1 \mod p$, p does not divide r_p, we must have $r_p = 1, q$.

But since $q < p$ and $q \neq 1 \mod p$, this forces $r_p = 1$.

So let P be a sylow p-subgroup, then $P < G$. Then gPg^{-1} is also a sylow, but there's only 1 of them, so P is normal.

■

Proposition: Let $|G| = 45$, then G is not simple.

Proof: Exercise.

■

Proposition: Let $|G| = p^n$, then G is not simple if $n > 1$.

Proof: By Sylow (1), there is a normal subgroup of order p^{n-1} in G.

■

Proposition: Let $|G| = 48$, then G is not simple.

Proof:

Note $48 = 2^4 3$, so consider r_2, the number of Sylow 2-subgroups. Then $r_2 \cong 1 \mod 2$ and $r_2 \mid 48$. So $r_2 = 1, 3$. If $r_2 = 1$, we're done, otherwise suppose $r_2 = 3$.

Let $H \neq K$ be Sylow 2-subgroups, so $|H| = |K| = 2^4 = 16$. Now consider $H \cap K$, which is a subgroup of G. How big is it?

Since $H \neq K, |H \cap K| < 16$. The order has to divides 16, so we in fact have $|H \cap K| \leq 8$. Suppose it is less than 4, towards a contradiction. Then

$$|HK| = \frac{|H||K|}{|H \cap K|} \geq \frac{(16)(16)}{4} = 64 > |G| = 48.$$

So we can only have $|H \cap K| = 8$. Since this is an index 2 subgroup in both H and K, it is in fact normal. But then

$$H, K \subseteq N_G(H \cap K) \coloneqq X.$$

But then $|X|$ must be a multiple of 16 *and* divide 48, so it's either 16 or 28. But $|X| > 16$, because $H \subseteq X$ and $K \subseteq X$. So then

$$N_G(H \cap K) = G \text{ and so } H \cap K \trianglelefteq G.$$

■

6 | Thursday August 29th

6.1 Classification of Groups of Certain Orders

We have a classification of some finite abelian groups.

Order of G	Number of Groups	List of Distinct Groups
1	1	$\{e\}$
2	1	$\mathbb{Z}_2$
3	1	$\mathbb{Z}_3$
4	2	$\mathbb{Z}_4, \mathbb{Z}_2^2$
5	1	$\mathbb{Z}_5$
6	2	$\mathbb{Z}_6, S_3$ (*)
7	1	$\mathbb{Z}_7$
8	5	$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2^3, D_4, Q$
9	2	$\mathbb{Z}_9, \mathbb{Z}_3^2$
10	2	$\mathbb{Z}_{10}, D_5$
11	1	$\mathbb{Z}_{11}$

Exercise: show that groups of order p^2 are abelian.

We still need to justify S_3, D_4, Q, D_5.

Recall that for any group A, we can consider the free group on the elements of A given by $F[A]$.

Note that we can also restrict A to just its generators.

There is then a homomorphism $F[A] \to A$, where the kernel is the relations.

Example:

$$\mathbb{Z} * \mathbb{Z} = \left\langle x, y \;\middle|\; xyx^{-1}y^{-1} = e \right\rangle \text{ where } x = (1,0),\; y = (0,1).$$

6.2 Groups of Order 6

Let G be nonabelian of order 6.

Idea: look at subgroups of index 2.

Let P be a Sylow 3-subgroup of G, then $r_3 = 1$ so $P \trianglelefteq G$. Moreover, P is cyclic since it is order 3, so $P = \langle a \rangle$.

But since $|G/P| = 2$, it is also cyclic, so $G/P = \langle bP \rangle$.

Note that $b \notin P$, but $b^2 \in P$ since $(bP)^2 = P$, so $b^2 \in \left\{e, a, a^2\right\}$.

If $b = a, a^2$ then b has order 6, but this would make $G = \langle b \rangle$ cyclic and thus abelian. So $b^2 = 1$.

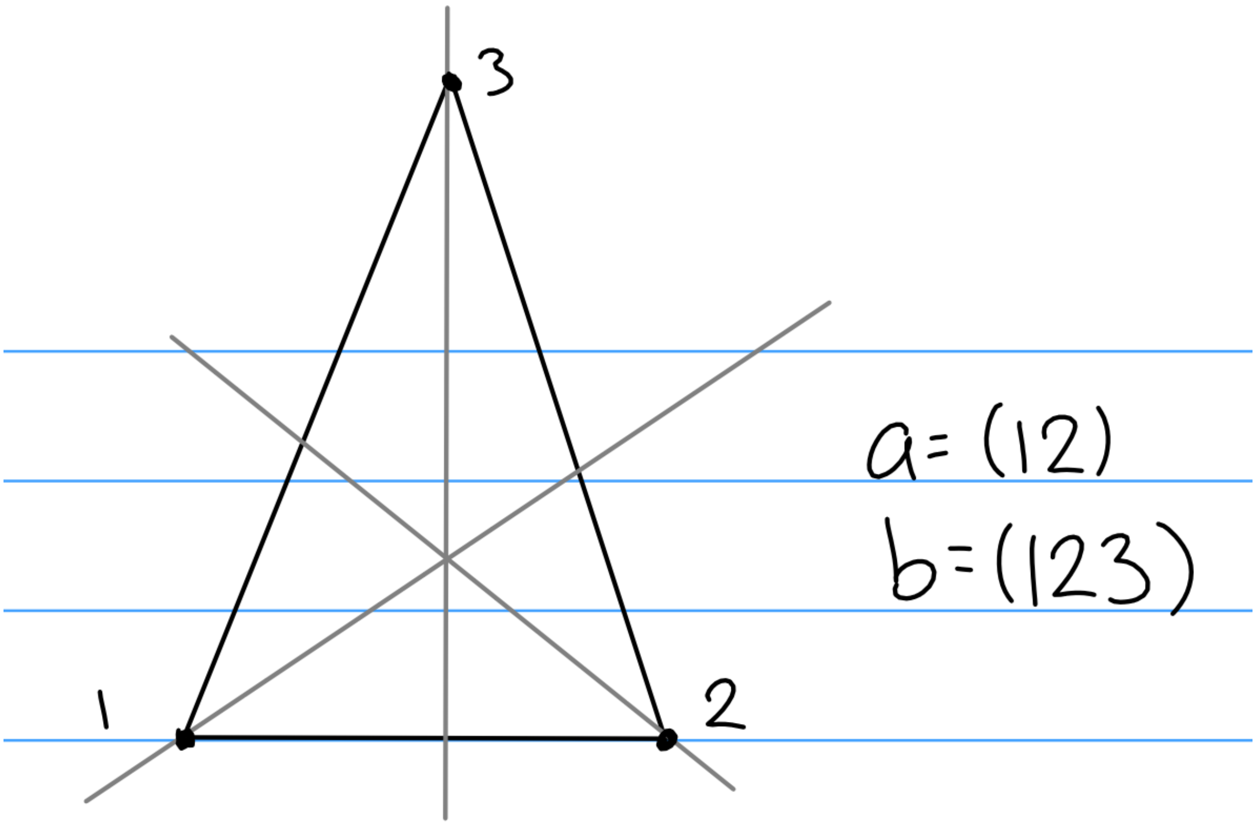
Since $P \trianglelefteq G$, we have $bPb^{-1} = P$, and in particular bab^{-1} has order 3.

So either $bab^{-1} = a$, or $bab^{-1} = a^2$. If $bab^{-1} = a$, then G is abelian, so $bab^{-1} = a^2$. So

$$G = \left\langle a, b \;\middle|\; a^3 = e, b^2 = e, bab^{-1} = a^2 \right\rangle.$$

We've shown that *if* there is such a nonabelian group, then it must satisfy these relations – we still need to produce some group that actually realizes this.

Consider the symmetries of the triangle:



You can check that a, b satisfy the appropriate relations.

6.3 Groups of Order 10

For order 10, a similar argument yields

$$G = \left\langle a, b \;\middle|\; a^5 = 1, b^2 = 1, ba = a^4 b \right\rangle,$$

and this is realized by symmetries of the pentagon where $a = (1\ 2\ 3\ 4\ 5), b = (1\ 4)(2\ 3)$.

6.4 Groups of Order 8

Assume G is nonabelian of order 8. G has no elements of order 8, so the only possibilities for orders of elements are 1, 2, or 4.

Assume all elements have order 1 or 2. Let $a, b \in G$, consider

$$(ab)^2 = abab \implies ab = b^{-1}a^{-1} = ba,$$

and thus G is abelian. So there must be an element of order 4.

So suppose $a \in G$ has order 4, which is an index 2 subgroup, and so $\langle a \rangle \trianglelefteq G$.

But $|G/\langle a\rangle| = 2$ is cyclic, so $G/\langle a\rangle = \langle bH\rangle$.

Note that $b^2 \in H = \langle a\rangle$.

If $b^2 = a, a^3$ then b will have order 8, making G cyclic. So $b^2 = 1, a^2$. These are both valid possibilities.

Since $H \trianglelefteq G$, we have $b\langle a\rangle b^{-1} = \langle a\rangle$, and since a has order 4, so does bab^{-1}.

So $bab^{-1} = a, a^3$, but a is not an option because this would make G abelian.

So we have two options:

$$G_1 = \left\langle a, b \;\middle|\; a^4 = 1, b^2 = 1, bab^{-1} = a^3 \right\rangle$$
$$G_2 = \left\langle a, b \;\middle|\; a^4 = 1, b^2 = a^2, bab^{-1} = a^3 \right\rangle.$$

Exercise: prove $G_1 \not\cong G_2$.

Now to realize these groups:

- G_1 is the group of symmetries of the square, where $a = (1\ 2\ 3\ 4), b = (1\ 3)$.
- $G_2 \cong Q$, the quaternions, where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, and there are relations (add picture here).

6.5 Some Nice Facts

- If and $\varphi : G \to G'$, then
  - $N \trianglelefteq G \implies N \trianglelefteq \varphi(G)$, although it is not necessarily normal in G.
  - $N' \trianglelefteq G' \implies \varphi^{-1}(N') \trianglelefteq G$

Definition: A *maximal normal subgroup* is a normal subgroup $M \trianglelefteq G$ that is properly contained in G, and if $M \leq N \trianglelefteq G$ (where N is proper) then $M = N$.

Theorem: M is a maximal normal subgroup of G iff G/M is simple.

6.6 Simple Groups

Definition: A group G is simple iff $N \trianglelefteq G \implies N = \{e\}, G$.

Note that if an abelian group has *any* subgroups, then it is not simple, so $G = \mathbb{Z}_p$ is the only simple abelian group. Another example of a simple group is A_n for $n \geq 5$.

Theorem (Feit-Thompson, 1964): Every finite nonabelian simple group has even order.

Note that this is a consequence of the "odd order theorem".

6.7 Series of Groups

A composition series is a descending series of pairwise normal subgroups such that each successive quotient is simple:

$$\begin{aligned} G_0 \trianglelefteq G_1 \trianglelefteq G_2 \cdots &\trianglelefteq \{e\} \\ G_i/G_{i+1} &\text{ simple.} \end{aligned}$$

Example:

$$\begin{aligned} \mathbb{Z}_9 \trianglelefteq \mathbb{Z}_3 &\trianglelefteq \{e\} \\ \mathbb{Z}_9/\mathbb{Z}_3 &= \mathbb{Z}_3, \\ \mathbb{Z}_3/\{e\} &= \mathbb{Z}_3. \end{aligned}$$

Example:

$$\begin{aligned} \mathbb{Z}_6 \trianglelefteq \mathbb{Z}_3 &\trianglelefteq \{e\} \\ \mathbb{Z}_6/\mathbb{Z}_3 &= \mathbb{Z}_2 \\ \mathbb{Z}_2/\{e\} &= \mathbb{Z}_2. \end{aligned}$$

but also

$$\begin{aligned} \mathbb{Z}_6 \trianglelefteq \mathbb{Z}_2 &\trianglelefteq \{e\} \\ \mathbb{Z}_6/\mathbb{Z}_2 &= \mathbb{Z}_3 \\ \mathbb{Z}_3/\{e\} &= \mathbb{Z}_3. \end{aligned}$$

Theorem (Jordan-Holder): Any two composition series are "isomorphic" in the sense that the same quotients appear in both series, up to a permutation.

Definition: A group is *solvable* iff it has a composition series where all factors are abelian.

Exercise: Show that any abelian group is solvable.

Example: S_n is *not* solvable for $n \geq 5$, since

$$\begin{aligned} S_n \trianglelefteq A_n &\trianglelefteq \{e\} \\ S_n/A_n &= \mathbb{Z}_2 \text{ simple} \\ A_n/\{e\} &= A_n \text{ simple} \iff n \geq 5. \end{aligned}$$

Example:

$$
\begin{aligned}
S_4 \trianglelefteq A_4 \trianglelefteq G \trianglelefteq \{e\} &\quad \text{where } |H| = 4 \\
S_4/A_4 &= \mathbb{Z}_2 \\
A_4/H &= \mathbb{Z}_3 \\
H/\{e\} &= \{a, b\}?.
\end{aligned}
$$

7 | August 30th

Recall the Sylow theorems:

- p groups exist for *every* p^i dividing $|G|$, and $H(p) \trianglelefteq H(p^2) \trianglelefteq \cdots H(p^n)$.
- All Sylow p-subgroups are conjugate.
- Numerical constraints
  - $r_p \cong 1 \mod p$,
  - $r_p \Bigm| |G|$ and $r_p \Bigm| m$,

7.1 Internal Direct Products

Suppose $H, K \leq G$, and consider the smallest subgroup containing both H and K. Denote this $H \vee K$.

If either H or K is normal in G, then we have $H \vee K = HK$.

There is a "recipe" for proving you have a direct product of groups:

Theorem (Recognizing Direct Products): Let G be a group, $H \trianglelefteq G$ and $K \trianglelefteq G$, and

1. $H \vee K = HK = G$,
2. $H \cap K = \{e\}$.

Then $G \cong H \times K$.

Proof: We first want to show that $hk = kh \;\forall k \in K, h \in H$. We then have

$$hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in K = h(kh^{-1}k^{-1}) \in H \implies hkh^{-1}k^{-1} \in H \cap K = \{e\}\,.$$

So define

$$
\begin{aligned}
\varphi : H \times K &\to G \\
(h, k) &\mapsto hk,
\end{aligned}
$$

Exercise: check that this is a homomorphism, it is surjective, and injective.

■

Applications:

Theorem: Every group of order p^2 is abelian.

Proof: If G is cyclic, then it is abelian and $G \cong \mathbb{Z}_{p^2}$. So suppose otherwise. By Cauchy, there is an element of order p in G. So let $H = \langle a \rangle$, for which we have $|H| = p$.

Then $H \trianglelefteq G$ by Sylow 1, since it's normal in $H(p^2)$, which would have to equal G.

Now consider $b \notin H$. By Lagrange, we must have $o(b) = 1, p$, and since $e \in H$, we must have $o(b) = p$. This uses fact that G is not cyclic.

Now let $K = \langle b \rangle$. Then $|K| = p$, and $K \trianglelefteq G$ by the same argument.

■

Theorem: Let $|G| = pq$ where $q \neq 1 \mod p$ and $p < q$. Then G is cyclic (and thus abelian).

Proof: Use Sylow 1. Let P be a sylow p-subgroup. We want to show that $P \trianglelefteq G$ to apply our direct product lemma, so it suffices to show $r_p = 1$.

We know $r_p = 1 \mod p$ and $r_p \,\Big|\, |G| = pq$, and so $r_p = 1, q$. It can't be q because $p < q$.

Now let Q be a sylow q-subgroup. Then $r_q \cong 1 \mod 1$ and $r_q \,\Big|\, pq$, so $r_q = 1, q$. But since $p < q$, we must have $r_q = 1$. So $Q \trianglelefteq G$ as well.

We now have $P \cap Q = \emptyset$ (why?) and

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = |P||Q| = pq,$$

and so $G = PQ$, and $G \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$.

■

Example: Every group of order $15 = 5^1 3^1$ is cyclic.

7.2 Determination of groups of a given order

Order of G	Number of Groups	List of Distinct Groups
1	1	$\{e\}$
2	1	$\mathbb{Z}_2$
3	1	$\mathbb{Z}_3$
4	2	$\mathbb{Z}_4, \mathbb{Z}_2^2$
5	1	$\mathbb{Z}_5$
6	2	$\mathbb{Z}_6, S_3$ (*)
7	1	$\mathbb{Z}_7$
8	5	$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2^3, D_8, Q$
9	2	$\mathbb{Z}_9, \mathbb{Z}_3^2$
10	2	$\mathbb{Z}_{10}, D_5$
11	1	$\mathbb{Z}_{11}$

We still need to justify 6, 8, and 10.

7.3 Free Groups

Define an *alphabet* $A = \{a_1, a_2, \cdots a_n\}$, and let a *syllable* be of the form a_i^m for some m. A *word* is any expression of the form $\prod_{n_i} a_{n_i}^{m_i}$.

We have two operations,

- Concatenation, i.e. $(a_1a_2) \star (a_3^2a_5) = a_1a_2a_3^2a_5$.
- Contraction, i.e. $(a_1a_2^2) \star (a_2^{-1}a_5) = a_1a_2^2a_2^{-1}a_5 = a_1a_2a_5$.

If we've contracted a word as much as possible, we say it is *reduced*.

We let $F[A]$ be the set of reduced words and define a binary operation

$$\begin{aligned} f : F[A] \times F[A] &\to F[A] \\ (w_1, w_2) &\mapsto w_1w_2 \text{ (reduced) }. \end{aligned}$$

Theorem: (A, f) is a group.

Proof: Exercise.

■

Definition: $F[A]$ is called the **free group generated by** A. A group G is called *free* on a subset $A \subseteq G$ iff $G \cong F[A]$.

Examples:

1. $A = \{x\} \implies F[A] = \left\{x^n \;\middle|\; n \in \mathbb{Z}\right\} \cong \mathbb{Z}$.
2. $A = \{x, y\} \implies F[A] = \mathbb{Z} * \mathbb{Z}$ (not defined yet!).

Note that there are not relations, i.e. $xyxyxy$ is *reduced*. To abelianize, we'd need to introduce the relation $xy = yx$.

Properties:

1. If G is free on A and free on B then we must have $|A| = |B|$.
2. Any (nontrivial) subgroup of a free group is free.

(See Fraleigh or Hungerford for possible Algebraic proofs!)

Theorem: Let G be generated by some (possibly infinite) subset $A = \left\{A_i \;\middle|\; i \in I\right\}$ and G' be generated by some $A_i' \subseteq A_i$.

Then

a. There is at most one homomorphism $a_i \to a_i'$.

b. If $G \cong F[A]$, there is exactly *one* homomorphism.

Corollary: Every group G' is a homomorphic image of a free group.

Proof: Let A be the generators of G' and $G = F[A]$, then define

$$\begin{aligned}\varphi : F[A] &\to G' \\ a_i &\mapsto a_i.\end{aligned}$$

This is onto exactly because $G' = \langle a_i \rangle$, and using the theorem above we're done.

■

7.4 Generators and Relations

Let G be a group and $A \subseteq G$ be a generating subset so $G = \left\langle a \;\middle|\; a \in A \right\rangle$. There exists a $\varphi : F[A] \twoheadrightarrow G$, and by the first isomorphism theorem, we have $F[A]/\ker\varphi \cong G$.

Let $R = \ker\varphi$, these provide the *relations*.

Examples:

Let $G = \mathbb{Z}_3 = \langle [1]_3 \rangle$. Let $x = [1]_3$, then define $\varphi : F[\{x\}] \twoheadrightarrow \mathbb{Z}_3$.

Then since $[1] + [1] + [1] = [0] \mod 3$, we have $\ker\varphi = \left\langle x^3 \right\rangle$.

Let $G = \mathbb{Z} \oplus \mathbb{Z}$, then $G \cong \left\langle x, y \;\middle|\; [x, y] = 1 \right\rangle$.

> We'll use this for groups of order 6 – there will be only one presentation that is nonabelian, and we'll exhibit such a group.

8 | September 9th

8.1 Series of Groups

Recall that a *simple* group has no nontrivial normal subgroups.

Example:

$$\begin{aligned}\mathbb{Z}_6 \trianglelefteq \langle [3] \rangle &\trianglelefteq \langle [0] \rangle \\ \mathbb{Z}_6 / \langle [3] \rangle &= \mathbb{Z}_3 \\ \langle [3] \rangle / \langle [0] \rangle &= \mathbb{Z}_2.\end{aligned}$$

Definition: A *normal series* (or an *invariant series*) of a group G is a finite sequence $H_i \le G$ such that $H_i \trianglelefteq H_{i+1}$ and $H_n = G$, so we obtain

$$H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

Definition: A normal series $\{K_i\}$ is a **refinement** of $\{H_i\}$ if $K_i \leq H_i$ for each i.

Definition: We say two normal series of the same group G are *isomorphic* if there is a bijection from

$$\{H_i/H_{i+1}\} \iff \{K_j/K_{j+1}\}$$

Theorem (Schreier): Any two normal series of G has isomorphic refinements.

Definition: A normal series of G is a **composition series** iff all of the successive quotients H_i/H_{i+1} are **simple**.

Note that every finite group has a composition series, because any group is a maximal normal subgroup of itself.

Theorem (Jordan-Holder): Any two composition series of a group G are isomorphic.

Proof: Apply Schreier's refinement theorem.

■

Example: Consider $S_n \trianglelefteq A_n \trianglelefteq \{e\}$. This is a composition series, with quotients Z_2, A_n, which are both simple.

Definition: A group G is **solvable** iff it has a composition series in which all of the successive quotients are **abelian**.

Examples:

- Any abelian group is solvable.
- S_n is not solvable for $n \geq 5$, since A_n is not abelian for $n \geq 5$.

Recall Feit-Thompson: Any nonabelian simple group is of *even* order.

Consequence: Every group of *odd* order is solvable.

8.2 The Commutator Subgroup

Let G be a group, and let $[G, G] \leq G$ be the subgroup of G generated by elements $aba^{-1}b^{-1}$, i.e. every element is a *product* of commutators. So $[G, G]$ is called *the commutator subgroup.*

Theorem: Let G be a group, then

1. $[G, G] \leq G$
2. $[G, G]$ is a normal subgroup
3. $G/[G, G]$ is abelian.
4. $[G, G]$ is the smallest normal subgroup such that the quotient is abelian,

    I.e., $H \trianglelefteq G$ and if G/N is abelian $\implies [G, G] \leq N$.

Proof of 1:

$[G, G]$ is a subgroup:

- Closure is clear from definition as generators.

- The identity is $e = ee^{-1}ee^{-1}$.
- So it suffices to show that $(aba^{-1}b^{-1})^{-1} \in [G, G]$, but this is given by $bab^{-1}a^{-1}$ which is of the correct form.

■

Proof of 2:

$[G, G]$ is normal.

Let $x_i \in [G, G]$, then we want to show $g \prod x_i g^{-1} \in [G, G]$, but this reduces to just showing $gxg^{-1} \in [G, G]$ for a single $x \in [G, G]$.

Then,

$$\begin{aligned} g(aba^{-1}b^{-1})g^{-1} &= (g^{-1}aba^{-1})e(b^{-1}g) \\ &= (g^{-1}aba^{-1})(gb^{-1}bg^{-1})(b^{-1}g) \\ &= [(g^{-1}a)b(g^{-1}a)^{-1}b^{-1}][bg^{-1}b^{-1}g] \\ &\in [G, G]. \end{aligned}$$

■

Proof of 3:

$G/[G, G]$ is abelian.

Let $H = [G, G]$. We have $aHbH = (ab)H$ and $bHaH = (ba)H$.

But $abH = baH$ because $(ba)^{-1}(ab) = a^{-1}b^{-1}ab \in [G, G]$.

■

Proof of 4:

$H \trianglelefteq G$ and if G/N is abelian $\implies [G, G] \leq N$.

Suppose G/N is abelian. Let $aba^{-1}b^{-1} \in [G, G]$.

Then $abN = baN$, so $aba^{-1}b^{-1} \in N$ and thus $[G, G] \subseteq N$.

■

8.3 Free Abelian Groups

Example: $\mathbb{Z} \times \mathbb{Z}$.

Take $e_1 = (1, 0), e_2 = (0, 1)$. Then $(x, y) \in \mathbb{Z}^2$ can be written $x(1, 0) + y(0, 1)$, so $\{e_i\}$ behaves like a basis for a vector space.

Definition: A group G is *free abelian* if there is a subset $X \subseteq G$ such that every $g \in G$ can be represented as

$$g = \sum_{i=1}^{r} n_i x_i, \quad x_i \in X,\ n_i \in \mathbb{Z}.$$

Equivalently, X generates G, so $G = \langle X \rangle$, and if $\sum n_i x_i = 0 \implies n_i = 0 \; \forall i$.

If this is the case, we say X is a **basis** for G.

Examples:

- $\mathbb{Z}^n$ is free abelian
- $\mathbb{Z}_n$ is not free abelian, since $n[1] = 0$ and $n \neq 0$.

> In general, you can replace $\mathbb{Z}_n$ by any finite group and replace n with the order of the group.

Theorem: If G is free abelian on X where $|X| = r$, then $G \cong \mathbb{Z}^r$.

Theorem: If $X = \{x_i\}_{i=1}^r$, then a basis for $\mathbb{Z}^r$ is given by

$$\{(1,0,0,\cdots),(0,1,0,\cdots),\cdots,(0,\cdots,0,1)\} \coloneqq \{e_1, e_2, \cdots, e_r\}$$

Proof: Use the map $\varphi : G \to \mathbb{Z}^r$ where $x_i \mapsto e_i$, and check that this is an isomorphism of groups.

Theorem: Let G be free abelian with two bases X, X', then $|X| = |X|'$.

Definition: Let G be free abelian, then if X is a basis then $|X|$ is called the *rank* of of G.

9 | Thursday September 5th

9.1 Rings

Recall the definition of a ring: A *ring* $(R, +, \times)$ is a set with binary operations such that

1. $(R, +)$ is a group,
2. $(R, \times)$ is a monoid.

Examples: $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or the ring of $n \times n$ matrices, or $\mathbb{Z}_n$.

A ring is *commutative* iff $ab = ba$ for every $a, b \in R$, and *a ring with unity* is a ring such that $\exists 1 \in R$ such that $a1 = 1a = a$.

Exercise: Show that 1 is unique if it exists.

In a ring with unity, an element $a \in R$ is a *unit* iff $\exists b \in R$ such that $ab = ba = 1$.

Definition: A ring with unity is a **division ring** $\iff$ every nonzero element is a unit.

Definition: A division ring is a *field* $\iff$ it is commutative.

Definition: Suppose that $a, b \neq 0$ with $ab = 0$. Then a, b are said to be *zero divisors*.

Definition: A commutative ring without zero divisors is an *integral domain*.

Example: In $\mathbb{Z}_n$, an element a is a zero divisor iff $\gcd(a, n) \neq 1$.

Fact: In a ring with no zero divisors, we have

$$ab = ac \text{ and } a \neq 0 \implies b = c.$$

Theorem: Every field is an integral domain.

Proof: Let R be a field. If $ab = 0$ and $a \neq 0$, then a^{-1} exists and so $b = 0$.

$\blacksquare$

Theorem: Any finite integral domain is a field.

Proof:

Idea: Similar to the pigeonhole principle.

Let $D = \{0, 1, a_1, \cdots, a_n\}$ be an integral domain. Let $a_j \neq 0, 1$ be arbitrary, and consider $a_j D = \left\{ a_j x \mathrel{\Big|} x \in D \setminus \{0\} \right\}$.

Then $a_j D = D \setminus \{0\}$ as sets. But

$$a_j D = \{a_j, a_j a_1, a_j a_2, \cdots, a_j a_n\}.$$

Since there are no zero divisors, 0 does not occur among these elements, so some $a_j a_k$ must be equal to 1.

$\blacksquare$

9.2 Field Extensions

If $F \leq E$ are fields, then E is a vector space over F, for which the dimension turns out to be important.

Definition: We can consider

$$\mathrm{Aut}(E/F) \coloneqq \left\{ \sigma : E \circlearrowleft \mathrel{\Big|} f \in F \implies \sigma(f) = f \right\},$$

i.e. the field automorphisms of E that fix F.

Examples of field extensions: $\mathbb{C} \to \mathbb{R} \to \mathbb{Q}$.

Let $F(x)$ be the smallest field containing both F and x. Given this, we can form a diagram

$$\begin{tikzcd}
& F(x,y) & \\
F(x) \arrow[ur] & & F(y) \arrow[ul] \\
& F \arrow[ul] \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Let $F[x]$ the polynomials with coefficients in F.

Theorem: Let F be a field and $f(x) \in F[x]$ be a non-constant polynomial. Then there exists an $F \to E$ and some $\alpha \in E$ such that $f(\alpha) = 0$.

Proof: Since $F[x]$ is a unique factorization domain, given $f(x)$ we can find an irreducible $p(x)$ such that $f(x) = p(x)g(x)$ for some $g(x)$. So consider $E = F[x]/(p)$.

Since p is irreducible, (p) is a prime ideal, but in $F[x]$ prime ideals are maximal and so E is a field.

Then define

$$
\begin{aligned}
\psi : F &\to E \\
a &\mapsto a + (p).
\end{aligned}
$$

Then ψ is a homomorphism of rings: supposing $\psi(\alpha) = 0$, we must have $\alpha \in (p)$. But all such elements are multiples of a polynomial of degree $d \geq 1$, and α is a scalar, so this can only happen if $\alpha = 0$.

Then consider $\alpha = x + (p)$; the claim is that $p(\alpha) = 0$ and thus $f(\alpha) = 0$. We can compute

$$
\begin{aligned}
p(x + (p)) &= a_0 + a_1(x + (p)) + \cdots + a_n(x + (p))^n \\
&= p(x) + (p) = 0.
\end{aligned}
$$

■

Example: $\mathbb{R}[x]/(x^2 + 1)$ over $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a field.

9.3 Algebraic and Transcendental Elements

Definition: An element $\alpha \in E$ with $F \to E$ is **algebraic** over F iff there is a nonzero polynomial in $f \in F[x]$ such that $f(\alpha) = 0$.

Otherwise, α is said to be **transcendental**.

Examples:

- $\sqrt{2} \in \mathbb{R} \leftarrow \mathbb{Q}$ is algebraic, since it satisfies $x^2 - 2$.
- $\sqrt{-1} \in \mathbb{C} \leftarrow \mathbb{Q}$ is algebraic, since it satisfies $x^2 + 1$.
- $\pi, e \in \mathbb{R} \leftarrow \mathbb{Q}$ are transcendental

  This takes some work to show.

An *algebraic number* $\alpha \in \mathbb{C}$ is an element that is algebraic over $\mathbb{Q}$.

Fact: The set of algebraic numbers forms a field.

Definition: Let $F \leq E$ be a field extension and $\alpha \in E$. Define a map

$$
\begin{aligned}
\varphi_\alpha : F[x] &\to E \\
\varphi_\alpha(f) &= f(\alpha).
\end{aligned}
$$

This is a homomorphism of rings and referred to as the *evaluation homomorphism*.

Theorem: Then φ_α is injective iff α is transcendental.

> Note: otherwise, this map will have a kernel, which will be generated by a single element that is referred to as the **minimal polynomial** of α.

9.4 Minimal Polynomials

Theorem: Let $F \leq E$ be a field extension and $\alpha \in E$ algebraic over F. Then

1. There exists a polynomial $p \in F[x]$ of minimal degree such that $p(\alpha) = 0$.
2. p is irreducible.
3. p is unique up to a constant.

Proof:

Since α is algebraic, $f(\alpha) = 0$. So write f in terms of its irreducible factors, so $f(x) = \prod p_j(x)$ with each p_j irreducible. Then $p_i(\alpha) = 0$ for some i because we are in a field and thus don't have zero divisors.

So there exists at least one $p_i(x)$ such that $p(\alpha) = 0$, so let q be one such polynomial of minimal degree.

Suppose that $\deg q < \deg p_i$. Using the Euclidean algorithm, we can write $p(x) = q(x)c(x) + r(x)$ for some c, and some r where $\deg r < \deg q$.

But then $0 = p(\alpha) = q(\alpha)c(\alpha) + r(\alpha)$, but if $q(\alpha) = 0$, then $r(\alpha) = 0$. So $r(x)$ is identically zero, and so $p(x) - q(x) = c(x) = c$, a constant.

$\blacksquare$

Definition: Let $\alpha \in E$ be algebraic over F, then the unique monic polynomial $p \in F[x]$ of minimal degree such that $p(\alpha) = 0$ is the **minimal polynomial** of α.

Example: $\sqrt{1+\sqrt{2}}$ has minimal polynomial $x^4 + x^2 - 1$, which can be found by raising it to the 2nd and 4th power and finding a linear combination that is constant.

10 | Tuesday September 10th

10.1 Vector Spaces

Definition: Let $\mathbb{F}$ be a field. A **vector space** is an abelian group V with a map $\mathbb{F} \times V \to V$ such that

- $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
- $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$,
- $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
- $1\mathbf{v} = \mathbf{v}$

Examples: $\mathbb{R}^n, \mathbb{C}^n, F[x] = \operatorname{span}(\left\{1, x, x^2, \cdots\right\}), L^2(\mathbb{R})$

Definition: Let V be a vector space over $\mathbb{F}$; then a set $W \subseteq V$ *spans* V iff for every $\mathbf{v} \in V$, one can write $\mathbf{v} = \sum \alpha_i \mathbf{w}_i$ where $\alpha_i \in \mathbb{F},\ \mathbf{w}_i \in W$.

Definition: V is *finite dimensional* if there exists a finite spanning set.

Definition: A set $W \subseteq V$ is *linearly independent* iff

$$\sum \alpha_i \mathbf{w}_i = \mathbf{0} \implies \alpha_i = 0 \text{ for all } i.$$

Definition: A *basis* for V is a set $W \subseteq V$ such that

1. W is linearly independent, and
2. W spans V.

A basis is a midpoint between a spanning set and a linearly independent set.

We can add vectors to a set until it is spanning, and we can throw out vectors until the remaining set is linearly independent. This is encapsulated in the following theorems:

Theorem: If W spans V, then some subset of W spans V.

Theorem: If W is a set of linearly independent vectors, then some superset of W is a basis for V.

Fact: Any finite-dimensional vector spaces has a finite basis.

Theorem: If W is a linearly independent set and B is a basis, then $|B| \leq |W|$.

Corollary: Any two bases have the same number of elements.

> So we define the dimension of V to be the number of elements in any basis, which is a unique number.

10.2 Algebraic Extensions

Definition: $E \geq F$ is an algebraic extension iff every $\alpha \in E$ is algebraic of F.

Definition: $E \geq F$ is a *finite extension* iff E is finite-dimensional as an F-vector space.

Notation: $[E : F] = \dim_F E$, the dimension of E as an F-vector space.

Observation: If $E = F(\alpha)$ where α is algebraic over F, then E is an algebraic extension of F.

Observation: If $E \geq F$ and $[E : F] = 1$, then $E = F$.

Theorem: If $E \geq F$ is a finite extension, then E is algebraic over F.

Proof: Let $\beta \in E$. Then the set $\left\{1, \beta, \beta^2, \cdots\right\}$ is not linearly independent. So $\sum_{i=0}^{n} c_i \beta^i = 0$ for some n and some c_i. But then β is algebraic. ■

> Note that the converse is not true in general. *Example*: Let $E = \overline{\mathbb{R}}$ be the algebraic numbers. Then $E \geq \mathbb{Q}$ is algebraic, but $[E : \mathbb{Q}] = \infty$.

Theorem: Let $K \geq E \geq F$, then $[K : F] = [K : E][E : F]$.

Proof: Let $\{\alpha_i\}^m$ be a basis for E/F Let $\{\beta_i\}^n$ be a basis for K/E. Then the RHS is mn.

Claim: $\{\alpha_i \beta_j\}^{m,n}$ is a basis for K/F.

Linear independence:

$$
\begin{aligned}
&\sum_{i,j} c_{ij}\alpha_i\beta_j = 0 \\
\implies &\sum_j \sum_i c_{ij}\alpha_i\beta_j = 0 \\
\implies &\sum_i c_{ij}\alpha_i = 0 \quad \text{since } \beta \text{ form a basis} \\
\implies &\sum c_{ij} = 0 \quad \text{since } \alpha \text{ form a basis.}
\end{aligned}
$$

Exercise: Show this is also a spanning set.

■

Corollary: Let $E_r \geq E_{r-1} \geq \cdots \geq E_1 \geq F$, then

$$[E_r : F] = [E_r : E_{r-1}][E_{r-1} : E_{r-2}]\cdots[E_2 : E_1][E_1 : F].$$

Observation: If $\alpha \in E \geq F$ and α is algebraic over F where $E \geq F(\alpha) \geq F$, then $F(\alpha)$ is algebraic (since $[F(\alpha) : F] < \infty$) and $[F(\alpha) : F]$ is the degree of the minimal polynomial of α over F.

Corollary: Let $E = F(\alpha) \geq F$ where α is algebraic. Then

$$\beta \in F(\alpha) \implies \deg\min(\beta, F) \;\Big|\; \deg\min(\alpha, F).$$

Proof: Since $F(\alpha) \geq F(\beta) \geq F$, we have $[F(\alpha) : F] = [F(\alpha) : F(\beta)][F(\beta) : F]$. But just note that

$$
\begin{aligned}
&[F(\alpha) : F] = \deg\min(\alpha, F) \text{ and} \\
&[F(\beta) : F] = \deg\min(\beta, F).
\end{aligned}
$$

■

Theorem: Let $E \geq F$ be algebraic, then

$$[E : F] < \infty \iff E = F(\alpha_1, \cdots, \alpha_n) \text{ for some } \alpha_n \in E.$$

10.3 Algebraic Closures

Definition: Let $E \geq F$, and define

$$\overline{F_E} = \left\{ \alpha \in E \;\middle|\; \alpha \text{ is algebraic over } F \right\}$$

to be the **algebraic closure of F in E.**

Example: $\mathbb{Q} \hookrightarrow \mathbb{C}$, while $\overline{\mathbb{Q}} = \mathbb{A}$ is the field of algebraic numbers, which is a dense subfield of $\mathbb{C}$.

Proposition: $\overline{F_E}$ is a always field.

Proof: Let $\alpha, \beta \in \overline{F_E}$, so $[F(\alpha,\beta) : F] < \infty$. Then $F(\alpha, \beta) \subseteq \overline{F_E}$ is algebraic over F and

$$\alpha \pm \beta, \quad \alpha\beta, \quad \frac{\alpha}{\beta} \quad \in F(\alpha, \beta).$$

So $\overline{F_E}$ is a subfield of E and thus a field.

Definition: A field F is **algebraically closed** iff every non-constant polynomial in $F[x]$ is a root in F. Equivalently, every polynomial in $F[x]$ can be factored into linear factors.

If F is algebraically closed and $E \geq F$ and E is algebraic, then $E = F$.

10.3.1 The Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra): $\mathbb{C}$ is an algebraically closed field.

Proof:

Liouville's theorem: A bounded entire function $f : \mathbb{C} \circlearrowleft$ is constant.

- *Bounded* means $\exists M \mid z \in \mathbb{C} \implies |f(z)| \leq M$.
- *Entire* means analytic everywhere.

Let $f(z) \in \mathbb{C}[z]$ be a polynomial without a zero which is non-constant.

Then $\dfrac{1}{f(z)} : \mathbb{C} \circlearrowleft$ is analytic and bounded, and thus constant, and contradiction.

■

10.4 Geometric Constructions:

Given the tools of a straightedge and compass, what real numbers can be constructed? Let $\mathcal{C}$ be the set of such numbers.

Theorem: C is a subfield of $\mathbb{R}$.

11 | Thursday September 12th

11.1 Geometric Constructions

Definition: A real number α is said to be **constructible** iff $|\alpha|$ is constructible using a ruler and compass. Let $\mathcal{C}$ be the set of constructible numbers.

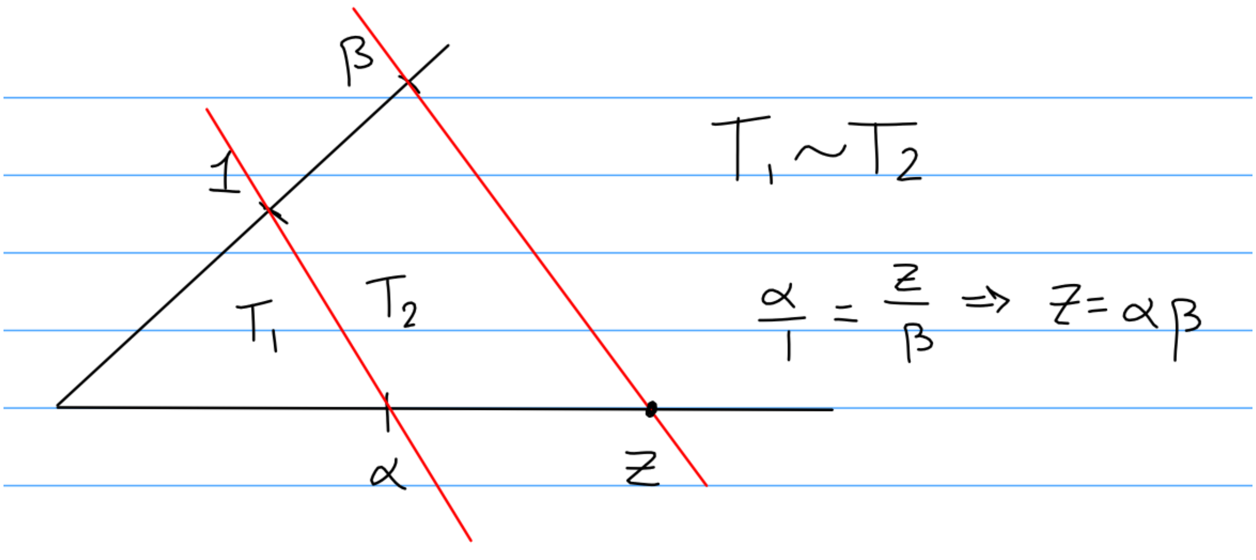
Note that ± 1 is constructible, and thus so is $\mathbb{Z}$.

Theorem: $\mathcal{C}$ is a field.

Proof: It suffices to construct $\alpha \pm \beta,\ \alpha\beta,\ \alpha/\beta$.

Showing $\pm$ and inverses: Relatively easy.

Showing closure under products:



Corollary: $\mathbb{Q} \leq \mathcal{C}$ is a subfield.

Can we get all of $\mathbb{R}$ with $\mathcal{C}$? The operations we have are

1. Intersect 2 lines (gives nothing new)
2. Intersect a line and a circle
3. Intersect 2 circles

Operation (3) reduces to (2) by subtracting two equations of a circle $(x^2 + y^2 + ax + by + c)$ to get an equation of a line.

Operation (2) reduces to solving quadratic equations.

Theorem: $\mathcal{C}$ contains precisely the real numbers obtained by adjoining finitely many square roots of elements in $\mathbb{Q}$.

Proof: Need to show that $\alpha \in \mathcal{C} \implies \sqrt{\alpha} \in \mathcal{C}$.

- Bisect PA to get B.
- Draw a circle centered at B.
- Let Q be intersection of circle with y axis and O be the origin.
- Note triangles 1 and 2 are similar, so

$$\frac{OQ}{OA} = \frac{PO}{OQ} \implies (OQ)^2 = (PO)(OA) = 1\alpha.$$

■

Corollary: Let $\gamma \in \mathcal{C}$ be constructible. Then there exist $\{\alpha_i\}_{i=1}^n$ such that

$$\gamma = \prod_{i=1}^n \alpha_i \quad \text{and} \quad [\mathbb{Q}(\alpha_1, \cdots, \alpha_j) : \mathbb{Q}(\alpha_1, \cdots, \alpha_{j-1})] = 2,$$

and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^d$ for some d.

Applications:

Doubling the cube: Given a cube of size 1, can we construct one of size 2? To do this, we'd need $x^3 = 2$. But note that $\min(\sqrt[3]{2}, \mathbb{Q}) = x^3 - 2 = f(x)$ is irreducible over $\mathbb{Q}$. So $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \neq 2^d$ for any d, so this can not be constructible.

Trisections of angles: We want to construct regular polygons, so we'll need to construct angles. We can get some by bisecting known angles, but can we get all of them?

Example: Attempt to construct 20° by trisecting the known angle 60°, which is constructible using a triangle of side lengths $1, 2, \sqrt{3}$.

If 20° were constructible, $\cos 20^\circ$ would be as well. There is an identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Letting $\theta = 20^\circ$ so $3\theta = 60^\circ$, we obtain

$$\frac{1}{2} = 4(\cos 20^\circ)^3 - 3\cos 20^\circ,$$

so if we let $x = \cos 20^\circ$ then x satisfies the polynomial $f(x) = 8x^3 - 6x - 1$, which is irreducible. But then $[\mathbb{Q}(20^\circ) : \mathbb{Q}] = 3 \neq 2^d$, so $\cos 20^\circ \notin \mathcal{C}$.

11.2 Finite Fields

Definition: The *characteristic* of F is the smallest $n \geq 0$ such that $n1 = 0$, or 0 if such an n does not exist.

Exercise: For a field F, show that char $F = 0$ or p a prime.

Note that if char $F = 0$, then $\mathbb{Z} \in F$ since $1,\ 1+1,\ 1+1+1, \cdots$ are all in F. Since inverses must also exist in F, we must have $\mathbb{Q} \in F$ as well. So char $F = 0 \iff F$ is infinite.

If char $F = p$, it follows that $\mathbb{Z}_p \subset F$.

Theorem:

$$\text{For } E \geq F \text{ where } [E : F] = n \text{ and } F \text{ finite}, \quad |F| = q \implies |E| = q^n.$$

Proof: E is a vector space over F. Let $\{v_i\}^n$ be a basis. Then $\alpha \in E \implies \alpha = \sum_{i=1}^n a_i v_i$ where each $a_i \in F$. There are q choices for each a_i, and n coefficients, yielding q^n distinct elements.

■

Corollary: Let E be a finite field where char $E = p$. Then $|E| = p^n$ for some n.

Theorem: Let $\mathbb{Z}_p \leq E$ with $|E| = p^n$. If $\alpha \in E$, then α satisfies

$$x^{p^n} - x \in \mathbb{Z}_p[x].$$

Proof: If $\alpha = 0$, we're done. So suppose $\alpha \neq 0$, then $\alpha \in E^\times$, which is a group of order $p^n - 1$. So $\alpha^{p^n-1} = 1$, and thus $\alpha\alpha^{p^n-1} = \alpha 1 \implies \alpha^{p^n} = \alpha$.

■

Definition: $\alpha \in F$ is an *nth root of unity* iff $\alpha^n = 1$. It is a *primitive* root of unity of n iff $k \leq n \implies \alpha^k \neq 1$ (so n is the smallest power for which this holds).

Fact: If F is a finite field, then $F^\times$ is a cyclic group.

Corollary: If $E \geq F$ with $[E : F] = n$, then $E = F(\alpha)$ for just a single element α.

Proof: Choose $\alpha \in E^\times$ such that $\langle \alpha \rangle = E^\times$. Then $E = F(\alpha)$.

■

Next time: Showing the existence of a field with p^n elements.

For now: derivatives.

Let $f(x) \in F[x]$ by a polynomial with a multiple zero $\alpha \in E$ for some $E \geq F$.

If it has multiplicity $m \geq 2$, then note that

$$f(x) = (x - \alpha)^m g(x) \implies f'(x) m(x - \alpha)^{m-1} g(x) + g'(x)(x - \alpha)^m \implies f'(\alpha) = 0.$$

So

$$\alpha \text{ a multiple zero of } f \implies f'(\alpha) = 0.$$

The converse is also useful.

Application: Let $f(x) = x^{p^n} - x$, then $f'(x) = p^n x^{p^n-1} - 1 = -1 \neq 0$, so all of the roots are distinct.

12 | Tuesday September 17th

12.1 Finite Fields and Roots of Polynomials

Recall from last time:

Let $\mathbb{F}$ be a finite field. Then $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ is *cyclic* (this requires some proof).

Let $f \in \mathbb{F}[x]$ with $f(\alpha) = 0$. Then α is a *multiple root* if $f'(\alpha) = 0$.

Lemma: Let $\mathbb{F}$ be a finite field with characteristic $p > 0$. Then

$$f(x) = x^{p^n} - x \in \mathbb{F}[x]$$

has p^n distinct roots.

Proof:

$$f'(x) = p^n x^{p^n - 1} - 1 = -1,$$

since we are in char p.

This is identically -1, so $f'(x) \neq 0$ for any x. So there are no multiple roots. Since there are at most p^n roots, this gives exactly p^n distinct roots.

■

Theorem: A field with p^n elements exists (denoted $\mathbb{GF}(p^n)$) for every prime p and every $n > 0$.

Proof: Consider $\mathbb{Z}_p \subseteq K \subseteq \overline{\mathbb{Z}}_p$ where K is the set of zeros of $x^{p^n} - x$. Then we claim K is a field.

Suppose $\alpha, \beta \in K$. Then $(\alpha \pm \beta)^{p^n} = \alpha^{p^n} \pm \beta^{p^n}$.

We also have

$$(\alpha\beta)^{p^n} = \alpha^{p^n} \beta^{p^n} - \alpha\beta \text{ and } \alpha^{-p^n} = \alpha^{-1}.$$

So K is a field and $|K| = p^n$.

■

Corollary: Let F be a finite field. If $n \in \mathbb{N}^+$, then there exists an $f(x) \in F[x]$ that is irreducible of degree n.

Proof: Let F be a finite field, so $|F| = p^r$. By the previous lemma, there exists a K such that $\mathbb{Z}_p \subseteq k \subseteq \overline{F}$.

K is defined as

$$K := \left\{ \alpha \in F \;\middle|\; \alpha^{p^n} - \alpha = 0 \right\}.$$

We also have

$$F = \left\{ \alpha \in \overline{F} \;\middle|\; \alpha^{p^n} - \alpha = 0 \right\}.$$

Moreover, $p^{rs} = p^r p^{r(s-1)}$. So let $\alpha \in F$, then $\alpha^{p^r} - \alpha = 0$.

Then

$$\alpha^{p^{rn}} = \alpha^{p^r p^{r(n-1)}} = (\alpha^{p^r})^{p^{r(n-1)}} = \alpha^{p^{r(n-1)}},$$

and we can continue reducing this way to show that this is yields to $\alpha^{p^r} = \alpha$.

So $\alpha \in K$, and thus $F \leq K$. We have $[K : F] = n$ by counting elements. Now K is simple, because $K^\times$ is cyclic. Let β be the generator, then $K = F(\beta)$. This the minimal polynomial of β in F has degree n, so take this to be the desired $f(x)$.

■

12.2 Simple Extensions

Let $F \leq E$ and

$$\begin{aligned} \varphi_\alpha : F[x] &\to E \\ f &\mapsto f(\alpha). \end{aligned}$$

denote the evaluation map.

Case 1: Suppose α is **algebraic** over F.

There is a kernel for this map, and since $F[x]$ is a PID, this ideal is generated by a single element – namely, the minimal polynomial of α.

Thus (applying the first isomorphism theorem), we have $F(\alpha) \supseteq E$ isomorphic to $F[x]/\min(\alpha, F)$. Moreover, $F(\alpha)$ is the smallest subfield of E containing F and α.

Case 2: Suppose α is **transcendental** over F.

Then $\ker \varphi_\alpha = 0$, so $F[x] \hookrightarrow E$. Thus $F[x] \cong F[\alpha]$.

Definition: $E \geq F$ is a *simple extension* if $E = F(\alpha)$ for some $\alpha \in E$.

Theorem: Let $E = F(\alpha)$ be a simple extension of F where α is algebraic over F.

Then every $\beta \in E$ can be uniquely expressed as

$$\beta = \sum_{i=0}^{n-1} c_i \alpha^i \text{ where } n = \deg \min(\alpha, F).$$

Proof:

Existence: We have

$$F(\alpha) = \left\{ \sum_{i=1}^{r} \beta_i \alpha^i \;\middle|\; \beta_i \in F \right\},$$

so all elements look like polynomials in α.

Using the minimal polynomial, we can reduce the degree of any such element by rewriting α^n in terms of lower degree terms:

$$\begin{aligned} f(x) = \sum_{i=0}^{n} a_i x^i, &\quad f(\alpha) = 0 \\ \implies \sum_{i=0}^{n} a_i \alpha^i &= 0 \\ \implies \alpha^n &= -\sum_{i=0}^{n-1} a_i \alpha^i. \end{aligned}$$

Uniqueness: Suppose $\sum c_i \alpha^i = \sum^{n-1} d_i \alpha^i$. Then $\sum^{n-1} (c_i - d_i) \alpha^i = 0$. But by minimality of the minimal polynomial, this forces $c_i - d_i = 0$ for all i.

■

> Note: if α is algebraic over F, then $\left\{1, \alpha, \cdots \alpha^{n-1}\right\}$ is a basis for $F(\alpha)$ over F where $n = \deg \min(\alpha, F)$. Moreover,
>
> $$[F(\alpha) : F] = \dim_F F(\alpha) = \deg \min(\alpha, F).$$

> Note: adjoining any root of a minimal polynomial will yield isomorphic (usually not *identical*) fields. These are distinguished as subfields of the algebraic closure of the base field.

Theorem: Let $F \leq E$ with $\alpha \in E$ algebraic over F.

If $\deg \min(\alpha, F) = n$, then $F(\alpha)$ has dimension n over F, and $\left\{1, \alpha, \cdots, \alpha^{n-1}\right\}$ is a basis for $F(\alpha)$ over F.

Moreover, any $\beta \in F(\alpha)$, is *also* algebraic over F,and $\deg \min(\beta, F) \;\Big|\; \deg \min(\alpha, F)$.

Proof of first part: Exercise.

Proof of second part: We want to show that β is algebraic over F.

We have

$$[F(\alpha) : F] = [F(\alpha) : F(\beta)][F(\beta) : F],$$

so $[F(\beta) : F]$ is less than n since this is a finite extension, and the division of degrees falls out immediately.

■

12.3 Automorphisms and Galois Theory

Let F be a field and $\overline{F}$ be its algebraic closure. Consider subfields of the algebraic closure, i.e. E such that $F \leq E \leq \overline{F}$. Then $E \geq F$ is an algebraic extension.

Definition: $\alpha, \beta \in E$ are *conjugates* iff $\min(\alpha, F) = \min(\beta, F)$.

Examples:

- $\sqrt[3]{3}$, $\sqrt[3]{3}\zeta$, $\sqrt[3]{3}\zeta^2$ are all conjugates, where $\zeta = e^{2\pi i/3}$.
- $\alpha = a + bi \in \mathbb{C}$ has conjugate $\overline{\alpha} = a - bi$, and

$$\min(\alpha, \mathbb{R}) = \min(\overline{\alpha}, \mathbb{R}) = x^2 - 2ax + (a^2 + b^2).$$

13 | Thursday September 19th

13.1 Conjugates

Let $E \geq F$ be a field extension. Then $\alpha, \beta \in E$ are *conjugate* $\iff \min(\alpha, F) = \min(\beta, F)$ in $F[x]$.

Example: $a + bi, a - bi$ are conjugate in $\mathbb{C}/\mathbb{R}$, since they both have minimal polynomial $x^2 - 2ax + (a^2 + b^2)$ over $\mathbb{R}$.

Theorem: Let F be a field and $\alpha, \beta \in E \geq F$ with $\deg \min(\alpha, F) = \deg \min(\beta, F)$, i.e.

$$[F(\alpha) : F] = [F(\beta) : F].$$

Then α, β are conjugates $\iff$ $F(\alpha) \cong F(\beta)$ under the map

$$\begin{aligned} \varphi : F(\alpha) &\to F(\beta) \\ \sum_i a_i \alpha^i &\mapsto \sum_i a_i \beta^i. \end{aligned}$$

Proof: Suppose φ is an isomorphism.

Let

$$f \coloneqq \min(\alpha, F) = \sum c_i x^i \text{ where } c_i \in F,$$

so $f(\alpha) = 0$.

Then

$$0 = f(\alpha) = f(\sum c_i \alpha^i) = \sum c_i \beta^i,$$

so β satisfies f as well, and thus $f = \min(\alpha, F) \,\Big|\, \min(\beta, F)$.

But we can repeat this argument with f^{-1} and $g(x) \coloneqq \min(\beta, F)$, and so we get an equality. Thus α, β are conjugates.

Conversely, suppose α, β are conjugates so that $f = g$. Check that φ is a homomorphism of fields, so that

$$\varphi(x + y) = \varphi(x) + \varphi(y) \text{ and } \varphi(xy) = \varphi(x)\varphi(y).$$

Then φ is clearly surjective, so it remains to check injectivity.

To see that φ is injective, suppose $f(z) = 0$. Then $\sum a_i \beta^i = 0$. But by linear independence, this forces $a_i = 0$ for all i, which forces $z = 0$. ■

Corollary: Let $\alpha \in \overline{F}$ be algebraic over F.

Then

1. $\varphi : F(\alpha) \hookrightarrow \overline{F}$ for which $\varphi(f) = f$ for all $f \in F$ maps α to one of its conjugates.
2. If $\beta \in \overline{F}$ is a conjugate of α, then there exists one isomorphism $\psi : F(\alpha) \to F(\beta)$ such that $\psi(f) = f$ for all $f \in F$.

Corollary: Let $f \in \mathbb{R}[x]$ and suppose $f(a + bi) = 0$. Then $f(a - bi) = 0$ as well.

Proof: We know $i, -i$ are conjugates since they both have minimal polynomial $f(x) = x^2 + 1$. By (2), we have an isomorphism $\mathbb{R}[i] \xrightarrow{\psi} \mathbb{R}[-i]$. We have $\psi(a + bi) = a - bi$, and $f(a + bi) = 0$.

This isomorphism commutes with f, so we in fact have

$$0 = \psi(f(a + bi)) = f(\psi(a - bi)) = f(a - bi).$$

■

13.2 Fixed Fields and Automorphisms

Definition: Let F be a field and $\psi : F\circlearrowleft$ is an *automorphism* iff ψ is an isomorphism.

Definition: Let $\sigma : E\circlearrowleft$ be an automorphism. Then σ is said to *fix* $a \in E$ iff $\sigma(a) = a$. For any subset $F \subseteq E$, σ fixes F iff σ fixes every element of F.

Example: Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \supseteq \mathbb{Q} = F$.

A basis for E/F is given by $\left\{1, \sqrt{2}, \sqrt{5}, \sqrt{10}\right\}$. Suppose $\psi : E\circlearrowleft$ fixes $\mathbb{Q}$. By the previous theorem, we must have $\psi(\sqrt{2}) = \pm\sqrt{2}$ and $\psi(\sqrt{5}) = \pm\sqrt{5}$.

What is fixed by ψ? Suppose we define ψ on generators, $\psi(\sqrt{2}) = -\sqrt{2}$ and $\psi(\sqrt{5}) = \sqrt{5}$.

Then

$$f(c_0 + c_1\sqrt{2} + c_2\sqrt{5} + c_3\sqrt{10}) = c_0 - c_1\sqrt{2} + c_2\sqrt{5} - c_3\sqrt{10}.$$

This forces $c_1 = 0, c_3 = 0$, and so ψ fixes $\left\{c_0 + c_2\sqrt{5}\right\} = \mathbb{Q}(\sqrt{5})$.

Theorem: Let I be a set of automorphisms of E and define

$$E_I = \left\{\alpha \in E \;\middle|\; \sigma(a) = a \;\forall\sigma \in I\right\}$$

Then $E_I \leq E$ is a subfield.

Proof: Let $a, b \in E_i$. We need to show $a \pm b, ab, b \neq 0 \implies b^{-1} \in I$.

We have $\sigma(a \pm b) = \sigma(a) \pm \sigma(b) = a + b \in I$ since σ fixes everything in I. Moreover

$$\sigma(ab) = \sigma(a)\sigma(b) = ab \in I \quad \text{and} \quad \sigma(b^{-1}) = \sigma(b)^{-1} = b^{-1} \in I.$$

$\blacksquare$

Definition: Given a set I of automorphisms of F, E_I is called the *fixed field* of E under I.

Theorem: Let E be a field and $A = \left\{\sigma : E\circlearrowleft \;\middle|\; \sigma \text{ is an automorphism }\right\}$. Then A is a group under function composition.

Theorem: Let E/F be a field extension, and define

$$G(E/F) = \left\{\sigma : E\circlearrowleft \;\middle|\; f \in F \implies \sigma(f) = f\right\}.$$

Then $G(E/F) \leq A$ is a subgroup which contains F.

Proof: This contains the identity function.

Now if $\sigma(f) = f$ then $f = \sigma^{-1}(f)$, and

$$\sigma, \tau \in G(E/F) \implies (\sigma \circ \tau)(f) = \sigma(\tau(f)) = \sigma(f) = f.$$

$\blacksquare$

Note $G(E/F)$ is called the group of automorphisms of E fixing F, i.e. **the Galois Group.**

Theorem (Isomorphism Extension): Suppose $F \leq E \leq \overline{F}$, so E is an algebraic extension of F.

Suppose similarly that we have $F' \leq E' \leq \overline{F}'$, where we want to find E'.

Then any $\sigma : F \to F'$ that is an isomorphism can be lifted to some $\tau : E \to E'$, where $\tau(f) = \sigma(f)$ for all $f \in F$.

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] & \overline{F}' \arrow[d, dash] \\
E \arrow[r, "\tau"] \arrow[d, dash] & E' \arrow[d, dash] \\
F \arrow[r, "\sigma"] & F
\end{tikzcd}$$

14 | Tuesday October 1st

14.1 Isomorphism Extension Theorem

Suppose we have $F \leq E \leq \overline{F}$ and $F' \leq E' \leq \overline{F}'$. Supposing also that we have an isomorphism $\sigma : F \to F'$, we want to extend this to an isomorphism from E to *some* subfield of $\overline{F}'$ over F'.

Theorem: Let E be an algebraic extension of F and $\sigma : F \to F'$ be an isomorphism of fields. Let $\overline{F}'$ be the algebraic closure of F'.

Then there exists a $\tau : E \to E'$ where $E' \leq F'$ such that $\tau(f) = \sigma(f)$ for all $f \in F$.

Proof: See Fraleigh. Uses Zorn's lemma.

$\blacksquare$

Corollary: Let F be a field and $\overline{F}, \overline{F}'$ be algebraic closures of F. Then $\overline{F} \cong \overline{F}'$.

Proof: Take the identity $F \to F$ and lift it to some $\tau : \overline{F} \to E = \tau(\overline{F})$ inside $\overline{F}'$.

$$\begin{tikzcd}
 & \overline{F}' \arrow[d, dash] \\
\overline{F} \arrow[r, "\tau"] \arrow[d, dash] & E = \tau(\overline{F}) \arrow[d, dash] \\
F \arrow[r, "\mathrm{id}"] & F
\end{tikzcd}$$

Then $\tau(\overline{F})$ is algebraically closed, and $\overline{F}' \geq \tau(\overline{F})$ is an algebraic extension. But then $\overline{F}' = \tau(\overline{F})$. $\blacksquare$

Corollary: Let $E \geq F$ be an algebraic extension with $\alpha, \beta \in E$ conjugates. Then the conjugation isomorphism that sends $\alpha \to \beta$ can be extended to E.

Proof:

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] & \overline{F} \arrow[d, dash] \\
E \arrow[r, "\tau"] \arrow[d, dash] & E \arrow[d, dash] \\
F(\alpha) \arrow[r, "\psi"] \arrow[d, dash] & F(\beta) \arrow[d, dash] \\
F \arrow[r, "\mathrm{id}"] & F
\end{tikzcd}$$

> Note: Any isomorphism needs to send algebraic elements to algebraic elements, and even more strictly, conjugates to conjugates.

Counting the number of isomorphisms:

Let $E \geq F$ be a finite extension. We want to count the number of isomorphisms from E to a subfield of $\overline{F}$ that leave F fixed.

I.e., how many ways can we fill in the following diagram?

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] & \overline{F} \arrow[d, dash] \\
E \arrow[r, "\tau"] \arrow[d, dash] & E \arrow[d, dash] \\
F \arrow[r, "\mathrm{id}"] & F
\end{tikzcd}$$

Let $G(E/F) \coloneqq \mathrm{Gal}(E/F)$; this will be a finite group if $[E : F] < \infty$.

Theorem: Let $E \geq F$ with $[E : F] < \infty$ and $\sigma : F \to F'$ be an isomorphism.

Then the number of isomorphisms $\tau : E \to E'$ extending σ is *finite.*

Proof: Since $[E : F]$ is finite, we have $F_0 \coloneqq F(\alpha_1, \alpha_2, \cdots, \alpha_t)$ for some $t \in \mathbb{N}$. Let $\tau : F_0 \to E'$ be an isomorphism extending σ.

Then $\tau(\alpha_i)$ must be a conjugate of α_i, of which there are only finitely many since $\deg \min(\alpha_j, F)$ is finite. So there are at most $\prod_i \deg \min(\alpha_i, F)$ isomorphisms.

Example: $f(x) = x^3 - 2$, which has roots $\sqrt[3]{2}$, $\sqrt[3]{2}\zeta$, $\sqrt[3]{\zeta}^2$.

Two other concepts to address:

- Separability (multiple roots)
- Splitting Fields (containing all roots)

Definition: Let

$$\{E : F\} \coloneqq \left| \left\{ \sigma : E \to E' \;\middle|\; \sigma \text{ is an isomorphism extending } \mathrm{id} : F \to F \right\} \right|,$$

and define this to be the *index*.

Theorem: Suppose $F \leq E \leq K$, then

$$\{K : F\} = \{K : E\} \{E : F\} .$$

Proof: Exercise.

■

Example: $\mathbb{Q}(\sqrt{2}, \sqrt{5})/\mathbb{Q}$, which is an extension of *degree* 4. It also turns out that

$$\left\{\mathbb{Q}(\sqrt{2}, \sqrt{5}) : \mathbb{Q}\right\} = 4.$$

Questions:

1. When does $[E : F] = \{E : F\}$? (This is always true in characteristic zero.)
2. When is $\{E : F\} = |\mathrm{Gal}(E/F)|$?

Note that in this example, $\sqrt{5} \mapsto \pm\sqrt{5}$ and likewise for $\sqrt{2}$, so any isomorphism extending the identity must in fact be an *automorphism*.

We have automorphisms

$$\begin{aligned}
\sigma_1 &: (\sqrt{2}, \sqrt{5}) \mapsto (-\sqrt{2}, \sqrt{5}) \\
\sigma_2 &: (\sqrt{2}, \sqrt{5}) \mapsto (\sqrt{2}, -\sqrt{5}),
\end{aligned}$$

as well as id and $\sigma_1 \circ \sigma_2$. Thus $\mathrm{Gal}(E/F) \cong \mathbb{Z}_2^2$.

14.2 Separable Extensions

Goal: When is $\{E : F\} = [E : F]$? We'll first see what happens for simple extensions.

Definition: Let $f \in F[x]$ and α be a zero of f in $\overline{F}$.

The maximum ν such that $(x - \alpha)^\nu \,\Big|\, f$ is called the *multiplicity* of f.

Theorem: Let f be irreducible.

Then all zeros of f in $\overline{F}$ have the same multiplicity.

Proof: Let α, β satisfy f, where f is irreducible. Then consider the following lift:

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] & \overline{F} \arrow[d, dash] \\
F(\alpha) \arrow[r, "\psi"] \arrow[d, dash] & F(\beta) \arrow[d, dash] \\
F \arrow[r, "\mathrm{id}"] & F
\end{tikzcd}$$

This induces a map

$$\begin{gathered} F(\alpha)[x] \xrightarrow{\tau} F(\beta)[x] \\ \sum c_i x^i \mapsto \sum \psi(c_i) x^i, \end{gathered}$$

so $x \mapsto x$ and $\alpha \mapsto \beta$, so $x \mapsto x$ and $\alpha \mapsto \beta$.

Then $\tau(f(x)) = f(x)$ and

$$\tau((x-\alpha)^\nu) = (x-\beta)^\nu.$$

So write $f(x) = (x-\alpha)^\nu h(x)$, then

$$\tau(f(x)) = \tau((x-\alpha)^\nu)\tau(h(x)).$$

Since $\tau(f(x)) = f(x)$, we then have

$$f(x) = (x-\beta)^\nu \tau(h(x)).$$

So we get $\mathrm{mult}(\alpha) \leq \mathrm{mult}(\beta)$. But repeating the argument with α, β switched yields the reverse inequality, so they are equal.

■

Observation: If $F(\alpha) \to E'$ extends the identity on F, then $E' = F(\beta)$ where β is a root of $f \coloneqq \min(\alpha, F)$. Thus we have

$$\{F(\alpha) : F\} = |\{\text{distinct roots of } f\}|.$$

Moreover,

$$[F(\alpha) : F] = \{F(\alpha) : F\}\, \nu$$

where ν is the multiplicity of a root of $\min(\alpha, F)$.

Theorem: Let $E \geq F$, then $\{E : F\} \,\Big|\, [E : F]$.

15 | Thursday October 3rd

When can we guarantee that there is a $\tau : E \circlearrowleft$ lifting the identity?

If E is *separable*, then we have $|\mathrm{Gal}(E/F)| = \{E : F\}\,[E : F]$.

Fact: $\{F(\alpha) : F\}$ is equal to number of *distinct* zeros of $\min(\alpha, F)$.

If F is algebraic, then $[F(\alpha) : F]$ is the degree of the extension, and $\{F(\alpha) : F\} \,\Big|\, [F(\alpha) : F]$.

Theorem: Let $E \geq F$ be finite, then $\{E : F\} \,\Big|\, [E : F]$.

Proof: If $E \geq F$ is finite, $E = F(\alpha_1, \cdots, \alpha_n)$.

So $\min(\alpha_i, F)$ has a_j as a root, so let n_j be the number of distinct roots, and v_j the respective multiplicities.

Then

$$[F : F(\alpha_1, \cdots, \alpha_{n-1})] = n_j v_j = v_j \{F : F(\alpha_1, \cdots, \alpha_{n-1})\}.$$

So $[E : F] = \prod_j n_j v_j$ and $\{E : F\} = \prod_j n_j$, and we obtain divisibility.

■

Definitions:

1. An extension $E \geq F$ is **separable** iff $[E : F] = \{E : F\}$
2. An element $\alpha \in E$ is **separable** iff $F(\alpha) \geq F$ is a separable extension.
3. A polynomial $f(x) \in F[x]$ is **separable** iff $f(\alpha) = 0 \implies \alpha$ is separable over F.

Lemma:

1. α is separable over F iff $\min(\alpha, F)$ has zeros of multiplicity one.
2. Any irreducible polynomial $f(x) \in F[x]$ is separable iff $f(x)$ has zeros of multiplicity one.

Proof of (1): Note that $[F(\alpha) : F] = \deg \min(\alpha, F)$, and $\{F(\alpha) : F\}$ is the number of distinct zeros of $\min(\alpha, F)$.

Since all zeros have multiplicity 1, we have $[F(\alpha) : F] = \{F(\alpha) : F\}$.

■

Proof of (2): If $f(x) \in F[x]$ is irreducible and $\alpha \in \overline{F}$ a root, then $\min(\alpha, F) \mathrel{\Big|} f(\alpha)$.

But then $f(x) = \ell \min(\alpha, F)$ for some constant $\ell \in F$, since $\min(\alpha, F)$ was monic and only had zeros of multiplicity one.

■

Theorem: If $K \geq E \geq F$ and $[K : F] < \infty$, then K is separable over F iff K is separable over E and E is separable over F.

Proof:

$$\begin{aligned} [K : F] &= [K : E][E : F] \\ &= \{K : E\}\{E : F\} \\ &= \{K : F\}. \end{aligned}$$

Corollary: Let $E \geq F$ be a finite extension. Then

$$E \text{ is separable over } F \iff \text{Every } \alpha \in E \text{ is separable over } F.$$

Proof:

$\Longrightarrow$: Suppose $E \geq F$ is separable.

Then $E \geq F(\alpha) \geq F$ implies that $F(\alpha)$ is separable over F and thus α is separable.

$\Longleftarrow$: Suppose every $\alpha \in E$ is separable over F.

Since $E = F(\alpha_1, \cdots, \alpha_n)$, build a tower of extensions over F. For the first step, consider $F(\alpha_1, \alpha_2) \to F(\alpha_1) \to F$.

We know $F(\alpha_1)$ is separable over F. To see that $F(\alpha_1, \alpha_2)$ is separable over $F(\alpha_1)$, consider α_2.

α_2 is separable over $F \iff \min(\alpha_2, F)$ has roots of multiplicity one.

Then $\min(\alpha_2, F(\alpha_1)) \bigm| \min(\alpha_2, F)$, so $\min(\alpha_2, F(\alpha))$ has roots of multiplicity one.

Thus $F(\alpha_1, \alpha_2)$ is separable over $F(\alpha_1)$.

$\blacksquare$

15.1 Perfect Fields

Lemma: $f(x) \in F[x]$ has a multiple root $\iff f(x), f'(x)$ have a nontrivial (multiple) common factor.

Proof:

$\Longrightarrow$: Let $K \geq F$ be an extension field of F.

Suppose $f(x), g(x)$ have a common factor in $K[x]$; then f, g also have a common factor in $F[x]$.

If f, g do not have a common factor in $F[x]$, then $\gcd(f, g) = 1$ in $F[x]$, and we can find $p(x), q(x) \in F[x]$ such that $f(x)p(x) + g(x)q(x) = 1$.

But this equation holds in $K[x]$ as well, so $\gcd(f, g) = 1$ in $K[x]$.

We can therefore assume that the roots of f lie in F. Let $\alpha \in F$ be a root of f. Then

$$\begin{aligned} f(x) &= (x-\alpha)^m g(x) \\ f'(x) &= m(x-\alpha)^{m-1} g(x) + (x-\alpha)^m g'(x). \end{aligned}$$

If α is a multiple root, $m > 2$, and thus $(x - \alpha) \bigm| f'$.

$\Longleftarrow$: Suppose f does not have a multiple root.

We can assume all of the roots are in F, so we can split f into linear factors.

So

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)$$
$$f'(x) = \sum_{i=1}^{n} \prod_{j \neq i} (x - \alpha_j).$$

But then $f'(\alpha_k) = \prod j \neq k(x - \alpha_j) \neq 0$. Thus f, f' can not have a common root.

■

Moral: we can thus test separability by taking derivatives.

Definition: A field F is *perfect* if every finite extension of F is separable.

Theorem: Every field of characteristic zero is perfect.

Proof: Let F be a field with $\operatorname{char}(F) = 0$, and let $E \geq F$ be a finite extension.

Let $\alpha \in E$, we want to show that α is separable. Consider $f = \min(\alpha, F)$. We know that f is irreducible over F, and so its only factors are $1, f$. If f has a multiple root, then f, f' have a common factor in $F[x]$. By irreducibility, $f \bigm| f'$, but $\deg f' < \deg f$, which implies that $f'(x) = 0$. But this forces $f(x) = c$ for some constant $c \in F$, which means f has no roots – a contradiction.

So α separable for all $\alpha \in E$, so E is separable over F, and F is thus perfect.

■

Theorem: Every finite field is perfect.

Proof: Let F be a finite field with $\operatorname{char} F = p > 0$ and let $E \geq F$ be finite. Then $E = F(\alpha)$ for some $\alpha \in E$, since E is a simple extension (look at E^*?) So E is separable over F iff $\min(\alpha, F)$ has distinct roots.

So $E^\times = E \setminus \{0\}$, and so $|E| = p^n \implies |E| = p^{n-1}$. Thus all elements of E satisfy

$$f(x) \coloneqq x^{p^n} - x \in \mathbb{Z}_p[x].$$

So $\min(\alpha, F) \bigm| f(x)$. One way to see this is that *every* element of E satisfies f, since there are exactly p^n distinct roots.

Another way is to note that

$$f'(x) = p^n x^{p^n - 1} - 1 = -1 \neq 0.$$

Since $f(x)$ has no multiple roots, $\min(\alpha, F)$ can not have multiple roots either.

■

Note that $[E : F] < \infty \implies F(\alpha_1, \cdots, \alpha_n)$ for some $\alpha_i \in E$ that are algebraic over F.

15.2 Primitive Elements

Theorem (Primitive Element): Let $E \geq F$ be a finite extension and separable.

Then there exists an $\alpha \in E$ such that $E = F(\alpha)$.

Proof: See textbook.

Corollary: Every finite extension of a field of characteristic zero is simple.

16 | Tuesday October 8th

16.1 Splitting Fields

For $\overline{F} \geq E \geq F$, we can use the lifting theorem to get a $\tau : E \to E'$. What conditions guarantee that $E = E'$?

If $E = F(\alpha)$, then $E' = F(\beta)$ for some β a conjugate of α. Thus we need E to contain conjugates of all of its elements.

Definition: Let $\left\{ f_i(x) \in F[x] \;\middle|\; i \in I \right\}$ be any collection of polynomials. We way that E is a **splitting field** $\iff$ E is the smallest subfield of $\overline{F}$ containing all roots of the f_i.

Examples:

- $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a splitting field for $\left\{ x^{-}2, x^2 - 5 \right\}$.
- $\mathbb{C}$ is a splitting field for $\left\{ x^2 + 1 \right\}$.
- $\mathbb{Q}(\sqrt[3]{2})$ is *not* a splitting field for any collection of polynomials.

Theorem: Let $F \leq E \leq \overline{F}$. Then E is a splitting field over F for some set of polynomials $\iff$ every isomorphism of E fixing F is in fact an automorphism.

Proof:

$\implies$: Let E be a splitting field of $\left\{ f_i(x) \;\middle|\; f_i(x) \in F[x], i \in I \right\}$.

Then $E = \left\langle \alpha_j \;\middle|\; j \in J \right\rangle$ where α_j are the roots of all of the f_i.

Suppose $\sigma : E \to E'$ is an isomorphism fixing F. Then consider $\sigma(\alpha_j)$ for some $j \in J$. We have

$$\min(\alpha, F) = p(x) = a_0 + a_1 x + \cdots a_{n-1} x^{n-1} + a_n x^n,$$

and so

$$p(x) = 0,\ 0 \in F \implies 0 = \sigma(p(\alpha_j)) = \sum_i a_i \sigma(\alpha_j)^i.$$

Thus $\sigma(\alpha_j)$ is a conjugate, and thus a root of some $f_i(x)$.

$\impliedby$: Suppose any isomorphism of E leaving F fixed is an automorphism.

Let $g(x)$ be an irreducible polynomial and $\alpha \in E$ a root.

$$\begin{tikzcd}
\overline{F} \arrow[d, dash] & \overline{F} \arrow[d, dash] \\
E \arrow[r, "\tau"] \arrow[d, dash] & E' = E \arrow[d, dash] \\
F(\alpha) \arrow[r, "\mathrm{id}"] \arrow[d, dash] & F(\beta) \arrow[d, dash] \\
F \arrow[r, "\mathrm{id}"] & F
\end{tikzcd}$$

Using the lifting theorem, where $F(\alpha \leq E$, we get a map $\tau : E \to E'$ lifting the identity and the conjugation homomorphism. But this says that E' must contain every conjugate of α.

Therefore we can take the collection

$$S = \left\{ g_i(x) \in F[x] \;\middle|\; g_i \text{ irreducible and has a root in } E \right\}.$$

This defines a splitting field for $\{g_j\}$, and we're done.

$\blacksquare$

Examples:

1. $x^2 + 1 \in \mathbb{R}[x]$ splits in $\mathbb{C}$, i.e. $x^2 + 1 = (x+i)(x-i)$.
2. $x^2 - 2 \in \mathbb{Q}[x]$ splits in $\mathbb{Q}(\sqrt{2})$.

Corollary: Let E be a splitting field over F. Then every **irreducible** polynomial in $F[x]$ with a root $\alpha \in E$ splits in $E[x]$.

Corollary: The index $\{E : F\}$ (the number of distinct lifts of the identity). If E is a splitting field and $\tau : E \to E'$ lifts the identity on F, then $E = E'$. Thus $\{E : F\}$ is the number of automorphisms, i.e. $|\mathrm{Gal}(E/F)|$.

Question: When is it the case that

$$[E : F] = \{E : F\} = |\mathrm{Gal}(E/F)|?$$

- The first equality occurs when E is separable.
- The second equality occurs when E is a splitting field.

Characteristic zero implies separability

Definition: If E satisfies both of these conditions, it is said to be a **Galois extension**.

Some cases where this holds:

- $E \geq F$ a finite algebraic extension with E characteristic zero.
- E a finite field, since it is a splitting field for $x^{p^n} - x$.

Example 1: $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ is

1. A degree 4 extension,

2. The number of automorphisms was 4, and
3. The Galois group was $\mathbb{Z}_2^2$, of size 4.

Example 2: E the splitting field of $x^3 - 3$ over $\mathbb{Q}$.

This polynomial has roots $\sqrt[3]{3},\ \zeta_3 \sqrt[3]{3},\ \zeta_3^2 \sqrt[3]{3}$ where $\zeta_3^3 = 1$.

Then $E = \mathbb{Q}(\sqrt[3]{3}, \zeta_3)$, where

$$\min(\sqrt[3]{3}, \mathbb{Q}) = x^3 - 3$$
$$\min(\zeta_3, \mathbb{Q}) = x^2 + x + 1,$$

so this is a degree 6 extension.

Since char $\mathbb{Q} = 0$, we have $[E : \mathbb{Q}] = \{E : \mathbb{Q}\}$ for free.

We know that any automorphism has to map

$$\sqrt[3]{3} \mapsto \sqrt[3]{3},\ \sqrt[3]{3}\zeta_3,\ \sqrt[3]{3}\zeta_3^2$$
$$\zeta_3 \mapsto \zeta_3,\ \zeta_3^2.$$

You can show this is nonabelian by composing a few of these, thus the Galois group is S^3.

Example 3 If $[E : F] = 2$, then E is automatically a splitting field.

Since it's a finite extension, it's algebraic, so let $\alpha \in E \setminus F$.

Then $\min(\alpha, F)$ has degree 2, and thus $E = F(\alpha)$ contains all of its roots, making E a splitting field.

16.2 The Galois Correspondence

There are three key players here:

$$[E : F], \quad \{E : F\}, \quad \mathrm{Gal}(E/F).$$

How are they related?

Definition: Let $E \geq F$ be a finite extension. E is **normal** (or Galois) over F iff E is a separable splitting field over F.

Examples:

1. $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is normal over $\mathbb{Q}$.
2. $\mathbb{Q}(\sqrt[3]{3})$ is not normal (not a splitting field of any irreducible polynomial in $\mathbb{Q}[x]$).
3. $\mathbb{Q}(\sqrt[3]{3}, \zeta_3)$ is normal

Theorem: Let $F \leq E \leq K \leq \overline{F}$, where K is a finite normal extension of F. Then

1. K is a normal extension of E as well,
2. $\mathrm{Gal}(K/E) \leq \mathrm{Gal}(K/F)$.

3. For $\sigma, \tau \in \mathrm{Gal}(K/F)$,

$$\sigma\Big|_E = \tau\Big|_E \iff \sigma, \tau \text{ are in the same left coset of } \frac{\mathrm{Gal}(K/F)}{\mathrm{Gal}(K/E)}.$$

Proof of (1): Since K is separable over F, we have K separable over E.

Then K is a splitting field for polynomials in $F[x] \subseteq E[x]$. Thus K is normal over E.

■

Proof of (2):

$$\begin{array}{ccc} K & \xrightarrow{\tau} & K \\ | & & | \\ E & \xrightarrow{\text{id}} & E \\ | & & | \\ F & \xrightarrow{\text{id}} & F \end{array}$$

So this follows by definition.

■

Proof of (3): Let $\sigma, \tau \in \mathrm{Gal}(K/F)$ be in the same left coset. Then

$$\tau^{-1}\sigma \in \mathrm{Gal}(K/E),$$

so let $\mu \coloneqq \tau^{-1}\sigma$.

Note that μ fixes E by definition.

So $\sigma = \tau\mu$, and thus

$$\sigma(e) = \tau(\mu(e)) = \tau(e) \text{ for all } e \in E.$$

■

Note: We don't know if the intermediate field E is actually a *normal* extension of F.

Standard example: $K \geq E \geq F$ where

$$K = \mathbb{Q}(\sqrt[3]{3}, \zeta_3) \quad E = \mathbb{Q}(\sqrt[3]{3}) \quad F = \mathbb{Q}.$$

Then $K \trianglelefteq E$ and $K \trianglelefteq F$, since $\mathrm{Gal}(K/F) = S_3$ and $\mathrm{Gal}(K/E) = \mathbb{Z}_2$. But $E \ntrianglelefteq F$, since $\mathbb{Z}_2 \ntrianglelefteq S_3$.

17 | Thursday October 10th

17.1 Computation of Automorphisms

Setup:

- $F \leq E \leq K \leq \overline{F}$
- $[K : F] < \infty$
- K is a normal extension of F

Facts:

- $\operatorname{Gal}(K/E) = \left\{ \sigma \in \operatorname{Gal}(K/F) \;\middle|\; \sigma(e) = e \ \forall e \in E \right\}$.
- $\sigma, \tau \in \operatorname{Gal}(K/F)$ and $\sigma|_E = \tau|_E \iff \sigma, \tau$ are in the same left coset of $\operatorname{Gal}(K/F)/\operatorname{Gal}(K/E)$.

Example: $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$.

Then $\operatorname{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_2^2$, given by the following automorphisms:

$$
\begin{aligned}
\mathrm{id} : \sqrt{2} &\mapsto \sqrt{2}, & \sqrt{5} &\mapsto \sqrt{5} \\
\rho_1 : \sqrt{2} &\mapsto \sqrt{2}, & \sqrt{5} &\mapsto -\sqrt{5} \\
\rho_2 : \sqrt{2} &\mapsto -\sqrt{2}, & \sqrt{5} &\mapsto \sqrt{5} \\
\rho_1 \circ \rho_2 : \sqrt{2} &\mapsto -\sqrt{2}, & \sqrt{5} &\mapsto -\sqrt{5}.
\end{aligned}
$$

We then get the following subgroup/subfield correspondence:

$$
\begin{tikzcd}
 & \mathbb{Z}_2^2 & \\
\{\mathrm{id}, \rho_1\} \arrow[ur] & \{\mathrm{id}, \rho_2\} \arrow[u] & \{\mathrm{id}, \rho_1 \circ \rho_2\} \arrow[ul] \\
 & \{\mathrm{id}\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Q}(\sqrt{2}, \sqrt{5}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr] & \mathbb{Q}(\sqrt{5}) \arrow[d] & \mathbb{Q}(\sqrt{10}) \arrow[dl] \\
 & \mathbb{R} &
\end{tikzcd}
$$

17.2 Fundamental Theorem of Galois Theory

Recall that $:= \operatorname{Gal}(K/E)$.

Theorem (Fundamental Theorem of Galois Theory):

Let $\mathcal{D}$ be the collection of subgroups of $\operatorname{Gal}(K/F)$ and $\mathcal{C}$ be the collection of subfields E such that $F \leq E \leq K$.

Define a map

$$
\begin{aligned}
&\lambda : \mathcal{C} \to \mathcal{D} \\
&\lambda(E) := \left\{ \sigma \in \operatorname{Gal}(K/F) \;\middle|\; \sigma(e) = e \ \forall e \in E \right\}.
\end{aligned}
$$

Then λ is a bijective map, and

1. $\lambda(E) = \mathrm{Gal}(K/E)$
2. $E = K_{\lambda(E)}$
3. If $H \leq \mathrm{Gal}(K/F)$ then
$$\lambda(K_H) = H$$
4. $[K : E] = |\lambda(E)|$ and
$$[E : F] = [\mathrm{Gal}(K/F) : \lambda(E)]$$
5. E is normal over $F \iff \lambda(E) \trianglelefteq \mathrm{Gal}(K/F)$, and in this case
$$\mathrm{Gal}(E/F) \cong \mathrm{Gal}(K/F)/\,\mathrm{Gal}(K/E).$$
6. λ is order-reversing, i.e.
$$E_1 \leq E_2 \implies \lambda(E_2) \leq \lambda(E_1).$$

Proof of 1: Proved earlier.

■

Proof of 2: We know that $E \leq L_{\mathrm{Gal}(K/E)}$. Let $\alpha \in K \setminus E$; we want to show that α is not fixed by all automorphisms in $\mathrm{Gal}(K/E)$.

We build the following tower:

$$\begin{tikzcd}
K \arrow[r, dotted, "\tau'"] & K \\
E(\alpha) \arrow[u] \arrow[r, "\tau"] & E(\beta) \arrow[u] \\
E \arrow[u] \arrow[r, "\mathrm{id}"] & E \arrow[u] \\
F \arrow[u] \arrow[r, "\mathrm{id}"] & F \arrow[u]
\end{tikzcd}$$

This uses the isomorphism extension theorem, and the fact that K is normal over F.

If $\beta \neq \alpha$, then β must be a conjugate of α, so $\tau'(\alpha) \neq \alpha$ while $\tau' \in \mathrm{Gal}(K/E)$.

■

Claim: λ is injective.
Proof: Suppose $\lambda(E_1) = \lambda(E_2)$. Then by (2), $E_1 = K_{\lambda(E_1)} = K_{\lambda(E_2)} = E_2$. ■

Proof of 3: We want to show that if $H \leq \mathrm{Gal}(K/F)$ then $\lambda(K_H) = H$.

We know $H \leq \lambda(K_H) = \mathrm{Gal}(K/K_H) \leq \mathrm{Gal}(K/F)$, so suppose $H \lneq \lambda(K_H)$.

Since K is a finite, separable extension, $K = K_H(\alpha)$ for some $\alpha \in K$.

Let

$$n = [K : K_H] = K : K_H = |\mathrm{Gal}(K/K_H)|.$$

Since $H \lneq \lambda(K_H)$, we have $|H| < n$. So denote $H = \{\sigma, \sigma_2, \cdots\}$ and let define

$$f(x) = \prod_i (x - \sigma_i(\alpha)).$$

We then have

- $\deg f = |H|$
- The coefficients of f are symmetric polynomials in the $\sigma_i(\alpha)$ and are fixed under any $\sigma \in H$
- $f(x) \in K_H(\alpha)[x]$
- $f(\alpha) = 0$ since $\sigma_i(\alpha) = \alpha$ for every i.

This is a contradiction, so we must have

$$[K_H : K] = n = \deg \min(\alpha, K_H) \leq \deg f = |H|.$$

■

Assuming (3), λ is surjective, so suppose $H < \mathrm{Gal}(K/F)$. Then $\lambda(K_H) = H \implies \lambda$ is surjective.

Proof of 4:

$$|\lambda(E)| = |\mathrm{Gal}(K/E)| =_{\text{splitting field}} [K : E]$$
$$[E : F] =_{\text{separable}} \{E : F\} =_{\text{previous part}} [\mathrm{Gal}(K/F) : \lambda(E)].$$

Proof of 5:

We have $F \leq E \leq K$ and E is separable over F, so E is normal over $F \iff E$ is a splitting field over F.

That is, every extension E'/E maps K to itself, since K is normal.

$$\begin{tikzcd}
K & K \\
E \arrow[u] & E' \arrow[u] \\
F \arrow[u] \arrow[r, "id"] & F \arrow[u]
\end{tikzcd}$$

So E is normal over $F \iff$ for all $\sigma \in \mathrm{Gal}(K/F), \sigma(\alpha) \in E$ for all $\alpha \in E$.

By a previous property, $E = K_{\mathrm{Gal}(K/E)}$, and so

$$\begin{aligned}
\sigma(\alpha) \in E &\iff \tau(\sigma(\alpha)) = \sigma(\alpha) && \forall \tau \in \mathrm{Gal}(K/E) \\
&\iff (\sigma^{-1}\tau\sigma)(\alpha) = \alpha S && \forall \tau \in \mathrm{Gal}(K/E) \\
&\iff \sigma^{-1}\tau\sigma \in \mathrm{Gal}(K/E) \\
&\iff \mathrm{Gal}(K/E) \trianglelefteq \mathrm{Gal}(K/F).
\end{aligned}$$

Now assume E is a normal extension of F, and let

$$\begin{aligned}\varphi : \mathrm{Gal}(K/F) &\to \mathrm{Gal}(E/F)\\ \sigma &\mapsto \sigma|_E.\end{aligned}$$

Then φ is well-defined precisely because E is normal over F, and we can apply the extension theorem:

$$\begin{tikzcd} K & K \\ E \arrow[u] \arrow[r, "\tau"] & E \arrow[u] \\ F \arrow[u] \arrow[r, "\mathrm{id}"] & F \arrow[u] \end{tikzcd}$$

φ is surjective by the extension theorem, and φ is a homomorphism, so consider $\ker \varphi$.

Let $\varphi(\sigma) = \sigma|_E = \mathrm{id}$. Then φ fixes elements of $E \iff \sigma \in \mathrm{Gal}(K/E)$, and thus $\ker \varphi = \mathrm{Gal}(K/E)$.

■

Proof of 6:

$$\begin{array}{cccc} E_1 \leq E_2 \iff & \mathrm{Gal}(K/E_2) \leq & & \mathrm{Gal}(K/E_1) \\ & \shortparallel & & \shortparallel \\ & \lambda(E_2) \leq & & \lambda(E_1). \end{array}$$

Example: $K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. Then $\min(\zeta, \mathbb{Q}) = x^2 + x + 1$ and $\mathrm{Gal}(K/\mathbb{Q}) = S_3$. There is a subgroup of order 2, $E = \mathrm{Gal}(K/\mathbb{Q}(\sqrt[3]{2})) \leq \mathrm{Gal}(K/\mathbb{Q})$, but E doesn't correspond to a normal extension of F, so this subgroup is not normal. On the other hand, $\mathrm{Gal}(\mathbb{Q}(\zeta_3), \mathbb{Q}) \trianglelefteq \mathrm{Gal}(K/\mathbb{Q})$.

18 | Tuesday October 15th

18.1 Cyclotomic Extensions

Definition: Let K denote the splitting field of $x^n - 1$ over F. Then K is called the **nth cyclotomic extension of F**.

If we set $f(x) = x^n - 1$, then $f'(x) = nx^{n-1}$.

So if char F does not divide n, then the splitting field is separable. So this splitting field is in fact normal.

Suppose that char F doesn't divide n, then $f(x)$ has n zeros, and let ζ_1, ζ_2 be two zeros. Then $(\zeta_1\zeta_2)^n = \zeta_1^n \zeta_2^n = 1$, so the product is a zero as well, and the roots of f form a subgroup in $K^\times$.

So let's specialize to $F = \mathbb{Q}$.

The roots of f are the nth roots of unity, i.e. $\zeta_n = e^{2\pi i/n}$, and are given by $\left\{\zeta_n, \zeta_n^2, \zeta_n^3, \cdots, \zeta_n^{n-1}\right\}$.

The *primitive* roots of unity are given by $\left\{\zeta_n^m \;\middle|\; \gcd(m, n) = 1\right\}$.

Definition: Let

$$\Phi_n(x) = \prod_{i=1}^{\varphi(n)} (x - \alpha_i),$$

where this product runs over all of the primitive nth roots of unity.

Let G be $\text{Gal}(K/\mathbb{Q})$. Then any $\sigma \in G$ will permute the primitive nth roots of unity. Moreover, it *only* permutes primitive roots, so every σ fixes $\Phi_n(x)$. But this means that the coefficients must lie in $\mathbb{Q}$.

Since ζ generates all of the roots of Φ_n, we in fact have $K = \mathbb{Q}(\zeta)$. But what is the group structure of G?

Since any automorphism is determined by where it sends a generator, we have automorphisms $\tau_m(\zeta) = \zeta^m$ for each m such that $\gcd(m, n) = 1$.

But then $\tau_{m_1} \circ \tau_{m_2} = \tau_{m_1+m_2}$, and so $G \cong G_m \leq \mathbb{Z}_n$ as a ring, where

$$G_m = \left\{ [m] \;\middle|\; \gcd(m, n) = 1 \right\}$$

and $|G| = \varphi(n)$.

Note that as a *set*, there are the units $\mathbb{Z}_n^\times$.

Theorem: The Galois group of the nth cyclotomic extension over $\mathbb{Q}$ has $\varphi(n)$ elements and is isomorphic to G_m.

Special case: $n = p$ where p is a prime.

Then $\varphi(p) = p - 1$, and

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Note that $\mathbb{Z}_p^\times$ is in fact cyclic, although this may not always happen. In this case, we have $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_p^\times$.

18.2 Construction of n-gons

To construct the vertices of an n-gon, we will need to construct the angle $2\pi/n$, or equivalently, ζ_n. Note that if $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] \neq 2^\ell$ for some $\ell \in \mathbb{N}$, then the n-gon is *not* constructible.

Example: An 11-gon. Noting that $[\mathbb{Q}(\zeta_{11}) : \mathbb{Q}] = 10 \neq 2^\ell$, the 11-gon is not constructible.

Since this is only a sufficient condition, we'll refine this.

Definition: A prime of the form $p = 2^{2^k} + 1$ are called **Fermat primes**.

Theorem: The regular n-gon is constructible $\iff$ all odd primes dividing n are *Fermat primes* p where p^2 does not divide n.

Example: Consider

$$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1.$$

Then take $\zeta = \zeta_5$; we then obtain the roots as $\left\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\right\}$ and $\mathbb{Q}(\zeta)$ is the splitting field.

Any automorphism is of the form $\sigma_r : \zeta \mapsto \zeta^r$ for $r = 1, 2, 3, 4$. So $|\mathrm{Gal}(K/\mathbb{Q})| = 4$, and is cyclic and thus isomorphic to $\mathbb{Z}_4$. Corresponding to $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4$, we have the extensions

$$\mathbb{Q} \to \mathbb{Q}(\zeta^2) \to \mathbb{Q}(\zeta).$$

How can we get a basis for the degree 2 extension $\mathbb{Q}(\zeta^2)/\mathbb{Q}$? Let

$$\lambda(E) = \left\{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \;\middle|\; \sigma(e) = e \ \forall e \in E\right\},$$

$\lambda(K_H) = H$ where H is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, and

$$K_H = \left\{x \in K \;\middle|\; \sigma(x) = x \ \forall \sigma \in H\right\}.$$

Note that if $\mathbb{Z}_4 = \langle \psi \rangle$, then $\mathbb{Z}_2 \leq \mathbb{Z}_4$ is given by $\mathbb{Z}_2 = \left\langle \psi^2 \right\rangle$.

We can compute that if $\psi(\zeta) = \zeta^2$, then

$$\begin{aligned}
\psi^2(\zeta) &= \zeta^{-1} \\
\psi^2(\zeta^2) &= \zeta^{-2} \\
\psi^2(\zeta^3) &= \zeta^{-3}.
\end{aligned}$$

Noting that ζ_4 is a linear combination of the other ζs, we have a basis $\left\{1, \zeta, \zeta^2, \zeta^3\right\}$.

Then you can explicitly compute the fixed field by writing out

$$\sigma(a + b\zeta + c\zeta^2 + d\zeta^3) = a + b\sigma(\zeta) + c\sigma(\zeta^2) + \cdots,$$

gathering terms, and seeing how this restricts the coefficients.

In this case, it yields $\mathbb{Q}(\zeta^2 + \zeta^3)$.

18.3 The Frobenius Automorphism

Definition: Let p be a prime and F be a field of characteristic $p > 0$. Then

$$\begin{aligned}
\sigma_p &: F \to F \\
\sigma_p(x) &= x^p
\end{aligned}$$

is denoted the *Frobenius map.*

Theorem: Let F be a finite field of characteristic $p > 0$. Then

1. φ_p is an automorphism, and
2. φ_p fixes $F_{\sigma_p} = \mathbb{Z}_p$.

Proof of part 1: Since σ_p is a field homomorphism, we have

$$\sigma_p(x+y) = (x+y)^p = x^p + y^p \text{ and } \sigma(xy) = (xy)^p = x^p y^p$$

Note that σ_p is injective, since $\sigma_p(x) = 0 \implies x^p = 0 \implies x = 0$ since we are in a field. Since F is finite, σ_p is also surjective, and is thus an automorphism.

Proof of part 2: If $\sigma(x) = x$, then

$$x^p = x \implies x^p - x = 0,$$

which implies that x is a root of $f(x) = x^p - x$. But these are exactly the elements in the prime ring $\mathbb{Z}_p$.

■

19 | Thursday October 17th

19.1 Example Galois Group Computation

Example: What is the Galois group of $x^4 - 2$ over $\mathbb{Q}$?

First step: find the roots. We can find directly that there are 4 roots given by

$$\left\{ \pm \sqrt[4]{2}, \pm i \sqrt[4]{2} \right\} \coloneqq \{r_i\}.$$

The splitting field will then be $\mathbb{Q}(\sqrt[4]{2}, i)$, which is separable because we are in characteristic zero. So this is a normal extension.

We can find some automorphisms:

$$\sqrt[4]{2} \mapsto r_i, \quad i \mapsto \pm i.$$

So $|G| = 8$, and we can see that G can't be abelian because this would require every subgroup to be abelian and thus normal, which would force every intermediate extension to be normal.

But the intermediate extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not a normal extension since it's not a splitting field.

So the group must be D_4.

■

19.2 Insolubility of the Quintic

19.2.1 Symmetric Functions

Let F be a field, and let

$$F(y_1, \cdots, y_n) = \left\{ \frac{f(y_1, \cdots, y_n)}{g(y_1, \cdots, y_n)} \;\middle|\; f, g \in F[y_1, \cdots, y_n] \right\}$$

be the set of *rational* functions over F.

Then $S_n \curvearrowright F(y_1, \cdots, y_n)$ by permuting the y_i, i.e.

$$\sigma\left(\frac{f(y_1,\cdots,y_n)}{g(y_1,\cdots,y_n)}\right) = \frac{f(\sigma(y_1),\cdots,\sigma(y_n))}{g(\sigma(y_1),\cdots,\sigma(y_n))}.$$

Definition: A function $f \in F(\alpha_1, \cdots, \alpha_n)$ is **symmetric** $\iff$ under this action, $\sigma \curvearrowright f = f$ for all $\sigma \in S_n$.

Examples:

1. $f(y_1, \cdots, y_n) = \prod y_i$
2. $f(y_1, \cdots, y_n) = \sum y_i$.

19.2.2 Elementary Symmetric Functions

Consider $f(x) \in F(y_1, \cdots, y_n)[x]$ given by $\prod(x - y_i)$. Then $\sigma f = f$, so f is a symmetric function. Moreover, all coefficients are fixed by S_n. So the coefficients themselves are symmetric functions.

Concretely, we have

Coefficient	Term
1	$(-1)^n$
x^{n-1}	$-y_1 - y_2 - \cdots - y_n$
x^{n-2}	$y_1y_2 + y_1y_3 + \cdots + y_2y_3 + \cdots$

The coefficient of x^{n-i} is referred to as the *ith elementary symmetric function*.

Consider an intermediate extension E given by joining all of the elementary symmetric functions:

$$\begin{tikzcd}
F(y_1, \cdots, y_n) \arrow[d, dash] \\
K \arrow[d, dash] \\
E = F(s_1, \cdots, s_n) \arrow[d, dash] \\
F
\end{tikzcd}$$

Let K denote the base field with *all* symmetric functions adjoined; then K is an intermediate extension, and we have the following results:

Theorem:

1. $E \leq K$ is a field extension.
2. $E \leq F(y_1, \cdots, y_n)$ is a finite, normal extension since it is the splitting field of $f(x) = \prod(x - y_i)$, which is separable.

We thus have

$$[F(y_1, \cdots, y_n) : E] \leq n! < \infty.$$

Proof:

We'll show that in fact $E = K$, so all symmetric functions are generated by the elementary symmetric functions.

By definition of symmetric functions, K is exactly the fixed field $F(y_1, \cdots, y_n)_{S_n}$, and $|S|_n = n!$.

So we have

$$\begin{aligned} n! &= |\mathrm{Gal}(F(y_1, \cdots, y_n/K))| \\ &\leq \{F(y_1, \cdots, y_n) : K\} \\ &\leq [F(y_1, \cdots, y_n) : K]. \end{aligned}$$

But now we have

$$n! \leq [F(y_1, \cdots, y_n) : K] \leq [F(y_1, \cdots, y_n) : E] \leq n!$$

which forces $K = E$.

■

Theorem:

1. Every symmetric function can be written as a combination of sums, products, and possibly quotients of elementary symmetric functions.
2. $F(y_1, \cdots, y_n)$ is a finite normal extension of $F(s_1, \cdots, s_n)$ of degree $n!$.
3. $\mathrm{Gal}(F(y_1, \cdots, y_n)/F(s_1, \cdots, s_n)) \cong S_n$.

We know that every group $G \hookrightarrow S_n$ by Cayley's theorem. So there exists an intermediate extension

$$F(s_1, \cdots, s_n) \leq L \leq F(y_1, \cdots, y_n)$$

such that $G = \mathrm{Gal}(F(y_1, \cdots, y_n)/L)$.

> Open question: which groups can be realized as Galois groups over $\mathbb{Q}$? Old/classic question, possibly some results in the other direction (i.e. characterizations of which groups *can't* be realized as such Galois groups).

19.2.3 Extensions by Radicals

Let $p(x) = \sum a_i x^i \in \mathbb{Q}[x]$ be a polynomial of degree n. Can we find a formula for the roots as a function of the coefficients, possibly involving radicals?

- For $n = 1$ this is clear
- For $n = 2$ we have the quadratic formula.
- For $n = 3$, there is a formula by work of Cardano.
- For $n = 4$, this is true by work of Ferrari.
- For $n \geq 5$, there can **not** be a general equation.

Definition: Let $K \geq F$ be a field extension. Then K is an **extension of** F **by radicals** (or a **radical extension**) $\iff$ $K = \alpha_1, \cdots, \alpha_n$ for some α_i such that

1. Each $\alpha_i^{m_i} \in F$ for some $m_i > 0$.
2. For each i, $\alpha_i^{\ell_i} \in F(\alpha_1, \cdots, \alpha_{i-1})$ for some $\ell_i < m_i$ (?).

Definition: A polynomial $f(x) \in F[x]$ is **solvable by radicals** over $F \iff$ the splitting field of f is contained in some radical extension.

Example: Over $\mathbb{Q}$, the polynomials $x^5 - 1$ and $x^3 - 2$ are solvable by radicals.

Recall that G is *solvable* if there exists a normal series

$$1 \trianglelefteq H_1 \trianglelefteq H_2 \cdots \trianglelefteq H_n \trianglelefteq G \text{ such that } H_n/H_{n-1} \text{ is abelian } \forall n.$$

19.2.4 The Splitting Field of $x^n - a$ is Solvable

Lemma: Let char $F = 0$ and $a \in F$. If K is the splitting field of $p(x) = x^n - a$, then $\mathrm{Gal}(K/F)$ is a solvable group.

Example: Let $p(x) = x^4 - 2/\mathbb{Q}$, which had Galois group D_4.

Proof: Suppose that F contains all nth roots of unity, $\left\{1, \zeta, \zeta^2, \cdots, \zeta[n-1]\right\}$ where ζ is a primitive nth root of unity. If β is any root of $p(x)$, then $\zeta^i\beta$ is also a root for any $1 \leq i \leq n-1$. This in fact yields n distinct roots, and is thus all of the them. Since the splitting field K is of the form $F(\beta)$, then if $\sigma \in \mathrm{Gal}(K/F)$, then $\sigma(\beta) = \zeta^i\beta$ for some i. Then if $\tau \in \mathrm{Gal}(K/F)$ is any other automorphism, then $\tau(\beta) = \zeta^k\beta$ and thus (exercise) the Galois group is abelian and thus solvable.

Suppose instead that F does not contain all nth roots of unity. So let $F' = F(\zeta)$, so $F \leq F(\zeta) = F' \leq K$. Then $F \leq F(\zeta)$ is a splitting field (of $x^n - 1$) and separable since we are in characteristic zero and this is a finite extension. Thus this is a normal extension.

We thus have $\mathrm{Gal}(K/F)/\mathrm{Gal}(K/F(\zeta)) \cong \mathrm{Gal}(F(\zeta)/F)$. We know that $\mathrm{Gal}(F(\zeta)/F)$ is abelian since this is a cyclotomic extension, and so is $\mathrm{Gal}(K/F(\zeta))$. We thus obtain a normal series

$$1 \trianglelefteq \mathrm{Gal}(K/F(\zeta)) \trianglelefteq \mathrm{Gal}(K/F)$$

Thus we have a solvable group.

$\blacksquare$

20 | Tuesday October 22nd

20.1 Certain Radical Extensions are Solvable

Recall the definition of an extension being *radical* (see above).

We say that a polynomial $f(x) \in K[x]$ is *solvable by radicals* iff its splitting field L is a radical extension of K.

Lemma: Let F be a field of characteristic zero.

If K is a splitting field of $f(x) = x^n - a \in F[x]$, then $\mathrm{Gal}(K/F)$ is a solvable group.

Theorem: Let F be characteristic zero, and suppose $F \leq E \leq K \leq \overline{F}$ be algebraic extension where E/F is normal and K a radical extension of F. Moreover, suppose $[K : F] < \infty$.

Then $\mathrm{Gal}(E/F)$ is solvable.

Proof: The claim is that K is contained in some L where $F \subset L$, L is a finite normal radical extension, and $\operatorname{Gal} L/F$ is solvable.

Since K is a radical extension of F, we have $F = K(\alpha_1, \cdots, \alpha_n)$ and $\alpha_i^{n_i} \in K(\alpha_1, \cdots, \alpha_{i-1})$ for each i and some $n_i \in \mathbb{N}$.

Let L_1 be the splitting field of $f_1(x) = x^{n_1} - \alpha_1^{n_1}$, then by the previous lemma, L_1 is a normal extension and $\operatorname{Gal}(L_1/F)$ is a solvable group.

Inductively continue this process, and letting

$$f_2(x) = \prod_{\sigma \in \operatorname{Gal}(L_1/F)} x^{n_2} - \sigma(\alpha_2)^{n_2} \in F[x].$$

Note that the action of the Galois group on this polynomial is stable. Let L_2 be the splitting field of f_2, then L_2 is a finite normal radical extension.

Then

$$\frac{\operatorname{Gal}(L_2/F)}{\operatorname{Gal}(L_2/L_1)} \cong \operatorname{Gal}(L_1/F),$$

which is solvable, and the denominator in this quotient is solvable, so the total group must be solvable as well. ■

20.2 Proof: Insolubility of the Quintic

Theorem (Insolubility of the quintic): Let $y_1, \cdots, y_n$ be independent transcendental elements in $\mathbb{R}$, then the polynomial $f(x) = \prod(x - y_i)$ is not solvable by radicals over $\mathbb{Q}(s_1, \cdots, s_n)$ where the s_i are the elementary symmetric polynomials in y_i.

So there are no polynomial relations between the transcendental elements.

Proof:

Let $n \geq 5$ and suppose y_i are transcendental over $\mathbb{R}$ and linearly independent over $\mathbb{Q}$. Then consider

$$\begin{aligned} s_1 &= \sum y_i \\ s_2 &= \sum_{i \leq j} y_i y_j \\ \cdots \\ s_n &= \prod_i y_i. \end{aligned}$$

Then $\mathbb{Q}(y_1, \cdots, y_n)/\mathbb{Q}(s_1, \cdots, s_n)$ would be a normal extension precisely if $A_n \trianglelefteq S_n$ (by previous theorem). For $n \geq 5$, A_n is simple, and thus S_n is not solvable in this range.

Thus the polynomial is not solvable by radicals, since the splitting field of $f(x)$ is $\mathbb{Q}(y_1, \cdots, y_n)$. ■

20.3 Rings and Modules

Recall that a ring is given by $(R, +, \cdot)$, where

1. $(R, +)$ is an abelian group,
2. $(R, \cdot)$ is a monoid,
3. The distributive laws hold.

An *ideal* is certain type of subring that allows taking quotients, and is defined by $I \trianglelefteq R \iff I \leq R$ and $RI, IR \subseteq I$. The quotient is given by $R/I = \left\{ r + I \mathrel{\Big|} r \in R \right\}$, and the ideal property is what makes this well-defined.

Much like groups, we have some notion of homomorphism $\varphi : R \to R'$, where $\varphi(ax + y) = \varphi(a)\varphi(x) + \varphi(y)$.

20.3.1 Modules

We want to combine the following two notions:

- Groups acting on sets, and
- Vector spaces

Definition: Let R be a ring and M an abelian group. Then if there is a map

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto rm. \end{aligned}$$

such that $\forall s, r_1, r_2 \in R$ and $m_1, m_2 \in M$ we have

- $(sr_1 + r_2)(m_1 + m_2) = sr_1m_1 + sr_1m_2 + r_2m_1 + r_2m_2$
- $1 \in R \implies 1m = m.$

then M is said to be an R**-module.**

Think of R like the group acting by scalar multiplication, and M the set of vectors with vector addition.

Examples:

1. $R = k$ a field, then a k-module is a vector space.
2. $R = G$ an abelian group, then R is a $\mathbb{Z}$-module where

$$n \curvearrowright a \coloneqq \sum_{i=1}^{n} a.$$

(In fact, these two notions are equivalent.)

3. $I \trianglelefteq R$, then $M \coloneqq R/I$ is an ring, which has an underlying abelian group, so M is an R-module where

$$M \curvearrowright R = r \curvearrowright (s + I) \coloneqq (rs) + I.$$

4. For M an abelian group, $R := \operatorname{End}(M) = \hom_{\text{AbGrp}}(M, M)$ is a ring, and M is a left R-module given by

$$f \curvearrowright m := f(m).$$

Definition: Let M, N be left R-modules. Then $f : M \to N$ is an R-module homomorphism $\iff$

$$f(rm_1 + m_2) = rf(m_1) + f(m_2).$$

Definition: *Monomorphisms* are injective maps, *epimorphisms* are surjections, and *isomorphisms* are both.

Definition: A *submodule* $N \leq M$ is a subset that is closed under all module operations.

We can consider images, kernels, and inverse images, so we can formulate homomorphism theorems analogous to what we saw with groups/rings:

Theorem:

1. If $M \xrightarrow{f} N$ in R-mod, then

$$M/\ker(f) \cong \operatorname{im}(()f).$$

2. Let $M, N \leq L$, then $M + N \leq L$ as well, and

$$\frac{M}{M \cap N} \cong \frac{M+N}{N}.$$

3. If $M \leq M \leq L$, then

$$\frac{M}{N} \cong \frac{L/M}{L/N}$$

Note that we can always quotient, since there's an underlying abelian group, and thus the "normality"/ideal condition is always satisfied for submodules. Just consider

$$M/N := \left\{ m + N \;\middle|\; m \in M \right\},$$

then $R \curvearrowright (M/N)$ in a well-defined way that gives M/N the structure of an R-module as well.

21 | Thursday October 24

21.1 Conjugates

Let $E \geq F$. Then $\alpha, \beta \in E$ are **conjugate** iff $\min(\alpha, F) = \min(\beta, F)$.

Example: $\alpha \pm bi \in \mathbb{C}$.

Theorem: Let F be a field and $\alpha, \beta \in F$ with $\deg \min(\alpha, F) = \deg \min(\beta, F)$, so

$$[F(\alpha) : F] = [F(\beta) : F].$$

Then α, β are conjugates $\iff F(\alpha) \cong F(\beta)$ under the *conjugation map*,

$$\begin{aligned} \psi : F(\alpha) &\to F(\beta) \\ \sum_{i=1}^{n-1} a_i \alpha^i &\mapsto \sum_{i=1}^{n-1} a_i \beta^i. \end{aligned}$$

Proof:

$\impliedby$:

Suppose that ψ is an isomorphism. Let $\min(\alpha, F) = p(x) = \sum c_i x^i$ where each $c_i \in F$. Then

$$0 = \psi(0) = \psi(p(\alpha)) = p(\beta) \implies \min(\beta, F) \mathrel{\Big|} \min(\alpha, F).$$

Applying the same argument to $q(x) = \min(\beta, F)$ yields $\min(\beta, F) = \min(\alpha, F)$.

$\implies$:

Suppose α, β are conjugates.

Exercise: Check that ψ is surjective and

$$\begin{aligned} \psi(x+y) &= \psi(x) + \psi(y) \\ \psi(xy) &= \psi(x)\psi(y). \end{aligned}$$

Let $z = \sum a_i \alpha^i$. Supposing that $\psi(z) = 0$, we have $\sum a_i \beta^i = 0$. By linear independence, this forces $a_i = 0$ for all i, and thus $z = 0$. So ψ is injective.

■

Corollary: Let $\alpha \in \overline{F}$ be algebraic. Then

1. Any $\varphi : F(\alpha) \hookrightarrow \overline{F}$ such that $\varphi(f) = f$ for all $f \in F$ must map α to a conjugate.
2. If $\beta \in \overline{F}$ is a conjugate of α, then there exists an isomorphism $\varphi : F(\alpha) \to F(\beta) \subseteq \overline{F}$ such that $\varphi(f) = f$ for all $f \in F$.

Proof of 1:

Let $\min(\alpha, F) = p(x) = \sum a_i x^i$. Note that $0 = \psi(p(\alpha)) = p(\psi(\alpha))$, and since p was irreducible, p must also be the minimal polynomial of $\psi(\alpha)$. Thus $\psi(\alpha)$ is a conjugate of α.

■

Proof of 2:

$F(\alpha)$ is generated by F and α, and ψ is completely determined by where it sends F and α. This shows uniqueness.

■

Corollary: Let $f(x) \in \mathbb{R}[x]$ and suppose $f(a + bi) = 0$. Then $f(a - bi) = 0$.

Proof: Both $i, -i$ are conjugates and $\min(i, \mathbb{R}) = \min(-i, \mathbb{R}) = x^2 + 1 \in \mathbb{R}[x]$. We then have a map

$$\begin{aligned} \psi : \mathbb{R}[i] &\to \mathbb{R}[-i] \\ \psi(a + bi) &= a + b(-i). \end{aligned}$$

So if $f(a + bi) = 0$, then $0 = \psi(f(a + bi)) = f(\psi(a + bi)) = f(a - bi)$.

■

22 | October 27th

22.1 Modules

Let R be a ring and M be an R-module.

Definition: For a subset $X \subseteq M$, we can define the *submodule generated by X* as

$$\langle X \rangle \coloneqq \cap_{X \subseteq N \leq M} N \subseteq M.$$

Then M is generated by X iff $M = \langle X \rangle$.

As a special case, when $X = \{m\}$ consists of a single element, we write

$$\langle m \rangle = Rm \coloneqq \left\{ rm \;\middle|\; r \in R \right\}.$$

In general, we have

$$\langle X \rangle = \left\{ \sum r_i x_i \;\middle|\; r_i \in R, x_i \in X \right\}.$$

22.2 Direct Products and Direct Sums

Definition: Let $\{M_i\}$ be a finite collection of R-modules, and let

$$N = \bigoplus M_i = \left\{ \sum m_i \;\middle|\; m_i \in M_i \right\}$$

with multiplication given by $\gamma \sum m_i = \sum \gamma m_i$ denote the **direct sum**.

For an infinite collection, we require that all but finitely many terms are zero.

Definition: Define $N = \prod M_i$ denote the **direct product**, where we now drop the condition that finitely many terms are zero.

When the indexing set is finite, $\bigoplus M_i \cong \prod M_i$. In general, $\bigoplus M_i \hookrightarrow \prod M_i$.

Note that the natural inclusions

$$\iota_j : M_j \hookrightarrow \prod M_i$$

and projections

$$\pi_j : \prod M_i \twoheadrightarrow M_j$$

are both R-module homomorphisms.

Theorem: $M \cong \bigoplus M_i$ iff there exist maps $\pi_j : M \to M_j$ and $\iota_j : M_j \to M$ such that

1. $$\pi_j \circ \iota_k = \begin{cases} 1m & j = k \\ 0 & \text{else} \end{cases}$$

2. $\sum\limits_j \iota_j \circ \pi_j = \mathrm{id}_M$

Remark: Let M, N be R-modules. Then $\hom_{R-\mathrm{mod}}(M, N)$ is an abelian group.

22.3 Internal Direct Sums

For a collection of submodules of M given by $\{M_i\}$, denote the *internal direct sum*

$$\sum M_i := \left\{ m_1 + m_2 + \cdots \;\middle|\; m_i \in M_i \right\}$$

iff it satisfies the following conditions:

1. $M = \sum\limits_i M_i$
2. $M_i \cap M_j = \{0\}$ for $i \neq j$.

22.4 Exact Sequences

Definition: A sequence of the form

$$0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{p} M_3 \to 0$$

where

- i is a monomorphism
- p is an empimorphism
- $\mathrm{im}(i) = \ker p$

is said to be **short exact**.

Examples:

- $$0 \to 2\mathbb{Z} \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

- For any epimorphism $\pi : M \to N$,

$$0 \to \ker \pi \to M \to N \to 0$$

-

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$$

In general, any sequence

$$\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is **exact** iff $\operatorname{im}(f)_i = \ker f_{i+1}$.

1. If α, γ are monomorphisms then β is a monomorphism.

23 | Tuesday October 29th

23.1 Exact Sequences

Lemma (Short Five):

Consider a diagram of the following form:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\alpha"] & N \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f'"] & N' \arrow[r, "g'"] & Q' \arrow[r] & 0
\end{tikzcd}$$

1. α, γ monomorphisms implies β is a monomorphism.
2. α, γ epimorphisms implies β is an epimorphism.
3. α, γ isomorphisms implies β is an isomorphism.

Moreover, (1) and (2) together imply (3).

Proof: Exercise.

Example proof of (2): Suppose α, γ are monomorphisms.

- Let $n \in N$ with $\beta(n) = 0$, then $g' \circ \beta(n) = 0$.

- $\implies \gamma \circ g(n) = 0$.
- $\implies g(n) = 0$
- $\implies \exists m \in M$ such that $f(m) = n$
- $\implies \beta \circ f(m) = \beta(n)$
- $\implies f'\alpha(m) = \beta(n) = 0$
- $\implies \alpha(m) = 0$
- $\implies f'$ is injective, so $m = 0$ and $n = f(m) = 0$.

■

Definition: Two exact sequences are *isomorphic* iff in the following diagram, f, g, h are all isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, "g"] & Q \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

Theorem: Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a SES. Then TFAE:

- There exists an R-module homomorphisms $h : M_3 \to M_2$ such that $g \circ h = \mathrm{id}_{M_3}$.
- There exists an R-module homomorphisms $k : M_2 \to M_1$ such that $k \circ f = \mathrm{id}_{M_1}$.
- The sequence is isomorphic to $0 \to M_1 \to M_1 \oplus M_3 \to M_3 \to 0$.

Proof: Define $\varphi : M_1 \oplus M_3 \to M_2$ by $\varphi(m_1 + m_2) = f(m_1) + h(m_2)$. We need to show that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & M_1 \oplus M_3 \arrow[r] & M_3 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] \arrow[u, "\varphi"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

We can check that

$$(g \circ \varphi)(m_1 + m_2) = g(f(m_1)) + g(h(m_2)) = m_2 = \pi(m_1 + m_2).$$

This yields $1 \implies 3$, and $2 \implies 3$ is similar.

To see that $3 \implies 1, 2$, we attempt to define k, h in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & M_1 \oplus M_3 \arrow[r] \arrow[l, bend right, "\pi_1"'] & M_3 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] \arrow[l, bend right, "\iota_2"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] \arrow[u, "\varphi"] \arrow[l, bend left, "k"] & M_3 \arrow[r] \arrow[l, bend left, "h"] & 0
\end{tikzcd}$$

So define $k = \pi_1 \circ \varphi^{-1}$ and $h = \varphi \circ \iota_2$. It can then be checked that

$$g \circ h = g \circ \varphi \circ \iota_2 = \pi_2 \circ \iota_2 = \mathrm{id}_{M_3}.$$

■

23.2 Free Modules

Moral: A *free module* is a module with a basis.

Definition: A subset $X = \{x_i\}$ is *linearly independent* iff

$$\sum r_i x_i = 0 \implies r_i = 0 \ \forall i.$$

Definition: A subset X *spans* M iff

$$m \in M \implies m = \sum_{i=1}^{n} r_i x_i \quad \text{for some } r_i \in R,\ x_i \in X.$$

Definition: A subset X is a basis $\iff$ it is a linearly independent spanning set.

Example: $\mathbb{Z}_6$ is an abelian group and thus a $\mathbb{Z}$-module, but not free because $3 \curvearrowright [2] = [6] = 0$, so there are torsion elements. This contradicts linear independence for any subset.

Theorem (Characterization of Free Modules): Let R be a unital ring and M a unital R-module (so $1 \curvearrowright m = m$).

TFAE:

- There exists a nonempty basis of M.
- $M = \oplus_{i \in I} R$ for some index set I.
- There exists a non-empty set X and a map $\iota : X \hookrightarrow M$ such that given $f : X \to N$ for N any R- module, $\exists ! \tilde{f} : M \to N$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[dr, dotted, "\exists ! \tilde{f}"] & \\ X \arrow[u, hook, "\iota"] \arrow[r, "f"] & N \end{tikzcd}$$

Definition: An R-module is *free* iff any of 1,2, or 3 hold.

Proof of $1 \implies 2$*:*

Let X be a basis for M, then define $M \to \oplus_{x \in X} Rx$ by $\varphi(m) = \sum r_i x_i$.

It can be checked that

- This is an R-module homomorphism,
- $\varphi(m) = 0 \implies r_j = 0 \ \forall j \implies m = 0$, so φ is injective,
- φ is surjective, since X is a spanning set.

So $M \cong \bigoplus_{x\in X} Rx$, so it only remains to show that $Rx \cong R$. We can define the map

$$\begin{aligned} \pi_x : R &\to Rx \\ r &\mapsto rx. \end{aligned}$$

Then π_x is onto, and is injective exactly because X is a linearly independent set. Thus $M \cong \oplus R$.

■

Proof of $1 \implies 3$*:*

Let X be a basis, and suppose there are two maps $X \xrightarrow{\iota} M$ and $X \xrightarrow{f} M$. Then define

$$\begin{aligned} \tilde f : M &\to N \\ \sum_i r_i x_i &\mapsto \sum_i r_i f(x_i). \end{aligned}$$

This is clearly an R-module homomorphism, and the diagram commutes because $(\tilde f \circ \iota)(x) = f(x)$. This is unique because $\tilde f$ is determined precisely by $f(X)$.

■

Proof of $3 \implies 2$*:*

We use the usual "2 diagram" trick to produce maps

$$\begin{aligned} \tilde f : M &\to \bigoplus_{x\in X} R \\ \tilde g : \bigoplus_{x\in X} R &\to M. \end{aligned}$$

Then commutativity forces

$$\tilde f \circ \tilde g = \tilde g \circ \tilde f = \mathrm{id}.$$

■

Proof of $2 \implies 1$*:*

We have $M = \oplus_{i\in I} R$ by (2). So there exists a map

$$\psi : \oplus_{i\in I} R \to M,$$

so let $X := \left\{ \psi(1_i) \;\middle|\; i \in I \right\}$, which we claim is a basis.

To see that X is a basis, suppose $\sum r_i \psi(1_i) = 0$. Then $\psi(\sum r_i 1_i) = 0$ and thus $\sum r_i 1_i = 0$ and $r_i = 0$ for all i.

Checking that it's a spanning set: Exercise.

$\blacksquare$

Corollary: Every R-module is the homomorphic image of a free module.

Proof: Let M be an R-module, and let X be any set of generators of R. Then we can make a map

$$M \to \bigoplus_{x \in X} R$$

and there is a map $X \hookrightarrow M$, so the universal property provides a map

$$\tilde{f} : \bigoplus_{x \in X} R \to M.$$

Moreover, $\bigoplus_{x \in X} R$ is free.

$\blacksquare$

Examples:

- $\mathbb{Z}_n$ is **not** a free $\mathbb{Z}$-module for any n.
- If V is a vector space over a field k, then V is a free k-module (even if V is infinite dimensional).
- Every nonzero submodule of a free module over a PID is free.

Some facts:

Let $R = k$ be a field (or potentially a division ring).

1. Every maximal linearly independent subset is a basis for V.
2. Every vector space has a basis.
3. Every linearly independent set is contained in a basis
4. Every spanning set contains a basis.
5. Any two bases of a vector space have the same cardinality.

Theorem (Invariant Dimension): Let R be a commutative ring and M a free R-module.

If X_1, X_2 are bases for R, then $|X_1| = |X_2|$.

Any ring satisfying this condition is said to have the **invariant dimension property**.

> Note that it's difficult to say much more about generic modules. For example, even a finitely generated module may *not* have an invariant number of generators.

24 | Tuesday November 5th

24.1 Free vs Projective Modules

Let R be a PID. Then any nonzero submodule of a free module over a PID is free, and any projective module over R is free.

Recall that a module M is **projective** $\iff$ M is a direct summand of a free module.

In general,

- Free $\implies$ projective, but
- Projective $\not\Longrightarrow$ free.

Example:

Consider $\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}$-module. Is this free as a $\mathbb{Z}$-module?

Note that $\mathbb{Z}_2$ is a submodule and thus projective, but $\mathbb{Z}_2$ is not free since it is not a free module over $\mathbb{Z}$. What fails here is that $\mathbb{Z}_6$ is not a PID, since it is not a domain.

24.2 Annihilators

Definition: Let $m \in M$ a module, then define

$$\mathrm{Ann}_m := \left\{ r \in R \;\middle|\; r.m = 0 \right\} \trianglelefteq R.$$

We can then define a map

$$\begin{aligned} \varphi : R &\to R.m \\ r &\mapsto r.m. \end{aligned}$$

Then $\ker \varphi = \mathrm{Ann}_m$, and $R/\mathrm{Ann} \cong R.m$.

We can also define

$$M_t := \left\{ m \in M \;\middle|\; \mathrm{Ann}_m \neq 0 \right\} \leq M.$$

Lemma: Let R be a PID and p a prime element. Then

- If $p^i m = 0$ then $\mathrm{Ann}_m = (p^j)$ where $0 \leq j \leq i$.
- If $\mathrm{Ann}_m = (p^i)$, then $p^j m \neq 0$ for any $j < m$.

Proof of (1): Since we are in a PID and the annihilator is an ideal, we have $\mathrm{Ann}_m := (r)$ for some $r \in M$. Then $p^i \in (r)$, so $r \;\middle|\; p^i$. But p was prime, to up to scaling by units, we have $r = p^j$ for some $j \leq i$.

$\blacksquare$

Proof of (2): Towards a contradiction, suppose that $\mathrm{Ann}_m = (p^i)$ and $p^j m = 0$ for some $j < i$. Then $p^j \in \mathrm{Ann}_m$, so $p^j \;\middle|\; p^i$. But this forces $j \leq i$, a contradiction.

$\blacksquare$

Some terminology:

- Ann_m is the **order ideal** of m.
- M_t is the **torsion submodule** of M.

- M is **torsion** iff $M = M_t$.
- M is **torsion free** iff $M_t = 0$.
- $\mathrm{Ann}_m = (r)$ is said to have **order** r.
- Rm is the **cyclic module** generated by m.

Theorem: A finitely generated *torsion-free* module over a PID is free.

Proof: Let $M = \langle X \rangle$ for some finite generating set.

We can assume $M \neq (0)$. If $m \neq 0 \in M$, with $rm = 0$ iff $r = 0$.

So choose $S = \{x_1, \cdots, x_n\} \subseteq X$ to be a maximal linearly independent subset of generators, so

$$\sum r_i x_i = 0 \implies r_i = 0 \ \forall i.$$

Consider the submodule $F \coloneqq \langle x_1, \cdots, x_n \rangle \leq M$; then S is a basis for F and thus F is free.

The claim is that $M \cong F$. Supposing otherwise, let $y \in X \setminus S$. Then $S \cup \{y\}$ can not be linearly independent, so there exists $r_y, r_i \in R$ such that

$$r_y y + \sum r_i x^i = 0.$$

Thus $r_y y = -\sum r_i x^i$, where $r_y \neq 0$.

Since $|X| < \infty$, let

$$r = \prod_{y \in X \setminus S} r_y.$$

Then $rX = \left\{ rx \;\middle|\; x \in X \right\} \subseteq F$, and $rM \leq F$.

Now using the particular r we've just defined, define a map

$$\begin{aligned} f : M &\to M \\ m &\mapsto rm. \end{aligned}$$

Then $\mathrm{im}(f) = r.M$, and since M is torsion-free, $\ker f = (0)$. So $M \cong rM \subseteq F$ and M is free.

$\blacksquare$

Theorem: Let M be a finitely generated module over a PID R. Then M can be decomposed as

$$M \cong M_t \oplus F$$

where M_t is torsion and F is free of finite rank, and $F \cong M/M_t$.

Note: we also have $M/F \cong F_t$ since this is a direct sum.

Proof:

Part 1: M/M_t is torsion free.

Suppose that $r(m + M_t) = M_t$, so that r acting on a coset is the zero coset. Then $rm + M_t = M_t$, so $rm \in M_t$, so there exists some r' such that $r'(rm) = 0$ by definition of M_t. But then $(r'r)m = 0$, so in fact $m \in M_t$ and thus $m + M_t = M_t$, making M/M_t torsion free.

Part 2: $F \cong M/M_t$.

We thus have a SES

$$0 \to M_t \to M \to M/M_t \coloneqq F \to 0,$$

and since we've shown that F is torsion-free, by the previous theorem F is free. Moreover, every SES with a free module in the right-hand slot splits:

$$\begin{tikzcd}
 & & & X \arrow[dl, "f"] \arrow[d, hook, "\iota"] & \\
0 \arrow[r] & M_t \arrow[r] & M \arrow[r, "f"'] & F \arrow[r] \arrow[l, dotted, bend right, "h"'] & 0
\end{tikzcd}$$

For $X = \{x_j\}$ a generating set of F, we can choose elements $\{y_i\} \in \pi^{-1}(\iota(X))$ to construct a set map $f : X \to M$. By the universal property of free modules, we get a map $h : F \to M$.

It remains to check that this is actually a splitting, but we have

$$\pi \circ h(x_j) = \pi(h(\iota(x_j))) = \pi(f(x_j)) = \pi(y_j) = x_j.$$

Lemma: Let R be a PID, and $r \in R$ factor as $r = \prod p_i^{k_i}$ as a prime factorization. Then

$$R/(r) \cong \bigoplus R/(p_i^{k_i}).$$

Since R is a UFD, suppose that $\gcd(s, t) = 1$. Then the claim is that

$$R/(st) = R/(s) \oplus R/(t),$$

which will prove the lemma by induction.

Define a map

$$\begin{aligned} \alpha : R/(s) \oplus R/(t) &\to R/(st) \\ (x + (s), y + (t)) &\mapsto tx + sy + (st). \end{aligned}$$

Exercise: Show that this map is well-defined.

Since $\gcd(s, t) = 1$, there exist u, v such that $su + vt = 1$. Then for any $r \in R$, we have

$$rsu + rvt = r,$$

so for any given $r \in R$ we can pick $x = tv$ and $y = su$ so that this holds. As a result, the map α is onto.

Now suppose $tx + sy \in (st)$; then $tx + sy = stz$. We have $su + vt = 1$, and thus

$$utx + usy = ustz \implies utx + (y - tvy) = ustz.$$

We can thus write

$$y = ustv - utx + tvy \in (t).$$

Similarly, $x \in (t)$, so $\ker \alpha = 0$.

■

24.3 Classification of Finitely Generated Modules Over a PID

Theorem (Classification of Finitely Generated Modules over a PID):

Let M be a finitely generated R-module where R is a PID. Then

1. $$M \cong F \bigoplus_{i=1}^{t} R/(r_i)$$

   where F is free of finite rank and $r_1 \,\Big|\, r_2 \,\Big|\, \cdots \,\Big|\, r_t$. The rank and list of ideals occurring is uniquely determined by M. The r_i are referred to as the **invariant factors**.

b. $$M \cong F \bigoplus_{i=1}^{k} R/(p_i^{s_i})$$

   where F is free of finite rank and p_i are primes that need not be distinct. The rank and ideals are uniquely determined by M. The $p_i^{s_i}$ are referred to as **elementary divisors**.

25 | Thursday November 7th

25.1 Projective Modules

Definition: A **projective** module P over a ring R is an R-module such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\exists\varphi"'] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

i.e. for every surjective map $g : M \twoheadrightarrow N$ and every map $f : P \to N$ there exists a lift $\varphi : P \to M$ such that that $g \circ \varphi = f$.

Theorem: Every free module is projective.

Proof: Suppose $M \twoheadrightarrow N \to 0$ and $F \xrightarrow{f} N$, so we have the following situation:

$$\begin{tikzcd}
& x \arrow[d, hook] \arrow[dl, dotted] & \\
& F \arrow[d, "f"] \arrow[dl, dashed, "\exists\varphi"'] & \\
M \arrow[r, two heads, "g"] & N \arrow[r, dotted] & 0
\end{tikzcd}$$

For every $x \in X$, there exists an $m_x \in M$ such that $g(m_x) = f(i(x))$. By freeness, there exists a $\varphi : F \to M$ such that this diagram commutes.

■

Corollary: Every R-module is the homomorphic image of a projective module.

Proof: If M is an R-module, then $F \twoheadrightarrow M$ where F is free, but free modules are surjective.

■

Theorem: Let P be an R-module. Then TFAE:

a. P is projective.

b. Every SES $0 \to M \to N \to P \to 0$ splits.

c. There exists a free module F such that $F = P \oplus K$ for some other module K.

Proof:

$a \implies b$:

We set up the following situation, where s is produced by the universal property:

$$\begin{tikzcd}
& & & P \arrow[dl, "\exists s"'] \arrow[d, hook, "\mathrm{id}"] & \\
0 \arrow[r] & M \arrow[r] & N \arrow[r, two heads] & P \arrow[r] & 0
\end{tikzcd}$$

■

$b \implies c$:

Suppose we have $0 \to M \to N \to P \to 0$ a SES which splits, then $N \cong M \oplus P$ by a previous theorem.

■

$c \implies a$:

We have the following situation:

$$
\begin{tikzcd}
 & F = P \oplus K \arrow[d, bend right, "\pi"'] \arrow[ddl, dotted, "\exists h"'] \\
 & P \arrow[u, bend right, "\iota"'] \arrow[d, "f"] \arrow[dl, dotted, "\varphi = \iota \circ h"] \\
M \arrow[r, two heads] & N
\end{tikzcd}
$$

By the previous argument, there exists an $h : F \to M$ such that $g \circ h = f \circ \pi$. Set $\varphi = h \circ \iota$.

Exercise: Check that $g \circ \varphi = f$.

■

Theorem: $\bigoplus P_i$ is projective $\iff$ each P_i is projective.

Proof:

$\implies$: Suppose $\oplus P_i$ is projective.

Then there exists some $F = K \oplus \bigoplus P_i$ where F is free. But then P_i is a direct summand of F, and is thus projective.

$\impliedby$: Suppose each P_i is projective.

Then there exists $F_i = P_i \oplus K_i$, so $F \coloneqq \bigoplus F_i = \bigoplus (P_i \oplus K_i) = \bigoplus P_i \oplus \bigoplus K_i$. So $\bigoplus P_i$ is a direct summand of a free module, and thus projective.

■

> Note that a direct sum has *finitely many* nonzero terms. Can use the fact that a direct sum of free modules is still free by taking a union of bases.

Example of a projective module that is not free:

Take $R = \mathbb{Z}_6$, which is not a PID and not a domain. Then $\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$, and $\mathbb{Z}_2, \mathbb{Z}_3$ are projective R-modules. By previous statements, we know these are torsion as $\mathbb{Z}$-modules, and thus not free.

25.2 Endomorphisms as Matrices

> See section 7.1 in Hungerford

Let $M_{m,n}(\mathbb{R})$ denote $m \times n$ matrices with coefficients in R. This is an R-R bimodule, and since R is not necessarily a commutative ring, these two module actions may not be equivalent.

If $m = n$, then $M_{n,n}(R)$ is a ring under the usual notions of matrix addition and multiplication.

Theorem: Let V, W be vector spaces where $\dim V = m$ and $\dim W = n$. Let $\hom_k(V, W)$ be the set of linear transformations between them.

Then $\hom_k(V, W) \cong M_{m,n}(k)$ as k-vector spaces.

Proof: Choose bases of V, W. Then consider

$$
\begin{aligned}
T : V &\to W \\
v_1 &\mapsto \sum_{i=1}^{n} a_{1,i}\ w_i \\
v_2 &\mapsto \sum_{i=1}^{n} a_{2,i}\ w_i \\
&\vdots
\end{aligned}
$$

This produces a map

$$
\begin{aligned}
f : \hom_k(V, W) &\to M_{m,n}(k) \\
T &\mapsto (a_{i,j}),
\end{aligned}
$$

which is a matrix.

Exercise: Check that this is bijective.

■

Theorem: Let M, N be free left R-modules of rank m, n respectively. Then $\hom_R(M, N) \cong M_{m,n}(R)$ as R-R bimodules.

Notation: Suppose M, N are free R-modules, then denote β_m, β_n be fixed respective bases. We then write $[T]_{\beta_m, \beta_n} \coloneqq (a_{i,j})$ to be its *matrix representation.*

Theorem: Let R be a ring and let V, W, Z be three free left R-modules with bases $\beta_v, \beta_w, \beta_z$ respectively. If $T : V \to W, S : W \to Z$ are R-module homomorphisms, then $S \circ T : V \to Z$ exists and

$$
[S \circ T]_{\beta_v, \beta_z} = [T]_{\beta_v, \beta_w} [S]_{\beta_w, \beta_z}
$$

Proof: Exercise.

Show that

$$
(S \circ T)(v_i) = \sum_{j}^{t} \sum_{k}^{m} a_{ik} b_{kj} z_j .
$$

■

25.3 Matrices and Opposite Rings

Suppose $\Gamma : \hom_R(V, V) \to M_n(R)$ and V is a free left R-module. By the theorem, we have $\Gamma(T \circ S) = \Gamma(S)\Gamma(T)$. We say that Γ is an **anti-homomorphism**.

To address this mixup, given a ring R we can define R^{op} which has the same underlying set of R but with the modified multiplication

$$
x \cdot y \coloneqq yx \in R.
$$

If R is commutative, then $R \cong R^{op}$.

$\blacksquare$

Theorem: Let R be a unital ring and V an R-module.

Then $\hom_R(V, V) \cong M_n(R^{op})$ as rings.

Proof: Since $\Gamma(S \circ T) = \Gamma(T)\Gamma(S)$, define a map

$$\begin{aligned} \Theta : M_{n,n}(R) &\to M_{n,n}(R^{op}) \\ A &\mapsto A^t. \end{aligned}$$

Then

$$\Theta(AB) = (AB)^t = B^t A^t = \Theta(B)\Theta(A),$$

so Θ is an anti-isomorphism.

Thus $\Theta \circ \Gamma$ is an anti-anti-homomorphism, i.e. a usual homomorphism.

$\blacksquare$

Definition: A matrix A is **invertible** iff there exists a B such that $AB = BA = \mathrm{id}_n$.

Proposition: Let R be a unital ring and V, W free R-modules with $\dim V = n, \dim W = m$. Then

1. $T \in \hom_R(V, W)$ is an isomorphisms iff $[T]_{\beta_v, \beta_w}$ is invertible.
2. $[T^{-1}]_{\beta_v, \beta_w} = [T]_{\beta_v, \beta_w}^{-1}$.

Definition: We'll say that two matrices A, B are **equivalent** iff there exist P, Q invertible such that $PAQ = B$.

26 Tuesday November 12th

26.1 Equivalence and Similarity

Recall from last time:

If V, W are free left R-modules of ranks m, n respectively with bases β_v, β_w respectively, then

$$\hom_R(V, W) \cong M_{m,n}(R).$$

Definition: Two matrices $A, B \in M_{m \times n}(R)$ are **equivalent** iff

$$\exists P \in \mathrm{GL}(m, R),\ \exists Q \in \mathrm{GL}(n, R) \quad \text{such that} \quad A = PBQ.$$

Definition: Two matrices $A, B \in M_m(R)$ are **similar** iff

$$\exists P \in \mathrm{GL}(m, R) \quad \text{such that} \quad A = P^{-1}BP.$$

Theorem: Let $T : V \to W$ be an R-module homomorphism.

Then T has an $m \times n$ matrix relative to other bases for $V, W \iff$

$$B = P[T]_{\beta_v, \beta_w} Q.$$

Proof: $\implies$:

Let β_v', β_w' be other bases. Then we want $B = [T]_{\beta_v', \beta_w'}$, so just let

$$P = [\mathrm{id}]_{\beta_v', \beta_v} \quad Q = [\mathrm{id}]_{\beta_w, \beta_w'}.$$

■

$\impliedby$:

Suppose $B = P[T]_{\beta_v, \beta_w} Q$ for some P, Q.

Let $g : V \to V$ be the transformation associated to P, and $h : W \to W$ associated to Q^{-1}.

Then

$$\begin{aligned} P &= [\mathrm{id}]_{g(\beta_v), \beta_v} \\ \implies Q^{-1} &= [\mathrm{id}]_{h(\beta_w), \beta_w} \\ \implies Q &= [\mathrm{id}]_{\beta_w, h(\beta_w)} \\ \implies B &= [T]_{g(\beta_v), h(\beta_w)}. \end{aligned}$$

■

Corollary: Let V be a free R-module and β_v a basis of size n.

Then $T : V \to V$ has an $n \times n$ matrix relative to β_v relative to another basis $\iff$

$$B = P[T]_{\beta_v, \beta_v} P^{-1}.$$

Note how this specializes to the case of linear transformations, particularly when B is diagonalizable.

26.2 Review of Linear Algebra:

Let D be a division ring. Recall the notions of rank and nullity, and the statement of the rank-nullity theorem.

Note that we can always factor a linear transformation $\varphi : E \to F$ as the following short exact sequence:

$$0 \to \ker \varphi \to E \xrightarrow{\varphi} \mathrm{im}(\varphi) \to 0,$$

and since every module over a division ring is free, this sequence splits and $E \cong \ker \varphi \oplus \mathrm{im}(\varphi)$. Taking dimensions yields the rank-nullity theorem.

Let $A \in M_{m,n}(D)$ and define

- $R(A) \in D^n$ is the span of the rows of A, and
- $C(A) \in D^m$ is the span of the columns of A.

Recall that finding a basis of the **row space** involves doing Gaussian Elimination and taking the rows which have nonzero pivots.

For a basis of the **column space**, you take the corresponding columns in the *original* matrix.

Note that in this case, $\dim R(A) = \dim C(A)$, and in fact these are always equal.

Theorem (Rank and Equivalence): Let $\varphi : V \to W$ be a linear transformation and A be the matrix of φ relative to β_v, β_v'.

Then $\dim \operatorname{im}(\pi) = \dim C(A) = \dim R(A)$.

Proof: Construct the matrix $A = [\varphi]_{\beta_v, \beta_w}$.

Then $\varphi : V \to W$ descends to a map $A : D^m \to D^n$. Writing the matrix A out and letting $v \in D^m$ a row vector act on A from the *left* yields a column vector $Av \in D^n$.

But then $\operatorname{im}(\varphi)$ corresponds to $R(A)$, and so

$$\dim \operatorname{im}(\varphi) = \dim R(A) = \dim C(A).$$

■

26.3 Canonical Forms

Let $1 \leq r \leq \min(m, n)$, and define E_r to be the $m \times n$ matrix with the $r \times r$ identity matrix in the top-left block.

Theorem: Let $A, B \in M_{m,n}(D)$. Then

1. A is equivalent to $E_r \iff \operatorname{rank} A = r$
   - That is, $\exists P, Q$ such that $E_r = PAQ$
2. A is equivalent to B iff $\operatorname{rank} A = \operatorname{rank} B$.
3. E_r for $r = 0, 1, \cdots, \min(m, n)$ is a complete set of representatives for the relation of matrix equivalence on $M_{m,n}(D)$.

Let $X = M_{m,n}(D)$ and $G = \mathrm{GL}_m(D) \times \mathrm{GL}_n(D)$, then

$$G \curvearrowright X \text{ by } (P, Q) \curvearrowright A := PAQ^{-1}.$$

Then the orbits under this action are exactly $\left\{ E_r \;\middle|\; 0 \leq r \leq \min(m, n) \right\}$.

Proof: Note that 2 and 3 follow from 1, so we'll show 1.

$\implies$:

Let A be an $m \times n$ matrix for some linear transformation $\varphi : D^m \to D^n$ relative to some basis. Assume $\operatorname{rank} A = \dim \operatorname{im}(\varphi) = r$. We can find a basis such that $\varphi(u_i) = v_i$ for $1 \leq i \leq r$, and $\varphi(u_i) = 0$ otherwise. Relative to this basis, $[\varphi] = E_r$. But then A is equivalent to E_r.

$\Longleftarrow$:

If $A = PE_rQ$ with P, Q invertible, then $\dim \operatorname{im}(A) = \dim \operatorname{im}(E)_r$, and thus $\operatorname{rank} A = \operatorname{rank} E_r = r$.

How do we do this? Recall the row operations:

- Interchange rows
- Multiply a row by a unit
- Add one row to another

But each corresponds to left-multiplication by an elementary matrix, each of which is invertible. If you proceed this way until the matrix is in RREF, you produce $P \prod P_i A$. You can now multiply on the *right* by elementary matrices to do column operations and move all pivots to the top-left block, which yields E_r.

■

Theorem: Let $A \in M_{m,n}(R)$ where R is a PID.

Then A is equivalent to a matrix with L_r in the top-left block, where L_r is a diagonal matrix with $L_{ii} = d_i$ such that $d_1 \,\Big|\, d_2 \,\Big|\, \cdots \,\Big|\, d_r$. Each (d_i) is uniquely determined by A.

27 | Thursday November 14th

27.1 Equivalence to Canonical Forms

Let D be a division ring and k a field.

Recall that a matrix A is *equivalent* to $B \iff \exists P, Q$ such that $PBQ = A$. From a previous theorem, if $\operatorname{rank}(A) = r$, then A is equivalent to a matrix with I_r in the top-left block.

Theorem: Let A be a matrix over a PID R. Then A is equivalent to a matrix with L_r in the top-left corner, where $L_r = \operatorname{diag}(d_1, d_2, \cdots, d_r)$ and $d_1 \,\Big|\, d_2 \,\Big|\, \cdots \,\Big|\, d_r$, and the d_i are uniquely determined.

Theorem: Let A be an $n \times n$ matrix over a division ring D. TFAE:

1. $\operatorname{rank} A = n$.
2. A is equivalent to I_n.
3. A is invertible.

$1 \implies 2$: Use Gaussian elimination.

$2 \implies 3$: $A = PI_nQ = PQ$ where P, Q are invertible, so $PQ = A$ is invertible.

$3 \implies 1$: If A is invertible, then $A : D^n \to D^n$ is bijective and thus surjective, so $\dim \operatorname{im}(A) = n$.

Note: the image is now *row space* because we are taking *left* actions.

■

27.2 Determinants

Definition: Let $M_1, \cdots, M_n$ be R-modules, and then $f : \prod M_i \to R$ is n-linear iff

$$\begin{aligned} f(m_1, m_2, \cdots, rm_k + sm_k', \cdots, m_n) = \\ rf(m_1, \cdots, m_k, \cdots m_k) + sf(m_1, \cdots, m_k', \cdots, m_n). \end{aligned}$$

Example: The inner product is a 2-linear form.

Definition: f is **symmetric** iff

$$f(m_1, \cdots, m_n) = f(m_{\sigma(1)}, \cdots, m_{\sigma(n)}) \ \forall \sigma \in S_n.$$

Definition: f is **skew-symmetric** iff

$$f(m_1, \cdots, m_n) = \operatorname{sgn}(\sigma) f(m_{\sigma(1)}, \cdots, m_{\sigma(n)}) \ \forall \sigma \in S_n,$$

where

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd} \end{cases}.$$

Definition: f is **alternating** iff

$$m_i = m_j \text{ for some pair } (i, j) \implies f(m_1, \cdots, m_n) = 0.$$

Theorem: Let f be an n-linear form. If f is alternating, then f is skew-symmetric.

Proof: It suffices to show the $n = 2$ case. We have

$$\begin{aligned} 0 &= f(m + 1 + m_2, m_1 + m_2) \\ &= f(m_1, m_1) + f(m_1, m_2) + f(m_2, m_1) + f(m_2, m_2) \\ &= f(m_1, m_2) + f(m_2, m_1) \\ \implies f(m_1, m_2) &= -f(m_2, m_1). \end{aligned}$$

■

Theorem: Let R be a unital commutative ring and let $r \in R$ be arbitrary.

Then

$$\exists! f : \bigoplus_{i=1}^{n} R^n \to R,$$

where f is an alternating R-form such that $f(\mathbf{e}_i) = r$ for all i, where $\mathbf{e}_i = [0, 0, \cdots, 0, 1, 0, \cdots, 0, 0]$.

R^n is a free module, so f can be identified with a matrix once a basis is chosen.

Proof:

Existence: Let $x_i = [a_{i1}, a_{i2}, \cdots, a_{in}]$ and define

$$f(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) r \prod_i a_{i\sigma(i)}.$$

Exercise: Check that $f(\mathbf{e}_1, \cdots, \mathbf{e}_n) = r$ and f is n-linear.

Moreover, f is alternating. Consider $f(x_1, \cdots, x_n)$ where $x_i = x_j$ for some $i \neq j$.

Letting $\varphi = (i, j)$, we can write $S_n = A_n \coprod A_n \rho$.

If σ is even, then the summand is

$$(+1) r a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

Since $x_i = x_j$, we'll have $\prod\limits_k a_{ik} = \prod a_{jk}$. Then consider applying $\sigma\rho$. We have

$$\begin{aligned} -r \prod a_{i\sigma(i)} &= -r a_{1\sigma(1)} \cdots \mathbf{a}_{j\sigma(j)} \cdots \mathbf{a}_{i\sigma(i)} \cdots a_{n,\sigma(n)} \\ &= -r \prod a_{i\sigma(i)} = -r a_{1\sigma(1)} \cdots \mathbf{a}_{i\sigma(i)} \cdots \mathbf{a}_{j\sigma(j)} \cdots a_{n,\sigma(n)}, \end{aligned}$$

which permutes the i, j terms. So these two terms cancel, the remaining terms are untouched.

Uniqueness: Let $x_i = \sum\limits_j a_{ij} \mathbf{e}_j$. Then

$$\begin{aligned} f(x_1, \cdots, x_n) &= f(\sum_{j_1} a_j^1 \mathbf{e}_j, \cdots, \sum_{j_n} a_j^n \mathbf{e}_j) \\ &= \sum_{j_1} \cdots \sum_{j_n} f(\mathbf{e}_{j_1}, \cdots, \mathbf{e}_{j_n}) a_{1,j_1} \cdots a_{n,j_n} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f(\mathbf{e}_1, \cdots, \mathbf{e}_n) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) r a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}. \end{aligned}$$

■

Definition: Let R be a commutative unital ring and define $\det : M_n(R) \to R$ is the unique n-alternating form with $\det(I) = 1$, and is called the *determinant*.

Theorem: Let $A, B \in M_n(R)$. Then

a. $|AB| = |A||B|$

b. A is invertible $\iff$ $|A| \in R^\times$

c. $A \sim B \implies |A| = |B|$.

d. $\left|A^t\right| = |A|$.

e. If A is triangular, then $|A|$ is the product of the diagonal entries.

Proof of a: Let B be fixed.

Let $\Delta_B : M_n(R) \to R$ be defined as $C \mapsto |CB|$. Then this is an alternating form, so by the theorem, $\Delta_B = r\det$. But then $\Delta_B(C) = r|C|$, so $r|C| = |CB|$. So pick $C = I$, then $r = |B|$.

■

Proof of b: Suppose A is invertible.

Then $AA^{-1} = I$, so $\left|AA^{-1}\right| = |A|\left|A^{-1}\right| = 1$, which shows that $|A|$ is a unit.

■

Proof of c: Let $A = PBP^{-1}$. Then

$$|A| = \left|PBP^{-1}\right| = |P||B|\left|P^{-1}\right| = |P|\left|P^{-1}\right||B| = |B|.$$

■

Proof of d: Let $A = (a_{ij})$, so $B = (b_{ij}) = (a_{ji})$. Then

$$\begin{aligned}
\left|A^t\right| &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_k b_{k\sigma(k)} \\
&= \sum_\sigma \operatorname{sgn}(\sigma) \prod_k a_{\sigma(k)k} \\
&= \sum_{\sigma^{-1}} \operatorname{sgn}(\sigma) \prod_k a_{k\sigma^{-1}(k)} \\
&= \sum_\sigma \operatorname{sgn}(\sigma) \prod_k a_{k\sigma(k)} \\
&= |A|.
\end{aligned}$$

■

Proof of e: Let A be upper-triangular. Then

$$|A| = \sum_\sigma \operatorname{sgn}(\sigma) \prod_k a_{k\sigma(k)} = a_{11}a_{22}\cdots a_{nn}.$$

■

Next time:

- Calculate determinants
  - Gaussian elimination
  - Cofactors
- Formulas for A^{-1}
- Cramer's rule

28 | Tuesday November 19th

28.1 Determinants

Let $A \in M_n(R)$, where R is a commutative unital ring.

Given $A = (a_{ij})$, recall that

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod a_{i,\sigma(i)}.$$

This satisfies a number of properties:

- $\det(AB) = \det A \det B$
- A invertible $\implies \det A$ is a unit in R
- $A \sim B \implies \det(A) = \det(B)$
- $\det A^t = \det A$
- A is triangular $\implies \det A = \prod a_{ii}$.

28.1.1 Calculating Determinants

1. **Gaussian Elimination**

   a. B is obtained from A by interchanging rows: $\det B = -\det A$

   b. B is obtained from A by multiplying $\det B = r \det A$

   c. B is obtained from A by adding a scalar multiple of one row to another: $\det B = \det A$.

2. **Cofactors** Let A_{ij} be the $(n-1) \times (n-1)$ minor obtained by deleting row i and column j, and $C_{ij} = (-1)^{i+j} \det A_{ij}$.

   Then **(theorem)** $\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ by expanding along either a row or column.

Theorem:

$$A \operatorname{Adj}(A) = \det(A) I_n,$$

where $\operatorname{Adj} = (C_{ij})^t$.

If A^{-1} is a unit, then $A^{-1} = \operatorname{Adj}(A)/\det(A)$.

28.1.2 Decomposition of a Linear Transformation:

Let $\varphi : V \to V$ be a linear transformation of vector spaces. and $R = \hom_k(V, V)$. Then R is a ring.

Let $f(x) = \sum a_j x^j \in k[x]$ be an arbitrary polynomial. Then for $\varphi \in R$, it makes sense to evaluate $f(\varphi)$ where φ^n denotes an n-fold composition, and $f(\varphi) : V \to V$.

Lemma:

- There exists a unique monic polynomial $q_\varphi(x) \in k[x]$ such that $q_\varphi(\varphi) = 0$ and $f(\varphi) = 0 \implies q_\varphi \,\Big|\, f$. q_φ is referred to as the **minimal polynomial** of φ.
- The exact same conclusion holds with φ replaced by a matrix A, yielding q_A.
- If A is the matrix of φ relative to a fixed basis, then $q_\varphi = q_A$.

Proof of a and b: Fix φ, and define

$$\begin{aligned}\Gamma : k[x] &\to \hom_k(V, V)\\ f &\mapsto f(\varphi).\end{aligned}$$

Since $\dim_k V^\vee = \dim_k V < \infty$ and $\dim_k k[x] = \infty$, we must have $\ker \Gamma \neq 0$.

Since $k[x]$ is a PID, we have $\ker \Gamma = (q)$ for some $q \in k[x]$. Then if $f(\varphi) = 0$, we have $f(x) \in \ker \Gamma \implies q \,\Big|\, f$. We can then rescale q to be monic, which makes it unique.

Note: for (b), just replace φ with A everywhere.

$\blacksquare$

Proof of c: Suppose $A = [\varphi]_{\mathcal{B}}$ for some fixed basis $\mathcal{B}$.

Then $\hom_k(V, V) \cong M_n(k)$, so we have the following commutative diagram:

$$\begin{tikzcd} k[x] \arrow[r, "\Gamma_\varphi"] \arrow[dr, "\Gamma_A"'] & \hom_k(V,V) \arrow[d, hook, two heads, "\cong"] \\ & M_n(k) \end{tikzcd}$$

$\blacksquare$

28.1.3 Finitely Generated Modules over a PID

Let M be a finitely generated module over R a PID. Then

$$\begin{aligned} M &\cong F \oplus \bigoplus_{i=1}^n R/(r_i) \quad r_1 \,\Big|\, r_2 \,\Big|\, \cdots r_n \\ M &\cong F \oplus \bigoplus_{i=1}^n R/(p_i^{s_i}) \quad p_i \text{ not necessarily distinct primes. } . \end{aligned}$$

Letting $R = k[x]$ and $\varphi : V \to V$ with $\dim_k V < \infty$, V becomes a $k[x]$-module by defining

$$f(x) \curvearrowright \mathbf{v} \coloneqq f(\varphi)(\mathbf{v})$$

Note that W is a $k[x]$-submodule iff $\varphi : W \to W$.

Let $v \in V$, and $\langle v \rangle = \left\{ \varphi^i(v) \;\middle|\; i = 0, 1, 2, \cdots \right\}$ is the **cyclic submodule generated by** v, and we write $\langle v \rangle = k[x].v$.

Theorem: Let $\varphi : V \to V$ be a linear transformation. Then

1. There exist cyclic $k[x]$-submodules V_i such that $V = \bigoplus_{i=1}^{t} V_i$, where for each i there exists a $q_i : V_i \to V_i$ such that $q_1 \;\Big|\; q_2 \;\Big|\; \cdots \;\Big|\; q_t$.

2. There exist cyclic $k[x]$-submodules V_j such that $V = \bigoplus_{j=1}^{\nu}$ and $p_j^{m_j}$ is the minimal polynomial of $\varphi : V_j \to V_j$.

Proof: Apply the classification theorem to write $V = \bigoplus R/(r_i)$ as an invariant factor decomposition.

Then $R/(q_i) \cong V_i$, some vector space, and since there is a direct sum decomposition, the invariant factors are minimal polynomials for $\varphi_i : V_i \to V_i$, and thus $k[x]/(q_i)$.

■

28.1.4 Canonical Forms for Matrices

We'll look at

- Rational Canonical Form
- Jordan Canonical Form

Theorem: Let $\varphi : V \to V$ be linear, then V is a cyclic $k[x]$-module and $\varphi : V \to V$ has minimal polynomial $q(x) = \sum_j a_j x^j$ iff $\dim V = n$ and V has an ordered basis of the form

$$[\varphi]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

with ones on the super-diagonal.

Proof:

$\Longleftarrow$:

Let $V = k[x].v = \left\langle v, \varphi(v), \cdots, \varphi^{n-1}(v) \right\rangle$ where $\deg q(x) = n$. The claim is that this is a linearly independent spanning set.

Linear independence: suppose $\sum_{j=0}^{n-1} k_j \varphi^j(v) = 0$ with some $k_j \neq 0$. Then $f(x) = \sum k_j x^j$ is a polynomial where $f(\varphi) = 0$, but this contradicts the minimality of $q(x)$.

But then we have n linearly independent vectors in V which is dimension n, so this is a spanning set.

$\implies$:

We can just check where basis elements are sent. Set $\mathcal{B} = \left\{v, \varphi(v), \cdots, \varphi^{n-1}(v)\right\}$. Then

$$\begin{aligned} v &\mapsto \varphi(v) \\ \varphi(v) &\mapsto \varphi^2(v) \\ &\vdots \\ \varphi n-1(v) &\mapsto \varphi^n(v) = -\sum a_i \varphi^i(v) \\ &. \end{aligned}$$

$\impliedby$ Fix a basis $B = \{v_1, \cdots, v_n\}$ and $A = [\varphi]_B$, then

$$\begin{aligned} v_1 &\mapsto v_2 = \varphi(v_1) \\ v_1 &\mapsto v_3 = \varphi^2(v_1) \\ v_{n-2} &\mapsto v_{n-1} = \varphi^2(v_1). \end{aligned}$$

and

$$\varphi^n(v) = -a_k v_1 \neq -a_1\varphi(v_1), \cdots - a_{n-1}\varphi^{n-1}(v_1).$$

Thus $V = k[x].v_1$, since $\dim V = n$ with $\left\{v_1, \varphi(v_1), \cdots, \varphi^{n-1}(v_1)\right\}$ as a basis.

■

29 | Thursday November 21

29.1 Cyclic Decomposition

Let $\varphi : V \to V$ be a linear transformation; then V is a $k[x]$ module under $f(x) \curvearrowright v \coloneqq f(\varphi)(v)$.

By the structure theorem, since $k[x]$ is a PID, we have an invariant factor decomposition $V = \bigoplus V_i$ where each V_i is a cyclic $k[x]$-module. If q_i is the minimal polynomial for $\varphi_i : V_i \to V_i$, then $q_i \,\Big|\, q_{i+1}$ for all i.

We also have an elementary divisor decomposition where $p_i^{m_i}$ are the minimal polynomials for φ_i.

Note: one is only for the restriction to the subspaces? Check.

Recall that if φ has minimal polynomial $q(x)$. Then if $\dim V = n$, there exists a basis of B if V such that $[\varphi]_B$ is given by the **companion matrix** of $q(x)$. This is the **rational canonical form**.

Corollary: Let $\varphi : V \to V$ be a linear transformation. Then V is a cyclic $k[x]$-module and φ has minimal polynomial $(x-b)^n \iff \dim V = n$ and there exists a basis such that

$$[\varphi]_B = \begin{bmatrix} b & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & b & 1 \end{bmatrix}.$$

This is the **Jordan Canonical form**.

> Note that if k is not algebraically closed, we can only reduce to RCF. If k *is* closed, we can reduce to JCF, which is slightly nicer.

Proof:

Let $\delta = \varphi - b \cdot \mathrm{id}_V$. Then

- $q(x)$ is the minimal polynomial for $\varphi \iff x^n$ is the minimal polynomial for δ.
- A priori, V has two $k[x]$ structures – one given by φ, and one by δ.
- *Exercise*: V is cyclic with respect to the φ structure $\iff$ V is cyclic with respect to the the δ structure.

Then the matrix $[\delta]_B$ relative to an ordered basis for δ is with only zeros on the diagonal and 1s on the super-diagonal, and $[\varphi]_B$ is the same but with b on the diagonal.

$\blacksquare$

Lemma: Let $\varphi : V \to V$ with $V = \bigoplus_i^t V_i$ as $k[x]$-modules. Then M_i is a matrix of $\varphi|_{V_i} : V_i \to V_i$ relative to some basis for $V_i \iff$ the matrix of φ wrt some ordered basis is given by

$$\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_t \end{bmatrix}.$$

Proof:

$\implies$: Suppose B_i is a basis for V_i and $[\varphi]_{B_i} = M_i$. Then let $B = \cup_i B_i$; then B is a basis for V and the matrix is of the desired form.

$\impliedby$: Suppose that we have a basis B and $[\varphi]_B$ is given by a block diagonal matrix filled with blocks M_i. Suppose $\dim M_i = n_i$. If $B = \{v_1, v_2, \cdots, v_n\}$, then take $B_1 = \{v_1, \cdots, v_{n_1}\}$ and so on. Then $[\varphi_i]_{B_i} = M_i$ as desired.

$\blacksquare$

Application: Let $V = \bigoplus V_i$ with q_i the minimal polynomials of $\varphi : V_i \to V_i$ with $q_i \,\Big|\, q_{i+1}$.

Then there exists a basis where $[\varphi]_B$ is block diagonal with blocks M_i, where each M_i is in rational canonical form with minimal polynomial $q_i(x)$. If k is algebraically closed, we can obtain elementary divisors $p_i(x) = (x - b_i)^{m_i}$. Then there exists a similar basis where now each M_i is a *Jordan block* with b_i on the diagonals and ones on the super-diagonal.

Moreover, in each case, there is a basis such that $A = P[M_i]P^{-1}$ (where M_i are the block matrices obtained). When A is diagonalizable, P contains the eigenvectors of A.

Corollary: Two matrices are similar $\iff$ they have the same invariant factors and elementary divisors.

Example: Let $\varphi : V \to V$ have invariant factors $q_1(x) = (x-1)$ and $q_2(x) = (x-1)(x-2)$.

Then $\dim V = 3$, $V = V_1 \oplus V_2$ where $\dim V_1 = 1$ and $\dim V_2 = 2$. We thus have

$$[\varphi]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{pmatrix}.$$

Moreover, we have

$$V \cong \frac{k[x]}{(x-1)} \oplus \frac{k[x]}{(x-1)(x-2)} \cong \frac{k[x]}{(x-1)} \oplus \frac{k[x]}{(x-1)} \oplus \frac{k[x]}{(x-2)},$$

so the elementary divisors are $x-1, x-1, x-2$.

Invariant factor decompositions should correspond to rational canonical form blocks, and elementary divisors should correspond to Jordan blocks.

Theorem: Let A be an $n \times n$ matrix over k. Then the matrix $xI_n - A \in M_n(k[x])$ is equivalent in $k[x]$ to a diagonal matrix D with non-zero entries $f_1, f_2, \cdots f_t \in k[x]$ such that the f_i are monic and $f_i \,\Big|\, f_{i+1}$. The non-constant polynomials among the f_i are the invariant factors of A.

Proof (Sketch): Let $V = k^n$ and $\varphi : k^n \to k^n$ correspond to A under the fixed standard basis $\{e_i\}$. Then V has a $k[x]$-module structure induced by φ.

Let F be the free $k[x]$ module with basis $\{u_i\}_{i=1}^n$, and define the maps

$$\begin{aligned} \pi : F &\to k^n \\ u_i &\mapsto e_i \end{aligned}$$

and

$$\begin{aligned} \psi : F &\to F \\ u_i &\mapsto xu_i - \sum_j a_{ij} u_j. \end{aligned}$$

Then ψ relative to the basis $\{u_i\}$ is $xI_n - A$.

Then *(exercise)* the sequence

$$F \xrightarrow{\psi} F \xrightarrow{\pi} k^n \to 0$$

is exact, $\operatorname{im}(\pi) = k^n$, and $\operatorname{im}(\psi) = \ker \pi$.

We then have $k^n \cong F/\ker \pi = F/\operatorname{im}(\psi)$, and since $k[x]$ is a PID,

$$xI_n - A \sim D := \begin{bmatrix} L_r & 0 \\ 0 & 0 \end{bmatrix}.$$

where L_r is diagonal with f_is where $f_i \mid f_{i+1}$.

However, $\det(xI_n - A) \neq 0$ because $xI_n - A$ is a monic polynomial of degree n.

But $\det xI_n - A = \det(D)$, so this means that L_r must take up the entire matrix of D, so there is no zero in the bottom-right corner. So $L_r = D$, and D is the matrix of ψ with respect to $B_1 = \{v_i\}$ and $B_2 = \{w_i\}$ with $\psi(v_i) = f_i w_i$.

Thus

$$\operatorname{im}(\psi) = \bigoplus_{i=1}^n k[x] f_i w_i.$$

But then

$$\begin{aligned} V = k^n \cong F/\operatorname{im}(\psi) &\cong \frac{k[x]w_1 \oplus \cdots \oplus k[x]w_n}{k[x]f_1 w_1 \oplus \cdots \oplus k[x] f_n w_n} \\ &\cong \bigoplus_{i=1}^n k[x]/(f_i). \end{aligned}$$

■

30 | Tuesday November 26th

30.1 Minimal and Characteristic Polynomials

Theorem

a. ? (Todo)

b. **(Cayley Hamilton)** If p is the minimal polynomial of a linear transformation φ, then $p(\varphi) = 0$

c. For any $f(x) \in k[x]$ that is irreducible, $f(x) \mid p_\varphi(x) \iff f(x) \mid q_\varphi(x)$.

Proof of (a): ?

■

Proof of (b):

If $q_\varphi(x) \mid p_\varphi(x)$ and $q_\varphi(\varphi) = 0$, then $p_\varphi(\varphi) = 0$ as well.

■

Proof of (c): We have $f(x) \mid q_\varphi(x) \implies f(x) \mid p_\varphi(x)$ and $f(x) \mid p_\varphi(x) \implies f(x) \mid q_i(x)$ for some i, and so $f(x) \mid q_\varphi(x)$.

■

30.2 Eigenvalues and Eigenvectors

Definition: Let $\varphi : V \to V$ be a linear transformation. Then

1. An **eigenvector** is a vector $\mathbf{v} = \mathbf{0}$ such that $\varphi(\mathbf{v}) = \lambda \mathbf{v}$ for some $\lambda \in k$.
2. If such a $\mathbf{v}$ exists, then λ is called an **eigenvalue** of φ.

Theorem: The eigenvalues of φ are the roots of $p_\varphi(x)$ in k.

Proof: Let $[\varphi]_B = A$, then

$$
\begin{aligned}
& p_A(\lambda) = p_\varphi(\lambda) = \det(\lambda I - A) = 0 \\
& \iff \exists \mathbf{v} \neq \mathbf{0} \text{ such that } (\lambda I - A)\mathbf{v} = \mathbf{0} \\
& \iff \lambda I \mathbf{v} = A\mathbf{v} \\
& \iff A\mathbf{v} = \lambda \mathbf{v} \\
& \iff \lambda \text{ is an eigenvalue and } \mathbf{v} \text{ is an eigenvector.}
\end{aligned}
$$

■

31 | Tuesday December 3rd

31.1 Similarity and Diagonalizability

Recall that $A \sim B \iff A = PBP^{-1}$.

Fact: If $T : V \to V$ is a linear transformation and $\mathcal{B}, \mathcal{B}'$ are bases where $[T]_\mathcal{B} = A$ and $[T]_{\mathcal{B}'}$, then $A \sim B$.

Theorem: Let A be an $n \times n$ matrix. Then

1. A is similar to a diagonal matrix / diagonalizable $\iff$ A has n linearly independent eigenvectors.
2. $A = PDP^{-1}$ where D is diagonal and $P = [\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}]$ with the $\mathbf{v_i}$ linearly independent.

Proof: Consider $AP = PD$, then AP has columns $A\mathbf{v_i}$ and PD has columns $\lambda_i \mathbf{v_i}$. ■

Corollary: If A has distinct eigenvalues, then A is diagonalizable.

Examples:

1. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

   A has eigenvalues $4, 5$, and it turns out that A is defective.

   Note that $\dim \Lambda_4 + \dim \Lambda_5 = 2 < 3$, so the eigenvectors can't form a basis of $\mathbb{R}^3$.

2.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

   A has eigenvalues $2, 8$. $\Lambda_2 = \operatorname{span}_{\mathbb{R}} \left\{[-1,1,0]^t, [-1,0,1]^t\right\}$ and $\Lambda_8 = \operatorname{span}_{\mathbb{R}} \left\{[1,1,1]^t\right\}$. These vectors become the columns of P, which is (by no coincidence!) an orthogonal matrix, since A was symmetric.

Exercise:

$$\begin{bmatrix} 0 & 4 & 2 \\ -1 & -4 & -1 \\ 0 & 0 & -2 \end{bmatrix}.$$

Find $J = JCF(A)$ (so $A = PJP^{-1}$) and compute P.

Definition: Let $A = (a_{ij})$, then define that *trace* of A by $\operatorname{Tr}(A) = \sum_i a_{ii}$.

The trace satisfies several properties:

- $\operatorname{Tr}(A + B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$,
- $\operatorname{Tr}(kA) = k \operatorname{Tr}(A)$,
- $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

Theorem: Let $T : V \to V$ be a linear transformation with $\dim V < \infty$, $A = [T]_{\mathcal{B}}$ with respect to some basis, and $p_T(x)$ be the characteristic polynomial of A.

Then

$$\begin{aligned} p_T(x) &= x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0, \\ c_0 &= (-1)^n \det(A), \\ c_{n-1} &= -\operatorname{Tr}(A). \end{aligned}$$

Proof: We have $p_T(0) = \det(0I_n - A) = \det(-A) = (-1)^n \det(A)$.

Compute $p_T(x)$ by expanding $\det xI - A$ along the first row. The first term looks like $\prod (x - a_{ii})$, and no other term contributes to the coefficient of x^{n-1}.

$\blacksquare$

Definition: A *Lie Algebra* is a vector space with an operation $[\,\cdot\,,\,\cdot\,] : V \times V \to V$ satisfying

1. Bilinearity,
2. $[x, x] = 0$,
3. The Jacobi identity $[x, [y, z]] = [y, [z, x]] + [z, [x, y]] = 0$.

Examples:

1. $L = \mathfrak{gl}(n, \mathbb{C}) = n \times n$ invertible matrices over $\mathbb{C}$ with $[A, B] = AB - BA$.
2. $L = \mathfrak{sl}(n, \mathbb{C}) = \left\{ A \in \mathfrak{gl}(n, \mathbb{C}) \,\middle|\, \operatorname{Tr}(A) = 0 \right\}$ with the same operation, and it can be checked that

$$\operatorname{Tr}([A, B]) = \operatorname{Tr}(AB - BA) = \operatorname{Tr}(AB) - \operatorname{Tr}(BA) = 0.$$

This turns out to be a *simple* algebra, and simple algebras over $\mathbb{C}$ can be classified using root systems and Dynkin diagrams – this is given by type A_{n-1}.

32 | Preface

These are notes live-tex'd from a graduate Algebra course taught by Dan Nakano at the University of Georgia in Fall 2019. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Thursday 22nd October, 2020
22:25

33 | Indices

List of Todos

Definitions

Theorems

Exercises

List of Figures