Title

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1.1 Regular Functions

> See chapter 3 in the notes.

Some examples:

- X a manifold or an open set in $\mathbb{R}^n$ has a ring of C^∞ functions.
- $X \subset \mathbb{C}$ has a ring of holomorphic functions.
- $X \subset \mathbb{R}$ has a ring of real analytic functions

These all share a common feature: it suffices to check if a function is a member on an arbitrary open set about a point, i.e. they are *local*.

Definition 1.0.1 (?).
Let X be an affine variety and $U \subseteq X$ open. A **regular function** on U is a function $\varphi : U \to k$ such that φ is "locally a fraction", i.e. a ratio of polynomial functions.
More formally, for all $p \in U$ there exists a U_p with $p \in U_p \subseteq U$ such that $\varphi(x) = g(x)/f(x)$ for all $x \in U_p$ with $f, g \in A(X)$.

Example 1.1.
For X an affine variety and $f \in A(X)$, consider the open set $U \coloneqq V(f)^c$. Then $\frac{1}{f}$ is a regular function on U, so for $p \in U$ we can take U_p to be all of U.

Example 1.2.
For $X = \mathbb{A}^1$, take $f = x - 1$. Then $\frac{x}{x-1}$ is a regular function on $\mathbb{A}^1 \setminus \{1\}$.

Example 1.3.
Let $X + V(x_1x_4 - x_2x_3)$ and

$$U := X \setminus V(x_2, x_4) = \left\{ [x_1, x_2, x_3, x_4] \;\middle|\; x_1x_4 = x_2x_3, x_2 \neq 0 \text{ or } x_4 \neq 0 \right\}.$$

Define

$$\begin{aligned} \varphi : U &\to K \\ [x_1, x_2, x_3, x_4] &\mapsto \begin{cases} \dfrac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \dfrac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}. \end{aligned}$$

This is well-defined on $\{x_2 \neq 0\} \cap \{x_4 \neq 0\}$, since $\dfrac{x_1}{x_2} = \dfrac{x_3}{x_4}$. Note that this doesn't define an element of k at $[0, 0, 0, 1] \in U$. So this is not globally a fraction.

Notation: we'll let $\mathcal{O}_X(U)$ is the ring of regular function on U.

Proposition 1.1 ***(?).***
Let $U \subset X$ be an affine variety and $\varphi \in \mathcal{O}_X(U)$. Then $V(\varphi) := \left\{ x \in U \;\middle|\; \varphi(x) = 0 \right\}$ is closed in the subspace topology on U.

Proof .
For all $a \in U$ there exists $U_a \subset U$ such that $\varphi = g_a/f_a$ on U_a with $f_a, g_a \in A(X)$ with $f_a \neq 0$ on U_a.
Then

$$\left\{ x \in U_a \;\middle|\; \varphi(x) \neq 0 \right\} = U_a \setminus V(g_a) \cap U_a$$

is an open subset of U_a, so taking the union over a again yields an open set. But this is precisely $V(\varphi)^c$.

■

Proposition 1.2.
Let $U \subset V$ be open in X an *irreducible* affine variety. If $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ agree on U, then they are equal.

Proof .
$V(\varphi_1 - \varphi_2)$ contains U and is closed in V. It contains $\overline{U} \cap V$, by an earlier lemma, X irreducible implies that $\overline{U} = X$ and so $V(\varphi_1 - \varphi_2) = V$.

■

Compare and contrast: Let $U \subset V \subset \mathbb{R}^n$ be open. If $\varphi_1, \varphi_2 \in C^\infty(V)$ such that φ_1, φ_2 are equal when restricted $U \subset V$. Does this imply $\varphi_1 = \varphi_2$?

For $\mathbb{R}^n$, no, there exist smooth bump functions. You can make a bump function on $V \setminus U$ and extend by zero to U. For $\mathbb{C}$ and holomorphic functions, the answer is yes, by the uniqueness of analytic continuation.

Definition 1.2.1 ((Important) Distinguished Opens)**.**
A **distinguished open set** in an affine variety is one of the form

$$D(f) \coloneqq X \setminus V(f) = \left\{ x \in X \;\middle|\; f(x) = 0 \right\}.$$

Proposition 1.3.
The distinguished open sets form a base of the zariski topology.

Proof .
Given $f, g \in A(X)$, we can check:

1. Closed under finite intersections: $D(f) \cap D(g) = D(fg)$.
2. $$U = X \setminus V(f_1, \cdots, f_k) = V \setminus \bigcap V(f_i) = \bigcup D(f_i),$$

   and any open set is a *finite* union of distinguished opens by the Hilbert basis theorem. ∎

Proposition 1.4*(?)*.
The regular functions on $D(f)$ are given by

$$\mathcal{O}_X(D(f)) = \left\{ \frac{g}{f^n} \;\middle|\; g \in A(X), n \in \mathbb{N} \right\} = A(X)_{\langle f \rangle},$$

the localization of $A(X)$ at $\langle f \rangle$.

Note that if $f = 1$, then $\mathcal{O}_X(X) = A(X)$.

Proposition 1.5*(?)*.
Note that $\frac{g}{f^n} \in \mathcal{O}_X(D(f))$ since $f^n \neq 0$ on $D(f)$. Let $\varphi : D(f) \to k$ be a regular function. By definition, for all $a \in D(f)$ there exists a local representation as a fraction $\varphi = g_a/f_a$ on $U_a \ni a$. Note that U_a can be covered by distinguished opens, one of which contains a. Shrink U_a if necessary to assume it is a distinguished open set $U_a = D(h_a)$.

Now replace

$$\varphi = \frac{g_a}{f_a} = \frac{g_a h_a}{f_a h_a},$$

which makes sense because $h_a \neq 0$ on U_a. We can assume wlog that $h_a = f_a$. Why? We have $\varphi = \frac{g_a}{f_a}$ on $D(f_a)$. Since f_a doesn't vanish on U_a, we have $V(f_a h_a) = V(h_a)$ since $V(f_a) \subset D(h_a)^c = V(h_a)$.
Consider $U_a = D(f_a)$ and $U_b = D(f_b)$, on which $\varphi = \frac{g_a}{f_a}$ and $\varphi = \frac{g_b}{f_b}$ respectively. On

$U_a \cap U_b = D(f_a f_b)$, these are equal, i.e. $f_b g_a = f_a g_b$ in the coordinate ring $A(X)$.

Then $D(f) = \bigcup_a D(f_a)$, so take the component $V(f) = \cap V(f_a)$ by the Nullstellensatz $f \in I(V(f_a)) = I(V(g_a, a \in D_f)) = \sqrt{f_a \mid a \in D_f}$.

Then there exists an expression $f^n = \sum k_a f_a$ as a finite sum, so set $g - \sum g_a k_a$.

Claim: $\varphi = g/f^n$ on $D(f)$.

This follows because on $D(f_b)$, we have $\varphi = \dfrac{g_b}{f_b}$, and so $g f_b = \sum k_a g_a f_b$.

Finish next class