

# Problem Set 10

D. Zack Garza

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## 1 Problem 1

Let  $\phi$  be an  $n$ -form. It suffices to show these statements for  $n = 2$ .

$\implies$  : Suppose  $\phi$  is alternating, then  $\phi(b, b) = 0$  for all  $b \in B$ .

Letting  $a, b \in B$  be arbitrary, we then have

$$\begin{aligned}\phi(a + b, a + b) &= \phi(a, a + b) + \phi(b, a + b) \\ &= \phi(a, a) + \phi(a, b) + \phi(b, a) + \phi(b, b) \\ &= \phi(a, b) + \phi(b, a) \\ &\implies \phi(a, b) = -\phi(b, a),\end{aligned}$$

which shows that  $\phi$  is skew-symmetric.

$\Leftarrow$  Suppose  $\phi$  is skew-symmetric, so  $\phi(a, b) = -\phi(b, a)$  for all  $a, b \in B$ . Then  $\phi(b, b) = -\phi(b, b)$  by transposing the terms, which says that  $\phi(b, b) = 0$  for all  $b \in B$  and thus  $\phi$  is alternating.

## 2 Problem 2

Let  $f(x) = \det(P + xQ) \in R[x]$ , then  $f$  is a polynomial in  $x$  which is not identically zero.

To see that  $f \neq 0$ , we can use that fact that  $P$  is invertible to evaluate  $f(0) = \det(P) \neq 0$ .

We can now note that  $f$  has finite degree, and thus finitely many zeroes in  $R$ .

### 3 Problem 3

Letting  $k[x] \curvearrowright_\phi E$  to yield a  $k[x]$ -module structure on  $E$  and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \mid q_2 \mid \cdots \mid q_t$$

where  $E_i = k[x]/(q_i)$ . Then  $q_t = q$ , the minimal polynomial of  $E$ .

In particular,  $E_t$  is a  $\phi$ -invariant subspace of  $E$ , and if  $\deg q_t = m$ , then  $E_t$  is in fact an  $m$ -dimensional cyclic module with basis  $\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \dots, \phi^{m-1}(\mathbf{v})\}$  for some  $\mathbf{v} \in E_t$ .

But since  $E_t \leq E$  is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

### 4 Problem 4

$\Rightarrow$  : Suppose  $A \sim D$  where  $D$  is diagonal. Then  $JCF(A) = JCF(D) = D$ , which means that every Jordan block of  $A$  has size exactly 1.

Since the elementary divisors of  $A$  are precisely the minimal polynomials of the Jordan blocks of  $A$ , and the minimal polynomial of any  $1 \times 1$  matrix  $[a_{ij}]$  is given by the linear polynomial  $x - a_{ij}$ , every elementary divisor of  $A$  must be linear.

$\Leftarrow$  : Suppose all of the elementary divisors of  $A$  are linear. Every elementary divisor is the minimal polynomial of a Jordan block of  $A$ , and so if we write  $JCF(A) = \bigoplus M_i$ , then the minimal polynomial of each  $M_i$  is linear.

Supposing that  $M_i$  has minimal polynomial  $p_i(x) = x - c$  for some scalar  $c$ , we have

$$p_i(M_i) = 0 \implies M_i - cI_n = 0 \implies M_i = cI_n,$$

which shows that  $M_i$  is a diagonal matrix with only  $c$  on its diagonal.

But if every Jordan block of  $A$  is diagonal, then  $JCF(A) = D$  is diagonal and  $A \sim D$ .

### 5 Problem 5

#### 5.1 Part 1

We'll use the fact that the minimal polynomial  $q$  is the invariant factor of highest degree, and so every other invariant factor must divide  $q$ .

Moreover,  $RCF(A) = C_1 \oplus C_2 \oplus \cdots \oplus C_k$  where each  $C_i$  is the companion matrix of the  $i$ th invariant factor if we write  $V \cong \bigoplus_{i=1}^k k[x]/(a_i)$ . So it suffices to determine all of the possible distinct combinations of invariant factors.

We can restrict this list by noting that the characteristic polynomial satisfies  $\chi_A(x) = \prod a_i$ , and in particular,  $\deg \chi_A(x) = 6$ . Noting that  $\deg q(x) = 3$ , the degrees of the remaining invariant factors must sum to 3.

These are:

$$\begin{array}{lll} a_1 = (x-2), & a_2 = (x-2)^2, & a_3 = q(x), \\ a_1 = (x-2), & a_2 = (x-2)(x-3), & a_3 = q(x), \\ a_1 = (x-3), & a_2 = (x-2)(x-3), & a_3 = q(x). \end{array}$$

Noting that

- $(x-2)^2 = x^2 - 4x + 4$ ,
- $(x-2)(x-3) = x^2 - 5x + 6$ , and
- $q(x) = x^3 - 7x^2 + 16x - 12$ ,

these choices correspond to the matrices

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & -16 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$