Notes on Lee's Manifolds

D. Zack Garza

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Contents

1 General Notes to Self

Interesting things to know:

- The Whitney embedding theorem
- The Jordan-Brouwer separation theorem
- The Poincare-Hopf theorem
- The Hopf degree theorem
- Generalized Stokes' Theorem
- Sard's Theorem
- The Frobenius Integrability Theorem

2 Preface: Point Set Review

2.1 Quotients

Definition 2.0.1 (Saturated)**.**
A subset $A \subseteq X$ is *saturated* with respect to $p : X \longrightarrow Y$ if whenever $p^{-1}(\{y\}) \bigcap A \neq \emptyset$, then $p^{-1}(\{y\}) \subseteq A$.
Equivalently, $A = p^{-1}(B)$ for some $B \subseteq Y$, i.e. it is a complete inverse image of some subset of Y, i.e. A is a union of fibers $p^{-1}(b)$.

Definition 2.0.2 (Quotient Map)**.**
A continuous surjective map $p : X \twoheadrightarrow Y$ is a *quotient map* if $U \subseteq Y$ is open **iff** $p^{-1}(U) \subset X$ is open.

> Note that $\implies$ comes from the definition of continuity of p, but $\impliedby$ is a stronger condition.

Equivalently, p maps saturated subsets of X to open subsets of Y.

Definition 2.0.3 (Universal Property of Quotients)**.**
For $\pi : X \longrightarrow Y$ a quotient map, if $g : X \longrightarrow Z$ is a map that is constant on each $p^{-1}(\{y\})$, then there is a unique map f making the following diagram commute:

$$\begin{tikzcd}
X \arrow[d, "\pi"] \arrow[dr, "g"] & \\
Y \arrow[r, dotted, "f"] & Z
\end{tikzcd}$$

Fact: an injective quotient map is a homeomorphism.

Fact: a product of quotient maps need not be a quotient map.

2.2 Subspaces

Definition 2.0.4 (The Subspace Topology)**.**
$U \subset A$ is open iff $U = V \bigcap A$ for some open $V \subseteq X$.

Proposition 2.1 ***(Universal Property of Subspaces).***
If X and $\iota_S : S \hookrightarrow Y$ is a subspace, then every continuous map $f : X \longrightarrow S$ lifts to a continuous map $\tilde{f} : X \longrightarrow Y$ where $\tilde{f} \coloneqq \iota_S \circ f$:

$$\begin{tikzcd}
& Y \\
X \arrow[r, "f"] \arrow[ur, dotted, "\exists ! \tilde{f}"] & S \arrow[u, hook, "\iota_S"]
\end{tikzcd}$$

Note that we can view $\iota_S \coloneqq \mathrm{id}_Y|_S$. The subspace topology is the unique topology for which this property holds.

Some properties of subspace:

- The inclusion ι_S is a topological embedding.
- Restricting a continuous map to a subspace is still continuous.

- A basis for the subspace topology for $A \subset X$ can be obtained by intersecting basis elements of X with A.
- If X is Hausdorff/first/second-countable, then so is A.

2.3 Products

Definition 2.1.1 (The Product Topology).
The coarsest topology such that every projection map $p_\alpha : \prod_\beta X_\beta \longrightarrow X_\alpha$ is continuous, i.e. for every $U_\alpha \subseteq X_\alpha$ open, $p_\alpha^{-1}(U_\alpha) \in \prod X_\beta$ is open. For finite index sets, we can take the box topology: the collection of sets of the form $\prod_{i=1}^N U_i$ with each U_i open in X_i forms a basis for the product topology on $\prod_{i=1}^N X_i$.

Why these differ: in $\mathbb{R}^\infty$, the set $S = \prod(-1,1)$ is open in the box topology but not the product topology, since $\{0\}^\infty$ is not contained in any basic open neighborhood contained in S.

Some properties of products:

- Projections π_i are continuous by definition.
- A basis for the product topology can be obtained by taking the product of bases.
- A map $f : X \longrightarrow \prod Y_i$ *into* a product is continuous iff each component function $F_i \coloneqq \pi_i \circ f : X \longrightarrow Y_i$ is continuous.
  - I.e. if we have continuous maps $f_i : X \longrightarrow Y_i$ then the composite map $F = [f_1, f_2, \cdots]$ is continuous.
- Separate continuity does not imply joint continuity: A map $f : \prod X_i \longrightarrow Y$ *out* of a product need not be continuous even if (defining $\iota_j : X_j \hookrightarrow \prod X_i$) the map $f \circ \iota_j : X_j \longrightarrow Y$ is continuous for all arbitrary inclusions ι_j.
- Any map of the form $f_{\mathbf{a}_j} : X_j \longrightarrow \prod_{i=1}^n X_i$ where $x \mapsto (a_1, \cdots, a_{j-1}, x, a_{j+1}, \cdots a_n)$ is a topological embedding.
- If X_i are Hausdorff/first/second-countable, then so is $\prod_{i=1}^n X_i$.

2.4 Misc

Definition 2.1.2 (Precompact).
A subset $A \subseteq X$ is *precompact* iff its closure $\operatorname{cl}_X(A)$ is compact in X.

Definition 2.1.3 (Locally Compact).
A space X is *locally compact* iff every $x \in X$ has a neighborhood which is contained in some compact subset of X.

2.5 Analysis Review

Definition 2.1.4 (Derivative, Real Valued)**.**
A function $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable at x iff there is a number $y \in \mathbb{R}$ such that

$$\left(\frac{f(x+h) - f(x)}{h} - y\right) \overset{h \longrightarrow 0}{\longrightarrow} 0$$

where $h \in \mathbb{R}$.
The number $f'(x) \coloneqq y$ is the *derivative* of f at x.
Note that this equivalently says

$$f(x+h) - f(x) = f'(x)h + r(h) \text{ where } \frac{r(h)}{h} \overset{h \longrightarrow 0}{\longrightarrow} 0.$$

Definition 2.1.5 (Derivative, Vector Valued)**.**
For $\mathbf{f} : (a, b) \longrightarrow \mathbb{R}^n$, $f'(x)$ is the vector $\mathbf{y} \in \mathbb{R}^n$ such that

$$\left(\frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} - \mathbf{y}\right) \overset{h \longrightarrow 0}{\longrightarrow} 0 \iff \frac{|\mathbf{f}(x+h) - \mathbf{f}(x) - hy|}{|h|} \overset{h \longrightarrow 0}{\longrightarrow} 0$$

where $h \in \mathbb{R}$.
The vector $\nabla f \coloneqq \mathbf{y}$ is the *derivative* (or *gradient*) of f at $\mathbf{x}$.
Note that this equivalently says

$$\mathbf{f}(x+h) - \mathbf{f}(x) = h\nabla\mathbf{f} + \mathbf{r}(h) \quad \text{ where } \frac{\mathbf{r}(h)}{h} \overset{h \longrightarrow 0}{\longrightarrow} \mathbf{0}.$$

Definition 2.1.6 (Derivative, General Case)**.**
A function $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable iff there exists a linear transformation $\mathbf{Y}$ such that

$$\frac{\|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{Y}\mathbf{h}\|_{\mathbb{R}^m}}{\|\mathbf{h}\|_{\mathbb{R}^n}} \overset{\mathbf{h} \longrightarrow \mathbf{0}}{\longrightarrow} 0.$$

The matrix $D_f(\mathbf{x}) \coloneqq \mathbf{Y}$ is the *total derivative* of f at $\mathbf{x}$.
Note that this equivalently says

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) = D_f\mathbf{h} + \mathbf{r}(\mathbf{h}) \quad \text{ where } \frac{\|\mathbf{r}(\mathbf{h})\|}{\|\mathbf{h}\|} \overset{\mathbf{h} \longrightarrow \mathbf{0}}{\longrightarrow} \mathbf{0}.$$

Note that we can write $(\nabla f)(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{e}_i$.

Theorem 2.2 ***(Chain Rule).***
If $E \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $E \xrightarrow{f} f(E) \xrightarrow{g} g(f(E))$ with f differentiable at $\mathbf{x}_0$ and g differentiable at $f(\mathbf{x}_0)$, then the map $F(\mathbf{x}) \coloneqq g(f(\mathbf{x}))$ is differentiable at $\mathbf{x}_0$ with derivative

$$D_F(\mathbf{x}_0) = D_g(f(\mathbf{x}_0)) \cdot D_f(\mathbf{x}_0).$$

Definition 2.2.1 (Components of a Function)**.**
If $\mathcal{B}_n := \{\mathbf{e}_i\} \subset \mathbb{R}^n$ and $\mathcal{B}_m := \{\mathbf{u}_i\} \subset \mathbb{R}^m$ are standard bases and $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then the *components* of $\mathbf{f}$ are the functions $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})\mathbf{u}_i = [f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})]_{\mathcal{B}_m}.$$

Definition 2.2.2 (Partial Derivative)**.**
For $\{\mathbf{e}_j\}$ the standard orthonormal basis of $\mathbb{R}^n$, define

$$\frac{\partial f_i}{\partial x_j} = (D_j f_i)(\mathbf{x}) = \lim_{t \longrightarrow 0} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t}.$$

Warning: f continuous and existence of all $\dfrac{\partial f_i}{\partial x_j}$ does not imply differentiability. If f is differentiable, however, then D_f is the Jacobian matrix.

Theorem 2.3 ***(Derivative Equals Jacobian).***
If f is differentiable at $\mathbf{x}_0$, then its derivative is an $m \times n$ matrix, its partial derivatives exist, and

$$D_f(\mathbf{x})\mathbf{e}_j = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j} \mathbf{u}_i$$

$$= \left[\frac{\partial \mathbf{f}}{\partial x_1}, \cdots, \frac{\partial \mathbf{f}}{\partial x_n}\right] = \begin{bmatrix} \nabla f_1 & \longrightarrow \\ \nabla f_2 & \longrightarrow \\ \vdots & \vdots \\ \nabla f_m & \longrightarrow \end{bmatrix} = \left[\nabla f_1^t, \cdots, \nabla f_m^t\right]^t = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

Remark This implies that

$$D_f(\mathbf{x})\mathbf{h} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} h_j \mathbf{u}_i.$$

Theorem 2.4 ***(Inverse Function Theorem).***
Suppose $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $D_f(\mathbf{a}) \in \mathrm{GL}(n, \mathbb{R})$ for some $\mathbf{a}$ and $\mathbf{b} = f(\mathbf{a})$.
Then there exist $U \ni \mathbf{a}$ and $V \ni \mathbf{b}$ such that $f(U) = V$ and $f|_U$ is bijective with inverse $g \in C^1(V)$.

Theorem 2.5.
If $f \in C^1(\mathbb{R}^n)$ and $D_f(\mathbf{x}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathbf{x} \in \mathbb{R}^n$, then f is an open map (and thus *locally injective*)

Theorem 2.6 ***(Implicit Function Theorem).***
Let $A : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ and suppose $A_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible.

Then for every $\mathbf{k} \in \mathbb{R}^m$ there exists a unique $\mathbf{h} \in \mathbb{R}^n$ such that

$$A(\mathbf{h}, \mathbf{k}) = \mathbf{0} \quad \text{and} \quad \mathbf{h} = -A_x^{-1} A_y \mathbf{k}.$$

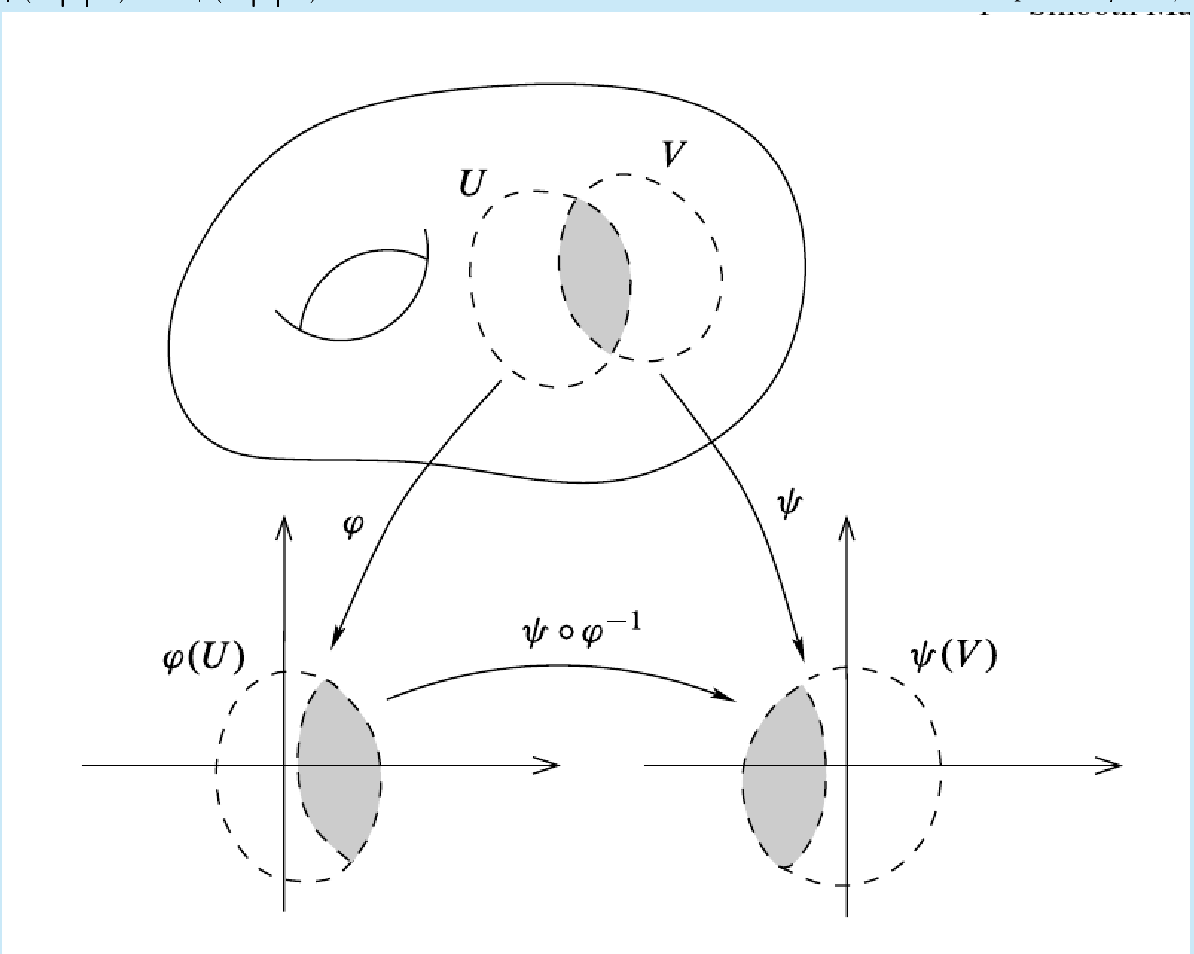
3 Chapter 1: Smooth Manifolds

Definition 3.0.1 (Smooth Functions)**.**
A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $[f_1(\mathbf{x}^n), f_2(\mathbf{x}^n), \cdots, f_m(\mathbf{x}^n)]$ (or any subsets thereof) is said to be C^∞ or **smooth** iff each f_i has continuous partial derivatives of all orders.

Definition 3.0.2 (Diffeomorphism)**.**
A smooth bijective map with a smooth inverse is a *diffeomorphism.*

Remark A diffeomorphism is necessarily a homeomorphism, but not conversely.

Definition 3.0.3 (Transition Maps)**.**
If $(U, \varphi), (V, \psi)$ are two charts on M such that $U \bigcap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1}$: $\varphi(U \bigcap V) \longrightarrow \psi(U \bigcap V)$ is a function $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and is called the *transition map* from φ to ψ.



Two charts are *smoothly compatible* iff $U \bigcap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Definition 3.0.4.
A collection of charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}$ is an *atlas* for M iff $\{U_\alpha\} \rightrightarrows M$, and is a *smooth atlas* iff all of the charts it contains are pairwise smoothly compatible.

Remark To show an atlas is smooth, it suffices to show that an arbitrary $\psi \circ \varphi^{-1}$ is smooth. This is because this immediately implies that its inverse is smooth, and these these are diffeomorphisms. Alternatively, one can show that $\psi \circ \varphi^{-1}$ is smooth, injective, and has nonsingular Jacobian at each point.

Remark Attempting to define a function $f : M \longrightarrow \mathbb{R}$ to be smooth iff $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth for each φ may not work because many atlases give the "same" smooth structure in the sense that they all determine the same collection of smooth functions on M.

What does "determine the same collection of smooth functions" mean?

For example, take the following two atlases on $\mathbb{R}^n$:

$$\begin{aligned} \mathcal{A}_1 &= \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\} \\ \mathcal{A}_2 &= \left\{ \left(\mathbb{D}_1(\mathbf{x}), \mathrm{id}_{\mathbb{D}_1(\mathbf{x})} \right) \;\middle|\; \mathbf{x} \in \mathbb{R}^n \right\} . \end{aligned}$$

Claim: a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth wrt either atlas iff it is smooth in the usual sense.

Definition 3.0.5 (Maximal or Complete Atlas)**.**
A smooth atlas on M is *maximal* iff it is not properly contained in any larger smooth atlas.

Remark Not every topological manifold admits a smooth structure. See Kervaire's 10-dimensional manifold from 1960.

Definition 3.0.6 (Smooth Structures and Smooth Manifolds)**.**
If M is a topological manifold, a maximal smooth atlas $\mathcal{A}$ is a *smooth structure* on M. The triple $(M, \tau, \mathcal{A})$ where $\mathcal{A}$ is a smooth structure is a *smooth manifold.*

Remark To show that two smooth structures are *distinct*, it suffices to show that they are not smoothly compatible, i.e. one of the transition functions $\psi \circ \varphi^{-1}$ is not smooth. This is because any maximal atlas $\mathcal{A}_1$ must contain ψ and likewise $\mathcal{A}_2$ contains φ^{-1}, but no maximal atlas can contain φ *and* ψ because all charts in a maximal atlas are smoothly compatible by definition.

Proposition 3.1.
Let M be a topological manifold.

1. Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the *smooth structure determined by* $\mathcal{A}$.
2. Two smooth atlases for M determine the same smooth structure $\iff$ their union is a smooth atlas.

Remark That we can place many requirements on the functions $\psi \circ \varphi^{-1}$ and get various other structures: C^k, real-analytic, complex-analytic, etc. C^0 structures recover topological manifolds.

Definition 3.1.1 (Smooth Charts, Maps, Domains)**.**
If $(M, \tau, \mathcal{A})$ is a smooth manifold, any chart $(U, \varphi) \in \mathcal{A}$ is a *smooth chart*, where U is a *smooth coordinate domain* and φ is a *smooth coordinate map*. A *smooth coordinate ball* is a smooth coordinate domain U such that $\varphi(U) = \mathbb{D}^n$.

Definition 3.1.2 (Regular Coordinate Ball)**.**
A set $B \subseteq M$ is a *regular coordinate ball* if there is a smooth coordinate ball B' such that $\mathrm{cl}_M(B) \subseteq B'$, and a smooth coordinate map $\varphi : B' \longrightarrow \mathbb{R}^n$ such that for some positive numbers $r < r'$,

- $\varphi(B) = \mathbb{D}_r(\mathbf{0})$,
- $\varphi(B') = \mathbb{D}_{r'}(\mathbf{0})$, and
- $\varphi(\mathrm{cl}_M(B)) = \mathrm{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$.

This says B "sits nicely" insane a larger coordinate ball.

Remark $\mathrm{cl}_M(B) \cong_{\mathrm{Top}} \mathrm{cl}_{\mathbb{R}^n}(\mathbb{D}_r(\mathbf{0}))$ which is closed and bounded and thus compact, so $\mathrm{cl}_M(B)$ is compact. Thus every regular coordinate ball in M is precompact.

Proposition 3.2.
Every smooth manifold has a countable basis of regular coordinate balls.

Remark There is only one 0-dimensional smooth manifold, up to equivalence of smooth structures.

Definition 3.2.1 (Standard Smooth Structure on $\mathbb{R}^n$)**.**
Define the atlas $\mathcal{A}_0 = \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\}$ and take the smooth structure it generates, this is the *standard smooth structure* on $\mathbb{R}^n$.

Proposition 3.3.
There are at least two distinct smooth structures on $\mathbb{R}^n$.

Proof .
Define $\psi(x) = x^3$; then $\mathcal{A}_1 \coloneqq \{(\mathbb{R}^n, \varphi)\}$ defines a smooth structure.
Then $\mathcal{A}_1 \neq \mathcal{A}_0$, which follows because $\left(\mathrm{id}_{\mathbb{R}^n} \circ \varphi^{-1}\right)(x) = x^{\frac{1}{3}}$, which is not smooth at $\mathbf{0}$. ■

4 Chapter 1 Problems

4.1 Recommended Problems

Note: helpful theorem, two smooth structures induced by two smooth atlases $\mathcal{A}_1, \mathcal{A}_2$ are equivalent iff $\mathcal{A}_1 \bigcup \mathcal{A}_2$ is again a smooth atlas. So it suffices to check pairwise compatibility of charts.

Exercise (Problem 1.6) Show that if $M^n \neq \emptyset$ is a topological manifold of dimension $n \geq 1$ and M has a smooth structure, then it has uncountably many distinct ones.

Recommended problem

Hint: show that for any $s > 0$ that $F_s(x) \coloneqq |x|^{s-1} x$ defines a homeomorphism $F_x : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ which is a diffeomorphism iff $s = 1$.

Solution:

Define

$$\begin{aligned} F_s : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\mapsto \|\mathbf{x}\|^{s-1}\mathbf{x}. \end{aligned}$$

Claim: F_s restricted to $\mathbb{D}^n$ is a continuous map $\mathbb{D}^n \longrightarrow \mathbb{D}^n$.

- Note that if $\|\mathbf{x}\| \leq \varepsilon < 1$ then

  $$\|F_s(\mathbf{x})\| = \|\|\mathbf{x}\|^s \widehat{\mathbf{x}}\| = \|\mathbf{x}\|^s \leq \|\mathbf{x}\| \leq \varepsilon < 1,$$

  so $F_s(\mathbb{D}^n) \subseteq \mathbb{D}^n$ and moreover $F_s(\mathbb{D}^n_\varepsilon) \subseteq \mathbb{D}^n_\varepsilon$.

  - We'll use the fact that $F_s^{-1} = F_{\frac{1}{s}}$ is of the same form, and thus $F_s^{-1}(\mathbb{D}^n) \subseteq \mathbb{D}^n$, forcing $F_s(\mathbb{D}^n) = \mathbb{D}^n$.

- This is a continuous function on the punctured disc $\mathbb{D}^n_0 \coloneqq \mathbb{D}^n \setminus \{\mathbf{0}\}$, since it can be written as a composition of smooth functions:

  $$\mathbb{D}^n_0 \xrightarrow{\Delta} \mathbb{D}^n_0 \times \mathbb{D}^n_0 \xrightarrow{(\|\cdot\|,\ \mathrm{id}_{\mathbb{D}^n_0})} \mathbb{D}^n_0 \times \mathbb{D}^n_0 \xrightarrow{((\cdot)^{s-1},\ \mathrm{id}_{\mathbb{D}^n_0})} \mathbb{D}^1_0 \times \mathbb{D}^n_0 \xrightarrow{(a,b)\mapsto ab} \mathbb{D}^n_0$$

  $$\mathbf{x} \longrightarrow (\mathbf{x}, \mathbf{x}) \longrightarrow (\|\mathbf{x}\|, \mathbf{x}) \longrightarrow (\|\mathbf{x}\|^{s-1}, \mathbf{x}) \longrightarrow \|\mathbf{x}\|^{s-1}\mathbf{x}$$

  For any $s \geq 0$, continuity at zero follows from the fact that $\|F_s(\mathbf{x})\| \leq \|\mathbf{x}\| \longrightarrow 0$, so $\lim_{\mathbf{x}\longrightarrow\mathbf{0}} F_s(\mathbf{x}) = \mathbf{0}$ and the sequential definition of continuity applies. So F_s is continuous on $\mathbb{D}^n$ for every s.

  > Here we are taking for granted the fact that taking norms, exponentiating, and multiplying are all smooth functions away from zero.

Claim: F_s is a bijection $\mathbb{D}^n \setminus \mathbf{0} \circlearrowleft$ that extends to a bijection $\mathbb{D}^n \circlearrowleft$.

We can note that

$$F_s(\mathbf{x}) = \begin{cases} \|\mathbf{x}\|^s \dfrac{\mathbf{x}}{\|\mathbf{x}\|} \coloneqq \|\mathbf{x}\|^s \widehat{\mathbf{x}} & \text{if } \|\mathbf{x}\| \neq 0 \\ \mathbf{0} & \text{if } \|\mathbf{x}\| = 0 \end{cases}$$

This follows because we can construct a two-sided inverse that composes to the identity, namely $F_{\frac{1}{s}}$, for $\mathbf{x} \neq \mathbf{0}$, and note that $F_s(\mathbf{0}) = \mathbf{0}$. Using the fact that $\|t\mathbf{x}\| = t\|\mathbf{x}\|$ for any scalar t, we can check that

$$
\begin{aligned}
\left(F_s \circ F_{\frac{1}{s}}\right)(\mathbf{x}) &= F_s(\|\mathbf{x}\|^{\frac{1}{s}}\widehat{\mathbf{x}}) \\
&= \left\|\|\mathbf{x}\|^{\frac{1}{s}}\widehat{\mathbf{x}}\right\|^s \cdot \widehat{\|\mathbf{x}\|^{\frac{1}{s}}\mathbf{x}} \\
&= \left(\|\mathbf{x}\|^{\frac{1}{s}}\right)^s \cdot \|\widehat{\mathbf{x}}\|^s \cdot \frac{\|\mathbf{x}\|^{\frac{1}{s}}\widehat{\mathbf{x}}}{\left\|\|\mathbf{x}\|^{\frac{1}{s}}\widehat{\mathbf{x}}\right\|} \\
&= \|\mathbf{x}\| \cdot 1^s \cdot \left(\frac{\|\mathbf{x}\|^{\frac{1}{s}}}{\|\mathbf{x}\|^{\frac{1}{s}}}\right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\
&= \|\mathbf{x}\|\widehat{\mathbf{x}} \\
&= \mathbf{x}.
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\left(F_{\frac{1}{s}} \circ F_s\right)(\mathbf{x}) &= F_{\frac{1}{s}}(\|\mathbf{x}\|^s\widehat{\mathbf{x}}) \\
&= \|\|\mathbf{x}\|^s\widehat{\mathbf{x}}\|^{\frac{1}{s}} \cdot \widehat{\|\mathbf{x}\|^s\widehat{\mathbf{x}}} \\
&= (\|\mathbf{x}\|^s)^{\frac{1}{s}}\|\widehat{\mathbf{x}}\|^{\frac{1}{s}} \cdot \frac{\|\mathbf{x}\|^s\widehat{\mathbf{x}}}{\|\|\mathbf{x}\|^s\widehat{\mathbf{x}}\|} \\
&= \|\mathbf{x}\| \cdot 1^{1-s} \cdot \left(\frac{\|\mathbf{x}\|^s}{\|\mathbf{x}\|^s}\right) \cdot \frac{\widehat{\mathbf{x}}}{\|\widehat{\mathbf{x}}\|} \\
&= \|\mathbf{x}\|\widehat{\mathbf{x}} \\
&= \mathbf{x}.
\end{aligned}
$$

Claim: F_s is a homeomorphism for all s.

This follows from the fact that the domain $\mathbb{D}^n$ is compact and the codomain $\mathbb{D}^n$ is Hausdorff, and a continuous bijection between such spaces is a homeomorphism.

Claim: F_s is a diffeomorphism iff $s = 1$.

If $s = 1$, $F_s = \mathrm{id}_{\mathbb{D}^n}$ which is clearly a diffeomorphism.

Otherwise, we claim that F_s is not a diffeomorphism because either F_s or F_s^{-1} will fail to be smooth at $\mathbf{x} = \mathbf{0}$.

- If $0 \le s < 1$, then F_s fails to be differentiable at zero. Why? Should boil down to $x \mapsto x^t$ for $0 \le t < 1$ failing to be differentiable at 0 in $\mathbb{R}$
- If $1 < s < \infty$ then $0 \le \dfrac{1}{s} < 1$ and the same argument applies to $F_s^{-1} \coloneqq F_{\frac{1}{s}}$.

We now show that we can produce infinitely many distinct maximal atlases on M. Let $\mathcal{A}$ by any smooth atlas on M and fix $p_0 \in M$.

Claim: We can modify $\mathcal{A}$ to obtain an atlas $\mathcal{A}'$ where p_0 is in exactly one chart (V, ψ) with $\psi(p_0) = \mathbf{0} \in \mathbb{R}^n$.

- Pick a chart containing p_0, say (U, φ) where $\varphi(p_0) \coloneqq \mathbf{p}$
- Since $\varphi(U) \subseteq \mathbb{R}^n$ is open, find a disc containing $\mathbf{p}$, say $\mathbb{D}_R(\mathbf{p}) \subset \varphi(U)$.
- Define $V \subseteq M$ as $V \coloneqq \varphi^{-1}(\mathbb{D}_R(\mathbf{p}))$.

- Define $\psi : U \longrightarrow \mathbb{R}^n$ by

$$\begin{aligned}\psi : U &\longrightarrow \mathbb{R}^n \\ x &\mapsto \frac{\varphi(x) - \varphi(p_0)}{R}.\end{aligned}$$

  - Note: this is constructed precisely so that $\psi(V) = \mathbb{D}_1(\mathbf{0}) \in \mathbb{R}^n$ and $\psi(p) = 0$.
  - This is a homeomorphism onto its image since we can write

$$\psi = \delta_{\frac{1}{R}} \circ \tau_{\mathbf{p}} \circ \varphi$$

    is a composition of continuous functions, where δ, τ are dilations/translations in $\mathbb{R}^n$ which are known to be continuous, and

$$\psi^{-1} = \varphi^{-1} \circ \tau_{-\mathbf{p}} \circ \delta_R$$

    is again a composition of smooth (and in particular, continuous) functions.
- Define $\mathcal{A}^1 := \mathcal{A} \bigcup \{(V, \psi|_V)\}$
  - This is a smooth atlas: any pair of charts coming from $\mathcal{A}$ are smoothly compatible, so it suffices to check that an arbitrary chart from $\mathcal{A}$ is smoothly compatible with the new chart.
  - Let (T, ξ) be any other chart, then if $T \bigcap V \neq \emptyset$, the transition function

$$\psi \circ \xi^{-1} = \delta_{\frac{1}{R}} \tau_{\mathbf{p}} \circ \varphi \circ \xi^{-1}$$

    is a composition of smooth functions and thus smooth, and similarly for $\xi \circ \psi^{-1}$.
  - Since the charts from $\mathcal{A}$ cover M, so do the charts of $\mathcal{A}^1$ since $\mathcal{A} \subseteq \mathcal{A}^1$.
- For every $(U_\alpha, \varphi_\alpha) \in \mathcal{A}^1$, define a new chart $(U_\alpha \setminus \{p\}, \varphi_\alpha|_{U_\alpha \setminus \{p\}})$ and define this set of charts as $\mathcal{A}^2$.
  - This still covers M: p is in the chart $(V, \psi \Big|_V)$, and if $q \neq p$, then $q \in U_\alpha$ for some α since $\mathcal{A}$ was an atlas, and $q \in U_\alpha \setminus \{p\}$.
  - The coordinate maps are still homeomorphisms onto their images, because the restriction of a homeomorphism is again a homeomorphism.
  - The transition functions are still smooth because the restriction of a smooth function is again smooth.

Claim: We can define a new atlas $\mathcal{A}_s$ from $\mathcal{A}^2$ by only replacing the single chart (V, ψ) with $(V, F_s \circ \psi)$.

- $\mathcal{A}_s$ still covers M, since we haven't changed the coordinate domains
- All coordinate functions are still a homeomorphisms onto their images, since the only change is ψ is replaced with $F_s \circ \psi$ and we've shown that F_s is a homeomorphism; a composition of homeomorphisms is again a homeomorphism.
- The chart $(V, F_s \circ \psi)$ is still a valid chart, since $F_s : \mathbb{D}_n \circlearrowleft$ and $\psi(V) \cong \mathbb{D}^n$ by construction.
- All charts in $\mathcal{A}_s$ are still smoothly compatible:
  - If suffices to check compatibility between an arbitrary $(U_\alpha, \varphi_\alpha)$ and $(V, F_s \circ \psi)$, so we consider $F_s \circ \psi \circ \varphi_\alpha^{-1}$
  - By construction, $p \notin U_\alpha$, and we know F_s is smooth away from $\mathbf{0}$, so this is a smooth function.

Claim: If $s \neq t$ then $\mathcal{A}_s$ and $\mathcal{A}_t$ are not smoothly compatible, and thus generate distinct maximal smooth atlases.

- If $\mathcal{A}_s, \mathcal{A}_t$ define the same smooth structure, then in particular $(V, F_s \circ \psi)$ must be smoothly compatible with $(V, F_t \circ \psi)$.
- We can compute the transition function

$$(F_s \circ \psi) \circ (F_t \circ \psi)^{-1} = F_s \circ \psi \circ \psi^{-1} \circ F_t^{-1} = F_s \circ F_t^{-1} = F_s \circ F_{\frac{1}{t}} = F_{\frac{s}{t}}.$$

- From above, we know this is smooth iff $\dfrac{s}{t} = 1$, i.e. $s = t$.
- So if $s \neq t$, then the maximal atlases correspond to $\mathcal{A}_s, \mathcal{A}_t$ each contain a chart that is not smoothly compatible with the other, and so these are distinct smooth structures.

■

Exercise (Problem 1.7) Let $N := [0, \cdots, 1] \in S^n$ and $S := [0, \cdots, -1]$ and define the stereographic projection

Recommended problem

$$\sigma : S^n \setminus N \longrightarrow \mathbb{R}^n$$
$$\left[x^1, \cdots, x^{n+1}\right] \mapsto \frac{1}{1 - x^{n+1}} \left[x^1, \cdots, x^n\right]$$

and set $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus S$ (projection from the South pole)

Note that the figure should say $\left\{x^{n+1} = 0\right\}$ instead of x^n.

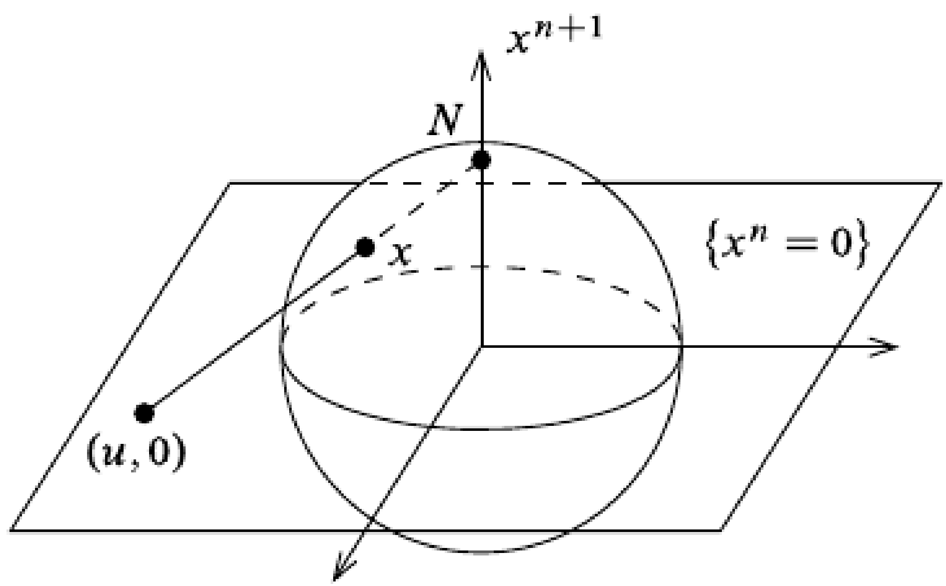


Fig. 1.13 Stereographic projection

1. For any $x \in S^n \setminus N$ show that $\sigma(x) = \mathbf{u}$ where $(\mathbf{u}, 0)$ is the point where the line through N and x intersects the linear subspace $H_{n+1} := \left\{x^{n+1} = 0\right\}$.

   Similarly show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects H_{n+1}.

2. Show that σ is bijective and

$$\sigma^{-1}(\mathbf{u}) = \sigma^{-1}\Big(\Big[u^1, \cdots, u^n\Big]\Big) = \frac{1}{\|\mathbf{u}\|^2 + 1}\Big[2u^1, \cdots, 2u^n, \|\mathbf{u}\|^2 - 1\Big].$$

3. Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas

$$\mathcal{A} \coloneqq \{(S^n \setminus N, \sigma), (S^n \setminus S, \tilde{\sigma})\}$$

define a smooth structure on S^n.

4. Show that this smooth structure is equivalent to the standard smooth structure: Put graph coordinates on S^n as outlined in 5.2 to obtain $\Big\{(U_i^{\pm}, \varphi_i^{\pm})\Big\}$.

For indices $i < j$, show that

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}\Big[u^1, \cdots, u^n\Big] = \Big[u^1, \cdots, \widehat{u^i}, \cdots, \pm\sqrt{1 - \|\mathbf{u}\|^2}, \cdots u^n\Big]$$

where the square root appears in the jth position. Find a similar formula for $i > j$. Show that if $i = j$, then

$$\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1} = \varphi_i^{-} \circ (\varphi_i^{+})^{-1} = \mathrm{id}_{\mathbb{D}^n}.$$

Show that these yield a smooth atlas.

Solution (1):

- Parameterize the line through $\mathbf{x} \in S^n$ and $\mathbf{N}$:

$$\begin{aligned}
\ell_{N,\mathbf{x}}(t) &= t\mathbf{x} + (1-t)\mathbf{N} \\
&= t\Big[x^1, \cdots, x^n, x^{n+1}\Big] + (1-t)[0, \cdots, 1] \\
&= \Big[tx^1, \cdots, x^n, tx^{n+1} + (1-t)\Big] \\
&= \Big[tx^1, \cdots, x^n, 1 - t\Big(1 - x^{n+1}\Big)\Big]
\end{aligned}$$

.

- Evaluate at $t = \dfrac{1}{1 - x^{n+1}}$ to obtain $\dfrac{1}{x^{n+1}}\Big[x^1, \cdots, x^n, 0\Big] = [\sigma(\mathbf{x}), 0]$.
- For $\tilde{\sigma}(\mathbf{x})$: Todo . Todo

Solution (2):

- How to derive this formula: no clue. Figure out how to invert.
  - Start with $\mathbf{u} \in \mathbb{R}^n$, parameterize the line $\ell_{N,\mathbf{u}}(t)$, solve for where $\|\ell_{N,\mathbf{u}}(t)\| = 1$ and $\mathbf{u} \neq N$
  - Should yield $t^2\|u\| + (1-t)^2 = 1$, solve for nonzero t; should get $t = \dfrac{2}{\|\mathbf{u}\| + 1}$, so $x^i = 2u^i/(\|\mathbf{u}\| + 1)$ and $x^{n+1} = \left(\dfrac{2}{\|\mathbf{u}\| + 1}\right) - 1$.
- Compute compositions $\sigma \circ \sigma^{-1}$: Todo. Messy computations that didn't work out.

Solution (3):

- Computing the transition maps:

$$
\begin{aligned}
(\tilde{\sigma} \circ \sigma^{-1})(\mathbf{u}) &= -\sigma\left(\left(\frac{-1}{\|\mathbf{u}\|^2+1}\right)\left[2u^1, \cdots, 2u^n, \|\mathbf{u}\|^2 - 1\right]\right) \\
&= -1 \cdot \left[\frac{\frac{-2u^1}{\|\mathbf{u}\|^2+1}}{1 - \frac{1-\|\mathbf{u}\|^2}{1+\|\mathbf{u}\|^2}}, \cdots_n\right] \\
&= \left[\frac{2u^1}{\|\mathbf{u}\|^2+1} \cdot \frac{1+\|\mathbf{u}\|^2}{1+\|\mathbf{u}\|^2 - (1-\|\mathbf{u}\|^2)}, \cdots_n\right] \\
&= \left[\frac{2u^1}{2\|\mathbf{u}\|^2}, \cdots_n\right] \\
&= \frac{\mathbf{u}}{\|\mathbf{u}\|^2} \\
&:= \widehat{\mathbf{u}},
\end{aligned}
$$

  which is a smooth function on $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

- Todo: computing $(\sigma \circ \tilde{\sigma}^{-1})(\mathbf{u}) = \widehat{\mathbf{u}}$

Computation.

- Todo: argue that it suffices that these are smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$

What are the actual domains and ranges of the transition functions? It seems like you pull back $\mathbb{R}^n$ to $S^n \setminus N$, then push $S^n \setminus \{N, S\}$ to $R^n \setminus 0$, but this yields $\mathbb{R}^n \longrightarrow \mathbb{R}^n \setminus 0$ where we haven't deleted zero in the domain (problem: not smooth!).

Solution (4):

We want to argue that these define the same maximal smooth atlas, for which it suffices to the charts from each are pairwise smoothly compatible.

- Define $\varphi_i\left(\left[x^1, \cdots, x^n\right]\right) = \left[x^1, \cdots, \widehat{x^i}, \cdots, x^n\right]$ and $\varphi_i^{-1}\left(\left[x^1, \cdots, x^{n-1}\right]\right) = \left[x^1, \cdots_i, \sqrt{1 - \|\mathbf{x}\|}, \cdots, x^n\right]$.
- Compute $(\varphi_i \circ \sigma^{-1})(\mathbf{u}) = \dfrac{1}{\|\mathbf{u}\|+1}\left[2u^1, \cdots \widehat{u^i}, \cdots, 2u^n, \|\mathbf{u}\|^2 - 1\right]$, which is (clearly) smooth?
- Compute $(\sigma \circ \varphi_i^{-1})(\mathbf{u}) = \sigma\left(\left[u^1, \cdots_i, \sqrt{1 - \|\mathbf{u}\|^2}, \cdots, u^n\right]\right)$, which is $\dfrac{1}{1-u^n}\left[u^1, \cdots_i, \sqrt{1-\|\mathbf{u}\|^2}, \cdots, u^{n-1}\right]$.
  - This is smooth if $u^n \neq 1$, but this corresponds to $\mathbf{N}$ in S^2, in which case $\varphi_i^{-1}(\mathbf{u})$ isn't in the domain of σ to begin with.

Exercise (Problem 1.8) Define an *angle function* on $U \subset S^1$ as any continuous function $\theta : U \longrightarrow \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

Show that U admits an angle function iff $U \neq S^1$, and for any such function θ, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

Note that $f : \mathbb{R} \longrightarrow S^1$ given by $f(x) = e^{ix}$ is a covering map (in fact the universal cover).

Some way to do this just with covering spaces?

$\implies$:

- Suppose there exists an angle function $\theta : U \longrightarrow \mathbb{R}$.
- Then $f \circ \theta|_U = \mathrm{id}_U$ by assumption, since $u \xrightarrow{\theta|_U} \theta(u) \xrightarrow{f} e^{i\theta(u)} = u$.
- So θ has a left-inverse and is thus injective.
- Suppose $U = S^1$, which is compact.
- Then θ is an injective continuous map on a compact set, so its image $\theta(S^1) \subseteq \mathbb{R}$ is compact.

- Lemma: a continuous map from a compact space to a Hausdorff space is a closed map.
- Since θ is injective and is surjective onto its image, since it is continuous it is a homeomorphism onto its image and $S^1 \cong \theta(S^1)$.
- Since S^1 is connected, $\theta(S^1)$ is connected, and the only connected subsets of $\mathbb{R}$ are intervals.
- Since $\theta(S^1)$ is compact, it must be a closed and bounded subset, so $\theta(S^1) = [a, b] \subset \mathbb{R}$.
- But this forces $S^1 \cong [a, b]$ is a homeomorphism, which is a contradiction: removing one point from S^1 yields one connected component, while removing $\frac{1}{2}(b-a)$ from $[a, b]$ produces a disconnected set.

$\Longleftarrow$:

- Suppose $U \neq S^1$, then there exists a point $p \in S^1 \setminus U$; wlog suppose $p = 1$.
- Then $U \subseteq S^1 \setminus \{1\}$
- Note that $f^{-1}(\{1\}) = \left\{2k\pi \;\middle|\; k \in \mathbb{Z}\right\}$.
- Take the interval $I = [0, 2\pi]$ and set $\tilde{f} = f|_I$.
- Since $U \neq S^1$, $\tilde{f}^{-1}(U) \subsetneq I$.
- Then $\tilde{f}$ restricted to $f^{-1}(U)$ is injective, since $\tilde{f}$ only fails injectivity at $0, 2\pi$.
- Then the restricted map $\widehat{f} := f|_{f^{-1}(U)} : f^{-1}(U) \longrightarrow U$ is a continuous injection and surjects onto its image, thus a bijection
- Claim: $\widehat{f}$ is a homeomorphism
  - Define a candidate inverse $\theta = \widehat{f}^{-1} : S^1 \longrightarrow \mathbb{R}$.
  - Then $f \circ \theta = \mathrm{id}_{S^1}$ implies $e^{i\theta(x)} = x$ for all $x \in U$.
  - Letting $V \subseteq f^{-1}(U)$ be open, we have $\theta^{-1}(V) = \widehat{f}(V)$ which (claim?) is open since ???
  - So θ is continuous.

Alternatively:

- Take $I = (0, 2\pi)$.
- Then $\tilde{f}(I) = S^1 \setminus \{1\}$, so $U \subseteq \tilde{f}(I)$.
- Claim: $f : S^1 \setminus \{1\} \longrightarrow I$ is a homeomorphism.
- Set $\theta(x) = \tilde{f}\Big|_I^{-1} U(x)$; the claim is that this works.
  - Taking a branch cut $\left\{x + iy \;\middle|\; x \in [0, \infty), y = 0\right\}$ for the complex logarithm defines an inverse. How to prove?

(U, θ) is a smooth coordinate chart:

- Let θ be arbitrary with $e^{i\theta(z)} = z$ and $\theta \subsetneq S^1$.
- $U \subseteq S^1$ is open by assumption.
- We need to show that $\theta : U \longrightarrow \varphi(U)$ is a homeomorphism

Exercise (Problem 1.9) Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to equip it with a smooth structure, using the correspondence Recommended problem

$$\mathbb{R}^{2n} \iff \mathbb{C}^n$$
$$\left[x^1, y^1, \cdots, x^n, y^n\right] \iff \left[x^1 + iy^1, \cdots, x^n + iy^n\right].$$

5 Chapter 1: Point-Set Properties of Topological Manifolds

Pages 1- 29.

5.1 Notes

Definition 5.0.1 (Topological Manifold)**.**
A topological space M that satisfies

1. M is Hausdorff, i.e. points can be separated by open sets
2. M is second-countable, i.e. has a countable basis
3. M is locally Euclidean, i.e. every point has a neighborhood homeomorphic to an open subset $\widehat{U}$ of $\mathbb{R}^n$ for some fixed n.

The last property says $p \in M \implies \exists U$ with $p \in U \subseteq M$, $\widehat{U} \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi : U \longrightarrow \widehat{U}$.

Note that second countability is primarily needed for existence of partitions of unity.

Exercise Show that the in the last condition, $\widehat{U}$ can equivalently be required to be an open ball or $\mathbb{R}^n$ itself.

Theorem 5.1 ***(Topological Invariance of Dimension).***
Two nonempty topological manifolds of different dimensions can not be homeomorphic.

Exercise Show that in a Hausdorff space, finite subsets are closed and limits of convergent sequences are unique.

Exercise Show that subspaces and finite products of Hausdorff (resp. second countable) spaces are again Hausdorff (resp. second countable).

Thus any open subset of a topological manifold with the subspace topology is again a topological manifold.

Exercise Give an example of a connected, locally Euclidean Hausdorff space that is not second countable.

Definition 5.1.1 (Charts)**.**
A chart on M is a pair (U, φ) where $U \subseteq M$ is open and $\varphi : U \longrightarrow \widehat{U}$ is a homeormohpsim from U to $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. If $p \in M$ and $\varphi(p) = 0 \in \overline{U}$, then the chart is said to be *centered* at p. Note that any chart about p can be modified to a chart $(\varphi_1, \widehat{U}_1)$ that is centered at p by defining $\varphi_1(x) = x - \varphi(v)$.

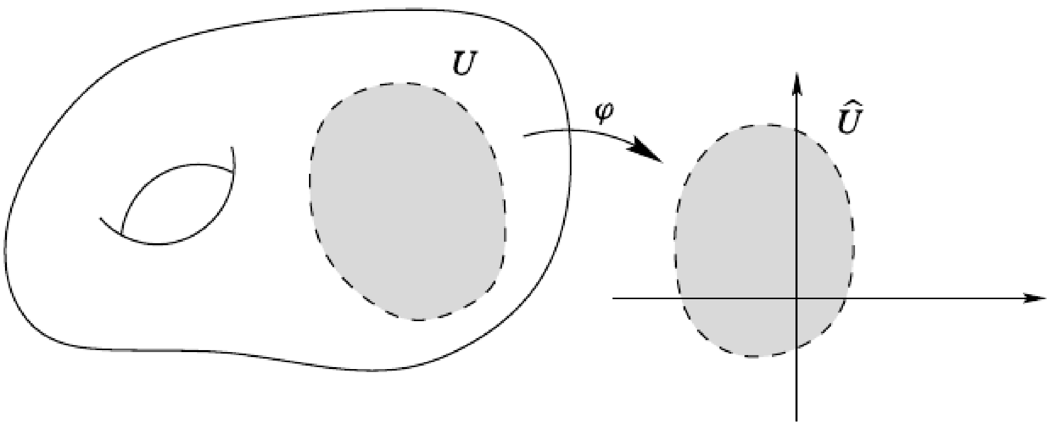


Fig. 1.2 A coordinate chart

U is the *coordinate domain* and φ is the *coordinate map.*

Note that we can write φ in components as $\varphi(p) = \left[x^1(p), \cdots, x^n(p)\right]$ where each x^i is a map $x^i : U \longrightarrow \mathbb{R}$. The component functions x^i are the *local coordinates* on U.

Shorthand notation: $\left[x^i\right] := \left[x^1, \cdots, x^n\right]$.

Example 5.1 (Graphs of Continuous Functions)**.**
Define

$$\Gamma(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \;\middle|\; x \in U,\; y = f(x) \in \widehat{U}\right\}.$$

This is a topological manifold since we can take $\varphi : \Gamma(f) \longrightarrow U$ by restricting $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ to the subspace $\Gamma(f)$. Projections are continuous, restrictions of continuous functions are continuous.

Thus graphs of continuous functions $f : \mathbb{R}^n \to \mathbb{R}^k$ are locally Euclidean?

This is a homeomorphism because the map $g : x \mapsto (x, f(x))$ is continuous and $g \circ \pi_1 = \mathrm{id}_{\mathbb{R}^n}$ is continuous with $\pi_1 \circ g = \mathrm{id}_{\Gamma(f)}$. Note that $U \cong \Gamma(f)$, and thus $(U, \varphi) = (\Gamma(f), \varphi)$ is a single *global* coordinate chart, called the *graph coordinates* of f.

Note that this works in greater generality:: "The same observation applies to any subset of $\mathbb{R}^{n+k}$ by setting *any* k of the coordinates equal to some continuous function of the other n."

Coordinates as numbers vs functions?

Example 5.2 (Spheres)**.**
S^n is a subspace of $\mathbb{R}^{n+1}$ and is thus Hausdorff and second-countable by exercise 5.1.

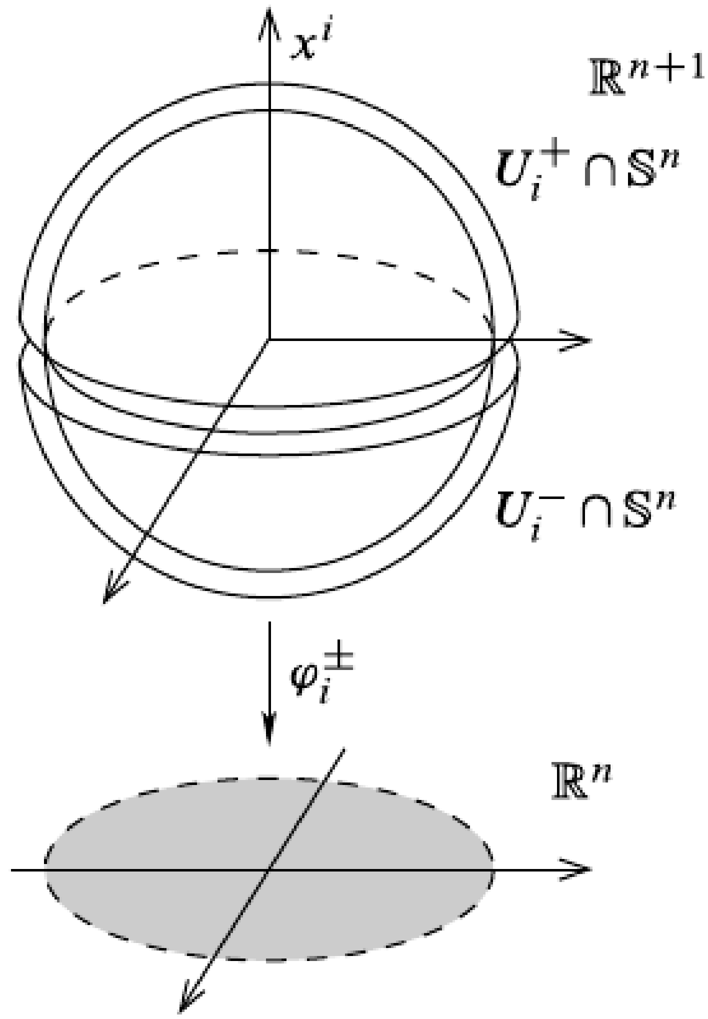


Fig. 1.3 Charts for $\mathbb{S}^n$

To see that it's locally Euclidean, take

$$\begin{aligned} U_i^+ &\coloneqq \left\{ \left[x^1, \cdots, x^n\right] \in \mathbb{R}^{n+1} \;\middle|\; x^i > 0 \right\} \quad \text{for} \quad 1 \le i \le n+1 \\ U_i^- &\coloneqq \left\{ \left[x^1, \cdots, x^n\right] \in \mathbb{R}^{n+1} \;\middle|\; x^i < 0 \right\} \quad \text{for} \quad 1 \le i \le n+1. \end{aligned}$$

Define

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R}^{\ge 0} \\ \mathbf{x} &\mapsto \sqrt{1 - \|\mathbf{x}\|^2}. \end{aligned}$$

Note that we immediately need to restrict the domain to $\mathbb{D}^n \subset \mathbb{R}^n$, where $\|x\|^2 \leq 1 \implies 1 - \|x\|^2 \geq 0$, to have a well-defined real function $f : \mathbb{D}^n \longrightarrow \mathbb{R}^{\geq 0}$.

Then (claim)

$$\begin{aligned} U_i^+ \bigcap S^n \quad &\text{is the graph of} \quad x^i = f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}) \\ U_i^- \bigcap S^n \quad &\text{is the graph of} \quad x^i = -f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}). \end{aligned}$$

This is because

$$\begin{aligned} \Gamma(x^i) &\coloneqq \{(\mathbf{x}, f(\mathbf{x})) \subseteq \mathbb{R}^n \times \mathbb{R}\} \\ &= \left\{ \left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right], f\left(\left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right]\right) \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \\ &= \left\{ \left[x_1, \cdots, \widehat{x^i}, \cdots, x^{n+1}\right], \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right)^{\frac{1}{2}} \subseteq \mathbb{R}^n \times \mathbb{R} \right\} \end{aligned}$$

and any vector in this set has norm satisfying

$$\|(\mathbf{x}, y)\|^2 = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 + \left(1 - \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (x^j)^2 \right) = 1$$

and is thus in S^n.

To see that any such point also has positive i coordinate and is thus in U_i^+, we can rearrange (?) coordinates to put the value of f in the ith coordinate to obtain

Seems like f is always the *last* coordinate in the graph

$$\Gamma(x_i) = \left\{ \left[x^1, \cdots, f(x^1, \cdots, \widehat{x^i}, \cdots, x^n), \cdots, x^n\right] \right\}$$

and note that the square root only takes on positive values.

Thus each $U_i^{\pm} \bigcap S^n$ is the graph of a continuous function and thus locally Euclidean, and we can define chart maps

$$\begin{aligned} \varphi_i^{\pm} : U_i^{\pm} \bigcap S^n &\longrightarrow \mathbb{D}^n \\ \left[x^1, \cdots, x^n\right] &\mapsto [x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}] \end{aligned}$$

yield $2(n+1)$ charts that are graph coordinates for S^n.

Example 5.3 (Projective Space)**.**
Define $\mathbb{RP}^n$ as the space of 1-dimensional subspaces of $\mathbb{R}^{n+1}$ with the quotient topology determined by the map

How is this map a quotient map?

$$\begin{aligned} \pi : \mathbb{R}^{n+1} \setminus \{0\} &\longrightarrow \mathbb{RP}^n \\ \mathbf{x} &\mapsto \operatorname{span}_{\mathbb{R}} \{\mathbf{x}\}. \end{aligned}$$

Notation: for $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\}$ write $[\mathbf{x}] \coloneqq \pi(\mathbf{x})$, the line spanned by $\mathbf{x}$.

Define charts:

$$\tilde{U}_i \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \setminus \{0\} \;\middle|\; x^i \neq 0 \right\}, \quad U_i = \pi(\tilde{U}_i) \subseteq \mathbb{RP}^n$$

.

and chart maps

$$\begin{aligned} \tilde{\varphi}_i : \tilde{U}_i &\longrightarrow \mathbb{R}^n \\ \left[x^1, \cdots, x^{n+1}\right] &\mapsto \left[\frac{x^1}{x^i}, \cdots \widehat{x^i}, \cdots \frac{x^{n+1}}{x^i}\right]. \end{aligned}$$

Then (claim) this descends to a continuous map $\varphi_i : U_i \longrightarrow \mathbb{R}^n$ by the universal property of the quotient:

$$\begin{tikzcd} \tilde{U}_i \arrow[d, "\pi_U"'] \arrow[dr, "\tilde{\varphi}_i"] & \\ U_i \arrow[r, dotted, "\varphi_i"] & \mathbb{R}^n \end{tikzcd}$$

- The restriction $\pi_U : \tilde{U}_i \longrightarrow U_i$ of π is still a quotient map because $\tilde{U}_i = \pi_U^{-1}(U_i)$ where $U_i \subseteq \mathbb{RP}^n$ is open in the quotient topology and thus $\tilde{U}_i$ is saturated.

  Thus π_U sends saturated sets to open sets and is thus a quotient map.

- $\tilde{\varphi}_i$ is constant on preimages under π_U: fix $y \in U_i$, then $\pi_U^{-1}(\{y\}) = \left\{ \lambda\mathbf{y} \;\middle|\; \lambda \in \mathbb{R} \setminus \{0\} \right\}$, i.e. the point $y \in \mathbb{RP}^n$ pulls back to every nonzero point on the line spanned by $\mathbf{y} \in \mathbb{R}^n$.

  But

$$\begin{aligned} \tilde{\varphi}_i(\lambda\mathbf{y}) &= \varphi_i\left(\left[\lambda y^1, \cdots, \lambda y^i, \cdots, \lambda y^n\right]\right) \\ &= \left[\frac{\lambda y^1}{\lambda y^i}, \cdots, \widehat{\lambda y^i}, \cdots, \frac{\lambda y^{n+1}}{\lambda y^i}\right] \\ &= \left[\frac{y^1}{y^i}, \cdots, \widehat{y^i}, \cdots, \frac{y^{n+1}}{y^i}\right] \\ &= \tilde{\varphi}_i(\mathbf{y}). \end{aligned}$$

So this yields a continuous map

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n.$$

We can now verify that φ is a homeomorphism since it has a continuous inverse given by

$$\begin{aligned} \varphi_i^{-1} : \mathbb{R}^n &\longrightarrow U_i \subseteq \mathbb{RP}^n \\ \mathbf{u} \coloneqq \left[u^1, \cdots, u^n\right] &\mapsto \left[u^1, \cdots, u^{i-1}, 1, u^{i+1}, \cdots, u^n\right]. \end{aligned}$$

It remains to check: Exercise

1. The $n+1$ sets $U_1, \cdots, U_{n+1}$ cover $\mathbb{RP}^n$.
2. $\mathbb{RP}^n$ is Hausdorff
3. $\mathbb{RP}^n$ is second-countable.

Exercise (1.6) Show that $\mathbb{RP}^n$ is Hausdorff and second countable.

Exercise (1.7) Show that $\mathbb{RP}^n$ is compact. (Hint: show that π restricted to S^n is surjective.)

Definition 5.1.2 (Topological Embedding)**.**
A continuous map $f : X \longrightarrow Y$ is a *topological embedding* iff it is injective and $\tilde{f} : X \longrightarrow f(X)$ is a homeomorphism.

Example 5.4 (Product Manifolds)**.**
Let $M := M_1 \times \cdots \times M_k$ be a product of manifolds of dimensions $n_1, \cdots, n_k$ respectively. A product of Hausdorff/second-countable spaces is still Hausdorff/second-countable, so just need to check that it's locally Euclidean.

- Let $\mathbf{p} \in \prod_{i=1}^{N} M_i$, so $p_i \in M_i$
- Choose a chart (U_i, φ_i) with $p_i \in U_i$ and assymble a product map:
$$\Phi := \prod \varphi_i : \prod U_i \longrightarrow \prod R^{n_i} \cong \mathbb{R}^{\Sigma n_i} := \mathbb{R}^N.$$
- Claim: Φ is a homeomorphism onto its image in R^N.
  - Each φ_i is a homeomorphism onto $\varphi_i(U_i)$ (by the definition of a chart on M_i)
  - It suffices to show that that Φ^{-1} exists and is continuous, where
$$\Phi^{-1}(V) := \left(\prod \varphi_i\right)^{-1}\left(\prod V_i\right).$$
  - Φ is a product of continuous functions and thus continuous.
  - $\Phi^{-1} := \left(\prod \varphi_i\right)^{-1} = \prod \varphi_i^{-1}$, which are all assumed continuous since φ_i were homeomorphisms.

Example 5.5 (Torii)**.**
$T^n := \prod_{i=1}^{n} S^1$ is a topological n-manifold.

Definition 5.1.3 (Precompact)**.**
A subset $A \subseteq X$ is *precompact* iff its closure $\mathrm{cl}_X(A)$ is compact in X.

Proposition 5.2.
Every topological manifold has a countable basis of precompact coordinate balls.

Proposition 5.3.
Let M be a topological manifold.

- M is locally path-connected.
- M is connected $\iff$ M is path-connected
- The connected components and path components of M coincide.
- $\pi_0(M)$ is countable and each component is open and a connected topological manifold.

Proposition 5.4.
Every topological manifold M is locally compact.

Proof .
M has a basis of precompact open sets. ■

Theorem 5.5 ***(Manifolds are Paracompact).***
Given any open cover $\mathcal{U} \rightrightarrows M$ of a topological manifold and any basis $\mathcal{B}$ for the topology on M, there exists a countable locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$.

Proposition 5.6.
$\pi_1(M)$ is countable.

6 Chapter 2

Definition 6.0.1 (Smooth Functionals on Manifolds)**.**
A function $f : M^n \longrightarrow \mathbb{R}^k$ is *smooth* iff for every $p \in M$ there exists a smooth chart (U, φ) about p such that $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}^k$ is smooth as a real function.

Fact: $C^\infty(M) \coloneqq \{f : M \longrightarrow \mathbb{R}\}$ is a vector space

Definition 6.0.2 (Coordinate Representations of Functions)**.**
Given a function $f : M \longrightarrow \mathbb{R}^k$, the function $\widehat{f} : \varphi(U) \longrightarrow \mathbb{R}^k$ where $\widehat{f}(x) = (f \circ \varphi^{-1})(x)$ is a *coordinate representation* of f.

Fact: f is smooth $\iff$ f is smooth (in the above sense) in *some* smooth chart about each point.

Definition 6.0.3 (Smooth Maps Between Manifolds)**.**
$F : M \longrightarrow N$ is *smooth* iff for every $p \in M$ there exists charts $p \in (U, \varphi)$ and $F(p) \in (V, \psi)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ is smooth.

Fact: taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{id}$ recovers the previous definition.

Proposition 6.1.
Every smooth map between manifolds is continuous.

Proposition 6.2 ***(Smoothness is Local).***
If $F : M \longrightarrow N$, then

1. If every $p \in M$ has a neighborhood $U \ni p$ such that F restricted to U is smooth, then F is smooth.
2. If F is smooth, then its restriction to every open subset is smooth.

Definition 6.2.1.
For $F : M \longrightarrow N$ and (U, φ), (V, ψ) smooth charts in M, N respectively, then $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is the *coordinate representation* of F.

Proposition 6.3.

1. Constant maps $c : M \longrightarrow N$, $c(x) = n_0$ are smooth
2. The identity is smooth
3. Inclusion of open submanifolds $U \hookrightarrow M$ is smooth
4. $F : M \longrightarrow N$ and $G : N \longrightarrow P$ smooth implies $G \circ F$ is smooth.

Proposition 6.4.
A map $F : N \longrightarrow \prod_{i=1}^{k} M_i$ with at most one i such that $\partial M_i \neq \emptyset$ is smooth iff each component map $\pi_i \circ F : N \longrightarrow M_i$ is smooth.

Proving a map between manifolds is smooth:

1. Write the map as a composition of known smooth functions.
2. Write in *smooth local coordinates* and recognize the component functions as compositions of smooth functions

Fact: projection maps from products are smooth

- Every closed subset $A \subseteq M$ of a smooth manifold is the level set of some smooth nonnegative functional $f : M \longrightarrow \mathbb{R}$, i.e. $f^{-1}(0) = A$.

7 Chapter 3

Definition 7.0.1.
For a fixed point $\mathbf{a} \in \mathbb{R}^n$, define the *geometric tangent space* at $\mathbf{a}$ to be the set

$$\mathbb{R}^n_{\mathbf{a}} := \{\mathbf{a}\} \times \mathbb{R}^n = \left\{ (\mathbf{a}, \mathbf{v}) \;\middle|\; \mathbf{p} \in \mathbb{R}^n \right\}.$$

Notation: $\mathbf{v}_a$ denotes the tangent vector at $\mathbf{v}$, i.e. the pair $(\mathbf{a}, \mathbf{v})$. Think of this as a vector with its base at the point $\mathbf{a}$.

Remark There is a natural isomorphism $\mathbb{R}^n_a \cong \mathbb{R}^n$ given by $(\mathbf{a}, \mathbf{v}) \mapsto \mathbf{v}$.

This map is not explicitly stated.

Proposition 7.1.
$D_v\Big|_a$ satisfies the product rule:

$$D_v\Big|_a(fg) = f(a)\cdot D_v\Big|_a g + D_v\Big|_a f\cdot g(a).$$

Picking the standard basis for $\mathbb{R}^n_a = \{\mathbf{e}_{i,a}\}_{i=1}^n$ and expanding $\mathbf{v} = \sum_{i=1}^n v^i\mathbf{e}_{i,a}$, we can explicitly write

$$D_v\Big|_a f = \sum_{i=1}^n v^i\frac{\partial f}{\partial x^i}(a).$$

Definition 7.1.1.
Denote the space of all derivations of $C^\infty(\mathbb{R}^n)$ at a as

$$T_a\mathbb{R}^n := \left\{w\in\hom_{\mathbb{R}\text{-mod}}(C^\infty(\mathbb{R}^n),\mathbb{R})\;\middle|\; w(fg) = f(a)w(g)+g(a)w(f)\right\},$$

i.e. a derivation w is an $\mathbb{R}$-linear map satisfying the Leibniz Rule (LR).

What does this equality mean? Is $w(fg)$ a real number? Does $wg = w(g)$, so this is a number too?

Facts:

1. If f is a constant function then $v(f) = 0\in\mathbb{R}$.
2. If $f(p) = g(p)$ for $p\in M$ then $v(fg) = 0\in\mathbb{R}$.

Example 7.1.
Claim: if $f\in C^\infty(\mathbb{R}^n)$ is constant, say $f(\mathbf{p}) = 1$ for all $\mathbf{p}\in\mathbb{R}^n$, then $w(f) = 0$ for any derivation w.

Proof: WLOG suppose $f(\mathbf{p}) = 1\in\mathbb{R}$. Note that $f(\mathbf{p}) = f(\mathbf{p})\cdot f(\mathbf{p})$, so

$$w(f) = w(f\cdot f)\overset{LR}{=} f(\mathbf{p})w(f) + w(f)f(\mathbf{p}) = 2f(\mathbf{p})w(f) = 2w(f)\quad\text{since } f(\mathbf{p}) = 1,$$

and thus $w(f) = 2w(f)\in\mathbb{R}$ forcing $w(f) = 0$.

Remark A geometric tangent vector provides a way of taking directional derivatives via the correspondence

$$\begin{aligned}\mathbb{R}^n_a &\longrightarrow C^\infty(\mathbb{R}^n)^\vee\\ \mathbf{v}_a &\mapsto D_{\mathbf{v}}|_a\end{aligned}$$

where

$$\begin{aligned}D_{\mathbf{v}}|_a : C^\infty(\mathbb{R}^n) &\longrightarrow \mathbb{R}\\ f &\mapsto D_{\mathbf{v}}f(\mathbf{a}) := \frac{\partial}{\partial t}\Big|_{t=0} f(\mathbf{a}+t\mathbf{v}).\end{aligned}$$

More precisely,

Proposition 7.2 ***(Space of Derivations is Isomorphic to Geometric Tangent Space).*** For each geometric tangent vector $\mathbf{v}_a \in \mathbb{R}^n_a$, the map $D_{\mathbf{v}}\Big|_a$ is a derivation at a, and the map $\mathbf{v}_a \mapsto D_{\mathbf{v}}\Big|_a$ is an isomorphism.

Theorem 7.3 ***(Basis of Tangent Space).***
For any $\mathbf{p} \in \mathbb{R}^n$, there is a basis for $T_{\mathbf{p}}\mathbb{R}^n$ given by $\left\{ \frac{\partial}{\partial x^i}\Big|_{\mathbf{p}} \;\middle|\; 1 \leq i \leq n \right\} \subset C^\infty(\mathbb{R}^n, \mathbb{R})$ which are defined as

$$\begin{aligned} \frac{\partial}{\partial x^i} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x^i}(\mathbf{p}). \end{aligned}$$

Definition 7.3.1 (Tangent Vector on a Manifold)**.**
Let M be smooth and $p \in M$, then

$$T_pM := \left\{ v : C^\infty(M) \longrightarrow \mathbb{R} \;\middle|\; v(fg) = f(p)vg + g(p)vg \right\}.$$

Definition 7.3.2 (Differential of a Map)**.**
For $F : M \longrightarrow N$ a smooth map, for each $p \in M$, we define the *differential* of f at p as

$$\begin{aligned} dF_p : T_pM &\longrightarrow T_{F(p)}N \\ v &\mapsto (DF_p(v) : f \mapsto v(f \circ F)). \end{aligned}$$

Note that $f \in C^\infty(N)$ implies that $f \circ F \in C^\infty(M)$, and since $v \in T_pM$ is a functional in $C^\infty(M)^\vee$, v can act on such objects.. Moreover, $dF_p(v)$ is in fact a derivation at $F(p)$, since

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) \\ &= v((f \circ F) \cdot (g \circ F)) && Why? \\ &= (f \circ F)(p) \cdot v(g \circ F) + v(f \circ F) \cdot (g \circ F)(p) && \text{since } v \text{ is a derivation} \\ &:= (f \circ F)(p) dF_p(v)(g) + (g \circ F)(p) dF_p(v)(f) \\ &:= f(F(p)) dF_p(v)(g) + g(F(p)) dF_p(v)(f), \end{aligned}$$

which puts it in the form $\partial(fg) = f(q)\partial(g) + \partial(f)g(q)$ where $q = F(p)$.

Facts:

- dF_p is a linear map.
- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$.
- If F is a diffeomorphism, then dF_p is an isomorphism with $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 7.4 ***(Tangent Vectors Act Locally).***
If $f, g \in C^\infty(M)$ agree on any neighborhood of $p \in M$, then $v(f) = v(g)$.

Warning: the action of a derivation depends only on the values of a function in arbitrarily small neighborhoods, and in particular, can only be applied to functions defined in a neighborhood of p (not necessarily on all of M).

Remark The tangent space of an n-manifolds is n-dimensional, even on boundary point.

For a vector space V, there is a natural smooth structure (Example 1.24) and for any $\mathbf{a}, \mathbf{v} \in V$ we can similarly define a map

$$\begin{aligned} D_{\mathbf{v}}\Big|_{\mathbf{a}} : C^\infty(V) &\longrightarrow \mathbb{R} \\ f &\mapsto D_{\mathbf{v}} f(\mathbf{a}) := \frac{\partial}{\partial t}\Big|_{t=0} f(\mathbf{a} + t\mathbf{v}). \end{aligned}$$

Proposition 7.5.
If V is a vector space, for any $\mathbf{a} \in V$, the map $\mathbf{a} \mapsto D_{\mathbf{v}}\Big|_{\mathbf{a}}$ yields an isomorphism $V \cong T_{\mathbf{a}} V$. Thus tangent vectors in V are routinely identified with elements of V.

Example 7.2.
$\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{Mat}(n \times n, \mathbb{R})$ is an open submanifold, and thus if $p \in \mathrm{GL}(n, \mathbb{R})$ then we can identify $T_p \mathrm{GL}(n, \mathbb{R}) \cong \mathrm{Mat}(n \times n, \mathbb{R})$.

Definition 7.5.1.
The *tangent bundle* of a manifold is defined as $TM := \coprod_{p \in M} T_p M$. Points in TM are often written as (p, v), and there is a natural projection map $\pi : TM \longrightarrow M$ given by $(p, v) \mapsto p$.

Proposition 7.6.
If $F : M \longrightarrow N$ is smooth with $p \in M$ and $v \in T_p M$, then $dF_p(v) = (F \circ \gamma)(0)$ where $\gamma : (-a, b) \longrightarrow M$ is any smooth curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Definition 7.6.1 (Germ of a Function)**.**
The *germ* of a function f at p is the equivalence class of ordered pairs (f, U) where $U \subseteq M$ is open and $f \in C^\infty(U, \mathbb{R})$, where $(f, U) \sim (g, V)$ iff there exists a neighborhood $N \subset U \bigcap V$ containing p such that $f|_N \equiv g|_N$. The set of germs of functions at p is denoted $C_p^\infty(M)$ and is an associative $\mathbb{R}$-algebra.

Remark This definition is the only one available for real or complex analytic manifolds, since there do not exist analytic bump functions.

Todo list