

# Problem Set 2

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## 1 Humphreys 1.5

Proposition: Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and  $M(\lambda), M(\mu)$  Verma modules. Then  $M(\lambda) \otimes M(\mu)$  can not lie in  $\mathcal{O}$ .

Useful facts:

- For any  $\lambda \in \mathfrak{h}^\vee$ ,  $\mathbb{C}_\lambda$  is a 1-dimensional  $\mathfrak{b}$ -bimodule with a trivial  $\mathfrak{n}$ -action.
- $M(\lambda) = \text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$  as a left  $U(\mathfrak{g})$ -module.
- $M(\lambda) = U(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda$  as a left  $U(\mathfrak{n}^-)$ -module.
- $M(\lambda)$  is generated as a  $U(\mathfrak{g})$ -module by the maximal vector  $v^+ = 1 \otimes 1$ .
- The set of weights of  $M(\lambda)$  is  $\lambda - \Gamma$  where  $\Gamma$  is the semigroup in  $\Lambda_r$  generated by  $\Phi^+$ .
- $M(\lambda)$  has weights  $\lambda, \lambda - 2, \lambda - 4, \dots$  each with multiplicity 1.

Questions

- What is the tensor product over? Guess:  $\otimes_{\mathbb{C}}$ .
- MSE: the product is no longer finitely generated.
  - Consider dimensions of weight spaces – eventually constant.
  - If  $\text{wt}(v) = \lambda$  and  $\text{wt}(u) = \mu$ , then  $\text{wt}(u \otimes v) = \lambda + \mu$ .
  - Consider a weight space  $N_\gamma$  of  $M$ . This must have the form  $\bigoplus_{a+b=\gamma} M_a \otimes_{\mathbb{C}} M_b$ .
  - Example: consider  $\lambda = \mu = 0$ . Then  $M = M(0) \otimes M(0)$  and  $N_{-2m}$  has dimension  $m + 1$  for every  $m \in \mathbb{Z}^+$ .

**Solution:**

Let  $M(\lambda), M(\mu)$  be arbitrary Verma modules with highest weight vectors  $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$  respectively. We can then consider the weight of  $v \otimes w$  in  $N := M(\lambda) \otimes_{\mathbb{C}} M(\mu)$ :

$$\begin{aligned}
h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\
&= \lambda(h)v \otimes w + v \otimes \mu(h)w \\
&= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\
&= (\lambda(h) + \mu(h))(v \otimes w).
\end{aligned}$$

Letting  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ , so  $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$ , the claim is that it is possible for  $N$  to *not* be finitely-generated as a  $U(\mathfrak{g})$ -module.

Let  $\{y, h, x\}$  be the usual basis for  $\mathfrak{g}$ , for which  $U(\mathfrak{g})$  has the usual associated PBW basis. We can use the fact that  $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$ , so if we pick  $\mu, \lambda \in \mathbb{Z}^{\leq 0}$  we have weight space decompositions

$$\begin{aligned}
M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda-2i} := \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i := \lambda-2i}} M(\lambda)_{\lambda_i} \\
M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu-2j} := \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_i := \mu-2j}} M(\mu)_{\mu_j}
\end{aligned}$$

where we can explicitly identify bases  $M(\lambda)_{\lambda_i} = \text{span}_{\mathbb{C}} \{y^i v\}$  and  $M(\mu)_{\mu_i} = \text{span}_{\mathbb{C}} \{y^i w\}$ .

By the initial observation, this yields a weight space decomposition for  $N$  given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left( \bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_{\nu}.$$

Since each weight space  $N_{\nu} \supseteq \text{span}_{\mathbb{C}} \{y^i v \otimes y^j w \mid i + j = \nu\}$  and there are infinitely many such weight spaces, extending to a PBW basis for  $U(\mathfrak{g})$  shows that no finite number of elements in  $U(\mathfrak{g})$  can generate  $N$ . ■

## 2 Humphreys 1.9

**Proposition:** Let  $\psi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h})$  be the Harish-Chandra homomorphism. Then  $\psi$  is independent of the choice of a simple system in  $\Phi$ .

Hint: any simple system has the form  $w\Delta$  for some  $w \in W$ .

Useful facts:

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