Problem Set 7

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Contents

1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence, so $\|x^j - x^k\|_{\ell^2} \to 0$. We want to produce some $\mathbf{x} := \lim\limits_{n\to\infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i := \lim_{n\to\infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left|x_i^j - x_i^k\right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left|x_i^j - x_i^k\right|^2$ is a Cauchy sequence of real numbers which necessarily converges in $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\|\lim_{k\to\infty} x^k - x^j\right\|_{\ell^2} = \lim_{k\to\infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$\begin{aligned}\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\ &\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\ &\to M < \infty,\end{aligned}$$

where $\|\mathbf{x}_i - x^j\|_{\ell^2} \to 0$ and the second sum is finite because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} \coloneqq M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot,\ \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, we have

$$\Re(\langle x,\ iy \rangle) = \Re(-i\langle x,\ y \rangle) = \Im(\langle x,\ y \rangle).$$

We can compute the following:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\ \Re\left(\langle x,\ y \rangle\right)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\ \Re\left(\langle x,\ y \rangle\right)$$

$$\begin{aligned}\|x+iy\|^2 &= \|x\|^2 + \|y\|^2 + 2\ \Re\left(\langle x,\ iy \rangle\right) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y \rangle)\end{aligned}$$

$$\begin{aligned}\|x-iy\|^2 &= \|x\|^2 + \|y\|^2 - 2\ \Re\left(\langle x,\ iy \rangle\right) \\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y \rangle)\end{aligned}$$

and summing these all

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\| &= 4\ \Re\left(\langle x,\ y \rangle\right) + 4i\ \Im\left(\langle x,\ y \rangle\right) \\ &= 4\langle x,\ y \rangle.\end{aligned}$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 \coloneqq \langle x,\ x \rangle = \langle Ux,\ Ux \rangle \coloneqq \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned}
\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2 + \|Ux-Uy\|^2 + i\|Ux+Uy\|^2 - i\|Ux+Uy\|^2\right)\\
&= \frac{1}{4}\left(\|U(x+y)\|^2 + \|U(x-y)\|^2 + i\|U(x+y)\|^2 - i\|U(x+y)\|^2\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2 + i\|x+y\|^2 - i\|x+y\|^2\right)\\
&= \langle x,\ y\rangle.
\end{aligned}$$

□

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$\begin{aligned}
|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle|\\
&= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle|\\
&\le \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\|\\
&\to 0\cdot M + C\cdot 0 < \infty,
\end{aligned}$$

where $\|y_n\| \to M$ since $y_n \to y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$\langle e,\ y\rangle = \left\langle \lim_n e_n,\ y\right\rangle = \lim_n \langle e_n,\ y\rangle = \lim_n 0 = 0,$$

so $e \in E^\perp$.

2.2 Part b:

Let $S := \mathrm{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

We first note that $S \subseteq E^{\perp\perp}$: let $y \in E^\perp$ be arbitrary. Then, if $e \in E$, we have $\langle e,\ y\rangle = 0$ since $y \in E^\perp$. But if e is orthogonal to every $y \in E^\perp$, then $e \in E^{\perp\perp}$ by definition. It then follows that $\overline{S} \subseteq \overline{E^{\perp\perp}} = E^{\perp\perp}$, since by (1), any orthogonal complement is a closed subspace.

We now want to show that $E^{\perp\perp} \subseteq \overline{S}$.