Title

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1 | Lecture 15: The L-Polynomial

Recall that we had $Z(t) + F(t) + G(t)$:

$$(q-1)F(t) = \sum_{0 \leq \deg C \leq 2g-2} q^{\ell(C)} t^{\deg(C)}$$
$$(q-1)G(t) = h\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t}\right).$$

Note that $F(t)$ is a polynomial of degree at most $2g-2$, and clearing denominators in $G(t)$ yields a polynomial of degree at most $2g$

Definition 1.0.1 (The L-polynomial)
The L-polynomial is defined as

$$L(t) := (1-t)(1-qt)Z(t) = (1-t)(1-qt)\sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[t].$$

It turns out that the degree bound of $2g$ is sharp, and the coefficients closer to the middle are most interesting:

Theorem 1.0.2 *(?).*
Let $K/\mathbb{F}_q$ be a function field of genus $g \geq 1$, then

a. $\deg L = 2g$.
b. $L(1) = h$
c. $L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right)$.
d. Writing $L(t) = \sum_{j=1}^{2g} a_j t^j$,

- $a_0 = 1$ and $a_{2g} = q^g$.
- For all $0 \leq j \leq g$, we have $a_{2g-j} = q^{g-j} a_j$.
- $a_1 = |\Sigma_1(K/\mathbb{F}_q)| - (q+1)$, which notably does not depend on g.
- Write $L(t) = \prod_{j=1}^{2g}(1 - \alpha_j t) \in \mathbb{C}[t]$ [a]

e. The $\alpha_j \in \overline{\mathbb{Z}}$ [b] (which were *a priori* in $\mathbb{C}$) and can be ordered such that for all $1 \leq j \leq g$, we have $a_j a_{g+j} = q$. [c]

f. If $L_r(t) = (1-t)(1-q^r t)Z_r(t)$ then $L_r(t) = \prod_{j=1}^{2g}(1-\alpha_j^r t)$, where K_r is the constant extension $K\mathbb{F}_{q^r}/\mathbb{F}_{q^r}$

[a]The polynomial isn't monic, but rather has a constant coefficient, so this expansion is somewhat more natural than (say) $\prod(t-\alpha)$.
[b]$\overline{\mathbb{Z}}$ denotes the algebraic integers.
[c]This is the first hint at the Riemann hypothesis: if for example they all had the same complex modulus, this would force $|a_j| = \sqrt{q}$. Thus proving that they all have the same absolute value is 99% of the content!

Note that the α_j are reciprocal roots.

Proof (of a).
We saw from $Z(t) = F(t) + G(t)$ that $\deg L \leq 2g$. Equality will follow from the proof of (d) part 1, since this would imply that $a_{2g} = q^g \neq 0$.

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Proof (of b).
Our formula $Z(t) = F(t) + G(t)$ and Schmidt's theorem (showing $\delta = 1$) gives

$$L(t) = (1-t)(1-qt)F(t) + \frac{h}{q-1}\left(q^g t^{2g-2}(1-t) - (1-qt)\right),$$

where we've expanded G but not F because it involves various $\ell(D)$ which are difficult to compute. It is some polynomial though, and we can evaluate L at 1 to get $L(1) = h$. Thus the class number is the sum of the coefficients!

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Proof (of c).
This follows easily from the functional equation for $Z(t)$, which we already established using the Riemann-Roch theorem:

$$Z(t) = q^{g-1}t^{2g-2}Z\left(\frac{1}{qt}\right).$$

We can compute

$$\begin{aligned} q^g t^{2g} L\left(\frac{1}{qt}\right) &= q^g t^{2g}\left(1-\frac{1}{qt}\right)\left(1-\frac{1}{t}\right)Z\left(\frac{1}{qt}\right) \\ &= q^{g-1}t^{2g-2}(1-t)(1-qt)Z\left(\frac{1}{qt}\right) \\ &= (1-t)(1-qt)Z(t) \\ &\coloneqq L(t), \end{aligned}$$

where we've distributed one q and two ts in the first steps.

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Proof (of d).
Using the functional equation from (c), we can write

$$L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right) = \left(\frac{a_{2g}}{q^g}\right) + \left(\frac{a_{2g-1}}{q^{g-1}}\right) t + \cdots + (a_0 q^g)\, t^{2g},$$

where we're correcting by enough in t but not enough in q and seeing what we get. Equating coefficients, for $0 \le j \le g$ we have

$$a_{2g-j} = q^{g-j} a_j. \tag{1}$$

Using the fact that A_0 is the number of effective degree zero divisors, which is only zero, we have $A_0 = 1$ and we can multiply formal power series to obtain

$$\begin{aligned} L(t) = a_0 + a_1 t + \cdots + a_{2g} t^{2g} &= (1-t)(1-qt) \sum_{n=0}^{\infty} A_n t^n \\ &= \left(1 - (q+1)t + qt^2\right)(1 + A_1 t + A_2 t^2 + \cdots) \\ &= 1 + (A_1 - (q+1))\, t + \cdots . \end{aligned}$$

From this, we can read off

- $L(0) = a_0 = 1$
- $a_1 = A_1 - (q+1) = \Sigma_1(K/k) - (q+1)$
- $a_{2g} = a_{2g-0} = q^{g-0} a_0 = a^g$ by taking $j = 0$ in eq. 1, and thus $\deg L = 2g$.

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Proof (of e (the most interesting!)).
Consider the **reciprocal polynomial**

$$L^\perp(t) \coloneqq t^{2g} L\left(\frac{1}{t}\right) = t^{2g} + a_1 t^{2g-1} + \cdots + q^g.$$

The original polynomial had $\mathbb{Z}$ coefficients and constant term 1, so this polynomial is monic and has a nonzero constant term. Thus its roots are patently nonzero algebraic integers in $\overline{\mathbb{Z}}^{\bullet}$. If $L^\perp(t) = \prod_{j=1}^{2g} (t - \alpha_j)$, then

$$L(t) = t^{2g} L^\perp\left(\frac{1}{t}\right) = \prod_{j=1}^{2g} (1 - \alpha_j t)$$

and if the roots of $L(t)$ are r_j, then the roots of $L^\perp(t)$ are the reciprocal roots $1/r_j$ and vice-versa. This shows the first assertion that $r_j \in \overline{\mathbb{Z}}$ as well.
The most interesting part is what follows. Making the substitution $t = qu$ and using (c) we

get

$$\begin{aligned} L^{\perp}(t) &= \prod_{j=1}^{2g}(t-\alpha_j) \\ &:= t^{2g}L\left(\frac{1}{t}\right) \\ &= q^{2g}u^{2g}L\left(\frac{1}{qu}\right) && \text{by (c).} \end{aligned}$$

Using $u = t/q$, we can write

$$\begin{aligned} q^g L(u) &= q^g \prod_{j=1}^{2g}(1-\alpha_j u) \\ &= q^g \prod_{j=1}^{2g}\left(1-\frac{\alpha_j}{q}t\right) \\ &= q^g \prod_{j=1}^{2g}\frac{\alpha_j}{q}\prod_{j=1}^{2g}\left(t-\frac{1}{\alpha_j}\right) \\ &= \prod_{j=1}^{2g}\left(t-\frac{q}{\alpha_j}\right), \end{aligned}$$

where we've pulled out a factor of $-\alpha_j/q$ and in the last step we've used that $\prod_{j=1}^{2g}\alpha_j = q^g$. This follows because the α_j are the roots of $L^{\perp}$, which has even degree, so the product of all of the roots is equal to the constant term of $L^{\perp}$, which is the leading term of L, which we showed was q^g.

This says that if we take these roots α_j as a multiset and replace each α_j with q/α_j, we get the same multiset back. I.e., this multiset is stable under the involution

$$\begin{aligned} \mathbb{C}^{\times} &\to \mathbb{C}^{\times} \\ z &\mapsto \frac{q}{z}. \end{aligned}$$

This almost pairs up the elements of this finite set of roots, except it may have fixed points. The complex numbers α such that $\alpha = q/\alpha$ are precisely $\pm\sqrt{q}$. So group the α_i^{-1} into

- k **pairs** of nonfixed points, where $\alpha_i \neq q/\alpha_i$,
- m points such that $\alpha_i = \sqrt{q}$,
- n points such that $\alpha_i = -\sqrt{q}$.

So we'd like to show that m and n are both even, so when we're pairing roots with reciprocals these get paired with themselves. We know $2k + m + n = 2g$, so $m + n$ is even. We also know that

$$\begin{aligned} q^g &= \prod_{j=1}^{2g} \alpha_j \\ &= q^k \left(\sqrt{q}\right)^m \left(-\sqrt{q}\right)^n \\ &= (-1)^n q^{k+\frac{m}{2}+\frac{n}{2}} \\ &= (-1)^n q^g. \end{aligned}$$

This forces n to be even, and since $m = 2g - 2k - n$, m must be even as well. ■

Proof (of f).
We used Dirichlet's character-style decomposition of $Z(t)$ in Schmidt's theorem, and we'll use it again here. Write

$$\begin{aligned} L_r(t^r) &= (1-t^r)(1-q^r t^r) Z_r(t^r) \\ &= (1-t^r)(1-q^r t^r) \prod_{\xi \in \mu_r} Z(\xi t) \\ &= (1-t^r)(1-q^r t^r) \prod_{\xi \in \mu_r} \frac{L(\xi t)}{(1-\xi t)(1-q\xi t)} \\ &= \prod_{\xi \in \mu_r} L(\xi t), \end{aligned}$$

where we've used that

$$\begin{aligned} \prod_{\xi \in \mu_r} \frac{1}{1-\xi t} &= 1 - t^r \\ \prod_{\xi \in \mu_r} \frac{1}{1-q\xi t} &= 1 - q^r t^r \end{aligned}$$

which leads to all of the denominators canceling. We can then expand $L_r(t^r)$ as a product to compute

$$\begin{aligned} L_r(t^r) &= \prod_{\xi \in \mu_r} L(\xi t) \\ &= \prod_{\xi \in \mu_r} \prod_{j=1}^{2g} (1 - \alpha_j q t) \\ &= \prod_{j=1}^{2g} \prod_{\xi \in \mu_r} (1 - \alpha_j q t) && \text{since these are finite products} \\ &= \prod_{j=1}^{2g} (1 - \alpha_j^r t^r). \end{aligned}$$

From this we can conclude that $L_r(t) = \prod_{j=1}^{2g}(1-\alpha_j^r t)$, since t^r is just an indeterminate and these are all identities of polynomials.

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Corollary 1.0.3 *(?)*.

Suppose $K/\mathbb{F}_q$ is genus $g \geq 1$ and $L(t) = \prod_{j=1}^{2g}(1-\alpha_j t)$. Then for all $r \in \mathbb{Z}^{\geq 0}$, we have a nice expression for N_r:

$$N_r := |\Sigma_1(K_r/\mathbb{F}_{q^r})| = q^r + 1 - \sum_{j=1}^{2g}\alpha_j^r.$$

Proof (?).

Let $L_r(t) = \sum_{j=1}^{2g} a_{j,r} = \prod_{j=1}^{2g}(1-\alpha_j^r t)$, so $a_{1,r} = -\sum_{j=1}^{2g}\alpha_j^r$. Then using (d) part 3, we can write

$$|\Sigma_1(K_r/\mathbb{F}_{q^r})| = q^r + 1 + a_{1,r} = q^r + 1 - \sum_{j=1}^{2g}\alpha_j^r.$$

This follows from consider $\prod(1-\alpha_j^r t)$, where extracting the t^1 coefficient involves choosing $-\alpha_j^r$ once and 1 from all of the remaining terms, and then you sum over the disjoint possibilities.

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Remark 1.0.4: We'd really like to compute the coefficients of the L polynomials, since we can solve a polynomial equation to get the roots. But the Galois groups of these polynomials may not be solvable, so the term $\sum \alpha_j^r$ will in general be some symmetric function in the complex roots. Note that any symmetric polynomial in the roots is also a symmetric polynomial in the coefficients.

Corollary 1.0.5 *(?)*.

For $K/\mathbb{F}_q$ a function field, define

$$S_r := N_r - (q^r + 1) = -\sum_{j=1}^{2g}\alpha_j^r.$$

Note that $N_r = |\Sigma(K_r/\mathbb{F}_{q^r})|$ is the number of $\mathbb{F}_{q^r}$-rational point. Then

a. $L'(t)/L(t) = \sum_{r=1}^{\infty} S_r t^{r-1}$.

b. $a_0 = 1$, and for all $1 \leq i \leq g$,

$$ia_i = S_i a_0 + S_{i-1}a_1 + \cdots + S_1 a_{i-1}.$$

Remark 1.0.6: What's the usefulness here? If you only have the coefficients of the L polynomials, taking the logarithmic derivative gives access to these quantities S_r. The second formula is a recursive expression for the a_i in terms of the S_i. So you can compute the coefficients of the L polynomial by counting $\mathbb{F}_{q^r}$-rational points on your curve (or places on your function field) for $r = 1, 2, \cdots, g$. Similarly, if you have all of the coefficients for a Z polynomial, you can solve for the S_i.

Proof (of a).

Essentially just a computation. Logarithmically differentiating both sides of $L(t) = \prod_{j=1}^{2g}(1-\alpha_j t)$ and expanding in a geometric series yields

$$\begin{aligned}\frac{L'(t)}{L(t)} &= \sum_{j=1}^{2g} \frac{-\alpha_j}{a - \alpha_j t} \\ &= \sum_{j=1}^{2g}(-\alpha_j)\sum_{r=0}^{\infty}(\alpha_j t)^r \\ &= \sum_{r=1}^{\infty}\left(\sum_{j=1}^{2g}(-\alpha_j^r)\right) t^{r-1} \\ &= \sum_{r=1}^{\infty} S_r t^{r-1}.\end{aligned}$$

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Proof (of b).

Clearing denominators and equating coefficients in $L'(t) = L(t)\sum_{r=1}^{\infty} S_r t^{r-1}$ yields the result immediately, since the ia_i are what appear as coefficients in the derivative of a formal power series, whereas the RHS is a Cauchy product.

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Remark 1.0.7: The moral: to compute zeta functions, you don't have to enumerate divisors and compute dimensions of Riemann-Roch spaces. Note that the Riemann-Roch theorem tells us something interesting about these dimensions, but doesn't compute the dimension outright! Instead, it suffices to compute $\mathbb{F}_{q^r}$-rational points for $r \leq g$.

A few lectures ago we discussed the places on a hyperelliptic function field, including a place at infinity. Computing the zeta function of a hyperelliptic curve involves plugging in x-values and determining if it is

- A nonzero non-square: no y-values,
- Zero: exactly one y-value,
- A nonzero square: two y-values.

This is what happens at the finite places. To handle the place at ∞, there is a recipe for the degree of the polynomial in terms of the coefficients. So for any hyperelliptic function field (and in particular, for any elliptic function field) we have a concrete algorithm for computing their zeta functions. Note that this is not necessarily a *good* algorithm: it still involves plugging in many values and checking if things are squares in finite values. It seems that most people who compute a lot of zeta functions mostly focus on hyperelliptic function fields.

How are you going to compute zeta functions or even places for more complicated function fields? The Riemann-Hurwitz formula says that since any function field is a finite degree extension of a rational function field,