

# Title

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### 0.1 Exercises

If  $\sigma = (i_1 i_2 \cdots i_r) \in S_n$  and  $\tau \in S_n$ , then show that  $\tau \sigma \tau^{-1} = (\tau(i_1) \tau(i_2) \cdots \tau(i_r))$ .

Show that  $S_n \cong \langle (12), (123 \cdots n) \rangle$  and also that  $S_n \cong \langle (12), (23 \cdots n) \rangle$

Let  $G$  be a finite abelian group that is not cyclic. Show that  $G$  contains a subgroup isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$  for some prime  $p$ .

Determine all abelian groups of order  $n$  for  $n \leq 20$ .

Let  $G$  be a group and  $A \trianglelefteq G$  be a normal abelian subgroup. Show that  $G/A$  acts on  $A$  by conjugation and construct a homomorphism  $\varphi : G/A \rightarrow \text{Aut}(A)$ .

Let  $Z(G)$  be the center of  $G$ . Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

*Note that Hungerford uses the notation  $C(G)$  for the center.*

Let  $G$  be a finite group and  $H \trianglelefteq G$  a normal subgroup of order  $p^k$ . Show that  $H$  is contained in every Sylow  $p$ -subgroup of  $G$ .

Let  $|G| = p^n q$  for some primes  $p > q$ . Show that  $G$  contains a unique normal subgroup of index  $q$ .

### 0.2 Qual Problems

Let  $G$  be a finite group and  $p$  a prime number. Let  $X_p$  be the set of Sylow- $p$  subgroups of  $G$  and  $n_p$  be the cardinality of  $X_p$ . Let  $\text{Sym}(X)$  be the permutation group on the set  $X_p$ .

1. Construct a homomorphism  $\rho : G \rightarrow \text{Sym}(X_p)$  with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that  $G$  is simple and the order of  $G$  divides  $n_p!$ .
3. Show that for any  $1 \leq a \leq 4$  and any prime power  $p^k$ , no group of order  $ap^k$  is simple.

Let  $G$  be a finite group and  $H < G$  a subgroup. Let  $n_H$  be the number of subgroups of  $G$  that are conjugate to  $H$ . Show that  $n_H$  divides the order of  $G$ .

Let  $G = S_5$ , the symmetric group on 5 elements. Identify all conjugacy classes of elements in  $G$ , provide a representative from each class, and prove that this list is complete.