

# Category $\mathcal{O}$ , Problem Set 4

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Thursday 16<sup>th</sup> April, 2020

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## 1 Humphreys 3.1

Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and identify  $\lambda \in \mathfrak{h}^\vee$  with a scalar. Let  $N$  be a 2-dimensional  $U(\mathfrak{b})$ -module defined by letting  $x$  act as 0 and  $h$  act as  $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ .

Show that the induced  $U(\mathfrak{g})$ -module structure  $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$  fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

### 1.1 Solution

Reference 1 Reference 2

Hence  $M \notin \mathcal{O}$ .

## 2 Humphreys 3.2

Show that for  $M \in \mathcal{O}$  and  $\dim L < \infty$ ,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

## 2.1 Solution

By theorem 3.2d, we have

$$M, N \in \mathcal{O} \implies (M \oplus N)^\vee \cong M^\vee \oplus N^\vee$$

and by definition,  $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$  is the direct sum of the duals of various weight spaces.

## 3 Humphreys 3.4

Show that  $\Phi_{[\lambda]} \cap \Phi^+$  is a positive system in the root system  $\Phi_{[\lambda]}$ , but the corresponding simple system  $\Delta_{[\lambda]}$  may be unrelated to  $\Delta$ .

For a concrete example, take  $\Phi$  of type  $B_2$  with a short simple root  $\alpha$  and a long simple root  $\beta$ . If  $\lambda := \alpha/2$ , check that  $\Phi_{[\lambda]}$  contains just the four short roots in  $\Phi$ .

### 3.1 Solution

We would like to show the following two propositions:

1.  $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \cap \Phi^+$  is a positive system in  $\Phi_{[\lambda]}$ ,
2. The simple system  $\Delta_{[\lambda]}$  corresponding to  $\Phi_{[\lambda]}^+$  is *not* generally given by  $\Delta_{[\lambda]} = \Phi_{[\lambda]} \cap \Delta$ , where  $\Delta$  is the simple system corresponding to  $\Phi$ .

We proceed by first showing (2) using the hinted counterexample when  $\Phi$  is of type  $B_2$  with  $\Delta = \{\alpha, \beta\}$  with  $\alpha$  a short root and  $\beta$  a long root.

Concretely, we can realize  $\Phi$  as a subset of  $\mathbb{R}^2$  in the following way:

$$\Phi = P_1 \amalg P_2 := \{[1, 0], [0, 1], [-1, 0], [0, -1]\} \amalg \{[1, 1], [-1, 1], [1, -1], [-1, -1]\},$$

where we note that  $P_1$  consists of short roots (of norm 1) and  $P_2$  of long roots (of norm  $\sqrt{2}$ ).

We can choose the simple system  $\Delta = \{\alpha := [1, 0], \beta := [-1, 1]\}$ , and then let

$$\Phi_{[\lambda]} := \left\{ \gamma \in \Phi \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\} \quad \gamma^\vee := \frac{2}{\|\gamma\|^2} \gamma.$$

Now choosing  $\lambda := \frac{\alpha}{2} = \left[ \frac{1}{2}, 0 \right]$ , a short calculation shows that for an arbitrary  $\gamma \in \Phi$ ,

$$\langle \lambda, \gamma^\vee \rangle := \left\langle \left[ \frac{1}{2}, 0 \right], \frac{2}{\|\gamma\|^2} \gamma \right\rangle.$$

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Thus

$$\begin{aligned}\gamma_1 \in P_2 &\implies \left\langle \left[ \frac{1}{2}, 0 \right], 2\gamma_1 \right\rangle = 2 \frac{1}{2} \langle [1, 0], [a, b] \rangle \\ \gamma_2 \in P_2 &\implies \langle \lambda, \gamma_2^\vee \rangle = \left\langle \left[ \frac{1}{2}, 0 \right], \frac{2}{(\sqrt{2})^2} [\pm 1, \pm 1] \right\rangle = \pm \frac{1}{2} \notin \mathbb{Z}.\end{aligned}$$

## 4 Humphreys 3.7

### 4.1 a

If a module  $M$  has a standard filtration and there exists an epimorphism  $\phi : M \longrightarrow M(\lambda)$ , prove that  $\ker \phi$  admits a standard filtration.

### 4.2 b

Show by example that when  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  that the existence of a monomorphism  $\phi : M(\lambda) \longrightarrow M$  where  $M$  has a standard filtration fails to imply that  $\operatorname{coker} \phi$  has a standard filtration.