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Last time: proved that if X, Y are affine varieties then there is a bijection

$$\begin{aligned} \left\{ \begin{array}{c} \text{Morphisms} \\ f: X \rightarrow Y \end{array} \right\} &\iff \left\{ \begin{array}{c} k\text{-algebra morphisms} \\ A(Y) \rightarrow A(X) \end{array} \right\} \\ f &\mapsto f^* : \mathcal{O}_Y(Y) \rightarrow \mathcal{O}_X(X). \end{aligned}$$

Remark 1.0.1.

A morphism $f : X \rightarrow Y$ is by definition a morphism of ringed spaces where $\mathcal{O}_X, \mathcal{O}_Y$ are the sheaves of regular functions.

Remark 1.0.2.

This shows $X \cong Y$ as ringed spaces iff $A(X) \cong A(Y)$ as k -algebras.

Example 1.0.1.

Take

$$\begin{aligned} f : \mathbb{A}^1 &\rightarrow V(y^2 - x^3) \subset \mathbb{A}^2 \\ t &\mapsto (t^2, t^3). \end{aligned}$$

This is a morphism by proposition 4.7.

We then get a map on algebras

$$\begin{aligned} f^* : A(V(y^2 - x^3)) &= k[x, y] / \langle y^2 - x^3 \rangle \rightarrow k[t] \\ x &\mapsto t^2 \\ y &\mapsto t^3, \end{aligned}$$

but even though f is a bijective morphism, it's not an isomorphism of ringed spaces. This can be seen from the fact that the image doesn't contain t .

Review of introductory category theory.

We'll define a category AffVar_k whose objects are affine varieties over k and morphisms in $\text{hom}(X, Y)$ will be morphisms of ringed spaces. There is a contravariant functor A into reduced finitely generated k -algebras which sends X to $A(X)$ and sends morphisms $f : X \rightarrow Y$ to their pullbacks $f^* : A(Y) \rightarrow A(X)$, where “reduced” denotes the fact that there are no nilpotents.

Review of the universal property of the product.

Remark 1.0.3.

If we have X, Y affine varieties, we take $X \times Y$ to be the categorical product instead of the underlying product of topological spaces. We have $A(X \times Y) \cong A(X) \otimes_k A(Y) \cong k[x_1, \dots, x_n, y_1, \dots, y_m]/I(X) \otimes 1 + 1 \otimes I(Y)$. This recovers the product, since if we have

