Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t) Y \end{aligned}$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$$
$$S(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t).$$

Outline

- Took $R^\pm : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$: symplectic paths associated to $S^\pm$
- These paths defined $\mu(x), \mu(y)$
- Section 8.7:

$$R^\pm \in \mathcal{S} := \left\{ R(t) \;\middle|\; R(0) = \mathrm{id},\ \det(R(1) - \mathrm{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\mathrm{Ind}(L) \overset{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know $\mathrm{Ind}(L) = \mathrm{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing $\mathrm{Ind}(L_1)$ by computing kernels.

$$
\begin{tikzcd}
\text{8.8.4: } \mathrm{Ind}(L_0) = \mathrm{Ind}(L) \arrow[d] & & \text{8.8.5: } \dim\ker F, F^* \arrow[d] \\
\text{8.8.2: } \mathrm{Ind}(L_1) = \mathrm{Ind}(L) \arrow[dr] & & \text{8.8.3: } \mathrm{Ind}(L_1) = k^- - k^+ \arrow[dl] \\
& \text{8.8.1: } \mathrm{Ind}(L) = k^- - k^+ = \mu(x) - \mu(y) &
\end{tikzcd}
$$

8.8.5: dim ker F, F^*

Recall

$$
\begin{aligned}
L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\
Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y
\end{aligned}
$$

$$
\begin{aligned}
L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\
Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y
\end{aligned}
$$

$$
\begin{aligned}
L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\
Z &\longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z
\end{aligned}
$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since coker $L_1 \cong \ker L_1^*$, it suffices to compute $\ker L_1^*$
- We have

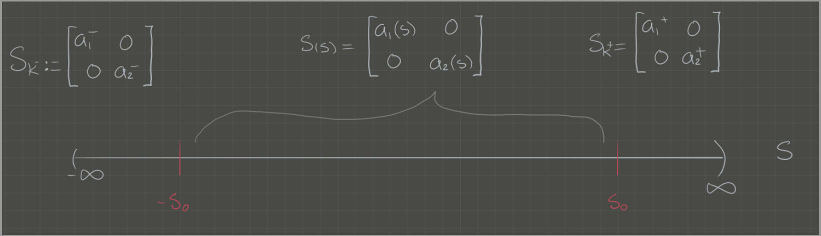
$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^{n} \mathrm{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the $n = 1$ case.

Setup

L_1 used a path of diagonal matrices constant near ∞:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



Statement of Later Lemma (8.8.5)

Let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s) Y. \end{aligned}$$

Note: F is L_1 for $n = 1$:

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s) Y. \end{aligned}$$

Statement of Lemma

$$F : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose $a_i^\pm \notin 2\pi\mathbb{Z}$.

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \operatorname{Ker} F = 2\cdot\#\left\{\ell\in\mathbb{Z} \;\middle|\; 2\pi\ell\in(a^-, a+)\subset\mathbb{R}\right\}$$
$$\dim \operatorname{Ker} F^* = 2\cdot\#\left\{\ell\in\mathbb{Z} \;\middle|\; 2\pi\ell\in(a^+, a^-)\subset\mathbb{R}\right\}.$$

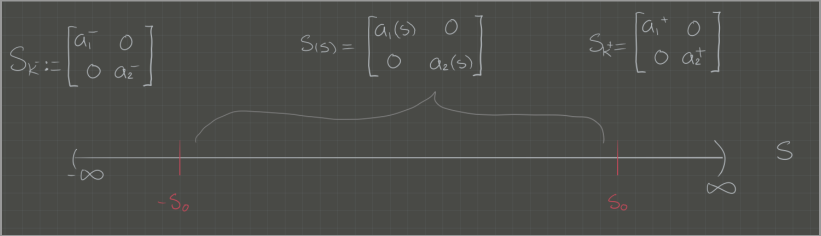
2. Suppose $\sup_{s\in\mathbb{R}}\|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \#\left\{i\in\{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0\right\}$$
$$\dim \operatorname{Ker} F^* = \#\left\{i\in\{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0\right\}.$$

Statement of Lemma

In words:

1. If $S(s)$ is a scalar matrix, set $a^{\pm} = a_1^{\pm} = a_2^{\pm}$ to the limiting scalars and count the integer multiples of 2π between a^- and a^+.
2. Otherwise, if S is uniformly bounded by 1, count the number of entries the flip from positive to negative as s goes from $-\infty \longrightarrow \infty$.



Proof of Assertion 1

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \operatorname{Ker} F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \operatorname{Ker} F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) := a_1(s) = a_2(s)$.
- Start with equation on $\mathbb{R}^2$,

$$\mathbf{Y}(s,t) = [Y_1(s,t), Y_2(s,t)].$$

- Replace with equation on $\mathbb{C}$:

$$\mathbf{Y}(s,t) = Y_1(s,t) + iY_2(s,t).$$

Proof of Assertion 1

– Rewrite the PDE $F(Y) = 0$ as $\bar{\partial} Y + S(s) Y = 0$, i.e.

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0$$

8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$

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