Title

Contents

0.1 Big Theorems / Tools:

Proposition 0.1.1 *(Fundamental Theorem of Calculus I).*

$$\frac{\partial}{\partial x}\int_a^x f(t)dt = f(x)$$

Proposition 0.1.2 *(Generalized Fundamental Theorem of Calculus).*

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)dt = f(x,t)\cdot\frac{\partial}{\partial x}(t)\Bigg|_{t=a(x)}^{t=b(x)}$$

$$= f(x,b(x))\cdot b'(x) - f(x,a(x))\cdot a'(x)$$

If $f(x,t) = f(t)$ doesn't depend on x, then $\dfrac{\partial f}{\partial x} = 0$ and the second integral vanishes:

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

Find examples.

Remark 0.1.1.

Note that you can recover the original FTC by taking

$$a(x) = c$$
$$b(x) = x$$
$$f(x, t) = f(t).$$

Corollary 0.1.1 *(?)*.

$$\frac{\partial}{\partial x} \int_1^x f(x, t)dt = \int_1^x \frac{\partial}{\partial x} f(x, t)dt + f(x, x)$$

Proposition 0.1.3 *(Extreme Value Theorem)*.
Todo

\todo[inline]{Todo}

Proposition 0.1.4 *(Mean Value Theorem)*.

$$f \in C^0(I) \implies \exists p \in I : f(b) - f(a) = f'(p)(b - a)$$
$$\implies \exists p \in I : \int_a^b f(x)\ dx = f(p)(b - a).$$

Proposition 0.1.5 *(Rolle's Theorem)*.

todo

Proposition 0.1.6 *(L'Hopital's Rule)*.
If

- $f(x)$ and $g(x)$ are differentiable on $I - \{\text{pt}\}$, and

$$\lim_{x \to \{\text{pt}\}} f(x) = \lim_{x \to \{\text{pt}\}} g(x) \in \{0, \pm\infty\}, \qquad \forall x \in I, g'(x) \neq 0, \qquad \lim_{x \to \{\text{pt}\}} \frac{f'(x)}{g'(x)} \text{ exists,}$$

Then it is necessarily the case that

$$\lim_{x \to \{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x \to \{\text{pt}\}} \frac{f'(x)}{g'(x)}.$$

Remark 0.1.2.

Note that this includes the following indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty, \quad \infty - \infty.$$

For $0 \cdot \infty$, can rewrite as $\dfrac{0}{\frac{1}{\infty}} = \dfrac{0}{0}$, or alternatively $\dfrac{\infty}{\frac{1}{0}} = \dfrac{\infty}{\infty}$.

For $1^\infty, \infty^0$, and 0^0, set

$$L := \lim f^g \implies \ln L = \lim g \ln(f)$$

to recover $\infty \cdot 0, 0 \cdot \infty$, or $0 \cdot 0$.

Proposition 0.1.7 (Taylor Expansion).

$$T(a, x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$= f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

$$+ \frac{1}{6}f'''(a)(x - a)^3 + \frac{1}{24}f^{(4)}(a)(x - a)^4 + \cdots$$

There is a bound on the error:

$$|f(x) - T_k(a, x)| \leq \left| \frac{f^{(k+1)}(a)}{(k + 1)!} \right|$$

where $T_k(a, x) = \displaystyle\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x - a)^n$ is the kth truncation.

Remark 0.1.3.
Approximating change: $\Delta y \approx f'(x)\Delta x$

0.2 Differential

0.2.1 Limits

0.2.2 Tools for finding limits

Examples

How to find $\lim\limits_{x \to a} f(x)$ in order of difficulty:

- Plug in: if f is continuous, $\lim\limits_{x \to a} f(x) = f(a)$.

- Check for indeterminate forms and apply L'Hopital's Rule.

- Algebraic rules

- Squeeze theorem

- Expand in Taylor series at a

- Monotonic + bounded

- One-sided limits: $\lim\limits_{x \to a^-} f(x) = \lim\limits_{\varepsilon \to 0} f(a - \varepsilon)$

- Limits at zero or infinity:

$$\lim_{x \to \infty} f(x) = \lim_{\frac{1}{x} \to 0} f\left(\frac{1}{x}\right) \text{ and } \lim_{x \to 0} f(x) = \lim_{x \to \infty} f\left(\frac{1}{x}\right)$$

  - Also useful: if $p(x) = p_n x^n + \cdots$ and $q(x) = q_n x^m + \cdots$,

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \deg p < \deg q \\ \infty & \deg p > \deg q \\ \dfrac{p_n}{q_n} & \deg p = \deg q \end{cases}$$

⚠️**Warning 0.1:** Be careful: limits may not exist!! Example : $\lim\limits_{x \to 0} \dfrac{1}{x} \neq 0$.

0.2.3 Asymptotes

- Vertical asymptotes: at values $x = p$ where $\lim\limits_{x \to p} = \pm\infty$
- Horizontal asymptotes: given by points $y = L$ where $L \lim\limits_{x \to \pm\infty} f(x) < \infty$
- Oblique asymptotes: for rational functions, divide - terms without denominators yield equation of asymptote (i.e. look at the asymptotic order or "limiting behavior").
  - Concretely:

$$f(x) = \frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{t(x)} \sim r(x)$$

0.2.4 Recurrences

- Limit of a recurrence: $x_n = f(x_{n-1}, x_{n-2}, \cdots)$
  - If the limit exists, it is a solution to $x = f(x)$

0.2.5 Derivatives

Proposition 0.2.1 *(Chain Rule).*

$$\frac{\partial}{\partial x}(f \circ g) = (f' \circ g) \cdot g'$$

Proposition 0.2.2 *(Product Rule).*

$$\frac{\partial}{\partial x} f \cdot g = f' \cdot g + g' \cdot f$$

Proposition 0.2.3 *(Quotient Rule).*

$$\frac{\partial}{\partial x} \frac{f(x)}{g(x)} = \frac{f'g - g'f}{g^2}$$

Mnemonic: Low d-high minus high d-low

Proposition 0.2.4 *(Inverse Rule).*

$$\frac{\partial f^{-1}}{\partial x} (f(x_0)) = \left(\frac{\partial f}{\partial x}\right)^{-1} (x_0) = 1/f'(x_0)$$

0.2.6 Implicit Differentiation

$$(f(x))' = f'(x) \ dx, (f(y))' = f'(y) \ dy$$

- Often able to solve for $\dfrac{\partial y}{\partial x}$ this way.

- Obtaining derivatives of inverse functions: if $y = f^{-1}(x)$ then write $f(y) = x$ and implicitly differentiate.

0.2.7 Related Rates

General series of steps: want to know some unknown rate y_t

- Lay out known relation that involves y
- Take derivative implicitly (say w.r.t t) to obtain a relation between y_t and other stuff.
- Isolate $y_t = $ known stuff
- Example: ladder sliding down wall
    - Setup: l, x_t and $x(t)$ are known for a given t, want y_t.
    - $x(t)^2 + y(t)^2 = l^2 \implies 2xx_t + 2yy_t = 2ll_t = 0$ (noting that l is constant)
    - So $y_t = -\dfrac{x(t)}{y(t)} x_t$
    - $x(t)$ is known, so obtain $y(t) = \sqrt{l^2 - x(t)^2}$ and solve.

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0.3 Integral

> **Proposition 0.3.1** *(Integral formula for average value).*
>
> $$\mu_f = \frac{1}{b-a} \int_a^b f(t)dt$$

Proof (?).
Apply MVT to $F(x)$.

∎

0.3.1 Area Between Curves

- Area in polar coordinates:

$$A = \int_{r_1}^{r_2} \frac{1}{2} r^2(\theta) \ d\theta$$

0.3.2 Solids of Revolution

$$\text{Disks} \qquad\qquad A = \int \pi r(t)^2 \ dt$$

$$\text{Cylinders} \qquad\qquad A = \int 2\pi r(t)h(t) \ dt.$$

0.3.3 Arc Lengths

$$L = \int \ ds \qquad\qquad\qquad ds = \sqrt{dx^2 + dy^2}$$

$$= \int_{x_0}^{x_1} \sqrt{1 + \frac{\partial y}{\partial x}} \ dx$$

$$= \int_{y_0}^{y_1} \sqrt{\frac{\partial x}{\partial y} + 1} \ dy$$

$$SA = \int 2\pi r(x) \ ds$$

0.3.4 Center of Mass

Given a density $\rho(\mathbf{x})$ of an object R, the x_i coordinate is given by

$$x_i = \frac{\displaystyle\int_R x_i \rho(x) \ dx}{\displaystyle\int_R \rho(x) \ dx}$$

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0.3.5 Big List of Integration Techniques

Given $f(x)$, we want to find an antiderivative $F(x) = \int f$ satisfying $\dfrac{\partial}{\partial x} F(x) = f(x)$

- Guess and check: look for a function that differentiates to f.

- u- substitution

  - More generally, any change of variables

$$x = g(u) \implies \int_a^b f(x) \ dx = \int_{g^{-1}(a)}^{g^{-1}(b)} (f \circ g)(x) \ g'(x) \ dx$$

- Integration by Parts:

  - The standard form:

$$\int u dv = uv - \int v du$$

  - A more general form for repeated applications: let $v^{-1} = \int v$, $v^{-2} = \int \int v$, etc.

$$\int_a^b uv = uv^{-1} \Big|_a^b - \int_a^b u^1 v^{-1}$$

$$= uv^{-1} - u^1 v^{-2} \Big|_a^b + \int_a^b u^2 v^{-2}$$

$$= uv^{-1} - u^1 v^{-2} + u^2 v^{-3} \Big|_a^b - \int_a^b u^3 v^{-3}$$

$$\vdots$$

$$\implies \int_a^b uv = \sum_{k=1}^n (-1)^k u^{k-1} v^{-k} \Big|_a^b + (-1)^n \int_a^b u^n v^{-n}$$

  - Generally useful when one term's nth derivative is a constant.

  - Shoelace method:

  - Note: you can choose u or v equal to 1! Useful if you know the derivative of the integrand.

- Differentiating under the integral

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x, t) dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt = f(x, \cdot) \frac{\partial}{\partial x} (\cdot) \Big|_{a(x)}^{b(x)}$$

$$= f(x, b(x)) \ b'(x) - f(x, a(x)) \ a'(x)$$

  - Proof: let $F(x)$ be an antiderivative and compute $F'(x)$ using the chain rule.
  - #todo for constants, this should allow differentiating under the integral when f, f_x are "jointly continuous"

– LIPET: Log, Inverse trig, Polynomial, Exponeitial, Trig: generally let u be whichever one comes first.

- The ridiculous trig sub: for any integrand containing only trig terms

    – Transforms *any* such integrand into a rational function of x

    – Let $u = 2\tan^{-1} x,\ du = \dfrac{2}{x^2+1}$, then

$$\int_a^b f(x)\ dx = \int_{\tan\frac{a}{2}}^{\tan\frac{b}{2}} f(u)\ du$$

◇ Example: $\displaystyle\int_0^{\pi/2} \dfrac{1}{\sin\theta}\ d\theta = 1/2$

Derivatives	Integrals	Signs	Result
u	v	NA	NA
u'	$\displaystyle\int v$	$+$	$u\displaystyle\int v$
u''	$\displaystyle\int\int v$	$-$	$-u'\displaystyle\int\int v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fill out until one column is zero (alternate signs). Get the result column by multiplying diagonally, then sum down the column.

- Trigonometric Substitution

$$\sqrt{a^2 - x^2} \qquad \Rightarrow \qquad x = a\sin(\theta) \qquad dx = a\cos(\theta)\ d\theta$$
$$\sqrt{a^2 + x^2} \qquad \Rightarrow \qquad x = a\tan(\theta) \qquad dx = a\sec^2(\theta)\ d\theta$$
$$\sqrt{x^2 - a^2} \qquad \Rightarrow \qquad x = a\sec(\theta) \qquad dx = a\sec(\theta)\tan(\theta)\ d\theta$$

- Partial Fractions
- Completing the Square #todo
- Trig Formulas
    – Double angle formulas:

$$\sin^2(x) \qquad = \qquad \frac{1}{2}(1 - 2\cos x)$$
$$=$$
$$=$$
$$=$$
$$=$$

- Products of trig functions
    – Setup: $\displaystyle\int \sin^a(x)\cos^b(x)\ dx$

$\diamondsuit$ Both a, b even: $\sin(x)\cos(x) = \dfrac{1}{2}\sin(x)$

$\diamondsuit$ a odd: $\sin^2 = 1 - \cos^2,\ u = \cos(x)$

$\diamondsuit$ b odd: $\cos^2 = 1 - \sin^2,\ u = \sin(x)$

– Setup: $\displaystyle\int \tan^a(x)\sec^b(x)\ dx$

$\diamondsuit$ a odd: $\tan^2 = \sec^2 - 1,\ u = \sec(x)$

$\diamondsuit$ b even: $\sec^2 = \tan^2 - 1, u = \tan(x)$

Other small but useful facts:

$$\int_0^{2\pi} \sin\theta\ d\theta = \int_0^{2\pi} \cos\theta\ d\theta = 0.$$

0.3.6 Optimization

- Critical points: boundary points and wherever $f'(x) = 0$

- Second derivative test:

  – $f''(p) > 0 \implies p$ is a min
  – $f''(p) < 0 \implies p$ is a max

- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)

- Inverse function theorem: The slope of the inverse is reciprocal of the original slope

- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.

- Fundamental theorem of Calculus: If

$$\int f(x)dx = F(b) - F(a) \implies F'(x) = f(x).$$

- Min/maxing - either derivatives of Lagranage multipliers!

- Distance from origin to plane: equation of a plane

$$P : ax + by + cz = d.$$

  – You can always just read off the normal vector $\mathbf{n} = (a, b, c)$. So we have $\mathbf{nx} = d$.

  – Since $\lambda\mathbf{n}$ is normal to P for all λ, solve $\mathbf{n}\lambda\mathbf{n} = d$, which is $\lambda = \dfrac{d}{\|\mathbf{n}\|^2}$

- A plane can be constructed from a point p and a normal n by the equation $np = 0$.

- In a sine wave $f(x) = \sin(\omega x)$, the period is given by $2\pi/\omega$. If $\omega > 1$, then the wave makes exactly ω full oscillations in the interval $[0, 2\pi]$.

- The directional derivative is the gradient dotted against a *unit vector* in the direction of interest

- Related rates problems can often be solved via implicit differentiation of some constraint function

- The second derivative of a parametric equation is not exactly what you'd intuitively think!

- For the love of god, remember the FTC!

$$\frac{\partial}{\partial x} \int_0^x f(y)dy = f(x)$$

- Technique for asymptotic inequalities: WTS $f < g$, so show $f(x_0) < g(x_0)$ at a point and then show $\forall x > x_0, f'(x) < g'(x)$. Good for big-O style problems too.

- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)

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