Lie Algebras

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Contents

1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

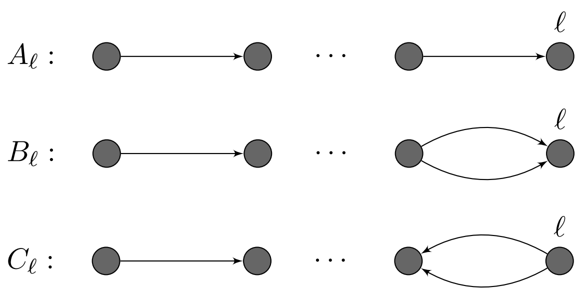
1.1.2 Chapter 3-4

We will describe the following series of correspondences:

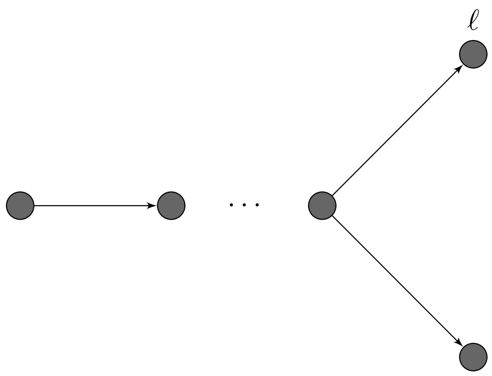
$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\bigoplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

1.2 Classification

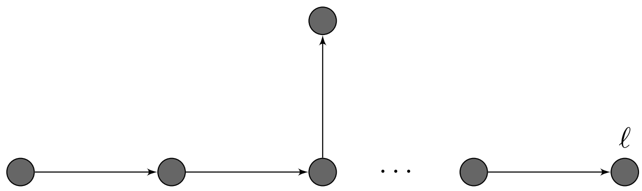
The classical Lie algebras can be essentially classified by certain classes of diagrams:



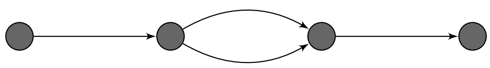
D_ℓ :



E_6, E_7, E_8 :



F_4:



1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Exercise 1. Show that $[x, y] = -[y, x]$.

Definition 2. Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\mathrm{End}(V) = \{f : V \to V \ni V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \ni \mathrm{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n, V)$.).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

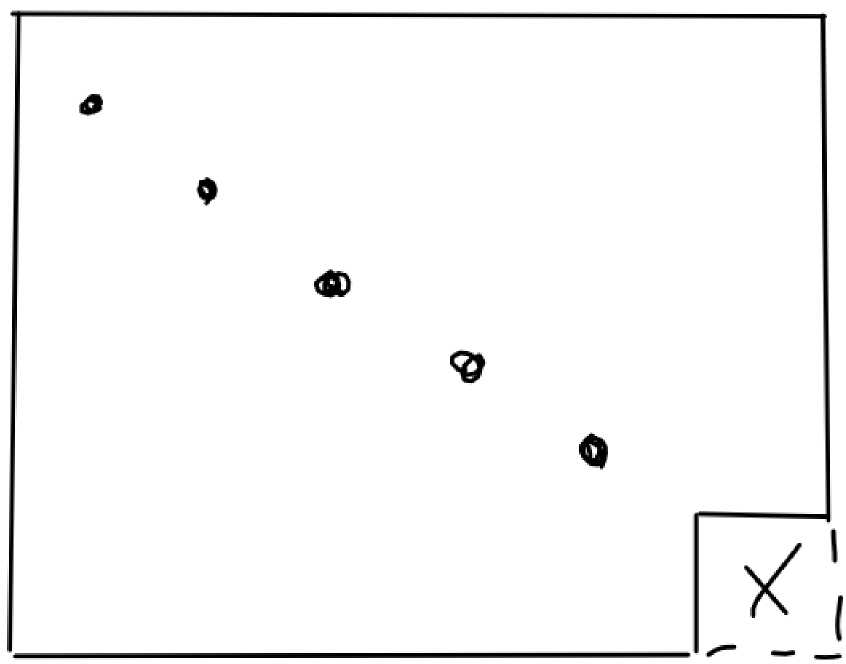
Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \end{array} \right) & n = 2\ell + 1 \text{ odd}, \\ \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

Proposition 7. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and



  zero elsewhere.
- For type A_ℓ, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell,\mathbb{F}) = \dim\left\{\begin{pmatrix} A & B=-B^t \\ C=-C^t & -A^t \end{pmatrix}\right\},$$

which turns out to be $2\ell^2-\ell$.

- For type B_ℓ, we have $\dim\mathfrak{so}(2\ell,\mathbb{F}) = 2\ell^2-\ell+2\ell = 2\ell^2+\ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C=C^t \\ -M^t & B=B^t & -A^t \end{array}\right).$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}$$

Theorem 8. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x,[y,z]]+[y,[x,z]]+[z,[x,y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell+1,F)$
- $B_\ell \iff \mathfrak{so}(2\ell+1,F)$
- $C_\ell \iff \mathfrak{sp}(2\ell,F)$
- $D_\ell \iff \mathfrak{so}(2\ell,F)$

Exercise 4. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 9. An F**-algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition 10. An algebra is **associative** if $x(yz) = (xy)z$.

> Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 11. Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 12. We define $\mathrm{Der}(A) = \{\delta \ni \delta \text{ is a derivation }\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\cdot, \cdot]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x, [y, z]] = 0$.

Exercise 5. Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 6 (Turn in)**.** Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x, y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 13. Any F-vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian.*

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\cdot, \cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 14. Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise 7. Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\text{ad }(e), \text{ad }(h), \text{ad }(g)$ with respect to this basis.

2.3 Ideals

Definition 15. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise 8. Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 16. A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 17. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \text{ad }_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I,$$
$$2f = [h, -f] \in I \implies f \in I.$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

$\square$

Note that there is a homework coming due next Monday, about 4 questions.

3 Lecture 3

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x + I, y + I] = [x, y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x, y] + I = [x', y'] + I$.

3.1 Homomorphisms and Representations

Definition 18. A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x, y] = [\phi(x), \phi()]$.

Remark. $\ker \phi \trianglelefteq \mathfrak{g}_1$ and im $\phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I + J)/J \cong I/(I \bigcap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V.

We call V a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad } &: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \\ &x \mapsto [x, \cdot\,]. \end{aligned}$$

Corollary 19. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) = \Big\{ x \in \mathfrak{g} \ni \text{ad }_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \Big\} \\ = \ker \text{ad }_x. \end{aligned}$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad } \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

3.2 Automorphisms

Definition: An automorphism of $\mathfrak{g}$ is an isomorphism $\mathfrak{g}\circlearrowleft$, and we define

$$\mathrm{Aut}(\mathfrak{g}) = \{\phi : \mathfrak{g}\circlearrowleft \ni \phi \text{ is an isomorphism } \}.$$

Proposition: If $\delta \in \mathrm{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) \coloneqq \sum \frac{\delta^n}{n!} \in \mathrm{Aut}(\mathfrak{g}).$$

This is well-defined because δ is nilpotent, and a binomial formula holds:

$$\frac{\delta^{n([x,y])}}{n!} = \sum_{i=0}^{n} [\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!}].$$

and for $n = 1, \delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp\delta)(x), (\exp\delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\mathrm{ad}\ _e) \exp(\mathrm{ad}\ _{-f}) \exp(\mathrm{ad}\ _e) \in \mathrm{Aut}\mathfrak{g}.$$

where e, f are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of s is 2 and $W = \{1, s\}$.

4 Lecture 4

4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\cdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ is *solvable* if there is some n for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let $\mathfrak{g}$ be a Lie algebra, then

1. If $\mathfrak{g}$ is solvable, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also solvable.
2. If $I \trianglelefteq \mathfrak{g}$ and both I and $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\mathrm{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

4.2 Nilpotency

$$
\begin{aligned}
\mathfrak{g}^0 &= \mathfrak{g} \\
\mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\
&\cdots \\
\mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i].
\end{aligned}
$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Claim: If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}_{\,x} \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof: This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \cdots [x_n, y]]] = 0$, and so $\mathrm{ad}_{\,x_1} \circ \mathrm{ad}_{\,x_2} \circ \cdots \mathrm{ad}_{\,x_n} = 0$ which implies that $\mathrm{ad}_{\,x}^{\,n} = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If $\mathrm{ad}_{\,x}$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while $\mathrm{ad}_{\,x}$ *is* nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V. If x is nilpotent as an endomorphism on V for all $x \in V$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases: - Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$. - Case $\dim g = 1$, left as an exercise.

Inductive step: Let A be a maximal proper subalgebra and define $\phi : A \to \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that ϕ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that $\text{im } \phi \leq \mathfrak{gl}(\mathfrak{g}/a)$ is a Lie subalgebra, and $\dim \text{im } \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \text{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\text{im } \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that A is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2 [y, y]$. The last term is zero, the middle two terms are in A, and because A is closed under the bracket, the first term is in A as well.

But then $A \oplus F_y$ is a larger subalgebra than A, which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in $A, A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in A.

Proof to be continued on Friday!

5 Lecture 5

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of V, constructing $\text{im } \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus Fy$. Now consider

$$W = \{v \in V \ni Av = 0\},$$

which is $\mathfrak{g}$-invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But V is killed by any element in A, and both of these terms are in A. In particular, the y appearing in Fy also satisfies $y \in W$. Consider $y|_W \in \text{End}(w)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \text{End}(V)$, which is true exactly because y is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \text{End}(\mathfrak{g}/A)$ is nilpotent. Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by a, $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since m_ℓ, m_r commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly ad _a. But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if $\text{ad}_x \in \text{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to $\text{ad}\, g \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that $\text{ad}\, \mathfrak{g}v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension. Since ad_x is nilpotent, so is $\text{ad}_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space. $\square$

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of F, then the matrices of $\mathfrak{g}$ with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on n, where the base case is easy. For the inductive step, we use the previous theorem to get a v_1 such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\overline{V} = F^n/Fv_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \to \mathfrak{gl}(\overline{V})$ where $x \mapsto (\overline{y} \mapsto \overline{y(x)})$.

Then $\text{im}\, \phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \text{End}(F^{n-1})$ is nilpotent, since x was nilpotent on the larger space. But (see notes) then x can be written as a strictly upper-triangular matrix.

5.1 Chapter 2: Semisimple Lie Algebras

We now assume $\text{char}\, F = 0$ and $\overline{F} = F$.

Theorem: If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V, then V contains a common eigenvector for a $x \in \mathfrak{g}$, i.e. a $\lambda : \mathfrak{g} \to F, x \mapsto \lambda(x)$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$.

Proof: We will use induction on the dimension of $\mathfrak{g}$. For the inductive step:

Claim 1: There is an ideal $A \trianglelefteq \mathfrak{g}$ such that $\mathfrak{g} = A \oplus Fy$ for some $y \neq 0$, so A is a subalgebra of a solvable Lie algebra $\mathfrak{g}$ and thus solvable itself. By hypothesis, we can produce a $w \in V \setminus \{0\}$, and thus a functional $\lambda : A \to F$ such that $aw = \lambda(a)w$ for all $a \in A$. So we define

$$V_\lambda = \{v \in V \ni av = \lambda(a)v \forall a \in A\}$$

where $w \in V_\lambda$.

Claim 2: $y(V_\lambda) \subseteq V_\lambda$, or $y|_{V_\lambda} \in \text{End}(V_\lambda)$.

Thus $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$ is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a $v \in V_\lambda$ and some $\mu \in F$ such that $y(v) = \mu v$. An arbitrary element $x \in \mathfrak{g}$ can be written as $x = a + cy$ for some $a \in A, c \in F$ and it acts by $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c)v \in V_\lambda$.

6 Lecture 6

Chapter 3: Theorems of Lie and Cartan

6.1 4.1: Lie's Theorem

Theorem: Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is finite-dimensional. If $V \neq 0$, then V contains a common eigenvector for all of the endomorphisms in L.

Proof: Use induction on $\dim L$. The case $\dim L = 0$ is trivial. We'll attempt to mimic the proof of Theorem 3.3. The idea is to

1. Locate an ideal of K of codimension 1,
2. Show by induction that common eigenvectors exist for k,
3. Verify that L stabilizes a space consisting of such eigenvectors,
4. Find in that space an eigenvector for a single $z \in L$ satisfying $L = K + Fz$.

Step (1): Since L is solvable and of positive dimension, then $L \subsetneq [L, L]$. Otherwise, if $L = [L, L]$, then $L^{(1)} = L \implies L^{(n)} = L$, which would contradict L being solvable.

Since $[L, L]$ is abelian, any subspace is automatically an ideal. So take a subspace of codimension one, then its inverse image $K \trianglelefteq L$ is an ideal satisfying $[L, L] \subseteq K$.

Step (2): Use induction to find a common eigenvector $v \in V$ for K. (K is solvable; if $K = 0$ then L is abelian of dimension 1 and any eigenvector for a basis vector of L finishes the proof.)

This means that $x \in K \implies x \curvearrowright v = \lambda(x)v$ for some $\lambda : K \to F$ a linear functional. Fix this λ, and let $W = \{w \in V \ni x \curvearrowright w = \lambda(x)w \forall x \in K\}$; note that $W \neq 0$.

Step (3): This will involve showing that L leaves W invariant. Assume for the moment that this is done, and proceed to step (4).

Step (4):

Write $L = K + Fz$. Since F is algebraically closed, we can find an eigenvector $v_0 \in W$ of z for some eigenvalue of z. Then v_0 is a common eigenvector for L, and λ can be extended to a linear function on L satisfying $x \curvearrowright v_0 = \lambda(x)v_0$ where $x \in L$.

It remains to show that L stabilizes W. Let $w \in W, x \in L$. To test whether or not $x \curvearrowright w \in W$, we take an arbitrary $y \in K$ and examine

$$yx \curvearrowright w = xy \curvearrowright w - [x, y] \curvearrowright w = \lambda(y)x \curvearrowright w - \lambda([x, y])w.$$

Note: the above equality is an important trick.

Thus we need to show that $\lambda([x, y]) = 0$. To this end, fix $w \in W, x \in L$. Let $n > 0$ be the smallest integer for which $w, x \curvearrowright w, \cdots x^n \curvearrowright w$ are all linearly *independent*. Let $W_i = \operatorname{span}(\{w, x \curvearrowright w, \cdots x^{i-1} \curvearrowright w\})$ and set $W_0 = 0$. Then $\dim W_n = n$, and $W_{n+i} = W_n$ for all $i \geq 0$. Moreover, x maps W_n into itself. It is easy to check that each $y \in K$ is represented by an upper-triangular matrix with diagonal entries equal to $\lambda(y)$. This follows immediately from the congruence

$$yx^i \curvearrowright w = \lambda(y)x^i \curvearrowright w \mod W_i,$$

which can by proved by induction on i. The case $i = 0$ is trivial. For the inductive step, write

$$yx^i \curvearrowright i = yxx^{i-1} \curvearrowright w = xyx^{i-1} \curvearrowright w = [x, y]x^{i-1} \curvearrowright w$$

By induction,

$$yx^{i-1} = \lambda(y)x^{i-1} \curvearrowright w + w',$$

where $w' \in W_{i-1}$. Since x maps W_{i-1} into W_i by construction, the congruence holds for all i.

According to our description of the action of $y \in K$ on W_n, we have $\mathrm{Tr}_{W_n}(y) = n\lambda(y)$. In particular, this is true for elements k of f of the special form $[x, y]$ where x is as it was above and y is in K.

But both x and y stabilize W_n, so $[x, y]$ acts on W_n as the commutator of two endomorphisms of W_n, and the trace is therefore zero.

We conclude that $n\lambda([x, y]) = 0$. Since $\mathrm{char}F = 0$, this forces $\lambda([x, y]) = 0$ as required. $\square$

Corollary A (Lie's Theorem): Let $L \leq \mathfrak{gl}(V)$ be a solvable subalgebra where $\dim V = n < \infty$. Then L stabilizes some flag in V, i.e. the matrices of L relative to a suitable basis of V are upper triangular.

Proof: Use the above theorem, along with induction on $\dim V$. This is similar to the proof of corollary 3.3.

6.2 4.2: Jordan-Chevalley Decomposition

Fact 1:

The Jordan Canonical Form of a single endomorphism x over F algebraically closed is an expression of x in matrix form as a sum of blocks: $\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & \ddots & 1 \\ 0 & & & a \end{pmatrix}$

Fact 2:

Call $x \in \mathrm{End}V$ *semisimple* if the roots of its minimal polynomial over F are all distinct. Equivalently, if F is algebraically closed, then x is semisimple iff x is diagonalizable.

Fact 3:

Two commuting semisimple endomorphisms can be simultaneously diagonalized. Therefore, their sum or difference is again semisimple.

Proposition: Let V be a finite dimensional vector space over F and $x \in \mathrm{End}V$. Then

a. There exist unique $x_s, x_n \in \mathrm{End}V$ satisfying the conditions $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and x_s, x_n commute.

b. There exists polynomials $p(t), g(t)$ such that $x_s = p(x)$ and $x_n = g(x)$. In particular, x_s, x_n commute with any endomorphism commuting with x.

c. If $A < B < V$ are subspaces and x maps B into A, then x_s, x_n also map B into A.

The decomposition $x = x_s + x_n$ is called the (additive) **Jordan-Chevalley decomposition** of x, or just the Jordan decomposition. x_s, x_n are respectively called the **semisimple part** and the **nilpotent part** of x.

Example:

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies x_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $x_s x_n = x_n = x_n x_s$, $x_s = 2x - x^2$, and $x_n = x^2 - x$. We thus have $p(t) = 2t - t^2$ and $q(t) = t^2 - t$.

7 Lecture n+2

Definition (Jordan Decomposition)

Let $X \in \operatorname{End}(V)$ for V finite dimensional. Then,

(a) There exists a unique $X_s, X_n \in \operatorname{End}(V)$ such that $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.

(b) There exists a $p(t), q(t) \in t\mathbb{F}[t]$ such that $X_s = p(X), X_n = q(X)$.

(Polynomials with no constant term.)

Proof of (a): Assume $X_s = X_s + X_n = X_s' + X_n'$, so both have bracket zero. Assuming that (b) holds, we have $X_s = p(X)$, and so

$$[X, X_s] = [X_s + X_n', X_s'] = [X_s', X_s'] + [X_s', X_n'] = 0 \implies [p(X), X_S'] = 0 = [X_s, X_s']$$

Using fact (c) from last time, then X_s, X_s' can be diagonalized simultaneously, and so $X_s - X_s'$ is semisimple.

On the other hand, if X_n', X_n are nilpotent, and since these commute, $X_n - X_n'$ is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X_s') + (X_n + X_n')$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$ be the minimal polynomial of X, where each $m_i \geq 1$ and the λ_i are distinct. Then the primary composition of V is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V) \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomail $p \in F[t]$ such that

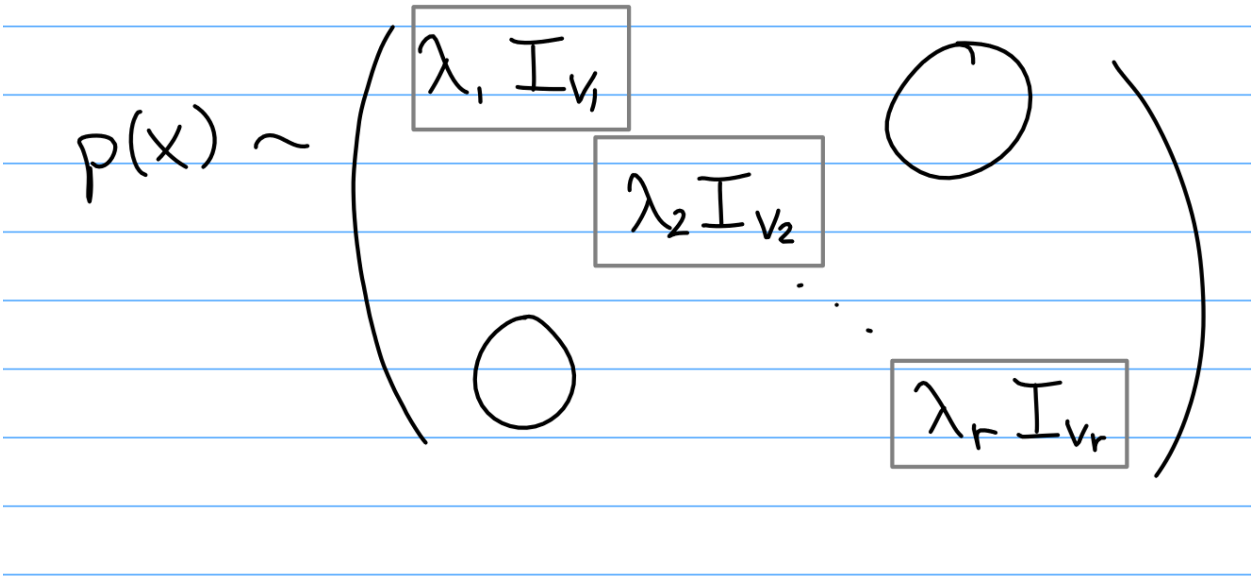


Figure 1: ???

$$\begin{aligned} p &= \lambda \mod (t-\lambda_i)^{m_i} \quad \forall i, \\ p &= 0 \mod t. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is $p(x) \curvearrowright V_i$? This acts by scalar multiplication by λ_i for all i. (Check). Because of the restrictive conditions, $p(x)$ has no constant term.

So $p(X) = X_s$ is the semisimple part we want. Now just set $q(t) = t - p(t)$, then $X_n \coloneqq q(X) = X - X_s$ is nilpotent.

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have $X = X_s + X_n$, then $\text{ad}_X \in \text{End}(\text{End}(V))$. It can be shown that $(\text{ad}_X)_s + (\text{ad}_X)_n = \text{ad}(X_s) + \text{ad}(X_n)$.

Let e_{ii} be the elementary matrix with a 1 in the i, j position. You can write ad_X as a 4×4 matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)$$

$$= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)$$

You can check that $(\text{ad }_X)_S = 0, \text{ad }(X_s) = 0$, and $(\text{ad } X)_n$ is the Jordan form given above.

Lemma:

(a) $x \in \text{End}(V) \implies \text{ad }(x)_s = \text{ad }(x_s)$ and $\text{ad }(x)_n = \text{ad }(x_n)$.

(b) If A is a finite dimensional $\mathbb{F}$-algebra, then $\delta \in \text{Der}(A) \implies \delta_s, \delta_n \in \text{Der}(A)$ as well.

Proof of (a):

Check that $\text{ad }(x) = \text{ad }(x_s) + \text{ad }(x_n)$. Then for $y \in \text{End}(V)$, we have

$$\begin{aligned}(\text{ad }(x))(y) &= [x, y] \\ &= [x_s + x_n, y] \\ &= [x_s, y] + [x_n, y] \\ &- (\text{ad }(x_s))(y) + (\text{ad }(x_n))(y).\end{aligned}$$

Using theorem 3.3, x_n nilpotent $\implies$ ad (x_n) is also nilpotent. So write $x_s = \sum \lambda_i e_{ii}$ with the eigenvalues on the diagonal. Then ad $x_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$ for all i, j. But then ad x_s is given

$$(\delta - (\lambda+\mu)I)^n([x,y])$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left[(\delta - \lambda I)^i(x), (\delta - \mu I)^{n-i}(y) \right]$$

Figure 2: Image

by a matrix with $\lambda_i - \lambda_j$ in the i, j position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

Proof of (b):

Since $\delta \in \mathrm{Der}(A)$, the primary decomposition with respect to δ is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \left\{ a \in A \ni (\delta - \lambda I)^k a = 0 \text{ for some } k >> 0 \right\}.$$

So $\delta_s \curvearrowright A_\lambda$ by scalar multiplication (by λ). Then for $\lambda, \mu \in \mathbb{F}$, we have

So $[A_x, A_y] \subseteq A_{\lambda+\mu}$ for all $x, y \in A$. But then

and so $\delta_s \in \mathrm{Der}(A)$, and $\delta_n = \delta - \delta_s \in \mathrm{Der}(A)$ as well.

8 Lecture n+3

Todo

9 Lecture n+4

Review of bilinear forms: let $V = \mathbb{F}^n$.

Definition: A bilinear form $\beta : V^2 \to \mathbb{F}$ can be represented by a matrix B with respect to a basis $\{\mathbf{v}_i\}$ such that

$$\beta\beta(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i) = (a_1 \ a_2 \ \cdots) B (b_1 \ b_2 \ \cdots)$$

- β is *symmetric* iff $\beta(a, b) = \beta(b, a)$.
- β is *symplectic* iff $\beta(a, b) = -\beta(b, a)$.
- β is *isotropic* iff $\beta(a, a) = 0$.

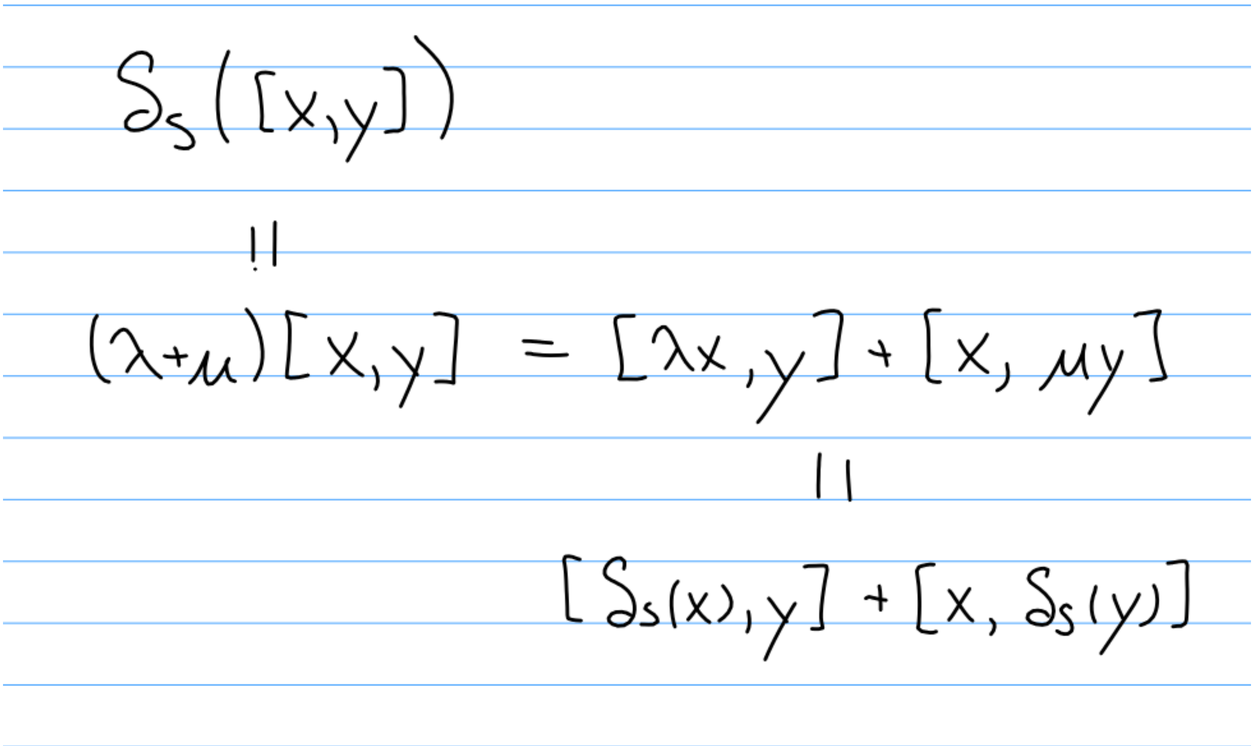


Figure 3: Image

For a subspace $U \leq V$, define

$$U^{\perp} := \{\mathbf{v} \in V \ni \beta(\mathbf{u}, \mathbf{v}) = \mathbf{0} \ \forall u \in U\}.$$

> Note: in general, left/right orthogonality are distinguished, but these will be identical when β is symmetric/symplectic.

The form β is said to be *non-degenerate* iff $V^{\perp} = 0$ iff $\det B \neq 0$.

Assume F is an algebraically closed field, so $\overline{F} = F$, and $\mathrm{char} F \neq 2$, then

- If β is non-degenerate and symmetric, then $B \sim I_n$
- If β is non-degenerate and symplectic, then $B \sim [0, I_{n/2}; I_{n/2}, 0]$.

Remark:

$\mathfrak{so}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, F) \ni \beta(x(u), v) = -\beta(u, x(v))\}$, where B has the matrix $[0, I; I, 0]$ if n is odd, or this matrix with a 1 in the top-left corner if n is odd.

Similarly, $\mathfrak{sp}(2m, \mathbb{F})$ can be described this way with the matrix $[0, -I_m; -I_m, 0]$.

Overview: The killing form is defined as $\kappa : \mathfrak{g}^2 \to \mathbb{F}$ where $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_{\,x} \circ \mathrm{ad}_{\,y})$.

Then we have **Cartan's Criteria**:

- $\mathfrak{g}$ solvable $\iff \kappa(x, y) = 0 \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$.
- $\mathfrak{g}$ semisimple $\iff \kappa$ is non-degenerate.

Note that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \bigoplus_i I_i$ with each $I_i \trianglelefteq \mathfrak{g}$ and simple.

9.1 Cartan's Criteria

Some facts:

1. κ is symmetric
2. If $\mathfrak{g}$ is finite dimensional, then κ is associative, i.e $\kappa([x,y],z) = \kappa(x,[y,z])$.

Exercise: Show that if $I \trianglelefteq \mathfrak{g}$, then $I^\perp \leq \mathfrak{g}$ is an ideal.

Proof of (2): In section 4.3, it was shown that $\operatorname{tr}([a,b] \circ c) = \operatorname{tr}(a \circ [b,c])$ for all $a,b,c \in \operatorname{End}(V)$ (provided V is finite dimensional).

So

$$\begin{aligned}
\kappa([x,y],z) &= \operatorname{tr}(\operatorname{ad}_{[x,y]} \circ \operatorname{ad}_z) \\
&= \operatorname{tr}([\operatorname{ad}_x, \operatorname{ad}_y] \circ \operatorname{ad}_z) \\
&= \operatorname{tr}(\operatorname{ad}_x \circ [\operatorname{ad}_y, \operatorname{ad}_z]) \\
&= \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_{[y,z]}) \\
&= \operatorname{tr}(x,[y,z])..
\end{aligned}$$

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Proof: $\implies$: We want to show that $\mathfrak{g}^\perp = 0$. Note that $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}$, and so for all $x \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$ and for any $y \in \mathfrak{g}^\perp$, we have

$$\kappa(x,y) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0$$

by the const(?) of $\mathfrak{g}^\perp$. This implies $\mathfrak{g}^\perp$ is solvable.

Using fact (2), we have $\mathfrak{g}^\perp \trianglelefteq \mathfrak{g}$ and thus $\mathfrak{g}^\perp \subseteq \operatorname{rad}(\mathfrak{g})$, which is 0 since because $\mathfrak{g}$ is semisimple. So either $\mathfrak{g}^\perp = 0$ or κ is nondegenerate.

Used the fact that the radical was a maximal solvable ideal.

$\impliedby$: We want to show that for all $I \trianglelefteq \mathfrak{g}$ where $[I,I] = 0$, we have $I^\perp \subseteq \mathfrak{g}^\perp$.

For $x \in I, y \in \mathfrak{g}$, we have

$$(\operatorname{ad}_x \circ \operatorname{ad}_y)^2 = \mathfrak{g} \xrightarrow{\operatorname{ad}_y} \mathfrak{g} \xrightarrow{\operatorname{ad}_x} I \xrightarrow{\operatorname{ad}_y} I \xrightarrow{\operatorname{ad}_x} 0$$

And thus $\operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0$ and $I \subseteq \mathfrak{g}^\perp$.

Suppose that $\mathfrak{g}$ is *not* semisimple. Then there exists a solvable ideal $J \neq 0$ such that the last term J^i in the derived series is an ideal $I \trianglelefteq \mathfrak{g}$ such that $[I,I] = 0$, forcing $J^i \subset \mathfrak{g}^\perp = 0$, which is a contradiction.

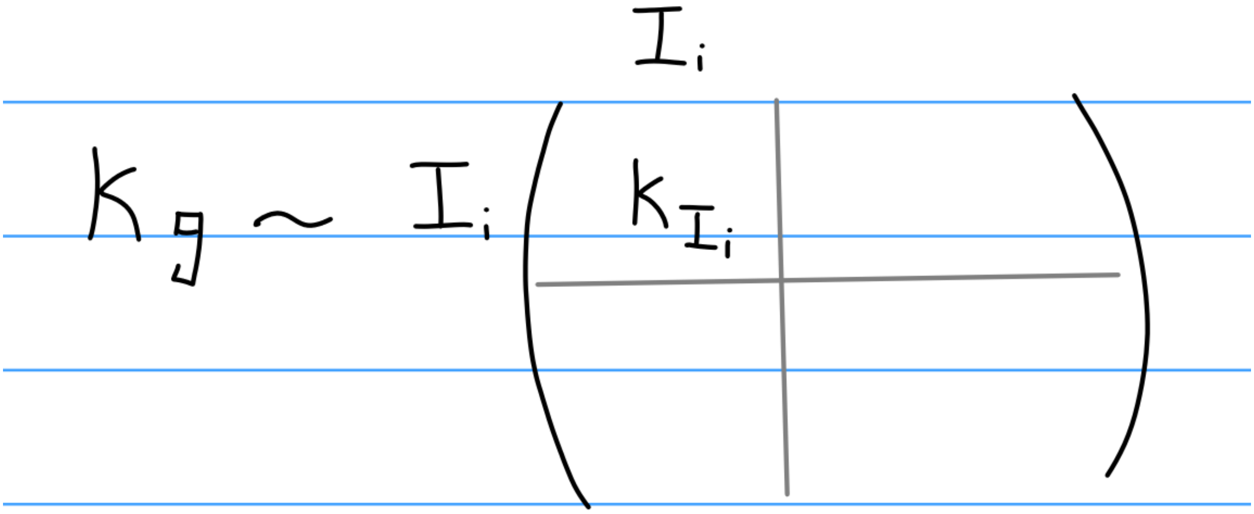


Figure 4: Image

9.2 Section 5.2

Theorem: If $\mathfrak{g}$ is semisimple, then

a. There exist ideals $I_i \trianglelefteq \mathfrak{g}$ which are simple Lie algebras satisfying $\mathfrak{g} = \bigoplus I_i$. Note that $[I_i, I_j] \subseteq I_i \bigcap I_j = 0$, since direct summands intersect only trivially.

b. Every simple $I \trianglelefteq \mathfrak{g}$ is one of these I_i.

c. $\kappa_{I_i} = \kappa_{\mathfrak{g}}|_{I_i \times I_i}$, so

Remark: $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g} = \bigoplus_i I_i$ for some simple Lie algebras I_i.

$\Longleftarrow$: For all $i, S \coloneqq \mathrm{rad}\mathfrak{g}, I_i \trianglelefteq I_i$ is a solvable ideal. This implies that it is 0, since I_i is simple.

By definition, simple Lie algebras are not abelian.

Supposing that $S = I_i$, we would then have $[S.S] \neq 0$ since $[I_i, I_i] \neq 0$ by definition. But $[S, S] \neq S$ because S is solvable, which says that S is not simple (a contradiction).

Note that $[\mathrm{rad}\mathfrak{g}, \mathfrak{g}] \subseteq \bigoplus [\mathrm{rad}\mathfrak{g}, I_i] = 0$, which forces $\mathrm{rad}\mathfrak{g} \subseteq Z(\mathfrak{g})$. Since I_i is simple, $Z(I_i) = 0$ for all i. But $Z(\mathfrak{g}) = \bigoplus Z(I_i) = 0$, and this forces $\mathrm{rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g}) \implies \mathrm{rad}\mathfrak{g} = 0$. So $\mathfrak{g}$ is semisimple.

Next time – starting the representation theory with $\mathfrak{sl}(2, \mathbb{F})$.

10 Lecture 10?

Recall the killing form:

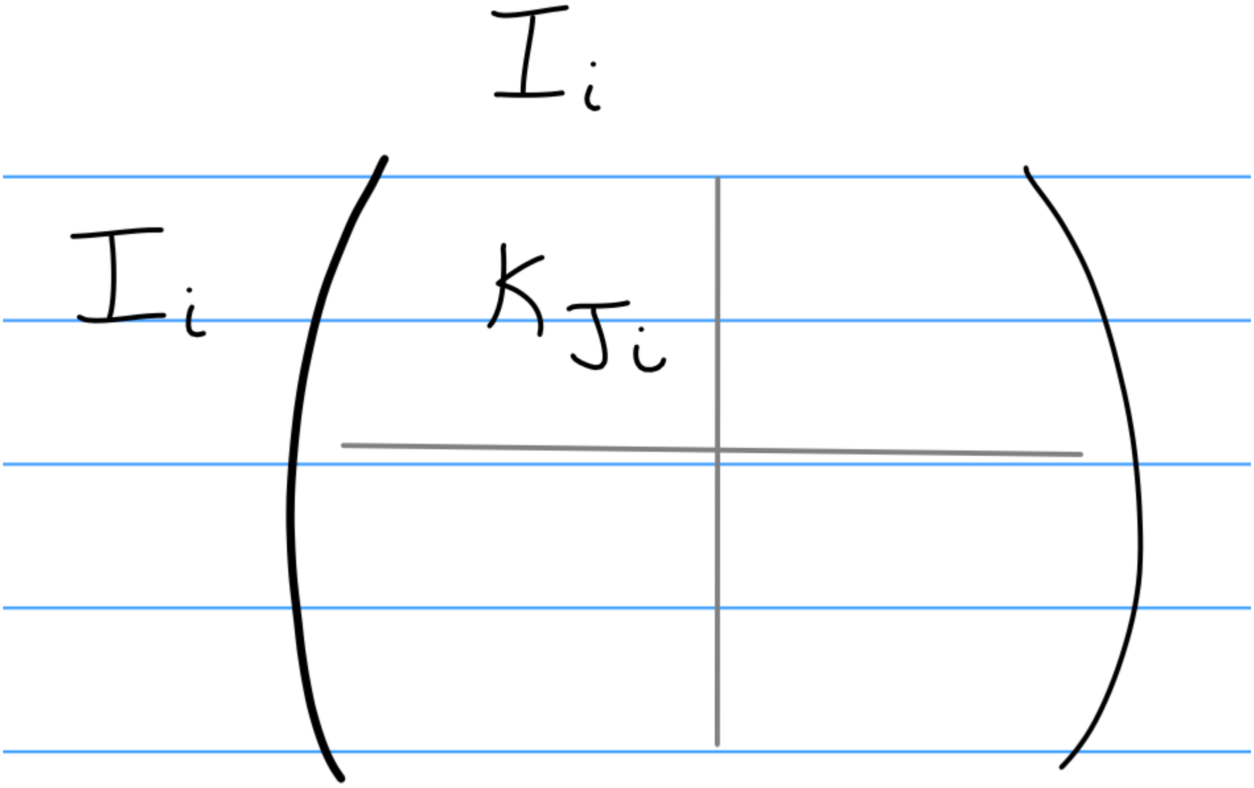


Figure 5: Image

$$\kappa : lieg^2 \to \mathbb{F}$$
$$(x, y) \mapsto \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y).$$

and Cartan's criteria:

1. $\mathfrak{g}$ is solvable $\iff \kappa(x, y) = 0 \ \forall x \in \mathfrak{g}, \mathfrak{g}], \ y \in \mathfrak{g}$.
2. $\mathfrak{g}$ is semisimple $\iff \kappa$ is non-degenerate.

Theorem: If $\mathfrak{g}$ is semisimple, then

a. $\mathfrak{g} = \bigoplus_{i=1}^n I_i$ for some $I_i \trianglelefteq \mathfrak{g}$ which are all simple.
b. Every simple ideal $I \trianglelefteq \mathfrak{g}$ is one of the I_i.
c. $\kappa_{I_i} = \kappa_{\mathfrak{g}} \mid_{I_i \times I_i}$.

Proof of (a): Use induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ has no nonzero proper ideals, then $\mathfrak{g}$ is simple and we're done.

Otherwise, let I_1 be a minimal nonzero ideal of $\mathfrak{g}$. Then $I_1^\perp \trianglelefteq \mathfrak{g}$ is also an ideal, and thus $I := I_1 \bigcap I_1^\perp \trianglelefteq \mathfrak{g}$ is as well. Then for all $x \in [I, I]$, we must have $\kappa(x, y) = 0$ for any $y \in I \subseteq I_1^\perp$. So I is solvable, and thus $I = 0$. So $\mathfrak{g} = I_1 \oplus I_1^\perp$.

$$\operatorname{ad} x \sim \left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right) \qquad \kappa_{\mathfrak{g}}(x,y) = \operatorname{tr}\left(\left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right)\right)$$

$$\operatorname{ad} y \sim \left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right) \qquad = \operatorname{tr}\left(\begin{array}{c|c} A_x A_y & B_x B_y \\ \hline 0 & 0 \end{array}\right)$$

$$= \operatorname{tr}(A_x A_y)$$

$$= \kappa_{I_i}(x,y)$$

Figure 6: Image

Note that any ideal of $I_1^\perp$ is also an ideal of $\mathfrak{g}$, which implies that $\operatorname{rad}(I_1^\perp) \subseteq \operatorname{rad}(\mathfrak{g})$, which is zero since $\mathfrak{g}$ is semisimple, and thus $I_1^\perp$ is semisimple as well.

By the inductive hypothesis, $I_1^\perp = I_2 \oplus \cdots \oplus I_n$ where each $I_j \trianglelefteq I_i^\perp$ is simple. Then $I_j \trianglelefteq \mathfrak{g} \implies [I_1, I_j] \subset I_1 \bigcap I_j$, since I_1 has no contribution. But this is a subset of $I_1 \bigcap I_1^\perp = 0$. $\square$

Proof of (b): If $I \trianglelefteq \mathfrak{g}$, then $[I, \mathfrak{g}] \trianglelefteq I$ because $[[I, \mathfrak{g}], I] \subseteq [I, I] \subseteq [I, \mathfrak{g}]$.

Since $\mathfrak{g}$ is semisimple, $0 = \operatorname{rad}(\mathfrak{g}) \supseteq Z(\mathfrak{g})$. So $[I, \mathfrak{g}] \neq 0$, and thus $[I, \mathfrak{g}] = I$ since I is simple. But then $[I, \mathfrak{g}] = \bigoplus [I, I_i]$ is simple as well. So only one direct summand can survive, since otherwise this would produce at least 2 nontrivial ideals, and $[I, \mathfrak{g}] = [I, I_i]$ for some i.

So for all $j \neq i$, we must have $I_j \bigcap I = I_j \bigcap [I, I_i] = 0$, and so $I \subseteq I_i$. But then $I = I_i$ since I_i itself is simple, and we're done.

Proof of (c):

(Without using the simplicity of I_i)

For $x, y \in I_i$, we have

10.1 Inner Derivations

Recall that $\operatorname{ad} \mathfrak{g} \subseteq \operatorname{Der} \mathfrak{g}$, and in fact (lemma) this is an ideal.

Theorem: If $\mathfrak{g}$ is semisimple, then $\operatorname{ad} \mathfrak{g} = \operatorname{Der} \mathfrak{g}$.

Proof of lemma:

For all $\delta \in \mathrm{Der}\mathfrak{g}$ and all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} [\delta, \mathrm{ad}_{\,x}](y) &= \delta([x,y]) - [x, \delta(y)] \\ &= [\delta(x), y] \\ &= [\mathrm{ad}_{\,\delta(x)}](y), \end{aligned}$$

and so $[\delta, \mathrm{ad}\ x] \subseteq \mathrm{ad}\ \mathfrak{g}$. $\square$

Proof of theorem:

If $\mathfrak{g}$ is semisimple, then $0 = \mathrm{rad}\mathfrak{g} \supseteq Z(\mathfrak{g}) = \ker \mathrm{ad}$. Thus $\mathrm{ad}\ \mathfrak{g} \cong g/\ker \mathrm{ad}\ \cong \mathfrak{g}$ is also semisimple.

This means that $\kappa_{\mathrm{ad}\ \mathfrak{g}}$ is non-degenerate, and thus $\mathrm{ad}\ \mathfrak{g} \bigcap (\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, where $(\mathrm{ad}\ \mathfrak{g})^{\perp} \trianglelefteq \mathrm{Der}(\mathfrak{g})$.

(Note that the non-degeneracy of κ already forces $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$.)

Then $[(\mathrm{ad}\ \mathfrak{g})^{\perp}, \mathrm{ad}\ \mathfrak{g}] = 0$, and so for all $\delta \in (\mathrm{ad}\ \mathfrak{g})^{\perp}$, we have $\delta(x) = [\delta, \mathrm{ad}\ x]$ by the lemma, but we've shown that this is zero.

But then δ must be zero because ad is an isomorphism, and in particular it is injective. This means that $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, and thus $\mathrm{ad}\ \mathfrak{g} = \mathfrak{g}$. $\square$

We can use this to define an abstract Jordan decomposition by pulling back decompositions on adjoints:

11 Wednesday September 4th

11.1 4.3: Cartan's Criterion

Lemma: Let $A \subset B$ be two subspaces of $\mathfrak{gl}(V)$ where $\dim V < \infty$. Set $M = \{x \in \mathfrak{gl}(V) \ni [x, B] \subset A\}$. Suppose that $x \in M$ satisfies $\mathrm{Tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

Proof: Let $x = s + n$ (where $s = x_s$ and $n = x_n$) be be the Jordan decomposition of x. Fix a basis $v_1 \cdots v_m$ of V relative to which s has matrix $\mathrm{diag}(a_1 \cdots a_m)$. Let E be the vector subspace of F over the prime field Q spanned by the eigenvalues $a_1 \cdots a_m$. We have to show that $s = 0$, or equivalently that $E = 0$, since E has finite Q-dimension by construction. It will suffice to show that the dual space $E\vee$ is 0, i.e. that any linear functional $f : E \to Q$ is zero.

Given f, let y be the element of $\mathfrak{gl}(V)$ whose matrix is given by $\mathrm{diag}(\mathrm{f(a_1)}, \cdots \mathrm{f(a_m)})$. If $\{e_{ij}\}$ is a basis of $\mathfrak{gl}(V)$, then $\mathrm{ad}_{\,s}(e_{ij}) = (a_i - a_j)e_{ij}$ and $\mathrm{ad}_{\,y}(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$.

Now let $r(t) \in F[t]$ be a polynomial with no constant term, satisfying $r(a_i - a_j) = f(a_i) - f(a_j)$ for all pairs i, j. The existence of such $r(t)$ follows from Lagrange interpolation, and the fact that if $a_i = a_j$ then $0 = r(a_j) - r(a_i) = r(a_i - a_j) = r(0)$, so r has no constant term. Thus there is no ambiguity in the assigned values, since $a_i - a_j = a_j - a_l$ would imply (by linearity of f) that $f(a_i) - f(a_j) = f(a_k) - f(a_l)$. Thus $\mathrm{ad}_{\,y} = r(\mathrm{ad}_{\,s})$.

> Note that Lagrange Interpolation is a special case of the Chinese Remainder Theorem for polynomials. If all x_is are distinct, then $p_i(x) = x - x_i$ are all pairwise coprime. Then dividing $\frac{p(x)}{p_i(x)} = p(x_i)$. So letting $A_1 \cdots A_k$ be constants in k, there is a unique polynomial of degree less than k such that $p(x_i) = A_i$. Thus there is a polynomial $p(x)$ such that $p(x) = A_i \mod p_i(x)$, and $p(x_i) = A_i$.

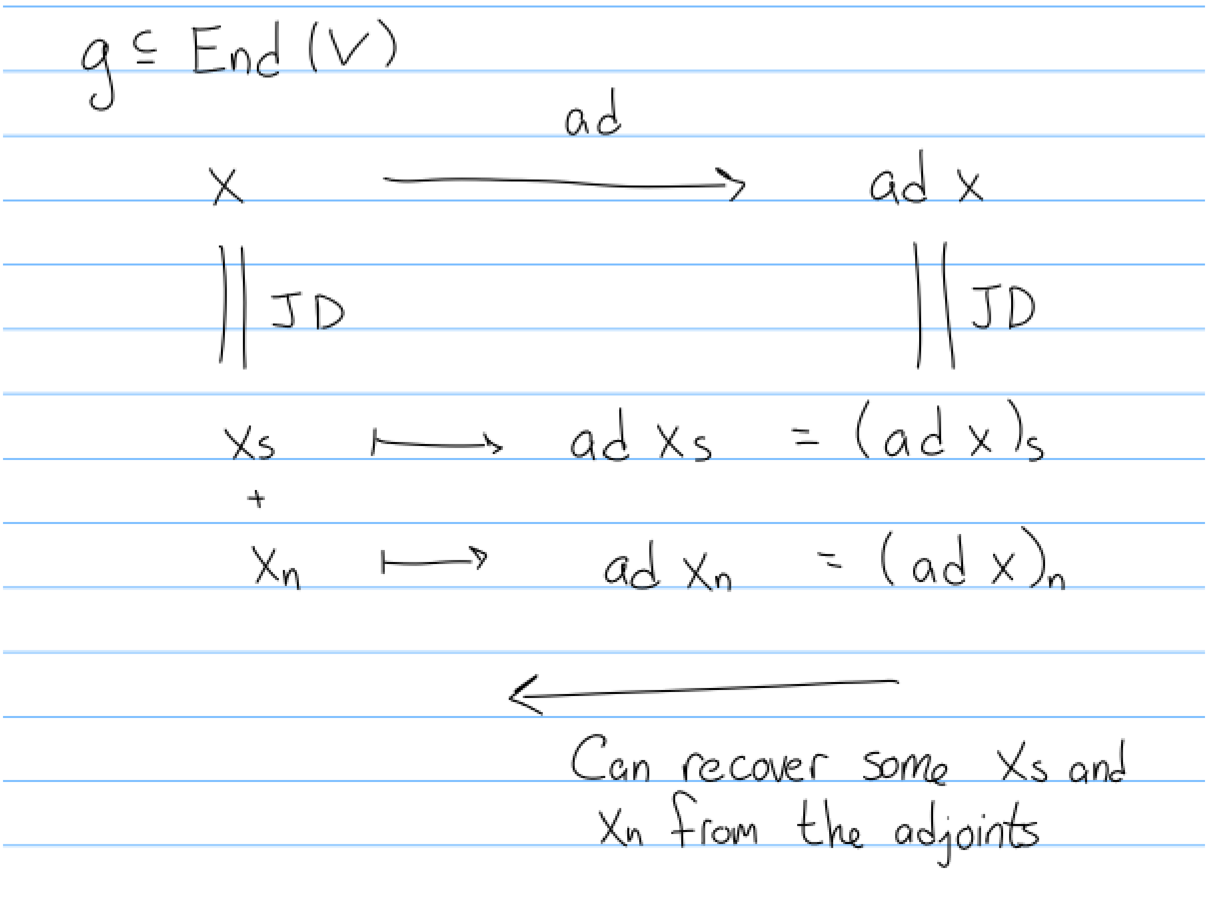


Figure 7: Image

Now ad_s is the semisimple part of ad_x. By lemma A of 4.2, ad_s can be written as a polynomial in ad_x without a constant term. Therefore ad_y is also a polynomial in ad_x without constant term. By hypothesis, ad_x maps B into A, so we have $\operatorname{ad}_y(B) \subset A$, and so $y \in M$. Using the hypothesis of the lemma, $\operatorname{Tr}(xy) = 0$, and so $\sum a_i f(a_i) = 0$. The left side is a Q-linear combination of elements of E. Applying f, we obtain $\sum f(a_i)^2 = 0$. But the numbers $f(a_i)$ are rational, so this forces all of them to be zero. Finally, f must be identically 0 because the a_i span E. □

> Note that $\operatorname{Tr}([x,y]z) = \operatorname{Tr}(x[y,z])$. To verify this, write $[x,y]z = xyz - yxz$ and $x[y,z] = xyz - xzy$, then use the fact that $\operatorname{Tr}(y(xz)) = \operatorname{Tr}((xz)y)$.

Theorem (Cartan's Criterion): Let $L \leq \mathfrak{gl}(V)$ be a subalgebra with V finite dimensional. Suppose $\operatorname{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. Then L is solvable.

Proof: It suffices to show that $[L,L]$ is nilpotent, or just that all $x \in [L,L]$ are nilpotent endomorphisms. We apply the above lemma, with V as given, $A = [L,L]$, and $B = L$, so $M = \{x \in \mathfrak{gl}(V) \ni [x,L] \subset [L,L]\}$. We have $L \subset M$. Our hypothesis is that $\operatorname{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. To use the lemma to reach the desired conclusion, we need a stronger result: that $\operatorname{Tr}(xy) = 0$ for $x \in [L,L]$ and $y \in M$.

If $[x,y]$ is a generator of $[L,L]$ and $z \in M$, then $\operatorname{Tr}([x,y]z) = \operatorname{Tr}(x[y,z]) = \operatorname{Tr}([y,z]x)$. By definition of M, $[y,z] \in [L,L]$, so the right side is 0 by hypothesis.

Corollary: Let L be a Lie algebraic such that $\operatorname{Tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0$ for all $x \in [L,L], y \in L$. Then L is solvable.

Proof: Apply the theorem to the adjoint representation of L. We then get $\operatorname{ad} L$ is solvable. Since $\ker \operatorname{ad} = Z(L)$ is also solvable, L itself is solvable.

11.2 Killing Form

11.2.1 Criterion for Semisimplicity

Let L be any lie algebra. If $x, y \in L$, then define $\kappa(x,y) = \operatorname{Tr}(\operatorname{ad}_x \circ \operatorname{ad}_y)$. Then k is a symmetric bilinear form on L, called the **killing form**.

Theorem: $\mathfrak{g}$ is solvable $\iff \kappa(x,y) = 0$ for all $x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$.

Proof: $\Longleftarrow$: By Cartan's Criterion.

$\Longrightarrow$: Exercise.

Example: The killing form of $\mathfrak{sl}(2,F)$.

We have

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then ad $_h = \text{diag}(2, 0, -2)$, and

$$\text{ad }_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\text{ad }_y == \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

and thus k has the matrix

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

where $k_{ij} = \kappa(x_i, x_j)$ where x_i is a basis of L.

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Theorem: If L is semisimple and $x \in L$, there exists a unique x_s, x_n in L such that $x = x_s + x_n$, $[x_n, x_s] = 0$, ad x_s is semisimple, and ad x_n is nilpotent.

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Todo

14 Monday September 16th

Let $S = \exp(\text{ad } e) \circ \exp(\text{ad } -f) \circ \exp(\text{ad } ei)$, which has the following matrix:

Where $\exp(\text{ad } e) = 1 + \text{ad } e + \frac{1}{2}(\text{ad } e)^2$, which would have the form

Theorem: If $\mathfrak{g}$ is semisimple, then any finite dimensional $\mathfrak{g}$-module V is completely reducible, i.e. it splits into a direct sum of simple modules.

14.1 Proof of Weyl's(?) Theorem

If V itself is simple, then we're done, so suppose it is not.

Assume there exists a nonzero submodule $U \subsetneq V$. It suffices to show that $V = U \oplus U'$ for some U'.

$$\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

Figure 8: Image

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} \cdot & 2 & \\ & \cdot & -1 \\ & & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & 1 \\ & \cdot & \cdot \\ & & \cdot \end{pmatrix}$$

Figure 9: Image

14.1.1 Step 1:

If $\dim V = 2$ and $\dim U = 1$.

Then $U, V/U$ are both trivial modules. So $g \curvearrowright u = 0$ for all $u \in U$. But then $g \curvearrowright (v + U) = U$ for all $v \in V$, since $g \curvearrowright v \in U$.

So for all $x, y \in lieg$ and all $v \in V$, we have $[x, y] \curvearrowright v = x \curvearrowright (y \curvearrowright v) - y \curvearrowright (x \curvearrowright v)$. But both of the terms in parenthesis are in U, and all elements in $\mathfrak{g}$ kill elements in U, so this is zero. So $[\mathfrak{g}, \mathfrak{g}] \curvearrowright V$ trivially.

> Exercise: If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

So $\mathfrak{g} \curvearrowright V$ trivially. Thus any U' that is a complementary subspace of U will be a submodule of V.

14.1.2 Step 2:

Suppose U is simple and $\dim U > 1$, so $\dim V/U = 1$.

Let Ω be the Casimir element on U (faithful representation?). Then $\Omega u = \check{c}$ for some $c \in \mathbb{F}$, and so $\Omega(U) \subseteq U$.

Since $\Omega : V \circlearrowleft$ is a homomorphism, $\ker \Omega \subseteq V$ is a $\mathfrak{g}$-submodule. Then $\dim V/U = 1 \implies V/U$ is a trivial module. So $\mathfrak{g} \curvearrowright V/U = 0$, i.e. $\mathfrak{g} \curvearrowright V \subseteq U$.

Then $\Omega(v) = \sum_i x_i \curvearrowright (y_i \curvearrowright v) \in U$ for all $v \in V$. What is the matrix of Ω?

In particular, $\operatorname{Tr}(\Omega \mid_{V/U}) = 0$. So $\operatorname{Tr}(\Omega) = \operatorname{Tr}(\Omega \mid_U)$. From 6.2, we know that $\operatorname{Tr}(\Omega) \neq 0 \implies c \neq 0$, where c is the scalar appearing above. So $\ker \Omega$ is 1-dimensional, and $\ker \Omega \bigcap U = \{0\}$.

So take $U' = \ker \Omega$.

14.1.3 Step 3:

Suppose U is *not* simple, but $\dim V/U = 1$.

We will induct on the dimension of U. Pick a proper nonzero submodule $\overline{U} \subsetneq U$, so that $\dim U/\overline{U} < \dim U$. Now $V/U \cong (V/\overline{U})/(U/\overline{U})$ by an isomorphism theorem. So $U/\overline{U}$ is a submodule of $V/\overline{U}$ of codimension 1. Applying the inductive hypothesis, we obtain $V/\overline{U} = U/\overline{U} \oplus \overline{V}/\overline{U}$ for some $\overline{V}$ such that $U \subseteq \overline{V} \subseteq V$.

In particular, since $U \subseteq \overline{V}$ has codimension 1, $\dim \overline{U} < \dim U$. So apply the inductive hypothesis again: $\overline{V} = \overline{U} \oplus U'$ for some U', and $V = U \oplus U'$.

14.1.4 Step 4: The general case

Recall that $\hom(V, U)$ is a $\mathfrak{g}$-module where

$$(g \curvearrowright \phi)(v) = g \curvearrowright \phi(v) - \phi(g \curvearrowright v).$$

$$
\begin{array}{cc}
 & \begin{array}{cc} \quad U \quad & \quad V/U \end{array} \\
\begin{array}{c} U \\ V/U \end{array} & \left(\begin{array}{c|c} cI & * \\ \hline 0 \cdots\cdots 0 & 0 \end{array}\right)
\end{array}
$$

Figure 10: Image

Define

$$S = \{\phi \in \hom(V, U) \ni \phi\mid_U \in F1_U\}.$$

Then $S \leq \hom(V, U)$ as a submodule. Define $T = \{\phi \in S \ni \phi\mid_U = 0\}$. Then $T \leq S$ as a submodule, and $\mathfrak{g}(S) \subseteq T$.

Now each $\phi \in S$is determined $\pmod T$ by the scalar $\phi\mid_U$. Note that $\dim(S/T) = 1$. By steps 1-3, we know that $S = T \oplus T'$ for some $T' \subseteq S$ of dimension 1. Then $T' = \operatorname{span}_{\mathbb{F}}(f)$ for some nonzero map $f : V \to U$ such that $f(u) = cu$ for some $c \neq 0$.

Then $\mathfrak{g}(T \oplus T') = \mathfrak{g}(S) \subseteq T \implies \mathfrak{g}(T') = 0$. So for all $g \in \mathfrak{g}$, we have $0 = (g \curvearrowright f)(v) = f \curvearrowright f(v) - f(g \curvearrowright v)$. Then $f : V \to U$ is a lie algebra homomorphism, $\ker f = U'$, and thus $V = U \oplus U'$. $\square$

Some consequences of Weyl's theorem:

14.2 Preservation of Jordan Decomposition

Recall that when $\mathfrak{g} \in \mathfrak{gl}(V)$ is a linear lie algebra, then for $x \in \mathfrak{g}$ we have:

Jordan Decomposition: $x = x_s + x_n$ where $x_s, x_n \in \operatorname{End}(V)$.

Abstract Jordan Decomposition:

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\text{ad}} \text{ad }(\mathfrak{g}) \\ x &\mapsto \text{ad } x \\ x_s &\leftarrow (\text{ad } x)_s \\ x_n &\leftarrow (\text{ad } x)_n. \end{aligned}$$

and so $x = x'_s + x'_n$ for some x'. The theorem will be that these recover the usual Jordan decomposition.

Theorem: If $\mathfrak{g} \in \mathfrak{gl}(V)$ is semisimple and V is finite dimensional, then $x_s, x_n \in \mathfrak{g}$, and $x_s = x'_s, x'_n$.

Corollary: If $\mathfrak{g}$ is semisimple and finite dimensional and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation, then if $x = x_s + x_n$ is the abstract Jordan decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the Jordan decomposition in $\mathfrak{gl}(V)$.

Example: If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is semisimple and finite dimensional, and h is diagonal, then by JD $h = h + 0, \phi(h) = \phi(h) + 0$. Then $h \curvearrowright V$ semisimply, or $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, where $V_\lambda = \{v \in V \ni h \curvearrowright v = \lambda v\}$ are the eigenspaces.

15 Wednesday Lecture

Last time: The abstract Jordan Decomposition coincides with the actual Jordan Decomposition.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Figure 11: Image

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x \mapsto \phi(x) = \phi(x)_s + \phi(x)_n &= \phi(x_n) + \phi(x_s) \\ x_s + x_n &\mapsto \phi(x_s) + \phi(x_n). \end{aligned}$$

Therefore $x_s \curvearrowright V$ semisimply. The example we saw last time was $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with a matrix $h = [1, 0; 0, -1]$ and $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$.

15.1 Finite Dimensional Representations of $\mathfrak{sl}(2, \mathbb{C})$

15.2 Weights and Maximal Vectors

Definition: If $V_\lambda \neq 0$, then V_λ is a *weight space* of V and $\lambda \in \mathbb{C}$ is a *weight* of h in V. We then define $W_t(V) = \{\text{weights in } V\}$.

Lemma: If $v \in V_\lambda$ then $e \curvearrowright v \in V_{\lambda+2}$ and $f \curvearrowright v \in V_{\lambda-2}$.

Proof:

$$\begin{aligned} h \curvearrowright (e \curvearrowright v) &= [h, e] \curvearrowright v + e \curvearrowright (h \curvearrowright v) \\ &= 2e \curvearrowright v + \lambda e \curvearrowright v \\ &= (\lambda + 2) e \curvearrowright v. \end{aligned}$$

and

$$\begin{aligned} h \curvearrowright (f \curvearrowright v) &= [h,f] \curvearrowright v + f \curvearrowright (h \curvearrowright v) \\ &= -2f \curvearrowright v + \lambda f \curvearrowright v \\ &= (\lambda - 2) f \curvearrowright v. \end{aligned}$$

So if V is a finite-dimensional $\mathfrak{g}$-module, then there exists a $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. Any nonzero $v \in V_\lambda$ is called a *maximal vector*.

Note: in category $\mathcal{O}$, these always exist?

Some computations:

- $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ Then $V = \mathbb{C}$ is the trivial module, and $g \curvearrowright V = 0$. So $W_t(V) = \{0\}$, and $V = V_0$.

If $V = \mathbb{C}^2$, then take the natural representation $\text{span}_\mathbb{C}\{v_1 = [1,0], v_2 = [0,1]\}$. Then $g \curvearrowright V$ by matrix multiplication, and if $h = [1,0;0,-1]$ then $h \curvearrowright v_1 = v_1$ and $h \curvearrowright v_2 = -v_2$ by just doing the matrix-vector multiplication. Then $\mathbb{C}([1,0]) = V_1, \mathbb{C}([0,1]) = V_{-1}$, so $W_t(V) = \{\pm 1\}$.

Taking $V = \mathbb{C}^3 = \text{ad}\, \mathfrak{g} = \text{span})_\mathbb{C}\{e, f, h\}$, then

$$\begin{aligned} h \curvearrowright f &= [h,f] = -2f \\ h \curvearrowright h &= [h,h] = 0h \\ h \curvearrowright e &= [h,e] = 2e. \end{aligned}$$

So $W_t(V) = \{2, 0, -2\}$ and $V_2 = \mathbb{C}e, V_0 = \mathbb{C}h, V_{-2} = \mathbb{C}f$.

Note the pattern: some largest value, then jumping by 2 to lower values, ending at negative the largest value. In some sense, the rest of the theory will reduce to the case of $\mathfrak{sl}(2, \mathbb{C})$.

Lemma: Let V be a finite dimensional simple $\mathfrak{sl}(2, \mathbb{C})$-module, and $V_0 \in V_\lambda$ a maximal vector.

Set $V_{-1} = 0, V_i = f^{(i)} \curvearrowright v_0$ (where $f^{(i)} = \frac{f^i}{i!}$). Then for all $i \geq 0$, we have

a. $h \curvearrowright v_i = (\lambda - 2i) v_i$
b. $f \curvearrowright v_i = (i+1) v_{i+1}$
c. $e \curvearrowright v_i = (\lambda - i + 1) v_{i-1}$

Proof of (a): By lemma 7.1, we have $f \curvearrowright v_0 \in V_{\lambda-2}$, and so inductively $f^{(i)} \curvearrowright v_0 \in V_{\lambda - 2i}$

Proof of (b): By definition.

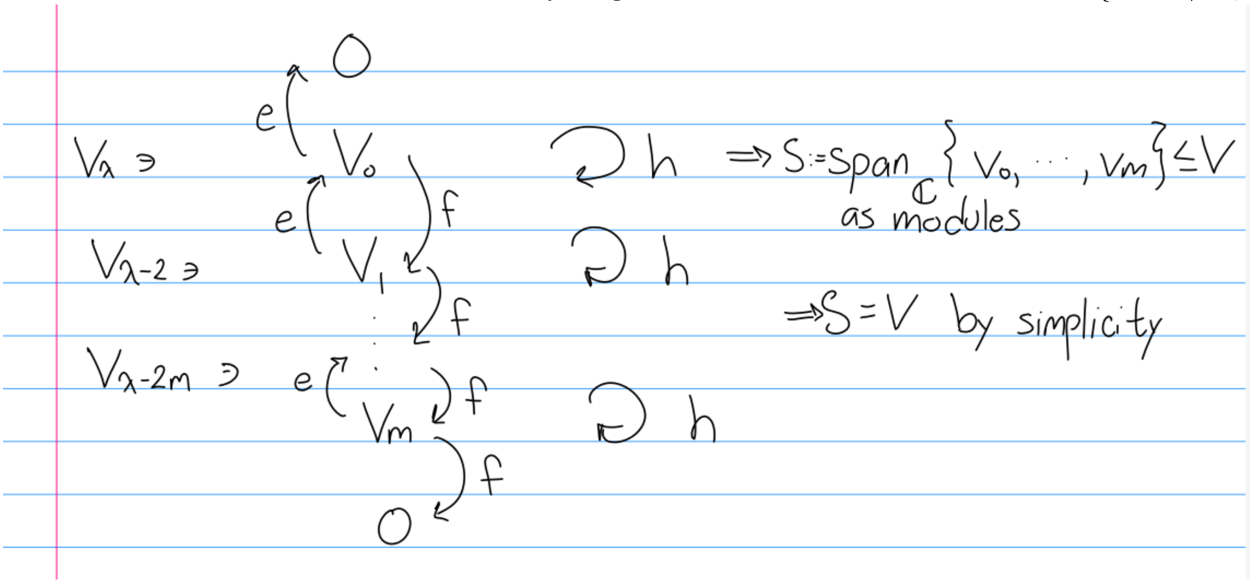
Proof of (c):

$$
\begin{aligned}
ie \curvearrowright v_i = ie \curvearrowright \frac{f^i \curvearrowright v_0}{i!} \\
= e \curvearrowright (f \curvearrowright v_{i-1}) \\
= [e, f] \curvearrowright v_{i-1} + f \curvearrowright (e \curvearrowright v_{i-1}) \\
= h \curvearrowright v_{i-1} + f \curvearrowright ((\lambda - i + 2) v_{i-2}) \text{ind} \\
= (\lambda - 2i + 2) v_{i-2} + (\lambda + i - 2)(i - 1 v_{i-1}) \\
= i(\text{RHS}).
\end{aligned}
$$

Theorem: If V is a finite dimensional and simple, then $V \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$ where $L(m) = \operatorname{span}_{\mathbb{C}} \{v_0, v)1, \cdots v_m\}$ where each v_i is of weight $m - 2i$.

Thus $L(m) = L(m)_m \oplus L(m)_{m-2} \oplus \cdots \oplus L(m)_{-m}$ where $\dim L(m)_\mu = 1$ for all μ and $\dim L(m = m + 1.)$

Proof: Pick a maximal vector $v_0 \in V_\lambda$ for any weight λ. Define v_i as usual. Let $m = \min \{i \ni V_i \neq 0,\ V_{i+1} = 0\}$



Definition: A module V is a *highest weight module* of weight λ if $V = \mathfrak{g} \curvearrowright v_0$ for some maximal vector $v_0 \in V_\lambda$.

Then λ is referred to as the *highest weight*, and v_0 is the *highest weight vector*.

Corollary: If V is finite-dimensional, then

a. $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$
b. The number of summands $= \dim V_0 + \dim V_1$.

Proof of (a): By Weyl's theorem, we know $V = \oplus W_i$ for some simple W_i. By theorem 7.2, this is equal to $\oplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\mu_m}$

Proof of (b): $\dim V_0 = \#\{\text{summands where } m \text{ is even}\}$ $\dim V_1 = \#\{\text{summands where } m \text{ is odd}\}$

Remark: Let $V_d = \{f \in \mathbb{C}[x, y] \ni f \text{ is homogeneous of total degree } d\} = \operatorname{span}_{\mathbb{C}} \left\{x^d, x^{d-1}y, \cdots, y^d\right\}$.

Then $\mathfrak{sl}(2, \mathbb{C}) \curvearrowright V_d$ by

$$
\begin{aligned}
e &\mapsto x\frac{\partial}{\partial y} \\
f &\mapsto y\frac{\partial}{\partial x} \\
h &\mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.
\end{aligned}
$$

Fact: For $L(m), \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(L(m))$, define

$$s = (\exp\phi(e)) \circ (\exp\phi(-f)) \circ (\exp\phi(e))$$

Then $s(v_i) = -v_{m-i}$.

16 Friday Lecture

Last time: Construction of simple finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ module.

Today: Root space decomposition for semisimple finite-dimensional $\mathfrak{g}$.

16.1 Root Space Decomposition

Let $\mathfrak{g}$ be semisimple and finite dimensional, and let $\mathbb{F} = \mathbb{C}$.

16.1.1 Maximal Toral subalgebra and roots

Definition: A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is *toral* if $\mathfrak{h} \neq 0$ and it consists of only semisimple elements (i.e. $x_n = 0 \forall x \in \mathfrak{h}$)

Lemma:

a. There exists a toral subalgebra of $\mathfrak{g}$, which is a nontrivial maximal toral subalgebra.
b. Any toral subalgebra is abelian.

Proof of (a): Want to show that there exists an $x \in \mathfrak{g}$ such that $x_s \neq 0$, which will imply that $\mathfrak{h} = \mathbb{C}x_s$ is toral.

Suppose $x_s = 0$ for all $x \in \mathfrak{g}$, then ad $x =$ ad x_n is nilpotent. By Engel's theorem, this means $\mathfrak{g}$ must be nilpotent. But this contradicts $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (since $\mathfrak{g}$ is semisimple) so the derived series can never reach zero.

Proof of (b): Fix $x \in \mathfrak{h}$, want to show that $[x, h] = 0 \forall h \in \mathfrak{h}$. Then $x = x_s$, and so ad $x : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. It suffices to show that ad $x|_{\mathfrak{h}} = 0$ for all $\mathfrak{h}$.

Suppose that $[x, h] = ah$ for some vector h where $a \neq 0$. Decompose $\mathfrak{h}$ into eigenspaces, so $\mathfrak{h} = \bigoplus_\lambda \mathfrak{h}_\lambda$ where $\mathfrak{h}_\lambda = \{y \in \mathfrak{h} \ni [h, y] = \lambda y\}$. But then $[h, x] \in \mathfrak{h}_0$, since $[h, [h, x]] = [h, -ah] = 0$.

So write $x = \sum_\lambda c_\lambda x_\lambda$, where $c_\lambda \in \mathbb{C}$ and $x_\lambda \in \mathfrak{h}_\lambda$. Then

$$
\begin{aligned}
[h,x] &= \sum_\lambda c_\lambda [h, x_\lambda] \\
&= \sum_\lambda c_\lambda \lambda x_\lambda \in \mathfrak{h}_0,
\end{aligned}
$$

so $\lambda c \lambda = 0 \forall \lambda \neq 0$, which means $c_\lambda = 0 \forall \lambda \neq 0$, and thus $x \in \mathfrak{h}_0$ and $[h,x] = 0$. But this contradicts $[x,h] = ah$.

Now $\forall x, h \in \mathfrak{h}, g \in \mathfrak{g}$, we have $[h,[x,y]] = [x,[h,y]] + [y,[x,h]] = [x,[h,y]]$. Thus $\text{ad } h \circ \text{ad } x = \text{ad } x \circ \text{ad } h$ as elements of $\text{End}(\mathfrak{g})$.

So $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ni [h,x] = \alpha(h)x \forall h \in \mathfrak{h}\}$.

Note that $\mathfrak{g}_0 = \{x \in \mathfrak{g} \ni [h,x] = 0 \forall h \in \mathfrak{h}\} = C_\mathfrak{g}(\mathfrak{h}) \supseteq \mathfrak{h}$, i.e. the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Definition: Fix a toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then a *root* is a nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. $\mathfrak{g}_\alpha$ is referred to as the *root space*.

We write $\Phi = \{\text{roots}\}$ and $\mathfrak{g} = C_\mathfrak{g}(\mathfrak{h}) \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$.

Example: $\mathfrak{sl}(3, \mathbb{C})$.

TODO: Insert image from phone.

Then $\Phi = \{\alpha : \mathfrak{h} \to \mathbb{C}, h_1 \mapsto \alpha(h_1) \in \{\pm 1, \pm 2\}\}$. So

- $\mathfrak{g}_0 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$
- $\mathfrak{g}_1 = \mathbb{C}f_2 \oplus \mathbb{C}e_3$
- $\mathfrak{g}_2 = \mathbb{C}e_1$
- $\mathfrak{g}_{-1}\mathbb{C}f_3 \oplus \mathbb{C}e_2$
- $\mathfrak{g}_{-2} = \mathbb{C}f_1$.

TODO: Insert second and third image from phone

From these computations, we collect the eigenvalues as ordered pairs. If we choose a larger toral subalgebra, we get a finer decomposition. And if we take a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$ and all $\dim \mathfrak{g}_\alpha = 1$.

Proposition (a): $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$.

Proposition (b): If $x \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$ then ad x is nilpotent.

Proposition (c): If $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta = 0$, then $\kappa(x,y) = 0 \forall x \in \mathfrak{g}_\alpha, y = \mathfrak{g}_\beta$.

Proof of (a): Easy exercise:

Proof of (b): For all $y \in \mathfrak{g}$, $y \in \mathfrak{g}_\mu$ for some $\mu \in \mathfrak{h}^*$. We have $\mathfrak{g}_u \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+\alpha} \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+2\alpha} \to \cdots$ by $y \mapsto [x,y] \mapsto \cdots$. Since $\mathfrak{g}$ is finite dimensional, this must terminate, so $(\text{ad } x)^n(y) = 0$ for some n.

Proof of (c): If $\alpha + \beta = 0$, then there exists an $h \in \mathfrak{h}$ such that $\alpha(h) + \beta(h) \neq 0$. Since the killing form is associative, we have

Corollary: $\kappa \mid_{\mathfrak{g}_0}$ is nondegenerate.

Proof: We want to show $\kappa(h,y) = 0 \forall y \in \mathfrak{g}_0 \implies h = 0$ holds for any choice of $y \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$.

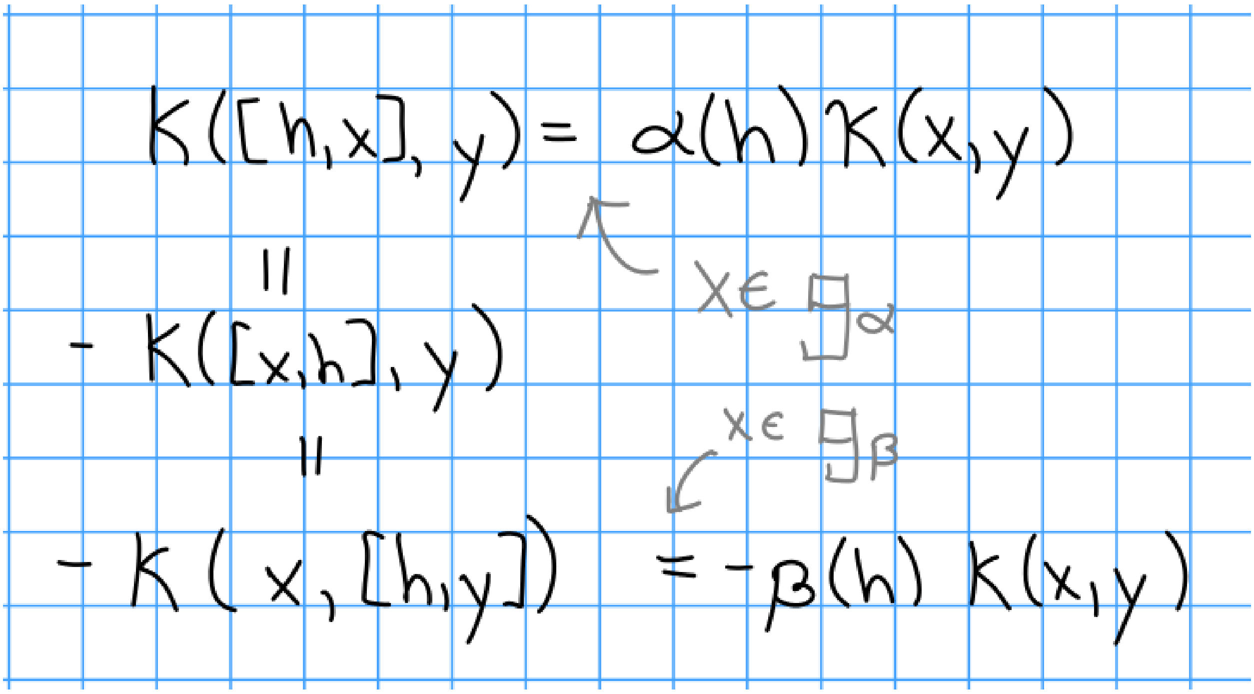


Figure 12: Image

By proposition (c), we have $\kappa(h, y) = 0$. Note that we have $\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha)$. This implies that $\kappa(h, y) = 0 \forall y \in \mathfrak{g}$. But then $h = 0$ because κ is nondegenerate and $\mathfrak{g}$ is semisimple.

17 Monday Lecture

Last time: $\mathfrak{h}$ is a *toral* subalgebra if it contains only semisimple elements, and implies that there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \ni [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ and $\Phi = \{\alpha : \mathfrak{h} \to \mathbb{C} \ni \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\}$ and $\mathfrak{g}_0 = C_\mathfrak{g}(\mathfrak{h})$.

Take larger $\mathfrak{h}$ yields finer decompositions, and a maximal $\mathfrak{h}$ gives $\dim \mathfrak{g}_\alpha = 1 \forall \alpha \in Phi$.

Corollary: $\kappa|_{\mathfrak{g}_0}$ is nondegenerate.

17.1 The Centralizer of $\mathfrak{h}$

If $x, y \in \mathrm{End}(V)$ where V is finite dimensional, $xy = yx$, and y is nilpotent, then xy is nilpotent and $\mathrm{Tr}(xy) = 0$.

Proposition: If $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$.

Proof:

Step 1: If $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$.

If $x \in \mathfrak{g}_0$, then ad $x(\mathfrak{h}) \subseteq 0$. By proposition 4.2, ad $x_s(\mathfrak{h}) \subseteq 0, \text{ad } x_n(\mathfrak{h})$, and so $x_s, x_n \in \mathfrak{g}_0$.

Step 2: $\{x_s \ni x \in \mathfrak{g}_0\} \subseteq \mathfrak{h}$.

If $x \in \mathfrak{g}_0$, then by step 1 we have $x_s \in \mathfrak{g}_0$ and so $\mathfrak{h} + \mathbb{C}x_s$ is toral, and thus $x_s \in \mathfrak{h}$.

Step 3: $\kappa|_{\mathfrak{h}}$ is non-degenerate.

We want to show that $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g} \implies h = 0$. By the corollary, it suffices to show that $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g}_0$. By step 2, it suffices to check this only for $x \in \mathfrak{g}_0$ such that $x = x_n$.

If $x = x_n$, then ad x_n is nilpotent and ad h commutes with ad x because $[h, x] = 0$ (since $x \in \mathfrak{g}_0$). By the lemma, $\text{Tr}(\text{ad } h \circ \text{ad } x) = 0$, since ad $h = \kappa(h, x)$.

Step 4: $\mathfrak{g}_0$ is nilpotent.

Pick $x \in \mathfrak{g}_0$. Then by step 2, $x_s \in \mathfrak{h}$, so ad $x_s : \mathfrak{g}_0 \circlearrowleft$ is a zero map and thus nilpotent.

So ad x_n is nilpotent, meaning that ad x is nilpotent. By Engel's theorem, this implies that $\mathfrak{g}_0$ itself is nilpotent.

Step 5: $\mathfrak{g}_0$ is abelian.

Suppose that $I := [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$. We have $I \trianglelefteq \mathfrak{g}_0$, and I is not nilpotent whereas $\mathfrak{g}_0$ is.

By Lemma 3.3, we have $I \bigcap Z(\mathfrak{g}_0) \neq 0$, so pick x in the intersection. Note that $\kappa(\mathfrak{h}, I) = \kappa(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0])$, which by associativity equals $\kappa([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = 0$.

By step 3, we have $\mathfrak{h} \bigcap I = 0$. By step 2, $x \neq x_s$, and thus $x_n \neq 0$. But we also have $x \in Z(\mathfrak{g}_0)$, so $[x, \mathfrak{g}_0] = 0$ and ad $x(\mathfrak{g}_0) \subseteq 0$. By Proposition 4.2, this holds for x_s, x_n as well, which are both in the center. So for all $y \in \mathfrak{g}_0$, ad y commutes with ad x_n, which is nilpotent.

By the lemma, this implies that $0 = \text{Tr}(\text{ad } y \circ \text{ad } x_n) = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$. So $x_n = 0$.

Step 6: Suppose $\mathfrak{g}_0 \not\subset \mathfrak{h}$. By step 2, there exists an $x \in \mathfrak{g}_0$ such that $x \notin \mathfrak{h}$, where $x_n \neq 0$. By step 5, $[x_n, y] = 0$ for all $y \in \mathfrak{g}_0$. Then ad x (which is nilpotent) commutes with ad y. By the lemma, $0 = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$, and thus $x_n = 0$. $\square$

Main idea: Choose a maximal toral subalgebra to get a nice root space decomposition, and so it coincides with $\mathfrak{g}_0$.

Corollary: $\kappa|_{\mathfrak{g}}$ is nondegenerate.

Thus for all $\alpha \in \mathfrak{h}^*$, there exists a unique $t_\alpha \in \mathfrak{h}$ such that $\alpha = \kappa(t_\alpha, \cdot\,) : \mathfrak{h} \to \mathbb{C}$.

In other words, there is an identification

$$\begin{aligned} \mathfrak{h} &\xrightarrow{1-1} \mathfrak{h}^* \\ h &\mapsto \kappa(h, \cdot\,) \\ t_\alpha &\leftarrow \alpha. \end{aligned}$$

Definition: A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \ni [x, h] \subseteq \mathfrak{h}\}.$$

Note that $N_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.

Remark: If $\mathfrak{g}$ is semisimple and finite dimensional with $\mathrm{char}(F) = 0$, we will have a correspondence:

$$\{\text{CSAs of } \mathfrak{g}\} \iff \{\text{maximal toral subalgebras of } \mathfrak{g}\}.$$

Maximal toral subalgebras advantages over Cartan subalgebra definition:

- Yields the finest root space decomposition
- $\mathfrak{h}^* = \mathfrak{h}$, Weyl group?
- Existence is easy compared to CSAs

On the other hand, CSA advantages:

- All CSAs are conjugate under G (some group to be defined)
- The dimensions of all CSAs are the same, giving a well-defined notion of dimension ($\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$).

17.2 8.3: Orthogonality Properties

From now on, $\mathfrak{h}$ will be a maximal toral subalgebra.

Proposition: Let $\alpha \in \Phi$. Then

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$
c. $\forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \kappa(x, y)t_\alpha$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$ (let the nonzero scalar be λ)
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
f. For any nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha := \frac{\lambda}{\kappa(t_\alpha,t_\alpha)} t_\alpha$. Moreover, $\langle e_\alpha, f_\alpha, h_\alpha \rangle = \mathfrak{sl}(2, \mathbb{C})$.

18 Lecture Wednesday

Today: Properties of the root space when the toral subalgebra is maximal.

Last time: We have $\mathfrak{g} = \mathfrak{g} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$ where $\kappa \mid_{\mathfrak{h}}$ is nondegenerate. We also have a correspondence

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}^* h \mapsto \kappa(\mathfrak{h}, \cdot\,) \\ t_\alpha &\leftarrow \alpha := \kappa(t_\alpha, \cdot\,). \end{aligned}$$

18.1 Orthogonality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$

c. $[x, y] = \kappa(x, y)t_\alpha$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$
f. For each nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha \coloneqq \frac{2}{\kappa(t_\alpha,t_\alpha)}t_\alpha$. Moreover, $\langle e_\alpha, t_\alpha, h_\alpha\rangle \cong \mathfrak{sl}(2, \mathbb{C})$.

Proof of (a): We want to show that $h \in \mathfrak{h}$ implies that if $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $h = 0$.

Take $x \in \mathfrak{g}_\alpha$. Then $[h, x] = \alpha(h)x = 0$. So $[\mathfrak{h}, \mathfrak{g}] = 0$ because $\mathfrak{h}$ is abelian. But then $[h, \mathfrak{g}] = 0$, or $h \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

Proof of (b): By Proposition 8.1c, we have $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$.

If $-\alpha \notin \Phi$, then $\mathfrak{g}_{-\alpha} = 0$. So $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$ by the non-degeneracy of κ.

Proof of (c): For all $h \in \mathfrak{h}$, we have

$$\begin{aligned}\kappa(h, [x, y]) &= \kappa([h, x], y)\\ &= \kappa(\alpha(h)x, y)\\ &= \kappa(t_\alpha, h)\kappa(x, y)\\ &= \kappa(\kappa(x, y)t_\alpha, h)\\ &= \kappa(h, \kappa(x, y)t_\alpha).\end{aligned}$$

which implies that $\kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0$, which forces the second argument to be zero by non-degeneracy.

Proof of (d): We will show that (d) implies (c), i.e. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}t_\alpha$.

We want to show $\kappa(x, y)$ is not always zero.

Pick any nonzero $x \in \mathfrak{g}_\alpha$. Then $\kappa(x, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$. If $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$, then $\kappa(x, \mathfrak{g}) = 0$. By non-degeneracy, this forces $x = 0$.

Proof of (e): We will skip this for now, and revisit with methods from later sections that make this proof simpler.

Proof of (f): Let $e_\alpha \neq 0$ in $\mathfrak{g}_\alpha$. Then there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, y) \neq 0$. Set $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, f_\alpha) = \frac{2}{\kappa(t_\alpha,t_\alpha)}$.

By (c), we have

$$\begin{aligned}[e_\alpha, f_\alpha] &= \kappa(e_\alpha, t_\alpha)t_\alpha\\ &= \frac{2}{\kappa(t_\alpha, t_\alpha)}t_\alpha\\ &= h_\alpha.\end{aligned}$$

and

$$
\begin{aligned}
[h_\alpha, e_\alpha] &= \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] \\
&= \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha)e_\alpha \\
&= 2e_\alpha.
\end{aligned}
$$

and similarly $[h_\alpha, f_\alpha] = -2f_\alpha$.

Definition:

Let $\mathfrak{sl}(2, \alpha) = \langle e_\alpha, f_\alpha, h_\alpha \rangle$ as in (f). A priori, this depends on a choice of $e_\alpha \neq 0$. We will show that this only depends on α.

18.2 Orthogonality/Integrality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. $\dim \mathfrak{g}_\alpha = 1$. (Note that in general, $\dim \mathfrak{g}_0 = \dim \mathfrak{h} \geq 1$)
b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
c. If $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
d. If $\beta \neq -\alpha \in \Phi$, then let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $\beta + q\alpha$ are both in Φ. Then $\beta + i\alpha \in \Phi$ for every $-p \leq i \leq q$, and

$$
\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = p - q \in \mathbb{Z}.
$$

Proof of (a) and (b):

Let $M = \mathfrak{h} \oplus \left(\bigoplus_{c \neq 0} \mathfrak{g}_{c\alpha} \right) \leq \mathfrak{g}$ as a subspace. By a routine check, M is an $\mathfrak{sl}(2, \alpha)$ submodule of $\mathfrak{g}$. Recall that $M = \bigoplus_{\lambda \in \mathbb{Z}}$ as a direct sum of vector spaces. Applying Weyl's theorem, we also have $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\oplus \mu_m}$ as a direct sum of (irreducible?) modules.

For $\mathfrak{h}$, if we have $[h_\alpha, h] = 0$ for all $h \in \mathfrak{h}$, then $h \in M_0$. For $\mathfrak{g}_{c\alpha}$, $[h_\alpha, x] = c\alpha(h_\alpha)x$ for all $x \in \mathfrak{g}_{c\alpha}$. But this equals $2cx$. So this implies that $\mathfrak{g}_{c\alpha} \subseteq M_{2c}$.

Thus $2c \in \mathbb{Z}$, and thus $c \in \frac{1}{2}\mathbb{Z}$, and $M_0 = \mathfrak{h}$.

We then have $\dim M_0 = \sum_{m \in 2\mathbb{Z}} \mu_m$. So write $h = \mathbb{C}t_\alpha \oplus \ker \alpha$ as vector spaces. Consider the action $\mathfrak{sl}(2, \alpha) \curvearrowright \ker \alpha$, which is trivial since $h \in \ker \alpha$. We $[h_\alpha, h] = 0$, $[e_\alpha, h] = -\alpha(h)e_\alpha = 0$ since $h \in \ker \alpha$, and similarly $[f_\alpha, h] = 0$.

Thus $\ker \alpha = L(0)^{\oplus \dim \mathfrak{h} - 1}$. Moreover, $\mathfrak{sl}(2, \alpha) = L(2) = \operatorname{span}(e_\alpha, t_\alpha, f_\alpha)^T$. But this forces the case that there is no other summand of the form $L(k)$ for k even in M.

Then $\mathfrak{g}_{2\alpha} \subseteq M_4$, which must be zero. So $2\alpha \notin \Phi$, so 2α is not a root. ("Twice a root is never a root")

So $\frac{1}{2}\alpha \notin \Phi$, otherwise we could apply this argument to conclude that α is not a root and reach a contradiction. Thus $M_1 = 0$, since $c \neq \frac{1}{2}$ implies that there is not summand of the form $L(k)$ for k odd in M. But this forces $M = \mathfrak{h} \oplus \mathfrak{sl}(2, \alpha)$.

Motto: reduce the complexity by using the $\mathfrak{sl}(2)$ module structure and its representation theory!

19 Lecture (Friday)

Last time, we saw $\Phi \subseteq \mathfrak{h}^* = \{\alpha : \mathfrak{h} \to \mathbb{C}\}$.

Suppose $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a maximal toral subalgebra and take $F = \mathbb{C}$.

We have the following propositions:

a. $\dim \mathfrak{g}_\alpha = 1\ \forall \alpha \in \Phi$
b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$, and $2\alpha \notin \Phi$ where $c\alpha : \mathfrak{h} \to \mathbb{C},\ h \mapsto c \curvearrowright \alpha(h)$. Moreover, $M = \mathfrak{h} \oplus \left(\bigoplus_{c\neq 0} \mathfrak{g}_{c\alpha}\right)$
c. If $\alpha, \beta \in \Phi$ and $\beta \neq -\alpha$ Let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $b + q\alpha$ are in Φ. Moreover, $\beta(h_\alpha) = \kappa(t_\beta, t_\alpha) = p - q \in \mathbb{Z}$.

Proof of (c):

Set $M = \sum_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$, which is an $\mathfrak{sl}(2,\alpha)$ module. By (a), we have $\dim \mathfrak{g}_{\beta+i\alpha} = 1 \iff \beta + i\alpha \in \Phi$. But for all $x \in \mathfrak{g}_{\beta+i\alpha}$, we have $[h,x] = (\beta + i\alpha)(h)x$ for all $h \in \mathfrak{h}$. But then $[h_\alpha, x] = (\beta(h_\alpha) + i\alpha(h_\alpha))x = (\beta(h_\alpha) + 2i)x$

Then $\mathfrak{g}_{\beta+i\alpha} \subseteq M_{\beta(h_\alpha)+2i}$, so $\beta(h_\alpha) \in \mathbb{Z}$.

Moreover, $\mathrm{Wt}(M) = 2\mathbb{Z}$ or $2\mathbb{Z}+1$, and in particular $\dim M_0 + \dim M_1 = 1$.

Thus M is irreducible, and $M \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$. Moreover, $\mathrm{Wt}(M) = \{m, m-2, \cdots - m\}$, and$simg\mathfrak{g}_{\beta+i\alpha} = 1$ for all $i \in [-p, q]$. Thus $\beta + i\alpha \in \Phi$.

Proof of 8.3(e): $\alpha(t_\alpha) \neq 0$. The claim is that for all $\beta \in Phi$, there exists an $r \in \mathbb{Q}$ such that $\beta(h) = r\alpha(h)$ for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

There are two cases: if $\beta = -\alpha$, then we're done by the previous argument.

Otherwise, $\beta \neq -\alpha$. Take $M = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$.

Then,

$$\begin{aligned}
\mathrm{Tr}_M(\mathrm{ad}\ h) = \sum_i \mathrm{Tr}_M((\mathrm{ad}\ e_i \circ \mathrm{ad}\ f_i) - (\mathrm{ad}\ f_i \circ \mathrm{ad}\ e_i)) &= \sum_i \mathrm{Tr}_{\mathfrak{g}_{\beta+i\alpha}}(\mathrm{ad}\ h) \\
&= \sum (\beta + i\alpha)(h) \dim \mathfrak{g}_{\beta+i\alpha} \\
&= \sum_i \dim \mathfrak{g}_{\beta+i\alpha}\beta(h) + \sum_i i \dim(\mathfrak{g}_{\beta+i\alpha}) \\
&\implies \beta(h) = \frac{-\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}{\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}\alpha(h).
\end{aligned}$$

Now consider the killing form $\kappa(t_\beta, t_\alpha) = \beta(t_\alpha) = r\alpha(t_\alpha)$, where the last equality is what we are claiming.

Suppose that $\alpha(t_\alpha) = 0$. Then $\kappa(t_\beta, t_\alpha) = 0$ for all $\beta \in Phi$. By the non-degeneracy of κ, we have $t_\alpha = 0$ and thus $\alpha = 0$.

19.1 Summary

We have $\mathfrak{g}$ semisimple, finite dimensional, and $\mathfrak{h}$ a maximal toral subalgebra (i.e. the Cartan subalgebra). This implies that κ is nondegenerate, and we have a correspondence

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}\vee \\ h &\mapsto \kappa(h, \cdot\,) \\ t_\alpha &\leftarrow \alpha. \end{aligned}$$

This gives a symmetric bilinear form $(\,\cdot\,,\,\cdot\,) : \mathfrak{h}\vee \to \mathfrak{h}\vee$.

For $\alpha \in Phi$, define its *coroot* $\alpha\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

Note that $(\,\cdot\,)\vee$ is not linear: note that

$$(2\alpha)\vee = \frac{2}{(2\alpha, 2\alpha)} 2\alpha = \frac{\alpha}{(\alpha,\alpha)} = \frac{\alpha\vee}{2}.$$

Assume that $\Phi = \{\alpha_i\}$. Define $E_{\mathbb{Q}} = \bigoplus_{i=1}^{\ell} \mathbb{Q}_{\alpha_i}$, and $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$.

Lemma: If $\alpha, \beta \in \Phi$, then

a. $(\beta, \alpha) \in \mathbb{Q}$,
b. $(\,\cdot\,,\,\cdot\,)$ on $E_{\mathbb{Q}}$ is positive definite, i.e. $x \neq 0 \implies (x,x) > 0$.

An immediate consequence of (b) is that $(\,\cdot\,, wait)$ on E is an inner product.

Proof: For all $\lambda, \mu \in \mathfrak{h}\vee$, we have

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\ t_\lambda \circ \mathrm{ad}\ t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(\ldots) + \sum_{\alpha\in\Phi} \mathrm{Tr}_{\mathfrak{g}_\alpha}(\ldots) = 0 + \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{[} \alpha \in \Phi] = \kappa(t_\alpha, t_\lambda)\kappa($$

So pick $\lambda = \mu = \alpha \in \Phi$. Then $(\alpha,\alpha) = \sum_{\beta\in\Phi} (\beta,\alpha)^2$.

Then

$$\frac{1}{(\alpha,\alpha)} = \sum_{\beta\in\Phi} \left(\frac{(\beta, \alpha\vee)}{2}\right)^2.$$

where $(\beta, \alpha\vee) = \cdots = \beta(h_\alpha) \in \mathbb{Z}$.

This means that $(\alpha,\alpha) \in \mathbb{Q}_{>0}$.

Summary of properties proved:

Let $\alpha, \beta \in \Phi$. Then

1. $0 \notin \Phi$ and Φ spans E
2. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\beta - (\beta, \alpha\vee)\alpha \in \Phi$
4. $(\beta, \alpha\vee) \in \mathbb{Z}$

Thus the assignment $(\mathfrak{g}, \mathfrak{h}) \mapsto (\Phi, E)$ defines a **root system**. This only works when $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is maximal toral.

Proof of (3):

We computed $(\beta, \alpha\vee) = p - q$. Then $-p \leq -(\beta, \alpha\vee) = q - p \leq q$. So this must be something on the root stream.

20 Monday September 23th

Last time: Let $\mathfrak{g}$ be finite dimensional and $\mathfrak{h}$ a maximal toral subalgebra.

Then (Φ, E) is a *root system*, and we obtain a bilinear product

$$\begin{aligned} \langle \cdot, \ \cdot \rangle : E \times E \to \mathbb{R} \\ (\alpha, \beta) \mapsto \kappa(t_\alpha, t_\beta). \end{aligned}$$

Examples: $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ where

Todo: Insert clip image h1, h2

$$\begin{aligned} \alpha_1 : \mathfrak{h} \to \mathbb{C} \\ h_1 \mapsto 2 h_2 \mapsto -1. \end{aligned}$$

$$\begin{aligned} \alpha_2 : \mathfrak{h} \to \mathbb{C} \\ h_1 \mapsto -1 h_2 \mapsto 2. \end{aligned}$$

To find t_{α_i}, we need to look at $\kappa \mid_{\mathfrak{h}}$.

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Since we only need the trace, this suffice, and we find

$$\left. \begin{array}{c|cc} h_1 & h_2 & \\ h_1 & 12 & -6 \\ h_2 & -6 & 12 \end{array} \right] .$$

We then get $t_{\alpha_1} = \frac{h_1}{6}$ and $t_{\alpha_2} = \frac{h_2}{6}$. Moreover

$$
\begin{aligned}
\langle \alpha_1,\ \alpha_1 \rangle = \kappa(t_{\alpha_1}, t_{\alpha_1}) = \frac{1}{3} \in \mathbb{Q} \\
\langle \alpha_1,\ \alpha_1 \rangle = \frac{1}{3} \\
\langle \alpha_1,\ \alpha_2 \rangle = -\frac{1}{6} \\
\langle \alpha_1,\ \alpha_2 \vee \rangle = \frac{2\langle \alpha_1,\ \alpha_2 \rangle}{\langle \alpha_2,\ \alpha_2 \rangle} = -1 \in \mathbb{Z} \langle \alpha_i,\ \alpha_i \vee \rangle = \frac{2\langle \alpha_i,\ \alpha_i \rangle}{\langle \alpha_i,\ \alpha_i \rangle} = 2 \in \mathbb{Z}.
\end{aligned}
$$

This leads to a nice fact: the matrix $\langle \alpha_i,\ \alpha_j \vee \rangle$ has $\mathbb{Z}$ entries, and this is called the *Cartan matrix*.

20.1 Ch.3: Root Systems

20.1.1 Axiomatics: Reflections

Fix a Euclidean space E.

Definition: A *hyperplane* in E is a subspace of codimension 1. A *reflection* in E is an element $s \in \mathfrak{gl}(E)$ such that

$$
\{E^s \coloneqq \{x \in E \ni sx = s\} \text{ is a hyperplane } H \text{ and } s(x) = -x \quad \forall x \in E \ni (x, H) = 0\}
$$

For nonzero $\alpha \in E$, its reflection is

$$
\begin{aligned}
S_\alpha : E \to E \\
\beta \mapsto \beta - \langle \beta,\ \alpha \vee \rangle \alpha.
\end{aligned}
$$

with respect to $H_\alpha = \{x \in E \ni \langle x,\ \alpha \rangle = 0\}$, where $\alpha \vee = \frac{2\alpha}{\langle \alpha,\ \alpha \rangle}$.

Lemma: Let $\Phi \subseteq E$ be finite such that $S_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

Suppose that $S \in \mathfrak{gl}(E)$ satisfies

1. $S(\Phi) = \Phi$,
2. $S(h) = h$ for all $h \in H$, and
3. $S(\alpha) = -\alpha$ for some $\alpha \in \Phi$,

then $S = S_\alpha$, i.e. this uniquely characterizes S

Proof:

Let $\tau = S \circ S_\alpha$. Then $\tau(\Phi) = \Phi$ and $\tau(\alpha) = \alpha$. This $\tau \curvearrowright \mathbb{R}\alpha$ by 1, and similarly $\tau \curvearrowright E/\mathbb{R}\alpha$ by 1 by picking a representative in H. Moreover, all eigenvalues of τ are 1. So the minimal polynomial of τ divides $(t-1)^{\dim E}$.

We want to show that $\tau \,\Big|\, (t-1)^N$ for some large N, which forces $\tau \,\Big|\, \gcd((t-1)^{\dim E}, t^N - 1) = 1$. For any $\beta \in \Phi$ and $k > |\Phi|$, not all vectors $\beta, \tau(\beta), \cdots \tau^k(\beta)$. So $\beta = \tau^{k_\beta}(\beta)$ for some k_β depending on β (noting that τ is invertible.)

Multiplying all of these k_βs together, we can get some k_Φ that is larger than $|\Phi|$, and so $\beta = \tau^{k_\Phi}$ for *all* $\beta \in \Phi$. But then $\tau^{k_\Phi} = 1$ in $\mathfrak{gl}(E)$.

20.1.2 Root Systems

Definition: A subset Φ of E a Euclidean space is called a *root system* iff

1. $|\Phi| < \infty, 0 \notin \Phi$, and $E = \bigoplus_{\alpha\in\Phi} \mathbb{R}\alpha$
2. $\alpha \in \phi \implies \mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\alpha \in \Phi \implies S_\alpha(\Phi) = \Phi$
4. $\alpha, \beta \in \Phi \implies \langle\beta,\ \alpha\vee\rangle \in \mathbb{Z}$.

Definition: The *rank* of a root system is the dimension on E.

Definition: The *Weyl Group* of Φ is defined as

$$W = \langle S_\alpha \mid \alpha \in \Phi\rangle \subseteq \mathfrak{gl}(E)$$

Note that $W \hookrightarrow \Sigma_{|\Phi|}$, a permutation group of size $|\Phi|$.

Lemma: If $g \in \mathfrak{gl}(E)$ and $g(\Phi) = \Phi$, then for all $\alpha, \beta \in \Phi$, we have

$$\begin{aligned} gs_\alpha g^{-1} &= s_{g(\alpha)}, \\ \langle\beta,\ \alpha\vee\rangle &= \langle g(\beta),\ g(\alpha)\vee\rangle, \\ \langle\beta,\ \alpha\vee\rangle &= \langle w(\beta),\ w(\alpha)\vee\rangle \quad \forall w \in W. \end{aligned}$$

Proof: Check 1-3 in Lemma 9.1.

Proof of 1: We have

$$gs_\alpha g^{-1}(g(\beta)) = gs_\alpha(\beta) \in g(\Phi) = \Phi \quad \forall\beta \in \Phi,$$
$$.$$

Proof of 2: We have

$$\{g(\beta) \ni \beta \in \Phi\} = \Phi \implies gs_\alpha g^{-1}(\Phi) = \Phi \quad \forall h \in gH_\alpha$$
$$.$$

and so $gs_\alpha g^{-1}(h) = gg^{-1}(h) = h$, so h is a fixed point of this map.

Proof of 3: We have $gs_\alpha g^{-1}(g(\alpha) = gs_\alpha(\alpha) = -g(\alpha)$, and so $gs_\alpha g^{-1} = s_{g(\alpha)})$ by Lemma 9.1.

Finally, we have

$$\begin{aligned} gs_\alpha g^{-1}(g(\beta)) = g(s_\alpha(\beta)) = g(\beta - \langle\beta,\ \alpha\vee\rangle\alpha) = g(\beta) - \langle\beta,\ \alpha\vee\rangle g(\alpha) \\ = \\ s_{g(\alpha)} = g(\beta) - \langle g(\beta),\ g(\alpha)\vee\rangle g(\alpha). \end{aligned}$$

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Recall from last time:

1. $|\Phi| < \infty$ and Φ spans E, where $0 \notin \Phi$
2. If $\alpha \in \Phi$, then $C\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\alpha \in \Phi$, then $S_\alpha(\Phi) = \Phi$.
4. If $\alpha, \beta \in \Phi$, then $\langle \beta,\ \alpha\vee \rangle \in \mathbb{Z}$ where $(E, \langle \cdot,\ \cdot \rangle)$is Euclidean and

$$S_\alpha : E \to E$$
$$\beta \mapsto \beta - \langle \beta,\ \alpha\vee \rangle \alpha, \quad \alpha\vee = \frac{2}{\langle \alpha,\ \alpha \rangle}\alpha.$$

Examples:

In Rank 1:

1. Prop 2 implies $\Phi = \{\pm\alpha\}$
2. Prop 1 implies $E = \mathbb{R}\alpha$
3. Prop 3: $S_\alpha(\alpha) = -\alpha$
4. Prop 4 implies $\langle \pm\alpha,\ \pm\alpha \rangle = \pm\frac{2\langle \alpha,\ \alpha \rangle}{\langle \alpha,\ \alpha \rangle} = \pm 2$

Rank 1 Diagram: Todo: Insert phone image

In Rank 2: Todo: Insert phone image

Exercise:

- Show that $\operatorname{ord}(S_\alpha, S_\beta) = 2, 3, 4, 6$ for types $A_1 \times A_1, B_2, G_2$.
- Show that $W(A_2) \cong \mathbb{Z}_3$ and $W(B_2) \cong D_8$.

21.1 Pairs of Roots

Lemma: Let $\alpha, \beta \in \Phi$ where $\beta \neq \pm\alpha$, then

1. $\langle \alpha,\ \beta\vee \rangle \langle \beta,\ \alpha\vee \rangle \in \{0, 1, 2, 3\}$ Moreover, assuming $|\beta| \geq |\alpha|$, we have the following table Todo: Insert table

2. If $\langle \alpha,\ \beta \rangle > 0$, then $\alpha - \beta \in \Phi$. Similarly, if $\langle \alpha,\ \beta \rangle > 0$, then $\alpha + \beta \in \Phi$.

3. Any root string is unbroken and has length greater than 4.

Proof of (1):

By the Law of Cosines, we can write $x \coloneqq \langle \beta,\ \alpha\vee \rangle \langle \alpha,\ \beta\vee \rangle = 4\cos^2(\theta) \in \mathbb{Z}$. This restricts the possibilities to $x \leq 4$. But $x = 4 \iff \alpha = c\beta$, i.e. $\theta = 0$, but we are assuming that $\alpha \neq \pm\beta$, so this can not happen.

Proof of (2):

Since $\langle \alpha,\ \beta \rangle > 0$ and $|\beta| \geq |\alpha|$, then $\langle \alpha,\ \beta\vee \rangle = 1$. But then $S_\beta(\alpha) = \alpha - \langle \alpha, \beta\vee,\ \beta \rangle \in \Phi$ by Prop 3. So this is equal to $\alpha - \beta$.

A similar argument works for $|\beta| \leq |\alpha|$.

Proof of (3): Let p, q be the largest integers such that $b - p\alpha, b + q\alpha \in \Phi$ respectively. Suppose that the root stream between these two is broken somewhere, say $\beta + s\alpha \in \Phi$ and $\beta + (s+1)\alpha \notin \Phi$ by counting up from $\beta - p\alpha$. Similarly, there is some t counting down from $b + q\alpha$ then $\beta + t\alpha \in \Phi$ but $\beta + (t-1)\alpha \notin \Phi$. In particular, $s < t$. From (2), we have $\langle \alpha,\ \beta + s\alpha \rangle \geq 0$, $\langle \alpha,\ \beta + t\alpha \rangle \leq 0$.

We have

$$\langle \alpha,\ \beta \rangle + t\langle \alpha,\ \alpha \rangle = \langle \alpha,\ \beta + t\alpha \rangle \leq 0 \leq \langle \alpha,\ beta + s\alpha \rangle = \langle \alpha,\ \beta \rangle + s\langle \alpha,\ \alpha \rangle$$

where we know that $\langle \alpha,\ \alpha \rangle > 0$.

Since $S_\alpha(\Phi) = \Phi$ and these $S_\alpha(\beta + i\alpha) = \beta - \mathbb{Z}\alpha$, we find that reflections permute the root string. We then find that $p = \langle \beta,\ \alpha\vee \rangle + q$, and so $\langle \beta,\ \alpha\vee \rangle = p - q \in [-3, 3]$.

21.2 Chapter 10: Simple Roots and Weyl Groups

Definition: A *base* of a root system Φ is a subset $\Pi \subseteq \Phi$ such that

1. Π is a basis for the underlying vector space E, and
2. Each $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Pi} \kappa_\alpha^\beta \alpha$ where all of the coefficients κ_α^β all have the same sign.

The roots in Π are called *simple*. A root β is *positive* (resp. *negative*) if the $\kappa_\alpha^\beta \geq 0$ for all $\beta in \Phi^+$ (resp ≤ 0 in Φ^-). The *height* of a β is the sum of the coefficients. Π defines a partial order on E where $\mu \leq \lambda \iff \lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$.

> Note that this is defined on the roots themselves, and can then be extended to all of E.

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Last time:

Lemma 10.2

a. ?
b. $\alpha \in \Pi \implies S_\alpha \curvearrowright \Phi^+ \setminus \{\alpha\}$ by permutation
c. $\alpha_i \in \Pi$ and $S_{\alpha_1}, \cdots, S_{\alpha_{j-1}}(\alpha_j) \in \Phi^-$ then $S_{\alpha_1} \cdots S_{\alpha_j} = S_{\alpha_1} \cdots S_{alpha_{t-1}} \cdots S_{\alpha_{j-1}}$ for some t, where the former has j terms and the latter has $j-2$ terms.

Proof of (a): ?

Proof of (b):

Suppose towards a contradiction that $w(\alpha_j) \in \Phi^+$. Then consider $WS(\alpha_j) = -W(\alpha_j) \in \Phi^-$.

By Lemma 10.2(c), we have $W = S_{\alpha_1} \cdots S_{alpha_{t-1}} S_{\alpha_{t+1}} \cdots S_{\alpha_{j-1}} S_{\alpha_j}$, where this is $j-1$ terms. So $w = S_\alpha \cdots S_{\alpha_j}$ is not reduced.

22.1 Weyl Groups

Recall that the *chambers* are given by the connected component of $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$.

Theorem: Fix Π of Φ. Then

a. $W \curvearrowright \{\text{chambers}\}$ transitively
b. $W \curvearrowright \{\text{bases}\}$ transitively
c. $\forall \alpha\Phi,\ \exists w \in W \ni w(\alpha) \in Pi$
d. $W := \{S_\alpha \ni \alpha \in \Phi\} = \langle S_\alpha \mid \alpha \in \Pi\rangle := W_0$
e. $W \curvearrowright \{\text{bases}\}$ simply transitively, i.e. $w(\Pi) = \Pi \implies w = e$.

I.e. we can describe the Weyl group using only simple reflections

Proof: We will prove (a) – (c)\$ for \$W_0#.

Proof of (a): Recall the fundamental chamber, $C(\Pi) = \{x \in E \ni (x, a) > 0\ \forall\alpha \in \Pi\}$. We want to show that any chamber C is equal to $wC(\Pi)$.

Pick $\gamma \in C$ and $g \in W_0$ such that $(g(\gamma), \rho) = \max\{(w(\gamma), \rho) \ni w \in W_0\}$, which exists because W_0 is a finite group.

For all $\alpha \in \Pi, S_\alpha g \in W_0$ and so by maximality we have

$$\begin{aligned}(g(\gamma), \rho) &\geq (s_\alpha g(\gamma)\rho) \\ &= (g(\gamma), S_\alpha(\rho)) \\ &= (g(\gamma), \rho - \alpha) \\ &= (g(\gamma), \rho) - (g(\gamma), \alpha).\end{aligned}$$

and so $(g(\gamma), 0) \geq 0$, because this can never be an equality since $\gamma \in C$. Thus $g(\gamma) \in C(\Pi)$.

Proof of (b):

This holds because there a correspondence between $\{C(\Pi)\} \iff \{\text{bases}\Pi\}$.

Proof of (c):

It suffices to show that $\alpha \in \Phi$ lies in some base $\Pi' = W(\Pi)$. Note that $\beta \neq \alpha \implies H_\beta \neq H_\alpha$, and so we can pick a $\gamma \in H_\alpha \bigcap H_\beta^c$ for every $\beta \in \Phi \setminus \pm\alpha$. Since $\langle\gamma,\ \alpha\rangle = 0$ but $\langle\gamma,\ \beta\rangle \neq 0$ for all $\beta \neq \pm\alpha$, we can choose some $\varepsilon > 0$ such that $|\langle\gamma',\ \beta\rangle| > \varepsilon$ for every $\beta \neq \pm\alpha$. Then $\gamma' \in C(\Pi')$ and thus $\alpha \in \Pi'$.

Proof of (d):

By definition, $W_0 \subseteq W$, so we need to show the reverse containment. For all $\alpha \in Phi$, we want to show $S_\alpha \in W_0$. By (c), there exists a $w \in W_0$ such that $w(\alpha) := \beta \in \Pi$, Then $S_\beta = S_{w(\alpha)} = ws_\alpha w^{-1}$. So $S_\alpha = w^{-1}S_\beta w$, where each term is in W_0, so the whole thing is in W_0 as well.

Proof of (e):

Suppose $W(\Pi) = \Pi$. Let $W = S_{\alpha_1} \cdots S_{\alpha_\ell}$ be a reduced expression, which exists by (d). By corollary 10.2b, we have $W(\alpha_\ell \in \Phi^-)$. But this forces $w = e$. $\square$

Remarks:

By (d), there is a well-defined notion of *length* for $w \in W$. We will now show that $\ell(w) = n(w) \coloneqq \# N_w \coloneqq \# \left\{ \alpha \in \Phi^+ \ni W(\alpha) \in \Phi^- \right\}$, i.e. the number of roots that get sent to a negative root.