Title

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1 Tuesday, November 24

1.1 The Veronese Embedding

Definition 1.1.1 (Veronese Embedding)
Let $n, d > 0$ and let $f_0, \cdots, f_n$ be the monomials of degree d in $k[x_1, \cdots, x_n]$. There is a morphism

$$\begin{aligned} \mathbb{P}^n \setminus V(f_0,\ codts, f_n) &\to \mathbb{P}^N \\ \mathbf{x} &\mapsto [f_0(\mathbf{x}), \cdots, f_N(\mathbf{x})], \end{aligned}$$

where $N+1$ is the number of monomials, and is equal to $\binom{n+d}{d}$.

Remark 1.1.2: It is true that $V(f_0, \cdots, f_N) \neq \emptyset$, since $V(x_0^d, x_1^d, \cdots, x_n^d) = V(x_0, \cdots, x_n)$. This will be the Veronese embedding, although we need to prove it is an embedding. On an open set $D(x_0) \subset \mathbb{P}^2$ one can define an inverse. Suppose we hyave a coordinate $z_j = x_i^{d-1} x_j$ and $z_i = x_i^d$ on $\mathbb{P}^N$. Then we can take the point

$$\left[\frac{z_1}{z_i}, \cdots, \frac{z_i}{z_i}, \cdots, \frac{z_n}{z_i}\right].$$

This defines an inverse on $D(z_i)$. Since the open sets $D(x_i)$ cover $\mathbb{P}^N$, we have an inverse on the entire image.

Remark 1.1.3: This embedding converts hypersurfaces of degree d into hyperplanes. The Veronese is an isomorphism onto its image. Consider some arbitrary degree d element of $S(\mathbb{P}^n)$. Consider $X \coloneqq V(\sum_{j=1}^N a_j f_j) \subset \mathbb{P}^n$, where $a_j \in k$, which is equal to $\varphi^{-1}(V(\sum_{j=1}^N a_j w_j))$.

Probably not right.

We have a picture: embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ in some curved way sends a hypersurface to the intersection of a hyperplane with the embedded image:

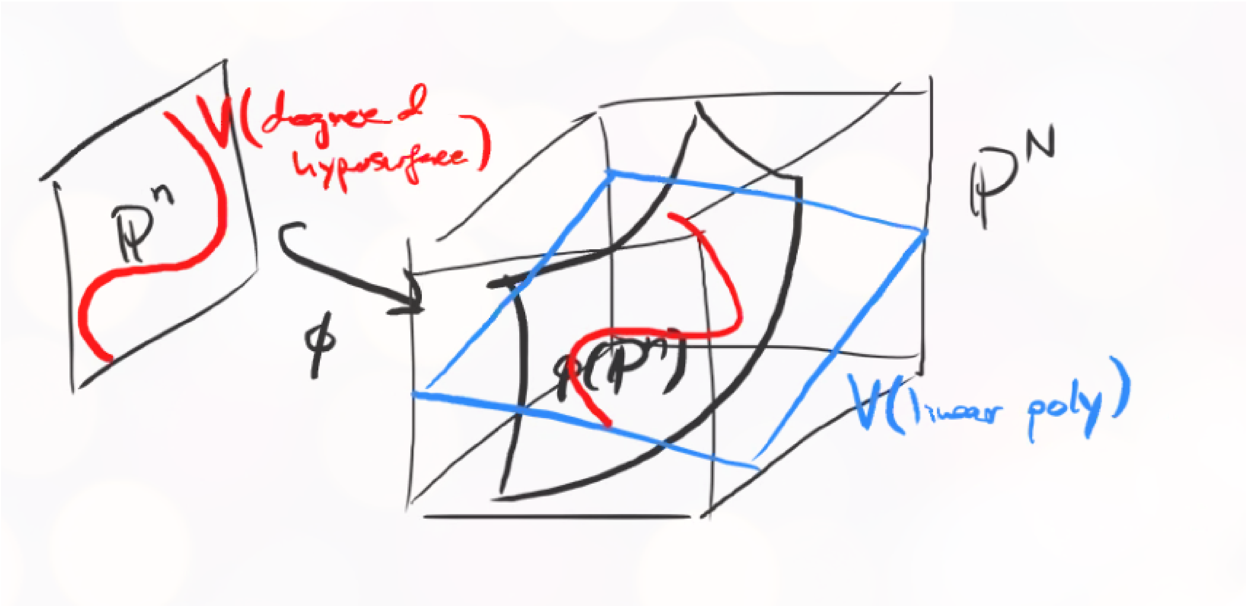

Figure 1: Image

> **Definition 1.1.4** (Hyperplane Sections)
> Let $X \subset \mathbb{P}^n$ be a projective variety. A **hyperplane section** is the intersection of X with some hyperplane $H := V(f)$ for f some linear homogeneous polynomial.

Example 1.1.5 ***(of the Veronese embedding):*** Let $n = 1$, then we get the embedding

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$$
$$[x_0 : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \cdots : x_0 x_1^{d-1} : x_1^d].$$

Note that there are $d+1$ such monomials, and not all can simultaneously vanish. The image of this $\mathbb{P}^1$ is called the *twisted normal curve.*

Example 1.1.6 ***(?):*** Take

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$
$$[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2].$$

What homogeneous polynomials cut out $\varphi(\mathbb{P}^1)$? I.e., what is $I(\varphi(\mathbb{P}^1)) \subset S(\mathbb{P}^2)$? Note that $w_0 w_2 - w_1^2|_{\varphi(\mathbb{P}^1)}$, so this is an element. Is it a generator? I.e., given any $p \in V(w_0 w_2 - w_1^2)$ is of the form $p = [x_0^2 : x_0 x_1 : x_1^2]$ for some $x_), x_1 \in k$? The answer is yes, by choosing signs of $\sqrt{w_0}, \sqrt{w_2}$.

Example 1.1.7 ***(?):*** Take

$$\varphi : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$[x_0 : x_1] \mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3].$$

What are some elements of this ideal?

- $w_0w_3 - w_1w_2$
- $w_0w_2 - w_1^2$
- $w_1w_3 - w_2^2$

Note that the first is not a k-linear combination of the other two. There is also a pattern: $w_0/w_1 = w_1/w_2 = w_2/w_3 = \cdots$. However, there will be issues when the denominators are zero.

In this case, $\varphi(\mathbb{P}^1)$ is the *twisted cubic*. What is $V(w_0w_2 - w_1^2, w_1w_3 - w_2^2) \setminus \varphi(\mathbb{P}^1)$? Note that being in $\varphi(\mathbb{P}^1)$ means $w_1, w_2, w_3 \neq 0$, and similarly if $w_0, w_1, w_2 \neq 0$. We can conclude that $V(w_1, w_2) \subset V(w_0w_2 - w_1^2, w_1w_3 - w_2^2)$:

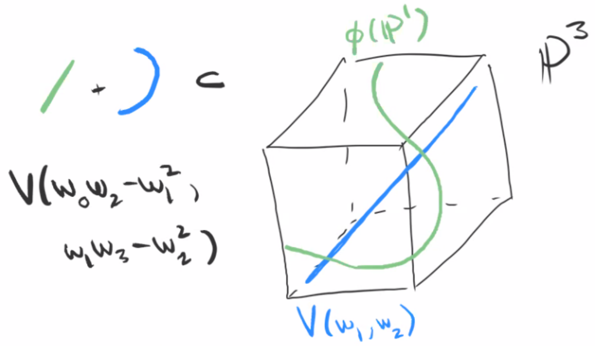

Figure 2: Image

This variety has two components: the twisted cubic, and a line. This variety has degree 4, since any generic hyperplane intersects it at 4 points. Why? Pulling back a hyperplane yields a cubic, which generally vanishes at three points in affine space.

Remark 1.1.8: $\varphi(\mathbb{P}^1)$ is a nice example of a curve in $\mathbb{P}^3$ that can not be cut out by two homogeneous polynomials.

Remark 1.1.9: This is usually used to embed intersections like $X \cap V(f)$ to $X \cap H$, exchanging a hypersurface section for a hyperplane section. This is useful for induction:

1. Prove for $\mathbb{P}^n$.
2. Induction: If it's true for $X \subset \mathbb{P}^n$, then it's true for $X \cap H$ for some hyperplane $H \subset \mathbb{P}^N$.

This will prove it for any projective variety by taking $X = V(f_1, \cdots, f_n)$ and embedding.

1.2 Chapter 10: Smoothness

Motivation: we want to distinguish between things like $V(xy)$ and $V(xy - 1)$.

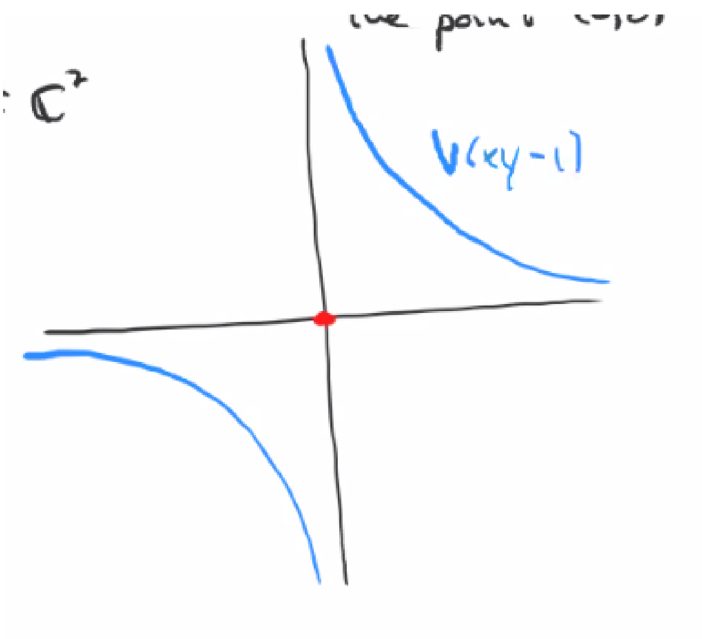

Figure 3: Image

Over $\mathbb{C}$, we can distinguish these: one is a complex manifold, and the other is not. This means we want each point to have a neighborhood biholomorphic to a disc.

Definition 1.2.1 (Tangent Space)
Let $a \in X$ be a point on a variety X. Choose an affine open set containing a and a chart such that a is the origin, then define

$$T_a X := V(f_1 \mid f \in I(X)),$$

where f_1 denotes the linear part of f.

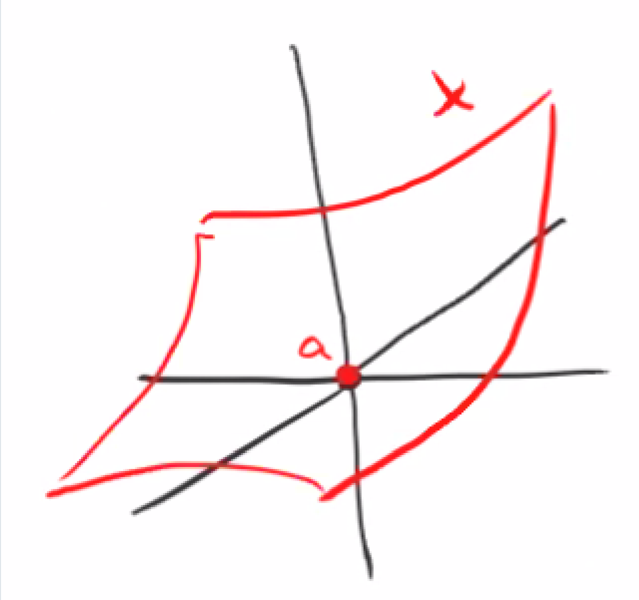

Figure 4: Image

Remark 1.2.2: Since $0 = a$, any $f \in I(X)$ has no constant term – otherwise f would not vanish at the origin.

Example 1.2.3 ***(?):*** Consider $T_{(1,1)}V(xy-1)$. First translate $(1,1)$ to the origin, so $T_{(1,1)}V(xy-1) = T_{(0,0)} = V((x-1)(y-1)-1) = T_{(0,0)}V(xy-x-y) = V(-x-y)$. On the other hand, $T_{(0,0)}V(xy) = V(0) = \mathbb{C}^2$.

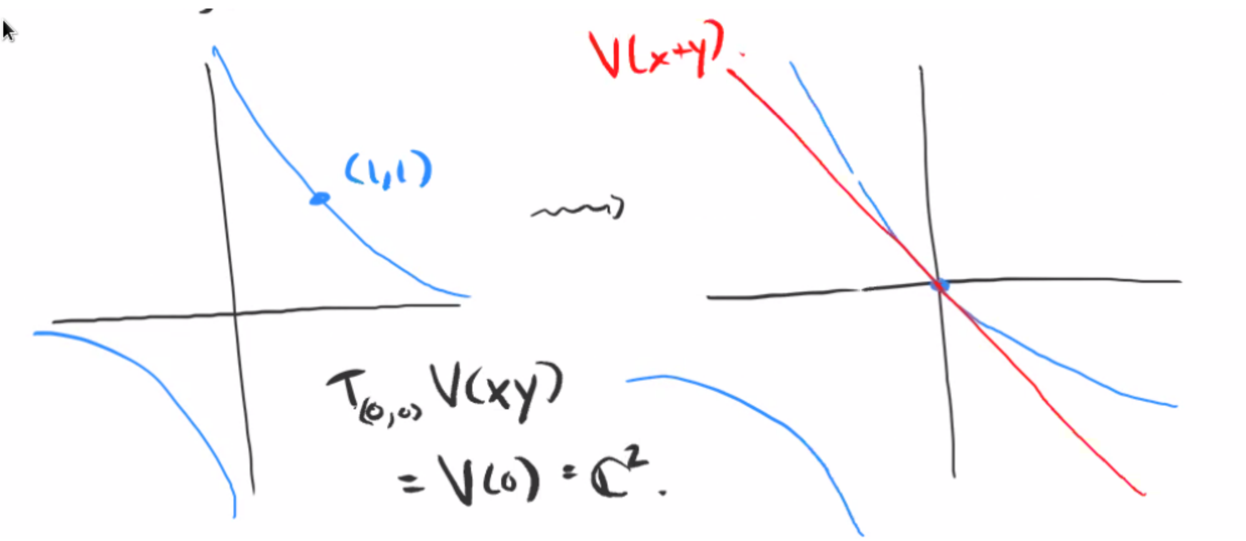

Figure 5: Image