Algebraic Geometry

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Tuesday 1st September, 2020

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These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Philip Engel at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Tuesday 1st September, 2020
20:42

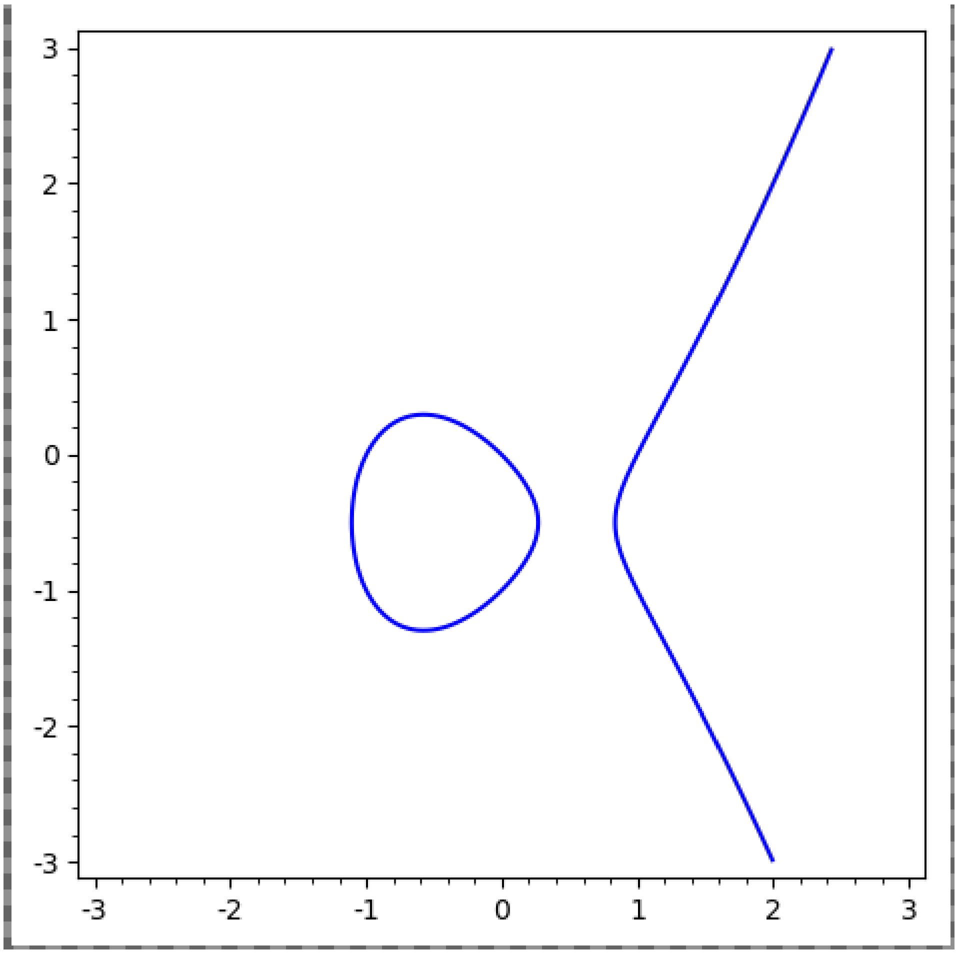
1 Friday, August 21

Reference:
https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf

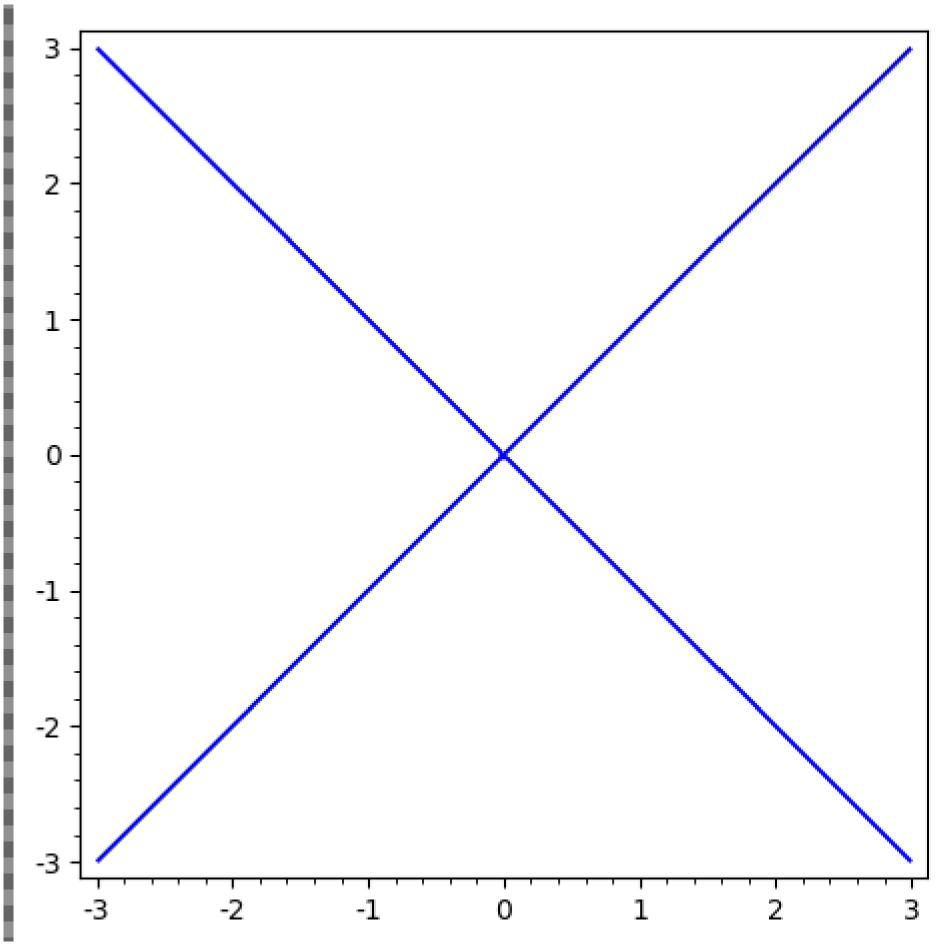
General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)
- $x^2 + 1$, which has no $\mathbb{R}$-points.
- $x^2 + y^2 + 1/\mathbb{R}$has vanishes nowhere, so ring of functions is not $\mathbb{R}[x, y]/\left\langle x^2 + y^2 + 1\right\rangle$ (problem: $\mathbb{R}$ is not algebraically closed)
- $x^2 - y^2 = 0$ over $\mathbb{C}$ is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$, which has 3 points over $\mathbb{F}_3^2$, but $f(x, y) = (x^3 - x)(y^3 - y)$ vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in $\mathbb{C}$?)
  - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x, y]/\langle f \rangle$ (addressed by scheme theory)

Theorem 1.1 ***(Harnack Curve Theorem).***
If $f \in \mathbb{R}[x, y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.
Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.1.
Show that the vanishing locus of the first two equations above is $X \cup L$ for L a line.

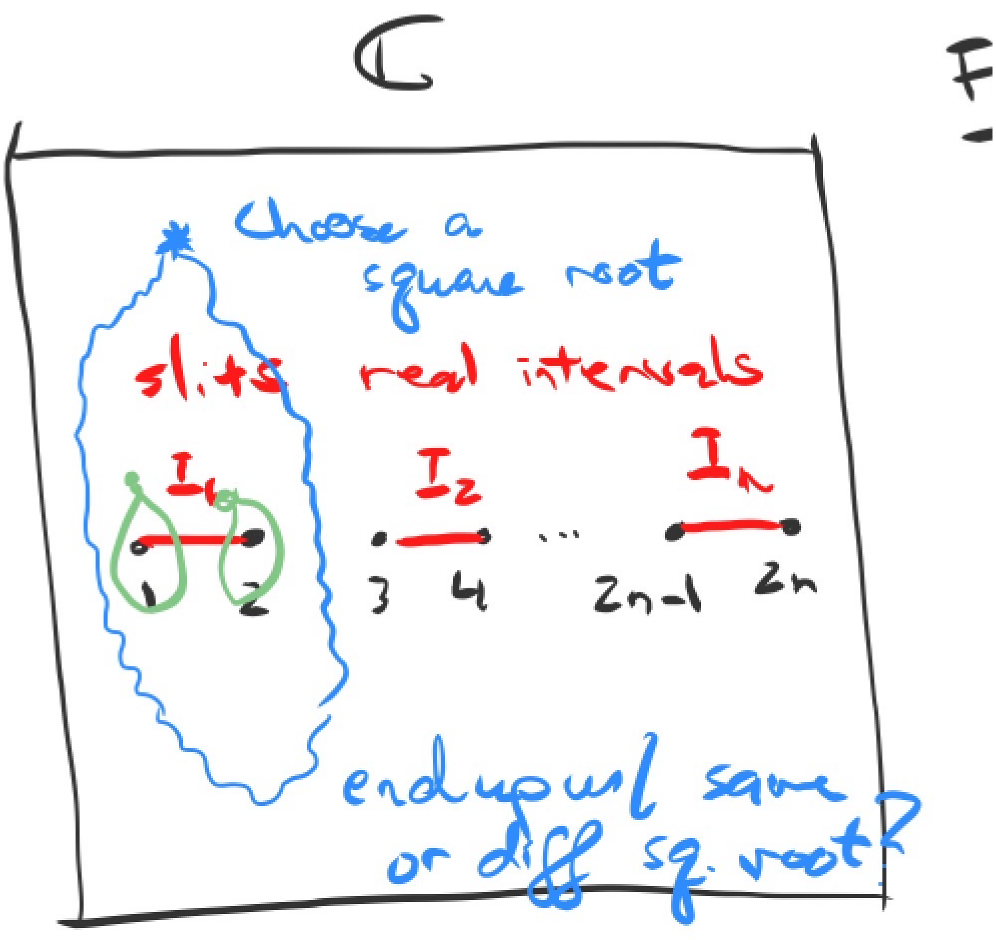
Compare to linear algebra: codimension d iff cut out by exactly d equations.

Example 1.3.
Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

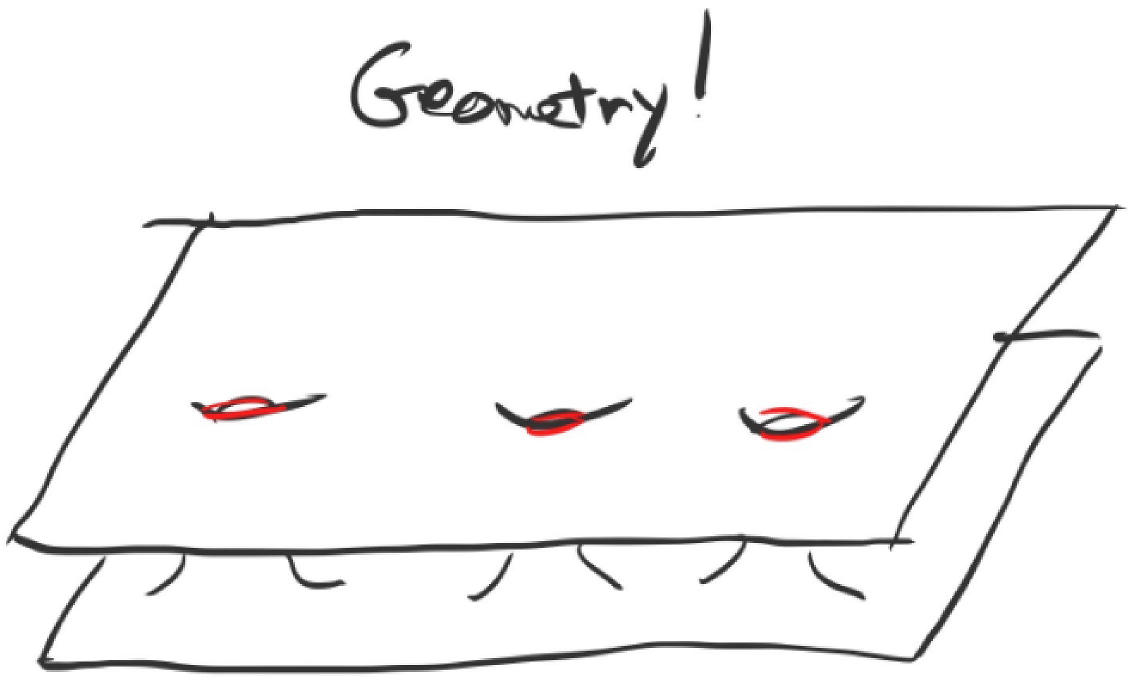
how to visualize the solution set?

Fact: on $\mathbb{C}$ with some slits, you can consistently choose a square root of the RHS.



Away from $x = 1, \cdots, 2n$, there are two solutions for y given x.

After gluing along strips, obtain:



2 Tuesday, August 25

Let $k = \bar{k}$ and R a ring containing ideals I, J.

Definition 2.0.1 (Radical)**.**
Recall that the *radical* of I is defined as

$$\sqrt{I} = \left\{ r \in R \;\middle|\; r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

Example 2.1.
Let $I = (x_1, x_2^2) \subset \mathbb{C}[x_1, x_2]$, so $I = \left\{ f_1 x_1 + f_2 x_2 \;\middle|\; f_1, f_2 \in \mathbb{C}[x_1, x_2] \right\}$. Then $\sqrt{I} = (x_1, x_2)$, since $x_2^2 \in I \implies x_2 \in \sqrt{I}$.

Given $f \in k[x_1, \cdots, x_n]$, take its value at $a = (a_1, \cdots, a_n)$ and denote it $f(a)$. Set $\deg(f)$ to be the largest value of $i_1 + \cdots + i_n$ such that the coefficient of $\prod x_j^{i_j}$ is nonzero.

Example 2.2.
$\deg(x_1 + x_2^2 + x_1 x_2^3 = 4)$

Definition 2.0.2 (Affine Variety).

1. Affine n-space $\mathbb{A}^n = \mathbb{A}^n_k$ is defined as $\left\{(a_1, \cdots, a_n) \;\middle|\; a_i \in k\right\}$.

   Remark: not k^n, since we won't necessarily use the vector space structure (e.g. adding points).

2. Let $S \subset k[x_1, \cdots, x_n]$ to be a set of polynomials. Then define $V(S) = \left\{x \in \mathbb{A}^n \;\middle|\; f(x) = 0\right\} \subset \mathbb{A}^n$ to be an *affine variety.*

Example 2.3.

- $\mathbb{A}^n = V(0)$.
- For any point $(a_1, \cdots, a_n) \in \mathbb{A}^n$, then $V(x_1 - a_1, \cdots, x_n - a_n) = \{a_1, \cdots, a_n\}$ uniquely determines the point.
- For any finite set $r_1, \cdots, r_k \in \mathbb{A}^1$, there exists a polynomial $f(x)$ whose roots are r_i.

Remark 1.
We may as well assume S is an ideal by taking the ideal it generates, $S \subseteq \langle S \rangle = \left\{\sum g_i f_i \;\middle|\; g_i \in k[x_1, \cdots, x_n],\ f_i \in S\right\}$. Then $V(\langle S \rangle) \subset V(S)$.

Conversely, if f_1, f_2 vanish at $x \in \mathbb{A}^n$, then $f_1 + f_2, gf_1$ also vanish at x for all $g \in k[x_1, \cdots, x_n]$. Thus $V(S) \subset V(\langle S \rangle)$.

Lemma 2.1.

1. If $S_1 \subseteq S_2$ then $V(S_1) \subseteq V(S_2)$.
2. $V(S_1 \cup S_2) = V(S_1 S_2) = V(S_1) \cap V(S_2)$.

We thus have a map

$$V : \{\text{Ideals in } k[x_1, \cdots, x_n]\} \longrightarrow \{\text{Affine varieties in } \mathbb{A}^n\}.$$

Definition 2.1.1 (The Ideal of a Set).
Let $X \subset \mathbb{A}^n$ be any set, then *the ideal of* X is defined as

$$I(X) \coloneqq \left\{f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X\right\}.$$

Example 2.4.
Let X be the union of the x_1 and x_2 axes in $\mathbb{A}^2$, then $I(X) = (x_1 x_2) = \left\{x_1 x_2 g \;\middle|\; g \in k[x_1, x_2]\right\}$.

Note that if $X_1 \subset X_2$ then $I(X_1) \subset I(X_2)$.

Proposition 2.2 ***(The Image of V is Radical).***
$I(X)$ is a radical ideal, i.e. $I(X) = \sqrt{I(X)}$.
This is because $f(x)^k = 0 \forall x \in X$ implies $f(x) = 0$ for all $x \in X$, so $f^k \in I(X)$ and thus $f \in I(X)$.

Our correspondence is thus

$$\left\{\text{Ideals in } k[x_1, \cdots, x_n]\right\} \xrightarrow{V} \left\{\text{Affine Varieties}\right\}$$
$$\left\{\text{Radical Ideals}\right\} \xleftarrow{I} \left\{?\right\}.$$

Proposition 2.3 ***(Hilbert Nullstellensatz (Zero Locus Theorem)).***

a. For any affine variety X, $V(I(X)) = X$.
b. For any ideal $J \subset k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

Thus there is a bijection between radical ideals and affine varieties.

2.1 Proof of Nullstellensatz

Remark 2.
Recall the Hilbert Basis Theorem: any ideal in a finitely generated polynomial ring over a field is again finitely generated.

We need to show 4 inclusions, 3 of which are easy.

a: $X \subset V(I(X))$:

- If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$.
- So $x \in V(I(X))$, since every $f \in I(X)$ vanishes at x.

b: $\sqrt{J} \subset I(V(J))$:

- If $f \in \sqrt{J}$ then $f^k \in J$ for some k.
- Then $f^k(x) = 0$ for all $x \in V(J)$.
- So $f(x) = 0$ for all $x \in V(J)$.
- Thus $f \in I(V(J))$.

c: $V(I(X)) \subset X$:

- Need to now use that X is an affine variety.
  - Counterexample: $X = \mathbb{Z}^2 \subset \mathbb{C}^2$, then $I(X) = 0$. But $V(I(X)) = \mathbb{C}^2$, but $\mathbb{C}^2 \not\subset \mathbb{Z}^2$.
- By (b), $I(V(J)) \supset \sqrt{J} \supset J$.
- Since $V(\cdot)$ is order-reversing, taking V of both sides reverses the containment.
- So $V(I(V(J))) \subset V(J)$, i.e. $V(I(X)) \subset X$.

d: $I(V(J)) \subset \sqrt{J}$ (hard direction)

Theorem 2.4 ***(1st Version of Nullstellensatz).***
Suppose k is algebraically closed and uncountable (still true in countable case by a different proof).
Then the maximal ideals in $k[x_1, \cdots, x_n]$ are of the form $(x_1 - a_1, \cdots, x_n - a_n)$.

Proof .
Let $\mathfrak{m}$ be a maximal ideal, then by the Hilbert Basis Theorem, $\mathfrak{m} = \langle f_1, \cdots, f_r \rangle$ is finitely generated.
Let $L = \mathbb{Q}[\{c_i\}]$ where the c_i are all of the coefficients of the f_i if char $(K) = 0$, **or** $\mathbb{F}_p[\{c_i\}]$ if char $(k) = p$. Then $L \subset k$.
Define $\mathfrak{m}_0 = \mathfrak{m} \cap L[x_1, \cdots, x_n]$. Note that by construction, $f_i \in \mathfrak{m}_0$ for all i, and we can write $\mathfrak{m} = \mathfrak{m}_0 \cdot k[x_1, \cdots, x_n]$.
Claim: $\mathfrak{m}_0$ is a maximal ideal.
If it were the case that

$$\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq L[x_1, \cdots, x_n],$$

then

$$\mathfrak{m}_0 \cdot k[x_1, \cdots, x_n] \subsetneq \mathfrak{m}_0' \cdot k[x_1, \cdots, x_n] \subsetneq k[x_1, \cdots, x_n].$$

So far: constructed a smaller polynomial ring and a maximal ideal in it.

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is a field that is finitely generated over either $\mathbb{Q}$ or $\mathbb{F}_p$.

Theorem 2.5 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Note: this theorem is perhaps more important than the Nullstellensatz!

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is finite over some $\mathbb{Q}(t_1, \cdots, t_n)$, and since k is uncountable, there exists an embedding $\mathbb{Q}(t_1, \cdots, t_n) \hookrightarrow k$.

Use the fact that there are only countably many polynomials over a countable field.

This extends to an embedding of $\varphi : L[x_1, \cdots, x_n]/\mathfrak{m}_0 \hookrightarrow k$ since k is algebraically closed. Letting a_i be the image of x_i under φ, then $f(a_1, \cdots, a_n) = 0$ by construction, $f_i \in (x_i - a_i)$ implies that $\mathfrak{m} = (x_i - a_i)$ by maximality.

■

3 Thursday, August 27

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{\text{Radical ideals } k[x_1, \cdots, x_n]\right\} \longrightarrow V(\,\cdot\,)I(\,\cdot\,) \left\{\text{Affine varieties in } \mathbb{A}^n\right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 3.1 ***(?).***
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty. ■

Returning to the proof that $I(V(J)) \subseteq \sqrt{J}$: let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. But $ft - 1 \neq 0$ if $f = 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

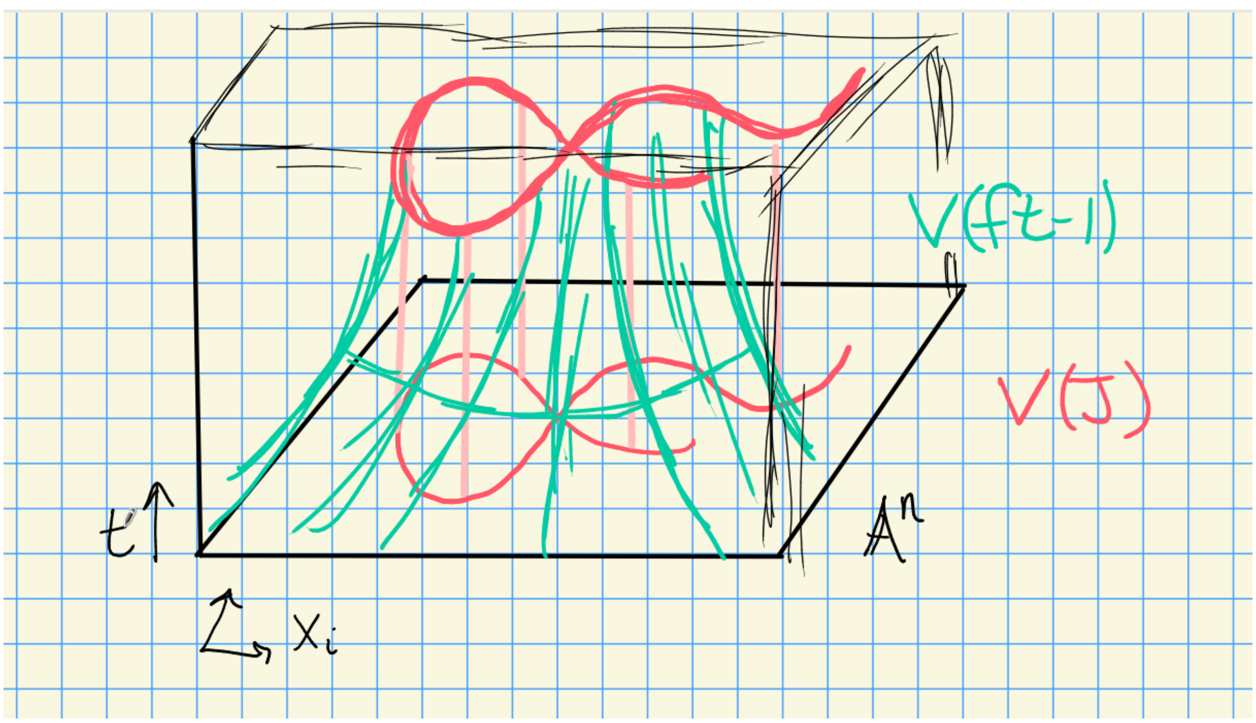


Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so $1 = \langle ft - 1 \rangle\, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$ with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N := (ft - 1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t.

Now take $k[x_1, \cdots, x_n, t]/\langle ft-1\rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t]/\langle ft-1\rangle .$$

Observation: there is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t]/\langle ft-1\rangle .$$

Exercise 3.1.
Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 3.1.

Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f\rangle$. We can factor $f(x) = \prod_{i=1}^{k}(x-a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then $I(V(f)) = \langle (x-a_1)(x-a_2)\cdots(x-a_k)\rangle = \sqrt{J} \subsetneq J$. Note that this loses information.

Example 3.2.
Let $J = \langle x-a_1, \cdots, x-a_n\rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\{\text{Points of } \mathbb{A}^n\} \iff \{\text{Maximal ideals of } k[x_1, \cdots, x_n]\} .$$

Theorem 3.2 ***(Properties of I).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) on the variety side.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

■

Example 3.3.
Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

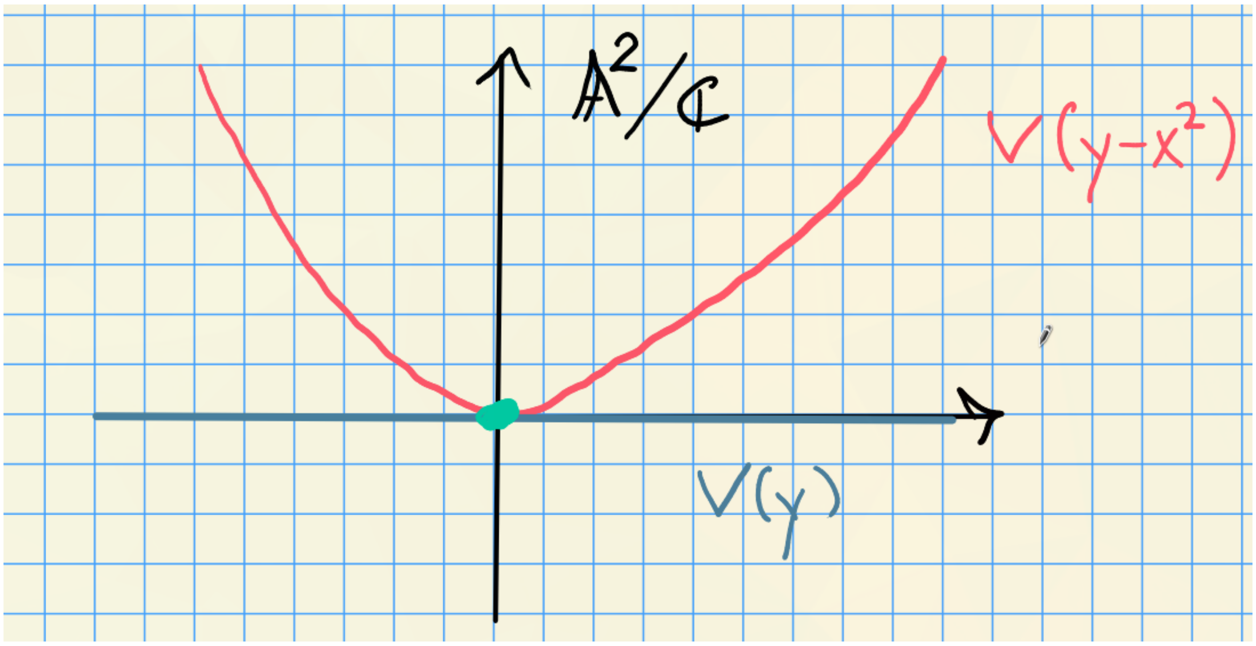


Figure 2: Image

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but $I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$.

Proposition 3.3 *(?)*.
If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof .
Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$. ∎

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

Definition 3.3.1 (Coordinate Ring).
For an affine variety X, the *coordinate ring of* X is

$$A(X) \coloneqq k[x_1, \cdots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation: The constructions $V(\,\cdot\,), I(\,\cdot\,)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) \coloneqq \left\{ x \in Y \;\middle|\; f(x) = 0 \ \forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) \coloneqq \left\{ f \in A(Y) \;\middle|\; f(x) = 0 \ \forall x \in X \right\} \subseteq A(Y).$$

Example 3.4.
For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$\begin{tikzcd}
A(\mathbb{A}^n) \arrow[r, two heads, "\cdot/I(Y)"] \arrow[rr, bend left, two heads, "\cdot/I(X)"] & A(Y) \arrow[r, two heads, "\cdot/I(X)"] & A(X)
\end{tikzcd}$$

.

> **Theorem 3.4** *(?)*.
> Let $X \subset Y$ be an affine subvariety, then
> a. $A(X) = A(Y)/I_Y(X)$
> b. There is a correspondence
>
> $$\begin{aligned} \left\{ \text{Affine subvarieties of } Y \right\} &\iff \left\{ \text{Radical ideals in } A(Y) \right\} \\ X &\mapsto I_Y(X) \\ V_Y(J) &\mapsfrom J. \end{aligned}$$

> *Proof* .
> Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ■

Example 3.5.
Let $Y = V)y - x^2 \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x-1, y-1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \langle x-1, y-1 \rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$\begin{tikzcd}
A(\mathbb{A}^n) \arrow[r] \arrow[d, equal] & A(Y) \arrow[r] \arrow[d, equal] & A(X) \arrow[d, equal] \\
k[x,y] \arrow[r] & k[x,y]/\left\langle y - x^2 \right\rangle \arrow[r] & k[x,y]/\langle x-1, y-1 \rangle
\end{tikzcd}$$

4 Tuesday, September 01

Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \, \forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \, \forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 4.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\displaystyle\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i \right)$.

Remark 3.
There are few closed sets, so this is a "weak" topology.

Example 4.1.
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A \coloneqq \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 4.2.
Let $f : \mathbb{A}^1/k \longrightarrow \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 4.3.
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\displaystyle\bigcup_{i=1}^{n} U_i \times V_i$ with U_i, V_i open in X, Y respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta := V(x - y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

Example 4.4.
Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1 x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 := V(x_1)$ and $X_2 := V(x_2)$. The Zariski topology can detect these decompositions.

Definition 4.0.2 (Irreducibility and Connectedness)**.**
Let X be a topological space.

a. X is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible.*
b. X is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, X is said to be *connected.*

Example 4.5.
$V(x_1 x_2)$ is reducible but connected.

Remark 4.
$\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

Proposition 4.1*(?)*.
Let X be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof .
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1 \emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \longrightarrow A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

■

Proposition 4.2*(?)*.
An affine variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof .
$\Longrightarrow$: By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\Longleftarrow$: Suppose X is reducible with $X = X_1 \cup X_2$ with X_i proper and closed. Define $J_i \coloneqq I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so f_i vanishes on X_i. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain. ■

Example 4.6.
Let $X = \{p_1, \cdots, p_d\}$ be a finite set in $\mathbb{A}^n$. The Zariski topology on X is the discrete topology, and $X = \coprod \{p_i\}$. So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^{d} A(\{p_i\}) = \prod_{i=1}^{d} k[x_1, \cdots, x_n] / \langle x_j - a_j(p_i) \rangle_{j=1}^{d}.$$

Example 4.7.
Set $V(x_1 x_2) = X$, then $A(X) = k[x_1, x_2] / \langle x_1 x_2 \rangle$. This not being a domain (since $x_1 x_2 = 0$) corresponds to $X = V(x_1) \cup V(x_2)$ not being irreducible.

Example 4.8.
$\mathbb{A}^2/k$ is irreducible since $k[x_1, \cdots x_n]$ is a domain.

Example 4.9.
Let X_1 be the xy plane and X_2 be the line parallel to the y-axis through $[0, 0, 1]$, and let $X = X_1 \coprod X_2$.
Then $X_1 = V(z)$ and $X_2 = V(x, z-1)$, and $I(X) = \langle z \rangle \cdots \langle x, z-1 \rangle = \left\langle xz, z^2 - z \right\rangle$.

Then the coordinate ring is given by $A(X) = \mathbb{C}[x, y, z] / \left\langle xz, z^2 - z \right\rangle = \mathbb{C}[x, y, z] / \langle z \rangle \oplus \mathbb{C}[x, y, z] / \langle x, z-1 \rangle$.

Image