

Interesting Topological Spaces in Algebraic Geometry

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1 Ideas for Spaces

- Curves
 - Elliptic Curves
 - Higher genus
 - Hyperelliptic curves
 - The modular curve
- Surfaces
 - Compact Riemann surfaces
 - * Bolza Surface (Genus 2)
 - * Klein Quartic (Genus 3)
 - * Hurwitz Surfaces
 - Kummer surfaces
- Compact Complex Surfaces
 - Rational ruled
 - Enriques Surfaces
 - $K3$

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- * Kahler Manifolds
 - Kodaira
 - Toric
 - Hyperelliptic
 - Properly quasi-elliptic
 - General type
 - Type VII
 - Fake projective planes
 - Conics
 - Calabi-Yau manifolds
 - Dimension 1: All elliptic curves (up to homeomorphism)
 - Dimension 2: $K3$ surfaces
 - Dimension 3 (threefolds): 500 million +, unknown if infinitely many
 - The bananafold
 - Hyperkähler
 - Hurwitz schemes
 - Topological galois groups, e.g. $G(\bar{F}/F)$ for $F = \mathbb{Q}, \mathbb{F}_p$.
 - $\text{Spec}(R)$ for R a DVR (a Sierpinski space)
 - Quiver Grassmannians
 - Rigid analytic spaces
 - Affine line with two origins
 - Moduli stack of elliptic curves $\mathcal{M}_{1,1}$.
 - Abelian Surface
 - Fano Varieties
 - Curves: isomorphic to \mathbb{P}^1
 - Surfaces: Del Pezzo surfaces
 - Weighted projective space
 - Toric Varieties
 - Grassmannian
 - Flag Varieties
 - Moduli Spaces

Due to Kunihiro Kodaira's classification of complex surfaces, we know that any compact hyperkähler 4-manifold is either a $K3$ surface or a compact torus T^4 . (Every Calabi–Yau manifold in 4 (real) dimensions is a hyperkähler manifold, because $SU(2)$ is isomorphic to $Sp(1)$.)

As was discovered by Beauville, the Hilbert scheme of k points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension $4k$. This gives rise to two series of compact examples: Hilbert schemes of points on a $K3$ surface and generalized Kummer varieties.

2 Intro/Motivation

Ursula Whitcher

Assume the universe is a “space”. Which one is it? What structures does it have? How many possible spaces *could* it be, and how can we test to find out?

3 Analogies

Notation: all dimensions are over \mathbb{R} .

Impossible goal: pick a category, understand all of the objects and all of the maps. Two main categories with a forgetful functor: **Diff** \longrightarrow **Top**. Question:

- What's in the “image” of this functor? (Manifolds that admit a differentiable structure.)
- What is the “fiber” above a given topological manifold? (Distinct differentiable structures)

Differentiable Manifolds: classified by geometric structure in low dimensions (≤ 4), algebraic data/methods in high dimensions

3.1 Topological Category

Identify objects up to homeomorphism

- Dimension 0: The point (terminal object)
- Dimensions 1: S^1, \mathbb{R}
- Dimension 2: $\langle \mathbb{S}, \mathbb{T}, \mathbb{RP} \mid \mathbb{S} = 0, 3\mathbb{RP} = \mathbb{RP} + \mathbb{T} \rangle$. Classified by π_1 (orientability and “genus”). Riemann, Poincare, Klein.
- Dimension 3: Can always be given a unique smooth structure, see uniformization.
- Dimension 4:
- Dimension $n \geq 5$:

3.2 Smooth Category

Generally expect things to split into more classes.

- 2-manifolds: Homeomorphic \iff diffeomorphic. Every surface admits a complex structure and a metric. Thus always orientable.
 - Uniformization: Holomorphically equivalent to a quotient of one of three spaces
 - * \mathbb{CP}^1 , positive curvature (spherical)
 - * \mathbb{C} , zero curvature (flat)
 - * \mathbb{H} (equiv. \mathbb{D}°), negative curvature (hyperbolic)
 - Stratified by genus:
 - * Genus 0: Only \mathbb{CP}^1
 - * Genus 1: All of the form \mathbb{C}/Λ , with a distinguished point $[0]$, i.e. an elliptic curve. Has a topological group structure!
- 3-manifolds: Thurston's Geometrization
 - Oriented prime 3-manifolds can be decomposed into geometric “pieces” of 8 possible types
 - Geometric structure: a diffeo $M \cong \tilde{M}/\Gamma$ where Γ is a discrete Lie group acting freely/transitively on X
- 4-manifolds: classified in the topological category by surgery, but not in the smooth category
- n -manifolds, $n \geq 5$: classified by surgery

4 Moduli Spaces

$$\begin{array}{ccc} \mathbb{k}^\ell & \longrightarrow & E \\ & & \downarrow \\ & & \mathrm{Gr}(\ell, n)/\mathbb{k} \end{array}$$

The canonical bundle:

$$\begin{array}{ccc} \mathbb{R}^{n+1} & \longrightarrow & E \\ & & \downarrow \\ & & \mathbb{RP}^n \end{array}$$

Here $E := \{(\mathbf{v}, L)\}$ where L is a line through $\mathbf{0}$ and \mathbf{v} is a vector on L .

5 Elliptic Curves

- Equivalently, Riemann surfaces with one marked point.
- Equivalently, \mathbb{C}/Λ a lattice, where homothetic lattices (multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$) are equivalent.
- Parameterized by a moduli space:
 - For $X = \mathbb{C}/\Lambda$ choose a positively oriented basis $\{z, w\}$ for Λ
 - * Note: push into meridians on a torus, generators of $H_1(X)$, and require that their intersection is $+1$.
 - Replace $z \mapsto 1, w \mapsto \frac{w}{z}$; the orientation condition forces $\Im(w) > 0$ so this yields a point $w \in \mathbb{H}$.

6 Kahlers

- For complex manifold, replace Riemannian metric with a Hermitian metric (positive definite sesquilinear inner product on tangent bundle)
- If skew-symmetric part is symplectic (closed and nondegenerate) then the metric is Kahler
- Includes smooth projective varieties, but not all complex manifolds
- Specialize to Calabi-Yaus: compact and Ricci-flat, or first Chern class vanishes

7 Calabi-Yaus

- As manifolds: Ricci-flat, i.e. Ricci curvature tensor vanishes (measures deviation of volumes of “geodesic balls” from Euclidean balls of the same radius).
- Applications: Physicists want to study G_2 manifolds (an exceptional Lie group, automorphisms of octonions), part of M -theory uniting several superstring theories, but no smooth or complex structures. Indirect approach: compactify an 11-dimension space, one small S^1 dimension \rightarrow 10 dimensions, 4 spacetime and 6 “small” Calabi-Yau.