

# Title

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We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for  $K/k$  of transcendence degree 1, all valuations on  $K$  which are trivial on  $k$  are discrete. We'll now want to go farther and describe the places  $\Sigma(K/k)$ , which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is  $K$ . Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if  $A$  is a Dedekind domain such that  $\text{ff}(A) = K$ , then for all  $\mathfrak{p} \in \text{mSpec}(A)$  there exists a discrete valuation  $v_{\mathfrak{p}}$  on  $K$ . I.e., every maximal ideal induces a discrete valuation that is  $A$ -regular, so the valuation ring will contain  $A$ . How is this obtained? Take a nonzero  $x \in K^{\times}$ , and take the corresponding principal fractional ideal  $\langle x \rangle := Ax$ , which we can factor in a Dedekind domain as  $Ax = \prod_{\mathfrak{p} \in \text{mSpec}(A)} \mathfrak{p}^{\alpha_{\mathfrak{p}}}$  with  $\alpha_{\mathfrak{p}} \in \mathbb{Z}$ . This looks like an infinite product, but for any fixed  $x$ , only finitely many  $\alpha$  are nonzero. Note that these  $\alpha$  are exactly what we're looking for: the  $\mathfrak{p}$ -adic evaluation of  $x$  is given precisely by  $v_{\mathfrak{p}}(x) := \alpha_{\mathfrak{p}}$ , where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$\begin{aligned} v. : \text{mSpec}(A) &\rightarrow \Sigma(K/A) \\ \mathfrak{p} &\mapsto v_{\mathfrak{p}}. \end{aligned}$$

So this sends a maximal ideal to a place that is  $A$ -regular, and it turns out to be a bijection.

## Proposition 1.0.1(?).

The map  $v$  is a bijection, and thus we may write

$$\Sigma(K/A) \cong \text{mSpec}(A).$$

*Proof* (?).

**Claim:**  $v$  is injective.

If  $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{mSpec}(A)$  are two different maximal ideals. Then there exists an element  $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ , and so  $x^{-1} \in A_{\mathfrak{p}_2} \setminus A_{\mathfrak{p}_1}$ . This follows since if  $x$  is not in  $\mathfrak{p}_2$ , its  $\mathfrak{p}_2$ -adic valuation is zero, and thus the  $\mathfrak{p}_2$ -adic valuation of  $x^{-1}$  is  $-0 = 0$  as well. On the other hand, since  $x \in \mathfrak{p}_1$ , its  $\mathfrak{p}_1$ -adic valuation is positive and therefore  $v_{\mathfrak{p}_1}(x^{-1}) < 0$  and  $x^{-1}$  is not in  $A_{\mathfrak{p}_1}$ .

**Claim:**  $v$  is surjective.

Let  $v \in \Sigma(K/A)$ , so  $A \subset R_v$ , i.e. take a valuation whose valuation ring contains  $A$ . Note that we're not assuming the valuation is discrete, this can be a general Krull valuation, but we're trying to show it's equal to a certain  $p$ -adic valuation. As always with a subring of a valuation ring, we can pull back the maximal ideal and consider  $\mathfrak{m}_v \cap A \in \text{Spec}(A)$ . We're hoping that this is a maximal ideal, since maximals correspond to valuations. Since we're in a Dedekind

domain, the only prime ideal we *don't* want this to be is the zero ideal of  $A$ , so suppose it were. Then  $A^\bullet \subset R_v^\times$ , and so  $K^\times \subset R_v^\times$ . This is because the only element of the maximal ideal that lies in  $A$  is zero, so every nonzero element of  $A$  is not in this maximal ideal and is thus a unit. But for any unit, its inverse is also a unit, yielding the inclusion  $K^\times \subset R_v^\times$ . The only way this could possibly happen is if  $R_v = K$ , which yields the trivial valuation ring. However, by definition, in  $\Sigma(K/A)$  we've excluded the trivial valuation, so this ideal can not be zero.

So we can conclude that the pullback  $\mathfrak{m}_v \cap A \in \text{mSpec}(A)$ , and so  $A_{\mathfrak{p}} \subset R_v$ . This is from viewing elements in  $A_{\mathfrak{p}}$  as quotients of elements in  $A$  whose denominator have  $\mathfrak{p}$ -adic valuation zero. Recall that we want to show that  $R_v = A_{\mathfrak{p}}$ . We know  $R_v \subset K$  is a proper containment, and we can use the fact that a *discrete* valuation ring is maximal among all proper subrings of its fraction field. In other words, for  $R$  a DVR, there is no ring  $R'$  such that  $R \subset R' \subset \text{ff}(R)$ . How do you prove this? This is similar to an early exercise in commutative algebra, where we looked at all rings between  $\mathbb{Z}$  and  $\mathbb{Q}$ , which generalized to looking at all rings between a PID and its fraction field, and a DVR is a local PID. So proving this statement is actually easier.

This is enough to show that  $A_{\mathfrak{p}} = R_v$ , and this  $v \sim v_{\mathfrak{p}}$ . ■

**Remark 1.0.2:** What the idea? For a general one variable function field  $K/k$ , we'll produce affine Dedekind domains  $R$  with  $k \subset R \subset K$  and  $\text{ff}(R) = K$ . This will give us subrings of this full ring of places that are  $\text{mSpec}$  of Dedekind domains. How many such domains will we need for their union to be the entire set of places? Just one won't work, since  $\Sigma(K/k)$  is like a complete or projective object, and a projective variety of dimension 1 can't be covered by a single affine variety. However, it turns out that you can always cover it with 2. In fact, if you take any Dedekind domain between  $k$  and  $\text{ff}(K)$ , the set of missing places (the ones that aren't regular for any of these domains) will be a nonempty finite set of places. So you can always cover it by finitely many, and two suffices: as a consequence of the Riemann-Roch theorem, after removing any nonempty finite set of places, you'll have the  $\text{mSpec}$  of a canonically associated Dedekind domain. We'll prove this by starting with the case of  $K = k(t)$ .

**Claim:**

$$|\Sigma(k(t)/k) \setminus \text{mSpec } k[t]| = 1.$$

Note that  $k \subset k[t] \subset k(t)$  and  $k[t]$  is a Dedekind domain, so this fits into the above framework, and moreover we know the maximal ideals of polynomial rings: irreducible monic polynomials. Taking all of these misses exactly one place. How do we describe this missing place?

Suppose  $v \in \Sigma(k(t)/k) \setminus \Sigma(k(t)/k[t])$ , so the valuation ring of  $v$  contains  $k$  but does not contain  $k[t]$ . Then the valuation ring can not contain  $t$ , and thus  $v(t) < 0$  and  $v(1/t) = -v(t) > 0$ . Since  $k[1/t]$  is a PID, so if the valuation wasn't *tdash*regular, it's  $1/t$ -regular by definition. So  $v \in \Sigma(k(t)/k[1/t])$ . Note that  $k[1/t] \cong k[t]$  as rings. How many valuations on this polynomial ring give positive valuation to  $1/t$ ? Exactly one, since this corresponds to a prime ideal, namely  $\langle 1/t \rangle$ , so this unique valuation is  $v = v_{\frac{1}{t}}$ , the  $1/t$ -adic valuation.

That is, if we write  $f \in k(t)$  as  $(1/t)^n a(1/t)/b(1/t)$  with  $a, b \in k[t]$  polynomials with nonzero

constant terms, then  $v_{\frac{1}{t}}(f) = n$ . Note that this process is the same as the one used to compute the  $t$ -adic valuation  $v_t$ .

Recall that a valuation on a domain can be uniquely extended to its fraction field by setting  $v(x/y) = v(x) - v(y)$ .

**Exercise 1.0.3(?):** Define  $v_{\infty} : k(t)^{\times} \rightarrow \mathbb{Z}$  by  $p(t)/q(t) \mapsto \deg q - \deg p$ .

- Show  $v_{\infty} \in \Sigma(k(t)/k[1/t])$ .
- Show  $v_{\infty} \sim v_{\frac{1}{t}}$  by showing they have the same valuation ring.
- Show that  $v_{\infty} = v_{\frac{1}{t}}$ .

Note that  $1/t$  is a uniformizer for  $v_{\infty}$

**Theorem 1.0.4(Complete description of places).**

$$\Sigma(k(t)/k) = \text{mSpec } k[t] \coprod \{v_{\infty}\}.$$

Note that we know the maximal ideals – the irreducible monic polynomials – but it takes some effort to write them down. If  $k$  is algebraically closed, however, every such polynomial is linear of the form  $t - \alpha$  for  $\alpha \in k$ . In this case,  $\text{mSpec } k(t) \cong k$ , and so  $\sigma(\bar{k}(t)/\bar{k}) = \bar{k} \coprod \{\infty\} = \mathbb{P}^1(\bar{k})$ . More generally, the set of places on a rational function field will yield the scheme-theoretic set of closed points on the projective line over  $k$ , which is more complicated if  $k \neq \bar{k}$  since not all closed points are  $k$ -rational. Another way to say this is that if you have a valuation, there is a residue field, and for any place on a one variable function field the residue field will be a finite degree extension of  $k$ . The degree 1 points will be the  $k$ -rational points, and so  $\Sigma(k(t)/k)$  will always contain a copy of  $k$  but may have closed points of larger degree, making things slightly more complicated. This complication is handled well in both the scheme-theoretic and this valuation-theoretic approach.

The next theorem is a fact from commutative algebra:

**Theorem 1.0.5(?).**

Let  $A$  be a domain with  $\text{ff}(A) = K$ . Suppose  $A$  is a finitely generated  $k$ -algebra, let  $L/K$  be a finite degree field extension, and let  $B$  be the integral closure of  $A$  in  $L$ . Then

- $B$  is finitely generated as an  $A$ -module.<sup>a</sup>
- $B$  is an integrally closed domain with  $\text{ff}(B) = L$  which is finitely generated as a  $k$ -algebra.
- $\dim A = \dim B$
- If  $A$  is Dedekind, so is  $B$ .

<sup>a</sup>See CA notes, “Second Normalization Theorem”, where normalization is a more geometric synonym for integral closure.

<sup>b</sup>Krull dimension, i.e. the supremum of lengths of chains of prime ideals.

*Proof (?)*.

See Pete’s CA notes sections 18 and 14. ■

**Remark 1.0.6:** We have a NTI square:

$$\begin{array}{ccc} B & \xhookrightarrow{\quad \subset \quad} & L \\ \uparrow & & \uparrow \\ A & \xhookrightarrow{\quad \subset \quad} & K \end{array}$$

We have a domain  $A$  with a fraction field  $K$ , we take a finite degree extension  $L/K$ , and to complete the square we let  $B$  be the integral closure of  $A$  in  $L$ : the collection of elements in  $L$  satisfying monic polynomials with coefficients in  $A$ .

In our case, we’re additionally assuming that  $A/k$  is finitely generated as a  $k$ -algebra. ✍

**Remark 1.0.7:**

On (b):  $B$  being finitely generated as a  $k$ -algebra follows from assuming  $A$  is, and additionally that  $B$  is finitely generated as an  $A$ -module, and finite generation as a module provides finite generation as an algebra. The result follows from transitivity of finite generation of algebras.

On (c): This is just a property of integral extensions.

On (d): Use the characterization of being Noetherian, integrally closed, and Krull dimension 1. The only thing to check is that  $B$  is Noetherian, which follows from  $B$  being finitely generated as a  $k$ -algebra and applying the Hilbert basis theorem. ✍

**Remark 1.0.8:** Note that we are not assuming that  $L/K$  is separable, which is an assumption that would simplify things. ✍