

1 Notes

1.1 Definitions

In these notes, C generally denotes some closed contour, \mathbb{H} is the upper half-plane, C_R is a semicircle of radius R in \mathbb{H} , f will denote a complex function.

1. Analytic

f is analytic at z_0 if it can be expanded as a convergent power series in some neighborhood of z_0 .

2. Holomorphic

A function f is holomorphic at a point z_0 if $f'(z_0)$ exists in a neighborhood of z_0 .

(Note - this is more than just being differentiable at a single point!)

Big Theorem: f is a holomorphic complex function iff f is analytic.

3. Meromorphic

Holomorphic, except for possibly a finite number of singularities.

4. Conformal

f is conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$.

5. Harmonic

A function $u(x, y)$ is harmonic if it satisfies Laplace's equation,

$$\Delta u = u_{xx} + u_{yy} = 0$$

1.2 What is the Complex Derivative?

In small neighborhoods, the derivative of a function at a point rotates it by an angle $\Delta\theta$ and scales it by a real number λ according to

$$\Delta\theta = \arg f'(z_0), \quad \lambda = |f'(z_0)|$$

1.3 n th roots of a complex number

The n th roots of z_0 are given by writing $z_0 = re^{i\theta}$, and are

$$\zeta = \left\{ \sqrt[n]{r} \exp \left[i \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right] \mid k = 0, 1, 2, \dots, n-1 \right\}$$

or equivalently

$$\zeta = \left\{ \sqrt[n]{r} \omega_n^k \mid k = 0, 1, 2, \dots, n-1 \right\} \text{ where } \omega_n = e^{\frac{2\pi i}{n}}$$

This can be derived by looking at $(re^{i\theta+2k\pi})^{\frac{1}{n}}$.

It is also useful to immediately recognize that $z^2 + a = (z - i\sqrt{a})(z + i\sqrt{a})$.

1.4 The Cauchy-Riemann Equations

If $f(x + iy) = u(x, y) + iv(x, y)$ or $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$, then f is complex differentiable if u, v satisfy

$$\begin{aligned} u_x &= v_y & u_y &= -v_x \\ ru_r &= v_\theta & u_\theta &= -rv_r \end{aligned}$$

In this case,

$$f'(x + iy) = u_x(x, y) + iv_x(x, y)$$

or in polar coordinates,

$$f'(re^{i\theta}) = e^{i\theta}(u_r(r, \theta) + iv_r(r, \theta))$$

1.5 The Residue Theorem

If f is meromorphic inside of a closed contour C , then

$$\oint_C f(z)dz = 2\pi i \sum_{z=z_k} \text{Res } f(z)$$

where $\text{Res}_{z=z_k} f(z)$ is the coefficient of z^{-1} in the Laurent expansion of f .

If f is analytic everywhere in the interior of C , then $\oint_C f(z)dz = 0$.

If f is meromorphic inside of a contour C and analytic everywhere else, one can equivalently calculate the residue at infinity

$$\oint_C f(z)dz = 2\pi i \sum_{z_k} \text{Res } z^{-2}f(z^{-1})$$

1.6 Computing Residues

1.6.1 Simple Poles

If z_0 is a pole of order m , define $g(z) := (z - z_0)^m f(z)$.

If $g(z)$ is analytic and $g(z_0) \neq 0$, then

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

In the case where $m = 1$, this reduces to

$$\operatorname{Res}_{z=z_0} f(z) = \phi(z_0)$$

To compute residues this way, attempt to write f in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where ϕ only needs to be analytic at z_0 .

1.6.2 Rational Functions

If $f(z) = \frac{p(z)}{q(z)}$ where

1. $p(z_0) \neq 0$
2. $q(z_0) = 0$
3. $q'(z_0) \neq 0$

then the residue can be computed as

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

1.7 Computing Integrals

When computing real integrals, the following contours can be useful:

One often needs bounds, which can come from the following lemmas

The Arc Length Bound If $|f(z)| \leq M$ everywhere on C , then

$$\left| \oint_C f(z) dz \right| \leq M L_C$$

where L_C is the length of C .

Jordan's Lemma: If f is analytic outside of a semicircle C_R and $|f(z)| \leq M_R$ on C_R where $M_R \rightarrow 0$, then

$$\int_{C_R} f(z) e^{iaz} dz \rightarrow 0$$

Can also be used for integrals of the form $\int f(z) \cos az dz$ or $\int f(z) \sin az dz$, just take real/imaginary parts of e^{iaz} respectively.

1.8 Conformal Maps

1. Linear Fractional Transformations:

$$f(z) = \frac{az + b}{cz + d} \quad f^{-1}(z) = \frac{-dz + b}{cz - a}$$

2. $[z_1, z_2, z_3] \mapsto [w_1, w_2, w_3]$

Every LFT is determined by its action on three points. Given 3 pairs points $z_i \mapsto w_i$, construct an LFT using the implicit equation

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

3. $z^k : \text{Wedge} \mapsto \mathbb{H}$

Just multiplies the angle by k . If a wedge makes angle θ , use $z^{\frac{\pi}{\theta}}$.

It is useful to know that $z \mapsto z^2$ is equivalent to $(x, y) \mapsto (x^2 - y^2, 2xy)$.

4. $e^z : \mathbb{C} \mapsto \mathbb{C}$

Horizontal lines	\mapsto	rays from origin
Vertical lines	\mapsto	circles at origin
Rectangles	\mapsto	portions of wedges/sectors

the boundaries of the two regions are marked.

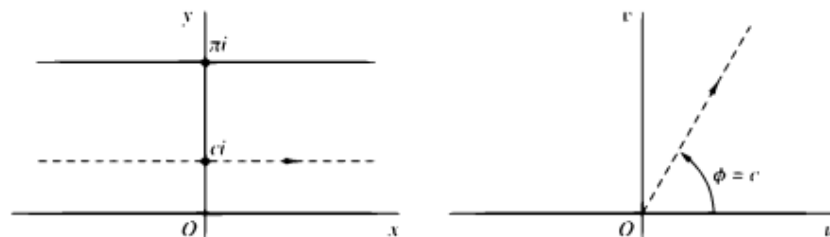


FIGURE 126

$w = \exp z$.

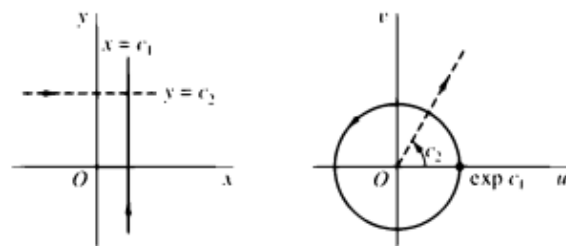


FIGURE 124

$w = \exp z$.

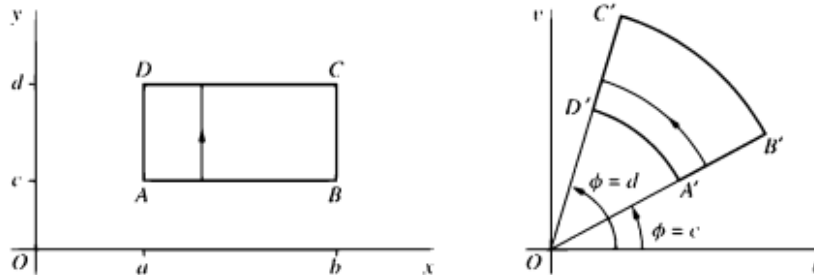


FIGURE 125

$w = \exp z$.

5. $\log : \mathbb{H} \mapsto \mathbb{R} + i[0, \pi]$

Just the inverse of what the exponential map does.

Rays \mapsto Horizontal Lines
Wedges \mapsto Horizontal Strips

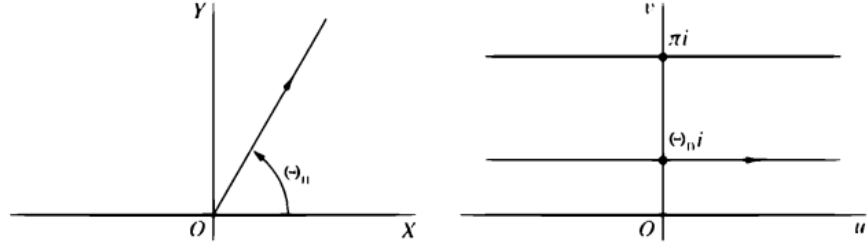


Figure 2: $z \mapsto \log z$

6. $\sin : [0, \pi/2] + i\mathbb{R} \mapsto \mathbb{H}_{\mathcal{R}(z) > 0}$

Maps the infinite strip to the first quadrant.

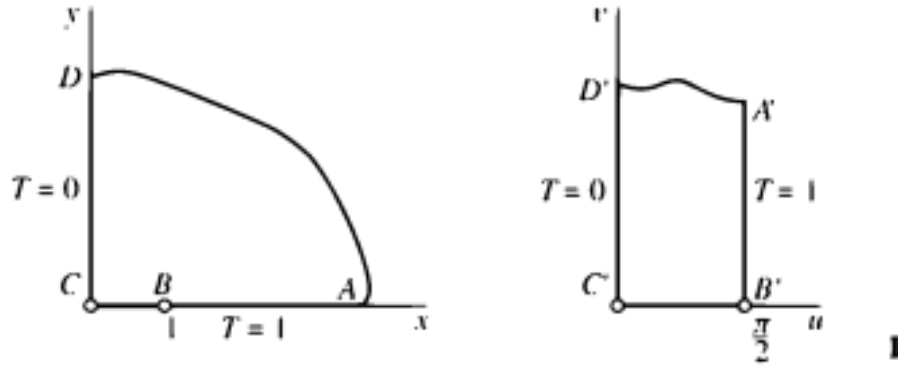


Figure 3: $z \leftarrow \sin w$

7. $z \mapsto \frac{i-z}{i+z} : \mathbb{H} \mapsto D^\circ$.

$\mathbb{R}_{>0} \mapsto$ Upper half of D°
 $\mathbb{R}_{<0} \mapsto$ Bottom half of D°

Has inverse $w \mapsto i \frac{1-w}{1+w}$

8. $z \mapsto z + z^{-1} : \partial D \mapsto \mathbb{R}$

The **lines of flow** are given by $S(x, y) = c$.

Any easy solution on the domain $\mathbb{R} \times i[0, \pi]$ in the u, v plane, where

$$\begin{aligned} T(x, 0) &= 0 \\ T(x, \pi) &= 1 \end{aligned}$$

is given by $T(u, v) = \frac{1}{\pi}v$.

It is harmonic, as the imaginary part of the analytic $F(u + iv) = \frac{1}{\pi}(u + iv)$, since every analytic function has harmonic component functions.

Similar methods work with different domains, just pick a smooth interpolation between the boundary conditions.

1.9.2 Fluid Flow

Write $F(z) = \phi(x, y) + i\psi(x, y)$. Then F is the complex potential of the flow, $\overline{F'}$ is the velocity, and setting $\psi(x, y) = c$ yields the streamlines.

A solution in \mathbb{H} is $F(z) = Az$ some some velocity A . Apply conformal mapping appropriately.

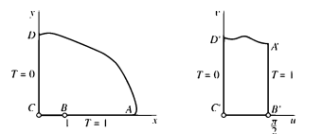


Figure 6: $T(u, v) = \frac{2}{\pi}u$

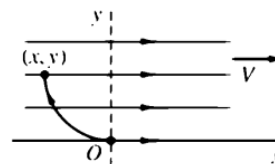


Figure 7: $T(u, v) = \frac{2}{\pi}u$

1.10 Theorems

1.10.1 General Theorems

1. Liouville's Theorem:

If f is entire and bounded on \mathbb{C} , then f is constant.

2. If f is continuous in a region D , f is bounded in D .
3. If f is differentiable at z_0 , f is continuous at z_0 .

Note - the converse need not hold!

4. If $f = u + iv$, where u, v satisfy the Cauchy-Riemann equations **and** have continuous partials, then f is differentiable.

Note - continuous partials are not enough, consider $f(z) = |z|^2$.

5. Rouché's Theorem

If $p(z) = f(z) + g(z)$ and $|g(z)| < |f(z)|$ everywhere on C , then f and p have the same number of zeros with C .

6. The Argument Principle

If f is analytic on a closed contour C and meromorphic within C , then

$$W := \frac{1}{2\pi} \Delta_C \arg f(z) = Z - P$$

Proof: Evaluate the integral $\oint_C \frac{f'(z)}{f(z)} dz$ first by parameterizing, changing to polar, and using the FTC, and second by using residues directly from the Laurent series.

7. The Main Story: The following are equivalent

- f is continuous
- f' exists
- f is analytic
- f is conformal
- f satisfies the Cauchy-Riemann equations

1.10.2 Theorems About Analytic Functions

1. If f is analytic on D , then $\oint_C f(z)dz = 0$ for any closed contour $C \subset D$.

Note: this does not require f to be f' to be continuous on C .

2. Maximum Modulus Principle

If f is analytic in a region D and not constant, then $|f(z)|$ attains its maximum on ∂D .

3. If f is analytic, then $f^{(n)}$ is analytic for every n . If $f = u(x, y) + iv(x, y)$, then all partials of u, v are continuous.
4. If f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0 .
5. If $f = u + iv$ is analytic, then u, v are harmonic conjugates.
6. If f is holomorphic, f is C_∞ (smooth).
7. If f is analytic, f is holomorphic.

Proof: Since f has a power series expansion at z_0 , its derivative is given by the term-by-term differentiation of this series.

1.11 Some Useful Formulae

$$f_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \dots$$

$$\frac{1}{1 - z} = \sum_k z^k$$

$$e^z = \sum_k \frac{1}{k!} z^k$$

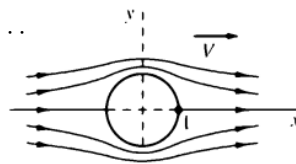


Figure 8: $T(u, v) = \frac{2}{\pi}u$

$$\left(\sum_i a_i z^i \right) \left(\sum_j b_j z^j \right) = \sum_n \left(\sum_{i+j=n} a_i b_j \right) z^n$$

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\
\cosh z &= \frac{1}{2}(e^z + e^{-z}) &= \cos iz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
\sinh z &= \frac{1}{2}(e^z - e^{-z}) &= -i \sin iz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots
\end{aligned}$$

Mnemonic: just remember that cosine is an even function, and that the even terms of e^z are kept. Similarly, sine is an odd function, so keep the odd terms of e^z .

Harmonic Conjugate

$$v(x, y) = \int_{(0,0)}^{(x,y)} -u_t(s, t)ds + u_s(s, t)dt$$

The Gamma Function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Useful to know: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

2 Question

1. True or False: If f is analytic and bounded in \mathbb{H} , then f is constant on \mathbb{H} .

False: Take $f(z) = e^{-z}$, where $|f(z)| \leq 1$ in \mathbb{H} .

2. Compute $\int_{-\infty}^\infty \frac{\sin x}{x(x^2+a^2)} dx$

Two semicircles needed to avoid singularity at zero. Limit equals the residue at zero, solution is $\pi(\frac{1}{a^2} - \frac{e^{-a}}{a^2})$.

3. Compute $\int_0^{2\pi} \frac{1}{2+\cos \theta} d\theta$

Cosine sub, solution is $\frac{2\pi}{\sqrt{3}}$

4. Find the first three terms of the Laurent expansion of $\frac{e^z+1}{e^z-1}$.

Equals $2z^{-1} + 0 + 6^{-1}z + \dots$

5. Compute $\int_{S_1} \frac{1}{z^2+z-1} dz$

Equals $i\frac{2\pi}{5}$

6. True or false: If f is analytic on the unit disk $E = \{z : |z| < 1\}$, then there exists an $a \in E$ such that $|f(a)| \geq |f(0)|$.

True, by the maximum modulus principal. Suppose otherwise. Then $f(0)$ is a maximum of f inside S_1 . But by the MMP, f must attain its maximum on ∂S_1 .

7. Prove that if $f(z)$ and $\overline{f(z)}$ are both analytic on a domain D , then f is constant on D

Analytic \Rightarrow Cauchy-Riemann equations are satisfied. Also have the identity $f' = u_x + iv_x$, and $\overline{f}' = 0 \Rightarrow f$ is constant.

8.

9.

10.