Homotopy Groups of Spheres

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Introduction

Spheres

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Graduate Student Seminar

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Outline

Homotopy Groups of Spheres

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- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

Intuition

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Homotopies of paths:



– Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \mathsf{hom}_{\mathsf{Top}}(I, X)$$
 $H \in \mathsf{hom}_{\mathsf{Top}}(I \times I, X).$

- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

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– Why care about path homotopies? Historically: contour integrals in $\ensuremath{\mathbb{C}}$



– By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_{\gamma} f(z) \ dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

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Generalize to a homotopy of morphisms:

$$f, g \in \mathsf{hom}_{\mathsf{Top}}(X, Y) \quad f \sim g \iff \exists F \in \mathsf{hom}_{\mathsf{Top}}(X \times I, Y)$$

- such that F(0) = f, F(1) = g.
- This yields an equivalence relation on morphisms, homotopy classes of maps

$$[X, Y] := \mathsf{hom}_{\mathsf{Top}}(X, Y) / \sim$$

Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \mathsf{hom}(X,Y) \\ g \in \mathsf{hom}(Y,X) \end{cases}$$
 such that $\begin{cases} f \circ g \sim \mathsf{id}_Y \\ g \circ f \sim \mathsf{id}_X \end{cases}$

Similarly write

$$[X] = \{ Y \in \mathsf{Top} \mid Y \sim X \}.$$

The Fundamental Group

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IIItroductioi

- $-\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

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Can now generalize to define

$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look \mathbb{A}^1 or \mathbb{P}^1 .

Classification

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- Holy grail: understand the topological category completely
 - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category DTop, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a homotopy type.

Example: Homotopy Equivalence is Useful

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Proposition: Let B be a CW complex; then isomorphism classes of \mathbb{R}^1 -bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G, the functor

$$h_G(\,\cdot\,):\mathsf{hoTop^{op}}\longrightarrow\mathsf{Set}$$

$$X\mapsto\{G\mathsf{-bundles\ over\ }X\}$$

is representable by a space called BG (Brown's representability theorem).

- I.e., let $Bun_G(X) = \{G-bundles/B\} / \sim$, there is an isomorphism

$$\operatorname{Bun}_G(X) \cong [X, BG]$$

- In general, identify $G = \operatorname{Aut}(F)$ the automorphism group of the fibers - for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

Example: Homotopy Equivalence is Useful

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Introduction Spheres Note that for a poset of spaces (M_i, \hookrightarrow) , the space $M^{\infty} := \varinjlim M_i$. These are infinite dimensional "Hilbert manifolds".

Proof:

$$\mathsf{Bun}_{\mathbb{R}^1}(X) = [X, B\mathrm{GL}(1, \mathbb{R})]$$

$$= [X, \mathsf{Gr}(1, \mathbb{R}^{\infty})]$$

$$= [X, \mathbb{RP}^{\infty}]$$

$$= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)]$$

$$= H^1(X; \mathbb{Z}/2\mathbb{Z})$$

Work being swept under the rug: identifying the homotopy type of the representing object.

Example: Homotopy Equivalence is Useful

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Introduction Spheres **Corollary:** There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.



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Setup

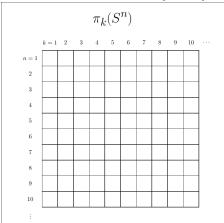
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- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate: $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:



Sphere 1

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