Title

Contents

0.1 Big Theorems / Tools:

Proposition 0.1.1 ***(Fundamental Theorem of Calculus I).***

$$\frac{\partial}{\partial x}\int_a^x f(t)dt = f(x)$$

Proposition 0.1.2 ***(Generalized Fundamental Theorem of Calculus).***

$$\begin{aligned}\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt &= f(x,t)\cdot\frac{\partial}{\partial x}(t)\Big|_{t=a(x)}^{t=b(x)} \\ &= f(x,b(x))\cdot b'(x) - f(x,a(x))\cdot a'(x)\end{aligned}$$

If $f(x,t) = f(t)$ doesn't depend on x, then $\dfrac{\partial f}{\partial x} = 0$ and the second integral vanishes:

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

Find examples.

Remark 0.1.1.

Note that you can recover the original FTC by taking

$$a(x) = c$$
$$b(x) = x$$
$$f(x,t) = f(t).$$

Corollary 0.1.1 ***(?).***

$$\frac{\partial}{\partial x} \int_1^x f(x,t)dt = \int_1^x \frac{\partial}{\partial x} f(x,t)dt + f(x,x)$$

Proposition 0.1.3 ***(Extreme Value Theorem).***
Todo

\todo[inline]{Todo}

Proposition 0.1.4 ***(Mean Value Theorem).***

$$\begin{aligned} f \in C^0(I) &\implies \exists p \in I : f(b) - f(a) = f'(p)(b-a) \\ &\implies \exists p \in I : \int_a^b f(x)\ dx = f(p)(b-a). \end{aligned}$$

Proposition 0.1.5 ***(Rolle's Theorem).***

todo

Proposition 0.1.6 ***(L'Hopital's Rule).***
If

- $f(x)$ and $g(x)$ are differentiable on $I - \{\text{pt}\}$, and

$$\lim_{x \to \{\text{pt}\}} f(x) = \lim_{x \to \{\text{pt}\}} g(x) \in \{0, \pm\infty\}, \qquad \forall x \in I, g'(x) \neq 0, \qquad \lim_{x \to \{\text{pt}\}} \frac{f'(x)}{g'(x)} \text{ exists},$$

Then it is necessarily the case that

$$\lim_{x \to \{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x \to \{\text{pt}\}} \frac{f'(x)}{g'(x)}.$$

Remark 0.1.2.
Note that this includes the following indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty, \quad \infty - \infty.$$

For $0 \cdot \infty$, can rewrite as $\frac{0}{\frac{1}{\infty}} = \frac{0}{0}$, or alternatively $\frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$.

For $1^\infty, \infty^0$, and 0^0, set

$$L := \lim f^g \implies \ln L = \lim g \ln(f)$$

to recover $\infty \cdot 0, 0 \cdot \infty$, or $0 \cdot 0$.

Proposition 0.1.7 ***(Taylor Expansion).***

$$\begin{aligned} T(a,x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &\quad + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{24}f^{(4)}(a)(x-a)^4 + \cdots \end{aligned}$$

There is a bound on the error:

$$|f(x) - T_k(a,x)| \leq \left| \frac{f^{(k+1)}(a)}{(k+1)!} \right|$$

where $T_k(a,x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is the kth truncation.

0.2 Differential

0.2.1 Limits

0.2.2 Tools for finding limits

How to find $\lim_{x\to a} f(x)$in order of difficulty:

- Plug in: if f is continuous, $\lim_{x\to a} f(x) = f(a)$.
- Check for indeterminate forms and apply L'Hopital's Rule.
- Algebraic rules
- Squeeze theorem
- Expand in Taylor series at a
- Monotonic + bounded
- One-sided limits: $\lim_{x\to a^-} f(x) = \lim_{\varepsilon\to 0} f(a-\varepsilon)$

- Limits at zero or infinity:

$$\lim_{x\to\infty} f(x) = \lim_{\frac{1}{x}\to 0} f\left(\frac{1}{x}\right) \text{ and } \lim_{x\to 0} f(x) = \lim_{x\to\infty} f\left(\frac{1}{x}\right)$$

  - Also useful: if $p(x) = p_n x^n + \cdots$ and $q(x) = q_n x^m + \cdots$,

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \deg p < \deg q \\ \infty & \deg p > \deg q \\ \frac{p_n}{q_n} & \deg p = \deg q \end{cases}$$

Warning 0.1: Be careful: limits may not exist!! Example : $\lim_{x\to 0} \frac{1}{x} \neq 0$.

0.2.3 Asymptotes

- Vertical asymptotes: at values $x = p$ where $\lim_{x\to p} = \pm\infty$
- Horizontal asymptotes: given by points $y = L$ where $L \lim_{x\to\pm\infty} f(x) < \infty$
- Oblique asymptotes: for rational functions, divide - terms without denominators yield equation of asymptote (i.e. look at the asymptotic order or "limiting behavior").
  - Concretely:

$$f(x) = \frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{t(x)} \sim r(x)$$

0.2.4 Recurrences

- Limit of a recurrence: $x_n = f(x_{n-1}, x_{n-2}, \cdots)$
  - If the limit exists, it is a solution to $x = f(x)$

0.2.5 Derivatives

- Chain rule: $\frac{\partial}{\partial x}(f \circ g)(x) = f'(g(x))g'(x)$
- Product rule: $\frac{\partial x}{\partial f}(x)g(x) = f'g + g'f$
  - Note for all rules: always prime the first thing!
- Quotient rule: $\frac{\partial}{\partial x}\frac{f(x)}{g(x)} = \frac{f'g - g'f}{g^2}$
  - Mnemonic: Low d-high minus high d-low
- Inverse rule: $\frac{\partial f^{-1}}{\partial x}(f(x_0)) = \left(\frac{\partial f}{\partial x}\right)^{-1}(x_0) = 1/f'(x_0)$
- Implicit differentiation: $(f(x))' = f'(x)\ dx, (f(y))' = f'(y)\ dy$
  - Often able to solve for $\frac{\partial[}{\partial y}]x$ this way.
- Obtaining derivatives of inverse functions: if $y = f^{-1}(x)$ then write $f(y) = x$ and implicitly differentiate.
- Approximating change: $\Delta y \approx f'(x)\Delta x$

0.2.6 Related Rates

General series of steps: want to know some unknown rate y_t

- Lay out known relation that involves y
- Take derivative implicitly (say w.r.t t) to obtain a relation between y_t and other stuff.
- Isolate $y_t =$ known stuff
- Example: ladder sliding down wall
  - Setup: l, x_t and $x(t)$ are known for a given t, want y_t.
  - $x(t)^2 + y(t)^2 = l^2 \implies 2xx_t + 2yy_t = 2ll_t = 0$ (noting that l is constant)
  - So $y_t = -\frac{x(t)}{y(t)} x_t$
  - $x(t)$ is known, so obtain $y(t) = \sqrt{l^2 - x(t)^2}$ and solve.

0.3 Integral

- Average values:

$$f_{\text{avg}}(x) = \frac{1}{b-a} \int_a^b f(t) dt$$

  - Proof: apply MVT to $F(x)$.
- Area Between Curves
  - Area in polar coordinates:

$$A = \int_{r_1}^{r_2} \frac{1}{2} r^2(\theta)\ d\theta$$

- Solids of Revolution
  - Disks: $A = \int \pi r(t)^2\ dt$
  - Cylinders: $A = \int 2\pi r(t) h(t)\ dt$
- Arc lengths

$$ds = \sqrt{dx^2 + dy^2} \qquad L = \int\ ds$$
$$= \int_{x_0}^{x_1} \sqrt{1 + \frac{\partial y}{\partial x}}\ dx$$
$$= \int_{y_0}^{y_1} \sqrt{\frac{\partial x}{\partial y} + 1}\ dy$$

  - $SA = \int 2\pi r(x)\ ds$
- Center of Mass
  - Given a density $\rho(\mathbf{x})$ of an object R, the x_i coordinate is given by

$$x_i = \frac{\int_R x_i \rho(x)\ dx}{\int_R \rho(x)\ dx}$$

0.3.1 Big List of Integration Techniques

Given $f(x)$, we want to find an antiderivative $F(x) = \int f$ satisfying $\frac{\partial}{\partial x}F(x) = f(x)$

- Guess and check: look for a function that differentiates to f.
- u- substitution
  - More generally, any change of variables

$$x = g(u) \implies \int_a^b f(x)\ dx = \int_{g^{-1}(a)}^{g^{-1}(b)} (f \circ g)(x)\ g'(x)\ dx$$

- Integration by Parts:
  - The standard form:

$$\int udv = uv - \int vdu$$

  - A more general form for repeated applications: let $v^{-1} = \int v$, $v^{-2} = \int\int v$, etc.

$$\begin{aligned}
\int_a^b uv &= uv^{-1}\Big|_a^b - \int_a^b u^1 v^{-1} \\
&= uv^{-1} - u^1 v^{-2}\Big|_a^b + \int_a^b u^2 v^{-2} \\
&= uv^{-1} - u^1 v^{-2} + u^2 v^{-3}\Big|_a^b - \int_a^b u^3 v^{-3} \\
&\vdots \\
\implies \int_a^b uv &= \sum_{k=1}^{n} (-1)^k u^{k-1} v^{-k}\Big|_a^b + (-1)^n \int_a^b u^n v^{-n}
\end{aligned}$$

  - Generally useful when one term's nth derivative is a constant.
  - Shoelace method:
  - Note: you can choose u or v equal to 1! Useful if you know the derivative of the integrand.
- Differentiating under the integral

$$\begin{aligned}
\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt &= f(x,\cdot)\frac{\partial}{\partial x}(\cdot)\Big|_{a(x)}^{b(x)} \\
&= f(x, b(x))\ b'(x) - f(x, a(x))\ a'(x)
\end{aligned}$$

  - Proof: let $F(x)$ be an antiderivative and compute $F'(x)$ using the chain rule.
  - #todo for constants, this should allow differentiating under the integral when f, f_x are "jointly continuous"

  - LIPET: Log, Inverse trig, Polynomial, Exponeitial, Trig: generally let u be whichever one comes first.
- The ridiculous trig sub: for any integrand containing only trig terms
  - Transforms *any* such integrand into a rational function of x
  - Let $u = 2\tan^{-1} x,\ du = \dfrac{2}{x^2+1}$, then

$$\int_a^b f(x)\ dx = \int_{\tan\frac{a}{2}}^{\tan\frac{b}{2}} f(u)\ du$$

♢ Example: $\displaystyle\int_0^{\pi/2} \frac{1}{\sin\theta}\ d\theta = 1/2$

Derivatives	Integrals	Signs	Result
u	v	NA	NA
u'	$\int v$	$+$	$u\int v$
u''	$\int\int v$	$-$	$-u'\int\int v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fill out until one column is zero (alternate signs). Get the result column by multiplying diagonally, then sum down the column.

- Trigonometric Substitution

$$\begin{aligned} \sqrt{a^2-x^2} &\Rightarrow & x &= a\sin(\theta) & dx &= a\cos(\theta)\ d\theta \\ \sqrt{a^2+x^2} &\Rightarrow & x &= a\tan(\theta) & dx &= a\sec^2(\theta)\ d\theta \\ \sqrt{x^2-a^2} &\Rightarrow & x &= a\sec(\theta) & dx &= a\sec(\theta)\tan(\theta)\ d\theta \end{aligned}$$

- Partial Fractions
- Completing the Square #todo
- Trig Formulas
  - Double angle formulas:

$$\begin{aligned} \sin^2(x) &= \frac{1}{2}(1-2\cos x) \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

- Products of trig functions
  - Setup: $\displaystyle\int \sin^a(x)\cos^b(x)\ dx$

- ♢ Both a, b even: $\sin(x)\cos(x) = \dfrac{1}{2}\sin(x)$
- ♢ a odd: $\sin^2 = 1 - \cos^2,\ u = \cos(x)$
- ♢ b odd: $\cos^2 = 1 - \sin^2,\ u = \sin(x)$

– Setup: $\displaystyle\int \tan^a(x)\sec^b(x)\ dx$

- ♢ a odd: $\tan^2 = \sec^2 - 1,\ u = \sec(x)$
- ♢ b even: $\sec^2 = \tan^2 - 1, u = \tan(x)$

Other small but useful facts:

$$\int_0^{2\pi} \sin\theta\ d\theta = \int_0^{2\pi} \cos\theta\ d\theta = 0.$$

0.3.2 Optimization

- Critical points: boundary points and wherever $f'(x) = 0$
- Second derivative test:
  - $f''(p) > 0 \implies p$ is a min
  - $f''(p) < 0 \implies p$ is a max
- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.
- Fundamental theorem of Calculus: If

$$\int f(x)dx = F(b) - F(a) \implies F'(x) = f(x).$$

- Min/maxing - either derivatives of Lagranage multipliers!
- Distance from origin to plane: equation of a plane

$$P : ax + by + cz = d.$$

  - You can always just read off the normal vector $\mathbf{n} = (a, b, c)$. So we have $\mathbf{nx} = d$.
  - Since $\lambda\mathbf{n}$ is normal to P for all λ, solve $\mathbf{n}\lambda\mathbf{n} = d$, which is $\lambda = \dfrac{d}{\|\mathbf{n}\|^2}$
- A plane can be constructed from a point p and a normal n by the equation $np = 0$.
- In a sine wave $f(x) = \sin(\omega x)$, the period is given by $2\pi/\omega$. If $\omega > 1$, then the wave makes exactly ω full oscillations in the interval $[0, 2\pi]$.
- The directional derivative is the gradient dotted against a *unit vector* in the direction of interest

- Related rates problems can often be solved via implicit differentiation of some constraint function
- The second derivative of a parametric equation is not exactly what you'd intuitively think!
- For the love of god, remember the FTC!

$$\frac{\partial}{\partial x} \int_0^x f(y)dy = f(x)$$

- Technique for asymptotic inequalities: WTS $f < g$, so show $f(x_0) < g(x_0)$ at a point and then show $\forall x > x_0, f'(x) < g'(x)$. Good for big-O style problems too.
- Inflection points of h occur where the *tangent* of h' changes sign. (Note that this is where h' itself changes sign.)
- Inverse function theorem: The slope of the inverse is reciprocal of the original slope
- If two equations are equal at exactly one real point, they are tangent to each other there - therefore their derivatives are equal. Find the x that satisfies this; it can be used in the original equation.
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