

# Real Analysis

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See Folland's Real Analysis, definitely a recommended reference.

Possible first day question: how can we “measure” a subset of  $\mathbb{R}$ ? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of  $\mathbb{R}$ . We'll come back to this in a few lectures.

#### Notions of “smallness” in $\mathbb{R}$

Definition: Let  $E$  be a set, then  $E$  is *countable* if it is in a one-to-one correspondence with  $E' \subseteq \mathbb{N}$ , which includes  $\emptyset, \mathbb{N}$ .

Definition:  $E$  is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of  $\mathbb{R}$  is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a  $X \subseteq Y$  is dense in  $Y$  iff the closure of  $X$  is all of  $Y$ . So we'll make the following definition.

Definition: A set  $A \subseteq \mathbb{R}$  is *nowhere dense* if every interval  $I$  contains a subinterval  $S \subseteq I$  such that  $S \subseteq A^c$ .

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. Example:  $\mathbb{Q}$  is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, -  $A^c$  contains a dense, open set. - The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than  $\varepsilon$ .

Definition:  $E$  is a *null set* (or has *measure zero*) if  $\forall \varepsilon > 0$ , there exists a sequence of intervals  $\{I_j\}_{j=1}^{\infty}$  such that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable  $\implies$  Meager, but not the converse.
- Countable  $\implies$  Null, but not the converse.

Exercise: Show that the “middle third” Cantor set is not countable, but is both null and meager.

Key point: the Cantor set does not contain any intervals.

Theorem: Every  $E \subseteq \mathbb{R}$  can be written as  $E = A \sqcup B$  where  $A$  is null and  $B$  is meager.

This gives some information about how nullity and meagerness interact – in particular,  $\mathbb{R}$  itself is neither meager nor null. Idea: if meager  $\implies$  null, this theorem allows you to write  $\mathbb{R}$  as the union of two null sets. This is bad!