Title

D. Zack Garza

November 26, 2019

Contents

1 Tuesday November 26th

Question: Let $f \in L^1([a,b])$ and $F(x) = \int_a^x f(y)\, dy$ – is F differentiable a.e. and $F' = f$?

If f is continuous, then absolutely yes.

Otherwise, we are considering

$$\frac{f(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(y)\, dy \to_? f(x)$$

so the more general question is

$$\lim_{m(I) \to 0, x \in I} \frac{1}{m(I)} \int_I f(y)\, dy =_? f(x)\ a.e.$$

Note that if f is continuous, since $[a,b]$ is compact, we have uniform continuity and $\frac{1}{m(I)} \int_I (f(y) - f(x))\, dy < \frac{1}{m(I)} \int_I \varepsilon$.

1.1 Lebesgue Differentiation Theorem

Theorem: If $f \in L^1(\mathbb{R}^n)$ then

$$\lim_{m(B) \to 0, x \in B} \int \frac{1}{m(B)} \int_B f(y)\, dy = f(x)\ a.e.$$

> Note: although it's not obvious at first glance, this really is a theorem about differentiation.

Corollary (Lebesgue Density Theorem): For any measurable set $E \subseteq \mathbb{R}^n$, we have

$$\lim_{r \to 0} \frac{m(E \bigcap B_r(x))}{m(B_r(x))} = 1\ a.e.$$

Proof: Let $f = \chi_E$ in the theorem.

Proof of theorem: We want to show

$$Df(x) \coloneqq \limsup_{m(B)\to 0, x\in B} \left| \frac{1}{m(B)} \int_B (f(y) - f(x))\ dy \right| \to 0$$