

Category \mathcal{O} , Problem Set 3

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February 23, 2020

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1 Humphreys 1.10

Prove that the transpose map τ fixes $Z(\mathfrak{g})$ pointwise.

Check that τ commutes with the Harish-Chandra morphism ξ and use the fact that ξ is injective.

1.1 Solution

We first note that after choosing a PBW basis for \mathfrak{g} , τ is defined on \mathfrak{g} in the following way:

$$\begin{aligned}\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha\end{aligned}$$

which lifts to an anti-involution $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by extending linearly over PBW monomials. We can note that since τ fixes \mathfrak{h} pointwise by definition, its lift also fixes $U(\mathfrak{h})$ pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned} \xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}. \end{aligned}$$

Proposition 1.1.

The following diagram commutes

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \\ \downarrow \tau & & \downarrow \tau \\ Z(\mathfrak{g}) & \xrightarrow{\xi} & U(\mathfrak{h}) \end{array}$$

Proof.

We will show that for all $z \in Z(\mathfrak{g})$, $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$. Expand z in a PBW basis as $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k}$. We then make the following computations:

$$\begin{aligned} (\xi \circ \tau)(z) &= (\xi \circ \tau) \left(\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} \right) \\ &= \xi \left(\prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_k} \right) \quad \text{since } \tau \text{ is an anti-homomorphism} \\ &= \prod_j h_j^{s_j} \end{aligned}$$

Similarly, we have

$$\begin{aligned} (\tau \circ \xi)(z) &= \tau \left(\prod_j h_j^{s_j} \right) \\ &= \prod_j h_j^{s_j} \end{aligned}$$

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of ξ , we have

$$\begin{aligned} (\xi \circ \tau)(z) &= \xi(z) \\ \implies \tau(z) &= z. \end{aligned}$$

■

2 Humphreys 1.12

Fix a central character χ and let $\{V^{(\lambda)}\}$ be a collection of modules in \mathcal{O} indexed by the weights λ for which $\chi = \chi_\lambda$ satisfying

1. $\dim V^{(\lambda)} = 1$
2. $\mu < \lambda$ for all weights μ of $V^{(\lambda)}$.

Then the symbols $[V^{(\lambda)}]$ form a \mathbb{Z} -basis for the Grothendieck group $K(\mathcal{O}_\chi)$.

For example take $V^{(\lambda)} = M(\lambda)$ or $L(\lambda)$.

2.1 Solution

Following a similar proof outlined here.

Fix a λ_0 such that $\chi = \chi_{\lambda_0}$ by Harish-Chandra's theorem, fix some order on the Weyl group $W = \{w_j \mid 1 \leq j \leq |W| < \infty\}$, and note that $\chi_{\lambda_0} = \chi_{w \cdot \lambda_0}$ for each $w \in W$.

Proposition 2.1.

The simple modules $\{L(w \cdot \lambda_0) \mid w \in W\}$ form a \mathbb{Z} -basis for \mathcal{O}_χ .

Proof.

Write $\mathcal{L} = \text{span}_{\mathbb{Z}} \{[L(w_j \cdot \lambda_0)] \mid 1 \leq j \leq |W|\} \subset K(\mathcal{O}_\chi)$.

Spanning: Let $M \in \mathcal{O}_\chi$ be arbitrary, and consider $[M] \in K(\mathcal{O}_\chi)$. By Humphreys Theorem 1.11, M has a finite composition series

$$M = M_1 > M_2 > \cdots > M_n$$

with simple quotients $M^{i+1}/M^i \cong L(\lambda_i)$ for some $\lambda_i \in \mathfrak{h}^\vee$. By collecting terms, we can write

$$[M] = \sum_{i=1}^n [L(\lambda_i)] = \sum_{i=1}^{n'} c_i [L(\lambda_i)] \in K(\mathcal{O}_\chi),$$

where each c_i is the multiplicity of $L(\lambda_i)$ in the above composition series.

By definition, $M \in \mathcal{O}_\chi \iff L(\lambda_i) \in \mathcal{O}_\chi$, i.e. M is in this block precisely when all of its composition factors are. But this forces each $L(\lambda_i) = L(w_j \cdot \lambda_0)$ for some j , and so we have

$$[M] = \sum_{i=j}^{n'} c_j [L(w_j \cdot \lambda_0)] \in \mathcal{L}.$$

Linear Independence: Define a family of maps

$$r_j : \mathcal{O}_\chi \longrightarrow \mathbb{Z}^{\geq 0}$$

$$M \mapsto \left| \left\{ M^{i+1}/M^i \mid M^{i+1}/M^i \cong L(w_j \cdot \lambda_0) \right\} \right|,$$

i.e. the map that counts the multiplicity of $L(w_j \cdot \lambda_0)$ appearing in any composition series of M for a fixed j .

This lifts to a group morphism $r_j : K(\mathcal{O}_\chi) \longrightarrow \mathbb{Z}^{\geq 0}$ which satisfies

$$r_j(L(w_i \cdot \lambda_0)) = \delta_{ij},$$

i.e. it takes the value 1 on the Verma modules in \mathcal{L} precisely when $i = j$ and zero otherwise.

Now suppose $\sum_{i=1}^n a_i [L(w_i \cdot \lambda_0)] = [0]$ in $K(\mathcal{O}_\chi)$. For each fixed j , we can then apply the above group morphism to obtain

$$\begin{aligned} r_j \left(\sum_{i=1}^n a_i [L(w_i \cdot \lambda_0)] \right) &= \sum_{i=1}^n a_i r_j([L(w_i \cdot \lambda_0)]) \\ &= \sum_{i=1}^n a_i r_j \delta_{ij} \\ &= a_j. \end{aligned}$$

Since group morphisms preserve equalities and $r_j([0]) = 0 \in \mathbb{Z}$, this forces $a_j = 0$ for each j . ■

Proposition 2.2.

An arbitrary set of the stated form $\mathcal{V} = \{V^{(\lambda_i)} \mid 1 \leq i < N < \infty\}$ is also a \mathbb{Z} -basis of $K(\mathcal{O}_\chi)$.

Proof.

We first note that we can similarly write $V^{(\lambda_i)} = V^{(w_j \cdot \lambda_0)}$ for some j , so wlog we reindex the λ_i to λ_j s. Similarly, fixing a V^{λ_j} , for $\mu < \lambda_j$, there is an i such that $\mu = w_i \cdot \lambda_0$, so we reindex all lower weights accordingly as well.

By the previous proposition, for each fixed $V^{(\lambda_i)}$, we can write

$$[V^{(\lambda_j)}] = [L(w_j \cdot \lambda_0) + \sum_{\mu_i < \lambda_j} a_{ij} [L(w_i \cdot \lambda_0)]].$$

The matrix $A = (a_{ij})$ is then strictly upper-triangular with ones on the diagonal, and is thus invertible, and so expresses a change of basis matrix $\mathcal{L} \rightarrow \mathcal{V}$. ■

3 Humphreys 1.13

Suppose $\lambda \notin \Lambda$, so the linkage class $W \cdot \lambda$ is the disjoint union of its nonempty intersections of various cosets of $\Lambda_r \in \mathfrak{h}^\vee$.

Prove that each $M \in \mathcal{O}_{\chi_\lambda}$ has a corresponding direct sum decomposition $M = \bigoplus M_i$ in which all weights of M_i lie in a single coset.

Recall exercise 1.1b.

3.1 Solution

Fix a nonintegral $\lambda \in \mathfrak{h}^\vee \setminus \Lambda$ and $M \in \mathcal{O}_{\chi_\lambda}$, and write

$$\mathfrak{h}^\vee / \Lambda = \{ \lambda_i + \Lambda \mid i \in I \} = \{ [\lambda_i] \mid i \in I \}$$

for some indexing set I . As in exercise 1.1, for each i we can define

$$M_i = M^{[\lambda_i]} := \sum_{\mu \in [\lambda_i]} M_\mu,$$

the sum of weight spaces M_μ for which $\mu \in [\lambda_i]$. Note that by construction, all of the weights of M_i lie in the single coset $[\lambda_i]$.

By the result of that exercise, $M = \bigoplus_{i=1}^N M^{[\lambda_i]}$ is a finite direct sum of such modules.

Let $W \cdot \lambda$ be the orbit of λ under the action of W , i.e. the linkage class of λ . When $\lambda \in \Lambda$ is integral, then $W \cdot \lambda$ intersects only one coset $[\lambda_i]$, and there is only one term in the above sum.