

Title

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Remark 1.

There is a natural action of $\text{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\text{MCG}(\Sigma)$:

$$\begin{aligned}\rho : \text{MCG}(\Sigma) &\rightarrow \text{Aut}_{\text{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_*.\end{aligned}$$

Definition 1.0.1 (Special Linear Group).

$$\text{SL}(n, \mathbb{k}) = \left\{ M \in \text{GL}(n, \mathbb{k}) \mid \det M = 1 \right\} = \ker \det_{\mathbb{G}_m}.$$

Remark 2.

$$\text{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that $S^2 = 1$ and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Moreover, if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $A \in \text{SL}(2, \mathbb{Z})$, we have $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. this preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \mid p, q \in \mathbb{Z} \right\} \cong \left\{ p\omega_1 + q\omega_2 \mid p, q \in \mathbb{Z} \right\} \simeq \left\{ p' + q'\tau \mid p', q' \in \mathbb{Z} \right\}.$$

where the ω_i, τ come from identifying \mathbb{R}^2 with \mathbb{C} , and in the last step we've rescaled the lattice by *homothety* to align one vector with the x -axis.

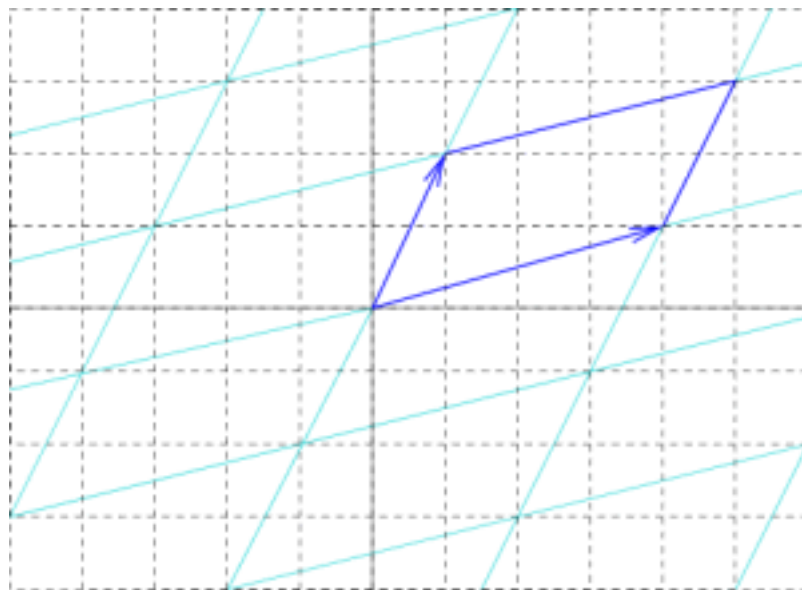


Figure 1: Lattice

Remark 3.

For any finite-index subgroup $G \leq \mathrm{SL}(2, \mathbb{Z})$, the orbits/left-quotient $G \backslash \mathbb{H}$ yields a complex curve (i.e. a torus).

Theorem 1.1 (Mapping Class Group of the Torus).

The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

Proof .

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Map}(X, X), \circ) &\rightarrow (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_* \end{aligned}$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff $\mathrm{Map}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$
 - This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned}\tilde{\sigma} : \text{MCG}(X) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: $\text{im}(\tilde{\sigma}) \subseteq \text{SL}(2, \mathbb{Z})$.

Proof.
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- Algebraic intersection numbers in Σ_2 correspond to determinants
- $f \in \text{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2

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- We can thus freely restrict the codomain to define the map

$$\begin{aligned}\sigma : \text{MCG}(X) &\rightarrow \text{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: σ is surjective.

- \mathbb{R}^2 is the universal cover of Σ_2 , with deck transformation group \mathbb{Z}^2 .
- Any $A \in \text{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \text{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

Claim: \tilde{A} is equivariant wrt \mathbb{Z}^2

Proof.
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- So \tilde{A} descends to a well-defined map $\psi_{\tilde{A}}$ on $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$, which is still a linear self-homeomorphism
- There is a correspondence

$$\{\text{Primitive vectors in } \mathbb{Z}^2\} \iff \left\{ \begin{array}{c} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{array} \right\} / \text{homotopy}.$$

- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

Claim: σ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.
- Thus there is a correspondence (Ref 91, Prop 1B.9)

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } \Sigma_2 \hookrightarrow \end{array} \right\} \iff \{\text{Group morphisms } \mathbb{Z}^2 \hookrightarrow\}.$$

- Claim: any element $f \in \text{MCG}(\Sigma_2)$ has a representative φ which fixes any given basepoint
- So if $f \in \ker \sigma$, then $f \simeq \varphi \simeq \text{id}$ are homotopic, so $\ker \sigma = 1$.

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