

Moduli Spaces

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1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
 - Slightly more detailed: Nitsure, . . . Hilbert schemes, Fundamentals of Algebraic Geometry
 - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

1.1 Representability

Last time: Fix an S -scheme, i.e. a scheme over S .

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \mathrm{hom}_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form} \end{aligned}$$

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & X \\ & \searrow & \nearrow \\ & T & \end{array} .$$

Theorem 1.1 (Yoneda).

$$\mathrm{hom}_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 1.2.

$$\mathrm{hom}_{\mathrm{Sch}/S}(x, y) \cong \mathrm{hom}_{\mathrm{Fun}}(h_x, h_y).$$

Definition 1.2.1 (Moduli Functor).

A **moduli functor** is a map

$$\begin{aligned} F : (\text{Sch}/S)^{\text{op}} &\longrightarrow \text{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 1.2.2 (Moduli Space).

A **moduli space** for that “something” appearing above is an $M \in \text{Obj}(\text{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \text{Spec}(k)$.

h_m is the functor of points over M .

Remark (1) $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{“families over Spec } k\text{”} = F(\text{Spec } k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M , and $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{U_{\text{niv}}}$ is the universal family.

Every family is uniquely the pullback of $\xi_{U_{\text{niv}}}$. This makes it much like a classifying space.

For $T \in \text{Sch}/S$,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{U_{\text{niv}}}). \end{aligned}$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\text{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\xi_M \qquad \qquad \xi_{M'}$$

$$M \xrightarrow{f} M'$$

$$M' \xrightarrow{g} M$$

$$\xi_{M'} \qquad \qquad \xi_M$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.

A length 2 subscheme of \mathbb{A}_k^1 (??) then

$$F(S) = \{V(x^2 + bx + c)\} \subset \mathbb{A}'_5$$

where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

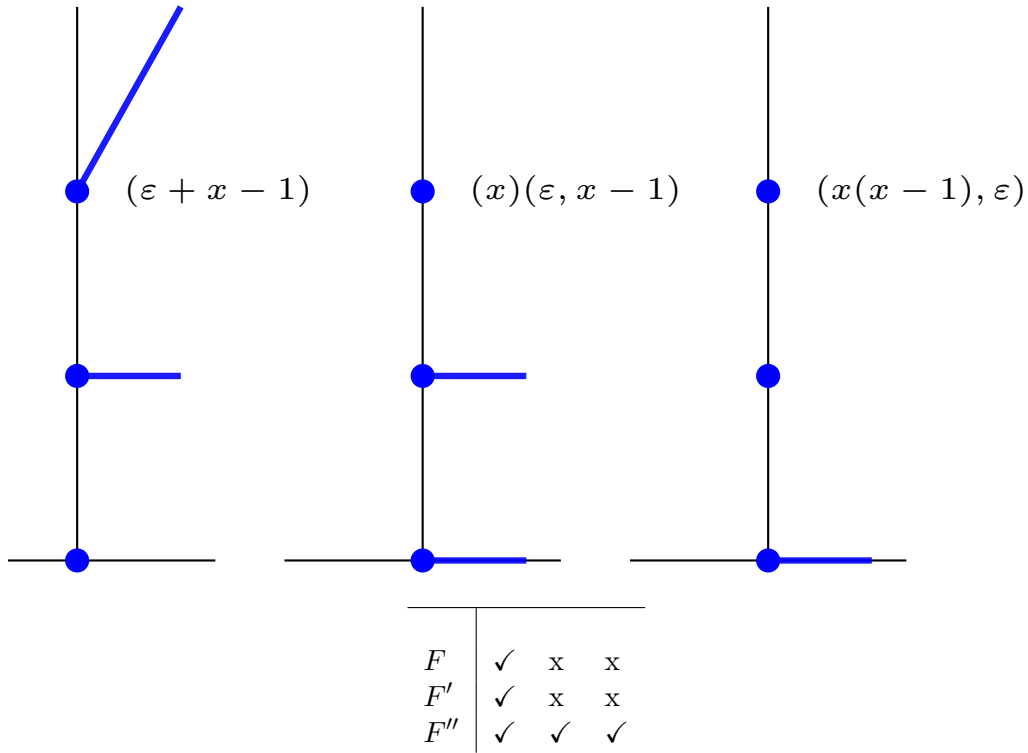
Then F is representable by $\mathbb{A}_k^2(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in \mathbb{A}'_5 which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of \mathbb{A}'_5 which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\text{Spec } k[\varepsilon]$.



$$\begin{array}{ccccc}
 \text{Spec } k & \xleftarrow{i} & \text{Spec } k[\varepsilon] & & \\
 & & & \nearrow & \\
 & & & & = F'(\text{Spec } k) \\
 F(\text{Spec } k[\varepsilon]) & \xrightarrow{F(i)} & F(\text{Spec } k) & & \\
 \uparrow \subset & & \uparrow \in & & \\
 T_p F' & & P = V(x(x-1)) & & \\
 & & & \searrow & \\
 & & & & = F''(\text{Spec } k)
 \end{array}$$

We think of $T_p F'$ as the tangent space at p .

If F is representable, then it is actually the Zariski tangent space.

$$\begin{array}{ccc}
 M(\mathrm{Spec} \, k[\varepsilon]) & \longrightarrow & M(\mathrm{Spec} \, k) \\
 \uparrow \subset & & \uparrow \subset \\
 T_p M & \longrightarrow & p \\
 & & \uparrow \\
 & & \mathrm{Spec} \, k \\
 & \swarrow & \searrow ? \\
 \mathrm{Spec} \, k[\varepsilon] & \longrightarrow & \mathrm{Spec} \, \mathcal{O}_{M,p} \subset M
 \end{array}$$

$$\begin{array}{ccc}
 & & k \\
 & \nearrow & \uparrow \\
 \mathcal{O}_{M,p} & \longrightarrow & k[\varepsilon] \\
 \uparrow & & \uparrow \\
 \mathfrak{m}_p & & (\varepsilon) \\
 \uparrow & & \uparrow \\
 \mathfrak{m}_p^2 & & 0
 \end{array}$$

Moreover, $T_p M = (\mathfrak{m}_p / \mathfrak{m}_p^2)^\vee$, and in particular this is a k -vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
 \lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
 \varepsilon &\mapsto \lambda \varepsilon
 \end{aligned}$$

$$\lambda^* : \mathrm{Spec} \, (k[\varepsilon]) \longrightarrow \mathrm{Spec} \, (k[\varepsilon])$$

$$\begin{aligned}
 \lambda : M(\mathrm{Spec} \, (k[\varepsilon])) &\longrightarrow M(\mathrm{Spec} \, (k[\varepsilon])) \\
 \cup & \qquad \cup \\
 T_p M &\longrightarrow T_p M.
 \end{aligned}$$

Conclusion: If F is representable, for each $p \in F(\mathrm{Spec} \, k)$ there exists a unique point of $T_p F$ that are invariant under scaling.

1. If $F, F', G \in \mathrm{Fun}((\mathrm{Sch}/S)^{\mathrm{op}}, \mathrm{Set})$, there exists a fiber product

$$\begin{array}{ccc}
 F \times_G F' & \cdots \longrightarrow & F' \\
 \downarrow & & \downarrow \\
 F & \longrightarrow & G
 \end{array}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \rightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
 4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\text{Sch}/T) &\rightarrow \text{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}_{\mathbb{Z}}^n$, i.e. “rank 1 quotient of an $n + 1$ dimensional free module”.

Proposition 1.3.

$\mathbb{P}_{\mathbb{Z}}^n$ represents the following functor

$$\begin{aligned} F : \text{Sch}^{\text{op}} &\rightarrow \text{Set} \\ F(S) &= \mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0 / \sim. \end{aligned}$$

where \sim identifies diagrams of the following form:

$$\begin{array}{ccccc} \mathcal{O}_S^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \parallel & & \downarrow \cong & & \\ \mathcal{O}_S^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and $F(f)$ is given by pullbacks.

Remark \mathbb{P}_S^n represents the following functor:

$$\begin{aligned} F_S : (\text{Sch}/S)^{\text{op}} &\rightarrow \text{Set} \\ T &\mapsto F_S(T) = \left\{ \mathcal{O}_T^{n+1} \rightarrow L \rightarrow 0 \right\} / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of Proposition).

Note: $\mathcal{O}^{n+1} \rightarrow L \rightarrow 0$ is the same as giving $n+1$ sections s_1, \dots, s_n of L , where surjectivity ensures that they are not the zero section.

So

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.

There is a natural transformation $F_i \rightarrow F$ by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

Claim: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \uparrow & & \uparrow \\ F_{ij} & \longrightarrow & F_j \end{array}$$

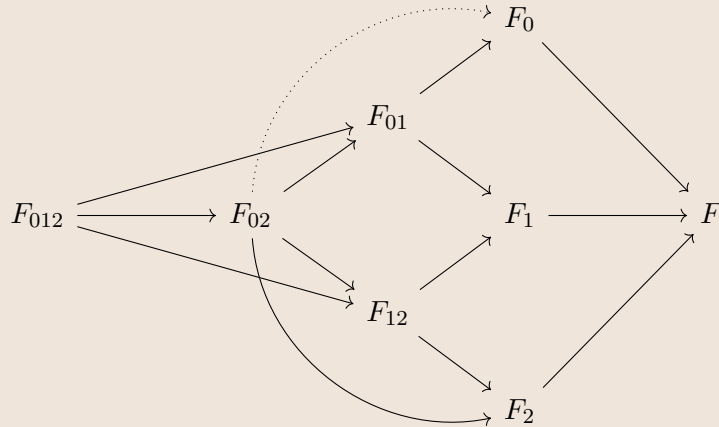
and each $F_{ij} \rightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example .

For $n = 1$, we can glue along open subschemes



For $n = 2$, we get overlaps of the following form:



This claim implies that we can glue together F_i to get a scheme M . We want to show that M represents F . $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S .

RHS to LHS comes from gluing. ■

Proof (of Claim).

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i .

So specifying $F_i(S)$ is equivalent to specifying $n - 1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by \mathbb{A}^n .

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. $\text{Spec } k[x_1, \dots, \widehat{x_i}, \dots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \dots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigcap_{i \in I} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$. ■

2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space $\mathbb{P}_{\mathbb{Z}}^n$ constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}_{\mathbb{Z}}^n} : \mathbb{P}_{\mathbb{Z}}^n \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ s &\mapsto \mathbb{P}_{\mathbb{Z}}^n(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for L a line bundle, up to isomorphisms of diagrams:

$$\begin{array}{ccccc} \mathcal{O}_s^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \Downarrow & & \downarrow \cong & & \\ \mathcal{O}_s^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

That is, line bundles with $n + 1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}_{\mathbb{Z}}^n$ where

$$F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \mid s_i \text{ is invertible} \right\}$$

are representable and can be glued together, and projective space represents this functor.

Remark Because projective space represents this functor, there is a universal object:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \\ & & \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1) & & \end{array}$$

and other functors are pullbacks of the universal one. (Moduli Space)

Exercise Show that $\mathbb{P}_{\mathbb{Z}}^n$ is proper over $\text{Spec } \mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$\begin{array}{ccc} \text{Spec } k & \longrightarrow & \mathbb{P}_{\mathbb{Z}}^n \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec } R & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

Definition 2.0.1 (Equalizer).

For a category C , we say a diagram $X \longrightarrow Y \rightrightarrows Z$ is an *equalizer* iff it is universal with respect to the property:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \rightrightarrows & Z \\ & \nwarrow \text{dashed } \exists! & \uparrow & \nearrow & \\ & & S & & \end{array}$$

Note that X is the universal object here.

Example 2.1.

For sets, $X = \{y \mid f(y) = g(y)\}$ for $Y \xrightarrow{f,g} Z$.

Definition 2.0.2 (Coequalizer).

A **coequalizer** is the dual notion,

$$\begin{array}{ccccc} & & S & & \\ & \nearrow & \uparrow & \nwarrow \text{dashed } \exists! & \\ Z & \rightrightarrows & Y & \longrightarrow & X \end{array}$$

Example 2.2.

Take $C = \text{Sch}/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$\begin{array}{ccc}
X_\alpha & \longrightarrow & X \\
\uparrow & & \uparrow \\
X_{\alpha\beta} & \longrightarrow & X_\beta
\end{array}
\qquad
\begin{array}{ccc}
X_\beta & \longrightarrow & X \\
\uparrow & & \uparrow \\
X_{\beta\alpha} & \longrightarrow & X_\alpha
\end{array}$$

These are canonically isomorphic.

In Sch/S , we have

$$\coprod_{\alpha\beta} X_{\alpha\beta} \begin{array}{c} \xrightarrow{f_{\alpha\beta}} \\ \xrightarrow{g_{\alpha\beta}} \end{array} \coprod_{\alpha} X_{\alpha} \longrightarrow X$$

where

$$\begin{aligned}
f_{\alpha\beta} : X_{\alpha\beta} &\longrightarrow X_{\alpha} \\
g_{\alpha\beta} : X_{\alpha\beta} &\longrightarrow X_{\beta};
\end{aligned}$$

this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the **cocycle condition**:

1.

$$\varphi_{\alpha\beta}^{-1}(X_{\beta\alpha} \cap X_{\beta\gamma}) = X_{\alpha\beta} \cap X_{\alpha\gamma},$$

noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

$$\begin{array}{ccc}
X_{\alpha\beta} \cap X_{\alpha\gamma} & \xrightarrow{\varphi_{\alpha\gamma}} & X_{\gamma\alpha} \cap X_{\gamma\beta} \\
& \searrow \varphi_{\alpha\beta} \quad \nearrow \varphi_{\beta\gamma} & \\
& X_{\beta\alpha} \cap X_{\beta\gamma} &
\end{array}$$

Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_{\alpha} \rightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 2.0.3 (Zariski Sheaf).

A functor $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_{α} , the following is an equalizer:

$$F(T) \rightarrow \prod F(T_{\alpha}) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 2.3.

Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \rightarrow \coprod_{\alpha} T_{\alpha} \rightarrow T,$$

since giving a local map to X that agrees on intersections is enough to specify a map from $T \rightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 2.0.4 (Subfunctors, Open/Closed Functors).

Suppose we have a morphism $F' \rightarrow F$ in the category $\text{Fun}(\text{Sch}/S, \text{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all T/S .
- ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over T , then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U .

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F' , which maps to F (noting a section is the same as a map to the functor of points), and since $T \rightarrow F$ and $F' \rightarrow F$, we can form the fiber product $F' \times_F T$:

$$\begin{array}{ccc}
F' & \longrightarrow & F \\
\uparrow & & \uparrow \xi \\
F' \times_F T & \xrightarrow{g} & T
\end{array}$$

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \rightarrow F$ is open/closed/locally closed iff $F' \times_F T$ is representable and g is open/closed/locally closed.

I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \rightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

Definition 2.0.5 (Open Covers).

A collection of open subfunctors $F_\alpha \subset F$ is an **open cover** iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \rightarrow F$, the T_α in the following diagram are an open cover of T :

$$\begin{array}{ccc}
F_\alpha & \longrightarrow & F \\
\uparrow & & \uparrow \xi \\
T_\alpha & \longrightarrow & T
\end{array}$$

Example 2.4.

Given

$$F(s) = \{ \mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0 \}$$

and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \rightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

Proposition 2.1.

A Zariski sheaf $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ with a representable open cover is representable.

Proof.

Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α . Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$

representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \rightarrow F_\alpha$ have the correct gluing maps.

This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \rightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x ; we'll show that x represents F .

First produce a map $x \rightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_{\alpha} F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \rightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

Lemma 2.2.

If $E \rightarrow F$ is a map of functors and E, F are zariski sheaves, where there are open covers $E_\alpha \rightarrow E, F_\alpha \rightarrow F$ with commutative diagrams

$$\begin{array}{ccc} E & \longrightarrow & F \\ \uparrow & & \uparrow \\ E_\alpha & \xrightarrow{\cong} & F_\alpha \end{array}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$\begin{array}{ccc} X & \longrightarrow & F \\ \uparrow & & \uparrow \\ X_\alpha & \xrightarrow{\cong} & F_\alpha \end{array}$$

Example 2.5.

For S and E a locally free coherent \mathcal{O}_S module,

$$\mathbb{P}E(T) = \{f^*E \rightarrow L \rightarrow 0\} / \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E .

Then the restriction $F_i \rightarrow \mathbb{P}E$ where $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i , $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 2.6.

For E locally free over S of rank n , take $r < n$ and consider the functor $\text{Gr}(k, E)(T) = \{f^*E \rightarrow Q \rightarrow 0\} / \sim$ (a Grassmannian) where Q is locally free of rank k .

Exercise

- Show that this is representable
- For the Plucker embedding

$$\text{Gr}(k, E) \rightarrow \mathbb{P} \wedge^k E,$$

a section over T is given by $f^*E \rightarrow Q \rightarrow 0$ corresponding to

$$\wedge^k f^*E \rightarrow \wedge^k Q \rightarrow 0,$$

noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \text{Spec } k$, then E is a k -vector space V , then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n .

Take the subfunctor $G_w \subset \text{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \rightarrow Q \rightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\text{hom}(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for \mathbb{P}^n of all flat families of subschemes.

3 Thursday January 16th

3.1 Subfunctors

A functor $F' \subset F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ is **open** iff for all $T \xrightarrow{\xi} F$ where $T = h_T$ and $\xi \in F(T)$.

We can take fiber products:

$$\begin{array}{ccc} F' & \xrightarrow{\quad} & F \\ \uparrow & & \uparrow \\ F' \times_F T & \xrightarrow{\text{Open}} & T \\ \text{Representable} & & \end{array}$$

So we can think of “inclusion in F ” as being an *open condition*: for all T/S and $\xi \in F(T)$, there exists an open $U \subset T$ such that for all covers $f : T' \rightarrow T$, we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff f factors through U .

Suppose $U \subset T$ in Sch/T , we then have

$$h_{U/T}(T') = \begin{cases} \emptyset & T' \longrightarrow T \text{ doesn't factor} \\ \{\text{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is $h_{U/T}(T') = \text{hom}_T(T', U)$.

By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{ (a, b) \in F'(T) \times T(T) \mid \xi(b) = \iota(a) \text{ in } F(T) \right\},$$

where $F' \xrightarrow{\iota} F$ and $T \xrightarrow{\xi} F$.

So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of F/T' as sections of F over T/T' (?).

$$\begin{array}{ccc} F' & \xrightarrow{\iota} & F \\ \uparrow & & \uparrow \xi \\ F' \times_F T & \longrightarrow & T \end{array} \quad \begin{array}{c} \swarrow f \circ \xi \\ T' \end{array}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U/S}(T'),$$

and so for $U \subset T$ in Sch/S we have $h_{U/S} \subset h_{T/S}$ is the functor of maps that factor through U . We just identify $h_{U/S}(T') = \text{hom}_S(T', U)$ and $h_{T/S}(T') = \text{hom}_S(T', T)$.

Example 3.1.

$\mathbb{G}_m, \mathbb{G}_a$. \mathbb{G}_a represents giving a global function, \mathbb{G}_m represents giving an invertible function.

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_a \\ \uparrow & & \uparrow f \in \mathcal{O}_T(T) \\ T' & \longrightarrow & T \end{array} \quad \text{L}$$

where $T' = \{f \neq 0\}$ and $\mathcal{O}_T(T)$ are global functions.

3.2 Actual Geometry: Hilbert Schemes

The best moduli space!

Want to parameterize families of subschemes over a fixed object. Fix k a field, X/k a scheme; we'll parameterize subschemes of X .

Definition 3.0.1 (Hilbert Functor).

The hilbert functor is given by

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

which sends T to closed subschemes $Z \subset X \times_S T \longrightarrow T$ which are flat over T .

Here flatness replaces the Cartier condition.

Definition 3.0.2 (Flatness).

For $X \xrightarrow{f} Y$ and \mathbb{F} a coherent sheaf on X , f is flat over Y iff for all $x \in X$ the stalk F_x is a flat $\mathcal{O}_{y,f(x)}$ -module.

Note that f is flat if \mathcal{O}_x is.

Flatness corresponds to varying continuously.

Warning: Unless otherwise stated, assume schemes are Noetherian.

Note that everything works out if we only path with finite covers.

Remark If X/k is projective, so $X \subset \mathbb{P}_k^n$, we have line bundles $\mathcal{O}_x(1) = \mathcal{O}(1)$. For any sheaf F over X , there is a hilbert polynomial $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$. (i.e. we twist by $\mathcal{O}(1)$ n times.)

The cohomology of F isn't changed by the pushforward into \mathbb{P}_n since it's a closed embedding, i.e.

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

Fact (First) For $n \gg 0$, $\dim_k H^0 = \dim M_n$, the n th graded piece of M , which is a graded module over the homogeneous coordinate ring whose $i_* F = \tilde{M}$.

In general, for L ample of X and F coherent on X , we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

Proposition 3.1.

For $f : X \longrightarrow S$ projective, i.e. there is a factorization:

$$\begin{array}{ccc}
 X & \hookrightarrow & \mathbb{P}^n \times S \ni \mathcal{O}(1) \\
 & \searrow f & \swarrow \\
 & S &
 \end{array}$$

If S is reduced, irreducible, locally Noetherian, then f is flat $\iff P_{\mathcal{O}_{x_s}}$ is constant for all $s \in S$.

To be more precise, look the base change to X_1 , and the pullback of the fiber? $\mathcal{O} \Big|_{x_i} ?$

Note: not using the word “integral” here! S is flat \iff the hilbert polynomial over the fibers are constant.

Example 3.2.

The zero-dimensional subschemes $Z \in \mathbb{P}_k^n$, then P_Z is the length of Z , i.e. $\dim_k(\mathcal{O}_Z)$, and

$$P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z).$$

For two closed points in \mathbb{P}^2 , $P_Z = 2$.

Consider the affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$, which is given by

$$\text{Spec } k[x, y]/(y, x^2) \cong k[x]/(x^2)$$

and $P_Z = 2$. I.e. in flat families, it has to record how the tangent directions come together.

Example 3.3.

Consider the flat family $xy = 1$ (flat because it’s an open embedding) over $k[x]$, here we have points running off to infinity.

Proposition 3.2 (Modified Characterization of Flatness for Sheaves).

A sheaf F is flat iff P_{F_S} is constant.

3.2.1 Proof

Assume $S = \text{Spec } A$ for A a local Noetherian domain.

Lemma 3.3.

For F a coherent sheaf on X/A is flat, we can take the cohomology via global sections $H^0(X; F(n))$. This is an A -module, and is a free A -module for $n \gg 0$.

Proof (of Lemma).

Assumed X was projective, so just take $X = \mathbb{P}_A^n$ and let F be the pushforward. There is a correspondence sending F to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module $\sum_{n \gg 0} H^0(\mathbb{P}_A^m; F(n))$. This is equal to $\oplus_{n \gg 0} H^0(\mathbb{P}_A^m; F(n))$ and taking the associated sheaf ($Y \mapsto \tilde{Y}$, as per Hartshorne's notation) which is free, and thus F is free.

See tilde construction in Hartshorne, essentially amounts to localizing free rings.

Conversely, take an affine cover U_i of X . We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume $H^i(\mathbb{P}^m; F(n)) = 0$ for $n \gg 0$, and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^m; F(n)) \longrightarrow \mathcal{C}^0(\underline{U}; F(n)) \longrightarrow \cdots \longrightarrow \mathcal{C}^m(\underline{U}; F(n)) \longrightarrow 0.$$

Assuming F is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right.

Finally, local Noetherian + finitely generated flat implies free. ■

By the lemma, we want to show $H^0(\mathbb{P}^m; F(n))$ is free for $n \gg 0$ iff the hilbert polynomials on the fibers P_{F_S} are all constant.

Claim 1 (1).

It suffices to show that for each point $s \in \text{Spec } A$, we have

$$H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$$

for $k(S)$ the residue field, for $n \gg 0$.

Note that P_{F_s} measures the rank of the LHS.

\implies : The dimension of RHS is constant, whereas the LHS equals $P_{F_S}(n)$.

\impliedby : If the dimension of the RHS is constant, so the LHS is free.

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank.

For M a finitely generated module over A , find $0 \longrightarrow A^n \longrightarrow M \longrightarrow Q$ is surjective after tensoring with $\text{Frac}(A)$, and tensoring with $k(S)$ for a closed point, if $\dim A^n = \dim M$ then $Q = 0$.

Proof (of Claim 1).

By localizing, we can assume s is a closed point. Since A is Noetherian, its ideal is f.g. and we have

$$A^m \longrightarrow A \longrightarrow k(S) \longrightarrow 0.$$

We can tensor with F (viewed as restricting to fiber) to obtain

$$F(n)^m \longrightarrow F(n) \longrightarrow F_S(n) \longrightarrow 0.$$

Because F is flat, this is still exact.

We can take $H^*(x, \cdot)$, and for $n \gg 0$ only H^0 survives. This is the same as tensoring with $H^0(x, F(n))$. ■

Definition 3.3.1 (Hilbert Polynomial Subfunctor).

Given a polynomial $P \in \mathbb{Z}[n]$ for X/S projective, we define a subfunctor by picking only those with Hilbert polynomial p fiberwise as $\text{Hilb}_{X/S}^P \subset \text{Hilb}_{X/S}$. This is given by $Z \subset X \times_S T$ with $P_Z = P$.

Theorem 3.4 (Grothendieck).

If S is Noetherian and X/S projective, then $\text{Hilb}_{X/S}^P$ is representable by a projective S -scheme.

See cycle spaces in analytic geometry.

4 Thursday January 23rd

Some facts about the Hilbert polynomial:

1. For a subscheme $Z \subset \mathbb{P}_k^n$ with $\deg P_Z = \dim Z = n$, then

$$p_Z(t) = \deg z t^n / (n!) + O(t^{n-1}).$$

2. We have $p_Z(t) = \chi(\mathcal{O}_Z(t))$, consider the sequence

$$0 \longrightarrow I_Z(t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{(t)} \longrightarrow \mathcal{O}_Z^{(t)} \longrightarrow 0,$$

then $\chi(I_Z(t)) = \dim H^0(\mathbb{P}^n, J_Z(t))$ for $t \gg 0$, and $p_Z(0)$ is the Euler characteristic of \mathcal{O}_Z .

Serre vanishing, Riemann-Roch, ideal sheaf.

Good example to keep in mind: the twisted cubic

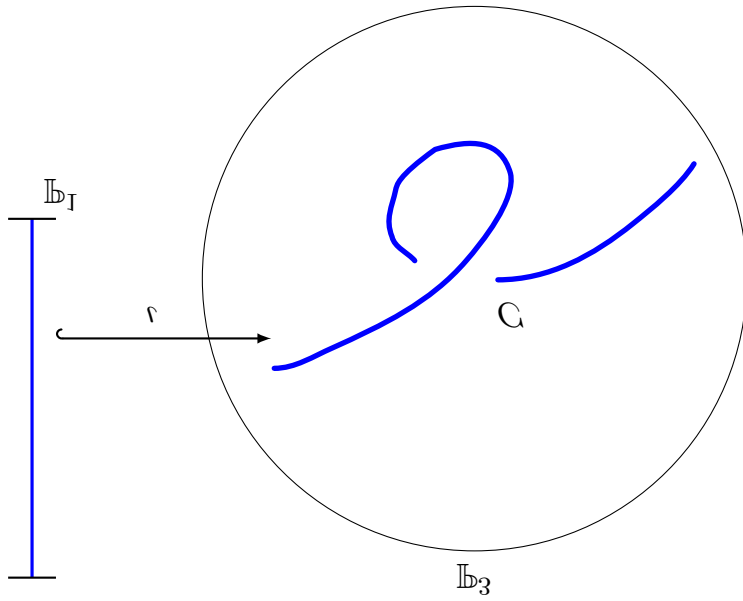
Then $p_C(t) = (\deg C)t + \chi(\mathcal{O}_{\mathbb{P}^1}) = 3t + 1$.

Hypersurfaces: Recall length 2 subschemes of \mathbb{P}^1 are the same as specifying quadratics that cut them out, each such $Z \subset \mathbb{P}^1$ satisfies $Z = V(f)$ where $\deg f = d$ and f is homogeneous. So we'll be looking at $\mathbb{P}H^0(\mathbb{P}_k^n, \mathcal{O}(d))^\vee$, and the guess would be that this is $\text{Hilb}_{\mathbb{P}_k^n}^P$

Resolve the structure sheaf

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

so we can twist to obtain



$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

Then $\chi(\mathcal{O}_D(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(t-d))$, which is $\binom{n+t}{n} - \binom{n+t-d}{n} = dt^{n-1}/(n-1)! + O(t^{n-2})$.

Lemma: Anything with the Hilbert polynomial of a degree d hypersurface is in fact a degree d hypersurface.

Want to write a morphism of functors

$$\mathrm{Hilb}_{\mathbb{P}_{\mathbb{k}}^n}^{P_{n,d}} \longrightarrow \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee.$$

which sends flat families to families of equations cutting them out.

Want $Z \subset \mathbb{P}^n \times S \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee \longrightarrow L \longrightarrow 0$. This happens iff $0 \longrightarrow L^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$ with torsion-free quotient. Note that we use L^\vee instead of \mathcal{O}_s because of scaling.

We have

$$\begin{aligned} 0 &\longrightarrow I_z \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S} \longrightarrow \mathcal{O}_z \longrightarrow 0 \\ 0 &\longrightarrow I_z(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \mathcal{O}_z(d) \longrightarrow 0 \quad \text{by twisting.} \end{aligned}$$

We then consider $\pi_s : \mathbb{P}^n \times S \longrightarrow S$, and apply the pushforward to the above sequence noting that it is not right-exact.

$$\begin{aligned}
0 &\longrightarrow \pi_{s*} I_z(d) \longrightarrow \pi_{s*} \mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \pi_{s*} \mathcal{O}_z(d) \longrightarrow 0 \\
&= 0 \longrightarrow L^\vee = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow \text{locally free.}
\end{aligned}$$

This equality follows from flatness, cohomology, and base change. In particular, we need the following facts

The scheme-theoretic fibers, given by $H^0(\mathbb{P}^n, I_z(d))$ and $H^0(\mathbb{P}^n, \mathcal{O}_z(d))$, are all the same dimension.

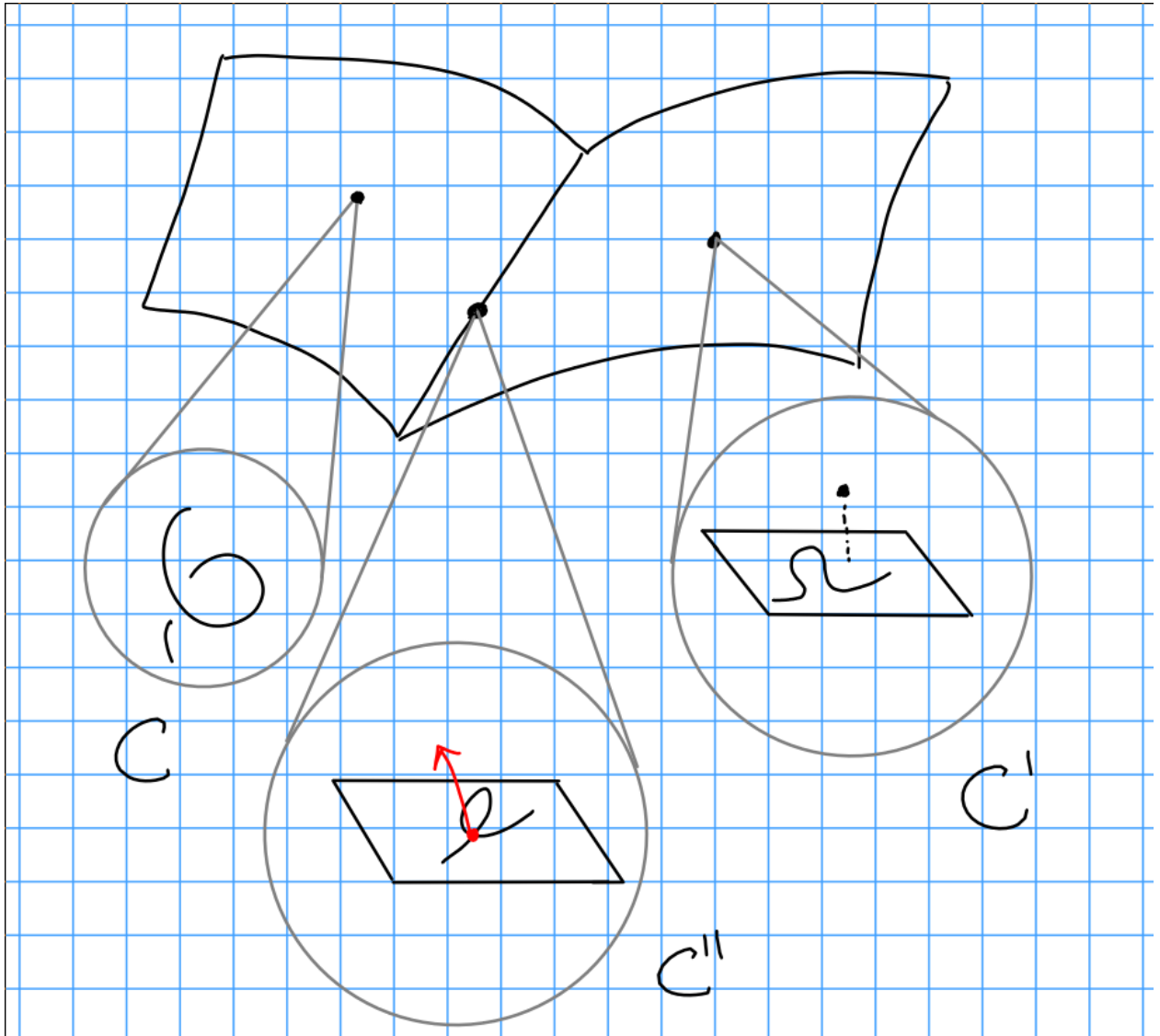
Using

1. Cohomology and base change, i.e. for $X \xrightarrow{f} Y$ a map of Noetherian schemes (or just finite-type) and F a sheaf on X which is flat over Y , there is a natural map (not usually an isomorphism) $R^i f_* f \otimes k(y) \longrightarrow H^i(x_y, F|_{x_y})$, but is an isomorphism if $\dim H^i(x_y, F|_{x_y})$ is constant, in which case $R^i f_* f$ is locally free.
2. If $Z \subset \mathbb{P}_k^n$ is a degree d hypersurface, then independently we know $\dim H^0(\mathbb{P}^n, I_z(d)) = 1$ and $\dim H^0(\mathbb{P}^n, \mathcal{O}_z(d)) = \binom{d+n}{n} - 1$.

To get a map going backwards, we take the universal degree 2 polynomial and form $V(a_{00}x_0^2 + a_{11}x_1^2 + a_{12}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2) \subset \mathbb{P}^2 \times \mathbb{P}^5$.

Next example: twisted cubics.

Consider a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ obtained by taking a basis of a homogeneous cubic polynomial. The canonical example is $(x, y) \longrightarrow (x^3, x^2y, xy^2, y^3)$. Then $P_C(t) = 3t + 1$, and $\mathrm{Hilb}_{\mathbb{P}_k^3}^{3t+1}$ has a component with generic point a twisted cubic, and another component with points a curve disjoint union a point, and the overlap are nodal curves with a “fat” 3-dimensional point:



Then $P_{C'} = 1 + \tilde{P}$, the hilbert polynomial of just the base without the disjoint point, so this equals $1 + P_{2,3} = 1 + (3t+0) = 3t+1$. For $P_{C''}$, we take the sequence $0 \rightarrow k \rightarrow \mathcal{O}_{C''} \rightarrow \mathcal{O}_{C''\text{reduced}} \rightarrow 0$, so $P_{C''} = 1 + P_{C''\text{red}} = 3t+1$.

Note: flat families have to have the same constant Hilbert polynomial.

Note that we can get paths in this space from $C \rightarrow C''$ and $C' \rightarrow C''$ by collapsing a twisted cubic onto a plane, and sending a disjoint point crashing into the node on a nodal cubic.

We're mapping $\mathbb{P}^1 \rightarrow \mathbb{P}^3$, and there is a natural action of $\mathbb{P}\text{GL}(4) \curvearrowright \mathbb{P}^3$, so we get a map

$$\mathbb{P}\text{GL}(4) \times \mathbb{P}^3 \rightarrow \mathbb{P}^3.$$

Let $c \in \mathbb{P}^3$ and let \mathcal{C} be the preimage. This induces (?) a map $\mathbb{P}\text{GL}(4) \rightarrow \mathrm{Hilb}_{\mathbb{P}^3}^{3t+1}$ where the fiber over $[C]$ in the latter is $\mathbb{P}\text{GL}(2) = \text{Aut}(\mathbb{P}^1)$. By dimension counting, we find that the

dimension of the twisted cubic component is $15 - 3 = 12$.

The 15 in the other component comes from 3-dim choices of plane, 3-dim choices of a disjoint point, and $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee \cong \mathbb{P}^9$, yielding 15 dimensions.

To show that these are actually different components, we use Zariski tangent spaces. Let T_1 be the tangent space of the twisted cubic component, then $\dim T_1 \mathrm{Hilb}_{\mathbb{P}_k^3}^{3t+1} = 12$, and similarly the dimension of the tangent space over the C' component is 15.

Algebra fact: Let A be Noetherian and local, then the dimension of the Zariski tangent space, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$, the Krull dimension. If this is an equality, then A is regular.

Thus dimensions of the tangent spaces give an upper bound.

Proposition: If X/k is projective and P is a Hilbert polynomial, then $[Z] \in \mathrm{Hilb}_{X/k}^P$, i.e. a closed subscheme of X with hilbert polynomial p (note there's an ample bundle floating around) then the tangent space is $\mathrm{hom}_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.