Title

D. Zack Garza

Contents

1 | Lecture 5A

We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for K/k of transcendence degree 1, all valuations on K which are trivial on k are discrete. We'll now want to go farther and describe the places $\Sigma(K/k)$, which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is K. Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if A is a Dedekind domain such that $\mathrm{ff}(A) = K$, then for all $\mathfrak{p} \in \mathrm{mSpec}(A)$ there exists a discrete valuation v_p on K. I.e., every maximal ideal induces a discrete valuation that is A-regular, so the valuation ring will contain A. How is this obtained? Take a nonzero $x \in K^\times$, and take the corresponding principal fractional ideal $\langle x \rangle \coloneqq Ax$, which we can factor in a Dedekind domain as $Ax = \prod_{\mathfrak{p} \in \mathrm{mSpec}(A)} \mathfrak{p}^{\alpha_\mathfrak{p}}$ with $\alpha_\mathfrak{p} \in \mathbb{Z}$. This looks like an infinite product, but for any fixed x, only finitely many α are nonzero. Note that these α are exactly what we're looking for: the $\mathfrak{p}$-adic evaluation of x is given precisely by $v_\mathfrak{p}(x) \coloneqq \alpha_\mathfrak{p}$, where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$\begin{aligned} v_\cdot : \mathrm{mSpec}(A) &\to \Sigma(K/A) \\ \mathfrak{p} &\mapsto v_\mathfrak{p}. \end{aligned}$$