Title

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Today: G_r-T modules.

Note that $G_r \trianglelefteq G_rT$, with $G_rT/G_r \cong T^{(r)}$. We consider G_rT-modules, which are G_r-modules with a T action given by

$$\begin{aligned} G_r \times M &\to M \\ (g, m) &\mapsto g \cdot m \end{aligned}$$

which are T-equivariant, i.e. $t(g \cdot m) = (t \cdot g)(t \cdot m)$ for $t \in T, g \in G_r$, and $m \in M$ is a G_rT-module. This essentially induces a grading on G_r.

1.1 Representations for G_rT and G_rB

Recall that we have a Frobenius map, for which we take the scheme-theoretic kernel:

$$\begin{aligned} F &: G \to G \\ F^r &:= F \circ F \circ \cdots \circ F \\ G_r &:= \ker F^r, \end{aligned}$$

and we then define

$$\begin{aligned} G_rT &:= (F^r)^{-1}(T) \\ G_rB &:= (F^r)^{-1}(B) \end{aligned}$$

taken as scheme-theoretic objects.

Noting that $B \subset G_rB$, for $\lambda \in X(T)$ we define

$$\begin{aligned} \widehat{Z}_r'(\lambda) &:= \mathrm{Ind}_B^{G_rB} \lambda \\ \widehat{Z}_r(\lambda) &:= \mathrm{Coind}_B^{G_rB} \lambda. \end{aligned}$$

These are enhancements of the baby Verma modules, in the sense that if we take restrictions we get

$$\widehat{Z}_r'(\lambda) \downarrow_{G_r} = \mathrm{Ind}_{B_r}^{G_r} \lambda.$$

We similarly have

$$\begin{aligned} Z_r'(\lambda)\downarrow_{G_r T} &= \operatorname{Ind}_{B_r T}^{G_r T}\lambda \\ \widehat{Z}_r'(\lambda)\downarrow_{G_r T} &= \operatorname{Coind}_{B_r T}^{G_r T}\lambda. \end{aligned}$$

Proposition 1.1.1 *(?).*

1. $\widehat{Z}_r(\lambda + p^r\mu) \cong \widehat{Z}_r(\lambda)\otimes p^r\mu$
2. $\widehat{Z}_r'(\lambda + p^r\mu) \cong \widehat{Z}_r'(\lambda)\otimes p^r\mu$
3. $\operatorname{ch}\widehat{Z}_r(\lambda) = \operatorname{ch}\widehat{Z}_r'(\lambda) = e^\lambda \prod_{\alpha\in\Phi^+}\dfrac{1-e^{-p^r\mu}}{1-e^{-\alpha}}.$

Proof (of 1 and 2).
From the definition, we have

$$\begin{aligned} \widehat{Z}_r'(\lambda + p^r\mu) &= \operatorname{Ind}_B^{G_r B}(\lambda + p^r\mu) \\ &= \operatorname{Ind}_B^{G_r B}(\lambda\otimes p^r\mu) \\ &\cong \left(\operatorname{Ind}_B^{G_r B}\lambda\right)\otimes p^r\mu. \end{aligned}$$

Where in the last equality we've applied the tensor identity, noting that $p^r\mu$ is a 1-dimensional $G_r B$-module, since

$$G_r B \to G_r B/G_r = B^{(r)} = B/B_r,$$

making it a representation by pullback. ■

Proof (of 3).
We can write

$$\widehat{Z}_r(\lambda) = \operatorname{dist}(U_r)\otimes\lambda,$$

and thus

$$\begin{aligned} \operatorname{ch}\widehat{Z}_r(\lambda) &= e^\lambda \operatorname{ch}\operatorname{dist}(U_r) \\ &= e^\lambda \prod_{\alpha\in\Phi^+}\left(1 + e^{-\alpha} + \cdots + e^{-(p^r-1)\alpha}\right) \\ &= e^\lambda \frac{1-e^{-p^r\alpha}}{1-e^{-\alpha}} && \text{as a geometric series.} \end{aligned}$$

■

The next theorem is related to the fact that when comparing these categories of modules, one is essentially a graded version of the other.

Theorem 1.1.1 *(?).*
Let $M \in G_rT\text{-}\operatorname{mod}$, then TFAE:

1. M is an injective G_rT-module.
2. M is an injective G_r-module.

Note that $G_r \trianglelefteq G_rT$, where the quotient is $T^{(r)}$ which is twisted by Frobenius r times.

Proof (?).
We'll apply the Lydon-Hoschild-Serre spectral sequence: for N a G_rT-module,

$$E_2^{i,j} = \operatorname{Ext}^i_{T^{(r)}}\left(K, \operatorname{Ext}^j_{G_r}(N, M)\right) \Rightarrow \operatorname{Ext}^{i+j}_{G_rT}(N, M).$$

$2 \implies 1$:
We first note that 2 implies $\operatorname{Ext}^{>0}_{G_r}(N, M) = 0$, so the spectral sequence collapses and we have

$$\operatorname{Ext}^i_{T^{(r)}}(k, \hom_{G_r}(M, N)) \cong \operatorname{Ext}^i_{G_rT}(N, M).$$

Since modules over $T^{(r)}$ are completely reducible, we have

$$\operatorname{Ext}^{>0}_{T^{(r)}}(k, \hom_{G_r}(N, M)) = 0,$$

and thus $\operatorname{Ext}^{>0}_{G_rT}(N, M) = 0$, making M an injective G_rT-module. ■

$1 \implies 2$:
The simple G_rT-modules are of the form $N := L_r(\lambda) \otimes p^r\sigma$ where $\lambda \in X_r(T)$ and $\sigma \in X(T)$. Note that $L_r(\lambda)$ is simple G_r-module. Applying the spectral sequence, there is a 5 term exact sequence. Letting $E_t := \operatorname{Ext}^t_{G_rT}(N, M)$.

$$0 \longrightarrow E_2^{1,0} \longrightarrow E_1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E_2$$

Everything is zero here except for the middle term: $E_1, E_2 = 0$ by assumption? $E_2^{1,0}, E_2^{2,0} = 0$ by ?.
We can thus conclude that

$$\begin{aligned} 0 = E_2^{0,1} &= \hom_{T^{(r)}}(k, \operatorname{Ext}^1_{G_r}(L_r(\lambda) \otimes p^r\sigma, M)) \\ &= \hom_{T^{(r)}}(p^r\sigma, \operatorname{Ext}^1_{G_r}(L_r(\lambda), M)), \end{aligned}$$

which holds for all $p^r\sigma$, and thus $\operatorname{Ext}^1_{G_r}(L_r(\lambda), M) = 0$ for all $\lambda \in X_1(T)$. So M is injective as a G_r-module.

■

Proposition 1.1.2 *(?).*
Let $\lambda \in X(T)$, then

1. $\widehat{Z}_r(\lambda) \downarrow_{B_rT}$ is the projective cover of λ and the injective hull of $\lambda - 2(p^r - 1)\rho$.
2. $\widehat{Z}'_r(\lambda) \downarrow_{B_r^+}$ is the projective cover of $\lambda - 2(p^r - 1)\rho$ and the injective hull of λ.

1.2 Summary: Classification of Simple G_rT-Modules

- $\operatorname{Soc}_{B_r^+} \widehat{Z}_r'(\lambda) = \lambda$
- $\widehat{Z}_r'(\lambda)^{U^+} = \lambda$, where the RHS denotes the U^+ invariants.
- Let $\widehat{L}_r(\lambda) \coloneqq \operatorname{Soc}_{G_rT} \widehat{Z}_r'(\lambda)$.
- Each simple G_rT-module is isomorphic to $\widehat{L}_r(\lambda)$ for some $\lambda \in X(T)$.
- $\widehat{L}_r(\lambda) \downarrow_{G_r} \cong L_r(\lambda)$ for all $\lambda \in X_1(T)$.
- Translation invariance: $\widehat{L}_r(\lambda + p^r\sigma) = \widehat{L}_r(\lambda) \otimes p^r\sigma$
- $\widehat{L}_r(\lambda + p^r\sigma) \downarrow_{G_r} = L_r(\lambda)$ for all $\lambda \in X_r(T)$.

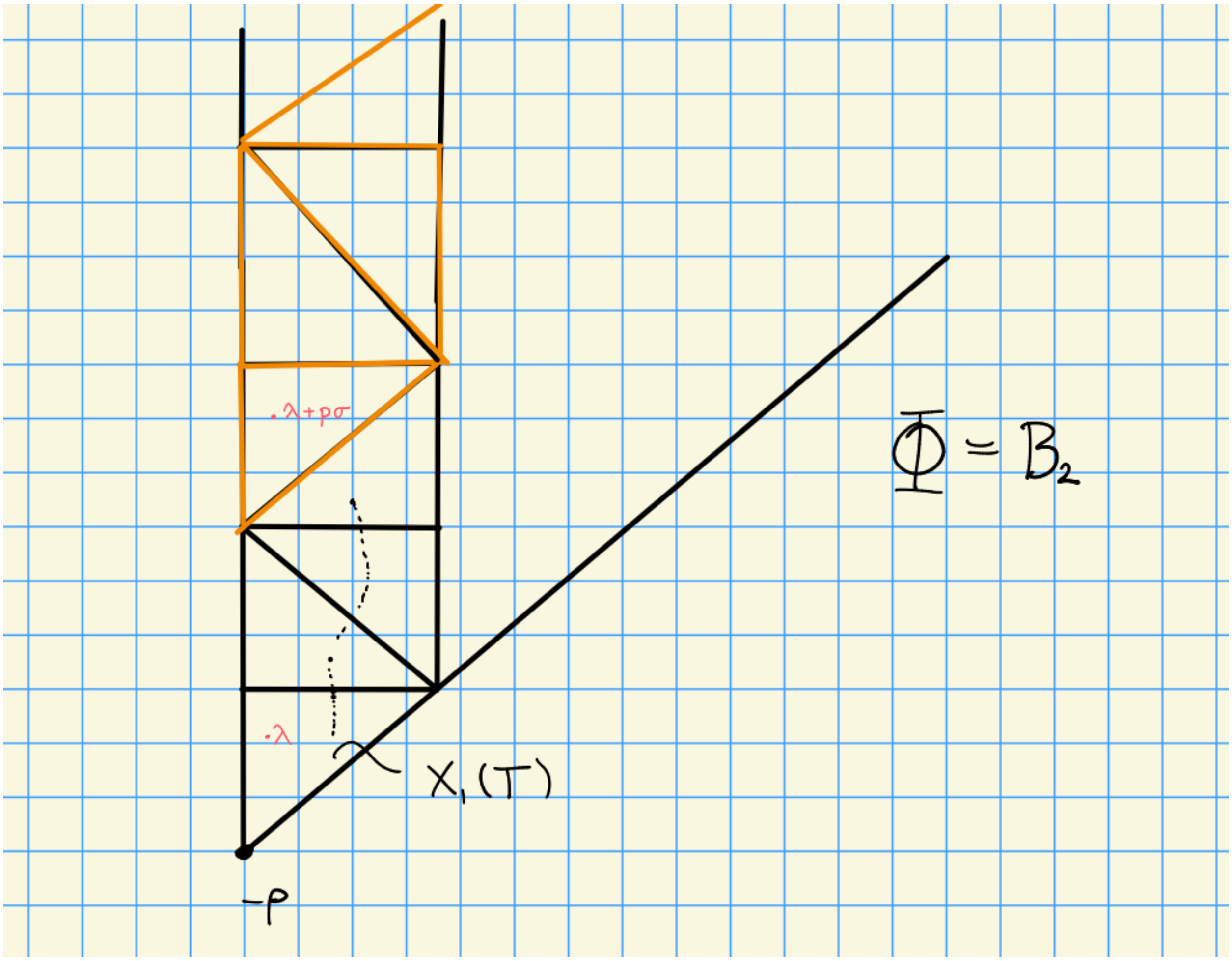

Figure 1: Example in $\Phi = B_2$

This essentially allows you to replace working mod p in characteristic p with working with integers instead, allowing the usual weight theory to be used.

Proposition 1.2.1 ***(?).***
Let $\lambda \in X(T)$, then there exists an isomorphism of G-modules

$$H^i(\lambda) = R^i \operatorname{Ind}_{G_r B}^{G} \widehat{Z}_r'(\lambda),$$

where $\widehat{Z}_r'(\lambda) = \operatorname{Ind}_B^{G_r B}(\lambda)$.