

## Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

# Summary/Outline

# Outline

What we're trying to prove:

- 8.1.5:  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .

What we have so far:

- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$

$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

# Outline

- Took  $R^\pm : I \longrightarrow \text{Sp}(2n; \mathbb{R})$ : symplectic paths associated to  $S^\pm$
- These paths defined  $\mu(x), \mu(y)$
- Section 8.7:

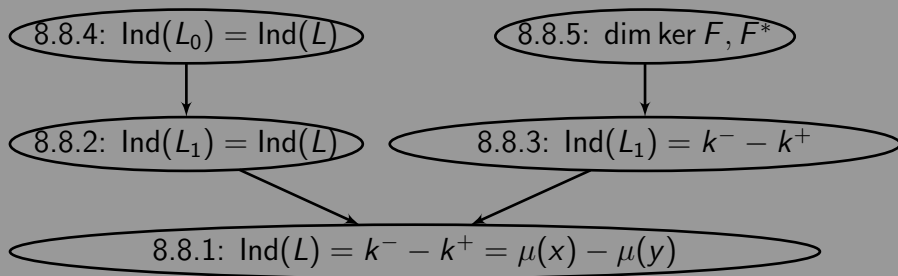
$$R^\pm \in \mathcal{S} := \left\{ R(t) \mid R(0) = \text{id}, \det(R(1) - \text{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\text{Ind}(L) \stackrel{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

# From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know  $\text{Ind}(L) = \text{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing  $\text{Ind}(L_1)$  by computing kernels.



$$8.8.3: \text{Ind}(L_1) = k^- - k^+$$

# Recall

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^* : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^q(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here  $\frac{1}{p} + \frac{1}{q} = 1$  are conjugate exponents.

# Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since  $\text{coker } L_1 \cong \ker L_1^*$ , it suffices to compute  $\ker L_1^*$ .
- We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & \\ & J_0^1 & \\ & & \ddots \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \text{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the  $n = 1$  case.



# Outline

$L_1$  used a path of diagonal matrices constant near  $\infty$ :

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



# Statement of Lemma

Let  $p > 2$  and define

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Note: this seems to be  $L_1$  for  $n = 1$ :

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

# Statement of Lemma

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose  $a_i^\pm \notin 2\pi\mathbb{Z}$ .

- ① Suppose  $a_1(s) = a_2(s)$  and set  $a^\pm := a_1^\pm = a_2^\pm$ . Then

$$\dim \text{Ker } F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \subset \mathbb{R} \right\}$$

$$\dim \text{Ker } F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

- ② Suppose  $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$ , then

$$\dim \text{Ker } F = \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$

$$\dim \text{Ker } F^* = \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

# Statement of Lemma

In words:

- 1 If  $S(s)$  is a scalar matrix, set  $a^\pm = a_1^\pm = a_2^\pm$  to the limiting scalar and count the integer multiples of  $2\pi$  between  $a^-$  and  $a^+$ .

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$

## 8.8.5: $\dim \ker F, F^*$

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