Title

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Last time: projective varieties $V(f_i) \subset \mathbb{P}^n_{/k}$ with f_i homogeneous. We proved the projective nullstellensatz: for any projective variety X, we have $V_p(I_p(X))$ and for any homogeneous ideal I with $\sqrt{I} \neq I_0$ the irrelevant ideal, $I_p(V_p(I)) = \sqrt{I}$. Recall that $I_0 = \langle x_0, \cdots, x_n \rangle$. We had a notion of a projective coordinate ring, $S(X) := k[x_1, \cdots, x_n]/I_p(X)$, which is a graded ring since $I_p(X)$ is a homogeneous ideal.

Note that $S(X)$ is not a ring of functions on X: e.g. for $X = \mathbb{P}^n$, $S(X) = k[x_1, \cdots, x_n]$ but x_0 is not a function on $\mathbb{P}^n$. This is because $f([x_0 : \cdots : x_n]) = f([\lambda x_0 : \cdots : \lambda x_n])$ but $x_0 \neq \lambda x_0$. It still makes sense to ask if f is zero, so $V_p(f)$ is a well-defined object.

Definition 1.0.1 (Dehomogenization of functions and ideals)**.**
Let $f \in k[x_1, \cdots, x_n]$ be a homogeneous polynomial, then we define its dehomogenization as

$$f^i := f(1, x_1, \cdots, x_n) \in k[x_1, \cdots, x_n].$$

For a homogeneous ideal, we define

$$J^i := \left\{ f^i \;\middle|\; f \in J \right\}.$$

Example 1.0.1 : This is usually not homogeneous. Take

$$\begin{aligned} f &= x_0^3 + x_0 x_1^2 + x_0 x_1 x_2 + x_0^2 + x_1 \\ \implies f' &= 1 + x_1^2 + x_1 x_2 + x_1, \end{aligned}$$

where has terms of mixed degrees.

Remark 1.0.1 :

- $(fg)^i = f^i g^i$,
- $(f+g)^i = f^i + g^i$

In other words, evaluating at $x_0 = 1$ is a ring morphism.

Definition 1.0.2 (Homogenization of a function)**.**
Let $f \in k[x_1, \cdots, x_n]$, then the **homogenization** of f is defined by

$$f^h := x_0^d f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

where $d := \deg(f)$.

Example 1.0.2 (?): Let $f(x_1, x_2) = 1 + x_1^2 + x_1x_2 + x_2^3$, then

$$f^h(x_0, x_1, x_2) = x_0^3 + x_0x_1^2 + x_0x_1x_2 + x_2^3,$$

which is a homogeneous polynomial of degree 3. Note that $(f^h)^i = f$.

Example 1.0.3 (?): It need not be the case that $(f^i)^h = f$. Take $f = x_0^3 + x_0x_1x_2$, then $f^i = 1 + x_1x_2$ and $(f^i)^h = x_0^2 + x_1x_2$. Note that the total degree dropped, since everything was divisible by x_0.

Remark 1.0.2 :

$$(f^i)^h = f \iff x_0 \nmid f.$$

Definition 1.0.3 (Homogenization of an ideal)**.**
Given $J \subset k[x_1, \cdots, x_n]$, define its **homogenization** as

$$J^h := \left\{ f^h \;\middle|\; f \in J \right\}.$$

Example 1.0.4 : This is not a ring morphism, since $(f+g)^h \neq f^h + g^h$ in general. Taking $f = x_0^2 + x_1$ and $g = -x_0^2 + x_2$, we have $f^h + g^h = x_0x_1 + x_0x_2$ while $(f+g)^h = x_12 + x_2$.

Remark 1.0.3 : What is the geometric significance? Set $U_0 := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n_{/k} \;\middle|\; x_0 \neq 0 \right\} \cong \mathbb{A}^n_{/k}$ with coordinates $\left[\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0} \right]$. If we define the Zariski topology on $\mathbb{P}^n$ as having closed sets $V_p(I)$, we would want to check that $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$ is closed in the subspace topology. This amounts to showing that $V_p(I) \cap U_0$ is closed in $\mathbb{A}^n \cong U_0$. We can check that

$$V_p(f, f \in I) = \left\{ [x_0 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0 \, \forall f \in I \right\}.$$

Intersecting with U_0 yields $\left\{ [x_1 : \cdots : x_n] \;\middle|\; f(\mathbf{x}) = 0, \, x_0 \neq 0 \right\}$. Equivalently, we can rewrite this set as

$$\left\{ [x_1 : \cdots : x_n] \;\middle|\; f\left(\left[1, \frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right] \right) = 0, \, f \text{ homogeneous} \right\}$$

Since these are coordinates on $\mathbb{A}^1$, we have $V_p(I) \cap U_0 = V_a(I^i)$ which is closed.

Conversely, given a closed set $V(I)$, we can write this as $V(I) = U_0 \cap V_p(I^h)$. The conclusion is thus that U_0 with the subspace topology is equal to $\mathbb{A}^n$ with the Zariski topology.