GENERAL TOPOLOGY PROBLEMS FROM OLD QUALS

1. General topology

1 (Spring '06). Suppose (X, d) is a metric space. State criteria for continuity of a function $f : X \to X$ in terms of:

(i) open sets;
(ii) ϵ's and δ's; and
(iii) convergent sequences.

Then prove that (iii) implies (i).

2 (Spring '12). Let X be a topological space.

(i) State what it means for X to be *compact*.
(ii) Let
$$X = \{0\} \cup \{1/n | n \in \mathbb{Z}_+\}.$$
Is X compact?
(iii) Let $X = (0, 1]$. Is X compact?

3 (Spring '09). Let (X, d) be a compact metric space, and let $f : X \to X$ be an isometry: for all $x, y \in X$, $d(f(x), f(y)) = d(x, y)$. Prove that f is a bijection.

4 (Spring '05). Suppose (X, d) is a compact metric space and $\mathcal{U}$ is an open covering of X. Prove that there is a number $\delta > 0$ such that for every $x \in X$, the ball of radius δ centered at x is contained in some element of $\mathcal{U}$.

5 (Fall '11). Let X be a topological space, and $B \subset A \subset X$. Equip A with the subspace topology, and write $cl_X(B)$ or $cl_A(B)$ for the closure of B as a subset of, respectively, X or A. Determine, with proof, the general relationshup between $cl_X(B) \cap A$ and $cl_A(B)$ (*i.e.*, are they always equal? is one always contained in the other but not conversely? neither?)

6 (Fall '05). Prove that the unit interval I is compact. Be sure to explicitly state any properties of real numbers that you use.

7 (Fall '06). A topological space is *sequentially compact* if every infinite sequence in X has a convergent subsequence. Prove that every compact metric space is sequentially compact.

8 (Fall '10). Show that for any two topological spaces X and Y, $X \times Y$ is compact if and only if both X and Y are compact.

9 (Spring '13). Recall that a topological space is said to be connected if there does not exist a pair U, V of disjoint nonempty subsets whose union is X.

(i) Prove that X is connected if and only if the only subsets of X that are both open and closed are X and the empty set.
(ii) Suppose that X is connected and let $f : X \to \mathbb{R}$ be a continuous map. If a and b are two points of X and r is a point of $\mathbb{R}$ lying between $f(a)$ and $f(b)$ show that there exists a point c of X such that $f(c) = r$.

10 (Fall '05). Let $X = \{(0, y) | -1 \le y \le 1\} \cup \{(x, \sin(1/x)) | 0 < x \le 1\}$. Prove that X is connected but not path connected.

11 (Fall '18). Let

$$X = \{(x, y) \in \mathbb{R}^2 | x > 0, y \geq 0, \text{ and } \frac{y}{x} \text{ is rational}\}$$

and equip X with the subspace topology induced by the usual topology on $\mathbb{R}^2$. Prove or disprove that X is connected.

12 (Spring '06). Write Y for the interval $[0, \infty)$, equipped with the usual topology. Find, with proof, all subspaces Z of Y which are retracts of Y.

13 (Fall '06).

(a) Prove that if the space X is connected and locally path connected then X is path connected.
(b) Is the converse true? Prove or give a counterexample.

14 (Fall '07). Let $\{X_\alpha | \alpha \in \mathcal{A}\}$ be a family of connected subspaces of a space X such that there is a point $p \in X$ which is in each of the X_α. Show that the union of the X_α is connected.

15 (Fall '04). Let X be a topological space.

(a) Prove that X is connected if and only if there is no continuous nonconstant map to the discrete two-point space $\{0, 1\}$.
(b) Suppose in addition that X is compact and Y is a connected Hausdorff sace. Suppose further that there is a continuous map $f : X \to Y$ such that every preimage $f^{-1}(y)$, $y \in Y$, is a connected subset of X. Show that X is connexted.
(c) Give an example showing that the conclusion of b) may be false if X is not compact.

16 (Spring '10). If X is a topological space and $S \subset X$, define, in terms of open subsets of X, what it means for S *not* to be connected. Show that if S is not connected there are nonempty subsets $A, B \subset X$ such that $A \cup B = S$ and $A \cap \bar{B} = \bar{A} \cap B = \varnothing$ (here $\bar{A}$ and $\bar{B}$ denote closure with respect to the topology on the ambient space X).

17 (Spring '11). A topological space is *totally disconnected* if its only connected subsets are one-point sets. Is it true that if X has the discrete topology, it is totally disconnected? Is the converse true? Justify your answers.

18 (Fall '07). Prove that if (X, d) is a compact metric space, $f : X \to X$ is a continuous map, and C is a constant with $0 < C < 1$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for all x, y, then f has a fixed point.

19 (Spring '15). Prove that the product of two connected topological spaces is connected.

20 (Fall '14). (a) define what it means for a topological space to be:

(i) connected
(ii) locally connected

(b) Give, with proof, an example of a space that is connected but not locally connected.

21 (Fall '14). Let X and Y be topological spaces and let $f : X \to Y$ be a function. Suppose that $X = A \cup B$ where A and B are closed subsets, and that the restrictions $f|_A$ and $f|_B$ are continuous (where A and B have the subspace topology). Prove that f is continuous.

22 (Fall '18). Let X be a compact space and let $f : X \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x, 0) > 0$ for all $x \in X$. Prove that there is $\epsilon > 0$ such

that $f(x,t) > 0$ whenever $|t| < \epsilon$. Moreover give an example showing that this conclusion may not hold if X is not assumed compact.

23 (Spring '15). Define a family $\mathcal{T}$ of subsets of $\mathbb{R}$ by saying that $A \in \mathcal{T}$ is if and only if $A = \varnothing$ or $\mathbb{R} \setminus A$ is a finite set. Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$, and that $\mathbb{R}$ is compact with respect to this topology.

24 (Spring '16). In each part of this problem X is a compact topological space. Give a proof or a counterexample for each statement.

(a) If $\{F_n\}_{n=1}^{\infty}$ is a sequence of nonempty closed subsets of X such that $F_{n+1} \subset F_n$ for all n then $\cap_{n=1}^{\infty} F_n$ is nonempty.

(b) If $\{O_n\}_{n=1}^{\infty}$ is a sequence of nonempty open subsets of X such that $O_{n+1} \subset O_n$ for all n then $\cap_{n=1}^{\infty} O_n$ is nonempty.

25 (Fall '16). Let $\mathcal{S}, \mathcal{T}$ be topologies on a set X. Show that $\mathcal{S} \cap \mathcal{T}$ is a topology on X. Give an example to show that $\mathcal{S} \cup \mathcal{T}$ need not be a topology.

26 (Fall '17). Let $f : X \to Y$ be a continuous function between topological spaces. Let A be a subset of X and let $f(A)$ be its image in Y. One of the following statements is true and one is false. Decide which is which, prove the true statement, and provide a counterexample to the false statement:

(1) If A is closed then $f(A)$ is closed.
(2) If A is compact then $f(A)$ is compact.

27 (Fall '17). A metric space is said to be **totally bounded** if for every $\epsilon > 0$ there exists a finite cover of X by open balls of radius ϵ.

(a) Show: a metric space X is totally bounded iff every sequence in X has a Cauchy subsequence.
(b) Exhibit a complete metric space X and a closed subset A of X that is bounded but not totally bounded. (You are not required to prove that your example has the stated properties.)

28 (Spring '19). Is every complete bounded metric space compact? If so, give a proof; if not, give a counterexample.

29 (Fall '14). Is every product (finite or infinite) of Hausdorff spaces Hausdorff? If yes, prove it. If no, give a counterexample.

30 (Spring '18). Suppose that X is a Hausdorff topological space and that $A \subset X$. Prove that if A is compact in the subspace topology then A is closed as a subset of X.

31 (Spring '09). (a) Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

(b) Give an example that shows that the "Hausdorff" hypothesis in part (a) is necessary.

32 (Fall '14). Let X be a topological space and let

$$\Delta = \{(x, y) \in X \times X | x = y\}.$$

Show that X is a Hausdorff space if and only if Δ is closed in $X \times X$.

33 (Fall '06). If f is a function from X to Y, consider the graph $G = \{(x, y) \in X \times Y | f(x) = y\}$.

(a) Prove that if f is continuous and Y is Hausdorff, then G is a closed subset of $X \times Y$.

(b) Prove that if G is closed and Y is compact, then f is continuous.

34 (Fall '04). Let X be a noncompact locally compact Hausdorff space, with topology $\mathcal{T}$. Let $\bar{X} = X \cup \{\infty\}$ (X with one point adjoined), and consider the

family $\mathcal{B}$ of subsets of $\bar{X}$ defined by

$$\mathcal{B} = \mathcal{T} \cup \{S \cup \{\infty\} | S \subset X,\ X \setminus S \text{ is compact}\}.$$

(a) Prove that $\mathcal{B}$ is a topology on $\bar{X}$, that the resulting space is compact, and that X is dense in $\bar{X}$.

(b) Prove that if $Y \supset X$ is a compact space such that X is dense in Y and $Y \setminus X$ is a singleton, then Y is homeomorphic to $\bar{X}$. (The space $\bar{X}$ is called the *one-point compactification* of X.)

(c) Find familiar spaces that are homeomorphic to the one point compactifications of (i) $X = (0, 1)$ and (ii) $X = \mathbb{R}^2$.

35 (Fall '16). Prove that a metric space X is normal, i.e. if $A, B \subset X$ are closed and disjoint then there exist open sets $A \subset U \subset X$, $B \subset V \subset X$ such that $U \cap V = \varnothing$.

36 (Spring '06). Prove that every compact, Hausdorff topological space is normal.

37 (Spring '09). Show that a connected, normal topological space with more than a single point is uncountable.

38 (Spring '08). Give an example of a quotient map in which the domain is Hausdorff, but the quotient is not.

39 (Fall '04). Let X be a compact Hausdorff space and suppose $R \subset X \times X$ is a closed equivalence relation. Show that the quotient space X/R is Hausdorff.

40 (Spring '18). Let $U \subset \mathbb{R}^n$ be an open set which is bounded in the standard Euclidean metric. Prove that the quotient space $\mathbb{R}^n/U$ is not Hausdorff.

41 (Fall '09). Let A be a closed subset of a normal topological space X. Show that both A and the quotient X/A are normal.

42 (Spring '10). Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Let X be the set of equivalence classes, endowed with the quotient topology induced by the canonical projection $\pi : \mathbb{R} \to X$. Describe, with proof, all open subsets of X with respect to this topology.

43 (Fall '12). Let A denote a subset of points of S^2 that looks exactly like the capital letter A. Let Q be the quotient of S^2 given by identifying all points of A to a single point. Show that Q is homeomorphic to a familiar topological space and identify that space.

44 (Spring '15). (a) Prove that a topological space that has a countable base for its topology also contains a countable dense subset.

(b) Prove that the converse to (a) holds if the space is a metric space.

45 (Spring '11). Recall that a topological space is *regular* if for every point $p \in X$ and for every closed subset $F \subset X$ not containing p, there exist disjoint open sets $U, V \subset X$ with $p \in U$ and $F \subset V$. Let X be a regular space that has a countable basis for its topology, and let U be an open subset of X.

(a) Show that U is a countable union of closed subsets of X.

(b) Show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.