

# Title

D. Zack Garza

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## 1 | Wednesday, September 16

### 1.1 Group Schemes

**Definition 1.0.1** (Representable Functors).

Let  $F :: k\text{-alg} \rightarrow \text{Set}$  be a functor, then  $F$  is **representable** iff  $F(R)$  corresponds to “solutions to equations in  $R$ ”.

**Example 1.1.**

Let  $F(\cdot) = \text{SL}(2, \cdot)$ , then the corresponding equations are  $\det(x_{ij}) = 1$ .

If  $F$  is representable, there is a correspondence  $F(R) \cong \text{hom}_R(A, R)$ . In the above example,

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}] / \langle x_{11}x_{22} - x_{12}x_{21} \rangle,$$

which is exactly the coordinate algebra.

**Definition 1.0.2** (Affine Group Scheme).

An *affine group scheme* is a representable functor  $F : k\text{-alg} \rightarrow \text{Groups}$ .

Suppose  $G$  is an affine group scheme, and let  $A = k[G]$  be the representing object. Then there is a correspondence

$$G\text{-modules} \iff k[G]^\vee\text{-modules}.$$

For  $G$  reductive, the RHS is equivalent to  $\text{Dist}(G)$ -modules.

**Definition 1.0.3** (Finite Group Schemes).

$G$  is a **finite** group scheme iff  $k[G]$  is finite dimensional.

If  $G$  is finite, then  $A^\vee \cong k[G]^\vee$  is a cocommutative Hopf algebras. Thus representations for *finite* group schemes are equivalent to representations for finite-dimensional cocommutative Hopf algebras.

On group scheme side: see reduction, spectral sequences, conceptual arguments. On the algebra side: have bases, underlying vector space, can do concrete computations. Can take  $\text{Spec}(k[G])^\vee$  to recover a group scheme.

**1.2 Hopf Algebras**

For  $A$  a  $k$ -alg, we have a multiplication and a unit, which can be defined in terms of diagrams. To categorically reverse arrows, we can ask for a comultiplication and a counit.

$$\Delta : A \rightarrow A \otimes A$$

$$\epsilon : A \rightarrow k.$$

We'll want another map, an *antipode*

$$s : A \rightarrow A.$$

The comultiplication should satisfy

$$\begin{array}{ccc} A^{\otimes 3} & \xleftarrow{1 \otimes A} & A^{\otimes 2} \\ \Delta \otimes 1 \uparrow & & \uparrow \Delta \\ A^{\otimes 2} & \xleftarrow{\Delta} & A \end{array}$$

The counit should satisfy

$$\begin{array}{ccc} k \otimes A & \xleftarrow{\epsilon \otimes 1} & A^{\otimes 2} \\ \downarrow \cong & & \uparrow \Delta \\ A & \xrightarrow{\cong} & A \end{array}$$

And the antipode should satisfy

$$\begin{array}{ccc} A & \xleftarrow{m(s \otimes 1)} & A \\ \uparrow & & \uparrow \Delta \\ A & \xleftarrow{\epsilon} & A \end{array}$$

**1.2.1 Module Constructions**

Let  $A$  be a Hopf algebra.

1. For  $A$ -modules  $M, N$ , we can form the  $A$ -module  $M \otimes_k N$  with

$$\Delta(a) = \sum a_i \otimes a_j$$

$$a(m \otimes n) = \sum a_1 m \otimes a_2 n.$$

2. If  $M$  is finite-dimensional over  $A$ , then  $M^\vee = \text{hom}_k(M, k) \ni f$  is an  $A$ -module, and we can define  $(af)(x) := f(s(a)x)$  for  $a \in A, x \in M$ .

**Example 1.2.**

$A = kG$  the group algebra on a finite group is a Hopf algebra:

$$\begin{aligned} \Delta : A &\rightarrow A^{\otimes 2} \\ g &\mapsto g \otimes g. \end{aligned}$$

The module action is diagonal, namely  $g(m \otimes n) = gm \otimes gn$ . The antipode is given by  $s(g) = g^{-1}$ , and the unit is  $\varepsilon(g) = 1$  for all  $g \in G$ .

**Example 1.3.**

Let  $A = U(\mathfrak{g})$ , the universal enveloping algebra for  $\mathfrak{g}$  a Lie algebra. Recall that  $\mathfrak{g}$ -modules are equivalent to  $U(\mathfrak{g})$ -modules (unitary representations, some big associative algebra). Then  $A$  is a Hopf algebra, with  $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$  for  $\ell \in \mathfrak{g}$ . The unit is  $\varepsilon(\ell) = 0$ , and the antipode is  $s(\ell) = -\ell$ .

**Example 1.4.**

Take the additive group  $\mathbb{G}_a$ , then  $A = k[\mathbb{G}_a] \cong k[x]$  is a commutative Hopf algebra with  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $\varepsilon(x) = 0$ ,  $s(x) = -x$ .

**Example 1.5.**

For  $\mathbb{G}_m$ , we have  $A = k[\mathbb{G}_m] \cong k[x, x^{-1}]$ ,  $\varepsilon(x) = 1$ ,  $s(x) = x^{-1}$ .

### 1.3 Frobenius Kernels

Let  $G$  be an algebraic group (scheme) over  $k$ , where  $\text{char}(k) = p$ . Let  $F : G \rightarrow G$  be the Frobenius, where e.g.

$$\begin{aligned} F : \text{GL}(n, \cdot) &\rightarrow \text{GL}(n, \cdot) \\ (x_{ij}) &\mapsto (x_{ij}^p). \end{aligned}$$

Then  $F$  is a map of group schemes.

**Definition 1.0.4** (Frobenius Kernels).

$G_r := \ker F^r$ , where  $F^r := F \circ F \circ \cdots \circ F$  is the  $r$ -fold composition of the Frobenius.

This yields a nesting  $G_1 \subseteq G_2 \subseteq G_3 \cdots \subseteq G$ .

Recall that

$$\text{Dist}(G) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \right\rangle.$$

We get a chain of finite dimensional algebras

$$\text{Dist}(G_1) \leq \text{Dist}(G_2) \leq \cdots \leq \text{Dist}(G)$$

where

$$\text{Dist}(G_1) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \mid 0 \leq n, m, k \leq p-1 \right\rangle,$$

where in general  $\text{Dist}(G_\ell)$  goes up to  $p^\ell - 1$ . Recall that  $G_r$  representations were equivalent to  $\text{Dist}(G_r)$  representations.