Title

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Contents

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p.

$$
\begin{aligned}
k[\mathbf{x}] &\coloneqq k[x_1, \cdots, x_n] && \text{The polynomial ring in } n \text{ indeterminates} \\
k(\mathbf{x}) &\coloneqq k(x_1, \cdots, x_n) && \text{The rational function field } \{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}),\ p, q, \in k[x_1, \cdots, x_n]\} \\
V(J), V_a(J) & && \text{The variety associated to an ideal } J \trianglelefteq k[x_1, \cdots, x_n] \\
& && V(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \;\middle|\; f(\mathbf{x}) = 0,\ \forall f \in J \right\} \\
I(S), I_a(S) & && \text{The ideal associated to a subset } S \subseteq \mathbb{A}^n_k \\
& && I(S) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(\mathbf{x}) = 0\, \forall \mathbf{x} \in X \right\} \\
A(X) &\coloneqq k[x_1, \cdots, x_n]/I(X) && \text{The coordinate ring of a variety} \\
\mathcal{O}_X & && \text{The structure sheaf } \left\{ f : U \to k \;\middle|\; f \in k(\mathbf{x}) \text{ locally} \right\}.
\end{aligned}
$$

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

0.4 Useful Algebra Facts

Fact 0.4.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff$ $R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff$ $R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f, then $\langle f \rangle$ is maximal

Proposition 0.4.2 ***(Finitely generated polynomial rings are Noetherian).***
A polynomial ring $k[x_1, \cdots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \cdots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \cdots, f_m \rangle$.

Proof (?).
A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.4.5), $k[x_1, \cdots, x_n]$ is thus Noetherian.

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Proposition 0.4.3 ***(Properties and Definitions of Ideal Operations).***

$$I + J := \left\{ f + g \;\middle|\; f \in I,\, g \in J \right\}$$

$$IJ := \left\{ \sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\, g_i \in J, N \in \mathbb{N} \right\}$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ \qquad \text{(coprime or comaximal)} \ \langle a \rangle + \langle b \rangle = \langle a, b \rangle \, .$$

Theorem 0.4.4 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Theorem 0.4.5 ***(Hilbert's Basis Theorem).***
If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k \coloneqq \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomi
	$I \mapsto V(I) \coloneqq \left\{ a \;\middle\vert\; f(a) = 0 \forall f \in I \right\}$
$I(X) \coloneqq \left\{ f \;\middle\vert\; f\vert_X = 0 \right\} \mapsfrom X$	
Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets