# Problem Set 1

# D. Zack Garza

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# 1 Problem 5

#### 1.1 Part 1

Let  $A \in \operatorname{Mat}(n, n)$  be a positive definite  $n \times n$  matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and  $B \in Math(n, n)$  be positive semi-definite, so

$$\langle v, Bv \rangle \ge 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A+B)v \rangle \ge 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\langle v, (A+B)v \rangle = \langle v, Av \rangle + \langle v, Bv \rangle$$
  
>  $\langle v, Av \rangle + 0$   
 $\geq 0 + 0$   
= 0.

#### 1.2 Part 2

Let M be a smooth manifold with a maximal smooth atlas  $\mathcal{A}$ , and choose a covering of M by charts  $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A} \text{ such that } M \subseteq \bigcup_{i \in I} U_i.$ 

Then choose a partition of unity  $\{f_i\}_{i\in I}$  subordinate to  $\mathcal{C}$ . In each copy of  $\phi_i(U_i) \cong \mathbb{R}^n$ , let  $g^i$  be the Euclidean metric given by the identity matrix, i.e.  $g^i_{jk} := \delta_{jk}$ . We thus have

$$g^{i}: T\phi_{i}(U_{i}) \times T\phi_{i}(U_{i}) \to \mathbb{R}$$

$$(\partial x_{i}, \partial x_{j}) \mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

which takes pairs in  $T\phi_i(U_i) \cong T\mathbb{R}^n \cong \mathbb{R}^n$ , which is spanned by  $\{\partial x_i\}$ , and is defined as the kronecker delta on basis vectors and extended linearly.

Let 
$$G^i := (\phi_i)_* g^i(p,q) == g^i(\phi_i(p), \phi_i(q))$$

## 2 Problem 6

#### 2.1 Part 1

Let  $M = S^2$  as a smooth manifold, and consider a vector field on M,

$$X: M \to TM$$

We want to show that there is a point  $p \in M$  such that X(p) = 0.

Every vector field on a compact manifold without boundary is complete, and since  $S^2$  is compact with  $\partial S^2 = \emptyset$ , X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi: M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} = X(\phi_t(p)),$$

$$\phi_0(p) = p$$

at every point  $p \in M$ .

This yields a one-parameter family

$$\phi_t: M \to M \in \text{Diff}(M, M).$$

In particular,  $\phi_0 = \mathrm{id}_M$ , and  $\phi_1 \in \mathrm{Diff}(M, M)$ . Moreover  $\phi_0$  is homotopic to  $\phi_1$  via the homotopy

$$H: M \times I \to M$$
  
 $(p,t) \mapsto \phi_t(p).$ 

We can now apply the Lefschetz fixed-point theorem to  $\phi_0$  and  $\phi_1$ . For an arbitrary map  $f: M \to M$ , we have

$$\Lambda(f) = \sum_{k} \operatorname{Tr} \left( f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where  $f_*: H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$  is the induced map on homology, and

 $\Lambda(f) \neq 0 \iff f$  has at least one fixed point.

In particular, we have

$$\Lambda(\mathrm{id}_M) = \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})})$$
$$= \sum_k \dim H_k(X;\mathbb{Q})$$
$$= \chi(M),$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have  $\Lambda(\phi_1) = \chi(M)$ .

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2\\ 0 & \text{otherwise} \end{cases}$$

we have  $\chi(S^2) = 2 \neq 0$ , and thus  $\phi_1$  has a fixed point  $p_0$ , thus  $\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$  so

$$\phi_t(p) = p$$

$$\Rightarrow \frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 \qquad \text{by differentiating wrt } t$$

$$\Rightarrow \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \Big|_{t=0} = 0 \qquad \text{by evaluating at } t = 0$$

$$\Rightarrow X(\phi_1(p_0)) := \frac{\partial}{\partial t} \phi_t(p) \Big|_{t=1} = 0 \qquad \text{by definition of } \phi_1$$

so  $X(\phi_1(p_0)) = 0$ , which shows that  $p_0$  is a zero of X. So X has at least one zero, as desired.  $\square$ 

### 2.2 Part 2

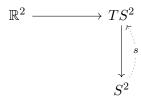
The trivial bundle

$$\mathbb{R}^2 \xrightarrow{S^2 \times \mathbb{R}^2} \int_{S^2}^{\pi}$$

has a nowhere vanishing section, namely

$$s: S^2 \to S^2 \times \mathbb{R}^2$$
  
 $\mathbf{x} \to (\mathbf{x}, [1, 1])$ 

which is the identity on the  $S^2$  component and assigns the constant vector [1, 1] to every point. However, as part 1 shows, the bundle



can not have a nowhere vanishing section.  $\square$