Math 174 HW #6

Wednesday, 15 November, 2017 03:48 PM

Daniel Zack Garza

1)

x_i	y_i	x_i^2	$x_i y_i$
-1	-1.1	1	1.1
-0.5	-1	-.25	.25
0	0.9	0	0
0.5	1.8	.25	.9
1	3.2	1	3.2
Σ: 0 $=\sum x_i$	3.8 $=\sum y_i$	2 $=\sum x_i^2$	5.45 $=\sum x_i y_i$

a)

$$a_0 = (\sum x_i^2)(\sum y_i) - (\sum x_i y_i)(\sum x_i) / m\sum x_i^2 - (\sum x_i)^2$$

$$= 2 \cdot 3.8 - \cancel{5.45 \cdot 0} / 5\, 2 - \cancel{0^2}$$

$$= 7.6/10 = \underline{0.76}$$

$$a_1 = m\sum x_i y_i - \sum x_i \sum y_i / m\sum x_i^2 - (\sum x_i)^2$$

$$= 5 \cdot 5.45 - \cancel{0 \cdot 3.8} / 5 \cdot 2 - 0^2$$

$$= 27.25/10 = \underline{2.725}$$

b)

$$\boxed{\to L(x) = a_1 x + a_0 = 2.725x + 0.76}$$

$\underline{n=2, m=5}$

$j=0$

$$a_0 m + a_1 \sum x_i + a_2 \sum x_i^2 = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x_i y_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x_i^2 y_i$$

c)

$$\to \begin{bmatrix} m & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{bmatrix} \qquad \boxed{\text{Eqn (1)}}$$

$$\to \begin{bmatrix} 5 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2.125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.8 \\ 5.45 \\ 2.3 \end{bmatrix}$$

2) Let $P(x) = a_0$, then

$$E = \sum_{i=1}^{n} (y_i - P(x_i))^2$$

$$= \sum_{i=1}^{n} (y_i - a_0)^2$$

$$= \sum_{i=1}^{n} y_i^2 - 2a_0 \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a_0^2 = \sum y_i^2 - 2a_0 \sum y_i + na_0^2$$

$$= \sum_{i=1}^{n} y_i^2 - 2a_0 \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a_0^2 = \sum y_i^2 - 2a_0 \sum y_i + n a_0^2$$

$$\rightarrow \partial E / \partial a_0 = -2 \sum_{i=1}^{n} y_i + 2 n a_0$$

And $\partial E / \partial a_0 = 0 \Rightarrow \sum_{i=1}^{n} y_i = n a_0$ $\boxed{\Rightarrow a_0 = \frac{1}{n} \sum_{i=1}^{n} y_i}$

So a_0 is the average/mean of the y_i.

3) Using eqn 1

$\sum x_i = 1$ $\quad \sum y_i = 5.3$

$\sum x_i^2 = 0.3$ $\quad \sum x_i y_i = 1.1$

$\sum x_i^3 = 0.1$ $\quad \sum x_i^2 y_i = 0.3354$

$\sum x_i^4 = 0.0354$

$$\rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & .3 \\ 1 & .3 & .1 \\ .3 & .1 & .0354 \end{bmatrix}^{-1} \begin{bmatrix} 5.3 \\ 1.1 \\ .3354 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \varepsilon \\ 1 \end{bmatrix} \text{ where } \varepsilon \approx 1.4211 \cdot 10^{-14}$$

$$\boxed{\rightarrow P(x) = x^2 + \varepsilon x + 1 \ (\approx x^2 + 1)}$$

4) Let $P(x) = \sum_{i=0}^{n} a_i (1+x)^i$, then $P_n(\bar{x}) = A\bar{x}$ for some A

Then

$$P(x_i) = y_i \Rightarrow a_0 + a_1(1+x_i) + a_2(1+x_i+x_i^2) = y_i$$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 1+x_1 & 1+x_1+x_1^2 \\ 1 & 1+x_2 & 1+x_2+x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & 1+x_n & 1+x_n+x_n^2 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{\bar{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\bar{y}}$$

$\rightarrow$ Solve $X\bar{a} = \bar{y}$ for $\bar{a}$.

So define $r(\bar{a}) = \bar{y} - X\bar{a}$, and let us minimize $\|r(\bar{a})\|^2 = \|X\bar{a} - \bar{y}\|^2$. where $\langle \bar{v}, \bar{w} \rangle_c = \sum_{i=1}^{n} c_i v_i w_i$ by defn and $\|\bar{v}\|_c = \sqrt{\langle \bar{v}, \bar{v} \rangle_c}$

We then have $\|r(\bar{a})\|_c^2 = \langle r(\bar{a}), r(\bar{a}) \rangle_c = \langle C r(\bar{a}), r(\bar{a}) \rangle$ $\rightarrow$ where $C_{ij} = \begin{cases} c_i & \text{if } i=j \\ 0 & \text{else} \end{cases} = \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \ c_n \end{bmatrix}$, note $C = C^T$

$$= \langle C^{1/2} r(\bar{a}), C^{1/2} r(\bar{a}) \rangle \quad \rightarrow \text{where } C^{1/2}_{ij} = \sqrt{C_{ij}}$$

$$= \| C^{1/2} r(\bar{a}) \|_2^2$$

$$= \| C^{1/2} (X\bar{a} - \bar{y}) \|_2^2$$

which is minimized when $X^T C^{1/2} X \hat{a} = X^T C^{1/2} \bar{y}$

$$\boxed{\rightarrow \hat{a} = (X^T C^{1/2} X)^{-1} X^T C^{1/2} \bar{y}}$$

5) a)

x_i	y_i	Back	Center	Forward
0	1	x	x	.05
0.1	1.01	.05	.2	.15
0.2	1.04	.15	.4	.25
0.3	1.09	.25	.6	.35
0.4	1.16	.35	x	x

$(h = 0.1, 2h = 0.2)$

Note the central dif has error in $O(h^2)$, so it's used when possible.

b) $f(x) = 1+x^2 \to \frac{\partial}{\partial x} f(x) = 2x$

x_i	$f'(x_i)$	Δ_x	$\lvert\Delta_x - f'(x)\rvert$
0	0	05	.05
0.1	.2	.2	0
0.2	.4	.4	0
0.3	.6	.6	0
0.4	.8	.35	.45

c) $f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h} = (1/2h)\left(f(x_i+h) - f(x_i-h)\right)$

$$= (1/2h)\left(\sum_{n=0}^{\infty} \frac{1}{n!} h^n f^{(n)}(x_i) - \sum_{n=0}^{\infty} \frac{1}{n!}(-h)^n f^{(n)}(x_i)\right)$$

$$= (1/2h) \sum_{n=0}^{\infty} \left(\frac{1}{n!} h^n f^{(n)}(x_i)\right) + (-1)^{n+1} h^n f^{(n)}(x_i)\Big)\ \Big]$$ when n is even, n+1 is odd, and the terms cancel

$$= (1/2h) \sum_{n\text{ odd}} 2 \cdot \frac{1}{n!} h^n f^{(n)}(x_i)$$

$$= \frac{1}{h}\left(h f'(x) + \tfrac{1}{6} h^3 f^{(3)}(x_i) + \dots\right)$$

$$= f'(x) + \underbrace{\tfrac{1}{6} h^2 f^{(3)}(x_i) + \dots}$$

But $f^{(3)}(x) = \frac{\partial^3}{\partial x^3} 1+x^2$

$= \frac{\partial^2}{\partial x^2} 2x$

$= \frac{\partial}{\partial x} 2 \quad = 0$ for all x, so these terms are all zero

and $\Delta_x = f'(x) + 0$
$= f(x)$

d) Using $f''(x_i) \approx \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2}$

$$\to f''(.2) = \frac{1.01 - 2(1.04) + 1.09}{(.1)^2} \boxed{= 2}$$

6) $h = 0.1$

$f'(1) \cong \frac{f(1+.1) + f(1-.1)}{.2} = \frac{f(1.1) - f(.9)}{2} \approx .539402252$

$h = 0.05$

$f'(1) \approx \frac{f(1.05) - f(.95)}{.1} \approx .540077208$

$h = .025$ $\quad f'(1) \approx \frac{f(1.025) - f(.975)}{.05} \approx .540246026$

b) $\frac{\partial}{\partial x} \sin(x) = \cos(x)$

$h = 0.1$ $\quad E = \lvert\Delta_x - \cos(1)\rvert \approx 8.798 \times 10^{-4}$

$h = .05$ $\quad E = 2.251 \times 10^{-4}$

$h = .025$ $\quad E = 5.627 \times 10^{-5}$

c) $h = 0.1 \longrightarrow E(h)/E(h/2) \approx 3.903$

$h = .05 \longrightarrow E(h)/E(h/2) \approx 4.0004$

$\leadsto$ Possibly converging to 4.

d) $E(x,h) \approx O(h^2)$, so choose h such that

$$h^2 \leq 10^{-10}$$

$$\rightarrow 2\log_{10} h \leq -10$$

$$\rightarrow \log_{10} h \leq -5$$

$$\rightarrow \boxed{h \leq 10^{-5}.}$$

7) $$(1/h^2)(f(x+h) - 2f(x) + f(x-h)) = (1/h^2)\left(\sum_{n=0}^{\infty} \frac{1}{n!} h^n f^{(n)}(x) - 2\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) + \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n h^n f^{(n)}(x) \right)$$

$$= (1/h^2) \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x) \left(h^n - 2 + (-1)^n h^n\right)$$

$$= (1/h^2)\left[0 - 2f'(x) + \tfrac{1}{2}(2h^2-2)f''(x) - \tfrac{1}{3} f'''(x) + \tfrac{1}{24}(2h^4-2) f^{(4)}(x) - \ldots \right]$$

$$= (-2/h^2) f'(x) + \frac{h^2-1}{h^2} f''(x) - \frac{1}{3h^2} f'''(x) + \frac{1}{12} \frac{h^4-1}{h^2} f^{(4)}(x) - \ldots$$

$$\rightarrow \left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| \leq \left| \frac{-2}{h^2} f'(x) - \frac{1}{h^2} f''(x) - \frac{1}{3h^2} f'''(x) + \underbrace{\frac{h^2 - 1/h^2}{12} f^{(4)}(x)}_{\text{First positive power of } h} \cdots \right|$$

$$\approx O(h^2)$$

8) See attached.