

Title

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1.1 Chapter 1

Let k be a field, not necessarily algebraically closed.

Definition 1.1.1 (Algebraic Function Field).

An one variable **algebraic function field** F/K is a field extension F of K which factors as



where $x \in \bar{k}$ is some element that is not algebraic over k .

Definition 1.1.2 (Field of Constants).

The subfield

$$\tilde{k} := \{z \in F \cap K^{\text{alg}}\} \leq F,$$

consisting of elements that are algebraic over F is denoted the **field of constants**.

Definition 1.1.3 (Algebraically Closed).

If $\tilde{k} = k$, we say that k is **algebraically closed** in F .

Definition 1.1.4 (Rational Function Field).

An extension F/k is **rational** iff $F = k(y)$ for some $y \in k^{\text{transc}}$ which is transcendental over k .

Definition 1.1.5 (Valuation Ring).

A ring $\mathcal{O} \subseteq F$ is a **valuation ring** for F iff $k \subset \mathcal{O} \subseteq F$ and $z \in F \implies z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$.

Definition 1.1.6 (Discrete Valuation Ring).

A ring local R (thus with a unique maximal ideal) which is a PID but not a field is a **discrete valuation ring**.

Definition 1.1.7 (Place).

A **place** of a function field F/K is the maximal ideal of a valuation ring of F/K .

Definition 1.1.8 (Discrete Valuation).

A **discrete valuation** of F/k is a function

$$v : F \rightarrow \mathbb{Z} \cup \{\infty\}$$

that is

1. Nondegenerate: $v(x) = \infty$ iff $x = 0$.
2. Multiplicative: $v(xy) = v(x) + v(y)$.
3. Ultrametric triangle inequality: $v(x + y) \geq \min(v(x), v(y))$.
4. Fiber over one: there exist a $z \in F$ with $v(z) = 1$.
5. $v|_k = 0$.

Definition 1.1.9 (Rational Place).

A place of degree one is said to be a **rational place**.

Definition 1.1.10 (Valuation Ring of a Place).

The **valuation ring of a place** is defined by

$$\mathcal{O}_P := \left\{ z \in F \mid z^{-1} \notin P \right\}.$$

Definition 1.1.11 (Degree of a Place).

The **degree** of a place P is defined by

$$\deg(P) := [F_P : k],$$

where $F_P = \mathcal{O}_P/P$.

Definition 1.1.12 (Discrete Valuation of a Place).

To any place P we associate the function

$$v_P : F \rightarrow \mathbb{Z} \cup \{\infty\}$$

defined by choosing any prime $t \in P$, writing any $x \in F$ as $x = t^n u$ with $u \in \mathcal{O}_P^\times$, and setting

$$v_P(x) = \begin{cases} n & \text{if } x = t^n u \\ 0 & x = \infty. \end{cases}$$

Note: from now on we assume $\tilde{K} = K$

Definition 1.1.13 (Divisor).

The **divisor group** of F/K is the free abelian group on the set of places of F/K , i.e. a formal sum

$$D = \sum_{p \in} .$$