Problem Set 1

D. Zack Garza

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1 Problem 1

We'll use the following definition of a smooth map between manifolds:

Definition: Let M, N be smooth manifolds of dimensions m, n respectively and $f : M \to N$ a continuous map. Then f is *smooth* iff for every $p \in M$, there exists a chart (U, ϕ) with $p \in U$ and a chart (V, ψ) with $f(p) \in V$ such that

- $f(U) \subseteq V$, and
- The induced map

$$\begin{aligned} \overline{f} &: \phi(U) \to \psi(V) \\ \overline{f} &= \psi \circ f \circ \phi^{-1} \end{aligned}$$

is smooth as a map from $\mathbb{R}^m \to \mathbb{R}^n$.

We will thus show that both $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ and $f^{-1} : \mathbb{CP}^1 \to \mathbb{CP}^1$ are both smooth bijections, from which we can conclude that f is a diffeomorphism.

So identify $0 = [0,1]$ and $\infty = [1,0]$ in $\mathbb{CP}^1$ and choose the following charts on $\mathbb{CP}^1$ in terms of homogeneous coordinates:

$$
\begin{aligned}
(U, \phi) &:= \\
U = \mathbb{CP}^1 \setminus \{\infty\} &= \{[x,y] \mid x, y \in \mathbb{C},\ y \neq 0\} \\
\phi : \mathbb{CP}^1 &\to \mathbb{C} \\
[x,y] &\mapsto x/y. \\
\\
\phi^{-1} : \mathbb{C} &\to \mathbb{CP}^1 \\
z &\mapsto [z,1].
\end{aligned}
$$

and

$$
\begin{aligned}
(V, \psi) &:= \\
V = \mathbb{CP}^1 \setminus \{0\} &= \{[x,y] \mid x, y \in \mathbb{C},\ x \neq 0\} \\
\psi : \mathbb{CP}^1 &\to \mathbb{C} \\
[x,y] &\mapsto y/x. \\
\\
\psi^{-1} : \mathbb{C} &\to \mathbb{CP}^1 \\
z &\mapsto [1,z].
\end{aligned}
$$

Now define

$$
\begin{aligned}
\tilde{f} : \mathbb{CP}^1 &\to \mathbb{CP}^1 \\
p &\mapsto \begin{cases} p, & p = \infty \\ p + c & \text{otherwise} \end{cases}
\end{aligned}
$$

We then need to determine a formula for $\tilde{f}$ in homogeneous coordinates. We compute

$$
\begin{aligned}
p \in U \implies p = [a,b],\ a, b, \in \mathbb{C},\ b \neq 0 \\
\implies \tilde{f}([a,b]) \Big|_U &= (\phi^{-1} \circ f \circ \phi)([a,b]) \\
&= (\phi^{-1} \circ f)(\frac{b}{a}) \\
&= \phi^{-1}(\frac{b}{a} + c) \\
&= [\frac{b}{a} + c, 1] \\
&= [b + ac, a]
\end{aligned}
$$

and

$$\begin{aligned} p \in V \implies p &= [a,b],\ a,b,\in \mathbb{C},\ a \neq 0 \\ \implies \tilde{f}([a,b])\Big|_V &= (\psi^{-1} \circ f \circ \psi)([a,b]) \\ &= (\psi^{-1} \circ f)(\frac{a}{b}) \\ &= (\psi^{-1})(\frac{a}{b} + c) \\ &= [1, \frac{a}{b} + c] \\ &= [b, a + bc] \end{aligned}$$

Since $\mathbb{CP}^1 = U \bigcup V$, we can note that if $p \in M$ then either $p \in U$ or $p \in V$. So in order for $\tilde{f}$ to be smooth, we just need to check that the following two maps are smooth

- $f_U : \mathbb{C} \to \mathbb{C}, f_U \coloneqq \phi \circ \tilde{f} \circ \phi^{-1}$.
- $f_V : \mathbb{C} \to \mathbb{C}, f_V \coloneqq \psi \circ \tilde{f} \circ \psi^{-1}$.

2 Problem 3

2.1 Part 1

Note: Throughout this question, we will identify $\{f : C^\infty(M) \to \mathbb{R}\} \cong C^\infty(M)^\vee$ as vector spaces.

Let M, N be smooth manifolds and $f : M \to N$ be a fixed smooth map, and define a map

$$\begin{aligned} \phi : C^\infty(N) \times TM &\to \mathbb{R} \\ (h, v) &\mapsto v(h \circ f) \end{aligned}$$

2.2 Part 1

Using the derivation definition, we can identify this assignment as a map

$$\begin{aligned} \phi : C^\infty(N) \times C^\infty(M)^\vee &\to \mathbb{R} \\ (h, v) &\mapsto v(h \circ f) \end{aligned}$$

We'd like to show that this yields a well-defined element of $T_pM = C^\infty(M)$. So for some fixed $v \in T_pM$, define a map

$$\begin{aligned} \phi_v : C^\infty(N) &\to \mathbb{R} \\ h &\mapsto v(h \circ f), \end{aligned}$$

which will be an element of TM if it is a derivation. For $x \in N$, we have

$$
\begin{aligned}
\phi_v(h_1 \cdot h_2)(x) &\coloneqq v((h_1 h_2) \circ f)(x) \\
&= v((h_1 \circ f)(h_2 \circ f))(x) \\
&= v(h_1 \circ f)(x) \cdot h_2(x) + h_1(x) \cdot v(h_2 \circ f)(x) \quad \text{since } v \text{ is a derivation} \\
&= \phi_v(h_1)(x) \cdot h_2(x) + h_1(x) \cdot \phi_v(h_2)(x).
\end{aligned}
$$

2.3 Part 2

Using the integral curve definition,

3 Problem 4

3.1 Part 1

Let $V = \mathbb{R}^n$ as a vector space, let g be a nonsingular matrix, and define a map

$$
\begin{aligned}
\phi : V &\to V^\vee \\
v &\mapsto (\phi_v : w \mapsto \langle v,\ gw \rangle)
\end{aligned}
$$

The claim is that ϕ is a natural isomorphism. It is clearly linear (following from the linearity of the inner product and matrix multiplication), so it remains to check that it is a bijection.

To see that $\ker \phi = 0$, so that only the zero gets sent to the zero map, we can suppose that $x \in \ker \phi$. Then $\phi_x : w \to \langle x,\ gw \rangle$ is the zero map. But the inner product is nondegenerate by definition, i.e. $\langle x,\ y \rangle = 0\ \forall y \implies x = 0$. So x could only have been the zero vector to begin with.

But $\dim V = \dim V^\vee$, so any injective linear map will necessarily be surjective as well.

3.2 Part 2

Let $g : TM \otimes TM \to \mathbb{R}$ be a metric, and consider the tangent space TM. By definition, the cotangent space $T_p^* M = (T_p M)^\vee$

4 Problem 5

4.1 Part 1

Let $A \in \mathrm{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \mathrm{Math}(n, n)$ be positive semi-definite, so

$$\langle v,\ Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v,\ (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned}\langle v,\ (A+B)v\rangle &= \langle v,\ Av\rangle + \langle v,\ Bv\rangle \\ &> \langle v,\ Av\rangle + 0 \\ &\geq 0 + 0 \\ &= 0.\end{aligned}$$

4.2 Part 2

Let M be a smooth manifold with tangent bundle TM and a maximal smooth atlas $\mathcal{A}$. Choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i\in I} U_i$.Then choose a partition of unity $\{f_i\}_{i\in I}$ subordinate to $\mathcal{C}$, so for each i we have

$$\begin{aligned} f_i &: M \to I \\ \forall p \in M, \quad &\sum_{i\in I} f_i(p) = 1 \end{aligned}$$

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} \coloneqq \delta_{jk}$. We then have

$$\begin{aligned} g^i &: T\phi_i(U_i) \otimes T\phi_i(U_i) \to \mathbb{R} \\ (\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Note that each coordinate function $\phi_i : U_i \to \mathbb{R}^n$ induces a map $\tilde{\phi}_i : TU_i \to T\mathbb{R}^n$.

Let G^i be the pullback of g^i along these induced maps $\tilde{\phi}_i$, so

$$\begin{aligned} G^i &: TU_i \otimes TU_i \to \mathbb{R} \\ G^i(x, y) \coloneqq \left(\left(\tilde{\phi}_i\right)^* g^i\right)(x, y) &\coloneqq g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)) \end{aligned}$$

Then, for a point $p \in M$, define the following map:

$$\begin{aligned} g_p &: T_pM \otimes T_pM \to \mathbb{R} \\ (x, y) &\mapsto \sum_{i\in I} f_i(p) G^i(x, y). \end{aligned}$$

The claim is that g_p defines a metric on M, and thus the family $\{g_p \mid p \in M\}$ yields a tensor field and thus a Riemannian metric on M. If we define the map

$$\begin{aligned} g : M &\to (TM \otimes TM)^\vee \\ p &\mapsto g_p \end{aligned}$$

then g can be expressed as

$$g = \sum_{i \in I} f_i G^i.$$

We can check that this is positive definite by considering $x \in T_pM$ and computing

$$\begin{aligned} g(x,x) &\coloneqq g_p(x,x) \\ &= \sum_{i \in I} f_i(p)\ G^i(v,v) \\ &= \sum_{i \in I} f_i(p)\ g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)), \end{aligned}$$

where each term is positive semi-definite, and *at least one term* is positive definite because $\sum_i f_i(p)$ must equal 1. By part 1, this means that the entire expression is positive definite, so g is a metric. $\square$

5 Problem 6

5.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s}\phi_s(p)\ \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\to M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \operatorname{Tr}\left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned}
\Lambda(\mathrm{id}_M) &= \sum_k \operatorname{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})}) \\
&= \sum_k \dim H_k(X;\mathbb{Q}) \\
&= \chi(M),
\end{aligned}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned}
\phi_t(p) &= p \\
\implies \frac{\partial}{\partial t}\phi_t(p) &= \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\
\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} &= 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\
\implies X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} &= 0 && \text{by definition of } \phi_1
\end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. $\square$

5.2 Part 2

The trivial bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & S^2 \times \mathbb{R}^2 \arrow[d] \\
& S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\to S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\to (\mathbf{x}, [1,1]) \end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1,1]$ to every point. However, as part 1 shows, the bundle

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & TS^2 \arrow[d] \\ & S^2 \arrow[u, bend right, dotted, "s"'] \end{tikzcd}$$

can *not* have a nowhere vanishing section. $\square$