# Homological Algebra Problem Sets

**Problem Set 1**

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# 1 | Weibel 1.1.2

*Problem* 1.0.1 (Weibel 1.1.2)
Show that a morphism $u : C \to D$ of chain complexes preserves boundaries and cycles respectively, hence inducing a map $H_n(C) \to H_n(D)$ for each $n$. Prove that $H_n : \mathrm{Ch}(R\text{-mod}) \to R\text{-mod}$ is a functor.

**Solution:**

**Claim 1:** The chain map $u$ induces the following well-defined maps:

$$Z_n(u) : Z_n(C) \to Z_n(D)$$
$$B_n(u) : B_n(C) \to B_n(D).$$

*Proof (of claim (1)).*
We'll use the convention that $Z_n \coloneqq \ker d_n$ and $B_n \coloneqq \operatorname{im} d_{n+1}$ where we index chain complexes as $C = \left( \cdots \to C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \to \cdots \right)$. Unraveling definitions, we would like to show the existence of maps

$$\begin{aligned} Z_n(u) &: \ker d^C_n \to \ker d^D_n \\ B_n(u) &: \operatorname{im} d^C_{n+1} \to \operatorname{im} d^D_{n+1}. \end{aligned}$$

It suffices to show

a. $x \in \ker d^C_n \implies u_n(x) \in \ker d^D_n$, and
b. $y \in \operatorname{im} d^C_{n+1} \implies u_n(y) \in \operatorname{im} d^D_{n+1}$.

Since $u$ is a morphism of chain complexes, we have a commuting ladder where $u_{n-1} \circ d^C_n = d^D_n \circ u_n$:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \arrow[r, "d^D_{n-1}"'] & \cdots \end{tikzcd}$$

*Link to Diagram*

To see that (a) holds, we use that fact that $R$-module morphisms send $0_R \to 0_R$ (using $R$-linearity) to compute

$$\begin{aligned} & x \in \ker d^C_n && \leq C_n && \\ \iff & d^C_n(x) = 0_R && \in C_{n-1} && \\ \iff & (u_{n-1} \circ d^C_n)(x) = 0_R && \in D_{n-1} && \text{since } u_n \text{ is } R\text{-linear} \\ \implies & (d^D_n \circ u_n)(x) = 0_R && \in D_{n-1} && \text{commutativity} \\ \implies & x \in \ker(d^D_n \circ u_n) && \leq D_{n-1} && \\ \iff & u_n(x) \in \ker d^D_n && \leq D_n. && \end{aligned}$$

Similarly, for (b) we have

$$\begin{aligned} y \in \operatorname{im} d^C_{n+1} &\iff \exists x \in C_{n+1} \text{ such that } d^C_{n+1}(x) = y && \\ &\implies u_{n+1}(x) \in D_{n+1} && \\ &\implies (d^D_{n+1} \circ u_{n+1})(x) \in \operatorname{im} d^D_{n+1} \leq D_n && \\ &\implies (u_n \circ d^C_{n+1})(x) \in \operatorname{im} d^D_{n+1} \leq D_n && \text{commutativity} \\ &\iff u_n(y) \in \operatorname{im} d^D_{n+1} && \text{using } d^C_{n+1}(x) = y. \end{aligned}$$

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# 2 | Weibel 1.1.4

*Problem* 2.0.1 (Weibel 1.1.4)
Show that for every $A \in R$-mod and $C \in \mathrm{Ch}(R\text{-mod})$ that $D. \coloneqq \mathrm{Hom}_{R\text{-mod}}(A, C.)$ is a chain complex of abelian groups. Taking $A \coloneqq Z_n$, show that $H_n(D.) = 0 \implies H_n(C.) = 0$. Is the converse true?

**Solution:**
We first show that if $A \in R$-mod and $C \in \mathrm{Ch}(R\text{-mod})$, then

$$D_n \coloneqq \mathrm{Hom}_{R\text{-mod}}(A, C_n).$$

defines a chain complex of abelian groups. Fixing notation, we write

$$C \coloneqq (\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots).$$

1. **$D_n$ is an abelian group for all $n$:** Define an operation

$$\begin{aligned} +_D : D_n \times D_n &\to D_n \\ (f, g) &\mapsto \left\{ \begin{aligned} f + g : A &\to C_n \\ x &\mapsto f(x) +_C g(x) \end{aligned} \right\}, \end{aligned}$$

   where $+_C$ is the addition on $C_n$ provided by its structure as an $R$-module. We can then check that this operation is commutative:

$$\begin{aligned} (f +_D g)(x) &\coloneqq f(x) +_C g(x) \\ &= g(x) +_C f(x) && \text{since the addition on } C_n \text{ is commutative} \\ &= (g +_D f)(x), \end{aligned}$$

   The additive inverse of $f$ is $-f$, there is an identity function $\mathrm{id}_{C_n}(x) \coloneqq x$, and the sum of two functions $A \to C_n$ is again a function $A \to C_n$, making $D_n$ an abelian group for all $n$.

2. **There exist differentials $D_n \xrightarrow{d_n^D} D_{n-1}$:** Noting that we have differentials $C_n \xrightarrow{d_n^C} C_{n-1}$, we can define

$$\begin{aligned} d_n^D : D_n &\to D_{n-1} \\ (A \xrightarrow{f} C_n) &\mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1}), \end{aligned}$$

   i.e. we send $f \mapsto d_n^C \circ f$ be precomposing with the differential from $C_*$.

3. $(d^D)^2 = 0$: We can explicitly write

$$\begin{aligned} (d^D)^2 : D_n &\to D_{n-2} \\ (A \xrightarrow{f} C_n) &\mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1} \xrightarrow{d_{n-1}^C} C_{n-2}), \end{aligned}$$

   and so $f \mapsto d_{n-1}^C \circ d_n^C \circ f$. The claim is that this is the zero map, which follows from writing this as $(d^C)^2 \circ f = 0 \circ f = 0$, using that $C_*$ is a chain complex.

Thus

$$D \coloneqq (\cdots \to D_{n+1} \xrightarrow{d^D_{n+1}} D_n \xrightarrow{d^D_n} D_{n-1} \to \cdots) \in \mathrm{Ch}(\mathrm{Ab}).$$

Writing $Z_n \coloneqq Z_n(C) \coloneqq \ker d^C_n$, we now show the following:

**Claim:**

$$H_n(\mathrm{Hom}_{R\text{-mod}}(Z_n, C) = 0 \implies H_n(C) = 0.$$

It suffices to show that $\ker d^C_n \subseteq \operatorname{im} d^C_{n+1}$, so let $y \in \ker d^C_n$; we want to produce the following:

$$x \in C_{n+1}, \quad d^C_{n+1}(x) = y.$$

We can start with the inclusion map

$$\iota : \ker d^C_n \hookrightarrow C_n,$$

which by definition is an element of $D_n \coloneqq \mathrm{Hom}_{R\text{-mod}}(Z_n, C_n)$. By assumption, the following complex is exact at $n$ since its homology vanishes at position $n$:

$$(\cdots \to D_{n+1} \to D_n \to D_{n-1} \to \cdots) \coloneqq$$
$$\cdots \to \mathrm{Hom}_R(Z_n, C_{n+1}) \xrightarrow{d^D_{n+1}} \mathrm{Hom}_R(Z_n, C_n) \xrightarrow{d^D_n} \mathrm{Hom}_R(Z_n, C_{n-1}) \to \cdots.$$

**Claim:** $d^D_n(\iota) = 0$.

This can be seen by writing this out as the composition

$$d^D_n(\ker d^C_n \xrightarrow{\iota} C_n) = (\ker d^C_n \xrightarrow{\iota} C_n \xrightarrow{d^C_n} C_{n-1}).$$

We can now use the general fact that the $f(\ker f) = 0$ for any map $f$, i.e. the image of the kernel is necessarily zero. Taking $f = d^C_n$ shows that this composition is zero. By exactness, $\ker d^D_n = \operatorname{im} d^D_{n+1}$ and we can thus pull $\iota$ back to some $f \in D_{n+1} \coloneqq \mathrm{Hom}_R(Z_n, C_{n+1})$, and since our original $y \in \ker d^C_n \coloneqq Z_n$, it makes sense to consider $x \coloneqq f(y) \in C_{n+1}$ and to identity $y = \iota(y) \in C_n$:

$$\begin{tikzcd}
 & & y \arrow[d, phantom, sloped, "\in"] & & \\
 & & Z_n \arrow[dl, dashed, "\exists f"'] \arrow[d, hook, "\iota"'] & & \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"'] \arrow[d, phantom, sloped, "\ni"] & C_n \arrow[r, "d^C_n"'] \arrow[d, phantom, sloped, "\ni"] & C_{n-1} \arrow[r] & \cdots \\
 & x \coloneqq f(y) & \iota(y) & &
\end{tikzcd}$$

Link to Diagram

Importantly, this $f$ satisfies $\iota = d^D_{n+1}(f) \coloneqq d^C_{n+1} \circ f$, and so we can write

$$y = \iota(y) = (d^C_{n+1} \circ f)(y) \coloneqq d_{n+1}(x),$$

which is what we wanted to show.

# 3 | Weibel 1.1.6

*Problem* 3.0.1 (Weibel 1.1.6: Homology of a graph)
Let $\Gamma$ be a finite graph with vertices $V \coloneqq \{v_1, \cdots, v_V\}$ and edge $E \coloneqq \{e_1, \cdots, e_E\}$. Define the **incidence matrix** of $\Gamma$ to be the $V \times E$ matrix $A$ where

$$A_{ij} = \begin{cases} 1 & e_j \text{ starts at } v_i, \\ -1 & e_j \text{ ends at } v_i, \\ 0 & \text{else.} \end{cases}$$

Define a chain complex by taking free $R$-modules:

$$C \coloneqq (\cdots \to 0 \to C_1 \to C_0 \to 0 \to \cdots) = (\cdots \to 0 \to R^E \xrightarrow{A} R^V \to 0 \to \cdots).$$

If $\Gamma$ is connected, show that $H_0(C)$ and $H_1(C)$ are free $R$-modules of dimensions 1 and $E - V + 1$ respectively.

*Hint: choose a basis $\{v_1, v_2 - v_1, \cdots, v_V - v_1\}$ and use a path from $v_1 \rightsquigarrow v_i$ to produce an element $e \in C_1$ with $d(e) = v_i - v_1$.*

**Solution:**
We first make the following two observations:

1. $H_0(C) = \operatorname{coker}(A) \cong R^V / \operatorname{im} A \implies \operatorname{rank} H_0(C) = V - \operatorname{rank} \operatorname{im} A$, and
2. $H_1(C) = \ker(A) \implies \operatorname{rank} H_1(C) = \operatorname{rank} \ker A$

**Claim:** $\operatorname{rank} \operatorname{im}(A) = V - 1$.
Given this claim, applying observation (1) we immediately obtain

$$\operatorname{rank} H_0(C) = V - (V - 1) = 1,$$

which is the first equality we want to show. For the second equality, we can use the first isomorphism theorem to get a SES of free $R$-modules

$$0 \to \ker(A) \hookrightarrow R^E \to \operatorname{im}(A) \to 0,$$

and since $\operatorname{im}(A)$ is free and thus projective, this sequence splits. So $R^E \cong \ker(A) \oplus \operatorname{im}(A)$, and taking free ranks yields

$$E = \operatorname{rank} \ker(A) + (V - 1) \implies \operatorname{rank} \ker(A) = E - V + 1,$$

and this yields the second equality by using observation (2) to identify the LHS with $\operatorname{rank} H_1(C)$.

*Proof (of claim).*
Using the fact that

$$\mathcal{B} \coloneqq \{v_1, \cdots, v_V\}$$

is a basis for $R^V$ as a free $R$-module, we can make a change of basis to

$$\mathcal{B}' \coloneqq \{v_1, v_2 - v_1, \cdots, v_V - v_1\}.$$

That this is again a basis follows from the fact that the change-of-basis matrix $M$ is upper-triangular with ones on the diagonal and thus satisfies $\det M = 1_R \in R^\times$ (i.e. it's a unit), so $M$ is nonsingular. We can then observe that if $e_i$ is an edge between two vertices $v_{i_1} \xrightarrow{e_i} v_{i_2}$, then $d(e_i) \coloneqq Ae_i = v_{i_2} - v_{i_2}$. By linearity, if $e_{i_1}, \cdots, e_{i_n}$ is a sequence of edges connecting $v_1$ to $v_j$ for any $1 \leq j \leq V$, then

$$d(e_{i_1} + \cdots + e_{i_n}) = v_j - v_1.$$

Since $\Gamma$ is connected, there always exists such a sequence of edges connecting each $v_j$ to $v_1$, and thus $v_j - v_1$ is in $\operatorname{im}(A)$. We can conclude that

$$V - 1 \leq \operatorname{rank} \operatorname{im}(A) \leq V.$$

To see that $\operatorname{rank} \operatorname{im}(A) \neq V$, note that if $e$ is any sequence of edges connecting $v_1$ to itself in a loop, then $d(e_1) = v_1 - v_1 = 0$. Any other path $e'$ must necessarily start or end at some $v_j \neq v_1$ and satisfies $d(e') = v_j - v_1 \neq v_1$, and so $v_1 \notin \operatorname{im}(A)$. Thus

$$\operatorname{rank} \operatorname{im}(A) = V - 1.$$

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# 4 Weibel 1.2.3

*Problem* 4.0.1 (Weibel 1.2.3)
Let $\mathcal{A}$ be the category $\mathrm{Ch}(R\text{-mod})$ and let $f$ be a chain map. Show that the complex $\ker f$ is a (categorical) kernel of $f$ and that $\operatorname{coker} f$ is a (categorical) cokernel of $f$.

**Solution:**
For a fixed map $f : A \to B$, the *kernel* of $f$ is an object $\ker f$ satisfying the following universal property: for any object $K$ with a morphism $K \xrightarrow{g} A$ making the following outer square commute, there is a unique morphism $u : K \to \ker f$ making the entire diagram commute:

$$\begin{tikzcd}
K \arrow[drr, bend left, "g"] \arrow[dr, dotted, "\exists ! u"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f \arrow[r, "\iota^f"] \arrow[d, "0"'] & A \arrow[d, "f"] \\
& 0 \arrow[r, "0"'] & B
\end{tikzcd}$$

We'll use without proof that kernels exist in $\mathcal{A} = R$-mod and are given by $\ker f \coloneqq \left\{a \in A \;\middle|\; f(a) = 0_B\right\}$ along with an inclusion map $\iota^f : \ker f \hookrightarrow A$.
Let $A, B \in \mathrm{Ch}(\mathcal{A})$ be chain complexes and $f : A \to B$ be a chain map. We will construct $\ker f$ as a chain complex and show it satisfies the correct universal property.

**Claim 1:** There are unique objects $\ker f_n \in R$-mod which can be assembled into a unique chain complex $(\ker f, \partial^f)$.
**Proof of Claim 1**:
Let $u : A \to B$ be a chain map, so that we have a commuting diagram of the following form:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

*Link to Diagram*

Appealing to the universal property of kernels in $R$-mod, we can produce unique objects $\ker f_n$ and morphisms $\iota^f_n : \ker f_n \to A_n$ satisfying $(\ker f_n \to A_n \to B_n) = 0$ for every $n$. We also claim that there are maps $\partial^f_n : \ker f_n \to \ker f_{n-1}$, yielding the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & \ker f_{n+1} \arrow[r, dotted, "\partial^f_{n+1}"] \arrow[d, "\iota^f_{n+1}"] \arrow[dr, phantom, "2"] & \ker f_n \arrow[r, dotted, "\partial^f_n"] \arrow[d, "\iota^f_n"] \arrow[dr, phantom, "3"] & \ker f_{n-1} \arrow[r, dotted] \arrow[d, "\iota^f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "u_{n+1}"] \arrow[dr, phantom, "1"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "u_n"] & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

*Link to Diagram*

**Why the $\partial^f_n$ exist:** this follows from the universal property of kernels in $\mathcal{A}$: Using the commutativity of square 1 we have

$$0 = (\ker f_{n+1} \to A_{n+1} \to B_{n+1} \to B_n) = (\ker f_{n+1} \to A_{n+1} \to A_n \to B_n),$$

where we've also used the fact that $(\ker f_{n+1} \to A_{n+1} \to B_{n+1} = 0)$ from the universal property of $\ker f_{n+1}$. So we can fit these into an appropriate diagram in $\mathcal{A}$, which supplies these differentials:

$$\begin{tikzcd}
\ker f_{n+1} \arrow[dr, dotted, "\exists!\partial^f_{n+1}"] \arrow[drr, bend left, "\partial^A_{n+1}\circ\iota^f_{n+1}"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f_n \arrow[r, "\iota^f"] \arrow[d, "0"'] & A_n \arrow[d, "f_n"] \\
& 0 \arrow[r, "0"'] & B_n
\end{tikzcd}$$

**Why the $\iota^f : \ker f \to A$ assemble into a chain map:** Note that everything here commutes, and we can break the northeast corner of this diagram up and rearrange things slightly to form the following diagram:

$$\begin{tikzcd}
\ker f_{n+1} \arrow[r, "\iota^f_{n+1}"] \arrow[d, dashed, "\exists!\partial^f_{n+1}"'] \arrow[dd, bend right] \arrow[dr, phantom, "2"] & A_{n+1} \arrow[d, "\partial^A_{n+1}"] \\
\ker f_n \arrow[r, "\iota^f_n"] \arrow[d, "0"'] & A_n \arrow[d, "f_n"] \\
0 \arrow[r, "0"'] & B_n
\end{tikzcd}$$

*Link to Diagram*

Here, square 2 is precisely the square 2 appearing in the original diagram, and commutativity of it for each $n$ is precisely what is required for $\iota^f$ to be a chain map.

**Why $(\partial^f)^2 = 0$:** Using the commutativity of square 3 and the fact that $(\partial^A)^2 = 0$, we have

$$\begin{aligned}
\iota^f_{n-1} \circ (\partial^f)^2 &\coloneqq (\ker f_{n+1} \to \ker f_n \to \ker f_{n-1} \to A_{n-1}) \\
&= (\ker f_{n+1} \to A_{n+1} \to A_n \to A_{n-1}) \\
&\coloneqq \iota^f_{n+1} \circ (\partial^A)^2 \\
&= 0,
\end{aligned}$$

and since $\iota^f_{n-1}$ is not the zero map, this forces $(\partial^f)^2 = 0$.

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**Claim 2:** The complex $\ker f$ satisfies the universal property of kernels in $\mathrm{Ch}(\mathcal{A})$, i.e. if $g^K : K \to A$ is a chain map satisfying $K \to A \to B = 0$, there is a unique chain map $u : K \to \ker f$ making the appropriate diagram commute.

*Proof (?).*
Again using the universal property of kernels in $R$-mod, for each $n$ we have a commutative diagram

$$\begin{tikzcd}
K_n \arrow[dr, dotted, "\exists! u_n"] \arrow[drr, bend left, "g_n^K"] \arrow[ddr, bend right, "0"'] & & \\
& \ker f_n \arrow[r, "\iota_n^f"] \arrow[d, "0"'] & A_n \arrow[d, "f"] \\
& 0 \arrow[r, "0"'] & B_n
\end{tikzcd}$$

This results in a diagram of the following form:

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n+1} \arrow[r, "\partial^K_{n+1}"] \arrow[d, dashed, "\exists u_{n+1}"] \arrow[dd, bend right, "g^K_{n+1}"'] \arrow[dr, phantom, "1"] & K_n \arrow[r, "\partial^K_{n+1}"] \arrow[d, dashed, "\exists u_n"] \arrow[dd, bend right, "g^K_n"'] & K_{n-1} \arrow[r] \arrow[d, dashed, "\exists u_{n-1}"] \arrow[dd, bend right, "g^K_{n-1}"'] & \cdots \\
\cdots \arrow[r] & \ker f_{n+1} \arrow[r, "\partial^f_{n+1}"] \arrow[d, "\iota^f_{n+1}"] \arrow[dr, phantom, "3"] & \ker f_n \arrow[r, "\partial^f_n"] \arrow[d, "\iota^f_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "\iota^f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial^A_{n+1}"] \arrow[d, "u_{n+1}"] & A_n \arrow[r, "\partial^A_n"] \arrow[d, "u_n"] & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial^B_{n+1}"] & B_n \arrow[r, "\partial^B_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

*Link to Diagram*

It only remains to check that the $u_n$ assemble to a chain map $K \to \ker f$, which would follow from the commutativity of e.g. square (1). However, if (1) were *not* commutative, then the rectangle formed by (1) and (3) together would not be commutative – but $g^K$ was assumed to be a chain map, so this rectangle commutes, yielding a contradiction.

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*Note: a proof of a similar flavor seems to work for the cokernel complex by reversing all of the arrows.*

# 5 | Exactness in the Snake Lemma

*Problem* 5.0.1 (?)
Verify exactness in the Snake Lemma in at least two other positions.

**Solution:**
This follows from the construction of the complex $\ker f$ above, specifically using the fact that the constructed differential $\partial^f$ satisfies $(\partial^f)^2 = 0$.