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References: <https://www.daniellitt.com/etale-cohomology>

Prerequisites:

- Homological Algebra
 - Abelian Categories
 - Derived Functors
 - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
 - Etale morphism
 - Grothendieck topologies
 - The etale topology
 - Etale cohomology and the basis theorems
 - Etale cohomology of curves
 - Comparison theorems to singular cohomology
 - Focused on the case where coefficients are a constructible sheaf.
 - Prove the Weil Conjectures (more than one proof)
 - Proving the Riemann Hypothesis for varieties over finite fields
- One of the greatest pieces of 20th century mathematics!
- Topics
 - Weil 2 (Strengthening of RH, used in practice)
 - Formality of algebraic varieties (topological features unique to varieties)
 - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose X/\mathbb{C} is a quasiprojective variety: a finite type separated integral \mathbb{C} -scheme.

If you take the complex points, it naturally has the structure of a complex analytic space $X(\mathbb{C})^{\text{an}}$: you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology $H^i(X(\mathbb{C})^{\text{an}}, \mathbb{Z})$, which satisfies several nice properties:

- Finitely generated \mathbb{Z} -modules
- Extra Hodge structure when tensored up to \mathbb{C} (same as \mathbb{C} coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general “nice” schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We’ll associate X a “nice scheme” $\rightsquigarrow H^i(X_{\text{et}}, \mathbb{Z}/\ell^n \mathbb{Z})$. Take the inverse limit over all n to obtain the ℓ -adic cohomology $H^i(X_{\text{et}}, \mathbb{Z}_\ell)$. You can tensor with \mathbb{Q} to get something with \mathbb{Q}_ℓ coefficients. And as in singular cohomology, you can a “twisted coefficient system”.

What are nice schemes:

- $X = \text{Spec } \mathcal{O}_k$, the ring of integers over a number field.
- X a variety over an algebraically closed field
 - Typical, most analogous to taking a variety over \mathbb{C} .
- X a variety over a non-algebraically closed field

Some comparisons between the last two cases:

- For \mathbb{C} - variety, H_{sing}^i will vanish above $i = 2d$.
- Over a finite field, H^i will vanish for $i > 2d + 1$ but generally not vanish for $i = 2d + 1$.

In good situations, these are finitely generated $\mathbb{Z}/\ell^n \mathbb{Z}$ -modules, have Mayer-Vietoris and excision sequences, spectral sequences, etc.

Related invariants: for a scheme with a geometric point $(X, \bar{x}) \rightsquigarrow \pi_1^{\text{étale}}(X, \bar{x})$, which is a profinite topological group, which is a profinite topological group.

Note: a geometric point is a map from $\text{Spec } X$ to an algebraically closed field.

More invariants beyond the scope of this course:

- Higher homotopy groups
- Homotopy type (equivalence class of spaces)

So we want homotopy-theoretic invariants for varieties.

Remark 1.

This cohomology theory is necessarily weird!

Theorem 1.1 (Serre).

There does not exist a cohomology theory for schemes over $\bar{\mathbb{F}}_q$ with the following properties:

1. Functorial
2. Satisfies the Kunneth formula
3. For E an elliptic curve, $H^1(E) = \mathbb{Q}^2$.

Slogan: No cohomology theory with \mathbb{Q} coefficients.

Proof .

Take E to be a supersingular elliptic curve. Then $\text{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra.

Fact: There are no algebra morphisms $R \rightarrow \text{Mat}_{2 \times 2}(\mathbb{Q})$

Exercise .

Functoriality and Kunneth implies that $\text{End}(E) \curvearrowright E$ yields an action on $H^1(E)$, which is precisely an algebra morphism $\text{End}(E) \rightarrow \text{Mat}_{2 \times 2}(\mathbb{Q})$, a contradiction.

The content: the sum of two endomorphisms act via their sum on H^1 .

Exercise .

Prove the same thing for \mathbb{Q}_p coefficients, where p divides the characteristic of the ground field.

Proof the same, just need to know what quaternion algebras show up.

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This forces using some funky type of coefficients.

What are the Weil Conjectures?

Suppose X/\mathbb{F}_q is a variety, then

$$\zeta_X(t) = \exp \left(\sum_{n>0} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n \right).$$

Some comments: