# Title

D. Zack Garza

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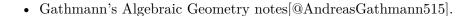
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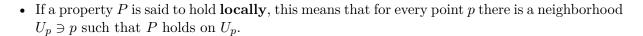
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## **Prologue**

#### 0.1 References



#### 0.2 Notation



Notation	Definition
$\overline{k[\mathbf{x}] = k[x_1, \cdots, x_n] \ k(\mathbf{x}) = k(x_1, \cdots, x_n) \ \mathcal{U} \rightrightarrows X \ \Delta_X}$	Polynomial ring
$\mathbb{A}^n_{/k} \mathbb{P}^n_{/k}$	in $n$
, . ,	indeterminates.
	Rational function
	field in $n$
	indeterminates
	An open cover
	$\mathcal{U} = \overline{}$
	$\{U_j \mid j \in J\}$
	The diagonal
	$\{(x,x) \mid x \in X\} \subseteq$
	$X \times X$ Affine
	<i>n</i> -space
	$\mathbb{A}^n_{/k}\coloneqq$
	$\left\{\mathbf{a} = [a_1, \cdots, a_n] \mid a\right\}$
	Projective
	n-space

 $- \mathbb{P}^n_{/k} \coloneqq \left(k^n \setminus \{0\}\right)/x \sim \lambda x - = \left\{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \; \middle|\; p,q,\in k[x_1,\cdots,x_n]\right\} \, V(J), V_a(J) \text{ Variety associated to an ideal } J \preceq k[x_1,\cdots,x_n] - \coloneqq \left\{\mathbf{x} \in \mathbb{A}^n \;\middle|\; f(\mathbf{x}) = 0, \; \forall f \in J\right\} \, I(S), I_a(S) \text{ Ideal associated to a subset } S \subseteq \mathbb{A}^n_k - \coloneqq \left\{f \in k[x_1,\cdots,x_n] \;\middle|\; f(\mathbf{x}) = 0 \;\forall \mathbf{x} \in X\right\} \, A(X) \text{ Coordinate ring of a variety, } k[x_1,\cdots,x_n]/I(X) \, V_p(J) \text{ Projective variety of an ideal } - \coloneqq \left\{\mathbf{x} \in \mathbb{P}^n_{/k} \;\middle|\; f(\mathbf{x}) = 0, \; \forall f \in J\right\} \, I_p(S) \text{ Projective ideal } (?) - \coloneqq \left\{f \in k[x_1,\cdots,x_n] \;\middle|\; f \text{ is homogeneous and } f(x) = 0 \;\forall x \in S\right\} \, S(X) \, \text{ Projective coordinate ring, } k[x_1,\cdots,x_n]/I_p(X) \, f^h \text{ Homogenization, } x_0^{\deg f} f\left(\frac{x_1}{x_0},\cdots,\frac{x_n}{x_0}\right) \, f^i \text{ Deho-}$ 

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mogenization,  $f(1, x_1, \dots, x_n)$   $J^h$  Homogenization of an ideal,  $\{f^j \mid f \in J\}$   $\overline{X}$  Projective closure of a subset  $- := V_p(J^h) := \{\mathbf{x} \in \mathbb{P}^n \mid f^h(\mathbf{x}) = 0 \,\forall f \in X\}$   $\mathcal{O}_X$  Structure sheaf  $\{f : U \to k \mid f \in k(\mathbf{x}) \text{ locally}\}$  D(f) Distinguished open set,  $D(f) = V(f)^c = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$ 

#### 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for X = D(f) a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

#### 0.4 Useful Examples



#### 0.4.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$  a hyperbola
- V(x) a coordinate axis
- V(x-p) a point.

#### 0.4.2 Presheaves / Sheaves

- $C^{\infty}(\cdot, \mathbb{R})$ , a sheaf of smooth functions
- $C^0(\cdot,\mathbb{R})$ , a sheaf of continuous functions
- $\mathcal{O}_X(\,\cdot\,)$ , the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$ , the constant sheaf associated to  $\mathbb{R}$  (locally constant real-valued functions)
- $\operatorname{Hol}(\cdot,\mathbb{C})$ , a sheaf of holomorphic functions
- $K_p$  the skyscraper sheaf:

$$K_p(U) \coloneqq \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

### 0.5 The Algebra-Geometry Dictionary



Let  $k = \bar{k}$ , we're setting up correspondences

Algebra	Geometry
$\frac{1}{k[x_1,\cdots,x_n]}$	$\mathbb{A}^n_{/k}$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \cdots, x_n - p_n$	Points $[a_1, \cdots, a_n]$
Radical ideals $J = \sqrt{J} \le k[x_1, \cdots, x_n]$	V(J) the zero locus
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets
I(S) the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
I + J	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
I(V):I(W)	$\overline{V\setminus W}$
$k[x_1,\cdots,x_n]/I(X)$	A(X) (Functions on $X$ )
A(X) a domain	X is irreducible
A(X) indecomposable	X is connected
Krull dimension $n$ (chaints of primes)	Topological dimension $n$ (chains of irreducibles)