# **Title**

## D. Zack Garza

## Thursday 10<sup>th</sup> September, 2020

### **Contents**

| 1 | Thu | ırsday, September 10                | 1 |
|---|-----|-------------------------------------|---|
|   | 1.1 | Proof of Dimension Proposition      | 1 |
|   |     | 1.1.1 Proof That $P_1$ is Principle |   |
|   | 1.2 | Using Dimension Theory              | 3 |

## 1 Thursday, September 10

Recall that the dimension of a ring R is the length of the longest chain of prime ideals. Similarly, for an affine variety X, we defined dim X to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking R = A(X), i.e. dim  $\mathbb{A}^n/k = n$ .

## Proposition 1.1(Dimensions).

Let  $k = \bar{k}$ .

- a. The dimension of  $k[x_1, \dots, x_n]$  is n.
- b. All maximal chains of prime ideals have length n.

### 1.1 Proof of Dimension Proposition

The case for n = 0 is trivial, just take  $P_0 = \langle 0 \rangle$ . For n = 1, easy to see since the only prime ideals in k[x] are  $\langle 0 \rangle$  and  $\langle x - a \rangle$ , since any polynomial factors into linear factors.

Let  $P_0 \subsetneq \cdots \subsetneq P_m$  be a maximal chain of prime ideals in  $k[x_1, \cdots, x_n]$ ; we then want to show that m = n. Assume  $P_0 = \langle 0 \rangle$ , since we can always extend our chain to make this true (using maximality). Then  $P_1$  is a minimal prime and  $P_m$  is a maximal ideal (and maximals are prime).

**Claim:**  $P_1$  is principle, i.e.  $P_1 = \langle f \rangle$  for some irreducible f.

#### 1.1.1 Proof That $P_1$ is Principle

**Claim:**  $k[x_1, \dots, x_n]$  is a unique factorization domain. This follows since k is a UFD since it's a field, and R a UFD  $\Longrightarrow R[x]$  is a UFD for any R.

See Gauss' lemma.

**Claim:** In a UFD, minimal primes are principal. Let  $r \in P$ , and write  $r = u \prod p_i^{n_i}$  with  $p_i$  irreducible and u a unit. So some  $p_i \in P$ , and  $p_i$  irreducible implies  $\langle p_i \rangle$  is prime. Since  $0 \subseteq \langle p_i \rangle \subset P$ , but P was prime and assumed minimal, so  $\langle p_i \rangle = P$ .

The idea is to now transfer the chain  $P_0 \subsetneq \cdots \subsetneq P_m$  to a maximal chain in  $k[x_1, \cdots, x_{n-1}]$ . The first step is to make a linear change of coordinates so that f is monic in the variable  $x_n$ .

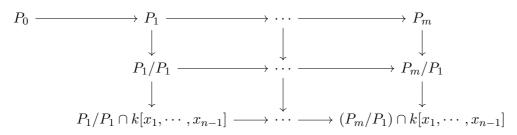
#### Example 1.1.

Take  $f = x_1 x_2 + x_3^2 x_4$  and map  $x_3 \mapsto x_3 + x_4$ .

So write

$$f(x_1, \dots, x_n) = x_n^d + f_1(x_1, \dots, x_{n-1})x_n^{d-1} + \dots + f_d(x_1, \dots, x_{n-1}).$$

We can then descend to  $k[x_1, \dots, x_n]$  to  $k[x_1, \dots, x_n]/\langle f \rangle$ :



The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

#### **Definition 1.1.1** (Integral Extension).

An *integral* ring extension  $R \hookrightarrow R'$  of R is one such that all  $r' \in R'$  satisfying a monic polynomial with coefficients in R, where R' is finitely generated.

In this case, also implies that R' is a finitely-generated R module.

In this case,  $k[x_1, \dots, x_{n-1}] \hookrightarrow k[x_1, \dots, x_n]/\langle f \rangle$  is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions, since it preserves primality.

#### Lemma 1.2.

Suppose  $P', Q' \subset R'$  are distinct prime ideals with  $R \hookrightarrow R'$  an integral extension. Then if  $P' \cap R = Q' \cap R$ , neither contains the other, i.e.  $P' \not\subset Q'$  and  $Q' \not\subset P'$ .

#### Proof.

Toward a contradiction, suppose  $P' \subset Q'$ , we then want to show that  $Q' \supset P'$ . Let  $a \in Q' \setminus P'$ 

(again toward a contradiction), then

$$R/(P'\cap R) \hookrightarrow R'/P'$$

is integral.

Then  $\bar{a} \neq 0$  in R'/P', and there exists a monic polynomial of minimal degree that  $\bar{a}$  satisfies,

$$p(x) = x^n + \sum_{i=2}^n \bar{c}_i x^{n-i}$$
. This implies  $\bar{c}_n \in Q'/P'$  (which will contradict  $c_n \in P'$ ), since if  $\bar{c}_n = 0$  then factoring out  $x$  yields a lower degree polynomial that  $\bar{a}$  satisfies.

But then  $\bar{a}_n \in Q' \cap R$ , so ????

Question: Given  $R \hookrightarrow R'$  is an integral extension, can we lift chains of prime ideals?

Answer: Yes, by the "Going Up" Theorem: given  $P \subset R$  prime, there exists  $P' \subset R'$  prime such that  $P' \cap R = P$ . Furthermore, we can lift  $P_1 \subset P_2$  to  $P_1' \subset P_2'$ , as well as "lifting sandwiches":

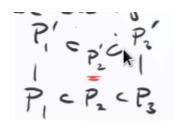


Figure 1: Image

In this process, the length of the chain decreased since  $\langle 0 \rangle$  was deleted, but otherwise the chains are in bijective correspondence. So the inductive hypothesis applies.

#### 1.2 Using Dimension Theory

Key fact used: the dimension doesn't change under integral extensions, i.e. if  $R \hookrightarrow R'$  is integral then  $\dim R = \dim R'$ .

Claim: Any affine variety has finite dimension.

Proof.

We have dim  $X = \dim A(X)$ , where  $A(X) := k[x_1, \dots, x_n]I$  for some  $I(X) = \sqrt{I(X)}$ . The noether normalization lemma (used in proof of nullstellensatz) shows that a finitely generated k-algebra is an integral extension of some polynomial ring  $k[y_1, \dots, y_d]$ . I.e., the following extension is integral:

$$k[y_1, \cdots, y_d] \hookrightarrow k[x_1, \cdots, x_n]/I.$$

We can conclude that  $\dim A(X) = d$ 

1 THURSDAY, SEPTEMBER 10