Title

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1 Lecture 13

Recall that we previously looked at the regular function fields: we took a function field in one variable and considered the class of function fields for which we could take any extension of the constant field that we wanted. As long as the ground field is perfect, being regular is equivalent to the constant subfield being k itself. However, we haven't done anything with them yet!

If you take an algebraic closure of the finite ground field $\mathbb{F}_q$, there is a unique subextension of degree r for every r, so we call that $\mathbb{F}_q^r$. The extension $\mathbb{F}_q^r/\mathbb{F}_q$ is cyclic galois, with a geometric Frobenius $x \to x^q$. Note that $\mathbb{F}_q^r$ is the fixed field of F^r, the rth power of the Frobenius map. We set $K_r := K\mathbb{F}_q^r$, which is a regular function field over $\mathbb{F}_q^r$. Note that we could view this as a function field just over $\mathbb{F}_q$, but it would not be regular. Then K_r/K is a degree r arithmetic extension of function fields.

Question: What happens to places when making this scalar extension? I.e., how to places in K decompose in K_r?

Remark 1.0.1 : This is related to an Algebraic Number Theory I problem: for $v \in \Sigma(K_{/\mathbb{F}_q})$ above an affine Dedekind domain R such that $v \in \Sigma(K/R)$, let S be the integral closure of K in K_r. Then we want to factor $p_v S$?

Not quite sure.

Lemma 1.1 ***(Key lemma about how places split.).***
Suppose $d := \deg(v)$. Then K_r/K is galois, so we have $efg = r$. In fact, $c = 1$, so $f = \dfrac{r}{\gcd(d, r)}$ and $g = \gcd(d, r)$ and each place $w \in \Sigma(K_r/\mathbb{F}_q^r)$ has degree $\dfrac{d}{\gcd(d, r)}$.

Remark 1.0.2 : We have the following cases:

- The extension is *inert* iff $\gcd(d, r) = 1$,
- The extension *splits completely* iff $r \Bigm| d$,
- All w dividing v have degree 1 iff $d \Bigm| r$.

The last thing we proved was that the degree zero divisor class group is finite when we're over a finite ground field. Why is this true? Whenever there is a divisor of degree n, then the set of degree n divisors is a coset of the degree zero divisors, all of which have the same cardinality. So it's enough to prove that