Title

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Thursday 8^{th} October, 2020

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Proposition 1.1 ***(Gluing).***
Let $f: X \to Y$ be a map of ringed spaces such that there exists an open cover $U_i \rightrightarrows X$ such that $f|_{U_i}$ is a morphism of ringed spaces. Then f itself is a morphism is a morphism of ringed spaces.

Recall that we proved part (a).

Proof (part b).
We want to show that f^* sends sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$ (e.g. regular functions pullback). Let $V \subset Y$ be open and $\varphi \in \mathcal{O}_Y(V)$, then

$$f^*\varphi|_{U_i \cap f^{-1}(V)} \left(f^*\varphi|_{U_i \cap f^{-1}(V)} \right)^* \varphi \in \mathcal{O}_X(U_i f^{-1}(V)).$$

Since pullback commutes with restriction, $f^*\varphi$ is the unique k-valued function for which

$$f^*\varphi|_{U_i \cap f^{-1} V} = f|_{U_i \cap f^{-1} V}{}^* \varphi.$$

and all of the latter functions agree on overlaps $U_i \cap U_j$. This by unique gluing, $f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$.

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Proposition 1.2 ***(?).***
Let $U \subset X$ be open in an affine variety and let $Y \subset \mathbb{A}^n$ be another affine variety. Then the morphisms $U \to Y$ of ringed spaces are the maps of the form $f = [f_1, \cdots, f_n] : U \to \mathbb{A}^n$ such that $f(U) \subset Y$ and $f_i \in \mathcal{O}_X(U)$ for all i.

Proof .
$\implies$: Assume that $f : U \to Y$ is a morphism. Then the coordinate functions $Y \xrightarrow{y_i} \mathbb{A}_1$ are regular functions, since they generate $\mathcal{O}_Y(Y) = k[y_1, \cdots, y_n]/I(Y)$. Then $f^* y_i$ is a regular function, so define $f_i \coloneqq f^* y_i$. But then $f = [f_1, \cdots, f_n]$.
$\impliedby$: Conversely suppose $f \coloneqq [f_1, \cdots, f_n] : U \to Y \subset \mathbb{A}^n$ is a map such that $f_i \in \mathcal{O}_U(U)$. We want to show that f is a morphism, i.e. that the pullback of every regular function is regular. We thus need to show

1. f is continuous, and
2. f^* pulls back regular functions.

For 1, suppose Z is closed, then it suffices to show $f^{-1}(Z)$ is closed. Then $Z = V(g_1, \cdots, g_n)$ for some $g_i \in A(Y)$. So we can write

$$f^{-1}(Z) = \left\{ x \in U \;\middle|\; g_i(f_1(x), \cdots, f_n(x)) = 0 \, \forall i \right\}.$$

The claim is that the functions g_i are regular, i.e. in $\mathcal{O}_U(U)$, because the g_i are polynomials in regular functions, which form a ring.
This is the common vanishing locus of m regular functions on U. By lemma 3.4, the vanishing locus of a regular function is closed, so $f^{-1}(Z)$ is closed.

For 2, let $\varphi \in \mathcal{O}_Y(W)$ be a regular function on $W \subset Y$ open. Then

$$\begin{aligned} f^* \varphi = \varphi \circ f : f^{-1}(W) &\to K \\ x &\mapsto \varphi(f_1(x), \cdots, f_n(x)). \end{aligned}$$

We want to show that this is a regular function. Since the f_i are regular functions, they are locally fractions, so for all $x \in f^{-1}(W)$ there is a neighborhood of $U_x \ni x$ such that (by intersecting finitely many neighborhoods) all of the f_i are fractions a_i/b_i.
Then at a point $p = [f_i(x)]$ in the image, there exists an open neighborhood W_p in W such that $\varphi = U/V$. But then $\varphi[a_i/b_i] = (U/V)([a_i/b_i])$, which is evaluation of a fraction of functions on fractions.

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Example 1.1.
Let $Y = V(xy - 1)$ and $U \subset \mathbb{A}^1$ be $D(x)$, so $U = \mathbb{A}^1 \setminus \{0\}$. Note that $A(Y) = k[x, y]/\langle xy - 1 \rangle$ and $A(\mathbb{A}^1) = k[t]$, and $f_1 = t, f_2 = t^{-1} \in \mathcal{O}_U(U)$. Then

$$\begin{aligned} [f_1, f_2] : U &\to Y \subset \mathbb{A}^2 \\ p &\mapsto \left[p, \frac{1}{p} \right]. \end{aligned}$$

Thus the image lies in Y.

Conversely, there is a map

$$\begin{aligned} V(xy - 1) &\to U = D(0) \subset \mathbb{A}^1 \\ [x, y] &\mapsto x. \end{aligned}$$

This a morphism from $V(xy - 1)$ to $\mathbb{A}^1$, since the coordinates are regular functions. Since the image is contained in U, the definitions imply that this is in fact a morphism of ringed spaces. We thus have maps $U \xrightarrow{[t,t^{-1}]} V(xy-1)$ and $V(xy-1) \xrightarrow{x} U$ which are mutually inverse, so these are isomorphic as ringed spaces.

Thus maps of affine varieties (or their open subsets) are given by functions whose coordinates are regular.

Corollary 1.3*(?)*.
Let X, Y be affine varieties, then there is a correspondence

$$\begin{aligned}\left\{\text{Morphisms } X \to Y\right\} &\iff \left\{k\text{-algebra morphisms } A(Y) \to A(X)\right\} \\ X \to Y &\mapsto A(Y) \to A(X) \\ f &\mapsto f^* \mathcal{O}_Y(Y) = \mathcal{O}_X(X).\end{aligned}$$

Thus there is an equivalence of categories between reduced k-algebras and ???.

Proof .
We have a map in the forward direction. Conversely, given a k-algebra morphism $g : A(Y) \to A(X)$, we need to construct a morphism f such that $f^* = g$. Let $Y \subset \mathbb{A}^n$ with coordinate functions $y_1, \cdots, y_n$. Then $f_i = g(y_i) \in A(X) = \mathcal{O}_X(X)$. Set $f = [f_1, \cdots, f_n]$. Then by the proposition, f is a morphism to $\mathbb{A}^n$.
Let $h \in A(\mathbb{A}^n)$, then

$$\begin{aligned}(f^*h)(x) &= h(f(x)) \\ &= h([f_1(x), \cdots, f_n(x)]) \\ &= h(g(y_1), \cdots, g(y_n)) \\ &= g(h)(x) \qquad \text{since } g \text{ is an algebra morphism, } h \text{ is a polynomial}\end{aligned}$$

which follows since $f_i(x) = g(y_i)(x)$, where $g : A(Y) \to A(X)$. So $f^*(h) = g(h)$ for all $h \in A(\mathbb{A}^n)$, so the pullback of f is g. We now need to check that it's contained in the image. Let $h \in I(Y)$, then $f^*(h) = g(h) = 0$ since $h = 0 \in A(Y)$. So $\operatorname{im}(f) \subset Y$. Since the coordinate f_i are regular, this is a morphism, and we have $f^* = g$ as desired.

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Example 1.2.
Isomorphisms are not necessarily bijective morphisms. Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$.

Then there is a morphism

$$\begin{aligned}\varphi : \mathbb{A}^1 &\to X \\ t &\mapsto \left[t^2, t^3\right],\end{aligned}$$

since the coordinates t^2, t^3 are regular functions. Then φ is a bijection, since we can define a piecewise inverse

$$\begin{aligned}
\varphi^{-1} : X &\to \mathbb{A}^1 \\
[x, y] &\mapsto \begin{cases} y/x & x \neq 0 \\ 0 & \text{else} \end{cases}.
\end{aligned}$$

However, φ^{-1} is not a morphism. For instance, pulling back the function t yields $\left(\varphi^{-1}\right)^* t \notin A(X)$, since it is equal to the map $[x, y] \mapsto y/x$ for $x \neq 0$ and 0 if $x = y = 0$, which is not a regular function. Since φ is a morphism, we can consider the corresponding map of k-algebras

$$\begin{aligned}
\varphi^* : A(X) &\to A(\mathbb{A}^1) \\
k[x, y]/\left\langle y^2 - x^3 \right\rangle &\mapsto k[t] \\
x &\mapsto t^2 \\
y &\mapsto t^3.
\end{aligned}$$