Homotopy Groups of Spheres

Graduate Student Seminar

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Big Points

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

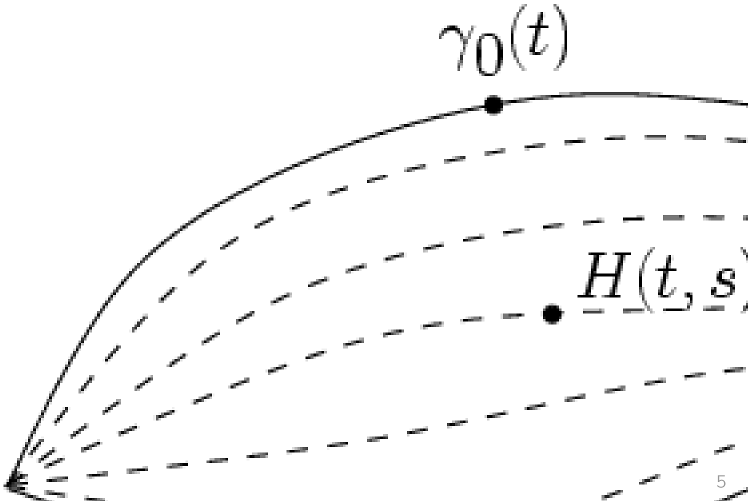
History

- 1860s-1890s: (Roughly) defined by Jordan for complex integration, "combinatorial topology"
  - Original motivation: when does a path integral depend on a specific path? (E.g. a contour integral in $\mathbb{C}$)
- 1895: Poincare, *Analysis situs* ("the analysis of position") in analogy to Euler *Geometria situs* in 1865 on the Kongisberg bridge problem Attempts to study spaces arising from gluing polygons, polyhedra, etc (surfaces!), first use of "algebraic invariant theory" for spaces by introducing π_1 and homology.
- 1920s: Rigorous proof of classification of surfaces (Klein, Möbius, Clifford, Dehn, Heegard), captured entirely by π_1 (equivalently, by genus and orientability).
- 1925-1928: Noether, Mayer, Vietoris develop general algebraic theory of homology, now "algebraic topology"
- 1931: Hopf discovers a nontrivial (not homotopic to identity) map $S^3 \longrightarrow S^2$
  - Compare to homology: $H^k S^n = 0$ for $k \geq n$ is an easy theorem!
- 1932/1935: Cech (resp Hurewicz) introduce higher homotopy

Actual Outline

- Definitions of spheres and balls
- Definition of homotopy of maps
  - Motivations from complex analysis
- Functoriality
- Examples of spaces that are homotopy equivalent and *aren't*.
- Example where homotopy distinguishes homologically equivalent spaces

Images



Application: SO3

Application: $\pi_1(\mathrm{SO}(n, \mathbb{R})$, the lie group of rigid rotations in 3-space. The fibration $\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$ yields a LES in homotopy:

which reduces to

and thus $\pi_1(\mathrm{SO}(3, \mathbb{R})) \cong \pi_1(\mathrm{SO}(4, \mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(\mathrm{SO}(3, \mathbb{R}))$. Use the fact that "accidental" homeomorphism in low dimension $\mathrm{SO}(3, \mathbb{R}) \cong_{\mathrm{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

> *Can also use the fact that $SU(2, \mathbb{R}) \longrightarrow SO(3, \mathbb{R})$ is a double cover from the universal cover.*

Important consequence: $\mathrm{SO}(3, \mathbb{R})$ is not simply connected! See "plate trick", there is a loop of rotations that is not contractible, but squares to the identity. Causes problems in robotics (leads to paths in configuration spaces that encounter singularities) and compute graphics (smoothly interpolating between e.g. quaternions for rotated camera views).

Example: Knowing Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.