Problem Set 1

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Contents

1 Problem 6

1.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} = X(\phi_t(p)),$$
$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$H : M \times I \to M$$
$$(p, t) \mapsto \phi_t(p).$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \text{Tr}\left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned}
\Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X;\mathbb{Q})}) \\
&= \sum_k \dim H_k(X;\mathbb{Q}) \\
&= \chi(M),
\end{aligned}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus $\phi_1(p_0) = p_0$ and we have $\frac{\partial}{\partial t}\phi_t(p_0) = \frac{\partial}{\partial t}p_0 = 0$

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned}
\phi_t(p) &= p \\
\implies \frac{\partial}{\partial t}\phi_t(p) &= \frac{\partial}{\partial t}p = 0 \\
\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} &= X(\phi_1(p))
\end{aligned}$$