Title

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1.1 1.a

Proof ($A \implies B$).

- Suppose $\{a_n\}$ is not bounded above.
- Then any $k \in \mathbb{N}$ is not an upper bound for $\{a_n\}$.
- So choose a subsequence $a_{n_k} > k$, then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k \to \infty} a_{n_k} > \liminf_{k \to \infty} k = \infty.$$

> Note that $\lim_{n \to \infty} a_n$ need not exist, but liminf/limsup always exist.

∎

Proof ($\cancel{A} \implies \cancel{B}$).

- Suppose $\{a_n\}$ is bounded by M, so $a_n < M < \infty$ for all $n \in \mathbb{N}$.
- Then if $\{a_{n_k}\}$ is a subsequence, we have $a_{n_k} \in \{a_n\}$, so $a_{n_k} < M$ for all $k \in \mathbb{N}$.
- But then

$$a_{n_k} < M \implies \limsup_{k \to \infty} a_{n_k} \leq M,$$

- Now note that if $\lim_{k \to \infty} a_{n_k}$ exists,

$$\lim_{k \to \infty} a_{n_k} < \limsup_{k \to \infty} a_{n_k} \leq M < \infty,$$

so every subsequence is bounded and thus can not converge to ∞.

$\blacksquare$

1.2 3.a

Proof (Using definition (i)).

- Suppose $x_n \leq M$ for all n, we will show that every subsequential limit is also bounded by M.
- Let

$$S := \left\{ x \in \mathbb{R} \ \middle| \ x \text{ is a subsequential limit of } \{x_n\} \right\}$$

  be the set of subsequential limits.
  - Note that $\inf S := \liminf_{n \to \infty} x_n$ by definition (i).
- Let $\{x_{n_k}\} \in S$ be an arbitrary convergent subsequence (since we are only concerned about subsequences with well-defined limits).
- Then for every k we have $x_{n_k} \in \{x_n\}$, so

$$|x_{n_k}| \leq M.$$

- By order limit laws,

$$|x_{n_k}| \leq M \implies \lim_{k \to \infty} |x_{n_k}| \leq M,$$

- Since the map $x \mapsto |x|$ is continuous, using the sequential definition of continuity we can pass the limit through the absolute value to obtain

$$\left| \lim_{k \to \infty} x_{n_k} \right| \leq M.$$

- Since the subsequence was arbitrary, we find that M is an upper bound for S.
- But $\inf S$ is a *lower* bound for S, so $\inf S \leq M$.

$\blacksquare$

Proof (Using definition (ii)).

- Suppose $|x_n| \leq M$ for every n, we will directly show that $\left| \liminf_{n \to \infty} x_n \right| \leq M$.

- Let $\{x_{n_k}\}$ be an arbitrary subsequence, then since $x_{n_k} \in \{x_n\}$ for all k, $|x_{n_k}| \leq M$ for all k.

- By order-limit laws, for every fixed n we have

$$|x_{n_k}| \leq M \iff -M \leq x_{n_k} \leq M \implies -M \leq \inf_{k > n} x_{n_k} \leq M.$$

- Again applying order limit laws,

$$-M \leq \inf_{k > n} x_{n_k} \leq M \implies -M \leq \lim_{n \to \infty} \inf_{k > n} x_{n_k} \leq M \iff \left| \lim_{n \to \infty} \inf_{k \geq n} x_{n_k} \right| \leq M.$$

- But by definition (i), this precisely says that $\left| \liminf\limits_{n \to \infty} x_n \right| \leq M$.

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