CRAG

The Weil Conjectures

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April 2020

Background: Generating Functions

Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\ n \in \mathbb{Z}^{\geq 2}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic* variety $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I} = (i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^\times.$$

Point Counts

- For a fixed variety X, we can consider its $\mathbb{F}$-points $X(\mathbb{F})$.
  - Note that $\#X(\mathbb{F}) < \infty$ is an integer
- For any $L/\mathbb{F}$, we can also consider $X(L)$
  - In particular, we can consider $X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
  - We again have $\#X(\mathbb{F}_{q^n}) < \infty$ and are integers for every such n.
- So we can consider the sequence

$$[N_1, N_2, \cdots, N_n, \cdots] := [\#X(\mathbb{F}),\ \#X(\mathbb{F}_{q^2}), \cdots,\ \#X(\mathbb{F}_{q^n}), \cdots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \cdots.$$

Why Generating Functions?

Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of F: we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n F(z) \Big|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} \, dz = N_n.$$

The latter form is very amenable to computer calculation.

Why Generating Functions?

An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \longrightarrow \mathbb{C}$ – in nice situations, we can hope for a closed-form representation.

A useful example: by integrating a geometric series we can derive

$$
\begin{aligned}
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n & \qquad (= 1 + z + z^2 + \cdots) \\
\implies \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \int z^n \quad for |z| < 1 \quad \text{by uniform convergence} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\
\implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} & \qquad \left(= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).
\end{aligned}
$$

For completeness, also recall that

$$
\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}
$$

Zeta Functions

Definition: Local Zeta Function

Problem: count points of a (smooth?) projective variety $X/\mathbb{F}$ in all (finite) degree n extensions of $\mathbb{F}$.

Definition (Local Zeta Function)

The *local zeta function* of an algebraic variety X is the following formal power series:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n}\right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note that

$$\begin{aligned} z\left(\frac{\partial}{\partial z}\right) \log Z_X(z) &= z\frac{\partial}{\partial z}\left(N_1 z + N_2\frac{z^2}{2} + N_3\frac{z^3}{3} + \cdots\right) \\ &= z\left(N_1 + N_2 z + N_3 z^2 + \cdots\right) \qquad \text{(unif. conv.)} \\ &= N_1 z + N_2 z^2 + \cdots = \sum_{n=1}^{\infty} N_n z^n, \end{aligned}$$

which is an *ordinary* generating function for the sequence (N_n).

Examples

Example: A Point

Take $X = \{\mathrm{pt}\} = V(\{f(x) = 0\})/\mathbb{F}$ a single point over $\mathbb{F}$, then

$$\begin{aligned} \#X(\mathbb{F}_q) &:= N_1 = 1 \\ \#X(\mathbb{F}_{q^2}) &:= N_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_{q^n}) &:= N_n = 1 \\ &\ddots \end{aligned}$$

and so

$$\begin{aligned} Z_{\{\mathrm{pt}\}}(z) &= \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \cdots\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) \\ &= \exp\left(-\log\left(1 - z\right)\right) \\ &= \frac{1}{1 - z}. \end{aligned}$$

Notice: Z admits a closed form **and** *is a rational function.*

Example: The Affine Line

Take $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^1(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1] \;\middle|\; x_1 \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned}
X(\mathbb{F}_q) &= q \\
X(\mathbb{F}_{q^2}) &= q^2 \\
&\vdots \\
X(\mathbb{F}_{q^n}) &= q^n.
\end{aligned}$$

Then

$$\begin{aligned}
Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} \right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(qz)^n}{n} \right) \\
&= \exp(-\log(1-qz)) \\
&= \frac{1}{1-qz}.
\end{aligned}$$

Example: Affine m-space

Take $X = \mathbb{A}^m/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^m(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned} X(\mathbb{F}_q) &= q^m \\ X(\mathbb{F}_{q^2}) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^{nm}. \end{aligned}$$

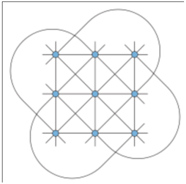


Figure: $\mathbb{A}^2/\mathbb{F}_3$ $(q = 3, m = 2, n = 1)$

Then

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^{nm} \frac{z^n}{n} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{(q^m z)^n}{n} \right) \\ &= \exp(-\log(1 - q^m z)) \\ &= \frac{1}{1 - q^m z}. \end{aligned}$$

Example: Projective Line

Take $X = \mathbb{P}^1/\mathbb{F}$, we can still count by enumerating coordinates:

$$\mathbb{P}^1(\mathbb{F}_{q^n}) = \left\{[x_1 : x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}_{q^n}\right\} / \sim \quad = \left\{[x_1 : 1] \;\middle|\; x_1 \in \mathbb{F}_{q^n}\right\} \coprod \{[1:0]\}.$$

Thus

$$\begin{aligned} X(\mathbb{F}_q) &= q+1 \\ X(\mathbb{F}_{q^2}) &= q^2+1 \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^n+1. \end{aligned}$$

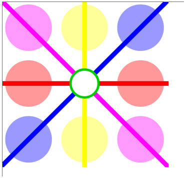


Figure: $\mathbb{P}^1/\mathbb{F}_3$ $(q=3, n=1)$

Thus

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} (q^n+1)\frac{z^n}{n}\right) \\ &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} + \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n}\right) \\ &= \frac{1}{(1-qz)(1-z)}. \end{aligned}$$

The Weil Conjectures

Weil 1

(Weil 1949)

Let X be a smooth projective variety of dimension N over $\mathbb{F}_q$ for q a prime, let $Z_X(z)$ be its zeta function, and define $\zeta_X(s) = Z_X(q^{-s})$.

1. (Rationality)
   $Z_X(z)$ is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$
$$P_{2N}(z) = 1 - q^N z$$
$$P_j(z) = \prod_{j=1}^{\beta_i} (1 - a_{j,k} z) \quad \text{for some reciprocal roots} \quad a_{j,k} \in \mathbb{C}$$

where we've factored each P_i using its reciprocal roots a_{ij}.
In particular, this implies the existence of a meromorphic continuation of the associated function $\zeta_X(s)$, which a priori only converges for $\Re(s) \gg 0$. This also implies that for n large enough, N_n satisfies a linear recurrence relation.

2. (Functional Equation and Poincare Duality)
Let $\chi(X)$ be the Euler characteristic of X, i.e. the self-intersection number of the diagonal embedding $\Delta \hookrightarrow X \times X$; then $Z_X(z)$ satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm\left(q^{\frac{N}{2}} z\right)^{\chi(X)} \ Z_X(z).$$

Equivalently,

$$\zeta_X(N-s) = \pm\left(q^{\frac{N}{2}-s}\right)^{\chi(X)} \zeta_X(s)$$

.

Note that when $N=1$, e.g. for a curve, this relates $\zeta_X(s)$ to $\zeta_X(1-s)$.

Equivalently, there is an involutive map on the (reciprocal) roots

$$z \iff \frac{q^N}{z}$$

$$\alpha_{j,k} \iff \alpha_{2N-j,k}$$

which sends roots of p_j to roots of p_{2N-j}.

3. (Riemann Hypothesis)
The reciprocal roots $a_{j,k}$ are *algebraic* integers (roots of some monic $p \in \mathbb{Z}[x]$) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}}, \qquad 1 \leq j \leq 2N-1, \ \forall k.$$

4. (Betti Numbers)
If X is a "good reduction mod q" of a nonsingular projective variety $\tilde{X}$ in characteristic zero, then the $\beta_i = \mathbf{deg}\, p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$.

Moral:

- The Diophantine properties of a variety's zeta function are governed by its (algebraic) topology.
- Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information.
- Langland's: similarly asks for every L function arising from an automorphic representation to satisfy Weil 2 and 3.

Why is (3) called the "Riemann Hypothesis"?

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

After modifying ζ to make it symmetric about $\Re(s) = \frac{1}{2}$ and eliminate the trivial zeros to obtain $\widehat{\zeta}(s)$, there are three relevant properties

- "Rationality": $\widehat{\zeta}(s)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$.
- "Functional equation": $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
- "Riemann Hypothesis": The only zeros of $\widehat{\zeta}$ have $\Re(s) = \frac{1}{2}$.

Why is (3) called the “Riemann Hypothesis”?

Suppose it holds. We can use the facts that

a. $\left|\exp(z)\right| = \exp(\Re(z))$ and
b. $a^z := \exp(z\mathrm{Log}(a))$,

and to replace the polynomials P_i with

$$L_j(s) := P_j(q^{-s}) = \prod_{k=1}^{\beta_j} \left(1 - \alpha_{j,k} q^{-s}\right).$$

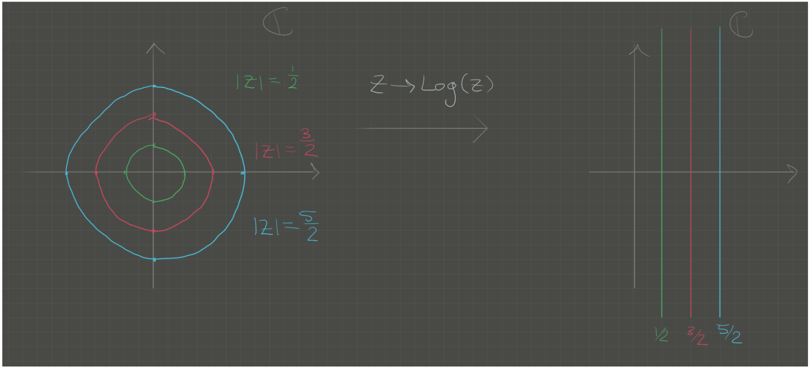
Analogy to Riemann Hypothesis

Now consider the roots of $L_j(s)$: we have

$$\begin{aligned}
& L_j(s_0) = 0 \\
& \iff q^{-s_0} = \frac{1}{\alpha_{j,k}} \quad \text{for some} \quad k \\
& \implies \left|q^{-s_0}\right| = \left|\frac{1}{\alpha_{j,k}}\right| \overset{\text{by assumption}}{=} q^{-\frac{j}{2}} \\
\implies q^{-\frac{j}{2}} \overset{(a)}{=} \exp\left(-\frac{j}{2}\cdot \mathrm{Log}(q)\right) &= \left|\exp\left(-s_0 \cdot \mathrm{Log}(q)\right)\right| \\
& \overset{(b)}{=} \left|\exp\left(-(\Re(s_0) + i\cdot \Im(s_0))\cdot \mathrm{Log}(q)\right)\right| \\
& \overset{(a)}{=} \exp\left(-(\Re(s_0))\cdot \mathrm{Log}(q)\right) \\
& \implies -\frac{j}{2}\cdot \mathrm{Log}(q) = -\Re(s_0)\cdot \mathrm{Log}(q) \quad \text{by injectivity} \\
& \implies \Re(s_0) = \frac{j}{2}.
\end{aligned}$$

Analogy with Riemann Hypothesis

Roughly speaking, realizing that we would need to apply a logarithm (a conformal map) to send the $\alpha_{j,k}$ to zeros of the L_j, this says that the zeros all must lie on the "critical lines" $\frac{j}{2}$.



In particular, the zeros of L_1 have real part $\frac{1}{2}$, analogous to the classical Riemann hypothesis.

Precise Relation

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Background: Generating Functions

Zeta Functions

Examples

The Weil Conjectures

Weil for Elliptic Curves

Weil for Projective m-space

Grassmannian

Weil's Proof

- Difficult to find in the literature! Idea: make a similar definition for schemes, then take $X = \operatorname{Spec} \mathbb{Z}$.
- Define the "reductions mod q" X_q for closed points q.
- Define the *local* zeta functions $\zeta_{X_p}(s) = Z_{X_p}(q^{-s})$.
- (Potentially incorrect) Evaluate to find $Z_{X_p}(z) = \frac{1}{1-z}$.
- Take a product over all closed points to define

$$
\begin{aligned}
L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\
&= \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}} \right) \\
&= \zeta(s),
\end{aligned}
$$

which is the Euler product expansion of the classical Riemann Zeta function.

If anyone knows a reference for this, let me know!

Weil for Elliptic Curves

Example: An Elliptic Curve

The Weyl conjectures take on a particularly nice form for curves. Let $X/\mathbb{F}_q$ be a smooth projective curve of genus g, then

1. (Rationality)

$$Z_X(z) = \frac{p(z)}{(1-z)(1-qz)}$$

2. (Functional Equation)

$$Z_X\left(\frac{1}{qz}\right) = (z\sqrt{q})^{2-2g} Z_X(z)$$

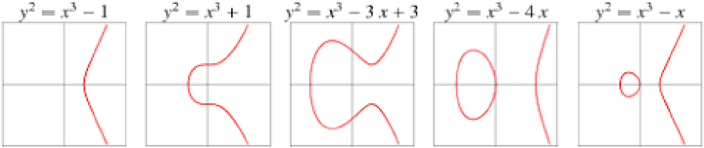
3. (Riemann Hypothesis)

$$p(z) = \prod_{i=1}^{2g} (z - a_i) \quad \text{where} \quad |a_i| = \frac{1}{\sqrt{q}}$$

Take $X = E/\mathbb{F}_q$.

Elliptic Curves

Figure: Some Elliptic Curves



- The number of points is given by

$$N_n := X(\mathbb{F}_{q^n}) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

  - Proof: Unsure! Maybe someone can point me to a reference. Involves trace (or eigenvalues?) of Frobenius.

- The Poincare polynomial is given by $P(x) = \sum \beta_i x^i = 1 + 2x + x^2$.
- The dimension of X over $\mathbb{C}$ is $N = 1$ and its genus is $g = 1$.

The WC say we should be able to write as

$$Z_E(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1-\alpha_{1,1}z)(1-\alpha_{1,2}z)}{(1-z)(1-qz)}.$$

Since we know the number of points, we can compute

$$
\begin{aligned}
Z_E(z) &= \exp \sum_{n=1}^{\infty} \#E(\mathbb{F}_{q^n}) \frac{z^n}{n} \\
&= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
&= \exp \left(\sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
&= \exp \left(-\log \left(1 - qz \right) \right) \cdot \exp \left(-\log \left(1 - z \right) \right) \cdot \exp \left(\log \left(1 - \alpha z \right) \right) \cdot \exp \left(\log \left(1 - \bar{\alpha} z \right) \right) \\
&= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),
\end{aligned}
$$

which is indeed a rational function (Weil 1).

Noting that $g = 1, \chi(E) = 0$, the functional equation reads $Z_E(z) = Z_E(\frac{1}{qz})$.

Not sure how to check!

Writing $p(z) = (1 - \alpha z)(1 - \bar{\alpha} z)$, note that $p(z) = 0 \iff z = 1/\alpha, 1/\bar{\alpha}$, so $|z| = 1/|\alpha| = 1/\sqrt{q}$, satisfying the RH (Weil 3).

Thus

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-s})}.$$

Originally conjectured for curves by Artin, proved for elliptic curves by Hasse in 1934. Proved for curves by Weil in 1949, proposed generalization to projective varieties Proof had work contributed by Dwork (rationality using p-adic analysis), Artin, Grothendieck (etale cohomology), with completion by Deligne in 1970s (RH)

Weil for Projective m-space

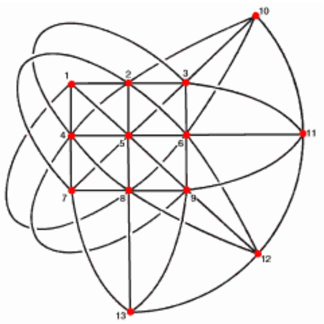
Setup

Take $X = \mathbb{P}^m/\mathbb{F}$ We can write

$$\mathbb{P}^m(\mathbb{F}_{q^n}) = \mathbb{A}^{m+1}(\mathbb{F}_{q^n}) \setminus \{\mathbf{0}\} / \sim \quad = \left\{ \mathbf{x} = [x_0, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\} / \sim$$

But how many points are actually in this space?

Figure: Points and Lines in $\mathbb{P}^2/\mathbb{F}_3$



A nontrivial combinatorial problem!

To illustrate, this can be done combinatorially: identify $\mathbb{P}_{\mathbb{F}}^m = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}_{\mathbb{F}}^{m+1}$.

Theorem

The number of k-dimensional subspaces of $\mathbb{A}_{\mathbb{F}_q}^N$ is the q-analog of the binomial coefficient:

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q^N - 1)(q^{N-1} - 1)\cdots(q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

Remark: Note $\lim_{q \longrightarrow 1} \begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k}$, *the usual binomial coefficient.*

Proof: To choose a k-dimensional subspace,

- Choose a nonzero vector $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$ in $q^N - 1$ ways.
  - For next step, note that $\#\mathrm{span}\,\{\mathrm{v}_1\} = \#\left\{\lambda \mathrm{v}_1 \;\middle|\; \lambda \in \mathbb{F}_q\right\} = \#\mathbb{F}_q = q$.
- Choose a nonzero vector $\mathbf{v}_2$ *not* in the span of $\mathbf{v}_1$ in $q^N - q$ ways.
  - Now note $\#\mathrm{span}\,\{\mathrm{v}_1, \mathrm{v}_2\} = \#\left\{\lambda_1 \mathrm{v}_1 + \lambda_2 \mathrm{v}_2 \;\middle|\; \lambda_i \in \mathbb{F}\right\} = q \cdot q = q^2$.

Proof continued

- Choose a nonzero vector $\mathbf{v}_3$ *not* in the span of $\mathbf{v}_1, \mathbf{v}_2$ in $q^N - q^2$ ways.
- $\cdots$ until $\mathbf{v}_k$ is chosen in

$$(q^N - 1)(q^N - q)\cdots(q^N - q^{k-1}) \quad \text{ways}$$

.

  - This yields a k-tuple of linearly independent vectors spanning a k-dimensional subspace V_k
- This overcounts because many linearly independent sets span V_k, we need to divide out by the number of ways to choose a basis inside of V_k.
- By the same argument, this is given by

$$(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$$

Thus

$$\#\text{subspaces} = \frac{(q^N - 1)(q^N - q)(q^N - q^2)\cdots(q^N - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2)\cdots(q^k - q^{k-1})}$$

$$= \frac{q^N - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right)\frac{q^{N-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right)\frac{q^{N-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right)\frac{q^{N-(k-1)} - 1}{q^{k-(k-1)-1}}$$

$$= \frac{(q^N - 1)(q^{N-1} - 1)\cdots(q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

Counting Points

Note that we've actually computed the number of points in any Grassmannian.

Identify $\mathbb{P}_{\mathbb{F}}^m = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}_{\mathbb{F}}^{m+1}$.

We obtain a nice simplification for the number of lines corresponding to setting $k = 1$:

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = \frac{q^{m+1} - 1}{q - 1} = q^m + q^{m-1} + \cdots + q + 1 = \sum_{j=0}^m q^j.$$

Thus

$$X(\mathbb{F}_q) = \sum_{j=0}^m q^j$$

$$X(\mathbb{F}_{q^2}) = \sum_{j=0}^m \left(q^2\right)^j$$

$$\vdots$$

$$X(\mathbb{F}_{q^n}) = \sum_{j=0}^m (q^n)^j.$$

So

$$\begin{aligned}
Z_X(z) &= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}(q^n)^j\frac{z^n}{n}\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}\frac{\left(q^j z\right)^n}{n}\right)\\
&= \exp\left(\sum_{j=0}^{m}\sum_{n=1}^{\infty}\frac{\left(q^j z\right)^n}{n}\right)\\
&= \exp\left(\sum_{j=0}^{m-1} - \log(1 - q^j z)\right)\\
&= \prod_{j=0}^{m}\left(1 - q^j z\right)^{-1}\\
&= \left(\frac{1}{1-z}\right)\left(\frac{1}{1-qz}\right)\left(\frac{1}{1-q^2 z}\right)\cdots\left(\frac{1}{1-q^m z}\right),
\end{aligned}$$

Miraculously, still a rational function!

An Easier Proof

Quick recap:

$$Z_{\{\mathrm{pt}\}} = \frac{1}{1-z} \qquad Z_{\mathbb{P}^1}(z) = \frac{1}{1-qz} \qquad Z_{\mathbb{A}^1}(z) = \frac{1}{(1-z)(1-qz)}.$$

Note that $\mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$ and correspondingly $Z_{\mathbb{P}^1}(z) = Z_{\mathbb{A}^1}(z) \cdot Z_{\{\mathrm{pt}\}}(z)$.
This works in general:

Lemma (Excision)

If $Y/\mathbb{F}_q \subset X/\mathbb{F}_q$ is a closed subvariety, for $U = X \setminus Y$,
$Z_X(z) = Z_Y(z) \cdot Z_U(z)$.

Proof: Let $N_n = \#Y(\mathbb{F}_{q^n})$ and $M_n = \#U(\mathbb{F}_{q^n})$, then

$$\begin{aligned}
\zeta_X(z) &= \exp\left(\sum_{n=1}^{\infty} (N_n + M_n)\frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n} + \sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right) = \zeta_Y(z) \cdot \zeta_U(z).
\end{aligned}$$

A Easier Proof

Note that geometry can help us here: we have a stratification $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$, and so inductively

$$\mathbb{P}^m = \coprod_{j=0}^m \mathbb{A}^j = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m,$$

and recalling that

$$Z_{X \coprod Y}(z) = Z_X(z) \cdot Z_Y(z)$$

and $Z_{\mathbb{A}^j}(z) = \frac{1}{1-q^j z}$ we have

$$Z_{\mathbb{P}^m}(z) = \prod_{j=0}^m Z_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1-q^j z}.$$

Notice that the highest degree is exactly m, and there is exactly one factor for each $j \leq m$. Note that $PP^m/\mathbb{F}_q$ can be though of as a mod q reduction of $\mathbb{R}\mathbb{P}^m$ or $\mathbb{C}\mathbb{P}^m$, and somehow Z "sees" its dimension.

Grassmannian

Motivation

Consider now $X = \mathrm{Gr}(k, m)/\mathbb{F}$ – by the previous computation, we know

$$X(\mathbb{F}_{q^n}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} := \frac{(q^{nm}-1)(q^{nm-1}-1)\cdots(q^{nm-n(k-1)}-1)}{(q^{nk}-1)(q^{n(k-1)}-1)\cdots(q^n-1)}$$

but the corresponding Zeta function is much more complicated than the previous examples:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} \frac{z^n}{n}\right) = \cdots ?.$$

Note that $\dim_{\mathbb{R}} \mathrm{Gr}_{\mathbb{R}}(k, m) = k(m-k)$ as a real manifold, so by Weil we should expect

$$Z_X(z) = \prod_{j=0}^{k(m-k)} \frac{p_{2(j+1)}(z)}{p_{2j}(z)}$$

with $\deg p_j = \beta_j$.

The Poincare polynomial of the complex Grassmannian is given by

$$P(x) = \sum_{i=1}^{k(m-k)} \lambda_{m,k}(i) x^i$$

, i.e. the number of integer partitions of of $[i]$ into at most $m-k$ parts, each of size at most k.

It turns out that (proof omitted) one can show

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \sum_{j=0}^{k(m-k)} \lambda_{m,k}(j) q^j \implies Z_X(z) = \prod_{j=0}^{k(m-k)} \left(\frac{1}{1-q^j x} \right)^{\lambda_{m,k}(j)}.$$

Weil's Proof

Very Hard Example: A Diagonal Hypersurface

Proof of rationality of $Z_X(T)$ for X a diagonal hypersurface.

- Set q to be a prime power and consider $X/\mathbb{F}_q$ defined by
$$X = V(a_0 x_0^{n_0} + \cdots + a_r x_r^{n_r}) \subset \mathbb{F}_q^{r+1}.$$
- We want to compute $N = \#X$.
- Set $d_i = \gcd(n_i, q-1)$.
- Define the character
$$\begin{aligned} \psi_q : \mathbb{F}_q &\longrightarrow \mathbb{C}^\times \\ a &\mapsto \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}\right). \end{aligned}$$
  - By Artin's theorem for linear independence of characters, $\psi_q \not\equiv 1$ and every additive character of $\mathbb{F}_q$ is of the form $a \mapsto \psi_q(ca)$ for some $c \in \mathbb{F}_q$.
- Fix an injective multiplicative map
$$\psi : \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times.$$
- Define
$$\begin{aligned} \chi_{\alpha,n} : \mathbb{F}_{q^n}^\times &\longrightarrow \mathbb{C}^\times \\ x &\mapsto \phi(x)^{\alpha(q^n-1)} \end{aligned}$$

- Now restrict to $n_0 = \cdots = n_r = n$ a constant, and we consider a point count

$$\overline{N}_\nu = \# \left\{ [x_0 : \cdots : x_r] \in \mathbb{P}^r_{\mathbb{F}_q^\nu} \;\middle|\; \sum_{i=0}^{r} a_i x_i^n = 0 \right\}.$$

- We have a relation $(q^\nu - 1)\overline{N}_\nu = N_\nu$.
- This lets us write

$$\overline{N}_\nu = \sum_{j=0}^{r-1} q^{j\nu} + \sum_{\substack{\sum \alpha_i \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \prod_{j=0}^{r} \overline{\chi}_{\alpha_{j,\nu}}(a_i) J_\nu(\alpha).$$

- Set

$$\mu(\alpha) = \min \left\{ \mu \;\middle|\; (q^\mu - 1)\alpha \sim 0 \right\}$$

$$C(\alpha) = (-1)^{r+1} \prod_{j=1}^{r} \overline{\chi}_{\alpha_0, \mu(\alpha)}(a_j) \cdot J_{\mu(\alpha)}(\alpha).$$

- Plugging into the zeta function Z yields

$$\exp\left(\sum_{\nu=1}^{\infty} \overline{N}_\nu \frac{T^\nu}{\nu} \right) = \frac{1}{(1-T)(1-qT)\cdots(1-q^{r-1}T)} \prod_{\sum \alpha_i \sim 0} \Big(1 - C($$