

# Title

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## Contents

<b>1</b>	<b>Sunday, September 13</b>	<b>1</b>
1.1	General Notes . . . . .	1
1.2	1.a . . . . .	2
1.3	3.a . . . . .	3
1.4	3.b . . . . .	4
1.5	4.a . . . . .	5
1.6	4.c . . . . .	6

## 1 | Sunday, September 13

### 1.1 General Notes

- Say what you're assuming at the start of the proof.
  - If flipping logic and not using a direct proof (contradiction, contrapositive, etc), then signpost/announce it near the beginning of the proof.
  - Examples: for  $P \implies Q$ ,
    - \* Direct proof: "Suppose  $P \dots$ "
    - \* Contradiction: "Suppose toward a contradiction  $P$  but not  $Q \dots$ "  
(Usually show  $\neg P$ . If you show  $Q$ , a direct proof might be simpler.)
    - \* Contrapositive: "Suppose by contrapositive that  $\neg Q$  holds,  $\dots$ "
- Put any important equations (i.e. major steps of the proof) on their own lines or in `displaymath` environments.
- Use some whitespace to separate parts of the proof and increase readability.
- Remember that limits of sequences need not exist, but  $\liminf$ s/ $\limsup$ s always do  
(just may be  $\pm\infty$ ).
- Try to avoid abbreviating the names of major theorems (example: "AP" can stand for many results, not just the Archimedean property!)
- It's not generally true that  $a \leq M \implies |a| \leq M$ , e.g. take  $a = -1$ . This only holds  $a \geq 0$ .

- A generic set **may not** contain its inf or sup. Example:  $\inf \left\{ \frac{1}{n} \right\} = 0$  and  $0 \notin \left\{ \frac{1}{n} \right\}$ , or  $\sup \left\{ 1 - \frac{1}{n} \right\} = 1$  with  $1 \notin \left\{ 1 - \frac{1}{n} \right\}$ .
- If there exists some element of a set or sequence with a given property, try to say where it comes from and why the property holds for it.
- Similarly, if a property holds for all elements of a set or sequence, try to say why.
- The crux of many proofs are certain inequalities, so try to justify every inequality that appears.
- If you use a theorem, be sure to mention it by its full name.
- Useful counterexamples:
  - $x_n = (-1)^n$
  - Literal lists of numbers:  $[0, 1, 0, 2, \dots]$ .

## 1.2 1.a

*Proof* ( $A \implies B$ ).

- Suppose  $\{a_n\}$  is not bounded above.
- Then any  $k \in \mathbb{N}$  is not an upper bound for  $\{a_n\}$ .
- So choose a subsequence  $a_{n_k} > k$ , then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k \rightarrow \infty} a_{n_k} > \liminf_{k \rightarrow \infty} k = \infty.$$

■

*Proof* ( $\neg A \implies \neg B$ ).

- Suppose  $\{a_n\}$  is bounded by  $M$ , so  $a_n < M < \infty$  for all  $n \in \mathbb{N}$ .
- Then if  $\{a_{n_k}\}$  is a subsequence, we have  $a_{n_k} \in \{a_n\}$ , so  $a_{n_k} < M$  for all  $k \in \mathbb{N}$ .
- But then

$$a_{n_k} < M \implies \limsup_{k \rightarrow \infty} a_{n_k} \leq M,$$

- Now note that if  $\lim_{k \rightarrow \infty} a_{n_k}$  exists,

$$\lim_{k \rightarrow \infty} a_{n_k} < \limsup_{k \rightarrow \infty} a_{n_k} \leq M < \infty,$$

so every subsequence is bounded and thus can not converge to  $\infty$ .

■

## 1.3 3.a

*Proof (Using definition (i)).*

- Suppose  $x_n \leq M$  for all  $n$ , we will show that every subsequential limit is also bounded by  $M$ .
- Let

$$S := \{x \in \mathbb{R} \mid x \text{ is a subsequential limit of } \{x_n\}\}$$

be the set of subsequential limits.

– Note that  $\inf S := \liminf_{n \rightarrow \infty} x_n$  by definition (i).

- Let  $\{x_{n_k}\} \in S$  be an arbitrary convergent subsequence (since we are only concerned about subsequences with well-defined limits).
- Then for every  $k$  we have  $x_{n_k} \in \{x_n\}$ , so

$$|x_{n_k}| \leq M.$$

- By order limit laws,

$$|x_{n_k}| \leq M \implies \lim_{k \rightarrow \infty} |x_{n_k}| \leq M,$$

- Since the map  $x \mapsto |x|$  is continuous, using the sequential definition of continuity we can pass the limit through the absolute value to obtain

$$\left| \lim_{k \rightarrow \infty} x_{n_k} \right| \leq M.$$

- Since the subsequence was arbitrary, we find that  $M$  is an upper bound for  $S$  and so  $\sup S \leq M$ .
- But

$$\inf S \leq \sup S \leq M \implies \inf S \leq M.$$

■

*Proof (Using definition (ii)).*

- Suppose  $|x_n| \leq M$  for every  $n$ , we will directly show that  $\left| \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k \right| \leq M$ .
- By order-limit laws, for every fixed  $n$  we have

$$|x_n| \leq M \iff -M \leq x_n \leq M \implies -M \leq \inf_{k \geq n} x_k \leq M,$$

where we've used the fact that  $x_n \geq -M$  for all  $n$  implies that  $\inf_{k \geq n} x_k \geq -M$ .

- Again applying order-limit laws,

$$-M \leq \inf_{k \geq n} x_k \leq M \implies -M \leq \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k \leq M \iff \left| \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k \right| \leq M.$$

■

## 1.4 3.b

*Proof (Using definition (i)).*

Note that here we define  $S$  to be the set of all subsequential limits of  $\{x_n\}$  and

$$\liminf_n x_n := \inf S.$$

- Suppose toward a contradiction that  $\beta < \liminf_n x_n$  **but** there does not exist any  $N$  such that  $n \geq N \implies x_n > \beta$ .
- Then for all  $N$  there exists an  $n > N$  with  $x_n \leq \beta$ , so the set

$$B := \{n \in \mathbb{N} \mid x_n \leq \beta\}$$

is countably infinite.

- Then by Bolzano-Weierstrass, since  $B$  is bounded it contains a convergent subsequence  $x_{n_k}$  which satisfies

$$x_{n_k} \leq \beta \quad \forall k \implies L := \lim_{k \rightarrow \infty} x_{n_k} \leq \beta$$

where we've used order-limit laws.

- We now have  $L \in S$ , a subsequential limit satisfying  $L \leq \beta$  and since  $\inf S$  is a lower bound for  $S$ ,

$$\inf S \leq L \leq \beta.$$

which contradicts  $\beta < \liminf_n x_n$ . ■

*Proof (Using definition (ii)).*

Note that here we define

$$\liminf_n x_n := \lim_{n \rightarrow \infty} S_n \quad \text{where} \quad S_n := \inf \{x_k \mid k \geq n\}.$$

- Write  $L := \lim_{n \rightarrow \infty} S_n$  and suppose  $\beta < L$ .
- Then we have

$$\forall \varepsilon > 0, \exists N \quad \text{such that} \quad n \neq N \implies |S_n - L| < \varepsilon.$$

- Since  $\beta < L \iff L - \beta > 0$ , we can set  $\varepsilon := L - \beta$  to produce an  $N$  such that

$$n \geq N \implies |L - S_n| < L - \beta \iff \beta - L < S_n - L < L - \beta.$$

- Just taking the first part of this composite inequality we have

$$n \geq N \implies \beta - L < S_n - L \iff \beta < S_n := \inf_{k \geq n} x_k \leq x_n,$$

supplying the  $N$  for which  $n \geq N \implies \beta < x_n$  as desired. ■

*Proof (Using definition (ii), alternative).*

- Suppose toward a contradiction that  $\beta < \liminf_n x_n$  **but** there is no  $N$  such that  $n \geq N \implies x_n > \beta$ .
- Then for all  $N$  there exists an  $n$  with  $x_n \leq \beta$ , so if we form the set

$$B_n := \left\{ k \in \mathbb{N} \mid k \geq n \text{ and } x_k \leq \beta \right\},$$

then  $B_n$  is countably infinite for every  $n$

- But then  $B_n \subseteq \left\{ k \in \mathbb{N} \mid k \geq n \right\}$  for every  $n$  implies that

$$\inf_{k \geq n} x_k \leq \inf_{k \in B_n} x_k \leq \beta \quad \forall n,$$

since an infimum over a larger set can only get smaller.

- Applying order-limit laws, we then have

$$\inf_{k \geq n} x_k \leq \beta \quad \forall n \implies \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k \leq \beta,$$

but this contradicts  $\liminf_n x_n > \beta$ .

■

## 1.5 4.a

*Proof .*

- Suppose  $\{x_n\}$  is bounded and  $\limsup |x_n| = 0$ .
- Then using the supremum definition,  $\lim_{n \rightarrow \infty} \sup_{k \geq n} |x_k| = 0$ .
- Note that

$$\lim_{n \rightarrow \infty} x_n = 0 \iff \forall \varepsilon \quad \exists N \text{ such that } n \geq N \implies |x_n| < \varepsilon.$$

- So let  $\varepsilon > 0$  be arbitrary.
- By the definition of the limit appearing in the  $\limsup$ , there exists an  $N_0$  such that

$$n \geq N_0 \implies \sup_{k \geq n} |x_k| < \varepsilon.$$

- But then taking  $N = N_0$  in the first equation yields the result, since

$$n \geq N_0 \implies |x_n| \leq \sup_{k \geq n} |x_k| < \varepsilon.$$

■

**1.6 4.c***Proof .*

- Note that  $-1 \leq \sin(x) \leq 1$  and  $\sin(x) = \pm 1 \iff x = 2k\pi \pm \frac{\pi}{2}$ .
- Since  $\pi$  is irrational,  $\sin(n)$  will never be of this form, so  $-1 < \sin(n) < 1$ .
- Taking floors, we have  $-1 \leq \lfloor \sin(n) \rfloor \leq 0$ , which in fact means that  $\sin(n) \in \{-1, 0\}$  can only take on one of two values.
- The set of subsequential limits is then just  $\{-1, 0\}$ , which is a finite set and thus achieves its sup and inf.
- Claim:  $\limsup \lfloor \sin(n) \rfloor = 0$ .
  - It suffices to show that  $\lfloor \sin(n) \rfloor = 0$  infinitely often.
  - But note that in any interval of the form  $(a, a + \frac{\pi}{2})$

