# Floer Talk

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Tuesday 14th April, 2020

## Contents



# 1 Background and Notation

From the text:

- $(W, \omega \in \Omega_2(W))$ is a (compact?) symplectic manifold
- $C^\infty(A, B)$ is the space of smooth maps with the $C^\infty$ topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C^\infty_{\text{Loc}}(A, B)$ is the space with the $C^\infty$ uniform convergence topology on compact subsets of $A$
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with $X_H$ its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$ given by $H_t \in C^\infty(W; \mathbb{R})$ is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  where $\mathcal{L}W$ is the contractible loop space of $W$, $u : \mathbb{D} \longrightarrow W$ is an extension of $x : S^1 \longrightarrow W$ to the disc with $u(\exp(2\pi i t)) = x(t)$.

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \displaystyle\int_0^1 (H_t\ dt - p\ dq)$.

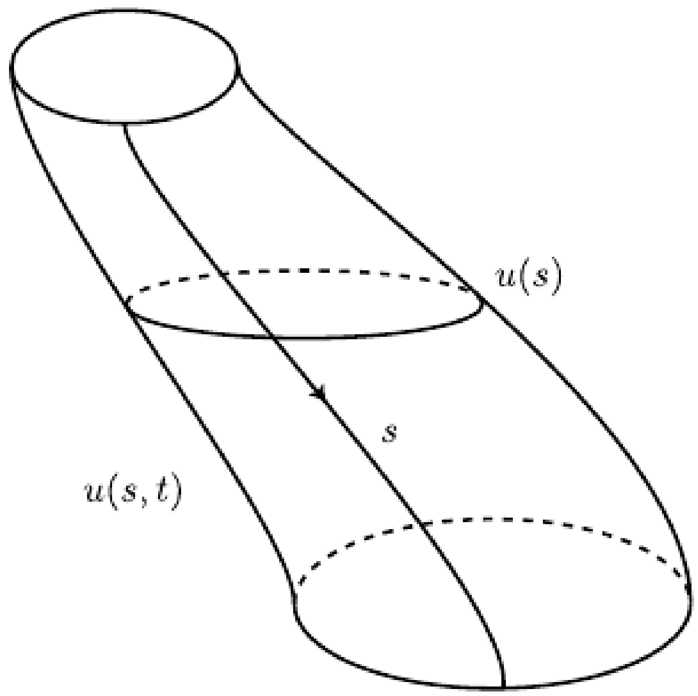
- Critical points of the action functional $\mathcal{A}_H$ are given by orbits, i.e. contractible loops $x, y \in \mathcal{L}W$
- In general, $x, y$ are two periodic orbits of $H$ of period 1.

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

  This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be $J$-holomorphic curves.

- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
  - They correspond to "embedded cylinders" with sides $u$ and contractible caps $x, y$ regarded as loops in $W$.
  - They also correspond to paths in $\mathcal{L}W$ from $x \longrightarrow y$ (precisely: trajectories of the vector field $-\mathrm{grad}\mathcal{A}_H$)



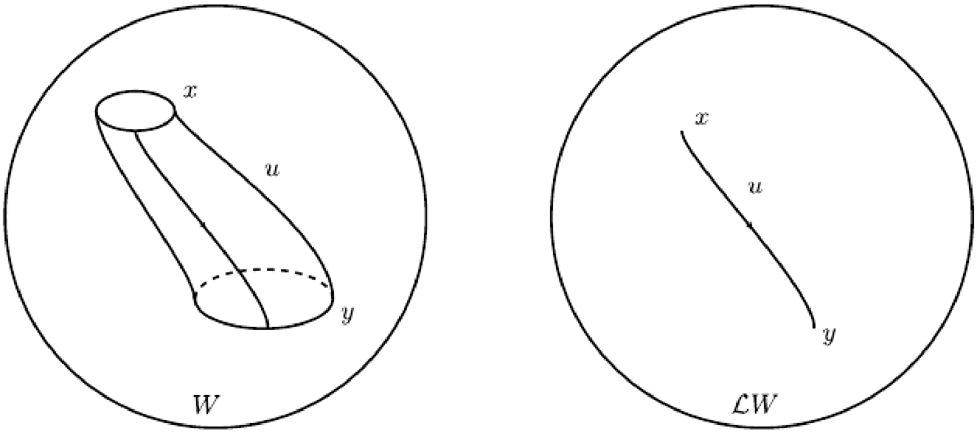



**Fig. 6.5**

Here $u(s) \in \mathcal{L}W$ is a loop with value at time $t$ given by $u(s,t)$, and $\lim_{s \longrightarrow -\infty} u_s(t) = x$, $\lim_{s \longrightarrow \infty} u_s(t) = y$.

- The energy of a solution is $E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 \, dsdt$.
- $\mathcal{M} = \left\{ u \in C^\infty(\mathbb{R}; \mathcal{L}W) \;\middle|\; E(u) < \infty \right\}$ (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x,y)$ is the space of solutions of the Floer equation connecting orbits $x$ and $y$.
- $C_\searrow(x,y)$:

$$C_\searrow(x,y) := \{ u \in C^\infty(\mathbb{R} \times S^1; W) \;\Big|\; \lim_{s \longrightarrow -\infty} u(s,t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s,t) = y(t), \\ \left| \frac{\partial u}{\partial s}(s,t) \right| \leq K e^{-\delta |s|}, \quad \left| \frac{\partial u}{\partial t}(s,t) - X_H(u) \right| \leq K e^{-\delta |s|} \}$$

where $K, \delta > 0$ are constants depending on $u$. So

$$|\partial_s u(s,t)|, |\partial_t u(s,t) - X_H(u)| \sim e^{|s|}.$$

From the Appendices

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator: $\dim \ker - \dim \operatorname{coker}$.
- Fredholm operators: those for which the index makes sense, i.e. $\dim \ker < \infty, \dim \operatorname{coker} < \infty$.
- Elliptic operators: generalize the Laplacian $\Delta$, coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset

- Sobolev spaces: in dimension 1, define $\|u(t)\|_{s,p} = \sum_{i=0}^{s} \left\| \partial_t^i u(t) \right\|_{L^p}$ on $C^\infty(\overline{U})$, then take the completion and denote $W^{s,p}(\overline{U})$. Yields a distribution space, elements are functions with weak derivatives.
- Distribution: $C_c^\infty(U)^\vee$, the dual of the space of smooth compactly supported functions on an open set $U \subset \mathbb{R}^n$.

# 2 Talk

Overview: Analyze the space $\mathcal{M}(x,y)$ of solutions to the Floer equation connecting two orbits $x, y$ of $H$. Show $\mathcal{M}(x,y)$ is in fact a manifold of dimension $\mu(x) - \mu(y)$.

Strategy:

1. Describe $\mathcal{M}(x,y)$ as the zero set of a section of a vector bundle over the Banach manifold $\mathcal{P}(x,y)$.
2. Apply the Sard-Smale theorem: perturb $H$ to make $\mathcal{M}(x,y)$ the inverse image of a regular value of some map.
3. Show that the tangent maps (?) are Fredholm operators of index $\mu(x) - \mu(y) = \dim \mathcal{M}(x,y)$.

Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of $\mathcal{F}$.

## 2.1 Review 8.2

What is $\mathcal{F}$?

We started with the unadorned Floer map:

$$\begin{aligned} \mathcal{F} : \mathcal{C}^\infty \left( \mathbf{R} \times S^1; W \right) &\longrightarrow \mathcal{C}^\infty \left( \mathbf{R} \times S^1; TW \right) \\ u &\mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \operatorname{grad}_u (H_t) \end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x,y) &\longrightarrow \mathcal{L}^p(x,y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \operatorname{grad} H_t(u). \end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned} \mathcal{P}^{1,p}(x,y) :? &\longrightarrow ? \\ (s,t) &\mapsto \exp_{w(s,t)} Y(s,t). \end{aligned}$$

where $Y \in W^{1,p}(w^* TW)$ and $w \in C^\infty_{\searrow}(x,y)$.

## 2.2 8.3: The Space of Perturbations of $H$

Goal: given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Start by trying to construct a subspace $\mathcal{C}^\infty_\varepsilon(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of $H$, and show it is dense.

Idea: define a norm $\|\cdot\|_\varepsilon$ on $C^\infty_\varepsilon(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$ denote a perturbation of $H$.
- Fix $\varepsilon = \left\{\varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0}\right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W, t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left|d^k h(\mathbf{x}, t)\right| = \max\left\{d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k\right\},$$

  the maximum over all sets of multi-indices $\alpha$ of length $k$.

  Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1}\, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

  the partial derivatives wrt the corresponding variables.

- Define a norm on $C^\infty_\varepsilon$:

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left|d^k h(x, t)\right|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

Then

$$\|h\|_\varepsilon = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} \left|d^k (h \circ \Psi_i^{-1})(z)\right|.$$

Where $\{\varepsilon_k\} \subset \mathbb{R}$ is chosen such that $\mathcal{C}^\infty_\varepsilon \hookrightarrow \mathcal{C}^\infty(W \times S^1)$ is dense for the $C^\infty$ topology, and the $\Psi_i : B_i \longrightarrow \overline{B(0,1)}$ is a fixed finite sequence of diffeomorphisms where $\bigcup_i B_i^\circ = W \times S^1$.

Note that we'll only use density for the $C^1$ topology in our case.

**Proposition 2.1.**
Such a sequence $\{\varepsilon_k\}$ can be chosen.

*Proof .*
Show that $C^\infty(W \times S^1)$ is separable, yielding a sequence $(f_n) \subset C^\infty(W \times S^1)$ that is dense in the $C^1$ topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k \le n} \|f_k\| C^n(W \times S^1)}$$

where the diffeomorphisms $\Psi_i$ are used to compute these norms. ■

Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$ and are nondegenerate.

## 2.3 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as $Z_i$, tangents to $W$ along $u$ or in a neighborhood $B$ of $u$) with actual vectors in $\mathbb{R}^m$.

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

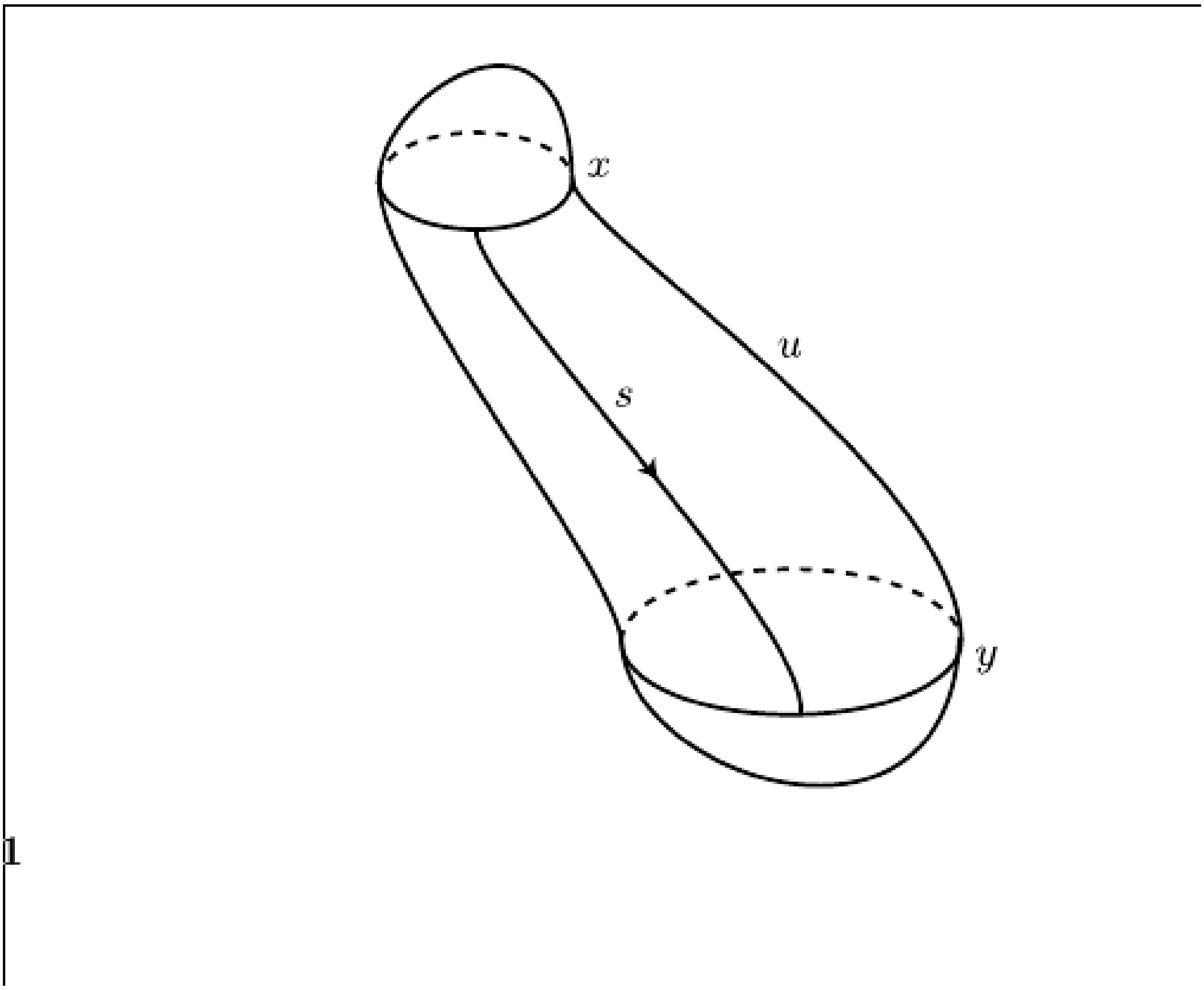
We can then identify $\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R} \times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that $x, y$ are contractible loops in $W$ that are nondegenerate critical points of the action functional $\mathcal{A}_H$ (i.e. solutions to the Floer equation), and $C_\searrow(x, y)$ was the set of maps $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions.

Fix a solution $u \in \mathcal{M}(x, y) \subset C^\infty_{\mathrm{Loc}}(\mathbb{R} \times S^1; W)$.

We lift each map to $\tilde{u} : S^2 \longrightarrow W$ in the following way: the loops $x, y$ are contractible, so they bound discs. So we extend according to:




Recall assumption 6.22: every smooth map $w : S^2 \longrightarrow W$ yields a symplectic trivialization of $w^*TW$ (e.g. when $\pi_2(W) = 0$, so every map from $S^2$ extends to $B^3$).

Trivialize the symplectic fiber bundle $\tilde{u}^*TW$ to obtain an orthonormal unitary frame $\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$ depending smoothly on $(s,t) \in S^2$, where $\lim\limits_{s\longrightarrow\infty} Z_i$ exists for each $i$. We also require that $\partial_s Z_i, \partial_s^2 Z_i, \partial_s\partial_t Z_i \overset{s\longrightarrow\pm\infty}{\longrightarrow} 0$ for each $i$.

This frame defines a chart about $u$ of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y) \\ \mathbf{y} = (y_1,\dots,y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

Since $(d\exp)_0 = \mathrm{id}$, we have $(d\iota)_0(\mathbf{y}) = \sum\limits_i y_i Z_i$.

We'll now consider and compute the differential of

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R}\times S^1;TW\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right) \\ u \longmapsto \frac{\partial u}{\partial s} + J(u)&\left(\frac{\partial u}{\partial t} - X_t(u)\right). \end{aligned}$$

Take the vector $Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$, where we view $Y$ as a vector in $\mathbb{R}^m$ tangent to $W$, given by $Y = \sum y_i Z_i$.

We write

$$\mathcal{F}(u+Y) = \frac{\partial(u+Y)}{\partial s} + J(u+Y)\frac{\partial(u+Y)}{\partial t} - J(u+Y)X_t(u+Y)$$

and extract the part that is linear in $Y$:

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y).$$

**Lemma 2.2** ***(Acting by Derivation).***
For any $J \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and $Y, v :? \longrightarrow \mathbb{R}^m$ we have

$$(dj)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

There is a proof.

For every such smooth map $u : \mathbb{R} \times S^1 \longrightarrow W$, $(d\mathcal{F})_u(Y) = O_1 + O_0$ where $O_i$ are differential operators of order $i$, and in fact $O_1$ can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$ where $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^n)$ is linear of order 0, and in fact we have

**Proposition 2.3.**
If $u$ solves Floer's equation, then $(d\mathcal{F})_u = \bar{\partial} + S(s,t)$ where $S$ is linear, tends to a symmetric operator as $s \longrightarrow \pm\infty$, and $\lim \partial_t S = 0$ uniformly in $t$.

There is a very long computational proof.

Denote the order 0 part of $(d\mathcal{F})_u$ as $Y \mapsto S \cdot Y$ so $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and define $S^{\pm} := \lim_{s \longrightarrow \pm\infty} S(s, \cdot)$.

**Proposition 2.4.**
The equation $\partial_t Y = J_0 S^{\pm} Y$ linearizes Hamilton's equation $\dot{z} = X_t(z)$ at $x = \lim_{s \longrightarrow \pm\infty} u$ for $S^+$ and $S^-$ respectively.

Proof: uses previous proposition.

Given a solution $u$, the product

$$\begin{aligned} u \cdot s :? &\longrightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and $\mathcal{F}(u \cdot s) = 0$ for all $s$.

**Punchline**:

Thus $\dfrac{\partial u}{\partial s}$ is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s}\mathcal{F}(u \cdot s) = (d\mathcal{F})_u\left(\frac{\partial u}{\partial s}\right).$$

Along any nonconstant solution connecting $x$ and $y$, $\dim \ker(d_{\mathcal{F}})_u \geq 1$.