

Assignment 6: The Fourier Transform

D. Zack Garza

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = \lim_{\xi' \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx$$

But as an immediate consequence, this yields

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \, dx \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\ &\rightarrow 0, \end{aligned}$$

which follows from continuity in L^1 since $f(x - \xi') \rightarrow f(x)$ as $\xi' \rightarrow 0$.

It thus only remains to show that the hint holds, and that $\xi' \rightarrow 0$ as $\xi \rightarrow \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned}
 \widehat{(f * g)}(\xi) &= \int \int f(x - y)g(y) \exp(-2\pi x \cdot \xi) \, dy \, dx \\
 &= \int \int f(x - y)g(y) \exp(-2\pi x \cdot \xi) \, dx \, dy \\
 &= \int \int f(t) \exp(-2\pi i(x - y) \cdot \xi)g(y) \exp(-2\pi i y \cdot \xi) \, dx \, dy \\
 &\quad (t = x - y, \, dt = \, dx) \\
 &= \int \int f(t) \exp(-2\pi i t \cdot \xi)g(y) \exp(-2\pi i y \cdot \xi) \, dt \, dy \\
 &= \int f(t) \exp(-2\pi i t \cdot \xi) \left(\int g(y) \exp(-2\pi i y \cdot \xi) \, dy \right) \, dt \\
 &= \int f(t) \exp(-2\pi i t \cdot \xi) \hat{g}(\xi) \, dt \\
 &= \hat{g}(\xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\
 &= \hat{g}(\xi) \hat{f}(\xi).
 \end{aligned}$$

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

2.2.2 (ii)

Similarly, we have

$$(\widehat{f * g}) * h = \widehat{f * g}\hat{h} = \hat{f}\hat{g}\hat{h} = \widehat{f\hat{g} * h} = f * (g * h).$$

3 Problem 3

4 Problem 4

5 Problem 5

6 Problem 6