Problem Set 6

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1 Humphreys 5.3

Let λ be regular, antidominant, and integral, and suppose $M(\lambda)^n \neq 0$ but $M(\lambda)^{n+1} = 0$. In the Jantzen filtration of $M(w \cdot \lambda)$, show that $n = \ell_\lambda(w)$ where ℓ_λ is the length function of the system $(W_{[\lambda]}, \Delta_{[\lambda]})$. Thus there are $\ell(w) + 1$ nonzero layers in this filtration.

> Use 0.3(2) to describe $\Phi^+_{w\cdot\lambda}$.

2 Humphreys 7.2

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and show that T_λ^μ need not take Verma modules to Verma modules.

> For example, let $\lambda = 1$ and $\mu = -3$.

2.1 Solution

Let $\lambda = 1$ and $\mu = -3$, noting that both are integral, μ is antidominant, and μ, λ are *compatible* as in the definition in 7.1. We can then consider $\nu \coloneqq \mu - \lambda = -3 - 1 = -4$, and to compute the $\bar\nu$ that appears in the definition of T_λ^μ, we consider the (usual) W-orbit of ν. In $\mathfrak{sl}(2, \mathbb{C})$, we identify $\Lambda = \mathbb{Z}$, $W = \{\mathrm{id}, s_\alpha\}$, and $s_\alpha \lambda = -\lambda$ as reflection about 0. Thus the orbit is given by $W\nu = \{-4, 4\}$, which contains the unique dominant weight $\bar\nu = 4$. We thus have

$$T_1^{-3}(\,\cdot\,) = \mathrm{pr}_{-3}(L(4) \otimes \mathrm{pr}_1(\,\cdot\,)).$$

We use the fact that we always have an exact sequence of the form

$$0 \longrightarrow N(\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

where in $\mathfrak{sl}(2,\mathbb{C})$ we can identify $N(\lambda) = L(-\lambda-2)$, thus we have

$$0 \longrightarrow L(-\lambda-2) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

Here we can identify

$$\begin{aligned} L(-\lambda-2) &= L(-1-2) \\ &= L(-3) \\ &= L(\mu) \\ &= M(\mu) \quad \text{since } \mu = -3 \text{ is integral and antidominant,} \end{aligned}$$

thus we can rewrite the exact sequence as

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & M(\mu) \arrow[r] \arrow[d, equal] & M(\lambda) \arrow[r] \arrow[d, equal] & L(\lambda) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & M(-3) \arrow[r] & M(1) \arrow[r] & L(1) \arrow[r] & 0 \end{tikzcd}$$

We know that the translation functor is exact, so applying T_λ^μ yields the following short exact sequence:

$$0 \longrightarrow T_1^{-3}M(-3) \longrightarrow T_1^{-3}M(1) \longrightarrow T_1^{-3}L(1) \longrightarrow 0$$

Since not *both* λ, μ are antidominant, we can not apply Theorem 7.6 to compute these, so we instead turn to the definition. We claim that

$$\begin{aligned} T_1^{-3}L(1) &= \mathrm{pr}_{-3}(L(4) \otimes \mathrm{pr}_1(L(1))) \\ &= \mathrm{pr}_{-3}(L(4) \otimes L(1)) \\ &= 0. \end{aligned}$$

This follows from the fact that any module in $\mathcal{O}_{\chi_{-3}}$ has a composition series for which *all* of the composition factors have highest weight in \$, but $L(4) \otimes L(1) \cong L(3) \oplus L(5)$ has only composition factors with highest weight 3 or 5.

This forces an isomorphism $T_1^{-3}M(-3) \xrightarrow{\sim} T_1^{-3}M(1)$, so it suffices to show that either of these is not a Verma module.

and considering the weights of the parenthesized term.

This follows by considering

$$\begin{aligned} T_1^{-3}M(-3) &= \mathrm{pr}_{-3}(L(4) \otimes \mathrm{pr}_1 M(-3)) \\ &= \mathrm{pr}_{-3}(L(4) \otimes M(-3)). \end{aligned}$$

We'll use the fact that

$$\begin{aligned} \Pi(M(-3)) &= \{-3, -5, \cdots\} \\ \Pi(L(4)) &= \{-4, -2, 0, 2, 4\}, \end{aligned}$$

and by Theorem 3.6, the parenthesized term has a finite filtration with quotients of the form

$$Q(\mu) \in \left\{ M(\lambda+\mu) \;\middle|\; \mu \in \Pi(L(4)) \right\} = \{\cdots, M(-3+2), M(-3+4), \cdots\} = \{\cdots, M(-1), M(3), \cdots\}$$

and since $W_{[\lambda]} = \{\lambda, -\lambda-2\} = \{1, -3\}$, we see that composition factors with linked weights appear in the subquotients above. Thus the projection onto $\mathcal{O}_{\chi_{-3}}$ has a composition series subquotients isomorphic to $M(-1)$ and $M(-3)$. But then the resulting projection must have at least *two distinct* simple quotients, whereas every Verma module has a unique simple quotient, so the projection is not a Verma module. ■

3 Exercise p.108

a. Work out the Jantzen filtration sections for $M(w_0 \cdot \lambda)$. List carefully any additional assumptions or facts needed to deduce $M(w_0 \cdot \lambda)^i$ uniquely.
b. Continue #4.11 for the case of singular λ, e.g. $(\lambda+\rho, \widehat{\alpha}) = 1$. If you didn't deduce the structure of all $M(w \cdot \lambda)$ there, can you complete it now?
c. Work out the non-integral case. (There are several different cases to consider.)