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Today: projective spaces. We defined $\mathbb{P}^n_{/k} \coloneqq k^{n+1} \setminus \{0\} / \sim$ where $x \sim \lambda x$ for all $x \in k^\times$, which we identified with lines through the origin in k^{n+1}. We have homogeneous coordinates $p = [x_0 : \cdots : x_n]$.

We say an ideal is *homogeneous* iff for all $f \in I$, the homogeneous part $f_d \in I$ for all d. In this case $V_p(I) \subset \mathbb{P}^n_{/k}$ defined as the vanishing locus of all homogeneous elements of I is well-defined. Think of this as the "projective version" of a vanishing locus.

Similarly we defined $I_p(S)$ defined as the ideal generated by all homogeneous $f \in k[x_1, \cdots, x_n]$ such that $f(x) = 0$ for all $x \in S$.

Remark 1.0.1.
Observe that $V_a(I)$ defined as the cone over $V_p(I)$ is the set of points in $\mathbb{A}^{n+1} \setminus \{0\} \cup \{0\}$ which map to $V_p(I)$.

We have an alternative definition of a cone in $\mathbb{A}^{n+1}$, characterized as a closed subset C which is closed under scaling, so $kC \subseteq C$.

Proposition 1.0.1.

- If $S \subset k[x_1, \cdots, x_n]$ is a set of homogeneous polynomials, then $V_a(S)$ is a cone since it is closed and closed under scaling. This follows from the fact that $f(x) = 0 \iff f(\lambda x) = 0$ for $\lambda \in k^\times$ when f is homogeneous.
- If C is a cone, then its affine ideal $I_a(C)$ is homogeneous.

Proof (?).
Let $f \in I_a(C)$, then $f(x) = 0$ for all $x \in C$. Since C is closed under scaling, $f(\lambda x) = 0$ for all $x \in C$ and $\lambda \in k^\times$. Decompose $f = \sum_d f_d$ into homogeneous pieces, then

$$x \in C \implies 0 = f(\lambda x) = \sum \lambda^d f_d(x).$$

Fixing $x \in C$, the quantities $f_d(x)$ are constants, so the resulting polynomial in λ vanishes for all λ. But since k is infinite, this forces $f_d(x) = 0$ for all d, which shows that $f_d \in I_a(C)$. ∎

Lemma 1.1 *(?).*

There is a bijective correspondence

$$\begin{aligned}\{\text{Cones}\} &\iff \{\text{Projective Varieties}\}\\ \mathbb{A}^{n+1} \supset X &\mapsto \mathbb{P}X \subset \mathbb{P}^n\\ \mathbb{A}^{n+1} \supset CX &\mapsfrom X \subset \mathbb{P}^n\end{aligned}$$

.

Proof (?).
$\mathbb{P}V_a(S) = V_p(S)$ for any set S of homogeneous polynomials, and $C(V_p(S)) = V_a(S)$, where $V_p(S)$ is a cone by part (a) of the previous proposition. Conversely, every cone is the variety associated to some homogeneous ideal.

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1.1 Projective Nullstellensatz

Definition 1.1.1 (Irrelevant Ideal)**.**
The homogeneous ideal $I_0 := (x_0, \cdots, x_n) \subset k[x_1, \cdots, x_n]$ is denoted the **irrelevant ideal**.

Proposition 1.1.1 ***(Projective Nullstellensatz).***

a. For all $X \subseteq \mathbb{P}^n$, $V_p(I_p(X)) = X$.
b. For all homogeneous ideal $J \subset k[x_1, \cdots, x_n]$ such that (importantly) $\sqrt{J} \neq I_0$, $I_p(V_p(J)) = \sqrt{J}$.

Proof (of a).
$\supset$: If we let I denote the ideal of all homogeneous polynomials vanishing on X, then this certainly contains X.
$\subset$: This follows from part (b), since $X = V_p(J)$ implies that $(V_p I_p V_p)(J) = V_p(\sqrt{J}) = V_p(J) = X$, since taking roots of homogeneous polynomials doesn't change the vanishing locus.

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Proof (of b).
That $I_p(V_p(J)) \supset \sqrt{J}$ is obvious, since $f \in \sqrt{J}$ vanishes on $V_p(J)$.

Check

It remains to show $\sqrt{J} \subset I_p(V_p(J))$, but we can write $I_p(V_p(J))$ as $\langle f \in k[x_1, \cdots, x_n]\rangle$ the set of homogeneous polynomials vanishing on $V_p(S)$, which is equal to those vanishing on $V_a(J) \setminus \{0\}$.
But since $I_p(\cdots)$ is closed, this is equal to the f that vanish on $\overline{V_a(J) \setminus \{0\}}$, which is only equal to $V_a(J)$ iff $V_a(J) \neq \{0\}$.

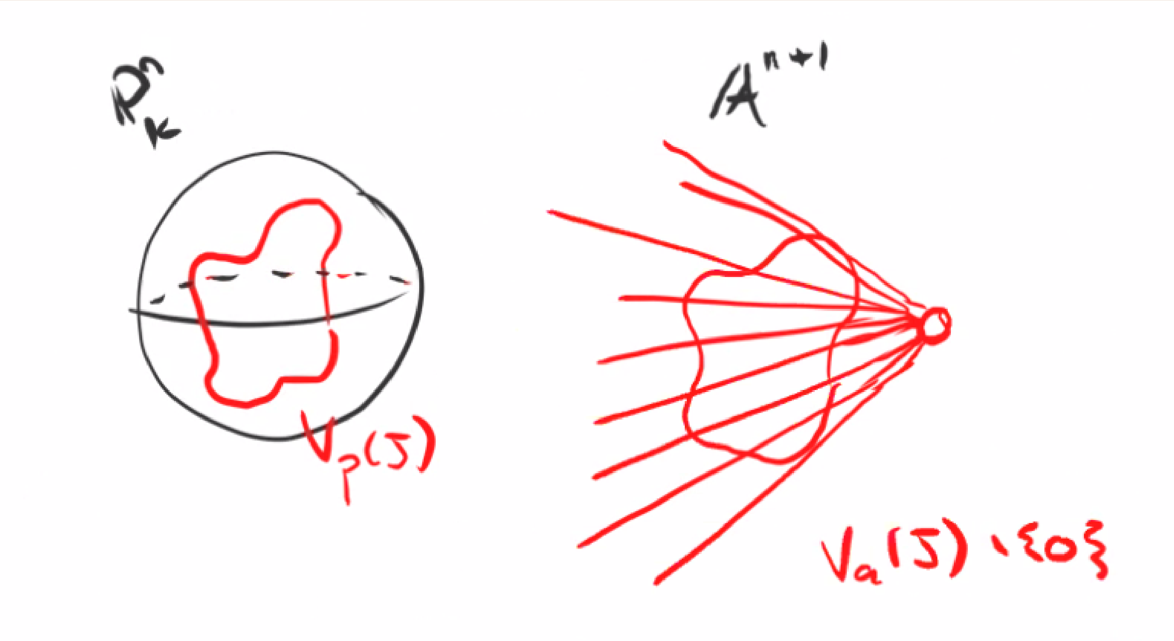

Figure 1: Image

By the affine Nullstellensatz,

$$V_a(J) = \{0\} \iff \sqrt{J} = I_0.$$

Thus $I_p(V_p(J)) = \left\langle f \;\middle|\; \text{homogeneous vanishing on } V_a(J) \right\rangle$. Using the fact that $V_a(J)$ is a cone, its ideal is homogeneous and thus generated by homogeneous polynomials by part (b) of the previous proposition. Thus

$$I_p(V_p(J)) = I_a(V_a(J)) = \sqrt{J},$$

where the last equality follows from the affine Nullstellensatz.

∎

Corollary 1.1.1 ***(?).***
There is an order-reversing bijection

$$\left\{ \substack{\text{Projective varieties} \\ X \subset \mathbb{P}^n} \right\} \iff \left\{ \substack{\text{Homog non-irrelevant radical ideals} \\ \in k[x_1, \cdots, x_n]} \right\}$$
$$X \mapsto I_p(X)$$
$$? \hookleftarrow ?.$$

Remark 1.1.1.

A better definition of a cone over $X \subset \mathbb{P}^n_{/k}$ is $\overline{\pi^{-1}(X)} \subset \mathbb{A}^{n+1}_{/k}$ where

$$\begin{aligned}\pi : \mathbb{A}^{n+1} \setminus \{0\} &\to \mathbb{P}^n \\ [x_0, \cdots, x_n] &\mapsto [x_0 : \cdots : x_n].\end{aligned}$$

Definition 1.1.2 (Projective coordinate ring)**.**
Given $X \subset \mathbb{P}^n$ a projective variety, the **projective coordinate ring** of X is given by

$$S(X) \coloneqq k[x_1, \cdots, x_n]/I_p(X).$$

Remark 1.1.2.
This is a graded ring since $I_p(X)$ is homogeneous. This follows since the quotient of a graded ring by a homogeneous ideal yields a grading on the quotient.

Remark 1.1.3.
We have relative versions of everything. Projective subvarieties of projective varieties are given by $Y \subset X \subset \mathbb{P}^n$ where X is a projective variety. We have a topology on X where the closed subsets are projective subvarieties.

Remark 1.1.4.
Given $J \subset S(X)$, where $S(X)$ is the projective coordinate ring of X and has a grading, we can take $V_p(J) \subset X$. Conversely, given a set $Y \subset S(X)$, we can take $I_p(Y) \subset S(X)$ those homogeneous elements vanishing on Y. Thus there is an order-reversing bijection

$$\left\{ \substack{\text{Projective subvarieties} \\ Y \subset X} \right\} \iff \left\{ \substack{\text{Homogeneous nonirrelevant radical ideals} \\ I \subset S(X)} \right\}$$

and $S(X) = k[x_1, \cdots, x_n]/J \subset \overline{I_0}$.

Remark 1.1.5.
Every nontrivial homogeneous ideal J contained in I_0. Why? Suppose $f \in J \setminus I_0$ and $f_0 \neq 0$. Then $f_0 \in J$ but $f_0 \in k \subset k[x_1, \cdots, x_n]$, implying that $1 \in J$ and thus $J = \langle 1 \rangle$.

Remark 1.1.6.
It is sometimes useful to know that a projective variety is cut out by homogeneous polynomials all of *equal* degree, so $X = V(f_1, \cdots, f_m)$ with each f_i homogeneous of degree d_i. Then there is some maximum degree d. We can write

$$\begin{aligned} V(f_1) = V(x_0^k f_1, \cdots, x_n^k f_1) \qquad \forall k \geq 0 \\ X = \bigcap V(f_1) \cup V(x_i). \end{aligned}$$

This follows because V of a product is a union of the vanishing loci, but $\bigcap V(x_i) = \emptyset$. The equality follows because for all points $[x_0, \cdots, x_n] \in \mathbb{P}^n$, some x_i is nonzero.

Next time: dehomogenization.