# Real Analysis

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# 1 Lecture 1 (Thu 15 Aug 2019 11:04)

> See Folland's Real Analysis, definitely a recommended reference.

Possible first day question: how can we "measure" a subset of $\mathbb{R}$? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of $\mathbb{R}$. We'll come back to this in a few lectures.

## 1.1 Notions of "smallness" in $\mathbb{R}$

Definition: Let $E$ be a set, then $E$ is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes $\emptyset, \mathbb{N}$.

Definition: $E$ is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of $\mathbb{R}$ is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in $Y$ iff the closure of $X$ is all of $Y$. So we'll make the following definition.

Definition: A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval $I$ contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. Example: $\mathbb{Q}$ is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, - $A^c$ contains a dense, open set. - The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than $\varepsilon$.

Definition: $E$ is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable $\implies$ Meager, but not the converse.
- Countable $\implies$ Null, but not the converse.

Exercise: Show that the "middle third" Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem: Every $E \subseteq \mathbb{R}$ can be written as $E = A \coprod B$ where $A$ is null and $B$ is meager.

> This gives some information about how nullity and meagerness interact – in particular, $\mathbb{R}$ itself is neither meager nor null. Idea: if meager $\implies$ null, this theorem allows you to write $\mathbb{R}$ as the union of two null sets. This is bad!

Proof: We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Around each $q_j$, put an interval around it of size $1/2^{j+k}$ where we'll allow $k$ to vary, yielding multiple intervals around $q_j$. To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that $A$ is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick $k$ such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that $A^c := B$ is meager. Note that $G_k$ covers the rationals, and is a countable union of open sets, so it is dense. So $G_k$ is an open and dense set. By one of the equivalent formulations of meagerness, this means that $G_k^c$ is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager.

## 1.2 $\mathbb{R}$ is not small

Theorem A (Cantor): $\mathbb{R}$ is not countable.

Theorem B (Baire): $\mathbb{R}$ is not meager. (Baire Category Theorem)

Theorem C (Borel): $\mathbb{R}$ is not null.

Note that theorems B and C imply theorem A. You can also replace $\mathbb{R}$ with any nonempty interval, which is a strictly stronger statement – if any subset of $\mathbb{R}$ is not countable, then certainly $\mathbb{R}$ isn't, and so on.