# Problem Set 10

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## 1 Problem 1

Let $\phi$ be an $n$-form. If suffices to show these statements for $n = 2$.

$\implies$ : Suppose $\phi$ is alternating, then $\phi(b, b) = 0$ for all $b \in B$.

Letting $a, b \in B$ be arbitrary, we then have

$$\begin{aligned}\phi(a+b, a+b) &= \phi(a, a+b) + \phi(b, a+b) \\ &= \phi(a,a) + \phi(a,b) + \phi(b,a) + \phi(b,b) \\ &= \phi(a,b) + \phi(b,a) \\ &\implies \phi(a,b) = -\phi(b,a),\end{aligned}$$

which shows that $\phi$ is skew-symmetric.

$\impliedby$ Suppose $\phi$ is skew-symmetric, so $\phi(a, b) = -\phi(b, a)$ for all $a, b \in B$. Then $\phi(b, b) = -\phi(b, b)$ by transposing the terms, which says that $\phi(b, b) = 0$ for all $b \in B$ and thus $\phi$ is alternating.

## 2 Problem 2

Let $f(x) = \det(P + xQ) \in R[x]$, then $f$ is a polynomial in $x$ which is not identically zero.

To see that $f \not\equiv 0$, we can use that fact that $P$ is invertible to evaluate $f(0) = \det(P) \neq 0$.

We can now note that $f$ has finite degree, and thus finitely many zeroes in $R$.

# 3 Problem 3

Letting $k[x] \curvearrowright_\phi E$ to yield a $k[x]$-module structure on $E$ and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \,\Big|\, q_2 \,\Big|\, \cdots \,\Big|\, q_t$$

where $E_i = k[x]/(q_i)$. Then $q_t = q$, the minimal polynomial of $E$.

In particular, $E_t$ is a $\phi$-invariant subspace of $E$, and if $\deg q_t = m$, then $E_t$ is in fact an $m$-dimensional cyclic module with basis $\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \cdots, \phi^{m-1}(\mathbf{v})\}$ for some $\mathbf{v} \in E_t$.

But since $E_t \leq E$ is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

# 4 Problem 4

$\implies$ : Suppose $A \sim D$ where $D$ is diagonal. Then $JCF(A) = JCF(D) = D$, which means that every Jordan block of $A$ has size exactly 1.

Since the elementary divisors of $A$ are precisely the minimal polynomials of the Jordan blocks of $A$, and the minimal polynomial of any $1 \times 1$ matrix $[a_{ij}]$ is given by the linear polynomial $x - a_{ij}$, every elementary divisor of $A$ must be linear.

$\impliedby$ : Suppose all of the elementary divisors of $A$ are linear. Every elementary divisor is the minimal polynomial of a Jordan block of $A$, and so if we write $JCF(A) = \bigoplus M_i$, then the minimal polynomial of each $M_i$ is linear.

Supposing that $M_i$ has minimal polynomial $p_i(x) = x - c$ for some scalar $c$, we have

$$p_i(M_i) = 0 \implies M_i - cI_n = 0 \implies M_i = cI_n,$$

which shows that $M_i$ is a diagonal matrix with only $c$ on its diagonal.

But if every Jordan block of $A$ is diagonal,then $JCF(A) = D$ is diagonal and $A \sim D$.

# 5 Problem 5

## 5.1 Part 1

We'll use the fact that the minimal polynomial $q$ is the invariant factor of highest degree, and so every other invariant factor must divide $q$.

Moreover, $RCF(A) = C_1 \oplus C_2 \oplus \cdots \oplus C_k$ where each $C_i$ is the companion matrix of the $i$th invariant factor if we write $V \cong \bigoplus_{i=1}^{k} k[x]/(a_i)$. So it suffices to determine all of the possible distinct combinations of invariant factors.

We can restrict this list by noting that the characteristic polynomial satisfies $\chi_A(x) = \prod a_i$, and in particular, $\deg \chi_A(x) = 6$. Noting that $\deg q(x) = 3$, the degrees of the remaining invariant factors must sum to 3.

These are:

$$
\begin{aligned}
a_1 &= (x-2), & a_2 &= (x-2)^2, & a_3 &= q(x), \\
a_1 &= (x-2), & a_2 &= (x-2)(x-3), & a_3 &= q(x), \\
a_1 &= (x-3), & a_2 &= (x-2)(x-3), & a_3 &= q(x).
\end{aligned}
$$

Noting that

- $(x-2)^2 = x^2 - 4x + 4$, and
- $(x-2)(x-3) = x^2 - 5x + 6$,

these choices correspond to the matrices

$$
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$