Title

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1 Chapter 1

1.1 Within Chapter

Proposition 1.1: Fix an ideal $\mathfrak{a} \trianglelefteq R$. There is a correspondence

$$\left\{ \mathfrak{b} \;\middle|\; \mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R \right\} \iff \left\{ \tilde{\mathfrak{b}} \trianglelefteq R/\mathfrak{a} \right\}.$$

Proof: Let $f : R \to T$ be any ring homomorphism and let $S(R), S(T)$ denote the sets of subrings of R, T respectively. Then f induces two maps:

$$\begin{aligned} \Phi : S(R) &\to S(T) \\ H &\mapsto f(H) \end{aligned}$$

$$\begin{aligned} \Psi : S(T) &\to S(R) \\ K &\mapsto f^{-1}(K). \end{aligned}$$

It can be shown that

- $\Phi \circ \Psi(K) = K \bigcap \operatorname{im} f$
- $\Psi \circ \Phi(H) = H \bigcap \ker f$.

Proposition 1.2: TFAE

- R is a field
- R is simple, i.e. the only ideals of R are $0, R$.

- Every homomorphism $\phi : R \to S$ for S an arbitrary ring is injective.

Proof: ?

Proposition: Maximal ideals are prime.

Proof: ?

Proposition: If $\mathfrak{p} \trianglelefteq R$ is prime, $R/\mathfrak{p}$ is a domain. If $\mathfrak{m} \trianglelefteq R$ is maximal, $R/\mathfrak{r}$ is a field.

Proof: ?

Theorem 1.3: Every ring R has a nontrivial maximal ideal $I \neq 0$, and every ideal is contained in a maximal ideal.

Proof: ?

Corollary 1.5: Every non-unit of R is contained in a maximal ideal.

Proof: ?

Proposition 1.6: If $A \setminus \mathfrak{m} \subset R^\times$, then A is a local ring with $\mathfrak{m}$ its maximal ideal. If $\mathfrak{m}$ is maximal and $1 + m \in R^\times$ for all $m \in \mathfrak{m}$, then A is a local ring.

Proof: ?

Proposition: If $f \in k[x_1, \cdots x_n]$ is irreducible over k, then (f) is prime.

Proposition: $\mathbb{Z}$ is a PID, and (p) is prime iff p is zero or a prime number, and every such ideal is maximal.

Proposition: $k[\{x_i\}]$ has maximal ideals that are not principal iff $n > 1$.

Exercise: Characterize the maximal and prime ideals of $k[x_1, \cdots, x_n]$? Is this a field, domain, PID, UFD, a local ring, ...?

Proposition: Every nonzero prime ideal in a PID is maximal.

Proof: ?

Definition: The set $\mathrm{nil}(A)$ of all nilpotent elements in a ring A is the nilradical of A. The set $J(A) = \bigcap_{\mathfrak{m} \in \mathrm{Spec}_{\max}(A)} \mathfrak{m}$ is the Jacobson radical.,

Proposition 1.7: $\mathrm{nil}(A) \trianglelefteq R$ is an ideal and $A/\mathfrak{N}$ has no nonzero nilpotent elements.

Proof: ?

Proposition 1.8: $\mathrm{nil}(A) = \bigcap \mathfrak{p} \in \mathrm{Spec}(A)\mathfrak{p}$ is the intersection of all prime ideals of A.

Proof: ?

Proposition 1.9: $x \in J(A)$ iff $1 - xa \in A^\times$ for all $a \in A$.

Proposition: If $(m), (n) \trianglelefteq \mathbb{Z}$ then $(m) \bigcap (n) = (\gcd(m, n))$ and $(m)(n) = (mn)$.

Exercise: If $\mathfrak{a} \trianglelefteq k[x_1, \cdots, x_m]$, characterize $\mathfrak{a}^n$.

Exercise: Show that $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$ are coprime iff there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$.

Proposition 1.10: Let $\{mfa_i\} \trianglelefteq A$ be a family of ideals and define $\phi : A \to \prod A/\mathfrak{a}_i$.

1. If $\{\mathfrak{a}_i\}$ are pairwise coprime, then $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$
2. ϕ is surjective iff $\{\mathfrak{a}_i\}$ are pairwise coprime.
3. ϕ is injective iff $\bigcap \mathfrak{a}_i = (0)$.

Exercise: Show that the union of ideals is not necessarily an ideal.

Proposition 1.11:

a. Let $\{\mathfrak{p}_i\}$ be a set of prime ideals and let $\mathfrak{a} \in \bigcup \mathfrak{p}$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.

b. Let $\{\mathfrak{a}_i\}$ be ideals and $\mathfrak{p} \supseteq \bigcap \mathfrak{a}_i$ be prime. $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i, and if $\mathfrak{p} = \bigcap \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

Exercise: Let $A = \mathbb{Z}$, and characterize the ideal quotient $(m : n)$.

Exercise 1.12:

1. $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$
2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$
3. $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
4. $(\bigcap \mathfrak{a}_i : \mathfrak{b}) = \bigcap (\mathfrak{a}_i : \mathfrak{b})$
5. $(\mathfrak{a} : \sum \mathfrak{b}_i) = \bigcap (\mathfrak{a} : \mathfrak{b}_i)$

Proposition: For $\mathfrak{a} \trianglelefteq A$, $\sqrt{\mathfrak{a}}$ is an ideal.

Exercise 1.13:

1. $\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$
3. $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \bigcap \mathfrak{b}} = \sqrt{\mathfrak{a}} \bigcap \sqrt{\mathfrak{b}}$
4. $\sqrt{\mathfrak{a}} = (1) \iff \mathfrak{a} = (1)$
5. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
6. For $\mathfrak{p}$ prime, $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n \geq 1$.

Proposition 1.14: $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p}$

Proposition 1.15: Let D be the set of zero-divisors in A. Then $D = \bigcup_{x \neq 0} \sqrt{\operatorname{Ann}(x)}$.

Exercise: Let $(m) \trianglelefteq \mathbb{Z}$ where $m = \prod p_i^{k_i}$, and show that $\sqrt{(m)} = (p_1 p_2 \cdots) = \bigcap (p_i)$.

Proposition 1.16: If $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$ are coprime then $\mathfrak{a}, \mathfrak{b}$ are coprime.

Exercise: Show that if $f : A \to B$ and $\mathfrak{a} \trianglelefteq A$, it is not necessarily the case that $f(\mathfrak{a}) \trianglelefteq B$.

Exercise: Show that if $\mathfrak{b}$ is prime then $A \cdot f^{-1}(\mathfrak{b})$ is prime, but if $\mathfrak{a}$ is prime then $B \cdot f(\mathfrak{a})$ need not be prime.

Exercise: Write $\mathfrak{a}^e \coloneqq \langle f(\mathfrak{a}) \rangle$ and $\mathfrak{b}^c = \left\langle f^{-1}(\mathfrak{b}) \right\rangle$. Let $f : \mathbb{Z} \to \mathbb{Z}[i]$ be the inclusion, and show that

- $(2)^e = \left\langle (1+i)^2 \right\rangle$, which is not prime in $\mathbb{Z}[i]$
- (Nontrivial) If $p = 1 \mod 4$, then $\mathfrak{p}^e$ is the product of two distinct prime ideals

- If $p = 3 \mod 4$ then $\mathfrak{p}^e$ is prime.

Proposition: Let $C = \left\{\mathfrak{b}^c \;\middle|\; \mathfrak{b} \trianglelefteq B\right\}$ and $E = \left\{\mathfrak{a}^e \;\middle|\; \mathfrak{a} \trianglelefteq A\right\}$. Then

1. $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ and $\mathfrak{b} \supset \mathfrak{b}^{ce}$,
2. $\mathfrak{b}^c = \mathfrak{b}^{cec}$ and $\mathfrak{a}^e = \mathfrak{a}^{ece}$
3. $C = \left\{\mathfrak{a} \trianglelefteq A \;\middle|\; \mathfrak{a}^{ec} = \mathfrak{a}\right\}$ and $E = \left\{\mathfrak{b} \trianglelefteq B \;\middle|\; \mathfrak{b}^{ce} = \mathfrak{b}\right\}$.
4. The map $\phi : C \to E$ given by $\phi(\mathfrak{a}) = \mathfrak{a}^{ec}$ is a bijection with inverse $\mathfrak{b} \mapsto \mathfrak{b}^c$.
5. If $\mathfrak{a} \in C$ then $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b}^{cec} = \mathfrak{a}^{ec}$, and if $\mathfrak{a} = \mathfrak{a}^{ec}$ then $\mathfrak{a}$ is the contraction of $\mathfrak{a}^e$.

Exercise 1.18:

$$\begin{array}{ll}
(\mathfrak{a}_1 + \mathfrak{a}_2)^{\mathfrak{e}} = \mathfrak{a}_1^{\mathfrak{e}} + \mathfrak{a}_2^{\mathfrak{e}}, & (\mathfrak{b}_1 + \mathfrak{b}_2)^c \geq \mathfrak{b}_1^{\mathfrak{c}} + \mathfrak{b}_2^{\mathfrak{c}} \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^{\mathfrak{e}} \cap \mathfrak{a}_2^{e}, & (\mathfrak{b}_1 \cap \mathfrak{b}_2)^{\mathfrak{c}} = \mathfrak{b}_1^{\mathfrak{c}} \cap \mathfrak{b}_{\mathfrak{z}}^{\mathfrak{c}} \\
(\mathfrak{a}_1 \mathfrak{a}_2)^{\mathfrak{e}} = \mathfrak{a}_1^{\mathfrak{e}} \mathfrak{a}_2^{\mathfrak{e}}, & (\mathfrak{b}_1 \mathfrak{b}_2)^{\mathfrak{c}} \supseteq \mathfrak{b}_1^{\mathfrak{c}} \mathfrak{b}_2^{\mathfrak{c}} \\
(\mathfrak{a}_1 : \mathfrak{a}_2)^{\mathfrak{e}} \subseteq \left(\mathfrak{a}_1^{\mathfrak{e}} : \mathfrak{a}_2^{\mathfrak{e}}\right), & (\mathfrak{b}_1 : \mathfrak{b}_2)^{\mathfrak{c}} \subseteq \left(\mathfrak{b}_1^{\mathfrak{c}} : \mathfrak{b}_2^{\mathfrak{c}}\right) \\
r(\mathfrak{a})^e \subseteq r\left(\mathfrak{a}^e\right), & r(\mathfrak{b})^c = r\left(\mathfrak{b}^c\right)
\end{array}.$$

1.2 End of Chapter Exercises