Discussion Notes

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1 Discussion 1

If X is an F_σ set, then

$$X = \bigcup_{i=1}^{\infty} F_i \quad \text{with each } F_i \text{ closed.}$$

If X is a G_δ set, then

$$X = \bigcap_{i=1}^{\infty} G_i \quad \text{with each } G_i \text{ open.}$$

A set A is *nowhere dense* iff $(\overline{A})^\circ = \emptyset$ iff for any interval I, there exists a subinterval S such that $S \bigcap A = \emptyset$. This is a set that is not dense in any nonempty open set. If the closure of a subset of $\mathbb{R}$ contains no open intervals, it will be nowhere dense.

A set A is *meager* or *first category* if it can be written as

$$A = \bigcup_{i \in \mathbb{N}} A_i \quad \text{with each } A_i \text{ nowhere dense}$$

A set A is *null* if for any ε, there exists a cover of A by countably many intervals of total length less than ε, i.e. there exists $\{I_k\}_{j \in \mathbb{N}}$ such that $A \subseteq \bigcup_{j \in \mathbb{N}} I_j$ and $\sum_{j \in \mathbb{N}} \mu(I_j) < \varepsilon$. If A is null, we say $\mu(A) = 0$.

Some facts:

- If $f_n \to f$ and each f_n is continuous, then D_f is meager.
- If $f \in \mathcal{R}(a, b)$ and f is bounded, then D_f is null.
- If f is monotone, then D_f is countable.
- If f is monotone and differentiable on (a, b), then D_f is null.

We define the *oscillation of f* as

$$\omega_f(x) \coloneqq \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

1.1 Uniform Convergence

We say that $f_n \to f$ *converges uniformly on A* if $\|f_n - f\|_\infty = \sup_{x \in A} |f_n(x) - f(x)| \to 0$. (Note that this defines a sequence of *numbers* in $\mathbb{R}$.)

This means that one can find an n large enough that that for every $x \in A$, we have $|f_n(x) - f(x)| \leq \varepsilon$ for any ε.

- Showing uniform convergence: find some M_n, independent of x, such that $|f_n(x) - f(x)| \leq M_n$ where $M_n \to 0$.
- Negating: Fix ε, let n be arbitrary, and find a bad x (which can depend on n) such that $|f_n(x) - f(x)| \geq \varepsilon$.

Example: $\frac{1}{1+nx} \to 0$ pointwise on $(0, \infty)$, which can be seen by fixing x and taking $n \to \infty$. To see the convergence is not uniform, choose $x = \frac{1}{n}$ and $\varepsilon = \frac{1}{2}$. Then

$$\sup_{x>0} \left| \frac{1}{1+nx} - 0 \right| \geq \frac{1}{2} \not\to 0.$$

Here, the problem is at small scales – note that the convergence *is* unform on $[a, \infty)$ for any $a > 0$. To see this, note that

$$x > a \implies \frac{1}{x} < \frac{1}{a} \implies \left| \frac{1}{1+nx} \right| \leq \left| \frac{1}{nx} \right| \leq \frac{1}{na} \to 0$$

since a is fixed.

1.2 Uniformly Cauchy

Let $C^0(([a,b], \|\cdot\|_\infty))$ be the metric space of continuous functions of $[a,b]$, endowed with the metric

$$d(f,g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|$$

This is a complete metric space, and

$$f_n \to^U f \iff \forall \varepsilon \exists N \ni m \geq n \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon \forall x \in X$$

$\implies$: Use the triangle inequality.

$\impliedby$: Find a candidate limit f: first fix an x, so that each $f_n(x)$ is just a number. Now we can consider the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$, which (by assumption) is a Cauchy sequence in $\mathbb{R}$ and thus converges. So define $f(x) \coloneqq \lim_n f_n(x)$. Aside: we note that if $a_n < \varepsilon$ for all n and $a_n \to a$, then $a \leq \varepsilon$.

So take $m \to \infty$, i.e.

$$|f_n(x) - f_m(x)| < \varepsilon \forall x \implies \lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \leq \varepsilon \forall x \implies f_n \to^U f.$$

Note: $f_n \to^U f$ does not imply that $f_n' \to^U f'$.

Counterexample: Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$, which converges to 0 uniformly, but $f_n'(x) = n\cos(n^2 x)$ does not even converge pointwise.

To make this work, the theorem is that if $f_n' \to^U g$ for some g and for at least 1 point x we have $f_n(x) \to f(x)$, then $g = \lim f_n'$.