A 2 Step Filtration

Goal:

We want to explicitly consider all of the objects, maps, and differentials in a particular spectral sequence arising from a space that admits a filtration that terminates in two steps. There are several concrete examples that should fit into this framework:

- $0 \hookrightarrow S^k \hookrightarrow S^n$ for any $k < n$
- $0 \hookrightarrow S^n \hookrightarrow \mathbb{CP}^n$
- $0 \hookrightarrow \mathbb{RP}^n \hookrightarrow S^n$
  - Using S^n as a double cover of $\mathbb{RP}^n$

Setup: Space and Filtration

Let X be a space and let $A \subset X$ be a subspace, inducing the inclusion $A \xrightarrow{i} X$, so we have the following inclusions of spaces:

$$0 \hookrightarrow A \hookrightarrow B$$

Then consider applying the "chain functor" $C_*(\cdot) : \mathbf{Top} \to \mathbf{Ab}$ that sends a space X to a singular chain complex

$$C_*(X) := \cdots \xrightarrow{\partial_{i-1}} C_i(X) \xrightarrow{\partial_i} C_{i+1}(X) \xrightarrow{\partial_{i+1}} \cdots$$

Applying this functor to the above inclusion induces an inclusion of chain complexes:

$0 \hookrightarrow C_*(A) \hookrightarrow C_*(X)$

We regard this as a two step filtration on $C^*(X)$ by making the following identifications:

- $F_0 C_*(X) := C_*(X)$
- $F_1 C_*(X) := C_*(A)$
- $F_2 C_*(X) := 0$

And we obtain the primary object of interest for this spectral sequence:

$0 = F_2 C_*(X) \hookrightarrow F_1 C_*(X) \hookrightarrow F_0 C_*(X) = C_*(X)$

This process is roughly summarized in the following diagram:

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d] & A \arrow[r, hook, "i"] \arrow[d, "C_*(\cdot)"] & X \arrow[d] \\
0 \arrow[r, hook] \arrow[d, equal] & C_*(A) \arrow[r, hook, "i_*"] \arrow[d, equal] & C_*(X) \arrow[d, equal] \\
F_2 C_*(X) \arrow[r, hook] & F_1 C_*(X) \arrow[r, hook, "i_*"] & F_0 C_*(X)
\end{tikzcd}$$

Setup: Spectral Sequence

A few definitions to recall:

$G_i C_*(X) := \frac{F_i C_*(X)}{F_{i+1} C_*(X)}$

$E_0^{p,q} = G_p C_{p+q}(X)$

$E_1^{p,q} = H(E_0^{p,q}, d_0)$

Computation of Pages

E_{-1}

Not standard usage, here I consider the "E_{-1} page" to be simply a presentation of the double complex itself. The formula works out to be something like
$E_{-1}^{p,q} = F_p C_q(X)$

$$
\begin{array}{l:r|ccc}
q=n & 0 & F_0C_n(X) & F_1C_n(X) & F_2C_n(X) \\
\vdots & \vdots & \vdots & \vdots & \\
q=3 & 0 & F_0C_3(X) & F_1C_3(X) & F_2C_3(X) \\
q=2 & 0 & F_0C_2(X) & F_1C_2(X) & F_2C_2(X) \\
q=1 & 0 & F_0C_1(X) & F_1C_1(X) & F_2C_1(X) \\
q=0 & 0 & F_0C_0(X) & F_1C_0(X) & F_2C_0(X) \\
\hline
q=-1 & 0 & 0 & 0 & 0 \\
q-2 & 0 & 0 & 0 & 0 \\
\hdashline
p=-2 & p=-1 & p=0 & p=1 & p=2
\end{array}
$$

For clarity, we unpack definitions here to show how the actual original chain complexes sit inside of this page:

$$
\begin{array}{l:r|ccc}
q=n & 0 & C_n(X) & C_n(A) & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
q=3 & 0 & C_3(X) & C_3(A) & 0 \\
q=2 & 0 & C_2(X) & C_2(A) & 0 \\
q=1 & 0 & C_1(X) & C_1(A) & 0 \\
q=0 & 0 & C_0(X) & C_0(A) & 0 \\
\hline
q=-1 & 0 & 0 & 0 & 0 \\
q-2 & 0 & 0 & 0 & 0 \\
\hdashline
p=-2 & p=-1 & p=0 & p=1 & p=2
\end{array}
$$

Focusing on the area $p, q >= -1$, we use the fact that the chain complexes come with natural boundary maps to define the differentials $d_{-1} := \partial_n : C_n(X) \to C_{n-1}(X)$.

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_n(X) \arrow[l] \arrow[d, "\partial_n"] & C_n(A) \arrow[l, "i_*"'] \arrow[d, "\partial_n|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_{n-1}(X) \arrow[l] \arrow[d, "\partial_{n-1}"] & C_{n-1}(A) \arrow[l, "i_*"'] \arrow[d, "\partial_{n-1}|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_{n-2}(X) \arrow[l] \arrow[d] \arrow[r, phantom, "\vdots"' yshift=-4ex] & C_{n-2}(A) \arrow[l, "i_*"'] \arrow[d] \arrow[r, phantom, "\vdots"' yshift=-4ex] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_2(X) \arrow[l] \arrow[d, "\partial_2"] & C_2(A) \arrow[l, "i_*"'] \arrow[d, "\partial_2|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_1(X) \arrow[l] \arrow[d, "\partial_1"] & C_1(A) \arrow[l, "i_*"'] \arrow[d, "\partial_1|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_0(X) \arrow[l] \arrow[d, "\partial_0"] & C_0(A) \arrow[l, "i_*"'] \arrow[d, "\partial_0|_A"] & 0 \arrow[l] \arrow[d] \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

E_0

Here we use the following formulas/facts:

- $G_iC_*(X) := \frac{F_iC_*(X)}{F_{i+1}C_*(X)}$
- $E_0^{p,q} := G_pC_{p+q}(X)$
- $C_n(X,A) := \frac{C_n(X)}{C_n(A)}$
    - This can be done because there is a SES
      $0 \to C_*(A) \to C_*(X) \to \frac{C_*(X)}{C_*(A)} \to 0$
      Then since $\partial_n : C_n(X) \to C_{n-1}(X)$ has the property that $\partial_n(C_*(A)) = C_*(A)$, it factors through the quotient $\frac{C_*(X)}{C_*(A)}$ to yield a map $\hat{\partial}_n : \frac{C_n(X)}{C_n(A)} \to \frac{C_{n-1}(X)}{C_{n-1}(A)}$. Shorten notation by calling $\frac{C_*(X)}{C_*(A)} := C_*(X,A)$ the relative chain complex; this yields relative homology with respect to $\hat{\partial}$,

i.e. $H_n(X,A) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \subset C_n(X,A)$.

which explicitly yields

$$G_0C_*(X) = \frac{F_0C_*(X)}{F_1C_*(X)} = \frac{C_*(X)}{C_*(A)} := C_*(X,A)$$
$$G_1C_*(X) = \frac{F_1C_*(X)}{F_2C_*(X)} = \frac{C_*(A)}{0} = C_*(A)$$
$$G_2C_*(X) = \frac{0}{0} = 0$$

$E_0^{p,q} := G_pC_q(X)$

$C_n(X,A) := \frac{C_n(X)}{C_n(A)}$

$q=n$	0	$G_0C_n(X)$	$G_1C_{n+1}(X)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$q=3$	0	$G_0C_3(X)$	$G_1C_4(X)$	0
$q=2$	0	$G_0C_2(X)$	$G_1C_3(X)$	0
$q=1$	0	$G_0C_1(X)$	$G_1C_2(X)$	0
$q=0$	0	$G_0C_0(X)$	$G_1C_1(X)$	0
$q=-1$	0	0	$G_1C_0(X)$	0
$q-2$	0	0	0	0
$p=-2$	$p=-1$	$p=0$	$p=1$	$p=2$

Which unpacks as

$$
\begin{array}{l|r|cccc}
q = n & 0 & \frac{F_0 C_n(X)}{F_1 C_n(X)} & \frac{F_1 C_{n+1}(X)}{F_2 C_{n+1}(X)} & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
q = 3 & 0 & \frac{F_0 C_3(X)}{F_1 C_3(X)} & \frac{F_1 C_4(X)}{F_2 C_4(X)} & 0 \\
q = 2 & 0 & \frac{F_0 C_2(X)}{F_1 C_2(X)} & \frac{F_1 C_3(X)}{F_2 C_3(X)} & 0 \\
q = 1 & 0 & \frac{F_0 C_1(X)}{F_1 C_1(X)} & \frac{F_1 C_2(X)}{F_2 C_2(X)} & 0 \\
q = 0 & 0 & \frac{F_0 C_0(X)}{F_1 C_0(X)} & \frac{F_1 C_1(X)}{F_2 C_1(X)} & 0 \\
\hline
q = -1 & 0 & 0 & \frac{F_1 C_0(X)}{F_2 C_0(X)} & 0 \\
q - 2 & 0 & 0 & 0 & 0 \\
\hdashline
p = -2 & p = -1 & p = 0 & p = 1 & p = 2
\end{array}
$$

Which further unpacks as

$$
\begin{array}{l|r|cccc}
q = n & 0 & \frac{C_n(X)}{C_n(A)} & \frac{C_{n+1}(A)}{0} & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
q = 3 & 0 & \frac{C_3(X)}{C_3(A)} & \frac{C_4(A)}{0} & 0 \\
q = 2 & 0 & \frac{C_2(X)}{C_2(A)} & \frac{C_3(A)}{0} & 0 \\
q = 1 & 0 & \frac{C_1(X)}{C_1(A)} & \frac{C_2(A)}{0} & 0 \\
q = 0 & 0 & \frac{C_0(X)}{C_0(A)} & \frac{C_1(A)}{0} & 0 \\
\hline
q = -1 & 0 & 0 & \frac{C_0(A)}{0} & 0 \\
q - 2 & 0 & 0 & 0 & 0 \\
\hdashline
p = -2 & p = -1 & p = 0 & p = 1 & p = 2
\end{array}
$$

Which by definition is

$$
\begin{array}{l:r|ccc}
q=n & 0 & C_n(X,A) & C_{n+1}(A) & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
q=3 & 0 & C_3(X,A) & C_4(A) & 0 \\
q=2 & 0 & C_2(X,A) & C_3(A) & 0 \\
q=1 & 0 & C_1(X,A) & C_2(A) & 0 \\
q=0 & 0 & C_0(X,A) & C_1(A) & 0 \\
\hline
q=-1 & 0 & 0 & C_0(A) & 0 \\
q-2 & 0 & 0 & 0 & 0 \\
\hdashline
p=-2 & p=-1 & p=0 & p=1 & p=2
\end{array}
$$

For any pair (X,A), there is a long exact sequence

$$
\cdots H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \cdots
$$

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_n(X,A) \arrow[l] \arrow[d, "\delta_n"] & C_{n+1}(A) \arrow[l, "i_*"] \arrow[d, "\delta_n|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_{n-1}(X,A) \arrow[l] \arrow[d, "\delta_{n-1}"] & C_n(A) \arrow[l, "i_*"] \arrow[d, "\delta_{n-1}|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_{n-2}(X,A) \arrow[l] \arrow[d] \arrow[dr, phantom, "\vdots"] & C_{n-1}(A) \arrow[l, "i_*"] \arrow[d] \arrow[dr, phantom, "\vdots"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_1(X,A) \arrow[l] \arrow[d, "\delta_2"] & C_2(A) \arrow[l, "i_*"] \arrow[d, "\delta_2|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & C_0(X,A) \arrow[l] \arrow[d, "\delta_1"] & C_1(A) \arrow[l, "i_*"] \arrow[d, "\delta_1|_A"] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d, "\delta_0"] & C_0(A) \arrow[l, "i_*"] \arrow[d, "\delta_0|_A"] & 0 \arrow[l] \arrow[d] \\
0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

E_1

E_2