Proof of Leray-Hirsch Theorem

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December 8, 2019

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1 Preliminaries

Definition: Fibre Bundle

Definition: Homology

Definition: Cup Product

Let R be an arbitrary ring, and let h denote the functor

$$\begin{aligned} h(\,\cdot\,; R) : \mathbf{Top} &\to \mathbf{Ring} \\ X &\mapsto H^*_{\text{sing}}(X; R) \\ (X \xrightarrow{f} Y) &\mapsto (H^*(Y; R) \xrightarrow{f^*} H^*(X; R)) \end{aligned}$$

2 Statement of the Theorem

Let

$$\begin{tikzcd} F \arrow[r, hook, "i"] & E \arrow[d, two heads, "p"] \\ & B \end{tikzcd}$$

be a fibre bundle. Taking cohomology induces maps

$$\begin{tikzcd} h(F; R) & h(E; R) \arrow[l, "i^*"'] \\ & h(B; R) \arrow[u, "p^*"'] \end{tikzcd}$$

If we then define the following group action

$$\begin{aligned} h(B;R) &\curvearrowright h(E;R) \\ b \curvearrowright e &\coloneqq p^*(b) \smile e. \end{aligned}$$

1. Both $h(E;R)$ and $h(F;R) \otimes h(B;R)$ are modules over the ring $h(B;R)$, and
2. In the category of $h(B;R)$-modules, the map

$$\begin{aligned} \varphi : h(F;R) \otimes h(B;R) \to &\quad h(E;R) \\ \alpha \otimes \beta \mapsto &\, s(\alpha) \smile \pi^*(\beta) \end{aligned}$$

Note: this map is not an isomorphism in the category of rings.