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1 | Friday, October 23

1.1 Facets

W_p has a dot action on $E := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 1.1.1 (Facet)**.**
We can write $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, and define the *facet* as

$$F := \left\{ \lambda \in E \;\middle|\; \langle \lambda + \rho,\ \alpha^\vee \rangle = n_\alpha p \ \forall \alpha \in \Phi_0^+(F),\ (n_\alpha - 1)p < \langle \lambda + \rho,\ \alpha^\vee \rangle < n_\alpha p \ \forall \alpha \in \Phi_1^+(F) \right\}.$$

The first condition corresponds to being on a vertex in the following diagram, while the second corresponds to being in the interior of a triangle:

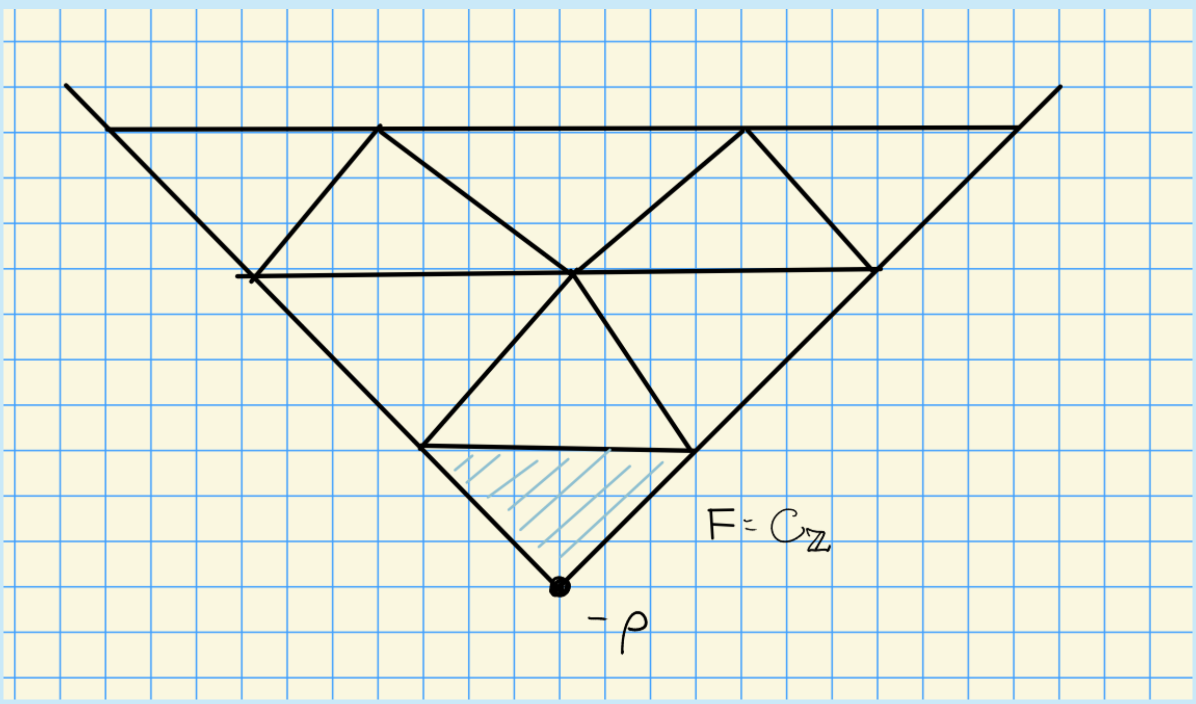

Figure 1: Image

Definition 1.1.2 (Closure of a Facet)**.**
The *closure* of a facet is defined by replacing the second condition with an inequality.

$$\overline{F} \coloneqq \left\{ \lambda \in E \;\middle|\; \langle \lambda + \rho,\ \alpha^\vee \rangle = n_\alpha p \ \forall \alpha \in \Phi_0^+(F),\ n_\alpha - 1 \le \langle \lambda + \rho,\ \alpha^\vee \rangle \le n_\alpha p \ \forall \alpha \in \Phi_1^+(F) \right\}.$$

This includes all of the walls of the triangle.

Definition 1.1.3 (Upper Closure of a Facet)**.**
Finally, we define the *upper closure* by replacing one inequality with a strict inequality:

$$\widehat{F} \coloneqq \left\{ \lambda \in E \;\middle|\; \langle \lambda + \rho,\ \alpha^\vee \rangle = n_\alpha p \ \forall \alpha \in \Phi_0^+(F),\ n_\alpha - 1 < \langle \lambda + \rho,\ \alpha^\vee \rangle \le n_\alpha p \ \forall \alpha \in \Phi_1^+(F) \right\}.$$

Definition 1.1.4 (Alcove)**.**
A facet is called an **alcove** for W_p iff $\Phi_0^+(F) = \emptyset$.

Remark 1.1.1.
Note that if F is an alcove for W_p, then $\widehat{F}$ is a fundamental domain for $W_p \curvearrowright E$ with the dot action.

1.2 Translation Functors

Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$, and define

$$\begin{gathered} T_\lambda^\mu(\,\cdot\,) \coloneqq \mathrm{pr}_\mu(L(\nu_1) \otimes \mathrm{pr}_\lambda(\,\cdot\,)) \\ \text{where } \nu_1 \in X(T)_+ \cap W(\mu - \lambda). \end{gathered}$$

This is exact as a composition of exact functors, since we're tensoring over a field and taking projections (which are themselves exact).

Lemma 1.1 *(?).*
Let $\lambda, \mu \in X(T)$ and M be a finite-dimensional G-module. Then the functors

$$\begin{aligned} F(\,\cdot\,) &\coloneqq \mathrm{pr}_\mu \circ (M \otimes_k \,\cdot\,) \circ \mathrm{pr}_\lambda \\ G(\,\cdot\,) &\coloneqq \mathrm{pr}_\lambda \circ (M^\vee \otimes_k \,\cdot\,) \circ \mathrm{pr}_\mu \end{aligned}$$

define an adjoint pair, i.e.

$$\begin{aligned} \hom_{\mathcal{C}}(G(\,\cdot\,), A) &= \hom_{\mathcal{D}}(\,\cdot\,, F(A)) \\ \hom_{\mathcal{C}}(\,\cdot\,, G(A)) &= \hom_{\mathcal{D}}(F(\,\cdot\,), \,\cdot\,) \end{aligned}$$

.

Proof .

Let V, V' be G-modules. Then

$$\begin{aligned}
\hom_G(FV, V') &= \hom_G(\mathrm{pr}_\mu(M \otimes \mathrm{pr}_\lambda V), V') \\
&= \hom_G(M \otimes \mathrm{pr}_\lambda V, \mathrm{pr}_\mu V') \\
&= \hom_G(\mathrm{pr}_\mu(M \otimes \mathrm{pr}_\lambda V), \mathrm{pr}_\mu V') \\
&= \hom_G(\mathrm{pr}_\lambda V, M^\vee \otimes_k \mathrm{pr}_\mu V') \\
&= \hom_G(\mathrm{pr}_\lambda V, \mathrm{pr}_\lambda \left(M^\vee \otimes_k \mathrm{pr}_\mu V'\right)) \\
&= \hom_G(V, \mathrm{pr}_\lambda \left(M^\vee \otimes_k \mathrm{pr}_\mu V'\right)) \\
&= \hom_G(V, GV').
\end{aligned}$$

Here we've used the fact that there no nontrivial homs between distinct blocks.

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Theorem 1.2.1 *(?).*
Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ are in the closure of the bottom alcove. Then $T_\lambda^\mu \leftrightarrows T_\mu \lambda$ form an adjoint pair.

Proof .
Applying the previous corollary, we just need to show the last equality in the following:

$$\begin{aligned}
T_\lambda^\mu(\,\cdot\,) &= \mathrm{pr}_\mu(L(\nu_1) \otimes \mathrm{pr}_\lambda(\,\cdot\,)) \\
&= \mathrm{pr}_\lambda\left(L(\nu_1)^\vee \otimes \mathrm{pr}_\mu(\,\cdot\,)\right) \\
&=_? T_\mu^\lambda.
\end{aligned}$$

This requires checking the highest weight condition on $L(\nu_1)^\vee = L(-w_0\nu_1)$. We know $\nu_1 \in X(T)_+ \cap W(\mu - \lambda)$, so if $\nu_1 = w(\mu - \lambda)$, we have $-w_0\nu_1 = w_0 w(\lambda - \mu) \in W(\lambda - \mu)$. Since $-w_0\nu_1 \in X(T)_+$, this verifies the condition.

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Remark 1.2.1.
The adjointness can be extended from homs to exts:

$$\mathrm{Ext}_G^i(T_\mu^\lambda V, V') \cong \mathrm{Ext}_G^i(V, T_\lambda^\mu V').$$

1.3 Technical Preliminaries

1. If $\lambda \in X(T)$ and

$$\sum_\mu a(\mu) e^\mu \in \mathbb{Z}[X(T)]^W$$

is W-invariant, then we proved that

$$\chi(\lambda)\left(\sum_\mu a(\mu) e^\mu\right) = \sum_\mu a(\mu)\chi(\lambda + \mu).$$

2. If $\mathrm{pr}_\lambda V = V$, then we have

$$\begin{aligned}\text{char }(M \otimes V) &= \text{char }(M)\,\text{char }(V) \\ &= \text{char }(M)\left(\sum_{w \in W_p} a_w \chi(w \cdot \lambda)\right) \\ &= \left(\sum_{\nu \in X(T)} \dim M_\nu e^\nu\right)\left(\sum_{w \in W_p} a_w \chi(w \cdot \lambda)\right).\end{aligned}$$

Proposition 1.3.1 ***(?).***
Let V be a finite dimensional G-module with $\mathrm{pr}_\lambda V = V$. Write

$$\text{char }(V) = \sum_{w \in W_p} a_w \chi(w \cdot \lambda) \quad a_w \in \mathbb{Z}, \text{ cofinitely zero.}$$

Then

$$\text{char }(\mathrm{pr}_\lambda(M \otimes V)) = \sum_{w \in W} a_w \left(\sum_{\substack{\nu \in X(T) \\ \lambda+\nu \in W_p \cdot \mu}} \dim M_\nu\right) \chi(w \cdot (\lambda + \nu)).$$

Proof .
Using (1) and (2), we can write

$$\text{char }(M \otimes V) = \sum_{w \in W_p} a_w \sum_\nu \dim M_\nu \chi(w \cdot \lambda + \nu).$$

Note that $w \cdot \lambda + \nu = w \cdot (\lambda + w_1 \nu)$ where $w_1 := w^{-1}$, using the fact that the dot action acts linearly on the second term. This comes from the following computation:

$$\begin{aligned} w \cdot (\mu_1 + \mu_2) &= w(\mu_1 + \mu_2 - \rho) + \rho \\ &= w(\mu_1 + \rho) - \rho + w\mu_2 \\ &= w \cdot \mu_1 + w\mu_2. \end{aligned}$$

We can thus write

$$\text{char }(M \otimes V) = \sum_{w \in W_p} a_w \left(\sum_\nu \dim M_\nu \chi(w \cdot (\lambda + \nu))\right),$$

since summing over ν is the same as summing over $w\nu$ for any w.
To get char $(\mathrm{pr}_\mu(M \otimes V))$, take $\chi(w(\lambda + \nu))$ and note that $\lambda + \nu \in W_p \cdot \mu$.

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Remark 1.3.1.
Given char V, one can write char $T_\lambda^\mu V$. What will be important here are stabilizers. If λ is on a wall, the stabilizer fixes the corresponding hyperplane.

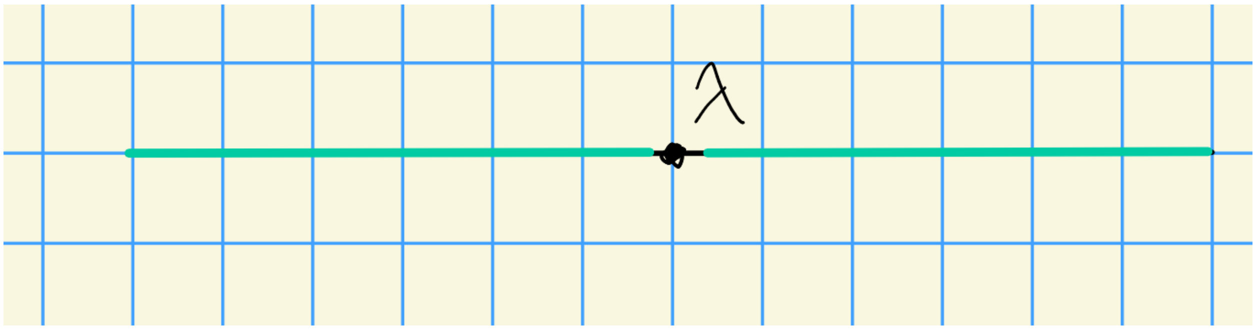

Figure 2: Image