Qual Problems #10

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1 Problem 1

1.1 Part 1

Since 0 is an eigenvalue, there exists an eigenvector $\mathbf{v}$ such that $L\mathbf{v} = 0\mathbf{v} = 0$. But then $\mathbf{v} \in \ker(L)$, so $\dim \ker(L) \geq 1$. Since $\ker(L) \neq 0$, L can not be injective.

By the rank-nullity theorem, we must also have $5 = \dim \ker(L) + \dim \operatorname{im}(L)$. But then $\dim \operatorname{im}(L) \leq 5 = \dim \mathbb{R}^5$, so L can not be surjective either.

1.2 Part 2

Since all eigenvalues are roots of the minimal polynomial and complex roots occur in conjugate pairs, we must have

$$\operatorname{Spec}(L) = \{0, 1 \pm i, 1 \pm 2i\}.$$

Moreover, since this is a 5×5 matrix and we have 5 eigenvalues, this is all of them, and we have the characteristic polynomial

$$\chi_L(x) = x(x^2 - 2x + 2)(x^2 - 2x + 5) \in \mathbb{R}[x]$$

Since the minimal polynomial $p_L(x)$ must divide the characteristic polynomial and have every eigenvalue as a root, this forces

$$p_L(x) = \chi_L(x).$$

1.3 Part 3

If $L\mathbf{x} = \mathbf{x}$, then $\mathbf{x}$ is an eigenvector with eigenvalue $\lambda = 1$. Since $1 \notin \mathrm{Spec}(L)$, such an $\mathbf{x}$ can not exist, so L has only one fixed point: namely $\mathbf{x} = \mathbf{0}$.

2 Problem 2

Let M be an $n \times n$ matrix such that $M_{ij} = 1$ for all i, j, and consider the possible eigenvectors of M.

We have

$$M[1, 1, \cdots, 1]^t = [n, n, \cdots, n]^t = n[1, 1, \cdots, 1]^t,$$

which exhibits $\mathbf{x} = [1, 1, \cdots, 1]$ as an eigenvector with eigenvalue $\lambda = n$.

Now consider

$$\mathbf{x}_j \coloneqq \mathbf{e}_1 - \mathbf{e}_j = [1, 0, 0, \cdots, 0, -1, 0, \cdots, 0]$$

which has a 1 in the 1st coordinate and a -1 in the jth coordinate.

Then

$$M\mathbf{x}_j = \begin{bmatrix} 1 + 0 + \cdots + 0 + (-1) + 0 + \cdots + 0 \\ 1 + 0 + \cdots + 0 + (-1) + 0 + \cdots + 0 \\ \vdots \end{bmatrix} = [0, 0, \cdots, 0]^t,$$

which exhibits each $\mathbf{x}_j$ as an eigenvector with eigenvalue $\lambda = 0$.

But the set $\left\{ \mathbf{x}_j \;\middle|\; 2 \leq j \leq n \right\}$ with eigenvalue 0 contains $n - 1$ distinct eigenvectors, and we have an additional 1 eigenvector with eigenvalue 1, which yields n distinct eigenvectors.

So M is fact diagonalizable and given by

$$JCF(M) = (n-1)J_0^1 \oplus J_n^1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

3 Problem 3

3.1 Part 1

Note that we can't have $T^j = 0$ for any $j \leq 4$, since then $T^5 = T^{5-k}T^k = T^{5-k}0 = 0$, contradicting $T^5 \neq 0$.

So in fact $p_T(x) = x^6$ is the minimal polynomial of T, and since V is 6 dimensional, the degree of the characteristic polynomial $\chi_T(x)$ is 6. Since $p_T \mid \chi_T$, and both are monic polynomials of degree 6, we in fact have

$$p_T(x) = \chi_T(x) = x^6.$$

But this means T has eigenvalue $\lambda = 0$ with multiplicity 6. This means

- The size of the largest Jordan block associated to $\lambda = 0$ is size 6, since 0 has multiplicity 6 in p_T, and
- The sum of the sizes of all Jordan blocks associated to $\lambda = 0$ is 6, since 0 has multiplicity 6 in χ_T

which forces $JCF(T)$ to have a single Jordan block of size 6, i.e.

$$JCF(T) = J_0^6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Part 2

By part (1), we know that these conditions uniquely specify their Jordan forms, so we have $M := JCF(T) = JCF(S)$.

Moreover, since $M = JCF(T)$, we know there is a matrix P such that $T = PMP^{-1}$.

Similarly, we know there is a matrix Q such that $S = QMQ^{-1}$.

But then $P^{-1}TP = M$, and so

$$S = QMQ^{-1} = Q(P^{-1}TP)Q^{-1} = (QP^{-1})T(QP^{-1})^{-1} := ATA^{-1}$$

where $A = QP^{-1}$ is a product of invertible matrices and thus invertible.