

# Title

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## 1 Friday February 21st

### 1.1 Singularities

Recall that there are three types of singularities:

- Removable
- Poles
- Essential

Recall that a function  $g$  is holomorphic at  $z_0$  iff

$$\lim_{z \rightarrow z_0} (z - z_0)g(z) = 0$$

#### **Theorem 1.1 (3.2).**

An isolated singularity  $z_0$  of  $f$  is a pole  $\iff \lim_{z \rightarrow z_0} f(z) = \infty$ .

#### **Theorem 1.2 (3.3, Casorati-Weierstrass).**

If  $f$  is holomorphic in  $D_r(z_0) \setminus \{z_0\}$  and has an essential singularity  $z_0$ , then there exists a radius  $r$  such that  $f(D_r(\{z_0\}) \setminus \{z_0\})$  is dense in  $\mathbb{C}$ .

*Proof .*

Proceed by contradiction. Suppose there exists a  $w \in \mathbb{C}$  and a  $\delta > 0$  such that

$$D_\delta(w) \cap f(D_r(\{z_0\}) \setminus \{z_0\}) = \emptyset.$$

If  $z \in D_r(w) \setminus z_0$ , then  $|f(z) - w| > \delta$ . Define  $g(z) = \frac{1}{f(z) - w}$  on  $D_r(z_0) \setminus \{z_0\}$ ; then  $|g(z)| < \frac{1}{\delta}$ .

Note that this implies that  $g(z)$  is holomorphic on  $D_r(z_0) \setminus \{z_0\}$ .  $g(z)$  being holomorphic here follows from  $f$  being holomorphic here.

Then  $g(z)$  has a removable singularity at  $z = z_0$  by theorem 3.1.

If  $g(z_0) \neq 0$ , then  $f(z) - w$  is holomorphic at  $z_0$ , contradicting the fact that  $z_0$  is an essential singularity.

If instead  $g(z_0) = 0$ , then  $z_0$  is a pole, again a contradiction. ■

Note: revisit why this is a contradiction.

## 1.2 Singularities at Infinity

The point  $z = \infty$  can be one of three types of singularities:

1. *Removable*  $\iff f(z) = \sum_{k=-1}^{\infty} c_k \frac{1}{z^k}$ .

- I.e. only one positive exponent.

2. *Pole*  $\iff f(z) = \sum_{k=-\infty}^n c_k z^k$

- I.e. there are finitely many positive exponents.

3. *Essential*  $\iff f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$

- There are infinitely many positive exponents.

**Definition 1.2.1** (Meromorphic).

A function  $f$  is **meromorphic** on  $\Omega$  iff there exists a sequence  $\{z_i\} \subset \Omega$  with no limit point in  $\Omega$  such that

1.  $f$  is holomorphic on  $\Omega \setminus \{z_i\}$ , and
2.  $f$  has poles at each  $z_i$ .

**Theorem 1.3(3.4, Meromorphic Functions are Rational).**

$f$  is meromorphic on  $\mathbb{CP}^1$  iff  $f$  is a rational function.

*Proof.*

$\implies$  : By part 1 of the definition above, the point  $z = 0$  is either a pole or a removable singularity of the function  $F(z) = f\left(\frac{1}{z}\right)$ . By part 2,  $F$  has finitely many poles  $\{z_k\}_{k=1}^N$ . So for each  $k$ , write

$$f(z) = f_k(z) + g_k(z)$$

where  $f_k$  is the principal part and  $g_k$  is holomorphic in a neighborhood of  $z_k$ . Then  $f_k(z)$  is a

polynomial in  $\left(\frac{1}{z - z_k}\right)$ , say of degree  $m_k$ . But then

$$F(z) := f\left(\frac{1}{z}\right) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

where  $\tilde{f}_\infty(z)$  is a polynomial in  $z$ , and  $\tilde{g}_\infty(z)$  is holomorphic near zero. Thus  $\tilde{f}_\infty\left(\frac{1}{z}\right)$  is a polynomial in  $\frac{1}{z}$ .

Define  $f_\infty(z) = \tilde{f}_\infty\left(\frac{1}{z}\right)$  and

$$H(z) = f(z) - f_\infty(z) - \sum_k f_k(z).$$

Then  $H$  is entire and bounded and thus constant, and since  $\lim_{z \rightarrow \infty} H(z) = 0$ ,  $H$  is identically zero. Thus

$$f(z) = f_\infty(z) + \sum_k f_k(z)$$

$\Leftarrow$  : asdsda

■