Category $\mathcal{O}$, Problem Set 3

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1 Humphreys 1.10

Prove that the transpose map τ fixes $Z(\mathfrak{g})$ pointwise.

> Check that τ commutes with the Harish-Chandra morphism ξ and use the fact that ξ is injective.

1.1 Solution

We first note that after choosing a PBW basis for $\mathfrak{g}$, τ is defined on $\mathfrak{g}$ in the following way:

$$\begin{aligned}
\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\
x_\alpha &\mapsto y_\alpha \\
h_\alpha &\mapsto h_\alpha \\
y_\alpha &\mapsto x_\alpha
\end{aligned}$$

which lifts to an anti-involution $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by extending linearly over PBW monomials. We can note that since τ fixes $\mathfrak{h}$ pointwise by definition, its lift also fixes $U(\mathfrak{h})$ pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned}
\xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\
\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}.
\end{aligned}$$

Proposition 1.1.
The following diagram commutes

$$\begin{tikzcd}
Z(\mathfrak{g}) \arrow[r, "\xi"] \arrow[d, "\tau"] & U(\mathfrak{h}) \arrow[d, "\tau"] \\
Z(\mathfrak{g}) \arrow[r, "\xi"] & U(\mathfrak{h})
\end{tikzcd}$$

Proof.
We will show that for all $z \in Z(\mathfrak{g})$, $(\xi \circ \tau)(z) = (\tau \circ \xi)(z)$. Expand z in a PBW basis as $z = \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}$. We then make the following computations:

$$\begin{aligned}
(\xi \circ \tau)(z) &= (\xi \circ \tau)\left(\prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_j}\right) \\
&= \xi\left(\prod_{i,j,k} y_i^{r_i} h_j^{s_j} x_k^{t_j}\right) \quad \text{since } \tau \text{ is an anti-homomorphism} \\
&= \prod_j h_j^{s_j}
\end{aligned}$$

.

Similarly, we have

$$\begin{aligned}
(\tau \circ \xi)(z) &= \tau\left(\prod_j h_j^{s_j}\right) \\
&= \prod_j h_j^{s_j}
\end{aligned}$$

,

where we note that the two resulting expressions are equal. ■

The above computation in fact shows that

$$(\xi \circ \tau)(z) = (\tau \circ \xi)(z) = \xi(z),$$

and using the injectivity of ξ, we have

$$\begin{aligned}
&(\xi \circ \tau)(z) = \xi(z) \\
&\implies \tau(z) = z.
\end{aligned}$$

■

2 Humphreys 1.12

Fix a central character χ and let $\left\{V^{(\lambda)}\right\}$ be a collection of modules in $\mathcal{O}$ indexed by the weights λ for which $\chi = \chi_\lambda$ satisfying

1. $\dim V^{(\lambda)} = 1$
2. $\mu < \lambda$ for all weights μ of $V^{(\lambda)}$.

Then the symbols $[V^{(\lambda)}]$ form a $\mathbb{Z}$-basis for the Grothendieck group $K(\mathcal{O}_\xi)$.

For example take $V^{(\lambda)} = M(\lambda)$ or $L(\lambda)$.

2.1 Solution

Following a similar proof outlined here.

Fix a λ_0 such that $\chi = \chi_{\lambda_0}$ by Harish-Chandra's theorem, fix some order on the Weyl group $W = \left\{w_j \;\middle|\; 1 \leq j \leq |W| < \infty\right\}$, and note that $\chi_{\lambda_0} = \chi_{w\cdot\lambda_0}$ for each $w \in W$.

Proposition 2.1.
The Verma modules $\left\{L(w \cdot \lambda_0) \;\middle|\; w \in W\right\}$ form a $\mathbb{Z}$-basis for $\mathcal{O}_\chi$.

Proof.
Write $\mathcal{L} = \operatorname{span}_{\mathbb{Z}} \left\{[L(w_j \cdot \lambda_0)] \;\middle|\; 1 \leq j \leq |W|\right\} \subset K(\mathcal{O}_\chi)$.
Spanning: Let $M \in \mathcal{O}_\chi$ be arbitrary, and consider $[M] \in K(\mathcal{O}_\chi)$. By Humphreys Theorem 1.11, M has a finite composition series

$$M = M_1 > M_2 > \cdots > M_n$$

with simple quotients $M^{i+1}/M^i \cong L(\lambda_i)$ for some λ_i. By collecting terms, we can write

$$[M] = \sum_{i=1}^{n} [L(\lambda_i)] = \sum_{i=1}^{n'} c_i [L(\lambda_i)] \in K(\mathcal{O}_\chi).$$

By definition, $M \in \mathcal{O}_\chi \iff L(\lambda_i) \in \mathcal{O}_\chi$, i.e. M is in this block precisely when all of its composition factors are. But this forces each $L(\lambda_i) = L(w_j \cdot \lambda_0)$ for some j, and so we have

$$[M] = \sum_{i=j}^{n'} c_j [L(w_j \cdot \lambda_0)] \in K(\mathcal{O}_\chi) \in \mathcal{L}.$$

■

3 Humphreys 1.13

Suppose $\lambda \notin \lambda$, so the linkage class $W \cdot \lambda$ is the disjoint union of its nonempty intersections of various cosets of $\Lambda_r \in \mathfrak{h}^\vee$.

Prove that each $M \in \mathcal{O}_{\chi_\lambda}$ has a corresponding direct sum decomposition $M = \bigoplus M_i$ in which all weights of M_i lie in a single coset.

Recall exercise 1.1b.