Title

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1 Lecture 10

Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of ??; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{\'et}}, X_{\text{fppf}}$, then the data of a GL_n -torsor split by a τ -cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$U \times_X U$$

$$\pi_1 \bigcup_{\pi_2} \pi_2$$

$$U$$

$$\downarrow$$

That U trivializes our torsor means that $\pi^*T = \pi^*G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\pi_1^*\pi^*T \xrightarrow{\sim} \pi_2^*\pi^*T$$

$$\downarrow \qquad \qquad \downarrow$$

$$\pi_1^*\pi^*G \xrightarrow{\sim} \pi_2^*\pi^*G$$

Both of the bottom objects are isomorphic to $G|_{U\times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n -torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{\'et}}, GL_n)$ is in bijection with n-dimensional vector bundles on X.

Exercise 1.0.3(?): See if Hilbert 90 is true for groups other than GL_n .

1.1 Representability and Local Triviality

Lecture 10 3

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).

You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:

$$T \times_X T \longrightarrow T$$

$$\downarrow \uparrow \exists \qquad \qquad \downarrow$$

$$T \xrightarrow{f} X$$

The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that



Here "slicing" means finding such a U, and this can be done using the structure theorem for smooth morphisms.

Example 1.1.5 (non-smooth group schemes):

- α_p , the kernel of Frobenius on \mathbb{A}^1 or \mathbb{G}_a ,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on \mathbb{G}_m ,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \operatorname{GL}_n$, etc.

1.1.1 What Hilbert 90 Means

Example 1.1.6(?): Let $X = \operatorname{Spec} k$, n = 1, so we're looking at $H^{\cdot}(\operatorname{Spec} k, \mathbb{G}_m)$.

$$H^{1}((\operatorname{Spec} k)_{\operatorname{zar}}, \mathbb{G}_{m}) = 0$$

$$= H^{1}((\operatorname{Spec} k)_{\operatorname{\acute{e}t}}, \mathbb{G}_{m})$$

$$= H^{1}(\operatorname{Gal}(k^{s}/k), \bar{k}^{\times}).$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.¹

Let's generalize this observation.

Example 1.1.7(?): Let X be any scheme and n = 1, then $H^1(X_{\text{\'et}}, \mathbb{G}_m) = \text{Pic}(X)$.

Example 1.1.8(?): Let's compute $H^1(X_{\text{\'et}}, \mu_{\ell})$ where ℓ is an invertible function on X. We have a SES of ℓ tale sheaves, the **Kummer sequence**,

$$1 \to \mu_{\ell} \to \mathbb{G}_m \xrightarrow{z \mapsto z^p} \mathbb{G}_m \to 1.$$

This is exact in the étale topology since adjoining an ℓ th power of any function gives an étale cover. We get a LES in cohomology

$$H^{0}(X_{\operatorname{\acute{e}t}}, \mu_{\ell}) \xrightarrow{H^{0}(X_{\operatorname{\acute{e}t}}, \mathbb{G}_{m})} H^{0}(X_{\operatorname{\acute{e}t}}, \mathbb{G}_{m})$$

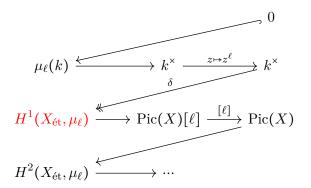
$$H^{1}(X_{\operatorname{\acute{e}t}}, \mu_{\ell}) \xrightarrow{Pic(X)} \operatorname{Pic}(X)$$

$$H^{2}(X_{\operatorname{\acute{e}t}}, \mu_{\ell}) \xrightarrow{} \cdots$$

We know that $H^0(X_{\text{\'et}}, \mathbb{G}_m)$ are invertible functions on X, and the red term is what we'd like to compute.

Suppose now $H^0(X, \mathcal{O}_X) = k = \overline{k}$, then $H^0(X_{\text{\'et}}, \mu_{\ell}) = \mu_{\ell}(k)$ since it is the kernel of the ℓ th power map. We can also compute $H^1(X_{\text{\'et}}, \mu_{\ell})$, since our diagram reduces to

¹This is called "90" since Hilbert numbered his theorems in at least one of his books.



where surjectivity of δ follows from the fact that $k = \bar{k}$ and thus every element has an ℓ th root, making H^1 the kernel of $[\ell]$.

Example 1.1.9(?): Let $X_{/k}$ with $k = \bar{k}$ with ℓ invertible in k, then (claim) $\mathbb{Z}/\ell\mathbb{Z} \cong \mu_{\ell}$ given by sending a generator to some choice of a primitive ℓ th root of unity. To be explicit, we have a representation $\mathbb{Z}/\ell\mathbb{Z} = \text{hom}(\cdot, \text{Spec}\,k[t]/t(t-1)\cdots(t-\ell+1))$ and $\mu_{\ell} = \text{Spec}\,k[t]/t^{\ell} - 1$. These are both disjoint unions of points, and hence schemes of dimension zero since ℓ is invertible in the base and the Chinese Remainder Theorem, so one can write down the isomorphism explicitly between the schemes and hence the functors they represent.

Corollary 1.1.10(?). If
$$\mu_{\ell} \subseteq k$$
, then
$$H^i(X_{\text{\'et}}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) = H^i(X_{\text{\'et}}, \mu_{\ell}).$$

Since the isomorphism depends on the choice of a primitive root, this will not be Galois equivariant, which will come up when we talk about Galois actions on étale cohomology. This already happens for H^0 , since $G \sim \mathbb{Z}/\ell\mathbb{Z}$ trivially but not on μ_{ℓ} .

1.1.2 Geometric Interpretations

Let X be an affine scheme, we now know $H^1(X_{\text{\'et}}, \mathbb{F}_p) = \operatorname{coker}(\mathcal{O}_X \xrightarrow{x^p-x} \mathcal{O}_x)$, the Artin-Schreier map, and these are \mathbb{F}_p -torsors. We also know $H^1(X_{\text{\'et}}, \mathbb{Z}/\ell\mathbb{Z})$ in terms of the LES if $k = \bar{k}$ and $\operatorname{ch}(k) = p$, and this is a $\mathbb{Z}/\ell\mathbb{Z}$ -torsor. Being torsors here geometrically means they're covering spaces with those groups as Galois groups.

Question 1.1.11: How does one write down these torsors/covering spaces?

Example 1.1.12(?): Given

$$[Y] \in H^1(X_{\operatorname{\acute{e}t}}, \mathbb{F}_p) = \operatorname{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$$

where we write [Y] to denote thinking of the torsor as some geometric object, how to we write down the covering space? Using Artin-Schreier, we can write $Y = \{y^p - y = a\}$ for some $a \in \mathcal{O}_X$, an **Artin-Schreier covering**.

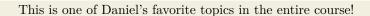
If $\ell \neq \operatorname{ch}(k)$ and $[Z] \in H^1(X_{\operatorname{\acute{e}t}}, \mu_{\ell})$ and assume $\operatorname{Pic}(X) = 0$. Then we can write

$$H^1(X_{\operatorname{\acute{e}t}},\mu_\ell) = \operatorname{coker}(\mathcal{O}_X \xrightarrow{x \mapsto x^\ell} \mathcal{O}_X^{\times})$$

In this case, $Z = \{z^{\ell} = f\}$ where $f \in \mathcal{O}_X^{\times}$ is an element representing the class in cohomology, and $\mu_{\ell} \sim Z$ by multiplication by z.

Remark 1.1.13: The process of explicitly writing down covers is called **explicit geometric class** field **theory**, which gives a recipe for writing down abelian covers of covers. In general, for $Pic(X) \neq 0$, the Picard group plays a crucial role.

1.2 Computing the Cohomology of Curves



Theorem 1.2.1(?).

Let $X_{/k}$ be a smooth curve over $k = \overline{k}$, then

$$H^{i}(X_{\text{\'et}}, \mathbb{G}_{m}) = \begin{cases} \mathcal{O}_{X}(X)^{\times} & i = 0\\ \operatorname{Pic}(X) & i = 1\\ 0 & \text{else}, \end{cases}$$

noting that $\mathcal{O}_X(X)^{\times}$ are the global sections of \mathbb{G}_m , i.e. invertible functions on X.

The first two cases we've done, i > 1 is the hard case.