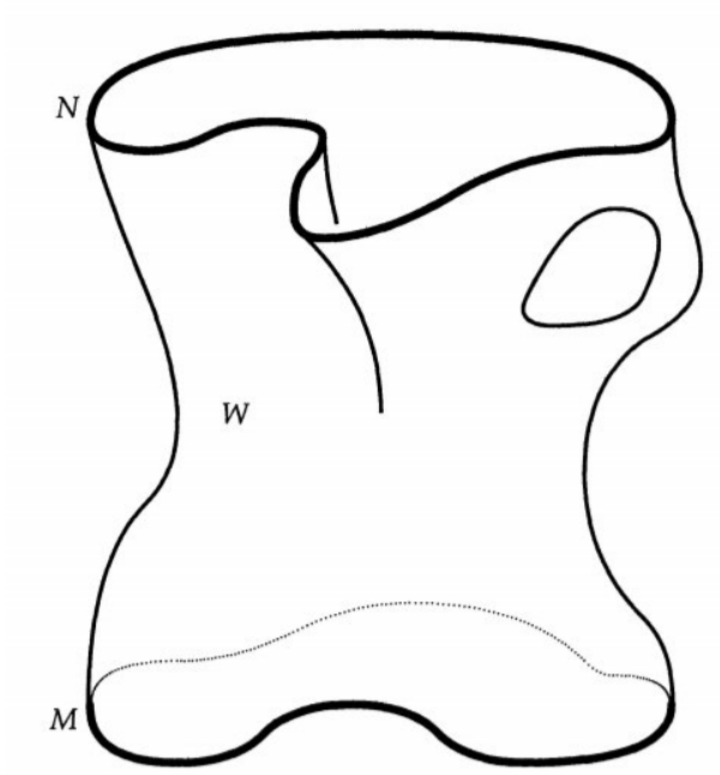

Notes: These are notes live-tex'd from a graduate course in Floer Homology taught by Akram Alishahi at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Floer Homology

Lectures by Akram Alishahi. University of Georgia, Spring 2021

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1 Lecture 1: Overview (Wednesday, January 13)

1.1 Course Logistics

Note (DZG): Everything in this section comes from Akram!

1.1.1 Description

"I am teaching a topics course about Heegaard Floer homology next semester. Heegaard Floer homology was defined by Peter Ozsváth and Zoltan Szabó around 2000. It is a package of powerful invariants of smooth 3- and 4-manifolds, knots/links and contact structures. Over the last two decades, it has become a central tool in low-dimensional topology. It has been used extensively to study and resolve important questions concerning unknotting number, slice genus, knot concordance and Dehn surgery. It has been employed in critical ways to study taut foliations, contact structures and smooth 4-manifolds. There are also many rich connections between Heegaard Floer homology and other manifold and knot invariants coming from gauge theory as well as representation theory. We will learn the basic construction of Heegaard Floer homology, starting with the definition of the 3-manifold and knot invariants. In the second half of this course, we will turn to computations and applications of the theory to low-dimensional topology and knot theory. In particular, several numerical invariants have been defined using this homological invariants. At the end of the semester, I would expect each one of you to learn the construction of one of these invariants (of course with my help) and present it to the class."

1.1.2 Expository Papers

- [**G**] J. Greene, Heegaard Floer homology
- [**H**] J. Hom, Lecture notes on Heegaard Floer homology
- [**L**] R. Lipshitz, Heegaard Floer homologies
- [**M**] C. Manolescu, An introduction to knot Floer homology
- [**OS-1**] P. Ozsváth and Z. Szabó, An introduction to Heegaard Floer homology
- [**OS-2**] P. Ozsváth and Z. Szabó, Lectures on Heegaard Floer homology
- [**OS-3**] P. Ozsváth and Z. Szabó, Heegaard diagrams and holomorphic disks

1.1.3 Research Papers

- [**OSz04a**] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and topological invariants for closed three-manifolds. Ann. of Math. (2) 159 (2004), no. 3, 1027–1158. arXiv:math/0101206

- [**OSz04b**] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and three-manifold invariants: properties and applications. Ann. of Math. (2) 159 (2004), no. 3, 1159–1245. arXiv:math/0105202
- [**OSz04c**] Peter Ozsváth and Zoltán Szabó, Holomorphic disks and knot invariants. Adv. Math. 186 (2004), no. 1, 58–116. arXiv:math/0209056
- [**OSz06**] Peter Ozsváth and Zoltán Szabó, Holomorphic triangles and invariants for smooth four manifolds. Adv. Math. 202 (2006), no. 2, 326–400. arXiv:math/0110169
- [**Per08**] Timothy Perutz, Hamiltonian handleslides for Heegaard Floer homology. Proceedings of Gökova Geometry-Topology Conference 2007, 15–35, Gökova Geometry/Topology Conference (GGT), Gökova, 2008. arXiv:0801.0564

1.1.4 Basic Morse Theory, Symplectic Geometry and Floer Homology

- [**Mi-1**] Milnor, Morse theory
- [**Mi-2**] Milnor, Lectures on the *h*-cobordism theorem
- [**Ca**] A. Cannas da Silva. Lectures on Symplectic Geometry
- [**Mc**] D. McDuff, Floer theory and low-dimensional topology
- [**AD**] M. Audin and M. Damian, Morse theory and Floer homology
- [**Hu**] M. Hutchings, Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)

1.1.5 Low-dimensional Topology

- [**S**] N. Saveliev, Lectures on the topology of 3-manifolds
- [**R**] D. Rolfsen, Knots and links
- [**GS**] R. Gompf and A. Stipsicz, 4-manifolds and Kirby calculus
- [**L**] R. Lickorish, An introduction to knot theory

1.1.6 Suggested Topics for Presentations

- [**SW**] S. Sarkar and J. Wang, [An algorithm for computing some Heegaard Floer homologies, Ann. of Math., 171 (2010), 1213–1236, arXiv:math/0607777.

- Grid homology from:

  - C. Manolescu and P. Ozsváth and S. Sarkar, A combinatorial description of knot Floer homology, Ann. of Math., 169 (2009), 633–660, arXiv:math/0607691.

  - P. Ozsváth and A. Stipsicz and Z. Szabó, Grid Homology for Knots and Links,

    - ◇ Also available here with comment: please go and buy a hard copy, too!

- J. Hom, A survey on Heegaard Floer homology and concordance J. of Knot Theo. and Its Ram.(2) 26 (2017) arXiv:1512.00383
- K. Honda and W. Kazez and G. Matić, On the contact class in Heegaard Floer homology, J. Differential Geom. (2) 83 (2009), 289-311, arXiv:math/0609734
- Sutured Floer homology from:
  - [**L**] Lipshitz expository paper listed above
  - A. Juhász Holomorphic discs and sutured manifolds Algebr. Geom. Topol., (3) 6 (2006), 1429-1457, arXiv:math/0601443
  - A. Juhász, Knot Floer homology and Seifert surfaces Algebr. Geom. Topol., (1) 8 (2008), 603-608 arxiv:math/0702514

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1.2 Intro and Motivation

We'll assume everything is smooth and oriented.

Proposition 1.2.1 ***(Osvath-Szabo (2000)).***
To closed 3-manifolds M we can assign a graded abelian group $\widehat{HF}(M)$, which can be computed combinatorially [a] . There are several variants:

- $HF^- \in \mathrm{grMod}(\mathbb{Z}_2[u])$, [b]
- $HF^+ \in \mathrm{Mod}(\mathbb{Z}_2[u, u^{-1}]/u\mathbb{Z}_2[u])$.
- $HF^\infty \in \mathrm{grMod}(\mathbb{Z}_2[u, u^{-1}])$,

HF^+ and HF^∞ can be computed using HF^-. In general, we'll write $HF^\cdot$ to denote constructions that work with any of the above variants.

[a] See Sarkour-Wang
[b] This is the strongest variant.

Remark 1.2.2: Note that $\mathbb{Z}_2$ can be replaced with $\mathbb{Z}$, but it's technical and we won't discuss it here. For the first half of the course, we'll just discuss $\widehat{HF}$, and we'll discuss the latter 3 in the second half.

1.2.1 Geometric Information

These invariants can be used to compute the **Thurston seminorm** of a 3-manifold:

Definition 1.2.3 (Thurston Seminorm)
A homology class $\alpha \in H_2(M)$ can be represented as $\alpha \in [S]$ for S a closed surface whose fundamental class represents α where $S = \bigcup_{i=1}^{n} S_i$ can be a union of closed embedded surfaces S_i. Then we first compute

$$\max\{0, -\chi(S_i)\} = \begin{cases} 0 & \text{if } S_i \cong \mathbb{S}^2, \mathbb{T}^2 \\ -\chi(S_i) = 2g(S_i) - 2 & \text{else.} \end{cases}.$$

Note that the max checks if χ is positive. Then define

$$\|\alpha\| := \min_S \left(\sum_{i=1}^{n} \max\{0, -\chi(S_i)\} \right),$$

where we sum over the embedded subsurfaces and check which overall surface gives the smallest norm.

Remark 1.2.4: Note that this can't be a norm, since if $\mathbb{S}^2, \mathbb{T}^2 \in [S] \implies \|\alpha\| = 0$.

Theorem 1.2.5 ***(Osvath-Szabo).***
HF detects [a] the Thurston seminorm, and there is a splitting as groups/modules

$$HF^{\cdot}(M) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M)} HF^{\cdot}(M, S)$$

where $S \in \mathrm{Spin}^c(M)$ is a **spin^c structure**: an oriented 2-dimensional vector bundle on M (up to some equivalence).

[a] What does "detect" mean? This is slightly technical.

Remark 1.2.6: The Thurston norm $\|a\|$ can be computed from this data by considering a perturbed version of $\widehat{HF}$, denoted $\underline{\widehat{HF}}$, in the following way: taking the first Chern class $c_1(\mathfrak{s}) \in H^2(M)$ (which can be associated to every 2-dimensional vector bundle), we have

$$\|\alpha\| = \max_{\underline{HF}(M,\mathfrak{s}) \neq 0} |\langle c_1(\mathfrak{s}),\ \alpha \rangle|.$$

Slogan 1.2.7
Floer homology groups split over these spin^c structures and can be used to compute Thurston norms.

Theorem 1.2.8 ***(Ni).***
Given $F \subseteq M$ with genus $g \geq 2$, HF detects if M *fibers* over S^1 with F as a fiber, i.e. there exists a fiber bundle

$$\begin{tikzcd} F \arrow[r, hook] & M \arrow[d, "\pi"] \\ & S^1 \end{tikzcd}$$

This uses the existence of the splitting over spin^c structures and uses HF^+ in the following way: such a bundles exists if and only if

$$\bigoplus_{\langle c_1(\mathfrak{s}),\ [F]\rangle = 2g-2} HF^+(M, \mathfrak{s}) = \mathbb{Z}.$$

Definition 1.2.9 (Contact Structure)
Equivalently,

- A smooth oriented nowhere integrable 2-plane field ξ, or
- A 2-plane field $\xi \coloneqq \ker(\alpha)$ where α is a 1-form such that $\alpha \wedge d\alpha > 0$. [a]

[a] Note that wedging to a nontrivial top form is equivalent to being nowhere integrable here.

Example 1.2.10*(?):* The standard contact structure on $\mathbb{R}^3$ is given by

$$\alpha \coloneqq dz - ydz,$$

which yields the following 2-plane field $\xi \coloneqq \ker \alpha$:

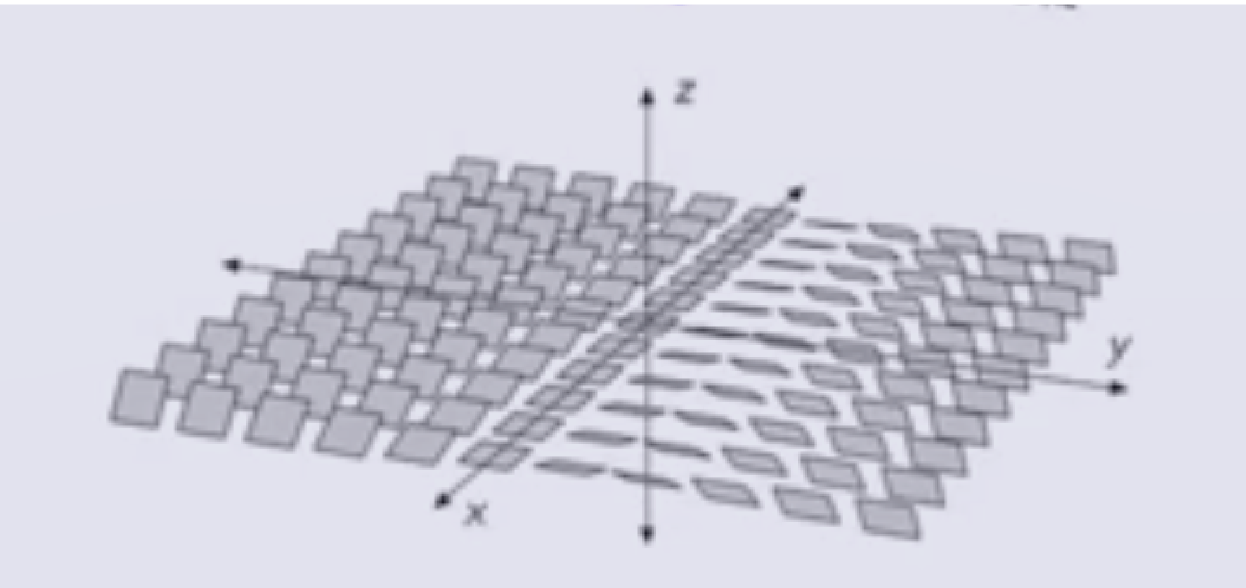

Figure 1: 2-Plane Field in $\mathbb{R}^3$

You can see that $z = 0 \implies y = 0$, so the xy-plane is in the kernel, yielding the flat planes down the middle:

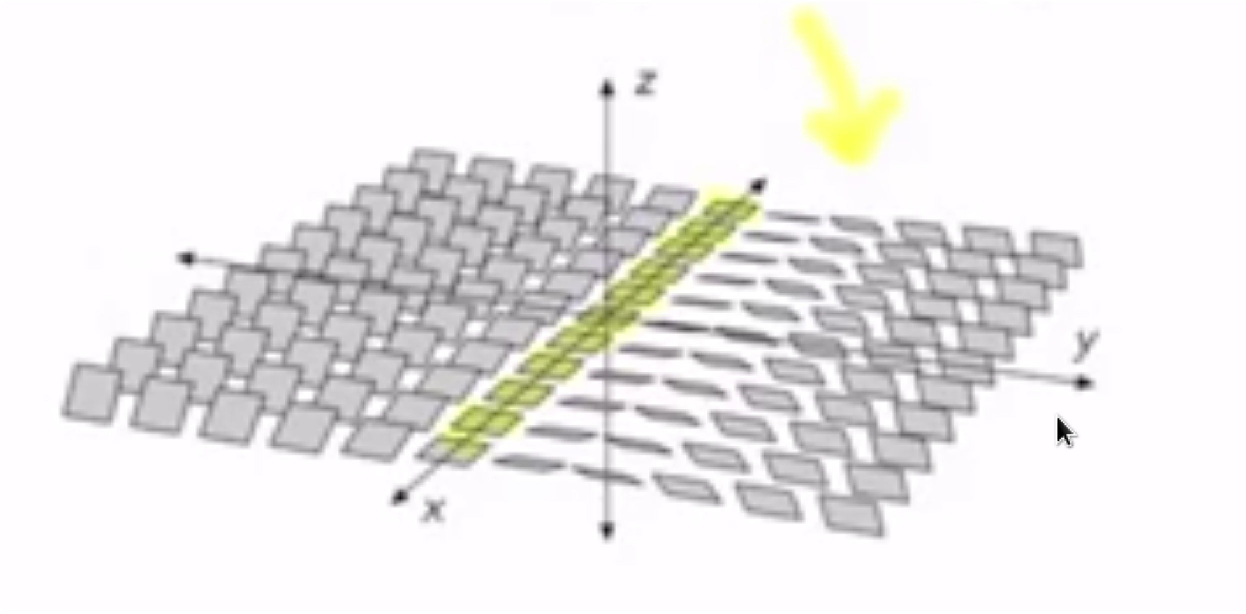

Figure 2: Flat Planes

Proposition 1.2.11 ***(Contact Class (Osvath-Szabo-Honda-Kazez-Matic)).***
To each such ξ one can associate a **contact class** $c(\xi) \in \widehat{HF}(-M)$, where $-M$ is M with the reversed orientation.

Remark 1.2.12: This gives obstructions for two of the following important properties of contact structures:

- Being **overtwisted**, or
- Being **Stein fillable**.

Theorem 1.2.13 ***(?).***

- If ξ is overtwisted, then $c(\xi) = 0$.
- If ξ is Stein fillable, then $c(\xi) \neq 0$.

We'll also discuss similar invariants for knots that were created after these invariants for manifolds.

Definition 1.2.14 (Knots)
Recall that a **knot** is an embedding $S^1 \hookrightarrow M$.

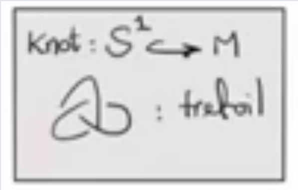

Figure 3: Example: the trefoil knot

Proposition 1.2.15 ***(Knot Floer Homology (Ozsváth-Szabó)).***
Given a knot $K \subseteq M$ a 3-manifold (e.g. $M = S^3$), there is extra algebraic structure on $\widehat{CF}(M)$: a filtration. These allow defining a new bigraded abelian group $\widehat{HFK}(M, K)$ (which is also a $\mathbb{Z}_2$-vector space) that takes includes the information of K. This yields a decomposition

$$\widehat{HFK}(M, K) = \bigoplus_{m,a} \widehat{HFK}_m(M, K, a).$$

This similarly works for other variants: there is a filtration on $CF^-(M)$ which yields $HFK^-(M, K)$, a bigraded $\mathbb{Z}_2[u]$-module.

Some properties of Knot Floer Homology:

Fact 1.2.16
$\widehat{HFK}(K)$ categorifies the Alexander polynomial $\Delta_K(t)$ of K, i.e. taking the graded Euler characteristic yields

$$\Delta_K(t) = \sum_{m,a} (-1)^m \left(\dim \widehat{HFK}_m(K, a)\right) t^a.$$

Fact 1.2.17
$\widehat{HFK}(K)$ detects the **Seifert genus** of a knot $g(K)$, defined as the smallest g such that there exists an embedded surface [1] F of genus g in S^3 that bounds K, so $\partial F = K$.

Example 1.2.18 ***(The Unknot):*** The unknot bounds a disc, so its genus is zero:

Figure 4: The genus of the unknot

[1]These are referred to as **Seifert surfaces**.

Exercise 1.2.19 (The Trefoil)
Using the "outside" disc on the trefoil, find 3 bands that show its genus is 1.

Figure 5: The genus of the trefoil

The genus can be computed by setting $\widehat{HFK}(K,a) \coloneqq \bigoplus_m \widehat{HFK}_m(K,a)$, which yields

$$g(k) = \max\left\{a \;\middle|\; \widehat{HFK}(K,a) \neq 0\right\}.$$

Note that the a grading here is referred to as the **Alexander grading**.

Fact 1.2.20
$\widehat{HFK}$ detects whether or not a knot is **fibered**, where K is fibered if and only if it admits an S^1 family F_t of Seifert surfaces such that $t \neq s \in S^1 \implies F_t \cap F_s = K$. I.e., there is a fibration on the knot complement where each fiber is a Seifert surface:

$$\begin{tikzcd}
\Sigma_g \arrow[r] & S^3 \setminus K \arrow[d, "\pi"] \\
& K
\end{tikzcd}$$

Example 1.2.21 ***(The Unknot):*** The unknot is fibered by $\mathbb{D}^2$s:

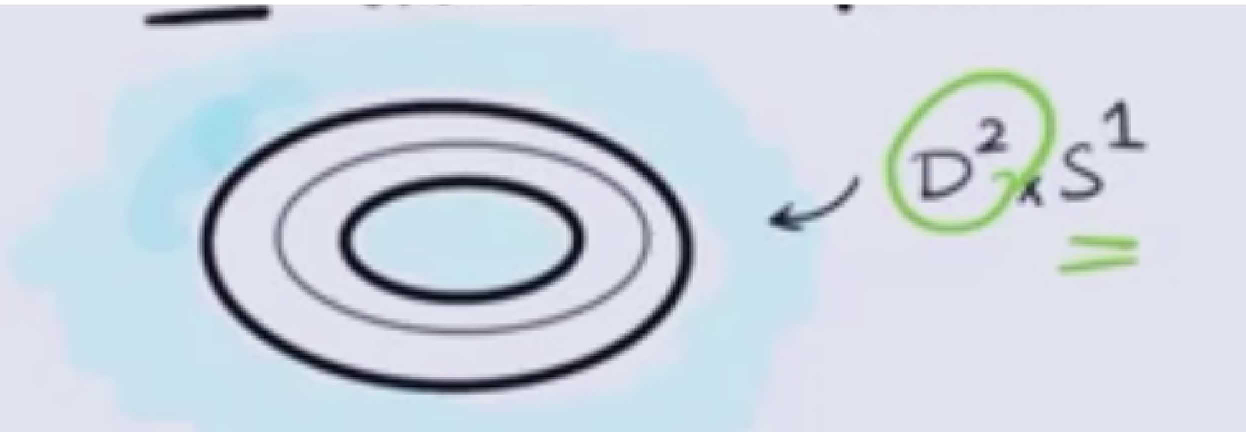

Figure 6: The unknot fibered by discs.

This is "detected" in the following sense: K is fibered if and only if

$$\widehat{HFK}(k, g(K)) = \mathbb{Z}_2.$$

Definition 1.2.22 (Slice Genus)
Let $K \subseteq S^3$. We know $S^3 = \partial B^4$, so we consider all of the smoothly properly embedded surfaces F in B^4 such that $\partial F = K$ and take the smallest genus:

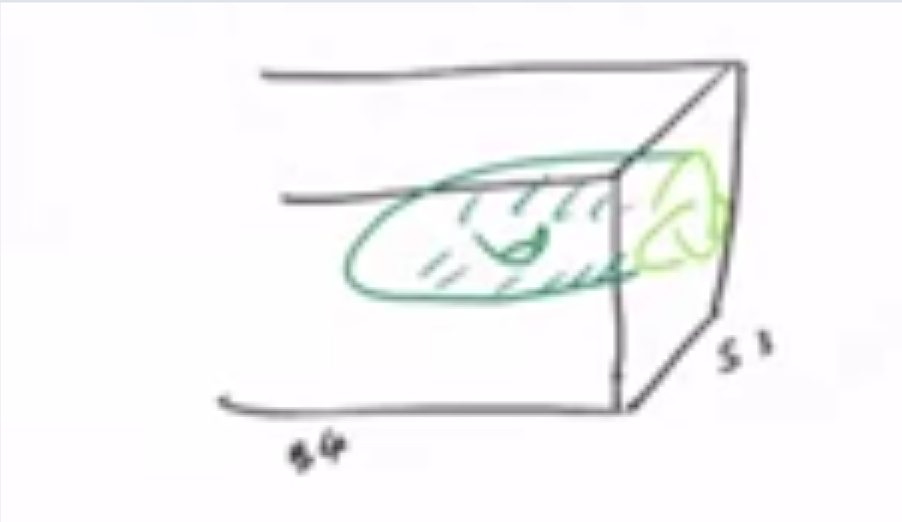

Figure 7: Knot in S^3 bounding a surface in B^4

We thus define the **slice genus** or **4-ball genus** as

$$g_S(K) \coloneqq g_4(K) \coloneqq \min\left\{ g(F) \;\middle|\; F \hookrightarrow B^4 \text{ smootherly, properly with } \partial F = K \right\}.$$

Exercise 1.2.23 (?)
Show that $g_4(K) \leq g(K)$.

Definition 1.2.24 (Unknotting number)
Define $u(K)$ the **unknotting number** of K as the minimum number of times that K must cross itself to become unknotted.

Example 1.2.25 ***(The Trefoil):*** Consider changing the bottom crossing of a trefoil:

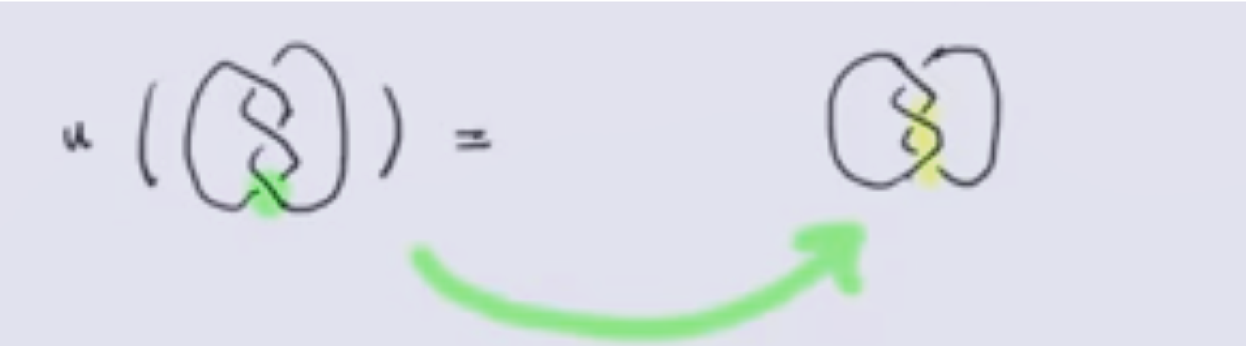

Figure 8: Changing one crossing in the trefoil

This in fact produces the unknot:

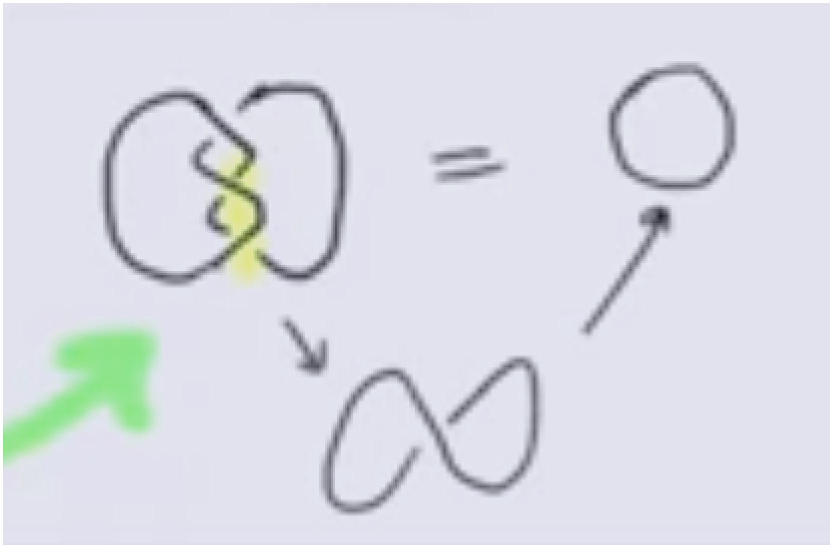

Figure 9: Unkink to yield the unknot

Thus $u(K) = 1$, assuming that we know $K \neq 0$ is not the unknot.

Exercise 1.2.26 (?)
Show that $g_f(K) \leq u(K)$.

Hint: each crossing change $K \to K'$ yields some surface that is a cobordism from K to K' in B^4, and you can use each step to build your surface.

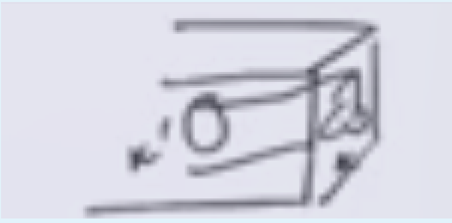

Figure 10: Surface between K and K'

Theorem 1.2.27 ***(Ozsváth-Szabó).***
Define an invariant $\tau(K) \in \mathbb{Z}$ from $\widehat{HFK}$ such that $|\tau(K)| \leq g_4(K) \leq u(K)$.

Definition 1.2.28 (Torus Knots $T_{p,q}$)
Recall that we can view $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ where the action is $(x, y) \xrightarrow{(m,n)} (x + m, y + m)$, i.e. we module out by integer translations. Then for $p, q > 0 $ coprime, $T_{p,q}$ is the image of the line $y = mx$ in $\mathbb{T}^2$ where $m = p/q$.

Example 1.2.29 *($T_{2,3}$):* The torus knot $T_{2,3}$ wraps 3 times around the torus in one direction and twice in the other:

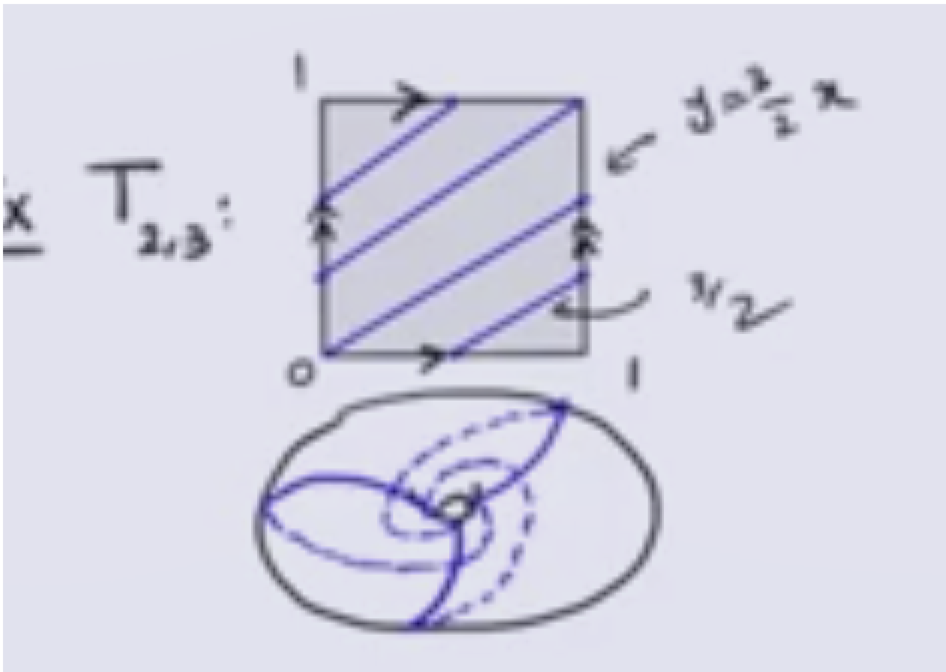

Figure 11: The torus knot $T_{2,3}$

Theorem 1.2.30 ***(Milnor).***

$$g_4(T_{p,q}) = u(T_{p,q}) = \frac{(p-1)(1-q)}{2}.$$

- First proved by Kronheimer-Mrowka
- Another proof by Osvath-Szabó using Heegard Floer homology.

Exercise 1.2.31 (?)
Show that $u(T_{p.q}) \leq \dfrac{(p-1)(q-1)}{2}$, i.e. torus knots can be unknotted with this many crossing changes.

Theorem 1.2.32 ***(Osvath-Szabó).***

$$\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2},$$

which implies

$$\frac{(p-1)(q-1)}{2} \leq g_4(T_{p,q}) \leq u(T_{p,q}) \leq \frac{(p-1)(q-1)}{2},$$

making all of these equal.

Remark 1.2.33: There are better lower bounds for $u(K)$ defined using $\widehat{HFK}$ which are *not* lower bounds for the slice genus. There are also other lower bounds for the slice genus with different names (see Jen Hom's survey), some of which are stronger than τ.

Remark 1.2.34: Another application of having these lower bounds is that we can construct exotic (or *fake*) $\mathbb{R}^4$s, i.e. 4-manifolds X homeomorphic to $\mathbb{R}^4$ but not diffeomorphic to $\mathbb{R}^4$.

Remark 1.2.35: All of these invariants work nicely in a $(3+1)$-TQFT: we have invariants of 3-manifolds M_i and knots in them, so we can talk about **cobordisms** between them: W^4 a compact oriented 4-manifold with $\partial W^4 = -M_1 \coprod M_2$.

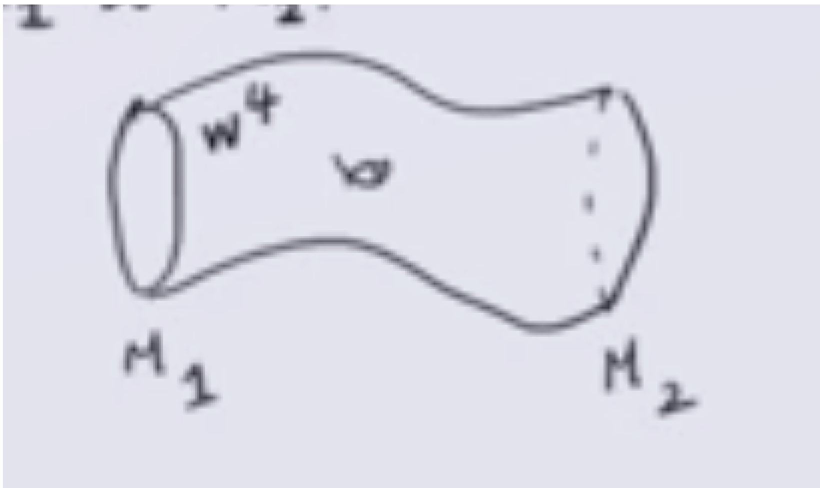

Figure 12: A cobordism

Osvath-Szabó define a map

$$F^{\cdot}_{W,t} : HF^{\cdot}(M_1, t|_{M_1}) \to HF^{\cdot}(M_2, t|_{M_2})$$

using t coming from the splitting of spin^c structure which yields an invariant of closed 4-manifolds referred to as **mixed invariants**.

Similarly, if we have knots in 3-manifolds we can define a cobordism $(M_1, K_1) \to (M_2, K_2)$ as (W^4, F) where W^4 is a cobordism $M_1 \to M_2$ and $F \hookrightarrow W$ is a smoothly embedded surface that is a cobordism from $K_1 \to K_2$ with $F \cap M_i = K_i$ and $\partial F = -K_1 \coprod K_2$.

Figure 13: A cobordism including knots

This similarly yields a map

$$F^{\cdot}_{W,Ft} : HF^{\cdot}(M_1, K_1, t|_{M_1}) \to HF^{\cdot}(M_2, K_2, t|_{M_2})$$

Remark 1.2.36: This smoothly embedded surface in the middle can be used to study other smoothly embedded surfaces in 4-manifolds, which has been done recently.

2 | Lecture 2 (Tuesday, January 19)

Copy in references recommended by Akram!

Remark 2.0.1: For Morse Theory, there are some good exercises in Audin's book – essentially anything other than the existence questions. The first 8 look good on p. 18.

Today:

1. Overview of the construction of HF, and
2. A discussion of Morse Theory.

2.1 Constructing Heegard Floer

First goal: discuss how the name "Heegard" fits in.

Definition 2.1.1 (Genus g handlebody)
A **genus** g **handlebody** H_g is a compact oriented 3-manifold with boundary obtained from B^3 by attaching g solid handles (a neighborhood of an arc).

Example 2.1.2 ***(Attaching*** $g = 2$ ***handles to a sphere):*** For $g = 2$ attached to a sphere, we glue $D^2 \times I$ by its boundary to S^2.

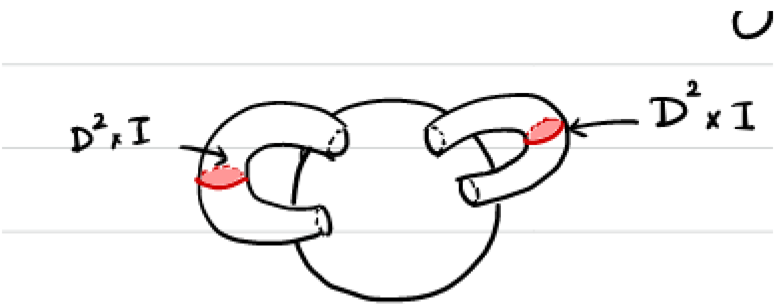

Figure 14: image_2021-01-19-00-35-48

In general, $\partial H_g = \Sigma_g$ is a genus g surface, and $H_g \setminus \coprod_{i=1}^g D_i = B^3$. We can keep track of the data by specifying $(\Sigma, \alpha_1, \alpha_2, \cdots, \alpha_g)$ where $\partial D_i = \alpha_i$.

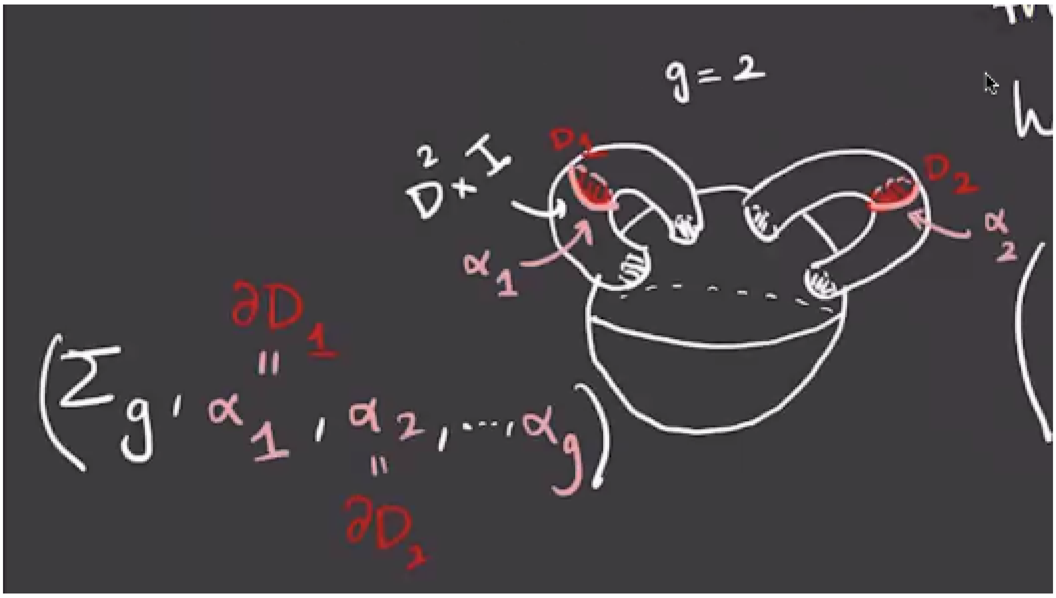

Figure 15: Attaching a handlebody

Definition 2.1.3 (Heegard Decomposition)
A **Heegard diagram** is $M = H_1 \cup_\partial H_2$ where H_i are genus g handlebodies and there is a diffeomorphism $\partial H_1 \to \partial H_2$.

Theorem 2.1.4 *(?).*
Every closed 3-manifold has a Heegard decomposition, although it is not unique.

Definition 2.1.5 (Heegard Diagram)
A **Heegard diagram** is the data $(\Sigma_g, \alpha = \{\alpha_1, \cdots, \alpha_g\}, \beta = \{\beta_1, \cdots, \beta_g\})$ where the α correspond to H_1 and β to H_2 and $\Sigma_g = \partial H_1 = \partial H_2$.

2.2 Lagrangian Floer Homology

This is essentially an infinite-dimensional version of Morse homology.

Definition 2.2.1 (Symplectic Manifold)
A **symplectic manifold** is a pair (M^{2n}, ω) such that

- ω is *closed*, i.e. $d\omega = 0$, and
- ω is *nondegenerate*, i.e. $\wedge^n \omega \neq 0$.

Definition 2.2.2 (Lagrangian)
A **Lagrangian submanifold** is an $L^n \subseteq M$ such that $\omega|_L = 0$.

If $L_1 \cap L_2$ is finitely many points, case we can define a chain complex

$$CF(M^{2n}, L_1, L_2) \coloneqq \mathbb{Z}_2[L_1 \cap L_2],$$

the $\mathbb{Z}_2$-vector space generated by the intersection points of the Lagrangian submanifolds. We'll define a differential by essentially counting discs between intersection points:

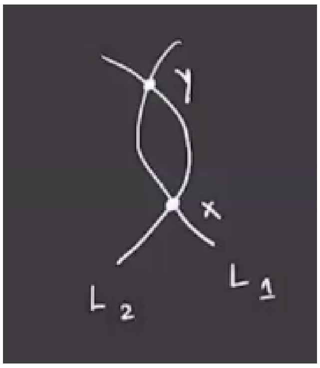

Figure 16: Two intersection points

We'll want to write $\partial x = c_y y + \cdots$ where c_y is some coefficient. How do we compute it? In this case, we have half of the boundary on L_1 and half is on L_2

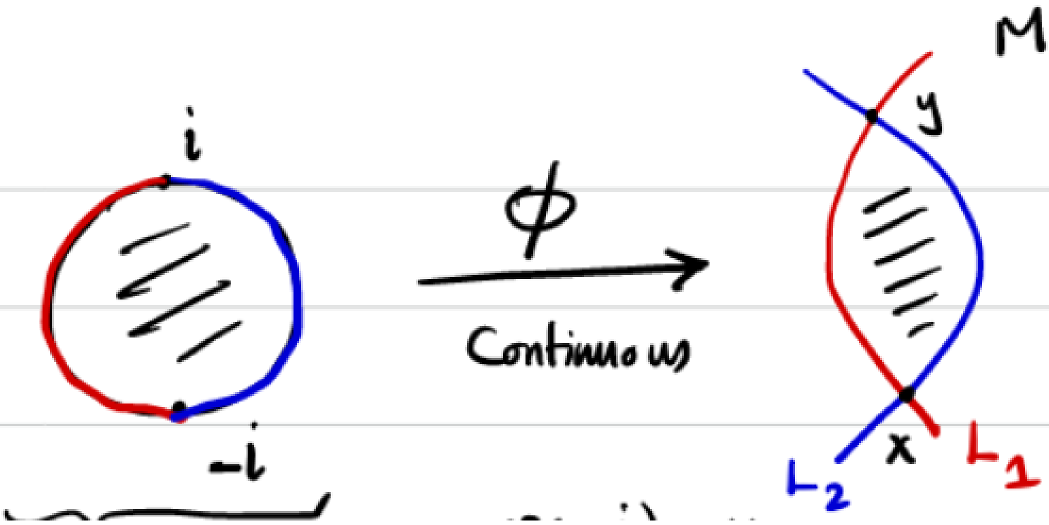

Figure 17: i

So we can the number of *holomorphic* discs from x to y. We'll get $\partial^2 = 0 \iff \operatorname{im}\partial \subset \ker\partial$, and

HF will be kernels modulo images. In more detail, we'll have

$$\partial x = \sum_y \sum_{\mu(\varphi)=1} \#\widehat{\mathcal{M}}(\varphi) y \qquad \widehat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R}$$

where $\widehat{\mathcal{M}}$ will (in good cases) be a 1-dimensional manifold with finitely many points. Note that it's not necessarily true that CF has a grading!

Given a 3-manifold M^3, we'll associate a Heegard diagram Σ, α, β. Note the g-element symmetric group acts on $\prod_{i=1}^g \Sigma$ by permuting the g coordinates, so we can define $\mathrm{Sym}^g(\Sigma) \coloneqq \prod_{i=1}^g \Sigma / S_g$.

Theorem 2.2.3 *(?).*
The space $\mathrm{Sym}^g(\Sigma)$ is a smooth complex manifold of $\mathbb{R}$-dimension $2g$.

Write $\mathbb{T}_\alpha \coloneqq \prod_{i=1}^g \alpha_i \subseteq \prod_{i=1}^g \Sigma$ for a g-dimensional torus; this admits a quotient map to $\mathrm{Sym}^g(\Sigma)$. We can repeat this to obtain $\mathbb{T}_\beta$. Then $HF^\cdot(M)$ will be a variation of Lagrangian Floer Homology for $(\mathrm{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta)$.

Example 2.2.4 *(?):* Consider constructing a genus $g = 1$ Heegard diagram. Recall that S^3 can be constructed by gluing two solid torii.

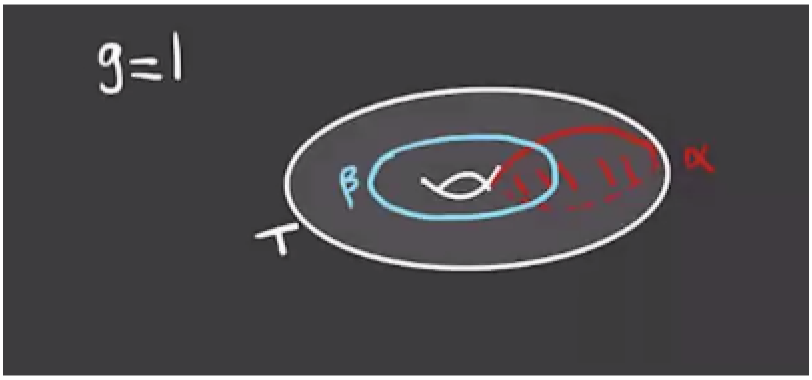

Figure 18: image_2021-01-19-12-20-16

Here (T, α, β) will be a Heegard diagram for S^3.

Exercise 2.2.5 (?)
Show that the following diagram with β defined as some perturbation of α is a Heegard diagram for $S^1 \times S^2$.

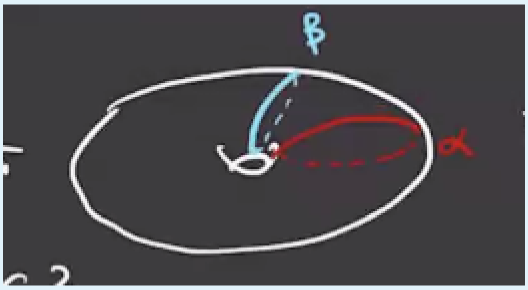

Figure 19: image_2021-01-19-12-21-56

Definition 2.2.6 (Dehn Surgery)
Consider M a 3-manifold containing a knot K, we can construct a new 3-manifold by first removing a neighborhood of K to yield $M \setminus N(K)$:

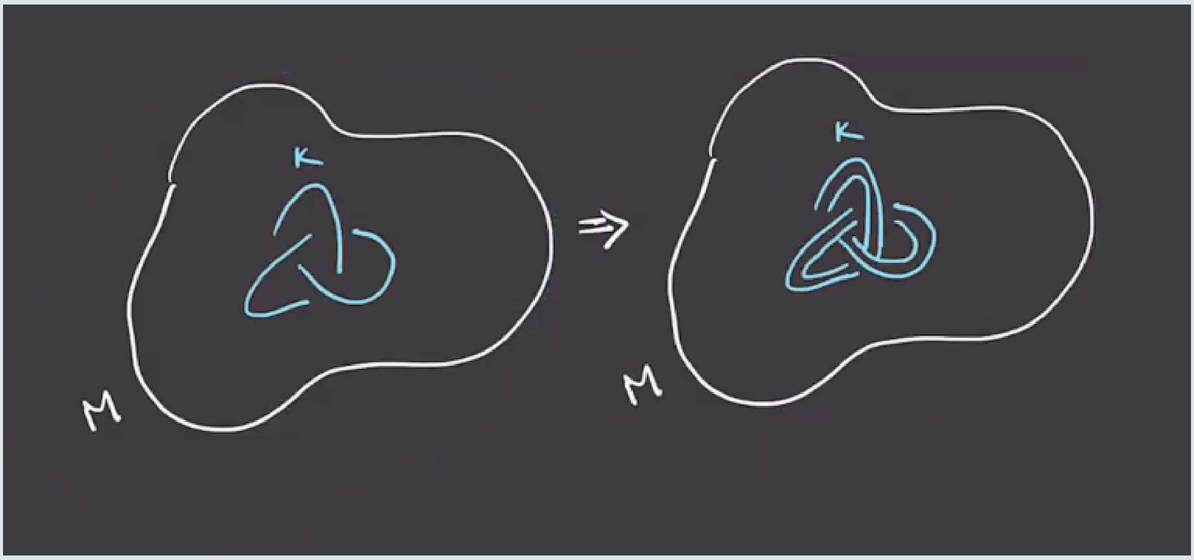

Figure 20: image_2021-01-19-12-23-16

Taking a new solid torus $S \coloneqq \mathbb{D}^2 \times S^1$ and a diffeomorphism $i : \partial S \to \partial(M \setminus N(K))$, this yields a new manifold $M_\varphi(K)$, a **surgery** along K.

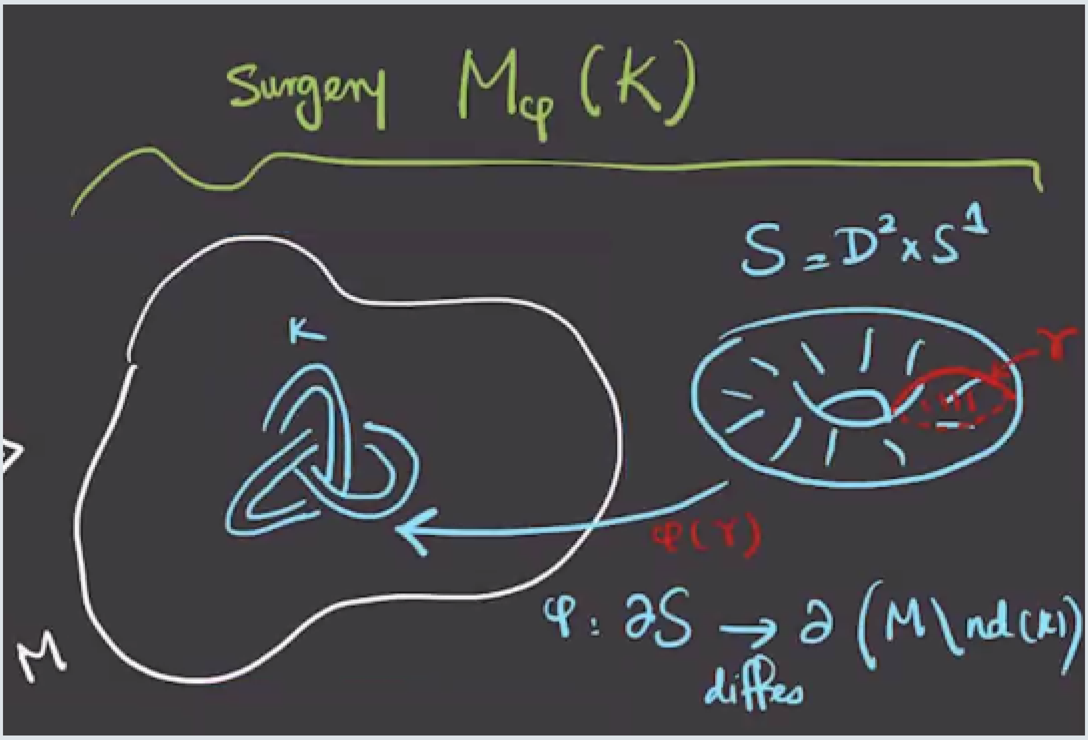

Figure 21: image_2021-01-19-12-25-25

Remark 2.2.7: Note that the diffeomorphism is entirely determined by the image of the curve α . The Knot Floer chain complex of K will allow us to compute any flavor $HF^{\cdot}M_{\varphi}(K)$ of Floer homology. Why is this important: any closed 3-manifold is surgery on a link in S^3. However there are many more computational tools available here and not in the other theories: combinatorial approaches to compute, exact sequences, bordered Floer homology.

Next time: we'll talk about "integer surgeries".

3 Lecture 3: Morse Theory (Thursday, January 19)

3.1 Intro to Morse Theory

Let M^n be a smooth closed manifold, then the goal is to study the topology of M by studying smooth functions $f \in C^{\infty}(M, \mathbb{R})$. We'll need f to be *generic* in a sense we'll discuss later.

Definition 3.1.1 (Critical Point)
A point $p \in M$ is called a **critical point** if and only if $(df)_p = 0$.

Definition 3.1.2 (Hessian / Second Derivative)
Fixing a critical point p for f, the **second derivative** or **Hessian** of f at p is a bilinear form on T_pM which is defined in the following way: for $v, w \in T_pM$, extend w to a vector field $\tilde{w}$ in a neighborhood of p and set

$$d^2 f_p(v, w) = v \cdot (\tilde{w} \cdot f)(p) \coloneqq v \cdot (df)(\tilde{w})(p).$$

where we take the derivative of f with respect to $\tilde{w}$, then take the derivative with respect to v, then evaluate at the point to get a number.

Remark 3.1.3: This is only well-defined at critical points (check!). Note that we need $\tilde{w}$ so that $\tilde{w} \cdot f$ is again a function (and not a number) which can be differentiated again. You can also take e.g. $\tilde{v} \cdot (\tilde{w} \cdot f)$, differentiating with respect to the vector field instead of just the vector v, but we're plugging in p in either case.

Claim: The second derivative is

1. Well-defined, and
2. Symmetric

Remark 3.1.4: If you fix a coordinate chart in a neighborhood of p, then the bilinear form is represented by a matrix given by

$$(d^2 f)_p = H_p = \left(\frac{\partial^2}{\partial x_j \partial x_i}(p) \right)_{ij}.$$

Proof (of 2).
We can compute

$$\begin{aligned} (d^2 f)_p(v, w) - (d^2 f)_p(w, v) &= v \cdot (\tilde{w} \cdot f)(p) - w \cdot (\tilde{v} \cdot f)(p) \\ &\coloneqq df_p([\tilde{v}, \tilde{w}]) \\ &= 0 && \text{since } p \text{ is a critical point and } df_p = 0. \end{aligned}$$

■

Proof (of 1).
This is now easier to prove: we are picking an extension of w to a vector field, so we need to show that the definition doesn't depend on that choice.

$$\begin{aligned} (d^2 f)(_p(v, w) &= v \cdot (\tilde{w} \cdot f)(p) && \text{which doesn't depend on } \tilde{v} \\ &= (d^2 f)_p(w, v) \\ &= w \cdot (\tilde{v} \cdot f)(p) && \text{which doesn't depend on } \tilde{w}, \end{aligned}$$

and thus this is independent of both $\tilde{v}$ and $\tilde{w}$.

■

Exercise 3.1.5 (?)
Show that the second derivative in local coordinates is given by the matrix H_p above.

Remark 3.1.6: In local coordinates, we can write $v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$ and $w = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$, and thus

$$(d^2 f)_p(v, w) = \mathbf{b}^t H_p \mathbf{a} = \sum_{1 \le i,j \le n} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}(p).$$

Definition 3.1.7 (Nondegenerate Critical Points)
A critical point $p \in M$ is called **nondegenerate** if the bilinear form $(d^2 f)_p$ is nondegenerate at p, i.e. for all $v \in T_pM$ there exists a $w \in T_pM \setminus \{\mathbf{0}\}$ such that $(d^2 f)_p(v, w) \neq 0$. This occurs if and only if H_p is invertible.

Definition 3.1.8 (Index of a critical point)
Given a nondegenerate critical point $p \in M$, define the **index** $\operatorname{Ind}(p)$ of f at p in the following way: since H_p is symmetric and nondegenerate, its eigenvalues are real and nonzero, so define the index as the number of *negative* eigenvalues of H_p.

Definition 3.1.9 (Morse Function)
A function $f \in C^\infty(M, \mathbb{R})$ is called a **Morse function** if and only if all of its critical points are nondegenerate.

Remark 3.1.10: We'll see that almost every smooth function is Morse, and these are preferable since they have a simple and predictable structure near critical points and don't do anything interesting elsewhere.

Theorem 3.1.11 ***(Morse Lemma).***
Let $p \in M$ be a nondegenerate critical point of f with $\operatorname{Ind}(p) = \lambda$. Then there exists charts $\varphi : (U, p) \to (\mathbb{R}^n, 0)$ such that writing f in local coordinates yields

$$(f \circ \varphi^{-1})(x) = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

Remark 3.1.12 ***(Observation 1):*** We have

$$H_p = \begin{bmatrix} -2 & & & & & & \\ & \ddots & & & & & \\ & & -2 & & & & \\ & & & 2 & & & \\ & & & & \ddots & & \\ & & & & & 2 & \\ & & & & & & 2 \end{bmatrix} = -2I_\lambda \oplus 2I_{n-\lambda}.$$

Remark 3.1.13 ***(Observation 2):*** If $\lambda = n$??

Remark 3.1.14 ***(Observation 3):*** ??

Example 3.1.15 ***(Sphere):*** Consider S^2 with a height function:

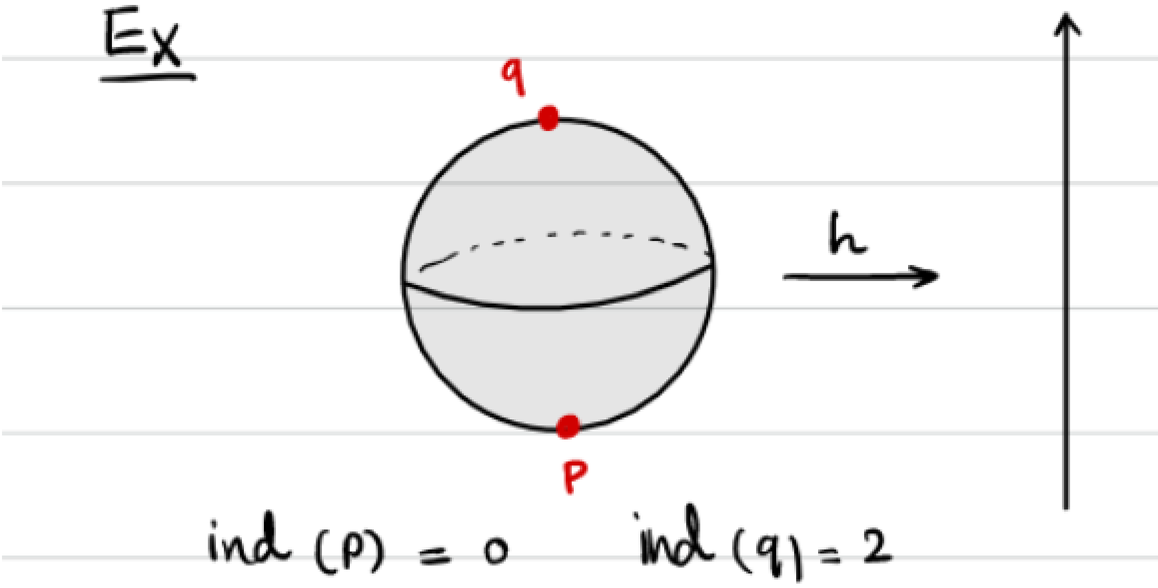

Figure 22: Sphere with a height function

Then we have a local minimum at the South pole p and a local max at the North pole q, where $\mathrm{Ind}(p) = 0$ and $\mathrm{Ind}(q) = 2$. Note that the critical points essentially occur where the tangent space is horizontal

Example 3.1.16 ***(Torus):*** Consider $\mathbb{T}^2$ with the height function:

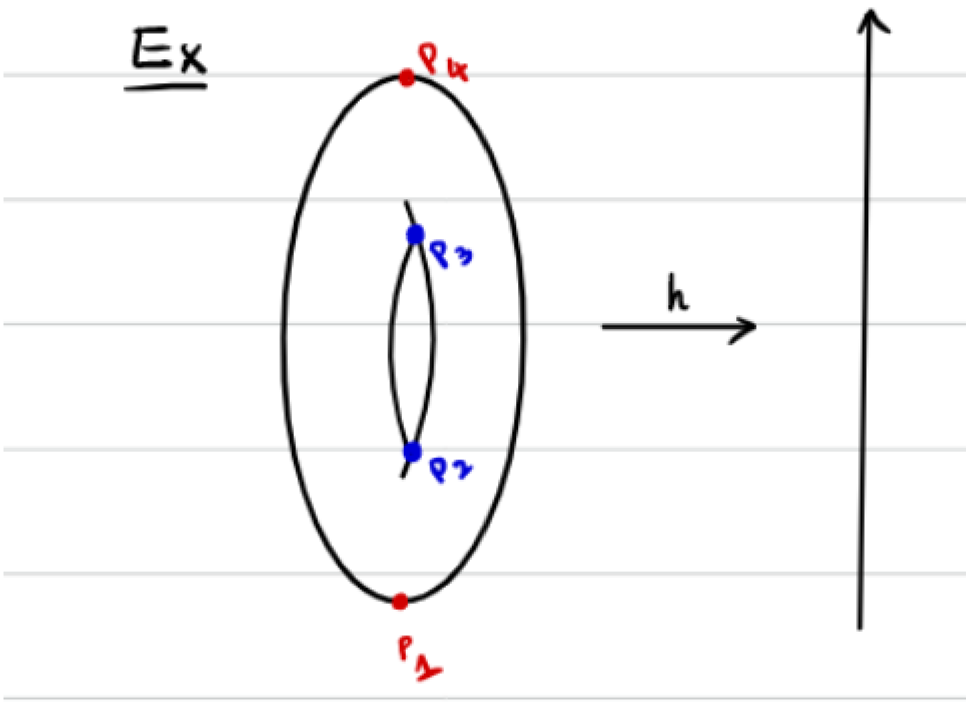

Figure 23: Torus with a height function

This has a similar max/min as the sphere, but also has two critical points in the middle that resemble saddles:

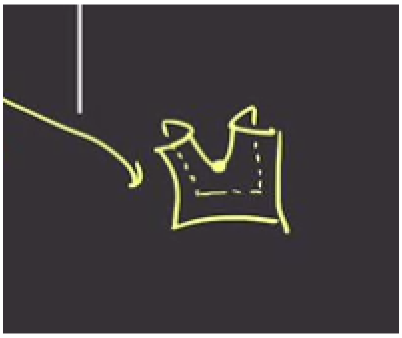

Figure 24: Saddle points

Remark 3.1.17: Define $M_a \coloneqq f^{-1}((-\infty, a])$; we then want to consider how M_a changes as a changes:

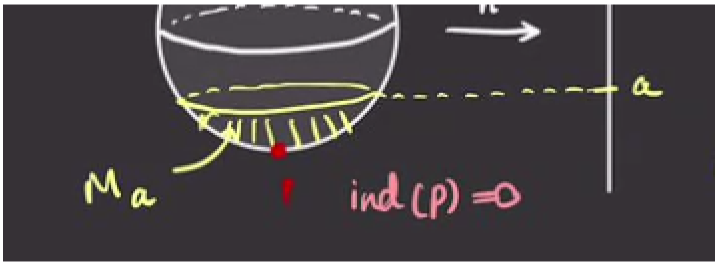

Figure 25: M_a on the sphere

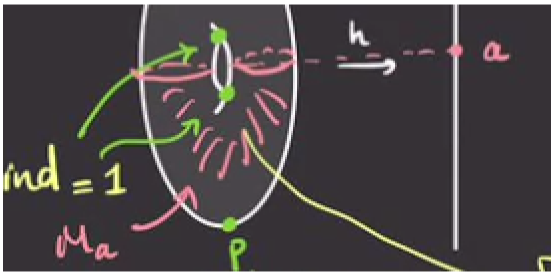

Figure 26: M_a on the torus

Lemma 3.1.18 ***(?).***
If $f^{-1}([a, b])$ contains no critical points, then

$$f^{-1}(a) \cong f^{-1}(b)$$
$$M_a \cong M_b.$$

Definition 3.1.19 (Gradients)
Choose a metric g on M, then the **gradient vector** of f is given by

$$g(\nabla f, v) = df(v).$$

Remark 3.1.20: We have

$$df(\nabla f) = g(\nabla f, \nabla f) = \|\nabla f\|^2.$$

Proof (?).
We have the following situation:

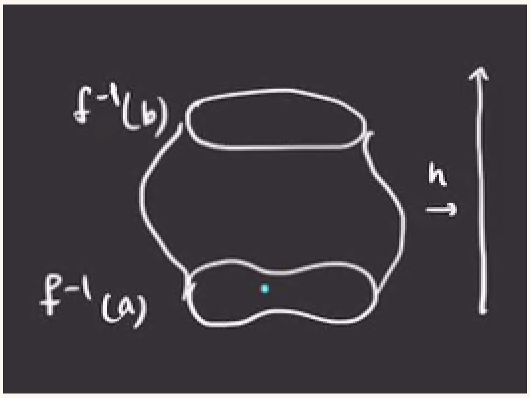

Figure 27: image_2021-01-21-12-11-16

The gradient vector is always tangent to the level sets, so we can consider the curve γ which satisfies $\dot{\gamma}(t) = -\nabla f(\gamma(t))$:

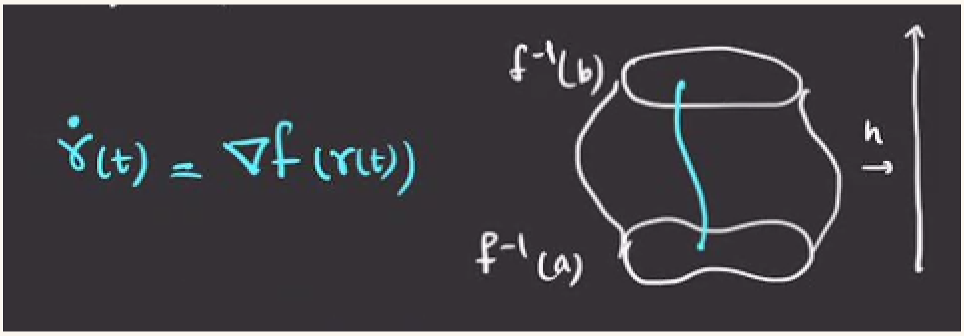

Figure 28: image_2021-01-21-12-12-42

For technical reasons, we want to end up with cohomology instead of homology and will take $-\nabla f$ instead of ∇f everywhere:

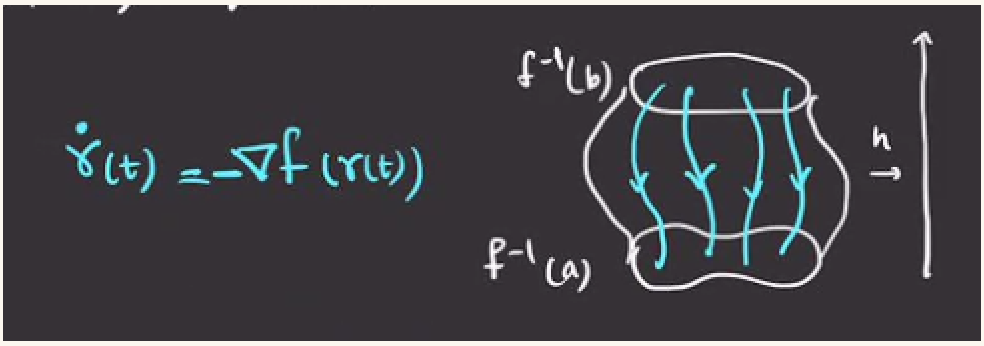

Figure 29: image_2021-01-21-12-13-35

So γ will be a trajectory of $-\nabla f$, and $f^{-1}[a, b] \cong f^{-1}(a) \times [0, 1]$. A problem is that following these trajectories may involve arriving at $f^{-1}(a)$ at different times:

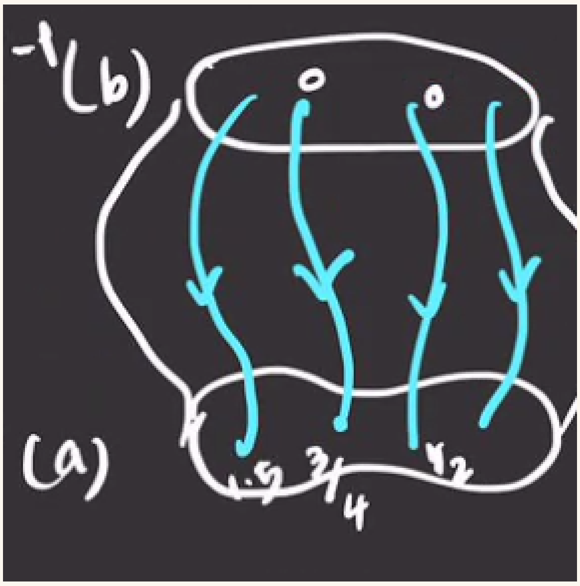

Figure 30: image_2021-01-21-12-15-10

We can fix this by normalizing:

$$V \coloneqq -\nabla f / \|\nabla f\|^2 \implies (df)(v) = \left\langle \nabla f,\ -\nabla f/\|\nabla f\|^2 \right\rangle = -1.$$

For every $p \in f^{-1}(b)$, if $\gamma(t)$ is the trajectory starting from p, i.e. $\gamma(0) = p$, then $\gamma(b-a) \in f^{-1}(a)$. So define

$$\begin{aligned} \Phi : f^{-1}(b) \times [0, b-a] &\to f^{-1}([a,b]) \\ (p, t) &\mapsto \gamma_p(t), \end{aligned}$$

which will be a diffeomorphism.

■

Theorem 3.1.21 *(?).*
Suppose $f^{-1}([a,b])$ contains exactly one critical point p with $\operatorname{Ind}(p) = \lambda$ and $f(p) = c$. Then

$$M_b = M_a \cup_\partial \left(D^\lambda \times D^{n-\lambda}\right)$$

where $n \coloneqq \dim M$.

Example 3.1.22 *(?):* For $\lambda = 1, n - \lambda = 2$:

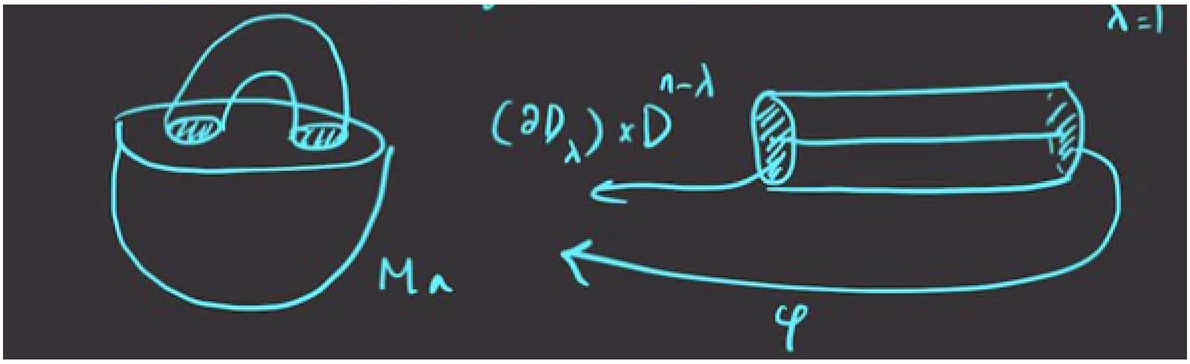

Figure 31: image_2021-01-21-12-32-38

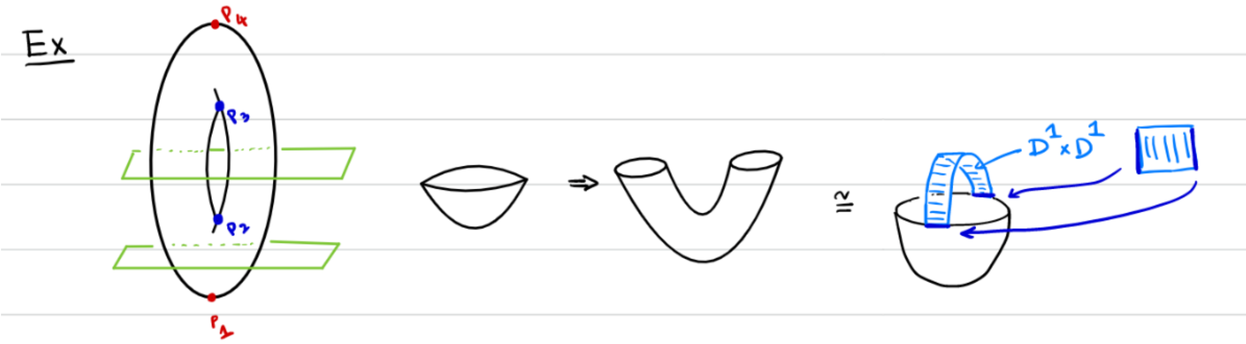

Figure 32: image_2021-01-19-00-53-07

Example 3.1.23 *(?):*

Definition 3.1.24 (Unstable Submanifold)

$$W_f^u(p) \coloneqq \{p\} \cup \left\{ \gamma(\dot{t}) = -\nabla f(\gamma(t)),\ \lim_{t \to -\infty} \gamma(t) = p,\ t \in \mathbb{R} \right\}.$$

Lemma 3.1.25 *(?).*
If $\text{Ind}(p) = \lambda$ then $W_f^u(p) \cong \mathbb{R}^\lambda$.

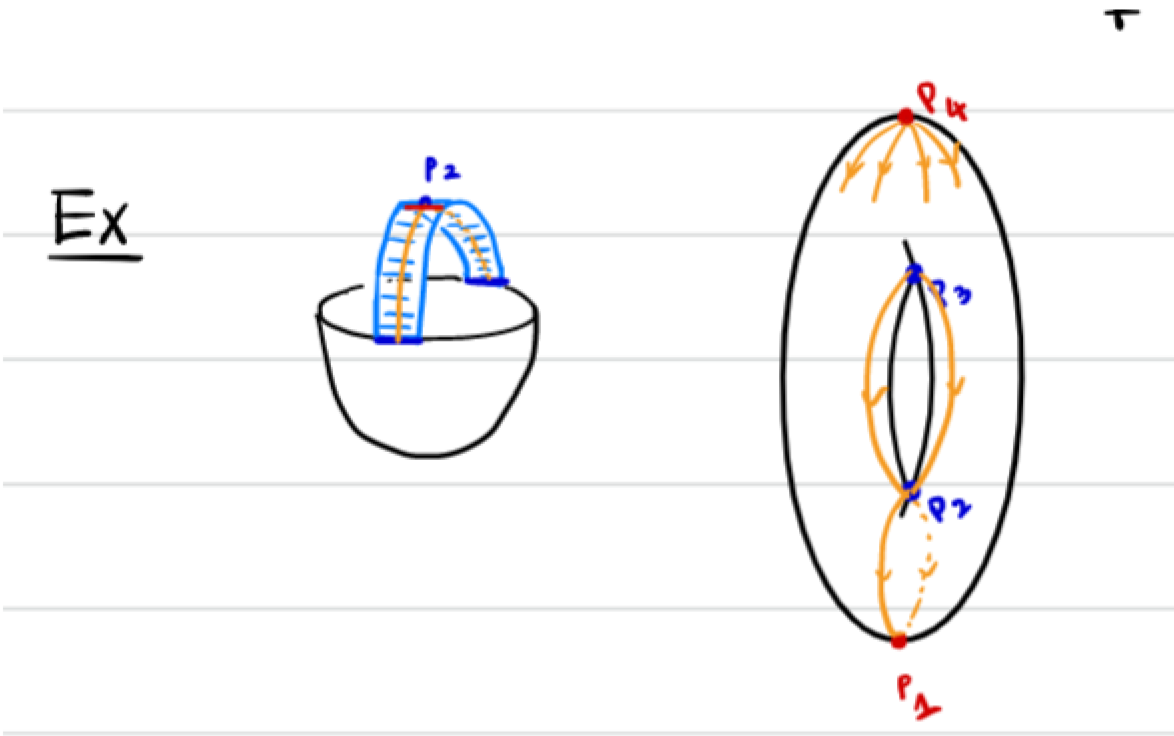

Figure 33: image_2021-01-19-00-55-24

Example 3.1.26*(?):*

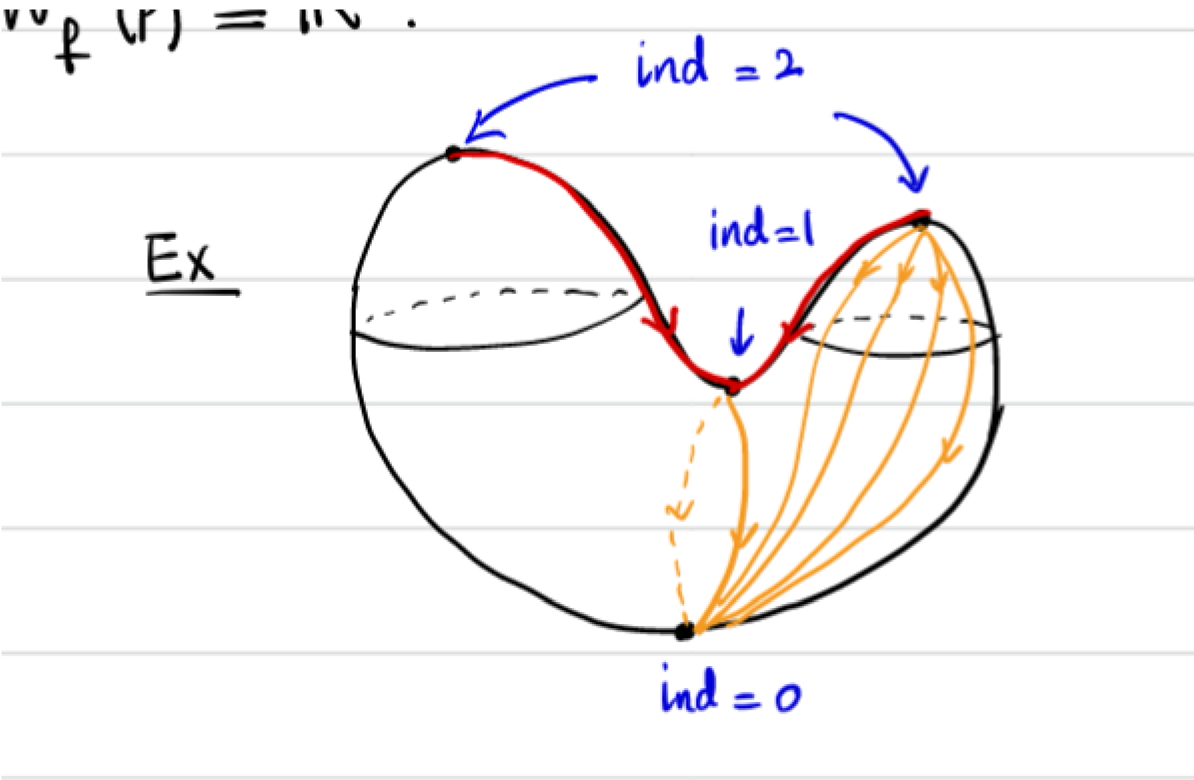

Figure 34: image_2021-01-19-00-55-41

Example 3.1.27 *(?):*

Definition 3.1.28 (Stable Manifold)

$$W_f^s(p) \coloneqq \{p\} \cup \left\{ \dot{\gamma(t)} = -\nabla f(\gamma(t)),\ \lim_{t\to+\infty} \gamma(t) = p,\ t\in\mathbb{R} \right\}.$$

Lemma 3.1.29 *(?).*
If $\operatorname{Ind}(p) = \lambda$ then $W_f^s(p) \cong \mathbb{R}^{n-\lambda}$.

Definition 3.1.30 (C^∞)
$C^\infty(M;\mathbb{R})$ is defined as smooth function $M \to |RR$, topologized as:

- ?
- ?

And a basis for open neighborhoods around p is given by

$$N_g(f) = \left\{ g : M \to \mathbb{R} \;\middle|\; \left| \frac{\partial^k g}{\partial\partial x_{i_1} \cdots \partial x_{i_k}}(p) - \frac{\partial^k f}{\partial\partial x_{i_1} \cdots \partial x_{i_k}}(p) \right| < \infty\ \forall\alpha,\ \forall p \in h_\alpha(C_\alpha) \right\}.$$

Theorem 3.1.31 *(?)*.
The set of Morse functions on M is open and dense in $C^\infty(M;\mathbb{R})$.

4 | Tuesday, January 26

4.1 Attaching Handles

Goal: we want to use Morse functions (smooth, nondegenerate critical points) to study the topology of M. Recall that the torus had 4 critical points,

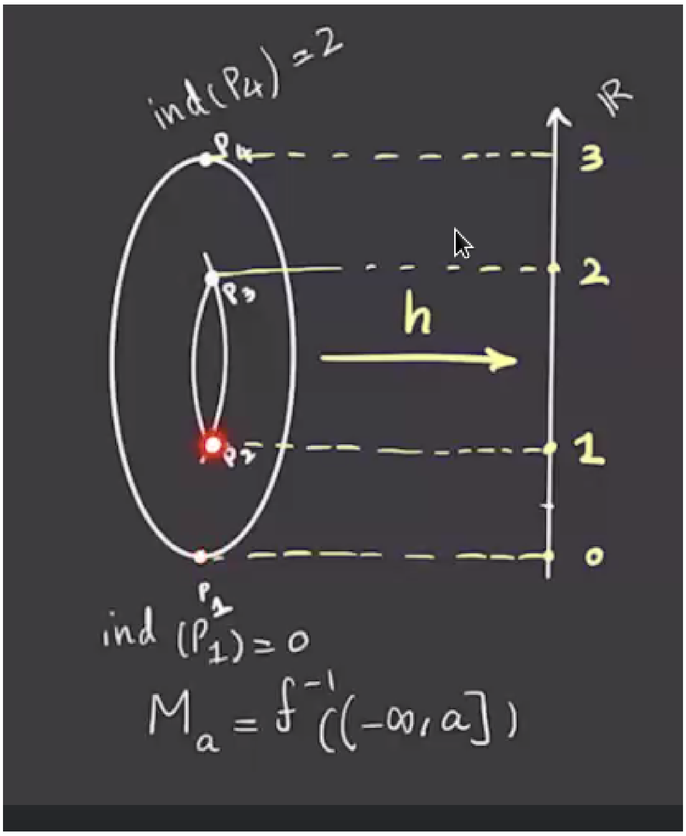

Figure 35: image_2021-01-26-11-14-32

We defined the index as the number of negative eigenvalues of the Hessian matrix. Here the highest index will be the dimension of the manifold, and by the Morse lemma the two intermediate critical points will be index 1.

Remark 4.1.1: We want to use the Morse function to decompose the manifold, so we consider

$M_a := f^{-1}((-\infty, a])$. If $f^{-1}[a,b]$ does not contain a critical point, then $M_a \cong M_b$ and $f^{-1}(a) \cong f^{-1}(b)$. So taking $M_{1/2}$ and $M_{3/4}$ here both yield discs:

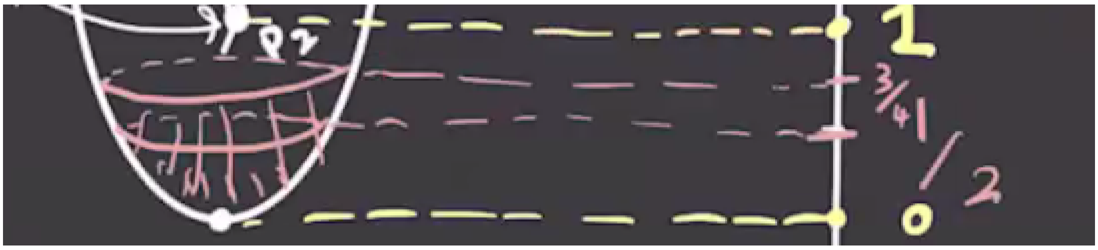

Figure 36: image_2021-01-26-11-17-46

Passing through critical points does change the manifold, though:

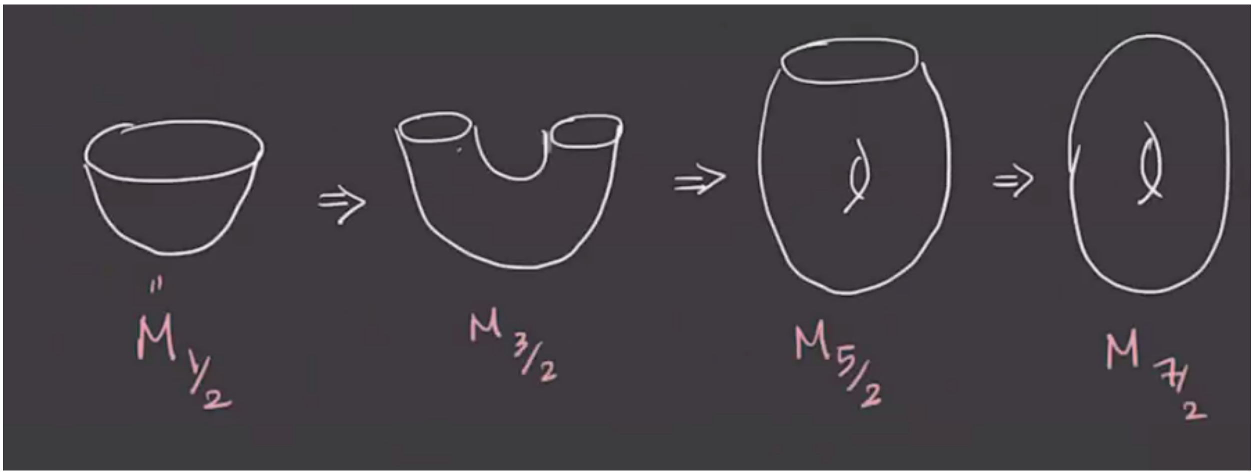

Figure 37: image_2021-01-26-11-19-01

Theorem 4.1.2 ***(?).***
Suppose $f^{-1}[a,b]$ contains exactly *one* critical point of index λ then

$$M_b \cong M_a \cup_\varphi (D_\lambda \times D_{n-\lambda}),$$

where $\varphi : (\partial D_\lambda \times D_{n-\lambda}) \hookrightarrow \partial M_a$.

Example 4.1.3 ***(?):*** For the case $\lambda = 1, n = 3$, we have the following situation:

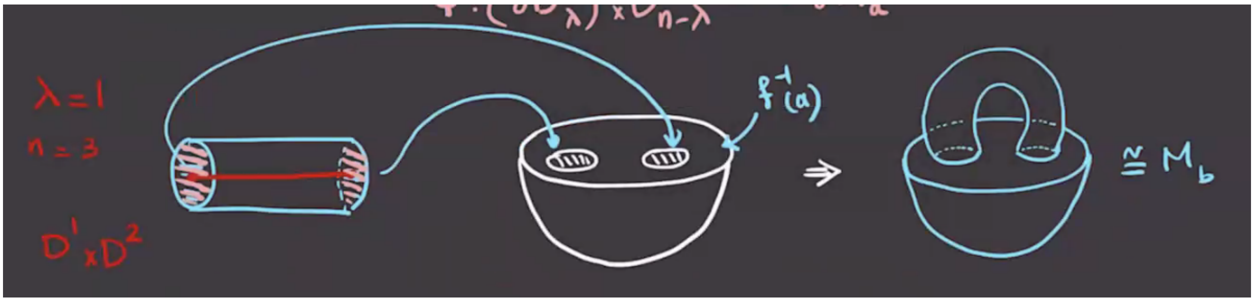

Figure 38: image_2021-01-26-11-24-46

Example 4.1.4 ***(?):*** Taking $\lambda = 1, n = 2$, we attach $D^1 \times D^1$ and get the following situation:

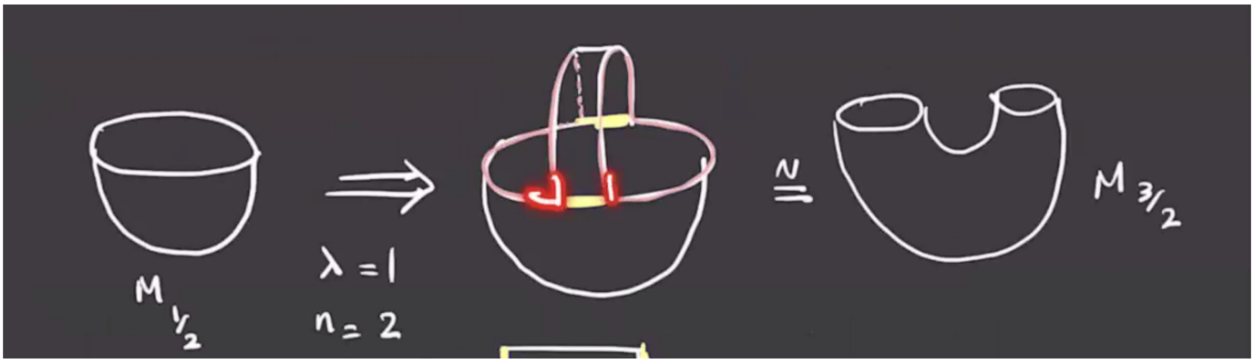

Figure 39: image_2021-01-26-11-27-16

Adding on another piece, the new boundary is given by the highlighted region:

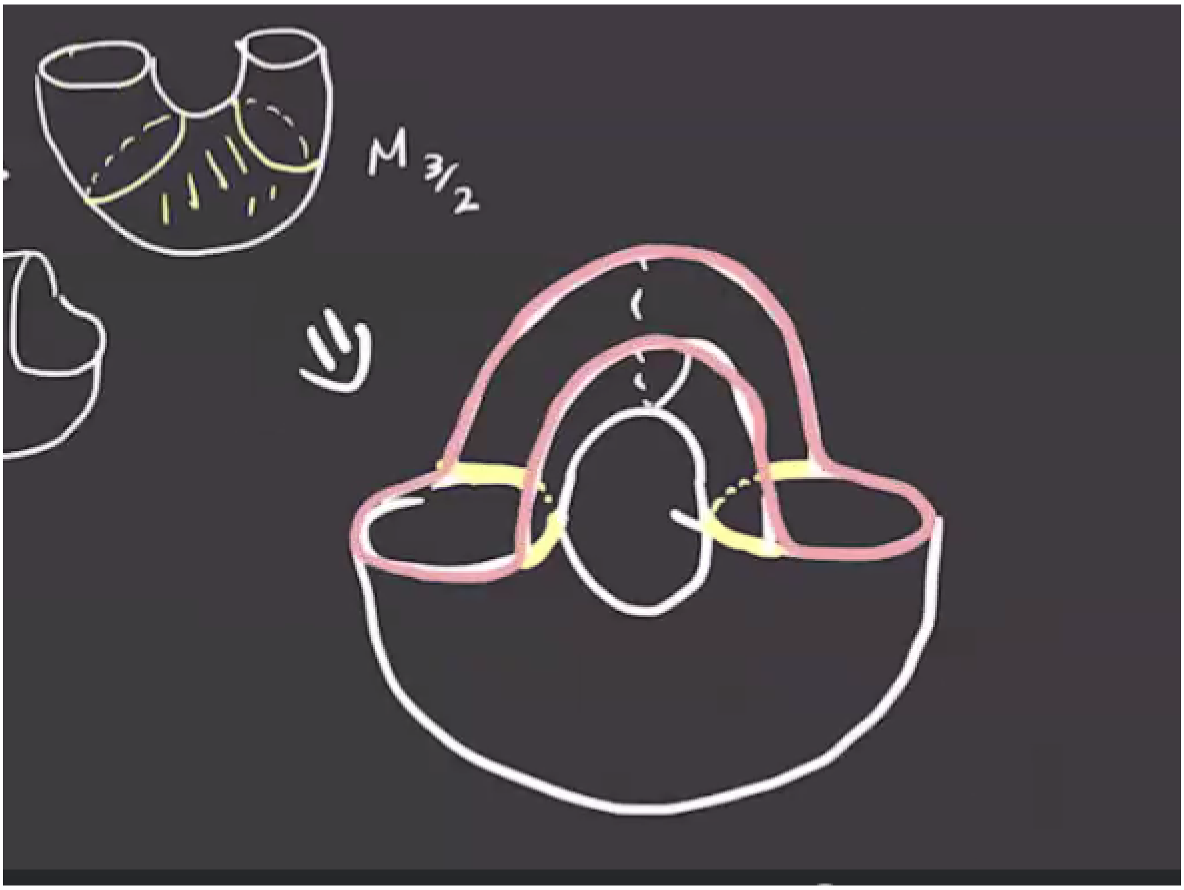

Figure 40: image_2021-01-26-11-32-27

And continuing to attach the last pieces yields the following:

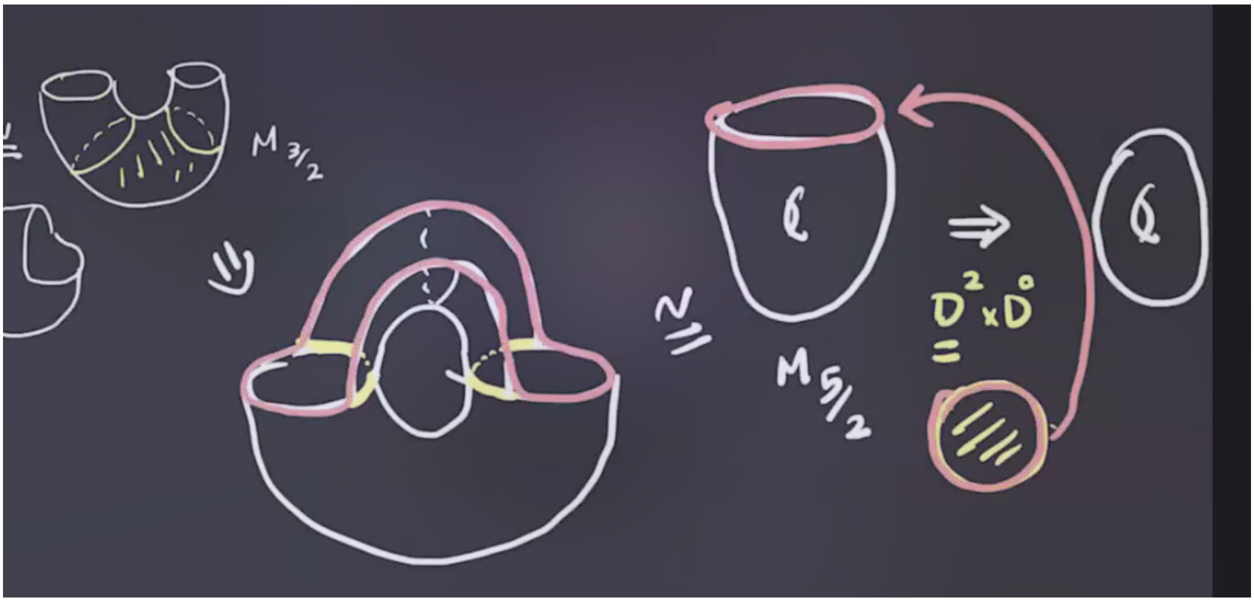

Figure 41: image_2021-01-26-11-33-31

Remark 4.1.5: There is a deformation retract $M_b \to M_a \cup C_\lambda$, where C_λ is a λ-cell given by $D_\lambda \times \{0\}$. For example:

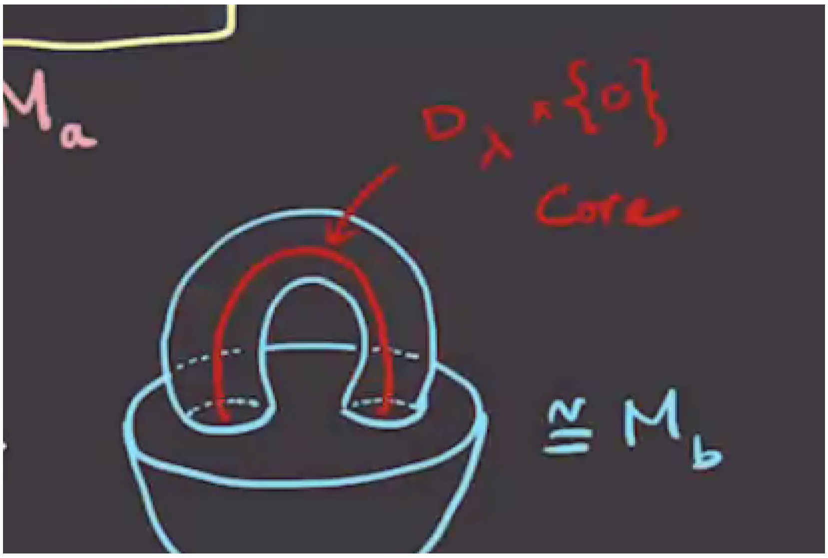

Figure 42: image_2021-01-26-11-36-35

4.2 Stable and Unstable Manifolds

Definition 4.2.1 (Unstable Manifold)
Given $-\nabla f$ for a fixed metric, the **unstable manifold** for a critical point p is defined as

$$W_f^u(p) \coloneqq \{p\} \cup \left\{ \gamma(t) \;\middle|\; \dot{\gamma}(t) = -\nabla f(\gamma(t)),\, \gamma(t) \overset{t\to -\infty}{\to} p \right\}.$$

Here $\gamma(t)$ is the trajectory of $-\nabla(f)$.

Example 4.2.2 ***(?):*** The unstable manifold is highlighted in blue here:

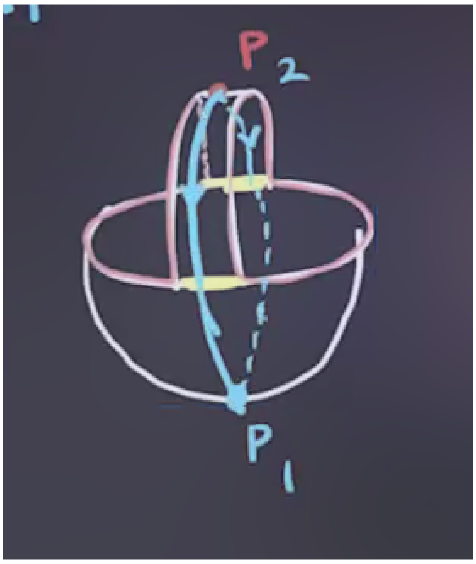

Figure 43: image_2021-01-26-11-42-01

The gradient trajectories for other points are given by the yellow lines in the following:

Figure 44: image_2021-01-26-11-44-13

Lemma 4.2.3 *(?)*.
If $\text{Ind}(p) = \lambda$, then the unstable manifold W_f^u at p is isomorphic to $\mathbb{R}^\lambda$.

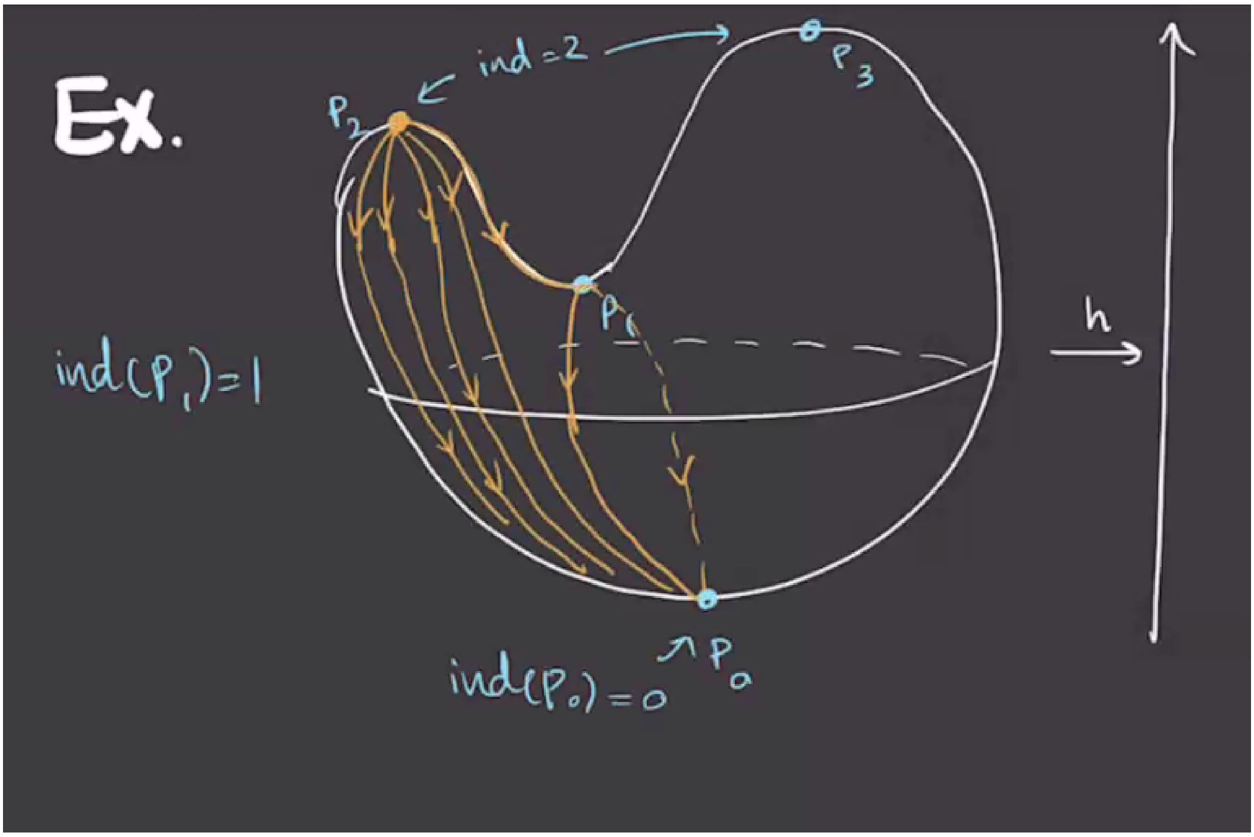

Figure 45: image_2021-01-26-11-46-46

Example 4.2.4 *(?):* Here the unstable manifold for p_2 will be 2-dimensional, with one flow line ending at p_1 and the rest ending at p_0.

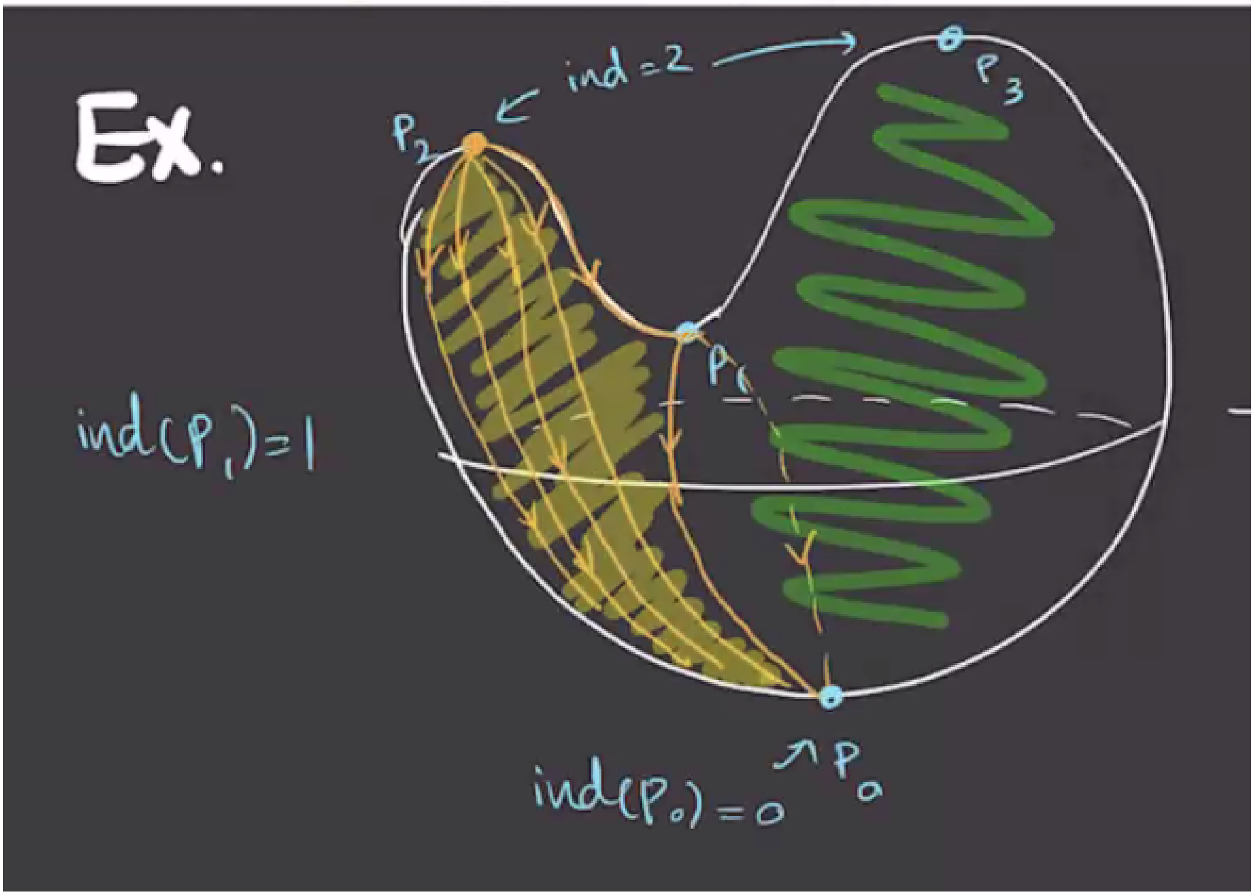

Figure 46: image_2021-01-26-11-47-24

Definition 4.2.5 (Stable Manifold)
The **stable manifold** for a critical point p is defined as

$$W_f^s(p) \coloneqq \{p\} \cup \left\{ \gamma(t) \;\middle|\; \dot{\gamma}(t) = -\nabla f(\gamma(t)),\, \gamma(t) \overset{t\to+\infty}{\to} p \right\}.$$

Example 4.2.6 *(?):* The stable manifold for p_0 above is every trajectory ending at p_0. $W^s(p) = S^2 \setminus W^s(p_1) \cup W_s(p_3)$? See video?

Which point p is this for?

4.3 Morse Functions

Theorem 4.3.1 ***(Existence of Morse Functions).***
The set of Morse functions is open and dense in $C^\infty(M;\mathbb{R})$ in a certain topology.[a]

[a]See Akram's notes for details.

Remark 4.3.2: We'll use this to define a chain complex $C_*(f,g)$ where g is a chosen metric, define a differential, and use this to define a homology theory. For notation, we'll write $\mathrm{crit}(f)$ as the set of critical points of f, and given $p,q \in \mathrm{crit}(f)$ with γ a trajectory running from p to q, we have

$$W^u(p) \cap W^s(q) = \left\{ \gamma(t) \;\middle|\; \gamma(t) \overset{t\to-\infty}{\to} p,\, \gamma(t) \overset{t\to+\infty}{\to} q \right\}.$$

Definition 4.3.3 (Transverse Intersections)
Two submanifolds $X, Y \subseteq M$ **intersect transversely** if and only if

$$T_pX + T_pY \coloneqq \left\{ v + w \;\middle|\; v \in T_pX, w \in T_pY \right\} = T_pM \qquad \forall p \in X \cap Y.$$

In this case, we write $X \pitchfork Y$.

Example 4.3.4 ***(?):*** An example of a transverse intersection:

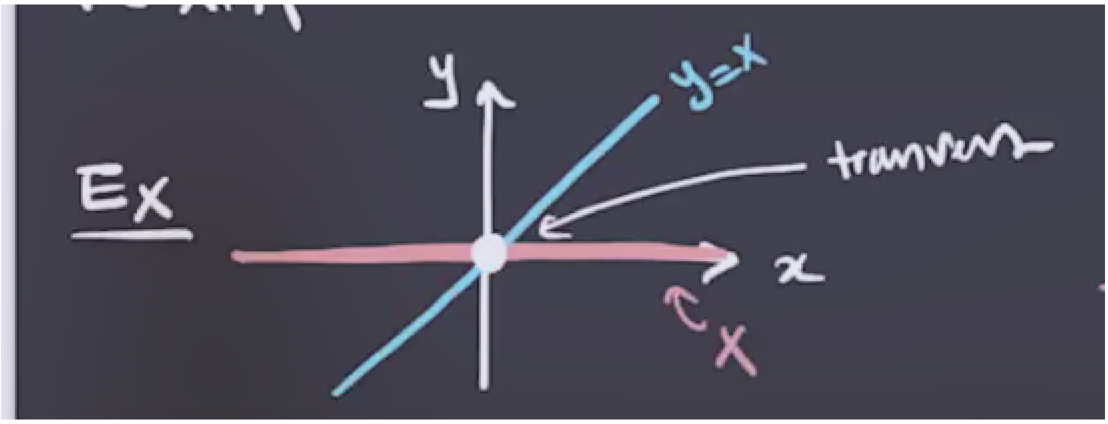

Figure 47: image_2021-01-26-12-02-29

Example 4.3.5 ***(?):*** An example of an intersection that is *not* transverse:

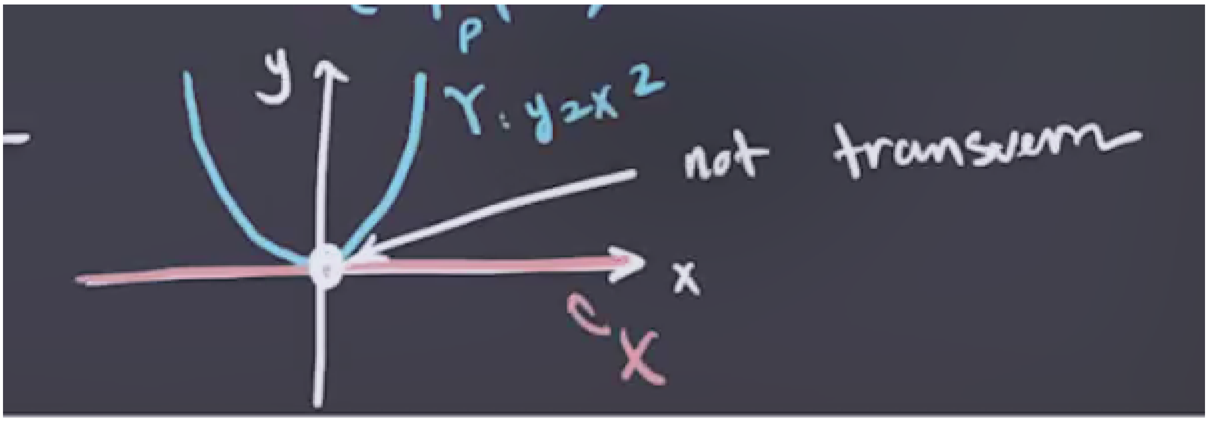

Figure 48: image_2021-01-26-12-03-13

Definition 4.3.6 (Morse-Smale)
A pair (f, g) with f a Morse function and g a metric is **Morse-Smale** if and only if

- f is a Morse function,
- $W^u(p)$ is *transverse* to $W^s(q)$ for all $p, q \in \text{crit}(f)$.

Theorem 4.3.7 ***(?).***
For a generic metric g, the pair (f, g) is Morse-Smale.

Remark 4.3.8: This means that metrics can be perturbed to become Morse-Smale.

Example 4.3.9 ***(?):*** The following is not Morse-Smale:

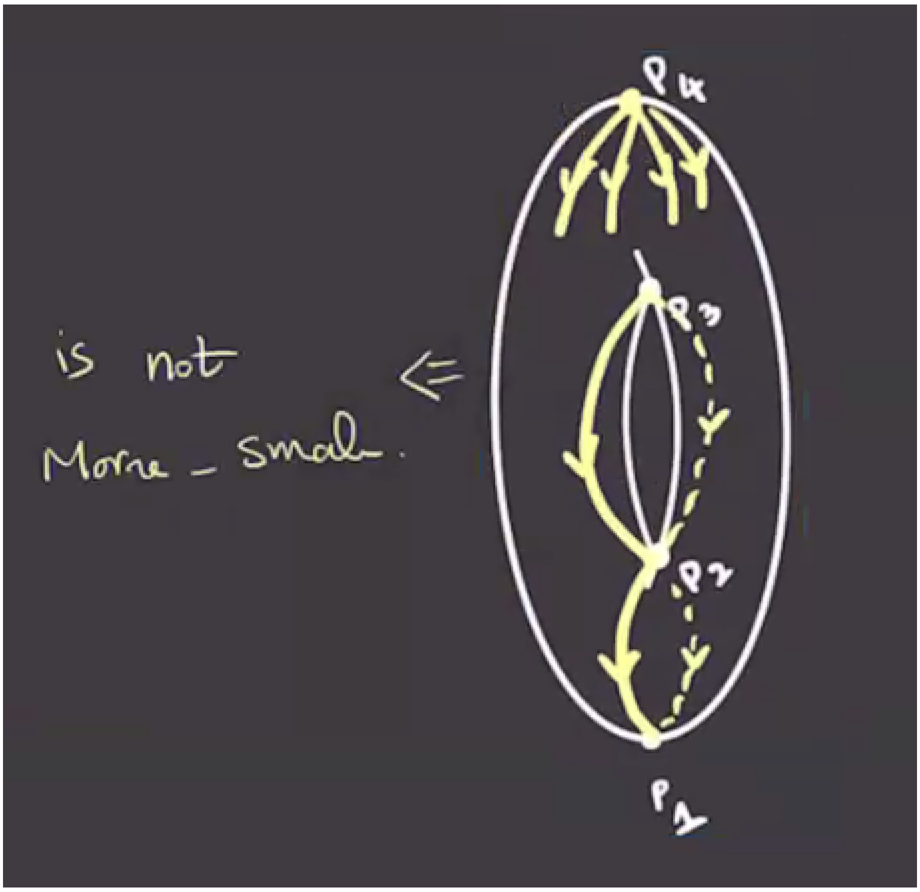

Figure 49: image_2021-01-26-12-06-06

Note that if $X^a \pitchfork Y^b$, then $X \cap Y \subseteq M^n$ is a smooth submanifold of dimension $a+b-n$. In general, we have $M^s(p) \cong \mathbb{R}^{n-\lambda}$ where $\lambda = \operatorname{Ind}(p)$.

Observation 4.3.10
If (f,g) is Morse-Smale, then $M^u(p) \pitchfork M^s(q)$. In this case,

$$\dim(M^u(p) \cap M^s(q)) = \operatorname{Ind}(p) + n - \operatorname{Ind}(q) - n = \operatorname{Ind}(p) - \operatorname{Ind}(q).$$

Thus if $\operatorname{Ind}(p) = \operatorname{Ind}(q)$ then $\dim M^s(p) \cap M^s(q) = 0$.

Remark 4.3.11: There is an $\mathbb{R}$-action of $M^s(p) \cap M^s(q)$:

$$\begin{aligned}(M^s(p) \times M^u(q)) \times \mathbb{R} &\to M^s(p) \cap M^u(q) \\ (\gamma(t), c) &\mapsto \gamma(t+c).\end{aligned}$$

If $p \neq q$, this action is free and we can thus quotient by it to obtain

$$\mathcal{M}(p,q) \coloneqq (M^s(p) \cap M^u(q))/\mathbb{R}.$$

This identifies all points on the same trajectory, yielding one point for every trajectory, and so this is called the **moduli space of trajectories from p to q**.

If $\operatorname{Ind}(p) = \operatorname{Ind}(q)$, we have $\dim M^u(p) \cap M^s(q) = 0$, making $\dim \mathcal{M}(p,q) = -1$ and thus $\mathcal{M}(p,q) = \varnothing$ and no gradient trajectories connect p to q. Referring back to the example, since $\operatorname{Ind}(p_3) = \operatorname{Ind}(p_2)$, if (f,g) were Morse-Smale then there would be no trajectory $p_3 \to p_2$, whereas in this case there is at least one.

Remark 4.3.12: If $\operatorname{Ind}(p) - \operatorname{Ind}(q) = 1$, then $\dim \mathcal{M}(p,q) = \operatorname{Ind}(p) - \operatorname{Ind}(q) - 1 = 0$, making $\mathcal{M}(p,q)$ a compact 0-dimensional manifold, which is thus finitely many points, meaning there are only finitely many trajectories connecting $p \to q$ and it becomes possible to define a Morse complex.

Definition 4.3.13 (Morse Complex)
Fix (f,g) a Morse-Smale pair, then define

$$C_i(f,g) \coloneqq \mathbb{Z}/2\mathbb{Z}\left[\left\{p \;\middle|\; \operatorname{Ind} p = i\right\}\right] = \bigoplus_{\operatorname{Ind}(p)=i} \mathbb{Z}/2\mathbb{Z}\left\langle p\right\rangle,$$

with a differential

$$\begin{aligned} \partial : C_i(f,g) &\to C_{i-1}(f,g) \\ p, \operatorname{Ind}(p) = i &\mapsto \sum_{\operatorname{Ind}(q)=i-1} \#\mathcal{M}(p,q)q, \end{aligned}$$

where we take the count mod 2.

Theorem 4.3.14 ***(?).***
$\partial^2 = 0$, and thus $(C(f,g), \partial)$ is a chain complex.

Next time we will work on proving this.

ToDos

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