Title

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Contents

0.1 Exercises

Problem 1.

Let C denote the Cantor set.

1. Show that C contains point that is not an endpoint of one of the removed intervals.

2. Show that C is nowhere dense, meager, and has measure zero.

3. Show that C is uncountable.

Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let C_n be the nTh stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:

$$C_n = [0, \frac{1}{3^n}]\bigcup[\frac{2}{2^n}, \frac{3}{3^n}]\bigcup \cdots \bigcup[\frac{3^n - 1}{n}, 1],$$

and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in C_n is never removed in any later step, and so all endpoints remaining in C are of this form where we allow $0 \leq n < \infty$.

Thus, our goal is to produce a number $x \in [0, 1]$ such that $x \neq \frac{k}{3^n}$ for any k or n, but also satisfies $x \in C$. So we will need a general characterization of all of the points in C.

Lemma: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

Proof: By induction on the index k in a_k, first consider note that if $x \in C$ then $x \in C_1 = [0,1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. So if $x \in C_1$, then $x \notin (\frac{1}{3}, \frac{2}{3})$. But note that a_1 is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \le x < \frac{1}{3}, \\ 1 & \frac{1}{3} \le x < \frac{2}{3}, \\ 2 & \frac{2}{3} \le x < 1. \end{cases}$$

Since the interval $(\frac{1}{3}, \frac{2}{3})$ is deleted in C_1, we find that $a_1 = 1 \iff x = \frac{1}{3}$. In this case, however, we claim that we can find a ternary expansion of x that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that $x = x'$, which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left(\sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left(\sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 (\frac{1}{1 - \frac{1}{3}}) - 2 - 2(3^{-1}) \\ &= 2 \left(\frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have $\frac{1}{3} = (0.1)_3 = (0.222 \cdots)_3$ as ternary expansions, and a similar proof shows that such an expansion without 1s can be found for any endpoint.

For the inductive step, consider a_n: the claim is that if $a_n = 1$, then $x \notin C_{n+1}$ – that is, it is contained in one of the intervals deleted at the $n + 1$st stage. Writing the deleted interval at this stage as (a, b), we find that $a_n = 1$ if and only if $x \in [a, b)$. Since $x \in C$, the only way a_n can be 1 is if x was in fact the endpoint a (since no previous digit was a 1, by hypothesis). However, as shown above, every such endpoint has a ternary expansion containing no 1s. $\quad \square$

Therefore, if we can produce an x that satisfies $x \neq \frac{k}{3^n}$ for any k, n **and** x has no 1s in its ternary expansion, we will have an $x \in C$ that is not an endpoint.

So take
$$x = (0.\overline{02})_3 = (0.020202 \cdots)_3.$$

This evidently has no 1s in its ternary expansion, and if we sum the corresponding geometric series, we find $x = \frac{1}{4}$. This is not of the form $\frac{k}{3^n}$ for any k, n, and thus fulfills both conditions.

2. We first show that C is nowhere dense by showing that the interior of its closure is empty, i.e. $(\overline{C})^\circ = \emptyset$.

To do so, we note that C is itself closed and so $C = \overline{C}$. To see why this is, consider C^c; we'll show that it is open. By construction, C_1^c is the open interval $(\frac{1}{3}, \frac{2}{3})$ that is deleted, and similarly C_n^c is the finite union of the open intervals that are deleted at the nth stage. But then
$$C^c = \left(\bigcap C_n \right)^c = \bigcup C_n^c$$
is an infinite union of open sets, which is also open. So C is closed.

It is also the case that C has empty interior, so $C^\circ = \emptyset$. Towards a contradiction, suppose $x \in C$ is an interior point; then there is some neighborhood $N_\varepsilon(x) \subset C$. Since we are on the real line, we can write this as an interval $(x - \varepsilon, x + \varepsilon)$, which has length $2\varepsilon > 0$. Moreover, we have the containment
$$(x - \varepsilon, x + \varepsilon) \subset C \subset C_n$$
for every n.

Claim: The length of C_n is $(\frac{2}{3})^n$ where we define $C_0 = [0, 1]$. Letting L_n be the length of C_n, one easy way to see that this is the case is to note that L_n satisfies the recurrence relation
$$L_{n+1} = \frac{2}{3} L_n,$$
since an interval of length $\frac{1}{3} L_n$ is removed at each stage. With the initial conditions $L_0 = 1$, it can be checked that $L_n = \left(\frac{2}{3} \right)^n$ solves this relation.

Now, since $x \in C = \bigcap C_n$, it is in every C_n. So we can choose n large enough such that
$$\left(\frac{2}{3} \right)^n \leq 2\varepsilon.$$

Letting $\mu(X)$ denote the length of an interval, we always have $C \subseteq C_n$ and so $\mu(C) \leq \mu(C_n)$.

Using the subadditivity of measures, we now have
$$(x - \varepsilon, x + \varepsilon) \subset C \subset C_n$$
$$\implies \mu(x - \varepsilon, x + \varepsilon) \leq \mu(C) \leq \mu(C_n)$$
$$\implies 2\varepsilon \leq \left(\frac{2}{3} \right)^n,$$

a contradiction. So C has no interior points.

But this means that
$$(\overline{C})^\circ = C^\circ = \emptyset,$$

and so C is nowhere dense.

To see that $\mu(C) = 0$, we can use the fact that for any sets, measures are additive over disjoint sets and we have

$$\mu(A) + \mu(X \setminus A) = \mu(X) \implies \mu(X \setminus A) = \mu(X) - \mu(A).$$

Here we will take $X = [0, 1]$, so $\mu(X) = 1$, and $A = C$ the Cantor set.

By tracing through the construction of the Cantor set, letting B_n be the length of the interval that is removed at each stage, we can deduce

$$B_1 = \frac{1}{3}$$
$$B_2 = \frac{2}{9}$$
$$\dots$$
$$B_n = \frac{2^n}{3^{n+1}}.$$

We can identify $B_n = \mu(C_n^c)$, and using the fact that $C_n^c \cap C_{>n}^c = \emptyset$ and the fact that measures are additive over disjoint sets, we can compute

$$\mu(C) = 1 - \mu(C^c)$$
$$= 1 - \mu((\bigcap_{n=0}^{\infty} C_n)^c)$$
$$= 1 - \mu(\coprod_{n=0}^{\infty} C_n^c)$$
$$= 1 - \sum_{n=0}^{\infty} \mu(C_n^c)$$
$$= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{-n}}$$
$$= 1 - \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$
$$= 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right)$$
$$= 1 - \frac{1}{3}(3) = 0,$$

which is what we wanted to show. $\square$

3. Let $y \in [0, 1]$ be arbitrary, we will construct an element $x \in C$ such that $y = f(x)$. We first note that every number has a binary expansion, and we can write

$$y = \sum_{k=1}^{\infty} y_k 2^{-k} \quad \text{where } y_k \in \{0, 1\}.$$

Now we construct

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k = 2y_k \implies a_k \in \{0, 2\}.$$

By the characterization given in part (1), we see that $x \in C$ because it has no 1s in its ternary expansion. Moreover, under f, we have $a_k \mapsto \frac{1}{2}a_k = \frac{1}{2}(2a_k) = a_k$, and so $f(x) = y$ by construction.

This shows that C surjects onto $[0, 1]$, and in particular, $\#C \geq \#[0, 1]$ holds for the cardinalities of these sets. Since $[0, 1]$ is uncountable (say, by Cantor's diagonalization argument), this shows that C is uncountable.

Problem 2.

1. Show that X is G_δ iff X^c is F_σ.

2. Show that X closed $\implies X$ is G_δ and X open $\implies X$ is F_σ.

3. Give an example of an F_σ set that is not G_δ, and a set that is neither.

Solution 2.

1. To show the forward direction, suppose X is a F_σ, so $X = \bigcup_{i \in \mathbb{N}} A_i$ with each A_i an closed set. By definition, each A_i^c is open, and we have

$$X^c = \left(\bigcup_{i \in \mathbb{N}} A_i \right)^c = \bigcap_{i \in \mathbb{N}} A_i^c,$$

which exhibits X^c as a countable intersection of closed sets, making it an G_δ.

The reverse direction proceeds analogously: supposing X^c is G_δ, we can write $X^c = \bigcap_{i \in \mathbb{N}} B_i$ with each B_i open, where B_i^c is closed by definition, and

$$X = (X^c)^c = (\bigcap B_i)^c = \bigcup B_i^c$$

which exhibits X as a union of closed sets, and thus an F_σ.

2. Suppose X is closed, we want to then write X as a countable intersection of open sets. For every $x \in X$ and every $n \in \mathbb{N}$, define

$$B_n(x) = \left\{ y \in \mathbb{R}^n \ni |x - y| \leq \frac{1}{n} \right\},$$

$$asdasdas.$$