Title

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Recall that we previously looked at the regular function fields: we took a function field in one variable and considered the class of function fields for which we could take any extension of the constant field that we wanted. As long as the ground field is perfect, being regular is equivalent to the constant subfield being k itself. However, we haven't done anything with them yet!

If you take an algebraic closure of the finite ground field $\mathbb{F}_q$, there is a unique subextension of degree r for every r, so we call that $\mathbb{F}_{q^r}$. The extension $\mathbb{F}_{q^r}/\mathbb{F}_q$ is cyclic galois, with a geometric Frobenius $x \to x^q$. Note that $\mathbb{F}_{q^r}$ is the fixed field of F^r, the rth power of the Frobenius map. We set $K_r := K\mathbb{F}_{q^r}$, which is a regular function field over $\mathbb{F}_{q^r}$. Note that we could view this as a function field just over $\mathbb{F}_q$, but it would not be regular. Then K_r/K is a degree r arithmetic extension of function fields.

Question: What happens to places when making this scalar extension? I.e., how to places in K decompose in K_r?

Remark 1.0.1 : This is related to an Algebraic Number Theory I problem: for $v \in \Sigma(K_{/\mathbb{F}_q})$ above an affine Dedekind domain R such that $v \in \Sigma(K/R)$, let S be the integral closure of K in K_r. Then we want to factor $p_v S$?

> Not quite sure.

> **Lemma 1.1** *(Key lemma about how places split.).*
> Suppose $d := \deg(v)$. Then K_r/K is galois, so we have $efg = r$. In fact, $c = 1$, so $f = \dfrac{r}{\gcd(d,r)}$
> and $g = \gcd(d,r)$ and each place $w \in \Sigma(K_r/\mathbb{F}_{q^r})$ has degree $\dfrac{d}{\gcd(d,r)}$.

Remark 1.0.2 : We have the following cases:

- The extension is *inert* iff $\gcd(d, r) = 1$,

- The extension *splits completely* iff $r \mid d$,

- All w dividing v have degree 1 iff $d \mid r$.

The last thing we proved was that the degree zero divisor class group is finite when we're over a finite ground field. Why is this true? Whenever there is a divisor of degree n, then the set of degree n divisors is a coset of the degree zero divisors, all of which have the same cardinality. We proved finiteness using the Riemann-Roch theorem, using the fact that the set of *effective* degree n divisors is finite for all n.

The next main topic will be the zeta function, which keeps track of three equivalent packets of

information: A_n, the number of effective divisors of degree n, the number of places of degree d (since an effective divisor is a linear combination of these), and N_r the number of degree 1 points in the degree r extension, i.e. the number of $\mathbb{F}_{q^r}$ rational points.

Lemma 1.2*(?).*
Suppose $C \in \mathrm{Cl}(K)$, then

- The number of effective divisors $D \in [C]$ is given by

$$\frac{q^{\ell(C)} - 1}{q - 1},$$

where $\ell(C)$ is the dimension of the linear system associated to the divisor class C, and this is the dimension of a projective space over $\mathbb{F}_q$.

- For all $n > 2g - 2$ with $\delta \mid n$, we have

$$A_n = h\left(\frac{q^{n+1-g} - 1}{q - 1}\right).$$

Proof (?).
Proof of (a): The set of effective divisors linearly equivalent to D is naturally viewed as the projectivization $\mathbb{PL}(D)$ of the one-dimensional subspaces of the linear system of that divisor class. It is then a fact that the number of elements in a d-dimensional vector space over $\mathbb{F}_q$ has dimension precisely $\frac{q^d - 1}{q-1}$ elements.
The projectivization comes in because two different functions have the same divisor if one of them is a constant multiple of the other. Note that the number of elements is computed as the number of nonzero elements divided by the number of nonzero scalars.
Proof of (b): This will come out of the Riemann-Roch theorem. In order to compute the number of divisors in a divisor class, you need to know the dimension of the linear system, which is not easy in general. However, if the divisor class has sufficiently large degree, the Riemann-Roch theorem tells you exactly what it is. As long as $n > 2g - 2$, there is no correction term, and the dimension of the linear system is equal to its degree minus the genus plus one. So by Riemann-Roch, since $\deg(D) > 2g - 2$, D is non-special and $\ell([D]) = n - g + 1$, which yields the desired formula for A_n. ∎

Remark 1.0.3 : This is the sharpest result possible: the canonical divisor has degree $2g - 2$ and is special, so this fails for the canonical class.

Upshot: there are three piece of information:

- N_r, the number of $\mathbb{F}_{q^r}$ rational points,

- $\left|\Sigma_d(K_{/\mathbb{F}_q})\right|$ the number of closed points / places of degree d,

- A_n the number of effective divisors of degree n,

and there are simple formulas relating these. Moreover, it is enough to know only finitely many of these quantities, where the number depends on g.

1.1 Hasse-Weil Zeta Functions

There is a general theory that will unify

- The Riemann zeta function, thought of as the zeta function of $\mathbb{Z}$,

- The Dedekind zeta function, attached to the ring of integers over a number field,

- The Hasse-Weil zeta function of a one variable function field over a finite field,

all of which will be special cases of a *Serre zeta function* which can be attached to a finite type scheme over $\mathbb{Z}$.

Note that we aren't specifically discussing schemes in this course, but you don't need to know much about what a scheme is to define the Hasse-Weil zeta function. Just note that an affine finite-type $\mathbb{Z}$-scheme corresponds to a finitely generated $\mathbb{Z}$-algebra, and a general finite-type $\mathbb{Z}$-scheme will be covered by finitely many affine ones, the zeta function will be determined by these finitely many $\mathbb{Z}$-algebras and some kind of inclusion-exclusion principle (since the scheme is a not necessarily disjoint union of affine schemes).

Recall that $A_n = A_n(K)$ is the number of effective divisors of degree n, which we've proved is finite. We have a formula when $n > 2g - 2$, namely

$$Z(t) = \sum_{n=0}^{\infty} A_n t^n = \sum_{D \in \mathrm{Div}^+(K)} t^{\deg(D)} \in \mathbb{Z}[[t]],$$

where Div^+ are the effective divisors and we've collected terms based on their degree. This is analogous to the Dedekind zeta function of a number field K, a formal Dirichlet series which is given by

$$\zeta_K(s) = \sum_{I \in \mathcal{I}(\mathbb{Z}_K^\bullet)} |\mathbb{Z}_K/I|^{-s}.$$

where the sum is now over all of the nonzero ideals of the ring of integers, where we measure the size using the *norm*, i.e. the size of the residue field. There's an analogy between integral ideals (vs fractional ideals) and effective divisors. We could get an Euler product decomposition for the Dedekind zeta function by only considering prime ideals, since in a Dedekind domain all ideals factor uniquely into prime ideals. In fact, any nonzero ideal is a linear combination of prime ideals. Similarly, the effective divisors are linear combinations of effective divisors, so an Euler product expansion is possible here too. If we take a prime ideal, since we're in a discrete valuation ring, we

can consider the local ring at that point. We can take the residue field, which in general won't be finite, but will be a finite extension. So a reasonable measure of the size of a prime divisor would be the dimension of its residue field as a vector space over K.

Note that if we wanted to make these look even more similar to each other, we could define a_n (depending on $\mathbb{Z}_K$) as

$$a_n = \left| \left\{ I \trianglelefteq \mathbb{Z}_K \mid |\mathbb{Z}_K/I| = n \right\} \right|,$$

which allows us to write

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Where we're going: How does $Z(t)$ depend on K? It turns out that it only depends on $A_0, A_1, \cdots, A_{2g-2}$, and thus $Z(t)$ depends on only finitely much information. We will

1. Show that $Z(t) \in \mathbb{Z}(t)$, i.e. it is a rational function and can be written $Z(t) = P(t)/Q(t)$.

   Note: the denominator will always be the same, $

2. Understand $\deg P$ and $\deg Q$ in terms of the genus g.