CRAG

The Weil Conjectures

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Background: Generating Functions

Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\ n \in \mathbb{Z}^{\geq 2}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I}=(i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^\times.$$

Problem: count points of a (smooth?) projective variety $X/\mathbb{F}$ in all degree n extensions of $\mathbb{F}$.

Definition

The *local zeta function* of X is the following formal power series:

$$Z_X(z) = \exp\left(\sum_{n=1}^\infty N_n \frac{z^n}{n}\right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note the following two properties:

$$Z_X(0) = 1$$

$$z\left(\frac{\partial}{\partial z}\right) \log Z_X(z) = z\left(\frac{Z_X'(z)}{Z_X(z)}\right) = \sum_{n=1}^\infty N_n z^n = N_1 z + N_2 z^2 + \cdots,$$

which is an *ordinary* generating function for the sequence (N_n).

> *The reason for defining this instead of an ordinary generating function will hopefully become more clear in examples. In particular, we are not losing information!*