

# Section 8.6: The Solutions of the Floer Equation are “Somewhere Injective”.

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## 0.1 Outline

Two Goals:

1. Prove critical points are discrete and regular points are open/dense.
2. Prove the continuation principle that was used in Proposition 8.1.4
3. Idea: A holomorphic function with all derivatives vanishing on a line is identically zero on  $\mathbb{C}$

## 0.2 Outline of Statements



- 8.1.5:  $(d\mathcal{F})_u$  is a Fredholm operator of index  $\mu(x) - \mu(y)$ .
- 8.1.4:  $\Gamma : W^{1,p} \times C_\varepsilon^\infty \longrightarrow L^p$  has a continuous right-inverse and is surjective
- 8.1.3:  $\mathcal{Z}(x, y, J)$  is a Banach manifold
- 8.1.1: For  $h \in \mathcal{H}_{\text{reg}}$ ,  $H_0 + h$  is nondegenerate and  $(d\mathcal{F})_u$  is surjective for every  $u \in \mathcal{M}(H_0 + h, J)$ .

- 8.1.2: For  $h \in \mathcal{H}_{\text{reg}}$  and all contractible orbits  $x, y$  of  $H_0$ ,  $\mathcal{M}(x, y, H_0 + h)$  is a manifold of dimension  $\mu(x) - \mu(y)$ .

### 0.3 Notation

- The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y \\ Y &\in u^*TW, \quad S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n})).\end{aligned}$$

- $z = s + it$
- $X$  is a vector field (time-dependent and periodic) on  $\mathbb{R}^{2n}$ ,  $J$  an almost complex structure  
–  $X, J$  are smooth
- $u \in C^\infty(\mathbb{R} \times S^1; W)$  is a solution to the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left( \frac{\partial u}{\partial t} - X(t, u) \right) = 0$$

- $C(u)$  the set of critical points and  $R(u)$  the set of regular points of  $u$ :

$$\begin{aligned}(s_0, t_0) \in C(u) \subseteq \mathbb{R} \times S^1 &\iff \frac{\partial u}{\partial s}(s_0, t_0) = 0 \\ (s_0, t_0) \in R(u) \subset \mathbb{R} \times S^1 &\iff (s_0, t_0) \notin C(u) \ \& \ s \neq s_0 \implies u(s_0, t_0) \neq u(s, t_0).\end{aligned}$$

### 0.4 Goal 1: Discrete Critical Points and Dense Regular Points

Goal 1: prove the following theorem

**Theorem 0.1(8.5.4).**

1.  $C(u)$  is discrete and
2.  $R(u) \hookrightarrow \mathbb{R} \times S^1$  is open and dense.

Outline of the proof:

- Prove 8.6.1 (direct, short) which transforms the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left( \frac{\partial u}{\partial t} - X(t, u) \right) = 0 \quad \text{where } u \in C^\infty(\mathbb{R} \times S^1; W)$$

to a Cauchy-Riemann equation on  $\mathbb{R}^2$ :

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0 \quad \text{where } v \in C^\infty(\mathbb{R}^2; W)$$

- Produces a map  $v$  which reduces Theorem 8.5.4 to two statements
  - 8.6.2:  $C(v)$  (and thus  $C(u)$ ) is discrete
  - \* Proved later using similarity principle.

- 8.6.3 (Injectivity): If  $v$  is a smooth periodic solution of CR with  $\frac{\partial v}{\partial s} \neq 0$  then  $R(v) \hookrightarrow \mathbb{R}^2$  is open and dense.
- Prove 8.6.3 (Injectivity)
  - Show open.
  - Show dense
- Prove 8.6.8 (similarity principle)
- Use similarity principle to prove 8.6.1, yields theorem.



## 0.5 8.6.1: Transform to Cauchy-Riemann

**Proposition 0.2(8.6.1, Transform to CR-equation on  $R^2$ ).**

If  $u$  is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t, u) \left( \frac{\partial u}{\partial t} - X(t, u) \right) = 0.$$

Then there exists

- An almost complex structure  $J_1$
- A diffeomorphism  $\varphi$  on  $W$  ?
- A map  $v \in C^\infty(\mathbb{R}^2; W)$

satisfying

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

where

1.  $v(s, t+1) = \varphi(v(s, t))$
2.  $C(u) = C(v)$ , i.e.  $u, v$  have the same critical points
3.  $R(u) = R(v)$ .

Proof

- Since  $W \times S^1$  is compact, the flow  $\psi_t$  of  $X_t$  is defined on all of  $W$ 
  - We thus have a map  $\psi_t : W \rightarrow W$  such that

$$\frac{\partial}{\partial t} \psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}$$

- Define the (important!) map

$$v(s, t) := (\psi_t^{-1} \circ u)(s, t)$$

- We can then compute

$$\begin{aligned} \frac{\partial u}{\partial s} &= (d\psi_t) \left( \frac{\partial v}{\partial s} \right) \\ \frac{\partial u}{\partial t} &= (d\psi_t) \left( \frac{\partial v}{\partial t} \right) + X_t(u). \end{aligned}$$

- Attempt at explanation: rearrange, use chain rule, and known derivative of  $\psi_t$ :

$$u(s, t) = (\psi_t \circ v)(s, t) \implies \frac{\partial u}{\partial s}(s, t) = \frac{\partial \psi_t}{\partial s}(v(s, t)) \cdot \frac{\partial v}{\partial s}(s, t)$$

$$? \implies \frac{\partial u}{\partial s} = (d\psi_t) \cdot \left( \frac{\partial v}{\partial s} \right)$$

and

$$\begin{aligned} \frac{\partial u}{\partial t}(s, t) &= \frac{\partial \psi_t}{\partial t}(v(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= (X_t \circ \psi_t)(v(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= (X_t \circ \psi_t \circ v)(s, t) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= X_t(u(s, t)) \cdot \frac{\partial v}{\partial t}(s, t) \\ &= X_t(u) \left( \frac{\partial v}{\partial t} \right) \dots???? \end{aligned}$$

- Continuing computations,

$$\begin{aligned}
0 &= \frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_t(u) \right) && \text{since } u \text{ is a solution} \\
&= \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} - J X_t(u) && \text{expanding terms} \\
&= \left( (d\psi_t) \left( \frac{\partial v}{\partial s} \right) \right) + J \left( (d\psi_t) \left( \frac{\partial v}{\partial t} \right) + X_t(u) \right) - J X_t(u) && \text{by substitution} \\
&= (d\psi_t) \left( \frac{\partial v}{\partial s} \right) + J(u) (d\psi_t) \left( \frac{\partial v}{\partial t} \right) && \text{cancelling} \\
&= (d\psi_t) \left( \frac{\partial v}{\partial s} + (d\psi_t)^{-1} J(u) (d\psi_t) \left( \frac{\partial v}{\partial t} \right) \right) && \text{collecting terms} \\
&:= (d\psi_t) \left( \frac{\partial v}{\partial s} + \psi_t^* J(v) \right) && \text{by definition.}
\end{aligned}$$

- Conclude that  $v$  is a solution of

$$\frac{\partial v}{\partial s} + \psi_t^* J(v) \frac{\partial v}{\partial t} = 0.$$

- Set  $\varphi := \psi_1$  and  $J_1(v) := \psi_1^* J(v)$  to obtain

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

of which  $v$  is still a solution

- Attempt to check directly, using  $\psi_{t+1} = \psi_t \circ \psi_1$ :

$$\begin{aligned}
v(s, t+1) &:= (\psi_t^{-1} \circ u)(s, t+1) \\
&= (\psi_1 \circ \psi_t^{-1} \circ u)(s, t) \\
&= \psi_1(v(s, t)) \\
&:= \varphi(v(s, t)),
\end{aligned}$$

which verifies property 1.

Note: just a guess from me!

- Recall definition of  $v$ :

$$v(s, t) := \psi_t^{-1}(u(s, t))$$

- Verifying that  $C(v) = C(u)$ : not spelled out. Property of flow?

– Need to check that

$$\frac{\partial u}{\partial s}(s_0, t_0) = 0 \implies \frac{\partial v}{\partial s}(s_0, t_0) = 0$$

where

$$\frac{\partial u}{\partial s} = (d\psi_t) \left( \frac{\partial v}{\partial s} \right)$$

- How: ?
- Verifying that  $R(v) = R(u)$ 
  - Need to check that for  $(s_0, t_0) \notin C(u)$  and  $s \neq s_0$  we have

$$u(s_0, t_0) \neq u(s, t_0) \implies v(s_0, t_0) \neq v(s, t_0)$$

- Follows directly:

$$\begin{aligned} v(s_0, t_0) \neq v(s, t_0) &\iff \psi_t^{-1}(u(s_0, t_0)) \neq \psi_t^{-1}(u(s, t_0)) \quad \text{by definition} \\ &\iff (\psi_t \circ \psi_t^{-1})(u(s_0, t_0)) \neq (\psi_t \circ \psi_t^{-1})(u(s, t_0)) \quad \text{injectivity of } \psi_t \\ &\iff u(s_0, t_0) \neq u(s, t_0). \end{aligned}$$

■

## 0.6 Splitting the Main Theorem

- The main theorem is equivalent to two upcoming statements

**Proposition 0.3 (8.6.2: Statement 1, Critical Points are Discrete).**

Let  $z = s + it$  where  $(s, t) \in \mathbb{R}^1 \times S^1$  (?). There exists a constant  $\delta > 0$  such that

$$0 < |z| < \delta \implies (dv)_z \neq 0.$$

*Proof.*

Postponed to p.264 because it relies on the 8.6.8 (Similarity Principle). ■

For the second statement, we set up some notation/definitions.

- $v \in C^\infty(\mathbb{R}^2; W)$  is a solution satisfying

$$\frac{\partial v}{\partial s} + J_1(v) \frac{\partial v}{\partial t} = 0$$

$$v(s, t+1) = \varphi(v(s, t))$$

$$C(v) = C(u), R(v) = R(u).$$

- The **regular points** are given by

$$R(v) = \left\{ (s, t) \in \mathbb{R}^2 \mid \frac{\partial v}{\partial s}(s, t) \neq 0, \quad v(s, t) \neq x^\pm(t), \quad v(s, t) \notin v(\mathbb{R} \setminus \{s\}, t) \right\}.$$

Note: last condition should be equivalent to  $s \neq s' \implies v(s, t) \neq v(s', t)$ . For reference, also equivalent to  $v(s, t) = v(s', t) \implies s = s'$ .

- **Multiple points** are defined as follows:

- Set  $\bar{\mathbb{R}} = \mathbb{R} \coprod \{\pm\infty\}$
- Extend  $v : \mathbb{R}^2 \rightarrow W$  to

$$v : \bar{\mathbb{R}} \times \mathbb{R} \rightarrow W$$

$$v(\pm\infty, t) = x^\pm(t).$$

- Define the set of *multiple points* as

$$M(s, t) := \left\{ (s', t) \in \mathbb{R}^2 \mid s \neq s' \in \bar{\mathbb{R}}, \quad v(s', t) = v(s, t) \right\}$$

Note that the same  $t$  is used throughout.

- Recast definition of  $R(v)$  as “points in the complement of  $C(v)$  that do not admit multiples”.
  - Potentially incorrect formulation:

$$R(v) = C(v)^c \bigcap_{(s, t) \in \bar{\mathbb{R}} \times \mathbb{R}} M(s, t)^c.$$

**Proposition 0.4(8.6.3: Regular Points Open/Dense, "Injectivity").**

Let  $v$  be a smooth solution of the Cauchy-Riemann equation, then

$$\left. \begin{array}{l} v(s, t+1) = \varphi(v(s, t)) \\ \frac{\partial v}{\partial s} \neq 0 \end{array} \right\} \implies R(v) \subseteq \mathbb{R}^2 \text{ is open and dense.}$$

*Proof (Long).*

Splits into two parts:

- Show  $R(v)$  is open (easy)
- Show  $R(v)$  is dense (delicate)

■

## 0.7 Regular Points Are Open

Proving the first part:  $R(v)$  is open.

- Want to show  $R(v)^c$  is closed.
- Toward a contradiction, suppose otherwise:  $R(v)^c$  is *open*.
  - Use limit point definition:  $R(v)^c$  is closed  $\iff$  it contains all of its limit points
  - So  $R(v)^c$  does *not* contain one of its limit points
  - Produces a sequence

$$R(v)^c \supseteq \{(s_n, t_n)\}_{n \in \mathbb{N}} \xrightarrow{n \rightarrow \infty} (s, t) \in R(v)$$

- The first two conditions defining  $R(v)$  are open conditions:
  - The two conditions:

$$\frac{\partial v}{\partial s}(s, t) \neq 0 \quad \text{Condition 1}$$

$$v(s, t) \neq x^\pm(t) \quad \text{Condition 2.}$$

- Thus for  $N \gg 1$  we have

$$n \geq N \implies \frac{\partial v}{\partial s}(s_n, t_n) \neq 0, \quad v(s_n, t_n) \neq x^\pm(t)$$

- But  $(s_n, t_n) \notin R(v)$  for such  $n$ , so they must fail the last condition: injectivity
  - Third condition:

$$s \neq s' \implies v(s, t) \neq v(s', t)$$

- Failing this conditions means:

$$\forall n > N, \exists s'_n \in \mathbb{R} \text{ s.t. } s'_n \neq s_n \text{ and } v(s_n, t_n) = v(s'_n, t_n).$$

- Produces a sequence  $\{s'_n\}_{n \in \mathbb{N}}$ , want to show it is bounded.



- Toward a contradiction, suppose not, then there is a subsequence

$$\{s_{n_k}\}_{n_k \in \mathbb{N}} \xrightarrow{n_k \rightarrow \infty} \pm\infty.$$

- This implies

$$\begin{aligned} v(s, t) &= \lim_{n_k \rightarrow \infty} v(s'_{n_k}, t'_{n_k}) \quad \text{using continuity of } v \\ &= v(\pm\infty, t) \\ &:= x^\pm(t). \end{aligned}$$

- Why? By definition, precisely because we extended  $v$  by setting  $v(\pm\infty, t) = x^\pm(t)$ .
- But condition 2 for points in  $R(v)$  says  $v(s, t) \neq x^\pm(t)$ , so this contradicts  $(s, t) \in R(v)$ .
- Sequence is bounded, so apply Bolzano-Weierstrass to extract a convergent subsequence converging to some limit:

$$\{s'_{n_j}\}_{n_j \in \mathbb{N}} \xrightarrow{n_j \rightarrow \infty} s'.$$

- Use the fact that injectivity failed:

$$\begin{aligned} \forall n, s'_n \neq s_n \quad \text{and} \quad v(s_n, t_n) &= v(s'_n, t_n) \\ \implies \lim_{n_k \rightarrow \infty} v(s_n, t_n) &= \lim_{n_k \rightarrow \infty} v(s'_n, t'_n) \\ \iff v(s, t) &= v(s', t) \quad \text{using continuity.} \end{aligned}$$

- Use the fact that because  $(s, t) \in R(v)$  we must have

$$s = s'.$$

- (Minor technical point) Now have  $\{s'_{n_j}\}_{n_j \in \mathbb{N}} \rightarrow s'$  and  $\{s_n\}_{n \in \mathbb{N}} \rightarrow s$ : since the latter sequence is convergent, every subsequence converges to the same limit, so  $\{s_{n_j}\}_{n_j \in \mathbb{N}} \rightarrow s$ .
- Again using failed injectivity, we have

$$s'_{n_j} \neq s_{n_j} \quad \text{and} \quad v(s_{n_j}, t_{n_j}) = v(s'_{n_j}, t_{n_j})$$

- By the last step, we do have

Lemma 8.6.4: For every  $r > 0$  there exists a  $\delta > 0$  such that

$$|t - t_0|, |s - s_0| < \delta \implies \exists s' \in B_r(s_j) \text{ s.t. } v(s, t) = v(s', t).$$

Proof: short.

Lemma 8.6.5: Let  $v_1, v_2$  be two solutions of the CR-equation with  $X_t \equiv 0$  on  $B_\varepsilon(0)$ ,  $v_1(0, 0) = v_2(0, 0)$  such that  $(dv_1)_0, (dv_2)_0 \neq 0$ . Also suppose

$$\begin{aligned} &\forall \varepsilon \exists \delta \text{ s.t.} \\ &\forall (s, t) \in B_\delta(0), \exists s' \in \mathbb{R} \begin{cases} (s', t) \in B_\varepsilon(0) \\ v_1(s, t) = v_2(s', t) \end{cases}. \end{aligned}$$

Then

$$\forall z \in B_\varepsilon(0), \quad v_1(s, t) = v_2(s, t).$$

Take perturbed CR equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Fix  $S \in C^\infty(\mathbb{R}^2; \text{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let  $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$  be a solution to the perturbed CR equation and let  $p > 2$ . Then there exists  $0 < \delta < \varepsilon$  and a map  $A \in W^{1,p}(B_\delta, \text{GL}(\mathbb{R}^{2n}))$  and a holomorphic map  $\sigma : B_\delta \rightarrow \mathbb{C}^n$  such that

$$\forall (s, t) \in B_\delta \quad Y(s, t) = A(s, t) \sigma(s + it) \quad \text{and} \quad J_0 A(s, t) = A(s, t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of  $\bar{\partial}$ ) For every  $p > 1$ , the following operator is surjective and Fredholm:

$$\bar{\partial} : W^{1,p}(S^2; \mathbb{C}^n) \rightarrow L^p(\Lambda^{0,1} T^* S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7

## 1 Goal 2: Continuation Principle

Goal 2: prove a continuation principle:

**Proposition 1.1 (8.6.6, Continuation Principle).**

On an open  $U \subset \mathbb{R}^2$ , let  $Y$  be a solution to the perturbed CR equation

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0$$

where  $J_0$  is the standard complex structure on  $\mathbb{R}^{2n}$  and  $S \in C^\infty(\mathbb{R}^2, \text{End}(\mathbb{R}^{2n}))$ .

Say that  $f$  has an *infinite-order zero* at  $z_0$  iff

$$\forall k \geq 0, \quad \sup_{|z-z_0| \leq t} \frac{|f(z)|}{r^k} \xrightarrow{r \rightarrow 0} 0.$$

For  $f$  smooth, equivalently  $f^{(k)}(z_0) = 0$  for all  $k$ .

Then the set

$$C := \{(s, t) \in U \mid Y \text{ has an infinite order zero at } (s, t)\}$$

is clopen. In particular, if  $U$  is connected and  $Y = 0$  on some nonempty  $V \subset U$ , then  $Y \equiv 0$ .

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**Proposition 1.2(8.1.4,).**

Define

$$\mathcal{Z}(x, y, J) := \{(u, H_0 + h) \mid h \in \mathcal{C}_\varepsilon^\infty(H_0) \text{ and } u \in \mathcal{M}(x, y, J, H)\}.$$

If  $(u, H_0 + h) \in \mathcal{Z}(x, y)$  then the following map admits a continuous right-inverse and is surjective:

$$\begin{aligned} \Gamma : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \times \mathcal{C}_\varepsilon^\infty(H_0) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \\ (Y, h) &\longmapsto \left(d\mathcal{F}^{H_0+h}\right)_u(Y) + \text{grad}_u h \end{aligned}$$

where  $\mathcal{F}^{H_0+h}$  is the Floer operator corresponding to  $H_+h$ .

Used to show (via the implicit function theorem) that  $\mathcal{Z}(x, y, J)$  is a Banach manifold when  $x \neq y$ .