Title

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We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for K/k of transcendence degree 1, all valuations on K which are trivial on k are discrete. We'll now want to go farther and describe the places $\Sigma(K/k)$, which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is K. Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if A is a Dedekind domain such that $\mathrm{ff}(A) = K$, then for all $\mathfrak{p} \in \mathrm{mSpec}(A)$ there exists a discrete valuation v_p on K. I.e., every maximal ideal induces a discrete valuation that is A-regular, so the valuation ring will contain A. How is this obtained? Take a nonzero $x \in K^\times$, and take the corresponding principal fractional ideal $\langle x \rangle := Ax$, which we can factor in a Dedekind domain as $Ax = \prod_{\mathfrak{p} \in \mathrm{mSpec}(A)} \mathfrak{p}^{\alpha_\mathfrak{p}}$ with $\alpha_\mathfrak{p} \in \mathbb{Z}$. This looks like an infinite product, but for any fixed x, only finitely many α are nonzero. Note that these α are exactly what we're looking for: the $\mathfrak{p}$-adic evaluation of x is given precisely by $v_\mathfrak{p}(x) := \alpha_\mathfrak{p}$, where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$\begin{aligned} v_\cdot : \mathrm{mSpec}(A) &\to \Sigma(K/A) \\ \mathfrak{p} &\mapsto v_\mathfrak{p}. \end{aligned}$$

So this sends a maximal ideal to a place that is A-regular, and it turns out to be a bijection.

Proposition 1.0.1 *(?).*
The map v is a bijection, and thus we may write

$$\Sigma(K/A) \cong \mathrm{mSpec}(A).$$

Proof (?).

Claim: v is injective.
If $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{mSpec}(A)$ are two different maximal ideals. Then there exists an element $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$, and so $x^{-1} \in A_{\mathfrak{p}_2} \setminus A_{\mathfrak{p}_1}$. This follows since if x is not in $\mathfrak{p}_2$, its $\mathfrak{p}_2$-adic valuation is zero, and thus the $\mathfrak{p}_2$-adic valuation of x^{-1} is $-0 = 0$ as well. On the other hand, since $x \in \mathfrak{p}_1$, its $\mathfrak{p}_1$-adic valuation is positive and therefore $v_{\mathfrak{p}_1}(x^{-1}) < 0$ and x^{-1} is not in $A_{\mathfrak{p}_1}$.

Claim: v is surjective.
Let $v \in \Sigma(K/A)$, so $A \subset R_v$, i.e. take a valuation whose valuation ring contains A. Note that we're not assuming the valuation is discrete, this can be a general Krull valuation, but we're trying to show it's equal to a certain p-adic valuation. As always with a subring of a valuation ring, we can pull back the maximal ideal and consider $\mathfrak{m}_v \cap A \in \mathrm{Spec}(A)$. We're hoping that this is a maximal ideal, since maximals correspond to valuations. Since we're in a Dedekind

domain, the only prime ideal we *don't* want this to be is the zero ideal of A, so suppose it were. Then $A^\bullet \subset R_v^\times$, and so $K^\times \subset R_v^\times$. This is because the only element of the maximal ideal that lies in A is zero, so every nonzero element of A is not in this maximal ideal and is thus a unit. But for any unit, its inverse is also a unit, yielding the inclusion $K^\times \subset R_v^\times$. The only way this could possibly happen is if $R_v = K$, which yields the trivial valuation ring. However, by definition, in $\Sigma(K/A)$ we've excluded the trivial valuation, so this ideal can not be zero.

So we can conclude that the pullback $\mathfrak{m}_v \cap A \in \mathrm{mSpec}(A)$, and so $A_\mathfrak{p} \subset R_v$. This is from viewing elements in $A_\mathfrak{p}$ as quotients of elements in A whose denominator have $\mathfrak{p}$-adic valuation zero. Recall that we want to show that $R_v = A_\mathfrak{p}$. We know $R_v \subset K$ is a proper containment, and we can use the fact that a *discrete* valuation ring is maximal among all proper subrings of its fraction field. In other words, for R a DVR, there is no ring R' such that $R \subset R' \subset \mathrm{ff}(R)$. How do you prove this? This is similar to an early exercise in commutative algebra, where we looked at all rings between $\mathbb{Z}$ and $\mathbb{Q}$, which generalized to looking at all rings between a PID and its fraction field, and a DVR is a local PID. So proving this statement is actually easier.

This is enough to show that $A_\mathfrak{p} = R_v$, and this $v \sim v_\mathfrak{p}$.

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Remark 1.0.2: What the idea? For a general one variable function field K/k, we'll produce Dede