Title

D. Zack Garza

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Lemma (Short Five):

Let R be a ring, then if we have the following diagram:

where

1. α, γ mono implies β is mono.
2. α, γ epi implies β is too.
3. α, γ is iso implies β is too.

Proof: Check

We say that two exact sequences are *isomorphic* if in the following diagram, f, g, h are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, "g"] & Q \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

Theorem: Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a SES. Then TFAE:

- There exists an R-module homomorphisms $h : M_3 \to M_2$ such that $g \circ h = \mathrm{id}_{M_3}$.
- There exists an R-module homomorphisms $k : M_2 \to M_1$ such that $k \circ f = \mathrm{id}_{M_1}$.
- The sequence is isomorphic to $0 \to M_1 \to M_1 \oplus M_3 \to M_3 \to 0$.

Proof: Define $\phi : M_1 \oplus M_3 \to M_2$ by $\phi(m_1 + m_2) = f(m_1) + h(m_2)$. We need to show that this diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & M_1 \oplus M_3 \arrow[r] & M_3 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] \arrow[u, "\phi"'] & M_3 \arrow[r] & 0
\end{tikzcd}
$$

We can check that $(g \circ \phi)(m_1 + m_2) = g(f(m_1)) + g(h(m_2)) = m_2 = \pi(m_1 + m_2)$. This yields $1 \implies 3$, and $2 \implies 3$ is similar.

To see that $3 \implies 1, 2$, we attempt to define k, h in the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] & M_1 \oplus M_3 \arrow[r] \arrow[l, bend right, "\pi_1"'] & M_3 \arrow[r] \arrow[d, leftrightarrow, "\mathrm{id}"] \arrow[l, bend right, "\iota_2"'] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] \arrow[u, "\phi"'] \arrow[l, bend left, "k"] & M_3 \arrow[r] \arrow[l, bend left, "h"] & 0
\end{tikzcd}
$$

So define $= \pi_1 \circ \phi^{-1}$ and $h = \phi \circ \iota_2$. It can then be checked that $g \circ h = g \circ \phi \circ \iota_2 = \pi_2 \circ \iota_2 = \mathrm{id}_{M_3}$.
$\square$

1.1 Free Modules

A free module is a module with a basis.

Definition: A subset $X = \{x_i\}$ is *linearly independent* iff $\sum r_i x_i = 0 \implies r_i = 0 \; \forall i$.

Definition: A subset X *spans* M iff $m \in M \implies m = \sum^n r_i x_i$.

Definition: A subset X is a basis

Example: $\mathbb{Z}_6$ is an abelian group and thus a $\mathbb{Z}$-module, but not free because $3 \curvearrowright [2] = [6] = 0$, so there are torsion elements.

This might contradict linear independence?

Theorem (Characterization of Free Modules): Let R be a unital ring and M a unital R-module (so $1 \curvearrowright m = m$). Then TFAE:

- There exists a nonempty basis of M.
- $M = \oplus_{i \in I} R$ for some index set I.
- There exists a non-empty set X and a map $\iota : X \hookrightarrow M$ such that given $f : X \to N$ for N any R- module, $\exists! \tilde{f} : M \to N$ such that the following diagram commutes.

$$
\begin{tikzcd}
M \arrow[dr, dotted, "\exists! \tilde{f}"] & \\
X \arrow[u, hook, "\iota"] \arrow[r, "f"] & N
\end{tikzcd}
$$

Definition: An R-module is *free* iff any of 1,2,3 hold.

Proof of 1 $\implies$ 2*:* Let X be a basis for M, then define $M \to \oplus_{x\in X} Rx$ by $\phi(m) = \sum r_i x_i$. It can be checked that

- This is an R-module homomorphism,
- $\phi(m) = 0 \implies r_j = 0 \; \forall j \implies m = 0$, so ϕ is injective,
- ϕ is surjective, since X is a spanning set.

So $M \cong \oplus_{x\in X} Rx$, so it only remains to show that $Rx \cong R$. We can define the map $R \xrightarrow{\pi_x} Rx$ by $r \mapsto rx$, then π_x is onto, and is injective exactly because X is a linearly independent set. Thus $M \cong \oplus R$.

Proof 1 $\implies$ 3*:* Let X be a basis, and suppose there are two maps $X \xrightarrow{\iota} M$ and $X \xrightarrow{f} M$. Then define $\tilde{f} : M \to N$ by $\sum r_i x_i \mapsto \sum r_i f(x_i)$. This is clearly an R-module homomorphism, and the diagram commutes because $(\tilde{f} \circ \iota)(x) = f(x)$. This is unique because $\tilde{f}$ is determined precisely by $f(X)$.o

Proof 3 $\implies$ 2*:* We use the usual "2 diagram" trick to produce a map $\tilde{f} : M \to \oplus_{x\in X} R$ and $\tilde{g} : \oplus_{x\in X} R \to M$, then commutativity forces $\tilde{f} \circ \tilde{g} = \tilde{g} \circ \tilde{f} = \mathrm{id}$.

Proof 2 $\implies$ 1*:* We have $M = \oplus_{i\in I} R$ by (2). So there exists a $\psi : \oplus_{i\in I} R \to M$, so let $X \coloneqq \{\psi(1_i) \mid i \in I\}$. The claim is that X is a basis. To see this is a basis, suppose $\sum r_i \psi(1_i) = 0$, then $\psi(\sum r_i 1_i) = 0$ and thus $\sum r_i 1_i = 0$ and $r_i = 0$ for all i. Checking that it's a spanning set: exercise. $\square$

Corollary: Every R-module is the homomorphic image of a free module.

Proof: Let M be an R-module, and let X be any set of generators of R. Then we can make a map $M \to \oplus x \in XR$ and there is a map $X \hookrightarrow M$, so the universal property provides $\tilde{f} : \oplus_{x\in X} R \to M$. Moreover, $\oplus_{x\in X} R$ is free.

Examples:

- $\mathbb{Z}_n$ is not a free $\mathbb{Z}$-module.
- If V is a vector space over a field k, then V is a free k-module (even if infinite dimensional).
- Every nonzero submodule of a free module over a PID is free.

Some facts:

Let $R = k$ be a field (or potentially a division ring).

1. Every maximal linearly independent subset is a basis for V.
2. Every vector space has a basis.
3. Every linearly independent set is contained in a basis
4. Every spanning set contains a basis.
5. Any two bases of a vector space have the same cardinality.

Theorem (Invariant Dimension): Let R be a commutative ring and M a free R-module. If X_1, X_2 are bases for R, then $|X_1| = |X_2|$.

Any ring satisfying this property is said to have the *invariant dimension property.*

> Note that it's difficult to say much more about generic modules, e.g. a finitely generated module may not have an invariant number of generators.