

Homological Algebra Problem Sets

Problem Set 3

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Problem 1.0.1 (Prove Corollary 2.3.2)

For R a PID, show that an R -module A is divisible if and only if A is injective.

Recall that a module is divisible if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.

Solution:

Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.

\Rightarrow : Suppose A is divisible, we then want to show every R -module morphism of the following form lifts, where we regard the ideal J and the ring R as R -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R -linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending R -linearly yields a well-defined R -module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

\Leftarrow : Suppose $A \in R\text{-Mod}$ is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \leq R$:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \rightarrow A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting \mathbf{a} range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)

Calculate $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q .

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).

Note that there is an injection

$$1 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \rightarrow \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \rightarrow 1$$

where π_n is the canonical quotient morphism, and applying the left-exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any \mathbb{Z} -module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since \mathbb{Z} is a free \mathbb{Z} -module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z} \\ \pi_n \downarrow & \nearrow \exists \tilde{f} & \\ \mathbb{Z}/n & & \end{array}$$

[Link to Diagram](#)

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0, 1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}.$$

Setting $f_i(1) := \left[\frac{i}{n} \right]$ (where we take the equivalence class mod \mathbb{Z}) yields n distinct morphisms $f_i : \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\begin{aligned} \Psi : \mathbb{Z} &\rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i, \end{aligned}$$

and using the fact that if $i = i' \pmod{n}$, write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left\lfloor \frac{i+kn}{n} \right\rfloor = \left\lfloor \frac{i}{n} + k \right\rfloor = \left\lfloor \frac{i}{n} \right\rfloor = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

Claim 2: \mathbb{Q}/\mathbb{Z} is an injective object in \mathbb{Z} -modules.

Proof (?).

By the previous exercise, it suffices to show that \mathbb{Q}/\mathbb{Z} is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That \mathbb{Q} is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b := \frac{a}{n} \in \mathbb{Q}$ works. Since \mathbb{Q} is an abelian group, \mathbb{Z} is automatically normal, and the result follows.

■

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \quad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).

Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M := \mathbb{Z}/m$,
- $A := \mathbb{Z}$,
- $I := n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n(\mathbb{Z}/m)} \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\rightarrow \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
 \ker \varphi &= \{x \in \mathbb{Z} \mid x(1 \otimes 1) = 0\} \\
 &= \{x \in \mathbb{Z} \mid n \mid x \text{ or } m \mid x\} \\
 &= \langle n, m \rangle \\
 &= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
 &:= \langle d \rangle.
 \end{aligned}$$

Now applying the first isomorphism theorem yields the result. ■

Solution:

We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\cdot) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \cdot)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of \mathbb{Z}/q .
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbb{Z}/q & \xrightarrow{d^{-1}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & \mathbb{Q}/\mathbb{Z} \xrightarrow{d^1} 1 \\
 & & & & & & \\
 & & [1]_q & \longrightarrow & \begin{bmatrix} 1 \\ q \end{bmatrix} & &
 \end{array}$$

$$[x] \longrightarrow [qx]$$

[Link to Diagram](#)

This is a chain complex by construction, since $d^2([1]_q) = \begin{bmatrix} q \left(\frac{1}{q} \right) \end{bmatrix} = [1] = [0]$. We now delete the augmentation and apply $F(\cdot)$:

$$\begin{array}{ccccccc}
1 & \longrightarrow & I^0 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & I^1 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^1} & 1 \\
& & \downarrow F(\cdot) & & & & \\
1 & \longrightarrow & F(I^0) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^0 := F(d^0)} & F(I^1) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^1 := F(d^1)} & 1 \\
\parallel & & \uparrow \Psi \cong & & \uparrow \Psi \cong & & \parallel \\
1 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^0} & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^1} & 1
\end{array}$$

[Link to Diagram](#)

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q , we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q .

We now take homology:

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\text{im } \partial^0} = \frac{\mathbb{Z}/p}{q(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)

Let $A \in \mathbf{Ab}$, and show that the following map is injective:

$$\begin{aligned}
\varepsilon_A : A &\rightarrow I(A) := \prod_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\
a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},
\end{aligned}$$

i.e. when looking at the image $\varepsilon_A(a)$ in the product, the component indexed by f is an element of \mathbb{Q}/\mathbb{Z} obtained by evaluating $f(a)$.

Hint: if $a \in A$, find a map $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$.

Solution:

By contrapositive, we'll suppose $a \neq 0$ and show $\varepsilon_A(a) \neq 0$. Following the hint, we first consider the cyclic subgroup $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$ and define a map

$$\begin{aligned}
f_a : a\mathbb{Z} &\rightarrow \mathbb{Z} \\
an &\mapsto n.
\end{aligned}$$

We now pick $\ell > 1 \in \mathbb{Z}$ to be any integer, and define a composition $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$:

$$\begin{array}{ccccccc}
 a\mathbb{Z} & \xrightarrow{f_a} & \mathbb{Z} & \xleftarrow{\iota} & \mathbb{Q} & \xrightarrow{x \mapsto \frac{x}{\ell}} & \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \\
 & \nearrow f & & & & & \nearrow f \\
 & & & & & & \\
 an & \longmapsto & n & \longmapsto & n & \longmapsto & \frac{n}{\ell} \longmapsto \left[\frac{n}{\ell} \right] \\
 & & & & & & \\
 a & \xrightarrow{\quad\quad\quad} & & & & & \left[\frac{1}{\ell} \right]
 \end{array}$$

[Link to Diagram](#)

By choice of ℓ , this map satisfies $f(a) = [1/\ell] \neq 0$, so the map is nonzero. Since \mathbb{Q}/\mathbb{Z} is injective, the universal property provides a lift \tilde{f} :

$$\begin{array}{ccc}
 A & & \\
 \uparrow & \searrow \exists \tilde{f} & \\
 a\mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z}
 \end{array}$$

[Link to Diagram](#)

Since \tilde{f} lifts f , it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) := f(a) \neq 0,$$

so the f component of the image of a is nonzero and thus $\mathbf{a} := \varepsilon_A(a) \neq 0$ in the product.

Problem 1.0.4 (Weibel 2.4.2)

If $U : \mathcal{B} \rightarrow \mathcal{C}$ is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Solution:

We'll show that $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$ for every object X . To compute left-derived functors, we'll take projective resolutions, so let P be a projective resolution of X . Fixing labeling, we have the following situation:

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X \xrightarrow{0} 0 \\
 & & & & & \downarrow F(\cdot) & & & \\
 \cdots & \longrightarrow & FP_2 & \xrightarrow{F(\partial_2)} & FP_1 & \xrightarrow{F(\partial_1)} & FP_0 & \xrightarrow{0} & 0
 \end{array}$$

[Link to Diagram](#)

We now have by definition

$$L_i F(X) := \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U(L_i F(X)) := U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

Problem 1.0.5 (Weibel 2.4.3)

If $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$ is exact with P projective or F -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

Show that $L_{m+1} F(A)$ is the kernel of $F(M_m) \rightarrow F(P_m)$. Conclude that if $P \rightarrow A$ is an F -acyclic resolution of A , then $L_i F(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

- a. A is a projective R -module.
- b. $\operatorname{Hom}_R(\cdot, A)$ is an exact functor.
- c. $\operatorname{Ext}_R^{i \neq 0}(A, B) = 0$ and for all B , i.e. A is $\operatorname{Hom}_R(\cdot, B)$ -acyclic for all B .
- d. $\operatorname{Ext}_R^1(A, B)$ vanishes for all B .

Problem 1.0.7 (Weibel 2.6.4)

Show that colim is left adjoint to Δ , and conclude that colim is right-exact when \mathcal{A} is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on \mathbf{Ab} .