

# Title

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January 3, 2020

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## 1 Appendix

An alternative characterization of **uniform continuity**:

$$\|\tau_y f - f\|_u \rightarrow 0 \text{ as } y \rightarrow 0$$

Lemma: Measurability is not preserved by homeomorphisms.

Counterexample: there is a homeomorphism that takes that Cantor set (measure zero) to a fat Cantor set

### 1.1 Undergraduate Analysis Review

- Some inclusions on the real line:

Differentiable with a bounded derivative  $\subset$  Lipschitz continuous  $\subset$  absolutely continuous  
 $\subset$  uniformly continuous  $\subset$  continuous  
Proofs: Mean Value Theorem, Triangle inequality, Definition of absolute continuity  
specialized to one interval, Definition of uniform continuity

- **Bolzano-Weierstrass**: Every bounded sequence has a convergent subsequence.
- **Heine-Borel**:

$$X \subseteq \mathbb{R}^n \text{ is compact} \iff X \text{ is closed and bounded.}$$

- **Baire Category Theorem**: If  $X$  is a complete metric space, then

- For any sequence  $\{U_k\}$  of open, dense sets,  $\bigcap_k U_k$  is also dense.
- $X$  is *not* a countable union of nowhere-dense sets
- **Nested Interval Characterization of Completeness:**  $\mathbb{R}$  being complete  $\implies$  for any sequence of intervals  $\{I_n\}$  such that  $I_{n+1} \subseteq I_n$ ,  $\bigcap I_n \neq \emptyset$ .
- **Convergence Characterization of Completeness:**  $\mathbb{R}$  being complete is equivalent to “absolutely convergent implies convergent” for sums of real numbers.
- Compact subsets  $K \subseteq \mathbb{R}^n$  are also *sequentially compact*, i.e. every sequence in  $K$  has a convergent subsequence.
- **Urysohn’s Lemma:** For any two sets  $A, B$  in a metric space or compact Hausdorff space  $X$ , there is a function  $f : X \rightarrow I$  such that  $f(A) = 0$  and  $f(B) = 1$ .
- Continuous compactly supported functions are
  - Bounded almost everywhere
  - Uniformly bounded
  - Uniformly continuous

*Proof:*

- Uniform convergence allows commuting sums with integrals
- Closed subsets of compact sets are compact.
- Every compact subset of a Hausdorff space is closed
- Showing that a series converges: (*Todo*)

## 1.2 Big Counterexamples

### 1.2.1 For Limits

- Differentiability  $\implies$  continuity but not the converse:
  - The Weierstrass function is continuous but nowhere differentiable.
- $f$  continuous does not imply  $f'$  is continuous:  $f(x) = x^2 \sin(1/x)$ .
- Limit of derivatives may not equal derivative of limit:

$$f(x) = \frac{\sin(nx)}{n^c} \text{ where } 0 < c < 1.$$

- Also shows that a sum of differentiable functions may not be differentiable.
- Limit of integrals may not equal integral of limit:

$$\sum \mathbb{1}[x = q_n \in \mathbb{Q}].$$

- A sequence of continuous functions converging to a discontinuous function:

$$f(x) = x^n \text{ on } [0, 1].$$

- The Thomae function (*todo*)

### 1.2.2 For Convergence

- Notions of convergence:
  1. Uniform
  2. Pointwise
  3. Almost everywhere
  4. In norm

Uniform  $\implies$  pointwise  $\implies$  almost everywhere.

See Section 17.3.

**Almost everywhere convergence does not imply  $L^p$  convergence for any  $1 \leq p \leq \infty$**

See notes section 1

Sequences  $f_k \xrightarrow{a.e.} f$  but  $f_k \not\xrightarrow{L^p} f$ :

- For  $1 \leq p < \infty$ : The skateboard to infinity,  $f_k = \chi_{[k, k+1]}$ .

Then  $f_k \xrightarrow{a.e.} 0$  but  $\|f_k\|_p = 1$  for all  $k$ .

Converges pointwise and a.e., but not uniformly and not in norm.

- For  $p = \infty$ : The sliding boxes  $f_k = k \cdot \chi_{[0, \frac{1}{k}]}$ .

Then similarly  $f_k \xrightarrow{a.e.} 0$ , but  $\|f_k\|_p = 1$  and  $\|f_k\|_\infty = k \rightarrow \infty$

Converges a.e., but not uniformly, not pointwise, and not in norm.

**The Converse to the DCT does not hold**

$L^p$  boundedness does not imply a.e. boundedness.

I.e. it is not true that  $\lim \int f_k = \int f$  implies that  $\exists g \in L^p$  such that  $f_k < g$  a.e. for every  $k$ .

Take

- $b_k = \sum_{j=1}^k \frac{1}{j} \rightarrow \infty$
- $f_k = \chi_{[b_k, b_{k+1}]}$

Then

- $f_k \xrightarrow{a.e.} f = 0$ ,
- $\int f_k = \frac{1}{k} \rightarrow 0 \implies \|f_k\|_p \rightarrow 0$ ,

- $0 = \int f = \lim \int f_k = 0$
- But  $g > f_k \implies g > \|f_k\|_\infty = 1$  a.e.  $\implies g \notin L^p(\mathbb{R})$ .

### 1.3 Errata

- **Equicontinuity:** If  $\mathcal{F} \subset C(X)$  is a family of continuous functions on  $X$ , then  $\mathcal{F}$  *equicontinuous* at  $x$  iff

$$\forall \varepsilon > 0 \exists U \ni x \text{ such that } y \in U \implies |f(y) - f(x)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

- **Arzela - Ascoli 1:** If  $\mathcal{F}$  is pointwise bounded and equicontinuous, then  $\mathcal{F}$  is totally bounded in the uniform metric and its closure  $\overline{\mathcal{F}} \in C(X)$  in the space of continuous functions is compact.
- **Arzela - Ascoli 2:** If  $\{f_k\}$  is pointwise bounded and equicontinuous, then there exists a continuous  $f$  such that  $f_k \xrightarrow{u} f$  on every compact set.

**Example:** Using Fatou to compute the limit of a sequence of integrals:

$$\lim_{n \rightarrow \infty} \int_0^\infty \frac{n^2}{1 + n^2 x^2} e^{-\frac{x^2}{n^3}} dx \stackrel{\text{Fatou}}{\geq} \int_0^\infty \lim_{n \rightarrow \infty} \frac{n^2}{1 + n^2 x^2} e^{-\frac{x^2}{n^3}} dx \rightarrow \int \infty.$$

Note that MCT might work, but showing that this is non-decreasing in  $n$  is difficult.

**Lemma:**

$$f_k \xrightarrow{a.e.} f, \quad \|f_k\|_p \leq M \implies f \in L^p \text{ and } \|f\|_p \leq M.$$

*Proof:* Apply Fatou to  $|f|^p$ :

$$\int |f|^p = \int \liminf |f_k|^p \leq \liminf \int |f_k|^p = M.$$

**Lemma:** If  $f$  is uniformly continuous, then

$$\|\tau_h f - f\|_p \xrightarrow{L^p} 0 \quad \text{for all } p.$$

**Lemma:**  $\|\tau_h f - f\|_p \rightarrow 0$  for every  $p$ .

- i.e. “Continuity in  $L^1$ ” holds for all  $L^p$ .
- i.e. Translation operators are continuous.

*Proof:* Take  $g_k \in C_c^0 \rightarrow f$ , then  $g$  is uniformly continuous, so

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p \rightarrow 0.$$

**Lemma:** For  $f \in L^p, g \in L^q$ ,  $f * g$  is uniformly continuous.

*Proof:* Use Young's inequality

$$\|\tau_h(f * g) - f * g\|_\infty = \|(\tau_h f - f) * g\|_\infty \leq \|\tau_h f - f\|_p \|g\|_q \rightarrow 0.$$

**Lemma:** If  $\int f \phi = 0$  for every  $\phi \in C_c^0$ , then  $f = 0$  almost everywhere.

*Proof:* Let  $A$  be an interval, choose  $\phi_k \rightarrow \chi_A$ , then  $\int f \chi_A = 0$  for all intervals. So this holds for any Borel set  $A$ . Then just take  $A_1 = \{f > 0\}$  and  $A_2 = \{f < 0\}$ , then  $\int_{\mathbb{R}} f = \int_{A_1} f + \int_{A_2} f = 0$ .

## 1.4 The Fourier Transform

Some Useful Properties:

$$\begin{aligned}\widehat{f * g}(\xi) &= \hat{f}(\xi) \cdot \hat{g}(\xi) \\ \widehat{\tau_h f}(\xi) &= e^{2\pi i \xi \cdot h} \hat{f}(\xi) \\ e^{2\pi i \xi \cdot h} \widehat{f}(\xi) &= \widehat{\tau_{-h} f}(\xi) \\ \widehat{f \circ T}(\xi) &= |\det T|^{-1} (\hat{f} \circ T^{-t})(\xi) \\ \frac{\partial}{\partial \xi} \hat{f}(\xi) &= -2\pi i \cdot \xi \hat{f}(\xi) \\ \widehat{\frac{\partial}{\partial \xi} f}(\xi) &= 2\pi i \xi \cdot \hat{f}(\xi).\end{aligned}$$

Some Useful Transform Pairs:

Dirichlet:.