

# Title

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## 1 Chapter 1

### 1.1 Within Chapter

**Proposition 1.1:** Fix an ideal  $\mathfrak{a} \trianglelefteq R$ . There is a correspondence

$$\left\{ \mathfrak{b} \mid \mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R \right\} \iff \left\{ \tilde{\mathfrak{b}} \trianglelefteq R/\mathfrak{a} \right\}.$$

*Proof:* Adapted from proof for groups here: <https://math.stackexchange.com/a/955413/147053>.

Let  $f : R \rightarrow T$  be any ring homomorphism and let  $S(R), S(T)$  denote the lattices of subrings of  $R, T$  respectively. Then  $f$  induces two maps:

$$\begin{aligned} F : S(R) &\rightarrow S(T) \\ H &\mapsto f(H) \end{aligned}$$

$$\begin{aligned} F^{-1} : S(T) &\rightarrow S(R) \\ K &\mapsto f^{-1}(K). \end{aligned}$$

It follows that

- $H \leq R \implies F(H) \leq \text{im } f$ , by the subring test
  - Subring test: contains 1, closed under multiplication/subtraction.
  - Properties of ring homomorphisms:  $f(sa + b) = sf(a) + f(b)$  and  $f(1) = 1$ .
  - Thus if  $f$  is not surjective,  $F$  is not surjective either.

- $K \leq T \implies \ker f \subseteq F^{-1}(K)$ .
  - Follows because subrings contain 0, and  $H \in \ker F \implies f(H) = 0_T \in K$ .
  - Thus if there is any subring  $H$  that *doesn't* contain  $\ker f$ ,  $F^{-1}$  is not surjective.

The claim is that if you restrict to

- $S'(R) := \{H \leq R \mid \ker f \subseteq H\}$  and
- $S'(T) := \{K \leq T \mid K \subseteq \operatorname{im} f\}$ ,

this is a bijection.

This follows from the fact that

- $(F \circ F^{-1})(K) = K \cap \operatorname{im} f \leq T$ 
  - No clear motivation for why it's *this* specific thing, but the inclusions are easy to check.
- $(F^{-1} \circ F)(H) = \langle H, \ker f \rangle \leq S$ .
  - Inclusions easy to check, need to take subring generated since  $F(H)$  is a pushforward/direct image, which don't preserve sub-structures in general.

So we take the projection  $f = \pi : R \rightarrow R/\mathfrak{a}$ , then

- $K \subseteq \operatorname{im} \pi \implies K \cap \operatorname{im} \pi = K \implies (F \circ F^{-1})(K) = K$ ,
- $\ker \pi \subseteq H \implies \langle H, \ker \pi \rangle = H \implies (F^{-1} \circ F)(H) = H$ ,

so both directions are surjections. Restricting to just those subrings that are ideals preserves this bijection. Moreover,  $\ker \pi = \mathfrak{a}$  so  $S'(R)$  is the set of ideals containing  $\mathfrak{a}$ , and  $\operatorname{im} \pi = R/\mathfrak{a}$ , so  $S'(T)$  is the set of ideals of the quotient. ■

### Proposition 1.2: TFAE

1.  $R$  is a field
2.  $R$  is simple, i.e. the only ideals of  $R$  are  $0, R$ .
3. Every nonzero homomorphism  $\phi : R \rightarrow S$  for  $S$  an arbitrary ring is injective.

*Proof:*

**Lemma:**  $I \trianglelefteq R$  and  $1 \in I \implies I = R$ . This is because  $RI \subseteq I$ , and  $r \in R \implies r \cdot 1 \in I \implies r \in I \implies R \subseteq I$ .

$1 \implies 2$ :

Let  $0 \neq I \trianglelefteq R$  for  $R$  a field, then pick any  $x \in I$ , since  $x^{-1} \in R$ , we have  $x^{-1}x = 1 \in I \implies I = R$ .

$\nexists \implies \exists$ :

If  $R$  is not a field, pick a non-unit element  $r$ ; then  $(r) \trianglelefteq R$  is a proper ideal.

$2 \implies 3$ :

$\ker \phi \trianglelefteq R$  is an ideal, so  $\ker \phi = 0$ .

$3 \implies 2$ :

Take  $\mathfrak{a} \trianglelefteq R$  a proper ideal and let  $S = R/\mathfrak{a}$  with  $\phi : R \rightarrow S$  the projection.  $\phi$  is a bijection, since it's always a surjection and assumed injective. So  $R \cong S = R/\mathfrak{a}$ , forcing  $\mathfrak{a} = (0)$ .

■

**Proposition:** If  $\mathfrak{m} \subseteq R$  is maximal iff  $R/\mathfrak{m}$  is a field.

*Proof:*

$R/\mathfrak{m}$  is a field  $\iff R/\mathfrak{m}$  is simple  $\iff$  there are no nontrivial ideals  $\mathfrak{a}$  such that  $\mathfrak{m} \subset \mathfrak{a}$   
(correspondence)  $\iff \mathfrak{m}$  is maximal.

**Proposition:**  $\mathfrak{p} \subseteq R$  is prime iff  $R/\mathfrak{p}$  is a domain.

*Proof:*

**Proposition:** Maximal ideals are prime.

*Proof:* Let  $\mathfrak{m} \subseteq A$  be maximal, then  $R/\mathfrak{m}$  is simple and thus a field, so  $\mathfrak{m}$  is prime.

**Proposition:** Prime does not imply maximal in general.

*Proof:* Take  $(0) \in \mathbb{Z}$ , then  $ab = 0 \implies a = 0$  or  $b = 0$ , so this is prime. It is not maximal, because  $(0) \in (n)$  for any  $n$ .

Theorem 1.3: Every ring  $R$  has a nontrivial maximal ideal  $I \neq 0$ , and every ideal is contained in a maximal ideal.

*Proof:* ?

Corollary 1.5: Every non-unit of  $R$  is contained in a maximal ideal.

*Proof:* ?

Proposition 1.6: If  $A \setminus \mathfrak{m} \subset R^\times$ , then  $A$  is a local ring with  $\mathfrak{m}$  its maximal ideal. If  $\mathfrak{m}$  is maximal and  $1 + m \in R^\times$  for all  $m \in \mathfrak{m}$ , then  $A$  is a local ring.

*Proof:* ?

Proposition: If  $f \in k[x_1, \dots, x_n]$  is irreducible over  $k$ , then  $(f)$  is prime.

Proposition:  $\mathbb{Z}$  is a PID, and  $(p)$  is prime iff  $p$  is zero or a prime number, and every such ideal is maximal.

Proposition:  $k[\{x_i\}]$  has maximal ideals that are not principal iff  $n > 1$ .

Exercise: Characterize the maximal and prime ideals of  $k[x_1, \dots, x_n]$ ? Is this a field, domain, PID, UFD, a local ring, ...?

Proposition: Every nonzero prime ideal in a PID is maximal.

*Proof:* ?

Definition: The set  $\text{nil}(A)$  of all nilpotent elements in a ring  $A$  is the nilradical of  $A$ . The set  $J(A) = \bigcap_{\mathfrak{m} \in \text{Spec}_{\max}(A)} \mathfrak{m}$  is the Jacobson radical.

Proposition 1.7:  $\text{nil}(A) \subseteq R$  is an ideal and  $A/\mathfrak{N}$  has no nonzero nilpotent elements.

*Proof:* ?

Proposition 1.8:  $\text{nil}(A) = \bigcap_{\mathfrak{p} \in \text{Spec}(A)} \mathfrak{p}$  is the intersection of all prime ideals of  $A$ .

*Proof:* ?

Proposition 1.9:  $x \in J(A)$  iff  $1 - xa \in A^\times$  for all  $a \in A$ .

Proposition: If  $(m), (n) \trianglelefteq \mathbb{Z}$  then  $(m) \cap (n) = (\gcd(m, n))$  and  $(m)(n) = (mn)$ .

Exercise: If  $\mathfrak{a} \trianglelefteq k[x_1, \dots, x_m]$ , characterize  $\mathfrak{a}^n$ .

Exercise: Show that  $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$  are coprime iff there exist  $a \in \mathfrak{a}, b \in \mathfrak{b}$  such that  $a + b = 1$ .

Proposition 1.10: Let  $\{mfa_i\} \trianglelefteq A$  be a family of ideals and define  $\phi : A \rightarrow \prod A/\mathfrak{a}_i$ .

1. If  $\{\mathfrak{a}_i\}$  are pairwise coprime, then  $\prod \mathfrak{a}_i = \bigcap \mathfrak{a}_i$
2.  $\phi$  is surjective iff  $\{\mathfrak{a}_i\}$  are pairwise coprime.
3.  $\phi$  is injective iff  $\bigcap \mathfrak{a}_i = (0)$ .

Exercise: Show that the union of ideals is not necessarily an ideal.

Proposition 1.11:

- a. Let  $\{\mathfrak{p}_i\}$  be a set of prime ideals and let  $\mathfrak{a} \in \bigcup \mathfrak{p}_i$ . Then  $\mathfrak{a} \subseteq \mathfrak{p}_i$  for some  $i$ .
- b. Let  $\{\mathfrak{a}_i\}$  be ideals and  $\mathfrak{p} \supseteq \bigcap \mathfrak{a}_i$  be prime.  $\mathfrak{p} \supseteq \mathfrak{a}_i$  for some  $i$ , and if  $\mathfrak{p} = \bigcap \mathfrak{a}_i$ , then  $\mathfrak{p} = \mathfrak{a}_i$  for some  $i$ .

Exercise: Let  $A = \mathbb{Z}$ , and characterize the ideal quotient  $(m : n)$ .

Exercise 1.12:

1.  $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$
2.  $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$
3.  $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
4.  $(\bigcap \mathfrak{a}_i : \mathfrak{b}) = \bigcap (\mathfrak{a}_i : \mathfrak{b})$
5.  $(\mathfrak{a} : \sum \mathfrak{b}_i) = \bigcap (\mathfrak{a} : \mathfrak{b}_i)$

Proposition: For  $\mathfrak{a} \trianglelefteq A$ ,  $\sqrt{\mathfrak{a}}$  is an ideal.

Exercise 1.13:

1.  $\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$
2.  $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$
3.  $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$
4.  $\sqrt{\mathfrak{a}} = (1) \iff \mathfrak{a} = (1)$
5.  $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$ .
6. For  $\mathfrak{p}$  prime,  $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$  for all  $n \geq 1$ .

Proposition 1.14:  $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p}$

Proposition 1.15: Let  $D$  be the set of zero-divisors in  $A$ . Then  $D = \bigcup_{x \neq 0} \sqrt{\text{Ann}(x)}$ .

Exercise: Let  $(m) \trianglelefteq \mathbb{Z}$  where  $m = \prod p_i^{k_i}$ , and show that  $\sqrt{(m)} = (p_1 p_2 \cdots) = \bigcap (p_i)$ .

Proposition 1.16: If  $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$  are coprime then  $\mathfrak{a}, \mathfrak{b}$  are coprime.

Exercise: Show that if  $f : A \rightarrow B$  and  $\mathfrak{a} \trianglelefteq A$ , it is not necessarily the case that  $f(\mathfrak{a}) \trianglelefteq B$ .

Exercise: Show that if  $\mathfrak{b}$  is prime then  $A \cdot f^{-1}(\mathfrak{b})$  is prime, but if  $\mathfrak{a}$  is prime then  $B \cdot f(\mathfrak{a})$  need not be prime.

Exercise: Write  $\mathfrak{a}^e := \langle f(\mathfrak{a}) \rangle$  and  $\mathfrak{b}^c = \langle f^{-1}(\mathfrak{b}) \rangle$ . Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}[i]$  be the inclusion, and show that

- $(2)^e = \langle (1+i)^2 \rangle$ , which is not prime in  $\mathbb{Z}[i]$
- (Nontrivial) If  $p \equiv 1 \pmod{4}$ , then  $\mathfrak{p}^e$  is the product of two distinct prime ideals
- If  $p \equiv 3 \pmod{4}$  then  $\mathfrak{p}^e$  is prime.

Proposition: Let  $C = \{\mathfrak{b}^c \mid \mathfrak{b} \trianglelefteq B\}$  and  $E = \{\mathfrak{a}^e \mid \mathfrak{a} \trianglelefteq A\}$ . Then

1.  $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$  and  $\mathfrak{b} \supset \mathfrak{b}^{ce}$ ,
2.  $\mathfrak{b}^c = \mathfrak{b}^{cec}$  and  $\mathfrak{a}^e = \mathfrak{a}^{ece}$
3.  $C = \{\mathfrak{a} \trianglelefteq A \mid \mathfrak{a}^{ec} = \mathfrak{a}\}$  and  $E = \{\mathfrak{b} \trianglelefteq B \mid \mathfrak{b}^{ce} = \mathfrak{b}\}$ .
4. The map  $\phi : C \rightarrow E$  given by  $\phi(\mathfrak{a}) = \mathfrak{a}^{ec}$  is a bijection with inverse  $\mathfrak{b} \mapsto \mathfrak{b}^c$ .
5. If  $\mathfrak{a} \in C$  then  $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b}^{cec} = \mathfrak{a}^{ec}$ , and if  $\mathfrak{a} = \mathfrak{a}^{ec}$  then  $\mathfrak{a}$  is the contraction of  $\mathfrak{a}^e$ .

Exercise 1.18:

$$\begin{array}{ll}
 (\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e, & (\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c \\
 (\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e, & (\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c \\
 (\mathfrak{a}_1 \mathfrak{a}_2)^e = \mathfrak{a}_1^e \mathfrak{a}_2^e, & (\mathfrak{b}_1 \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c \mathfrak{b}_2^c \\
 (\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e), & (\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c) \\
 r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e), & r(\mathfrak{b})^c = r(\mathfrak{b}^c)
 \end{array}$$

## 1.2 End of Chapter Exercises