

# Title

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## 1 Thursday, September 10

Recall that the dimension of a ring  $R$  is the length of the longest chain of prime ideals. Similarly, for an affine variety  $X$ , we defined  $\dim X$  to be the length of the longest chain of irreducible closed subsets.

These notions of dimension of the same when taking  $R = A(X)$ , i.e.  $\dim \mathbb{A}^n/k = n$ .

### Proposition 1.1 (*Dimensions*).

Let  $k = \bar{k}$ .

- The dimension of  $k[x_1, \dots, x_n]$  is  $n$ .
- All maximal chains of prime ideals have length  $n$ .

### 1.1 Proof of Dimension Proposition

The case for  $n = 0$  is trivial, just take  $P_0 = \langle 0 \rangle$ . For  $n = 1$ , easy to see since the only prime ideals in  $k[x]$  are  $\langle 0 \rangle$  and  $\langle x - a \rangle$ , since any polynomial factors into linear factors.

Let  $P_0 \subsetneq \dots \subsetneq P_m$  be a maximal chain of prime ideals in  $k[x_1, \dots, x_n]$ ; we then want to show that  $m = n$ . Assume  $P_0 = \langle 0 \rangle$ , since we can always extend our chain to make this true (using maximality). Then  $P_1$  is a minimal prime and  $P_m$  is a maximal ideal (and maximals are prime).

**Claim:**  $P_1$  is principle, i.e.  $P_1 = \langle f \rangle$  for some irreducible  $f$ .

*Proof.*

**Claim:**  $k[x_1, \dots, x_n]$  is a unique factorization domain. This follows since  $k$  is a UFD since it's a field, and  $R$  a UFD  $\implies R[x]$  is a UFD for any  $R$ .

See Gauss' lemma.

**Claim:** In a UFD, minimal primes are principal. Let  $r \in P$ , and write  $r = u \prod p_i^{n_i}$  with  $p_i$  irreducible and  $u$  a unit. So some  $p_i \in P$ , and  $p_i$  irreducible implies  $\langle p_i \rangle$  is prime. Since  $0 \subsetneq \langle p_i \rangle \subset P$ , but  $P$  was prime and assumed minimal, so  $\langle p_i \rangle = P$ .

The idea is to now transfer the chain  $P_0 \subsetneq \cdots \subsetneq P_m$  to a maximal chain in  $k[x_1, \dots, x_{n-1}]$ . The first step is to make a linear change of coordinates so that  $f$  is monic in the variable  $x_n$ .

**Example .**

Take  $f = x_1x_2 + x_3^2x_4$  and map  $x_3 \mapsto x_3 + x_4$ .

So write

$$f(x_1, \dots, x_n) = x_n^d + f_1(x_1, \dots, x_{n-1})x_n^{d-1} + \cdots + f_d(x_1, \dots, x_{n-1}).$$

We can then descend to  $k[x_1, \dots, x_n]$  to  $k[x_1, \dots, x_n]/\langle f \rangle$ :

$$\begin{array}{ccccccc} P_0 & \longrightarrow & P_1 & \longrightarrow & \cdots & \longrightarrow & P_m \\ & & \downarrow & & & & \\ & & P_1/P_1 & \longrightarrow & \cdots & \longrightarrow & P_m/P_1 \\ & & \downarrow & & \downarrow & & \\ & & P_1/P_1 \cap k[x_1, \dots, x_{n-1}] & \longrightarrow & \cdots \cap k[x_1, \dots, x_{n-1}] & \longrightarrow & P_m/P_1 \end{array}$$

The first set of downward arrows denote taking the quotient, and the upward is taking inverse images, and this preserves strict inequalities.

**Definition (Integral Extension).**

An *integral* ring extension  $R \hookrightarrow R'$  of  $R$  is one such that all  $r' \in R'$  satisfying a monic polynomial with coefficients in  $R$ , where  $R'$  is finitely generated.

In this case, also implies that  $R'$  is a finitely-generated  $R$  module.

In this case,  $k[x_1, \dots, x_{n-1}] \hookrightarrow k[x_1, \dots, x_n]/\langle f \rangle$  is an integral extension. We want to show that the intersection step above also preserves strictness of inclusions.

**Lemma 1.2.**

Suppose  $P', Q' \subset R'$  are distinct prime ideals with  $R \hookrightarrow R'$  an integral extension. Then if  $P' \cap R = Q' \cap R$ , neither contains the other, i.e.  $P' \not\subset Q'$  and  $Q' \not\subset P'$ .

*Proof .*

Toward a contradiction, suppose  $P' \subset Q'$ , we then want to show that  $Q' \supset P'$ . Let  $a \in Q' \setminus P'$  (again toward a contradiction), then

$$R/(P' \cap R) \hookrightarrow R'/P'$$

is integral.

Then  $\bar{a} \neq 0$  in  $R'/P'$ , and there exists a monic polynomial of minimal degree that  $\bar{a}$  satisfies,  $p(x) = x^n + \sum_{i=2}^n \bar{c}_i x^i$ . This implies  $\bar{c}_n \in Q'/P'$

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