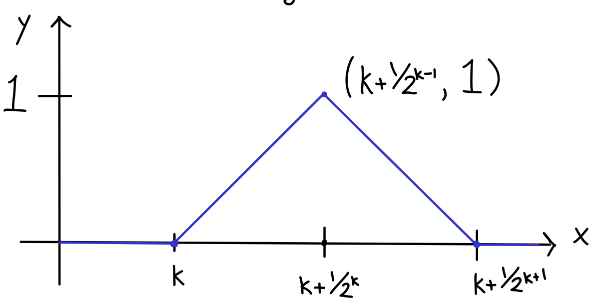
Analysis HW #4

Zack Garza

1a) Let f_k be the following function:



Note that this yields a triangle of area $\frac{1}{2}bh = \frac{1}{2}(k+1/2^{k+1}-k)\cdot 1 = 2^{-k}$, so we have $\int_{\mathbb{R}} f_k = \int_k^{k+1/2^{k+1}} f_k = 2^{-k}$. Moreover, $k \neq j \Rightarrow [k, k+1/2^{k+1}] \cap [j, j+1/2^{j+1}] = \emptyset$, so let $g_N = \sum_{k=0}^N f_k$ and $g = \lim_{N\to\infty} g_N = \sum_{k=0}^\infty f_k$. Then $g_N \nearrow g$, so we can apply the MCT to obtain

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} \lim_{N\to\infty} g_N \overset{\text{MCT}}{=} \lim_{N\to\infty} \int_{\mathbb{R}} g_N = \lim_{N\to\infty} \int_{\mathbb{R}} \sum_{k=0}^N f_k \underset{\text{Integral commutes with finite sums}}{=} \lim_{N\to\infty} \sum_{k=0}^N \int_{\mathbb{R}} f_k = \lim_{N\to\infty} \sum_{k=0}^N 2^{-k} = 1$$

However, $\limsup_{x\to\infty} g(x) = 1 > 0$, so $\lim_{x\to\infty} g(x) \neq 0$. ▨

1b) Towards a contradiction, suppose $f \in L^+$ is uniformly cts and $\limsup_{x\to\infty} f(x) = \varepsilon > 0$. Choose a sequence $\{x_n\} \nearrow \infty$ such that for all i,j we have $|x_i - x_j| > 1$. Then, for any $\delta < 1$ and any x_i, x_j, we have $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset$. Now by uniform continuity of f, choose δ such that $\delta < 1$ and

$$y \in B_\delta(x) \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall x, y \in \mathbb{R}^N$$

Now let n be fixed, and consider some $x \in B_\delta(x_n)$. We have $|f(x) - f(x_n)| < \varepsilon$; note that $|f(x_n)| > 0$ for all n large enough; otherwise the limsup would be zero. It also must be the case that $|f(x)| > \varepsilon$; otherwise $|f(x)| < \varepsilon \Rightarrow \big||f(x_n)| - |f(x)|\big| > |0 - \varepsilon| = \varepsilon$, so

$$\varepsilon < \big||f(x_n)| - |f(x)|\big| \leq |f(x_n) - f(x)| < \varepsilon \quad \#$$

So $|f(x)| > \varepsilon$. But then

$$\int_{B_\delta(x_n)} |f| \geq \int_{B_\delta(x_n)} \varepsilon = \varepsilon \cdot m(B_\delta(x_n)) = \varepsilon \cdot 2\delta,$$

and so if we let

$$X = \bigsqcup_{n=1}^{\infty} B_{\delta}(x_n) \subsetneq \mathbb{R}^N,$$

we have

$$\int_{\mathbb{R}^N} |f| \geq \int_X |f| = \sum_{n=1}^{\infty} \int_{B_\delta(x_n)} |f| \leq \sum_{n=1}^{\infty} \varepsilon \cdot 2\delta \longrightarrow \infty,$$

contradicting $f \in L^1$. ■

2a) Let $X = \{ x \in \mathbb{R}^n \,|\, |f(x)| = \infty \}$, then $X \cap X^c = \emptyset$ and $\mathbb{R}^n = X \sqcup X^c$, so

$$\int_{\mathbb{R}^n} |f| = \int_X |f| + \int_{X^c} |f| = \infty \cdot m(X) + \int_{X^c} |f| < \infty$$

since $f \in L^1$; but if $m(X) > 0$ this yields a contradiction. So we must have $m(X) = 0$. ▨

2b) We'll use the fact that $A \subseteq B$ and $\int_B |f| < \infty$, then $\int_B |f| - \int_A |f| = \int_{B \setminus A} |f|$. Noting that

$$\int_E |f| > \left(\int_{\mathbb{R}^n} |f| \right) - \varepsilon \iff \int_{\mathbb{R}^n} |f| - \int_E |f| < \varepsilon \iff \int_{E^c} |f| < \varepsilon,$$

we will produce an E s.t. E^c satisfies this condition. Write $\mathbb{R}^n = \lim_{k \to \infty} B(k, \vec{0})$, the n-ball of radius k centered at $\vec{0} \in \mathbb{R}^n$. Since the map $(A \mapsto \int_A |f|)$ is a measure, it satisfies continuity from below, and since $B(k, \vec{0}) \nearrow \mathbb{R}^n$, we have $\lim_{k \to \infty} \int_{B(k,\vec{0})} |f| = \int_{\mathbb{R}^n} |f|$. Since this limit exists, let $\varepsilon > 0$ and choose N such that

$$\int_{\mathbb{R}^n} |f| - \int_{B(N,\vec{0})} |f| < \varepsilon \implies \varepsilon > \int_{\mathbb{R}^n} |f| - \int_{B(N,\vec{0})} |f| = \int_{B(N,\vec{0})^c} |f|,$$

so $E := B(N, \vec{0})$ satisfies the desired property. ■

---

(3) We want to show a iff b iff c, where

a) $\int f < \infty$

b) $\sum_{k \in \mathbb{Z}} 2^k m(E_k) < \infty$, $\quad E_k = \{ x \,|\, f(x) > 2^k \}$

c) $\sum_{k \in \mathbb{Z}} 2^k m(F_k) < \infty$, $\quad F_k = \{ x \,|\, 2^k < f(x) \leq 2^{k+1} \}$

Note that $F_i \cap F_j = \emptyset$ if $i \neq j$, and $F_k = E_k \setminus E_{k+1}$

(b) iff (c): We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^k m(F_k) &= \sum_{k\in\mathbb{Z}} 2^k\left[m(E_k)-m(E_{k+1})\right]\\ &= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \sum_{k\in\mathbb{Z}} 2^k m(E_{k+1})\\ &= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^{k+1} m(E_{k+1})\\ &= \sum_{k\in\mathbb{Z}} 2^k m(E_k) - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^k m(E_k)\\ &= \sum_{k\in\mathbb{Z}} (1-\tfrac{1}{2})\, 2^k m(E_k)\\ &= \tfrac{1}{2}\sum_{k\in\mathbb{Z}} 2^k m(E_k),\end{aligned}$$

Might need to use absolute convergence of these sums for this to work.

and so either sum is finite iff the other is.

(a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a):

Write $X := \{x \mid f(x)>0\} = \bigsqcup_{k\in\mathbb{Z}} F_k$, then $\int_X f = \sum_{k\in\mathbb{Z}} \int_{F_k} f$ and we have

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) \le \sum_{k\in\mathbb{Z}} \int_{F_k} f \le \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) = \sum_{k\in\mathbb{Z}} 2^k m(E_k)$$

So

$$\int_X f < \infty \Rightarrow \sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$$

and

$$\sum_{k\in\mathbb{Z}} 2^k m(E_k) < \infty \Rightarrow \int_X f < \infty.$$

$\blacksquare$

---

4) Let $A_k = \{x\in\mathbb{R}^n \mid 2^k < \|x\| \le 2^{k+1}\}$, so we have

$$A := \{x\in\mathbb{R}^n \mid \|x\|\le 1\} = \bigsqcup_{k=1}^{\infty} A_{-k}$$

$$B := \{x\in\mathbb{R}^n \mid \|x\| > 1\} = \bigsqcup_{k=0}^{\infty} A_k$$

$$\underbrace{\omega_n}_{\text{Volume of unit n-ball.}} 2^{nk} \le m(A_k) \le \omega_n 2^{n(k+1)}, \qquad \omega_n 2^{-nk} \le m(A_{(-k)}) \le \omega_n 2^{-n(k-1)}$$

Then noting that

$$x\in A_k \Rightarrow 2^k < \|x\| \le 2^{k+1} \Rightarrow 2^{-p(k+1)} \le \|x\|^{-p} < 2^{-kp},$$

$$x\in A_{(-k)} \Rightarrow 2^{-k} < \|x\| \le 2^{-(k-1)} \Rightarrow 2^{p(k-1)} \le \|x\|^{-p} < 2^{pk}$$

Raise to $-p$ power for $p>0$.

we define

(4a)

$$I_A = \int_A \|\vec{x}\|^{-p}, \quad I_B = \int_B \|\vec{x}\|^{-p}$$

and find

$$I_A \le \sum_{k=1}^{\infty} 2^{pk} m(A_{(-k)}) \le \sum_{k=1}^{\infty} 2^{pk} 2^{-n(k-1)} = \omega_n \sum_{k=1}^{\infty} (2^{-k})^{n-p} < \infty \quad \text{iff } p<n,$$

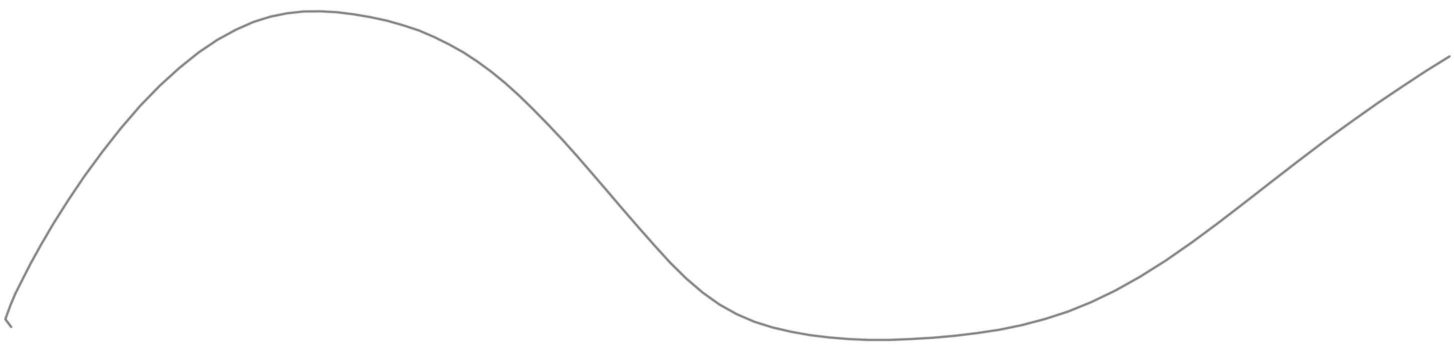
$$\text{and } \infty > I_A \ge \sum_{k=1}^{\infty} 2^{p(k-1)} m(A_{(-k)}) \ge \sum_{k=1}^{\infty} 2^{p(k-1)} \omega_n 2^{-nk} = \omega_n 2^{-p} \sum_{k=1}^{\infty} (2^{-k})^{n-p} \quad \text{iff } p<n$$

(4b)

Similarly

$$I_B \le \sum_{k=0}^{\infty} 2^{-kp} \omega_n 2^{n(k+1)} = \omega_n 2^n \sum_{k=0}^{\infty} (2^{-k})^{p-n} < \infty \quad \text{iff } p>n,$$

$$\text{and } \infty > I_B \ge \sum_{k=0}^{\infty} 2^{-p(k+1)} \omega_n 2^{nk} = \omega_n 2^{-p} \sum_{k=0}^{\infty} (2^{-k})^{p-n} \quad \text{iff } p>n. \quad \blacksquare$$



---

⑤ To see that $\hat{f}$ is bounded, supposing that $f \in L^1(\mathbb{R}^n)$, we have

$$|\hat{f}(\xi)| \le \int |f(x)| \cdot \underbrace{|e^{2\pi i x\cdot\xi}|}_{\le 1} \le \int_{\mathbb{R}^n} |f| < \infty.$$

To see that it is cts, we will use the sequential defn. of continuity. So let $\{\xi_n\} \to \xi$ be any sequence converging to ξ. Then

$$\lim_{n\to\infty} |\hat{f}(\xi_n) - \hat{f}(\xi)| = \lim_{n\to\infty} \left| \int f(x) [e^{2\pi i x\cdot\xi_n} - e^{2\pi i x\cdot\xi}] \right|$$

$$= \lim_{n\to\infty} \left| \int f(x) e^{2\pi i x\cdot\xi} [e^{2\pi i x\cdot(\xi_n-\xi)} - 1] \right|$$

$$\le \lim_{n\to\infty} \int |f(x) e^{2\pi i x\cdot\xi}| \cdot |e^{2\pi i x\cdot(\xi_n-\xi)} - 1|$$

$$\overset{DCT}{=} \int \lim_{n\to\infty} \left|f(x) e^{2\pi i x\cdot\xi}\right| \cdot \left|e^{2\pi i x\cdot(\xi_n-\xi)}-1\right|$$

Since

$$= \int \underbrace{\left|f(x) e^{2\pi i x\cdot\xi}\right|}_{\text{no } n \text{ involved}} \cdot \lim_{n\to\infty} \left|e^{2\pi i x\cdot(\xi_n-\xi)}-1\right|$$

$$= \int \left|f(x) e^{2\pi i x\cdot\xi}\right| \cdot 0$$

$$= 0$$

Where the DCT can be applied by letting

$$f_n = f(x) e^{2\pi i x\cdot\xi}\left(e^{2\pi i x\cdot(\xi_n-\xi)}-1\right)$$

$$\Rightarrow |f_n| = \left|f(x) e^{2\pi i x\cdot\xi}\right| \cdot \left|e^{2\pi i x\cdot(\xi_n-\xi)}-1\right|$$

$$\le \left|f(x) e^{2\pi i x\cdot\xi}\right| \cdot \left(\underbrace{\left|e^{2\pi i x\cdot(\xi_n-\xi)}\right|}_{\le 1} + |-1|\right)$$

$$\le \left|f(x) e^{2\pi i x\cdot\xi}\right| \cdot 2$$

$$\le 2|f| \in L^1.$$

But this says $\lim_{n\to\infty} |\hat{f}(\xi_n) - \hat{f}(\xi)| = 0$, so $\hat{f}$ is continuous. ■

6a.i) Let $g_n = |f_n| - |f_n - f|$; then $g_n \to |f|$ and

$$|g_n| = \big||f_n| - |f_n - f|\big| \le |f_n - (f_n - f)| = |f| \in L^1,$$

↖ Reverse Δ-ineq

so $\lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int |f| = B$ by the DCT. We can then write

$$\lim_{n\to\infty} \int |f_n - f| = \lim_{n\to\infty} \int |f_n - f| - |f_n| + |f_n|$$

$$= \lim_{n\to\infty} \int |f_n| - \underbrace{(|f_n| - |f_n - f|)}_{:= g_n}$$

$$= \lim_{n\to\infty} \int |f_n| - g_n$$

$$= \lim_{n\to\infty} \int |f_n| - \lim_{n\to\infty} \int g_n = A - B$$ ■

6a.ii) Let $f_n = n\cdot\chi_{(0,\frac{1}{n}]}$, then $f_n \to 0 := f$ a.e., so $\int f = \int 0 = 0 \Rightarrow B = 0$, but $\int f_n = 1$ for all n, so $\lim_{n\to\infty}\int|f_n| = 1 = A \neq B$. $\blacksquare$

6b) ($\Rightarrow$) $\lim_{k\to\infty}\int|f_k - f| = 0 = A - B \Rightarrow A = B \Rightarrow \lim\int|f_k| = \int|f|$.

($\Leftarrow$) $\lim\int|f_k| = \int|f| \Rightarrow A = B \Rightarrow A - B = 0 \Rightarrow \int|f_k - f| = A - B = 0$. $\blacksquare$

7a) Let $\{t_n\}\to t$ and define

$$g_n(x) = f(x)\left(\frac{\cos(t_n x)-\cos(tx)}{t_n - t}\right).$$

Then $\lim_{n\to\infty} g_n(x) = f(x)\,\partial/\partial t(\cos(tx)) = f(x)\,x\sin(tx)$, and applying the Mean Value Theorem, we have

$$\frac{\cos(t_n x)-\cos(tx)}{t_n-t} = x\sin(tx)\Big|_{x=\xi} = \xi\sin(t\xi) \text{ for some } \xi, \text{ so}$$

$$|g_n| = |f(x)\,x\sin(tx)| = |f(x)\,\xi\underbrace{\sin(t\xi)}_{\le 1}| \le \xi|f| \in L^1,$$

so $\lim_{n\to\infty}\int g_n \overset{DCT}{=} \int \lim_{n\to\infty} g_n = \int g = \int f(x)\,x\sin(tx)\,dx$, which is integrable because

$\int |f(x)\,x\underbrace{\sin(tx)}_{\le 1}| \le \int |xf(x)| < \infty$ since $xf \in L^1$.

Thus $F'(t) = \int_{\mathbb{R}} f(x)\,x\sin(tx)\,dx$. $\blacksquare$

7b) $$\lim_{t\to 0}\int_0^1 \frac{e^{t\sqrt{x}}-1}{t}\,dx = \lim_{t\to 0}\int_0^1 \frac{e^{t\sqrt{x}}-e^{0\sqrt{x}}}{t-0}\,dx \overset{DCT}{=} \int_0^1 \lim_{t\to 0}\left(\frac{e^{t\sqrt{x}}-e^{0\sqrt{x}}}{t-0}\right)dx$$

$$:= \int_0^1 \frac{\partial}{\partial t} e^{t\sqrt{x}}\Big|_{t=0}\,dx = \int_0^1 \sqrt{x}\,e^{t\sqrt{x}}\Big|_{t=0}\,dx = \int_0^1 \sqrt{x}\,dx = (2/3)\,x^{3/2}\Big|_0^1 = 2/3.$$

The DCT here is justified by letting $\{t_n\}\to 0$ and setting $g_n(t) = \dfrac{e^{t\sqrt{x}}-e^{t_n\sqrt{x}}}{t-t_n}$

Then by the MVT, for each n we have $g_n(t) = \partial/\partial t\, e^{t\sqrt{x}}\Big|_{t=c}$ for some $c\in[0,t_n]\subseteq[0,1]$.

But $\partial/\partial t\, e^{t\sqrt{x}}\Big|_{t=c} = \sqrt{x}\,e^{t\sqrt{x}}\Big|_{t=c} = \sqrt{x}\,e^{c\sqrt{x}} \le \sqrt{1}\,e^{c\cdot\sqrt{1}} = e^c \le e^1$, so $|g_n| \le e^1 \in L^1([0,1])$,

since $\int_0^1 e\,dx = e < \infty$, so $f(x) = e$ is a dominating function. $\blacksquare$