Title

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Remark 1.
There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma;\mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned}\rho : \mathrm{MCG}(\Sigma) &\to \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma;\mathbb{Z})) \\ f &\mapsto f_*.\end{aligned}$$

Definition 1.0.1 (Special Linear Group)**.**

$$\mathrm{SL}(n,\mathbb{k}) = \left\{ M \in \mathrm{GL}(n,\mathbb{k}) \;\middle|\; \det M = 1 \right\} = \ker \underset{\mathbb{G}_m}{\det}\,.$$

Remark 2.

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle .$$

Note that $S^2 = 1$ and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Moreover, if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $A \in \mathrm{SL}(2,\mathbb{Z})$, we have $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. this preserves the integer lattice

$$\mathbb{Z}^2 = \{\}\,.$$

Theorem 1.1 ***(Mapping Class Group of the Torus).***
The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

Proof .

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Map}(X, X), \circ) &\to (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*. \end{aligned}$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff $\mathrm{Map}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$
  - This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.
- So we have a (now injective) map

$$\begin{aligned} \tilde{\sigma} : \mathrm{MCG}(X) &\to \mathrm{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

Claim: $\mathrm{im}(\tilde{\sigma}) \subseteq \mathrm{SL}(2, \mathbb{Z})$.

- We can thus freely restrict the codomain to define the map

$$\begin{aligned} \sigma : \mathrm{MCG}(X) &\to \mathrm{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*. \end{aligned}$$

Claim: σ is surjective.

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