Title

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1 | Thursday, August 27

1.1 Consequence of the Nullstellensatz

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{ \text{Radical ideals } k[x_1,\cdots,x_n] \right\} \to V(\,\cdot\,)I(\,\cdot\,) \left\{ \text{Affine varieties in } \mathbb{A}^n \right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 1.1.1 ***(?).***
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

Remark 1.1.2: This is because no common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Slogan 1.1.3: The only ideals that vanish nowhere are trivial.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty. ■

We now return to the remaining hard part of the proof of the Nullstellensatz:

Proof ($I(V(J)) \subseteq \sqrt{J}$). ■

let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation 1.1.4: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$.

However, if $f = 0$, then $ft - 1 \neq 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

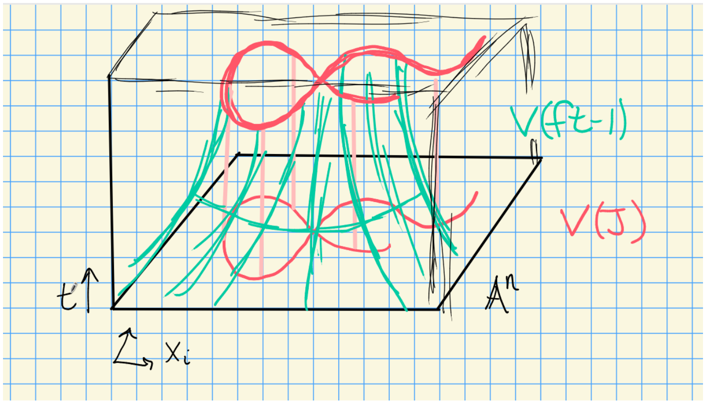

Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so

$$1 = \langle ft - 1 \rangle\, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$$

with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N := (ft - 1) G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t. Now take $k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Observation 1.1.5: There is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Exercise 1.1.6: Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 1.1.7: Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then

$$I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J,$$

so this loses information.

Example 1.1.8: Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\left\{ \text{Points of } \mathbb{A}^n \right\} \iff \left\{ \text{Maximal ideals of } k[x_1, \cdots, x_n] \right\} .$$

Theorem 1.1.9 ***(Properties of*** I***).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.

b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) on the variety side.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I\left(VI(X_1) \cap VI(X_2)\right) \\ &= IV\left(I(X_1) + I(X_2)\right) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

■

Example 1.1.10: Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

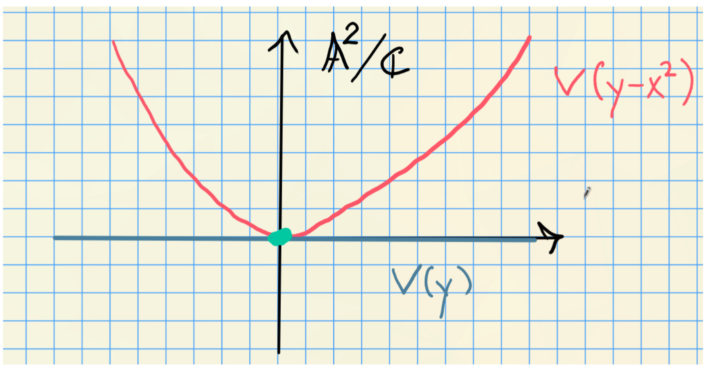

Figure 2: Intersecting $V(y - x^2)$ and $V(y)$

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but

$$I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$$

Proposition 1.1.11 ***(?).***
If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof .
Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$.

■

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

Definition 1.1.12 (Coordinate Ring)
For an affine variety X, the *coordinate ring of* X is

$$A(X) := k[x_1, \cdots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation 1.1.13: The constructions $V(\,\cdot\,), I(\,\cdot\,)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \left\{ x \in Y \;\middle|\; f(x) = 0 \;\forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \left\{ f \in A(Y) \;\middle|\; f(x) = 0 \;\forall x \in X \right\} \subseteq A(Y).$$

Example 1.1.14: For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$\begin{tikzcd} A(\mathbb{A}^n) \arrow[r, two heads, "\cdot/I(Y)"] \arrow[rr, bend left, two heads, "\cdot/I(X)"] & A(Y) \arrow[r, two heads, "\cdot/I(X)"] & A(X) \end{tikzcd}$$

Theorem 1.1.15 *(?).*
Let $X \subset Y$ be an affine subvariety, then

a. $A(X) = A(Y)/I_Y(X)$

b. There is a correspondence

$$\begin{aligned} \left\{\text{Affine subvarieties of } Y\right\} &\iff \left\{\text{Radical ideals in } A(Y)\right\} \\ X &\mapsto I_Y(X) \\ V_Y(J) &\hookleftarrow J. \end{aligned}$$

Proof .
Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ■

Example 1.1.16: Let $Y = V(y - x^2) \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x-1, y-1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \langle x - 1, y - 1 \rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$\begin{tikzcd}
A(\mathbb{A}^n) \arrow[r] \arrow[d, equal] & A(Y) \arrow[r] \arrow[d, equal] & A(X) \arrow[d, equal] \\
k[x, y] \arrow[r] & k[x, y]/\left\langle y - x^2 \right\rangle \arrow[r] & k[x, y]/\langle x - 1, y - 1 \rangle
\end{tikzcd}$$