Problem Set 3

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Exercise 0.1 (Gathmann 2.33)**.**
Define

$$X := \left\{ M \in \operatorname{Mat}(2 \times 3, k) \;\middle|\; \operatorname{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that X is an irreducible variety, and find its dimension.

Solution:
We'll use the following fact from linear algebra: :::{.theorem title="Rank is a Function of Minors"}

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Exercise 0.2 (Gathmann 2.34)**.**
Let X be a topological space, and show

a. If $\{U_i\} \rightrightarrows X$, then $\dim X = \sup_{i \in I} \dim U_i$.
b. If X is an irreducible affine variety and $U \subset X$ is a nonempty subset, then $\dim X = \dim U$. Does this hold for any irreducible topological space?

Exercise 0.3 (Gathmann 2.36)**.**
Prove the following:

a. Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.
b. A complex affine variety of dimension at least 1 is never compact in the classical topology.

Exercise 0.4 (Gathmann 2.40)**.**
Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1 x_4 - x_2 x_3 \rangle$$

and show the following:

a. R is an integral domain of dimension 3.

b. $x_1, \cdots, x_4$ are irreducible but not prime in R, and thus R is not a UFD.

c. x_1x_4 and x_2x_3 are two decompositions of the same element in R which are nonassociate.

d. $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

Exercise 0.5 (Problem 5)**.**
Consider a set U in the complement of $(0,0) \in \mathbb{A}^2$. Prove that any regular function on U extends to a regular function on all of $\mathbb{A}^2$.