Title

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Contents

1 Appendix

An alternative characterization of **uniform continuity**:

$$\|\tau_y f - f\|_u \to 0 \text{ as } y \to 0$$

Lemma: Measurability is not preserved by homeomorphisms.

> Counterexample: there is a homeomorphism that takes that Cantor set (measure zero) to a fat Cantor set

1.1 Undergraduate Analysis Review

- Some inclusions on the real line:

  > Differentiable with a bounded derivative $\subset$ Lipschitz continuous $\subset$ absolutely continuous $\subset$ uniformly continuous $\subset$ continuous
  > Proofs: Mean Value Theorem, Triangle inequality, Definition of absolute continuity specialized to one interval, Definition of uniform continuity

- **Bolzano-Weierstrass**: Every bounded sequence has a convergent subsequence.
- **Heine-Borel**:

$$X \subseteq \mathbb{R}^n \text{ is compact} \iff X \text{ is closed and bounded.}$$

- **Baire Category Theorem:** If X is a complete metric space, then

- For any sequence $\{U_k\}$ of open, dense sets, $\bigcap\limits_k U_k$ is also dense.
- X is *not* a countable union of nowhere-dense sets
- **Nested Interval Characterization of Completeness:** $\mathbb{R}$ being complete $\implies$ for any sequence of intervals $\{I_n\}$ such that $I_{n+1} \subseteq I_n$, $\bigcap I_n \neq \emptyset$.
- **Convergence Characterization of Completeness:** $\mathbb{R}$ being complete is equivalent to "absolutely convergent implies convergent" for sums of real numbers.
- Compacts subsets $K \subseteq \mathbb{R}^n$ are also *sequentially compact*, i.e. every sequence in K has a convergent subsequence.
- **Urysohn's Lemma:** For any two sets A, B in a metric space or compact Hausdorff space X, there is a function $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$.
- Continuous compactly supported functions are
  - Bounded almost everywhere
  - Uniformly bounded
  - Uniformly continuous

    Proof:

- Uniform convergence allows commuting sums with integrals
- Closed subsets of compact sets are compact.
- Every compact subset of a Hausdorff space is closed
- Showing that a series converges: *(Todo)*

1.2 Big Counterexamples

1.2.1 For Limits

- Differentiability $\implies$ continuity but not the converse:
  - The Weierstrass function is continuous but nowhere differentiable.
- f continuous does not imply f' is continuous: $f(x) = x^2 \sin(x)$.
- Limit of derivatives may not equal derivative of limit:

$$f(x) = \frac{\sin(nx)}{n^c} \text{ where } 0 < c < 1.$$

  - Also shows that a sum of differentiable functions may not be differentiable.
- Limit of integrals may not equal integral of limit:

$$\sum \mathbb{1}\left[x = q_n \in \mathbb{Q}\right].$$

- A sequence of continuous functions converging to a discontinuous function:

$$f(x) = x^n \text{ on } [0, 1].$$

- The Thomae function *(todo)*

1.2.2 For Convergence

- Notions of convergence:
  1. Uniform
  2. Pointwise
  3. Almost everywhere
  4. In norm

Uniform $\implies$ pointwise $\implies$ almost everywhere.

See Section 17.3.

Almost everywhere convergence does not imply L^p convergence for any $1 \leq p \leq \infty$

See notes section 1

Sequences $f_k \overset{a.e.}{\to} f$ but $f_k \overset{L^p}{\not\to} f$:

- For $1 \leq p < \infty$: The skateboard to infinity, $f_k = \chi_{[k,k+1]}$.

  Then $f_k \overset{a.e.}{\to} 0$ but $\|f_k\|_p = 1$ for all k.

  Converges pointwise and a.e., but not uniformly and not in norm.

- For $p = \infty$: The sliding boxes $f_k = k \cdot \chi_{[0,\frac{1}{k}]}$.

  Then similarly $f_k \overset{a.e.}{\to} 0$, but $\|f_k\|_p = 1$ and $\|f_k\|_\infty = k \to \infty$

  Converges a.e., but not uniformly, not pointwise, and not in norm.

The Converse to the DCT does not hold

L^p boundedness does not imply a.e. boundedness.

I.e. it is not true that $\lim \int f_k = \int f$ implies that $\exists g \in L^p$ such that $f_k < g$ a.e. for every k.

Take

- $b_k = \sum_{j=1}^{k} \frac{1}{j} \to \infty$
- $f_k = \chi_{[b_k, b_{k+1}]}$

Then

- $f_k \overset{a.e.}{\to} f = 0$,
- $\int f_k = \frac{1}{k} \to 0 \implies \|f_k\|_p \to 0$,

- $0 = \int f = \lim \int f_k = 0$
- But $g > f_k \implies g > \|f_k\|_\infty = 1$ a.e. $\implies g \notin L^p(\mathbb{R})$.

1.3 Errata

- **Equicontinuity**: If $\mathcal{F} \subset C(X)$ is a family of continuous functions on X, then $\mathcal{F}$ *equicontinuous* at x iff

$$\forall \varepsilon > 0 \; \exists U \ni x \text{ such that } y \in U \implies |f(y) - f(x)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

- **Arzela - Ascoli 1**: If $\mathcal{F}$ is pointwise bounded and equicontinuous, then $\mathcal{F}$ is totally bounded in the uniform metric and its closure $\overline{\mathcal{F}} \in C(X)$ in the space of continuous functions is compact.
- **Arzela - Ascoli 2**: If $\{f_k\}$ is pointwise bounded and equicontinuous, then there exists a continuous f such that $f_k \overset{u}{\to} f$ on every compact set.

Example: Using Fatou to compute the limit of a sequence of integrals:

$$\lim_{n\to\infty} \int_0^\infty \frac{n^2}{1+n^2x^2} e^{-\frac{x^2}{n^3}}\, dx \overset{\text{Fatou}}{\geq} \int_0^\infty \lim_{n\to\infty} \frac{n^2}{1+n^2x^2} e^{-\frac{x^2}{n^3}}\, dx \to \int \infty.$$

> Note that MCT might work, but showing that this is non-decreasing in n is difficult.

Lemma:

$$f_k \overset{a.e.}{\to} f, \quad \|f_k\|_p \leq M \implies f \in L^p \text{ and } \|f\|_p \leq M.$$

> *Proof:* Apply Fatou to $|f|^p$:
>
> $$\int |f|^p = \int \liminf |f_k|^p \leq \liminf \int |f_k|^p = M.$$

Lemma: If f is uniformly continuous, then

$$\|\tau_h f - f\|_p \overset{L^p}{\to} 0 \quad \text{for all } p.$$

Lemma: $\|\tau_h f - f\|_p \to 0$ for every p.

> - i.e. "Continuity in L^1" holds for all L^p.
> - i.e. Translation operators are continuous.

> *Proof:* Take $g_k \in C_c^0 \to f$, then g is uniformly continuous, so
>
> $$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p \to 0.$$

Lemma: For $f \in L^p, g \in L^q$, $f * g$ is uniformly continuous.

Proof: Use Young's inequality

$$\|\tau_h(f * g) - f * g\|_\infty = \|(\tau_h f - f) * g\|_\infty \leq \|\tau_h f - f\|_p \|g\|_q \to 0.$$

Lemma: If $\displaystyle\int f\phi = 0$ for every $\phi \in C_c^0$, then $f = 0$ almost everywhere.

Proof: Let A be an interval, choose $\phi_k \to \chi_A$, then $\displaystyle\int f\chi_A = 0$ for all intervals. So this holds for any Borel set A. Then just take $A_1 = \{f > 0\}$ and $A_2 = \{f < 0\}$, then $\displaystyle\int_{\mathbb{R}} f = \int_{A_1} f + \int_{A_2} f = 0$.

1.4 The Fourier Transform

Some Useful Properties:

$$\begin{aligned}
\widehat{f * g}(\xi) &= \hat{f}(\xi) \cdot \hat{g}(\xi) \\
\widehat{\tau_h f}(\xi) &= e^{2\pi i \xi \cdot h} \widehat{f}(\xi) \\
\widehat{e^{2\pi i \xi \cdot h} f}(\xi) &= \tau_{-h} \widehat{f}(\xi) \\
\widehat{f \circ T}(\xi) &= |\det T|^{-1} (\hat{f} \circ T^{-t})(\xi) \\
\frac{\partial}{\partial \xi} \widehat{f}(\xi) &= -2\pi i \cdot \widehat{\xi f}(\xi) \\
\widehat{\frac{\partial}{\partial \xi} f}(\xi) &= 2\pi i \xi \cdot \widehat{f}(\xi).
\end{aligned}$$