SPECTRAL SEQUENCES AND AN APPLICATION

A THESIS

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By

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ABSTRACT

This thesis is an exposition on spectral sequences, starting with the very basics. We present this mathematical tool in full detail with all applicable proofs included. The reader is not expected to have any background in homological algebra; all terms beyond basic algebra are defined. Some topics covered are limit terms, convergence, collapsing, and boundedness. Detailed examples of spectral sequences are included. The paper ends with a chapter dedicated to a description of another example—the Grothendieck spectral sequence.

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INTRODUCTION

Weibel [1995], Cartan and Eilenberg [1956], and Kostrikin and Shafarevich [1994] all agree that Jean Leray invented spectral sequences by 1945. Weibel explains that Leray did so in order to compute the homology of a complex, while Cartan and Eilenberg state that spectral sequences emerged from a study of fiber bundles. Consequently, spectral sequences are an important part of homological algebra. Lang explains that they have many applications in topology, differential geometry, and algebraic geometry. Spectral sequences are mainly viewed as a tool in calculations; we will try to show that in this paper.

In Chapter 1, we define what a spectral sequence is. Then we give three examples of how one finds a spectral sequence. In Chapter 2, we discuss three properties of spectral sequences. We define what it means for a spectral sequence to have a limit term, to be bounded, and lastly what it means for a spectral sequence to converge. These properties are referred to later in the paper. Chapter 3 covers another property a spectral sequence may have, collapsing. In Chapter 4 we give an application of a spectral sequence. Chapter 5 describes the Grothendieck spectral sequence, which is a kind of spectral sequence.

Let A, B be monoids, i.e. sets with an associative operation and a zero element. By a *map* we will mean an assignment $f : A \to B$ such that $f(0) = 0$, f is well defined (i.e. $a = a'$ implies $f(a) = f(a')$), and $f(a + a')$ equals $f(a) + f(a')$.

CHAPTER 1

THE SPECTRAL SEQUENCE

In this Chapter we will define a spectral sequence and give three examples of spectral sequences.

1.1 Definition of the Spectral Sequence

We start with the definition of a spectral sequence. It involves objects from an *abelian category*. For the definition of an abelian category, see Appendix A.

Definition 1.1.1 *A* ***bigraded object*** $\mathcal{O} = \{\mathcal{O}_{i,j}\}$ *is a collection of double indexed objects* $\mathcal{O}_{i,j}$, $(i,j) \in \mathcal{Z} \times \mathcal{Z}$.

Definition 1.1.2 *A* ***spectral sequence*** *is a sequence* $\{\mathcal{E}^r, d^r\}, r \geq 0$, *of bigraded objects* $\mathcal{E}^r = \{\mathcal{E}^r_{p,q}\}$, $\mathcal{E}^r_{p,q}$ *from an abelian category* $\mathcal{A}$, *and families of morphisms* $d^r = \{d^r_{p,q}\}, p, q \in \mathbb{Z}$, *where* $d^r_{p,q} \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{E}^r_{p,q}, \mathcal{E}^r_{p-r,q+r-1})$ *(See Appendix A) and* $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$. *We also require that the homology of* $\mathcal{E}^r$ *is* $\mathcal{E}^{r+1}$, *i.e. for all p,q,* $H(\mathcal{E}^r_{p,q}) \overset{dfn}{=} \ker(d^r_{p,q})/\operatorname{im}(d^r_{p+r,q-r+1})$ *is isomorphic to* $\mathcal{E}^{r+1}_{p,q}$.

We often denote $\mathcal{Z}^{r+1}_{p,q} = \ker(d^r_{p,q})$ and $\mathcal{B}^{r+1}_{p,q} = \operatorname{im}(d^r_{p+r,q-r+1})$. Since

$$\mathcal{Z}^{r+1}_{p,q} = \ker(d^r_{p,q} : \mathcal{E}^r_{p,q} \to \mathcal{E}^r_{p-r,q+r-1}) \subset \mathcal{E}^r_{p,q}$$

and

$$\mathcal{B}^{r+1}_{p,q} = \operatorname{im}(d^r_{p+r,q-r+1} : \mathcal{E}^r_{p+r,q-r+1} \to \mathcal{E}^r_{p,q}) \subset \mathcal{E}^r_{p,q},$$

we have that $\mathcal{Z}^{r+1} = \{\mathcal{Z}^{r+1}_{p,q}\}$ and $\mathcal{B}^{r+1} = \{\mathcal{B}^{r+1}_{p,q}\}$ are families of bigraded subobjects of $\mathcal{E}^r = \{\mathcal{E}^r_{p,q}\}$. We say that $\mathcal{E}^r_{p,q}$ has *total degree* $p+q$ and that $d^r_{p,q}$ has *bidegree* $(-r, r-1)$. Then $d^r_{p,q}$ decreases the total degree by one. One can think of each bigraded object $\mathcal{E}^r$ as a lattice:

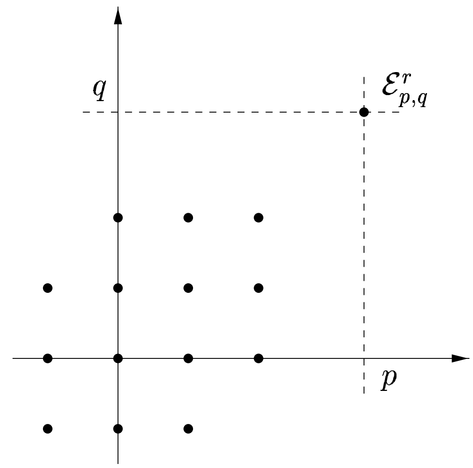

Figure 1.1: Viewing the bigraded object $\mathcal{E}^r = \{\mathcal{E}^r_{p,q}\}$ as a lattice

Note that the maps $d^r_{p,q}$ have lines of slope $\frac{r-1}{-r}$.

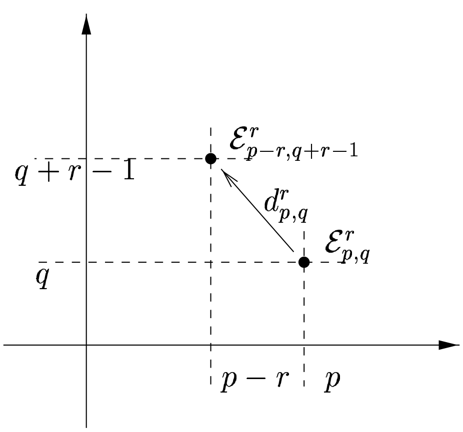

Figure 1.2: Viewing the morphisms $d^r = \{d^r_{p,q}\}$

1.2 Example 1: Obtained From an Exact Couple

In this section we introduce the notion of an exact couple and show how we construct a spectral sequence from it.

Definition 1.2.1 *An **exact triangle** consists of three objects $\mathcal{A}, \mathcal{B}, \mathcal{C}$ from an abelian category, together with three morphisms $\mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \xrightarrow{c} \mathcal{A}$ such that* $\operatorname{im} a = \ker b$*,* $\operatorname{im} b = \ker c$*, and* $\operatorname{im} c = \ker a$*.*

We call an exact triangle an *exact couple* when $\mathcal{A} = \mathcal{B}$. Consider an exact couple of bigraded objects $\mathcal{D}^1 = \{\mathcal{D}^1_{p,q}\}$ and $\mathcal{E}^1 = \{\mathcal{E}^1_{p,q}\}$ with morphisms $\alpha^1 = \{\alpha^1_{p,q}\}, \beta^1 = \{\beta^1_{p,q}\}, \gamma^1 = \{\gamma^1_{p,q}\}$, i.e.

$$\mathcal{D}^1_{p,q} \xrightarrow{\alpha^1_{p,q}} \mathcal{D}^1_{p+1,q-1} \xrightarrow{\beta^1_{p+1,q-1}} \mathcal{E}^1_{p+1,q-1} \xrightarrow{\gamma^1_{p+1,q-1}} \mathcal{D}^1_{p,q-1},$$

where α^1, β^1, and γ^1 have bidegrees $(1,-1)$, $(0,0)$, and $(-1,0)$, respectively. It would actually be more accurate to write this exact couple as

$$\{\mathcal{D}^1_{p,q}\} \xrightarrow{\{\alpha^1_{p,q}\}} \{\mathcal{D}^1_{p+1,q-1}\} \xrightarrow{\{\beta^1_{p+1,q-1}\}} \{\mathcal{E}^1_{p+1,q-1}\} \xrightarrow{\{\gamma^1_{p+1,q-1}\}} \{\mathcal{D}^1_{p,q-1}\}.$$

Better yet we can put this information in the following diagram:

$$\begin{tikzcd}
\mathcal{D}^1 = \{\mathcal{D}^1_{p,q}\} \arrow[rr, "\alpha^1"] & & \mathcal{D}^1 = \{\mathcal{D}^1_{p,q}\} \arrow[dl, "\beta^1"] \\
& \mathcal{E}^1 = \{\mathcal{E}^1_{p,q}\} \arrow[ul, "\gamma^1"] &
\end{tikzcd}$$

Figure 1.3: An exact couple

Of interest in this exact couple is the map:

$$d^1_{p,q} \overset{dfn}{=} \beta^1_{p-1,q} \circ \gamma^1_{p,q} : \mathcal{E}^1_{p,q} \xrightarrow{\gamma^1_{p,q}} \mathcal{D}^1_{p-1,q} \xrightarrow{\beta^1_{p-1,q}} \mathcal{E}^1_{p-1,q}. \tag{1.1}$$

This map decreases the total degree by one and satisfies the condition

$$d^1_{p,q} \circ d^1_{p+1,q} = 0,$$

since $\gamma^1_{p,q} \circ \beta^1_{p,q} = 0$ implies $d^1_{p,q} \circ d^1_{p+1,q}(\mathcal{E}^1_{p+1,q}) = \beta^1_{p-1,q} \circ \gamma^1_{p,q} \circ \beta^1_{p,q} \circ \gamma^1_{p+1,q}(\mathcal{E}^1_{p+1,q}) = \beta^1_{p-1,q}(0) = 0$. This allows us to define $\mathcal{E}^2 = \{\mathcal{E}^2_{p,q}\}$ where

$$\mathcal{E}^2_{p,q} = \ker(d^1_{p,q}) / \operatorname{im}(d^1_{p+1,q}).$$

Next define $\mathcal{D}^2 = \{\mathcal{D}^2_{p,q}\}$, where

$$\mathcal{D}^2_{p,q} = \alpha^1_{p-1,q+1}(\mathcal{D}^1_{p-1,q+1}).$$

The morphism α^1 has bidegree $(1, -1)$, hence $\mathcal{D}^2_{p,q} \subset \mathcal{D}^1_{p,q}$. So far, we have d^1, $\mathcal{E}^1$, and $\mathcal{E}^2$ exactly as needed in the spectral sequence definition. Having defined $\mathcal{D}^2$ and $\mathcal{E}^2$, we are on our way to another exact couple and hence will be able to construct d^2. Define three maps

$$\begin{aligned} \alpha^2_{p,q} &: \mathcal{D}^2_{p,q} \to \mathcal{D}^2_{p+1,q-1}, \\ \beta^2_{p,q} &: \mathcal{D}^2_{p,q} \to \mathcal{E}^2_{p-1,q+1}, \\ \gamma^2_{p,q} &: \mathcal{E}^2_{p,q} \to \mathcal{D}^2_{p-1,q}, \end{aligned}$$

by

$$\begin{aligned} \alpha^2_{p,q} &= \alpha^1_{p,q}|_{\mathcal{D}^2_{p,q}}, \\ \beta^2_{p,q}(a) &= [\beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(a)] \\ &\overset{dfn}{=} \beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(a) \mod \operatorname{im}(d^1_{p,q+1}), \\ \gamma^2_{p,q}([m]) &= \gamma^1_{p,q}(m). \end{aligned}$$

Note that in the definition of $\beta^2_{p,q}$, we quotient out by $\operatorname{im}(d^1_{p,q+1})$;

$$\mathcal{E}^2_{p-1,q+1} = \ker(d^1_{p-1,q+1}) / \operatorname{im}(d^1_{p,q+1}).$$

To show that $\beta^2_{p,q}(a)$ is well defined, let $r = (\alpha^1_{p-1,q+1})^{-1}(a)$ and $r' = (\alpha^1_{p-1,q+1})^{-1}(a)$ and we will show that $[\beta^1_{p-1,q+1}(r)] = [\beta^1_{p-1,q+1}(r')]$. Since

$$\alpha^1_{p-1,q+1}(r - r') = \alpha^1_{p-1,q+1}(r) - \alpha^1_{p-1,q+1}(r') = a - a = 0,$$

we have that $r - r'$ is in $\ker(\alpha^1_{p-1,q+1}) = \operatorname{im}(\gamma^1_{p,q+1}) = \gamma^1_{p,q+1}(\mathcal{E}^1_{p,q+1})$. So $\beta^1_{p-1,q+1}(r) - \beta^1_{p-1,q+1}(r') = \beta^1_{p-1,q+1}(r - r')$ is in $\beta^1_{p-1,q+1} \circ \gamma^1_{p,q+1}(\mathcal{E}^1_{p,q+1}) = \operatorname{im}(d^1_{p,q+1})$, and hence $[\beta^1_{p-1,q+1}(r)] = [\beta^1_{p-1,q+1}(r')]$.

To show that $\gamma^2_{p,q}([m])$ is well defined, let $[m] = [n]$ and we will show that $\gamma^1_{p,q}(m) = \gamma^1_{p,q}(n)$. Since $m - n \in \operatorname{im}(d^1_{p+1,q}) = d^1_{p+1,q}(\mathcal{E}^1_{p+1,q}) = \beta^1_{p,q} \circ \gamma^1_{p+1,q}(\mathcal{E}^1_{p+1,q})$, we have that $m - n = \beta^1_{p,q} \circ \gamma^1_{p+1,q}(r)$ for some r in $\mathcal{E}^1_{p+1,q}$. So $\gamma^1_{p,q}(m) - \gamma^1_{p,q}(n) = \gamma^1_{p,q}(m - n) = \gamma^1_{p,q} \circ \beta^1_{p,q} \circ \gamma^1_{p+1,q}(r) = 0$, since $\gamma^1_{p,q} \circ \beta^1_{p,q} = 0$, and hence $\gamma^1_{p,q}(m) = \gamma^1_{p,q}(n)$. Let us verify the exactness at each point (see Figure 1.4) and then we will have another exact couple. We need to check three things:

i. $\operatorname{im} \alpha^2_{p,q} = \ker \beta^2_{p+1,q-1}$,

ii. $\operatorname{im} \beta^2_{p,q} = \ker \gamma^2_{p-1,q+1}$,

iii. $\operatorname{im} \gamma^2_{p,q} = \ker \alpha^2_{p-1,q}$.

To prove these equalities, we will show containment in both directions.

i.a. First let us show that $\operatorname{im} \alpha^2_{p,q} \subset \ker \beta^2_{p+1,q-1}$. Let x be in $\operatorname{im} \alpha^2_{p,q} = \alpha^2_{p,q}(D^2_{p,q}) = \alpha^1_{p,q}(D^2_{p,q})$. So $x = \alpha^1_{p,q}(s)$, where s is in $D^2_{p,q}$. Since $D^2_{p,q} = \alpha^1_{p-1,q+1}(D^1_{p-1,q+1})$, we have $s = \alpha^1_{p-1,q+1}(u)$, for u in $D^1_{p-1,q+1}$. Note that x is in $\alpha^2_{p,q}(D^2_{p,q}) \subset D^2_{p+1,q-1}$, so x is in the domain of $\beta^2_{p+1,q-1}$ and all we need to show is that $\beta^2_{p+1,q-1}(x) = 0$.

$$\begin{aligned}
\beta^2_{p+1,q-1}(x) &= [\beta^1_{p,q}(\alpha^1_{p,q})^{-1}(x)];\\
&= [\beta^1_{p,q}(\alpha^1_{p,q})^{-1} \circ \alpha^1_{p,q}(s)];\\
&= [\beta^1_{p,q}(s)], \text{ since we already saw that } \beta^2 \text{ is well defined};\\
&= [\beta^1_{p,q} \circ \alpha^1_{p-1,q+1}(u)];\\
&= [0], \text{ since } \beta^1_{p,q} \circ \alpha^1_{p-1,q+1} = 0.
\end{aligned}$$

i.b. Now let us check that $\ker \beta^2_{p+1,q-1} \subset \operatorname{im} \alpha^2_{p,q}$. Let x be in

$\ker \beta^2_{p+1,q-1} = \ker(D^2_{p+1,q-1} \xrightarrow{\beta^2_{p+1,q-1}} \mathcal{E}^2_{p,q}) = \ker(\alpha^1_{p,q}(D^1_{p,q}) \xrightarrow{\beta^2_{p+1,q-1}} \ker(d^1_{p,q})/\operatorname{im}(d^1_{p+1,q}))$.

Then $\beta^2_{p+1,q-1}(x) = 0$, which means that $\beta^1_{p,q}(\alpha^1_{p,q})^{-1}(x)/\operatorname{im}(d^1_{p+1,q}) = 0$. So

$$\beta^1_{p,q}(\alpha^1_{p,q})^{-1}(x) \subset d^1_{p+1,q}(\mathcal{E}^1_{p+1,q}) = \beta^1_{p,q} \circ \gamma^1_{p+1,q}(\mathcal{E}^1_{p+1,q}). \tag{1.2}$$

Let a be in $(\alpha^1_{p,q})^{-1}(x)$. Then by (1.2), $\beta^1_{p,q}(a) = \beta^1_{p,q} \circ \gamma^1_{p+1,q}(e)$ for some e in $\mathcal{E}^1_{p+1,q}$. So $\beta^1_{p,q}(a - \gamma^1_{p+1,q}(e)) = 0$, which means that

$$a - \gamma^1_{p+1,q}(e) \in \ker \beta^1_{p,q} = \operatorname{im} \alpha^1_{p-1,q+1} = \alpha^1_{p-1,q+1}(D^1_{p-1,q+1}).$$

So then

$$\alpha^1_{p,q}(a) - \alpha^1_{p,q} \circ \gamma^1_{p+1,q}(e) \in \alpha^1_{p,q} \circ \alpha^1_{p-1,q+1}(D^1_{p-1,q+1}).$$

So $x - 0 = x$ is in $\alpha^1_{p,q} \circ \alpha^1_{p-1,q+1}(D^1_{p-1,q+1}) = \alpha^1_{p,q}(D^2_{p,q}) = \alpha^2_{p,q}(D^2_{p,q}) = \operatorname{im} \alpha^2_{p,q}$.

ii.a. Let us show that $\operatorname{im} \beta^2_{p,q} \subset \ker \gamma^2_{p-1,q+1}$. Let

$$x/\operatorname{im} d^1_{p,q+1} \in \operatorname{im} \beta^2_{p,q} = \beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(D^2_{p,q})/\operatorname{im} d^1_{p,q+1},$$

where $x = \beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(n)$ for some n in $D^2_{p,q}$. We show that $x/\operatorname{im} d^1_{p,q+1}$ is in the domain of $\ker \gamma^2_{p-1,q+1} =$

$$\ker(\mathcal{E}^2_{p-1,q+1} \xrightarrow{\gamma^2_{p-1,q+1}} D^2_{p-2,q+1}) = \ker(\ker(d^1_{p-1,q+1})/\operatorname{im}(d^1_{p,q+1}) \to D^2_{p-2,q+1}),$$

and that $\gamma^2_{p-1,q+1}(x/\operatorname{im} d^1_{p,q+1}) = 0$. To show the first thing, we need to show that $x/\operatorname{im} d^1_{p,q+1}$ is in $\ker(d^1_{p-1,q+1})/\operatorname{im}(d^1_{p,q+1})$. So we will show that x is in $\ker(d^1_{p-1,q+1})$:

$$\begin{aligned} d^1_{p-1,q+1}(x) &= \beta^1_{p-2,q+1} \circ \gamma^1_{p-1,q+1}(x); \\ &= \beta^1_{p-2,q+1} \circ \gamma^1_{p-1,q+1} \circ \beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(n); \\ &= 0, \text{ since } \gamma^1_{p-1,q+1} \circ \beta^1_{p-1,q+1} = 0. \end{aligned}$$

The second thing holds because

$$\begin{aligned} \gamma^2_{p-1,q+1}(x/\operatorname{im} d^1_{p,q+1}) &= \gamma^1_{p-1,q+1}(x); \\ &= \gamma^1_{p-1,q+1} \circ \beta^1_{p-1,q+1}(\alpha^1_{p-1,q+1})^{-1}(n); \\ &= 0, \text{ since } \gamma^1_{p-1,q+1} \circ \beta^1_{p-1,q+1} = 0. \end{aligned}$$

ii.b Now let us show that $\ker \gamma^2_{p-1,q+1} \subset \operatorname{im} \beta^2_{p,q}$. Let $x/\operatorname{im} d^1_{p,q+1}$ be in $\ker \gamma^2_{p-1,q+1}$. So $\gamma^1_{p-1,q+1}(x) = 0$. This implies that x is in $\ker \gamma^1_{p-1,q+1} = \operatorname{im} \beta^1_{p-1,q+1}$, so that $x = \beta^1_{p-1,q+1}(m)$, for some m in $D^1_{p-1,q+1} \subset (\alpha^1_{p-1,q+1})^{-1}(D^2_{p,q})$. Hence x is in $\beta^1_{p-1,q+1} \circ (\alpha^1_{p-1,q+1})^{-1}(D^2_{p,q})$, and thus $x/\operatorname{im} d^1_{p,q+1}$ is in $\beta^1_{p-1,q+1} \circ (\alpha^1_{p-1,q+1})^{-1}(D^2_{p,q})/\operatorname{im} d^1_{p,q+1} = \operatorname{im} \beta^2_{p,q}$.

iii.a. First let us check that $\operatorname{im} \gamma^2_{p,q} \subset \ker \alpha^2_{p-1,q}$. Let x be in $\operatorname{im} \gamma^2_{p,q} = \gamma^2_{p,q}(E^2_{p,q}) = \gamma^2_{p,q}(\ker d^1_{p,q}/\operatorname{im} d^1_{p+1,q}) = \gamma^1_{p,q}(\ker d^1_{p,q})$. So $x = \gamma^1_{p,q}(w)$, for some w in $\ker d^1_{p,q}$. Then

$$\begin{aligned}
\alpha^2_{p-1,q}(x) &= \alpha^1_{p-1,q}(x);\\
&= \alpha^1_{p-1,q} \circ \gamma^1_{p,q}(w);\\
&= 0, \text{ since } \alpha^1_{p-1,q} \circ \gamma^1_{p,q} = 0.
\end{aligned}$$

So x is in $\ker \alpha^2_{p-1,q}$. Note that indeed x is in $\gamma^2_{p,q}(E^2_{p,q}) \subset D^2_{p-1,q}$, the domain of $\alpha^2_{p-1,q}$.

iii.b Lastly, we will check that $\ker \alpha^2_{p-1,q} \subset \operatorname{im} \gamma^2_{p,q}$. Let x be in $\ker \alpha^2_{p-1,q} = \ker(\alpha^1_{p-1,q})|_{D^2_{p-1,q}}$. Then x is in $D^2_{p-1,q}$ and $\alpha^2_{p-1,q}(x) = 0$, which means that x is in $\alpha^1_{p-2,q+1}(D^1_{p-2,q+1})$ and $\alpha^1_{p-1,q}(x) = 0$, respectively. So we have that for some v in $D^1_{p-2,q+1}$,

$$x = \alpha^1_{p-2,q+1}(v). \tag{1.3}$$

Now x in $\ker \alpha^1_{p-1,q}$ implies that x is in $\operatorname{im} \gamma^1_{p,q}$, by the exactness of Figure 1.3. So we have that $x = \gamma^1_{p,q}(m)$, for some m in $E^1_{p,q}$. So

$$m \in (\gamma^1_{p,q})^{-1}(x). \tag{1.4}$$

We want to show that x is in

$$\operatorname{im} \gamma^2_{p,q} = \gamma^2_{p,q}(E^2_{p,q}) = \gamma^2_{p,q}(\ker d^1_{p,q}/\operatorname{im} d^1_{p+1,q}) = \gamma^1_{p,q}(\ker d^1_{p,q}).$$

Since $x = \gamma^1_{p,q}(m)$, we only have left to show that m is in $\ker d^1_{p,q}$.

$$\begin{aligned}
d^1_{p,q}(m) &\in d^1_{p,q} \circ (\gamma^1_{p,q})^{-1}(x), \text{ by } (1.4);\\
&= \beta^1_{p-1,q} \circ \gamma^1_{p,q} \circ (\gamma^1_{p,q})^{-1}(x), \text{ by definition of } d^1_{p,q};\\
&= \beta^1_{p-1,q}(x);\\
&= \beta^1_{p-1,q} \circ \alpha^1_{p-2,q+1}(v), \text{ by } (1.3);\\
&= 0, \text{ since } \beta^1_{p-1,q} \circ \alpha^1_{p-2,q+1} = 0.
\end{aligned}$$

So we have shown the exactnes at all three points in Figure 1.4.:

$$\begin{tikzcd}
\mathcal{D}^2 = \{\mathcal{D}^2_{p,q}\} \arrow[rr, "\alpha^2"] & & \mathcal{D}^2 = \{\mathcal{D}^2_{p,q}\} \arrow[dl, "\beta^2"] \\
& \mathcal{E}^2 = \{\mathcal{E}^2_{p,q}\} \arrow[ul, "\gamma^2"] &
\end{tikzcd}$$

Figure 1.4: The exact couple derived from the original exact couple

Then just as before, we can define $d^2 = \{d^2_{p,q}\}$, where

$$d^2_{p,q} \overset{dfn}{=} \beta^2_{p-1,q} \circ \gamma^2_{p,q} : \mathcal{E}^2_{p,q} \xrightarrow{\gamma^2_{p,q}} \mathcal{D}^2_{p-1,q} \xrightarrow{\beta^2_{p-1,q}} \mathcal{E}^2_{p-2,q+1}.$$

This will allow us to construct $\mathcal{E}^3$, $\mathcal{D}^3$, then d^3, and so on. Doing this over and over, we have in general Figure 1.5:

$$
\begin{tikzcd}
\mathcal{D}^r = \{\mathcal{D}^r_{p,q}\} \arrow[rr, "{(1,-1)}"] & & \mathcal{D}^r = \{\mathcal{D}^r_{p,q}\} \arrow[dl, "{(1-r,r-1)}"] \\
& \mathcal{E}^r = \{\mathcal{E}^r_{p,q}\} \arrow[ul, "{(-1,0)}"] &
\end{tikzcd}
$$

Figure 1.5: The general exact couple shown with bidegrees

More specifically we have bigraded objects:

$$\mathcal{E}^r_{p,q} = \ker(d^{r-1}_{p,q}) / \operatorname{im}(d^{r-1}_{p+r-1,q-r+2}),$$

$$\mathcal{D}^r_{p,q} = \alpha^{r-1}_{p-1,q+1}(\mathcal{D}^{r-1}_{p-1,q+1}) \subset \mathcal{D}^{r-1}_{p,q},$$

and maps:

$$\begin{aligned}
\alpha^r_{p,q} &: \mathcal{D}^r_{p,q} \to \mathcal{D}^r_{p+1,q-1}, \\
\beta^r_{p,q} &: \mathcal{D}^r_{p,q} \to \mathcal{E}^r_{p+1-r,q+r-1}, \\
\gamma^r_{p,q} &: \mathcal{E}^r_{p,q} \to \mathcal{D}^r_{p-1,q}, \\
d^r_{p,q} = \beta^r_{p-1,q} \circ \gamma^r_{p,q} &: \mathcal{E}^r_{p,q} \xrightarrow{\gamma^r_{p,q}} \mathcal{D}^r_{p-1,q} \xrightarrow{\beta^r_{p-1,q}} \mathcal{E}^r_{p-r,q+r-1},
\end{aligned}$$

such that

$$\begin{aligned}
\alpha^r_{p,q} &= \alpha^{r-1}_{p,q}|_{\mathcal{D}^r_{p,q}}, \\
\beta^r_{p,q}(a) &= [\beta^{r-1}_{p-1,q+1}(\alpha^{r-1}_{p-1,q+1})^{-1}(a)], \\
&\overset{dfn}{=} \beta^{r-1}_{p-1,q+1}(\alpha^{r-1}_{p-1,q+1})^{-1}(a) \mod \operatorname{im}(d^{r-1}_{p,q+1}) \\
\gamma^r_{p,q}([m]) &= \gamma^{r-1}_{p,q}(m),
\end{aligned}$$

where α^r, β^r, γ^r, and d^r have bidegrees $(1,-1)$, $(1-r,r-1)$, $(-1,0)$, and $(-r,r-1)$ respectively. Therefore we have constructed a sequence $\{\mathcal{E}^r, d^r\}$ which satisfies all the requirements to be a spectral sequence.

Remark. The numeration here begins at $r = 1$; the numeration in the definition of a spectral sequence begins at $r = 0$.

1.3 Example 2: Obtained from a Filtration of a Complex

In this Section we introduce the notion of a filtration of a complex and show how to construct a spectral sequence from it.

1.3.1 Preliminary Definitions and the Construction of the First Two Bigraded Objects

Let $\mathcal{A}$ be an abelian category. Refer to Appendix A for the definition of an abelian category.

Definition 1.3.1 *A* ***complex*** *C in $\mathcal{A}$ is a sequence of objects from $\mathcal{A}$ with morphisms* $\ldots \xrightarrow{d_{i+1}} \mathcal{C}_i \xrightarrow{d_i} \mathcal{C}_{i-1} \xrightarrow{d_{i-1}} \ldots$ *where* $d_i \circ d_{i+1} = 0$, *i.e.* $\operatorname{im}(d_{i+1}) \subset \ker(d_i)$.

The i^{th} spot is called *exact* if $\operatorname{im}(d_{i+1}) = \ker(d_i)$. The *homology* of the i^{th} spot, $\ker(d_i)/\operatorname{im}(d_{i+1})$, measures the deviation from exactness at the i^{th} spot.

Definition 1.3.2 *Let $\mathcal{F}$ be an object with a morphism $d : F \to F$ such that $d^2 = 0$. A* ***filtration*** *of $\mathcal{F}$ is a tower,*

$$\cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots ,$$

of subobjects of $\mathcal{F}$, such that $d_p(\mathcal{F}_p) \subset \mathcal{F}_p$, where $d_p = d|_{\mathcal{F}_p}$.

An example of a filtration. A tower of subcomplexes is a filtration. Suppose we have a complex $\mathcal{C}$ with family of morphisms $d = \{d_i\}$, and a tower of subcomplexes $\{\mathcal{F}_n\mathcal{C}\}_n$ of the complex $\mathcal{C}$:

$$\cdots \subset \mathcal{F}_{n-1}\mathcal{C} \subset \mathcal{F}_n\mathcal{C} \subset \mathcal{F}_{n+1}\mathcal{C} \subset \cdots . \tag{1.5}$$

Let $d_n = \{d_{n,i}\}$ be the family of morphisms corresponding to the subcomplex $\mathcal{F}_n\mathcal{C}$,

$$\cdots \xrightarrow{d_{n,i+1}} \mathcal{F}_n\mathcal{C}_i \xrightarrow{d_{n,i}} \mathcal{F}_n\mathcal{C}_{i-1} \xrightarrow{d_{n,i-1}} \cdots .$$

Since $\mathcal{F}_n\mathcal{C}_i \subset \mathcal{C}_i$, we define the morphisms of a subcomplex to be $d_{n,i} = d_i|_{\mathcal{F}_n\mathcal{C}_i}$. This is precisely $d_n = d|_{\mathcal{F}_n\mathcal{C}}$. We have $d_n(\mathcal{F}_n\mathcal{C}) \subset \mathcal{F}_n\mathcal{C}$ and since $d^2 = 0$, i.e. $d_i \circ d_{i+1} = 0$ because $\mathcal{C}$ is a complex, we get that the tower of subcomplexes of $\mathcal{C}$, (1.5), is a filtration of $\mathcal{C}$.

Definition 1.3.3 *A* ***subquotient*** *of an object A is an object of the form A'/A'', where $A'' \subset A'$ are subobjects of A.*

Now we will show that a filtration $\{\mathcal{F}_n\mathcal{C}\}$ of a complex $\mathcal{C}$ gives rise to a spectral sequence. We will need to construct $\mathcal{E}^r = \{\mathcal{E}^r_{p,q}\}, (r \geq 0)$, as in Definition 1.1.2. Let

$$\mathcal{E}^0_{p,q} = \mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q}, \tag{1.6}$$

a subquotient of $\mathcal{C}_{p+q}$ (see Figure 1.6). This object $\mathcal{E}^0_{p,q}$ is the object in the $p+q$ spot of the complex $\mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C}$.

$$\begin{array}{ccccccccc}
 & & \Big\downarrow d_{p-1,p+q+1} & & \Big\downarrow d_{p,p+q+1} & & \Big\downarrow d_{p+1,p+q+1} & & \\
\cdots \subset & & \mathcal{F}_{p-1}\mathcal{C}_{p+q} & \subset & \mathcal{F}_p\mathcal{C}_{p+q} & \subset & \mathcal{F}_{p+1}\mathcal{C}_{p+q} & \subset \cdots & \\
 & & \Big\downarrow d_{p-1,p+q} & & \Big\downarrow d_{p,p+q} & & \Big\downarrow d_{p+1,p+q} & & \\
\cdots \subset & & \mathcal{F}_{p-1}\mathcal{C}_{p+q-1} & \subset & \mathcal{F}_p\mathcal{C}_{p+q-1} & \subset & \mathcal{F}_{p+1}\mathcal{C}_{p+q-1} & \subset \cdots & \\
 & & \Big\downarrow d_{p-1,p+q-1} & & \Big\downarrow d_{p,p+q-1} & & \Big\downarrow d_{p+1,p+q-1} & &
\end{array}$$

Figure 1.6: Tower of subcomplexes of $\mathcal{C}$

Note that $\mathcal{E}^0_{p,*}$ represents the complex

$$\cdots \to \mathcal{E}^0_{p,q+1} \to \mathcal{E}^0_{p,q} \to \mathcal{E}^0_{p,q-1} \to \cdots ,$$

which by (1.6) is the complex

$$\cdots \to \mathcal{F}_p\mathcal{C}_{p+q+1}/\mathcal{F}_{p-1}\mathcal{C}_{p+q+1} \xrightarrow{\widehat{d}_{p,p+q+1}}$$
$$\mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q} \xrightarrow{\widehat{d}_{p,p+q}} \mathcal{F}_p\mathcal{C}_{p+q-1}/\mathcal{F}_{p-1}\mathcal{C}_{p+q-1} \to \cdots . \tag{1.7}$$

The maps in (1.7) are just the maps induced by the maps of the complex

$$\cdots \xrightarrow{d_{p,p+q+2}} \mathcal{F}_p\mathcal{C}_{p+q+1} \xrightarrow{d_{p,p+q+1}} \mathcal{F}_p\mathcal{C}_{p+q} \xrightarrow{d_{p,p+q}} \cdots ,$$

which is found as a column in Figure 1.6. So now we can define our second bigraded object in the spectral sequence, which we denote $\mathcal{E}^1 = \{\mathcal{E}^1_{p,q}\}$, as

$$\mathcal{E}^1_{p,q} = \mathcal{H}_{p+q}(\mathcal{E}^0_{p,*}) = \ker(\widehat{d}_{p,p+q})/\operatorname{im}(\widehat{d}_{p,p+q+1}). \tag{1.8}$$

This second object, $\mathcal{E}^1_{p,q}$, is just the homology at the $p+q$ spot of the complex $\mathcal{E}^0_{p,*}$.

1.3.2 Useful Notation

Next we need to make some definitions. Let n_p be the surjection

$$n_p : \mathcal{F}_p\mathcal{C}_{p+q+i} \to \mathcal{F}_p\mathcal{C}_{p+q+i}/\mathcal{F}_{p-1}\mathcal{C}_{p+q+i} \; (= \mathcal{E}^0_{p,q+i}) \tag{1.9}$$

from the object in the $p+q+i$ spot of the complex $\mathcal{F}_p\mathcal{C}$ to the object in the $p+q+i$ spot of the complex $\mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C}$. Define

$$\mathcal{A}^r_{p,q} = \{c \in \mathcal{F}_p\mathcal{C}_{p+q} : d_{p,p+q}(c) \in \mathcal{F}_{p-r}\mathcal{C}_{p+q-1}\} \subset \mathcal{F}_p\mathcal{C}_{p+q} \tag{1.10}$$

and define

$$\mathcal{Z}^r_{p,q} = n_p(\mathcal{A}^r_{p,q}) \subset \mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q} = \mathcal{E}^0_{p,q}. \tag{1.11}$$

Next define

$$\mathcal{B}^r_{p,q} = n_p(d_{p+r-1,p+q}(\mathcal{A}^{r-1}_{p+r-1,q-r+1})) \tag{1.12}$$

which is contained in $n_p(\mathcal{F}_{(p+r-1)-(r-1)}\mathcal{C}_{p+q-1}) = n_p(\mathcal{F}_p\mathcal{C}_{p+q-1})$ by (1.10), and this equals $\mathcal{F}_p\mathcal{C}_{p+q-1}/\mathcal{F}_{p-1}\mathcal{C}_{p+q-1}$ by (1.9), which is $\mathcal{E}^0_{p,q-1}$.

Let us look closer at these objects.

By (1.10) we have

$$\cdots \subset A^{r+1}_{p,q} \subset \mathcal{A}^{r}_{p,q} \subset \mathcal{A}^{r-1}_{p,q} \subset \cdots \subset \mathcal{A}^{1}_{p,q} \subset \mathcal{A}^{0}_{p,q} = \mathcal{F}_p\mathcal{C}_{p+q}.$$

So then

$$\cdots \subset n_p(\mathcal{A}^{r+1}_{p,q}) \subset n_p(\mathcal{A}^{r}_{p,q}) \subset n_p(\mathcal{A}^{r-1}_{p,q}) \subset \cdots \subset n_p(\mathcal{A}^{1}_{p,q}) \subset n_p(\mathcal{F}_p\mathcal{C}_{p+q}),$$

which is

$$\cdots \subset \mathcal{Z}^{r+1}_{p,q} \subset \mathcal{Z}^{r}_{p,q} \subset \mathcal{Z}^{r-1}_{p,q} \subset \cdots \subset \mathcal{Z}^{1}_{p,q} \subset \mathcal{Z}^{0}_{p,q} = \mathcal{E}^{0}_{p,q}.$$

Now let us show that $\mathcal{B}^r_{p,q+1} \subset \mathcal{B}^{r+1}_{p,q+1}$, where

$$\mathcal{B}^r_{p,q+1} = n_p(d_{p+r-1,p+q+1}(A^{r-1}_{p+r-1,q-r+2})) \text{ and } \mathcal{B}^{r+1}_{p,q+1} = n_p(d_{p+r,p+q+1}(A^{r}_{p+r,q-r+1})).$$

Let $x \in \mathcal{B}^r_{p,q+1}$. So $x = n_p(d_{p+r-1,p+q+1}(a))$, where

$$a \in A^{r-1}_{p+r-1,q-r+2} = \{c \in \mathcal{F}_{p+r-1}\mathcal{C}_{p+q+1} : d_{p+r-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\}. \tag{1.13}$$

Since $n_p(d_{p+r-1,p+q+1}(a)) = n_p(d_{p+r,p+q+1}(a))$, all we have to show is that

$$a \in A^{r}_{p+r,q-r+1} = \{c \in \mathcal{F}_{p+r}\mathcal{C}_{p+q+1} : d_{p+r,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\}.$$

This is true because: first, $a \in \mathcal{F}_{p+r-1}\mathcal{C}_{p+q+1} \subset \mathcal{F}_{p+r}\mathcal{C}_{p+q+1}$, and second, $d_{p+r,p+q+1}(a)$ $= d_{p+r-1,p+q+1}(a) \in \mathcal{F}_p\mathcal{C}_{p+q}$. So $\mathcal{B}^r_{p,q+1} \subset \mathcal{B}^{r+1}_{p,q+1}$. Consider the set $d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \subset$ $\mathcal{F}_p\mathcal{C}_{p+q}$. For any r,

$$d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \subset \mathcal{A}^{r}_{p,q} \tag{1.14}$$

since

$$d_{p,p+q} \circ d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) = d_{p+r-1,p+q} \circ d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) = 0.$$

Apply n_p to (1.14) and we obtain $n_p(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2})) \subset n_p(\mathcal{A}^{r}_{p,q})$, which is

$$\mathcal{B}^r_{p,q+1} \subset \mathcal{Z}^r_{p,q}$$

for any r. Also notice that

$$
\begin{aligned}
B^0_{p,q+1} &= n_p(d_{p-1,p+q+1}(\mathcal{A}^{-1}_{p-1,q+2})); \\
&= n_p(d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\})); \\
&= d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\})/ \\
&\quad (d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\}) \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q}), \\
&\quad \text{by definition of } n_p; \\
&= d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\})/ \\
&\quad d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\}), \\
&\quad \text{since } d_{p-1,p+q+1}(\{c \in \mathcal{F}_{p-1}\mathcal{C}_{p+q+1} : d_{p-1,p+q+1}(c) \in \mathcal{F}_p\mathcal{C}_{p+q}\}) \\
&\quad \subset d_{p-1,p+q+1}(\mathcal{F}_{p-1}\mathcal{C}_{p+q+1}) \subset \mathcal{F}_{p-1}\mathcal{C}_{p+q}; \\
&= 0.
\end{aligned}
$$

So we have a tower of sets

$$
\begin{aligned}
0 = \mathcal{B}^0_{p,q+1} \subset \mathcal{B}^1_{p,q+1} \subset \cdots \subset \mathcal{B}^r_{p,q+1} \subset \mathcal{B}^{r+1}_{p,q+1} \subset \cdots \\
\cdots \subset \mathcal{Z}^{r+1}_{p,q} \subset \mathcal{Z}^r_{p,q} \subset \cdots \subset \mathcal{Z}^1_{p,q} \subset \mathcal{Z}^0_{p,q} = \mathcal{E}^0_{p,q}.
\end{aligned}
$$

1.3.3 Constructing the Maps and the Remaining Bigraded Objects

We will frequently use the fact that

$$\mathcal{A}^r_{p,q} \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q} = \mathcal{A}^{r-1}_{p-1,q+1}, \tag{1.15}$$

which immediately follows from (1.10). Also we will want to use the following isomorphism:

$$\mathcal{Z}^r_{p,q} \simeq \mathcal{A}^r_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1}. \tag{1.16}$$

This isomorphism holds because

$$\mathcal{Z}^r_{p,q} = n_p(\mathcal{A}^r_p) \text{ by definition of } \mathcal{Z}^r_{p,q};$$

$$\simeq \mathcal{A}^r_{p,q}/(\mathcal{A}^r_{p,q} \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q}), \text{ by the first isomorphism theorem applied to}$$

$$n_p|_{\mathcal{A}^r_{p,q}} : \mathcal{A}^r_{p,q} \to \mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q};$$

$$\simeq \mathcal{A}^r_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1}, \text{ by (1.15).}$$

See Appendix B for a description of the first, second, and third isomorphism theorems.

Let us now define $\mathcal{E}^r_p$:

$$\mathcal{E}^r_{p,q} \stackrel{dfn}{=} \mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q+1};$$

$$= \mathcal{Z}^r_{p,q}/(n_p(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}))), \text{ by definition of } \mathcal{B}^r_{p,q+1};$$

$$\simeq (\mathcal{A}^r_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1})/(n_p(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}))), \text{ by (1.16);}$$

$$= (\mathcal{A}^r_{p,q}/(\mathcal{A}^r_{p,q} \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q}))/(n_p(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}))), \text{ by (1.15);}$$

$$= (\mathcal{A}^r_{p,q}/(\mathcal{A}^r_{p,q} \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q}))$$

$$/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2})/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q})),$$

by definition of n_p since

$$d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \subset \mathcal{F}_{p-1}\mathcal{C}_{p+q};$$

$$\simeq ((\mathcal{A}^r_{p,q} + \mathcal{F}_{p-1}\mathcal{C}_{p+q})/\mathcal{F}_{p-1}\mathcal{C}_{p+q})$$

$$/((d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{F}_{p-1}\mathcal{C}_{p+q})/\mathcal{F}_{p-1}\mathcal{C}_{p+q}),$$

by the second isomorphism theorem;

$$\simeq (\mathcal{A}^r_{p,q} + \mathcal{F}_{p-1}\mathcal{C}_{p+q})/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{F}_{p-1}\mathcal{C}_{p+q}),$$

by the third isomorphism theorem;

$$\simeq \mathcal{A}^r_{p,q}/(\mathcal{A}^r_{p,q} \bigcap (d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{F}_{p-1}\mathcal{C}_{p+q})),$$

by the second isomorphism theorem since $d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \subset \mathcal{A}^r_{p,q}$;

$$= \mathcal{A}^r_{p,q}/((\mathcal{A}^r_{p,q} \bigcap d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2})) + (\mathcal{A}^r_{p,q} \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q}));$$

$$= \mathcal{A}^r_{p,q}/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}),$$

since $d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \subset \mathcal{A}^r_{p,q}$ and by (1.15).

So now we can define

$$\mathcal{E}^r_{p,q} = \mathcal{A}^r_{p,q}/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}). \tag{1.17}$$

We are on our way to having a spectral sequence. So far we have constructed the bigraded objects. Consider the map $d_{p,p+q} : \mathcal{F}_p\mathcal{C}_{p+q} \to \mathcal{F}_p\mathcal{C}_{p+q-1}$. It induces a map

$$d^r_{p,q} : \mathcal{E}^r_{p,q} \to \mathcal{E}^r_{p-r,q+r-1},$$

i.e.

$$d^r_{p,q} : \mathcal{A}^r_{p,q}/(d(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}) \to \mathcal{A}^r_{p-r,q+r-1}/(d(\mathcal{A}^{r-1}_{p-1,q+1}) + \mathcal{A}^{r-1}_{p-r-1,q+r})$$

which takes $[c] \mapsto [d_{p,p+q}(c)]$.

Next we will prove that $\mathcal{B}^{r+1}_{p-r,q+r}/\mathcal{B}^r_{p-r,q+r} \simeq \mathcal{Z}^r_{p,q}/\mathcal{Z}^{r+1}_{p,q}$, an isomorphism which will be useful in finding the image of $d^r_{p,q}$, which in turn is used to prove that $H(\mathcal{E}^r) = \mathcal{E}^{r+1}$. Notice that

$$\begin{aligned}
\mathcal{B}^r_{p,q} &= n_p(d_{p+r-1,p+q}(\mathcal{A}^{r-1}_{p+r-1,q-r+1}));\\
&\simeq d_{p+r-1,p+q}(\mathcal{A}^{r-1}_{p+r-1,q-r+1})/(d_{p+r-1,p+q}(\mathcal{A}^{r-1}_{p+r-1,q-r+1}) \bigcap \mathcal{F}_{p-1}\mathcal{C}_{p+q-1}),\\
&\quad \text{by definition of } n_p;\\
&\simeq d_{p+r-1,p+q}(\mathcal{A}^{r-1}_{p+r-1,q-r+1})/d_{p+r-1,p+q}(\mathcal{A}^r_{p+r-1,q-r+1}),\ \text{by a check of definitions.}
\end{aligned} \tag{1.18}$$

And similarly that

$$\begin{aligned}
\mathcal{B}^{r+1}_{p-r,q+r} &= n_{p-r}(d_{p,p+q}(\mathcal{A}^r_{p,q}));\\
&\simeq d_{p,p+q}(\mathcal{A}^r_{p,q})/(d_{p,p+q}(\mathcal{A}^r_{p,q}) \bigcap \mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}),\ \text{by definition of } n_{p-r};\\
&\simeq d_{p,p+q}(\mathcal{A}^r_{p,q})/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}),\ \text{by a check of definitions.}
\end{aligned} \tag{1.19}$$

It will also be useful to show that

$$d_{p-1,p+q}(\mathcal{A}^r_{p-1,q+1}) = d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}) \bigcap d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1}). \tag{1.20}$$

Let us prove that (1.20) is true. First we will show that the left hand side is contained in the right hand side. Since

$$\mathcal{A}^r_{p-1,q+1} \subset \mathcal{A}^{r-1}_{p-1,q+1}$$

implies that $d_{p-1,p+q}(\mathcal{A}^r_{p-1,q+1}) \subset d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1})$, all we have left to show is that $d_{p-1,p+q}(\mathcal{A}^r_{p-1,q+1}) \subset d_{p,p+q}(\mathcal{A}^{r+1}_{p,q})$. Let

$$x \in d_{p-1,p+q}(\mathcal{A}^r_{p-1,q+1}) \subset \mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}. \tag{1.21}$$

Then $x = d_{p-1,p+q}(e)$, where e is in $\mathcal{A}^r_{p-1,q+1} \subset \mathcal{F}_{p-1}\mathcal{C}_{p+q} \subset \mathcal{F}_p\mathcal{C}_{p+q}$. Now $x = d_{p-1,p+q}(e) = d_{p,p+q}(e)$, by definition of a subcomplex, and this is in $\mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}$ by (1.21). Hence e is in $\mathcal{A}^{r+1}_{p,q}$, and so x is in $d_{p,p+q}(\mathcal{A}^{r+1}_{p,q})$.

Now we need to show that the right hand side is contained in the left hand side. Let y be in the right hand side. Then y in $d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1})$ implies that $y = d_{p-1,p+q}(u)$, for some u in $\mathcal{A}^{r-1}_{p-1,q+1} \subset \mathcal{F}_{p-1}\mathcal{C}_{p+q}$. So

$$\begin{aligned} y &= d_{p-1,p+q}(u); \\ &= d_{p,p+q}(u), \text{ by definition of a subcomplex;} \\ &\in \mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}, \text{ since } y \in d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}) \subset \mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}. \end{aligned}$$

Therefore u is in $\mathcal{A}^r_{p-1,q+1}$ and hence y is in $d_{p-1,p+q}(\mathcal{A}^r_{p-1,q+1})$.

Using the above isomorphisms, we obtain:

$$\begin{aligned}
&\mathcal{B}^{r+1}_{p-r,q+r}/\mathcal{B}^{r}_{p-r,q+r}\\
&\simeq (d_{p,p+q}(\mathcal{A}^{r}_{p,q})/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}))/(d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1})/d_{p-1,p+q}(\mathcal{A}^{r}_{p-1,q+1})),\\
&\quad \text{by (1.18) and (1.19);}\\
&\simeq (d_{p,p+q}(\mathcal{A}^{r}_{p,q})/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}))/\\
&\quad (d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1})/(d_{p,p+q}(\mathcal{A}^{r+1}_{p,q})\bigcap d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1}))),\ \text{by (1.20);}\\
&\simeq (d_{p,p+q}(\mathcal{A}^{r}_{p,q})/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}))/(d_{p,p+q}(\mathcal{A}^{r+1}_{p,q})+d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1}))/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}),\\
&\quad \text{by the second isomorphism theorem;}\\
&\simeq d_{p,p+q}(\mathcal{A}^{r}_{p,q})/(d_{p,p+q}(\mathcal{A}^{r+1}_{p,q})+d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1})),\\
&\quad \text{by the third isomorphism theorem;}\\
&= d_{p,p+q}(\mathcal{A}^{r}_{p,q})/d_{p,p+q}(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{Z}^{r}_{p,q}/\mathcal{Z}^{r+1}_{p,q} &\simeq (\mathcal{A}^{r}_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1})/(\mathcal{A}^{r+1}_{p,q}/\mathcal{A}^{r}_{p-1,q+1}),\ \text{by (1.16);}\\
&\simeq (\mathcal{A}^{r}_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1})/(\mathcal{A}^{r+1}_{p,q}/(\mathcal{A}^{r+1}_{p,q}\bigcap \mathcal{A}^{r-1}_{p-1,q+1})),\ \text{by (1.15);}\\
&\simeq (\mathcal{A}^{r}_{p,q}/\mathcal{A}^{r-1}_{p-1,q+1})/((\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1})/\mathcal{A}^{r-1}_{p-1,q+1}),\\
&\quad \text{by the second isomorphism theorem;}\\
&\simeq \mathcal{A}^{r}_{p,q}/(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1}),\ \text{by the third isomorphism theorem.}
\end{aligned}$$

Now we will prove that

$$\mathcal{B}^{r+1}_{p-r,q+r}/\mathcal{B}^{r}_{p-r,q+r}\simeq \mathcal{Z}^{r}_{p,q}/\mathcal{Z}^{r+1}_{p,q}. \tag{1.22}$$

Define $f:\mathcal{A}^{r}_{p,q}/(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1})\to d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(\mathcal{A}^{r}_{p,q})/d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1})$ by $[a]\mapsto[d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(a)]$. Since $\mathcal{A}^{r+1}_{p,q}\subset\mathcal{A}^{r}_{p,q}$ and $\mathcal{A}^{r-1}_{p-1,q+1}\subset\mathcal{A}^{r}_{p,q}$, we will define f using the restriction $d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}$. It follows immediately that f is surjective. Let us show that f is injective. Let $[d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(a)]=0$. Then $d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(a)\in d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1})$. This implies that a is in $(d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}})^{-1}\circ d_{p,p+q}|_{\mathcal{A}^{r}_{p,q}}(\mathcal{A}^{r+1}_{p,q}+\mathcal{A}^{r-1}_{p-1,q+1})$, which implies that a is in $\mathcal{A}^{r+1}_{p,q} + \mathcal{A}^{r-1}_{p-1,q+1} + \ker(d_{p,p+q}|_{\mathcal{A}^r_{p,q}}) = \mathcal{A}^{r+1}_{p,q} + \mathcal{A}^{r-1}_{p-1,q+1} + \ker(d_{p,p+q} : \mathcal{A}^r_{p,q} \to \mathcal{F}_{p-r}\mathcal{C}_{p+q-1})$. This is just $\mathcal{A}^{r+1}_{p,q} + \mathcal{A}^{r-1}_{p-1,q+1}$ because: if a is in $\ker(d_{p,p+q} : \mathcal{A}^r_{p,q} \to \mathcal{F}_{p-r}\mathcal{C}_{p+q-1})$, then $d_{p,p+q}(a) = 0$ and so a is in $\mathcal{A}^{r+1}_{p,q}$ and hence $\ker(d_{p,p+q}|_{\mathcal{A}^r_{p,q}}) \subset \mathcal{A}^{r+1}_{p,q} \subset \mathcal{A}^{r+1}_{p,q} + \mathcal{A}^{r-1}_{p-1,q+1}$. Therefore a is in $\mathcal{A}^{r+1}_{p,q} + \mathcal{A}^{r-1}_{p-1,q+1}$ and so $[a] = 0$. So we have shown that f is injective, hence f is an isomorphism and (1.22) is true.

Consider again the map

$$d^r_{p,q} : \mathcal{E}^r_{p,q} \to \mathcal{E}^r_{p-r,q+r-1},$$

$$d^r_{p,q} : \mathcal{A}^r_{p,q}/(d(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}) \to \mathcal{A}^r_{p-r,q+r-1}/(d(\mathcal{A}^{r-1}_{p-1,q+1}) + \mathcal{A}^{r-1}_{p-r-1,q+r}).$$

For the kernel of $d^r_{p,q}$, we have

$$\begin{aligned}
\ker(d^r_{p,q}) &= \{z \in \mathcal{A}^r_{p,q} : d_{p,p+q}(z) \in d_{p-1,p+q}(\mathcal{A}^{r-1}_{p-1,q+1}) + \mathcal{A}^{r-1}_{p-r-1,q+r}\}/ \\
&\quad (d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}); \\
&= (\mathcal{A}^{r-1}_{p-1,q+1} + \mathcal{A}^{r+1}_{p,q})/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1}), \\
&\quad \text{by a lengthy check of definitions ;} \\
&\simeq ((\mathcal{A}^{r-1}_{p-1,q+1} + \mathcal{A}^{r+1}_{p,q})/\mathcal{A}^{r-1}_{p-1,q+1})/ \\
&\quad ((d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) + \mathcal{A}^{r-1}_{p-1,q+1})/\mathcal{A}^{r-1}_{p-1,q+1}), \\
&\quad \text{by the third isomorphism theorem;} \\
&\simeq (\mathcal{A}^{r+1}_{p,q}/(\mathcal{A}^{r-1}_{p-1,q+1} \bigcap \mathcal{A}^{r+1}_{p,q}))/ \\
&\quad (d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2})/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2}) \bigcap \mathcal{A}^{r-1}_{p-1,q+1})), \\
&\quad \text{by the second isomorphism theorem;} \\
&\simeq (\mathcal{A}^{r+1}_{p,q}/\mathcal{A}^r_{p-1,q+1})/(d_{p+r-1,p+q+1}(\mathcal{A}^{r-1}_{p+r-1,q-r+2})/d_{p+r-1,p+q+1}(\mathcal{A}^r_{p+r-1,q-r+2})), \\
&\quad \text{by a check of definitions;} \\
&= \mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}, \text{ by (1.16) and (1.18).}
\end{aligned}$$

So

$$\ker(d^r_{p,q}) = \mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}. \tag{1.23}$$

Computing the image of $d^r_{p,q}$, we obtain

$$\begin{aligned}
\operatorname{im}(d^r_{p,q}) &\simeq \mathcal{E}^r_{p,q}/\ker(d^r_{p,q}), \text{ by the first isomorphism theorem;}\\
&\simeq \mathcal{E}^r_{p,q}/(\mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}), \text{ by (1.23);}\\
&\simeq (\mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q+1})/(\mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}), \text{ by the definition of } \mathcal{E}^r_{p,q};\\
&\simeq \mathcal{Z}^r_{p,q}/\mathcal{Z}^{r+1}_{p,q};\\
&\simeq \mathcal{B}^{r+1}_{p-r,q+r}/\mathcal{B}^r_{p-r,q+r} \text{ by (1.22).}
\end{aligned}$$

Replace p with $p+r$ and q with $q-r+1$ and obtain

$$\operatorname{im}(d^r_{p+r,q-r+1}) = \mathcal{B}^{r+1}_{p,q+1}/\mathcal{B}^r_{p,q+1}. \tag{1.24}$$

Finally,

$$\begin{aligned}
\mathcal{E}^{r+1}_{p,q} &= \mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^{r+1}_{p,q+1}, \text{ by definition;}\\
&\simeq (\mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1})/(\mathcal{B}^{r+1}_{p,q+1}/\mathcal{B}^r_{p,q+1});\\
&\simeq \ker(d^r_{p,q})/\operatorname{im}(d^r_{p+r,q-r+1}), \text{ by (1.23) and (1.24);}\\
&= H(\mathcal{E}^r_{p,q}).
\end{aligned}$$

Remark. $\mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}, \mathcal{B}^{r+1}_{p,q+1}/\mathcal{B}^r_{p,q+1}$ in this section correspond to $\mathcal{Z}^{r+1}_{p,q}, \mathcal{B}^{r+1}_{p,q}$, respectively, in the definition of a spectral sequence.

1.4 Example 3: Obtained From a Double Complex

In this section we define the notions of a double complex and a total complex and show how to obtain a spectral sequence from the filtration of the total complex.

Definition 1.4.1 *A* ***double complex*** *is a family $\{\mathcal{C}_{p,q}\}$ of objects of an abelian category together with maps $d^h = \{d^h_{p,q}\}$ and $d^v = \{d^v_{p,q}\}$ where*

$$d^h_{p,q} : \mathcal{C}_{p,q} \to \mathcal{C}_{p-1,q} \text{ and } d^v_{p,q} : \mathcal{C}_{p,q} \to \mathcal{C}_{p,q-1},$$

such that $d^h \circ d^h = 0, d^v \circ d^v = 0,$ and $d^v d^h + d^h d^v = 0$.

Each row and each column of the double complex $\mathcal{C}_{**}$ is a complex. The last property $d^v_{p-1,q} \circ d^h_{p,q} = -d^h_{p,q-1} \circ d^v_{p,q}$ means that each square as in Figure 1.7 *anti-commutes*. A double complex is called *bounded* if each diagonal line $p+q=n$ has only finitely many nonzero terms. Since $p, q \in \mathbb{Z}$, we can view $\mathcal{C}_{**}$ as a lattice. Figure 1.7 shows a portion of the lattice $\mathcal{C}_{**}$.

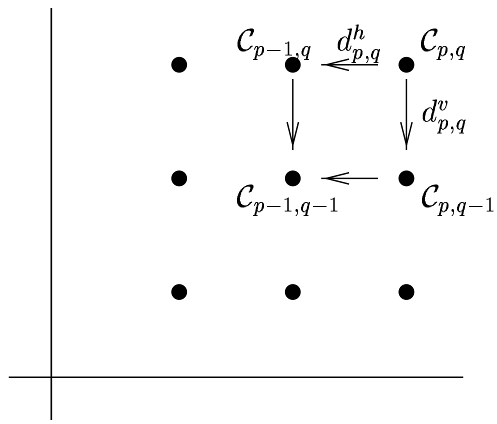

Figure 1.7: Viewing the double complex C_{**}

Definition 1.4.2 *We define the complex associated with the double complex $\mathcal{C}_{**}$ to be the total complex* $\mathrm{Tot}(\mathcal{C})$. *Here,* $\mathrm{Tot}(\mathcal{C}) = \{\mathrm{Tot}(\mathcal{C})_n\}$, $n \in \mathbb{Z}$, *where* $\mathrm{Tot}(\mathcal{C})_n = \prod_{p+q=n} \mathcal{C}_{p,q}$. *We define* $d = \prod(d^v + d^h) : \mathrm{Tot}(\mathcal{C})_n \to \mathrm{Tot}(\mathcal{C})_{n-1}$.

The total complex is easily viewed to be a complex of objects, each object is just the direct product of objects of $\mathcal{C}_{**}$ along a diagonal. See Figure 1.8.
The map

$$d = \prod (d^v + d^h) : \mathrm{Tot}(\mathcal{C})_n \to \mathrm{Tot}(\mathcal{C})_{n-1}$$

is really a family of maps $\{d_{p,q} = d^v_{p,q} + d^h_{p+1,q-1}\}$, each map taking an element of $\mathcal{C}_{p,q} \times \mathcal{C}_{p+1,q-1}$ to $\mathcal{C}_{p,q-1}$. See Figure 1.9. (Note that the maps are without their subscripts).

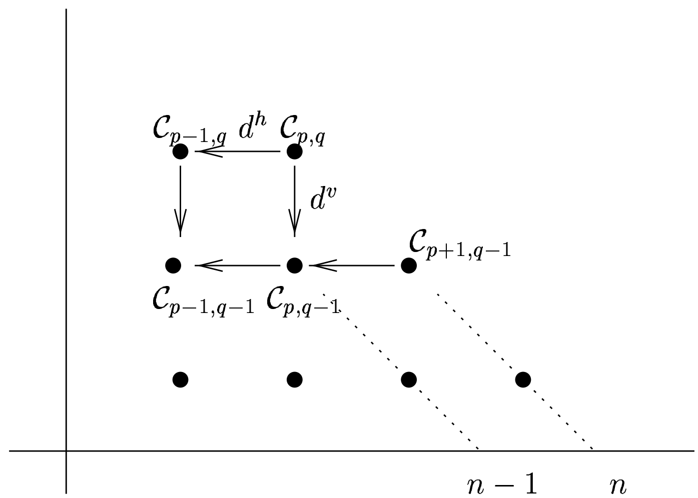

Figure 1.8: In the total complex, view the cross product along each diagonal

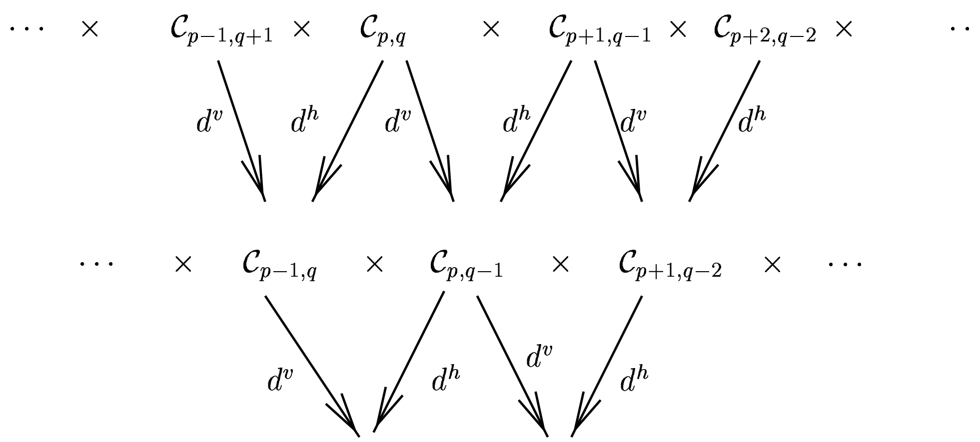

Figure 1.9: Viewing $\mathrm{Tot}(\mathcal{C})_n \xrightarrow{d} \mathrm{Tot}(\mathcal{C})_{n-1} \xrightarrow{d} \mathrm{Tot}(\mathcal{C})_{n-2}$

Let us verify that $\mathrm{Tot}(\mathcal{C})$ with the map d described above is a complex. Notice that

$$d = \prod (d^v_{p,q} + d^h_{p+1,q-1})$$

means

$$d = \prod_{i \in \mathbb{Z}} (d^v_{p+i,q-i} + d^h_{p+i+1,q-i-1}).$$

So

$$
\begin{aligned}
d \circ d &= \prod_{i \in \mathbb{Z}} (d^v_{p+i,q-i} + d^h_{p+i+1,q-i-1}) \circ \prod_{i \in \mathbb{Z}} (d^v_{p+i,q-i} + d^h_{p+i+1,q-i-1}); \\
&= [\cdots \times (d^v_{p-1,q} + d^h_{p,q-1}) \times (d^v_{p,q-1} + d^h_{p+1,q-2}) \times (d^v_{p+1,q-2} + d^h_{p+2,q-3}) \times \cdots] \\
&\quad \circ [\cdots \times (d^v_{p-1,q+1} + d^h_{p,q}) \times (d^v_{p,q} + d^h_{p+1,q-1}) \times (d^v_{p+1,q-1} + d^h_{p+2,q-2}) \times \cdots]; \\
&= \cdots \times d^v_{p-1,q}(d^v_{p-1,q+1} + d^h_{p,q}) + d^h_{p,q-1}(d^v_{p,q} + d^h_{p+1,q-1}) \times \\
&\quad d^v_{p,q-1}(d^v_{p,q} + d^h_{p+1,q-1}) + d^h_{p+1,q-2}(d^v_{p+1,q-1} + d^h_{p+2,q-2}) \times \cdots; \\
&= \cdots \times d^v_{p,q-1} d^v_{p-1,q+1} + d^v_{p-1,q} d^h_{p,q} + d^h_{p,q-1} d^v_{p,q} + d^h_{p,q-1} d^h_{p+1,q-1} \times \cdots; \\
&= 0 + d^v_{p-1,q} d^h_{p,q} + d^h_{p,q-1} d^v_{p,q} + 0, \text{ since each column and row of } \mathcal{C}_{**} \text{ is a complex;} \\
&= 0, \text{ by the anti-commutativity of the double complex.}
\end{aligned}
$$

So $\operatorname{Tot}(\mathcal{C})$ is indeed a complex. Recall from Example 2 of this Chapter that if we have a filtration of a complex, we can construct a spectral sequence. We will associate two different filtrations with the total complex.

But let us restrict ourselves to the case of the total complex $\operatorname{Tot}(\mathcal{M})$, where $\mathcal{M}$ is a *first quadrant double complex*. The reason for this restriction will be explained in the next Chapter. A *first quadrant double complex* $\mathcal{M} = \{\mathcal{M}_{p,q}\}$ is one where $\mathcal{M}_{p,q}$ is zero if p or $q \leq 0$. We will construct a filtration of $\operatorname{Tot}(\mathcal{M})$,

$$0 \subset \cdots \subset \mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}) \subset \mathcal{F}_p \operatorname{Tot}(\mathcal{M}) \subset \mathcal{F}_{p+1} \operatorname{Tot}(\mathcal{M}) \subset \cdots \subset \operatorname{Tot}(\mathcal{M}),$$

that will include the subcomplexes 0 and $\operatorname{Tot}(\mathcal{M})$. Describe the subcomplex $\mathcal{F}_p \operatorname{Tot}(\mathcal{M})$ by

$$(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n = \prod_{i \leq p} \mathcal{M}_{i,n-i}.$$

This subcomplex $\mathcal{F}_p \operatorname{Tot}(\mathcal{M})$ is a complex of objects, each object a direct product of particular objects of $\mathcal{M}$ along a diagonal $p + q = n$. See Figure 1.10.

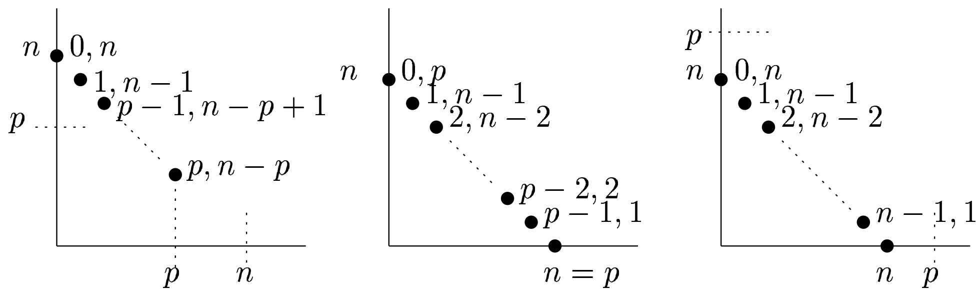

Figure 1.10: Viewing $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n$, an object in the subcomplex $\mathcal{F}_p \operatorname{Tot}(\mathcal{M})$

The map from $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n$ to $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_{n-1}$ is the restriction of the map from $\operatorname{Tot}(\mathcal{M})_n$ to $\operatorname{Tot}(\mathcal{M})_{n-1}$, so we indeed have a tower of subcomplexes. This is made clear by diagram (1.25):

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
0 \subset \cdots & \subset & (\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_n & \subset & (\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n & \subset \cdots \subset & \operatorname{Tot}(\mathcal{M})_n \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \subset \cdots & \subset & (\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_{n-1} & \subset & (\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_{n-1} & \subset \cdots \subset & \operatorname{Tot}(\mathcal{M})_{n-1}. \\
 & & \downarrow & & \downarrow & & \downarrow
\end{array} \tag{1.25}$$

Indeed we have a filtration of $\operatorname{Tot}(\mathcal{M})$ since we already saw that $d^2 = 0$ for a total complex. Hence by the previous section, this filtration of a complex gives a spectral sequence. We call this filtration the *first filtration*.

Remark. We could also define another filtration of $\operatorname{Tot}(\mathcal{M})$, which we call the *second filtration*, by

$$(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n = \prod_{j \leq p} \mathcal{M}_{n-j,j}.$$

1.5 A Filtration of a Complex Gives an Exact Couple

In this section, we will show that given any complex with a filtration, we obtain an exact couple.

Definition 1.5.1 *A* ***morphism of complexes*** $f : A \to B$ *is a sequence of morphisms* $\{f_n : A_n \to B_n\}$ *such that each square*

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ \downarrow & & \downarrow \\ A_{n+1} & \xrightarrow[f_{n+1}]{} & B_{n+1} \end{array}$$

commutes. Similarly, in a sequence of complexes

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots,$$

all squares will commute.

Definition 1.5.2 *In an* ***exact sequence of complexes*** $\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$,

$$\cdots \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to \cdots$$

is exact for all n.

Take a filtration of $\mathcal{C}$

$$\cdots \subset \mathcal{F}_{p-1}\mathcal{C} \subset \mathcal{F}_p\mathcal{C} \subset \mathcal{F}_{p+1}\mathcal{C} \subset \cdots.$$

Then we obtain a short exact sequence (of complexes) for each p,

$$0 \to \mathcal{F}_{p-1}\mathcal{C} \to \mathcal{F}_p\mathcal{C} \to \mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C} \to 0, \tag{1.26}$$

since $\mathcal{F}_{p-1}\mathcal{C} \to \mathcal{F}_p\mathcal{C}$ is the inclusion map. Hence we obtain the following diagram

$$\begin{array}{ccccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{p-1}\mathcal{C}_i & \longrightarrow & \mathcal{F}_p\mathcal{C}_i & \longrightarrow & \mathcal{F}_p\mathcal{C}_i/\mathcal{F}_{p-1}\mathcal{C}_i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_{p-1}\mathcal{C}_{i-1} & \longrightarrow & \mathcal{F}_p\mathcal{C}_{i-1} & \longrightarrow & \mathcal{F}_p\mathcal{C}_{i-1}/\mathcal{F}_{p-1}\mathcal{C}_{i-1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{array} \tag{1.27}$$

which leads us to the corresponding long exact homology sequence (see Appendix C for a proof of this):

$$\begin{aligned}
&\cdots \to H_{p+q}(\mathcal{F}_{p-1}\mathcal{C}) \to H_{p+q}(\mathcal{F}_p\mathcal{C}) \to H_{p+q}(\mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C}) \\
&\to H_{p+q-1}(\mathcal{F}_{p-1}\mathcal{C}) \to H_{p+q-1}(\mathcal{F}_p\mathcal{C}) \to H_{p+q-1}(\mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C}) \to \cdots .
\end{aligned} \tag{1.28}$$

The maps in this long exact sequence are induced by the obvious maps from the corresponding short exact sequence. For example, the map $F_{p-1}C \xrightarrow{i} F_pC$ from the short exact sequence (1.26), which is really a family $\{i_n\}$ of inclusion maps where $i_n : F_{p-1}C_n \longrightarrow F_pC_n$, induces a map from $H_{p+q}(F_{p-1}C)$ to $H_{p+q}(F_pC)$ in the long exact sequence (1.28). At this point it is extremely helpful to consult Figure 1.6 on page 12.

Let us define

$$\mathcal{D}_{p,q} = H_{p+q}(\mathcal{F}_p\mathcal{C}) \text{ and } \mathcal{E}_{p,q} = H_{p+q}(\mathcal{F}_p\mathcal{C}/\mathcal{F}_{p-1}\mathcal{C}). \tag{1.29}$$

Then the particular induced map we are using as an example goes from $D_{p-1,q+1}$ to $D_{p,q}$. We will conveniently label this map $\alpha_{p-1,q+1}$. Doing the replacements from (1.29) in the long exact sequence (1.28) and inserting the induced maps gives

$$\cdots \to \mathcal{D}_{p-1,q+1} \xrightarrow{\alpha_{p-1,q+1}} \mathcal{D}_{p,q} \xrightarrow{\beta_{p,q}} \mathcal{E}_{p,q} \xrightarrow{\gamma_{p,q}} \mathcal{D}_{p-1,q} \xrightarrow{\alpha_{p-1,q}} \mathcal{D}_{p,q-1} \xrightarrow{\beta_{p,q-1}} \cdots .$$

The maps α, β, γ have bidegrees $(1,-1), (0,0), (-1,0)$ respectively. So we recognize that this gives us the type of exact couple that will lead us to a spectral sequence, as in Example 1 of this Chapter.

Remark. We did not use superscripts in this exact couple of bigraded objects. Before we use this exact couple to get a spectral sequence, we would put in the superscript 1, i.e. rename $D_{p,q}$ as $D^1_{p,q}$.

CHAPTER 2

LIMIT, BOUNDEDNESS, AND CONVERGENCE OF THE SPECTRAL SEQUENCE

In this Chapter we define the limit term, boundedness and convergence of a spectral sequence, and give some examples.

2.1 Limit Term

In this section we will define the limit term.

Definition 2.1.1 *Consider a spectral sequence $\{\mathcal{E}^r, d^r\}$. Suppose for each p, q there is some $r = r(p,q)$ such that*

$$\mathcal{E}^r_{p,q} = \mathcal{E}^{r+1}_{p,q} = \mathcal{E}^{r+2}_{p,q} = \cdots .$$

We call this stable value $\mathcal{E}^\infty_{p,q}$, and say that the spectral sequence abuts to $\mathcal{E}^\infty = \{\mathcal{E}^\infty_{p,q}\}$. The bigraded object $\mathcal{E}^\infty$ is called the ***limit term*** *of the sequence. (So $\mathcal{E}^\infty_{p,q} = \mathcal{E}^{r(p,q)}_{p,q}$.)*

In order to give an example of a spectral sequence that has a limit term, it will be helpful to define the notion of a *bounded* filtration.

Definition 2.1.2 *A filtration $\mathcal{F}_p\mathcal{C}$ of a complex $\mathcal{C}$ is called* ***bounded*** *if for each n, there is $s = s(n) < t = t(n)$ such that*

$$\mathcal{F}_s\mathcal{C}_n = 0 \text{ and } \mathcal{F}_t\mathcal{C}_n = \mathcal{C}_n.$$

Here are some examples of spectral sequences that have a limit term.

Example 1. If we make the assumption that the filtration in Example 2 of Chapter 1 is bounded, then we will have a limit term. Take any p, q. We want to find an r_0 such that $\mathcal{E}^r_{p,q} = \mathcal{E}^{r+1}_{p,q}$ for $r \geq r_0$. Let us first look at $\mathcal{E}^r_{p,q}$ for any r. We get $d^r_{p,q}(\mathcal{E}^r_{p,q}) \subset \mathcal{E}^r_{p-r,q+r-1}$ since $d^r_{p,q}$ has bidegree $(-r, r-1)$. Now choose r such that $p - r < s(n)$, where $n = p + q - 1$. Then $\mathcal{F}_{s(n)}\mathcal{C}_{p+q-1} = 0$, so for $p - r < s(n)$,

$$\cdots = \mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1} = \mathcal{F}_{p-r}\mathcal{C}_{p+q-1} = 0. \tag{2.1}$$

Therefore we have

$$\begin{aligned} \mathcal{E}^1_{p-r,q+r-1} &= H_{p+q-1}(E^0_{p-r,*}) \text{ by (1.8);} \\ &= \text{ homology at the spot } \mathcal{F}_{p-r}\mathcal{C}_{p+q-1}/\mathcal{F}_{p-r-1}\mathcal{C}_{p+q-1}, \text{ by (1.7);} \\ &= 0, \text{ by (2.1).} \end{aligned}$$

Since $\mathcal{E}^r_{p-r,q+r-1}$ is a subquotient of $\mathcal{E}^1_{p-r,q+r-1}$, we have that $\mathcal{E}^r_{p-r,q+r-1} = 0$. (Suppose $\mathcal{E}^1_{p,q} = 0$. Then $\mathcal{E}^2_{p,q} = (\text{subobject of } \mathcal{E}^1_{p,q})/(\text{subobject of } \mathcal{E}^1_{p,q}) = 0$. Do this $r - 2$ more times to get $\mathcal{E}^r_{p,q} = 0$.) So $d^r_{p,q}(\mathcal{E}^r_{p,q}) \subset \mathcal{E}^r_{p-r,q+r-1} = 0$ and hence $\mathcal{E}^r_{p,q} = \ker(d^r_{p,q})$.

Next look at $\mathcal{E}^{r+1}_{p,q}$. We have $\mathcal{E}^{r+1}_{p,q} = \ker(d^r_{p,q})/\operatorname{im}(d^r_{p+r,q-r+1})$. Make r even larger if necessary so that $p + r > t(n)$, where $n = p + q + 1$. Then $\mathcal{F}_{t(n)}\mathcal{C}_{p+q+1} = \mathcal{C}$, so for $p + r - 1 > t(n)$,

$$\mathcal{F}_{p+r-1}\mathcal{C}_{p+q+1} = \mathcal{F}_{p+r}\mathcal{C}_{p+q+1} = \cdots = \mathcal{C}. \tag{2.2}$$

Then

$$\begin{aligned} \mathcal{E}^1_{p+r,q-r+1} &= H_{p+q+1}(\mathcal{E}^0_{p+r,*}), \text{ by (1.8);} \\ &= \mathcal{F}_{p+r}\mathcal{C}_{p+q+1}/\mathcal{F}_{p+r-1}\mathcal{C}_{p+q+1}, \text{ by (1.7);} \\ &= 0, \text{ by (2.2).} \end{aligned}$$

Since $\mathcal{E}^r_{p+r,q-r+1}$ is a subquotient of $\mathcal{E}^1_{p+r,q-r+1}$, we have $\mathcal{E}^r_{p+r,q-r+1} = 0$. Therefore

$$\begin{aligned} \operatorname{im}(d^r_{p+r,q-r+1}) &= d^r_{p+r,q-r+1}(\mathcal{E}^r_{p+r,q-r+1}); \\ &= d^r_{p+r,q-r+1}(0); \\ &= 0. \end{aligned}$$

Then $\mathcal{E}^{r+1}_{p,q} = \ker d^r_{p,q}/0 = \ker d^r_{p,q}$. So for r larger than $t(n) - p$,

$$\mathcal{E}^r_{p,q} = \ker(d^r_{p,q}) = \mathcal{E}^{r+1}_{p,q}.$$

So Example 2 of Chapter 1, with the assumption that the filtration is bounded, has a limit term.

Example 2. Here we show that Example 3 of Chapter 1 with $\mathcal{M}$ a first quadrant double complex has a limit term. The assumption that $\mathcal{M}$ is a first quadrant double complex implies that both filtrations mentioned in the example under consideration are bounded: Let n be given. Use $s(n) = -1$ and $t(n) = n$ for both filtrations in the definition of a bounded filtration. This gives $(\mathcal{F}_{-1}\operatorname{Tot}(\mathcal{M}))_n = 0$ and $(\mathcal{F}_n \operatorname{Tot}(\mathcal{M}))_n = \operatorname{Tot}(\mathcal{M})_n$. This satisfies the conditions in Example 1 of this Section and thus we obtain a spectral sequence with a limit term.

Example 3. Let us consider again Example 2 of Chapter 1, the example of a spectral sequence obtained from a filtration of a complex. Recall we had defined objects $\{\mathcal{Z}^r_{p,q}\}$ and $\{\mathcal{B}^r_{p,q}\}$ such that $\mathcal{E}^r_{p,q} = \mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q+1}$, $\ker(d^r_{p,q}) = \mathcal{Z}^{r+1}_{p,q}/\mathcal{B}^r_{p,q+1}$, and $\operatorname{im}(d^r_{p+r,q-r+1}) = \mathcal{B}^{r+1}_{p,q+1}/\mathcal{B}^r_{p,q+1}$. Then we had proven that we had a tower of sets

$$0 = \mathcal{B}^0_{p,q+1} \subset \mathcal{B}^1_{p,q+1} \subset \cdots \subset \mathcal{B}^r_{p,q+1} \subset \mathcal{B}^{r+1}_{p,q+1} \subset \cdots$$
$$\cdots \subset \mathcal{Z}^{r+1}_{p,q} \subset \mathcal{Z}^r_{p,q} \subset \cdots \subset \mathcal{Z}^1_{p,q} \subset \mathcal{Z}^0_{p,q} = \mathcal{E}^0_{p,q}.$$

Let

$$\mathcal{Z}^\infty_{p,q} = \bigcap_{r=1}^{\infty} \mathcal{Z}^r_{p,q} \text{ and } \mathcal{B}^\infty_{p,q+1} = \bigcup_{r=1}^{\infty} \mathcal{B}^r_{p,q+1}, \tag{2.3}$$

and let

$$\mathcal{E}^\infty_{p,q} = \mathcal{Z}^\infty_{p,q}/\mathcal{B}^\infty_{p,q+1}.$$

Suppose that

$$\mathcal{Z}^\infty_{p,q} = \bigcap_{r=1}^{N_1} \mathcal{Z}^r_{p,q} \text{ and } \mathcal{B}^\infty_{p,q+1} = \bigcup_{r=1}^{N_2} \mathcal{B}^r_{p,q+1}$$

for some numbers N_1, N_2. Let $r = \max\{N_1, N_2\}$. Then

$$\mathcal{Z}^r_{p,q} = \mathcal{Z}^{r+1}_{p,q} = \cdots = \mathcal{Z}^\infty_{p,q} \text{ and } \mathcal{B}^r_{p,q+1} = \mathcal{B}^{r+1}_{p,q+1} = \cdots = \mathcal{B}^\infty_{p,q+1}.$$

So $\mathcal{E}^\infty_{p,q} = \mathcal{Z}^\infty_{p,q}/\mathcal{B}^\infty_{p,q+1} = \mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q+1} = \mathcal{E}^r_{p,q}$ for $r = \max\{N_1, N_2\}$, we obtain $\mathcal{E}^r_{p,q} = \mathcal{E}^{r+1}_{p,q} = \cdots = \mathcal{E}^\infty_{p,q}$, and hence we have shown that the spectral sequence has a limit term.

2.2 Boundedness

In this section we will define boundedness of a spectral sequence and give two examples. Also, we will see that a bounded spectral sequence has a limit term.

Definition 2.2.1 *A spectral sequence is called* ***bounded*** *if for each n, the number of nonzero terms of total degree n in $\mathcal{E}^a_{**}$ is finite for each a.*

This means if one looks at the lattice $\mathcal{E}^a_{**}$ for any a, along the diagonal $p + q = n$ for any n one will see only finitely many terms that are not equal to zero.

Example 4. A *first quadrant spectral sequence* is a spectral sequence with $\mathcal{E}^a_{p,q} = 0$ if p or $q < 0$. This is clearly a bounded spectral sequence since in any lattice $\mathcal{E}^a_{**}$, for each n the number of nonzero terms of total degree n is at most $n + 1$.

Example 5. Example 2 of Chapter 1 with the filtration bounded will produce a bounded spectral sequence. To check it, take $n = p+q$. The filtration being bounded implies that for this n there exist $s(n), t(n)$ such that $\mathcal{F}_{s(n)}\mathcal{C}_n = 0$ and $\mathcal{F}_{t(n)}\mathcal{C}_n = \mathcal{C}_n$. Now consider $\mathcal{E}^0_{**}$. Recall that $\mathcal{E}^0_{p,q} = \mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q}$, so the objects along the diagonal $p + q = n$ in $\mathcal{E}^0_{**}$ are:

$$\ldots, \mathcal{E}^0_{p-1,q+1}, \mathcal{E}^0_{p,q}, \mathcal{E}^0_{p+1,q-1}, \ldots,$$

which is

$$\ldots, \mathcal{F}_{p-1}\mathcal{C}_{p+q}/\mathcal{F}_{p-2}\mathcal{C}_{p+q}, \mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q}, \mathcal{F}_{p+1}\mathcal{C}_{p+q}/\mathcal{F}_p\mathcal{C}_{p+q}, \ldots \quad .$$

Now $\mathcal{F}_p\mathcal{C}_{p+q} = 0$ for $p < s(n)$ implies that $\mathcal{F}_p\mathcal{C}_{p+q}/\mathcal{F}_{p-1}\mathcal{C}_{p+q} = 0$ for $p < s(n)$. And $\mathcal{F}_p\mathcal{C}_{p+q} = \mathcal{C}_{p+q}$ for $p > t(n)$ implies that $\mathcal{F}_{p+1}\mathcal{C}_{p+q}/\mathcal{F}_p\mathcal{C}_{p+q} = 0$ for $p > t(n)$. So along the diagonal $n = p+q$ in $\mathcal{E}^0_{**}$, there are no more than $t(n)-s(n)$ nonzero terms. Hence along the diagonal $n = p+q$ in $\mathcal{E}^a_{**}$ for any a, there are no more than $t(n)-s(n)$ nonzero terms, according to the following lemma:

Lemma 2.2.2 *If in $\mathcal{E}^a_{**}$ there are no more than N nonzero terms along $p+q=n$, then in $\mathcal{E}^{a+1}_{**}$ there are no more than N nonzero terms along $p+q=n$.*

Proof Consider the diagonal $n = p+q$ in $\mathcal{E}^{a+1}_{**}$:

$$\ldots, \mathcal{E}^{a+1}_{p-1,q+1}, \mathcal{E}^{a+1}_{p,q}, \mathcal{E}^{a+1}_{p+1,q-1}, \ldots,$$

which is

$$\ldots, H(\mathcal{E}^a_{p-1,q+1}), H(\mathcal{E}^a_{p,q}), H(\mathcal{E}^a_{p+1,q-1}), \ldots \quad .$$

There are only finitely many nonzero terms in this sequence, since there were only finitely many nonzero terms along the $\mathcal{E}^a_{**}$ diagonal $n = p+q$. ■

2.2.1 A Bounded Spectral Sequence has a Limit Term

We will look at two methods to show that a bounded spectral sequence has a limit term.

Method 1. Take p, q. For a bounded spectral sequence, along the diagonal $p+q-1=n$, there is a finite number of nonzero terms. This implies that there exists $S(n)$ such that for all a, $\mathcal{E}^a_{p,n-p} = 0$ for all $p < S(n)$. Similarly, for $p+q+1 = m$, there exists $T(m)$ such that for all a, $\mathcal{E}^a_{m-q,q} = 0$ for all $q < T(m)$. We can find such numbers $S(n)$ and $T(m)$ by Lemma 2.2.2.
Choose

$$r > \max\{p - S(n), q+1-T(m)\}. \tag{2.4}$$

Then

$$r > p - S(n) \text{ implies } p - r < S(n) \text{ implies } \mathcal{E}^r_{p-r,q+r-1} = 0$$
$$\text{implies } d^r_{p,q}(\mathcal{E}^r_{p,q}) = 0 \text{ implies } \ker(d^r_{p,q}) = \mathcal{E}^r_{p,q} \tag{2.5}$$

and

$$r > q + 1 - T(m) \text{ implies } q - r + 1 < T(m) \text{ implies } \mathcal{E}^r_{p+r,q-r+1} = 0$$
$$\text{implies } d^r_{p+r,q-r+1}(\mathcal{E}^r_{p+r,q-r+1}) = 0 \text{ implies } \operatorname{im}(d^r_{p+r,q-r+1}) = 0. \tag{2.6}$$

So $\mathcal{E}^{r+1}_{p,q} = \ker(d^r_{p,q}) / \operatorname{im}(d^r_{p+r,q-r+1}) = \ker(d^r_{p,q})/0 = \mathcal{E}^r_{p,q}$ for r from (2.4).

Method 2. Suppose the spectral sequence is bounded. Let us use the notation $\mathcal{Z}^{r+1}_{p,q} = \ker(d^r_{p,q})$ and $\mathcal{B}^{r+1}_{p,q} = \operatorname{im}(d^r_{p+r,q-r+1})$. Using $S(n)$ from (2.5), we have

$$r - 1 > p - S(n) \text{ implies } \ker(d^{r-1}_{p,q}) = \mathcal{E}^{r-1}_{p,q} \text{ implies } \mathcal{Z}^r_{p,q} = \mathcal{E}^{r-1}_{p,q}$$

and

$$r > p - S(n) \text{ implies } \ker(d^r_{p,q}) = \mathcal{E}^r_{p,q} \text{ implies } \mathcal{Z}^{r+1}_{p,q} = \mathcal{E}^r_{p,q}.$$

Using $T(m)$ from (2.6), we have

$$r - 1 > q + 1 - T(m) \text{ implies } \operatorname{im}(d^{r-1}_{p+r-1,q-r+2}) = 0 \text{ implies } \mathcal{B}^r_{p,q} = 0$$

and

$$r > q + 1 - T(m) \text{ implies } \operatorname{im}(d^r_{p+r,q-r+1}) = 0 \text{ implies } \mathcal{B}^{r+1}_{p,q} = 0.$$

So for r from (2.4), $\mathcal{B}^r_{p,q} = \mathcal{B}^{r+1}_{p,q}$, and therefore

$$\mathcal{Z}^{r+1}_{p,q} = \mathcal{E}^r_{p,q} = \mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q} = \mathcal{Z}^r_{p,q}/0 = \mathcal{Z}^r_{p,q}.$$

Then $\mathcal{Z}^r_{p,q} = \mathcal{Z}^{r+1}_{p,q} = \cdots = \mathcal{Z}^\infty_{p,q}$ and $\mathcal{B}^r_{p,q} = \mathcal{B}^{r+1}_{p,q} = \cdots = \mathcal{B}^\infty_{p,q}$ and so the spectral sequence has a limit term since $\mathcal{E}^\infty_{p,q} = \mathcal{Z}^\infty_{p,q}/\mathcal{B}^\infty_{p,q} = \mathcal{Z}^r_{p,q}/\mathcal{B}^r_{p,q} = \mathcal{E}^r_{p,q}$ for r from (2.4).

Remark. An unbounded spectral sequence (of objects from an abelian category), does not have to have a limit term in general. But one could of course restrict oneself to objects from an abelian category with additional properties to ensure that the spectral sequence will have a limit term. These properties are listed at the end of Appendix A.

2.3 Convergence

In this Section, we will define convergence and discuss when a spectral sequence converges.

Definition 2.3.1 *We say that a spectral sequence $\{\mathcal{E}^r, d^r\}$* ***converges*** *to a graded object H_* and write $\mathcal{E}^a_{p,q} \Longrightarrow \mathcal{H}_*$ if $\mathcal{E}^\infty_{p,q} = \mathcal{F}_p H_{p+q}/\mathcal{F}_{p-1}H_{p+q}$ for all p, q, where $\{\mathcal{F}_p H_*\}_p$ is a bounded filtration of H_*. (i.e. there would be a bounded filtration $\{\mathcal{F}_p H_n\}_p$ for each H_n.)*

We will look at two examples of spectral sequences which converge, by looking closely at the construction of the spectral sequence examples from the previous Chapter.

Example 6. Example 2 of Chapter 1 with the bounded filtration assumption converges. Take a complex $\mathcal{C}$ and a bounded filtration of it:

$$\{0\} \subset \cdots \subset \mathcal{F}_{p-1}\mathcal{C} \subset \mathcal{F}_p\mathcal{C} \subset \cdots \subset \mathcal{C}.$$

We will show that the spectral sequence associated to this filtration converges to $H_*(\mathcal{C}) = \{H_n(\mathcal{C})\}_n$, the homology of $\mathcal{C}$.

In the bounded filtration above, we have the inclusion map $\mathcal{F}_p\mathcal{C} \hookrightarrow \mathcal{C}$. Hence we have a map for each n, $H_n(\mathcal{F}_p\mathcal{C}) \to H_n(\mathcal{C})$. Now define $\Phi_p H_n(\mathcal{C})$ to be the image of $H_n(\mathcal{F}_p\mathcal{C})$ in $H_n(\mathcal{C})$. Then we obtain a filtration of the graded object $H_*(\mathcal{C})$,

$$0 \subset \cdots \subset \Phi_{p-1}H_*(\mathcal{C}) \subset \Phi_p H_*(\mathcal{C}) \subset \cdots \subset H_*(\mathcal{C}),$$

where $\Phi_p H_*(\mathcal{C})$ is the graded subobject $\{\Phi_p H_n(\mathcal{C})\}_n$. The filtration $\{\mathcal{F}_p\mathcal{C}\}_p$ of $\mathcal{C}$ is bounded, so the filtration $\{\Phi_p H_*(\mathcal{C})\}_p$ of $H_*(\mathcal{C})$ is bounded. We saw at the end of the previous Chapter that the filtration $\{\mathcal{F}_p\mathcal{C}\}_p$ gives an exact couple with exact sequence

$$\cdots \to \mathcal{D}_{p-1,q+1} \xrightarrow{\alpha_{p-1,q+1}} \mathcal{D}_{p,q} \xrightarrow{\beta_{p,q}} \mathcal{E}_{p,q} \xrightarrow{\gamma_{p,q}} \mathcal{D}_{p-1,q} \xrightarrow{\alpha_{p-1,q}} \mathcal{D}_{p,q-1} \xrightarrow{\beta_{p,q-1}} \cdots .$$

Then we get the r^{th} derived couple of that exact couple with exact sequence

$$\cdots \to \mathcal{D}^r_{p+r-2,q-r+2} \xrightarrow{\alpha^r_{p+r-2,q-r+2}} \mathcal{D}^r_{p+r-1,q-r+1} \xrightarrow{\beta^r_{p+r-1,q-r+1}} \mathcal{E}^r_{p,q} \xrightarrow{\gamma^r_{p,q}} \mathcal{D}^r_{p-1,q} \to \cdots . \quad (2.7)$$

Notice that

$$\begin{aligned}
\mathcal{D}^r_{p+r-1,q-r+1} &= \alpha^{r-1}_{p+r-2,q-r+2} \cdots \alpha^4_{p+3,q-3}\alpha^3_{p+2,q-2}\alpha^2_{p+1,q-1}\alpha_{p,q}\mathcal{D}_{p,q}, \text{ by definition;} \\
&= \alpha^{r-1}_{p+r-2,q-r+2} \cdots \alpha^3_{p+2,q-2}\alpha^2_{p+1,q-1}\alpha_{p,q}H_{p+q}(\mathcal{F}_p\mathcal{C}), \text{ by (1.29);} \\
&= \operatorname{im}(H_{p+q}(\mathcal{F}_p\mathcal{C}) \to H_{p+q}(\mathcal{F}_{p+r-1}\mathcal{C})).
\end{aligned}$$

The boundedness of $\{\mathcal{F}_p\mathcal{C}\}_p$ implies that $\mathcal{F}_{p+r-1}\mathcal{C} = \mathcal{C}$ for some r, and we obtain for r at least this big that

$$\begin{aligned}
H_{p+q}(\mathcal{F}_{p+r-1}\mathcal{C}) &= H_{p+q}(\mathcal{C}) \\
\text{implies } \operatorname{im}(H_{p+q}(\mathcal{F}_p\mathcal{C}) \to H_{p+q}(\mathcal{F}_{p+r-1}\mathcal{C})) &= \operatorname{im}(H_{p+q}(\mathcal{F}_p\mathcal{C}) \to H_{p+q}(\mathcal{C})) \\
\text{implies } \alpha^{r-1}_{p+r-2,q-r+2} \cdots \alpha^3_{p+2,q-2}\alpha^2_{p+1,q-1}\alpha_{p,q}H_{p+q}(\mathcal{F}_p\mathcal{C}) &= \Phi_p H_{p+q}(\mathcal{C}) \\
\text{implies } \alpha^{r-1}_{p+r-2,q-r+2} \cdots \alpha^3_{p+2,q-2}\alpha^2_{p+1,q-1}\alpha_{p,q}\mathcal{D}_{p,q} &= \Phi_p H_{p+q}(\mathcal{C}) \text{ by (1.29)} \\
\text{implies } \mathcal{D}^r_{p+r-1,q-r+1} &= \Phi_p H_{p+q}(\mathcal{C}).
\end{aligned}$$

Similarly,

$$\mathcal{D}^r_{p+r-2,q-r+2} = \Phi_{p-1} H_{p+q}(\mathcal{C}) \text{ for } r \text{ at least this big.}$$

Also note that

$$\begin{aligned}
\mathcal{D}^r_{p-1,q} &= \alpha^{r-1}_{p-2,q+1} \cdots \alpha^3_{p-r+2,q+r-3}\alpha^2_{p-r+1,q+r-2}\alpha_{p-r,q+r-1}\mathcal{D}_{p-r,q+r-1}; \\
&= \alpha^{r-1}_{p-2,q+1} \cdots \alpha^3_{p-r+2,q+r-3}\alpha^2_{p-r+1,q+r-2}\alpha_{p-r,q+r-1}H_{p+q-1}(\mathcal{F}_{p-r}\mathcal{C}), \text{ by (1.29);} \\
&= \alpha^{r-1}_{p-2,q+1} \cdots \alpha^3_{p-r+2,q+r-3}\alpha^2_{p-r+1,q+r-2}\alpha_{p-r,q+r-1}H_{p+q-1}(0) \text{ for } r \text{ big enough;} \\
&= 0.
\end{aligned}$$

Then (2.7) becomes

$$\cdots \to \Phi_{p-1}H_{p+q}(\mathcal{C}) \to \Phi_p H_{p+q}(\mathcal{C}) \to \mathcal{E}^r_{p,q} \to 0 \to \cdots. \tag{2.8}$$

This sequence is exact *and* $\Phi_{p-1}H_{p+q}(\mathcal{C}) \to \Phi_p H_{p+q}(\mathcal{C})$ is an injection, so

$$\Phi_p H_{p+q}(\mathcal{C})/\Phi_{p-1}H_{p+q}(\mathcal{C}) \simeq \mathcal{E}^r_{p,q}. \tag{2.9}$$

Also, $\mathcal{E}^r_{p,q} = \mathcal{E}^\infty_{p,q}$ by the previous section since we are working with a *bounded* filtration. Therefore we have satisfied all the requirements to claim that

$$\mathcal{E}^a_{p,q} \Longrightarrow H_*(\mathcal{C}),$$

since $\Phi_p H_{p+q}(\mathcal{C})/\Phi_{p-1}H_{p+q}(\mathcal{C}) \simeq \mathcal{E}^\infty_{p,q}$ for all p, q and $\{\Phi_p H_*(\mathcal{C})\}_p$ is a bounded filtration of $H_*(\mathcal{C})$.

Example 7. Example 3 of Chapter 1 with $\mathcal{M}$ a first quadrant double complex converges to $H_*(\mathrm{Tot}(\mathcal{M}))$. Recall that both filtrations associated to the complex $\mathrm{Tot}(\mathcal{M})$ were *bounded* by letting $s(n) = -1$ and $t(n) = n$. This is precisely the case of Example 6 of this Chapter, hence we have that both spectral sequences (one for each filtration) in Example 3 of Chapter 1 converge to the graded object $H_*(\mathrm{Tot}(\mathcal{M}))$.

CHAPTER 3

PRELIMINARIES TO AN APPLICATION

In this Chapter we look more closely at the terms $\mathcal{E}^2_{p,q}$ in Example 3 of Chapter 1, which is the spectral sequence that we obtained from a double complex. Then we define the notion of collapsing.

3.1 A More Detailed Look at the Second Term of a Certain Type of Spectral Sequence

In this Section, we will see more explicitely what the terms $\mathcal{E}^2_{p,q}$ look like in the spectral sequence that arises from a first quadrant double complex $\mathcal{M} = \mathcal{M}_{**}$. Recall that we had a filtration of $\operatorname{Tot}(\mathcal{M})$,

$$0 \subset \cdots \subset \mathcal{F}_{p-1}\operatorname{Tot}(\mathcal{M}) \subset \mathcal{F}_p \operatorname{Tot}(\mathcal{M}) \subset \cdots \subset \operatorname{Tot}(\mathcal{M}),$$

where $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n = \prod_{i \leq p} \mathcal{M}_{i,n-i}$. We called this filtration the *first filtration*. Consider

$$(\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1}\operatorname{Tot}(\mathcal{M}))_n = (\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n/(\mathcal{F}_{p-1}\operatorname{Tot}(\mathcal{M}))_n.$$

Since $n = p+q$, we must have $p < n$. A look at Figure 3.1 shows that all one has left in $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M}))_n/(\mathcal{F}_{p-1}\operatorname{Tot}(\mathcal{M}))_n$ is the term $\mathcal{M}_{p,n-p} = \mathcal{M}_{p,q}$. So we have that

$$(\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1}\operatorname{Tot}(\mathcal{M}))_n = \mathcal{M}_{p,q}. \tag{3.1}$$

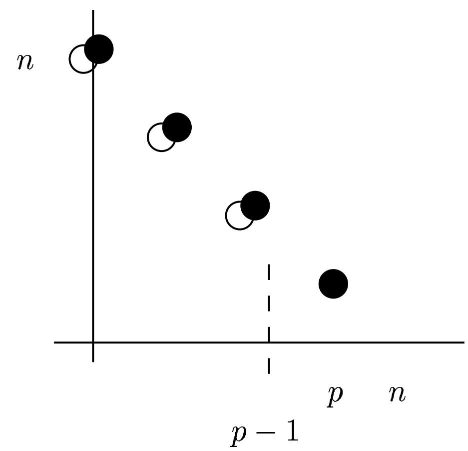

Figure 3.1: What the object $(\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_n$ looks like

Now notice:

$$\begin{aligned}
\mathcal{E}^1_{p,q} &= H_{p+q=n}(\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M})), \text{ by (1.8);} \\
&= \ker((\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_n \to (\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_{n-1})/ \\
&\quad \operatorname{im}((\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_{n+1} \to (\mathcal{F}_p \operatorname{Tot}(\mathcal{M})/\mathcal{F}_{p-1} \operatorname{Tot}(\mathcal{M}))_n); \\
&= \ker(\mathcal{M}_{p,q} \to \mathcal{M}_{p,q-1})/\operatorname{im}(\mathcal{M}_{p,q+1} \to \mathcal{M}_{p,q}), \text{ by (3.1);} \\
&= H_q(\mathcal{M}_{p,*}) = \text{ the homology at the } q^{th} \text{ spot of the } p^{th} \text{ column of } \mathcal{M}_{**}.
\end{aligned}$$

(Note: The p^{th} column of $\mathcal{M}_{**}$ is denoted by $\mathcal{M}_{p,*}$.) So we obtain the new lattice $\mathcal{E}^1_{**}$. The maps in the lattice $\mathcal{E}^r_{**}$ have slope $(r-1)/(-r)$; in our case $r = 1$, hence the maps are horizontal. So $\mathcal{E}^2_{p,q} = H(\mathcal{E}^1_{p,q}) = H_p(\mathcal{E}^1_{*,q})$, where $\mathcal{E}^1_{*,q} = \{\mathcal{E}^1_{p,q}\}_p$ denotes the q^{th} row of $\mathcal{E}^1_{*,*}$. $H_p(\mathcal{E}^1_{*,q})$ is the horizontal homology at the p^{th} spot of the row

$$\cdots \to \mathcal{E}^1_{p-1,q} \to \mathcal{E}^1_{p,q} \to \mathcal{E}^1_{p+1,q} \to \cdots$$

or

$$\cdots \to H_q(\mathcal{M}_{p-1,*}) \to H_q(\mathcal{M}_{p,*}) \to H_q(\mathcal{M}_{p+1,*}) \to \cdots.$$

Therefore

$$\mathcal{E}^2_{p,q} = H_p(\{H_q(\mathcal{M}_{p,*})\}_p) \tag{3.2}$$

in the spectral sequence resulting from the first filtration. Similarly

$$\mathcal{E}^2_{p,q} = H_p(\{H_q(\mathcal{M}_{*,p})\}_p) \tag{3.3}$$

in the spectral sequence resulting from the second filtration.

3.2 Collapsing

In this Section, we give the definition of collapsing, and show some consequences of it.

Definition 3.2.1 *A spectral sequence* $\{\mathcal{E}^r\}$ ***collapses*** *at* $\mathcal{E}^a$ *if there is exactly one nonzero row or column in the lattice* $\mathcal{E}^a_{**}$.

Remark. Suppose a spectral sequence collapses at $\mathcal{E}^2$, with the nonzero terms lying only on the p axis. We will show that $\mathcal{E}^2_{p,q} = \mathcal{E}^\infty_{p,q}$ for all p, q. Having nonzero terms only on the p axis in $\mathcal{E}^2_{**}$ means having nonzero terms only on the p axis in $\mathcal{E}^a_{**}$ for all $a \geq 2$. So for all $r \geq 2$, either $\mathcal{E}^r_{p,q} = 0$ or $\mathcal{E}^r_{p-r,q+r-1} = 0$. And hence $d^r_{p,q} = 0$ for all $r \geq 2$. So

$$\begin{aligned}
\mathcal{E}^{r+1}_{p,q} = H(\mathcal{E}^r_{p,q}) &= \ker(d^r_{p,q})/\operatorname{im}(d^r_{p+r,q-r+1}); \\
&= \ker(d^r_{p,q})/0 \text{ for } r \geq 2, \text{ since } d^r_{p,q} = 0; \\
&= \ker(d^r_{p,q}); \\
&= \mathcal{E}^r_{p,q}, \text{ because } d^r_{p,q}(\mathcal{E}^r_{p,q}) = 0 \text{ since } d^r_{p,q} = 0.
\end{aligned}$$

So we have that

$$\mathcal{E}^2_{p,q} = \mathcal{E}^3_{p,q} = \cdots = \mathcal{E}^\infty_{p,q}. \tag{3.4}$$

Lemma 3.2.2 *If the spectral sequence collapses at* $\mathcal{E}^2$*, then*

$$H_n(\mathrm{Tot}(\mathcal{M})) = \mathcal{E}^2_{n,0}$$

in Example 3 of Chapter 1, the spectral sequence we obtained from a double complex, assuming that $\mathcal{M}$ *is first quadrant.*

Proof. Let the spectral sequence collapse at $\mathcal{E}^2$, i.e. $\mathcal{E}^2_{p,q} = 0$ for $q \neq 0$. Fix some n. Recall that since $\mathcal{M}$ is first quadrant, both filtrations of $\mathrm{Tot}(\mathcal{M})$ are bounded. Then the filtration of $H_n(\mathrm{Tot}(\mathcal{M}))$ corresponding to the first filtration of $\mathrm{Tot}(\mathcal{M})$ is bounded. And so for our fixed n, we have a finite tower

$$0 = \Phi_{-1}H_n(\mathrm{Tot}(\mathcal{M})) \subset \cdots \subset \Phi_n H_n(\mathrm{Tot}(\mathcal{M})) = H_n(\mathrm{Tot}(\mathcal{M})). \tag{3.5}$$

Recall from Example 7 of Chapter 2 that

$$\Phi_m H_n(\mathrm{Tot}(\mathcal{M}))/\Phi_{m-1}H_n(\mathrm{Tot}(\mathcal{M})) \simeq \mathcal{E}^{\infty}_{m,n-m=q}. \tag{3.6}$$

And this equals $\mathcal{E}^2_{m,q}$ by (3.4). Consider $m < n$. This implies $n - m = q > 0$, which implies that $\mathcal{E}^2_{m,q} = 0$ since $\mathcal{E}^2_{m,q}$ for $q \neq 0$. So $\Phi_m H_n(\mathrm{Tot}(\mathcal{M})) = \Phi_{m-1}H_n(\mathrm{Tot}(\mathcal{M}))$ for $m < n$. So considering all $m < n$, we obtain

$$\begin{aligned}\Phi_m H_n(\mathrm{Tot}(\mathcal{M})) &= \Phi_{m-1}H_n(\mathrm{Tot}(\mathcal{M})) = \cdots \\ &\cdots = \Phi_0 H_n(\mathrm{Tot}(\mathcal{M})) = \Phi_{-1}H_n(\mathrm{Tot}(\mathcal{M})) = 0.\end{aligned} \tag{3.7}$$

Now consider $m = n$. Then

$$\begin{aligned}\mathcal{E}^2_{n,q} &= \Phi_n H_n(\mathrm{Tot}(\mathcal{M}))/\Phi_{n-1}H_n(\mathrm{Tot}(\mathcal{M})), \text{ by (3.4) and (3.6);} \\ &= \Phi_n H_n(\mathrm{Tot}(\mathcal{M}))/0, \text{ by (3.7) since } n-1 < n; \\ &= H_n(\mathrm{Tot}(\mathcal{M})), \text{ by (3.5).}\end{aligned}$$

Hence for $q = 0$, $n = m + q = m$, and

$$\mathcal{E}^2_{n,0} = H_n(\mathrm{Tot}(\mathcal{M})). \quad \blacksquare \tag{3.8}$$

CHAPTER 4

AN APPLICATION

To show an application of spectral sequences, we will use spectral sequences as the main element of a proof. This will be done using the two results found at the end of the previous two sections.

4.1 Some Needed Definitions

In this Section, we will introduce some definitions that will be useful in the application.

Definition 4.1.1 *A (left)* ***resolution*** *of an R-module A is an exact sequence*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to A \to 0.$$

(Hence the sequence is a complex, too.) We denote

$$P_A = \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to 0.$$

Definition 4.1.2 *An R-module F is* ***flat*** *if given an exact sequence $E' \to E \to E''$, then the sequence $F \otimes E' \to F \otimes E \to F \otimes E''$ is exact.*

Now we define the tensor product of two complexes.

Definition 4.1.3 *The* ***tensor product*** *of P and Q is a complex $P \otimes Q$, where*

$$(P \otimes Q)_n = \bigoplus_{p+q=n} P_p \otimes Q_q.$$

We have maps $d = \{d_n\}, d_n : (P \otimes Q)_n \to (P \otimes Q)_{n-1}$.

To see the map d_n in more detail, let us take an element of $(P \otimes Q)_n$,

$$\left(\sum r_0 \otimes s_0, \sum r_1 \otimes s_1, \sum r_2 \otimes s_2, \cdots, \sum r_n \otimes s_n\right),$$

with $r_i \otimes s_i \in P_i \otimes Q_{n-i}$. The map d_n takes this element to the following element of $(P \otimes Q)_{n-1}$:

$$(\sum(-1)^0 r_0 \otimes d(s_0) + d(r_1) \otimes s_1, \sum(-1)^1 r_1 \otimes d(s_1) + d(r_2) \otimes s_2, \cdots),$$

where $(-1)^i r_i \otimes d(s_i) + d(r_{i+1}) \otimes s_{i+1} \in P_i \otimes Q_{n-1-i}$.

4.2 Using the Spectral Sequence in a Proof

In this Section, we will prove the following lemma.

Lemma 4.2.1 *Let A be a right R-module, and B be a left R-module. Let P and Q be the resolutions*

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to B \to 0,$$

respectively, where P^i and Q^i are flat for all i. Let P_A represent $\cdots \to P_2 \to P_1 \to P_0 \to 0$ and Q_B represent $\cdots \to Q_2 \to Q_1 \to Q_0 \to 0$. Then

$$H_n(P_A \otimes B) \simeq H_n(P_A \otimes Q_B) \simeq H_n(A \otimes Q_B). \tag{4.1}$$

Proof If we define $\{\mathcal{M}_{p,q}\} = \{P_p \otimes Q_q\}$, then by the definition of tensor product of complexes given above, we immediately see that

$$P_A \otimes Q_B = \mathrm{Tot}(\mathcal{M}). \tag{4.2}$$

Notice that this double complex $\{\mathcal{M}_{p,q}\}$ is a first quadrant double complex. Also, (4.2) gives us

$$H(P_A \otimes Q_B) = H(\mathrm{Tot}(\mathcal{M})). \tag{4.3}$$

Recall from (3.8) that if the spectral sequence collapses at $\mathcal{E}^2$ with nonzero terms lying only on the p axis, then

$$H_n(\mathrm{Tot}(\mathcal{M})) = \mathcal{E}^2_{n,0}.$$

Let us show that

$$\mathcal{E}^2_{n,0} = H_n(P_A \otimes B) \text{ and } \mathcal{E}^2_{n,0} = H_n(A \otimes Q_B).$$

The proofs of these two equalities are the same, so we will just show the first equality.

Case 1: $q = 0$.

$$\begin{aligned}
H_0(\mathcal{M}_{p,*}) = &\text{ the homology at the } 0^{th} \text{ spot of the } p^{th} \text{ column of } \mathcal{M}_{**};\\
&= \ker(\mathcal{M}_{p,0} \to 0)/\operatorname{im}(\mathcal{M}_{p,1} \to \mathcal{M}_{p,0});\\
&= \ker(P_p \otimes Q_0 \to 0)/\operatorname{im}(P_p \otimes Q_1 \to P_p \otimes Q_0);\\
&= P_p \otimes Q_0/\operatorname{im}(P_p \otimes Q_1 \to P_p \otimes Q_0);\\
&= P_p \otimes Q_0/\ker(P_p \otimes Q_0 \to P_p \otimes B), \text{ by the remark that follows;}\\
&\simeq P_p \otimes B, \text{ by the first isomorphism theorem applied to}\\
&\quad f : P_p \otimes Q_0 \to P_p \otimes B.
\end{aligned}$$

Remark. The R-module P_p is flat means that if $Q_1 \to Q_0 \to B$ is exact, then $P_p \otimes Q_1 \to P_p \otimes Q_0 \to P_p \otimes B$ is exact. Hence,

$$\operatorname{im}(P_p \otimes Q_1 \to P_p \otimes Q_0) = \ker(P_p \otimes Q_0 \to P_p \otimes B).$$

Case 2: $q > 0$.

$$\begin{aligned}
H_q(\mathcal{M}_{p,*}) = &\text{ the homology at the } q^{th} \text{ spot of the } p^{th} \text{ column of } \mathcal{M}_{**};\\
&= \ker(P_p \otimes Q_q \to P_p \otimes Q_{q-1})/\operatorname{im}(P_p \otimes Q_{q+1} \to P_p \otimes Q_q);\\
&= \operatorname{im}(P_p \otimes Q_{q+1} \to P_p \otimes Q_q)/\operatorname{im}(P_p \otimes Q_{q+1} \to P_p \otimes Q_q) \text{ for } q > 0,\\
&\quad \text{since } P_p \text{ is flat;}\\
&= 0 \text{ for } q > 0.
\end{aligned}$$

Now, H'_p denotes taking the horizontal homology at the p^{th} spot; i.e. for a fixed q,

$$H'_p H_q(\mathcal{M}_{p,*}) = \ker(H_q(\mathcal{M}_{p,*}) \to H_q(\mathcal{M}_{p-1,*}))/\operatorname{im}(H_q(\mathcal{M}_{p+1,*}) \to H_q(\mathcal{M}_{p,*})). \quad (4.4)$$

The condition $q = 0$ in Case 1 causes (4.4) to become

$$H'_p H_0(\mathcal{M}_{p,*}) = \ker(P_p \otimes B \to P_{p-1} \otimes B)/\operatorname{im}(P_{p+1} \otimes B \to P_p \otimes B). \quad (4.5)$$

And since $H_p(P_A \otimes B) = H_p(\cdots \to P_{p+1} \otimes B \to P_p \otimes B \to P_{p-1} \otimes B \to \cdots) = \ker(P_p \otimes B \to P_{p-1} \otimes B)/\operatorname{im}(P_{p+1} \otimes B \to P_p \otimes B)$, we have that $H'_p H_q(\mathcal{M}_{p,*}) = H_p(P_A \otimes B)$ when $q = 0$.

The condition $q > 0$ in Case 2 causes (4.4) to become

$$H'_p H_q(\mathcal{M}_{p,*}) = \ker(0 \to 0)/\operatorname{im}(0 \to 0) = 0. \quad (4.6)$$

We saw in (3.2) that the spectral sequence associated with the *first* filtration has $\mathcal{E}^2_{p,q} = H'_p H_q(\mathcal{M}_{p,*})$ for its second term. Since (4.5) implies that $\mathcal{E}^2_{p,0} = H_p(P_A \otimes B)$ and (4.6) implies that $\mathcal{E}^2_{p,q} = 0$ for $q > 0$, then the spectral sequence associated with the first filtration collapses at $\mathcal{E}^2$. Hence $\mathcal{E}^2_{n,0} = H_n(\operatorname{Tot}(\mathcal{M}))$ by (3.8).

In summary, for each n,

$$\begin{aligned} H_n(P_A \otimes B) &= \mathcal{E}^2_{n,0}, \text{ by Case 1 above;} \\ &= H_n(\operatorname{Tot}(\mathcal{M})) \text{ by (3.8).} \end{aligned}$$

So $H(P_A \otimes B) = H(\operatorname{Tot}(\mathcal{M})) = H(P_A \otimes Q_B)$. (Again, a similar proof shows that $H(A \otimes Q_B) = H(P_A \otimes Q_B)$.) ■

CHAPTER 5

THE GROTHENDIECK SPECTRAL SEQUENCE

In this Chapter we give the necessary definitions and lemmas that enable us to describe the Grothendieck spectral sequence.

5.1 Lemmas, Proofs, and Definitions

In this Section, we prove several lemmas and define several terms applicable to the Grothendieck Spectral Sequence.

Lemma 5.1.1 *Suppose we have a commutive diagram of modules:*

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & C' \arrow[u] \arrow[r, "\gamma'"] & C \arrow[u] \arrow[r, "\gamma"] & C'' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & B' \arrow[u, "r'"] \arrow[r, "\beta'"] & B \arrow[u, "r"] \arrow[r, "\beta"] & B'' \arrow[u, "r''"] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[u, "d'"] \arrow[r, "\alpha'"] & A \arrow[u, "d"] \arrow[r, "\alpha"] & A'' \arrow[u, "d''"] \arrow[r] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Suppose the columns are exact and the bottom two rows are exact. Then the top row is exact.

Proof We need to show three things:

i. $\operatorname{im}\gamma = \ker(C'' \longrightarrow 0) = C''$,

ii. $\operatorname{im}\gamma' = \ker\gamma$,

iii. $0 = \operatorname{im}(0 \longrightarrow C') = \ker\gamma'$.

i. Since it is clear that $\operatorname{im}\gamma \subset C''$, we only need to show $C'' \subset \operatorname{im}\gamma$. Let x be in C''. We just need to find a c in C such that $\gamma(c) = x$. Since column 3 is exact and hence r'' is surjective, there exists a b'' in B'' such that $r''(b'') = x$. Since row 2 is exact and hence β is surjecive, there exists a b in B such that $\beta(b) = b''$. So we have $x = r''(\beta(b))$, which equals $\gamma(r(b))$ since the diagram commutes. So our c is $r(b)$.

ii.a. First let us see that $\operatorname{im}\gamma' \subset \ker\gamma$, i.e. that $\gamma\circ\gamma' = 0$. Let x be an element in C'. Then x is in $\operatorname{im} r'$ since r' is surjective. So $x = r'(b)$ for some b in B'. Then $\gamma'(x) = \gamma'\circ r'(b) = r\circ\beta'(b)$ since the diagram is commutative, so $\gamma\circ\gamma'(x) = \gamma\circ r\circ\beta'(b) = r''\circ\beta\circ\beta'(b) = r''(0) = 0$. So we have shown that $\operatorname{im}\gamma' \subset \ker\gamma$.

ii.b. Next, let us check that $\ker\gamma \subset \operatorname{im}\gamma'$. First notice that

$$\begin{aligned} r''\circ\beta\circ r^{-1}(\ker\gamma) &= \gamma\circ r\circ r^{-1}(\ker\gamma), \text{ since the diagram commutes;} \\ &= \gamma(\ker\gamma) = 0. \end{aligned}$$

Note, we can take r^{-1} since r is surjective. So

$$\begin{aligned} \beta\circ r^{-1}(\ker\gamma) &\subset \ker(r''); \\ &= \operatorname{im}(d''), \text{ since column 3 is exact;} \\ &= \operatorname{im}(d''\circ\alpha), \text{ since row 3 is exact, } \alpha \text{ is surjective;} \\ &= \operatorname{im}(\beta\circ d), \text{ since the diagram commutes;} \\ &= \beta\circ d(A) = \beta(d(A)); \\ &= \beta(\operatorname{im} d). \end{aligned}$$

Therefore $\beta \circ r^{-1}(\ker \gamma) \subset \beta(\operatorname{im} d)$ implies $r^{-1}(\ker \gamma) \subset (\beta)^{-1} \circ \beta(\operatorname{im} d) = \operatorname{im} d + \ker \beta$. And so

$$\begin{aligned}
r \circ r^{-1}(\ker \gamma) &\subset r(\operatorname{im}(d) + \ker(\beta)); \\
\text{implies } \ker(\gamma) &\subset r(\operatorname{im} d) + r(\ker \beta); \\
&= 0 + r(\ker(\beta)), \text{ since column 2 is exact;} \\
&= r(\operatorname{im}(\beta')), \text{ since row 2 is exact;} \\
&= r \circ \beta'(B'); \\
&= \gamma' \circ r'(B'), \text{ since the diagram commutes;} \\
&= \gamma'(C'), \text{ since column 1 is exact, hence } r' \text{ is surjective;} \\
&= \operatorname{im}(\gamma').
\end{aligned}$$

iii. Let x be in $\ker \gamma' \subset C'$. We will show that $x = 0$. Since r' is surjective, $r'(b') = x$ for some b' in B'. Now $\gamma'(x) = 0$ implies that $\gamma' \circ r'(b') = 0$, since $x = r'(b')$. This implies that $r \circ \beta'(b') = 0$ since the diagram commutes. Then $\beta'(b')$ is in $\ker r$ and hence $\beta'(b')$ is in $\operatorname{im} d$. So

$$\beta'(b') = d(a) \text{ for some } a \in A. \tag{5.1}$$

Then

$$b' = (\beta')^{-1} \circ d(a), \tag{5.2}$$

since β' is injective. Let us notice that

$$\begin{aligned}
d'' \circ \alpha(a) &= \beta \circ d(a), \text{ since the diagram commutes;} \\
&= \beta \circ \beta'(b'), \text{ by (5.1);} \\
&= 0.
\end{aligned}$$

Then $\alpha(a) = 0$ since d'' is injective. So a is in $\ker \alpha = \operatorname{im} \alpha'$. Hence

$$\begin{aligned} x = r'(b') &= r' \circ (\beta')^{-1} \circ d(a), \text{ by (5.2);} \\ &= r' \circ d' \circ (\alpha')^{-1}(a), \text{ by } (D.1) \text{ in Appendix D since } a \text{ is in } \operatorname{im} \alpha'; \\ &= 0, \text{ since } r' \circ d' = 0. \end{aligned}$$

Therefore γ' is injective. ■

Definition 5.1.2 *A (right)* ***resolution*** *of an object M is an exact sequence*

$$0 \to M \to E^0 \to E^1 \to E^2 \to \cdots \to E^n \to E^{n+1} \to \cdots.$$

Definition 5.1.3 *An object E is* ***injective*** *if given the following diagram*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B, \end{tikzcd}$$

where the row is exact and f is a map from A to E, then f can be extended to a map $\widetilde{f} : B \to E$, where $\widetilde{f} \circ i = f$.

Definition 5.1.4 *A resolution is called an* ***injective resolution*** *if each $E^i, i \geq 0$, is injective.*

Definition 5.1.5 *An abelian category $\mathcal{A}$ has* ***enough injectives*** *if given any object M in $\mathcal{A}$, there exists an injection $M \to I$ into an injective object I.*

Lemma 5.1.6 *Suppose we are in an abelian category with enough injectives. Then every object M has an injective resolution.*

Proof By the hypothesis, we have an injection $M \xrightarrow{\epsilon} I^0$ into some injective object I^0. Let $M^0 = \epsilon(M) \subset I^0$. Consider the object I^0/M^0. Again by the hypothesis, we have an injection $I^0/M^0 \xrightarrow{d^0} I^1$. The following diagram is helpful:

$$\begin{tikzcd} & I^1 & \\ 0 \arrow[r] & I^0/M^0 \arrow[u, "d^0"] \arrow[r] & I^0. \end{tikzcd}$$

Since I^1 is injective, d^0 extends to $\widetilde{d^0} : I^0 \to I^1$, where $\ker(\widetilde{d^0}) = M^0$.

Do the same procedure again, let $M^1 = d^0(I^0) \subset I^1$. By hypothesis, we have an injection $I^1/M^1 \xrightarrow{d^1} I^2$ into some injective object I^2. Since I^2 is injective, d^1 extends to $\widetilde{d^1} : I^1 \to I^2$, where $\ker(\widetilde{d^1}) = M^1$. We continue this procedure, and get an injective resolution of M,

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots. \qquad \blacksquare$$

Lemma 5.1.7 *Suppose we have the following diagram*

$$\begin{tikzcd}
& {} & & {} & \\
& I^1 \arrow[u, "\alpha^1"] & & J^1 \arrow[u, "\gamma^1"'] & \\
& I^0 \arrow[u, "\alpha^0"] & & J^0 \arrow[u, "\gamma^0"'] & \\
0 \arrow[r] & A' \arrow[u, "\alpha"] \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[u, "\gamma"'] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the row is exact and the two columns are injective resolutions. Then there exists an injective resolution of A that fits in the middle with maps so that the diagram commutes and each row is exact.

Proof Consider the initial diagram

$$\begin{tikzcd}
& 0 & & 0 & \\
& I^0/\alpha(A') \arrow[u] & & J^0/\gamma(A'') \arrow[u] & \\
& I^0 \arrow[u] & & J^0 \arrow[u] & \\
0 \arrow[r] & A' \arrow[u, "\alpha"] \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[u, "\gamma"'] \arrow[r] & 0. \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In this diagram, the columns and the row are exact, and the objects I^0, J^0 are injective. Let $K^0 = I^0 \oplus J^0$. Define maps $i^0 : I^0 \to K^0$ by $x \mapsto (x, 0)$, and $p^0 : K^0 \to J^0$ by $(x, y) \mapsto y$.

Let us see that $K^0 = I^0 \oplus J^0$ is injective. Consider the diagram

$$\begin{tikzcd} & I^0 \oplus J^0 & \\ 0 \arrow[r] & U \arrow[u, "f"] \arrow[r, "i"'] & B. \end{tikzcd} \tag{5.3}$$

Suppose we put I^0 in the upper right corner of diagram (5.3), with the map $I^0 \oplus J^0 \xrightarrow{P_{I^0}} I^0$, $(x, y) \mapsto x$. Then I^0 injective implies that $P_{I^0} \circ f$ extends to $g_{I^0} : B \to I^0$, such that $g_{I^0} \circ i = P_{I^0} \circ f$. Similarly, suppose we put J^0 in the upper right corner of diagram (5.3), with the map $I^0 \oplus J^0 \xrightarrow{P_{J^0}} J^0$, $(x, y) \mapsto y$. Then J^0 injective implies that $P_{J^0} \circ f$ extends to $g_{J^0} : B \to J^0$, such that $g_{J^0} \circ i = P_{J^0} \circ f$. Now define $h : B \to I^0 \oplus J^0$ by $b \mapsto (g_{I^0}(b), g_{J^0}(b))$. All we need to check is that the map h is an extension of f, i.e. that $h \circ i = f$. Let u be an element of U. Then $h \circ i(u) = (g_{I^0}(i(u)), g_{J^0}(i(u))) = (P_{I^0} \circ f(u), P_{J^0} \circ f(u)) = f(u)$. So $K^0 = I^0 \oplus J^0$ is injective. At this stage we have

$$\begin{tikzcd} & & & & \\ & I^0 \arrow[u] \arrow[r, "i^0"] & K^0 \arrow[r, "p^0"] & J^0 \arrow[u] & \\ 0 \arrow[r] & A' \arrow[u, "\alpha"] \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[u, "\gamma"'] \arrow[r] & 0. \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} \tag{5.4}$$

Let us find an appropriate map from A to K^0. Since I^0 is injective, there is a map $\sigma : A \to I^0$ with $\sigma \circ i = \alpha$. Define $\epsilon : A \to K^0$ by $a \mapsto (\sigma(a), \gamma \circ p(a))$.

Now add $K^0/\epsilon(A)$ into diagram (5.4), along with the maps $I^0/\alpha(A') \to K^0/\epsilon(A)$ and $K^0/\epsilon(A) \to J^0/\gamma(A'')$ induced by i^0 and p^0, respectively. Then our diagram becomes:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & I^0/\alpha(A') \arrow[u] \arrow[r, "\widetilde{i^0}"] & K^0/\epsilon(A) \arrow[u] \arrow[r, "\widetilde{p^0}"] & J^0/\gamma(A'') \arrow[u] & \\
 & I^0 \arrow[u] \arrow[r, "i^0"] & K^0 \arrow[u] \arrow[r, "p^0"] & J^0 \arrow[u] & \\
0 \arrow[r] & A' \arrow[u, "\alpha"] \arrow[r, "i"'] & A \arrow[u, "\epsilon"] \arrow[r, "p"'] & A'' \arrow[u, "\gamma"'] \arrow[r] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5.5}$$

Now let's check that diagram (5.5) commutes. We have four squares to check. We will use the notation [ ] to represent all quotient maps, e.g. [ ] will represent the map from I^0 to $I^0/\alpha(A')$.

i. The bottom left square of diagram (5.5) commutes because

$$\begin{aligned}
i^0 \circ \alpha(A') &= (\alpha(A'), 0), \text{ by the definition of } i^0 \\
\text{and } \epsilon \circ i(A') &= \sigma \circ i(A'), \gamma \circ p(i(A'))), \text{ by the definition of } \epsilon; \\
&= (\sigma \circ i(A'), 0), \text{ since } p \circ i \equiv 0; \\
&= (\alpha(A'), 0); \text{ since } \sigma \circ i \equiv \alpha.
\end{aligned}$$

ii. The bottom right square of diagram (5.5) commutes because

$$\begin{aligned}
p^0 \circ \epsilon(A) &= p^0(\sigma(A), \gamma \circ p(A)), \text{by the definition of } \epsilon; \\
&= \gamma \circ p(A), \text{ by the definition of } p^0.
\end{aligned}$$

iii. The top left square of diagram (5.5) commutes because

$$\widetilde{i^0} \circ [\ ](I^0) = \widetilde{i^0}(I^0/\alpha(A')) = (I^0/\sigma(A), 0)$$

$$\text{and} \qquad [\ ] \circ i^0(I^0) = [\ ](I^0, 0) = (I^0, 0)/\epsilon(A) = (I^0/\sigma(A), 0).$$

iv. The top right square of diagram (5.5) commutes because

$$\widetilde{p^0} \circ [\ ](K^0) = \widetilde{p^0}(K^0/\epsilon(A)) = J^0/\gamma(A'')$$

$$\text{and} \qquad [\ ] \circ p^0(K^0) = [\ ](J^0) = J^0/\gamma(A'').$$

So diagram (5.5) is commutative.

Let us verify that the three columns of diagram (5.5) are exact. The proof of the first column will carry over to the other two columns. For the first column we have

$$\begin{aligned} \operatorname{im}(0 \to A') &= 0 \\ \text{and } \ker(A' \xrightarrow{\alpha} I^0) &= 0, \text{ by hypothesis ;} \\ \operatorname{im}(A' \xrightarrow{\alpha} I^0) &= \alpha(A') \\ \text{and } \ker(I^0 \longrightarrow I^0/\alpha(A')) &= \alpha(A'); \\ \operatorname{im}(I^0 \longrightarrow I^0/\alpha(A')) &= I^0/\alpha(A') \\ \text{and } \ker(I^0/\alpha(A') \longrightarrow 0) &= I^0/\alpha(A'). \end{aligned}$$

Let us put zeroes into diagram (5.5):

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I^0/\alpha(A') \arrow[u] \arrow[r, "\widetilde{i^0}"] & K^0/\epsilon(A) \arrow[u] \arrow[r, "\widetilde{p^0}"] & J^0/\gamma(A'') \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & I^0 \arrow[u] \arrow[r, "i^0"] & K^0 \arrow[u] \arrow[r, "p^0"] & J^0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[u, "\alpha"] \arrow[r, "i"'] & A \arrow[u, "\epsilon"] \arrow[r, "p"'] & A'' \arrow[u, "\gamma"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5.6}$$

so that we can use Lemma 5.1.1. The bottom row of diagram (5.6) is exact by hypothesis. Let us look at the second row:

$$0 \to I^0 \xrightarrow{i^0} K^0 \xrightarrow{p^0} J^0 \to 0.$$

It is exact because

$$\begin{aligned} \operatorname{im}(0 \to I^0) &= 0 \\ \text{and } \ker(I^0 \xrightarrow{i^0} K^0) &= \ker(I^0 \to (I^0, 0)) = 0; \\ \operatorname{im}(I^0 \xrightarrow{i^0} K^0) &= i^0(I^0) = (I^0, 0) \\ \text{and } \ker(K^0 \xrightarrow{p^0} J^0) &= \ker((I^0, J^0) \xrightarrow{p^0} J^0) = (I^0, 0); \\ \operatorname{im}(K^0 \xrightarrow{p^0} J^0) &= p^0(K^0) = J^0 \\ \text{and } \ker(J^0 \to 0) &= J^0. \end{aligned}$$

Now we have satisfied all the conditions of Lemma 5.1.1. So the top row of diagram (5.6),

$$0 \to I^0/\alpha(A') \to K^0/\epsilon(A) \to J^0/\gamma(A'') \to 0,$$

is exact. We will use the fact this top row is exact later in this proof.

Let us prove that $K^0/\epsilon(A)$ is injective. Consider the diagram

$$\begin{tikzcd} & K^0/\epsilon(A) & \\ 0 \arrow[r] & U \arrow[u, "f"] \arrow[r, "i"'] & V, \end{tikzcd}$$

with the row exact. We want to show that f can be extended to a map $\widetilde{f} : V \longrightarrow K^0/\epsilon(A)$ such that $\widetilde{f} \circ i = f$.

Suppose we put K^0 in the upper right hand corner of our diagram, with the map

$$K^0/\epsilon(A) \xrightarrow{s} K^0,$$

where s is an injective inclusion map. Then K^0 injective implies that $s \circ f$ extends to $w : V \longrightarrow K^0$ such that $w \circ i = s \circ f$. Now define $m : V \longrightarrow K^0/\epsilon(A)$ by

$v \mapsto w(v)/\epsilon(A)$. We just need to check that m is an extension of f, i.e. that $m \circ i = f$. Let u be an element in U. Then

$$\begin{aligned} m \circ i(u) &= (w \circ i(u))/\epsilon(A), \text{ by the definition of } m; \\ &= (s \circ f(u))/\epsilon(A), \text{ since } w \circ i = s \circ f; \\ &= f(u), \text{ by the definition of } s. \end{aligned}$$

Let us describe this last equality in detail. The element $f(u)$ in $K^0/\epsilon(A)$ is a conjugacy class. No matter which conjugacy class representative we decide to have s take $f(u)$ to, it in turn gets quotiented by $\epsilon(A)$ again, and we are back with the same conjugacy class, $f(u)$.

So far, we have K^0 only. (It is injective, gives the exact row $0 \to I^0 \to K^0 \to J^0$, and commutes with everything below it.) We can do the same procedure to obtain K^1, using the initial diagram

$$\begin{tikzcd} & 0 & & 0 & \\ & I^1/\alpha^0(I^0) \arrow[u] & & J^1/\gamma^0(J^0) \arrow[u] & \\ & I^1 \arrow[u] & & J^1 \arrow[u] & \\ 0 \arrow[r] & I^0/\alpha(A') \arrow[r] \arrow[u] & K^0/\epsilon(A) \arrow[r] & J^0/\gamma(A'') \arrow[r] \arrow[u] & 0. \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

As before, in this diagram, the columns and the row are exact, and the objects I^1, J^1 are injective. We obtain $K^1 = I^1 \oplus J^1$ injective, we have the exact sequence

$0 \to I^1 \to K^1 \to J^1 \to 0$, and we have that diagram (5.7)

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I^1/\alpha^0(I^0) \arrow[r, "\widetilde{i^1}"] \arrow[u] & K^1/\epsilon^0(K^0/\epsilon(A)) \arrow[r, "\widetilde{p^1}"] \arrow[u] & J^1/\gamma^0(J^0) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^1 \arrow[r, "i^1"] \arrow[u] & K^1 \arrow[r, "p^1"] \arrow[u] & J^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I^0/\alpha(A') \arrow[r, "\widetilde{i^0}"] \arrow[u, "R"] & K^0/\epsilon(A) \arrow[r, "\widetilde{p^0}"] \arrow[u, "\epsilon^0"] & J^0/\gamma(A'') \arrow[r] \arrow[u, "S"'] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5.7}$$

commutes. The maps in the above diagram have the following properties:

1. $R : I^0/\alpha(A') \to I^1$ takes $x/\alpha(A')$ to $\alpha^0(x)$, since $I^0/\ker \alpha^0 \simeq \alpha^0(I^0) \subset I^1$ by the first isomorphism theorem. $S : J^0/\gamma(A'') \to J^1$ takes $y/\gamma(A'')$ to $\gamma^0(y)$ since $J^0/\ker \gamma^0 \simeq \gamma^0(J^0)$.

2. Since I^1 is injective, there exists $\sigma^0 : K^0/\epsilon(A) \to I^1$ such that $\sigma^0 \circ \widetilde{i^0} = R$.

3. Define $\epsilon^0 : K^0/\epsilon(A) \to K^1$ by $k^0/\epsilon(A) \mapsto (\sigma^0(k^0/\epsilon(A)), S \circ \widetilde{p^0}(k^0/\epsilon(A)))$. (We will define $\widetilde{\epsilon^0} : K^0 \to K^1$ to be $\widetilde{\epsilon^0} = \epsilon^0 \circ [\,] : K^0 \xrightarrow{[\,]} K^0/\epsilon(A) \xrightarrow{\epsilon^0} K^1$.

Let us show that the bottom two squares of diagram (5.7) indeed commute.

i. The lower left square of diagram (5.7) commutes because

$$\begin{aligned} i^1 \circ R(x/\alpha(A')) &= (R(x/\alpha(A')), 0); \\ &= (\alpha^0(x), 0). \end{aligned}$$

and

$$\begin{aligned}
\epsilon^0 \circ \widetilde{i^0}(x/\alpha(A')) &= \epsilon^0(x/\sigma(A), 0), \text{ by the definition of } \widetilde{i^0};\\
&= (\sigma^0(x/\sigma(A), 0), S \circ \widetilde{p^0}(x/\sigma(A), 0));\\
&= (\sigma^0(x/\sigma(A), 0), S(0));\\
&= (\sigma^0(x/\sigma(A), 0), 0;\\
&= (\sigma^0(\widetilde{i^0}(x/\alpha(A'))), 0);\\
&= (\sigma^0 \circ \widetilde{i^0}(x/\alpha(A')), 0);\\
&= (R(x/\alpha(A')), 0);\\
&= (\alpha^0(x), 0).
\end{aligned}$$

ii. The lower right square of diagram (5.7) commutes because

$$\begin{aligned}
p^1 \circ \epsilon^0(k^0/\epsilon(A)) &= p^1(\sigma^0(k^0/\epsilon(A)), S \circ \widetilde{p^0}(k^0/\epsilon(A)));\\
&= S \circ \widetilde{p^0}(k^0/\epsilon(A)).
\end{aligned}$$

To prove the lemma, we need these two squares commute:

$$\begin{array}{ccccc}
I^1 & \xrightarrow{i^1} & K^1 & \xrightarrow{p^1} & J^1 \\
{\scriptstyle \alpha^0}\uparrow & & {\scriptstyle \widetilde{\epsilon^0}}\uparrow & & \uparrow{\scriptstyle \gamma^0} \\
I^0 & \xrightarrow[i^0]{} & K^0 & \xrightarrow[p^0]{} & J^0.
\end{array} \tag{5.8}$$

i. The left square of diagram (5.8) commutes because $i^1 \circ \alpha^0(x) = (\alpha^0(x), 0)$ and

$$\begin{aligned}
\widetilde{\epsilon^0} \circ i^0(x) &= \widetilde{\epsilon^0}(x, 0); \\
&= \epsilon^0 \circ [\ ](x, 0); \\
&= \epsilon^0(x/\sigma(a), 0), \text{ by the definition of } [\ ]; \\
&= (\sigma^0(x/\sigma(A), 0), S \circ \widetilde{p^0}(x/\sigma(A), 0)), \text{ by the definition of } \epsilon^0; \\
&= (\sigma^0 \circ \widetilde{i^0}(x/\alpha(A')), S \circ \widetilde{p^0}(x/\sigma(A), 0)), \text{ by the definition of } \widetilde{i^0}; \\
&= (R(x/\alpha(A')), S \circ \widetilde{p^0}(x/\sigma(A), 0)), \text{ since } \sigma^0 \circ \widetilde{i^0} = R; \\
&= (\alpha^0(x), S \circ \widetilde{p^0}(x/\sigma(A), 0)), \text{ since } \sigma^0 \circ \widetilde{i^0} = R; \\
&= (\alpha^0(x), S(0)), \text{ by the definition of } \widetilde{p^0}; \\
&= (\alpha^0(x), 0).
\end{aligned}$$

ii. The right square of diagram (5.8) commutes because $\gamma^0 \circ p^0(x, y) = \gamma^0(y)$ and

$$\begin{aligned}
p^1 \circ \widetilde{\epsilon^0}(x, y) &= p^1 \circ \epsilon^0 \circ [\ ](x, y); \\
&= p^1 \circ \epsilon^0(x/\sigma(A), y/\gamma \circ p(A)), \text{ by the definition of } [\ ]; \\
&= p^1(\sigma^0(x/\sigma(A), y/\gamma \circ p(A)), S \circ \widetilde{p^0}(x/\sigma(A), y/\gamma \circ p(A))), \\
&\quad \text{by the definition of } \epsilon^0; \\
&= S \circ \widetilde{p^0}(x/\sigma(A), y/\gamma \circ p(A)), \text{ by the definition of } p^1; \\
&= S(y/\gamma(A'')), \text{ by the definition of } \widetilde{p^0}; \\
&= \gamma^0(y), \text{ by the definition of } S.
\end{aligned}$$

Thus we have an appropriate K^1. We can continue to get K^r, $r > 1$, and the appropriate maps. ■

Remark. As a result of this lemma, we also get that the maps i^n, $p^n, n \geq 0$, are injective and surjective, respectively.

Definition 5.1.8 *A complex is called* ***injective*** *if each object in the complex is injective.*

Recall that in an exact sequence of complexes $\cdots \to A \to B \to C \to \cdots$,

$$\cdots \to A_n \to B_n \to C_n \to \cdots$$

is exact for all n and the corresponding diagram commutes.

Definition 5.1.9 *An* ***injective resolution of a complex*** *C is a resolution of C made up of injective complexes.*

Here is an injective resolution of a complex C in detail:

$$\begin{tikzcd}
{} & & & {} & {} & {} & \\
M^1 \arrow[u] & & \cdots \arrow[r] & M^{0,1} \arrow[r] \arrow[u] & M^{1,1} \arrow[r] \arrow[u] & M^{2,1} \arrow[r] \arrow[u] & \cdots \\
M^0 \arrow[u] & = & \cdots \arrow[r] & M^{0,0} \arrow[r, "d^{0,0}"] \arrow[u, "f^{0,0}"'] & M^{1,0} \arrow[r, "d^{1,0}"] \arrow[u, "f^{1,0}"'] & M^{2,0} \arrow[r, "d^{2,0}"] \arrow[u, "f^{2,0}"'] & \cdots \\
C \arrow[u] & & \cdots \arrow[r] & C^0 \arrow[r, "\alpha^0"] \arrow[u] & C^1 \arrow[r, "\alpha^1"] \arrow[u] & C^2 \arrow[r, "\alpha^2"] \arrow[u] & \cdots \\
0 \arrow[u] & & & 0 \arrow[u] & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd} \tag{5.9}$$

In diagram (5.9), each $M^{p,q}$ is injective, each row is a complex, and each column is exact. Since the far left column is an exact sequence of complexes, each column in diagram (5.9) is exact. Since a resolution of a complex is an exact sequence of complexes, diagram (5.9) commutes. Let us make a double complex: Let $e^{p,q} = (-1)^p f^{p,q}$. We still have $d^{p,q} \circ d^{p-1,q} = 0$ since our rows are complexes, and

$$\begin{aligned} e^{p,q+1} \circ e^{p,q} &= (-1)^p f^{p,q+1} \circ (-1)^p f^{p,q}, \text{ by the definition of } e^{p,q}; \\ &= 0. \end{aligned}$$

Also we have

$$\begin{aligned} e^{p+1,q} \circ d^{p,q} &= (-1)^p f^{p+1,q} \circ d^{p,q}, \text{ by the definition of } e^{p,q}; \\ &= (-1)^{p+1} d^{p,q+1} \circ f^{p,q}, \text{ since the diagram commutes ;} \\ &= (-1)(-1)^p d^{p,q+1} \circ f^{p,q} \end{aligned}$$

and

$$d^{p,q+1} \circ e^{p,q} = d^{p,q+1} \circ (-1)^p f^{p,q}, \text{ by the definition of } e^{p,q}.$$

So $e^{p+1,q} \circ d^{p,q} + d^{p,q+1} \circ e^{p,q} = 0$ and we have a double complex. Remove the complex C from diagram (5.9) (so the columns are then deleted resolutions) and label this new double complex $M^{*,*}$.

Let $Z^{p,q} = \ker(d^{p,q})$, $B^{p,q} = \operatorname{im}(d^{p-1,q})$, and $H^{p,q} = Z^{p,q}/B^{p,q}$. Let $Z^i(C) = \ker(\alpha^i)$, $B^i(C) = \operatorname{im}(\alpha^{i-1})$, and $H^i(C) = Z^i(C)/B^i(C)$. Then we obtain three diagrams:

$$\begin{tikzcd}
{} & & & {} & {} & {} & \\
Z^1 \arrow[u] & & \cdots \arrow[r] & Z^{0,1} \arrow[r] \arrow[u] & Z^{1,1} \arrow[r] \arrow[u] & Z^{2,1} \arrow[r] \arrow[u] & \cdots \\
Z^0 \arrow[u] & = & \cdots \arrow[r] & Z^{0,0} \arrow[r] \arrow[u] & Z^{1,0} \arrow[r, "0"] \arrow[u, "e^{1,0}|_{Z^{1,0}}"'] & Z^{2,0} \arrow[r, "0"] \arrow[u, "e^{2,0}|_{Z^{2,0}}"'] & \cdots \\
Z(C) \arrow[u] & & \cdots \arrow[r] & Z^0(C) \arrow[r, "0"] \arrow[u] & Z^1(C) \arrow[r, "0"] \arrow[u] & Z^2(C) \arrow[r, "0"] \arrow[u] & \cdots \\
0 \arrow[u] & & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

of kernels,

$$\begin{tikzcd}
{} & & & {} & {} & {} & \\
B^1 \arrow[u] & & \cdots \arrow[r] & B^{0,1} \arrow[r] \arrow[u] & B^{1,1} \arrow[r] \arrow[u] & B^{2,1} \arrow[r] \arrow[u] & \cdots \\
B^0 \arrow[u] & = & \cdots \arrow[r] & B^{0,0} \arrow[r] \arrow[u] & B^{1,0} \arrow[r, "0"] \arrow[u, "e^{1,0}|_{B^{1,0}}"'] & B^{2,0} \arrow[r, "0"] \arrow[u, "e^{2,0}|_{B^{2,0}}"'] & \cdots \\
B(C) \arrow[u] & & \cdots \arrow[r] & B^0(C) \arrow[r, "0"] \arrow[u] & B^1(C) \arrow[r, "0"] \arrow[u] & B^2(C) \arrow[r, "0"] \arrow[u] & \cdots \\
0 \arrow[u] & & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

of images, and

$$
\begin{array}{ccccccccccc}
\uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
H^1 & & \cdots & \longrightarrow & H^{0,1} & \longrightarrow & H^{1,1} & \longrightarrow & H^{2,1} & \longrightarrow & \cdots \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
H^0 & = & \cdots & \longrightarrow & H^{0,0} & \longrightarrow & H^{1,0} & \xrightarrow{0} & H^{2,0} & \xrightarrow{0} & \cdots \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
H(C) & & \cdots & \longrightarrow & H^0(C) & \xrightarrow{0} & H^1(C) & \xrightarrow{0} & H^2(C) & \xrightarrow{0} & \cdots \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

of homologies.

In these three diagrams, the columns are no longer exact, they are complexes. Also these three diagrams are double complexes since the rows and columns are complexes and since the horizontal zero maps in all three diagrams clearly give us anti-commutativity.

Definition 5.1.10 *An injective resolution of a complex C is* ***proper*** *if all the columns in the previous three diagrams are injective resolutions, i.e. all the columns are exact and made up of injective objects.*

Lemma 5.1.11 *Every complex C, made up of objects from an abelian category with enough injectives, has a proper injective resolution.*

Proof We need to come up with an injective resolution $M^{*,*}$ of C such that when we form the three corresponding diagrams from it, the columns of these diagrams are injective resolutions. We will do this in reverse. Take our complex C

$$\cdots \to C^0 \xrightarrow{\alpha^0} C^1 \xrightarrow{\alpha^1} C^2 \xrightarrow{\alpha^2} \cdots .$$

We will come up with injective resolutions of $Z^p(C) = \ker(\alpha^p), B^p(C) = \mathrm{im}(\alpha^{p-1})$, and $H^p(C) = Z^p(C)/B^p(C)$ for all p, and hence right away have *injective resolutions* of the complexes

$$Z(C) = \cdots \to Z^0(C) \to Z^1(C) \to Z^2(C) \to \cdots ,$$

$$B(C) = \cdots \to B^0(C) \to B^1(C) \to B^2(C) \to \cdots ,$$

and

$$H(C) = \cdots \to H^0(C) \to H^1(C) \to H^2(C) \to \cdots .$$

(Note: the horizontal zero maps ensure that the injective resolutions of these complexes are made up of *complexes*.) Then we will define $M^{*,*}$, which will be an injective resolution of C, and which will correspond to *our* $Z(C), B(C)$, and $H(C)$.

For each p, we have the exact sequence

$$0 \to B^p(C) \to Z^p(C) \to H^p(C) \to 0, \tag{5.10}$$

since

$$\begin{aligned}
&\mathrm{im}(0 \to B^p(C)) = 0 \\
\text{and} \quad &\ker(B^p(C) \to Z^p(C)) = \ker(\mathrm{im}\,\alpha^{p-1} \xrightarrow{inj} \ker \alpha^p) = 0; \\
&\mathrm{im}(B^p(C) \xrightarrow{inj} Z^p(C)) = B^p(C) \\
\text{and} \quad &\ker(Z^p(C) \to H^p(C)) = \ker(Z^p(C) \to Z^p(C)/B^p(C)) = B^p(C); \\
&\mathrm{im}(Z^p(C) \xrightarrow{surj} H^p(C)) = H^p(C) \\
\text{and} \quad &\ker(H^p(C) \to 0) = H^p(C).
\end{aligned}$$

And we have the exact sequence

$$0 \to Z^p(C) \to C^p \to B^{p+1}(C) \to 0, \tag{5.11}$$

since

$$\begin{aligned}
&\mathrm{im}(0 \to Z^p(C)) = 0 \\
\text{and} \quad &\ker(Z^p(C) \to C^p) = \ker(\ker \alpha^p \xrightarrow{inj} C^p) = 0; \\
&\mathrm{im}(Z^p(C) \xrightarrow{inj} C^p) = Z^p(C) \\
\text{and} \quad &\ker(C^p \to B^{p+1}(C)) = \ker(C^p \xrightarrow{\alpha^p} \mathrm{im}\,\alpha^p) = Z^p(C); \\
&\mathrm{im}(C^p \xrightarrow{surj} B^{p+1}(C)) = B^{p+1}(C) \\
\text{and} \quad &\ker(B^{p+1}(C) \to 0) = B^{p+1}(C).
\end{aligned}$$

By Lemma 5.1.6, for each p we can find injective resolutions of $H^p(C)$ and $B^p(C)$. Let $H^{p,*}$ denote the injective resolution of $H^p(C)$, and $B^{p,*}$ denote the injective resolution of $B^p(C)$. Applying Lemma 5.1.7 to (5.10) with injective resolutions $H^{p,*}$ and $B^{p,*}$ we obtain an injective resolution of $Z^p(C)$ for each p; denote this injective resolution by $Z^{p,*}$. We also obtain maps $B^{p,*} \xrightarrow{l^{p,*}} Z^{p,*}$ and $Z^{p,*} \xrightarrow{g^{p,*}} H^{p,*}$, $l^{p,*}$ are injective and $g^{p,*}$ are surjective. These injective resolutions are shown in the following diagram:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & B^{p,1} & \xrightarrow{l^{p,1}} & Z^{p,1} & \xrightarrow{g^{p,1}} & H^{p,1} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & B^{p,0} & \xrightarrow{l^{p,0}} & Z^{p,0} & \xrightarrow{g^{p,0}} & H^{p,0} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B^p(C) & \longrightarrow & Z^p(C) & \longrightarrow & H^p(C) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.12}$$

Again, apply Lemma 5.1.7 to (5.11) with injective resolutions $Z^{p,*}$ and $B^{p+1,*}$ and we obtain an injective resolution of C^p for each p; denote it by $M^{p,*}$. We also obtain maps $Z^{p,*} \xrightarrow{h^{p,*}} M^{p,*}$, and $M^{p,*} \xrightarrow{k^{p,*}} B^{p+1,*}$, $h^{p,*}$ are injective and $k^{p,*}$ are surjective; see the follwing diagram:

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & Z^{p,1} & \xrightarrow{h^{p,1}} & M^{p,1} & \xrightarrow{k^{p,1}} & B^{p+1,1} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & Z^{p,0} & \xrightarrow{h^{p,0}} & M^{p,0} & \xrightarrow{k^{p,0}} & B^{p+1,0} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z^p(C) & \longrightarrow & C^p & \longrightarrow & B^{p+1}(C) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.13}$$

So we have injective resolutions of the objects $Z^p(C), B^p(C)$, and $H^p(C)$ for all p, and hence we have injective resolutions of the complexes $Z(C), B(C)$, and $H(C)$. All we have left is to check is that these resolutions indeed consist of the kernels, images, and homologies of this injective resolution $M^{*,*}$ of C.

Let $d^{p,q} : M^{p,q} \to M^{p+1,q}$ be defined by

$$M^{p,q} \xrightarrow{k^{p,q}} B^{p+1,q} \xrightarrow{l^{p+1,q}} Z^{p+1,q} \xrightarrow{h^{p+1,q}} M^{p+1,q},$$

i.e. $d^{p,q} = h^{p+1,q} \circ l^{p+1,q} \circ k^{p,q}$. Then

$$\begin{aligned} \ker(d^{p,q}) &\simeq \ker(k^{p,q}), \text{ since } l^{p,*} \text{ and } h^{p,*} \text{ are injective;} \\ &= \operatorname{im}(h^{p,q}) \text{ since the rows in diagram 5.13} \\ &\quad \text{are exact by Lemma (5.1.7);} \\ &= h^{p,q}(Z^{p,q}); \\ &\simeq Z^{p,q}, \text{ since } h^{p,q} \text{ is injective.} \end{aligned}$$

Also, $\operatorname{im}(d^{p,q}) \simeq \operatorname{im}(k^{p,q}) = k^{p,q}(M^{p,q}) = B^{p+1,q}$. So $\operatorname{im}(d^{p-1,q}) = B^{p,q}$. Lastly, $\ker(d^{p,q})/\operatorname{im}(d^{p-1,q}) = Z^{p,q}/B^{p,q}$ which equals $H^{p,q}$ since the rows diagram (5.12) are exact by Lemma (5.1.7). So we have constructed a proper injective resolution $M^{*,*}$ of C. ■

5.2 Theorem due to Grothendieck

In this Section we give several definitions relating to covariant functors and then finish with the Grothendieck spectral sequence, given in the form of a theorem.

Definition 5.2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. (see Appendix A for the definition of a category) We call F a* ***covariant functor*** *from $\mathcal{A}$ to $\mathcal{B}$ if*

1. *To each object A in $\mathcal{A}$, we get an object $F(A)$ in $\mathcal{B}$.*

2. *To each morphism $f : A \longrightarrow A'$ in $\mathcal{A}$, we get a morphism $F(f) : F(A) \longrightarrow F(A')$ in $\mathcal{B}$ such that:*

   i. *For each object A in $\mathcal{A}$, we have $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$; i.e. F takes the identity morphism $\mathrm{Id}_A : A \longrightarrow A$ in $\mathcal{A}$ to the identity morphism $\mathrm{Id}_{F(A)} : F(A) \longrightarrow F(A)$ in $\mathcal{B}$.*

   ii. *If $f : A \longrightarrow A'$ and $g : A' \longrightarrow A''$ are two morphisms in $\mathcal{A}$, then $F(g \circ f) = F(g) \circ F(f)$ in $\mathcal{B}$.*

Definition 5.2.2 *The covariant functor F is **additive** if the map*

$$F : \mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$$

is additive for all A, A'' in $\mathcal{A}$, i.e. $F(f+g) = F(f) + F(g)$ for all maps f, g in $\mathrm{Hom}_{\mathcal{A}}(A, A')$.

Definition 5.2.3 *A covariant additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called **left exact** if it takes an exact sequence of objects from $\mathcal{A}$,*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'',$$

into an exact sequence

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

of objects from $\mathcal{B}$.

Definition 5.2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough injectives. Given a covariant additive functor F from $\mathcal{A}$ to $\mathcal{B}$, we can make a new functor R^nF from $\mathcal{A}$ to $\mathcal{B}$, called the **right-derived functor**. Let us describe where R^nF takes an object A of $\mathcal{A}$. By (5.1.6), there exists an injective resolution of A,*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots .$$

Let I represent the complex

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

called the deleted resolution of A. Apply F to get the complex $F(I)$,

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots. \tag{5.14}$$

We define $R^nF(A)$ to be $H^n(F(I))$, the n^{th} homology of (5.14).

Remark. The definition of R^nF is independent of the injective resolution that is chosen. (This is proven in Appendix E)

Let us describe what R^nF does to a morphism $f : A \to A'$ in $\mathrm{Hom}_{\mathcal{A}}(A, A')$. Let I be $0 \to I^0 \xrightarrow{i^0} I^1 \xrightarrow{i^1} I^2 \xrightarrow{i^2} \cdots$, a deleted resolution of A, and let J be $0 \to J^0 \xrightarrow{j^0} J^1 \xrightarrow{j^1} J^2 \xrightarrow{j^2} \cdots$, a deleted resolution of A'. Now by Lemma E.1, $f : A \to A'$ leads us to a set of maps $\{f^n : I^n \to J^n\}$ (See Appendix E). We will have $R^nF(f) : R^nF(A) \to R^nF(A')$ which is $R^nF(f) : H^n(F(I)) \to H^n(F(J))$, i.e.

$$R^nF(f) : \ker Fi^n / \operatorname{im} Fi^{n-1} \to \ker Fj^n / \operatorname{im} Fj^{n-1}. \tag{5.15}$$

We define

$$R^nF(f) : x / \operatorname{im} Fi^{n-1} \mapsto Ff^n(x) / \operatorname{im} Fj^{n-1}. \tag{5.16}$$

By the lemmas in Appendix E the map f is unique up to homotopy; hence $R^nF(f)$ is unique.

Definition 5.2.5 *Let F be a covariant additive functor from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories with enough injectives. An object A in $\mathcal{A}$ is called **F-acyclic** if $R^nF(A) = 0$ for all $n > 0$.*

Lemma 5.2.6 *Let F be a covariant additive functor from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}$ has enough injectives. Let A be an injective object in $\mathcal{A}$. Then A is F-acyclic.*

Proof Consider the injective resolution of A:

$$0 \to A \to A \to 0 \to 0 \to 0 \to \cdots.$$

Let I represent the corresponding deleted resolution. Then $R^nF(A) = H^n(FI) = H^n(0 \to FA \to 0 \to 0 \to \cdots) = 0/0 = 0$ for $n > 0$. ■

Now we are prepared to prove the following theorem, which is due to Grothendieck:

Theorem 5.2.7 *Let $G : \mathcal{U} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be covariant additive functors, F left exact, where $\mathcal{U}, \mathcal{B}$, and $\mathcal{C}$ are abelian categories. Let $\mathcal{U}$ and $\mathcal{B}$ have enough injectives. Also assume that if A is an injective object in $\mathcal{U}$ then the object GA is injective in $\mathcal{B}$. Then for each object A in $\mathcal{U}$,*
(i) we have a spectral sequence which converges to $\{R^n(FG)(A)\}_n$, and
(ii) this spectral sequence has $\mathcal{E}^2_{p,q} = R^pF(R^qG(A))$.

Proof of (i) Let A be an object in $\mathcal{U}$. By Lemma 5.1.6 we can find an injective resolution of A,

$$0 \to A \to E^0 \to E^1 \to E^2 \to \cdots.$$

Let E be the deleted resolution of A,

$$0 \to E^0 \to E^1 \to E^2 \to \cdots.$$

Now apply the functor G to the complex E to get the complex GE,

$$0 \to GE^0 \to GE^1 \to GE^2 \to \cdots.$$

By Lemma 5.1.11, we have a proper injective resolution of GE,

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M^{0,0} & \longrightarrow & M^{1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & GE^0 & \longrightarrow & GE^1 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0. & &
\end{array}$$

By the definition of an exact sequence of complexes, this diagram commutes. Let $M = M^{*,*}$ represent:

$$\begin{tikzcd}
 & {} & {} & \\
0 \arrow[r] & M^{0,1} \arrow[r] \arrow[u] & M^{1,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & M^{0,0} \arrow[r] \arrow[u] & M^{1,0} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd} \tag{5.17}$$

As done on page 58, we can turn commutative diagram (5.17) into a double complex by adjusting the vertical maps. Then apply the functor F to obtain the double complex $FM^{*,*} = FM$,

$$\begin{tikzcd}
 & {} & {} & \\
0 \arrow[r] & FM^{0,1} \arrow[r] \arrow[u] & FM^{1,1} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & FM^{0,0} \arrow[r] \arrow[u] & FM^{1,0} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0. \arrow[u] &
\end{tikzcd}$$

The diagram FM remains a double complex since F is a covariant additive functor, i.e. $F(a) \circ F(b) + F(c) \circ F(d) = F(a \circ b) + F(c \circ d) = F(a \circ b + c \circ d) = F(0) = 0$, if $a \circ b + c \circ d = 0$. We can then obtain the total complex $\mathrm{Tot}(FM)$, where FM is first quadrant. At this point we can obtain the spectral sequence associated with the first filtration of $\mathrm{Tot}(FM)$, as in Example 3 of Chapter 1. (We call this spectral sequence the *first spectral sequence*.)

Notice that in $M^{*,*}$, the p^{th} column is the deleted resolution of GE^p. Recall that $M^{i,j}$ is injective for all i, j. Let us look at the vertical homology of $FM^{*,*}$. The (vertical) homology at the q^{th} spot of the p^{th} column, $H^q(FM^{p,*})$, is

$$\ker(FM^{p,q} \to FM^{p,q+1}) / \operatorname{im}(FM^{p,q-1} \to FM^{p,q}).$$

This is exactly $(R^qF)(GE^p)$.

Now let us show that

$$H^q(FM^{p,*}) = \begin{cases} FG(E^p) \text{ if } q = 0 \\ 0 \text{ if } q > 0. \end{cases}$$

Notice that E^p injective implies that GE^p is right F-acyclic by Lemma 5.2.6, hence $(R^qF)(GE^p) = 0$ for $q > 0$. So $H^q(FM^{p,*}) = 0$ for $q > 0$. In the case $q = 0$ we need to look at $(R^0F)(GE^p)$, which is the homology at the 0 spot of the p^{th} column of $FM^{*,*}$,

$$0 \to FM^{p,0} \to FM^{p,1} \to \cdots .$$

We have that $(R^0F)(GE^p) = \ker(FM^{p,0} \to FM^{p,1})/0 = \ker(FM^{p,0} \to FM^{p,1})$. Recall that F left exact means if $0 \to GE^p \to M^{p,0} \to M^{p,1} \to \cdots$ is exact, then $0 \to FGE^p \to FM^{p,0} \to FM^{p,1} \to \cdots$ is exact. Hence, $(R^0F)(GE^p) = \ker(FM^{p,0} \to FM^{p,1}) = \mathrm{im}(FGE^p \overset{injection}{\longrightarrow} FM^{p,0}) \simeq FGE^p$. We have shown that

$$H^0(FM^{p,*}) = FG(E^p). \tag{5.18}$$

So doing vertical homology to $FM^{*,*}$, we have zero everywhere except possibly at $q = 0$; we are left with just one nonzero row in the lattice $\{H^q(FM^{p,*})\}_{p,q}$,

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & FGE^0 & \longrightarrow & FGE^1 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0. & &
\end{array}$$

This nonzero bottom row is

$$0 \longrightarrow FG(E^0) \longrightarrow FG(E^1) \longrightarrow FG(E^2) \longrightarrow \cdots .$$

Now we will do horizontal homology, $H^p(\{H^q(FM^{p,*})\}_p)$, i.e. the homology at the p^{th} spot of

$$\cdots \to H^q(FM^{p-1,*}) \to H^q(FM^{p,*}) \to H^q(FM^{p+1,*}) \to \cdots .$$

This is by definition $\mathcal{E}_2^{p,q}$, by (3.2) on page 38.

For $q > 0$, we clearly have $H^p(\{H^q(FM^{p,*})\}_p) = H^p(\{0\}) = 0$. For $q = 0$,

$$\begin{aligned} H^p(\{H^0(FM^{p,*})\}_p) &= H^p(\{FG(E^p)\}_p), \text{ by (5.18);} \\ &= (R^p(FG))(A), \text{ by definition of right derived functor.} \end{aligned}$$

So now we have that

$$\mathcal{E}_2^{p,q} = \begin{cases} R^p(FG)(A) \text{ if } q = 0 \\ 0 \text{ if } q > 0. \end{cases} \tag{5.19}$$

This means that our spectral sequence collapses at $r = 2$. Therefore by Lemma 3.2.2, $\mathcal{E}_2^{n,0} = H^n(\mathrm{Tot}(FM^{*,*}))$. Recall from Section 2.3 that our spectral sequence converges to $H^n(\mathrm{Tot}(FM^{*,*}))$. In (5.19) with $p = n$ and $q = 0$, $\mathcal{E}_2^{n,0} = R^n(FG)(A)$, so we proved that our spectral sequence converges to $\{R^n(FG)(A)\}_n$. ■

Let us have a few definitions and lemmas before doing part (ii) of the proof.

Definition 5.2.8 *Let $0 \to B \xrightarrow{h} C \xrightarrow{f} D \xrightarrow{g} 0$ be an exact sequence. We say the sequence is split if there exist a map $\widetilde{h} : C \to B$ such that $\widetilde{h} \circ h = id_B$.*

Lemma 5.2.9 *If B is injective in the exact sequence $0 \to B \xrightarrow{h} C \xrightarrow{f} D \xrightarrow{g} 0$, then the sequence is split.*

Proof Consider the diagram

$$\begin{tikzcd} & B & \\ 0 \arrow[r] & B \arrow[u, "id_B"] \arrow[r, "h"'] & C. \end{tikzcd}$$

Since B is injective, we obtain a map $\widetilde{h}$ such that $\widetilde{h} \circ h = id_B$. ■

Lemma 5.2.10 *Let $0 \to B \xrightarrow{h} C \xrightarrow{f} D \xrightarrow{g} 0$ be an exact sequence. If the sequence is split, then there exists a map $\widetilde{f} : D \to C$ such that $f \circ \widetilde{f} = id_D$.*

Proof Since the sequence is exact, we certainly have the isomorphic map

$$D = f(C) \xrightarrow{m} C/h(B), \quad f(z) \mapsto [z].$$

Consider

$$D = f(C) \xrightarrow{m} C/h(B) \overset{i}{\hookrightarrow} C,$$

where i takes $[z]$ to some representative of the class $[z]$.

Let $\widetilde{f} = i \circ m$, and let x be in D. Then $x = f(c)$ for some c in C, and $f \circ \widetilde{f}(x) = f \circ i \circ m(x) = f \circ i \circ m \circ f(c) = f \circ i([c])$. Since for any representative u of $[c]$, $u - c \in h(B) = \ker f$ and so $f(u) = f(c)$. Therefore $f \circ i([c]) = f(u) = f(c) = x$, and hence we have $f \circ \widetilde{f} = id_D$. ■

Lemma 5.2.11 *Let* $0 \to B \xrightarrow{h} C \xrightarrow{f} D \xrightarrow{g} 0$ *be an exact sequence that is split. Apply any functor* F *to obtain* $0 \to FB \xrightarrow{Fh} FC \xrightarrow{Ff} FD \xrightarrow{Fg} 0$. *Then there exists* $\widetilde{Fh} : FC \to FB$ *such that* $\widetilde{Fh} \circ Fh = id_{FB}$.

Proof We have the map $h' : C \to B$ where $h' \circ h = id_B$, since the exact sequence splits. Let $\widetilde{Fh} = Fh' : FC \to FB$. Then $Fh' \circ Fh = F(h' \circ h)$, by the definition of a functor. This equals $F(id_B)$ which equals id_{FB}, again by the definition of a functor. ■

Remark. Similarly, if there exists $f' : D \to C$ such that $f \circ f' = id_D$, then $Ff' \circ Ff = id_{FD}$.

Proof of (ii) As expected, the *second spectral sequence* is the spectral sequence associated with the second filtration of $\mathrm{Tot}(FM)$. It was explained in Example 7 of Chapter 2 that both the first and second spectral sequences converge to the same object. Let us look at the term $\mathcal{E}_2^{p,q}$ of the second spectral sequence, which is $H^p(H^q(FM^{*,p}))$ by (3.3) on page 39. We will show that $H^p(H^q(FM^{*,p})) = R^pF(R^qG(A))$.

Consider our proper injective resolution of GE, where E is the injective resolution of A. It has q^{th} row

$$0 \to M^{0,q} \to M^{1,q} \to M^{2,q} \to M^{3,q} \to \cdots .$$

Using the notation from pages 59-61, we have for each p injective resolutions

$$0 \to Z^p(GE) \to Z^{p,0} \to Z^{p,1} \to Z^{p,2} \to \cdots ,$$

$$0 \to B^p(GE) \to B^{p,0} \to B^{p,1} \to B^{p,2} \to \cdots ,$$

and

$$0 \to H^p(GE) \to H^{p,0} \to H^{p,1} \to H^{p,2} \to \cdots ,$$

since the injective resolution of GE is *proper*. (These injective resolutions for each p can be found as columns from the diagrams on pages 59–60.) From Lemma 5.1.11, diagram (5.13) gives us an exact sequence of objects $0 \to Z^{p,q} \to M^{p,q} \to B^{p+1,q} \to 0$. Transposing indices gives us

$$0 \to Z^{q,p} \to M^{q,p} \to B^{q+1,p} \to 0. \tag{5.20}$$

The object $Z^{q,p}$ is injective, so by Lemma 5.2.9, (5.20) splits. Apply F to get the sequence

$$0 \to FZ^{q,p} \xrightarrow{\phi} FM^{q,p} \xrightarrow{Fd^{q,p}} FB^{q+1,p} \to 0. \tag{5.21}$$

Then we have

$$\begin{aligned} \ker Fd^{q,p} &= \operatorname{im} \phi, \text{ since } F \text{ is left exact;} \\ &= \phi(FZ^{q,p}) \simeq FZ^{q,p}. \end{aligned} \tag{5.22}$$

Now replace q with $q-1$ in (5.21) to obtain

$$0 \to FZ^{q-1,p} \xrightarrow{\phi} FM^{q-1,p} \xrightarrow{Fd^{q-1,p}} FB^{q,p} \to 0. \tag{5.23}$$

We will show that $\operatorname{im}(Fd^{q-1,p}) = FB^{q,p}$. In one direction, the containment is clear, $Fd^{q-1,p}(FM^{q-1,p}) \subset FB^{q,p}$. For the other direction, let $Fx \in FB^{q,p}$, where $x \in B^{q,p}$.

Since $d^{q-1,p}$ is surjective, we have that $x = d^{q-1,p}(m)$, for some $m \in M^{q-1,p}$. By the remark following Lemma 5.2.11, there exists $l : FB^{q,p} \to FM^{q-1,p}$ such that $Fd^{q-1,p} \circ l = id_{FB^{q,p}}$. Now since Fx is in $FB^{q,p}$, we have that $Fx = Fd^{q-1,p} \circ l(Fx)$. And since $l(Fx)$ is in $FM^{q-1,p}$, $Fx \subset Fd^{q-1,p}(FM^{q-1,p})$. So we have shown that

$$\operatorname{im}(Fd^{q-1,p}) = FB^{q,p}. \tag{5.24}$$

Recall, $H^q(FM^{*,p})$ denotes taking the homology at the q^{th} spot of the row $FM^{*,p}$. Therefore

$$\begin{aligned} H^q(FM^{*,p}) &= \ker(Fd^{q,p}) / \operatorname{im}(Fd^{q-1,p}); \\ &= FZ^{q,p} / \operatorname{im}(Fd^{q-1,p}), \text{ by (5.22)}; \\ &= FZ^{q,p}/FB^{q,p}, \text{ by (5.24)}. \end{aligned} \tag{5.25}$$

From Lemma 5.1.11, diagram (5.12) gives us an exact sequence of objects

$$0 \to B^{p,q} \to Z^{p,q} \to H^{p,q} \to 0.$$

Transposing indices gives us

$$0 \to B^{q,p} \to Z^{q,p} \to H^{q,p} \to 0. \tag{5.26}$$

The object $B^{q,p}$ is injective, so by Lemma 5.2.9, (5.26) is split. Apply F to (5.26) to obtain

$$0 \to FB^{q,p} \xrightarrow{m} FZ^{q,p} \xrightarrow{n} FH^{q,p} \to 0. \tag{5.27}$$

Then we have

$$\begin{aligned} \operatorname{im} n &\simeq FZ^{q,p} / \ker n, \text{ by the first isomorphism theorem}; \\ &= FZ^{q,p} / \operatorname{im} m, \text{ since } F \text{ is left exact}; \\ &= FZ^{q,p}/m(FB^{q,p}) \simeq FZ^{q,p}/FB^{q,p}, \text{ since } m \text{ is injective}. \end{aligned}$$

The object $\operatorname{im} n = FH^{q,p}$ by the exact same proof we used to show (5.24). So we have that

$$FZ^{q,p}/FB^{q,p} = FH^{q,p}. \tag{5.28}$$

So

$$
\begin{aligned}
\mathcal{E}_2^{p,q} &= H^p(\{H^q(FM^{*,p})\}_p); \\
&= H^p(\{FZ^{q,p}/FB^{q,p}\}_p), \text{ by (5.25)}; \\
&= H^p(\{FH^{q,p}\}_p), \text{ by (5.28)}.
\end{aligned} \tag{5.29}
$$

Recall that

$$
\begin{aligned}
R^qG(A) &= \text{homology at the } q \text{ spot of} \\
&\quad 0 \to GE^0 \to GE^1 \to \cdots \to GE^q \to GE^{q+1} \to \cdots; \\
&= H^q(GE).
\end{aligned} \tag{5.30}
$$

And (for each q) we have an injective resolution of $H^q(GE)$,

$$
0 \to H^q(GE) \to H^{q,0} \to H^{q,1} \to H^{q,2} \to \cdots. \tag{5.31}
$$

Hence

$$
\begin{aligned}
R^pF(R^qG(A)) &= R^pF(H^q(GE)), \text{ by (5.30)}; \\
&= \text{homology at the } p \text{ spot of} \\
&\quad 0 \to FH^{q,0} \to FH^{q,1} \to FH^{q,2} \to \cdots; \\
&= H^p(\{FH^{q,p}\}_p); \\
&= \mathcal{E}_2^{p,q}, \text{ by (5.29)}.
\end{aligned}
$$

So it is the second spectral sequence which converges to $\{R^n(FG(A))\}_n$ and has $\mathcal{E}_2^{p,q} = R^pF(R^qG(A))$. ■

APPENDIX A

CATEGORIES

The following came from Lang [1993] and Weibel [1995]:

A *category* $\mathcal{C}$ consists of *objects* and *morphisms*, where a morphism is an assignment from an object to another object. The collection of all objects in $\mathcal{C}$ is denoted by $Ob(\mathcal{C})$. Taking two objects A, B in $Ob(\mathcal{C})$, we denote the set of morphisms from A to B by $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Let A, B, C be any three objects from $Ob(\mathcal{C})$, and take arbitrary morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C), h \in \mathrm{Hom}_{\mathcal{C}}(C, D), j \in \mathrm{Hom}_{\mathcal{C}}(B, A)$. A category must satisfy the following properties:

1. We must be able to compose f and g and obtain $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$. This is commonly viewed as

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \text{ by } g \times f \mapsto g \circ f.$$

2. There exists a map Id_A in $\mathrm{Hom}_{\mathcal{C}}(A, A)$ such that $f \circ \mathrm{Id}_A = f$ and $\mathrm{Id}_A \circ j = j$.

3. $(h \circ g) \circ f = h \circ (g \circ f)$.

4. $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is disjoint from $\mathrm{Hom}_{\mathcal{C}}(C, D)$ unless both $A = C$ and $B = D$, in which case the two sets are equal.

There are a few types of categories that will be important to us.

The Additive Category. An *additive* category is a category $\mathcal{C}$ with the additional properties:

1. Every set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group under addition, i.e. for h, h' in $\mathrm{Hom}_{\mathcal{C}}(A, B)$, $h + h' = h' + h$ and $h + 0 = h$ where 0 is the zero map from A to B.

2. The map

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \text{ by } g \times f \mapsto g \circ f.$$

   is bilinear, i.e. $g \circ (f + f') = g \circ f + g \circ f'$ and $(g + g') \circ f = g \circ f + g \circ f'$.

3. There exists a zero object 0 in the category such that $\mathrm{Hom}_{\mathcal{C}}(A, 0)$ and $\mathrm{Hom}_{\mathcal{C}}(0, B)$ each have exactly one element.

4. The product of A and B exists. Suppose we have $\alpha \in \mathrm{Hom}_{\mathcal{C}}(P, A)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}}(P, B)$ such that when we are given any two morphisms $\gamma \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ and $\delta \in \mathrm{Hom}_{\mathcal{C}}(C, B)$ there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(C, P)$ such that $\gamma = \alpha \circ h$ and $\delta = \beta \circ h$. Then we say the product of A and B exists and write $A \times B = (P, \alpha, \beta)$.

The Abelian Category. An *abelian* category is an additive category $\mathcal{A}$ with the following properties:

1. Every morphism $f \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ has a kernel i. (A *kernel* i is a morphism in $\mathrm{Hom}_{\mathcal{A}}(A, B)$ such that $f \circ i = 0$ and given any $e \in \mathrm{Hom}_{\mathcal{A}}(A', B)$ with $f \circ e = 0$ then $e = i \circ e'$ for a unique $e' \in \mathrm{Hom}_{\mathcal{A}}(A', A)$.)

2. Every morphism $f \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ has a cokernel p. (A *cokernel* p is a morphism in $\mathrm{Hom}_{\mathcal{A}}(C, D)$ such that $p \circ f = 0$ and given any $g \in \mathrm{Hom}_{\mathcal{A}}(C, D')$ with $g \circ f = 0$ then $g = g' \circ p$ for a unique $g' \in \mathrm{Hom}_{\mathcal{A}}(D, D')$.)

3. Every morphism that is monic is the kernel of its cokernel. (A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ is *monic* if for any $e_1, e_2 \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, $e_1 \neq e_2$ implies $f \circ e_1 \neq f \circ e_2$.)

4. Every morphism that is epi is the cokernel of its kernel. (A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ is *epi* if for any $g_1, g_2 \in \mathrm{Hom}_{\mathcal{A}}(C, D)$, $g_1 \neq g_2$ implies $g_1 \circ f \neq g_2 \circ f$.)

Note. If $0 \to A \xrightarrow{f} B$ is exact, f is called a monomorphism; we will refer to f as injective. If $A \xrightarrow{f} B \to 0$ is exact, f is called an epimorphism; we will refer to f as surjective.

The Needed Category. The *needed* category is an abelian category $\mathcal{A}$ with the additional four properties (the first and third were introduced by Grothendieck):

1. For every set $\{A_i\}$ of objects in $\mathcal{A}$, the direct sum $\oplus A_i$ exists in $\mathcal{A}$. Then we call $\mathcal{A}$ cocomplete.

2. The direct sum of monic morphisms is a monic morphism.

3. For every set $\{A_i\}$ of objects in $\mathcal{A}$, the product $\prod A_i$ exists in $\mathcal{A}$. Then we call $\mathcal{A}$ complete.

4. The product of epi morphisms is an epi morphism.

APPENDIX B

ISOMORPHISM THEOREMS

A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ is called an *isomorphism* if there exists $g \in \mathrm{Hom}_{\mathcal{A}}(A', A)$ such that $g \circ f$ equals the identity map in $\mathrm{Hom}_{\mathcal{A}}(A, A)$ and $f \circ g$ equals the identity map in $\mathrm{Hom}_{\mathcal{A}}(A', A')$.

Let A be an abelian category $\mathcal{A}$. Let $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$, where $A, A' \in Ob(\mathcal{A})$. The first isomorphism theorem states that

$$A/\ker(f) \simeq f(A).$$

In addition, let A and A' both be subobjects of some object G in $\mathcal{A}$ (A is a subobject of G if $0 \to A \to B$ is exact). Then we have the second isomorphism theorem, which states that

$$(A + A')/A' \simeq A/(A \bigcap A').$$

Furthermore, assume that $A \subset A'$. The third isomorphism theorem states that

$$(G/A)/(A'/A) \simeq G/A'.$$

See Dummit and Foote [1991] for proofs of the ismorphism theorems.

APPENDIX C

AN EXACT SEQUENCE OF COMPLEXES GIVES A LONG EXACT SEQUENCE

In this appendix, we will prove the following lemma in detail:

Lemma C.1 *Let us be in an abelian category. Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be an exact sequence of complexes, (so $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \to 0$ *is exact for all* n*.) Here is the diagram corresponding to this exact sequence of complexes:*

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{i_{n+1}} & B_{n+1} & \xrightarrow{p_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle g_{n+1}}\downarrow & & {\scriptstyle h_{n+1}}\downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \longrightarrow & 0 \\
 & & {\scriptstyle f_n}\downarrow & & {\scriptstyle g_n}\downarrow & & {\scriptstyle h_n}\downarrow & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i_{n-1}} & B_{n-1} & \xrightarrow{p_{n-1}} & C_{n-1} & \longrightarrow & 0. \\
 & & {\scriptstyle f_{n-1}}\downarrow & & {\scriptstyle g_{n-1}}\downarrow & & {\scriptstyle h_{n-1}}\downarrow & &
\end{array}$$

Then there is an exact sequence of objects

$$\cdots \to H_n(A) \xrightarrow{i_n^*} H_n(B) \xrightarrow{p_n^*} H_n(C) \xrightarrow{\delta_n} H_{n-1} \xrightarrow{i_{n-1}^*} H_{n-1}(B) \xrightarrow{p_{n-1}^*} \cdots . \qquad \text{(C.1)}$$

Proof The maps i_n^* and p_n^* are induced from i_n, p_n, respectively. Let us describe δ_n. Define

$$\delta_n : H_n(C) \to H_{n-1}(A)$$

which is

$$\ker h_n / \operatorname{im} h_{n+1} \to \ker f_{n-1} / \operatorname{im} f_n,$$

by

$$z / \operatorname{im} h_{n+1} \mapsto i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z) / \operatorname{im} f_n,$$

where z is in $\ker h_n$. We will show that δ_n is indeed a map by showing that it satisfies the following properties:

I. δ_n takes zero to zero. Take $x / \operatorname{im} h_{n+1}$ in $\ker h_n / \operatorname{im} h_{n+1}$, where x is in $\operatorname{im} h_{n+1}$. Then $x = h_{n+1}(c)$ for some c in C_{n+1}, where $c = p_{n+1}(b)$ since p_{n+1} is surjective. We want to show that $i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(x) = i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ h_{n+1}(c)$ is in $\operatorname{im} f_n$. Notice that

$$\begin{aligned} i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ h_{n+1}(c) &= i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ h_{n+1} \circ p_{n+1}(b); \\ &= i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ p_n \circ g_{n+1}(b), \text{ since the diagram commutes.} \end{aligned}$$

Now let u be any element in the set $p_n^{-1} \circ p_n \circ g_{n+1}(b)$. Of course, $g_{n+1}(b)$ is in $p_n^{-1} \circ p_n \circ g_{n+1}(b)$. The difference $i_{n-1}^{-1} \circ g_n(u) - i_{n-1}^{-1} \circ g_n(g_{n+1}(b))$ is in $\operatorname{im} f_n$ by the definition of δ_n. So $g_n(u) - g_n(g_{n+1}(b))$ is in $i_{n-1}(\operatorname{im} f_n)$ which implies that $g_n(u)$ is in $i_{n-1}(\operatorname{im} f_n)$ since $g_n \circ g_{n+1} = 0$. Hence $i_{n-1}^{-1} \circ g_n(u)$ is in $\operatorname{im} f_n$, where u is any element in $p_n^{-1} \circ p_n \circ g_{n+1}(b)$. Therefore

$$i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ p_n \circ g_{n+1}(b) \subset \operatorname{im} f_n;$$

$$\text{which implies } i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ h_{n+1} \circ p_{n+1}(b) \subset \operatorname{im} f_n, \text{ since the diagram commutes;}$$

$$\text{which implies } i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ h_{n+1}(c) \subset \operatorname{im} f_n;$$

$$\text{which implies } i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(x) \subset \operatorname{im} f_n.$$

II. δ_n is well defined. Since p_n is surjective, we can get some element b_n in B_n such that $p_n(b_n) = z$. Since

$$p_{n-1} \circ g_n(b_n) = h_n \circ p_n(b_n) = h_n(z) = 0,$$

$$g_n(b_n) \text{ is in } \ker p_{n-1} = \operatorname{im} i_{n-1}.$$

So we can apply i_{n-1}^{-1}, which is injective, to $g_n(b_n)$ and get $i_{n-1}^{-1} \circ g_n(b_n) = a_{n-1}$. To complete the verification that δ_n is well defined, suppose we had obtained b_n'' in B_n such that $p_n(b_n'') = z$. Then doing as above, we would have $i_{n-1}^{-1} \circ g_n(b_n'') = a_{n-1}''$.

We need to show that $a_{n-1}/\operatorname{im} f_n = a_{n-1}''/\operatorname{im} f_n$, i.e. that $a_{n-1} - a_{n-1}''$ is in $\operatorname{im} f_n$. Now since $p_n(b_n - b_n'') = p_n(b_n) - p_n(b_n'') = z - z = 0$, $b_n - b_n''$ is in $\ker p_n = \operatorname{im} i_n$. So $b_n - b_n'' = i_n(a_n)$ for some a_n in A_n. Hence

$$\begin{aligned}
a_{n-1} - a_{n-1}'' &= i_{n-1}^{-1} \circ g_n(b_n) - i_{n-1}^{-1} \circ g_n(b_n''); \\
&= i_{n-1}^{-1} \circ g_n(b_n - b_n''); \\
&= i_{n-1}^{-1} \circ g_n \circ i_n(a_n); \\
&= i_{n-1}^{-1} \circ i_{n-1} \circ f_n(a_n), \text{ since the diagram commutes;} \\
&= f_n(a_n).
\end{aligned}$$

III. δ_n preserves the operation of summation.

$$\begin{aligned}
\delta_n(x/\operatorname{im} h_{n+1} + y/\operatorname{im} h_{n+1}) &= \delta_n((x+y)/\operatorname{im} h_{n+1}); \\
&= i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(x+y)/\operatorname{im} f_n; \\
&= i_{n-1}^{-1} \circ g_n(u+v))/\operatorname{im} f_n, \text{ for any } u \in p_n^{-1}(x) \\
&\quad \text{ and any } v \in p_n^{-1}(y) \text{ by } (II.); \\
&= i_{n-1}^{-1} \circ g_n(u) + i_{n-1}^{-1} \circ g_n(v); \\
&= i_{n-1}^{-1} \circ g_n(p_n^{-1}(x)) + i_{n-1}^{-1} \circ g_n(p_n^{-1}(y)), \text{ by } (II.); \\
&= \delta_n(x/\operatorname{im} h_{n+1}) + \delta_n(y/\operatorname{im} h_{n+1}).
\end{aligned}$$

IV. The range given for δ_n is appropriate. Let us verify that $i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z)$ is in $\ker f_{n-1}$.

$$\begin{aligned} f_{n-1} \circ i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z) &= i_{n-2}^{-1} \circ g_{n-1} \circ g_n \circ p_n^{-1}(z), \text{ by Lemma } D.1 \\ &\quad \text{since } g_{n-1} \circ g_n \circ p_n^{-1}(z) \text{ is in } \operatorname{im} i_{n-1}; \\ &= 0, \text{ since } g_{n-1} \circ g_n = 0. \end{aligned}$$

To show that long sequence (C.1) is exact, it suffices to check the exactness at three spots:

i. $H_n(B)$,

ii. $H_n(C)$,

iii. $H_{n-1}(A)$.

To prove i., ii., and iii., we will show containment in both directions.

i.a. We have $\operatorname{im} i_n^* \subset \ker p_n^*$ since

$$\begin{aligned} \operatorname{im} i_n^* &= i_n^*(H_n(A) \to H_n(B)); \\ &= i_n^*(\ker f_n / \operatorname{im} f_{n+1} \to \ker g_n / \operatorname{im} g_{n+1}); \\ &= i_n(\ker f_n) / \operatorname{im} g_{n+1} \end{aligned}$$

and so

$$\begin{aligned} p_n^*(\operatorname{im} i_n^*) &= p_n^*(i_n(\ker f_n) / \operatorname{im} g_{n+1}), \text{ by the above}; \\ &= p_n \circ i_n(\ker f_n) / \operatorname{im} h_{n+1}; \\ &= 0 / \operatorname{im} h_{n+1} = 0. \end{aligned}$$

i.b. To show that $\ker p_n^* \subset \operatorname{im} i_n^*$, let $z/\operatorname{im} g_{n+1}$ be in $\ker p_n^*$. Then $p_n^*(z/\operatorname{im} g_{n+1}) = 0$ which is $p_n(z)/\operatorname{im} h_{n+1} = 0$, and so $p_n(z)$ is in $\operatorname{im} h_{n+1}$. So $p_n(z) = h_{n+1}(c_{n+1})$ for some c_{n+1} in C_{n+1}. The map p_{n+1} is surjective, so $c_{n+1} = p_{n+1}(b_{n+1})$ for some b_{n+1} in B_{n+1}. So

$$\begin{aligned} p_n(z) &= h_{n+1} \circ p_{n+1}(b_{n+1}); \\ &= p_n \circ g_{n+1}(b_{n+1}), \text{ since the diagram commutes.} \end{aligned}$$

This implies that

$$\begin{gathered} p_n(z) - p_n \circ g_{n+1}(b_{n+1}) = 0; \\ \text{which implies } p_n(z - g_{n+1}(b_{n+1})) = 0; \\ \text{which implies } z - g_{n+1}(b_{n+1}) \text{ is in } \ker p_n = \operatorname{im} i_n; \\ \text{which implies } z - g_{n+1}(b_{n+1}) = i_n(a_n) \text{ for some } a_n \text{ in } A_n. \end{gathered}$$

Actually, a_n is in $\ker f_n$ since

$$\begin{aligned} i_{n-1} \circ f_n(a_n) &= g_n \circ i_n(a_n), \text{ since the diagram commutes;} \\ &= g_n(z - g_{n+1}(b_{n+1})), \text{ from the above;} \\ &= g_n(z) - g_n \circ g_{n+1}(b_{n+1}); \\ &= 0 - g_n \circ g_{n+1}(b_{n+1}), \text{ since } z \text{ is in } \operatorname{im} g_{n+1} \subset \ker g_n \text{ by hypothesis;} \\ &= 0, \text{ since } g_n \circ g_{n+1} = 0. \end{aligned}$$

So we obtain that $f_n(a_n) = 0$ since i_{n-1} is injective. Therefore

$$\begin{aligned} i_n^*(a_n/\operatorname{im} f_{n+1}) &= i_n(a_n)/\operatorname{im} g_{n+1}; \\ &= z - g_{n+1}(b_n)/\operatorname{im} g_{n+1}; \\ &= z/\operatorname{im} g_{n+1}. \end{aligned}$$

ii.a. Let us show that $\operatorname{im} p_n^* \subset \ker \delta_n$. Let $p_n^*(z/\operatorname{im} g_{n+1})$ be in $\operatorname{im} p_n^*$, which is $p_n(z)/\operatorname{im} h_{n+1}$, where z is in $\ker g_n$. We need to show that $\delta_n(p_n(z)/\operatorname{im} h_{n+1})$ which equals $(i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ p_n(z))/\operatorname{im} f_n$ is equal to zero. Le us look at $g_n \circ p_n^{-1} \circ p_n(z)$. Let c be any element in the set $p_n^{-1} \circ p_n(z)$. Of course, z is in $p_n^{-1} \circ p_n(z)$. From the definition of δ_n, we know that $i_{n-1}^{-1} \circ g_n(c) - i_{n-1}^{-1} \circ g_n(z)$ is in $\operatorname{im} f_n$. This implies that

$$g_n(c) - g_n(z) \text{ is in } i_{n-1}(\operatorname{im} f_n);$$

$$\text{which implies that } g_n(c) \text{ is in } i_{n-1}(\operatorname{im} f_n), \text{ since } g_n(z) = 0;$$

$$\text{which implies that } i_{n-1}^{-1} \circ g_n(c) \text{ is in } \operatorname{im} f_n.$$

So $i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ p_n(z)$ is in $\operatorname{im} f_n$ for any choice of c in $p_n^{-1} \circ p_n(z)$. Hence $(i_{n-1}^{-1} \circ g_n \circ p_n^{-1} \circ p_n(z))/\operatorname{im} f_n = 0$.

ii.b. Let us show that $\ker \delta_n \subset \operatorname{im} p_n^*$, where the domain of p_n^* is $\ker g_n/\operatorname{im} g_{n+1}$. Let $z/\operatorname{im} h_{n+1}$ be in $\ker \delta_n$, where z is in $\ker h_n$. Then

$$\delta_n(z/\operatorname{im} h_{n+1}) = (i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z))/\operatorname{im} f_n = 0/\operatorname{im} f_n,$$

i.e. $i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z)$ is in $\operatorname{im} f_n$ and so it equals $f_n(a)$ for some a in A_n. Apply i_{n-1} to obtain

$$i_{n-1} \circ i_{n-1}^{-1} \circ g_n \circ p_n^{-1}(z) = i_{n-1} \circ f_n(a)$$

which implies

$$g_n \circ p_n^{-1}(z) = g_n \circ i_n(a),$$

since the diagram commutes. Then we have $g_n(p_n^{-1}(z) - i_n(a)) = 0$ which implies $p_n^{-1}(z) - i_n(a)$ is in $\ker g_n$. So the element $(p_n^{-1}(z) - i_n(a))/\operatorname{im} g_{n+1}$ is in the domain of p_n^*. Hence we have

$$\begin{aligned} p_n^*((p_n^{-1}(z) - i_n(a))/\operatorname{im} g_{n+1}) &= (p_n \circ p_n^{-1}(z) - p_n \circ i_n(a))/\operatorname{im} h_{n+1}, \\ &\qquad \text{by the definition of } p_n^*; \\ &= (z - 0)/\operatorname{im} h_{n+1} = z/\operatorname{im} h_{n+1}. \end{aligned}$$

So $z/\operatorname{im} h_{n+1}$ is in $\operatorname{im} p_n^*$.

iii.a. Let us show that $\operatorname{im}\delta_n \subset \ker i^*_{n-1}$. Let $\delta_n(z/\operatorname{im}h_{n+1})$ be in $\operatorname{im}\delta_n$, where z is in $\ker h_n$. This is in $\ker i^*_{n-1}$ because:

$$\begin{aligned}
i^*_{n-1}\circ\delta_n(z/\operatorname{im}h_{n+1}) &= i^*_{n-1}((i^{-1}_{n-1}\circ g_n\circ p_n^{-1}(z))/\operatorname{im}f_n), \text{ by the definition of } \delta_n;\\
&= (i_{n-1}\circ i^{-1}_{n-1}\circ g_n\circ p_n^{-1}(z))/\operatorname{im}g_n, \text{ by the definition of } i^*_{n-1};\\
&= (g_n\circ p_n^{-1}(z))/\operatorname{im}g_n;\\
&= 0/\operatorname{im}g_n, \text{ since } g_n\circ p_n^{-1}(z) \text{ is in } \operatorname{im}g_n.
\end{aligned}$$

iii.b. Let us show that $\ker i^*_{n-1} \subset \operatorname{im}\delta_n$, where the domain of δ_n is $\ker h_n/\operatorname{im}h_{n+1}$. Let $z/\operatorname{im}f_n$ be in $\ker i^*_{n-1}$, where z is in $\ker f_{n-1}$. Then $i^*_{n-1}(z/\operatorname{im}f_n) = i_{n-1}(z)/\operatorname{im}g_n$ $= 0$, which implies that $i_{n-1}(z)$ is in $\operatorname{im}g_n$ and so

$$i_{n-1}(z) = g_n(b) \tag{C.2}$$

for some b in B_n. Now

$$\begin{aligned}
h_n\circ p_n(b) &= p_{n-1}\circ g_n(b), \text{ since the diagram commutes;}\\
&= p_{n-1}\circ i_{n-1}(z), \text{ by } (C.2);\\
&= 0, \text{ since the rows are exact.}
\end{aligned}$$

So $p_n(b)$ is in $\ker h_n$. The element $p_n(b)/\operatorname{im}h_{n+1}$ is in the domain of δ_n. All we have to show is that $\delta_n(p_n(b)/\operatorname{im}h_{n+1}) = z/\operatorname{im}f_n$. Note that $\delta_n(p_n(b)/\operatorname{im}h_{n+1}) =$ $(i^{-1}_{n-1}\circ g_n\circ p_n^{-1}\circ p_n(b))/\operatorname{im}f_n$ by the definition of δ_n, so we only need to show that $i^{-1}_{n-1}\circ g_n\circ p_n^{-1}\circ p_n(b) - z$ is in $\operatorname{im}f_n$.

Consider $g_n\circ p_n^{-1}\circ p_n(b)$. Let c be any element in $p_n^{-1}\circ p_n(b)$, and of course b is in $p_n^{-1}\circ p_n(b)$. From the definition of δ_n, we know that $i^{-1}_{n-1}\circ g_n(c) - i^{-1}_{n-1}\circ g_n(b)$ is in $\operatorname{im}f_n$ which implies that $i^{-1}_{n-1}\circ g_n(c) - i^{-1}_{n-1}\circ i_{n-1}(z)$ is in $\operatorname{im}f_n$, by (C.2). Then $i^{-1}_{n-1}\circ g_n(c) - z$ is in $\operatorname{im}f_n$ which implies that $i^{-1}_{n-1}\circ g_n\circ p_n^{-1}\circ p_n(b) - z$ is in $\operatorname{im}f_n$. So for any c in $p_n^{-1}\circ p_n(b)$, $(i^{-1}_{n-1}\circ g_n\circ p_n^{-1}\circ p_n(b))/\operatorname{im}f_n = z/\operatorname{im}f_n$.

APPENDIX D

A COMMUTING LEMMA

The following lemma is helpful in proofs that involve commuting diagrams.

Lemma D.1 *Let*

$$\begin{tikzcd} 0 \arrow[r] & B^1 \arrow[r, "\beta^1"] & B \\ 0 \arrow[u] \arrow[r] & A^1 \arrow[u, "d^1"] \arrow[r, "\alpha^1"'] & A \arrow[u, "d"'] \end{tikzcd}$$

be a commutative diagram with exact rows. Then $d^1 \circ (\alpha^1)^{-1}(a) = (\beta^1)^{-1} \circ d(a)$ *for a in* A *if* a *is in* $\operatorname{im} \alpha^1$ *and* $d(a)$ *is in* $\operatorname{im} \beta^1$.

Proof. Since a is in $\operatorname{im} \alpha^1$, we have $\alpha^1(a^1) = a$ for some a^1 in A^1. Since $d(a)$ is in $\operatorname{im} B^1$, we have $\beta^1(b^1) = d(a)$ for some b^1 in B^1. Now since the diagram is commutative, we have $\beta^1 \circ d^1(a^1) = d \circ \alpha^1(a^1)$. Apply $(\beta^1)^{-1}$ to obtain

$$d^1(a^1) = (\beta^1)^{-1} \circ d \circ \alpha^1(a^1);$$

which implies $d^1(a^1) = (\beta^1)^{-1} \circ d(a),$ since $a = \alpha^1(a^1)$;

which implies $d^1 \circ (\alpha^1)^{-1}(a) = (\beta^1)^{-1} \circ d(a),$ since $(\alpha^1)^{-1}(a) = a^1$.

APPENDIX E

THE RIGHT DERIVED FUNCTOR IS WELL DEFINED

The following lemmas prove that the right-derived functor is well defined.

Lemma E.1 *Let us have two injective resolutions of M and M', with a map from M to M':*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{d^{\#}} & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & E^2 & \xrightarrow{d^2} & \cdots \\ & & \varphi\big\downarrow & & & & & & & & \\ 0 & \longrightarrow & M' & \xrightarrow[i^{\#}]{} & I^0 & \xrightarrow[i^0]{} & I^1 & \xrightarrow[i^1]{} & I^2 & \xrightarrow[i^2]{} & \cdots \end{array} \quad . \tag{E.1}$$

Then there exists $\{f^n : E^n \to I^n\}$ such that the diagram commutes.

Proof. Since I^0 is injective in

$$\begin{array}{ccccc} & & I^0 & & \\ & & i^{\#}\big\uparrow & & \\ & & M^1 & & \\ & & \varphi\big\uparrow & & \\ 0 & \longrightarrow & M & \xrightarrow[d^{\#}]{} & E^0, \end{array}$$

we can extend $i^{\#} \circ \varphi$ to $f^0 : E^0 \to I^0$ where $f^0 \circ d^{\#} = i^{\#} \circ \varphi$. Hence the first square of diagram (E.1),

$$\begin{array}{ccc} M & \xrightarrow{d^{\#}} & E^0 \\ \varphi\big\downarrow & & f^0\big\downarrow \\ M^1 & \xrightarrow[i^{\#}]{} & I^0 \end{array}$$

commutes.

Let $\widetilde{f^0}$ be something which takes $x/d^{\#}(M)$ to $f^0(x)$ (it does not necessarily take zero to zero). Let us look at $i^0 \circ \widetilde{f^0} : E^0/d^{\#}(M) \xrightarrow{\widetilde{f^0}} I^0 \xrightarrow{i^0} I^1$. This is not a composite of maps, since $\widetilde{f^0}$ is not a real map. But we will show that $i^0 \circ \widetilde{f^0}$ itself is a map by showing that it satisfies the following three properties:

I. $i^0 \circ \widetilde{f^0}$ **takes zero to zero.** Let x be in $d^{\#}(M)$, so $x = d^{\#}(a)$. Then $x/d^{\#}(M)$ is zero in $E^0/d^{\#}(M)$. Then $i^0 \circ \widetilde{f^0}(x/d^{\#}(M)) = i^0 \circ f^0(x) = i^0 \circ f^0 \circ d^{\#}(a) = i^0 \circ i^{\#} \circ \varphi(a) = 0$, since $i^0 \circ i^{\#}$ equals zero.

II. $i^0 \circ \widetilde{f^0}$ **is well defined.** Let $x/d^{\#}(M) = y/d^{\#}(M)$. Then $x - y$ is in $d^{\#}(M)$. We need to show that $i^0 \circ \widetilde{f^0}(x/d^{\#}(M)) = i^0 \circ \widetilde{f^0}(y/d^{\#}(M))$, which is to show that $i^0 \circ f^0(x) = i^0 \circ f^0(y)$. So we just need to show that $i^0 \circ f^0(x - y) = 0$. This is true because we already saw that $x - y$ is in $d^{\#}(M)$.

III. $i^0 \circ \widetilde{f^0}$ **preserves the operation of summation.**

$$\begin{aligned}
(i^0 \circ \widetilde{f^0})(x/d^{\#}(M) + y/d^{\#}(M)) &= (i^0 \circ \widetilde{f^0})((x+y)/d^{\#}(M));\\
&= (i^0 \circ f^0)(x+y);\\
&= i^0(f^0(x) + f^0(y)), \text{ since } f^0 \text{ is a map;}\\
&= i^0(\widetilde{f^0}(x/d^{\#}(M)) + \widetilde{f^0}(y/d^{\#}(M)));\\
&= i^0 \circ \widetilde{f^0}(x/d^{\#}(M)) + i^0 \circ \widetilde{f^0}(y/d^{\#}(M)).
\end{aligned}$$

Now let us do the second square. Consider the diagram with exact row

$$\begin{tikzcd}
 & I^1 & \\
 & I^0 \arrow[u, "i^0"] & \\
0 \arrow[r] & E^0/d^{\#}(M) \arrow[u, "\widetilde{f^0}"] \arrow[r, "\widetilde{d^0}"'] & E^1,
\end{tikzcd}$$

where $\widetilde{d^0}(x/d^{\#}(M)) = d^0(x)$. Let us prove that $\widetilde{d^0}$ is a map by showing that it satisfies the following three properties:

I. $\widetilde{d^0}$ takes zero to zero. Let x be in $d^{\#}(M)$. Then $x = d^{\#}(a)$. So $\widetilde{d^0}(x/d^{\#}(a)) = d^0(x) = d^0(d^{\#}(a)) = 0$, since the row is exact.

II. $\widetilde{d^0}$ is well defined. Let $x/d^{\#}(M) = y/d^{\#}(M)$. Then $x - y$ is in $d^{\#}(M)$. We need to show that $\widetilde{d^0}(x/d^{\#}(M)) = \widetilde{d^0}(y/d^{\#}(M))$, which is to show that $d^0(x) = d^0(y)$. This holds since $x - y$ in $d^{\#}(M)$ implies that $x - y$ is in $\ker d^0$, since the top row of the original diagram is exact. So $d^0(x - y) = 0J$ which is $d^0(x) = d^0(y)$, so we are done.

III. $\widetilde{d^0}$ preserves the operation of summation.

$$\begin{aligned}
\widetilde{d^0}(x/d^{\#}(M) + y/d^{\#}(M)) &= \widetilde{d^0}((x+y)/d^{\#}(M)); \\
&= d^0(x+y); \\
&= d^0(x) + d^0(y); \\
&= \widetilde{d^0}(x/d^{\#}(M)) + \widetilde{d^0}(y/d^{\#}(M)).
\end{aligned}$$

Now since I^1 is injective, we can extend $i^0 \circ \widetilde{f^0}$ to $f^1 : E^1 \to I^1$, where

$$f^1 \circ \widetilde{d^0} = i^0 \circ \widetilde{f^0}. \tag{E.2}$$

Let us verify that the second square of diagram (E.1) commutes. Let x be in E^0. Then

$$\begin{aligned}
f^1 \circ d^0(x) &= f^1 \circ \widetilde{d^0}(x/d^{\#}(M)), \text{ by the definition of } \widetilde{d^0}; \\
&= i^0 \circ \widetilde{f^0}(x/d^{\#}(M)), \text{ by } (E.2); \\
&= i^0 \circ f^0(x), \text{ by the definition of } \widetilde{f^0}.
\end{aligned}$$

We can repeat this procedure to get maps $\{f^n\}$ such that every square commutes. ■

Lemma E.2 *Let $\{f^n\}$ and $\{g^n\}$ be two such maps of our injective resolutions. Then there exists $\{h^n : E^n \to I^{n-1}\}$ such that*

$$f^n - g^n = i^{n-1} \circ h^n + h^{n+1} \circ d^n \tag{E.3}$$

for all $n \geq -1$. See the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & E^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}, g^{n-1}"'] & E^n \arrow[r, "d^n"] \arrow[d, "f^n, g^n"] & E^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "f^{n+1}, g^{n+1}"] & \cdots \\
\cdots \arrow[r] & I^{n-1} \arrow[r, "i^{n-1}"] & I^n \arrow[r, "i^n"] & I^{n+1} \arrow[r, "i^{n+1}"] & \cdots .
\end{tikzcd}$$

The maps $\{f^n\}$ and $\{g^n\}$ are then called homotopic.

Proof. Let $h^\# : M \to 0$ and $h^0 : E^0 \to M'$ be the zero maps. Then for $n = -1$, the right hand side of (E.3) is $i^{-2} \circ h^{-1} + h^0 \circ d^{-1}$, where $i^{-2} : 0 \to M'$, $h^{-1} = h^\#$, and $d^{-1} = d^\#$. So we have $0 \circ h^\# + h^0 \circ d^\# = 0 + 0 = 0$. For $n = -1$, the left hand side of (E.3) is $f^{-1} - g^{-1} = \varphi - \varphi = 0$. So equation (E.3) is satisfied for $n = -1$.

Now for $n = 0$, consider the diagram

$$\begin{tikzcd}
 & I^0 & \\
0 \arrow[r] & E^0/d^\#(M) \arrow[u, "\widetilde{f^0} - \widetilde{g^0}"] \arrow[r, "\widetilde{d^0}"'] & E^1,
\end{tikzcd}$$

In order to show that $\widetilde{f^0} - \widetilde{g^0} : E^0 \to I^0$ is a map we prove that if satisfies the following three properties:

I. $\widetilde{f^0} - \widetilde{g^0}$ takes zero to zero. We need to show that $(\widetilde{f^0} - \widetilde{g^0})(x/d^\#(M)) = 0$ for x in $d^\#(M)$.

$$\begin{aligned}
(\widetilde{f^0} - \widetilde{g^0})(x/d^\#(M)) &= f^0(x) - g^0(x); \\
&= f^0 \circ d^\#(a) - g^0 \circ d^\#(a), \text{ for some } a \text{ in } M; \\
&= i^\# \circ \varphi(a) - i^\# \circ \varphi(a), \text{ since the diagram commutes;} \\
&= 0.
\end{aligned}$$

II. $\widetilde{f^0} - \widetilde{g^0}$ is well defined. Let $x/d^{\#}(M) = y/d^{\#}(M)$. Then $x - y$ is in $d^{\#}(M)$. We need to show that $(\widetilde{f^0} - \widetilde{g^0})(x/d^{\#}(M)) = (\widetilde{f^0} - \widetilde{g^0})(y/d^{\#}(M))$, which is to show that $(f^0 - g^0)(x) = (f^0 - g^0)(y)$. So we just need to show that $(f^0 - g^0)(x - y) = 0$. This is true because $x - y$ is in $d^{\#}(M)$, and in part (I.) we saw that $(f^0 - g^0)(u) = 0$ for u in $d^{\#}(M)$.

III. $\widetilde{f^0} - \widetilde{g^0}$ preserves the operation of summation.

$$\begin{aligned}
(\widetilde{f^0} - \widetilde{g^0})(x/d^{\#}(M) + y/d^{\#}(M)) &= (\widetilde{f^0} - \widetilde{g^0})((x+y)/d^{\#}(M));\\
&= (f^0 - g^0)(x+y);\\
&= (f^0 - g^0)(x) + (f^0 - g^0)(y);\\
&= (\widetilde{f^0} - \widetilde{g^0})(x/d^{\#}(M)) + (\widetilde{f^0} - \widetilde{g^0})(y/d^{\#}(M)).
\end{aligned}$$

Since I^0 is injective, we can extend $f^0 - g^0$ to $h^1 : E^1 \to I^0$, where $h^1 \circ \widetilde{d^0} = \widetilde{f^0} - \widetilde{g^0}$. For $n = 0$, the right hand side of (E.3) for x in E^0 is

$$\begin{aligned}
(i^{-1} \circ h^0 + h^1 \circ d^0)(x) &= (i^{\#} \circ h^0 + h^1 \circ d^0)(x);\\
&= h^1 \circ d^0(x), \text{ since } h^0 = 0;\\
&= h^1 \circ \widetilde{d^0}(x/d^{\#}(M)), \text{ by the definition of } \widetilde{d^0};\\
&= (\widetilde{f^0} - \widetilde{g^0})(x/d^{\#}(M));\\
&= \widetilde{f^0}(x/d^{\#}(M)) - \widetilde{g^0}(x/d^{\#}(M));\\
&= f^0(x) - g^0(x);\\
&= \text{the left hand side of } (E.3).
\end{aligned}$$

So equation (E.3) is satisfied for $n = 0$.

Suppose equation (E.3) is satisfied for n, i.e. we have

$$f^n - g^n = i^{n-1} \circ h^n + h^{n+1} \circ d^n. \tag{E.4}$$

We will show that equation (E.3) is true for $n + 1$. Consider the map

$$f^{n+1} - g^{n+1} - i^n \circ h^{n+1} : E^{n+1} \to I^{n+1}. \tag{E.5}$$

Let us see that for x in $d^n(E^n) \subset E^{n+1}$, $(f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(x) = 0$.

$$\begin{aligned}
(f^{n+1}-&g^{n+1} - i^n \circ h^{n+1})(x) \\
&= (f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(d^n(e)), \text{ since } x \text{ is in } {}^n(E^n); \\
&= (f^{n+1} - g^{n+1})(d^n(e) - i^n \circ h^{n+1}) \circ d^n(e); \\
&= (f^{n+1} - g^{n+1})(d^n(e)) - i^n(f^n - g^n - i^{n-1} \circ h^n)(e), \text{ by } (E.4); \\
&= (f^{n+1} - g^{n+1})(d^n(e)) - i^n(f^n - g^n)(e), \text{ since } i^n \circ i^{n-1} = 0; \\
&= f^{n+1} \circ d^n(e) - g^{n+1} \circ d^n(e) - i^n \circ f^n(e) + i^n \circ g^n(e); \\
&= (f^{n+1} \circ d^n(e) - i^n \circ f^n(e)) + (i^n \circ g^n(e) - g^{n+1} \circ d^n(e)); \\
&= 0 + 0, \text{ since the diagram commutes.}
\end{aligned}$$

Now consider the diagram

$$\begin{tikzcd}
 & I^{n+1} & \\
0 \arrow[r] & E^{n+1}/d^n(E^n) \arrow[u, "\widetilde{f^{n+1}-g^{n+1}-i^n\circ h^{n+1}}"] \arrow[r, "\widetilde{d^{n+1}}"'] & E^{n+2},
\end{tikzcd}$$

where $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}$ takes $x/d^n(E^n)$ to $(f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(x)$. Let us verify that $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}$ is a map, by showing that it satisfies the following three properties:

I. $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}$ **takes zero to zero.** We already saw above that it does.

II. $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}$ **is well defined.** Let $x/d^n(E^n) = y/d^n(E^n)$. Then $x - y$ is in $d^n(E^n)$. We need to show that $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}(x/d^n(E^n)) = \widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}(y/d^n(E^n))$, which is to show that $(f^{n+1}-g^{n+1}-i^n\circ h^{n+1})(x) = (f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(y)$. But this holds by the above argument because $(f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(x - y) = 0$ since $x - y$ is in $d^n(E^n)$.

III. $\widetilde{f^{n+1} - g^{n+1} - i^n \circ h^{n+1}}$ **preserves the operation of summation.** It does this because it is the difference and composition of maps with this property.

So we obtain $h^{n+2} : E^{n+2} \to I^{n+1}$ such that $h^{n+2} \circ \widetilde{d^{n+1}} = f^{n+1} - \widetilde{g^{n+1} - i^n} \circ h^{n+1}$. So for $n+1$, the right hand side of (E.3) for x in E^{n+1} is

$$\begin{aligned}
&(i^n \circ h^{n+1} + h^{n+2} \circ d^{n+1}) \\
&\quad = i^n \circ h^{n+1}(x) + h^{n+2} \circ \widetilde{d^{n+1}}(x/d^n(E^n)), \text{ by the definition of } \widetilde{d^{n+1}}; \\
&\quad = i^n \circ h^{n+1}(x) + f^{n+1} - \widetilde{g^{n+1} - i^n} \circ h^{n+1}(x/d^n(E^n)); \\
&\quad = i^n \circ h^{n+1}(x) + (f^{n+1} - g^{n+1} - i^n \circ h^{n+1})(x); \\
&\quad = (f^{n+1} - g^{n+1})(x); \\
&\quad = \text{the left hand side of } (E.3).
\end{aligned}$$

Lemma E.3 *Let $\{f^n\}$ and $\{g^n\}$ be homotopic. Then $H(f^n) = H(g^n)$ on $H^n(E)$ for all n.*

Proof We want to show that $H(f^n) = H(g^n) : H^n(E) \to H^n(I)$, which is $H(f^n) = H(g^n) : \ker d^n / \operatorname{im} d^{n-1} \to \ker i^n / \operatorname{im} i^{n-1}$. Let $x/\operatorname{im} d^{n-1}$ be in $\ker d^n / \operatorname{im} d^{n-1}$. Then

$$\begin{aligned}
(H(f^n) - H(g^n))(x/\operatorname{im} d^{n-1}) &= H(f^n)(x/\operatorname{im} d^{n-1}) - H(g^n)(x/\operatorname{im} d^{n-1}); \\
&= f^n(x)/\operatorname{im} i^{n-1} - g^n(x)/\operatorname{im} i^{n-1}; \\
&= (f^n - g^n)(x)/\operatorname{im} i^{n-1}; \\
&= (i^{n-1} \circ h^n + h^{n+1} \circ d^n)(x)/\operatorname{im} i^{n-1}, \\
&\qquad \text{by the homotopy condition;} \\
&= (i^{n-1} \circ h^n(x))/\operatorname{im} i^{n-1}, \text{ since } x \text{ is in } \ker d^n; \\
&= 0, \text{ since } i^{n-1}(h^n(x)) \text{ is in } \operatorname{im} i^{n-1}. \quad \blacksquare
\end{aligned}$$

Lemma E.4 *$R^nF(M)$ is independent of the choice of injective resolution of M.*

Proof Let E and I be two injective resolutions of M. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "d^{\#}"] \arrow[d, "1_M"'] & E^0 \arrow[r, "d^0"] & E^1 \arrow[r, "d^1"] & E^2 \arrow[r, "d^2"] & \cdots \\
0 \arrow[r] & M \arrow[r, "i^{\#}"'] & I^0 \arrow[r, "i^0"'] & I^1 \arrow[r, "i^1"'] & I^2 \arrow[r, "i^2"'] & \cdots
\end{tikzcd}$$

By Lemma E.1, there exist maps $\{f^n : E^n \to I^n\}$ that fit in with 1_M to make the diagram commute. Apply the functor F to obtain $\{Ff^n : FE^n \to FI^n\}$ and the induced maps $\{H(Ff^n) : H^n(FE) \to H^n(FI)\}$. We only need to show that each map $H(Ff^n)$ is an isomorphism. Consider the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{i^{\#}} & I^0 & \xrightarrow{i^0} & I^1 & \xrightarrow{i^1} & I^2 & \xrightarrow{i^2} & \cdots \\
 & & {\scriptstyle 1_M}\big\downarrow & & & & & & & & \\
0 & \longrightarrow & M & \xrightarrow[d^{\#}]{} & E^0 & \xrightarrow[d^0]{} & E^1 & \xrightarrow[d^1]{} & E^2 & \xrightarrow[d^2]{} & \cdots
\end{array} .$$

Again by Lemma E.1, there exists maps $\{g^n : I^n \to E^n\}$ that fit in with 1_M to make the diagram commute. Consider the composites $\{g^n \circ f^n : E^n \to E^n\}$ and the identity maps $\{\mathrm{Id}^n : E^n \to E^n\}$. Both these sets of maps fit in the following diagram, making it commute:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow[d^{\#}]{} & E^0 & \xrightarrow[d^0]{} & E^1 & \xrightarrow[d^1]{} & E^2 & \xrightarrow[d^2]{} & \cdots \\
 & & {\scriptstyle 1_M}\big\downarrow & & & & & & & & \\
0 & \longrightarrow & M & \xrightarrow[d^{\#}]{} & E^0 & \xrightarrow[d^0]{} & E^1 & \xrightarrow[d^1]{} & E^2 & \xrightarrow[d^2]{} & \cdots
\end{array} .$$

Lemma E.2 says that $\{g^n \circ f^n\}$ and $\{\mathrm{Id}^n\}$ satisfy the homotopy condition. Then Lemma E.3 says that $H(g^n \circ f^n) = H(\mathrm{Id}^n) : H(E^n) \to H(E^n)$ for all n. So $H(g^n) \circ H(f^n) =$ the identity map on $H(E^n)$. Similary, we can get $H(f^n) \circ H(g^n) =$ the identity map on $H(I^n)$. So $H(f^n)$ is an isomorphism, and hence $H(Ff^n)$ is an isomorphism from $H^n(FE)$ to $H^n(FI)$. ■

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