Spectral Sequences in Topology

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What is a Spectral Sequence?

Definition. A (first quadrant) **spectral sequence of homological type** consists of R-modules $E^r_{p,q}$ ($r \in \mathbb{N}, p, q \in \mathbb{Z}$) and maps

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that:

- $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$,
- $E^{r+1}_{p,q} = \ker(d^r_{p,q})/\mathrm{Im}(d^r_{p+r,q-r+1})$,
- $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$.

Observe that for each fixed pair (p, q), there exists N (depending on p and q) such that

$$E^N_{p,q} \cong E^{N+1}_{p,q} \cong \cdots ,$$

since eventually every arrow coming into the (p, q) spot will come from a zero and every arrow coming out of the (p, q) spot will go to a zero, and in this case taking homology does nothing. We call this R-module $E^\infty_{p,q}$.

Definition. Assume temporarily that R is a field, so that the entries in a spectral sequence are vector spaces. Then a spectral sequence $\{E^r_{p,q}\}$ is said to **converge** to a graded vector space $V = \bigoplus_{n=0}^\infty V_n$ if

$$V_n = \bigoplus_{p+q=n} E^\infty_{p,q}$$

for all n.

This is saying that the entries on the E^∞ page "fit together" to form the vector space V. Unfortunately, the way vector spaces fit together to form larger vector spaces is much more straightforward than the way arbitrary modules can fit together. For example, if V and W are vector spaces and we know that $U \cong V/W$, then it follows that $V \cong U \oplus W$. This is not true for arbitrary modules; consider $\mathbb{Z} \ncong (\mathbb{Z}/2) \oplus (2\mathbb{Z})$. In light of this, the notion of convergence for general spectral sequences of R-modules is more complicated than the above.

How Do Spectral Sequences Arise?

Suppose C is a chain complex of finite-dimensional vector spaces,

$$\cdots \to C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \to \cdots ,$$

and each C_d has a filtration

$$0 = C_{d,0} \subset C_{d,1} \subset C_{d,2} \subset \cdots \subset C_{d,n} = C_d$$

which is respected by ∂ in the sense that $\partial C_{d,p} \subset C_{d-1,p}$ for all d, p. Can we compute $H_*(C)$ in pieces by way of these filtrations?

One thing we can do is write

$$C_d \cong \frac{C_{d,n}}{C_{d,n-1}} \oplus C_{d,n-1} \cong \frac{C_{d,n}}{C_{d,n-1}} \oplus \frac{C_{d,n-1}}{C_{d,n-2}} \oplus C_{d,n-2} \cong \cdots ,$$

so if $E^0_{d,p} := C_{d,p}/C_{d,p-1}$, we have

$$C_d \cong \bigoplus_p E^0_{d,p}.$$

Moreover, ∂ induces well-defined maps $\partial^0 : E^0_{d,p} \to E^0_{d-1,p}$, and these fit together into chain complexes as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & E^0_{d+1,n} & \longrightarrow & E^0_{d,n} & \longrightarrow & E^0_{d-1,n} & \longrightarrow & \cdots \\
\\
\cdots & \longrightarrow & E^0_{d+1,n-1} & \longrightarrow & E^0_{d,n-1} & \longrightarrow & E^0_{d-1,n-1} & \longrightarrow & \cdots \\
\\
 & & \vdots & & \vdots & & \vdots & & \\
\\
\cdots & \longrightarrow & E^0_{d+1,1} & \longrightarrow & E^0_{d,1} & \longrightarrow & E^0_{d-1,1} & \longrightarrow & \cdots ,
\end{array}$$

where the direct sum of all the vector spaces in the d^{th} column is the original vector space C_d.

We might hope that $H_d(C) = \bigoplus_p H_d(E^0_{*,p})$, but this is not the case. To see why, and to get an idea of how to fix the problem, let's reduce the simplest nontrivial case, where $n = 2$. In this case, we have a chain complex in which each C_d has a filtration

$$0 = C_{d,0} \subset C_{d,1} \subset C_{d,2} = C_d.$$

So $E^0_{d,1} = C_{d,1}$ and $E^0_{d,2} = C_d/C_{d,1}$, and the above diagram becomes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{d+1}/C_{d+1,1} & \longrightarrow & C_d/C_{d,1} & \longrightarrow & C_{d-1}/C_{d-1,1} & \longrightarrow & \cdots \\
\\
\cdots & \longrightarrow & C_{d+1,1} & \longrightarrow & C_{d,1} & \longrightarrow & C_{d-1,1} & \longrightarrow & \cdots .
\end{array}$$

Now, we do have a decomposition:

$$H_d(C) \cong \frac{Z_d/B_d}{(Z_d \cap C_{d,1})/(B_d \cap C_{d,1})} \oplus \frac{Z_d \cap C_{d,1}}{B_d \cap C_{d,1}} \cong \frac{Z_d/C_{d,1}}{B_d/C_{d,1}} \oplus \frac{Z_d \cap C_{d,1}}{B_d \cap C_{d,1}}.$$

The second of these two terms looks sort of like the d^{th} homology of the bottom chain complex above.... but when we compute the homology of that chain complex, we mod out only by the things that are boundaries of something in $C_{d+1,1}$, which isn't enough! Similarly, the first of the two terms in the above decomposition looks sort of like the d^{th} homology of the top chain complex above... but when we compute the homology of that chain complex, we include not only all the things which map to zero but also all the things which map to $C_{d-1,1}$, which is too much!

Let $E^1_{p,d} = H_d(E^0_{*,p})$. We've seen that $E^1_{1,d}$ has a too-small denominator and $E^1_{2,d}$ has a too-large numerator. Fortunately, it turns out that ∂ descends to a map $\partial^1 : E^1_{d+1,2} \to E^1_{d,1}$ for all d, and this map precisely encapsulates the discrepancy between $E^1_{p,d}$ and the homology groups we want. Specifically, although

$$H_d(C) \ncong E^1_{d,1} \oplus E^1_{d,2},$$

we have

$$H_d(C) \cong \frac{E^1_{d,1}}{\mathrm{Im}(\partial^1 : E^1_{d+1,2} \to E^1_{d,1})} \oplus \ker(\partial^1 : E^1_{d,2} \to E^1_{d-1,1}).$$

A more algebraic/graphical representation of what we've done is we've formed a new array of chain complexes:

$$\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[dr] & 0 \arrow[dr] & & \\
\cdots & E^1_{d+1,2} \arrow[dr] & E^1_{d,2} \arrow[dr] & E^1_{d-1,2} \arrow[dr] & \cdots \\
\cdots & E^1_{d+1,1} \arrow[dr] & E^1_{d,1} \arrow[dr] & E^1_{d-1,1} \arrow[dr] & \cdots, \\
 & & 0 & 0 & 0
\end{tikzcd}$$

and, unlike our first array of chain complexes, the homology of these actually *do* add up to give $H_*(C)$, as desired.

As you might imagine, this simple fix doesn't work when n (the number of pieces in the filtrations) is larger than two, since it only handles the interaction between adjacent filtered pieces. However, in this more general situation, we can define yet another array of chain complexes, this time with boundary maps $\partial^2 : E^2_{d+1,p+2} \to E^2_{d,p}$. One can prove that eventually, we wind up with an array of chain complexes whose homology groups, when summed down columns, give the homology of C.

The Serre Spectral Sequence:

Theorem. Let $F \to E \to B$ be a fibration with B path-connected. If $\pi_1(B)$ acts trivially on $H_*(F; A)$, then there is a spectral sequence $\{E^r_{p,q}\}$ with

$$E^2_{p,q} \cong H_p(B; H_q(F; A))$$

that converges to $H_*(E; A)$.

Definition. A **fibration** is a map $p : E \to B$ with the property that, given a homotopy $g_t : X \to B$ of maps from some other space into B, and given a lift $\tilde{g}_0 : X \to E$ of g_0, there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t and extending the given lift $\tilde{g}_0$.

Notation. A fact that I will assume without proof is that if $p : E \to B$ is a fibration, then every fiber $p^{-1}(b)$ is homotopy equivalent to every other fiber. So, by a slight abuse of notation, we will refer to "the" fiber of a fibration. We will write $F \to E \to B$ for a fibration $E \to B$ with fiber F.

You might worry that the Serre Spectral Sequence, since it only applies to fibrations, is not very useful for computing homology groups. But in fact, we can produce fibrations very easily. Here's a standard way to do "turn any map into a fibration":

Definition. If $f : A \to B$ is a map, define

$$E_f = \{(a, \gamma) \mid a \in A, \gamma : I \to B, \gamma(0) = f(a)\}.$$

There is a map $p : E_f \to B$ such that $p(a, \gamma) = \gamma(1)$, and one can check directly that this is a fibration. This is known as the **pathspace fibration**.

There is a copy of A sitting inside E_f, namely the subspace of pairs (a, γ) such that γ is the constant path at $f(a)$. Contraction of paths gives a deformation retraction from E_f to A, and up to this homotopy equivalence, the map p is the same as the original map f.

In the special case where $A = \{*\}$, this construction gives a fibration $F \to P \to B$ with P contractible. This turns out to be a useful way to study the homology of B, as we demonstrate below.

Example: $H_*(K(\mathbb{Z},2))$

Let B be a $K(\mathbb{Z},2)$. (One example is $\mathbb{CP}^\infty$, and all the other examples are homotopy equivalent.) Consider the pathspace fibration $F \to P \to B$. From the long exact sequence of the fibration, we get

$$\pi_i(F) \cong \pi_{i+1}(K(\mathbb{Z},2)),$$

so F is a $K(\mathbb{Z},1)$. Up to homotopy, then, F is S^1. Therefore, the E^2 page of the Serre spectral sequence for this fibration has:

$$E^2_{p,q} = H_p(K(\mathbb{Z},2); H_q(S^1)) = \begin{cases} H_p(K(\mathbb{Z},2)) & \text{if } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

Pictorially, this is:

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} = H_0(K(\mathbb{Z},2)) & H_1(K(\mathbb{Z},2)) & H_2(K(\mathbb{Z},2)) & H_3(K(\mathbb{Z},2)) & \cdots \\
\mathbb{Z} = H_0(K(\mathbb{Z},2)) & H_1(K(\mathbb{Z},2)) & H_2(K(\mathbb{Z},2)) & H_3(K(\mathbb{Z},2)) & \cdots
\end{array}$$

This spectral sequence must converge to the homology of P, which is trivial. But this is the only page on which there can be nonzero differentials, so all of the differentials on this page must be isomorphisms in order to kill all these groups. This gives us

$$H_i(K(\mathbb{Z},2)) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

What Next?

There are many ways in which to generalize and build upon the ideas discussed today. Two important examples are:

- The Serre spectral sequence in cohomology. This turns out to be much more powerful than the Serre spectral sequence in homology, because the cup product structure on the E^2-page of the spectral sequence extends to a product structure on every page of the spectral sequence. The differentials d satisfy a Leibniz rule with respect to these products, and the induced product on the E^∞ page coincides with the cup product structure on $H^*(E)$. Sample applications:
  - Complete computation of $H^*(K(\mathbb{Z}, n))$.
  - The groups $\pi_i(S^n)$ are finite for $i > n$, except for $\pi_{4k-1}(S^{2k})$, which is the direct sum of $\mathbb{Z}$ with a finite group.
- Other spectral sequences in topology:
  - Mayer-Vietoris for a cover by more than two sets gives the Mayer-Vietoris spectral sequence.
  - Universal Coefficients theorem for rings other than $\mathbb{Z}$ gives the Universal Coefficients spectral sequence.
  - Cellular homology for generalized homology theories gives the Atiyah-Hirzebruch spectral sequence.