Problem Set 9

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1 Problem 1

1.1 Part 1

Let $A = (a_{ij})$ and consider ϵ_{ij}, the matrix with a 1 in the ith row and jth column and zeros elsewhere.

Then, for a fixed (i, j), if we write $A = [\mathbf{a}_1^t, \mathbf{a}_2^t, \cdots, \mathbf{a}_n^t]$ as a block matrix of column vectors, we have

$$A\mathbf{e}_{ij} = [0, 0, \cdots, \mathbf{a}_i^t, 0, \cdots, 0]$$

as a block matrix where $\mathbf{a}_i^t$ occurs as the jth column.

In other words, right-multiplication by $\mathbf{e}_{ij}$ selects column i from A, placing it in column j of a matrix of zeros.

For example, for $(i,j) = (3,2)$ we have

$$A\mathbf{e}_{32} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{13} & 0 \\ 0 & a_{23} & 0 \\ 0 & a_{33} & 0 \end{pmatrix},$$

which is a matrix that contains column 3 of A (the i value) as its 2nd column (the j value).

On the other hand, *left* multiplication by $\mathbf{e}_{ij}$ selects the jth **row** of A and places it the ith **row** of a zero matrix, so for example we have

$$\mathbf{e}_{32}A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

In general, these two products will not be equal, since the first has a nontrivial column and the latter has a nontrivial row. If $A \in Z(M_n(R))$, these two must be equal, so we can equate corresponding entries to find that

- $a_{21} = 0$, from comparing entries in row 3, column 1,
- $a_{23} = 0$, from comparing entries in row 3, column 3
- $a_{22} = a_{33}$ by comparing entries in row 3, column 2.

Letting the multiplication run over all possibilities for $\mathbf{e}_{ij}$ yields $a_{ii} = a_{jj}$ for every pair i, j and $a_{ij} = 0$ whenever $i \neq j$. Setting $r = a_{ii} = a_{jj}$ for all $1 \leq i, j \leq n$ forces A to be a matrix of the form

$$A = \begin{pmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & r & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r \end{pmatrix} := rI_n.$$

To see that we must have $r \in Z(G)$, let $sI_n \in Z(M_n(R))$ be arbitrary, where s is not assumed to be in $Z(R)$. Then $(rI_n)(sI_n) = (sI_n)(rI_n)$ by assumption, since these are matrices in the center of $M_n(R)$. But $M_n(R)$ is an R-module, and so the scalars r, s commute with the module elements I_n. This means that we in fact have

$$(rI_n)(sI_n) = (rs)I_n^2 = (rs)I_n,$$
$$(sI_n)(rI_n) = (sr)I_n^2 = (sr)I_n$$
$$\implies (rs)I_n = (sr)I_n$$
$$\implies (rs - sr)I_n = 0_n,$$

the $n \times n$ zero matrix.

But then by equating (for example) the $1, 1$ entry of the matrix $(rs - sr)I_n$ with the corresponding entry in 0_n, we find $rs - sr = 0_R$, which means $rs = sr \in R$.

Now since $s \in R$ was arbitrary, we find that $r \in Z(R)$ as desired.

1.2 Part 2

Define a map

$$\phi : Z(R) \to Z(M_n(R)$$
$$r \mapsto rI_n.$$

By part 1, this map is surjective. To see that it is also injective, we can consider $\ker \phi = \{r \in Z(r) \ni rI_n = 0_n\}$, which clearly forces $r = 0$. It is also a homomorphism of R-modules, since $\phi(rx + y) = (rx + y)I_n = r(xI_n) + yI_n$.

Thus by the first isomorphism theorem, we have $Z(R) \cong Z(M_n(R))$.

2 Problem 2

2.1 Part 1

If A, B are (skew)-symmetric, then $A^t = \pm A$ and $B^t = \pm B$ respectively. But then

$$(A + B)^t = A^t + B^t = \pm A + \pm B = \pm(A + B),$$

which shows that $A + B$ is (skew)-symmetric.

2.2 Part 2

$\implies$: Suppose that whenever A, B are symmetric then AB is symmetric as well.

We then have $(AB)^t = AB$ by assumption, and then by calculation we have $(AB^t) = B^t A^t = BA$, so $AB = BA$.

$\impliedby$: Suppose that $AB = BA$ and A, B are symmetric. We want to show that AB is also symmetric, so we compute

$$(AB)^t = B^t A^t = BA = BA.$$

$\square$

Now let $B \in M_n(R)$ be arbitrary. We have

- $(BB^t)^t = (B^t)^t B^t = BB^t$, so BB^t is symmetric,
- $(B + B^t)^t = B^t + (B^t)^t = B^t + B = B + B^t$, so $B + B^t$ is symmetric,
- $(B - B^t)^t = B^t - B = -(B + B^t)$, so $B - B^t$ is skew-symmetric

3 Problem 3

Definition: We say $A \sim B$ in $M_n(R) \iff$ there exists an invertible P such that $B = PAP^{-1}$.

- Reflexive, $A \sim A$:

  Take $P = I_n$ the identity matrix.

- Symmetric, $A \sim B \implies B \sim A$:

  $B = PAP^{-1} \implies BP = PA \implies P^{-1}BP = A$, so we can take $Q = P^{-1}$ to yield $A = QBQ^{-1}$.

- Transitive, $A \sim B \& B \sim C \implies A \sim C$:

  If $B = PAP^{-1}, C = QBQ^{-1}$, then $C = Q(PAP^{-1})Q^{-1} = (QP)A(QP)^{-1}$, so take $L = QP$ to yield $C = LAL^{-1}$.

Definition: We say $A \sim B$ in $M(n \times n, R) \iff B = PAQ$ with $P \in \mathrm{GL}(n, R), Q \in \mathrm{GL}(m, R)$.

- Reflexive, $A \sim A$:

  Take $P = I_{m,n}$ the matrix with 1s on the diagonal and zeros elsewhere, and $Q = P^t$.

- Symmetric, $A \sim B \implies B \sim A$:

  $B = PAQ \implies BQ^{-1} = PA \implies P^{-1}BQ^{-1} = A$, so we can take $S = P^{-1}, T = Q^{-1}$ to yield $A = QBT$.

- Transitive, $A \sim B \& B \sim C \implies A \sim C$:

  If $B = PAQ, C = RBS$, then $C = R(PAQ)S = (RP)A(QS)$, so take $L = RP, M = QS$ to yield $C = LAM$.

4 Problem 4

1. $A \in M(n \times m, D)$ has a left inverse $B \iff \mathrm{rank}(A) = m$:

$\implies$: Suppose toward the contrapositive that $\mathrm{rank}(A) < m$, so A has at least one pair of linearly dependent columns. So wlog write $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m]$ in block form with each $\mathbf{a}_i$ a column vector, and we can assume that $\mathbf{a}_1, \mathbf{a}_2$ are linearly dependent.

Now suppose such a left inverse B were to exists. Write it in block form as $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n]^t$, so each $\mathbf{b}_i$ is a row of B.

Now if $BA = I_m$ is to hold, noting that $(BA)_{ij} = \langle \mathbf{b}_i, \ \mathbf{a}_j \rangle$, we must have

$$I_{1,1} = \langle \mathbf{b}_1, \ \mathbf{a}_1 \rangle = 1$$
$$I_{1,2} = \langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = 0$$
$$I_{1,3} = \langle \mathbf{b}_1, \ \mathbf{a}_3 \rangle = 0$$
$$\vdots$$
$$I_{2,1} = \langle \mathbf{b}_2, \ \mathbf{a}_1 \rangle = 0$$
$$I_{2,2} = \langle \mathbf{b}_2, \ \mathbf{a}_2 \rangle = 1$$
$$I_{2,3} = \langle \mathbf{b}_2, \ \mathbf{a}_3 \rangle = 0$$
$$\ddots$$

But the claim is that this can *not* happen if $\mathbf{a}_1, \mathbf{a}_2$ are linearly dependent. To see why, note that the linear dependence supplies elements $d_1, d_2 \neq 0 \in D$ such that $d_1 \mathbf{a}_1 + d_2 \mathbf{a}_2 = \mathbf{0}$. But then taking inner products against, for example, $\mathbf{b}_1$, we obtain

$$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 = \mathbf{0} \implies d_1 \langle \mathbf{b}_1, \ \mathbf{a}_1 \rangle + d_2 \langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = \langle \mathbf{b}_1, \ \mathbf{0} \rangle = 0$$
$$\implies d_1 \langle \mathbf{b}_1, \ \mathbf{a}_1 \rangle + d_2 \langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = 0$$
$$\implies d_1 + d_2 \langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = 0$$
$$\implies \langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = -\frac{d_1}{d_2} \neq 0,$$

which contradicts $\langle \mathbf{b}_1, \ \mathbf{a}_2 \rangle = 0$ as required by the previous equations.

$\impliedby$: Suppose $\mathrm{rank}(A) = m$, so A has m linearly independent columns – note that this is *all* of its columns. Since row ranks equal column ranks, this also says that A has m linearly independent rows, so we must have $n \geq m$. Then viewing A as a map from $D^m \to D^n$, we find that $\mathrm{rank\,im}\, A = m \leq n$. In particular, $\ker A = \{\mathbf{0}\}$; otherwise this would force $\mathrm{rank\,im}\, A < m$. So A represents an injective map $f_A : D^m \to D^n$.

But any injective set map $f : S_1 \to S_2$ has a left-inverse g such that $g \circ f = \mathrm{id}_{S_1}$. So $f_A : D^m \to D^n$ as a set map has a left inverse $g_B : D^n \to D^m$ set map satisfying $g_B \circ f_A = \mathrm{id}_{D^m}$. But then taking the matrix associated to g_B yields a matrix $B \in M(m \times n, D)$ such that $BA = I_m$ as desired. $\quad \square$

2. A has a right inverse $B \iff \mathrm{rank}(A) = n$:

$\implies$: By a similar argument, supposing that $\mathrm{rank}\, A < n$ but $AB = I_n$ for some B, we find that A has at least two linearly dependent *rows* this time, say $\mathbf{a}_1, \mathbf{a}_2$, whereas we obtain a system of equations of the form $\langle a_i, \ \mathbf{b}_k \rangle = \delta_{ik}$ where $\mathbf{b}_i$ are now the columns of B.

In a similar manner, the linear dependence forces, say, $\langle \mathbf{a}_2, \ \mathbf{b}_1 \rangle \neq 0$, which is a contradiction.

$\impliedby$: By another similar argument, we find that A represents a map $f_A : D^m \to D^n$, and since $\mathrm{rank}\, A = \dim \mathrm{im}\, A = n$, we find that A represents a surjective map f_A. Surjective set maps have *right* inverses, so there is some $g_B : D^n \to D^m$ such that $f_A \circ g_B = \mathrm{id}_{D^n}$, and when translated to matrices this yields $AB = I_n$. $\quad \square$

5 Problem 5

5.1 Part 1

$\Longleftarrow$: Suppose that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$.

Write $A = [\mathbf{a}_i]$ in block form with each $\mathbf{a}_i$ a row of A. By definition, a solution to this equation is a $\mathbf{x} = (x_i)$ such that for each i, we have $\langle \mathbf{a}_i,\ \mathbf{x} \rangle = b_i$ (by carrying out the matrix multiplication).

But

$$\langle \mathbf{a}_i,\ \mathbf{x} \rangle = b_i$$
$$\Longrightarrow \sum_{j=1}^{m} a_{ij} x_j = b_i,$$

which says that the collection $x_1, \cdots, x_n$ solves the equation

$$a_{i1} x_1 + a_{i2} x_2 + \cdots a_{im} = b_i$$

for every i, which is exactly the statement that the x_i simultaneously solve the given system.

$\Longrightarrow$: Suppose that the given system has a simultaneous solutions $x_1, x_2, \cdots, x_n$, and consider the matrix equation $A\mathbf{x} = \mathbf{b}$.

Letting $\mathbf{x} = [x_1, x_2, \cdots, x_n]$, we can rewrite

$$b_i = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{im} x_m = \langle \mathbf{a}_i,\ \mathbf{x} \rangle,$$

where $\mathbf{a}_i = [a_{i1}, a_{i2}, \cdots, a_{im}]$.

But then $\mathbf{a}_i$ is the ith row of A, and $A\mathbf{x} = \mathbf{b}$ has a solution iff there is a $\mathbf{x}$ such that $\langle \mathbf{a}_i,\ \mathbf{x} \rangle = b_i$ for all i, which is exactly what we've constructed.

5.2 Part 2

Noting that applying a row operation to A is the same as taking the product EA for some elementary matrix E, we can write $A_1 = \left(\prod_{i=1}^{\ell} E_i \right) A$ and $B_1 = \left(\prod_{i=1}^{\ell} E_i \right) B$,

thus

$$A\mathbf{x} = \mathbf{b}$$
$$\Longrightarrow E_\ell A\mathbf{x} = E_\ell \mathbf{b}$$
$$\Longrightarrow E_{\ell-1} E_\ell A\mathbf{x} = E_{\ell-1} E_\ell \mathbf{b}$$
$$\vdots$$
$$\Longrightarrow E_1 E_2 \cdots E_\ell A\mathbf{x} = E_1 E_2 \cdots E_\ell A\mathbf{b}$$
$$\Longrightarrow A_1 \mathbf{x} = B_1$$

5.3 Part 3

1. $AX = B$ has a solution $\iff$ rank$(A) = $ rank(C):

Note that we can only have rank $C \geq$ rank A.

$\implies$:

Suppose that $AX = B$ has a solution; then $\mathbf{b}$ is in the column space of A. But this says that

$$\text{span}(\{\mathbf{a}_i\}) = \text{span}(\{\mathbf{a}_i\} \bigcup \{\mathbf{b}\}),$$

where $\mathbf{a}_i$ are the columns of A. But then taking dimensions on both sides yields rank $A = $ rank C, since the rank of the dimension of the column space.

$\impliedby$:

Suppose rank $A = $ rank C; then the

$$\dim \text{span}(\{\mathbf{a}_i\}) = \dim \text{span}(\{\mathbf{a}_i\} \bigcup \{\mathbf{b}\}),$$

which says that $\mathbf{b}_i$ is in the column space of A, and thus $AX = B$ has a solution. $\square$

2. The solution is unique $\iff$ rank$(A) = m$.

$\impliedby$:

Suppose that rank$(A) = m$ and a solution to $AX = B$ exists. Then rank$(C) = m$ as well

5.4 Part 4

Todo

6 Problem 6