# Algebraic Groups

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Thursday $24^{\text{th}}$ September, 2020

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These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Dan Nakano at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Thursday $24^{\text{th}}$ September, 2020
15:34

# 1 Friday, August 21

Reference: Carter's "Finite Groups of Lie Type".
Reference: Humphrey's "Linear Algebraic Groups" (Springer)

## 1.1 Intro and Definitions

**Definition 1.0.1** (Affine Variety).
Let $k = \bar{k}$ be algebraically closed (e.g. $k = \mathbb{C}, \overline{\mathbb{F}_p}$). A variety $V \subseteq k^n$ is an *affine $k$-variety* iff $V$ is the zero set of a collection of polynomials in $k[x_1, \cdots, x_n]$.

Here $\mathbb{A}^n := k^n$ with the Zariski topology, so the closed sets are varieties.

**Definition 1.0.2** (Affine Algebraic Group).
An *affine algebraic $k$-group* is an affine variety with the structure of a group, where the multiplication and inversion maps

$$\begin{aligned} \mu : G \times G \to G \\ \iota : G \to G \end{aligned}$$

are continuous.

**Example 1.1.**
$G = \mathbb{G}_a \subseteq k$ the *additive group* of $k$ is defined as $\mathbb{G}_a := (k, +)$. We then have a *coordinate ring* $k[\mathbb{G}_a] = k[x]/I = k[x]$.

**Example 1.2.**
$G = \mathrm{GL}(n, k)$, which has coordinate ring $k[x_{ij}, T] / \left\langle \det(x_{ij}) \cdot T = 1 \right\rangle$.

**Example 1.3.**
Setting $n = 1$ above, we have $\mathbb{G}_m := \mathrm{GL}(1, k) = (k^\times, \cdot)$. Here the coordinate ring is $k[x, T] / \left\langle xT = 1 \right\rangle$.

**Example 1.4.**
$G = \mathrm{SL}(n, k) \leq \mathrm{GL}(n, k)$, which has coordinate ring $k[G] = k[x_{ij}] / \left\langle \det(x_{ij}) = 1 \right\rangle$.

**Definition 1.0.3** (Irreducible).
A variety $V$ is *irreducible* iff $V$ can not be written as $V = \cup_{i=1}^n V_i$ with each $V_i \subsetneq V$ a proper subvariety.

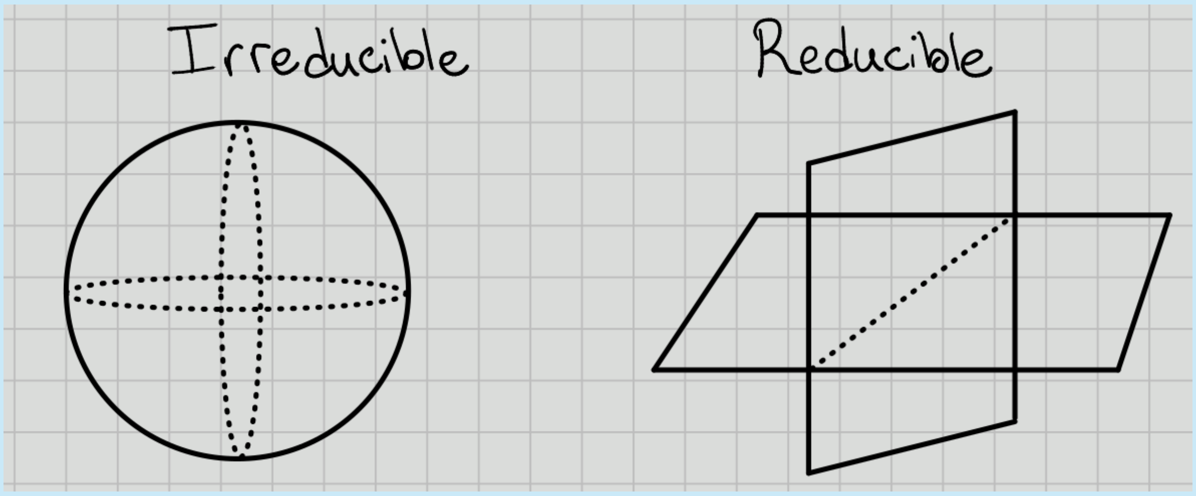


Figure 1: Reducible vs Irreducible

**Proposition 1.1*(?).***
There exists a unique irreducible component of $G$ containing the identity $e$. Notation: $G^0$.

**Proposition 1.2*(?).***
$G$ is the union of translates of $G^0$, i.e. there is a decomposition

$$G = \coprod_{g\in\Gamma} g \cdot G^0,$$

where we let $G$ act on itself by left-translation and define $\Gamma$ to be a set of representatives of distinct orbits.

**Proposition 1.3*(?).***
One can define solvable and nilpotent algebraic groups in the same way as they are defined for finite groups, i.e. as having a terminating derived or lower central series respectively.

## 1.2 Jordan-Chevalley Decomposition

**Proposition 1.4*(Existence and Uniqueness of Radical).***
There is a maximal connected normal solvable subgroup $R(G)$, denoted the *radical of* $G$.
- $\{e\} \subseteq R(G)$, so the radical exists.
- If $A, B \leq G$ are solvable then $AB$ is again a solvable subgroup.

**Definition 1.4.1** (Unipotent)**.**
An element $u$ is *unipotent* $\iff$ $u = 1 + n$ where $n$ is nilpotent $\iff$ its the only eigenvalue is $\lambda = 1$.

**Proposition 1.5** ***(JC Decomposition).***
For any $G$, there exists a closed embedding $G \hookrightarrow \mathrm{GL}(V) = \mathrm{GL}(n, k)$ and for each $x \in G$ a unique decomposition $x = su$ where $s$ is semisimple (diagonalizable) and $u$ is unipotent.

Define $R_u(G)$ to be the subgroup of unipotent elements in $R(G)$. :::{.definition title="Semisimple and Reductive"} Suppose $G$ is connected, so $G = G^0$, and nontrivial, so $G \neq \{e\}$. Then

- $G$ is semisimple iff $R(G) = \{e\}$.
- $G$ is reductive iff $R_u(G) = \{e\}$. :::

**Example 1.5.**
$G = \mathrm{GL}(n, k)$, then $R(G) = Z(G) = kI$ the scalar matrices, and $R_u(G) = \{e\}$. So $G$ is reductive and semisimple.

**Example 1.6.**
$G = \mathrm{SL}(n, k)$, then $R(G) = \{I\}$.

**Exercise 1.1.**
Is this semisimple? Reductive? What is $R_u(G)$?

**Definition 1.5.1** (Torus)**.**
A *torus* $T \subseteq G$ in $G$ an algebraic group is a commutative algebraic subgroup consisting of semisimple elements.

**Example 1.7.**
Let

$$T := \left\langle \begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_n \end{bmatrix} \subseteq \mathrm{GL}(n, k) \right\rangle.$$

**Remark 1.**
Why are torii useful? For $g = \mathrm{Lie}(G)$, we obtain a root space decomposition

$$g = \left( \bigoplus_{\alpha \in \Phi_-} g_\alpha \right) \oplus t \oplus \left( \bigoplus_{\alpha \in \Phi_+} g_\alpha \right).$$

When $G$ is a simple algebraic group, there is a classification/correspondence:

$$(G, T) \iff (\Phi, W).$$

where $\Phi$ is an irreducible root system and $W$ is a Weyl group.

# 2 | Monday, August 24

## 2.1 Review and General Setup

- $k = \bar{k}$ is algebraically closed
- $G$ is a reductive algebraic group
- $T \subseteq G$ is a *maximal split torus*

> Split: $T \cong \bigoplus \mathbb{G}_m$.

We'll associate to this a root system, not necessarily irreducible, yielding a correspondence

$$(G, T) \iff (\Phi, W)$$

with $W$ a Weyl group.

This will be accomplished by looking at $\mathfrak{g} = \mathrm{Lie}(G)$. If $G$ is simple, then $\mathfrak{g}$ is "simple", and $\Phi$ irreducible will correspond to a Dynkin diagram.

There is this a 1-to-1 correspondence

$$G \text{ simple}/\sim \iff A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$$

where $\sim$ denotes *isogeny*.

Taking the Zariski tangent space at the identity "linearizes" an algebraic group, yielding a Lie algebra.

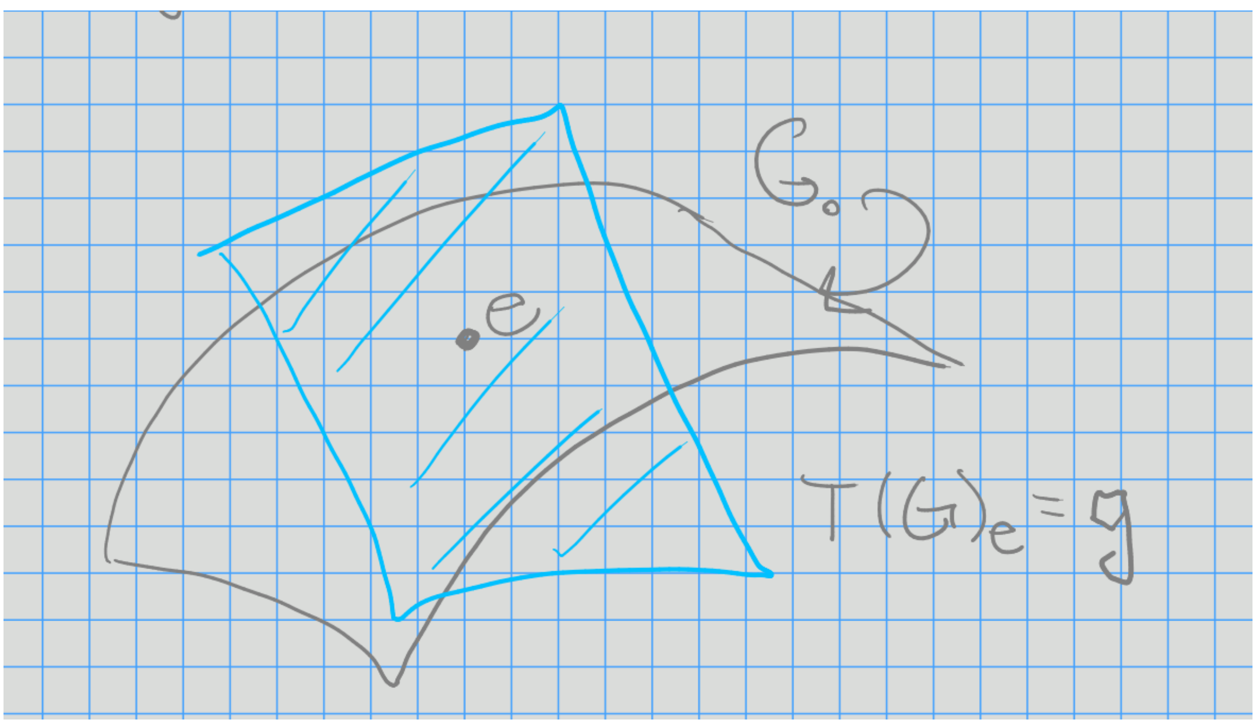


Figure 2: Image

We have the coordinate ring $k[G] = k[x_1, \cdots, x_n]/\mathcal{I}(G)$ where $\mathcal{I}(G)$ is the zero set. This is equal to $\{f : G \to k\}$,

## 2.2 The Associated Lie Algebra

**Definition 2.0.1** (The Lie Algebra of an Algebraic Group)**.**
Define *left translation* is

$$\begin{aligned} \lambda_x : k[G] &\to k[G] \\ y &\mapsto f(x^{-1}y). \end{aligned}$$

Define *derivations* as

$$\text{Der } k[G] = \left\{ D : k[G] \to k[G] \;\middle|\; D(fg) = D(f)g + fD(g) \right\}.$$

We can then realize the Lie algebra as

$$\mathfrak{g} = \text{Lie}(G) = \left\{ D \in \text{Der}k[G] \;\middle|\; \lambda_x \circ D = D \circ \lambda_x \right\},$$

the left-invariant derivations.

**Example 2.1.**

- $G = \text{GL}(n, k) \implies \mathfrak{g} = \mathfrak{gl}(n, k)$
- $G = \text{SL}(n, k) \implies \mathfrak{g} = \mathfrak{sl}(n, k)$

Let $G$ be reductive and $T$ be a split torus. Then $T$ acts on $\mathfrak{g}$ via an *adjoint action*. (For $\text{GL}_n, \text{SL}_n$, this is conjugation.)

There is a decomposition into eigenspaces for the action of $T$,

$$\mathfrak{g} = \left( \bigoplus_{\alpha \in \Phi} g_\alpha \right) \oplus t$$

where $t = \text{Lie}(T)$ and $g_\alpha \coloneqq \left\{ x \in \mathfrak{g} \;\middle|\; t.x = \alpha(t)x \;\forall t \in T \right\}$ with $\alpha : T \to K^\times$ a rational function (a *root*).

In general, take $\alpha \in \hom_{\text{AlgGrp}}(T, \mathbb{G}_m)$.

**Example 2.2.**
Let $G = \text{GL}(n, k)$ and

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \;\middle|\; a_j \in k^\times \right\}.$$

Then check the following action:

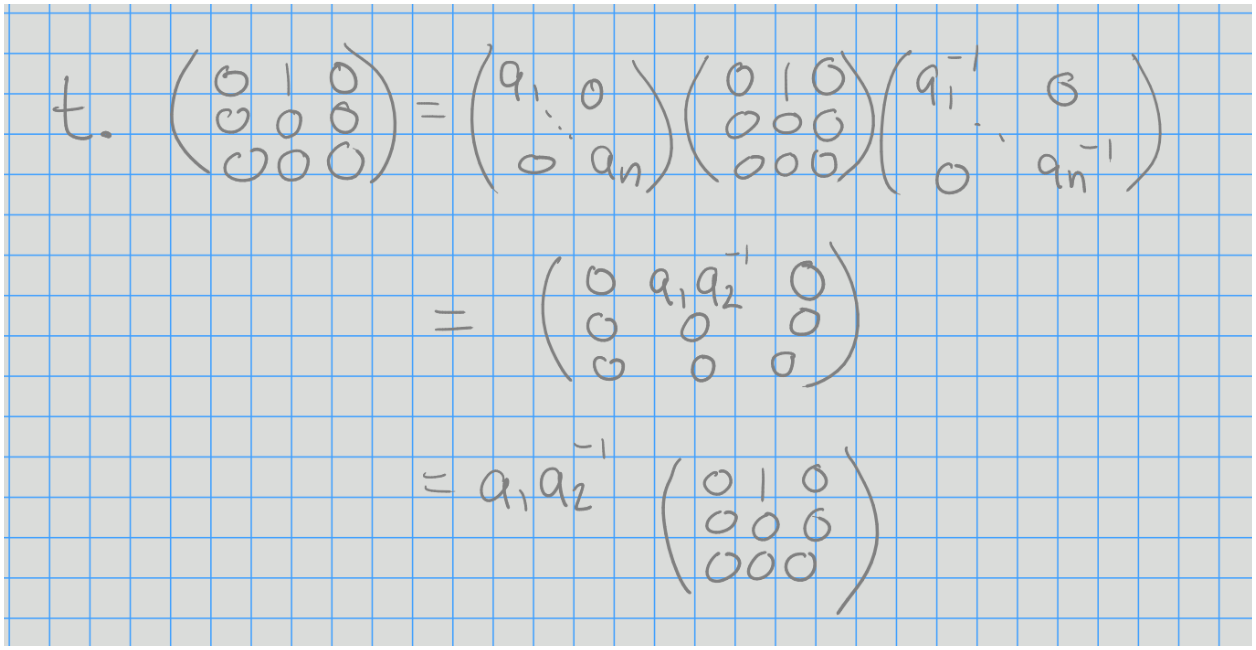

Figure 3: Action

which indeed acts by a rational function.

Then

$$g_\alpha = \operatorname{span}\left\{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\} = g_{(1,-1,0)}.$$

For $\mathfrak{g} = \mathfrak{gl}(3,k)$, we have

$$\begin{aligned}\mathfrak{g} = t &\oplus g_{(1,-1,0)} \oplus g_{(-1,1,0)}\\ &\oplus g_{(0,1,-1)} \oplus g_{(0,-1,1)}\\ &\oplus g_{(1,0,-1)} \oplus g_{(-1,0,1)}.\end{aligned}$$

## 2.3 Representations

Let $\rho : G \to \mathrm{GL}(V)$ be a group homomorphisms, then equivalently $V$ is a (rational) $G$-module.

For $T \subseteq G$, $T \curvearrowright G$ semisimply, so we can simultaneously diagonalize these operators to obtain a *weight space decomposition* $V = \bigoplus_{\lambda \in X(T)} V_\lambda$, where

$$\begin{aligned} V_\lambda &\coloneqq \left\{ v \in V \;\middle|\; t.v = \lambda(t)v \ \forall t \in T \right\}\\ X(T) &\coloneqq \hom(T, \mathbb{G}_m).\end{aligned}$$

**Example 2.3.**
Let $G = \mathrm{GL}(n, k)$ and $V$ the $n$-dimensional natural representation as column vectors,

$$V = \left\{ [v_1, \cdots, v_n] \;\middle|\; v_j \in k \right\}.$$

Then

$$T = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \;\middle|\; a_j \in k^\times \right\}.$$

Consider the basis vectors $\mathbf{e}_j$, then

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_j \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = a_1^0 a_2^0 \cdots a_j^0 \cdots a_n^0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Here the weights are of the form $\varepsilon_j \coloneqq [0, 0, \cdots, 1, \cdots, 0]$ with a 1 in the $j$th spot, so we have

$$V = V_{\varepsilon_1} \oplus V_{\varepsilon_2} \oplus \cdots \oplus V_{\varepsilon_n}.$$

**Example 2.4.**
For $V = \mathbb{C}$, we have $t.v = (a_1^0 \cdots a_n^0)v$ and $V = V_{(0,0,\cdots,0)}$.

## 2.4 Classification

Let $G$ be a simple algebraic group (ano closed, connected, normal subgroups other than $\{e\}, G$) that is nonabelian that is nonabelian.

**Example 2.5.**
Let $G = \mathrm{SL}(3, k)$. Then

$$T = \left\{ t = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 a_2^{-1} & 0 \\ 0 & 0 & a_2^{-1} \end{bmatrix} \;\middle|\; a_1, a_2 \in k^\times \right\}$$

and

$$t. \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = a_1^2 a_2^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and $\alpha_1 = (2, -1)$.

What is $\alpha_1$? Note that you can recover the Cartan something here?

Then

$$\mathfrak{g} = \mathfrak{g}_{(2,-1)} \oplus \mathfrak{g}_{(-2,1)} \oplus \mathfrak{g}_{(-1,2)} \oplus \mathfrak{g}_{(1,-2)} \oplus \mathfrak{g}_{(1,1)} \oplus \mathfrak{g}_{(-1,-1)}.$$

Then $\alpha_2 = (-1, 2)$ and $\alpha_1 + \alpha_2 = (1, 1)$.

This gives the root space decomposition for $\mathfrak{sl}_3$:

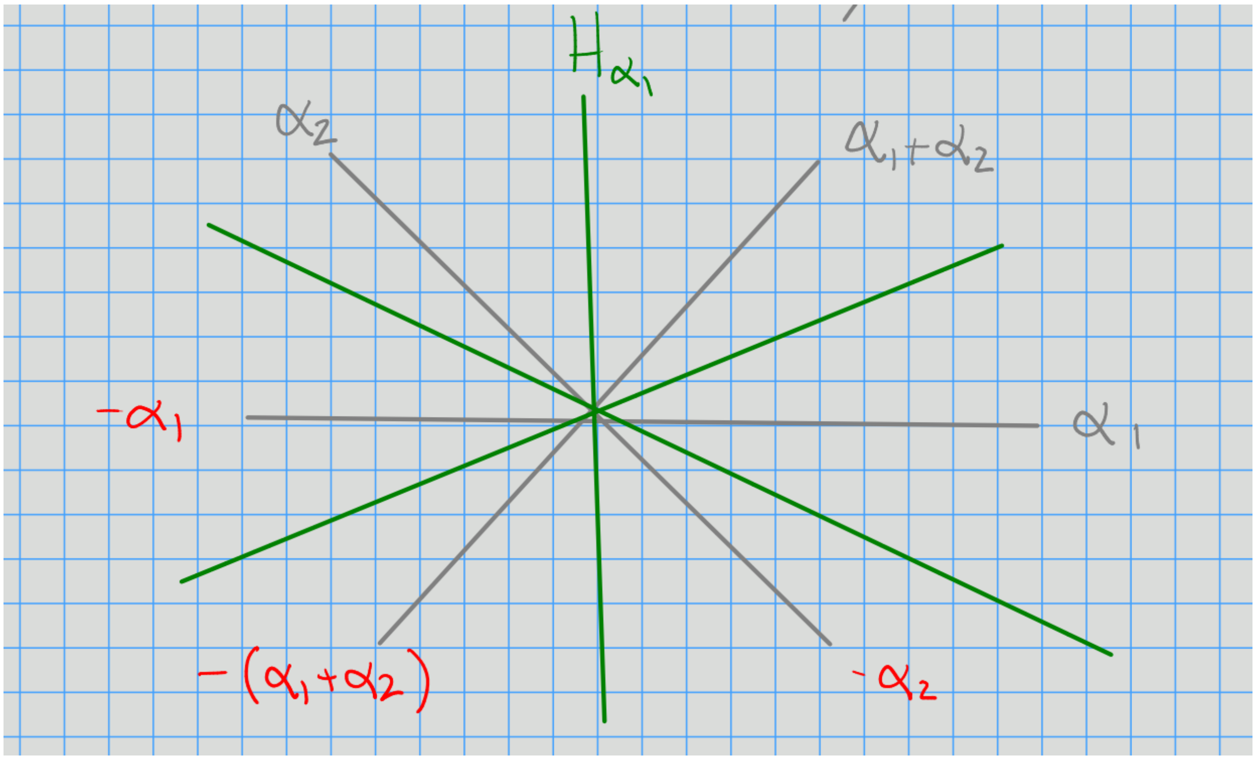


Figure 4: Image

Then the Weyl group will be generated by reflections through these hyperplanes.

# 3 | Wednesday, August 26

## 3.1 Review

- $G$ a reductive algebraic group over $k$
- $T = \prod_{i=1}^{n} \mathbb{G}_m$ a maximal split torus
- $\mathfrak{g} = \mathrm{Lie}(G)$
- There's an induced root space decomposition $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$
- When $G$ is simple, $\Phi$ is an *irreducible* root system
  - There is a classification of these by Dynkin diagrams

**Example 3.1.**
$A_n$ corresponds to $\mathfrak{sl}(n+1, k)$ (mnemonic: $A_1$ corresponds to $\mathfrak{sl}(2)$)

- We have representations $\rho : G \to \mathrm{GL}(V)$, i.e. $V$ is a $G$-module
- For $T \subseteq G$, we have a weight space decomposition: $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ where $X(T) = \hom(T, \mathbb{G}_m)$.

  Note that $X(T) \cong \mathbb{Z}^n$, the number of copies of $\mathbb{G}_m$ in $T$.

## 3.2 Root Systems and Weights

**Example 3.2.**
Let $\Phi = A_2$, then we have the following root system:

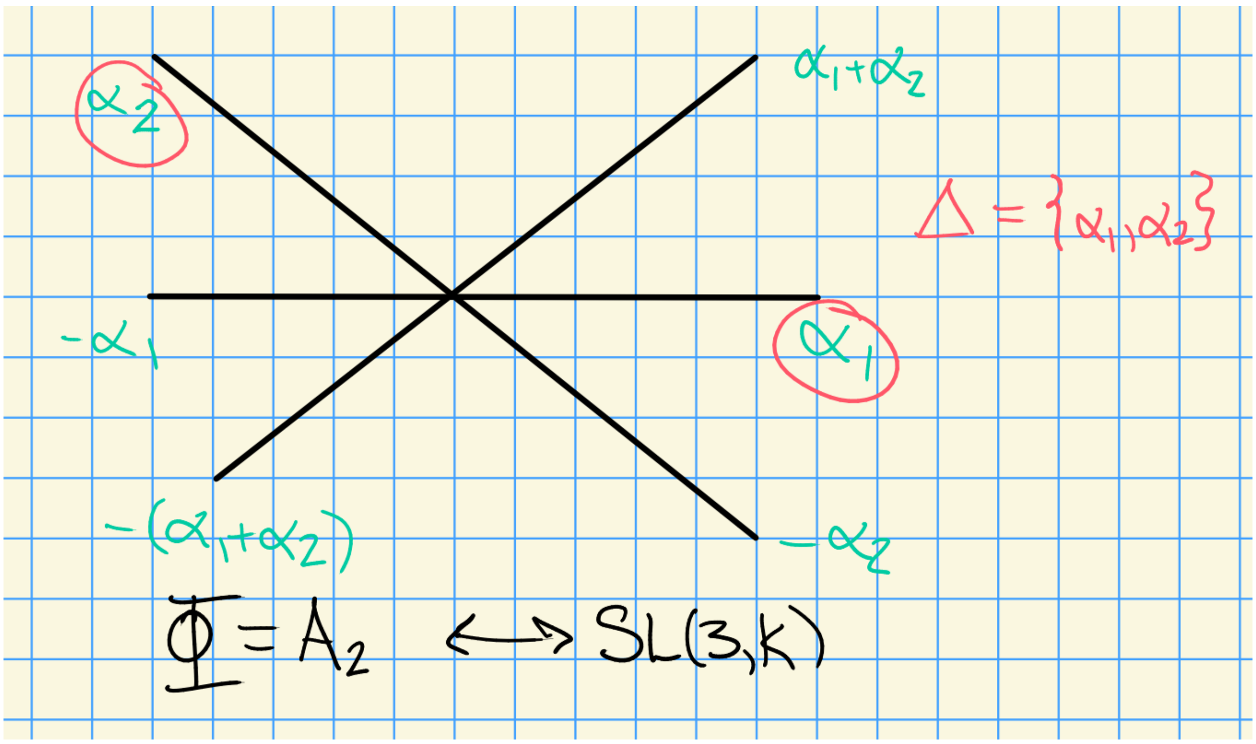


Figure 5: Image

In general, we'll have $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a basis of *simple roots*.

**Remark 2.**
Every root $\alpha \in I$ can be expressed as either positive integer linear combination (or negative) of simple roots.

For any $\alpha \in \Phi$, let $s_\alpha$ be the reflection across $H_\alpha$, the hyperplane orthogonal to $\alpha$. Then define the *Weyl group* $W = \left\{ s_\alpha \;\middle|\; \alpha \in \Phi \right\}$.

**Example 3.3.**
Here the Weyl group is $S_3$:

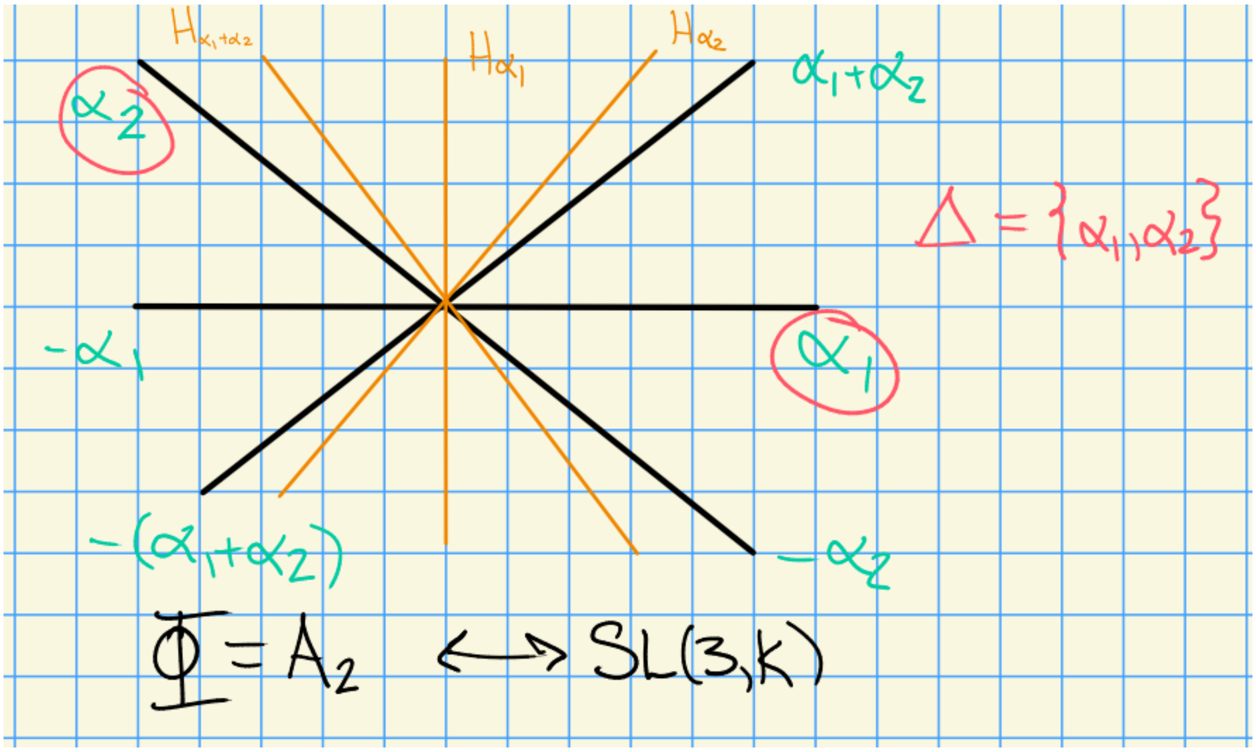


Figure 6: Image

**Remark 3.**
$W$ acts transitively on bases.

**Remark 4.**
$X(T) \subseteq \mathbb{Z}\Phi$, recalling that $X(T) = \hom(T, \mathbb{G}_m) = \mathbb{Z}^n$ for some $n$. Denote $\mathbb{Z}\Phi$ the *root lattice* and $X(T)$ the *weight lattice*.

**Example 3.4.**
Let $G = \mathfrak{sl}(2, \mathbb{C})$ then $X(T) = \mathbb{Z}\omega$ where $\omega = 1$, $\mathbb{Z}\Phi = \mathbb{Z}\{\alpha\}$ Then there is one weight $\alpha$, and the root lattice $\mathbb{Z}\Phi$ is just $2\mathbb{Z}$. However, the weight lattice is $\mathbb{Z}\omega = \mathbb{Z}$, and these are not equal in general.

**Remark 5.**
There is partial ordering on $X(T)$ given by $\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha \in \Delta} n_\alpha \alpha$ where $n_\alpha \geq 0$. (We say $\lambda$ *dominates* $\mu$.)

**Definition 3.0.1** (Fundamental Dominant Weights)**.**
We extend scalars for the weight lattice to obtain $E := X(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, a Euclidean space with an inner product $\langle \cdot, \ \cdot \rangle$.
For $\alpha \in \Phi$, define its *coroot* $\alpha^\vee := \dfrac{2\alpha}{\langle \alpha, \ \alpha \rangle}$. Define the *simple coroots* as $\Delta^\vee := \{\alpha_i^\vee\}_{i=1}^n$, which has a dual basis $\Omega := \{\omega_i\}_{i=1}^n$ the *fundamental weights.* These satisfy $\left\langle \omega_i, \ \alpha_j^\vee \right\rangle = \delta_{ij}$.

What is the notation for fundamental weights? Definitely not $\Omega$ usually!

Important because we can index irreducible representations by fundamental weights.

A weight $\lambda \in X(T)$ is *dominant* iff $\lambda \in \mathbb{Z}^{\geq 0}\Omega$, i.e. $\lambda = \sum n_i \omega_i$ with $n_i \in \mathbb{Z}^{\geq 0}$.

If $G$ is simply connected, then $X(T) = \bigoplus \mathbb{Z}\omega_i$.

See Jantzen for definition of simply-connected, $\mathrm{SL}(n+1)$ is simply connected but its adjoint $PGL(n+1)$ is not simply connected.

## 3.3 Complex Semisimple Lie Algebras

When doing representation theory, we look at the Verma modules $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda \twoheadrightarrow L(\lambda)$.

**Theorem 3.1** *(?)***.**
$L(\lambda)$ as a finite-dimensional $U(\mathfrak{g})$-module $\iff$ $\lambda$ is dominant, i.e. $\lambda \in X(T)_+$.

Thus the representations are indexed by lattice points in a particular region:

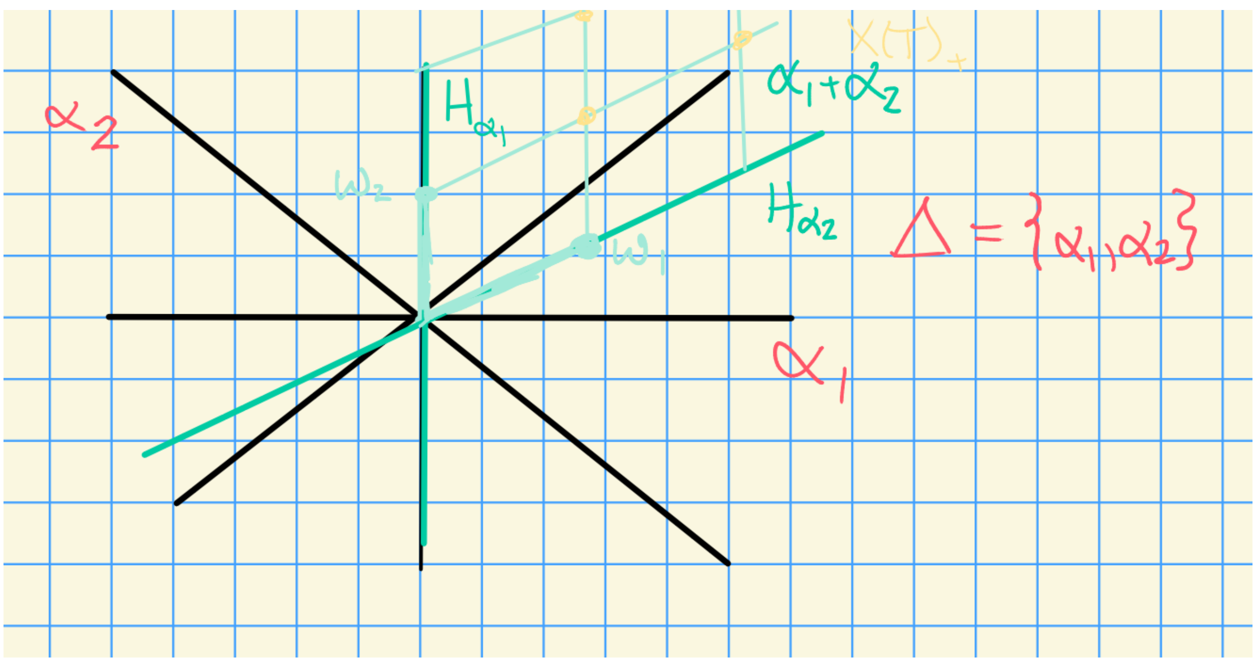


Figure 7: Image

**Question 1**:

Suppose $G$ is a simple (simply connected) algebraic group. How do you parameterize *irreducible* representations?

For $\rho : G$
*to*$\mathrm{GL}(V)$, $V$ is a *simple module* (an *irreducible representation*) iff the only proper $G$-submodules of $V$ are trivial.

**Answer 1**: They are also parameterized by $X(T)_+$. We'll show this using the induction functor $\mathrm{Ind}_B^G \lambda = H^0(G/B, \mathcal{L}(\lambda))$ (sheaf cohomology of the flag variety with coefficients in some line bundle).

> We'll define what $B$ is later, essentially upper-triangular matrices.

**Question 2**: What are the dimensions of the irreducible representations for $G$?

**Answer 2**: Over $k = \mathbb{C}$ using Weyl's dimension formula.

For $k = \overline{\mathbb{F}_p}$: conjectured to be known for $p \geq h$ (the *Coxeter number*), but by Williamson (2013) there are counterexamples. Current work being done!

# 4 | Friday, August 28

## 4.1 Representation Theory

Review: let $\mathfrak{g}$ be a semisimple lie algebra $/\mathbb{C}$. There is a decomposition $\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^- = \mathfrak{n}^+ \oplus t \oplus \mathfrak{n}^-$, where $t$ is a torus. We associate $U(\mathfrak{g})$ the universal enveloping algebra, and representations of $\mathfrak{g}$ correspond with representations of $U(\mathfrak{g})$.

Let $\lambda \in X(T)$ be a weight, then $\lambda$ is a $U(\mathfrak{b}^+)$-module. We can write $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda$.

**Remark 6.**
There exists a unique maximal submodule of $Z(\lambda)$, say $RZ(\lambda)$ where $Z(\lambda)/RZ(\lambda) \cong L(\lambda)$ is an irreducible representation of $\mathfrak{g}$.

**Theorem 4.1** ***(?).***
Let $L = L(\lambda)$ be a finite-dimensional irreducible representation for $\mathfrak{g}$. Then
1. $L \cong Z(\lambda)/RZ(\lambda)$ for some $\lambda$.
2. $\lambda \in X(T)_+$ is a dominant integral weight.

### 4.1.1 Induction

Let $\mathfrak{g}$ be an algebraic group $/k$ with $k = \bar{k}$, and let $H \leq G$. Let $M$ be an $H$-module, we'll eventually want to produce a $G$-modules.

Step 1: Make $M$ into a $G \times H$ where the first component $(g, 1)$ acts trivially on $M$.

Taking the coordinate algebra $k[G]$, this is a $(G-G)$-bimodule, and thus becomes a $G \times H$-module. Let $f \in k[G]$, so $f : G \to K$, and let $y \in G$. The explicit action is

$$[(g,h)f](y) \coloneqq f(g^{-1}yh).$$

Note that we can identify $H \cong 1 \times H \leq G \times H$. We can form $(M \otimes_k k[G])^H$, the $H$-fixed points.

**Exercise 4.1.**
Let $N$ be an $A$-module and $B \trianglelefteq A$, then $N^B$ is an $A/B$-module.

Hint: the action of $B$ is trivial on $N^B$. Here $N^B \coloneqq \left\{ n \in N \mathrel{\Big|} b.n = n \forall b \in B \right\}$

**Definition 4.1.1** (Induction)**.**
The *induced module* is defined as

$$\operatorname{Ind}_H^G(M) \coloneqq (M \otimes k[G])^H.$$

### 4.1.2 Properties of Induction

1. $(\cdot \otimes_k k[G]) = \hom_H(k, \cdot \otimes_k k[G])$ is only *left-exact*, i.e.

$$(0 \to A \to B \to C \to 0) \mapsto (0 \to FA \to FB \to FC \to \cdots).$$

2. By taking right-derived functors $R^j F$, you can take cohomology.

Note that in this category, we won't have enough projectives, but we will have enough injectives.

3. This functor commutes with direct sums and direct limits.
4. (**Important**) Frobenius Reciprocity: there is an adjoint, *restriction*, satisfying

$$\hom_G(N, \operatorname{Ind}_H^G M) = \hom_H(N \downarrow_H, M).$$

5. (Tensor Identity) If $M \in \operatorname{Mod}(H)$ and additionally $M \in \operatorname{Mod}(G)$, then $\operatorname{Ind}_H^G = M \otimes_k \operatorname{Ind}_H^G k$.

If $V_1, V_2 \in \operatorname{Mod}(G)$ then $V_1 \otimes_k V_2 \in \operatorname{Mod}(G)$ with the action given by $g(v_1 \otimes v_2) = gv_1 \otimes gv_2$.

6. Another interpretation: we can write

$$\operatorname{Ind}_H^G(M) = \left\{ f \in \operatorname{Hom}(G, M_a) \mathrel{\Big|} f(gh) = h^{-1} \cdot f(g) \forall g \in G, h \in H \right\} \qquad M_a = M \coloneqq \mathbb{A}^{\dim M}.$$

I.e., equivariant wrt the $H$-action.

Then $G$ acts on $\operatorname{Ind}_H^G M$ by left-translation: $(gf)(y) = f(g^{-1}y)$.

7. There is an evaluation map:

$$\begin{aligned} \varepsilon : \operatorname{Ind}_H^G(M) &\to M \\ f &\mapsto f(1). \end{aligned}$$

This is an $H$-module morphism. Why? We can check

$$\begin{aligned}
\varepsilon(h.f) &\coloneqq (h.f)(a) \\
&= f(h^{-1}) \\
&= hf(1) \\
&= h(\varepsilon(f)).
\end{aligned}$$

We can write the isomorphism in Frobenius reciprocity explicitly:

$$\begin{aligned}
\hom_G(N, \operatorname{Ind}_H^G M) &\xrightarrow{\cong} \hom_H(N, M) \\
\varphi &\mapsto \varepsilon \circ \varphi.
\end{aligned}$$

8. Transitivity of induction: for $H \leq H' \leq G$, there is a natural transformation (?) of functors:

$$\operatorname{Ind}_H^G(\,\cdot\,) = \operatorname{Ind}_{H'}^G\left(\operatorname{Ind}_H^{H'}(\,\cdot\,)\right).$$

Equality as a composition of functors?

## 4.2 Classification of Simple $G$-modules

Suppose $G$ is a connected reductive algebraic group $/k$ with $k = \bar{k}$.

**Example 4.1.**
Let $G = \mathrm{GL}(n, k)$. There is a decomposition:

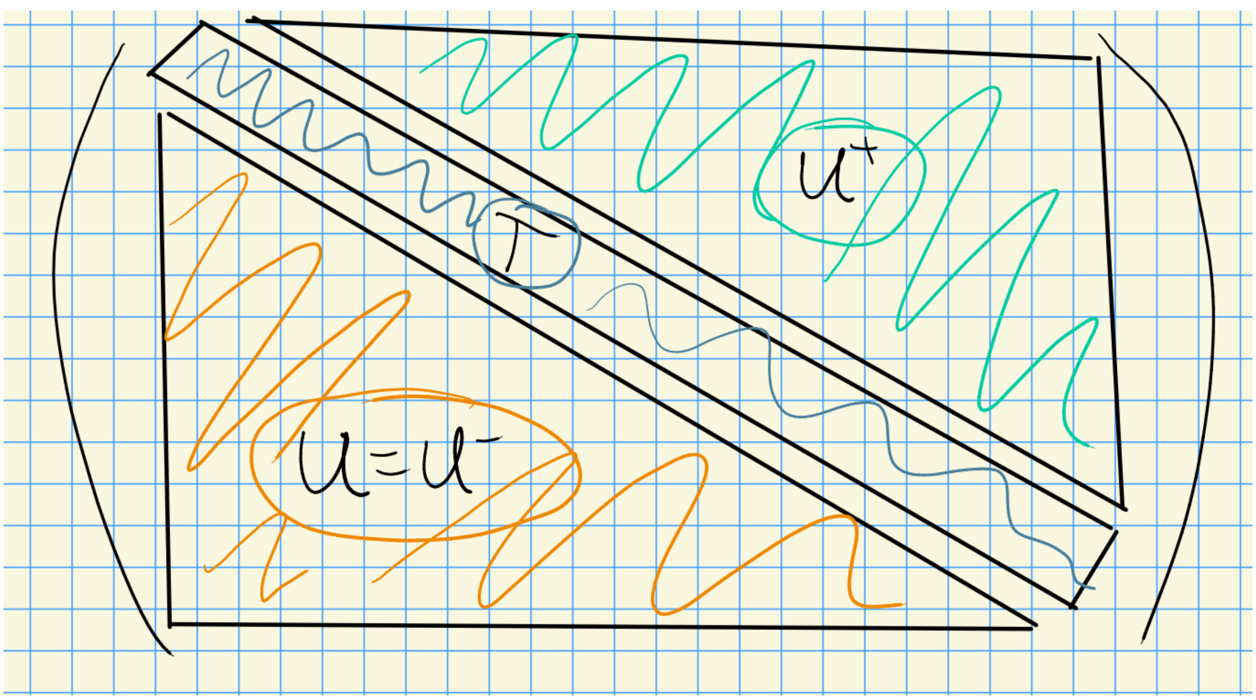


Figure 8: Image

**Step 1**: Getting modules for $U$.

Then there's a general fact: $U^+TU \hookrightarrow G$ is dense in the Zariski topology for any reductive algebraic group. We can form

- $B^+ := T \rtimes U^+$, the *positive borel*,
- $B^- := T \rtimes U$, the *negative borel*,

Suppose we have a $U$-module, i.e. a representation $\rho : U \to \mathrm{GL}(V)$. We can find a basis such that $\rho(u)$ is upper triangular with ones on the diagonal. In this case, there is a composition series with 1-dimensional quotients, and the composition factors are all isomorphic to $k$.

Moral: for unipotent groups, there are only trivial representations, i.e. the only simple $U$-modules are isomorphic to $k$.

**Step 2**: Getting modules for $B$.

Modules for $B$ are solvable, in which case we can find a flag. In this case, $\rho(b)$ embeds into upper triangular matrices, where the diagonal action may now not be trivial (i.e. diagonal is now arbitrary).

Thus simple $B$-modules arise by taking $\lambda \in X(T) = \hom(T, \mathbb{G}_m) = \hom(T, \mathrm{GL}(1, k))$, then letting $u$ act trivially on $\lambda$, i.e. $u.v = v$. Here we have $B \to B/U = T$, so any $T$-module can be pulled back to a $B$-module.

**Step 3**: Getting modules for $G$.

Let $\lambda \in X(T)$, then $H^0(\lambda) = \mathrm{Ind}_B^G \lambda = \nabla(\lambda)$.

# 5 | Monday, August 31

## 5.1 Review of Representation Theory of Modules

Take $R$ a ring, then consider $M$ an $R$-module to be a "vector space" over $M$. Note that $M$ is an $R$-module $\iff$ there exists a ring morphism $\rho : R \to \hom_{\mathrm{AbGrp}}(M, M)$.

Now let $G$ be a group and consider $G$-modules $M$. Then a $G$-module will be defined by taking $M/k$ a vector space and a $G$-action on $M$. This is equivalent to having a group morphism $\rho : G \to \mathrm{GL}(M)$.

For $M$ a $G$-module, given a group action, define

$$\begin{aligned}\rho : G &\to \mathrm{GL}(M) \\ \rho(g)(m) &= g.m\end{aligned}$$

where $\rho(h) : M \to M$.

Similarly, for $\rho : G \to \mathrm{GL}(M)$ a group morphism, define the group action $g.m := \rho(g)m$. Thus representations of $G$ and $G$-modules are equivalent.

**Definition 5.0.1** (?).
Let $M$ be a $G$-module.

1. $M$ is a *simple* $G$-module (equivalently an *irreducible representation*) $\iff$ the only $G$-submodules (equiv. $G$-invariant subspaces) are $0, M$.
2. $M$ is *indecomposable* $\iff$ $M$ can not be written as $M = M_1 \oplus M_2$ with $M_i < M$ proper

submodules.

**Example 5.1.**
For $G = \mathrm{SL}(n, \mathbb{C})$, there is a natural $n$-dimensional representation $M = V$, and this is irreducible.

What is $V$?

**Example 5.2.**
Let $R = \mathbb{Z}$, so we're considering $\mathbb{Z}$-modules. For $M = \mathbb{Z}$, $M$ is not simple since $2\mathbb{Z} < \mathbb{Z}$ is a proper submodule. However $M$ is indecomposable.

Recall from last time: we defined a functor $\mathrm{Ind}_H^G(\,\cdot\,) : H\text{-mod} \to G\text{-mod}$, where $\mathrm{Ind}_H^G = (k[G] \otimes M)^H$, the $H$-invariants. This functor is left-exact but not right-exact, so we have cohomology $R^j \mathrm{Ind}_H^G$ by taking right-derived functors.

Goal: classify simple $G$-modules for $G$ a reductive connected algebraic group.

**Example 5.3.**
For $G = \mathrm{GL}(n, k)$, we have a decomposition

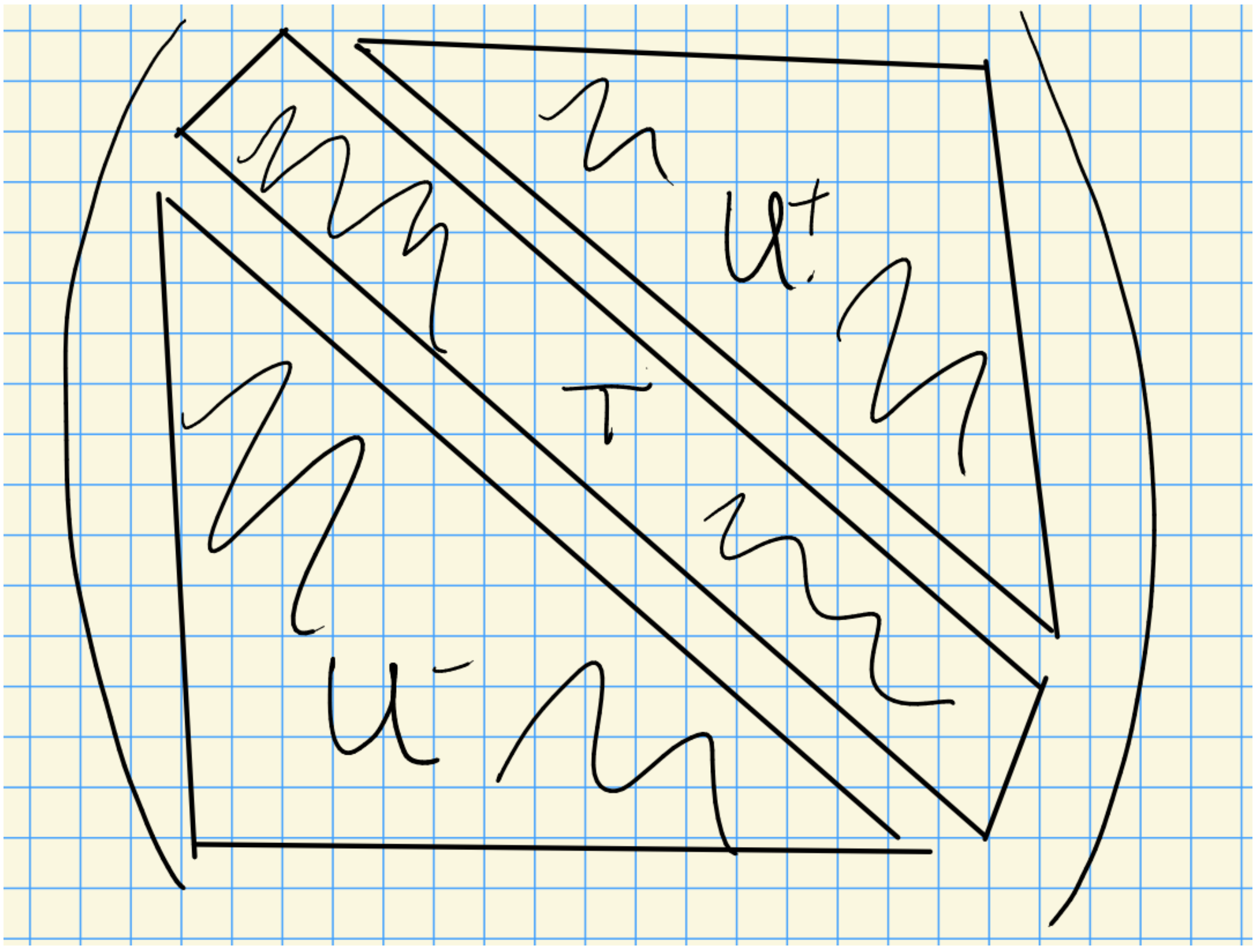


Figure 9: Image

We have

- $B = T \rtimes U$ the negative Borel,
- $B = T \rtimes U^+$ the Borel

For $U$-modules: $k$ is the only simple $U$-module. Importantly, if $V$ is a $U$-module, then the fixed points are never zero, i.e. $V^U = \hom_{U\text{-Mod}}(k, V) \neq 0$.

For $B$-modules: let $X(T) \coloneqq \hom(T, \mathbb{G}_m) = \hom(T, \mathrm{GL}(1, k))$. These are the simple representations for the torus $T$. Thus $\lambda \in X(T)$ represents a simple $T$-module.

We have a map $B \to B/U = T$, so we can pullback $T$-representations to $B$-representations ("inflation"), since we have a map $T \to \mathrm{GL}(1, k)$ and we can just compose. So $\lambda$ is a 1-dimensional (simple) $B$-module where $U$ acts trivially.

Lee's theorem: all irreducible representations for $B$ are one-dimensional. Thus these are the simple $B$-modules.

For $G$-modules: define $\nabla(\lambda) \coloneqq \mathrm{Ind}_B^G(\lambda) = H^0(\lambda)$.

Questions:

1. When does $H^0(\lambda) = 0$?
2. What is $\dim_{k\text{-Vect}} H^0(\lambda)$?
3. What are the composition factors of $H^0(\lambda)$?

Known in characteristic zero, wildly open in positive characteristic.

**Remark 7.**
Another interpretation: look at the flag variety $G/B$ and take global sections, then $H^0(\lambda) = H^0(G/B, \mathcal{L}(\lambda))$ where $\mathcal{L}$ is given by projecting the fiber product $G \times_B \lambda \twoheadrightarrow G/B$ onto the first factor.

**Remark 8.**

1. $H^0(k) = H^0([0, \cdots, 0]) = k[G/B] = k$.
2. $H^0(M) = M$ if $M$ is a $G$-module.
3. A $G$-module $M$ is *semisimple* iff $M = \bigoplus_{i \in I} M_i$ with each $M_i$ are simple.
4. Can consider the largest semisimple submodule, the *socle* $\operatorname{Soc}_G(M)$.

$$\begin{tikzcd}
L_4 \arrow[dr] & & L_5 \oplus L_7 \arrow[dl] \\
& (L_1 \oplus L_2 \oplus L_3) = \operatorname{Soc}_G(M)) &
\end{tikzcd}$$

Goal: classify simple $G$-modules. Strategy: used dominant highest weights.

As opposed to Verma modules, the irreducibles will be a dual situation where they sit at the bottom of the module. Indicated by the notation $\nabla$ pointing down!

**Proposition 5.1** *(?)*.
Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$.

1. $\dim H^0(\lambda)^{U^+} = 1$ and $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$.
2. Every weight of $H^0(\lambda)$ satisfies $w_u\lambda \leq \mu \leq \lambda$, where $w_0$ is the longest Weyl group element and $\alpha \leq \beta \iff \alpha - \beta \in \mathbb{Z}^+\Phi$.

Note that in fact $\ell(w_0) = \left|\Phi^+\right|$.

**Example 5.4.**
Take $A_2$ with simple reflections $s_{\alpha_1}, s_{\alpha_2}$ and $\Delta = \{\alpha_1, \alpha_2\}$.

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
s_{\alpha_1} \arrow[d] & & s_{\alpha_2} \arrow[d] \\
s_{\alpha_1}s_{\alpha_2} \arrow[dr] & & s_{\alpha_2}s_{\alpha_1} \arrow[dl] \\
 & s_{\alpha_2}s_{\alpha_1}s_{\alpha_1} = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} = w_0 &
\end{tikzcd}$$

*Proof ((Sketch)).*
We can write

$$H^0(\lambda) = \left\{ f \in k[G] \;\middle|\; f(gb) = \lambda(b)^{-1} f(g)\, b \in B, g \in G \right\}.$$

Suppose $f \in H^0(\lambda)^{U^+}$ and $u_+ \in U^+, t \in T, u \in U$. Then

$$\begin{aligned}\left(u_+^{-1} f\right)(tu) &= f(tu) \\ &= \lambda(t)^{-1} f(1).\end{aligned}$$

On the other hand,

$$\left(u_+^{-1} f\right)(tu) = f(u_+ tu).$$

So by density, $f(1)$ is determined by $f(u_+ tu)$ and $\dim H^0(\lambda)^{U^+} \leq 1$. But since this can't be zero, the dimension must be equal to 1.

■

**Proposition 5.2** *(?).*
Let

$$\varepsilon : H^0(\lambda) \to \lambda$$

be the evaluation morphism.
This is a morphism of $B$-modules, and in particular is a morphism of $T$-modules. Thus the image of a weight $\mu \neq \lambda$ is zero, so $\varepsilon$ is injective.

*Proof .*
We have

$$f(u_+ tu) = \lambda(t)^{-1} f(1) = \lambda(t)^{-1} \varepsilon(f).$$

Suppose $f \in H^0(\lambda)^{U^+}$ and $\varepsilon(f) = 0$. Then $f(u_+ tu) = 0$, and by density $f \equiv 0$, showing injectivity.
Therefore $H^0(\lambda)^{U^+} \subset H^0(\lambda)_\lambda$. Suppose $\mu$ is maximal among weights in $H^0(\lambda)$. Then

$$H^0(\lambda)_\mu \subseteq H^0(\lambda)^{U^+}$$

because $U^+$ raises weights.
But $H^0(\lambda)^{U^+} \subseteq H^0(\lambda)_\lambda$ implies $\mu = \lambda$. Thus the maximal weight in $H^0(\lambda)$ is $\lambda$.

Recall the situation in lie algebras: $g_\alpha v \in V_{\lambda+\alpha}$ when $v$
$inV_\lambda$.

Since $\lambda$ is maximal, any other weight $\mu$ satisfies $\mu \leq \lambda$. Thus

$$H^0(\lambda)_\lambda \subseteq H^0(\lambda)^{U^+} \subseteq H^0(\lambda)_\lambda,$$

forcing these to be equal and finishing part 1.

■

# 6 | Friday, September 04

Some concepts used in the proof of other theorems: Let $G$ be a reductive algebraic group and $\mathfrak{g}$ its lie algebra. There is an associative algebra $U(\mathfrak{g})$ which reflects the representation theory of $G$.

Fact: $\mathfrak{g}$-mod $\equiv U(\mathfrak{g})$-modules which are unitary, i.e. $1.m = m$.

We can write a basis

$$\mathfrak{g} = \left\langle e_\alpha, h_i, f_\beta \;\middle|\; \alpha \in \Phi^+,\, \beta \in \Phi^-,\, i = 1, 2, \cdots, n \right\rangle,$$

the *Chevalley basis*. It turns out that the structure constants are all in $\mathbb{Z}$.

**Example 6.1.**
Take $\mathfrak{g} = \mathfrak{sl}(2, k)$, then

$$e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We want to form a $\mathbb{Z}$-lattice in $U(\mathfrak{g})$, denoted

$$U(\mathfrak{g})_\mathbb{Z} = \left\langle e_\alpha^{[n]} = \frac{e_\alpha^n}{n!}, f_\beta^{[n]} = \frac{f_\beta^n}{n!}, \binom{h_i}{m} \right\rangle.$$

We then form the *distribution algebra* (or *hyperalgebra* in earlier literature) as $\mathrm{Dist})G) := U(\mathfrak{g})_\mathbb{Z} \otimes_\mathbb{Z} k$ for $k$ any field (e.g. char $(k) = p$).

**Theorem 6.1** ***(?).***
$G$-modules $\equiv \mathrm{Dist}(G)$-modules which are

- *Weight modules*
- *Locally finite*: $\dim \mathrm{Dist}(G).m < \infty$ for all $m \in M$.

**Remark 9.**
In characteristic zero, $\mathrm{Dist}(G) = U(\mathfrak{g})$. Thus there is a correspondence

$$\left\{ G\text{-modules} \right\} \iff \left\{ U(\mathfrak{g})\text{-modules} \right\}.$$

If char $(k) = p$, e.g. $k = \bar{\mathbb{F}}_p$, and if the Frobenius map $F : G \to G$ satisfies $G_1 \coloneqq \ker F$ (thinking of $G_1$ as a group scheme), then $\mathrm{Dist}(G_1) < \mathrm{Dist}(G)$ is a proper submodule. In this case, $\mathfrak{g} \subseteq \mathrm{Dist}(G_1)$ is a finite dimensional Hopf algebra, and $k[G_1] = \mathrm{Dist}(G_1)^\vee$. Importantly, the lie algebra does *not* generate $\mathrm{Dist}(G)$ if $k = \bar{\mathbb{F}}_p$.

**Example 6.2.**
Take $G = \mathbb{G}_a$, then $\mathrm{Dist}(\mathbb{G}_a) = \left\langle T^k \;\middle|\; k = 0, 1, \cdots \right\rangle$ is an infinite dimensional algebra. In this case, $T^k T^\ell = \binom{k+\ell}{\ell} T^{k+\ell}$. For $k = \mathbb{C}$, $\mathrm{Dist}(\mathbb{G}_a) = \left\langle T^1 \right\rangle$ has one generator.

In the case $k = \bar{\mathbb{F}}_p$, we have $\mathrm{Dist}((\mathbb{G}_a)_1) = \left\langle T^k \;\middle|\; 0 \le k \le p-1 \right\rangle$.

Note that taking duals yields a truncated polynomial algebra: $k[(\mathbb{G}_a)_1] = k[x]/\langle x^p \rangle$.

## 6.1 Review

Recall that $H^0(\lambda) \coloneqq \mathrm{Ind}_B^G \lambda$. Proved in last (missed) class: :::{.remark} Let $H^0(\lambda) \neq 0$. Then

a. $\dim H^0(\lambda)_\lambda = 1$ where $H^0(\lambda) = H^0(\lambda)^{U^+}$.
b. Each weight $\mu$ of $H^0(\lambda)$ satisfies $w_0 \lambda \le \mu \le \lambda$, where $w_0$ is the longest Weyl group element. :::

**Remark 10.**
Let $H^0(\lambda)_\lambda \neq 0$, then $L(\lambda) = \mathrm{Soc}_G H^0(\lambda)$ is simple.

**Remark 11.**
If $\mu$ is a weight of $L(\lambda)$, then $w_0 \lambda \le \mu \le \lambda$, $\dim L(\lambda)_\lambda = 1$, and $L(\lambda)_\lambda = L(\lambda)^{U+}$.

**Remark 12.**
Any simple $G$-module is isomorphic to $L(\lambda)$ where $H^0(\lambda) \neq 0$.

Goal: We now want to classify simple $G$-modules. So we need an iff criterion for when $H^0(\lambda) \neq 0$.

We look at the set of dominant weights

$$X(T)_+ = \left\{ \lambda \in X(T) \;\middle|\; \langle \lambda,\ \alpha^\vee \rangle \ge 0 \forall \alpha \in \Delta \right\} \qquad = \left\{ \lambda \in X(T) \;\middle|\; \lambda = \sum_{i=1}^{\ell} n_i w_i,\ n_i \ge 0 \right\}.$$

**Theorem 6.2** ***(?).***
TFAE:

1. $H^0(\lambda) \neq 0$.
2. $\lambda \in X(T)_+$, i.e. $\lambda$ is a dominant weight.

*Proof .*
$1 \implies 2$: Suppose (1), then consider a simple reflection $s_\alpha$ for some $\alpha \in \Delta$. We know $H^0(\lambda)_\lambda \neq 0$, thus $H^0(\lambda)_{s_\alpha \lambda} \neq 0$. Therefore

$$\begin{aligned} s_\alpha \lambda = \lambda - \langle \lambda,\ \alpha^\vee \rangle \alpha &\leq \lambda \\ \implies 0 &\leq \langle \lambda,\ \alpha^\vee \rangle \alpha \\ \implies \langle \lambda,\ \alpha^\vee \rangle &\geq 0 \qquad \forall \alpha \in \Delta. \end{aligned}$$

$2 \implies 1$: For a detailed proof, see Jantzen 2.6 in Part II.

- Let $\lambda \in X(T)_+$, then (by the intro lie algebras course) there exists an $L(\lambda)$: a simple finite dimensional $U(\mathfrak{g})$-module over $\mathbb{C}$.
- $L(\lambda)$ has an integral basis which is compatible with $U(\mathfrak{g})_\mathbb{Z}$ (Kostant's $\mathbb{Z}$-form).
- Thus we can base change to get $\tilde{L}(\lambda) \coloneqq L(\lambda) \otimes_\mathbb{Z} k$, which is a $\mathrm{Dist}(G)$-module. Note that $\tilde{L}(\lambda)$ still has highest weight $\lambda$, so consider $\hom_B(\tilde{L}(\lambda), \lambda) \neq 0$.
- Apply frobenius reciprocity: $\hom_B(\tilde{L}(\lambda), \lambda) = \hom_G(\tilde{L}(\lambda), \mathrm{Ind}_B^G \lambda) = \hom_G(\tilde{L}(\lambda), H^0(\lambda))$. But then $H^0(\lambda) \neq 0$ (since otherwise this would imply the original hom was zero).

■

**Theorem 6.3** *(?).*
Let $G$ be a reductive connected algebraic group over $k$. Then there exists a 1-to-1 correspondence between dominant weights and irreducible $G$-representations:

$$\left\{ \text{Dominant weights: } X(T)_+ \right\} \iff \left\{ \text{Irreducible representations: } \left\{ L(\lambda) \mathrel{\Big|} \lambda \in X(T)_+ \right\} \right\}.$$

### 6.2 Characters of $G$-modules

Let $G$ be reductive, so (importantly) it has a maximal torus $T$. Let $M \in G$-mod, so (importantly) $M \in T$-mod.

Then there is a weight space decomposition $M = \bigoplus_{\lambda \in X(T)} M_\lambda$. We then write the character of $M$ as

$$\mathrm{char}\ M \coloneqq \sum_{\lambda \in X(T)} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X(T)].$$

Next time: more characters, and Weyl's dimension formula.

# 7 | Wednesday, September 09

Todo

# 8 | Wednesday, September 16

## 8.1 Group Schemes

**Definition 8.0.1** (Representable Functors)**.**
Let $F :: k\text{-alg} \to \text{Set}$ be a functor, then $F$ is **representable** iff $F(R)$ corresponds to "solutions to equations in $R$".

**Example 8.1.**
Let $F(\,\cdot\,) = \mathrm{SL}(2, \,\cdot\,)$, then the corresponding equations are $\det(x_{ij}) = 1$.

If $F$ is representable, there is a correspondence $F(R) \cong \hom_R(A, R)$. In the above example,

$$A = k[x_{11}, x_{12}, x_{21}, x_{22}] / \langle x_{11}x_{22} - x_{12}x_{21} \rangle \,,$$

which is exactly the coordinate algebra.

**Definition 8.0.2** (Affine Group Scheme)**.**
An *affine group scheme* is a representable functor $F : k\text{-alg} \to \text{Groups}$.

Suppose $G$ is an affine group scheme, and let $A = k[G]$ be the representing object. Then there is a correspondence

$$G\text{-modules} \iff k[G]^\vee\text{-modules}.$$

For $G$ reductive, the RHS is equivalent to $\mathrm{Dist}(G)$-modules.

**Definition 8.0.3** (Finite Group Schemes)**.**
$G$ is a **finite** group scheme iff $k[G]$ is finite dimensional.

If $G$ is finite, then $A^\vee \cong k[G]^\vee$ is a cocommutative Hopf algebras. Thus representations for *finite* group schemes are equivalent to representations for finite-dimensional cocommutative Hopf algebras.

> On group scheme side: see reduction, spectral sequences, conceptual arguments. On the algebra side: have bases, underlying vector space, can do concrete computations. Can take $\mathrm{Spec}\,(k[G])^\vee$ to recover a group scheme.

## 8.2 Hopf Algebras

For $A$ a $k$-alg, we have a multiplication and a unit, which can be defined in terms of diagrams. To categorically reverse arrows, we can ask for a comultiplication and a counit.

$$\Delta : A \to A^{\otimes 2}$$

$$\epsilon : A \to k.$$

We'll want another map, an *antipode*

$$s : A \to A.$$

The comultiplication should satisfy

$$\begin{tikzcd}
A^{\otimes 3} & A^{\otimes 2} \arrow[l, "1\otimes A"'] \\
A^{\otimes 2} \arrow[u, "\Delta\otimes 1"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

The counit should satisfy

$$\begin{tikzcd}
k \otimes A \arrow[d, "\cong"] & A^{\otimes 2} \arrow[l, "\varepsilon\otimes 1"'] \\
A \arrow[r, "\cong"] & A \arrow[u, "\Delta"']
\end{tikzcd}$$

And the antipode should satisfy

$$\begin{tikzcd}
A & A \arrow[l, "m(s\otimes 1)"'] \\
A \arrow[u] & A \arrow[l, "\varepsilon"] \arrow[u, "\Delta"']
\end{tikzcd}$$

### 8.2.1 Module Constructions

Let $A$ be a Hopf algebra.

1. For $A$-modules $M, N$, we can form the $A$-module $M \otimes_k N$ with
$$\Delta(a) = \sum a_i \otimes a_j$$
$$a(m \otimes n) = \sum a_1 m \otimes a_2 n.$$
2. If $M$ is finite-dimensional over $A$, then $M^\vee = \hom_k(M, k) \ni f$ is an $A$-module, and we can define $(af)(x) \coloneqq f(s(a)x)$ for $a \in A, x \in M$.

**Example 8.2.**
$A = kG$ the group algebra on a group is a Hopf algebra:

$$\begin{aligned}\Delta : A &\to A^{\otimes 2} \\ g &\mapsto g \otimes g.\end{aligned}$$

The module action is diagonal, namely $g(m \otimes n) = gm \otimes gn$. The antipode is given by $s(g) = g^{-1}$, and the unit is $\varepsilon(g) = 1$ for all $g \in G$.

**Example 8.3.**
Let $A = U(\mathfrak{g})$, the universal enveloping algebra for $\mathfrak{g}$ a Lie algebra. Recall that $\mathfrak{g}$-modules are equivalent to $U(\mathfrak{g})$-modules (unitary representations, some big associative algebra). Then $A$ is a Hopf algebra, with $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ for $\ell \in \mathfrak{g}$. The unit is $\varepsilon(\ell) = 0$, and the antipode is $s(\ell) = -\ell$.

**Example 8.4.**
Take the additive group $\mathbb{G}_a$, then $A = k[\mathbb{G}_a] \cong k[x]$ is a commutative Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0, s(x) = -x$.

**Example 8.5.**
For $\mathbb{G}_m$, we have $A = k[\mathbb{G}_m] \cong k[x, x^{-1}], \varepsilon(x) = 1, s(x) = x^{-1}$.

## 8.3 Frobenius Kernels

Let $G$ be an algebraic group (scheme) over $k$, where char $(k) = p$. Let $F : G \to G$ be the Frobenius, where e.g.

$$\begin{aligned} F : \mathrm{GL}(n, \cdot\,) &\to \mathrm{GL}(n, \cdot\,) \\ (x_{ij}) &\mapsto (x_{ij}^p). \end{aligned}$$

Then $F$ is a map of group schemes.

**Definition 8.0.4** (Frobenius Kernels)**.**
$G_r \coloneqq \ker F^r$, where $F^r \coloneqq F \circ F \circ \cdots \circ F$ is the $r$-fold composition of the Frobenius.
This yields a nesting $G_1 \trianglelefteq G_2 \trianglelefteq G_3 \cdots \leq G$.

Recall that

$$\mathrm{Dist}(G) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \right\rangle.$$

We get a chain of finite dimensional algebras

$$\mathrm{Dist}(G_1) \leq \mathrm{Dist}(G_2) \leq \cdots \leq \mathrm{Dist}(G)$$

where

$$\mathrm{Dist}(G_1) = \left\langle \frac{x_\alpha^n}{n!}, \frac{y_\beta^m}{m!}, \binom{H_i}{k} \;\middle|\; 0 \leq n, m, k \leq p-1 \right\rangle,$$

where in general $\mathrm{Dist}(G_\ell)$ goes up to $p^\ell - 1$. Recall that $G_r$ representations were equivalent to $\mathrm{Dist}(G_r)$ representations.

Some basic questions (Curtis, Steinberg, 1960s):

1. What are the simple modules for Frobenius kernels? I.e., what are the irreducible representations for $G_r$?
2. How are the representations for $G_r$ related to those for $G$?

It turns out the representations for $G_r$ will lift to representations to $G$. Use "twisted tensor product" (Steinberg).

**Remark 13.**
It turns out that $G_1$ is special.

$$\mathrm{Dist}(G_1) \cong u(\mathfrak{g}) \coloneqq U(\mathfrak{g}) / \left\langle x^p - x^{[p]} \right\rangle,$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ is a *restricted lie algebra* (N. Jacobson). Note that for $D \in \mathfrak{g}$ a derivation, we define $D^{[p]} \coloneqq D \circ \cdots \circ D$ is the $p$-fold composition.

$G_1$-modules are equivalent to $\mathfrak{g}$-modules which are *restricted* in the sense that

$$\begin{aligned} \rho : g &\to \mathfrak{gl}(V) \\ x^{[p]} &\mapsto \rho(x)^p. \end{aligned}$$

# 9 | Friday, September 18

## 9.1 Frobenius Kernels

Let char $(k)p > 0$ and let $G$ be an algebraic group scheme. We have a Frobenius map $F : G \to G$ given by $F((x_{ij})) = (x_{ij}^p)$, which we can iterate to get $F^r$ for $r \in \mathbb{N}$. Setting $G_r = \ker F^r$ the $r$th Frobenius kernel, we get a normal series of group schemes

$$G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G.$$

There is an associated chain of finite dimensional Hopf algebras

$$\mathrm{Dist}(G_1) \leq \mathrm{Dist}(G_2) \leq \cdots \leq \mathrm{Dist}(G).$$

Then $k[G]^\vee = \mathrm{Dist}(G_r)$, and we get an equivalence of representations for $G_r$ to representations for $\mathrm{Dist}(G_r)$.

A special case will be when $G$ is a reductive algebraic group scheme. We'll start by finding a basis for $\mathrm{Dist}(G_r)$.

Recall the PBW theorem: we have a basis for $\mathfrak{g}$ given by

$$\begin{aligned} \left\{ x_\alpha \,\middle|\, \alpha \in \Phi^+ \right\} &\text{ Positive root vectors} \\ \left\{ h_i \,\middle|\, i = 1, \cdots, n \right\} &\text{ A basis for } t \\ \left\{ x_\alpha \,\middle|\, \alpha \in \Phi^- \right\} &\text{ Negative root vectors} \\ &. \end{aligned}$$

We can then obtain a basis for $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \left\langle \prod_{\alpha \in \Phi^+} x_\alpha^{n(\alpha)} \prod_{i=1}^n h_i^{k_i} \prod_{\alpha \in \Phi^+} x_{-\alpha}^{m(\alpha)} \right\rangle.$$

We can similarly obtain a basis for the distribution algebra

$$\mathrm{Dist}(G) = \left\langle \prod_{\alpha\in\Phi^+} \frac{x_\alpha^{n(\alpha)}}{n!} \prod_{i=1}^n \binom{h_i}{k_i} \prod_{\alpha\in\Phi^+} \frac{x_{-\alpha}^{n(\alpha)}}{n!} \right\rangle,$$

and we can similar get $\mathrm{Dist}(G_r)$ by restricting to $0 \le n(\alpha), k_i, m(\alpha) \le p^r - 1$. Above the $k_i$ are allowed to be any integers. This yields a triangular decomposition

$$\mathrm{Dist}(G_r) = \mathrm{Dist}(U_r^+)\mathrm{Dist}(T_r)\mathrm{Dist}(U_r^-),$$

where we'll denote the first two terms $\mathrm{Dist}(B_r^+)$ and the last two as $\mathrm{Dist}(B_r)$.

## 9.2 Induced and Coinduced Modules

Goal: Classify simple $G_r$-modules. We know the classification of simple $G$-modules, so we'll follow similar reasoning. We started by realizing $L(\lambda) \hookrightarrow \mathrm{Ind}_B^G \lambda$ as a submodule (the socle) of some "universal" module.

Let $M$ be a $B_r$-module, we can then define

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r},$$

where we're now taking the $B_r$-invariants. We get a decomposition as vector spaces,

$$k[G_r] = k[U_r^+] \otimes_k k[B_r]$$

and thus an isomorphism

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes M$$

since $k[B_r] \otimes M \cong \mathrm{Ind}_{B_r}^{B_r} M \cong M$.

We then define

$$\mathrm{Coind}_{B_r}^{G_r} = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \otimes M,$$

which is an analog of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M$.

We have $\mathrm{Dist}(U_r^+) \otimes \mathrm{Dist}(B_r) \cong \mathrm{Dist}(G_r)$, so

$$\mathrm{Coind}_{B_r}^{G_r} = \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \otimes M \cong \mathrm{Dist}(U_r^+) \otimes_k \mathrm{Dist}(B_r) \otimes_{\mathrm{Dist}(B_r)} M \cong \mathrm{Dist}(U_r^+) \otimes_k M,$$

which we'll define as the **coinduced module**.

We can compute the dimension:

$$\dim \mathrm{Ind}_{B_r}^{G_r} M = \dim \mathrm{Coind}_{B_r}^{G_r} M = (\dim M) p^{r\left|\Phi^+\right|}.$$

Open: don't know how to compute composition factors.

**Proposition 9.1** *(?).*

1. $$\operatorname{Coind}_{B_r}^{G_r} M \equiv \operatorname{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho,$$

   where the last term is a one-dimensional $B_r$-module and $\rho$ is the *Weyl weight*.
2. $$\operatorname{Coind}_{B_r^+}^{G_r} M \cong \operatorname{Ind}_{B_r^+}^{G_r} M \otimes -2(p^r - 1)\rho$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} w_i.$$

*Proof (Sketch for (1)).*
Since the tensor product satisfies a universal property, we have a map

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "B_r"'] & & \operatorname{Dist}(G_r) \otimes_{\operatorname{Dist}(B_r)} M \arrow[dl, dotted, "\exists \psi"] \\ & N = M \operatorname{Ind}_{B_r}^{G_r} \otimes 2(p^r - 1)\rho & \end{tikzcd}$$

1. We need to find a $B_r$ morphism $f : M \to N$.
2. We need to show that $f$ generates $N$ as a $G_r$-module.

Note that if (1) and (2) hold, then $\psi$ is surjective, but since $\dim \operatorname{Coind}_{B_r}^{G_r} M = \dim N$ this forces $\psi$ to be an isomorphism.
We can write

$$\begin{aligned} \operatorname{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho &= (k[G_r] \otimes M \otimes 2(p^r - 1)\rho)^{B_r} \\ &\cong \hom_{B_r}(\operatorname{Dist}(G_r), M \otimes 2(p^r - 1)\rho). \end{aligned}$$

Let $g_m(x) := m \otimes 2(p^r - 1)\rho$ for any $x = \displaystyle\prod_{\alpha \in \Phi^+} \frac{x_\alpha^{p^r - 1}}{(p^r - 1)!}$, and $g_m(x) = 0$ for any other $x$.
Now define $f(m) = g_m$, and check that $\operatorname{im} f$ generates $N$. ■

## 9.3 Verma Modules

Recall that $W(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda$ were the *Verma modules* for lie algebras.

Let $\lambda \in X(T)$, we have $T_r \leq T$ and restriction yields a map $X(T) \to X(T_r)$. Given a weight $\lambda$, we can write it $p$-adically as

$$\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2 + \cdots + \lambda_{r-1} + \cdots.$$

This yields an exact sequence

$$0 \to p^r X(T) \to X(T) \to X(T_r) \to 0,$$

and thus $X(T)/p^r X(T) \cong X(T_r)$.

Let $\lambda \in X(T_r)$, then $\lambda$ becomes a $B_r$-module by letting $U_r$ act trivially, since we have

$$\cdots U_r \to B_r \twoheadrightarrow T_r \to 0.$$

Set $Z(r) = \operatorname{Coind}_{B_r}^{G_r} \lambda$, and set $Z(r)' = \operatorname{Ind}_{B_r}^{G_r} \lambda$. Then $\dim Z_r(\lambda) = \dim Z_r'(\lambda) = p^{r\left|\Phi^+\right|}$. We'll then think of

- Coind $\twoheadrightarrow L_r(\lambda)$ being in the head,
- $L_r(\lambda) \hookrightarrow$ Ind being the socle.

Note that the dimensions aren't known, nor are the projective covers or injective hulls.

We have a form of translation invariance, namely

$$\begin{aligned} Z_r(\lambda + p^r \nu) &= Z_r(\lambda) \qquad \forall \nu \in X(T) \\ Z_r'(\lambda + p^r \nu) &= Z_r'(\lambda) \qquad \forall \nu \in X(T). \end{aligned}$$

**Proposition 9.2** ***(?).***
Let $\lambda \in X(T)$.

1. $Z_r(\lambda)\downarrow_{B_r}$ is the projective cover of $\lambda$ and the injective hull of $\lambda - 2(p^r - 1)\rho$.
2. $Z_r'(\lambda)\downarrow_{B_r^+}$ is the injective hull of $\lambda$ and the projective hull of $\lambda - 2(p^r - 1)\rho$.

# 10 | Monday, September 21

Let $G$ be a reductive algebraic group scheme, $k = \overline{\mathbb{F}}_p$ with $p > 0$, equipped with the Frobenius map $F : G \to G$ with $F^r$ its $r$-fold composition. We defined *Frobenius kernels* $G_r := \ker F^r$, which are in correspondence with the cocommutative Hopf algebras $\operatorname{Dist}(G_r)$.

Goal: We want to classify simple $G_r$-modules, and to do this we'll use socles.

We have a maximal torus $T \subseteq G$ and thus $T_r \subseteq G_r$ after acting by Frobenius. This yields a SES

$$0 \to p_r X(T) \to X(T) \to X(T)/p^r X(T) = X(T_r) \to 0.$$

How to think about this: take $\lambda \in X(T_r)$, then we can write $\lambda = \lambda + p^r \sigma$ in $X(T_r)$ for some other weight $\sigma \in X(T)$. We'll define the "baby Verma modules"

$$\begin{aligned} Z_r(\lambda) &:= \operatorname{Coind}_{B_r^+}^{G_r} \lambda \\ Z_r'(\lambda) &:= \operatorname{Ind}_{B_r^+}^{G_r} \lambda, \end{aligned}$$

and we have $\dim Z_r(\lambda) = \dim Z_r'(\lambda) = p^{r\left|\Phi^+\right|}$.

**Proposition 10.1** ***(?).***
Let $\lambda \in X(T)$ be a weight.

1. $Z_r(\lambda)\downarrow_{B_r}$ is the *projective cover* of $\lambda$ and the *injective hull* of $\lambda - 2(p^r-1)\rho$.
2. $Z_r'(\lambda)\downarrow_{B_r^+}$ is the *injective hull* of $\lambda$ and the *projective cover* of $\lambda - 2(p^r-1)\rho$.

Note the latter are $T_r$-modules, so we let $U^+$ act trivially.

*Proof (of 1).*
What we need to do:

1. Show $Z_r(\lambda)\downarrow_{B_r}$ is projective.
2. Show $Z_r(\lambda)$ is the smallest projective module such that $Z_r(\lambda) \twoheadrightarrow \lambda$.

For (1), we can write

$$\mathrm{Dist}(G_r) = \mathrm{Dist}(U_r^+)\mathrm{Dist}(B_r) = \mathrm{Dist}(B_r^+)\mathrm{Dist}(U_r), ,$$

and so

$$\begin{aligned} Z_r(\lambda) &= \mathrm{Coind}_{B_r^+}^{G_r} \lambda \\ &= \left(\mathrm{dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} \lambda\right) \downarrow_{B_r^+} \\ &= \mathrm{Dist}(U_r^+) \otimes \lambda \\ &= \mathrm{Dist}(B_r^+) \otimes_{\mathrm{Dist}(T_r)} \lambda \\ &= \mathrm{Coind}_{T_r}^{B_r^+} \lambda. \end{aligned}$$

Why is this projective? Look at cohomology, suffices to show that higher Exts vanish. So consider

$$\begin{aligned} \mathrm{Ext}^n_{B_r^+}(\mathrm{Coind}_{T_r}^{B_r^+}, M) &= \mathrm{Ext}^n_{T_r}(\lambda, M) \qquad \text{by Frobenius reciprocity} \\ &= 0 \qquad \text{for } n \geq 0, \end{aligned}$$

since representations for $T_r$ are completely reducible, and we've used the fact that $\mathrm{Coind}_{T_r}^{B_r^+}(\,\cdot\,)$ is exact.

Note: general algebra fact that higher exts vanish for projective modules.

For (2), we can write

$$\begin{aligned} \hom_{B_r^+}(Z_r(\lambda), \mu) &= \hom_{B_r^+}(\mathrm{Coind}_{T_r}^{B_r^+} \lambda, \mu) \\ &= \hom_{T_r}(\lambda, \mu) \qquad \text{by Frobenius reciprocity} \\ &= \begin{cases} k\&\lambda = \mu \\ 0\&\text{else}. \end{cases} \end{aligned}$$

Thus $Z_r(\lambda)/\mathrm{rad}\, Z_r(\lambda) \downarrow B_r^+ = \lambda$.

If we now write $A = \mathrm{Dist}(B_r^+)$ and $\mathfrak{g} = \mathfrak{n}^+ \oplus t \oplus \mathfrak{n}$ with $\mathfrak{b}^+ := \mathfrak{n}^+ \oplus t$,

$$\begin{aligned}\sum_S (\dim P(S))(\dim(S)) \\ &= \sum_{\lambda \in X(T_r)} (\dim Z_r(\lambda))(\dim \lambda) \\ &= \sum_{\lambda \in X(T_r)} p^{r\left|\Phi^+\right|} \cdot 1 \\ &= |X(T_r)| p^{r\left|\Phi^+\right|} \\ &= p^{rn} p^{r\left|\Phi^+\right|} \qquad n = \dim t \\ &= p^{r \dim \mathfrak{b}^+} \\ &= \dim A\end{aligned}$$

■

## 10.1 Simple $G$-modules

We know that after taking fixed points, $Z_r(\lambda)^{U_r}$ and $Z_r'(\lambda)^{U_r^+}$ are one-dimensional, and thus

$$Z_r(\lambda)/\mathrm{rad}\ Z_r(\lambda) \cong L_r(\lambda) \qquad \mathrm{Soc}_{G_r} Z_r'(\lambda) = L_r(\lambda)$$

following the same argument considering $H_0(\lambda)$.

For any $\lambda \in X(T_r)$ we have $0 \neq L_r = \mathrm{Soc}_{G_r} Z_r'(\lambda)$. By the one-dimensionality above, we know

$$L_r(\mu) = L_r(\lambda) \iff \lambda = \mu \in X(T_r).$$

Letting $N$ be a simple $G_r$-module, we can consider it as a $B_r$-module, and the simple $B_r$-modules are one dimensional and obtained from simple $T_r$-modules. We then know that for some $\lambda \in X(T_r)$,

$$\begin{aligned}0 \neq \hom_{B_r}(N, \lambda) \\ &= \hom_{G_r}(N, \mathrm{Ind}_{B_r}^{G_r} \lambda),\end{aligned}$$

which implies that $N \hookrightarrow \mathrm{Ind}_{B_r}^{G_r} \lambda = Z_r'(\lambda)$ as a submodule, and thus $N = L_r(\lambda)$.

**Theorem 10.2** ***(Main Theorem).***
Let $\Lambda$ be a set of representatives of $XX(T)/p^r X(T) \cong X(T_r)$. Then there exists a one-to-one correspondence

$$\Lambda \iff \{L_r(\lambda)\lambda \in \Lambda\},$$

where the RHS are simple $G_r$-modules.

How to think about this: **restricted regions**. Choose dominant weights as representatives

$$X_r(T) = \left\{ \lambda \in X(T)_+ \;\middle|\; 0 \leq \langle \lambda,\ \alpha^\vee \rangle < p^r \,\forall \alpha \in \Delta \right\}$$
$$= \left\{ \lambda \in X(T)_+ \;\middle|\; \lambda = \sum_{i=1}^{\ell} n_i w_i,\, 0 \leq n_j \leq p^r - 1 \,\forall j \right\}$$

.

Pictures:

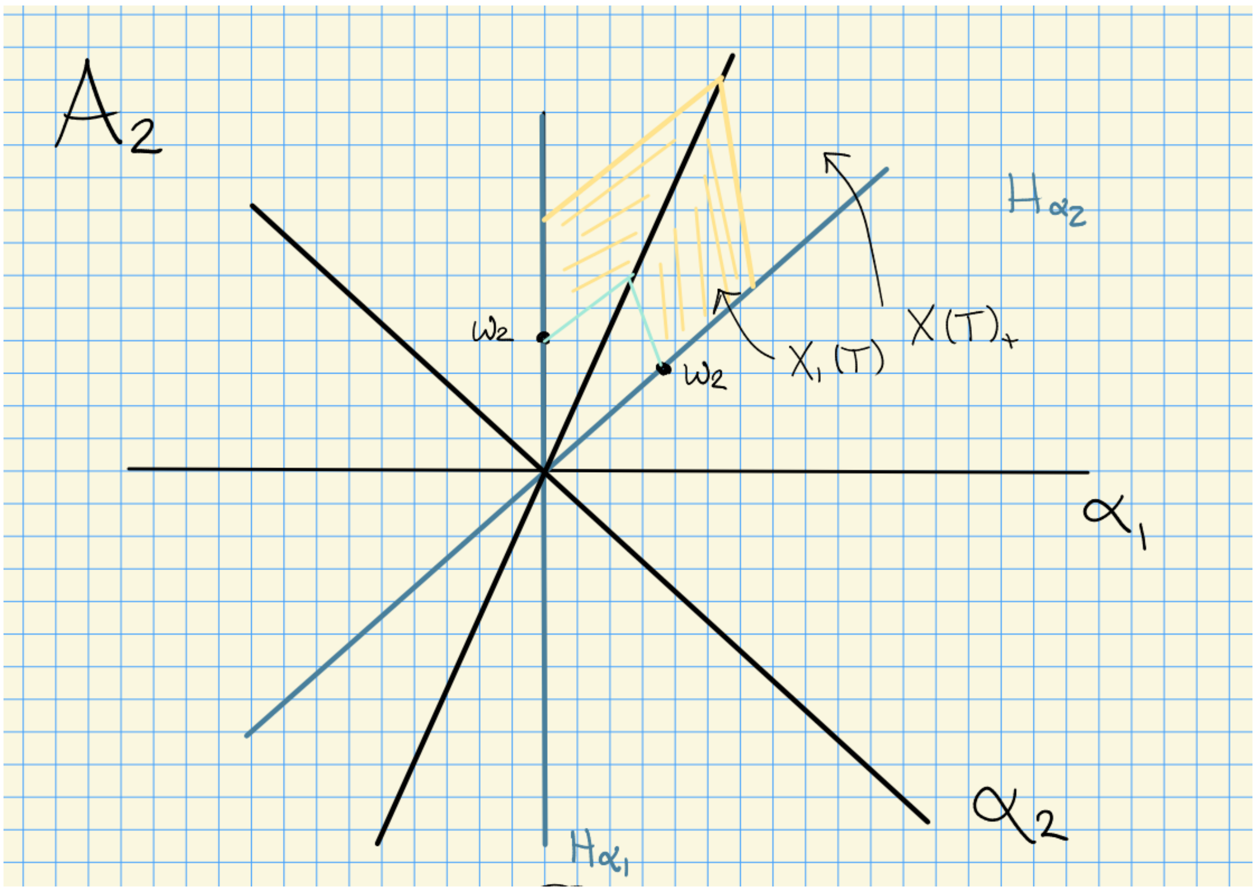


Figure 10: Root systems, chambers formed by dominant weights

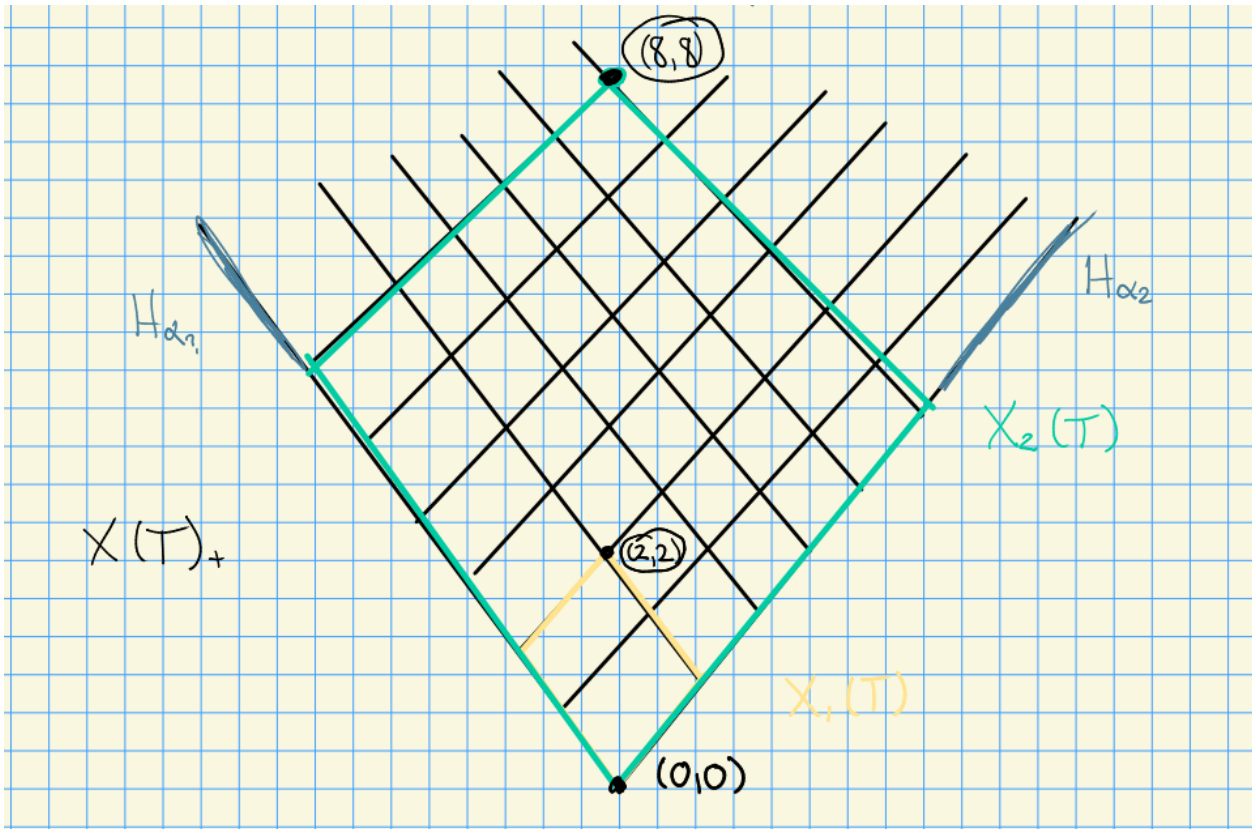


Figure 11: Restricted regions

Some facts:

If $\lambda \in X(T)_+$, then $L(\lambda)$ is a simple $G$-module.

**Question 1**: What happens when we restrict $L(\lambda) \downarrow_{G_r}$?

**Answer**: This remains irreducible over $G_r$ iff $\lambda \in X_r(T)$, i.e. if $L(\lambda) \downarrow_G \cong L_r(\lambda)$ when $\lambda \in X_r(T)$.

**Question 2**: Given $L(\lambda)$ for $\lambda \in X(T)_+$, can we express $L(\lambda)$ in terms of simple $G_r$-modules?

**Answer**: Yes, can be formulated in terms of *Steinberg's twisted tensor product.*