Title

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1 Friday, August 28

1.1 Representation Theory

Review: let $\mathfrak{g}$ be a semisimple lie algebra $/\mathbb{C}$. There is a decomposition $\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^- = \mathfrak{n}^+ \oplus t \oplus \mathfrak{n}^-$, where t is a torus. We associate $U(\mathfrak{g})$ the universal enveloping algebra, and representations of $\mathfrak{g}$ correspond with representations of $U(\mathfrak{g})$.

Let $\lambda \in X(T)$ be a weight, then λ is a $U(\mathfrak{b}^+)$-module. We can write $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda$.

Remark 1.
There exists a unique maximal submodule of $Z(\lambda)$, say $RZ(\lambda)$ where $Z(\lambda)/RZ(\lambda) \cong L(\lambda)$ is an irreducible representation of $\mathfrak{g}$.

Theorem 1.1 ***(?).***
Let $L = L(\lambda)$ be a finite-dimensional irreducible representation for $\mathfrak{g}$. Then
1. $L \cong Z(\lambda)/RZ(\lambda)$ for some λ.
2. $\lambda \in X(T)_+$ is a dominant integral weight.

1.1.1 Induction

Let $\mathfrak{g}$ be an algebraic group $/k$ with $k = \bar{k}$, and let $H \leq G$. Let M be an H-module, we'll eventually want to produce a G-modules.

Step 1: Make M into a $G \times H$ where the first component $(g, 1)$ acts trivially on M.

Taking the coordinate algebra $k[G]$, this is a $(G-G)$-bimodule, and thus becomes a $G \times H$-module. Let $f \in k[G]$, so $f : G \longrightarrow K$, and let $y \in G$. The explicit action is

$$[(g,h)f](y) \coloneqq f(g^{-1}yh).$$

Note that we can identify $H \cong 1 \times H \leq G \times H$. We can form $(M \otimes_k k[G])^H$, the H-fixed points.

Exercise 1.1.
Let N be an A-module and $B \trianglelefteq A$, then N^B is an A/B-module.

Hint: the action of B is trivial on N^B. Here $N^B \coloneqq \left\{ n \in N \;\middle|\; b.n = n \,\forall b \in B \right\}$

Definition 1.1.1 (Induction)**.**
The *induced module* is defined as

$$\operatorname{Ind}_H^G(M) \coloneqq (M \otimes k[G])^H.$$

1.1.2 Properties of Induction

1. $(\cdot \otimes_k k[G]) = \hom_H(k, \cdot \otimes_k k[G])$ is only *left-exact*, i.e.

$$(0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0) \mapsto (0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow \cdots).$$

2. By taking right-derived functors $R^j F$, you can take cohomology.

Note that in this category, we won't have enough projectives, but we will have enough injectives.

3. This functor commutes with direct sums and direct limits.
4. (**Important**) Frobenius Reciprocity: there is an adjoint, *restriction*, satisfying

$$\hom_G(N, \operatorname{Ind}_H^G M) = \hom_H(N \downarrow_H, M).$$

5. (Tensor Identity) If $M \in \operatorname{Mod}(H)$ and additionally $M \in \operatorname{Mod}(G)$, then $\operatorname{Ind}_H^G = M \otimes_k \operatorname{Ind}_H^G k$.

If $V_1, V_2 \in \operatorname{Mod}(G)$ then $V_1 \otimes_k V_2 \in \operatorname{Mod}(G)$ with the action given by $g(v_1 \otimes v_2) = gv_1 \otimes gv_2$.

6. Another interpretation: we can write

$$\operatorname{Ind}_H^G(M) = \left\{ f \in \operatorname{Hom}(G, M_a) \;\middle|\; f(gh) = h^{-1} \cdot f(g) \,\forall g \in G, h \in H \right\} \qquad M_a = M \coloneqq \mathbb{A}^{\dim M}.$$

I.e., equivariant wrt the H-action.

Then G acts on $\operatorname{Ind}_H^G M$ by left-translation: $(gf)(y) = f(g^{-1}y)$.

7. There is an evaluation map:

$$\begin{aligned} \varepsilon : \operatorname{Ind}_H^G(M) &\longrightarrow M \\ f &\mapsto f(1). \end{aligned}$$

This is an H-module morphism. Why? We can check

$$\begin{aligned} \varepsilon(h.f) &:= (h.f)(a) \\ &= f(h^{-1}) \\ &= hf(1) \\ &= h(\varepsilon(f)). \end{aligned}$$

We can write the isomorphism in Frobenius reciprocity explicitly:

$$\begin{aligned} \hom_G(N, \operatorname{Ind}_H^G M) &\xrightarrow{\cong} \hom_H(N, M) \\ \varphi &\mapsto \varepsilon \circ \varphi. \end{aligned}$$

8. Transitivity of induction: for $H \leq H' \leq G$, there is a natural transformation (?) of functors:

$$\operatorname{Ind}_H^G(\,\cdot\,) = \operatorname{Ind}_{H'}^G\left(\operatorname{Ind}_H^{H'}(\,\cdot\,)\right).$$

Equality as a composition of functors?

1.2 Classification of Simple G-modules

Suppose G is a connected reductive algebraic group $/k$ with $k = \bar{k}$.

Example 1.1.
Let $G = \mathrm{GL}(n, k)$. There is a decomposition:

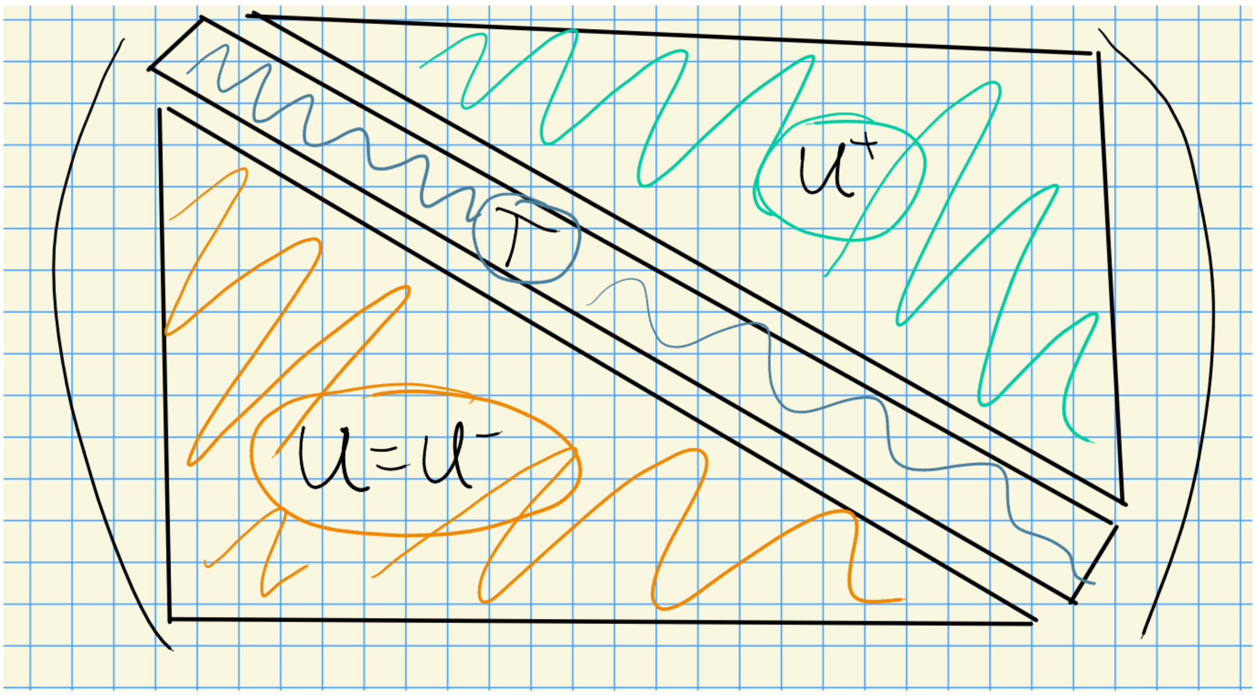

Figure 1: Image

Step 1: Getting modules for U.

Then there's a general fact: $U^+TU \hookrightarrow G$ is dense in the Zariski topology for any reductive algebraic group. We can form

- $B^+ \coloneqq T \rtimes U^+$, the *positive borel*,
- $B^- \coloneqq T \rtimes U$, the *negative borel*,

Suppose we have a U-module, i.e. a representation $\rho : U \longrightarrow \mathrm{GL}(V)$. We can find a basis such that $\rho(u)$ is upper triangular with ones on the diagonal. In this case, there is a composition series with 1-dimensional quotients, and the composition factors are all isomorphic to k.

Moral: for unipotent groups, there are only trivial representations, i.e. the only simple U-modules are isomorphic to k.

Step 2: Getting modules for B.

Modules for B are solvable, in which case we can find a flag. In this case, $\rho(b)$ embeds into upper triangular matrices, where the diagonal action may now not be trivial (i.e. diagonal is now arbitrary).

Thus simple B-modules arise by taking $\lambda \in X(T) = \hom(T, \mathbb{G}_m) = \hom(T, \mathrm{GL}(1, k))$, then letting u act trivially on λ, i.e. $u.v = v$. Here we have $B \longrightarrow B/U = T$, so any T-module can be pulled back to a B-module.

Step 3: Getting modules for G.

Let $\lambda \in X(T)$, then $H^0(\lambda) = \mathrm{Ind}_B^G \lambda = \nabla(\lambda)$.