

# Full Notes

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## 1 Friday January 10

Recall that  $\mathbb{C}$  is a field, where  $z = x + iy \implies \bar{z} = x - iy$ , and if  $z \neq 0$  then  $z^{-1} = \bar{z}/|z|^2$ .

Lemma (Triangle Inequality):  $|z + w| \leq |z| + |w|$

Proof:

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\bar{w}| - \Re z\bar{w}) \geq 0.$$

Lemma (Reverse Triangle Inequality):  $||z| - |w|| \leq |z - w|$ .

Proof:

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

Claim:  $(\mathbb{C}, |\cdot|)$  is a normed space.

Definition:  $\lim z_n = z \iff |z_n - z| \rightarrow 0 \in \mathbb{R}$ .

Definition: A disc is defined as  $D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$ , and a subset is open iff it contains a disc. By convention,  $D_r$  denotes a disc about  $z_0 = 0$ .

Definition:  $\sum_k z_k$  converges iff  $S_N := \sum_{|k| < N} z_k$  converges.

Note that  $z_n \rightarrow z$  and  $z_n = x_n + iy_n$ , if  $|z_n - z| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$  then  $|x - x_n|, |y - y_n| < \varepsilon$ .

Since  $\mathbb{R}$  is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

Note: This is useful precisely when you don't know the limiting term.

Note that  $\sum_k z_k$  thus converges if  $\left| \sum_{k=m}^n z_k \right| < \varepsilon$  for  $m, n$  large enough, so sums converges iff they have small tails.

Definition:  $\sum z_k$  converges absolutely iff  $\sum |z_k|$  converges.

Note that the partial sums  $\sum_{k=1}^N |z_k|$  are monotone, so  $\sum |z_k|$  converges iff the partial sums are bounded above.

Definition: A sum of the form  $\sum_{k=0}^{\infty} a_k z_k$  is a power series.

Examples

$$\sum x^k = \frac{1}{1-x}$$

$$\sum (-x^2)^k = \frac{1}{1+x^2}.$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at  $x = 1$  and the second as a pole at  $x = i$ .

## 2 Monday January 13th

Recall that  $\sum z_k$  converges iff  $s_n = \sum_{k=1}^n z_k$  converges.

**Lemma:** Absolute convergence implies convergence.

The most interesting series:  $f(z) = \sum a_k z^k$ , i.e. power series.

**Divergence lemma:** If  $\sum z_k$  converges, then  $\lim z_k = 0$ .

*Corollary:* If  $\sum z_k$  converges,  $\{z_k\}$  is uniformly bounded by a constant  $C > 0$ , i.e.  $|z_k| < C$  for all  $k$ .

**Proposition:** If  $\sum a_k z_k$  converges at some point  $z_0$ , then it converges for all  $|z| < |z_0|$ .

The inequality is necessarily strict. For example,  $\sum \frac{z^{n-1}}{n}$  converges at  $z = -1$  (alternating harmonic series) but not at  $z = 1$  (harmonic series).

Proof: Suppose  $\sum a_k z_1^k$  converges. The terms are uniformly bounded, so  $|a_k z_1^k| \leq C$  for all  $k$ . Then we have  $|a_k| \leq C/|z_1|^k$ , so if  $|z| < |z_1|$ , we have  $|a_k z^k| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k$ . So if  $|z| < |z_1|$ , the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting  $r = |z|/|z_1|$ , we have

$$\sum |a_k z^k| \leq \sum C r^k = \frac{C}{1-r},$$

and so we have absolute convergence. ■

*Exercise (future problem set):* Show that  $\sum \frac{1}{k} z^{k-1}$  converges for all  $|z| = 1$  except for  $z = 1$ . (Use summation by parts.)

**Definition** The radius of convergence is the real number  $R$  such that  $f(z) = \sum a_k z^k$  converges precisely for  $|z| < R$  and diverges for  $|z| > R$ . We denote a disc of radius  $R$  centered at zero by  $D_R$ .

If  $R = \infty$ , then  $f$  is said to be *entire*.

**Proposition:** Suppose that  $\sum a_k z^k$  converges for all  $|z| < R$ . Then  $f(z) = \sum a_k z^k$  is continuous on  $D_R$ , i.e. using the sequential definition of continuity,  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$  for all  $z_0 \in D_R$ .

Recall that  $S_n(z) \rightarrow S(z)$  uniformly on  $\Omega$  iff  $\forall \varepsilon > 0$ , there exists a  $M \in \mathbb{N}$  such that  $n > M \implies |S_n(z) - S(z)| < \varepsilon$  for all  $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample:  $f_n(x) = x^n$  on  $[0, 1]$ ; then  $f_n \rightarrow \delta(1)$ . Note that it uniformly converges on  $[0, 1 - \varepsilon]$  for any  $\varepsilon > 0$ .

*Exercise:* Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

Proof of proposition: Write  $f(z) = \sum_{k=0}^N a_k z^k + \sum_{k=N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$ . Note that if  $|z| < R$ , then there exists a  $T$  such that  $|z| < T < R$  where  $f(z)$  converges uniformly on  $D_T$ .

Check!

We need to show that  $|R_N(z)|$  is uniformly small for  $|z| < s < T$ . Note that  $\sum a_k z^k$  converges on  $D_T$ , so we can find a  $C$  such that  $|a_k z^k| \leq C$  for all  $k$ . Then  $|a_k| \leq C/T^k$  for all  $k$ , and so

$$\begin{aligned}
\left| \sum_{k=N+1}^{\infty} a_k z^k \right| &\leq \sum_{k=N+1}^{\infty} |a_k| |z|^k \\
&\leq \sum_{k=N+1}^{\infty} (c/T^k) s^k \\
&= c \sum_{k=N+1}^{\infty} |s/T|^k \\
&= c \frac{r^{N+1}}{1-r} = C\varepsilon_n \rightarrow 0,
\end{aligned}$$

which follows because  $0 < r = s/T < 1$ .

So  $S_N(z) \rightarrow f(z)$  uniformly on  $|z| < s$  and  $S_N(z)$  are all continuous, so  $f(z)$  is continuous.

There are two ways to compute the radius of convergence:

- Root test:  $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$ .
- Ratio test:  $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$ .

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

### 3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

#### 3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function  $f : \Omega \rightarrow \mathbb{C}$
2. Complex-differentiability: For  $\Omega \subset \mathbb{C}$  open and  $z_0 \in \Omega$ , there exists  $\varepsilon > 0$  such that  $D_\varepsilon = \{z \mid |z - z_0| < \varepsilon\} \subset \Omega$ , and  $f$  is **holomorphic** (complex-differentiable) at  $z_0$  iff

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(z_0 + h) - f(z_0))$$

exists; if so we denote it by  $f'(z_0)$ .

*Example:*  $f(z) = z$  is holomorphic, since  $f(z+h) - f(z) = z+h - z = h$ , so  $f'(z_0) = \frac{h}{h} = 1$  for all  $z_0$ .

*Example:* Given  $f(z) = \bar{z}$ , we have  $f(z+h) - f(z) = \bar{h}$ , so the ratio is  $\frac{\bar{h}}{h}$  and the limit doesn't exist (?).

We say  $f$  is holomorphic on an open set  $\Omega$  iff it is holomorphic at every point, and is holomorphic on a closed set  $C$  iff there exists an open  $\Omega \supset C$  such that  $f$  is holomorphic on  $\Omega$ .

If  $f$  is holomorphic, writing  $h = h_1 + ih_2$ , then the following two limits exist and are equal:

$$\begin{aligned} \lim_{h_1 \rightarrow 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\ \lim_{h_2 \rightarrow 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i} \frac{\partial f}{\partial y}(x_0, y_0) \\ &\implies \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}. \end{aligned}$$

So if we write  $f(z) = u(x, y) + iv(x, y)$ , we have

$$\left. \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} = \frac{1}{i} \left( \left. \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right|_{(x_0, y_0)} \right),$$

and equating real and imaginary parts yields the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ \iff \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{aligned}$$

The usual rules of derivatives apply:

1.  $(\sum f)' = \sum f'$

Proof: Direct.

2.  $(\prod f)' = \text{product rule}$

Proof: Consider  $(f(z+h)g(z+h) - f(z)g(z))/h$  and use continuity of  $g$  at  $z$ .

3. Quotient rule

Proof: Nice trick, write  $q = \frac{f}{g}$  so  $qg = f$ , then  $f' = q'g + qg'$  and  $q' = \frac{f'}{g} - \frac{fg'}{g^2}$ .

4. Chain rule

Proof: Use the fact that if  $f'(g(z)) = a$ , then  $f(z+h) - f(z) = ah + r(z, h)$  with  $|r(z, h)| = o(|h|) \rightarrow 0$ . Write  $b = g'(z)$ , then  $f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$  by considering error terms, and so  $\frac{1}{h}(f(g(z+h)) - f(g(z))) \rightarrow f'(g(z))g'(z)$ .

Standard forms of conic sections:

- Circle:  $x^2 + y^2 = r^2$
- Ellipse:  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola:  $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
- Rectangular Hyperbola:  $xy = \frac{c^2}{2}$ .
- Parabola:  $-4ax + y^2 = 0$ .

Mnemonic: Write  $f(x, y) = Ax^2 + Bxy + Cy^2 + \dots$ , then consider the discriminant  $\Delta = B^2 - 4AC$ :

- $\Delta < 0 \iff$  ellipse  
     –  $\Delta < 0$  and  $A = C, B = 0 \iff$  circle
- $\Delta = 0 \iff$  parabola
- $\Delta > 0 \iff$  hyperbola

Completing the square:

$$x^2 - bx = (x - s)^2 - s^2 \quad \text{where } s = \frac{b}{2}$$

$$x^2 + bx = (x + s)^2 - s^2 \quad \text{where } s = \frac{b}{2}.$$

Properties of complex numbers

- $\Re(z) = \frac{1}{2}(z + \bar{z})$  and  $\Im(z) = \frac{1}{2i}(z - \bar{z})$ .
- $z\bar{z} = |z|^2$