# Title

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**1 Joint GT/UGA Topology Seminar: Monday January 13th** **1**

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# 1 Joint GT/UGA Topology Seminar: Monday January 13th

## 1.1 Talk 1: Knot Floer Homology and Cosmetic Surgeries

Dehn surgery: fundamental procedure for building 3-manifolds.

Outline

- Background on problem
- Results (known and new)
- Tools
- Proof

### 1.1.1 Basics

Let $K \hookrightarrow S^3$ be a knot. Pick a rational number $p/q$ or $\infty$. Can perform $p/q$-surgery (i.e. Dehn surgery) to obtain $S^3_{p/q}(K) \coloneqq (S^3 \setminus K)\coprod_f (D^2 \times S^1)$ where $\partial D^2 \times \{\text{pt}\} \mapsto -p\mu + q\lambda$ where $\mu = ?, \lambda$ is the Seifert fiber (?).

Is this a unique operation? I.e. do different knots yield different 3-manifolds?

Question: Can different surgeries on the same knot yield different 3-manifolds?

**Definition:** Two surgeries are *purely cosmetic* iff there is an orientation-preserving diffeomorphism between them. If there is an orientation *reversing* diffeomorphism, they are said to be chirally cosmetic.

**Conjecture:** There are no purely cosmetic surgeries.

Remark: The conjecture can be stated for $K \hookrightarrow Y^3$

Note: don't know what $Y^3$ is.

Remark: There exist chirally cosmetic surgeries.

Example: $+9$ and $+\frac{9}{2}$ surgery on $T_{2,3}$, or $+r, -r$ on any ampichiral knot.

Remark: Meant to generalize the "knot complement problem", i.e. are knots determined by their complements?

Theorem (Gordon-Luecke 89): If $S^3_r(K) = S^3 = S^3_\infty(K)$, then $r = \infty$. (I.e. the only trivial surgery really is the trivial surgery?)

### 1.1.2 Known Results

Suppose $K \subset S^3$is nontrivial, and two Dehn surgeries with different slopes are diffeomorphic.

- Computing $H^1 = \mathbb{Z}_p$ forces $p = p'$.
- By Boyer-Lines '91, the Alexander polynomial satisfies $\Delta_k''(1) = 0$.
- By Osvath-Szabo-Wu ('05, '09), $q$ and $q'$ have opposite signs (not necessarily $q = -q'$).
  - So there at most two ways of getting the same manifold from cosmetic surgeries.
- By Wang '06, $g \neq 1$
- By Ni-Wu '10, $\tau(K) = 0, q' = -q$, and $q^2 = -1 \mod p$.

**Theorem:** Let $q > 0$, so $q' < 0$. Then

- $p = 1, 2$
- If $p = 2$, then $q = 1$ and $g = 2$.
- If $p = 1$, then $q \leq \frac{t + 2g}{2g(g-1)}$

where $g$ is the genus and $t$ is the *Heegard-Floer thickness*.

Moreover, the knot Floer homology satisfies some further conditions (stronger than e.g. $\tau(K) = 0$).

Note that if $t < 4$, then the last condition forces $q = 1, g = 2$. We then only have to consider two slopes.

Corollary: The corollary holds for *thin knots* (i.e. thickness zero), e.g. all alternating and quasialternating knots.

For knots up to 16 crossings, $t \leq 2$ (from computations of knot-Floer homology on 1.6 million knots?) When $K$ is thin, this condition can be stated in terms of the Alexander polynomial.

**Theorem:** If $K$ is thin and has purely cosmetic surgery, then

- $g(K) = 2$
- The slopes are $\pm 1, \pm 2$
- The coefficients of the Alexander polynomial occur in ratios: $\Delta_k(t) = nt^2 - 4nt + (6n + 1) - 4nt^{-1} + nt^{-2}$ for some $n$,

This is computationally effective:

- Number of prime knots with at most 16 crossings: 1.7 million
- Number with $\tau = 0$: 450,000
- Number satisfying the conditions in the theorem: 337

For each of these, need to consider $\pm 1, \pm 2$. Noting from $HF$ will distinguish these surgeries. Can use SnapPea to compute hyperbolic invariants – most are distinguished by hyperbolic volume, or Chern-Simons invariant.

Could potentially take connect sums of the above knots, but eventually they stop satisfying the necessary condition. In fact, the conjecture holds if all prime summands of $K$ have less than 16 crossings.

**Theorem (Ichihara-Song-Mattman-Saito):** The conjecture holds for all 2-bridge knots.

**Theorem (Tao):** Conjecture holds for arbitrary connected sums.

> So if there is a knot with cosmetic surgery, it is not prime.

Remark: Futer-Purcell-Schleimer independently proved a similar result using hyperbolic techniques.

### 1.1.3 Tools

What we'll want

1. Need some 3-manifold invariant to distinguish different surgeries
2. A knot invariant
3. A surgery formula computing (1) from (2) and the slope $p, q$.

Previously used

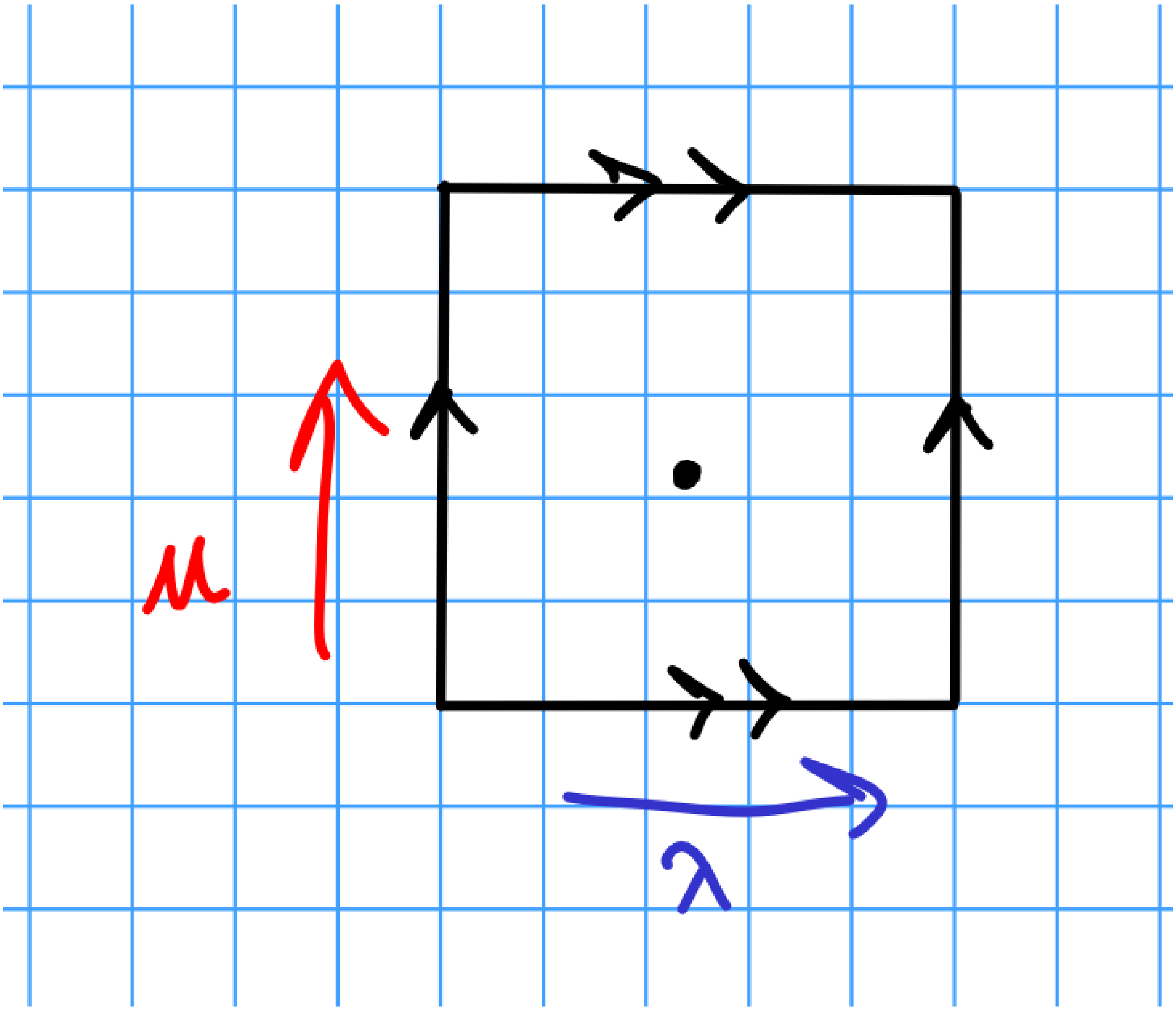
- Cassem-Gordon and Cassem-Walker invariants, and
- Alexander polynomial ( $\implies \Delta''(1) = 0$)

For (1), we'll use Heegard-Floer homology for the 3-manifold invariant: Associated to a closed oriented 3-manifold $Y$ a graded vector space $\widehat{HF}(Y)$. In our case, it will be over $\mathbb{Z}/(2)$, and splits over $\mathrm{Spin}^c$ structures as $\widehat{HF}(Y) = \bigoplus_{s \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y; s)$.

> Note that $\mathrm{Spin}^c(Y)$ can be put in correspondence with $H^1(Y)$.

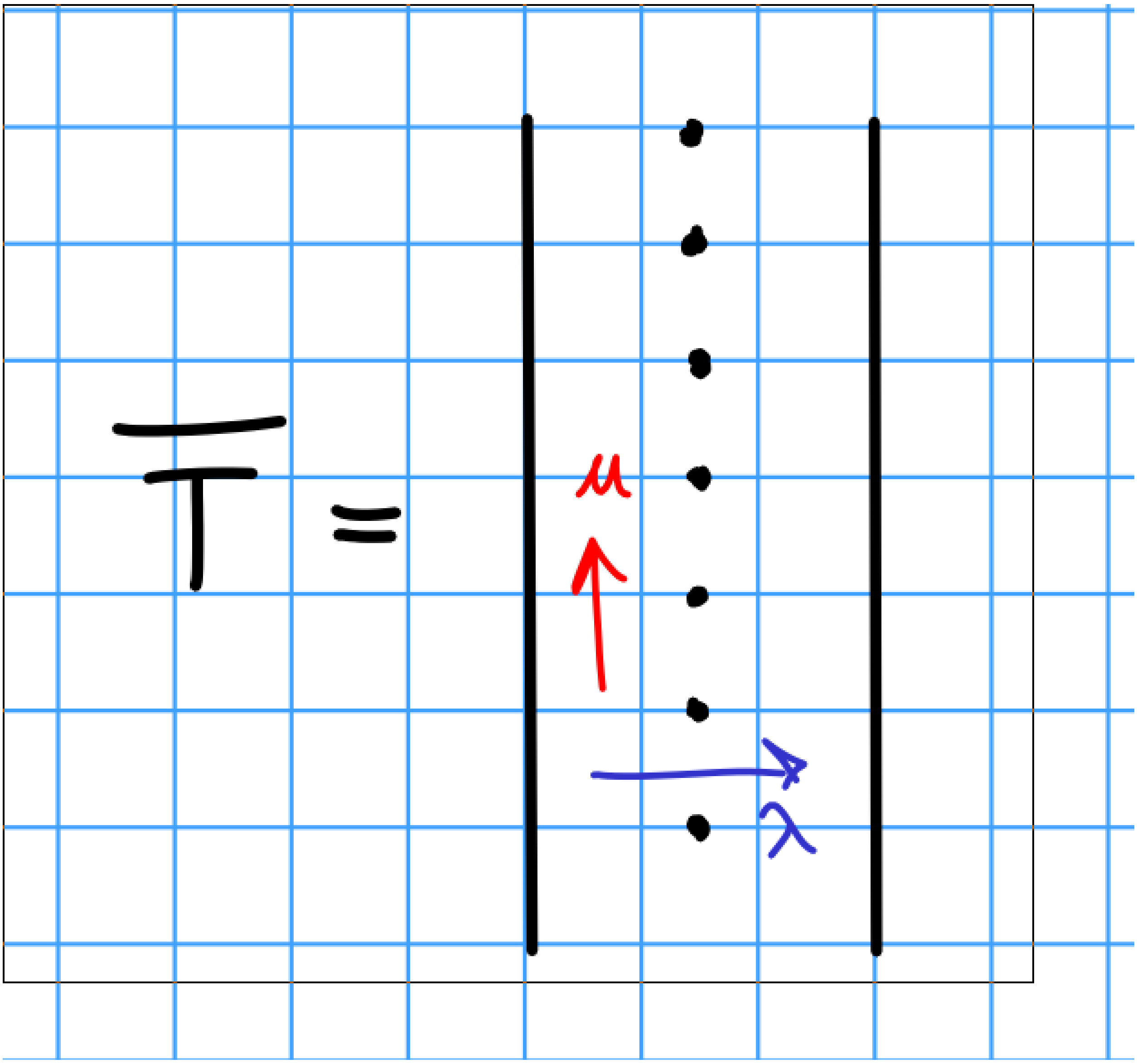
For (2), we'll use knot Floer homology, namely a reformulation following H-Rasmussen-Watson. To a knot $K \subset S^3$, associate

- An immersed collection of closed curves $\Gamma = (\gamma_0, \cdots, \gamma_n)$ in the punctured torus $T$:
  - I.e. a graded immersed Lagrangian
  - Defined up to homotopy equivalence, where homotopies can't cross the puncture.




- Some grading data (×2, Alexander and Maslov) amounting to labeling each component of $\Gamma$ with an integer. (Important to proof!)
- (From world of immersed Lagrangians) A bounding cochain, i.e. a subset of self-intersection points of $\Gamma$.

Interpret the Alexander grading as specifying a lift $\overline{\Gamma}$ of $\Gamma$ from $T$ to $\overline{T}$, a $\mathbb{Z}$-fold covering space of $T$:




Examples (These are curves wrapped around cylinders):

Somehow, this last example is representative.

A surgery formula: $\widehat{HF}(S^3_{p/q}(K))$ is floer homology in $T$ of $\Gamma$ with $\ell_{p,q}$ a line of slope $p/q$, i.e. it is generated by minimal intersection points $\Gamma \bigcap \ell_{p,q}$. (This gives a chain complex, count bi-gons.)

For the spin$^c$ decomposition, look at $\overline{T}$ with different lifts of $\ell_{p,q}$.

## 1.2 Talk 2: Branched Covers Bounding $\mathbb{Q}HB^4$

Joint work with Aceto, Meier, A. Miller, M. Miller, Stipsicz.

**Definition:** Two knots $K_0, K_1$ are **concordant** iff they cobound an annulus, i.e. there exists a smooth cylinder $S^1 \times I$ embedded in $S^3 \times I$ such that $S^1 \times \{i\}$ in $S^3 \times \{i\}$ is $K_i$.

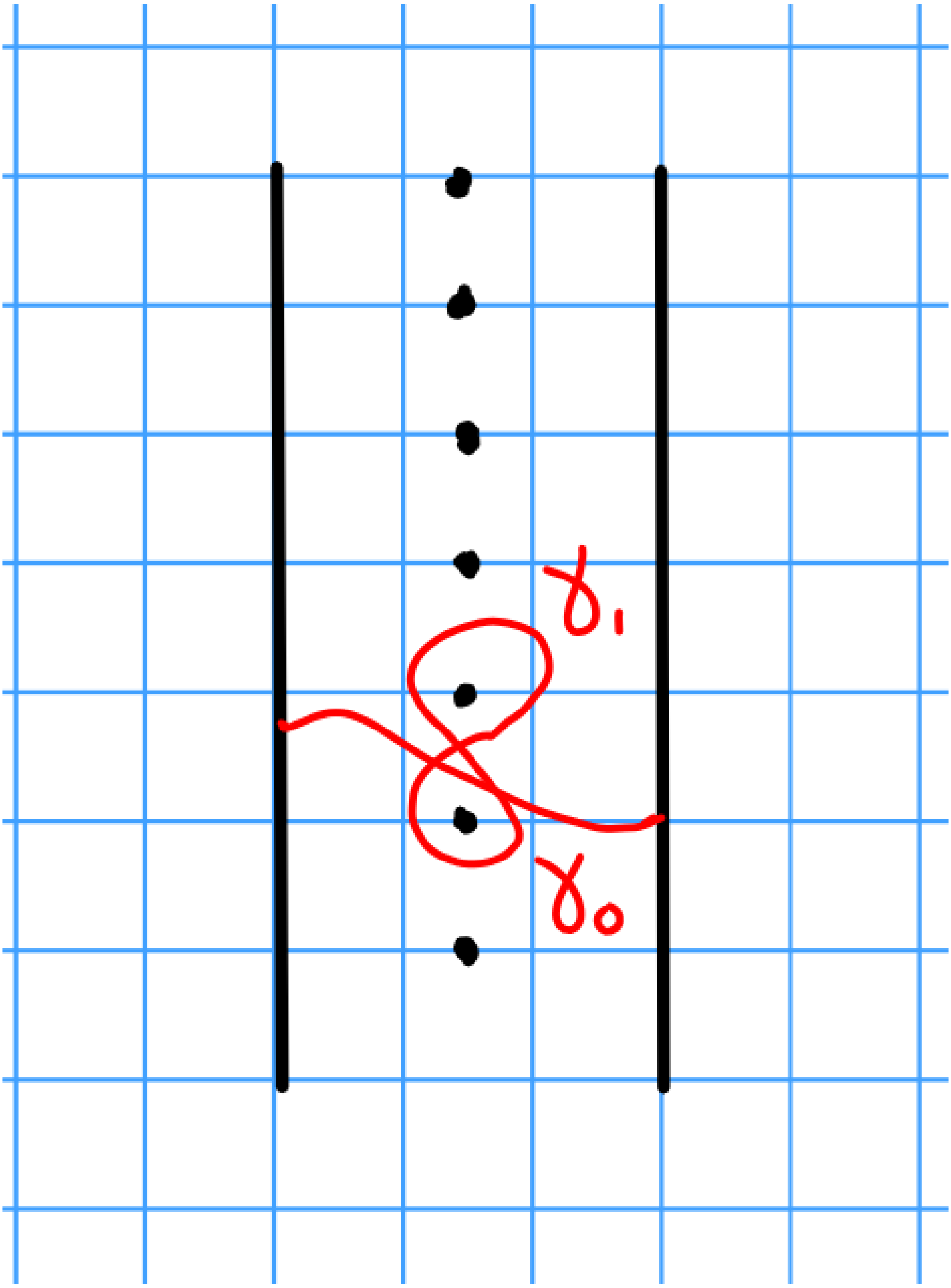



Figure 1: The unknot

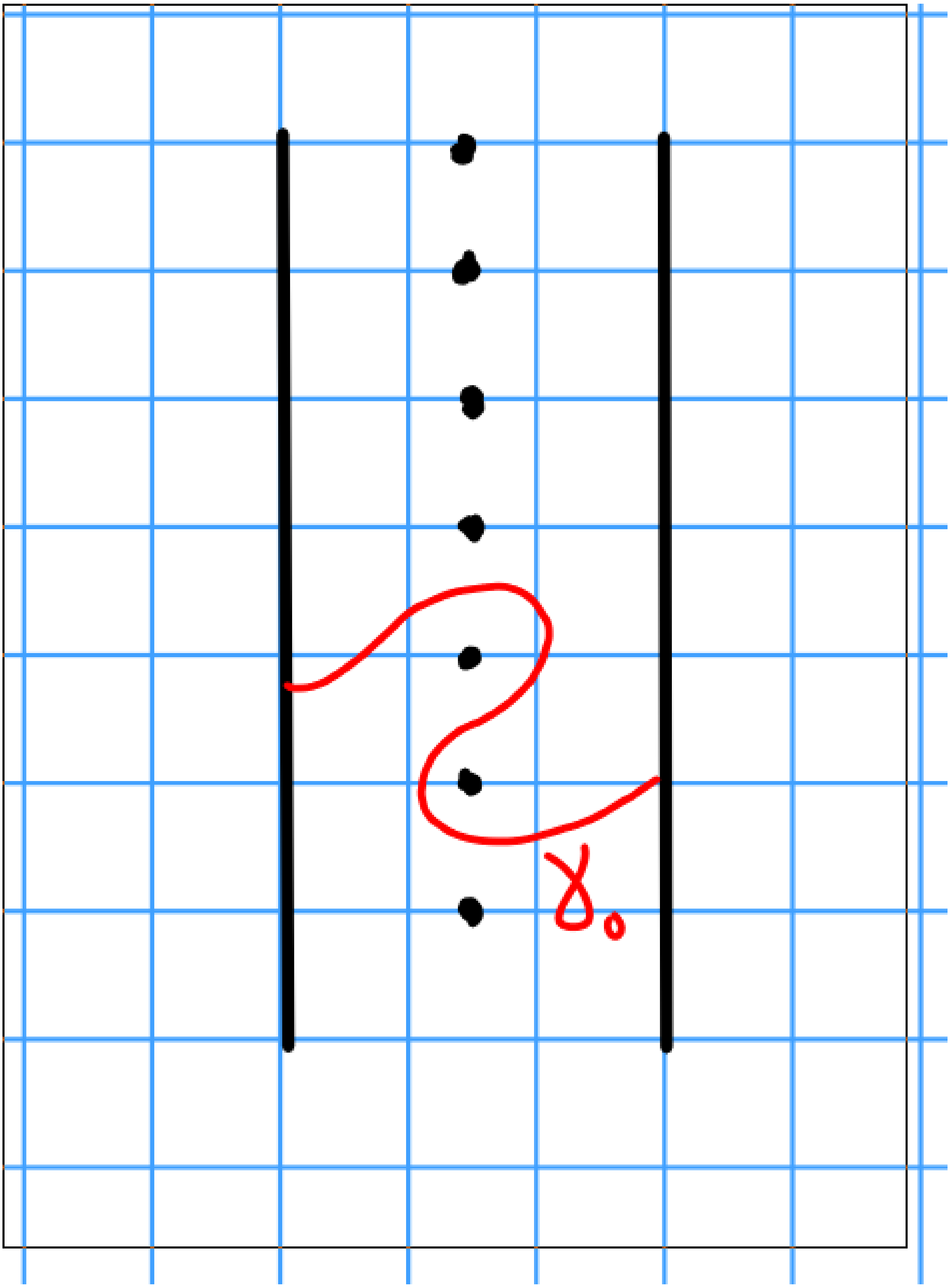



Figure 2: $T_{2,3}$

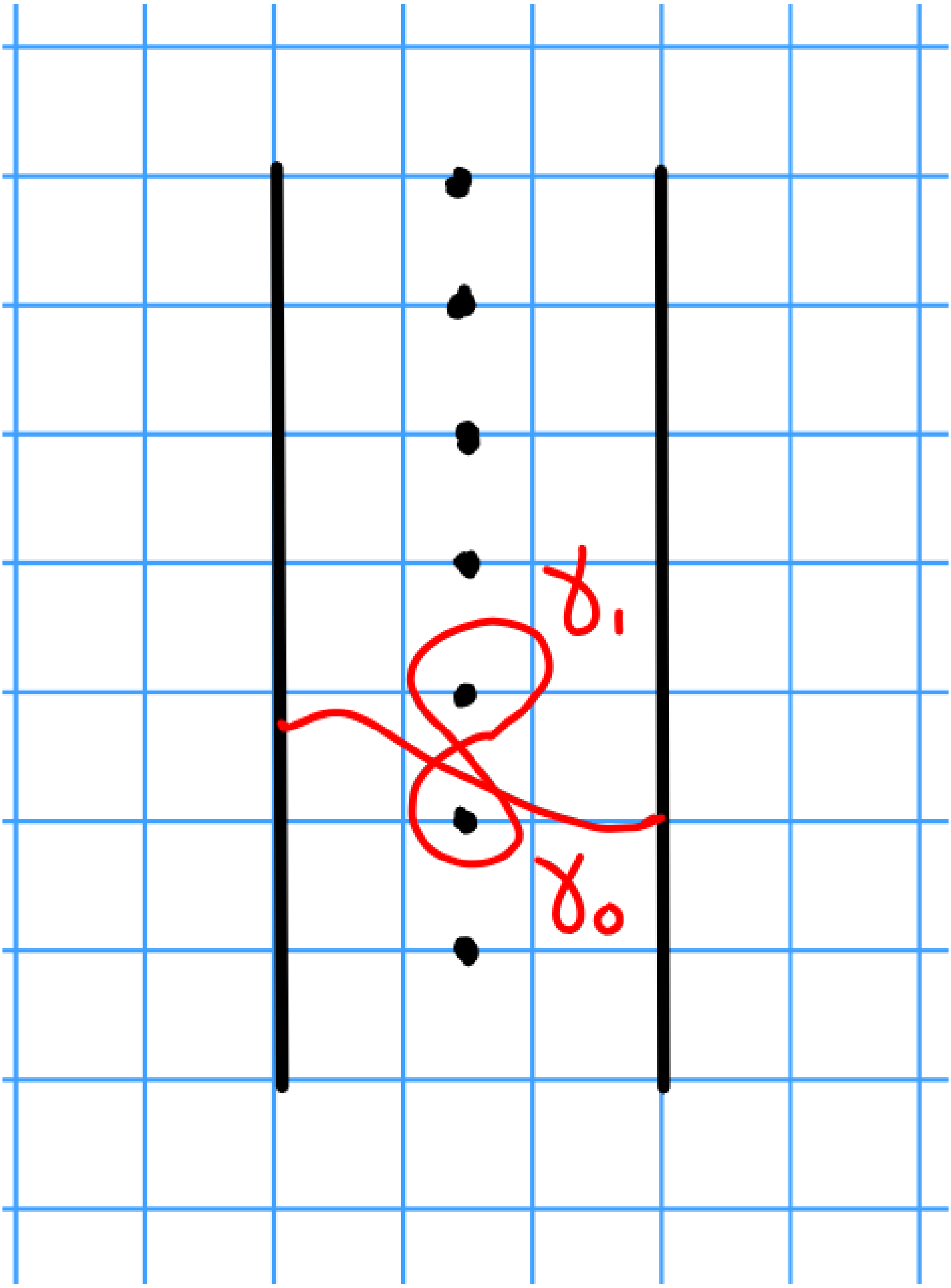



Figure 3: Figure 8

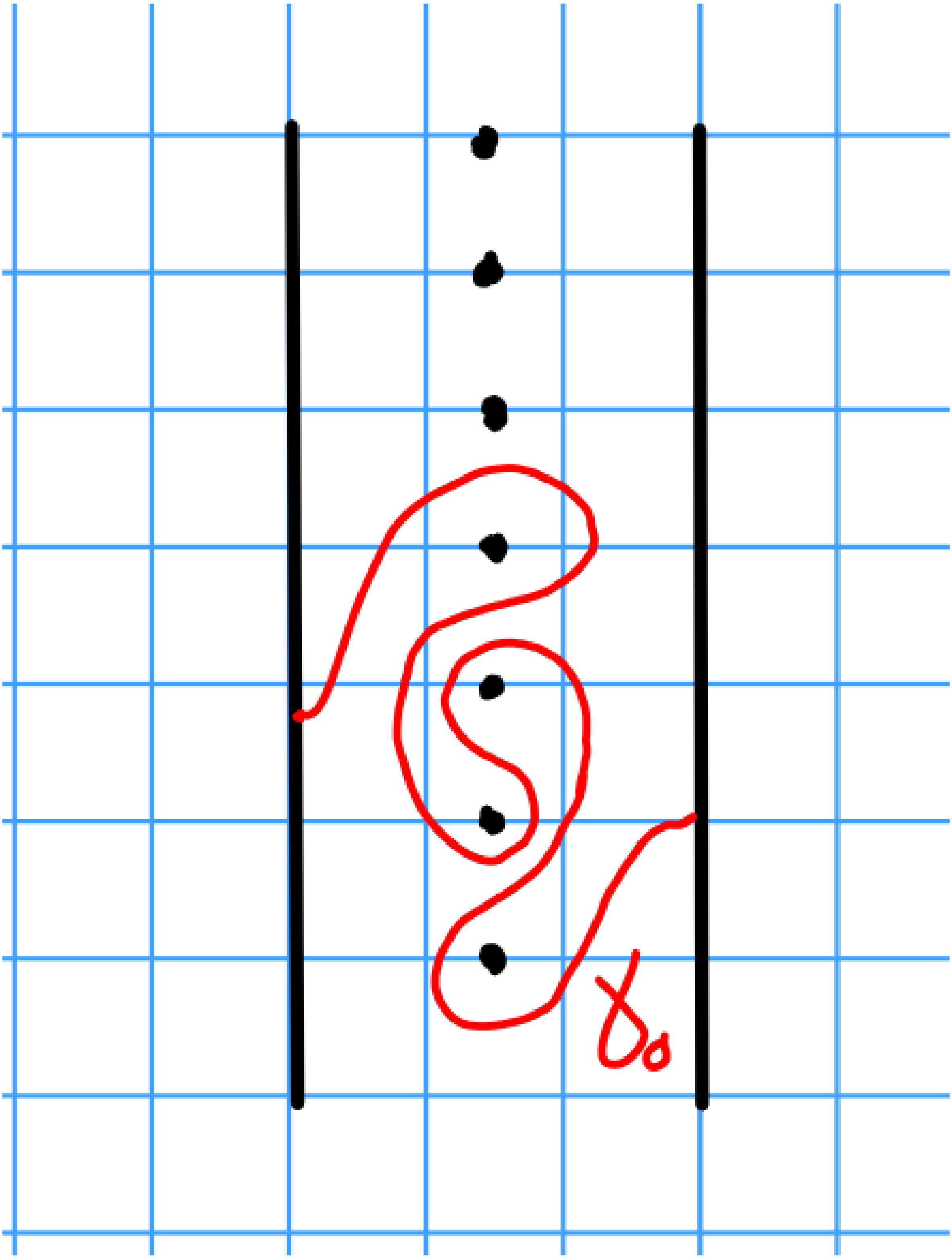



Figure 4: $C_{2,1}(T_{2, 3})

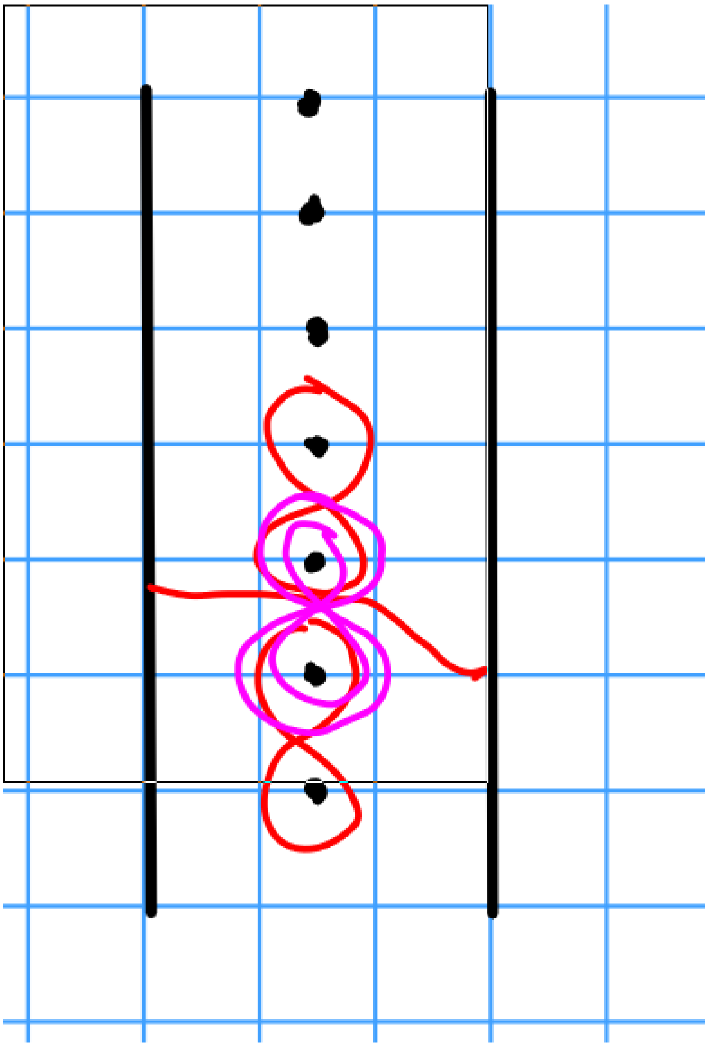


Figure 5: $9_{44}$

The concordance group is an abelian group defined by $C = \left\{\text{knots in } S^3\right\} / \sim$ where we identify knots that are concordant.

Theorem (Fox-Milnor 66): If $K$ is slice, then $\Delta_K(t) = f(t)f(t^{-1}) \in \mathbb{Z}[t^{\pm 1}]$.

Remark: Define $A(k) \coloneqq H_1(\widetilde{S^3 \setminus K}, \mathbb{Z})$ as the integral homology of the infinite cyclic cover as a $\mathbb{Z}[t^{\pm 1}]$-module. This is equal to $H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$. Then $\Delta_k(t) \coloneqq \operatorname{ord}(A(k))$. Given an element of $M_{n,m}$ we get $\mathbb{Z}[t^{\pm 1}]^m \longrightarrow \mathbb{Z}[t^{\pm 1}]^n \longrightarrow A(k) \longrightarrow 0$. We can consider the ideal generated by all the minors (the order ideal), and if this ideal is principal we call the generator $\Delta_k(t)$.

$V - tV^t$ (the Seifert matrix?) is a square presentation matrix for $A(k)$, so $\Delta_k(t) = \det(V - tV^t)$. Note that this is easy to compute. Example: for the figure 8, $\Delta_{4,1}(t) = \det([1-t, t; -1, -1+t]) = -t^2 + 3t - 1$.

There is a notion of algebraically slice, and an algebraically slice knot implies Fox-Milnor.

Theorem (Casson-Gordon, 78): If $K$ is slice and $p$ prime then the $p^r$-fold branched cover $\Sigma_{p^r}(K)$ is a rational homology 3-sphere $\mathbb{Q}HS^3$ and bounds a rational homology 4-ball $\mathbb{Q}HB^4$.

Remark: $\Sigma_{p^r}(K_1 \# K_2) = \Sigma_{p^r} K_1 \# \Sigma_{p^r} K_2$. The following map measures the obstruction to being slice: $\beta_{p^r} : \rho \longrightarrow \Theta^3_{\mathbb{Q}}$, where $[K] \longrightarrow [\Sigma_{p^r} K]$.

Question: How good is $\beta_{p^r}$ as a slice obstruction?

$\beta_2$ is pretty good for 2-bridge knots, i.e.

Theorem (Lisca 07): If $K$ is a connected sum of 2-bridge knots, then $\beta_2(K) = 0 \implies K$ is slice.

Note: there are non-slice knots with $\beta_2 = 0$.

Theorem (Casson-Haner 81): For each $s > 0$, $\Sigma(2, 2s-1, 2s+1) \cong \Sigma_2(T_{2s-1,2s+1})$ bounds a contractible manifold.

Theorem (Litherland 78): Torus knots are $L_\delta I_0$ in $\rho$. (??)

These together imply that $\ker \beta_2 \geq \mathbb{Z}^\infty$.

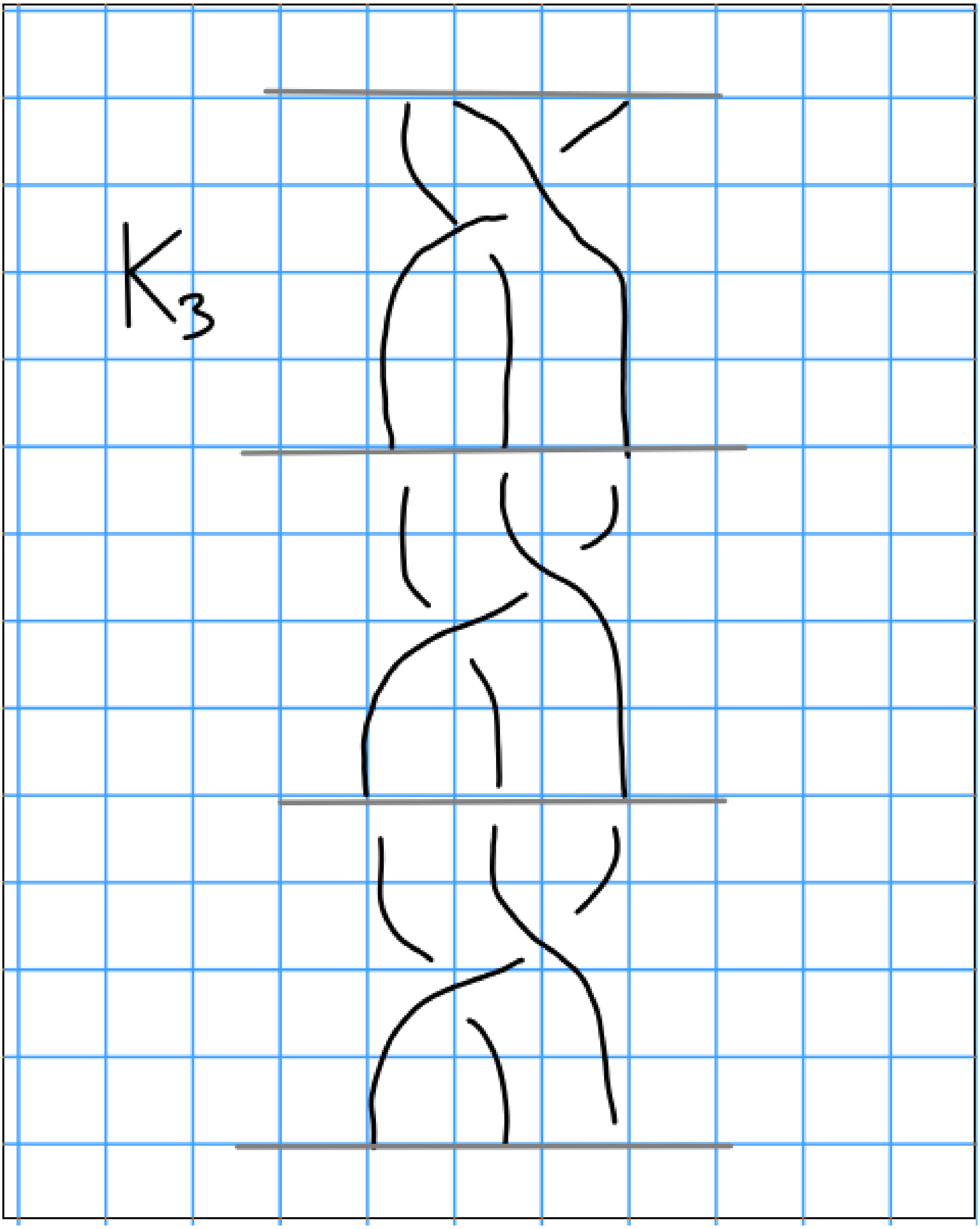
Theorem (Aceto-Larson 18): $\ker \beta_2 \cong \mathbb{Z}^\infty \oplus G$.

Main Theorem: $\displaystyle\bigcap_{p \text{ prime}, r \in \mathbb{N}} \ker \beta_{p^r} \geq (\mathbb{Z}/(2))^4$.

Knots in here have arf invariant zero, and are torsion in the concordance group.

**Step 1: Construction** Define $K_n \coloneqq (\sigma_1 \cdot \sigma_2^{-1})^n$ for $\sigma_i$ in the braid group $B_3$.

Example:




Definition: $K$ is strongly negative ampichiral if there exists an orientation reversing involution $\tau : S^3 \longrightarrow S^3$ such that $\tau(K) = K$.

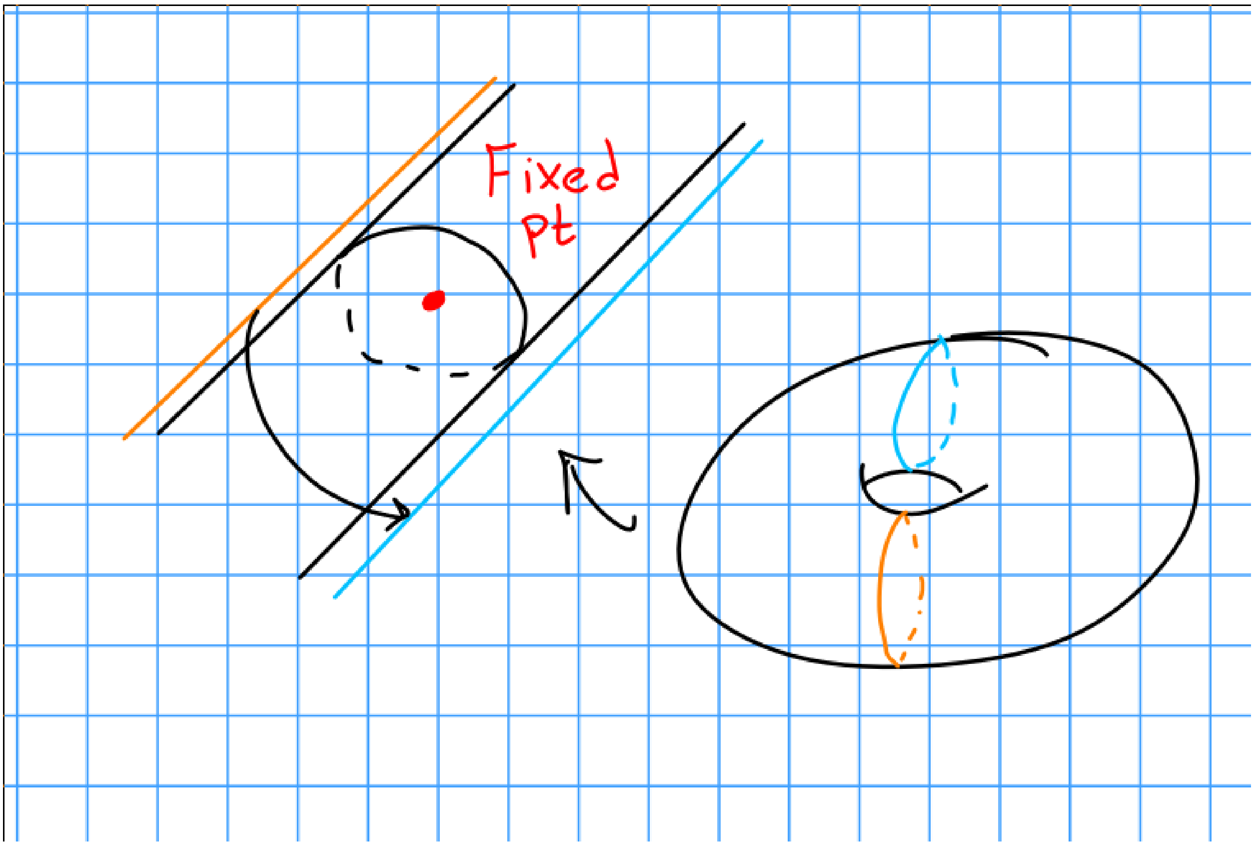



Figure 6: Image

Lemma (Kawauchi 09): If $K$ is strongly negative ampichiral, then $K$ bounds a disc in a $X = \mathbb{Q}HB^4$ with only 2-torsion in $H_1(X;\mathbb{Z})$.

*Proof (sketch):* Let $M_k = S^3_0(K)$ be zero surgery on the knot.

Then $\tau : M_k \longrightarrow M_k$ is fixed-point free.

Can then consider the map $\pi : M_k \longrightarrow M_k/\tau$ and the associated twisted $I$-bundle $I \longrightarrow Z \longrightarrow M_k \longrightarrow M_k/\tau$. Then:

Theorem: If $K$ is slice, the $\Sigma_{p^r}(K)$ bounds a $\mathbb{Q}HB^4$ if $p$ is prime.

Proof (Milnor 68).

There is an exact sequence

$$\begin{aligned} \tilde{H}_i(\tilde{X};\mathbb{Z}_p) \xrightarrow{t^{p^r}-\mathrm{id}} \tilde{H}_i(\tilde{X};\mathbb{Z}_p) \longrightarrow \tilde{H}(\Sigma_{p^r}(D);\mathbb{Z}_p)(=0) \longrightarrow 0 \\ \tilde{H}_i(\tilde{X};\mathbb{Z}_p) \xrightarrow{t-\mathrm{id}} \tilde{H}_i(\tilde{X};\mathbb{Z}_p) \longrightarrow \tilde{H}(B^4;\mathbb{Z}_p)(=0) \longrightarrow 0, \end{aligned}$$

where if $t - \mathrm{id}$ is an isomorphism, $(t^{p^r} - \mathrm{id}) = (t - \mathrm{id})^{p^r}$ is an isomorphism as well (note we're in $\mathbb{Z}_p$.)

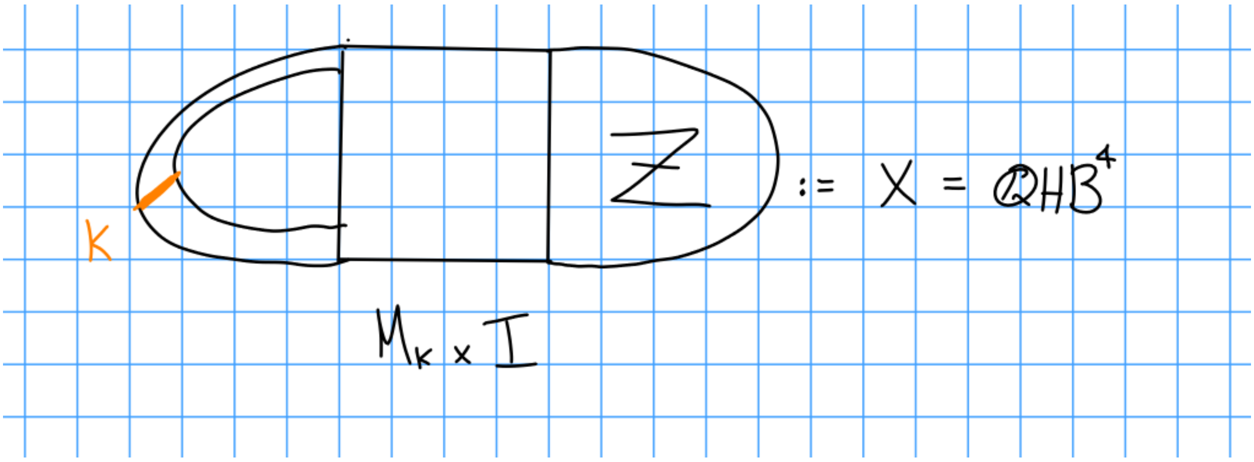



Figure 7: Image

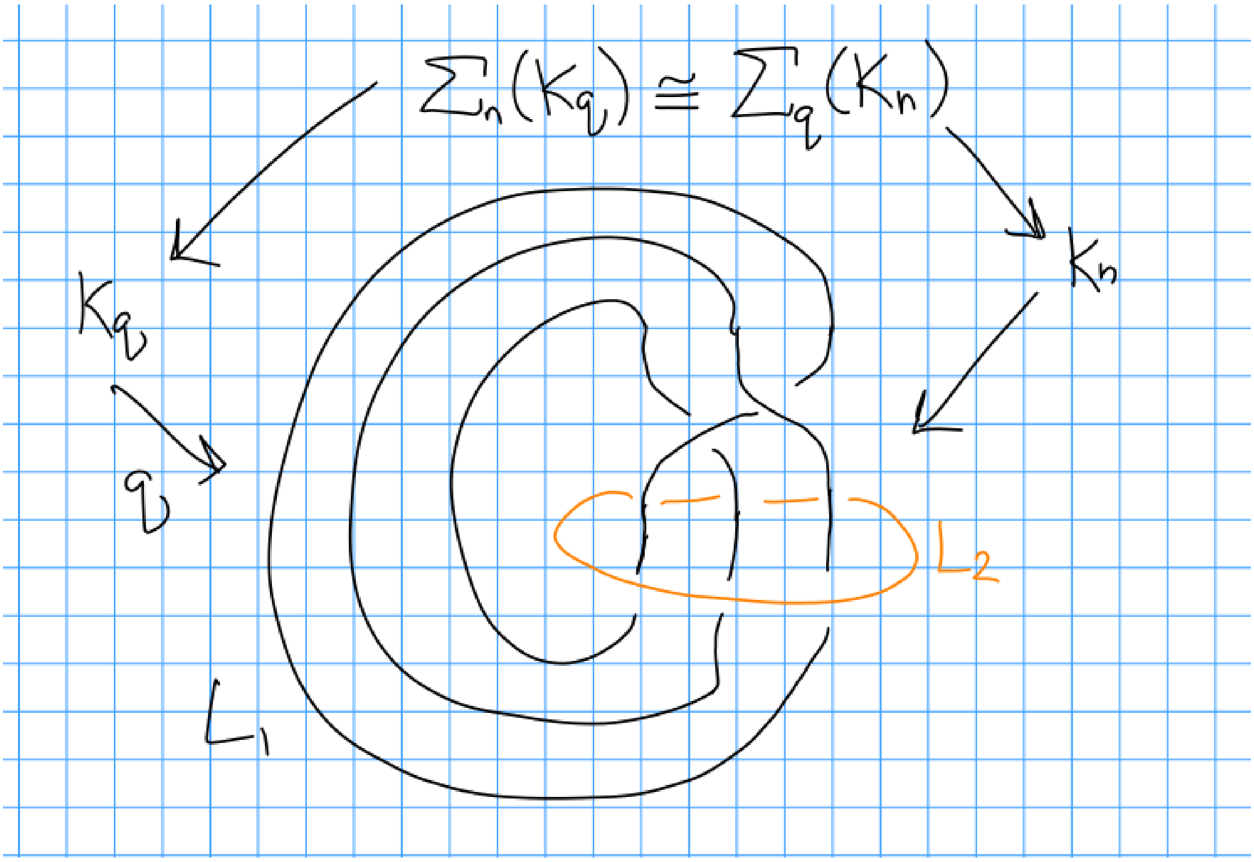



Figure 8: Image

Corollary: If $p$ is an odd prime, then $\Sigma_{p^r}(K)$ bounds a $\mathbb{Q}HB^4$.

Thus $\Sigma_{2^r}(K_n) \cong \Sigma_n(K_{2^r})$ by this symmetry.

In conclusion, if $n$ is an odd prime power, then $K_n \in \bigcap \ker \beta_{p^r}$.

**Step 2: Obstruction** Theorem (Brandenbursky 16): $K_n$ is algebraically slice iff $n$ odd.

Uses a twisted Alexander polynomial: Take $M_k = S^3_0(K)$ a zero surgery, $G = \pi_1(M_k)$, and $A(k) = G^{(1)}/G^{(2)}$ (?).

The input is a map $X : H_1(\Sigma_{p^r}(K); \mathbb{Z}) \longrightarrow \mathbb{Z}_q$. This lets you define a character

$$\alpha(X) : G \longrightarrow G/G^{(1)} \cong \mathbb{Z} \rtimes A(k) \longrightarrow \mathbb{Z} \rtimes A(t)/t^{p^r} - \mathrm{id} \cong \mathbb{Z} \rtimes H_1(\Sigma_{p^r}(K)) \longrightarrow \mathrm{GL}(K, \mathbb{Q}(\zeta_q)[t^{\pm 1}]).$$

Then $\tilde{M}_k$ is the universal cover of $M_k$. Consider $C_*(\tilde{M}_k) \otimes \mathbb{Z}[G]Y$ for $Y = (\mathbb{Q}(\zeta_q)[t^{\pm 1}])^k$, then define the twisted Alexander polynomial $\Delta_k^{\tilde{X}}(t) = \mathrm{ord}(H_1(M_k, Y))$.

Theorem: If $K$ is slice, then there exists some $X$ such that $\delta_K^{\tilde{X}}(t) = f(t)f(t^{-1})$ in $\mathbb{Q}(\zeta_q)[t^{\pm 1}]$.

Open question: Are there infinite order elements in this group?