

# Problem Set 1

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## 1 Problem 6

### 1.1 Part 1

Let  $M = S^2$  as a smooth manifold, and consider a vector field on  $M$ ,

$$X : M \rightarrow TM$$

We want to show that there is a point  $p \in M$  such that  $X(p) = 0$ .

Every vector field on a compact manifold without boundary is complete, and since  $S^2$  is compact with  $\partial S^2 = \emptyset$ ,  $X$  is necessarily a complete vector field.

Thus every integral curve of  $X$  exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point  $p \in M$ .

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular,  $\phi_0 = \text{id}_M$ , and  $\phi_1 \in \text{Diff}(M, M)$ . Moreover  $\phi_0$  is homotopic to  $\phi_1$  via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to  $\phi_0$  and  $\phi_1$ . For an arbitrary map  $f : M \rightarrow M$ , we have

$$\Lambda(f) = \sum_k \text{Tr} \left( f_* \big|_{H_k(X; \mathbb{Q})} \right).$$

where  $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$  is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of  $M$ .

Since homotopic maps induce equal maps on homology, we also have  $\Lambda(\phi_1) = \chi(M)$ .

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have  $\chi(S^2) = 2 \neq 0$ , and thus  $\phi_1$  has a fixed point, so

$$\begin{aligned} &\phi_1(p) = p \\ \implies &\frac{\partial}{\partial t} \phi_1(p) = \frac{\partial}{\partial t} p = 0 \\ \implies &\frac{\partial}{\partial t} \phi_1(p) \big|_{t=0} = X(p) \end{aligned}$$