Computation of $H^*(\mathbb{CP}^2)$

Theorem

Suppose $F \to E \to B$ is a fibration satisfying (conditions).

Then there exists a spectral sequence E_* such that

1. $E_2^{p,q} = H^p(B, H^q(F;\mathbb{Z})) = H^p(B;\mathbb{Z}) \otimes H^q(F;\mathbb{Z})$
2. $E_\infty^{p,q} \Rightarrow H^{p+q}(E)$

Computation

Use the above theorem with the fibration $S^1 \to S^5 \to \mathbb{CP}^2$, as well as the following facts:

1. $H^*(S^1) = \mathbb{Z}\delta_0 + \mathbb{Z}\delta_1$
2. $H^*(S^5) = \mathbb{Z}\delta_0 + \mathbb{Z}\delta_5$
3. $H^0(\mathbb{CP}^2) = \mathbb{Z}$ (i.e. it is simply connected)
4. $d_2 : E_2^{p,q} \to E_2^{p-2,q+1}$

By the theorem, we have

$$E_2^{p,q} = H^p(\mathbb{CP}^2) \otimes H^q(S^1)$$

Thus the E_2 page of the spectral sequence looks like this:

S^1							
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^5(S^1)$	
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^4(S^1)$	
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^3(S^1)$	
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^2(S^1)$	
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^1(S^1)$	
0	$H^0(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} H^0(S^1)$	
	0	1	2	3	4	5	$\mathbb{CP}_2$

Using the facts above, we can substitute in some known groups:

S^1							
0	$\mathbb{Z} \otimes_{\mathbb{Z}} 0$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	
0	$\mathbb{Z} \otimes_{\mathbb{Z}} 0$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	
0	$\mathbb{Z} \otimes_{\mathbb{Z}} 0$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	
0	$\mathbb{Z} \otimes_{\mathbb{Z}} 0$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} 0$	
0	$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	
0	$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^1(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^2(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^3(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^4(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	$H^5(\mathbb{CP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}$	
	0	1	2	3	4	5	$\mathbb{CP}_2$

Now recalling two useful properties of the tensor product:

1. $A \otimes_{\mathbb{Z}} 0 = 0$, and
2. $A \otimes_{\mathbb{Z}} \mathbb{Z} = A$,

we obtain the following simplified version of the E_2 page, with several of the potentially non-trivial differentials indicated:

S^1							
5	0	0	0	0	0	0	
4	0	0	0	0	0	0	
3	0	0	0	0	0	0	
2	0	0	0	0	0	0	
1	$\mathbb{Z}$	$H^1(\mathbb{CP}^2)$	$H^2(\mathbb{CP}^2)$	$H^3(\mathbb{CP}^2)$	$H^4(\mathbb{CP}^2)$	$H^5(\mathbb{CP}^2)$	$\cdots$
0	$\mathbb{Z}$	$H^1(\mathbb{CP}^2)$	$H^2(\mathbb{CP}^2)$	$H^3(\mathbb{CP}^2)$	$H^4(\mathbb{CP}^2)$	$H^5(\mathbb{CP}^2)$	$\cdots$
	0	1	2	3	4	5	$\mathbb{CP}_2$

Now we use the fact that the spectral sequence converges to make several deductions:

Claim:

$$H^1(S^5) = 0 \implies H^2(\mathbb{CP}^2) \cong \mathbb{Z} \text{ and } H^1(\mathbb{CP}^2) = 0$$

(This will be a template argument for most of the rest, so I will spell out more details here and gloss over them later.)

- This means that $E_\infty^{0,1} \oplus E_\infty^{1,0} = 0$.
- Consider the process of obtaining the E_3 page:
  - $E_3^{0,1}$ is obtained from the homology of the complex $0 \to \mathbb{Z} \xrightarrow{\partial_1} H^2(\mathbb{CP}^2) \to 0$, i.e. we have $E_3^{0,1} = \frac{\ker \partial_1}{\operatorname{im} 0} = \ker \partial_1$
    - Note that all differentials after the E_3 page extend into the $p < 0$ and $q < 0$ quadrants, so there is stabiization here and $E_3^{0,1} = E_\infty^{0,1}$
      - But if the homology of this sequence is not zero, then $E_3^{1,0} \neq 0$, so $E_\infty^{0,1} \neq 0$ and $E_\infty^{0,1} \oplus E_\infty^{1,0} \neq 0$, a contradiction.
    - So this is an acyclic complex, and thus an exact sequence.
    - **So ∂_1 is an isomorphism, and $H^2(\mathbb{CP}^2) \cong \mathbb{Z}$**
  - $E_3^{1,0}$ is obtained from the homology of $0 \to H^1(\mathbb{CP}^2) \to 0$
    - By the same argument, this spot stabilizes at E_3 and so this complex must have trivial homology.
    - **But this can only happen if $H^1(\mathbb{CP}^2) = 0$**

Claim:

$$H^2(S^5) = 0 \implies H^1(\mathbb{CP}^2) \cong H^3(\mathbb{CP}^2) \text{ and } H^2(\mathbb{CP}^2) = \mathbb{Z}$$

We have $H^2(S^5) = E_\infty^{0,2} \oplus E_\infty^{1,1} \oplus E_\infty^{2,0}$.

Note that $E_2^{0,2} = 0$, so $E_\infty^{0,2} = 0$ there are only two contributing terms to consider.

$E_\infty^{1,1}$: This involves looking at the complex $0 \to H^1(\mathbb{CP}^2) \xrightarrow{\partial_2} H^3(\mathbb{CP}^2) \to 0$. All differentials extend into zero quadrants starting at E_3, so this entry stabilizes at E_3. But any homology in this complex would contribute a nonzero contribution to $H^2(S^5)$, so this complex is acyclic/exact and ∂_2 is an isomorphism.

$E_\infty^{2,0}$: This involves $0 \to \mathbb{Z} \xrightarrow{f} H^2(\mathbb{CP}^2) \to 0$, where the E^3 differentials extend into zero quadrants and thus this entry stabilizes at E^3. Any nonzero homology here yields a nonzero contribution to $H^2(S^5)$, so this complex is acyclic/exact and thus f is an isomorphism.

Claim:

$$H^3(S^5) = 0 \implies H^2(\mathbb{CP}^4) \cong H^4(\mathbb{CP}^2) \cong H^6(\mathbb{CP}^2),\ H^1(\mathbb{CP}^2) \cong H^3(\mathbb{CP}^2) \cong H^5(\mathbb{CP}^2)$$

Note: this is the first spot where the differentials may not extend into zero quadrants, but since the total homology is zero, this is not a real issue yet.

We have $H^3(S^5) = \bigoplus_{p+q=n} E_\infty^{p,q} = E_\infty^{0,3} \oplus E_\infty^{1,2} \oplus E_\infty^{2,1} \oplus E_\infty^{3,0}$. Every summand must be zero, so we examine them individually.

$E_\infty^{0,3}$: We have $E_2^{0,3} = 0$ and is involved in a complex of the form
$0 \to E_2^{0,3} \to E_2^{2,2} \to E_2^{4,1} \to E_2^{6,0} \to 0$, which we can identify as
$0 \to 0 \to 0 \to H^4(\mathbb{CP}^2) \to H^6(\mathbb{CP}^2) \to 0$, which must be exact, so we have $H^4(\mathbb{CP}^2) \cong H^6(\mathbb{CP}^2)$.

$E_\infty^{1,2}$: We have the complex $0 \to E_2^{1,2} \to E_2^{3,1} \to E_2^{5,0} \to 0$ which equals
$0 \to 0 \to H^3(\mathbb{CP}^2) \xrightarrow{f} H^5(\mathbb{CP}^2) \to 0$, which must be exact and so f is an isomorphism yielding $H^3(\mathbb{CP}^2) \cong H^5(\mathbb{CP}^2)$.

$E_\infty^{2,1}$: We have the complex $0 \to E_2^{0,2} \to E_2^{2,1} \to E_2^{4,0} \to 0$ which equals
$0 \to 0 \to H^2(\mathbb{CP}^2) \to H^4(\mathbb{CP}^2) \to 0$, so $H^2(\mathbb{CP}^2) \cong H^4(\mathbb{CP}^2)$.

(Here we are using the fact that $E_2^{0,2} = H^2(S^1) = 0$ instead of the automatic zeros from the differentials extending into zero quadrants.)

$E_\infty^{3,0}$: We have $0 \to E_2^{1,1} \to E_2^{3,0} \to 0$ which equals $0 \to H^1(\mathbb{CP}^2) \to H^3(\mathbb{CP}^2) \to 0$ which must be exact and so $H^1(\mathbb{CP}^2) \cong H^3(\mathbb{CP}^2)$

Note that $H^4(S^5) = 0$ doesn't give any new information at this point.

Claim

$$H^5(S^5) = \mathbb{Z} \implies H^6(\mathbb{CP}^2) = 0$$

We have $H^5(S^5) = \bigoplus_{p+q=n} E_2^{p,q}$, and so there must now be a nonzero term in this sum.

Since $q > 1$ stabilizes to zero on E_2, the nonzero term must come from $E_2^{5,0}$ or $E_2^{4,1}$.

$E_2^{5,0}$: The complex is $0 \to H^3(\mathbb{CP}^2) \to H^5(\mathbb{CP}^2) \to 0$

$E_2^{4,1}$: The complex is $0 \to H^4(\mathbb{CP}^2) \to H^6(\mathbb{CP}^2) \to 0$

In order for an E_3 term to be nonzero, one of these complexes must have nonzero homology. But by the previous claim, $0 \to H^3(\mathbb{CP}^2) \to H^5(\mathbb{CP}^2) \to 0$ does have zero homology, so we consider the second complex instead.

We know from our current results that $0 \to H^4(\mathbb{CP}^2) \to H^6(\mathbb{CP}^2) \to 0$ is equal to
$0 \to \mathbb{Z} \xrightarrow{f} H^6(\mathbb{CP}^2) \to 0$, and we know that $\frac{\ker f}{\operatorname{im} 0} = \ker f \cong H^5(S^5) = \mathbb{Z}$, since this is the only possible nonzero term in the above sum.

(Not sure how to use $\ker f = 0$ to show $H^6(\mathbb{CP}^2) = 0$, or how to inductively compute $H^*(\mathbb{CP}^n)$.)