

Mapping Class Groups

Last updated:

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1 | Setup

- All manifolds:
 - Connected
 - Oriented
 - 2nd countable (countable basis)
 - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
 - With boundary (possibly empty)
- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Curves: simple, closed, oriented
- Surfaces: these guys

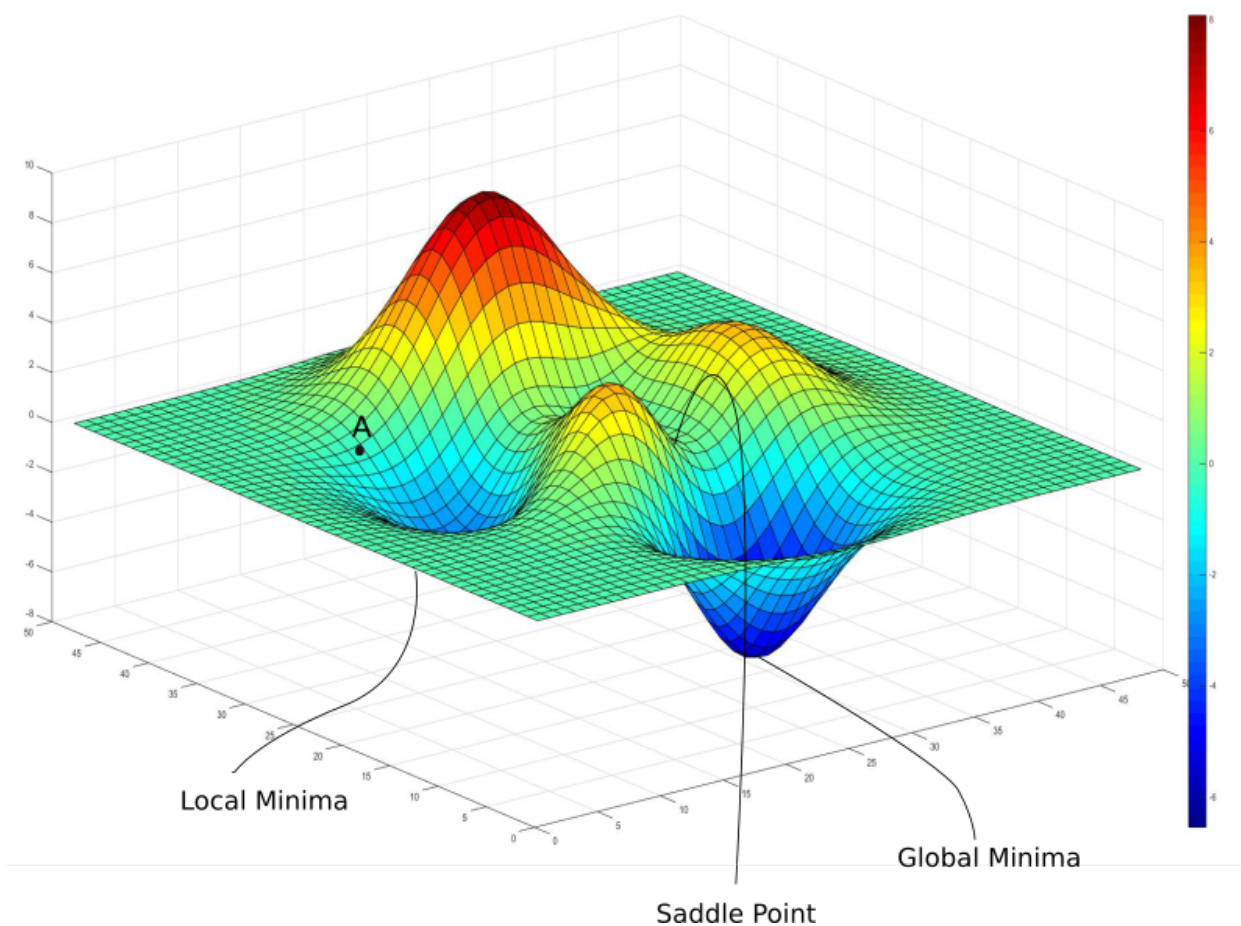


Figure 1: A surface

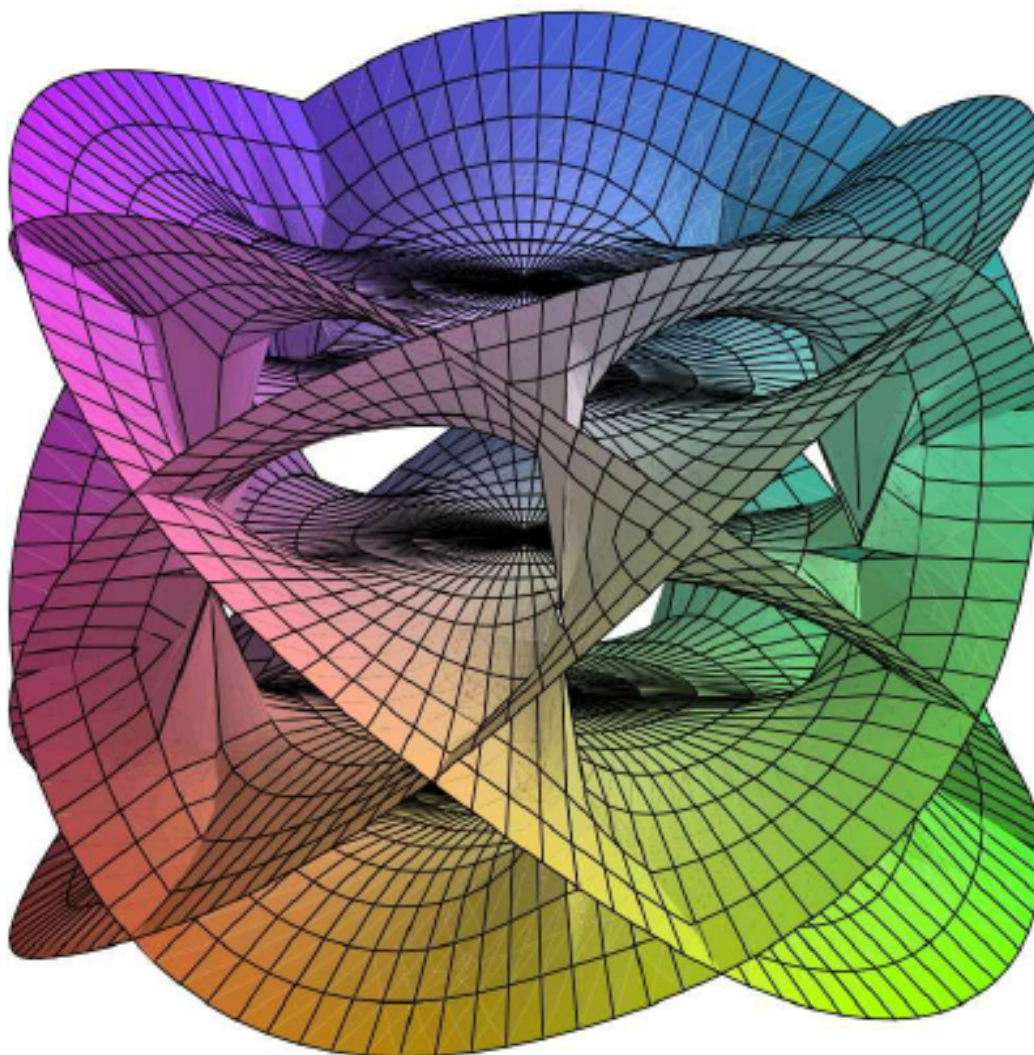


Figure 2: More surfaces

- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \left\{ f : X \rightarrow Y \mid f \text{ is continuous} \right\}.$$

1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it
 - Slogan: we'd like homs to be spaces.
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
- Topologize with the *compact-open* topology \mathcal{O}_{CO} :

$$U \in \mathcal{O}_{CO} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

1.1.1 Mapping Spaces

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{CO}) \quad \text{where } \mathcal{O}_{CO} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:
 - $\text{Homeo}(X, Y)$ the subspace of homeomorphisms
 - $\text{Imm}(X, Y)$, the subspace of immersions (injective map on tangent spaces)
 - $\text{Emb}(X, Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
 - $C^k(X, Y)$, the subspace of $k \times$ differentiable maps
 - $C^\infty(X, Y)$ the subspace of smooth maps
 - $\text{Diffeo}(X, Y)$ the subspace of diffeomorphisms
 - $C^\omega(X, Y)$ the subspace of analytic maps
 - $\text{Isom}(X, Y)$ the subspace of isometric maps (for Riemannian metrics)
 - $[X, Y]$ homotopy classes of maps

1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of “uniform convergence on compact sets”: for $f_n \rightarrow f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact.}$$

- In words: $f_n \rightarrow f$ uniformly on every compact set.
- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\text{Map}(\cdot, Y)$:

$$\begin{aligned}X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}.\end{aligned}$$

- Adjoint property: there is a homeomorphism

$$\begin{aligned}\text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, Y^X) \\H : X \times Z \rightarrow Y &\iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\(x, z) \mapsto H(x, z) &\iff z \mapsto H(\cdot, z).\end{aligned}$$

- Categorically, $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top .
- A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space PY , the based loop space ΩY .
- Importance in homotopy theory: the path space fibration

$$\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y$$

- Plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences.
- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps f, g are homotopic \iff they are connected by a path in $\text{Map}(X, Y)$.

Picture!

2.1.1 Proof

$$\mathcal{P}\mathrm{Map}(X, Y) = \mathrm{Map}(I, Y^X) \cong \mathrm{Map}(X \times I, Y),$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \rightarrow Y$, the LHS are paths in the space of maps.
-

3 | Defining the Mapping Class Group

3.1 Isotopy

- Define a homotopy between $f, g : X \rightarrow Y$ as a map $F : X \times I \rightarrow Y$ restricting to f, g on the ends.
 - Equivalently: a *path*, an element of $\text{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \rightarrow Y$ to be homeomorphisms for every t .
 - Equivalently: a path in the subspace of homeomorphisms, an element of $\text{Map}(I, \text{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\text{Map}(X, Y)$, subspace of homeomorphisms.

3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.
 - In a general category \mathcal{C} , we can always define the group $\text{Aut}_{\mathcal{C}}(X)$.
 - ◇ If the group has a topology, we can consider $\pi_0\text{Aut}_{\mathcal{C}}(X)$, the set of path components.
 - ◇ Since groups have identities, we can consider $\text{Aut}_{\mathcal{C}}^0(X)$, the path component containing the identity.
 - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X)/\text{Aut}_{\mathcal{C}}^0(X).$$

- Here the \pm indicates that we take both orientation preserving and non-preserving automorphisms.
- Has an index 2 subgroup of orientation-preserving automorphisms, $\text{MCG}^+(X)$.
- Can define $\text{MCG}_{\partial}(X)$ as those that restrict to the identity on ∂X .

Picture: quotienting out by identity component

3.3 Definitions in Several Categories

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with \mathcal{O}_{CO} .

- Taking $\text{MCG}_{\text{Top}}^{\pm}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking $\text{MCG}_{C^{\infty}}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f] \in [X, Y]\}.$$

- Taking $\text{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences*.

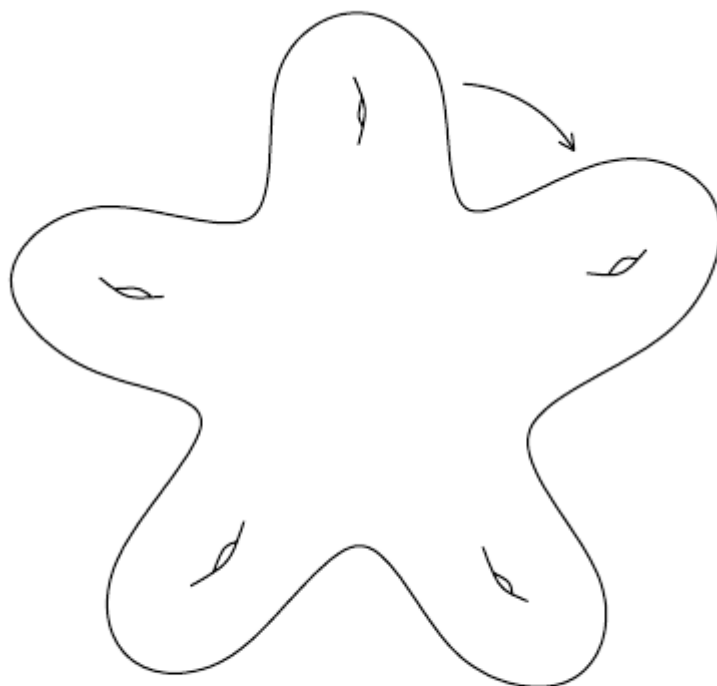


Figure 2.1 An order 5 element of $\text{Mod}(S_5)$.

Figure 3: Image

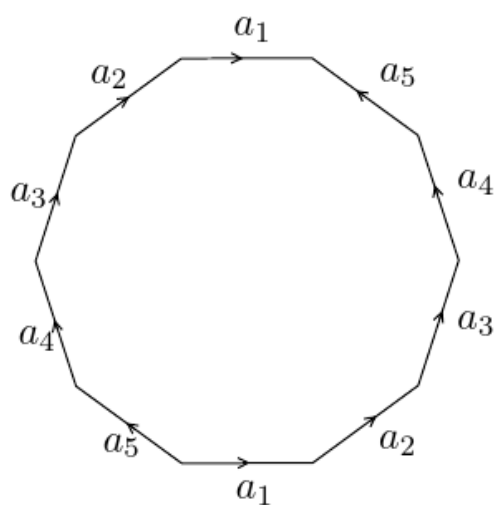


Figure 2.2 Rotation by $2\pi/10$ gives an order 10 element of $\text{Mod}(S_2)$.

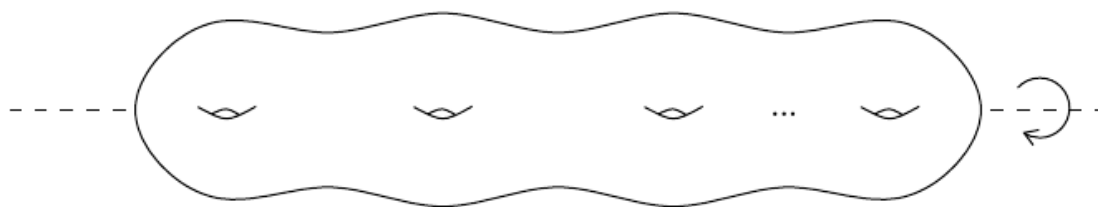


Figure 2.3 The rotation by π about the indicated axis is a hyperelliptic involution.

Figure 4: Image

3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms
 - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi_0 \text{Aut}(X)$.
- For surfaces: For Σ a genus g surface, $\text{MCG}(S)$ acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \rightarrow \text{MCG}(\Sigma) \rightarrow T(\Sigma) \rightarrow \mathcal{M}_g \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to X .

- $T(S)$ is the moduli space of complex structures on S , up to the action of homeomorphisms that are isotopic to the identity:
 - Points are isomorphism classes of marked Riemann surfaces
 - Equivalently the space of hyperbolic metrics
- Used in the Nielsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:
 - Periodic,
 - Reducible (preserves some simple closed curves), or
 - Pseudo-Anosov (has directions of expansion/contraction)

Picture: \mathcal{M}_g .

4 | Examples of MCG

4.1 The Plane: Straight Lines

- $\text{MCG}_{\text{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, take the straight-line homotopy:

$$F : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2$$
$$F(x, t) = tf(x) + (1 - t)x.$$

Picture: parameterize line between x and $f(x)$ and flow along it over time.

4.2 The Closed Disc: The Alexander Trick

- $\text{MCG}_{\text{Top}}(\mathbb{D}^2) = 1$: for any $f : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that $f|_{\partial\mathbb{D}^2} = \text{id}$, take

$$F : \mathbb{D}^2 \times I \rightarrow \mathbb{D}^2$$

$$F(x, t) := \begin{cases} tf\left(\frac{x}{t}\right) & \|x\| \in [0, t) \\ x & \|x\| \in [1-t, 1] \end{cases}.$$

- This is an isotopy from f to the identity.
- Interpretation: “cone off” your homeomorphism over time:

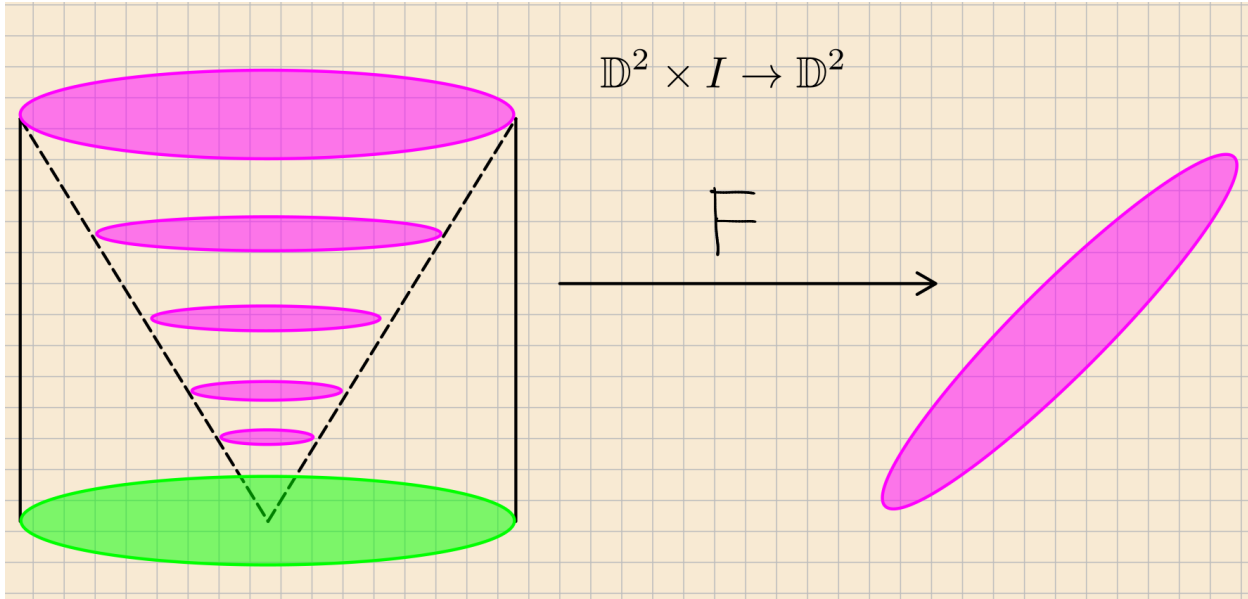


Figure 5: Image

- Note that this won't work in the smooth category: singularity at origin

4.3 Overview of Big Results

- The word problem in $\text{MCG}(\Sigma_g)$ is solvable
- Any finite group is $\text{MCG}(X)$ for some compact hyperbolic 3-manifold X .
- For $g \geq 3$, the center of $\text{MCG}(\Sigma_g)$ is trivial and $H_1(\text{MCG}(\Sigma_g); \mathbb{Z}) = 1$
 - Why care: same as abelianization of the group.

Theorem 4.3.1 (Dehn-Neilsen-Baer).

Let Σ_g be compact and oriented with $\chi(\Sigma_g) < 0$. Then

$$\text{MCG}_{\partial}^+(\Sigma_g) \cong \text{Out}_{\partial}(\pi_1(\Sigma_g)) \cong_{\text{Grp}} \pi_0 \text{ho}_{\partial}(\Sigma_g).$$

- For $g \geq 4$, $H_2(\text{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$
 - Why care: used to understand surface bundles

$$\begin{array}{ccc} \Sigma_g & \longrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$

- Find the classifying space $B\text{Diffo}$
- Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\text{Diffo}(\Sigma_g)] \cong [S^{n-1}, \text{Diffo}(\Sigma_g)]$$

- Theorem (Earle-Ells): For $g \geq 2$, $\text{Diffo}_0(\Sigma_g)$ is contractible. As a consequence, $\text{Diffo}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_g)$ is a homotopy equivalence, and there is a correspondence:

$$\left\{ \begin{array}{c} \text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B \end{array} \right\} / \text{Bundle isomorphism} \iff \left\{ \begin{array}{c} \text{Monodromy Representations} \\ \rho: \pi_1(B) \rightarrow \text{MCG}(\Sigma_g) \end{array} \right\} / \text{Conjugacy}.$$

5 | Dehn Twists

- $\text{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation

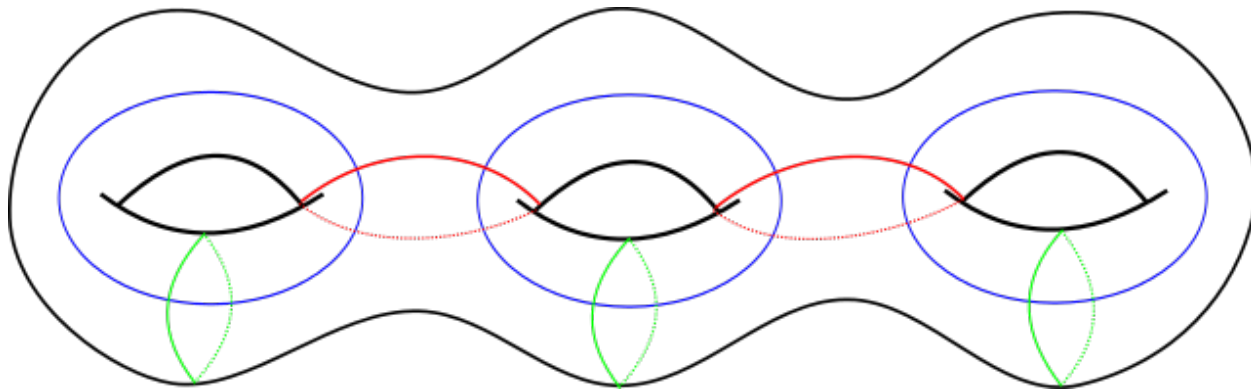


Figure 6: $3g - 1$ generators for Dehn Twist theorem

Claim: Let $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, then $\text{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto \exp(2\pi i|z|) z. \end{aligned}$$

See complex function plotter

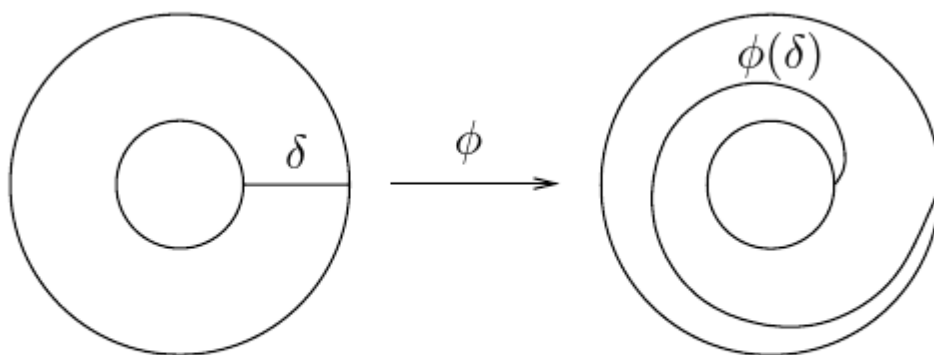


Figure 2.5 A generator for $\text{Mod}(A)$.

Figure 7: Image

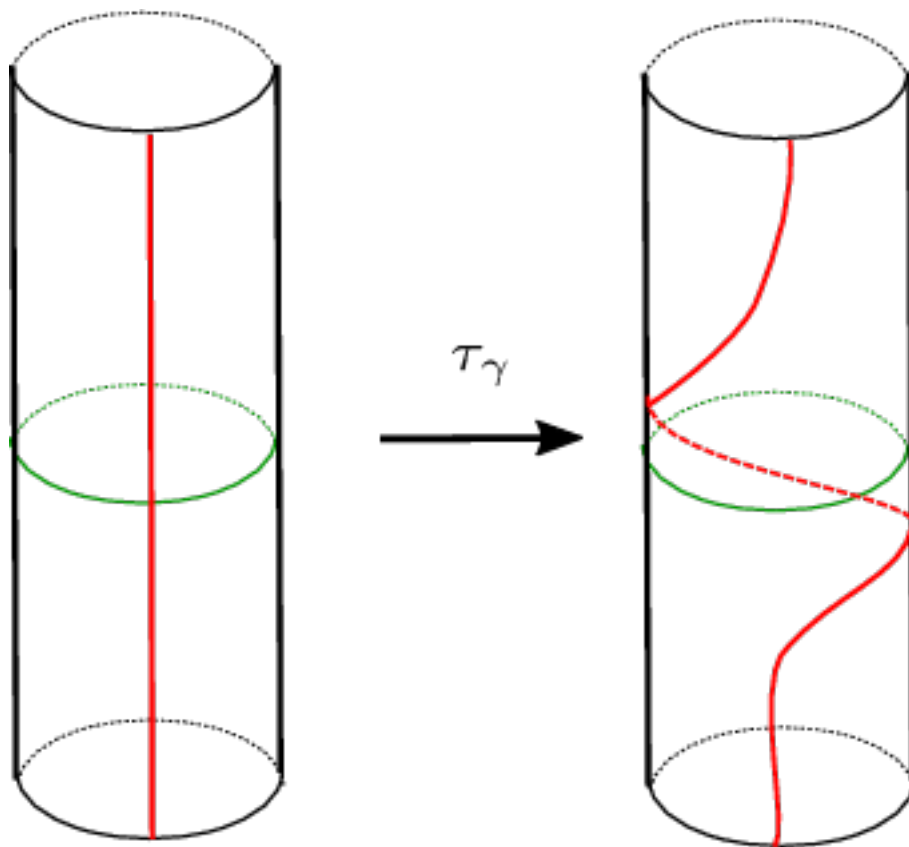


Figure 8: Dehn Twist Cylinder

6 | MCG of the Torus

6.1 Setup

Definition 6.1.1 (Special Linear Group).

$$\mathrm{SL}(n, \mathbb{k}) = \left\{ M \in \mathrm{GL}(n, \mathbb{k}) \mid \det M = 1 \right\} = \ker \det_{\mathbb{G}_m}.$$

Definition 6.1.2 (Symplectic Group).

$$\mathrm{Sp}(2n, \mathbb{k}) = \left\{ M \in \mathrm{GL}(2n, \mathbb{k}) \mid M^t \Omega M = \Omega \right\} \leq \mathrm{SL}(2n, \mathbb{k})$$

where Ω is a nondegenerate skew-symmetric bilinear form on \mathbb{k} .

Example:

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Definition 6.1.3 (Algebraic Intersection).

A bilinear antisymmetric form on middle homology:

$$\hat{\iota} : H_1(\Sigma_g; \mathbb{Z}) \otimes H_1(\Sigma_g; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

Note that this is a symplectic pairing.

- There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned} \rho : \mathrm{MCG}(\Sigma) &\rightarrow \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_* \end{aligned}$$

- For a surface of finite genus $g \geq 1$, elements in $\text{im}(\rho)$ preserve the *algebraic intersection form*

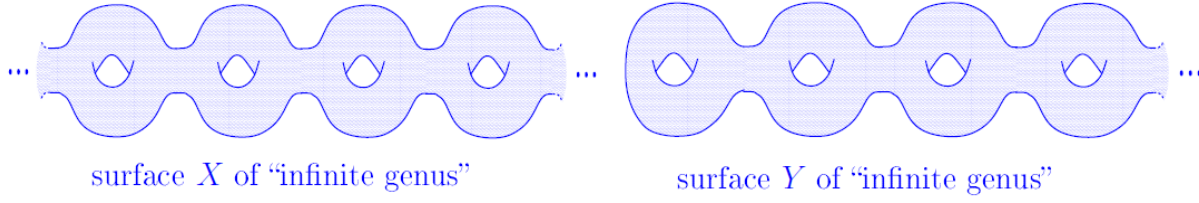


Figure 9: Jacob’s Ladder and Loch Ness

- Thus there is an interesting surjective representation:

$$0 \rightarrow \text{Tor}(\Sigma_g) \hookrightarrow \text{MCG}(\Sigma_g) \twoheadrightarrow \text{Sp}(2g; \mathbb{Z}) \rightarrow 0.$$

- Kernel is the *Torelli group*, interesting because the symplectic group is well understood, so questions about MCG reduce to questions about Tor.

Theorem 6.1.1 (Mapping Class Group of the Torus).

The homology representation of the torus induces an isomorphism

$$\sigma : \text{MCG}(\Sigma_2) \xrightarrow{\cong} \text{SL}(2, \mathbb{Z})$$

6.2 Proof

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned}\tilde{\sigma} : (\text{Homeo}(X, X), \circ) &\rightarrow (\text{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*.\end{aligned}$$

- This will descend to the quotient $\text{MCG}(X)$ iff

$$\text{Homeo}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\text{id})$$

- This is true here, since any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.
- So we have a (now injective) map

$$\begin{aligned}\tilde{\sigma} : \text{MCG}(X) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: $\text{im}((\tilde{\sigma})) \subseteq \text{SL}(2, \mathbb{Z})$.

Proof .

- Algebraic intersection numbers in Σ_2 correspond to determinants
- $f \in \text{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2 of Farb and Margalit

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- We can thus freely restrict the codomain to define the map

$$\begin{aligned}\sigma : \text{MCG}(X) &\rightarrow \text{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

6.2.1 Surjectivity**Claim:** σ is surjective.

- \mathbb{R}^2 is the universal cover of Σ_2 , with deck transformation group \mathbb{Z}^2 .
- Any $A \in \mathrm{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \mathrm{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

Claim: \tilde{A} is equivariant wrt \mathbb{Z}^2 *Proof.*

$$\mathrm{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that

$$S^2 = 1, \quad T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

So if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $\tilde{A} \in \mathrm{SL}(2, \mathbb{Z})$, we have $\tilde{A}\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. A preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \mid p, q \in \mathbb{Z} \right\}.$$

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- So \tilde{A} descends to a well-defined map

$$\psi_{\tilde{A}} : \Sigma_2 \circlearrowleft = \mathbb{R}^2 / \mathbb{Z}^2 \circlearrowleft$$

which is still a linear self-homeomorphism.

- There is a correspondence

$$\left\{ \begin{array}{c} \text{Primitive curves in} \\ \pi_1(\Sigma_2) \cong \mathbb{Z}^2 \end{array} \right\} \iff \{ \text{Primitive vectors in } \mathbb{Z}^2 \} \iff \left\{ \begin{array}{c} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{array} \right\} /_{\text{homotopy}},$$

where an element x is *primitive* iff it is not a multiple of another element.

- By changing basis, you can associate a unique primitive vector to M (all components coprime)
- By the correspondence, changing a map by a homotopy corresponds to the same primitive vector
- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

6.2.2 Injectivity

Claim: σ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.

Proposition 6.2.1 (Hatcher 1B.9).

Let X be a connected CW complex and Y a $K(G, 1)$. Then there is a map

$$\text{hom}_{\text{Grp}}(\pi_1(X; x_0), \pi_1(Y; y_0)) \rightarrow \text{hom}_{\text{Top}}((X; x_0), (Y; y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map.
 Moreover, the map is unique up to homotopies fixing x_0 .

- Thus there is a correspondence

$$\left\{ \begin{array}{c} \text{Homotopy classes of} \\ \text{maps } \Sigma_2 \hookrightarrow \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Group morphisms } \mathbb{Z}^2 \hookrightarrow \end{array} \right\}.$$

- Claim: any element $f \in \text{MCG}(\Sigma_2)$ has a representative φ which fixes any given basepoint
- So if $f \in \ker \sigma$, then $f \simeq \varphi \simeq \text{id}$ are homotopic, so $\ker \sigma = 1$.