## Title

## **Contents**

## 1 Monday April 20th

2

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Today: torsors.

Let k be a (perfect, separable) field and G/k a commutative algebraic group (a finite type reduced group?).

**Definition** A variety X/k is a torsor under G is  $\mu: G \times X \to X$  a group action such that the map

$$G \times X \to X \times X$$
  
 $(g, x) \mapsto (\mu(gx), x)$ 

is an isomorphism.

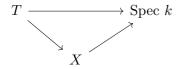
For  $\ell/k$  any field extension, the base change to  $X/\ell$  induces  $\mu_{\ell}$  making  $X/\ell$  a  $G/\ell$  torsor. X is trivial iff it is isomorphic to

$$\mu: G \times G \to G$$
  
 $(g, hg) \mapsto gh.$ 

**Claim** X is trivial iff  $X(k) \neq \emptyset$ .

First "proof": for  $p \in X(k)$ , define  $\mu(\cdot, p) : G \to X$ . Want to get a map  $G(k^{\text{sep}}) \to X(k^{\text{sep}})$ , when does this happen? In characteristic zero, we have some map  $G \to X$  (???) which is surjective with trivial kernel and thus an isogeny but has not  $k^{\text{sep}}$  points. But this doesn't work in positive characteristic.

Second proof: the map  $G \times X \to X \times X$  being an isomorphism says that upon base change on  $X \to \operatorname{Spec} k$ , X becomes isomorphic to G. But then it also becomes isomorphic over base change for which X is intermediate. So if we have



which factors through Y, if  $p \in X(k)$  then Spec  $k \to X$  and thus  $X/k \cong G/k$ .

The form of the assumed isomorphic implies that the base change of the G-torsor X from Spec k to X is trivial as a  $G \times X$  torsor over X.

For k a field, G/k, an equivalent definition would be that a G torsor is X/k with a G action that becomes trivial over  $k^{\text{sep}}$ . Therefore A G torsor X is a  $k^{\text{sep}}/k$  twisted form of X where  $X/k^{\text{sep}} \cong G/k^{\text{sep}}$ .

Example: Let G = E an elliptic curve, and X/k is a nice curve of genus 1, but X(k) is likely empty. Conversely, given such a curve of genus 1, we can take the Picard variety  $\underline{\text{Pic}}^0 X$ , i.e. the Jacobian. Then there is an isomorphism

$$X \xrightarrow{\cong} \underline{\operatorname{Pic}}^1 X$$
$$p \mapsto [p].$$

So every nice curve is a torsor for its Jacobian (?). Note that in higher dimensions, we'd need to take the albanese, and the same statement would work: every abelian variety is a torsor over its albanese.

For G/k commutative, we can make the set of torsors X for G/k modulo equivalence into a commutative group. We define the Weil-Chatelet group of G/k as WC(k,G). For two torsors, we can define the  $Baer\ sum\ X_1 \oplus X_2$  by first defining a map

$$\mu_{\pm}: G \times (X_1 \times X_2) \to X_1 \times X_2$$

$$(g, x_1, x_2) \mapsto (\mu_1(g, x_1), \mu_2([-1]g, x_2))$$

and defining  $X_1 \oplus X_2 = (X_1 \times X_2)/\mu_{\pm}$ . Then the action  $\mu_{\pm}$  on  $X_1 \oplus X_2$  is a G torsor.

This makes WC(k,G) into a commutative group where  $\mu: G \times G \to G$  defines  $[-1](X,\mu) := (X,\mu([-1] \cdot))$ .

**Exercise** For C/k a nice genus one curve,  $G = E = \underline{\operatorname{Pic}}^0 C$  and  $C = \underline{\operatorname{Pic}}^1 C$ . Show that  $n[C]\underline{\operatorname{Pic}}^n C$ .

Note that by adding divisor classes, there is a map  $\underline{\text{Pic}}^1C \times \underline{\text{Pic}}^1C \to \underline{\text{Pic}}^2E$ .

**Corollary** For E/k an elliptic curve, WC(k, E) is a torsion abelian group iff for all genus 1 curves C, there exists an  $n \in \mathbb{Z}^{\geq 0}$  such that  $(\underline{\operatorname{Pic}}^n C)(k) \neq \emptyset$ .

We can define the period of an elliptic curve as the least n for which the torsor becomes trivial, this is an interesting numerical invariant.

Next up: cocycles and descent.