Real Analysis

D. Zack Garza

August 20, 2019

Contents

---

1 Lecture 1 (Thu 15 Aug 2019 11:04)

> See Folland's Real Analysis, definitely a recommended reference.

Possible first day question: how can we "measure" a subset of $\mathbb{R}$? We'd like bigger sets to have a higher measure, we wouldn't want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of $\mathbb{R}$. We'll come back to this in a few lectures.

1.1 Notions of "smallness" in $\mathbb{R}$

Definition 1. Let E be a set, then E is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes $\emptyset, \mathbb{N}$.

Definition 2. A set E is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

Exercise 1. Show that any finite subset of $\mathbb{R}$ is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in Y iff the closure of X is all of Y. So we'll make the following definition:

Definition 3. A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval I contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we're giving a name to such a countable union above. For example, $\mathbb{Q}$ is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently,

- A^c contains a dense, open set.
- The interior of the closure is empty.

We'd like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than ε.

Definition 4. Definition: E is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} \text{ and } \sum |I_j| < \varepsilon.$$

Exercise 2. Show that a countable union of null sets is null.

We have several relationships

- Countable $\implies$ Meager, but not the converse.
- Countable $\implies$ Null, but not the converse.

Exercise 3. Show that the "middle third" Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem 1. Every $E \subseteq \mathbb{R}$ can be written as $E = A \coprod B$ where A is null and B is meager.

> This gives some information about how nullity and meagerness interact – in particular, $\mathbb{R}$ itself is neither meager nor null. Idea: if meager $\implies$ null, this theorem allows you to write $\mathbb{R}$ as the union of two null sets. This is bad!

Proof. We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Around each q_j, put an interval around it of size $1/2^{j+k}$ where we'll allow k to vary, yielding multiple intervals around q_j. To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that A is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick k such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that A^c :
B is meager. Note that G_k covers the rationals, and is a countable union of open sets, so it is dense. So G_k is an open and dense set. By one of the equivalent formulations of meagerness, this means that G_k^c is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager. $\square$

1.2 $\mathbb{R}$ is not small

Theorem 2 (A, Cantor)**.** $\mathbb{R}$ is not countable.

Theorem 3 (B, Baire)**.** $\mathbb{R}$ is not meager. (Baire Category Theorem)

Theorem 4 (C, Borel)**.** $\mathbb{R}$ is not null.

Note that theorems B and C imply theorem A. You can also replace $\mathbb{R}$ with any nonempty interval $I = [a, b]$ where $a < b$. This is a strictly stronger statement – if any subset of $\mathbb{R}$ is not countable, then certainly $\mathbb{R}$ isn't, and so on.

Proof of (A). Begin by thinking of $I = [0, 1]$, then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry. $\square$

A second proof of (A). Take an interval I, and suppose it is countable so $I = \{x_i\}$. Choose $I_1 \subseteq I$ that avoids x_1, so $x_1 \notin I_1$. Choose $I_2 \subseteq I_1$ avoiding x_2 and so on to produce a nested sequence of closed intervals. Since $\mathbb{R}$ is complete, the intersection $\bigcap_{n=1}^{\infty} I_n$ is nonempty, so say it contains x. But then $x \in I_1 \in I$, for example, but $x \neq x_i$ for any i, so $x \notin I$, a contradiction. $\square$ $\square$

Proof of (B). Suppose $I = \bigcup_{i=1}^{\infty} A_n$ where each A_n is nowhere dense. We'll again construct a nested sequence of closed sets. Let $I_1 \subseteq I$ be a subinterval that misses all of A_1, so $A_1 \bigcap I_1 = \emptyset$ using the fact that A_1 is nowhere dense. Repeat the same process, let $I_2 \subset I_1 \setminus A_2$. By the nested interval property, there is some $x \in \bigcap A_i$. $\square$

> Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of $\mathbb{R}$ is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

Consider the Thomae function: It is continuous on $\mathbb{Q}$, but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Can this be switched to get some function f that is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$? The answer is no. The set of discontinuities of a function is *always* an F_σ set, and $\mathbb{R} \setminus \mathbb{Q}$ is not one. Equivalently, the rationals are not a G_δ set.

Some facts:

- The pointwise limit of continuous functions has a meager set of discontinuities.
- If f is integrable, the set of discontinuities is null.
- If f is monotone, they are countable.
- There is a continuous nowhere differentiable function: let $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$, and in fact *most* functions are like this.
- If f is continuous and monotone, the discontinuities are null.

Theorem 5. Let $I = [a, b]$. If $I \subseteq \bigcup_{i=1}^{\infty} I_i$, then $|I| \leq \sum_{i=1}^{\infty} |I_i|$.

Proof. The proof is by induction. Assume $I \subseteq \bigcup_n^{N+1} I_n$, where wlog we can assume that $a < a_{N+1} < b \leq b_{N+1}$, then $[a, a_{N+1}] \subset \bigcup_{n=1}^{N} I_n$ so the inductive hypothesis applies. But then $b - a \leq b_{N+1} - a = (b_{N+1} - a_{N+1}) + (a_{N+1} - a) \leq \sum_{n=1}^{N+1} |I_n|$. $\square$

> Note that this proves that the reals are uncountable!

2 Lecture 2

(Find notes for first 15 mins)

- Countable $\implies$ Cantor, all intervals are not countable
- Meager $\implies$ Baire, all intervals are not meager
- Null $\implies$ Borel, all intervals are not null.

Exercise: Verify that f is continuous at$x$ iff $\lim f(x_n) = f(x)$ for every sequence $\{x_n\} \to x$.

Definition: If $f : X \to \mathbb{R}$, the *oscillation of f at $x \in X$ is given as

$$\omega_f(x) = \lim_{\delta\to\infty} \sup_{y\in B_\delta(x)} |f(y) - f(z)|.$$

Exercise: Show that f is continuous at $x \iff \omega_f(x) = 0$.

We can then define points of discontinuity as

$$D_f = \{x \in X \ni \omega_f(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in X \ni \omega_f(x) \geq \frac{1}{n}\right\}$$

Exercise: show that D_f is closed.

Theorem 1: f is monotone $\implies$ D_f is countable.

> Hint: we can't cover $\mathbb{R}$ by uncountable many disjoint intervals.

Theorem 2: D_f is always an F_σ set.

> $\mathbb{R} - \mathbb{Q}$ is not at F_σ set, i.e. one can not construct a function that is discontinuous on exactly this set.

Theorem 3: f is "1st class" $\implies$ D_f is meager.

> f is first class if $f(x) = \lim_{n\to\infty} f_n(x)$ pointwise and each f_n is continuous.

Theorem 4 (Lebesgue Criterion): Let $f : [a, b] \to \mathbb{R}$ be bounded, then f is Riemann integrable iff D_f is null.

> So the Dirichlet function is not Riemann integrable.

Exercise: Prove theorems 1 and 2.

2.1 Proof of Theorem 3

Proof of theorem 3: Want to show that D_f is meager. We know it's some countable union of some sets, and it suffices to show that they are nowhere dense.

So let $F_n = \{x \ni \omega_f(x) \geq 0\}$ for some fixed n. Let I be an arbitrary closed interval, we will show that there exists a subinterval $J \subseteq I$ with $J \subseteq F_n^c$.

Consider

$$E_k = \bigcap_{i,j \leq k} \left\{ x \ni |f_i(x) - f_j(x)| \leq \frac{1}{5n} \right\}$$

Motivation: this comes from working backwards from 4-5 triangle inequalities that will appear later.

Some observations: E_k is closed by the continuity of the f_i (good exercise). We also have $E_k \subseteq E_{k+1}$. Moreover, $\bigcup_k E_k = \mathbb{R}$ because the $f_i \to f$ are Cauchy.

We'll now look for an interval entirely contained in the complement. Let $I \subset \mathbb{R}$ be an interval, then write $I = \bigcup_k (I \bigcap E_k)$. Baire tells us that I is not meager, so at least one term appearing in this union is *not* nowhere dense, i.e. there is some k for which $I \bigcap E_k$ is not nowhere dense, i.e. it contains an open interval (it has a nonempty interior, and its already closed, and thus it contains an interval).

So let J be this open interval. We want to show that $J \subseteq F_n^c$. If $x \in J$, then $x \in E_k$ as well, and so $|f_i(x) - f_j(x)| \leq 1/5n$ for all $i, j \geq k$. So let $i \to \infty$, so $|f(x) - f_j(x)| \leq 1/5n$ for all $j \geq k$.

Now for any $x \in J$, there exists some interval $I(x) \subseteq J$ depending on x such that $|f(y) - f_k(x)| \leq 2/5n$. (where we will rewrite this as $f(y) - f_k(y) + f_k(y) - f_k(x)$).

This implies that $\omega_f(x) \leq 4/5n$. $\square$.

2.2 Proof of Theorem 4

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Recall that f is Riemann integrable iff for any ε there exists a partition $P_\varepsilon = \{a = x_1 \leq x_2 \leq \cdots x_n = b\}$ of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, \varepsilon) \leq \varepsilon$, where this expression is equal to

$$\sum_n \sup_{y,z \in [x_n, x_{n+1}]} |f(y) - f(z)|(x_{n+1} - x_n)$$

($\Rightarrow$): Let $\varepsilon > 0$ and n be fixed, and produce a partition P_ε so that this sum is less than ε/n.

Recall that we want to show that F_n is null.

Now exclude from this sum all intervals that miss F_n, making it no bigger. We also know that in F_n, the sups are no greater than $1/n$,

$$\varepsilon/n \geq \sum \text{stuff} \geq \sum \frac{1}{n}(x_{n+1} - x_n)$$

($\Leftarrow$): Suppose D_f is null and let $\varepsilon > 0$ be arbitrary, we want to construct P_ε. Choose $n > 1/\varepsilon$ and $F_n \subseteq D_f$ is closed and bounded and thus compact. But a compact measure zero interval can in fact be covered by *finitely* many open intervals. So F_n is covered by finitely many intervals $\{I_n\}^N$ with $\sum |I_n| \leq \varepsilon$.

Now if $x \notin F_n$, then $\exists \delta(x) > 0$ where $\sup_{y,z \in B_\delta(x)} |f(y) - f(z)| < \frac{1}{n} < \varepsilon$

Since $(\bigcup_j I_j)^c$ is compact, there's a finite cover $I_{N+1}, \cdots I_{N'}$ covering F_n^c.