Title

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1 | Monday August 12

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

1.1.2 Chapter 3-4

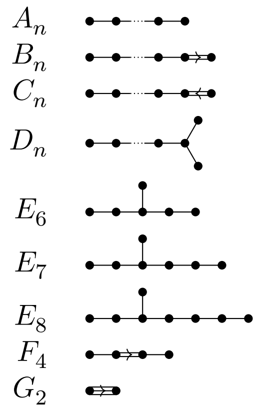
We will describe the following series of correspondences:

$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\oplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:

Figure 1: The Dynkin diagrams of the simple root systems



1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1.4.1.
A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned}[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y].\end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Exercise 1.4.1 : Show that $[x, y] = -[y, x]$.

Definition 1.4.2.
Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\mathrm{End}(V) = \left\{ f : V \to V \;\middle|\; V \text{ is linear} \right\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 1.5.1 : Verify that V is a Lie algebra.

Definition 1.5.1.
Define

$$\mathfrak{sl}(n, V) = \left\{ f \in \mathfrak{gl}(n, V) \;\middle|\; \mathrm{Tr}(f) = 0 \right\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n, V)$.).

Definition 1.5.2.
A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 1.5.3.
The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \;\middle|\; MA - A^T M = 0 \right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

Definition 1.5.4.
The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \;\middle|\; MA - A^T M = 0 \right\} \text{ where}$$
$$M = \begin{cases} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \end{array} \right) & n = 2\ell + 1 \text{ odd}, \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

Proposition 1.5.1.
The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and zero elsewhere.

x is determined to force the trace to be zero

- For type A_ℓ, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2 \left(\frac{\ell(\ell + 1)}{2} \right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type B_ℓ, we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array}\right).$$

Exercise 1.5.2 : Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}$$

Theorem 1.5.1.
These are *all* of the isomorphisms between any of these types of algebras, in any dimension.