Interesting Topological Spaces in Algebraic Geometry

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> Due to Kunihiko Kodaira's classification of complex surfaces, we know that any compact hyperkähler 4-manifold is either a K3 surface or a compact torus T^{4}. (Every Calabi–Yau manifold in 4 (real) dimensions is a hyperkähler manifold, because SU(2) is isomorphic to Sp(1).)

> As was discovered by Beauville, the Hilbert scheme of k points on a compact hyperkähler 4-manifold is a hyperkähler manifold of dimension 4k. This gives rise to two series of compact examples: Hilbert schemes of points on a K3 surface and generalized Kummer varieties.

2 Intro/Motivation

> Ursula Whitcher

Assume the universe is a “space”. Which one is it? What structures does it have? How many possible spaces *could* it be, and how can we test to find out?

3 Analogies

Notation: all dimensions are over $\mathbb{R}$.

Impossible goal: pick a category, understand all of the objects and all of the maps. Two main categories with a forgetful functor: **Diff** $\longrightarrow$ **Top**. Question:

- What's in the "image" of this functor? (Manifolds that admit a differentiable structure.)
- What is the "fiber" above a given topological manifold? (Distinct differentiable structures)

Differentiable Manifolds: classified by geometric structure in low dimensions (≤ 4), algebraic data/methods in high dimensions

3.1 Topological Category

Identify objects up to homeomorphism

- Dimension 0: The point (terminal object)
- Dimeions 1: $S^1, \mathbb{R}$
- Dimension 2: $\left\langle \mathbb{S}, \mathbb{T}, \mathbb{RP} \;\middle|\; \mathbb{S} = 0, 3\mathbb{RP} = \mathbb{RP} + \mathbb{T} \right\rangle$. Classified by π_1 (orientability and "genus"). Riemann, Poincare, Klein.
- Dimension 3: Can always be given a unique smooth structure, see uniformization.
- Dimension 4:
- Dimension $n \geq 5$:

3.2 Smooth Category

Generally expect things to split into more classes.

- 2-manifolds: Homeomorphic $\iff$ diffeomorphic. Every surface admits a complex structure and a metric. Thus always orientable.
  - Uniformization: Holomorphically equivalent to a quotient of one of three spaces
    - $\mathbb{CP}^1$, positive curvature (spherical)
    - $\mathbb{C}$, zero curvature (flat)
    - $\mathbb{H}$ (equiv. $\mathbb{D}^\circ$), negative curvature (hyperbolic)
  - Stratified by genus:
    - Genus 0: Only $\mathbb{CP}^1$
    - Genus 1: All of the form $\mathbb{C}/\Lambda$, with a distinguished point $[0]$, i.e. an elliptic curve. Has a topological group structure!
- 3-manifolds: Thurston's Geometrization
  - Oriented prime 3-manifolds can be decomposed into geometric "pieces" of 8 possible types
  - Geometric structure: a diffeo $M \cong \tilde{M}/\Gamma$ where Γ is a discrete Lie group acting freely/transitively on X
- 4-manifolds: classified in the topological category by surgery, but not in the smooth category
- n-manifolds, $n \geq 5$: classified by surgery

4 Moduli Spaces

$$
\begin{tikzcd}
\mathbb{k}^{\ell} \arrow[r] & E \arrow[d] \\
& \mathrm{Gr}(\ell, V)/\mathbb{k}
\end{tikzcd}
$$

The canonical bundle:

$$
\begin{tikzcd}
\mathbb{R}^{n+1} \arrow[r] & E \arrow[d] \\
& \mathbb{RP}^{n}
\end{tikzcd}
$$

Here $E \coloneqq \{(\mathbf{v}, L)\}$ where L is a line through $\mathbf{0}$ and $\mathbf{v}$ is a vector on L.

5 Elliptic Curves

- Equivalently, Riemann surfaces with one marked point.
- Equivalently, $\mathbb{C}/\Lambda$ a lattice, where homothetic lattices (multiplication by $\lambda \in \mathbb{C} \setminus \{0\}$) are equivalent.
- Parameterized by a moduli space:
  - For $X = \mathbb{C}/\Lambda$ choose a positively oriented basis $\{z, w\}$ for Λ
    * Note: push into meridians on a torus, generators of $H_1(X)$, and require that their intersection is $+1$.
  - Replace $z \mapsto 1, w \mapsto \dfrac{w}{z}$; the orientation condition forces $\Im(w) > 0$ so this yields a point $w \in \mathbb{H}$.

6 Kahlers

- For complex manifold, replace Riemannian metric with a Hermitian metric (positive definite sesquilinear inner product on tangent bundle)
- If skew-symmetric part is symplectic (closed and nondegenerate) then the metric is Kahler
- Includes smooth projective varieties, but not all complex manifolds
- Specialize to Calabi-Yaus: compact and Ricci-flat, or first Chern class vanishes

7 Calabi-Yaus

- As manifolds: Ricci-flat, i.e. Ricci curvature tensor vanishes (measures deviation of volumes of “geodesic balls” from Euclidean balls of the same radius).
- Applications: Physicists want to study G_2 manifolds (an exceptional Lie group, automorphisms of octonions), part of M-theory uniting several superstring theories, but no smooth or complex structures. Indirect approach: compactify an 11-dimension space, one small S^1 dimension $\longrightarrow$ 10 dimensions, 4 spacetime and 6 “small” Calabi-Yau.