Floer Talk

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Tuesday 14^{th} April, 2020

Contents

1 Background and Notation

From the text:

- $(W, \omega \in \Omega_2(W))$ is a (compact?) symplectic manifold
- $C^\infty(A, B)$ is the space of smooth maps with the C^∞ topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C^\infty_{\text{loc}}(A, B)$ is the space with the C^∞ uniform convergence topology on compact subsets of A
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$ given by $H_t \in C^\infty(W; \mathbb{R})$ is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  where $\mathcal{L}W$ is the contractible loop space of W, $u : \mathbb{D} \longrightarrow W$ is an extension of $x : S^1 \longrightarrow W$ to the disc with $u(\exp(2\pi i t)) = x(t)$.

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \displaystyle\int_0^1 (H_t\ dt - p\ dq)$.

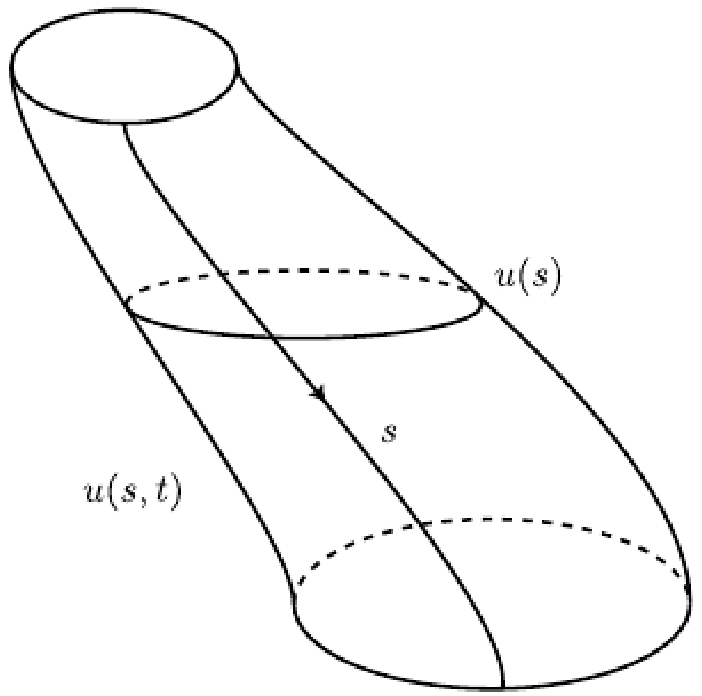
- Critical points of the action functional $\mathcal{A}_H$ are given by orbits, i.e. contractible loops $x, y \in \mathcal{L}W$
- In general, x, y are two periodic orbits of H of period 1.

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

  This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be J-holomorphic curves.

- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
  - They correspond to "embedded cylinders" with sides u and contractible caps x, y regarded as loops in W.
  - They also correspond to paths in $\mathcal{L}W$ from $x \longrightarrow y$ (precisely: trajectories of the vector field $-\mathrm{grad}\mathcal{A}_H$)



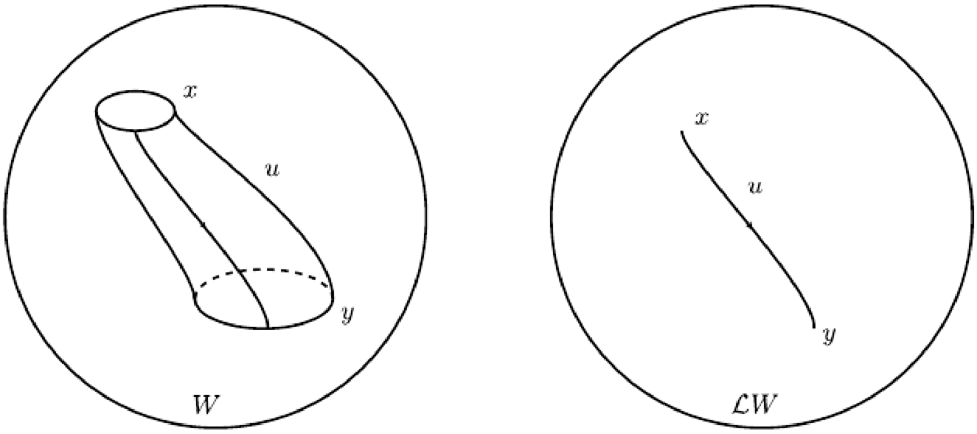


Fig. 6.5

Here $u(s) \in \mathcal{L}W$ is a loop with value at time t given by $u(s,t)$, and $\lim_{s \longrightarrow -\infty} u_s(t) = x$, $\lim_{s \longrightarrow \infty} u_s(t) = y$.

- The energy of a solution is $E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2 \, dsdt$.
- $\mathcal{M} = \left\{ u \in C^\infty(\mathbb{R}; \mathcal{L}W) \;\middle|\; E(u) < \infty \right\}$ (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x,y)$ is the space of solutions of the Floer equation connecting orbits x and y.
- $C_{\searrow}(x,y)$:

$$C_{\searrow}(x,y) \coloneqq \{ u \in C^\infty(\mathbb{R}\times S^1; W) \;\Big|\; \lim_{s\longrightarrow -\infty} u(s,t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s,t) = y(t), \\ \left|\frac{\partial u}{\partial s}(s,t)\right| \leq Ke^{-\delta|s|}, \quad \left|\frac{\partial u}{\partial t}(s,t) - X_H(u)\right| \leq Ke^{-\delta|s|} \}$$

where $K, \delta > 0$ are constants depending on u. So

$$|\partial_s u(s,t)|, |\partial_t u(s,t) - X_H(u)| \sim e^{|s|}.$$

From the Appendices

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator: $\dim\ker - \dim\operatorname{coker}$.
- Fredholm operators: those for which the index makes sense, i.e. $\dim\ker < \infty, \dim\operatorname{coker} < \infty$.
- Elliptic operators: generalize the Laplacian Δ, coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset

- Sobolev spaces: in dimension 1, define $\|u(t)\|_{s,p} = \sum_{i=0}^{s} \left\| \partial_t^i u(t) \right\|_{L^p}$ on $C^\infty(\overline{U})$, then take the completion and denote $W^{s,p}(\overline{U})$. Yields a distribution space, elements are functions with weak derivatives.
- Distribution: $C_c^\infty(U)^\vee$, the dual of the space of smooth compactly supported functions on an open set $U \subset \mathbb{R}^n$.

2 Talk

Overview: Analyze the space $\mathcal{M}(x, y)$ of solutions to the Floer equation connecting two orbits x, y of H. Show $\mathcal{M}(x, y)$ is in fact a manifold of dimension $\mu(x) - \mu(y)$.

Strategy:

1. Describe $\mathcal{M}(x, y)$ as the zero set of a section of a vector bundle over the Banach manifold $\mathcal{P}(x, y)$.
2. Apply the Sard-Smale theorem: perturb H to make $\mathcal{M}(x, y)$ the inverse image of a regular value of some map.
3. Show that the tangent maps (?) are Fredholm operators of index $\mu(x) - \mu(y) = \dim \mathcal{M}(x, y)$.

Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of $\mathcal{F}$.

2.1 Review 8.2

What is $\mathcal{F}$?

We started with the unadorned Floer map:

$$\begin{aligned} \mathcal{F} : \mathcal{C}^\infty \left(\mathbf{R} \times S^1; W \right) &\longrightarrow \mathcal{C}^\infty \left(\mathbf{R} \times S^1; TW \right) \\ u &\mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \operatorname{grad}_u (H_t) \end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \operatorname{grad} H_t(u). \end{aligned}$$

What is the LHS? It is the space of maps

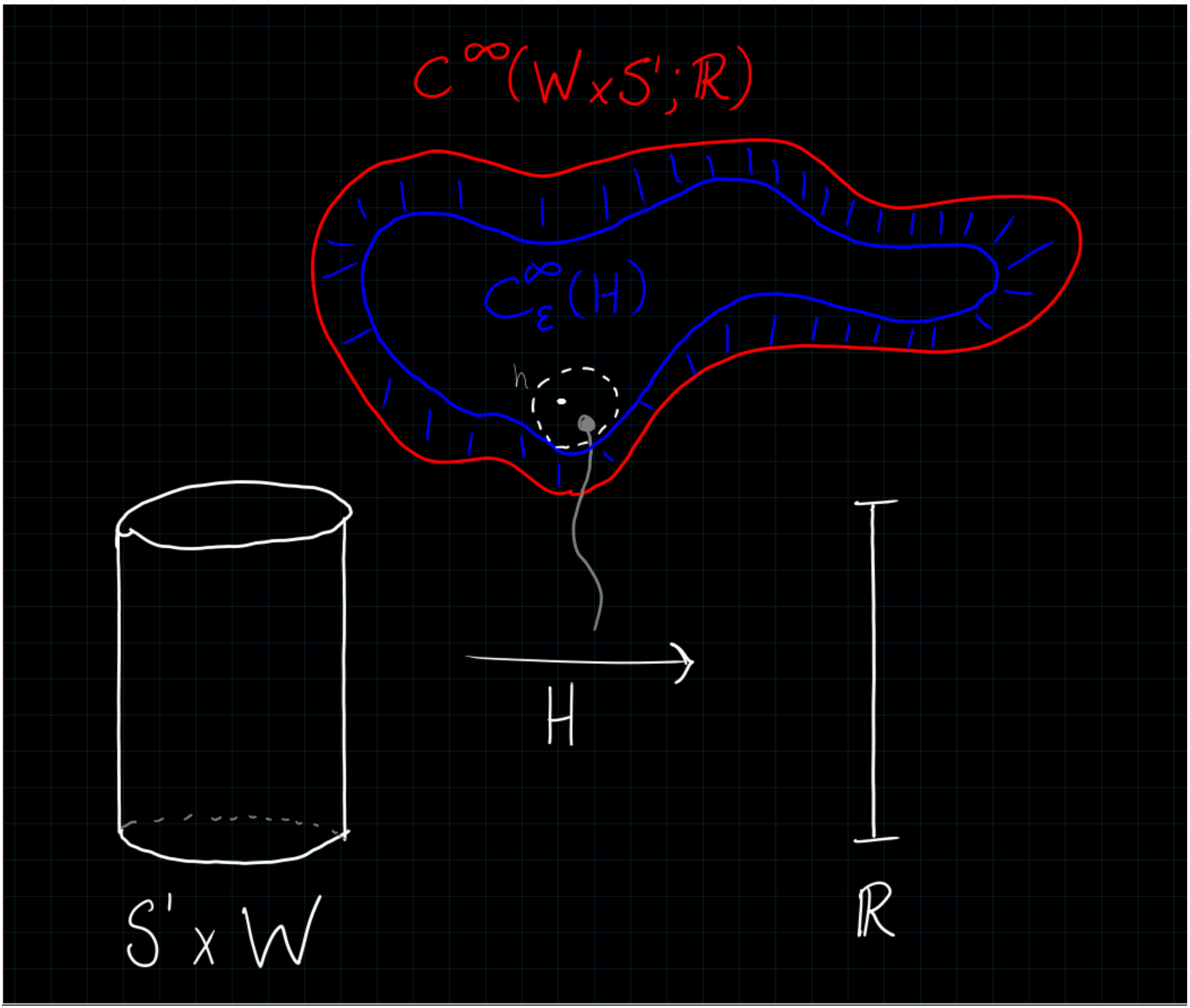
$$\begin{aligned} \mathcal{P}^{1,p}(x, y) :? &\longrightarrow ? \\ (s, t) &\mapsto \exp_{w(s,t)} Y(s, t). \end{aligned}$$

where $Y \in W^{1,p}(w^* TW)$ and $w \in C^\infty_{\searrow}(x, y)$.

2.2 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Start by trying to construct a subspace $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C_\varepsilon^\infty(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C_\varepsilon^\infty(H)$ denote a perturbation of H.
- Fix $\varepsilon = \left\{\varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0}\right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W, t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left|d^k h(\mathbf{x}, t)\right| = \max\left\{d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k\right\},$$

the maximum over all sets of multi-indices α of length k.

Note: I interpret this as

$$d^{\alpha_1,\alpha_2,\cdots,\alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1}\, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

the partial derivatives wrt the corresponding variables.

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_\varepsilon = \sum_{k\geq 0} \varepsilon_k \sup_{(x,t)\in W\times S^1} \left| d^k h(x,t) \right|.$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

- Obtain the computable form

$$\|h\|_\varepsilon = \sum_{k\geq 0} \varepsilon_k \sup_{(x,t)\in W\times S^1} \sup_{i, z\in B(0,1)} \left| d^k (h \circ \Psi_i^{-1})(z) \right|.$$

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \;\middle|\; \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

  which is a Banach space (normed and complete).

- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Proposition 2.1.
Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma 2.2.
$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Proof (of Lemma, Sketch).

First prove for C^0:

- **Idea:** reduce to polynomials in $\mathbb{R}^m$.
- Embed $W \times S^1 \hookrightarrow [-M, M]^m \cong I^m \subset \mathbb{R}^m$ for some large m, reduces to proving it for $C^\infty(I^m; \mathbb{R})$.

- Recall Stone-Weierstrass:

  For $A \leq C^0(X;\mathbb{R})$ a subalgebra with X compact Hausdorff and A containing a nonzero constant function, A is dense iff it separates points (for all $a \neq b \in X$ there exists $f \in A$ such that $f(a) \neq f(b)$)
- Apply to $A = \mathbb{Q}[x_1, \cdots, x_m]$ the subalgebra of polynomial functions, the nonzero constant function $c(x) = 1$, and show it separates points via $f(x) = x - a$, then $f(a) = 0$ and $f(b) = a - b \neq 0$ by assumption.
- Thus A is a countable dense subset.

Then prove for C^1:

- **Idea**: Take polynomials convolved with a countable sequence of bump functions, which is still a countable dense subset.
- Choose a smooth bump function χ supported on $B(0,1)$
- Define the sequence $\chi_k(x) := k^m \chi(kx)$.
- Prove that $(f * \chi_k) \overset{k\longrightarrow\infty}{\Longrightarrow} f$ in the C^0_{loc} sense (?)
- Show that for a fixed k, any other sequence $g_\ell \longrightarrow f$ in C^∞_{loc}, we have $g_\ell * \chi_k \longrightarrow f * \chi_k$ in the C^0_{loc} sense using

$$|g_\ell - f| \longrightarrow 0 \implies \sup_K \left| \frac{\partial}{\partial x_i}(g_\ell - f) * \chi_k \right| \leq \sup_k |g_\ell - f| \cdot (\cdots) \longrightarrow 0 \quad \forall i$$

- Conclude $\lim_\ell \lim_k g_\ell * \chi_k = f$.
- Taking g_ℓ to be polynomial approximations, the following subset is countable and dense:

$$\bigcup_{k \in \mathbb{Z}^{\geq 0}} \left\{ P * \chi_k \;\middle|\; P \in \mathbb{Q}[x_1, \cdots, x_m] \right\}$$

  which are pushed through the charts Ψ_i to actually compute.

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The second part of this proof generalizes to C^∞.

Proof (of Proposition, Sketch).

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1;\mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1;\mathbb{R})$ for the f_n, define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x,t)| \leq 2^{-n}$$

  which is summable.

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Why does this imply density? I don't know.

The next proposition establishes a version of this theorem with compact support:

Proposition 2.3.
For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the C^1 topology by functions in C^∞_ε supported in U.

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C^\infty_\varepsilon(H_0) \;\middle|\; h(x,t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Go on to show that for $\|h\|_\varepsilon \ll 1$, the $\mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$ and are nondegenerate.

2.3 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Embed $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

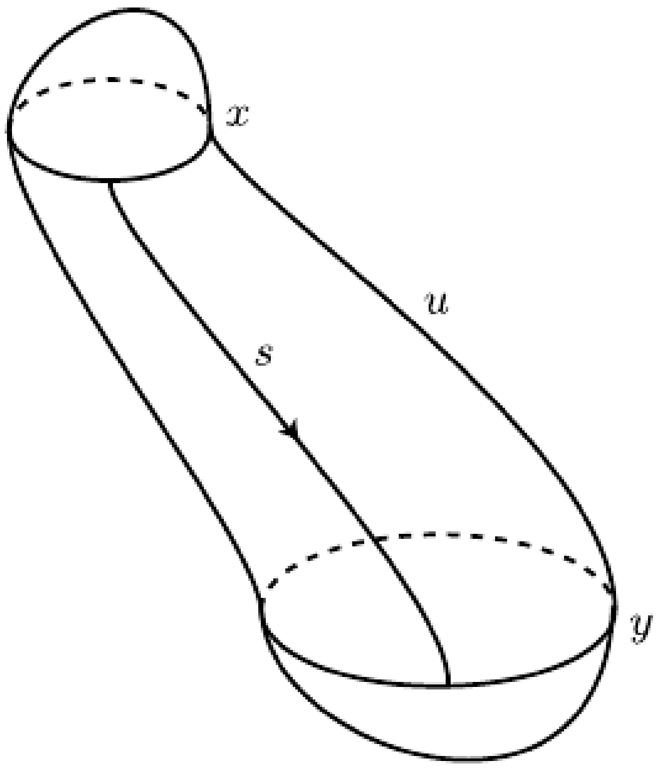
We can then identify im $\mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$ or $L^p(\mathbb{R} \times S^1; W)$, and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the target spaces here.

Recall that x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$ (i.e. solutions to the Floer equation), and $C_\searrow(x, y)$ was the set of maps $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions.

Fix a solution $u \in \mathcal{M}(x, y) \subset C^\infty_{\text{loc}}(\mathbb{R} \times S^1; W)$.

We lift each map to $\tilde{u} : S^2 \longrightarrow W$ in the following way: the loops x, y are contractible, so they bound discs. So we extend according to:



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Recall assumption 6.22: every smooth map $w : S^2 \longrightarrow W$ yields a symplectic trivialization of w^*TW (e.g. when $\pi_2(W) = 0$, so every map from S^2 extends to B^3).

Trivialize the symplectic fiber bundle $\tilde{u}^*TW$ to obtain an orthonormal unitary frame $\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$ depending smoothly on $(s,t) \in S^2$, where $\lim_{s\longrightarrow\infty} Z_i$ exists for each i. We also require that $\partial_s Z_i, \partial_s^2 Z_i, \partial_s\partial_t Z_i \overset{s\longrightarrow\pm\infty}{\longrightarrow} 0$ for each i.

This frame defines a chart about u of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R}\times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y) \\ \mathbf{y} = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

Since $(d\exp)_0 = \mathrm{id}$, we have $(d\iota)_0(\mathbf{y}) = \sum_i y_i Z_i$.

We'll now consider and compute the differential of

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R}\times S^1; TW\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1; \mathbb{R}^m\right) \\ u &\longmapsto \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right). \end{aligned}$$

Take the vector $Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$, where we view Y as a vector in $\mathbb{R}^m$ tangent to W, given by $Y = \sum y_i Z_i$.

We write

$$\mathcal{F}(u+Y) = \frac{\partial(u+Y)}{\partial s} + J(u+Y)\frac{\partial(u+Y)}{\partial t} - J(u+Y)X_t(u+Y)$$

and extract the part that is linear in Y:

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y).$$

Lemma 2.4 ***(Acting by Derivation).***
For any $J \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and $Y, v :? \longrightarrow \mathbb{R}^m$ we have

$$(dj)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

There is a proof.

For every such smooth map $u : \mathbb{R} \times S^1 \longrightarrow W$, $(d\mathcal{F})_u(Y) = O_1 + O_0$ where O_i are differential operators of order i, and in fact O_1 can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$ where $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^n)$ is linear of order 0, and in fact we have

Proposition 2.5.
If u solves Floer's equation, then $(d\mathcal{F})_u = \bar{\partial} + S(s,t)$ where S is linear, tends to a symmetric operator as $s \longrightarrow \pm\infty$, and $\lim \partial_t S = 0$ uniformly in t.

There is a very long computational proof.

Denote the order 0 part of $(d\mathcal{F})_u$ as $Y \mapsto S \cdot Y$ so $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{End}(\mathbb{R}^m)$ and define $S^{\pm} := \lim_{s \longrightarrow \pm\infty} S(s, \cdot)$.

Proposition 2.6.
The equation $\partial_t Y = J_0 S^{\pm} Y$ linearizes Hamilton's equation $\dot{z} = X_t(z)$ at $x = \lim_{s \longrightarrow \pm\infty} u$ for S^+ and S^- respectively.

Proof: uses previous proposition.

Given a solution u, the product

$$\begin{aligned} u \cdot s &:? \longrightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and $\mathcal{F}(u \cdot s) = 0$ for all s.

Punchline:

Thus $\dfrac{\partial u}{\partial s}$ is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s}\mathcal{F}(u \cdot s) = (d\mathcal{F})_u\left(\frac{\partial u}{\partial s}\right).$$

Along any nonconstant solution connecting x and y, $\dim \ker(d_{\mathcal{F}})_u \geq 1$.