Linearization Continued
Section 8.4 Follow-Up

D. Zack Garza

April 2020

Review

– The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

– We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

– We use the assumption:
   For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W) \rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

– We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

– We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow \mathcal{P}^{1,p}(x, y)$$

$$Y = (y_1, \ldots, y_{2n}) \longmapsto \exp_u\left(\sum y_i Z_i\right).$$

– We regard $Y(s, t)$ as a tangent vector to W in some Euclidean embedding.

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– We seek to compute the composite map in charts:

$$\begin{array}{ccc}
& \mathcal{F}_u & \\
& \mathcal{F} & \\
W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \xrightarrow{\ i\ } \mathcal{P}^{1,p}(x,y) \xrightarrow{\ \mathcal{F}\ } L^p\left(\mathbb{R} \times S^1; TW\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right) \\
\\
u \xrightarrow{\qquad\qquad \mathcal{F} \qquad\qquad} \dfrac{\partial u}{\partial s} + J(u)\left(\dfrac{\partial u}{\partial t} - X_t(u)\right) \\
\\
(y_1, \ldots, y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)
\end{array}$$

Add a Tangent

$$\mathcal{F}(u) = \qquad \frac{\partial u}{\partial s} \qquad + J(u)\frac{\partial u}{\partial t} \qquad - J(u)X_t(u)$$

$$\mathcal{F}(u + Y) = \quad \frac{\partial (u+Y)}{\partial s} \quad + J(u + Y)\frac{\partial (u+Y)}{\partial t} \quad - J(u + Y)X_t(u + Y)$$

Extract the part that is linear in Y and collect terms:

$$(d\mathcal{F})_u(Y)$$
$$= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y)$$
$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right)$$
$$\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y) \right)$$

.

Review

Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - J dv(Y)$$

$$(d\mathcal{F})_u(Y) = \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right)$$
$$+ \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u (Y) \right)$$
$$= \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$
$$+ \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right.$$
$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

Use the fact that this is $O_1 + O_0$ in Y.

Review

Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

1. *S is linear*
2. *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s, t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$O_0 = S(y_1, \cdots, y_{2n}) = \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right.$$

$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right]$$

$$= \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t \right) \right.$$

$$\left. + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right].$$

- Claim: the terms in blue and orange vanish in the limit $s \longrightarrow \pm\infty$, so it suffices to prove that the red term limits to a symmetric operator.

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$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since u is a solution and

$$\frac{\partial u}{\partial s} \stackrel{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \stackrel{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

This seems to be the full argument for the blue term.

Proof: Orange Term Vanishes (1 and 3)

$$\frac{\partial Z_i}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

Follows since the frame Z_i was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s\, \partial t} \quad \curvearrowright Z_i \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0.$$

This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in t?

Proof: Symmetry

Write the remaining red term as

$$A\left(y_1, \ldots, y_{2n}\right) = \sum y_i \left(J(u)\frac{\partial Z_i}{\partial t} - J(u)\left(dX_t\right)_u \left(Z_i\right) \right).$$