

# Moduli Spaces

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## List of Theorems

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## 1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, ... Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

## 1.1 Representability

Last time: Fix an  $S$ -scheme, i.e. a scheme over  $S$ .

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \mathrm{hom}_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where  $T' \xrightarrow{f} T$  is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form} \end{aligned}$$

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & X \\ & \searrow & \nearrow \\ & T & \end{array} .$$

**Theorem 1.1 (Yoneda).**

$$\mathrm{hom}_{\mathrm{Fun}}(h_x, F) = F(x).$$

**Corollary 1.2.**

$$\mathrm{hom}_{\mathrm{Sch}/S}(x, y) \cong \mathrm{hom}_{\mathrm{Fun}}(h_x, h_y).$$

**Definition 1.2.1 (Moduli Functor).**

A **moduli functor** is a map

$$\begin{aligned}
F : (\text{Sch}/S)^{\text{op}} &\longrightarrow \text{Set} \\
F(x) &= \text{"Families of something over } x\text{"} \\
F(f) &= \text{"Pullback"}.
\end{aligned}$$

**Definition 1.2.2** (Moduli Space).

A **moduli space** for that “something” appearing above is an  $M \in \text{Obj}(\text{Sch}/S)$  such that  $F \cong h_M$ .

Now fix  $S = \text{Spec}(k)$ .

$h_m$  is the functor of points over  $M$ .

**Remark (1)**  $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{“families over Spec } k\text{”} = F(\text{Spec } k)$ .

**Remark (2)**  $h_M(M) \cong F(M)$  are families over  $M$ , and  $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{\text{Univ}}$  is the universal family.

Every family is uniquely the pullback of  $\xi_{\text{Univ}}$ . This makes it much like a classifying space.

For  $T \in \text{Sch}/S$ ,

$$\begin{aligned}
h_M &\xrightarrow{\cong} F \\
f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\text{Univ}}).
\end{aligned}$$

where  $T \xrightarrow{f} M$  and  $f = h_M(f)(\text{id}_M)$ .

**Remark (3)** If  $M$  and  $M'$  both represent  $F$  then  $M \cong M'$  up to unique isomorphism.

$$\xi_M \qquad \qquad \xi_{M'}$$

$$M \xrightarrow{f} M'$$

$$M' \xrightarrow{g} M$$

$$\xi_{M'} \qquad \qquad \xi_M$$

which shows that  $f, g$  must be mutually inverse by using universal properties.

**Example 1.1.**

A length 2 subscheme of  $\mathbb{A}_k^1$  (??) then

$$F(S) = \{V(x^2 + bx + c)\} \subset \mathbb{A}'_5$$

where  $b, c \in \mathcal{O}_s(s)$ , which is functorially bijective with  $\{b, c \in \mathcal{O}_s(s)\}$  and  $F(f)$  is pullback.

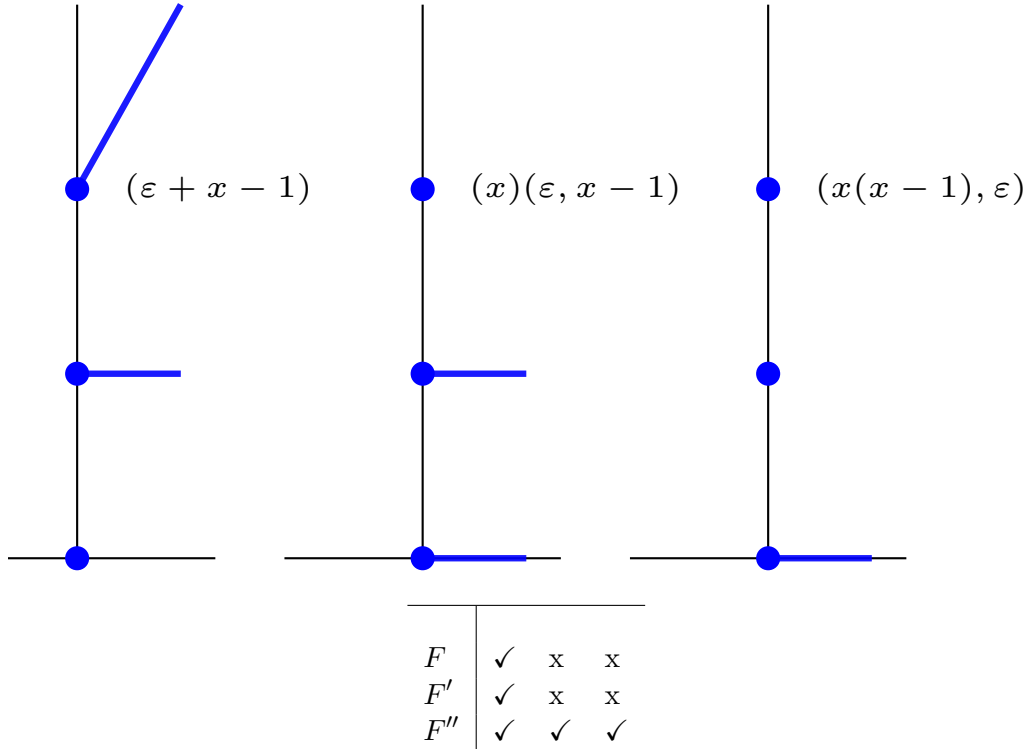
Then  $F$  is representable by  $\mathbb{A}_k^2(b, c)$  and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where  $b, c \in k[b, c]$ .

Moreover,  $F'(S)$  is the set of effective Cartier divisors in  $\mathbb{A}'_5$  which are length 2 for every geometric fiber.  $F''(S)$  is the set of subschemes of  $\mathbb{A}'_5$  which are length 2 on all geometric fibers. In both cases,  $F(f)$  is always given by pullback.

Problem:  $F''$  is not a good moduli functor, as it is not representable. Consider  $\text{Spec } k[\varepsilon]$ .



$$\begin{array}{ccccc}
 \text{Spec } k & \xleftarrow{i} & \text{Spec } k[\varepsilon] & & \\
 & & & \nearrow & \\
 F(\text{Spec } k[\varepsilon]) & \xrightarrow{F(i)} & F(\text{Spec } k) & & = F'(\text{Spec } k) \\
 \uparrow \subset & & \uparrow \in & & \searrow \\
 T_p F','' & & P = V(x(x-1)) & & = F''(\text{Spec } k)
 \end{array}$$

We think of  $T_p F',''$  as the tangent space at  $p$ .

If  $F$  is representable, then it is actually the Zariski tangent space.

$$\begin{array}{ccc}
M(\mathrm{Spec} \, k[\varepsilon]) & \longrightarrow & M(\mathrm{Spec} \, k) \\
\uparrow \subset & & \uparrow \subset \\
T_p M & \longrightarrow & p \\
& \nearrow & \searrow \\
& \mathrm{Spec} \, k & \\
& \nwarrow & \nearrow \\
\mathrm{Spec} \, k[\varepsilon] & \longrightarrow & \mathrm{Spec} \, \mathcal{O}_{M,p} \subset M
\end{array}$$
  

$$\begin{array}{ccc}
& & k \\
& \nearrow & \uparrow \\
\mathcal{O}_{M,p} & \longrightarrow & k[\varepsilon] \\
\uparrow & & \uparrow \\
\mathfrak{m}_p & & (\varepsilon) \\
\uparrow & & \uparrow \\
\mathfrak{m}_p^2 & & 0
\end{array}$$

Moreover,  $T_p M = (\mathfrak{m}_p / \mathfrak{m}_p^2)^\vee$ , and in particular this is a  $k$ -vector space. To see the scaling structure, take  $\lambda \in k$ .

$$\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda \varepsilon
\end{aligned}$$

$$\lambda^* : \mathrm{Spec} \, (k[\varepsilon]) \longrightarrow \mathrm{Spec} \, (k[\varepsilon])$$

$$\begin{aligned}
\lambda : M(\mathrm{Spec} \, (k[\varepsilon])) &\longrightarrow M(\mathrm{Spec} \, (k[\varepsilon])) \\
\cup & \qquad \cup \\
T_p M &\longrightarrow T_p M.
\end{aligned}$$

**Conclusion:** If  $F$  is representable, for each  $p \in F(\mathrm{Spec} \, k)$  there exists a unique point of  $T_p F$  that are invariant under scaling.

1. If  $F, F', G \in \mathrm{Fun}((\mathrm{Sch}/S)^{\mathrm{op}}, \mathrm{Set})$ , there exists a fiber product

$$\begin{array}{ccc}
F \times_G F' & \cdots \longrightarrow & F' \\
\downarrow & & \downarrow \\
F & \longrightarrow & G
\end{array}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes  $X \times_T Y$  for  $X, Y \rightarrow T$ , where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If  $F, F', G$  are representable, then so is the fiber product  $F \times_G F'$ .  
 4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set},$$

for any  $T \xrightarrow{f} S$  there is an induced functor

$$\begin{aligned} F_T : (\text{Sch}/T) &\rightarrow \text{Set} \\ x &\mapsto F(x). \end{aligned}$$

5.  $F$  is representable by  $M/S$  implies that  $F_T$  is representable by  $M_T = M \times_S T/T$ .

## 1.2 Projective Space

Consider  $\mathbb{P}_{\mathbb{Z}}^n$ , i.e. “rank 1 quotient of an  $n + 1$  dimensional free module”.

### Proposition 1.3.

$\mathbb{P}_{\mathbb{Z}}^n$  represents the following functor

$$\begin{aligned} F : \text{Sch}^{\text{op}} &\rightarrow \text{Set} \\ F(S) &= \mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0 / \sim. \end{aligned}$$

where  $\sim$  identifies diagrams of the following form:

$$\begin{array}{ccccc} \mathcal{O}_S^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \parallel & & \downarrow \cong & & \\ \mathcal{O}_S^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and  $F(f)$  is given by pullbacks.

**Remark**  $\mathbb{P}_S^n$  represents the following functor:

$$\begin{aligned} F_S : (\text{Sch}/S)^{\text{op}} &\rightarrow \text{Set} \\ T &\mapsto F_S(T) = \left\{ \mathcal{O}_T^{n+1} \rightarrow L \rightarrow 0 \right\} / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

*Proof (of Proposition).*

Note:  $\mathcal{O}^{n+1} \rightarrow L \rightarrow 0$  is the same as giving  $n+1$  sections  $s_1, \dots, s_n$  of  $L$ , where surjectivity ensures that they are not the zero section.

So

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0 \right\} / \sim,$$

with the additional condition that  $s_i \neq 0$  at any point.

There is a natural transformation  $F_i \rightarrow F$  by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$  is a subfunctor iff  $F(s) \hookrightarrow G(s)$ .

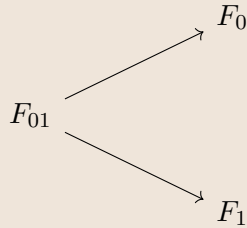
**Claim:** It is enough to show that each  $F_i$  and each  $F_{ij}$  are representable, since we have natural transformations:

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \uparrow & & \uparrow \\ F_{ij} & \longrightarrow & F_j \end{array}$$

and each  $F_{ij} \rightarrow F_i$  is an open embedding (on the level of their representing schemes).

**Example .**

For  $n = 1$ , we can glue along open subschemes



For  $n = 2$ , we get overlaps of the following form:



This claim implies that we can glue together  $F_i$  to get a scheme  $M$ . We want to show that  $M$  represents  $F$ .  $F(s)$  (LHS) is equivalent to an open cover  $U_i$  of  $S$  and sections of  $F_i(U_i)$  satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for  $\mathcal{O}_S^{n+1} \rightarrow L \rightarrow 0$ ,  $U_i$  is the locus where  $s_i \neq 0$  and by surjectivity, this gives a cover of  $S$ .

RHS to LHS comes from gluing. ■

*Proof (of Claim).*

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than  $s_i$ .

So specifying  $F_i(S)$  is equivalent to specifying  $n - 1$  functions  $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$  with  $f_k \neq 0$ . We know this is representable by  $\mathbb{A}^n$ .

We also know  $F_{ij}$  is obviously the same set of sequences, where now  $s_j \neq 0$  as well, so we need to specify  $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$  with  $f_j \neq 0$ . This is representable by  $\mathbb{A}^{n-1} \times \mathbb{G}_m$ , i.e.  $\text{Spec } k[x_1, \dots, \widehat{x_i}, \dots, x_n, x_j^{-1}]$ . Moreover,  $F_{ij} \hookrightarrow F_i$  is open.

What is the compatibility we are using to glue? For any subset  $I \subset \{0, \dots, n\}$ , we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigcap_{i \in I} F_i,$$

and  $F_I \longrightarrow F_J$  when  $I \supset J$ . ■

## 2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space  $\mathbb{P}_{\mathbb{Z}}^n$  constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}_{\mathbb{Z}}^n} : \mathbb{P}_{\mathbb{Z}}^n \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ s &\mapsto \mathbb{P}_{\mathbb{Z}}^n(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for  $L$  a line bundle, up to isomorphisms of diagrams:

$$\begin{array}{ccccc} \mathcal{O}_s^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \parallel & & \downarrow \cong & & \\ \mathcal{O}_s^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

That is, line bundles with  $n + 1$  sections that globally generate it, up to isomorphism.

The point was that for  $F_i \subset \mathbb{P}_{\mathbb{Z}}^n$  where

$$F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \mid s_i \text{ is invertible} \right\}$$

are representable and can be glued together, and projective space represents this functor.



**Remark** Because projective space represents this functor, there is a universal object:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \\ & & \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1) & & \end{array}$$

and other functors are pullbacks of the universal one. (Moduli Space)

**Exercise** Show that  $\mathbb{P}_{\mathbb{Z}}^n$  is proper over  $\text{Spec } \mathbb{Z}$ . Use the evaluative criterion, i.e. there is a unique lift

$$\begin{array}{ccc} \text{Spec } k & \longrightarrow & \mathbb{P}_{\mathbb{Z}}^n \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec } R & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

**Definition 2.0.1** (Equalizer).

For a category  $C$ , we say a diagram  $X \longrightarrow Y \rightrightarrows Z$  is an *equalizer* iff it is universal with respect to the property:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \rightrightarrows & Z \\ & \nwarrow \text{dashed } \exists! & \uparrow & \nearrow & \\ & & S & & \end{array}$$

Note that  $X$  is the universal object here.

**Example 2.1.**

For sets,  $X = \{y \mid f(y) = g(y)\}$  for  $Y \xrightarrow{f,g} Z$ .

**Definition 2.0.2** (Coequalizer).

A **coequalizer** is the dual notion,

$$\begin{array}{ccccc} & & S & & \\ & \nearrow & \uparrow & \nwarrow \text{dashed } \exists! & \\ Z & \rightrightarrows & Y & \longrightarrow & X \end{array}$$

**Example 2.2.**

Take  $C = \text{Sch}/S$ ,  $X/S$  a scheme, and  $X_\alpha \subset X$  an open cover. We can take two fiber products,  $X_{\alpha\beta}, X_{\beta,\alpha}$ :

---


$$\begin{array}{ccc}
X_\alpha & \longrightarrow & X \\
\uparrow & & \uparrow \\
X_{\alpha\beta} & \longrightarrow & X_\beta
\end{array}
\qquad
\begin{array}{ccc}
X_\beta & \longrightarrow & X \\
\uparrow & & \uparrow \\
X_{\beta\alpha} & \longrightarrow & X_\alpha
\end{array}$$

These are canonically isomorphic.

In  $\text{Sch}/S$ , we have

$$\coprod_{\alpha\beta} X_{\alpha\beta} \begin{array}{c} \xrightarrow{f_{\alpha\beta}} \\ \xrightarrow{g_{\alpha\beta}} \end{array} \coprod_{\alpha} X_{\alpha} \longrightarrow X$$

where

$$\begin{aligned}
f_{\alpha\beta} : X_{\alpha\beta} &\longrightarrow X_{\alpha} \\
g_{\alpha\beta} : X_{\alpha\beta} &\longrightarrow X_{\beta};
\end{aligned}$$

this is a coequalizer.

Conversely, we can glue schemes. Given  $X_\alpha \longrightarrow X_{\alpha\beta}$  (schemes over open subschemes), we need to check triple intersections:



Then  $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$  must satisfy the **cocycle condition**:

1.

$$\varphi_{\alpha\beta}^{-1}(X_{\beta\alpha} \cap X_{\beta\gamma}) = X_{\alpha\beta} \cap X_{\alpha\gamma},$$

noting that the intersection is exactly the fiber product  $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$ .

2. The following diagram commutes:

$$\begin{array}{ccc}
X_{\alpha\beta} \cap X_{\alpha\gamma} & \xrightarrow{\varphi_{\alpha\gamma}} & X_{\gamma\alpha} \cap X_{\gamma\beta} \\
& \searrow \varphi_{\alpha\beta} \quad \nearrow \varphi_{\beta\gamma} & \\
& X_{\beta\alpha} \cap X_{\beta\gamma} &
\end{array}$$

Then there exists a scheme  $X/S$  such that  $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_{\alpha} \longrightarrow X$  is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

**Definition 2.0.3** (Zariski Sheaf).

A functor  $F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set}$  is a *Zariski sheaf* iff for any scheme  $T/S$  and any open cover  $T_{\alpha}$ , the following is an equalizer:

$$F(T) \longrightarrow \prod F(T_{\alpha}) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

**Example 2.3.**

Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_{\alpha} T_{\alpha} \longrightarrow T,$$

since giving a local map to  $X$  that agrees on intersections is enough to specify a map from  $T \longrightarrow X$ .

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

**Definition 2.0.4** (Subfunctors, Open/Closed Functors).

Suppose we have a morphism  $F' \longrightarrow F$  in the category  $\text{Fun}(\text{Sch}/S, \text{Set})$ .

- This is a **subfunctor** if  $\iota(T)$  is injective for all  $T/S$ .
- $\iota$  is **open/closed/locally closed** iff for any scheme  $T/S$  and any section  $\xi \in F(T)$  over  $T$ , then there is an open/closed/locally closed set  $U \subset T$  such that for all maps of schemes  $T' \xrightarrow{f} T$ , we can take the pullback  $f^*\xi$  and  $f^*\xi \in F'(T')$  iff  $f$  factors through  $U$ .

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

**Note** This is the same as asking if the subfunctor  $F'$ , which maps to  $F$  (noting a section is the same as a map to the functor of points), and since  $T \longrightarrow F$  and  $F' \longrightarrow F$ , we can form the fiber product  $F' \times_F T$ :

$$\begin{array}{ccc}
F' & \longrightarrow & F \\
\uparrow & & \uparrow \xi \\
F' \times_F T & \xrightarrow{g} & T
\end{array}$$

and  $F' \times_F T \cong U$ .

Note: this is almost tautological!

Thus  $F' \rightarrow F$  is open/closed/locally closed iff  $F' \times_F T$  is representable and  $g$  is open/closed/locally closed.

I.e. base change is representable, and (?).

### Exercise (Tautologous)

1. If  $F' \rightarrow F$  is open/closed/locally closed and  $F$  is representable, then  $F'$  is representable as an open/closed/locally closed subscheme
2. If  $F$  is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

#### Definition 2.0.5 (Open Covers).

A collection of open subfunctors  $F_\alpha \subset F$  is an **open cover** iff for any  $T/S$  and any section  $\xi \in F(T)$ , i.e.  $\xi : T \rightarrow F$ , the  $T_\alpha$  in the following diagram are an open cover of  $T$ :

$$\begin{array}{ccc}
F_\alpha & \longrightarrow & F \\
\uparrow & & \uparrow \xi \\
T_\alpha & \longrightarrow & T
\end{array}$$

#### Example 2.4.

Given

$$F(s) = \{ \mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0 \}$$

and  $F_i(s)$  given by those where  $s_i \neq 0$  everywhere, the  $F_i \rightarrow F$  are an open cover. Because the sections generate everything, taking the  $T_i$  yields an open cover.

#### Proposition 2.1.

A Zariski sheaf  $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$  with a representable open cover is representable.

*Proof.*

Let  $F_\alpha \subset F$  be an open cover, say each  $F_\alpha$  is representable by  $x_\alpha$ . Form the fiber product  $F_{\alpha\beta} = F_\alpha \times_F F_\beta$ . Then  $x_\beta$  yields a section (plus some openness condition?), so  $F_{\alpha\beta} = x_{\alpha\beta}$

representable. Because  $F_\alpha \subset F$ , the  $F_{\alpha\beta} \rightarrow F_\alpha$  have the correct gluing maps.

This follows from Yoneda (schemes embed into functors), and we get maps  $x_{\alpha\beta} \rightarrow x_\alpha$  satisfying the gluing conditions. Call the gluing scheme  $x$ ; we'll show that  $x$  represents  $F$ .

First produce a map  $x \rightarrow F$  from the sheaf axioms. We have a map  $\xi \in \prod_{\alpha} F(x_\alpha)$ , and because we can pullback, we get a unique element  $\xi \in F(X)$  coming from the diagram

$$F(x) \rightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

**Lemma 2.2.**

If  $E \rightarrow F$  is a map of functors and  $E, F$  are zariski sheaves, where there are open covers  $E_\alpha \rightarrow E, F_\alpha \rightarrow F$  with commutative diagrams

$$\begin{array}{ccc} E & \longrightarrow & F \\ \uparrow & & \uparrow \\ E_\alpha & \xrightarrow{\cong} & F_\alpha \end{array}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$\begin{array}{ccc} X & \longrightarrow & F \\ \uparrow & & \uparrow \\ X_\alpha & \xrightarrow{\cong} & F_\alpha \end{array}$$

**Example 2.5.**

For  $S$  and  $E$  a locally free coherent  $\mathcal{O}_S$  module,

$$\mathbb{P}E(T) = \{f^*E \rightarrow L \rightarrow 0\} / \sim$$

is a generalization of projectivization, then  $S$  admits a cover  $U_i$  trivializing  $E$ .

Then the restriction  $F_i \rightarrow \mathbb{P}E$  where  $F_i(T)$  is the above set if  $f$  factors through  $U_i$  and empty otherwise. On  $U_i$ ,  $E \cong \mathcal{O}_{U_i}^{n_i}$ , so  $F_i$  is representable by  $\mathbb{P}_{U_i}^{n_i-1}$  by the proposition. (Note that this is clearly a sheaf.)

---

**Example 2.6.**

For  $E$  locally free over  $S$  of rank  $n$ , take  $r < n$  and consider the functor  $\text{Gr}(k, E)(T) = \{f^*E \rightarrow Q \rightarrow 0\} / \sim$  (a Grassmannian) where  $Q$  is locally free of rank  $k$ .

**Exercise**

- Show that this is representable
- For the Plucker embedding

$$\text{Gr}(k, E) \rightarrow \mathbb{P} \wedge^k E,$$

a section over  $T$  is given by  $f^*E \rightarrow Q \rightarrow 0$  corresponding to

$$\wedge^k f^*E \rightarrow \wedge^k Q \rightarrow 0,$$

noting that the left-most term is  $f^* \wedge^k E$ .

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take  $S = \text{Spec } k$ , then  $E$  is a  $k$ -vector space  $V$ , then sections of the Grassmannian are quotients of  $V \otimes \mathcal{O}$  that are free of rank  $n$ .

Take the subfunctor  $G_w \subset \text{Gr}(k, V)$  where

$$G_w(T) = \{\mathcal{O}_T \otimes V \rightarrow Q \rightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting  $V = W \oplus U$ , then  $G_W = \mathbb{A}(\text{hom}(U, W))$ . If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for  $\mathbb{P}^n$  of all flat families of subschemes.

### 3 Thursday January 16th

#### 3.1 Subfunctors

A functor  $F' \subset F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$  is **open** iff for all  $T \xrightarrow{\xi} F$  where  $T = h_T$  and  $\xi \in F(T)$ .

We can take fiber products:

$$\begin{array}{ccc} F' & \xrightarrow{\quad} & F \\ \uparrow & & \uparrow \\ F' \times_F T & \xrightarrow{\text{Open}} & T \\ \text{Representable} & & \end{array}$$

So we can think of “inclusion in  $F$ ” as being an *open condition*: for all  $T/S$  and  $\xi \in F(T)$ , there exists an open  $U \subset T$  such that for all covers  $f : T' \rightarrow T$ , we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff  $f$  factors through  $U$ .

Suppose  $U \subset T$  in  $\text{Sch}/T$ , we then have

$$h_{U/T}(T') = \begin{cases} \emptyset & T' \longrightarrow T \text{ doesn't factor} \\ \{\text{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is  $h_{U/T}(T') = \text{hom}_T(T', U)$ .

By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{ (a, b) \in F'(T) \times T(T) \mid \xi(b) = \iota(a) \text{ in } F(T) \right\},$$

where  $F' \xrightarrow{\iota} F$  and  $T \xrightarrow{\xi} F$ .

So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of  $F/T'$  as sections of  $F$  over  $T/T'$  (?).

$$\begin{array}{ccc} F' & \xrightarrow{\iota} & F \\ \uparrow & & \uparrow \xi \\ F' \times_F T & \longrightarrow & T \end{array} \quad \begin{array}{c} \swarrow f \circ \xi \\ T' \end{array}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U/S}(T'),$$

and so for  $U \subset T$  in  $\text{Sch}/S$  we have  $h_{U/S} \subset h_{T/S}$  is the functor of maps that factor through  $U$ . We just identify  $h_{U/S}(T') = \text{hom}_S(T', U)$  and  $h_{T/S}(T') = \text{hom}_S(T', T)$ .

**Example 3.1.**

$\mathbb{G}_m, \mathbb{G}_a$ .  $\mathbb{G}_a$  represents giving a global function,  $\mathbb{G}_m$  represents giving an invertible function.

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{G}_a \\ \uparrow & & \uparrow f \in \mathcal{O}_T(T) \\ T' & \longrightarrow & T \end{array} \quad \text{L}$$

where  $T' = \{f \neq 0\}$  and  $\mathcal{O}_T(T)$  are global functions.

## 3.2 Actual Geometry: Hilbert Schemes

The best moduli space!

Want to parameterize families of subschemes over a fixed object. Fix  $k$  a field,  $X/k$  a scheme; we'll parameterize subschemes of  $X$ .

**Definition 3.0.1** (Hilbert Functor).

The hilbert functor is given by

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

which sends  $T$  to closed subschemes  $Z \subset X \times_S T \longrightarrow T$  which are flat over  $T$ .

Here flatness replaces the Cartier condition.

**Definition 3.0.2** (Flatness).

For  $X \xrightarrow{f} Y$  and  $\mathbb{F}$  a coherent sheaf on  $X$ ,  $f$  is flat over  $Y$  iff for all  $x \in X$  the stalk  $F_x$  is a flat  $\mathcal{O}_{y,f(x)}$ -module.

Note that  $f$  is flat if  $\mathcal{O}_x$  is.

Flatness corresponds to varying continuously.

**Warning:** Unless otherwise stated, assume schemes are Noetherian.

Note that everything works out if we only path with finite covers.

**Remark** If  $X/k$  is projective, so  $X \subset \mathbb{P}_k^n$ , we have line bundles  $\mathcal{O}_x(1) = \mathcal{O}(1)$ . For any sheaf  $F$  over  $X$ , there is a hilbert polynomial  $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$ . ( i.e. we twist by  $\mathcal{O}(1)$   $n$  times.)

The cohomology of  $F$  isn't changed by the pushforward into  $\mathbb{P}_n$  since it's a closed embedding, i.e.

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

**Fact (First)** For  $n \gg 0$ ,  $\dim_k H^0 = \dim M_n$ , the  $n$ th graded piece of  $M$ , which is a graded module over the homogeneous coordinate ring whose  $i_* F = \tilde{M}$ .

In general, for  $L$  ample of  $X$  and  $F$  coherent on  $X$ , we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

**Proposition 3.1.**

For  $f : X \longrightarrow S$  projective, i.e. there is a factorization:



$$\begin{array}{ccc}
X & \xrightarrow{\quad} & \mathbb{P}^n \times S \ni \mathcal{O}(1) \\
& \searrow f & \swarrow \\
& S &
\end{array}$$

If  $S$  is reduced, irreducible, locally Noetherian, then  $f$  is flat iff  $P_{\mathcal{O}_{x_s}}$  is constant for all  $s \in S$ .

To be more precise, look the base change to  $X_1$ , and the pullback of the fiber?  $\mathcal{O} \Big|_{x_i} ?$

Note: not using the word “integral” here!  $S$  is flat iff the hilbert polynomial over the fibers are constant.

Example: The zero-dimensional subschemes  $Z \in \mathbb{P}_k^n$ , then  $P_Z$  is the length of  $Z$ , i.e.  $\dim_k(\mathcal{O}_Z)$ . And  $P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z)$ .

For two closed points in  $\mathbb{P}^2$ ,  $P_Z = 2$ .

Consider the affine chart  $\mathbb{A}^2 \subset \mathbb{P}^2$ , which is given by  $\text{Spec } k[x, y]/(y, x^2) \cong k[x]/(x^2)$  and  $P_Z = 2$ . I.e. in flat families, it has to record how the tangent directions come together.

Example: Consider the flat family  $xy = 1$  (flat because it’s an open embedding) over  $k[x]$ , here we have points running off to infinity.

**Modified proposition:** A sheaf  $F$  is flat iff  $P_{F_S}$  is constant.

*Proof of proposition:* Assume  $S = \text{Spec } A$  for  $A$  a local Noetherian domain.

**Lemma:** For  $F$  a coherent sheaf on  $X/A$  is flat, we can take the cohomology via global sections  $H^0(X; F(n))$ . This is an  $A$ -module, and is a free  $A$ -module for  $n \gg 0$ .

*Proof of Lemma:* Assumed  $X$  was projective, so just take  $X = \mathbb{P}_A^n$  and let  $F$  be the pushforward. There is a correspondence sending  $F$  to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module  $\sum_{n \gg 0} H^0(\Pi_A^m; F(n))$ . This is equal to  $\oplus_{n \gg 0} H^0(\mathbb{P}_A^m; F(n))$  and taking the associated sheaf ( $Y \mapsto \tilde{Y}$ , as per Hartshorne’s notation) which is free, and thus  $F$  is free.

See tilde construction in Hartshorne, essentially amounts to localizing free tings.

Conversely, take an affine cover  $U_i$  of  $X$ . We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume  $H^i(\mathbb{P}^m; F(n)) = 0$  for  $n \gg 0$ , and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^m; F(n)) \longrightarrow \mathcal{C}^0(\underline{U}; F(n)) \longrightarrow \cdots \longrightarrow \mathcal{C}^m(\underline{U}; F(n)) \longrightarrow 0.$$

Assuming  $F$  is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right.

Finally, local Noetherian + finitely generated flat implies free. ■

By the lemma, we want to show  $H^0(\mathbb{P}^m; F(n))$  is free for  $n \gg 0$  iff the hilbert polynomials on the fibers  $P_{F_S}$  are all constant.

Claim 1: It suffices to show that for each point  $s \in \text{Spec } A$ , we have  $H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$  for  $k(S)$  the residue field, for  $n \gg 0$ .

Note that  $P_{F_S}$  measures the rank of the LHS.

$\implies$  : The dimension of RHS is constant, whereas the LHS equals  $P_{F_S}(n)$ .

$\impliedby$  : If the dimension of the RHS is constant, so the LHS is free.

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank.

For  $M$  a finitely generated module over  $A$ , find  $0 \rightarrow A^n \rightarrow M \rightarrow Q$  is surjective after tensoring with  $\text{Frac}(A)$ , and tensoring with  $k(S)$  for a closed point, if  $\dim A^n = \dim M$  then  $Q = 0$ .

Proof of Claim 1: By localizing, we can assume  $s$  is a closed point. Since  $A$  is Noetherian, its ideal is f.g. and we have  $A^m \rightarrow A \rightarrow k(S) \rightarrow 0$ . We can tensor with  $F$  (viewed as restricting to fiber) to obtain  $F(n)^m \rightarrow F(n) \rightarrow F_S(n) \rightarrow 0$ . Because  $F$  is flat, this is still exact.

We can take  $H^*(x, \cdot)$ , and for  $n \gg 0$  only  $H^0$  survives. This is the same as tensoring with  $H^0(x, F(n))$ .

Definition: Given a polynomial  $P \in \mathbb{Z}[n]$  for  $X/S$  projective, we define a subfunctor by picking only those with hilbert polynomial  $p$  fiberwise as  $\text{Hilb}_{X/S}^P \subset \text{Hilb}_{X/S}$ . This is given by  $Z \subset X \times_S T$  with  $P_Z = P$ .

Theorem (Grothendieck): If  $S$  is Noetherian and  $X/S$  projective, then  $\text{Hilb}_{X/S}^P$  is representable by a projective  $S$ -scheme.

See cycle spaces in analytic geometry.