Title

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Remark 1.
There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\begin{aligned} \rho : \mathrm{MCG}(\Sigma) &\to \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_*. \end{aligned}$$

Definition 1.0.1 (Special Linear Group)**.**

$$\mathrm{SL}(n, \Bbbk) = \left\{ M \in \mathrm{GL}(n, \Bbbk) \;\middle|\; \det M = 1 \right\} = \ker \underset{\mathbb{G}_m}{\det}.$$

Remark 2.

$$\mathrm{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that $S^2 = 1$ and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Moreover, if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $A \in \mathrm{SL}(2, \mathbb{Z})$, we have $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. this preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \;\middle|\; p, q \in \mathbb{Z} \right\} \cong \left\{ p\omega_1 + q\omega_2 \;\middle|\; p, q \in \mathbb{Z} \right\} \simeq \left\{ p' + q'\tau \;\middle|\; p', q' \in \mathbb{Z} \right\}.$$

where the ω_i, τ come from identifying $\mathbb{R}^2$ with $\mathbb{C}$, and in the last step we've rescaled the lattice by *homothety* to align one vector with the x-axis.

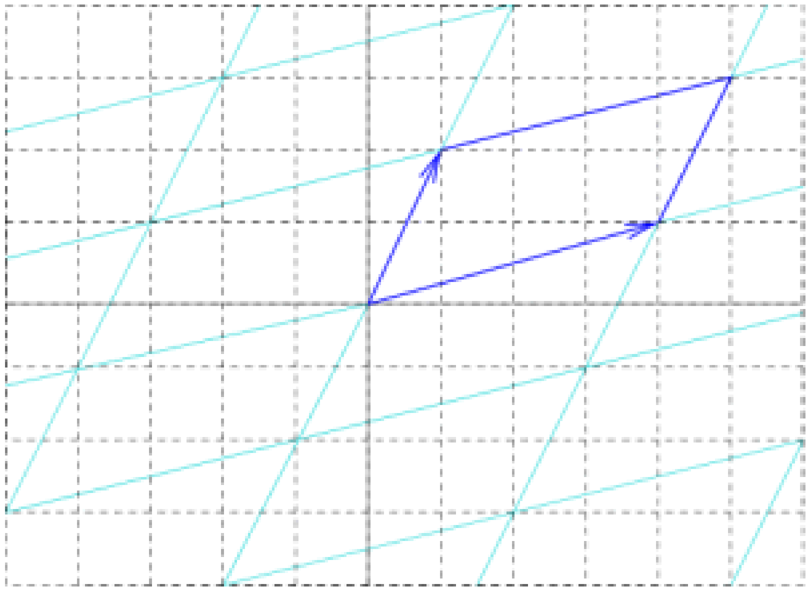

Figure 1: Lattice

Remark 3.
For any finite-index subgroup $G \leq \mathrm{SL}(2, \mathbb{Z})$, the orbits/left-quotient ${}_G\backslash^{\mathbb{H}}$ yields a complex curve (i.e. a torus).

Theorem 1.1 ***(Mapping Class Group of the Torus).***
The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

Proof .

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Map}(X, X), \circ) &\to (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_*. \end{aligned}$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff $\mathrm{Map}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$
  - This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned}\tilde{\sigma}: \mathrm{MCG}(X) &\to \mathrm{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: $\mathrm{im}(\tilde{\sigma}) \subseteq \mathrm{SL}(2, \mathbb{Z})$.

Proof .
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- Algebraic intersection numbers in Σ_2 correspond to determinants
- $f \in \mathrm{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2

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- We can thus freely restrict the codomain to define the map

$$\begin{aligned}\sigma: \mathrm{MCG}(X) &\to \mathrm{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

Claim: σ is surjective.

- $\mathbb{R}^2$ is the universal cover of Σ_2, with deck transformation group $\mathbb{Z}^2$.
- Any $A \in \mathrm{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \mathrm{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

Claim: $\tilde{A}$ is equivariant wrt $\mathbb{Z}^2$

Proof .
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- So $\tilde{A}$ descends to a well-defined map $\psi_{\tilde{A}}$ on $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$, which is still a linear self-homeomorphism
- There is a correspondence

$$\left\{\text{Primitive vectors in } \mathbb{Z}^2\right\} \iff \left\{\begin{array}{c}\text{Oriented simple closed} \\ \text{curves in } \Sigma_2\end{array}\right\} /\text{homotopy}.$$

- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

Claim: σ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.

Proposition 1.2 ***(Hatcher 1B.9).***
Let X be a connected CW complex and Y a $K(G, 1)$. Then there is a map

$$\hom_{\mathrm{Grp}}(\pi_1(X; x_0), \pi_1(Y; y_0)) \to \hom_{\mathrm{Top}}((X; x_0), (Y; y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map. Moreover, the map is unique up to homotopies fixing x_0.

- Thus there is a correspondence

$$\left\{\begin{array}{c}\text{Homotopy classes of} \\ \text{maps } \Sigma_2 \circlearrowleft\end{array}\right\} \iff \left\{\text{Group morphisms } \mathbb{Z}^2 \circlearrowleft\right\}.$$

- Claim: any element $f \in \mathrm{MCG}(\Sigma_2)$ has a representative φ which fixes any given basepoint
- So if $f \in \ker \sigma$, then $f \simeq \varphi \simeq \mathrm{id}$ are homotopic, so $\ker \sigma = 1$.

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