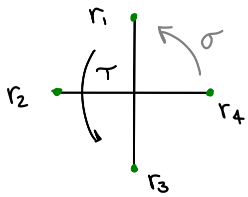
(5a) $f(x) = x^4 - 5$ over

- $\mathbb{Q}$
- $\mathbb{Q}(\sqrt{5})$
- $\mathbb{Q}(i\sqrt{5})$

Let $\omega = 5^{1/4}$, $\zeta = e^{2\pi i/4}$, then f splits in $F := \mathbb{Q}(\omega, \zeta)$ as $f(x) = \prod_{j=1}^{4} (x - \omega\zeta^j)$.

We can embed these roots in $\mathbb{C}$ to find some automorphisms of $F/\mathbb{Q}$:



where $r_j = \omega\zeta^j$, so we can define

$$\begin{array}{ll} \tau : F \to F & \sigma : F \to F \\ \quad i \mapsto -i & \quad i \mapsto i \\ \quad \omega \mapsto \omega & \quad \omega \mapsto i\omega \end{array}$$

Then τ corresponds to the cycle $(1,3)$ in $\mathrm{Sym}(\{r_j\}) \cong S_4$, which has order 2, and σ corresponds to $(1,2,3,4)$, which has order 4; thus $G := \langle \tau, \sigma \rangle \Rightarrow |G| = 8$.

$\mathbb{Q}$

Claim: $G = \mathrm{Gal}(F/\mathbb{Q})$ & $G \cong D_4 = \langle s, r \mid s^2 = r^4 = e,\ (sr)^2 = e \rangle$.

Since F splits $f(x)$ by construction, $F/\mathbb{Q}$ is separable, and since (claim) $[F:\mathbb{Q}] = 8 < \infty$, it is also normal & thus a Galois extension, so we have $[F:\mathbb{Q}] = \{F:\mathbb{Q}\} = \#\mathrm{Gal}(F/\mathbb{Q}) = 8$.

Since $\langle \tau, \sigma \rangle \leq \mathrm{Gal}(F/\mathbb{Q})$, it must be the entire group. To see that $[F:\mathbb{Q}] = 8$, we can note that

$$[\mathbb{Q}(\omega,\zeta):\mathbb{Q}] = \underbrace{[\mathbb{Q}(\omega,\zeta):\mathbb{Q}(\omega)]}_{=2,\ \text{since } \mathbb{Q}(\omega) \subseteq \mathbb{R} \text{ but } \zeta \notin \mathbb{R},\ \text{so } \min(\zeta, \mathbb{Q}(\omega)) = x^2+1.} \underbrace{[\mathbb{Q}(\omega):\mathbb{Q}]}_{=4,\ \text{since } \min(\omega,\mathbb{Q}) = x^4 - 5}$$

We can immediately note that $\tau\sigma = (13)(1234) = (12)(34) \neq (14)(23) = \sigma\tau$, so G is non-abelian.

Moreover, G contains 2 elts of order 2, namely τ & $\sigma\tau$, so $G \not\cong Q_8$, so we must have $G \cong D_4$.

(This follows since they have the same # of generators, satisfy the same relations, & are the same size.)

$\boxed{\text{So } \mathrm{Gal}(F/\mathbb{Q}) \cong D_4.}$

$\mathbb{Q}(\omega)$

$\omega = 5^{1/4} \Rightarrow \omega^2 = \sqrt{5}$

$\min(\sqrt{5}, \mathbb{Q}) = x^2 - 5$

Noting that $[\mathbb{Q}(\omega^2):\mathbb{Q}] = 2$, by the Galois correspondence, $[\mathrm{Gal}(F/\mathbb{Q}) : \overbrace{\mathrm{Gal}(F/\mathbb{Q}(\omega))}^{:=G}] = 4$, so we are looking for an index 4 subgroup of $\langle \tau, \sigma \rangle$ that fixes $\mathbb{Q}(\omega)$. Noting that τ corresponds to complex conjugation and $\text{order}(\tau)=2$, we have $\langle\tau\rangle \subseteq G$. We also find that σ^2 fixes $\mathbb{Q}(\omega^2)$, since

$$\sigma^2(a+b\omega^2) = a+b\sigma(\sigma(\omega)^2) = a+b\sigma((i\omega)^2) = a+b\sigma(-\omega^2) = a-b\sigma(\omega)^2 = a-b(i\omega)^2 = a+b\omega^2$$

and since $\text{order}(\sigma^2)=2$, we have $|\langle\tau,\sigma^2\rangle|=4$, so $G:=\langle\tau,\sigma\rangle$ has index 2 & fixes $\mathbb{Q}(\omega)$, so we must have

$$\boxed{\text{Gal}(F/\mathbb{Q}(\omega)) = \langle\tau,\sigma^2\rangle.} \quad (\cong \mathbb{Z}_2\times\mathbb{Z}_2)$$

$\mathbb{Q}(i\omega)$

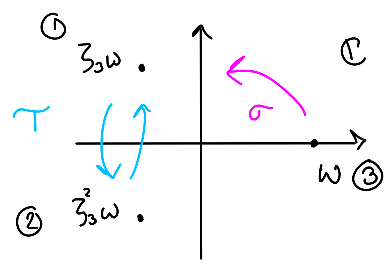
Noting that $[\mathbb{Q}(i\omega):\mathbb{Q}]=4$ since $\min(i\omega,\mathbb{Q}) = x^4-5$, we look for a subgroup of $\text{Gal}(F/\mathbb{Q})$ of index 4 (& thus order 2) that fixes $\mathbb{Q}(i\omega)$. The subgroup $\langle\tau\sigma^2\rangle$ does the trick, since $\tau\sigma^2(a+bi\omega) = a+b(-i)(i^2\omega) = a+bi\omega$.

$$\boxed{\text{Thus } \text{Gal}(F/\mathbb{Q}(i\omega)) = \langle\tau\sigma^2\rangle \cong \mathbb{Z}_2}$$

$f(x)=x^3-2$ over $\mathbb{Q}$

$\omega = 2^{1/3}$

Factor $f(x) = (x-\omega)(x-\zeta_3\omega)(x-\zeta_3^2\omega)$ where $\zeta_3 = e^{2\pi i/3}$, then $F:=\mathbb{Q}(\omega,\zeta_3)$ is the splitting field of $f(x)$, and $[F:\mathbb{Q}] = [F:\mathbb{Q}(\omega)][\mathbb{Q}(\omega):\mathbb{Q}]$



- $[\mathbb{Q}(\omega):\mathbb{Q}] = 3$, since $\min(\omega,\mathbb{Q}) = x^3-2$.
- $[F:\mathbb{Q}(\omega)] = 2$ since $\min(\zeta_3,\mathbb{Q}(\omega)) = \Phi_3 = x^2+x+1$.

So $[F:\mathbb{Q}] = 6 = |G| := |\text{Gal}(F/\mathbb{Q})| \Rightarrow G \in \{\mathbb{Z}_6, S_3\}$.

We can produce at least two automorphisms fixing $\mathbb{Q}$: $\rightsquigarrow$ $\tau: \begin{cases} \omega \mapsto \omega \\ \zeta_3 \mapsto \zeta_3^2 \end{cases} \rightsquigarrow (12)$

And we can check $\quad \sigma: \begin{cases} \omega \mapsto \zeta_3\omega \\ \zeta_3 \mapsto \zeta_3^2 \end{cases} \rightsquigarrow (123)$

$(12)(123) = (1)(23)$

$(123)(12) = (13)(2) \neq (12)(123)$

So G contains a non-abelian subgroup $\langle\tau,\sigma\rangle$ & thus $\boxed{G \cong S_3}$

$f(x) = (x^3-2)(x^2-5)\ /\ \mathbb{Q}$

Noting that $x^2-5 = (x+\omega_5)(x-\omega_5)$ where $\omega_5 = 5^{1/2}$, the splitting field of $f(x)$ will be

$$L := \mathbb{Q}(\omega,\zeta_3,\omega_5) = \mathbb{Q}(2^{1/3}, e^{2\pi i/3})(\sqrt{5}).$$

Claim: $[L:\mathbb{Q}] = [L:\mathbb{Q}(\omega,\zeta_3)][\mathbb{Q}(\omega,\zeta_3):\mathbb{Q}] = 2\cdot 6 = 12$.

The only new content is that $[L:\mathbb{Q}(\omega,\zeta_3)] = 2$, i.e. $\min(\sqrt{5},\mathbb{Q}(\omega,\zeta_3)) = x^2-5$.

The degree could not be higher, since $E \subseteq F \Rightarrow \min(\alpha, F) \mid \min(\alpha, E)$ and $\min(\sqrt{5}, \mathbb{Q}) = x^2 - 5$.
But it could not be 1, since $\sqrt{5} \notin \mathbb{Q}(3^{1/3}, \zeta_3)$.

So $G := \mathrm{Gal}(L/\mathbb{Q}) \geq S_3$ as a subgroup by the previous problem, and is thus a nonabelian group of order 12. We can produce a new automorphism $\gamma: \begin{cases} \sqrt{5} \mapsto -\sqrt{5} \\ \zeta_4 \mapsto \zeta_4 \\ \omega \mapsto \omega \end{cases}$

Thus $\langle\gamma\rangle$ is a subgroup of order 2, $\langle\gamma\rangle \cap \langle\tau,\sigma\rangle = \{e\}$, and $|\langle\gamma\rangle| \cdot |\langle\sigma,\tau\rangle| = 2 \cdot 6 = 12 = |G|$, and $G = \underbrace{\langle\gamma\rangle\langle\tau,\sigma\rangle}_{\text{product of subgroups}} \Rightarrow \boxed{G = \langle\gamma\rangle \times \langle\tau,\sigma\rangle \cong \mathbb{Z}_2 \times S_3}$

■

5

a) Noting that $g(x) \mid f(x)$ and f splits in F, g must split in F as well. (Otherwise, g would have an irreducible nonlinear factor in F and thus f would as well.)

b) The irreducible factors of g are separable in E and F/E is a splitting field for g, so by (3.3) above, F/E is Galois.

c) $K \subseteq E \Rightarrow \mathrm{Aut}(F/E) \subseteq \mathrm{Aut}(F/K)$, and to see $\mathrm{Aut}(F/K) \subseteq \mathrm{Aut}(F/E)$, letting $\sigma \in \mathrm{Aut}(F/K)$ we must have $\sigma \in \mathrm{Sym}(\{u_1, \cdots, u_n\})$ and so

$$\sigma(g(x)) = g(\sigma(x)) = \prod(\sigma(x) - u_i) = \sum v_i \sigma(x)^i$$
$$\sigma(g(x)) = \sigma\left(\sum v_i x^i\right) = \sum \sigma(v_i)\sigma(x)^i$$

so $\sigma(v_i) = v_i$ & $\sigma \in \mathrm{Aut}(F/E)$.

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6 Suppose f is irreducible & not separable, so $\gcd(f, f') > 1$. Since $\deg f' < \deg f$, and f is irreducible, we have $f'(x) \equiv 0$ in $K[x]$. But if $f(x) = \sum_{j=0}^{m} a_j x^j$ $(a_m \neq 0 \in K)$ $f'(x) = m a_m x^{m-1} + \ldots + a_1 \equiv 0$. So in particular, $m a_m = 0$ in K, forcing $m = 0$ in K and since $m \neq 0 \in \mathbb{N}$, we must have $\mathrm{char}(K) \mid m$. ■