Discussion Notes

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1 Discussion 1

If X is an F_σ set, then

$$X = \bigcup_{i=1}^\infty F_i \quad \text{with each } F_i \text{ closed.}$$

If X is a G_δ set, then

$$X = \bigcap_{i=1}^\infty G_i \quad \text{with each } G_i \text{ open.}$$

A set A is *nowhere dense* iff $(\overline{A})^\circ = \emptyset$ iff for any interval I, there exists a subinterval S such that $S \bigcap A = \emptyset$. This is a set that is not dense in any nonempty open set. If the closure of a subset of $\mathbb{R}$ contains no open intervals, it will be nowhere dense.

A set A is *meager* or *first category* if it can be written as

$$A = \bigcup_{i \in \mathbb{N}} A_i \quad \text{with each } A_i \text{ nowhere dense}$$

A set A is *null* if for any ε, there exists a cover of A by countably many intervals of total length less than ε, i.e. there exists $\{I_k\}_{j\in\mathbb{N}}$ such that $A \subseteq \bigcup_{j\in\mathbb{N}} I_j$ and $\sum_{j\in\mathbb{N}} \mu(I_j) < \varepsilon$. If A is null, we say $\mu(A) = 0$.

Some facts:

- If $f_n \to f$ and each f_n is continuous, then D_f is meager.

- If $f \in \mathcal{R}(a, b)$ and f is bounded, then D_f is null.
- If f is monotone, then D_f is countable.
- If f is monotone and differentiable on (a, b), then D_f is null.

We define the *oscillation of* f as

$$\omega_f(x) := \lim_{\delta \to 0^+} \sup_{y,z \in B_\delta(x)} |f(y) - f(z)|$$

1.1 Uniform Convergence

We say that $f_n \to f$ *converges uniformly on* A if $\|f_n - f\|_\infty = \sup_{x \in A} |f_n(x) - f(x)| \to 0$. (Note that this defines a sequence of *numbers* in $\mathbb{R}$.)

This means that one can find an n large enough that that for every $x \in A$, we have $|f_n(x) - f(x)| \leq \varepsilon$ for any ε.

- Showing uniform convergence: find some M_n, independent of x, such that $|f_n(x) - f(x)| \leq M_n$ where $M_n \to 0$.
- Negating: Fix ε, let n be arbitrary, and find a bad x (which can depend on n) such that $|f_n(x) - f(x)| \geq \varepsilon$.

Example: $\frac{1}{1+nx} \to 0$ pointwise on $(0, \infty)$, which can be seen by fixing x and taking $n \to \infty$. To see the convergence is not uniform, choose $x = \frac{1}{n}$ and $\varepsilon = \frac{1}{2}$. Then

$$\sup_{x>0} \left| \frac{1}{1+nx} - 0 \right| \geq \frac{1}{2} \not\to 0.$$

Here, the problem is at small scales – note that the convergence *is* unform on $[a, \infty)$ for any $a > 0$. To see this, note that

$$x > a \implies \frac{1}{x} < \frac{1}{a} \implies \left| \frac{1}{1+nx} \right| \leq \left| \frac{1}{nx} \right| \leq \frac{1}{na} \to 0$$

since a is fixed.

1.2 Uniformly Cauchy

Let $C^0(([a, b], \|\cdot\|_\infty))$ be the metric space of continuous functions of $[a, b]$, endowed with the metric

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|$$

This is a complete metric space, and

$$f_n \to^U f \iff \forall \varepsilon \exists N \ni m \geq n \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon \forall x \in X$$

$\implies$: Use the triangle inequality.

$\impliedby$: Find a candidate limit f: first fix an x, so that each $f_n(x)$ is just a number. Now we can consider the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$, which (by assumption) is a Cauchy sequence in $\mathbb{R}$ and thus converges. So define $f(x) := \lim_n f_n(x)$. Aside: we note that if $a_n < \varepsilon$ for all n and $a_n \to a$, then $a \leq \varepsilon$.

So take $m \to \infty$, i.e.

$$|f_n(x) - f_m(x)| < \varepsilon \forall x \implies \lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \leq \varepsilon \forall x \implies f_n \to^U f.$$

Note: $f_n \to^U f$ does not imply that $f_n' \to^U f'$.

Counterexample: Let $f_n(x) = \frac{1}{n}\sin(n^2 x)$, which converges to 0 uniformly, but $f_n'(x) = n\cos(n^2 x)$ does not even converge pointwise.

To make this work, the theorem is that if $f_n' \to^U g$ for some g and for at least 1 point x we have $f_n(x) \to f(x)$, then $g = \lim f_n'$.

Exercise: Let $f(x) = \sum_{n=1}^\infty \frac{nx^2}{n^3+x^3}$.

Does it converge at all, say on $(0,\infty)$?

We can check pointwise convergence by fixing x, say $x = 1$, and noting that

$$x = 1 \implies \left|\frac{nx^2}{n^3+x^2}\right| \leq \left|\frac{n}{n^3+1}\right| \leq \frac{1}{n^2} := M_n,$$

where $\sum M_n < \infty$. To see why it does not converge uniformly, we can let $x = n$. Then,

$$x = n \implies \left|\frac{nx^2}{n^3+x^2}\right| = \frac{n^3}{2n^3} = \frac{1}{2} \not\to 0,$$

so there is a problem at large values of x.

However, if we restrict attention to $(0, b)$ for some fixed b, we have $x < b$ and so

$$\left|\frac{nx^2}{n^3+x^2}\right| \leq \frac{nb^2}{n^3+b^2} \leq b^2\left(\frac{n}{n^3}\right) = b^2\frac{1}{n^2} \to 0.$$

Note that this actually tells us that f is *continuous* on $(0,\infty)$, since if we want continuity at a specific point x, we can take $b > x$. Since each term is a continuous function of x, and we have uniform convergence, the limit function is the uniform limit of continuous functions on this interval and thus also continuous here. Checking $x = 0$ separately, we find that f is in fact continuous on $[0,\infty)$.

1.3 Series of Functions

Let f_n be a function of x, then we say $\sum_{n=1}^\infty f_n$ converges uniformly to S on A iff the partial sums $s_n = f_1 + f_2 + \cdots$ converges to S uniformly on A.

This equivalently requires that

$$\forall\varepsilon\exists N \ni n \geq m \geq N \implies |s_n - s_m| = \left|\sum_{k=m}^{n} f_k(x)\right| \leq \varepsilon \quad \forall x \in A.$$

> Showing uniform convergence of a series: **Always use the M-test**!!! I.e. if $|f_n(x)| \leq M_n$, which doesn't depend on x, and $\sum M_n < \infty$, then $\sum f_n$ converges uniformly.

Example: Let $f(x) = \sum \frac{1}{x^2+n^2}$.

Does it converge at all? Fix $x \in \mathbb{R}$, say $x = 1$, then $\frac{1}{1+n^2} \leq \frac{1}{n^2}$ which is summable. So this converges pointwise. But since $x^2 > 0$, we generally have $\frac{1}{x^2+n^2} \leq \frac{1}{n^2}$ for any x, so this actually converges uniformly.

1.3.1 Negating Uniform Convergence for Series

???

1.4 Misc

A useful inequality:

$$(1+x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k = 1 + nx + n^2 x \geq 1 + nx + nx^2 > 1 + nx$$

A summary of convergence results:

- Functions $f_n \to f$:
  - Uniform: M test. Produce a bound $\|f_n - f\|_\infty < M_n$ which doesn't depend on n, where $M_n \to 0$.
  - Negating:
    * If f_n is continuous but f is not,
    * Let n be arbitrary, then find a bad x (which can depend on n) such that $\sup |f_n(x) - f(x)|$ is bounded below.
- Series of function $\sum f_n \to f$:
  - Uniform: M test. Produce a bound $\|f_n\|_\infty < M_n$ where $\sum M_n < \infty$.
  - Negating:
    * If each partial sum is continuous, but f is not.
    * If $f_n \not\to^U 0$.