Problem Set 8

D. Zack Garza

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1 Problem 1

1.1 Part a

It follows from the definition that $\|f\|_\infty = 0 \iff f = 0$ almost everywhere, and if $\|f\|_\infty$ is the best upper bound for f almost everywhere, then $\|cf\|_\infty$ is the best upper bound for cf almost everywhere.

So it remains to show the triangle inequality. Suppose that $|f(x)| \leq \|f\|_\infty$ a.e. and $|g(x)| \leq \|g\|_\infty$ a.e., then by the triangle inequality for the $|\cdot|_{\mathbb{R}}$ we have

$$\begin{aligned} |(f+g)(x)| &\leq |f(x)| + |g(x)| \quad a.e. \\ &\leq \|f\|_\infty + \|g\|_\infty \quad a.e., \end{aligned}$$

which means that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ as desired.

1.2 Part b

$\implies$: Suppose $\|f_n - f\|_\infty \to 0$, then for every ε, N_ε can be chosen large enough such that $|f_n(x) - f(x)| < \varepsilon$ a.e., which precisely means that there exist sets E_ε such that $x \in E_\varepsilon \implies |f_n(x) - f(x)|$ and $m(E_\varepsilon^c) = 0$.

But then taking the sequence $\varepsilon_n := \frac{1}{n} \to 0$, we have $f_n \rightrightarrows f$ uniformly on $E := \bigcap_n E_n$ by definition, and $E^c = \bigcup_n E_n^c$ is still a null set.

$\impliedby$: Suppose $f_n \rightrightarrows f$ uniformly on some set E and $m(E^c) = 0$. Then for any ε, we can choose N large enough such that $|f_n(x) - f(x)| < \varepsilon$ on E; but then ε is an upper bound for $f_n - f$ almost everywhere, so $\|f_n - f\|_\infty < \varepsilon \to 0$.

1.3 Part c

To see that simple functions are dense in $L^\infty(X)$, we can use the fact that $f \in L^\infty(X) \iff$ there exists a g such that $f = g$ a.e. and g is bounded.

Then there is a sequence s_n of simple functions such that $\|s_n - g\|_\infty \to 0$, which follows from a proof in Folland:

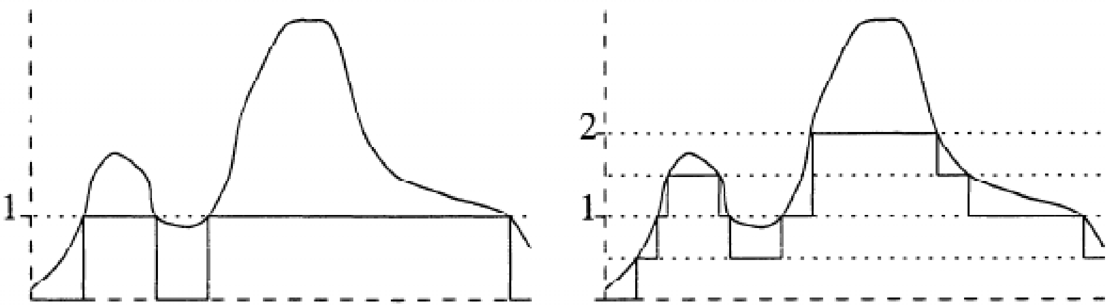
Proof. (a) For $n = 0, 1, 2, \ldots$ and $0 \le k \le 2^{2n} - 1$, let

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \text{and} \quad F_n = f^{-1}((2^n, \infty]),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n \chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that $\phi_n \le \phi_{n+1}$ for all n, and $0 \le f - \phi_n \le 2^{-n}$ on the set where $f \le 2^n$. The result therefore follows.



However, $C_c^0(X)$ is dense $L^\infty(X) \iff$ every $f \in L^\infty(X)$ can be approximated by a sequence $\{g_k\} \subset C_c^0(X)$ in the sense that $\|f - g_n\|_\infty \to 0$. To see why this can *not* be the case, let $f(x) = 1$, so $\|f\|_\infty = 1$ and let $g_n \to f$ be an arbitrary sequence of C_c^0 functions converging to f pointwise.

Since every g_n has compact support, say $\operatorname{supp}(g_n) \coloneqq E_n$, then $g_n|_{E_n^c} \equiv 0$ and $m(E_n^c) > 0$. In particular, this means that $\|f - g_n\|_\infty = 1$ for every n, so g_n can not converge to f in the infinity norm.

2 Problem 2

2.1 Part a

2.1.1 Part i

Lemma: $\|1\|_p = m(X)^{1/p}$

This follows from $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$ and taking pth roots. $\square$

By Holder with $p = q = 2$, we can now write

$$\begin{aligned} \|f\|_1 &= \|1 \cdot f\|_1 \leq \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\ \implies \|f\|_1 &\leq m(X)^{1/2} \|f\|_2. \end{aligned}$$

Letting $M \coloneqq \|f\|_\infty$, We also have

$$\begin{aligned} \|f\|_2^2 &= \int_X |f|^2 \leq \int_X |M|^2 = M^2 \int_X 1 = M^2 m(X) \\ \implies \|f\|_2 &\leq m(X)^{1/2} \|f\|_\infty \\ \implies m(X)^{1/2} \|f\|_2 &\leq m(X) \|f\|_\infty, \end{aligned}$$

and combining these yields

$$\|f\|_1 \leq m(X)^{1/2} \|f\|_2 \leq m(X) \|f\|_\infty,$$

from which it immediately follows

$$m(X) < \infty \implies L^\infty(X) \subseteq L^2(X) \subseteq L^1(X).$$

The Inclusions Are Strict:

1. $\exists f \in L^1(X) \setminus L^2(X)$:

Let $X = [0, 1]$ and consider $f(x) = x^{-\frac{1}{2}}$. Then

$$\|f\|_1 = \int_0^1 x^{-\frac{1}{2}} < \infty \qquad \text{by the } p \text{ test},$$

while

$$\|f\|_2^2 = \int_0^1 x^{-1} \to \infty \qquad \text{by the } p \text{ test}.$$

2. $\exists f \in L^2(X) \setminus L^\infty(X)$:

Take $X = [0,1]$ and $f(x) = x^{-\frac{1}{4}}$. Then

$$\|f\|_2^2 = \int_0^1 x^{-\frac{1}{4}} < \infty \qquad \text{by the } p \text{ test,}$$

while $\|f\|_\infty > M$ for any finite M, since f is unbounded in neighborhoods of 0, so $\|f\|_\infty = \infty$.

2.1.2 Part ii

1. $\exists f \in L^2(X) \setminus L^1(X)$ when $m(X) = \infty$:

Take $X = [1, \infty)$ and let $f(x) = x^{-1}$, then

$$\begin{aligned} \|f\|_2^2 &= \int_0^\infty x^{-2} < \infty \qquad \text{by the } p \text{ test,} \\ \|f\|_1 &= \int_0^\infty x^{-1} \to \infty \qquad \text{by the } p \text{ test.} \end{aligned}$$

2. $\exists f \in L^\infty(X) \setminus L^2(X)$ when $m(X) = \infty$:

Take $X = \mathbb{R}$ and $f(x) = 1$. then

$$\begin{aligned} \|f\|_\infty &= 1 \\ \|f\|_2^2 &= \int_{\mathbb{R}} 1 \to \infty. \end{aligned}$$

3. $L^2(X) \subseteq L^1(X) \implies m(X) < \infty$:

Let $f = \chi_X$, by assumption we can find a constant M such that $\|\chi_X\|_2 \leq M\|\chi_X\|_1$.

Then pick a sequence of sets $E_k \nearrow X$ such that $m(E_k) < \infty$ for all k, $\chi_{E_k} \nearrow \chi_X$, and thus $\|\chi_{E_k}\|_p \leq M\|\chi_E\|_p$. By the lemma, $\|\chi_{E_k}\|_p = m(E_k)^{1/p}$, so we have

$$\begin{aligned} \|\chi_{E_k}\|_2 \leq M\|\chi_{E_k}\|_1 &\implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \leq M \\ &\implies \frac{m(E_k)^{1/2}}{m(E_k)} \leq M \\ &\implies m(E_k)^{-1/2} \leq M \\ &\implies m(E_k) \leq M^2 < \infty. \end{aligned}$$

and by continuity of measure, we have $\lim_K m(E_k) = m(X) \leq M^2 < \infty$. $\square$

2.2 Part b

1. $L_1(X) \bigcap L^\infty(X) \subset L^2(X)$:

Let $f \in L^1(X) \bigcap L^\infty(X)$ and $M := \|f\|_\infty$, then

$$\|f\|_2^2 = \int_X |f|^2 = \int_X |f||f| \leq \int_X M|f| = M \int |f| := \|f\|_\infty \|f\|_1 < \infty. \tag{1}$$

The inclusion is strict, since we know from above that there is a function in $L^2(X)$ that is not in $L^\infty(X)$.

Note that taking square roots in (1) immediately yields

$$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2} \|f\|_{L^\infty(X)}^{1/2}.$$

2. $L^2(X) \subset L^1(X) + L^\infty(X)$:

Let $f \in L^2(X)$. Noting that continuous functions with compact support are dense in $L^2(X)$, take an approximating sequence $\{g_n\} \subseteq C_c^0(X)$ with $\|g_n - f\|_2 \to 0$.

The claim is that we can choose N large enough such that when we write $f = (f - g_N) + g_N$, we will have $\|f - g_N\|_1 < \infty$ and $\|g_N\|_\infty < \infty$, which establishes the desired result.

To see that $g_N \in L^\infty(X)$, we can just note that since each g_n is C_c^0, they are all **bounded**, say by $M < \infty$, in which case $\|g_N\|_\infty \leq M < \infty$.

To see that $g_N \in L^1(X)$, we can use the fact that $\|f - g_N\|_2 \leq \|f\|_2 + \|g_N\|_2 < \infty$, so $f - g_N \in L^2(X)$.

TODO

3 Problem 3

For notational convenience, it suffices to prove this for $\ell^p(\mathbb{N})$, where we re-index each sequence in $\ell^p(\mathbb{Z})$ using a bijection $\mathbb{Z} \to \mathbb{N}$.

> Note: this technically reorders all sums appearing, but since we are assuming absolute convergence everywhere, this can be done. One can also just replace $\sum_{j=n}^m |a_j|^p$ with $\sum_{n \leq |j| \leq m} |a_j|^p$ in what follows.

1. $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$:

Suppose $\sum_j |a|_j < \infty$, then its tails go to zero, so choose N large enough so that

$$j \geq N \implies |a_j| < 1.$$

But then

$$j \geq N \implies |a_j|^2 < |a_j|,$$

and

$$
\begin{aligned}
\sum_j |a_j|^2 &= \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j|^2 \\
&\leq \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=1}^{\infty} |a_j| \\
&< \infty.
\end{aligned}
$$

where we just note that the first portion of the sum is a finite sum of finite numbers and thus bounded.

To see that the inclusion is strict, take $\mathbf{a} \coloneqq \{j^{-1}\}_{j=1}^{\infty}$; then $\|\mathbf{a}\|_2 < \infty$ by the p-test by $\|\mathbf{a}\|_1 = \infty$ since it yields the harmonic series.

2. $\ell^2(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$:

This follows from the contrapositive: if $\mathbf{a}$ is a sequence with unbounded terms, then $\|\mathbf{a}\|_2 = \sum |a_j|^2$ can not be finite, since convergence would require that $|a_j|^2 \to 0$ and thus $|a_j| \to 0$.

To see that the inclusion is strict, take $\mathbf{a} = \{1\}_{j=1}^{\infty}$. Then $\|\mathbf{a}\|_\infty = 1$, but the corresponding sum does not converge.

3. $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1$:

Let $M = \|\mathbf{a}\|_1$, then

$$
\|\mathbf{a}\|_2^2 \leq \|\mathbf{a}\|_1^2 \iff \frac{\|\mathbf{a}\|_2^2}{M^2} \leq 1 \iff \sum_j \left|\frac{a_j}{M}\right|^2 \leq 1.
$$

But then we can use the fact that

$$
\left|\frac{a_j}{M}\right| \leq 1 \implies \left|\frac{a_j}{M}\right|^2 \leq \left|\frac{a_j}{M}\right|
$$

to obtain

$$
\sum_j \left|\frac{a_j}{M}\right|^2 \leq \sum_j \left|\frac{a_j}{M}\right| = \frac{1}{M} \sum_j |a_j| \coloneqq 1.
$$

4. $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_2$:

This follows from the fact that for any n, we have

$$
\|\mathbf{a}\|_\infty^n \coloneqq \left(\sup_j |a_j|\right)^n = \sup_j |a_j|^n \leq \sum_j |a_j|^n = \|\mathbf{a}\|_2^n
$$

and taking nth roots yields the desired inequality.

> Note: the middle inequality follows from the fact that if the sum were any smaller than the sup, then every term would have to be smaller.

4 Problem 4

5 Problem 5

6 Problem 6