Problem Set 1

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1 Problem 5

1.1 Part 1

Let $A \in \text{Mat}(n,n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av\rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \text{Math}(n,n)$ be positive semi-definite, so

$$\langle v,\ Bv\rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v,\ (A+B)v\rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned}
\langle v,\ (A+B)v\rangle &= \langle v,\ Av\rangle + \langle v,\ Bv\rangle \\
&> \langle v,\ Av\rangle + 0 \\
&\geq 0 + 0 \\
&= 0.
\end{aligned}$$

1.2 Part 2

Let M be a smooth manifold with a maximal smooth atlas $\mathcal{A}$, and choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$.

Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to $\mathcal{C}$. In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} \coloneqq \delta_{jk}$. We thus have

$$\begin{aligned} g^i : T\phi_i(U_i) \times T\phi_i(U_i) &\to \mathbb{R} \\ (\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n \cong \mathbb{R}^n$, which is spanned by $\{\partial x_i\}_{i=1}^n$, and is defined on basis vectors as the Kronecker delta and extended linearly.

Let $G^i \coloneqq (\phi_i)^* \, g^i(p, q) == g^i(\phi_i(p), \phi_i(q))$

2 Problem 6

2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\to M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \operatorname{Tr}\left(f_* \Big|_{H_k(X;\mathbb{Q})} \right).$$

where $f_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned}
\Lambda(\mathrm{id}_M) &= \sum_k \operatorname{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})}) \\
&= \sum_k \dim H_k(X;\mathbb{Q}) \\
&= \chi(M),
\end{aligned}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned}
\phi_t(p) =& p \\
\implies \frac{\partial}{\partial t}\phi_t(p) =& \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\
\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& 0\Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\
\implies X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& 0 && \text{by definition of } \phi_1
\end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. □

2.2 Part 2

The trivial bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & S^2 \times \mathbb{R}^2 \arrow[d] \\
 & S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\to S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\to (\mathbf{x}, [1,1]) \end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1,1]$ to every point. However, as part 1 shows, the bundle

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & TS^2 \arrow[d] \\ & S^2 \arrow[u, bend right, dotted, "s"'] \end{tikzcd}$$

can *not* have a nowhere vanishing section. $\square$