Complex Analysis Qual Prep Week 1: Preliminaries

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1 Week 1: Preliminaries

1.1 Topics

- Complex arithmetic and geometry, conic section equations
- Uniform (continuity, differentiability, convergence)
- Inverse and implicit function theorems
- Green's theorem, Stokes theorem
- Complex plane, Riemann sphere

1.2 Warmup

- State the Cauchy-Riemann equations.
- Define what it means for a function to be
  - Holomorphic
  - Meromorphic
  - Analytic
  - Harmonic
  - Uniformly continuous
  - Uniformly bounded
  - Entire
- What does it mean for a sequence or series to uniformly converge?
- State the Laplace equation.
- What is the Dirichlet problem?
- Discuss how to carry out partial fraction decomposition
- Determine the radius of convergence of the power series for $\sqrt{z}$ expanded at $z_0 = 4 + 3i$.
- What is the logarithmic derivative?
- Find a function f such that f^2 is analytic on the open unit disc but f is not.

20. Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

3.3.3 c

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and give a necessary and sufficient condition for a real-differentiable function at (a, b) to be complex differentiable at the point $a + ib$.

3.4 4 🚩

Let $f = u + iv$ be complex-differentiable with continuous partial derivatives at a point $z = re^{i\theta}$ with $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

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9.1 1 ✨

Suppose f is analytic on a region Ω such that $\mathbb{D} \subseteq \Omega \subseteq \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence exactly 1.

9.1.1 a ✨

Give an example of such an f that converges at every point of S^1.

9.1.2 b 🚩

Give an example of such an f which is analytic at 1 but $\sum_{n=0}^{\infty} a_n$ diverges.

9.1.3 c 🚩

Prove that f can not be analytic at *every* point of S^1.

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1.3 3 🔨

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)^2}$$

about $z = 0$ and $z = 1$ respectively.

Hint: recall that power series can be differentiated.

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1.3 Exercises

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3. Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \cdots \sin \frac{(n-1)\pi}{n} = n \, .$$

Hint: $1 - \cos 2\theta = 2 \sin^2 \theta, \ \sin 2\theta = 2 \sin \theta \cos \theta$.

2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r_0 < 1$, let $\overline{D}_{r_0}$ be the closed disk $|z| \leq r_0$. Show that the area A of $f(\overline{D}_{r_0})$ is finite and is given by

$$A = \pi \sum_{n=1}^{\infty} n |c_n|^2 r_0^{2n}.$$

[Hint: First find a formula in terms of polar coordinates in xy-plane for the area element $dudv$ using complex analysis, where $f = u + iv$. Note that $dxdy = rdrd\theta$.]

4. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for any two complex numbers z_1, z_2, and explain the geometric meaning of this identity.

1.1 1 ✨

Find the Laurent expansion of

$$f(z) = \frac{z+1}{z(z-1)}$$

about $z = 0$ and $z = 1$ respectively.

5. Prove the following:

   (a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

   (b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

   (c) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at every point of the unit circle except at $z = 1$.

6. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

1. COMPLEX ANALYSIS PRACTICE PROBLEMS 2.0

Complex 2.0 #9.2
Let D be a domain which contains in its interior the closed unit disk $|z| \le 1$. Let $f(z)$ be analytic in D except at a finite number of points $z_1, \ldots, z_k$ on the unit circle $|z| = 1$ where $f(z)$ has first order poles with residues $s_1, \ldots, s_k$. Let the Taylor series of $f(z)$ at the origin be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that there exists a positive constant M such that $|a_n| \le M$.

Additional Problem

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

(1) f is continuous on $[0, \infty)$.
(2) $f'(x)$ exists for all $x \ge 0$.
(3) $f(0) = 0$.
(4) f' is increasing.

For $x > 0$, define $g(x) = \frac{f(x)}{x}$. Prove that g is increasing.

Problem: Prove or disprove that there is a sequence of analytic polynomials $\{p_n(z)\}, n \in \mathbb{N}$, so that $p_n(z) \to \bar{z}^4$ as $n \to \infty$ uniformly for $z \in \partial D(0,1)$.

Problem: Show that for $R > 0$, there is N_R such that when $n > N_R$, the function

$$P_n(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} \ne 0, \quad \forall\, |z| \le R.$$

Problem: Let $f(z)$ be analytic in the disk $U = \{|z| < 1\}$, with $f(0) = f'(0) = 0$. Show that $g(z) = \sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$ defines an analytic function on U. Moreover, show that the above function $g(z)$ satisfies

$$g(z) = f(z) \sum_{n=1}^{\infty} \frac{1}{n^2}$$

if and only if $f(z) = cz^2$.

1.4 Qual Problems

1. Find the Laurent series expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ valid in the annulus $1 < |z| < 2$.

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2. Prove that the distinct complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

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1. (a) Prove that if $|w_1| = c|w_2|$ where $c > 0$, then $|w_1 - c^2 w_2| = c|w_1 - w_2|$.

   (b) Prove that if $c > 0$, $c \neq 1$ and $z_1 \neq z_2$, then $\left|\dfrac{z - z_1}{z - z_2}\right| = c$ represents a circle. Find its center and radius.

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2. Expand $\dfrac{1}{1 - z^2} + \dfrac{1}{z - 3}$ in a series of the form $\sum_{-\infty}^{\infty} a_n z^n$ so it converges for (a) $|z| < 1$, (b) $1 < |z| < 3$; and (c) $|z| > 3$.

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1. Let z_1 and z_2 be two complex numbers.

   (a) Show that $|z_1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|)$.

   (b) Show that if $|z_1| < 1$ and $|z_2| < 1$, then $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$.

   (c) Assume that $z_1 \neq z_2$. Show that $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$ if only if $|z_1| = 1$ or $|z_2| = 1$.

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6. Suppose $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on the unit disk $\mathbb{D}$, and $f(z)$ is a holomorphic function on the unit disk $\mathbb{D}$. Show that the following are equivalent.

   (a) $\{f_n(z)\}$ converges to $f(z)$ uniformly on compact subsets in $\mathbb{D}$.

   (b) $\int_{|z|=r} |f_n(z) - f(z)|\,|dz|$ converges to 0 if $0 < r < 1$.

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1. Let $n \geq 2$ be an integer. Show that $2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n$.

   [Hint: Use n-th roots of unity i.e., solutions of $z^n - 1 = 0$]

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2. Let $u(x, y)$ be a harmonic functions defined in an open disk of radius $R > 0$. Suppose that $u(x, y)$ has continuous partial derivatives of order two in its domain.

   a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

$$v(x, y) = \int_{(a,b)}^{(x,y)} (-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy).$$

   b) (i) Prove that $u(x, y) + iv(x, y)$ is an analytic function in this disk.

   (ii) Prove that $v(x, y)$ is harmonic in this disk.

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3. a) $f : D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain. Let $\alpha : [a, b] \to D$ be a smooth curve.

   a) Define the *complex line integral* $\int_\alpha f$.

   b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \text{Image}(\alpha)$. Prove that

$$|\int_\alpha f| \leq M \times \text{length}(\alpha).$$

   c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $|\int_{C_R} \frac{\log(z)}{z^2}|$, which depends only on R and (possibly) other constants.

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