Problem Set 8

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1 Problem 1

1.1 Part a

It follows from the definition that $\|f\|_\infty = 0 \iff f = 0$ almost everywhere, and if $\|f\|_\infty$ is the best upper bound for f almost everywhere, then $\|cf\|_\infty$ is the best upper bound for cf almost everywhere.

So it remains to show the triangle inequality. Suppose that $|f(x)| \leq \|f\|_\infty$ a.e. and $|g(x)| \leq \|g\|_\infty$ a.e., then by the triangle inequality for the $|\cdot|_{\mathbb{R}}$ we have

$$
\begin{aligned}
|(f+g)(x)| &\leq |f(x)| + |g(x)| \quad a.e. \\
&\leq \|f\|_\infty + \|g\|_\infty \quad a.e.,
\end{aligned}
$$

which means that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ as desired.

1.2 Part b

$\implies$: Suppose $\|f_n - f\|_\infty \to 0$, then for every ε, N_ε can be chosen large enough such that $|f_n(x) - f(x)| < \varepsilon$ a.e., which precisely means that there exist sets E_ε such that $x \in E_\varepsilon \implies |f_n(x) - f(x)|$ and $m(E_\varepsilon^c) = 0$.

But then taking the sequence $\varepsilon_n := \frac{1}{n} \to 0$, we have $f_n \rightrightarrows f$ uniformly on $E := \bigcap_n E_n$ by definition, and $E^c = \bigcup_n E_n^c$ is still a null set.

$\impliedby$: Suppose $f_n \rightrightarrows f$ uniformly on some set E and $m(E^c) = 0$. Then for any ε, we can choose N large enough such that $|f_n(x) - f(x)| < \varepsilon$ on E; but then ε is an upper bound for $f_n - f$ almost everywhere, so $\|f_n - f\|_\infty < \varepsilon \to 0$.

1.3 Part c

To see that simple functions are dense in $L^\infty(X)$, we can use the fact that $f \in L^\infty(X) \iff$ there exists a g such that $f = g$ a.e. and g is bounded.

Then there is a sequence s_n of simple functions such that $\|s_n - g\|_\infty \to 0$, which follows from a proof in Folland:

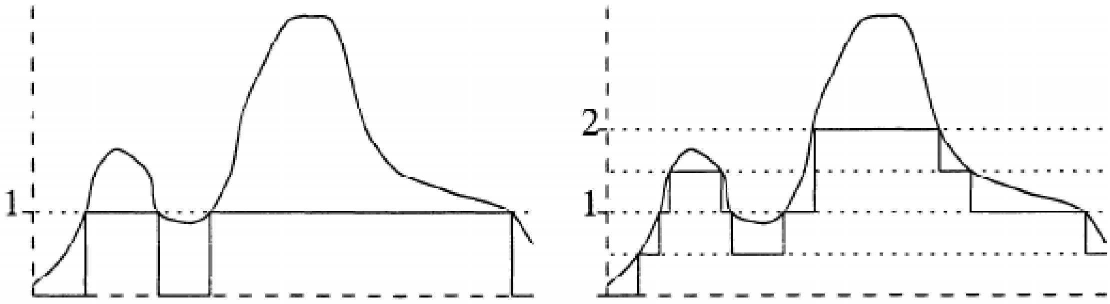
Proof. (a) For $n = 0, 1, 2, \ldots$ and $0 \leq k \leq 2^{2n} - 1$, let

$$E_n^k = f^{-1}\big((k2^{-n}, (k+1)2^{-n}]\big) \quad \text{and} \quad F_n = f^{-1}\big((2^n, \infty]\big),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n\chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that $\phi_n \leq \phi_{n+1}$ for all n, and $0 \leq f - \phi_n \leq 2^{-n}$ on the set where $f \leq 2^n$. The result therefore follows.



However, $C_c^0(X)$ is dense $L^\infty(X) \iff$ every $f \in L^\infty(X)$ can be approximated by a sequence $\{g_k\} \subset C_c^0(X)$ in the sense that $\|f - g_n\|_\infty \to 0$. To see why this can *not* be the case, let $f(x) = 1$, so $\|f\|_\infty = 1$ and let $g_n \to f$ be an arbitrary sequence of C_c^0 functions converging to f pointwise.

Since every g_n has compact support, say $\operatorname{supp}(g_n) \coloneqq E_n$, then $g_n|_{E_n^c} \equiv 0$ and $m(E_n^c) > 0$. In particular, this means that $\|f - g_n\|_\infty = 1$ for every n, so g_n can not converge to f in the infinity norm.

2 Problem 2

2.1 Part a

2.1.1 Part i

Lemma: $\|1\|_p = m(X)^{1/p}$

This follows from $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$ and taking pth roots. $\square$

By Holder with $p = q = 2$, we can now write

$$\begin{aligned}
&\|f\|_1 = \|1 \cdot f\|_1 \le \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\
\implies &\|f\|_1 \le m(X)^{1/2} \|f\|_2.
\end{aligned}$$

Letting $M \coloneqq \|f\|_\infty$, We also have

$$\begin{aligned}
\|f\|_2^2 &= \int_X |f|^2 \le \int_X |M|^2 = M^2 \int_X 1 = M^2 m(X) \\
\implies &\|f\|_2 \le m(X)^{1/2} \|f\|_\infty \\
\implies &m(X)^{1/2} \|f\|_2 \le m(X) \|f\|_\infty,
\end{aligned}$$

and combining these yields

$$\|f\|_1 \le m(X)^{1/2} \|f\|_2 \le m(X) \|f\|_\infty,$$

from which it immediately follows

$$m(X) < \infty \implies L^\infty(X) \subseteq L^2(X) \subseteq L^1(X).$$

The Inclusions Are Strict:

1. $\exists f \in L^1(X) \setminus L^2(X)$:

Let $X = [0,1]$ and consider $f(x) = x^{-\frac{1}{2}}$. Then

$$\|f\|_1 = \int_0^1 x^{-\frac{1}{2}} < \infty \qquad \text{by the } p \text{ test},$$

while

$$\|f\|_2^2 = \int_0^1 x^{-1} \to \infty \qquad \text{by the } p \text{ test}.$$

2. $\exists f \in L^2(X) \setminus L^\infty(X)$:

Take $X = [0,1]$ and $f(x) = x^{-\frac{1}{4}}$. Then

$$\|f\|_2^2 = \int_0^1 x^{-\frac{1}{4}} < \infty \qquad \text{by the } p \text{ test},$$

while $\|f\|_\infty > M$ for any finite M, since f is unbounded in neighborhoods of 0, so $\|f\|_\infty = \infty$.

2.1.2 Part ii

1. $\exists f \in L^2(X) \setminus L^1(X)$ when $m(X) = \infty$:

Take $X = [1, \infty)$ and let $f(x) = x^{-1}$, then

$$\|f\|_2^2 = \int_0^\infty x^{-2} < \infty \qquad \text{by the } p \text{ test},$$
$$\|f\|_1 = \int_0^\infty x^{-1} \to \infty \qquad \text{by the } p \text{ test}.$$

2. $\exists f \in L^\infty(X) \setminus L^2(X)$ when $m(X) = \infty$:

Take $X = \mathbb{R}$ and $f(x) = 1$. then

$$\|f\|_\infty = 1$$
$$\|f\|_2^2 = \int_{\mathbb{R}} 1 \to \infty.$$

3. $L^2(X) \subseteq L^1(X) \implies m(X) < \infty$:

Let $f = \chi_X$, by assumption we can find a constant M such that $\|\chi_X\|_2 \leq M\|\chi_X\|_1$.

Then pick a sequence of sets $E_k \nearrow X$ such that $m(E_k) < \infty$ for all k, $\chi_{E_k} \nearrow \chi_X$, and thus $\|\chi_{E_k}\|_p \leq M\|\chi_E\|_p$. By the lemma, $\|\chi_{E_k}\|_p = m(E_k)^{1/p}$, so we have

$$\begin{aligned}
\|\chi_{E_k}\|_2 \leq M\|\chi_{E_k}\|_1 &\implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \leq M \\
&\implies \frac{m(E_k)^{1/2}}{m(E_k)} \leq M \\
&\implies m(E_k)^{-1/2} \leq M \\
&\implies m(E_k) \leq M^2 < \infty.
\end{aligned}$$

and by continuity of measure, we have $\lim_K m(E_k) = m(X) \leq M^2 < \infty$. $\square$

2.2 Part b

1. $L_1(X) \bigcap L^\infty(X) \subset L^2(X)$:

Let $f \in L^1(X) \bigcap L^\infty(X)$ and $M := \|f\|_\infty$, then

$$\|f\|_2^2 = \int_X |f|^2 = \int_X |f||f| \leq \int_X M|f| = M \int |f| := \|f\|_\infty \|f\|_1 < \infty. \tag{1}$$

The inclusion is strict, since we know from above that there is a function in $L^2(X)$ that is not in $L^\infty(X)$.

Note that taking square roots in (1) immediately yields

$$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2} \|f\|_{L^\infty(X)}^{1/2}.$$

2. $L^2(X) \subset L^1(X) + L^\infty(X)$:

Let $f \in L^2(X)$, then write $S = \{x \ni |f(x)| \geq 1\}$ and $f = \chi_S f + \chi_{S^c} f := g + h$.

Since $x \geq 1 \implies x^2 \geq x$, we have

$$\|g\|_1^2 = \int_X |g| = \int_S |f| \leq \int_S |f|^2 \leq \int_X |f|^2 = \|f\|_2^2 < \infty,$$

and so $g \in L^1(X)$.

To see that $h \in L^\infty(X)$, we just note that h is bounded by 1 by construction, and so $\|h\|_\infty \leq 1 < \infty$.

3 Problem 3

For notational convenience, it suffices to prove this for $\ell^p(\mathbb{N})$, where we re-index each sequence in $\ell^p(\mathbb{Z})$ using a bijection $\mathbb{Z} \to \mathbb{N}$.

> Note: this technically reorders all sums appearing, but since we are assuming absolute convergence everywhere, this can be done. One can also just replace $\sum_{j=n}^m |a_j|^p$ with $\sum_{n \leq |j| \leq m} |a_j|^p$ in what follows.

1. $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$:

Suppose $\sum_j |a|_j < \infty$, then its tails go to zero, so choose N large enough so that

$$j \geq N \implies |a_j| < 1.$$

But then

$$j \geq N \implies |a_j|^2 < |a_j|,$$

and

$$\begin{aligned}
\sum_j |a_j|^2 &= \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j|^2 \\
&\leq \sum_{j=1}^{N} |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=1}^{\infty} |a_j| \\
&< \infty.
\end{aligned}$$

where we just note that the first portion of the sum is a finite sum of finite numbers and thus bounded.

To see that the inclusion is strict, take $\mathbf{a} \coloneqq \{j^{-1}\}_{j=1}^{\infty}$; then $\|\mathbf{a}\|_2 < \infty$ by the p-test by $\|\mathbf{a}\|_1 = \infty$ since it yields the harmonic series.

2. $\ell^2(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$:

This follows from the contrapositive: if $\mathbf{a}$ is a sequence with unbounded terms, then $\|\mathbf{a}\|_2 = \sum |a_j|^2$ can not be finite, since convergence would require that $|a_j|^2 \to 0$ and thus $|a_j| \to 0$.

To see that the inclusion is strict, take $\mathbf{a} = \{1\}_{j=1}^{\infty}$. Then $\|\mathbf{a}\|_\infty = 1$, but the corresponding sum does not converge.

3. $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1$:

Let $M = \|\mathbf{a}\|_1$, then

$$\|\mathbf{a}\|_2^2 \leq \|\mathbf{a}\|_1^2 \iff \frac{\|\mathbf{a}\|_2^2}{M^2} \leq 1 \iff \sum_j \left| \frac{a_j}{M} \right|^2 \leq 1.$$

But then we can use the fact that

$$\left| \frac{a_j}{M} \right| \leq 1 \implies \left| \frac{a_j}{M} \right|^2 \leq \left| \frac{a_j}{M} \right|$$

to obtain

$$\sum_j \left|\frac{a_j}{M}\right|^2 \leq \sum_j \left|\frac{a_j}{M}\right| = \frac{1}{M}\sum_j |a_j| := 1.$$

4. $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_2$:

This follows from the fact that, we have

$$\|\mathbf{a}\|_\infty^2 := \left(\sup_j |a_j|\right)^2 = \sup_j |a_j|^2 \leq \sum_j |a_j|^2 = \|\mathbf{a}\|_2^2$$

and taking square roots yields the desired inequality.

Note: the middle inequality follows from the fact that the supremum S is the least upper bound of all of the a_j, so for all j, we have $a_j + \varepsilon > S$ for every $\varepsilon > 0$. But in particular, $a_k + a_j > a_j$ for any pair a_j, a_k where $a_k \neq 0$, so $a_k + a_j > S$ and thus so is the entire sum.

4 Problem 4

4.1 Part a

Let $\{f_k\}$ be a Cauchy sequence, then $\|f_k - f_j\|_u \to 0$. Define a candidate limit by fixing x, then using the fact that $|f_j(x) - f_k(x)| \to 0$ as a Cauchy sequence in $\mathbb{R}$, which converges to some $f(x)$.

We want to show that and $\|f_n - f\|_u \to 0$ and $f \in C([0,1])$.

This is immediate though, since $f_n \to f$ uniformly by construction, and the uniform limit of continuous functions is continuous.

4.2 Part b

It suffices to produce a Cauchy sequence of continuous functions f_k such that $\|f_j - f_j\|_1 \to 0$ but if we define $f(x) := \lim f_k(x)$, we have either $\|f\|_1 = \infty$ or f is not continuous.

To this end, take $f_k(x) = x^k$ for $k = 1, 2, \cdots, \infty$.

Then pointwise we have

$$f_k \to \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases},$$

which has a clear discontinuity, but

$$\|f_k - f_j\|_1 := \int_0^1 x^k - x^j = \frac{1}{k+1} - \frac{1}{j+1} \to 0.$$

5 Problem 5

5.1 Part a

$\Longleftarrow$: It suffices to show that the map

$$\begin{aligned}
&H \twoheadrightarrow \ell^2(\mathbb{N}) \\
&\mathbf{x} \mapsto \{\langle \mathbf{x},\ \mathbf{u}_n\rangle\}_{n=1}^{\infty} := \{a_n\}_{n=1}^{\infty}
\end{aligned}$$

is a surjection, and for every $\mathbf{a} \in \ell^2(\mathbb{N})$, we can pull back to some $\mathbf{x} \in H$ such that $\|\mathbf{x}\|_H = \|\mathbf{a}\|_{\ell^2(\mathbb{N})}$.

Following the proof in Neil's notes, let $\mathbf{a} \in \ell^2(\mathbb{N})$ be given by $\mathbf{a} = \{a_j\}$, and define $S_N = \sum_{n=1}^{N} a_n \mathbf{u}_n$. We then have

$$\begin{aligned}
\|S_N - S_M\|_H &= \left\| \sum_{n=M+1}^{N} a_n \mathbf{u}_n \right\|_H \\
&= \sum_{n=M+1}^{N} \|a_n \mathbf{u}_n\|_H && \text{by Pythagoras, since the } \mathbf{u}_n \text{ are orthogonal} \\
&= \sum_{n=M+1}^{N} |a_n|_{\mathbb{C}}\ \|\mathbf{u}_n\|_H \\
&= \sum_{n=M+1}^{N} |a_n|_{\mathbb{C}} && \text{since the } \mathbf{u}_n \text{ are orthonormal} \\
&\to 0 && \text{as } N, M \to \infty,
\end{aligned}$$

which goes to zero because it is the tail of a convergent sum in $\mathbb{R}$.

Since H is complete, every Cauchy sequence converges, and in particular $S_N \to \mathbf{x} \in H$ for some $\mathbf{x}$.

We now have

$$\begin{aligned}
|\langle \mathbf{x},\ \mathbf{u}_n\rangle| &= |\langle \mathbf{x} - S_N + S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&= |\langle \mathbf{x} - S_N,\ \mathbf{u}_n\rangle + \langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&\leq \|\mathbf{x} - S_N\|_H \|\mathbf{u}_n\|_H + |\langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \text{ by Cauchy-Schwartz} \\
&= \|\mathbf{x} - S_N\|_H + |\langle S_N,\ \mathbf{u}_n\rangle| && \forall n, N \\
&= \|\mathbf{x} - S_N\|_H + |a_n| && \forall N \geq n \\
&\to 0 + |a_n| && \text{as } N \to \infty,
\end{aligned}$$

where we just note that

$$\langle S_N,\ \mathbf{u}_n\rangle = \left\langle \sum_{j=1}^{N} a_j \mathbf{u}_j,\ \mathbf{u}_n \right\rangle = \sum_{j=1}^{N} a_j \langle \mathbf{u}_j,\ \mathbf{u}_n\rangle = a_n \iff N \geq n$$

since $\langle \mathbf{u}_j,\ \mathbf{u}_n \rangle = \delta_{j,n}$ and so the a_n term is extracted iff $\mathbf{u}_n$ actually appears as a summand.

We thus have

$$\langle \mathbf{x},\ \mathbf{u}_n \rangle = |a_n| \quad \forall n,$$

and since $\{\mathbf{u}_n\}$ is a basis, we can apply Parseval's identity to obtain

$$\|\mathbf{x}\|_H^2 = \sum_{n=1}^{\infty} |\langle \mathbf{x},\ \mathbf{u}_n \rangle| \coloneqq \sum_{n=1}^{\infty} |a_n|.$$

$\implies$: Given a vector $\mathbf{x} = \sum_n a_n \mathbf{u}_n$, we can immediately note that both $\|\mathbf{x}\|_H < \infty$ and $\langle \mathbf{x},\ \mathbf{u}_n \rangle = a_n$. Since $\{\mathbf{u}_n\}$ being a basis is equivalent to Parseval's identity holding, we immediately obtain

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |\langle \mathbf{x},\ \mathbf{u}_n \rangle| = \|\mathbf{x}\|_H^2 < \infty.$$

5.2 Part b

In both cases, suppose such a linear functional exists.

1. Using part (a), we know that H is isometrically isomorphic to $\ell^2(\mathbb{N})$, and thus $H_f^\vee \cong (\ell^2(\mathbb{N}))^\vee \cong_d \ell^2(\mathbb{N})$.

Note: this follows since $\ell^p(\mathbb{N})^\vee \cong \ell^q(\mathbb{N})$ where p, q are Holder conjugates.

But then, since $L \in H^\vee$, under the isometry f it maps to the functional

$$\begin{aligned} L_\ell : \ell^2(\mathbb{Z}) &\to \mathbb{C} \\ \mathbf{a} = \{a_n\} &\mapsto \sum_{n \in \mathbb{N}} a_n n^{-1}, \end{aligned}$$

which under the identification of dual spaces g identifies L_ℓ with the vector $\mathbf{b} \coloneqq \{n^{-1}\}_{n \in \mathbb{N}}$.

Most importantly, these are all isometries, so we have the equalities

$$\|L\|_H = \|L_\ell\|_{\ell^2(\mathbb{N})^\vee} = \|\mathbf{b}\|_{\ell^2(\mathbb{N})},$$

so it suffices to compute the ℓ^2 norm of the sequence $b_n = \frac{1}{n}$. To this end, we have

$$\begin{aligned} \|\mathbf{b}\|_{\ell^2(\mathbb{N})}^2 &= \sum_n \left| \frac{1}{n} \right|^2 \\ &= \sum_n \frac{1}{n^2} \\ &= \frac{\pi^2}{6}, \end{aligned}$$

which shows that $\|L\|_H = \pi/\sqrt{6}$.

2. Using the same argument, we obtain $\mathbf{b} = \left\{n^{-1/2}\right\}_{n\in\mathbb{N}}$, and thus

$$\|L\|_H^2 = \|\mathbf{b}\|_{\ell^2(\mathbb{N})}^2 = \sum_n \left|n^{-1/2}\right|^2 \to \infty.$$

which shows that L is unbounded, and thus can not be a continuous linear functional. $\square$

6 Problem 6

We can use the fact that $\Lambda_p \in (L^p)^\vee \cong L^q$, where this is an isometric isomorphism given by the map

$$\begin{aligned} I : L^q &\to (L^p)^\vee \\ g &\mapsto (f \mapsto \int fg). \end{aligned}$$

Under this identification, for any $\Lambda \in (L^p)^\vee$, to any $\Lambda \in (L^p)^\vee$ we can associate a $g \in L^q$, where we have

$$\|\Lambda\|_{(L^p)^\vee} = \|g\|_{L^q}.$$

In this case, we can identify $\Lambda_p = I(g)$, where $g(x) = x^2$ and we can verify that $g \in L^q$ by computing its norm:

$$\begin{aligned} \|g\|_{L^q}^q &= \int_0^1 (x^2)^q \, dx \\ &= \left.\frac{x^{2q+1}}{2q+1}\right|_0^1 \\ &= \frac{1}{2q+1} \\ &= \frac{p-1}{3p-1} < \infty, \end{aligned}$$

where we identify $q = \frac{p}{p-1}$, and note that this is finite for all $1 \leq p \leq \infty$ since it limits to $\frac{1}{3}$.

But then

$$\|\Lambda_p\|_{(L^p)^\vee} = \|g\|_{L^q} = \left(\frac{p-1}{3p-1}\right)^{\frac{1}{q}} = \left(\frac{p-1}{3p-1}\right)^{\frac{p-1}{p}},$$

which shows that Λ_p is bounded and thus a continuous linear functional. $\square$