Title

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Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \,\forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 1.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i \right)$.

Remark 1.
There are few closed sets, so this is a "weak" topology.

Example 1.1.
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A \coloneqq \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 1.2.
Let $f : \mathbb{A}^1/k \longrightarrow \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 1.3.
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\cup_{i=1}^{n} U_i \times V_i$ with $U_i