Title

D. Zack Garza

Tuesday 13$^{\text{th}}$ October, 2020

Contents

1 | **Tuesday, October 13**

Last time: proved that if X, Y are affine varieties then there is a bijection

$$\left\{ \begin{matrix} \text{Morphisms} \\ f:X \to Y \end{matrix} \right\} \iff \left\{ \begin{matrix} k\text{-algebra morphisms} \\ A(Y) \to A(X) \end{matrix} \right\}$$
$$f \mapsto f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).$$

Remark 1.0.1.
A morphism $f : X \to Y$ is by definition a morphism of ringed spaces where $\mathcal{O}_X, \mathcal{O}_Y$ are the sheaves of regular functions.

Remark 1.0.2.
This shows $X \cong Y$ as ringed spaces iff $A(X) \cong A(Y)$ as k-algebras.

Example 1.0.1.
Take

$$f : \mathbb{A}^1 \to V(y^2 - x^3) \subset \mathbb{A}^2$$
$$t \mapsto (t^2, t^3).$$

This is a morphism by proposition 4.7.

We then get a map on algebras

$$f^* : A(V(y^2 - x^3)) = k[x, y]/\left\langle y^2 - x^3 \right\rangle \to k[t]$$
$$x \mapsto t^2$$
$$y \mapsto t^3,$$

but even though f is a bijective morphism, it's not an isomorphism of ringed spaces. This can be seen from the fact that the image doesn't contain t.

We'll define a category AffVar_k whose objects are affine varieties over k and morphisms in $\hom(X, Y)$ will be morphisms of ringed spaces. There is a contravariant functor A into reduced finitely generated k-algebras which sends X to $A(X)$ and sends morphisms $f : X \to Y$ to their pullbacks $f^* : A(Y) \to A(X)$, where "reduced" denotes the fact that there are no nilpotents.
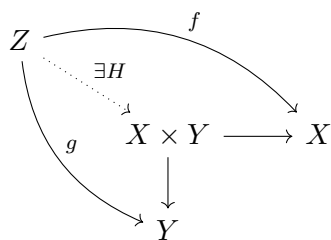
Remark 1.0.3.

If we have X, Y affine varieties, we take $X \times Y$ to be the categorical product instead of the underlying product of topological spaces. We have

$$A(X \times Y) \cong A(X) \otimes_k A(Y) \cong k[x_1, \cdots, x_n, y_1, \cdots, y_m]/I(X) \otimes 1 + 1 \otimes I(Y).$$

This recovers the product, since if we have

where $H = (f, g)$.

Remark 1.0.4.

Products of spaces are sent to the tensor product of k-algebras, i.e. pullbacks are sent to pushouts.