Title

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Contents

1 Ordinary Differential Equations

1.1 Techniques Overview

$p(y)y' = q(x)$	separable
$y' + p(x)y = q(x)$	integrating factor
$y' = f(x,y), f(tx,ty) = f(x,y)$	$y = xV(x)$ COV reduces to separable
$y' + p(x)y = q(x)y^n$	Bernoulli, divide by y^n and COV $u = y^{1-n}$
$M(x,y)dx + N(x,y)dy = 0$	$M_y = N_x : \varphi(x,y) = c(\varphi_x = M, \varphi_y = N)$
$P(D)y = f(x,y)$	$x^k e^{rx}$ for each root

Where e^{zx} *yields* $e^{ax}\cos bx, e^{ax}\sin bx$

1.2 Types of Equations

- Separable equations:

$$p(y)\frac{dy}{dx} - q(x) = 0 \implies \int p(y)dy = \int q(x)dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

  - Population growth:

$$\frac{dP}{dt} = kP \implies \qquad P = P_0 e^{kt}$$

  - Logistic growth:
    - ♢ General form: $\dfrac{dP}{dt} = (B(t) - D(t))P(t)$
    - ♢ Assume birth rate is constant $B(t) = B_0$ and death rate is proportional to instantaneous population $D(t) = D_0 P(t)$. Then let $r = B_0, C = B_0/D_0$ be the *carrying capacity*:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P \implies \qquad P(t) = \frac{P_0}{\frac{P_0}{C} + e^{-rt}(1 - \frac{P_0}{C})}$$

- First order linear:

$$\frac{dy}{dx} + p(x)y = q(x) \implies I(x) = e^{\int p(x)dx}, \qquad y(x) = \frac{1}{I(x)}\left(\int q(x)I(x)dx + C\right)$$

- Exact:
  - $M(x,y)dx + N(x,y)dy = 0$ is exact $\iff \exists\varphi : \dfrac{\partial\varphi}{\partial x} = M(x,y),\ \dfrac{\partial\varphi}{\partial y} = N(x,y)$
    $\iff \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{x}$
  - General solution:

$$\varphi(x,y) = \int^x M(s,y)ds + \int^y N(x,t)dt - \int^y \frac{\partial}{\partial t}\left(\int^x M(s,t)ds\right)dt$$

    (where $\int^x f(t)dt$ means take the antiderivative of f and consider it a function of x)

- Cauchy Euler: #todo
- Bernoulli: todo

1.3 Linear Homogeneous

General form:

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + cy' + cy = 0$$
$$p(D)y = \prod (D - r_i)^{m_i} y = 0$$

where p is a polynomial in the differential operator D with roots r_i:

- Real roots: contribute m_i solutions of the form

$$e^{rx}, xe^{rx}, \cdots, x^{m_i-1}e^{rx}$$

- Complex conjugate roots: for $r = a + bi$, contribute $2m_i$ solutions of the form

$$\begin{aligned} & e^{(a\pm bi)x}, xe^{(a\pm bi)x}, \ \cdots, \ x^{m_i-1}e^{(a\pm bi)x} \\ = & e^{ax}\cos(bx), e^{ax}\sin(bx), \ xe^{ax}\cos(bx), xe^{ax}\sin(bx), \ \cdots, \end{aligned}$$

Example: by cases, second order equation of the form

$$ay'' + by' + cy = 0$$

- Two distinct roots: $c_1e^{r_1x} + c_2e^{r_2x}$ - One real root: $c_1e^{rx} + c_2xe^{rx}$ - Complex conjugates $\alpha \pm i\beta$: $e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

1.4 Linear Inhomogeneous

General form:

$$\begin{aligned} y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2y'' + cy' + cy = F(x) \\ p(D)y = \prod(D - r_i)^{m_i}y = 0 \end{aligned}$$

Then solutions are of the form $y_c + y_p$, where y_c is the solution to the associated homogeneous system and y_p is a particular solution.

Methods of obtaining particular solutions

1.4.1 Undetermined Coefficients

- Find an operator $p(D)$ the annihilates $F(x)$ (so $q(D)F = 0$)
- Find solution of $q(D)p(D) = 0$, subtract of known solutions from homogeneous part to obtain the form of the trial solution $A_0f(x)$, where A_0 is the undetermined coefficient
- Substitute trial solution into original equation to determine A_0

Useful Annihilators:

$$\begin{aligned} F(x) &= p(x): && D^{\deg(p)+1} \\ F(x) &= p(x)e^{ax}: && (D-a)^{\deg(p)+1} \\ F(x) &= \cos(ax) + \sin(ax): && D^2 + a^2 \\ F(x) &= e^{ax}(a_0\cos(bx) + b_0\sin(bx)): && (D-z)(D-\overline{z}) = D^2 - 2aD + a^2 + b^2 \\ F(x) &= p(x)e^{ax}\cos(bx) + p(x)e^{ax}\cos(bx): && ((D-z)(D-\overline{z}))^{\max(\deg(p),\deg(q))+1} \end{aligned}$$

1.4.2 Variation of Parameters

todo

1.4.3 Reduction of Order

todo

1.5 Systems of Differential Equations

General form:

$$\frac{\partial \mathbf{x}(t)}{\partial t} = A\mathbf{x}(t) + \mathbf{b}(t) \iff \mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

General solution to homogeneous equation:

$$c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + \cdots + c_n\mathbf{x_n}(t) = \mathbf{X}(t)\mathbf{c}$$

If A is a matrix of constants: $\mathbf{x}(t) = e^{\lambda_i t}\ \mathbf{v}_i$ is a solution for each eigenvalue/eigenvector pair $(\lambda_i, \mathbf{v}_i)$
- If A is defective, you'll need generalized eigenvectors.

Inhomogeneous Equation: particular solutions given by

$$\mathbf{x}_p(t) = \mathbf{X}(t)\int^t \mathbf{X}^{-1}(s)\mathbf{b}(s)\ ds$$

1.6 Laplace Transforms

Definitions:

$$H_a(t) := \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

$$\delta(t) : \int_{\mathbb{R}} \delta(t-a)f(t)\ dt = f(a), \quad \int_{\mathbb{R}} \delta(t-a)\ dt = 1$$

$$(f * g)(t) = \int_0^t f(t-s)g(s)\ ds$$

$$L[f(t)] = L[f] = \int_0^\infty e^{-st}f(t)dt = F(s).$$

Useful property: for $a \le b$, $H_a(t) - H_b(t) = \mathbb{1}\left[[a,b]\right]$.

$$
\begin{aligned}
t^n, n \in \mathbb{N} \quad &\Longleftrightarrow \quad \frac{n!}{s^{n+1}}, \quad s > 0 \\
t^{-\frac{1}{2}} \quad &\Longleftrightarrow \quad \sqrt{\pi} s^{-\frac{1}{2}} \quad s > 0 \\
e^{at} \quad &\Longleftrightarrow \quad \frac{1}{s-a}, \quad s > a \\
\cos(bt) \quad &\Longleftrightarrow \quad \frac{s}{s^2+b^2}, \quad s > 0 \\
\sin(bt) \quad &\Longleftrightarrow \quad \frac{b}{s^2+b^2}, \quad s > 0 \\
\cosh(bt) \quad &\Longleftrightarrow \quad \frac{s}{s^2-b^2}, \quad ? \\
\sinh(bt) \quad &\Longleftrightarrow \quad \frac{b}{s^2-b^2}, \quad ? \\
\delta(t-a) \quad &\Longleftrightarrow \quad e^{-as} \\
H_a(t) \quad &\Longleftrightarrow \quad s^{-1}e^{-as} \\
e^{at}f(t) \quad &\Longleftrightarrow \quad F(s-a) \\
H_a(t)f(t-a) \quad &\Longleftrightarrow \quad e^{-as}F(s) \\
f'(t) \quad &\Longleftrightarrow \quad sL(f) - f(0) \\
f''(t) \quad &\Longleftrightarrow \quad s^2L(f) - sf(0) - f'(0) \\
f^{(n)}(t) \quad &\Longleftrightarrow \quad s^nL(f) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0) \\
f(t)g(t) \quad &\Longleftrightarrow \quad F(s) * G(s)
\end{aligned}
$$

- For f periodic with period T, $L(f) = \frac{1}{1+e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

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Theorem 1.6.1 ***(First Shifting Theorem).***

$$?L[e^{at}f(t)] = \int_0^\infty e^{(a-s)}f(t)dt = F(s-a),.$$