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1 | Wednesday, October 28

1.1 Review of Last Time

Suppose we have two weights in the same facet, i.e. they're in the same stabilizer under the action of the affine Weyl group:

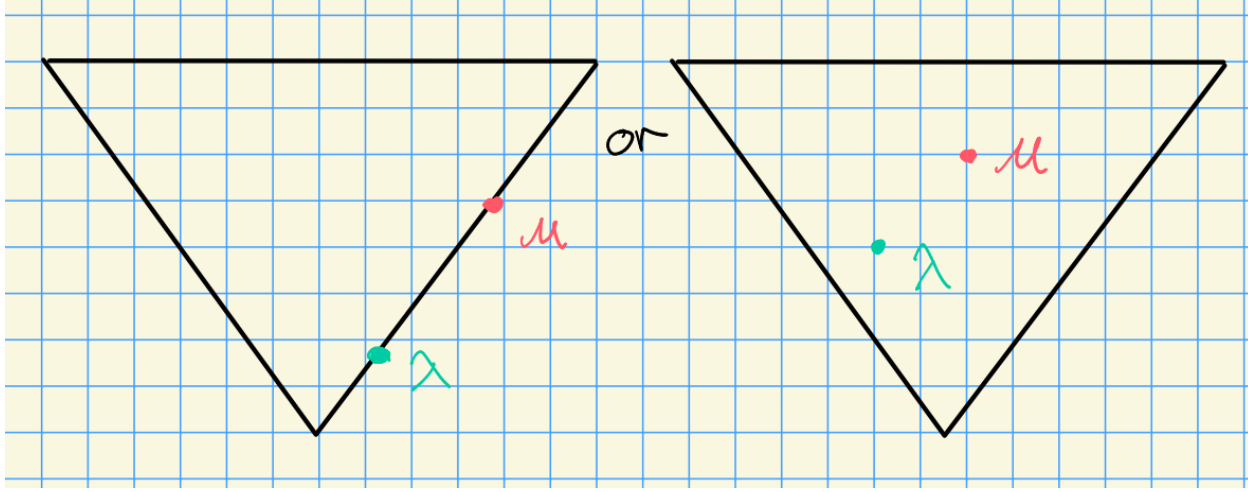


Figure 1: Weights in the same facet

We had a theorem: if λ, μ are in the same facet, then $\mathcal{B}_\lambda \cong \mathcal{B}_\mu$ is an equivalence of categories, where the map is via the translation functors.

1.2 Description of $T_\lambda^\mu(H^i(w \cdot \lambda))$

We can write

$$\begin{aligned}
 T_\lambda^\mu(H^i(w \cdot \lambda)) &= \text{pr}_\mu(L(\nu_1) \otimes \text{pr}_\lambda(H^i(w \cdot \lambda))) \\
 &= \text{pr}_\mu(L(\nu_1) \otimes H^i(w \cdot \lambda)) \\
 &= \text{pr}_\mu(L(\nu_1) \otimes R^i \text{Ind}_B^G w \cdot \lambda) \\
 &= \text{pr}_\mu(R^i \text{Ind}_B^G (L(\nu_1) \otimes w \cdot \lambda)).
 \end{aligned}$$

Take a composition series by B -modules of $L(\nu_1) \otimes w \cdot \lambda$, say

$$0 = M_0 \subseteq M_1 \cdots \subseteq M_r = L(\nu_1) \otimes w \cdot \lambda.$$

where $M_j/M_{j-1} \cong \lambda + j + w \cdot \lambda$ and $\lambda_j < \lambda_{j'} \implies j < j'$, i.e. we can order them in a decreasing way.

Consider the SES

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow M_j/M_{j-1} \longrightarrow 0$$

where applying $\text{pr}_\mu(\cdot)$ induces the LES

$$\cdots \longrightarrow \text{pr}_\mu M_{j-1} \longrightarrow \text{pr}_\mu M_j \longrightarrow \text{pr}_\mu(M_j/M_{j-1}) \longrightarrow \cdots$$

We know that

$$\text{pr}_\mu H^i(\lambda_j + w \cdot \lambda) = \begin{cases} H^i(\lambda_j + w \cdot \lambda) & \lambda + j + w \cdot \lambda \in W_p \cdot \mu \\ 0 & \text{else} \end{cases},$$

i.e. this projection is the identity for weights linked to μ and zero otherwise. We also have

$$\text{pr}_\mu H^i(M_r) = T_\lambda^\mu H^i(w \cdot \lambda).$$

Theorem 1.2.1 (?).

Let $\lambda, \mu \in \bar{C}_\mathbb{Z}$ and F be a facet with $\lambda \in F$. If $\mu \in \bar{F}$, then we have

$$T_\lambda^\mu(H^i(w \cdot \lambda)) = H^i(w \cdot \mu) \quad \forall w \in W_p.$$

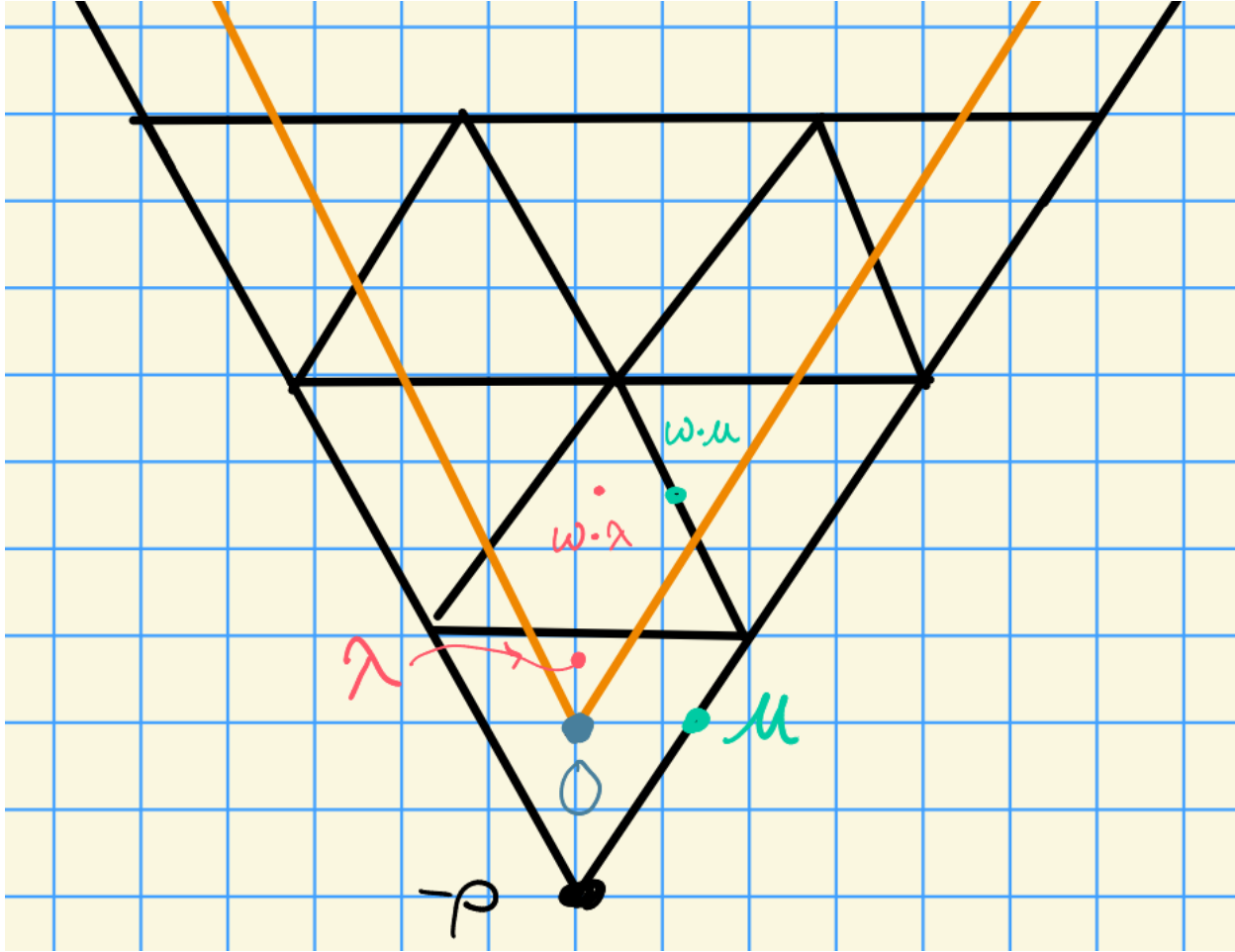


Figure 2: Image

Example 1.2.1 (?).

Here consider $H_0(\lambda) \xrightarrow{T_\lambda^\mu} H_0(\mu) = 0$, since μ is outside of the dominant region (in orange.) We also have $H^0(w \cdot \lambda) \rightarrow H^0(w \cdot \mu) \neq 0$, since this falls *into* the dominant region.

Proof (?).

Let $\lambda \in F$ and $\mu \in \bar{F}$. Then $\text{Stab}_{W_p}(\lambda) \subseteq \text{Stab}_{W_p}(\mu)$. By a previous technical lemma, we had a formula for computing $\text{ch } T_\lambda^\mu V$, which involved considering

$$w_1 \in \frac{\text{Stab}_{W_p}(\lambda)}{\text{Stab}_{W_p}(\lambda) \cap \text{Stab}_{W_p}(\mu)}.$$

In this case, we get $w_1 = \text{id}$, since the top and bottom are equal.

By that lemma, there exists a unique ℓ such that $w \cdot \lambda + \lambda_\ell \in W_p \cdot \mu$, where λ_ℓ is a weight of $L(\nu_1)$. From the LES, we have

$$\cdots \longrightarrow \text{pr}_\mu M_{j-1} \longrightarrow \text{pr}_\mu M_j \longrightarrow \text{pr}_\mu (M_j / M_{j-1}) = \lambda_j + w \cdot \lambda \longrightarrow \cdots$$

where the last term will only be nonzero in restricted cases. We can thus conclude that

$$\mathrm{pr}_\mu(H^i(M_j)) = \begin{cases} 0 & j < \ell \\ H^i(w \cdot \mu) & j \geq \ell. \end{cases}$$

Setting $j = r$, we have

$$T_\lambda^\mu(H^i(w \cdot \lambda)) = \mathrm{pr}_\mu H^j(M_r) = H^i(w \cdot \mu).$$

■

Suppose $\lambda \in \overline{C}_\mathbb{Z}$ and $\mu \in C_\mathbb{Z}$. What happens when you translate λ (blue) off of a wall? $T_\lambda^\mu(H^0(w \cdot \lambda))$ has a filtration with factors $H^0(w_1 \cdot \mu)$ and $H^0(w_2 \cdot \mu)$ (shown in green).

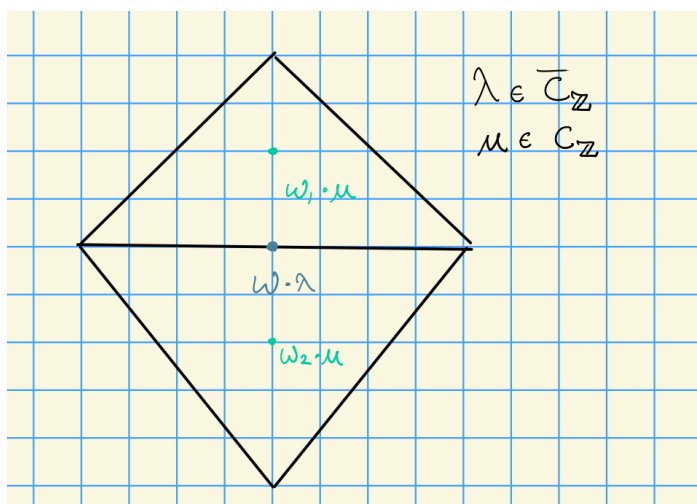


Figure 3: Image

If $w\lambda$ is a vertex with $\mu \in C_\mathbb{Z}$, then $T_\lambda^\mu(H^0(w \cdot \lambda))$ can have six factors:

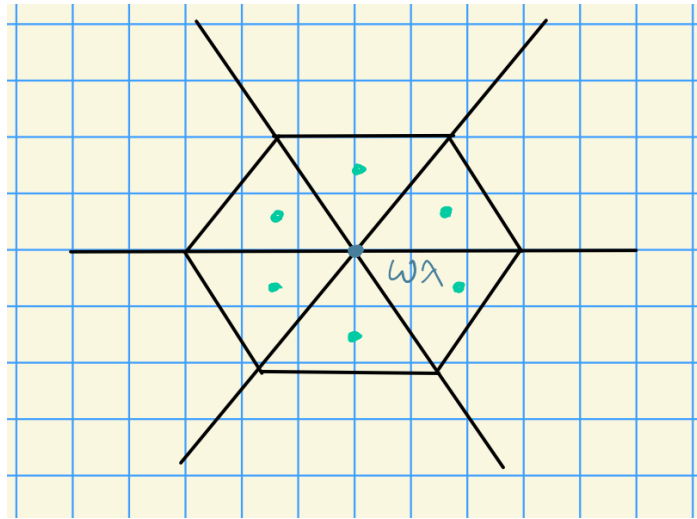


Figure 4: Weight where the translation has six factors

Proposition 1.2.1(?).

Suppose $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\mu \in C_{\mathbb{Z}}$, and let $w \in W_p$ where $w \cdot \lambda \in X(T)_+$. Then $T_\lambda^\mu(H^0(w \cdot \lambda))$ has a filtration such that all of the composition factors are of the form $H^0(ww_1 \cdot \mu)$ where $w_1 \in \text{Stab}_{W_p}(\lambda)$ and each of the factors occurs at most once.