# Title

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## 1.1 Stalks and Localizations

Recall that a sheaf of rings on a topological space $X$ is a ring $\mathcal{F}(U)$ for all open sets $U \subset X$ satisfying four properties:

1. The empty set is mapped to zeor.
2. The morphism $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity.
3. Given $W \subset V \subset U$ we have
4. Gluing: given sections $s_i \in \mathcal{F}(U_i)$ which agree on overlaps (restrict to the same function on $U_i \cap U_j$), there is a unique $s \in \mathcal{F}(\cup U_i)$.

**Example 1.1.**
If $X$ is an affine variety with the zariski topology, $\mathcal{O}_X$ is a sheaf of regular functions, where we recall $\mathcal{O}_X(U)$ are the functions $\varphi : U \to k$ that are locally a fraction.

Recall that the *stalk* of a sheaf $\mathcal{F}$ at a point $p \in X$, is defined as

$$\mathcal{F}_p \coloneqq \left\{ (U, \varphi) \;\middle|\; p \in U \text{ open }, \varphi \in \mathcal{F}(U) \right\} / \sim .$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists a $p \in W \subset U \cap U'$ such $\varphi, \varphi'$ restricted to $W$ are equal.

Recall that a *local ring* is a ring with a unique maximal ideal $\mathfrak{m}$. Given a prime ideal $\mathfrak{p} \in R$, so $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$, the complement $R \setminus P$ is closed under multiplication. So we can localize to obtain $R_\mathfrak{p} = \left\{ a/s \;\middle|\; s \in R \setminus P, a \in R \right\} / \sim$ where $a'/s' \sim a/s$ iff there exists a $t \in R \setminus P$ such that $t(a's - as') = 0$.

⚠ **Warning:** Note that $R_f$ is localizing at the powers of $f$, whereas $R_{\mathfrak{p}}$ is localizing at the *complement* of $\mathfrak{p}$.

Since maximal ideals are prime, we can localize any ring $R$ at a maximal ideal $R_{\mathfrak{m}}$, and this will be a local ring. Why? The ideals in $R_{\mathfrak{m}}$ biject with ideals in $R$ contained in $\mathfrak{m}$. Thus all ideals in $R_{\mathfrak{m}}$ are contained in the maximal ideal generated by $\mathfrak{m}$, i.e. $\mathfrak{m}R_{\mathfrak{m}}$.

**Lemma 1.1** *(?).*
Let $X$ be an affine variety. The stalk of the sheaf of regular functions $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$ is isomorphic to the localization $A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p := I(\{p\})$.

*Proof .*
We can write

$$A(X)_{\mathfrak{m}_p} := \left\{ \frac{g}{f} \;\middle|\; g \in A(X),\, f \in A(X) \setminus \mathfrak{m}_p \right\} / \sim$$
$$\text{where } g_1/f_1 \sim g_2/f_2 \iff \exists h(p) \neq 0 \text{ where } 0 = h(f_2 g_1 - f_1 g_2).$$

where the $f$ are regular functions on $X$ such that $f(p) = 0$.
We can also write

$$\mathcal{O}_{X,p} := \left\{ (U, \varphi) \;\middle|\; p \in U,\, \varphi \in \mathcal{O}_X(U) \right\} / \sim$$
$$\text{where } (U, \varphi) \sim (U', \varphi') \iff \exists p \in W \subset U \cap U' \text{ s.t. } \varphi|_W = \varphi'|_W.$$

So we can define a map

$$\begin{aligned} \Phi : A(X)_{\mathfrak{m}_p} &\to \mathcal{O}_{X,p} \\ \frac{g}{f} &\mapsto \left( D_f, \frac{g}{f} \right). \end{aligned}$$

**Step 1:** There are equivalence relations on both sides, so we need to check that things are well-defined.
We have

$$\begin{aligned} g/f \sim g'/f' &\iff \exists g \text{ such that } h(p) \neq 0,\, h(gf' - g'f) = 0 \in A(X) \\ &\iff \text{the functions } \frac{g}{f}, \frac{g'}{f'} \text{ agree on } W := D(f) \cap D(f') \cap D(h) \\ &\implies (D_f, g/f) \sim (D_{f'}, g'/f'), \end{aligned}$$

since there exists a $W \subset D_f \cap D_{f'}$ such that $g/f, g'/f'$ are equal.

**Step 2:** Surjectivity, since this is clearly a ring map with pointwise operations.
Any germ can be represented by $(U, \varphi)$ with $\varphi \in \mathcal{O}_X(U)$. Since the sets $D_f$ form a base for the topology, there exists a $D_f \subset U$ containing $p$. By definition, $(U, \varphi) = (D_f, \varphi|_{D_f})$ in $\mathcal{O}_{X,p}$. Using the proposition that $\mathcal{O}_X(D(f)) = A(X)_f$, this implies that $\varphi|_{D_f} = g/f^n$ for some $n$ and $f(p) \neq 0$, so $(U, \varphi)$ is in the image of $\Phi$.

**Step 3**: Injectivity. We want to show that $g/f \mapsto 0$ implies that $g/f = 0 \in A(X)_{\mathfrak{m}_p}$.

Suppose that $(D_f, g/f) = 0 \in \mathcal{O}_{X,p}$ and $(U, \varphi) = 0 \in \mathcal{O}_{X,p}$, then there exists an open $W \subset D_f$ containing $p$ such that after passing to some distinguished open $D_h \ni p$ such that $\varphi = 0$ on $D_h$. Wlog we can assume $\varphi = 0$ on $U$, since we could shrink $U$ (staying in the same equivalence class) to make this true otherwise. Then $\varphi = g/f$ on $D_h$, using that $\mathcal{O}_X(D_f) = A(X)_f$, so $g/f = 0$ here. So there exists a $k$ such that $f^k(g \cdot 1 - 0 \cdot f) = 0$ in $A(X)$, so $f^k g = 0 \in A(X)_{\mathfrak{m}_p}$. ■

Conclusion:

$$\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}.$$

**Example 1.2.**
Let $X = \{p, q\}$ with the discrete topology with the sheaf $\mathcal{F}$ given by $p \mapsto R, q \mapsto S, X \mapsto R \times S$.

Then $\mathcal{F}_p = R$, since if $U$ is open and $p \in U$ then either $U = \{p\}$ or $U = X$. We can check that for $(r, s)$ a section of $\mathcal{F}$, we have an equivalence of germs $(X, (r, s)) \sim (\{p\}, r)$ since $\{p\} \subset X \cap \{p\}$. Here $X$ plays the role of $U$, $\{p\}$ of $U'$, and the last $\{p\}$ the role of $W \subset U \cap U'$.

$$\begin{aligned} \mathcal{O}_{X,p} &\to A(X) \\ (\{p\}, r) &\mapsto r \\ \mathcal{F}_p &\cong R. \end{aligned}$$

**Example 1.3.**
Let $M$ be a manifold and consider the sheaf $C^\infty$ of smooth functions on $M$. Then the stalk $C^\infty_p$ at $p$ is defined as the set of smooth functions in a neighborhood of $p$ modulo functions being equivalent if they agree on a small enough ball $B_\varepsilon(p)$. This contains a maximal ideal $\mathfrak{m}_p$, the smooth functions vanishing at $p$.

Then $\mathfrak{m}_p^2$ is again an ideal, equal to the set $\left\{ f \;\middle|\; \partial_i \partial_j f \,\Big|_p = 0, \, \forall i, j \right\}$. Thus $\mathfrak{m}_p / \mathfrak{m}_p^2 \cong \{\partial_v\}^\vee$, the dual of the set of directional derivatives.

## 1.2 What's the Point!

Problem: what should a map of affine varieties be? A bad definition would be just taking the continuous maps: for example, any bijection $\mathbb{A}^1_{\mathbb{C}}$ is a homeomorphism in the zariski topology. Why? This coincides with the cofinite topology, and the preimage of a cofinite set is cofinite.

How do we fix this?

1. $f : X \to Y$ is continuous, i.e. $f^{-1}(U)$ is open whenever $U$ is open.
2. Given $U \subset Y$ open and $\varphi \in \mathcal{O}_Y(U)$, the function $\varphi \circ f : f^{-1}(U) \to k$ is regular.

We'll take this to be the definition of a morphism $X \to Y$.

**Example 1.4.**
For smooth manifolds, we also require that there is a pullback that preserves smooth functions:

$$f^* : C^\infty(U) \to C^\infty(f^{-1}(U)).$$