

# Problem Set 2

D. Zack Garza

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## 1 Humphreys 1.5

**Proposition:** Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and  $M(\lambda), M(\mu)$  Verma modules. Then  $M(\lambda) \otimes M(\mu)$  may not lie in  $\mathcal{O}$ .

**Proof:**

Let  $M(\lambda), M(\mu)$  be arbitrary Verma modules with highest weight vectors  $v = 1 \otimes 1_\lambda, w = 1 \otimes 1_\mu$  respectively. We can then consider the weight of  $v \otimes w$  in  $N := M(\lambda) \otimes_{\mathbb{C}} M(\mu)$ :

$$\begin{aligned} h \cdot (v \otimes w) &= h \cdot v \otimes w + v \otimes h \cdot w \\ &= \lambda(h)v \otimes w + v \otimes \mu(h)w \\ &= \lambda(h)(v \otimes w) + \mu(h)(v \otimes w) \\ &= (\lambda(h) + \mu(h))(v \otimes w). \end{aligned}$$

Letting  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ , so  $\lambda, \mu \in \mathfrak{h}^* \cong \mathbb{C}$ , the claim is that it is possible for  $N$  to *not* be finitely-generated as a  $U(\mathfrak{g})$ -module.

Let  $\{y, h, x\}$  be the usual basis for  $\mathfrak{g}$ , for which  $U(\mathfrak{g})$  has the usual associated PBW basis. We can use the fact that  $\dim M(z) < \infty \iff z \in \mathbb{Z}^+$ , so if we pick  $\mu, \lambda \in \mathbb{Z}^{\leq 0}$  we have weight space decompositions

$$\begin{aligned} M(\lambda) &= \bigoplus_{i \in \mathbb{Z}^+} M(\lambda)_{\lambda-2i} := \bigoplus_{\substack{i \in \mathbb{Z}^+ \\ \lambda_i := \lambda-2i}} M(\lambda)_{\lambda_i} \\ M(\mu) &= \bigoplus_{j \in \mathbb{Z}^+} M(\mu)_{\mu-2j} := \bigoplus_{\substack{j \in \mathbb{Z}^+ \\ \mu_j := \mu-2j}} M(\mu)_{\mu_j} \end{aligned}$$

where we can explicitly identify  $\mathbb{C}$ -bases  $M(\lambda)_{\lambda_i} = \text{span}_{\mathbb{C}} \{y^i v^+\}$  and  $M(\mu)_{\mu_i} = \text{span}_{\mathbb{C}} \{y^i w^+\}$  where  $v^+, w^+$  are maximal weight vectors for  $M(\lambda), M(\mu)$  respectively.

By the initial observation, this yields a weight space decomposition for  $N$  given by

$$N = M(\lambda) \otimes_{\mathbb{C}} M(\mu) = \bigoplus_{\nu \in \mathbb{Z}^+} \left( \bigoplus_{\lambda_i + \mu_i = \nu} M(\lambda)_{\lambda_i} \otimes_{\mathbb{C}} M(\mu)_{\mu_i} \right) := \bigoplus_{\nu \in \mathbb{Z}^+} N_{\nu}.$$

Since each weight space  $N_{\nu} = \text{span}_{\mathbb{C}} \{y^i v^+ \otimes y^j w^+ \mid i + j = \nu\}$  has dimension  $p_2(\nu)$ , the (combinatorial) number of partitions of  $\nu$  into two parts. In particular,  $p_2(\nu)$  takes on arbitrarily large values as  $\nu$  ranges over  $\mathbb{Z}^+$ , and thus  $N$  has weight spaces of arbitrarily large dimension.

Now suppose toward a contradiction that  $N$  is finitely generated as a  $U(\mathfrak{g})$ -module, say by the  $n$  generators  $\{m_1, \dots, m_n\}$ . Then the  $\mathbb{C}$ -vector spaces spanned by the  $m_i$  is of dimension no larger than  $n^2$  – however, picking  $\nu > n^2$  yields  $p_2(\nu) > n^2$ , and thus there is a  $\mathbb{C}$ -subspace of dimension greater than  $n^2$  by the above argument – a contradiction. ■

## 2 Humphreys 1.9

**Proposition:** Let  $\psi : Z(\mathfrak{g}) \rightarrow S(\mathfrak{h})$  be the twisted Harish-Chandra homomorphism. Then  $\psi$  is independent of the choice of a simple system in  $\Phi$ .

Hint: any simple system has the form  $w\Delta$  for some  $w \in W$ .

**Proof:**

For a given simple root system  $\Delta_1 = \{\alpha_1, \dots, \alpha_{\ell}\}$ , we can choose a PBW basis  $\{h_i^{t_j} \mid 1 \leq i \leq \ell, j \in \mathbb{Z}^+\}$  for  $U(\mathfrak{h})$ . Then if  $z \in Z(\mathfrak{g})$ , we can write  $z = \sum_{i,j} c_{ij} h_i^{t_j}$  for some  $c_{ij} \in \mathbb{C}$ . We can then identify the (twisted) Harish-Chandra morphism as follows:

$$\begin{aligned} \psi : Z(\mathfrak{g}) & \xrightarrow{\xi} U(\mathfrak{h}) \rightarrow S(\mathfrak{h}) = \mathbb{C}[\{h_i\}] = P(\mathfrak{h}^*) & \xrightarrow{\tau_{\rho}} \mathbb{C}[\{h_i\}] \\ z & \mapsto z = \sum_{i,j} c_{ij} h_i^{t_j} & \mapsto \left( \lambda \mapsto \sum_{i,j} c_{ij} \lambda(h_i)^{t_j} \right) & \mapsto \psi(z) = \sum_{i,j} (\lambda - \rho)(h_i)^{t_j}, \end{aligned}$$

where  $\xi$  is the Harish-Chandra morphism and  $\tau_{\rho}$  is the twist sending  $f(\lambda)$  to  $f(\lambda - \rho)$ . We thus find that  $\psi$  explicitly depends only on  $\rho$  and potentially the basis  $\{h_i\}$

The claim is that if an alternative simple root system  $\Delta_2 = \{\alpha'_1, \dots, \alpha'_{\ell}\}$  is chosen,  $\psi(z)$  does not change. By the hint, there exists some uniform  $w \in W$  such that  $w\alpha_i = \alpha'_i$ .

We can denote the positive root system induced by  $\Delta_1$  as  $\Phi_1^+$  and similarly  $\Delta_2$  induces  $\Phi_2^+$ . From this, a priori we may have two distinct weyl vectors:

$$\begin{aligned}\rho_1 &= \sum_{\beta \in \Phi_1^+} \beta \\ \rho_2 &= \sum_{\beta' \in \Phi_2^+} \beta' \\ &\cdot\end{aligned}$$

However, since  $W$  acts transitively on the Weyl chambers, it only permutes the elements in such a sum, and since  $\Delta_1 = w\Delta_2$  we in fact obtain  $\rho_1 = \rho_2 := \rho$ .