

Title

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1 History

Poincare, *Analysis Situs* papers in 1895. Coined “homeomorphism”, defined homology, gave rigorous definition of homotopy, established “method of invariants” and essentially kicked off algebraic topology.

2 Motivation

Generalized Topological Poincare Conjecture: When is a homotopy sphere also a topological sphere? i.e. when does $\pi_* X \cong_{Grp} \pi_* S^n \implies X \cong_{Top} S^n$?

- $n = 1$: True. Trivial
- $n = 2$: True. Proved by Poincare, classical
- $n = 3$: True. Perelman (2006) using Ricci flow + surgery
- $n = 4$: True. Freedman (1982), Fields medal!
- $n = 5$: True. Zeeman (1961)
- $n = 6$: True. Stallings (1962)
- $n \geq 7$: True. Smale (1961) using h-cobordism theorem, uses handle decomposition + Morse functions

Smooth Poincare Conjecture: When is a homotopy sphere a *smooth* sphere?

- $n = 1$: True. Trivial
- $n = 2$: True. Proved by Poincare, classical

- $n = 3$: True. (Top = PL = Smooth)
- $n = 4$: **Open**
- $n = 5$: Zeeman (1961)
- $n = 6$: Stalling (1962)
- $n \geq 7$: False in general (Milnor and Kervaire, 1963), Exotic S^n , 28 smooth structures on S^7

It is unknown whether or not B^4 admits an exotic smooth structure. If not, the smooth 4-dimensional Poincare conjecture would have an affirmative answer.

Current line of attack: Gluck twists on S^4 . Yield homeomorphic spheres, *suspected* not to be diffeomorphic, but no known invariants can distinguish smooth structures on S^4 .

Relation to homotopy: Define a monoid G_n with

- Objects: smooth structures on the n sphere (identified as oriented smooth n -manifolds which are homeomorphic to S^n)
- Binary operation: Connect sum

For $n \neq 4$, this is a group. Turns out to be isomorphic to Θ_n , the group of h -cobordism classes of “homotopy S^n s”

Recently (almost) resolved question: what is Θ_n for all n ?

Application: what spheres admit unique smooth structures?

- Define $bP_{n+1} \leq \Theta_n$ the subgroup of spheres that bound *parallelizable* manifolds (define in a moment).
- The Kervaire invariant is an invariant of a framed manifold that measures whether the manifold could be surgically converted into a sphere. 0 if true, 1 otherwise.
- Hill/Hopkins/Ravenel (2016): $= 0$ for $n \geq 254$.
- Kervaire invariant $= 1$ only in 2, 6, 14, 30, 62. Open case: 126.
- Punchline: there is a map $\Theta_n/bP_{n+1} \rightarrow \pi_n^S/J$, (to be defined) and the Kervaire invariant influences the size of bP_{n+1} . This reduces the differential topology problem of classifying smooth structures to (essentially) computing homotopy groups of spheres.
- Open question: is there a manifold of dimension 126 with Kervaire invariant 1?

Parallelizable/framed: Trivial tangent bundle, i.e. the principal frame bundle has a smooth global section. Parallelizable spheres S^0, S^1, S^3, S^7 corresponding to $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

- Framed Bordism Classes of manifolds – $\Omega_n^{fr} \cong \pi_n^S$ > Note: bordism is one of the coarsest equivalence relations we can put on manifolds. Hope to understand completely!

3 Background

Definition (Homotopy) Given two paths $P_1, P_2 : I \rightarrow X$ (where we identify the paths with their images under these maps), then a *homotopy* from P_1 to P_2 is a function

$$\begin{aligned} H : I &\rightarrow (I \rightarrow X) \\ H(0, \cdot) &= x_0 \\ H(1, \cdot) &= x_1 \end{aligned}$$

$$\begin{aligned} H(\cdot, 0) &= P_1(\cdot) \\ H(\cdot, 1) &= P_2(\cdot) \end{aligned}$$

such that the associated “partially applied” function $H_t : I \rightarrow X$ is continuous.

Definition (Homotopic Maps) Given two maps $f, g : X \rightarrow Y$, we say f is *homotopic* to g and write $f \sim g$ if there is a homotopy

$$\begin{aligned} H : I &\rightarrow (X \rightarrow Y) \\ H(0, \cdot) &= f(\cdot) \\ H(1, \cdot) &= g(\cdot) \end{aligned}$$

such that $H_t : X \rightarrow Y$ is continuous.

Can think of this as a map from the cylinder on X into Y , or deformations through continuous functions.

Note: This is an equivalence relation. If $f : X \rightarrow Y$ is a map, we write $[X, Y]$ to denote the homotopy classes of maps X to Y .

Definition (Fundamental Group)

$$\pi_1(X) := [S^1, X].$$

Note that this actually measures homotopy classes of *loops* in X .

Example: $\pi_1 T^2 = \mathbb{Z}^{*2}$, a *free* abelian group of rank 2.

Definition (Higher Homotopy Groups)

$$\pi_n(X) := [S^n, X].$$

Introduced by Cech in 1932, Alexandrov reportedly told him to withdraw because it couldn’t possibly be the right generalization due to the following theorem:

Theorem:

$$n \geq 2 \implies [S^n, X] \in \text{Ab}.$$

In words, higher homotopy groups are abelian. We have a complete classification of abelian groups, so we know $\pi_n(X) = F \oplus T$ for some free and torsion parts.

Theorem (Hopf, 1931):

$$[S^3, S^2] = \mathbb{Z} \neq 0$$

Recall that homology vanishes above the dimension of a given manifold!

This group is generated by the *Hopf fibration*, and provides infinitely many ways of “wrapping” a 3-sphere around a 2-sphere nontrivially! This was surprising and unexpected

Definition (CW Complex) A CW complex is any space built from the following inductive process:

Denote X_n the n -skeleton.

- Let X_0 by a discrete set of points.
- Let X_{n+1} be obtained from X_n by taking a collection of n - balls and glue them to X_n by maps

$$\phi : \partial B^n \rightarrow X_n$$

.

- If infinitely many stages, let $X = \bigcup X_n$ with the weak topology
(i.e. a set $A \subset X$ is open iff $A \cap X_n$ is open for all n)

Example: Every graph is a 1-dimensional CW complex

Example: Identification polyhedra for surfaces

Example: $S_n = e_0 + e_n$ by gluing B^{n+1} to a point by a map $\phi : \partial B^{n+1} \rightarrow \{\text{pt}\}$, i.e. $B^{n+1}/B^n \cong S^n$.
Can also attach two hemispheres at each $i \leq n$ to get $S^n = e_0 + e_1 + 2e_2 + \cdots + 2e_n$.

Note: Cellular homology is very easy to compute!

Note: Replacing ϕ with a homotopic map yields an equivalent CW complex. So understanding CW complexes boils down to understanding $[S^n, S^m]$ for $m < n$, i.e. higher homotopy groups of spheres.

Definition (Cellular Map) A map between $f : X \rightarrow Y$ between CW complex is *cellular* if $f(X_{(k)}) \subseteq Y_{(k)}$ for every k .

Theorem (Cellular Approximation): Any map $f : X \rightarrow Y$ is homotopic to a cellular map.

Application: $\pi_k S^n = 0$ if $k < n$. Use $f \in \pi_k S^n \iff f \in [S_k, S_n]$, deform f to be cellular, then $f(S_{(k)}^k) \hookrightarrow S_{(k)}^n = \{\text{pt}\}$, so $f \simeq c_0$, a constant map.

Definition (Homotopy Equivalence) Two spaces X, Y are said to be *homotopy equivalent* if there exists a maps $f : X \rightarrow Y$ and $f^{-1} : Y \rightarrow X$ such that

$$\begin{aligned} f \circ f^{-1} &\simeq \text{id}_Y \\ f^{-1} \circ f &\simeq \text{id}_X \end{aligned}$$

Definition (Weak Equivalence) A continuous map

$$f : X \rightarrow Y$$

is called a *weak homotopy equivalence* if the induced map

$$f_* : \pi_*(X) \rightarrow \pi_*(Y)$$

is a graded isomorphism.

Note that this is a strictly weaker notion than homotopy equivalence – we don't require an explicit inverse.

Note that a weak homotopy equivalence also induces isomorphisms on all homology and cohomology.

Theorem (Whitehead): If $f : X \rightarrow Y$ is a weak equivalence between CW complexes, then it is a homotopy equivalence.

Corollary (Relative Whitehead): If $f : X \rightarrow Y$ between CW complexes induces an isomorphism $H_*X \cong H_*Y$, then f is a weak equivalence.

Theorem (CW Approximation): For every topological space X , there exists a CW complex \tilde{X} and a weak homotopy equivalence $f : X \rightarrow \tilde{X}$.

Note: Weak equivalences = equivalences for CW complexes, which means we can essentially throw out the distinction!

Note: This says that if we understand CW complexes, we essentially understand the category hoTop completely. Moreover, we only have to understand spaces up to *weak* equivalence, i.e. we just need to check induced maps on π_* instead of checking for inverse maps.

Definition (Connectedness): A space is said to be n -connected if $\pi_{\leq n}X = 0$.

Recall that a space is *simply connected* iff $\pi_1X = 0$.

Theorem (Hurewicz): Given a fixed space X , any map $f \in \pi_kX = [S^k, X]$ has the type $f : S^k \rightarrow X$. This induces a map $f_* : H_*S^k \rightarrow H_*X$. Since $H_kS^k \cong \mathbb{Z} \cong \langle \mu \rangle$, define a family of maps

$$\begin{aligned} h_k : \pi_kX &\rightarrow H_kX \\ [f] &\mapsto f_*(\mu) \end{aligned}$$

If $n \geq 2$ and X is $n - 1$ connected, then h_k is an isomorphism for all $k \leq n$.

Note: If $k = 1$, then h_1 is the abelianization of π_1 .

3.1 Application

If X a simply connected, closed 3-manifold is a homology sphere, then it is a homotopy sphere.

- $H_0X = \mathbb{Z}$ since X is path-connected
- $H_1X = 0$ since X is simply-connected
- $H_3X = \mathbb{Z}$ since X is orientable

- $H_2X = H^1X$ by **Poincare duality**. What group is this?
 - $0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_0(X; \mathbb{Z}), \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_1(X; \mathbb{Z}), \mathbb{Z}) \rightarrow 0$ yields
 - $0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) \rightarrow H^1(X; \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(0, \mathbb{Z}) \rightarrow 0$
 - Then $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) = 0$ because \mathbb{Z} is a projective \mathbb{Z} -module, so $H^1X = 0$.
- So $H_*(X) = [\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \dots]$
- So $h_3 : \pi_3X \rightarrow H_3X$ is an isomorphism by **Hurewicz**. Pick some $f \in \pi_3X \cong \mathbb{Z}$. By partial application, this induces an isomorphism $H_*S^3 \rightarrow H_*X$.
- Taking **CW approximations** for S^3, X , we find that f is a homotopy equivalence.

4 Other Topics

Theorem (Freudenthal Suspension): If X is an n -connected CW complex, then there is a map

$$\Sigma : \pi_i X \rightarrow \pi_{i+1} \Sigma X$$

which is an isomorphism for $i \leq 2n$ and a surjection for $i = 2n + 1$.

Note: $[S^k, X] \mapsto [\Sigma S^k, \Sigma X] = [S^{k+1}, \Sigma X]$

Application: $\pi_2 S^2 = \pi_3 S^3 = \dots$ since 2 is already in the stable range.

A consequence: $\pi_1 X \rightarrow \pi_2 \Sigma X \rightarrow \pi_3 \Sigma^2 X \rightarrow \dots$ is eventually constant, we say the homotopy groups *stabilize*. So define the **stable homotopy groups*

$$\pi_i^S := \lim_{k \rightarrow \infty} \pi_{i+k} X$$

$X = S^n$ yields *stable homotopy groups of spheres*, ties back to initial motivation.

Noting that $\Sigma S^n = S^{n+1}$, we could alternatively define $\mathbb{S} := \lim_k \Sigma^k S^0$, then it turns out that $\pi_n \mathbb{S} = \pi_n^S$.

This object is a *spectrum*, which vaguely resembles a chain complex with a differential:

$$X_0 \xrightarrow{\Sigma} X_2 \xrightarrow{\Sigma} X_3 \xrightarrow{\Sigma} \dots$$

Spectra *represent* invariant theories (like cohomology) in a precise way. For example,

$$HG := \left(K(G, 1) \xrightarrow{\Sigma} K(G, 2) \xrightarrow{\Sigma} \dots \right)$$

then $H^n(X; G) \cong [X, K(G, 1)]$, and we can similarly extract $H^*(X; G)$ from (roughly) $\pi_* HG := [\mathbb{S}, HG \wedge X]$.

Note: this glosses over some important details! Also, smash product basically just looks like the tensor product in the category of spectra.

A modern direction is cooking up spectra that represent *extraordinary* cohomology theories. There are Eilenberg–Steenrod axioms that uniquely characterize homology on spaces; if we drop $H^*\{\text{pt}\} = 0$, we get generalized alternatives.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

Figure 1: Image

5 Other Topics

- The homotopy hypothesis
- Generalized Cohomology theories
- Stable Homotopy Theory
- Infinity Categories
- Higher Homotopy Groups of Spheres
- Eilenberg MacLane and Moore Spaces
- Below jagged line: Zero by cellular approximation, or stable by Freudenthal suspension.
- Above line: Unstable range. Need to throw everything in the book at these guys to compute!