Title

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Last time:

$$\begin{aligned}
\mathbb{Z}\Lambda &\iff \{\mathfrak{h}^* \to \mathbb{Z}_{\geq 0} \mid \sim\} \\
& e(\mu) \mapsto e_\mu \\
e(\lambda)e(\mu) = e(\lambda+\mu) \mapsto f \star g(\lambda) &= \sum_{a+b=\lambda} f(a)g(b)
\end{aligned}$$

and $\mathrm{ch} L(\lambda) = \sum_{\mu\in\Lambda} \dim L(\lambda)_\mu e(\mu)$.

We have the Kostant function $p(\lambda) = \#\left\{(k_\alpha)_\alpha \mid -\lambda = \sum_{\alpha\in\Phi^+} k_\alpha \alpha\right\}$ and the Weyl function $q = e_\rho \star \prod_{\alpha\in\Phi^+}(1-e_{-\alpha}) = \prod_{\alpha\in\Phi^+}(e_{\alpha/2} - e_{-\alpha/2})$.

Lemma: $p \star e_\lambda = \mathrm{ch} M(\lambda)$, so $q \star \mathrm{ch} M(\lambda) = e_{\lambda+\rho}$ and $q \star p = e_\rho$.

1.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of W is given by $w \cdot \lambda = w(\lambda+\rho) - \rho$, i.e. a reflection for hyperplanes passing through $-\rho$.

E.g. for type $A2$, where $W(0) = 0$, we have:

Type A2

And for the dot action, we have

Image

where $W \cdot 0 = 0$ and $s(\alpha_1) = -\alpha_1$.

Theorem (Harish-Chandra): If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$ for $\mu \leq \lambda$.

Proof: Postponed.

ch are characters, $L(\lambda)$ is a Verma module.

Remark: if we sum over $\mu \leq \lambda$, we obtain

$$\begin{aligned} \mathrm{ch}M(\lambda) &= \sum_{\mu \in W \cdot \lambda} a_{\lambda\mu} \mathrm{ch}L(\mu) \\ \mathrm{ch}L(\lambda) &= \sum_{\mu \in W \cdot \lambda} b_{\lambda\mu} \mathrm{ch}M(\mu) \\ &= \sum_{W \cdot \lambda \in \Lambda} c_{\lambda W} \mathrm{ch}M(w \cdot \lambda). \end{aligned}$$

Theorem (Weyl's Character Formula): If $\lambda \in \Lambda^+$, then

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)}$$

Proof:

We have $\mathrm{ch}L(\lambda) = \sum_w c_{\lambda w} \mathrm{ch}M(w \cdot \lambda)$, and so by the lemma,

$$q * \mathrm{ch}L(\lambda) = \sum c_{\lambda w} q \star \mathrm{ch}M(W(\lambda+\rho)-\rho) = \sum_w c_{\lambda w} e_{W(\lambda+p)}$$

Thus for all $\alpha \in \Phi^+$, we have

$$s_\alpha(q \star \mathrm{ch}L(\lambda)) = \sum_w c_{\lambda, s_\alpha w} e_{w(\lambda+\rho)}$$

On the other hand, by part (c) of the lemma, we have

which implies that $c_{\lambda, s_\alpha w} = -c_{\lambda, w}$ by comparing term-by-term, and thus $c_{\lambda, w} = (-1)^{\ell(w)}$ because $c_{\lambda e} = 1$.

In particular, $q = q \star e(0) = q \star \mathrm{ch}L(0) = \sum_{w \in W} (-1)^{\ell(w)} e_{w(\rho)}$, and thus

$$\begin{aligned} \mathrm{ch}L(\lambda) &= \frac{\sum_w (-1)^{\ell(w)} e_{w(\lambda+p)}}{\sum_w (-1)^{\ell(w)} e_{w(p)}} \\ &= \frac{\sum_w (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_w (-1)^{\ell(w)} e(w \cdot 0)}. \end{aligned}$$

□

Example: For type $A1$, we have $W = \Sigma_2 = \{\mathbf{1}, s\}$. Take $\lambda = 3$ under

$$\begin{aligned} \Lambda &\equiv \mathbb{Z} \\ \alpha_1 &\to 2 \\ w_1 = \rho &\to 1, \end{aligned}$$

from which we obtain

$$\begin{aligned}\mathrm{ch}L(3) &= \frac{e(\mathbf{1}\cdot 3)-e(s\cdot 3)}{e(\mathbf{1}\cdot 0)-e(s\cdot 0)}\\ &= \frac{e(3)-e(-5)}{e(0)-e(-2)}\\ &= e(3)+e(1)+e(-1)+e(-3) \quad \text{by long division.}\end{aligned}$$

Corollary (Kostant's Dimension Formula):

If $\mu \le \lambda \in \Lambda^+$, then

$$\dim L(\lambda)_\mu = \sum_{w\in W}(-1)^{\ell(w)}P(w\cdot\lambda-\mu).$$

Proof: $p \star e_\mu(w\cdot\lambda) = \sum_{a+b=w\cdot\lambda} p(a)e_\mu(h) = p(w\cdot\lambda-\mu)$, since this is the only term that survives.

Then $p(w\cdot\lambda-\mu)$ is the coefficient for $e(\mu)$ in $\mathrm{ch}M(w\cdot\lambda) = \dim M(\lambda)_\mu$. Thus $\dim L(\lambda)_\mu = \sum_{w\in W}(-1)^{\ell(w)}\dim M(w\cdot\lambda)_\mu$.

Corollary (Weyl's Dimension Formula):

If $\lambda\in\Lambda^+$, then

$$\dim L(\lambda) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha^\vee)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha^\vee)}$$

Proof (sketch):

Define an operator $\partial = \prod_{\alpha\in\Phi^+}\partial_a$, where $\partial_a : e(\mu)\mapsto (u,\alpha^\vee)e(\mu)$. Then ∂ is well-defined since $\partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha$ for all α,β, and (exercise) ∂ is a derivation.

Define an evaluation homomorphism $\nu : \sum_\mu c_\mu e(\mu) \mapsto \prod_\mu c_\mu$. Note that $\nu(\mathrm{ch}L(\lambda)) = \dim L(\lambda)$, and $\nu(q)=0$ because $\nu(e_{\alpha_i - 1}) = 0$.

Claim:

$$\nu(\partial(q\star \mathrm{ch}L(\mu-\rho))) = |w|\prod_{\alpha\in\Phi^+}(\mu,\alpha^\vee)$$

This is relatively straightforward once you know that you have a derivation and a homomorphism.

With this claim, we have

$$\nu(\partial(q\star\mathrm{ch}L(\lambda))) = \nu(\partial q)\nu(\mathrm{ch}L(\lambda)) + \nu(q)\nu(\partial\mathrm{ch}L(\lambda))$$

where we can identify a number of terms, and then taking ratios yields Weyl's dimension formula.