# Title

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## 1 Thursday, September 03

Recall that the Zariski topology is defined on an affine variety $X = V(J)$ with $J \trianglelefteq k[x_1, \cdots, x_n]$ by describing the closed sets.

**Proposition 1.1** ***(?).***
$X$ is irreducible if its coordinate ring $A(X)$ is a domain.

**Proposition 1.2** ***(?).***
There is a 1-to-1 correspondence

$$\left\{ \substack{\text{Irreducible subvarieties} \\ \text{of } X} \right\} \iff \left\{ \substack{\text{Prime ideals} \\ \text{in } A(X)} \right\}.$$

*Proof .*
Suppose $Y \subset X$ is an affine subvariety. Then

$$A(X)/I_X(Y) = A(Y).$$

By NSS, there is a bijection between subvarieties of $X$ and radical ideals of $A(X)$ where $Y \mapsto I_X(Y)$. A quotient is a domain iff quotienting by a prime ideal, so $A(Y)$ is a domain iff $I_X(Y)$ is prime.

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Recall that $\mathfrak{p} \trianglelefteq R$ is prime when $fg \in \mathfrak{p} \iff f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus $\bar{f}\bar{g} = 0$ in $R/\mathfrak{p}$ implies $\bar{f} = 0$ or $\bar{g} = 0$ in $R/\mathfrak{p}$, i.e. $R/\mathfrak{p}$ is a domain.

Finally note that prime ideals are radical (easy proof).

**Example 1.1.**
Consider $\mathbb{A}^2/\mathbb{C}$ and some subvarieties $C_i$:

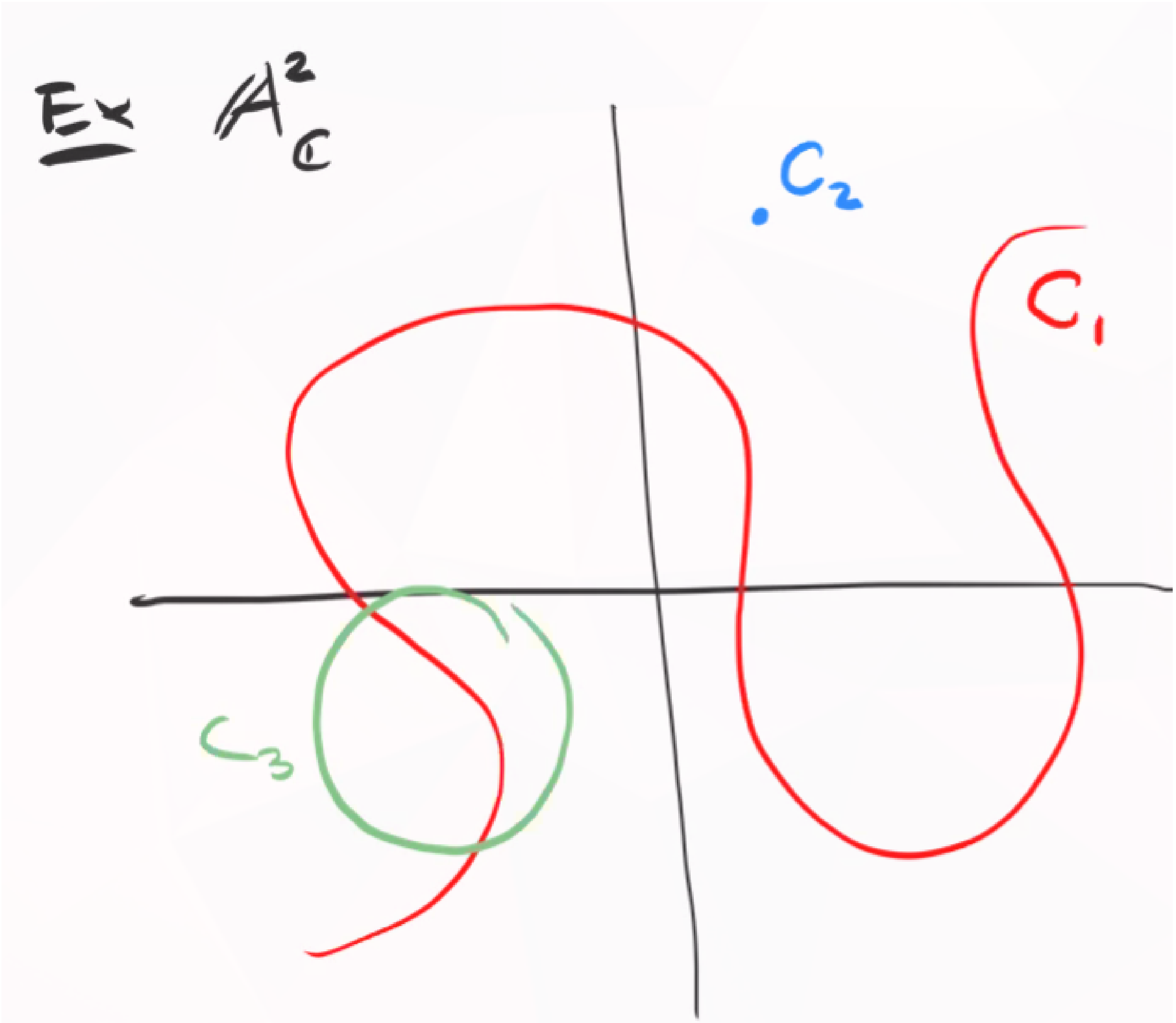


Figure 1: Subvarieties

Then irreducible subvarieties correspond to prime ideals in $\mathbb{C}[x, y]$. Here $C_1, C_3$ correspond to $V(f), V(g)$ for $f, g$ irreducible polynomials, whereas $C_2$ corresponds to a maximal ideal, i.e. $V(x_1 - a_1, x_2 - a_2)$.

Note that $I(C_1 \cup C_2 \cup C_3)$ is not a prime ideal, since the variety is reducible as the union of 3 closed subsets.

**Example 1.2.**
A finite set is irreducible iff it contains only one point.

**Example 1.3.**

Any irreducible topological space is connected, since irreducible requires a union but connectedness requires a *disjoint* union.

**Example 1.4.**
$A^n/k$ is irreducible: by prop 2.8, its irreducible iff the coordinate ring is a domain. However $A(\mathbb{A}^n) = k[x_1, \cdots, x_n]$, which is a domain.

**Example 1.5.**
$V(x_1x_2)$ is not irreducible, since it's equal to $V(x_1) \cup V(x_2)$.

**Definition 1.2.1** (Noetherian Space)**.**
A *Noetherian* topological space $X$ is a space with no infinite strictly decreasing sequence of closed subsets.

**Proposition 1.3*(?)*.**
An affine variety $X$ with the zariski topology is a noetherian space.

*Proof .*
Let $X_0 \supsetneq X_1 \supsetneq \cdots$ be a decreasing sequence of closed subspaces. Then $I(X_0) \subsetneq I(X_1) \subsetneq$. Note that these containments are strict, otherwise we could use $V(I(X_1)) = X_1$ to get an equality in the original chain.
Recall that a ring $R$ is Noetherian iff every ascending chain of ideals terminates. Thus it suffices to show that $A(X)$ is Noetherian.
We have $A(X) = k[x_1, \cdots, x_n]/I(X)$, and if this had an infinite chain $I_1 \subsetneq I_2 \subsetneq \cdots$ lifts to a chain in $k[x_1, \cdots, x_n]$, which is Noetherian. A useful fact: $R$ noetherian implies that $R[x]$ is noetherian, and fields are always noetherian.

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**Remark 1.**
Any subspace $A \subset X$ of a noetherian space is noetherian. To see why, suppose we have a chain of closed sets in the subspace topology,

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq \cdots .$$

Then $X_0 \supsetneq X_1 \supsetneq \cdots$ is a strictly decreasing chain of closed sets in $X$. Why strictly decreasing: $\cap^n X_i = \cap^{n+1} X_i \implies A \cap^n X_i = A \cap^{n+1} X_i$, a contradiction.

**Proposition 1.4*(Important)*.**

Every noetherian space $X$ is a finite union of irreducible closed subsets, i.e. $X = \bigcup_{i=1}^{k} X_i$. If we further assume $X_i \not\subset X_j$ for all $i, j$, then the $X_i$ are unique up to permutation.

**Remark 2.**
The $X_i$ are the **components** of $X$. In the previous example $C_1 \cup C_2 \cup C_3$ has three components.

*Proof .*
If $X$ is irreducible, then $X = X$ and this holds.
Otherwise, write $X = X_1 \cup X_2$ with $X_i$ proper closed subsets. If $X_1$ and $X_1'$ are irreducible, we're done, so otherwise suppose wlog $X_1'$ is not irreducible.
Then we can express $X = X_1 \cup (X_2 \cup X_2')$ with $X_2, X_2' \subset X_1'$ closed and proper.
Thus we can obtain a tree whose leaves are proper closed subsets:

Figure 2: Image

This tree terminates because $X$ is Noetherian: if it did not, this would generate an infinite decreasing chain of subspaces.
We now want to show that the decomposition is unique if no two components are contained in the other.
Suppose

$$X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=1}^{\ell} X_j'.$$

Note that $X_i \subset X$ implies that $X_i = \bigcup_{j=1}^{\ell} X_i \cap X_j'$. But $X_i$ is irreducible and this would express $X_i$ as a union of proper closed subsets, so some $X_i \cap X_j'$ is *not* a proper closed subset.
Thus $X_i = X_i \cap X_j'$ for some $j$, which forces $X_i \subset X_j'$. Applying the same argument to $X_j'$ to obtain $X_j' \subset X_k$ for some $k$.
Then $X_i \subset X_j' \subset X_k$, but $X_i \not\subset X_j$ when $j \neq i$. Thus $X_i = X_j' = X_k$, forcing the $X_i$ to be unique up to permutation.

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Recall from ring theory: for $I \subset R$ and $R$ noetherian, $I$ has a *primary decomposition* $I = \bigcap_{i=1}^{k} Q_i$ with $\sqrt{Q_i}$ prime. Assuming the $Q_i$ are minimal in the sense that $\sqrt{Q_i} \not\subset \sqrt{Q_j}$ for any $i, j$, this decomposition is unique.

Applying this to $I(X) \trianglelefteq k[x_1, \cdots, x_n] = R$ yields

$$I(X) = \bigcap_{i=1}^{k} Q_i \implies X = V(I(X)) = \bigcup_{i=1}^{k} V(Q_i).$$

Letting $P_i = \sqrt{Q_i}$, noting that the $P_i$ are prime and thus radical, we have $V(Q_i) = V(P_i)$. Writing $X = \bigcup V(P_i)$, we have $I(V(P_i)) = P_i$ and thus $A(V(P_i)) = R/P_i$ is a domain, meaning $V(P_i)$ are irreducible affine varieties.

Conversely, if we express $X = \bigcup X_i$, we have $I = I\left(\bigcup X_i\right) = \bigcap I(X_i) = \bigcap P_i$ which are irreducible since they are prime.

**Remark 3.**
There is a correspondence

$$\left\{\begin{matrix}\text{Irreducible components} \\ \text{of } X\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Minimal prime ideals} \\ \text{in } A(X)\end{matrix}\right\},$$

where here *minimal* is the condition that no pair of ideals satisfies a subset containment.

**Remark 4.**
Let $X$ be an irreducible topological space.

**Proposition 1.5** ***(1).***
The intersection of nonempty two open sets is *never* empty.

*Proof .*
Let $U, U'$ be open and $X \setminus U, X \setminus U'$ closed. Then $U \cap U' = \emptyset \iff (X \setminus U) \cup (X \setminus U') = X$, but this is not possible since $X$ is irreducible.

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Irreducible iff any two nonempty open sets intersect.