Homological Algebra Problem Sets

Problem Set 3

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> *Problem* 1.0.1 (Prove Corollary 2.3.2)
> For R a PID, show that an R-module A is divisible if and only if A is injective.
>
> > *Recall that a module is* divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.
$\implies$: Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:

$$
\begin{array}{ccccc}
0 & \longrightarrow & J & \overset{\iota}{\lhook\joinrel\longrightarrow} & R \\
& & \downarrow{\scriptstyle f} & {\scriptstyle \exists \tilde{f}} & \\
& & A & &
\end{array}
$$

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$
\begin{array}{ccccc}
0 & \longrightarrow & jR & \overset{\iota}{\lhook\joinrel\longrightarrow} & R \\
& & \downarrow{\scriptstyle f} & {\scriptstyle \exists \tilde{f}} & \\
& & A & &
\end{array}
$$

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R-linearity and the fact that j is left-cancelable, we have

$$ jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'. $$

Thus we can set

$$
\begin{aligned}
\tilde{f} : R &\to A \\
1_R &\mapsto \mathbf{a}',
\end{aligned}
$$

and extending R-linearly yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

$\Longleftarrow$: Suppose $A \in R\text{-}\mathbf{Mod}$ is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$0 \longrightarrow J \lhook\joinrel\longrightarrow R$$
$$\downarrow \qquad \swarrow$$
$$A$$

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$f_a : J = jR \to R$$
$$x = jr \mapsto r\mathbf{a}.$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$0 \longrightarrow jR \overset{\iota}{\lhook\joinrel\longrightarrow} R$$
$$f_a \downarrow \qquad \overset{\exists \tilde{f}_a}{\swarrow}$$
$$A$$

We can now use the fact that

$$
\begin{aligned}
r\mathbf{a} &= f_a(jr) \\
&= \tilde{f}_a(\iota(jr)) \\
&= \tilde{f}_a(jr) \\
&= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\
&= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\
\Longrightarrow \mathbf{a} &= j\tilde{f}_a(1_R) \in jA,
\end{aligned}
$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting $\mathbf{a}$ range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).
Note that there is an injection

$$1 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \to \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \to 1$$

where π_m is the canonical quotient morphism, and applying the left-exact contravariant functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any $\mathbb{Z}$-module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module with generator 1, f is entirely determined by $f(1)$, and

2. f descends to a map $\tilde{f} : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\operatorname{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\quad f \quad} & \mathbb{Q}/\mathbb{Z} \\
{\scriptstyle \pi_m} \downarrow & \nearrow{\scriptstyle \exists \tilde{f}} & \\
\mathbb{Z}/n & &
\end{array}
$$

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0, 1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\right\}.$$

Setting $f_i(1) := \left[\dfrac{i}{n}\right]$ (where we take the equivalence class mod $\mathbb{Z}$) yields n distinct morphisms $f_i : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\Psi : \mathbb{Z} \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$$
$$i \mapsto f_i,$$

and using the fact that if $i = i'$ (mod n), write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left[\frac{i+kn}{n}\right] = \left[\frac{i}{n} + k\right] = \left[\frac{i}{n}\right] = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

∎

Claim 2: $\mathbb{Q}/\mathbb{Z}$ is an injective object in $\mathbb{Z}$-modules.

Proof (?).
By the previous exercise, it suffices to show that $\mathbb{Q}/\mathbb{Z}$ is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That $\mathbb{Q}$ is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b := \dfrac{a}{n} \in \mathbb{Q}$ works. Since $\mathbb{Q}$ is an abelian group, $\mathbb{Z}$ is automatically normal, and the result follows.

∎

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \qquad\qquad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).
Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M := \mathbb{Z}/m$,
- $A := \mathbb{Z}$,
- $I := n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n\,(\mathbb{Z}/m)} \qquad\qquad \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\varphi : \mathbb{Z} \to \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n$$
$$x \mapsto x(1 \otimes 1)$$

and compute

$$
\begin{aligned}
\ker \varphi &= \left\{ x \in \mathbb{Z} \ \middle| \ x(1 \otimes 1) = 0 \right\} \\
&= \left\{ x \in \mathbb{Z} \ \middle| \ n \mid x \text{ or } m \mid x \right\} \\
&= \langle n, m \rangle \\
&= \langle \gcd(n, m) \rangle \qquad\qquad\qquad\text{by Bezout's theorem} \\
&:= \langle d \rangle \,.
\end{aligned}
$$

Now applying the first isomorphism theorem yields the result.

∎

Solution:
We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\,\cdot\,) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \,\cdot\,)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of $\mathbb{Z}/q$.
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$
1 \xrightarrow{} \mathbb{Z}/q \xrightarrow{\ d^{-1}\ } \mathbb{Q}/\mathbb{Z} \xrightarrow{\ d^0\ } \mathbb{Q}/\mathbb{Z} \xrightarrow{\ d^1\ } 1
$$

$$
[1]_q \xrightarrow{} \left[\frac{1}{q} \right]
$$

$$
[x] \xrightarrow{} [qx]
$$

Link to Diagram

This is a chain complex by construction, since $d^2([1]_q) = \left[q\left(\dfrac{1}{q} \right) \right] = [1] = [0]$. We now delete the augmentation and apply $F(\,\cdot\,)$:

$$1 \longrightarrow I^0 := \mathbb{Q}/\mathbb{Z} \xrightarrow{\quad d^0 \quad} I^1 := \mathbb{Q}/\mathbb{Z} \xrightarrow{\quad d^1 \quad} 1$$

with the vertical map $F(\,\cdot\,)$, and

$$1 \longrightarrow F(I^0) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial^0 := F(d^0)} F(I^1) := \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial^1 := F(d^1)} 1$$

with vertical isomorphisms $\Psi \cong$, and

$$1 \longrightarrow \mathbb{Z}/p \xrightarrow{\quad \tilde{\partial}^0 \quad} \mathbb{Z}/p \xrightarrow{\quad \tilde{\partial}^1 \quad} 1$$

Link to Diagram

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q, we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q.

We now take homology:

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\operatorname{im} \partial^0} = \frac{\mathbb{Z}/p}{q\,(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)
Let $A \in \mathbf{Ab}$, and show that the following map is injective:

$$\varepsilon_A : A \to I(A) := \prod_{f \in \operatorname{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

$$a \mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},$$

i.e. when looking at the image $\varepsilon_A(a)$ in the product, the component indexed by f is an element of $\mathbb{Q}/\mathbb{Z}$ obtained by evaluating $f(a)$.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Solution:
By contrapositive, we'll suppose $a \neq 0$ and show $\varepsilon_A(a) \neq 0$. Following the hint, we first consider the cyclic subgroup $a\mathbb{Z} = \left\{ an \mid n \in \mathbb{Z} \right\}$ and define a map

$$f_a : a\mathbb{Z} \to \mathbb{Z}$$
$$an \mapsto n.$$

We now pick $\ell > 1 \in \mathbb{Z}$ to be any integer, and define a composition $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$:

$$
\begin{array}{ccccccccc}
a\mathbb{Z} & \xrightarrow{f_a} & \mathbb{Z} & \overset{\iota}{\hookleftarrow} & \mathbb{Q} & \xrightarrow{x \mapsto \frac{x}{\ell}} & \mathbb{Q} & \xrightarrow{\pi} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

with f the arc over the top.

$$
an \longmapsto n \longmapsto n \longmapsto \frac{n}{\ell} \longmapsto \left[\frac{n}{\ell}\right]
$$

$$
a \dashrightarrow \left[\frac{1}{\ell}\right]
$$

By choice of ℓ, this map satisfies $f(a) = [1/\ell] \neq 0$, so the map is nonzero. Since $\mathbb{Q}/\mathbb{Z}$ is injective, the universal property provides a lift $\tilde{f}$:

$$
\begin{array}{ccc}
A & & \\
\uparrow & \overset{\exists \tilde{f}}{\nwarrow} & \\
a\mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z}
\end{array}
$$

Since $\tilde{f}$ lifts f, it is also nonzero. But now we can check that

$$
\varepsilon_A(a)(f) := f(a) \neq 0,
$$

so the f component of the image of a is nonzero and thus $\mathbf{a} := \varepsilon_A(a) \neq 0$ in the product.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is right-exact functor, show that

$$
U(L_i F) \cong L_i(UF).
$$

Solution:
We'll show that $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$ for every object X. Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let $P \to X$ be a projective resolution of X. Fixing labeling, we have the following situation:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X & \xrightarrow{0} & 0 \\
& & & & & & \Big\Downarrow{\scriptstyle F(\cdot)} & & & & \\
\cdots & \longrightarrow & FP_2 & \xrightarrow{F(\partial_2)} & FP_1 & \xrightarrow{F(\partial_1)} & FP_0 & \xrightarrow{0} & 0 & &
\end{array}
$$

We now have by definition

$$L_i F(X) := \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U\left(L_i F(X)\right) := U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution $P \to X$, and apply a similar process:

$$\cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} X \xrightarrow{0} 0$$

$$\Big\Downarrow {\scriptstyle (U \circ F)(\cdot)}$$

$$\cdots \longrightarrow UFP_2 \xrightarrow{(UF)(\partial_2)} UFP_1 \xrightarrow{(UF)(\partial_1)} UFP_0 \xrightarrow{0} 0$$

<div style="text-align:center">*Link to Diagram*</div>

Again, by definition,

$$(L_i(UF))(X) := \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor U to the following SES to produce a new SES, from which we'll produce the desired isomorphism f:

$$0 \longrightarrow \operatorname{im} F(\partial_{i+1}) \xrightarrow{\iota_i} \ker F(\partial_i) \xrightarrow{\pi_i} \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \longrightarrow 0$$

$$\Big\Downarrow {\scriptstyle U}$$

$$0 \longrightarrow U\left(\operatorname{im} F(\partial_{i+1})\right) \xrightarrow{U(\iota_i)} U\left(\ker F(\partial_i)\right) \xrightarrow{U(\pi_i)} U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \longrightarrow 0$$

$$\Big\| \qquad\qquad\qquad \Big\| \qquad\qquad\qquad \Big\downarrow {\scriptstyle f}$$

$$0 \longrightarrow U\left(\operatorname{im} F(\partial_{i+1})\right) \xrightarrow{U(\iota_i)} U\left(\ker F(\partial_i)\right) \xrightarrow{\tilde{\pi}_i} \frac{U\left(\ker F(\partial_i)\right)}{U\left(\operatorname{im} F(\partial_{i+1})\right)} \longrightarrow 0$$

<div style="text-align:center">*Link to Diagram*</div>

Here $\tilde{\pi}_i$ is the natural quotient map, whose image is $\operatorname{coker} U(\iota_i)$. Finally, the map f exists an any abelian category, using that whenever $0 \to A \xrightarrow{g_1} B \xrightarrow{g_2} C \to 0$ is exact, there is an isomorphism $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$.

Problem 1.0.5 (Weibel 2.4.3)

- If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} FM \qquad\qquad i \geq 2.$$

- Show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with P_i projective or F-acyclic, then

$$L_i F(A) \cong L_{i-m-1} F(M_m) \qquad\qquad i \geq m+2.$$

  - Moreover show that $L_{m+1} F(A)$ is the kernel of $F(M_m) \to F(P_m)$.

- Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Solution:

Claim:

$$L_i F(A) \cong L_{i-1} FM \qquad\qquad i \geq 2.$$

Proof (of claim).

Following the proof of Weibel Theorem 2.4.6, let $P_M \to M$ and $P_A \to A$ be projective resolutions of M and A respectively. Then applying the Horseshoe Lemma, there is a projective resolution $P_P \to P$ of P such that the following is a short exact sequence of chain complexes:

$$0 \to P_M \to P_P \to P_A \to 0,$$

where in fact in each degree n piece, this is induces a *split* exact sequence. Using that F is additive and additive functors preserve split exact sequences, the following is a SES for every n:

$$0 \to FP_M^n \to FP_P^n \to FP_A^n \to 0,$$

which implies that there is a SES of chain complexes

$$0 \to FP_M \to FP_P \to FP_A \to 0.$$

Thus there is an associated LES of derived functors:

Link to Diagram

Using that P is F-acyclic, the middle terms $L_iFP = 0$ for all $i > 0$, and thus this splits into a collection of SESs:

$$0 \to L_2FA \xrightarrow{\partial_2} L_1FM \to 0$$
$$0 \to L_3FA \xrightarrow{\partial_3} L_2FM \to 0$$
$$\vdots$$
$$0 \to L_iFA \xrightarrow{\partial_3} L_{i-1}FM \to 0.$$

This makes every ∂_i for $i \geq 2$ an isomorphism. ∎

Claim:

$$L_iFA \cong \ker(FM \to FP).$$

Proof (?).

Using the same argument as above, consider the lower order terms of the associated LES:

$$\cdots$$

$$L_1FM \xleftarrow{\hspace{2cm}} L_1FP = 0 \xrightarrow{\hspace{2cm}} L_1FA$$

$$\partial_2$$

$$\partial_1$$

$$L_0FM \xrightarrow{\hspace{2cm}} L_0FP \xrightarrow{\hspace{2cm}} L_0FA \xrightarrow{\hspace{1cm}} 0$$

Link to Diagram

Noting that $L_1FP = 0$ by F-acyclicity, the highlighted portion forms a four term exact sequence. We can form another exact sequence and compare the two:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & L_1FA & \xrightarrow{\partial_1} & L_0FM & \longrightarrow & L_0FP & \longrightarrow & L_0FA & \longrightarrow & 0 \\
& & \downarrow & & \| & & \| & & \| & & \\
0 & \longrightarrow & \ker(L_0FM \to L_0FP) & \longrightarrow & L_0FM & \longrightarrow & L_0FP & \longrightarrow & L_0FA & \longrightarrow & 0
\end{array}
$$

Link to Diagram

That the map indicated by the dotted line exists and is an isomorphism holds in any abelian category, using that fact that whenever $0 \to A \to B \xrightarrow{f} C \to 0$ is a SES we have $A \cong \ker f$. ∎

Claim: If $P \to A$ is an F-acyclic resolution of A, then there is an isomorphism

$$L_iFA \cong H_i(FP).$$

Problem 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

     a. A is a projective R-module.

     b. $\operatorname{Hom}_R(A, \cdot)$ is an exact functor.

     c. $\operatorname{Ext}_R^{i \geq 1}(A, B) = 0$ and for all B, i.e. A is $\operatorname{Hom}_R(\cdot, B)$-acyclic for all B.

     d. $\operatorname{Ext}_R^1(A, B)$ vanishes for all B.

Solution:

We'll show

- $a \iff b$
- $b \implies c$
- $c \iff d$:
- $d \implies b$

Proof ($a \iff b$).

Let ξ be the following SES:

$$\xi : 0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

and define the functor $F(\,\cdot\,) := \operatorname{Hom}_R(A, \,\cdot\,)$. This is a covariant left-exact functor, and so applying it to the above sequence yields

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\ f\ } & M_2 & \xrightarrow{\ g\ } & M_3 & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & \text{\scriptsize $F(\,\cdot\,):=\operatorname{Hom}_R(A,\,\cdot\,)$} & & & & \\
 & & & & \Downarrow & & & & \\
0 & \longrightarrow & FM_1 & \xrightarrow[F(f):\lambda \mapsto f\circ\lambda]{} & FM_2 & \xrightarrow[F(g):\lambda \mapsto g\circ\lambda]{} & FM_3 & &
\end{array}
$$

<div align="center" style="background:#d0cde0">

Link to Diagram

</div>

$\implies$:

For F to be exact, it suffices to show it is right-exact, i.e. that $F(g)$ is surjective. This amounts to asking that every $\varphi \in FM_3 := \operatorname{Hom}_R(A, M_3)$ lifts to a preimage $\tilde{\varphi} \in FM_2 := \operatorname{Hom}_R(A, M_2)$ satisfying $F(g)(\tilde{\varphi}) = \varphi$. Unwinding definitions, this requires that $g \circ \tilde{\varphi} = \varphi$, which is precisely the lift required for the universal property of projective objects:

$$
\begin{array}{ccccc}
 & & A & & \\
 & {\scriptstyle \exists\tilde{\varphi}}\ \nearrow & \downarrow {\scriptstyle \varphi} & & \\
M_2 & \xrightarrow{\ g\ } & M_3 & \longrightarrow & 0
\end{array}
$$

<div align="center" style="background:#d0cde0">

Link to Diagram

</div>

If A is projective, this lift always exists, so $\operatorname{Hom}_R(A, \,\cdot\,)$ is an exact functor. Conversely, if $\operatorname{Hom}_R(A, \,\cdot\,)$ is exact, this lift always exists, so A satisfies the universal property of a projective object.

∎

Proof ($b \implies c$).

Suppose $F(\,\cdot\,) := \operatorname{Hom}_R(A, \,\cdot\,)$ is exact, then since F is left-exact covariant it has right-derived functors $\operatorname{Ext}^i_R(A, B) := R^i F(B)$ which are computed in the following way

1. Taking an *injective* resolution of

$$B \to I := (I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots).$$

2. Applying $\mathrm{Hom}_R(A, \cdot)$ to get the complex

$$FI := (0 \to \mathrm{Hom}_R(A, I_0) \xrightarrow{F(\partial_0)} \mathrm{Hom}_R(A, I_1) \xrightarrow{F(\partial_1)} \cdots).$$

3. Defining

$$R^i F(B) := \ker F(\partial_i)/\operatorname{im} F(\partial_{i-1}).$$

Note that in step (2), if $\mathrm{Hom}_R(A, \cdot)$ is an exact functor, then since I is an acyclic complex, FI is again acyclic and so $\ker F(\partial_i) = \operatorname{im} F(\partial_{i-1}) = 0$ for $i \geq 1$. So

$$\mathrm{Ext}_R^{\geq 1}(A, B) := R^{\geq 1}F(B) = 0.$$

∎

Proof (c $\iff$ d).
$\implies$: This direction is clear, since if $\mathrm{Ext}_R^i(A, B) = 0$ for all B, then taking $i = 1$ is the statement of (d).

$\impliedby$: This follows from the dimension-shifting isomorphism in a previous exercise. Let $F(\cdot) := \mathrm{Hom}_R(A, \cdot)$ and suppose $\mathrm{Ext}_R^1(A, B) := L_1 F(B) = 0$ for all B. Let B' be arbitrary, it then suffices to show that $\mathrm{Ext}^i(A, B') := L_i(B') = 0$ for all $i > 1$, since we can take B' as one such B in the assumption for the $i = 1$ case.
The dimension shifting results states that if P_i are F-acyclic, then for every exact sequence

$$0 \to M_m \to P_m \to \cdots \to P_0 \to B' \to 0$$

we obtain an isomorphism

$$L_i F(B') \cong L_{i-m-1} F(M_m) \iff L_i F(M_m) \cong L_{i+m+1} F(B').$$

So take any F-acyclic resolution of P, say

$$B' \xrightarrow{\partial_{-1}} I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \cdots,$$

then consider truncating it at the mth stage:

$$0 \to B' \xrightarrow{\partial_{i-1}} I_0 \to I_1 \to \cdots \xrightarrow{\partial_{m-1}} I_m \to M_m := \operatorname{coker} \partial_{m-1} \to 0$$

By assumption, we have $L_1 F(M_m) = 0$ for every m, and thus

$$0 = L_1 F(B') \quad \text{by assumption}$$
$$0 = L_1 F(M_0) \cong L_2 F(B')$$
$$0 = L_1 F(M_1) \cong L_3 F(B')$$
$$0 = L_1 F(M_2) \cong L_4 F(B')$$
$$\vdots$$
$$0 = L_1(M_m) \cong L_{m+2}(B') \quad \forall m \geq 0.$$

and so $L_i(B') = 0$ for all $i \geq 1$.

∎

Proof (d $\implies$ b).
Take an arbitrary SES

$$\xi : 0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$$

and consider applying the left-exact covariant functor $F(\cdot) := \operatorname{Hom}_R(A, \cdot)$ and taking the associated LES:

$$
\begin{array}{c}
0 \\
\downarrow \\
\operatorname{Hom}_R(A, B') \xrightarrow{f^*} \operatorname{Hom}_R(A, B) \xrightarrow{g^*} \operatorname{Hom}_R(A, B'') \longrightarrow \\
\xrightarrow{\delta_0} \\
\operatorname{Ext}^1_R(A, B') \longrightarrow \operatorname{Ext}^1_R(A, B) \longrightarrow \operatorname{Ext}^1_R(A, B'') \longrightarrow \\
\xrightarrow{\delta_1} \\
\hookrightarrow \cdots
\end{array}
$$

Link to Diagram

By assumption, all of the higher Ext terms vanish, and in particular the red term $\operatorname{Ext}^1_R(A, B') = 0$. This implies that g^* is surjective, making the following sequence exact:

$$0 \to \operatorname{Hom}_R(A, B') \xrightarrow{f^*} \operatorname{Hom}_R(A, B) \xrightarrow{g^*} \operatorname{Hom}_R(A, B'') \to 0,$$

making $\operatorname{Hom}_R(A, \cdot)$ an exact functor.

∎

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is

abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on **Ab**.