

# Title

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## 1 Sunday, August 30

### 1.1 Base Extension

Given some object  $A/k$  and  $k \hookrightarrow \ell$  is a field extension, we would like some extended object  $A/\ell$ .

#### Example 1.1.

An *affine variety*  $V/k$  is given by finitely many polynomials in  $p_i \in k[t_1, \dots, t_n]$ , and base extension comes from the map  $k[t_1, \dots, t_n] \hookrightarrow \ell[t_1, \dots, t_n]$ .

More algebraically, we have the affine coordinate ring over  $k$  given by  $k[V] = k[t_1, \dots, t_n]/\langle p_i \rangle$ , the ring of polynomial functions on the zero locus corresponding to this variety. We can similarly replace  $k$  by  $\ell$  in this definition. Here we can observe that  $\ell[V] \cong k[V] \otimes_k \ell$ .

In general we have a map

$$\begin{aligned} \cdot \otimes_k \ell \\ \{k\text{-vector space}\} &\longrightarrow \{\ell\text{-vector spaces}\} \\ \{k\text{-algebras}\} &\longrightarrow \{\ell\text{-algebras}\}. \end{aligned}$$

Note that this will be an exact functor on the category  $k\text{-Vect}$ , i.e.  $\ell$  is a flat module. Here everything is free, and free  $\implies$  flat, so things work out nicely.

What about for function fields?

Since  $k$  is a  $k$ -algebra, we can consider  $k \otimes_k \ell$ , however this need not be a field.

Note: tensor products of fields come up very often, but don't seem to be explicitly covered in classes! We'll broach this subject here.

#### Exercise 1.1.

If  $\ell/k$  is algebraic and  $\ell \otimes_k \ell$  is a domain, the  $\ell = k$ .

I.e. this is rarely a domain. Hint: start with the monogenic case, and also reduce to the case where the extension is not just algebraic but finite.

Tensor products of field extensions are still interesting: if  $\ell/k$  is finite, it is galois  $\iff \ell \otimes_k \ell \cong \ell^{[\ell:k]}$ . So its dimension as an  $\ell$ -algebra is equal to the degree of  $\ell/k$ , so it splits as a product of copies of  $\ell$ .

**Remark 1.**

We'd like the tensor product of a field to be a field, or at least a domain where we can take the fraction field and get a field. This hints that we should not be tensoring algebraic extensions, but rather transcendental ones.

**Exercise 1.2.**

For  $\ell/k$  a field extension,

- Show  $k(t) \otimes_k \ell$  is a domain with fraction field  $\ell(t)$ .
- Show it is a field  $\iff \ell/k$  is algebraic.

**Proposition 1.1 (FT 12.7, 12.8).**

Let  $k_1, k_2/k$  are field extensions, and suppose  $k_1 \otimes_k k_2$  is a domain. Then this is a field  $\iff$  at least one of  $k_1/k$  or  $k_2/k$  is algebraic.

Reminder: for  $\ell/k$  and  $\alpha \in \ell$  algebraic over  $k$ , then  $k(\alpha) = k[\alpha]$ .

So we'll concentrate on when  $K \otimes_k \ell$  is a domain. What's the condition on a function field  $K/k$  that guarantees this, i.e. when extending scalars from  $k$  to  $\ell$  still yields a domain? If this remains a domain, we'll take the fraction field and call it the *base change*.

**Exercise 1.3.**

If  $K/k$  is finitely generated (i.e. a function field) and  $K \otimes_k \ell$  is a domain, then  $ff(K \otimes_k \ell)/\ell$  is finitely generated.

The point: if taking a function field and extending scalars still results in a domain, we'll call the result a function field as well.

Most of all, we want to base change to the algebraic closure. We'll have issues if the constant field is not just  $k$  itself:

**Lemma 1.2.**

If  $K \otimes_k \bar{k}$  is a domain, then the constant field  $\kappa(K) = k$ .

*Proof.*

Use the fact that  $\cdot \otimes_k V$  is exact. We then get an injection

$$\begin{array}{ccc}
 \kappa(K) \otimes_k \kappa(K) & \xrightarrow{\quad} & K \otimes_k \bar{k} \\
 & \searrow \quad \swarrow & \\
 & \kappa(K) \otimes_k \bar{k} &
 \end{array}$$

Here we use the injections  $\kappa(K) \hookrightarrow \bar{k}$  and  $\kappa(K) \hookrightarrow K$ .

We now have an injection of  $k$ -algebras, and subrings of domains are domains. So apply the first exercise: the only way this can happen is if  $\kappa(K) = k$ . ■

#### Exercise 1.4.

The simplest possible case: describe  $\mathbb{C}(t) \otimes_{\mathbb{R}} \mathbb{C}$ , tensored as  $\mathbb{R}$ -algebras.

Won't be a domain by the lemma, some  $\mathbb{C}(t)$ -algebra of dimension 2.

In order to have a good base change for our function fields, we want to constant extension to be trivial, i.e.  $\kappa(K) = k$ . This requires that the ground field be algebraically closed.

In this case, you might expect that extending scalars to the algebraic closure would yield a field again. This is true in characteristic zero, but false in positive characteristic.

A more precise question: if  $\kappa(K) = k$ , must  $K \otimes_K \bar{k}$  be a field? If that's true and we're in positive characteristic, recalling that for an algebraic extension this being a field is equivalent to it being a domain. But if that's a domain, the tensor product of every algebraic extension must be a domain, which is why this is an important case.

If so, then  $K \otimes_k k^{\frac{1}{p}}$  is a field, where  $k^{\frac{1}{p}} := k(\{\zeta\})$