Title

D. Zack Garza

Table of Contents

Contents

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p.

Notation	Definition
$k[\mathbf{x}] = k[x_1, \cdots, x_n]$	Polynomial ring in n indeterminates
$k(\mathbf{x}) = k(x_1, \cdots, x_n)$	Rational function field in n indeterminates
$\mathcal{U} \rightrightarrows X$	An open cover $\mathcal{U} = \left\{ U_j \;\middle\vert\; j \in J \right\}$
Δ_X	The diagonal $\Delta_X \coloneqq \left\{ (x, x) \;\middle\vert\; x \in X \right\} \subseteq X \times X$
$\mathbb{A}^n_{/k}$	Affine n-space $\mathbb{A}^n_{/k} \coloneqq \left\{ \mathbf{a} = [a_1, \cdots, a_n] \;\middle\vert\; a_j \in k \right\}$
$\mathbb{P}^n_{/k}$	Projective n-space $\mathbb{P}^n_{/k} \coloneqq \left(k^n \setminus \{0\} \right) / x \sim \lambda x$ $\mathbb{P}^n_{/k} = \left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \;\middle\vert\; p, q, \in k[x_1, \cdots, x_n] \right\}$
$V(J), V_a(J)$	Variety associated to an ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ $V_a(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \;\middle\vert\; f(\mathbf{x}) = 0, \, \forall f \in J \right\}$
$I(S), I_a(S)$	Ideal associated to a subset $S \subseteq \mathbb{A}^n_k$ $I_a(S) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \;\middle\vert\; f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in X \right\}$
$A(X)$	Coordinate ring of a variety $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$
$V_p(J)$	Projective variety of an ideal $V_p(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n_{/k} \;\middle\vert\; f(\mathbf{x}) = 0, \, \forall f \in J \right\}$
$I_p(S)$	Projective ideal (?) $I_p(S) \coloneqq$ $\left\{ f \in k[x_1, \cdots, x_n] \;\middle\vert\; f \text{ is homogeneous and } f(x) = 0 \, \forall x \in S \right\}$
$S(X)$	Projective coordinate ring, $S(X) \coloneqq k[x_1, \cdots, x_n]/I_p(X)$
f^h	Homogenization, $f^h \coloneqq x_0^{\deg f} f\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0} \right)$

Notation	Definition
f^i	Dehomogenization, $f^i \coloneqq f(1, x_1, \cdots, x_n)$
J^h	Homogenization of an ideal, $J^h \coloneqq \left\{ f^j \;\middle|\; f \in J \right\}$
$\overline{X}$	Projective closure of a subset $\overline{X} \coloneqq V_p(J^h) \coloneqq \left\{ \mathbf{x} \in \mathbb{P}^n \;\middle|\; f^h(\mathbf{x}) = 0 \,\forall f \in X \right\}$
$\mathcal{O}_X$	Structure sheaf $\mathcal{O}_X(U) \coloneqq \left\{ f : U \to k \;\middle|\; f \in k(\mathbf{x}) \text{ locally} \right\}$
$D(f)$	Distinguished open set, $D(f) \coloneqq V(f)^c = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) \neq 0 \right\}$

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\,\cdot\,)$ and $I(\,\cdot\,)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples

0.4.1 Varieties

- $V(xy-1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x-p)$ a point.

0.4.2 Presheaves / Sheaves

- $C^\infty(\,\cdot\,, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\,\cdot\,, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\,\cdot\,)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\,\cdot\,)$, the constant sheaf associated to $\mathbb{R}$ (locally constant real-valued functions)
- $\text{Hol}(\,\cdot\,, \mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

Algebra	Geometry
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n_{/k}$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \cdots, x_n - p_n$	Points $[a_1, \cdots, a_n]$
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
$k[x_1, \cdots, x_n]/I(X)$	$A(X)$ (Functions on X)
$A(X)$ a domain	X is irreducible
$A(X)$ indecomposable	X is connected
Krull dimension n (chaints of primes)	Topological dimension n (chains of irreducibles)