Title

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1 | Friday, November 13

1.1 Review

Review: we're considering G_rT-modules, with several associated modules of interest:

- Simple modules $\widehat{L}_r(\lambda)$ for $\lambda \in X(T)$
- Intermediate modules $\nabla(\lambda) = \widehat{Z}_r'(\lambda)$ and $\Delta(\lambda) = \widehat{Z}_r(\lambda)$.
- Injective and projective modules $\widehat{Q}_r(\lambda)$

Theorem 1.1.1 *(?).*
Let M be a G_rT-module of finite dimension. Then M has a $\widehat{Z}_r$ filtration $\iff$ $M \downarrow_{B_r}$ is projective.

Remark 1.1.1 : From this, the multiplicity $[M : \widehat{Z}_r(\mu)]$ (the number of times $\widehat{Z}_r(\mu)$ appears in a $\widehat{Z}_r$ filtration) is well-defined. Moreover, we have a decomposition

$$M \downarrow_{B_r} = \bigoplus_{\mu} Z_r(\mu) \downarrow_{B_r},$$

where the sum contains as many terms as the number of factors that appear. We have $Z_r(\mu) \downarrow_{B_r} \twoheadrightarrow \mu$, making is the projective cover of μ and thus indecomposable. We can then apply the Krull-Schmidt theorem.

1.2 Reciprocity

Consider $\widehat{Q}_r(\lambda)$, a projective G_rT-module. Note that it also happens to be injective. We saw before that the functor $\mathrm{Coind}_{B_rT}^{G_rT}(\,\cdot\,)$ is exact, and thus $\widehat{Q}_r(\lambda) \downarrow_{B_rT}$ being projective implies that $\widehat{Q}_r(\lambda) \downarrow B_r$ is also projective. This implies that $\widehat{Q}_r(\lambda)$ has a $\widehat{Z}_r$-filtration.

Thus the multiplicity can be computed as

$$\begin{aligned}
[\widehat{Q}_r(\lambda) : \widehat{Z}_r(\mu)] &= [\widehat{Q}_r \downarrow B_rT : \widehat{Z}_r(\mu)] \\
&= \dim \mathrm{Hom}_{B_rT}\left(\widehat{Q}_r(\lambda), \mu\right) \\
&= \dim \mathrm{Hom}_{B_rT}\left(\widehat{Q}_r(\lambda), \mathrm{Ind}_{B_rT}^{G_rT} \mu\right) && \text{by Frobenius reciprocity}
\end{aligned}$$

Exercise 1.2.1 (?): Show that

$$[M : S] = \dim \mathrm{Hom}_A(P(S), \mu) = [\mathrm{Ind}_{B_rT}^{B_rT} \mu : \widehat{L}_r(\lambda)].$$

We can thus continue this computation as

$$\begin{aligned} \cdots &= [\widehat{Z}'_r(\mu) : \widehat{L}_r(\mu)] \\ &= [\widehat{Z}_r(\mu) : \widehat{L}_r(\mu)], \end{aligned}$$

since $\operatorname{ch} \widehat{Z}_r(\mu) = \operatorname{ch} \widehat{Z}'_r(\mu)$.

Thus we have the following reciprocity theorem

Theorem 1.2.1 ***(Humphreys).***

$$[\widehat{Q}_r(\lambda) : \widehat{Z}_r(\mu)] = [\widehat{Z}_r(\mu) : \widehat{L}_r(\lambda)].$$

Remark 1.2.1 : This is hard to prove in the G_r category, need to work in the G_rT category and descend. However, this reciprocity does also work for G_r.

Example 1.2.1 (?): For $G = \mathrm{SL}_2$, consider G_1T or G_1 where $\lambda = 0, 1, 2, \cdots, (p-1)$. We have a notion of *linkage*: λ, μ are in the same G_1 block iff $\lambda + \mu = p - 2$. Note that $\lambda = p - 1$ is in its own block.

We have

$$Z_r(\lambda) = \operatorname{Coind}_{B_1^+}^{G_1} \lambda \twoheadrightarrow L(\lambda).$$

If $\lambda + \mu = p - 2$, then we have the following situation:

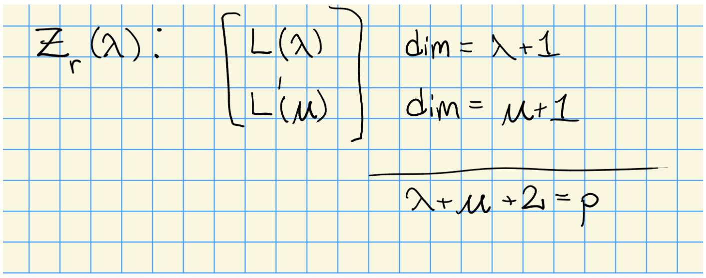

Figure 1: Image

Taking $\lambda = p - 1$, we have $Z_r((p-1)\rho) = L(p-1) = \mathrm{St}_1$.

Applying reciprocity, we gave

$$[Q_1(0) : Q_1(\mu)] = [Q_1(\mu) : L(0)].$$

Since $Q_1(0)$ has factors $Z_1(0)$ and $Z_1(p-2)$, we have

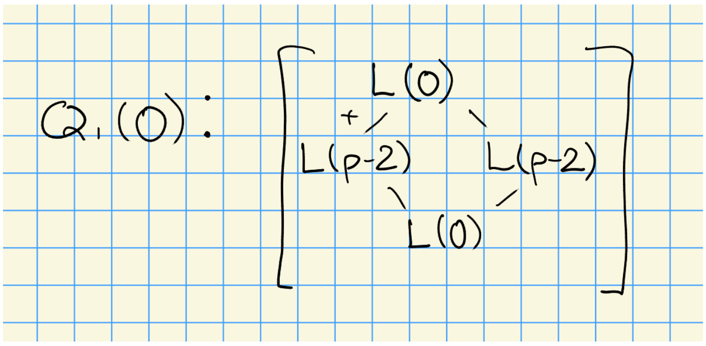

Figure 2: Image

We can identify the two filtrations here:

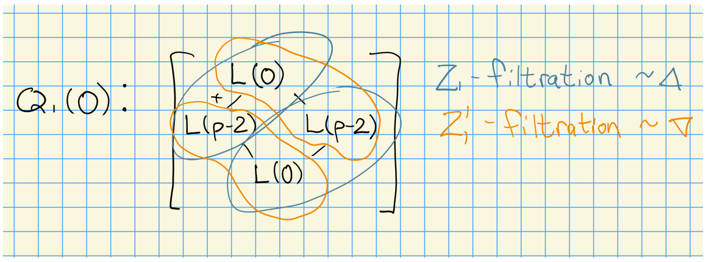

Figure 3: Image

Similarly, for $Q_1(p-2)$ we have

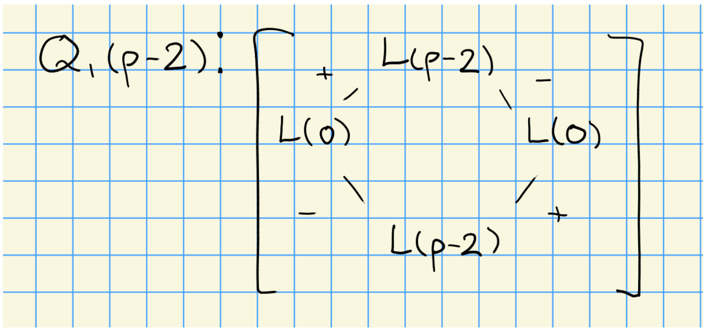

Figure 4: Image

We have

- $\dim \widehat{Q}_1(\lambda) = 2p$ for $\lambda \neq p-1$
- $\dim \widehat{Q}_1(p-1) = p$ for $\lambda = p-1$.

Remark 1.2.2 (Some historical background on reciprocity laws): Some work predated the BGG Category $\mathcal{O}$. For finite groups, a notion of CDE triangles was worked out.

1. Pollack (1967) computed the structure of projectives for G_1 in $G = \mathrm{SL}_2$.
2. Humphreys (1971) proved reciprocity for G_1. (They were students together.)
3. Bernstein-Gelfand-Gelfand (1976): developed machinery for Category $\mathcal{O}$, crediting Humphreys.
4. Roche-Caridi (1980): Proved reciprocity for generalized Verma modules.
5. BGG Algebra, Irving: A more axiomatic approach.
6. CPS (1988): Generalized to highest weight categories, also attributed to Humphreys.
7. Holmes-Nakano (1987): Proved when there is a triangular decomposition $A = A^- A_0 A^+$, looked at filtrations and reciprocity, applies to Lie algebras of Cartan type.[1]

1.3 Toward Lifting Conjectures

Recall that $G_r T \subseteq G$.

Question: Given $\widehat{Q}_r(\lambda)$ for a restricted weight $\lambda \in X_r(T)$, does $\widehat{Q}_r(\lambda)$ *lift* to G? I.e., does there exist a G-module $M(\lambda)$ such that $M(\lambda) \downarrow_{G_r T} = \widehat{Q}_r(\lambda)$?

Remark 1.3.1 : Note that $L_r(\lambda)$ for $\lambda \in X_r(T)$ lifts to G, since $L(\lambda) \downarrow_{G_r T} = \widehat{L}_r(\lambda)$.

Theorem 1.3.1 *(?).*
Let $p > 2h - 2$ and $\lambda \in X_r(T)$, then $\widehat{Q}_r(\lambda)$ has a lift to a G structure.

Remark 1.3.2 (Some history):

- One can prove that the G structure is unique, since this turns out to be a projective module in an appropriate category (which we won't get into).
- Ballard (1970s) proved the theorem for $p > 3h - 3$.
- Jantzen (late 1970s) lowered the bound to $p > 2h - 2$
- Amazingly, no one has been able to lower this bound! This is currently an open question.
- For $G = \mathrm{SL}_2, \mathrm{SL}_3$, it is known that $\widehat{Q}_r(\lambda)$ has a G structure for all p.

[1] Simple Lie algebras in characteristic p with a triangular decomposition which is highly non-symmetric (negative part is typically smaller).

1.3.1 Donkin's Tilting Module Conjecture

From MSRI, 1990. Some notation first: for $\lambda \in X_r(T)$, define

$$\widehat{\lambda} \coloneqq 2(p-1)\rho + w_0\lambda.$$

Conjecture 1.1 ***(?).***
Let G be a semisimple simply connected algebraic group over $k = \overline{F}_p$ for some p. Then

$$T(\widehat{\lambda}) \downarrow_{G_rT} \cong \widehat{Q}_r(\lambda).$$

Something about DTilt conjecture being true for $p > 2h - 2$.

Next time:

- Proof of theorem
- $\widehat{Q}_r(\lambda) \mid \mathrm{St}_r \otimes L(\sigma)$ as G-modules, and is also projective as a G_rT-module.
- Find a G-summand $M(\lambda)$ such that $M(\lambda) \downarrow_{G_rT} = \widehat{Q}_r(\lambda)$.
- More with injective modules.
- Possibly something about cohomology of Frobenius kernels.