

# Linearization Continued

## Section 8.4 Follow-Up

D. Zack Garza

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# Review

Linearization  
Continued

D. Zack Garza

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:

*For every  $w \in C^\infty(S^2, W)$  there exists a symplectic trivialization of the fiber bundle  $w^*TW$ , i.e.  $\langle c_1(TW), \pi_2(W) \rangle = 0$  where  $c_1$  denotes the first Chern class of the bundle  $TW$ .*

- We use this to trivialize the pullback  $\tilde{u}^*TW$  to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

# Review

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- We used the chosen frame  $\{Z_i\}$  to define a chart centered at  $u$  of  $\mathcal{P}^{1,p}(x, y)$  given by

$$\begin{aligned}\iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ Y = (y_1, \dots, y_{2n}) &\longmapsto \exp_u \left( \sum y_i Z_i \right).\end{aligned}$$

- We regard  $Y(s, t)$  as a tangent vector to  $W$  in some Euclidean embedding.

# Review

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- We seek to compute the composite map in charts:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & \mathcal{F}_u & & & \\
 & & \nearrow & & \searrow & & \\
 W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) & \xrightarrow{\iota} & \mathcal{P}^{1,p}(x, y) & \xrightarrow{\mathcal{F}} & L^p(\mathbb{R} \times S^1; TW) & \longrightarrow & L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\
 & & \nwarrow & & \nearrow & & \\
 & & & \mathcal{F} & & & 
 \end{array} \\
 \\
 u & \xrightarrow{\mathcal{F}} & \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} - X_t(u) \right) \\
 \\
 (y_1, \dots, y_{2n}) & \longrightarrow & \exp_u \left( \sum y_i Z_i \right)
 \end{array}$$

# Add a Tangent

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$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u) \\ \mathcal{F}(u + Y) &= \frac{\partial(u+Y)}{\partial s} + J(u+Y) \frac{\partial(u+Y)}{\partial t} - J(u+Y) X_t(u+Y)\end{aligned}$$

Extract the part that is linear in  $Y$  and collect terms:

$$\begin{aligned}(d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right)\end{aligned}$$

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Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}(d\mathcal{F})_u(Y) &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \\ &= \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) \\ &\quad + \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u) (dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).\end{aligned}$$

Use the fact that this is  $O_1 + O_0$  in  $Y$ .

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Study  $O_1$  first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \dots, y_{2n}).$$

where  $J_0$  is the standard complex structure on  $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial}Y + SY$$

where  $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^n))$  is a linear operator of order 0.

# Order 0 Symmetry in the Limit

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## Theorem (8.4.4, CR + Symmetric in the Limit)

*If  $u$  solves Floer's equation, then*

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

*where*

- $S$  is linear*
- $S$  tends to a symmetric operator as  $s \rightarrow \pm\infty$ , and*

$$\frac{\partial S}{\partial s}(s, t) \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{uniformly in } t$$



# Proof

Collect terms in the order zero part:

$$\begin{aligned} O_0 = S(y_1, \dots, y_{2n}) &= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right] \\ &= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i) X_t \right) \right. \\ &\quad \left. + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right]. \end{aligned}$$

- Claim: the terms in blue and orange vanish in the limit  $s \rightarrow \pm\infty$ , so it suffices to prove that the remaining yields a symmetric operator.

# Proof

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$$(dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i)X_t \right) \rightarrow 0$$

The term in blue vanishes: since  $u$  is a solution and

$$\frac{\partial u}{\partial s} \xrightarrow{s \rightarrow \pm \infty} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left( \frac{\partial u}{\partial t} - X_t(u) \right) \xrightarrow{s \rightarrow \pm \infty} 0$$

– Thus

$$\frac{\partial S}{\partial s} \xrightarrow{s \rightarrow \pm \infty} 0.$$

– The term in blue vanishes as  $s \rightarrow \pm \infty$