Title

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1 Monday, August 31

1.1 Review of Representation Theory of Modules

Take R a ring, then consider M an R-module to be a "vector space" over M. Note that M is an R-module $\iff$ there exists a ring morphism $\rho : R \longrightarrow \hom_{\mathrm{AbGrp}}(M, M)$.

Now let G be a group and consider G-modules M. Then a G-module will be defined by taking M/k a vector space and a G-action on M. This is equivalent to having a group morphism $\rho : G \longrightarrow \mathrm{GL}(M)$.

For M a G-module, given a group action, define

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}(M) \\ \rho(g)(m) &= g.m \end{aligned}$$

where $\rho(h) : M \longrightarrow M$.

Similarly, for $\rho : G \longrightarrow \mathrm{GL}(M)$ a group morphism, define the group action $g.m \coloneqq \rho(g)m$. Thus representations of G and G-modules are equivalent.

Definition 1.0.1 (?).
Let M be a G-module.

1. M is a *simple* G-module (equivalently an *irreducible representation*) $\iff$ the only G-submodules (equiv. G-invariant subspaces) are $0, M$.
2. M is *indecomposable* $\iff$ M can not be written as $M = M_1 \oplus M_2$ with $M_i < M$ proper submodules.

Example 1.1.
For $G = \mathrm{SL}(n, \mathbb{C})$, there is a natural n-dimensional representation $M = V$, and this is irreducible.

What is V?

Example 1.2.
Let $R = \mathbb{Z}$, so we're considering $\mathbb{Z}$-modules. For $M = \mathbb{Z}$, M is not simple since $2\mathbb{Z} < \mathbb{Z}$ is a proper submodule. However M is indecomposable.

Recall from last time: we defined a functor $\operatorname{Ind}_H^G(\,\cdot\,) : H\text{-mod} \longrightarrow G\text{-mod}$, where $\operatorname{Ind}_H^G = (k[G] \otimes M)^H$, the H-invariants. This functor is left-exact but not right-exact, so we have cohomology $R^j \operatorname{Ind}_H^G$ by taking right-derived functors.

Goal: classify simple G-modules for G a reductive connected algebraic group.

Example 1.3.
For $G = \mathrm{GL}(n, k)$, we have a decomposition

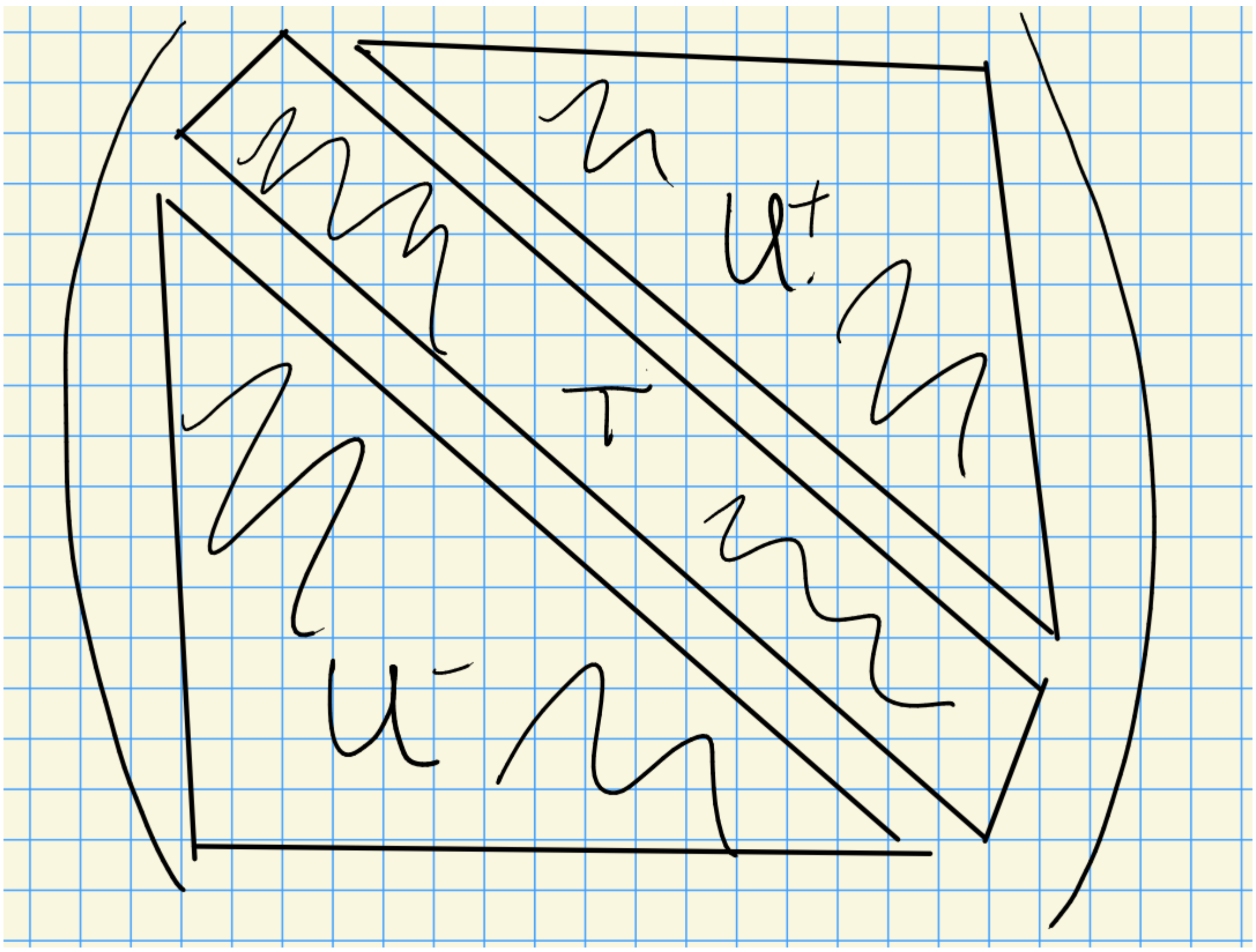

Figure 1: Image

We have

- $B = T \rtimes U$ the negative Borel,
- $B = T \rtimes U^+$ the Borel

For U-modules: k is the only simple U-module. Importantly, if V is a U-module, then the fixed points are never zero, i.e. $V^U = \hom_{U\text{-Mod}}(k, V) \neq 0$.

For B-modules: let $X(T) \coloneqq \hom(T, \mathbb{G}_m) = \hom(T, \mathrm{GL}(1,k))$. These are the simple representations for the torus T. Thus $\lambda \in X(T)$ represents a simple T-module.

We have a map $B \longrightarrow B/U = T$, so we can pullback T-representations to B-representations ("inflation"), since we have a map $T \longrightarrow \mathrm{GL}(1,k)$ and we can just compose. So λ is a 1-dimensional (simple) B-module where U acts trivially.

Lee's theorem: all irreducible representations for B are one-dimensional. Thus these are the simple B-modules.

For G-modules: define $\nabla(\lambda) \coloneqq \mathrm{Ind}_B^G(\lambda) = H^0(\lambda)$.

Questions:

1. When does $H^0(\lambda) = 0$?
2. What is $\dim_{k\text{-Vect}} H^0(\lambda)$?
3. What are the composition factors of $H^0(\lambda)$?

Known in characteristic zero, wildly open in positive characteristic.

Remark 1.
Another interpretation: look at the flag variety G/B and take global sections, then $H^0(\lambda) = H^0(G/B, \mathcal{L}(\lambda))$ where $\mathcal{L}$ is given by projecting the fiber product $G \times_B \lambda \twoheadrightarrow G/B$ onto the first factor.

Remark 2.

1. $H^0(k) = H^0([0, \cdots, 0]) = k[G/B] = k$.
2. $H^0(M) = M$ if M is a G-module.
3. A G-module M is *semisimple* iff $M = \bigoplus_{i \in I} M_i$ with each M_i are simple.
4. Can consider the largest semisimple submodule, the *socle* $\mathrm{Soc}_G(M)$.

$$
\begin{tikzcd}
L_4 \arrow[dr, hook] & & L_5 \oplus L_7 \arrow[dl, hook'] \\
& (L_1 \oplus L_2 \oplus L_3) = \mathrm{Soc}_G(M)) &
\end{tikzcd}
$$

Goal: classify simple G-modules. Strategy: used dominant highest weights.

As opposed to Verma modules, the irreducibles will be a dual situation where they sit at the bottom of the module. Indicated by the notation ∇ pointing down!

Proposition 1.1 ***(?).***
Let $\lambda \in X(T)$ with $H^0(\lambda) \neq 0$.

1. $\dim H^0(\lambda)^{U^+} = 1$ and $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$.
2. Every weight of $H^0(\lambda)$ satisfies $w_u\lambda \leq \mu \leq \lambda$, where w_0 is the longest Weyl group element and $\alpha \leq \beta \iff \alpha - \beta \in \mathbb{Z}^+\Phi$.

Note that in fact $\ell(w_0) = \left|\Phi^+\right|$.

Example 1.4.
Take A_2 with simple reflections $s_{\alpha_1}, s_{\alpha_2}$ and $\Delta = \{\alpha_1, \alpha_2\}$.

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
s_{\alpha_1} \arrow[d] & & s_{\alpha_2} \arrow[d] \\
s_{\alpha_1}s_{\alpha_2} \arrow[dr] & & s_{\alpha_2}s_{\alpha_1} \arrow[dl] \\
 & s_{\alpha_2}s_{\alpha_1}s_{\alpha_1} = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1} = w_0 &
\end{tikzcd}$$

Proof ((Sketch)).
We can write

$$H^0(\lambda) = \left\{ f \in k[G] \;\middle|\; f(gb) = \lambda(b)^{-1} f(g)\, b \in B, g \in G \right\}.$$

Suppose $f \in H^0(\lambda)^{U^+}$ and $u_+ \in U^+, t \in T, u \in U$. Then

$$\begin{aligned}
\left(u_+^{-1} f\right)(tu) &= f(tu) \\
&= \lambda(t)^{-1} f(1).
\end{aligned}$$

On the other hand,

$$\left(u_+^{-1} f\right)(tu) = f(u_+ tu).$$

So by density, $f(1)$ is determined by $f(u_+ tu)$ and $\dim H^0(\lambda)^{U^+} \leq 1$. But since this can't be zero, the dimension must be equal to 1.

∎

Proposition 1.2 *(?).*
Let

$$\varepsilon : H^0(\lambda) \longrightarrow \lambda$$

be the evaluation morphism.
This is a morphism of B-modules, and in particular is a morphism of T-modules. Thus the image of a weight $\mu \neq \lambda$ is zero, so ε is injective.

Proof .
We have

$$f(u_+tu) = \lambda(t)^{-1} f(1) = \lambda(t)^{-1}\varepsilon(f).$$

Suppose $f \in H^0(\lambda)^{U^+}$ and $\varepsilon(f) = 0$. Then $f(u_+tu) = 0$, and by density $f \equiv 0$, showing injectivity.
Therefore $H^0(\lambda)^{U^+} \subset H^0(\lambda)_\lambda$. Suppose μ is maximal among weights in $H^0(\lambda)$. Then

$$H^0(\lambda)_\mu \subseteq H^0(\lambda)^{U^+}$$

because U^+ raises weights.
But $H^0(\lambda)^{U^+} \subseteq H^0(\lambda)_\lambda$ implies $\mu = \lambda$. Thus the maximal weight in $H^0(\lambda)$ is λ.

Recall the situation in lie algebras: $g_\alpha v \in V_{\lambda+\alpha}$ when v $in V_\lambda$.

Since λ is maximal, any other weight μ satisfies $\mu \leq \lambda$. Thus

$$H^0(\lambda)_\lambda \subseteq H^0(\lambda)^{U^+} \subseteq H^0(\lambda)_\lambda,$$

forcing these to be equal and finishing part 1.

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