

# Title

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### 0.1 Exercises

#### Problem 1.

Let  $C$  denote the Cantor set.

1. Show that  $C$  contains point that is not an endpoint of one of the removed intervals.
2. Show that  $C$  is nowhere dense, meager, and has measure zero.
3. Show that  $C$  is uncountable.

#### Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let  $C_n$  be the  $n$ Th stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:

$$C_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup \dots \cup [\frac{3^n - 1}{3^n}, 1],$$

and so the endpoints are precisely the numbers of the form  $\frac{k}{3^n}$  where  $0 \leq k \leq 3^n$ . Moreover, any endpoint appearing in  $C_n$  is never removed in any later step, and so all endpoints remaining in  $C$  are of this form where we allow  $0 \leq n < \infty$ .

Thus, our goal is to produce a number  $x \in [0, 1]$  such that  $x \neq \frac{k}{3^n}$  for any  $k$  or  $n$ , but also satisfies  $x \in C$ . So we will need a general characterization of all of the points in  $C$ .

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Lemma: If  $x \in C$ , then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

Proof: By induction on the index  $k$  in  $a_k$ , first consider note that if  $x \in C$  then  $x \in C_1 = [0, 1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . So if  $x \in C_1$ , then  $x \notin (\frac{1}{3}, \frac{2}{3})$ . But note that  $a_1$  is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \leq x < \frac{1}{3}, \\ 1 & \frac{1}{3} \leq x < \frac{2}{3}, \\ 2 & \frac{2}{3} \leq x < 1. \end{cases}$$

Since the interval  $(\frac{1}{3}, \frac{2}{3})$  is deleted in  $C_1$ , we find that  $a_1 = 1 \iff x = \frac{1}{3}$ . In this case, however, we claim that we can find a ternary expansion of  $x$  that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that  $x = x'$ , which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left( \sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left( \sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left( \frac{1}{1 - \frac{1}{3}} \right) - 2 - 2(3^{-1}) \\ &= 2 \left( \frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have  $\frac{1}{3} = (0.1)_3 = (0.222\cdots)_3$  as ternary expansions, and a similar proof shows that such an expansion without 1s can be found for any endpoint.

For the inductive step, consider  $a_n$ : the claim is that if  $a_n = 1$ , then  $x \notin C_{n+1}$  – that is, it is contained in one of the intervals deleted at the  $n + 1$ st stage. Writing the deleted interval at this stage as  $(a, b)$ , we find that  $a_n = 1$  if and only if  $x \in [a, b)$ . Since  $x \in C$ , the only way  $a_n$  can be 1 is if  $x$  was in fact the endpoint  $a$  (since no previous digit was a 1, by hypothesis). However, as shown above, every such endpoint has a ternary expansion containing no 1s.  $\square$

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Therefore, if we can produce an  $x$  that satisfies  $x \neq \frac{k}{3^n}$  for any  $k, n$  **and**  $x$  has no 1s in its ternary expansion, we will have an  $x \in C$  that is not an endpoint.

So take

$$x = (0.\overline{02})_3 = (0.020202\cdots)_3.$$

This evidently has no 1s in its ternary expansion, and if we sum the corresponding geometric series, we find  $x = \frac{1}{4}$ . This is not of the form  $\frac{k}{3^n}$  for any  $k, n$ , and thus fulfills both conditions.

2. We first show that  $C$  is nowhere dense by showing that the interior of its closure is empty, i.e.  $(\overline{C})^\circ = \emptyset$ .

To do so, we note that  $C$  is itself closed and so  $C = \overline{C}$ . To see why this is, consider  $C^c$ ; we'll show that it is open. By construction,  $C_1^c$  is the open interval  $(\frac{1}{3}, \frac{2}{3})$  that is deleted, and similarly  $C_n^c$  is the finite union of the open intervals that are deleted at the  $n$ th stage. But then

$$C^c = \left(\bigcap C_n\right)^c = \bigcup C_n^c$$

is an infinite union of open sets, which is also open. So  $C$  is closed.

It is also the case that  $C$  has empty interior, so  $C^\circ = \emptyset$ . Towards a contradiction, suppose  $x \in C$  is an interior point; then there is some neighborhood  $N_\varepsilon(x) \subset C$ . Since we are on the real line, we can write this as an interval  $(x - \varepsilon, x + \varepsilon)$ , which has length  $2\varepsilon > 0$ . Moreover, we have the containment

$$(x - \varepsilon, x + \varepsilon) \subset C \subset C_n$$

for every  $n$ .

Claim: The length of  $C_n$  is  $(\frac{2}{3})^n$  where we define  $C_0 = [0, 1]$ . Letting  $L_n$  be the length of  $C_n$ , one easy way to see that this is the case is to note that  $L_n$  satisfies the recurrence relation

$$L_{n+1} = \frac{2}{3}L_n,$$

since an interval of length  $\frac{1}{3}L_n$  is removed at each stage. With the initial conditions  $L_0 = 1$ , it can be checked that  $L_n = \left(\frac{2}{3}\right)^n$  solves this relation.

Now, since  $x \in C = \bigcap C_n$ , it is in every  $C_n$ . So we can choose  $n$  large enough such that

$$\left(\frac{2}{3}\right)^n \leq 2\varepsilon.$$

Letting  $\mu(X)$  denote the length of an interval, we always have  $C \subseteq C_n$  and so  $\mu(C) \leq \mu(C_n)$ .

Using the subadditivity of measures, we now have

$$\begin{aligned} (x - \varepsilon, x + \varepsilon) &\subset C \subset C_n \\ \implies \mu(x - \varepsilon, x + \varepsilon) &\leq \mu(C) \leq \mu(C_n) \\ &\implies 2\varepsilon \leq \left(\frac{2}{3}\right)^n, \end{aligned}$$

a contradiction. So  $C$  has no interior points.

But this means that

$$(\overline{C})^\circ = C^\circ = \emptyset,$$

and so  $C$  is nowhere dense.

To see that  $\mu(C) = 0$ , we can use the fact that for any sets, measures are additive over disjoint sets and we have

$$\mu(A) + \mu(X \setminus A) = \mu(X) \implies \mu(X \setminus A) = \mu(X) - \mu(A).$$

Here we will take  $X = [0, 1]$ , so  $\mu(X) = 1$ , and  $A = C$  the Cantor set.

By tracing through the construction of the Cantor set, letting  $B_n$  be the length of the interval that is removed at each stage, we can deduce

$$\begin{aligned} B_1 &= \frac{1}{3} \\ B_2 &= \frac{2}{9} \\ &\dots \\ B_n &= \frac{2^n}{3^{n+1}}. \end{aligned}$$

We can identify  $B_n = \mu(C_n^c)$ , and using the fact that  $C_n^c \cap C_{>n}^c = \emptyset$  and the fact that measures are additive over disjoint sets, we can compute

$$\begin{aligned} \mu(C) &= 1 - \mu(C^c) \\ &= 1 - \mu\left(\left(\bigcap_{n=0}^{\infty} C_n\right)^c\right) \\ &= 1 - \mu\left(\bigcup_{n=0}^{\infty} C_n^c\right) \\ &= 1 - \sum_{n=0}^{\infty} \mu(C_n^c) \\ &= 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} \\ &= 1 - \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ &= 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) \\ &= 1 - \frac{1}{3}(3) = 0, \end{aligned}$$

which is what we wanted to show.  $\square$

- Let  $y \in [0, 1]$  be arbitrary, we will construct an element  $x \in C$  such that  $y = f(x)$ . We first note that every number has a binary expansion, and we can write

$$y = \sum_{k=1}^{\infty} y_k 2^{-k} \quad \text{where } y_k \in \{0, 1\}.$$

Now we construct

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k = 2y_k \implies a_k \in \{0, 2\}.$$

By the characterization given in part (1), we see that  $x \in C$  because it has no 1s in its ternary expansion. Moreover, under  $f$ , we have  $a_k \mapsto \frac{1}{2}a_k = \frac{1}{2}(2a_k) = a_k$ , and so  $f(x) = y$  by construction.

This shows that  $C$  surjects onto  $[0, 1]$ , and in particular,  $\#C \geq \#[0, 1]$  holds for the cardinalities of these sets. Since  $[0, 1]$  is uncountable (say, by Cantor's diagonalization argument), this shows that  $C$  is uncountable.

## Problem 2.

1. Show that  $X$  is  $G_\delta$  iff  $X^c$  is  $F_\sigma$ .
2. Show that  $X$  closed  $\implies X$  is  $G_\delta$  and  $X$  open  $\implies X$  is  $F_\sigma$ .
3. Give an example of an  $F_\sigma$  set that is not  $G_\delta$ , and a set that is neither.

## Solution 2.

1. To show the forward direction, suppose  $X$  is a  $F_\sigma$ , so  $X = \bigcup_{i \in \mathbb{N}} A_i$  with each  $A_i$  an closed set. By definition, each  $A_i^c$  is open, and we have

$$X^c = \left( \bigcup_{i \in \mathbb{N}} A_i \right)^c = \bigcap_{i \in \mathbb{N}} A_i^c,$$

which exhibits  $X^c$  as a countable intersection of closed sets, making it an  $G_\delta$ .

The reverse direction proceeds analogously: supposing  $X^c$  is  $G_\delta$ , we can write  $X^c = \bigcap_{i \in \mathbb{N}} B_i$  with each  $B_i$  open, where  $B_i^c$  is closed by definition, and

$$X = (X^c)^c = \left( \bigcap_{i \in \mathbb{N}} B_i \right)^c = \bigcup_{i \in \mathbb{N}} B_i^c$$

which exhibits  $X$  as a union of closed sets, and thus an  $F_\sigma$ .

2. Suppose  $X$  is closed, we want to then write  $X$  as a countable intersection of open sets. For every  $x \in X$  and every  $n \in \mathbb{N}$ , define

$$\begin{aligned} B_n(x) &= \left\{ y \in \mathbb{R}^n \mid |x - y| \leq \frac{1}{n} \right\}, \\ V_n &= \bigcup_{x \in X} B_n(x), \\ W &= \bigcap_{n \in \mathbb{N}} V_n. \end{aligned}$$

Explicitly, we have

$$W = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in X} B_n(x),$$

and the claim is that  $W$  is a  $G_\delta$  and  $W = X$ .

To see that the  $V_n$  are open, note that  $n$  is fixed and each  $B_n(x)$  is an open ball around a point  $x$ . Any union of open sets is open, and thus so is  $V_n$ . By construction,  $W$  is then a countable intersection of open sets, and thus  $W$  is a  $G_\delta$  by definition.

We show  $W = X$  in two parts. To see that  $X \subseteq W$ , note that if  $x \in X$ , then  $x \in B_n(x)$  for every  $n$  and thus  $x \in V_n$  for every  $n$  as well. But this means that  $x \in \bigcap_n V_n$ , and so  $x \in W$ .

To see that  $W \subseteq X$ , let  $w \in W$  be arbitrary. If  $w \in X$ , there is nothing to check, so suppose  $w \notin X$  towards a contradiction.

Since  $w \in \bigcap_n V_n$ , it is in  $V_n$  for every  $n$ . But this means that there is some particular  $x_0$  such that  $w \in B_n(x_0)$  for every  $n$  as well, and moreover since we assumed  $w \notin X$ , we have  $w \neq x_0$ .

Then, letting  $N_\varepsilon(w)$  be an arbitrary neighborhood of  $w$ , we can find an  $n$  large enough such that  $B_n(x) \subset N_\varepsilon(w)$ . This means that  $x_0 \neq w$  can be found in every neighborhood of  $w$ , which makes  $w$  a limit point of  $X$ . However, since we assumed  $X$  was closed, it contains all of its limit points, which would force  $w \in X$ , a contradiction.  $\square$

Now suppose  $X$  is an open set, we want to show it is an  $F_\sigma$  and can thus be written as a countable union of closed sets. We can use the fact that  $X^c$  is closed, and by the previous result,  $X^c$  is thus a  $G_\delta$ . But by an earlier result,  $X^c$  is a  $G_\delta \iff (X^c)^c = X$  is an  $F_\sigma$ , and we are done.

3. We want to construct a set that can be written as a countable union of closed sets, but not as a countable intersection of open sets. Note that in  $\mathbb{R}$  with the usual topology, singletons are closed, and so  $\{p\}^c$  is an open set for any point  $p$ .

With this motivation, consider  $X = \mathbb{Q}$  and  $X^c = \mathbb{R} \setminus \mathbb{Q}$ . We can write

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\},$$

which exhibits  $X$  as a countable union of closed sets because  $\mathbb{Q}$  itself is countable. So  $\mathbb{Q}$  is an  $F_\sigma$  set. Suppose towards a contradiction that  $\mathbb{Q}$  is also  $G_\delta$ , so we have  $\mathbb{Q} = \bigcap_{i \in \mathbb{N}} O_i$  with each  $O_i$  open. So each  $O_i$  covers  $\mathbb{Q}$ , i.e.  $\mathbb{Q} \subseteq O_i$ , which (importantly!) forces each  $O_i$  to be dense in  $\mathbb{R}$ .

But now note that we can also write

$$\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{q \in \mathbb{Q}} \{q\} = \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\},$$

where we can note that  $\mathbb{R} \setminus \{q\}$  is an open, dense subset of  $\mathbb{R}$  for each  $q$ . We can appeal to the Baire category theorem twice, which tells us that any countable intersection of *open* dense sets will also be dense. This first tells us that the above intersection, and thus  $\mathbb{R} \setminus \mathbb{Q}$ , is dense in  $\mathbb{R}$ . Then, writing

$$\left( \bigcap_{i \in \mathbb{N}} O_i \right) \cap \left( \bigcap_{q \in \mathbb{Q}} \mathbb{R} \setminus \{q\} \right) = \mathbb{Q} \cap \mathbb{R} \setminus \mathbb{Q} = \emptyset,$$

we produce what is still just a countable intersection of open dense sets, and by Baire, the result would need to be dense as well. Since the empty set is *not* dense in  $\mathbb{R}$ , so we arrive at a contradiction.

### Problem 3.

1. Let  $r_n$  be an enumeration of the rationals, define  $f(r_n) = \frac{1}{n}$  and  $f(x) = 0$  for  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Show that  $\lim_{x \rightarrow c} f(x) = 0$  for every  $c \in I$ , and  $D_f = \mathbb{Q} \cap I$ .
2. Supposing  $f$  is bounded, show that  $\omega_f$  is (in general) well-defined, and that  $f$  is continuous at  $x \iff \omega_f(x) = 0$ .
3. Show that for every  $\varepsilon > 0$ , the set  $A(\varepsilon) = \{x \in \mathbb{R} \mid \omega_f(x) > \varepsilon\}$  is closed, and thus  $D_f$  is an  $F_\sigma$  set.

### Solution 3.

1. We need to show that

$$\forall c \in I, \forall \varepsilon > 0, \exists \delta \ni |x - c| \leq \delta \implies |f(x) - 0| \leq \varepsilon.$$

To that end, let  $\{r_n\}$  be an arbitrary enumeration of  $\mathbb{Q} \cap I$ , let  $\varepsilon$  be fixed, and let  $c \in I$  be arbitrary. If  $c \in I \setminus \mathbb{Q}$ , then  $f(c) = 0 < \varepsilon$  and there's nothing to prove. Otherwise,  $c \in \mathbb{Q}$ , so  $c = r_n$  for some  $n$ , and  $f(c) = \frac{1}{n}$ . Let  $S = \{r_i \mid i \in \mathbb{N}, \frac{1}{i} > \varepsilon\} \subset \mathbb{Q}$ , and note that  $S$  is finite by the archimedean property of  $\mathbb{R}$ . So choose

$$\delta < \min \{|c - s| \mid s \in S\},$$

so that  $S \cap B_\delta(c) = \emptyset$ .

This means that if  $x \in B_\delta(c) \cap \mathbb{Q}$ , then  $x = r_m$  where  $\frac{1}{m} < \varepsilon$  by construction. But then  $|f(x)| = \frac{1}{m} < \varepsilon$ , and we are done.

By the sequential definition of continuity,  $f$  is continuous iff  $\lim_{x \rightarrow c} f(x) = f(c)$ . As we have shown, if  $c \in I \setminus \mathbb{Q}$ , then  $\lim_{x \rightarrow c} f(x) = 0 = f(c)$ , and so  $f$  is continuous there. However, for  $c \in I \cap \mathbb{Q}$ , since  $\lim_{x \rightarrow r_n} f(x) = 0 \neq \frac{1}{n}$ ,  $f$  fails to be continuous there. Taken together, this says that  $D_f = I \setminus \mathbb{Q}$  as desired.

2. To show that this is well-defined, we need to prove that the limit exists. By definition, since  $f$  is bounded, there exists some  $M$  that is independent of  $x$  such that  $x \in \mathbb{R} \implies |f(x)| \leq M$ . In particular, for any fixed  $\delta$ , it is certainly the case that  $B_\delta(x) \subset \mathbb{R}$ , and so  $x \in B_\delta(x) \implies |f(x)| \leq M$  as well.

We can then say that if  $y, z \in B_\delta(x)$ , then

$$|f(y) - f(z)| \leq |f(y)| + |f(z)| \leq 2M,$$

and thus the set  $\{|f(y) - f(z)| \mid y, z \in B_\delta(x)\}$  is bounded above and thus has a least upper bound (since  $\mathbb{R}$  has the least upper bound property). Thus the following supremum exists:

$$S(x, \delta) = \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|.$$

We now just need to show that  $\lim_{\delta \rightarrow 0^+} S(x, \delta)$  exists. To this end, we can note that if  $\delta_1 < \delta_2$ , then  $B_{\delta_1} \subset B_{\delta_2}$ , and so  $S$  is a monotonically decreasing function of  $\delta$  that is bounded below by 0 (since  $B_0(x) = \{x\} \implies y = z = x$  are the only choices), and is thus convergent by the monotone convergence theorem. So  $\omega_f$  is well-defined.

To see that  $f$  continuous at  $x \implies \omega_f(x) = 0$ , let  $\varepsilon$  be arbitrary; we will show that  $\omega_f(x) < \varepsilon$ . Since  $f$  is continuous, we can pick a  $\delta$  such that  $y, z \in B_\delta(x) \implies f(y), f(z) \in B_{\varepsilon/2}(f(x))$ . Thus we have

$$\begin{aligned} |y - x| < \delta &\implies |f(y) - f(x)| < \varepsilon/2 \\ |z - x| < \delta &\implies |f(z) - f(x)| < \varepsilon/2 \end{aligned}$$

Moreover, we can write

$$|f(y) - f(z)| = |f(y) - f(x) + f(x) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| \leq \varepsilon,$$

and thus we also have

$$\sup_{y, z \in B_\delta(x)} |f(y) - f(z)| < \varepsilon.$$

We now want to take the limit as  $\delta \rightarrow 0^+$ ; again since  $\delta_1 \leq \delta_2 \implies B_{\delta_1} \subseteq B_{\delta_2}$ , this can only make the left-hand-side of the above inequality smaller, and thus  $\omega_f(x) \leq \varepsilon$ . Taking  $\varepsilon \rightarrow 0$  completes the proof.

To see that  $\omega_f(x) = 0 \implies f$  is continuous at  $x$ , let  $x$  be fixed and  $\varepsilon > 0$  be arbitrary; we want to produce a  $\delta$  to use in the definition of continuity. Since  $\omega_f(x) = 0$ , we can find a  $\delta$  such that

$$\sup_{y, z \in B_\delta(x)} |f(y) - f(z)| < \varepsilon.$$

In particular, we can fix  $x \in B_\delta(x)$  and let  $y$  vary to obtain

$$\sup_{y \in B_\delta(x)} |f(y) - f(x)| < \varepsilon.$$

But for any particular choice  $y_0$  such that  $|y_0 - x| < \delta$ , we have

$$|f(x) - f(y_0)| \leq \sup_{y \in B_\delta(x)} |f(y) - f(x)| < \varepsilon,$$

which is exactly the condition that

$$|y_0 - x| < \delta \implies |f(x) - f(y_0)| < \varepsilon,$$

which says that  $f$  is continuous at  $x$ .  $\square$



3. Note that if  $A_\varepsilon$  is closed, we can write  $D_f = \bigcup_{n=1}^{\infty} A_{\frac{1}{n}}$ , which is a countable union of closed sets, making  $D_f$  an  $F_\sigma$  set.

We will proceed by fixing  $\varepsilon$  and showing that  $A_\varepsilon^c = \{x \in \mathbb{R} \mid \omega_f(x) < \varepsilon\}$  is open. To do so, let  $x \in A_\varepsilon^c$ , we then want to produce a  $\delta$  such that

$$B_\delta(x) = (x - \delta, x + \delta) \subset A_\varepsilon^c,$$

which can be equivalently stated as

$$y \in B_\delta(x) \implies |y - x| < \delta \implies \omega_f(y) < \varepsilon \implies \lim_{\delta \rightarrow 0^+} \sup_{a, b \in B_\delta(y)} |f(a) - f(b)| < \varepsilon$$