

*Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.*

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# 4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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*Last updated: 2021-02-19*

# Table of Contents

## Contents

<b>Table of Contents</b>	<b>2</b>
<b>1 Tuesday, January 12</b>	<b>4</b>
1.1 Background . . . . .	4
1.2 Introduction . . . . .	4
<b>2 Friday, January 15</b>	<b>8</b>
<b>3 Main Theorems for the Course</b>	<b>10</b>
3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure . . . . .	12
3.1.1 Sketch of Proof . . . . .	13
<b>4 Lecture 3 (Wednesday, January 20)</b>	<b>14</b>
4.1 Sheaves . . . . .	14
4.2 Bundles . . . . .	16
<b>5 Lecture 4 (Friday, January 22)</b>	<b>19</b>
5.1 The Exponential Exact Sequence . . . . .	19
<b>6 Principal <math>G</math>-Bundles and Connections (Monday, January 25)</b>	<b>22</b>
<b>7 Wednesday, January 27</b>	<b>26</b>
7.1 Bundles and Connections . . . . .	26
7.2 Sheaf Cohomology . . . . .	29
<b>8 Sheaf Cohomology (Friday, January 29)</b>	<b>30</b>
<b>9 Monday, February 01</b>	<b>34</b>
<b>10 Wednesday, February 03</b>	<b>36</b>
<b>11 Friday, February 05</b>	<b>43</b>
11.1 Characteristic Classes . . . . .	46
<b>12 Monday, February 08</b>	<b>47</b>
<b>13 Wednesday, February 10</b>	<b>49</b>
<b>14 Friday, February 12</b>	<b>52</b>
14.1 Section 5: Riemann-Roch and Generalizations . . . . .	54
<b>15 Monday, February 15</b>	<b>55</b>
15.1 Riemann-Roch . . . . .	56

<b>16 Friday, February 19</b>	<b>60</b>
16.1 Applications of Riemann-Roch . . . . .	60
<b>ToDoS</b>	<b>62</b>
<b>Definitions</b>	<b>63</b>
<b>Theorems</b>	<b>64</b>
<b>Exercises</b>	<b>65</b>
<b>Figures</b>	<b>66</b>
<b>Bibliography</b>	<b>67</b>

# 1 | Tuesday, January 12

## 1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [5]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [2]
  - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic  $\mathbb{R}^4$  was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [1]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [4]
- Akhil Mathew: The Dirac Operator [3]
- Tom Weston: An Introduction to Cobordism Theory [6]

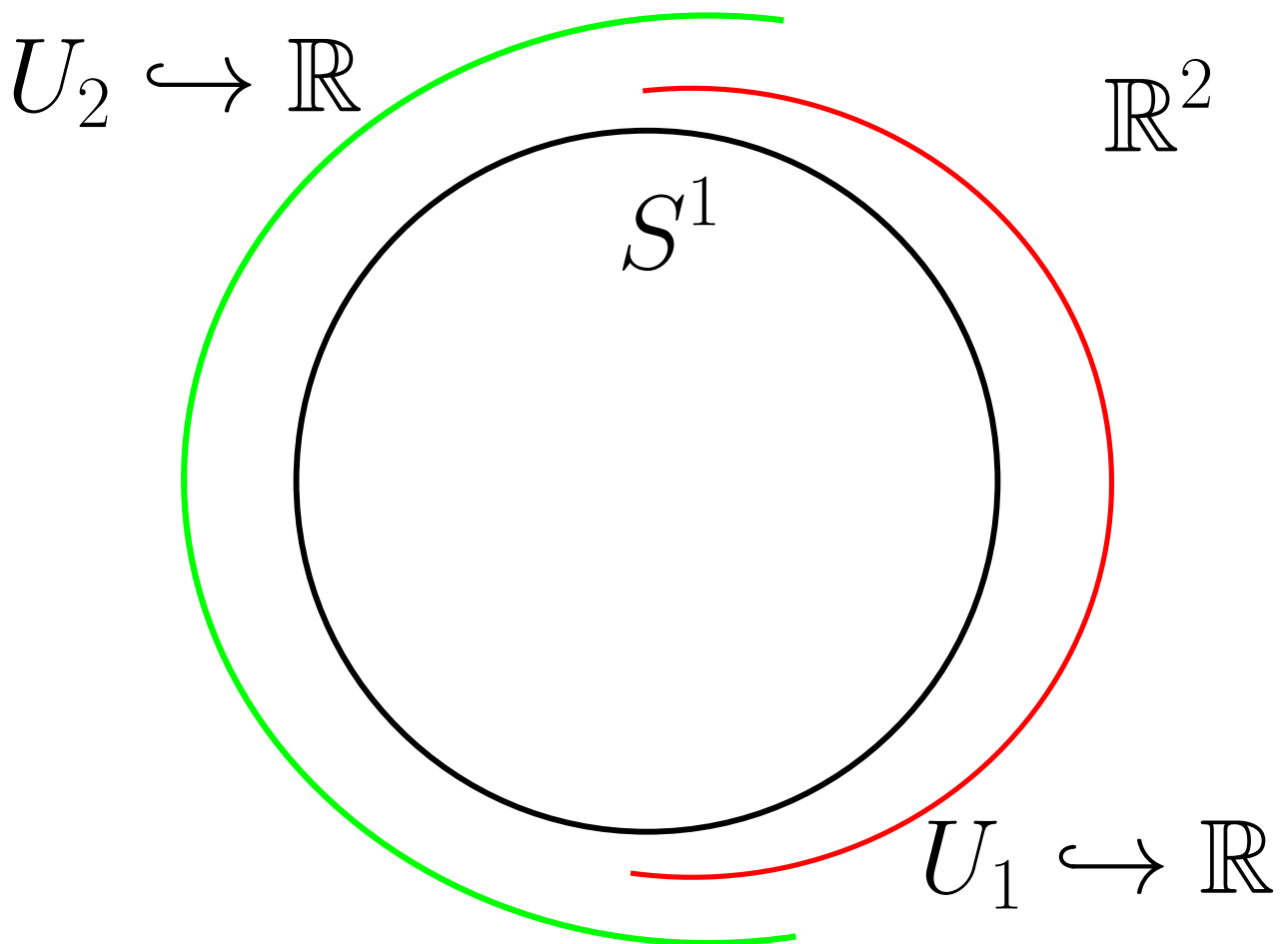
A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

## 1.2 Introduction

### Definition 1.2.1 (Topological Manifold)

Recall that a **topological manifold** (or  $C^0$  manifold)  $X$  is a Hausdorff topological space *locally homeomorphic* to  $\mathbb{R}^n$  with a countable topological base, so we have charts  $\varphi_u : U \rightarrow \mathbb{R}^n$  which are homeomorphisms from open sets covering  $X$ .

**Example 1.2.2 (The circle):**  $S^1$  is covered by two charts homeomorphic to intervals:



**Remark 1.2.3:** Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} := \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V).$$

**Definition 1.2.4** (Restricted Structures on Manifolds)

- We say  $X$  is a **PL manifold** if and only if  $t_{UV}$  are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say  $X$  is a  $C^k$  **manifold** if they are  $k$  times continuously differentiable, and **smooth** if infinitely differentiable.
- We say  $X$  is **real-analytic** if they are locally given by convergent power series.
- We say  $X$  is **complex-analytic** if under the identification  $\mathbb{R}^n \cong \mathbb{C}^{n/2}$  if they are holomorphic, i.e. the differential of  $t_{UV}$  is complex linear.
- We say  $X$  is a **projective variety** if it is the vanishing locus of homogeneous polynomials on  $\mathbb{CP}^N$ .

**Remark 1.2.5:** Is this a strictly increasing hierarchy? It's not clear e.g. that every  $C^k$  manifold is PL.

### Question 1.2.6

Consider  $\mathbb{R}^n$  as a topological manifold: are any two smooth structures on  $\mathbb{R}^n$  diffeomorphic?

**Remark 1.2.7:** Fix a copy of  $\mathbb{R}$  and form a single chart  $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$ . There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\rightarrow \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on  $X$ , since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form  $t_{VU} : t \mapsto t^{1/3}$ , which is not smooth at zero. However, the map

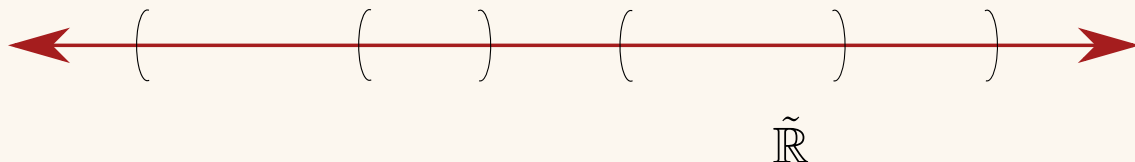
$$\begin{aligned} X &\rightarrow X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

**Claim:**  $\mathbb{R}$  admits a unique smooth structure.

*Proof (sketch).*

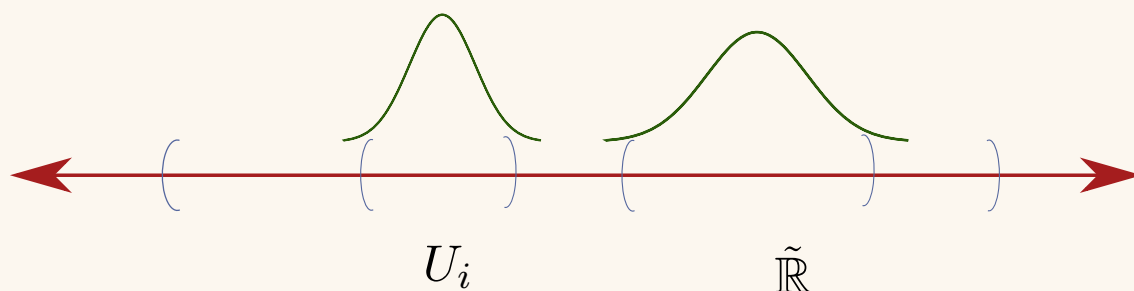
Let  $\tilde{\mathbb{R}}$  be some exotic  $\mathbb{R}$ , i.e. a smooth manifold homeomorphic to  $\mathbb{R}$ . Cover this by coordinate charts to the standard  $\mathbb{R}$ :



### Fact

There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions  $f_i : U_i \rightarrow \mathbb{R}$  with  $f_i \geq 0$  and  $\text{supp } f \subseteq U_i$  such that  $\sum f_i = 1$  (i.e., *bump functions*). It is also a purely topological fact that  $\tilde{\mathbb{R}}$  is orientable.

So we have bump functions:



Take a smooth vector field  $V_i$  on  $U_i$  everywhere aligning with the orientation. Then  $\sum f_i V_i$  is a smooth nowhere vector field on  $X$  that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\rightarrow \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that  $\varphi'(t) = V(\varphi(t))$ . Then  $\varphi$  is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

#### Slogan

To understand smooth structures on  $X$ , we should try to solve differential equations on  $X$ .



**Remark 1.2.10:** Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g.  $S^2$ .

#### Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

#### Answer 1.2.12

1. (Munkres) Every  $C^1$  structure gives a unique  $C^k$  and  $C^\infty$  structure.<sup>1</sup>
2. (Grauert) Every  $C^\infty$  structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in  $\dim X \leq 7$ , and it's unique in  $\dim X \leq 6$ , and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let  $X$  be a topological manifold of  $\dim X \geq 5$ , then there exists a cohomology class  $ks(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$  which is 0 if and only if  $X$  admits a PL structure.

<sup>1</sup>Note that this doesn't start at  $C^0$ , so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if  $\text{ks}(X) = 0$ , then (up to concordance) the set of PL structures is given by  $H^3(X; \mathbb{Z}/2\mathbb{Z})$ .

5. (Moise) Every topological manifold in  $\dim X \leq 3$  admits a unique smooth structure.
6. (Smale et al.): In  $\dim X \geq 5$ , the number of smooth structures on a topological manifold  $X$  is finite. In particular,  $\mathbb{R}^n$  for  $n \neq 4$  has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes)  $\mathbb{R}^4$  admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface  $\Sigma$ , the space of complex-analytic structures is a complex orbifold<sup>2</sup> of dimension  $3g - 2$  where  $g$  is the genus of  $\Sigma$ , up to biholomorphism (i.e. *moduli*).

**Remark 1.2.13:** Kervaire-Milnor:  $S^7$  admits 28 smooth structures, which form a group. 

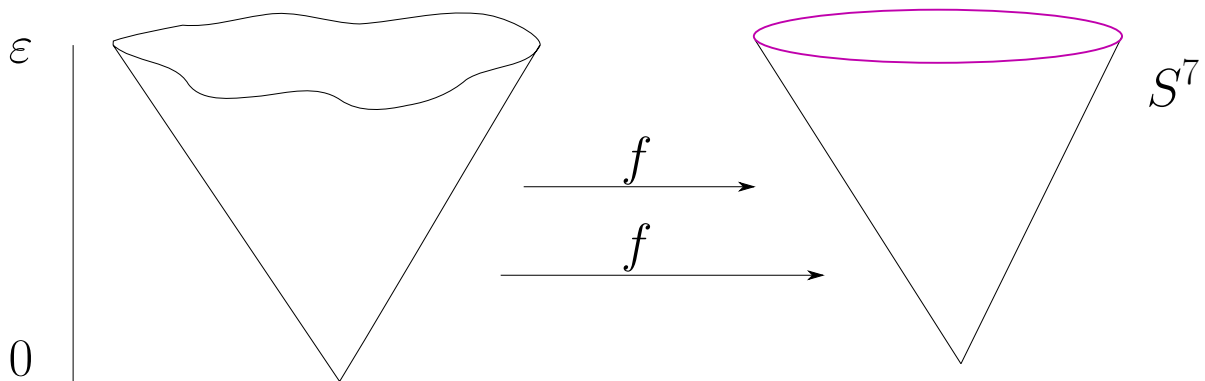
## 2 | Friday, January 15


**Remark 2.0.1:** Let

$$V := \{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\} \subseteq \mathbb{C}^5$$

$$S_\varepsilon := \{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\}.$$

Then  $V_k \cap S_\varepsilon \cong S^7$  is a homeomorphism, and taking  $k = 1, 2, \dots, 28$  yields the 28 smooth structures on  $S^7$ . Note that  $V_k$  is the cone over  $V_k \cap S_\varepsilon$ .



? Admits a smooth structure, and  $\bar{V}_k \subseteq \mathbb{CP}^5$  admits no smooth structure. 

<sup>2</sup>Locally admits a chart to  $\mathbb{C}^n/\Gamma$  for  $\Gamma$  a finite group.

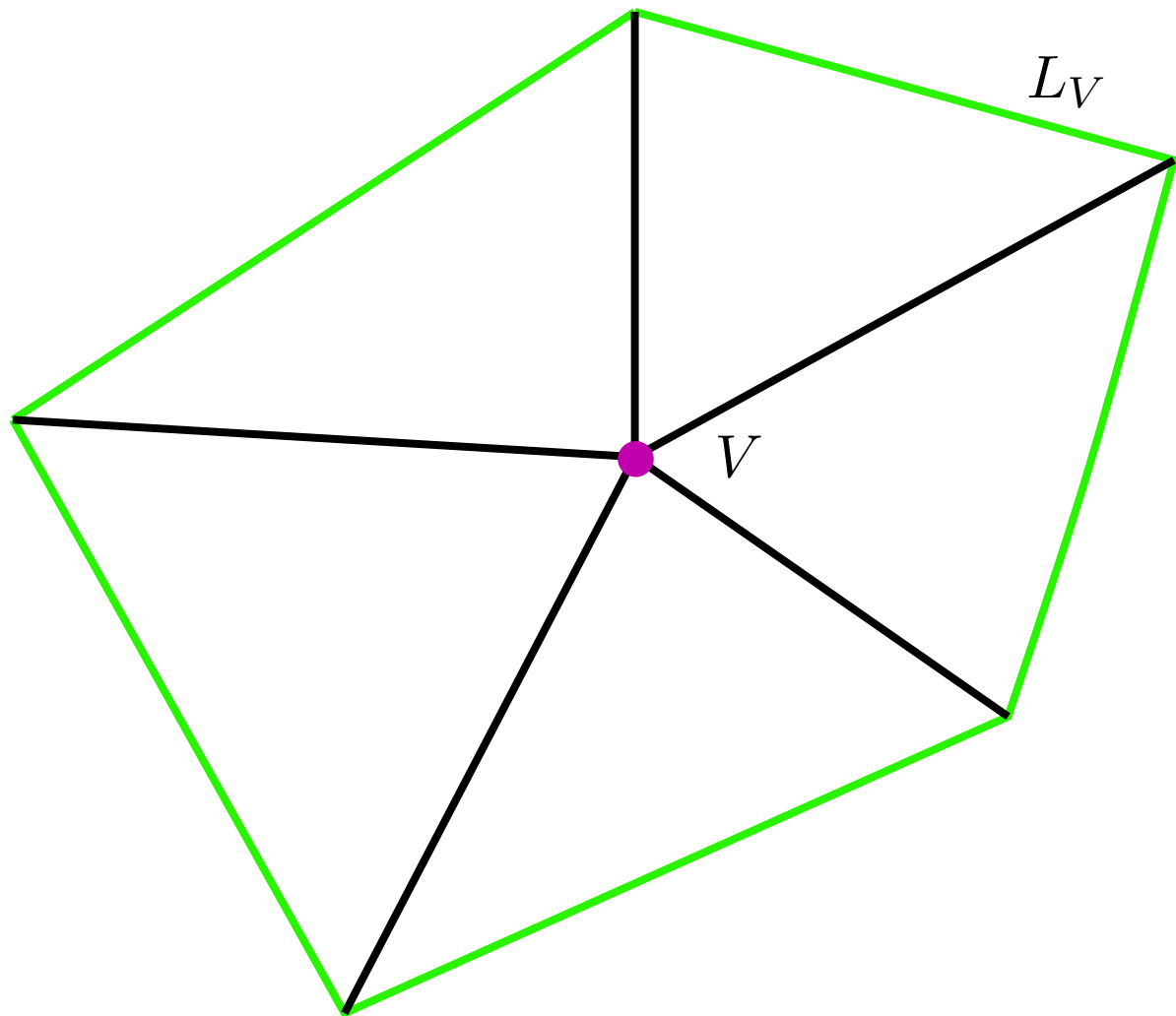


**Question 2.0.2**

Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

**Answer 2.0.3**

No! Given a simplicial complex, there is a notion of the **combinatorial link**  $L_V$  of a vertex  $V$ :



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a “PL triangulation” where the links are spheres.

**Remark 2.0.4:** What’s special in dimension 4? Recall the **Kirby-Siebenmann** invariant  $ks(x) \in H^4(X; \mathbb{Z}_2)$  for  $X$  a topological manifold where  $ks(X) = 0 \iff X$  admits a PL structure, with the caveat that  $\dim X \geq 5$ . We can use this to cook up an invariant of 4-manifolds.

**Definition 2.0.5** (Kirby-Siebenmann Invariant of a 4-manifold)

Let  $X$  be a topological 4-manifold, then

$$\text{ks}(X) := \text{ks}(X \times \mathbb{R}).$$

**Remark 2.0.6:** Recall that in  $\dim X \geq 7$ , every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\text{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} . \end{cases}$$

**Remark 2.0.7:**  $\text{ks}(X) \neq 0$  implies that  $X$  has no smooth structure, since  $X \times \mathbb{R}$  doesn't. Note that it was not known if this invariant was nonzero for a while!

**Remark 2.0.8:** Note that  $H^2(X; \mathbb{Z})$  admits a symmetric bilinear form  $Q_X$  defined by

$$\langle \alpha, \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where  $[X]$  is the fundamental class.

## 3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

**Theorem 3.0.1 (Freedman).**

If  $X, Y$  are compact oriented topological 4-manifolds, then  $X \cong Y$  are homeomorphic if and only if  $\text{ks}(X) = \text{ks}(Y)$  and  $Q_X \cong Q_Y$  are isometric, i.e. there exists an isometry

$$\varphi : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z}).$$

that preserves the two bilinear forms in the sense that  $\langle \varphi\alpha, \varphi\beta \rangle = \langle \alpha, \beta \rangle$ .

Conversely, every **unimodular** bilinear form appears as  $H^2(X; \mathbb{Z})$  for some  $X$ , i.e. the pairing induces a map

$$\begin{aligned} H^2(X; \mathbb{Z}) &\rightarrow H^2(X; \mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

**Remark 3.0.2:** Note that preservation of a bilinear form is a stand-in for “being an element of the orthogonal group”, where we only have a lattice instead of a full vector space.

**Remark 3.0.3:** There is a map  $H^2(X; \mathbb{Z}) \xrightarrow{PD} H_2(X; \mathbb{Z})$  from Poincaré, where we can think of elements in the latter as closed surfaces  $[\Sigma]$ , and

$$\langle \Sigma_1, \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \cap \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

**Definition 3.0.4 (Signature)**

The **signature** of a topological 4-manifold is the signature of  $Q_X$ , where we note that  $Q_X$  is a symmetric nondegenerate bilinear form on  $H^2(X; \mathbb{R})$  and for some  $a, b$

$$(H^2(X; \mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where  $a$  is the number of +1s appearing in the matrix and  $b$  is the number of -1s. This is  $\mathbb{R}^{ab}$  where  $e_i^2 = 1, i = 1 \cdots a$  and  $e_i^2 = -1, i = a + 1, \cdots b$ , and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is  $a - b$ , the dimension of the positive-definite space minus the dimension of the negative-definite space.

**Theorem 3.0.5 (Rokhlin's Theorem).**

Suppose  $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$  and  $\alpha \in H^2(X; \mathbb{Z})$  and  $X$  a simply connected **smooth** 4-manifold. Then 16 divides  $\text{sig}(X)$ .

**Remark 3.0.6:** Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

**Theorem 3.0.7 (Donaldson).**

Let  $X$  be a smooth simply-connected 4-manifold. If  $a = 0$  or  $b = 0$ , then  $Q_X$  is diagonalizable and there exists an orthonormal basis of  $H^2(X; \mathbb{Z})$ .

**Remark 3.0.8:** This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

### 3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

**Remark 3.1.1:** Last time we showed  $\mathbb{R}^1$  had a unique smooth structure, so now we'll do this for  $\mathbb{R}^2$ . The strategy of solving a differential equation, we'll now sketch the proof.

**Definition 3.1.2** (Riemannian Metrics)

A **Riemannian metric**  $g \in \text{Sym}^2 T^*X$  for  $X$  a smooth manifold is a metric on every  $T_pX$  given by

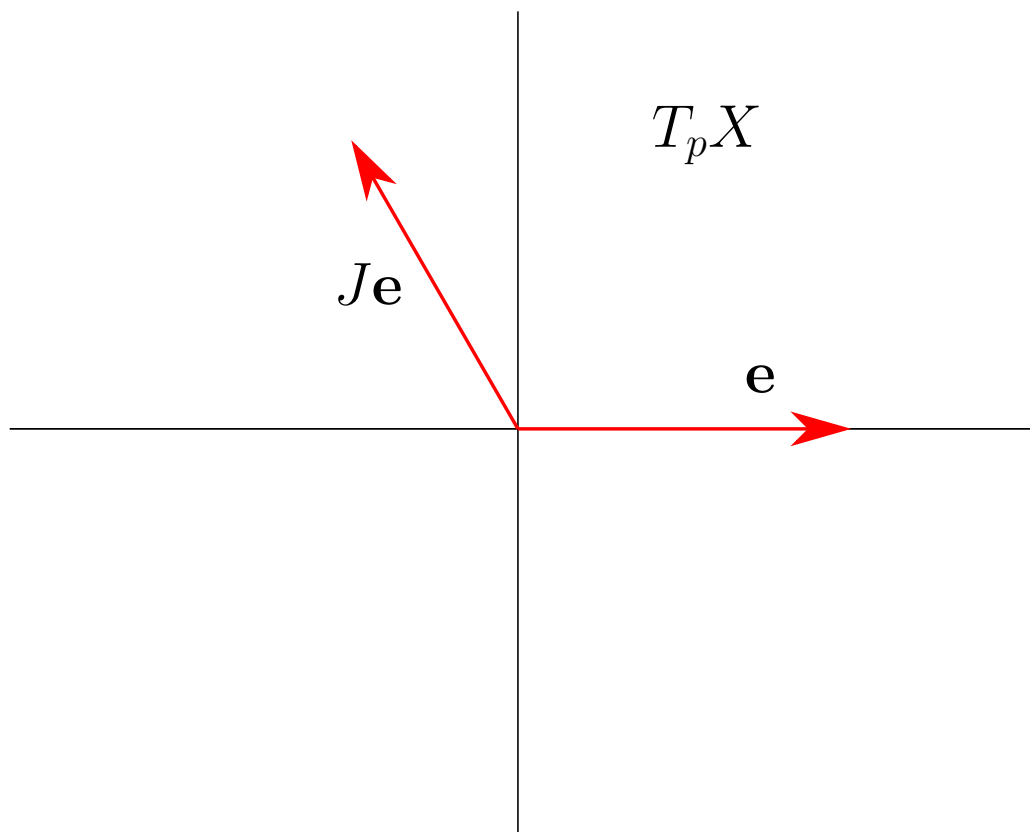
$$g_p : T_pX \times T_pX \rightarrow \mathbb{R}$$

$$g(v, v) \geq 0, g(v, v) = 0 \iff v = 0.$$

**Definition 3.1.3** (Almost complex structure)

An **almost complex structure** is a  $J \in \text{End}(TX)$  such that  $J^2 = -\text{id}$ .

**Remark 3.1.4:** Let  $e \in T_pX$  and  $e \neq 0$ , then if  $X$  is a surface then  $\{e, Je\}$  is a basis of  $T_pX$ .



This is a basis because if  $Je$  and  $e$  are parallel, then ??? In particular,  $J_p$  is determined by a point in  $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

### 3.1.1 Sketch of Proof

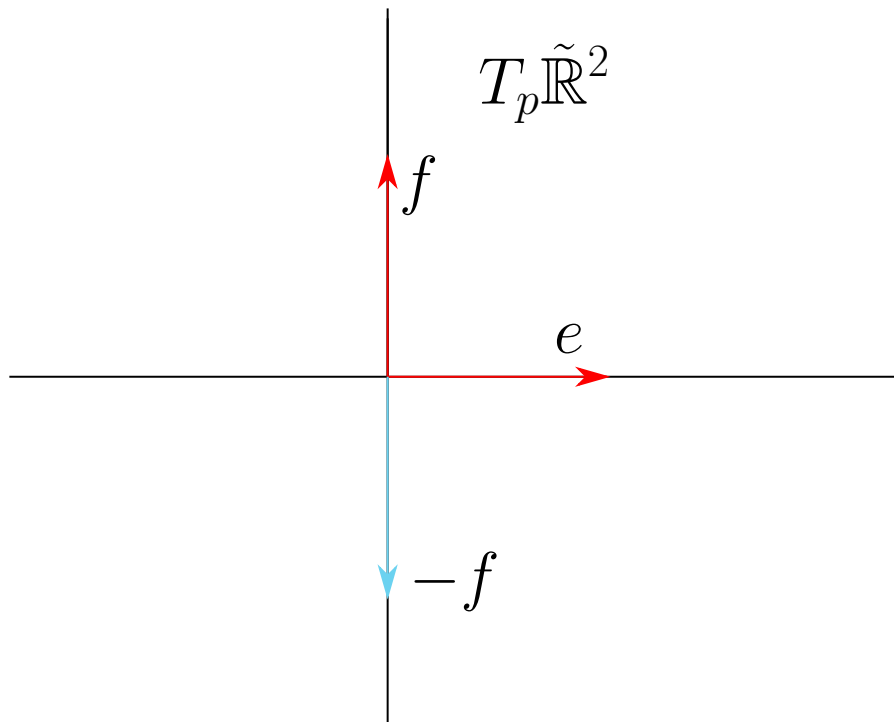
Let  $\tilde{\mathbb{R}}^2$  be an exotic  $\mathbb{R}^2$ .

**Step 1** Choose a metric on  $\tilde{\mathbb{R}}^2$   $g := \sum f_I g_I$  with  $g_I$  metrics on coordinate charts  $U_i$  and  $f_i$  a partition of unity.

**Step 2** Find an almost complex structure on  $\tilde{\mathbb{R}}^2$ . Choosing an orientation of  $\tilde{\mathbb{R}}^2$ ,  $g$  defines a unique almost complex structure  $J_p e := f \in T_p \tilde{\mathbb{R}}^2$  such that

- $g(e, e) = g(f, f)$
- $g(e, f) = 0$ .
- $\{e, f\}$  is an oriented basis of  $T_p \tilde{\mathbb{R}}^2$

This is because after choosing  $e$ , there are two orthogonal vectors, but only one choice yields an *oriented* basis.



**Step 3** We then apply a theorem:

**Theorem 3.1.5(?).**

Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts  $\varphi_i : U_i \rightarrow \mathbb{C}$  such that  $J$  is multiplication by  $i$ .

So  $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$ , and  $(\tilde{\mathbb{R}}^2, J)$  is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to  $\mathbb{D}$  or  $\mathbb{C}$ , both of which are diffeomorphic to  $\mathbb{R}^2$ .

*See the Newlander-Nirenberg theorem, a result in complex geometry.*

## 4 | Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

### 4.1 Sheaves

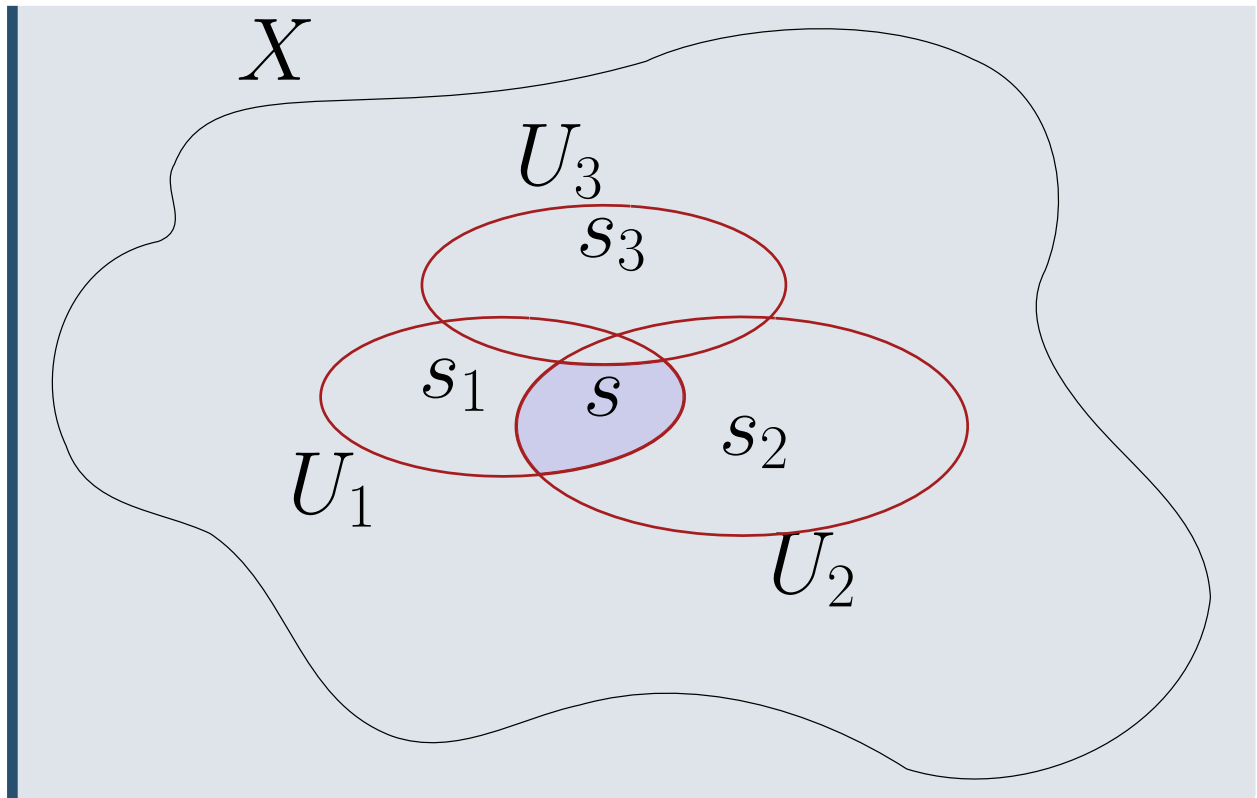
#### Definition 4.1.1 (Presheaves and Sheaves)

Recall that if  $X$  is a topological space, a **presheaf** of abelian groups  $\mathcal{F}$  is an assignment  $U \rightarrow \mathcal{F}(U)$  of an abelian group to every open set  $U \subseteq X$  together with a restriction map  $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  for any inclusion  $V \subseteq U$  of open sets. This data has to satisfying certain conditions:

- a.  $\mathcal{F}(\emptyset) = 0$ , the trivial abelian group.
- b.  $\rho_{UU} : \mathcal{F}(U) \rightarrow \mathcal{F}(U) = \text{id}_{\mathcal{F}(U)}$
- c. Compatibility if restriction is taken in steps:  $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$ .

We say  $\mathcal{F}$  is a **sheaf** if additionally:

- d. Given  $s_i \in \mathcal{F}(U_i)$  such that  $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$  implies that there exists a unique  $s \in \mathcal{F}(\bigcup_i U_i)$  such that  $\rho_{U_i}(s) = s_i$ .



**Example 4.1.2(?):** Let  $X$  be a topological manifold, then  $\mathcal{F} := C^0(\cdot, \mathbb{R})$  the set of continuous functionals form a sheaf. We have a diagram

$$\begin{array}{ccc}
 U & \xrightarrow{\mathcal{F}} & C^0(U; \mathbb{R}) \\
 \uparrow & & \downarrow \text{restrict cts. functions} \\
 V & \xrightarrow{\mathcal{F}} & C^0(V; \mathbb{R})
 \end{array}$$

[Link to diagram](#)

Property (d) holds because given sections  $s_i \in C^0(U_i; \mathbb{R})$  agreeing on overlaps, so  $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ , there exists a unique  $s \in C^0(\bigcup_i U_i; \mathbb{R})$  such that  $s|_{U_i} = s_i$  for all  $i$  – continuous functions glue.

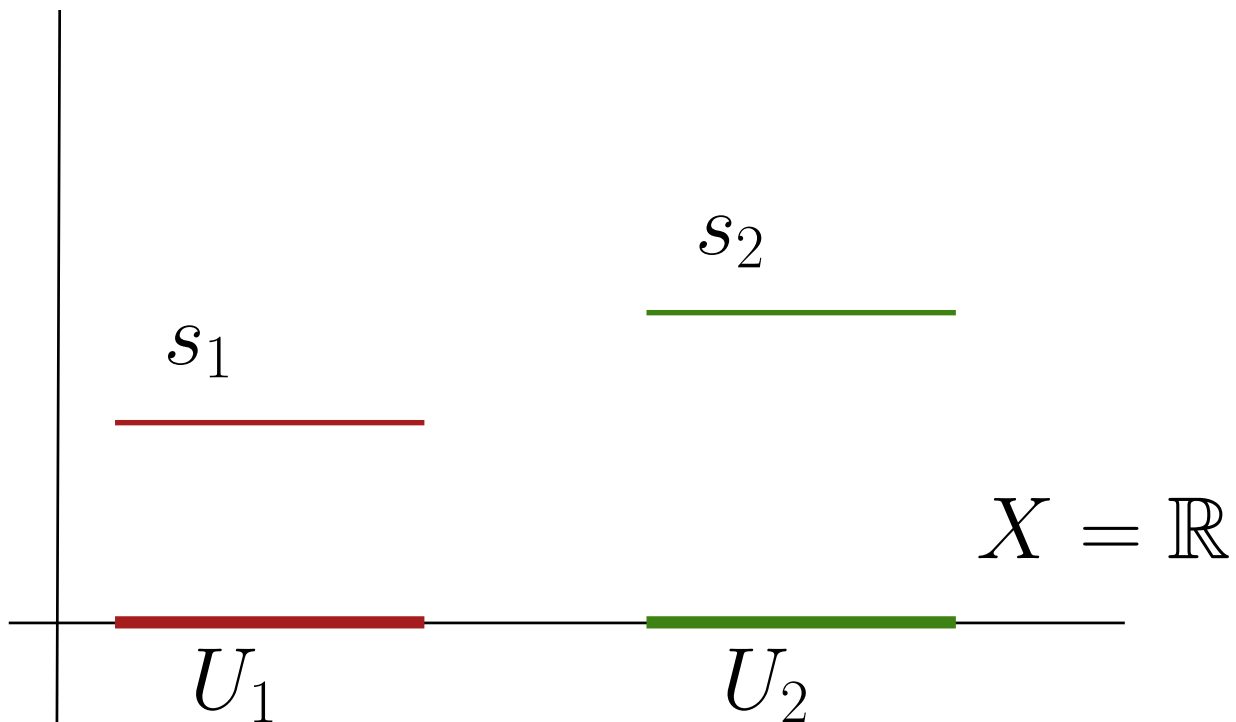
**Remark 4.1.3:** Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote  $\mathcal{O}$ . For example,  $\mathcal{O} := C^0(\cdot; \mathbb{R})$  for topological manifolds, and  $\mathcal{O} := C^\infty(\cdot; \mathbb{R})$  is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set  $\mathcal{O}$  to be the sheaf of holomorphic functions.

**Example 4.1.4(Locally Constant Sheaves):** Let  $A \in \mathbf{Ab}$  be an abelian group, then  $\underline{A}$  is the

sheaf defined by setting  $\underline{A}(U)$  to be the locally constant functions  $U \rightarrow A$ . E.g. let  $X \in \mathbf{Mfd}_{\mathbf{Top}}$  be a topological manifold, then  $\underline{\mathbb{R}}(U) = \mathbb{R}$  if  $U$  is connected since locally constant  $\implies$  globally constant in this case.

**Warning 4.1.5**

Note that the presheaf of constant functions doesn't satisfy (d)! Take  $\mathbb{R}$  and a function with two different values on disjoint intervals:



Note that  $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$  since the intersection is empty, but there is no constant function that restricts to the two different values.

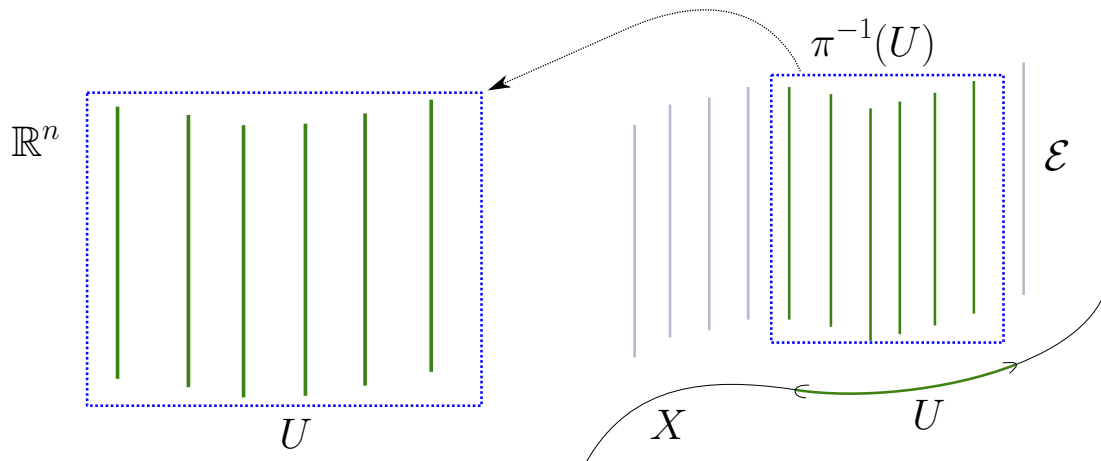
## 4.2 Bundles

**Remark 4.2.1:** Let  $\pi : \mathcal{E} \rightarrow X$  be a **vector bundle**, so we have local trivializations  $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$  where we take either  $Y = \mathbb{R}, \mathbb{C}$ , such that  $h_v \circ h_u^{-1}$  preserves the fibers of  $\pi$  and acts linearly on each fiber of  $Y \times (U \cap V)$ . Define

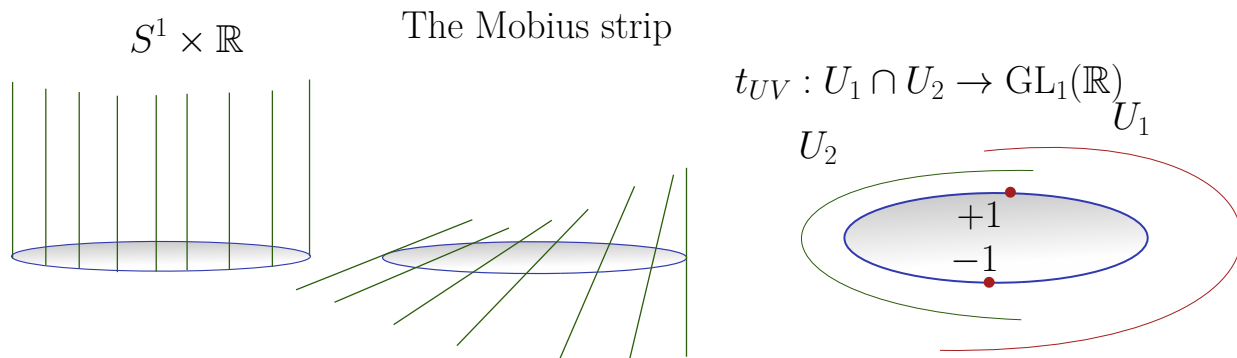
$$t_{UV} : U \cap V \rightarrow \mathrm{GL}_d(Y)$$

where we require that  $t_{UV}$  is continuous, smooth, complex-analytic, etc depending on the context.





**Example 4.2.2 (Bundles over  $S^1$ ):** There are two  $\mathbb{R}^1$  bundles over  $S^1$ :



Note that the Möbius bundle is not trivial, but can be locally trivialized.

**Remark 4.2.3:** We abuse notation:  $\mathcal{E}$  is also a sheaf, and we write  $\mathcal{E}(U)$  to be the set of sections  $s : U \rightarrow \mathcal{E}$  where  $s$  is continuous, smooth, holomorphic, etc where  $\pi \circ s = \text{id}_U$ . I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

**Example 4.2.4(?):** The trivial line bundle gives the sheaf  $\mathcal{O} : \text{maps } U \xrightarrow{s} U \times Y \text{ for } Y = \mathbb{R}, \mathbb{C}$  such that  $\pi \circ s = \text{id}$  are the same as maps  $U \rightarrow Y$ .

#### Definition 4.2.5 ( $\mathcal{O}$ -modules)

An  $\mathcal{O}$ -module is a sheaf  $\mathcal{F}$  such that  $\mathcal{F}(U)$  has an action of  $\mathcal{O}(U)$  compatible with restriction.

**Example 4.2.6(?):** If  $\mathcal{E}$  is a vector bundle, then  $\mathcal{E}(U)$  has a natural action of  $\mathcal{O}(U)$  given by  $f \curvearrowright s := fs$ , i.e. just multiplying functions.

**Example 4.2.7 (Non-example):** The locally constant sheaf  $\underline{\mathbb{R}}$  is not an  $\mathcal{O}$ -module: there isn't natural action since the sections of  $\mathcal{O}$  are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

**Definition 4.2.8** (Morphisms of Sheaves)

A **morphism** of sheaves  $\mathcal{F} \rightarrow \mathcal{G}$  is a group morphism  $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$  for all opens  $U \subseteq X$  such that the diagram involving restrictions commutes:


$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_{UV} & & \downarrow \rho_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

**Example 4.2.9** (*An  $\mathcal{O}$ -module that is not a vector bundle.*): Let  $X = \mathbb{R}$  and define the skyscraper sheaf at  $p \in \mathbb{R}$  as

$$\mathbb{R}_p(U) := \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases}$$

The  $\mathcal{O}(U)$ -module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\rightarrow \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since  $\mathbb{R}_p(U)$  is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of  $p$  for which the sheaf makes trivial assignments. 

**Example 4.2.10** (*of morphisms*): Let  $X = \mathbb{R} \in \mathbf{Mfd}_{\text{Sm}}$  viewed as a smooth manifold, then multiplication by  $x$  induces a morphism of structure sheaves:

$$\begin{aligned} (x \cdot) : \mathcal{O} &\rightarrow \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$


for any  $x \in \mathcal{O}(U)$ , noting that  $x \cdot s \in \mathcal{O}(U)$  again.

**Exercise 4.2.11** (?)

Check that  $\ker \varphi$  is naturally a sheaf and  $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open  $U$  we have  $(x \cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$  is injective. Taking the cokernel  $\text{coker}(x \cdot)$  as a presheaf, this assigns to  $U$  the quotient presheaf  $\mathcal{O}(U)/x\mathcal{O}(U)$ , which turns out to be equal to  $\mathbb{R}_0$ . So  $\mathcal{O} \rightarrow \mathbb{R}_0$  by restricting to the value at 0, and there is an exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{(x \cdot)} \mathcal{O} \rightarrow \mathbb{R}_0 \rightarrow 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles. 

# 5 | Lecture 4 (Friday, January 22)

## 5.1 The Exponential Exact Sequence

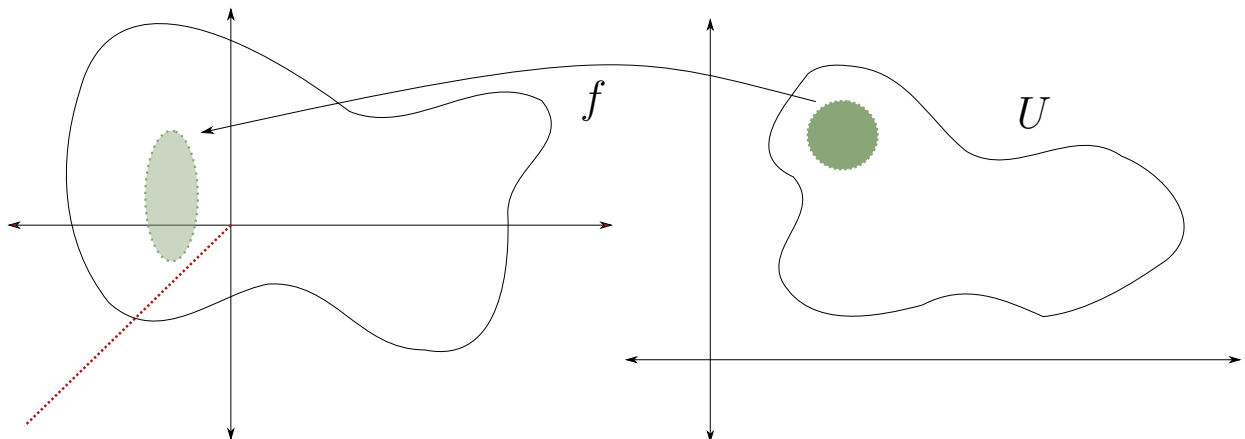
Let  $X = \mathbb{C}$  and consider  $\mathcal{O}$  the sheaf of holomorphic functions and  $\mathcal{O}^\times$  the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map  $\exp : \mathcal{O} \rightarrow \mathcal{O}^\times$ , the **exponential map**, which is the data  $\exp(U) : \mathcal{O}(U) \rightarrow \mathcal{O}^\times(U)$  on every open  $U$  given by  $f \mapsto e^f$ . There is a kernel sheaf  $2\pi i\mathbb{Z}$ , and we get an exact sequence

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \rightarrow \text{coker}(\exp) \rightarrow 0.$$

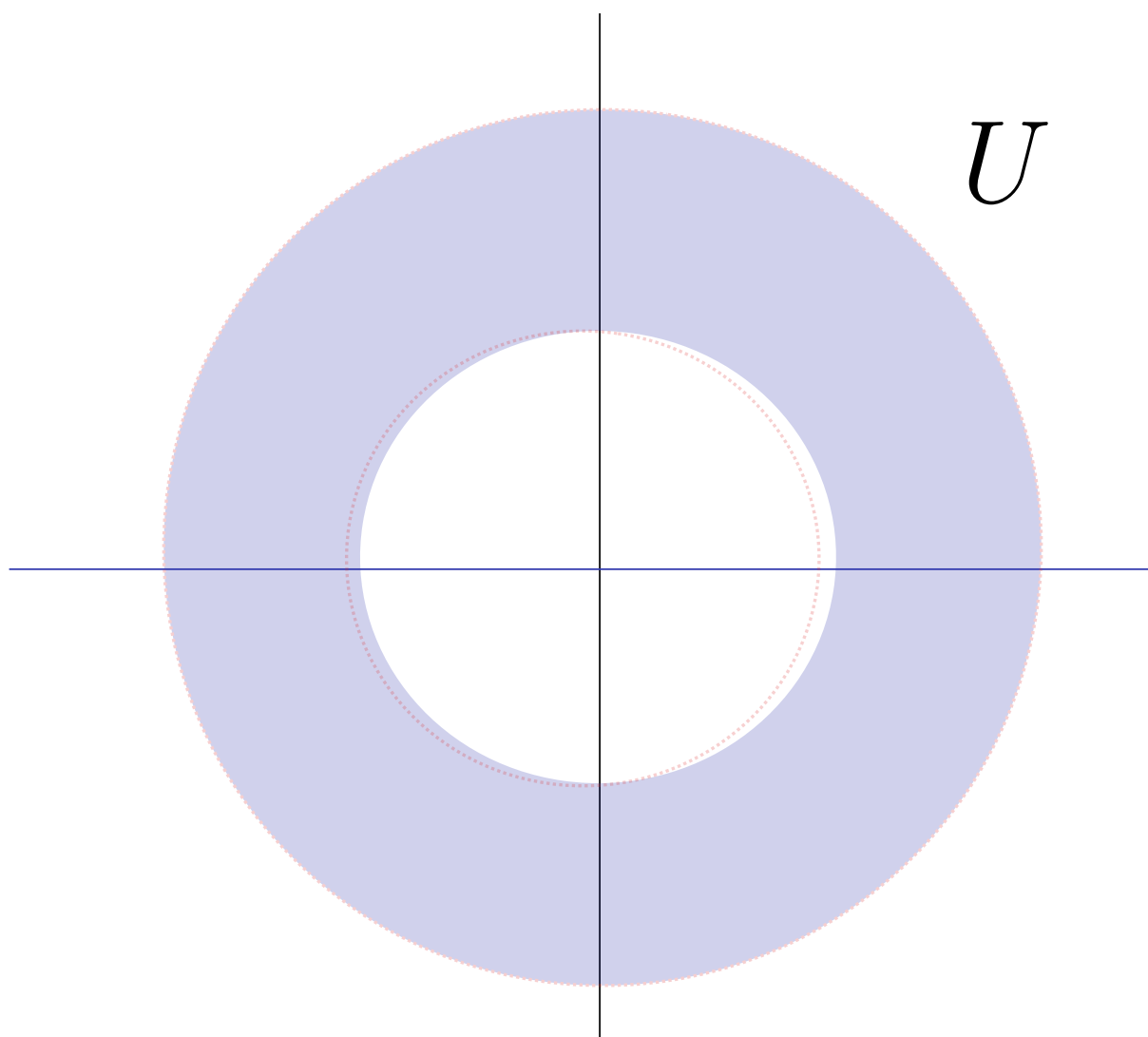
### Question 5.1.1

What is the cokernel sheaf here?

Let  $U$  be a contractible open set, then we can identify  $\mathcal{O}^\times(U)/\exp(\mathcal{O}(U)) = 1$ .



Any  $f \in \mathcal{O}^\times(U)$  has a logarithm, say by taking a branch cut, since  $\pi_1(U) = 0 \implies \log f$  has an analytic continuation. Consider the annulus  $U$  and the function  $z \in \mathcal{O}^\times(U)$ , then  $z \notin \exp(\mathcal{O}(U))$  – if  $z = e^f$  then  $f = \log(z)$ , but  $\log(z)$  has monodromy on  $U$ :



Thus on any sufficiently small open set,  $\text{coker}(\exp) = 1$ . This is only a presheaf: there exists an open cover of the annulus for which  $z|_{U_i}$ , and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to  $z$ , which is nontrivial.

**Exercise 5.1.2 (?)**

Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property  $\text{Hom}_{\text{Presheaf}}(\mathcal{G}, \mathcal{F}^{\text{Presheaf}}) = \text{Hom}_{\text{Sheaf}}(\mathcal{G}, \mathcal{F}^{\text{Sh}})$  for any sheaf  $\mathcal{G}$ .

**Definition 5.1.3 (Global Sections Sheaf)**

The **global sections** sheaf of  $\mathcal{F}$  on  $X$  is given by  $H^0(X; \mathcal{F}) = \mathcal{F}(X)$ .

**Example 5.1.4(?):**

- $C^\infty(X) = H^0(X, C^\infty)$  are the smooth functions on  $X$
- $VF(X) = H^0(X; T)$  are the smooth vector fields on  $X$  for  $T$  the tangent bundle

- If  $X$  is a complex manifold then  $\mathcal{O}(X) = H^0(X; \mathcal{O})$  are the globally holomorphic functions on  $X$ .
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$  are ??

**Remark 5.1.5:** Given vector bundles  $V, W$ , we have constructions  $V \oplus W, V \otimes W, V^\vee, \text{Hom}(V, W) = V^\vee \otimes W, \text{Sym}^n V, \bigwedge^p V$ , and so on. Some of these work directly for sheaves:

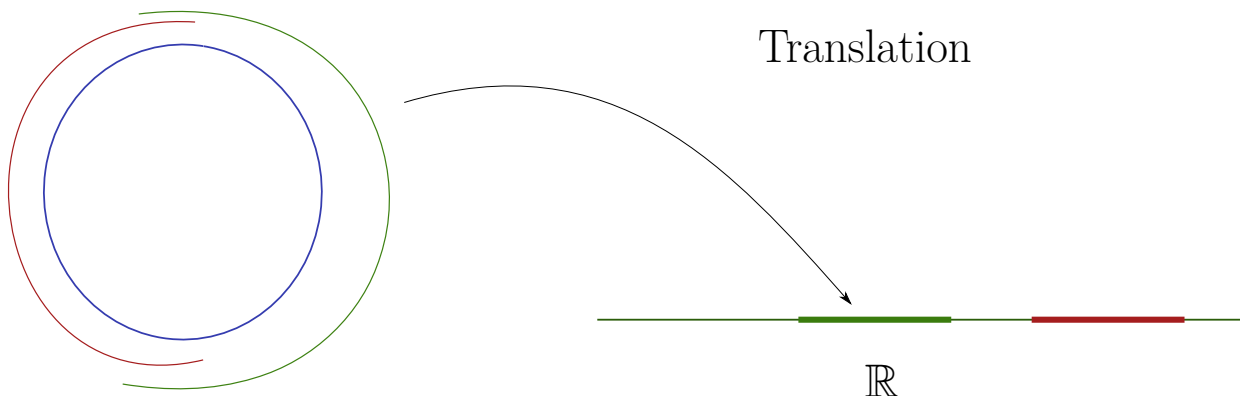
- $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs  $\mathcal{H}\text{om}(V, W)$  we only get presheaves, so we need to sheafify.

**⚠ Warning 5.1.6**

$\text{Hom}(V, W)$  will denote the *global* homomorphisms  $\mathcal{H}\text{om}(V, W)(X)$ , which is a sheaf.

**Example 5.1.7(?):** Let  $X^n \in \mathbf{Mfd}_{\text{sm}}$  and let  $\Omega^p$  be the sheaf of smooth  $p$ -forms, i.e.  $\bigwedge^p T^\vee$ , i.e.  $\Omega^p(U)$  are the smooth  $p$  forms on  $U$ , which are locally of the form  $\sum f_{i_1, \dots, i_p}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_p}$  where the  $f_{i_1, \dots, i_p}$  are smooth functions.

**Example 5.1.8(Sub-example):** Take  $X = S^1$ , writing this as  $\mathbb{R}/\mathbb{Z}$ , we have  $\Omega^1(X) \ni dx$ . There are two coordinate charts which differ by a translation on their overlaps, and  $dx(x+c) = dx$  for  $c$  a constant:



**Exercise 5.1.9(?)**

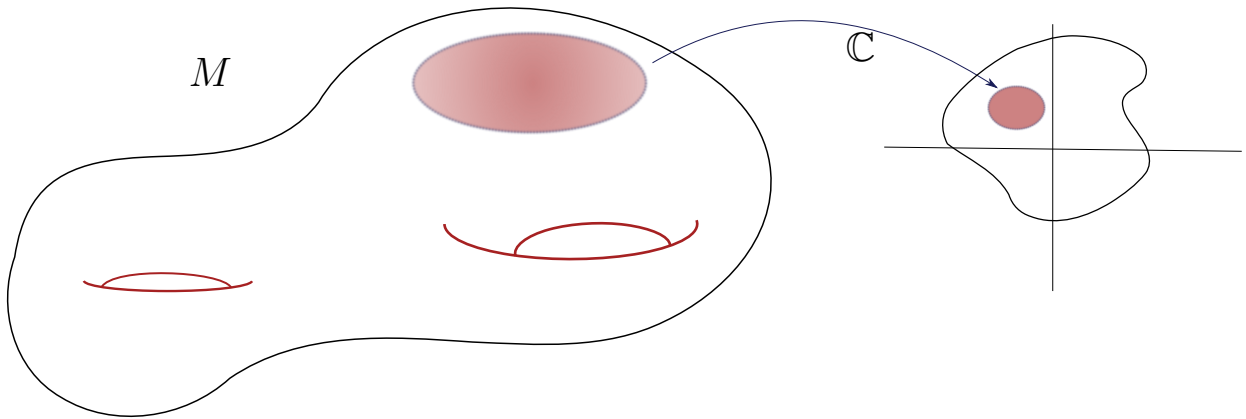
Check that on a torus,  $dx_i$  is a well-defined 1-form.

**Remark 5.1.10:** Note that there is a map  $d : \Omega^p \rightarrow \Omega^{p+1}$  where  $\omega \mapsto d\omega$ .

**⚠ Warning 5.1.11**

$d$  is **not** a map of  $\mathcal{O}$ -modules:  $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$ , where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since  $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$ , making  $d$  a sheaf morphism.

Let  $X \in \mathbf{Mfd}_{\mathbb{C}}$ , we'll use the fact that  $TX$  is complex-linear and thus a  $\mathbb{C}$ -vector bundle.



**Remark 5.1.12 (Subtlety 1):** Note that  $\Omega^p$  for complex manifolds is  $\bigwedge^p T^\vee$ , and so if we want to view  $X \in \mathbf{Mfd}_\mathbb{R}$  we'll write  $X_\mathbb{R}$ .  $TX_\mathbb{R}$  is then a real vector bundle of rank  $2n$ .

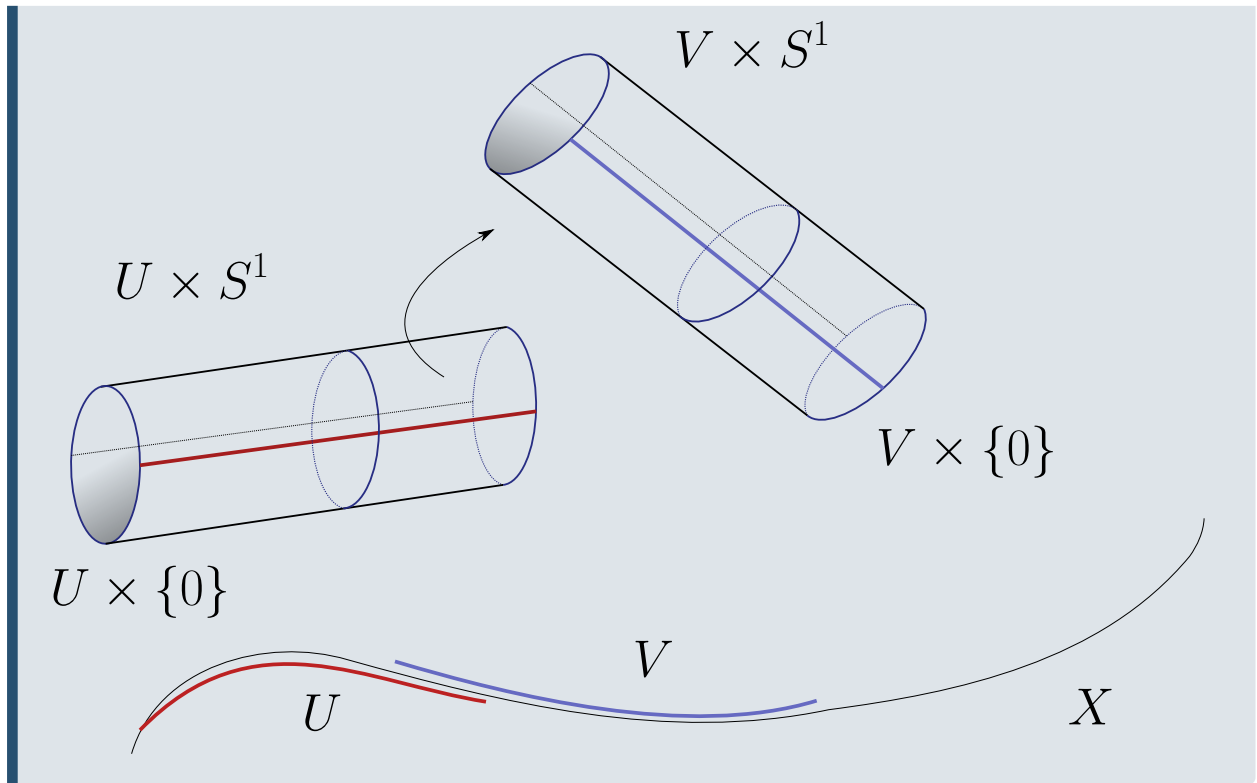
**Remark 5.1.13 (Subtlety 2):**  $\Omega^p$  will denote *holomorphic*  $p$ -forms, i.e. local expressions  $\sum f_I(z_1, \dots, z_n) \bigwedge dz_I$ . For example,  $e^z dz \in \Omega^1(\mathbb{C})$  but  $z\bar{z}dz$  is not, where  $dz = dx + idy$ . We'll use a different notation when we allow the  $f_I$  to just be smooth:  $A^{p,0}$ , the sheaf of  $(p,0)$ -forms. Then  $z\bar{z}dz \in A^{1,0}$ .

**Remark 5.1.14:** Note that  $T^\vee X_\mathbb{R} \otimes_\mathbb{C} = A^{1,0} \oplus A^{0,1}$  since there is a unique decomposition  $\omega = f dz + g d\bar{z}$  where  $f, g$  are smooth. Then  $\Omega^d X_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$ . Note that  $\Omega^p \neq A^{p,q}$  and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover  $\dim \Omega^p(X) < \infty$ , whereas  $\Omega^1$  is infinite-dimensional.

## 6 | Principal $G$ -Bundles and Connections (Monday, January 25)

### Definition 6.0.1 (Principal Bundles)

Let  $G$  be a (possibly disconnected) Lie group. Then a **principal  $G$ -bundle**  $\pi : P \rightarrow X$  is a space admitting local trivializations  $h_u : \pi^{-1}(U) \rightarrow G \times U$  such that the transition functions are given by left multiplication by a continuous function  $t_{UV} : U \cap V \rightarrow G$ .



**Remark 6.0.2:** Setup: we'll consider  $TX$  for  $X \in \mathbf{Mfd}_{\text{Sm}}$ , and let  $g$  be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \rightarrow \mathbb{R},$$

a symmetric bilinear form with  $g_p(u, v) \geq 0$  with equality if and only if  $v = 0$ .

**Definition 6.0.3** (The Frame Bundle)

Define  $\text{Frame}_p(X) := \{\text{bases of } T_p X\}$ , and  $\text{Frame}X := \bigcup_{p \in X} \text{Frame}_p X$ .

**Remark 6.0.4:** More generally,  $\text{Frame}\mathcal{E}$  can be defined for any vector bundle  $\mathcal{E}$ , so  $\text{Frame}X := \text{Frame}TX$ . Note that  $\text{Frame}X$  is a principal  $\text{GL}_n(\mathbb{R})$ -bundle where  $n := \text{rank}(\mathcal{E})$ . This follows from the fact that the transition functions are fiberwise in  $\text{GL}_n(\mathbb{R})$ , so the transition functions are given by left-multiplication by matrices.

**Remark 6.0.5 (Important):** A principal  $G$ -bundle admits a  $G$ -action where  $G$  acts by *right* multiplication:

$$\begin{aligned} P \times G &\rightarrow P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point:** the actions of left and right multiplication commute.

**Definition 6.0.6** (Orthogonal Frame Bundle)

The **orthogonal frame bundle** of a vector bundle  $\mathcal{E}$  equipped with a metric  $g$  is defined as  $\text{OFrame}_p \mathcal{E} := \{\text{orthonormal bases of } \mathcal{E}_p\}$ , also written  $O_r(\mathbb{R})$  where  $r := \text{rank}(\mathcal{E})$ .

**Remark 6.0.7:** The fibers  $P_x \rightarrow \{x\}$  of a principal  $G$ -bundle are naturally **torsors** over  $G$ , i.e. a set with a free transitive  $G$ -action.

**Definition 6.0.8** (?)

Let  $\mathcal{E} \rightarrow X$  be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \rightarrow \mathbb{C}.$$

where  $h_p(v, \bar{v}) \geq 0$  with equality if and only if  $v = 0$ . Here we define  $\overline{\mathcal{E}_p}$  as the fiber of the complex vector bundle  $\overline{\mathcal{E}}$  whose transition functions are given by the complex conjugates of those from  $\mathcal{E}$ .

**Remark 6.0.9:** Note that  $\mathcal{E}, \overline{\mathcal{E}}$  are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e.  $L(cv) = \bar{c}L(v)$ , where the canonical example is

$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\mapsto (\bar{z}_1, \dots, \bar{z}_n). \end{aligned}$$

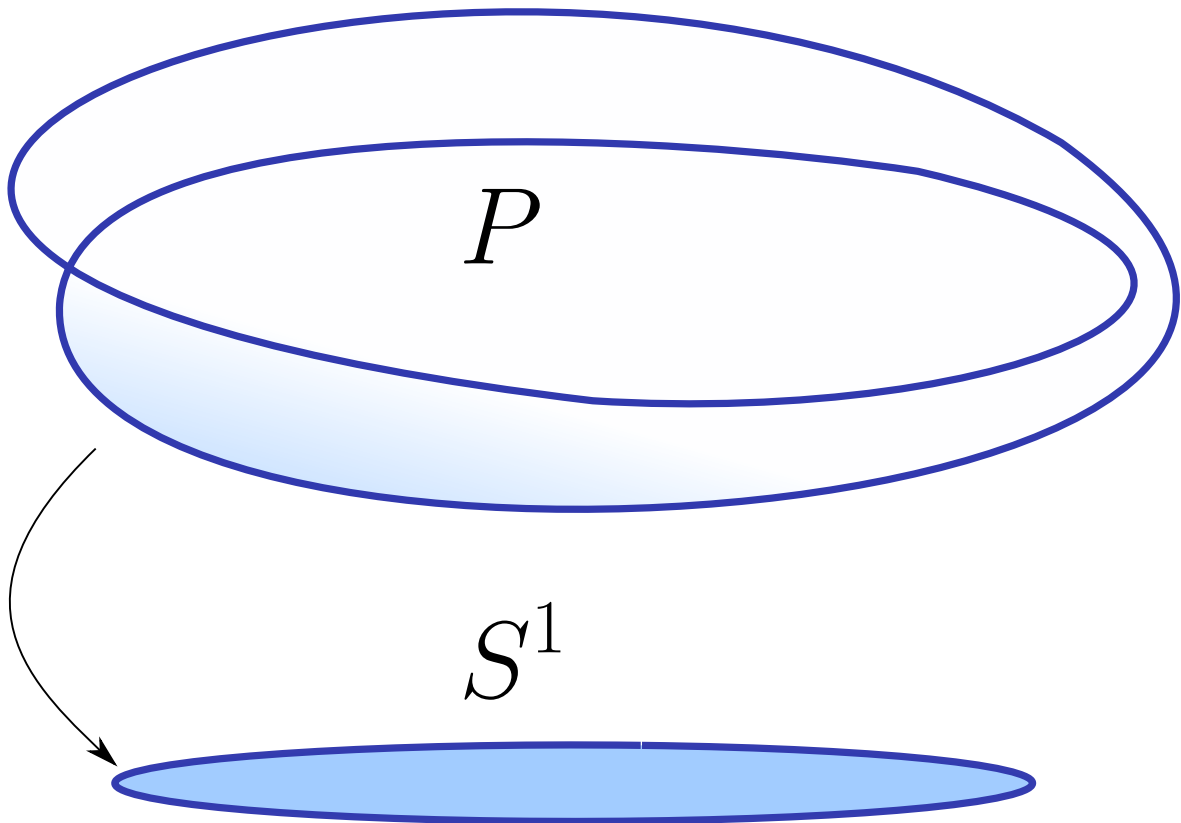
**Definition 6.0.10** (Unitary Frame Bundle)

We define the **unitary frame bundle**  $\text{UFrame}(\mathcal{E}) := \bigcup_p \text{UFrame}(\mathcal{E})_p$ , where at each point this is given by the set of orthogonal frames of  $\mathcal{E}_p$  given by  $(e_1, \dots, e_n)$  where  $h(e_i, \bar{e}_j) = \delta_{ij}$ .

**Remark 6.0.11:** This is a principal  $G$ -bundle for  $G = U_r(\mathbb{C})$ , the invertible matrices  $A_{/\mathbb{C}}$  satisfy  $A\bar{A}^t = \text{id}$ .

**Example 6.0.12 (of more principal bundles):** For  $G = \mathbb{Z}/2\mathbb{Z}$  and  $X = S^1$ , the Möbius band is a principal  $G$ -bundle:





**Example 6.0.13 (more principal bundles):** For  $G = \mathbb{Z}/2\mathbb{Z}$ , for any (possibly non-oriented) manifold  $X$  there is an **orientation principal bundle**  $P$  which is locally a set of orientations on  $U$ , i.e.

$$P := \{(x, O) \mid x \in X, O \text{ is an orientation of } T_p X\}.$$

Note that  $P$  is an oriented manifold,  $P \rightarrow X$  is a local isomorphism, and has a canonical orientation. (?) This can also be written as  $P = \text{Frame} X / \text{GL}_n^+(\mathbb{R})$ , since an orientation can be specified by a choice of  $n$  linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

**Definition 6.0.14** (Associated Bundles)

Let  $P \rightarrow X$  be a principal  $G$ -bundle and let  $G \rightarrow \text{GL}(V)$  be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \{(p, v) \mid p \in P, v \in V\} / \sim \quad \text{where } (p, v) \sim (pg, g^{-1}v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

**Example 6.0.15 (?)**: Note that  $\text{Frame}(\mathcal{E})$  is a  $\text{GL}_r(\mathbb{R})$ -bundle and the map  $\text{GL}_r(\mathbb{R}) \xrightarrow{\text{id}} \text{GL}(\mathbb{R}^r)$  is

a representation. At every fiber, we have  $G \times_G V = (p, v) / \sim$  where there is a unique representative of this equivalence class given by  $(e, pv)$ . So  $P \times_G V_p \rightarrow \{p\} \cong V_x$ .

### Exercise 6.0.16(?)

Show that  $\text{Frame}(\mathcal{E}) \times_{\text{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$ . This follows from the fact that the transition functions of  $P \times_G V$  are given by left multiplication of  $t_{UV} : U \cap V \rightarrow G$ , and so by the equivalence relation,  $\text{im } t_{UV} \in \text{GL}(V)$ .

**Remark 6.0.17:** Suppose that  $M^3$  is an oriented Riemannian 3-manifold. Then  $TM \rightarrow \text{Frame}(M)$  which is a principal  $\text{SO}(3)$ -bundle. The universal cover is the double cover  $\text{SU}(2) \rightarrow \text{SO}(3)$ , so can the transition functions be lifted? This shows up for spin structures, and we can get a  $\mathbb{C}^2$  bundle out of this.

## 7 | Wednesday, January 27

### 7.1 Bundles and Connections

#### Definition 7.1.1 (Connections)

Let  $\mathcal{E} \rightarrow X$  be a vector bundle, then a **connection** on  $\mathcal{E}$  is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens  $U$  with  $f \in \mathcal{O}(U)$  and  $s \in \mathcal{E}(U)$ . Note that this works in the category of complex manifolds, in which case  $\nabla$  is referred to as a **holomorphic connection**.

**Remark 7.1.2:** A connection  $\nabla$  induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\rightarrow \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge \omega + s \otimes d\omega. \end{aligned}$$

where  $\wedge : \Omega^p \otimes \Omega^1 \rightarrow \Omega^{p+1}$ . The standard example is

$$\begin{aligned} d : \mathcal{O} &\rightarrow \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

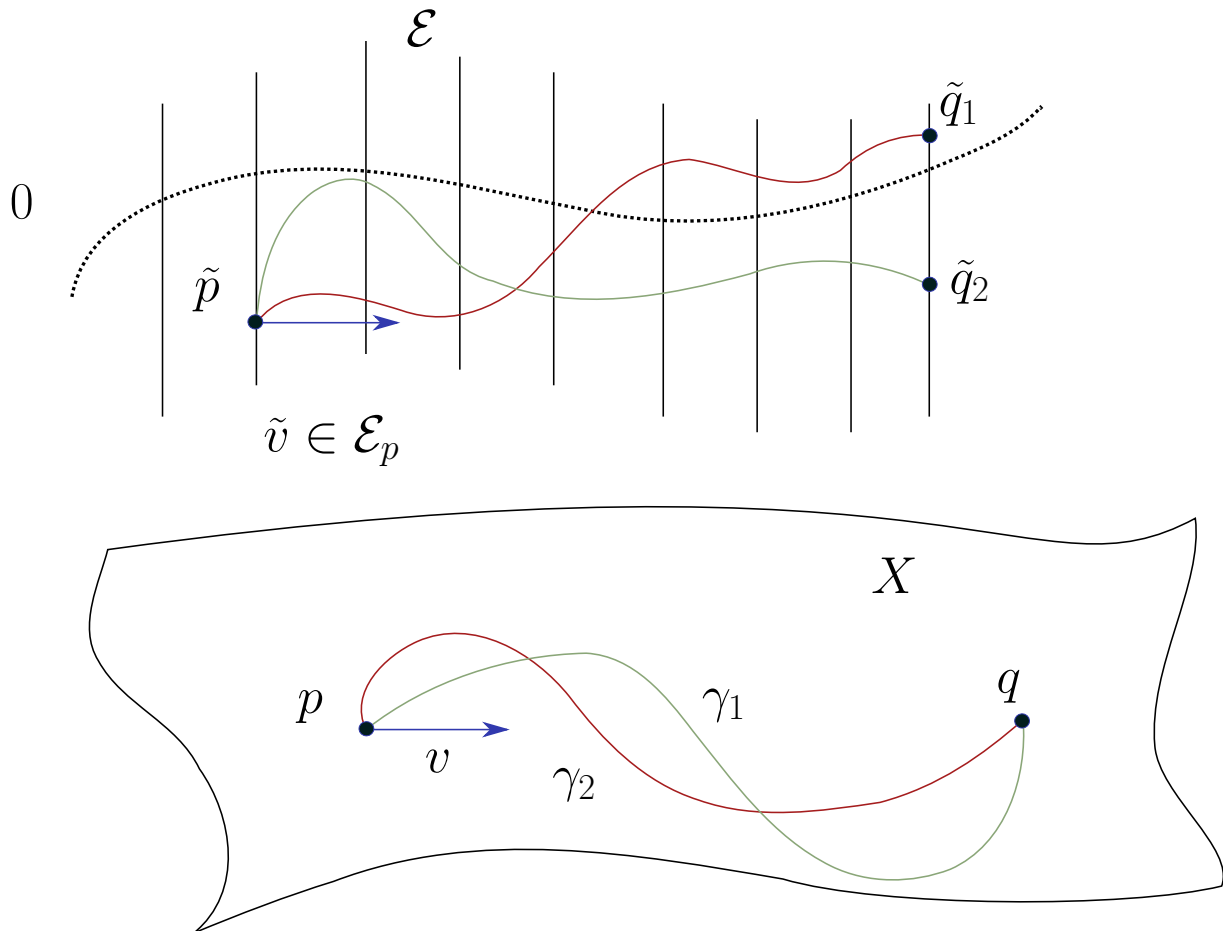
### Exercise 7.1.3 (?)

Prove that the *curvature* of  $\nabla$ , i.e. the map

$$F_{\nabla} := \nabla \circ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^2$$

is  $\mathcal{O}$ -linear, so  $F_{\nabla}(fs) = f\nabla \circ \nabla(s)$ . Use the fact that  $\nabla s \in \mathcal{E} \otimes \Omega^1$  and  $\omega \in \Omega^p$  and so  $\nabla s \otimes \omega \in \mathcal{E} \otimes \Omega^{p+1}$  and thus reassociating the tensor product yields  $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$ .

**Remark 7.1.4:** Why is this called a connection?

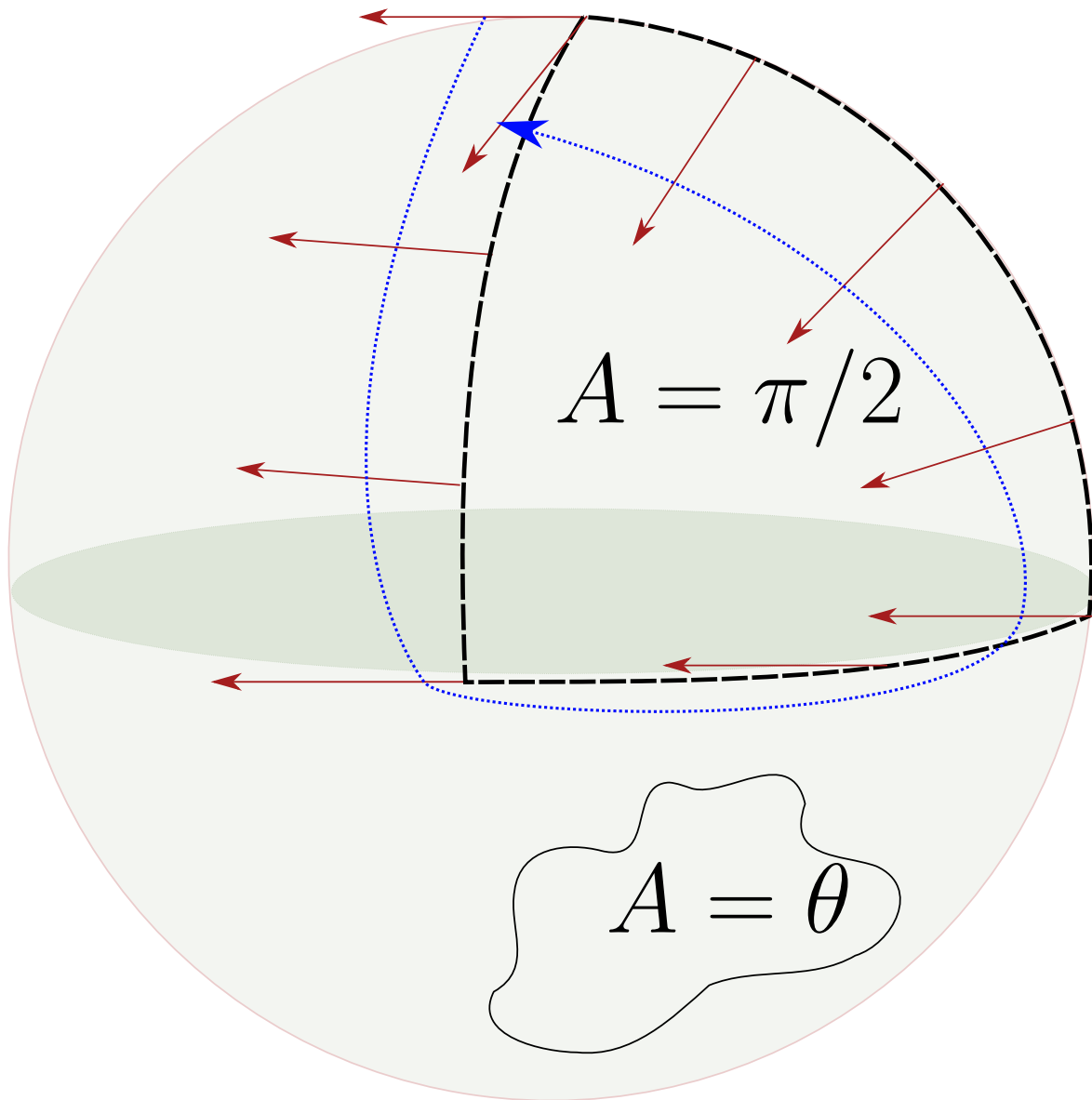


This gives us a way to transport  $v \in \mathcal{E}_p$  over a path  $\gamma$  in the base, and  $\nabla$  provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing  $\gamma'(t) \in T_{\gamma(t)}X$  with  $\Omega^1$ , yielding  $\nabla s = (\gamma'(t)) = s(\gamma(t))$  which are sections of  $\gamma$ .

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and  $F_{\nabla} = 0$  if and only if the parallel transport from  $p$  to  $q$  depends only on the homotopy class of  $\gamma$ .

*Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!*

**Example 7.1.5 (A classic example):** The Levi-Cevita connection  $\nabla^{LC}$  on  $TX$ , which depends on a metric  $g$ . Taking  $X = S^2$  and  $g$  is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

**Definition 7.1.6** (Flat Connection and Flat Sections)

A connection is **flat** if  $F_\nabla = 0$ . A section  $s \in \mathcal{E}(U)$  is **flat** if it is given by

$$L(U) := \left\{ s \in \mathcal{E}(U) \mid \nabla s = 0 \right\}.$$

**Exercise 7.1.7** (?)

Show that if  $\nabla$  is flat then  $L$  is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf  $\underline{\mathbb{C}}^d$ . Furthermore  $\text{rank}(L) = \text{rank}(\mathcal{E})$ .

**Remark 7.1.8:** Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take  $L \otimes_{\mathbb{R}} \mathcal{O}$ , and  $L \otimes 1$  form flat sections of a connection.

## 7.2 Sheaf Cohomology

**Definition 7.2.1** (?)

Let  $\mathcal{F}$  be a sheaf of abelian groups on a topological space  $X$ , and let  $\mathfrak{U} := \{U_i\} \rightrightarrows X$  be an open cover of  $X$ . Let  $U_{i_1, \dots, i_p} := U_{i_1} \cap U_{i_2} \cap \dots \cap U_{i_p}$ . Then the **Čech Complex** is defined as

$$C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1 < \dots < i_p} \mathcal{F}(U_{i_1, \dots, i_p})$$

with a differential

$$\begin{aligned} \partial^p : C_{\mathfrak{U}}^p(X, \mathcal{F}) &\rightarrow C_{\mathfrak{U}}^{p+1}(X, \mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \dots, i_p} := \prod_j (-1)^j \sigma_{i_0, \dots, \widehat{i_j}, \dots, i_p} \Big|_{U_{i_0, \dots, i_p}} \end{aligned}$$

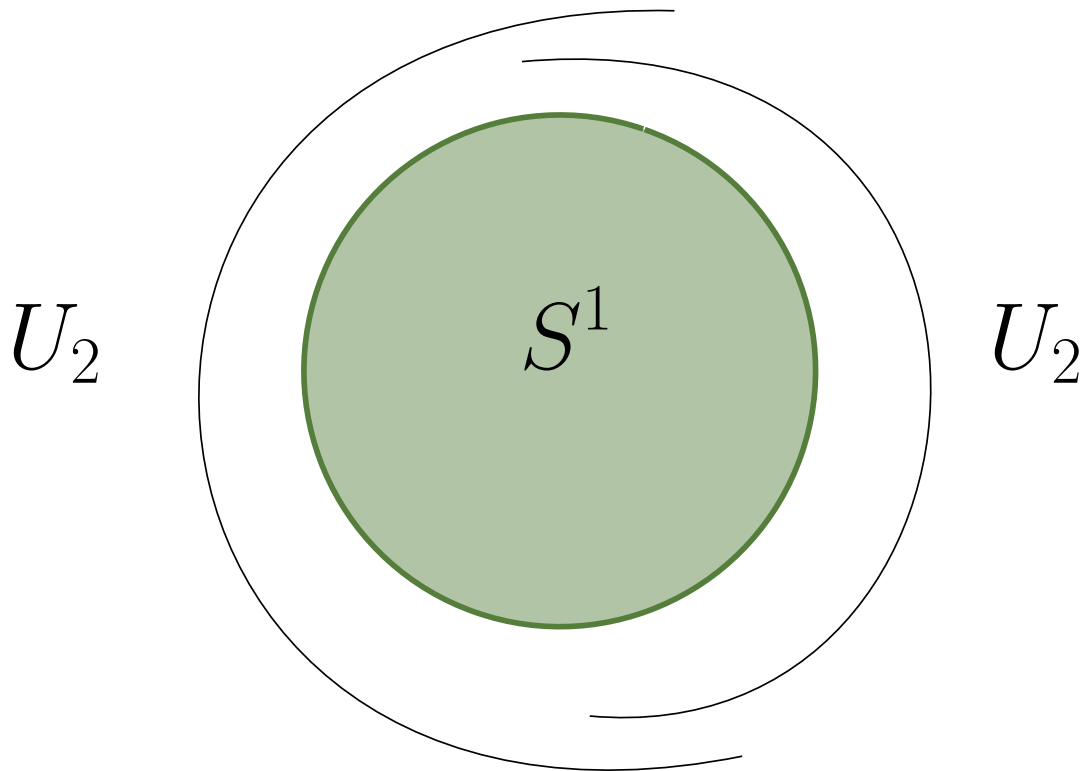
where we've defined this just on one given term in the product, i.e. a  $p$ -fold intersection.

**Exercise 7.2.2** (?)

Check that  $\partial^2 = 0$ .

**Remark 7.2.3:** The Čech cohomology  $H_{\mathfrak{U}}^p(X, \mathcal{F})$  with respect to the cover  $\mathfrak{U}$  is defined as  $\ker \partial^p / \text{im } \partial^{p-1}$ . It is a difficult theorem, but we write  $H^p(X, \mathcal{F})$  for the Čech cohomology for any sufficiently refined open cover when  $X$  is assumed paracompact.

**Example 7.2.4(?)**: Consider  $S^1$  and the constant sheaf  $\underline{\mathbb{Z}}$ :



ere we have

$$C^0(S^1, \mathbb{Z}) = \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) = \mathbb{Z} \oplus \mathbb{Z},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\substack{\text{double} \\ \text{intersections}}} \mathbb{Z}(U_{ij})\mathbb{Z}(U_{12}) = \mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}.$$

We then get

$$\begin{aligned} C^0(S^1, \mathbb{Z}) &\xrightarrow{\partial} C^1(S^1, \mathbb{Z}) \\ \mathbb{Z} \oplus \mathbb{Z} &\rightarrow \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields  $H^*(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$ .

## 8 | Sheaf Cohomology (Friday, January 29)

Last time: we defined the Čech complex  $C_{\mathfrak{U}}^p(X, \mathcal{F}) := \prod_{i_1, \dots, i_p} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_p})$  for  $\mathfrak{U} := \{U_i\}$  is an open cover of  $X$  and  $\mathcal{F}$  is a sheaf of abelian groups.

**Fact 8.0.1**

If  $\mathfrak{U}$  is a sufficiently fine cover then  $H_{\mathfrak{U}}^p(X, \mathcal{F})$  is independent of  $\mathfrak{U}$ , and we call this  $H^p(X; \mathcal{F})$ .

**Remark 8.0.2:** Recall that we computed  $H^p(S^1, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0, \dots]$ .

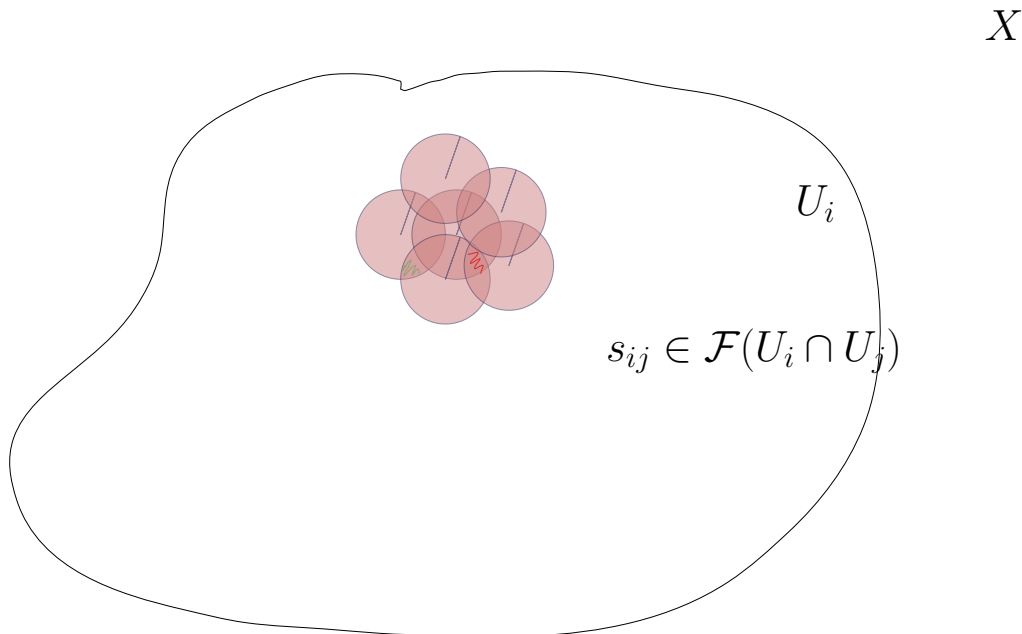
**Theorem 8.0.3(?)**

Let  $X$  be a paracompact and locally contractible topological space. Then  $H^p(X, \mathbb{Z}) \cong H_{\text{Sing}}^p(X, \mathbb{Z})$ . This will also hold more generally with  $\mathbb{Z}$  replaced by  $\underline{A}$  for any  $A \in \mathbf{Ab}$ .

**Definition 8.0.4** (Acyclic Sheaves)

We say  $\mathcal{F}$  is *acyclic* on  $X$  if  $H^{>0}(X; \mathcal{F}) = 0$ .

**Remark 8.0.5:** How to visualize when  $H^1(X; \mathcal{F}) = 0$ :



On the intersections, we have  $\text{im } \partial^0 = \{(s_i - s_j)_{ij} \mid s_i \in \mathcal{F}(U_i)\}$ , which are *cocycles*. We have  $C^1(X; \mathcal{F})$  are collections of sections of  $\mathcal{F}$  on every double overlap. We can check that  $\ker \partial^1 = \{(s_{ij}) \mid s_{ij} - s_{ik} + s_{jk} = 0\}$ , which is the cocycle condition. From the exercise from last class,  $\partial^2 = 0$ .

**Theorem 8.0.6 ((Important!)).**

Let  $X$  be a paracompact Hausdorff space and let

$$0 \rightarrow \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

be a SES of sheaves of abelian groups, i.e.  $\mathcal{F}_3 = \text{coker}(\varphi)$  and  $\varphi$  is injective. Then there is a LES in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \mathcal{F}_1) & \longrightarrow & H^0(X; \mathcal{F}_2) & \longrightarrow & H^0(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & H^1(X; \mathcal{F}_1) & \longrightarrow & H^1(X; \mathcal{F}_2) & \longrightarrow & H^1(X; \mathcal{F}_3) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

**Example 8.0.7(?)**: For  $X$  a manifold, we can define a map and its cokernel sheaf:

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2} \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0.$$

Using that cohomology of constant sheaves reduces to singular cohomology, we obtain a LES in homology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}}) & \longrightarrow & H^0(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}}) & \longrightarrow & H^1(X; \underline{\mathbb{Z}/2\mathbb{Z}}) \\ & & & & & \searrow & \\ & & \dots & & & & \end{array}$$

**Corollary 8.0.8(of theorem).**

Suppose  $0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} I_2 \xrightarrow{d_2} \dots$  is an exact sequence of sheaves, so on any sufficiently small set kernels equal images., and suppose  $I_n$  is acyclic for all  $n \geq 0$ . This is referred to as an **acyclic resolution**. Then the homology can be computed at  $H^p(X; \mathcal{F}) = \ker(I_p(X) \rightarrow I_{p+1}(X)) / \text{im}(I_{p-1}(X) \rightarrow I_p(X))$ .

*Note that locally having kernels equal images is different than satisfying this globally!*



*Proof (of corollary).*

This is a formal consequence of the existence of the LES. We can split the LES into a collection of SESs of sheaves:

$$\begin{array}{ll} 0 \rightarrow \mathcal{F} \rightarrow I_0 \xrightarrow{d_0} \operatorname{im}(d_0) \rightarrow 0 & \operatorname{im}(d_0) = \ker(d_1) \\ 0 \rightarrow \ker(d_1) \hookrightarrow I_1 \rightarrow I_1/\ker(d_1) = \operatorname{im}(d_1) & \operatorname{im}(d_1) = \ker(d_2) \\ & \cdot \end{array}$$

Note that these are all exact sheaves, and thus only true on small sets. So take the associated LESs. For the SES involving  $I_0$ , we obtain:

$$\begin{array}{ccccc} & & & & \dots \\ & & & \nearrow & \\ H^{p-1}(\mathcal{F}) & \xleftarrow{\quad} & H^{p-1}(\mathcal{I}_i) = 0 & \xrightarrow{\quad} & H^{p-1}(\operatorname{im}(\lceil \cdot \rceil)) \\ & & \searrow \cong & & \\ & & H^p(\mathcal{F}) & \xleftarrow{\quad} & \dots = 0 \end{array}$$

The middle entries vanish since  $I_*$  was assumed acyclic, and so we obtain  $H^p(\mathcal{F}) \cong H^{p-1}(\operatorname{im} d_0) \cong H^{p-1}(\ker d_1)$ . Now taking the LES associated to  $I_1$ , we get  $H^{p-1}(\ker d_1) \cong H^{p-2}(\operatorname{im} d_1)$ . Continuing this inductively, these are all isomorphic to  $H^p(\mathcal{F}) \cong H^0(\ker d_p)/d_{p-1}(H^0(I_{p-1}))$  after the  $p$ th step. ■

**Corollary 8.0.9 (of the previous corollary).**

Suppose  $\mathfrak{U} \rightrightarrows X$ , then if  $\mathcal{F}$  is acyclic on each  $U_{i_1, \dots, i_p}$ , then  $\mathfrak{U}$  is sufficiently fine to compute Čech cohomology, and  $H_{\mathfrak{U}}^p(X; \mathcal{F}) \cong H^p(X; \mathcal{F})$ .

*Proof (?)*.

See notes. ■

**Corollary 8.0.10 (of corollary).**

Let  $X \in \mathbf{Mfd}_{\setminus}$ , then  $H^p(X, \mathbb{R}) = H_{\mathrm{dR}}^p(X; \mathbb{R}R)$ .

*Proof (?)*.

Idea: construct an acyclic resolution of the sheaf  $\mathbb{R}$  on  $M$ . The following exact sequence works:

$$0 \rightarrow \underline{\mathbb{R}} \rightarrow \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \rightarrow \dots$$

So we start with locally constant functions, then smooth functions, then smooth 1-forms, and so on. This is an exact sequence of sheaves, but importantly, not exact on the total space. To check this, it suffices to show that  $\ker d^p = \operatorname{im} d^{p-1}$  on any contractible coordinate chart. In other words, we want to show that if  $d\omega = 0$  for  $\omega \in \Omega^p(\mathbb{R}^n)$  then  $\omega = d\alpha$  for some  $\alpha \in \Omega^{p-1}(\mathbb{R}^n)$ . This is true by integration! Using the previous corollary,  $H^p(X; \underline{\mathbb{R}}) = \ker(\Omega^p(X) \xrightarrow{d} \Omega^{p+1}(X)) / \operatorname{im}(\Omega^{p-1}(X) \xrightarrow{d} \Omega^p(X))$ . ■

*Check Hartshorne to see how injective resolutions line up with derived functors!*

## 9 | Monday, February 01

**Remark 9.0.1:** Last time  $\underline{\mathbb{R}}$  on a manifold  $M$  has a resolution by vector bundles:

$$0 \rightarrow \underline{\mathbb{R}} \hookrightarrow \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

This is an exact sequence of sheaves of any smooth manifold, since locally  $d\omega = 0 \implies \omega = d\alpha$  (by the *Poincaré d-lemma*). We also want to know that  $\Omega^k$  is an acyclic sheaf on a smooth manifold. ✍

### Exercise 9.0.2 (?)

Let  $X \in \mathbf{Top}$  and  $\mathcal{F} \in \mathbf{Sh}(\mathbf{Ab})_X$ . We say  $\mathcal{F}$  is **flasque** if and only if for all  $U \supseteq V$  the map  $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(V)$  is surjective. Show that  $\mathcal{F}$  is acyclic, i.e.  $H^i(X; \mathcal{F}) = 0$ . This can also be generalized with a POU.

**Example 9.0.3(?):** The function  $1/x \in \mathcal{O}(\mathbb{R} \setminus \{0\})$ , but doesn't extend to a continuous map on  $\mathbb{R}$ . So the restriction map is not surjective. ✍

**Remark 9.0.4:** Any vector bundle on a smooth manifold is acyclic. Using the fact that  $\Omega^k$  is acyclic and the above resolution of  $\underline{\mathbb{R}}$ , we can write  $H^k(X; \mathbb{R}) = \ker(d_k) / \operatorname{im} d_{k-1} := H_{dR}^k(X; \mathbb{R})$ . ✍

**Remark 9.0.5:** Now letting  $X \in \mathbf{Mfd}_{\mathbb{C}}$ , recalling that  $\Omega^p$  was the sheaf of holomorphic  $p$ -forms. Locally these are of the form  $\sum_{|I|=p} f_I(\mathbf{z}) dz^I$  where  $f_I(\mathbf{z})$  is holomorphic. There is a resolution

$$0 \rightarrow \Omega^p \rightarrow A^{p,0},$$

where in  $A^{p,0}$  we allowed also  $f_I$  are *smooth*. These are the same as bundles, but we view sections differently. The first allows only holomorphic sections, whereas the latter allows smooth sections. What can you apply to a smooth  $(p, 0)$  form to check if it's holomorphic? ✍

**Example 9.0.6 (?)**: For  $p = 0$ , we have

$$0 \rightarrow \mathcal{O} \rightarrow A^{0,0}.$$

where we have the sheaf of holomorphic functions mapping to the sheaf of smooth functions. We essentially want a version of checking the Cauchy-Riemann equations.

**Definition 9.0.7 (?)**

Let  $\omega \in A^{p,q}(X)$  where

$$d\omega = \sum \frac{\partial f_I}{\partial z_j} dz^j \wedge dz^I \wedge d\bar{z}^J + \sum_j \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}^j \wedge dz^I d\bar{z}^J := \partial + \bar{\partial}$$

with  $|I| = p, |J| = q$ .

**Example 9.0.8 (?)**: The function  $f(z) = z\bar{z} \in A^{0,0}(\mathbb{C})$  is smooth, and  $df = \bar{z}dz + z d\bar{z}$ . This can be checked by writing  $z^j = x^j + iy^j$  and  $\bar{z}^j = x^j - iy^j$ , and  $\frac{\partial}{\partial \bar{z}} g = 0$  if and only if  $g$  is holomorphic. Here we get  $\partial\omega \in A^{p+1,q}(X)$  and  $\bar{\partial} \in A^{p,q+1}(X)$ , and we can write  $d(z\bar{z}) = \partial(z\bar{z}) + \bar{\partial}(z\bar{z})$ .

**Definition 9.0.9 (Cauchy-Riemann Equations)**

Recall the Cauchy-Riemann equations:  $\omega$  is a holomorphic  $(p,0)$ -form on  $\mathbb{C}^n$  if and only if  $\bar{\partial}\omega = 0$ .

**Remark 9.0.10**: Thus to extend the previous resolution, we should take

$$0 \rightarrow \Omega^p \hookrightarrow A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} A^{p,2} \rightarrow \dots$$

The fact that this is exact is called the *Poincaré  $\bar{\partial}$ -lemma*.

**Remark 9.0.11**: There are no bump functions in the holomorphic world, and since  $\Omega^p$  is a holomorphic bundle, it may not be acyclic. However, the  $A^{p,q}$  are acyclic (since they are smooth vector bundles and thus admit POUs), and we obtain

$$H^q(X; \Omega^p) = \ker(\bar{\partial}_q) / \text{im}(\bar{\partial}_{q-1}).$$

Note the similarity to  $H_{\text{dR}}$ , using  $\bar{\partial}$  instead of  $d$ . This is called **Dolbeault cohomology**, and yields invariants of complex manifolds: the **Hodge numbers**  $h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X; \Omega^p)$ . These are analogies:

Smooth	Complex
$\mathbb{R}$	$\Omega^p$
$\Omega^k$	$A^{p,q}$
Betti numbers $\beta_k$	Hodge numbers $h^{p,q}$

Note the slight overloading of terminology here!

**Theorem 9.0.12 (Properties of Singular Cohomology).**

Let  $X \in \mathbf{Top}$ , then  $H_{\text{Sing}}^i(X; \mathbb{Z})$  satisfies the following properties:

- Functoriality: given  $f \in \text{Hom}_{\mathbf{Top}}(X, Y)$ , there is a pullback  $f^* : H^i(Y; \mathbb{Z}) \rightarrow H^i(X; \mathbb{Z})$ .
- The cap product: a pairing

$$H^i(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H_j(X; \mathbb{Z}) \rightarrow H_{j-i}(X; \mathbb{Z})$$

$$\varphi \otimes \sigma \mapsto \varphi \left( \sigma|_{\Delta_{0, \dots, j}} \right) \sigma|_{\Delta_{i, \dots, j}}.$$

This makes  $H_*$  a module over  $H^*$ .

- There is a ring structure induced by the cup product:

$$H^i(X; \mathbb{R}) \times H^j(X; \mathbb{R}) \rightarrow H^{i+j}(X; \mathbb{R}) \quad \alpha \cup \beta = (-1)^{ij} \beta \cup \alpha.$$

- Poincaré Duality: If  $X$  is an oriented manifold, there exists a fundamental class  $[X] \in H_n(X; \mathbb{Z}) \cong \mathbb{Z}$  and  $(\cdot) \cap X : H^i \rightarrow H_{n-i}$  is an isomorphism.

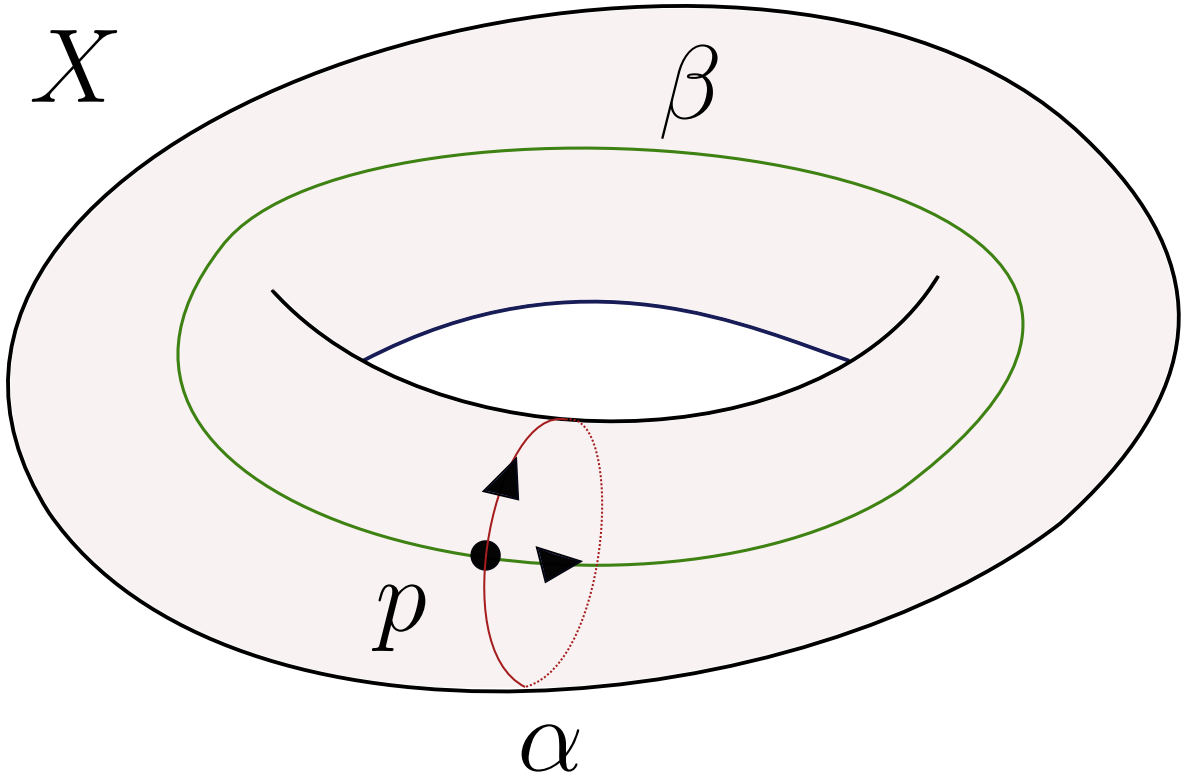
**Remark 9.0.13:** Let  $M \subset X$  be a submanifold where  $X$  is a smooth oriented  $n$ -manifold. Then  $M \hookrightarrow X$  induces a pushforward  $H_n(M; \mathbb{Z}) \xrightarrow{\iota_*} H_n(X; \mathbb{Z})$  where  $\sigma \mapsto \iota \circ \sigma$ . Using Poincaré duality, we'll identify  $H_{\dim M}(X; \mathbb{Z}) \rightarrow H^{\text{codim } M}(X; \mathbb{Z})$  and identify  $[M] = PD(\iota_*([M]))$ . In this case, if  $M \pitchfork N$  then  $[M] \cap [N] = [M \cap N]$ , i.e. the cap product is given by intersecting submanifolds.

**⚠ Warning 9.0.14**

This can't always be done! There are counterexamples where homology classes can't be represented by submanifolds.

# 10 | Wednesday, February 03

Consider an oriented surface, and take two oriented submanifolds

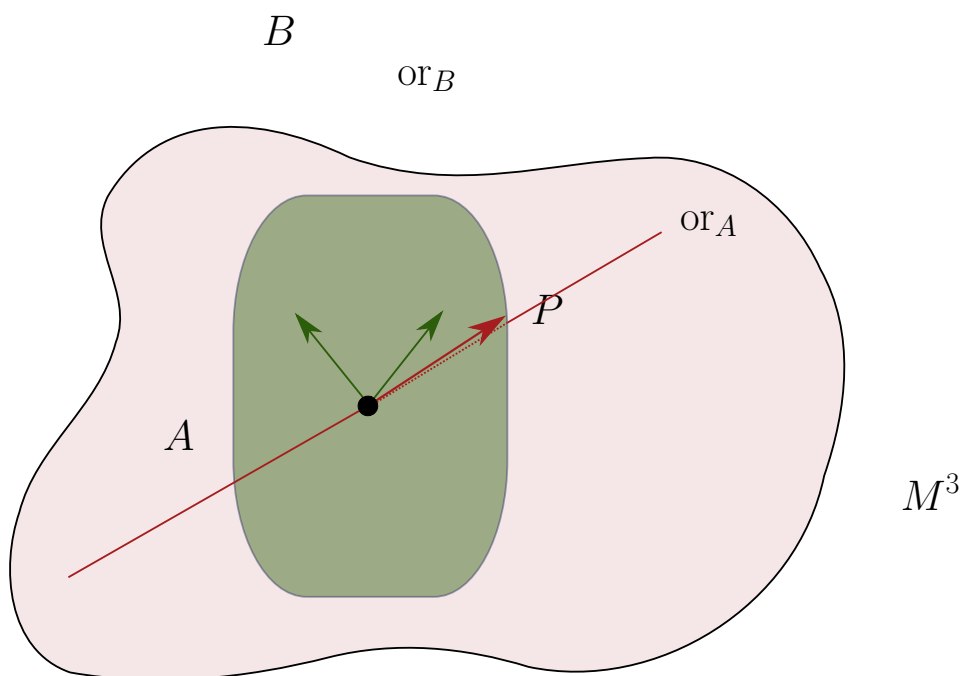


We can then take the fundamental classes of the submanifolds, say  $[\alpha], [\beta] \in H^1(X; \mathbb{Z}) \xrightarrow{PD} H^1(X, \mathbb{Z})$ . Here  $T_p\alpha \oplus T_p\beta = T_pX$ , since the intersections are transverse. Since  $\alpha, \beta$  are oriented, let  $\{e\}$  be a basis of  $T_p\alpha$  (up to  $\mathbb{R}^+$ ) and similarly  $\{f\}$  a basis of  $T_p\beta$ . We can then ask if  $\{e, f\}$  constitutes an *oriented* basis of  $T_pX$ . If so, we write  $\alpha \cdot_p \beta := +1$  and otherwise  $\alpha \cdot_p \beta = -1$ . We thus have

$$[\alpha] \smile [\beta] \in H^2(X; \mathbb{Z}) \xrightarrow{PD} H_0(X; \mathbb{Z}) = \mathbb{Z}$$

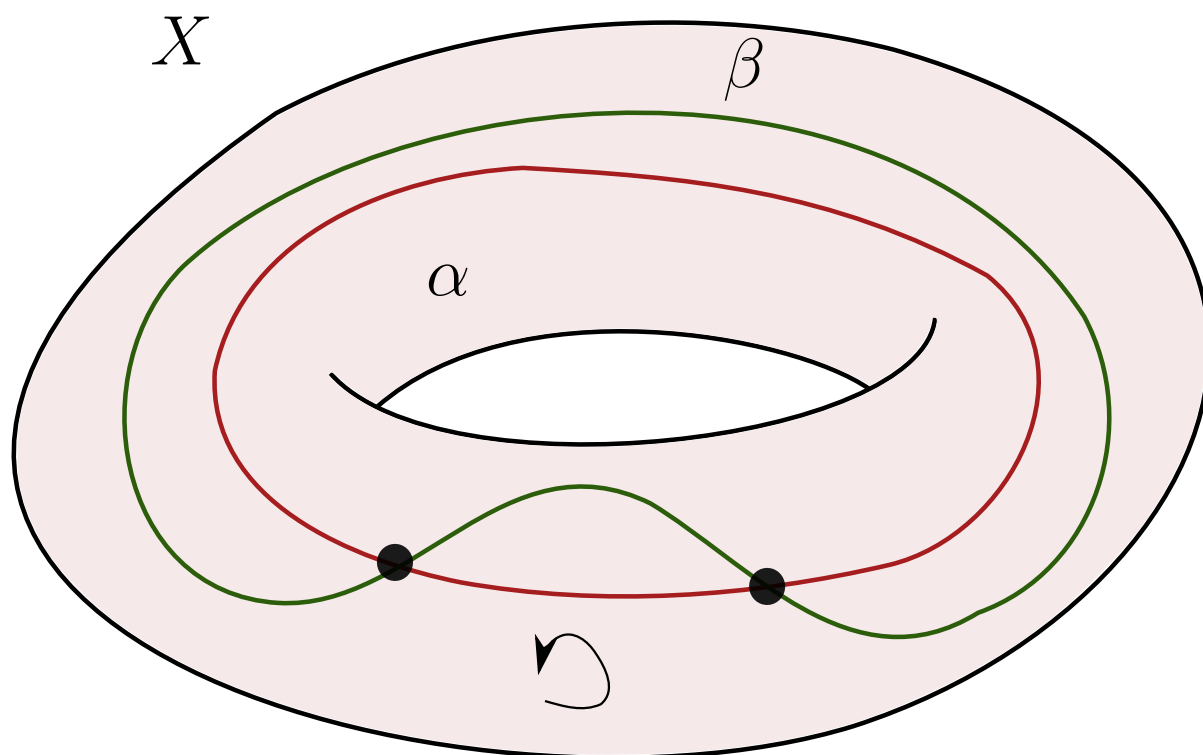
since  $X$  is connected. We can thus define the **intersection form**  $\alpha \cdot \beta := [\alpha] \smile [\beta]$ . In general if  $A, B$  are oriented transverse submanifolds of  $M$  which are themselves oriented, we'll have  $[A] \smile [B] = [A \cap B]$ . We need to be careful: how do we orient the intersection? This is given by comparing the orientations on  $A$  and  $B$  as before.

**Example 10.0.1(?)**: If  $\dim M = \dim A + \dim B$ , then any  $p \in A \cap B$  is oriented by comparing  $\{\text{or}_A, \text{or}_B\}$  to  $\text{or}_M$ .



Here it suffices to check that  $\{e, f_1, f_2\}$  is an oriented basis of  $T_p M$ .

**Example 10.0.2(?):** In this case,  $[\alpha] \smile [\beta] = 0$  and so  $\alpha \cdot \beta = 0$ :



**Remark 10.0.3:** Note that cohomology with  $\mathbb{Z}$  coefficients can be defined for any topological space, and Poincaré duality still holds.

**Remark 10.0.4:** We'll be considering  $M = M^4$ , smooth 4-manifolds. How to visualize: take a 3-manifold and cross it with time!

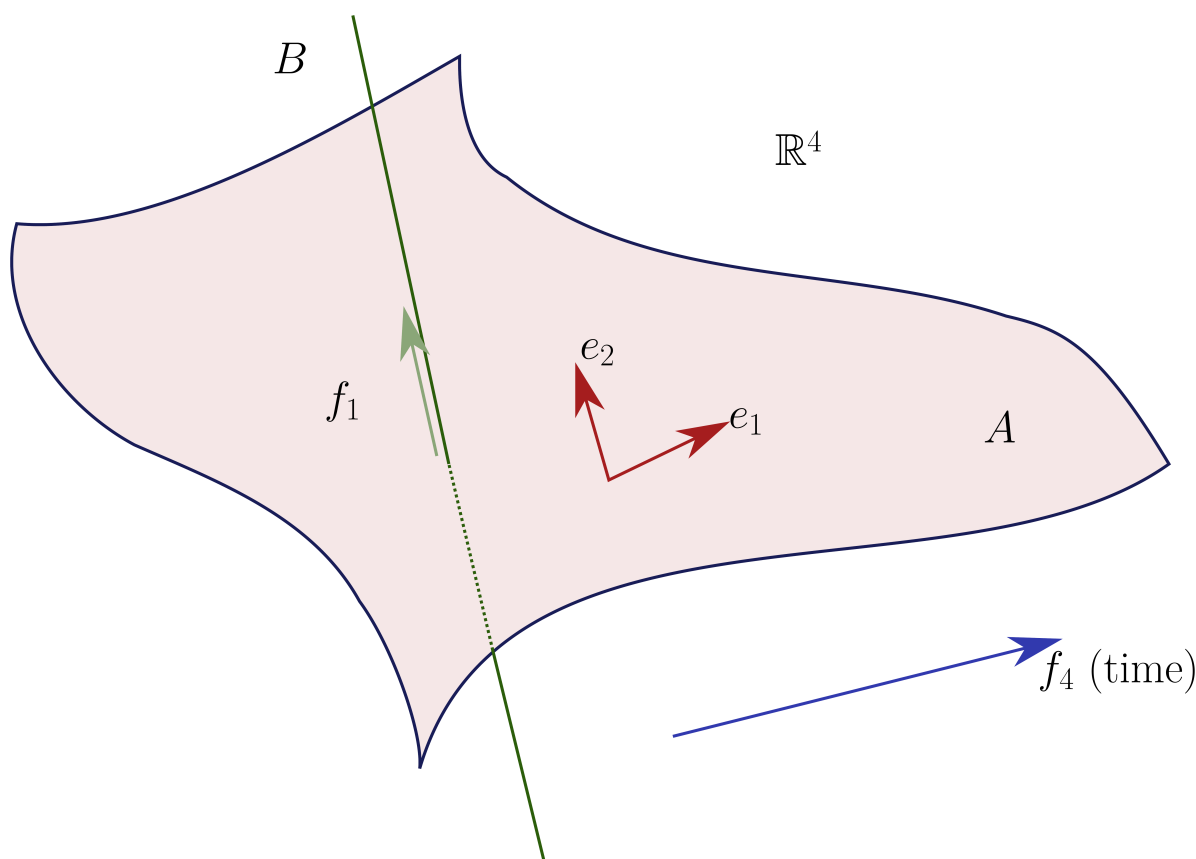


Figure 1: Picking one basis element in the time direction

Here  $\alpha$  is oriented in the “forward time” direction, and this is a surface at time  $t = 0$ . Where  $A \cdot B = +1$ , since  $\{e_1, e_2, f_1, f_2\} = \{e_x, e_y, e_z, e_t\}$  is a oriented basis for  $\mathbb{R}^4$ . For  $\alpha^2$ , switching the order of  $\alpha, \beta$  no longer yields an oriented basis, but in this case it is  $\alpha$  and  $A \cdot B = B \cdot A$ . This is because

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies \det(A) = -1 \qquad \det \begin{bmatrix} A & \\ & A \end{bmatrix} = 1.$$

**Remark 10.0.5:** Let  $M^{2n}$  be an oriented manifold, then the cup product yields a bilinear map  $H^n(M; \mathbb{Z}) \otimes H^n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$  which is symmetric when  $n$  is odd and antisymmetric (or symplectic) when  $n$  is even. This is a **perfect** (or **unimodular**) pairing (potentially after modding out by torsion) which realizes an isomorphism:

$$(H^n(M; \mathbb{Z})/\text{tors})^\vee \xrightarrow{\sim} H^n(M; \mathbb{Z})/\text{tors}$$

$$\alpha \smile \cdot \mapsto \alpha,$$

where the LHS are linear functionals on cohomology.



**Remark 10.0.6:** Recall the universal coefficients theorem:

$$H^i(X; \mathbb{Z})/\text{tors} \cong (H_i(X; \mathbb{Z})/\text{tors})^\vee.$$

The general theorem shows that  $H^i(X; \mathbb{Z})_{\text{tors}} = H_{i-1}(X; \mathbb{Z})_{\text{tors}}$ .

**Remark 10.0.7:** Note that if  $M$  is an oriented 4-manifold, then

	tors	torsionfree		tors	torsionfree	
$H^0$	0	$\mathbb{Z}$		$H_0$	0	$\mathbb{Z}$
$H^1$	0	$\mathbb{Z}^{\beta_1}$		$H_1$	$A$	$\mathbb{Z}^{\beta_1}$
$H^2$	$A$	$\mathbb{Z}^{\beta_2}$	$\xrightarrow{PD}$	$H_2$	$A$	$\mathbb{Z}^{\beta_2}$
$H^3$	$A$	$\mathbb{Z}^{\beta_1}$		$H_3$	0	$\mathbb{Z}^{\beta_1}$
$H^4$	0	$\mathbb{Z}$		$H_4$	0	$\mathbb{Z}$

In particular, if  $M$  is simply connected, then  $H_1(M) = \mathbf{Ab}(\pi_1(M)) = 0$ , which forces  $A = 0$  and  $\beta_1 = 0$ .

**Definition 10.0.8** (Lattice)

A **lattice** is a finite-dimensional free  $\mathbb{Z}$ -module  $L$  together with a symmetric bilinear form

$$\begin{aligned} \cdot : L^{\otimes 2} &\rightarrow \mathbb{Z} \\ \ell \otimes m &\mapsto \ell \cdot m. \end{aligned}$$

The lattice  $(L, \cdot)$  is **unimodular** if and only if the following map is an isomorphism:

$$\begin{aligned} L &\rightarrow L^\vee \\ \ell &\mapsto \ell \cdot (\cdot). \end{aligned}$$

**Remark 10.0.9:** How to determine if a lattice is unimodular: take a basis  $\{e_1, \dots, e_n\}$  of  $L$  and form the *Gram matrix*  $M_{ij} := (e_i \cdot e_j) \in \text{Mat}(n \times n, \mathbb{Z})^{\text{Sym}}$ . Then  $(L, \cdot)$  is unimodular if and only if  $\det(M) = \pm 1$  if and only if  $M^{-1}$  is integral. In this case, the rows of  $M^{-1}$  will form a basis of the dual basis.

**Definition 10.0.10** (?)

The **index** of a lattice is  $|\det M|$ .

**Exercise 10.0.11** (?)

Prove that  $|\det M| = |L^\vee/L|$ .

**Remark 10.0.12:** In general, for  $M^{4k}$ , the  $H^{2k}/\text{tors}$  is unimodular. For  $M^{4k+2}$ , the  $H^{2k+1}/\text{tors}$  is a unimodular *symplectic* lattice, which is obtained by replacing the word “symmetric” with “antisymmetric” everywhere above.

**Example 10.0.13(?)**: For the torus, since the dimension is 2 (mod 4), you get the skew-symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check!

**Definition 10.0.14** (?)

A lattice is **nondegenerate** if  $\det M \neq 0$ .

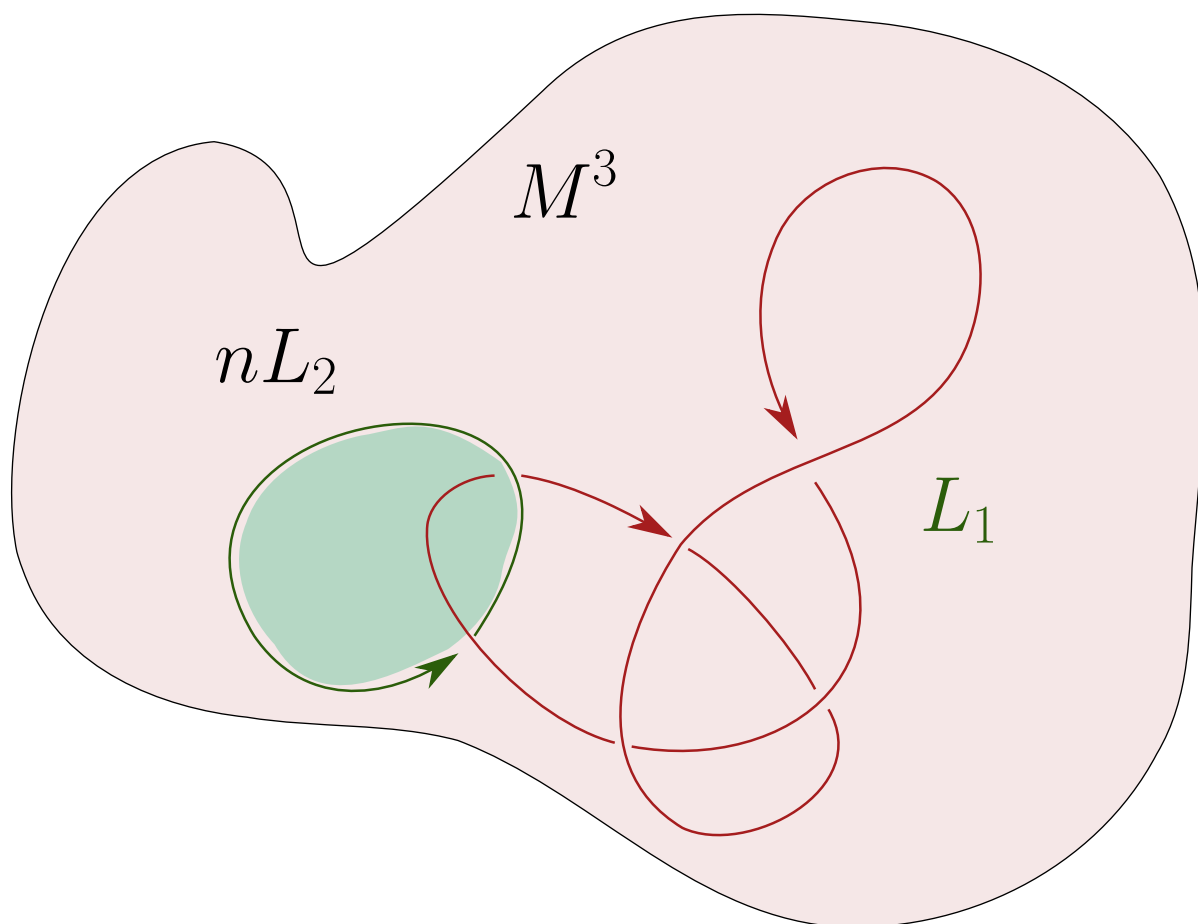
**Definition 10.0.15** (?)

The tensor product  $L \otimes_{\mathbb{Z}} \mathbb{R}$  is a vector space with an  $\mathbb{R}$ -valued symmetric bilinear form. This allows extending the lattice from  $\mathbb{Z}^n$  to  $\mathbb{R}^n$ .

**Remark 10.0.16:** If  $(L, \cdot)$  is nondegenerate, then Gram-Schmidt will yield an orthonormal basis  $\{v_i\}$ . The number of positive norm vectors is an invariant, so we obtain  $\mathbb{R}^{p,q}$  where  $p$  is the number of +1s in the Gram matrix and  $q$  is the number of -1s. The **signature** of  $(L, \cdot)$  is  $(p, q)$ , or by abuse of notation  $p - q$ . This is an invariant of the 4-manifold, as is the lattice itself  $H^2(X; \mathbb{Z})/\text{tors}$  equipped with the intersection form.

**Remark 10.0.17:** There is a perfect pairing called the **linking pairing**:

$$H^i(X; \mathbb{Q}/\mathbb{Z}) \otimes H^{n-i-1}(X; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$



**Remark 10.0.18:**  $A \cdot B := \sum_{p \in A \cap B} \text{sgn}_p(A, B)$ , where  $A \pitchfork B$  and this turns out to be equal to the cup product. This works for topological manifolds – but there are no tangent spaces there, so taking oriented bases doesn't work so well! You can also view

$$[A] \smile [\omega] = \int_A \omega.$$

# 11 | Friday, February 05

**Remark 11.0.1:** Recall that a lattice is **unimodular** if the map  $L \rightarrow L^\vee := \text{Hom}(L, \mathbb{Z})$  is an isomorphism, where  $\ell \mapsto \ell \cdot (\cdot)$ . To check this, it suffices to check if the Gram matrix  $M$  of a basis  $\{e_i\}$  satisfies  $|\det M| = 1$ .

**Example 11.0.2 (Determinant 1 Integer Matrices):** The matrices  $[1]$  and  $[-1]$  correspond to the lattice  $\mathbb{Z}e$  where either  $e^2 := e \cdot e = 1$  or  $e^2 = -1$ . If  $M_1, M_2$  both have absolute determinant 1,

then so does

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}.$$

So if  $L_1, L_2$  are unimodular, then taking an orthogonal sum  $L_1 \oplus L_2$  also yields a unimodular lattice. So this yields diagonal matrices with  $p$  copies of  $+1$  and  $q$  copies of  $-1$ . This is referred to as  $rm1_{p,q}$ , and is an *odd* unimodular lattice of signature  $(p, q)$  (after passing to  $\mathbb{R}$ ). Here *odd* means that there exists a  $v \in L$  such that  $v^2$  is odd.

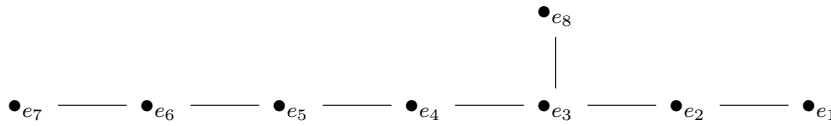
**Example 11.0.3 (Even unimodular lattices):** An even lattice must have no vectors of odd norm, so all of the diagonal elements are in  $2\mathbb{Z}$ . This is because  $(\sum n_i e_i)^2 = \sum_i n_i^2 e_i^2 + \sum_{i < j} 2n_i n_j e_i \cdot e_j$ .

Note that the matrix must be symmetric, and one example that works is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We'll refer to this lattice as  $H$ , sometimes referred to as the *hyperbolic cell* or *hyperbolic plane*.

**Example 11.0.4 (A harder even unimodular lattice):** This is built from the  $E_8$  Dynkin diagram:



The rule here is

$$e_i \cdot e_j = \begin{cases} -2 & i = j \\ 1 & e_i \rightarrow e_j \\ 0 & \text{if not connected.} \end{cases}$$

So for example,  $e_2 \cdot e_6 = 0, e_1 \cdot e_3 = 1, e_2^2 = -2$ . You can check that  $\det(e_i \cdot e_j) = 1$ , and this is referred to as the  $E_8$  lattice. This is of signature  $(0, 8)$ , and it's negative definite if and only if  $v^2 < 0$  for all  $v \neq 0$ . One can also negate the intersection form to define  $-E_8$ . Note that any simply-laced Dynkin diagram yields some lattice. For example,  $E_{10}$  is unimodular of signature  $(1, 9)$ , and it turns out that  $E_{10} \cong E_8 \oplus H$ .

#### Definition 11.0.5 (?)

Take

$$\Pi_{a,a+8b} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b E_8,$$

which is an even unimodular lattice since the diagonal entries are all  $-2$ , and using the fact

that the signature is additive, is of signature  $(a, a + 8b)$ . Similarly,

$$\mathbf{II}_{a+8b,a} := \bigoplus_{i=1}^a H \oplus \bigoplus_{j=1}^b (-E_8),$$

which is again even and unimodular.

**Remark 11.0.6:** Thus

- $\mathbf{I}_{p,q}$  is odd, unimodular, of signature  $(p, q)$ .
- $\mathbf{II}_{p,q}$  is even, unimodular, of signature  $(p, q)$  only for  $p \equiv q \pmod{8}$ .

**Theorem 11.0.7 (Serre).**

Every unimodular lattice which is not positive or negative definite is isomorphic to either  $\mathbf{I}_{p,q}$  or  $\mathbf{II}_{p,q}$  with  $8 \mid p - q$ .

**Remark 11.0.8:** So there are obstructions to the existence of even unimodular lattices. Other than that, the number of (say) positive definite even unimodular lattices is

Dimension	Number of Lattices
8	1: $E_8$
16	2: $E_8^{\oplus 2}, D_{16}^+$
24	24: The Neimeir lattices (e.g. the Leech lattice)
32	$> 8 \times 10^{16}!!!!$

Note that the signature of a definite lattice must be divisible by 8.

**Remark 11.0.9:** There is an isometry:  $f : E_8 \rightarrow E_8$  where  $f \in O(E_8)$ , the linear maps preserving the intersection form (i.e. the Weyl group  $W(E_8)$ , given by  $v \mapsto v + (v, e_i)e_i$ . The Leech lattice also shows up in the sphere packing problems for dimensions 2, 4, 8, 24. See Hale's theorem / Kepler conjecture for dimension 3! This uses an identification of  $L$  as a subset of  $\mathbb{R}^n$ , namely  $L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{24}$  for example, and the map  $L \hookrightarrow (\mathbb{R}^{24}, \cdot)$  is an isometric embedding into  $\mathbb{R}^n$  with the standard form. Connection to classification of Lie groups: root lattices.

**Remark 11.0.10:** If  $M^4$  is a compact oriented 4-manifold and if the intersection form on  $H^2(M; \mathbb{Z})$  is indefinite, then the only invariants we can extract from that associated lattice are

- Whether it's even or odd, and
- Its signature

If the lattice is even, then the signature satisfies  $8 \mid p - q$ . So Poincaré duality forces unimodularity,

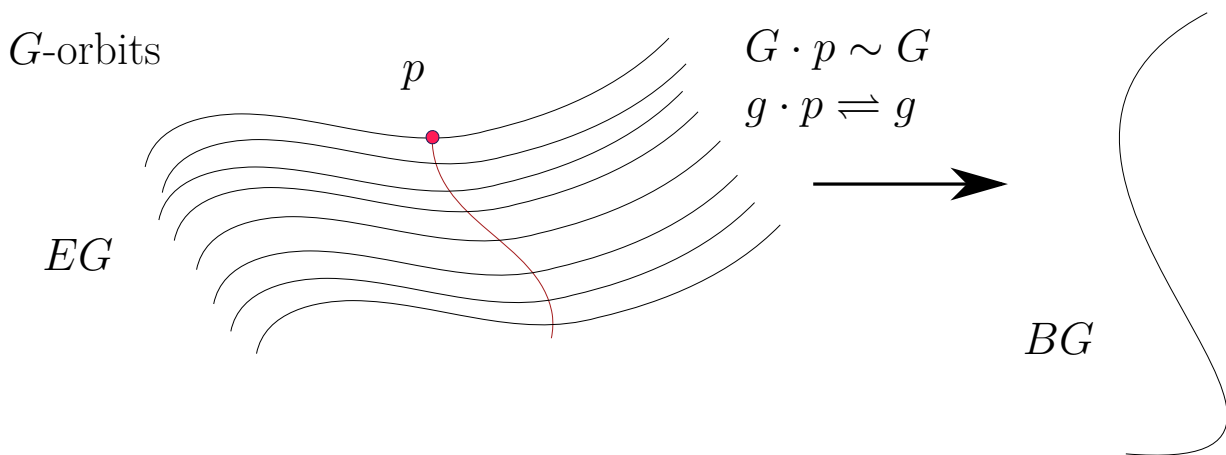
and then there are further number-theoretic restrictions. E.g. this prohibits  $\beta_2 = 7$ , since then the signature couldn't possibly be 8 if the intersection form is even.

## 11.1 Characteristic Classes

### Definition 11.1.1 (?)

Let  $G$  be a topological group, then a **classifying space**  $EG$  is a contractible topological space admitting a free continuous  $G$ -action with a “nice” quotient.

**Remark 11.1.2:** Thus there is a map  $EG \rightarrow BG := EG/G$  which has the structure of a principal  $G$ -bundle.



Here we use a point  $p$  depending on  $U$  in an orbit to identify orbits  $g \cdot p$  with  $g$ , and we want to take transverse slices to get local trivializations of  $U \in BG$ . It suffices to know where  $\pi^{-1}(U) \cong U \times G$ , and it suffices to consider  $U \times \{e\}$ . Moreover,  $EG \rightarrow BG$  is a universal principal  $G$ -bundle in the sense that if  $P \rightarrow X$  is a universal  $G$ -bundle, there is an  $f : X \rightarrow BG$ .

$$\begin{array}{ccc} P & \xrightarrow{\quad\quad\quad} & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & BG \end{array}$$

[Link to Diagram](#)

Here bundles will be classified by homotopy classes of  $f$ , so

$$\{\text{Principal } G\text{-bundles}/_X\} \cong [X, BG].$$

**⚠ Warning 11.1.3**

This only works for paracompact Hausdorff spaces! The line  $\mathbb{R}$  with the doubled origin is a counterexample, consider complex line bundles.

Revisit this last section, had to clarify a few things for myself!

# 12 | Monday, February 08

Last time:  $BG$  and  $EG$ . See Milnor and Stasheff.

**Example 12.0.1(?)**: Let  $G := \mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^\times$ , then we can take

$$EG = \mathbb{R}^\infty := \left\{ (a_1, a_2, \dots) \mid a_i \in \mathbb{R}, a_i \gg 0 = 0, a_i \text{ not all zero} \right\}.$$

Then  $\mathbb{R}^\times$  acts on  $EG$  by scaling, and we can take the quotient  $\mathbb{R}^\infty \setminus \{0\} / \mathbb{R}^\times$ , where  $\mathbf{a} \sim \lambda \mathbf{a}$  for all  $\lambda \in \mathbb{R}^\times$ . This yields  $\mathbb{RP}^\infty$  as the quotient. You can check that  $EG$  is contractible: it suffices to show that  $S^\infty := \left\{ \sum |a_i| = 1 \right\}$  is contractible. This works by decreasing the last nonzero coordinate and increasing the first coordinate correspondingly. Moreover, local lifts exist, so we can identify  $\mathbb{RP}^\infty \cong B\mathbb{R}^\times = BG$ . Similarly  $BC^\times \cong \mathbb{CP}^\infty$  with  $EC^\times := \mathbb{C}^\infty \setminus \{0\}$ .

**Example 12.0.2(?)**: Consider  $G = \mathrm{GL}_n(\mathbb{R})$ . It turns out that  $BG = \mathrm{Gr}(d, \mathbb{R}^\infty)$ , which is the set of linear subspaces of  $\mathbb{R}^\infty$  of dimension  $d$ . This is spanned by  $d$  vectors  $\{e_i\}$  in some large enough  $\mathbb{R}^N \subseteq \mathbb{R}^\infty$ , since we can take  $N$  to be the largest nonvanishing coordinate and include all of the vectors into  $\mathbb{R}^\infty$  by setting  $a_{>N} = 0$ . For any  $L \in \mathrm{Gr}_d(\mathbb{R}^\infty)$ , since  $\mathbb{R}^d$  has a standard basis, there is a natural  $\mathrm{GL}_d$  torsor: the set of ordered bases of linear subspaces. So define

$$EG := \{\text{bases of linear subspaces } L \in \mathrm{Gr}_d(\mathbb{R}^\infty)\},$$

then any  $A \in \mathrm{GL}_d(\mathbb{R})$  acts on  $EG$  by sending  $(L, \{e_i\}) \mapsto (L, \{Le_i\})$ . We can identify  $EG$  as  $d$ -tuples of linearly independent elements of  $\mathbb{R}^\infty$ , and there is a map

$$\begin{aligned} EG &\rightarrow BG \\ \{e_i\} &\mapsto \mathrm{span}_{\mathbb{R}} \{e_i\}. \end{aligned}$$

Thus there is a universal vector bundle over  $B\mathrm{GL}_d$ :

$$\begin{array}{ccc} \mathcal{E}_L := L & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & B\mathrm{GL}_d \end{array}$$

So  $\mathcal{E} \subseteq B\mathrm{GL}_d \times \mathbb{R}^\infty$ , where we can define  $\mathcal{E} := \{(L, p) \mid p \in L\}$ . In this case,  $EG = \mathrm{Frame}(\mathcal{E})$  is the frame bundle of this universal bundle. The same setup applies for  $G := \mathrm{GL}_d(\mathbb{C})$ , except we take  $\mathrm{Gr}_d(\mathbb{C}^\infty)$ .

**Example 12.0.3(?)**: Consider  $G = O_d$ , the set of orthogonal transformations of  $\mathbb{R}^d$  with the standard bilinear form, and  $U_d$  the set of unitary such transformations. To be explicit:

$$U_d := \left\{ A \in \text{Mat}(d \times d, \mathbb{C}) \mid \langle Av, Av \rangle = \langle v, v \rangle \right\},$$

where

$$\langle [v_1, \dots, v_n], [v_1, \dots, v_n] \rangle = \sum |v_i|^2.$$

Alternatively,  $A^t A = I$  for  $O_d$  and  $\overline{A}^t A = I$  for  $U_d$ . In this case,  $BO_d = \text{Gr}_d(\mathbb{R}^\infty)$  and  $BU_d = \text{Gr}_d(\mathbb{C}^\infty)$ , but we'll make the fibers smaller: set the fiber over  $L$  to be

$$(EO_d)_L := \{\text{orthogonal frames of } L\}$$

and similarly  $(EU_d)_L$  the unitary frames of  $L$ . That there are related comes from the fact that  $\text{GL}_d$  retracts onto  $O_d$  using the Gram-Schmidt procedure.

**Remark 12.0.4**: Recall that there is a bijective correspondence

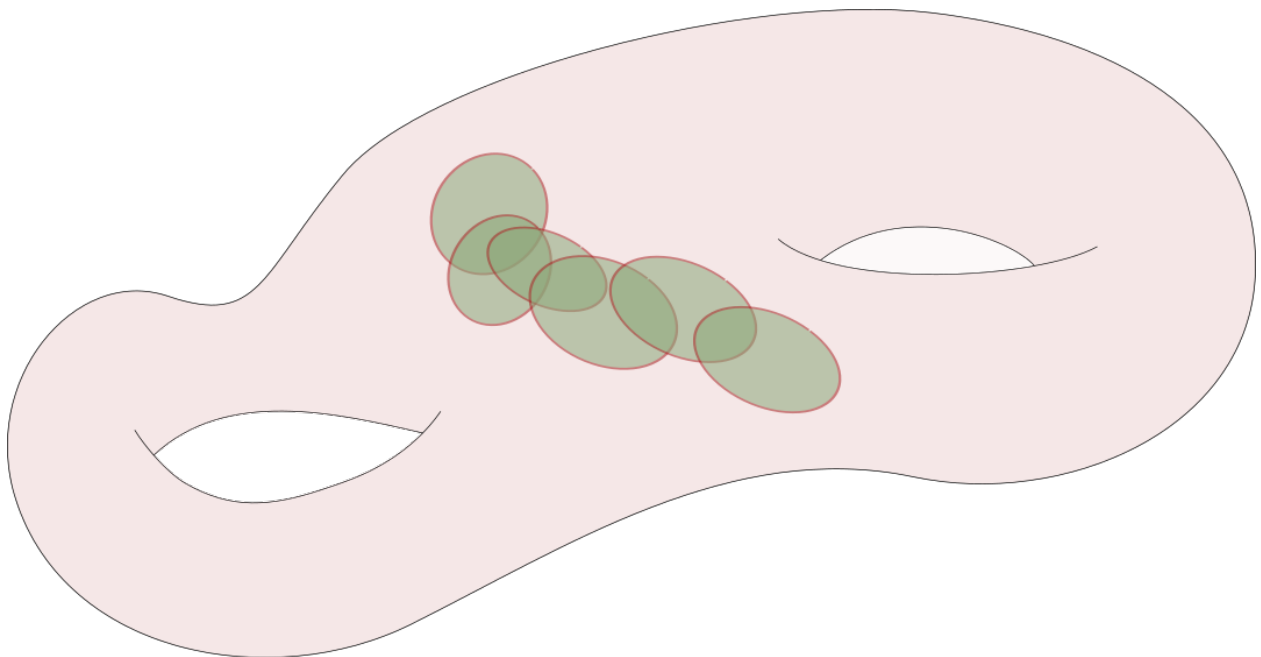
$$\left\{ \begin{array}{c} \text{Principal } G\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons [X, BG]$$

and there is also a correspondence

$$\left\{ \begin{array}{c} \text{Principal } \text{GL}_d\text{-bundles} \\ \text{on } X \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{Principal } O_d\text{-bundles} \\ \text{on } X \end{array} \right\}$$

Using the associated bundle construction, on the LHS we obtain vector bundles  $\mathcal{E} \rightarrow X$  of rank  $d$ , and on the RHS we have bundles with a metric. In local trivializations  $U \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , the metric is the standard one on  $\mathbb{R}^d$ . This is referred to as a **reduction of structure group**, i.e. a principal  $\text{GL}_d$  bundle admits possibly different trivializations for which the transition functions lie in the subgroup  $O_d$ .

**Example 12.0.5(?)**: Given any trivial principal  $G$ -bundle, it has a reduction of structure group to the trivial group. But the fact that the bundle is trivial may not be obvious.





**Remark 12.0.6:** We want to compute  $H^*(BU_d; \mathbb{Z})$ . Why is this important? Given any complex vector bundle  $\mathcal{E} \rightarrow X$  there is an associated principal  $U_d$  bundle by choosing a metric, so we get a homotopy class  $[X, BU_d]$ . Given any  $f \in [X, BU_d]$  and any  $\alpha \in H^k(BU_d; \mathbb{Z})$ , we can take the pullback  $f^*\alpha \in H^k(X; \mathbb{Z})$ , which are **Chern classes**.

**Exercise 12.0.7 (?)**

Show that  $H^*(BU_d; \mathbb{Z})$  stabilizes as  $d \rightarrow \infty$  to an infinitely generated polynomial ring  $\mathbb{Z}[c_1, c_2, \dots]$  with each  $c_i$  in cohomological degree  $2i$ , so  $c_i \in H^{2i}(BU_d, \mathbb{Z})$ .

**Definition 12.0.8 (?)**

There is a map  $BU_{d-1} \rightarrow BU_d$ , which we can identify as

$$\begin{aligned} \text{Gr}_{d-1}(C^\infty) &\rightarrow \text{Gr}_d(\mathbb{C}^\infty) \\ \{v_1, \dots, v_{d-1}\} &\mapsto \text{span}\{(1, 0, 0, \dots), sv_1, \dots, sv_{d-1}\}. \end{aligned}$$

This is defined by sending a basis where  $s: \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$  is the map that shifts every coordinate to the right by one.

Question: does  $\text{Gr}_d(\mathbb{C}^\infty)$  deformation retract onto the image of this map?

This will yield a fiber sequence

$$S^{2d-1} \rightarrow BU_{d-1} \rightarrow BU_d$$

and using connectedness of the sphere and the LES in homotopy this will identify

$$H^*(BU_d) = H^*(BU_{d-1})[c_d] \quad \text{where } c_d \in H^{2d}(BU_d).$$

The **Chern class** of a vector bundle  $\mathcal{E}$ , denoted  $c_k(\mathcal{E})$ , will be defined as the pullback  $f^*c_k$ .

# 13 | Wednesday, February 10

**Theorem 13.0.1 (?)**

As  $n \rightarrow \infty$ , we have

$$H^*(BO_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \dots] \quad w_i \in H^i.$$

**Definition 13.0.2 (?)**

Given any principal  $O_n$ -bundle  $P \rightarrow X$ , there is an induced map  $X \xrightarrow{f} BO_n$ , so we can pull back the above generators to define the **Stiefel-Whitney classes**  $f^*w_i$ .

**Remark 13.0.3:** If  $P := \text{OFrame}TX$ , then  $f^*w_1$  measures whether  $X$  has an orientation, i.e.  $f^*w_1 = 0 \iff X$  can be oriented. We also have  $f^*w_i(P) = w_i(\mathcal{E})$  where  $P = \text{OFrame}(\mathcal{E})$ . In general, we'll just write  $w_i$  for Stiefel-Whitney classes and  $c_i$  for Chern classes.

**Definition 13.0.4** (Pontryagin Classes)

The **Pontryagin classes** of a real vector bundle  $\mathcal{E}$  are defined as

$$p_i(\mathcal{E}) = (-1)^i c_{2i}(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

Note that the complexified bundle above is a complex vector bundle with the same transition functions as  $\mathcal{E}$ , but has a reduction of structure group from  $\mathrm{GL}_n(\mathbb{C})$  to  $\mathrm{GL}_n(\mathbb{R})$ .

**Observation 13.0.5**

$\mathbb{R}\mathbb{P}^\infty$  and  $\mathbb{C}\mathbb{P}^\infty$  are examples of  $K(\pi, n)$  spaces, which are the unique-up-to-homotopy spaces defined by

$$\pi_k K(\pi, n) = \begin{cases} \pi & k = n \\ 0 & \text{else.} \end{cases}$$

**Theorem 13.0.6** (*Brown Representability*).

$$H^n(X; \pi) \cong [X, K(\pi, n)].$$

**Example 13.0.7** (?):

$$[X, \mathbb{R}\mathbb{P}^\infty] \cong H^1(X; \mathbb{Z}/2\mathbb{Z})$$

$$[X, \mathbb{C}\mathbb{P}^\infty] \cong H^2(X; \mathbb{Z}).$$

**Proposition 13.0.8** (?).

There is a correspondence

$$\{\text{Complex line bundles}\} \rightleftharpoons [X, \mathbb{C}\mathbb{P}^\infty] = [X, BC^\times] \rightleftharpoons H^2(X; \mathbb{Z})$$

Importantly, note that for  $X \in \mathbf{Mfd}_{\mathbb{C}}$ ,  $H^2(X; \mathbb{Z})$  measures *smooth* complex line bundles and not holomorphic bundles.

*Proof* (?).

We'll take an alternate direct proof. Consider the exponential exact sequence on  $X$ :

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times.$$

Note that  $\underline{\mathbb{Z}}$  consists of locally constant  $\mathbb{Z}$ -valued functions,  $\mathcal{O}$  consists of smooth functions, and  $\mathcal{O}^\times$  are ???.

Can't read screenshot! :(

This yields a LES in homology:

$$\begin{array}{ccccccc}
H^0(X; \mathbb{Z}) & \longrightarrow & H^0(X; \mathcal{O}) & \longrightarrow & H^0(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^1(X; \mathbb{Z}) & \longrightarrow & H^1(X; \mathcal{O}) & \longrightarrow & H^1(X; \mathcal{O}^\times) & \longrightarrow & \\
\searrow & & & & & & \\
\hookrightarrow H^2(X; \mathbb{Z}) & \longrightarrow & H^2(X; \mathcal{O}) & \longrightarrow & H^2(X; \mathcal{O}^\times) & \longrightarrow & 
\end{array}$$

[Link to Diagram](#)

Since  $\mathcal{O}$  admits a partition of unity,  $H^{>0}(X; \mathcal{O}) = 0$  and all of the red terms vanish. For complex line bundles  $L$ ,  $H^1(X; \mathcal{O}^\times) \cong H^2(X; \mathbb{Z})$ . Taking a local trivialization  $L|_U \cong U \times \mathbb{C}$ , we obtain transition functions

$$t_{UV} \in C^\infty(U \cap V, \mathrm{GL}_1(\mathbb{C}))$$

where we can identify  $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$ . We then have

$$(t_{U_{ij}}) \in \prod_{i < j} \mathcal{O}^\times(U_i \cap U_j) = C^1(X; \mathcal{O}^\times).$$

Moreover,

$$(t_{U_{ij}} t_{U_{ik}}^{-1} t_{U_{jk}})_{i,j,k} = \partial(t_{U_{ij}})_{i,j} = 0,$$

since transition functions satisfy the cocycle condition. So in fact  $(t_{U_{ij}}) \in Z^1(X; \mathcal{O}^\times) = \ker \partial^1$ , and we can take its equivalence class  $[(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times) = \ker \partial^1 / \mathrm{im} \partial^0$ . Changing trivializations by some  $s_i \in \prod_i \mathcal{O}^\times(U_i)$  yields a composition which is a different trivialization of the same bundle:

$$\begin{array}{ccccc}
L|_{U_i} & \xrightarrow{h_i} & U_i \times \mathbb{C} & \xrightarrow{\cdot s_i} & U_i \times \mathbb{C} \\
& & \searrow & & \nearrow \\
& & & & 
\end{array}$$

[Link to Diagram](#)

So the  $(t_{U_{ij}})$  change *exactly* by an  $\partial^0(s_i)$ . Thus the following map is well-defined:

$$L \mapsto [(t_{U_{ij}})] \in H^1(X; \mathcal{O}^\times).$$

There is another construction of the map

$$\begin{aligned}
\{L\} &\rightarrow H^2(X; \mathbb{Z}) \\
L &\mapsto c_1(L).
\end{aligned}$$

Take a smooth section of  $L$  and  $s \in H^0(X; L)$  that intersects an  $\mathcal{O}$ -section of  $L$  transversely. Then

$$V(s) := \{x \in X \mid s(x) = 0\}$$

is a submanifold of real codimension 2 in  $X$ , and  $c_1(L) = [V(s)] \in H^2(X; \mathbb{Z})$ . ■

**Theorem 13.0.9 (Splitting Principle for Complex Vector Bundles).**

1. Suppose that  $\mathcal{E} = \bigoplus_{i=1}^r L_i$  and let  $c(\mathcal{E}) := \sum c_i(\mathcal{E})$ . Then

$$c(\mathcal{E}) = \prod_{i=1}^r (1 + c_i(L_i)).$$

2. Given any vector bundle  $\mathcal{E} \rightarrow X$ , there exists some  $Y$  and a map  $Y \rightarrow X$  such that  $f^* : H^k(X; \mathbb{Z}) \hookrightarrow H^k(Y; \mathbb{Z})$  is injective and  $f^*\mathcal{E} = \bigoplus_{i=1}^r L_i$ .

**Slogan 13.0.10**

To verify any identities on characteristic classes, it suffices to prove them in the case where  $\mathcal{E}$  splits into a direct sum of line bundles.

**Example 13.0.11 (?) :**


$$c(\mathcal{E} \oplus \mathcal{F}) = c(\mathcal{E})c(\mathcal{F}).$$


To prove this, apply the splitting principle. Choose  $Y, Y'$  splitting  $\mathcal{E}, \mathcal{F}'$  respectively, this produces a space  $Z$  and a map  $f : Z \rightarrow X$  where both split. We can write

$$\begin{aligned} f^*\mathcal{E} &= \bigoplus L_i & c(f^*\mathcal{E}) &= \prod (1 + c_1(L_i)) \\ f^*\mathcal{F} &= \bigoplus M_j & c(f^*\mathcal{F}) &= \prod (1 + c_1(M_j)). \end{aligned}$$


We thus have

$$\begin{aligned} c(f^*\mathcal{E} \oplus f^*\mathcal{F}) &= \prod (1 + c_1(L_i)) (1 + c_1(M_j)) \\ &= c(f^*\mathcal{E})c(f^*\mathcal{F}), \end{aligned}$$

and  $f^*(c(\mathcal{E} \oplus \mathcal{F})) = f^*(c(\mathcal{E})c(\mathcal{F}))$ . Since  $f^*$  is injective, this yields the desired identity. 

**Example 13.0.12 (?) :** We can compute  $c(\text{Sym}^2 \mathcal{E})$ , and really any tensorial combination involving  $\mathcal{E}$ , and it will always yield some formula in the  $c_i(\mathcal{E})$ . 

# 14 | Friday, February 12

**Remark 14.0.1:** Last time: the splitting principle. Suppose we have  $\mathcal{E} = L_1 \oplus \cdots \oplus L_r$  and let  $x_i := c_1(L_i)$ . Then  $c_k(\mathcal{E})$  is the degree  $2k$  part of  $\prod_{i=1}^r (1 + x_i)$  where each  $x_i$  is in degree 2. This is equal to  $e_k(x_1, \dots, x_r)$  where  $e_k$  is the  $k$ th elementary symmetric polynomial. 

**Example 14.0.2(?)**: For example,

- $e_1 = x_1 + \cdots x_r$ .
- $e_2 = x_1x_2 + x_1x_3 + \cdots = \sum_{i<j} x_ix_j$
- $e_3 = \sum_{i<j<k} x_ix_jx_k$ , etc.

**Remark 14.0.3**: The theorem is that any symmetric polynomial is a polynomial in the  $e_i$ . For example,  $p_2 = \sum x_i^2$  can be written as  $e_1^2 - 2e_2$ . Similarly,  $p_3 = \sum x_i^3 = e_1^3 - 3e_1e_2 - 3e_3$ . Note that the coefficients of these polynomials are important for representations of  $S_n$ , see *Schur polynomials*.

**Remark 14.0.4**: Due to the splitting principle, we can pretend that  $x_i = c_i(L_i)$  exists even when  $\mathcal{E}$  doesn't split. If  $\mathcal{E} \rightarrow X$ , the individual symbols  $x_i$  don't exist, but we can write

$$x_1^3 + \cdots + x_r^3 = e_1^3 - 3e_1e_2 - 3e_3 := c_1(\mathcal{E})^3 + 3c_1(\mathcal{E})c_2(\mathcal{E}) + \cdots,$$

which is a well-defined element of  $H^6(X; \mathbb{Z})$ . So this polynomial defines a characteristic class of  $\mathcal{E}$ , and this can be done for any symmetric polynomial. We can change basis in the space of symmetric polynomials to now define different characteristic classes.

**Definition 14.0.5** (Chern Character)

The **Chern character** is defined as

$$\begin{aligned} \text{ch}(\mathcal{E}) &:= \sum_{i=1}^r e^{x_i} \in H^*(X; \mathbb{Q}) \\ &:= \sum_{i=1}^r \sum_{k=0}^{\infty} \frac{x_i^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{p_k(x_1, \dots, x_r)}{k!} \\ &= \text{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{c_1(\mathcal{E})^2 - c_2(\mathcal{E})}{2!} + \frac{c_1(\mathcal{E})^3 - 3c_1(\mathcal{E})c_2(\mathcal{E}) - 3c_3(\mathcal{E})}{3!} + \cdots \\ &\quad \in H^0 + H^2 + H^4 + H^6 \\ &= \text{ch}_0(\mathcal{E}) + \text{ch}_1(\mathcal{E}) + \text{ch}_2(\mathcal{E}) + \cdots, \\ &\quad \text{ch}_i(\mathcal{E}) \in H^{2i}(X; \mathbb{Q}). \end{aligned}$$

**Definition 14.0.6** (Todd Class)

The **total Todd class**

$$\text{td}(\mathcal{E}) := \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}.$$

Note that

$$\frac{x_i}{1 - e^{-x_i}} = 1 + \frac{x_i}{2} + \frac{x_i^2}{12} + \frac{x_i^4}{720} + \cdots = 1 + \frac{x_i}{2} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{(2i)!} x_i^{2i}.$$

where L'Hopital shows that the derivative at  $x_i = 0$  exists, so it's analytic at zero and the expansion makes sense, and the  $B_i$  are Bernoulli numbers.

**Remark 14.0.7 (Very important and useful!!):**  $\text{ch}(\mathcal{E} \oplus \mathcal{F}) = \text{ch}(\mathcal{E}) + \text{ch}(\mathcal{F})$  and  $\text{ch}(\mathcal{E} \otimes \mathcal{F}) = \sum_{i,j} e^{x_i + y_j} = \text{ch}(\mathcal{E}) \text{ch}(\mathcal{F})$  using the fact that  $c_1(L_1 \otimes L_2) = c_1(L_1)c_1(L_2)$ . So  $\text{ch}$  is a “ring morphism” in the sense that it preserves multiplication  $\otimes$  and addition  $\oplus$ , making the Chern character even better than the total Chern class.

**Definition 14.0.8** (Todd Class)

Let  $X \in \mathbf{Mfd}_{\mathbb{C}}$ , then define the **Todd class** of  $X$  as  $\text{td}_{\mathbb{C}}(X) := \text{td}(TX)$  where  $TX$  is viewed as a complex vector bundle. If  $X \in \mathbf{Mfd}_{\mathbb{R}}$ , define  $\text{td}_{\mathbb{R}} = \text{td}(TX \otimes_{\mathbb{R}} \mathbb{C})$ .

## 14.1 Section 5: Riemann-Roch and Generalizations

**Remark 14.1.1:** Let  $X \in \mathbf{Top}$  and let  $\mathcal{F}$  be a sheaf of vector spaces. Suppose  $h^i(X; \mathcal{F}) := \dim H^i(X; \mathcal{F}) < \infty$  for all  $i$  and is equal to 0 for  $i \gg 0$ .

**Definition 14.1.2** (Euler Characteristic of a Sheaf)

The **Euler characteristic** of  $\mathcal{F}$  is defined as

$$\chi(X; \mathcal{F}) := \chi(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i h_i(X; \mathcal{F}).$$

 **Warning 14.1.3**

This is not always well-defined!

**Example 14.1.4(?)**: Let  $X \in \mathbf{Mfd}_{\text{cpt}}$  and take  $\mathcal{F} := \underline{\mathbb{R}}$ , we then have

$$\chi(X; \underline{\mathbb{R}}) = h^0(X; \underline{\mathbb{R}}) - h^1(X; \underline{\mathbb{R}}) + \cdots = b_0 - b_1 + b_2 - \cdots := \chi_{\mathbf{Top}}(X).$$


**Example 14.1.5(?)**: Let  $X = \mathbb{C}$  and take  $\mathcal{F} := \mathcal{O} := \mathcal{O}^{\text{holo}}$  the sheaf of holomorphic functions. We then have  $h^{>0}(X; \mathcal{O}) = 0$ , but  $H^0(X; \mathcal{O})$  is the space of all holomorphic functions on  $\mathbb{C}$ , making  $\dim_{\mathbb{C}} h^0(X; \mathcal{O})$  infinite.

**Example 14.1.6(?)**: Take  $X = \mathbb{P}^1$  with  $\mathcal{O}$  as above,  $h^0(\mathbb{P}^1; \mathcal{O}) = 1$  since  $\mathbb{P}^1$  is compact and the maximum modulus principle applies, so the only global holomorphic functions are constant. We can write  $\mathbb{P}^1 = \mathbb{C}_1 \cup \mathbb{C}_2$  as a cover and  $h^i(\mathbb{C}, \mathcal{O}) = 0$ , so this is an acyclic cover and we can use it to compute  $h^1(\mathbb{P}^1; \mathcal{O})$  using Čech cohomology. We have


- $C^0(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1) \oplus \mathcal{O}(\mathbb{C}_2)$
- $C^1(\mathbb{P}^1; \mathcal{O}) = \mathcal{O}(\mathbb{C}_1 \cap \mathbb{C}_2) = \mathcal{O}(\mathbb{C}^\times)$ .
- The boundary map is given by

$$\begin{aligned} \partial_0 : C^0 &\rightarrow C^1 \\ (f(z), g(z)) &\mapsto g(1/z) - f(z) \end{aligned}$$

and there are no triple intersections.

Is every holomorphic function on  $\mathbb{C}^\times$  of the form  $g(1/z) - f(z)$  with  $f, g$  holomorphic on  $\mathbb{C}$ . The answer is yes, by Laurent expansion, and thus  $h^1 = 0$ . We can thus compute  $\chi(\mathbb{P}^1; \mathcal{O}) = 1 - 0 = 1$ . 

# 15 | Monday, February 15


**Remark 15.0.1:** Last time: we saw that  $\chi(\mathbb{P}^1, \mathcal{O}) = 1$ , and we'd like to generalize to holomorphic line bundles on a Riemann surface. This will be the main ingredient for Riemann-Roch. 

## Theorem 15.0.2(?).

Let  $X \in \mathbf{Mfd}_{\mathbb{C}}$  be compact and let  $\mathcal{F}$  be a holomorphic vector bundle on  $X$ .<sup>a</sup> Then  $\chi$  is well-defined and

$$h^{>\dim_{\mathbb{C}} X}(X; \mathcal{F}) = 0.$$

<sup>a</sup>Or more generally a finitely-generated  $\mathcal{O}$ -module, i.e. a coherent sheaf.

**Remark 15.0.3:** The locally constant sheaf  $\underline{\mathbb{C}}$  is not an  $\mathcal{O}$ -module, i.e.  $\underline{\mathbb{C}}(U) \notin \mathcal{O}(U)\text{-Mod}$ . In fact,  $h^{2i}(X, \underline{\mathbb{C}}) = \mathbb{C}$  for all  $i$ . 

*Proof (?).*

We'll resolve  $\mathcal{F}$  as a sheaf by first mapping to its smooth sections and continuing in the following way:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \rightarrow \dots,$$

where  $\bar{\partial}f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$ . Suppose we have a holomorphic trivialization of  $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$  and we have sections  $(s_1, \dots, s_r) \in C^\infty \mathcal{F}(U)$ , which are smooth functions on  $U$ . In local coordinates we have

$$\bar{\partial}s := (\bar{\partial}s_1, \dots, \bar{\partial}s_r),$$

but is this well-defined globally? Given a different trivialization over  $V \subseteq X$ , the  $s_i$  are related by transition functions, so the new sections are  $t_{UV}(s_1, \dots, s_r)$  where  $t_{UV} : U \cap V \rightarrow \mathrm{GL}_r(\mathbb{C})$ .

Since  $t_{UV}$  are holomorphic, we have

$$\bar{\partial}(t_{UV}(s_1, \dots, s_r)) = t_{UV}\bar{\partial}(s_1, \dots, s_r).$$

This makes  $\bar{\partial} : C^\infty \mathcal{F} \rightarrow F \otimes A^{0,1}$  a well-defined (but not  $\mathcal{O}$ -linear) map. We can thus continue this resolution using the Leibniz rule:

$$0 \rightarrow \mathcal{F} \rightarrow C^\infty \mathcal{F} \xrightarrow{\bar{\partial}} F \otimes A^{0,1} \xrightarrow{\bar{\partial}} \dots F \otimes A^{0,2} \xrightarrow{\bar{\partial}} \dots,$$

which is an exact sequence of sheaves since  $(A^{0,\cdot}, \bar{\partial})$  is exact.

Why? Split into line bundles?

We can identify  $C^\infty \mathcal{F} = \mathcal{F} \otimes A^{0,0}$ , and  $\mathcal{F} \otimes A^{0,q}$  is a smooth vector bundle on  $X$ . Using partitions of unity, we have that  $\mathcal{F} \otimes A^{0,q}$  is acyclic, so its higher cohomology vanishes, and

$$H^i(X; \mathcal{F}) \cong \frac{\ker(\bar{\partial} : \mathcal{F} \otimes A^{0,i} \rightarrow \mathcal{F} \otimes A^{0,i+1})}{\operatorname{im}(\bar{\partial} : \mathcal{F} \otimes A^{0,i-1} \rightarrow \mathcal{F} \otimes A^{0,i})}.$$

However, we know that  $A^{0,p} = 0$  for all  $p > n := \dim_{\mathbb{C}} X$ , since any wedge of  $p > n$  forms necessarily vanishes since there are only  $n$  complex coordinates. ■

#### ⚠ Warning 15.0.4

This only applies to holomorphic vector bundles or  $\mathcal{O}$ -modules!

## 15.1 Riemann-Roch

### Theorem 15.1.1 (*Riemann-Roch*).

Let  $C$  be a compact connected Riemann surface, i.e.  $X \in \mathbf{Mfd}_{\mathbb{C}}$  with  $\dim_{\mathbb{C}}(X) = 1$ , and let  $\mathcal{L} \rightarrow C$  be a holomorphic line bundle. Then

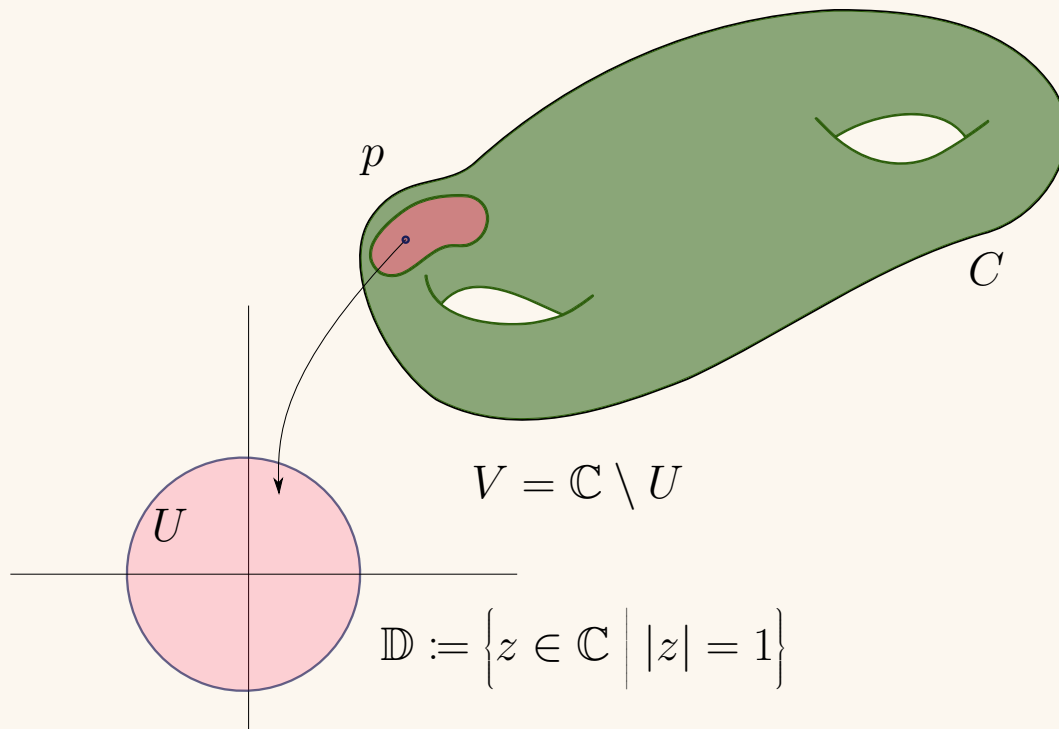
$$\chi(C, \mathcal{L}) = \deg(L) + (1 - g) \quad \text{where } \int_C c_1(\mathcal{L})$$

and  $g$  is the genus of  $C$ .

*Proof* (?).

We'll introduce the notion of a “point bundle”, which are particularly nice line bundles, denoted  $\mathcal{O}(p)$  for  $p \in \mathbb{C}$ .





Taking  $\mathbb{D}$  to be a disc of radius  $1/2$  and  $V$  to be its complement, we have  $t_{uv}(z) = z^{-1} \in \mathcal{O}^*(U \cap V)$ . We can take a holomorphic section  $s_p \in H^0(C, \mathcal{O}(p))$ , where  $s_p|_U = z$  and  $s_p|_V = 1$ . Then  $t_{uv}(s_p|_U) = s_p|_V$  on the overlaps. We have a function which precisely vanishes to first order at  $p$ . Recall that  $c_1(\mathcal{O}(p))$  is represented by  $[V(s)] = [p]$ , and moreover  $\int_C c_1(\mathcal{O}(p)) = 1$ .

We now want to generalize this to a **divisor**: a formal  $\mathbb{Z}$ -linear combination of points.

**Example 15.1.2(?)**: Take  $p, q, r \in C$ , then a divisor can be defined as something like  $D := 2[p] - [q] + 3[r]$ .

Define  $\mathcal{O}(D) := \bigotimes_i \mathcal{O}(p_i)^{\otimes n_i}$  for any  $D = \sum n_i [p_i]$ . Here tensoring by negatives means taking duals, i.e.  $\mathcal{O}(-[p]) := \mathcal{O}^{\otimes -1} := \mathcal{O}(p)^\vee$ , the line bundle with inverted transition functions.  $\mathcal{O}(D)$  has a meromorphic section given by

$$s_D := \prod s_{p_i}^{n_i} \in \text{Mero}(C, \mathcal{O}(D))$$

where we take the sections coming from point bundles. We can compute

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i := \deg(D).$$

**Example 15.1.3(?)**:

$$\deg(2[p] - [q] + 3[r]) = 4.$$

**Remark 15.1.4**: Assume our line bundle  $L$  is  $\mathcal{O}(D)$ , we'll prove Riemann-Roch in this case by induction on  $\sum |n_i|$ . The base case is  $\mathcal{O}$ , which corresponds to taking an empty divisor. Then either

- Take  $D = D_0 + [p]$  with  $\deg(D_0) < \sum |n_i|$  (for which we need some positive coefficient),  
or
- Take  $D_0 = D + [p]$ .

**Claim:** There is an exact sequence

$$0 \rightarrow \mathcal{O}(D_0) \rightarrow \mathcal{O}(D) \rightarrow \mathbb{C}_p \rightarrow 0$$

$$s \in \mathcal{O}(D_0)(U) \mapsto s \cdot s_p \in \mathcal{O}(D_0 + [p])(U),$$

where the last term is the skyscraper sheaf at  $p$ .

*Proof (?)*.

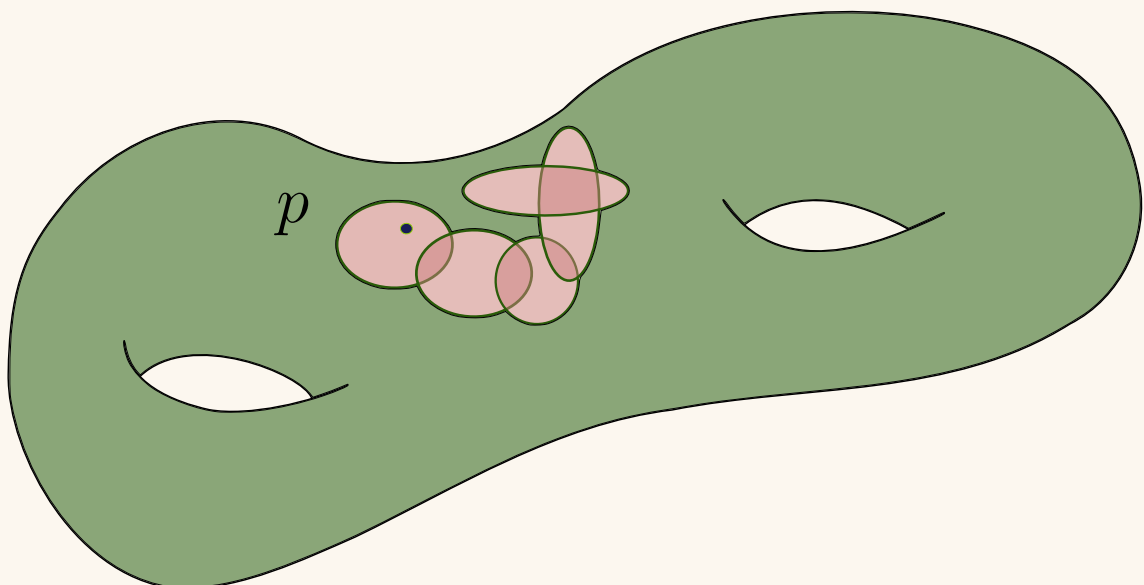
The given map is  $\mathcal{O}$ -linear and injective, since  $s_p \neq 0$  and  $ss_p = 0$  forces  $s = 0$ . Recall that we looked at  $\mathcal{O} \xrightarrow{z} \mathcal{O}$  on  $\mathbb{C}$ , and this section only vanishes at  $p$  (and to first order). The same situation is happening here. ■

Thus there is a LES

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & & & \searrow & \\
 \rightarrow & H^0(\mathcal{O}(D_0)) & \longrightarrow & H^0(\mathcal{O}(D)) & \longrightarrow & H^0(\mathcal{O}(\mathbb{C}_p)) & \rightarrow \\
 & & & & & \searrow & \\
 \rightarrow & H^1(\mathcal{O}(D_0)) & \longrightarrow & H^1(\mathcal{O}(D)) & \longrightarrow & H^1(\mathcal{O}(\mathbb{C}_p)) = 0 & \rightarrow \\
 & & & & & \searrow & \\
 & & & & & \rightarrow & 0
 \end{array}$$

[Link to Diagram](#)

We also have  $h^1(\mathbb{C}_p) = 0$  by taking a sufficiently fine open cover where  $p$  is only in one open set. So just checking Čech cocycles yields  $C_U^1(C, \mathbb{C}_p) := \prod_{i < j} \mathbb{C}_p(U_i \cap U_j) = 0$  since  $p$  is in no intersection.



$X$

$p$

We obtain  $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D_0)) + 1$ , using that it is additive in SESs

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_3 \rightarrow 0 \implies \chi(\mathcal{E}_2) = \chi(\mathcal{E}_\infty) + \chi(\mathcal{E}_3)$$


and thus

$$\int_C c_1(\mathcal{O}(D)) = \sum n_i = \deg(D) = \deg D_0 + 1.$$

The last step is to show that  $\chi(C, \mathcal{O}) = 1 - g$ , so just define  $g$  so that this is true! ■

**Remark 15.1.5:** Why is every  $L \cong \mathcal{O}(D)$  for some  $D$ ? Easy to see if  $L$  has meromorphic sections: if  $s$  is a meromorphic section of  $L$ , then the following works:

$$D = \text{Div}(s) = \sum_p \text{Ord}_p(s)[p].$$

Then  $\mathcal{O} \cong L \otimes \mathcal{O}(-D)$  has a meromorphic section  $ss_{-D}$ , a global nonvanishing section with  $\text{Div}(ss_{-D}) = \emptyset$ . Proving that every holomorphic line bundle has a meromorphic section is hard! 

# 16 | Friday, February 19

## 16.1 Applications of Riemann-Roch

### Definition 16.1.1 (Curves)

A **curve** is a compact complex manifold of complex dimension 1.

**Example 16.1.2(?)**: Let  $C$  be a curve, then  $\Omega_C^1$  is the sheaf of holomorphic 1-forms, and  $\Omega_C^{>1} = 0$ . We also have the sheaves  $A^{1,0}, A^{0,1}, A^{1,1}$ , the sheaves of smooth  $(p, q)$ -forms. Here the only nonzero combinations are  $(0, 0), (0, 1), (1, 0), (1, 1)$  by dimensional considerations. Let  $L$  be a holomorphic line bundle on  $C$ , then

$$\chi(C, L) = h^0(L) - h^1(L) = \deg(L) + 1 - g.$$

**Remark 16.1.3**: In general it can be hard to compute  $h^1(L)$ , since this is sheaf cohomology (sections over double overlaps, cocycle conditions, etc). On the other hand,  $h^0$  is easy to understand, since  $h^0(\Omega_C^1)$  is the dimension of the global holomorphic sections  $H^0(C, L) = L(C)$ . A key tool here is the following:

### Proposition 16.1.4 (Serre Duality).

$$H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega_C^1)^\vee,$$

noting that these are both global sections of a line bundle.

*Proof (?)*.

Recall that we had a resolution of the sheaf  $L$  given by smooth vector bundles:

$$0 \rightarrow L \hookrightarrow L \otimes A^{0,0} \xrightarrow{\bar{\partial}} L \otimes A^{0,1} \xrightarrow{\bar{\partial}} 0.$$

So we know that  $H^1(C, L) = H^0(L \otimes A^{0,1}) / \bar{\partial} H^0(L \otimes A^{0,0})$ . Choose a Hermitian metric  $h$  on  $L$ , i.e. a map  $h : L \otimes \bar{L} \rightarrow \mathcal{O}$ . On fibers, we have  $h_p : L_p \otimes \bar{L}_p \rightarrow \mathbb{C}$ . We'll also choose a metric on  $C$ , say  $g$ . Since  $C$  is a Riemann surface, we have an associated volume form  $\nu$  on  $C$  (essentially the determinant), so we can define a pairing between sections of  $L \otimes A^{0,0}$ :

$$\langle s, t \rangle := \int_C h(s, \bar{t}) d\nu.$$

Note that  $\langle s, s \rangle = \int_C h(s, \bar{s}) d\nu \geq 0$  since  $h(s, \bar{s})(p) = 0 \iff s_p = 0$ , and moreover this integral is zero if and only if  $s = 0$ . So we have an inner product on  $H^0(L \otimes A^{0,0})$ . We can also define a pairing on sections of  $L \otimes A^{0,1}$ , say

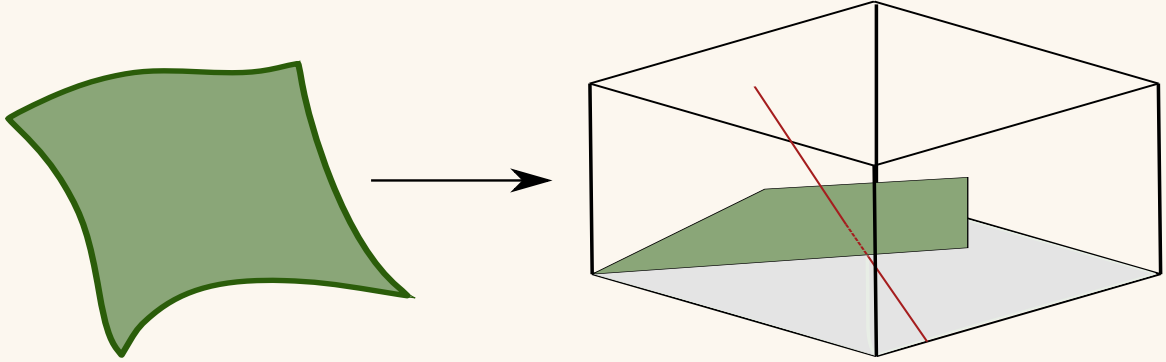
$$\langle s \otimes \alpha, t \otimes \beta \rangle = \int_C h(s, \bar{t}) \alpha \wedge \bar{\beta}.$$

Note that  $h$  is a smooth function and  $\alpha \wedge \bar{\beta}$  is a  $(1,1)$ -form. Moreover, this is positive and nondegenerate. We want to understand the cokernel of the linear map

$$H^0(L \otimes A^{0,0}) \xrightarrow{\bar{\partial}} H^0(L \otimes A^{0,1}).$$

To compute  $\text{coker}(\bar{\partial})$ , we can look at the kernel of the adjoint, and it suffices to find the orthogonal complement of  $\text{im}(\bar{\partial})$ , i.e.

$$\text{coker}(\bar{\partial}) = \left\{ t \in H^0(L \otimes A^{0,1}) \mid \langle \bar{\partial}s, t \rangle = 0 \forall s \right\}.$$



So we want to understand sections  $t \in H^0(L \otimes A^{0,1})$  such that

$$\int_C (\bar{\partial}s) \bar{t} = 0 \quad \forall s \in H^0(L \otimes A^{0,0}),$$

where  $\partial C = \emptyset$ . We'll basically want to do integration by parts on this. Note that  $h(s, t) = hst$  here where we view  $h$  as a certain section. Note that  $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$ , so we can replace  $\partial$  with  $d = \bar{\partial} + \partial$  and apply Stokes' theorem:

$$\begin{aligned} \int_C sd(h\bar{t}) &= 0 & \forall s \in H^0(L \otimes A^{0,0}) \\ 0 &= \int_C s\bar{\partial}(h\bar{t}) \\ &= \int_C s \frac{\bar{\partial}(h\bar{t})}{d\nu} d\nu \\ &= \left\langle s, \frac{\bar{\partial}(h\bar{t})}{d\nu} \right\rangle \end{aligned}$$

where  $h \in C^\infty(L^{-1} \otimes \bar{L}^{-1})$  and  $h\bar{t} \in C^\infty(L^{-1} \otimes A^{1,0})$ . But the right-hand side is in  $H^0(L \otimes A^{0,0})$  and by nondegeneracy we can conclude

$$\frac{\bar{\partial}(h\bar{t})}{d\nu} = 0 \iff \bar{\partial}(h\bar{t}) = 0.$$

We thus have  $h\bar{t} \in H^0(L^{-1} \otimes A^{1,0})$  which is a holomorphic line bundle tensored with  $A^{0,0}$ . Thus  $\text{coker}(\bar{\partial}) \cong_h H^0(L^{-1} \otimes \Omega^1)$ . ■

**Remark 16.1.5:** We showed  $\langle \bar{\partial}s, t \rangle = \langle s, Y(t) \rangle$  where  $Y$  is the adjoint given above. Then the kernel of  $Y$  wound up being where  $\bar{\partial}$  vanishes, i.e. holomorphic sections of a separate bundle. Here we had

- $t \in H^0(L \otimes A^{0,1})$
- $\bar{t} \in H^0(\bar{L} \otimes A^{1,0})$
- $h \in H^0(L^{-1} \otimes \overline{L^{-1}})$



## ToDos

### List of Todos

Check! . . . . .	42
Revisit this last section, had to clarify a few things for myself! . . . . .	47
Question: does $\text{Gr}_d(\mathbb{C}^\infty)$ deformation retract onto the image of this map? . . . . .	49
Can't read screenshot! :( . . . . .	50
Why? Split into line bundles? . . . . .	56

# Definitions

1.2.1	Definition – Topological Manifold . . . . .	4
1.2.4	Definition – Restricted Structures on Manifolds . . . . .	5
2.0.5	Definition – Kirby-Siebenmann Invariant of a 4-manifold . . . . .	10
3.0.4	Definition – Signature . . . . .	11
3.1.2	Definition – Riemannian Metrics . . . . .	12
3.1.3	Definition – Almost complex structure . . . . .	12
4.1.1	Definition – Presheaves and Sheaves . . . . .	14
4.2.5	Definition – $\mathcal{O}$ -modules . . . . .	17
4.2.8	Definition – Morphisms of Sheaves . . . . .	18
5.1.3	Definition – Global Sections Sheaf . . . . .	20
6.0.1	Definition – Principal Bundles . . . . .	22
6.0.3	Definition – The Frame Bundle . . . . .	23
6.0.6	Definition – Orthogonal Frame Bundle . . . . .	24
6.0.8	Definition – ? . . . . .	24
6.0.10	Definition – Unitary Frame Bundle . . . . .	24
6.0.14	Definition – Associated Bundles . . . . .	25
7.1.1	Definition – Connections . . . . .	26
7.1.6	Definition – Flat Connection and Flat Sections . . . . .	29
7.2.1	Definition – ? . . . . .	29
8.0.4	Definition – Acyclic Sheaves . . . . .	31
9.0.7	Definition – ? . . . . .	35
9.0.9	Definition – Cauchy-Riemann Equations . . . . .	35
10.0.8	Definition – Lattice . . . . .	41
10.0.10	Definition – ? . . . . .	41
10.0.14	Definition – ? . . . . .	42
10.0.15	Definition – ? . . . . .	42
11.0.5	Definition – ? . . . . .	44
11.1.1	Definition – ? . . . . .	46
12.0.8	Definition – ? . . . . .	49
13.0.2	Definition – ? . . . . .	49
13.0.4	Definition – Pontryagin Classes . . . . .	50
14.0.5	Definition – Chern Character . . . . .	53
14.0.6	Definition – Todd Class . . . . .	53
14.0.8	Definition – Todd Class . . . . .	54
14.1.2	Definition – Euler Characteristic of a Sheaf . . . . .	54
16.1.1	Definition – Curves . . . . .	60

# Theorems

3.0.1	Theorem – Freedman . . . . .	10
3.0.5	Theorem – Rokhlin's Theorem . . . . .	11
3.0.7	Theorem – Donaldson . . . . .	11
3.1.5	Theorem – ? . . . . .	13
8.0.3	Theorem – ? . . . . .	31
8.0.6	Theorem – (Important!) . . . . .	31
9.0.12	Theorem – Properties of Singular Cohomology . . . . .	36
11.0.7	Theorem – Serre . . . . .	45
13.0.1	Theorem – ? . . . . .	49
13.0.6	Theorem – Brown Representability . . . . .	50
13.0.8	Proposition – ? . . . . .	50
13.0.9	Theorem – Splitting Principle for Complex Vector Bundles . . . . .	52
15.0.2	Theorem – ? . . . . .	55
15.1.1	Theorem – Riemann-Roch . . . . .	56
16.1.4	Proposition – Serre Duality . . . . .	60



## Exercises

4.2.11	Exercise – ?	18
5.1.2	Exercise – ?	20
5.1.9	Exercise – ?	21
6.0.16	Exercise – ?	26
7.1.3	Exercise – ?	26
7.1.7	Exercise – ?	29
7.2.2	Exercise – ?	29
9.0.2	Exercise – ?	34
10.0.11	Exercise – ?	42
12.0.7	Exercise – ?	49

## Figures

### List of Figures

1	Picking one basis element in the time direction . . . . .	40
---	---	----

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