Title

D. Zack Garza

Table of Contents

Contents

1 Lecture 12

1.1 Brauer Groups

Goal: for C a curve over $k = \bar{k}$, we've computed

$$H^i(C, \mathbb{G}_m) = \begin{cases} \mathcal{O}_C^\times(C) & i = 0 \\ \operatorname{Pic}(C) & i = 1 \\ 0 & i > 1 \end{cases}.$$

Currently $i > 1$ is a mystery, so today we'll look at $i = 2$. Recall that we've reduced this to the Galois cohomology of the function field $H^i(k(C), \mathbb{G}_m)$ and of the strict Henselization [1] $H^i(K_{\bar{x}}, \mathbb{G}_m)$.

Today we'll try to understand the Galois cohomology of a field with coefficient in $\bar{k}^\times$, or $\mathbb{G}_m$ thought of as a sheaf on the étale site. We'll discuss $i = 2$, and a general principle in group cohomology is that if one understands $i = 1, 2$ then one can often understand all degrees.

In general, H^1 has a geometric interpretation: torsors. H^2 is much harder: they classify more general objects called **gerbes**. A miracle is that $H^2(\mathbb{G}_m)$ has real meaning, and is very closely related to real physical objects (certain torsors). Recall that we defined the *cohomological Brauer group of* X (**??**) as

$$\operatorname{Br}^{\text{coh}} \coloneqq \operatorname{Br}'(X) \coloneqq H^i(X_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}.$$

We also started defining the Brauer group by considering

$$\bigcup_n \{\text{étale locally trivial } \operatorname{PGL}_n\text{-torsors}\} \xrightarrow{\delta} H^2(X_{\text{ét}}, \mathbb{G}_m),$$

and defining $\operatorname{Br}(X) \coloneqq \operatorname{im} f$ as a set, which is a reasonably concrete geometric object. This map came from a LES in cohomology, coming from a SES of sheaves, not all of which were abelian. The definition of δ was the boundary map of

$$\bigcup_n H^1(X_{\text{ét}}, \operatorname{PGL}_n) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mathbb{G}_m) \tag{1}$$

arising from the SES of sheaves of groups on $X_{\text{ét}}$,

$$1 \to \mathbb{G}_m \to \operatorname{GL}_m \to \operatorname{PGL}_n \to 1.$$

We argued last time that this was exact in the Zariski topology since the RHS map was a $\mathbb{G}_m$-torsor and thus Zariski locally trivial. What does δ mean? [2]

[1] The stalk of the structure sheaf, $\mathcal{O}_{C,x}$.
[2] Best reference: Giraud, "Cohomologie non Abelienne".

Remark 1.1.1: Making the LES here is a little subtle. You get a long exact sequence of *sets* here which terminates at the H^2 we're interested in, although one usually doesn't get a map of the form $H^1(C) \to H^2(B)$ for a SES $A \to B \to C$, you need that A is abelian (or in the center).

We'll now try to make δ explicit in terms of Čech cohomology, which is the only way we have to make sense of the LHS set in equation (1)