

Title

D. Zack Garza

Table of Contents

Contents

Table of Contents	2
Prologue	3
0.1 References	3
0.2 Notation	3
0.3 Summary of Important Concepts	4
0.4 Useful Examples	5
0.4.1 Varieties	5
0.4.2 Presheaves / Sheaves	5
0.5 The Algebra-Geometry Dictionary	5

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p .

Notation	Definition
$k[\mathbf{x}] = k[x_1, \dots, x_n]$	Polynomial ring in n indeterminates

$+ \mid k(\mathbf{x}) = k(x_1, \dots, x_n) \mid$	Rational function field in n indeterminates
$\mid \mathcal{U} \rightrightarrows X \mid$	An open cover $\mathcal{U} = \{U_j \mid j \in J\}$
$\mid \mid \Delta_X \mid$	The diagonal $\{(x, x) \mid x \in X\} \subseteq X \times X$
$\mid \mid \mathbb{A}_k^n \mid$	Affine n -space
$\mathbb{A}_k^n := \{\mathbf{a} = [a_1, \dots, a_n] \mid a_j \in k\}$	
$\mid \mid \mathbb{P}_k^n \mid$	Projective n -space
$\mid \mid \mathbb{P}_k^n := (k^n \setminus \{0\}) / x \sim \lambda x \mid$	
$\mid \mid = \{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \mid p, q, \in k[x_1, \dots, x_n]\}$	
$\mid \mid + \mid V(J), V_a(J) \mid$	Variety associated to an ideal $J \leq k[x_1, \dots, x_n]$
$\mid \mid - \mid := \{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$	
$\mid \mid I(S), I_a(S) \mid$	Ideal associated to a subset $S \subseteq \mathbb{A}_k^n$
$\mid \mid - \mid := \{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in X\}$	
$\mid \mid A(X) \mid$	Coordinate ring of a variety, $k[x_1, \dots, x_n]/I(X)$
$\mid \mid V_p(J) \mid$	Projective variety of an ideal
$\mid \mid - \mid := \{\mathbf{x} \in \mathbb{P}_k^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$	
$\mid \mid I_p(S) \mid$	Projective ideal (?)
$\mid \mid - \mid := \{f \in k[x_1, \dots, x_n] \mid f \text{ is homogeneous and } \deg f = d\}$	
$\mid \mid S(X) \mid$	Projective coordinate ring, $k[x_1, \dots, x_n]/I_p(X)$
$\mid \mid f^h \mid$	Homogenization, $x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$
$\mid \mid f^i \mid$	Dehomogenization, $f(1, x_1, \dots, x_n)$
$\mid \mid J^h \mid$	Homogenization of an ideal, $\{f^h \mid f \in J\}$
$\mid \mid \bar{X} \mid$	Projective closure of a subset
$\mid \mid - \mid := V_p(J^h) := \{\mathbf{x} \in \mathbb{P}^n \mid f^h(\mathbf{x}) = 0 \forall f \in J\}$	
$\mid \mid \mathcal{O}_X \mid$	Structure sheaf $\{f : U \rightarrow k \mid f \in k(\mathbf{x}) \text{ locally}\}$
$\mid \mid D(f) \mid$	Distinguished open set, $D(f) = V(f)^c = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
 - The Zariski topology?
 - Irreducibility?
 - Connectedness?
 - Dimension?
- What is the definition of a presheaf?
 - What are some examples and counterexamples?
- What is the definition of sheaf?
 - What are some examples?
 - What are some presheaves that are not sheaves?
- What is the definition of \mathcal{O}_X , the sheaf of regular functions?
 - How does one compute \mathcal{O}_X for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
 - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
 - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples

0.4.1 Varieties

- $V(xy - 1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x - p)$ a point.

0.4.2 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\cdot, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$, the constant sheaf associated to \mathbb{R} (locally constant real-valued functions)
- $\text{Hol}(\cdot, \mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

Algebra	Geometry
$k[x_1, \dots, x_n]$	$\mathbb{A}_{/k}^n$
Maximal ideals $\mathfrak{m} = x_1 - p_1, \dots, x_n - p_n$	Points $[a_1, \dots, a_n]$
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \dots, x_n]$	$V(J)$ the zero locus
Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$	Irreducible subsets
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$
$k[x_1, \dots, x_n]/I(X)$	$A(X)$ (Functions on X)
$A(X)$ a domain	X is irreducible
$A(X)$ indecomposable	X is connected
Krull dimension n (chains of primes)	Topological dimension n (chains of irreducibles)