Category O

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Wednesday 15^{th} April, 2020

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1 Definitions

- Indecomposable: doesn't decompose as $AoplusB$. Weaker than irreducible.
- Irreducible: simple, i.e. no nontrivial proper submodules. Implies indecomposable.
- Solvable: Obtained as a tower of extensions of abelian groups; derived series terminates; composition series has cyclic sections;
- Borel: maximal solvable subalgebra
- Radical: Largest solvable ideal.
- Semisimple: Direct sum of simple; radical equals 0.
  - Acts in a diagonalizable way.
- Reductive: Radical equals center.
- Artinian: ?
- Completely reducible: ?

2 List of Notation

- $M(\lambda)$: Verma Modules
- $L(\lambda)$: Unique simple quotients of Verma modules.

3 SL2 Theory

Definition 3.0.1.
The group and the algebra:

$$\mathfrak{sl}(n, \mathbb{C}) = \left\{ M \in \mathrm{GL}(n, \mathbb{C}) \;\middle|\; \det(M) = 1 \right\}$$
$$\mathfrak{sl}(n, \mathbb{C}) = \left\{ M \in \mathfrak{gl}(n, \mathbb{C}) \;\middle|\; \mathrm{Tr}(M) = 0 \right\}.$$

Generated by

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with relations

$$[xy] = h$$
$$[hx] = 2x$$
$$[hy] = -2y.$$

Facts:

- $\dim_{\mathbb{C}} \mathfrak{sl}(n, \mathbb{C}) = n^2 - 1$.
- $\mathfrak{sl}(n, \mathbb{C}) \sim A_{n-1}$
- $\mathfrak{sl}(2, \mathbb{C})$ is simple.
    - Proof: show that if $I \trianglelefteq \mathfrak{g}$ then any of $x, y, h \in I$, using commutation relations.

Irreducible (?) representations (i.e. simple modules) of $\mathfrak{sl}(2, \mathbb{C})$ are parameterized by $L(\lambda),\ n \in \mathbb{Z}^+$ and have basis given by

$$h \cdot v_i = (\lambda - 2i)v_i$$
$$x \cdot v_i = (\lambda - i + 1)v_{i-1}$$
$$y \cdot v_i = v_{i+1},$$

setting $v_{-1} = v_{\lambda+1}$ where v_0 is the unique vector in $L(\lambda)$ annihilated by x.

- The Weyl group is given by $W \cong \mathbb{Z}/2\mathbb{Z}$ where $\lambda - 2i \mapsto -(\lambda - 2i)$.
- The integral weight lattice is given by $\Lambda \cong \mathbb{Z}$.
- The integral root lattice (?) is given by $\Lambda_R \cong 2\mathbb{Z}$.
- The usual representation on $\mathbb{C}^2$: h has eigenvalues ± 1, yields $L(1)$.
- The adjoint representation on $\mathbb{C}^3$: ad $h = \mathrm{diag}(2, 0, -2)$ with eigenvalues $0, \pm 2$, yields $L(2)$.
- Each $L(\lambda)$ has 1-dimensional weight spaces, parameterized by weights $\lambda, \lambda - 2, \cdots, -\lambda$.
- $\dim L(\lambda) = \lambda + 1$.

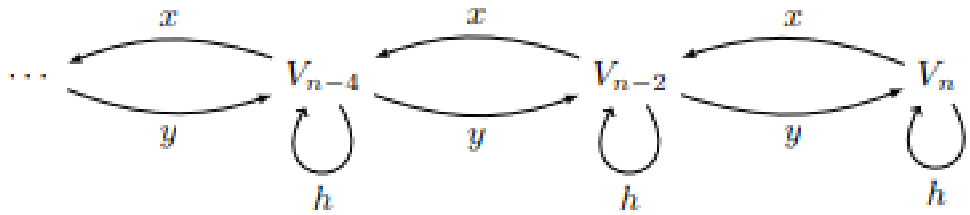


FIGURE 2.2. The action of x and y on the eigenspaces of an irreducible $\mathfrak{sl}_2$-module.

Basis

4 Wednesday January 8

> Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$".
> Course Website: `https://faculty.franklin.uga.edu/brian/math-8030-spring-2020`

4.1 Chapter Zero: Review

> Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase mathfrak characters!

In this course, we'll take $k = \mathbb{C}$.

Recall that a Lie Algebra is a vector space $\mathfrak{g}$ with a bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying

- $[xx] = 0$ for all $x \in \mathfrak{g}$
  - Exercise: this implies $[xy] = -[yx]$.
    > Hint: Consider $[x + y, x + y]$. Note that the converse holds iff char $k \neq 2$.

    > Exercise: This implies Lie Algebras never have an identity.
- $[x[yz]] = [[xy]z] + [y[xz]]$ (The Jacobi identity)
  - This says x acts as a derivation.

> **Definition 4.0.1** (Abelian).
> $\mathfrak{g}$ is *abelian* iff $[xy] = 0$ for all $x, y \in \mathfrak{g}$.

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
  - A vector subspace that is closed under brackets.
- Homomorphisms
  - I.e. a linear transformation ϕ that commutes with the bracket, i.e. $\phi([xy]) = [\phi(x)\phi(y)]$.
- Ideals

Exercise Given a vector space (possibly infinite-dimensional) over k, then (exercise) $\mathfrak{gl}(V) := \mathrm{End}_k(V)$ is a Lie algebra when equipped with $[fg] = f \circ g - g \circ f$.

Definition 4.0.2 (Representation)**.**
A *representation* of $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \longrightarrow \mathrm{GL}(V)$ for some V.

Example 4.1.
The adjoint representation is $\mathrm{ad}\ : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, where $\mathrm{ad}\ (x)(y) := [xy]$.

Representations give $\mathfrak{g}$ the structure of a module over V, where $x \cdot v := \phi(x)(v)$. All of the usual module axioms hold, where now $[xy] \cdot v = x \cdot (\cdot v) - y \cdot (x \cdot v)$.

Example 4.2.
The trivial representation $V = k$ where $x \cdot a = 0$.

Definition 4.0.3 (Irreducible)**.**
V is *irreducible* (or *simple*) iff V as exactly two $\mathfrak{g}$-invariant subspaces, namely $0, V$.

Definition 4.0.4 (Completely Reducible)**.**
V is *completely reducible* iff V is a direct sum of simple modules, and *indecomposable* iff V can not be written as $V = M \oplus N$, a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragradient/dual* $V^\vee := \hom_k(V, k)$ where $(x \cdot f) = -f(x \cdot v)$ for $f \in V^\vee, x \in \mathfrak{g}, v \in V$.
- The direct sum $V \oplus W$ where $x \cdot (v, w) = (x \cdot v, x \cdot w)$ and $x \cdot (v + w) = x \cdot v_x \cdot w$.
- The tensor product where $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.
- $\hom_k(V, W)$ where $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$.
  - Note that if we take $W = k$ then the first term vanishes and this recovers the dual.

4.2 Semisimple Lie Algebras

Definition 4.0.5 (Derived Ideal)**.**
The *derived ideal* is given by $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \mathrm{span}_k\left(\left\{[xy] \;\middle|\; x, y \in \mathfrak{g}\right\}\right)$.

This is the analog of the commutator subgroup.

Lemma 4.1.
$\mathfrak{g}$ is abelian iff $\mathfrak{g}^{(1)} = \{0\}$, and 1-dimensional algebras are always abelian.

This follows because if $[xy] := xy = yx$ then $[xy] = 0 \iff xy = yx$.

Definition 4.1.1 (Simple)**.**
A lie group $\mathfrak{g}$ is *simple* iff the only ideals of $\mathfrak{g}$ are $0, \mathfrak{g}$ and $\mathfrak{g}^{(1)} \neq \{0\}$.

Note that thus rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

Definition 4.1.2 (Derived Series)**.**
The *derived series* is defined by $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$, continuing inductively. $\mathfrak{g}$ is said to be **solvable** if $\mathfrak{g}^{(n)} = 0$ for some n.

Lemma 4.2.
Abelian implies solvable.

Review definition of nilpotent algebras.

Definition 4.2.1 (Semisimple)**.**
$\mathfrak{g}$ is *semisimple* (s.s.) iff $\mathfrak{g}$ has no nonzero solvable ideals.

Exercise Simple implies semisimple.

Some remarks:

1. Semisimple algebras $\mathfrak{g}$ will usually have solvable subalgebras.
2. $\mathfrak{g}$ is semisimple iff $\mathfrak{g}$ has no nonzero abelian ideals.

Definition 4.2.2 (Killing Form)**.**
The *Killing form* is given by $\kappa : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow k$ where $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$, which is a symmetric bilinear form.

Lemma 4.3.
$\kappa([xy], z) = \kappa(x, [yz])$.

Recall that if $\beta : V^{\otimes 2} \longrightarrow k$ is any symmetric bilinear form, then its radical is defined by

$$\mathrm{rad}\beta = \left\{ v \in V \;\middle|\; \beta(v, w) = 0 \ \forall w \in V \right\}.$$

Definition 4.3.1 (Nondegenerate Bilinear Form)**.**
A bilinear form β is *nondegenerate* iff $\mathrm{rad}\beta = 0$.

Lemma 4.4.
$\mathrm{rad}\kappa \trianglelefteq \mathfrak{g}$ is an ideal, which follows by the above associative property.

Theorem 4.5 ***(Characterization of Semisimplicity Using the Killing Form).***
$\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Example 4.3.
The standard example of a semisimple lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \;\middle|\; \mathrm{Tr}(x) = 0 \right\}$.

Note: from now on, $\mathfrak{g}$ will denote a semisimple lie algebra over $\mathbb{C}$.

Theorem 4.6 ***(Weyl's Complete Reducibility Criterion).***
Every finite dimensional representation of a semisimple $\mathfrak{g}$ is completely reducible.

In other words, the category of finite-dimensional representations is relatively uninteresting – there are no extensions, so everything is a direct sum. Thus once you classify the simple algebras (which isn't terribly difficult), you have complete information.

5 Friday January 10th

Let $\mathfrak{g}$ be a finite dimensional semisimple lie algebra over $\mathbb{C}$.

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the trace zero $n \times n$ matrices.

Let $\mathfrak{h}$ be a maximal toral subalgebra, where $x \in \mathfrak{g}$ is *toral* if x is semisimple, i.e. $\mathrm{ad}\, x$ is semisimple (i.e. diagonalizable).

Example 5.1.
$\mathfrak{h}$ is the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$.

Fact $\mathfrak{h}$ is abelian, so $\mathrm{ad}\, \mathfrak{h}$ consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \alpha(h)x \;\forall h \in \mathfrak{h} \right\}$ where $\alpha \in \mathfrak{h}^\vee$ is a linear functional.

Here $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, so $[hx] = 0$ corresponds to zero eigenvalues, and (fact) it turns out that $\mathfrak{h}$ is its own centralizer.

We then obtain a set of roots of $\mathfrak{h}, \mathfrak{g}$ given by $\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \;\middle|\; \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\} \right\}$.

Example 5.2.
$\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$ for some $i \neq j$, the matrix with a 1 in the i,j position and zero elsewhere.

Fact The restriction $\kappa|_{\mathfrak{h}}$ is nondegenerate, so we can identify $\mathfrak{h}, \mathfrak{h}^\vee$ via κ (can always do this with vector spaces with a nondegenerate bilinear form), where κ maps to another bilinear form $(\,\cdot\,,\,\cdot\,)$.

$$\mathfrak{h}^\vee \ni \lambda \iff t_\lambda \in \mathfrak{h}$$
$$\lambda(h) = \kappa(t_\lambda, h) \quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

5.1 Facts About Φ and Root Spaces

Let $\alpha, \beta \in \Phi$ be roots.

1. Φ spans $\mathfrak{h}^\vee$ and does not contain zero.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other scalar multiple of α is in Φ.

Aside:

- $\dim \mathfrak{g}_\alpha = 1$.
- If $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ then there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$ spans a 3-dimensional subalgebra in $\mathfrak{sl}_2$, given by $x_\alpha = [0,1;0,0], y_\alpha = [0,0;1,0], h_\alpha = [1,0;0,-1]$.
- Under the correspondence $\mathfrak{h} \iff \mathfrak{h}^\vee$ induced by κ, $h_\alpha \iff \alpha^\vee := \dfrac{2\alpha}{(\alpha,\alpha)}$. Thus for all $\lambda \in \mathfrak{h}^\vee$,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}.$$

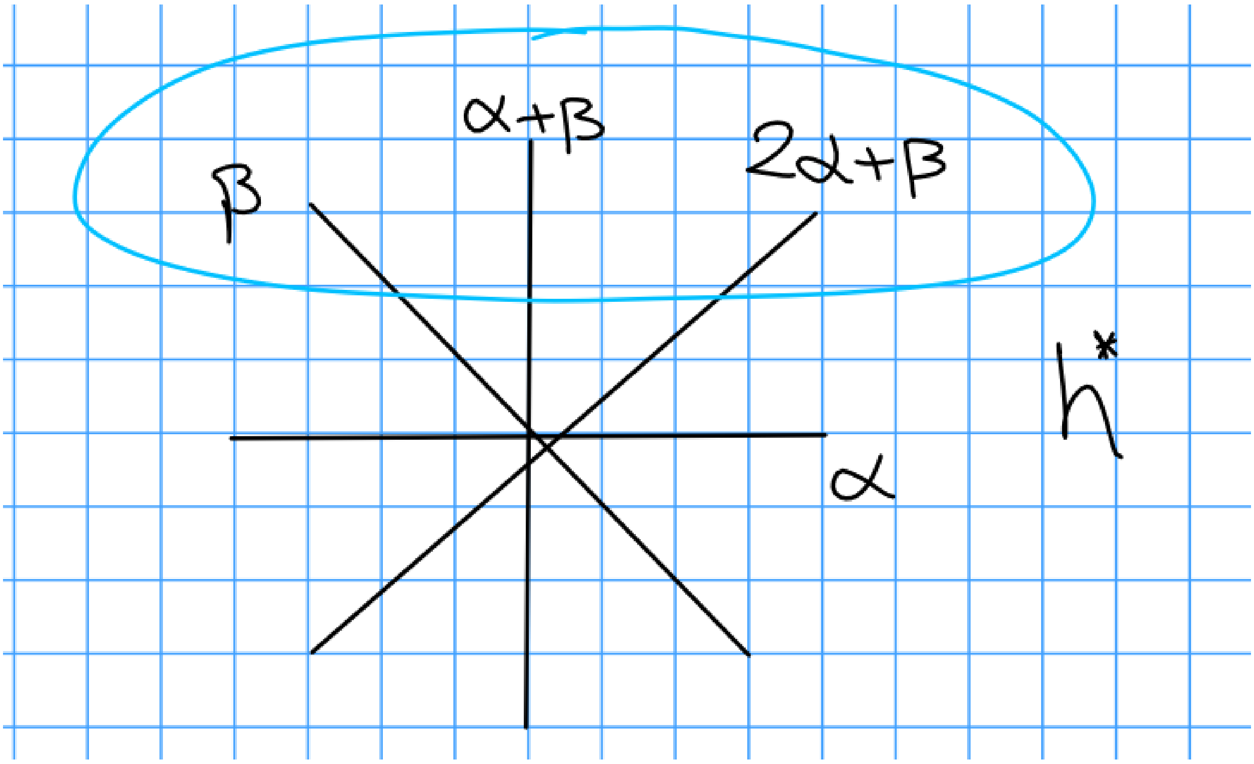
- If $\alpha + \beta \neq 0$, then $\kappa(g_\alpha, g_\beta) = 0$.

3. $(\beta, \alpha^\vee) \in \mathbb{Z}$
4. $S_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in \Phi$.

If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Example: If $\alpha = E_{ij}, \beta = E_{jk}$ where $k \neq i$, then $[E_{ij}, E_{jk}] = E_{ik}$.

- $\mathfrak{g}$ is generated as an algebra by the root spaces $\mathfrak{g}_\alpha$
- Root strings: If $\beta \neq \pm\alpha$, then the roots of the form $\alpha + k\beta$ for $k \in \mathbb{Z}$ form an unbroken string $\alpha - r\beta, \cdots, \alpha - \beta, \alpha, \alpha + \beta, \cdots, \alpha + \ell\beta$ consisting of at most 4 roots where $r - s = (\alpha, \beta^\vee)$.

Example: The circled roots below form the root string for β:



In general, a subset Φ of a real euclidean space E satisfying conditions (1) through (4) is an *(abstract) root system*.

When Φ comes from a $\mathfrak{g}$, $E := \mathbb{R}\Phi$.

5.1.1 The Root System

There exists a subset $\Delta \subseteq \Phi$ such that

- Δ is a $\mathbb{C}$-basis for $\mathfrak{g}^\vee$
- $\beta \in \Phi$ implies that $\beta = \sum\limits_{\alpha \in \Delta} c_\alpha \alpha$ with either
  - All $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$ or $\beta < 0$.
  - All $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$ or $\beta > 0$.

Δ is called a **simple system**.

If $\Delta = \{a_1, \cdots, a_\ell\}$ then $\Phi^+$are the *positive roots*, and $\Phi^+ \ni \beta = \sum\limits_{\alpha \in \Delta} c_\alpha \alpha$, then the *height* of β is defined as $\sum c_\alpha \in \mathbb{Z}_{>0}$.

Note that $\mathbb{Z}\Phi := \Lambda_r$ is a lattice, and is referred to as the *root lattice*, and $\Lambda_r \subset E = \mathbb{R}\Phi$. We also have $\Phi^+ = \left\{ \beta^\vee \mathrel{\Big|} \beta \in \Phi \right\}$, the *dual root system*, is a root system with simple system $\Delta^\vee$.

Important subalgebras of $\mathfrak{g}$:

- Upper triangular with zero diagonal $\mathfrak{n} = \mathfrak{n}^+ = \sum\limits_{\beta > 0} \mathfrak{g}_\beta$

- Lower triangular with zero diagonal $\mathfrak{n}^- = \sum_{\beta>0} \mathfrak{g}_{-\beta}$
- Upper triangular, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra
- Lower triangular, $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

There is thus a triangular (Cartan) decomposition, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Fact If $\beta \in \Phi^+ \setminus \Delta$, and if $\alpha \in \Delta$ such that $(\beta, \alpha^\vee) > 0$, then $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$ has height strictly less than the height of β.

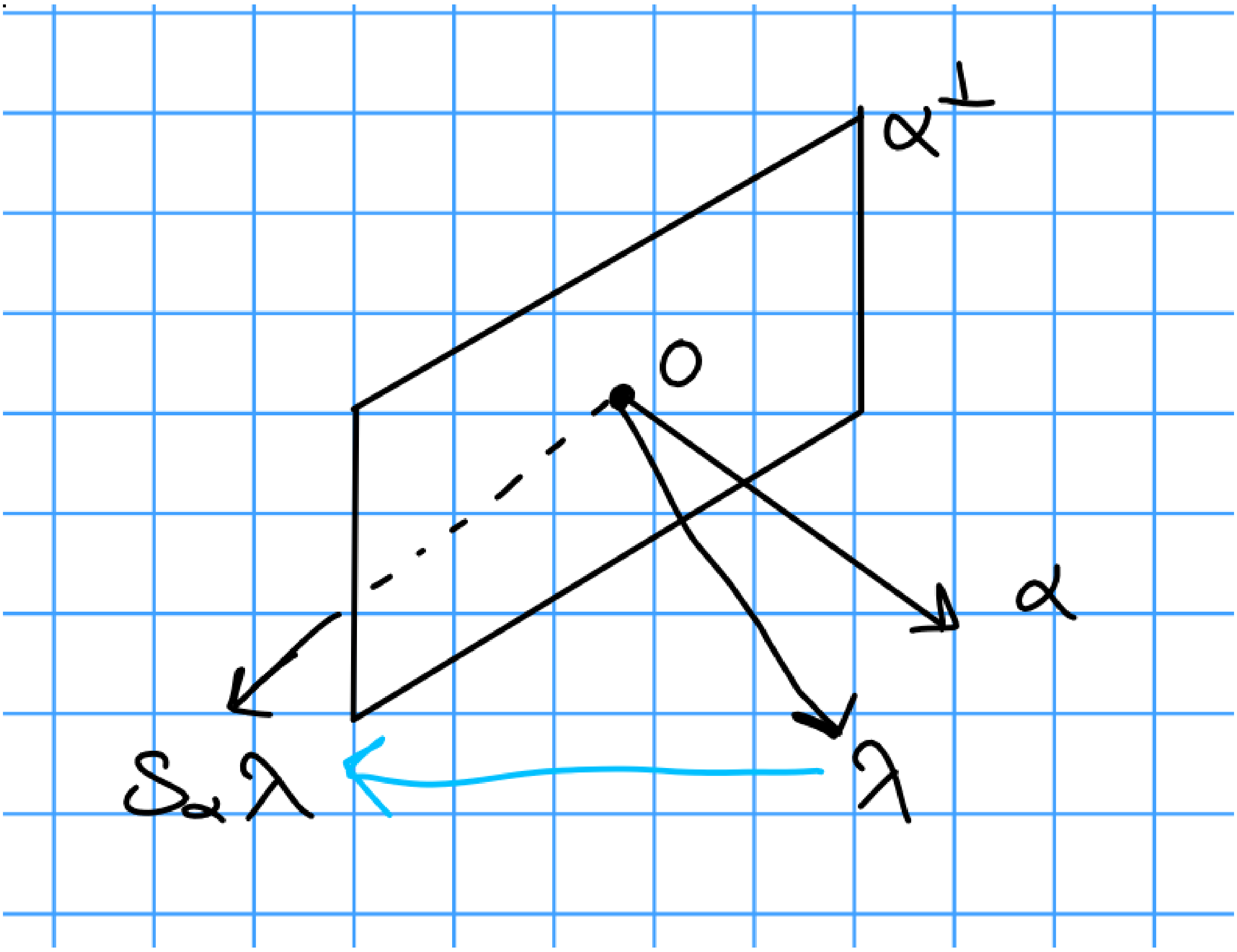
By root strings, $\beta - \alpha \in \Phi^+$ is positive root of height one less than β, yielding a way to induct on heights (useful technique).

5.1.2 Weyl Groups

For $\alpha \in \Phi$, define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\longrightarrow \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in E perpendicular to α:



Note that $S_\alpha^2 = \mathrm{id}$.

Define W as the subgroup of $\mathrm{GL}(E)$ generated by all s_α for $\alpha \in \Phi$, this is the *Weyl group* of $\mathfrak{g}$ or Φ, which is finite and $W = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$ is generated by simple reflections.

By (4), W leaves Φ invariant. In fact W is a finite Coxeter group with generators $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$ and defining relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ for $\alpha, \beta \in \Delta$ where $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$ and $m(\alpha, \alpha) = 1$.

Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group*.

6 Monday January 13th

6.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over $\mathbb{C}$. We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group

$$W = \left\langle s_\beta \;\middle|\; \beta \in \Phi \right\rangle = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$$

where $\Delta = \{a_i\}$ are the simple roots.

For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) \coloneqq n$ where $\ell(1) \coloneqq 0$.

Facts:

1. $\ell(w)$ is the size of the set $\left\{ \beta \in \Phi^+ \;\middle|\; w\beta < 0 \right\}$

- The above set is equal to $\Phi^+ \bigcap w^{-1}\Phi^{-1}$.
- In particular, for $\beta \in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
- Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.

2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \bigcap w\Phi$ (replacing w^{-1} with w)

3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = \left|\Phi^+\right|$ and $w_0(\Phi^+) = \Phi^-$.

- Also $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$\begin{aligned} \ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0 \end{aligned}$$

.

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

6.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \left\{ s_\alpha \,\middle|\, \alpha \in \Delta \right\}$. Then define

$$T := \bigcup_{w \in W} wSw^{-1} = \left\{ s_\beta \,\middle|\, \beta \in \Phi^+ \right\}.$$

This is the set of *all* reflections in W through hyperplanes in E.

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w, and refer to this as "the covering relation". So $w' \longrightarrow w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that

$$w' = w_0 \longrightarrow w_1 \longrightarrow \cdots \longrightarrow w_n = w.$$

This is called the **Bruhat-Chevalley order** on W.

Corollary 6.1.
$w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order*. In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively.

> Note that this is because conjugating a simple reflection may not yield a simple reflection again.

Recall that lie algebras yield finite crystallographic coxeter groups.

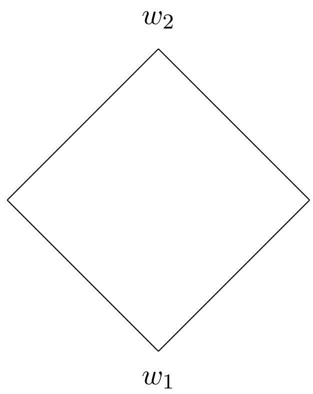
Properties For (W, S) a coxeter group,

a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \cdots s_n$ for w, where a subexpression is any subcollection of s_i in the same order.

> Note that this implies that 1 is not only a minimal element in this order, but an infimum.

b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.

c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\left\{ w \in W \,\middle|\, w_1 < w < w_2 \right\}$ is either 0 or 2.



6.3 Properties of Universal Enveloping Algebras

Let $\mathfrak{g}$ be any lie algebra, and $\phi : \mathfrak{g} \longrightarrow A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, dotted, "\tilde{\phi}"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[r, "\phi"'] & A \end{tikzcd}$$

Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \longrightarrow U(\mathfrak{g}_2)$, where $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \left\langle [x,y] - x \otimes y - y \otimes x \;\middle|\; x, y \in \mathfrak{g} \right\rangle .$$

Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\mathrm{ad}\ x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Theorem 6.2 ***(Poincaré-Birkhoff-Witt, i.e. PBW).***
If $\{x_1, \cdots x_n\}$ is a basis for $\mathfrak{g}$, then $\left\{ x_1^{t_1}, \cdots, x_n^{t_n} \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$ (noting that $x^n = x \otimes x \otimes \cdots x$ and $\mathbb{Z}^+$ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary 6.3.
$i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If $\mathfrak{g}$ is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for $\mathfrak{g}$, then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \longrightarrow \mathrm{GL}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

A note on notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

6.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the $\mathbb{R}$-span of the roots.

Definition 6.3.1 (Integral Weight Lattice)**.**
We also have the **integral weight lattice**

$$\Lambda = \left\{ \lambda \in E \;\middle|\; (\lambda, \alpha^\vee) \in \mathbb{Z} \; \forall \alpha \in \Phi (\text{or } \Phi^+ \text{ or } \Delta) \right\}.$$

There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and $\mathfrak{h}^\vee$. We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+\Delta = \mathbb{Z}^+\Phi^+$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a $\mathbb{Z}$-basis for Λ. Then Λ is a free abelian group of rank ℓ, and $\Lambda^+ = \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.

7 Wednesday January 15th

7.1 Review

The Weyl vector is given by

$$\rho = \bar{\omega}_1 + \cdots + \bar{\omega}_\ell = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \Lambda^+.$$

Some properties:

- If $\alpha \in \Delta$ then $(\rho, \alpha^\vee) = 1$
- $s_\alpha(\rho) = \rho - \alpha$.

Let $\lambda \in \Lambda^+$; a few facts:

1. The size of $\left\{ \mu \in \Lambda^+ \;\middle|\; \mu \leq \lambda \right\}$ (with the partial order from last time) is finite.
2. $w\lambda < \lambda$ for all $w \in W$.

The Weyl chamber (for a fixed root, $E =$ Euclidean space) is $C = \left\{ \lambda \in E \;\middle|\; (\lambda, \alpha) > 0 \; \forall \alpha \in \Delta \right\}$ (Note that the hyperplane splits E into connected components, we mark this component as distinguished.)

- A connected component of the union of hyperplanes is orthogonal to roots

- They're in bijection with Δ
- They're permuted simply transitively by W.

And $\overline{C}$ denotes the fundamental domain.

7.2 Weight Representations

For $\lambda \in \mathfrak{h}^\vee$, we let $M_\lambda = \left\{ v \in M \;\middle|\; h \cdot v = \lambda(h)v \ \forall h \in \mathfrak{h} \right\}$ denote a *weight space* of M if $M_\lambda \neq 0$. In this case, λ is a *weight* of M. The dimension of M_λ is the *multiplicity* of λ in M, and we define the set of weights as $\Pi(M) = \left\{ \lambda \in \mathfrak{h}^\vee \;\middle|\; M_\lambda \neq 0 \right\}$.

Example if $M = \mathfrak{g}$ under the adjoint action, then $\Pi(M) = \Phi \bigcup \{0\}$.

Remark The weight vectors for distinct weights are linearly independent. Thus there is a $\mathfrak{g}$-submodule given by $\sum_\lambda M_\lambda$, which is in fact a direct sum.

Note: It may not be the case that $\sum_\lambda M_\lambda = M$, and can in fact be zero, although this is an odd situation.

See Humphreys #1, #20.2, p. 110.

In our case, we'll have a *weight module* $M = \bigoplus_\lambda M_\lambda$, so $\mathfrak{h} \curvearrowright M$ semisimply.

7.3 Finite Dimensional Modules

Recall Weyl's complete reducibility theorem, which implies that any finite dimensional $\mathfrak{g}$-module is a weight module. In fact, $\mathfrak{n}, \mathfrak{n}^- \curvearrowright M$ nilpotently.

Some facts:

- $\Pi(M) \subset \Lambda$ is a subset of the integral lattice.
- $\Pi(M)$ is W-invariant.
- $\dim M_\lambda = \dim M_{W\lambda}$ for any $\lambda \in \Pi(M), w \in W$.

7.4 Simple Finite Dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules

Fix the standard basis $\{x, h, y\}$ of $\mathfrak{sl}(2, \mathbb{C})$ with $[hx] = 2x, [hy] = -2y, [xy] = h$. Since $\dim \mathfrak{h} = 1$, there is a bijection $\mathfrak{h}^\vee \leftrightarrow \mathbb{C}$, $\Lambda \leftrightarrow \mathbb{Z}$, and $\Lambda_r \leftrightarrow 2\mathbb{Z}$ with $\alpha \longrightarrow 2$ and $\rho \longrightarrow 1$.

There is a correspondence between weights and simple modules:

$$\begin{aligned} \left\{\text{Isomorphism classes of simple modules}\right\} &\iff \Lambda^+ = \{0, 1, 2, 3, \cdots\} \\ L(\lambda) &\iff \lambda. \end{aligned}$$

Moreover, $L(\lambda)$ has a 1-dimensional weight spaces with weights $\lambda, \lambda - 2, \cdots, -\lambda$ and thus $\dim L(\lambda) = \lambda + 1$.

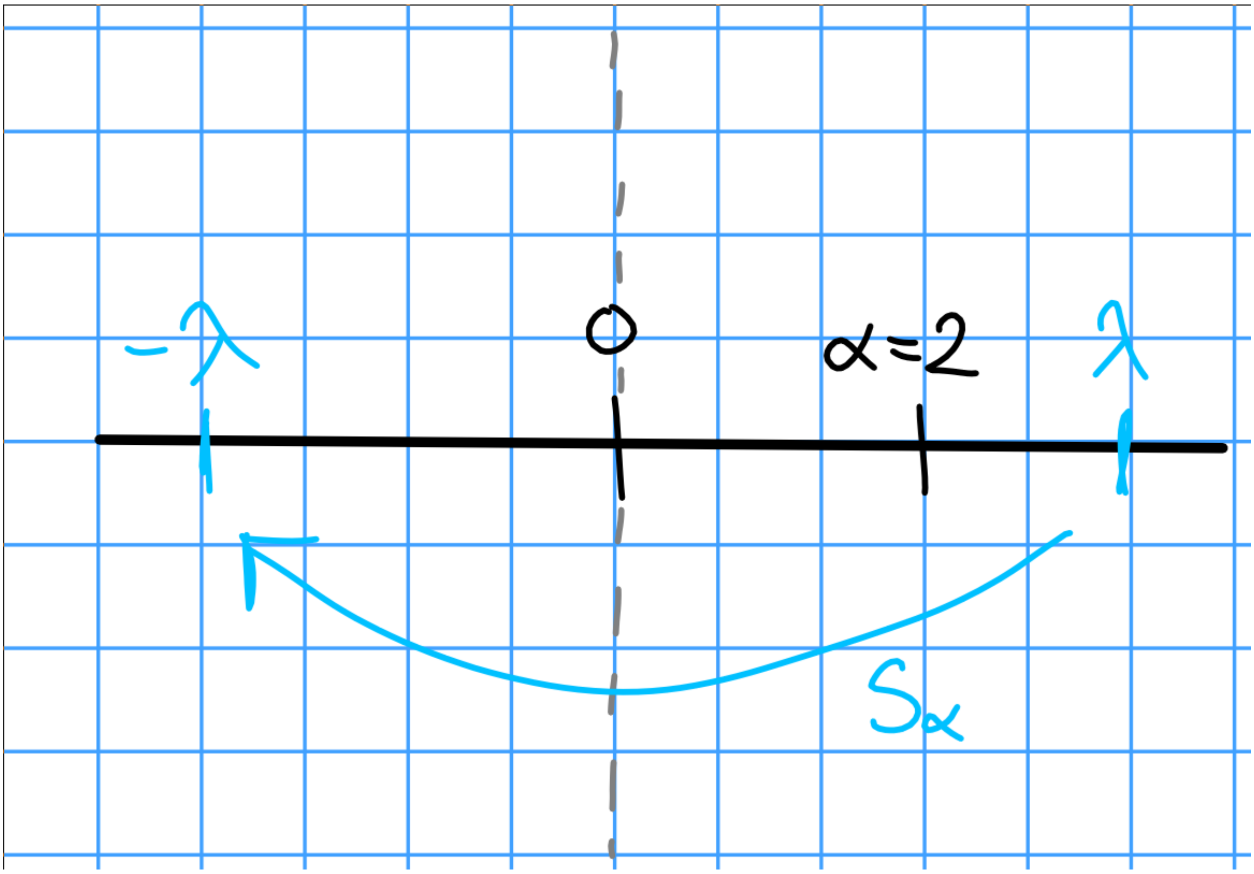
Examples:

- $L(0) = \mathbb{C}$, the trivial representation,
- $L(1) = \mathbb{C}^2$, the natural representation where $\mathfrak{sl}(2, \mathbb{C})$ acts by matrix multiplication,
- $L(2) = \mathfrak{g}$, the adjoint representation.

Choose a basis $\{v_1, \cdots, v_\lambda\}$ for $L(\lambda)$ so that $\mathbb{C}v_0 = M_\lambda$, $\mathbb{C}v_1 = M_{\lambda-2}$, $\cdots \mathbb{C}v_\lambda M_{-\lambda}$. Then

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$, where $v_{-1} := 0$
- $y \cdot v_i = (i + 1)v_{i+1}$ where $v_{\lambda+1} := 0$.

We then say $L(\lambda)$ is a highest weight module, since it is generated by a highest weight vector λ. Then $W = \{1, s_\alpha\}$, where s_α is reflection through 0 by the identification $\alpha = 2$.



8 Chapter 1: Category $\mathcal{O}$ Basics

The category of $U(\mathfrak{g})$-modules is too big. Motivated by work of Verma in 60s, started by Bernstein-Gelfand-Gelfand in the 1970s. Used to solve the Kazhdan-Lusztig conjecture.

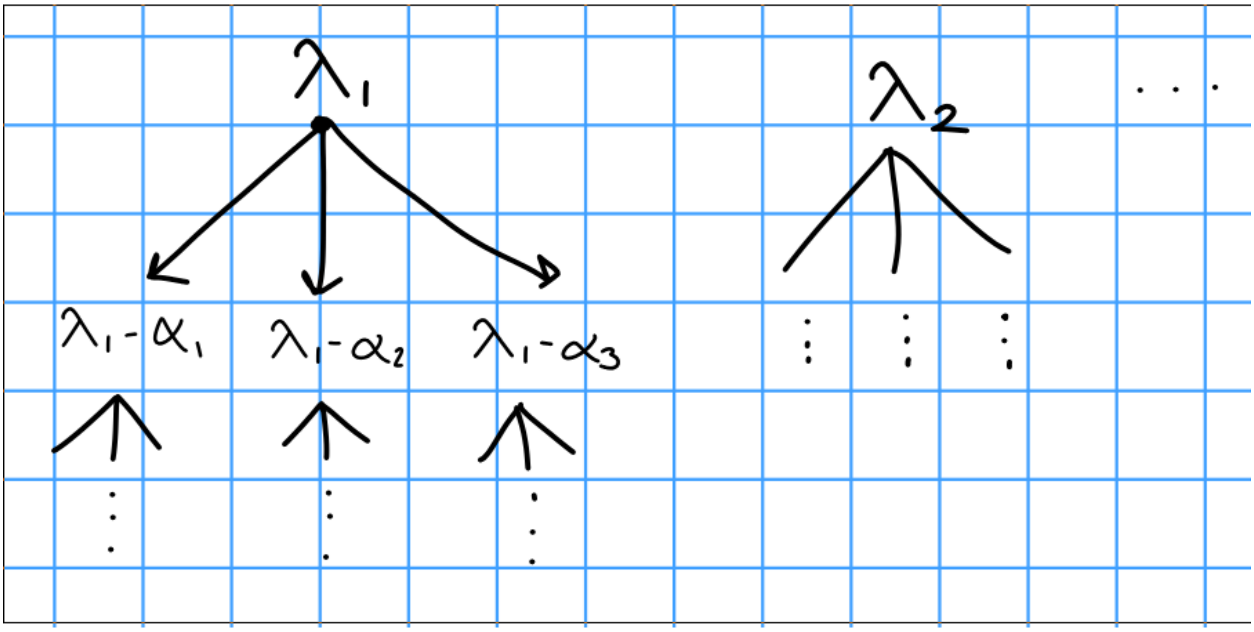


Figure 1: Image

8.1 Axioms and Consequences

Definition 8.0.1 (Category O)**.**
$\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$ modules consisting of M such that

1. M is finitely generated as a $U(\mathfrak{g})$-module.
2. M is $\mathfrak{h}$-semisimple, i.e. M is a weight module $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$.
3. M is locally n-finite, i.e. the dimension of $U(\mathfrak{n})v < \infty$ for all $v \in M$.

Example 8.1.
If $\dim M < \infty$, then M is $\mathfrak{h}$-semisimple, and axioms 1, 3 are obvious.

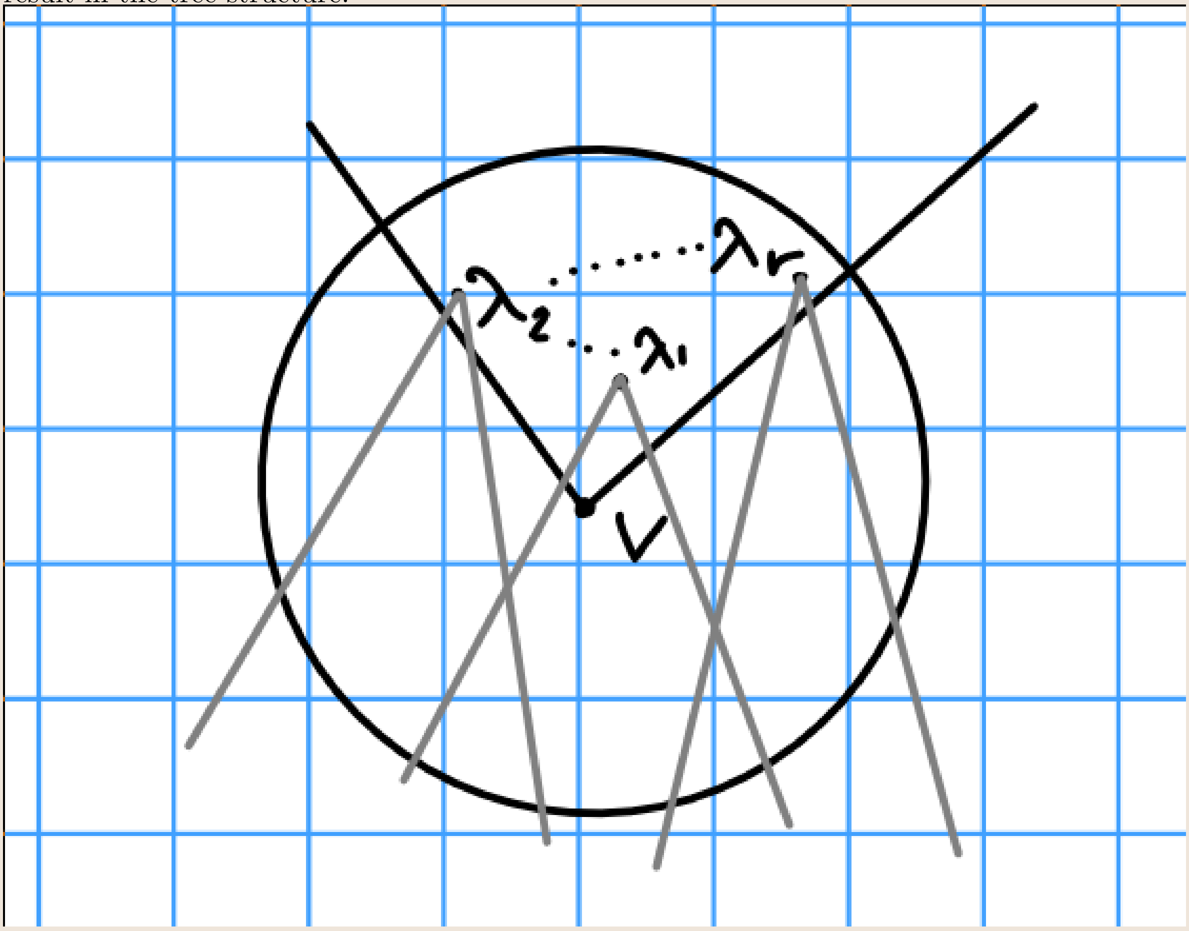
Lemma 8.1.
Let $M \in \mathcal{O}$, then

4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$.
5. There exist $\lambda_1, \cdots \lambda_r \in \mathfrak{h}^\vee$ such that $\Pi(M) \subset \bigcup_{i=1}^{\lambda} (\lambda_i - \mathbb{Z}^+\Phi^+)$.

Proof .
By axiom 2, every $v \in M$ is a finite sum of weight vectors in M. We can thus assume that our finite generating set consists of weight vectors. We can then reduce to the case where M is

generated by a single weight vector v. So consider $U(\mathfrak{g}) \cdot v$. By the PBW theorem, there is a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$.

By axiom 3, $U(\mathfrak{n}) \cdot v$ is finite dimensional, so there are finitely many weights of finite multiplicity in the image. Then $U(\mathfrak{h})$ acts by scalar multiplication, and $U(\mathfrak{n}^-)$ produces the "cones" that result in the tree structure:



A weight of the form $\mu = \lambda_i - \sum n_i \alpha_i$ can arise from $y_{n_1}^{n_1} \cdots$.

■

9 Friday January 17th

Let M

1. Be finitely generated,
2. Semisimple $M = \oplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$,
3. Locally finite
4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$,
5. Satisfy the forest condition for weights.

Theorem 9.1 ***(Properties of O).***

a. $\mathcal{O}$ is Noetherian (ascending chain condition on submodules, i.e. no infinite filtrations by submodules)
b. $\mathcal{O}$ is closed under quotients, submodules, finite direct sums
c. $\mathcal{O}$ is abelian (similar to a category of modules)
d. If $M \in \mathcal{O}$, $\dim L < \infty$, then $L \otimes M \in \mathcal{O}$ and the endofunctor $M \mapsto L \otimes M$ is exact
e. If $M \in \mathcal{O}$ is locally $Z(\mathfrak{g})$-finite (recall: this is the center of $U(\mathfrak{g})$), i.e. $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ for all $v \in M$.
f. $M \in \mathcal{O}$ is finitely generated module. (?)

Proofs of a and b: See BA II, page 103.

Proof of c: Implied by (b), BA II Page 330.

Proof (of (d)).
Can check that $L \otimes M$ satisfies 2 and 3 above. Need to check first condition. Take a basis $\{v_i\}$ for L and $\{w_j\}$ a finite set of generators for M. The claim is that $B = \{v_i \otimes w_j\}$ generates $L \otimes M$. Let N be the submodule generated by B.
For any $v \in V$, $v \otimes w_j \in N$. For arbitrary $x \in \mathfrak{g}$, compute $x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + x \otimes (v \cdot w_j)$. Since the LHS is in N and the first term on the RHS is in N, the entire RHS is in N. By iterating, we find that $v \otimes (u \cdot w_j) \in N$ for all PBW monomials u. So $L \otimes M \in \mathcal{O}$.
■

Proof (of (e)).
Since $v \in M$is a sum of weight vectors, wlog we can assume $v \in M_\lambda$ is a weight vector (where $\lambda \in \mathfrak{h}^\vee$). For any central element $z \in Z(\mathfrak{g})$, we can compute $h \cdot (z \cdot v) = z \cdot (h \cdot v) = z \cdot \lambda(h)v = \lambda(h) z \cdot v$. Thus $z \cdot v \in M_\lambda$. By (4), we know that $\dim M_\lambda < \infty$, so $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ as well.
■

Proof (of (f)).
By 5, M is generated by a finite dimensional $U(\mathfrak{b})$ submodule N. Since we have a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{b})$, there is a basis of weight vectors for N that generates M as a $U(\mathfrak{n}^-)$ module.
■

9.1 Highest Weight Modules

Definition 9.1.1 (Maximal Vector)**.**
A **maximal vector** $v^+ \in M \in \mathcal{O}$ is a nonzero vector such that $\mathfrak{n} \cdot v^+ = 0$.

Note: By 2 and 3, every nonzero $M \in \mathcal{O}$ has a maximal vector.

Definition 9.1.2 (Highest Weight Modules)**.**
A **highest weight module** M of highest weight λ is a module generated by a maximal vector

of weight λ, i.e. $M = U(\mathfrak{g})v^+ = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})v^+ = U(\mathfrak{n}^-)v^+$.

Theorem 9.2 ***(Properties of Highest Weight Modules).***
Let $M = U(\mathfrak{n}^-)v^+$ be a highest weight module, where $v^+ \in M_\lambda$. Fix $\Phi^+ = \{\beta_1, \cdots, \beta_n\}$ with root vectors $y_i \in \mathfrak{g}_{\beta_i}$.

a. M is the $\mathbb{C}$-span of PBW monomials $\left\langle y_1^{t_1} \cdots y_m^{t_m} \right\rangle$ of weight $\lambda - \sum t_i \beta_i$. Thus M is a module.
b. All weights μ of M satisfy $\mu \leq \lambda$
c. $\dim M_\mu < \infty$ for all $\mu \in T(M)$, and $\dim M_\lambda = 1$. In particular, property (3) holds and $M \in \mathcal{O}$.
d. Every nonzero quotient of M is a highest-weight module of highest weight λ.
e. Every submodule of M is a weight module, and any submodule generated by a maximal vector with $\mu < \lambda$ is proper. If M is semisimple, then the set of maximal weight vectors equals $\mathbb{C}^\times v^+$.
f. M has a unique maximal submodule N and a unique simple quotient L, thus M is indecomposable.
g. All simple highest weight modules of highest weight λ are isomorphic.

For such M, $\dim \operatorname{End}(M) = 1$. (Category $\mathcal{O}$ version of Schur's Lemma, generalizes to infinite dimensional case)

Proofs of a to e: Either obvious or follows from previous results. First few imply M is in $\mathcal{O}$, and we know the latter hold for such modules.

Proof (of (f)).
N is a sum of submodules, so $N = \sum M_i$, proper submodules of M. So take $L = M/N$. To see indecomposability, there exists a better proof in section 1.3. ∎

Proof (of (g)).
Let $M_1 = U(\mathfrak{n}^-)v_1^+$ and M_2 be define similarly, where the $v_i \in (M_i)_\lambda$ have the same weight. Then $M_0 := M_1 \oplus M_2$ implies that $v^+ := (v_1^+, v_2^+)$ is a maximal vector for M_0. So $N := U(\mathfrak{n}^-)v^+$ is a highest weight module of highest weight λ.
We have the following diagram:

$$\begin{tikzcd}
 & N \arrow[dl, tail] \arrow[d, hook] \arrow[dr, tail] & \\
 & M_1 \oplus M_2 \arrow[dl] \arrow[dr] & \\
M_1 & & M_2
\end{tikzcd}$$

and since e.g. $N \longrightarrow M_1$ is not the zero map, it is a surjection.
By (f), N is a unique simple quotient, so this forces $M_1 \cong M_2$. Since M is simple, any nonzero $\mathfrak{g}$-endomorphism ϕ must be an isomorphism, and so we take $v^+ \mapsto cv^+$ for some $c \neq 0$. Note that since ϕ is also a $\mathfrak{h}$-morphism, we have $\dim M_\lambda = 1$.
Since v^+ generates M and $\phi(u \cdot v^+) = u\phi(v^+) = cu \cdot v^+$, thus ϕ is multiplication by a constant.

■

10 Wednesday January 22nd

Note: Try problems 1.1 and 1.3*.

Recall: In category $\mathcal{O}$, we have finite dimensional, semisimple modules over $\mathbb{C}$ with triangular decompositions.

If M is any $U(\mathfrak{g})$ module than a $v^+ \in M_\lambda$ a weight vector (so $\lambda \in \mathfrak{h}^\vee$) is primitive iff $\mathfrak{n} \cdot v^+ = 0$. Note: it doesn't have to be of maximal weight. M is a highest weight module of highest weight λ iff it's generated over $U(\mathfrak{g})$ as an associative algebra by a maximal vector v^+ of weight λ. Then $M = U(\mathfrak{g}) \cdot v^+$.

See structure of highest weight modules, and irreducibility.

Corollary 10.1.
If $0 \neq M \in \mathcal{O}$, then M has a finite filtration with quotients highest weight modules, i.e. $M_0 \subset M_1 \subset \cdots \subset M_n = M$ with M_i/M_{i-1} highest weight modules. Note that the quotients are not necessarily simple, so this isn't a composition series, although we'll show such a series exists later.

Theorem 10.2 ***(Module Span of Weight Vectors is Finite Dimensional).***
Let V be the $\mathfrak{n}$ submodule of M generated by a finite set of weight vectors which generate M as a $U(\mathfrak{g})$ module. (i.e. take the finite set of weight vectors and act on them by $U(\mathfrak{n})$.) Then $\dim_{\mathbb{C}} V < \infty$ since M is locally $\mathfrak{n}$-finite.

Proof .
Induction on $n = \dim V$. If $n = 1$, M itself is a highest weight module.

Note that $\mathfrak{n}$ increases weights.

For $n > 1$, choose a weight vector $v_1 \in V$ of weight λ which is maximal among all weights of V. Set $M_1 := U(\mathfrak{g})v_1$; this is a highest weight submodule of M of highest weight λ. ($\mathfrak{n}$ has to kill v_1, otherwise it increases weight and v_1 wouldn't be maximal.)
Let $\overline{M} = M/M_1 \in \mathcal{O}$, this is generated by the image of $\overline{V}$ of V and thus $\dim \overline{V} < \dim V$. By the IH, $\overline{M}$ has the desired filtration, say $0 \subset \overline{M}_2 \subset \overline{M}_{n-1} \subset \overline{M}_n = \overline{M}$. Let $\pi : M \longrightarrow M/M_1$, then just take the preimages $\pi^{-1}(\overline{M}_i)$ to be the filtration on M.

■

Note: by isomorphism theorems, the quotients in the series for M are isomorphic to the quotients for $\overline{M}$.

10.1 Verma and Simple Modules

Constructing *universal* highest weight modules using "algebraic induction". Start with a nice subalgebra of $\mathfrak{g}$ then "induce" via $\otimes$ to a module for $\mathfrak{g}$.

Recall $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, here $\mathfrak{h} \oplus \mathfrak{n}$ is the Borel subalgebra $\mathfrak{b}$, and $\mathfrak{n}$ corresponds to a fixed choice of positive roots in Φ^+ with basis Δ. Then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$. Given any $\lambda \in \mathfrak{h}^\vee$, let $\mathbb{C}_\lambda$ be the

1-dimensional $\mathfrak{h}$-module (i.e. 1-dimensional $\mathbb{C}$-vector space)on which $\mathfrak{h}$ acts by λ.

Let $\{1\}$ be the basis for $\mathbb{C}$, so $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. Then there is a map $\mathfrak{b} \longrightarrow \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$, so make C_λ a $\mathfrak{b}$-module via this map. This "inflate" C_λ into a 1-dimensional $\mathfrak{b}$-module.

Note that $\mathfrak{h}$ is solvable, and by Lie's Theorem, every finite dimensional irreducible $\mathfrak{b}$-module is of the form $\mathbb{C}_\lambda$ for some $\lambda \in \mathfrak{h}^\vee$.

Definition 10.2.1 (Verma Module of Highest Weight)**.**
$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda := \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ is the *Verma module of highest weight* λ.

This process is called algebraic/tensor induction. This is a $U(\mathfrak{g})$ module via left multiplication, i.e. acting on the first tensor factor.

Since $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{h})$, we have $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$, but at what level of structure?

- As a vector space (clear)
- As a $\mathfrak{n}^-$-module via left multiplication
- As a $\mathfrak{h}^-$-module via the $\otimes$ action.

In particular, $M(\lambda)$ is a *free* $U(\mathfrak{n}^-)$-module of rank 1.

Note: this always happens when tensoring with a vector space?

Consider $v^+ := 1 \otimes 1 \in M(\lambda)$ (note that $U(\mathfrak{n}^-)$ is not homogeneous, so not graded, but does have a filtration). Then v^+ is nonzero, and freely generates $M(\lambda)$ as a $U(\mathfrak{n}^-)$-module. Moreover $\mathfrak{n} \cdot v^+ = 0$ since for $x \in \mathfrak{g}_\beta$ for $\beta \in \Phi^+$, we have

$$\begin{aligned} x(1 \otimes 1) &= x \otimes 1 \\ &= 1 \otimes x \cdot 1 \quad \text{since } x \in \mathfrak{b} \\ &= 1 \otimes 0 \implies x \in \mathfrak{n} \\ &= 0, \end{aligned}$$

and for $h \in \mathfrak{h}$,

$$\begin{aligned} h(1 \otimes 1) &= h1 \otimes 1 \\ &= 1 \otimes h1 \\ &= 1 \otimes \lambda(h)1 \\ &= \lambda(h)v^+. \end{aligned}$$

So $M(\lambda)$ is a highest weight module of highest weight λ, and thus $M(\lambda) \in \mathcal{O}$.

Observation Any weight $\lambda \in \mathfrak{h}^\vee$ is the highest weight of some $M \in \mathcal{O}$. Let $\Pi(M)$ denote the set of weights of a module, then $\Pi(M(\lambda)) = \lambda - \mathbb{Z}^+\Phi^+$.

By PBW, we can obtain a basis for $M(\lambda)$ as $\left\{ y_1^{t_1} \cdots y_m^{t_m} v^+ \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$. Taking a fixed ordering $\{\beta_1, \cdots, \beta_m\} = \Phi^+$, then $0 \neq y_i \in \mathfrak{g}_{-\beta_i}$. Then every weight of this form is a weight of some $M(\lambda)$, and every weight of $M(\lambda)$ is of this form: $\lambda - \sum t_i \beta_i$.

Remark: The functor $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\,\cdot\,) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \,\cdot\,$ from the category of finite-dimensional $\mathfrak{g}$-semisimple $\mathfrak{b}$-modules to $\mathcal{O}$ is an exact functor, since it is naturally isomorphic to $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \,\cdot\,$ (which is clearly exact?)

Alternate construction of $M(\lambda)$: Let I by a left ideal of $U(\mathfrak{g})$ which annihilates v^+, so

$$I = \left\langle \mathfrak{n}, h - \lambda(h) \cdot 1 \;\middle|\; h \in \mathfrak{h} \right\rangle .$$

Since v^+ generates $M(\lambda)$ as a $U(\mathfrak{g})$-module, then (by a standard ring theory result) $M(\lambda) = U(\mathfrak{g})/I$, since I is the annihilator of $M(\lambda)$.

Theorem 10.3 ***(Universal property of Verma Modules).***
Let M be any highest weight module of highest weight λ generated by v. Then $I \cdot v = 0$, so I is the annihilator of v and thus M is a quotient of $M(\lambda)$. Thus $M(\lambda)$ is universal in the sense that every other highest weight module arises as a quotient of $M(\lambda)$.

By theorem 1.2, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ (nonstandard notation) and a unique simple quotient $L(\lambda)$ (standard notation).

Theorem 10.4 ***(Characterization of Simple Modules and Schur's Lemma).***
Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$ and is determined uniquely up to isomorphism by its highest weight. Moreover, there is an analog of Schur's lemma: $\dim \hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$, i.e. it's 1 iff $\lambda = \mu$ and 0 otherwise.

Note: up to isomorphism, we've found all of the simple modules in $\mathcal{O}$, and most are finite-dimensional.

Proof: Next class.

11 Friday January 24th

A standard theorem about classifying simple modules in category $\mathcal{O}$:

Theorem 11.1 ***(Classification of Simple Modules).***
Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$, and is determined uniquely up to isomorphism by its highest weight. Moreover, $\dim \hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\lambda\mu}$.

Proof .
Let $L \in \mathcal{O}$ be irreducible. As observed in 1.2, L has a maximal vector v^+ of some weight λ.
Recall: can increase weights and reach a maximal in a finite number of steps.
Since L is irreducible, L is generated by that weight vector, i.e. $LU(\mathfrak{g}) \cdot v^+$, so L must be a highest weight module.

Standard argument: use triangular decomposition.

By the universal property, L is a quotient of $M(\lambda)$. But this means $L \cong L(\lambda)$, the unique irreducible quotient of $M(\lambda)$.
By Theorem 1.2 part g (see last Friday), $\dim \mathrm{End}_{\mathcal{O}}(L(\lambda)) = 1$ and $\hom_{\mathcal{O}}(L(\mu), L(\lambda)) = 0$ since both entries are irreducible.

■

Theorem 11.2 ***(1.2 f, Highest Weight Modules are Indecomposable).***
A highest weight module M is indecomposable, i.e. can't be written as a direct sum of two nontrivial proper submodules.

Proof (of Theorem 1.2 f).
Suppose $M = M_1 \oplus M_2$ where M is a highest weight module of highest weight λ. Category $\mathcal{O}$ is closed under submodules, so M_i are weight modules and have weight-space decompositions. But M_λ is 1-dimensional (triangular decomposition, only $\mathbb{C}$ acts), and thus $M_\lambda \subset M_1$. Since M_λ is a highest weight module, it generates the entire module, so $M \subset M_1$. The reverse holds as well, so $M = M_1$ and this forces $M_2 = 0$.

■

11.1 1.4: Maximal Vectors in Verma Modules

1.5: Examples in the case $\mathfrak{sl}(2)$, over $\mathbb{C}$ as usual.

First, some review from lie algebras.

Let $\mathfrak{g}$ be any lie algebra, and take $u, v \in U(\mathfrak{g})$. Recall that we have the formula $uv = [uv] + vu$, where we use the definition $[uv] = uv - vu$.

Let x, y_1, y_2 be in $\mathfrak{g}$, what is $[x, y_1 y_2]$? Use the fact that ad $x(y_1, y_2)$ acts as a derivation, and so $[x, y_1 y_2] = [xy_1]y_2 + y_1[xy_2]$, which is a bracket entirely in the Lie algebra. This extends to an action on $U(\mathfrak{g})$ by the product rule.

Recall that $\mathfrak{sl}(2)$ is spanned by $y = [0, 0; 1, 0], h = [1, 0; 0, -1], x = [0, 1; 0, 0]$, where each basis vector spans $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}$ respectively. Then $[xy] = h, [hx] = 2x, [hy] = -2y$, so $E_{ij}E_{kl} = \delta_{jk}E_{il}$ (should be able to compute easily!).

Then $\mathfrak{h} = \mathbb{C}$, so $\mathfrak{h}^\vee \cong \mathbb{C}$ where $\lambda \mapsto \lambda(h)$. So we identify λ with a complex number, this is kind of like a bundle of Verma modules over $\mathbb{C}$.

Consider $M(1)$, then $\lambda = 1$ will denote $\lambda(h) = 1$. As in any Verma module, $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_\mathbb{C} \mathbb{C}_\lambda$. We can think of v^+ as $1 \otimes 1$, with the action $yv^+ = y1 \otimes 1$. Note that y has weight -2.

Weight	Basis
1	v^+
-1	yv^+
-3	y^2v^+
-5	y^3v^+

Consider how $x \curvearrowright y^2v^+$. Note that x has weight $+2$. We have

$$\begin{aligned}
x \cdot y^2 v^+ &= x \cdot y^2 \otimes 1_\lambda \\
&= ([xy^2] + y^2 x) \otimes 1 \\
&= ([xy]y + y[xy]) \otimes 1 + y^2 \otimes x \cdot 1 \quad \text{moving } x \text{ across the tensor because ?} \\
&= ([xy]y + y[xy]) \otimes 1 + 0 \quad \text{since } x \text{ is maximal} \\
&= (hy + yh) \otimes 1 \\
&= hy \otimes 1 + y \otimes h \cdot 1 \\
&= hy \otimes 1 + \lambda(h)1 \\
&= hy \otimes 1 + 1 \\
&= ([xy] + yh) \otimes 1 + y \otimes 1 \\
&= -2y \otimes 1 + y \otimes 1 + y \otimes 1 \\
&= 0.
\end{aligned}$$

So y moves us downward through the table, and x moves upward, except when going from $-3 \longrightarrow -1$ in which case the result is zero!

Thus there exists a morphism $\phi : M(-3) \longrightarrow M(1)$, with image $U(\mathfrak{g})y^2v^+ = U(\mathfrak{n}^-)y^2v^+$. So the image of ϕ is everything spanned by the bases in the rows $-3, -5, \cdots$, which is exactly $M(-3)$. So $M(-3) \hookrightarrow M(1)$ as a submodule.

Motivation for next section: we want to find Verma modules which are themselves submodules of Verma modules.

It turns out that $\operatorname{im} \phi \cong N(1)$. We should have $M(1)/N(1) \cong L(1)$. What is the simple module of weight 1 for $\mathfrak{sl}(2)$? The weights of $L(n)$ are $n, n-2, n-4, \cdots, -n$, so the representations are parameterized by $n \in \mathbb{Z}^+$. These are the Verma modules for $\mathfrak{sl}(2)$. What happens is that $y \curvearrowright -n \longrightarrow -n-2$ gives a maximal vector, so the calculation above roughly goes through the same way. So we'll have a similar picture with $L(n)$ at the top.

11.2 Back to 1.4

Question 1: What are the submodules of $M(\lambda)$?

Question 2: What are the Verma submodules $M(\mu) \subset M(\lambda)$? Equivalently, when do maximal vectors of weight $\mu < \lambda$ (the interesting case) lie in $M(\lambda)$?

Question 3: As a special case, when do maximal vectors of weight $\lambda - k\alpha$ for $\alpha \in \Delta$ lie in $M(\lambda)$ for $k \in \mathbb{Z}^+$?

Fix a Chevalley basis for $\mathfrak{g}$ (see section 0.1) $h_1, \cdots, h_\ell \in \mathfrak{h}$ and $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi^+$. Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ and let $x_i = x_{\alpha_i}, y_i = y_{\alpha_i}$ be chosen such that $[x_i y_i] = h_i$.

Lemma 11.3.
For $k \geq 0$ and $1 \leq i, j \leq \ell$, then

a. $[x_j, y_i^{k+1}] = 0$ if $j \neq i$
b. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$.
c. $[x_i, y_i^{k+1}] = -(k+1)y_i(k \cdot 1 - h_i)$.

Proof (sketch).
Both easy to prove by induction since $[x_j, y_i] \longrightarrow \alpha_j - \alpha_i \notin \Phi$ is a difference of simple roots. For $k = 0$, all identities are easy. For $k > 0$, an inductive formula that uses the derivation property, which we'll do next class.

■

12 Monday January 27th

12.1 Section 1.4

Fix $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$, $x_i \in g_{\alpha_i}$ and $y_i \in g_{-\alpha_i}$ with $h_i = [x_i y_i]$.

Lemma 12.1.
For $k \geq 0$ and $1 \leq i, j \leq \ell$,

a. $[x_j y_i^{k+1}] = 0$ if $j \neq i$
b. $[h_j y_i^{k+1}] = -(k+1)\alpha_i(h_j) y_i^{k+1}$
c. $[x_i y_i^{k+1}] = (k+1) y_i^k (k \cdot 1 - h_i)$.

Proof (Sketch for (c)).
By induction, where $k = 0$ is clear.

$$\begin{aligned}
[x + i y_i^{k+1}] &= [x_i y_i] y_i^k + y_i [x_i y_i^k] \\
&= h_i y_i^k + y_i(-k y_i^{k-1}((k-1)1 - h_i)) \quad \text{by I.H.} \\
&= (k+1) y_i^k h_i - (k^2 - k + 2k) y_i^k \\
&= -(k+1) y_i^k (k \cdot 1 - h_i)
\end{aligned}$$

.

■

Proposition 12.2.
Suppose $\lambda \in \mathfrak{h}^\vee, \alpha \in \Delta$, and $n := (\lambda, \alpha^\vee) \in \mathbb{Z}^+$. Then in $M(\lambda)$, $y_\alpha^{n+1} v^+$ is a maximal weight vector of weight $\mu := \lambda - (n+1)\alpha < \lambda$.

Note this is free as an $U(\mathfrak{n}^-)$-module, so $v^+ \neq 0$. Note that $n = \lambda(h_\alpha)$.

By the universal property, there is a nonzero homomorphism $M(\mu) \longrightarrow M(\lambda)$ with image contained in $N(\lambda)$, the unique maximal proper submodules of $M(\lambda)$.

Proof .
Say $\alpha = \alpha_i$. Fix $j \neq i$.

$$\begin{aligned}
x_i y_\alpha^{n+1} \otimes 1 &= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes x_j \cdot 1 \\
&= [x_j y_i^{n+1}] \otimes 1 + y_i^{n+1} \otimes 0 \quad \text{by a} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
x_i y_i^{n+1} \otimes 1 &= [x_i y_i^{n+1} \otimes 1] \\
&= -(n+1) y_i^n (n \cdot 1 - h_i) \otimes 1 \\
&= -(n+1)(n - \lambda(h_i)) 1 \otimes 1 \\
&\coloneqq -(n+1)(\lambda(h_i) - \lambda(h_i)) 1 \otimes 1 \\
&= 0.
\end{aligned}$$

Since g_{α_j} generate $\mathfrak{n}$ as a Lie algebra, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. This shows that $\mathfrak{n} \cdot y_i^{n+1} v^+ = 0$, and the weight of $y_i^{n+1} v^+$ is $\lambda - (n+1)\alpha_i$. So y_i^{n+1} is a maximal vector of weight μ. The universal property implies there is a nonzero map $M(\mu) \longrightarrow M(\lambda)$ sending highest weight vectors to highest weight vectors and preserving weights. The image is proper since all weights of M_μ are less than or equal to $\mu < \lambda$.

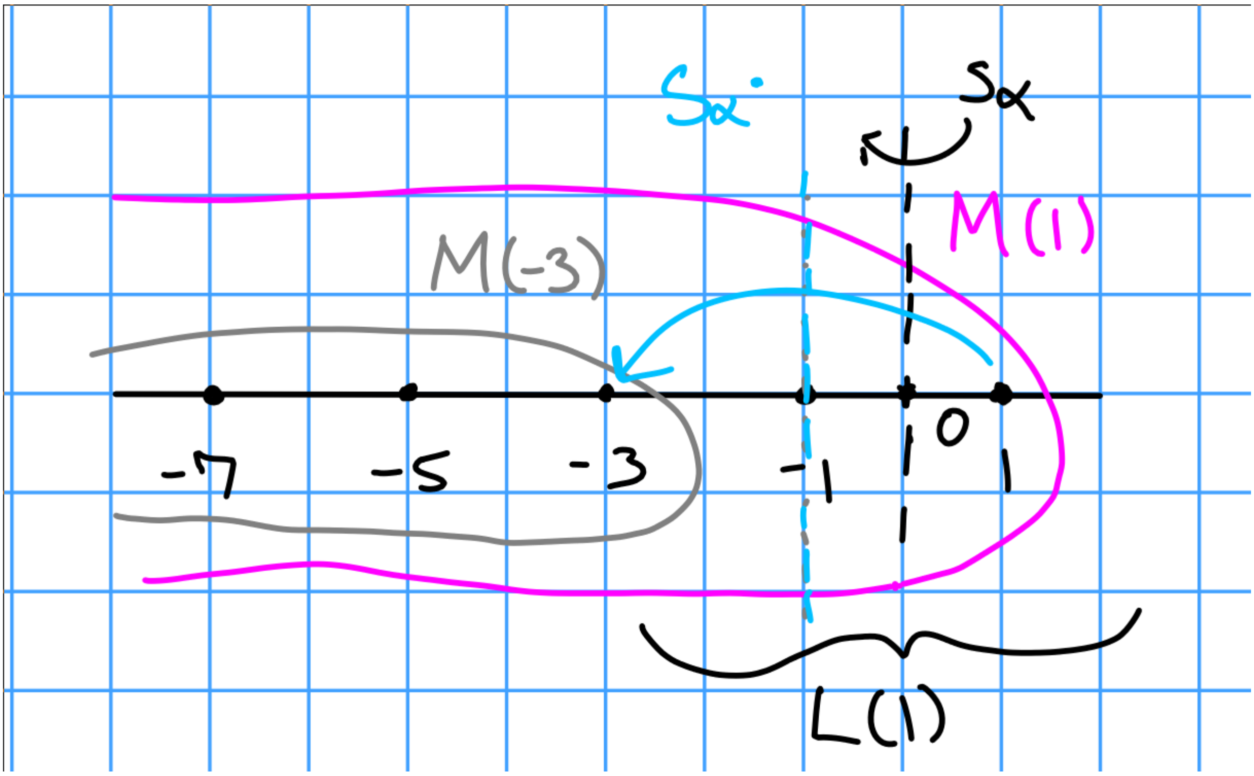
■

Consider $\mathfrak{sl}(2)$, then $M(1) \supset M(-3)$. Note that reflecting through 0 doesn't send 1 to -3, but shifting the origin to -1 and reflecting about that with $s_\alpha \cdot$ fixes this problem. Note that $L(1)$ is the quotient.

For $\lambda \in \mathfrak{h}^\vee$ and $\alpha \in \Delta$, we can compute $s_\alpha \cdot \lambda \coloneqq s_\alpha(\lambda + \rho) - \rho$ where $\rho = \sum_{j=1}^{\ell} e_i$. Then $(w_j, \alpha_i^\vee) = \delta_{ij}$ and $(\rho, \alpha_i^\vee) = 1$.

$$\begin{aligned}
s\alpha \cdot \lambda &= s_\alpha(\lambda + \rho) - \rho \\
&= (\lambda + \rho) - (\lambda + \rho, \alpha^\vee)\alpha - \rho \\
&= \lambda + \rho - ((\lambda < \alpha^\vee) + 1)\alpha - \rho \\
&= \lambda - (n+1)\alpha \\
&= \mu.
\end{aligned}$$

So this gives a well-defined, nonzero map $M(s_\alpha \cdot \lambda) \longrightarrow M(\lambda)$ for $s_\alpha \cdot \lambda < \lambda$.



Corollary 12.3.
Let λ, α, n be as in the above proposition. Let $\bar{v}^+$ now be a maximal vector of weight λ in $L(\lambda)$. Then $y_\alpha^{n+1}\bar{v}^+ = 0$.

Proof .
If not, then this would be a maximal vector, since it's the image of the vector $y_i^{n+1}v^+ \in M(\lambda)$ under the map $M(\lambda) \longrightarrow L(\lambda)$ of weight $\mu < \lambda$. Then it would generate a proper submodules of $L(\lambda)$, but this is a contradiction since $L(\lambda)$ is irreducible.

■

12.2 Section 1.5

Example: $\mathfrak{sl}(2)$. What do Verma modules $M(\lambda)$ and their simple quotients $L(\lambda)$ look like?

Fix a Chevalley basis $\{y, h, x\}$ and let $\lambda \in \mathfrak{h}^\vee \cong \mathbb{C}$.

Fact 1 For $v^+ = 1 \otimes 1_\lambda$, we have $M(\lambda) = U(\mathfrak{n}^-)v^+ = \mathbb{C}\left\langle y^i v^+ \,\middle|\, i \in \mathbb{Z}^+ \right\rangle$ is a basis for $M(\lambda)$ with weights $\lambda - 2i$ where α corresponds to 2. So the weights of $M(\lambda)$ are $\lambda, \lambda - 2, \lambda - 4, \cdots$ each with multiplicity 1.

Letting $v_i = \dfrac{1}{i!}y^i v^+$ for $i \in \mathbb{Z}^+$; this is a basis for $M(\lambda)$. Using the lemma, we have

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i \\ y \cdot v_i &= (i+1)v_{i+1} \\ x \cdot v_i &= (\lambda - i + 1)v_{i-1}. \end{aligned}$$

Note that these are the same for *finite-dimensional* $\mathfrak{sl}(2)$-modules, see section 0.9.

Fact (2) We know from the proposition that if $\lambda \in \mathbb{Z}^+$, i.e. $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$, then $M(\lambda)$ has a maximal vector of weight $\lambda - (n+1)\alpha = \lambda - (\lambda+1)2 = -\lambda - 2 = s_\alpha \cdot \lambda$.

Exercise Check that this maximal vector generates the maximal proper submodule

$$N(\lambda) = M(-\lambda - 2).$$

So the quotient $L(\lambda) = M(\lambda)/N(\lambda) = M(\lambda)/M(-\lambda-2)$ has weights $\lambda, \lambda-2, \cdots, -\lambda+2, -\lambda$. So when $\lambda \in \mathbb{Z}^+$, $L(\lambda)$ is the familiar simple $\mathfrak{sl}(2)$-module of highest weight λ.

Fact (3) When $\lambda \notin \mathbb{Z}^+$,

- $N(\lambda) = \{0\}$,
- $M(\lambda) = L(\lambda)$,
- $M(\lambda)$ is irreducible
- $L(\lambda)$ is infinite dimensional.

Proof .
Argue by contradiction. If not, $M(\lambda) \supset M \neq 0$ is a proper submodule. So $M \in \mathcal{O}$, and thus M has a maximal weight vector w^+, and by the restriction of weights for modules in $\mathcal{O}$, we know w^+ has height $\lambda - 2m$ for some $m \in \mathbb{Z}^+$. Then $w^+ = cv_i$ where $0 \neq c \in \mathbb{C}$, and taking $v_{-1} := 0$ and $x \cdot v_i = (\lambda - i + 1)v_{i-1}$ for $i \geq 1$, so $\lambda = i - 1 \implies \lambda \in \mathbb{Z}^+$.

■

13 Friday January 31st

Theorem 13.1 ***(Duals of Simple Quotients of Vermas).***
A useful formula: $L(\lambda)^\vee \cong L(-w_0)$.

Proof .
$L(\lambda)^\vee$ is a finite dimensional module, and $(x \cdot f)(v) = -f(x \cdot v)$, so $L(\lambda)^\vee \cong L(\nu)$ for some $\nu \in \Lambda^+$. The weights of $L(\lambda)^\vee$ are the negatives of the weight of $L(\lambda)$. Thus the lowest weight of $L(\lambda)$ is $w_0\lambda$, since w_o reverses the partial order on $\mathfrak{h}^\vee$, i.e $w_0\Phi^+ = \Phi^-$
Then

$$\mu \in \Pi(L(\mu)) \implies w_0\mu \in \Pi(L(\lambda)) \implies w_0\mu \leq \lambda.$$

This shows that the lowest weight of $L(\lambda)$ is $w_0\lambda$, thus the highest weight $L(\lambda)^\vee$ is $-w_0\lambda$ by reversing this inequality.
The inner product is W invariant and is its own inverse, so we can move it to the other side.

■

13.1 1.7: Action of $Z(\mathfrak{g})$

Next big goal: Every module in $\mathcal{O}$ has a *finite* composition series (Jordan-Holder series, the quotients are simple). Leads to Kazhdan-Lustzig conjectures from 1979/1980, which were solved, but are still open in characteristic p case.

The technique we'll use the Harish-Chandra homomorphism, which identifies $\mathcal{Z}(\mathfrak{g})$ explicitly.

It's commutative, a subalgebra of a Noetherian algebra, no zero divisors – could be a quotient, but then it'd have zero divisors, so this seems to force it to be a polynomial algebra on some unknown.

> Also note that $\mathcal{Z}(\mathfrak{g}) \coloneqq Z(U(\mathfrak{g}))$.

Recall: $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on any $M \in \mathcal{O}$ – this is by theorem 1.1e, i.e. $v \in M_\mu$ and $z \in \mathcal{Z}(\mathfrak{g})$ implies that $zv \in M_\mu$. (The calculation just follows by computing the weight and commuting things through.)

Let $\lambda \in \mathfrak{h}^\vee$ and $M = U(\mathfrak{g})v^+$ a highest weight module of highest weight λ. Then for $z \in \mathcal{Z}(\mathfrak{g})$, $z \cdot v^+ \in M_\lambda$ which is 1-dimensional. Thus z acts by scalar multiplication here, and $z \cdot v^+ = \chi_\lambda(z)v^+$. Now if $u \in U(\mathfrak{u}^-)$, we have $z \cdot (u \cdot v^+) = u \cdot (z \cdot v^+) = u(\chi_\lambda(z)v^+) = \chi_\lambda(z)u \cdot v^+$. Thus z acts on *all* of M by the scalar $\chi_\lambda(z)$.

Exercise Show that χ_λ is a nonzero additive and multiplicative function, so $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is linear and thus a morphism of algebras. Conclude that $\ker \chi_\lambda$ is a maximal ideal of $\mathcal{Z}(\mathfrak{g})$.

Note: this is called the *infinitesimal character*.

Note that χ_λ doesn't depend on which highest weight module M_λ was chosen, since they're all quotients of $M(\lambda)$. In fact, every submodule and subquotient of $M(\lambda)$ is the same infinitesimal character.

> **Definition 13.1.1** (Central/Infinitesimal Character).
> χ_λ is called the *central (or infinitesimal) character*, and $\widehat{\mathcal{Z}}(\mathfrak{g})$ denotes the set of all central characters. More generally, any algebra morphism $\chi : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is referred to as a central character. Central characters are in one-to-one correspondence with maximal ideals of $\mathcal{Z}(\mathfrak{g})$, where $\chi \iff \ker \chi$ and $\mathbb{C}[x_1, \cdots, x_n] \iff \langle x_1 - a_1, \cdots, x_n - a_n \rangle$ where $(\mathbf{a_i}) \in \mathbb{C}^n$.

Next goal: Describe $\chi_\lambda(z)$ more explicitly.

Using PBW, we can write $z \in \mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$. Some observations:

1. Any PBW monomial in z ending with a factor in $\mathfrak{n}$ will kill v^+, and hence can not contribute to $\chi_\lambda(z)$.
2. Any PBW monomial in z beginning with a factor in $\mathfrak{n}^-$ will send v^+ to a lower weight space, so it also can't contribute.

So we only need to see what happens in the $\mathfrak{h}$ part. A relevant decomposition here is

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \left(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+\right).$$

Exercise Why is this sum direct?

Let $\mathrm{pr} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$ be the projection onto the first factor. Then $\chi_\lambda(z) = \lambda(\mathrm{pr} z)$ for all $z \in \mathcal{Z}(\mathfrak{g})$. Then if $\mathrm{pr}(z) = h_1^{m_1} \cdots h_\ell^{m_\ell}$, we can extend the action on $\mathfrak{h}$ to all polynomials in elements of $\mathfrak{h}$ (which is in fact evaluation on these monomials), and thus $\chi_\lambda(z) = \lambda(h_1)^{m_1} \cdots \lambda(h_\ell)^{m_\ell}$.

Note that for $\lambda \in \mathfrak{h}^\vee$, we've extended this to the "evaluation map" $\lambda : U(\mathfrak{g}) \cong S(\mathfrak{h})$, the symmetric algebra on $\mathfrak{h}$.

Why is this the correct identification? The RHS is $T(\mathfrak{h})/\langle x \otimes y - y \otimes x - [xy] \rangle$, but notice that the bracket vanishes since $\mathfrak{h}$ is abelian, and this is the exact definition of the symmetric algebra.

Thus $\chi_\lambda = \lambda \circ \mathrm{pr}$.

Observation:

$$\begin{aligned}
\lambda(\mathrm{pr}(z_1 z_2)) &= \chi_\lambda(z_1 z_1) \\
&= \chi_\lambda(z_1)\chi_\lambda(z_2) \\
&= \cdots \\
&= \lambda(\mathrm{pr}(z_1)\mathrm{pr}(z_2)).
\end{aligned}$$

Exercise Show $\bigcap\limits_{\lambda \in \mathfrak{h}^\vee} \ker \lambda = \{0\}$.

Definition 13.1.2 (Harish-Chandra Morphism)**.**
Let $\xi = \mathrm{pr}|_{\mathcal{Z}(\mathfrak{g})} : \mathcal{Z}(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$.

Definition 13.1.3 (Twisted Harish-Chandra Morphism)**.**
ξ is an algebra morphism, and is referred to as the *Harish-Chandra homomorphism.*

See page 23 for interpretation of ξ without reference to representations.

Questions:

1. Is ξ injective?
2. What is im $\xi \subset U(\mathfrak{h})$?

When does $\chi_\lambda = \chi_\mu$? Proved last time: we introduced the $\cdot$ action and proved that $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ where $\alpha \in \Delta$. It'll turn out that the statement holds for all $\lambda \in W$.

Wednesday: Section 1.8.

14 Wednesday February 5th

Recall the Harish-Chandra morphism ξ:

$$\begin{tikzcd}
\mathcal{Z}(\mathfrak{g}) \arrow[r, hook] \arrow[dr, dashed, "\xi"] & U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \arrow[d, "\mathrm{pr}"] \\
& U(\mathfrak{h})
\end{tikzcd}$$

If M is a highest weight module of highest weight λ then $z \in \mathcal{Z}(\mathfrak{g})$ acts on M by scalar multiplication. Note that if we have $\chi_\lambda(z)$ where $z \cdot v = \chi_\lambda(z)v$ for all $v \in M$, we can identify $\lambda(\mathrm{pr}(z)) = \lambda(\xi(z))$.

14.1 Central Characters and Linkage

The χ_λ are not all distinct – for example, if $M(\mu) \subset M(\lambda)$, then $\chi_\mu = \chi_\lambda$. More generally, if $L(\mu)$ is a subquotient of $M(\lambda)$ then $\chi_\mu = \chi_\lambda$. So when do we have equality $\chi_\mu = \chi_\lambda$?

Given $\mathfrak{g} \supset \mathfrak{h}$ with $\Phi \supset \Phi^+ \supset \Delta$, then define

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathfrak{h}^\vee.$$

Note that $\alpha \in \Delta \implies s_\alpha \rho = \rho - \alpha$.

Definition 14.0.1 (Dot Action)**.**
The *dot action* of W on $\mathfrak{h}^\vee$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, which implies $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Delta$. Then $\rho = \sum_{i=1}^{\ell} w$.

Exercise Check that this gives a well-defined group action.

Definition 14.0.2 (Linkage Class)**.**
μ is *linked* to λ iff $\mu = w \cdot \lambda$ for some $w \in W$. Note that this is an equivalence relation, with equivalence classes/orbits where the orbit of λ is $\left\{ w \cdot \lambda \;\middle|\; w \in W \right\}$ is called the *linkage class* of λ.

Note that this is a finite subset, since W is finite. Orbit-stabilizer applies here, so bigger stabilizers yield smaller orbits and vice-versa.

Example 14.1.
$w \cdot (-\rho) = w(-\rho + \rho) - \rho = -\rho$, so $-\rho$ is in its own linkage class.

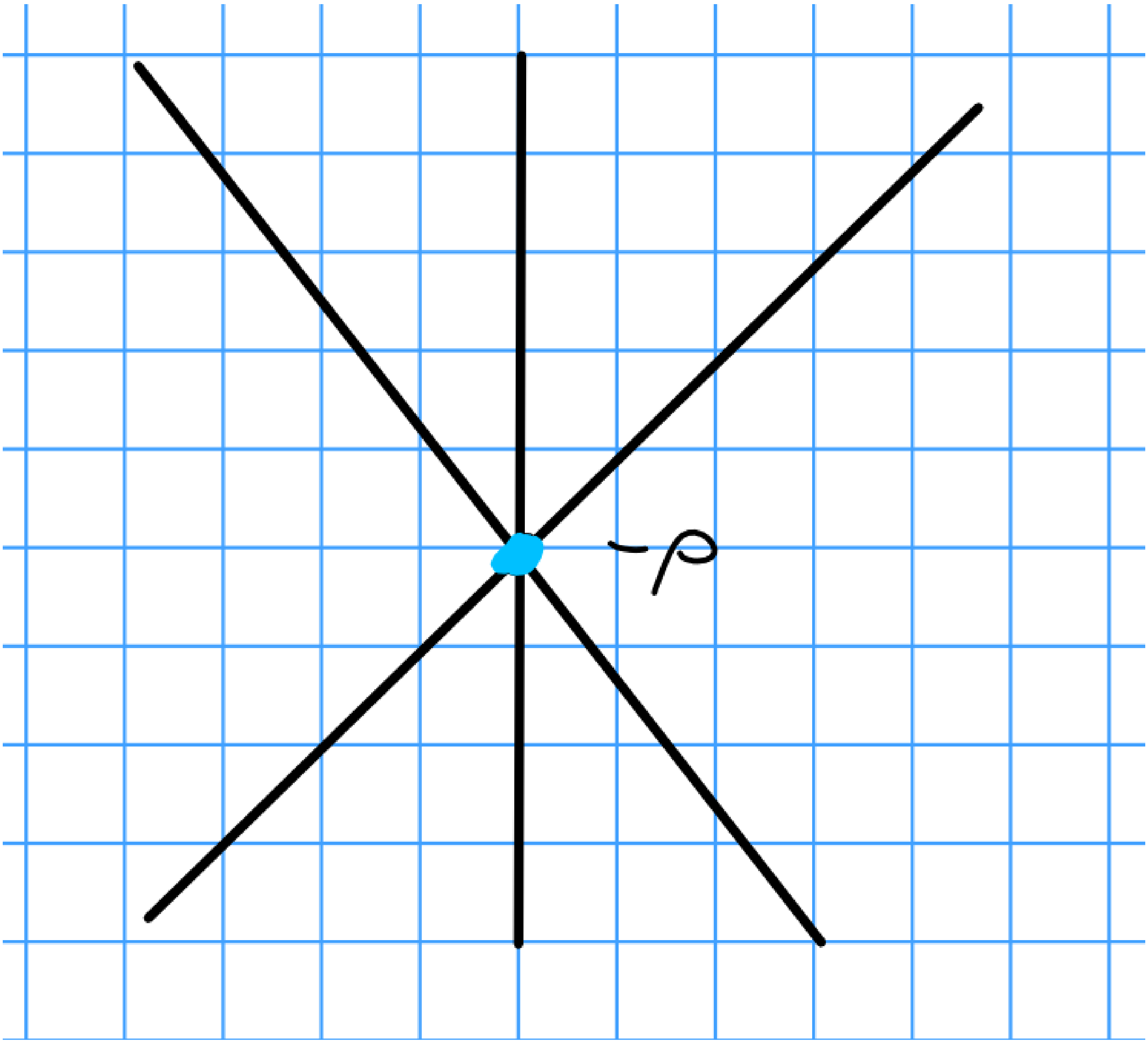
Definition 14.0.3 (Dot-Regular)**.**
$\lambda \in \mathfrak{h}^\vee$ is *dot-regular* iff $|W \cdot \lambda| = |W|$, or equivalently if $(\lambda + \rho, \beta^\vee) \neq 0$ for all $\beta \in \Phi$.

To think about: does this hold if Φ is replaced by Δ?

We also say λ is *dot-singular* if λ is not dot-regular, or equivalently $\mathrm{Stab}_{W \cdot} \lambda \neq \{1\}$.

I.e. lying on root hyperplanes.

Exercise If $0 \in \mathfrak{h}^\vee$ is regular, then $-\rho$ is singular.



Proposition 14.1.
If $\lambda \in \Lambda$ and $\mu \in W \cdot \lambda$, then $\chi_\mu = \chi_\lambda$.

Proof .
Start with $\alpha \in \Delta$ and consider $\mu = s_\alpha \cdot \lambda$. Since $\lambda \in \Lambda$, we have $n \coloneqq (\lambda, \alpha^\vee) \in \mathbb{Z}$ by definition. There are three cases:

1. $n \in \mathbb{Z}^+$, then $M(s_\alpha \cdot \lambda) \subset M(\lambda)$. By Proposition 1.4, we have $\chi_\mu = \chi_\lambda$.
2. For $n = -1$, $\mu = s_\alpha \cdot \lambda = \lambda + \rho - (\lambda + \rho, \alpha^\vee)\alpha - \rho = \lambda + n + 1 = \lambda + 0$. So $\mu = \lambda$ and thus $M_\mu = M_\lambda$.
3. For $n \leq -2$,

$$\begin{aligned}(\mu, \alpha^\vee) &= (s_\alpha \cdot \lambda, \alpha^\vee) \\ &= (\lambda i(n+1)\alpha, \alpha^\vee) \\ &= n - 2(n+1) \\ &= -n-2 \\ &\geq 0,\end{aligned}$$

■

so $\chi_\mu = \chi_{s_\alpha \cdot \mu} = \chi_{s\alpha \cdot (s_\alpha \cdot \lambda)} = \chi_\lambda$. Since W is generated by simple reflections and the linkage property is transitive, the result follows by induction on $\ell(w)$.

Exercise (1.8) See book, show that certain properties of the dot action hold (namely nonlinearity).

14.2 1.9: Extending the Harish-Chandra Morphism

We want to extend the previous proposition from $\lambda \in \Lambda$ to $\lambda \in \mathfrak{h}^\vee$. We'll use a density argument from affine algebraic geometry, and switch to the Zariski topology on $\mathfrak{h}^\vee \subset \mathbb{C}^n$.

Fix a basis $\Delta = \{a_1, \cdots, a_\ell\}$ and use the Killing form to identify these with a basis for $\mathfrak{h} = \{h_1, \cdots, h_\ell\}$. Similarly, take $\{w_1, \cdots, w_\ell\}$ as a basis for $\mathfrak{h}^\vee$, and we'll use the identification

$$\begin{aligned}\mathfrak{h}^\vee &\iff \mathbb{A}^\ell \\ \lambda &\iff (\lambda(h_1), \cdots, \lambda(h_\ell)).\end{aligned}$$

We identify $U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[h_1, \cdots, h_\ell]$ with $P(\mathfrak{h}^\vee)$ which are polynomial functions on $\mathfrak{h}^\vee$. Fix $\lambda \in \mathfrak{h}^\vee$, extended λ to be a multiplicative function on polynomials. For $f \in \mathbb{C}[h_1, \cdots, h_\ell]$, we defined $\lambda(f)$. Under the identification, we send this to $\tilde{f}$ where $\tilde{f}(\lambda) = \lambda(f)$.

Note: we'll identify f and $\tilde{f}$ notationally going forward and drop the tilde everywhere.

Then W acts on $P(\mathfrak{h}^\vee)$ by the dot action: $(w \cdot \tilde{f})(\lambda) = \tilde{f}(w^{-1} \cdot \lambda)$.

Exercise Check that this is a well-defined action.

Under this identification, we have

$$\begin{aligned}\mathfrak{h}^\vee &\iff \mathbb{A}^\ell \\ \Lambda &\iff \mathbb{Z}^\ell.\end{aligned}$$

Note that Λ is discrete in the analytic topology, but is *dense* in the Zariski topology.

Proposition 14.2.
A polynomial f on $\mathbb{A}^\ell$ vanishing on $\mathbb{Z}^\ell$ must be identically zero.

Proof .
For $\ell = 1$: A nonzero polynomial in one variable has only finitely many zeros, but if f vanishes on $\mathbb{Z}$ it has infinitely many zeros.
For $\ell > 1$: View $f \in \mathbb{C}[h_1, \cdots, h_{\ell-1}][h_\ell]$. Substituting any fixed integers for the h_i for $i \leq \ell - 1$ yields a polynomial in one variable which vanishes on $\mathbb{Z}$. By the first case, $f \equiv 0$, so the coefficients must all be zero and the coefficient polynomials in $\mathbb{C}[h_1, \cdots, h_{\ell-1}]$ vanish on $\mathbb{Z}^{\ell-1}$. By induction, these coefficient polynomials are identically zero.

$\blacksquare$

Corollary 14.3.
The only Zariski-closed subset of $\mathbb{A}^\ell$ containing $\mathbb{Z}^\ell$ is $\mathbb{A}^\ell$ itself, so the Zariski closure $\overline{\mathbb{Z}^\ell} = \mathbb{A}^\ell$ and $\mathbb{Z}^\ell$ is dense in $\mathbb{A}^\ell$.

15 Friday February 7th

So far, we have $\chi_\lambda = \chi_{w \cdot \lambda}$ if $\lambda \in \Lambda$ and $w \in W$. We have $\mathfrak{h}^\vee \supset \Lambda$ which is topologically equivalent to $\mathbb{A}^\ell \supset \mathbb{Z}^\ell$, where $\mathbb{Z}^\ell$ is dense in the Zariski topology.

For $z \in \mathcal{Z}(\mathfrak{g})$, we have $\chi_\lambda(z) = \chi_{W \cdot \lambda}(z)$ and so $\lambda(\xi(z)) = (w \cdot \lambda)(\xi(z))$ where $\xi : \mathcal{Z}(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h}) \cong P(\mathfrak{h}^\vee)$ where we send $\lambda(f)$ to $f(\lambda)$.

Then $\xi(z)(\lambda) = \xi(z)(w \cdot \lambda)$ for all $\lambda \in \Lambda$, and so $\xi(z) = w^{-1}\xi(z)$ on Λ. But both sides here are polynomials and thus continuous, and $\Lambda \subset \mathfrak{h}^\vee$ is dense, so $\xi(z) = w^{-1}\xi(z)$ on all of $\mathfrak{h}^\vee$. I.e., $\chi_\lambda = \chi_{w \cdot} \lambda$ for all $\lambda \in \mathfrak{h}^\vee$.

This in fact shows that the image of $\mathcal{Z}(\mathfrak{g})$ under ξ consists of W-invariant polynomials.

It's customary to state this in terms of the natural action of W on polynomials without the row-shift. We do this by letting $\tau_\rho : S(\mathfrak{h}) \xrightarrow{\cong} S(\mathfrak{h})$ be the algebra automorphism induced by $f(\lambda) \mapsto f(\lambda - \rho)$. This is clearly invertible by $f(\lambda) \mapsto f(\lambda + \rho)$. We then define $\psi : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\xi} S(\mathfrak{h}) \xrightarrow{\tau_\rho} S(\mathfrak{h})$ as this composition; this is referred to as the Harish-Chandra (HC) homomorphism.

Exercise Show $\chi_\lambda(z) = (\lambda + \rho)(\psi(z))$ and $\chi_{w \cdot \lambda}(w(\lambda + \rho))(\psi(z))$, where $w(\,\cdot\,)$ is the usual w-action.
Replacing λ by $\lambda + \rho$ and w by w^{-1}, we get

$$w\psi(z) = \psi(z)$$

for all $z \in \mathcal{Z}(\mathfrak{g})$ and all $w \in W$ where $(w\psi(z))(\lambda) = \psi(z)(w^{-1}\lambda)$.

We've proved that

Theorem 15.1 ***(Character Linkage and Image of the HC Morphism).***

a. If $\lambda, \mu \in \mathfrak{h}^\vee$ that are linked, then $\chi_\lambda = \chi_\mu$.
b. The image of the twisted HC homomorphism $\psi : \mathcal{Z}(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ lies in the subalgebra $S(\mathfrak{h})^W$.

Example 15.1.
Let $\mathfrak{g} = \mathfrak{sl}_2$. Recall from finite-dimensional representations there is a canonical element $c \in \mathcal{Z}(\mathfrak{g})$ called the Casimir element. For $\mathcal{O}$, we need information about the full center $\mathcal{Z}(\mathfrak{g})$ (hence introducing infinitesimal characters).

Expressing c in the PBW basis yields $c = h^2 + 2h + 4yx$, where $h^2 + 2h \in U(\mathfrak{h})$ and $4yx \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$.

Enveloping algebra convention: xs, hs, ys

Then $\xi(c) = \mathfrak{p}(c) = h^2 + 2h$, and under the identification $\mathfrak{h}^\vee \iff \mathbb{C}$ where $\lambda \iff \lambda(h)$, we can identify $\rho \iff \rho(h) = 1$. The row shift is given by $\psi(c) = (h-1)^2 + 2(h-1) = h^2 - 1$. This is W-invariant, since $s_{\alpha_h} = -h$. But $W = \langle s_\alpha, 1 \rangle$, so s_α generates W.

We also have $\chi_\lambda(c) = (\lambda + \rho)(\psi(c)) = (\lambda + 1)^2 - 1$. Then

$$\chi_\lambda(c) = \chi_\mu(c) \iff (\lambda+1)^2 - 1 = (\mu+1)^2 \iff \mu = \lambda \text{ or } \mu = -\lambda - 2$$

But $\lambda = 1 \cdot \lambda$ and $-\lambda - 2 = s_\alpha \cdot \lambda$, so $\mathcal{Z}(\mathfrak{g}) = \langle c \rangle \coloneqq \mathbb{C}[c]$ as an algebra. So these characters are equal iff $\mu = w \cdot \lambda$ for $w \in W$.

16 Section 1.10: Harish-Chandra's Theorem

Goal: prove the converse of the previous theorem.

Theorem 16.1 ***(Harish-Chandra).***
Let $\psi : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ be the twisted HC homomorphism. Then

a. ψ is an *isomorphism* of $\mathcal{Z}(\mathfrak{g})$ onto $S(\mathfrak{h})^W$.
b. For all $\lambda, \mu \in \mathfrak{h}^\vee$, $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.
c. Every central character $\chi : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is a χ_λ.

Proof (of (a)).
Relies heavily on the *Chevalley Restriction Theorem* (which we won't prove here).
Initially we have a restriction map on polynomial functions $\theta : P(\mathfrak{g}) \longrightarrow P(\mathfrak{h})$. We identified $P(\mathfrak{g}) = S(\mathfrak{g}^\vee)$, the formal polynomials on $\mathfrak{g}^\vee$. However, for $\mathfrak{g}$ semisimple, we can identify $S(\mathfrak{g}^\vee) \cong S(\mathfrak{g})$ via the Killing form.
By the Chinese Remainder Theorem, $\theta : S(\mathfrak{g})^G \longrightarrow S(\mathfrak{h})^W$ is an isomorphism, where the subgroup $G \leq \mathrm{Aut}(\mathfrak{g})$ is the *adjoint group* generated by $\left\{ \exp \mathrm{ad}_{\,x} \;\middle|\; x \text{ is nilpotent} \right\}$.
It turns out that $S(\mathfrak{g})^G$ is very close to $\mathcal{Z}(\mathfrak{g})$ – it is the associated graded of a natural filtration on $\mathcal{Z}(\mathfrak{g})$. This is enough to show that ψ is a bijection.

■

Proof (of (b)).
We'll prove the contrapositive of the converse.
Suppose $W \cdot \lambda \bigcap W \cdot \mu = \emptyset$ and both are in $\mathfrak{h}^\vee$. Since these are finite sets, Lagrange interpolation yields a polynomial that is 1 on $W \cdot \lambda$ and 0 on $W \cdot \mu$. Let $g = \dfrac{1}{|W|} \displaystyle\sum_{w \in W} w \cdot f$.

Note: definitely the dot action here, may be a typo in the book.

Then g is a $W\cdot$ invariant polynomial with the same properties. By part (a), we can get rid of the row shift to obtain an isomorphism $\xi : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^(W\cdot)$, the $W\cdot$ invariant polynomials. Choose $z \in \mathcal{Z}(\mathfrak{g})$ such that $\xi(z) = g$, then $\chi_\lambda(z) = \lambda(\xi(z)) = \lambda(g) = g(\lambda) = 1$. So $\chi_\mu(z) = 0$ similarly, and $\chi_\lambda = \chi_\mu$.

■

Proof (of (c)).
This follows from some commutative algebra, we won't say much here. Look at maximal ideals in $\mathbb{C}[x, y, \cdots]$ which correspond to evaluating on points in $\mathbb{C}^\ell$.

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Remark Chevalley actually proved that $S(\mathfrak{h})^W \cong \mathbb{C}(p_1, \cdots, p_\ell)$ where the p_i are homogeneous polynomials of degrees $d_1 \leq \cdots \leq d_\ell$. These numbers satisfy some remarkable properties: $\prod d_i = |W|$ and $d_1 = 2$ (these are called the *degrees of* W)

17 Section 1.11

Theorem 17.1 ***(Category O is Artinian).***
Category $\mathcal{O}$ is *artinian*, i.e. every $M \in \mathcal{O}$ is Artinian (DCC) and $\dim \hom_{\mathfrak{g}}(M, N) < \infty$ for every M, N.

Recall that $\mathcal{O}$ is known to be Noetherian from an earlier theorem. This will imply that **every** M **has a composition/Jordan-Holder series**, so we can take composition factors and multiplicities.

Most interesting question: what are the factors/multiplicities of the simple modules and Verma modules?

18 Wednesday February 12th

18.1 Infinitesimal Blocks

We'll break up category $\mathcal{O}$ into smaller subcategories (blocks).

Recall theorem 1.1 (e): $\mathcal{Z}(\mathfrak{g})$ acts locally finitely on $M \in \mathcal{O}$, and M has a *finite* filtration with highest weight sections, so M should involve only a finite number of central characters χ_λ (where $\lambda \in \mathfrak{h}^\vee$).

Note: an analog of Jordan decomposition works here because of this finiteness condition. This discussion will parallel the JCF of a simple operator on a finite dimensional $\mathbb{C}$-vector space. However, this involves the *entire* center instead of just scalar matrices, so the analogy is diagonalizing a family of operators simultaneously.

Let $\chi \in \widehat{\mathcal{Z}}(\mathfrak{g})$ and $M \in \mathcal{O}$, and

$$M^\chi := \left\{ v \in M \;\middle|\; \forall z \in \mathcal{Z}(\mathfrak{g}),\ \exists n > 0 \text{ s.t. } (z - \chi(z))^n \cdot v = 0 \right\}$$

Idea: write

$$z = \chi(z) \cdot 1 + (z - \chi(z) \cdot 1),$$

where the first is a scalar operator and the second is (locally) nilpotent on M^χ. Thus we can always arrange for z to act by a sum of "Jordan blocks":

$$\begin{bmatrix} \chi(z) & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \chi(z) \end{bmatrix} + \cdots$$

Observations:

- M^χ are $U(\mathfrak{g})$-submodules of M.
- The subspaces M^χ are linearly independent
- $\mathcal{Z}(\mathfrak{g})$ stabilizes each M_μ since $\mathcal{Z}(\mathfrak{g})$ and $U(\mathfrak{h})$ are a commuting family of operators on M_μ.
- We can write $M_\mu = \bigoplus_{\chi \in \widehat{\mathcal{Z}}(\mathfrak{g})} (M_\mu \bigcap M^\chi)$, and since M is generated by a finite sum of weight spaces, $M = \bigoplus_{\chi \in \widehat{\mathcal{Z}}(\mathfrak{g})} M^\chi$.
- By Harish-Chandra's theorem, every χ is χ_λ for some $\lambda \in \mathfrak{h}^\vee$.

Let $\mathcal{O}_\chi$ be the full subcategory of modules M such that $M = M^\chi$; we refer to this as a *block*.

Note: full subcategory means keep all of the hom sets.

Proposition 18.1.
$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^\vee} \mathcal{O}_{\chi_\lambda}$. Each indecomposable module in $\mathcal{O}$ lies in a *unique* $\mathcal{O}_\chi$. In particular, any highest weight module of highest weight λ lies in $\mathcal{O}_{\chi_\lambda}$.

Thus we can reduce to studying $\mathcal{O}_{\chi_\lambda}$.

Remark: $\mathcal{O}_{\chi_\lambda}$ has a finite number of simple modules $\left\{ L(w \cdot \lambda) \;\middle|\; w \in W \right\}$ and a finite number of Verma modules $\left\{ M(w \cdot \lambda) \;\middle|\; w \in W \right\}$.

18.2 Blocks

Let $\mathcal{C}$ be a category with is artinian and noetherian, with L_1, L_2 simple modules. We say $L_1 \sim L_2$ if there exists a non-split extension $0 \longrightarrow L_1 \longrightarrow M \longrightarrow L_2 \longrightarrow 0$, i.e. $\mathrm{Ext}^1_{\mathcal{O}}(L_2, L_1) \neq 0$. In particular, M equivalently needs to be indecomposable. We then extend $\sim$ to be reflexive/symmetric/transitive to obtain an equivalence relation.

> L_1 ends up being the socle here.

This partitions the simple modules in $\mathcal{C}$ into *blocks* $\mathcal{B}$. More generally, we say $M \in \mathcal{C}$ belongs to $\mathcal{B}$ iff all of the composition factors of M belong to $\mathcal{B}$. Although not obvious, there are no nontrivial extensions between modules in different blocks. Thus each simple module (generally, just an object) $M \in \mathcal{C}$ decomposes as a direct sum of submodules (subobjects) with each belonging to a single block.

Question: Is $\mathcal{O}_\chi$ a block of $\mathcal{O}$? The answer is not always. Because each indecomposable module in $\mathcal{O}$ lives in a simple $\mathcal{O}_\chi$ By the definition, it's clear that each block is contained in a single simple infinitesimal block $\mathcal{O}_\chi$.

> The block containing L_1, L_2 will be contained in the same infinitesimal block, and continuing the composition series puts all composition factors in a single block.

Proposition 18.2.
If λ is an *integral* weight, so $\lambda \in \Lambda$, then $\mathcal{O}_{\chi_\lambda}$ is a (simple) block of $\mathcal{O}$.

Proof .
It suffices to show that all $L(w \cdot \lambda)$ for $w \in W$ lie in a single block. We'll induct on the length of w. Start with $2 = s_\alpha$ for some $\alpha \in \Delta$. Let $\mu = s_\alpha \cdot \lambda$. If $\mu = \lambda$, i.e. λ is in the stabilizer, then we're done.
Otherwise, assume WLOG $\mu < \lambda$ in the partial order, using the fact that $\lambda \in \Lambda$. (The difference between these is just an integer multiple of α.)
By proposition 1.4, we have the following maps:

$$\begin{tikzcd} M(\mu) \arrow[r, "\phi\neq 0"] & N(\lambda) \arrow[r, hook] & M(\lambda) \\ N(\mu) \arrow[u, hook] \arrow[r, tail] & N = \phi(N(\mu)) \arrow[u, hook] & \end{tikzcd}$$

Then ϕ induces a map $L(\mu) \xrightarrow{\bar{\phi}} M(\lambda)/N$, where the codomain here is a highest weight module with quotient $L(\lambda)$. Since highest weight modules are indecomposable and thus lie in a single bloc, $L(\mu)$ and $L(\lambda)$ are in the same block.

> Note that if v^+ generates $M(\lambda)$, $v^+ + N$ generated the quotient.

Now inducting on $\ell(w)$, iterating this argument yields all $L(w \cdot \lambda)$ (as w varies) in the same block.

■

Example 18.1.
This isn't true for non-integral weights. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ and $\lambda > -1$. Then

$$
\begin{aligned}
\mu &= s_\alpha \lambda \\
&= -\lambda - 2 \\
&<_{\mathbb{R}} -1
\end{aligned}
$$

with the usual ordering on $\mathbb{R}$, but $\mu \not> \lambda$ in the ordering on $\mathfrak{h}^\vee$: we have $\lambda - \mu = 2\lambda + 2$, but $\alpha \equiv 2$ and thus these don't differ by an element of $2\mathbb{Z}$.

Thus μ, λ are in different cosets of $\mathbb{Z}\Phi = \Lambda_r$ in $\mathfrak{h}^\vee$. However, $M(\lambda), M(\mu)$ are simple since λ, μ are not non-negative integers.

By exercise 1.13, there can be no nontrivial extension, so they're in different homological blocks but in the same $\mathcal{O}_{\chi_\lambda}$ since $\mu = s_\alpha \cdot \lambda$. So this infinitesimal block splits into multiple homological blocks.

Friday: 1.14 and 1.15.

19 Friday February 14th

Recall that we have a decomposition

$$
\mathcal{O} = \bigoplus_{\chi \in \widehat{\mathcal{Z}}(\mathfrak{g})} \mathcal{O}_\chi
$$

into infinitesimal blocks, where $\mathcal{O}_0 := \mathcal{O}_{\chi_0}$ is the principal block. Since $0 \in \mathfrak{h}^\vee$, we can associate $\chi_0.M_0, L(0) = \mathbb{C}$ the trivial module for $\mathfrak{g}$.

19.1 1.14 – 1.15: Formal Characters

Some background from finite dimensional representation theory of a finite group G over $\mathbb{C}$. The hope is to find matrices for each element of G, but this isn't basis invariant. Instead, we take traces of these matrices, which is less data and basis-independent This is referred to as the *character* of the representation, and in nice situations, the characters determine the irreducible representations.

For a semisimple lie algebra $\mathfrak{g}$ and a finite dimensional representation M, it's enough to keep track of weight multiplicities when $\mathfrak{g}$ is the lie algebra associated to a compact lie group G. From this data, the characters can be recovered. So the data of all pairs $(\dim M_\lambda, \lambda \in \mathfrak{h}^\vee)$ suffices. To track this information, we introduce a *formal character.*

Remark: If G is a group and k is a commutative ring, kG is the group ring of G. This has the following properties:

- $\sum a_i g_i + \sum b_i g_i = \sum (a_i + b_i) g_i$
- $\left(\sum a_i g_i\right)\left(\sum b_i g_i\right) = \sum_{i,j} a_i b_j g_i g_j$

Let $\mathbb{Z}\Lambda$ be the integral group ring of the lattice. Since Λ is an abelian group, and the additive notation would be more difficult. So we write Λ multiplicatively and introduce $e(\lambda)$ for $\lambda \in \mathfrak{h}^\vee$, where $e(\lambda)e(\mu) = e(\lambda+\mu)$. For M a finite dimensional $\mathfrak{g}$-module, the formal character of M is given by

$$\text{char } M = \sum_{\lambda \in \Lambda} (M(\lambda))e(\lambda) \quad \in \mathbb{Z}\Lambda.$$

This satisfies

- char $(M \oplus N) =$ char $(M) +$ char (N)
- char $(M \otimes N) =$ char (M)char (N)
- For char $(M) = \sum a_\mu e(\mu)$ and char $(N) = \sum b_\nu e(\nu)$, we have char (M)char $(N) = \sum_\lambda \left(\sum_{\mu+\nu=\lambda} a_\mu b_\nu \right) e(\lambda)$

By Weyl's complete reducibility theorem, any semisimple module decomposes into a sum of simple modules. Thus it suffices to determine that characters of simple modules $L(\lambda)$ for $\lambda \in \Lambda^+$, corresponding to dominant integral weights. Then we can reconstruct char (M) from char $L(\lambda)$ for $M \in \mathcal{O}$.

Specifying the weight spaces dimensions is equivalent to a function char $_M : \mathfrak{h}^\vee \longrightarrow \mathbb{Z}^+$ where char $_M(\lambda) = \dim M_\lambda$. The analogy of $e(\lambda)$ in this setting is the characteristic function e_λ where $e_\lambda(\mu) = \delta_{\lambda\mu}$ for $\mu \in \mathfrak{h}^\vee$. We can thus write the function

$$\text{char }_M = \sum_{\lambda \in \mathfrak{h}^\vee} (\dim M_\lambda) e_\lambda.$$

When $\dim M < \infty$, char $_M$ has finite support, although we generally don't have this in $\mathcal{O}$. In this setting, multiplication of formal characters corresponds to convolution of functions, i.e. $(f * g)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu)$. Define

$$\mathcal{X} = \left\{ f : \mathfrak{h}^\vee \longrightarrow \mathbb{Z} \;\middle|\; \text{supp}(f) \subset \bigcup_{i \leq n} \left(\lambda_i - \mathbb{Z}^+\Phi^+ \right) \text{ for some } \lambda_1, \cdots, \lambda_n \in \mathfrak{h}^\vee \right\}$$

Idea: this is a "cone" below some weights.

This makes $\mathcal{X}$ into a $\mathbb{Z}$-module with a well-defined convolution, thus $\mathcal{X}$ is a commutative ring where

- $e_\lambda \in \mathcal{X}$ for all λ
- $e_0 = 1$
- $e_\lambda * e_\mu = e_{\lambda+\mu}$.

If $M \in \mathcal{O}$, then char $_M \in \mathcal{X}$ by axiom O5 (local finiteness).

Example: char $L(\lambda) = e(\lambda) + \sum_{\mu < \lambda} m_{\lambda\mu} e(\mu)$, where $m_{\lambda\mu} = \dim L(\lambda)_\mu \in \mathbb{Z}^\pm$.

Definition 19.0.1 (Principal Block?).
Let $\mathcal{X}_0$ be the additive subgroup of $\mathcal{X}$ generated by all char M for $M \in \mathcal{O}$.

Proposition 19.1.
In parts:
a. If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a SES in $\mathcal{O}$, then $\text{char } M = \text{char } M' + \text{char } M''$.
b. There is a 1-to-1 correspondence

$$\begin{aligned} \mathcal{X}_0 &\iff K(\mathcal{O}) \\ \text{char } M &\iff [M], \end{aligned}$$

where K is the Grothendieck group.
c. If $M \in \mathcal{O}$ and $\dim L < \infty$, then $\text{char } (L \otimes M) = \text{char } L * \text{char } M$.

Remark: (a) implies that char M is the sum of the formal characters of its composition factors with multiplicities. Thus

$$\text{char } M = \sum_{L \text{ simple}} [M : L] \text{ char } L.$$

Proof (of a).
Use the fact that $\dim M_\lambda = \dim M'_\lambda + \dim M''_\lambda$ ■

Proof (of b).
Check that the obvious maps are well-defined and mutually inverse. ■

Proof (of c).
Because of the module structure we've put on the tensor product $(L \otimes M)_\lambda = \sum_{\mu+\nu=\lambda} L_\mu \otimes M_\nu$. ■

Remark: The natural action of W on Λ or on $\mathfrak{h}^\vee$ extends to $\mathbb{Z}\Lambda$ and $\mathcal{X}$ if we define

$$w \cdot e(\lambda) := e(w\lambda) \quad w \in W,\ \lambda \in \Lambda \text{ or } \mathfrak{h}^\vee.$$

If $\lambda \in \Lambda^+$, then $w(\text{char } L(\lambda)) = \text{char } L(\lambda)$ since $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$. Thus the characters of simple finite-dimensional modules are W-invariant.

19.2 1.16: Formal Characters of Verma Modules

1: We have a similar formula

$$\text{char } M(\lambda) = \text{char } L(\lambda) + \sum_{\mu < \lambda} a_{\lambda\mu} \text{char } L(\mu)$$
$$\text{with } a_{\lambda\mu} \in \mathbb{Z}^+ \text{ and } a_{\lambda\mu} = [M(\lambda) : L(\mu)].$$

This all happens in a single block of $\mathcal{O}$, which has finitely many simple and Verma modules. In fact, the sum will be over $\left\{ \mu \in W \cdot \lambda \;\middle|\; \mu < \lambda \right\}$. But computing $L(\mu)$ is difficult in general.

Since the set of weights $W \cdot \lambda$ is finite, we can totally order it in a way that's compatible with the partial order on $\mathfrak{h}^\vee$ (so $\leq$ in the partial order implies $\leq$ in the total order). So if we order the weights μ_i indexing the Verma modules in columns and indexing the simple modules in the rows, this is an upper triangular matrix with 1s on the diagonal. This can inverted since it's unipotent, with the inverse of same upper triangular form.

2: We can write

$$\text{char } L(\lambda) = \text{char } M(\lambda) + \sum_{\mu < \lambda,\ \mu \in W \cdot \lambda} b_{\lambda\mu} \text{char } M(\mu) \quad b_{\lambda\mu} \in \mathbb{Z}$$

This expresses the character in terms of Verma modules, which are easier to compute.

Next time: formulas for the characters

20 Monday February 17th

20.1 Character Formulas

Last time: The second character formula (equation (2)),

$$\text{char } L(\lambda) = \text{char } M(\lambda) + \sum_{\mu < \lambda,\ \mu \in W \cdot \lambda} b_{\lambda,\mu} \text{ char } M(\mu).$$

Note that $b_{\lambda,\mu} \in \mathbb{Z}$, and this formula comes from inverting the previous one.

> Holy grail: characters of simple modules!

We can write $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as a $\mathfrak{h}$-module. Define $p : \mathfrak{h}^\vee \longrightarrow \mathbb{Z}$ where $p(\gamma)$ is the number of tuples $(t_\beta)_{\beta \in \Phi^+}$ where $t_\beta \in \mathbb{Z}^+$ and $\gamma = -\sum_{\beta \in \Phi^+} t_\beta \beta$. We have $\text{supp}(p) = -\mathbb{Z}^+\Phi^+$, which gives us something like a negative quadrant of the lattice.

The function p is essentially the *Kostant partition function.* The advantage here is that $p \in \mathcal{X}$ (defined last time, support is less than some finite weights?).

Observation: $p = \text{char }_{M(0)}$ since $U(\mathfrak{n}^-) \otimes \mathbb{C}_{\lambda=0}$ has PBW basis

$$\left\{ \prod_{\beta \in \Phi^+} y_\beta^{t_\beta} \otimes 1_{\lambda=0} \;\middle|\; t_\beta \in \mathbb{Z}^+ \right\}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(3)$, then $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Then $\gamma = -(\alpha_1 + 2\alpha_2)$ corresponds to $(1,2,0), (0,1,1)$ so $p(\gamma) = 2$. If $\gamma = -(2\alpha_1 + 2\alpha_2)$, this corresponds to $(2,2,0), (1,1,1), (0,0,2)$ so $p(\gamma) = 3$.

Note: just the number of ways of obtaining $-\gamma$ as a linear combinations of roots.

In general, $\dim M(0)_\gamma = p(\gamma)$.

Proposition 20.1.
For any $\lambda \in \mathfrak{h}^\vee$, we have char $_{M_\lambda} = p * e_\gamma$, taking the convolution product. In particular, char $_{M(0)} = p$.

Proof (of proposition).
We have the following computation:

$$\begin{aligned}(p * e_\lambda)(\lambda + \gamma) &= p(\gamma) e_\lambda(\lambda) \\ &= p(\gamma) 1 \\ &= p(\gamma) \\ &= \dim M(\lambda)_{\lambda+\gamma} \quad \text{ as a weight space }.\end{aligned}$$

■

Note that we can also write equation (2) as

$$\text{char } L(\lambda) = \sum_{w \cdot \lambda \leq \lambda} b_{\lambda, w} \text{char } M(w \cdot \lambda).$$

Here $b_{\lambda,w} \in \mathbb{Z}$ and in fact $b_{\lambda,1} = 1$.

Example: Let $\mathfrak{g} = \mathfrak{sl}(2)$. We know

$$\begin{aligned}\text{char } M(\lambda) &= \text{char } L(\lambda) + \text{char } L(s_\alpha \cdot \lambda) \\ chM(s_\alpha \cdot \lambda) &= \text{char } L(s_\alpha \cdot \lambda).\end{aligned}$$

We can think of this pictorially as the 'head' on top of the socle:

$$M(\lambda) = \frac{L(\lambda)}{L(s_\alpha \cdot \lambda)}.$$

The formula above corresponds to the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We can invert the formula to get equation (2), which corresponds to inverting this matrix:

$$\begin{aligned} \text{char } L(\lambda) &= \text{char } M(\lambda) - \text{char } M(s_\alpha \cdot \lambda) \\ \text{char } L(s_\alpha \cdot \lambda) &= \text{char } M(s_\alpha \cdot \lambda). \end{aligned}$$

Note that the coefficients $b_{w,\lambda} \in \{0, \pm 1\}$ in this equation are independent of $\lambda \in \Lambda^+$.

If $\lambda \notin \Lambda^+$, then char $L(\lambda) =$ char $M(\lambda)$ and $b_{\lambda,1} = 1, b_{\lambda,s_\alpha} = 0$ are again independent of $\lambda \in \mathfrak{h}^\vee \setminus \Lambda^+$.

Question: To what extent to $b_{\lambda,w}$ depend on λ? The answer is seemingly "not much".

20.2 Category $\mathcal{O}$ Methods

Note: skipping chapter 2 since we're focusing on infinite dimensional representations.

20.2.1 Hom and Ext

Recall that $\hom(\,\cdot\,,\,\cdot\,)$ is left exact but not exact, and is either covariant or contravariant depending on which variable is fact. So taking hom of a SES yields a LES involving the derived functors Ext^n. Convention: $\mathrm{Ext}^0 \coloneqq \hom$ and $\mathrm{Ext}^1 \coloneqq \mathrm{Ext}$.

Let A, C be $U(\mathfrak{g})$-modules. Consider two short exact sequences

$$\begin{aligned} 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \\ 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0. \end{aligned}$$

where B, B' are extensions of C by A.

We say two such sequences are equivalent iff there is an isomorphism making this diagram commute:

$$\begin{tikzcd} & B \arrow[rd] \arrow[dd, "\cong"] & \\ A \arrow[ru] \arrow[rd] & & C \\ & B' \arrow[ru] & \end{tikzcd}$$

The set $\mathrm{Ext}_{U(\mathfrak{g})}(C, A)$ of equivalence classes of extensions is a group under an operation called "Baer sum" (see Wikipedia) in which the identity is the class of the split SES

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0.$$

It turns out that the first right-derived functor of hom defined using projective resolutions, namely Ext_1, is isomorphic to Ext. In particular, each SES leads to a pair of LESs given by applying $\hom(\,\cdot\,, D)$ and $\hom(E, \,\cdot\,)$ for $D, E \in U(\mathfrak{g})$-mod.

Warning: Even if $A, C \in \mathcal{O}$, there's no guarantee $B \in \mathcal{O}$ for B an extension. In this case, we define $\mathrm{Ext}_{\mathcal{O}}(C, A)$ to be only those extensions lying in $\mathcal{O}$.

Proposition 20.2.
Let $\lambda, \mu \in \mathfrak{h}^\vee$.

a. If M is a highest weight module of highest weight μ and $\lambda \not< \mu$, then $\mathrm{Ext}_{\mathcal{O}}(M(\lambda), M) = 0$. Contrapositive: nontrivial extensions force the strict inequality $\mu < \lambda$. In particular, $\mathrm{Ext}_{\mathcal{O}}(M(\lambda), X) = 0$ for $X = L(\lambda), M(\lambda)$.
b. If $\mu \leq \lambda$, then $\mathrm{Ext}_{\mathcal{O}}(M(\lambda), L(\mu)) = 0$.
c. If $\mu < \lambda$, then $\mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\mu)) \cong \hom_{\mathcal{O}}(N(\lambda), L(\mu))$.

   (c) is useful, homs can be easier to compute. Can just look at radical structure of N, i.e. just the head.
d. $\mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$.

Proof (of (a)).
Given an extension $0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} M(\lambda) \longrightarrow 0$ where M is a highest weight module of highest weight $\mu \not< \lambda$. We want to show it splits.
Claim: Let v^+ be a maximal vector of $M(\lambda)$, let v be its preimage under g, then v is a maximal vector of weight λ in E. For $x \in \mathfrak{n}$, we can think of the RHS as a quotient and identify

$$\begin{aligned} x \cdot v + M &= x \cdot (v + M) \\ &= x \cdot v^+ \\ &= 0 \\ &= 0 + M, \end{aligned}$$

and for these to be equal this implies $x \cdot v \in M$. But $x \cdot v$ has weight $> \lambda$; since $\mu \not> \lambda$, M has no such weights. So we must have $x \cdot v = 0 \in E$, and v is a maximal vector.
It's also the case that $U(\mathfrak{n}^-)$ acts freely on v, since it acts freely on its image in the quotient $M(\lambda)$. So v generates a submodule $\langle v \rangle \leq E$ isomorphic to $M(\lambda)$. This defines a splitting (because of the freeness of this action) given by $h(v^+) = v$.

■

Proof (of (b)).
Follows from (a).

■

Proof (of (c)).
Look at the SES $0 \longrightarrow N(\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$. Apply $\hom_{\mathcal{O}}(\,\cdot\,, L(\mu))$ to get the LES

$$\begin{aligned} \cdots \longrightarrow \hom_{\mathcal{O}}(M(\lambda), L(\mu)) &\longrightarrow \hom_{\mathcal{O}}(N(\lambda), L(\mu)) \\ \longrightarrow \mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\mu)) &\longrightarrow \mathrm{Ext}_{\mathcal{O}}(M(\lambda), L(\mu)) \longrightarrow \cdots \end{aligned}$$

and since $L(\lambda)$ is the only simple quotient of $M(\lambda)$, so the first hom is zero. Similarly, the last $\mathrm{Ext}_{\mathcal{O}}$ is zero by (b), and the middle is an isomorphism.

■

Proof (of (d)).
Replace μ by λ in the LES, now term 2 above is zero since $\Pi(L(\lambda)) < \lambda$. Term 4 is zero by (b), and thus term 3 is zero.

■

Next section: duality in category $\mathcal{O}$.

21 Monday February 24th

21.1 Antidominant Weights

Recall that for $\lambda \in \mathfrak{h}^\vee$, we can associate $\Phi_{[\lambda]}$ and $W_{[\lambda]}$ and consider $W_{[\lambda]} \cdot \lambda$. When $\lambda \in \Lambda$ is integral and $\mu \in W\lambda \bigcap \Lambda^+$, we have $M(\mu) \longrightarrow L(\mu)$ its simple quotient, which is finite-dimensional.

Definition 21.0.1 (Antidominant)**.**
$\lambda \in \mathfrak{h}^\vee$ is *antidominant* if $(\lambda + \rho, \alpha^\vee) \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$. Dually, λ is *dominant* if $(\lambda + \rho, \alpha^\vee) \notin \mathbb{Z}^{<0}$ for all $\alpha \in \Phi^+$.

Note that most weights are both dominant and antidominant. Example: take $\lambda = -\rho$. We won't use the dominant condition often.

Remark For $\lambda \in \mathfrak{h}^\vee$, $W \cdot \lambda$ and $W_{[\lambda]} \cdot \lambda$ contain at least one antidominant weight. Let μ be minimal in either set with respect to the usual ordering on $\mathfrak{h}^\vee$. If $(\mu + \rho, \alpha^\vee) \in \mathbb{Z}^{>0}$ for some $\alpha > 0$, then $s_\alpha \cdot \mu < \mu$, which is a contradiction. So any minimal weight will be antidominant.

Proposition 21.1.
Fix $\lambda \in \mathfrak{h}^\vee$, as well as $W_{[\lambda]}, \Phi_{[\lambda]}$, Then define $\Phi^+_{[\lambda]} \coloneqq \Phi_{[\lambda]} \bigcap \Phi^+ \supset \Delta_{[\lambda]}$. TFAE:

a. λ is antidominant.
b. $(\lambda + \rho, \alpha^\vee) \leq 0$ for all $\alpha \in \Delta_{[\lambda]}$.
c. $\lambda \leq s_\alpha \cdot \lambda$ for all $\alpha \in \Delta_{[\lambda]}$.
d. $\lambda \leq w \cdot \lambda$ for all $w \in W_{[\lambda]}$.

In particular, there is a unique antidominant weight in $W_{[\lambda]} \cdot \lambda$.

Proof (a implies b).
$(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Delta_{[\lambda]}$ or $\Phi^+[\lambda]$.

■

Proof (b implies a).
Suppose (b) and $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Phi^+$. Then $\alpha \in \Phi^+ \bigcap \Phi_{[\lambda]}$, which is equal to $\Phi^+_{[\lambda]}$ by the homework problem. So $\alpha \in \mathbb{Z}^+ \Delta_{[\lambda]}$, and thus (claim) $(\lambda + \rho, \alpha^\vee) \leq 0$ by (b). Why? Replace $\alpha^\vee$ with a bunch of other $\alpha_i^\vee$ for which $(\lambda + \rho, \alpha_i^\vee) < 0$ and sum.

■

Proof (b iff c).
$s_\alpha \cdot \lambda = \lambda - (\lambda + \rho, \alpha^\vee)\alpha$.

■

Proof (d implies c).
Trivial due to definitions.

■

Proof (c implies d).
Use induction on $\ell(w)$ in $W_{[\lambda]}$. Assume (c), and hence (b), and consider $\ell(w) = 0 \implies w = 1$. For the inductive step, if $\ell(w) > 0$, write $w = w's_\alpha$ in $W_{[\lambda]}$ with $\alpha \in \Delta_{[\lambda]}$. Then $\ell(w') = \ell(w) - 1$, and by Proposition 0.3.4, $w(\alpha) < 0$.
We can then write

$$\lambda - w \cdot \lambda = (\lambda - w' \cdot \lambda) + (w' \cdot \lambda - w \cdot \lambda).$$

The first term is ≤ 0 by hypothesis, so noting that the w action is not linear but still an action, we have

$$\begin{aligned} w' \cdot \lambda - w \cdot \lambda &= w \cdot s_\alpha \cdot \lambda - w \cdot \lambda \\ &= w(s_\alpha \lambda - \lambda) \quad \text{by 1.8b} \\ &= w(-(\lambda + \rho, \alpha^\vee)\alpha) \\ &= -(\lambda + \rho, \alpha^\vee)(w\alpha) \\ &= -1(\in \mathbb{Z}^-)(< 0), \end{aligned}$$

which is a product of three negatives and thus negative.

■

A remark from page 56: Even when $\lambda \notin \Lambda$, we can decompose $\mathcal{O}_\chi$ into subcategories $\mathcal{O}_\lambda$. We then recover $\mathcal{O}_\chi$ as the sum over $\mathcal{O}_\lambda$ for antidominant λ's in the intersection of the linkage class with cosets of Λ_r. These are the homological blocks.

21.2 Tensoring Verma and Finite Dimensional Modules

First step toward understanding translation functors, which help with calculations.

By Corollary 1.2, we know that every $N \in \mathcal{O}$ has a filtration with every section being a highest weight module. We will improve this result to show that if M is finite-dimensional and V is a Verma module, then $V \otimes M$ has a filtration whose sections are all Verma modules. This is important for studying projectives in a couple of sections.

Theorem 21.2 ***(Sections of Finite-Dimensional Tensor Verma are Verma).***
Let M be a finite dimensional $U(\mathfrak{g})$-module. Then $T := M(\lambda) \otimes M$ has a finite filtration with sections $M(\lambda + \mu)$ for $\mu \in \Pi(M)$, occuring with the same multiplicities.

Proof .
Use the tensor identity

$$\Big(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L\Big) \otimes_{\mathbb{C}} M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes_{\mathbb{C}} M),$$

where

- $L \in U(\mathfrak{b})$-mod.
- $M \in U(\mathfrak{g})$-mod.
- $L \otimes M \in U(\mathfrak{b})$-mod via the tensor action.

The LHS is a $U(\mathfrak{g})$-module via the tensor action, and the RHS has an induced $U(\mathfrak{g})$-action.

> See proof in Knapp's "Lie Groups, Lie Algebras, and Cohomology". This is true more generally if $\mathfrak{g}$ is any lie algebra and $\mathfrak{b} \leq \mathfrak{g}$ any lie-subalgebra.

Recall from page 18 that the functor $\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}$ is exact on finite-dimensional $\mathfrak{b}$-modules. Assume L, M are finite-dimensional, and set $N := L \otimes_{\mathbb{C}} M$. Take a basis $v_1, \cdots, v_n$ of weight vectors for N of weights $\nu_1, \cdots, \nu_n$. Order these such that $\nu_i \leq \nu_j \iff i < j$.

Set N_k to be the $U(\mathfrak{b}) \langle v_k, \cdots, v_n \rangle$ for $1 \leq k \leq n$, which is a decreasing filtration since acting by $U(\mathfrak{b})$ moves along this list of vectors/weights to the right. By induction on n, this filtration induces a filtration on $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ whose sections are Verma modules.
This yields

$$\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N_k / \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N_{k+1} \cong M(\nu_k).$$

The intermediate quotients will be 1-dimensional submodules, which will induce up to highest weight modules. We'll finish the proof next time.

■

22 Wednesday February 26th

We want to show the following identity:

$$\Big(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L\Big) \otimes_{\mathbb{C}} M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes_{\mathbb{C}} M).$$

Assume L and M are finite dimensional. Then for $N = L \otimes M$, there is a basis of weight vectors $v_1, \cdots, v_n, \nu_1, \cdots, \nu_m$ with $\nu_i \leq \nu_j \iff i \leq j$. Moreover

$$N_k = \mathbb{C} \langle v_k, \cdots, v_n \rangle = U(\mathfrak{b}) \langle v_k, \cdots, v_n \rangle,$$

and we have a natural filtration

$$0 \subset N_n \subset \cdots \subset N_1 = N$$

with $N_i/N_{i+1} \cong \mathbb{C}_{v_i}$ as $\mathfrak{b}$-modules.

We thus obtain

$$\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N_i / \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N_{i+1} \cong \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_{v_i} = M(v_i)$$

by exactness of the Ind functor. Apply this to $L = \mathbb{C}_\lambda$, then the LHS is $M(\lambda) \otimes_{\mathbb{C}} M$, where M is finite dimensional. On the RHS, $N = \mathbb{C}_\lambda \otimes M$ has the same dimension as M with weights $\lambda + \mu$ for $\mu \in \Pi(M)$. Thus $M(\lambda) \otimes M$ has filtration with quotients $M(\lambda + \mu)$ over $\mu \in \Pi(M)$, which was the theorem we had last time.

Remark The proof shows that $M(\lambda) \otimes M$ has a submodules $M(\lambda+\mu)$ for any maximal weight μ of M, and a quotient $M(\lambda+\nu)$ where ν is any minimal weight of M. We knew that every $M \in \mathcal{O}$ has a finite filtration, but here the quotients are now Verma modules. This will help us study projectives later, which we need to study higher Exts.

22.1 Standard Filtrations

There are several main players in the theory of highest-weight categories, of which $\mathcal{O}_{\chi_\lambda}$ is one:

- Simple modules: $L(\lambda)$
- Standard modules $M(\lambda)$
- Costandard modules $M(\lambda)^\vee$
- Indecomposable projectives $P(\lambda)$
- Tilting modules $T(\lambda)$.

Definition 22.0.1 (Standard filtration/Verma flag)**.**
A *standard filtration* of $M \in \mathcal{O}$ is a filtration with subquotients isomorphic to Verma modules.

Note that when M has a standard filtration, the submodules are *not* unique, but the length, subquotients, and multiplicities are unique. We can thus use $K(\mathcal{O})$ or formal characters as an invariant, since the multiplicities $(M : M(\lambda))$ are well-defined.

If M, N have standard filtration, then so does $M \oplus N$ by concatenation. In this case, $(M \oplus N : M(\lambda)) = (M : M(\lambda)) + (N : M(\lambda))$.

Proposition 22.1.
Let $M \in \mathcal{O}$ have a standard filtration. Then

a. If λ is maximal in $\Pi(M)$, then M has a submodule isomorphic to $M(\lambda)$ and $M/M(\lambda)$ has a standard filtration
$$0 = M_0 \subset \cdots \subset M_n = M.$$
b. If $M = M' \oplus M''$, then M' and M'' have standard filtrations.
c. M is free as a $U(\mathfrak{n}^-)$-module.

Proof (of (a)).
By assumption on λ, M has a maximal vector of weight λ, and thus the universal property yields a nonzero morphism $\phi : M(\lambda) \longrightarrow M$.
The claim is that ϕ is injective, from which the proof follows. Proof of claim: choose a minimal index i such that $\phi(M(\lambda)) \subset M_i$ in the filtration. Follow this with the subquotient map to yield
$$\psi : M(\lambda) \longrightarrow M^i := M_i/M_{i-1} \cong M(\mu),$$
which is nonzero by minimality of i.
Thus $\lambda \leq \mu$, and by our assumption, this implies $\lambda = \mu$. But then ψ sends highest weight vectors to highest weight vectors and is free, so ψ is an isomorphism. Thus ϕ is injective and $M(\lambda) \subset M$.
We can now write $M(\lambda) \bigcap M_{i-1} = \ker \psi = 0$, so we obtain a direct sum decomposition

$M_i \cong M_{i-1} \oplus M(\lambda)$. We thus obtain a SES

$$0 \longrightarrow M_{i-1} \longrightarrow M/M(\lambda) \longrightarrow M/M_i \longrightarrow 0.$$

We can easily construct standard filtrations for M_{i-1} and M/M_i, so the middle term also has a standard filtration. Thus $M/M(\lambda)$ has a standard filtration of length one less than that of M. ■

Proof (of(b)).
By induction of the filtration length n of M. If $n = 0$, M is a Verma module and thus indecomposable and there's nothing to show.
For $n \geq 1$, let $\pi \in \Pi(M)$ be maximal (which we can always find for $M \in \mathcal{O}$)) and WLOG $M'_\lambda \neq 0$.
By the universal property, we have a nonzero composition

$$\begin{tikzcd} M(\lambda) \arrow[r] \arrow[rr, dashed, bend right, "\neq 0"] & M' \arrow[r, hook] & M \end{tikzcd}$$

Applying (a) to this composite map,

1. It must be injective, so $M(\lambda) \hookrightarrow M'$
2. $M/M(\lambda) \cong M'/M(\lambda) \oplus M''$ has a standard filtration of length $n-1$.

By induction, $M'/M(\lambda)$ and M'' have standard filtrations, and thus so does M'. ■

Proof (of (c)).
By induction on n: if $n = 1$, then $M \cong M(\lambda)$ is $U(\mathfrak{n}^-)$-free. Otherwise, if $n > 1$, by (a) $M(\lambda) \subset M$ and $M/M(\lambda)$ has a standard filtration of length $n-1$. By induction, $M/M(\lambda)$ is $U(\mathfrak{n}^-)$-free, and hence so is M. ■

Theorem 22.2 ***(Multiplicities of Vermas).***
If M has a standard filtration, then $(M : M(\lambda)) = \dim \hom_{\mathcal{O}}(M, M(\lambda)^\vee)$.

Proof .
By induction on the filtration length n. If $n = 1$, M is a Verma module, and $(M(\mu) : M(\lambda)) = \delta_{\mu\lambda} = \dim \hom_{\mathcal{O}}(M(\mu), M(\lambda)^\vee)$ by Theorem 3.3c. ■

For $n > 1$, consider

$$0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow M(\mu) \longrightarrow 0.$$

Apply the left-exact contravariant functor $\hom_{\mathcal{O}}(\,\cdot\,, M(\lambda)^\vee)$ to obtain

$$
\begin{tikzcd}
0 \arrow[r] & \hom(M(\mu), M(\lambda)^\vee) \arrow[r] \arrow[d, Rightarrow] & \hom(M, M(\lambda)^\vee) \arrow[r] \arrow[d, Rightarrow] & \hom(M_{n-1}, M(\lambda)^\vee) \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}(M(\mu), M(\lambda)^\vee) \arrow[r] & \cdots \\
& \delta_{\mu\lambda} & (M : M(\lambda)) = (M_{n-1}:M(\lambda)) + \delta_{\mu\lambda} & \substack{(M_{n-1}:M(\lambda)) \\ \text{by induction}} & \text{0 by Thm 3.3d} \arrow[u, Rightarrow] &
\end{tikzcd}
$$

22.2 Projectives in $\mathcal{O}$

We want to show that $\mathcal{O}$ has *enough projectives*, i.e. every $M \in \mathcal{O}$ is a quotient of a projective object. We'll also want to show $\mathcal{O}$ has *enough injectives*, i.e. every modules embeds into an injective object.

Definition 22.2.1 (Projective Objects)**.**
If $\mathcal{A}$ is an abelian category, an object $P \in \mathcal{A}$ is *projective* iff the left-exact functor $\hom_{\mathcal{A}}(P, \cdot\,)$ is exact, or equivalently

$$
\begin{tikzcd}
 & P \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "f"] & \\
M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}
$$

In other words, there is a SES

$$\hom(P, M) \longrightarrow \hom(P, N) \longrightarrow 0,$$

which precisely says that every f in the latter has a lift $\tilde{f}$ in the former by surjectivity.

Definition 22.2.2 (Injective Objects)**.**
An object $Q \in \mathcal{A}$ is *injective* iff $\hom_{\mathcal{A}}(\,\cdot\,, Q)$ is exact, i.e.

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "g"] & M \arrow[dl, "\exists \tilde{g}"] \\
 & Q &
\end{tikzcd}
$$

i.e.,

$$\hom_{\mathcal{A}}(M, Q) \longrightarrow \hom_{\mathcal{A}}(N, Q) \longrightarrow 0$$

so every g in the latter has a lift to $\tilde{g}$ in the former.

In $\mathcal{O}$, having enough projectives is equivalent to having enough injectives because $(\,\cdot\,)^\vee$ is an exact contravariant endofunctor, which sends projectives to injectives and vice-versa. Thus we'll focus on projectives.

23 Friday February 28th

Recall that $\lambda \in \mathfrak{h}^\vee$ is *dominant* iff for all $\alpha \in \Phi^+$, we have $(\lambda + \rho, \alpha^\vee) \notin \mathbb{Z}^{<0}$. Equivalently, as in Proposition 3.5c, λ is maximal in its $W_{[\lambda]}\cdot$ orbit.

23.1 Constructing Projectives

Proposition 23.1. a. If $\lambda \in \mathfrak{h}^\vee$ is dominant, then $M(\lambda)$ is projective in $\mathcal{O}$.
b. If $P \in \mathcal{O}$ is projective and $\dim L < \infty <$ then $P \otimes_{\mathbb{C}} L$ is projective.

Proof . a. We want to find a ψ making this diagram commute:

$$\begin{tikzcd} & v^+ \in M(\lambda) \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ M \arrow[r, "\pi"] & p(v^+) \in N \arrow[r] & 0 \end{tikzcd}$$

Assume $\phi \neq 0$. Since $M(\lambda) \in \mathcal{O}_{\chi_\lambda}$, we have $\phi(M(\lambda)) \subset N^{\chi_\lambda}$. WLOG, we can assume $M, N \in \mathcal{O}_{\chi_\lambda}$, and if v^+ is maximal, $p(v^+)$ is maximal. By surjectivity of π, there exists a $v \in M$ such that $v \mapsto p(v^+)$. Then $M \supset U(\mathfrak{n})v$ is finite dimensional, so it contains a maximal vector whose weight is linked to λ since $M \in \mathcal{O}_{\chi_\lambda}$.
But since λ is dominant, there is no such weight greater than λ, so v itself must be this maximal vector. Then by the universal property of $M(\lambda)$, there is a map $\psi : M(\lambda) \longrightarrow M$ where $v^+ \mapsto v$ making the diagram commute.

Note nice property: Vermas are projective iff maximal in orbit.

b. We want to show $F = \hom_{\mathcal{O}}(P \otimes L, \cdot\,)$ is exact. But this is isomorphic to

$$\hom_{\mathcal{O}}(P, \hom_{\mathbb{C}}(L, \cdot\,)) \cong \hom_{\mathcal{O}}(P, L^\vee \otimes_{\mathbb{C}} \cdot\,).$$

Thus F is the composition of two exact functors: first do $L^\vee \otimes_{\mathbb{C}} \cdot$, which is exact since $\mathbb{C}$ is a field, and $\hom_{\mathcal{O}}(P, \cdot\,)$ is exact since P is projective.

■

Example 23.1.
Let $M(-\rho)$ be the Verma of highest weight ρ. This is irreducible because $-\rho$ is antidominant, and projective since $-\rho$ is dominant. In fact $W \cdot (-\rho) = \{-\rho\}$ by a calculation. Thus

$$L(-\rho) = M(-\rho) = P(-\rho) = M(-\rho)^\vee,$$

so all 4 members of the highest weight category here are equal.

By convention, there is notation $M(-\rho) = \Delta(-\rho)$ and $M(-\rho)^\vee = \nabla(-\rho)$.

Note that we always have $\mathrm{Ext}_0(L(-\rho), L(-\rho)) = 0$, and every $\mathcal{O}_{\chi_{-\rho}} \in M$ is equal to $\bigoplus L(-\rho)^{\oplus n}$.

So this is referred to as a *semisimple category.*

Theorem 23.2 ***(O has Enough Projectives and Injectives).***
$\mathcal{O}$ has enough projectives and injectives.

Proof .
Step 1
For all $\lambda \in \mathfrak{h}^\vee$, there exists a projective mapping onto $L(\lambda)$. Clearly $\mu := \lambda + n\rho$ is dominant for $n \gg 0$, i.e. for n large enough there are no negative integers resulting from inner products with coroots. Thus $M(\mu)$ is projective, and since $n\rho \in \Lambda^+$, we have $\dim L(n\rho) < \infty$. This implies $P := M(\mu) \otimes L(n\rho)$ is projective by the previous proposition.
Apply w_0 reverses the weights, so $w_0(n\rho) = -n\rho$. Note that this doesn't happen for all weights, so this property is somewhat special for ρ. In particular, since $n\rho$ was a highest weight, $-n\rho$ is a lowest weight.
By remark 3.6, P has a quotient isomorphic to $M(\mu - n\rho) = M(\lambda)$. Thus $P \twoheadrightarrow M(\lambda) \longrightarrow L(\lambda)$, and $L(\lambda)$ is a quotient of a projective. This establishes the result for simple modules.

Remark: By theorem 3.6, P has a standard filtration with sections $M(\mu + \nu)$ for $\nu \in \Pi(L(n\rho))$. In particular $M(\lambda)$ occurs just once since

$$\dim L(n\rho)_{-n\rho} = \dim L(n\rho)_{w_0(n\rho)} = \dim L(n\rho)_{n\rho} = 1,$$

with all $\mu + \nu > \lambda$.

Step 2
Use induction on Jordan-Hilder length to prove that any $0 \neq M \in \mathcal{O}$ is a quotient of a projective. For $\ell = 1$, M is simple, and by Step 1 this case holds.
Assume $\ell > 1$, then M has a submodule $L(\lambda)$ obtained by taking the bottom of a Jordan-Holder series, so there is a SES

$$0 \longrightarrow L(\lambda) \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0.$$

By induction, since $\ell(N) = \ell(M) - 1$, there exists a projective module $Q \xrightarrow{\phi} N$ which extends to a map $\psi : Q \longrightarrow M$.
If ψ is surjective, we are done. Otherwise, then the composition length forces $\psi(Q) \cong N$, and by commutativity there is a section $\gamma : N \longrightarrow \psi(Q)$ splitting this SES. Thus $M \cong L(\lambda) \oplus N$, and by 1, there are projectives $P \oplus Q$ projecting onto each factor, so M is projective.

$$\begin{tikzcd}
0 \arrow[r] & L(\lambda) \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r] \arrow[l, dotted, bend left, "\gamma"] & 0 \\
 & & & P \arrow[u, "\varphi"'] \arrow[ul, dashed, "\psi"] &
\end{tikzcd}$$

■

23.2 3.9 Indecomposable Projectives

Definition 23.2.1 (A Projective Cover)**.**
A *projective cover* of $M \in \mathcal{O}$ is a map $\pi : P_M \longrightarrow M$ where P_M is projective and π is an *essential epimorphism*, i.e. no proper submodule of P_M is mapped surjectively onto M by π_M.

It is an algebraic fact that in an Artinian (abelian) category with enough projectives, every module has a projective cover that is unique up to isomorphism.

See Curtis and Reiner, Section 6c.

Definition 23.2.2 (The Projective Cover for a Weight)**.**
For $\lambda \in \mathfrak{h}^\vee$, denote $\pi_\lambda : P(\lambda) \twoheadrightarrow L(\lambda)$ to be a fixed projective cover of $L(\lambda)$.

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$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] & \\
P(\lambda) \arrow[r, "\pi_\lambda ?"'] & L(\lambda) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d, Rightarrow] & \\
P \arrow[r, "\psi"'] & P(\lambda) \arrow[r] & 0
\end{tikzcd}$$

$$\frac{L(\lambda)}{L(\mu)} = P(\lambda).$$

$$\left.\begin{array}{c} L(\mu) \\ \\ \hline \end{array}\right\} M(\mu)$$

$$\left.\begin{array}{c} L(\lambda) \\ \hline L(\mu) \end{array}\right\} M(\lambda)$$

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Proposition 26.1.
Suppose $\lambda + \rho$ is dominant integral, then

- $M(w \cdot \lambda) \subset M(\lambda)$ for all $w \in W$
- $[M(\lambda) : L(w \cdot \lambda)] > 0$ for all $w \in W$

More precisely, if $w = s_1 \cdots s_\ell$ is reduced with $s_i = s_{\alpha_i}$ with $\alpha_i \in \Delta$ and $\lambda_k = s_k \cdots s_1 \cdot \lambda$, then

$$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_0) = M(\lambda)$$

Moreover $(\lambda_k + \rho, \alpha_{k+1}^\vee) \in \mathbb{Z}^+$ for $0 \leq k \leq n-1$ and so

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0.$$

Proof .
By induction on $n = \ell(w)$. The $n = 0$ case is obvious. For $\ell(w) = k+1$, write $w' = s_k \cdots s_1$. From section 0.3, $(w')^{-1}\alpha_{k+1} > 0$. We can compute

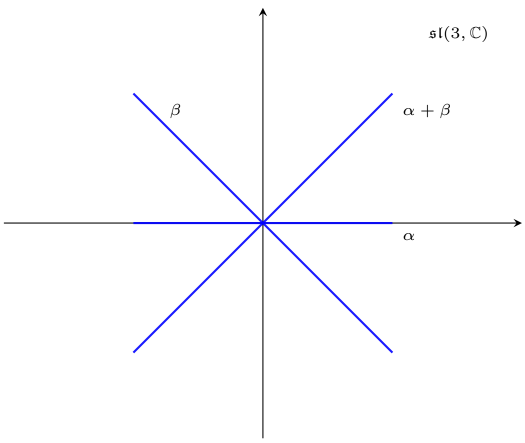
$$\begin{aligned}
(\lambda_k + \rho, \alpha_{k+1}^\vee) &= (w' \cdots \lambda + \rho, \alpha_{k+1}^\vee) \\
&= (w'(\lambda + \rho), \alpha_{k+1}^\vee) \\
&= (\lambda + \rho, (w')^{-1}\alpha_{k+1}^\vee) \\
&= (\lambda + \rho, ((w')^{-1}\alpha_{k+1})^\vee) \\
&\in \mathbb{Z}^+
\end{aligned}$$

since $\lambda + \rho \in \Lambda^+$ and $(w')^{-1}\alpha_{k+1} \in \Phi^+$.
This means that $\lambda_{k+1} = s_{k+1}\lambda_k \leq \lambda_k$. By proposition 1.4, reformulated in terms of the dot action, we have a map $M(\lambda_{k+1}) \hookrightarrow M(\lambda_k)$, and nonzero morphisms are injective by 4.2a. ■

Exercise (4.3) If $\lambda + \rho \in \Lambda^+$, $\mathrm{Soc} M(\lambda) = M(w_o \cdot \lambda)$, and moreover if $\lambda \in \Lambda_0^+$ then the inclusions in the proposition are all proper.

Remark For general $\mu \in \Lambda$, it is not so easy to decide when $M(w \cdot \mu) \subset M(\mu)$. The basic problem is that Proposition 1.4 only works for *simple* roots, whereas we can have $s_\gamma \cdot \mu < \mu$ for $\gamma \in \Phi^+ \setminus \Delta$ with no obvious way to constrct an embedding $M(s_\gamma \cdot \mu) \subset M(\mu)$. See the following example.

Example 26.1.
Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$.



We don't know if there's a diagonal map indicated by the question mark in the following diagram:

$$\begin{tikzcd}
& \lambda \in \Lambda^{+} & \\
\mu = s_\alpha \cdot \lambda \arrow[ur] & & s_\beta \cdot \lambda \arrow[ul] \\
& s_\alpha s_\beta \cdot \lambda = s_{\alpha+\beta}\lambda = s_\gamma \lambda \arrow[ul, dashed, "?"] \arrow[ur] &
\end{tikzcd}$$

Next few sections: any root reflection that moves downward through the ordering induces a containment of Verma modules.

26.1 (4.4) Simplicity Criterion: The Integral Case

Theorem 26.2 ***(Vermas Equal Quotients iff Antidominant Weight).***
Let $\lambda \in \mathfrak{h}^\vee$ be any weight. Then $M(\lambda) = L(\lambda) \iff \lambda$ is antidominant.

The proof for λ integral is fairly easy, because antidominance reduces to a condition involving simple roots, where we can use our Verma module embedding criterion from Proposition 1.4.

Proof (Integral Case).
Assume $\lambda \in \Lambda$.
$\implies$: Assume $M(\lambda)$ is simple but λ is not antidominant. Then since $\lambda \in \Lambda$, $(\lambda + \rho, \alpha^\vee)$ is a positive integer for some $\alpha \in \Delta$. But then $s_\alpha \lambda < \lambda$ so $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ by 1.4 and 4.2. But then $N(\lambda) \neq 0$, which contradicts irreducibility.
$\impliedby$: Assume λ is antidominant. By proposition 3.5, $\lambda < w \cdot \lambda$ for all $w \in W$. Since all composition factors of $M(\lambda)$ and $L(w \cdot \lambda)$ where $w \cdot \lambda \leq \lambda$. This can only happen if $w \cdot \lambda = \lambda$, and so the only possible composition factor is $L(\lambda)$. Since $[M(\lambda) : L(\lambda)]$ is always equal to one, $M(\lambda)$ is simple.

■

Remark The reverse implication still works in general if W is replaced by $W_{[\lambda]}$. To extend the forward implication, we need to understand embeddings $M(s_\beta \cdots \lambda) \hookrightarrow M(\lambda)$ when β is not simple.

26.2 Existence of Embeddings (Preliminaries)

Lemma 26.3 ***(Commuting Nilpotents).***
Let $\mathfrak{a}$ be a nilpotent Lie algebra (e.g. $\mathfrak{n}^-$) and $x \in \mathfrak{a}, u \in U(\mathfrak{a})$, then for every $n \in \mathbb{Z}^+$ there exists a $t \in \mathbb{Z}^+$ such that $x^t u \in U(\mathfrak{a}) x^n$.

See Engel's theorem

Proof .
Use the fact that ad x acts nilpotently on $U(\mathfrak{a})$, so there exists a $q \geq 0$ such that $(\text{ad } x)^{q+1} u = 0$. Let ℓ_x, r_x be left and right multiplication by x on $U(\mathfrak{a})$. Then ad $x = \ell_x - r_x$, and $\ell_x, r_x \text{ad } x$ all commute.
Choosing $t \geq q + n$, we have

$$\begin{aligned} x^t u &= \ell_x^t u \\ &= (r_x + \text{ad } x)^t u \\ &= \sum_{i=0}^{t} \binom{t}{i} r_x^{t-i} (\text{ad } x)^i u \\ &= \sum_{i=0}^{q} \binom{t}{i} \left((\text{ad } x)^i u \right) x^{t-i} \\ &\in U(\mathfrak{a}) x^{t-q} \\ &\subset U(\mathfrak{a}) x^n \end{aligned}$$

■

This will be useful when moving things around by positive roots that are not simple.

27 Monday March 30th

Reminder of what we did already: we started on chapter 4, going into more detail on the strucutre of Verma modules and morphisms between them. We showed that the socle is an irreducible Verma modules, any nonzero morphism is injective, and the dimension of the hom space is at most 1. We ended showing a proposition about how to commute elements.

Proposition 27.1 ***(Key Result).***
Let $\lambda, \mu \in \mathfrak{h}^\vee$ and $\alpha \in \Delta$ be simple. Assume that $n := (\lambda + \rho, \alpha^\vee) \in \mathbb{Z}$ and $M(s_\alpha \cdot \mu) \subset M(\mu) \subset M(\lambda)$. Then either

a. $n \leq 0$ and $M(\lambda) \subset M(s_\alpha \cdot \lambda)$, or
b. $n > 0$ and $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda) \subset M(\lambda)$.

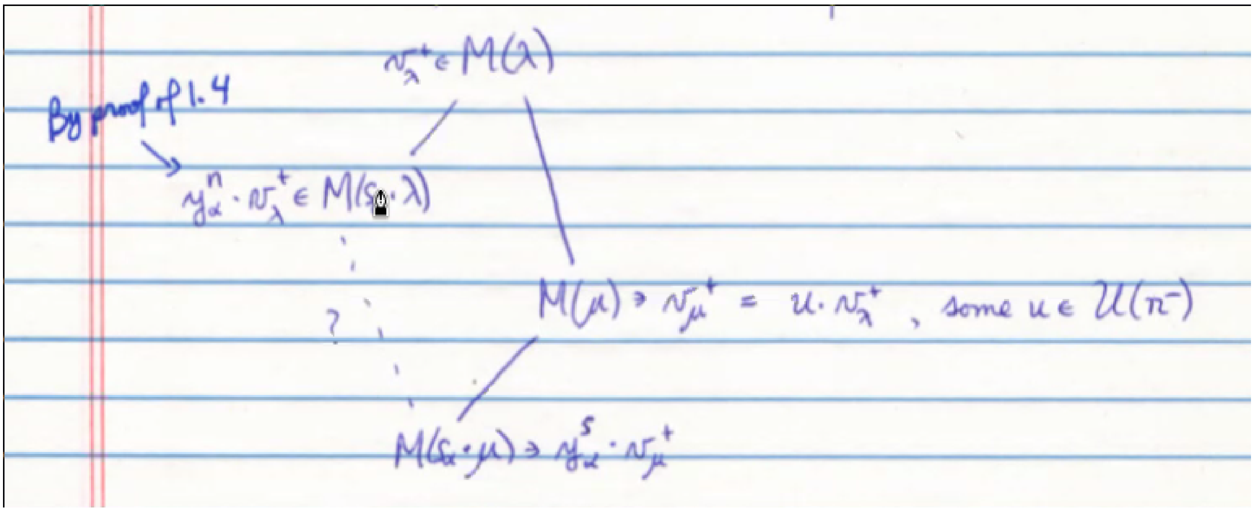
In either case, $M(s_\alpha \cdot \mu) \subset M(s_\alpha \cdot \lambda)$.

Proof (of (a)).
Use proposition 1.4 (exchanging λ and $s_\alpha \cdot \lambda$).

■

27.1 Proof (of (b))

Assume $n > 0$. Then $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ by proposition 1.4. Set $s = (\mu + \rho, \alpha^\vee) \in \mathbb{Z}^+$. Denote maximal vectors as follows:



Apply the lemma about nilpotent lie algebras to $\mathfrak{n}^-, y_\alpha, u$, and n, then there exists a $t > 0$ such that $y_\alpha^t u \in U(\mathfrak{n}^-) y_\alpha^n$. Then

$$y_\alpha^t \cdot v_\lambda^+ = y_\alpha^t u \cdot v_\lambda^+ \in U(\mathfrak{n}^-) y_\alpha^n \cdot v_\lambda^+ \subseteq M(s_\alpha \cdot \lambda).$$

Now there are two cases.

Case 1:

If $t \leq s$, we can apply y_α^{s-t} to equation star to obtain $y_\alpha^s \cdot v_\lambda^+ \in M(s_\alpha \cdot \lambda)$. Thus we have the containment we wanted to prove.

Case 2:

Suppose $t > s$. We can't divide in the enveloping algebra, but recall the identity in lemma 1.4(c):

$$[x_\alpha y_\alpha^t] = t y_\alpha^{t-1}(h_\alpha - t + 1).$$

Thus

$$[x_\alpha y_\alpha^t] \cdot v_\mu^+ = t(s-t) y_\alpha^{t-1} \cdot v_\mu^+.$$

Calculating the bracket another way, the LHS is equal to $x_\alpha y_\alpha^t \cdot v_\mu^+ - y_\alpha^t x_\alpha \cdot v_\mu^+$ and the second term is zero, so this is in $M(s_\alpha \cdot \lambda)$ by equation star. We can then iterate if $t - 1 > s$, reducing the power of y_α until we get down to $y_\alpha^s \cdot v_\mu^+ \in M(s_\alpha \cdot \lambda)$, in which case we are done by case 1.

■

27.2 4.6: Existence of Embeddings

Theorem 27.2 ***(Verma's Thesis).***
Let $\lambda \in \mathfrak{h}^\vee$ and $\alpha \in \Phi^+$ and assume $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then $M(\mu) \subset M(\lambda)$.

27.2.1 Proof

Assume $\lambda \in \Lambda$ is integral and μ is linked to λ, all weights involved are integral. Without loss of generality,$\mu < \lambda$, since we can apply a Weyl group element to place it in the dominant Weyl chamber.

1. Since μ is integral, choose $w \in W$ such that $\mu' \coloneqq w^{-1} \cdot \mu \in \Lambda^+ - \rho$. Following the notation in proposition 4.3, write $w = s_n \cdots s_1, \mu_k = s_k \cdots s_1 \cdot \mu'$. Then $\mu' = \mu_0 \geq \mu_1 \geq \cdots \geq \mu_n = \mu$, which yields a chain of inclusions of Verma modules $M(\mu_0) \supset M(\mu_1) \supset \cdots$.
2. Set $\lambda' = w^{-1}\lambda$ and $\lambda_k = s_k \cdots s_1 \cdot \lambda'$ so $\lambda_0 = \lambda'$ and $\lambda_n = \lambda$. Note that since $\mu \neq \lambda$, we have $\mu_k \neq \lambda_k$.
3. How are μ_k and λ_k related? Set $w_k = s_n \cdots s_{k+1}$. It can be checked that $\mu_k = w_k^{-1} s_\alpha w_k \cdot \lambda_k = s_{\beta_k}\lambda_k$ where $\beta_k = \left| w_k^{-1} \right| \in \Phi^+$ is the choice of whichever is positive by Humphreys 1, Lemma 9.2. In particular, $\mu_k - \lambda_k \in \mathbb{Z}\beta_k$.
4. We have $\mu' = \mu_0 \geq \cdots \geq \mu_k \geq \mu_{k+1} \geq \cdots \geq \mu_n = \mu$. Since $\lambda' < \mu'$ (because μ' is the unique dominant weight in $W \cdot \lambda$ but $\mu < \lambda$, so the inequalities must switch at some k. So say $\lambda_k < \mu_k$ but $\lambda_{k+1} > \mu_{k+1}$, where k is chosen to be the smallest index for which this happens. Note that all of the weights are linked to λ.
5. We want to show that $M_{\mu_{k+1}} \subset M(\lambda_{k+1}), \cdots, M(\mu) = M(\mu_n) \subset M(\lambda_n) = M(\lambda)$.
6. First, $\mu_{k+1} - \lambda_{k+1} = s_{k+1} \cdot \mu_k - s_{k+1} \cdot \lambda_k$, where the LHS is some negative times β_{k+1}, and the RHS is equal to $s_{k+1}(\mu_k - \lambda_k)$, which is a positive times β_k by exercise 1.8. Since s_{k+1} permutes the positive roots other than α_{k+1}, this forces $\beta_k = \beta_{k+1} = \alpha_{k+1}$. So we have $\mu_{k+1} = s_{\alpha_{k+1}}\lambda_{k+1}$ which by proposition 1.4 implies that $M(\mu_{k+1}) \subset M(\lambda_{k+1})$.
7. Combining 1 and 6, we have $M(\mu_{k+2}) = M(s_{k+2} \cdot \mu_{k+1}) \subset tM(\mu_{k+1}) \subset M(\lambda_{k+1})$. This is the setup of proposition 4.5 and wither alternative leads to $M(\mu_{k+2}) \subset M(\lambda_{k+2}) = M(s_{\alpha_{k+2}} \cdot \lambda_{k+1})$,
8. Since this increases the index, we can iterate step 7 to complete step 5 and get the desired containment.

28 Wednesday April 1st

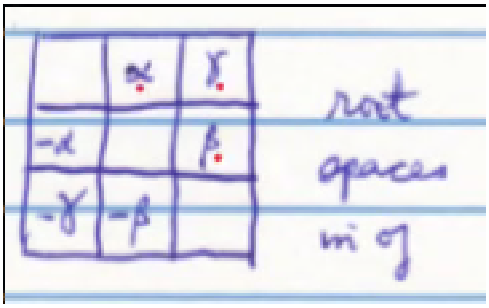
Exercise Work through the steps for $\mathfrak{sl}(3)$, due next Thursday.

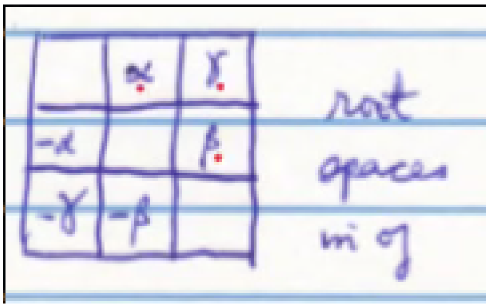
Preview of next sections:

- 4.8: Simplicity Criterion, General Case
  - Now that we know $M(s_\alpha \cdot \lambda) \subset M(\lambda)$ whenever $s_\alpha\lambda \leq \lambda$ for $\alpha \in \Phi^+$ (and not just $\alpha \in \Delta$) we can easily complete the proof of theorem 4.4 by copying the argument from the integral case.
- 4.9: Blocks of $\mathcal{O}$, revisited
  - Skip, mainly relevant for nonintegral weights (c.f. Proposition 1.13 for the description of integral blocks)
- 4.10: Example: Antidominant Projectives
  - Skip, at least for now

28.1 4.11: Application to $\mathfrak{sl}(3, \mathbb{C})$

The simplest nontrivial case, what can we say about the Verma modules?





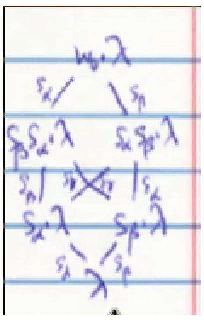
We have $\Delta = \{\alpha, \beta\}$ and $\Phi^+ = \{\alpha, \beta, \gamma := \alpha + \beta\}$. The Weyl group is

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_0 := s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta\} .$$

We first consider an integral regular linkage class $W \cdot \lambda$, and we way choose an antidominant λ and assume

$$(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}^{<0} \quad \forall \alpha \in \Phi^+ \quad \text{e.g. } \lambda = -2\rho$$

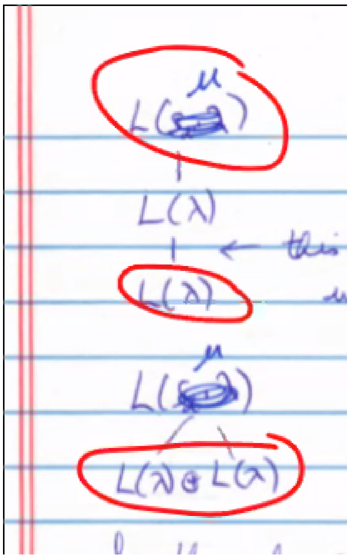
Then $W_\lambda = \{1\}$, given by the stabilizer of the isotropy subgroup, and $|W \cdot \lambda| = 6$. So there are 6 Verma modules to understand, and we have the following diamond:



The middle two edges are s_γ, and each edge corresponds to an inclusion of Verma modules (with the inclusions going upward). By Verma's theorem, the Bruhat order corresponds to these inclusions.

1. $M(\lambda) = L(\lambda)$ since λ is antidominant by Theorem 4.4

2. By the same theorem, no other $M(w \cdot \lambda)$ is simple. Then by Proposition 4.18, Theorem 4.2c, we have $\mathrm{Soc} M(w \cdot \lambda) = L(\lambda)$ for all $w \in W$
3. Consider $M(s_\alpha \cdot \lambda)$ and set $\mu := s_\alpha \cdot \lambda$ and the only possible composition factors are $L(\mu)$ and $L(\lambda)$ and $[M(\mu) : L(\mu)] = 1$. If we use Theorem 4.10, this multiplicity is 1 in the socle and we're done. If we don't, could it be larger than 1? Since $\mathrm{Ext}_{\mathcal{O}}(L(\lambda), L(\lambda)) = 0$, we can not have the following situation:

   The first extension doesn't exist, since the higher $L(\lambda)$ would drop down to give the bottom diagram, which contradicts $\mathrm{Soc} M(\mu) = L(\lambda)$.

   So the only possibilities are multiplicity 1, and $M(s_\alpha \cdot \lambda) = L(s_\alpha \cdot \lambda)$ which lives over $L(\lambda)$, so $\mathrm{char}\ L(s_\alpha \cdot \lambda) = \mathrm{char}\ M(s_\alpha \cdot \lambda) - \mathrm{char}\ M(\lambda)$.

   Similary for $M(s_\beta \cdot \lambda)$.
4. For the higher weights in the orbit, we need more theory. We know there are inclusions $x \leq w \implies M(x \cdot \lambda) \subset M(w \cdot \lambda)$ according to the Bruhat order - so every edge in the weight poset is a reflection, so use Verma's theorem.

$$[M(w \cdot \lambda) : L(x \cdot \lambda)] = \begin{cases} \geq 1 & ? \\ 0? & ? \end{cases}.$$

We'll skip 4.12,4.13, 4.14.

28.2 Chapter 5: Highest Weight Modules II

Development by BGG after 1970s, based on partly incorrect proof in Verma's thesis.

28.2.1 5.1: The BGG Theorem

Which simple modules occur as composition factors of $M(\lambda)$?

Definition 28.0.1.
For $\mu, \lambda \in \mathfrak{h}^\vee$, write $\mu \uparrow \lambda$ if $\mu = \lambda$ or there exists an $\alpha \in \Phi^+$ such that $\mu = s_\alpha \cdot \lambda < \lambda$, i.e. $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}^{>0}$. Extend this relation transitively: if there exists $\alpha_1, \cdots, \alpha_r \in \Phi^+$ such that $\mu = (s_{\alpha_1} \cdots s_{\alpha_r}) \cdot \lambda \uparrow (s_{\alpha_2} \cdots s_{\alpha_r} \uparrow \cdots \uparrow s_{\alpha_r} \lambda \uparrow \lambda$, we again write $\mu \uparrow \lambda$ and say μ is *strongly linked* to λ, yielding a partial order on $\mathfrak{h}^\vee$.

Theorem 28.1.
Let $\mu, \lambda \in \mathfrak{h}^\vee$. - (Verma) If $\mu \uparrow \lambda$ then $M(\mu) \hookrightarrow M(\lambda)$. In particular, $[M(\lambda) : L(\mu)] > 0$. - ??? ???

Corollary 28.2.
$[M(\lambda) : L(\mu)] \neq 0 \iff M(\mu) \hookrightarrow M(\lambda)$.

The situation is not as straightforward as it might appear (and as Verma believed). Namely, if $0 = M_0 \subset \cdots \subset M_n = M(\lambda)$ is a composition series and if $M_i / M_{i-1} \cong L(\mu) \ni \overline{v}_\mu^+$, there need not be any preimage of v_μ^+ which is a maximal vector in $M(\lambda)$, leading to a map $M(\mu) \hookrightarrow M(\lambda)$.

However, when this happens, there will always be some other $M_j / M_{j-1} \cong L(\mu)$ where a preimage of a maximal vector **is** maximal in $M(\lambda)$, leading to the required embedding.

28.2.2 5.2 Bruhat Ordering

In the case of "ρ-regular" integral weights, the BGG theorem has a nice reformulation in terms of W and the Bruhat ordering. Fix $\lambda \in \Lambda$ antidominant and ρ-regular, so $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}^{<0}$ for all $\alpha \in \Phi^+$.

As in the discussion of $\mathfrak{sl}(3)$, this means that $|W \cdot \lambda| = |W|$ and $[M(w \cdot \lambda) : L(\mu)] \neq 0$ implying that $\mu = x \cdot \lambda$ for some $x \in W$. What can we say about the relative positions of w and x?

Suppose that $w \in W, \alpha \in \Phi^+$ and $s_\alpha \cdot (w \cdot \lambda) < w \cdot \lambda$ so that $M(s_\alpha w \cdot \lambda) \hookrightarrow M(w \cdot \lambda)$. Our assumption is equivalent to

$$\mathbb{Z}^{>0}(w \cdot \lambda + \rho, \alpha^\vee) = (w(\lambda + \rho), \alpha^\vee) = (\lambda + \rho, w^{-1}\alpha^\vee) = (\lambda + \rho, (w^{-1}\alpha)^\vee) \iff w^{-1}\alpha \in \Phi^- \iff (w^{-1}s_\alpha)\alpha \in$$

29 Friday April 3rd

Recall from last time that we defined a new partial order for all positive roots generated by "reflecting down", namely *strong linkage*. We had a theorem: $\mu \uparrow \lambda \implies M(\mu)$ occurs as a composition factor of $M(\lambda)$. We also have a side-arrow notation $w' \xrightarrow{s_\alpha} w$ indicates that $w' = s_\alpha w$ and w' is shorter than w. We conclude that $x \cdot \lambda \uparrow w \cdot \lambda \iff x \leq w$ for $x, w \in W$, where the RHS is the usual Bruhat order and is notably independent of λ.

Corollary 29.1.
Let $\lambda \in \Lambda$ be antidominant and ρ-regular and $x, w \in W$. Then

$$[M(w \cdot \lambda) : L(x \cdot \lambda)] \neq 0 \iff M(x \cdot \lambda) \hookrightarrow M(w \cdot \lambda) \iff x \leq w$$

Note that this statement is why we use antidominant instead of dominant, since this equation now goes in the right direction.

29.1 Jantzen Filtration

Theorem 29.2.
Given $\lambda \in \mathfrak{h}^\vee$, $M(\lambda)$ has a terminating descending filtration satisfying

a. Each nonzero quotient has a certain nondegenerate contravariant form (3.14)
b. $M(\lambda)^i = N(\lambda)$
c. $\sum_{i>0} \operatorname{char} M(\lambda)^i = \sum_{\alpha>0, s_\alpha \cdot \lambda < \lambda} \operatorname{char} M(s_\alpha \cdot \lambda)$ (the Integer sum formula, very important)

Note that the sum on the RHS is over $\left\{ \alpha \in \Phi^+_{[\lambda]} \;\middle|\; s_\alpha \cdot \lambda < \lambda \right\} := \Phi^+_\lambda$.

Fact $\operatorname{Soc} M(\lambda) = L(\mu)$ for the unique antidominant μ in $W_{[\lambda]} \cdot \lambda$. Moreover, $[M(\lambda) : L(\mu)] = 1$.

Now suppose $M(\lambda)^n \neq 0$ but $M(\lambda)^{n+1} = 0$. Each $M(\lambda)^i \supset \operatorname{Soc} M(\lambda) = L(\mu)$, since they're submodules, and each $M(s_\alpha \cdot \lambda) \supset L(\mu)$, using the uniqueness of μ. By looking at coefficients of char $L(\mu)$ on each side of the sum formula, we obtain $n = \left| \Phi^+ \right|$.

Exercise (5.3) When λ is antidominant, integral, and ρ-regular, then $n = \ell(w)$. More generally, for nonintegral, $n = \ell_\lambda(w)$ where ℓ_λ is the length function of the system $(W_{[\lambda]}, \Delta_{[\lambda]})$.

Some natural questions:

1. Is the Jantzen filtration unique for properties (a)-(c)?
2. What are the "layer multiplicities" $[M(\lambda) : L(\mu)]$?
3. Are the layer $M(\lambda)$ semisimple? If so, is the Jantzen filtration the same as the canonical filtrations with semisimple quotients (the radical or socle filtrations)?
4. When $M(\mu) \subset M(\lambda)$, how to the respective Jantzen filtrations compare?

A guess for (4): Assume $\mu \uparrow \lambda$, set $r = \left| \Phi^+_\lambda \right| - \left| \Phi^+_\mu \right|$, which is the difference in lengths of the two Jantzen filtrations.

Is it true that:

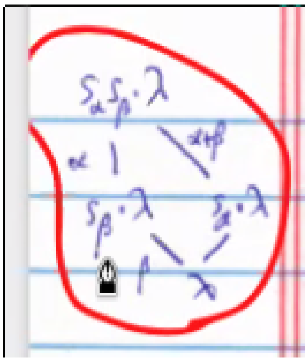
$$
\begin{array}{ccccccccccccc}
M(\lambda) = M(\lambda)^0 & \supset & \cdots & \supset & M(\lambda)^r & \supset & M(\lambda)^{r+1} & \supset & \cdots & \supset & M(\lambda)^n & \supset & 0 \\
 & & & & \cup & & \cup & & & & \cup & & \\
 & & & & M(\mu) = M(\mu)^0 & \supset & M(\mu)^1 & \supset & \cdots & \supset & M(\mu)^{n-r} & \supset & 0
\end{array}
$$

with $M(\mu) \bigcap M(\lambda)^i = M(\mu)^{i-r}$ for $i \geq r$?

This is called the *Jantzen conjecture* and turns out to be true.

> Thought equivalent to KL-conjecture, but turned out to be deeper. See decomposition theorem, sheaves on flag varieties, no simple algebraic proof until recently. See chapter 8.

Recall that we obtained a hexagon:



We have

$$\Phi^+_{w\cdot\lambda} = \left\{\gamma \in \Phi^+ | s_\gamma \cdot (w \cdot \lambda) < w \cdot \lambda\right\} = \{\alpha, \alpha + \beta\}$$

with corresponding weights $s_\gamma(w \cdot \lambda) = s_{|}beta \cdot \lambda, s_\alpha \cdot \lambda$. Thus we have a two-step filtration, and we've worked out the characters of the pieces previously.

Thus

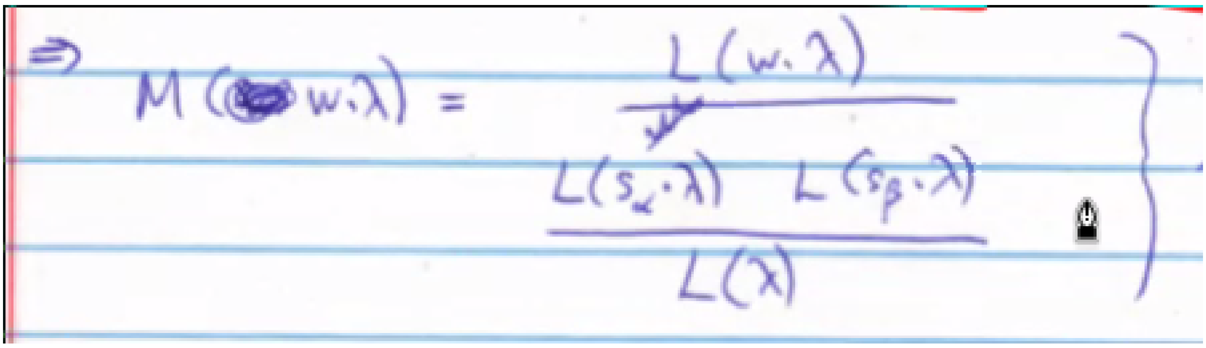
$$\sum_{i=1}^{n} \text{char } M(w \cdot \lambda)^i = \text{char } M(s_\alpha \cdot \lambda) + \text{char } M(s_\beta \cdot \lambda) = \text{char } L(s_\alpha \cdot \lambda) + \text{char } L(s_\beta \cdot \lambda) + 2\text{char } L(\lambda)$$

where we know the last expression explicitly. Since n has the be the number of $L(\lambda)$s occurring on the RHS, we must have $n = 2$.

We can reason that $M(w \cdot \lambda)^2 = L(\lambda)$, since any composition factor in $M(w \cdot \lambda)^2$ recurs in $M(w \cdot \lambda)^1$, and so

$$\text{char } (w \cdot \lambda)^1 = \text{char } N(w \cdot \lambda) = \text{char } L(s_\alpha \cdot \lambda) + \text{char } L(s_\beta \cdot \lambda) + \text{char } L(\lambda).$$

We then obtain the following structure on the sections/subquotients of the Jantzen filtration



where the subquotients move upward through the diagram, e.g. the middle is $M(w \cdot \lambda)^1 / M(w \cdot \lambda)^2$.

Exercise (Last Assignment) 1. Try to work on the Jantzen filtration sections for $M(w_0 \cdot \lambda)$ List completely any additional assumptions or facts needed to deduce $M(w_0 \cdot \lambda)^i$ uniquely. 2. Continue 4.11 in the case where λ is singular Does this allow you to deduce that structure of all $M(w \cdot \lambda)$ using the Jantzen sum formula? 3. Work out the non-integral case for $\mathfrak{sl}(3, \mathbb{C})$. (There are four different cases to consider here.)

29.2 Showing Jantzen Implies BGG

We'll prove that $[M(\lambda) : L(\mu)] \neq 0 \implies \mu \uparrow \lambda$.

Proof .
By induction on the number of linked weights $\mu \leq \lambda$
If λ is minimal in its linkage class, then $M(\lambda) = L(\lambda)$ so $\mu = \lambda$ and $\lambda \uparrow \lambda$ trivially.
Otherwise, inductively suppose $[M(\lambda) : L(\mu)] > 0$ with $\mu < \lambda$. Then $[M(\lambda)^1 : L(\mu)] > 0$ since $M(\lambda)^1 = N(\lambda)$. By the sum formula, $[M(s_\alpha \cdot \lambda) : L(\mu)] > 0$ for some $\alpha \in \Phi^+_\lambda$. Then $s_\alpha \cdot \lambda < \lambda$ so the number of linked weights $\nu \leq s_\alpha \cdot \lambda$ is *smaller* than for λ.
So by induction,

$$\begin{cases} \exists\, \alpha_1, \dots, \alpha_r \in \Phi^+ \text{ st} \\ \mu = (s_{\alpha_1} \cdots s_{\alpha_r}) s_\alpha \cdot \lambda \uparrow \cdots \uparrow s_{\alpha_r} s_\alpha \cdot \lambda \uparrow s_\alpha \cdot \lambda \overset{\text{assumption on } \alpha}{\uparrow} \lambda, \end{cases}$$

and $\mu \uparrow \lambda$ as required. ■

Example: $\mathfrak{sl}(4, \mathbb{C})$ with Dynkin diagram $\cdot \longrightarrow \cdot \longrightarrow \cdot$.

If $\lambda = (0, -1, 0) \in \Lambda^+ - \rho$ with coordinates with respect to the fundamental weight basis for Λ or $\mathfrak{h}^\vee$. Then take $w = s_2 s_3, x = s_3 s_2 s_3 s_1 s_2$, then $\mu = w \cdot \lambda = (1, -2, -1)$ and $\mu - x \cdot \mu = \alpha_1 + \alpha_3$ so $x \cdot \mu < \mu$.

However, Verma's direct calculations in $U(\mathfrak{sl}(4, \mathbb{C}))$ showed that $M(x \cdot \mu) \not\subset M(\mu)$, so $x \cdot \mu \not\uparrow \mu$.

The explanation (due to Verma) is that $x \cdot \mu = xw \cdot \lambda$, using the fact that s_1, s_3 commute,

$$\begin{aligned} xw &= (s_3 s_2 s_3 s_1 s_2) s_2 s_3 \\ &= s_3 s_2 s_3 s_3 s_1 \\ &= s_3 s_2 s_1. \end{aligned}$$

and $s_2 s_3, s_3 s_2 s_1$ are not related in the Bruhat order.

> This is because there is no root reflection relating the two. Note that this can be seen by considering subexpressions: $a < b$ iff a occurs as some deleted subexpression of b.

So it's possible to have one weight less than another with no inclusion of the Verma modules.

30 Monday April 6th

Note that most of the theory thus far has not relied on the properties of $\mathbb{C}$, so Jantzen's strategy was to extend the base field to $K = \mathbb{C}(T)$, rational functions in T, then setting $g_K := K \otimes_{\mathbb{C}} \mathfrak{g}$. The theory over K adapts to $A = \mathbb{C}[T]$, the PID of polynomials in one variable T with K as its fraction field and the "Lie algebra" $g_A = A \otimes_{\mathbb{C}} \mathfrak{g}$.

Setup: Let A be any PID, for example $\mathbb{Z}$ or $\mathbb{C}[T]$, and M a free A-module of finite rank r. Let $e, f \in M$ and suppose M has an A-valued symmetric bilinear form denoted $(\,\cdot\,, \,\cdot\,)$. Since M is finite rank, we can choose a basis $\{e_i\}^r$, so the matrix F of this form relative to this basis has nonzero determinant D depending on the choice of basis. A change of basis is realized by some $P \in \mathrm{GL}(r, A)$, giving $F' = P^t F P$ (note that forms change by a transpose instead of an inverse) and $\det P \in A^\times$. Thus D only changes by a unit $u = (\det P)^2$.

We can define the dual module $M^* = \hom_A(M, A)$ which is also free of rank r, and contains a submodule $M^\vee$ consisting of functions $e^\vee : M \longrightarrow A$ given by $e^\vee(f) = (e, f)$ for any fixed $e \in M$. Surprisingly, this doesn't give every hom: e.g. if the form only has even outputs. Since $(\,\cdot\,, \cdot\,)$ is nondegenerate, the map $\phi : M \longrightarrow M^\vee$ sending e to $e^\vee$ is an isomorphism.

We'll now invoke the structure theory of modules over a PID: There exists a basis of M^* given by $\{e + i^*\}^r$ where $M^\vee$ has a basis $\{d_i e_i^*\}^r$ for some scalars $0 \neq d_i \in A$. We can define a dual basis of M given by $\{e_i\}^r$ where $e_i^*(e_j) = \delta_{ij}$. We can similarly get a separate dual basis $\{f_i\}$ where $f_i^\vee = d_i e_i^*$.

We can compare these two bases:

$$(e_i, f_j) = f_j^\vee(e_i) = d_j e_j^*(e_i) = d_j \delta_{ij}$$

(Formula 1)

Thus up to units, $D = \prod_{i=1}^{r} d_i$, so this hybrid matrix is one way to compute this determinant.

Fix a prime element in A, then there is an associated valuation $v_p : A \longrightarrow \mathbb{Z}^+$ where $v_p(a) = n$ if $p^n \mathrel{\Big|} a$ but $p^{n+1} \nmid a$. Since p is prime, $\overline{M} \coloneqq M/pM$ makes sense and is a finitely generated module over the field $\overline{A} = A/pA$; thus $\overline{M}$ is a vector space.

We'll now define a filtration: for $n \in \mathbb{Z}^+$, define $M(n) = \left\{ e \in M \mathrel{\Big|} (e, M) \subset p^n A \right\}$. Then

$$M = M(0) \supset M(1) \supset \cdots$$

is a decreasing filtration, with corresponding images $\overline{M(n)}$ that are vector spaces.

Lemma 30.1.
For A a PID, $p \in A$ prime, $\overline{A} = A/pA$ with valuation v_p and M a free A-module with a nondegenerate symmetric bilinear form wrt some basis of M having nonzero determinant D Then

a. $v_p(D) = \sum_{n>0} \dim_{\overline{A}} \overline{M(n)}$.
b. For each n, the modified bilinear form $(e, f)_n \coloneqq p^{-n}(e, f)$ on $M(n)$ induces a nondegenerate form on $\overline{M(n)}/\overline{M(n+1)}$.

Proof (of (a)).

1. For $f \in M$, write $f = \sum a_i j f_j$ in terms of the given basis, and $(e_i, f) = a_i d_i$. For $n > 0$, we have

$$
\begin{aligned}
f \in M(n) &\iff v_p((e_i, f)) \geq n \forall i \\
&\iff v_p(a_i d_i) \geq n \\
&\iff v_p(a_i) + v_p(d_i) \geq n \\
&\iff v_p(a_i) \geq n - n_i \quad n_i := v_p(d_i)
\end{aligned}
$$

This a_i must be divisibly by p at least $n - n_i$ times. This $M(n)$ is spanned by $\left\{f_i \;\middle|\; n_i \geq n\right\} \bigcup \left\{p^{n-n_i} f_i \;\middle|\; n_i < n\right\}$.

2. So $\overline{M(n)}$ has basis $\left\{\bar{f}_i \;\middle|\; n_i \geq n\right\}$ and $\dim \overline{M(n)} = \# \left\{i \;\middle|\; n_i \geq n\right\}$. In particular, $\overline{M(n)} = 0$ for $n \gg 0$ since there are only finitely many n_i. Thus

$$
\begin{aligned}
\sum_{n>0} \dim \overline{M(n)} &= \sum_{n>0} \# \left\{i \;\middle|\; n_i > n\right\} \\
&= \sum_{i=1}^r n_i \\
&= \sum_{i=1}^r v_p(d_i) \\
&= v_p\left(\prod_{i=1}^r d_i\right) \\
&= v_p(D)
\end{aligned}
$$

■

Proof (of (b).
)

1. Note that $(e, f) \in p^n A \implies (e, f)_n \in A$, so this is well-defined on $M(n)$. To see that it's well-defined on $\overline{M(n)}$ we must show that $e \in pM(n)$.

$$
\begin{aligned}
&(e, pM(n))_n \subset pA \\
\implies (e, M(n))_n \subset p^{-n}(pM, M(n)) \subset p^{-n+1}(M, M(n)) \subset p^{-n+1} p^n A = pA
\end{aligned}
$$

So there is an induced form $(\bar{e}, \bar{f})_n$ on $\overline{M(n)}$.

To show it's nondegenerate, need to compute the radical.

- If $f \in M(n+1)$ then

$$
(f, M(n))_n = p^{-n}(f, M(n)) \in p^{-n}(f, M) e p^{-n} p^{n+1} A = pA
$$

so $\bar{f} \in \operatorname{rad}(\,\cdot\,, \cdot\,)_n$

- See notes.

■

31 Wednesday April 8th

Recall that we are setting up Jantzen's filtration. Let A be a PID, $\mathfrak{p} \in A$ prime, $\overline{A} = A/pA$, M a free A-module of rank r, with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ having nonzero determinant wrt some basis of M. Define $M(n), \overline{M(n)}$ as before

Lemma 31.1.

1. $v_p(D) = \sum_{n>0} \dim_{\overline{A}} \overline{M(n)}$
2. For each n, the modified bilinear form induces a nondegenerate form on $\overline{M(n)}/p\overline{M(n)}$?

31.1 Proof of Jantzen's Theorem

Let $A = \mathbb{C}[T]$ and $K = \mathbb{C}(T)$ its fraction field, and let $\mathfrak{g}_A = A \otimes_{\mathbb{C}} \mathfrak{g}$ and $\mathfrak{g}_K = K \otimes_{\mathbb{C}} \mathfrak{g}$, which is a Lie algebra that is split over K, i.e. every $h \in \mathfrak{h}_K = K \otimes_{\mathbb{C}} \mathfrak{h}$ has all eigenvalues of ad h in K.

The theory we need carries over to the extended setting. The plan is the following:

- Construct and look at basic properties of Verma modules (1.3-1.4)
- Look at properties of their contravariant forms (3.14 - 3.15)
- Find a simplicity criterion (4.8)

We'll use Lemma 5.6 to construct filtrations on the weight spaces of the extended Verma module, then reduce mod T (using evaluation morphisms) to assemble the Jantzen filtration for the original $\mathbb{C}$-module.

Given $\lambda \in \mathfrak{h}^\vee$, set $\lambda_T = \lambda + T_\rho \in \mathfrak{h}_K^\vee$. For all $\alpha \in \Phi$, we have

$$(\lambda_T + \rho, \alpha^\vee) = (\lambda + \rho, \alpha^\vee) + T(\rho, \alpha^\vee) \notin \mathbb{Z},$$

since this is a linear polynomial. So λ_T is antidominant.

Therefore $M(\lambda_T)$ is simple, and equivalently (unique up to scalars) its nonzero contravariant form is nondegenerate.

The module $U(\mathfrak{g}_A) \cong A \otimes_{\mathbb{C}} U(\mathfrak{g})$ is a natural "A-form" of $U(\mathfrak{g}_K) \cong K \otimes_{\mathbb{C}} U(\mathfrak{g})$. This yields $M(\lambda_T)_A$, an A-form of $M(\lambda_T)$, where each weight space is an A-module of finite rank.

Some remarks about contravariant forms on highest weight modules: given M and such a form $(\cdot, \cdot) : M \times M \longrightarrow \mathbb{C}$, the transpose serves as an adjoint and we have $(uv, v') = (v, \tau(u)v')$.

Distinct weight spaces are orthogonal, i.e. $(M_\mu, M_\nu) = 0$ since

$$\begin{aligned} (hv, v') &= \mu(h)(v, v') \\ = (v, hv') &= \nu(h)(v, v') \end{aligned}$$

where $\mu(h) \neq \nu(h)$, implying $(v, v') = 0$.

We can compute

$$(uv^+ \in M_\mu, u'v^+ \in M_\mu) = (v^+, \tau(u)u'v') = a(v^+, v^+)$$

for some $a \in A$, since $\lambda_T = \lambda + T_p$ maps $\mathfrak{h}_A \longrightarrow A$. We can use the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$, where $\mathfrak{n}^+$ kills v^+ and $\mathfrak{n}^-$ lowers into an orthogonal weight space, and so this pairing only depends on (v^+, v^+).

Note that the radical of this bilinear form is a maximal submodule.

The weights are of the form $\lambda_T - \nu$ for $\nu \in \mathfrak{n}^+\Phi^+ = \Lambda_r^+$. Apply lemma 5.6 to the A-form $M_{\lambda-\nu}$ of $M(\lambda_T)_{\lambda_T - \nu}$ to get a decreasing finite filtration of A-submodules

$$M_{\lambda-\nu}(0) = M_{\lambda_T - v}(1) \supset \cdots$$

where $M_{\lambda_T-\nu}(i) = \left\{ e \in M_{\lambda_T-\nu} \;\middle|\; (e, M_{\lambda_T-\nu}) \subset T^i A \right\}$.

For each $i \geq 0$, set $M(\lambda_T)_A^i = \sum_{\nu \in \Lambda_r^+} M_{\lambda_T-\nu}(i)$. This yields a decreasing filtration of A-submodules.

Next we want to "set $T = 0$": formally, pass to the quotient $\overline{M} = M/TM$ over the field $\overline{A} = A/TA \cong \mathbb{C}$. Since $\lambda_T = \lambda + T_\rho \xrightarrow{T=0} \lambda$, we have $M(\lambda_T)_A \cong M(\lambda)$ and the filtration becomes a decreasing filtration of $M(\lambda)$.

By the lemma, the sections of this filtration inherit nondegenerate contravariant forms, proving (a). By the proof of that lemma, the filtration on each individual weight space terminates at 0.

Claim: Some $M(\lambda)^{n+1} = 0$.

Proof .
If not, since $M(\lambda)$ has finite length, we must have $0 \neq M(\lambda)^n = M(\lambda)^{n+1} = \cdots$ for some n. Choose some $u \in \mathfrak{h}^\vee$ such that $M(\lambda)^n_\mu = 0$, but then $0 \neq M(\lambda)^n_\mu = M(\lambda)^n_\mu = \cdots$, a contradiction.

∎

For a proof of (c), we want to show $\sum_{i>0} \operatorname{char} M(\lambda)^i = \sum_{\alpha \in \Phi^+} \operatorname{char} M(s_\alpha \cdot \lambda)$. We can show that the LHS is given by

$$\begin{aligned}
\cdots &= \sum_{i>0} \sum_{\nu \in \Lambda_r^+} \dim M(\lambda)^i_{\lambda-\nu} \\
&= \sum_{i>0} \sum_{\nu} \dim \left(\overline{M(\Lambda_T)^i_A} \right)_{\lambda_T - \nu} e(\lambda - v) \\
&= \sum_{i} \sum_{\nu} \dim \overline{M_{\lambda_T-\nu}(i)} e(\lambda - \nu) \\
&= \sum_{\nu} \sum_{i} \dim \overline{M_{\lambda_T-\nu}(i)} e(\lambda - \nu) \\
&= \sum_{\nu} v_T(D_\nu(\lambda_T)) e(\lambda - v) \quad \text{Lemma 5.6a.}
\end{aligned}$$

where $D_\nu(\lambda_T)$ is the determinant of the matrix of the contravariant form on the $\lambda_T - \nu$ weight space of $M(\lambda_T)$.

Fact (Jantzen, Shapovalov): Up to a nonzero scalar multiple depending on a choice of basis of $U(\mathfrak{n}^-)$,

$$D_\nu(\lambda_T) = \prod_{\alpha>0} \prod_{r\in\mathbb{Z}^{>0}} \left((\lambda_T, \rho, \alpha^\vee) - r\right)^{P(\nu - r_\alpha)}$$

where P is the Kostant partition function.

We can calculate v_T of this, which doesn't depend on the scalar:

$$\begin{aligned}
(\lambda_T + \rho, \alpha^\vee) - r &= (\lambda + \rho, \alpha^\vee) - r + T(\rho, \alpha^\vee) \\
\implies v_T(\cdots) &= \begin{cases} 1 & (\lambda+\rho,\alpha^\vee) = r > 0 \iff \alpha \in \Phi_\lambda^+ \\ 0 & \text{else} \end{cases}.
\end{aligned}$$

We then have

$$v_T(D_\nu(\lambda_T)) = \sum_{\alpha\in\Phi_\lambda^+} P(\nu - (\lambda+\rho,\alpha^\vee)\alpha).$$

Thus the LHS is given by

$$\begin{aligned}
\cdots &= \sum_{\nu\in\Lambda_r^+} \sum_{\alpha\in\Phi_\lambda^+} P(\nu - (\lambda+\rho,\alpha^\vee)\alpha) e(\lambda - \nu) \\
&= \sum_{\alpha\in\Phi_\lambda^+} \sum_{\sigma\in\Lambda_r^+} P(\sigma) e(\lambda - (\lambda+\rho,\alpha^\vee)\alpha - \sigma) \\
&= \sum_{\alpha\in\Phi_\lambda^+} \operatorname{char} M(s_\alpha \cdot \lambda),
\end{aligned}$$

where we've used what we know about characters of Verma modules.

Note that the proof of the determinant formula is technical.

We'll skip chapter 6 on KL theory.

32 Friday April 10th

32.1 Translation Functors

Extremely important, allow mapping functorially between blocks (recalling $\mathcal{O} = \bigoplus \mathcal{O}_{\chi_\lambda}$) and in good situations gives an equivalence of categories.

Definition 32.0.1.
A *projection functor* $\mathfrak{p}_\lambda : \mathcal{O} \longrightarrow \mathcal{O}_{\chi_\lambda}$ where $M = \bigoplus_\mu M^{\chi_\mu} \mapsto M^{\chi_\lambda}$.

Convention: From now on, all weights will be integral

Proposition 32.1.

1. $\mathfrak{p}_\lambda$ is an exact functor 2.$\hom(M, N) \cong \bigoplus_\lambda \hom(\mathfrak{p}_\lambda M, \mathfrak{p}_\lambda N)$
2. $\mathfrak{p}_\lambda(M^\vee) = (\mathfrak{p}_\lambda M)^\vee$
3. $\mathfrak{p}_\lambda$ maps projectives to projectives
4. $\mathfrak{p}_\lambda$ is self-adjoint

Proof .

1. Given $0 \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$, we can decompose this as $0 \longrightarrow \bigoplus_\lambda M^{\chi_\lambda} \xrightarrow{\oplus f_\lambda} \bigoplus_\lambda N^{\chi_\lambda} \xrightarrow{\oplus g_\lambda} \bigoplus_\lambda P^\lambda \longrightarrow 0$, which gives exactness on each factor.
2. We can move direct sums out of homs.
3. Write $\mathfrak{p}_\lambda\left(\left(\bigoplus M^{\chi_\lambda}\right)^\vee\right)$ and use theorem 3.2b to write as $(M^{\chi_\lambda})^\vee$.
4. $\mathfrak{p}_\lambda(P)$ is a direct summand of a projective and thus projective.
5. We have $\hom(\mathfrak{p}_\lambda M, N) = \hom(\mathfrak{p}_\lambda M, \mathfrak{p}_\lambda N) = \hom(M, \mathfrak{p}_\lambda N)$.

■

Definition 32.1.1.
Let $\lambda, \mu \in \Lambda$ with $\nu = \mu - \lambda$ integral. Then there exists $w \in W$ such that $\tilde{\nu} := w\nu \in \Lambda^+$ is in the dominant chamber. Define the *translation functor* $T_\lambda^\mu = \mathfrak{p}_\mu(L(\tilde{\nu}) \otimes_{\mathbb{C}} \mathfrak{p}_\lambda(M))$, where we use the fact that $\tilde{\nu}$ dominant makes $L(\tilde{\nu})$ finite-dimensional.

This is a functor $\mathcal{O}^{\chi_\lambda} \longrightarrow \mathcal{O}^{\chi_\mu}$.

Proposition 32.2.

1. The translation functor is exact.
2. $T_\lambda^\mu(M^\vee) = (T_\lambda^\mu M)^\vee$
3. It maps projects to projectives.

Proof .

1. It is a composition of exact functors, noting that tensoring over a field is always exact.
2. Use proposition 12, $L(\tilde{\nu})$ is self-dual, and $A^\vee \otimes B^\vee \cong (A \otimes B)^\vee$.
3. Use proposition 1 and previous results, e.g. $L \otimes_{\mathbb{C}} \cdot$ preserves projectives if $\dim L < \infty$ (Prop 3.8b).

■

Proposition 32.3.
$\hom(T_\lambda^\mu M, N) \cong \hom(M, T_\mu^\lambda N)$, which also holds for every Ext^n.

Proof .
We have

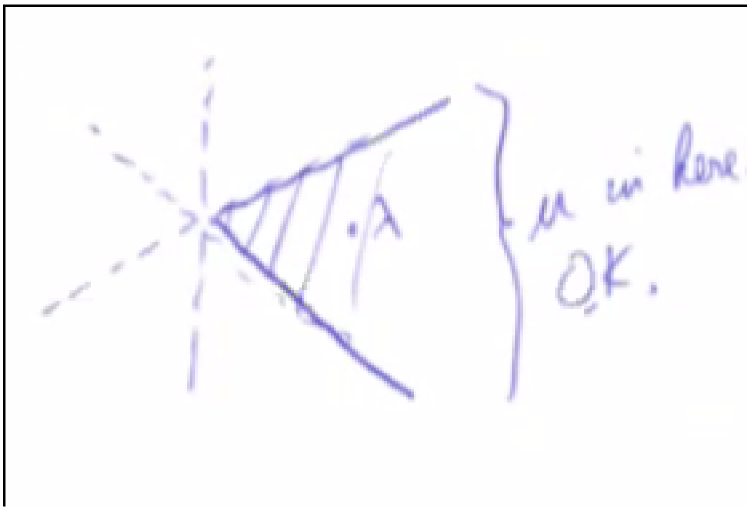
$$\mathrm{Hom}(T_\lambda^\mu M, N) = \mathrm{Hom}(pr_\mu(L(\tilde\nu) \otimes pr_\lambda(M)), N)$$
$$\simeq \mathrm{Hom}(L(\tilde\nu) \otimes pr_\lambda(M), pr_\mu(N))$$
$$\simeq \mathrm{Hom}(pr_\lambda(M), L(\tilde\nu)^* \otimes pr_\mu(N))$$
$$\simeq \mathrm{Hom}(M, pr_\lambda(L(\tilde\nu)^* \otimes pr_\mu(N)))$$

But $L(\tilde\nu)^\vee \cong L(-w_0\tilde\nu)$ and $-w_0\tilde\nu = w_0 w(\lambda - \mu)$ is the dominant weight in the orbit of $\lambda - \mu$ used to define T_μ^λ.
For the second part, use a long exact sequence – if two functors are isomorphic, then their right-derived functors are isomorphic.

■

Does this functor take Vermas to Vermas? I.e. do we have $M(w \cdot \lambda) \mapsto M(w \cdot \mu)$ when $T_\lambda^\mu \mathcal{O}_{\chi_\lambda} \longrightarrow \mathcal{O}_{\chi_\mu}$?

Picture for $\mathfrak{sl}(3, \mathbb{C})$:



This doesn't always happen, and depends on the geometry.

32.1.1 Weyl Group Geometry – Facets

Definition 32.3.1.
Given a partition of $\Phi^+ = \Phi_F^0 \bigcup \phi_F^+ \bigcup \Phi_F^-$, a *facet* associated to this partition is a nonempty set consisting of solutions $\lambda \in E$ to the following equations:

(i) $(\lambda+\rho, \check{\alpha}) = 0 \quad \forall \alpha \in \Phi_F^0$
(ii) $(\lambda+\rho, \check{\alpha}) > 0 \quad \forall \alpha \in \Phi_F^+$
(iii) $(\lambda+\rho, \check{\alpha}) < 0 \quad \forall \alpha \in \Phi_F^-$

The <u>upper closure</u> $\hat{F}$ of F : replace $<$ by $\leq$ in (iii)
The <u>lower closure</u> $\underline{F}$ of F : replace $>$ by $\geq$ in (ii)
The closure $\bar{F}$ of F : replace $<$ by $\leq$ in (iii) and $>$ by $\geq$ in (ii).

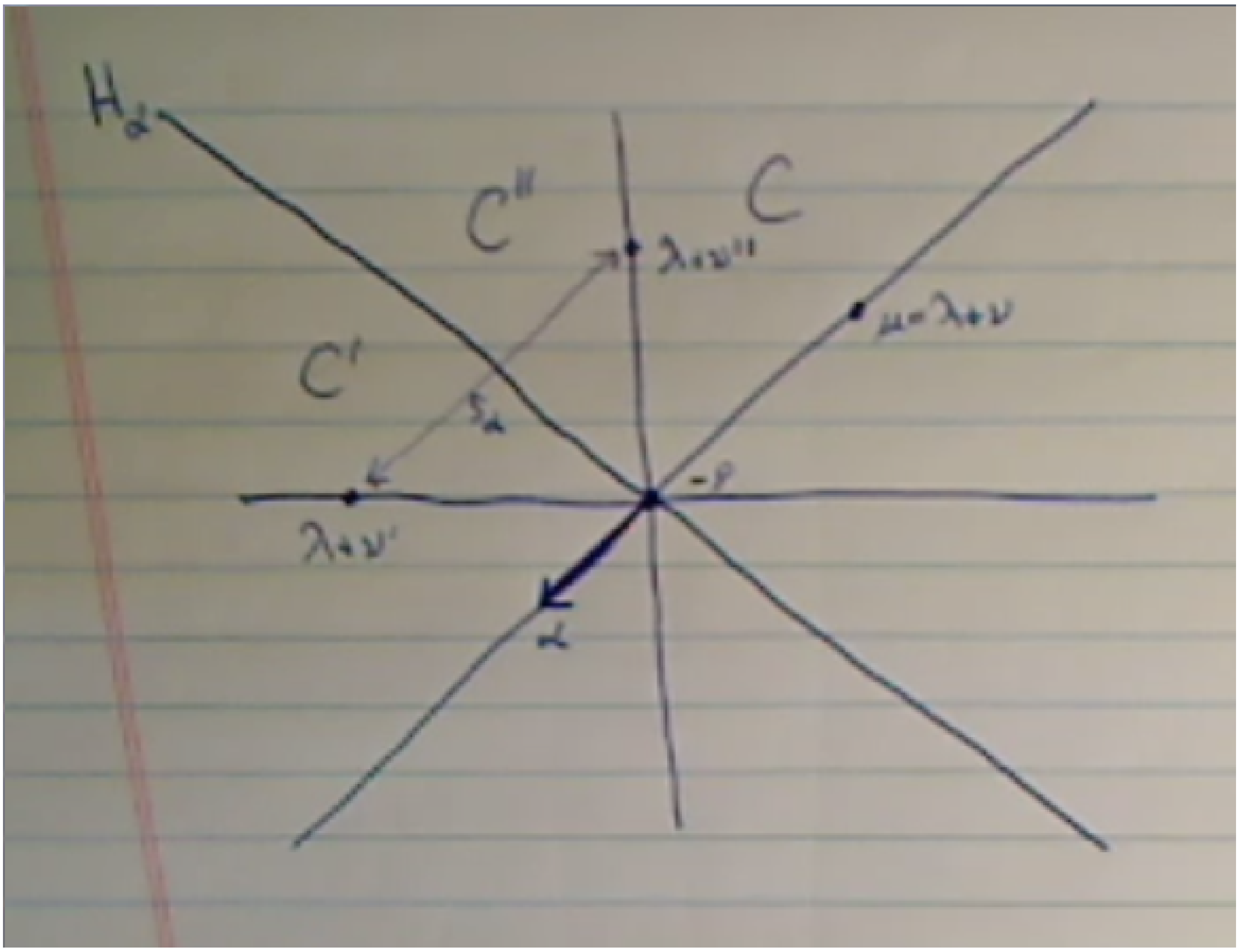
Example: A_2, where $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$.

1. Take $\Phi_F^0 = \Phi^+$, and by the orthogonality conditions, $F = \{-\rho\}$ since it must be orthogonal to all 3 roots. So the origin is a facet.
2. Take $\Phi_F^0\,\{\alpha, \beta\}$ and $\Phi_F^+ = \{\alpha+\beta\}$, so $F = \emptyset$ can not be a facet.
3. See notes
4. see notes

Note that $\overline{F} \supset \widehat{F} \bigcup \underline{\mathrm{F}}$ in general.

33 Monday April 13th

Reviewing the definition of *facets*. We partitioned Φ into 3 sets $\Phi_F^{0,\pm}$, some of which could be empty. We had notion of upper and lower closure given by replacing the strict inequalities with inequalities in condition (3) and (2) respectively.



Definition 33.0.1.
If F is a facet with $\Phi_F^0 = \emptyset$, then F is called a *chamber*.

A facet with exactly 1 root in Φ_F^0, then this is called a *wall*.

Observations:

1. $\Phi^+ = \Phi_F^+$ always defines a chamber called the *fundamental chamber* and is denoted C_0.
2. If F is any chamber, then $F = w \cdot C_0$ for some $w \in W$.

Proposition 33.1.

a. Every facet F is the upper closure of some unique chamber C.
b. If $F \subset \widehat{C}$ then $\widehat{F} \subset \widehat{C}$.

Proof .

a. If F is given by $\Phi_F^0 \bigcup \Phi_F^+ \bigcup \Phi_F^-$ and C pairs with $\Phi_C^+ = \Phi_F^+$ and thus $\Phi_C^- = \Phi_F^- \bigcup \Phi_F^0$. To see that $C \neq \emptyset$, use remark (1) on page 132.
b. Obvious from above description of C.

■

33.1 Key Lemma from 7.5

We're focusing only on integral weights, and we want to calculate the translation functor of a Verma $T_\lambda^\mu M(\lambda)$. First step: project onto λ block, but $M(\lambda)$ is in that block already. Then tensor with $L(\tilde{\nu})$, then the product has a standard filtration with certain Verma section $M(\lambda + w\tilde{\nu})$, each occurring with multiplicity one. The weight $\tilde{\nu}$ is the unique dominant weight in the orbit of $\mu - \lambda$, one of the Verma sections is in $M(\mu)$. We plan to show that $T_\lambda^\mu M(\lambda) = M(\mu)$ in "good" situations.

Lemma 33.2.
Let $\lambda, \mu \in \Lambda$ be integral weights and $\nu = \mu - \lambda$ and $\tilde{\nu} \in \Lambda^+ \bigcap W\nu$ (which is unique). Assume there is a facet F with $\lambda \in F, \mu \in \overline{F}$. Then for all weights $\nu' \neq \nu$ of $L(\tilde{\nu})$, the weight $\lambda + \nu'$ is *not* linked to $\lambda + \nu = \mu$ under W.

Proof .

Toward a contradiction, suppose there exists $\nu \neq \nu$ in $\Pi(L(\tilde{\nu}))$ with $\lambda + \nu' \in W \cdot (\lambda + \nu)$. Define the *distance* between two chambers C, C' as the number of root hyperplanes separating them. Under the correspondence between chambers and W given by picking a fundamental chamber, the distance corresponds to the difference in lengths between the corresponding Weyl group elements.

So choose chambers C, C' with $F \subset \overline{C}$, $\lambda + \nu' \in \overline{C}'$, and $d(C, C')$ is minimal. We now go through 14 easy steps.

1. The case $d(C, C') = 0$ is impossible, since this would force $C = C'$/ But C is a fundamental domain for the dot action, where $C' \ni \lambda + \nu' \neq \lambda + \nu = \mu \in \overline{F} \subset \overline{C}$. This contradicts C being a fundamental domain, since each ? will be conjugate to a *unique* element.
2. The case $d(C, C') > 0$ implies there's a hall $H_\alpha \bigcap \overline{C}'$ of C' separating C' from C. Wlog assume C' is on the positive side of H_α and $\alpha > 0$ and C is on the negative side. Since $\overline{F} \subset \overline{C}$, we have $(\xi + \phi, \alpha^\vee) \leq 0$ for all $\xi \in \overline{F}$.
3. Reflect and set $C'' := s_\alpha C'$, then $d(C, C'') < d(C, C')$ and we will be able to apply the induction hypothesis.
4. By (2), $(\lambda + \nu' + \rho, \alpha^\vee) \geq 0$ since $\lambda + \nu'$ was on the positive side.
5. By (2), $(\lambda + \rho, \alpha^\vee) \leq 0$ since $\lambda + \nu'$ was on the negative side.
6. By (4), $\lambda + \nu' \geq s_\alpha \cdot (\lambda + \nu') = s_\alpha \cdot \lambda + s_\alpha \nu' = \lambda - (\lambda + \rho, \alpha^\vee)\alpha + s_\alpha \nu' := \nu''$ by just applying the formula for the dot action.
7. By (5) and (6), $s_\alpha \nu' \leq s_\alpha \nu' - (\lambda + \rho, \alpha^\vee)\alpha \leq \nu'$, where the first and last terms are weights of $L(\tilde{\nu})$, so $\nu'' \leq \nu'$. In fact, this inequality is obtained by cancelling λ and adding/substracting multiples of α, so these come from an α root string.
8. Rewriting (6), we have $s_\alpha(\lambda + \nu') = \lambda + \nu'' \in s_\alpha \overline{C'} = \overline{C''}$.
9. By 1.6 bullet (2) in Humphreys, $\nu'' \in \Pi(L(\tilde{\nu}))$.
10. By the minimality assumed for ν', along with (3), (8), (9), we have $\nu'' = \nu$.

11. Rewriting (7) and using the hypothesis $\nu \neq \nu'$, we can write $s_\alpha \nu' \leq \nu < \nu'$ where the inequality is strict because they are not equal. This is still an α root string of weights in the simple module $L(\tilde{\nu})$ with $\nu \in W\tilde{\nu}$. The first inequality can *not* be strict, otherwise $v \pm \alpha$ would both be weights of $L(\tilde{\nu})$, contradicting Humphreys 1.6 bullet 1. So $s_\alpha \nu' = \nu$.
12. By (10), (11), and (6), $s_\alpha \nu' = \nu = \nu'' = s_\alpha \nu' - (\lambda + \rho, \alpha^\vee)\alpha$, so $(\lambda + \rho, \alpha^\vee) = 0$.
13. Since $\lambda \in F$ by assumption, if $\alpha \in \Phi_F^0$ then $(\xi + \rho, \alpha^\vee) = 0$ for all $\xi \in \overline{F}$. In particular, for $\xi = \mu = \lambda + \nu$, and combined with (12), this say $(\nu, \alpha^\vee) = 0$ since the pairing is linear in the first slot.
14. We're now done: combining (11), (13) yields $\nu' = s_\alpha \nu = \nu - (\nu, \alpha^\vee)\alpha = \nu$, which contradicts $\nu \neq \nu'$.

■

33.2 7.6: Translation Functors and Verma Modules

Theorem 33.3.
Let $\lambda, \mu \in \Lambda$ be antidominant. Assume there is a facet F relative to the dot action of W with $\lambda \in F$ and $\mu \in \overline{F}$. Then for all $w \in W$, we have

$$T_\lambda^\mu M(w \cdot \lambda) = M(w \cdot \mu)$$
$$T_\lambda^\mu M(w \cdot \lambda)^\vee \cong M(w \cdot \mu)^\vee.$$

Proof .
Apply the previous lemma to $w \cdot \lambda, w \cdot \mu$ and the facet $w \cdot F$ using $\nu = w \cdot \mu - w \cdot \lambda$. To compute T_λ^μ, first consider $L(\tilde{\nu}) \otimes M(w \cdot \lambda)$. By Theorem 3.6, this has a standard filtration with quotients $M(w \cdot \lambda + \nu')$ for $\nu' \in \Pi(L(\tilde{\nu}))$, potentially with multiplicity.

In particular, $M(w \cdot \mu) = M(w \cdot \lambda + \nu)$ appears exactly once. By the lemma, none of the other highest weights $w \cdot \lambda + \nu'$ are linked to μ. Thus decomposing the tensor product into direct summands in infinitesimal blocks, the only summand in $\mathcal{O}_{\chi_\mu}$ is $M(w \cdot \mu)$. Therefore $T_\lambda^\mu M(w \cdot \lambda) = \mathfrak{p}_\mu(L(\tilde{\nu}) \otimes M(w \cdot \lambda)) = M(w \cdot \mu)$. The statement about duals follows from translation functors commuting with taking duals.

■

34 Wednesday April 15th

Section 7.6, proved theorem about translation functors on Verma modules. We fixed an antidominant weight, since we can apply elements of W to obtain the rest. We proved that translation functors take Verma modules to Verma modules.

Corollary 34.1.
If $M \in \mathcal{O}_\chi$ has a standard filtration, then so does $T_\lambda^\mu M \in \mathcal{O}_\mu$.

Proof .
By induction on the length of the filtration, where the length 1 case is handled by the theorem. In general we have $0 \longrightarrow N \longrightarrow M \longrightarrow M(w \cdot \lambda) \longrightarrow 0$ and since T_λ^μ is exact, we can apply it to get another exact sequence. ??? See notes.

■

34.1 Translation Functors and Simple Modules

Proposition 34.2.
Let $\lambda, \mu \in \Lambda$ be antidominant with a facet F such that $\lambda \in F$ and $\mu \in \overline{F}$. Then for all $w \in W$, $T_\lambda^\mu L(w \cdot \lambda)$ is either $L(w \cdot \mu)$ or 0.

Idea: we're pushing λ to something more singular.

Proof .
Apply the exact functor T_λ^μ to the surjection $M(w \cdot \lambda) \twoheadrightarrow L(w \cdot \lambda)$ so obtain $M(w \cdot \mu) \twoheadrightarrow M$ for some M. Since M is a quotient of a Verma module, it is a highest weight module of highest weight $w \cdot \mu$. Suppose $M \neq 0$, we can apply T_λ^μ to $L(w \cdot \lambda) \hookrightarrow M(w \cdot \lambda)^\vee$ to obtain $M(w \cdot \mu) \twoheadrightarrow M \hookrightarrow M(w \cdot \mu)^\vee$, a nonzero map. By Theorem 3.3c, the image is the socle, so we obtain $M \cong L(w \cdot \mu)$.

■

It turns out that $T_\lambda^\mu \cong L(w \cdot \mu)$ precisely when $w \cdot \mu \in \widehat{w \cdot F}$ (the upper closure, see Theorem 7.9 and example 7.7 for $\mathfrak{sl}(2, \mathbb{C})$).