# Title

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## 1.1 Abstract Measure Theory

**Definition**: Let $X$ be a set and $\mathcal{M}$ be a $\sigma$-algebra of subsets of $X$.

Then $(X, \mathcal{M})$ is referred to as a *measurable* space, noting that we have not yet equipped it with a measure $\mu$.

**Definition:** A *measure* $\mu$ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

- The silly condition, $\mu(\emptyset) = 0$
- The important condition, $\mu(\coprod_{i\in\mathbb{N}} E_i) = \displaystyle\sum_{i\in\mathbb{N}} \mu(E_i)$.

Then $(X, \mathcal{M}, \mu)$ is called a *measure space.* Things can be measured in this setting, but more importantly, an integral can be defined.

**Definition:** A measure is $\sigma$*-finite* iff $X = \bigcup E_j$ with $m(E_j) < \infty$ for each $j$.

> Note: most measures encountered in practice seem to be $\sigma$-finite, so we could just as well incorporate this into our definition.

*Examples:*

- The Lebesgue measure
- Let $X = \{x_n\}_{n=1}^\infty$ a countable collection of objects, $\{\mu_n \in [0, \infty]\}$, and define $\mu(x_n) \coloneqq \mu_n$.

  Then we can take the $\sigma$-algebra $\mathcal{M} = \mathcal{P}(X)$, so

$$\begin{aligned}\mu : \mathcal{P}(X) &\to [0, \infty] \\ E &\mapsto \sum_{\left\{n \mid x_n \in E\right\}} \mu_n.\end{aligned}$$

In the special case $\mu_n = 1$ for all $n$, we have $\mu(E) = \#E$, the number of elements in $E$, which is the counting measure.

- Let $X = \mathbb{R}^n$ and let $\mathcal{M}$ be the Lebesgue measurable subsets, and let $\mu(E) = \int_E f$ for some fixed $f \in L^+$.

  Exercise: Show that this defines a measure.

  In the special case $f \equiv 1$, we get the usual Lebesgue measure $\mu = m$. We write $d\mu \coloneqq f dx$. Note that

  $$m(E) = 0 \iff \mu(E) = 0,$$

  which is referred to as *absolute continuity*.

  Note that all absolutely continuous measures occur in this way! But there are more exotic measures. Thinking about representability theorems, this says that measures are like "generalized integrable functions", but the collection of measures is richer.

- The Dirac mass:

  $$\delta_0(E) = \begin{cases} 1 & 0 \in E \\ 0 & \text{else} \end{cases}.$$

## 1.2 Basic Properties of Measures

Fix a measure space $(X, \mathcal{M}, \mu)$.

1. **Monotonicity**:

   $$E_1 \subseteq E_2 \implies \mu(E_1) \leq \mu(E_2).$$

   This follows from writing

   $$E_2 = E_1 \coprod (E_2 \setminus E_1)$$

   and taking measures, which are always $\geq 0$.

2. **Subadditivity**:

   $$\mu(\bigcup E_i) \leq \sum \mu(E_i)$$

3. **Continuity from above and below:**

   $$\begin{aligned} E_j \nearrow E &\implies \mu(E_j) \to \mu(E) \text{ and} \\ E_j \searrow E,\ \mu(E_1) < \infty &\implies \mu(E_j) \to \mu(E). \end{aligned}$$

**Definition:** A measure space is *complete* iff when $F \in \mathcal{M}$ is measurable, $\mu(F) = 0$, and $E \subseteq F$, we have $E \in \mathcal{M}$.

> Recall that the Lebesgue measure is complete, and the Borel measure is *not*. Review why this is the case!

### 1.3 Construction of Measures

Given an $(X, \mathcal{M})$, we construct $\mu$ in the following way:

1. Define an outer measure (or premeasure) $\mu^*$ on $\mathcal{P}(X)$.
2. Caratheodory:

$$E \subseteq X \text{ is measurable} \iff \mu_*(A) = \mu_*(A \bigcap E) + \mu_*(A \bigcap E^c) \quad \forall A \subseteq X.$$

> Note: it is worth recalling why this is equivalent to the usual "open set" definition, i.e. $\exists G$ open such that $\mu_*(G \setminus E < \varepsilon$, where we really needed a topology to talk about open sets.

3. Defining

$$\mathcal{M} \coloneqq \{\text{Caratheodory measurable sets}\}$$

yields a $\sigma$-algebra and $\mu_*|_{\mathcal{M}}$ is a measure.

### 1.4 Measurable Functions

Next up: define integrability, by first defining what it means for a function to be measurable.

**Definition:** A function $f : X \to \overline{\mathbb{R}}$ is *measurable* $\iff$ $\mu(\left\{x \in X \;\middle|\; f(x) > a\right\}) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

We say two functions are *equal almost everywhere* if they disagree on a measure zero set, and we can define simple functions in a similar way.

**Definition:** If $\phi$ is simple, i.e. $\phi = \sum_{j=1}^{N} a_j \chi_{E_j} \in L^+$ (is non-negative), then

$$\int \phi d\mu \coloneqq \sum a_j \mu(E_j).$$

Then if $f \in L^+$, we define

$$\int f d\mu = \sup \left\{ \int \phi d\mu \;\middle|\; 0 \leq \phi \leq f, \phi \text{ is simple.} \right\}.$$

For $f$ arbitrary and measurable, write $f = f_+ - f_-$, and define

$$\int f d\mu \coloneqq \int f_+ d\mu - \int f_- d\mu$$

whenever it makes sense (i.e. both are not infinite)

Consider an earlier example: given $(X, \mathcal{M}, \mu)$ and $f \in L^+(X, \mu)$, we can define

$$\mu_f(E) \coloneqq \int_E f d\mu \in \overline{\mathbb{R}}.$$

This always yields a measure, and moreover has the property $\mu(E) = 0 \implies \mu_f(E) = 0$.

> Note that we can actually generalize and let $f \in L^+$. Then the measure defined here can take on negative or even complex numbers, which turns out to be a useful (see "*signed measures*").

This is closely related to the usual notion of signed area between a curve and the $x$-axis we deal with in Calculus.

### 1.5 Absolute Continuity and Radon-Nikodym

**Definition** Let $\mu, \nu$ be two measures on $(X, \mathcal{M})$. Then we say $\nu \ll \mu \iff \nu(E) = 0$ whenever $E \in \mathcal{M}$ and $\mu(E) = 0$, and that $\nu$ *is* absolutely continuous with respect to $\mu$*.

*Exercise:* If $\nu$ is finite, i.e. $\nu(X) < \infty$, then

$$\nu \ll \mu \iff \forall \varepsilon > 0 \, \exists \delta > 0 \; \middle| \; \mu(E) < \delta \implies \nu(E) < \varepsilon,$$

which explains the terminology.

> Worth looking at more in-depth. Should be in textbook.

**Theorem (Partial Radon-Nikodym):** If $\mu, \nu$ are two $\sigma$-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$, then there exists a unique non-negative function $f \in L^1(X, \mu)$ such that

$$d\nu = f d\mu \text{ and } \nu(E) = \int_E f d\mu \text{ for all } E \in \mathcal{M}.$$

> Note: this is a representation theorem. This somehow all traces back to the Riesz Representation theorem for Hilbert spaces, which was a trivial proof! Worth recalling.

*Proof (Sketch)*: We can assume $\mu, \nu$ are $\sigma$-finite (there are standard techniques to do this).

Now define the measure $\rho : \nu + \mu$ and $L(\psi) = \int_X \psi d\nu$ for all $\psi \in L^2(X, \rho)$. Then $L$ turns out to be a *continuous* linear functional on $L^2(X, \rho)$, which isn't completely obvious. This follows because it is bounded, since for all $\psi \in L^2(\rho)$ we have

$$\begin{aligned} \int |\psi| d\nu &\leq \int |\psi| d\rho \\ &\leq \|\psi\|_{L^2(\rho)} \rho(X)^{1/2} \\ &\leq C \|\psi\|_{L^2(\rho)}, \end{aligned}$$

which follows from an application of Cauchy-Schwarz.

Then there exists a $g \in L^1(\rho)$ such that

$$\int \psi d\nu = \int \psi g d\rho = \int \psi g d\nu + \int \psi g d\mu.$$

By collecting terms, we obtain

$$\int_X \psi(1 - g) d\nu = \int_X \psi g d\mu \quad \forall \psi \in L^2(\rho).$$

Now consider letting $\psi = \chi_E$ for some set. Then $\nu(E) = \int_E g d\rho$, from which it can be deduced that $0 \leq g \leq 1$ almost everywhere.

Since $\nu \ll \mu$, we actually have $0 \leq g < 1$ $\rho$-a.e. instead. This follows from taking $B = \{g(x) = 1\}$ and $\psi = \chi_B$ and using the above identity we found to deduce that $\mu(B) = 0$ and thus $\nu(B) = 0$ and $\rho(B) = 0$.

Now let

$$\psi = \chi_E(1)g + g^2 + \cdots g^n),$$

yielding

$$\begin{aligned} \int_E (1 - g^{n+1}) d\nu &= \int_E (1 + g + \cdots + g^n) d\mu \\ \to_{DCT} \nu(E) &= \int_E \frac{g}{1-g} d\mu, \end{aligned}$$

so we can take the integrand on the RHS to be our $f$.

■

Beautiful proof! Due to Von Neumann.

To show: the fundamental theorem of Calculus for measures, i.e. the Lebesgue differentiation theorem, which looks like

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\ dy = f(x) \quad \text{a.e.}$$

and specializes when $f = \chi_E$.