

Title

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Table of Contents

Contents

| | |
|---|----------|
| Table of Contents | 2 |
| Prologue | 3 |
| 0.1 References | 3 |
| 0.2 Notation | 3 |
| 0.3 Summary of Important Concepts | 4 |
| 0.4 Useful Examples | 5 |
| 0.4.1 Varieties | 5 |
| 0.4.2 Presheaves / Sheaves | 5 |
| 0.5 The Algebra-Geometry Dictionary | 5 |

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

- If a property P is said to hold **locally**, this means that for every point p there is a neighborhood $U_p \ni p$ such that P holds on U_p .

| Notation | Definition |
|--------------------------------------|---|
| $k[\mathbf{x}] = k[x_1, \dots, x_n]$ | Polynomial ring in n indeterminates. |
| $k(\mathbf{x}) = k(x_1, \dots, x_n)$ | Rational function field in n indeterminates |
| $\mathcal{U} \rightrightarrows X$ | An open cover $\mathcal{U} = \{U_j \mid j \in J\}$ |
| Δ_X | The diagonal $\{(x, x) \mid x \in X\} \subseteq X \times X$ |
| $\mathbb{A}_{/k}^n$ | Affine n -space |
| — | $\mathbb{A}_{/k}^n := \{\mathbf{a} = [a_1, \dots, a_n] \mid a_j \in k\}$ |
| $\mathbb{P}_{/k}^n$ | Projective n -space |
| — | $\mathbb{P}_{/k}^n := (k^n \setminus \{0\}) / x \sim \lambda x$ |
| — | $= \{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \mid p, q, \in k[x_1, \dots, x_n]\}$ |
| $V(J), V_a(J)$ | Variety associated to an ideal $J \trianglelefteq k[x_1, \dots, x_n]$ |
| — | $:= \{\mathbf{x} \in \mathbb{A}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$ |
| $I(S), I_a(S)$ | Ideal associated to a subset $S \subseteq \mathbb{A}_k^n$ |
| — | $:= \{f \in k[x_1, \dots, x_n] \mid f(\mathbf{x}) = 0 \forall \mathbf{x} \in X\}$ |
| $A(X)$ | Coordinate ring of a variety, $k[x_1, \dots, x_n]/I(X)$ |
| $V_p(J)$ | Projective variety of an ideal |
| — | $:= \{\mathbf{x} \in \mathbb{P}_{/k}^n \mid f(\mathbf{x}) = 0, \forall f \in J\}$ |
| $I_p(S)$ | Projective ideal (?) |
| — | $:= \{f \in k[x_1, \dots, x_n] \mid f \text{ is homogeneous and } f(x) = 0 \forall x \in S\}$ |
| $S(X)$ | Projective coordinate ring, $k[x_1, \dots, x_n]/I_p(X)$ |
| f^h | Homogenization, $x_0^{\deg f} f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ |
| f^i | Dehomogenization, $f(1, x_1, \dots, x_n)$ |
| J^h | Homogenization of an ideal, $\{f^j \mid f \in J\}$ |
| \bar{X} | Projective closure of a subset |
| — | $:= V_p(J^h) := \{\mathbf{x} \in \mathbb{P}^n \mid f^h(\mathbf{x}) = 0 \forall f \in X\}$ |

| Notation | Definition |
|-----------------|---|
| \mathcal{O}_X | Structure sheaf $\left\{ f : U \rightarrow k \mid f \in k(\mathbf{x}) \text{ locally} \right\}$ |
| $D(f)$ | Distinguished open set, $D(f) = V(f)^c = \left\{ x \in \mathbb{A}^n \mid f(x) \neq 0 \right\}$ |

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\cdot)$ and $I(\cdot)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
 - The Zariski topology?
 - Irreducibility?
 - Connectedness?
 - Dimension?
- What is the definition of a presheaf?
 - What are some examples and counterexamples?
- What is the definition of sheaf?
 - What are some examples?
 - What are some presheaves that are not sheaves?
- What is the definition of \mathcal{O}_X , the sheaf of regular functions?
 - How does one compute \mathcal{O}_X for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
 - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
 - What are some examples and counterexamples of complete spaces?

0.4 Useful Examples

0.4.1 Varieties

- $V(xy - 1) \subseteq \mathbb{A}^2$ a hyperbola
- $V(x)$ a coordinate axis
- $V(x - p)$ a point.

0.4.2 Presheaves / Sheaves

- $C^\infty(\cdot, \mathbb{R})$, a sheaf of smooth functions
- $C^0(\cdot, \mathbb{R})$, a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$, the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$, the constant sheaf associated to \mathbb{R} (locally constant real-valued functions)
- $\text{Hol}(\cdot, \mathbb{C})$, a sheaf of holomorphic functions
- K_p the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

| Algebra | Geometry |
|--|--|
| $k[x_1, \dots, x_n]$ | $\mathbb{A}_{/k}^n$ |
| Maximal ideals $\mathfrak{m} = x_1 - p_1, \dots, x_n - p_n$ | Points $[a_1, \dots, a_n]$ |
| Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \dots, x_n]$ | $V(J)$ the zero locus |
| Prime ideals $\mathfrak{p} \in \text{Spec}(k[x_1, \dots, x_n])$ | Irreducible subsets |
| $I(S)$ the ideal of a set | $S \subseteq \mathbb{A}^n$ a subset |
| $I + J$ | $V(I) \cap V(J)$ |
| $\sqrt{I(V) + I(W)}$ | $V \cap W$ |
| $I \cap J, IJ$ | $V(I) \cup V(J)$ |
| $I(V) \cap I(W), \sqrt{I(V)I(W)}$ | $V \cup W$ |
| $I(V) : I(W)$ | $\overline{V \setminus W}$ |
| $k[x_1, \dots, x_n]/I(X)$ | $A(X)$ (Functions on X) |
| $A(X)$ a domain | X is irreducible |
| $A(X)$ indecomposable | X is connected |
| Krull dimension n (chains of primes) | Topological dimension n (chains of irreducibles) |