

Title

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1.1 Strong Linkage

We have two categories:

- $G_r T$, with a notion of *strong linkage*, and
- G_r , which instead only has *linkage*.

We'll restate a few theorems.

Theorem 1.1.1(?).

Let $\lambda, \mu \in X(T)$.

1. If $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_r T} \neq 0$, then $\mu \uparrow \lambda$ are strongly linked.
2. If $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$, then $\mu \in W_p \cdot \lambda + p^r X(T)$.

Example 1.1.1(?): In the case of $\Phi = A_2$, we'll consider the two different categories.

We have the following picture for \widehat{Z} :

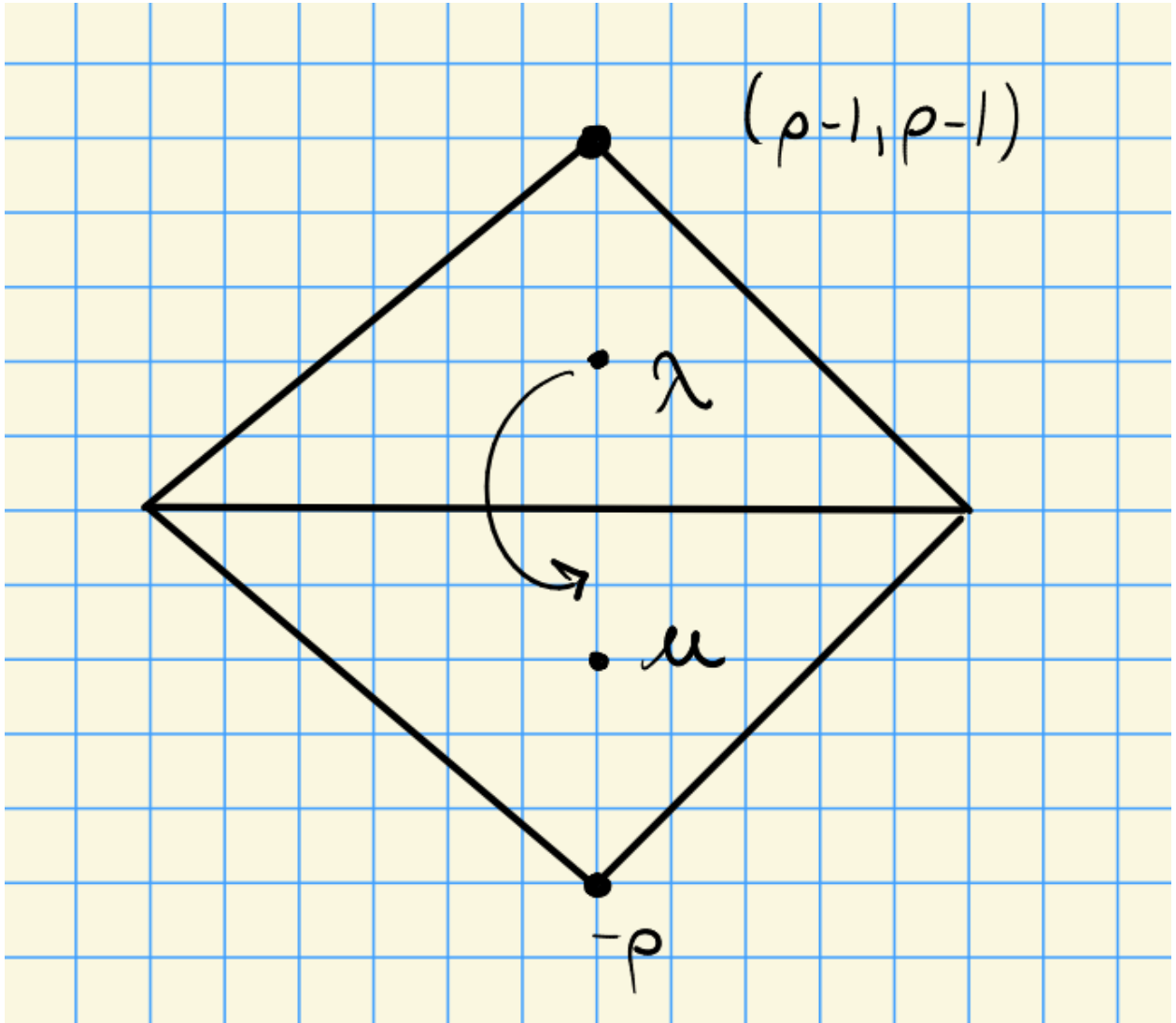


Figure 1: Image

Considering $X_1(T)$ and $[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] \neq 0$, and $\hat{Z}_1(\lambda)$ has 6 composition factors as G_1T -modules.

On the other hand, for Z , we have the following:

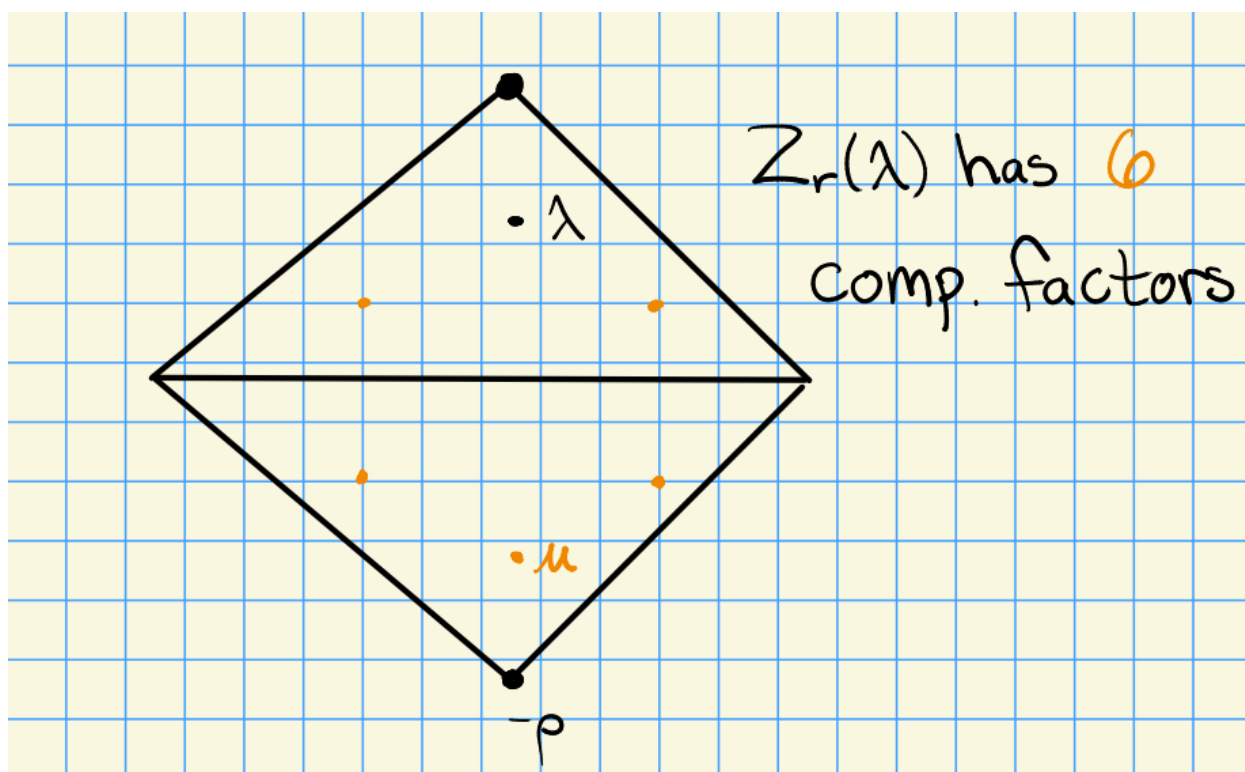


Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

1.2 Extensions

Let $\lambda, \mu \in X(T)$. We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\mathrm{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) = \mathrm{Ext}_{G_r}^j(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)) \quad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu)) = \bigoplus_{\gamma \in X(T)} \mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu))_{\gamma}$$

where we are taking the fixed points under the torus T action on the first factor (for which T_r acts

trivially). We can write this as

$$\begin{aligned}
 \cdots &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu) \otimes \gamma) \\
 &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r T}^j(L_r(\lambda), L_r(\mu) \otimes p^r v) \\
 &= \bigoplus_{v \in X(T)} \text{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)).
 \end{aligned}$$

So if we know extensions in the G_r category, we know them in the $G_r T$ category.

There is an isomorphism

$$\text{Ext}_{G_r T}^1(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) \cong \text{Hom}_{G_r T}(\text{rad}_{G_r T} \widehat{Z}_r(\lambda), \widehat{L}_r(\mu)).$$

Finally, for $\lambda, \mu \in X(T)$, if the above Ext^1 vanishes, then $\lambda \in W_p \cdot \mu$ (i.e. λ and μ are linked).

1.3 The Steinberg Modules

Example 1.3.1(Steinberg): Consider A_2 :

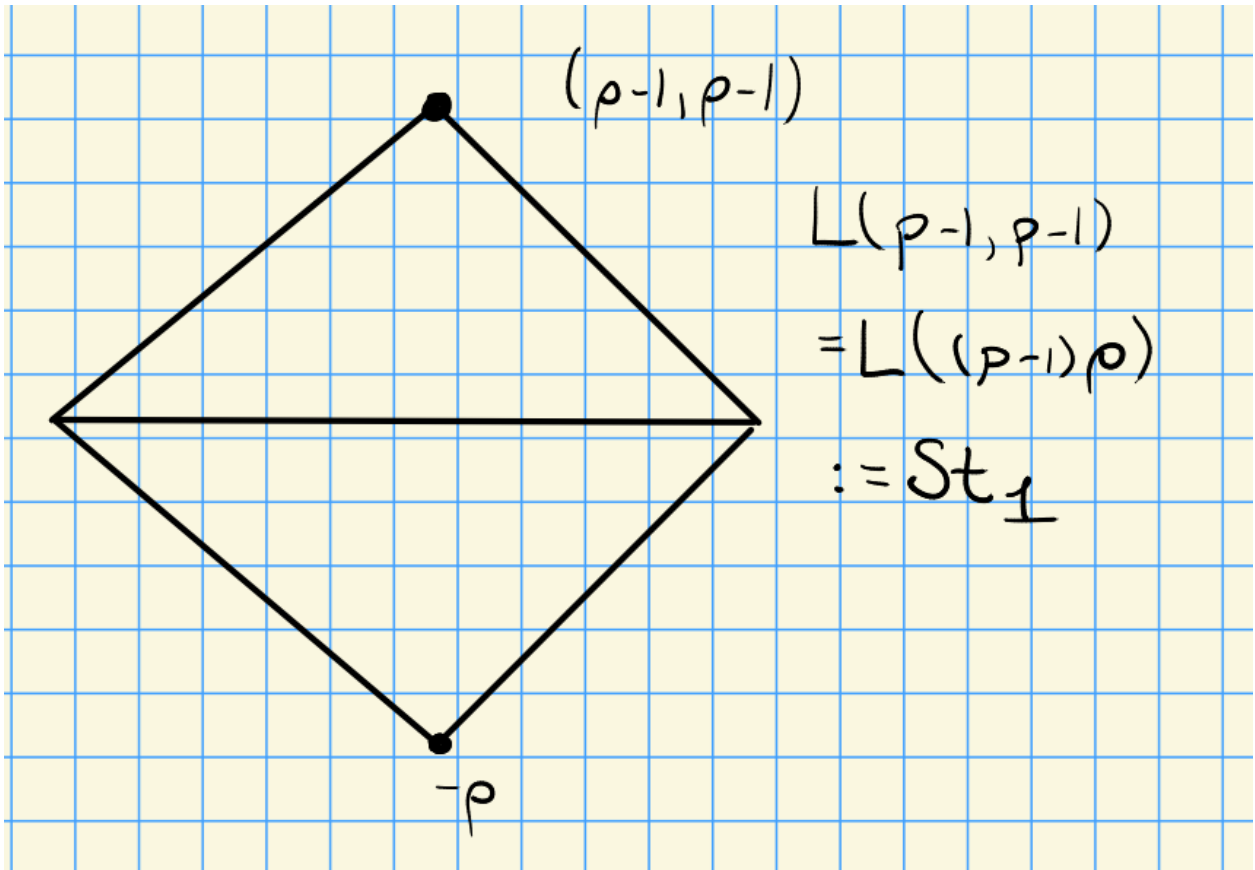


Figure 3: Image

Taking the representation corresponding to $(p-1, p-1)$ yields the “first Steinberg module”

$$L(p-1, p-1) = L((p-1)\rho) := St_1.$$

In this case, we have an equality of many modules:

$$H^0((p-1)\rho) = L((p-1)\rho) = V((p-1)\rho) = T((p-1)\rho).$$

Definition 1.3.1 (Steinberg Modules).

The r th **Steinberg module** is defined to be $L((p^r - 1)\rho)$.

Remark 1.3.1: In general, we have

$$L((p^r - 1)\rho) = H^0((p^r - 1)\rho) = V((p^r - 1)\rho).$$

We also have

$$\widehat{Z}_r((p^r - 1)\rho) \cong L((p^r - 1)\rho) \downarrow_{G_r T}.$$

Theorem 1.3.1 (?).

The Steinberg module is both injective and projective as both a G_r -module and a $G_r T$ -module.

Proof (?).

It suffices to prove that St_r is projective over $G_r T$, then by a previous theorem, it will also be projective over G_r . Let $\widehat{L}_r(\mu)$ be a simple $G_r T$ -module, and consider

$$\text{Ext}_{G_r T}^1(\text{St}_r, \widehat{L}_r(\mu)) = \text{Ext}_{G_r T}^1(\widehat{L}_r((p^r - 1)\rho), \widehat{L}_r(\mu)).$$

If we show this is zero for every simple module, the result will follow.

Suppose $(p^r - 1)\rho \not\prec \mu$. In this case, the RHS above is zero.

Missed why: something to do with radical of the first term?

Otherwise, we have

$$\text{Ext}_{G_r T}^1(\widehat{L}_r(\mu), \text{St}_r) = \text{Hom}_{G_r T}(\text{rad}(\widehat{Z}_r(\mu)), \text{St}_r).$$

Suppose that the RHS is nonzero. Then $\text{rad}(\widehat{Z}_r(\mu)) \rightarrow \text{St}_r$, and thus

$$\dim \text{rad}(\widehat{Z}_r(\mu)) \geq \dim \text{St}_r = p^{r|\Phi^+|}$$

But we know that

$$\dim \text{rad}(\widehat{Z}_r(\mu)) < \dim \widehat{Z}_r(\mu) = p^{r|\Phi^+|},$$

so we've reached a contradiction and the hom must be zero. ■

Proposition 1.3.1 (Open Conjecture, Donkin, MSRI 1990: 'DFilt Conjecture').

Let G be a reductive group and M a finite-dimensional G -module. Then M has a good (p, r) -filtration iff $\text{St}_r \otimes M$ has a good filtration.

Remark 1.3.2: See Nakano-Kildetoft (2015) and Bendel-Nakano-Pillen-Subaje (2018-). The forward direction is equivalent to the following: