

8.8 Part 2, Computing the Index of L

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What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$

$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
 - (not the same kernel/cokernel)
- Compute $\text{Ind } L_1$: explicitly describe $\ker L_1, \text{coker } L_1$.
- Replace in two steps:
 - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s .
 - * Replace $S(s, t)$ by a matrix

$$\tilde{S}(s, t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

- * Idea: approximate by cylinders at infinity.

- * Use invariance of index under small perturbations.
- $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
- * Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\text{Ind}(L) = \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

where $S : \mathbb{R} \longrightarrow \text{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\text{Ind}(R^\pm(t))$; then

$$\text{Ind}(L) = \text{Ind}(L_1) = \text{Ind}(R^-(t)) - \text{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing L_1 to R^\pm . Let $k^\pm := \text{Ind}(R^\pm)$; then $\text{Ind}(L_1) = k^- - k^+$.
- Lemma 8.8.4: $\text{Ind}(L_0) = \text{Ind}(L)$.
- Han's Talk:
 - Prop 8.8.3, using Lemma 8.8.5
- Me
 - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\text{Ind}(L_1) = k^- - k^+$.

Setup:

We identified for $n = 1$

$$J_0^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and thus for general n ,

$$J_0^1 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix}.$$

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) = \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$

Statement: let $p > 2$ and define

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \text{Ker } F = 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\}$$

$$\dim \text{Ker } F^* = 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^+ < 2\pi\ell < a^- \right\}.$$

2. Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\dim \text{Ker } F = \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$

$$\dim \text{Ker } F^* = \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

0.3 Proof

0.3.1 Assertion 1

Assertion 1: Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \text{Ker } F = 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\}$$

$$\dim \text{Ker } F^* = 2 \cdot \# \left\{ \ell \in \mathbf{Z} \mid a^+ < 2\pi\ell < a^- \right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) = a_1(s) = a_2(s)$.
- Start with equation on \mathbb{R}^2 ,

$$Y(s, t) = (Y_1(s, t), Y_2(s, t))$$

- Replace with equation on \mathbb{C} :

$$Y(s, y) = Y_1(s, t) + iY_2(s, t)$$

- Rewrite the PDE $F(Y) = 0$ as $\bar{\partial}Y + S(s)Y = 0$, i.e.

$$\frac{\partial}{\partial s} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let $Y = B\tilde{Y}$ where $B \in \text{GL}(1, \mathbb{C})$ satisfies $(\bar{\partial} + S)B = 0$ to obtain $\bar{\partial}\tilde{Y} = 0$.

- Can choose $B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix}$ where $\frac{\partial b}{\partial s} = -a(s)b(s)$.
- Explicitly, we can take the integral $b(s) = e^{\int_0^s -a(t) dt} = e^{-A(s)}$

- Remark: for some constants C_i , we have

$$A(s) = \begin{cases} C_1 + a^- s & s \leq -\sigma_0 \\ C_2 + a^+ s & s \geq \sigma_0 \end{cases}.$$

- Remark: the new \tilde{Y} satisfies CR. It is continuous and L^1_{loc} and thus by elliptic regularity C^∞ . Its real/imaginary parts are C^∞ and harmonic.

Step 2: ?

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$
- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of \tilde{Y} in z .
- Deduce that the solutions of the system are given by

$$\tilde{Y}(s + it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $c_\ell \in \mathbb{C}$ and this sequence converges for all s, t .

- Write in real coordinates as

$$\tilde{Y}(s, t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell} \left(\alpha_\ell \begin{pmatrix} \cos 2\pi\ell t \\ \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -\sin 2\pi\ell t \\ \cos 2\pi\ell t \end{pmatrix} \right).$$

- Return to $Y = B\tilde{Y}$:

$$Y(s, t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell} \left(\alpha_\ell \begin{pmatrix} e^{-A(s)} \cos 2\pi\ell t \\ e^{-A(s)} \sin 2\pi\ell t \end{pmatrix} + \beta_\ell \begin{pmatrix} -e^{-A(s)} \sin 2\pi\ell t \\ e^{-A(s)} \cos 2\pi\ell t \end{pmatrix} \right).$$

- For $s \geq s_0$, for some constants K_i we can write

$$Y(s, t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^-)s + K} (\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t) \\ e^{(2\pi\ell - a^-)s + K'} (\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t) \end{pmatrix}.$$

and for $s \geq s_0$

$$Y(s, t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^+)s+C} (\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t) \\ e^{(2\pi\ell - a^+)s+C'} (\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t) \end{pmatrix}.$$

- Then $Y \in L^p \iff$ the exponential terms die at infinity. Forces the conditions:
 - $\ell \neq 0 \implies \alpha_\ell = \beta_\ell = 0$ or $2\pi\ell < a^+$.
 - $\ell = 0 \implies (a_0 = 0 \text{ or } a^+ > 0)$ and $(\beta_0 = 0 \text{ or } a^+ > .0)$.

This further forces

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } a^- < 2\pi\ell < a^+ & \ell \neq 0 \\ \left(\alpha_0 = 0 \text{ or } a^- < 0 < a^+ \right) \text{ and } \left(\beta_0 = 0 \text{ or } a^- < 0 < a^+ \right) & \ell = 0 \end{cases}.$$

- Finitely many such ℓ that satisfy these conditions
- Sufficient conditions for $Y(s, t) \in W^{1,p}$.

$$\begin{aligned} F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

I.e. $F = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)$.

- Compute dimension of space of solutions:

$$\dim \text{Ker } F = 2\# \left\{ \ell \in \mathbf{Z}^* \mid a^- < 2\pi\ell < a^+ \right\} = 2\# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\}.$$

Test:

$$\mathbb{1}[\{x\}].$$

Use this to deduce $\dim \ker F^*$:

- $Y \in \ker F^* \iff Z(s, t) := Y(-s, t)$ is in the kernel of the operator

$$\begin{aligned} \tilde{F} : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Z &\mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(\textcolor{red}{-s})Y. \end{aligned}$$

- Obtain $\ker F^* \cong \ker \tilde{F}$.

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0.3.2 Assertion 2

Assertion 2: Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\begin{aligned} \dim \operatorname{Ker} F &= \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\} \\ \dim \operatorname{Ker} F^* &= \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}. \end{aligned}$$

We use the following lemma

- Lemma 8.8.7:

$$\sup_{s \in \mathbb{R}} \|S(s)\| < 1 \implies \text{the elements in } \ker F, \ker F^* \text{ are independent of } t.$$

- Proof: see Proposition 10.1.7, in subsection 10.4.a.
- We know (?)

$$\mathbf{a}(s) := \begin{bmatrix} a_1(s) \\ a_2(s) \end{bmatrix}, \quad \mathbf{Y} := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \ker F \implies \frac{\partial \mathbf{Y}}{\partial s} = -\mathbf{a}(s)\mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c} \exp(-\mathbf{A}(s)), \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma) d\sigma.$$

- Explicitly,

$$\begin{cases} \frac{\partial Y_1}{\partial s} = -a_1(s)Y_1 \\ \frac{\partial Y_2}{\partial s} = -a_2(s)Y_2 \end{cases} \implies Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma) d\sigma.$$

- As before,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

- Thus $Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+)$, establishing the formula for $\dim \ker F$.
- Get formula for $\dim \ker F^*$ using \tilde{F} as before.

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(End of Chapter)