

Homological Algebra Problem Sets

Problem Set 3

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

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Problem 1.0.1 (Prove Corollary 2.3.2)

For R a PID, show that an R -module A is divisible if and only if A is injective.

Recall that a module is divisible if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.

Solution:

Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.

\Rightarrow : Suppose A is divisible, we then want to show every R -module morphism of the following form lifts, where we regard the ideal J and the ring R as R -modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f & \nearrow \exists \tilde{f} & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R -linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\rightarrow A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending R -linearly yields a well-defined R -module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

\Leftarrow : Suppose $A \in R\text{-Mod}$ is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \leq R$:

$$\begin{array}{ccccc} 0 & \longrightarrow & J & \hookrightarrow & R \\ & & \downarrow & \nearrow & \\ & & A & & \end{array}$$

[Link to Diagram](#)

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \rightarrow A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\rightarrow A \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{array}{ccccc} 0 & \longrightarrow & jR & \xhookrightarrow{\iota} & R \\ & & \downarrow f_a & \nearrow \exists \tilde{f}_a & \\ & & A & & \end{array}$$

[Link to Diagram](#)

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting \mathbf{a} range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)

Calculate $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q .

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).

Note that there is an injection

$$1 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \rightarrow \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \rightarrow 1$$

where π_n is the canonical quotient morphism, and applying the left-exact contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any \mathbb{Z} -module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since \mathbb{Z} is a free \mathbb{Z} -module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z} \\ \pi_n \downarrow & \nearrow \exists \tilde{f} & \\ \mathbb{Z}/n & & \end{array}$$

[Link to Diagram](#)

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0, 1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0, 1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n} \right\}.$$

Setting $f_i(1) := \left[\frac{i}{n} \right]$ (where we take the equivalence class mod \mathbb{Z}) yields n distinct morphisms $f_i : \mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \rightarrow \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\begin{aligned} \Psi : \mathbb{Z} &\rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i, \end{aligned}$$

and using the fact that if $i = i' \pmod{n}$, write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left\lfloor \frac{i+kn}{n} \right\rfloor = \left\lfloor \frac{i}{n} + k \right\rfloor = \left\lfloor \frac{i}{n} \right\rfloor = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

Claim 2: \mathbb{Q}/\mathbb{Z} is an injective object in \mathbb{Z} -modules.

Proof (?).

By the previous exercise, it suffices to show that \mathbb{Q}/\mathbb{Z} is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That \mathbb{Q} is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b := \frac{a}{n} \in \mathbb{Q}$ works. Since \mathbb{Q} is an abelian group, \mathbb{Z} is automatically normal, and the result follows.

■

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \quad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).

Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M := \mathbb{Z}/m$,
- $A := \mathbb{Z}$,
- $I := n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n(\mathbb{Z}/m)} \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\rightarrow \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
 \ker \varphi &= \{x \in \mathbb{Z} \mid x(1 \otimes 1) = 0\} \\
 &= \{x \in \mathbb{Z} \mid n \mid x \text{ or } m \mid x\} \\
 &= \langle n, m \rangle \\
 &= \langle \gcd(n, m) \rangle && \text{by Bezout's theorem} \\
 &:= \langle d \rangle.
 \end{aligned}$$

Now applying the first isomorphism theorem yields the result. ■

Solution:

We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\cdot) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \cdot)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of \mathbb{Z}/q .
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\text{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/p, \mathbb{Z}/q) := R^i F(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathbb{Z}/q & \xrightarrow{d^{-1}} & \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & \mathbb{Q}/\mathbb{Z} \xrightarrow{d^1} 1 \\
 & & & & & & \\
 & & [1]_q & \longrightarrow & \begin{bmatrix} 1 \\ q \end{bmatrix} & &
 \end{array}$$

$$[x] \longrightarrow [qx]$$

[Link to Diagram](#)

This is a chain complex by construction, since $d^2([1]_q) = \left[q \left(\frac{1}{q} \right) \right] = [1] = [0]$. We now delete the augmentation and apply $F(\cdot)$:

$$\begin{array}{ccccccc}
1 & \longrightarrow & I^0 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^0} & I^1 := \mathbb{Q}/\mathbb{Z} & \xrightarrow{d^1} & 1 \\
& & \downarrow F(\cdot) & & & & \\
1 & \longrightarrow & F(I^0) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^0 := F(d^0)} & F(I^1) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial^1 := F(d^1)} & 1 \\
\parallel & & \uparrow \Psi \cong & & \uparrow \Psi \cong & & \parallel \\
1 & \longrightarrow & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^0} & \mathbb{Z}/p & \xrightarrow{\tilde{\partial}^1} & 1
\end{array}$$

[Link to Diagram](#)

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q , we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q .

We now take homology:

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/q) := R^1 F(\mathbb{Z}/q) := \frac{\ker \partial^1}{\text{im } \partial^0} = \frac{\mathbb{Z}/p}{q(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)

Let $A \in \mathbf{Ab}$, and show that the following map is injective:

$$\begin{aligned}
\varepsilon_A : A &\rightarrow I(A) := \prod_{f \in \text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} \\
a &\mapsto \mathbf{a} \text{ where } \mathbf{a}(f) := f(a) \in \mathbb{Q}/\mathbb{Z},
\end{aligned}$$

i.e. when looking at the image $\varepsilon_A(a)$ in the product, the component indexed by f is an element of \mathbb{Q}/\mathbb{Z} obtained by evaluating $f(a)$.

Hint: if $a \in A$, find a map $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$.

Solution:

By contrapositive, we'll suppose $a \neq 0$ and show $\varepsilon_A(a) \neq 0$. Following the hint, we first consider the cyclic subgroup $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$ and define a map

$$\begin{aligned}
f_a : a\mathbb{Z} &\rightarrow \mathbb{Z} \\
an &\mapsto n.
\end{aligned}$$

We now pick $\ell > 1 \in \mathbb{Z}$ to be any integer, and define a composition $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$:

$$\begin{array}{ccccccc}
 a\mathbb{Z} & \xrightarrow{f_a} & \mathbb{Z} & \xleftarrow{\iota} & \mathbb{Q} & \xrightarrow{x \mapsto \frac{x}{\ell}} & \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \\
 & \nearrow f & & & & & \nearrow f \\
 & & & & & & \\
 an & \longmapsto & n & \longmapsto & n & \longmapsto & \frac{n}{\ell} \longmapsto \left[\frac{n}{\ell} \right] \\
 & & & & & & \\
 a & \xrightarrow{\quad\quad\quad} & & & & & \left[\frac{1}{\ell} \right]
 \end{array}$$

[Link to Diagram](#)

By choice of ℓ , this map satisfies $f(a) = [1/\ell] \neq 0$, so the map is nonzero. Since \mathbb{Q}/\mathbb{Z} is injective, the universal property provides a lift \tilde{f} :

$$\begin{array}{ccc}
 A & & \\
 \uparrow & \searrow \exists \tilde{f} & \\
 a\mathbb{Z} & \xrightarrow{f} & \mathbb{Q}/\mathbb{Z}
 \end{array}$$

[Link to Diagram](#)

Since \tilde{f} lifts f , it is also nonzero. But now we can check that

$$\varepsilon_A(a)(f) := f(a) \neq 0,$$

so the f component of the image of a is nonzero and thus $\mathbf{a} := \varepsilon_A(a) \neq 0$ in the product.

Problem 1.0.4 (Weibel 2.4.2)

If $U : \mathcal{B} \rightarrow \mathcal{C}$ is right-exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Solution:

We'll show that $(U \circ L_i F)(X) \cong (L_i(U \circ F))(X)$ for every object X . Starting with the left-hand side, to compute left-derived functors, we'll need projective resolutions, so let $P \rightarrow X$ be a projective resolution of X . Fixing labeling, we have the following situation:

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X \xrightarrow{0} 0 \\
 & & & & & \downarrow F(\cdot) & & & \\
 \cdots & \longrightarrow & FP_2 & \xrightarrow{F(\partial_2)} & FP_1 & \xrightarrow{F(\partial_1)} & FP_0 & \xrightarrow{0} & 0
 \end{array}$$

[Link to Diagram](#)

We now have by definition

$$L_i F(X) := \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \implies U(L_i F(X)) := U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right).$$

For the right-hand side, we can take the same projective resolution $P \rightarrow X$, and apply a similar process:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & X & \xrightarrow{0} & 0 \\ & & & & & \Downarrow (U \circ F)(\cdot) & & & & & \\ \cdots & \longrightarrow & UFP_2 & \xrightarrow{(UF)(\partial_2)} & UFP_1 & \xrightarrow{(UF)(\partial_1)} & UFP_0 & \xrightarrow{0} & 0 & & \end{array}$$

[Link to Diagram](#)

Again, by definition,

$$(L_i(UF))(X) := \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})},$$

and thus it suffices to show that there is an isomorphism

$$U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \xrightarrow{\sim} \frac{\ker(UF)(\partial_i)}{\operatorname{im}(UF)(\partial_{i+1})}.$$

To show this, we apply the exact functor U to the following SES to produce a new SES, from which we'll produce the desired isomorphism f :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{im} F(\partial_{i+1}) & \xrightarrow{\iota_i} & \ker F(\partial_i) & \xrightarrow{\pi_i} & \frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})} \longrightarrow 0 \\ & & & & \Downarrow U & & \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{U(\pi_i)} & U\left(\frac{\ker F(\partial_i)}{\operatorname{im} F(\partial_{i+1})}\right) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow f \\ 0 & \longrightarrow & U(\operatorname{im} F(\partial_{i+1})) & \xrightarrow{U(\iota_i)} & U(\ker F(\partial_i)) & \xrightarrow{\tilde{\pi}_i} & \frac{U(\ker F(\partial_i))}{U(\operatorname{im} F(\partial_{i+1}))} \longrightarrow 0 \end{array}$$

[Link to Diagram](#)

Here $\tilde{\pi}_i$ is the natural quotient map, whose image is $\operatorname{coker} U(\iota_i)$. Finally, the map f exists in any abelian category, using that whenever $0 \rightarrow A \xrightarrow{g_1} B \xrightarrow{g_2} C \rightarrow 0$ is exact, there is an isomorphism $C \xrightarrow{\sim} B/\operatorname{im}(g_1)$.

Problem 1.0.5 (Weibel 2.4.3)

- If $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$ is exact with P projective or F -acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

- Show that if

$$0 \rightarrow M_m \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

is exact with P_i projective or F -acyclic, then

$$L_i F(A) \cong L_{i-m-1} F(M_m) \quad i \geq m+2.$$

– Moreover show that $L_{m+1} F(A)$ is the kernel of $F(M_m) \rightarrow F(P_m)$.

- Conclude that if $P \rightarrow A$ is an F -acyclic resolution of A , then $L_i F(A) = H_i(F(P))$.

Solution:

Claim:

$$L_i F(A) \cong L_{i-1} F M \quad i \geq 2.$$

Proof (of claim).

Following the proof of Weibel Theorem 2.4.6, let $P_M \rightarrow M$ and $P_A \rightarrow A$ be projective resolutions of M and A respectively. Then applying the Horseshoe Lemma, there is a projective resolution $P_P \rightarrow P$ of P such that the following is a short exact sequence of chain complexes:

$$0 \rightarrow P_M \rightarrow P_P \rightarrow P_A \rightarrow 0,$$

where in fact in each degree n piece, this induces a *split* exact sequence. Using that F is additive and additive functors preserve split exact sequences, the following is a SES for every n :

$$0 \rightarrow FP_M^n \rightarrow FP_P^n \rightarrow FP_A^n \rightarrow 0,$$

which implies that there is a SES of chain complexes

$$0 \rightarrow FP_M \rightarrow FP_P \rightarrow FP_A \rightarrow 0.$$

Thus there is an associated LES of derived functors:

$$\begin{array}{ccccccc} & & & & \dots & & \\ & & & & \nearrow & & \\ & & & \partial_{i+1} & & & \\ \hookrightarrow & L_i FM & \longrightarrow & L_i FP & \longrightarrow & L_i FA & \longrightarrow \dots \\ & & & \searrow & & & \\ & & & \partial_i & & & \\ \hookrightarrow & L_{i-1} FM & \longrightarrow & L_{i-1} FP & \longrightarrow & \dots & \end{array}$$

[Link to Diagram](#)

Using that P is F -acyclic, the middle terms $L_i FP = 0$ for all $i > 0$, and thus this splits into a collection of SESs:

$$\begin{array}{l} 0 \rightarrow L_2 FA \xrightarrow{\partial_2} L_1 FM \rightarrow 0 \\ 0 \rightarrow L_3 FA \xrightarrow{\partial_3} L_2 FM \rightarrow 0 \\ \vdots \\ 0 \rightarrow L_i FA \xrightarrow{\partial_i} L_{i-1} FM \rightarrow 0. \end{array}$$

This makes every ∂_i for $i \geq 2$ an isomorphism. ■

Claim:

$$L_i FA \cong \ker(FM \rightarrow FP).$$

Proof (?).

Using the same argument as above, consider the lower order terms of the associated LES:

$$\begin{array}{ccccccc}
 & & & & & & \dots \\
 & & & & \nearrow \partial_2 & & \\
 L_1 FM & \xrightarrow{\quad} & L_1 FP = 0 & \xrightarrow{\quad} & L_1 FA & & \\
 & & \nwarrow \partial_1 & & & & \\
 L_0 FM & \xrightarrow{\quad} & L_0 FP & \xrightarrow{\quad} & L_0 FA & \longrightarrow & 0
 \end{array}$$

[Link to Diagram](#)

Noting that $L_1 FP = 0$ by F -acyclicity, the highlighted portion forms a four term exact sequence. We can form another exact sequence and compare the two:

$$\begin{array}{ccccccccccc}
 0 & \longrightarrow & L_1 FA & \xrightarrow{\partial_1} & L_0 FM & \longrightarrow & L_0 FP & \longrightarrow & L_0 FA & \longrightarrow & 0 \\
 & & \downarrow \text{dotted} & & \parallel & & \parallel & & \parallel & & \\
 0 & \longrightarrow & \ker(L_0 FM \rightarrow L_0 FP) & \longrightarrow & L_0 FM & \longrightarrow & L_0 FP & \longrightarrow & L_0 FA & \longrightarrow & 0
 \end{array}$$

[Link to Diagram](#)

That the map indicated by the dotted line exists and is an isomorphism holds in any abelian category, using that fact that whenever $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ is a SES we have $A \cong \ker f$. ■

Claim: If $P \rightarrow A$ is an F -acyclic resolution of A , then there is an isomorphism

$$L_i FA \cong H_i(FP).$$

Problem 1.0.6 (Weibel 2.5.2)

Show that the following are equivalent:

- A is a projective R -module.
- $\text{Hom}_R(A, \cdot)$ is an exact functor.
- $\text{Ext}_R^{i \geq 1}(A, B) = 0$ and for all B , i.e. A is $\text{Hom}_R(\cdot, B)$ -acyclic for all B .
- $\text{Ext}_R^1(A, B)$ vanishes for all B .

Solution:

We'll show

- $a \iff b$
- $b \implies c$
- $c \iff d$:
- $d \implies b$

Proof ($a \iff b$).Let ξ be the following SES:

$$\xi : 0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$$

and define the functor $F(\cdot) := \text{Hom}_R(A, \cdot)$. This is a covariant left-exact functor, and so applying it to the above sequence yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & F(\cdot) := \text{Hom}_R(A, \cdot) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & FM_1 & \xrightarrow{F(f): \lambda \mapsto f \circ \lambda} & FM_2 & \xrightarrow{F(g): \lambda \mapsto g \circ \lambda} & FM_3 \end{array}$$

[Link to Diagram](#) \implies :

For F to be exact, it suffices to show it is right-exact, i.e. that $F(g)$ is surjective. This amounts to asking that every $\varphi \in FM_3 := \text{Hom}_R(A, M_3)$ lifts to a preimage $\tilde{\varphi} \in FM_2 := \text{Hom}_R(A, M_2)$ satisfying $F(g)(\tilde{\varphi}) = \varphi$. Unwinding definitions, this requires that $g \circ \tilde{\varphi} = \varphi$, which is precisely the lift required for the universal property of projective objects:

$$\begin{array}{ccc} & A & \\ & \downarrow \varphi & \\ M_2 & \xrightarrow{g} & M_3 \longrightarrow 0 \end{array}$$

$\exists \tilde{\varphi}$ (dashed arrow from M_2 to A)

[Link to Diagram](#)

If A is projective, this lift always exists, so $\text{Hom}_R(A, \cdot)$ is an exact functor. Conversely, if $\text{Hom}_R(A, \cdot)$ is exact, this lift always exists, so A satisfies the universal property of a projective object. ■

Proof ($b \implies c$).

Suppose $F(\cdot) := \text{Hom}_R(A, \cdot)$ is exact, then since F is left-exact covariant it has right-derived functors $\text{Ext}_R^i(A, B) := R^i F(B)$ which are computed in the following way

1. Taking an *injective* resolution of

$$B \rightarrow I := (I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \dots).$$

2. Applying $\text{Hom}_R(A, \cdot)$ to get the complex

$$FI := (0 \rightarrow \text{Hom}_R(A, I_0) \xrightarrow{F(\partial_0)} \text{Hom}_R(A, I_1) \xrightarrow{F(\partial_1)} \dots).$$

3. Defining

$$R^i F(B) := \ker F(\partial_i) / \text{im } F(\partial_{i-1}).$$

Note that in step (2), if $\text{Hom}_R(A, \cdot)$ is an exact functor, then since I is an acyclic complex, FI is again acyclic and so $\ker F(\partial_i) = \text{im } F(\partial_{i-1}) = 0$ for $i \geq 1$. So

$$\text{Ext}_R^{\geq 1}(A, B) := R^{\geq 1} F(B) = 0.$$

■

Proof ($c \iff d$).

\implies : This direction is clear, since if $\text{Ext}_R^i(A, B) = 0$ for all B , then taking $i = 1$ is the statement of (d).

\impliedby : This follows from the dimension-shifting isomorphism in a previous exercise. Let $F(\cdot) := \text{Hom}_R(A, \cdot)$ and suppose $\text{Ext}_R^1(A, B) := L_1 F(B) = 0$ for all B . Let B' be arbitrary, it then suffices to show that $\text{Ext}_R^i(A, B') := L_i(B') = 0$ for all $i > 1$, since we can take B' as one such B in the assumption for the $i = 1$ case.

The dimension shifting results states that if P_i are F -acyclic, then for every exact sequence

$$0 \rightarrow M_m \rightarrow P_m \rightarrow \dots \rightarrow P_0 \rightarrow B' \rightarrow 0$$

we obtain an isomorphism

$$L_i F(B') \cong L_{i-m-1} F(M_m) \iff L_i F(M_m) \cong L_{i+m+1} F(B').$$

So take any F -acyclic resolution of P , say

$$B' \xrightarrow{\partial_{-1}} I_0 \xrightarrow{\partial_0} I_1 \xrightarrow{\partial_1} \dots,$$

then consider truncating it at the m th stage:

$$0 \rightarrow B' \xrightarrow{\partial_{i-1}} I_0 \rightarrow I_1 \rightarrow \dots \xrightarrow{\partial_{m-1}} I_m \rightarrow M_m := \text{coker } \partial_{m-1} \rightarrow 0$$

By assumption, we have $L_1 F(M_m) = 0$ for every m , and thus

$$\begin{aligned}
 0 &= L_1 F(B') \quad \text{by assumption} \\
 0 &= L_1 F(M_0) \cong L_2 F(B') \\
 0 &= L_1 F(M_1) \cong L_3 F(B') \\
 0 &= L_1 F(M_2) \cong L_4 F(B') \\
 &\vdots \\
 0 &= L_1 F(M_m) \cong L_{m+2} F(B') \quad \forall m \geq 0.
 \end{aligned}$$

and so $L_i(B') = 0$ for all $i \geq 1$. ■

Proof ($d \implies b$).

Take an arbitrary SES

$$\xi : 0 \rightarrow B' \xrightarrow{f} B \xrightarrow{g} B'' \rightarrow 0$$

and consider applying the left-exact covariant functor $F(\cdot) := \text{Hom}_R(A, \cdot)$ and taking the associated LES:

$$\begin{array}{ccccccc}
 0 & & & & & & \\
 \downarrow & & & & & & \\
 \text{Hom}_R(A, B') & \xrightarrow{f^*} & \text{Hom}_R(A, B) & \xrightarrow{g^*} & \text{Hom}_R(A, B'') & \rightarrow & 0 \\
 & \searrow \delta_0 & & & & & \\
 & \text{Ext}_R^1(A, B') & \longrightarrow & \text{Ext}_R^1(A, B) & \longrightarrow & \text{Ext}_R^1(A, B'') & \longrightarrow 0 \\
 & \searrow \delta_1 & & & & & \\
 & \dots & & & & &
 \end{array}$$

[Link to Diagram](#)

By assumption, all of the higher Ext terms vanish, and in particular the red term $\text{Ext}_R^1(A, B') = 0$. This implies that g^* is surjective, making the following sequence exact:

$$0 \rightarrow \text{Hom}_R(A, B') \xrightarrow{f^*} \text{Hom}_R(A, B) \xrightarrow{g^*} \text{Hom}_R(A, B'') \rightarrow 0,$$

making $\text{Hom}_R(A, \cdot)$ an exact functor. ■

Problem 1.0.7 (Weibel 2.6.4)

Show that colim is left adjoint to Δ , and conclude that colim is right-exact when \mathcal{A} is

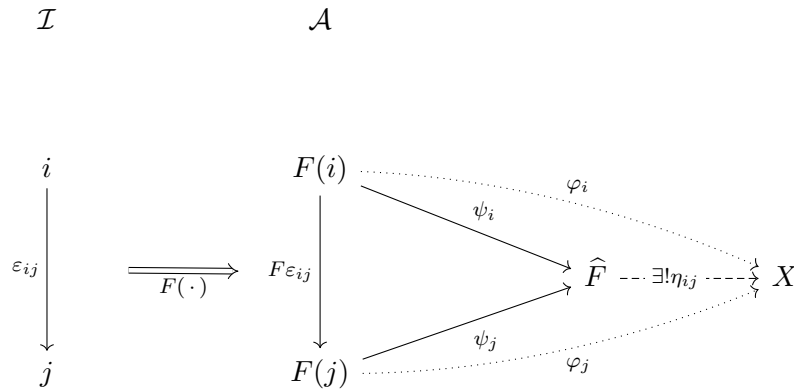
abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on **Ab**.

Solution:

Fixing some index category \mathcal{I} and a functor $F : \mathcal{I} \rightarrow \mathcal{A}$, so $F \in \mathcal{A}^{\mathcal{I}}$, write $\hat{A} := \text{colim}_{i \in \mathcal{I}} F(i)$. We want to show that $\mathcal{A}^{\mathcal{I}} \xrightleftharpoons[\Delta]{\text{colim}} \mathcal{A}$ defines an adjoint pair, so that colim is a left-adjoint and Δ is a right-adjoint. By definition, this is equivalent to showing the existence of natural bijections of sets

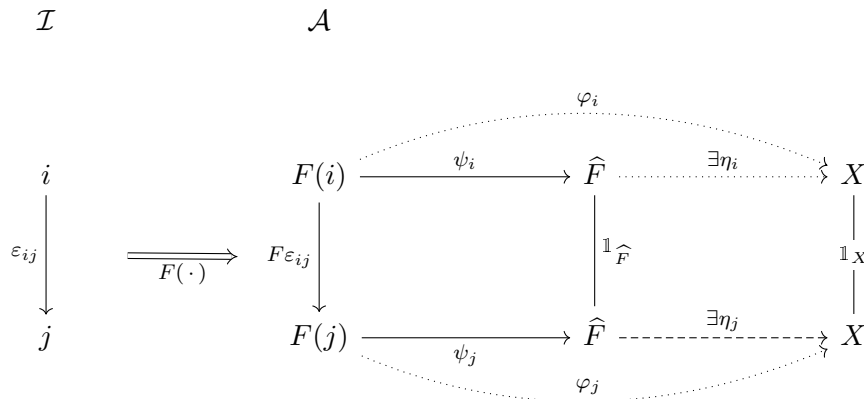
$$\tau_{FX} : \text{Hom}_{\mathcal{A}}(\hat{F}, X) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \quad \forall X \in \mathcal{A}, F \in \mathcal{A}^{\mathcal{I}}.$$

We first note that the data of \hat{F} is equivalent to the following universal property:



[Link to Diagram](#)

That is, $\hat{F} \in \text{Ob}(\mathcal{A})$ is an object equipped with structure maps ψ_i for every object $F(i)$ in the image of F such that the solid triangle commutes, and for any object X with maps $\varphi_i : F(i) \rightarrow X, \varphi_j : F(j) \rightarrow X$ making the outer triangle commute, there is a unique map $\eta_{ij} : \hat{F} \rightarrow X$ making the entire diagram commute. We can rewrite this condition in a more suggestive way:



[Link to Diagram](#)

Applying the Δ functor, we can view this as a simpler universal property in $\mathcal{A}^{\mathcal{I}}$, since the above data is precisely the data of a natural transformation:

$$\begin{array}{ccc}
 & & \Delta F \\
 & \nearrow \Psi & \uparrow \exists! \eta \\
 F & \xrightarrow{\Phi} & \Delta X
 \end{array}$$

[Link to Diagram](#)

That is, the functor $\Delta\hat{F}$ is equipped with structure maps $\Phi : F \rightarrow \Delta\hat{F}$ which assemble into a natural transformation (i.e. a morphism in $\mathcal{A}^{\mathcal{I}}$) such that any other natural transformation from F to a diagonal object ΔX produces a unique natural transformation $\eta : \Delta X \rightarrow \Delta F$. This provides exactly the data needed to specify τ :

$$\begin{aligned}
 \tau_{FX} : \text{Hom}_{\mathcal{A}}(\hat{F}, X) &\xrightarrow{\sim} \text{Hom}_{\mathcal{A}^{\mathcal{I}}}(F, \Delta X) \\
 \left(\hat{F} \xrightarrow{f} X \right) &\mapsto \left(F \xrightarrow{\Psi} \Delta\hat{F} \xrightarrow{\Delta(f)} \Delta X \right),
 \end{aligned}$$

i.e. we take the image $\Delta(f)$ and pre-compose with the structure morphism Ψ .

Claim: This is a bijection of sets.

Proof (?).

This is surjective by the universal property: any morphism $F \xrightarrow{g} \Delta X$ in $\mathcal{A}^{\mathcal{I}}$ factors through $\Delta\hat{F}$, and so all such morphisms are of this form.

That this is injective: ???

■

Claim: This is a natural isomorphism, i.e. for all $X \xrightarrow{f} Y \in \text{Mor}(\mathcal{A})$ and all $F \xrightarrow{\eta} G \in \text{Mor}(\mathcal{A}^{\mathcal{I}})$, there is a commuting diagram

$$\begin{array}{ccccc}
 \text{Hom}(\hat{G}, X) & \xrightarrow{\hat{\eta}_*} & \text{Hom}(\hat{F}, X) & \xrightarrow{f_*} & \text{Hom}(\hat{F}, Y) \\
 \downarrow \tau_{GX} & & \downarrow \tau_{FX} & & \downarrow \tau_{FY} \\
 \text{Hom}(G, \Delta X) & \xrightarrow{\eta_*} & \text{Hom}(F, \Delta X) & \xrightarrow{(\Delta f)_*} & \text{Hom}(F, \Delta Y)
 \end{array}$$

[Link to Diagram](#)

Proof (?).

Todo.

■

Claim: If \mathcal{A} is abelian and \mathcal{I} is an index category such that $\text{colim}_{i \in \mathcal{I}} F(i)$ exists for all $F \in \mathcal{A}^{\mathcal{I}}$, then the functor $\text{colim} : \mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$ is right-exact.

Proof (Sketch).

A sketch of the proof proceeds by showing every right adjoint is left-exact:

- Since $\text{Hom}(LA, \cdot)$ is left-exact, we can apply it to a SES $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$.
- Applying the natural isomorphisms coming from the adjunction, this is isomorphic to a sequence involving terms $\text{Hom}(\cdot, RB)$.
- This sequence is exact, so applying Yoneda yields an exact sequence $0 \rightarrow RB' \rightarrow RB \rightarrow RB''$, making R left-exact.

Finally, pushing a left adjoint L into the opposite category makes L^{op} a right adjoint. Thus L^{op} is left-exact, making L right-exact. ■

Claim: Let $\mathcal{I} := (\bullet \leftarrow \bullet \rightarrow \bullet)$ and define the pushout as $\text{colim} : \mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$. Then taking $\mathcal{A} := \mathbf{Ab}$, the pushout does not define an exact functor $\mathcal{A}^{\mathcal{I}} \rightarrow \mathcal{A}$.