

Title

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For a $R \subset T$ a subring of a ring, the set of intermediate rings is a large/interesting class of rings. Recall: uncountably many rings between \mathbb{Z} and \mathbb{Q} ! Taking R a PID and T its fraction field, a similar result will hold.

Define $I \trianglelefteq R$ as the kernel of a ring morphism. This implies that $I \subset (R, +)$ with the absorption property $RI \subset I$. Conversely, any I satisfying these two properties is the kernel of a ring morphism: namely $R \rightarrow R/I$. This makes sense as a group morphism.

Exercise Define $xy + I = (x + I)(y + I)$, need to check well-definedness. Write out

$$(x + i_1)(y + i_2) = \dots$$

Need to check that

$$i_1y + i_2x + i_1i_2 \in I,$$

but the absorption property does precisely this.

Note that if we were in a non-commutative setting, this would define a left ideal. These don't have to coincide with right ideals – there are rings where the former satisfy properties that the latter does not.

Example: The subrings of $R = \mathbb{Z}$ are of the form $n\mathbb{Z}$ for $n \geq 0$, with the usual quotient.

Definition 1.0.1 (Proper Ideals).

An ideal $I \trianglelefteq R$ is *proper* iff $I \subsetneq R$.

Exercise An ideal I is not proper iff I contains a unit.

Exercise R is a field iff the only ideals are $0, R$.

Definition 1.0.2 (Lattice Structure of Ideals).

Let $\mathcal{I}(R)$ be the set of all ideals in R . What structure does it have? It is partially ordered under

inclusion. It is a complete lattice, i.e. every element has an infimum (GLB) and a supremum (LUB).

Namely, for a family of ideals $\{I_j\}$, the **infimum** is the intersection and **supremum** is the ideal generated by the union.

$$\sum_{i=1}^n r_i y_i \text{ such that } n \in \mathbb{N}_{>0}, r_i \in R, y_i \in \{y_j\}$$

Exercise For $I_1, I_2 \trianglelefteq R$, it is the case that $I_1 + I_2 := \{i_1 + i_2\} = \langle I_1, I_2 \rangle$.

Theorem 1.1 (Lattice Isomorphism Theorem for Rings).

Let $I \trianglelefteq R$ and $\phi : R \rightarrow R/I$, and define $\ell(I) = \{I \subset J \trianglelefteq R\}$. Then we can define maps

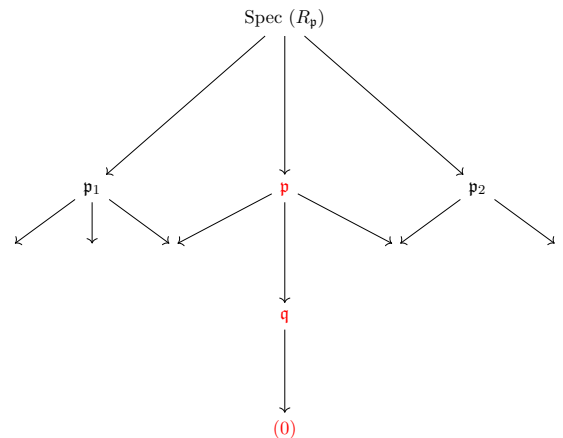
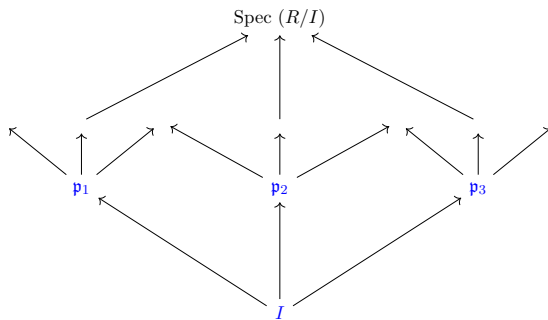
$$\begin{aligned} \Phi : \ell(R) &\longrightarrow \ell(R/I) \\ J &\mapsto \frac{I + J}{I}, \end{aligned}$$

and

$$\begin{aligned} \Psi : \ell(R/I) &\longrightarrow \ell(R) \\ J/I &\mapsto \phi^{-1}(J). \end{aligned}$$

We can check that $\Psi \circ \Phi(J) = I + J$, and $\Phi \circ \Psi(J/I) = J/I$. So Ψ has a left inverse and is thus injective. Its image is the collection of ideals that contain I , and $\Psi : \ell(R/I) \rightarrow \ell_I(R)$ is a **bijection** and is in fact a lattice isomorphism with $\ell_I(R) \subset \ell(R)$.

Note that this gives us everything above a given ideal in the ideal lattice (blue); the dual notion will come from localization (red):



Here we take \mathfrak{p} to be a maximal ideal.

Remarks The ideal lattice $\ell(R)$ is

- A complete lattice under subset inclusion,

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- A commutative monoid under addition
 - A commutative monoid under *multiplication*, which we'll define.

Definition 1.1.1 (Product of Ideals).

For $I, J \trianglelefteq R$, we define

$$IJ = \langle ij \mid i \in I, j \in J \rangle.$$

Note that we have to take the ideal generated by products here.

For $\langle x \rangle = (x)$ a principal ideal and $\langle y \rangle$ principal, we do have $(x)(y) = (xy)$. Note that $IJ \subset I \cap J$, whereas the sum was larger than I, J .

Exercise Note that $(\ell(R), \cdot)$ has an absorbing element, namely $(0)I = (0)$. For $(M, +)$ a commutative monoid and $M \hookrightarrow G$ a group, then multiplication by x is injective and so for all $y \in M$, $xz = yz \implies x = y$, so M is cancellative.

Question: what if we consider $\mathcal{I}^\bullet(R)$ the set of nonzero ideals of R . Does this help? We will see next time.