Homological Algebra Problem Sets

Problem Set 1

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1 | Weibel 1.1.2

Problem 1.0.1 (Weibel 1.1.2)

Show that a morphism $u: C \to D$ of chain complexes preserves boundaries and cycles respectively, hence inducing a map $H_n(C) \to H_n(D)$ for each n. Prove that $H_n: \operatorname{Ch}(R\operatorname{-mod}) \to R\operatorname{-mod}$ is a functor.

Solution:

Claim 1: The chain map u induces the following well-defined maps:

$$Z_n(u): Z_n(C) \to Z_n(D)$$

$$B_n(u): B_n(C) \to B_n(D).$$

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Proof (of claim (1)).

We'll use the convention that $Z_n := \ker d_n$ and $B_n := \operatorname{im} d_{n+1}$ where we index chain complexes as $C = \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots \right)$. Unraveling definitions, we would like to show the existence of maps

$$Z_n(u) : \ker d_n^C \to \ker d_n^D$$

 $B_n(u) : \operatorname{im} d_{n+1}^C \to \operatorname{im} d_{n+1}^D.$

It suffices to show

a.
$$x \in \ker d_n^C \Longrightarrow u_n(x) \in \ker d_n^D$$
, and
b. $y \in \operatorname{im} d_{n+1}^C \Longrightarrow u_n(y) \in \operatorname{im} d_{n+1}^D$.

Since u is a morphism of chain complexes, we have a commuting ladder where $u_{n-1} \circ d_n^C = d_n^D \circ u_n$:

Link to Diagram

To see that (a) holds, we use that fact that R-module morphisms send $0_R \to 0_R$ (using R-linearity) to compute

$$x \in \ker d_n^C \qquad \leq C_n$$

$$\iff d_n^C(x) = 0_R \qquad \in C_{n-1}$$

$$\iff (u_{n-1} \circ d_n^C)(x) = 0_R \qquad \in D_{n-1} \qquad \text{since } u_n \text{ is } R\text{-linear}$$

$$\iff (d_n^D \circ u_n)(x) = 0_R \qquad \in D_{n-1} \qquad \text{commutativity}$$

$$\iff x \in \ker(d_n^D \circ u_n) \qquad \leq D_{n-1}$$

$$\iff u_n(x) \in \ker d_n^D \qquad \leq D_n.$$

Similarly, for (b) we have

$$y \in \operatorname{im} d_{n+1}^{C} \iff \exists x \in C_{n+1} \text{ such that } d_{n+1}^{C}(x) = y$$

$$\implies u_{n+1}(x) \in D_{n+1}$$

$$\implies (d_{n+1}^{D} \circ u_{n+1})(x) \in \operatorname{im} d_{n+1}^{D} \leq D_{n}$$

$$\implies (u_{n} \circ d_{n+1}^{C})(x) \in \operatorname{im} d_{n+1}^{D} \leq D_{n} \qquad \text{commutativity}$$

$$\iff u_{n}(y) \in \operatorname{im} d_{n+1}^{D} \qquad \text{using } d_{n+1}^{C}(x) = y.$$

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2 | Weibel 1.1.4

Problem 2.0.1 (Weibel 1.1.4)

Show that for every $A \in R$ -mod and $C \in Ch(R\text{-mod})$ that $D := \operatorname{Hom}_{R\text{-mod}}(A, C)$ is a chain complex of abelian groups. Taking $A := Z_n$, show that $H_n(D) = 0 \implies H_n(C) = 0$. Is the converse true?

Solution:

We first show that if $A \in R$ -mod and $C \in Ch(R$ -mod), then

$$D_n := \operatorname{Hom}_{R\text{-}\mathrm{mod}}(A, C_n).$$

defines a chain complex of abelian groups. Fixing notation, we write

$$C \coloneqq (\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots).$$

1. D_n is an abelian group for all n: Define an operation

$$(f,g) \mapsto \begin{cases} f+g: A \to C_n \\ x \mapsto f(x) +_C g(x) \end{cases},$$

where $+_C$ is the addition on C_n provided by its structure as an R-module. We can then check that this operation is commutative:

$$(f +_D g)(x) \coloneqq f(x) +_C g(x)$$

= $g(x) +_C f(x)$ since the addition on C_n is commutative
= $(g +_D f)(x)$,

The additive inverse of f is -f, there is an identity function $\mathrm{id}_{C_n}(x) = x$, and the sum of two functions $A \to C_n$ is again a function $A \to C_n$, making D_n an abelian group for all n.

2. There exist differentials $D_n \xrightarrow{d_n^D} D_{n-1}$: Noting that we have differentials $C_n \xrightarrow{d_n^C} C_{n-1}$, we can define

$$d_n^D: D_n \to D_{n-1}$$

$$(A \xrightarrow{f} C_n) \mapsto (A \xrightarrow{f} C_n \xrightarrow{d_n^C} C_{n-1}),$$

i.e. we send $f \mapsto d_n^C \circ f$ be precomposing with the differential from C_* .

3. $(d^D)^2 = 0$: We can explicitly write

$$(d^{D})^{2}: D_{n} \to D_{n-2}$$

$$(A \xrightarrow{f} C_{n}) \mapsto (A \xrightarrow{f} C_{n} \xrightarrow{d_{n}^{C}} C_{n-1} \xrightarrow{d_{n-1}^{C}} C_{n-2}),$$

and so $f \mapsto d_{n-1}^C \circ d_n^C \circ f$. The claim is that this is the zero map, which follows from writing this as $(d^C)^2 \circ f = 0 \circ f = 0$, using that C_* is a chain complex.

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Thus

$$D := (\cdots \to D_{n+1} \xrightarrow{d_{n+1}^D} D_n \xrightarrow{d_n^D} D_{n-1} \to \cdots) \in \operatorname{Ch}(\operatorname{Ab}).$$

Writing $Z_n = Z_n(C) = \ker d_n^C$, we now show the following:

Claim:

$$H_n(\operatorname{Hom}_{R\operatorname{-mod}}(Z_n,C)=0 \implies H_n(C)=0.$$

It suffices to show that $\ker d_n^C \subseteq \operatorname{im} d_{n+1}^C$, so let $y \in \ker d_n^C$; we want to produce the following:

$$x \in C_{n+1}, \quad d_{n+1}^C(x) = y.$$

We can start with the inclusion map

$$\iota : \ker d_n^C \hookrightarrow C_n,$$

which by definition is an element of $D_n := \operatorname{Hom}_{R\text{-}\mathrm{mod}}(Z_n, C_n)$. By assumption, the following complex is exact at n since its homology vanishes at position n:

$$(\cdots \to D_{n+1} \to D_n \to D_{n-1} \to \cdots) \coloneqq$$

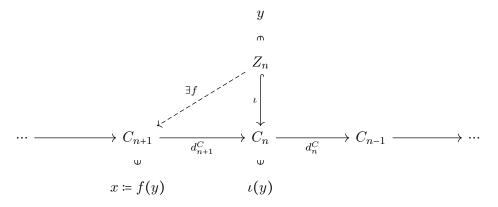
$$\cdots \to \operatorname{Hom}_R(Z_n, C_{n+1}) \xrightarrow{d_{n+1}^D} \operatorname{Hom}_R(Z_n, C_n) \xrightarrow{d_n^D} \operatorname{Hom}_R(Z_n, C_{n-1}) \to \cdots$$

Claim: $d_n^D(\iota) = 0$.

This can be seen by writing this out as the composition

$$d_n^D(\ker d_n^C \xrightarrow{\iota} C_n) = (\ker d_n^C \xrightarrow{\iota} C_n \xrightarrow{d_n^C} C_{n-1}).$$

We can now use the general fact that the $f(\ker f) = 0$ for any map f, i.e. the image of the kernel is necessarily zero. Taking $f = d_n^C$ shows that this composition is zero. By exactness, $\ker d_n^D = \operatorname{im} d_{n+1}^D$ and we can thus pull ι back to some $f \in D_{n+1} := \operatorname{Hom}_R(Z_n, C_{n+1})$, and since our original $y \in \ker d_n^C := Z_n$, it makes sense to consider $x := f(y) \in C_{n+1}$ and to identity $y = \iota(y) \in C_n$:



Link to Diagram

Importantly, this f satisfies $\iota = d_{n+1}^D(f) = d_{n+1}^C \circ f$, and so we can write

$$y = \iota(y) = (d_{n+1}^C \circ f)(y) \coloneqq d_{n+1}(x),$$

which is what we wanted to show.

Weibel 1.1.4

$\mathbf{3}$ | Weibel 1.1.6

Problem 3.0.1 (Weibel 1.1.6: Homology of a graph)

Let Γ be a finite graph with vertices $V = \{v_1, \dots, v_V\}$ and edge $E = \{e_1, \dots, e_E\}$. Define the **incidence matrix** of Γ to be the $V \times E$ matrix A where

$$A_{ij} = \begin{cases} 1 & e_j \text{ starts at } v_i, \\ -1 & e_j \text{ ends at } v_i, \\ 0 & \text{else.} \end{cases}$$

Define a chain complex by taking free R-modules:

$$C \coloneqq (\cdots \to 0 \to C_1 \to C_0 \to 0 \to \cdots) = (\cdots \to 0 \to R^E \xrightarrow{A} R^V \to 0 \to \cdots).$$

If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and E-V+1 respectively.

Hint: choose a basis $\{v_1, v_2 - v_1, \dots, v_V - v_1\}$ and use a path from $v_1 \rightsquigarrow v_i$ to produce an element $e \in C_1$ with $d(e) = v_i - v_1$.

Solution:

We first make the following two observations:

1.
$$H_0(C) = \operatorname{coker}(A) \cong R^V / \operatorname{im} A \Longrightarrow \operatorname{rank} H_0(C) = V - \operatorname{rank} \operatorname{im} A$$
, and

2.
$$H_1(C) = \ker(A) \implies \operatorname{rank} H_1(C) = \operatorname{rank} \ker A$$

Claim: $\operatorname{rankim}(A) = V - 1$.

Given this claim, applying observation (1) we immediately obtain

$$rank H_0(C) = V - (V - 1) = 1,$$

which is the first equality we want to show. For the second equality, we can use the first isomorphism theorem to get a SES of free R-modules

$$0 \to \ker(A) \hookrightarrow R^E \to \operatorname{im}(A) \to 0,$$

and since $\operatorname{im}(A)$ is free and thus projective, this sequence splits. So $R^E \cong \ker(A) \oplus \operatorname{im}(A)$, and taking free ranks yields

$$E = \operatorname{rank} \ker(A) + (V - 1) \implies \operatorname{rank} \ker(A) = E - V + 1,$$

and this yields the second equality by using observation (2) to identify the LHS with rank $H_1(C)$.

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Proof (of claim). Using the fact that

$$\mathcal{B} \coloneqq \{v_1, \cdots, v_V\}$$

is a basis for \mathbb{R}^V as a free \mathbb{R} -module, we can make a change of basis to

$$\mathcal{B}' \coloneqq \{v_1, v_2 - v_1, \dots, v_V - v_1\}.$$

That this is again a basis follows from the fact that the change-of-basis matrix M is upper-triangular with ones on the diagonal and thus satisfies $\det M = 1_R \in R^{\times}$ (i.e. it's a unit), so M is nonsingular. We can then observe that if e_i is an edge between two vertices $v_{i_1} \xrightarrow{e_i} v_{i_2}$, then $d(e_i) := Ae_i = v_{i_2} - v_{i_2}$. By linearity, if e_{i_1}, \dots, e_{i_n} is a sequence of edges connecting v_1 to v_i for any $1 \le j \le V$, then

$$d(e_{i_1} + \dots + e_{i_n}) = v_j - v_1.$$

Since Γ is connected, there always exists such a sequence of edges connecting each v_j to v_1 , and thus $v_j - v_1$ is in im(A). We can conclude that

$$V-1 \le \operatorname{rankim}(A) \le V$$
.

To see that rank im(A) $\neq V$, note that if e is any sequence of edges connecting v_1 to itself in a loop, then $d(e_1) = v_1 - v_1 = 0$. Any other path e' must necessarily start or end at some $v_i \neq v_1$ and satisfies $d(e') = v_i - v_1 \neq v_1$, and so $v_1 \notin \text{im}(A)$. Thus

$$\operatorname{rankim}(A) = V - 1.$$

4 Weibel 1.2.3

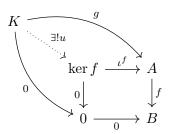
Problem 4.0.1 (Weibel 1.2.3)

Let \mathcal{A} be the category Ch(R-mod) and let f be a chain map. Show that the complex ker f is a (categorical) kernel of f and that coker f is a (categorical) cokernel of f.

Solution:

For a fixed map $f: A \to B$, the *kernel* of f is an object ker f satisfying the following universal property: for any object K with a morphism $K \xrightarrow{g} A$ making the following outer square commute, there is a unique morphism $u: K \to \ker f$ making the entire diagram commute:

Weibel 1.2.3



We'll use without proof that kernels exist in $\mathcal{A} = R$ -mod and are given by $\ker f := \{a \in A \mid f(a) = 0_B\}$ along with an inclusion map $\iota^f : \ker f \to A$.

Let $A, B \in Ch(A)$ be chain complexes and $f : A \to B$ be a chain map. We will construct ker f as a chain complex and show it satisfies the correct universal property.

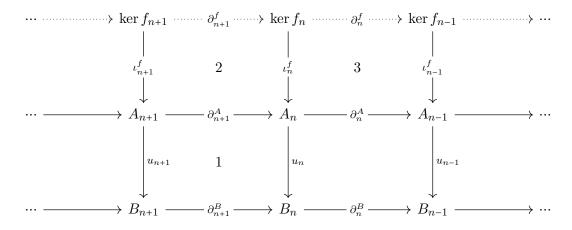
Claim 1: There are unique objects $\ker f_n \in R$ -mod which can be assembled into a unique chain complex $(\ker f, \partial^f)$.

Proof of Claim 1:

Let $u: A \to B$ be a chain map, so that we have a commuting diagram of the following form:

Link to Diagram

Appealing to the universal property of kernels in R-mod, we can produce unique objects $\ker f_n$ and morphisms $\iota_n^f : \ker f_n \to A_n$ satisfying $(\ker f_n \to A_n \to B_n) = 0$ for every n. We also claim that there are maps $\partial_n^f : \ker f_n \to \ker f_{n-1}$, yielding the following diagram:



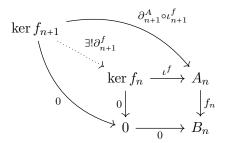
Link to Diagram

Why the ∂_n^f exist: this follows from the universal property of kernels in \mathcal{A} : Using the commutativity of square 1 we have

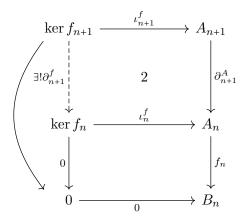
$$0 = (\ker f_{n+1} \to A_{n+1} \to B_{n+1} \to B_n) = (\ker f_{n+1} \to A_{n+1} \to A_n \to B_n),$$

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where we've also used the fact that (ker $f_{n+1} \to A_{n+1} \to B_{n+1} = 0$) from the universal property of ker f_{n+1} . So we can fit these into an appropriate diagram in \mathcal{A} , which supplies these differentials:



Why the ι^f : ker $f \to A$ assemble into a chain map: Note that everything here commutes, and we can break the northeast corner of this diagram up and rearrange things slightly to form the following diagram:



Link to Diagram

Here, square 2 is precisely the square 2 appearing in the original diagram, and commutativity of it for each n is precisely what is required for ι^f to be a chain map.

Why $(\partial^f)^2 = 0$: Using the commutativity of square 3 and the fact that $(\partial^A)^2 = 0$, we have

$$\iota_{n-1}^{f} \circ (\partial^{f})^{2} \coloneqq (\ker f_{n+1} \to \ker f_{n} \to \ker f_{n-1} \to A_{n-1})$$

$$= (\ker f_{n+1} \to A_{n+1} \to A_{n} \to A_{n-1})$$

$$\coloneqq \iota_{n+1}^{f} \circ (\partial^{A})^{2}$$

$$= 0,$$

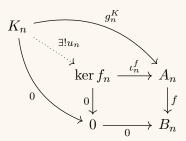
and since ι_{n-1}^f is not the zero map, this forces $(\partial^f)^2=0$.

Claim 2: The complex ker f satisfies the universal property of kernels in $Ch(\mathcal{A})$, i.e. if $g^K: K \to A$ is a chain map satisfying $K \to A \to B = 0$, there is a unique chain map $u: K \to \ker f$ making the appropriate diagram commute.

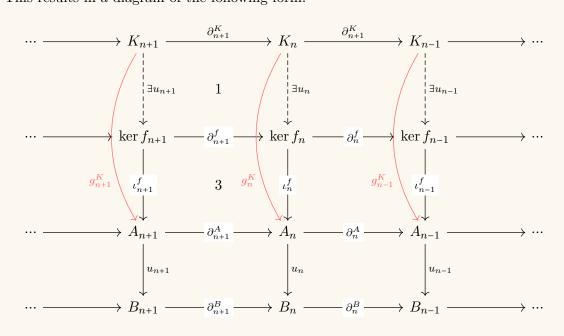
Weibel 1.2.3

Proof(?).

Again using the universal property of kernels in R-mod, for each n we have a commutative diagram



This results in a diagram of the following form:



Link to Diagram

It only remains to check that the u_n assemble to a chain map $K \to \ker f$, which would follow from the commutativity of e.g. square (1). However, if (1) were *not* commutative, then the rectangle formed by (1) and (3) together would not be commutative – but g^K was assumed to be a chain map, so this rectangle commutes, yielding a contradiction.

Note: a proof of a similar flavor seems to work for the cokernel complex by reversing all of the arrows.

Weibel 1.2.3

5 | Exactness in the Snake Lemma

Problem 5.0.1 (?)

Verify exactness in the Snake Lemma in at least two other positions.

Solution:

This follows from the construction of the complex ker f above, specifically using the fact that the constructed differential ∂^f satisfies $(\partial^f)^2 = 0$.