Title

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Friday 21st August, 2020

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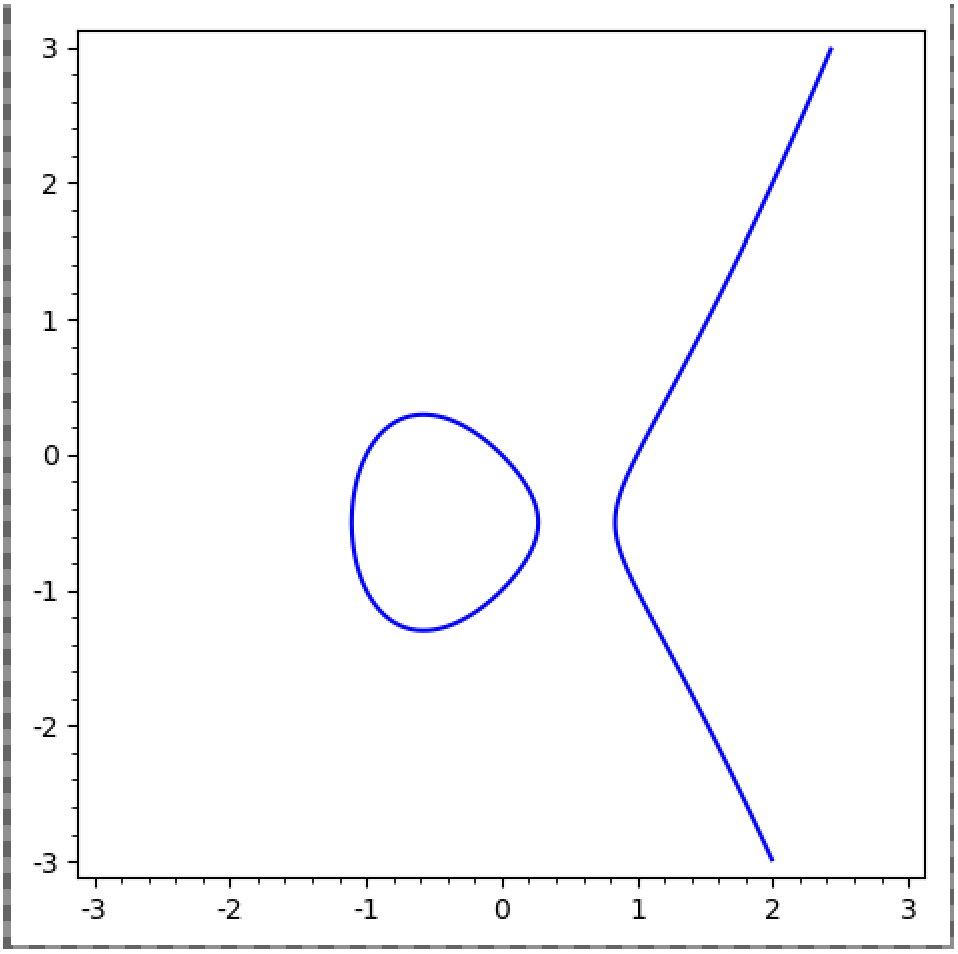
> Reference:
> `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

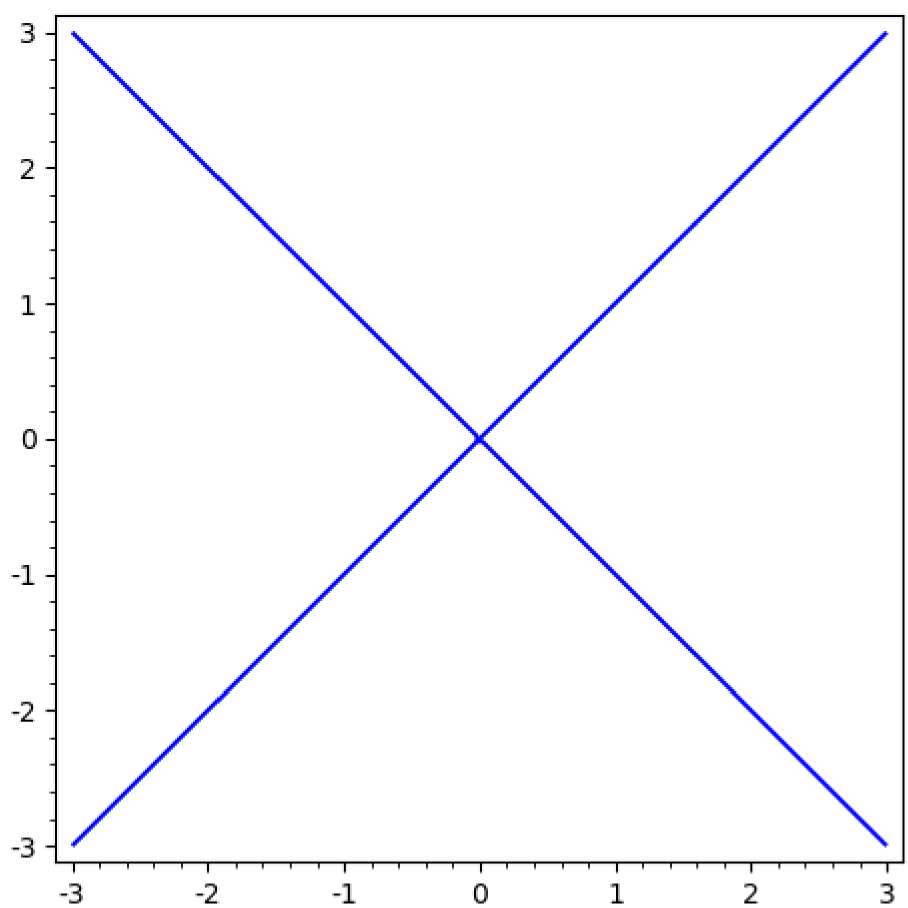
General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)
- $x^2 + 1$, which has no $\mathbb{R}$-points.
- $x^2 + y^2 + 1/\mathbb{R}$has vanishes nowhere, so ring of functions is not $\mathbb{R}[x, y]/\left\langle x^2 + y^2 + 1 \right\rangle$ (problem: $\mathbb{R}$ is not algebraically closed)
- $x^2 - y^2 = 0$ over $\mathbb{C}$ is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$, which has 3 points over $\mathbb{F}_3^2$, but $f(x,y) = (x^3 - x)(y^3 - y)$ vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in $\mathbb{C}$?)
  - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x,y]/\langle f \rangle$ (addressed by scheme theory)

Theorem 1.1 ***(Harnack Curve Theorem).***
If $f \in \mathbb{R}[x,y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.
Take the curve

$$X = \left\{ (x,y,z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.1.
Show that the vanishing locus of the first two equations above is $X \bigcup L$ for L a line.

Compare to linear algebra: codimension d iff cut out by exactly d equations.

Example 1.3.
Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

how to visualize the solution set?

Fact: on $\mathbb{C}$ with some slits, you can consistently choose a square root