Title

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Note: the sheaf of locally constant functions valued in a set S is written $\underline{\mathbf{S}}$.

1.1 Gathmann Chapter 4

Definition 1.0.1 (Ringed Spaces)**.**
A **ringed space** is a topological space X together with a sheaf $\mathcal{O}_X$ of rings.

Example 1.1.

1. X an affine variety and $\mathcal{O}_X$ its ring of regular functions.
2. X a manifold over $\mathbb{R}^n$ with $\mathcal{O}_X$ a ring of smooth or continuous functions on X.
3. $X = \{p, q\}$ with the discrete topology and $\mathcal{O}_X$ given by $p \mapsto R, q \mapsto S$.
4. Let $U \subset X$ an open subset of X an affine variety. Then declare $\mathcal{O}_U$ to be $OO_X|_U$.

Recall that the restriction of a sheaf $\mathcal{F}$ to an open subset $U \subset X$ is defined by $\mathcal{F}|_U(V) = \mathcal{F}(V)$.

Example 1.2.
Let X be a topological space and $p \in X$ a point. The *skyscraper sheaf at p* is defined by

$$K_p(U) := \begin{cases} K & p \in U \\ 0 & p \notin U \end{cases}.$$

Convention: we'll always assume that $\mathcal{O}_X$ is a sheaf of functions, so $\mathcal{O}_X(U)$ is a subring of all K-valued functions on U. Moreover, Res_{UV} is restriction of K-valued functions.

Definition 1.0.2 (Morphisms)**.**
A *morphism of ringed spaces*

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $X \to Y$ such that for all opens $U \subset Y$ and any $\varphi \in \mathcal{O}_Y(U)$, the pullback satisfies $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$, i.e. the pullback of a regular function is regular.

Note: need convention that $\mathcal{O}_X$ is a sheaf of K-valued functions in order to make sense of pullbacks. In general, for schemes, need some analog of $f^* : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Example 1.3.
If $(X, \mathcal{O}_X)$ is a ringed space associated to an affine variety, ?

Example 1.4.
Let $X = \mathbb{A}^1/K$ and $U = D(f)$ for $f(x) = x$, then $D(f) = \mathbb{A}^1 \setminus \{0\}$. Then $U \hookrightarrow X$ is continuous. Given an open set $D(f) \subset \mathbb{A}^1$, we have

$$\mathcal{O}_{\mathbb{A}^1}(D(f)) \coloneqq \left\{ g/f^n \;\middle|\; g \in K[x] \right\}.$$

We want to show that $\iota : (U, \mathcal{O}_U) \hookrightarrow (X, \mathcal{O}_X)$ is a morphism of ringed spaces where $\mathcal{O}_U(V) = \mathcal{O}_X(V)$. Does ι^* pull back regular functions to regular functions? Yes, since $\iota^{-1}(D(f)) = D(xf)$ and $g/f^n \in \mathcal{O}_U(\iota^{-1}(D(f)))$.

Example 1.5.
A non-example: take

$$\begin{aligned} h : \mathbb{A}^1 &\to \mathbb{A}^1 \\ x &\mapsto \begin{cases} x & x \neq \pm 1 \\ -x & x = \pm 1 \end{cases}. \end{aligned}$$

This is continuous because the zariski topology on $\mathbb{A}^1$ is the cofinite topology (since the closed sets are finite), so any injective map is continuous since inverse images of cofinite sets are again cofinite.

Question: Does h define a morphism of ringed spaces? I.e., is the pullback of a regular function on an open still regular? Take $U = \mathbb{A}^1$ and the regular function $x \in \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1)$. Then $h^*x = x \circ h$, so

$$(x \circ h)(p) = \begin{cases} p & p \neq \pm 1 \\ -p & p = \pm 1 \end{cases} \notin K[x]$$

since this is clearly not a polynomial: if two polynomials agree on an infinite set of points, they are equal.

Example 1.6.
Consider $\iota : (\mathbb{R}^2, C^\infty) \hookrightarrow (\mathbb{R}^3, C^\infty)$ is the inclusion of a coordinate hyperplane. To say that this is a morphism of ringed spaces, we need that for all $U \subset \mathbb{R}^3$ open and $f : U \to \mathbb{R}$ a smooth function, we want $i^* f \in C^\infty(\iota^{-1}(U))$. But this is the same as $f \circ \iota \in C^\infty(\mathbb{R}^2 \cap U)$, which is true.

Proposition 1.1 ***(Properties of Morphisms of Ringed Spaces).***

1. They can be composed.