Title

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Contents

1 | Tuesday, December 01

Last time: we started discussing smoothness.

Definition 1.0.1 (Tangent Space)
The **tangent space** T_pX of a variety X at a point $p \in X$ is defined as

$$V\left(\left\{f_1 \;\middle|\; f \in I(U_i),\, U_i \ni p = 0 \text{ affine } \right\}\right)$$

where f_1 denotes the degree 1 part.

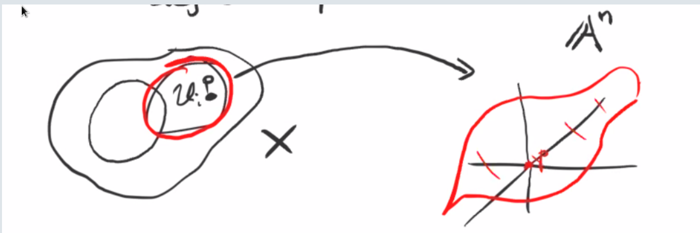

Figure 1: Image

Remark 1.0.2: We've really only defined it for affine varieties and $p = 0$, but this is a local definition. Note that this is also not a canonical definition, since it depends on the affine chart U_i.

Example 1.0.3 ***(?):*** Consider $T_0V(xy) = V(f_1 \mid f \in \langle xy \rangle) = V(0) = \mathbb{A}^2$, since every polynomial in this ideal has degree at least 2. Letting $X = V(xy)$, note that we could embed $X \hookrightarrow \mathbb{A}^3$ as $X \cong V(xy, z)$. In this case we have $T_0X = V(f_1 \mid f \in \langle xy, z \rangle) = V(z) \cong \mathbb{A}^2$. So we get a vector space of a different dimension from this different affine embedding, but $\dim T_0X$ is the same.

Example 1.0.4 ***(?):*** Let $X = V_p(xy - z^2) \subset \mathbb{P}^2$, which is a projective curve. What is T_pX for $p = [0:1:0]$? Take an affine chart $\{y \neq 0\} \cap X$, noting that $\{y \neq 0\} \cong \mathbb{A}^2$. We could dehomogenize the ideal $\left\langle xy - z^2 \right\rangle\Big|_{y=1} = \left\langle x - z^2 \right\rangle$. Thus $X \cap D(y) = V(x - z^2) \subset \mathbb{A}^2$ and the point $[0:1:0] \in X$ gives $(0,0)$ in this affine chart. Then $T_pX = V(f_1 \mid f \in \left\langle x - z^2 \right\rangle) = V(x)$. Then $f = (x - z^2)g$ implies that $f_1 = (xg)_1 = g_0 x$, the constant term of g multiplied by x, since z^2 kills any degree 1 part of g. So T_pX is a line.

Example 1.0.5 ***(?):*** Take X to be the union of the coordinate axes in $\mathbb{A}^3$.

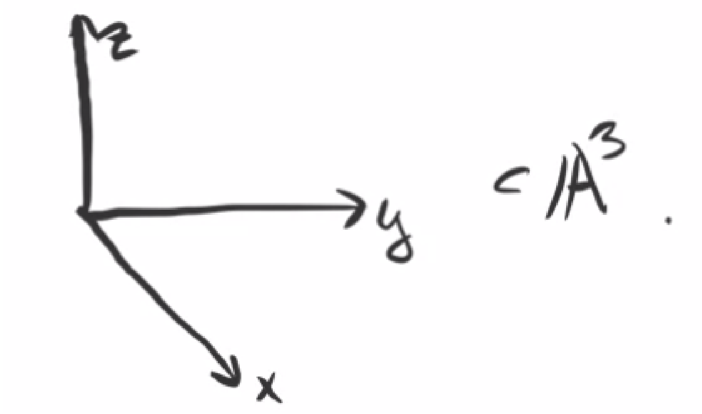

Figure 2: Image

Then $I(X) = \langle xy, yz, xz \rangle$ and $T_0 X = V(f_1 \mid f \in I(X)) = V(0) = \mathbb{A}^3$, since the minimal degree of any such polynomial is 2. Note that $\dim X = 1$ but $\dim T_0 X = 3$

Example 1.0.6 ***(?):*** Take $Y = V(xy(x-y)) \subset \mathbb{A}^2$. Then $T_0 X = V(0) = \mathbb{A}^2$:

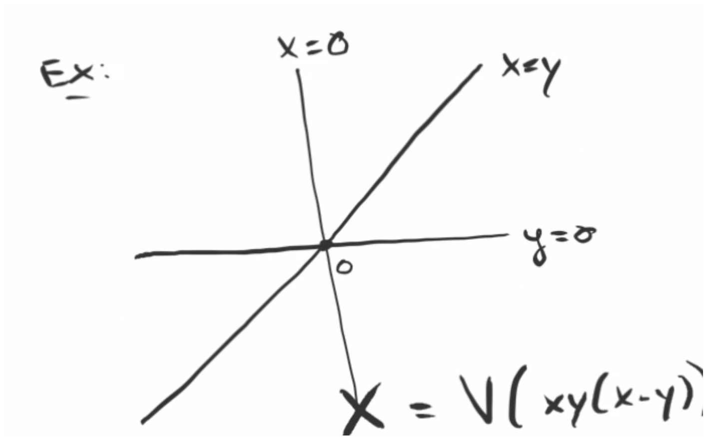

Figure 3: Image

Remark 1.0.7: Note that X and Y both consists of 3 copies of $\mathbb{A}^1$ intersecting at a single point.

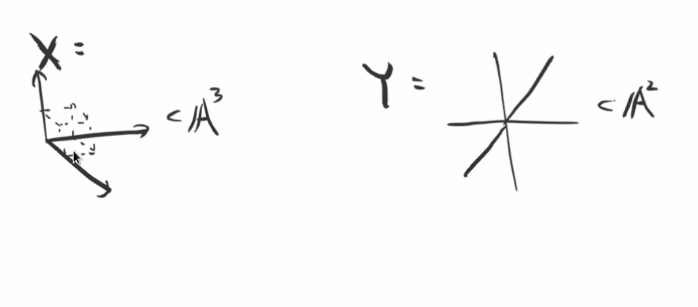

Figure 4: Image

Note that $\dim T_0 X = 3$ but $\dim T_0 Y = 3$, and interestingly $X \ncong Y$ as affine varieties. There is a bijective morphism that is not invertible.

Remark 1.0.8: We will prove that $\dim T_p X$ is invariant under choice of affine embedding.

Example 1.0.9 ***(?):*** How to compute $T_{(1,0,0)} V(xy, yz, xz)$: first move $(1,0,0)$ to the origin, yielding $T_{(0,0,0)} V((x+1)y, yz, (x+1)z)$. This is a different choice of affine embedding into $\mathbb{A}^3$ which sends $(1,0,0) \mapsto (0,0,0)$. Taking the vanishing locus of linear parts, it suffices to take the linear parts of the generators, which yields the x-axis $V(y,z)$, making the dimension of the tangent space 1.

Lemma 1.0.10 ***(?).***
Let $X \subset \mathbb{A}^n$ be an affine variety and let $0 = p \in X$. Then

$$T_0(X)^\vee := \hom_k(T_0 X, k) \cong I_X(p)/I_X(p)^2$$

Remark 1.0.11: Note that the hom involves an affine embedding, but the quotient of ideals does not. We know that the category of affine varieties is equivalent to the category of reduced k-algebras, since the points of X biject with the maximal ideals of the coordinate ring $A(X)$. $I_X(p)$ is the maximal ideal in $A(X)$ of regular functions vanishing at p.

Proof (?).
Consider the map

$$\begin{aligned}\varphi : I_X(p) &\to T_0(X)^\vee \\ \overline{f} &\mapsto f_1|_{T_0(X)}.\end{aligned}$$

E.g. given $\bar{x}_1 - \bar{x}_2^2 \in A(X)$, we first lift to $x_1 - x_2^2 \in A(\mathbb{A}^n)$, restrict to the linear part x_1, then restrict to $T_0(X)$. Note that $I_X(p) = \langle \bar{x}_1, \cdots, \bar{x}_n \rangle \in k[x_1, \cdots, x_n]/I(X)$, and we need to check that this well-defined since there is ambiguity in choosing the above lift.

Claim: φ is well-defined.
Consider two lifts f, f' of $\overline{f} \in A(X) = k[x_1, \cdots, x_n]/I(X)$. Then $f - f' \in I(X)$, so $(f - f')_1 = f_1 - f_1'$ is the linear part of some element in $I(X)$. The definition of $T_0(X)$ was the vanishing locus of linear parts of elements in $I(X)$, which contains $f_1 - f_1'$, and thus $(f_1 - f_1')|_{T_0(X)} = 0$. So $f_1 = f_1'$ on $T_0(X)$.

Claim: $I_X(p)^2 \to 0$.

We know $I_X(p) = \langle \bar{x}_1, \cdots, \bar{x}_n \rangle$, and so $I_X(p)^2 = \langle \bar{x}_i \bar{x}_j \rangle$. Giving any $\overline{f} \in I_X(p)^2$, we can lift this to some $f \in \langle x_i x_j \rangle$, in which case $f_1 = 0$.
So φ descends to

$$\overline{\varphi} : I_X(p)/I_X(p^2) \to T_0(X)^\vee$$

Claim: φ is injective and surjective.

That $\overline{\varphi}$ is surjective follows from the fact that if $\bar{x}_1, \cdots, \bar{x}_n \in I_X(p)$, then the restrictions $x_1|_{T_0 X}, \cdots, x_n|_{T_0 X}$ are in $\operatorname{im} \overline{\varphi}$ These elements generate $T_0(X)^\vee$, since $T_0(X) \subset \mathbb{A}^n$. For injectivity, suppose $\overline{\varphi}(\overline{f}) = 0$, then $f_1|_{T_0(X)} = 0$, so f_1 is the linear part of some $f' \in I(X)$. Then $f' \in I(X)$ and f, f' have the same linear part f_1, and $f - f'$ has no linear part. Thus $f - f' \in \langle x_i x_j \rangle$, which implies that $\overline{f} - \overline{f}' \in I_P(X)^2$ and $\overline{f} \equiv \overline{f}' \in I_p(X)/I_p(X)^2$. But $f' \equiv 0$ since $f' \in I(X)$.

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Remark 1.0.12: So for X an affine variety, the cotangent space has a more intrinsic description, and we can recover the tangent space by dualizing:

$$T_p(X) := \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^\vee$$

where $\mathfrak{m}_p = I_X(p)$ is the maximal ideal of regular functions vanishing at p. So how can we get rid of the word affine? Given X any variety, we can define $T_p(X) := \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$. This allows us to work on affine patches and localize. Moreover, this will be left invariant under the localization.