

# Problem Set 8

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## 1 Problem 1

### 1.1 Part a

It follows from the definition that  $\|f\|_\infty = 0 \iff f = 0$  almost everywhere, and if  $\|f\|_\infty$  is the best upper bound for  $f$  almost everywhere, then  $\|cf\|_\infty$  is the best upper bound for  $cf$  almost everywhere.

So it remains to show the triangle inequality. Suppose that  $|f(x)| \leq \|f\|_\infty$  a.e. and  $|g(x)| \leq \|g\|_\infty$  a.e., then by the triangle inequality for the  $|\cdot|_{\mathbb{R}}$  we have

$$\begin{aligned} |(f+g)(x)| &\leq |f(x)| + |g(x)| \quad a.e. \\ &\leq \|f\|_\infty + \|g\|_\infty \quad a.e., \end{aligned}$$

which means that  $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$  as desired.

## 1.2 Part b

$\Rightarrow$  : Suppose  $\|f_n - f\|_\infty \rightarrow 0$ , then for every  $\varepsilon$ ,  $N_\varepsilon$  can be chosen large enough such that  $|f_n(x) - f(x)| < \varepsilon$  a.e., which precisely means that there exist sets  $E_\varepsilon$  such that  $x \in E_\varepsilon \Rightarrow |f_n(x) - f(x)| < \varepsilon$  and  $m(E_\varepsilon^c) = 0$ .

But then taking the sequence  $\varepsilon_n := \frac{1}{n} \rightarrow 0$ , we have  $f_n \Rightarrow f$  uniformly on  $E := \bigcap_n E_n$  by definition, and  $E^c = \bigcup_n E_n^c$  is still a null set.

$\Leftarrow$  : Suppose  $f_n \Rightarrow f$  uniformly on some set  $E$  and  $m(E^c) = 0$ . Then for any  $\varepsilon$ , we can choose  $N$  large enough such that  $|f_n(x) - f(x)| < \varepsilon$  on  $E$ ; but then  $\varepsilon$  is an upper bound for  $f_n - f$  almost everywhere, so  $\|f_n - f\|_\infty < \varepsilon \rightarrow 0$ .

## 1.3 Part c

To see that simple functions are dense in  $L^\infty(X)$ , we can use the fact that  $f \in L^\infty(X) \iff$  there exists a  $g$  such that  $f = g$  a.e. and  $g$  is bounded.

Then there is a sequence  $s_n$  of simple functions such that  $\|s_n - g\|_\infty \rightarrow 0$ , which follows from a proof in Folland:

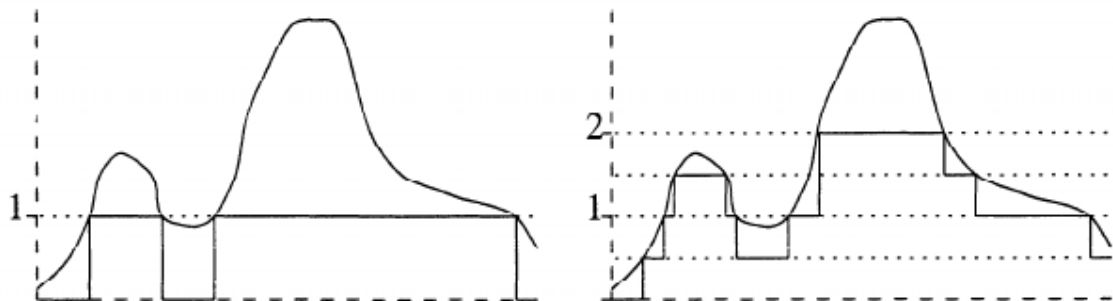
*Proof.* (a) For  $n = 0, 1, 2, \dots$  and  $0 \leq k \leq 2^{2^n} - 1$ , let

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \text{and} \quad F_n = f^{-1}((2^n, \infty]),$$

and define

$$\phi_n = \sum_{k=0}^{2^{2^n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}.$$

(This formula is messy in print but easily understood graphically; see Figure 2.1.) It is easily checked that  $\phi_n \leq \phi_{n+1}$  for all  $n$ , and  $0 \leq f - \phi_n \leq 2^{-n}$  on the set where  $f \leq 2^n$ . The result therefore follows.



However,  $C_c^0(X)$  is dense  $L^\infty(X) \iff$  every  $f \in L^\infty(X)$  can be approximated by a sequence  $\{g_k\} \subset C_c^0(X)$  in the sense that  $\|f - g_n\|_\infty \rightarrow 0$ . To see why this can *not* be the case, let  $f(x) = 1$ , so  $\|f\|_\infty = 1$  and let  $g_n \rightarrow f$  be an arbitrary sequence of  $C_c^0$  functions converging to  $f$  pointwise.

Since every  $g_n$  has compact support, say  $\text{supp}(g_n) := E_n$ , then  $g_n|_{E_n^c} \equiv 0$  and  $m(E_n^c) > 0$ . In particular, this means that  $\|f - g_n\|_\infty = 1$  for every  $n$ , so  $g_n$  can not converge to  $f$  in the infinity norm.

## 2 Problem 2

### 2.1 Part a

#### 2.1.1 Part i

**Lemma:**  $\|1\|_p = m(X)^{1/p}$

This follows from  $\|1\|_p^p = \int_X |1|^p = \int_X 1 = m(X)$  and taking  $p$ th roots.  $\square$

By Holder with  $p = q = 2$ , we can now write

$$\begin{aligned} \|f\|_1 &= \|1 \cdot f\|_1 \leq \|1\|_2 \|f\|_2 = m(X)^{1/2} \|f\|_2 \\ \implies \|f\|_1 &\leq m(X)^{1/2} \|f\|_2. \end{aligned}$$

Letting  $M := \|f\|_\infty$ , We also have

$$\begin{aligned} \|f\|_2^2 &= \int_X |f|^2 \leq \int_X |M|^2 = M^2 \int_X 1 = M^2 m(X) \\ \implies \|f\|_2 &\leq m(X)^{1/2} \|f\|_\infty \\ \implies m(X)^{1/2} \|f\|_2 &\leq m(X) \|f\|_\infty, \end{aligned}$$

and combining these yields

$$\|f\|_1 \leq m(X)^{1/2} \|f\|_2 \leq m(X) \|f\|_\infty,$$

from which it immediately follows

$$m(X) < \infty \implies L^\infty(X) \subseteq L^2(X) \subseteq L^1(X).$$

**The Inclusions Are Strict:**

1.  $\exists f \in L^1(X) \setminus L^2(X)$ :

Let  $X = [0, 1]$  and consider  $f(x) = x^{-\frac{1}{2}}$ . Then

$$\|f\|_1 = \int_0^1 x^{-\frac{1}{2}} < \infty \quad \text{by the } p \text{ test,}$$

while

$$\|f\|_2^2 = \int_0^1 x^{-1} \rightarrow \infty \quad \text{by the } p \text{ test.}$$

2.  $\exists f \in L^2(X) \setminus L^\infty(X)$ :

Take  $X = [0, 1]$  and  $f(x) = x^{-\frac{1}{4}}$ . Then

$$\|f\|_2^2 = \int_0^1 x^{-\frac{1}{2}} < \infty \quad \text{by the } p \text{ test,}$$

while  $\|f\|_\infty > M$  for any finite  $M$ , since  $f$  is unbounded in neighborhoods of 0, so  $\|f\|_\infty = \infty$ .

### 2.1.2 Part ii

1.  $\exists f \in L^2(X) \setminus L^1(X)$  when  $m(X) = \infty$ :

Take  $X = [1, \infty)$  and let  $f(x) = x^{-1}$ , then

$$\begin{aligned} \|f\|_2^2 &= \int_1^\infty x^{-2} < \infty && \text{by the } p \text{ test,} \\ \|f\|_1 &= \int_1^\infty x^{-1} \rightarrow \infty && \text{by the } p \text{ test.} \end{aligned}$$

2.  $\exists f \in L^\infty(X) \setminus L^2(X)$  when  $m(X) = \infty$ :

Take  $X = \mathbb{R}$  and  $f(x) = 1$ . then

$$\begin{aligned} \|f\|_\infty &= 1 \\ \|f\|_2^2 &= \int_{\mathbb{R}} 1 \rightarrow \infty. \end{aligned}$$

3.  $L^2(X) \subseteq L^1(X) \implies m(X) < \infty$ :

Let  $f = \chi_X$ , by assumption we can find a constant  $M$  such that  $\|\chi_X\|_2 \leq M\|\chi_X\|_1$ .

Then pick a sequence of sets  $E_k \nearrow X$  such that  $m(E_k) < \infty$  for all  $k$ ,  $\chi_{E_k} \nearrow \chi_X$ , and thus  $\|\chi_{E_k}\|_p \leq M\|\chi_{E_k}\|_1$ . By the lemma,  $\|\chi_{E_k}\|_p = m(E_k)^{1/p}$ , so we have

$$\begin{aligned} \|\chi_{E_k}\|_2 \leq M\|\chi_{E_k}\|_1 &\implies \frac{\|\chi_{E_k}\|_2}{\|\chi_{E_k}\|_1} \leq M \\ &\implies \frac{m(E_k)^{1/2}}{m(E_k)} \leq M \\ &\implies m(E_k)^{-1/2} \leq M \\ &\implies m(E_k) \leq M^2 < \infty. \end{aligned}$$

and by continuity of measure, we have  $\lim_K m(E_k) = m(X) \leq M^2 < \infty$ .  $\square$

## 2.2 Part b

1.  $L^1(X) \cap L^\infty(X) \subset L^2(X)$ :

Let  $f \in L^1(X) \cap L^\infty(X)$  and  $M := \|f\|_\infty$ , then

$$\|f\|_2^2 = \int_X |f|^2 = \int_X |f| |f| \leq \int_X M |f| = M \int_X |f| := \|f\|_\infty \|f\|_1 < \infty. \quad (1)$$

The inclusion is strict, since we know from above that there is a function in  $L^2(X)$  that is not in  $L^\infty(X)$ .

Note that taking square roots in (1) immediately yields

$$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2} \|f\|_{L^\infty(X)}^{1/2}.$$

2.  $L^2(X) \subset L^1(X) + L^\infty(X)$ :

Let  $f \in L^2(X)$ . Noting that continuous functions with compact support are dense in  $L^2(X)$ , take an approximating sequence  $\{g_n\} \subseteq C_c^0(X)$  with  $\|g_n - f\|_2 \rightarrow 0$ .

The claim is that we can choose  $N$  large enough such that when we write  $f = (f - g_N) + g_N$ , we will have  $\|f - g_N\|_1 < \infty$  and  $\|g_N\|_\infty < \infty$ , which establishes the desired result.

To see that  $g_N \in L^\infty(X)$ , we can just note that since each  $g_n$  is  $C_c^0$ , they are all **bounded**, say by  $M < \infty$ , in which case  $\|g_N\|_\infty \leq M < \infty$ .

To see that  $g_N \in L^1(X)$ , we can use the fact that  $\|f - g_N\|_2 \leq \|f\|_2 + \|g_N\|_2 < \infty$ , so  $f - g_N \in L^2(X)$ .

TODO

## 3 Problem 3

For notational convenience, it suffices to prove this for  $\ell^p(\mathbb{N})$ , where we re-index each sequence in  $\ell^p(\mathbb{Z})$  using a bijection  $\mathbb{Z} \rightarrow \mathbb{N}$ .

Note: this technically reorders all sums appearing, but since we are assuming absolute convergence everywhere, this can be done. One can also just replace  $\sum_{j=n}^m |a_j|^p$  with  $\sum_{n \leq |j| \leq m} |a_j|^p$  in what follows.

1.  $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$ :

Suppose  $\sum_j |a_j| < \infty$ , then its tails go to zero, so choose  $N$  large enough so that

$$j \geq N \implies |a_j| < 1.$$

But then

$$j \geq N \implies |a_j|^2 < |a_j|,$$

and

$$\begin{aligned}
\sum_j |a_j|^2 &= \sum_{j=1}^N |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j|^2 \\
&\leq \sum_{j=1}^N |a_j|^2 + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=N+1}^{\infty} |a_j| \\
&\leq M + \sum_{j=1}^{\infty} |a_j| \\
&< \infty.
\end{aligned}$$

where we just note that the first portion of the sum is a finite sum of finite numbers and thus bounded.

To see that the inclusion is strict, take  $\mathbf{a} := \{j^{-1}\}_{j=1}^{\infty}$ ; then  $\|\mathbf{a}\|_2 < \infty$  by the  $p$ -test by  $\|\mathbf{a}\|_1 = \infty$  since it yields the harmonic series.

2.  $\ell^2(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ :

This follows from the contrapositive: if  $\mathbf{a}$  is a sequence with unbounded terms, then  $\|\mathbf{a}\|_2 = \sum |a_j|^2$  can not be finite, since convergence would require that  $|a_j|^2 \rightarrow 0$  and thus  $|a_j| \rightarrow 0$ .

To see that the inclusion is strict, take  $\mathbf{a} = \{1\}_{j=1}^{\infty}$ . Then  $\|\mathbf{a}\|_\infty = 1$ , but the corresponding sum does not converge.

3.  $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_1$ :

Let  $M = \|\mathbf{a}\|_1$ , then

$$\|\mathbf{a}\|_2^2 \leq \|\mathbf{a}\|_1^2 \iff \frac{\|\mathbf{a}\|_2^2}{M^2} \leq 1 \iff \sum_j \left| \frac{a_j}{M} \right|^2 \leq 1.$$

But then we can use the fact that

$$\left| \frac{a_j}{M} \right| \leq 1 \implies \left| \frac{a_j}{M} \right|^2 \leq \left| \frac{a_j}{M} \right|$$

to obtain

$$\sum_j \left| \frac{a_j}{M} \right|^2 \leq \sum_j \left| \frac{a_j}{M} \right| = \frac{1}{M} \sum_j |a_j| := 1.$$

4.  $\|\mathbf{a}\|_\infty \leq \|\mathbf{a}\|_2$ :

This follows from the fact that, we have

$$\|\mathbf{a}\|_\infty^2 := \left( \sup_j |a_j| \right)^2 = \sup_j |a_j|^2 \leq \sum_j |a_j|^2 = \|\mathbf{a}\|_2^2$$

and taking square roots yields the desired inequality.

Note: the middle inequality follows from the fact that the supremum  $S$  is the least upper bound of all of the  $a_j$ , so for all  $j$ , we have  $a_j + \varepsilon > S$  for every  $\varepsilon > 0$ . But in particular,  $a_k + a_j > a_j$  for any pair  $a_j, a_k$  where  $a_k \neq 0$ , so  $a_k + a_j > S$  and thus so is the entire sum.

## 4 Problem 4

### 4.1 Part a

Let  $\{f_k\}$  be a Cauchy sequence, then  $\|f_k - f_j\|_u \rightarrow 0$ . Define a candidate limit by fixing  $x$ , then using the fact that  $|f_j(x) - f_k(x)| \rightarrow 0$  as a Cauchy sequence in  $\mathbb{R}$ , which converges to some  $f(x)$ .

We want to show that  $\|f_n - f\|_u \rightarrow 0$  and  $f \in C([0, 1])$ .

This is immediate though, since  $f_n \rightarrow f$  uniformly by construction, and the uniform limit of continuous functions is continuous.

### 4.2 Part b

It suffices to produce a Cauchy sequence of continuous functions  $f_k$  such that  $\|f_j - f_k\|_1 \rightarrow 0$  but if we define  $f(x) := \lim f_k(x)$ , we have either  $\|f\|_1 = \infty$  or  $f$  is not continuous.

To this end, take  $f_k(x) = x^k$  for  $k = 1, 2, \dots, \infty$ . Then

$$f_k \rightarrow \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases},$$

which has a clear discontinuity, but  $\int_0^1 x^k - x^j \, dx = \frac{1}{k+1} - \frac{1}{j+1}$ .

## 5 Problem 5

## 6 Problem 6