

Real Analysis

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1 Lecture 1 (Thu 15 Aug 2019 11:04)

See Folland’s Real Analysis, definitely a recommended reference.

Possible first day question: how can we “measure” a subset of \mathbb{R} ? We’d like bigger sets to have a higher measure, we wouldn’t want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of \mathbb{R} . We’ll come back to this in a few lectures.

1.1 Notions of “smallness” in \mathbb{R}

Definition: Let E be a set, then E is *countable* if it is in a one-to-one correspondence with $E' \subseteq \mathbb{N}$, which includes \emptyset, \mathbb{N} .

Definition: E is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of \mathbb{R} is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a $X \subseteq Y$ is dense in Y iff the closure of X is all of Y . So we’ll make the following definition.

Definition: A set $A \subseteq \mathbb{R}$ is *nowhere dense* if every interval I contains a subinterval $S \subseteq I$ such that $S \subseteq A^c$.

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we’re giving a name to such a countable union above. Example: \mathbb{Q} is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, - A^c contains a dense, open set. - The interior of the closure is empty.

We’d like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than ε .

Definition: E is a *null set* (or has *measure zero*) if $\forall \varepsilon > 0$, there exists a sequence of intervals $\{I_j\}_{j=1}^\infty$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable \implies Meager, but not the converse.
- Countable \implies Null, but not the converse.

Exercise: Show that the “middle third” Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem: Every $E \subseteq \mathbb{R}$ can be written as $E = A \cup B$ where A is null and B is meager.

This gives some information about how nullity and meagerness interact – in particular, \mathbb{R} itself is neither meager nor null. Idea: if meager \implies null, this theorem allows you to write \mathbb{R} as the union of two null sets. This is bad!

Proof: We can assume $E = \mathbb{R}$. Take an enumeration of the rationals, so $\mathbb{Q} = \{q_j\}_{j=1}^\infty$. Around each q_j , put an interval around it of size $1/2^{j+k}$ where we’ll allow k to vary, yielding multiple intervals around q_j . To do this, define $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 1/2^{j+k})$. Now let $G_k = \bigcup_j I_{j,k}$. Finally, let $A = \bigcap_k G_k$; we claim that A is null.

Note that $\sum_j |I_{j,k}| = \frac{1}{2^k}$, so just pick k such that $\frac{1}{2^k} < \varepsilon$.

Now we need to show that $A^c := B$ is meager. Note that G_k covers the rationals, and is a countable union of open sets, so it is dense. So G_k is an open and dense set. By one of the equivalent formulations of meagerness, this means that G_k^c is nowhere dense. But then $B = \bigcup_k G_k^c$ is meager.

1.2 \mathbb{R} is not small

Theorem A (Cantor): \mathbb{R} is not countable.

Theorem B (Baire): \mathbb{R} is not meager. (Baire Category Theorem)

Theorem C (Borel): \mathbb{R} is not null.

Note that theorems B and C imply theorem A. You can also replace \mathbb{R} with any nonempty interval $I = [a, b]$ where $a < b$. This is a strictly stronger statement – if any subset of \mathbb{R} is not countable, then certainly \mathbb{R} isn’t, and so on.

Proof of (A): begin by thinking of $I = [0, 1]$, then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry.

A second proof: Take an interval I , and suppose it is countable so $I = \{x_i\}$. Choose $I_1 \subseteq I$ that avoids x_1 , so $x_1 \notin I_1$. Choose $I_2 \subseteq I_1$ avoiding x_2 and so on to produce a nested sequence of closed intervals. Since \mathbb{R} is complete, the intersection $\bigcap_{n=1}^\infty I_n$ is nonempty, so say it contains x . But then $x \in I_1 \in I$, for example, but $x \neq x_i$ for any i , so $x \notin I$, a contradiction. \square

Proof of (B): Suppose $I = \bigcup_{i=1}^{\infty} A_n$ where each A_n is nowhere dense. We'll again construct a nested sequence of closed sets. Let $I_1 \subseteq I$ be a subinterval that misses all of A_1 , so $A_1 \cap I_1 = \emptyset$ using the fact that A_1 is nowhere dense. Repeat the same process, let $I_2 \subset I_1 \setminus A_2$. By the nested interval property, there is some $x \in \bigcap A_i$.

Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of \mathbb{R} is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

Thomae function: continuous on \mathbb{Q} , but discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Can this be switched to get some f that is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on \mathbb{Q} ? The answer is no. The set of discontinuities of a function is *always* an F_{σ} set, and $\mathbb{R} \setminus \mathbb{Q}$ is not one. Equivalently, the rationals are not a G_{δ} set.

The pointwise limit of continuous functions has a meager set of discontinuities. If f is integrable, the set of discontinuities is null. If f is monotone, they are countable. There is a continuous nowhere differentiable function: let $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$, and in fact *most* functions are like this. If f is continuous and monotone, the discontinuities are null.

Fact: If $I \subseteq \bigcup_{i=1}^{\infty} I_i$, then $|I| \leq \sum_{i=1}^{\infty} |I_i|$. The proof is by induction. Assume $I \subseteq \bigcup_{n=1}^{N+1} I_n$, where wlog we can assume that $a < a_{N+1} < b \leq b_{N+1}$, then $[a, a_{N+1}] \subset \bigcup_{n=1}^N I_n$ so the inductive hypothesis applies. But then $b - a \leq b_{N+1} - a = (b_{N+1} - a_{N+1}) + (a_{N+1} - a) \leq \sum_{n=1}^{N+1} |I_n|$.

Note that this proves that the reals are uncountable!