Title

D. Zack Garza

Tuesday 13th October, 2020

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Last time: proved that if X, Y are affine varieties then there is a bijection

$$\begin{aligned}
\left\{ \substack{\text{Morphisms} \\ f: X \to Y} \right\} &\iff \left\{ \substack{k\text{-algebra morphisms} \\ A(Y) \to A(X)} \right\} \\
f &\mapsto f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X).
\end{aligned}$$

Remark 1.0.1.
A morphism $f : X \to Y$ is by definition a morphism of ringed spaces where $\mathcal{O}_X, \mathcal{O}_Y$ are the sheaves of regular functions.

Remark 1.0.2.
This shows $X \cong Y$ as ringed spaces iff $A(X) \cong A(Y)$ as k-algebras.

Example 1.0.1.
Take

$$\begin{aligned}
f : \mathbb{A}^1 &\to V(y^2 - x^3) \subset \mathbb{A}^2 \\
t &\mapsto (t^2, t^3).
\end{aligned}$$

This is a morphism by proposition 4.7.

We then get a map on algebras

$$\begin{aligned}
f^* : A(V(y^2 - x^3)) = k[x, y]/ \left\langle y^2 - x^3 \right\rangle &\to k[t] \\
x &\mapsto t^2 \\
y &\mapsto t^3,
\end{aligned}$$

but even though f is a bijective morphism, it's not an isomorphism of ringed spaces. This can be seen from the fact that the image doesn't contain t.

Review of introductory category theory.

We'll define a category AffVar_k whose objects are affine varieties over k and morphisms in $\hom(X, Y)$ will be morphisms of ringed spaces. There is a contravariant functor A into reduced finitely generated k-algebras which sends X to $A(X)$ and sends morphisms $f : X \to Y$ to their pullbacks $f^* : A(Y) \to A(X)$, where "reduced" denotes the fact that there are no nilpotents.

Review of the universal property of the product.

Remark 1.0.3.
If we have X, Y affine varieties, we take $X \times Y$ to be the categorical product instead of the underlying product of topological spaces. We have $A(X \times Y) \cong A(X) \otimes_k A(Y) \cong k[x_1, \cdots, x_n, y_1, \cdots, y_m]/I(X) \otimes 1 + 1 \otimes I(Y)$. This recovers the product, since if we have

$$\begin{tikzcd}
Z \arrow[dr, dotted, "\exists H"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

where $H = (f, g)$.