

# Title

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# 1 | Thursday, October 01

Recall that a sheaf of rings on a topological space  $X$  is a ring  $\mathcal{F}(U)$  for all open sets  $U \subset X$  satisfying four properties:

1. The empty set is mapped to zero.
2. The morphism  $\mathcal{F}(U) \rightarrow \mathcal{F}(U)$  is the identity.
3. Given  $W \subset V \subset U$  we have
4. Gluing: given sections  $s_i \in \mathcal{F}(U_i)$  which agree on overlaps (restrict to the same function on  $U_i \cap U_j$ ), there is a unique  $s \in \mathcal{F}(\cup U_i)$ .

### Example 1.1.

If  $X$  is an affine variety with the zariski topology,  $\mathcal{O}_X$  is a sheaf of regular functions, where we recall  $\mathcal{O}_X(U)$  are the functions  $\varphi : U \rightarrow k$  that are locally a fraction.

Recall that the *stalk* of a sheaf  $\mathcal{F}$  at a point  $p \in X$ , is defined as

$$\mathcal{F}_p := \left\{ (U, \varphi) \mid p \in U \text{ open}, \varphi \in \mathcal{F}(U) \right\} / \sim.$$

where  $(U, \varphi) \sim (U', \varphi')$  if there exists a  $p \in W \subset U \cap U'$  such  $\varphi, \varphi'$  restricted to  $W$  are equal.

Recall that a *local ring* is a ring with a unique maximal ideal  $\mathfrak{m}$ . Given a prime ideal  $\mathfrak{p} \in R$ , so  $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$ , the complement  $R \setminus \mathfrak{p}$  is closed under multiplication. So we can localize to obtain  $R_{\mathfrak{p}} = \left\{ a/s \mid s \in R \setminus \mathfrak{p}, a \in R \right\} / \sim$  where  $a'/s' \sim a/s$  iff there exists a  $t \in R \setminus \mathfrak{p}$  such that  $t(a's - as') = 0$ .

**Warning:** Note that  $R_f$  is localizing at the powers of  $f$ , whereas  $R_{\mathfrak{p}}$  is localizing at the *complement* of  $\mathfrak{p}$ .

Since maximal ideals are prime, we can localize any ring  $R$  at a maximal ideal  $R_{\mathfrak{m}}$ , and this will be a local ring. Why? The ideals in  $R_{\mathfrak{m}}$  biject with ideals in  $R$  contained in  $\mathfrak{m}$ . Thus all ideals in  $R_{\mathfrak{m}}$  are contained in the maximal ideal generated by  $\mathfrak{m}$ , i.e.  $\mathfrak{m}R_{\mathfrak{m}}$ .

**Lemma 1.1 (?)**.

Let  $X$  be an affine variety. The stalk of the sheaf of regular functions  $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$  is isomorphic to the localization  $A(X)_{\mathfrak{m}_p}$  where  $\mathfrak{m}_p := I(\{p\})$ .

*Proof .*

We can write

$$A(X)_{\mathfrak{m}_p} := \left\{ \frac{g}{f} \mid g \in A(X), f \in A(X) \setminus \mathfrak{m}_p \right\} / \sim$$

$$\text{where } g_1/f_1 \sim g_2/f_2 \iff \exists h(p) \neq 0 \text{ where } 0 = h(f_2g_1 - f_1g_2).$$

where the  $f$  are regular functions on  $X$  such that  $f(p) \neq 0$ .

We can also write

$$\mathcal{O}_{X,p} := \left\{ (U, \varphi) \mid p \in U, \varphi \in \mathcal{O}_X(U) \right\} / \sim$$

$$\text{where } (U, \varphi) \sim (U', \varphi') \iff \exists p \in W \subset U \cap U' \text{ s.t. } \varphi|_W = \varphi'|_W.$$

So we can define a map

$$\begin{aligned} \Phi : A(X)_{\mathfrak{m}_p} &\rightarrow \mathcal{O}_{X,p} \\ \frac{g}{f} &\mapsto \left( D_f, \frac{g}{f} \right). \end{aligned}$$

**Step 1:** There are equivalence relations on both sides, so we need to check that things are well-defined.

We have

$$\begin{aligned} g/f \sim g'/f' &\iff \exists g \text{ such that } h(p) \neq 0, h(gf' - g'f) = 0 \in A(X) \\ &\iff \text{the functions } \frac{g}{f}, \frac{g'}{f'} \text{ agree on } W := D(f) \cap D(f') \cap D(h) \\ &\implies (D_f, g/f) \sim (D_{f'}, g'/f'), \end{aligned}$$

since there exists a  $W \subset D_f \cap D_{f'}$  such that  $g/f, g'/f'$  are equal.

**Step 2:** Surjectivity, since this is clearly a ring map with pointwise operations.

Any germ can be represented by  $(U, \varphi)$  with  $\varphi \in \mathcal{O}_X(U)$ . Since the sets  $D_f$  form a base for the topology, there exists a  $D_f \subset U$  containing  $p$ . By definition,  $(U, \varphi) = (D_f, \varphi|_{D_f})$  in  $\mathcal{O}_{X,p}$ . Using the proposition that  $\mathcal{O}_X(D(f)) = A(X)_f$ , this implies that  $\varphi|_{D_f} = g/f^n$  for some  $n$  and  $f(p) \neq 0$ , so  $(U, \varphi)$  is in the image of  $\Phi$ .

**Step 3:** Injectivity. We want to show that  $g/f \mapsto 0$  implies that  $g/f = 0 \in A(X)_{\mathfrak{m}_p}$ .

Suppose that  $(D_f, g/f) = 0 \in \mathcal{O}_{X,p}$  and  $(U, \varphi) = 0 \in \mathcal{O}_{X,p}$ , then there exists an open  $W \subset D_f$  containing  $p$  such that after passing to some distinguished open  $D_h \ni p$  such that  $\varphi = 0$  on  $D_h$ . Wlog we can assume  $\varphi = 0$  on  $U$ , since we could shrink  $U$  (staying in the same equivalence class) to make this true otherwise. Then  $\varphi = g/f$  on  $D_h$ , using that  $\mathcal{O}_X(D_f) = A(X)_f$ , so  $g/f = 0$  here. So there exists a  $k$  such that  $f^k(g \cdot 1 - 0 \cdot f) = 0$  in  $A(X)$ , so  $f^k g = 0 \in A(X)_{\mathfrak{m}_p}$ . ■

Conclusion:

$$\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}.$$

**Example 1.2.**

Let  $X = \{p, q\}$  with the discrete topology with the sheaf  $\mathcal{F}$  given by  $p \mapsto R, q \mapsto S, X \mapsto R \times S$ .

Then  $\mathcal{F}_p = R$ , since if  $U$  is open and  $p \in U$  then either  $U = \{p\}$  or  $U = X$ . We can check that for  $(r, s)$  a section of  $\mathcal{F}$ , we have an equivalence of germs  $(X, (r, s)) \sim (\{p\}, r)$  since  $\{p\} \subset X \cap \{p\}$ . Here  $X$  plays the role of  $U$ ,  $\{p\}$  of  $U'$ , and the last  $\{p\}$  the role of  $W \subset U \cap U'$ .

$$\begin{aligned} \mathcal{O}_{X,p} &\rightarrow A(X) \\ (\{p\}, r) &\mapsto r \\ \mathcal{F}_p &\cong R. \end{aligned}$$

**Example 1.3.**

Let  $M$  be a manifold and consider the sheaf  $C^\infty$  of smooth functions on  $M$ . Then the stalk  $C_p^\infty$  at  $p$  is defined as the set of smooth functions in a neighborhood of  $p$  modulo functions being equivalent if they agree on a small enough ball  $B_\varepsilon(p)$ . This contains a maximal ideal  $\mathfrak{m}_p$ , the smooth functions vanishing at  $p$ .

Then  $\mathfrak{m}_p^2$  is again an ideal, equal to the set  $\left\{ f \mid \partial_i \partial_j f \Big|_p = 0, \forall i, j \right\}$ . Thus  $\mathfrak{m}_p / \mathfrak{m}_p^2 \cong \{\partial_v\}^\vee$ , the dual of the set of directional derivatives.

## 1.1 What's the Point!

Problem: what should a map of affine varieties be?