Homotopy Groups of Spheres

Graduate Student Seminar

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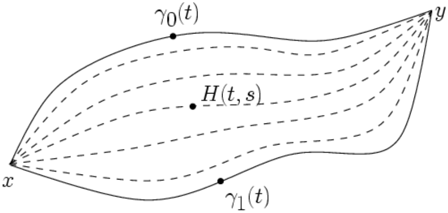
Introduction

Outline

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

Intuition

Homotopies of paths:



- Regard paths γ in X and homotopies of paths H as morphisms

$$\gamma \in \mathbf{hom}_{\text{Top}}(I, X)$$
$$H \in \mathbf{hom}_{\text{Top}}(I \times I, X).$$

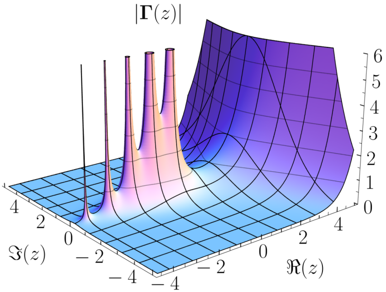
- Yields an equivalence relation: write

$$\gamma_0 \sim \gamma_1 \iff \exists H \text{ with } H(0) = \gamma_0, H(1) = \gamma(1)$$

- Write $[\gamma]$ to denote a homotopy class of paths.

Intuition

- Why care about path homotopies? Historically: contour integrals in $\mathbb{C}$



- By the residue theorem, for a meromorphic function f with simple poles $P = \{p_i\}$ we know that

$$\oint_\gamma f(z)\ dz \text{ is determined by } [\gamma] \in \pi_1(\mathbb{C} \setminus P)$$

Definitions

- Generalize to a homotopy of *morphisms*:

$$f, g \in \hom_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \hom_{\text{Top}}(X \times I, Y)$$

  such that $F(0) = f, F(1) = g$.
- This yields an equivalence relation on morphisms, *homotopy classes of maps*

$$[X, Y] \coloneqq \hom_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \hom(X, Y) \\ g \in \hom(Y, X) \end{cases} \text{such that} \begin{cases} f \circ g \sim \mathrm{id}_Y \\ g \circ f \sim \mathrm{id}_X \end{cases}$$

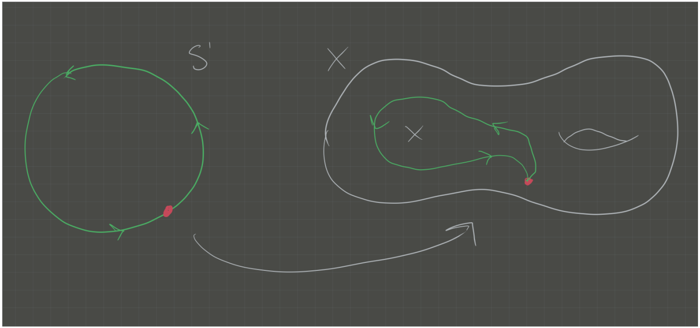
- Similarly write

$$[X] = \left\{ Y \in \text{Top} \;\middle|\; Y \sim X \right\}.$$

The Fundamental Group

- $\pi_1(X)$ is the group of homotopy classes of loops:
- Can recover this definition by finding a (co)representing object:

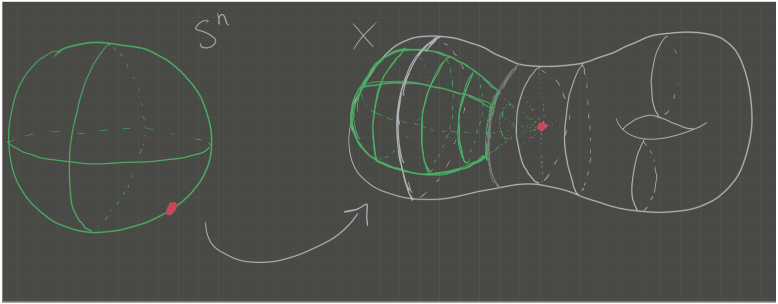
$$\pi_1(X) = [S^1, X]$$



Higher Homotopy Groups

- Can now generalize to define

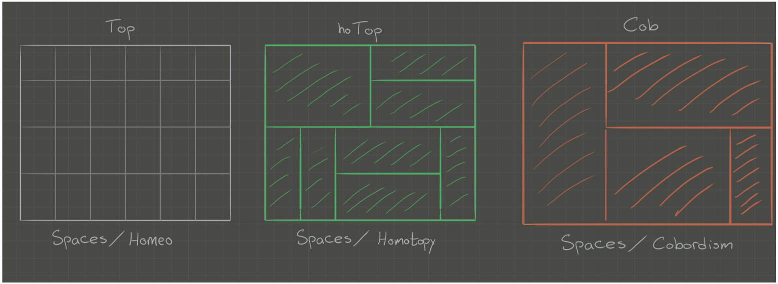
$$\pi_k(X) := [S^k, X]$$



Fun side note: this kind of definition generalizes to AG, see Motivic Homotopy Theory – the (co)representing objects look $\mathbb{A}^1$ *or* $\mathbb{P}^1$.

Classification

- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



Also have the derived category DTop, its interplay with hoTop is the subject of e.g. the Poincare conjecture(s).

- Any representative from a green box: a *homotopy type*.

Example: Homotopy Equivalence is Useful

Proposition: Let B be a CW complex; then isomorphism classes of $\mathbb{R}^1$-bundles over B are given by $H^1(X, \mathbb{Z}/2\mathbb{Z})$.

- Use the fact that for any fixed group G, the functor

$$\begin{aligned} h_G(\,\cdot\,) : \mathrm{hoTop}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ X &\mapsto \{G\text{-bundles over } X\} \end{aligned}$$

  is representable by a space called BG (Brown's representability theorem).

- I.e., let $\mathrm{Bun}_G(X) = \{G\text{-bundles}/B\} \,/\sim$, there is an isomorphism

$$\mathrm{Bun}_G(X) \cong [X, BG]$$

- In general, identify $G = \mathrm{Aut}(F)$ the automorphism group of the fibers – for vector bundles of rank n, take $G = GL(n, \mathbb{R})$.

Example: Homotopy Equivalence is Useful

Note that for a poset of spaces $(M_i, \hookrightarrow)$, the space $M^\infty \coloneqq \varinjlim M_i$. These are infinite dimensional "Hilbert manifolds".

Proof:

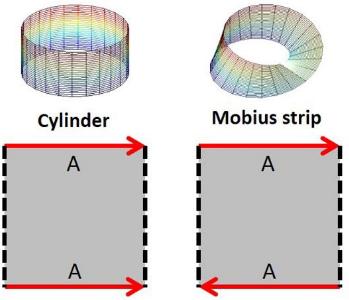
$$\begin{aligned}
\mathrm{Bun}_{\mathbb{R}^1}(X) &= [X, B\mathrm{GL}(1, \mathbb{R})] \\
&= [X, \mathrm{Gr}(1, \mathbb{R}^\infty)] \\
&= [X, \mathbb{RP}^\infty] \\
&= [X, K(\mathbb{Z}/2\mathbb{Z}, 1)] \\
&= H^1(X; \mathbb{Z}/2\mathbb{Z})
\end{aligned}$$

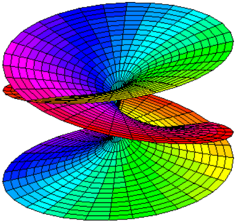
Work being swept under the rug: identifying the homotopy type of the representing object.

Example: Homotopy Equivalence is Useful

Corollary: There are 2 distinct line bundles over $X = S^1$ (the cylinder and the mobius strip), since $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary: A Riemann surface Σ_g satisfies $H^1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$ and thus there are 2^{2g} distinct real line bundles over it.





Example: Higher Homotopy Groups are Useful

- Application: computing $\pi_1(\mathrm{SO}(n, \mathbb{R}))$, the lie group of rigid rotations in 3-space.
- The fibration $\mathrm{SO}(n, \mathbb{R}) \longrightarrow \mathrm{SO}(n+1, \mathbb{R}) \longrightarrow S^n$ yields a LES in homotopy:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_2(SO(n, \mathbb{R})) \arrow[r] & \pi_2(SO(n, \mathbb{R})) \arrow[r] & \pi_2(S^n) \arrow[dll] \\
& \pi_1(SO(n, \mathbb{R})) \arrow[r] & \pi_1(SO(n, \mathbb{R})) \arrow[r] & \pi_1(S^n)
\end{tikzcd}$$

Uses of Higher Homotopy

Knowing $\pi_k S^n$, this reduces to

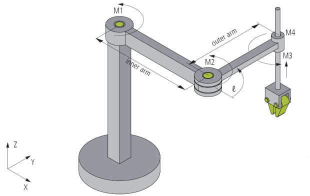
$$\begin{tikzcd}
\cdots 0 \arrow[r] & \pi_2(SO(n,\mathbb{R})) \arrow[r] & \pi_2(SO(n,\mathbb{R})) \arrow[r] \arrow[d, phantom, ""{coordinate, name=Z}] & 0 \arrow[dll, rounded corners, to path={ -- ([xshift=0ex]\tikztotarget)}] \\
& \pi_1(SO(n,\mathbb{R})) \arrow[r] & \pi_1(SO(n,\mathbb{R})) \arrow[r] & 0
\end{tikzcd}$$

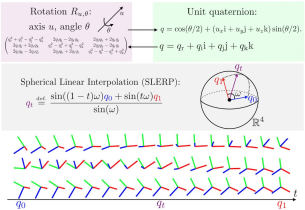
- Thus $\pi_1(\mathrm{SO}(3,\mathbb{R})) \cong \pi_1(\mathrm{SO}(4,\mathbb{R})) \cong \cdots$ and it suffices to compute $\pi_1(\mathrm{SO}(3,\mathbb{R}))$.
- Use the fact that "accidental" homeomorphism in low dimension $\mathrm{SO}(3,\mathbb{R}) \cong_{\mathrm{Top}} \mathbb{RP}^3$, and algebraic topology I yields $\pi_1 \mathbb{RP}^3 \cong \mathbb{Z}/2\mathbb{Z}$.

  Can also use the fact that $SU(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R})$ is a double cover from the universal cover.

Uses of Higher Homotopy

- Important consequence: $SO(3, \mathbb{R})$ is not simply connected! See "plate trick", there is a loop of rotations that is not contractible, but squares to the identity.
- Causes problems in robotics (leads to paths in configuration spaces that encounter singularities) and compute graphics (smoothly interpolating between e.g. quaternions for rotated camera views).

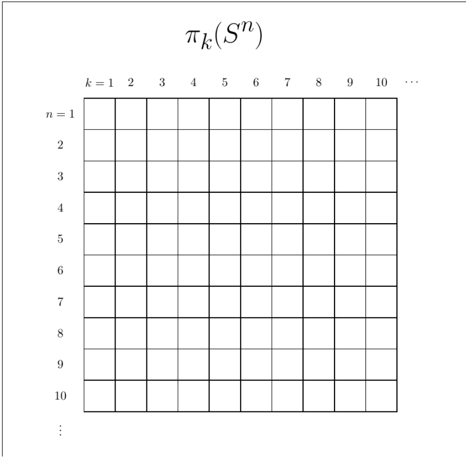




Spheres

Setup

- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate: $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:



Sphere 1