# 8005 Qual Problems

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(7) **Defn** $A/B$ is normal iff every lift of $id_B$ is an $A$-automorphism.

We have
$$\begin{array}{c} E \\ | \\ K \\ | \\ F \end{array}$$
with $E/F$ Galois

$\Rightarrow$ Suppose $K/F$ is normal, then if $\sigma \in Gal(E/F)$, then $\sigma|_F = id_F$, so $\sigma$ is a lift of $id_F$ and thus $\sigma(K) = K$. But by the fundamental theorem of Galois theory, we have

$$K \overset{\lambda}{\longleftrightarrow} Gal(E/K)$$

$$\sigma(K) \longleftrightarrow \tau \, Gal(E/K) \tau^{-1} \text{ for some } \tau \in Gal(E/F)$$

So if $\sigma(K) = K$ for every $\sigma$, then $\tau \, Gal(E/K) \tau^{-1} = Gal(E/K)$ $\forall \tau \in Gal(E/F)$, so $Gal(E/K) \trianglelefteq Gal(E/F)$

$\Leftarrow$: Running this argument backwards shows

$Gal(E/K) \trianglelefteq Gal(E/F) \Rightarrow \sigma(K) = K \;\; \forall \sigma \in Gal(E/F)$

$\Rightarrow$ Every lift of $id_F$ is a $K$-automorphism

$\Rightarrow$ $K/F$ is normal. $\blacksquare$

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⑧ By the Galois correspondence, we have

$$\begin{array}{c} L \\ \uparrow \\ K \\ \uparrow \\ E \\ \uparrow \\ F \end{array} \quad \overset{\lambda}{\rightsquigarrow} \quad \begin{array}{c} \{e\} \\ \downarrow \\ H_2 \\ \downarrow \\ H_1 \\ \downarrow \\ D_4 \end{array}$$

and

- $E/F$ normal iff $H_1 \trianglelefteq D_4$
- $K/E$ normal iff $H_2 \trianglelefteq H_2$
- $K/F$ not normal iff $H_2 \ntrianglelefteq D_4$

So writing $D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e,\ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$

we can take $H_1 = \langle \sigma^2, \tau \rangle = \{e, \tau, \sigma^2, \tau\sigma^2\}$, then $[D_4 : H_1] = 2$ so $H_1 \trianglelefteq D_4$. We can then take $H_2 = \langle \tau \rangle = \{e, \tau\} \leq H_1$. We have $H_1 \ntrianglelefteq D_4$, since e.g. if we write

$$\sigma = (1234),\ \tau = (24) \in S_n,\ \sigma\tau\sigma^{-1} = (13) \notin \langle \tau \rangle.$$

But $H_1 \trianglelefteq H_2$, since $H_1 \cong \{e, \tau, \sigma^2, \tau\sigma^2\} = \{(), (24), (13)(24), (13)\}$, while

- $\tau\tau\tau^{-1} = \tau \in H_1$
- $\sigma^2\tau\sigma^{-2} = (\sigma\tau\sigma^{-1})^2 = (13)(13) = e \in H_1$
- $\tau\sigma^2\tau(\tau\sigma^2)^{-1} = (13)(24)(13) = (24) = \tau \in H_1$

So $hH_2h^{-1} = H_2\ \forall h \in H_1$ and thus $H_2 \trianglelefteq H_1$. So taking

$$\boxed{\begin{aligned} H_1 &= \langle \sigma^2, \tau \rangle \\ H_2 &= \langle \tau \rangle \end{aligned}}$$

suffices. ■

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⑨ If $f(x) = x^3 - 7$, the splitting field of $f$ is $\mathbb{Q}(7^{1/3}, \zeta_3)$ where $\zeta_3 = e^{2\pi i/3}$.

1) Since $\min(7^{1/3}, \mathbb{Q}) = x^3 - 7$, $\min(\zeta_3, \mathbb{Q}(7^{1/3})) = \frac{x^3-1}{x-1} = x^2 + x + 1$, we have $[\mathbb{Q}(7^{1/3}, \zeta_3) : \mathbb{Q}] = 6$.

Since $\mathbb{Q}(7^{1/3}, \zeta_3)$ is a splitting field, and separable since char $\mathbb{Q} = 0$. Thus $|\mathrm{Gal}(\mathbb{Q}(7^{1/3}, \zeta_3)/\mathbb{Q})| = 6$ as well, so $\mathrm{Gal}(\mathbb{Q}(7^{1/3}, \zeta_3) \cong \mathbb{Z}_6$ or $S_3$. Since $\mathbb{Q}(7^{1/3})/\mathbb{Q}$ is not a normal extension, the galois group can not be abelian, so $\boxed{\mathrm{Gal}(\mathbb{Q}(7^{1/3}, \zeta_3)/\mathbb{Q} \cong S_3}$.

We have the correspondence $\sigma = \begin{cases} 7^{1/3} \mapsto \zeta_3 7^{1/3} \\ \zeta_3 \mapsto \zeta_3 \end{cases} \to (123) \in S_n$, $\tau = \begin{cases} 7^{1/3} \mapsto 7^{1/3} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases} \to (23) \in S_n$, we have

$$\begin{tikzcd}[column sep=small]
 & & S_3 \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle (123),(132) \rangle \arrow[drr, dash] & \langle (12) \rangle \arrow[dr, dash] & \langle (23) \rangle \arrow[d, dash] & \langle (13) \rangle \arrow[dl, dash] \\
 & & \{e\} &
\end{tikzcd}
\cong
\begin{tikzcd}[column sep=small]
 & \langle \tau, \sigma \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\langle \sigma \rangle \arrow[dr, dash] & \langle \sigma^2 \tau \rangle \arrow[d, dash] & \langle \tau \rangle \arrow[dl, dash] & \langle \sigma \tau \rangle \arrow[dll, dash] \\
 & \{e\} & &
\end{tikzcd}
\xleftrightarrow{\lambda}
\begin{tikzcd}[column sep=small]
 & & \mathbb{Q} \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\zeta_3) \arrow[drr, dash] & \mathbb{Q}(\zeta_3 7^{1/3}) \arrow[dr, dash] & \mathbb{Q}(7^{1/3}) \arrow[d, dash] & \mathbb{Q}(\zeta_3 7^{1/3}) \arrow[dl, dash] \\
 & & \{e\} &
\end{tikzcd}$$

$A_3$

Noting that $A_3 \trianglelefteq S_3$ is the only normal subgroup, the green extensions are not galois over $\mathbb{Q}$.

2) Since $7^{1/3} \in \mathbb{R}$, $L = \mathbb{Q}(\zeta_3) \Rightarrow |Gal(\mathbb{Q}(\zeta_3)/\mathbb{Q})| = \deg \min(\zeta_3, \mathbb{R}) = 2 \Rightarrow \boxed{Gal(\mathbb{Q}(\zeta_3)/\mathbb{R}) \cong \mathbb{Z}_2}$ generated by $\tau = \{\zeta_3 \mapsto \overline{\zeta_3} = \zeta_3^2\}$.

3) Since $f(x)$ is irreducible over $\mathbb{Z}$, it is irreducible over $\mathbb{F}_p$ for all $p$. So the splitting field is $\mathbb{F}_{13}(7^{1/3}, \zeta_3) := \mathbb{F}_{13}[x] / (x^2+x+1, x^3-7)$ which is a finite extension & separable since $\mathbb{F}_{13}$ is a finite field, and is thus separable. So the galois group is order 6, and by the same argument used in (1), $\boxed{Gal(\mathbb{F}_{13}(7^{1/3}, \zeta_3)/\mathbb{F}_{13}) \cong S_3.}$ $\blacksquare$