Problem Set 7

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Contents

1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence. We then have $\|x^j - x^k\|_{\ell^2} \to 0$, and we want to produce some $\mathbf{x} := \lim_{n\to\infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i \coloneqq \lim_{n\to\infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left|x_i^j - x_i^k\right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left|x_i^j - x_i^k\right|^2$ is a Cauchy sequence of real numbers which necessarily converges by the completeness of $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \left\|\lim_{k\to\infty} x^k - x^j\right\|_{\ell^2} = \lim_{k\to\infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$\begin{aligned}
\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\
&\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\
&\to M < \infty,
\end{aligned}$$

where $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$ by the previous argument, and the second term is bounded because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} \coloneqq M < \infty$. $\square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \ \cdot \rangle$ and induced norm $\|\cdot\|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on H takes values in $\mathbb{C}$, we have

$$\Re(\langle x, \ iy \rangle) = \Re(-i\langle x, \ y \rangle) = \Im(\langle x, \ y \rangle).$$

We can compute the following:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ y\rangle\right)$$

$$\begin{aligned}\|x+iy\|^2 &= \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x,\ iy\rangle\right)\\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)\end{aligned}$$

$$\begin{aligned}\|x-iy\|^2 &= \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x,\ iy\rangle\right)\\ &= \|x\|^2 + \|y\|^2 + \Im(\langle x,\ y\rangle)\end{aligned}$$

and summing these all

$$\begin{aligned}\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\| &= 4\,\Re\left(\langle x,\ y\rangle\right) + 4i\,\Im\left(\langle x,\ y\rangle\right)\\ &= 4\langle x,\ y\rangle.\end{aligned}$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 \coloneqq \langle x,\ x\rangle = \langle Ux,\ Ux\rangle \coloneqq \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned}\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2 + \|Ux-Uy\|^2 + i\|Ux+Uy\|^2 - i\|Ux+Uy\|^2\right)\\ &= \frac{1}{4}\left(\|U(x+y)\|^2 + \|U(x-y)\|^2 + i\|U(x+y)\|^2 - i\|U(x+y)\|^2\right)\\ &= \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2 + i\|x+y\|^2 - i\|x+y\|^2\right)\\ &= \langle x,\ y\rangle.\end{aligned}$$

□

2 Problem 2

Lemma: The map $\langle\,\cdot\,,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$
\begin{aligned}
|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle| \\
&= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle| \\
&\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\
&\to 0 \cdot M + C \cdot 0 < \infty,
\end{aligned}
$$

where $\|y_n\| \to M$ since $y_n \to y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

We want to show that sequences in $E^\perp$ converge to elements of $E^\perp$. Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$
\langle e,\ y\rangle = \left\langle \lim_n e_n,\ y \right\rangle = \lim_n \langle e_n,\ y\rangle = \lim_n 0 = 0,
$$

so $e \in E^\perp$.

2.2 Part b:

Let $S \coloneqq \operatorname{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n \to x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$
y \in E^\perp \implies \langle x_n,\ y\rangle = \left\langle \sum_i a_i e_i,\ y \right\rangle = \sum_i a_i \langle e_i,\ y\rangle = 0 \implies x_n \in (E^\perp)^\perp.
$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$
y \in E^\perp \implies \langle x,\ y\rangle = \left\langle \lim_n x_n,\ y \right\rangle = \lim_n \langle x_n,\ y\rangle = 0 \implies x \in (E^\perp)^\perp,
$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notation convenience, we'll just write S for $\overline{S}$. Let $x \in E^{\perp\perp}$. Noting that S is closed, we can define P, the operator projecting elements onto S, and write

$$
x = Px + (x - Px) \in S \oplus S^\perp
$$

But since $\langle x,\ x - Px\rangle = 0$ because $x - Px \in E^{\perp}$ and $x \in (E^{\perp})^{\perp}$, we can rewrite the first term in this inner product to obtain

$$0 = \langle x,\ x - Px\rangle = \langle Px + (x - Px),\ x - Px\rangle = \langle Px,\ x - Px\rangle + \langle x - Px,\ x - Px\rangle,$$

where we can note that the first term is zero because $Px \in S$ and $x - Px \in S^{\perp}$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\begin{aligned}
\|e_0\|^2 &= \int_0^1 1^2\ dx = 1 \\
\|e_1\|^2 &= \int_0^1 3(2x-1)^2 = \frac{1}{2}(2x-1)^2\bigg|_0^1 = 1 \\
\langle e_0,\ e_1\rangle &= \int_0^1 \sqrt{3}(2x-1)\ dx = \frac{\sqrt{3}}{4}(2x-1)\bigg|_0^1 = 0.
\end{aligned}$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0,1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$
\begin{aligned}
f(x) &= \sum_i \left\langle x^3,\ e_i \right\rangle e_i \\
&= \left\langle x^3,\ 1 \right\rangle 1 + \left\langle x^3,\ \sqrt{3}(2x-1) \right\rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2\ dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1)\ dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1)\frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}
$$

We can also compute

$$
\begin{aligned}
\|f-g\|_2^2 &= \int_0^1 (x^2 - x + \frac{1}{6})^2\ dx \\
&= \frac{1}{180} \\
\implies \|f-g\|_2 &= \frac{1}{\sqrt{180}}.
\end{aligned}
$$

4 Problem 4

4.1 Part a

4.1.1 i

We can first note that $\left\langle 1/\sqrt{2},\ \cos(2\pi nx) \right\rangle = \left\langle 1/\sqrt{2},\ \sin(2\pi mx) \right\rangle = 0$ for any n or m, since this involves integrating either sine or cosine over an integer multiple of its period.

Letting $m, n \in \mathbb{Z}$, we can then compute

$$
\begin{aligned}
\langle \cos(2\pi nx),\ \sin(2\pi mx) \rangle &= \int_0^1 \cos(2\pi nx)\sin(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x)\ dx \\
&= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2}\int_0^1 \sin(2\pi(n-m)x)\ dx \\
&= 0,
\end{aligned}
$$

which again follows from integration of sine over a multiple of its period (where we use the fact that $m+n, m-n \in \mathbb{Z}$).

Similarly,

$$
\begin{aligned}
\langle \cos(2\pi nx),\ \cos(2\pi mx)\rangle &= \int_0^1 \cos(2\pi nx)\cos(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x)\ dx \\
&= \begin{cases} \frac{1}{2}\int_0^1 \cos(4\pi nx) + 1\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
\langle \sin(2\pi nx),\ \sin(2\pi mx)\rangle &= \int_0^1 \sin(2\pi nx)\sin(2\pi mx)\ dx \\
&= \frac{1}{2}\int_0^1 \cos(2\pi(m-n)x) + \cos(2\pi(m+n)x)\ dx \\
&= \begin{cases} \frac{1}{2}\int_0^1 1 + \cos(4\pi nx)\ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.
\end{aligned}
$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

4.1.2 ii

We have

$$
\begin{aligned}
\left\langle e^{2\pi kx},\ e^{-2\pi i\ell x}\right\rangle &= \int_0^1 e^{2\pi ikx}\overline{e^{2\pi i\ell x}}\ dx \\
&= \int_0^1 e^{2\pi ikx}e^{-2\pi i\ell x}\ dx \\
&= \int_0^1 e^{2\pi i(k-\ell)x}\ dx \\
(&= \int_0^1 1\ dx = 1 \quad \text{if } k = \ell \text{, otherwise:}) \\
&= \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)}\bigg|_0^1 \\
&= \frac{e^{2\pi i(k-\ell)} - 1}{2\pi i(k-\ell)} \\
&= 0,
\end{aligned}
$$

since $e^{2\pi ik} = 1$ for every $k \in Z$, and $k - \ell \in \mathbb{Z}$. Thus this set is orthonormal.

4.2 Part b

4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials $P_n(x)$ such that $\|f - P_n\|_\infty \to 0$, i.e. the P_n uniformly approximate f on $[0, 1]$.

Letting $\varepsilon > 0$, we can thus choose a P such that $\|f - P\|_\infty < \varepsilon$, which necessarily implies that $\|f - P\|_{L^1} < \varepsilon$ since we have

$$\int_0^1 |f(x) - P(x)|\ dx \leq \int_0^1 \varepsilon\ dx = \varepsilon.$$

Thus we can write

$$f(x) = P(x) + (f(x) - P(x))$$

where $h(x) \coloneqq f(x) - P(x)$ satisfies $\|h\|_{L^1} < \varepsilon$. It only remains to show that $P \in L^2([0,1])$, but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say $|P(x)| \leq M < \infty$ for all $x \in [0,1]$, and thus

$$\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2\ dx \leq \int_0^1 M^2\ dx = M^2 < \infty.$$

It follows that we can let $g = P$ and $h = f - P$ to obtain the desired result.

4.2.2 ii

By part (i), the claim is that it suffices to show this is true for $f \in L^2$. In this case, we can identify

$$\begin{aligned} \int_0^1 f(x)\cos(2\pi kx)\ dx &\coloneqq \Re(\hat{f}(k)) \\ \int_0^1 f(x)\sin(2\pi kx)\ dx &\coloneqq \Im(\hat{f}(k)), \end{aligned}$$

the real and imaginary parts of the kth Fourier coefficient of f respectively.

By Bessel's inequality, we know that $\left\{\hat{f}(k)\right\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$, and so $\sum_k \left|\hat{f}(k)\right| < \infty$.

But this is a convergent sequence of real numbers, which necessarily implies that $\left|\hat{f}(k)\right| \to 0$. In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have $f \in L^1$, write $f = g + h$ where $g \in L^2$ and $\|h\|_{L^1} \to 0$. Then

$$\begin{aligned} \left|\int_0^1 f(x)\cos(2\pi kx)\ dx\right| &= \left|\int_0^1 (g(x) + h(x))\cos(2\pi kx)\ dx\right| \\ &\leq \left|\int_0^1 g(x)\cos(2\pi kx)\ dx\right| + \left|\int_0^1 h(x)\cos(2\pi kx)\ dx\right| \\ &\leq \left|\int_0^1 g(x)\cos(2\pi kx)\ dx\right| + \int_0^1 |h(x)||\cos(2\pi kx)|\ dx \\ &= |\hat{g}(k)| + \varepsilon \\ &\to 0, \end{aligned}$$

with a similar computation for $\int f(x)\sin(2\pi kx)$. $\square$

5 Problem 5

5.1 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e_1} = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k\leq i-1} \langle v_i,\ \hat{e_i}\rangle \hat{e_i}$

The result set $\{\hat{e_i}\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$\begin{aligned}
e_2 &= x - \langle x,\ \hat{e}_1\rangle \hat{e}_1 \\
&= x - \langle x,\ 1\rangle 1 \\
&= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x\ dx \\
&= x - \int \text{odd function} \\
&= x,
\end{aligned}$$

and so $e_2 = x$. We can then check that

$$\|e_2\| = \left(\int_{-1}^{1} x^2\ dx\right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}}x$.

We continue to compute

$$\begin{aligned}
e_3 &= x^2 - \left\langle x^2,\ \hat{e}_1\right\rangle \hat{e}_1 - \left\langle x^2,\ \hat{e}_2\right\rangle \hat{e}_2 \\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\ dx - \frac{3}{2}x\int_{-1}^{1} x^3\ dx \\
&= x^2 - \left(\frac{1}{6}x^3\right)\Big|_{-1}^{1} + \frac{3}{2}x\int_{-1}^{1} \text{odd function} \\
&= x^2 - \frac{1}{3}.
\end{aligned}$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$
\begin{aligned}
\hat{e}_3 &= \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \\
&= \frac{1}{2}\sqrt{\frac{45}{2}}\frac{1}{3}(3x^2 - 1) \\
&= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right).
\end{aligned}
$$

In summary, this yields

$$
\begin{aligned}
\hat{e}_1 &= \frac{1}{\sqrt{2}} \\
\hat{e}_2 &= x \\
\hat{e}_3 &= \frac{1}{3}\sqrt{\frac{45}{2}}\left(\frac{3x^2 - 1}{2}\right),
\end{aligned}
$$

which are scalar multiples of the first three Legendre polynomials.

5.2 Part b

Let $p(x) = a + bx + cx^2$, we are then looking for p such that $\|x^3 - p(x)\|_2^2$ is minimized. Noting that

$$
p(x) \in \operatorname{span}\left\{1, x, x^2\right\} = \operatorname{span}\left\{P_0(x), P_1(x), P_2(x)\right\} := S,
$$

we can conclude that $p(x)$ will be the projection of x^3 onto S. Thus $p(x) = \sum_{i=0}^{2} \langle x^3,\ \hat{e}_i \rangle \hat{e}_i$.

Proceeding to compute the terms in this expansion, we can note that $\langle x^3,\ f \rangle$ for any f that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$
\left\langle x^3,\ x \right\rangle x = x \int_{-1}^{1} x^4 \ dx = \frac{2}{5} x
$$

.

And thus $p(x) = \frac{2}{5}x$ is the minimizer.

5.3 Part c

The first three conditions necessitate $g \in S^\perp$ and $\|g\| = 1$. Since S is a closed subspace, we can write $x^3 = p(x) + (x^3 - p(x)) \in S \oplus S^\perp$, and so $x^3 - p(x) \in S^\perp$.

The claim is that $g(x) := x^3 - p(x)$ is a scalar multiple of the desired maximizer. This follows from the fact that

$$
\left| \left\langle x^3 - p,\ g \right\rangle \right| \leq \|x^3 - p\| \|g\|
$$

by Cauchy-Schwarz, with equality precisely when $g = \lambda(x^3 - p)$ for some scalar λ. However, the restriction $\|g\| = 1$ forces $\lambda = \|x^3 - p\|^{-1}$.

A computation shows that

$$\|x^3 - p\|^2 = \int_0^1 (x^3 - \frac{2}{5}x)^2 \, dx = \frac{19}{525},$$

and so we can take

$$g(x) := \frac{25}{\sqrt{19}} \left(x^3 - \frac{2}{5}x \right).$$

6 Problem 6

6.1 Part a

To see that $g \in \mathcal{C}$, we can compute

$$\begin{aligned} \langle g,\ 1 \rangle &= \int_0^1 18x^2 - 5 \, dx = 6 - 5 = 1 \\ \langle g,\ x \rangle &= \int_0^1 18x^3 - 5x \, dx = \frac{18}{4} - \frac{5}{2} = 2. \end{aligned}$$

To see that $\mathcal{C} = g + S^\perp$, let $f \in \mathcal{C}$, so $\langle f,\ 1 \rangle = 1$ and $\langle f,\ x \rangle = 2$. We can then conclude that $f - g \in S^\perp$, since we have

$$\begin{aligned} \langle f - g,\ 1 \rangle &= \langle f,\ 1 \rangle - \langle g,\ 1 \rangle = 1 - 1 = 0 \\ \langle f - g,\ x \rangle &= \langle f,\ x \rangle - \langle g,\ x \rangle = 2 - 2 = 0. \end{aligned}$$

6.2 Part b

Note that this equivalent to finding an $f_0 \in \mathcal{C}$ such that $\|f_0\|$ is minimized.

Letting $f_0 \in \mathcal{C}$, be arbitrary and noting that by part (a) we have $f_0 = g + s$ where $s \in S^\perp$, we can compute

$$\begin{aligned} \|f_0\|^2 &= \langle f_0,\ f_0 \rangle \\ &= \langle g + s,\ g + s \rangle \\ &= \|g\|^2 + 2\Re\langle g,\ s \rangle + \|s\|^2, \end{aligned}$$

which can be minimized by taking $s = 0$, which forces $\|s\|^2 = 0$ and $\langle g,\ s \rangle = 0$. But this imposes the condition $f_0 = g + 0 = g$. $\square$