

# Title

D. Zack Garza

Saturday 26<sup>th</sup> September, 2020

## Contents

1 Saturday, September 26

1

# 1 | Saturday, September 26

**Remark 1.**

There is a natural action of  $\text{MCG}(\Sigma)$  on  $H_1(\Sigma; \mathbb{Z})$ , i.e. a *homology representation* of  $\text{MCG}(\Sigma)$ :

$$\begin{aligned} \rho : \text{MCG}(\Sigma) &\rightarrow \text{Aut}_{\text{Grp}}(H_1(\Sigma; \mathbb{Z})) \\ f &\mapsto f_* . \end{aligned}$$

**Definition 1.0.1** (Special Linear Group).

$$\text{SL}(n, \mathbb{k}) = \left\{ M \in \text{GL}(n, \mathbb{k}) \mid \det M = 1 \right\} = \ker \det_{\mathbb{G}_m} .$$

**Remark 2.**

$$\text{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle .$$

Note that  $S^2 = 1$  and

$$T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Moreover, if  $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$  and  $A \in \text{SL}(2, \mathbb{Z})$ , we have  $A\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$ , i.e. this preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \mid p, q \in \mathbb{Z} \right\} \cong \left\{ p\omega_1 + q\omega_2 \mid p, q \in \mathbb{Z} \right\} \simeq \left\{ p' + q'\tau \mid p', q' \in \mathbb{Z} \right\} .$$

where the  $\omega_i, \tau$  come from identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ , and in the last step we've rescaled the lattice by *homothety* to align one vector with the  $x$ -axis.

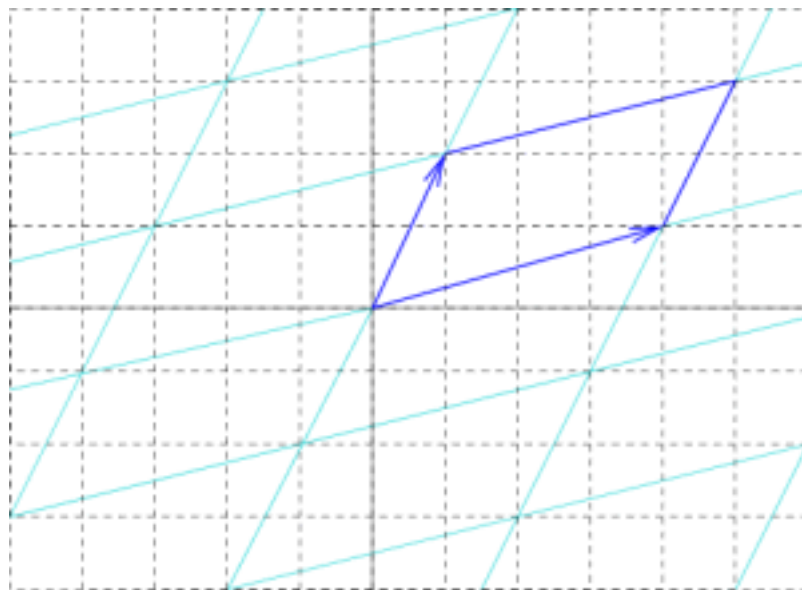


Figure 1: Lattice

**Remark 3.**

For any finite-index subgroup  $G \leq \mathrm{SL}(2, \mathbb{Z})$ , the orbits/left-quotient  $G \backslash \mathbb{H}$  yields a complex curve (i.e. a torus).

**Theorem 1.1 (Mapping Class Group of the Torus).**

The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

*Proof .*

- For  $f$  any automorphism, the induced map  $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  is a group automorphism, so we can consider the group morphism

$$\begin{aligned} \tilde{\sigma} : (\mathrm{Map}(X, X), \circ) &\rightarrow (\mathrm{GL}(2, \mathbb{Z}), \circ) \\ f &\mapsto f_* \end{aligned}$$

- This will descend to the quotient  $\mathrm{MCG}(X)$  iff  $\mathrm{Map}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$ 
  - This holds because any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\begin{aligned}\tilde{\sigma} : \text{MCG}(X) &\rightarrow \text{GL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

**Claim:**  $\text{im}(\tilde{\sigma}) \subseteq \text{SL}(2, \mathbb{Z})$ .

*Proof.*

?

- Algebraic intersection numbers in  $\Sigma_2$  correspond to determinants
- $f \in \text{Homeo}^+(X)$  preserve algebraic intersection numbers.
- See section 1.2

■

- We can thus freely restrict the codomain to define the map

$$\begin{aligned}\sigma : \text{MCG}(X) &\rightarrow \text{SL}(2, \mathbb{Z}) \\ f &\mapsto f_*.\end{aligned}$$

**Claim:**  $\sigma$  is surjective.

- $\mathbb{R}^2$  is the universal cover of  $\Sigma_2$ , with deck transformation group  $\mathbb{Z}^2$ .
- Any  $A \in \text{SL}(2, \mathbb{Z})$  extends to  $\tilde{A} \in \text{GL}(2, \mathbb{R})$ , a linear self-homeomorphism of the plane that is orientation-preserving.

**Claim:**  $\tilde{A}$  is equivariant wrt  $\mathbb{Z}^2$

*Proof.*

?

■

- So  $\tilde{A}$  descends to a well-defined map  $\psi_{\tilde{A}}$  on  $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$ , which is still a linear self-homeomorphism
- There is a correspondence

$$\{\text{Primitive vectors in } \mathbb{Z}^2\} \iff \left\{ \begin{array}{c} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{array} \right\} / \text{homotopy}.$$

- Thus  $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$ .

■