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1 | Tuesday, October 20

1.1 Gluing Two Opens

Recall that a *prevariety* is a ringed space that is locally isomorphic to an affine variety, where we recall that $(X, \mathcal{O}_X)$ is *locally isomorphic* to an affine variety iff there exists an open cover $U_i \rightrightarrows X$ such that $(U_i, \mathcal{O}_{U_i})$.

We found one way of producing these: the gluing construction. Given two ringed spaces $(X_1, \mathcal{O}_{X_1})$ and $(X_2, \mathcal{O}_{X_2})$ and open sets $U_{12} \in X_1$ and $U_{21} \in X_2$ and an isomorphism $(U_{12}, \mathcal{O}_{U_{12}}) \xrightarrow{f} (U_{21}, \mathcal{O}_{U_{21}})$, we defined

- The topological space as $X_1 \coprod_f X_2$
- The sheaf of rings as $\mathcal{O}_X = \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U \cap X_i} \text{ is regular for } i = 1, 2 \right\}$.

Example 1.1.1.
$\mathbb{P}^1/k = X_1 \cup X_2$ where $X_1 \cong \mathbb{A}^1, X_2 \cong \mathbb{A}^2$. Take $U_{12} = D(x)$ and $U_{21} = D(y)$ with

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto \frac{1}{x} = y. \end{aligned}$$

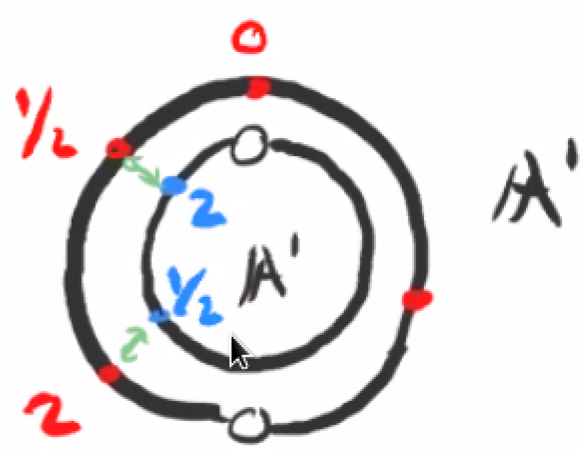

Figure 1: Supposing char $(k) \neq 2$. Note that for $\mathbb{C}$ this recovers S^2 in the classical topology.

Example 1.1.2.
Let $X_i = \mathbb{A}^1$ and $U_{12} = D(x), U_{21} = D(y)$ with

$$\begin{aligned} f : U_{12} &\to U_{21} \\ x &\mapsto x = y. \end{aligned}$$

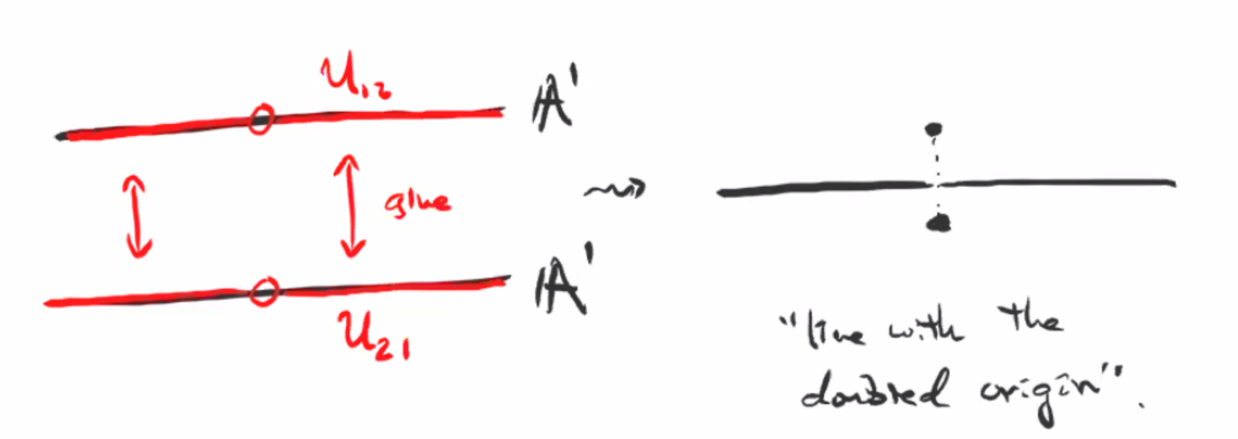

Figure 2: Line with the doubled origin.

Then $\mathcal{O}_X = \left\{ \varphi : X \to k \;\middle|\; \varphi|_{X_i} \text{ is regular} \right\} \cong k[x]$.

1.2 More General Gluing

Now we want to glue more than two open sets. Let I be an indexing set for prevarieties X_i. Suppose that for an ordered pair (i, j) we have open sets $U_{ij} \subset X_i$ and isomorphisms $f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ such that

a. $f_{ji} = f_{ij}^{-1}$

b. $f_{jk} \circ f_{ij} = f_{ik}$ (cocycle condition)

Figure 3: Opens with isomorphisms.

Then the gluing construction is given by

1. $X \coloneqq \coprod X_i / \sim$ where $x \sim f_{ij}(x)$ for all i, j and all $x \in U_{ij}$.
2. $\mathcal{O}_x(U) \coloneqq \left\{ \varphi : U \to k \;\middle|\; \varphi|_{U \cap X_i} \in \mathcal{O}_{X_i} \right\}$.

Every prevariety arises from the gluing construction applied to X_i affine varieties, since a prevariety $(X, \mathcal{O}_X)$ by definition has an open affine cover $X_i \rightrightarrows X$ and X is the result of gluing the X_is by the identity.

Example 1.2.1.
Let $X_1 = X_2 = X_3 = \mathbb{A}^2/k$. Glue by the following instructions:

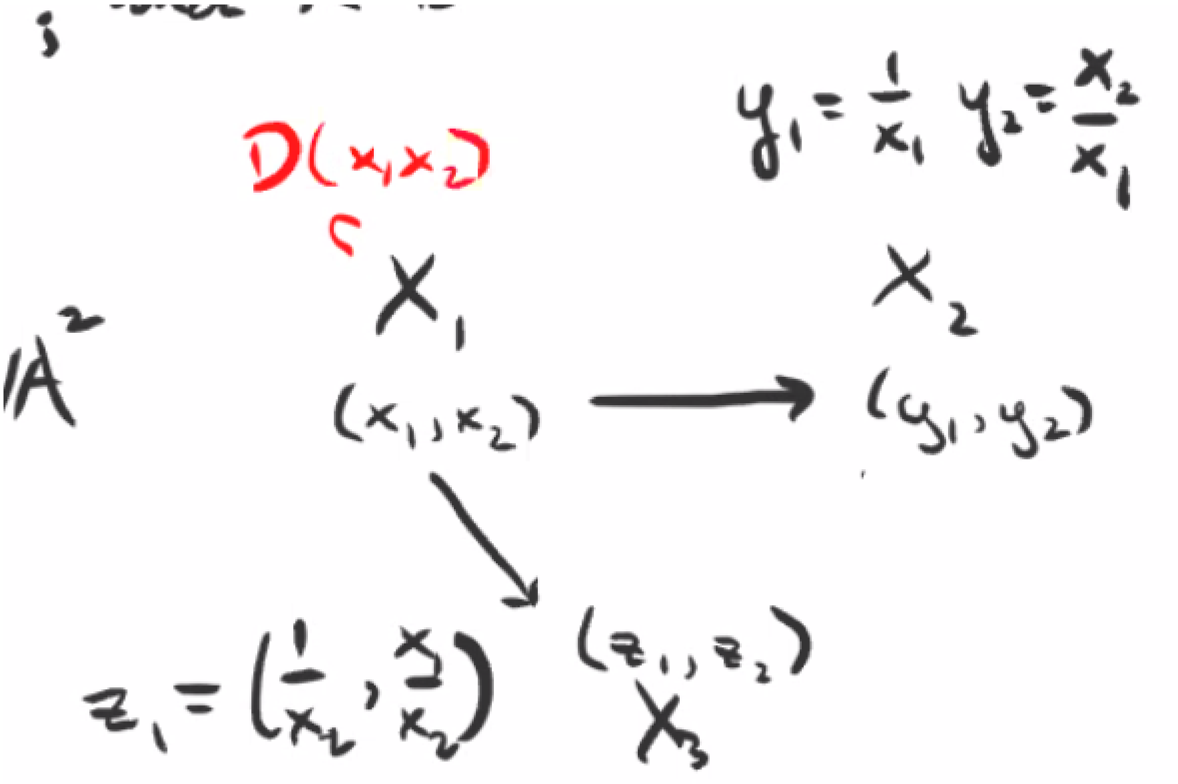

Figure 4: The map not shown is whatever formula is necessary to make the diagram commute.

Here

- $(y_1, y_2) = (1/x_1, x_2/x_1)$
- $(z_1, z_2) = (1/x_2, x_1/x_2)$
- $U_{12} = D(x_1)$
- $U_{21} = D(x_2)$.

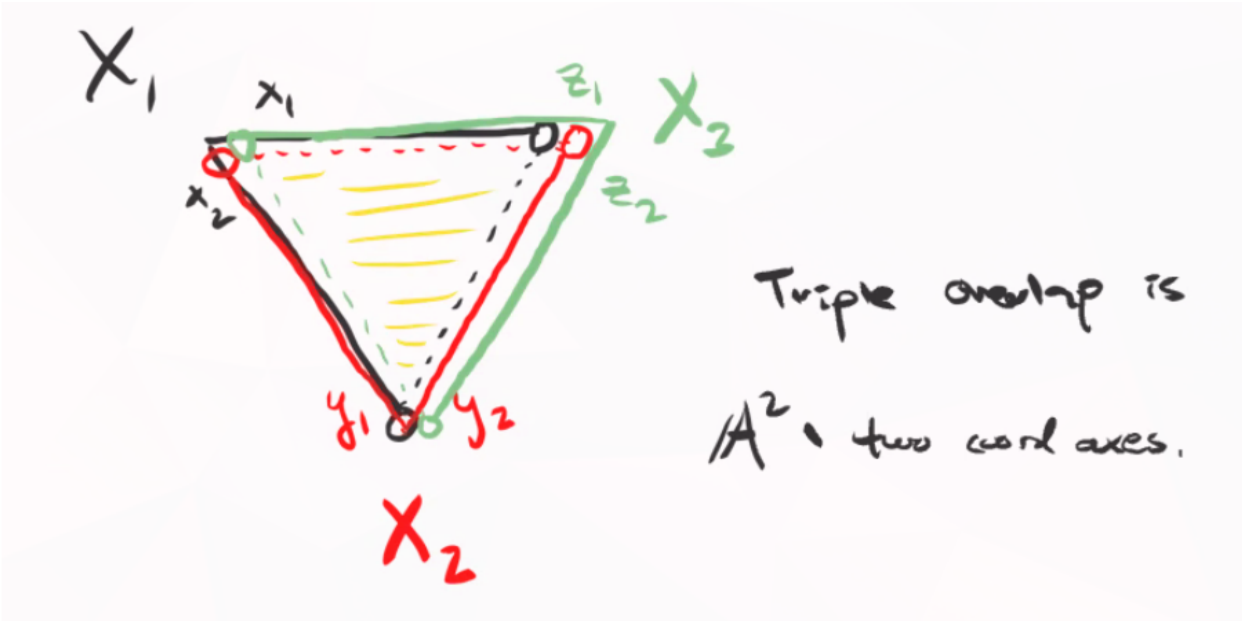

Figure 5: Yields $\mathbb{P}^2$

Here $X_1 = [1 : y/x : z/x]$, $X_2 = [x/y : 1 : z/y]$.

Example 1.2.2.
From Gathmann 5.10, open and closed subprevarieties. Let X be a prevariety and suppose $U \subset X$ is open. Then $(U, \mathcal{O}_U)$ is a prevariety where $\mathcal{O}_U = \mathcal{O}_X|_U$. How can we write U as (locally) an affine variety?

Since the U_i are covered by distinguished opens D_{ij} in X_i where $X = \cup X_i$ with X_i affine varieties, we can write $U = \bigcup\limits_i U_i = \bigcup\limits_{i,j} D_{ij}$.

Example 1.2.3.
Let $Y \subset X$ be a closed subset of a prevariety X. We need to define $\mathcal{O}_Y(U)$ for all $U \subset Y$ open, so we set

$$\mathcal{O}_Y(U) = \left\{ \varphi : U \to k \;\middle|\; \forall p \in U,\, \exists V_p \text{ with } p \in V_p \subset_{\text{open}} X \text{ and } \psi \in \mathcal{O}_X(V_p) \text{ s.t. } \psi|_{U \cap V}\varphi \right\}.$$

What's the picture?

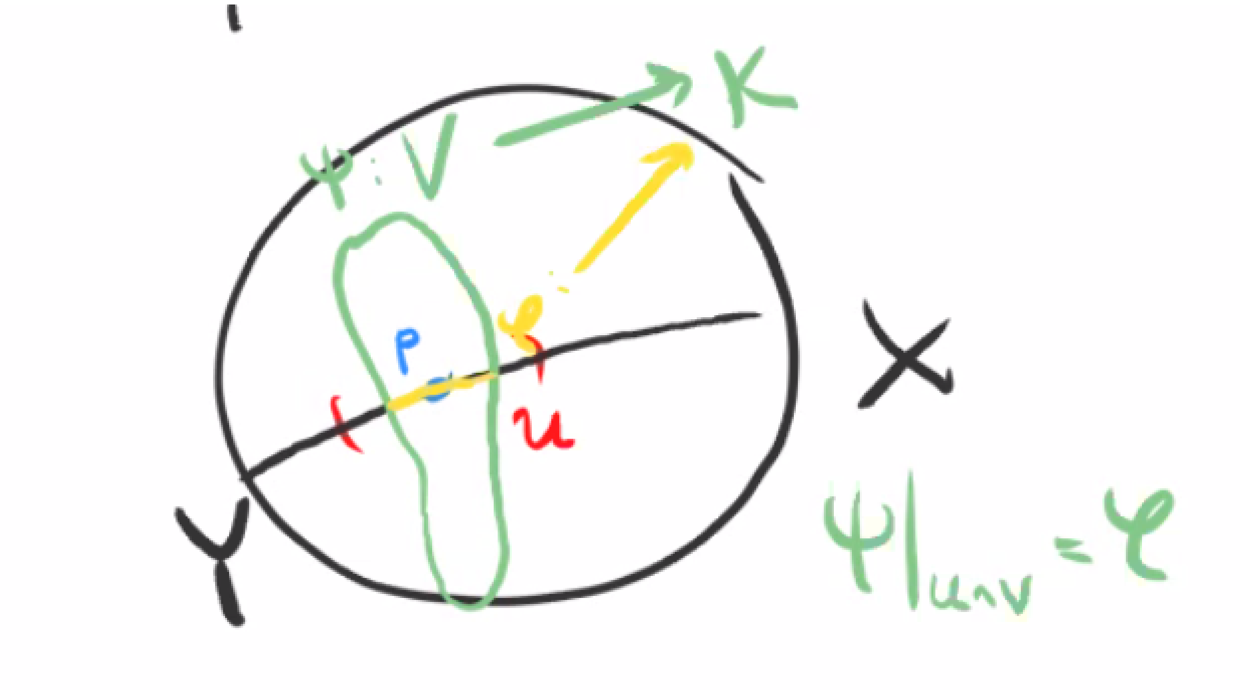

Figure 6: Sheaf for a closed subset.

It's an exercise to show that this is a prevariety.

Remark 1.2.1.
If $U \subset X$ is an open subprevariety or $Y \subset X$ is a closed subprevariety, then the inclusions are morphisms. We'd need to show that a pullback of a function is regular, but this is set up by definition.

Remark 1.2.2.
Define $\tilde{\mathcal{O}}_X(U)$ as the set of *all* functions $U \to k$. Then the inclusion $(X, \mathcal{O}_X) \hookrightarrow (X, \tilde{\mathcal{O}}_X)$ given by the identity on X is a morphism, but the reverse