

# Real Analysis

D. Zack Garza

August 15, 2019

## Contents

<b>1 Lecture 1 (Thu 15 Aug 2019 11:04)</b>	<b>1</b>
1.1 Notions of “smallness” in $\mathbb{R}$ . . . . .	1
1.2 $\mathbb{R}$ is not small . . . . .	2

---

## 1 Lecture 1 (Thu 15 Aug 2019 11:04)

See Folland’s Real Analysis, definitely a recommended reference.

Possible first day question: how can we “measure” a subset of  $\mathbb{R}$ ? We’d like bigger sets to have a higher measure, we wouldn’t want removing points to increase the measure, etc. This is not quite possible, at least something that works on *all* subsets of  $\mathbb{R}$ . We’ll come back to this in a few lectures.

### 1.1 Notions of “smallness” in $\mathbb{R}$

Definition: Let  $E$  be a set, then  $E$  is *countable* if it is in a one-to-one correspondence with  $E' \subseteq \mathbb{N}$ , which includes  $\emptyset, \mathbb{N}$ .

Definition:  $E$  is *meager* (or of *1st category*) if it can be written as a countable union of **nowhere dense** sets.

You can show that any finite subset of  $\mathbb{R}$  is meager.

Intuitively, a set is *nowhere dense* if it is full of holes. Recall that a  $X \subseteq Y$  is dense in  $Y$  iff the closure of  $X$  is all of  $Y$ . So we’ll make the following definition.

Definition: A set  $A \subseteq \mathbb{R}$  is *nowhere dense* if every interval  $I$  contains a subinterval  $S \subseteq I$  such that  $S \subseteq A^c$ .

Note that a finite union of nowhere dense sets is also nowhere dense, which is why we’re giving a name to such a countable union above. Example:  $\mathbb{Q}$  is an infinite, countable union of nowhere dense sets that is not itself nowhere dense.

Equivalently, -  $A^c$  contains a dense, open set. - The interior of the closure is empty.

We’d like to say something is measure zero exactly when it can be covered by intervals whose lengths sum to less than  $\varepsilon$ .

Definition:  $E$  is a *null set* (or has *measure zero*) if  $\forall \varepsilon > 0$ , there exists a sequence of intervals  $\{I_j\}_{j=1}^\infty$  such that

$$E \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum |I_j| < \varepsilon.$$

Exercise: show that a countable union of null sets is null.

We have several relationships

- Countable  $\implies$  Meager, but not the converse.
- Countable  $\implies$  Null, but not the converse.

Exercise: Show that the “middle third” Cantor set is not countable, but is both null and meager. Key point: the Cantor set does not contain any intervals.

Theorem: Every  $E \subseteq \mathbb{R}$  can be written as  $E = A \cup B$  where  $A$  is null and  $B$  is meager.

This gives some information about how nullity and meagerness interact – in particular,  $\mathbb{R}$  itself is neither meager nor null. Idea: if meager  $\implies$  null, this theorem allows you to write  $\mathbb{R}$  as the union of two null sets. This is bad!

Proof: We can assume  $E = \mathbb{R}$ . Take an enumeration of the rationals, so  $\mathbb{Q} = \{q_j\}_{j=1}^\infty$ . Around each  $q_j$ , put an interval around it of size  $1/2^{j+k}$  where we’ll allow  $k$  to vary, yielding multiple intervals around  $q_j$ . To do this, define  $I_{j,k} = (q_j - 1/2^{j+k}, q_j + 1/2^{j+k})$ . Now let  $G_k = \bigcup_j I_{j,k}$ . Finally, let  $A = \bigcap_k G_k$ ; we claim that  $A$  is null.

Note that  $\sum_j |I_{j,k}| = \frac{1}{2^k}$ , so just pick  $k$  such that  $\frac{1}{2^k} < \varepsilon$ .

Now we need to show that  $A^c := B$  is meager. Note that  $G_k$  covers the rationals, and is a countable union of open sets, so it is dense. So  $G_k$  is an open and dense set. By one of the equivalent formulations of meagerness, this means that  $G_k^c$  is nowhere dense. But then  $B = \bigcup_k G_k^c$  is meager.

## 1.2 $\mathbb{R}$ is not small

Theorem A (Cantor):  $\mathbb{R}$  is not countable.

Theorem B (Baire):  $\mathbb{R}$  is not meager. (Baire Category Theorem)

Theorem C (Borel):  $\mathbb{R}$  is not null.

Note that theorems B and C imply theorem A. You can also replace  $\mathbb{R}$  with any nonempty interval  $I = [a, b]$  where  $a < b$ . This is a strictly stronger statement – if any subset of  $\mathbb{R}$  is not countable, then certainly  $\mathbb{R}$  isn’t, and so on.

Proof of (A): begin by thinking of  $I = [0, 1]$ , then every number here has a unique binary expansion. So we are reduced to showing that the set of all Bernoulli sequences (infinite length strings of 0 or 1) is uncountable. Then you can just apply the usual diagonalization argument by assuming they are countable, constructing the table, and flipping the diagonal bits to produce a sequence differing from every entry.

A second proof: Take an interval  $I$ , and suppose it is countable so  $I = \{x_i\}$ . Choose  $I_1 \subseteq I$  that avoids  $x_1$ , so  $x_1 \notin I_1$ . Choose  $I_2 \subseteq I_1$  avoiding  $x_2$  and so on to produce a nested sequence of closed intervals. Since  $\mathbb{R}$  is complete, the intersection  $\bigcap_{n=1}^\infty I_n$  is nonempty, so say it contains  $x$ . But then  $x \in I_1 \in I$ , for example, but  $x \neq x_i$  for any  $i$ , so  $x \notin I$ , a contradiction.  $\square$

Proof of (B): Suppose  $I = \bigcup_{i=1}^{\infty} A_n$  where each  $A_n$  is nowhere dense. We'll again construct a nested sequence of closed sets. Let  $I_1 \subseteq I$  be a subinterval that misses all of  $A_1$ , so  $A_1 \cap I_1 = \emptyset$  using the fact that  $A_1$  is nowhere dense. Repeat the same process, let  $I_2 \subset I_1 \setminus A_2$ . By the nested interval property, there is some  $x \in \bigcap A_i$ .

Note that we've constructed a meager set here, so this argument shows that the complement of any meager subset of  $\mathbb{R}$  is nonempty. Setting up this argument in the right way in fact shows that this set is dense! Taking the contrapositive yields the usual statement of Baire's Category Theorem.

Thomae function: continuous on  $\mathbb{Q}$ , but discontinuous on  $\mathbb{R} \setminus \mathbb{Q}$ . Can this be switched to get some  $f$  that is continuous on  $\mathbb{R} \setminus \mathbb{Q}$  and discontinuous on  $\mathbb{Q}$ ? The answer is no. The set of discontinuities of a function is *always* an  $F_{\sigma}$  set, and  $\mathbb{R} \setminus \mathbb{Q}$  is not one. Equivalently, the rationals are not a  $G_{\delta}$  set.

The pointwise limit of continuous functions has a meager set of discontinuities. If  $f$  is integrable, the set of discontinuities is null. If  $f$  is monotone, they are countable. There is a continuous nowhere differentiable function: let  $f(x) = \sum_n \frac{\|10^n x\|}{10^n}$ , and in fact *most* functions are like this. If  $f$  is continuous and monotone, the discontinuities are null.