Assignment 6: The Fourier Transform

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \to \infty} \hat{f}(\xi) = \lim_{\xi' \to 0} \frac{1}{2} \int_{\mathbb{R}^n} (f()) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \ dx$$

.

But as an immediate consequence, this yields

$$\begin{aligned}
\left| \hat{f}(\xi) \right| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \ dx \right| \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \ dx \\
&\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \ dx \\
&\to 0,
\end{aligned}$$

which follows from continuity in L^1 since $f(x - \xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds, and that $\xi' \to 0$ as $\xi \to \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$
\begin{aligned}
\widehat{(f * g)}(\xi) &= \int \int f(x-y)g(y)\exp(-2\pi x \cdot \xi)\ dy\ dx \qquad =_? \int \int f(x-y)g(y)\exp(-2\pi x \cdot \xi)\ dx\ dy \\
&= \int \int f(t)\exp(-2\pi i(x-y)\cdot \xi)g(y)\exp(-2\pi i y \cdot \xi)\ dx\ dy \\
&\quad (t = x - y,\ dt = \ dx) \\
&= \int \int f(t)\exp(-2\pi i t \cdot \xi)g(y)\exp(-2\pi i y \cdot \xi)\ dt\ dy \\
&= \int f(t)\exp(-2\pi i t \cdot \xi)\left(\int g(y)\exp(-2\pi i y \cdot \xi)\ dy\right)\ dt \\
&= \int f(t)\exp(-2\pi i t \cdot \xi)\hat{g}(\xi)\ dt \\
&= \hat{g}(\xi)\int f(t)\exp(-2\pi i t \cdot \xi)\ dt \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}
$$

3 Problem 3

4 Problem 4

5 Problem 5

6 Problem 6