CRAG

The Weil Conjectures

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Background: Generating Functions

Varieties

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall \; n \in \mathbb{Z}^{\geq 2}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \cdots, f_M \rangle \;\trianglelefteq\; k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I}=(i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^{\times}.$$

Point Counts

- For a fixed variety X, we can consider its $\mathbb{F}$-points $X(\mathbb{F})$.
  - Note that $\#X(\mathbb{F}) < \infty$ is an integer
- For any $L/\mathbb{F}$, we can also consider $X(L)$
  - In particular, we can consider $X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
  - We again have $\#X(\mathbb{F}_{q^n}) < \infty$ and are integers for every such n.
- So we can consider the sequence

$$[N_1, N_2, \cdots, N_n, \cdots] := [\#X(\mathbb{F}),\ \#X(\mathbb{F}_{q^2}), \cdots,\ \#X(\mathbb{F}_{q^n}), \cdots].$$

- Idea: associate some generating function (a formal power series) encoding sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \cdots.$$

Why Generating Functions?

Note that for such an ordinary generating functions, the coefficients are related to the real-analytic properties of F: we can easily recover the coefficients in the following way:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n F(z) \Big|_{z=0} = N_n.$$

They are also related to the complex analytic properties: using the Residue theorem,

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} \, dz = N_n.$$

The latter form is very amenable to computer calculation.

Why Generating Functions?

An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \longrightarrow \mathbb{C}$ – in nice situations, we can hope for a closed-form representation.

A useful example: by integrating a geometric series we can derive

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \qquad\qquad (= 1 + z + z^2 + \cdots) \\
\implies \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \int z^n \quad for |z| < 1 \quad \text{by uniform convergence} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\
\implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad\qquad \left(= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).
\end{aligned}$$

For completeness, also recall that

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Zeta Functions

Definition: Local Zeta Function

Problem: count points of a (smooth?) projective variety $X/\mathbb{F}$ in all (finite) degree n extensions of $\mathbb{F}$.

Definition (Local Zeta Function)

The *local zeta function* of an algebraic variety X is the following formal power series:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n}\right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note that

$$\begin{aligned} z\left(\frac{\partial}{\partial z}\right) \log Z_X(z) &= z\frac{\partial}{\partial z}\left(N_1 z + N_2 \frac{z^2}{2} + N_3 \frac{z^3}{3} + \cdots\right) \\ &= z\left(N_1 + N_2 z + N_3 z^2 + \cdots\right) \qquad\qquad \text{(unif. conv.)} \\ &= N_1 z + N_2 z^2 + \cdots = \sum_{n=1}^{\infty} N_n z^n, \end{aligned}$$

which is an *ordinary* generating function for the sequence (N_n).

Examples

Example: A Point

Take $X = \{\mathrm{pt}\} = V(\{f(x) = 0\})/\mathbb{F}$ a single point over $\mathbb{F}$, then

$$\begin{aligned}
\#X(\mathbb{F}_q) &:= N_1 = 1 \\
\#X(\mathbb{F}_{q^2}) &:= N_2 = 1 \\
&\vdots \\
\#X(\mathbb{F}_{q^n}) &:= N_n = 1 \\
&\ddots
\end{aligned}$$

and so

$$\begin{aligned}
Z_{\{\mathrm{pt}\}}(z) &= \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \cdots\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) \\
&= \exp\left(-\log\left(1 - z\right)\right) \\
&= \frac{1}{1 - z}.
\end{aligned}$$

Notice: Z admits a closed form **and** *is a rational function.*

Example: The Affine Line

Take $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^1(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1] \;\middle|\; x_1 \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned} X(\mathbb{F}_q) &= q \\ X(\mathbb{F}_{q^2}) &= q^2 \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^n. \end{aligned}$$

Then

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} \right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(qz)^n}{n} \right) \\ &= \exp(-\log(1 - qz)) \\ &= \frac{1}{1 - qz}. \end{aligned}$$

Take $X = \mathbb{A}^m/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^m(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned} X(\mathbb{F}_q) &= q^m \\ X(\mathbb{F}_{q^2}) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^{nm}. \end{aligned}$$

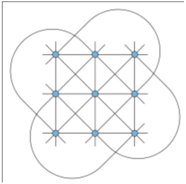


Figure: $\mathbb{A}^2/\mathbb{F}_3$ $(q = 3, m = 2, n = 1)$

Then

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^{nm} \frac{z^n}{n}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{(q^m z)^n}{n}\right) \\ &= \exp(-\log(1 - q^m z)) \\ &= \frac{1}{1 - q^m z}. \end{aligned}$$

Example: Projective Line

Take $X = \mathbb{P}^1/\mathbb{F}$, we can still count by enumerating coordinates:

$$\mathbb{P}^1(\mathbb{F}_{q^n}) = \left\{ [x_1 : x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}_{q^n} \right\} / \sim \quad = \left\{ [x_1 : 1] \;\middle|\; x_1 \in \mathbb{F}_{q^n} \right\} \coprod \{[1:0]\}.$$

Thus

$$\begin{aligned} X(\mathbb{F}_q) &= q + 1 \\ X(\mathbb{F}_{q^2}) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^n + 1. \end{aligned}$$

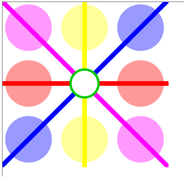


Figure: $\mathbb{P}^1/\mathbb{F}_3$ $(q = 3, n = 1)$

Thus

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} (q^n + 1) \frac{z^n}{n} \right) \\ &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} + \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \\ &= \frac{1}{(1 - qz)(1 - z)}. \end{aligned}$$

The Weil Conjectures

Weil 1

(Weil 1949)

Let X be a smooth projective variety of dimension N over $\mathbb{F}_q$ for q a prime, let $Z_X(z)$ be its zeta function, and define $\zeta_X(s) = Z_X(q^{-s})$.

1. (Rationality)
   $Z_X(z)$ is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$

$$P_{2N}(z) = 1 - q^N z$$

$$P_j(z) = \prod_{j=1}^{\beta_i} (1 - a_{j,k} z) \quad \text{for some reciprocal roots} \quad a_{j,k} \in \mathbb{C}$$

where we've factored each P_i using its reciprocal roots a_{ij}.
In particular, this implies the existence of a meromorphic continuation of the associated function $\zeta_X(s)$, which a priori only converges for $\Re(s) \gg 0$. This also implies that for n large enough, N_n satisfies a linear recurrence relation.

2. (Functional Equation and Poincare Duality)
Let $\chi(X)$ be the Euler characteristic of X, i.e. the self-intersection number of the diagonal embedding $\Delta \hookrightarrow X \times X$; then $Z_X(z)$ satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm\left(q^{\frac{N}{2}} z\right)^{\chi(X)} \; Z_X(z).$$

Equivalently,

$$\zeta_X(N-s) = \pm\left(q^{\frac{N}{2}-s}\right)^{\chi(X)} \zeta_X(s)$$

.

Note that when $N = 1$, e.g. for a curve, this relates $\zeta_X(s)$ to $\zeta_X(1-s)$.

Equivalently, there is an involutive map on the (reciprocal) roots

$$z \iff \frac{q^N}{z}$$
$$\alpha_{j,k} \iff \alpha_{2N-j,k}$$

which sends roots of p_j to roots of p_{2N-j}.

3. (Riemann Hypothesis)
   The reciprocal roots $a_{j,k}$ are *algebraic* integers (roots of some monic $p \in \mathbb{Z}[x]$) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}}, \qquad 1 \leq j \leq 2N-1, \ \forall k.$$

4. (Betti Numbers)
   If X is a "good reduction mod q" of a nonsingular projective variety $\tilde{X}$ in characteristic zero, then the $\beta_i = \mathbf{deg}\, p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$.

Moral:

- The Diophantine properties of a variety's zeta function are governed by its (algebraic) topology.
- Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information.

Why is (3) called the "Riemann Hypothesis"?

Suppose it holds. We can use the facts that

a. $|\exp(z)| = \exp(\Re(z))$ and
b. $a^z := \exp(z\mathrm{Log}(a))$,

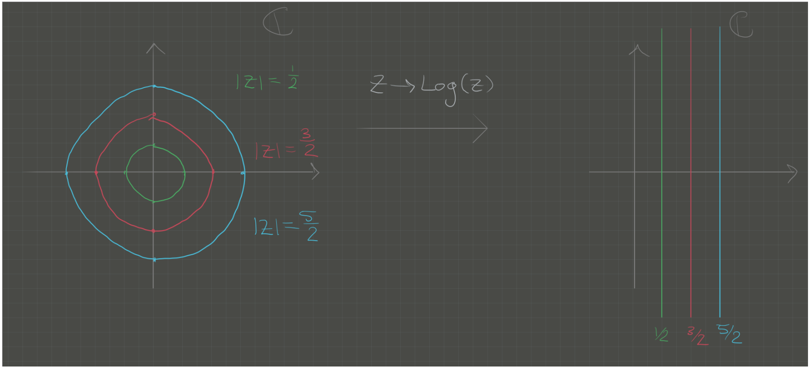
and to replace the polynomials P_i with

$$L_j(s) := P_j(q^{-s}) = \prod_{k=1}^{\beta_j} \left(1 - \alpha_{j,k} q^{-s}\right).$$

Now consider the roots of $L_j(s)$: we have

$$
\begin{aligned}
& L_j(s_0) = 0 \\
\iff \quad & q^{-s_0} = \frac{1}{\alpha_{j,k}} \quad \text{for some} \quad k \\
\implies \quad & \left|q^{-s_0}\right| = \left|\frac{1}{\alpha_{j,k}}\right| \overset{\text{by assumption}}{=} q^{-\frac{j}{2}} \\
\implies \quad & q^{-\frac{j}{2}} \overset{(a)}{=} \exp\left(-\frac{j}{2} \cdot \mathrm{Log}(q)\right) = \left|\exp\left(-s_0 \cdot \mathrm{Log}(q)\right)\right| \\
& \overset{(b)}{=} \left|\exp\left(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \mathrm{Log}(q)\right)\right| \\
& \overset{(a)}{=} \exp\left(-(\Re(s_0)) \cdot \mathrm{Log}(q)\right) \\
\implies \quad & -\frac{j}{2} \cdot \mathrm{Log}(q) = -\Re(s_0) \cdot \mathrm{Log}(q) \quad \text{by injectivity} \\
\implies \quad & \Re(s_0) = \frac{j}{2}.
\end{aligned}
$$

Roughly speaking, realizing that we would need to apply a logarithm (a conformal map) to send the $\alpha_{j,k}$ to zeros of the L_j, this says that the zeros all must lie on the "critical lines" $\frac{j}{2}$.



In particular, the zeros of L_1 have real part $\frac{1}{2}$, analogous to the classical Riemann hypothesis.

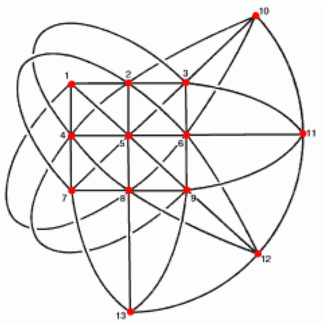
Projective m-space

Take $X = \mathbb{P}^m/\mathbb{F}$ We can write

$$\mathbb{P}^m(\mathbb{F}_{q^n}) = \mathbb{A}^{m+1}(\mathbb{F}_{q^n}) \setminus \{\mathbf{0}\} / \sim \quad = \left\{ \mathbf{x} = [x_0, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\} / \sim$$

But how many points are actually in this space?

Figure: Points and Lines in $\mathbb{P}^2/\mathbb{F}_3$



A nontrivial combinatorial problem!

To illustrate, this can be done combinatorially: identify $\mathbb{P}_{\mathbb{F}}^m = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}_{\mathbb{F}}^{m+1}$.

Theorem

The number of k-dimensional subspaces of $\mathbb{A}_{\mathbb{F}_q}^N$ is the q-analog of the binomial coefficient:

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q^N-1)(q^{N-1}-1)\cdots(q^{N-(k-1)}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

Remark: Note $\lim_{q\longrightarrow 1} \begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k}$, *the usual binomial coefficient.*

Proof: To choose a k-dimensional subspace,

- Choose a nonzero vector $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$ in $q^N - 1$ ways.
  - For next step, note that $\#\mathrm{span}\{v_1\} = \#\left\{\lambda v_1 \;\middle|\; \lambda \in \mathbb{F}_q\right\} = \#\mathbb{F}_q = q$.
- Choose a nonzero vector $\mathbf{v}_2$ *not* in the span of $\mathbf{v}_1$ in $q^N - q$ ways.
  - Now note $\#\mathrm{span}\{v_1, v_2\} = \#\left\{\lambda_1 v_1 + \lambda_2 v_2 \;\middle|\; \lambda_i \in \mathbb{F}\right\} = q \cdot q = q^2$.

Proof continued

- Choose a nonzero vector $\mathbf{v}_3$ *not* in the span of $\mathbf{v}_1, \mathbf{v}_2$ in $q^N - q^2$ ways.
- $\cdots$ until $\mathbf{v}_k$ is chosen in

$$(q^N - 1)(q^N - q)\cdots(q^N - q^{k-1}) \quad \text{ways}$$

.

  - This yields a k-tuple of linearly independent vectors spanning a k-dimensional subspace V_k
- This overcounts because many linearly independent sets span V_k, we need to divide out by the number of ways to choose a basis inside of V_k.
- By the same argument, this is given by

$$(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$$

Thus

$$\#\text{subspaces} = \frac{(q^N - 1)(q^N - q)(q^N - q^2)\cdots(q^N - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2)\cdots(q^k - q^{k-1})}$$

$$= \frac{q^N - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{N-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{N-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{N-(k-1)} - 1}{q^{k-(k-1)-1}}$$

$$= \frac{(q^N - 1)(q^{N-1} - 1)\cdots(q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

Counting Points

Note that we've actually computed the number of points in any Grassmannian.

Identify $\mathbb{P}^m_{\mathbb{F}} = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}^{m+1}_{\mathbb{F}}$.

We obtain a nice simplification for the number of lines corresponding to setting $k = 1$:

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = \frac{q^{m+1}-1}{q-1} = q^m + q^{m-1} + \cdots + q + 1 = \sum_{j=0}^m q^j.$$

Thus

$$\begin{aligned} X(\mathbb{F}_q) &= \sum_{j=0}^m q^j \\ X(\mathbb{F}_{q^2}) &= \sum_{j=0}^m \left(q^2\right)^j \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= \sum_{j=0}^m (q^n)^j. \end{aligned}$$

Computing the Zeta Function

So

$$\begin{aligned}
Z_X(z) &= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}(q^n)^j\frac{z^n}{n}\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}\frac{\left(q^j z\right)^n}{n}\right)\\
&= \exp\left(\sum_{j=0}^{m}\sum_{n=1}^{\infty}\frac{\left(q^j z\right)^n}{n}\right)\\
&= \exp\left(\sum_{j=0}^{m-1}-\log(1-q^j z)\right)\\
&= \prod_{j=0}^{m}\left(1-q^j z\right)^{-1}\\
&= \left(\frac{1}{1-z}\right)\left(\frac{1}{1-qz}\right)\left(\frac{1}{1-q^2 z}\right)\cdots\left(\frac{1}{1-q^m z}\right),
\end{aligned}$$

Miraculously, still a rational function!

An Easier Proof

Quick recap:

$$Z_{\{\mathrm{pt}\}} = \frac{1}{1-z} \qquad Z_{\mathbb{P}^1}(z) = \frac{1}{1-qz} \qquad Z_{\mathbb{A}^1}(z) = \frac{1}{(1-z)(1-qz)}.$$

Note that $\mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$ and correspondingly $Z_{\mathbb{P}^1}(z) = Z_{\mathbb{A}^1}(z) \cdot Z_{\{\mathrm{pt}\}}(z)$. This works in general:

Lemma (Excision)

If $Y/\mathbb{F}_q \subset X/\mathbb{F}_q$ is a closed subvariety, for $U = X \setminus Y$, $Z_X(z) = Z_Y(z) \cdot Z_U(z)$.

Proof: Let $N_n = \#Y(\mathbb{F}_{q^n})$ and $M_n = \#U(\mathbb{F}_{q^n})$, then

$$\begin{aligned}
\zeta_X(z) &= \exp\left(\sum_{n=1}^\infty (N_n + M_n)\frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^\infty N_n \cdot \frac{z^n}{n} + \sum_{n=1}^\infty M_n \cdot \frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^\infty N_n \cdot \frac{z^n}{n}\right) \cdot \exp\left(\sum_{n=1}^\infty M_n \cdot \frac{z^n}{n}\right) = \zeta_Y(z) \cdot \zeta_U(z).
\end{aligned}$$

A Easier Proof

Note that geometry can help us here: we have a stratification $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$, and so inductively

$$\mathbb{P}^m = \coprod_{j=0}^m \mathbb{A}^j = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m,$$

and recalling that

$$Z_{X \coprod Y}(z) = Z_X(z) \cdot Z_Y(z)$$

and $Z_{\mathbb{A}^j}(z) = \frac{1}{1-q^j z}$ we have

$$Z_{\mathbb{P}^m}(z) = \prod_{j=0}^m Z_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1-q^j z}.$$

Notice that the highest degree is exactly m, and there is exactly one factor for each $j \leq m$. Note that $PP^m/\mathbb{F}_q$ can be though of as a mod q reduction of $\mathbb{RP}^m$ or $\mathbb{CP}^m$, and somehow Z "sees" its dimension.

Grassmannian

Motivation

Consider now $X = \mathrm{Gr}(k, m)/\mathbb{F}$ – by the previous computation, we know

$$X(\mathbb{F}_{q^n}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} := \frac{(q^{nm}-1)(q^{nm-1}-1)\cdots(q^{nm-n(k-1)}-1)}{(q^{nk}-1)(q^{n(k-1)}-1)\cdots(q^n-1)}$$

but the corresponding Zeta function is much more complicated than the previous examples:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} \frac{z^n}{n}\right) = \cdots ?.$$

Note that $\dim_{\mathbb{R}} \mathrm{Gr}_{\mathbb{R}}(k, m) = k(m-k)$ as a real manifold, so