

# Commutative Algebra

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# 1 Wednesday January 8

Course text: <http://math.uga.edu/~pete/integral2015.pdf>

Summary: The study of commutative rings, ideals, and modules over them.

The chapters we'll cover:

- 1 (Intro),
- 2 (Modules, partial),
- 3 (Ideals, CRT),
- 7 (Localization),
- 8 (Noetherian Rings),
- 11 (Nullstellensatz),
- 12 (Hilbert-Jacobson rings),
- 13 (Spectrum),
- 14 (Integral extensions),
- 17 (Valuation rings),
- 18 (Normalization),
- 19 (Picard groups),
- 20 (Dedekind domains),
- 22 (1-dim Noetherian domains)

In number theory, arises in the study of  $\mathbb{Z}_k$ , the ring of integers over a number field  $k$ , along with *localizations* and *orders* (both preserve the fraction field?).

In algebraic geometry, consider  $R = k[t_1, \dots, t_n]/I$  where  $k$  is a field and  $I$  is an ideal.

Some preliminary results:

1. In  $\mathbb{Z}_k$ , ideals factor uniquely into primes (i.e. it is a Dedekind domain).
2.  $\mathbb{Z}_k$  has an integral basis (i.e. as a  $\mathbb{Z}$ -modules,  $\mathbb{Z}_k \cong \mathbb{Z}^{[k:\mathbb{Q}]}$ ).
3. The Nullstellansatz: there is a bijective correspondence

$$\{\text{Irreducible Zariski closed subsets of } \mathbb{C}^n\} \iff \{\text{Prime ideals in } \mathbb{C}[t_1, \dots, t_n]\}.$$

4. Noether normalization (a structure theorem for rings of the form  $R$  above).

All of these results concern particularly “nice” rings, e.g.  $\mathbb{Z}_k, \mathbb{C}[t_1, \dots, t_n]$ . These rings are

- Domains
- Noetherian
- Finitely generated over other rings
- Finite Krull dimension (supremum of length of chains of prime ideals)
  - In particular,  $\dim \mathbb{Z}_k = 1$  since nonzero prime ideals are maximal in a Dedekind domain
- Regular (nonsingularity condition, can be interpreted in scheme-theoretic language)

Note: schemes will have “local charts” given by commutative rings, analogous to building a manifold from Euclidean  $n$ -space. General philosophy (Grothendieck): Every commutative ring is the ring of functions on some space, so we should study the category of commutative rings as a whole (i.e. let the rings be arbitrary). This does not hold for non-commutative rings! I.e. we can’t necessarily associate a geometric space to every non-commutative ring. A common interesting example:  $k[G]$ , the group ring of an arbitrary group. Good references: Lam, ‘Lectures on Modules and Rings’.

*Example:* Let  $X$  be a topological space and  $C(X)$  be the continuous functions  $f : X \rightarrow \mathbb{R}$ . This is a ring under pointwise addition/multiplication. (This generally holds for the hom set into any commutative ring.)

*Example:* Take  $X = [0, 1]$  and  $C(X)$  as a ring.

**Exercise:**

1. Show that  $C(X)$  is not a domain.

Hint: find two nonzero functions whose product is identically zero, e.g. bump functions. Note that they are not analytic/holomorphic.

2. Show that it is not Noetherian (i.e. there is an ideal that is *not* finitely generated).
3. Fix a point  $x \in [0, 1]$  and show that the ideal  $\mathfrak{m}_x = \{f \mid f(x) = 0\}$  is maximal.
4. Are all maximal ideals of this form?

Hint: See textbook chapter 5, or Gilman and Jerison ‘Rings of Continuous Functions’.

**Theorem of Swan:** A theorem about topological vector bundles over  $C([0, 1])$ , see textbook. There is a categorical equivalence between vector bundles on a compact space and f.g. projective modules over this ring.

So commutative algebra has something to say about other branches of Mathematics!

**Definition:** A topological space is called *boolean* (or a *Stone space*) iff it is compact, hausdorff, and totally disconnected.

*Example:* A projective variety over  $p$ -adics with  $\mathbb{Q}_p$  points plugged in.

**Definition:** A ring is boolean if every element is idempotent, i.e.  $x \in R \implies x^2 = x$ .

**Exercise:** If  $R$  is a boolean domain, then it is isomorphic to the field with 2 elements.

**Lemma:** There is a categorical equivalence between Boolean spaces, Boolean rings, and so-called “Boolean algebras”.

## 2 Monday January 13

### 2.1 Logistics

Some topics for final projects

- The cardinal Krull dimension of  $\text{Hol}(X)$ .
- Galois connections
- Ordinal filtrations

- Lam-Reyes prime ideal principal
- $C(X)$
- $\text{Hol}(X)$
- Semigroup rings
- Swan's Theorem
  - Vector bundles on a compact space
- Boolean rings and Stone duality
- More Nullstellansatz
  - Beyond Hilbert's usual one
- Hochster's Theorem
  - Characterizes  $\text{Spec} R$  as a topological space, i.e. when is a topological space homeomorphic to the spectrum of some commutative ring.
- Invariant theory (quotients of rings under finite group actions, i.e.  $R^G$  for  $|G| < \infty$ )
  - For  $R = k$  a field, this is Galois theory
  - Easy case of geometric invariant theory, when  $G$  is infinite
- UFDs
  - What conditions does a ring need to have to ensure unique factorization?
- Euclidean rings
- Claborn (Leedham-Green-Clark): Every commutative group is (up to isomorphism) the class group of some Dedekind domain.
  - A type of inverse problem, class group measures deviation from being a UFD
  - Uses ordinal filtrations, transfinite induction
  - See proof in elliptic curves course

## 2.2 Rings of Functions

Let  $k$  be a field,  $X$  a set of cardinality  $|X| \geq 2$ , and define  $k^X := \text{Maps}(X, k) = \{f : X \rightarrow k\}$  is a ring under pointwise addition and multiplication. As a ring, this is a (big!) cartesian product.

*Some facts:*

- $k^X$  is not a domain (**exercise**), and there are nontrivial idempotents ( $e^2 = e$ )
- Note: it could be worse and have nilpotents.
- $k^X$  is *reduced*, i.e. it has no nonzero nilpotents, where  $z \in R$  is nilpotent iff  $\exists n \geq 1$  such that  $z^n = 0$ .
    - Note: domains are reduced, cartesian products of reduced rings are reduced.
  - Every subring of  $k^X$  is reduced.

Moral: should be viewing every ring as functions on some space, but this can't literally be true because of the above restrictions. Nilpotent elements are "hard to view as functions".

- For  $X$  a topological space,  $C(X)$  the ring of continuous functionals to  $\mathbb{R}$ , then  $C(X) \subset \mathbb{R}^X$ .

**Exercise:** When is  $C(X)$  a domain? (Note that we can have products of nonzero functions being identically zero.)

*Example:* Let  $R$  be the ring of holomorphic functions  $\mathbb{C}^\circ$ , i.e.  $\text{Hol}(\mathbb{C}, \mathbb{C}) := \{f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is holomorphic}\}$ .

The set of zeros of such an  $f$  must be discrete, the example of bump functions doesn't go through holomorphically.

This is a domain, not Noetherian, not a PID, but every f.g. ideal is principal (thus this is a Bezout domain, a non-Noetherian analog of a PID).

It has infinite Krull dimension: recall that ideals are prime iff  $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$  or  $y \in \mathfrak{p}$  iff  $R/\mathfrak{p}$  is a domain, and the Krull dimension is the supremum  $S$  of lengths of chains of prime ideals (only when  $S$  is finite).

If  $C \subset (X, \leq)$  is a finite-length chain in a totally ordered set, then the length  $\ell(C) = |C| - 1$  (1 less than the number of elements appearing). The *cardinal Krull dimension* of a ring  $R$  is the actual supremum.

Note: in Noetherian rings, there can still be finite but unbounded length chains.

Letting  $X$  be a complex manifold (i.e. covered by subsets of  $\mathbb{C}^n$  with holomorphic transition functions) and let  $\text{Hol}(X)$  be the holomorphic functionals  $f : X \rightarrow \mathbb{C}$ . Then  $\text{Hol}(X)$  is a domain iff  $X$  is connected.

Note that if  $X$  is disconnected, we can take a function that is constant on one component and zero on another, then switch, then multiply to get a zero function.

If  $X$  is a compact connected projective variety, then  $\text{Hol}(X)$  is just constant functions by the open mapping functions. So  $\text{Hol}(X) = \mathbb{C}$ , and  $\text{carddim}\mathbb{C} = 0$  because for any field there are only two ideals, and here  $(0)$  is prime. Moreover,  $\text{carddim}\text{Hol}(\mathbb{C}) \geq \alpha_0$ .

Note that for complex manifolds,  $X$  is either compact or supports many holomorphic functions.

**Theorem:** If  $X$  is a connected complex manifold which has a nontrivial holomorphic function, i.e.  $\text{Hol}(X) \supset \mathbb{C}$ , then there exists a chain of prime ideals in  $\text{Hol}(X)$  of length  $|\mathbb{R}| > \aleph_0$ , i.e. it has at least the cardinality of the continuum.

Note: the cardinality could be even bigger!

Maximals are prime: equivalent to fields are integral domains.

## 2.3 Rings

Take all rings to be unital, i.e. containing 1. A ring without identity is referred to as an *rng*. In this course, all rings are commutative.

*Example:* This is a fairly special restriction. Take  $(A, +)$  a commutative group and define  $\text{End}(A) = \{f : A \rightarrow A\}$  the ring of group homomorphisms under pointwise addition and composition. This is generally not commutative, i.e.  $\text{End}(\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)) = M_2(\mathbb{Z}/(2))$  the ring of matrices with  $\mathbb{Z}/(2)$  entries, which is not commutative.

**Exercise:** Given  $(A, +)$ , show that  $\text{End}(\bigoplus^n A) = M_n(\text{End}(A))$ .

Generally, if  $R$  is a ring and  $M$  is an  $R$ -module, then  $\text{End}_R(M) = \{f : M \rightarrow M\}$  of  $R$ -module homomorphisms is always a ring under pointwise addition and composition, and is (probably) non-commutative.

### 3 Wednesday January 15th

**Cayley's theorem:** For  $G$  a group, then there is a canonical injective group homomorphism  $\Phi : G \hookrightarrow \text{Sym}(G) \cong S_n$  for  $n = |G|$ . The map is given by  $g \mapsto g \cdot$ , i.e. multiplying on the left.

Is there an analog for rings?

Take a similar map:

$$\begin{aligned} R &\rightarrow \text{End}_{\mathbb{Z}}(R, +) \\ r &\mapsto (x \mapsto rx). \end{aligned}$$

Unfortunately there is no specialization for commutative groups/rings –  $\text{Sym}(G)$  for example is noncommutative when  $|G| \geq 2$ . Similarly, even if  $R$  is commutative,  $\text{End}(R, +)$  is probably not. As per the Grothendieck philosophy, we find that all rings are a ring of functions on something – namely themselves, since this map is injective.

All rings are commutative here, so take  $R^\times = \{x \in R \mid \exists y \text{ s.t. } xy = 1\}$ . For  $R$  a group,  $R^\times$  is a commutative group, so this is an interesting invariant.

Another interesting invariant: the class group.

*Notation:* Let  $R^\bullet = R \setminus 0$ . An element  $x \in R$  is a zero divisor iff there exists  $y \in R^\bullet$  such that  $xy = 0$ . For  $x, y \in R$  we write  $x \mid y$  iff  $\exists z \in R$  such that  $xz = y$ .

$R$  is a domain iff 0 is the only zero divisors, i.e.  $xy = 0 \implies x = 0$  or  $y = 0$ .  $(R^\bullet, \cdot)$  is a commutative monoid (group without inverses) iff  $R$  is a domain. Observe that  $R$  is a field iff  $R^\bullet = R^\times$ .

For rings  $R, S$  we have the usual definition of ring homomorphism, additionally requiring  $f(1) = 1$ . Note that  $f(0) = 0$  follows from  $f(x+y) = f(x) + f(y)$ , but  $f(1) = 1$  does not. Rings have products  $R_1 \times R_2$  which is again a ring under coordinate-wise operations. Note that there are canonical projections  $\pi_i R_1 \times R_2 \rightarrow R_i$ . There is a dual inclusion  $\iota_1 : R_1 \rightarrow R_1 \times R_2$  given by  $x \mapsto (x, 0)$ , but these are not ring homomorphisms (although everything is a group homomorphism). This is because  $\iota_1(1) = (1, 0) \neq (1, 1)$ , the identity of  $R_1 \times R_2$ . Note that 1 always has to map to an idempotent element, i.e.  $e^2 = e$ , and idempotents are always zero divisors. Also note that the map  $x \mapsto 0$  is not a ring homomorphism unless  $S = 0$ .

**Definition:** A ring homomorphism is a map  $f : R \rightarrow S$  is an isomorphism iff it has a two-sided inverse, i.e. there exists a morphism  $g : S \rightarrow R$  with  $g \circ f = \text{id}_R$  and  $f \circ g = \text{id}_S$ .

**Exercise:** Check that this is equivalent to  $f$  being a bijection.

**Exercise:** Check that the zero ring is the final object in the category of rings. Show that  $\mathbb{Z}$  is the initial object in this category?

$R$  is a subring of  $S$  iff  $R \subset S$  and the inclusion  $R \hookrightarrow S$  is a morphism.

Adjoining elements: Suppose  $R \leq S$  is a subring and  $X \subset S$  is just a subset. Then there exists a ring  $R[X]$  such that

- Top-down description:  $R[X] \leq S$  is a subring containing  $R$  and  $X$ , and is minimal with respect to this property (obtained by intersecting all such subrings)

- Bottom-up description: things resembling  $\sum r_i x_i$

**Exercise 1.6:** Take  $R = \mathbb{Z}$ ,  $S = \mathbb{Q}$ ,  $P$  a arbitrary set of prime numbers. Let  $\mathbb{Z}_{\mathcal{P}} = \mathbb{Z}[\{\frac{1}{p} \mid p \in P\}]$ .

- a. When do we have  $\mathbb{Z}_{\mathcal{P}_1} \cong \mathbb{Z}_{\mathcal{P}_2}$ ?

Hint: take  $P_1 = \{3, 7, 11\}$ ,  $P_2 = \{5\}$ . Need  $P_1 = P_2$ !

- b. Show that every subring  $T$  such that  $\mathbb{Z} \leq T \leq \mathbb{Q}$  is of the form  $\mathbb{Z}_{\mathcal{P}}$  for some unique set of primes  $P$ .

Note that if  $T$  is any intermediate ring between  $R$  and  $S$ , then  $R[T] = T$ .

### 3.1 Ideals and Quotients

For  $f : R \rightarrow S$  a ring homomorphism, define  $I = \ker f = f^{-1}(\{0\})$ . Then  $I$  is a subgroup of  $(R, +)$ , and for all  $i \in I$  and all  $r \in R$  we have  $ri \in I$ , since  $f(ri) = f(r)f(i) = f(r)0 = 0$ . In other words,  $RI \subseteq I$ .

By definition, an ideal  $I$  of  $R$  is an additive subgroup of  $R$  that satisfies these properties. Is every ideal the kernel of a ring homomorphism? The answer is yes, namely the quotient  $\pi : R \rightarrow R/I$ .

**Theorem:** Let  $I \subset (R, +)$ , then TFAE:

- $I$  is an ideal of  $R$ , written  $I \trianglelefteq R$ .
- There exists a ring structure on the quotient group  $R/I$  such that the projection  $r \mapsto r + I$  is a ring morphism.

When these conditions hold, the ring structure on  $R/I$  is *unique* and we refer to this as the *quotient ring*.

## 4 Friday January 17th

For a  $R \subset T$  a subring of a ring, the set of intermediate rings is a large/interesting class of rings. Recall: uncountably many rings between  $\mathbb{Z}$  and  $\mathbb{Q}$ ! Taking  $R$  a PID and  $T$  its fraction field, a similar result will hold.

Define  $I \trianglelefteq R$  as the kernel of a ring morphism. This implies that  $I \subset (R, +)$  with the absorption property  $RI \subset I$ . Conversely, any  $I$  satisfying these two properties is the kernel of a ring morphism: namely  $R \rightarrow R/I$ . This makes sense as a group morphism.

**Exercise:** Define  $xy + I = (x + I)(y + I)$ , need to check well-definedness. Write out  $(x + i_1)(y + i_2) = \dots$ , need to check that  $i_1y + i_2x + i_1i_2 \in I$ , but the absorption property does precisely this.

Note that if we were in a non-commutative setting, this would define a left ideal. These don't have to coincide with right ideals – there are rings where the former satisfy properties that the latter does not.

*Example:* The subrings of  $R = \mathbb{Z}$  are of the form  $n\mathbb{Z}$  for  $n \geq 0$ , with the usual quotient.

**Definition:** An ideal  $I \trianglelefteq R$  is *proper* iff  $I \subsetneq R$ .

**Exercise:** An ideal  $I$  is not proper iff  $I$  contains a unit.

**Exercise:**  $R$  is a field iff the only ideals are  $0, R$ .

**Definition:** Let  $\mathcal{I}(R)$  be the set of all ideals in  $R$ . What structure does it have? It is partially ordered under inclusion. It is a complete lattice, i.e. every element has an infimum (GLB) and a supremum (LUB). Namely, for a family of ideals  $\{I_j\}$ , the infimum is the intersection and supremum is defined as  $\langle I_j \mid j \in J \rangle$ , the smallest ideal containing all of the  $I_j$ , i.e.  $\langle y \rangle = \left\{ \sum_{i=1}^n r_i y_i \mid n \in \mathbb{N}_{>0}, r_i \in R, y_i \in y \right\}$ .

**Exercise:** For  $I_1, I_2 \trianglelefteq R$ , it is the case that  $I_1 + I_2 := \{i_1 + i_2\} = \langle I_1, I_2 \rangle$ .

**Theorem:** Let  $I \trianglelefteq R$  and  $\phi : R \rightarrow R/I$ , and define  $\ell(I) = \{I \subset J \trianglelefteq R\}$ . Then we can define maps

$$\begin{aligned} \Phi : \ell(R) &\rightarrow \ell(R/I) \\ J &\mapsto \frac{I + J}{J}, \end{aligned}$$

and

$$\begin{aligned} \Psi : \ell(R/I) &\rightarrow \ell(R) \\ J \trianglelefteq R/I &\mapsto \phi^{-1}(J). \end{aligned}$$

We can check that  $\Psi \circ \Phi(J) = I + J$ , and  $\Phi \circ \Psi(J) = J (= J/I?)$ . So  $\Psi$  has a left inverse and is thus injective. Its image is the collection of ideals that contain  $J$ , and  $\Psi : \ell(R/I) \rightarrow \ell_I(R)$  is a bijection and is in fact a lattice isomorphism with  $\ell_I(R) \subset \ell(R)$ .

Note that this gives us everything above (?) an ideal in the ideal lattice; the dual notion will come from localization.

*Remarks:* The ideal lattice  $\ell(R)$  is

- A complete lattice under subset inclusion,
- A commutative monoid under addition
- A commutative monoid under *multiplication*, which we'll define.

**Definition:** For  $I, J \trianglelefteq R$ , we define  $IJ = \langle ij \mid i \in I, j \in J \rangle$ . Note that we have to take the ideal generated by products here.

For  $\langle x \rangle = (x)$  a principal ideal and  $\langle y \rangle$  principal, we do have  $(x)(y) = (xy)$ . Note that  $IJ \subset I \cap J$ , whereas the sum was larger than  $I, J$ .

**Exercise:** Note that  $(\ell(R), \cdot)$  has an absorbing element, namely  $(0)I = (0)$ . For  $(M, +)$  a commutative monoid and  $M \hookrightarrow G$  a group, then multiplication by  $x$  is injective and so for all  $y \in M$ ,  $xz = yz \implies x = y$ , so  $M$  is cancellative.

Question: what if we consider  $\mathcal{I}^\bullet(R)$  the set of nonzero ideals of  $R$ . Does this help? We will see next time.



## 5 Wednesday January 22nd

Let  $R$  be a ring and let  $\mathcal{I}(R)$  be the set of ideals  $I \trianglelefteq R$ . This algebraic structure is

- Partially ordered under inclusion
- Forms a complete lattice with sup the ideal generated by a family and inf the intersection.
- Forms a commutative monoid under  $I + J$
- Forms a commutative monoid under  $IJ$

For any commutative monoid  $(M, +)$ , there exists a group completion  $G(M)$  such that

- $G(M)$  is a commutative group
- $g : M \rightarrow G(M)$  is a monoid homomorphism
- For any map  $\phi : (M, +) \rightarrow (G, +)$  into a commutative group, we have the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\forall \phi} & G \\ & \searrow g \quad \nearrow \exists! \Phi & \\ & M(G) & \end{array}$$

So  $\phi$  factors through  $M(G)$ .

If this exists, it is unique up to unique isomorphism (as are all objects defined by universal properties). It remains to construct it.

**Exercise:** For  $(M, +)$  a commutative monoid, show that TFAE

1. There exists an injective  $\iota : M \hookrightarrow G$  monoid homomorphism for  $G$  some commutative group.
2. The map  $g : M \rightarrow G(M)$  is an injection.
3.  $M$  is cancellative, i.e.  $\forall x, y, z \in M$  we have  $x + z = y + z \implies x = y$ , i.e. the map  $p_z(x) = x + z$  is injective.

The content here is in  $3 \implies 1$ .

A commutative monoid is *reduced* iff  $M^\times = (0)$ , i.e. if “ $\forall m \in M \exists n$  such that  $m + n = 0$ ”  $\implies m = 0$

*Example:*  $(\mathbb{N}, +)$  and  $(\mathbb{Z}^+, \cdot)$  are cancellative and reduced.

**Definition:**  $z \in M$  is a zero element iff  $z + x = z$  for all  $x \in M$ .

*Remark:* If  $M$  has a zero element, then  $G(M) = \{0\}$ .

$(0)$  is a zero element of  $(\mathcal{I}(R), \cdot)$ , so this is not cancellative. If we take  $\mathcal{I}^\bullet$  the set of nonzero ideals with multiplication, then this is a submonoid of  $\mathcal{I}(R)$  iff  $R$  is a domain.

For  $R$  a domain, let  $\mathcal{I}_1(R)$  be the set of nonzero principal ideals of  $R$ , then  $\mathcal{I}_1(R) = R^\bullet / R^\times$ , so this is reduced and cancellative.

What is the group completion? In this case, it will consist of fractional ideals.

If  $R$  is a PID, then  $\mathcal{I}_1^\bullet(R) = \mathcal{I}^\bullet(R)$  is reduced and cancellative.

*Example:*  $\mathcal{I}^\bullet \cong (\mathbb{Z}^+, \cdot)$ .

**Warning:** If  $R$  is not a PID, then  $\mathcal{I}^\bullet(R)$  need not be cancellative.

**Exercise:** Take  $R = \mathbb{Z}[\sqrt{-3}]$  and  $p_2 := \langle 1 + \sqrt{-3}, 1 - \sqrt{-3} \rangle$ . Show that  $|R/p_2| = 2$ ,  $|R/(2)| = 4$ , and  $p_2^2 = p_2(2)$  and  $|R/p_2^2| = 8$ . Conclude that  $\mathcal{I}^\bullet(R)$  is not cancellative.

What went wrong here? Take  $K = \mathbb{Q}[\sqrt{-3}]$ , then  $\mathbb{Z}_K[\frac{1 + \sqrt{-3}}{2}]$  is the integral closure of  $\mathbb{Z}$  in  $K$ .  $\mathbb{Z}_K$  is a Dedekind domain, and there are inclusions

$$\mathbb{Z} \subset \mathbb{Z}[\sqrt{-3}] \subsetneq \mathbb{Z}[\frac{1 + \sqrt{-3}}{2}] \subseteq K.$$

Here the problem is that  $\mathbb{Z}[\sqrt{-3}]$  is not a Dedekind domain. If  $R$  is a Dedekind domain, then  $\mathcal{I}^\bullet(R)$  is cancellative.

**Exercise:** Does the converse hold?

Things that are too small to be the full rings of integers, and things tend to wrong.

## 5.1 Pushing / Pulling

Let  $f : R \rightarrow S$  be a ring homomorphism.

We can define a pushforward on the set of ideals  $\mathcal{I}(R)$ :

$$\begin{aligned} f_* : \mathcal{I}_R &\rightarrow \mathcal{I}(S) \\ I &\mapsto \langle f(I) \rangle. \end{aligned}$$

and a pullback

$$\begin{aligned} f^* : \mathcal{I}(S) &\rightarrow \mathcal{I}(R) \\ J &\mapsto f^{-1}(J). \end{aligned}$$

**Exercise:** Show that  $f^{-1}(J) \trianglelefteq R$ .

For  $I \trianglelefteq R$  and  $J \trianglelefteq S$ , then

$$\begin{aligned} f^* f_*(I) &\supseteq I \\ f_* f^*(J) &\subseteq J. \end{aligned}$$

**Exercise:** These are not equal in general, and give examples where equality does and does not hold.

If  $f$  is surjective,  $f_* f^* J = J$ .

Will also hold for localization, which is dual to taking a quotient.

Define  $\bar{I} := f^*f_*(I)$  and  $J^\circ := f_*f^*(J)$ , the closure and interior respectively. Show that these operations are idempotent.

**Definition:** An ideal  $\mathfrak{p}$  is *prime* iff  $ab \in \mathfrak{p} \implies a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ .

**Exercise:**  $I$  is prime iff  $R/I$  is a domain.

**Definition:**  $\text{Spec}(R) = \{\mathfrak{p} \trianglelefteq R\}$  the collection of prime ideals is the spectrum.

**Exercise:** Show that for  $I \trianglelefteq R$ , if we define

$$V(I) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq I\} \subseteq \text{Spec}(R),$$

then  $\{V(I) \mid I \in \mathcal{I}(R)\}$  are the closed sets for a topology on  $\text{Spec}(R)$  (the Zariski topology).

**Exercise:** If  $f : R \rightarrow S$  and  $J \in \text{Spec}(S)$  then  $f^*(J) \in \text{Spec}(R)$ . Show that  $f^* : \text{Spec}(S) \rightarrow \text{Spec}(R)$  is a continuous map. Conclude that  $\text{Spec}(\cdot)$  is a functor.

**Definition:**  $I \trianglelefteq R$  is maximal iff  $I$  is proper and is not contained in any other proper ideal.

**Exercise:**  $I$  is maximal iff  $R/I$  is a field.

**Exercise:** Show that maximal ideals are prime.

**Definition:** Let  $\text{Spec}_{\max}(R)$  be the set of maximal ideals and define  $V(I) = \{\mathfrak{m} \in \text{Spec}_{\max}(R) \mid \mathfrak{m} \supseteq I\}$ .

**Exercise:** Show that these form the closed sets for a topology, and that this is the subspace topology for the Zariski topology.

**Exercise:** Show that if  $f : R \rightarrow S$  and  $\mathfrak{m} \in \text{Spec}_{\max}(S)$  that  $f^*(\mathfrak{m})$  is prime but need not be maximal.

**Exercise:** Show that if  $f$  is an integral extension, then maximals do pull back to maximals.

## 6 Friday January 24th

### 6.1 Ideals and Products

Recall: Prime and maximal ideals.

**Fact:** If  $I \trianglelefteq R$  then there exists a maximal ideal  $I \subset \mathfrak{m} \trianglelefteq R$ .

*Proof:* Use Zorn's lemma.

**Corollary:**  $\max\text{Spec } R \neq \emptyset \iff R \neq 0$ .

Later: Multiplicative avoidance, if  $S \subset R$  is nonempty with  $SS \subset S$ , let  $I \trianglelefteq R$  with  $I \cap S = \emptyset$ , then

- There exists an ideal  $J \supseteq I$  with  $J \cap S = \emptyset$  which is maximal with respect to being disjoint from  $S$ .
- Any such ideal  $J$  is prime.

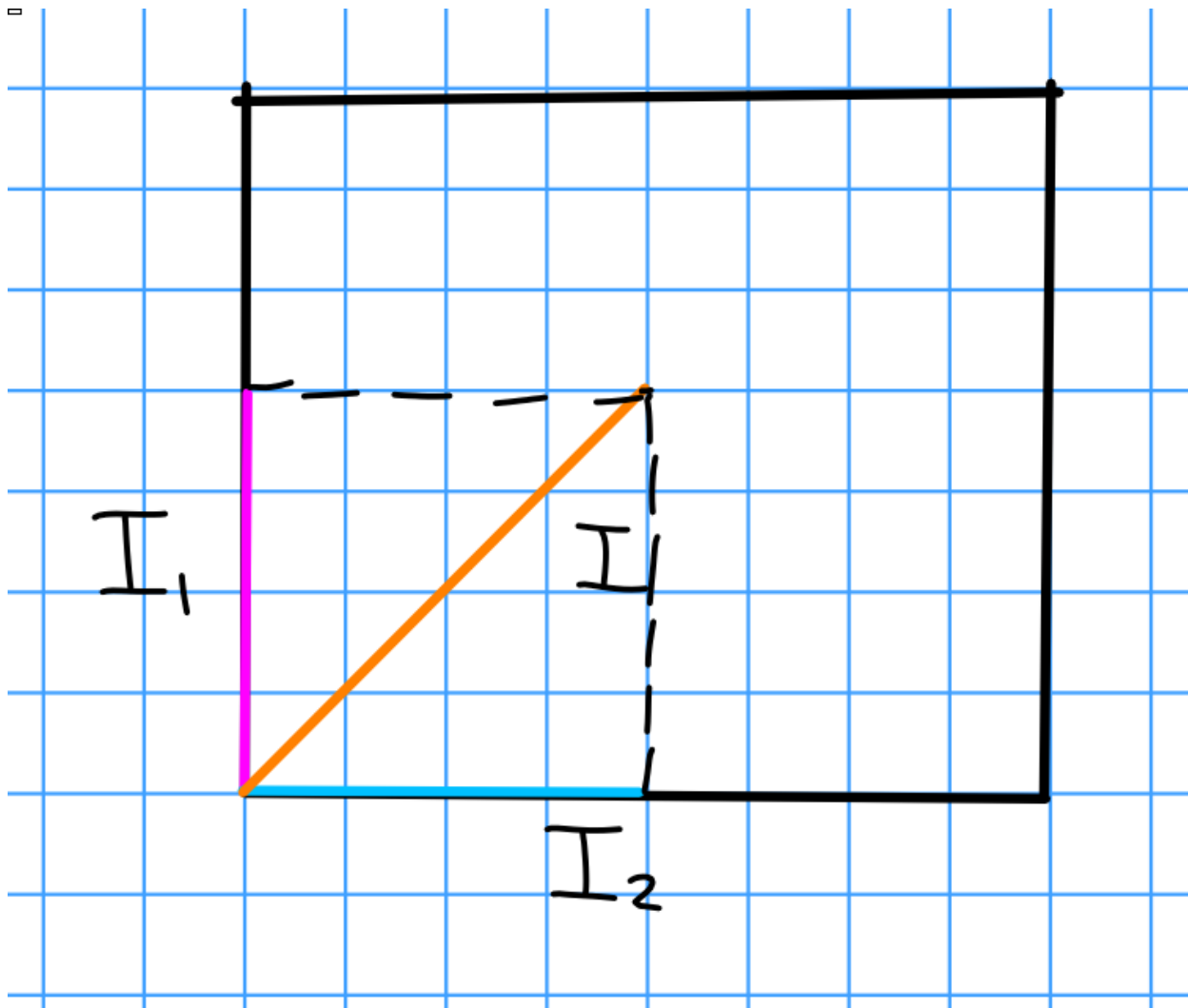


Figure 1: Image

Taking  $S = \{1\}$  recovers the previous fact.

**Exercise:** Let  $f : R \rightarrow S$  be a ring homomorphism and  $\mathfrak{p} \in \text{Spec}(R)$ . Show that  $f_*(\mathfrak{p})$  need not be prime in  $S$ .

We can consider products of rings, and correspondingly  $\mathcal{I}(R_1 \times R_2)$ .

**Exercise:** Show that if  $\phi$  is surjective,  $\phi(I)$  is an ideal.

**Proposition:** Let  $I \in \mathcal{I}(R_1 \times R_2)$ . Take  $\pi_i \rightarrow R_i$  the projections, and let  $I_i$  be the corresponding images of  $I$ . Then  $I = I_1 \times I_2$ .

Note: a suspiciously strong result! Not every group is the cartesian product of some subgroups.

It's clear that  $I \subset I_1 \times I_2$ .

*Proof:* Showing  $I_1 \times I_2 \trianglelefteq R_1 \times R_2$  is an ideal, since it equals  $\langle I_1 \times \{0\}, \{0\} \times I_2 \rangle$ .

To show  $I_1 \times I_2 \subseteq I$ , show that  $I_1 \times 0, 0 \times I_2 \subseteq I$ . E.g.  $I_1 \times 0 \subseteq I$ : take  $(x, 0) \in I_1 \times 0$  such that

there exists a  $y \in R_2$  with  $(x, y) \in I$ . Then  $(x, y) \cdot (1, 0) = (x, 0) \in I$ , then similarly  $0 \times I_s \subseteq I$ . ■

**Exercise:** Use  $\mathcal{I}(R_1 \times R_2) = \mathcal{I}(R_1) \times \mathcal{I}(R_2)$  to describe  $\text{Spec}(R_1 \times R_2)$  in terms of  $\text{Spec}(R_1)$  and  $\text{Spec}(R_2)$ .

Question: For a ring  $R$ , when is  $R \cong R_1 \times R_2$  for some nonzero  $R_1, R_2$ ?

**Exercise:** Show that comaximal ideals correspond with coprime ideals when  $R = \mathbb{Z}$ .

**Theorem (Chinese Remainder):** If  $I_1, I_2$  are comaximal, so  $I_1 + I_2 = R$ , then the map

$$\begin{aligned}\Phi : R &\rightarrow R/I_1 \times R/I_2 \\ x &\mapsto (x + I_1, x + I_2).\end{aligned}$$

Then  $\ker \Phi = I_1 \cap I_2 \stackrel{\text{CRT}}{=} I_1 I_2$  and  $\Phi$  is surjective, and

$$R/(I_1 \cap I_2) = R/I_1 I_2 \cong R/I_1 \times R/I_2.$$

Case 1: Let  $I_1 + I_2 = R$  and  $I_1 \cap I_2 = 0$  (equivalently  $I_1 I_2 = (0)$ ), then  $R \cong R/I_1 \times R/I_2$ .

Conversely, let  $R = R_1 \times R_2$  with  $R_1, R_2$  nonzero. Let  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ . Then  $e_1 e_2 = 0$  and  $e_2 = (1 - e_1)$ , so  $0 = e_1(1 - e_1) = e_1 - e_1^2$  and  $e_1$  is idempotent. So  $e_1, e_2$  are complementary nontrivial idempotents. Then  $I_1 I_2 = e_1 e_2 = (0)$ ,  $I_1 + I_2 = \langle e_1, e_2 \rangle = R$ , and thus  $R = R/e_2 R \times R/e_1 R$ . Note that  $e_2 R = 0 \times R_2$  and  $e_1 R = R_1 \times 0$ , thus

$$\begin{aligned}R/e_2 R &= \frac{R_1 \times R_2}{0 \times R_2} = R_1 \\ R/e_1 R &= \frac{R_1 \times R_2}{R_1 \times 0} = R_2.\end{aligned}$$
■

We thus have a correspondence

$$\{\text{Nontrivial product decompositions } R = R_1 \times R_2\} \iff \{I_1, I_2 \trianglelefteq R \text{ such that } I_1 I_2 = 0 \text{ and } I_1 + I_2 = R\} \iff \{\text{Idempotents } e \neq 0, 1\}.$$

Thus a ring can be decomposed as a product iff it contains nontrivial idempotents.

**Definition:**  $R$  is connected iff there do not exist nonzero  $R_1, R_2$  such that  $R \cong R_1 \times R_2$  iff  $R$  does not contain an idempotents  $e \neq 0, 1$ .

*Exercise:* Show that  $R$  is connected iff  $\text{Spec}(R)$  is connected as a topological space.

Note: Not every ring is a finite product of connected rings.

## 6.2 Modules

For  $(M, +)$  a commutative group, we want an action  $R \curvearrowright M$  for  $R$  a ring. Recall that  $\text{End}(M)$  for a group is a (potentially noncommutative) ring. An  $R$ -module structure is a ring homomorphism  $R \rightarrow \text{End}(M)$ . Equivalently, it is a function  $R \times M \rightarrow M$  with  $rs(x) = r(sx)$ ,  $r(x + y) = rx + ry$ , and  $1 \cdot x = x$  for all  $x \in M$ .

Note that this defines a left  $R$ -module, but right/left modules coincide for commutative rings.

*Exercise:* Let  $M$  be an  $R$ -module and for  $m \in M$  define  $\text{Ann}(m) = \{r \in R \mid xm = 0\} \trianglelefteq R$ ; show this is in fact an ideal.

Note: skipped chapter on Galois connections, i.e. some binary relation on a pair of sets. This is an instance of such a connection, where  $x \sim m \iff xm = 0$ .

For any subset  $S \subset M$ , define  $\text{Ann}(S) := \{x \in R \mid xm = 0 \forall m \in S\}$ . Show that  $\text{Ann}(S) = \bigcap_{m \in S} \text{Ann}(m)$  and  $\text{Ann}(M) = \{x \in R \mid xM = 0\} = \ker(R \rightarrow \text{End}(M))$ .

**Definition:**  $M$  is faithful iff  $\text{Ann}(M) = 0$  iff  $R \hookrightarrow \text{End}(M)$  is an injection.

*Exercise:* Any  $M$  is naturally a faithful  $R/\text{Ann}(M)$ -module.

## 7 Monday January 27th

### 7.1 Localization

Consider rings  $T$  such that  $\mathbb{Z} \subseteq T \subseteq \mathbb{Q}$ , and let  $P$  be a set of prime numbers. We've shown that if  $P, Q$  are two sets of prime numbers, then  $\mathbb{Z}_P = \mathbb{Z}_Q \iff \mathbb{Z}_P \cong \mathbb{Z}_Q \iff P = Q$ .

Let  $R$  be a domain with fraction field  $K$ . Let  $P$  be a set of mutually nonassociate prime elements. Note that  $p \in R$  is a prime element iff  $(p)$  is a prime ideal. We say  $x, y$  are associates iff there exists a  $u \in R^\times$  such that  $y = ux$ . Since we're in a domain, (exercise) this is equivalent to  $(x) = (y)$ .

**Fact:** We can then consider  $R_P := R[\{\frac{1}{p} \mid p \in P\}]$ , and the fact is that the previous statement still holds.

But if  $R = \mathbb{Z}$ , we also have (exercise) if  $Z \subset T \subset \mathbb{Q}$  then  $T = \mathbb{Z}_P$  for a unique  $P$ .

**Exercise:** How do we find such a  $P$ ? This comes down to looking at  $\frac{a}{b} \in T$  with  $\gcd(a, b) = 1$ , then  $\frac{1}{b} \in T$ .

Hint: In a PID,  $\gcd(a, b)$  exists and is a  $\mathbb{Z}$ -linear combination of  $a$  and  $b$ . The solution should work for an arbitrary PID.

Let  $R$  be a domain and  $S$  multiplicatively closed (so  $(S, \cdot) \leq (R, \cdot)$  is a submonoid). Then  $S$  is *primal* if  $S$  is generated as a monoid by its prime elements. Suppose that  $S$  is *saturated*, i.e. if  $s \in S$  and  $r \in R$  with  $r \mid s$ , then  $r \in S$ .

Can always add in all divisors.

We can then define the localization of  $R$  at  $S$ ,

$$R_s := \left\{ \frac{a}{s} \mid a \in R, s \in S \right\}.$$

This satisfies  $R \subset R_S \subset K$ , and is a multiplicative partial group completion. If we took nonzero elements, this would yield exactly the fraction field.

**Theorem (Negata):** Let  $R$  be a Noetherian domain with  $S \subset R$  primal as above. If  $R_S$  is a UFD, then  $R$  is a UFD.

**Exercise:** Show that the converse holds.

Fraction fields are always UFDs? Localizing makes it easier for irreducibles to be prime. This helps prove that some interesting rings are UFDs.

## 7.2 Modules

If  $M$  is an  $R$ -module, then an  $R$ -submodule  $N \leq M$  is a subgroup of  $(M, +)$  such that  $R \curvearrowright N \subset N$ .

Every ring  $R$  is an  $R$ -module over itself, and the  $R$ -submodules of  $R$  are precisely the ideals of  $R$ .

Can express certain concepts about rings/commutative algebra in the language of modules.

A morphism of  $R$ -modules  $f : M \rightarrow N$  is a homomorphism  $(M, +) \rightarrow (N, +)$  such that  $f(r \curvearrowright m) = r \curvearrowright f(m)$ .

**Exercise:** Any module morphism that is a bijection is an isomorphism. (Usually true in algebraic settings.)

We can form quotient modules  $\frac{M}{N}$  which is an  $R$ -module with  $r \curvearrowright (m + N) = (r \curvearrowright m) + N$ , and  $M \rightarrow \frac{M}{N}$  is a surjective morphism.

If  $I \trianglelefteq R$  is an  $R$ -submodule of  $R$ , then  $R/I$  is an  $R$ -module. We have  $\text{Ann}(R/I) = I$ .

**Fact:** Every ideal in  $R$  is the annihilator of some  $R$ -module.

**Fact:** Suppose  $R$  is a ring such that every nonzero  $R$ -module is faithful, then  $R$  is a field. The converse also holds.

General idea: we study rings by looking at modules over them.

For an  $R$ -module  $M$  and  $S \subset M$ , then we can consider  $\langle S \rangle$  the  $R$ -submodule generated by  $S$ . We can write this as

$$\bigcap_{N \text{ s.t. } S \subset N \subseteq RM} N = \left\{ \sum_{i=1}^n r_i s_i \mid r_i \in R, s_i \in S \right\}.$$

We say  $R$  is finitely generated iff there exists a finite generating set  $S \subset M$ . We say  $M$  is cyclic iff it is generated by a single element, i.e.  $M = \langle s \rangle$ .

Let  $\{M_i\}_{i \in I}$  be a family of  $R$ -modules. Let  $\prod_{i \in I} M_i$  be the cartesian product with a coordinate-wise  $R$ -action be the direct product. Let

$$\bigoplus_{i \in I} M_i = \left\{ (x_i) \in \prod_{i \in I} M_i \mid x_i \neq 0 \text{ for finitely many } i \right\},$$

which is a submodule of  $\prod_{i \in I} M_i$ . When  $I$  is finite, these are equal.

Recall: If  $R$  is a PID and  $M$  is a finitely generated  $R$ -module, then there exist finitely many cyclic  $R$ -modules  $\{C_1, \dots, C_n\}$ , then  $M \cong \bigoplus C_i$ .

**Exercise:** Let  $R$  be a ring and  $C$  a cyclic  $R$ -module, then show that  $C \cong R/\text{Ann}(C)$  as  $R$ -modules.

We'll later see that the class of rings  $R$  such that every  $R$ -module is free are exactly fields.

*Remark:* Let  $I \trianglelefteq R$ , then  $I$  is cyclic as an  $R$ -module iff  $I$  is principal.

**Exercise:**

- a. Let  $I \trianglelefteq R$  for  $R$  a domain, then  $I$  is indecomposable, i.e.  $I \neq M_1 \oplus M_2$  for any nonzero  $M_1, M_2$   $R$ -modules.
- b. If  $R$  is additionally Noetherian and not a PID, then there exists an  $I \trianglelefteq R$  where  $I$  is finitely generated, not principal, and so  $I$  is not a cyclic  $R$ -module.

Converse to structure theorem! Mild assumptions negate cyclic direct sum decomposition.

## 8 Friday January 31st

### 8.1 Tensor and Hom

Let  $M, N$  be  $R$ -modules, then we define

$$\text{hom}_R(M, N) := \left\{ f : M \rightarrow N \mid f \text{ is an } R\text{-module map} \right\}.$$

Recall that  $R$ -module maps satisfy

- $f : (M, +) \rightarrow (N, +)$  a morphism of abelian groups
- For all  $r \in R$ , for all  $m \in M$ ,  $f(rm) = rf(m)$ .

Note that  $\text{hom}_R$  is a commutative group, and is in fact an  $R$ -module with structure given by  $(r \cdot f) \cdot m \mapsto rf(m) = f(rm)$ .

Note that the proof of this fact uses commutativity in a key way.

Facts:

$$\begin{aligned} \text{hom}_R(R, N) &= N \\ \text{hom}_R\left(\bigoplus_{s \in S} R_s, N\right) &= N^S \\ \text{hom}_R(M, R) &:= M^\vee. \end{aligned}$$



Note: Infinite dimensional vector spaces over fields are never isomorphic to its dual.

**Exercise:** Think about  $M^\vee$  and  $(M^\vee)^\vee$ .

Recall the map

$$\begin{aligned}\iota : M &\rightarrow (M^\vee)^\vee = \text{hom}_R(\text{hom}_R(M, R), R) \\ x &\mapsto (\ell : M \rightarrow R \mapsto \ell(x) \in R).\end{aligned}$$

**Exercise:** If  $R = k$  is a field, then show that  $\iota$  is injective iff  $\dim M$  is finite.

Is this always injective? No! Counterexample: Take  $R = \mathbb{Z}$  and  $M = \mathbb{Z}/p\mathbb{Z}$ , then  $M^\vee = \text{hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$ .

It can also fail to be surjective in the infinite dimensional case – the space  $M^\vee$  is strictly larger than  $M$ .

**Definition:**  $M$  is *reflexive* if  $\iota : M \xrightarrow{\sim} (M^\vee)^\vee$  is an isomorphism.

**Exercise:** Show the following:

- If  $M$  is free and finitely generated, then  $M$  is reflexive.
- If  $R = k$  is a field, then  $M$  is reflexive iff  $M$  is finitely generated.
- There exists a ring  $R$  and a reflexive  $R$ -module  $M$  that is not finitely generated.

## 8.2 Free Torsion Modules

Let  $R$  be a domain, and for all  $a \in R^\bullet$  the map  $[a] : R \rightarrow R$  is injective, and  $[a] \in \text{hom}_R(R, R) = R$ .

**Definition:**  $M[\text{tors}] := \{m \in M \mid \text{Ann}(m) \neq (0)\} \leq M$  is the torsion submodule of  $M$ .

**Definition:**  $M$  is *torsion* iff  $M = M[\text{tors}]$ , and  $M$  is *torsion-free* iff  $M[\text{tors}] = (0)$ .

**Exercise:** Show that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , then

- Show that if  $B$  is torsion then  $A, C$  are torsion.
- If  $A, C$  are torsion, must  $B$  be torsion?
- Show that if  $B$  is torsion-free then  $A$  is torsion-free but  $C$  need not be torsion-free.
- If  $A, C$  are torsion-free, must  $B$  be torsion-free?

Note:  $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$  is an extension that isn't a semidirect product!

**Fact:** Free modules are torsion-free.

Note that we need to be in a domain to even talk about torsion.

**Proposition:** Let  $R$  be a domain and  $M$  an  $R$ -module. Then

- $M/M[\text{tors}]$  is torsion-free.
- If  $M$  is finitely generated, then  $M$  is torsion free iff  $M$  is isomorphic to a submodule of a finitely-generated free module.

**Proposition:** Free  $\implies$  projective  $\implies$  flat  $\implies$   $R$  a domain torsion-free.

*Proof of a:* Let  $x \in M/M[\text{tors}]$  such that  $\exists r \in R^\bullet$  such that  $rx = 0$ . Lift  $x$  to  $\tilde{x} \in M$ , then  $r\tilde{x} \in M[\text{tors}]$ . Then  $\exists r' \in R^\bullet$  such that  $0 = r'(r\tilde{x}) = (r'r)\tilde{x} := r_2\tilde{x}$  for some  $r_2 \neq 0$ . But then  $\tilde{x} \in M[\text{tors}]$ , and so  $x = 0$  in  $M/M[\text{tors}]$ .

*Proof of b:* Let  $M = \langle x_1, \dots, x_r \rangle$  with  $r \geq 1$  and  $x_i \neq 0$ . After reordering, there exists some  $s$  with  $1 \leq s \leq r$  such that  $x_1, \dots, x_s$  are  $R$ -linearly independent, and for all  $i > s$ ,  $\{x_j\}_{j \leq s} \cup x_i$  is linearly dependent. Then define  $F := \langle x_1, \dots, x_s \rangle$ ; this is a finitely generated free module. If  $r = s$ , we are done.

Otherwise,  $r < s$ , then  $\forall i > r$  there exists an  $a_i \in R^\bullet$  such that  $a_i x_i \in F$ . So we can take  $a := a_{s+1} \cdots a_r \neq 0$ ; then  $aM \subset F$ . Since  $M$  is torsion-free, the multiplication maps are injective, so  $[a] : M \xrightarrow{\cong} M \subset F$ , so  $M \hookrightarrow F$  embeds  $M$  into a free module. ■

Does this work with  $M$  not finitely generated? No, we can't take an infinite product for  $a$ . Is every torsion-free module a submodule of a free module? No.

*Remark:* This fails without finite generation, see Theorem 3.56 on ordinal filtration. If  $R$  is a PID and  $F$  is a free  $R$ -module and  $M \leq F$  as an  $R$ -submodule, then  $M$  is free.

Thus if  $R$  is a PID, “subfree”  $\iff$  free. Does torsion-free imply free? No, take  $R = \mathbb{Z}$  and  $M = (\mathbb{Q}, +)$ , this is not finitely generated and torsion-free but not a free  $\mathbb{Z}$ -module.

**Definition:** For  $R$  a domain,  $M$  is *divisible* if  $\forall a \in M^\bullet$  iff  $[a] : M \twoheadrightarrow M$  is a surjection.  $M$  is *uniquely divisible* if for all  $a \in M^\bullet$ ,  $[a] : M \xrightarrow{\cong} M$  is an isomorphism, i.e.  $M$  is torsion-free and divisible.

**Exercise:** Show that  $(\mathbb{Q}, +)$  is a uniquely divisible  $\mathbb{Z}$ -module.

**Exercise:** Let  $R$  be a domain with fraction field  $K$ , with  $R \neq K$ . Show that a nonzero free  $R$ -module is not divisible but  $(K, +)$  is a divisible torsion-free  $R$ -module. Thus  $(K, +)$  is a torsion-free module  $R$ -module that is not free.

*Remark:* Finitely generated torsion free modules are embedded in free modules. Note that in the spectrum of properties earlier (projective, free, etc), the two extremes are equal for f.g. PIDs.

**Exercise:** Let  $R$  be a Noetherian domain which is not a PID. Then an ideal  $I \leq R$  with  $I$  f.g., not principal, and a torsion-free  $R$ -module. Show that since  $I$  is not principal,  $I$  is not free as an  $R$ -module.

So ideals can't contain linearly independent elements, so they have to be free of rank 1 and thus principal. So f.g. torsion-free is strictly weaker than free in this setting.

## 9 Monday February 3rd

Some module topics from Chapter 8.

### 9.1 Noetherian and Artinian Modules

**Definition:** A poset  $(X, \leq)$  is said to satisfy the ACC or to be Noetherian iff there does not exist an infinite sequence (a chain)  $\{x_n\}$  with strict inequalities  $x_1 < x_2 < \dots$ . Equivalently,

every weakly ascending chain  $x_1 \leq x_2 \leq \dots$  eventually stabilizes, i.e. there exists an  $N$  such that  $x_N = x_{N+1} = \dots$ .

**Definition:** Similarly, a poset satisfies the DCC or is *Artinian* iff there does not exist an infinite decreasing sequence  $x_1 > x_2 > \dots$ .

**Definition:** For  $(X, \leq)$ , define the order dual  $(X^\vee, \leq)$  where  $x \leq y \in X^\vee \iff y \leq x \in X$ .

**Proposition:**  $X$  is Noetherian iff  $X^\vee$  is Artinian.

**Lemma:** The ACC holds iff every nonempty subset has a maximum (and similarly the DCC with minimums).

**Proof:** Otherwise use AOC to pick elements  $x_i$ ; if  $x_i$  isn't the maximum then there is some  $x_{i+1} > x_i$ , and this yields an infinite ascending chain iff no maximum.

Let  $M$  be an  $R$ -module, and define  $\text{Sub}_R M = \{(R\text{-submodules of } M, \leq)\}$ .

**Lemma:**  $M$  is Noetherian  $\iff$  every submodule  $N \leq M$  is finitely generated.

*Proof:* Apply DCC. ■

**Exercise:** Let  $M' \subset M$  and  $q : M \rightarrow M/M'$ . Let  $N_1 \subset N_2 \subset M$  such that

- $N_1 \cap M' = N_2 = M'$ , and
- $q(N_1) = q(N_2)$ .

Then  $N_1 = N_2$ .

**Proposition:** If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is exact and  $M$  is Noetherian (resp. Artinian) then  $M', M''$  are both Noetherian (resp. Artinian).

*Proof:* Note that  $\text{Sub}_R M', \text{Sub}_R M'' \hookrightarrow \text{Sub}_R M$  in an order-preserving manner. If we then have  $N_1 \subset N_2 \subset \dots$  with  $N_i \leq M$  submodules of  $M$ , we can consider  $N_n = \frac{N_n + M'}{M'}$ , which is weakly increasing in  $M'$ .

**Note:** this is how we push forward into quotients.

Thus this chain stabilizes, so for  $i, j \gg 0$  we have  $N_i + M' = N_j + M'$ . So then  $N_i \cap M' = N_j \cap M'$ , and by the exercise,  $N_i = N_j$  for all  $i, j \gg 0$ . ■

**Corollary:**  $R$  is Noetherian (resp. Artinian) iff every finitely-generated  $R$ -module is Noetherian (resp. Artinian)

*Proof:*

$\implies$  : Suppose  $R$  is Noetherian. Note that  $0 \rightarrow R \rightarrow R^2 \rightarrow R \rightarrow 0$  since  $R^2$  is an extension of  $R$  by  $R$ . Thus  $R^2$  is Noetherian, and inductively  $R^n$  is a Noetherian  $R$ -module.

If  $M$  is a finitely-generated  $R$ -module, it is a quotient of a finitely-generated free  $R$ -module, and in particular  $0 \rightarrow K \rightarrow R^n \rightarrow M \rightarrow 0$  is exact. So  $M$  is Noetherian, by the previous proposition (middle of a SES Noetherian  $\implies$  ends are Noetherian). ■

## 9.2 Tensor Products

Motivation from Representation Theory: For  $G$  finite,  $H \leq G$ , and  $\rho : G \rightarrow V$  a finite-dimensional  $\mathbb{C}$ -representation, this data is equivalent to a  $\mathbb{C}[G]$ -module structure on  $V$ . If  $W$  is a representation on  $H$ , then  $\text{Ind}_H^G W$  is a representation of  $G$  given by  $V = \text{Ind}_H^G W = W \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$ .

Definition: Let  $M, N$  be  $R$ -modules, then the tensor product  $M \otimes_R N$  is an object characterized up to canonical isomorphism by the following universal property: If  $P$  is an  $R$ -module and  $\Phi : M \times N \rightarrow P$  is any bilinear map, then there exists a unique lift such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_R N & & \\ \uparrow \iota & \searrow \exists! \psi & \\ M \times N & \xrightarrow{\Phi} & P \end{array}$$

where  $\iota : M \times N \rightarrow M \otimes_R N$  is  $R$ -bilinear and for all  $(m, n) \in M \times N$ , we denote  $m \otimes n := \iota(m, n)$ .

By dimension counting in the finite-dimensional case of vector space, it's clear that  $\iota$  need not be surjective. In general, elements in  $M \otimes_R N$  are *finite sums* of simple tensors, not just simple tensors, i.e.  $M \otimes_R N = \langle \iota(m, n) \rangle$ .

Proof (existence): Let  $F$  be the free  $R$ -module on  $M \times N$  with basis  $\{(m, n) \mid m \in M, n \in N\}$ . Mod out by the following relations: for all  $m, m_1, m_2 \in M$  and for all  $n, n_1, n_2 \in N$  and all  $r \in R$ ,

- $(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$
- $m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$
- $r(m \otimes n) - (rm) \otimes n$
- $r(m \otimes n) - m \otimes (rn)$

Let  $\mathcal{R}$  be the ideal generated by these relations, then define  $M \otimes_R N = F/\mathcal{R}$  by  $(m, n) \mapsto (m, n) + \mathcal{R}$ . Then (straightforward check) the universal mapping property holds. ■

How do we work with tensor products? Namely, how do we even know whether an arbitrary element is zero or not in this complicated quotient.

- To show  $m \otimes n = 0$ , use bilinear relations (reduce to relations above)
- To show  $m \otimes n \neq 0$ , find an  $R$ -module and a bilinear map  $\psi : M \otimes_R N \rightarrow P$  such that  $\text{im}(m \otimes n) \neq 0$ .
- To show  $M \otimes_R N \cong X$ , show that  $X$  satisfies the universal property.

**Exercise:**  $R \otimes_R M \cong M$  by  $(r, m) \mapsto r \cdot m$ , with  $\cdot$  the  $R$ -module action on  $M$ . Let  $P$  be arbitrary, let  $\phi : R \times M \rightarrow P$  be arbitrary, and define  $\psi : M \rightarrow P$  by  $m \mapsto \phi(1, m)$ .

**Exercise:**  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}$ . Show that every element is both  $n$ -torsion and  $m$ -torsion.

**Proposition:** For  $M$  and  $R$ -module and  $f : R \rightarrow S$ , we can create an  $S$ -module  $S \otimes_R M$  by *base change*.

Definitely the most important concept thus far!

## 10 Wednesday February 5th

Recall that if  $M, N$  are  $R$ -modules then there is a map  $M \times N \xrightarrow{\Phi} M \otimes_R N$  where  $(r, m) \mapsto r \otimes m$  which is universal wrt the property that any bilinear map  $\phi : M \times N \rightarrow A$  factors through  $\Phi$  uniquely.

We have a notion of *pullback*, where if  $i : R \rightarrow S$  and  $N$  is an  $S$ -module then  $i^*N$  is an  $R$  module with action given by composition  $R \xrightarrow{i} S \rightarrow \text{End}_{\mathbb{Z}}(N)$ .

Dually, we have a notion of *base change*, where for  $M$  an  $R$ -module we can form  $i_*M := S \otimes_R M$  an  $S$ -module where  $s(\sum s_i \otimes m_i) = \sum ss_i \otimes m_i$ .

An  $R$ -algebra is  $i : R \rightarrow S$  a ring morphism, where algebra morphisms  $f : S_1 \rightarrow S_2$  are given by commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{i_1} & S_1 \\ & \searrow i_2 & \downarrow f \\ & & S_2 \end{array}$$

For  $S, T$   $R$ -algebras, the tensor product  $S \otimes_R T$  is an  $R$ -algebra with  $(s_1 \otimes m_1) \cdot (s_2 \otimes m_2) = s_1 s_2 \otimes m_1 m_2$ . Note that the tensor product satisfies the universal property of the direct sum or coproduct:

$$\begin{array}{ccccc} & & t & & \\ & & \vdots & & \\ & & \psi & & \\ & & \searrow & & \\ & & 1 \otimes t & & \\ & & \downarrow & & \\ R & \nearrow & T & \searrow & \\ & & S \otimes_R T & \xrightarrow{\exists!} & W \\ & \searrow & \uparrow & & \\ & & S & \nearrow & \\ & & s & & \\ & & \phi & & \end{array}$$

**Exercise:** Verify the following identities

*One:* Let  $M$  be an  $R$ -module and  $N$  an  $S$ -module with  $\iota : R \rightarrow S$ .  $\text{hom}_R(M, \iota^*N) = \text{hom}_S(\iota_*M, N) = \text{hom}_S(S \otimes_R M, N)$ . What's the map?  $s \otimes m \mapsto sf(m)$ .

*Two:* For  $P$  and  $R$ -module and  $M, N$   $S$ -modules, we have  $M \otimes_X (i^*N \otimes_R P) = i^*(M \otimes_S N) \otimes_R P$ . So for  $N = S$ , then  $M \otimes_S (S \otimes_R P) = M \otimes_R P$ .

**Three (Good Exercise! Very important!):** For  $M$  an  $R$ -module and  $I \trianglelefteq R$ , we have  $IM \subset_R M$ . Show that we can identify the base change as  $R/I \otimes_R M = M/I$ .

Show that the RHS satisfies the appropriate universal property.

*Four:*

- $(\oplus M_i) \otimes_R N = \oplus (M_i \otimes_R N)$ .
- The tensor product of free modules is free.

- If  $F$  is a free  $R$ -module and we base change with  $\iota : R \rightarrow S$  then  $S \otimes_R F$  is a free  $S$ -module.

**Definition:** Let  $R$  be a ring, then  $R$  satisfies the *invariant basis number property* (IBN) iff any two bases for a free left  $R$ -module have the same cardinality.

**Definition:**  $R$  satisfies the *rank condition* iff whenever there exists a  $q : R^m \rightarrow R^n$ ,  $n \leq m$ .  $R$  satisfies the *strong rank condition* iff whenever  $q : R^m \hookrightarrow R^n$  then  $n \leq m$ .

**Facts:** If  $R$  is commutative or (left)-Noetherian, then strong rank condition  $\implies$  rank condition  $\implies$  IBN.

Note: this is not obvious, since if  $R$  is not Noetherian there are submodules that aren't finitely generated but can still have bounded rank.

*Exercise (Non-Commutative):* Let  $k$  be a field and  $V$  an infinite dimensional  $k$ -vector space, i.e.  $V \cong V \oplus V$ . Let  $R := \text{End}_k(V)$ ; then  $R$  does not satisfy the IBN.

**Proposition:** If  $R$  is nonzero and commutative then  $R$  satisfies IBN.

*Proof:* Suppose there exist  $I, J$  such that  $\bigoplus_{i \in I} R \cong_R \bigoplus_{j \in J} R$ . We want to show that  $|I| = |J|$ . Since  $R \neq 0$ , there is a maximal ideal  $\mathfrak{m} \in \text{maxSpec}(R)$ . Since  $R/\mathfrak{m}$  is a field, we base change to it to obtain  $R/\mathfrak{m} \otimes_R (\bigoplus_{i \in I} R) = \bigoplus_{i \in I} R/\mathfrak{m}$ . We know this equals  $R/\mathfrak{m} \otimes_R (\bigoplus_{j \in J} R) = \bigoplus_{j \in J} R/\mathfrak{m}$ . So  $I, J$  are bases of isomorphic vector spaces and thus  $|I| = |J|$  by linear algebra.

**Definition:** A module  $M$  is *Noetherian* iff ACC on submodules, and *Artinian* iff DCC on submodules.

**Exercise:** If  $R = k$  is a field and  $V$  is a  $k$ -vector space, then  $V$  is Noetherian iff Artinian iff infinite-dimensional.

**Exercise:** If  $R = \mathbb{Z}$ ,  $R$  is Noetherian but not Artinian. Find a  $\mathbb{Z}$ -module that is Artinian but not Noetherian.

Try all  $2^n$  possibilities for adjectives.

**Exercise:** If  $R$  is finite, it is both Artinian and Noetherian, and moreover has only finitely many ideals.

Artinian is much stronger, and implies Noetherian? Converse iff every ideal is maximal. The only Artinian integral domains are fields. Very small class of rings. It's not true that Artinian alone implies finitely many ideals.

**Exercise (8.29 in Notes):** Let  $I = (x^2, xy, y^2) = (xy)^2 \trianglelefteq \mathbb{C}[x, y]$  and take  $R = \mathbb{C}[x, y]/I$ .

- Show that a  $\mathbb{C}$ -basis for  $R$  is given by  $\{1 + I, x + I, y + I\}$ .
- Deduce that  $R$  is Noetherian and Artinian.
- Show proper ideals of  $R$  are precisely the  $\mathbb{C}$ -subspaces of  $\langle x, y \rangle + I$ .
- Deduce that  $\mathbb{R}$  has continuum many ideals.