

Moduli Spaces

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1 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
 - Slightly more detailed: Nitsure, . . . Hilbert schemes, Fundamentals of Algebraic Geometry
 - Mumford, Curves on Surfaces

- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

1.1 Representability

Last time: Fix an S -scheme, i.e. a scheme over S .

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \mathrm{hom}_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ T &\mapsto x \longrightarrow \text{triangles} \end{aligned}$$

of the form

$$\begin{array}{ccc} T' & \xrightarrow{\quad} & X \\ & \searrow & \nearrow \\ & T & \end{array} .$$

Theorem 1.1 (Yoneda).

$$\mathrm{hom}_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 1.1.

$$\mathrm{hom}_{\mathrm{Sch}/S}(x, y) \cong \mathrm{hom}_{\mathrm{Fun}}(h_x, h_y).$$

Definition 1.1.

A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 1.2.

A **moduli space** for that “something” appearing above is an $M \in \text{Obj}(\text{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \text{Spec}(k)$.

h_m is the functor of points over M .

Remark (1) $h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{“families over Spec } k\text{”} = F(\text{Spec } k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M , and $\text{id}_M \in \text{Mor}_{\text{Sch}/S}(M, M) = \xi_{U_{\text{niv}}}$ is the universal family

Every family is uniquely the pullback of $\xi_{U_{\text{niv}}}$. This makes it much like a classifying space.

For $T \in \text{Sch}/S$,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{U_{\text{niv}}}). \end{aligned}$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\text{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\xi_M \qquad \qquad \xi_{M'}$$

$$M \xrightarrow{f} M'$$

$$M' \xrightarrow{g} M$$

$$\xi_{M'} \qquad \qquad \xi_M$$

which shows that f, g must be mutually inverse by using universal properties.

Example 1.1.

A length 2 subscheme of \mathbb{A}_k^1 then $F(S) = \{V(x^2 + bx + c)\} \subset \mathbb{A}'_5$ where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

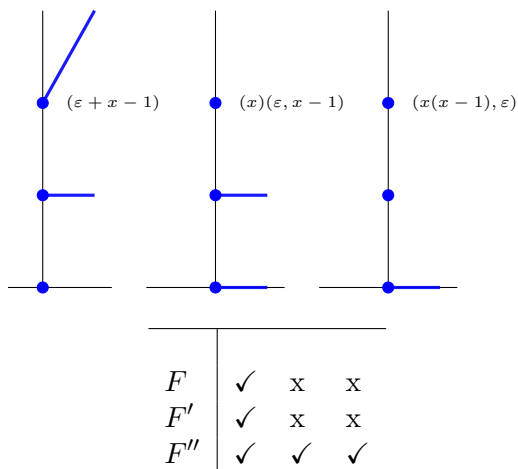
Then F is representable by $\mathbb{A}_k^2(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in \mathbb{A}'_5 which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of \mathbb{A}'_5 which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\text{Spec } k[\varepsilon]$.



$$\begin{array}{ccccc}
 \text{Spec } k & \xleftarrow{i} & \text{Spec } k[\varepsilon] & & \\
 & & & \nearrow & \\
 & & & = F'(\text{Spec } k) & \\
 F(\text{Spec } k[\varepsilon]) & \xrightarrow{F(i)} & F(\text{Spec } k) & & \\
 \uparrow \subset & & \uparrow \in & \searrow & \\
 T_p F','' & & P = V(x(x-1)) & & = F''(\text{Spec } k)
 \end{array}$$

We think of $T_p F',''$ as the tangent space at p .

If F is representable, then it is actually the Zariski tangent space.

$$\begin{array}{ccc}
 M(\text{Spec } k[\varepsilon]) & \longrightarrow & M(\text{Spec } k) \\
 \uparrow \subset & & \uparrow \subset \\
 T_p M & \longrightarrow & p
 \end{array}$$

$$\begin{array}{ccc}
& \text{Spec } k & \\
\swarrow & & \searrow \text{?} \\
\text{Spec } k[\varepsilon] & \xrightarrow{\quad\quad\quad} & \text{Spec } \mathcal{O}_{M,p} \subset M
\end{array}$$

$$\begin{array}{ccc}
& & k \\
& \nearrow & \uparrow \\
\mathcal{O}_{M,p} & \xrightarrow{\quad\quad\quad} & k[\varepsilon]
\end{array}$$

$$\begin{array}{ccc}
m_p & & (\varepsilon) \\
m_p^2 & & 0
\end{array}$$

Moreover, $T_p M = (m_p/m_p^2)^\vee$, and in particular this is a k -vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda \varepsilon
\end{aligned}$$

$$\lambda^* : \text{Spec } (k[\varepsilon]) \longrightarrow \text{Spec } (k[\varepsilon])$$

$$\begin{aligned}
\lambda : M(\text{Spec } (k[\varepsilon])) &\longrightarrow M(\text{Spec } (k[\varepsilon])) \\
\cup &\quad \cup \\
T_p M &\longrightarrow T_p M.
\end{aligned}$$

Conclusion: If F is representable, for each $p \in F(\text{Spec } k)$ there exists a unique point of $T_p F$ that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$\begin{array}{ccc}
F \times_G F' & \xrightarrow{\quad\quad\quad} & F' \\
\downarrow & & \downarrow \\
F & \xrightarrow{\quad\quad\quad} & G
\end{array}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$,

where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\text{Sch}/S)^{\text{op}} \longrightarrow \text{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\text{Sch}/T) &\longrightarrow \text{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

1.2 Projective Space

Consider $\mathbb{P}_{\mathbb{Z}}^n$, i.e. “rank 1 quotient of an $n + 1$ dimensional free module”.

Proposition 1.1.

$\mathbb{P}_{\mathbb{Z}}^n$ represents the following functor

$$\begin{aligned} F : \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ F(S) &= \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 / \sim. \end{aligned}$$

where \sim identifies diagrams of the following form:

$$\begin{array}{ccccc} \mathcal{O}_s^{n+1} & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow = & & \downarrow \cong & & \\ \mathcal{O}_s^{n+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and $F(f)$ is given by pullbacks.

Remark \mathbb{P}_S^n represents the following functor:

$$\begin{aligned} F_S : (\text{Sch}/S)^{\text{op}} &\longrightarrow \text{Set} \\ T &\mapsto F_S(T) = \left\{ \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of Theorem).

Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n + 1$ sections s_1, \dots, s_n of L , where surjectivity ensures that they are not the zero section.

So

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.

There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

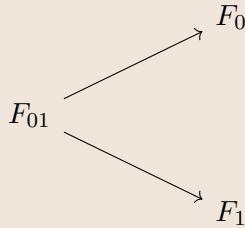
Claim: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{array}{ccc} F_i & \longrightarrow & F \\ \uparrow & & \uparrow \\ F_{ij} & \longrightarrow & F_j \end{array}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example .

For $n = 1$, we can glue along open subschemes



For $n = 2$, we get overlaps of the following form:



This claim implies that we can glue together F_i to get a scheme M . We want to show that M represents F . $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).

Going from LHS to RHS isn't difficult, since for $\mathcal{O}_S^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S .

RHS to LHS comes from gluing.

Proof (of Claim).

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i .

So specifying $F_i(S)$ is equivalent to specifying $n - 1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by \mathbb{A}^n .

We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. $\text{Spec } k[x_1, \dots, \widehat{x_i}, \dots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.

What is the compatibility we are using to glue? For any subset $I \subset \{0, \dots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

2 Tuesday January 14th

Last time: Representability of functors, and specifically projective space $\mathbb{P}_{\mathbb{Z}}^n$ constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}_{\mathbb{Z}}^n} : \mathbb{P}_{\mathbb{Z}}^n \text{Sch}^{\text{op}} &\longrightarrow \text{Set} \\ s &\mapsto \mathbb{P}_{\mathbb{Z}}^n(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

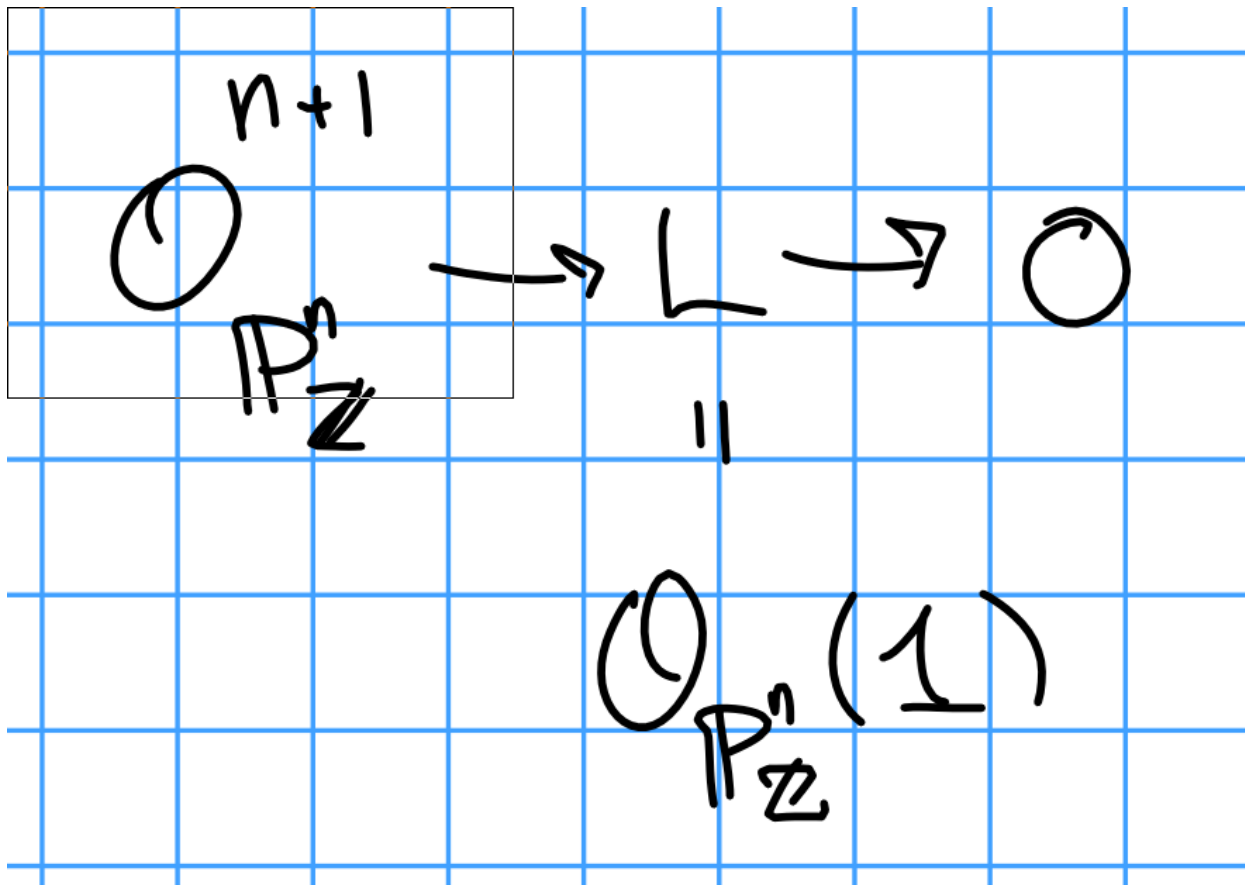
for L a line bundle, up to isomorphisms of diagrams:



That is, line bundles with $n + 1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}_{\mathbb{Z}}^n$ where $F_i(s) = \{ \mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0 \mid s_i \text{ invertible} \}$ are representable and can be glued together, and projective space represents this functor.

Remark: Because projective space represents this functor, there is a universal object:



and other functors are pullbacks of the universal one.

Example 2.1.

Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over $\text{Spec } \mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift



Definition 2.1 (Equalizer).

For a category C , we say a diagram $x \rightarrow y \rightrightarrows z$ is an *equalizer* iff it is universal wrt the property:



(Here X is the universal object).

Example 2.2.

For sets, $X = \{y \mid f(y) = g(y)\}$ for $Y \xrightarrow{f,g} Z$.

Definition 2.2.

A coequalizer is the dual notion,

**Example 2.3.**

Take $C = \text{Sch}/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta\alpha}$:



These are canonically isomorphic.

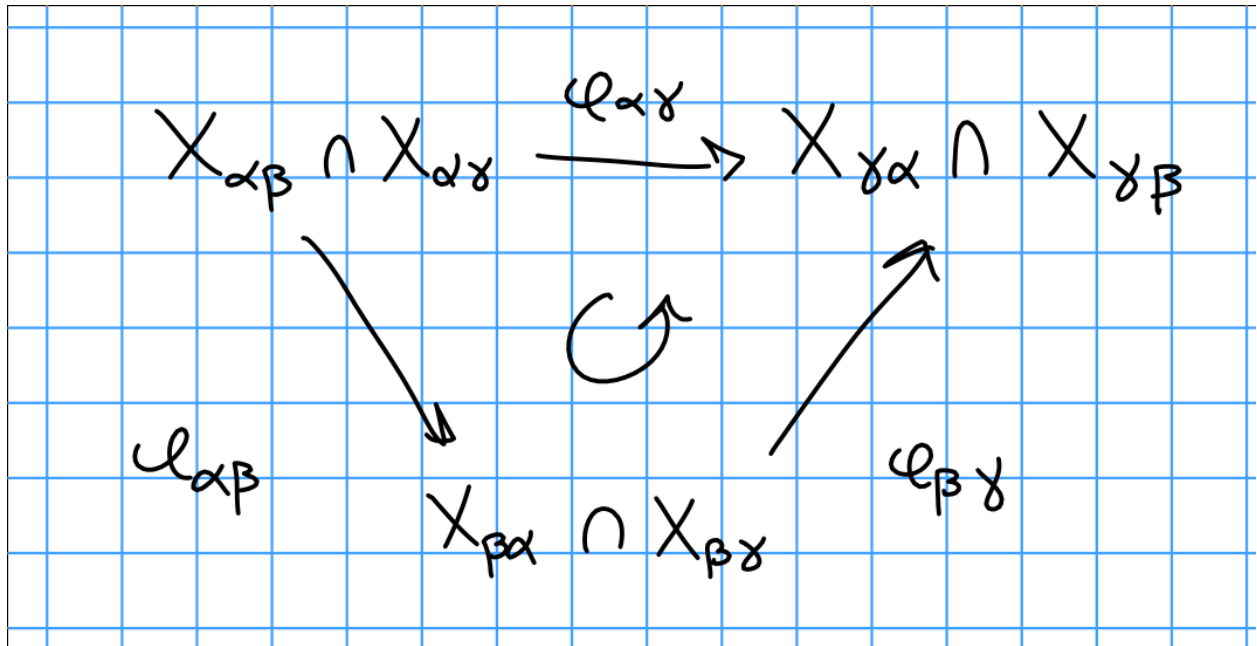
In Sch/S , we have $\coprod_{\alpha\beta} X_{\alpha\beta} \xrightarrow{f_{\alpha\beta}, g_{\alpha\beta}} \coprod X_{\alpha} \longrightarrow X$ where $f_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_{\alpha}$ and $g_{\alpha\beta} : X_{\alpha\beta} \longrightarrow X_{\beta}$; this is a coequalizer.

Conversely, we can glue schemes. Given $X_{\alpha} \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\phi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the cocycle condition:

1. $\phi_{\alpha\beta}^{-1}(X_{\beta\alpha} \cap X_{\beta\gamma}) = X_{\alpha\beta} \cap X_{\alpha\gamma}$, noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_{\beta}} X_{\beta\gamma}$.
2. The following diagram commutes:



Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_{\alpha} \rightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 2.3 (Zariski Sheaf).

A functor $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_{α} , the following is an equalizer:

$$F(T) \rightarrow \prod F(T_{\alpha}) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 2.4.

Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover $\coprod_{\alpha\beta} T_{\alpha\beta} \rightarrow \coprod_{\alpha} T_{\alpha} \rightarrow T$, since giving a local map to X that agrees on intersections is enough to specify a map from $T \rightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 2.4 (Subfunctors, Open/Closed Functors).

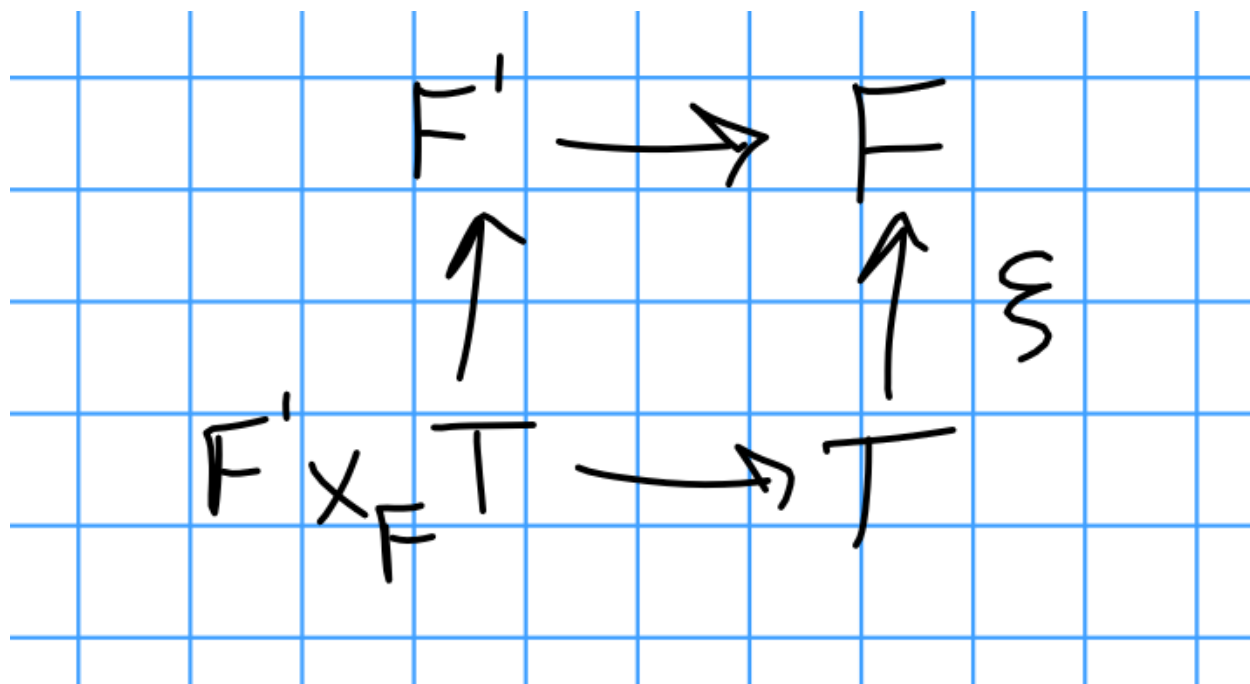
Suppose we have a morphism $F' \rightarrow F$ in the category $\text{Fun}(\text{Sch}/S, \text{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all T/S .
- ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over

T , then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U .

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F' , which maps to F (noting a section is the same as a map to the functor of points), and since $T \rightarrow F$ and $F' \rightarrow F$, we can form the fiber product $F' \times_F T$:



and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \rightarrow F$ is open/closed/locally closed iff



and g is open/closed/locally closed.

I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \rightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

Definition 2.5 (coverings).

A collection of open subfunctors $F_\alpha \subset F$ is an open cover iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \rightarrow F$, the T_α in the following diagram are an open cover of T :



Example 2.5.

Given $F(s) = \{\mathcal{O}_s^{n+1} \rightarrow L \rightarrow 0\}$ and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \rightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

Proposition 2.1.

A Zariski sheaf $F : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ with a representable open cover is representable.

Proof.

Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α . Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \rightarrow F_\alpha$ have the correct gluing maps. This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \rightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x ; we'll show that x represents F .

First produce a map $x \rightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_{\alpha} F(x_\alpha)$, and

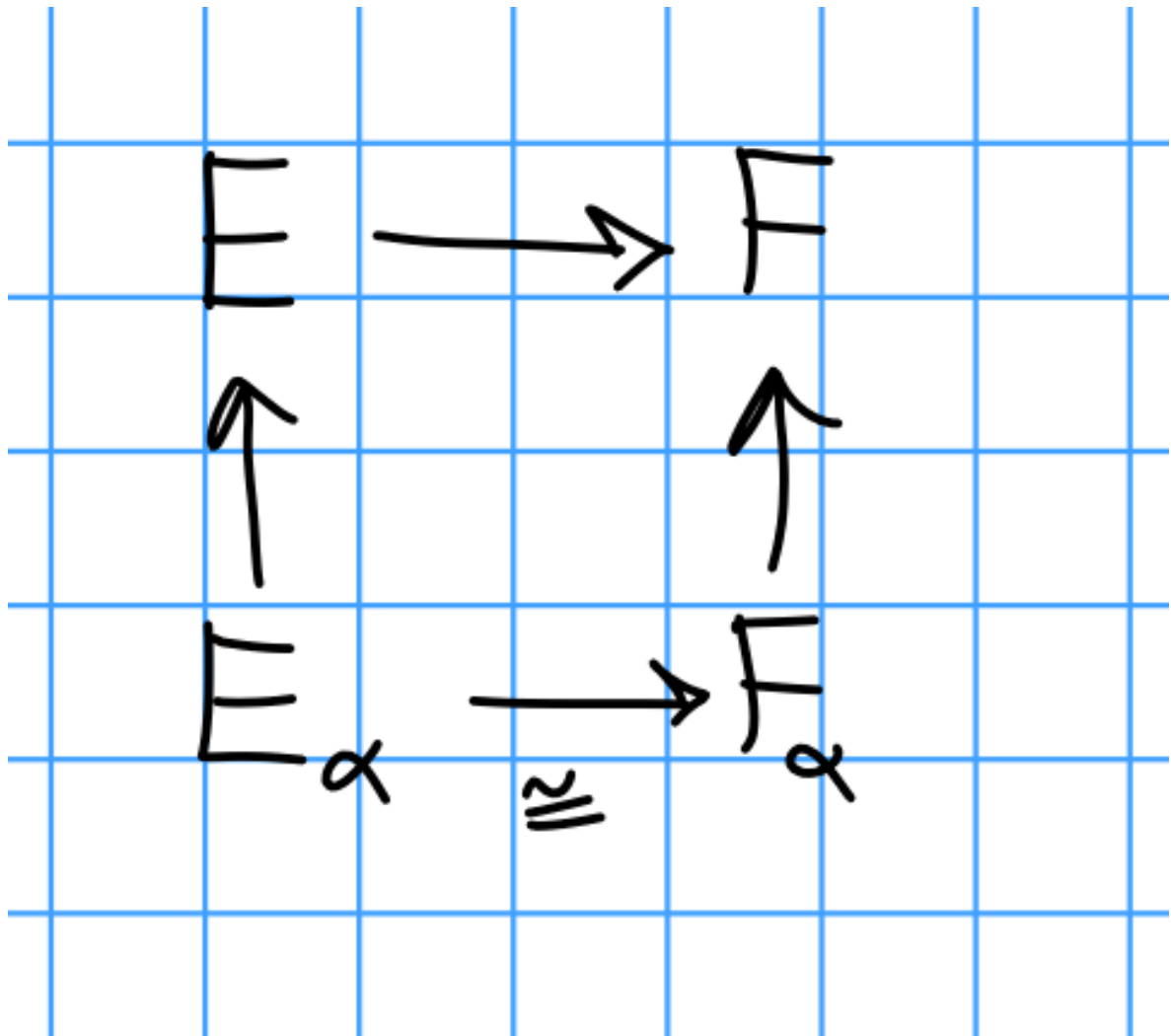
because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

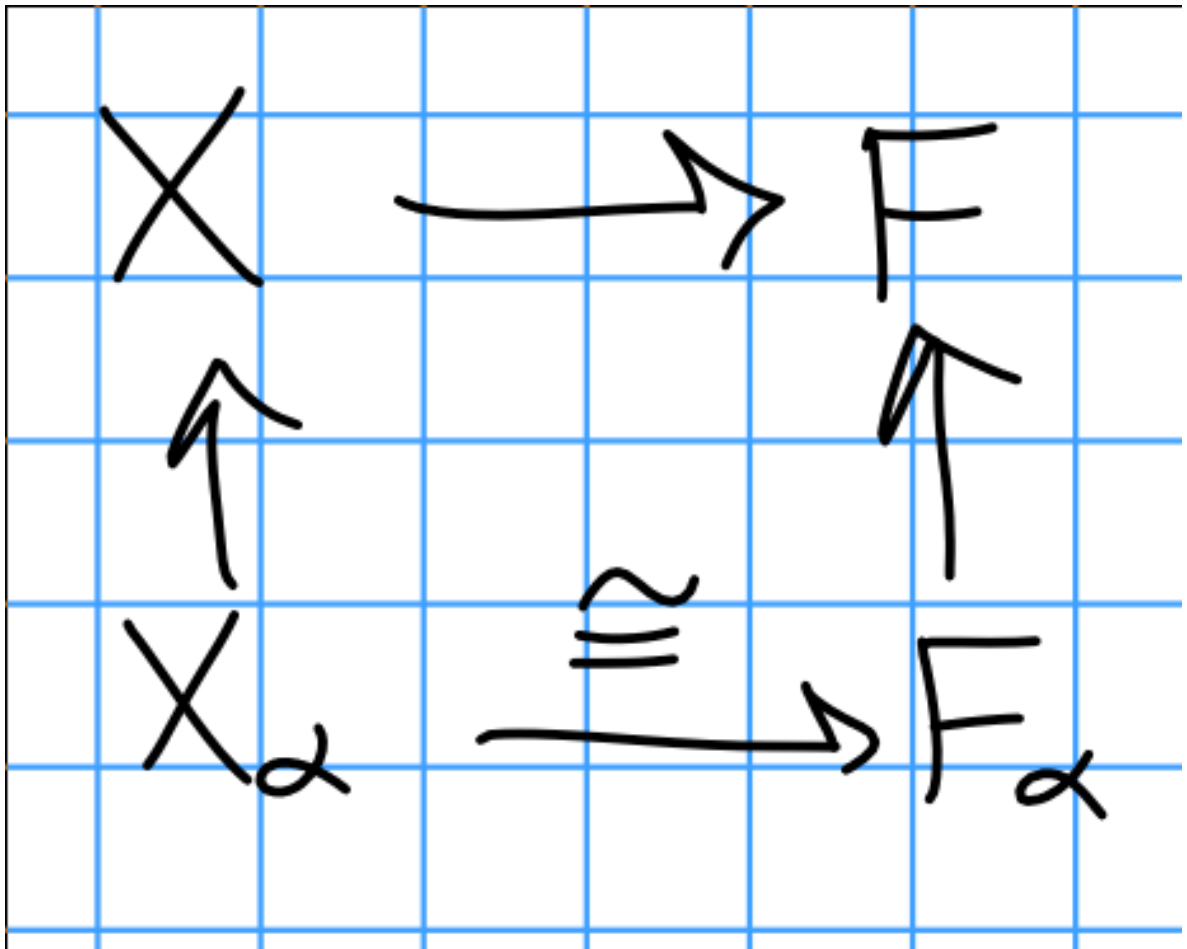
Lemma 2.1.

If $E \rightarrow F$ is a map of functors and E, F are zariski sheaves, where there are open covers $E_\alpha \rightarrow E, F_\alpha \rightarrow F$ with commutative diagrams



(i.e. these are isomorphisms locally) then the map is an isomorphism.

With the following diagram, we're done by the lemma:



Example 2.6.

For S and E a locally free coherent \mathcal{O}_S module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} / \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E .

Then the restriction $F_i \longrightarrow \mathbb{P}E$ where $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i , $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 2.7.

For E locally free over S of rank n , take $r < n$ and consider the functor $\text{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} / \sim$ (a Grassmannian) where Q is locally free of rank k .

Exercise

a. Show that this is representable

b. For the plucker embedding $\mathrm{Gr}(k, E) \rightarrow \mathbb{P} \wedge^k E$, then a section over T is given by $f^*E \rightarrow Q \rightarrow 0$ corresponding to $\wedge^k f^*E \rightarrow \wedge^k Q \rightarrow 0$, noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \mathrm{Spec} k$, then E is a k -vector space V , then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n .

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \rightarrow Q \rightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\mathrm{hom}(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for \mathbb{P}^n of all flat families of subschemes.