# Problem Set 1

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## 1 Problem 5

### 1.1 Part 1

Let $A \in \mathrm{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \mathrm{Math}(n, n)$ be positive semi-definite, so

$$\langle v,\ Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v,\ (A+B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v,\ (A+B)v \rangle &= \langle v,\ Av \rangle + \langle v,\ Bv \rangle \\ &> \langle v,\ Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

### 1.2 Part 2

Let $M$ be a smooth manifold with tangent bundle $TM$ and a maximal smooth atlas $\mathcal{A}$. Choose a covering of $M$ by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$.Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to $\mathcal{C}$, so for each $i$ we have

$$
\begin{aligned}
f_i &: M \to I \\
\forall p \in M, \quad \sum_{i \in I} f_i(p) &= 1
\end{aligned}
$$

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let $g^i$ be the Euclidean metric given by the identity matrix, i.e. $g^i_{jk} \coloneqq \delta_{jk}$. We then have

$$
\begin{aligned}
g^i &: T\phi_i(U_i) \times T\phi_i(U_i) \to \mathbb{R} \\
(\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n \cong \mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Let $G^i$ be the pullback of $g^i$ along the coordinate functions $\phi_i$, so

$$
\begin{aligned}
G^i &: TU_i \times TU_i \to \mathbb{R} \\
G^i(p, q) \coloneqq \left( (\phi_i)^* \, g^i \right) (p, q) &\coloneqq g^i(\phi_i(p), \phi_i(q))
\end{aligned}
$$

Then, for a point $\in M$, define the following map:

$$
\begin{aligned}
g_p &: T_pM \times T_pM \to \mathbb{R} \\
(x, y) &\mapsto \sum_{i \in I} f_i(p) G^i(x, y).
\end{aligned}
$$

The claim is that $g_p$ defines a metric on $M$, and thus the family $\{g_p \mid p \in M\}$ yields a tensor field and thus a Riemannian metric on $M$.

## 2 Problem 6

### 2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on $M$,

$$
X : M \to TM
$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since $S^2$ is compact with $\partial S^2 = \emptyset$, $X$ is necessarily a complete vector field.

Thus every integral curve of $X$ exists for all time, yielding a well-defined flow

$$
\phi : M \times \mathbb{R} \to M
$$

given by solving the initial value problems

$$
\begin{aligned}
\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} &= X(\phi_t(p)),\\
\phi_0(p) &= p
\end{aligned}
$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover $\phi_0$ is homotopic to $\phi_1$ via the homotopy

$$
\begin{aligned}
H : M \times I &\to M\\
(p, t) &\mapsto \phi_t(p).
\end{aligned}
$$

We can now apply the Lefschetz fixed-point theorem to $\phi_0$ and $\phi_1$. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \mathrm{Tr}\left(f_* \Big|_{H_k(X;\mathbb{Q})}\right).$$

where $f_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$
\begin{aligned}
\Lambda(\mathrm{id}_M) &= \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})})\\
&= \sum_k \dim H_k(X;\mathbb{Q})\\
&= \chi(M),
\end{aligned}
$$

the Euler characteristic of $M$.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2\\ 0 & \text{otherwise}\end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus $\phi_1$ has a fixed point $p_0$, thus

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned}
\phi_t(p) =& p \\
\implies \frac{\partial}{\partial t}\phi_t(p) =& \frac{\partial}{\partial t}p = 0 && \text{by differentiating wrt } t \\
\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& \ 0\Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\
\implies X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& \ 0 && \text{by definition of } \phi_1
\end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that $p_0$ is a zero of $X$. So $X$ has at least one zero, as desired. $\square$

## 2.2 Part 2

The trivial bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & S^2 \times \mathbb{R}^2 \arrow[d] \\
& S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

has a nowhere vanishing section, namely

$$\begin{aligned}
s : S^2 &\to S^2 \times \mathbb{R}^2 \\
\mathbf{x} &\to (\mathbf{x}, [1, 1])
\end{aligned}$$

which is the identity on the $S^2$ component and assigns the constant vector $[1, 1]$ to every point. However, as part 1 shows, the bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & TS^2 \arrow[d] \\
& S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

can *not* have a nowhere vanishing section. $\square$