

# Lie Algebras

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## 1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

### 1.1 Overview

Here is a short overview of the topics we expect to cover:

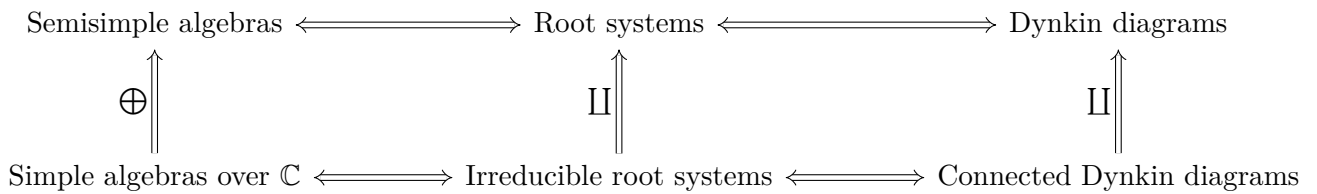
#### 1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory

- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

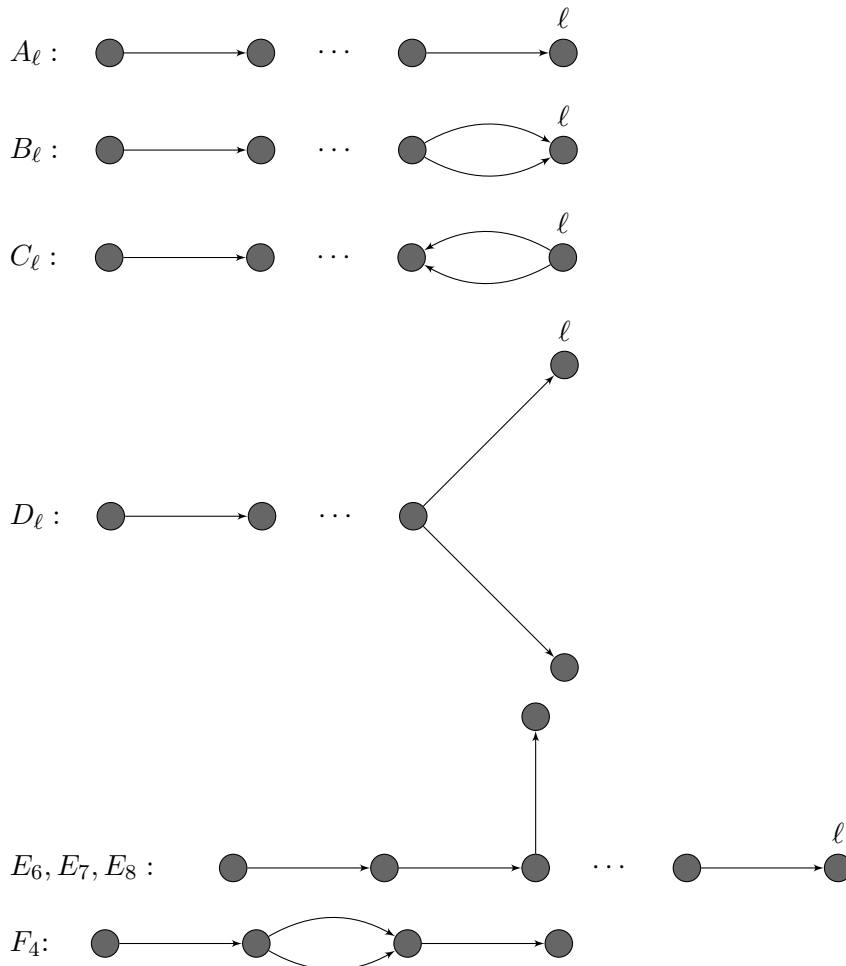
### 1.1.2 Chapter 3-4

We will describe the following series of correspondences:



## 1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:



## 1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

### 1.3.1 Chapter 6

Some important topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type  $A_\ell$  algebra used for the first time; however, it differs from the other types in several important/significant ways.

### 1.3.2 Chapter 7

Skip!

### 1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat- $\mathcal{O}$  [Humphrey 08]

## 1.4 Content

Fix  $F$  a field of characteristic zero – note that prime characteristic is closer to a research topic.

**Definition 1.** A **Lie Algebra**  $\mathfrak{g}$  over  $F$  is an  $F$ -vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\cdot, \cdot]$  is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

**Exercise 1.** Show that  $[x, y] = -[y, x]$ .

**Definition 2.** Two Lie algebras  $\mathfrak{g}, \mathfrak{g}'$  are said to be isomorphic if  $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ .

## 1.5 Linear Lie Algebras

Let  $V = \mathbb{F}^n$ , and define  $\text{End}(V) = \{f : V \rightarrow V \mid f \text{ is linear}\}$ . We can then define  $\mathfrak{gl}(n, V)$  by setting  $[x, y] = (x \circ y) - (y \circ x)$ .

**Exercise 2.** Verify that  $V$  is a Lie algebra.

**Definition 3.** Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \mid \text{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group*  $\text{SL}(n, V)$ .)

**Definition 4.** A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

**Definition 5.** The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where } M = \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

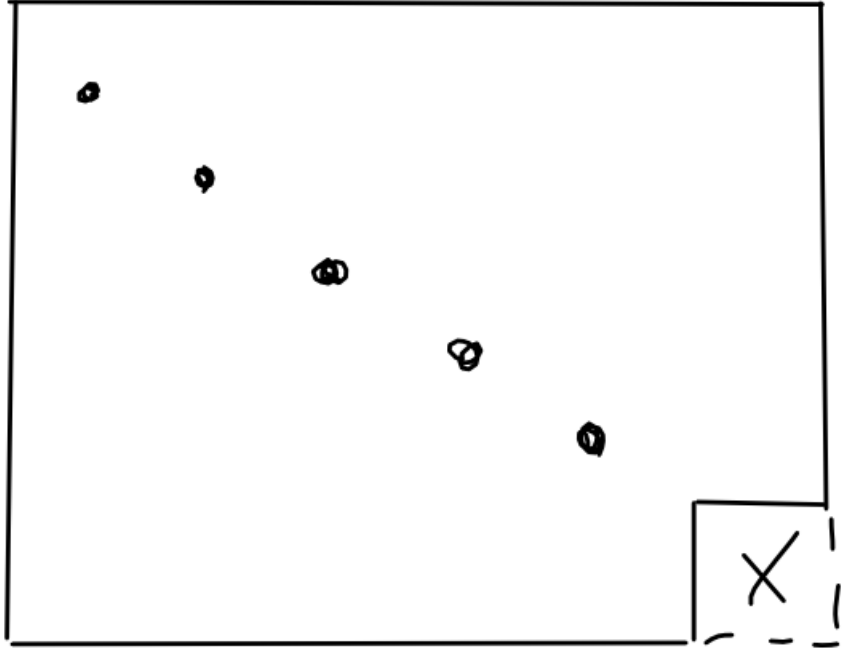
**Definition 6.** The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left( \begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & I_n \\ & -I_n & 0 \end{array} \right) & n = 2\ell + 1 \text{ odd,} \\ \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

**Proposition 1.** The dimensions of these algebras can be computed;

- The dimension of  $\mathfrak{gl}(n, \mathbb{F})$  is  $n^2$ , and has basis  $\{e_{i,j}\}$  the matrices if a 1 in the  $i, j$  position and



zero elsewhere.

- For type  $A_\ell$ , we have  $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$ .
- For type  $C_\ell$ , we have  $\dim \mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2 \left( \frac{\ell(\ell+1)}{2} \right)$ , and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type  $D_\ell$  we have

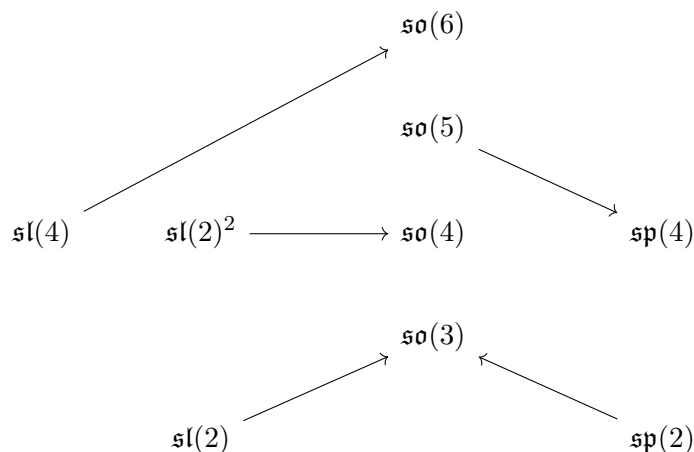
$$\dim \mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be  $2\ell^2 - \ell$ .

- For type  $B_\ell$ , we have  $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$ , with elements of the form

$$\left( \begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

**Exercise 3.** Use the relation  $MA = A^{tM}$  to reduce restrictions on the blocks.



**Theorem 1.** These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

## 2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

**Exercise 4.** Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

### 2.1 Lie Algebras of Derivations

**Definition 7.** An  $F$ -algebra  $A$  is an  $F$ -vector space endowed with a bilinear map  $A^2 \rightarrow A$ ,  $(x, y) \mapsto xy$ .

**Definition 8.** An algebra is **associative** if  $x(yz) = (xy)z$ .

Modern interest: simple Lie algebras, which have a good representation theory. Take a look at Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

**Definition 9.** Any map  $\delta : A^2 \rightarrow A$  that satisfies the Leibniz rule is called a **derivation** of  $A$ , where the rule is given by  $\delta(xy) = \delta(x)y + x\delta(y)$ .

**Definition 10.** We define  $\text{Der}(A) = \{\delta \mid \delta \text{ is a derivation}\}$ .

Any Lie algebra  $\mathfrak{g}$  is an  $F$ -algebra, since  $[\cdot, \cdot]$  is bilinear. Moreover,  $\mathfrak{g}$  is associative iff  $[x, [y, z]] = 0$ .

**Exercise 5.** Show that  $\text{Derg} \leq \mathfrak{gl}(\mathfrak{g})$  is a Lie subalgebra. One needs to check that  $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$ .

**Exercise 6** (Turn in). Define the adjoint by  $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$ . Show that  $\text{ad}_x \in \text{Der}(\mathfrak{g})$ .

## 2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of  $\mathfrak{gl}(V)$ . Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

**Example 1.** Any  $F$ -vector space can be made into a Lie algebra by setting  $[x, y] = 0$ ; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write  $\mathfrak{g} = Fx$ , and so  $[x, x] = 0 \implies [\cdot, \cdot] = 0$ . So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write  $\mathfrak{g} = Fx \oplus Fy$ , the only nontrivial bracket here is  $[x, y]$ . Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$  is abelian.
  - $[x, y] = ax + by \neq 0$ . Assume  $a \neq 0$  and set  $x' = ax + by, y' = \frac{y}{a}$ . Now compute  $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$ . Punchline:  $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$ .

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	$x'$	$y'$
$x'$	0	$x'$
$y'$	$-x'$	0

**Example 2.** Let  $V = \mathbb{R}^3$ , and define  $[a, b] = a \times b$  to be the usual cross product.

**Exercise 7.** Look at notes for basis elements of  $\mathfrak{sl}(2, F)$ ,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of  $\text{ad}(e), \text{ad}(h), \text{ad}(f)$  with respect to this basis.

## 2.3 Ideals

**Definition 11.** A subspace  $I \subseteq \mathfrak{g}$  is called an **ideal**, and we write  $I \trianglelefteq \mathfrak{g}$ , if  $x, y \in I \implies [x, y] \in I$ .

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using  $[x, y] = [-y, x]$ .

**Exercise 8.** Check that the following are all ideals of  $\mathfrak{g}$ :

- $\{0\}, \mathfrak{g}$ .
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra  $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \mid x_i, y_i \in \mathfrak{g}\}$ .  
– Moreover,  $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$ .

Fact: If  $I, J \trianglelefteq \mathfrak{g}$ , then

- $I + J = \{x + y \mid x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \cap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \mid x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

**Definition 12.** A Lie algebra is **simple** if  $[\mathfrak{g}, \mathfrak{g}] \neq 0$  (i.e. when  $\mathfrak{g}$  is not abelian) and has no non-trivial ideals. Note that this implies that  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .

**Theorem 2.** Suppose that  $\text{char } F \neq 2$ , then  $\mathfrak{sl}(2, F)$  is not simple.

*Proof.* Recall that we have a basis of  $\mathfrak{sl}(2, F)$  given by  $B = \{e, h, f\}$  where

- $[e, f] = h$ ,
- $[h, e] = 2e$ ,
- $[h, f] = -2f$ .

So think of  $[h, e] = \text{ad}_h$ , so  $h$  is an eigenvector of this map with eigenvalues  $\{0, \pm 2\}$ . Since  $\text{char } F \neq 2$ , these are all distinct. Suppose  $\mathfrak{sl}(2, F)$  has a nontrivial ideal  $I$ ; then pick  $x = ae + bh + cf \in I$ . Then  $[e, x] = 0 - 2be + ch$ , and  $[e, [e, x]] = 0 - 0 + 2ce$ . Again since  $\text{char } F \neq 2$ , then if  $c \neq 0$  then  $e \in I$ . Now you can show that  $h \in I$  and  $f \in I$ , but then  $I = \mathfrak{sl}(2, F)$ , a contradiction. So  $c = 0$ .

Then  $x = bh \neq 0$ , so  $h \in I$ , and we can compute

$$\begin{aligned} 2e &= [h, e] \in I \implies e \in I, \\ 2f &= [h, -f] \in I \implies f \in I. \end{aligned}$$

which implies that  $I = \mathfrak{sl}(2, F)$  and thus it is simple. □

Note that there is a homework coming due next Monday, about 4 questions.

### 3 Lecture 3

Last time, we looked at ideals such as  $0, \mathfrak{g}, Z(\mathfrak{g})$ , and  $[\mathfrak{g}, \mathfrak{g}]$ .

**Definition:** If  $I \trianglelefteq \mathfrak{g}$  is an ideal, then the quotient  $\mathfrak{g}/I$  also yields a Lie algebra with the bracket given by  $[x + I, y + I] = [x, y] + I$ .

**Exercise:** Check that this is well-defined, so that if  $x + I = x' + I$  and  $y + I = y' + I$  then  $[x, y] + I = [x', y'] + I$ .



### 3.1 Homomorphisms and Representations

**Definition 13.** A linear map  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a *Lie homomorphism* if  $\phi[x, y] = [\phi(x), \phi(y)]$ .

**Remark.**  $\ker \phi \trianglelefteq \mathfrak{g}_1$  and  $\text{im } \phi \leq \mathfrak{g}_2$  is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence  $\{I \trianglelefteq \mathfrak{g}\} \iff \{\text{hom } \phi : \mathfrak{g} \rightarrow \mathfrak{g}'\}$  where  $I \mapsto (x \mapsto x + I)$  and the inverse is given by  $\phi \mapsto \ker \phi$ .

Theorem (Isomorphism theorem for Lie algebras):

- If  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a Lie algebra homomorphism, then  $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If  $I, J \trianglelefteq \mathfrak{g}$  are ideals and  $I \subset J$  then  $J/I \trianglelefteq \mathfrak{g}/I$  and  $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$ .
- If  $I, J \trianglelefteq \mathfrak{g}$  then  $(I + J)/J \cong I/(I \cap J)$ .

Definition: A *representation* of a Lie algebra  $\mathfrak{g}$  is a Lie algebra homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  into a linear Lie algebra for some vector space  $V$ .

<!--> Idea: Define a semisimple Lie algebra-->

<!--> Remark: This can be confusing if  $\mathfrak{g}$  is a linear algebra, we can consider elements  $x \in \mathfrak{g}$  and ask if it is the case  $x$  being nilpotent (as an endomorphism) iff  $\text{ad } x$  is nilpotent? False, a counterexample is  $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ , where there exists an  $x$  which is *not* nilpotent while  $\text{ad } x$  is nilpotent, which contradicts the above theorem.-->