Title

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1.1 One Variable Function Fields (Ch. 1)

Since we have the field-theoretic preliminaries out of the way, we now start studying one-variable function fields in earnest. The main technique that we use to extract the geometry will be the theory of valuations. These may be familiar from NTII, but we will cover them in more generality here.

1.1.1 Valuation Rings and Krull Valuations

Recall that NTII approach to valuations:

> **Definition 1.1.1** (Valuation)
> A **valuation** on a field K is a map $v : K \to \mathbb{R} \cup \{\infty\}$ such that $v(K^\times) \subset \mathbb{R}$, $v(0) = \infty$, and v is of the form $-\log(|\cdot|)$ where $|\cdot| : K \to [0, \infty)$ is an *ultrametric norm.*[a] Recall that an *ultrametric norm* satisfies not only the triangle inequality but the ultrametric triangle inequality, i.e. $d(x, z) \leq \max(x, z)$.
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> [a]In other words, $e^{-v(\,\cdot\,)}$ is an ultrametric norm.

We now take an algebraic approach to this definition, where we'll end up replacing $\mathbb{R}$ with something more general.

> **Definition 1.1.2** (Valuation Ring)
> A subring R of a field K is a **valuation ring** if for all $x \in K^\times$, at least one of x or x^{-1} is in R.

Remark 1.1.3: This is a "largeness" property. It also implies that $K = \mathrm{ff}(R)$.

> **Definition 1.1.4** (Group of Divisibility)
> Given any integral domain R with fraction field K, the **group of divisibility** $G(R)$ is defined as the *partially ordered commutative group*[a]
>
> $$G(R) \coloneqq K^\times / R^\times.$$
>
> We will write the group law here additively. The ordering is given by $x \leq y \iff y/x \in R$.
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> [a]This means that the two structures are compatible.

Remark 1.1.5: Note that the way the partial order is written, it's a relation on $K^\times$, but it is

not quite a partial ordering there. It is reflexive and transitive, but need not be antireflexive: if $x/y, y/x \in R$ then x, y differ by an element of $u \in R^\times$ so that $x = uy$. In particular, they need not be equal. This gives a structure of a *quasiordering*, and if you set $x \sim y \iff x \leq y$ and $y \leq x$, this leads to an equivalence relation, and modding out by it yields a partial order. Here this is accomplished by essentially trivializing units.

Another way to think of $G(R)$ is as the nonzero principal fractional ideals of K, since any two generators of an ideal will differ by a unit.

Remark 1.1.6: Inside this group there is a *positive cone* $G(R)^+$ of elements that are "nonnegative": since we're in a commutative setting, the zero element is equal to 1, and the positive cone is given by $\{y \geq 0\} = \{y \in R\}$, and is thus given by the group $G(R)^+ = (R, \cdot)$.

This is very general: if you're studying factorization in integral domains, many properties are reflected in $G(R)$. E.g. being a UFD (the most important factorization property!) implies that $G(R)$ is a free commutative group.

Remark 1.1.7: In general this is only a *partially* ordered group and not totally ordered. For example, take $R = \mathbb{Z}$ and $x = 2, y = 3$, then neither of $2/3, 3/2$ are in $\mathbb{Z}$, so $x \not\leq y$ and $y \not\leq x$. On the other hand, if we do have a total order, then either x or x^{-1} is in the ring, which are exactly valuation subring of a field.

Claim: R is a valuation ring $\iff$ $G(R)$ is totally ordered.

Remark 1.1.8: Note that $\mathbb{R}$ is a totally ordered group.

This makes $G(R)$ the "target group" of a generalized analytic valuation. Whenever we have a valuation ring, we have a totally ordered commutative group, and the valuation $v : K^\times \to G(R)$ is a quotient map which we can extend to K by $v(0) := \infty$. This has some familiar properties:

- (VRK1) For all $x, y \in K^\times$,[1]

$$v(xy) = v(x) + v(y).$$

- (VRK2) For all $x, y \in K^\times$ such that $x + y \neq 0$,

$$v(x + y) \geq \min(v(x), v(y)).$$

For ultrametric norms, all triangles are isosceles: is that true for this type of function? The answer is yes, by the following exercise:

Exercise 1.1.9*(?):* If $v(x) \neq v(y)$, then $v(x + y) = \min(v(x), v(y))$.

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[1] This follows from the fact that the quotient map is a group morphism. Note that the additive notation makes this more suggestive of what an original valuation satisfied.

So the properties here are formally identical to the NTII notion of valuation, with $(\mathbb{R}, +, \leq)$ replaced by $(G(R), +, \leq)$.

Exercise 1.1.10*(?):* Conversely, if $v : K^\times \to G$ is a map into a totally ordered commutative group satisfying VRK1 and VRK2[2], then

$$R_v := \left\{ x \in K^\times \;\middle|\; v(x) \geq 0 \right\} \cup \{0\}$$

is a valuation ring.[3] We can thus extract valuation rings in this situation.

Exercise 1.1.11*(?):* A valuation ring is **local**, i.e. there is a unique maximal ideal

$$\mathfrak{m}_v := \left\{ x \in K^\times \;\middle|\; v(x) > 0 \right\} \cup \{0\}.$$

Remark 1.1.12: These two constructions are morally mutually inverse. This doesn't hold on the nose, since there is extraneous data in the new analytic valuation. Recall that in NTII we have a notion of equivalence of norms, and two distinct norms that are equivalent can give rise to the same valuation. For example, given a valuation, one can scale it by $\alpha \in \mathbb{R}$, and it's easy to check that this gives the same valuation. It is possible for the valuation not to surject onto $\mathbb{R}$, but this doesn't happen in practice. The image is usually infinite cyclic, what we call a *discrete valuation*, and so one is led to the definition of the *value group* of the valuation as its image. If you have a notion of equivalence of Krull valuations, you want to allow for isomorphisms of the value group. The cleanest notion of equivalence is thus the following:

> **Definition 1.1.13** (Equivalence of Krull valuations)
> Two Krull valuations on a field K are **equivalent** iff their valuation rings are *equal*.

Remark 1.1.14: Going back to NTII, if you have two nonarchimedean norms on a field, then there are many equivalent conditions for equivalence, and this is one of them.

Some general valuation theory:

- Every totally ordered commutative group is a group of divisibility.[4]

- A totally ordered group has **rank 1** if it is nontrivial and embeds into $\mathbb{R}$

    - If the value group is trivial, $R = K$

- A Krull valuation of rank at most 1 is the NTII notion of a valuation.

Exercise 1.1.15*(?):* For $n \geq 2$, put the lexicographic order on $\mathbb{Z}^n$, and show this has rank strictly larger than 1. Thus $\mathbb{Z}^n \hookrightarrow \mathbb{R}$ as a commutative group, but not as a totally ordered commutative group.

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[2]Any such map satisfying these two properties is a **Krull valuation**, Krull's generalization of classical valuations.
[3]Note that in a totally ordered group, either $v(x) \geq 0$ or $-v(x) \geq 0$, so we get the property that either $x, x^{-1} \in R_v$.
[4]Pete's Commutative Algebra Notes, Ch. 17.10

Remark 1.1.16: In fact, for any ordered group G, one can attach a rank: a cardinal number $r(G)$. Here, $r((\mathbb{Z}^n, \text{lex})) = n$. This is useful when studying $\text{Spec}(R)$ for R a DVR.

A valuation of rank bigger than 1 does not induce a norm on K in the metric sense, although this is not so important. A closer notion is expanding the notion of a metric space by allowing the metric to be defined on X as $d: X \times X \to R$ for some R more general than $\mathbb{R}$, like a totally ordered group or a nonarchimedean field. This would yield a class of topological spaces that are reminiscent of metric spaces.

Definition 1.1.17 (Important: Regular and Centered)

Let $v: K^\times \to (G, +)$ be a Krull valuation and let $A \subset K$ be a subring of K. Then v is A-**regular** or **centered in** A if A is a subset of some valuation ring R_v. In this case, $\mathfrak{p} := \mathfrak{m}_v \cap A \in \text{Spec}(A)$ is denoted the **center of** v **in** A.[a]

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[a]Here $\mathfrak{m}_v$ denotes pulling back the maximal ideal along this morphism of rings.

Remark 1.1.18: The term regularity here arises because we'll want to think of elements of A as functions and the valuation as a type of point, then the notion of being a regular function at a point will carry over. The center is the subset of A with strictly positive valuation. Also recall that pulling back prime ideals yields prime ideals, and maximal ideals are a special kind of prime ideal, but in general pulling back a maximal ideal may not result in another maximal ideal. So somehow the valuation affects every subring on which it is regular.

Definition 1.1.19 (Key: Zariski-Riemann Space)

For $A \subset K$, define

$$\Sigma(K/A) := \left\{ \text{valuation rings } A \subset R \subsetneq K \mid K = \text{ff}(R) \right\}$$

$$\tilde{\Sigma}(K/A) := \left\{ \text{valuation rings } A \subset R \subseteq K \mid K = \text{ff}(R) \right\}.$$

The set $\tilde{\Sigma}(K/A)$ is the **Zariski-Riemann space**.

Remark 1.1.20: Note that in this definition, we're taking all A-regular valuation rings R in K. If someone says R is a valuation ring of K, they likely mean that $K = \text{ff}(R)$. Note that fields are valuation rings, so otherwise, any subfield of K would also be a valuation ring of K. Here, K itself plays the role of a generic point. (?) The only difference in these two definitions is that in the first, the trivial valuation ring is being excluded.

Definition 1.1.21 (Key: Places, Points of a Curve)

If K/k is a one variable function field[a] , then $\Sigma(K/k)$ will be the **points of the associated algebraic curve** or **places**. These can be thought of as valuation rings, or equivalence classes of Krull valuations, where two valuations are equivalent if they have the same valuation ring.

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[a]Finitely generated field extension of transcendence degree one.

Remark 1.1.22: In terms of scheme theory, these will be the closed points of our algebraic curve. We will view elements $f \in K$ as meromorphic functions on $\Sigma(K/k)$.

> **Definition 1.1.23** (Zariski Topology)
> The **Zariski topology** on $\Sigma(K/A)$ has a sub-base
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> $$\left\{ U(f) \mid f \in K \right\} \qquad\qquad U(f) := \left\{ v \in \tilde{\Sigma}(K/A) \mid v(f) \geq 0 \right\} = \tilde{\Sigma}(K/A[f]).$$
>
> and we thus take the minimal topology such that all of these sets are open. In other words, every open set is a finite intersection and/or arbitrary unions, including empty intersections/unions. The last term is precisely the subring generated by A and f. Thus a base is $U(f_1, \cdots, f_n) = \tilde{\Sigma}(K/A[f_1, \cdots, f_n])$. The Zariski topology on $\Sigma(K/A)$ is defined the same way and/or via the subspace topology, since this removes a single point.

Remark 1.1.24: We thus get the subrings of K that contain A and are finitely generated as A-algebras. We'll be specifically looking at the case where A is a field and K is a one variable function field.

> **Theorem 1.1.25** *(Zariski).*
> $\tilde{\Sigma}(K/A)$ is quasi-compact.

> *Proof (?).*
> See Mazamara (?) in the chapter discussing valuation rings.
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Note that by definition, $v_n \notin \Sigma(K/A)$. In $\tilde{\Sigma}(K/A)$, we have a trivial valuation v_n whose value group is trivial and valuation ring is K itself, and v_n is a generic point of $\Sigma(K/A)$: its closure is the entire space. In other words, it is in every nonempty open subset. Since we have at least one generic point, and in general there may be many, if $\left| \tilde{\Sigma}(K/A) > 1 \right|$ then this is not a separated (T_1) space since the point is not closed.[5] Another example of such a space would be $\mathrm{Spec}(R)$ for R a commutative ring with positive Krull dimension, which will be Kolmogorov (T_0) but not separated. Such a spectrum is the underlying topological space of some affine scheme, and in general, schemes will have these kinds of properties that are bad (but not *too* bad).

In our case of interest, when K/k is finitely generated of transcendence degree one, we'll see that this is the cofinite topology on an infinite space: the proper closed subsets are precisely the finite subsets, or equivalently every nonempty open subset has finite complement. This is far from Hausdorff: the intersection of two open subsets will still have finite complement, so any two nonempty open subsets must intersect.

It's not generally true that just removing the generic point v_n will make the space separated, but in our case, it will be. So if we restrict to nontrivial valuation rings, then the underlying set will be infinite and we'll get the cofinite topology. This will be the coarsest separated topology, i.e. if you

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[5]Note that in French, separated may be interpreted as Hausdorff, but here we mean points are closed or equivalently any two distinct points admit open neighborhoods that do not meet the other point.

want singletons to be closed, finite subsets must be closed. If $k \subset A \subset K$ where A is a Dedekind domain with fraction field K, we will see that if we consider not the k-regular elements but the A-regular ones, we'll get $\Sigma(K/A) = \mathrm{maxSpec}(A)$ and both Zariski topologies are cofinite. Note that in a Dedekind domain, trading in a prime spectrum for a max spectrum is removing a generic point, so this matches up. The moral: the topology of $\Sigma(K/k)$ is not doing anything interesting and we won't need it much.

When K/k instead has transcendence degree *bigger* than 1, then $\tilde{\Sigma}(K/k)$ is much more interesting. If we were doing things scheme-theoretically, we could try to define a structure sheaf: attaching a sheaf whose stalks are local commutative rings to make it a locally ringed space.[6] Here, the choice of a ring is straightforward: literally $\tilde{\Sigma}(A, A[f_1, \cdots, f_n])$. There's an exercise that shows that although defining a sheaf on the entire space is somewhat annoying, defining it on a basis suffices.

Exercise 1.1.26*(?):* Endow $\tilde{\Sigma}(K/k)$ with the structure of a locally ringed space.

Remark 1.1.27: In dimension 1 (the case we're studying), the corresponding Zariski-Riemann space will be the scheme associated to the complete nonsingular model of the curve. So this valuation-theoretic approach will take you from the function field back to a nice model of the scheme itself. But note that in larger dimensions, there is no unique complete nonsingular model – for example, you can blow any one up to get another – so this pattern can't possibly continue to hold. In fact, it's not clear if we even know if there's *one* such model!

Remark 1.1.28: Thus in dimension > 1, you get something that is decidedly not a scheme, but is still relevant to the study of resolution of singularities for your function field. Where do these come up? Zariski used $\Sigma(K/A)$ to prove resolution of singularities [7] in characteristic zero and dimensions 2 and 3 in 1944, although dimension 2 was classical by the Italian school. Later, Hironaka (1984) got the Fields medal for proving resolution of singularities for all dimensions in characteristic zero using an ingenious inductive argument that avoided Zariski-Riemann spaces entirely. It remarkably doesn't use any new objects/tools, just uses existing ones in a clever way. So why talk about Zariski-Riemann spaces at all? In the last 10 years or so, work of Ternkin and Conrad has revived and generalized them. They study *relative* such spaces.

> *Problem. (Open)*
> In positive characteristic, resolution of singularities is only known up to dimension ≤ 3.

The following is an extremely important result from commutative algebra:

> **Theorem 1.1.29***(CA 17.17).*
> Let $A \subset K$ be a subring of a field, then
>
> $$\bigcap_{v \in \tilde{\Sigma}(K/A)} R_v,$$
>
> the intersection of all valuation subrings of the field, is the integral closure of A in K.

The proof is mostly a Zorn's lemma type of argument. Note that each R_v is generally big, contains

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[6]Schemes are a full subcategory of the much larger category of locally ringed spaces.

[7]Resolving means given K/k, we want to find a complete nonsingular affine variety whose function field is K.

A, and $\mathrm{ff}(R_v) = K$. Moreover, each valuation ring is integrally closed, although we haven't proved this yet.

> **Corollary 1.1.30***(?).*
> For K/k function field, $\cap_{v \in \Sigma(K/k)} R_v = \kappa(K)$, the constant subfield of K.

> *Proof (?).*
> Note that $\kappa(K)$ is the integral (algebraic) closure of k in K. Applying the theorem directly almost works, except the theorem involves $\tilde{\Sigma}$. Can we remove the tilde? Suppose not, this can only happen if $\Sigma(K/k) = \emptyset$ and the intersection is just K itself, the largest thing in the intersection. But can the integral closure of k in K be K itself? No, since the transcendence degree of the function field is positive. So K/k is transcendental, while $\kappa(K)/k$ is an algebraic extension, a contradiction.
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Remark 1.1.31: Note that $\Sigma(K/k)$ is nonempty: there is a nontrivial valuation ring between k and K in great generality, and there are often many.

Claim Key: If $\mathrm{trdeg}(K/k) = 1$, then every $v \in \Sigma(K/k)$ is discrete and thus has value group isomorphic to $\mathbb{Z}$.

So despite the fact that we've introduced a more general notion of higher rank valuations, in dimension 1, every single valuation is discrete.

> *Proof (?).*
> Let $v \in \Sigma(K/k)$ be a place, so its a valuation ring with fraction field K that is not K, then R_v is not a field. So its maximal ideal $\mathfrak{m}_v$ is nonzero, so choose a nonzero element $t \in \mathfrak{m}_v$. Then $t \in R_v$ and R_v contains k, so $k[t] \subset R_v$. Note that $k[t]$ is a PID sitting inside a valuation ring. So restrict this maximal ideal down: $\mathfrak{m}_v \cap k[t]$ is a prime ideal of $k[t]$ containing t, and thus the center $\mathfrak{m}_v \cap k[t] = \langle t \rangle$. This follows because a prime ideal in the polynomial ring $k[t]$ which contains t is necessarily generated by t, since there's exactly one such ideal.
>
> Now restricting the valuation on K to $k(t) \subset K$, $K/k(t)$ will be a finite extension (from the first lecture). We know $k(t) \subset K$, and we can now check that $v|_{k(t)}$ is the t-adic valuation v_t. Note that $\mathfrak{m}_v$ can not contain any other monic irreducible polynomials, since distinct such polynomials are coprime. Since we're in a PID, this ideal would contain any linear combination of them and thus contain 1. So consider the map
>
> $$k[t] \hookrightarrow R_b \to G(R_v) = K^\times / R^\times.$$
>
> Note that the units of $k[t]$ map trivially, using the fact that any element in $k[t]$ can be written as $u \prod p_i^{a_i}$ with the p_i monic irreducible polynomials. The unit maps to zero, along with all of the other monic irreducibles, and thus the image is determined entirely by the image of powers of t. This whole term goes to zero unless some $p_i \mapsto t$, in which case it maps to some power of t.

So suppose $t \mapsto \gamma \neq 0 \in G(R)$, which is nonzero because t was not a unit (since it was in the maximal ideal). Then the image is exactly $\gamma^{\mathbb{N}}$, the non-negative integer powers of the image of t. But if we know goes on this domain, taking denominators shows where it goes on the fraction field (of a UFD), so the image is the cyclic group generated by γ, i.e. the powers of t are literally the only valuations we get. So the image of $k(t)^{\times}$ in $G(R_v)$ is $\gamma^{\mathbb{Z}}$, yielding a discrete valuation. This proves that the restriction to the rational function field $k(t)$ is discrete, and we want to use this to deduce that the original valuation is discrete.

We can now use NTII:[a] since $K/k(t)$ is finite, it follows that v is discrete iff $v|_{k(t)}$ is discrete, and thus v is discrete.

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[a]See NTII, Corollary 1.60: a valuation on a field whose restriction to a finite index subfield is discrete is itself discrete.

So every place of K/k is a discrete valuation as long as the transcendence degree is one, but this is far from the case for degree ≥ 2! In the following example, we'll have a rational function field, which makes things easier. You need a theory of extending Krull valuations, since we'll define a non-rank 1 valuation on the rational function field. But an arbitrary finitely generated field extension of degree d over k is a finite degree extension of the rational function field, and valuation theory will tell you that every valuation downstairs can be extended in full generality to a finite degree field extension, and the rank will not change.

Exercise 1.1.32*(?):* If K/k is finitely generated of trdeg ≥ 2, then $\Sigma(K/k)$ has valuations of rank d.

Note that the Zariski-Riemann space only consists of discrete valuations, which is a characteristic property of one variable function fields. So these higher rank valuations may look weird, but when studying a function field of higher transcendence degree (e.g. for an algebraic surface), these occur.

Exercise 1.1.33*(Constructing valuations of arbitrary rank and value group):* Let k be a field and $K = k(t_1, \cdots, t_n)$. Set $G = \mathbb{Z}^n$ with the lex order, so $G^{\geq 0} = \mathbb{N}^n$.

- Show that $k[t_1, \cdots, t_n] = k[G^{\geq 0}]$, where the RHS is the associated semigroup ring.

- Define $v : k[G^{\geq 0}]^{\bullet} \to G^{\geq 0}$ by mapping each polynomial the minimal index of a monomial in its support. For example,

$$v(a_1 t_1^3 t_2 + a_2 t_1^2 t_2^{10}) = (2, 10),$$

which has support $(3, 1)$ and $(2, 10)$, and we take the min in the lex order.

- Extend v to $v : K^{\bullet} \twoheadrightarrow G$ satisfying VRK1 and VRK2. Show that $R_v := v^{-1}(G^{\geq 0}) \cup \{0\}$ is a valuation ring with value group G, and in particular, the rank is n.

Note that doing this for $n = 1$ reduces to the t-adic valuation, which just keeps track of the smallest power of t appearing. Here you can extend to fraction fields by defining $v(x/y) = v(x) - v(y)$. The semigroup ring can't *be* the valuation ring, since polynomial rings are not local rings, so it's much

bigger. Note also that $\mathbb{Z}$ can be replaced with any group G, since it's never used in anything but a psychological fashion.

Slogan 1.1.34: There is a huge difference between trdeg $= 1$ and trdeg > 1, and so we'll only be working with the former case in this course.