Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Summary/Outline

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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Outline

What we're trying to prove:

– 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

– Define

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$
$$S(s,t) \xrightarrow{s \longrightarrow \pm\infty} S^\pm(t).$$

Outline

- Took $R^\pm : I \longrightarrow \mathrm{Sp}(2n; \mathbb{R})$: symplectic paths associated to $S^\pm$
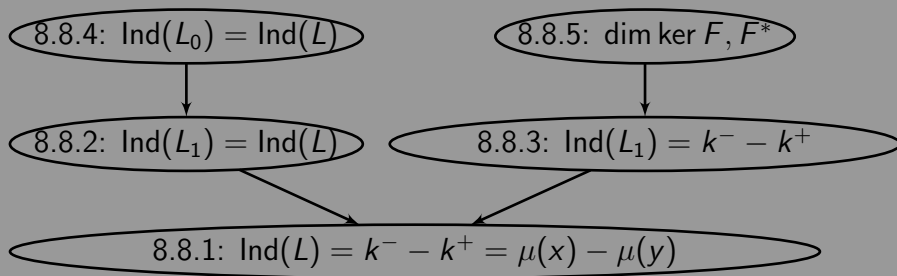
- These paths defined $\mu(x), \mu(y)$

- Section 8.7:

  $$R^\pm \in \mathcal{S} := \left\{ R(t) \;\middle|\; R(0) = \mathrm{id}, \;\; \det(R(1) - \mathrm{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

  $$\mathrm{Ind}(L) \overset{\mathsf{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
    - So we know $\text{Ind}(L) = \text{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
    - Computing $\text{Ind}(L_1)$ by computing kernels.

8.8.4: $\text{Ind}(L_0) = \text{Ind}(L)$ 8.8.5: dim ker F, F^*

8.8.2: $\text{Ind}(L_1) = \text{Ind}(L)$ 8.8.3: $\text{Ind}(L_1) = k^- - k^+$

8.8.1: $\text{Ind}(L) = k^- - k^+ = \mu(x) - \mu(y)$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Recall

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$
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Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

– Since coker $L_1 \cong \ker L_1^*$, it suffices to compute $\ker L_1^*$

– We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \mathrm{Mat}(2; \mathbb{R}).$$

– This allows us to reduce to the $n = 1$ case.

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: Ind(L_1) = k^- $-$ k^+
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Setup

L_1 used a path of diagonal matrices constant near ∞:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases} .$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: Ind(L_1) = $k^- - k^+$
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Statement of Later Lemma (8.8.5)

Let $p > 2$ and define

$$F : W^{1,p} \left(\mathbb{R} \times S^1; \mathbb{R}^2 \right) \longrightarrow L^p \left(\mathbb{R} \times S^1; \mathbb{R}^2 \right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Note: F is L_1 for $n = 1$:

$$L_1 : W^{1,p} \left(\mathbb{R} \times S^1; \mathbb{R}^{2n} \right) \longrightarrow L^p \left(\mathbb{R} \times S^1; \mathbb{R}^{2n} \right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\operatorname{Ind}(L_1) = k^- - k^+$
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Statement of Lemma

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose $a_i^\pm \notin 2\pi\mathbb{Z}$.

1. Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \operatorname{Ker} F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \operatorname{Ker} F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

2. Suppose $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \# \left\{ i \in \{1, 2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$
$$\dim \operatorname{Ker} F^* = \# \left\{ i \in \{1, 2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: Ind(L_1) = k^- − k^+
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Statement of Lemma

In words:

1. If $S(s)$ is a scalar matrix, set $a^\pm = a_1^\pm = a_2^\pm$ to the limiting scalars and count the integer multiples of 2π between a^- and a^+.

2. Otherwise, if S is uniformly bounded by 1, count the number of entries the flip from positive to negative as s goes from $-\infty \longrightarrow \infty$.

$$S_K := \begin{bmatrix} a_1^- & 0 \\ 0 & a_2^- \end{bmatrix} \qquad S(s) = \begin{bmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{bmatrix} \qquad S_K = \begin{bmatrix} a_1^+ & 0 \\ 0 & a_2^+ \end{bmatrix}$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$
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Proof of Assertion 1

1. Suppose $a_1(s) = a_2(s)$ and set $a^{\pm} := a_1^{\pm} = a_2^{\pm}$. Then

$$\dim \mathrm{Ker}\, F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \subset \mathbb{R} \right\}$$
$$\dim \mathrm{Ker}\, F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

– Write $a(s) := a_1(s) = a_2(s)$.
– Start with equation on $\mathbb{R}^2$,

$$\mathbf{Y}(s, t) = [Y_1(s, t), Y_2(s, t)].$$

– Replace with equation on $\mathbb{C}$:

$$\mathbf{Y}(s, t) = Y_1(s, t) + iY_2(s, t).$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Assertion 1, Step 1: Reduce to CR

– Expand definition of the PDE

$$F(\mathbf{Y}) = 0 \rightsquigarrow \bar{\partial}\mathbf{Y} + S\mathbf{Y} = 0$$

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\mathbf{Y} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix}\mathbf{Y} = 0.$$

– Change of variables: want to reduce to $\bar{\partial}\tilde{Y} = 0$
– Choose $B \in \text{GL}(1, \mathbb{C})$ such that $\bar{\partial}B + SB = 0$
– Set $Y = B\tilde{Y}$, which (?) reduces the previous equation to

$$\bar{\partial}\tilde{Y} = 0.$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: Ind(L_1) = k^- − k^+
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Assertion 1, Step 1: Reduce to CR

Can choose (and then solve)

$$B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix} \quad \text{where} \quad \frac{\partial b}{\partial s} = -a(s)b(s)$$

$$\implies b(s) = \exp\left(\int_0^s -a(\sigma) \, d\sigma\right) := \exp\left(-A(s)\right).$$

Remarks:

– For some constants C_i, we have

$$A(s) = \begin{cases} C_1 + a^- s, & s \leq -\sigma_0 \\ C_2 + a^+ s, & s \geq \sigma_0 \end{cases}.$$

– The new $\tilde{Y}$ satisfies CR, is continuous and L^1_{loc}, so elliptic regularity $\implies C^\infty$.
– The real/imaginary parts of $\tilde{Y}$ are C^∞ and harmonic.

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$
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Assertion 1, Step 2: Solve CR

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$
- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of $\tilde{Y}$ in z.
- Deduce that the solutions of the system are given by

$$\tilde{Y}(u) = \sum_{\ell \in \mathbf{Z}} c_\ell u^\ell \implies \tilde{Y}(s + it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $\{c_\ell\}_{\ell \in \mathbb{Z}} \subset \mathbb{C}$ converges for all s, t.

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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$$8.8.3: \; \text{Ind}(L_1) = k^- - k^+$$

Summary/Outline
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8.8.5: dim ker F, F^*
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8.8.3: $\mathsf{Ind}(L_1) = k^- - k^+$
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