Title

D. Zack Garza

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1 | Lecture 25: Differential Pullback Theorem

This will recover the Riemann-Hurwitz formula by taking degrees.

Lemma 1.0.1 *(?)*.
Let $K/k \subset L/\ell$ be a finite degree extension of function fields, and suppose K/k is regular and L/K is separable. Then ℓ/k and L/ℓ are separable and $L\ell$ is regular.

$$\begin{tikzcd}
 & L \arrow[dl, dash, "\text{separable}"'] \arrow[dr, dash] & \\
K \arrow[dr, dash, "\text{regular}"'] & & \ell \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

Link to diagram

Recall some facts/definitions:

- The **adele ring of** K is defined as

$$\mathcal{A}_K := \prod_{v \in \Sigma(K/k)}' K$$

  which is a *restricted direct product*, i.e. each element $\alpha \in \mathcal{A}_K$ has the property that for almost every p, the p-adic valuation of the pth coordinate $v_p(\alpha_p) \geq 0$. There is a diagonal embedding

$$\begin{aligned} K &\hookrightarrow \mathcal{A}_K \\ f &\mapsto (f, f, \cdots). \end{aligned}$$

- For any divisor $D \in \operatorname{Div} K$, define

$$\mathcal{A}_K(D) := \left\{ \alpha \in \mathcal{A}_K \;\middle|\; v_p(\alpha_p) \geq -v_p(D) \ \forall p \right\},$$

  the adelic analog of the Riemann-Roch space.

- A space of linear forms

$$\Omega(D) := \left\{ \omega : \mathcal{A}_K \to A \;\middle|\; \ker \omega \supseteq K + \mathcal{A}_K(D) \right\}$$

  where $D_1 \leq D_2 \implies \Omega_K(D_2) \leq \Omega_K(D_1)$.

- $\Omega_K := \varinjlim_{D} \Omega_K(D)$.

- For any $\omega \in \Omega_K^\bullet$, $(\omega) := \max \left\{ D \;\middle|\; \omega = 0 \text{ on } \mathcal{A}_K(D) + K \right\}$.

- $\mathcal{A}_{L/K} = \left\{ \alpha \in \mathcal{A}_L \;\middle|\; \alpha q_1 = \alpha q_2 \text{ if } Q_1, Q_2/p \right\} \leq_{\mathrm{Vect}_\ell} A_L$
- The **adelic trace map**

$$\begin{aligned} \mathrm{Tr}_{L/K} : \mathcal{A}_{L/K} &\to \mathcal{A}_K \\ \alpha &\mapsto \mathrm{Tr}_{L/K}(\alpha)/p = \mathrm{Tr}_{L/K}(\alpha_Q) \qquad \text{for any } Q/p. \end{aligned}$$

Theorem 1.0.2 ***(?).***
Let $\omega \in \Omega_K$, then there exists a unique $\omega^* \in \mathcal{A}_L$ such that

- For all $\alpha \in \mathcal{A}_{L/K}$, we have $\mathrm{Tr}_{\ell/k}\, \omega^*(\alpha) = \omega(\mathrm{Tr}_{L/K}(\alpha))$.

ω^* is formally denoted $\mathrm{Cotr}_{L/K}(\omega)$ and called the **cotrace** of ω.