Homological Algebra Problem Sets

Problem Set 3

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2021-02-24

Table of Contents

Contents

1 | Wednesday, February 17

Problem 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.
$\implies$: Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R-linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned} \tilde{f} : R &\to A \\ 1_R &\mapsto \mathbf{a}', \end{aligned}$$

and extending R-linearly yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

$\Longleftarrow$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\
 & A &
\end{tikzcd}$$

Link to Diagram

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle := jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) := r\mathbf{a}$. To summarize, we have

$$\begin{aligned}
f_a : J = jR &\to R \\
x = jr &\mapsto r\mathbf{a}.
\end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f_a"] & R \arrow[dl, dashed, "\exists \tilde{f}_a"] \\
 & A &
\end{tikzcd}$$

Link to Diagram

We can now use the fact that

$$\begin{aligned}
r\mathbf{a} &= f_a(jr) \\
&= \tilde{f}_a(\iota(jr)) \\
&= \tilde{f}_a(jr) \\
&= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\
&= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\
\implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA,
\end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' := \tilde{f}_a(1_R),$$

and letting $\mathbf{a}$ range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

Solution:
We'll use the following facts:

- $\varphi : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) \xrightarrow{\sim} \mathbb{Z}/n$, where $\varphi(g) := g(1)$.
  - That this is an isomorphism follows from
  - Surjectivity: for each $\ell \in \mathbb{Z}/n$ define a map

$$\begin{aligned}\psi_y : \mathbb{Z} &\to \mathbb{Z}/n \\ 1 &\mapsto [\ell]_n.\end{aligned}$$

  - Injectivity: if $g(1) = [0]_n$, then

$$g(x) = xg(1) = x[0]_n = [0]_n.$$

  - $\mathbb{Z}$-module morphism:

$$\varphi(gf) := \varphi(g \circ f) := (g \circ f)(1) = g(f(1)) = f(1)g(1) = \varphi(g)\varphi(f),$$

    where we've used the fact that $\mathbb{Z}/n$ is commutative.

We can start by taking a resolution of $\mathbb{Z}/p$ by projective $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \xrightarrow{m_p} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/p \to 0$$

$$:= 0 \to P_{-1} \xrightarrow{m_p} P_0 \xrightarrow{\varepsilon} \mathbb{Z}/p \to 0,$$

where $m_p(x) := px$ is multiplication by p. This is a well-defined resolution since $\mathbb{Z}$ is a free $\mathbb{Z}$-module and hence projective. We now apply the contravariant hom $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z}/q)$ to the resolution $0 \to P_{-1} \to P_0 \to 0$ yields

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/q) \xrightarrow{m_p^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/q) \xrightarrow{\varepsilon^*} 0,$$

where $m_p^*(g) := g \circ m_p$ Thus

$$\begin{aligned}\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) &:= H_0(\mathbb{Z}/p, \mathbb{Z}/q) = \ker(m_p^*) \\ \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) &:= H_1(\mathbb{Z}/p, \mathbb{Z}/q) = \mathrm{coker}(m_p^*).\end{aligned}$$

By exactness at the start, the map m_p^* is injective and thus.

Problem 1.0.3 (Weibel 2.3.2)
For $A \in \mathbf{Ab}$, define $I(A) := \bigoplus_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \to I(A)$. Show that e_A is injective.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_i F) \cong L_i(UF).$$

Problem 1.0.5 (Weibel 2.4.3)
If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$L_i F(A) \cong L_{i-1} F M \qquad i \geq 2.$$

Show that $L_{m+1} F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

a. A is a projective R-module.

b. $\mathrm{Hom}_R(\,\cdot\,, A)$ is an exact functor.

c. $\mathrm{Ext}_R^{i \neq 0}(A, B) = 0$ and for all B, i.e. A is $\mathrm{Hom}_R(\,\cdot\,, B)$-acyclic for all B.

d. $\mathrm{Ext}_R^1(A, B)$ vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on **Ab**.