Title

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Table of Contents

Contents

Table of Contents 2

1 Tuesday, January 26 3

1 | Tuesday, January 26

Problem 1.0.1 (Weibel 1.3.3)
Prove the 5-lemma. Suppose the following rows are exact:

$$\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}$$

Link to Diagram

a. Show that if f_2, f_4 are monic and f_1 is an epi, then f_3 is monic.
b. Show that if f_2, f_4 are epi and f_5 is monic, then f_3 is an epi.
c. Conclude that if f_1, f_2, f_4, f_5 are isomorphisms then f_4 is an isomorphism.

Solution (Part (a)):
Appealing to the Freyd-Mitchell Embedding Theorem, we proceed by chasing elements. For this part of the result, only the following portion of the diagram is relevant, where monics have been labeled with "$\hookrightarrow$" and the epis with "$\twoheadrightarrow$":

$$\begin{tikzcd}
A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] & A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[d, hook, "f_4"] \\
B_1 \arrow[r, "\partial_1^B"] & B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4
\end{tikzcd}$$

Link to Diagram

It suffices to show that f_3 is an injection, and since these can be thought of as R-module morphisms, it further suffices to show that $\ker f_3 = 0$, so $f_3(x) = 0 \implies x = 0$. The following outlines the steps of the diagram chase, with references to specific square and element names in a diagram that follows:

- Suppose $x \in A_3$ and $f(x) = 0 \in B_3$.
- Then under $A_3 \to B_3 \to B_4$, x maps to zero.
- Letting y_1 be the image of x under $A_3 \to A_4$, commutativity of square 1 and injectivity of f_4 forces $y_1 = 0$.
- Exactness of the top row allows pulling this back to some $y_2 \in A_2$.
- Under $A_2 \to B_2$, y_2 maps to some unique $y_3 \in B_2$, using injectivity of f_2.
- Commutativity of square 2 forces $y_3 \to 0$ under $B_2 \to B_3$.
- Exactness of the bottom row allows pulling this back to some $y_3 \in B_1$
- Surjectivity of f_1 allows pulling this back to some $y_5 \in A_1$.

- Commutativity of square 3 yields $y_5 \mapsto y_2$ under $A_1 \to A_2$ and $y_5 \mapsto x$ under $A_1 \to A_2 \to A_3$.
- But exactness in the top row forces $y_5 \mapsto 0$ under $A_1 \to A_2 \to A_3$, so $x = 0$.

$$\begin{tikzcd}
y_5 \in A_1 \arrow[r, "\partial_1^A"] \arrow[d, two heads, "f_1"] \arrow[dr, phantom, "3"] & y_2 \in A_2 \arrow[r, "\partial_2^A"] \arrow[d, hook, "f_2"] \arrow[dr, phantom, "2"] & x \in A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "1"] & y_1 \in A_4 \arrow[d, hook, "f_4"] \\
y_4 \in B_1 \arrow[r, "\partial_1^B"] & y_3 \in B_2 \arrow[r, "\partial_2^B"] & 0 \in B_3 \arrow[r, "\partial_3^B"] & 0 \in B_4
\end{tikzcd}$$

Link to Diagram

Solution (Part (b)):
Similarly, it suffices to consider the following portion of the diagram:

$$\begin{tikzcd}
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5
\end{tikzcd}$$

Link to Diagram

We'll proceed by starting with an element in B_3 and constructing an element in A_3 that maps to it. We'll complete this in two steps: first by tracing around the RHS rectangle with corners B_3, B_5, A_5, A_3 to produce an "approximation" of a preimage, and second by tracing around the LHS square to produce a "correction term". Various names and relationships between elements are summarized in a diagram following this argument.

Step 1 (the right-hand side approximation):

- Let $y \in B_3$ and y_1 be its image under $B_3 \to B_4$.
- By exactness of the bottom row, under $B_4 \to B_5$, $y_1 \mapsto 0$.
- By surjectivity of f_4, pull y_1 back to an element $y_2 \in A_4$.
- By commutativity of square 1, $y_2 \mapsto 0$ under $A_4 \to A_5 \to B_5$.
- By injectivity of f_5, the preimage of zero must be zero and thus $y_2 \mapsto 0$ under $A_4 \to A_5$.
- Using exactness of the top row, pull y_2 back to obtain some $y_3 \in A_3$

Step 2 (the left-hand correction term):

- Let z be the image of y_3 under $A_3 \to B_3$, noting that $z \neq y$ in general.
- By commutativity of square 2, $z \mapsto y_1$ under $B_3 \to B_4$
- Thus $z - y \mapsto y_1 - y_1 = 0$ under $B_3 \to B_4$, using that $d(z-y) = d(z) - d(y)$ since these are R-module morphisms.

- By exactness of the bottom row, pull $z-y$ back to some $\zeta_1 \in B_2$.
- By surjectivity of f_2, pull this back to $\zeta_2 \in A_2$. Note that by construction, $\zeta_2 \mapsto z-y$ under $A_2 \to B_2 \to B_3$.
- Let ζ_3 be the image of ζ_2 under $A_2 \to A_3$.
- By commutativity of square 3, $\zeta_4 \mapsto z-y$ under $A_3 \to B_3$.
- But then $y_3 - \zeta_3 \mapsto z-(z-y) = y$ under $A_3 \to B_3$ as desired.

$$
\begin{tikzcd}
& \zeta_3 \arrow[dddddd, dashed] & & \\
\zeta_2 \arrow[ur, dotted] \arrow[ddd, dotted, bend right] & y_3 \arrow[r, dotted] \arrow[dddd, dashed, bend left] & y_2 \arrow[r, dotted] \arrow[ddd, dotted, bend left] & 0 \arrow[ddd, dotted, bend left] \\
A_2 \arrow[r, "\partial_2^A"] \arrow[d, two heads, "f_2"] \arrow[dr, phantom, "3"] & A_3 \arrow[r, "\partial_3^A"] \arrow[d, "f_3"] \arrow[dr, phantom, "2"] & A_4 \arrow[r, "\partial_4^A"] \arrow[d, two heads, "f_4"] \arrow[dr, phantom, "1"] & A_5 \arrow[d, hook, "f_5"] \\
B_2 \arrow[r, "\partial_2^B"] & B_3 \arrow[r, "\partial_3^B"] & B_4 \arrow[r, "\partial_4^B"] & B_5 \\
\zeta_1 \arrow[ddr, dotted] & y \arrow[r, dotted] & y_1 \arrow[r, dotted] & 0 \\
& z & & \\
& z-y & &
\end{tikzcd}
$$

Link to Diagram

Problem 1.0.2 (Weibel 1.4.2)
Let C be a chain complex. Show that C is split if and only if there are R-module decompositions

$$
\begin{aligned}
C_n &\cong Z_n \oplus B'_n \\
Z_n &= B_n \oplus H'_n.
\end{aligned}
$$

Show that C is split exact if and only if $H'_n = 0$.

Problem 1.0.3 (Weibel 1.4.3)
Show that C is a split exact chain complex if and only if $\mathbb{1}_C$ is nullhomotopic.

Problem 1.0.4 (Weibel 1.4.5)
Show that chain homotopy classes of maps form a quotient category K of $\mathrm{Ch}(R\text{-mod})$ and that the functors H_n factor through the quotient functor $\mathrm{Ch}(R\text{-mod}) \to K$ using the following steps:

1. Show that chain homotopy equivalence is an equivalence relation on

$\left\{f : C \to D \;\middle|\; f \text{ is a chain map}\right\}$. Define $\mathrm{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.

2. Let $f \simeq g : C \to D$ be two chain homotopic maps. If $u : B \to C, v : D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).

3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $\mathrm{Ch}(R\text{-mod}) \to K$ is an additive functor.

4. Is K an abelian category? Explain.

Try at least two parts.

Problem 1.0.5 (Weibel 1.5.1)
Let $\mathrm{cone}(C) \coloneqq \mathrm{cone}(\mathbb{1}_C)$, so

$$\mathrm{cone}(C)_n = C_{n-1} \oplus C_n.$$

Show that $\mathrm{cone}(C)$ is split exact, with splitting map given by $(b, c) \mapsto (-c, 0)$.

Problem 1.0.6 (Weibel 1.5.2)
Let $f : C \to D \in \mathrm{Mor}(\mathrm{Ch}(\mathcal{A}))$ and show that f is nullhomotopic if and only if f lifts to a map

$$(s, f) : \mathrm{cone}(C) \to D.$$

Problem 1.0.7 (Extra)

a. Show that free implies projective.

b. Show that $\mathrm{Hom}_R(M, \cdot)$ is left-exact.

c. Show that P is projective if and only if $\mathrm{Hom}_R(P, \cdot)$ is exact.