

# Title

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Recall  $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} dx$ . Define  $\mathcal{F}_a = ??$ .

**Definition 1.1** (Decay).

$f \in \mathcal{F}_a$  iff 1.  $f$  is holomorphic in the strip  $S_a = \{z = x + iy \mid |y| < a\}$ . 2. There exists an  $A > 0$  such that  $|f(x + iy)| \frac{A}{1 + x^2}$ .

Examples:

- $e^{-z^2} \in \mathcal{F}_a$  for all  $a$
- $\frac{1}{c^2 + z^2} \in \mathcal{F}_a$  for all  $a > c$
- $\frac{1}{\cosh(\pi z)} \in \mathcal{F}_a$  for  $a < \frac{1}{2}$ .

**Lemma 1.1.**

If  $f \in \mathcal{F}_a$ , then  $f^{(n)}(z) \in \mathcal{F}_b$  for all  $b < a$ .

**Theorem 1.2.**

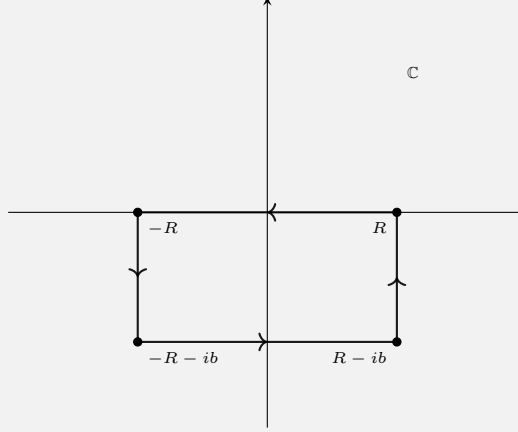
If  $f \in \mathcal{F}_a$ , then  $|\hat{f}(\xi)| \leq B e^{-2\pi b |\xi|}$  for some constants  $b, B$ .

*Proof.*

If  $\xi = 0$ ,

$$\begin{aligned}
 |\widehat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x) e^{2\pi i x \cdot \xi} \right| \\
 &\leq \int_{\mathbb{R}} |f(x)| \, dx \\
 &\leq A \int_{\mathbb{R}} \frac{1}{1+x^2} \, dx \\
 &= A\pi.
 \end{aligned}$$

For  $\xi > 0$ , integrate over the box  $[-R, R] \times i[-b, 0]$ :



Define  $g(z) = f(z)e^{-2\pi i z \cdot \xi}$ . The integral over the rectangle is zero, since  $g$  is holomorphic, so we can equate

$$\int_R^{R-ib} f(z) e^{-2\pi i z \cdot \xi} \, dz = \int_0^b f(R - it) e^{-2\pi i (R-it) \cdot \xi} (-i) \, dt$$

We can use the estimate in  $\mathcal{F}_a$  to obtain

$$\begin{aligned}
 \int_0^b \dots &\leq \int_0^b \frac{A}{1+R^2} e^{-2\pi s \xi} \, ds \\
 &\leq O(R^{-2}).
 \end{aligned}$$

Then

$$\begin{aligned}
 \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} \, d\xi &= \int_{-\infty-ib}^{\infty-ib} \dots \, dz \\
 &= \int_{\mathbb{R}} f(x - ib) e^{2\pi i (x-ib) \cdot \xi} \, dx \\
 &\leq \int_{\mathbb{R}} \frac{A}{1+x^2} e^{-2\pi b \xi} \, dx \\
 &= A\pi e^{-2\pi b \xi},
 \end{aligned}$$

so we can take  $B = A\pi$ .

For  $\xi > 0$ , the same argument works with the rectangle above the axis.

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**Theorem 1.3.**

If  $f \in \mathcal{F}_a$ , then  $f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$ .

*Proof.*

Letting  $L_1 = \{x - ib\}$  and  $L_2 = \{x + ib\}$

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$$\begin{aligned}
I &= \int_0^\infty \widehat{f} \cdots + \int_{-\infty}^0 \widehat{f} \cdots \\
&= \int_0^\infty e^{2\pi i x \cdot \xi} \left( \int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi + \int_{-\infty}^0 e^{2\pi i x \cdot \xi} \left( \int_{L_1} f(z) + e^{-2\pi i z \cdot \xi} dz \right) d\xi \\
&= \int_{L_1} \int_0^\infty e^{2\pi i x \xi - 2\pi i (s-ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i x \cdot \xi - 2\pi i (s+ib)\xi} d\xi ds \\
&\quad \text{by absolute convergence, where } z = s - ib \\
&= \int_{L_1} f(z) \int_0^\infty e^{2\pi i (x-s+ib)\xi} d\xi ds + \int_{L_2} f(z) \int_{-\infty}^0 e^{2\pi i (x-s+ib)\xi} d\xi ds \\
&= \int_{L_1} f(z) \frac{1}{2\pi i (x - i + ib)} ds + \int_{L_2} f(z) \frac{1}{2\pi i (x - s - ib)} \\
&= \frac{1}{2\pi i} \int \frac{f(z)}{z - x} dz \\
&= f(x).
\end{aligned}$$

Noting that

$$\int_0^\infty e^{as} ds = \frac{1}{a} \quad \text{for } \Re(a) > 0.$$

Note the similar trick: for  $\xi < 0$ , move up, and  $\xi > 0$  move down to form a rectangle. Use the fact that integration along the vertical edges is zero.