

Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$
$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

Outline

- Took $R^\pm : I \longrightarrow \text{Sp}(2n; \mathbb{R})$: symplectic paths associated to S^\pm
- These paths defined $\mu(x), \mu(y)$
- Section 8.7:

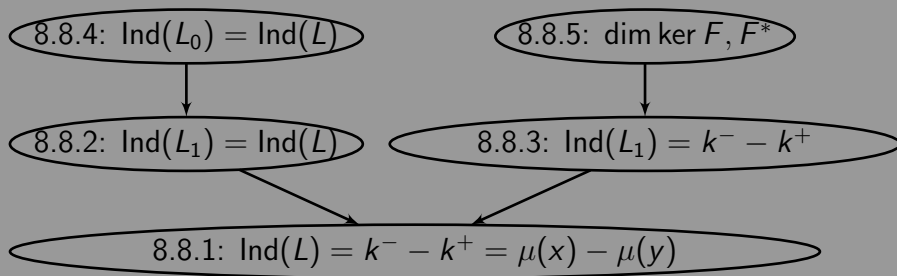
$$R^\pm \in \mathcal{S} := \left\{ R(t) \mid R(0) = \text{id}, \det(R(1) - \text{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\text{Ind}(L) \stackrel{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
 - So we know $\text{Ind}(L) = \text{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
 - Computing $\text{Ind}(L_1)$ by computing kernels.



8.8.5: $\dim \ker F, F^*$

Recall

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^* : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^q(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since $\text{coker } L_1 \cong \ker L_1^*$, it suffices to compute $\ker L_1^*$.
- We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \text{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the $n = 1$ case.

Setup

L_1 used a path of diagonal matrices constant near ∞ :

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) := \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$



Statement of Later Lemma (8.8.5)

Let $p > 2$ and define

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Note: F is L_1 for $n = 1$:

$$L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Statement of Lemma

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$

$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

Suppose $a_i^\pm \notin 2\pi\mathbb{Z}$.

- ① Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \text{Ker } F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \subset \mathbb{R} \right\}$$

$$\dim \text{Ker } F^* = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.$$

- ② Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\dim \text{Ker } F = \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$

$$\dim \text{Ker } F^* = \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Statement of Lemma

In words:

- 1 If $S(s)$ is a scalar matrix, set $a^\pm = a_1^\pm = a_2^\pm$ to the limiting scalars and count the integer multiples of 2π between a^- and a^+ .
- 2 Otherwise, if S is uniformly bounded by 1, count the number of entries the flip from positive to negative as s goes from $-\infty \rightarrow \infty$.

$$S_K := \begin{bmatrix} a_1^- & 0 \\ 0 & a_2^- \end{bmatrix} \quad S(s) = \begin{bmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{bmatrix} \quad S_K^* = \begin{bmatrix} a_1^+ & 0 \\ 0 & a_2^+ \end{bmatrix}$$

The diagram shows a horizontal axis for s ranging from $-\infty$ to ∞ . A bracket above the axis indicates a region where $S(s)$ is used. Two vertical red lines are marked on the axis at $-S_0$ and S_0 .

Proof of Assertion 1

- ① Suppose $a_1(s) = a_2(s)$ and set $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\begin{aligned}\dim \ker F &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \subset \mathbb{R} \right\} \\ \dim \ker F^* &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^+, a^-) \subset \mathbb{R} \right\}.\end{aligned}$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) := a_1(s) = a_2(s)$.
- Start with equation on \mathbb{R}^2 ,

$$\mathbf{Y}(s, t) = [Y_1(s, t), Y_2(s, t)].$$

- Replace with equation on \mathbb{C} :

$$\mathbf{Y}(s, t) = Y_1(s, t) + iY_2(s, t).$$

Assertion 1, Step 1: Reduce to CR

- Expand definition of the PDE

$$F(\mathbf{Y}) = 0 \rightsquigarrow \bar{\partial}\mathbf{Y} + S\mathbf{Y} = 0$$

$$\frac{\partial}{\partial s}\mathbf{Y} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\mathbf{Y} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix} \mathbf{Y} = 0.$$

- Change of variables: want to reduce to $\bar{\partial}\tilde{Y} = 0$
- Choose $B \in \text{GL}(1, \mathbb{C})$ such that $\bar{\partial}B + SB = 0$
- Set $Y = B\tilde{Y}$, which (?) reduces the previous equation to

$$\bar{\partial}\tilde{Y} = 0.$$

Assertion 1, Step 1: Reduce to CR

Can choose (and then solve)

$$B = \begin{bmatrix} b(s) & 0 \\ 0 & b(s) \end{bmatrix} \quad \text{where} \quad \frac{\partial b}{\partial s} = -a(s)b(s)$$

$$\implies b(s) = \exp \left(\int_0^s -a(\sigma) d\sigma \right) := \exp(-A(s)).$$

Remarks:

- For some constants C_i , we have

$$A(s) = \begin{cases} C_1 + a^- s, & s \leq -\sigma_0 \\ C_2 + a^+ s, & s \geq \sigma_0 \end{cases}.$$

- The new \tilde{Y} satisfies CR, is continuous and L^1_{loc} , so elliptic regularity $\implies C^\infty$.
- The real/imaginary parts of \tilde{Y} are C^∞ and harmonic.

Assertion 1, Step 2: Solve CR

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$
- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of \tilde{Y} in z .
- Deduce that the solutions of the system are given by

$$\tilde{Y}(u) = \sum_{\ell \in \mathbb{Z}} c_\ell u^\ell \implies \tilde{Y}(s + it) = \sum_{\ell \in \mathbb{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

where $\{c_\ell\}_{\ell \in \mathbb{Z}} \subset \mathbb{C}$ converges for all s, t .

Assertion 1, Step 2: Solve CR

Use $e^{s+it} = e^s(\cos(t) + i \sin(t))$ to write in real coordinates:

$$\tilde{Y}(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s \ell} \begin{bmatrix} \cos(2\pi \ell t) & -\sin(2\pi \ell t) \\ \sin(2\pi \ell t) & \cos(2\pi \ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

Use

$$Y = B\tilde{Y} = \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \tilde{Y}$$

to write

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi s \ell} \begin{bmatrix} e^{-A(s)} & 0 \\ 0 & e^{-A(s)} \end{bmatrix} \begin{bmatrix} \cos(2\pi \ell t) & -\sin(2\pi \ell t) \\ \sin(2\pi \ell t) & \cos(2\pi \ell t) \end{bmatrix} \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}.$$

For $s \leq s_0$ this yields for some constants K, K' :

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{2\pi \ell - a^-} \begin{bmatrix} e^K (\alpha_\ell \cos(2\pi \ell t) - \beta_\ell \sin(2\pi \ell t)) \\ e^{K'} (\alpha_\ell \sin(2\pi \ell t) + \beta_\ell \cos(2\pi \ell t)) \end{bmatrix}.$$

Condition on L^p Solutions

For $s \leq s_0$ we had

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^K (\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'} (\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

and similarly for $s \geq s_0$, for some constants C, C' we have:

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^+)s} \begin{bmatrix} e^C (\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'} (\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

Then

$$Y \in L^p \iff \text{exponential terms} \xrightarrow{\ell \rightarrow \infty} 0.$$

Condition on L^p Solutions: Small Tails

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^-)s} \begin{bmatrix} e^K(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{K'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}$$

- $\ell \neq 0$: Need $\alpha_\ell = \beta_\ell = 0$ **or** $2\pi\ell > a^-$
- $\ell = 0$: Need both
 - $\alpha_0 = 0$ or $a^- < 0$ and
 - $\beta_0 = 0$ or $a^- < 0$.

$$Y(s, t) = \sum_{\ell \in \mathbb{Z}} e^{(2\pi\ell - a^+)s} \begin{bmatrix} e^C(\alpha_\ell \cos(2\pi\ell t) - \beta_\ell \sin(2\pi\ell t)) \\ e^{C'}(\alpha_\ell \sin(2\pi\ell t) + \beta_\ell \cos(2\pi\ell t)) \end{bmatrix}.$$

- $\ell \neq 0$: Need $\alpha_\ell = \beta_\ell = 0$ **or** $2\pi\ell < a^+$
- $\ell = 0$: Need both
 - $\alpha_0 = 0$ or $a^+ > 0$ and
 - $\beta_0 = 0$ or $a^+ > 0$.

Counting Solutions

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } 2\pi\ell \in (a^-, a^+) & \ell \neq 0 \\ (\alpha_0 = 0 \text{ or } 0 \in (a^-, a^+)) \text{ and } (\beta_0 = 0 \text{ or } 0 \in (a^-, a^+)) & \ell = 0 \end{cases}.$$

- Finitely many such ℓ that satisfy these conditions
- Sufficient conditions for $Y(s, t) \in W^{1,p}$.

Compute dimension of space of solutions:

$$\begin{aligned} \dim \text{Ker } F &= 2 \cdot \# \left\{ \ell \in \mathbb{Z}^* \mid 2\pi\ell \in (a^-, a^+) \right\} + 2 \cdot \mathbb{1} [0 \in (a^-, a^+)] \\ &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\pi\ell \in (a^-, a^+) \right\}. \end{aligned}$$

Note: not sure what \mathbb{Z}^ is: most likely $\mathbb{Z} \setminus \{0\}$.*

Counting Solutions

Use this to deduce $\dim \ker F^*$:

- $Y \in \ker F^* \iff Z(s, t) := Y(-s, t)$ is in the kernel of the operator

$$\begin{aligned} \tilde{F} : W^{1,q}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Z &\mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y. \end{aligned}$$

- Obtain $\ker F^* \cong \ker \tilde{F}$.
- Formula for $\dim \ker \tilde{F}$ almost identical to previous formula, just swapping a^- and a^+ .

Assertion 2

Assertion 2: Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\begin{aligned} \dim \ker F &= \# \left\{ i \in \{1, 2\} \mid a_i^- < 0 < a_i^+ \right\} \\ \dim \ker F^* &= \# \left\{ i \in \{1, 2\} \mid a_i^+ < 0 < a_i^- \right\}. \end{aligned}$$

We use the following:

- Lemma 8.8.7:

$$\sup_{s \in \mathbb{R}} \|S(s)\| < 1 \implies \text{the elements in } \ker F, \ker F^* \text{ are independent of } t.$$

- Proof: in subsection 10.4.a.

Proof of Assertion 2

$$F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

- Given as a fact:

$$\mathbf{Y} \in \ker F \implies \frac{\partial}{\partial s} \mathbf{Y} = \mathbf{a}(s) \mathbf{Y} \quad := \begin{bmatrix} -a_1(s) & 0 \\ 0 & -a_2(s) \end{bmatrix} \mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c}_0 \exp(-\mathbf{A}(s)) \quad \text{where} \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma) d\sigma.$$

Proof of Assertion 2

- Explicitly in components:

$$\begin{cases} \frac{\partial Y_1}{\partial s} = -a_1(s)Y_1 \\ \frac{\partial Y_2}{\partial s} = -a_2(s)Y_2 \end{cases} \implies Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma) d\sigma.$$

- As before, for some constants $C_{j,i}$,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

- Thus

$$Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+),$$

establishing

$$\dim \ker F = \# \left\{ i \in \{1, 2\} \mid 0 \in (a_i^-, a_i^+) \right\}.$$

$$8.8.3: \text{Ind}(L_1) = k^- - k^+$$

Statement and Outline

Statement: let $k^\pm := \text{Ind}(R^\pm)$; then $\text{Ind}(L_1) = k^- - k^+$.

- Consider four cases, depending on parity of $k^\pm - n$
- Show all 4 lead to $\text{Ind}(L_1) = k^- - k^+$

- 1 $k^- \equiv k^+ \equiv n \pmod{2}$.
- 2 $k^- \equiv n, k^+ \equiv n - 1 \pmod{2}$
- 3 $k^- \equiv n - 1, k^+ \equiv n \pmod{2}$.
- 4 $k^- \equiv k^+ \equiv n - 1 \pmod{2}$

k^-	k^+	n
✓	✓	✓
✓		✓
	✓	✓
✓	✓	

Case 1: $k^+ \equiv k^- \equiv n \pmod{2}$

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^+)\pi & \\ & & & & & & (n-1-k^+)\pi \end{bmatrix}.$$

Case 1: $k^- \equiv k^+ \equiv n \pmod 2$

- Take $a_1(s) = a_2(s)$ so $a_1^\pm = a^\pm$
- Apply the proved lemma to obtain

$$\begin{aligned} \dim \ker L_1 &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\ell \in (n-1-k^-, n-1-k^+) \right\} \\ &= \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$\implies \text{Ind}(L_1) = k^- - k^+.$$

Case 2: $k^+ \not\equiv k^- \equiv n \pmod{2}$

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\varepsilon\pi & & & \\ & & & & -\varepsilon\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & \varepsilon & & & \\ & & & & -\varepsilon & & \\ & & & & & (n-2-k^+)\pi & \\ & & & & & & (n-2-k^+)\pi \end{bmatrix}.$$

Case 2: $k^+ \not\equiv k^- \equiv n \pmod{2}$

- Take $a_1(s) = a_2(s)$ everywhere except the $n - 1$ st block, where we can assume $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$.
- Assertion 2 applies and we get

$$\begin{aligned} \dim \ker L_1 &= 2 \cdot \# \left\{ \ell \in \mathbb{Z} \mid 2\ell \in (n - 1 - k^-, n - 2 - k^+) \right\} + 1 \\ &= \begin{cases} (k^- - k^+ - 1) + 1 & k^- > k^+ \\ 1 & \text{otherwise} \end{cases} \implies \text{Ind}(L_1) \end{aligned}$$