Algebraic Topology 2: Smooth Manifolds

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1 Lecture 1

The key point of this class will be a discussion of *smooth structures*. As you may recall, a sensational result of Milnor's exhibited exotic spheres with smooth structures – i.e., a differentiable manifold M which is homeomorphic but *not* diffeomorphic to a sphere.

Summary of this result: Look at bundles $S^3 \to X \to S^4$, then one can construct some $X \cong S^7 \in \mathbf{Top}$ but $X \ncong S^7 \in \mathbf{Diff}^\infty$. There are in fact 7 distinct choices for X.

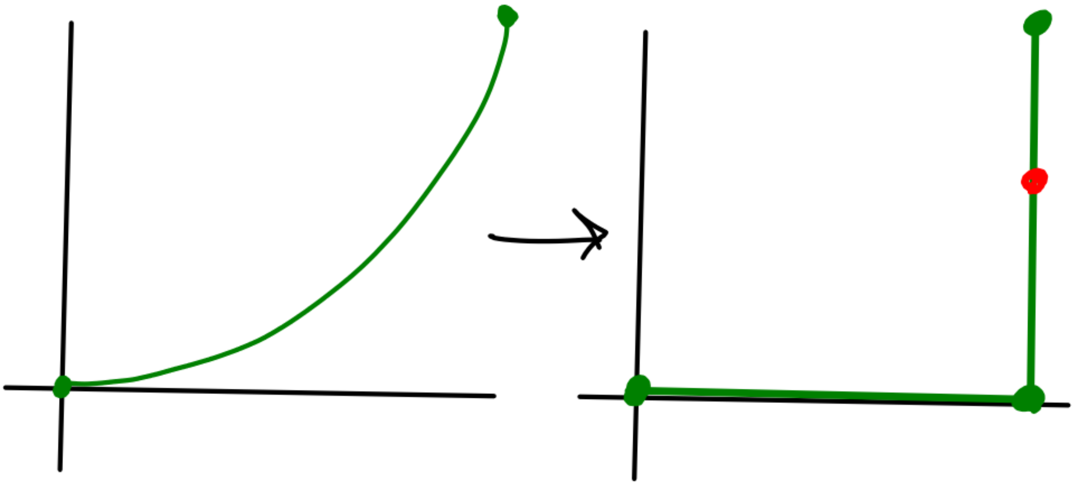
> It is not known if there are exotic smooth structures on S^4. The Smooth Poincare' conjecture is that these do not exist; this is believed to be false.

The other key point of this course is to show that $X \in \mathbf{Diff}^\infty \implies X \hookrightarrow \mathbb{R}^n$ for some n, and is in fact a topological subspace.

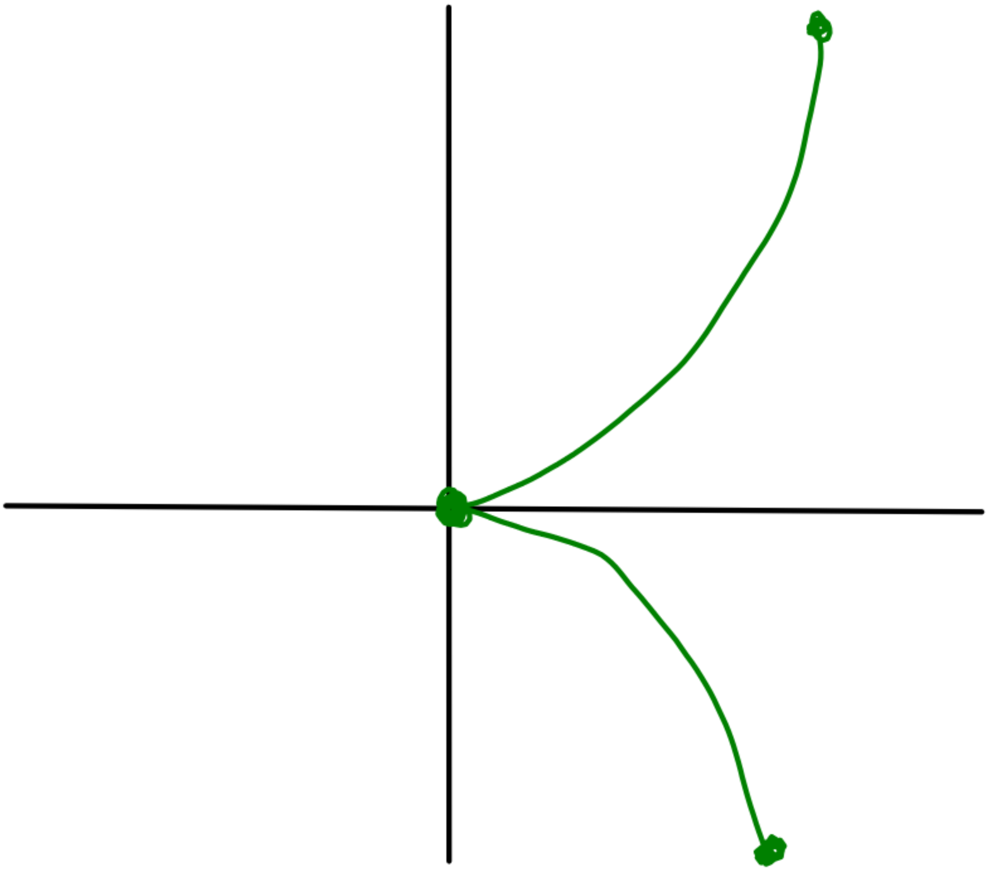
A short list of words/topics we hope to describe:

- Differentiable manifolds
- Local charts
- Submanifolds
- Projective spaces
- Lie groups
- Tangent spaces
- Vector fields
- Cotangent spaces
- Differentials of smooth maps G
- Differential forms
- de Rham's theorem

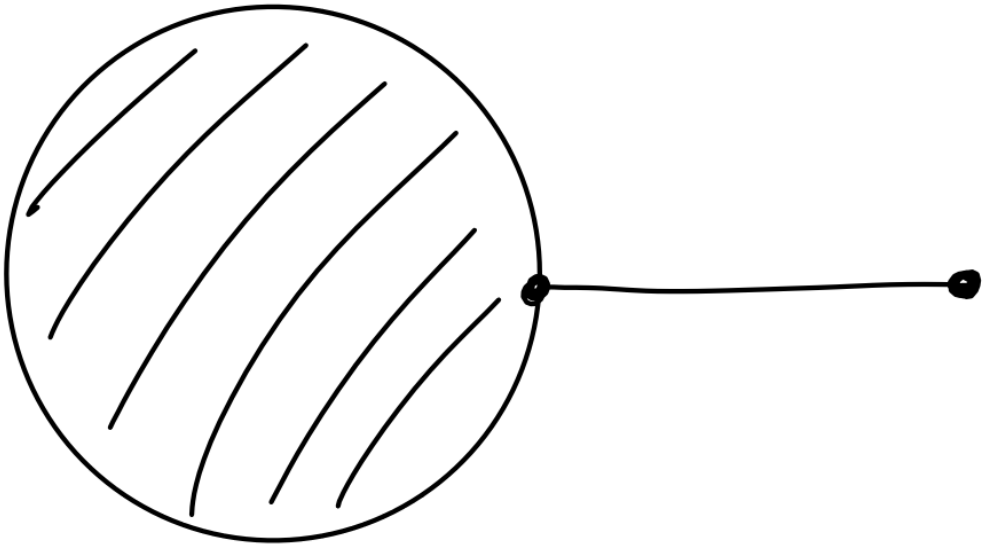
We'd like a notion of "convergence" for, say, curves in $\mathbb{R}^2$. Consider the following examples.



Note the problematic point on the bottom right, as well as the fact that neither of the usual notions of pointwise or uniform convergence will yield a point on the LHS that converges to the red point on the RHS.



Note the problematic point at the origin.



Note the problematic point in the middle, for which all neighborhoods of it are not homeomorphic to either a 2-dimensional nor a 1-dimensional space.

Definition 1. A topological space M is said to be a **topological manifolds** when

- M is Hausdorff, so $p \neq q \in M \implies \exists N(p), N(q)$ such that $N(p) \bigcap N(q) = \emptyset$.
- $x \in M \implies$ there exists some $U_x \subseteq M$ and a $\varphi : U_x \to \mathbb{R}^n$ for some n which is a homeomorphism.
- M is 2nd countable

> There are somewhat technical conditions – most of the theory goes through without M being Hausdorff or 2nd countable, but these are needed to construction *partitions of unity* later.

> Also note that these conditions exclude spaces such as the copy of $D_2 \vee I$ from above.

The intuition here is that we'd like spaces that "locally look like $\mathbb{R}^n$", and we introduce the additional structure of smoothness in the following way:

Definition 2. A family of coordinate systems $\{U_\alpha, \varphi_\alpha)\}$ is a **smooth atlas** on M exactly when the change-of-coordinate maps $f_{\alpha,\beta}$ are C^∞.

Exercise 1. Show that S^n is a smooth manifolds for every n.

Supposing that $f : M^n \to M^n$ is a map, then locally there is a map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_n$. Moreover, we can write

$$f(x_1, x_2, \cdots x_n) = [f_1(x_1, x_2, \cdots x_n), f_2(x_1, x_2, \cdots x_n), \cdots f_n(x_1, x_2, \cdots x_n)]$$

Proposition 1. If M and N are smooth manifolds, then the product $M \times N$ is also a smooth manifolds.

Being Hausdorff and 2nd countable can be checked on the basis elements, and it is indeed true that $\mathcal{B}_1 \times \mathcal{B}_2$ furnishes a basis that satisfies these conditions.

Example 1. The n-fold copy of 1-dimensional sphere is given by $(S^1)^n = \prod_n S^1 :$
$\mathbb{TT}^n$.