Math 200A Homework Question Compendium

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Contents

1 One

1. Given: $\forall x \in G, x^2 = e$

   Show: $G \in \mathbf{Ab}$

2. Given: $|G| < \infty, |G| = 0 \mod 2$

   Show: $\exists g \in G \ni o(g) = 2$

3. Given: $G \in \mathbf{Ab}$

   Show: $T(G) \leq G$ (where $T(G) = \{g \in G : |g| < \infty\}$

4. Show: Every finite group is finitely generated.

   - Show: $\mathbb{Z}$ is finitely generated
   - Show: $H \leq (\mathbb{Q}, +) \implies H$ is cyclic
   - Show: $\mathbb{Q}$ is not finitely generated

5. Show: $\mathbb{Q}/\mathbb{Z}$ has, for each coset, exactly one representative in $[0, 1) \cap \mathbb{Q}$

   - Show: Every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

- Show: There are elements in $\mathbb{Q}/\mathbb{Z}$ of arbitrarily large order.
- Show: $\mathbb{Q}/\mathbb{Z} = T(\mathbb{R}/\mathbb{Z})$
- Show: $\mathbb{Q}/\mathbb{Z} \cong \mathbb{C}^x$

6. Given: $G/Z(G)$ is cyclic

   Show: G is abelian

7. Given: $H \trianglelefteq G, K \trianglelefteq G, H \cap K = e$

   Show: $\forall h \in H, \forall k \in K, hk = kh$

8. Given: $|G| < \infty, \quad H \leq G, \quad N \trianglelefteq G, (|H|, [G : N]) = 1$

   Show: $H \leq N$

9. Given: $|G| < \infty, N \trianglelefteq G, (|N|, [G : N]) = 1$

   Show: N is the unique subgroup of order $|N|$

2 Two

1. Given: For every triplet in G, two elements commute

   Show: G is abelian

2. Given: $H_1, H_2, H_3 \leq G, G = H_1 \cup H_2$

   Show: $G = H_1 \vee G = H_2$

3. Given: $G = H_1 \cup H_2 \cup H_3, G$ finite

   Show: $G = H_i \vee \forall i, [G : H_i] = 2$

4. Show: TFAE; $\operatorname{clos}(H)$ satisfies:

   - It is the smallest normal subgroup of G containing H.
   - It is the subgroup generated by all conjugates of H.
   - $\operatorname{clos}(H) = \bigcap_{H \leq N \trianglelefteq G} N$
   - If $\phi : G \to X$ and $\phi(H) = e$, then ϕ factors through $G/\operatorname{clos}(H)$

5. Given: $H, K \trianglelefteq HK \leq G$

   Show:

$$\frac{HK}{H \cap K} \cong \frac{HK}{H} \times \frac{HK}{K}$$

6. Given: $H \leq G, N \trianglelefteq G, H \in \operatorname{Hall}(G)$

   Show:

$$H \cap N \in \operatorname{Hall}(N) \text{ and } \frac{HN}{N} \in \operatorname{Hall}\left(\frac{G}{N}\right)$$

7. Given: $|G| = n, G$ cyclic, $\sigma_i : G \to G \ni x \mapsto x^i$

- Show $\sigma_i \in End(G)$
  - Show $\sigma_i \in Aut(G)$ iff $(i, n) = 1$
  - $\sigma_i = \sigma_j$ iff $i = j \mod n$
  - $\tau \in Aut(G) \implies \exists i \ni \tau = \sigma_i$
  - $\sigma_i \circ \sigma_j = \sigma_{ij}$
- The map

$$\begin{aligned} \psi : Z_n^\times &\to Aut(G) \\ i &\mapsto \sigma_i \end{aligned}$$

is an isomorphism.

8. Given: G is cyclic

   Show: $Aut(G)$ is abelian of order $\phi(n)$

9. Show: $D_\infty \cong \langle a, b \mid b^2 = e, ba = a^{-1}b \rangle$

10. Show: $Q_8 \cong \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1} \rangle$

11. Show: $\langle x, y \mid xy^2 = y^3, yx^2 = x^3y \rangle = \langle e \rangle$

3 Three

1. Given: $G \sim X$ transitively, $H \trianglelefteq G$
   - Show: $H \sim X$, but possibly not transitively
   - Show: G acts transitively on $\left\{ \mathcal{O}_\langle : h \ \in H \right\}$
   - Show: $\forall i, j, |\mathcal{O}_{h_i}| = |\mathcal{O}_{h_j}|$
   - Given: $x \in \mathcal{O}_h$ Show: $|\mathcal{O}_h| = |H : H \cap G_x|$
   - Show: $|\{\mathcal{O}_h\}_{h \in H}| = [G : HG_x]$

2. Given: $\mathcal{K}$ a conjugacy class in S_n, $\{\mathcal{O}_s : s \in S_n\}$ orbits of an A_n-action on S_n

   Show: $\mathcal{K} = \mathcal{O}_s \vee \mathcal{K} = \mathcal{O}_{s_i} \cup \mathcal{O}_{s_j}$

   Show: Case 2 occurs iff $\{k_i\}$, the cycle lengths in disjoint cycle form, are odd and distinct

3. i: $|G| < \infty, H < G$
   - Show: $\{gHg^{-1} : g \in G\} = [G : N_G(H)]$
   - Show:

$$G \neq \bigcup_{g \in G} gHg^{-1}$$

4. Prove Cauchy's Theorem. Given $p \mid o(G) < \infty$

$$X = \left\{ (a_i)_{i=1}^p \in G^p \ni \prod_{i=1}^p a_i = e \right\}$$

   - Show: $(a_1 a_2 \cdots a_p) = e \implies (a_2 a_3 \cdots a_p a_1) = e$

- Show: $(Z_p, +) \sim X$ and $\bar{1} \sim (a_1 a_2 \cdots a_p) = (a_2 a_3 \cdots a_p a_1)$
- Show: $|X| = |G|^{p-1}$
- Show: $\{\mathcal{O}_x : |\mathcal{O}_x| = 1\} > 1$ and $\exists a \in G \ni a^p = e$

5. Given: $G \sim X, \quad |G| < \infty, \quad 1 < |X| < \infty$

- Show: $\exists g \; inG \ni \forall x \in X, g \sim x \neq x$
- Show: This holds if $|G| = \infty$, but not if $|X| = \infty$ as well.

6. Given: $H \leq G$. Show: $\text{core}(H)$ is

   - The largest $N \trianglelefteq G, N \subseteq H$
   - Generated by all normal subgroups contained in H
   - Given by $\bigcap_{g \in G} gHg^{-1}$
   - The kernel of $G \sim \frac{G}{H} \ni x \sim gH = (xg)H$

7. Given: $[H : G] = n < \infty$

   - Show: $[\text{core}(H) : G]$ divides $n!$
   - Show: G simple $\implies o(G) \mid n! \wedge |G| < \infty$

8. Given: A_n is simple for $n \geq 5$

   Show: $\nexists H \in A_n \ni [H : A_n] < n$

   Show: $\exists H [H : A_n] = n$

9. Given: r beads of n colors

   Show: How many distinct circular bracelets can be made.

4 Four

1. Given: H char G

   Show: $H \trianglelefteq G$

2. Given: H char $K \trianglelefteq G$

   Show: $H \trianglelefteq G$

3. Given: $K = \langle k \rangle \trianglelefteq G$

   Show: $H \leq K \implies H \trianglelefteq G$

4. Show $H \trianglelefteq K \trianglelefteq G \not\Longrightarrow H \trianglelefteq G$

5. Given: $P \leq H \leq K \leq G < \infty, P \in \text{Syl}_p(G)$

   Show: $P, H \trianglelefteq K \implies P \trianglelefteq K$

6. Show: $N_G(N_G(P)) = N_G(P)$

7. Given: $\sigma \in Aut(G)$

   Show: $\sigma Inn(G) \sigma^{-1} = Inn(G)$ iff $\forall g \in G, g^{-1}\sigma(g) \in Z(G)$

8. Show: $Inn(G)$ char $Aut(G)$

9. Given: $H \subseteq G, P \in \mathrm{Syl}_p(G)$

   Show: $\exists g \in G \ni gPg^{-1} \in Syl_p(H)$

   Given: $H \trianglelefteq G$

   Show: $P \cap H \in \mathrm{Syl}_p(H)$

   Given: $P \trianglelefteq G$

   Show: $P \cap H \in \mathrm{Syl}_p(H)$ and $|\mathrm{Syl}_p(H)| = 1$

10. Given: $|G| = pqr, p < q < r$

    Show: $\exists P_i \in \mathrm{Syl}_i(G) \trianglelefteq G$

11. Given: $|G| = 595$

    Show: All sylow subgroups are normal

12. Given: $|G| = p(p+1)$

    Show: $\exists N \trianglelefteq G$ where $|N| = p$ or $p+1$

5 Five

1. Given: $G = H \rtimes_\psi K$

$$\begin{aligned} \psi : K &\to Aut(H) \\ k &\mapsto \psi(k) \end{aligned}$$

   $\theta \in Aut(H)$ $\rho : K \to K$

$$\begin{aligned} \phi_\theta : Aut(H) &\to Aut(H) \\ \rho &\mapsto \theta \circ \rho \circ \theta^{-1} \end{aligned}$$

$$\begin{aligned} \psi_2 : K &\to Aut(H) \\ k &\mapsto (\phi_\theta \circ \psi)(k) \end{aligned}$$

$$\begin{aligned} \psi_3 : K &\to Aut(H) \\ k &\mapsto (\psi \circ \rho)(k) \end{aligned}$$

   Show: $H \rtimes_\psi K \cong H \rtimes_{\psi_2} K \cong H \rtimes_{\psi_3} K$

2. Classify groups of order $pq, p < q, p \mid q-1$

3. Classify groups of order 20.

4. Classify groups of order 75.

5. Show: $|G| < 60 \implies G$ is not simple.

6. Show: $|G| < 60 \implies G$ is solvable

7. Given: $|G| < \infty$, $H \leq G$ maximal $\implies [G : H] = p$, a prime.

- Show: $|G|$ is solvable
  - Given: $P \in Syl_p(G) \wedge \exists H \ni N_G(P) \leq H \leq G$ Show: $[G : H] = 1 \mod p$
  - Given: $p \mid o(G)$, the largest such prime Show: $\exists P \trianglelefteq G \in Syl_p(G)$,

8. $|G| < \infty$
   - Given: G is characteristically simple Show: $\exists H$ (simple) $\ni G \cong H^n$. Show: Whether or not the converse holds
   - Given: $N \trianglelefteq G$ minimal Show: N is characteristically simple, $N \cong H^n$

6 Six

1. Given: G is nilpotent

   Show: $H \leq G \implies H, G/H$ are nilpotent

2. Show: $G/Z(G)$ is nilpotent $\implies G$ is nilpotent

3. Given: $|G| < \infty$

   Show: $|G|$ is nilpotent iff $a, b \in G, (a, b) = 1 \implies ab = ba$

4. Show: D_{2n} is nilpotent iff $n = 2^i$

5. Given: $|G| < \infty$

   - Show $\Phi(G)$ char G
   - Show $\Phi(G)$ is nilpotent
   - Given: $|P| = p^e$
     - Show: $P/\Phi(P)$ is an elementary abelian p-group
     - Show:$N \trianglelefteq P, P/N$ is elementary abelian $\implies \Phi(P) \subseteq N$

6. Given: R a commutative ring, $x, y \in R$ nilpotent

   - Show: $x + y$ is nilpotent

- Show: $\{x \in R : x \text{ is nilpotent}\} \trianglelefteq R$
  - Given: $u \in R^\times, x \in R$ nilpotent
  - Show: $u + x \in R^\times$
  - Show: An counterexample to 1 when R is noncommutative.

7. Given: R a commutative ring, $R[[x]]$ its formal power series
   - Show: $\sum_{i=0}^{\infty} a_i x^i \in R[[x]]^\times \iff a_0 \in R^\times$
   - Show: R a domain $\implies R[[x]]$ a domain
   - Given: R a field
     - Show: $I = \{r \in R[[x]] : r_0 = 0\}$ is a maximal ideal of $R[[x]]$
     - Show: I is the unique maximal ideal
8. Given: R a commutative ring, G a finite group, RG a group ring.
   - Given: $\mathcal{K} = \{k_1, k_2, \cdots k_m\}$ a conjugacy class in G
     - Show:

$$K = \sum_{i=1}^{m} k_i \in RG \implies K \in Z(RG)$$

  - Given: $\mathcal{K}_1 \cdots \mathcal{K}_r$ distinct conjugacy classes in G, $K_i = \sum_j k_j \ni k_j \in \mathcal{K}_i$
    - Show: $Z(RG) = \{\sum a_l K_l : \forall 1 \leq l \leq r, a_l \in R\}$ (All R-linear combinations of the $\mathcal{K}_i$)
9. Given: R a ring, $M_n(R)$ its matrix ring
  - Given: $I \trianglelefteq R$ (two-sided)
    - Show: $M_n(I) \trianglelefteq M_n(R)$
    - Show:
$$\frac{M_n(R)}{M_n(I)} \cong M_n(\frac{R}{I})$$
  - Show: $\forall I_M \trianglelefteq M_n(R), I$ is of the form $M_n(I)$ for some $I \trianglelefteq R$
    - Show: R a division ring $\implies$ $M_n(R)$ is a simple ring.

7 Seven

8 Eight