Title

D. Zack Garza

Contents

1 | Lecture 15: The L-Polynomial

Recall that we had $Z(t) + F(t) + G(t)$:

$$
\begin{aligned}
(q-1)F(t) &= \sum_{0 \le \deg C \le 2g-2} q^{\ell(C)} t^{\deg(C)} \\
(q-1)G(t) &= h\left(\frac{q^g t^{2g-1}}{1-qt} - \frac{1}{1-t}\right).
\end{aligned}
$$

Note that $F(t)$ is a polynomial of degree at most $2g-2$, and clearing denominators in $G(t)$ yields a polynomial of degree at most $2g$

Definition 1.0.1 (The L-polynomial)
The L-polynomial is defined as

$$L(t) := (1-t)(1-qt)Z(t) = (1-t)(1-qt)\sum_{n=0}^{\infty} A_n t^n \in \mathbb{Z}[t].$$

It turns out that the degree bound of $2g$ is sharp, and the coefficients closer to the middle are most interesting:

Theorem 1.0.2 *(?).*
Let $K/\mathbb{F}_q$ be a function field of genus $g \ge 1$, then

- $\deg L = 2g$.
- $L(1) = h$
- $L(t) = q^g t^{2g} L\left(\frac{1}{qt}\right)$.
- Writing $L(t) = \sum_{j=1}^{2g} a_j t^j$,
  - $a_0 = 1$ and $a_{2g} = q^g$.
  - For all $0 \le j \le g$, we have $a_{2g-j} = q^{g-j} a_j$.
  - $a_1 = |\Sigma(K/\mathbb{F}_q)| - (q+1)$, which notably does not depend on g.
- Write $L(t) = \prod_{j=1}^{2g} (1-\alpha_j t) \in \mathbb{C}[t]$ [a]
- The $\alpha_j \in \overline{\mathbb{Z}}$ [b] (which were *a priori* in $\mathbb{C}$) and can be ordered such that for all $1 \le j \le g$, we have $a_j a_{g+j} = q$. [c]

•

[a] The polynomial isn't monic, but rather has a constant coefficient, so this expansion is somewhat more natural than (say) $\prod(t-\alpha)$.

[b] $\overline{\mathbb{Z}}$ denotes the algebraic integers.

[c] This is the first hint at the Riemann hypothesis: if for example they all had the same complex modulus, this would force $|a_j| = \sqrt{q}$.Thus proving that they all have the same absolute value is 99% of the content!