Title

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1.1 Compact-Open Topology

- For X, Y topological spaces, consider

$$Y^X = C(X,Y) = \hom_{\text{Top}}(X,Y) \coloneqq \left\{ f : X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

  - General idea: it's nice to *cartesian closed* categories, which require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it (i.e. here we'd like the homs to again be spaces).
  - Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
    * Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
    * Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
  - Topologize with the *compact-open* topology: $U \in \hom_T(X,X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
    * If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} \coloneqq \sup \left\{ d(f_n(x), f(x)) \;\middle|\; x \in K \right\} \overset{n\to\infty}{\to} 0 \quad \forall K \subseteq X \text{ compact}.$$

      In words: $f_n \to f$ uniformly on every compact set.
  - If X itself is compact and Y is a metric space, $C(X,Y)$ can be promoted to a metric space with $d(f,g) = \sup_{x \in X}(f(x), g(x))$.
  - Useful in analysis: when is a family of functions $\mathcal{F} = \{f_\alpha\} \subset \hom_{\text{Top}}(X,Y)$ compact? Essentially answered by Arzela-Ascoli

Theorem 1.1 ***(Ascoli).***
If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \hom_{\text{Top}}(X, Y)$ has compact closure $\iff$ $\mathcal{F}$ is equicontinuous and $F_x := \left\{ f(x) \mid f \in \mathcal{F} \right\} \subset Y$ has compact closure.

Corollary 1.2 ***(Arzela).***
If $\{f_n\} \subset \hom_{\text{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every X, it contains a uniformly convergent subsequence.

- Useful in Number Theory / Rep Theory / Fourier Series: can take G to be a locally compact abelian topological group and define its Pontryagin dual $\widehat{G} := \hom_{\text{TopGrp}}(G, S^1)$ where we consider $S^1 \subset \mathbb{C}$.
  * Can integrate with respect to the Haar measure μ, define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) := \int_G f(x)\overline{\chi(x)} d\mu(x).$$

- So define $\text{Map}(X, Y) = \hom_{\text{Top}}(X, Y)$ equipped with the compact-open topology.
  - Can immediately consider a lot of interesting spaces by considering $\text{Map}(\,\cdot\,, Y)$:

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \quad \text{Map}(\{\text{pt}\}, X) \cong X \\ X = I &\rightsquigarrow \quad \mathcal{P}Y := \{f : I \to Y\} = Y^I \\ X = S^1 &\rightsquigarrow \quad \mathcal{L}Y := \left\{ f : S^1 \to Y \right\} = Y^{S^1}. \end{aligned}$$

Note: take basepoints to obtain the base path space PY, the based loop space ΩY.

  - Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences).
  - Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, \text{Map}(X, Y)) \\ H : X \times Z \to Y &\iff \tilde{H} : Z \to \text{Map}(X, Y) \\ (x, z) \mapsto H(x, z) &\iff z \mapsto H(\,\cdot\,, z). \end{aligned}$$

  Categorically, $\hom(X, \cdot\,) \leftrightarrow (X \times \,\cdot\,)$ form an adjoint pair in Top.

A form of this adjunction holds in any cartesian closed category.

  - Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},$$

  i.e. two maps f, g are homotopic $\iff$ they are connected by a path in $\text{Map}(X, Y)$.
    * Proof:

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, \text{Map}(X, Y)) \cong \text{Map}(Y \times I, X),$$

    and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.
    * Note that we can interpret the RHS as the space of paths

  - Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I, \cdot)$: define

$$\mathrm{Map}_I^1(X,Y) \coloneqq \mathrm{Map}(I, \mathrm{Map}(X,Y)) = \mathcal{P}\mathrm{Map}(X,Y)$$

    and then

$$\begin{aligned}\mathrm{Map}_I^2(X,Y) &\coloneqq \mathrm{Map}(I, \mathrm{Map}_I^1(X,Y)) \\ &\cong \mathrm{Map}(I, \mathrm{Map}(I, \mathrm{Map}(X,Y))) \quad = \mathcal{P}(\mathcal{P}(X,Y)) \\ &\cong \mathrm{Map}(I, \mathrm{Map}(Y \times I, X)) \\ &\coloneqq \mathcal{P}\mathrm{Map}(Y \times I, X).\end{aligned}$$

    Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

1.2 Self-Homeomorphisms

- Now restrict attention to

$$\mathrm{Homeo}(X) \coloneqq \left\{ f \in \mathrm{Map}(X,X) \;\middle|\; f \text{ is invertible} \right\}.$$

  - Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.