Algebra Notes

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Contents

1 Group Theory

1.1 Big List of Notation

$$
\begin{aligned}
C(x) = && \left\{ g \in G \;\middle|\; gxg^{-1} = x \right\} && \subseteq G && \text{Centralizer} \\
C_G(h) = && \left\{ ghg^{-1} \;\middle|\; g \in G \right\} && \subseteq G && \text{Conjugacy Class} \\
Gx = && \left\{ g.x \;\middle|\; x \in X \right\} && \subseteq X && \text{Orbit} \\
G_x = && \left\{ g \in G \;\middle|\; g.x = x \right\} && \subseteq G && \text{Stabilizer} \\
X_g = && \left\{ x \in X \;\middle|\; \forall g \in G,\ g.x = x \right\} && \subseteq X && \text{Fixed Points} \\
Z(G) = && \left\{ x \in G \;\middle|\; \forall g \in G,\ gxg^{-1} = x \right\} && \subseteq G && \text{Center} \\
\mathrm{Inn}(G) = && \left\{ \phi_g(x) = gxg^{-1} \right\} && \subseteq \mathrm{Aut}(G) && \text{Inner Aut.} \\
\mathrm{Out}(G) = && \mathrm{Aut}(G)/\mathrm{Inn}(G) && \hookrightarrow \mathrm{Aut}(G) && \text{Outer Aut.} \\
N(H) = && \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\} && \subseteq G && \text{Normalizer}
\end{aligned}
$$

1.2 Basics

Definition (Centralizer):

$$C_G(H) = \left\{ g \in G \;\middle|\; ghg^{-1} = h \ \forall h \in H \right\}$$

Definition (Normalizer):

$$N_G(H) = \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\}$$

Lemma: $C_G(H) \trianglelefteq N_G(H)$

Lemma: The size of the conjugacy class of H is the index of its centralizer, i.e.

$$\left| \left\{ gHg^{-1} \;\middle|\; g \in G \right\} \right| = [G : C_G(H)].$$

Proof: Orbit-stabilizer.

Lemma ("The Fundamental Theorem of Cosets"):

$$aH = bH \iff a^{-1}b \in H \text{ or } aH \bigcap bH = \emptyset$$

Definition: $[x, y] = x^{-1}y^{-1}xy$ is the **commutator**, and $[G, G] \coloneqq \left\{ [x, y] \;\middle|\; x, y \in G \right\}$ is the **commutator subgroup**.

Lemma:

$$[G, G] \leq H \text{ and } H \trianglelefteq G \implies G/H \text{ is abelian.}$$

Lemmas:

- Every subgroup of a cyclic group is itself cyclic.
- Intersections of subgroups are still subgroups
  - Intersections of distinct coprime-order subgroups are trivial
  - Intersections of subgroups of the same prime order are either trivial or equality
- The Quaternion group has only one element of order 2, namely -1.
  - They also have the presentation

$$\begin{aligned} Q &= \left\langle x, y, z \;\middle|\; x^2 = y^2 = z^2 = xyz = -1 \right\rangle \\ &= \left\langle x, y \;\middle|\; x^4 = y^4 = e, x^2 = y^2, yxy^{-1} = x^{-1} \right\rangle. \end{aligned}$$

- A dihedral group always has a presentation of the form

$$D_n = \left\langle x, y \;\middle|\; x^n = y^2 = (xy)^2 = e \right\rangle,$$

  yielding at least 2 distinct elements of order 2.

1.3 Finitely Generated Abelian Groups

Invariant factor decomposition:

$$G \cong \mathbb{Z}^r \times \prod_{j=1}^{m} \mathbb{Z}/(n_j) \quad \text{where } n_1 \;\Big|\; \cdots \;\Big|\; n_m.$$

Going from invariant divisors to elementary divisors:

- Take prime factorization of each factor
- Split into coprime pieces

Example:

$$\begin{aligned} &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3 \cdot 5^2 \cdot 7) \\ \cong &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(7) \\ . \end{aligned}$$

Going from elementary divisors to invariant factors:

- Bin up by primes occurring (keeping exponents)
- Take highest power from each prime as *last* invariant factor
- Take highest power from all remaining primes as next, etc

Example: Given the invariant factor decomposition

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25},.$$

$p = 2$	$p = 3$	$p = 5$
$2, 2, 2$	$3, 3$	5^2

$\implies n_m = 5^2 \cdot 3 \cdot 2$

$p = 2$	$p = 3$	$p = 5$
$2, 2$	3	$\emptyset$

$\implies n_{m-1} = 3 \cdot 2$

$p = 2$	$p = 3$	$p = 5$
2	$\emptyset$	$\emptyset$

$\implies n_{m-2} = 2$

and thus

$$G \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3 \cdot 2) \oplus \mathbb{Z}/(5^2 \cdot 3 \cdot 2).$$

1.4 The Symmetric Group

Definitions:

- A cycle is **even** $\iff$ product of an *even* number of transpositions.
  - A cycle of even *length* is **odd**
  - A cycle of odd *length* is **even**

Definition The **alternating group** is the subgroup of **even** permutations, i.e. $A_n := \left\{ \sigma \in S_n \;\middle|\; \mathrm{sign}(\sigma) = 1 \right\}$ where $\mathrm{sign}(\sigma) = (-1)^m$ where m is the number of cycles of even length.

> *Corollary:* Every $\sigma \in A_n$ has an even number of *odd* cycles (i.e. an even number of *even-length* cycles).

Example:

$$\begin{aligned}A_4 = \{&\mathrm{id}, \\ &(1,3)(2,4), (1,2)(3,4), (1,4)(2,3), \\ &(1,2,3), (1,3,2), \\ &(1,2,4), (1,4,2), \\ &(1,3,4), (1,4,3), \\ &(2,3,4), (2,4,3)\}.\end{aligned}$$

Lemmas:

- The transitive subgroups of S_3 are S_3, A_3
- The transitive subgroups of S_4 are $S_4, A_4, D_4, \mathbb{Z}_2^2, \mathbb{Z}_4$.
- S_4 has two normal subgroups: $A_4, \mathbb{Z}_2^2$.
- $S_{n\geq 5}$ has one normal subgroup: A_n.
- $Z(S_n) = 1$ for $n \geq 3$
- $Z(A_n) = 1$ for $n \geq 4$
- $[S_n, S_n] = A_n$
- $[A_4, A_4] \cong \mathbb{Z}_2^2$
- $[A_n, A_n] = A_n$ for $n \geq 5$, so $A_{n\geq 5}$ is nonabelian.
- $A_{n\geq 5}$ is *simple*.

1.5 Counting Theorems

Lagrange's Theorem:

$$H \leq G \implies |H| \,\Big|\, |G|.$$

Corollary: The order of every element divides the size of G, i.e.

$$g \in G \implies o(g) \,\Big|\, o(G) \implies g^{|G|} = e.$$

Warning: There does **not** necessarily exist $H \leq G$ with $|H| = n$ for every $n \,\Big|\, |G|$.
Counterexample: $|A_4| = 12$ but has no subgroup of order 6.

Cauchy's Theorem:

For every prime p dividing $|G|$. there is an element (and thus a subgroup) of order p.

This is a partial converse to Lagrange's theorem, and strengthened by Sylow's theorem.

Notation: For a group G acting on a set X,

- $G \cdot x = \left\{ g \curvearrowright x \,\middle|\, g \in G \right\} \subseteq X$ is the orbit
- $G_x = \left\{ g \in G \,\middle|\, g \curvearrowright x = x \right\} \subseteq G$ is the stabilizer
- $X/G \subset \mathcal{P}(X)$ is the set of orbits

- $X^g = \left\{ x \in X \,\middle|\, g \curvearrowright x = x \right\} \subseteq X$ are the fixed points

Orbit-Stabilizer:

$$|G \cdot x| = [G : G_x] = |G|/|G_x| \quad \text{if } G \text{ is finite}$$

> Mnemonic: $G/G_x \cong G \cdot x$.

1.5.1 Examples of Orbit-Stabilizer

1. Let G act on itself by conjugation.

- $G \cdot x$ is the **conjugacy class** of x
- $G_x = Z(x) \coloneqq C_G(x) = \left\{ g \,\middle|\, [g, x] = e \right\}$, the **centralizer** of x.
- G^g (the fixed points) is the **center** $Z(G)$.

> *Corollary:* The number of conjugates of an element (i.e. the size of its conjugacy class) is the index of its centralizer, $[G : C_G(x)]$.
> *Corollary*: the **Class Equation**:
>
> $$|G| = |Z(G)| + \sum_{\substack{\text{One } x_i \text{ from} \\ \text{each conjugacy} \\ \text{class}}} [G : Z(x_i)]$$

1. Let G act on S, its set of *subgroups*, by conjugation.

- $G \cdot H = \left\{ gHg^{-1} \right\}$ is the **set of conjugate subgroups** of H
- $G_H = N_G(H)$ is the **normalizer** of in G of H
- S^G is the set of **normal subgroups** of G

> Corollary: Given $H \leq G$, the number of conjugate subgroups is $[G : N_G(H)]$.

1. For a fixed proper subgroup $H < G$, let G act on its cosets $G/H = \left\{ gH \,\middle|\, g \in G \right\}$ by left-multiplication.

- $G \cdot gH = G/H$, i.e. this is a *transitive* action.
- $G_{gH} = gHg^{-1}$ is a *conjugate subgroup* of H
- $(G/H)^G = \emptyset$

> *Application:* If G is simple, $H < G$ proper, and $[G : H] = n$, then there exists an injective map $\phi : G \hookrightarrow S_n$.
> *Proof:* This action induces ϕ; it is nontrivial since $gH = H$ for all g implies $H = G$; $\ker \phi \trianglelefteq G$ and G simple implies $\ker \phi = 1$.

Burnside's Formula:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

1.5.2 Sylow Theorems

Notation: For any p, let $\mathrm{Syl}_p(G)$ be the set of Sylow-p subgroups of G.

Write

- $|G| = p^n m$ where $(m, p) = 1$,
- S_p a Sylow-p subgroup, and
- n_p the number of Sylow-p subgroups.

Definition: A p-group is a group G such that every element is order p^k for some k. If G is a finite p-group, then $|G| = p^j$ for some j.

Lemma: p-groups have nontrivial centers.

Some useful facts:

- Coprime order subgroups are disjoint, or more generally $\mathbb{Z}_p, \mathbb{Z}_q \subset G \implies \mathbb{Z}_p \bigcap \mathbb{Z}_q = \mathbb{Z}_{(p,q)}$.
- The Chinese Remainder theorem: $(p, q) = 1 \implies \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

1.5.3 Sylow 1 (Cauchy for Prime Powers)

$$\forall p^n \text{ dividing } |G| \quad \text{there exists a subgroup of size } p^n.$$

If $|G| = \prod p_i^{\alpha_i}$, then there exist subgroups of order $p_i^{\beta_i}$ for every i and every $0 \leq \beta_i \leq \alpha_i$. In particular, Sylow p-subgroups always exist.

1.5.4 Sylow 2 (Sylows are Conjugate)

All sylow-p subgroups S_p are conjugate, i.e.

$$S_p^1, S_p^2 \in \mathrm{Syl}_p(G) \implies \exists g \text{ such that } g S_p^1 g^{-1} = S_p^2.$$

Corollary: $n_p = 1 \iff S_p \trianglelefteq G$

1.5.5 Sylow 3 (Numerical Constraints)

1. $n_p \mathrel{\Big|} m$ (in particular, $n_p \leq m$),
2. $n_p \equiv 1 \mod p$,
3. $n_p = [G : N_G(S_p)]$ where N_G is the normalizer.

Corollary: p does not divide n_p.

Lemma: Every p-subgroup of G is contained in a Sylow p-subgroup.

Proof: Let $H \leq G$ be a p-subgroup. If H is not *properly* contained in any other p-subgroup, it is a Sylow p-subgroup by definition.
Otherwise, it is contained in some p-subgroup H^1. Inductively this yields a chain $H \subsetneq H^1 \subsetneq \cdots$, and by Zorn's lemma $H := \bigcup_i H^i$ is maximal and thus a Sylow p-subgroup.

Fratini's Argument: If $H \trianglelefteq G$ and $P \in \mathrm{Syl}_p(G)$, then $HN_G(P) = G$ and $[G : H]$ divides $|N_G(P)|$.

1.6 Products

Characterizing direct products: $G \cong H \times K$ when

- $G = HK = \left\{hk \;\middle|\; h \in H, k \in K\right\}$
- $H \bigcap K = \{e\} \subset G$
- $H, K \trianglelefteq G$

Can relax to only $H \trianglelefteq G$ to get a semidirect product instead

Characterizing semidirect products: $G = N \rtimes_\psi H$ when

- $G = NH$
- $N \trianglelefteq G$
- $H \curvearrowright N$ by conjugation via a map

$$\begin{aligned}\psi : H &\to \operatorname{Aut}(N) \\ h &\mapsto h(\,\cdot\,)h^{-1}.\end{aligned}$$

Useful Facts

- If $\sigma \in \operatorname{Aut}(H)$, then $N \rtimes_\psi H \cong N \rtimes_{\psi \circ \sigma} H$.
- $\operatorname{Aut}((\mathbb{Z}/(p)^n) \cong \operatorname{GL}(n, \mathbb{F}_p)$, which has size \$- $|\operatorname{Aut}(\mathbb{Z}/(p)^n)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.
  - If this occurs in a semidirect product, it suffices to consider similarity classes of matrices (i.e. just use canonical forms)
- $\operatorname{Aut}(\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)^\times \cong \mathbb{Z}/(\varphi(n))$ where φ is the totient function.
  - $\varphi(p^k) = p^{k-1}(p-1)$
- If G, H have coprime order then $\operatorname{Aut}(G \oplus H) \cong \operatorname{Aut}(G) \oplus \operatorname{Aut}(H)$.

1.7 Isomorphism Theorems

Lemma: If $H, K \leq G$ and $H \leq N_G(K)$ (or $K \trianglelefteq G$) then $HK \leq G$ is a subgroup.

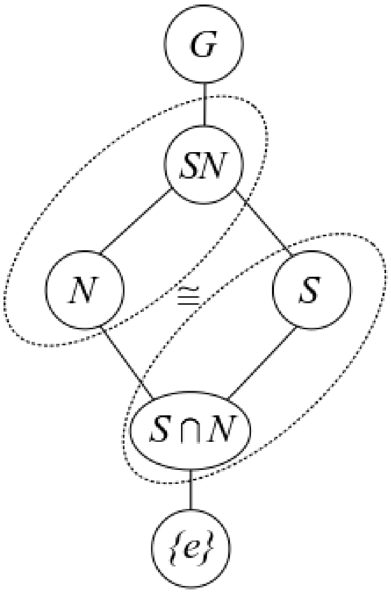
Note that this implies that HK is not always a subgroup.

Diamond Theorem / 2nd Isomorphism Theorem:

If $S \leq G$ and $N \trianglelefteq G$, then

$$\frac{SN}{N} \cong \frac{S}{S \bigcap N} \quad \text{and} \quad |SN| = \frac{|S||N|}{|S \bigcap N|}$$

Mnemonic:



Note: for this to make sense, we also have

- $SN \leq G$,
- $S \bigcap N \trianglelefteq S$,

Cancellation / 3rd Isomorphism Theorem

If $H, K \trianglelefteq G$ with $H \trianglelefteq K$, then

$$\frac{G/H}{G/K} \cong \frac{G}{K}$$

Note: for this to make sense, we also have $G/K \trianglelefteq G/H$.

The Correspondence Theorem / 4th Isomorphism Theorem: Suppose $N \trianglelefteq G$, then there exists a correspondence:

$$\left\{H < G \;\middle|\; N \subseteq H\right\} \iff \left\{H \;\middle|\; H < \frac{G}{N}\right\}$$
$$\left\{\begin{array}{c}\text{Subgroups of } G\\ \text{containing } N\end{array}\right\} \iff \left\{\begin{array}{c}\text{Subgroups of the}\\ \text{quotient } G/N\end{array}\right\}.$$

In words, subgroups of G containing N correspond to subgroups of the quotient group G/N. This is given by the map $H \mapsto H/N$.

Note: $N \trianglelefteq G$ and $N \subseteq H < G \implies N \trianglelefteq H$.

1.8 Special Classes of Groups

Definition: The **"2 out of 3 property"** is satisfied by a class of groups $\mathcal{C}$ iff whenever $G \in \mathcal{C}$, then $N, G/N \in \mathcal{C}$ for any $N \trianglelefteq G$.

Definition: If $|G| = p^k$, then G is a **p-group.**

Facts about p-groups:

- If $k = 1$ then G is cyclic
- If $k = 2$, then $G \cong \mathbb{Z}/(p)^2$ or $\mathbb{Z}/(p^2)$.
- p-groups have nontrivial centers
  - Proof: Use class equation.
- Every normal subgroup is contained in the center
- Normalizers grow
- Every maximal is normal
- Every maximal has index p
- p-groups are *nilpotent*
- p-groups are *solvable*

Facts about other special order groups:

General strategy: find a normal subgroup (usually a Sylow) and use recognition of semidirect products.

- $|G| = pq$: Two possibilities. By cases:
  - If p divides $q - 1$, two cases:
    - $G \cong \mathbb{Z}/(pq)$ or $\mathbb{Z}(p) \times \mathbb{Z}/(q)$
  - Otherwise, $G \cong \mathbb{Z}/(pq)$

  Proof: Sylow theorems. Note: Such groups are never simple.
- $|G| = p^2 q$:
  - $q \mid p^2 - 1$: Two abelian possibilities, $\mathbb{Z}/(p) \times \mathbb{Z}/(q^2)$, or $\mathbb{Z}/(pq) \times \mathbb{Z}/(q)$.
  - Otherwise, the sylow-q subgroup H is normal and order q^2, so either $\mathbb{Z}/(q)^2$ or $\mathbb{Z}/(q^2)$.
    - Case 2: $\left|\mathrm{Aut}(\mathbb{Z}/(q)^2)\right| = q(q-1)$, so only trivial action
    - Case 1: $\left|\mathrm{Aut}(\mathbb{Z}/(q^2))\right| = q(q-1)^2(q+1)$
      - If p doesn't divide $q + 1$, noting new
      - Otherwise, a nontrivial semidirect product.

Definition: A group G is **simple** iff $H \trianglelefteq G \implies H = \{e\}, G$, i.e. it has no non-trivial proper subgroups.

> **Lemma:** If G is *not* simple, then for any $N \trianglelefteq G$, it is the case that $G \cong E$ for an extension of the form $N \to E \to G/N$. >

Definition: A group G is **solvable** iff G has a terminating normal series with abelian factors, i.e.

$$G \to G^1 \to \cdots \to \{e\} \text{ with } G^i/G^{i+1} \text{ abelian for all } i.$$

Lemmas:

- G is solvable iff G has a terminating *derived series.*

- Solvable groups satisfy the 2 out of 3 property
- Abelian $\implies$ solvable
- Every group of order less than 60 is solvable.

Definition: A group G is **nilpotent** iff G has a terminating central series, upper central series, or lower central series.

> Moral: the adjoint map is nilpotent.

Lemma: For G a finite group, TFAE:

- G is nilpotent
- Normalizers grow (i.e.$H < N_G(H)$ whenever H is proper)
- Every Sylow-p subgroup is normal
- G is the direct product of its Sylow p-subgroups
- Every maximal subgroup is normal
- G has a terminating *Lower* Central Series
- G has a terminating *Upper* Central Series

Lemmas:

- G nilpotent $\implies$ G solvable
- Nilpotent groups satisfy the 2 out of 3 property.
- G has normal subgroups of order d for *every* d dividing $|G|$
- G nilpotent $\implies$ $Z(G) \neq 0$
- Abelian $\implies$ nilpotent
- p-groups $\implies$ nilpotent

1.9 Series of Groups

Definition: A **normal series** of a group G is a sequence $G \to G^1 \to G^2 \to \cdots$ such that $G^{i+1} \trianglelefteq G_i$ for every i.

Definition A **composition series** of a group G is a finite normal series such that G^{i+1} is a *maximal proper* normal subgroup of G^i.

Theorem (Jordan-Holder): Any two composition series of a group have the same length and isomorphic factors (up to permutation).1

Definition A **derived series** of a group G is a normal series $G \to G^1 \to G^2 \to \cdots$ where $G^{i+1} = [G^i, G^i]$ is the commutator subgroup.

> The derived series terminates iff G is *solvable.*

Definition: A **central series** for a group G is a terminating normal series $G \to G^1 \to \cdots \to \{e\}$ such that each quotient is **central**, i.e. $[G, G^i] \leq G^{i-1}$ for all i.

Definition: A **lower central series** is a terminating normal series $G \to G^1 \to \cdots \to \{e\}$ such that $G^{i+1} = [G^i, G]$

> Moral: Iterate the adjoint map $[\,\cdot\,, G]$.

> G is nilpotent $\iff$ the LCS terminates.

Definition: An **upper central series** is a terminating normal series $G \to G^1 \to \cdots \to \{e\}$ such that $G^1 = Z(G)$ and G^{i+1} is defined such that $G^{i+1}/G^i = Z(G^i)$.

> Moral: Iterate taking "higher centers".

2 Rings

2.1 Definitions

Lemma: Intersections, products, and sums (but not necessarily unions) of ideals are ideals.

Theorem (Krull): Every ring has proper maximal ideals, and any proper ideal is contained in a maximal ideal.

Definition: A ring R is **simple** iff every ideal $I \trianglelefteq R$ is either 0 or R.

Definition: An element $r \in R$ is **irreducible** iff $r = ab \implies a$ is a unit or b is a unit.F

Definition: An element $r \in R$ is **prime** iff $ab \mathrel{\Big|} r \implies a \mathrel{\Big|} r$ or $b \mathrel{\Big|} r$ whenever a, b are nonzero and not units.

Definition: $\mathfrak{p}$ is a **prime** ideal $\iff ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition: $\operatorname{Spec}(R) = \left\{\mathfrak{p} \trianglelefteq R \;\middle|\; \mathfrak{p} \text{ is prime}\right\}$ is the **spectrum** of R.

Definition: $\mathfrak{m}$ is **maximal** $\iff I \ntrianglelefteq R \implies I \subseteq \mathfrak{m}$.

> Example: Maximal ideals of $R[x]$ are of the form $I = (x - a_i)$ for some $a_i \in R$.

Definition: $\operatorname{Spec}_{\max}(R) = \left\{\mathfrak{m} \trianglelefteq R \;\middle|\; \mathfrak{m} \text{ is maximal}\right\}$ is the **max-spectrum** of R.

> Note: nonstandard notation / definition.

Lemmas (Quotienting):

- R/I is a domain $\iff I$ is prime,
- R/I is a field $\iff I$ is maximal.
- For R a PID, I is prime $\iff I$ is maximal.

Lemma (Characterizations of Rings):

- R a commutative division ring $\implies R$ is a field
- R a finite integral domain $\implies R$ is a field.
- $\mathbb{F}$ a field $\implies \mathbb{F}[x]$ is a Euclidean domain.
- $\mathbb{F}$ a field $\implies \mathbb{F}[x]$ is a PID.
- $\mathbb{F}$ is a field $\iff \mathbb{F}$ is a commutative simple ring.
- R is a UFD $\iff R[x]$ is a UFD.
- R a PID $\implies R[x]$ is a UFD
- R a PID $\implies R$ Noetherian
- $R[x]$ a PID $\implies R$ is a field.

Lemma: Fields $\subset$ Euclidean domains $\subset$ PIDs $\subset$ UFDs $\subset$ Integral Domains $\subset$ Rings

- A Euclidean Domain that is not a field: $\mathbb{F}[x]$ for $\mathbb{F}$ a field

- *Proof*: Use previous lemma, and x is not invertible
- A PID that is not a Euclidean Domain: $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$.
  - *Proof*: complicated.
- A UFD that is not a PID: $\mathbb{F}[x, y]$.
  - *Proof*: $\langle x, y\rangle$ is not principal
- An integral domain that is not a UFD: $\mathbb{Z}[\sqrt{-5}]$
  - *Proof*: $(2+\sqrt{-5})(2-\sqrt{-5}) = 9 = 3 \cdot 3$, where all factors are irreducible (check norm).
- A ring that is not an integral domain: $\mathbb{Z}/(4)$
  - *Proof*: $2 \mod 4$ is a zero divisor.

Lemma: In R a UFD, an element $r \in R$ is prime $\iff$ r is irreducible.

> Note: For R an integral domain, prime $\implies$ irreducible, but generally not the converse.
> *Example of a prime that is not irreducible:* $x^2 \mod (x^2+x) \in \mathbb{Q}[x]/(x^2+x)$. Check that x is prime directly, but $x = x \cdot x$ and x is not a unit.
> *Example of an irreducible that is not prime:* $3 \in \mathbb{Z}[\sqrt{-5}]$. Check norm to see irreducibility, but $3 \bigm| 9 = (2+\sqrt{-5})(2-\sqrt{-5})$ and doesn't divide either factor.

Lemma: If R is a PID, then every element in R has a unique prime factorization.

Definition: A nonzero unital ring R is **semisimple** iff $R \cong \bigoplus_{i=1}^{n} M_i$ with each M_i a simple module.

Theorem (Artin-Wedderubrn): If R is a nonzero, unital, *semisimple* ring then $R \cong \bigoplus_{i=1}^{m} \mathrm{Mat}(n_i, D_i)$, a finite sum of matrix rings over division rings.

> *Corollary:* If M is a simple ring over R a division ring, the M is isomorphic to a matrix ring.

2.2 Nontrivial Properties

Lemma: Every $a \in R$ for a finite ring is either a unit or a zero divisor.

> *Proof:* Let $a \in R$ and define $\phi(x) = ax$. If ϕ is injective, then it is surjective, so $1 = ax$ for some $x \implies x^{-1} = a$. Otherwise, $ax_1 = ax_2$ with $x_1 \neq x_2 \implies a(x_1 - x_2) = 0$ and $x_1 - x_2 \neq 0$, so a is a zero divisor.

2.3 Ideals

2.3.1 Maximal and Prime Ideals

Lemma: Maximal $\implies$ prime, but generally not the converse.

> *Counterexample*: $(0) \in \mathbb{Z}$ is prime since $\mathbb{Z}$ is a domain, but not maximal since it is properly contained in any other ideal.

Proof: Suppose $\mathfrak{m}$ is maximal, $ab \in \mathfrak{m}$, and $b \notin \mathfrak{m}$. Then there is a containment of ideals $\mathfrak{m} \subsetneq \mathfrak{m} + (b) \implies \mathfrak{m} + (b) = R$.
So

$$1 = m + rb \implies a = am + r(ab),$$

but $am \in \mathfrak{m}$ and $ab \in \mathfrak{m} \implies a \in \mathfrak{m}$.

■

Lemma: If x is not a unit, then x is contained in some maximal ideal $\mathfrak{m}$.

Proof: Zorn's lemma.

Lemma: $R/\mathfrak{m}$ is a field $\iff$ $\mathfrak{m}$ is maximal.

Lemma: $R/\mathfrak{p}$ is an integral domain $\iff$ $\mathfrak{p}$ is prime.

2.3.2 Nilradical and Jacobson Radical

Definition: $\mathfrak{N} := \left\{ x \in R \;\middle|\; x^n = 0 \text{ for some } n \right\}$ is the **nilradical** of R.

Lemma: The nilradical is the intersection of all **prime** ideals, i.e.

$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$$

Proof:
$\mathfrak{N} \subseteq \bigcap \mathfrak{p}$: $x \in \mathfrak{N} \implies x^n = 0 \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $x^{n-1} \in \mathfrak{p}$.
$\mathfrak{N}^c \subseteq \bigcup \mathfrak{p}^c$: Define $S = \left\{ I \trianglelefteq R \;\middle|\; a^n \notin I \text{ for any } n \right\}$. Then apply Zorn's lemma to get a maximal ideal $\mathfrak{m}$, and maximal $\implies$ prime.

Lemma: $R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.

Proof:

$$\begin{aligned} a + \mathfrak{N}(R) \text{ nilpotent } &\implies (a + \mathfrak{N}(R))^n := a^n + \mathfrak{N}(R) = \mathfrak{N}(R) \\ &\implies a^n \in \mathfrak{N}(R) \\ &\implies \exists \ell \text{ such that } (a^n)^\ell = 0 \\ &\implies a \in \mathfrak{N}(R). \end{aligned}$$

Definition: The **Jacobson radical** is the intersection of all **maximal** ideals, i.e.

$$J(R) = \bigcap_{\mathfrak{m} \in \operatorname{Spec}_{\max}} \mathfrak{m}$$

Lemma: $\mathfrak{N}(R) \subseteq J(R)$.

Proof: Maximal $\implies$ prime, and so if x is in every prime ideal, it is necessarily in every maximal ideal as well.

2.3.3 Zorn's Lemma

Lemma: A field has no nontrivial proper ideals.

Lemma: If $I \trianglelefteq R$ is a proper ideal $\iff$ I contains no units.

> *Proof:* $r \in R^\times \bigcap I \implies r^{-1}r \in I \implies 1 \in I \implies x \cdot 1 \in I \quad \forall x \in R.$

Lemma: If $I_1 \subseteq I_2 \subseteq \cdots$ are ideals then $\bigcup_j I_j$ is an ideal.

Example Application of Zorn's Lemma: Every proper ideal is contained in a maximal ideal.

> *Proof:* Let $0 < I < R$ be a proper ideal, and consider the set
>
> $$S = \left\{ J \;\middle|\; I \subseteq J < R \right\}.$$
>
> Note $I \in S$, so S is nonempty. The claim is that S contains a maximal element M.
> S is a poset, ordered by set inclusion, so if we can show that every chain has an upper bound, we can apply Zorn's lemma to produce M.
> Let $C \subseteq S$ be a chain in S, so $C = \{C_1 \subseteq C_2 \subseteq \cdots\}$ and define $\hat{C} = \bigcup_i C_i$.
>
> **$\hat{C}$ is an upper bound for C:**
> This follows because every $C_i \subseteq \hat{C}$.
> **$\hat{C}$ is in S:**
> Use the fact that $I \subseteq C_i < R$ for every C_i and since no C_i contains a unit, $\hat{C}$ doesn't contain a unit, and is thus proper.
>
> ■

3 Fields

Let k denote a field.

Lemmas:

- The characteristic of $\mathbb{F}$ is either 0 or p a prime.
- All fields are simple rings
- Any homomorphism of fields is either 0 or injective
- If L/k is algebraic, then $\min(\alpha, L)$ divides $\min(\alpha, k)$.

Lemma: Every finite extension is algebraic.

Eisenstein's Criterion: If $f(x) = \sum_{i=0}^{n} \alpha_i x^i \in \mathbb{Q}[x]$ and $\exists p$ such that

- p divides every coefficient *except* a_n and
- p^2 does not divide a_0,

then f is irreducible.

Definition: For R a UFD, a polynomial $p \in R[x]$ is **primitive** iff the greatest common divisors of its coefficients is a unit.

Gauss' Lemma: Let R be a UFD and F its field of fractions. Then a primitive $p \in R[x]$ is irreducible in $R[x] \iff p$ is irreducible in $F[x]$.

Corollary: A primitive polynomial $p \in \mathbb{Q}[x]$ is irreducible iff p is irreducible in $\mathbb{Z}[x]$.

3.1 Finite Fields

Definition: The prime subfield of a field F is the subfield generated by 1.

Lemma (Characterization of Prime Subfields): The prime subfield of any field is isomorphic to either $\mathbb{Q}$ or $\mathbb{F}_p$ for some p.

Lemma ("Freshman's Dream"): If $\operatorname{char} k = p$ then $(a+b)^p = a^p + b^p$ and $(ab)^p = a^p b^p$.

Theorem (Construction of Finite Fields): $\mathbb{GF}(p^n) \cong \dfrac{\mathbb{F}_p}{(f)}$ where $f \in \mathbb{F}_p[x]$ is any irreducible of degree n, and $\mathbb{GF}(p^n) \cong \mathbb{F}[\alpha] \cong \operatorname{span}_{\mathbb{F}} \left\{1, \alpha, \cdots, \alpha^{n-1}\right\}$ for any root α of f.

Lemma (Prime Subfields of Finite Fields): Every finite field F is isomorphic to a unique field of the form $\mathbb{GF}(p^n)$ and if $\operatorname{char} F = p$, it has prime subfield $\mathbb{F}_p$.

Lemma (Containment of Finite Fields): $\mathbb{GF}(p^\ell) \leq \mathbb{GF}(p^k) \iff \ell$ divides k.

Lemma (Identification of Finite Fields as Splitting Fields): $\mathbb{GF}(p^n)$ is the splitting field of $\rho(x) = x^{p^n} - x$, and the elements are exactly the roots of ρ.

Every element is a root by Cauchy's theorem, and the p^n roots are distinct since its derivative is identically -1.

Lemma (Splits Product of Irreducibles): Let $\rho_n \coloneqq x^{p^n} - x$. Then $f(x) \mid \rho_n(x) \iff \deg f \mid n$ and f is irreducible.

Corollary: $x^{p^n} - x = \prod f_i(x)$ over all irreducible monic $f_i \in \mathbb{F}_p[x]$ of degree d dividing n.
Proof:
$\Longleftarrow$: Suppose f is irreducible of degree d. Then $f \mid x^{p^d} - x$ (consider $F[x]/\langle f \rangle$) and $x^{p^d} - x \mid x^{p^n} - x \iff d \mid n$.
$\Longrightarrow$:

- $\alpha \in \mathbb{GF}(p^n) \iff \alpha^{p^n} - \alpha = 0$, so every element is a root of ϕ_n and $\deg \min(\alpha, \mathbb{F}_p) \mid n$ since $\mathbb{F}_p(\alpha)$ is an intermediate extension.
- So if f is an irreducible factor of ϕ_n, f is the minimal polynomial of some root α of ϕ_n, so $\deg f \mid n$.

$\phi_n'(x) = p^n x^{p^{n-1}} \neq 0$, so ϕ_n has distinct roots and thus no repeated factors. So ϕ_n is the product of all such irreducible f.

Lemma: No finite field is algebraically closed.

3.2 Galois Theory

Definition: A field extension L/k is **algebraic** iff every $\alpha \in L$ is the root of some polynomial $f \in k[x]$.

Definition: Let L/k be a finite extension. Then TFAE:

- L/k is **normal**.
- Every irreducible $f \in k[x]$ that has one root in L has *all* of its roots in L

- i.e. every polynomial splits into linear factors

- Every embedding $\sigma : L \hookrightarrow \overline{k}$ that is a lift of the identity on k satisfies $\sigma(L) = L$.
- If L is separable: L is the splitting field of some irreducible $f \in k[x]$.

Definition: Let L/k be a field extension, $\alpha \in L$ be arbitrary, and $f(x) \coloneqq \min(\alpha, k)$. TFAE:

- L/k is **separable**
- f has no repeated factors/roots
- $\gcd(f, f') = 1$, i.e. f is coprime to its derivative
- $f' \not\equiv 0$

Lemma: If char $k = 0$ or k is finite, then every *algebraic* extension L/k is separable.

Definition: $\operatorname{Aut}(L/k) = \left\{ \sigma : L \to L \;\middle|\; \sigma|_k = \mathrm{id}_k \right\}$.

Lemma: If L/k is algebraic, then $\operatorname{Aut}(L/k)$ permutes the roots of irreducible polynomials.

Lemma: $|\operatorname{Aut}(L/k)| \leq [L : k]$ with equality precisely when L/k is normal.

Definition: If L/k is Galois, we define $\operatorname{Gal}(L/k) \coloneqq \operatorname{Aut}(L/k)$.

3.2.1 Lemmas About Towers

Let $L/F/k$ be a finite tower of field extensions

- Multiplicativity: $[L : k] = [L : F][F : k]$
- L/k normal/algebraic/Galois $\implies$ L/F normal/algebraic/Galois.
  - *Proof (normal):* $\min(\alpha, F) \;\Big|\; \min(\alpha, k)$, so if the latter splits in L then so does the former.
  - *Corollary:* $\alpha \in L$ algebraic over $k \implies \alpha$ algebraic over F.
  - *Corollary:* E_1/k normal and E_2/k normal $\implies E_1E_2/k$ normal and $E_1 \bigcap E_2/k$ normal.

$$\begin{tikzcd}
L \arrow[d, bend right, dotted] \arrow[dd, bend left] \\
F \\
k
\end{tikzcd}$$

- F/k algebraic and L/F algebraic $\implies$ L/k algebraic.
- If L/k is algebraic, then F/k separable and L/F separable $\iff$ L/k separable

$$\begin{tikzcd}
L \arrow[d, bend right] \arrow[dd, bend left, dashed] \\
F \arrow[d, bend right] \\
k
\end{tikzcd}$$

- F/k Galois and L/K Galois $\implies$ F/k Galois **only if** $\operatorname{Gal}(L/F) \trianglelefteq \operatorname{Gal}(L/k)$

- $\implies \mathrm{Gal}(F/k) \cong \dfrac{\mathrm{Gal}(L/k)}{\mathrm{Gal}(L/F)}$

$$
\begin{tikzcd}
L \arrow[d, bend right, dotted] \arrow[dd, bend left] \\
F \arrow[d, bend right] \\
k
\end{tikzcd}
$$

Common Counterexamples:

- $\mathbb{Q}(\zeta_3, 2^{1/3})$ is normal but $\mathbb{Q}(2^{1/3})$ is not since the irreducible polynomial $x^3 - 2$ has only one root in it.

Definition (Characterizations of Galois Extensions): Let L/k be a finite field extension. TFAE:

- L/k is **Galois**
- L/k is finite, normal, and separable.
- L/k is the splitting field of a separable polynomial
- $|\mathrm{Aut}(L/k)| = [L : k]$
- The fixed field of $\mathrm{Aut}(L/k)$ is exactly k.

Fundamental Theorem of Galois Theory: Let L/k be a Galois extension, then there is a correspondence:

$$
\begin{aligned}
\left\{\text{Subgroups } H \leq \mathrm{Gal}(L/k)\right\} &\iff \left\{\begin{matrix}\text{Fields } F \text{ such} \\ \text{that } L/F/k\end{matrix}\right\} \\
H &\to \left\{E^H := \text{The fixed field of } H\right\} \\
\left\{\mathrm{Gal}(L/F) := \left\{\sigma \in \mathrm{Gal}(L/k) \;\middle|\; \sigma(F) = F\right\}\right\} &\leftarrow F.
\end{aligned}
$$

- This is contravariant with respect to subgroups/subfields.
- $[F : k] = [G : H]$, so degrees of extensions over the base field correspond to indices of subgroups.
- $[K : F] = |H|$
- L/F is Galois and $Gal(K/F) = H$
- F/k is Galois $\iff$ H is normal, and $\mathrm{Gal}(F/k) = \mathrm{Gal}(L/k)/H$.
- The compositum F_1F_2 corresponds to $H_1 \bigcap H_2$.
- The subfield $F_1 \bigcap F_2$ corresponds to H_1H_2.

3.2.2 Examples

1. $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/(n)^\times$ and is generated by maps of the form $\zeta_n \mapsto \zeta_n^j$ where $(j, n) = 1$.

I.e., the following map is an isomorphism:

$$\begin{aligned}\mathbb{Z}/(n)^\times &\to \operatorname{Gal}(\mathbb{Q}(\zeta_n), \mathbb{Q}) \\ r \mod n &\mapsto (\phi_r : \zeta_n \mapsto \zeta_n^r).\end{aligned}$$

2. $\operatorname{Gal}(\mathbb{GF}(p^n)/\mathbb{F}_p) \cong \mathbb{Z}/(n)$, a cyclic group generated by powers of the Frobenius automorphism:

$$\begin{aligned}\varphi_p : \mathbb{GF}(p^n) &\to \mathbb{GF}(p^n) \\ x &\mapsto x^p.\end{aligned}$$

Lemma: Every quadratic extension is Galois.

Lemma: If K is the splitting field of an irreducible polynomial of degree n, then $\operatorname{Gal}(K/\mathbb{Q}) \leq S_n$ is a transitive subgroup.

Corollary: n divides the order $|\operatorname{Gal}(K/\mathbb{Q})|$.

Definition: TFAE

- k is a **perfect** field.
- Every irreducible polynomial $p \in k[x]$ is separable
- Every finite extension F/k is separable.
- If char $k > 0$, the Frobenius is an automorphism of k.

Theorem:

- If char $k = 0$ or k is finite, then k is perfect.
- $k = \mathbb{Q}, \mathbb{F}_p$ are perfect, and any finite normal extension is Galois.
- Every splitting field of a polynomial over a perfect field is Galois.

Lemma (Composite Extensions): If F/k is finite and Galois and L/k is arbitrary, then FL/L is Galois and

$$\operatorname{Gal}(FL/L) = \operatorname{Gal}(F/F \bigcap L) \subset \operatorname{Gal}(F/k).$$

3.3 Cyclotomic Polynomials

Definition: Let $\zeta_n = e^{2\pi i/n}$, then

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (j,n)=1}}^{n} \left(x - \zeta_n^k\right),$$

which is a product over primitive roots of unity.

Lemma: $\deg \Phi_n(x) = \phi(n)$ for ϕ the totient function.

Computing Φ_n:

1.

$$\Phi_n(z) = \prod_{d|n, d>0} \left(z^d - 1\right)^{\mu\left(\frac{n}{d}\right)}$$

where

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \end{cases}$$

2.

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \implies \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d<n}} \Phi_d(x)},$$

so just use polynomial long division.

Lemma:

$$\begin{aligned} \Phi_p(x) &= x^{p-1} + x^{p-2} + \cdots + x + 1 \\ \Phi_{2p}(x) &= x^{p-1} - x^{p-2} + \cdots - x + 1. \end{aligned}$$

Lemma:

$$k \mid n \implies \Phi_{nk}(x) = \Phi_n\left(x^k\right)$$

Definition: An extension F/k is **simple** if $F = k[\alpha]$ for a single element α.

Theorem (Primitive Element): Every finite separable extension is simple.

Corollary: $\mathbb{GF}(p^n)$ is a simple extension over $\mathbb{F}_p$.

4 Modules

4.1 General Modules

Definition: A module is **simple** iff it has no nontrivial proper submodules.

Definition: A **free** module is a module with a basis (i.e. a spanning, linearly independent set).

> *Example*: $\mathbb{Z}/(6)$ is a $\mathbb{Z}$-module that is *not* free.

Definition: A module M is **projective** iff M is a direct summand of a free module $F = M \oplus \cdots$.

> Free implies projective, but not the converse.

Definition: A sequence of homomorphisms $0 \xrightarrow{d_1} A \xrightarrow{d_2} B \xrightarrow{d_3} C \to 0$ is *exact* iff $\operatorname{im} d_i = \ker d_{i+1}$.

Lemma: If $0 \to A \to B \to C \to 0$ is a short exact sequence, then

- C free $\implies$ the sequence splits
- C projective $\implies$ the sequence splits
- A injective $\implies$ the sequence splits

Moreover, if this sequence splits, then $B \cong A \oplus C$.

4.2 Classification of Modules over a PID

Let M be a finitely generated modules over a PID R. Then there is an invariant factor decomposition

$$M \cong F \bigoplus R/(r_i) \quad \text{where } r_1 \,\Big|\, r_2 \,\Big|\, \cdots,$$

and similarly an elementary divisor decomposition.

4.3 Minimal / Characteristic Polynomials

Fix some notation:

$$\begin{aligned} \min_A(x) :& \quad \text{The minimal polynomial of } A \\ \chi_A(x) :& \quad \text{The characteristic polynomial of } A. \end{aligned}$$

Definition: The minimal polynomial is the unique polynomial $\min_A(x)$ of minimal degree such that $\min_A(A) = 0$.

Definition: The **characteristic polynomial** of A is given by

$$\chi_A(x) = \det(A - xI)) = \det(SNF(A - xI)).$$

Useful lemma: If A is upper triangular, then $\det(A) = \prod_i a_{ii}$

Theorem (Cayley-Hamilton): The minimal polynomial divides the characteristic polynomial, and in particular $\chi_A(A) = 0$.

Lemma: Writing

$$\begin{aligned} \min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i} \end{aligned}$$

- $a_i \leq b_i$
- The roots both polynomials are precisely the eigenvalues of A.

Proof: By Cayley-Hamilton, $\min_A$ divides χ_A. Every λ_i is a root of μ_M:
Let $(\mathbf{v}_i, \lambda_i)$ be a nontrivial eigenpair. Then by linearity,

$$\min_A(\lambda_i)\mathbf{v}_i = \min_A(A)\mathbf{v}_i = \mathbf{0},$$

which forces $\min_A(\lambda_i) = 0$.

Definition: Two matrices A, B are **similar** (i.e. $A = PBP^{-1}$) $\iff$ A, B have the same Jordan Canonical Form (JCF).

Definition: Two matrices A, B are **equivalent** (i.e. $A = PBQ$) $\iff$

- They have the same rank,
- They have the same invariant factors, *and*
- They have the same (JCF)

Finding the minimal polynomial:

Let $m(x)$ denote the minimal polynomial A.

1. Find the characteristic polynomial $\chi(x)$; this annihilates A by Cayley-Hamilton. Then $m(x) \mathrel{\Big|} \chi(x)$, so just test the finitely many products of irreducible factors.
2. Pick any $\mathbf{v}$ and compute $T\mathbf{v}, T^2\mathbf{v}, \cdots T^k\mathbf{v}$ until a linear dependence is introduced. Write this as $p(T) = 0$; then $\min_A(x) \mathrel{\Big|} p(x)$.

Definition: Given a monic $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$, the **companion matrix** of p is given by

$$C_p \coloneqq \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

4.4 Canonical Forms

4.4.1 Rational Canonical Form

Corresponds to the **Invariant Factor Decomposition** of T.

Lemma: $RCF(A)$ is a block matrix where each block is the companion matrix of an invariant factor of A.

Derivation:

- Let $k[x] \curvearrowright V$ using T, take invariant factors a_i,
- Note that $T \curvearrowright V$ by multiplication by x
- Write $\overline{x} = \pi(x)$ where $F[x] \xrightarrow{\pi} F[x]/(a_i)$; then span $\{\overline{x}\} = F[x]/(a_i)$.
- Write $a_i(x) = \sum b_i x^i$, note that $V \to F[x]$ pushes $T \curvearrowright V$ to $T \curvearrowright k[x]$ by multiplication by $\overline{x}$
- WRT the basis $\overline{x}$, T then acts via the companion matrix on this summand.
- Each invariant factor corresponds to a block of the RCF.

4.4.2 Jordan Canonical Form

Corresponds to the **Elementary Divisor Decomposition** of T.

Lemma: The elementary divisors of A are the minimal polynomials of the Jordan blocks.

Lemma: Writing

$$\min_A(x) = \prod (x - \lambda_i)^{a_i}$$
$$\chi_A(x) = \prod (x - \lambda_i)^{b_i}$$

- $a_i \leq b_i$
- a_i tells you the size of the **largest** Jordan block associated to λ_i,
- b_i is the **sum of sizes** of all Jordan blocks associated to λ_i
- $\dim E_{\lambda_i}$ is the **number of Jordan blocks** associated to λ_i

4.5 Using Canonical Forms

Lemma: The characteristic polynomial is the *product of the invariant factors*, i.e.

$$\chi_A(x) = \prod_{j=1}^{n} f_j(x).$$

Lemma: The minimal polynomial of A is the *invariant factor of highest degree*, i.e.

$$\min_A(x) = f_n(x).$$

Lemma: For a linear operator on a vector space of nonzero finite dimension, TFAE:

- The minimal polynomial is equal to the characteristic polynomial.
- The list of invariant factors has length one.
- The Rational Canonical Form has a single block.
- The operator has a matrix similar to a companion matrix.
- There exists a *cyclic vector* $\mathbf{v}$ such that $\operatorname{span}_k \left\{ T^j \mathbf{v} \;\middle|\; j = 1, 2, \cdots \right\} = V$.
- T has $\dim V$ distinct eigenvalues

4.6 Diagonalizability

Notation: A^* denotes the conjugate tranpose of A.

Lemma: Let V be a vector space over k an algebraically closed and $A \in \operatorname{End}(V)$. Then if $W \subseteq V$ is an invariant subspace, so $A(W) \subseteq W$, the A has an eigenvector in W.

Theorem (The Spectral Theorem):

1. Hermitian matrices (i.e. $A^* = A$) are diagonalizable over $\mathbb{C}$.
2. Symmetric matrices (i.e. $A^t = A$) are diagonalizable over $\mathbb{R}$.

Proof: Suppose A is Hermitian. Since V itself is an invariant subspace, A has an eigenvector $\mathbf{v}_1 \in V$. Let $W_1 = \text{span}_k \{\mathbf{v}_1\}^\perp$. Then for any $\mathbf{w}_1 \in W_1$,

$$\langle \mathbf{v}_1,\ A\mathbf{w}_1 \rangle = \langle A\mathbf{v}_1,\ \mathbf{w}_1 \rangle = \lambda \langle \mathbf{v}_1,\ \mathbf{w}_1 \rangle = 0,$$

so $A(W_1) \subseteq W_1$ is an invariant subspace, etc.
Suppose now that A is symmetric. Then there is an eigenvector of norm 1, $\mathbf{v} \in V$.

$$\lambda = \lambda \langle \mathbf{v},\ \mathbf{v} \rangle = \langle A\mathbf{v},\ \mathbf{v} \rangle = \langle \mathbf{v},\ A\mathbf{v} \rangle = \overline{\lambda} \implies \lambda \in \mathbb{R}.$$

Lemma: $\{A_i\}$ pairwise commute $\iff$ they are all simultaneously diagonalizable.

Proof: By induction on number of operators

- A_n is diagonalizable, so $V = \bigoplus E_i$ a sum of eigenspaces
- Restrict all $n-1$ operators A to E_n.
- The commute in V so they commute in E_n
- **(Lemma)** They were diagonalizable in V, so they're diagonalizable in E_n
- So they're simultaneously diagonalizable by I.H.
- But these eigenvectors for the A_i are all in E_n, so they're eigenvectors for A_n too.
- Can do this for each eigenspace. ■

Full details here

Theorem (Characterizations of Diagonalizability)

M is diagonalizable over $\mathbb{F}$ $\iff$ $\min_M(x, \mathbb{F})$ splits into distinct linear factors over $\mathbb{F}$, or equivalently iff all of the roots of $\min_M$ lie in $\mathbb{F}$.

Proof: $\implies$: If $\min_A$ factors into linear factors, so does each invariant factor, so every elementary divisor is linear and $JCF(A)$ is diagonal.
$\impliedby$: If A is diagonalizable, every elementary divisor is linear, so every invariant factor factors into linear pieces. But the minimal polynomial is just the largest invariant factor.

4.7 Matrix Counterexamples

1. A matrix that is:

- Not diagonalizable over $\mathbb{R}$ but diagonalizable over $\mathbb{C}$
- No eigenvalues in $\mathbb{R}$ but distinct eigenvalues over $\mathbb{C}$
- $\min_M(x) = \chi_M(x) = x^2 + 1$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \left[\begin{array}{r|r} -1\sqrt{-1} & 0 \\ \hline 0 & 1\sqrt{-1} \end{array}\right].$$

2.

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Not diagonalizable over $\mathbb{C}$

- Eigenvalues $[1, 1]$ (repeated, multiplicity 2)
- $\min_M(x) = \chi_M(x) = x^2 - 2x + 1$

3. Non-similar matrices with the same characteristic polynomial

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

4. A full-rank matrix that is not diagonalizable:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Matrix roots of unity:

$$\sqrt{I_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\sqrt{-I_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

4.8 Miscellaneous

Lemma: $I \trianglelefteq R$ is a free R-module iff I is a principal ideal.

Proof: $\Longrightarrow$:
Suppose I is free as an R-module, and let $B = \{\mathbf{m}_j\}_{j \in J} \subseteq I$ be a basis so we can write $M = \langle B \rangle$.
Suppose that $|B| \geq 2$, so we can pick at least 2 basis elements $\mathbf{m}_1 \neq \mathbf{m}_2$, and consider

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1,$$

which is also an element of M .
Since R is an integral domain, R is commutative, and so

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_1\mathbf{m}_2 = \mathbf{0}_M$$

However, this exhibits a linear dependence between $\mathbf{m}_1$ and $\mathbf{m}_2$, namely that there exist $\alpha_1, \alpha_2 \neq 0_R$ such that $\alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 = \mathbf{0}_M$; this follows because $M \subset R$ means that we can take $\alpha_1 = -m_2, \alpha_2 = m_1$. This contradicts the assumption that B was a basis, so we must have $|B| = 1$ and so $B = \{\mathbf{m}\}$ for some $\mathbf{m} \in I$. But then $M = \langle B \rangle = \langle \mathbf{m} \rangle$ is generated by a single element, so M is principal.
$\Longleftarrow$:
Suppose $M \trianglelefteq R$ is principal, so $M = \langle \mathbf{m} \rangle$ for some $\mathbf{m} \neq \mathbf{0}_M \in M \subset R$.
Then $x \in M \implies x = \alpha\mathbf{m}$ for some element $\alpha \in R$ and we just need to show that $\alpha\mathbf{m} = \mathbf{0}_M \implies \alpha = 0_R$ in order for $\{\mathbf{m}\}$ to be a basis for M, making M a free R-module.
But since $M \subset R$, we have $\alpha, m \in R$ and $\mathbf{0}_M = 0_R$, and since R is an integral domain, we have $\alpha m = 0_R \implies \alpha = 0_R$ or $m = 0_R$.
Since $m \neq 0_R$, this forces $\alpha = 0_R$, which allows $\{m\}$ to be a linearly independent set and thus a basis for M as an R-module.

■