Summary/Outline
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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Summary/Outline

Outline

What we're trying to prove:

– 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

– Define

$$L : W^{1,p} \left(\mathbb{R} \times S^1 ; \mathbb{R}^{2n} \right) \longrightarrow L^p \left(\mathbb{R} \times S^1 ; \mathbb{R}^{2n} \right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t) Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$$
$$S(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t).$$

Outline

- Took $R^{\pm} : I \longrightarrow \text{Sp}(2n; \mathbb{R})$: symplectic paths associated to $S^{\pm}$
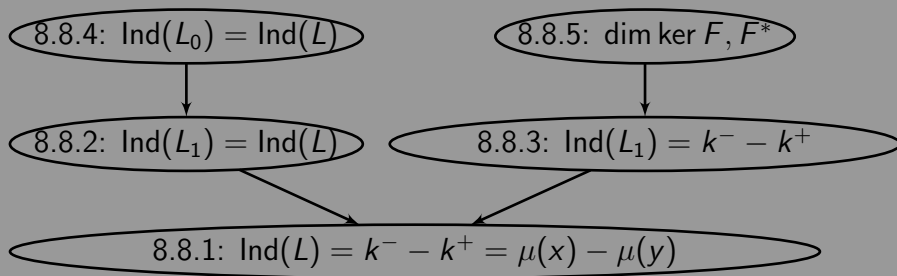
- These paths defined $\mu(x), \mu(y)$

- Section 8.7:

  $$R^{\pm} \in \mathcal{S} := \left\{ R(t) \ \Big| \ R(0) = \text{id}, \ \det(R(1) - \text{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

  $$\text{Ind}(L) \stackrel{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

Summary/Outline
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8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$
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From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know $\mathrm{Ind}(L) = \mathrm{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing $\mathrm{Ind}(L_1)$ by computing kernels.

Summary/Outline
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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$$8.8.3: \ \text{Ind}(L_1) = k^- - k^+$$

Summary/Outline
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8.8.3: Ind(L_1) = $k^- - k^+$
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8.8.5: dim ker F, F^*
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Recall

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

$$L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(s)^t Z$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Summary/Outline
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8.8.3: $\mathsf{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

– Since coker $L_1 \cong \ker L_1^*$, it suffices to compute $\ker L_1^*$

– We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \mathrm{Mat}(2; \mathbb{R}).$$

– This allows us to reduce to the $n = 1$ case.

Summary/Outline
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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L_1 used a path of diagonal matrices constant near ∞:

$$S(s) := \left(\begin{array}{cc} a_1(s) & 0 \\ 0 & a_2(s) \end{array} \right), \quad \text{with } a_i(s) := \left\{ \begin{array}{ll} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{array} \right. .$$

Summary/Outline
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8.8.3: $\mathsf{Ind}(L_1) = k^- - k^+$
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8.8.5: dim ker F, F^*
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Statement of Lemma

Let $p > 2$ and define

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y.$$

Note: this seems to be L_1 for $n = 1$:

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y.$$

Summary/Outline
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8.8.3: Ind(L_1) = k^- − k^+
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8.8.5: dim ker F, F^*
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Statement of Lemma

1. Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \operatorname{Ker} F = 2 \cdot \# \left\{ \ell \in \mathbb{Z} \;\middle|\; 2\pi\ell \in (a^-, a+) \right\}$$
$$\dim \operatorname{Ker} F^\star = 2 \cdot \# \left\{ \ell \in \mathbf{Z} \,\middle|\, a^+ < 2\pi\ell < a^- \right\}.$$

2. Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \# \left\{ i \in \{1, 2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$
$$\dim \operatorname{Ker} F^\star = \# \left\{ i \in \{1, 2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Summary/Outline
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8.8.3: Ind(L_1) = k^- − k^+
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8.8.5: dim ker F, F^*
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8.8.3: $\text{Ind}(L_1) = k^- - k^+$
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Summary/Outline
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8.8.3: Ind(L_1) = k^- − k^+
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Summary/Outline
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8.8.3: Ind(L_1) = k^- − k^+
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