

# Problem Set 7

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## 1 Regular Problems

### 1.1 Problem 1

Note that if either  $p = 1$  or  $q = 1$ ,  $G$  is a  $p$ -group, which is a nontrivial center that is always normal. So assume  $p \neq 1$  and  $q \neq 1$ .

We want to show that  $G$  has a non-trivial normal subgroup. Noting that  $\#G = p^2q$ , we will proceed by showing that either  $n_p$  or  $n_q$  must be 1.

We immediately note that

$$\begin{array}{ll} n_p \equiv 1 \pmod{p} & n_q \equiv 1 \pmod{q} \\ n_p \mid q & n_q \mid p^2, \end{array}$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \{1, p, p^2\}.$$

If either  $n_p = 1$  or  $n_q = 1$ , we are done, so suppose  $n_p \neq 1$  and  $n_1 \neq 1$ . This forces  $n_p = q$ , and we proceed by cases:

#### 1.1.1 Case 1: $p = q$ .

Then  $\#G = p^3$  and  $G$  is a  $p$ -group. But every  $p$ -group has a non-trivial center  $Z(G) \leq G$ , and the center is always a normal subgroup.

### 1.1.2 Case 2: $p > q$ .

Here, since  $n_p \mid q$ , we must have  $n_p < q$ . But if  $n_p < q < p$  and  $n_p = 1 \pmod p$ , then  $n_p = 1$ .

### 1.1.3 Case 3: $q > p$ .

Since  $n_p \neq 1$  by assumption, we must have  $n_p = q$ . Now consider sub-cases for  $n_q$ :

- $n_q = p$ : If  $n_q = p = 1 \pmod q$  and  $p < q$ , this forces  $p = 1$ .
- $n_q = p^2$ : We will reach a contradiction by showing that this forces

$$\left| P := \bigcup_{S_p \in \text{Syl}(p, G)} S_p \setminus \{e\} \right| + \left| Q := \bigcup_{S_q \in \text{Syl}(q, G)} S_q \setminus \{e\} \right| + |\{e\}| > |G|.$$

We have

$$\begin{aligned} |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2-1) + 1 \\ &= p^2(q-1) + q(p^2-1) + 1 \\ &= p^2(q-1) + 1(p^2-1) + (q-1)(p^2-1) + 1 && (\text{since } q > 1) \\ &= (p^2q - p^2) + (p^2-1) + (q-1)(p^2-1) + 1 \\ &= p^2q + (q-1)(p^2-1) \\ &\geq p^2q + (2-1)(2^2-1) && (\text{since } p, q \geq 2) \\ &= p^2q + 3 \\ &> p^2q = |G|, \end{aligned}$$

which is a contradiction.

## 2 Qual Problems