# Title

D. Zack Garza

Thursday 20th August, 2020

## Contents



## 1 Thursday, August 20

References: `https://www.daniellitt.com/tale-cohomology`

Prerequisites:

- Homological Algebra
  - Abelian Categories
  - Derived Functors
  - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
  - Etale morphism
  - Grothendieck topologies
  - The etale topology
  - Etale cohomology and the basis theorems
  - Etale cohomology of curves
  - Comparison theorems to singular cohomology
  - Focused on the case where coefficients are a constructible sheaf.
- Prove the Weil Conjectures (more than one proof)
  - Proving the Riemann Hypothesis for varieties over finite fields

  One of the greatest pieces of 20th century mathematics!
- Topics
  - Weil 2 (Strengthening of RH, used in practice)
  - Formality of algebraic varieties (topological features unique to varieties)
  - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose $X/\mathbb{C}$ is a quasiprojective variety: a finite type separated integral $\mathbb{C}$-scheme.

If you take the complex points, it naturally has the structure of a complex analytic space $X(\mathbb{C})^{\text{an}}$: you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology $H^i(X(\mathbb{C})^{\text{an}}, \mathbb{Z})$, which satisfies several nice properties:

- Finitely generated $\mathbb{Z}$-modules
- Extra Hodge structure when tensored up to $\mathbb{C}$ (same as $\mathbb{C}$ coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general "nice" schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We'll associate $X$ a "nice scheme" $\rightsquigarrow H^i(X_{\text{et}}, \mathbb{Z}/\ell^n\mathbb{Z})$. Take the inverse limit over all $n$ to obtain the $\ell$-adic cohomology $H^i(X_{\text{et}}, \mathbb{Z}_\ell)$. You can tensor with $\mathbb{Q}$ to get something with $\mathbb{Q}_\ell$ coefficients. And as in singular cohomology, you can a "twisted coefficient system".

What are nice schemes:

- $X = \text{Spec } \mathcal{O}_k$, the ring of integers over a number field.
- $X$ a variety over an algebraically closed field
  - Typical, most analogous to taking a variety over $\mathbb{C}$.
- $X$ a variety over a non-algebraically closed field

Some comparisons between the last two cases:

- For $\mathbb{C}$- variety, $H^i_{\text{sing}}$ will vanish above $i = 2d$.
- Over a finite field, $H^i$ will vanish for $i > 2d+1$ but generally not vanish for $i = 2d+1$.

In good situations, these are finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, have Mayer-Vietoris and excision sequences, spectral sequences, etc.

Related invariants: for a scheme with a geometric point $(X, \bar{x}) \rightsquigarrow \pi_1^{\text{étale}}(X, \bar{x})$, which is a profinite topological group, which is a profinite topological group.

Note: a geometric point is a map from Spec $X$ to an algebraically closed field.

More invariants beyond the scope of this course:

- Higher homotopy groups
- Homotopy type (equivalence class of spaces)

So we want homotopy-theoretic invariants for varieties.

**Remark 1.**
This cohomology theory is necessarily weird!

**Theorem 1.1** ***(Serre).***
There does not exists a cohomology theory for schemes over $\bar{\mathbb{F}}_q$ with the following properties:

1. Functorial
2. Satisfies the Kunneth formula
3. For $E$ an elliptic curve, $H^1(E) = \mathbb{Q}^2$.

Slogan: No cohomology theory with $\mathbb{Q}$ coefficients.

*Proof .*
Take $E$ to be a supersingular elliptic curve. Then $\mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra.
Fact: There are no algebra morphisms $R \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$

**Exercise .**
Functoriality and Kunneth implies that $\mathrm{End}(E) \curvearrowright E$ yields an action on $H^1(E)$, which is precisely an algebra morphism $\mathrm{End}(E) \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$, a contradiction.

The content: the sum of two endomorphisms act via their sum on $H^1$.

**Exercise .**
Prove the same thing for $\mathbb{Q}_p$ coefficients, where $p$ divides the characteristic of the ground field.

Proof the same, just need to know what quaternion algebras show up.

■

This forces using some funky type of coefficients.

What are the Weil Conjectures?

Suppose $X/\mathbb{F}_q$ is a variety, then

$$\zeta_X(t) = \exp\left(\sum_{n>0} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n\right).$$

Some comments:

- $\dfrac{\partial}{\partial t} \log \zeta_X(t)$ is an ordinary generating function for the number of rational points.
- Slogan: locations of zeros and poles of a meromorphic function control the growth rate of the coefficients of the Taylor series of the logarithmic derivative.

**Exercise 1.3.**
Make this slogan precise for rational functions, i.e. ratios of two polynomials.

The conjectures:

1. $\zeta_x(t)$ is a rational function.
2. (Functional equation) For $X$ smooth and proper

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t).$$

3. (RH) All roots and poles of $\zeta_X(t)$ have absolute value $q^{\frac{i}{2}}$ with $i \in \mathbb{Z}$, and these are equal to the $i$th Betti numbers if $X$ lifts to characteristic zero.

Note: we'll generalize betti numbers so this makes sense in general.

All theorems! Proofs:

1. Dwork, using $p$-adic methods. Proof here will follow from the fact that $H^i_{\text{étale}}$ are finite-dimensional. Related to Lefschetz Trace Formula (how Grothendieck thought about it).
2. Grothendieck, follows from some version of Poincaré duality.
3. (and 4) Deligne.

Euler Product:

Let $|X|$ denote the closed points of $X$, then there is an Euler product:

$$\begin{aligned}
\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t) &= \prod_{x \in |X|} \exp\left( t^{\deg(x)} + \frac{t^{2\deg(x)}}{2} + \cdots \right) \\
&= \prod_{x \in |X|} \exp\left( -\log(1 - t^{\deg(x)}) \right) \\
&= \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}.
\end{aligned}$$

**Exercise 1.4.**
Check the first equality. If you have a point of $\deg(x) = n$, how many $\mathbb{F}_{q^n}$ points does this contribute? I.e., how many maps are there $\text{Spec}\,(\mathbb{F}_{q^n}) \longrightarrow \text{Spec}\,(\mathbb{F}_{1^n})$ over $\mathbb{F}_q$?

There are exactly $n$: it's $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. But then division by $n$ drops this contribution to one.

We can keep going by expanding and multiplying out the product:

$$\begin{aligned}
\prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}} &= \prod_{x \in |X|} (1 + t^{\deg(x)} + t^{2\deg(x)}) \\
&= \sum_{n \geq 0} \left( \# \text{ of Galois-stable subset of } X(\overline{\mathbb{F}}_{q^n}) \text{ of size } n \right) t^n.
\end{aligned}$$

Why? If you have a degree $x$ point, it contributes a stable subset of size $x$: namely the $\mathbb{F}_{q^n}$ points of $\mathbb{F}_{q^n}$. Taking Galois orbits like that correspond to multiplying this product.

But these are the points of some algebraic variety:

$$\cdots = \sum_{n \geq 0} |\text{Sym}^n(X)(\mathbb{F}_q)| t^n,$$

where $\text{Sym}^n(X) \coloneqq X^n/\Sigma_n$, the action of the symmetric group. Why is that? A $\overline{\mathbb{F}}_q$ point of $\text{Sym}^n(X)$ is an unordered $n$-tuple of $\overline{\mathbb{F}}_q$ points without an ordering, and asking them to be Galois stable is the same as saying that they are an $\mathbb{F}_q$ point.

**Theorem 1.2** ***(First Weil Conjecture for Curves).***
For $X$ a smooth proper curve over $\mathbb{F}_q$, $\zeta_X(t)$ is rational.

*Proof* .
Claim: there is a set map

$$\mathrm{Sym}^n X \longrightarrow \mathrm{Pic}^{\,n} X$$
$$D \mapsto \mathcal{O}(D).$$

Here the divisor is an $n$-tuple of points.

What are the fibers over a line bundle $\mathcal{O}(D)$? A linear system, i.e. the projectivization of global sections $\mathbb{P}\Gamma(X, \mathcal{O}(D))$. What is the expected dimension? To compute the dimension of the space of line bundles on a curve, use Riemann-Roch:

$$\dim \mathbb{P}\Gamma(X, \mathcal{O}(D)) = \deg(D) + 1 - g + \dim H^1(X, \mathcal{O}(D)) - 1.$$

where the last $-1$ comes from the fact that this is a projective space.
Claim: if $\deg(D) = 2g - 2$, then $H^1(X, \mathcal{O}(D)) = 0$.
This is because it's dual to $H^0(X, \mathcal{O}(K - D))^\vee$, but this has negative degree and a line bundle of negative degree can never have sections.

Note: should check to make sure you know why this is true!

Thus the fibers are isomorphic to $\mathbb{P}^{n-g}$ for $n > 2g - 2$. Now make a reduction (exercise: justify why):
Wlog assume $X(\mathbb{F}_q) \neq \emptyset$. In this case, $\mathrm{Pic}^{\,n}(X) \cong \mathrm{Pic}^{\,n+1}(X)$ for all $n$, since you can take $\mathcal{O}(P)$ for $P$ a point, a degree 1 line bundle, and tensor with it. It's an isomorphism because you can tensor with the dual bundle to go back.
Thus for all $n > 2g - 2$,

$$|\mathrm{Sym}^n(X)(\mathbb{F}_q)| = |\mathbb{P}^{n-g}(\mathbb{F}_q)| \cdot |\mathrm{Pic}^{\,n}(X)(\mathbb{F}_q)| = |\mathbb{P}^{n-g}(\mathbb{F}_q)| \cdot \left|\mathrm{Pic}^{\,0}(X)(\mathbb{F}_q)\right|.$$

Thus $\zeta_X(t)$ is a polynomial plus $\displaystyle\sum_{n>2g-2} |\mathrm{Pic}^{\,n}(X)(\mathbb{F}_q)| \left(1 + q + q^2 + \cdots + q^{n-g}\right) t^n$.

**Exercise .**
Show that this is a rational function using the formula for a geometric series.

■

**Theorem 1.3** ***(Functional Equation).***
The functional equation in the case of curves:

$$\zeta_X(q^{-1}t^{-1}) = \pm q^{\frac{2-2g}{2}} t^{2-2g} \zeta_X(t).$$

**Exercise 1.6** (Important)**.**
Where it comes from in terms of $\mathrm{Sym}^n$: Serre duality.

We'll show the RH later.

**Theorem 1.4** ***(Dwork).***
Suppose $X/\mathbb{F}_q$ is any variety, then $\zeta_X(t)$ is rational function.

Roughly known to Weil, hinted at in original paper

*Proof (Grothendieck).*
Idea: take Frobenius (intentionally vague, arithmetic vs geometric vs ...) $F : X \longrightarrow X$, then $X(\mathbb{F}_q)$ are the fixed points of $F$ acting on $X_{\bar{\mathbb{F}}_q}$, and the $\mathbb{F}_{q^n}$ points are the fixed points of $F^n$. Uses the Lefschetz fixed point formula, which will say for $\ell \neq \text{char } (\mathbb{F}_q)$,

$$|X(\mathbb{F}_{q^n})| = \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}(F^n) H_c^i(X_{\mathbb{F}_q}, \mathbb{Q}_\ell).$$

Here $H_c^i$ is compactly supported cohomology, we'll define this later in the course.

**Lemma 1.5.**

$$\exp\left( \sum \frac{\mathrm{Tr}(F^n)}{n} t^n \right) \quad \text{is rational.}$$

This lemma implies the result, because if you plug the trace formula into the zeta function, you'll get an alternating product of functions of the form in the lemma, which is still rational.

■