Title

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1.1 1.a

Proof ($A \implies B$).

- Suppose $\{a_n\}$ is not bounded above.
- Then any $k \in \mathbb{N}$ is not an upper bound for $\{a_n\}$.
- So choose a subsequence $a_{n_k} > k$, then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k \longrightarrow \infty} a_{n_k} > \liminf_{k \longrightarrow \infty} k = \infty.$$

Note that $\lim_{n \longrightarrow \infty} a_n$ need not exist, but liminf/limsup always exist.

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Proof ($\not A \implies \not B$).

- Suppose $\{a_n\}$ is bounded by M, so $a_n < M < \infty$ for all $n \in \mathbb{N}$.
- Then if $\{a_{n_k}\}$ is a subsequence, we have $a_{n_k} \in \{a_n\}$, so $a_{n_k} < M$ for all $k \in \mathbb{N}$.
- But then

$$a_{n_k} < M \implies \limsup_{k \longrightarrow \infty} a_{n_k} \leq M,$$

- Now note that if $\lim_{k \longrightarrow \infty} a_{n_k}$ exists,

$$\lim_{k \longrightarrow \infty} a_{n_k} < \limsup_{k \longrightarrow \infty} a_{n_k} \leq M < \infty,$$

so every subsequence is bounded and thus can not converge to ∞.

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1.2 3.a

Proof .

- Suppose $x_n \leq M$ for all n and let $S \coloneqq \liminf_{n \longrightarrow \infty} x_n$ be the infimum of subsequential limits.
- Let $\{x_{n_k}\}$ be an arbitrary convergent subsequence, then for every k we have $x_{n_k} \in \{x_n\}$, so $|x_{n_k}| \leq M$.
- By order limit laws,

$$|x_{n_k}| \leq M \implies \lim_{k \longrightarrow \infty} |x_{n_k}| \leq M.$$

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Proof (Using definition (ii)).

- Suppose $|x_n| \leq M$ for every n.
- Let $\{x_{n_k}\}$ be an arbitrary subsequence, then since $x_{n_k} \in \{x_n\}$ for all k, $|x_{n_k}| \leq M$ for all k.
- By order-limit laws, for every fixed n we have

$$|x_{n_k}| \leq M \implies \inf_{k>n} |x_{n_k}| \leq M.$$

- Again applying order limit laws,

$$\inf_{k>n} |x_{n_k}| \leq M \implies \lim_{n \longrightarrow \infty} \inf_{k>n} |x_{n_k}| \leq M.$$

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