

# Title

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# 1 | Friday, September 25

## 1.1 Compact-Open Topology

- For  $X, Y$  topological spaces, consider

$$Y^X = C(X, Y) = \text{hom}_{\text{Top}}(X, Y) := \{f : X \rightarrow Y \mid f \text{ is continuous}\}.$$

- General idea: it's nice to *cartesian closed* categories, which require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it (i.e. here we'd like the homs to again be spaces).
- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.
  - \* Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.
  - \* Compactly generated: sets are closed iff their intersection with every compact subspace is closed.
- Topologize with the *compact-open* topology:  $U \in \text{hom}_T(X, X)$  open iff for every  $f \in U$ ,  $f(K)$  is open for every compact  $K \subseteq X$ .
  - \* If  $Y = (Y, d)$  is a metric space, this is the topology of “uniform convergence on compact sets”: for  $f_n \rightarrow f$  in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \{d(f_n(x), f(x)) \mid x \in K\} \xrightarrow{n \rightarrow \infty} 0 \quad \forall K \subseteq X \text{ compact}.$$

In words:  $f_n \rightarrow f$  uniformly on every compact set.

- If  $X$  itself is compact and  $Y$  is a metric space,  $C(X, Y)$  can be promoted to a metric space with  $d(f, g) = \sup_{x \in X} (f(x), g(x))$ .
- Useful in analysis: when is a family of functions  $\mathcal{F} = \{f_\alpha\} \subset \text{hom}_{\text{Top}}(X, Y)$  compact? Essentially answered by Arzela-Ascoli

**Theorem 1.1 (Ascoli).**

If  $X$  is locally compact Hausdorff and  $(Y, d)$  is a metric space, a family  $\mathcal{F} \subset \text{hom}_{\text{Top}}(X, Y)$  has compact closure  $\iff \mathcal{F}$  is equicontinuous and  $F_x := \{f(x) \mid f \in \mathcal{F}\} \subset Y$  has compact closure.

**Corollary 1.2 (Arzela).**

If  $\{f_n\} \subset \text{hom}_{\text{Top}}(X, Y)$  is an equicontinuous sequence and  $F_x := \{f_n(x)\}$  is bounded for every  $x$ , it contains a uniformly convergent subsequence.

- Useful in Number Theory / Rep Theory / Fourier Series: can take  $G$  to be a locally compact abelian topological group and define its Pontryagin dual  $\hat{G} := \text{hom}_{\text{TopGrp}}(G, S^1)$  where we consider  $S^1 \subset \mathbb{C}$ .
  - \* Can integrate with respect to the Haar measure  $\mu$ , define  $L^p$  spaces, and for  $f \in L^p(G)$  define a Fourier transform  $\hat{f} \in L^p(\hat{G})$ .

$$\hat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

- So define

$$\text{Map}(X, Y) := (\text{hom}_{\text{Top}}(X, Y), \mathcal{O}_{\text{CO}}) \quad \text{where } \mathcal{O}_{\text{CO}} \text{ is the compact-open topology.}$$

$\text{Map}(X, Y) = \text{hom}_{\text{Top}}(X, Y)$  equipped with the compact-open topology.

- Can immediately consider some interesting spaces via the functor  $\text{Map}(\cdot, Y)$ :

$$\begin{aligned} X = \{\text{pt}\} &\rightsquigarrow \text{Map}(\{\text{pt}\}, Y) \cong Y \\ X = I &\rightsquigarrow \mathcal{P}Y := \{f : I \rightarrow Y\} = Y^I \\ X = S^1 &\rightsquigarrow \mathcal{L}Y := \{f : S^1 \rightarrow Y\} = Y^{S^1}. \end{aligned}$$

Note: take basepoints to obtain the base path space  $PY$ , the based loop space  $\Omega Y$ .

- Importance in homotopy theory: the path space fibration  $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$  (plays a role in “homotopy replacement”, allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property: there is a homeomorphism

$$\begin{aligned} \text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, \text{Map}(X, Y)) \\ H : X \times Z &\rightarrow Y \iff \tilde{H} : Z \rightarrow \text{Map}(X, Y) \\ (x, z) &\mapsto H(x, z) \iff z \mapsto H(\cdot, z). \end{aligned}$$

Categorically,  $\text{hom}(X, \cdot) \leftrightarrow (X \times \cdot)$  form an adjoint pair in  $\text{Top}$ .

A form of this adjunction holds in any cartesian closed category.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \text{Map}(X, Y) = \{[f], \text{homotopy classes of maps } f : X \rightarrow Y\},$$

i.e. two maps  $f, g$  are homotopic  $\iff$  they are connected by a path in  $\text{Map}(X, Y)$ .

\* Proof:

$$\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, \text{Map}(X, Y)) \cong \text{Map}(Y \times I, X),$$

and just check that  $\gamma(0) = f \iff H(x, 0) = f$  and  $\gamma(1) = g \iff H(x, 1) = g$ .

- \* Note that we can interpret the RHS as the space of paths
- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor*  $\text{Map}(I, \cdot)$ : define

$$\text{Map}_I^1(X, Y) := \text{Map}(I, \text{Map}(X, Y)) = \mathcal{P}\text{Map}(X, Y)$$

and then

$$\begin{aligned} \text{Map}_I^2(X, Y) &:= \text{Map}(I, \text{Map}_I^1(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(I, \text{Map}(X, Y))) = \mathcal{P}(\mathcal{P}(X, Y)) \\ &\cong \text{Map}(I, \text{Map}(Y \times I, X)) \\ &:= \mathcal{P}\text{Map}(Y \times I, X). \end{aligned}$$

Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

## 1.2 Self-Homeomorphisms

- Now restrict attention to

$$\text{Homeo}(X) := \left\{ f \in \text{Map}(X, X) \mid f \text{ is invertible} \right\}.$$

- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.