Problem Set 2

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1 | Exercises

Exercise 1.1 (Gathmann 2.17)**.**
Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

Solution:
Since $x = yz$ for all points in X, we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V\Big(x - yz, y(z^2 - y)\Big) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &\coloneqq X_1 \cup X_2. \end{aligned}$$

Claim: These two subvarieties are irreducible.
It suffices to show that the $A(X_i)$ are integral domains. We have

$$A(X_1) \coloneqq \mathbb{C}[x, y, z]/\left\langle x - yz, y \right\rangle \cong \mathbb{C}[y, z]/\left\langle y \right\rangle \cong \mathbb{C}[z],$$

which is an integral domain since $\mathbb{C}$ is a field and thus an integral domain, and

$$A(X_2) \coloneqq \mathbb{C}[x, y, z]/\left\langle x - yz, z^2 - y \right\rangle \cong \mathbb{C}[y, z]/\left\langle z^2 - y \right\rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

Exercise 1.2 (Gathmann 2.18)**.**
Let $X \subset \mathbb{A}^n$ be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

Solution:

$\overline{X} \subseteq V(I(X))$:
We have $X \subseteq V(I(X))$ and since $V(J)$ is closed in the Zariski topology for any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ by definition, $V(I(X))$ is closed. Thus

$$X \subseteq V(I(X)) \text{ and } V(I(X)) \text{ closed } \implies \overline{X} \subseteq V(I(X)),$$

since $\overline{X}$ is the intersection of all closed sets containing X.

$V(I(X)) \subseteq \overline{X}$:
Noting that $V(\,\cdot\,), I(\,\cdot\,)$ are individually order-reversing, we find that $V(I(\,\cdot\,))$ is order-*preserving* and thus

$$X \subseteq \overline{X} \implies V(I(X)) \subseteq V(I(\overline{X})) = \overline{X},$$

where in the last equality we've used part (i) of the Nullstellensatz: if X is an affine variety, then $V(I(X)) = X$. This applies here because $\overline{X}$ is always closed, and the closed sets in the Zariski topology are precisely the affine varieties.

Exercise 1.3 (Gathmann 2.21)**.**
Let $\{U_i\}_{i\in I} \rightrightarrows X$ be an open cover of a topological space with $U_i \cap U_j \neq \emptyset$ for every i, j.

a. Show that if U_i is connected for every i then X is connected.

b. Show that if U_i is irreducible for every i then X is irreducible.

Solution ***(a):***
Suppose toward a contradiction that $X = X_1 \coprod X_2$ with X_i proper, disjoint, and open. Since $\{U_i\} \rightrightarrows X$, for each $j \in I$ this would force one of $U_j \subseteq X_1$ or $U_j \subseteq X_2$, since otherwise $U_j \cap X_1 \cap X_2$ would be nonempty.
So without loss of generality (relabeling if necessary), assume $U_j \in X_1$ for some fixed j. But then for every $i \neq j$, we have $U_i \cap U_j$ nonempty by assumption, and so in fact $U_i \subseteq X_1$ for every $i \in I$. But then $\cup_{i\in I} U_i \subseteq X_1$, and since $\{U_i\}$ was a cover, this forces $X \subseteq X_1$ and thus $X_2 = \emptyset$.

Solution ***(b):***

Claim: X is irreducible $\iff$ any two open subsets intersect.
This follows because otherwise, if $U, V \subset X$ are open and disjoint then $X \setminus U$, $X \setminus V$ are proper and closed. But then we can write $X = (X \setminus U) \coprod (X \setminus V)$ as a union of proper closed subsets, forcing X to not be irreducible.

So it suffices to show that if $U, V \subset X$ then $U \cap V$ is nonempty. Since $\{U_i\} \rightrightarrows X$, we can find a pair i, j such that there is at least one point in $U \cap U_i$ and one point in $V \cap U_j$.

But by assumption $U_i \cap U_j$ is nonempty, so both $U \cap U_i$ and $U_j \cap U_i$ are open nonempty subsets of U_i. Since U_i was assumed irreducible, they must intersect, so there exists a point

$$x_0 \in (U \cap U_i) \cap (U_j \cap U_i) = U \cap (U_i \cap U_j) := \tilde{U}.$$

We can now similarly note that $\tilde{U} \cap V$ and $U_j \cap V$ are nonempty open subsets of V, and thus intersect. So there is a point

$$\tilde{x}_0 \in \left(\tilde{U} \cap V\right) \cap (U_j \cap V) = \tilde{U} \cap V = U \cap V \cap (U_i \cap U_j),$$

and in particular $\tilde{x}_0 \in U \cap V$ as desired.

Exercise 1.4 (Gathmann 2.22)**.**
Let $f : X \to Y$ be a continuous map of topological spaces.

a. Show that if X is connected then $f(X)$ is connected.

b. Show that if X is irreducible then $f(X)$ is irreducible.

Solution*(a):*
Toward a contradiction, if $f(X) = Y_1 \coprod Y_2$ with Y_1, Y_2 nonempty and open in Y, then

$$f^{-1}(f(X)) \subseteq X$$

on one hand, and

$$f^{-1}(f(X)) = f^{-1}(Y_1) \coprod f^{-1}(Y_2)$$

on the other. If f is continuous, the preimages $f^{-1}(Y_i)$ are open (and nonempty), so X contains a disconnected subset. However, every subset of a connected set must be connected, so this contradicts the connectedness of X.

Solution*(b):*
Suppose $f(X) = Y_1 \cup Y_2$ with Y_i proper closed subsets of Y. Then $f^{-1}(Y_1) \cup f^{-1}(Y^2) = (f^{-1} \circ f)(X) \subseteq X$ are closed in X, since f is continuous. Since X is irreducible, without loss of generality (by relabeling), this forces $X_1 = \emptyset$. But then $f(X_1) = \emptyset$, forcing $f(X) = Y_2$.

Definition 1.0.1 (Ideal Quotient)**.**
For two ideals $J_1, J_2 \trianglelefteq R$, the *ideal quotient* is defined by

$$J_1 : J_2 := \left\{ f \in R \;\middle|\; f J_2 \subset J_1 \right\}.$$

Exercise 1.5 (Gathmann 2.23)**.**
Let X be an affine variety.

a. Show that if $Y_1, Y_2 \subset X$ are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

b. If $J_1, J_2 \trianglelefteq A(X)$ are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

Solution:
?

Exercise 1.6 (Gathmann 2.24)**.**
Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible affine varieties, and show that $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible.

Solution:
That $X \times Y$ is again an affine variety follows from writing $X = V(I)$, $Y = V(J)$, then $X \times Y = V(I + J)$ where $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$. So let

$$X \times Y = U \cup V$$

with U, V proper and closed, and let π_X, π_Y be the projections onto the factors.

Claim: For each $x \in X$, $\pi^{-1}(x) \cong Y$ is contained in only one of U or V.
Note that if this is true, we can write $X = G_U \cup G_V$ where

$$G_U := \left\{ x \in X \;\middle|\; \pi_X^{-1}(x) \subseteq U \right\}$$

are the points for which the entire fiber lies in U, and similarly G_V are those for which the fiber lies in V. If we can then show that G_U, G_V are closed, by irreducibility of X this will force (wlog) $G_V = \emptyset$ and $X = G_U$. But then

$$\pi_X^{-1}(X) = X \times Y \text{ and } \pi_X^{-1}(G_U) = U \implies X \times Y = U.$$

which shows that $X \times Y$ is irreducible.

Proof (Every fiber is contained in one irreducible component).
For any fixed x, we can write

$$\pi_X^{-1}(x) = \left(\pi_X^{-1}(x) \cap U\right) \cup \left(\pi_X^{-1}(x) \cap V\right).$$

Since points are closed in the Zariski topology and π_X is continuous, each $\pi_X^{-1}(x)$ is closed. and thus $\pi_X^{-1}(x) \cap U$ is closed (and similarly for V). Noting that $\pi_X^{-1}(x) \cong \{x\} \times Y \cong Y$, where we've assumed Y to be irreducible, we can conclude wlog that $\pi_X^{-1}(x) \cap V = \emptyset$. ■

Proof (G_U, G_V are closed).
Wlog consider $G_U \subseteq X$. Fixing any point $y_0 \in Y$, we have

$$X \cong X_{y_0} \coloneqq X \times \{y_0\} \subseteq X \times Y,$$

so we can identify $G_U \subset X$ with $G_U \subset X_{y_0}$ inside a Y-fiber the product. But then

$$G_U = X_{y_0} \cap U \subseteq X \times Y,$$

where U is closed in $X \times Y$ and thus closed in X_{y_0}, and X_{y_0} is trivially closed in itself. This exhibits G_U as the intersection of two sets that are closed in $X_{y_0} \cong X$.

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