Title

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Contents

0.1 Exercises

Problem 1.

Let C denote the Cantor set.

1. Show that C contains point that is not an endpoint of one of the removed intervals.
2. Show that C is nowhere dense, meager, and has measure zero.
3. Show that C is uncountable.

Solution 1.

1. First we will characterize the endpoints of the removed intervals. Let C_n be the nth stage of the deletion process that is used to define the Cantor set; then what remains is a union of intervals:
$$C_n = [0, \frac{1}{3^n}] \bigcup [\frac{2}{2^n}, \frac{3}{3^n}] \bigcup \cdots \bigcup [\frac{3^n - 1}{n}, 1],$$
and so the endpoints are precisely the numbers of the form $\frac{k}{3^n}$ where $0 \leq k \leq 3^n$. Moreover, any endpoint appearing in C_n is never removed in any later step, and so all endpoints remaining in C are of this form where we allow $0 \leq n < \infty$.

   Thus, our goal is to produce a number $x \in [0, 1]$ such that $x \neq \frac{k}{3^n}$ for any k or n, but also satisfies $x \in C$.

   Claim: If $x \in C$, then one can find a ternary expansion for which all of the digits are either 0 or 2, i.e.
$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_k \in \{0, 2\}.$$

   Proof: By induction on the index k in a_k, first consider note that if $x \in C$ then $x \in C_1 = [0, 1] \setminus [\frac{1}{3}, \frac{2}{3}] = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$. So if $x \in C_1$, and thus $x \notin [\frac{1}{3}, \frac{2}{3}]$. But note that a_1 is computed in the following way:

$$a_1 = \begin{cases} 0 & 0 \le x < \frac{1}{3}, \\ 1 & \frac{1}{3} \le x < \frac{2}{3}, \\ 2 & \frac{2}{3} \le x < 1. \end{cases}$$

Since the interval $(\frac{1}{3}, \frac{2}{3})$ is deleted in C_1, we find that $a_1 = 1 \iff x = \frac{1}{3}$. In this case, however, we claim that we can find a ternary expansion of x that does not contain a 1. We first write

$$x = \frac{1}{3} = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{where } a_1 = 1, a_{k>1} = 0,$$

and then define

$$x' = \sum_{k=1}^{\infty} b_k 3^{-k} \quad \text{where } b_1 = 0, b_{k>1} = 2.$$

The claim now is that $x = x'$, which follows from the fact that this is a geometric sum that can be written in closed form:

$$\begin{aligned} x' &= \sum_{k=2}^{\infty} (2) 3^{-k} \\ &= \left(\sum_{k=0}^{\infty} (2) 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2 \left(\sum_{k=0}^{\infty} 3^{-k} \right) - 2 - 2(3^{-1}) \\ &= 2(\frac{1}{1 - \frac{1}{3}}) - 2 - 2(3^{-1}) \\ &= 2 \left(\frac{3}{2} \right) - 2 - 2(3^{-1}) \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} = x. \end{aligned}$$

In short, we have $\frac{1}{3} = (0.1)_3 = (0.222\cdots)_3$ as ternary expansions.

For the inductive step, suppose $a_n = 1$ and $a_{k<n} \neq 1$. We claim that