Algebraic Geometry

D. Zack Garza

Sunday 6th September, 2020

Contents

1 Friday, August 21 **2**

2 Tuesday, August 25 **6**
2.1 Proof of Nullstellensatz 8

3 Thursday, August 27 **10**

4 Tuesday, September 01 **15**

5 Thursday, September 03 **18**

Todo list

List of Definitions

2.0.1 Definition – Radical . . . 6
2.0.2 Definition – Affine Variety . . . 7
2.2.1 Definition – The Ideal of a Set . . . 7
3.3.1 Definition – Coordinate Ring . . . 13
4.0.1 Definition – Zariski Topology . . . 15
4.0.2 Definition – Irreducibility and Connectedness . . . 16
5.2.1 Definition – Noetherian Space . . . 20

List of Theorems

1.1 Theorem – Harnack Curve Theorem . . . 4
2.1 Proposition – Properties and Definitions of Ideal Operations . . . 7
2.2 Proposition – Properties of V . . . 7
2.3 Proposition – The Image of V is Radical . . . 8
2.4 Proposition – Hilbert Nullstellensatz (Zero Locus Theorem) . . . 8
2.5 Theorem – 1st Version of Nullstellensatz . . . 9
2.6 Theorem – Noether Normalization . . . 9
3.2 Theorem – Properties of I . . . 12
3.3 Proposition – ? . . . 13
3.4 Theorem – ? . . . 14
4.1 Proposition – ? . . . 16
4.2 Proposition – ? . . . 16
5.1 Proposition – ? . . . 18
5.2 Proposition – ? . . . 18
5.3 Proposition – ? . . . 20
5.4 Proposition – Important . . . 21
5.5 Proposition – 1 . . . 22
5.6 Proposition – ? . . . 22

These are notes live-tex'd from a graduate course in Algebraic Geometry taught by Philip Engel at the University of Georgia in Fall 2020. As such, any errors or inaccuracies are almost certainly my own.

D. Zack Garza, Sunday 6th September, 2020
22:43

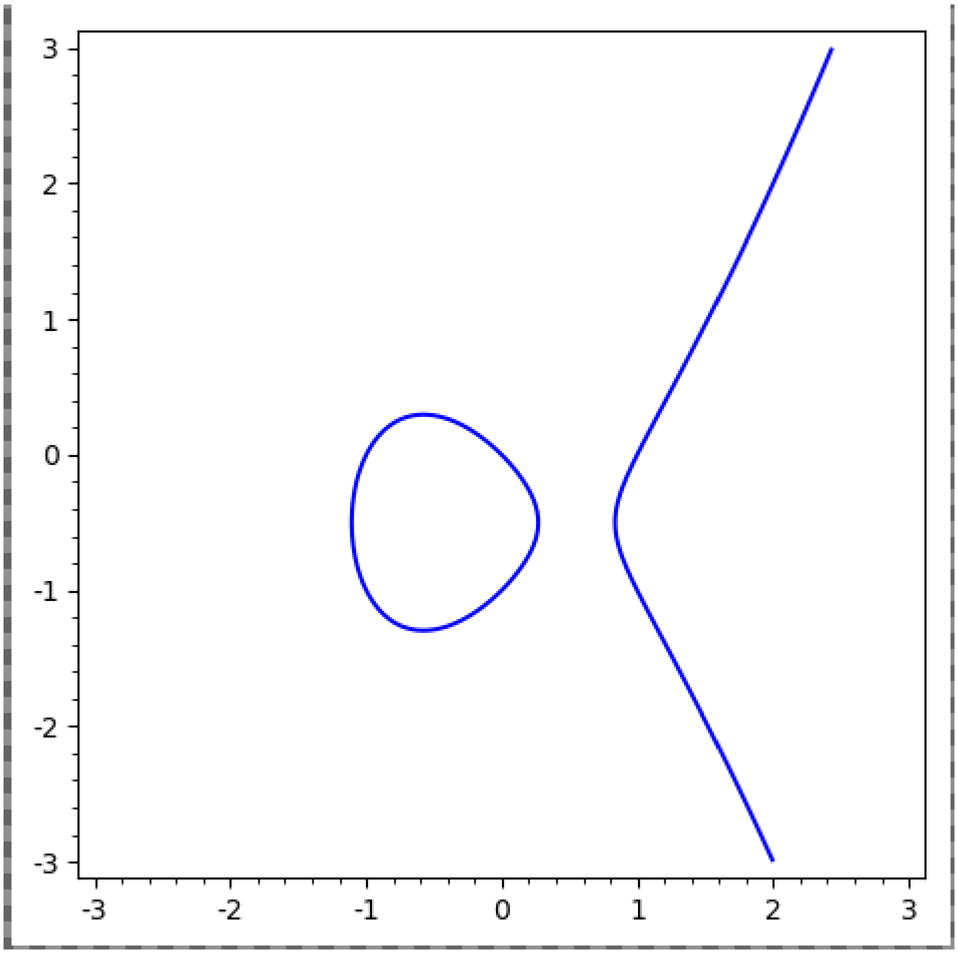
1 Friday, August 21

> Reference:
> `https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2019/alggeom-2019.pdf`

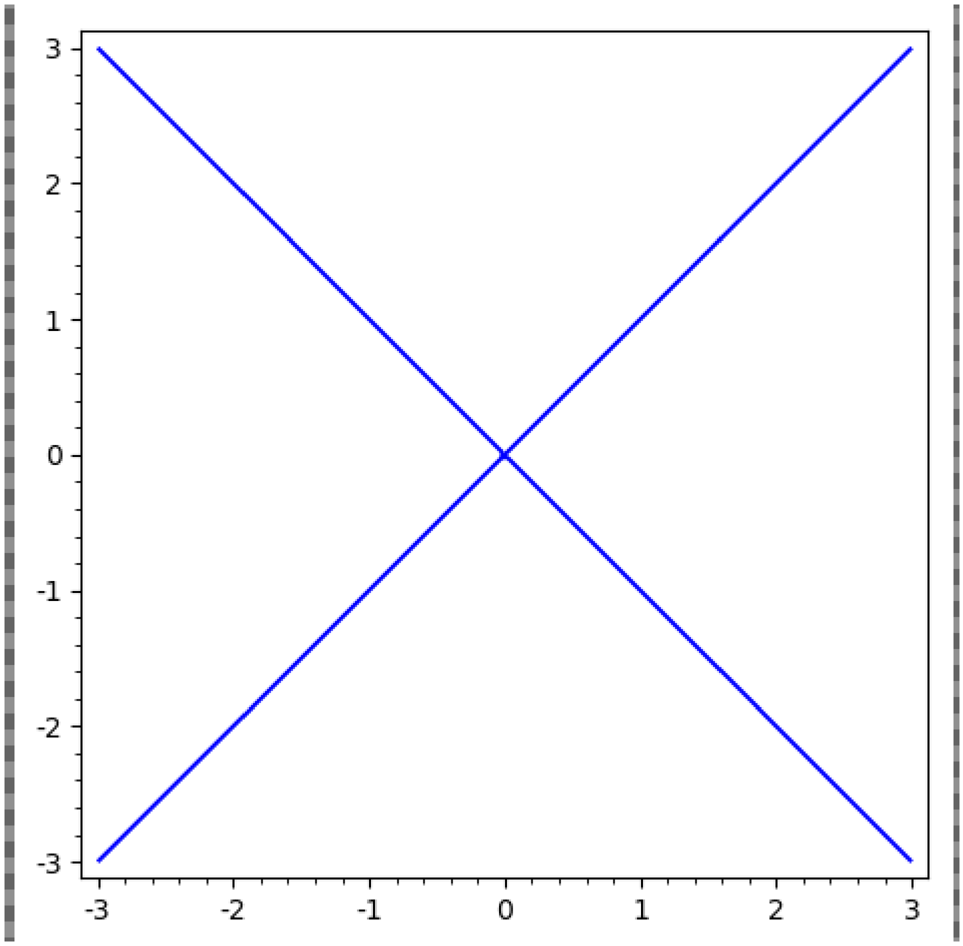
General idea: functions a coordinate ring $R[x_1, \cdots, x_n]/I$ will correspond to the geometry of the variety cut out by I.

Example 1.1.

- $x^2 + y^2 - 1$ defines a circle, say, over $\mathbb{R}$
- $y^2 = x^3 - x$ gives an elliptic curve:



- $x^n + y^n - 1$: does it even contain a $\mathbb{Q}$-point? (Fermat's Last Theorem)
- $x^2 + 1$, which has no $\mathbb{R}$-points.
- $x^2 + y^2 + 1/\mathbb{R}$ vanishes nowhere, so its ring of functions is not $\mathbb{R}[x, y]/\left\langle x^2 + y^2 + 1 \right\rangle$ (problem: $\mathbb{R}$ is not algebraically closed)
- $x^2 - y^2 = 0$ over $\mathbb{C}$ is not a manifold (no chart at the origin):



- $x + y + 1/\mathbb{F}_3$, which has 3 points over $\mathbb{F}_3^2$, but $f(x, y) = (x^3 - x)(y^3 - y)$ vanishes at every point
  - Not possible when algebraically closed (is there nonzero polynomial that vanishes on every point in $\mathbb{C}$?)
  - $V(f) = \mathbb{F}_3^2$, so the coordinate ring is zero instead of $\mathbb{F}_3[x, y]/\langle f\rangle$ (addressed by scheme theory)

Theorem 1.1 ***(Harnack Curve Theorem).***
If $f \in \mathbb{R}[x, y]$ is of degree d, then

$$\pi_1 V(f) \subseteq \mathbb{R}^2 \leq 1 + \frac{(d-1)(d-2)}{2}$$

Actual statement: the number of connected components is bounded above by this quantity.

Example 1.2.
Take the curve

$$X = \left\{ (x, y, z) = (t^3, t^4, t^5) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}.$$

Then X is cut out by three equations:

- $y^2 = xz$
- $x^2 = yz$
- $z^2 = x^2 y$

Exercise 1.1.
Show that the vanishing locus of the first two equations above is $X \cup L$ for L a line.

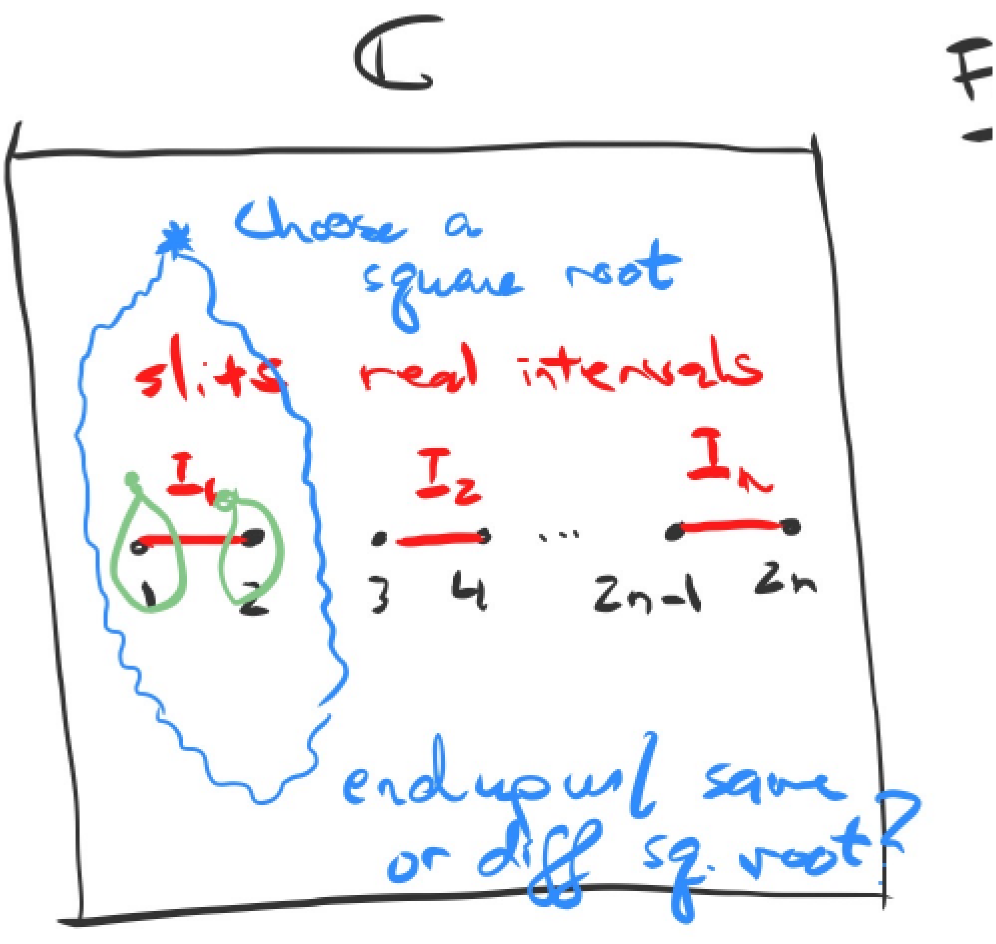
Compare to linear algebra: codimension d iff cut out by exactly d equations.

Example 1.3.
Given the Riemann surface

$$y^2 = (x-1)(x-2)\cdots(x-2n),$$

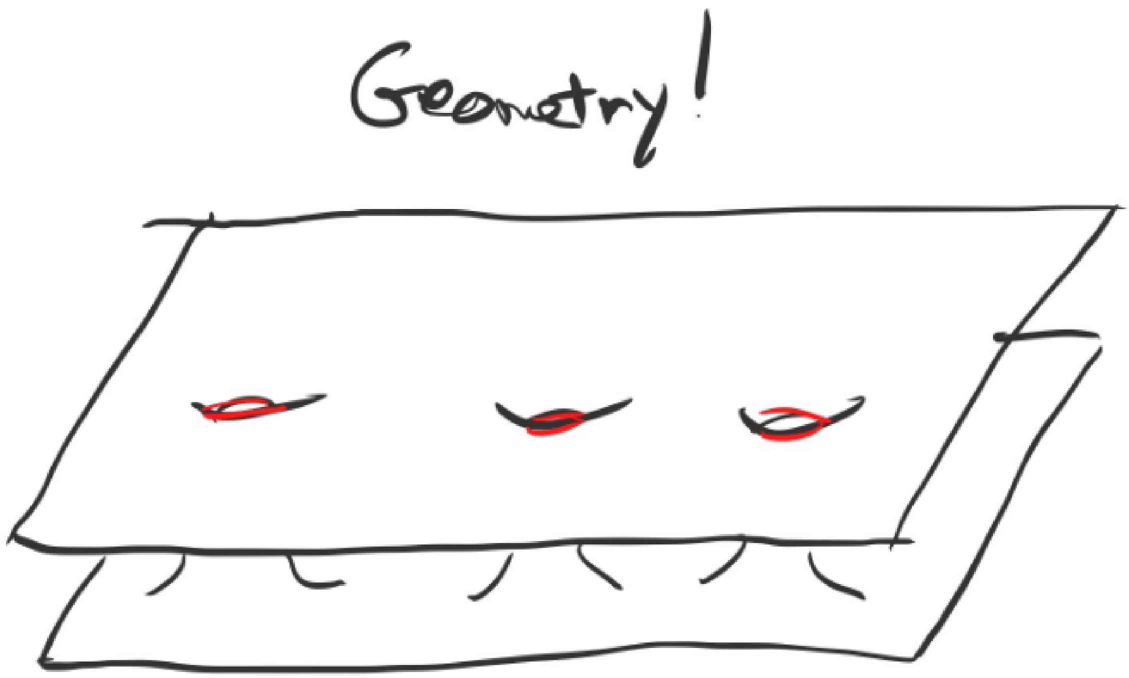
how to visualize the solution set?

Fact: on $\mathbb{C}$ with some slits, you can consistently choose a square root of the RHS.



Away from $x = 1, \cdots, 2n$, there are two solutions for y given x.

After gluing along strips, obtain:



2 Tuesday, August 25

Let $k = \bar{k}$ and R a ring containing ideals I, J.

Definition 2.0.1 (Radical)**.**
Recall that the *radical* of I is defined as

$$\sqrt{I} = \left\{ r \in R \;\middle|\; r^k \in I \text{ for some } k \in \mathbb{N} \right\}.$$

Example 2.1.
Let $I = (x_1, x_2^2) \subset \mathbb{C}[x_1, x_2]$, so $I = \left\{ f_1 x_1 + f_2 x_2 \;\middle|\; f_1, f_2 \in \mathbb{C}[x_1, x_2] \right\}$. Then $\sqrt{I} = (x_1, x_2)$, since $x_2^2 \in I \implies x_2 \in \sqrt{I}$.

Given $f \in k[x_1, \cdots, x_n]$, take its value at $a = (a_1, \cdots, a_n)$ and denote it $f(a)$. Set $\deg(f)$ to be the largest value of $i_1 + \cdots + i_n$ such that the coefficient of $\prod x_j^{i_j}$ is nonzero.

Example 2.2.
$\deg(x_1 + x_2^2 + x_1 x_2^3 = 4)$

Definition 2.0.2 (Affine Variety)**.**

1. Affine n-space $\mathbb{A}^n = \mathbb{A}^n_k$ is defined as $\left\{(a_1, \cdots, a_n) \;\middle|\; a_i \in k\right\}$.

   Remark: not k^n, since we won't necessarily use the vector space structure (e.g. adding points).

2. Let $S \subset k[x_1, \cdots, x_n]$ to be a set of polynomials. Then define $V(S) = \left\{x \in \mathbb{A}^n \;\middle|\; f(x) = 0\right\} \subset \mathbb{A}^n$ to be an *affine variety*.

Example 2.3.

- $\mathbb{A}^n = V(0)$.
- For any point $(a_1, \cdots, a_n) \in \mathbb{A}^n$, then $V(x_1 - a_1, \cdots, x_n - a_n) = \{a_1, \cdots, a_n\}$ uniquely determines the point.
- For any finite set $r_1, \cdots, r_k \in \mathbb{A}^1$, there exists a polynomial $f(x)$ whose roots are r_i.

Remark 1.
We may as well assume S is an ideal by taking the ideal it generates, $S \subseteq \langle S \rangle = \left\{\sum g_i f_i \;\middle|\; g_i \in k[x_1, \cdots, x_n],\ f_i \in S\right\}$. Then $V(\langle S \rangle) \subset V(S)$.

Conversely, if f_1, f_2 vanish at $x \in \mathbb{A}^n$, then $f_1 + f_2, gf_1$ also vanish at x for all $g \in k[x_1, \cdots, x_n]$. Thus $V(S) \subset V(\langle S \rangle)$.

Proposition 2.1 ***(Properties and Definitions of Ideal Operations).***

- $I + J := \left\{f + g \;\middle|\; f \in I,\ g \in J\right\}$.
- $IJ := \left\{\sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\ g_i \in J, N \in \mathbb{N}\right\}$.
- If $I + J = \langle 1 \rangle$ then $I \cap J = IJ$ (coprime or comaximal)

Proposition 2.2 ***(Properties of V).***

1. If $S_1 \subseteq S_2$ then $V(S_1) \supseteq V(S_2)$.
2. $V(S_1) \cup V(S_2) = V(S_1 S_2) = V(S_1 \cap S_2)$.
3. $\bigcap V(S_i) = V\left(\bigcup S_i\right)$.

We thus have a map

$$V : \{\text{Ideals in } k[x_1, \cdots, x_n]\} \longrightarrow \{\text{Affine varieties in } \mathbb{A}^n\}.$$

Definition 2.2.1 (The Ideal of a Set)**.**

Let $X \subset \mathbb{A}^n$ be any set, then *the ideal of* X is defined as

$$I(X) := \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \, \forall x \in X \right\}.$$

Example 2.4.
Let X be the union of the x_1 and x_2 axes in $\mathbb{A}^2$, then $I(X) = (x_1 x_2) = \left\{ x_1 x_2 g \;\middle|\; g \in k[x_1, x_2] \right\}$.

Note that if $X_1 \subset X_2$ then $I(X_1) \subset I(X_2)$.

Proposition 2.3 ***(The Image of*** V ***is Radical).***
$I(X)$ is a radical ideal, i.e. $I(X) = \sqrt{I(X)}$.
This is because $f(x)^k = 0 \forall x \in X$ implies $f(x) = 0$ for all $x \in X$, so $f^k \in I(X)$ and thus $f \in I(X)$.

Our correspondence is thus

$$\left\{ \text{Ideals in } k[x_1, \cdots, x_n] \right\} \xrightarrow{V} \left\{ \text{Affine Varieties} \right\}$$
$$\left\{ \text{Radical Ideals} \right\} \xleftarrow{I} \left\{ ? \right\}.$$

Proposition 2.4 ***(Hilbert Nullstellensatz (Zero Locus Theorem)).***

a. For any affine variety X, $V(I(X)) = X$.
b. For any ideal $J \subset k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

Thus there is a bijection between radical ideals and affine varieties.

2.1 Proof of Nullstellensatz

Remark 2.
Recall the Hilbert Basis Theorem: any ideal in a finitely generated polynomial ring over a field is again finitely generated.

We need to show 4 inclusions, 3 of which are easy.

a: $X \subset V(I(X))$:

- If $x \in X$ then $f(x) = 0$ for all $f \in I(X)$.
- So $x \in V(I(X))$, since every $f \in I(X)$ vanishes at x.

b: $\sqrt{J} \subset I(V(J))$:

- If $f \in \sqrt{J}$ then $f^k \in J$ for some k.
- Then $f^k(x) = 0$ for all $x \in V(J)$.
- So $f(x) = 0$ for all $x \in V(J)$.
- Thus $f \in I(V(J))$.

c: $V(I(X)) \subset X$:

- Need to now use that X is an affine variety.
  - Counterexample: $X = \mathbb{Z}^2 \subset \mathbb{C}^2$, then $I(X) = 0$. But $V(I(X)) = \mathbb{C}^2$, but $\mathbb{C}^2 \not\subset \mathbb{Z}^2$.
- By (b), $I(V(J)) \supset \sqrt{J} \supset J$.
- Since $V(\,\cdot\,)$ is order-reversing, taking V of both sides reverses the containment.
- So $V(I(V(J))) \subset V(J)$, i.e. $V(I(X)) \subset X$.

d: $I(V(J)) \subset \sqrt{J}$ (hard direction)

Theorem 2.5 ***(1st Version of Nullstellensatz).***
Suppose k is algebraically closed and uncountable (still true in countable case by a different proof).
Then the maximal ideals in $k[x_1, \cdots, x_n]$ are of the form $(x_1 - a_1, \cdots, x_n - a_n)$.

Proof .
Let $\mathfrak{m}$ be a maximal ideal, then by the Hilbert Basis Theorem, $\mathfrak{m} = \langle f_1, \cdots, f_r \rangle$ is finitely generated.
Let $L = \mathbb{Q}[\{c_i\}]$ where the c_i are all of the coefficients of the f_i if char $(K) = 0$, **or** $\mathbb{F}_p[\{c_i\}]$ if char $(k) = p$. Then $L \subset k$.
Define $\mathfrak{m}_0 = \mathfrak{m} \cap L[x_1, \cdots, x_n]$. Note that by construction, $f_i \in \mathfrak{m}_0$ for all i, and we can write $\mathfrak{m} = \mathfrak{m}_0 \cdot k[x_1, \cdots, x_n]$.
Claim: $\mathfrak{m}_0$ is a maximal ideal.
If it were the case that

$$\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq L[x_1, \cdots, x_n],$$

then

$$\mathfrak{m}_0 \cdot k[x_1, \cdots, x_n] \subsetneq \mathfrak{m}_0' \cdot k[x_1, \cdots, x_n] \subsetneq k[x_1, \cdots, x_n].$$

So far: constructed a smaller polynomial ring and a maximal ideal in it.

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is a field that is finitely generated over either $\mathbb{Q}$ or $\mathbb{F}_p$.

Theorem 2.6 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Note: this theorem is perhaps more important than the Nullstellensatz!

Thus $L[x_1, \cdots, x_n]/\mathfrak{m}_0$ is finite over some $\mathbb{Q}(t_1, \cdots, t_n)$, and since k is uncountable, there exists an embedding $\mathbb{Q}(t_1, \cdots, t_n) \hookrightarrow k$.

Use the fact that there are only countably many polynomials over a countable field.

This extends to an embedding of $\varphi : L[x_1, \cdots, x_n]/\mathfrak{m}_0 \hookrightarrow k$ since k is algebraically closed.
Letting a_i be the image of x_i under φ, then $f(a_1, \cdots, a_n) = 0$ by construction, $f_i \in (x_i - a_i)$ implies that $\mathfrak{m} = (x_i - a_i)$ by maximality.

■

3 Thursday, August 27

Recall Hilbert's Nullstellensatz:

a. For any affine variety, $V(I(X)) = X$.

b. For any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\left\{\text{Radical ideals } k[x_1, \cdots, x_n]\right\} \longrightarrow V(\,\cdot\,)I(\,\cdot\,) \left\{\text{Affine varieties in } \mathbb{A}^n\right\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \cdots, x_n]$ are of the form $\langle x - a_1, \cdots, x - a_n \rangle$.

Corollary 3.1 ***(?).***
If $V(I)$ is empty, then $I = \langle 1 \rangle$.

> Slogan: the only ideals that vanish nowhere are trivial. No common vanishing locus $\implies$ trivial ideal, so there's a linear combination that equals 1.

Proof .
By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, these exists a maximal ideals $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\,\cdot\,)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \cdots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \cdots, a_n\}$ is nonempty. ■

Returning to the proof that $I(V(J)) \subseteq \sqrt{J}$: let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal $\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \cdots, x_n, t]$.

Observation: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. But $ft - 1 \neq 0$ if $f = 0$, so $V(\tilde{J}) = V(G) \cap V(ft - 1) = \emptyset$.

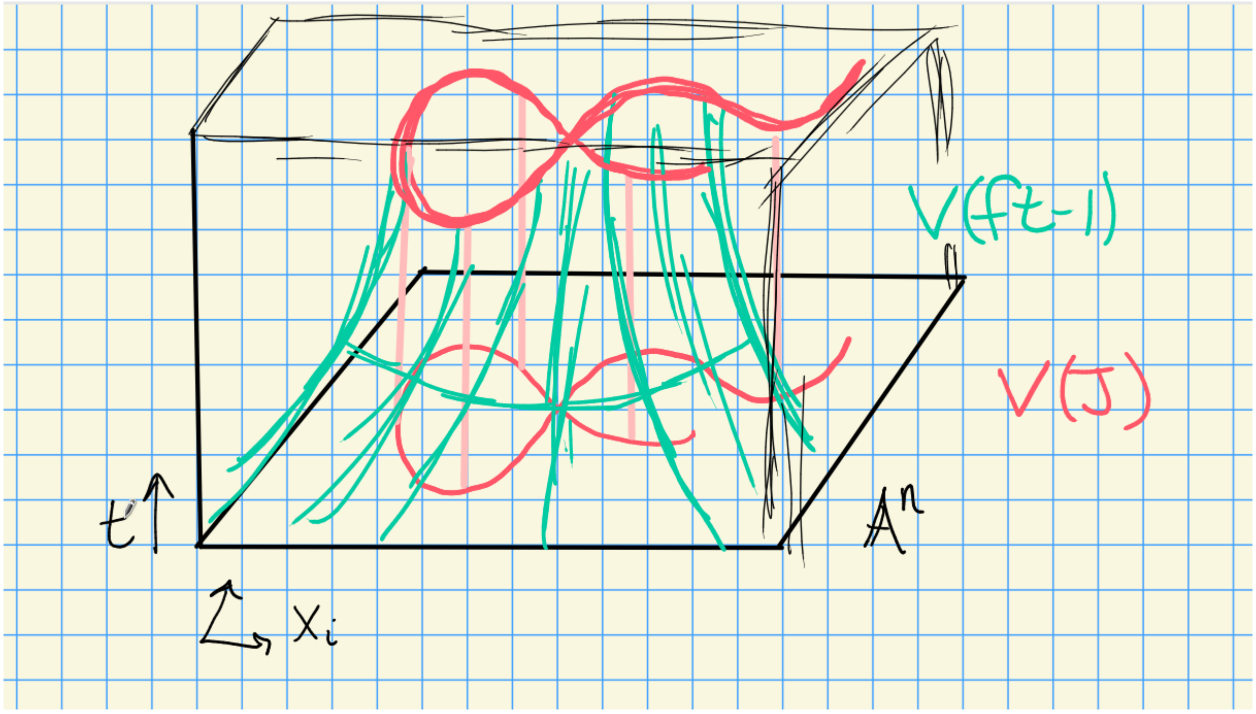


Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so $1 = \langle ft - 1 \rangle \, g_0(x_1, \cdots, x_n, t) + \sum f_i g_i(x_1, \cdots, x_n, t)$ with $f_i \in J$. Let t^N be the largest power of t in any g_i. Thus for some polynomials G_i, we have

$$f^N \coloneqq (ft - 1)G_0(x_1, \cdots, x_n, ft) + \sum f_i G_i(x_1, \cdots, x_n, ft)$$

noting that f does not depend on t.

Now take $k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Observation: there is an inclusion

$$k[x_1, \cdots, x_n] \hookrightarrow k[x_1, \cdots, x_n, t] / \langle ft - 1 \rangle .$$

Exercise 3.1.
Why is this true?

Since this is injective, this identity also holds in $k[x_1, \cdots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$.

Example 3.1.

Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ and $V(f) = \{a_1, \cdots, a_k\}$. Then $I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J$. Note that this loses information.

Example 3.2.
Let $J = \langle x - a_1, \cdots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\left\{\text{Points of } \mathbb{A}^n\right\} \iff \left\{\text{Maximal ideals of } k[x_1, \cdots, x_n]\right\}.$$

Theorem 3.2 ***(Properties of I).***

a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof .
We proved (a) on the variety side.
For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

∎

Example 3.3.
Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x-axis.

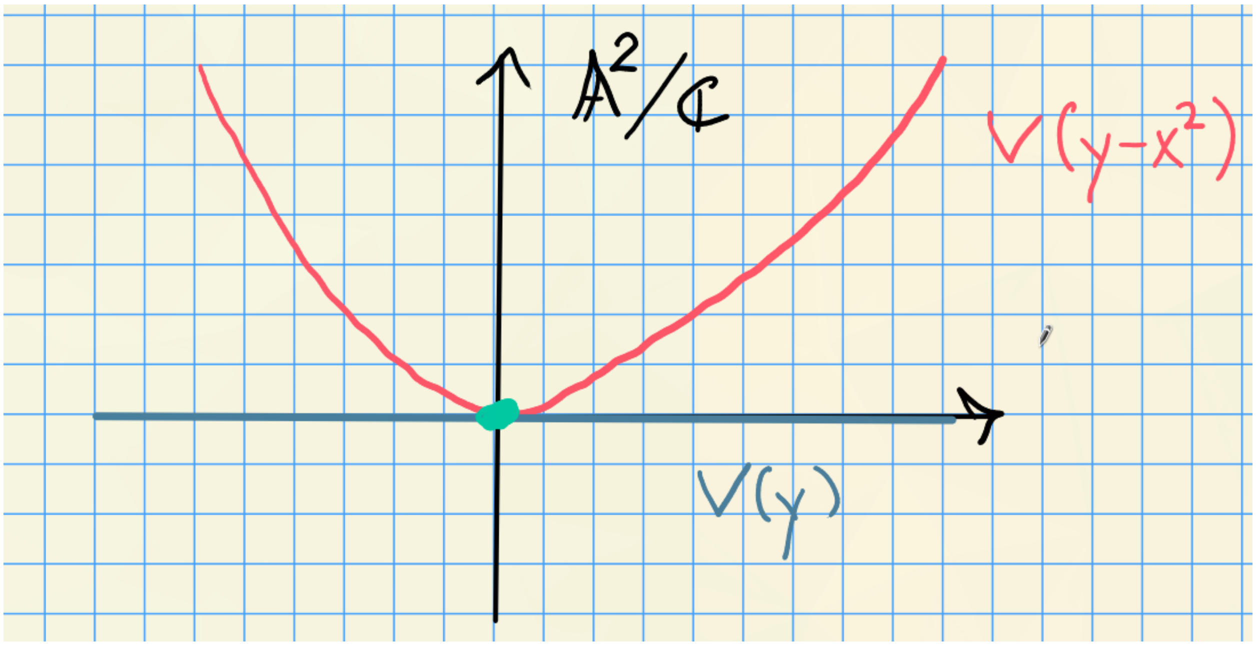


Figure 2: Image

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \left\langle y - x^2, y \right\rangle = \left\langle x^2, y \right\rangle$, but $I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$.

Proposition 3.3 ***(?).***
If $f, g \in k[x_1, \cdots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof .
Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$. ■

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a "natural" space of functions on X is $k[x_1, \cdots, x_n]/I(X)$.

Definition 3.3.1 (Coordinate Ring)**.**
For an affine variety X, the *coordinate ring of* X is

$$A(X) \coloneqq k[x_1, \cdots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X.

Observation: The constructions $V(\,\cdot\,), I(\,\cdot\,)$ work just as well for $A(X)$ and X.

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \left\{ x \in Y \;\middle|\; f(x) = 0 \;\forall f \in S \right\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \left\{ f \in A(Y) \;\middle|\; f(x) = 0 \;\forall x \in X \right\} \subseteq A(Y).$$

Example 3.4.
For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$\begin{tikzcd} A(\mathbb{A}^n) \arrow[r, two heads, "\cdot/I(Y)"] \arrow[rr, bend left, two heads, "\cdot/I(X)"] & A(Y) \arrow[r, two heads, "\cdot/I(X)"] & A(X) \end{tikzcd}$$

.

Theorem 3.4*(?)*.
Let $X \subset Y$ be an affine subvariety, then

a. $A(X) = A(Y)/I_Y(X)$
b. There is a correspondence

$$\begin{aligned} \left\{ \text{Affine subvarieties of } Y \right\} &\iff \left\{ \text{Radical ideals in } A(Y) \right\} \\ X &\mapsto I_Y(X) \\ V_Y(J) &\leftarrow J. \end{aligned}$$

Proof .
Properties are inherited from the case of $\mathbb{A}^n$, see exercise in Gathmann. ■

Example 3.5.
Let $Y = V)y - x^2 \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1,1)\} = V(x-1, y-1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\left\langle y - x^2 \right\rangle \subset \langle x-1, y-1 \rangle$ (e.g. by Taylor expanding about the point $(1,1)$), and there is a map

$$\begin{tikzcd} A(\mathbb{A}^n) \arrow[r] \arrow[d, equal] & A(Y) \arrow[r] \arrow[d, equal] & A(X) \arrow[d, equal] \\ k[x,y] \arrow[r] & k[x,y]/\left\langle y - x^2 \right\rangle \arrow[r] & k[x,y]/\langle x-1, y-1 \rangle \end{tikzcd}$$

4 Tuesday, September 01

Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \,\forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \,\forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 4.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\displaystyle\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i \right)$.

Remark 3.
There are few closed sets, so this is a "weak" topology.

Example 4.1.
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A \coloneqq \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 4.2.
Let $f : \mathbb{A}^1/k \longrightarrow \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 4.3.
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\displaystyle\bigcup_{i=1}^{n} U_i \times V_i$ with U_i, V_i open in X, Y respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta \coloneqq V(x-y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

Example 4.4.
Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 \coloneqq V(x_1)$ and $X_2 \coloneqq V(x_2)$. The Zariski topology can detect these decompositions.

Definition 4.0.2 (Irreducibility and Connectedness)**.**
Let X be a topological space.

a. X is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible*.
b. X is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, X is said to be *connected*.

Example 4.5.
$V(x_1x_2)$ is reducible but connected.

Remark 4.
$\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

Proposition 4.1 ***(?).***
Let X be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof .
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1 \emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \longrightarrow A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

∎

Proposition 4.2 ***(?).***
An affine variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof .
$\implies$: By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\impliedby$: Suppose X is reducible with $X = X_1 \cup X_2$ with X_i proper and closed. Define $J_i \coloneqq I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so f_i vanishes on X_i. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain.
$\blacksquare$

Example 4.6.
Let $X = \{p_1, \cdots, p_d\}$ be a finite set in $\mathbb{A}^n$. The Zariski topology on X is the discrete topology, and $X = \coprod \{p_i\}$. So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^{d} A(\{p_i\}) = \prod_{i=1}^{d} k[x_1, \cdots, x_n] / \langle x_j - a_j(p_i) \rangle_{j=1}^{d} .$$

Example 4.7.
Set $V(x_1 x_2) = X$, then $A(X) = k[x_1, x_2] / \langle x_1 x_2 \rangle$. This not being a domain (since $x_1 x_2 = 0$) corresponds to $X = V(x_1) \cup V(x_2)$ not being irreducible.

Example 4.8.
$\mathbb{A}^2 / k$ is irreducible since $k[x_1, \cdots x_n]$ is a domain.

Example 4.9.
Let X_1 be the xy plane and X_2 be the line parallel to the y-axis through $[0, 0, 1]$, and let $X = X_1 \coprod X_2$.
Then $X_1 = V(z)$ and $X_2 = V(x, z-1)$, and $I(X) = \langle z \rangle \cdots \langle x, z-1 \rangle = \left\langle xz, z^2 - z \right\rangle$.

Then the coordinate ring is given by $A(X) = \mathbb{C}[x, y, z] / \left\langle xz, z^2 - z \right\rangle = \mathbb{C}[x, y, z] / \langle z \rangle \oplus \mathbb{C}[x, y, z] / \langle x, z-1 \rangle$.

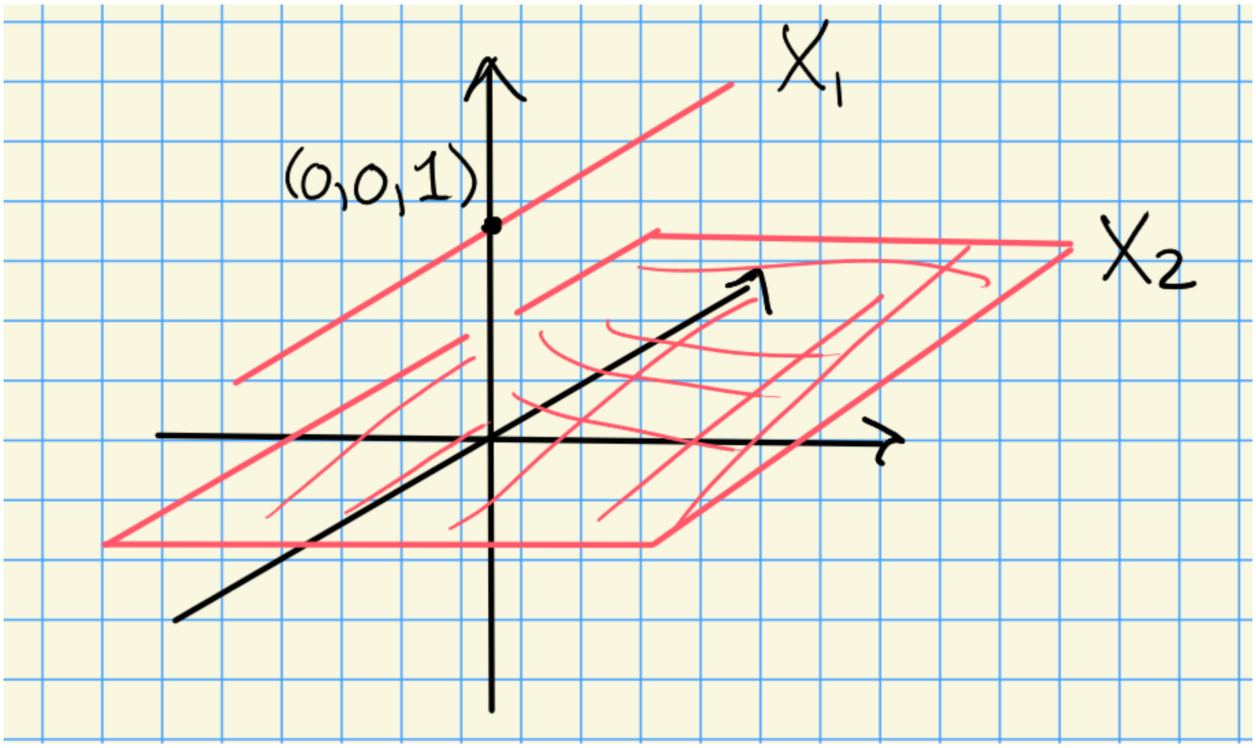


Figure 3: Image

5 Thursday, September 03

Recall that the Zariski topology is defined on an affine variety $X = V(J)$ with $J \trianglelefteq k[x_1, \cdots, x_n]$ by describing the closed sets.

Proposition 5.1 *(?).*
X is irreducible if its coordinate ring $A(X)$ is a domain.

Proposition 5.2 *(?).*
There is a 1-to-1 correspondence

$$\left\{ \begin{matrix} \text{Irreducible subvarieties} \\ \text{of } X \end{matrix} \right\} \iff \left\{ \begin{matrix} \text{Prime ideals} \\ \text{in } A(X) \end{matrix} \right\}.$$

Proof .
Suppose $Y \subset X$ is an affine subvariety. Then

$$A(X)/I_X(Y) = A(Y).$$

By NSS, there is a bijection between subvarieties of X and radical ideals of $A(X)$ where $Y \mapsto I_X(Y)$. A quotient is a domain iff quotienting by a prime ideal, so $A(Y)$ is a domain iff $I_X(Y)$ is prime.

■

Recall that $\mathfrak{p} \trianglelefteq R$ is prime when $fg \in \mathfrak{p} \iff f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus $\bar{f}\bar{g} = 0$ in $R/\mathfrak{p}$ implies $\bar{f} = 0$ or $\bar{g} = 0$ in $R/\mathfrak{p}$, i.e. $R/\mathfrak{p}$ is a domain.

Finally note that prime ideals are radical (easy proof).

Example 5.1.
Consider $\mathbb{A}^2/\mathbb{C}$ and some subvarieties C_i:

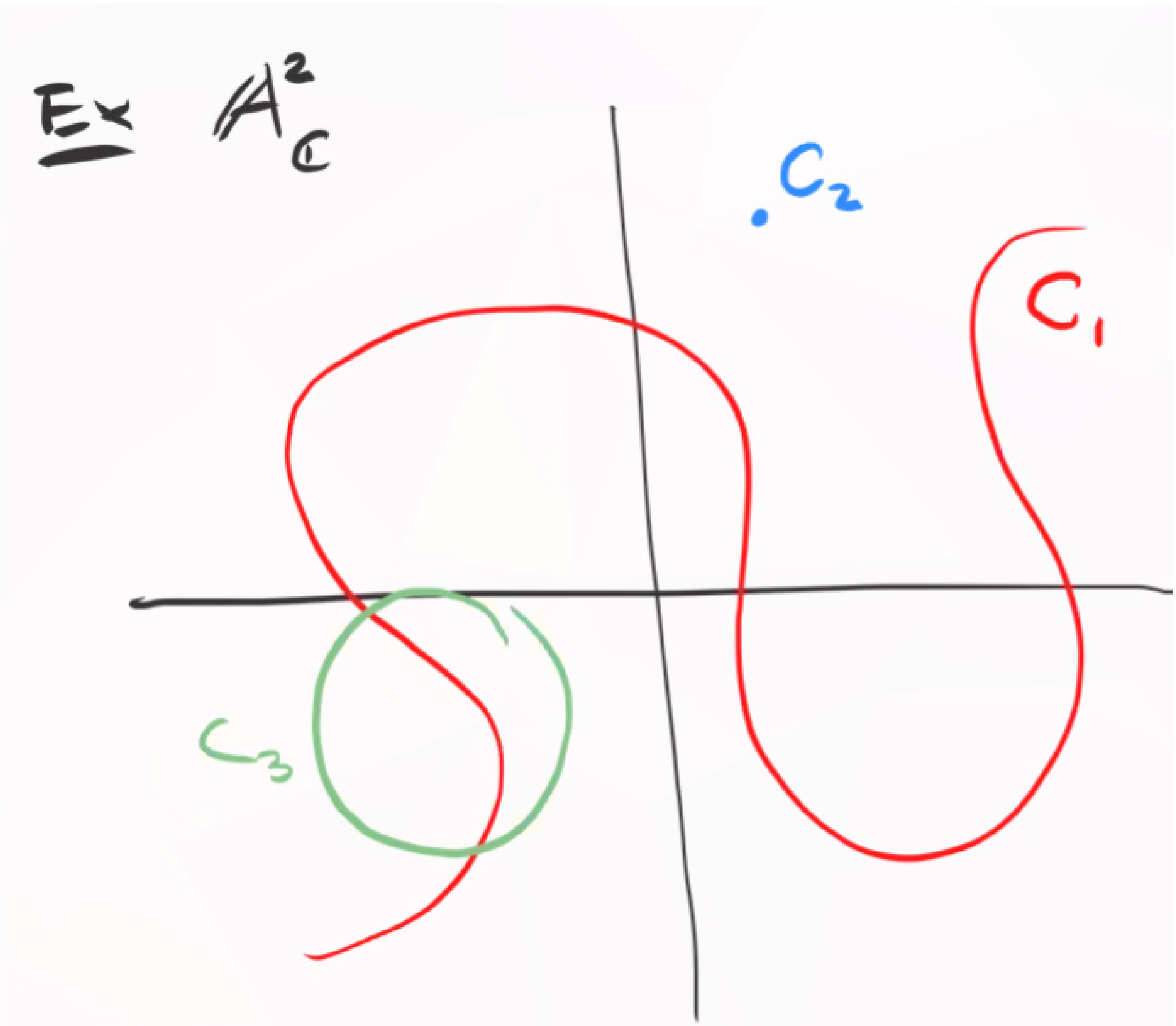


Figure 4: Subvarieties

Then irreducible subvarieties correspond to prime ideals in $\mathbb{C}[x, y]$. Here C_1, C_3 correspond to $V(f), V(g)$ for f, g irreducible polynomials, whereas C_2 corresponds to a maximal ideal, i.e. $V(x_1 - a_1, x_2 - a_2)$.

Note that $I(C_1 \cup C_2 \cup C_3)$ is not a prime ideal, since the variety is reducible as the union of 3 closed subsets.

Example 5.2.
A finite set is irreducible iff it contains only one point.

Example 5.3.
Any irreducible topological space is connected, since irreducible requires a union but connectedness requires a *disjoint* union.

Example 5.4.
A^n/k is irreducible: by prop 2.8, its irreducible iff the coordinate ring is a domain. However $A(\mathbb{A}^n) = k[x_1, \cdots, x_n]$, which is a domain.

Example 5.5.
$V(x_1 x_2)$ is not irreducible, since it's equal to $V(x_1) \cup V(x_2)$.

Definition 5.2.1 (Noetherian Space)**.**
A *Noetherian* topological space X is a space with no infinite strictly decreasing sequence of closed subsets.

Proposition 5.3 ***(?).***
An affine variety X with the zariski topology is a noetherian space.

Proof .
Let $X_0 \supsetneq X_1 \supsetneq \cdots$ be a decreasing sequence of closed subspaces. Then $I(X_0) \subsetneq I(X_1) \subsetneq$. Note that these containments are strict, otherwise we could use $V(I(X_1)) = X_1$ to get an equality in the original chain.
Recall that a ring R is Noetherian iff every ascending chain of ideals terminates. Thus it suffices to show that $A(X)$ is Noetherian.
We have $A(X) = k[x_1, \cdots, x_n]/I(X)$, and if this had an infinite chain $I_1 \subsetneq I_2 \subsetneq \cdots$ lifts to a chain in $k[x_1, \cdots, x_n]$, which is Noetherian. A useful fact: R noetherian implies that $R[x]$ is noetherian, and fields are always noetherian.

■

Remark 5.
Any subspace $A \subset X$ of a noetherian space is noetherian. To see why, suppose we have a chain of closed sets in the subspace topology,

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq \cdots .$$

Then $X_0 \supsetneq X_1 \supsetneq \cdots$ is a strictly decreasing chain of closed sets in X. Why strictly decreasing: $\cap^n X_i = \cap^{n+1} X_i \implies A \cap^n X_i = A \cap^{n+1} X_i$, a contradiction.

Proposition 5.4 ***(Important).***

Every noetherian space X is a finite union of irreducible closed subsets, i.e. $X = \bigcup_{i=1}^{k} X_i$. If we further assume $X_i \not\subset X_j$ for all i, j, then the X_i are unique up to permutation.

Remark 6.
The X_i are the **components** of X. In the previous example $C_1 \cup C_2 \cup C_3$ has three components.

Proof .
If X is irreducible, then $X = X$ and this holds.
Otherwise, write $X = X_1 \cup X_2$ with X_i proper closed subsets. If X_1 and X_1' are irreducible, we're done, so otherwise suppose wlog X_1' is not irreducible.
Then we can express $X = X_1 \cup (X_2 \cup X_2')$ with $X_2, X_2' \subset X_1'$ closed and proper.
Thus we can obtain a tree whose leaves are proper closed subsets:

Figure 5: Image

This tree terminates because X is Noetherian: if it did not, this would generate an infinite decreasing chain of subspaces.
We now want to show that the decomposition is unique if no two components are contained in the other.
Suppose

$$X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=1}^{\ell} X_j'.$$

Note that $X_i \subset X$ implies that $X_i = \bigcup_{j=1}^{\ell} X_i \cap X_j'$. But X_i is irreducible and this would express X_i as a union of proper closed subsets, so some $X_i \cap X_j'$ is *not* a proper closed subset.

Thus $X_i = X_i \cap X'_j$ for some j, which forces $X_i \subset X'_j$. Applying the same argument to X'_j to obtain $X'_j \subset X_k$ for some k.
Then $X_i \subset X'_j \subset X_k$, but $X_i \not\subset X_j$ when $j \neq i$. Thus $X_i = X'_j = X_k$, forcing the X_i to be unique up to permutation.

■

Recall from ring theory: for $I \subset R$ and R noetherian, I has a *primary decomposition* $I = \bigcap_{i=1}^{k} Q_i$ with $\sqrt{Q_i}$ prime. Assuming the Q_i are minimal in the sense that $\sqrt{Q_i} \not\subset \sqrt{Q_j}$ for any i, j, this decomposition is unique.

Applying this to $I(X) \trianglelefteq k[x_1, \cdots, x_n] = R$ yields

$$I(X) = \bigcap_{i=1}^{k} Q_i \implies X = V(I(X)) = \bigcup_{i=1}^{k} V(Q_i).$$

Letting $P_i = \sqrt{Q_i}$, noting that the P_i are prime and thus radical, we have $V(Q_i) = V(P_i)$. Writing $X = \bigcup V(P_i)$, we have $I(V(P_i)) = P_i$ and thus $A(V(P_i)) = R/P_i$ is a domain, meaning $V(P_i)$ are irreducible affine varieties.

Conversely, if we express $X = \bigcup X_i$, we have $I = I\left(\bigcup X_i\right) = \bigcap I(X_i) = \bigcap P_i$ which are irreducible since they are prime.

Remark 7.
There is a correspondence

$$\left\{\begin{matrix}\text{Irreducible components} \\ \text{of } X\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Minimal prime ideals} \\ \text{in } A(X)\end{matrix}\right\},$$

where here *minimal* is the condition that no pair of ideals satisfies a subset containment.

Remark 8.
Let X be an irreducible topological space.

Proposition 5.5 ***(1).***
The intersection of nonempty two open sets is *never* empty.

Proof .
Let U, U' be open and $X \setminus U, X \setminus U'$ closed. Then $U \cap U' = \emptyset \iff (X \setminus U) \cup (X \setminus U') = X$, but this is not possible since X is irreducible.

■

Irreducible iff any two nonempty open sets intersect.

Proposition 5.6 ***(?).***
Any nonempty open set is dense, i.e. if $U \subset X$ is open then its closure $\mathrm{cl}_X(U)$ is dense in X.

Proof .
Write $X = \mathrm{cl}_X(U) \cup (X \setminus U)$. Since $X \setminus U \neq X$ and X is irreducible, we have $\mathrm{cl}_X(U) = X$. ■