

Algebra

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1 Major Theorems

Theorem 1 (Cauchy). For any prime p dividing the order of G , there is an element x of order p (and thus a subgroup $H = \langle x \rangle$).

Theorem 2 (Lagrange). If $H \leq G$ is a subgroup, then $|H| \mid |G|$.

Theorem 3 (Sylow 1). If $|G| = n = \prod p_i^{a_i}$ as a prime factorization, then G has subgroups of order $p_i^{a_i}$ for every i . Moreover, this holds for any $1 \leq r \leq a_i$.

Theorem 4 (Classification of finitely generated abelian groups). If G is a finitely generated abelian group, then $G \cong F \oplus T$, where F is free abelian and T is a torsion group. If $|T| = n$, then $T \cong \bigoplus \mathbb{Z}_{p_i^{\alpha_i}}$ where $n = \prod p_i^{\alpha_i}$ is some factorization of n with the p_i **not necessarily distinct**.

Some nice lemmas:

- Every subgroup of a cyclic group is itself cyclic.

2 Lecture 1 (Thu 15 Aug 2019)

We'll be using Hungerford's Algebra text.

2.1 Definitions

The following definitions will be useful to know by heart:

- The order of a group
- Cartesian product
- Relations
- Equivalence relation
- Partition
- Binary operation
- Group
- Isomorphism
- Abelian group
- Cyclic group
- Subgroup
- Greatest common divisor
- Least common multiple
- Permutation
- Transposition
- Orbit
- Cycle
- The symmetric group S^n
- The alternating group A_n
- Even and odd permutations
- Cosets
- Index
- The direct product of groups
- Homomorphism
- Image of a function
- Inverse image of a function
- Kernel
- Normal subgroup
- Factor group
- Simple group

Here is a rough outline of the course:

- Group Theory
 - Groups acting on sets
 - Sylow theorems and applications
 - Classification
 - Free and free abelian groups
 - Solvable and simple groups
 - Normal series
- Galois Theory

- Field extensions
- Splitting fields
- Separability
- Finite fields
- Cyclotomic extensions
- Galois groups
- Solvability by radicals
- Module theory
 - Free modules
 - Homomorphisms
 - Projective and injective modules
 - Finitely generated modules over a PID
- Linear Algebra
 - Matrices and linear transformations
 - Rank and determinants
 - Canonical forms
 - Characteristic polynomials
 - Eigenvalues and eigenvectors

2.2 Preliminaries

Definition 1. A **group** is an ordered pair $(G, \cdot : G \times G \rightarrow G)$ where G is a set and \cdot is a binary operation, which satisfies the following axioms:

- Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$,
- Identity: $\exists e \in G \ni ge = eg = g$,
- Inverses: $g \in G \implies \exists h \in G \ni gh = gh = e$.

Example 1.

- $(\mathbb{Z}, +)$
- $(\mathbb{Q}, +)$
- $(\mathbb{Q}^\times, \times)$
- $(\mathbb{R}^\times, \times)$
- $(\text{GL}(n, \mathbb{R}), \times) = \{A \in \text{Mat}_n \ni \det(A) \neq 0\}$
- (S_n, \circ)

Definition 2. A subset $S \subseteq G$ is a **subgroup** of G iff

1. $s_1, s_2 \in S \implies s_1 s_2 \in S$
2. $e \in S$
3. $s \in S \implies s^{-1} \in S$

We denote such a subgroup $S \leq G$.

Examples of subgroups:

- $(\mathbb{Z}, +) \leq (\mathbb{Q}, +)$
- $\text{SL}(n, \mathbb{R}) \leq \text{GL}(n, \mathbb{R})$, where $\text{SL}(n, \mathbb{R}) = \{A \in \text{GL}(n, \mathbb{R}) \ni \det(A) = 1\}$

2.3 Cyclic Groups

Definition 3. A group G is **cyclic** iff G is generated by a single element.

Exercise 1. Show $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\} \cong \bigcap \{H \leq G \mid g \in H\}$.

Theorem 5. Let G be a cyclic group, so $G \cong \langle g \rangle$.

- If $|G| = \infty$, then $G \cong \mathbb{Z}$.
- If $|G| = n < \infty$, then $G \cong \mathbb{Z}_n$.

Definition 4. Let $H \leq G$, and define a **right coset** of G by $aH = \{ah \mid H \in H\}$. A similar definition can be made for **left cosets**.

Then $aH = bH \iff b^{-1}a \in H$ and $Ha = Hb \iff ab^{-1} \in H$.

Some facts:

- Cosets partition H , i.e. $b \notin H \implies aH \cap bH = \{e\}$.
- $|H| = |aH| = |Ha|$ for all $a \in G$.

Theorem 6 (Lagrange). If G is a finite group and $H \leq G$, then $|H| \mid |G|$.

Definition 5. A subgroup $N \leq G$ is **normal** iff $gN = Ng$ for all $g \in G$, or equivalently $gNg^{-1} \subseteq N$. I denote this $N \trianglelefteq G$.

When $N \trianglelefteq G$, the set of left/right cosets of N themselves have a group structure. So we define

$$G/N = \{gN \mid g \in G\} \text{ where } (g_1N)(g_2N) = (g_1g_2)N.$$

Given $H, K \leq G$, define $HK = \{hk \mid h \in H, k \in K\}$. We have a general formula,

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

2.4 Homomorphisms

Definition 6. Let G, G' be groups, then $\varphi : G \rightarrow G'$ is a **homomorphism** if $\varphi(ab) = \varphi(a)\varphi(b)$.

Example 2. • $\exp : (\mathbb{R}, +) \rightarrow (\mathbb{R}^{>0}, \cdot)$ where $\exp(a+b) = e^{a+b} = e^a e^b = \exp(a)\exp(b)$.

- $\det : (\text{GL}(n, \mathbb{R}), \times) \rightarrow (\mathbb{R}^\times, \times)$ where $\det(AB) = \det(A)\det(B)$.
- Let $N \trianglelefteq G$ and $\varphi G \rightarrow G/N$ given by $\varphi(g) = gN$.
- Let $\varphi : \mathbb{Z} \rightarrow \mathbb{Z}_n$ where $\varphi(g) = [g] = g \bmod n$ where $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$

Definition 7. Let $\varphi : G \rightarrow G'$. Then φ is a **monomorphism** iff it is injective, an **epimorphism** iff it is surjective, and an **isomorphism** iff it is bijective.

2.5 Direct Products

Let G_1, G_2 be groups, then define

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\} \text{ where } (g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2).$$

We have the formula $|G_1 \times G_2| = |G_1||G_2|$.

2.6 Finitely Generated Abelian Groups

Definition 8. We say a group is **abelian** if G is commutative, i.e. $g_1, g_2 \in G \implies g_1 g_2 = g_2 g_1$.

Definition 9. A group is **finitely generated** if there exist $\{g_1, g_2, \dots, g_n\} \subseteq G$ such that $G = \langle g_1, g_2, \dots, g_n \rangle$.

This generalizes the notion of a cyclic group, where we can simply intersect all of the subgroups that contain the g_i to define it.

We know what cyclic groups look like – they are all isomorphic to \mathbb{Z} or \mathbb{Z}_n . So now we'd like a structure theorem for abelian finitely generated groups.

Theorem 7. Let G be a finitely generated abelian group. Then

$$G \cong \mathbb{Z}^r \times \prod_{i=1}^s \mathbb{Z}_{p_i}^{\alpha_i}$$

for some finite $r, s \in \mathbb{N}$ and p_i are (not necessarily distinct) primes.

Example 3. Let G be a finite abelian group of order 4. Then $G \cong \mathbb{Z}_4$ or \mathbb{Z}_2^2 , which are not isomorphic because every element in \mathbb{Z}_2^2 has order 2 where \mathbb{Z}_4 contains an element of order 4.

2.7 Fundamental Homomorphism Theorem

Let $\varphi : G \rightarrow G'$ be a group homomorphism and define $\ker \varphi = \{g \in G \mid \varphi(g) = e'\}$.

2.7.1 The First Homomorphism Theorem

Theorem 8. There exists a map $\varphi' : G/\ker \varphi \rightarrow G'$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \eta \downarrow & \nearrow \varphi' & \\ G/\ker \varphi & & \end{array}$$

That is, $\varphi = \varphi' \circ \eta$, and φ' is an isomorphism onto its image, so $G/\ker \varphi = \text{im } \varphi$. This map is given by $\varphi'(g(\ker \varphi)) = \varphi(g)$.

Exercise 2. Check that φ is well-defined.

2.7.2 The Second Theorem

Theorem 9. Let $K, N \leq G$ where $N \trianglelefteq G$. Then

$$\frac{K}{N \cap K} \cong \frac{NK}{N}$$

Proof. Define a map $K \xrightarrow{\varphi} NK/N$ by $\varphi(k) = kN$. You can show that φ is onto by looking at $\ker \varphi$; note that $kN = \varphi(k) = N \iff k \in N$, and so $\ker \varphi = N \cap K$. \square

3 Lecture 2