Homological Algebra

Problem Set 5

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2021-04-02

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Problem 1.0.1 (Weibel 6.1.1)
Let $A \in \mathsf{R\text{-}Mod}$. Show that $A^G \leq A$ is the maximal G-trivial submodule, and conclude that if Triv is defined as the trivial module functor, there is an adjunction

$$\mathsf{Ab} \underset{(\,\cdot\,)^G}{\overset{\mathrm{Triv}}{\underset{\perp}{\rightleftarrows}}} \mathsf{G\text{-}Mod},$$

so taking invariants is a right-adjoint. Conclude that $(\,\cdot\,)^G$ is left-exact.

Solution:
That $A^G \leq A$ is the maximal trivial G-trivial submodule is clear: if there were any A' such that $A^G \leq A' \leq A$ with A' being G-trivial, we would have $ga' = a'$ for all $a' \in A'$. This would imply $a' \in A^G$ by definition, so $A' \leq A^g$ and this forces $A^G = A'$.
Moreover, since right adjoints are left-exact, it suffices to show the adjunction exists. The claim is that there is a natural isomorphism between the sets of morphisms

$$\begin{gathered}\mathsf{Ab}(A, M^G) \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} \mathsf{G\text{-}Mod}(\mathrm{Triv}(A), M) \\ (A \xrightarrow{f} M^G) \mapsto (A \xrightarrow{\tilde{f}} M^G \hookrightarrow M) \\ (A \xrightarrow{\tilde{w}} M^G) \leftarrow\!\shortmid (\mathrm{Triv}(A) \xrightarrow{g} M).\end{gathered}$$

Claim: Θ is well-defined:

Proof (?).
We use the fact that as sets, $A \cong \mathrm{Triv}(A)$, and so there is a set map $\tilde{f} : \mathrm{Triv}(A) \to M^G$. Since $M_G \leq M$ also has the structure of a sub G-module, there is a canonical inclusion $\iota \in \mathrm{Hom}_G(M^G, M)$, and so we can define the set map $\mathrm{Triv}(A) \xrightarrow{F := \iota \circ \tilde{f}} M$.
To see that $F := \Theta(f) \in \mathrm{Hom}_G(\mathrm{Triv}(A), M)$ is actually a morphism in $\mathsf{G\text{-}Mod}$, we check:

- $f(a_1 + a_2) = f(a_1) + f(a_2) \in M$: This is true since the equation already holds in Ab, and
- $f(ga) = gf(a)$: This holds because -Considering the RHS, since $\operatorname{im} f \subseteq M^G \implies \operatorname{im} F \subseteq M^G$, we have $gf(a) = f(a)$.
  - Considering the LHS, since $\mathrm{Triv}(A)$ has a trivial G-action, we have $ga = a$ and thus $f(ga) = f(a)$.

■

Claim: Ψ is well-defined.

Proof (?).
This follows from the fact that if $h \in \mathrm{Hom}_G(\mathrm{Triv}(A), M)$ we must have $h(ga) = gh(a)$ by virtue of being a G-module morphism. But since $a \in \mathrm{Triv}(A)$, we have $ga = a$, so $h(ga) = h(a)$. Combining these we have $gh(a) = h(a)$, and so $h(a) \in M^G$ and $\mathrm{im}\, h \subseteq M^G$, so restricting the codomain yields a set map $\tilde{h} : A \to M^G$. That this is a morphism of abelian groups follows from the fact that the equation $h(a_1 + a_2) = h(a_1) + h(a_2)$ holds for the original h, so $\tilde{h}$ satisfies this equation as well.

■

Claim: Θ, Ψ are mutually inverse, yielding an isomorphism of sets.

Proof (?).
We can characterize Θ as first regarding the set map $f : A \to M^G$ as a G-module map $f : \mathrm{Triv}(A) \to M^G$ and then extending the codomain to M. If $f \in \mathsf{Ab}(A, M^G)$, noting that its codomain is already M^G, taking $(\Psi\Theta)(f)$ is extending codomain to M followed by restricting codomain back M^G, so this recovers f. Similarly, if $h \in \mathrm{Hom}_G(\mathrm{Triv}(A), M)$, the composition $(\Theta\Psi)(h)$ is restricting codomain to M^G and then extending it back to M, which recovers h. Thus these two sets of morphisms are isomorphic.

■

Claim: The isomorphism is natural, so for $f \in \mathrm{Hom}_{\mathsf{Ab}}(A, B)$ and $g \in \mathrm{Hom}_G(M, N)$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_G(\mathrm{Triv}(B), M) \arrow[r, "(Lf)^*"] \arrow[d, "\Psi_{BM}"] & \mathrm{Hom}_G(\mathrm{Triv}(A), M) \arrow[r, "g_*"] \arrow[d, "\Psi_{AM}"] & \mathrm{Hom}_G(\mathrm{Triv}(A), N) \arrow[d, "\Psi_{AN}"] \\
\mathrm{Hom}_{\mathsf{Ab}}(B, M^G) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathsf{Ab}}(A, M^G) \arrow[r, "(Rg)_*"] & \mathrm{Hom}_{\mathsf{Ab}}(A, N^G)
\end{tikzcd}$$

Link to Diagram

Here $f^*(h) := h \circ f$ and $g_*(h) := g \circ h$, $L(\,\cdot\,) := \mathrm{Triv}(\,\cdot\,)$, and $R(\,\cdot\,) := (\,\cdot\,)^G$.

Proof (?).
That the first square commutes follows from tracing out the two paths: taking $h : \mathrm{Triv}(B) \to M$ and first going down and right yields

$$h \mapsto \Psi_{BM}(h) \mapsto f \circ \Psi_{BM}(h) \in \mathrm{Hom}_{\mathsf{Ab}}(A, M^G),$$

and now going right and down yields

$$h \mapsto Lf \circ h \mapsto \Psi_{AM}(Lf \circ h) \in \mathrm{Hom}_{\mathsf{Ab}}(A, M^G).$$

We want to show that these functions are equal, so it suffices to check that they take the same value on every element. This is clear from the fact that all of the Ψ maps were given by restriction of codomain, and do not alter the underlying set functions, and so we have

$$(f \circ \Psi_{BM}(h))(a) = f(\Psi_{BM}(h)(a)) = f(h(a)) \qquad \forall a \in A$$

and

$$(\Psi_A M(Lf \circ h))(a) = Lf(h(a) = f(h(a)) \qquad \forall a \in A,$$

where in the last equality we've used that if $f \in \mathrm{Hom}_{\mathsf{Ab}}(A, B)$ then $Lf \in \mathrm{Hom}_G(\mathrm{Triv}(A), \mathrm{Triv}(B))$ is the same underlying set map.
The right-hand square commutes by a similar argument: for $g \in \mathrm{Hom}_G(M, N)$, the map $Rg \in \mathrm{Hom}_{\mathsf{Ab}}(M^G, N^G)$ is given by restriction of g to a submodule. Fixing $h \in \mathrm{Hom}_G(\mathrm{Triv}(A), M)$, we trace out

$$h \mapsto \Psi_{AM}(h) \mapsto \Psi_{AM}(h) \circ Rg$$

and

$$h \mapsto h \circ g \mapsto \Psi_{AN}(h \circ g),$$

so we want to show

$$(\Psi_{AM}(h) \circ Rg)(a) = (\Psi_{AN}(h \circ g))(a) \qquad \forall a \in \mathrm{Triv}(A).$$

We again use that Ψ doesn't alter set-level values, so

$$(\Psi_{AM}(h) \circ Rg)(a) = \Psi_{AM}(h)(Rg(a)) = h(Rg(a)).$$

and

$$(\Psi_{AN}(h \circ g))(a) = (h \circ g)(a) = h(g(a)).$$

Finally, we note that since $a \in \mathrm{Triv}(A)$, every $g(a)$ is actually in M^G, and so $g(a) = Rg(a)$ at the level of sets. So these maps are equal.

■

Problem 1.0.2 (Weibel 6.1.2)
Let $A \in \mathsf{R\text{-}Mod}$. Show that A_G is the largest trivial quotient module of A, and conclude that there is an adjunction

$$\begin{tikzcd} \mathsf{G\text{-}Mod} \arrow[r, shift left, "(\cdot)_G"] \arrow[r, phantom, "\perp"] & \mathsf{R\text{-}Mod}. \arrow[l, shift left, "\mathrm{Triv}"] \end{tikzcd}$$

Solution:
Define a map

$$\begin{aligned} \mathrm{Hom}_G(M, \mathrm{Triv}(A)) &\to \mathrm{Hom}_{\mathsf{Ab}}(M_G, A) \\ f &\mapsto \overline{f} \\ \tilde{h} &\mapsfrom h \end{aligned}$$

where we realize $M_G := M / \langle gm - m \rangle$ and take $\overline{f}$ to be the natural quotient map:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[d, "\pi"'] & & A \\ M_G := M/R := M/\langle gm - m \rangle \arrow[urr, "\tilde{f}"'] & & \end{tikzcd}$$

Link to Diagram

and given $h : M_G \to A$, we define $\tilde{h} = h \circ \pi$, so

$$\begin{aligned} \tilde{h} : M &\to \mathrm{Triv}(A) \\ m &\mapsto h(\pi(m)). \end{aligned}$$

That f descends to a well-defined map on the quotient follows from the fact that $ga = a$ for all g and all $a \in \mathrm{Triv}(A)$, so $f(gm - m) = gf(m) - f(m)$, which is of the form $ga - a$, and so $\ker f \subseteq \langle ga - a \rangle$. The inverse map is a well-defined G-module morphism, using the fact that since g acts trivially on both sides we have $gh(m) = h(m) = h(gm)$.
In words, the forward map is descending morphisms to the quotient, and the inverse map is lifting maps from the quotient by composing with the projection. That these compose to the identity in other order and yield a bijection follows from the commutativity of the triangle for the quotient. That this is an adjunction follows from naturality of this isomorphism, for which the proof is similar to the one for the previous problem.

Problem 1.0.3 (Weibel 6.1.3)
Define

$$\begin{aligned} \varepsilon : \mathbb{Z}G &\to \mathbb{Z} \\ \sum n_g g &\mapsto \sum n_g \end{aligned}$$

and $\mathcal{J} := \ker \varepsilon$.

1. Show that if G is an infinite group, then $H^0(G; \mathbb{Z}G) \cong (\mathbb{Z}G)^G = 0$.

2. When G is a finite group, show that the map $\mathbb{Z} \cdot N := (\mathbb{Z}G)^G \to (\mathbb{Z}G)_G \cong \mathbb{Z}$ sends the norm $N := \sum_{g \in G} g$ to $\#G$, and that this is an injection.

3. Conclude that when G is finite,

$$\mathcal{J} \cong \ker(\mathbb{Z}G \xrightarrow{x \mapsto Nx} \mathbb{Z}G) := \left\{ a \in \mathbb{Z}G \;\middle|\; Na = 0 \right\}.$$

Solution (Part 1):
By definition we have $H^0(G;A) := A^G$, and lemma 6.1.1, we also have $H^0(G;A) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ for any A. Taking $A := \mathbb{Z}G$ yields the first isomorphism appearing in (1), so it suffice to show $(\mathbb{Z}G)^G = 0$. In this case, we can write

$$(\mathbb{Z}G)^G := \left\{ x := \sum_{j \in J} f(g_j) g_j \;\middle|\; h \sum_{j \in J} f(g_j) g_j = \sum_{j \in J} f(g_j) g_j \; \forall h \in G \right\} \subseteq \bigoplus_{g \in G} \mathbb{Z}$$

where $f : G \to \mathbb{Z}$ is some coefficient function, and every such sum has only finitely many nonzero terms. Since the g_α form a basis of the free $\mathbb{Z}$-module $\mathbb{Z}G$, we can equate coefficients. The coefficient of hg_j in the first expression is $f(hg_j)$, and is $f(g_j)$ in the second, so we must have $f(g_j) = f(hg_j)$ for all $h \in G$. Toward a contradiction fix an $x \in (\mathbb{Z}G)^G$ and suppose $x \neq 0$. Then there is at least one j for which $f(g_j) \neq 0$, then noting that G acts on itself transitively by left-multiplication, we can write any g_i as $g_i = h(i)g_j$ for some $h = h(i)$ depending on i. We then have

$$x := \sum f(g_i) g_i = \sum f(h(i)g_j) h(i) g_j,$$

and noting that

$$0 \neq f(g_j) = f(h(i)g_j) \qquad \forall i,$$

if G is infinite then this produces a sum with infinitely terms with nonzero coefficients. $\lightning$

Solution (Part 2):
We first identify $\mathbb{Z}G^G = \mathbb{Z}N$ and $\mathbb{Z}G_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G$ where we regard $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. We then identify the natural map $\mathbb{Z}G^G \to \mathbb{Z}$ as the composition

$$\begin{aligned} F : \mathbb{Z}G^G &\to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G && \to \mathbb{Z} \\ nN &\mapsto n \otimes N && \mapsto N \curvearrowright n := \sum_{g \in G} n, \end{aligned}$$

where we've used the trivial action $\mathbb{Z}G \curvearrowright \mathbb{Z}$ to write $(g+h) \curvearrowright n = gn + hn := n + n = 2n$ and extended this over the sum. So we have

$$nN \mapsto \sum_{g \in G} n = n \sum_{g \in G} 1 = n(\#G),$$

and taking $n = 1$ sends $N \mapsto \#G$. From this construction, we find that $\ker F = \left\{ n \in \mathbb{Z} \;\middle|\; n(\#G) = 0 \right\} = \{0\}$, so F is injective.

Solution (Part 3):
It suffices to show $Nx = 0 \iff \varepsilon(x) = 0$, since this yields both subset containments.
$\implies$: Let $N : \mathbb{Z}G \to \mathbb{Z}G$ by multiplication by N, then $\ker N := \left\{ \sum n_g g \in \mathbb{Z}G \;\middle|\; N \sum n_g g = 0 \right\}$. Letting $x := \sum n_g g \in \ker N$, supposing that $Nx = 0$, we can expand this product using the multiplication in $\mathbb{Z}G$:

$$\left(\sum_h n_h h \right) \left(\sum_g n_g g \right) = \sum_{g,h} n_h n_g hg,$$

and so

$$Nx := \left(\sum_h h \right) \left(\sum_g n_g g \right) = \sum_{h,g} n_g hg = 0.$$

We can now consider the coefficient of the identity 1_G. For any fixed g, there is some h such that $h = g^{-1}$, and this will contribute a term of the form $n_g \cdot 1_G$. Letting g run over G yields

$$Nx = \left(\sum_g n_g \right) \cdot 1_G + \cdots = 0.$$

Equating coefficients, we must have $\sum_g n_g = 0$, and noting that $\varepsilon(x) = \varepsilon(\sum n_g g) = \sum n_g = 0$, this means that $x \in \ker \varepsilon := \mathcal{J}$.
$\impliedby$: Conversely, if $\varepsilon(x) = 0$, then we can use an alternative characterization of the product:

$$\left(\sum_h n_h h \right) \left(\sum_g n_g g \right) = \sum_g \left(\sum_k n_{k^{-1}g} \right) g.$$

Applying this to $N := \sum_h h$, we have

$$Nx := \left(\sum_h h \right) \left(\sum_g n_g g \right) = \sum_g \left(\sum_k n_{k^{-1}g} \right) g.$$

But the inner sum runs over all elements of G, so for each fixed g it is equal to $\sum n_g := \varepsilon(x)$ and we in fact have

$$Nx = \sum_g \varepsilon(x) g = \sum_g 0 \cdot g = 0.$$

Problem 1.0.4 (Weibel 5.1.1)
Suppose that E is a double complex supported only in columns p and $p-1$ and define $T := \mathrm{Tot}^{\Pi}(E)$. Fix n and set $q := n - p$, so that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{p,q}$. Show that the previous argument (Weibel p. 120-121) computes the homology "up to extensions" in the sense that there is a SES

$$0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E_{p,q} \to 0.$$

Solution:

Following an argument from here

Without loss of generality, reindex E so that $p-1=0$ and $p=1$. Then since n is fixed, we have $q=n-p$ so that $q=n,n-1$; we then want to show that there is a SES

$$0 \to E_{0,n} \to H_nT \to E_{1,n-1} \to 0.$$

Noting that the differentials on E_r have bidegree $(-r, r-1)$ since this is a homological spectral sequence, when $r \geq 2$, fix an element on E_r in column 1 or 2 and consider the following two cases for the differentials:

- For outgoing differentials, the source is in column 0 or 1, so the target is in column $j \leq -1$
- For incoming differentials, the target is in column 0 or 1, so the source is in column $j \geq 2$.

Thus the spectral sequence collapses at E^2, and we have $E^{\infty}_{p,q} = E^2_{p,q}$ for all p,q Since this converges to $E^2_{p,q} \Rightarrow H_{p+q}T \coloneqq H_n(T)$, there is increasing exhaustive filtration $F_\bullet$ on $H_n(T)$ such that

$$E^{\infty}_{p,q} = \frac{F_pH_n(T)}{F_{p-1}H_n(T)}.$$

Thus the filtration has the form

$$0 \subseteq F_0H_n(T) \subseteq F_1H_n(T) = F_2H_n(T) = \cdots = H_n(T),$$

where we've used that if $i \neq 0,1$ we have

$$E^{\infty}_{i,q} = 0 = \frac{F_iH_n(T)}{F_{i-1}H_n(T)} \implies F_iH_n(T) = F_{i-1}H_n(T).$$

Setting $i=0,1$, we have

$$\begin{aligned} E^2_{0,q} &\cong F_0H_n(T) \\ E^2_{1,q-1} &\cong F_1H_n(T)/F_0H_n(T), \end{aligned}$$

and so if we write the following exact sequence

$$0 \to F_0H_n(T) \to F_1H_n(T) \to \frac{F_1H_n(T)}{F_0H_n(T)} \to 0$$

we can replace each term by an isomorphic term to get an isomorphic exact sequence

$$0 \to E^2_{0,q} \to H_n(T) \to E^2_{1,q-1} \to 0.$$

Problem 1.0.5 (Weibel 5.2.1)
Suppose that a spectral sequence converging to H_* has $E^2_{p,q} = 0$ for $p \neq 0,1$. Show that there

is a family of SESs

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0.$$

Solution:

This was solved in the problem above – which probably indicates that the previous problem should have been solved another way!

Problem 1.0.6 (Weibel 5.1.2)

1. Show that $E^2_{p,q}$ can be presented as

$$\frac{\left\{(a,b) \in E_{p-1,q+1} \times E_{p,q} \;\middle|\; d^v b = d^v a + d^h b = 0\right\}}{\left\langle (a,0), \left\{(d^h x, d^v x) \;\middle|\; x \in E_{p,q+1}\right\}, \left\{(0, d^h c) \;\middle|\; c \in E_{p+1,q},\ d^v c = 0\right\}\right\rangle}.$$

2. Show that if $d^h(a) = 0$, any such pair (a,b) determines an element $H_{p+q}(T)$.
3. Show that the formula $d(a,b) \coloneqq (0, d^h(a))$ yields a well-defined map

$$d : E^2_{p,q} \to E^2_{p-2,q+1}.$$

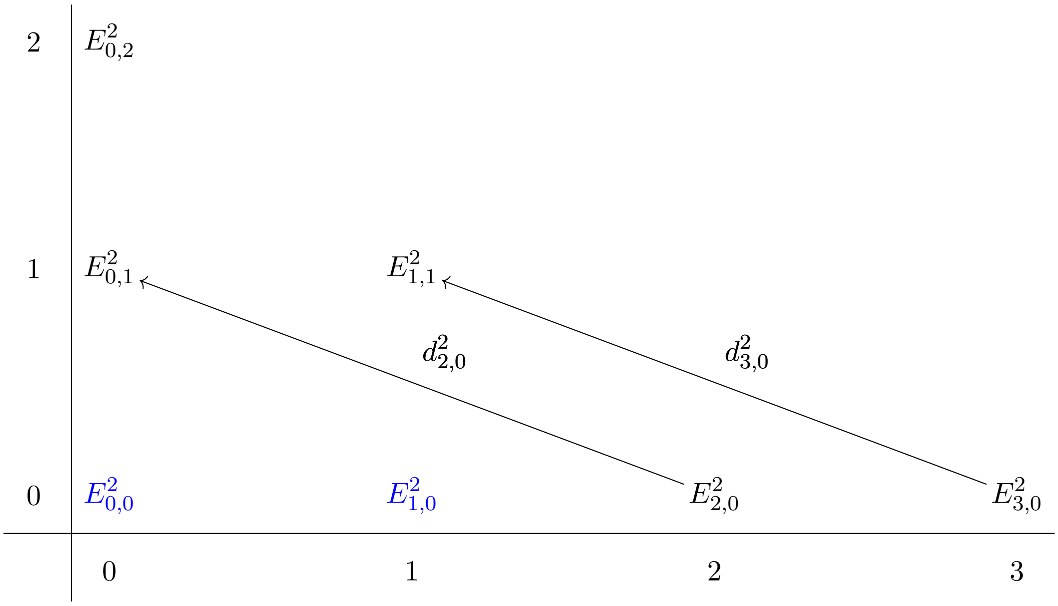
Solution:

Todo

Problem 1.0.7 (Weibel 5.1.3)
Let E be a first quadrant spectral sequence. By diagram chasing, show that $E^2_{0,0} = H_0(T)$ and that there is an exact sequence

$$H_2(T) \to E^2_{2,0} \to E^2_{0,1} \to H_1(T) \to E^2_{1,0} \to 0.$$

Solution:
Consider passing from E^2 to E^3. On the 2nd page, we have the following situation:



Here the blue terms have stabilized, since either their domain or codomain is a zero. If a first quadrant spectral sequence converges, for each n there will be a filtration

$$0 = F_{-1}H_n(T) \subseteq F_0H_n(T) \subseteq F_1H_n(T) \subseteq \cdots \subseteq H_n(T),$$

where $E^\infty_{p,q} \cong F_pH_{p+q}(T)/F_{p-1}H_{p+q}(T)$. Thus the E^∞ page is the following:

$$
\begin{array}{c|cccc}
\vdots & & & & \\
F_0H_2 & & & & \\
 & & \iddots & & \\
F_0H_1 & \dfrac{F_1H_2}{F_0H_2} & & & \\
F_0H_0 \cong H_0 & \dfrac{F_1H_1}{F_0H_1} \cong H_1 & & \dfrac{F_2H_1}{F_1H_1} \cong H_2 & \cdots \\
\hline
 & & & &
\end{array}
$$

Link to Diagram

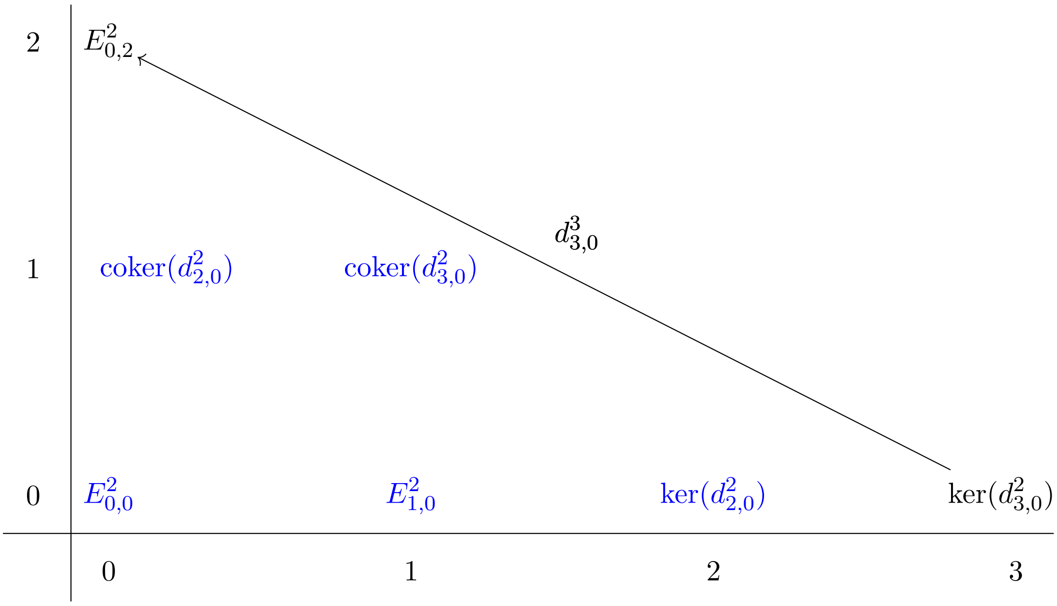
We note that for $H_0(T)$, since E is first-quadrant, this filtration must be of the form

$$0 = F_{-1}H_0(T) \subseteq F_0H_0(T) = F_1H_0(T) = \cdots = H_0(T),$$

i.e. there are no nontrivial terms. This follows because the degree zero part of the associated graded complex will be given by $\bigoplus\limits_{p+q=0} E^{\infty}_{p,q}$, so $p = q = 0$ will contribute a nonzero term, and every other term must involve either $p < 0$ or $q < 0$ and thus be zero, and $F_j/F_{j-1} = 0$ for some j forces $F_j \cong F_{j-1}$. We thus have

$$
\begin{aligned}
E^2_{0,0} &= E^{\infty}_{0,0} \\
&\cong \frac{F_0H_{0+0}(T)}{F_{-1}H_{0+0}(T)} \\
&= F_0H_0(T) \\
&= H_0(T),
\end{aligned}
$$

which yields part 1.
Now passing to the 3rd page yields the following, where several new terms have stabilized:



Starting in total degree 2, we can identify

- $E^\infty_{0,1} = F_0H_1$, and by stabilization this is isomorphic to $\mathrm{coker}(d^2_{2,0})$.
- $E^\infty_{1,0} = \dfrac{F_1H_1}{F_0H_1} = \dfrac{H_1}{F_0H_1}$, and by stabilization this is isomorphic to $E^2_{1,0}$.

From the second fact, we can extract the canonical quotient SES

$$0 \to F_0H_1 \to H_1 \to \frac{H_1}{F_0H_1} \to 0,$$

and using the above identifications, this is isomorphic to a SES of the form

$$0 \to E^\infty_{0,1} \to H_1 \to E^\infty_{1,0} \to 0,$$

which we will splice to form the desired 5-term exact sequence.
In total degree 2, we can identify

- $E^\infty_{0,2} = F_0H_2$: this is unknown at this stage, but we won't need it.
- $E^\infty_{1,1} = \dfrac{F_1H_2}{F_0H_2} \cong \mathrm{coker}(d^2_{3,0} : E^2_{3,0} \to E^2_{1,1})$ by stabilization, and
- $E^\infty_{2,0} \cong \dfrac{F_2H_2}{F_1H_2} = \dfrac{H_2}{F_1H_2} \cong \ker(d^2_{2,0} : E^2_{2,0} \to E^2_{0,1})$ by stabilization.

We will only need this last fact: we can construct the composition

$$\ker(d^2_{2,0} : E^2_{2,0} \to E^2_{0,1}) \hookrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \twoheadrightarrow \mathrm{coker}(d^2_{2,0} : E^2_{2,0} \to E^2_{0,1}).$$

Noting that the first term is isomorphic to $E^\infty_{2,0}$ by the analysis in degree 2 and the last is isomorphic to $E^\infty_{0,1}$ by the analysis in degree 1, there is a sequence

$$E^\infty_{2,0} \hookrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \twoheadrightarrow E^\infty_{0,1}.$$

We can now use that $E^\infty_{2,0}$ is a quotient of H_2 to splice onto the left, and splice the previous SES on the right to obtain

$$H_2 \twoheadrightarrow E^\infty_{2,0} \hookrightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \twoheadrightarrow E^\infty_{0,1} \rightarrow 0 \rightarrow H_1 \to E^\infty_{1,0} \to 0.$$

This isn't an *exact* sequence as written, so we collapse several maps to get a new sequence

$$H_2 \rightarrow E^2_{2,0} \xrightarrow{d^2_{2,0}} E^2_{0,1} \rightarrow H_1 \to E^\infty_{1,0} \to 0.$$

This new sequence is exact:

- Exactness at the 1st term $E^2_{2,0}$: this follows because in the first sequence, H_2 surjected onto $E^\infty_{2,0}$ which was isomorphic to the kernel of $d^2_{2,0}$, which is precisely the next map in the sequence.
- Exactness at the 2nd term $E^2_{0,1}$: this follows because $E^\infty_{0,1} \cong \operatorname{coker} d^2_{2,0}$, which is precisely the previous map in the sequence. So applying this d maps onto its image, which is quotiented out in $E^\infty_{0,1}$ before mapping to H_1.
- Exactness at the 3rd term H_1: this follows because we've factored the composite map through zero by construction.
- Exactness at the remaining terms: this follows from exactness of the original SES.