

Title

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1.1 Compact-Open Topology

1.2 Isotopy

- Define a homotopy between $f, g : X \rightarrow Y$ as a map $F : S \times I \rightarrow S$ restricting to f, g on the ends.
 - Equivalently: a *path* in $\text{Map}(X, Y)$.
- Isotopy: require the partially-applied function $F_t : S \rightarrow S$ to be homeomorphisms for every t .
 - Equivalently: a path in $\text{Map}(X, Y)$

1.3 Self-Homeomorphisms

- In any category, the automorphisms form a group.
 - In a general category \mathcal{C} , we can always define the group $\text{Aut}_{\mathcal{C}}(X)$.
 - * If the group has a topology, we can consider $\pi_0 \text{Aut}_{\mathcal{C}}(X)$, the set of path components.
 - * Since groups have identities, we can consider $\text{Aut}_{\mathcal{C}}^0(X)$, the path component containing the identity.
 - So we make a general definition, the *extended mapping class group*:

$$\text{MCG}_{\mathcal{C}}^{\pm}(X) := \text{Aut}_{\mathcal{C}}(X) / \text{Aut}_{\mathcal{C}}^0(X).$$

- Here the \pm indicates that we take both orientation preserving and non-preserving automorphisms.

- Has an index 2 subgroup of orientation-preserving automorphisms, $\text{MCG}^+(X)$.

- Now restrict attention to

$$\text{Homeo}(X) := \text{Aut}_{\text{Top}}(X) = \left\{ f \in \text{Map}(X, X) \mid f \text{ is an isomorphism} \right\}$$

equipped with \mathcal{O}_{CO} .

- Taking $\text{MCG}_{\text{Top}}^{\pm}(X)$ yields ??

- Similarly, we can do all of this in the smooth category:

$$\text{Diffeo}(X) := \text{Aut}_{C^{\infty}}(X).$$

- Taking $\text{MCG}_{C^{\infty}}(X)$ yields ??

- Similarly, we can do this for the homotopy category of spaces:

$$\text{ho}(X) := \{[f]\}.$$

- Taking $\text{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences*.

- For topological manifolds: Isotopy classes of homeomorphisms

- In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi \text{Aut}(X)$.

- For surfaces: $\text{MCG}(S)$ on the Teichmüller space $T(S)$, yielding a SES

$$0 \rightarrow \text{MCG}(S) \rightarrow T(S) \rightarrow \widetilde{\mathcal{M}}_g(S) \rightarrow 0$$

where the last term is the moduli space of Riemann surfaces homeomorphic to X .

- $T(S)$ is the moduli space of complex structures on S , up to the action of homeomorphisms that are isotopic to the identity:

* Points are isomorphism classes of marked Riemann surfaces

- Used in the Nielsen-Thurston Classification (for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of: periodic: reducible (preserves some simple closed curves), or pseudo-Anosov (has directions of expansion/contraction))

- Generated by Dehn twists: a self homeomorphism
- Any finite group is $\text{MCG}(X)$ for some compact hyperbolic 3-manifold X .

Theorem 1.1 (Dehn-Nielsen-Baer).

$$\text{MCG}^{\pm}(\Sigma_g) \cong \text{Out}(\pi_1(\Sigma_g)).$$

1.4 Dehn Twists

Claim: Let $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, then $\text{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$\begin{aligned} \tau_0 : \mathbb{C} &\rightarrow \mathbb{C} \\ z &\mapsto \exp(2\pi i |z|) z. \end{aligned}$$