Title

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Tuesday 15$^{\text{th}}$ September, 2020

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1.1 Review

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomials

$$I \mapsto V(I) := \left\{ a \mid f(a) = 0 \, \forall f \in I \right\}$$

$$I(X) := \left\{ f \mid f|_X = 0 \right\} \hookleftarrow X$$

Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets).

> Recall that we defined the coordinate ring $A(X) := k[x_1, \cdots, x_n]/I(X)$, which contained no nilpotents.

We had some results about dimension

1. $\dim X < \infty$ and $\dim \mathbb{A}^n = n$.
2. $\dim Y + \operatorname{codim}_X Y = \dim X$ when $Y \subset X$ is irreducible.
3. Only over $\bar{k} = k$, $\operatorname{codim}_X V(f) = 1$.

Example 1.1.
Take $V(x^2 + y^2) \subset \mathbb{A}^2/\mathbb{R}$

> **Definition 1.0.1** (?)**.**
> An affine variety Y of
> - $\dim Y = 1$ is a **curve**,
> - $\dim Y = 2$ is a **surface**,
> - $\mathrm{codim}_X Y = 1$ is a **hypersurface in** X

Question: Is every hypersurface the vanishing locus of a *single* polynomials $f \in A(X)$?

Answer: This is true iff $A(X)$ is a UFD.

> **Definition 1.0.2** (Codimension in a Ring)**.**
> $\mathrm{codim}_R \mathfrak{p}$ is the length of the longest chain
> $$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}$$

> **Proposition 1.1***(?)***.**
> Let R be a Noetherian domain, then TFAE
> a. All prime ideals of codimension 1 are principal.
> b. R is a UFD.

> *Proof .*
> $a \implies b$:
> Let f be a nonzero unit.
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