Title

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1 │ **Lecture 5A**

We saw an interesting example of a function field in more than one variable which showed that valuations of rank larger than 1 can arise, but this does not happen for one variable function fields. That is, for K/k of transcendence degree 1, all valuations on K which are trivial on k are discrete. We'll now want to go farther and describe the places $\Sigma(K/k)$, which will be the set of points on an algebraic curve. Scheme-theoretically, this will literally be the set of closed points on a certain projective curve whose function field is K. Note that a priori, finding closed points on a curve over an arbitrary field is hard!

Recall that if A is a Dedekind domain such that $\mathrm{ff}(A) = K$, then for all $\mathfrak{p} \in \mathrm{mSpec}(A)$ there exists a discrete valuation v_p on K. I.e., every maximal ideal induces a discrete valuation that is A-regular, so the valuation ring will contain A. How is this obtained? Take a nonzero $x \in K^\times$, and take the corresponding principal fractional ideal $\langle x \rangle := Ax$, which we can factor in a Dedekind domain as $Ax = \prod_{\mathfrak{p} \in \mathrm{mSpec}(A)} \mathfrak{p}^{\alpha_\mathfrak{p}}$ with $\alpha_\mathfrak{p} \in \mathbb{Z}$. This looks like an infinite product, but for any fixed x, only finitely many α are nonzero. Note that these α are exactly what we're looking for: the $\mathfrak{p}$-adic evaluation of x is given precisely by $v_\mathfrak{p}(x) := \alpha_\mathfrak{p}$, where we are using unique factorization of ideals in Dedekind domains. Thus we have a map

$$v_\cdot : \mathrm{mSpec}(A) \to \Sigma(K/A)$$
$$\mathfrak{p} \mapsto v_\mathfrak{p}.$$

So this sends a maximal ideal to a place that is A-regular, and it turns out to be a bijection.

Proposition 1.0.1*(?).*
The map v is a bijection, and thus we may write

$$\Sigma(K/A) \cong \mathrm{mSpec}(A).$$

Proof (?).

Claim: v is injective.
If $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{mSpec}(A)$ are two different maximal ideals. Then there exists an element $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$, and so $x^{-1} \in A_{\mathfrak{p}_2} \setminus A_{\mathfrak{p}_1}$. This follows since if x is not in $\mathfrak{p}_2$, its $\mathfrak{p}_2$-adic valuation is zero, and thus the $\mathfrak{p}_2$-adic valuation of x^{-1} is $-0 = 0$ as well. On the other hand, since $x \in \mathfrak{p}_1$, its $\mathfrak{p}_1$-adic valuation is positive and therefore $v_{\mathfrak{p}_1}(x^{-1}) < 0$ and x^{-1} is not in $A_{\mathfrak{p}_1}$.

Claim: v is surjective.
Let $v \in \Sigma(K/A)$, so $A \subset R_v$, i.e. take a valuation whose valuation ring contains A. Note that we're not assuming the valuation is discrete, this can be a g ∎