Title

D. Zack Garza

Monday 12^{th} October, 2020

Contents

1 | Monday, October 12

Chapter 10: From Floer to Morse

- To be precise with notation, define
  - $CM_*(H, J)$ will be the morse complex associated with a Morse function H, its vector field ∇H the gradient for the metric defined by J, ω.
  - $CF_*(H, J)$

Theorem 1.0.1*(Main Goal).*
There exists a nondegenerate Hamiltonian that is sufficiently small in the C^2 topology for which both the Floer and Morse complexes are well-defined, and

$$CF_*(H, J) \cong CM_{*+n}(H, J) = CM(H, J)[n].$$

- Can start with an H_0 and rescale to define $H := H_0/k$

Why?

- When sufficiently small, periodic trajectories are constant
  - Thus $\operatorname{crit}(\mathcal{A}_H) = \operatorname{crit}(H)$
  - Implies that H is a Morse function
  - Implies the for the Hessian $\operatorname{Spec}(\nabla^2 H) \cap 2\pi\mathbb{Z} = \emptyset$

- Allows comparing Morse index of critical point to Maslov index of corresponding constant trajectory using

$$\mathrm{Ind}_H(x) = \mu(x) + n.$$

- Gives an isomorphism of vector spaces, up to a dimension shift.
- Next need to show both differentials ∂_M, ∂_F can be defined, and they coincide
- Defining ∂_M:
  - Need a vector field $X \in \Gamma(T * M)$ adapted to H
  - X needs to satisfy Smale condition

\todo[inline]{What is the Smale condition?}

- Then relate trajectories of X to solutions of Floer equation, i.e. relate

$$\left\{ \begin{matrix} \text{Solutions to} \\ \frac{\partial u}{\partial s} + X(u) = 0 \end{matrix} \right\} \iff \left\{ \begin{matrix} \text{Solutions to} \\ \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u) = 0 \end{matrix} \right\}.$$

  In other words: want $X = \nabla H$ for the metric induced by J, ω.

Theorem 1.0.2*(Theorem to Prove)*.
Let H be Morse on (W, ω). Then there exists a dense subset $\mathcal{J}_{\mathrm{reg}}(H)$ of almost complex structures J calibrated by ω such that $(H, -JX_H)$ is Morse-Smale.

Note: transversality result analogous to ones in 8.5

What is Morse-Smale?

- Big idea: running ideas backwards, getting theorems for Morse functions similar to what we did when linearizing the Floer operator
- Proof in two steps:
  - Step 1: Morse Side, arbitrary morse functions
    Linearize the Morse equation $\dfrac{\partial u}{\partial s} + X(u) = 0$ of the flow of $-X$ along one of its solutions $L_u Y = 0$.
    Show that whenever H is Morse and u is a trajectory connecting critical points, L_u is Fredholm and $\mathrm{Ind}(L_u) = \mathrm{Ind}_H(y) - \mathrm{Ind}_H(x)$.
    Show that for H a nondegenerate Hamiltonian and u a trajectory of JX_H, the operators $(d\mathcal{F})_u$ and L_u are Fredholm of equal index.
    Show that X is Smale $\iff$ L_u is surjective.
  - Step 2: Floer Side, specific case of Hamiltonian
    Prove the actual result.
- Now fix an almost complex structure to obtain a Smale vector field X
- Compare solutions to Floer equation and trajectories of X
  - Solutions to Floer equation that *do not* depend on t are precisely trajectories of $X = -\nabla H$.
- Next show that elements in $\ker(d\mathcal{F}_u)$ do not depend on t.
- Corollary: $d\mathcal{F}_u$ is surjective along every trajectory of ∇H.

- Then show that replacing $H_k := H/k$ for $k \gg 0$ preserves all critical points and all indices
- Punch line: all the solutions of the Floer equation that we need are time-independent.
  - Statement: For $k \gg 0$, solutions to the Floer equation for H_k connecting $x \to y$ with $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$ are independent of t.

Goal by end of Ch. 10:

- Show that all Floer solutions connecting two consecutive critical points are *also* Morse trajectories, and $d\mathcal{F}_v$ is surjective along these trajectories
- Yields equality of complexes

1.1 Summary

What is X_H

- Take H_k for $k \gg 0$ and $J \in \mathcal{J}_{\mathrm{reg}}$ (dense)
- Then when $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$, trajectories of Floer equation for (H, J) connecting critical points x, y are trajectories of the Smale vector field $X = -JX_H$.
  - x, y will be critical points for both H and $\mathcal{A}_H$
- Regularity? The linearized Floer operator is surjective along these trajectories
- Implies that $\mathcal{M}^{(H,J)}(x, y)$ is a manifold, so CF_* can be defined.
- Claim: this shows the differentials coincide, and we're done.

1.2 Linearizing the Morse Equation

- Let f be morse on $V \hookrightarrow \mathbb{R}^m$ ($m \gg 0$) with adapted pseudo-gradient field X, then

$$\left\{ \substack{\text{Trajectories} \\ \text{of } X} \right\} \iff \left\{ \substack{\text{Solutions of} \\ \frac{\partial u}{\partial s} + X(u(s)) = 0} \right\}.$$

- Fix a metric g on V such that $X = \nabla_g f$.
- Define the space of solutions of finite energy:

$$E(u) := \int_{\mathbb{R}} \left\| \frac{\partial u}{\partial s} \right\|^2 ds$$

$$\mathcal{M} := \left\{ u \in C^\infty(\mathbb{R}, V) \mid \frac{\partial u}{\partial s} + \nabla f = 0, \qquad E(u) < \infty \right\}.$$

- Then $\mathcal{M}$ is compact and equal to $\cup_{x,y} \mathcal{M}(x, y)$, using the fact that if V is compact, *all* trajectories are of finite energy
- Now go to coordinates and linearize the equation of the flow along the solution u to get a linear differential equation
- Yields an equation

$$\begin{aligned} L_u : W^{1,2}(\mathbb{R}, \mathbb{R}^n) &\to L^2(\mathbb{R}, \mathbb{R}^n) \\ Y &\mapsto \frac{\partial Y}{\partial s} + A(s)Y := L_u Y, \end{aligned}$$

where A is a matrix limiting to $\nabla_y^2 f$ and $\nabla_x^2 f$ at $s = \pm\infty$

  - Limiting to Hessians of nondegenerate critical points will yield symmetric invertible matrices
  - We then consider $\ker L_u \subseteq \ker(d\mathcal{F}_u)$. Note: we have exponential decay.

- Note: the space of solutions to equation linearized at u is $T_u\mathcal{M}(x, y)$.

1.2.1 Showing L_u is Fredholm

- Bootstrapping: $Y \in \ker\check{}_u$ in $W^{1,2}$ is continuous, thus C^1, this C^∞ and form a finite-dimensional vector space.
- Behavior at infinity: reduces to $L_u Y = 0 \iff \dfrac{\partial Y}{\partial s} = -AY$ where A is a constant diagonal matrix
  - This is a linear system, so solutions are $Y(s) = e^{-As}Y(0)$, i.e. $y_i(s) = y_i e^{-\lambda_i s}$.
- Will prove that if u is a trajectory of ∇f connecting $x \to y$ then L_u is Fredholm
  - Proof: involves bounding $W^{1,2}$ norm of Y by L^2 norms of $Y, L_u Y$.
  - Lots of integral estimates: Fourier transform, Plancherel, Cauchy-Schwarz
- Integral bound yields: $\dim\ker L_u < \infty$ and $\mathrm{im}(L)_u$ is closed.
- Lemma: $\dim\mathrm{coker} < \infty$.
  - Proof: computer kernel of adjoint $L_u^* = -\dfrac{\partial}{\partial s} + A^*$ where the matrix is transposed.
  - Use the fact that $Z \in \mathrm{coker}(L_u) \iff Z \in \ker(L_u^*)$, i.e. $L_u^* Z = 0$ in the sense of distributions

1.2.2 Computing $\mathrm{Ind}\, L_u$

- Unsurprisingly, will show $\mathrm{Ind}(L_u) = \mathrm{Ind}_f(x) - \mathrm{Ind}_f(y)$.
- Ideas in proof:
  - Will choose two real numbers σ, s to plug into u, and consider *resolvent*: map between tangent spaces to V at $u(\sigma), u(s)$.
  - Look at the tangent spaces at $u(\sigma)$ of the stable and unstable manifolds

$$\begin{aligned} E^{\mathrm{u}}(\sigma) &:= T_{u(\sigma)}W^{\mathrm{u}}(x) \\ E^{\mathrm{s}}(\sigma) &:= T_{u(\sigma)}W^{\mathrm{s}}(x) \end{aligned}$$
.

  - Then $\ker L_u$ is isomorphic to the intersection for all σ.

1.2.3 Smale Condition

- Recall $X = \nabla_g f$ for g a metric.
- Statement: the vector field X satisfies the Smale condition $\iff$ all L_u are surjective.

Proof .

- L_u is surjective $\iff$ $\operatorname{coker}(L_u) = 0$ $\iff$ $\ker(L_u^*)$ is injective
- This is equivalent to

$$T_{u(\sigma)} W^{\mathrm{u}}(x) + T_{u(\sigma)} W^{\mathrm{s}}(x) = T_{u(\sigma)} V.$$

- This is exactly the transversality condition for the stable and unstable manifolds
  - We want this for all critical points

■

1.3 10.4: Morse and Floer Trajectories Coincide

1.3.1 Comparing Kernels

- Note $\ker(L_u) \subset \ker(d\mathcal{F}_u)$ since

$$\left(\frac{\partial}{\partial s} + S(s)\right) Y = 0 \implies \left(\frac{\partial}{\partial s} J \frac{\partial}{\partial t} + S(s)\right) Y = 0,$$

  so just need to show reverse inclusion.

- Use a lemma: for $f : [0,1] \to \mathbb{R}$,

$$\|f\|_{L^p([0,1])} \left\| \frac{\partial f}{\partial t} \right\|_{L^p([0,1])},$$

  then apply this to $f(t) := Y(s,t)$ and $p = 2$.

- Yields an equation

$$\|\partial_s Y\|_{L^2}^2 + \|\partial_t Y\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\mathrm{op}}^2 \|Y\|_{L^2}^2 \implies \|Y\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\mathrm{op}} \|Y\|^2 L^2 \|\|$$

  where the sup term being small forces $Y = 0$.

1.3.2 Trajectories are Independent of t

- WTS: trajectories of H_k appearing in the Floer complex are exactly those appearing in the Morse complex.
  - I.e. proving 10.1.9

Idea of proof:

- Contradiction: suppose there exists a sequence $n_k \to \infty$ with time-dependent solutions u_{n_k} connecting $x \to y$ which solve the Floer equation
- Consider case where indices differ by 1: using broken trajectories theorem, extract a subsequence converging to some $v \in \mathcal{M}(x, y, H)$.
  - Show v doesn't depend on t

  - Since $d\mathcal{F}_v$ is surjective, v is in a 1-dim component, and thus an isolated point of $\mathcal{L}(x, y)$
  - Get a contradiction from taking $k \gg 0$ and using $v_{n_k}(s,t) = v(s+\sigma_k, t) = v(s+\sigma_k)$, which does *not* depend on time
- Consider case where indices differ by 2
  - Use Smale property of the gradient $-JX_H$ of H: trajectories $x \to y$ form a 2-manifold
  - Since trajectories are also in $\mathcal{M}(x, y, H)$, parameterizes a submanifold in a neighborhood of v.
- Show that convergence toward broken orbits in Morse setting corresponds to converges toward broken trajectories in Floer setting
- Use gluing from last chapter: $\hat{v}_{n_k} \in \mathrm{im}(()\hat{\phi})$ for $k \gg 0$, contradicting the fact that v_{n_k} doesn't depend on t