# Title

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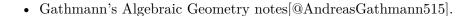
# **Contents**

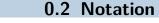
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## **Prologue**







• If a property P is said to hold **locally**, this means that for every point p there is a neighborhood  $U_p \ni p$  such that P holds on  $U_p$ .

$$k[\mathbf{x}] \coloneqq k[x_1, \cdots, x_n] \qquad \qquad \text{The polynomial ring in $n$ indeterminates} \\ k(\mathbf{x}) \coloneqq k(x_1, \cdots, x_n) \qquad \text{The rational function field } \{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \ p, q, \in k[x_1, \cdots, x_n] \} \\ V(J), V_a(J) \qquad \qquad \text{The variety associated to an ideal } J \leq k[x_1, \cdots, x_n] \\ V(J) \coloneqq \left\{\mathbf{x} \in \mathbb{A}^n \ \middle| \ f(\mathbf{x}) = 0, \ \forall f \in J \right\} \\ I(S), I_a(S) \qquad \qquad \text{The ideal associated to a subset } S \subseteq \mathbb{A}^n_k \\ I(S) \coloneqq \left\{f \in k[x_1, \cdots, x_n] \ \middle| \ f(\mathbf{x}) = 0 \ \forall \mathbf{x} \in X \right\} \\ A(X) \coloneqq k[x_1, \cdots, x_n]/I(X) \qquad \qquad \text{The coordinate ring of a variety} \\ \mathcal{O}_X \qquad \qquad \text{The structure sheaf } \left\{f : U \to k \ \middle| \ f \in k(\mathbf{x}) \text{ locally} \right\}.$$

Lots of notation to fill in

| Algebra  | Geometry                            |
|--|-------------------------------------|
| Radical ideals $J = \sqrt{J} \le k[x_1, \dots, x_n]$                     | V(J) the zero locus                 |
| I(S) the ideal of a set  | $S \subseteq \mathbb{A}^n$ a subset |
| I+J  | $V(I) \cap V(J)$                    |
| $\sqrt{I(V) + I(W)}$   | $V \cap W$                          |
| $I \cap J, IJ$   | $V(I) \cup V(J)$                    |
| $I(V) \cap I(W), \sqrt{I(V)I(W)}$  | $V \cup W$                          |
| I(V):I(W)  | $\overline{V\setminus W}$           |
| Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$ |                                     |

## 0.3 Summary of Important Concepts

0.2 Notation 3

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for X = D(f) a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

### 0.4 Useful Algebra Facts



#### Fact 0.4.1:

- $\mathfrak{p} \subseteq R$  is prime  $\iff R/\mathfrak{p}$  is a domain.
- $\mathfrak{p} \leq R$  is maximal  $\iff R/\mathfrak{p}$  is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and  $\langle f \rangle \leq R$  is generated by an irreducible element f, then  $\langle f \rangle$  is maximal

### Proposition 0.4.2 (Finitely generated polynomial rings are Noetherian).

A polynomial ring  $k[x_1, \dots, x_n]$  on finitely many generators is Noetherian. In particular, every ideal  $I \subseteq k[x_1, \dots, x_n]$  has a finite set of generators and can be written as  $I = \langle f_1, \dots, f_m \rangle$ .

#### Proof (?).

A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.4.5),  $k[x_1, \dots, x_n]$  is thus Noetherian.

#### Proposition 0.4.3 (Properties and Definitions of Ideal Operations).

$$I + J := \left\{ f + g \mid f \in I, g \in J \right\}$$

$$IJ := \left\{ \sum_{i=1}^{N} f_i g_i \mid f_i \in I, g_i \in J, N \in \mathbb{N} \right\}$$

 $I+J=\langle 1\rangle \implies I\cap J=IJ \qquad \qquad \text{(coprime or comaximal)} \ \langle a\rangle + \langle b\rangle = \langle a,b\rangle \ .$ 

#### Theorem 0.4.4 (Noether Normalization).

Any finitely-generated field extension  $k_1 \hookrightarrow k_2$  is a finite extension of a purely transcendental extension, i.e. there exist  $t_1, \dots, t_\ell$  such that  $k_2$  is finite over  $k_1(t_1, \dots, t_\ell)$ .

#### Theorem 0.4.5 (Hilbert's Basis Theorem).

If R is a Noetherian ring, then R[x] is again Noetherian.

## 0.5 The Algebra-Geometry Dictionary



Let  $k = \bar{k}$ , we're setting up correspondences

Ring Theory

Geometry/Topology of Affine Varieties

Polynomial functions

Affine space

 $k[x_1,\cdots,x_n]$ 

 $\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$ 

Maximal ideals  $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$ 

Points  $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$ 

Radical ideals  $I \leq k[x_1, \cdots, x_n]$ 

Affine varieties  $X \subset \mathbb{A}^n/k$ , vanishing locii of polynomi

$$I \mapsto V(I) \coloneqq \left\{ a \ \middle| \ f(a) = 0 \forall f \in I \right\}$$

$$I(X) \coloneqq \left\{ f \ \middle| \ f|_X = 0 \right\} \hookleftarrow X$$

Radical ideals containing I(X), i.e. ideals in A(X)

closed subsets of X, i.e. affine subvarieties

A(X) is a domain

X irreducible

A(X) is not a direct sum

X connected

Prime ideals in A(X)

Irreducible closed subsets of X

Krull dimension n (longest chain of prime ideals)

 $\dim X = n$ , (longest chain of irreducible closed subsets