

Problem Set 1

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1 Problem 1

We'll use the following definition of a smooth map between manifolds:

Definition: Let M, N be smooth manifolds of dimensions m, n respectively and $f : M \rightarrow N$ a continuous map. Then f is *smooth* iff for every $p \in M$, there exists a chart (U, ϕ) with $p \in U$ and a chart (V, ψ) with $f(p) \in V$ such that $f(U) \subseteq V$, and the induced map

$$\begin{aligned}\bar{f} : \phi(U) &\rightarrow \psi(V) \\ \bar{f} &= \psi \circ f \circ \phi^{-1}\end{aligned}$$

is smooth as a map from $\mathbb{R}^m \rightarrow \mathbb{R}^n$.

We will thus show that both $\tilde{f} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ and $\tilde{f}^{-1} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ are both smooth bijections, from which we can conclude that f is a diffeomorphism.

So identify $0 = [0, 1]$ and $\infty = [1, 0]$ in \mathbb{CP}^1 and choose the following charts on \mathbb{CP}^1 in terms of homogeneous coordinates:

$$\begin{aligned}(U, \phi) &:= \\ U = \mathbb{CP}^1 \setminus \{\infty\} &= \{[x, y] \mid x, y \in \mathbb{C}, y \neq 0\} \\ \phi : \mathbb{CP}^1 &\rightarrow \mathbb{C} \\ [x, y] &\mapsto x/y.\end{aligned}$$

$$\begin{aligned}\phi^{-1} : \mathbb{C} &\rightarrow \mathbb{CP}^1 \\ z &\mapsto [z, 1].\end{aligned}$$

and

$$\begin{aligned}(V, \psi) &:= \\ V = \mathbb{CP}^1 \setminus \{0\} &= \{[x, y] \mid x, y \in \mathbb{C}, x \neq 0\} \\ \psi : \mathbb{CP}^1 &\rightarrow \mathbb{C} \\ [x, y] &\mapsto y/x.\end{aligned}$$

$$\begin{aligned}\psi^{-1} : \mathbb{C} &\rightarrow \mathbb{CP}^1 \\ z &\mapsto [1, z].\end{aligned}$$

Now define

$$\begin{aligned}\tilde{f} : \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1 \\ p &\mapsto \begin{cases} p, & p = \infty \\ p + c & \text{otherwise} \end{cases}\end{aligned}$$

We then need to determine a formula for \tilde{f} in homogeneous coordinates. We compute

$$\begin{aligned}p \in U &\implies p = [a, b], \quad a, b \in \mathbb{C}, b \neq 0 \\ \implies \tilde{f}([a, b]) \Big|_U &= (\phi^{-1} \circ f \circ \phi)([a, b]) \\ &= (\phi^{-1} \circ f)\left(\frac{a}{b}\right) \\ &= \phi^{-1}\left(\frac{a}{b} + c\right) \\ &= \left[\frac{a}{b} + c, 1\right] \\ &= [a + bc, b]\end{aligned}$$

and

$$\begin{aligned}
p \in V &\implies p = [a, b], \quad a, b \in \mathbb{C}, \quad a \neq 0 \\
&\implies \tilde{f}([a, b]) \Big|_V = (\psi^{-1} \circ f \circ \psi)([a, b]) \\
&= (\psi^{-1} \circ f)\left(\frac{b}{a}\right) \\
&= (\psi^{-1})\left(\frac{b}{a} + c\right) \\
&= \left[1, \frac{b}{a} + c\right] \\
&= [a, b + ac]
\end{aligned}$$

Since $\mathbb{CP}^1 = U \cup V$, we can note that if $p \in M$ then either $p \in U$ or $p \in V$. Moreover, $p \in U \implies \tilde{f}(p) \in \tilde{f}(U) = U$, since $p = [a, b]$ with $b \neq 0 \implies f(p) = [a + bc, b]$ where $b \neq 0$ as well, so $\tilde{f}(p) \in U$ and $\tilde{f}(U) \subseteq U$. Similarly, $\tilde{f}(V) \subseteq V$. So it only remains to check that the following two compositions are smooth:

- $f_U : \mathbb{C} \rightarrow \mathbb{C}, f_U := \phi \circ \tilde{f} \circ \phi^{-1}$, and
- $f_V : \mathbb{C} \rightarrow \mathbb{C}, f_V := \psi \circ \tilde{f} \circ \psi^{-1}$.

We can compute

$$\begin{aligned}
f_U(z) &:= (\phi \circ \tilde{f} \circ \phi^{-1})(z) \\
&= (\phi \circ \tilde{f})([z, 1]) \\
&= \phi([z + c, 1]) \\
&= z + c
\end{aligned}$$

$$\begin{aligned}
f_V(z) &:= (\psi \circ \tilde{f} \circ \psi^{-1})(z) \\
&= (\psi \circ \tilde{f})([1, z]) \\
&= \psi([1, z + c]) \\
&= z + c
\end{aligned}$$

And $\frac{\partial}{\partial z} f_U(z) = \frac{\partial}{\partial z} f_V(z) = 1$, so these are smooth maps on their domains.

To Summarize: Let $p \in M$ be arbitrary. The map \tilde{f} will be smooth iff there are charts $(U_\alpha, \varphi_\alpha : U_\alpha \rightarrow \mathbb{C}), (U_\beta, \varphi_\beta : U_\beta \rightarrow \mathbb{C})$ with $U_\alpha, \beta \subseteq \mathbb{CP}^1$ such that

- $p \in U_\alpha$,
- $\tilde{f}(p) \in U_\beta$,
- $\tilde{f}(U_\alpha) \subseteq U_\beta$
- $\varphi_\beta \circ \tilde{f} \circ \varphi_\alpha^{-1}$ is smooth.

2 Problem 3

2.1 Part 1

Note: Throughout this question, we will identify $\{f : C^\infty(M) \rightarrow \mathbb{R}\} \cong C^\infty(M)^\vee$ as vector spaces.

Let M, N be smooth manifolds and $f : M \rightarrow N$ be a fixed smooth map, and define a map

$$\begin{aligned}\phi : C^\infty(N) \times TM &\rightarrow \mathbb{R} \\ (h, v) &\mapsto v(h \circ f)\end{aligned}$$

2.2 Part 1

Using the derivation definition, we can identify this assignment as a map

$$\begin{aligned}\phi : C^\infty(N) \times C^\infty(M)^\vee &\rightarrow \mathbb{R} \\ (h, v) &\mapsto v(h \circ f)\end{aligned}$$

We'd like to show that this yields a well-defined element of $T_p M = C^\infty(M)$. So for some fixed $v \in T_p M$, define a map

$$\begin{aligned}\phi_v : C^\infty(N) &\rightarrow \mathbb{R} \\ h &\mapsto v(h \circ f),\end{aligned}$$

which will be an element of TM if it is a derivation. For $x \in N$, we have

$$\begin{aligned}\phi_v(h_1 \cdot h_2)(x) &:= v((h_1 h_2) \circ f)(x) \\ &= v((h_1 \circ f)(h_2 \circ f))(x) \\ &= v(h_1 \circ f)(x) \cdot h_2(x) + h_1(x) \cdot v(h_2 \circ f)(x) \quad \text{since } v \text{ is a derivation} \\ &= \phi_v(h_1)(x) \cdot h_2(x) + h_1(x) \cdot \phi_v(h_2)(x).\end{aligned}$$

2.3 Part 2

Using the integral curve definition,

3 Problem 4

3.1 Part 1

Let $V = \mathbb{R}^n$ as a vector space, let g be a nonsingular matrix, and define a map

$$\begin{aligned}\phi : V &\rightarrow V^\vee \\ v &\mapsto (\phi_v : w \mapsto \langle v, gw \rangle)\end{aligned}$$

The claim is that ϕ is a natural isomorphism. It is clearly linear (following from the linearity of the inner product and matrix multiplication), so it remains to check that it is a bijection.

To see that $\ker \phi = 0$, so that only the zero gets sent to the zero map, we can suppose that $x \in \ker \phi$. Then $\phi_x : w \rightarrow \langle x, gw \rangle$ is the zero map. But the inner product is nondegenerate by definition, i.e. $\langle x, y \rangle = 0 \ \forall y \implies x = 0$. So x could only have been the zero vector to begin with.

But $\dim V = \dim V^\vee$, so any injective linear map will necessarily be surjective as well.

3.2 Part 2

Let $g : TM \otimes TM \rightarrow \mathbb{R}$ be a metric, and consider the tangent space TM . By definition, the cotangent space $T_p^*M = (T_pM)^\vee$

4 Problem 5

4.1 Part 1

Let $A \in \text{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \text{Mat}(n, n)$ be positive semi-definite, so

$$\langle v, Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v, (A + B)v \rangle &= \langle v, Av \rangle + \langle v, Bv \rangle \\ &> \langle v, Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

4.2 Part 2

Let M be a smooth manifold with tangent bundle TM and a maximal smooth atlas \mathcal{A} . Choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$. Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to \mathcal{C} , so for each i we have

$$\begin{aligned} f_i &: M \rightarrow \mathbb{R} \\ \forall p \in M, \quad \sum_{i \in I} f_i(p) &= 1 \end{aligned}$$

In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, let g^i be the Euclidean metric given by the identity matrix, i.e. $g_{jk}^i := \delta_{jk}$. We then have

$$\begin{aligned} g^i &: T\phi_i(U_i) \otimes T\phi_i(U_i) \rightarrow \mathbb{R} \\ (\partial x_i, \partial x_j) &\mapsto \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is defined for pairs of vectors in $T\phi_i(U_i) \cong T\mathbb{R}^n = \text{span}_{\mathbb{R}} \{\partial x_i\}_{i=1}^n$ on basis vectors as the Kronecker delta and extended linearly.

Note that each coordinate function $\phi_i : U_i \rightarrow \mathbb{R}^n$ induces a map $\tilde{\phi}_i : TU_i \rightarrow T\mathbb{R}^n$.

Let G^i be the pullback of g^i along these induced maps $\tilde{\phi}_i$, so

$$G^i : TU_i \otimes TU_i \rightarrow \mathbb{R}$$

$$G^i(x, y) := \left((\tilde{\phi}_i)^* g^i \right) (x, y) := g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y))$$

Then, for a point $p \in M$, define the following map:

$$g_p : T_p M \otimes T_p M \rightarrow \mathbb{R}$$

$$(x, y) \mapsto \sum_{i \in I} f_i(p) G^i(x, y).$$

The claim is that g_p defines a metric on M , and thus the family $\{g_p \mid p \in M\}$ yields a tensor field and thus a Riemannian metric on M . If we define the map

$$g : M \rightarrow (TM \otimes TM)^\vee$$

$$p \mapsto g_p$$

then g can be expressed as

$$g = \sum_{i \in I} f_i G^i.$$

We can check that this is positive definite by considering $x \in T_p M$ and computing

$$g(x, x) := g_p(x, x)$$

$$= \sum_{i \in I} f_i(p) G^i(v, v)$$

$$= \sum_{i \in I} f_i(p) g^i(\tilde{\phi}_i(x), \tilde{\phi}_i(y)),$$

where each term is positive semi-definite, and *at least one term* is positive definite because $\sum_i f_i(p)$ must equal 1. By part 1, this means that the entire expression is positive definite, so g is a metric. \square

5 Problem 6

5.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M ,

$$X : M \rightarrow TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \text{id}_M$, and $\phi_1 \in \text{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f : M \rightarrow M$, we have

$$\Lambda(f) = \sum_k \text{Tr} \left(f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of M .

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0 , thus

$$\left. \frac{\partial}{\partial t} \phi_t(p_0) \right|_{t=1} \text{ so}$$

$$\begin{aligned} & \phi_t(p) = p \\ \implies & \frac{\partial}{\partial t} \phi_t(p) = \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies & \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies & X(\phi_1(p_0)) := \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 && \text{by definition of } \phi_1 \end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X . So X has at least one zero, as desired. \square

5.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned} s : S^2 &\rightarrow S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\rightarrow (\mathbf{x}, [1, 1]) \end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1, 1]$ to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & TS^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

can *not* have a nowhere vanishing section. \square