8000: Homework 3

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1. Suppose G is finitely generated, then take a minimal generating set so $G = \langle g_1, g_2, \cdots, g_n \rangle$ and $g \in G \Rightarrow g = \sum_{i=1}^n a_i g_i$ is a unique representation of g.

Let $\mathbb{Z}^n := \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ copies}}$, and let $\vec{e}_i = (0, 0, \cdots, \underset{\uparrow\, i^{th} \text{ component}}{1}, \cdots, 0) \in \mathbb{Z}^n$. Then

since $\mathbb{Z} = \langle 1 \rangle$, we have $\mathbb{Z}^n = \langle \vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n \rangle$, so define a map on generators:

$$\varphi : \mathbb{Z}^n \to G$$
$$\vec{e}_i \mapsto g_i$$

<u>Claim</u>: $\ker \varphi = \vec{0} \in \mathbb{Z}^n$, $\operatorname{im} \varphi = G$, and so $\mathbb{Z}^n = \mathbb{Z}^n / \ker \varphi \cong \operatorname{im} \varphi = G$ by the 1^{st} isomorphism theorem.

- <u>$\ker \varphi = \vec{0}$</u>: We have

$$\begin{aligned}
\ker \varphi &= \{ \vec{x} = \sum_{i=1}^n a_i \vec{e}_i \in \mathbb{Z}^n \mid \varphi(\vec{x}) = 0 \in G \} \\
&= \{ \sum_{i=1}^n a_i \vec{e}_i \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i g_i = 0 \in G \} \\
&\overset{(\text{Claim})}{=} \{ \sum_{i=1}^n a_i \vec{e}_i \in \mathbb{Z} \mid a_i = 0 \ \forall i \} = \vec{0} \in \mathbb{Z}^n.
\end{aligned}$$

where the claim is that $\sum_{i=1}^n a_i g_i = 0 \in G \Rightarrow a_i = 0 \ \forall i$.
Supposing otherwise, we would have some $a_j \neq 0$. If $a_j g_j = 0$, g_j is an element of finite order. Otherwise $a_j g_j \neq 0$, an d

$\sum_{i=1}^n a_i g_i = 0 \Rightarrow \sum_{i \neq j}^n a_i g_i + a_j g_j = 0 \Rightarrow \sum_{i \neq j}^n a_i g_i = -a_j g_j$, which yields

two distinct representations of $g := -a_j g_j$, violating uniqueness.

- <u>$\operatorname{Im} \varphi = G$</u>:

If $g = \sum_{i=1}^n a_i g_i \in G$, take $\vec{x} = (a_1, a_2, \cdots, a_n) = \sum_{i=1}^n a_i \vec{e}_i \in \mathbb{Z}^n$, then $\varphi(\vec{x}) = g$ by construction.

- φ is a group homomorphism:

Let $\vec{x}=\sum x_i\vec{e}_i$, $\vec{y}=\sum y_i\vec{e}_i \in \mathbb{Z}^n$, then

$$\begin{aligned}\varphi(\vec{x}+\vec{y}) &= \varphi\left(\sum_{i=1}^n (x_i+y_i)\vec{e}_i\right)\\ &= \sum_{i=1}^n (x_i+y_i)g_i\\ &= \sum_{i=1}^n x_i g_i + \sum_{i=1}^n y_i g_i = \varphi(\vec{x})+\varphi(\vec{y}). \quad \blacksquare\end{aligned}$$

(2a) Suppose $\mathbb{Q}=\langle q_1, q_2, \cdots, q_n\rangle$ for some finite generating set. Then $q_i = p_i/s_i$ for some coprime pair $p_i, s_i \in \mathbb{Z}$. Then for any $x\in\mathbb{Q}$, we can write

$$x=\sum_{i=1}^n \alpha_i (p_i/s_i) = \sum_{i=1}^n \alpha_i p_i/s_i = \sum_{i=1}^n \left(\alpha_i p_i \prod_{j\neq i} s_j \Big/ \prod_{k=1}^n s_k\right)$$

By clearing denominators

so

$$x\prod_{k=1}^n s_k = \sum_{i=1}^n \left(\alpha_i p_i \prod_{j\neq i} s_j\right) \Rightarrow \underbrace{\prod_{k=1}^n s_k}_{\in\mathbb{Z}} = (1/x)\underbrace{\sum_{i=1}^n \left(\alpha_i p_i \prod_{j\neq i} s_j\right)}_{\in\mathbb{Z}}.$$

But we can choose x such that $1/x$ is an integer not dividing any s_k, a contradiction. ※

(2b) Suppose $\mathbb{Q}$ were free on some generating set S indexed by I, so $\sum_{i\in I} \alpha_i s_i = 0 \Rightarrow a_i = 0 \ \forall i\in I$.

$\in\mathbb{Z}$ $\in\mathbb{Q}$

However, let $s_1, s_2\in S$, so $s_1 = p_1/q_1$ and $s_2 = p_2/q_2$ for some $p_i, q_i\in\mathbb{Z}$. Then let

$x = q_1 p_2$, $y = -q_2 p_1$,

then

$$x s_1 + y s_2 = (q_1 p_2)\left(\frac{p_1}{q_1}\right) + (-q_2 p_1)\left(\frac{p_2}{q_2}\right) = p_2 p_1 - p_1 p_2 = 0$$

while $x, y \neq 0$, a contradiction. ※

(2c) Noting that if $q\in\mathbb{Q}$ had finite order, $nq=0$ for some n, while $|nq|>|q|$ for every n, which forces $q=0$. So no nonzero elt has finite order and $\mathbb{Q}$ is abelian but <u>not</u> finitely generated and <u>not</u> free. So the fin. gen. hypothesis is necessary. $\blacksquare$

③ <u>Claim 1:</u> If $S_p \in Syl(p,G)$, $S_q \in Syl(q,G)$ with p and q coprime and both $S_p \trianglelefteq G$, $S_q \trianglelefteq G$, then $a \in S_p, b \in S_q \Rightarrow ab = ba$ in G.

<u>Claim 2:</u> Let $\#G = \prod_{i=1}^{n} p_i^{k_i}$, then letting S_{p_i} be the corresponding sylow p_i-subgroups, the following map is an isomorphism:

$$f: \prod_{i=1}^{n} S_{p_i} \longrightarrow G$$
$$(s_1, s_2, \cdots, s_n) \longmapsto \prod_{i=1}^{n} s_i$$

<u>Proof of claim 1:</u> We have $aba^{-1}b^{-1} = \underbrace{(aba^{-1})}\,b^{-1} = b_0 b^{-1}$ for some $b_0 \in S_q$

So $aba^{-1}b^{-1} \in S_q$.

conjugate of b, so in some Sylow q-subgroup, but there's only one.

Similarly, $aba^{-1}b^{-1} = a(ba^{-1}b) = aa_0 \in S_p$. So $aba^{-1}b^{-1} \in S_p \cap S_q$

But since q, p are coprime, $S_p \cap S_q = \{e\}$. ▨

<u>Proof of claim 2:</u>

- <u>f is a homomorphism</u>

If $\vec{s} = (s_1, \cdots, s_n)$ and $\vec{t} = (t_1, \cdots, t_n) \in$ Domain f, then

$$\begin{aligned} f(\vec{s}\vec{t}) &= f((s_1t_1, s_2t_2, s_3t_3, \cdots, s_nt_n)) \\ &= (s_1t_1)(s_2t_2)(s_3t_3) \cdots (s_nt_n) \\ &= (s_1s_2)(t_1t_2)(s_3t_3) \cdots (s_nt_n) \\ &= (s_1s_2s_3)(t_1t_2t_3) \cdots (s_nt_n) \\ &= (s_1 s_2 s_3 \cdots s_n)(t_1t_2t_3 \cdots t_n) \\ &= f(\vec{s})f(\vec{t}). \end{aligned}$$

Since
- $s_i s_j = s_j s_i$
- $t_i t_j = t_i t_j$
- $s_i t_j = t_j s_i$

$\forall i \neq j$

- <u>$\ker f = \vec{e} = (e, e, \cdots, e)$:</u>

If $\vec{s} \in$ domain f and $f(\vec{s}) = s_1 s_2 \cdots s_n = e \in G$, then the order of $s_1 \cdots s_n$ is 1.

However, if any $s_i \neq e$, we have $o(s_i) = p_i^{k_i}$, so $o(s_1 \cdots s_n) = \operatorname{lcm}(p_i^{k_i}) = \prod p_i^{k_i} > 1$, so this forces $s_i = e$ $\forall i$.

- $\underline{\mathrm{Im}\, f = G}$: This follows because $|G| = |H|$ and $\varphi: G \to H$ injective $\Rightarrow \varphi$ surjective.

(Here, $\left|\prod_{i=1}^{n} S_{p_i}\right| = \prod_{i=1}^{n} p_i^{k_i} = |G|$ since $n_{p_i} = 1\ \forall p_i$ and $S_{p_i} \cap S_{p_j} = \{e\}$.) ■

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④ We have $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, and $Z(Q_8) = \pm 1$. No other elements are in the center because $ij = k \neq -k = ji$. Since $|Q_8/Z(Q_8)| = 8/2 = 4$, the quotient is either $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$, and both are abelian.

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⑤ $\underline{\text{Order } 18}$ $= 3^2 \cdot 2$

Noting $n_3 = 1 \bmod 3$ & $n_3 | 2 \Rightarrow n_3 = 1$, there is one sylow 3-subgroup $Q_3 \trianglelefteq G$. If Q_2 is the sylow 2-subgroup, we then have $G \cong Q_2 \ltimes_\psi Q_3$ for some $\psi: Q_2 \to \mathrm{Aut}\, Q_3$.

$\underline{\text{Case 1: } Q_3 \cong \mathbb{Z}_9}$

Then $\mathrm{Aut}\, \mathbb{Z}_9 \cong \mathbb{Z}_6$, which has only one nontrivial element of order $|\mathbb{Z}_2| = 2$, the map $(1 \to -1) \in \mathrm{Aut}\, \mathbb{Z}_9$. This yields

$$G \cong \langle a, b \mid a^2 = b^9 = e,\ aba^{-1} = b^8 \rangle \cong D_9, \text{ a dihedral group.}$$

$\underline{\text{Case 2: } Q_3 \cong \mathbb{Z}_3^2}$

Then $\mathrm{Aut}(\mathbb{Z}_3^2) \cong GL(2, \mathbb{Z}_3)$, and any such matrix A where $A^2 = I$ satisfies either $x+1$ or $x^2 - 1$, so $A \sim \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so we obtain

1) $G \cong \boxed{\langle a, b, c \mid a^2 = b^3 = c^3 = e,\ [b,c] = e,\ aba^{-1} = b^2,\ aca^{-1} = c^2 \rangle} := \mathbb{Z}_2 \ltimes \mathbb{Z}_3^2$

2) $G \cong \langle a, b, c \mid a^2 = b^3 = c^3 = e,\ [b,c] = e,\ aba^{-1} = b,\ aca^{-1} = c^2 \rangle \cong \mathbb{Z}_3 \times D_6$

Adding in the abelian groups yields

$$G \in \{\mathbb{Z}_{18}, \mathbb{Z}_3^2 \times \mathbb{Z}_2, D_9, \mathbb{Z}_3 \times D_6, \mathbb{Z}_2 \ltimes \mathbb{Z}_3^2\}.$$ ▨

$\underline{\text{Order } 20}$ $= 2^2 \cdot 5$

We have $n_5 = 1$, so $Q_5 \trianglelefteq G$ and $G \cong Q_2 \ltimes_\psi Q_5$.

Case 1: $Q_2 \cong \mathbb{Z}_4$, then $\{f \in \mathrm{Aut}\, \mathbb{Z}_5 \text{ s.t. } o(f) | 4\} = \{id, x \mapsto -x, x \mapsto 2x\}$

$$\boxed{\begin{array}{l} G_1 \cong \langle a, b \mid a^5 = b^4 = e,\ bab^{-1} = a^4 \rangle \\ G_2 \cong \langle a, b \mid a^5 = b^4 = e,\ bab^{-1} = a^2 \rangle \end{array}}$$

Case 2: $Q_2 \cong \mathbb{Z}_2^2$, and $\{f \in \mathrm{Aut}\, \mathbb{Z}_5 \text{ s.t. } o(f) | 2\} = \{id, x \mapsto -x\}$, so we have

- $G_3 \cong \langle a,b,c \mid a^5=b^2=c^2=[b,c]=e,\ bab^{-1}=a^4,\ cac^{-1}=a \rangle$

- $G_4 \cong \langle a,b,c \mid a^5=b^2=c^2=[b,c]=e,\ bab^{-1}=a,\ cac^{-1}=a^4 \rangle \cong G_3$ by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Aut}\, \mathbb{Z}_2^2$

- $G_5 \cong \langle a,b,c \mid a^5=b^2=c^2=[b,c]=e,\ bab^{-1}=a^4,\ cac^{-1}=a^4 \rangle \cong G_3$ by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathrm{Aut}\, \mathbb{Z}_2^2$

So in total we have $G \in \{ \mathbb{Z}_4 \times \mathbb{Z}_5,\ \mathbb{Z}_2^2 \times \mathbb{Z}_5,\ G_1, G_2, G_3 \}$ ∎

Order 30

We have $n_3 \in \{1, 10\}$, $n_5 \in \{1, 6\}$, but one must be 1 otherwise they contribute $6(4) + 10(2) + 1 = 45 > 20$ distinct elements. So one of Q_3, Q_5 is normal, so $H_{15} := Q_3 Q_5 \leq G$. Since $[G:H]=2$, H is normal, and $HQ_2 = G$ with $H \cap Q_2 = \{e\}$, so $G \cong Q_2 \rtimes_\psi H_{15}$.

We have $\{ f \in \mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_3) \text{ s.t. } o(f) \mid 2 \} = \left\{ id, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} = \left\{ \begin{matrix} id,\ (1,1) \mapsto (4,1) \\ (1,1) \mapsto (1,2),\ (1,1) \mapsto (4,2) \end{matrix} \right\}$

These yield

$G_1 \cong \langle a,b,c \mid a^5=b^3=c^2=[a,b]=e,\ cac^{-1}=a^4,\ cbc^{-1}=b \rangle$

$G_2 \cong \langle a,b,c \mid a^5=b^3=c^2=[a,b]=e,\ cac^{-1}=a,\ cbc^{-1}=b^2 \rangle$

$G_3 \cong \langle a,b,c \mid a^5=b^3=c^2=[a,b]=e,\ cac^{-1}=a^4,\ cbc^{-1}=b^2 \rangle$

along with the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. ∎

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⑥ Suppose S is free on $S = \{s_i\}_{i \in I}$, then if $x \in F$ then $x = \prod_{i=1}^{m} s_i^{\pm 1}$, some finite reduced word in the symbols s_i, s_i^{-1}. If $x^n = e$, then

$$x^n = \left(\prod_{i=1}^{m} s_i^{\pm 1} \right)^n = (s_1 s_2 \cdots s_n)(s_1 s_2 \cdots s_n) \cdots (s_1 s_2 \cdots s_n) = e.$$

But x is reduced, so the only cancellation that can happen is $s_n s_1 = e$. But then

$$(s_1 s_2 \cdots s_{n-1})(s_2 \cdots s_{n-1}) \cdots (s_2 \cdots s_n) = e,$$

so $s_{n-1} s_2 = e$ must happen. Continuing in this way, we obtain $s_1 s_2 \cdots s_n = e$, so x must be the identity in F. ∎

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⑦ Let g be arbitrary and $x^n \in H_n$. Then

$g x^n g^{-1} = (g x g^{-1})^n := y^n \in H_n$ by definition, so $g H_n g^{-1} = H_n$. ∎