

# Title

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## 0.1 Exercises

**Problem 1** (Hungerford 1.6.3).

If  $\sigma = (i_1 i_2 \cdots i_r) \in S_n$  and  $\tau \in S_n$ , then show that  $\tau\sigma\tau^{-1} = (\tau(i_1)\tau(i_2) \cdots \tau(i_r))$ .

**Solution 1.** Let

$$\sigma = (s_1 s_2 \cdots s_r) \in S_n$$

in cycle notation, and  $\tau \in S_n$  be arbitrary. Define  $t_j = \tau(s_j)$ ; we would then like to show that

$$\tau\sigma\tau^{-1} = (t_1 t_2 \cdots t_r) := (\tau(s_1)\tau(s_2) \cdots \tau(s_r))$$

To this end, it suffices to show that  $t_i$  maps to  $t_{i+1 \bmod r}$ , under  $\tau\sigma\tau^{-1}$ , which is to say

$$\tau\sigma\tau^{-1}(t_i) = \begin{cases} t_{i+1} & i+1 \leq r, \\ t_1 & i = r \end{cases}.$$

Bearing this in mind, we will immediately suppress notation and take all indices  $\bmod r$  for the rest of this problem.

The following then follows simply by definitions:

$$\begin{aligned} \tau\sigma\tau^{-1}(t_i) &= \tau\sigma(s_i) \\ &= \tau(s_{i+1}) \\ &= t_{i+1}. \end{aligned}$$

□

**Problem 2** (Hungerford 1.6.4).

Show that  $S_n \cong \langle (12), (123 \cdots n) \rangle$  and also that  $S_n \cong \langle (12), (23 \cdots n) \rangle$

**Solution 2.** Let  $\sigma = (12)$  and  $\tau = (123 \cdots n)$ .

Claim:  $S_n$  is generated by swaps  $F = \{f_{i,k} = (i \ i+k) \mid 1 \leq i, k \leq n\} = \langle f_{i,k} \rangle$ , and moreover each such swap can be written as a product in  $\sigma$  and  $\tau$ , and thus

$$S_n = F \subseteq \langle \sigma, \tau \rangle \subseteq S_n$$

which forces  $\langle \sigma, \tau \rangle = S_n$  as desired.

To see that  $F = S_n$  is generated by swaps in  $F$ , let  $(s_1 s_2 \cdots s_r) \in S_n$  be arbitrary. We then construct the swaps  $(s_1 s_2), (s_1, s_3), \dots (s_1 s_r)$ , and note that taking their product yields

$$(s_1 s_2)(s_1, s_3) \cdots (s_1 s_r) = (s_1 s_2 s_3 \cdots s_r)$$

and so  $F = S_n$ .

To see that  $F \subseteq \langle \sigma, \tau \rangle$ , we produce a way to write any swap as a product of powers of these generators. We can first note that for  $1 \leq i \leq n$ , we have  $\sigma(i) = i+1$  and  $\sigma^k(i) = i+k$  (where again everything is taken  $\mod n$ ).

By problem (1), we have

$$\sigma \tau \sigma^{-1} = \sigma (12) \sigma^{-1} = (\sigma(1) \sigma(2)) = (23),$$

and in general,

$$\sigma^k \tau \sigma^{-k} = (\sigma^k(1) \sigma^k(2)) = (k \ k+1).$$

So the cycles  $(k \ k+1)$  are products of powers of  $\tau, \sigma$  and thus contained in the group they generate, and we have  $F \subseteq \langle \sigma, \tau \rangle$ .

If we then define the cycle  $\gamma_k = (k \ k+1)$ , we can observe

$$\begin{aligned} \gamma_k \gamma_{k+1} \gamma_k^{-1} &= (k \ k+1) (k+1 \ k+2) (k+1 \ k) \\ &= (k \ k+2), \end{aligned}$$

and so  $\langle \gamma_k \rangle$  also contains all cycles of the form  $(k \ k+i)$  for any  $i$ . In particular, any swap can be written as such a cycle – explicitly, given a swap  $(s_1 s_2)$  (where without loss of generality  $s_1 \leq s_2$ ), let  $k = s_1$  and  $i = s_2 - s_1$ .

This implies that

**Problem 3** (Hungerford 2.2.1).

Let  $G$  be a finite abelian group that is not cyclic. Show that  $G$  contains a subgroup isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_p$  for some prime  $p$ .

**Problem 4** (Hungerford 2.2.12.b).

Determine all abelian groups of order  $n$  for  $n \leq 20$ .

**Problem 5** (Hungerford 2.4.1).

Let  $G$  be a group and  $A \trianglelefteq G$  be a normal abelian subgroup. Show that  $G/A$  acts on  $A$  by conjugation and construct a homomorphism  $\varphi : G/A \rightarrow \text{Aut}(A)$ .

**Problem 6** (Hungerford 2.4.9).

Let  $Z(G)$  be the center of  $G$ . Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

*Note that Hungerford uses the notation  $C(G)$  for the center.*

**Problem 7** (Hungerford 2.5.6).

Let  $G$  be a finite group and  $H \trianglelefteq G$  a normal subgroup of order  $p^k$ . Show that  $H$  is contained in every Sylow  $p$ -subgroup of  $G$ .

**Problem 8** (Hungerford 2.5.9).

Let  $|G| = p^n q$  for some primes  $p > q$ . Show that  $G$  contains a unique normal subgroup of index  $q$ .

## 0.2 Qual Problems

**Problem 9.**

Let  $G$  be a finite group and  $p$  a prime number. Let  $X_p$  be the set of Sylow- $p$  subgroups of  $G$  and  $n_p$  be the cardinality of  $X_p$ . Let  $\text{Sym}(X)$  be the permutation group on the set  $X_p$ .

1. Construct a homomorphism  $\rho : G \rightarrow \text{Sym}(X_p)$  with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that  $G$  is simple and the order of  $G$  divides  $n_p!$ .
3. Show that for any  $1 \leq a \leq 4$  and any prime power  $p^k$ , no group of order  $ap^k$  is simple.

**Problem 10.**

Let  $G$  be a finite group and  $H < G$  a subgroup. Let  $n_H$  be the number of subgroups of  $G$  that are conjugate to  $H$ . Show that  $n_H$  divides the order of  $G$ .

**Problem 11.**

Let  $G = S_5$ , the symmetric group on 5 elements. Identify all conjugacy classes of elements in  $G$ , provide a representative from each class, and prove that this list is complete.