Title

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1 Thursday January 9th

Recall: For M^n a closed smooth manifold, consider a smooth map $f : M^n \to \mathbb{R}$.

Definition: A critical point p of f is *non-degenerate* iff $\det(H \coloneqq \frac{\partial^i f}{\partial x_i \partial x_j}(p)) \neq 0$ in some coordinate system U.

Lemma (The Morse Lemma): For any non-degenerate critical point p there exists a coordinate system around p such that

$$f(x_1, \cdots, x_n) = f(p) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

λ is called the *index of f at p*.

Lemma: λ is equal to the number of *negative* eigenvalues of $H(p)$.

Proof: A change of coordinates sends $H(p) \to A^t H(p) A$, which (exercise) has the same number of positive and negative values.

Exercise: show this assuming that A is invertible and not necessarily orthogonal.

This means that f can be written as the quadratic form

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Proof of Morse Lemma:

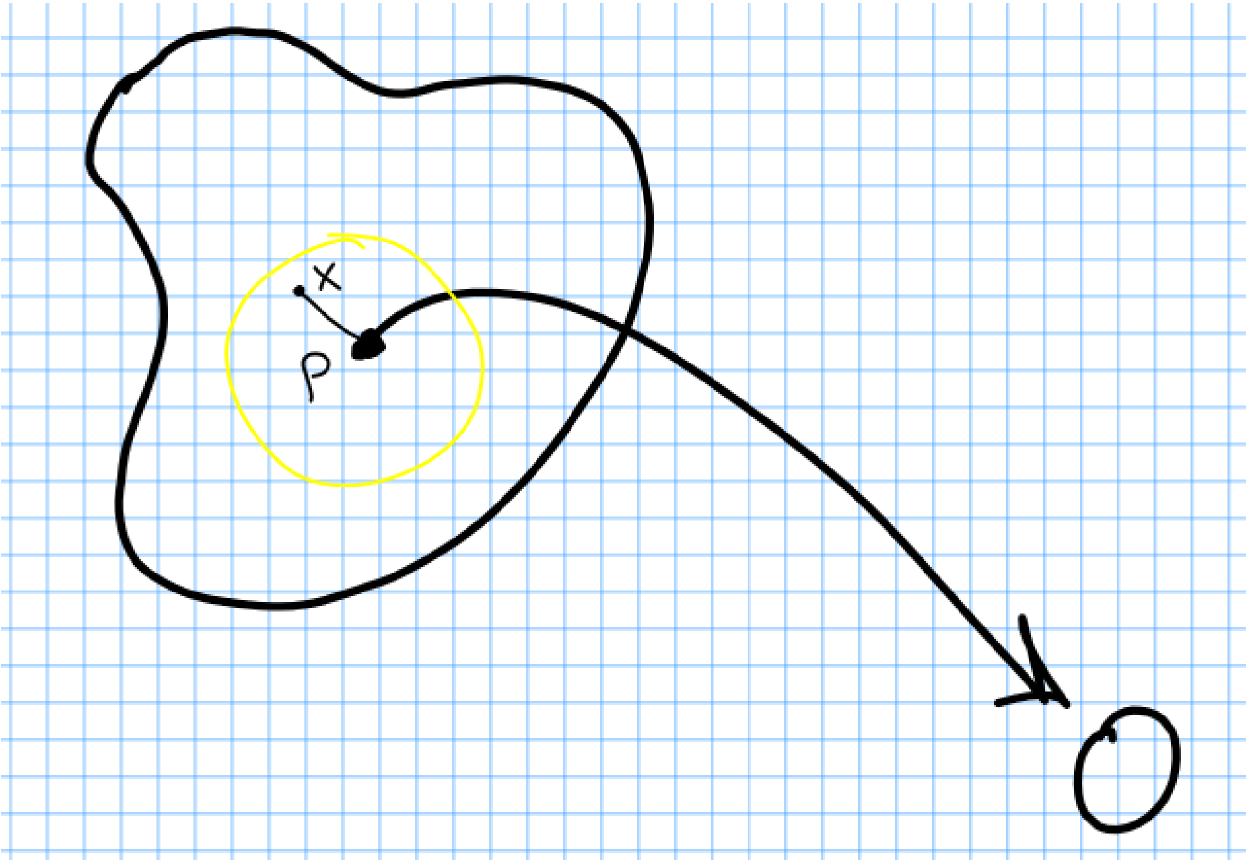
Suppose that we have a coordinate chart U around p such that $p \mapsto 0 \in U$ and $f(p) = 0$.

Step 1 – Claim: There exists a coordinate system around p such that

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x),$$

where $h_{ij}(x) = h_{ji}(x)$.

Proof: Pick a convex neighborhood V of $0 \in \mathbb{R}^n$.



Restrict f to a path between x and 0, and by the FTC compute

$$I = \int_0^1 \frac{df(tx_1, tx_2, \cdots, tx_n)}{dt}\, dt = f(x_1, \cdots, x_n) - f(0) = f(x_1, \cdots, x_n).$$

since $f(0) = 0$.

We can compute this in a second way,

$$I = \int_0^1 \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \cdots \frac{\partial f}{\partial x_n} x_n \, dt \implies \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}\, dt = f(x).$$