

# Title

D. Zack Garza

Tuesday 1<sup>st</sup> September, 2020

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Last time:  $V(I) = \{x \in \mathbb{A}^n \mid f(x) = 0 \forall x \in I\}$  and  $I(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in X\}$ .

We proved the Hilbert Nullstellensatz  $I(V(J)) = \sqrt{J}$ , defined the coordinate ring of an affine variety  $X$  as  $A(X) := k[x_1, \dots, x_n]/I(X)$ , the ring of “regular” (polynomial) functions on  $X$ .

Recall that a *topology* on  $X$  can be defined as a collection of “closed” subsets of  $X$  that are closed under arbitrary intersections and finite unions. A subset  $Y \subset X$  inherits a subspace topology with closed sets of the form  $Z \cap Y$  for  $Z \subset X$  closed.

**Definition 1.0.1** (Zariski Topology).

Let  $X$  be an affine variety. The closed sets are affine subvarieties  $Y \subset X$ .

We have  $\emptyset, X$  closed, since

1.  $V_X(1) = \emptyset$ ,
2.  $V_X(0) = X$

Closure under finite unions: Let  $V_X(I), V_X(J)$  be closed in  $X$  with  $I, J \subset A(X)$  ideals. Then  $V_X(IJ) = V_X(I) \cup V_X(J)$ .

Closure under intersections: We have  $\bigcap_{i \in \sigma} V_X(J_i) = V_X\left(\sum_{i \in \sigma} J_i\right)$ .

**Remark 1.**

There are few closed sets, so this is a “weak” topology.

**Example 1.1.**

Compare the classical topology on  $\mathbb{A}^1/\mathbb{C}$  to the Zariski topology.

Consider the set  $A := \{x \in \mathbb{A}^1/\mathbb{C} \mid \|x\| \leq 1\}$ , which is closed in the classical topology.

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But  $A$  is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

**Example 1.2.**

Let  $f : \mathbb{A}^1/k \rightarrow \mathbb{A}^1/k$  be any injective map. Then  $f$  is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

**Example 1.3.**

Consider  $X \times Y$  a product of affine varieties. Then there is a product topology where open sets are of the form  $\bigcup_{i=1}^n U_i \times V_i$  with  $U_i, V_i$  open in  $X, Y$  respectively.

This is the wrong topology! On  $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ , the diagonal  $\Delta := V(x - y)$  is closed in the Zariski topology on  $\mathbb{A}^2$  but not in the product topology.

**Example 1.4.**

Consider  $\mathbb{A}^2/\mathbb{C}$ , so the closed sets are curves and points. Observation:  $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$  decomposed into the union of the coordinate axes  $X_1 := V(x_1)$  and  $X_2 := V(x_2)$ . The Zariski topology can detect these decompositions.

**Definition 1.0.2** (Irreducibility and Connectedness).

Let  $X$  be a topological space.

- a.  $X$  is *reducible* iff there exist nonempty proper closed subsets  $X_1, X_2 \subset X$  such that  $X = X_1 \cup X_2$ . Otherwise,  $X$  is said to be *irreducible*.
- b.  $X$  is *disconnected* if there exist  $X_1, X_2 \subset X$  such that  $X = X_1 \amalg X_2$ . Otherwise,  $X$  is said to be *connected*.

**Example 1.5.**

$V(x_1x_2)$  is reducible but connected.