Title

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Question: how do we define $h_{V,D}$?

Answer: write $D = D_1 - D_2$ which are (very) ample divisors and basepoint free. We then obtain embeddings

$$\varphi_1 : V \hookrightarrow \mathbb{P}_K^{n_1}$$
$$\varphi_2 : V \hookrightarrow \mathbb{P}_K^{n_2}.$$

So write

$$h_{V,D}(p) = h(\varphi_1(p)) - h(\varphi_2(p)) + O(1)$$

Example 1.1.
For E/K an elliptic curve,

- $2[0]$ is an ample divisor
- $3[0]$ is a very ample divisor.

Let K be a local field (i.e. $\mathbb{C}, \mathbb{R}$, a p-adic field, or $\mathbb{F}_q((t))$ formal Laurent series) and A/K be an abelian variety; we want to understand $A(K)$. We know this has the structure of compact abelian K-analytic Lie group.

- Question 1: What does Lie theory say?
- Question 2: What extra information comes from A/K being a g-dimensional abelian variety?

If $K=\mathbb{C}$, then $A(K)\cong(\mathbb{R}/\mathbb{Z})^{2g}$. If $K=\mathbb{R}$, then $A(K)\cong(\mathbb{R}/\mathbb{Z})^{g}\oplus\prod_{i=1}^{d}\mathbb{Z}/2\mathbb{Z}$ where $0\leq d\leq g$.

Fix d, then

- Let $E_1/\mathbb{R}$ with $\Delta>0$ (and thus 3 real roots), then $E_1(\mathbb{R})[2]=(\mathbb{Z}/2\mathbb{Z})^2$.
- Let $E_2/\mathbb{R}$ with $\Delta<0$ (and 1 real root), then $E_2(\mathbb{R})[2]=\mathbb{Z}/2\mathbb{Z}$.

By taking products of E_1 and E_2, i.e. $A=(E_1)^d\times(E_2)^{g-d}$.

Todo: find reference in Silverman?

Fact $A(K)$ is totally disconnected and homeomorphic to a Cantor set.

Fact (From Lie Theory, Serre p.116) If G is an abelian compact K-analytic Lie group, then there exists a filtration by open finite index subgroups

$$G=G^0\supset G^1\supset\cdots\supset G^n\supset\cdots$$

such that

1. The successive quotients are finite, and each G^i is *standard*, i.e. obtained by evaluating a formal group law on $\left(\mathfrak{m}^i\right)^g$.
2. $\bigcap_i G^i=(0)$.
3. G^i/G^{i+1} has exponent p, i.e. it is a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space.
4. $G'[\text{tors}]=G'[p^\infty]$, all of the prime-to-$p$ torsion is p-primary.

Todo: define p-primary torsion, $\mathbb{Q}_p$.

What structure theorem does this give?

Theorem 1.1 ***(C-Lacy).***
Let G be a compact, second countable, K-analytic abelian Lie group of dimension $g\geq 1$. Then

a. If char $K=0$ and $d=[K:\mathbb{Q}_p]$, then

$$G\cong_{\text{TopGrp}}\mathbb{Z}_p^{dg}\oplus G[\text{tors}]$$

where $G[\text{tors}]$ is finite.

b. If char $K=p$, i.e. $K=\mathbb{F}_q((t))$, and if it is true that $G[\text{tors}]$ is finite iff $G[p]$ finite, then

$$G\cong_{\text{TopGrp}}\prod_{i=1}^{\infty}\mathbb{Z}_p\oplus G[\text{tors}]$$

For any $g\geq 1$, $(T,+)$ a finite discrete abelian group $(R,+)\cong(\mathbb{Z}_p^d,+)$ and $R^g\oplus T$ is a compact abelian K-analytic Lie group isomorphic to $\mathbb{Z}_p^{dg}\oplus T$ (?).

Question: what does this mean for $G=S^1$? Ask Pete!

Theorem 1.2 ***(Cartan).***
Let K be a local field, $\mathbb{Q} \hookrightarrow K$ dense (so $K = \mathbb{R}, \mathbb{Q}_p$). Then if G_1, G_2 are K-analytic, and $\varphi \in \hom_{\text{TopGrp}}(G_1, G_2)$, then $\varphi \in \hom_{k\text{-analytic}}(G_1, G_2)$.

Example 1.2.
For $R = \mathbb{F}_q[[t]]$, $(R,+)^g[p] = (R,+)^g$.

Example 1.3.
Take $G = \mathbb{G}_a^g(K)$ the additive group or A/K a g-dimensional abelian variety (i.e. $G = A(K)$) then $G[p] \subsetneq (\mathbb{Z}/p\mathbb{Z})^{2g}$ and is finite.

1.1 Proof of Cartan's Theorem

1.1.1 Step 1

We want to show that $G[p] < \infty$, then $G[\text{tors}] < \infty$. We'll use the filtration in Serre's result; then for $i \gg 1$, we'll have $G^i[p] = 0$. Thus for $i \gg 1$, we'll have $G^i[p^\infty] = 0$; but this is the only torsion that can appear. We'll then obtain an injection $G[\text{tors}] \hookrightarrow G/G^i < \infty$.

Lemma 1.3.
Let H be an abelian torsionfree pro-p group (e.g. $\prod \mathbb{Z}_p$). Then $H \cong \prod_{i \in I} \mathbb{Z}_p$, and if H is second-countable, then I is countable.

Proof .
Omitted. Idea: use Pontrayagin duality $G^\vee := \hom_{\text{Top}}(G, \mathbb{R}/\mathbb{Z})$. ■

Note: look up pro-p groups. Is the Pontrayagin dual a cohomotopy group?