

Title

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1 Chapter 1

1.1 Within Chapter

Proposition 1.1: Fix an ideal $\mathfrak{a} \trianglelefteq R$. There is a correspondence

$$\left\{ \mathfrak{b} \mid \mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R \right\} \iff \left\{ \tilde{\mathfrak{b}} \trianglelefteq R/\mathfrak{a} \right\}.$$

Proof: Let $\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}$ be the quotient map, this induces a map

$$\begin{aligned} \Phi : S_A(R) &\rightarrow S(R/\mathfrak{a}) \\ T &\mapsto \pi_A(T). \end{aligned}$$

where $S_A(R) = \left\{ \mathfrak{b} \mid \mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R \right\}$ is the set of ideals of R containing \mathfrak{a} and $S(R/\mathfrak{a})$ is the set of ideals of R/\mathfrak{a} .

- Injectivity: If $I_1, I_2 \trianglelefteq R/\mathfrak{a}$ then write $I_1 = H_1 + \mathfrak{a}, I_2 = H_2 + \mathfrak{a}$.

Proposition 1.2: TFAE

- R is a field
- R is simple, i.e. the only ideals of R are $0, R$.
- Every homomorphism $\phi : R \rightarrow S$ for S an arbitrary ring is injective.

Proof: ?

Proposition: Maximal ideals are prime.

Proof: ?

Proposition: If $\mathfrak{p} \trianglelefteq R$ is prime, R/\mathfrak{p} is a domain. If $\mathfrak{m} \trianglelefteq R$ is maximal, R/\mathfrak{m} is a field.

Proof: ?

Theorem 1.3: Every ring R has a nontrivial maximal ideal $I \neq 0$, and every ideal is contained in a maximal ideal.

Proof: ?

Corollary 1.5: Every non-unit of R is contained in a maximal ideal.

Proof: ?

Proposition 1.6: If $A \setminus \mathfrak{m} \subset R^\times$, then A is a local ring with \mathfrak{m} its maximal ideal. If \mathfrak{m} is maximal and $1 + m \in R^\times$ for all $m \in \mathfrak{m}$, then A is a local ring.

Proof: ?

Proposition: If $f \in k[x_1, \dots, x_n]$ is irreducible over k , then (f) is prime.

Proposition: \mathbb{Z} is a PID, and (p) is prime iff p is zero or a prime number, and every such ideal is maximal.

Proposition: $k[\{x_i\}]$ has maximal ideals that are not principal iff $n > 1$.

Exercise: Characterize the maximal and prime ideals of $k[x_1, \dots, x_n]$? Is this a field, domain, PID, UFD, a local ring, ...?

Proposition: Every nonzero prime ideal in a PID is maximal.

Proof: ?

Definition: The set $\text{nil}(A)$ of all nilpotent elements in a ring A is the nilradical of A . The set $J(A) = \bigcap_{\mathfrak{m} \in \text{Spec}_{\max}(A)} \mathfrak{m}$ is the Jacobson radical.

Proposition 1.7: $\text{nil}(A) \trianglelefteq A$ is an ideal and $A/\text{nil}(A)$ has no nonzero nilpotent elements.

Proof: ?

Proposition 1.8: $\text{nil}(A) = \bigcap_{\mathfrak{p} \in \text{Spec}(A)} \mathfrak{p}$ is the intersection of all prime ideals of A .

Proof: ?

Proposition 1.9: $x \in J(A)$ iff $1 - xa \in A^\times$ for all $a \in A$.

Proposition: If $(m), (n) \trianglelefteq \mathbb{Z}$ then $(m) \cap (n) = (\gcd(m, n))$ and $(m)(n) = (mn)$.

Exercise: If $\mathfrak{a} \trianglelefteq k[x_1, \dots, x_m]$, characterize \mathfrak{a}^n .

Exercise: Show that $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$ are coprime iff there exist $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$.

Proposition 1.10: Let $\{\mathfrak{m}_i\} \trianglelefteq A$ be a family of ideals and define $\phi : A \rightarrow \prod A/\mathfrak{m}_i$.

1. If $\{\mathfrak{m}_i\}$ are pairwise coprime, then $\prod \mathfrak{m}_i = \bigcap \mathfrak{m}_i$
2. ϕ is surjective iff $\{\mathfrak{m}_i\}$ are pairwise coprime.
3. ϕ is injective iff $\bigcap \mathfrak{m}_i = (0)$.

Exercise: Show that the union of ideals is not necessarily an ideal.

Proposition 1.11:

- a. Let $\{\mathfrak{p}_i\}$ be a set of prime ideals and let $\mathfrak{a} \in \bigcup \mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i .
- b. Let $\{\mathfrak{a}_i\}$ be ideals and $\mathfrak{p} \supseteq \bigcap \mathfrak{a}_i$ be prime. $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i , and if $\mathfrak{p} = \bigcap \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i .

Exercise: Let $A = \mathbb{Z}$, and characterize the ideal quotient $(m : n)$.

Exercise 1.12:

1. $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$
2. $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$
3. $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$
4. $(\bigcap \mathfrak{a}_i : \mathfrak{b}) = \bigcap (\mathfrak{a}_i : \mathfrak{b})$
5. $(\mathfrak{a} : \sum \mathfrak{b}_i) = \bigcap (\mathfrak{a} : \mathfrak{b}_i)$

Proposition: For $\mathfrak{a} \leq A$, $\sqrt{\mathfrak{a}}$ is an ideal.

Exercise 1.13:

1. $\sqrt{\mathfrak{a}} \supseteq \mathfrak{a}$
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$
3. $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \bigcap \mathfrak{b}} = \sqrt{\mathfrak{a}} \bigcap \sqrt{\mathfrak{b}}$
4. $\sqrt{\mathfrak{a}} = (1) \iff \mathfrak{a} = (1)$
5. $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.
6. For \mathfrak{p} prime, $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n \geq 1$.

Proposition 1.14: $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p} \in \text{Spec}(A)} \mathfrak{p}$

Proposition 1.15: Let D be the set of zero-divisors in A . Then $D = \bigcup_{x \neq 0} \sqrt{\text{Ann}(x)}$.

Exercise: Let $(m) \leq \mathbb{Z}$ where $m = \prod p_i^{k_i}$, and show that $\sqrt{(m)} = (p_1 p_2 \cdots) = \bigcap (p_i)$.

Proposition 1.16: If $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$ are coprime then $\mathfrak{a}, \mathfrak{b}$ are coprime.

Exercise: Show that if $f : A \rightarrow B$ and $\mathfrak{a} \leq A$, it is not necessarily the case that $f(\mathfrak{a}) \leq B$.

Exercise: Show that if \mathfrak{b} is prime then $A \cdot f^{-1}(\mathfrak{b})$ is prime, but if \mathfrak{a} is prime then $B \cdot f(\mathfrak{a})$ need not be prime.

Exercise: Write $\mathfrak{a}^e := \langle f(\mathfrak{a}) \rangle$ and $\mathfrak{b}^c = \langle f^{-1}(\mathfrak{b}) \rangle$. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}[i]$ be the inclusion, and show that

- $(2)^e = \langle (1+i)^2 \rangle$, which is not prime in $\mathbb{Z}[i]$
- (Nontrivial) If $p \equiv 1 \pmod{4}$, then \mathfrak{p}^e is the product of two distinct prime ideals
- If $p \equiv 3 \pmod{4}$ then \mathfrak{p}^e is prime.

Proposition: Let $C = \{\mathfrak{b}^c \mid \mathfrak{b} \leq B\}$ and $E = \{\mathfrak{a}^e \mid \mathfrak{a} \leq A\}$. Then

1. $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$ and $\mathfrak{b} \supset \mathfrak{b}^{ce}$,

2. $\mathfrak{b}^c = \mathfrak{b}^{cec}$ and $\mathfrak{a}^e = \mathfrak{a}^{ece}$
3. $C = \left\{ \mathfrak{a} \trianglelefteq A \mid \mathfrak{a}^{ec} = \mathfrak{a} \right\}$ and $E = \left\{ \mathfrak{b} \trianglelefteq B \mid \mathfrak{b}^{ce} = \mathfrak{b} \right\}$.
4. The map $\phi : C \rightarrow E$ given by $\phi(\mathfrak{a}) = \mathfrak{a}^{ec}$ is a bijection with inverse $\mathfrak{b} \mapsto \mathfrak{b}^c$.
5. If $\mathfrak{a} \in C$ then $\mathfrak{a} = \mathfrak{b}^c = \mathfrak{b}^{cec} = \mathfrak{a}^{ec}$, and if $\mathfrak{a} = \mathfrak{a}^{ec}$ then \mathfrak{a} is the contraction of \mathfrak{a}^e .

Exercise 1.18:

$$\begin{array}{ll}
(\mathfrak{a}_1 + \mathfrak{a}_2)^c = \mathfrak{a}_1^c + \mathfrak{a}_2^c, & (\mathfrak{b}_1 + \mathfrak{b}_2)^c \geq \mathfrak{b}_1^c + \mathfrak{b}_2^c \\
(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e, & (\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c \\
(\mathfrak{a}_1 \mathfrak{a}_2)^c = \mathfrak{a}_1^c \mathfrak{a}_2^c, & (\mathfrak{b}_1 \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c \mathfrak{b}_2^c \\
(\mathfrak{a}_1 : \mathfrak{a}_2)^c \subseteq (\mathfrak{a}_1^c : \mathfrak{a}_2^c), & (\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c) \\
r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e), & r(\mathfrak{b})^c = r(\mathfrak{b}^c)
\end{array}$$

1.2 End of Chapter Exercises