Section 8.6: The Solutions of the Floer Equation are "Somewhere Injective".

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Wednesday 13th May, 2020

Contents

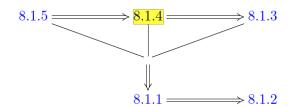
Goa	1.2: Continuation Principle	5
0.6	Regular Points Are Open and Dense	4
	8.6.1: Transform to Cauchy-Riemann	
0.4	Goal 1: Discrete Critical Points and Dense Regular Points	2
0.3	Notation	2
0.2	Outline of Statements	1
0.1	Outline	1

0.1 Outline

Two Goals:

- 1. Prove critical points are discrete and regular points are open/dense.
- 2. Prove the continuation principle that was used in Proposition 8.1.4

0.2 Outline of Statements



- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) \mu(y)$.
- 8.1.4: $\Gamma: W^{1,p} \times C_{\varepsilon}^{\infty} \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x,y,J)$ is a Banach manifold
- 8.1.1: For $h \in \mathcal{H}_{reg}$, $H_0 + h$ is nondegenerate and $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(H_0 + h, J)$.
- 8.1.2: For $h \in \mathcal{H}_{reg}$ and all contractible orbits x, y of H_0 , $\mathcal{M}(x, y, H_0 + h)$ is a manifold of dimension $\mu(x) \mu(y)$.

0.3 Notation

• The Floer equation and its linearization:

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \operatorname{grad}_{u}(H) = 0$$
$$(d\mathcal{F})_{u}(Y) = \frac{\partial Y}{\partial s} + J_{0} \frac{\partial Y}{\partial t} + S \cdot Y$$
$$Y \in u^{*}TW, \ S \in C^{\infty}(\mathbb{R} \times S^{1}; \operatorname{End}(\mathbb{R}^{2n})).$$

- z = s + it
- X is a vector field (time-dependent and periodic) on \mathbb{R}^{2n} , J an almost complex structure -X, J are smooth
- $u \in C^{\infty}(\mathbb{R} \times S^1; W)$ is a solution to the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0$$

• C(u) the set of critical points and R(u) the set of regular points of u:

$$(s_0, t_0) \in C(u) \subseteq \mathbb{R} \times S^1 \iff \frac{\partial u}{\partial s}(s_0, t_0) = 0$$

$$(s_0, t_0) \in R(u) \subset \mathbb{R} \times S^1 \iff (s_0, t_0) \notin C(u) \& s \neq s_0 \implies u(s_0, t_0) \neq u(s, t_0).$$

0.4 Goal 1: Discrete Critical Points and Dense Regular Points

Goal 1: prove the following theorem

Theorem 0.1(8.5.4).

- 1. C(u) is discrete and 2. $R(u) \hookrightarrow \mathbb{R} \times S^1$ is open and dense.

Outline of the proof:

• Prove 8.6.1 (direct, short) which transforms the equation

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0 \text{ where } u \in C^{\infty}(\mathbb{R} \times S^1; W)$$

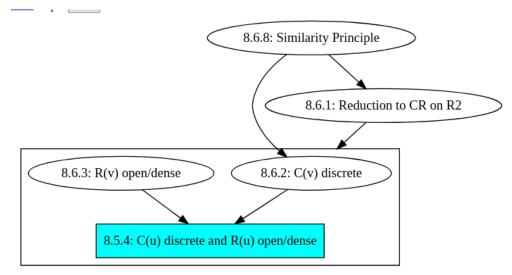
to a Cauchy-Riemann equation on \mathbb{R}^2 :

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0$$
 where $v \in C^{\infty}(\mathbb{R}^2; W)$

- Produces a map v which reduces Theorem 8.5.4 to two statements
 - 8.6.2: C(v) (and thus C(u)) is discrete
 - * Proved later using similarity principle.
 - 8.6.3 (Injectivity): If v is a smooth periodic solution of CR with $\frac{\partial v}{\partial s} \neq 0$ then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.
- Prove 8.6.3 (Injectivity)

Contents 2

- Show open.
- Show dense
- Prove 8.6.8 (similarity principle)
- Use similarity principle to prove 8.6.1, yields theorem.



0.5 8.6.1: Transform to Cauchy-Riemann

Proposition 0.2(8.6.1, Transform to CR-equation on R2).

If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t, u) \left(\frac{\partial u}{\partial t} - X(t, u) \right) = 0.$$

Then there exists

- An almost complex structure J
- A diffeomorphism ϕ on W?
- A map $v \in C^{\infty}(\mathbb{R}^2; W)$

satisfying

$$\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0$$

where

- $v(s,t+1) = \varphi(v(s,t))$
- C(u) = C(v), i.e. u, v have the same critical points
- R(u) = R(v).

Proof

• Since $W \times S^1$ is compact, the flow ψ_t of X_t is defined on all of W

- We thus have a map
$$\psi_t:W\longrightarrow W$$
 such that

$$* \frac{\partial}{\partial t} \psi_t = X_t \circ \psi_t$$

Contents 3

$$* \psi_0 = id$$

• Define the (important!) map

$$v(s,t) \coloneqq \left(\psi_t^{-1} \circ u\right)(s,t)$$

• We can then compute

$$\frac{\partial u}{\partial s} = (d\psi_t) \left(\frac{\partial v}{\partial s}\right)$$
$$\frac{\partial u}{\partial t} = (d\psi_t) \left(\frac{\partial v}{\partial t}\right) + X_t(u)$$

Lemma 8.6.2: The set of critical points of v above is discrete. Precisely: There exists a constant $\delta > 0$ such that $(dv)_z \neq 0$ for any $0 << |z| < \delta$.

Proof: Postponed to p.264.

Definition: Multiple points

Proposition 8.6.3: Injectivity result. Let v be a smooth 1-periodic (in t) solution of the CR equation, i.e. $v(s,t+1) = \phi(v(s,t))$ for some smooth ϕ ? and $\frac{\partial v}{\partial s} \neq 0$. Then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.

0.6 Regular Points Are Open and Dense

Proof (BIG):

- Show R(v) is open (easy)
- Show R(v) is dense (delicate)

Long proof.

Lemma 8.6.4: For every r > 0 there exists a $\delta > 0$ such that

$$|t - t_0|, |s - s_0| < \delta \implies \exists s' \in B_r(s_i) \text{ s.t. } v(s, t) = v(s', t).$$

Proof: short.

Lemma 8.6.5: Let v_1, v_2 be two solutions of the CR-equation with $X_t \equiv 0$ on $B_{\varepsilon}(0), v_1(0,0) = v_2(0,0)$ such that $(dv_1)_0, (dv_2)_0 \neq 0$. Also suppose

$$\forall \varepsilon \; \exists \delta \; \text{s.t.}$$

$$\forall (s,t) \in B_{\delta}(0), \ \exists s' \in \mathbb{R} \begin{cases} (s',t) \in B_{\varepsilon}(0) \\ v_1(s,t) = v_2(s',t) \end{cases}.$$

Then

$$\forall z \in B_{\varepsilon}(0), \quad v_1(s,t) = v_2(s,t).$$

Take perturbed CR equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Contents 4

Fix $S \in C^{\infty}(\mathbb{R}^2; \operatorname{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let $Y \in C^{\infty}(B_{\varepsilon}; \mathbb{C}^n)$ be a solution to the perturbed CR equation and let p > 2. Then there exists $0 < \delta < \varepsilon$ and a map $A \in W^{1,p}(B_{\delta}, \mathrm{GL}(\mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_{\delta} \longrightarrow \mathbb{C}^n$ such that

$$\forall (s,t) \in B_{\delta} \quad Y(s,t) = A(s,t) \ \sigma(s+it) \quad \text{and} \quad J_0 A(s,t) = A(s,t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of $\bar{\partial}$) For every p > 1, the following operator is surjective and Fredholm:

$$\bar{\partial}: W^{1,p}(S^2; \mathbb{C}^n) \longrightarrow L^p(\Lambda^{0,1}T^*S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7

1 Goal 2: Continuation Principle

Goal 2: prove a continuation principle:

Proposition 1.1(8.6.6, Continuation Principle).

On an open $U \subset \mathbb{R}^2$, let Y be a solution to the perturbed CR equation

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0$$

where J_0 is the standard complex structure on \mathbb{R}^{2n} and $S \in C^{\infty}(\mathbb{R}^2, \operatorname{End}(\mathbb{R}^{2n}))$. Say that f has an *infinite-order zero* at z_0 iff

$$\forall k \ge 0, \quad \sup_{|z-z_0| \le t} \frac{|f(z)|}{r^k} \stackrel{r \longrightarrow 0}{\longrightarrow} 0.$$

For f smooth, equivalently $f^{(k)}(z_0) = 0$ for all k.

Then the set

$$C \coloneqq \left\{ (s,t) \in U \mid Y \text{ has an infinite order zero at } (s,t) \right\}$$

is clopen. In particular, if U is connected and Y=0 on some nonempty $V\subset U$, then $Y\equiv 0$.

Proposition 1.2(8.1.4,).

Define

$$\mathcal{Z}(x, y, J) \coloneqq \{(u, H_0 + h) | h \in \mathcal{C}_{\varepsilon}^{\infty}(H_0) \text{ and } u \in \mathcal{M}(x, y, J, H)\}.$$

If $(u, H_0 + h) \in \mathcal{Z}(x, y)$ then the following map admits a continuous right-inverse and is

surjective:

$$\Gamma: W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \times \mathcal{C}_{\varepsilon}^{\infty}\left(H_0\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$(Y,h) \longmapsto \left(d\mathcal{F}^{H_0+h}\right)_u(Y) + \operatorname{grad}_u h$$

where \mathcal{F}^{H_0+h} is the Floer operator corresponding to H_+h .

Used to show (via the implicit function theorem) that $\mathcal{Z}(x,y,J)$ is a Banach manifold when $x \neq y$.