

# Category $\mathcal{O}$ , Problem Set 3

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## 1 Humphreys 1.10

Prove that the transpose map  $\tau$  fixes  $Z(\mathfrak{g})$  pointwise.

Check that  $\tau$  commutes with the Harish-Chandra morphism  $\xi$  and use the fact that  $\xi$  is injective.

### 1.1 Solution

We first note that after choosing a PBW basis for  $\mathfrak{g}$ ,  $\tau$  is defined on  $\mathfrak{g}$  in the following way:

$$\begin{aligned}\tau : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x_\alpha &\mapsto y_\alpha \\ h_\alpha &\mapsto h_\alpha \\ y_\alpha &\mapsto x_\alpha\end{aligned}$$

which lifts to an anti-involution  $\tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$  by extending linearly over PBW monomials. We can note that since  $\tau$  fixes  $\mathfrak{h}$  pointwise by definition, its lift also fixes  $U(\mathfrak{h})$  pointwise.

Using this basis, we can explicitly identify the Harish-Chandra morphism:

$$\begin{aligned}\xi : Z(\mathfrak{g}) &\longrightarrow U(\mathfrak{h}) \\ \prod_{i,j,k} x_i^{r_i} h_j^{s_j} y_k^{t_k} &\mapsto \prod_j h_j^{s_j}.\end{aligned}$$

## 2 Humphreys 1.12

Fix a central character  $\chi$  and let  $\{V^{(\lambda)}\}$  be a collection of modules in  $\mathcal{O}$  indexed by the weights  $\lambda$  for which  $\chi = \chi_\lambda$  satisfying

1.  $\dim V^{(\lambda)} = 1$
2.  $\mu < \lambda$  for all weights  $\mu$  of  $V^{(\lambda)}$ .

Then the symbols  $[V^{(\lambda)}]$  form a  $\mathbb{Z}$ -basis for the Grothendieck group  $K(\mathcal{O}_x)$ .

For example take  $V^{(\lambda)} = M(\lambda)$  or  $L(\lambda)$ .

## 3 Humphreys 1.13

Suppose  $\lambda \neq \mu$ , so the linkage class  $W \cdot \lambda$  is the disjoint union of its nonempty intersections of various cosets of  $\Lambda_r \in \mathfrak{h}^\vee$ .

Prove that each  $M \in \mathcal{O}_{\chi_\lambda}$  has a corresponding direct sum decomposition  $M = \bigoplus M_i$  in which all weights of  $M_i$  lie in a single coset.

Recall exercise 1.1b.