

# Lie Algebras

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September 6, 2019

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# 1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

## 1.1 Overview

Here is a short overview of the topics we expect to cover:

### 1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

### 1.1.2 Chapter 3-4

We will describe the following series of correspondences:



## 1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:





### 1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

#### 1.3.1 Chapter 6

Some important topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type  $A_\ell$  algebra used for the first time; however, it differs from the other types in several important/significant ways.

#### 1.3.2 Chapter 7

Skip!

#### 1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat- $\mathcal{O}$  [Humphrey 08]

## 1.4 Content

Fix  $F$  a field of characteristic zero – note that prime characteristic is closer to a research topic.

**Definition 1.** A **Lie Algebra**  $\mathfrak{g}$  over  $F$  is an  $F$ -vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\cdot, \cdot]$  is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0.$$

**Exercise 1.** Show that  $[x, y] = -[y, x]$ .

**Definition 2.** Two Lie algebras  $\mathfrak{g}, \mathfrak{g}'$  are said to be isomorphic if  $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ .

## 1.5 Linear Lie Algebras

Let  $V = \mathbb{F}^n$ , and define  $\text{End}(V) = \{f : V \rightarrow V \mid f \text{ is linear}\}$ . We can then define  $\mathfrak{gl}(n, V)$  by setting  $[x, y] = (x \circ y) - (y \circ x)$ .

**Exercise 2.** Verify that  $V$  is a Lie algebra.

**Definition 3.** Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \mid \text{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group*  $\text{SL}(n, V)$ ).

**Definition 4.** A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

**Definition 5.** The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \mid MA - A^T M = 0 \right\} \text{ where } M = \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

**Definition 6.** The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left( \begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & I_n \\ \hline & -I_n & 0 \end{array} \right) & n = 2\ell + 1 \text{ odd,} \\ \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else.} \end{cases}$$

**Proposition 7.** The dimensions of these algebras can be computed;

- The dimension of  $\mathfrak{gl}(n, \mathbb{F})$  is  $n^2$ , and has basis  $\{e_{i,j}\}$  the matrices if a 1 in the  $i, j$  position and



zero elsewhere.

- For type  $A_\ell$ , we have  $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$ .
- For type  $C_\ell$ , we have  $\dim \mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2 \left( \frac{\ell(\ell+1)}{2} \right)$ , and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type  $D_\ell$  we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be  $2\ell^2 - \ell$ .

- For type  $B_\ell$ , we have  $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$ , with elements of the form

$$\left( \begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

**Exercise 3.** Use the relation  $MA = A^{tM}$  to reduce restrictions on the blocks.



**Theorem 8.** These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

## 2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

**Exercise 4.** Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

## 2.1 Lie Algebras of Derivations

**Definition 9.** An  $F$ -algebra  $A$  is an  $F$ -vector space endowed with a bilinear map  $A^2 \rightarrow A$ ,  $(x, y) \mapsto xy$ .

**Definition 10.** An algebra is **associative** if  $x(yz) = (xy)z$ .

Modern interest: simple Lie algebras, which have a good representation theory. Take a look at Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

**Definition 11.** Any map  $\delta : A^2 \rightarrow A$  that satisfies the Leibniz rule is called a **derivation** of  $A$ , where the rule is given by  $\delta(xy) = \delta(x)y + x\delta(y)$ .

**Definition 12.** We define  $\text{Der}(A) = \{\delta \mid \delta \text{ is a derivation}\}$ .

Any Lie algebra  $\mathfrak{g}$  is an  $F$ -algebra, since  $[\cdot, \cdot]$  is bilinear. Moreover,  $\mathfrak{g}$  is associative iff  $[x, [y, z]] = 0$ .

**Exercise 5.** Show that  $\text{Derg} \leq \mathfrak{gl}(\mathfrak{g})$  is a Lie subalgebra. One needs to check that  $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$ .

**Exercise 6** (Turn in). Define the adjoint by  $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}, y \mapsto [x, y]$ . Show that  $\text{ad}_x \in \text{Der}(\mathfrak{g})$ .

## 2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of  $\mathfrak{gl}(V)$ . Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

**Example 13.** Any  $F$ -vector space can be made into a Lie algebra by setting  $[x, y] = 0$ ; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write  $\mathfrak{g} = Fx$ , and so  $[x, x] = 0 \implies [\cdot, \cdot] = 0$ . So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write  $\mathfrak{g} = Fx \oplus Fy$ , the only nontrivial bracket here is  $[x, y]$ . Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$  is abelian.
  - $[x, y] = ax + by \neq 0$ . Assume  $a \neq 0$  and set  $x' = ax + by, y' = \frac{y}{a}$ . Now compute  $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$ . Punchline:  $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$ .

We can fill in a table with all of the various combinations of brackets:

$[\cdot, \cdot]$	$x'$	$y'$
$x'$	0	$x'$
$y'$	$-x'$	0

**Example 14.** Let  $V = \mathbb{R}^3$ , and define  $[a, b] = a \times b$  to be the usual cross product.

**Exercise 7.** Look at notes for basis elements of  $\mathfrak{sl}(2, F)$ ,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of  $\text{ad}(e)$ ,  $\text{ad}(h)$ ,  $\text{ad}(g)$  with respect to this basis.

### 2.3 Ideals

**Definition 15.** A subspace  $I \subseteq \mathfrak{g}$  is called an **ideal**, and we write  $I \trianglelefteq \mathfrak{g}$ , if  $x, y \in I \implies [x, y] \in I$ .

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using  $[x, y] = [-y, x]$ .

**Exercise 8.** Check that the following are all ideals of  $\mathfrak{g}$ :

- $\{0\}, \mathfrak{g}$ .
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra  $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \mid x_i, y_i \in \mathfrak{g}\}$ .  
– Moreover,  $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$ .

Fact: If  $I, J \trianglelefteq \mathfrak{g}$ , then

- $I + J = \{x + y \mid x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \cap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \mid x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

**Definition 16.** A Lie algebra is **simple** if  $[\mathfrak{g}, \mathfrak{g}] \neq 0$  (i.e. when  $\mathfrak{g}$  is not abelian) and has no non-trivial ideals. Note that this implies that  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .

**Theorem 17.** Suppose that  $\text{char } F \neq 2$ , then  $\mathfrak{sl}(2, F)$  is not simple.

*Proof.* Recall that we have a basis of  $\mathfrak{sl}(2, F)$  given by  $B = \{e, h, f\}$  where

- $[e, f] = h$ ,
- $[h, e] = 2e$ ,
- $[h, f] = -2f$ .

So think of  $[h, e] = \text{ad}_h$ , so  $h$  is an eigenvector of this map with eigenvalues  $\{0, \pm 2\}$ . Since  $\text{char } F \neq 2$ , these are all distinct. Suppose  $\mathfrak{sl}(2, F)$  has a nontrivial ideal  $I$ ; then pick  $x = ae + bh + cf \in I$ . Then  $[e, x] = 0 - 2be + ch$ , and  $[e, [e, x]] = 0 - 0 + 2ce$ . Again since  $\text{char } F \neq 2$ , then if  $c \neq 0$  then  $e \in I$ . Now you can show that  $h \in I$  and  $f \in I$ , but then  $I = \mathfrak{sl}(2, F)$ , a contradiction. So  $c = 0$ .

Then  $x = bh \neq 0$ , so  $h \in I$ , and we can compute

$$2e = [h, e] \in I \implies e \in I,$$

$$2f = [h, -f] \in I \implies f \in I.$$



which implies that  $I = \mathfrak{sl}(2, F)$  and thus it is simple.

□

Note that there is a homework coming due next Monday, about 4 questions.

### 3 Lecture 3

Last time, we looked at ideals such as  $0, \mathfrak{g}, Z(\mathfrak{g})$ , and  $[\mathfrak{g}, \mathfrak{g}]$ .

**Definition:** If  $I \trianglelefteq \mathfrak{g}$  is an ideal, then the quotient  $\mathfrak{g}/I$  also yields a Lie algebra with the bracket given by  $[x + I, y + I] = [x, y] + I$ .

**Exercise:** Check that this is well-defined, so that if  $x + I = x' + I$  and  $y + I = y' + I$  then  $[x, y] + I = [x', y'] + I$ .

#### 3.1 Homomorphisms and Representations

**Definition 18.** A linear map  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a *Lie homomorphism* if  $\phi[x, y] = [\phi(x), \phi(y)]$ .

**Remark.**  $\ker \phi \trianglelefteq \mathfrak{g}_1$  and  $\text{im } \phi \leq \mathfrak{g}_2$  is a subalgebra.

**Fact:** There is a canonical way to set up a 1-to-1 correspondence  $\{I \trianglelefteq \mathfrak{g}\} \iff \{\text{hom } \phi : \mathfrak{g} \rightarrow \mathfrak{g}'\}$  where  $I \mapsto (x \mapsto x + I)$  and the inverse is given by  $\phi \mapsto \ker \phi$ .

**Theorem** (Isomorphism theorem for Lie algebras):

- If  $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  is a Lie algebra homomorphism, then  $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If  $I, J \trianglelefteq \mathfrak{g}$  are ideals and  $I \subset J$  then  $J/I \trianglelefteq \mathfrak{g}/I$  and  $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$ .
- If  $I, J \trianglelefteq \mathfrak{g}$  then  $(I + J)/J \cong I/(I \cap J)$ .

**Definition:** A *representation* of a Lie algebra  $\mathfrak{g}$  is a Lie algebra homomorphism  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  into a linear Lie algebra for some vector space  $V$ .

We call  $V$  a  $\mathfrak{g}$ -module with action  $g \cdot v = \phi(g)(v)$ .

**Example:** The *adjoint representation*:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot]. \end{aligned}$$

**Corollary 19.** Any simple Lie algebra is isomorphic to a linear Lie algebra.

**Proof:** Since  $\mathfrak{g}$  is simple, the center  $Z(\mathfrak{g}) = 0$ . We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) &= \left\{ x \in \mathfrak{g} \mid \text{ad}_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \right\} \\ &= \ker \text{ad}_x. \end{aligned}$$

Using the first isomorphism theorem, we have  $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad} \subseteq \mathfrak{gl}(\mathfrak{g})$ . But  $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$  here, so we are done.

## 3.2 Automorphisms

Definition: An automorphism of  $\mathfrak{g}$  is an isomorphism  $\mathfrak{g} \rightarrow \mathfrak{g}$ , and we define

$$\text{Aut}(\mathfrak{g}) = \{ \phi : \mathfrak{g} \rightarrow \mathfrak{g} \mid \phi \text{ is an isomorphism} \}.$$

Proposition: If  $\delta \in \text{Der}(\mathfrak{g})$  is nilpotent, then

$$\exp(\delta) := \sum \frac{\delta^n}{n!} \in \text{Aut}(\mathfrak{g}).$$

This is well-defined because  $\delta$  is nilpotent, and a binomial formula holds:

$$\frac{\delta^n([x, y])}{n!} = \sum_{i=0}^n \left[ \frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!} \right].$$

and for  $n = 1$ ,  $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ .

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!}.$$

Example: Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$  and define

$$s = \exp(\text{ad}_e) \exp(\text{ad}_{-f}) \exp(\text{ad}_e) \in \text{Aut} \mathfrak{g}.$$

where  $e, f$  are defined as (todo, see written notes).

Then define the Weyl group  $W = \langle s \rangle$ .

Exercise: Check that  $s(e) = -f, s(f) = -e, s(h) = -h$ , and so the order of  $s$  is 2 and  $W = \{1, s\}$ .

## 4 Lecture 4

### 4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for  $\mathfrak{g}$  is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\vdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra  $\mathfrak{g}$  is *solvable* if there is some  $n$  for which  $\mathfrak{g}^{(n)} = 0$ .

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in  $\mathfrak{gl}(n, \mathbb{F})$ .

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let  $\mathfrak{g}$  be a Lie algebra, then

1. If  $\mathfrak{g}$  is solvable, then all subalgebras and all homomorphic images of  $\mathfrak{g}$  are also solvable.
2. If  $I \trianglelefteq \mathfrak{g}$  and both  $I$  and  $\mathfrak{g}/I$  are solvable, then so is  $\mathfrak{g}$ .
3. If  $I, J \trianglelefteq \mathfrak{g}$  are solvable, then so is  $I + J$ .

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical*  $\text{Rad}(\mathfrak{g})$ .

Definition: A Lie algebra is semisimple if  $\text{Rad}(\mathfrak{g}) = 0$ .

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any  $\mathfrak{g}$ , the quotient  $\mathfrak{g}/\text{Rad}(\mathfrak{g})$  is semisimple.

## 4.2 Nilpotency

$$\begin{aligned}\mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\ &\vdots \\ \mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i].\end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If  $\mathfrak{g}$  is nilpotent, then all subalgebras and all homomorphic images of  $\mathfrak{g}$  are also nilpotent.
2. If  $\mathfrak{g}/Z(\mathfrak{g})$  is nilpotent, then so is  $\mathfrak{g}$ .
3. If  $\mathfrak{g} \neq 0$  is nilpotent, then  $Z(\mathfrak{g}) \neq 0$ .

Claim: If  $\mathfrak{g}$  is nilpotent, then  $\text{ad}_x \in \text{End}(\mathfrak{g})$  is nilpotent for all  $x \in \mathfrak{g}$ .

Proof: This is because  $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \dots]]] = 0$ , and so for every  $x_i, y \in \mathfrak{g}$  we have  $[x_1, [x_2, \dots [x_n, y]]] = 0$ , and so  $\text{ad}_{x_1} \circ \text{ad}_{x_2} \circ \dots \text{ad}_{x_n} = 0$  which implies that  $\text{ad}_x^n = 0$  for all  $x \in \mathfrak{g}$ .

Theorem [Engel]: If  $\text{ad}_x$  is nilpotent for all  $x \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent.

Remark: This can be confusing if  $\mathfrak{g}$  is a linear algebra, we can consider elements  $x \in \mathfrak{g}$  and ask if it is the case  $x$  being nilpotent (as an endomorphism) iff  $\mathfrak{g}x$  is nilpotent? False, a counterexample is  $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ , where there exists an  $x$  which is *not* nilpotent while  $\text{ad}_x$  is nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  be a Lie subalgebra for some finite dimensional vector space  $V$ . If  $x$  is nilpotent as an endomorphism on  $V$  for all  $x \in \mathfrak{g}$ , then there exists a nonzero vector  $v \in V$  such that  $\mathfrak{g}v = 0$ , so  $x \in \mathfrak{g} \implies x(v) = 0$ .

Proof of lemma Use induction on  $\dim \mathfrak{g}$ , splitting into two separate base cases: - Case  $\dim \mathfrak{g} = 0$ , then  $\mathfrak{g} = \{0\}$ . - Case  $\dim \mathfrak{g} = 1$ , left as an exercise.

Inductive step: Let  $A$  be a maximal proper subalgebra and define  $\phi : A \rightarrow \mathfrak{gl}(\mathfrak{g}/A)$  where  $a \mapsto (x + A \mapsto [a, x] + A)$ . We need to check that  $\phi$  is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that  $\text{im } \phi \leq \mathfrak{gl}(\mathfrak{g}/A)$  is a Lie subalgebra, and  $\dim \text{im } \phi < \dim \mathfrak{g}$ . The claim is that  $\phi(a) \in \text{End}(\mathfrak{g}/A)$  is nilpotent for all  $a \in A$ . By the inductive hypothesis, there is a nonzero coset  $y + A \in \mathfrak{g}/A$  such that  $(\text{im } \phi) \cdot (y + A) = A$ . Since  $y \notin A$ , then  $\phi(a)(y + A) = A$  for all  $a \in A$ , and so  $[a, y] \in A$ .

We want to show that  $A$  is a subalgebra of codimension 1, and  $A \oplus F_y \leq \mathfrak{g}$  is a Lie subalgebra. This is because  $[a_1 + c_1y, a_2 + c_2y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1c_2[y, y]$ . The last term is zero, the middle two terms are in  $A$ , and because  $A$  is closed under the bracket, the first term is in  $A$  as well.

But then  $A \oplus F_y$  is a larger subalgebra than  $A$ , which was maximal, so it must be everything. So  $A \oplus F_y = \mathfrak{g}$ . So  $A \trianglelefteq \mathfrak{g}$  because  $[a_1, a_2 + cy]$  is in  $A$ ,  $A \oplus F_y = \mathfrak{g}$  respectively, and this equals  $[a_1, a_2] + c[a_1, y]$ , where both terms are in  $A$ .

Proof to be continued on Friday!

## 5 Lecture 5

Last time: we had a theorem that said that if  $\mathfrak{g} \in \mathfrak{gl}(V)$  and every  $x \in \mathfrak{g}$  is nilpotent, then there exists a nonzero  $v \in V$  such that  $\mathfrak{g}v = 0$ .

We proceeded by induction on the dimension of  $V$ , constructing  $\text{im } \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$ , and showed that  $\mathfrak{g} = A \oplus F_y$ . Now consider

$$W = \{v \in V \mid Av = 0\},$$

which is  $\mathfrak{g}$ -invariant, so  $\mathfrak{g}(W) \subseteq W$ , or for all  $a \in A, x \in \mathfrak{g}, v \in W$ , we have  $a \curvearrowright x(v) = 0$ . This is true because  $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$ . But  $V$  is killed by any element in  $A$ , and both of these terms are in  $A$ . In particular, the  $y$  appearing in  $Fy$  also satisfies  $y \in W$ . Consider  $y|_W \in \text{End}(W)$ , and we want to apply the inductive hypothesis to  $Fy|_W \subseteq \mathfrak{gl}(V)$ .

We need to check that  $y|_W \in \text{End}(W)$ , which is true exactly because  $y$  is nilpotent. So we can construct a nonzero  $v \in W \subset V$  such that  $y(v) = 0$ , and so  $\mathfrak{g}v = 0$ .

Claim:  $\phi(a) \in \text{End}(\mathfrak{g}/A)$  is nilpotent. Each  $a \in A \subset \mathfrak{g}$  is nilpotent by assumption. Define the maps for left multiplication by  $a$ ,  $m_\ell : x \mapsto ax$ , and the right multiplication  $m_r : x \mapsto xa$ . These are nilpotent, and since  $m_\ell, m_r$  commute, the difference  $m_\ell - m_r$  is nilpotent, and this is exactly  $\text{ad}_a$ . But then  $\phi(a)$  is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if  $\text{ad}_x \in \text{End}(\mathfrak{g})$  is nilpotent for every  $x \in \mathfrak{g}$ , then  $\mathfrak{g}$  is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to  $\text{ad}_g \subset \mathfrak{gl}(\mathfrak{g})$ . So we can produce the nonzero  $v \in \mathfrak{g}$  such that  $\text{ad}_g v = 0$ . Then  $[x, v] = 0$  for all  $x \in \mathfrak{g}$ , so either  $v \in Z(\mathfrak{g})$  or  $Z(\mathfrak{g}) \neq 0$ . In either case,  $\mathfrak{g}/Z(\mathfrak{g})$  has smaller dimension. Since  $\text{ad}_x$  is nilpotent, so is  $\text{ad}_x + Z(\mathfrak{g})$ , and so  $\mathfrak{g}/Z(\mathfrak{g})$  is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space.  $\square$

Let  $\mathfrak{N}(F)$  be the subalgebra of  $\mathfrak{gl}(F)$  consisting of strictly upper triangular matrices. We have a corollary: if  $\mathfrak{g} \subset \mathfrak{gl}(n, F)$  is a Lie subalgebra such every  $x \in \mathfrak{g}$  is nilpotent as an endomorphism of  $F$ , then the matrices of  $\mathfrak{g}$  with respect to some bases of in  $\mathfrak{N}(n, F)$ .

The proof is by induction on  $n$ , where the base case is easy. For the inductive step, we use the previous theorem to get a  $v_1$  such that  $x(v_1) = 0$  for all  $x \in \mathfrak{g}$ . Let  $\bar{V} = F^n/Fv_1 \cong F^{n-1}$ , and define  $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(\bar{V})$  where  $x \mapsto (\bar{y} \mapsto \overline{y(x)})$ .

Then  $\text{im } \phi \leq \mathfrak{gl}(n-1, F)$  as a subalgebra, and every  $\phi(x) \in \text{End}(F^{n-1})$  is nilpotent, since  $x$  was nilpotent on the larger space. But (see notes) then  $x$  can be written as a strictly upper-triangular matrix.

## 5.1 Chapter 2: Semisimple Lie Algebras

We now assume  $\text{char } F = 0$  and  $\bar{F} = F$ .

Theorem: If  $\mathfrak{g}$  is a solvable Lie subalgebra of  $\mathfrak{gl}(V)$  for some finite dimensional  $V$ , then  $V$  contains a common eigenvector for a  $x \in \mathfrak{g}$ , i.e. a  $\lambda : \mathfrak{g} \rightarrow F, x \mapsto \lambda(x)$  such that  $x(v) = \lambda(x)v$  for all  $x \in \mathfrak{g}$ .

Proof: We will use induction on the dimension of  $\mathfrak{g}$ . For the inductive step:

Claim 1: There is an ideal  $A \trianglelefteq \mathfrak{g}$  such that  $\mathfrak{g} = A \oplus Fy$  for some  $y \neq 0$ , so  $A$  is a subalgebra of a solvable Lie algebra  $\mathfrak{g}$  and thus solvable itself. By hypothesis, we can produce a  $w \in V \setminus \{0\}$ , and thus a functional  $\lambda : A \rightarrow F$  such that  $aw = \lambda(a)w$  for all  $a \in A$ . So we define

$$V_\lambda = \{v \in V \mid av = \lambda(a)v \forall a \in A\}$$

where  $w \in V_\lambda$ .

Claim 2:  $y(V_\lambda) \subseteq V_\lambda$ , or  $y|_{V_\lambda} \in \text{End}(V_\lambda)$ .

Thus  $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$  is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a  $v \in V_\lambda$  and some  $\mu \in F$  such that  $y(v) = \mu v$ . An arbitrary element  $x \in \mathfrak{g}$  can be written as  $x = a + cy$  for some  $a \in A, c \in F$  and it acts by  $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c\mu)v \in V_\lambda$ .

## 6 Lecture n+1

Todo

## 7 Lecture n+2

Definition (Jordan Decomposition)

Let  $X \in \text{End}(V)$  for  $V$  finite dimensional. Then,

- (a) There exists a unique  $X_s, X_n \in \text{End}(V)$  such that  $X = X_s + X_n$  where  $X_s$  is semisimple,  $X_n$  is nilpotent, and  $[X_s, X_n] = 0$ .
- (b) There exists a  $p(t), q(t) \in t\mathbb{F}[t]$  such that  $X_s = p(X), X_n = q(X)$ .

(Polynomials with no constant term.)

Proof of (a): Assume  $X_s = X_s + X_n = X'_s + X'_n$ , so both have bracket zero. Assuming that (b) holds, we have  $X_s = p(X)$ , and so

$$[X, X_s] = [X_s + X'_n, X'_s] = [X'_s, X'_s] + [X'_s, X'_n] = 0 \implies [p(X), X'_s] = 0 = [X_s, X'_s]$$

Using fact (c) from last time, then  $X_s, X'_s$  can be diagonalized simultaneously, and so  $X_s - X'_s$  is semisimple.

On the other hand, if  $X'_n, X_n$  are nilpotent, and since these commute,  $X_n - X'_n$  is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X'_s) + (X_n + X'_n)$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let  $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$  be the minimal polynomial of  $X$ , where each  $m_i \geq 1$  and the  $\lambda_i$  are distinct. Then the primary composition of  $V$  is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V)^{m_i} \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomial  $p \in \mathbb{F}[t]$  such that

$$\begin{aligned} p &= \lambda \pmod{(t - \lambda_i)^{m_i}} \quad \forall i, \\ p &= 0 \pmod{t}. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is  $p(x) \curvearrowright V_i$ ? This acts by scalar multiplication by  $\lambda_i$  for all  $i$ . (Check). Because of the restrictive conditions,  $p(x)$  has no constant term.

So  $p(X) = X_s$  is the semisimple part we want. Now just set  $q(t) = t - p(t)$ , then  $X_n := q(X) = X - X_s$  is nilpotent.

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have  $X = X_s + X_n$ , then  $\text{ad}_X \in \text{End}(\text{End}(V))$ . It can be shown that  $(\text{ad}_X)_s + (\text{ad}_X)_n = \text{ad}(X_s) + \text{ad}(X_n)$ .

$$p(x) \sim \begin{pmatrix} \boxed{\lambda_1 I_{v_1}} & & & \\ & \boxed{\lambda_2 I_{v_2}} & & \\ & & \ddots & \\ & & & \boxed{\lambda_r I_{v_r}} \end{pmatrix}$$

Figure 1: ???

Let  $e_{ij}$  be the elementary matrix with a 1 in the  $i, j$  position. You can write  $\text{ad}_X$  as a  $4 \times 4$  matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$= \begin{pmatrix} e_{11} & 0 & 0 & 1 & 0 \\ e_{12} & -1 & 0 & 0 & 1 \\ e_{21} & 0 & 0 & 0 & 0 \\ e_{22} & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \left( \begin{array}{c|ccc} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{array} \right)$$



$$= \begin{matrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rightarrow \text{JNF} \begin{pmatrix} 0 & & & \\ \hline & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots & 0 \end{pmatrix}$$

You can check that  $(\text{ad}_X)_S = 0$ ,  $\text{ad}(X_S) = 0$ , and  $(\text{ad}_X)_n$  is the Jordan form given above.

Lemma:

- (a)  $x \in \text{End}(V) \implies \text{ad}(x)_S = \text{ad}(x_S)$  and  $\text{ad}(x)_n = \text{ad}(x_n)$ .
- (b) If  $A$  is a finite dimensional  $\mathbb{F}$ -algebra, then  $\delta \in \text{Der}(A) \implies \delta_S, \delta_n \in \text{Der}(A)$  as well.

Proof of (a):

Check that  $\text{ad}(x) = \text{ad}(x_S) + \text{ad}(x_n)$ . Then for  $y \in \text{End}(V)$ , we have

$$\begin{aligned} (\text{ad}(x))(y) &= [x, y] \\ &= [x_S + x_n, y] \\ &= [x_S, y] + [x_n, y] \\ &= (\text{ad}(x_S))(y) + (\text{ad}(x_n))(y). \end{aligned}$$

Using theorem 3.3,  $x_n$  nilpotent  $\implies \text{ad}(x_n)$  is also nilpotent. So write  $x_S = \sum \lambda_i e_{ii}$  with the eigenvalues on the diagonal. Then  $\text{ad}x_S(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$  for all  $i, j$ . But then  $\text{ad}x_S$  is given by

$$\begin{aligned}
 & (\delta - (\lambda + \mu)I)^n ([x, y]) \\
 &= \sum_{i=0}^n \binom{n}{i} \left[ (\delta - \lambda I)^i (x), (\delta - \mu I)^{n-i} (y) \right]
 \end{aligned}$$

Figure 2: Image

a matrix with  $\lambda_i - \lambda_j$  in the  $i, j$  position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

Proof of (b):

Since  $\delta \in \text{Der}(A)$ , the primary decomposition with respect to  $\delta$  is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \left\{ a \in A \mid (\delta - \lambda I)^k a = 0 \text{ for some } k \gg 0 \right\}.$$

So  $\delta_s \sim A_\lambda$  by scalar multiplication (by  $\lambda$ ). Then for  $\lambda, \mu \in F$ , we have

So  $[A_x, A_y] \subseteq A_{\lambda+\mu}$  for all  $x, y \in A$ . But then

and so  $\delta_s \in \text{Der}(A)$ , and  $\delta_n = \delta - \delta_s \in \text{Der}(A)$  as well.

## 8 Lecture n+3

Todo

## 9 Lecture n+4

Review of bilinear forms: let  $V = \mathbb{F}^n$ .

Definition: A bilinear form  $\beta : V^2 \rightarrow \mathbb{F}$  can be represented by a matrix  $B$  with respect to a basis  $\{\mathbf{v}_i\}$  such that

$$\beta\left(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i\right) = (a_1 \ a_2 \ \cdots) B (b_1 \ b_2 \ \cdots)$$

- $\beta$  is *symmetric* iff  $\beta(a, b) = \beta(b, a)$ .
- $\beta$  is *symplectic* iff  $\beta(a, b) = -\beta(b, a)$ .
- $\beta$  is *isotropic* iff  $\beta(a, a) = 0$ .

$$S_S([x, y])$$

||

$$(\lambda + \mu)[x, y] = [\lambda x, y] + [x, \mu y]$$

||

$$[S_S(x), y] + [x, S_S(y)]$$

Figure 3: Image

For a subspace  $U \leq V$ , define

$$U^\perp := \{v \in V \mid \beta(u, v) = 0 \forall u \in U\}.$$

Note: in general, left/right orthogonality are distinguished, but these will be identical when  $\beta$  is symmetric/symplectic.

The form  $\beta$  is said to be *non-degenerate* iff  $V^\perp = 0$  iff  $\det B \neq 0$ .

Assume  $F$  is an algebraically closed field, so  $\bar{F} = F$ , and  $\text{char} F \neq 2$ , then

- If  $\beta$  is non-degenerate and symmetric, then  $B \sim I_n$
- If  $\beta$  is non-degenerate and symplectic, then  $B \sim [0, I_{n/2}; I_{n/2}, 0]$ .

Remark:

$\mathfrak{so}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, F) \mid \beta(x(u), v) = -\beta(u, x(v))\}$ , where  $B$  has the matrix  $[0, I; I, 0]$  if  $n$  is odd, or this matrix with a 1 in the top-left corner if  $n$  is even.

Similarly,  $\mathfrak{sp}(2m, \mathbb{F})$  can be described this way with the matrix  $[0, -I_m; -I_m, 0]$ .

Overview: The Killing form is defined as  $\kappa : \mathfrak{g}^2 \rightarrow \mathbb{F}$  where  $\kappa(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y)$ .

Then we have **Cartan's Criteria**:

- $\mathfrak{g}$  solvable  $\iff \kappa(x, y) = 0 \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$ .
- $\mathfrak{g}$  semisimple  $\iff \kappa$  is non-degenerate.

Note that if  $\mathfrak{g}$  is semisimple, then  $\mathfrak{g} = \bigoplus_i I_i$  with each  $I_i \trianglelefteq \mathfrak{g}$  and simple.

## 9.1 Cartan's Criteria

Some facts:

1.  $\kappa$  is symmetric
2. If  $\mathfrak{g}$  is finite dimensional, then  $\kappa$  is associative, i.e  $\kappa([x, y], z) = \kappa(x, [y, z])$ .

Exercise: Show that if  $I \trianglelefteq \mathfrak{g}$ , then  $I^\perp \leq \mathfrak{g}$  is an ideal.

Proof of (2): In section 4.3, it was shown that  $\text{tr}([a, b] \circ c) = \text{tr}(a \circ [b, c])$  for all  $a, b, c \in \text{End}(V)$  (provided  $V$  is finite dimensional).

So

$$\begin{aligned} \kappa([x, y], z) &= \text{tr}(\text{ad}_{[x, y]} \circ \text{ad}_z) \\ &= \text{tr}([\text{ad}_x, \text{ad}_y] \circ \text{ad}_z) \\ &= \text{tr}(\text{ad}_x \circ [\text{ad}_y, \text{ad}_z]) \\ &= \text{tr}(\text{ad}_x \circ \text{ad}_{[y, z]}) \\ &= \text{tr}(x, [y, z]).. \end{aligned}$$

Theorem:  $\mathfrak{g}$  is semisimple iff  $\kappa$  is nondegenerate.

Proof:  $\implies$  : We want to show that  $\mathfrak{g}^\perp = 0$ . Note that  $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}$ , and so for all  $x \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$  and for any  $y \in \mathfrak{g}^\perp$ , we have

$$\kappa(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y) = 0$$

by the const(?) of  $\mathfrak{g}^\perp$ . This implies  $\mathfrak{g}^\perp$  is solvable.

Using fact (2), we have  $\mathfrak{g}^\perp \trianglelefteq \mathfrak{g}$  and thus  $\mathfrak{g}^\perp \subseteq \text{rad}(\mathfrak{g})$ , which is 0 since because  $\mathfrak{g}$  is semisimple. So either  $\mathfrak{g}^\perp = 0$  or  $\kappa$  is nondegenerate.

Used the fact that the radical was a maximal solvable ideal.

$\impliedby$  : We want to show that for all  $I \trianglelefteq \mathfrak{g}$  where  $[I, I] = 0$ , we have  $I^\perp \subseteq \mathfrak{g}^\perp$ .

For  $x \in I, y \in \mathfrak{g}$ , we have

$$(\text{ad}_x \circ \text{ad}_y)^2 = \mathfrak{g} \xrightarrow{\text{ad}_y} \mathfrak{g} \xrightarrow{\text{ad}_x} I \xrightarrow{\text{ad}_y} I \xrightarrow{\text{ad}_x} 0$$

And thus  $\text{tr}(\text{ad}_x \circ \text{ad}_y) = 0$  and  $I \subseteq \mathfrak{g}^\perp$ .