

# Floer Talk

D. Zack Garza

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Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of  $\mathcal{F}$ .

What is  $\mathcal{F}$ ?

We started with the unadorned Floer map:

$$\begin{aligned} \mathcal{F} : \mathcal{C}^\infty(\mathbf{R} \times S^1; W) &\longrightarrow \mathcal{C}^\infty(\mathbf{R} \times S^1; TW) \\ u &\mapsto \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \text{grad}_u(H_t) \end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u). \end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned} \mathcal{P}^{1,p}(x, y) & \stackrel{?}{\longrightarrow} ? \\ (s, t) & \mapsto \exp_{w(s,t)} Y(s, t). \end{aligned}$$

where  $Y \in W^{1,p}(w^*TW)$  and  $w \in C_\infty^\infty(x, y)$ .

## 1 8.3: The Space of Perturbations of $H$

Goal: given a fixed Hamiltonian  $H$ , perturb (without modifying the periodic orbits) so that  $\mathcal{M}(x, y)$  are manifolds of the right dimension.

Start by construction  $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty$ , the space of perturbations of  $H$ . Idea: define a norm  $\|\cdot\|_\varepsilon$  and take the subspace of finite-norm elements.

$$\begin{aligned}\|h\|_\varepsilon &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} |d^k h(x,t)| \\ &= \sum_{k \geq 0} \varepsilon k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} |d^k (h \circ \Psi_i^{-1})(z)|.\end{aligned}$$

Where  $\{\varepsilon_k\} \subset \mathbb{R}$  is chosen such that  $\mathcal{C}_\varepsilon^\infty \hookrightarrow \mathcal{C}^\infty(W \times S^1)$  is dense for the  $C^\infty$  topology, and the  $\Psi_i : B_i \rightarrow \overline{B(0,1)}$  is a fixed finite sequence of diffeomorphisms where  $\bigcup_i B_i^\circ = W \times S^1$ .

Note that we'll only use density for the  $C^1$  topology in our case.

**Proposition 1.1.**

Such a sequence  $\{\varepsilon_k\}$  can be chosen.

*Proof.*

Show that  $C^\infty(W \times S^1)$  is separable, yielding a sequence  $(f_n) \subset C^\infty(W \times S^1)$  that is dense in the  $C^1$  topology, then

$$\varepsilon_n = \frac{1}{2^n \max_{k \leq n} \|f_k\| C^n(W \times S^1)}$$

where the diffeomorphisms  $\Psi_i$  are used to compute these norms. ■

Go on to show that for  $\|h\|_\varepsilon \ll 1$ , the  $\text{Per}(H_0 + h) = \text{Per}(H_0)$  and are nondegenerate.

### 1.1 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Embed  $TW \hookrightarrow \mathbb{R}^m \times \mathbb{R}^m$  to identify tangent vectors (such as  $Z_i$ , tangents to  $W$  along  $u$  or in a neighborhood  $B$  of  $u$ ) with actual vectors in  $\mathbb{R}^m$ .

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

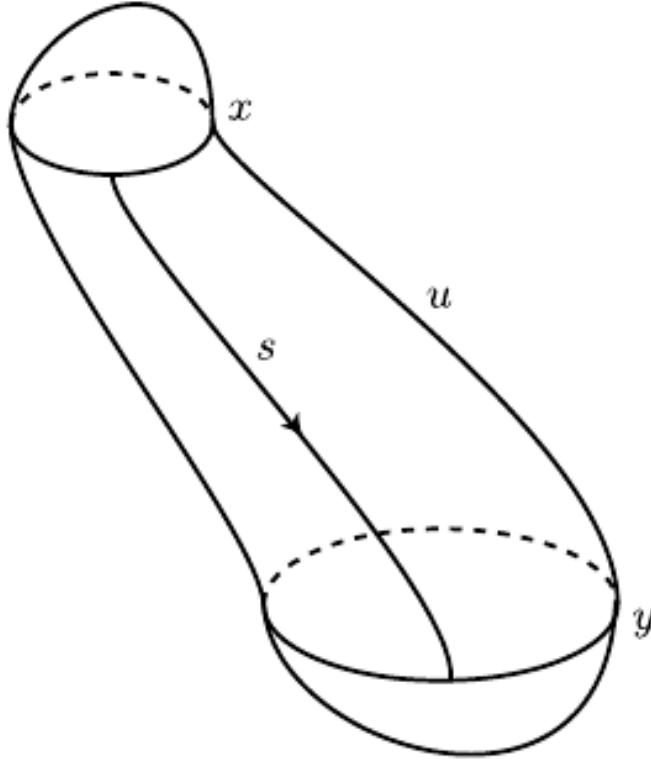
We can then identify  $\text{im } \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m)$  or  $L^p(\mathbb{R} \times S^1; W)$ , and we seek to compute its differential  $d\mathcal{F}$ .

We've just replaced the target spaces here.

Recall that  $x, y$  are contractible loops in  $W$  that are nondegenerate critical points of the action functional  $\mathcal{A}_H$  (i.e. solutions to the Floer equation), and  $C_{\searrow}(x, y)$  was the set of maps  $u : \mathbb{R} \times S^1 \rightarrow W$  satisfying some conditions.

Fix a solution  $u \in \mathcal{M}(x, y) \subset C_{\text{Loc}}^\infty(\mathbb{R} \times S^1; W)$ .

We lift each map to  $\tilde{u} : S^2 \rightarrow W$  in the following way: the loops  $x, y$  are contractible, so they bound discs. So we extend according to:



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Recall assumption 6.22: every smooth map  $w : S^2 \rightarrow W$  yields a symplectic trivialization of  $w^*TW$  (e.g. when  $\pi_2(W) = 0$ , so every map from  $S^2$  extends to  $B^3$ ).

Trivialize the symplectic fiber bundle  $\tilde{u}^*TW$  to obtain an orthonormal unitary frame  $\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$  depending smoothly on  $(s, t) \in S^2$ , where  $\lim_{s \rightarrow \pm\infty} Z_i$  exists for each  $i$ . We also require that  $\partial_s Z_i, \partial_s^2 Z_i, \partial_s \partial_t Z_i \xrightarrow{s \rightarrow \pm\infty} 0$  for each  $i$ .

This frame defines a chart about  $u$  of  $\mathcal{P}^{1,p}(x, y)$  given by

$$\begin{aligned} \iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\rightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \dots, y_{2n}) &\mapsto \exp_u \left( \sum y_i Z_i \right). \end{aligned}$$

Since  $(d\exp)_0 = \text{id}$ , we have  $(d\iota)_0(\mathbf{y}) = \sum_i y_i Z_i$ .

We'll now consider and compute the differential of

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y) &\xrightarrow{\mathcal{F}} L^p(\mathbb{R} \times S^1; TW) \rightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\ u &\mapsto \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} - X_t(u) \right). \end{aligned}$$

Take the vector  $Y(s, t) := (y_1(s, t), \dots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$ , where we view  $Y$  as a vector in  $\mathbb{R}^m$  tangent to  $W$ , given by  $Y = \sum y_i Z_i$ .

We write

$$\mathcal{F}(u + Y) = \frac{\partial(u + Y)}{\partial s} + J(u + Y) \frac{\partial(u + Y)}{\partial t} - J(u + Y) X_t(u + Y)$$

and extract the part that is linear in  $Y$ :

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y).$$

**Lemma 1.2 (Acting by Derivation).**

For any  $J \rightarrow \text{End}(\mathbb{R}^m)$  and  $Y, v : ? \rightarrow \mathbb{R}^m$  we have

$$(dj)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

There is a proof.

For every such smooth map  $u : \mathbb{R} \times S^1 \rightarrow W$ ,  $(d\mathcal{F})_u(Y) = O_1 + O_0$  where  $O_i$  are differential operators of order  $i$ , and in fact  $O_1$  can be chosen to be a Cauchy-Riemann operator. In this specific chart, we can in fact decompose  $(d\mathcal{F})_u(Y) = \bar{\partial}Y + SY$  where  $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^n)$  is linear of order 0, and in fact we have

**Proposition 1.3.**

If  $u$  solves Floer's equation, then  $(d\mathcal{F})_u = \bar{\partial} + S(s, t)$  where  $S$  is linear, tends to a symmetric operator as  $s \rightarrow \pm\infty$ , and  $\lim \partial_t S = 0$  uniformly in  $t$ .

There is a very long computational proof.

Denote the order 0 part of  $(d\mathcal{F})_u$  as  $Y \mapsto S \cdot Y$  so  $S : \mathbb{R} \times S^1 \rightarrow \text{End}(\mathbb{R}^m)$  and define  $S^\pm := \lim_{s \rightarrow \pm\infty} S(s, \cdot)$ .

**Proposition 1.4.**

The equation  $\partial_t Y = J_0 S^\pm Y$  linearizes Hamilton's equation  $\dot{z} = X_t(z)$  at  $x = \lim_{s \rightarrow \pm\infty} u$  for  $S^+$  and  $S^-$  respectively.

Proof: uses previous proposition.

Given a solution  $u$ , the product

$$\begin{aligned} u \cdot s : ? &\rightarrow ? \\ (\sigma, t) &\mapsto u(\sigma + s, t) \end{aligned}$$

is also a solution and  $\mathcal{F}(u \cdot s) = 0$  for all  $s$ .

**Punchline:**

Thus  $\frac{\partial u}{\partial s}$  is a solution of the linearized equation, since

$$0 = \frac{\partial}{\partial s} \mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left( \frac{\partial u}{\partial s} \right).$$

Along any nonconstant solution connecting  $x$  and  $y$ ,  $\dim \ker(d\mathcal{F})_u \geq 1$ .