Assignment 6: The Fourier Transform

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1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \lim_{\xi'\to 0} \frac{1}{2}\int_{\mathbb{R}^n} (f()) - f(x-\xi')) \exp(-2\pi i x\cdot\xi)\ dx$$

$$.$$

But as an immediate consequence, this yields

$$\begin{aligned}
\left|\hat{f}(\xi)\right| &= \left|\int_{\mathbb{R}^n} (f(x) - f(x-\xi')) \exp(-2\pi i x\cdot\xi)\ dx\right| \\
&\le \int_{\mathbb{R}^n} |f(x) - f(x-\xi')||\exp(-2\pi i x\cdot\xi)|\ dx \\
&\le \int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\ dx \\
&\to 0,
\end{aligned}$$

which follows from continuity in L^1 since $f(x-\xi') \to f(x)$ as $\xi' \to 0$.

It thus only remains to show that the hint holds, and that $\xi' \to 0$ as $\xi \to \infty$.

2 Problem 2

2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned}
\widehat{(f*g)}(\xi) &:= \int\int f(x-y)g(y)\exp(-2\pi x\cdot\xi)\ dy\ dx \\
&=_? \int\int f(x-y)g(y)\exp(-2\pi x\cdot\xi)\ dx\ dy \\
&= \int\int f(t)\exp(-2\pi i(x-y)\cdot\xi)g(y)\exp(-2\pi i y\cdot\xi)\ dx\ dy \\
&\quad (t = x-y,\ dt = \ dx) \\
&= \int\int f(t)\exp(-2\pi i t\cdot\xi)g(y)\exp(-2\pi i y\cdot\xi)\ dt\ dy \\
&= \int f(t)\exp(-2\pi i t\cdot\xi)\left(\int g(y)\exp(-2\pi i y\cdot\xi)\ dy\right)\ dt \\
&= \int f(t)\exp(-2\pi i t\cdot\xi)\hat{g}(\xi)\ dt \\
&= \hat{g}(\xi)\int f(t)\exp(-2\pi i t\cdot\xi)\ dt \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}$$

It thus remains to show that this swap is justified.

2.2 Part (b)

We'll use the following lemma: if $\hat{f} = \hat{g}$, then $f = g$ almost everywhere.

2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f}\hat{g} = \hat{g}\hat{f} = \widehat{g * f},$$

and so by the lemma, $f * g = g * f$.

Similarly, we have

$$\widehat{(f * g) * h} = \widehat{f * g}\ \hat{h} = \hat{f}\ \hat{g}\ \hat{h} = \hat{f}\ \widehat{g * h} = f * (g * h).$$

2.2.2 (ii)

Suppose that there exists some $I \in L^1$ such that $f * I = f$. Then $\widehat{f * I} = \hat{f}$ by the lemma, so $\hat{f}\ \hat{I} = \hat{f}$ by the above result.

But this says that $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$ almost everywhere, and thus $\hat{I}(\xi) = 1$ almost everywhere. Then $\lim_{|\xi| \to \infty} \hat{I}(\xi) \neq 0$, which by Problem 1 shows that I can not be in L^1, a contradiction.

3 Problem 3

3.1 (a)

3.1.1 (i)

Let $g(x) = f(x - y)$. We then have

$$\begin{aligned}
\hat{g}(\xi) &:= \int g(x) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x - y) \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int f(x - y) \exp(-2\pi i (x - y) \cdot \xi) \exp(-2\pi i y \cdot \xi)\ dx \\
&= \exp(-2\pi i y \cdot \xi) \int f(x - y) \exp(-2\pi i (x - y) \cdot \xi)\ dx \\
&\quad (t = x - y, dt = dx) \\
&= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi)\ dt \\
&= \exp(-2\pi i y \cdot \xi) \hat{f}(\xi).
\end{aligned}$$

3.1.2 (ii)

Let $h(x) = \exp(2\pi ix\ cdoty)f(x)$. We then have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \exp(2\pi ix \cdot y) f(x) \exp(-2\pi ix \cdot \xi)\ dx \\
&= \int \exp(2\pi ix \cdot y - 2\pi ix \cdot \xi) f(x)\ dx \\
&= \int f(\xi - y) \exp(-2\pi ix \cdot (\xi - y))\ dx \\
&= \hat{f}(\xi - y).
\end{aligned}$$

3.2 (b)

We'll use the fact that if $\langle \cdot,\ \ \cdot \rangle$ is an inner product on a vector space V and A is an invertible linear transformation, then for all $\mathbf{x}, \mathbf{y} \in V$ we have

$$\langle A\mathbf{x},\ \mathbf{y} \rangle = \left\langle \mathbf{x},\ A^T \mathbf{y} \right\rangle$$

where A^{-T} denotes the transpose of the inverse of A (or $(A^{-1})^*$ if V is complex).

We then have

$$\begin{aligned}
\frac{1}{|\det T|} \hat{f}(T^{-T}\xi) &= \frac{1}{|\det T|} \int f(x) \exp(-2\pi ix \cdot T^{-T}\xi)\ dx \\
& \qquad x \mapsto Tx,\ \ dx \mapsto |\det T|\ dx \\
&= \frac{1}{|\det T|} \int f(Tx) \exp(-2\pi i Tx \cdot T^{-T}\xi) |\det T|\ dx \\
&= \int f(Tx) \exp(-2\pi ix \cdot \xi)\ dx \\
& \qquad \text{since } Tx \cdot T^{-T}\xi = T^{-1}Tx \cdot \xi = x \cdot \xi \\
&= \widehat{(f \circ T)}(\xi).
\end{aligned}$$

4 Problem 4

4.1 (a)

4.1.1 (i)

Let $g(x) = xf(x)$. Then if an interchange of the derivative and the integral is justified, we have

$$
\begin{aligned}
\frac{\partial}{\partial \xi}\hat{f}(\xi) &:= \frac{\partial}{\partial \xi}\int f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&=_? \int f(x)\frac{\partial}{\partial \xi}\exp(-2\pi i x\cdot\xi)\ dx \\
&= \int f(x) 2\pi i x \exp(-2\pi i x\cdot\xi)\ dx \\
&= 2\pi i \int x f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&:= 2\pi i \hat{g}(\xi).
\end{aligned}
$$

It thus remains to show that this interchange is justified. TODO

4.1.2 (ii)

We have

$$
\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&= f(x)\exp(-2\pi i x\cdot\xi)\Big|_{x=-\infty}^{x=\infty} - \int f(x)(2\pi i\xi)\exp(-2\pi i x\cdot\xi)\ dx \\
&\qquad \text{(integrating by parts)} \\
&= -\int f(x)(-2\pi i\xi)\exp(-2\pi i x\cdot\xi)\ dx \\
&\qquad \text{(since } f(\infty) = f(-\infty) = 0) \\
&= 2\pi i\xi \int f(x)\exp(-2\pi i x\cdot\xi)\ dx \\
&:= 2\pi i\xi \hat{f}(\xi).
\end{aligned}
$$

4.2 (b)

Let $G(x) = \exp(-\pi x^2)$ and ∂_ξ be the operator that differentiates with respect to ξ.

Then

$$
\partial_\xi\left(\frac{\hat{G}(\xi)}{G(\xi)}\right) = \frac{G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi)}{G(\xi)^2},
$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$
G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = 0.
$$

Using the following facts,

- $\partial_\xi G(\xi) = -2\pi\xi G(\xi)$ by computing directly,
- $\partial_\xi \hat{G}(\xi) = -2\pi\xi\hat{G}(\xi)$, which follows from the following computation

$$
\begin{aligned}
\partial_\xi \hat{G}(\xi) &\coloneqq \partial_\xi \int G(x)\exp(-2\pi i x \cdot \xi)\ dx \\
&= \int G(x)\partial_\xi \exp(-2\pi i x \cdot \xi)\ dx \\
&= \int G(x)(-2\pi i x)\exp(-2\pi i x \cdot \xi)\ dx \\
&= i\int 2\pi x G(x)\exp(-2\pi i x \cdot \xi)\ dx \\
&= i\int \partial_x G(x)\exp(-2\pi i x \cdot \xi)\ dx \\
&\coloneqq i\ \widehat{\partial_x G(x)}(\xi) \\
&= i\ (2\pi i\xi\hat{G}(\xi)) \\
&= -2\pi\xi\hat{G}(\xi),
\end{aligned}
$$

we can thus write

$$
G(\xi)\partial_\xi\hat{G}(\xi) - \hat{G}(\xi)\partial_\xi G(\xi) = G(\xi)(-2\pi\xi\hat{G}(\xi)) - \hat{G}(\xi)(-2\pi\xi G(\xi)),
$$

which is patently zero.

It follows that $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$ for some constant c_0, from which it follows that $\hat{G}(\xi) = c_0 G(\xi)$.

Using the fact that $G(0) = 1$ by direct evaluation and $\hat{G}(0) = \int G(x)\ dx = 1$, we can conclude that $c_0 = 1$ and thus $\hat{G}(\xi) = G(\xi)$.

5 Problem 5

5.1 (a)

By a direct computation. we have

$$
\begin{aligned}
\hat{D}(\xi) &\coloneqq \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-2\pi i x\cdot\xi}\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi) + i\sin(-2\pi x\cdot\xi)\ dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi)\ dx \\
&\qquad \text{(since sin is odd and the domain is symmetric about 0)} \\
&= 2\int_{0}^{\frac{1}{2}} \cos(-2\pi x\cdot\xi)\ dx \\
&\qquad \text{(since cos is even and the domain is symmetric about 0)} \\
&= 2\left(\frac{1}{2\pi\xi}\sin(-2\pi x\cdot\xi)\Big|_{x=0}^{x=\frac{1}{2}}\right) \\
&= \frac{\sin(\pi\xi)}{\pi\xi}.
\end{aligned}
$$

5.2 (b)

5.2.1 (i)

Since $F(x) = D(x) * D(x)$, we have $\hat{F}(\xi) = (\hat{D}(\xi))^2$ by question 2a, and so $\hat{F}(\xi) = \left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2$.

5.2.2 (ii)

Letting $\mathcal{F}$ denote the Fourier transform operator, we have $\mathcal{F}^2(h)(\xi) = h(-\xi)$ for any $h \in L^1$. In particular, if f is an even function, then $f(\xi) = -f(\xi)$ and $\mathcal{F}^2(f) = f$.

In this case, letting F be the box function, F can be seen to be even from its definition. Since $f \coloneqq \mathcal{F}(F)$ by part (i), we have

$$
\hat{f} \coloneqq \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,
$$

which says that $\hat{f}(x) = F(x)$, the original box function.

5.3 (c)

By a direct computation of the integral in question, we have

$$
\begin{aligned}
I(x) &\coloneqq \int e^{-2\pi|\xi|}e^{2\pi i x\xi}\ d\xi \\
&= \int_{-\infty}^{0} e^{-2\pi(-\xi)}e^{-2\pi i x\xi}\ d\xi + \int_0^\infty e^{2\pi\xi}e^{2\pi i x\xi}\ d\xi \\
&= \int_0^\infty e^{-2\pi\xi}e^{-2\pi i x\xi}\ d\xi + \int_0^\infty e^{2\pi\xi}e^{2\pi i x\xi}\ d\xi \\
&\qquad \text{by the change of variables } \xi \mapsto -\xi,\ d\xi \mapsto -d\xi \text{ and swapping integration bounds} \\
&= \int_0^\infty e^{-2\pi\xi}e^{-2\pi i x\xi} + e^{2\pi\xi}e^{2\pi i x\xi}\ d\xi \\
&= \frac{1}{2\pi}\int_0^\infty e^{-u}e^{-ixu} + e^{-u}e^{ixu}\ du \\
&= \frac{1}{2\pi}\int_0^\infty e^{-u(1+ix)} + e^{-u(1-ix)}\ du \\
&= \frac{1}{2\pi}\left(\frac{-e^{-u(1+ix)}}{1+ix}\Big|_{u=0}^{u=\infty} + \frac{-e^{-u(1-ix)}}{1+ix}\Big|_{u=0}^{u=\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right) \\
&= \frac{1}{2\pi}\frac{2}{1+x^2} \\
&= \frac{1}{\pi}\frac{1}{1+x^2},
\end{aligned}
$$

so $P(x) = I(x)$.

Then, by the Fourier inversion formula, we have

$$
\begin{aligned}
I(x) = P(x) &= \int \hat{P}(\xi)e^{-2\pi i x\xi}\ dx \\
\implies \int e^{-2\pi|\xi|}e^{2\pi i x\xi} &= \int \hat{P}(\xi)e^{-2\pi i x\xi}\ dx \\
\implies \int e^{-2\pi|\xi|}e^{2\pi i x\xi} - \hat{P}(\xi)e^{-2\pi i x\xi}\ dx &= 0 \\
\implies \int \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right)e^{-2\pi i x\xi}\ dx &= 0 \\
\implies \left(e^{-2\pi|\xi|} - \hat{P}(\xi)\right)e^{-2\pi i x\xi} &=_{a.e.} 0 \\
\implies e^{-2\pi|\xi|} &=_{a.e} \hat{P}(\xi),
\end{aligned}
$$

where equality is almost everywhere and follows from the fact that if $\int f = 0$ then $f = 0$ almost everywhere.

6 Problem 6

We first note that if $G_t(x) \coloneqq t^{-n}e^{-\pi|x|^2/t^2}$, then $\hat{G}_t(\xi) = e^{-\pi t^2|\xi|^2}$.

Moreover, if an interchange of integrals is justified, we have have

$$\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \right| \, dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \, dx \\
&\quad \text{since the integrand and thus integral is positive.} \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \, dx \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \left(\int_{\mathbb{R}^n} G_t(x) \, dx \right) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, (1) \, dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt,
\end{aligned}$$

which we claim is finite, so $f \in L^1$.

To see that the norm is finite, we note that

$$t \in [0, 1] \implies e^{-\pi t^2} < 1$$

and if we take $\varepsilon < \frac{1}{2}$, we have $2\varepsilon - 1 < 0$ and thus

$$t \in [1, \infty) \implies t^{2\varepsilon - 1} \leq 1.$$

Thus

$$\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon - 1} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_1^\infty e^{-\pi t^2} \, dt \\
&\leq \int_0^1 t^{2\varepsilon - 1} \, dt + \int_0^\infty e^{-\pi t^2} \, dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

> Justifying the interchange: we note that the integrand $G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1}$ is non-negative, so Tonelli will apply if the integrand is measurable. But $G_t(x)$ is a continuous function on $\mathbb{R}^n$ and the remaining terms are continuous on $\mathbb{R}$, so they are all measurable on $\mathbb{R}^n$ and $\mathbb{R}$ respectively But then taking cylinders on them all still yields measurable functions, and the product of measurable functions is measurable. Since we also showed that the integrand was absolutely integrable, Tonelli applies.

If an interchange of integrals is justified, we can compute

$$
\begin{aligned}
\hat{f}(\xi) &\coloneqq \int_{\mathbb{R}^n} \left(\int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon - 1} \; dt \right) e^{-2\pi i x \cdot \xi} \; dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x\cdot\xi} \; dt \; dx \\
&=_? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x\cdot\xi} \; dx \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left(\int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x\cdot\xi} \; dx \right) \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \hat{G}_t(\xi) \; dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} e^{-\pi t^2 |\xi|^2} \; dt \\
&= \int_0^\infty e^{-\pi t^2 (1+|\xi|^2)} t^{2\varepsilon-1} \; dt \\
&= \int_0^\infty e^{-\pi (t\sqrt{1+|\xi|^2})^2} t^{2\varepsilon-1} \; dt \\
&\qquad s = t\sqrt{1+|\xi|^2}, \; ds = \sqrt{1+|\xi|^2} dt \\
&= \int_0^\infty e^{-\pi s^2} \left(\frac{s}{\sqrt{1-|\xi|^2}} \right)^{2\varepsilon-1} \frac{1}{\sqrt{1+|\xi|^2}} \; ds \\
&= (1+|\xi|^2)^{-\frac{2\varepsilon-1}{2}} (1+|\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon-1} \; ds \\
&= (1+|\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon-1} \; dt \\
&\coloneqq F(\xi) \|f\|_1 .
\end{aligned}
$$

To see that the interchange is justified, note that

$$
\int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x\cdot\xi} \right| \; dt \; dx = \int_{\mathbb{R}^n} \int_0^\infty \left| G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} \right| \; dt \; dx,
$$

where the integrand is now precisely what we showed was measurable when computed $\|f\|_1$ above. So Tonelli applies.

Thus $F(\xi)$ is the Fourier transform of the function $g(x) \coloneqq f(x)/\|f\|_1$. $\square$