# Analysis Qual Solutions

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December 27, 2019

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# 1 Fall 2019

## 1.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

## 1.2 a

Prove a stronger result:

$$a_n \to A \implies \frac{1}{N}\sum^{N} a_k \to A.$$

Idea: once $N$ is large enough, $a_k \approx A$, and all smaller terms will die off as $N \to \infty$.
See this MSE answer.

Suppose $S_k \to S$. Choose $\ell$ large enough such that

$$k \geq \ell \implies |S_k - S| < \varepsilon.$$

With $\ell$ fixed, choose $N$ large enough such that

$$k \leq \ell \implies \frac{|S_k - S|}{N} < \varepsilon.$$

Then

$$\begin{aligned}
\left| \left( \frac{1}{N} \sum_{k=1}^{N} S_k \right) - S \right| &= \frac{1}{N} \left| \sum_{k=1}^{N} (S_k - S) \right| \\
&\leq \frac{1}{N} \sum_{k=1}^{N} |S_k - S| \\
&= \sum_{k=1}^{\ell} \frac{|S_k - S|}{N} + \sum_{k=\ell+1}^{N} \frac{|S_k - S|}{N} \\
&\to 0.
\end{aligned}$$

## 1.3 b

Define

$$\Gamma_n := \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

Then $\Gamma_1 = \sum_k \frac{a_k}{k}$ and each $\Gamma_n$ is a tail of this series, so by assumption $\Gamma_n \to 0$.

Then

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^{n} a_k &= \frac{1}{n} (\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&\to 0.
\end{aligned}$$

This comes from consider the following summation:

$$\begin{array}{l|ccccccc}
\Gamma_1 : & a_1 & +\dfrac{a_2}{2} & +\dfrac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \dfrac{a_2}{2} & +\dfrac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \dfrac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n} \Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\dfrac{a_{n+1}}{n+1} & +\cdots
\end{array}$$

■

### 1.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

Use the fact that $\int_0^1 \cos(tx)\ dt = \sin(x)/x$, then

$$\begin{aligned}
\left|\frac{\partial^n}{\partial x}\sin(x)/x\right| &= \left|\frac{\partial^n}{\partial x}\int_0^1 \cos(tx)\ dt\right| \\
&=_{?} \left|\int_0^1 \frac{\partial^n}{\partial x}\cos(tx)\ dt\right| \\
&= \left|\int_0^1 -t^n\sin(tx)\ dt\right| \quad \text{for } n \text{ odd} \\
&\le \int_0^1 |t^n \sin(tx)|\ dt \\
&\le \int_0^1 t^n\ dt \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}$$

Where the DCT is justified by noting that $f(t) = \cos(tx)$ is dominated by $g(t) = 1$ on $[0, 1]$, which integrates to 1.

■

## 1.5 3

Borel-Cantelli.
Use the following observation: for a sequence of sets $X_n$,

$$\limsup_n X_n = \{x \ni x \in X_n \text{ for infinitely many } n\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} X_n$$
$$\liminf_n X_n = \{x \ni x \in X_n \text{ for all but finitely many } n\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} X_n.$$

And recall

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

### 1.5.1 a

The Borel $\sigma$-algebra is closed under countable unions/intersections/complements, and $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

### 1.5.2 b

We'll use the fact that tails of convergent sums go to zero, so $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\to\infty} 0$, and $B_M := \bigcap_{m=1}^{M} \bigcup_{n\geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \\
&\to 0,
\end{aligned}$$

and the result follows by continuity of measure.

### 1.5.3 c

To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$
\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{K} B_n^c\right) \\
&\le \sum_{m=1}^{\infty} \mu\left(\bigcap_{n=m}^{K} B_n^c\right) \qquad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty} \prod_{n=m}^{K} 1 - \mu(B_n) \\
&\le \sum_{m=1}^{\infty} \prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} e^{-\sum_{n=m}^{K} \mu(B_n^c)} \\
&\to 0
\end{aligned}
$$

since $\sum_{n=m}^{K} \mu(B_n^c) \to \infty$, and we can apply continuity of measure since $B_K^c \xrightarrow{K\to\infty} B^c$.

■

## 1.6 4

Bessel's Inequality, surjectivity of Riesz map, and Parseval's Identity.
Trick – remember to write out finite sum $S_N$, and consider $\|x - S_N\|$.

### 1.6.1 a

**Claim:**

$$
\begin{aligned}
0 \le \left\| x - \sum_{n=1}^{N} \langle x, u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n \rangle|^2 \\
&\implies \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \le \|x\|^2.
\end{aligned}
$$

*Proof:* Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$. Then

$$
\begin{aligned}
0 &\le \|x - S_N\|^2 \\
&= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&\xrightarrow{N\to\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

### 1.6.2 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x := \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=1}^{N} a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and $H$ complete $\implies x \in H$.
4. 

$$\langle x,\ u_n \rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n \rangle = a_n \quad \forall n \in \mathbb{N}$$

   since the $u_k$ are all orthogonal.

5. 

$$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

   by Pythagoras since the $u_k$ are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to $x$. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n \rangle = 0\ \forall n$) then the Fourier series *does* converge to $x$ and $\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 = \|x\|^2$ for all $x \in H$.

■

### 1.7 5

Continuity in $L^1$ (recall that DCT won't work! Notes 19.4, prove it for a dense subset first). Lebesgue differentiation in 1-dimensional case. See HW 5.6.

### 1.8 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \to 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \to 0$.

Write

$$\begin{aligned} \|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\le \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\to \|\tau_h g - g\|, \end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \to 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of $g$ to $K$ such that

$$|h| \leq 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of $g$, pick $\delta \leq 1$ small enough such that

$$x \in K, \; |h| \leq \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \leq \int_K \varepsilon = \varepsilon \cdot m(K) \to 0.$$

### 1.9 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)| \; dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y) \; dy \right| \; dx \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)| \; dy \; dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)| \; \mathbf{dx \; dy} \\
&= \int_{\mathbb{R}} |f(y)| \; dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)| \; dx &= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - f(x) \right| \; dx \\
&= \int_{\mathbb{R}} \left| \left( \frac{1}{2h} \int_{B(h,x)} f(y) \; dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x) \; dy \right| \; dx \\
&\leq_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)| \; \mathbf{dx \; dy} \\
&\leq \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1 \; dy \\
&\to 0 \quad \text{by (a)}.
\end{aligned}$$

$\blacksquare$

# 2 Spring 2019

## 2.1 1