Introduction to Differential Geometry

Lecture Notes for MAT367

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Chapter 1
Introduction

1.1 Some history

In the words of S.S. Chern, "*the fundamental objects of study in differential geometry are manifolds.*" [1] Roughly, an n-dimensional manifold is a mathematical object that "locally" looks like $\mathbb{R}^n$. The theory of manifolds has a long and complicated history. For centuries, manifolds have been studied as subsets of Euclidean space, given for example as level sets of equations. The term 'manifold' goes back to the 1851 thesis of Bernhard Riemann, *"Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"* ("foundations for a general theory of functions of a complex variable") and his 1854 habilitation address *"Über die Hypothesen, welche der Geometrie zugrunde liegen"* ("on the hypotheses underlying geometry").

[2] However, in neither reference Riemann makes an attempt to give a precise definition of the concept. This was done subsequently by many authors, including Rie-

[1] Page 332 of Chern, Chen, Lam: Lectures on Differential Geometry, World Scientific

[2] `http://en.wikipedia.org/wiki/Bernhard_Riemann`

mann himself. [3] Henri Poincaré in his 1895 work *analysis situs*, introduces the idea of a *manifold atlas*. [4]

The first rigorous axiomatic definition of manifolds was given by Veblen and Whitehead only in 1931.

We will see below that the concept of a manifold is really not all that complicated; and in hindsight it may come as a bit of a surprise that it took so long to evolve. Quite possibly, one reason is that for quite a while, the concept as such was mainly regarded as just a change of perspective (away from level sets in Euclidean spaces, towards the 'intrinsic' notion of manifolds). Albert Einstein's theory of General Relativity from 1916 gave a major boost to this new point of view; In his theory, space-time was regarded as a 4-dimensional 'curved' manifold with no distinguished coordinates (not even a distinguished separation into 'space' and 'time'); a local observer may want to introduce local *xyzt* coordinates to perform measurements, but all physically meaningful quantities must admit formulations that are coordinate-free. At the same time, it would seem unnatural to try to embed the 4-dimensional curved space-time continuum into some higher-dimensional flat space, in the absence of any physical significance for the additional dimensions. Some years later, *gauge theory* once again emphasized coordinate-free formulations, and provided physics motivations for more elaborate constructions such as fiber bundles and connections.

Since the late 1940s and early 1950s, differential geometry and the theory of manifolds has developed with breathtaking speed. It has become part of the basic education of any mathematician or theoretical physicist, and with applications in other areas of science such as engineering or economics. There are many sub-branches, for example complex geometry, Riemannian geometry, or symplectic geometry, which further subdivide into sub-sub-branches.

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[3] See e.g. the article by Scholz http://www.maths.ed.ac.uk/ aar/papers/scholz.pdf for the long list of names involved.

[4] `http://en.wikipedia.org/wiki/Henri_Poincare`

1.2 The concept of manifolds: Informal discussion

To repeat, an n-dimensional manifold is something that “locally” looks like $\mathbb{R}^n$. The prototype of a manifold is the surface of planet earth:

It is (roughly) a 2-dimensional sphere, but we use local charts to depict it as subsets of 2-dimensional Euclidean spaces.[5]

To describe the entire planet, one uses an atlas with a collection of such charts, such that every point on the planet is depicted in at least one such chart.

This idea will be used to give an ‘intrinsic’ definition of manifolds, as essentially a collection of charts glued together in a consistent way. One can then try to develop analysis on such manifolds – for example, develop a theory of integration and differentiation, consider ordinary and partial differential equations on manifolds, by working in charts; the task is then to understand the ‘change of coordinates’ as one leaves the domain of one chart and enters the domain of another.

[5] Note that such a chart will always give a somewhat ‘distorted’ picture of the planet; the distances on the sphere are never quite correct, and either the areas or the angles (or both) are wrong. For example, in the standard maps of the world, Canada always appears somewhat bigger than it really is. (Even more so Greenland, of course.)

1.3 Manifolds in Euclidean space

In multivariable calculus, you will have encountered manifolds as solution sets of equations. For example, the solution set of an equation of the form $f(x,y,z)=a$ in $\mathbb{R}^3$ defines a 'smooth' hypersurface $S\subseteq\mathbb{R}^3$ provided the gradient of f is non-vanishing at all points of S. We call such a value of f a *regular value*, and hence $S=f^{-1}(a)$ a *regular level set*. Similarly, the joint solution set C of two equations

$$f(x,y,z)=a,\quad g(x,y,z)=b$$

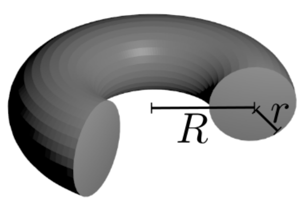
defines a smooth curve in $\mathbb{R}^3$, provided (a,b) is a regular value of (f,g) in the sense that the gradients of f and g are linearly independent at all points of C. A familiar example of a manifold is the 2-dimensional sphere S^2, conveniently described as a level surface inside $\mathbb{R}^3$:

$$S^2=\{(x,y,z)\in\mathbb{R}^3|\ x^2+y^2+z^2=1\}.$$

There are many ways of introducing local coordinates on the 2-sphere: For example, one can use spherical polar coordinates, cylindrical coordinates, stereographic projection, or orthogonal projections onto the coordinate planes. We will discuss some of these coordinates below. More generally, one has the n-dimensional sphere S^n inside $\mathbb{R}^{n+1}$,

$$S^n=\{(x^0,\ldots,x^n)\in\mathbb{R}^{n+1}|\ (x^0)^2+\ldots+(x^n)^2=1\}.$$

The 0-sphere S^0 consists of two points, the 1-sphere S^1 is the *unit circle*. Another example is the *2-torus*, T^2. It is often depicted as a surface of revolution: Given real numbers r,R with $0<r<R$, take a circle of radius r in the $x-z$ plane, with center at $(R,0)$, and rotate about the z-axis.



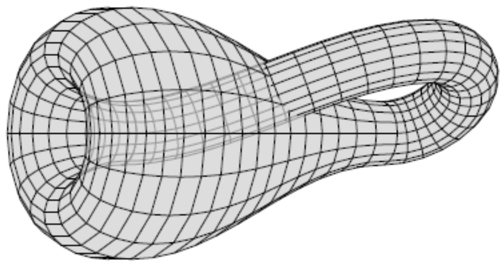
The resulting surface[6] is given by an equation,

$$T^2=\{(x,y,z)|\ \left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2\}. \tag{1.1}$$

Not all surfaces can be realized as 'embedded' in $\mathbb{R}^3$; for non-orientable surfaces one needs to allow for self-intersections. This type of realization is referred to as an

[6] http://calculus.seas.upenn.edu/?n=Main.CentroidsAndCentersOfMass.

immersion: We don't allow edges or corners, but we do allow that different parts of the surface pass through each other. An example is the *Klein bottle*[7]

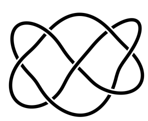


The Klein bottle is an example of a *non-orientable surface*: It has only one side. (In fact, the Klein bottle contains a Möbius band – see exercises.) It is not possible to represent it as a regular level set $f^{-1}(0)$ of a function f: For any such surface one has one side where f is positive, and another side where f is negative.

1.4 Intrinsic descriptions of manifolds

In this course, we will mostly avoid concrete embeddings of manifolds into any $\mathbb{R}^N$. Here, the term 'embedding' is used in an intuitive sense, for example as the realization as the level set of some equations. (Later, we will give a precise definition.) There are a number of reasons for why we prefer developing an 'intrinsic' theory of manifolds.

1. Embeddings of simple manifolds in Euclidean space can look quite complicated. The following one-dimensional manifold[8]

   is intrinsically, 'as a manifold', just a closed curve, that is, a circle. The problem of distinguishing embeddings of a circle into $\mathbb{R}^3$ is one of the goals of *knot theory*, a deep and difficult area of mathematics.
2. Such complications disappear if one goes to higher dimensions. For example, the above knot (and indeed any knot in $\mathbb{R}^3$) can be disentangled inside $\mathbb{R}^4$ (with $\mathbb{R}^3$ viewed as a subspace). Thus, in $\mathbb{R}^4$ they become *unknots*.
3. The intrinsic description is sometimes much simpler to deal with than the extrinsic one. For instance, the equation describing the torus $T^2 \subseteq \mathbb{R}^3$ is not especially

[7] http://www.map.mpim-bonn.mpg.de/2-manifolds

[8] http://math201s09.wdfiles.com/local--files/medina-knot/alternating.jpg

simple or beautiful. But once we introduce the following parametrization of the torus

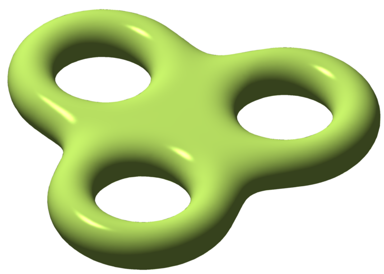
$$x = (R + r\cos\varphi)\cos\theta, \;\; y = (R + r\cos\varphi)\sin\theta, \;\; z = r\sin\varphi,$$

where θ, φ are determined up to multiples of 2π, we recognize that T^2 is simply a product:

$$T^2 = S^1 \times S^1. \tag{1.2}$$

That is, T^2 consists of ordered pairs of points on the circle, with the two factors corresponding to θ, φ. In contrast to (1.1), there is no distinction between 'small' circle (of radius r) and 'large circle' (of radius R). The new description suggests an embedding of T^2 into $\mathbb{R}^4$ which is 'nicer' then the one in $\mathbb{R}^3$. (But does it help?)

4. Often, there is no natural choice of an embedding of a given manifold inside $\mathbb{R}^N$, at least not in terms of concrete equations. For instance, while the triple torus [9]

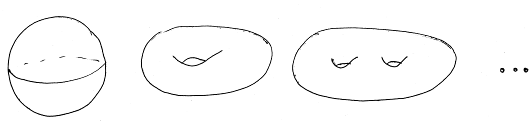


is easily pictured in 3-space $\mathbb{R}^3$, it is hard to describe it concretely as the level set of an equation.

5. While many examples of manifolds arise naturally as level sets of equations in some Euclidean space, there are also many examples for which the initial construction is different. For example, the set M whose elements are all affine lines in $\mathbb{R}^2$ (that is, straight lines that need not go through the origin) is naturally a 2-dimensional manifold. But some thought is required to realize it as a surface in $\mathbb{R}^3$.

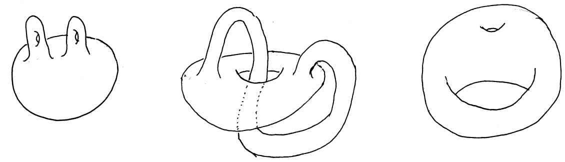
1.5 Surfaces

Let us briefly give a very informal discussion of *surfaces*. A surface is the same thing as a 2-dimensional manifold. We have already encountered some examples: The sphere, the torus, the double torus, triple torus, and so on:

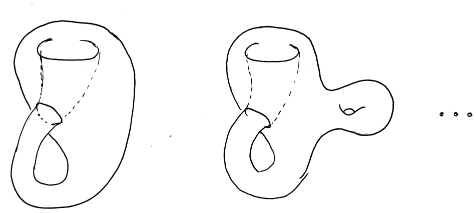


[9] http://commons.wikimedia.org/wiki/File:Triple_torus_illustration.png

All of these are 'orientable' surfaces, which essentially means that they have two sides which you might paint in two different colors. It turns out that these are all orientable surfaces, if we consider the surfaces 'intrinsically' and only consider surfaces that are compact in the sense that they don't go off to infinity and do not have a boundary (thus excluding a cylinder, for example). For instance, each of the following drawings depicts a double torus:

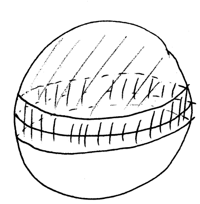


We also have one example of a non-orientable surface: The Klein bottle. More examples are obtained by attaching handles (just like we can think of the torus, double torus and so on as a sphere with handles attached).

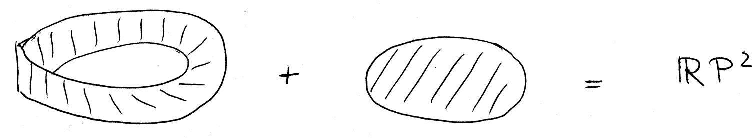


Are these *all* the non-orientable surfaces? In fact, the answer is *no*. We have missed what is in some sense the simplest non-orientable surface. Ironically, it is the surface which is hardest to visualize in 3-space. This surface is called the *projective plane* or *projective space*, and is denoted $\mathbb{R}\mathrm{P}^2$. One can define $\mathbb{R}\mathrm{P}^2$ as the set of all lines (i.e., 1-dimensional subspaces) in $\mathbb{R}^3$. It should be clear that this is a 2-dimensional manifold, since it takes 2 parameters to specify such a line. We can label such lines by their points of intersection with S^2, hence we can also think of $\mathbb{R}\mathrm{P}^2$ as the set of antipodal (i.e., opposite) points on S^2. In other words, it is obtained from S^2 by identifying antipodal points. To get a better idea of how $\mathbb{R}\mathrm{P}^2$ looks like, let us subdivide the sphere S^2 into two parts:

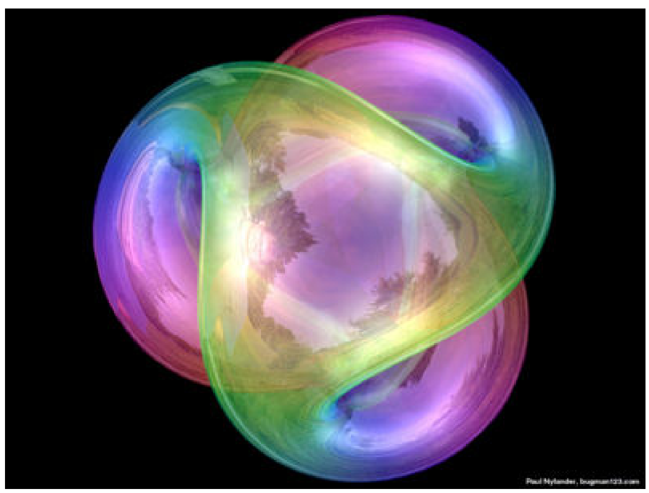
(i) points having distance $\le \varepsilon$ from the equator,
(ii) points having distance $\ge \varepsilon$ from the equator.



If we perform the antipodal identification for (i), we obtain a Möbius strip. If we perform antipodal identification for (ii), we obtain a 2-dimensional disk (think of it as the points of (ii) lying in the upper hemisphere). Hence, $\mathbb{R}P^2$ can also be regarded as gluing the boundary of a Möbius strip to the boundary of a disk:



Now, the question arises: *Is it possible to realize* $\mathbb{R}P^2$ *smoothly as a surface inside* $\mathbb{R}^3$*, possibly with self-intersections* (similar to the Klein bottle)? Simple attempts of joining the boundary circle of the Möbius strip with the boundary of the disk will always create sharp edges or corners – try it. Around 1900, David Hilbert posed this problem to his student Werner Boy, who discovered that the answer is *yes*. The following picture of *Boy's surface* was created by Paul Nylander. [10]



There are some nice videos illustrating the construction of the surface: See in particular

`https://www.youtube.com/watch?v=9gRx66xKXek`

and

`www.indiana.edu/~minimal/archive/NonOrientable/NonOrientable/Bryant-anim/web/`

While these pictures are very beautiful, it certainly makes the projective space appear more complicated than it actually is. If one is only interested in $\mathbb{R}P^2$ itself,

[10] http://mathforum.org/mathimages/index.php/Boy' s_Surface

rather than its realization as a surface in $\mathbb{R}^3$, it is much simpler to work with the definition (as a sphere with antipodal identification).

Going back to the classification of surfaces: It turns out that all closed, connected surfaces are obtained from either the 2-sphere S^2, the Klein bottle, or $\mathbb{R}\mathrm{P}^2$, by attaching handles.

Remark 1.1. Another operation for surfaces, generalizing the procedure of 'attaching handles', is the *connected sum*. Given two surfaces Σ_1 and Σ_2, remove small disks around given points $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$, to create two surfaces with boundary circles. Then glue-in a cylinder connecting the two boundary circles, without creating edges. The resulting surface is denoted

$$\Sigma_1 \sharp \Sigma_2.$$

For example, the connected sum $\Sigma \sharp T^2$ is Σ with a handle attached. You may want to think about the following questions: What is the connected sum of two $\mathbb{R}\mathrm{P}^2$'s? And what is the connected sum of $\mathbb{R}\mathrm{P}^2$ with a Klein bottle? Both must be in the list of 2-dimensional surfaces given above.

Chapter 2
Manifolds

It is one of the goals of these lectures to develop the theory of manifolds in intrinsic terms, although we may occasionally use immersions or embeddings into Euclidean space in order to illustrate concepts. In physics terminology, we will formulate the theory of manifolds in terms that are 'manifestly coordinate-free'.

2.1 Atlases and charts

As we mentioned above, the basic feature of manifolds is the existence of 'local coordinates'. The transition from one set of coordinates to another should be *smooth*. We recall the following notions from multivariable calculus.

Definition 2.1. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets. A map $F: U \to V$ is called *smooth* if it is infinitely differentiable. The set of smooth functions from U to V is denoted $C^\infty(U,V)$. The map F is called a *diffeomorphism* from U to V if it is invertible, and the inverse map $F^{-1}: V \to U$ is again smooth.

Example 2.1. The exponential map $\exp: \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(x) = e^x$ is smooth. It may be regarded as a map onto $\mathbb{R}_{>0} = \{y | y > 0\}$, and is a diffeomorphism

$$\exp: \mathbb{R} \to \mathbb{R}_{>0}$$

with inverse $\exp^{-1} = \log$ (the natural logarithm). Similarly,

$$\tan: \{x \in \mathbb{R} | -\pi/2 < x < \pi/2\} \to \mathbb{R}$$

is a diffeomorphism, with inverse arctan.

Definition 2.2. For a smooth map $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and any $x \in U$, one defines the *Jacobian matrix* $DF(x)$ to be the $n \times m$-matrix of partial derivatives

$$(DF(x))^i_j = \frac{\partial F^i}{\partial x^j}$$

Its determinant is called the *Jacobian matrix* of F at x.

The *inverse function theorem* states that F is a diffeomorphism if and only if it is invertible, and for all $x \in U$, the Jacobian matrix $DF(x)$ is invertible. (That is, one does not actually have to check smoothness of the inverse map!)

The following definition formalizes the concept of introducing local coordinates.

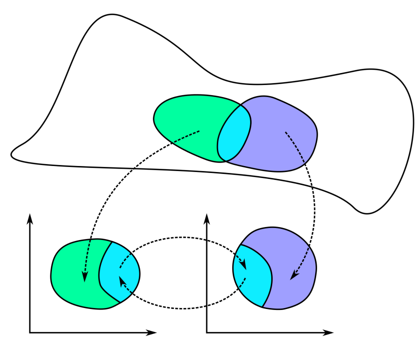
Definition 2.3 (Charts). Let M be a set.

1. An m-dimensional *(coordinate) chart* (U,φ) on M is a subset $U \subseteq M$ together with a map $\varphi: U \to \mathbb{R}^m$, such that $\varphi(U) \subseteq \mathbb{R}^m$ is open and φ is a bijection from U to $\varphi(U)$.
2. Two charts (U,φ) and (V,ψ) are called *compatible* if the subsets $\varphi(U\cap V)$ and $\psi(U\cap V)$ are open, and the *transition map*
$$\psi\circ\varphi^{-1}: \ \varphi(U\cap V) \to \psi(U\cap V)$$
is a diffeomorphism.

As a special case, charts with $U\cap V=\emptyset$ are always compatible.

Question: Is compatibility of charts an equivalence relation? (See the appendix to this section for a reminder of equivalence relations.)

Let (U,φ) be a coordinate chart. Given a point $p\in U$, and writing $\varphi(p) = (u^1,\ldots,u^m)$, we say that the u^i are the *coordinates of* p in the given chart. (Letting p vary, these become real-valued functions $p\mapsto u^i(p)$.) The transition maps $\psi\circ\varphi^{-1}$ amount to a *change of coordinates*. Here is a picture[1] of a 'coordinate change':



Definition 2.4 (Atlas). Let M be a set. An m-dimensional *atlas* on M is a collection of coordinate charts $\mathscr{A} = \{(U_\alpha,\varphi_\alpha)\}$ such that

1. The U_α cover all of M, i.e., $\bigcup_\alpha U_\alpha = M$.

[1] http://en.wikipedia.org/wiki/Differentiable_manifold

2. For all indices α, β, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible.

Example 2.2 (An atlas on the 2-sphere). Let $S^2 \subseteq \mathbb{R}^3$ be the unit sphere, consisting of all $(x,y,z) \in \mathbb{R}^3$ satisfying the equation $x^2+y^2+z^2=1$. We shall define an atlas with two charts (U_+, φ_+) and (U_-, φ_-). Let $n=(0,0,1)$ be the north pole, let $s=(0,0,-1)$ be the south pole, and put

$$U_+ = S^2 - \{s\}, \quad U_- = S^2 - \{n\}.$$

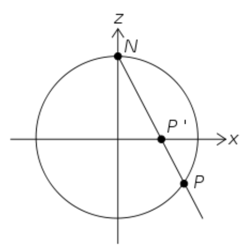
Regard $\mathbb{R}^2$ as the coordinate subspace of $\mathbb{R}^3$ on which $z=0$. Let

$$\varphi_+ : U_+ \to \mathbb{R}^2, \quad p \mapsto \varphi_+(p)$$

be *stereographic projection from the south pole*. That is, $\varphi_+(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and s. Similarly,

$$\varphi_- : U_- \to \mathbb{R}^2, \quad p \mapsto \varphi_-(p)$$

is *stereographic projection from the north pole*, where $\varphi_-(p)$ is the unique point of intersection of $\mathbb{R}^2$ with the affine line passing through p and n. A picture of φ_-, with $p' = \varphi_-(p)$ (the picture uses capital letters): [2]



A calculation shows that for $p=(x,y,z)$,

$$\varphi_+(x,y,z) = \Big(\frac{x}{1+z}, \frac{y}{1+z}\Big), \quad \varphi_-(x,y,z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big).$$

Exercise Verify these formulas.

Both $\varphi_\pm : U_\pm \to \mathbb{R}^2$ are bijections onto $\mathbb{R}^2$. Let us verify this in detail for the map φ_+. Given (u,v) we may solve the equation $(u,v) = \varphi_+(x,y,z)$, using the condition that $x^2+y^2+z^2=1$ and $z>-1$. One has

$$u^2+v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{1-z^2}{(1+z)^2} = \frac{(1-z)(1+z)}{(1+z)^2} = \frac{1-z}{1+z},$$

from which one obtains

$$z = \frac{1-(u^2+v^2)}{1+(u^2+v^2)},$$

[2] `http://en.wikipedia.org/wiki/User:Mgnbar/Hemispherical_projection`

and since $x = u(1+z),\ y = v(1+z)$ one obtains

$$\varphi_+^{-1}(u,v) = \Big(\frac{2u}{1+(u^2+v^2)}, \frac{2v}{1+(u^2+v^2)}, \frac{1-(u^2+v^2)}{1+(u^2+v^2)}\Big).$$

For the map φ_-, we obtain by a similar calculation

$$\varphi_-^{-1}(u,v) = \Big(\frac{2u}{1+(u^2+v^2)}, \frac{2v}{1+(u^2+v^2)}, \frac{(u^2+v^2)-1}{1+(u^2+v^2)}\Big).$$

(Actually, it is also clear from the geometry that $\varphi_+^{-1}, \varphi_-^{-1}$ only differ by the sign of the z-coordinate.) Note that $\varphi_+(U_+ \cap U_-) = \mathbb{R}^2 \backslash \{(0,0)\}$. The transition map on the overlap of the two charts is

$$(\varphi_- \circ \varphi_+^{-1})(u,v) = \Big(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\Big)$$

which is smooth on $\mathbb{R}^2 \backslash \{(0,0)\}$ as required. □

Here is another simple, but less familiar example where one has an atlas with two charts.

Example 2.3 (Affine lines in $\mathbb{R}^2$). A *line* in a vector space E is the same as a 1-dimensional subspace. By an *affine line*, we mean a subset $\mathsf{l} \subseteq E$, such that the set of differences $\{v - w |\ v, w \in \mathsf{l}\}$ is a 1-dimensional subspace. Put differently, l is obtained by adding a fixed vector v_0 to all elements of a 1-dimensional subspace. In plain terms, an affine line is simply a straight line that does not necessarily pass through the origin.

Let M be a set of affine lines in $\mathbb{R}^2$. Let $U \subseteq M$ be the subset of lines that are not vertical, and $V \subseteq M$ the lines that are not horizontal. Any $\mathsf{l} \in U$ is given by an equation of the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept. The map $\varphi :\ U \to \mathbb{R}^2$ taking l to (m,b) is a bijection. On the other hand, lines in V are given by equations of the form

$$x = ny + c,$$

and we also have the map $\psi :\ V \to \mathbb{R}^2$ taking such l to (n,c). The intersection $U \cap V$ are lines l that are neither vertical nor horizontal. Hence, $\varphi(U \cap V)$ is the set of all (m,b) such that $m \neq 0$, and similarly $\psi(U \cap V)$ is the set of all (n,c) such that $n \neq 0$. To describe the transition map $\psi \circ \varphi^{-1} :\ \varphi(U \cap V) \to \psi(U \cap V)$, we need to express (n,c) in terms of (m,b). Solving $y = mx + b$ for x we obtain

$$x = \frac{1}{m} y - \frac{b}{m}.$$

Thus, $n = \frac{1}{m}$ and $c = -\frac{b}{m}$, which shows that the transition map is

$$(\psi \circ \varphi^{-1})(m,b) = (\frac{1}{m},\ -\frac{b}{m}).$$

Note that this is smooth; similarly, $\varphi \circ \psi^{-1}$ is smooth; hence U,V define an 2-dimensional atlas on M.

Question: What is the resulting surface?

As a first approximation, we may take an m-dimensional manifold to be a set with an m-dimensional atlas. This is almost the right definition, but we will make a few adjustments. A first criticism is that we may not want any *particular* atlas as part of the definition: For example, the 2-sphere with the atlas given by stereographic projections onto the $x-y$-plane, and the 2-sphere with the atlas given by stereographic projections onto the $y-z$-plane, should be one and the same manifold S^2. To resolve this problem, we will use the following notion.

Definition 2.5. Suppose $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ is an m-dimensional atlas on M, and let (U,φ) be another chart. Then (U,φ) is said to be *compatible* with $\mathscr{A}$ if it is compatible with all charts $(U_\alpha, \varphi_\alpha)$ of $\mathscr{A}$.

Example 2.4. On the 2-sphere S^2, we had constructed the atlas

$$\mathscr{A} = \{(U_+, \varphi_+),\, (U_-, \varphi_-)\}$$

given by stereographic projection. Consider on the chart (V,ψ), with domain V the set of all $(x,y,z) \in S^2$ such that $y < 0$, with $\psi(x,y,z) = (x,z)$. To check that it is compatible (U_+, φ_+), note that $U_+ \cap V = V$, and

$$\varphi_+(U_+ \cap V) = \{(u,v) |\ v < 0\}, \quad \psi(U_+ \cap V) = \{(x,z) |\ x^2 + z^2 < 1\}$$

Expressing the coordinates $(u,v) = \psi(x,y,z)$ in terms of x,z and vice versa, we find

$$(x,z) = (\psi \circ \varphi_+^{-1})(u,v) = \Big(\frac{2u}{1+(u^2+v^2)},\ \frac{1-(u^2+v^2)}{1+(u^2+v^2)}\Big)$$
$$(u,v) = (\varphi_+ \circ \psi^{-1})(x,z) = \Big(\frac{x}{1+z},\ -\frac{\sqrt{1-(x^2+z^2)}}{1+z}\Big)$$

Both maps are smooth, proving that (V,ψ) is compatible with (U_+, φ_+).

Note that (U,φ) is compatible with the atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ if and only if the union $\mathscr{A} \cup \{(U,\varphi)\}$ is again an atlas on M. This suggests defining a bigger atlas, by using *all* charts that are compatible with the given atlas. In order for this to work, we need that the new charts are also compatible not only with the charts of $\mathscr{A}$, but also with each other.

Lemma 2.1. *Let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a given atlas on the set M. If two charts (U,φ), (V,ψ) are compatible with $\mathscr{A}$, then they are also compatible with each other.*

Proof. For every chart U_α, the sets $\varphi_\alpha(U \cap U_\alpha)$ and $\varphi_\alpha(V \cap U_\alpha)$ are open, hence their intersection is open. Since the map φ_α is injective, this intersection is

$$\varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(V \cap U_\alpha) = \varphi_\alpha(U \cap V \cap U_\alpha),$$

see the exercise below. Since $\varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \to \varphi(U \cap U_\alpha)$ is a diffeomorphism, it follows that

$$\varphi(U \cap V \cap U_\alpha) = (\varphi \circ \varphi_\alpha^{-1})\big(\varphi_\alpha(U \cap V \cap U_\alpha)\big)$$

is open. Taking the union over all α, we see that

$$\varphi(U \cap V) = \bigcup_\alpha \varphi(U \cap V \cap U_\alpha)$$

is open. A similar argument applies to $\psi(U \cap V)$. The transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth since for all α, its restriction to $\varphi(U \cap V \cap U_\alpha)$ is a composition of two smooth maps $\varphi_\alpha \circ \varphi^{-1} : \varphi(U \cap V \cap U_\alpha) \longrightarrow \varphi_\alpha(U \cap V \cap U_\alpha)$ and $\psi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap V \cap U_\alpha) \longrightarrow \psi(U \cap V \cap U_\alpha)$. Likewise, the composition $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. □

Exercise: Show that if $f : X \to Y$ is an injective map between sets, and $A, B \subseteq X$ are two subsets, then

$$f(A \cap B) = f(A) \cap f(B). \tag{2.1}$$

Show that (2.1) is not true, in general, if f is not injective.

Theorem 2.1. *Given an atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M, let $\widetilde{\mathscr{A}}$ be the collection of all charts (U, φ) that are compatible with $\mathscr{A}$. Then $\widetilde{\mathscr{A}}$ is itself an atlas on M, containing $\mathscr{A}$. In fact, $\widetilde{\mathscr{A}}$ is the largest atlas containing $\mathscr{A}$.*

Proof. Note first that $\widetilde{\mathscr{A}}$ contains $\mathscr{A}$, since the set of charts compatible with $\mathscr{A}$ contains the charts from the atlas $\mathscr{A}$ itself. In particular, the charts in $\widetilde{\mathscr{A}}$ cover M. By the Lemma, any two charts in $\widetilde{\mathscr{A}}$ are compatible. Hence $\widetilde{\mathscr{A}}$ is an atlas. If (U, φ) is a chart compatible with all charts in $\widetilde{\mathscr{A}}$, then in particular it is compatible with all charts in $\mathscr{A}$; hence $(U, \varphi) \in \widetilde{\mathscr{A}}$ by the definition of $\widetilde{\mathscr{A}}$. This shows that $\widetilde{\mathscr{A}}$ cannot be extended to a larger atlas.

Definition 2.6. An atlas $\mathscr{A}$ is called *maximal* if it is not properly contained in any larger atlas. Given an arbitrary atlas $\mathscr{A}$, one calls $\widetilde{\mathscr{A}}$ (as in Theorem 2.1) the *maximal atlas determined by $\mathscr{A}$*.

Remark 2.1. Although we will not need it, let us briefly discuss the notion of equivalence of atlases. (For background on equivalence relations, see the appendix to this chapter, Section 2.7.2.) Two atlases $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ and $\mathscr{A}' = \{(U'_\alpha, \varphi'_\alpha)\}$ are called *equivalent* if every chart of $\mathscr{A}$ is compatible with every chart in $\mathscr{A}'$. For example, the atlas on the 2-sphere given by the two stereographic projections to the

$x-y$-plane is equivalent to the atlas $\mathscr{A}'$ given by the two stereographic projections to the $y-z$-plane. Using Lemma 2.1, one sees that equivalence of atlases is indeed an equivalence relation. (In fact, two atlases are equivalent if and only if their union is an atlas.) Furthermore, two atlases are equivalent if and only if they are contained in the same maximal atlas. That is, *any maximal atlas determines an equivalence class of atlases, and vice versa*.

2.2 Definition of manifold

As a next approximation towards definition of manifolds, we can take an m-dimensional manifold to be a set M together with an m-dimensional *maximal* atlas. This is already quite close to what we want, but for technical reasons we would like to impose two further conditions.

First of all, we insist that M can be covered by *countably many* coordinate charts. In most of our examples, M is in fact covered by finitely many coordinate charts. This countability condition is used for various arguments involving a proof by induction.

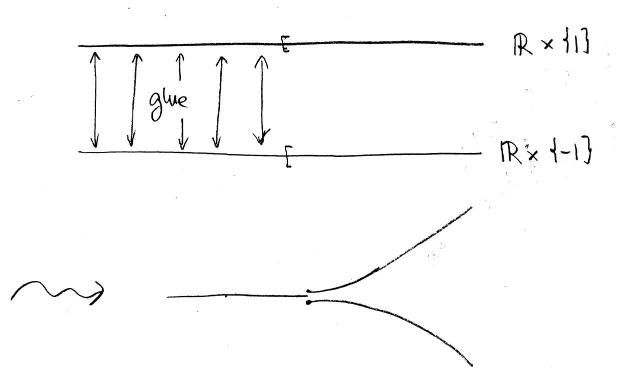
Example 2.5. A simple example that is not countable: Let $M = \mathbb{R}$, with $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ the 0-dimensional maximal (!) atlas, where each U_α consists of a single point, and $\varphi_\alpha: U_\alpha \to \{0\}$ is the unique map to $\mathbb{R}^0 = \{0\}$. Compatibility of charts is obvious. But M cannot be covered by countably many of these charts. Thus, we will not allow to consider $\mathbb{R}$ as a zero-dimensional manifold.

Secondly, we would like to avoid the following type of example.

Example 2.6. Let X be a disjoint union of two copies of the real line $\mathbb{R}$. We denote the two copies by $\mathbb{R}\times\{1\}$ and $\mathbb{R}\times\{-1\}$, just so that we can tell them apart. Define an equivalence relation on X generated by

$$(x,1) \sim (x',-1) \Leftrightarrow x' = x < 0,$$

and let $M = X/\sim$ the set of equivalence classes. That is, we 'glue' the two real lines along their negative real axes (taking care that no glue gets on the origins of the axes). Here is a (not very successful) attempt to sketch the resulting space:



As a set, M is a disjoint union of $\mathbb{R}_{<0}$ with two copies of $\mathbb{R}_{\geq 0}$. Let $\pi: X \to M$ be the quotient map, and let

$$U = \pi(\mathbb{R} \times \{1\}), \quad V = \pi(\mathbb{R} \times \{-1\})$$

the images of the two real lines. The projection map $X \to \mathbb{R}$, $(x, \pm 1) \mapsto x$ is constant on equivalence classes, hence it descends to a map $f: M \to \mathbb{R}$; let $\varphi: U \to \mathbb{R}$ be the restriction of f to U and $\psi: V \to \mathbb{R}$ the restriction to V. Then

$$\varphi(U) = \psi(V) = \mathbb{R}, \quad \varphi(U \cap V) = \psi(U \cap V) = \mathbb{R}_{<0},$$

and the transition map is the identity map. Hence, $\mathscr{A} = \{(U, \varphi), (V, \psi)\}$ is an atlas for M. A strange feature of M with this atlas is that the points

$$p = \varphi^{-1}(\{0\}), \quad q = \psi^{-1}(\{0\})$$

are 'arbitrarily close', in the sense that if $I, J \subseteq \mathbb{R}$ are any open subsets containing 0, the intersection of their pre-images is non-empty:

$$\varphi^{-1}(I) \cap \psi^{-1}(J) \neq \emptyset.$$

Yet, $p \neq q$! There is no really satisfactory way of drawing M (our picture above is inadequate), since it cannot be realized as a submanifold of any $\mathbb{R}^n$.

Since such a behaviour is inconsistent with the idea of a manifold that 'locally looks like $\mathbb{R}^n$', we shall insist that for two distinct points $p, q \in M$, there are always disjoint coordinate charts containing the two points. This is called the *Hausdorff condition*, after *Felix Hausdorff* (1868-1942). [3]

[3] http://en.wikipedia.org/wiki/Felix_Hausdorff

Definition 2.7. An m-dimensional manifold is a set M, together with a maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ with the following properties:

1. **(Countability condition)** M is covered by countably many coordinate charts in $\mathscr{A}$. That is, there are indices $\alpha_1, \alpha_2, \ldots$ with
$$M = \bigcup_i U_{\alpha_i}.$$
2. **(Hausdorff condition)** For any two distinct points $p, q \in M$ there are coordinate charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in $\mathscr{A}$ such that $p \in U_\alpha$, $q \in U_\beta$, with
$$U_\alpha \cap U_\beta = \emptyset.$$

The charts $(U, \varphi) \in \mathscr{A}$ are called (coordinate) *charts on the manifold* M.

Before giving examples, let us note the following useful fact concerning the Hausdorff condition:

Lemma 2.2. *Let M be a set with a maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$, and suppose $p, q \in M$ are distinct points contained in a single coordinate chart $(U, \varphi) \in \mathscr{A}$. Then we can find indices α, β such that $p \in U_\alpha$, $q \in U_\beta$, with $U_\alpha \cap U_\beta = \emptyset$.*

Proof. We begin with the following remark, which we leave as an exercise. Suppose (U, φ) is a chart, with image $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^m$. Let $V \subseteq U$ be a subset such that $\widetilde{V} = \varphi(V) \subseteq \widetilde{U}$ is open, and let $\psi = \varphi|_V$ be the restriction of φ. Then (V, ψ) is again a chart, and is compatible with (U, φ). If (U, φ) is a chart from an atlast $\mathscr{A}$, then (V, ψ) is compatible with that atlas.

Now let (U, φ) be as in the Lemma. Since
$$\widetilde{p} = \varphi(p), \quad \widetilde{q} = \varphi(q)$$
are distinct points in $\widetilde{U} \subseteq \mathbb{R}^m$, we can choose disjoint open subsets $\widetilde{U}_\alpha$ and $\widetilde{U}_\beta \subseteq \widetilde{U}$ containing $\widetilde{p} = \varphi(p)$ and $\widetilde{q} = \varphi(q)$, respectively.[4] Let U_α, $U_\beta \subseteq U$ be their preimages, and take $\varphi_\alpha = \varphi|_{U_\alpha}$, $\varphi_\beta = \varphi|_{U_\beta}$. Then $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are charts in $\mathscr{A}$,

[4] For instance, take these subsets to be the elements in $\widetilde{U}$ of distance less than $||\widetilde{p} - \widetilde{q}||/2$ from $\widetilde{p}$ and $\widetilde{q}$, respectively.

with disjoint chart domains, and by construction we have that $p \in U_\alpha$ and $q \in U_\beta$. $\square$

Example 2.7. Consider the 2-sphere S^2 with the atlas given by the two coordinate charts (U_+, φ_+) and (U_-, φ_-). This atlas extends uniquely to a maximal atlas. The countability condition is satisfied, since S^2 is already covered by two charts. The Hausdorff condition is satisfied as well: Given distinct points $p, q \in S^2$, if both are contained in U_+ or both in U_-, we can apply the Lemma. The only case to be considered is thus if one point (say p) is the north pole and the other (say q) the south pole. But here we can construct U_α, U_β by replacing U_+ and U_- with the open upper hemisphere and open lower hemisphere, respectively. Alternatively, we can use the chart given by stereographic projection to the $x-z$ plane, noting that this is also in the maximal atlas.

Remark 2.2. As we explained above, the Hausdorff condition rules out some strange examples that don't quite fit our idea of a space that is locally like $\mathbb{R}^n$. Nevertheless, the so-called *non-Hausdorff manifolds* (with non-Hausdorff more properly called *not necessarily Hausdorff*) do arise in some important applications. Much of the theory can be developed without the Hausdorff property, but there are some complications: For instance, the initial value problems for vector fields need not have unique solutions, in general.

Remark 2.3 (Charts taking values in 'abstract' vector spaces). In the definition of an m-dimensional manifold M , rather than letting the charts $(U_\alpha, \varphi_\alpha)$ take values in $\mathbb{R}^m$ we could just as well let them take values in m-dimensional real vector spaces E_α:

$$\varphi_\alpha : \; U_\alpha \to E_\alpha .$$

Transition functions are defined as before, except they now take an open subset of E_β to an open subset of E_α. The choice of basis identifies $E_\alpha = \mathbb{R}^m$, and takes us back to the original definition.

As far as the definition of manifolds is concerned, nothing has been gained by adding this level of abstraction. However, it often happens that the E_α's are given to us 'naturally'. For example, if M is a surface inside $\mathbb{R}^3$, one would typically use $x-y$ coordinates, or $x-z$ coordinates, or $y-z$ coordinates on appropriate chart domains. It can then be useful to regard the $x-y$ plane, $x-z$ plane, and $y-z$ plane as the target space of the coordinate maps, and for notational reasons it may be convenient to not associate them with a single $\mathbb{R}^2$.

2.3 Examples of Manifolds

We will now discuss some basic examples of manifolds. In each case, the manifold structure is given by a finite atlas; hence the countability property is immediate. We will not spend too much time on verifying the Hausdorff property; while it may be done 'by hand', we will later have some better ways of doing this.

2.3.1 Spheres

The construction of an atlas for the 2-sphere S^2, by stereographic projection, also works for the n-sphere

$$S^n = \{(x^0, \dots, x^n) |\ (x^0)^2 + \dots + (x^n)^2 = 1\}.$$

Let $U_\pm$ be the subsets obtained by removing $(\mp 1, 0, \dots, 0)$. Stereographic projection defines bijections $\varphi_\pm :\ U_\pm \to \mathbb{R}^n$, where $\varphi_\pm(x^0, x^1, \dots, x^n) = (u^1, \dots, u^n)$ with

$$u^i = \frac{x^i}{1 \pm x^0}.$$

For the transition function one finds (writing $u = (u^1, \dots, u^n)$)

$$(\varphi_- \circ \varphi_+^{-1})(u) = \frac{u}{||u||^2}.$$

We leave it as an exercise to check the details. An equivalent atlas, with $2n+2$ charts, is given by the subsets $U_0^+, \dots, U_n^+, U_0^-, \dots, U_n^-$ where

$$U_j^+ = \{x \in S^n |\ x^j > 0\}, \quad U_j^- = \{x \in S^n |\ x^j < 0\}$$

for $j = 0, \dots, n$, with $\varphi_j^\pm :\ U_j^\pm \to \mathbb{R}^n$ the projection to the j-th coordinate plane (in other words, omitting the j-th component x^j):

$$\varphi_j^\pm(x^0, \dots, x^n) = (x^0, \dots, x^{i-1}, x^{i+1}, \dots, x^n).$$

2.3.2 Products

Given manifolds M, M' of dimensions m, m', with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U'_\beta, \varphi'_\beta)\}$, the cartesian product $M \times M'$ is a manifold of dimension $m + m'$. An atlas is given by the product charts $U_\alpha \times U'_\beta$ with the product maps $\varphi_\alpha \times \varphi'_\beta :\ (x, x') \mapsto (\varphi_\alpha(x), \varphi'_\beta(x'))$. For example, the 2-torus $T^2 = S^1 \times S^1$ becomes a manifold in this way, and likewise for the n-torus

$$T^n = S^1 \times \cdots \times S^1.$$

2.3.3 Real projective spaces

The n-dimensional projective space $\mathbb{RP}^n$, also denoted $\mathbb{RP}^n$, is the set of all lines $\mathsf{l} \subseteq \mathbb{R}^{n+1}$. It may also be regarded as a quotient space[5]

$$\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$$

for the equivalence relation

$$x \sim x' \Leftrightarrow \exists \lambda \in \mathbb{R}\backslash\{0\} :\ x' = \lambda x.$$

Indeed, any $x \in \mathbb{R}^{n+1}\backslash\{0\}$ determines a line, and two points x, x' determine the same line if and only if they agree up to a non-zero scalar multiple. The equivalence class of $x = (x^0, \ldots, x^n)$ under this relation is commonly denoted

$$[x] = (x^0 : \ldots : x^n).$$

$\mathbb{RP}^n$ has a *standard atlas*

$$\mathscr{A} = \{(U_0, \varphi_0), \ldots, (U_n, \varphi_n)\}$$

defined as follows. For $j = 0, \ldots, n$, let

$$U_j = \{(x^0 : \ldots : x^n) \in \mathbb{RP}^n |\ x^j \neq 0\}$$

be the set for which the j-th coordinate is non-zero, and put

$$\varphi_j :\ U_j \to \mathbb{R}^n,\ \ (x^0 : \ldots : x^n) \mapsto \big(\frac{x^0}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^n}{x^j}\big).$$

This is well-defined, since the quotients do not change when all x^i are multiplied by a fixed scalar. Put differently, given an element $[x] \in \mathbb{RP}^n$ for which the j-th component x^j is non-zero, we first rescale the representative x to make the j-th component equal to 1, and then use the remaining components as our coordinates. As a random example (with $n = 2$),

$$\varphi_1(7:3:2) = \varphi_1\big(\frac{7}{3} : 1 : \frac{2}{3}\big) = \big(\frac{7}{3}, \frac{2}{3}\big).$$

From this description, it is immediate that φ_j is a bijection from U_j onto $\mathbb{R}^n$, with inverse map

$$\varphi_j^{-1}(u^1, \ldots, u^n) = (u^1 : \ldots : u^j : 1 : u^{j+1} : \ldots : u^n).$$

Geometrically, viewing $\mathbb{RP}^n$ as the set of lines in $\mathbb{R}^{n+1}$, the subset $U_j \subseteq \mathbb{RP}^n$ consists of those lines l which intersect the affine hyperplane

[5] See the appendix to this chapter for some background on quotient spaces.

$$H_j = \{x \in \mathbb{R}^{n+1} | \, x^j = 1\},$$

and the map φ_j takes such a line l to its unique point of intersection $\mathsf{l} \cap H_j$, followed by the identification $H_j \cong \mathbb{R}^n$ (dropping the coordinate $x^j = 1$).

Let us verify that $\mathscr{A}$ is indeed an atlas. Clearly, the domains U_j cover $\mathbb{R}\mathrm{P}^n$, since any element $[x] \in \mathbb{R}\mathrm{P}^n$ has at least one of its components non-zero. For $i \neq j$, the intersection $U_i \cap U_j$ consists of elements x with the property that both components x^i, x^j are non-zero. There are two cases:

Case 1: $0 \leq i < j \leq n$. We have

$$\varphi_i \circ \varphi_j^{-1}(u^1, \ldots, u^n) = \Big(\frac{u^1}{u^{i+1}}, \ldots, \frac{u^i}{u^{i+1}}, \frac{u^{i+2}}{u^{i+1}}, \ldots, \frac{u^j}{u^{i+1}}, \frac{1}{u^{i+1}}, \frac{u^{j+1}}{u^{i+1}}, \ldots, \frac{u^n}{u^{i+1}}\Big),$$

defined on $\varphi_j(U_i \cap U_j) = \{u \in \mathbb{R}^n | \, u^{i+1} \neq 0\}$.

Case 2: If $0 \leq j < i \leq n$. We have [6]

$$\varphi_i \circ \varphi_j^{-1}(u^1, \ldots, u^n) = \Big(\frac{u^1}{u^i}, \ldots, \frac{u^j}{u^i}, \frac{1}{u^i}, \frac{u^{j+1}}{u^i}, \ldots, \frac{u^{i-1}}{u^i}, \frac{u^{i+1}}{u^i}, \ldots, \frac{u^n}{u^i}\Big),$$

defined on $\varphi_j(U_i \cap U_j) = \{u \in \mathbb{R}^n | \, u^i \neq 0\}$.

In both cases, we see that $\varphi_i \circ \varphi_j^{-1}$ is smooth. To complete the proof that this atlas (or the unique maximal atlas containing it) defines a manifold structure, it remains to check the Hausdorff property.

This can be done with the help of Lemma 2.2, but we postpone the proof since we will soon have a simple argument in terms of smooth functions. See Proposition 3.1 below.

Remark 2.4. In low dimensions, we have that $\mathbb{R}\mathrm{P}^0$ is just a point, while $\mathbb{R}\mathrm{P}^1$ is a circle.

Remark 2.5. Geometrically, U_i consists of all lines in $\mathbb{R}^{n+1}$ meeting the affine hyperplane H_i, hence its complement consists of all lines that are parallel to H_i, i.e., the lines in the coordinate subspace defined by $x^i = 0$. The set of such lines is $\mathbb{R}\mathrm{P}^{n-1}$. In other words, the complement of U_i in $\mathbb{R}\mathrm{P}^n$ is identified with $\mathbb{R}\mathrm{P}^{n-1}$.

Thus, as sets, $\mathbb{R}\mathrm{P}^n$ is a disjoint union

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}\mathrm{P}^{n-1},$$

where $\mathbb{R}^n$ is identified (by the coordinate map φ_i) with the open subset U_n, and $\mathbb{R}\mathrm{P}^{n-1}$ with its complement. Inductively, we obtain a decomposition

$$\mathbb{R}\mathrm{P}^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R} \sqcup \mathbb{R}^0,$$

where $\mathbb{R}^0 = \{0\}$. At this stage, it is simply a decomposition into subsets; later it will be recognized as a decomposition into submanifolds.

[6] If $i = n$, the last entry in the following expression is u^{n-1}/u^n.

Exercise: Find an identification of the space of rotations in $\mathbb{R}^3$ with the 3-dimensional projective space $\mathbb{R}\mathrm{P}^3$. **Hint:** Associate to any $v \in \mathbb{R}^3$ a rotation, as follows: If $v = 0$, take the trivial rotation, if $v \neq 0$, take the rotation by an angle $||v||$ around the oriented axis determined by v. Note that for $||v|| = \pi$, the vectors v and $-v$ determine the same rotation.

2.3.4 Complex projective spaces

Similar to the real projective space, one can define a *complex projective space* $\mathbb{C}\mathrm{P}^n$ as the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$. We identify $\mathbb{C}$ with $\mathbb{R}^2$, thus $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. Thus

$$\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\sim$$

where the equivalence relation is $z \sim z'$ if and only if there exists a complex λ with $z' = \lambda z$. (Note that the scalar λ is then unique, and is non-zero.) Alternatively, letting $S^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ be the 'unit sphere' consisting of complex vectors of length $||z|| = 1$, we have

$$\mathbb{C}\mathrm{P}^n = S^{2n+1}/\sim,$$

where $z' \sim z$ if and only if there exists a complex number λ with $z' = \lambda z$. (Note that the scalar λ is then unique, and has absolute value 1.) One defines charts (U_j, φ_j) similar to those for the real projective space:

$$U_j = \{(z^0 : \ldots : z^n) |\; z^j \neq 0\}, \;\; \varphi_j : \; U_j \to \mathbb{C}^n = \mathbb{R}^{2n},$$

$$\varphi_j(z^0 : \ldots : z^n) = \Big(\frac{z^0}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j}\Big).$$

The transition maps between charts are given by similar formulas as for $\mathbb{R}\mathrm{P}^n$ (just replace x with z); they are smooth maps between open subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$. Thus $\mathbb{C}\mathrm{P}^n$ is a smooth manifold of dimension $2n$. [7] Similar to $\mathbb{R}\mathrm{P}^n$ there is a decomposition

$$\mathbb{C}\mathrm{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \mathbb{C}^0.$$

2.3.5 Grassmannians

The set $\mathrm{Gr}(k,n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *Grassmannian of k-planes in* $\mathbb{R}^n$. (Named after *Hermann Grassmann* (1809-1877).) [8]

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[7] The transition maps are not only smooth but even holomorphic, making $\mathbb{C}\mathrm{P}^n$ into an example of a *complex manifold* (of complex dimension n).

[8] http://en.wikipedia.org/wiki/Hermann_Grassmann

As a special case, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$.

We will show that for general k, the Grassmannian is a manifold of dimension

$$\dim(\mathrm{Gr}(k,n)) = k(n-k).$$

An atlas for $\mathrm{Gr}(k,n)$ may be constructed as follows. The idea is to present linear subspaces of dimension k as graphs of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Here $\mathbb{R}^k$ is viewed as the coordinate subspace corresponding to a choice of k components from $x = (x^1,\ldots,x^n) \in \mathbb{R}^n$, and $\mathbb{R}^{n-k}$ the coordinate subspace for the remaining coordinates. To make it precise, we introduce some notation. For any subset $I \subseteq \{1,\ldots,n\}$ of the set of indices, let

$$I' = \{1,\ldots,n\}\backslash I$$

be its complement. Let $\mathbb{R}^I \subseteq \mathbb{R}^n$ be the coordinate subspace

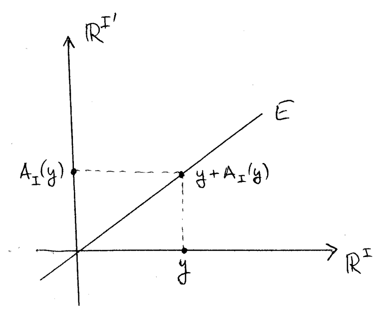
$$\mathbb{R}^I = \{x \in \mathbb{R}^n |\, x^i = 0 \text{ for all } i \in I'\}.$$

If I has cardinality $|I| = k$, then $\mathbb{R}^I \in \mathrm{Gr}(k,n)$. Note that $\mathbb{R}^{I'} = (\mathbb{R}^I)^\perp$. Let

$$U_I = \{E \in \mathrm{Gr}(k,n) |\, E \cap \mathbb{R}^{I'} = \{0\}\,\}.$$

Each $E \in U_I$ is described as the graph of a unique linear map $A_I : \mathbb{R}^I \to \mathbb{R}^{I'}$, that is,

$$E = \{y + A_I(y) |\, y \in \mathbb{R}^I\}.$$



This gives a bijection

$$\varphi_I : U_I \to L(\mathbb{R}^I, \mathbb{R}^{I'}),\ E \mapsto \varphi_I(E) = A_I,$$

where $L(F, F')$ denotes the space of linear maps from a vector space F to a vector space F'. Note $L(\mathbb{R}^I, \mathbb{R}^{I'}) \cong \mathbb{R}^{k(n-k)}$, because the bases of $\mathbb{R}^I$ and $\mathbb{R}^{I'}$ identify the space of linear maps with $(n-k) \times k$-matrices, which in turn is just $\mathbb{R}^{k(n-k)}$ by listing the matrix entries. In terms of A_I, the subspace $E \in U_I$ is the range of the injective linear map

$$\begin{pmatrix} 1 \\ A_I \end{pmatrix} :\ \mathbb{R}^I \to \mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n \tag{2.2}$$

where we write elements of $\mathbb{R}^n$ as column vectors.

To check that the charts are compatible, suppose $E \in U_I \cap U_J$, and let A_I and A_J be the linear maps describing E in the two charts. We have to show that the map

$$\varphi_J \circ \varphi_I^{-1} :\ L(\mathbb{R}^I, \mathbb{R}^{I'}) \to L(\mathbb{R}^J, \mathbb{R}^{J'}),\ \ A_I = \varphi_I(E) \mapsto A_J = \varphi_J(E)$$

is smooth. By assumption, E is described as the range of (2.2) and also as the range of a similar map for J. Here we are using the identifications $\mathbb{R}^I \oplus \mathbb{R}^{I'} \cong \mathbb{R}^n$ and $\mathbb{R}^J \oplus \mathbb{R}^{J'} \cong \mathbb{R}^n$. It is convenient to describe everything in terms of $\mathbb{R}^J \oplus \mathbb{R}^{J'}$. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ \mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

be the matrix corresponding to the identification $\mathbb{R}^I \oplus \mathbb{R}^{I'} \to \mathbb{R}^n$ followed by the inverse of $\mathbb{R}^J \oplus \mathbb{R}^{J'} \to \mathbb{R}^n$. For example, c is the inclusion $\mathbb{R}^I \to \mathbb{R}^n$ as the corresponding coordinate subspace, followed by projection to the coordinate subspace $\mathbb{R}^{J'}$. [9] We then get the condition that the injective linear maps

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ A_I \end{pmatrix} :\ \mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}, \quad \begin{pmatrix} 1 \\ A_J \end{pmatrix} :\ \mathbb{R}^J \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$$

have the same range. In other words, there is an isomorphism $S :\ \mathbb{R}^I \to \mathbb{R}^J$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ A_I \end{pmatrix} = \begin{pmatrix} 1 \\ A_J \end{pmatrix} S$$

as maps $\mathbb{R}^I \to \mathbb{R}^J \oplus \mathbb{R}^{J'}$. We obtain

$$\begin{pmatrix} a + bA_I \\ c + dA_I \end{pmatrix} = \begin{pmatrix} S \\ A_J S \end{pmatrix}$$

Using the first row of this equation to eliminate the second row of this equation, we obtain the formula

$$A_J = (c + dA_I)\,(a + bA_I)^{-1}.$$

[9] Put differently, the matrix is the permutation matrix 'renumbering' the coordinates of $\mathbb{R}^n$.

The dependence of the right hand side on the matrix entries of A_I is smooth, by Cramer's formula for the inverse matrix. It follows that the collection of all $\varphi_I : U_I \to \mathbb{R}^{k(n-k)}$ defines on $\mathrm{Gr}(k,n)$ the structure of a manifold of dimension $k(n-k)$. The number of charts of this atlas equals the number of subsets $I \subseteq \{1,\dots,n\}$ of cardinality k, that is, it is equal to $\binom{n}{k}$. (The Hausdorff property may be checked similar to that for $\mathbb{R}\mathrm{P}^n$. Alternatively, given distinct $E_1, E_2 \in \mathrm{Gr}(k,n)$, choose a subspace $F \in \mathrm{Gr}(k,n)$ such that $F^\perp$ has zero intersection with both E_1, E_2. (Such a subspace always exists.) One can then define a chart (U,φ), where U is the set of subspaces E transverse to $F^\perp$, and φ realizes any such map as the graph of a linear map $F \to F^\perp$. Thus $\varphi : U \to L(F,F^\perp)$. As above, we can check that this is compatible with all the charts (U_I,φ_I). Since both E_1, E_2 are in this chart U, we are done by Lemma 2.2.)

Remark 2.6. As already mentioned, $\mathrm{Gr}(1,n) = \mathbb{R}\mathrm{P}^{n-1}$. One can check that our system of charts in this case is the standard atlas for $\mathbb{R}\mathrm{P}^{n-1}$.

Exercise: This is a preparation for the following remark. Recall that a linear map $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection onto some subspace $E \subseteq \mathbb{R}^n$ if $\Pi(x) = x$ for $x \in E$ and $\Pi(x) = 0$ for $x \in E^\perp$. Show that a square matrix $P \in \mathrm{Mat}_{\mathbb{R}}(n)$ is the matrix of an orthogonal projection if and only if it has the properties

$$P^\top = P,\ PP = P,$$

where the superscript $\top$ indicates 'transpose'. What is the matrix of the orthogonal projection onto $E^\perp$?

Remark 2.7. For any k-dimensional subspace $E \subseteq \mathbb{R}^n$, let $\Pi^E : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by orthogonal projection onto E, and let $P_E \in \mathrm{Mat}_{\mathbb{R}}(n)$ be its matrix. By the exercise,

$$P_E^\top = P_E,\ \ P_E P_E = P_E,$$

Conversely, any square matrix P with the properties $P^\top = P$, $PP = P$ with $\mathrm{rank}(P) = k$ is the orthogonal projection onto a subspace $\{Px \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$. This identifies the Grassmannian $\mathrm{Gr}(k,n)$ with the set of orthogonal projections of rank k. In summary, we have an inclusion

$$\mathrm{Gr}(k,n) \hookrightarrow \mathrm{Mat}_{\mathbb{R}}(n) \cong \mathbb{R}^{n^2}, \quad E \mapsto P_E.$$

By construction, this inclusion take values in the subspace $\mathrm{Sym}_{\mathbb{R}}(n) \cong \mathbb{R}^{n(n+1)/2}$ of *symmetric* $n \times n$-matrices.

Remark 2.8. For all k, there is an identification $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$ (taking a k-dimensional subspace to the orthogonal subspace).

Remark 2.9. Similar to $\mathbb{R}\mathrm{P}^2 = S^2/\sim$, the quotient modulo antipodal identification, one can also consider

$$M = (S^2 \times S^2)/\sim$$

the quotient space by the equivalence relation

$$(x,x') \sim (-x,-x').$$

It turns out that this manifold M is the same as $\mathrm{Gr}(2,4)$, where 'the same' is meant in the sense that there is a bijection of sets identifying the atlases.

2.3.6 Complex Grassmannians

Similar to the case of projective spaces, one can also consider the *complex Grassmannian* $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of complex k-dimensional subspaces of $\mathbb{C}^n$. It is a manifold of dimension $2k(n-k)$, which can also be regarded as a complex manifold of complex dimension $k(n-k)$.

2.4 Oriented manifolds

The compatibility condition between charts (U,φ) and (V,ψ) on a set M is that the map $\psi\circ\varphi^{-1}:\ \varphi(U\cap V)\to\psi(U\cap V)$ is a diffeomorphism. In particular, the Jacobian matrix $D(\psi\circ\varphi^{-1})$ of the transition map is invertible, and hence has non-zero determinant. If the determinant is >0 everywhere, then we say $(U,\varphi),(V,\psi)$ are *oriented-compatible*. An *oriented atlas* on M is an atlas such that any two of its charts are oriented-compatible; a *maximal oriented atlas* is one that contains every chart that is oriented-compatible with all charts in this atlas. An *oriented manifold* is a set with a maximal oriented atlas, satisfying the Hausdorff and countability conditions as in definition 2.7. A manifold is called *orientable* if it admits an oriented atlas.

The notion of an orientation on a manifold will become crucial later, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Example 2.8. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+,φ_+) and (U_-,φ_-), given by stereographic projections. (Section 2.3.1.) Here $\varphi_-(U_+\cap U_-)=\varphi_+(U_+\cap U_-)=\mathbb{R}^n\backslash\{0\}$, with transition map $\varphi_-\circ\varphi_+^{-1}(u)=u/||u||^2$. The Jacobian matrix $D(\varphi_-\circ\varphi_+^{-1})(u)$ has entries

$$\Big(D(\varphi_-\circ\varphi_+^{-1})(u)\Big)_{ij}=\frac{\partial}{\partial u^j}\big(\frac{u^i}{||u||^2}\big)=\frac{1}{||u||^2}\delta_{ij}-\frac{2u_iu_j}{||u||^4}. \tag{2.3}$$

Its determinant is $-||u||^{-2n}$ (see exercise below).[10] Hence, the given atlas is *not* an oriented atlas. But this is easily remedied: Simply compose one of the charts, say U_-, with the map $(u_1, u_2, \ldots, u_n) \mapsto (-u_1, u_2, \ldots, u_n)$; then with the resulting new coordinate map $\widetilde{\varphi_-}$ the atlas $(U_+, \varphi_+), (U_-, \widetilde{\varphi_-})$ will be an oriented atlas.

Exercise: Calculate the determinant of the matrix with entries (2.3). Hint: Check that u is an eigenvector of the matrix, as is any vector orthogonal to u. Alternatively, use that the Jacobian determinant must be invariant under rotations in u-space.

Example 2.9. One can show that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd or $n = 0$. More generally, $\mathrm{Gr}(k,n)$ is orientable if and only if n is even or $n = 1$. The complex projective spaces $\mathbb{C}\mathrm{P}^n$ and complex Grassmannians $\mathrm{Gr}_{\mathbb{C}}(k,n)$ are all orientable. This follows because the transition maps for their standard charts, as maps between open subsets of $\mathbb{C}^m$, are actually complex-holomorphic, and this implies that as real maps, their Jacobian has positive determinant. See the following exercise.

Exercise: Let $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ be a complex $n \times n$-matrix, and $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ the same matrix regarded as a real-linear transformation of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Show that

$$\det_{\mathbb{R}}(A_{\mathbb{R}}) = |\det_{\mathbb{C}}(A)|^2.$$

You may want to start with the case $n = 1$, and next consider the case that A is upper triangular.

2.5 Open subsets

Let M be a set equipped with an m-dimensional maximal atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$.

Definition 2.8. A subset $U \subseteq M$ is *open* if and only if for all charts $(U_\alpha, \varphi_\alpha) \in \mathscr{A}$ the set $\varphi_\alpha(U \cap U_\alpha)$ is open.

To check that a subset U is open, it is not actually necessary to verify this condition for all charts. As the following proposition shows, it is enough to check for any collection of charts whose union contains U. In particular, we may take $\mathscr{A}$ in definition 2.8 to be any atlas, not necessarily a maximal atlas.

Proposition 2.1. *Given $U \subseteq M$, let $\mathscr{B} \subseteq \mathscr{A}$ be any collection of charts whose union contains U. Then U is open if and only if for all charts (U_β, φ_β) from $\mathscr{B}$, the sets $\varphi_\beta(U \cap U_\beta)$ are open.*

---

[10] Actually, to decide the sign of the determinant, one does not have to compute the determinant everywhere. If $n > 1$, since $\mathbb{R}^n \backslash \{0\}$, it suffices to compute the determinant at just one point, e.g. $u = (1, 0, \ldots, 0)$.

Proof. In what follows, we reserve the index β to indicate charts (U_β, φ_β) from $\mathscr{B}$. Suppose $\varphi_\beta(U \cap U_\beta)$ is open for all such β. Let $(U_\alpha, \varphi_\alpha)$ be a given chart in the maximal atlas $\mathscr{A}$. We have that

$$\begin{aligned}\varphi_\alpha(U \cap U_\alpha) &= \bigcup_\beta \varphi_\alpha(U \cap U_\alpha \cap U_\beta)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U \cap U_\alpha \cap U_\beta)\big)\\ &= \bigcup_\beta (\varphi_\alpha \circ \varphi_\beta^{-1})\big(\varphi_\beta(U_\alpha \cap U_\beta) \cap \varphi_\beta(U \cap U_\beta)\big).\end{aligned}$$

Since $\mathscr{B} \subseteq \mathscr{A}$, all $\varphi_\beta(U_\alpha \cap U_\beta)$ are open. Hence the intersection with $\varphi_\beta(U \cap U_\beta)$ is open, and so is the pre-image under the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$. Finally, we use that a union of open sets is again open. This proves the 'if' part; the 'only if' part is obvious. ☐

If $\mathscr{A}$ is an atlas on M, and $U \subseteq M$ is open, then U inherits an atlas by restriction:

$$\mathscr{A}_U = \{(U \cap U_\alpha, \varphi_\alpha|_{U \cap U_\alpha})\}.$$

Exercise: Verify that if $\mathscr{A}$ is a maximal atlas, then so is $\mathscr{A}_U$, and if this maximal atlas $\mathscr{A}$ satisfies the countability and Hausdorff properties, then so does $\mathscr{A}_U$.

This then proves:

Proposition 2.2. *An open subset of a manifold is again a manifold.*

The collection of open sets of M with respect to an atlas has properties similar to those for $\mathbb{R}^n$:

Proposition 2.3. *Let M be a set with an m-dimensional maximal atlas. The collection of all open subsets of M has the following properties:*

- *$\emptyset, M$ are open.*
- *The intersection $U \cap U'$ of any two open sets U, U' is again open.*
- *The union $\bigcup_i U_i$ of an arbitrary collection $U_i,\ i \in I$ of open sets is again open.*

Proof. All of these properties follow from similar properties of open subsets in $\mathbb{R}^m$. For instance, if U, U' are open, then

$$\varphi_\alpha((U \cap U') \cap U_\alpha) = \varphi_\alpha(U \cap U_\alpha) \cap \varphi_\alpha(U' \cap U_\alpha)$$

is an intersection of open subsets of $\mathbb{R}^m$, hence it is open and therefore $U \cap U'$ is open. ☐

These properties mean, by definition, that the collection of open subsets of M define a *topology* on M. This allows us to adopt various notions from topology:

1. A subset $A \subseteq M$ is called *closed* if its complement $M \backslash A$ is open.
2. M is called *connected* if the only subsets $A \subseteq M$ that are both closed and open are $A = \emptyset$ and $A = M$.
3. If U is an open subset and $p \in U$, then U is called an *open neighborhood of* p. More generally, if $A \subseteq U$ is a subset contained in M, then U is called an *open neighborhood of* A.

The Hausdorff condition in the definition of manifolds can now be restated as the condition that *any two distinct points p,q in M have disjoint open neighborhoods*. (It is not necessary to take them to be domains of coordinate charts.)

It is immediate from the definition that domains of coordinate charts are open. Indeed, this gives an alternative way of defining the open sets:

Exercise: Let M be a set with a maximal atlas. Show that a subset $U \subseteq M$ is open if and only if it is either empty, or is a union $U = \bigcup_{i \in I} U_i$ where the U_i are domains of coordinate charts.

2.6 Compact subsets

Another important concept from topology that we will need is the notion of *compactness*. Recall (e.g. Munkres, Chapter 1 § 4) that a subset $A \subseteq \mathbb{R}^m$ is *compact* if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of $\mathbb{R}^m$ whose union contains A, the set A is already covered by finitely many subsets from that collection. One then proves the important result (see Munkres, Theorems 4.2 and 4.9)

Theorem 2.2 (Heine-Borel). *A subset $A \subseteq \mathbb{R}^m$ is compact if and only if it is closed and bounded.*

While 'closed and bounded' is a simpler characterization of compactness to work with, it does not directly generalize to manifolds (or other topological spaces), while the original definition does:

Definition 2.9. Let M be a manifold.[11] A subset $A \subseteq M$ is *compact* if it has the following property: For every collection $\{U_\alpha\}$ of open subsets of M whose union contains A, the set A is already covered by finitely many subsets from that collection.

In short, $A \subseteq M$ is compact if every open cover admits a finite subcover.

Proposition 2.4. *If $A \subseteq M$ is contained in the domain of a coordinate chart (U, φ), then A is compact in M if and only if $\varphi(A)$ is compact in $\mathbb{R}^n$.*

Proof. Suppose $\varphi(A)$ is compact. Let $\{U_\alpha\}$ be an open cover of A. Taking intersections with U, it is still an open cover (since $A \subseteq U$). Hence

[11] More generally, the same definition is used for arbitrary topological spaces – e.g., sets with an atlas.

$$A \subseteq \bigcup_\alpha (U \cap U_\alpha),$$

and therefore

$$\varphi(A) \subseteq \bigcup_\alpha \varphi(U \cap U_\alpha).$$

Since $\varphi(A)$ is compact, there are indices $\alpha_1, \ldots, \alpha_N$ such that

$$\varphi(A) \subseteq \varphi(U \cap U_{\alpha_1}) \cup \ldots \cup \varphi(U \cap U_{\alpha_N}).$$

But then

$$A \subseteq (U \cap U_{\alpha_1}) \cup \ldots \cup (U \cap U_{\alpha_N}) \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}.$$

The converse is proved similarly. □

Exercise: Complete the proof, by working out the details for the other direction.

The proposition is useful, since we can check compactness of $\varphi(A)$ by using the Heine-Borel criterion. For more general subsets of M, we can often decide compactness by combining this result with the following:

Proposition 2.5. *If $A_1, \ldots, A_k \subseteq M$ is a finite collection of compact subsets, then their union $A = A_1 \cup \ldots \cup A_k$ is again compact.*

Proof. If $\{U_\alpha\}$ is an open cover of A, then in particular it is an open cover of each of the sets $A_1, \ldots, A_k$. For each A_i, we can choose a finite subcover. The collection of all U_α's such that appear in at least one of these subcovers, for $i = 1, \ldots k$ are then a finite subcover for A.

Example 2.10. Let $M = S^n$. The closed upper hemisphere $\{x \in S^n |\ x^0 \geq 0\}$ is compact, because is contained in the coordinate chart (U_+, φ_+) for stereographic projection, and its image under φ_+ is the closed and bounded subset $\{u \in \mathbb{R}^n |\ ||u|| \leq 1\}$. Likewise the closed lower hemisphere is compact, and hence S^n itself (as the union of upper and lower hemispheres) is compact.

Example 2.11. Let $\{(U_i, \varphi_i) |\ i = 0, \ldots, n\}$ be the standard atlas for $\mathbb{R}\mathrm{P}^n$. Let

$$A_i = \{(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n |\ ||x||^2 \leq (n+1)x_i^2\}.$$

Then $A_i \subseteq U_i$ (since necessarily $x^i \neq 0$ for elements of A_i). Furthermore, $\bigcup_{i=0}^n A_i = \mathbb{R}\mathrm{P}^n$: Indeed, given any $(x^0 : \cdots : x^n) \in \mathbb{R}\mathrm{P}^n$, let i be an index for which $|x^i|$ is maximal. Then $||x||^2 \leq (n+1)x_i^2$ (since the right hand side is obtained from the left hand side by replacing each $(x^j)^2$ with $(x^i)^2 \geq (x^j)^2$)), hence $(x^0 : \cdots : x^n) \in A_i$. Finally, one checks that $\varphi_i(A_i) \subseteq \mathbb{R}^n$ is a closed ball of radius $\sqrt{n+1}$, and in particular is compact.

In a similar way, one can prove compactness of $\mathbb{C}\mathrm{P}^n$, $\mathrm{Gr}(k,n)$, $\mathrm{Gr}_\mathbb{C}(k,n)$. However, soon we will have a simpler way of verifying compactness, by showing that they are closed and bounded subsets of $\mathbb{R}^N$ for a suitable N.

Proposition 2.6. *Let M be a set with a maximal atlas. If $A \subseteq M$ is compact, and $C \subseteq M$ is closed, then $A \cap C$ is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of $A \cap C$. Together with the open subset $M \backslash C$, these cover A. Since A is compact, there are finitely many indices $\alpha_1, \ldots, \alpha_N$ with

$$A \subseteq (M \backslash C) \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}.$$

Hence $A \cap C \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}$. □

The following fact uses the Hausdorff property (and holds in fact for any Hausdorff topological space).

Proposition 2.7. *If M is a manifold, then every compact subset $A \subseteq M$ is closed.*

Proof. Suppose $A \subseteq M$ is compact. Let $p \in M \backslash A$ be given. For any $q \in A$, there are disjoint open neighborhoods V_q of q and U_q of p. The collection of all V_q for $q \in A$ are an open cover of A, hence there exists a finite subcover $V_{q_1}, \ldots, V_{q_k}$. The intersection $U = U_{q_1} \cap \ldots \cap U_{q_k}$ is an open subset of M with $p \in M$ and not meeting $V_{q_1} \cup \ldots \cup V_{q_k}$, hence not meeting A. We have thus shown that every $p \in M \backslash A$ has an open neighborhood $U \subseteq M \backslash A$. The union over all such open neighborhoods for all $p \in M \backslash A$ is all of $M \backslash A$, which hence is open. It follows that A is closed. □

Exercise: Let M be the non-Hausdorff manifold from Example 2.6. Find a compact subset $A \subseteq M$ that is not closed.

2.7 Appendix

2.7.1 Countability

A set X is *countable* if it is either finite (possibly empty), or there exists a bijective map $f: \mathbb{N} \to X$. We list some basic facts about countable sets:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable, $\mathbb{R}$ is not countable.
- If X_1, X_2 are countable, then the cartesian product $X_1 \times X_2$ is countable.
- If X is countable, then any subset of X is countable.
- If X is countable, and $f: X \to Y$ is surjective, then Y is countable.
- If $(X_i)_{i \in I}$ are countable sets, indexed by a countable set I, then the (disjoint) union $\sqcup_{i \in I} X_i$ is countable.

2.7.2 Equivalence relations

We will make extensive use of *equivalence relations*; hence it may be good to review this briefly. A *relation* from a set X to a set Y is simply a subset

$$R \subseteq Y \times X.$$

We write $x \sim_R y$ if and only if $(y,x) \in R$. When R is understood, we write $x \sim y$. If $Y = X$ we speak of a *relation on* X.

Example 2.12. Any map $f:\ X \to Y$ defines a relation, given by its *graph* $\mathrm{Gr}_f = \{(f(x),x) | x \in X\}$. In this sense relations are generalizations of maps; for example, they are often used to describe 'multi-valued' maps.

Remark 2.10. Given another relation $S \subseteq Z \times Y$, one defines a composition $S \circ R \subseteq Z \times X$, where

$$S \circ R = \{(z,x) |\ \exists y \in Y:\ (z,y) \in S,\ (y,x) \in R\}.$$

Our conventions are set up in such a way that if $f:\ X \to Y$ and $g:\ Y \to Z$ are two maps, then $\mathrm{Gr}_{g \circ f} = \mathrm{Gr}_g \circ \mathrm{Gr}_f$.

Example 2.13. On the set $X = \mathbb{R}$ we have relations $\geq, >, <, \leq, =$. But there is also the relation defined by the condition $x \sim x' \Leftrightarrow x' - x \in \mathbb{Z}$, and many others.

A relation $\sim$ on a set X is called an *equivalence relation* if it has the following properties,

1. Reflexivity: $x \sim x$ for all $x \in X$,
2. Symmetry: $x \sim y \Rightarrow y \sim x$,
3. Transitivity: $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Given an equivalence relation, we define the *equivalence class* of $x \in X$ to be the subset

$$[x] = \{y \in X |\ x \sim y\}.$$

Note that X is a disjoint union of its equivalence classes. We denote by $X/\sim$ the set of equivalence classes. That is, all the elements of a given equivalence class are lumped together and represent a single element of $X/\sim$. One defines the *quotient map*

$$q:\ X \to X/\sim, \quad x \mapsto [x].$$

By definition, the quotient map is surjective.

Remark 2.11. There are two other useful ways to think of equivalence relations:

- An equivalence relation R on X amounts to a decomposition $X = \sqcup_{i \in I} X_i$ as a disjoint union of subsets. Given R, one takes X_i to be the equivalence classes; given the decomposition, one defines $R = \{(y,x) \in X \times X | \exists i \in I:\ x,y \in X_i\}$.
- An equivalence relation amounts to a surjective map $q:\ X \to Y$. Indeed, given R one takes $Y := X/\sim$ with q the quotient map; conversely, given q one defines $R = \{(y,x) \in X \times X |\ q(x) = q(y)\}$.

Remark 2.12. Often, we will not write out the entire equivalence relation. For example, if we say "*the equivalence relation on* S^2 *given by* $x \sim -x$", then it is understood that we also have $x \sim x$, since reflexivity holds for any equivalence relation. Similarly, when we say "*the equivalence relation on* $\mathbb{R}$ *generated by* $x \sim x+1$", it is understood that we also have $x \sim x+2$ (by transitivity: $x \sim x+1 \sim x+2$) as well as $x \sim x-1$ (by symmetry), hence $x \sim x+k$ for all $k \in \mathbb{Z}$. (Any relation $R_0 \subseteq X \times X$ extends to a unique smallest equivalence relation R; one says that R is the equivalence relation generated by R_0.)

Example 2.14. Consider the equivalence relation on S^2 given by

$$(x,y,z) \sim (-x,-y,-z).$$

The equivalence classes are pairs of antipodal points; they are in 1-1 correspondence with lines in $\mathbb{R}^3$. That is, the quotient space $S^2/\sim$ is naturally identified with $\mathbb{R}\mathrm{P}^2$.

Example 2.15. The quotient space $\mathbb{R}/\sim$ for the equivalence relation $x \sim x+1$ on $\mathbb{R}$ is naturally identified with S^1. If we think of S^1 as a subset of $\mathbb{R}$, the quotient map is given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

Example 2.16. Similarly, the quotient space for the equivalence relation on $\mathbb{R}^2$ given by $(x,y) \sim (x+k, y+l)$ for $k,l \in \mathbb{Z}$ is the 2-torus T^2.

Example 2.17. Let E be a k-dimensional real vector space. Given two ordered bases $(e_1,\ldots,e_k)$ and $(e'_1,\ldots,e'_k)$, there is a unique invertible linear transformation $A : E \to E$ with $A(e_i) = e'_i$. The two ordered bases are called *equivalent* if $\det(A) > 0$. One checks that equivalence of bases is an equivalence relation. There are exactly two equivalence classes; the choice of an equivalence class is called an *orientation* on E. For example, $\mathbb{R}^n$ has a standard orientation defined by the standard basis $(e_1,\ldots,e_n)$. The opposite orientation is defined, for example, by $(-e_1,e_2,\ldots,e_n)$. A permutation of the standard basis vectors defines the standard orientation if and only if the permutation is even.

Chapter 3
Smooth maps

3.1 Smooth functions on manifolds

A real-valued function on an open subset $U \subseteq \mathbb{R}^n$ is called smooth if it is infinitely differentiable. The notion of smooth functions on open subsets of Euclidean spaces carries over to manifolds: A function is smooth if its expression in local coordinates is smooth.

Definition 3.1. A function $f: M \to \mathbb{R}$ on a manifold M is called *smooth* if for all charts (U, φ) the function
$$f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}$$
is smooth. The set of smooth functions on M is denoted $C^\infty(M)$.

Remarks 3.1. 1. Since transition maps are diffeomorphisms, it suffices to check the condition for the charts from any given atlas $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$, which need not be the maximal atlas. Indeed, if the condition on f holds for all charts from the atlas $\mathscr{A}$, and if (U, φ) is another chart compatible with $\mathscr{A}$, then the functions

$$f \circ \varphi^{-1}\big|_{\varphi(U \cap U_\alpha)} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1}): \varphi(U \cap U_\alpha) \to \mathbb{R}$$

are smooth, and since the open sets $\varphi(U \cap U_\alpha)$ cover $\varphi(U)$ this implies smoothness of $f \circ \varphi^{-1}$.

2. Given an open subset $U \subseteq M$, we say that a function f is smooth on U if its restriction $f|_U$ is smooth. (Here we are using that U itself is a manifold.) Given $p \in M$, we say that f is smooth at p if it is smooth on some open neighborhood of p.

Example 3.1. The 'height function'

$$f: S^2 \to \mathbb{R}, \quad (x, y, z) \mapsto z$$

is smooth. In fact, we see that for any smooth function $h \in C^\infty(\mathbb{R}^3)$ (for example the coordinate functions), the restriction $f = h|_{S^2}$ is again smooth. This may be checked using the atlas with 6 charts given by projection to coordinate planes: E.g., in the chart $U = \{(x,y,z) |\ z > 0\}$ with $\varphi(x,y,z) = (x,y)$, we have

$$(f \circ \varphi^{-1})(x,y) = h\big(x,y,\sqrt{1-(x^2+y^2)}\big)$$

which is smooth on $\varphi(U) = \{(x,y) |\ x^2+y^2 < 1\}$. (The argument for the other charts in this atlas is similar.)

Of course, if h is not smooth, it might still happen that its restriction to S^2 is smooth. On the other hand, the map $f:\ S^2 \to \mathbb{R},\ (x,y,z) \mapsto \sqrt{1-z^2}$ is smooth only on $S^2 \backslash \{(0,0,1),(0,0,-1)\}$. To analyse the situation near the north pole, use the coordinate chart (U,φ) as above. In these coordinates, $z = \sqrt{1-(x^2+y^2)}$, hence $\sqrt{1-z^2} = \sqrt{x^2+y^2}$ which is not smooth near $(x,y) = (0,0)$.

Example 3.2. Let

$$\pi:\ \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}\mathrm{P}^n$$

be the quotient map. Given $f:\ \mathbb{R}\mathrm{P}^n \to \mathbb{R}$, the function

$$\widehat{f} = f \circ \pi:\ \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}$$

satisfies $\widehat{f}(\lambda x) = \widehat{f}(x)$ for $\lambda \neq 0$; conversely any $\widehat{f}$ with this property descends to a function f on the projective space. We claim that f is smooth if and only if $\widehat{f}$ is smooth. To see this, note that in the standard coordinate chart (U_i, φ_i) for $\mathbb{R}\mathrm{P}^n$, the function $f \circ \varphi_i^{-1}$ may be written as the smooth map

$$U_i \to \mathbb{R}^{n+1} \backslash \{0\},\ (u^1, \dots, u^n) \mapsto (u^1, \dots, u^i, 1, u^{i+1}, \dots, u^n)$$

followed by $\widehat{f}$. Hence, if $\widehat{f}$ is smooth then so is f. (The converse is similar.) As a special case, we see that for all $0 \leq j \leq k \leq n$ the functions

$$f:\ \mathbb{R}\mathrm{P}^n \to \mathbb{R},\ (x^0 : \dots : x^n) \mapsto \frac{x^j x^k}{||x||^2} \tag{3.1}$$

are well-defined and smooth. By a similar argument, the functions

$$f:\ \mathbb{C}\mathrm{P}^n \to \mathbb{C},\ (z^0 : \dots : z^n) \mapsto \frac{\overline{z^j} z^k}{||z||^2} \tag{3.2}$$

(where the bar denotes complex conjugation) are well-defined and smooth, in the sense that both the real and imaginary part are smooth.

Lemma 3.1. *Smooth functions $f \in C^\infty(M)$ are* continuous: *For every open subset $J \subseteq \mathbb{R}$, the pre-image $f^{-1}(J) \subseteq M$ is open.*

Proof. We have to show that for every (U,φ), the set $\varphi(U \cap f^{-1}(J)) \subseteq \mathbb{R}^m$ is open. But this subset coincides with the pre-image of J under the map $f \circ \varphi^{-1}:\ \varphi(U) \to \mathbb{R}$, which is a smooth function on an open subset of $\mathbb{R}^m$, and these are (by definition) continuous.

Exercise Show that $f:\ M \to \mathbb{R}$ *continuous* (i.e., the pre-image of any open subset $J \subseteq \mathbb{R}$ under f is open) if and only if for all charts (U,φ) the function $f \circ \varphi^{-1}$ is continuous.

From the properties of smooth functions on $\mathbb{R}^m$, one immediately gets the following properties of smooth functions on manifolds M:

- If $f,g \in C^\infty(M)$ and $\lambda,\mu \in \mathbb{R}$, then $\lambda f + \mu g \in C^\infty(M)$.
- If $f,g \in C^\infty(M)$, then $fg \in C^\infty(M)$.
- $1 \in C^\infty(M)$ (where 1 denotes the constant function $p \mapsto 1$).

These properties say that $C^\infty(M)$ is an *algebra* with unit 1. (See the appendix to this chapter for some background information on algebras.) Below, we will develop many of the concepts of manifolds in terms of this algebra of smooth functions.

Suppose M is any set with a maximal atlas $\{(U_\alpha,\varphi_\alpha)\}$. The definition of $C^\infty(M)$ does not use the Hausdorff or countability conditions; hence it makes sense in this more general context. We may use functions to check the Hausdorff property:

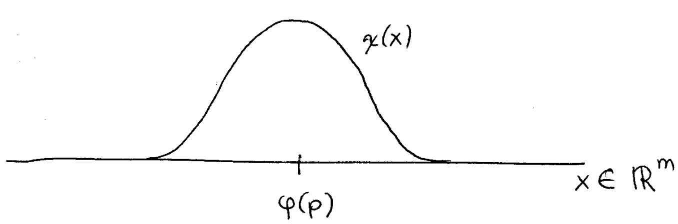
Proposition 3.1. *Suppose M is any set with a maximal atlas, and $p \neq q$ are two points in M. Then the following are equivalent:*

(i) There are open subsets $U,V \subseteq M$ with $p \in U,\ q \in V,\ U \cap V = \emptyset$,
(ii) There exists $f \in C^\infty(M)$ with $f(p) \neq f(q)$.

Proof. "$(i) \Rightarrow (ii)$". Suppose (i) holds. As explained in Section 2.5, we may take U,V to be the domains of coordinate charts (U,φ) and (V,ψ) around p,q. Choose $\varepsilon > 0$ such that the closed ε-ball

$$\overline{B_\varepsilon(\varphi(p))} = \left\{x \in \mathbb{R}^m \middle|\ ||x-\varphi(p)|| \leq \varepsilon\right\}$$

is contained in $\varphi(U)$; let $A \subseteq U$ be its pre-image under φ. Let $\chi \in C^\infty(\mathbb{R}^m)$ be a 'bump function' centered at $\varphi(p)$, with $\chi(\varphi(p)) = 1$ and $\chi(x) = 0$ for $||x-\varphi(p)|| \geq \varepsilon$. (For the existence of such a function see Munkres, Lemma 16.1, or Lemma A.2 in the appendix.)



The function $f:\ M \to \mathbb{R}$ such that $f = \chi \circ \varphi$ on U and $f = 0$ on $M \backslash A$ is smooth, and satisfies $f(p) = 1,\ f(q) = 0$.

"$(ii) \Leftarrow (i)$". Suppose (ii) holds. Let $\delta = |f(q) - f(p)|/2$, and put

$$U = \{x \in M | \; |f(x) - f(p)| < \delta\}, \tag{3.3}$$
$$V = \{x \in M | \; |f(x) - f(q)| < \delta\} \tag{3.4}$$

By the Lemma 3.1, U, V are open, and clearly $p \in U, \; q \in V, U \cap V = \emptyset$.

A direct consequence of this result is:

Corollary 3.1 (Criterion for Haudorff condition). *A set M with an atlas satisfies the Hausdorff condition if and only if for any two distinct points $p, q \in M$, there exists a smooth function $f \in C^\infty(M)$ with $f(p) \neq f(q)$. In particular, if there exists a smooth injective map $F: \; M \to \mathbb{R}^N$, then M is Hausdorff.*

Remark 3.2. One may replace 'smooth' with continuous in Proposition 3.1 and Corollary 3.1.

Example 3.3 (Projective spaces). Write vectors $x \in \mathbb{R}^{n+1}$ as column vectors, hence $x^\top$ is the corresponding row vector. The matrix product $xx^\top$ is a square matrix with entries $x^j x^k$. The map

$$\mathbb{R}\mathrm{P}^n \to \mathrm{Mat}_\mathbb{R}(n+1), \;\; (x^0 : \ldots : x^n) \mapsto \frac{x\,x^\top}{||x||^2} \tag{3.5}$$

is a smooth; indeed, its matrix components are the functions (3.1). For any given $(x^0 : \ldots : x^n) \in \mathbb{R}\mathrm{P}^n$, at least one of these components is non-zero. Identifying $\mathrm{Mat}_\mathbb{R}(n+1) \cong \mathbb{R}^N$, where $N = (n+1)^2$, this gives the desired smooth injective map from projective space into $\mathbb{R}^N$; hence the criterion applies, and the Hausdorff condition follows. For the complex projective space, one similarly has a smooth and injective map

$$\mathbb{C}\mathrm{P}^n \to \mathrm{Mat}_\mathbb{C}(n+1), \;\; (z^0 : \ldots : z^n) \mapsto \frac{z\,z^\dagger}{||x||^2} \tag{3.6}$$

(where $z^\dagger = \bar{z}^\top$ is the conjugate transpose of the complex column vector z) into $\mathrm{Mat}_\mathbb{C}(n+1) = \mathbb{R}^N$ with $N = 2(n+1)^2$.

Exercise: Verify that the map

$$\mathrm{Gr}(k,n) \to \mathrm{Mat}_\mathbb{R}(n), \;\; E \mapsto P_E, \tag{3.7}$$

taking a subspace E to the matrix of the orthogonal projection onto E, is smooth and injective, hence $\mathrm{Gr}(k,n)$ is Hausdorff. Discuss a similar map for the complex Grassmannian $\mathrm{Gr}_\mathbb{C}(k,n)$.

In the opposite direction, the criterion tells us that for a set M with an atlas, if the Hausdorff condition does not hold then no smooth injective map into $\mathbb{R}^N$ exists.

Example 3.4. Consider the non-Hausdorff manifold M from Example 2.6. Here, there are two points p, q that do not admit disjoint open neighborhoods. We see directly that any smooth function on M must take on the same values at p and q: With the coordinate charts $(U, \varphi), (V, \psi)$ in that example,

$$f(p) = f(\varphi^{-1}(0)) = \lim_{t \to 0^-} f(\varphi^{-1}(t)) = \lim_{t \to 0^-} f(\psi^{-1}(t)) = f(\psi^{-1}(0)) = f(q),$$

since $\varphi^{-1}(t) = \psi^{-1}(t)$ for $t < 0$.

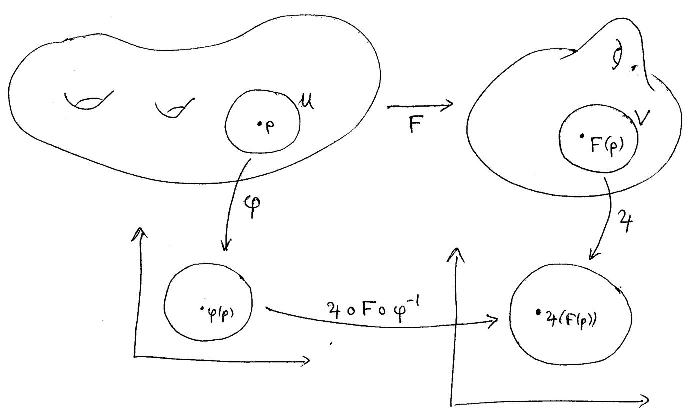
3.2 Smooth maps between manifolds

The notion of smooth maps from M to $\mathbb{R}$ generalizes to smooth maps between manifolds.

Definition 3.2. A map $F: M \to N$ between manifolds is *smooth at* $p \in M$ if there are coordinate charts (U, φ) around p and (V, ψ) around $F(p)$ such that $F(U) \subseteq V$ and such that the composition

$$\psi \circ F \circ \varphi^{-1}: \varphi(U) \to \psi(V)$$

is smooth. The function F is called a *smooth map from M to N* if it is smooth at all $p \in M$.



Remarks 3.3. 1. 1. The condition for smoothness at p does not depend on the choice of charts: Given a different choice of charts (U', φ') and (V', ψ') with $F(U') \subseteq V'$, we have

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ (\varphi)^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

on $\varphi'(U \cap U')$.

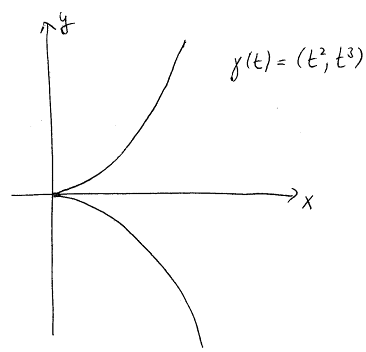
2. To check smoothness of F, it suffices to take *any* atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M with the property that $F(U_\alpha) \subseteq V_\alpha$ for *some* chart (V_α, ψ_α) of N, and then check smoothness of the maps

$$\psi_\alpha \circ F \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U_\alpha) \to \psi_\alpha(V_\alpha).$$

3. Smooth maps $M \to \mathbb{R}$ are the same thing as smooth functions on M:

$$C^\infty(M,\mathbb{R}) = C^\infty(M).$$

Smooth functions $\gamma : \ J \to M$ from an open interval $J \subseteq \mathbb{R}$ to M are called *(smooth) curves in* M. Note that the image of a smooth curve need not look smooth. For instance, the image of $\gamma\colon \ \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^2,t^3)$ has a 'cusp singularity' at $(0,0)$.



Example 3.5. a) Consider the map $F : \ \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ given by

$$(t:1) \mapsto (e^{t^2}:1) \text{ for } t \in \mathbb{R}, \ \ (1:0) \mapsto (1:0).$$

In the chart U_1, we have $F(U_1) \subseteq U_1$, and $\varphi_1 \circ F \circ \varphi_1^{-1}(t) = e^{t^2}$, which is smooth. It remains to check smoothness at the point $p = (1:0)$. Since $F(p) = p$, we will verify this using the chart U_0 around p. We have, for $u \neq 0$

$$F(\varphi_0^{-1}(u)) = F(1:u) = F(\frac{1}{u}:1) = (e^{\frac{1}{u^2}}:1) = (1:e^{-\frac{1}{u^2}}),$$

Hence $\varphi_0 \circ F \circ \varphi_0^{-1}$ is the map

$$u \mapsto e^{-\frac{1}{u^2}} \quad \text{for} \quad u \neq 0, \quad\ 0 \mapsto 0.$$

As is well-known, this map is smooth even at $u = 0$.

b) The same calculation applies for the map $F : \ \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$, given by the same formulas. However, the conclusion is different: The map

$$z \mapsto e^{-\frac{1}{z^2}} \quad \text{for} \quad u \neq 0, \quad 0 \mapsto 0.$$

is *not* smooth (or even continuous) at $z = 0$. (For a non-zero complex number a, consider the limit of this function for $z = sa$ as $s \to 0$. If $a = 1$, the limit is 0. If

$z = 1 + i$, the absolute value is always one, and the limit doesn't exist. If $a = i$, the limit is ∞.)

Proposition 3.2. *Suppose $F_1 : M_1 \to M_2$ and $F_2 : M_2 \to M_3$ are smooth maps. Then the composition*

$$F_2 \circ F_1 : M_1 \to M_3$$

is smooth.

Proof. Given $p \in M_1$, choose charts (U_1, φ_1) around p, (U_2, φ_2) around $F_1(p)$, and (U_3, φ_3) around $F_2(F_1(p))$, with $F_2(U_2) \subseteq U_3$ and $F_1(U_1) \subseteq U_2$. (This is always possible – see exercise below.) Then $F_2(F_1(U_2)) \subseteq U_3$, and we have:

$$\varphi_3 \circ (F_2 \circ F_1) \circ \varphi_1^{-1} = (\varphi_3 \circ F_2 \circ \varphi_2^{-1}) \circ (\varphi_2 \circ F_1 \circ \varphi_1^{-1}),$$

a composition of smooth maps between open subsets of Euclidean spaces. □

Exercise: Suppose $F \in C^\infty(M,N)$.

1. Let (U, φ) be a coordinate chart for M and (V, ψ) a coordinate chart for N, with $F(U) \subseteq V$. Show that for all open subsets $W \subseteq N$ the set $U \cap F^{-1}(W)$ is open.
   Hint: Show that $\varphi(U \cap F^{-1}(W))$ is the pre-image of the open set $\psi(V \cap W)$ under the smooth map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$.
2. Show that F is continuous: For every open $W \subseteq N$ the pre-image $F^{-1}(W)$ is open.
3. Given $p \in M$ and *any* chart (V, ψ) around $F(p)$, show that there exists a chart (U, φ) around p such that $F(U) \subseteq V$.
   Hint: Start with any open chart (U_1, φ_1) around p, and replace U_1 with $U = U_1 \cap F^{-1}(V)$.

Exercise: Using the previous exercise, show that smooth maps $F \in C^\infty(M,N)$ are *continuous:* The pre-images of every open subsets of N is open in M.

3.2.1 Diffeomorphisms of manifolds

Definition 3.3. A smooth map $F : M \to N$ is called a *diffeomorphism* if it is invertible, with a smooth inverse $F^{-1} : N \to M$. Manifolds M,N are called diffeomorphic if there exists a diffeomorphism from M to N.

In other words, a diffeomorphism of manifolds is a bijection of the underlying sets that identifies the maximal atlases of the manifolds. Manifolds that are diffeomorphic are therefore considered 'the same manifolds'.

Similarly, a continuous map $F : M \to N$ is called a *homeomorphism* if it is invertible, with a continuous inverse. Manifolds that are homeomorphic are considered 'the same topologically'. Since every smooth map is continuous, every diffeomorphism is a homeomorphism.

Example 3.6. By definition, every coordinate chart (U,φ) on a manifold M gives a diffeomorphism $\varphi:\ U\to\varphi(U)$ onto an open subset of $\mathbb{R}^m$.

Example 3.7. The standard example of a homeomorphism of smooth manifolds that is not a diffeomorphism is the map

$$\mathbb{R}\to\mathbb{R},\quad x\mapsto x^3.$$

Indeed, this map is smooth and invertible, but the inverse map $y\mapsto y^{\frac{1}{3}}$ is not smooth.

Example 3.8. Give a manifold M, with maximal atlas $\mathscr{A}$, then any homeomorphism $F:\ M\to M$ can be used to define a new atlas $\mathscr{A}'$ on M, with charts $(U',\varphi')\in\mathscr{A}'$ obtained from charts $(U,\varphi)\in\mathscr{A}$ as $U'=F(U),\ \varphi'=\varphi\circ F^{-1}$. One can verify (please do) that $\mathscr{A}'=\mathscr{A}$ if and only if F is a diffeomorphism. Thus, if F is a homeomorphism of M which is not a diffeomorphism, then F defines a new atlas $\mathscr{A}'\neq\mathscr{A}$.

However, the new manifold structure on M is not genuinely different from the old one. Indeed, while $F:\ M\to M$ is not a diffeomorphism relative to the atlas $\mathscr{A}$ on the domain M and target M, it *does* define a diffeomorphism if we use the atlas $\mathscr{A}$ on the domain and the atlas $\mathscr{A}'$ on the target. Hence, even though $\mathscr{A}$ and $\mathscr{A}'$ are different atlases, the resulting manifold structures are still diffeomorphic.

Remark 3.4. In the introduction, we explained (without proof) the classification of 1-dimensional and 2-dimensional connected compact manifolds up to diffeomorphism. This classification coincides with their classification up to homeomorphism. This means, for example, that for any maximal atlas $\mathscr{A}'$ on S^2 which induces the same system of open subsets as the standard maximal atlas $\mathscr{A}$, there exists a homeomorphism $F:\ S^2\to S^2$ taking $\mathscr{A}$ to $\mathscr{A}'$, in the sense that $(U,\varphi)\in\mathscr{A}$ if and only if $(U',\varphi')\in\mathscr{A}'$, where $U'=F(U)$ and $\varphi'\circ F=\varphi$. In higher dimensions, it becomes much more complicated: .

It is quite possible for two manifolds to be homeomorphic but not diffeomorphic (unlike example 3.8). The first example of 'exotic' manifold structures was discovered by John Milnor in 1956, who found that the 7-sphere S^7 admits manifold structures that are not diffeomorphic to the standard manifold structure, but induce the standard topology. Kervaire and Milnor in 1963, proved that there are exactly 28 distinct manifold structures on S^7, and in fact classified all manifold structures on all spheres S^n with the exception of the case $n=4$. For example, they showed that S^3,S^5,S^6 do not admit exotic (i.e., non-standard) manifold structures, while S^{15} has 16256 different manifold structures. For S^4 the existence of exotic manifold structures is an open problem; this is known as the *smooth Poincare conjecture*.

Around 1982, Michael Freedman (using results of Simon Donaldson) discovered the existence of exotic manifold structures on $\mathbb{R}^4$; later Clifford Taubes showed that there are uncountably many such. For $\mathbb{R}^n$ with $n\neq 4$, it is known that there are no exotic manifold structures on $\mathbb{R}^n$.

3.3 Examples of smooth maps

3.3.1 Products, diagonal maps

a) If M,N are manifolds, then the projection maps

$$\mathrm{pr}_M : M\times N \to M, \ \ \mathrm{pr}_N : M\times N \to N$$

are smooth. (This follows immediately by taking product charts $U_\alpha \times V_\beta$.)

b) The diagonal inclusion

$$\Delta_M : M \to M\times M$$

is smooth. (In a coordinate chart (U,φ) around p and the chart $(U\times U, \varphi\times\varphi)$ around (p,p), the map is the restriction to $\varphi(U) \subseteq \mathbb{R}^n$ of the diagonal inclusion $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$.)

c) Suppose $F : M \to N$ and $F' : M' \to N'$ are smooth maps. Then the direct product

$$F\times F' : M\times M' \to N\times N'$$

is smooth. This follows from the analogous statement for smooth maps on open subsets of Euclidean spaces.

3.3.2 The diffeomorphism $\mathbb{R}\mathrm{P}^1 \cong S^1$

We have stated before that $\mathbb{R}\mathrm{P}^1 \cong S^1$. To obtain an explicit diffeomorphism, we construct a bijection identifying the standard atlas for $\mathbb{R}\mathrm{P}^1$ with (essentially) the standard atlas for S^1. Recall that the atlas for $\mathbb{R}\mathrm{P}^1$ is given by

$$U_1 = \{(u:1) |\ u\in\mathbb{R}\}, \ \ \varphi_1(u:1) = u,$$
$$U_0 = \{(1:u) |\ u\in\mathbb{R}\}, \ \ \varphi_0(1:u) = u$$

with $\varphi_i(U_i) = \mathbb{R}$ and $\varphi_0(U_0\cap U_1) = \varphi_1(U_0\cap U_1) = \mathbb{R}\backslash\{0\}$, with the transition map $\varphi_1\circ\varphi_0^{-1} : u\mapsto u^{-1}$. Similarly, the atlas for S^1 is

$$U_+ = \{(x,y)\in S^1 |\ y \neq -1\} \ \ \varphi_+(x,y) = \frac{x}{1+y},$$
$$U_- = \{(x,y)\in S^1 |\ y \neq +1\} \ \ \varphi_-(x,y) = \frac{x}{1-y}.$$

again with $\varphi_\pm(U_\pm) = \mathbb{R}$, $\varphi_\pm(U_+\cap U_-) = \mathbb{R}\backslash\{0\}$, and transition map $u\mapsto u^{-1}$.

Hence, there is a well-defined diffeomorphism $F : \mathbb{R}\mathrm{P}^1 \to S^1$ which identifies the chart (U_-,φ_-) with (U_1,φ_1) and (U_+,φ_+) with (U_0,φ_0), in the sense that both

$$\varphi_- \circ F \circ \varphi_1^{-1} : \ \mathbb{R} \to \mathbb{R}, \quad \varphi_+ \circ F \circ \varphi_0^{-1} : \ \mathbb{R} \to \mathbb{R}$$

are the identity $\mathrm{id}_{\mathbb{R}}$. Namely, the restriction of F to U_1 is $F_{U_1} = \varphi_-^{-1} \circ \varphi_1 : \ U_1 \to U_-$, the restriction to U_0 is $F|_{U_0} = \varphi_+^{-1} \circ \varphi_0 : \ U_0 \to U_+$. The inverse map $G = F^{-1}$: $S^1 \to \mathbb{R}\mathrm{P}(1)$ is similarly given by $\varphi_0^{-1} \circ \varphi_+$ over U_+ and by $\varphi_1^{-1} \circ \varphi_-$ over U_-. A calculation gives

$$F : \ \mathbb{R}\mathrm{P}^1 \to S^1, \ \ (w^0 : w^1) \mapsto \frac{1}{||w||^2}(2w^1 w^0, \ \ (w^0)^2 - (w^1)^2);$$

with inverse.

$$\begin{aligned} G(x,y) &= (1+y : \ x), \quad y \neq -1, \\ G(x,y) &= (x : \ 1-y), \quad y \neq 1 \end{aligned}$$

(note that the two expressions agree if $-1 < y < 1$). For example, to get the formula for $G(x,y)$ for $y \neq -1$, i.e. $(x,y) \in U_+$, we calculate as follows:

$$\varphi_0^{-1} \circ \varphi_+(x,y) = \varphi_0^{-1}\Big(\frac{x}{1+y}\Big) = \Big(1 : \frac{x}{1+y}\Big) = (1+y : \ x).$$

Exercise: Work out the details of the calculation of $F(w^0 : w^1)$.

3.3.3 The diffeomorphism $\mathbb{C}\mathrm{P}^1 \cong S^2$

By a similar reasoning, we find $\mathbb{C}\mathrm{P}^1 \cong S^2$. For S^2 we use the atlas given by stereographic projection.

$$\begin{aligned} U_+ &= \{(x,y,z) \in S^2 |\ z \neq -1\} \quad \varphi_+(x,y,z) = \frac{1}{1+z}\ (x,y), \\ U_- &= \{(x,y,z) \in S^2 |\ z \neq +1\} \quad \varphi_-(x,y,z) = \frac{1}{1-z}\ (x,y). \end{aligned}$$

The transition map is $u \mapsto \frac{u}{||u||^2}$, for $u = (u^1, u^2)$. Regarding u as a complex number $u = u^1 + iu^2$, the norm $||u||$ is just the absolute value of u, and the transition map becomes

$$u \mapsto \frac{u}{|u|^2} = \frac{1}{\overline{u}}.$$

Note that it is not quite the same as the transition map for the standard atlas of $\mathbb{C}\mathrm{P}^1$, which is given by $u \mapsto u^{-1}$. We obtain a unique diffeomorphism $F : \ \mathbb{C}\mathrm{P}^1 \to S^2$ such that $\varphi_+ \circ F \circ \varphi_0^{-1}$is the identity, while $\varphi_- \circ F \circ \varphi_1^{-1}$ is complex conjugation. A calculation shows that this map is

$$F(w^0 : w^1) = \frac{1}{|w^0|^2 + |w^1|^2}\Big(2\mathrm{Re}(w^1\overline{w^0}),\ 2\mathrm{Im}(w^1\overline{w^0}),\ |w^0|^2 - |w^1|^2\Big);$$

the inverse map $G = F^{-1} :\ S^2 \to \mathbb{C}\mathrm{P}(1)$ is

$$G(x,y,z) = (1+z :\ x+iy), \quad z \neq -1,$$
$$G(x,y,z) = (x-iy :\ 1-z), \quad z \neq 1$$

(note that the two expressions agree if $-1 < z < 1$).

Exercise: Work out the details of the calculation.

3.3.4 Maps to and from projective space

The quotient map

$$\pi :\ \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n,\ x = (x^0,\ldots,x^n) \mapsto (x^0 : \ldots : x^n)$$

is smooth, as one verifies by checking in the standard atlas for $\mathbb{R}\mathrm{P}^n$. Indeed, on the open subset where $x^i \neq 0$, we have $\pi(x) \in U_i$, and

$$(\varphi_i \circ \pi)(x^0,\ldots,x^n) = (\frac{x^0}{x^i},\ldots,\frac{x^{i-1}}{x^i},\frac{x^{i+1}}{x^i},\ldots,\frac{x^n}{x^i}).$$

which is a smooth function on the open set of x's for which $x^i \neq 0$.

Given a map $F :\ \mathbb{R}\mathrm{P}^n \to N$ to a manifold N, let $\widetilde{F} = F \circ \pi :\ \mathbb{R}^{n+1}\backslash\{0\} \to N$ be its composition with the projection map $\pi :\ \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n$. That is,

$$\widetilde{F}(x^0,\ldots,x^n) = F(x^0 : \ldots : x^n).$$

Note that $\widetilde{F}(\lambda x^0 : \ldots : \lambda x^n) = \widetilde{F}(x^0,\ldots,x^n)$ for all non-zero λ; conversely, every map $\widetilde{F}$ with this property descends to a map F on projective space. We claim that the map F is smooth if and only the corresponding map $\widetilde{F}$ is smooth. One direction is clear: If F is smooth, then $\widetilde{F} = F \circ \pi$ is a composition of smooth maps. For the other direction, assuming that $\widetilde{F}$ is smooth, note that for the standard chart (U_j, φ_j), the maps

$$(F \circ \varphi_j^{-1})(u^1,\ldots,u^n) = \widetilde{F}(u^1,\ldots,u^i,1,u^{i+1},\ldots,u^n),$$

are smooth.

An analogous argument applies to the complex projective space $\mathbb{C}\mathrm{P}^n$, taking the x^i to be complex numbers z^i. That is, the quotient map $\pi :\ \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathrm{P}^n$ is smooth, and a map $F :\ \mathbb{C}\mathrm{P}^n \to N$ is smooth if and only if the corresponding map $\widetilde{F} :\ \mathbb{C}^{n+1}\backslash\{0\} \to N$ is smooth.

As an application, we can see that the map

$$\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2,\ \ (z^0 : z^1) \mapsto ((z^0)^2 : (z^1)^2 : z^0 z^1)$$

is smooth, starting with the (obvious) fact that the lifted map

$$\mathbb{C}^2\backslash\{0\} \to \mathbb{C}^3\backslash\{0\}, \ (z^0, z^1) \mapsto ((z^0)^2, (z^1)^2, z^0 z^1)$$

is smooth.

3.3.5 The quotient map $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$

As we explained above, the quotient map $q:\ \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathrm{P}^n$ is smooth. Since any class $[z] = (z^0 : \ldots : z^n)$ has a representative with $|z^0|^2 + \ldots + |z^n|^2 = 1$, and $|z^i|^2 = (x^i)^2 + (y^i)^2$ for $z^i = x^i + \sqrt{-1}y^i$, we may also regard $\mathbb{C}\mathrm{P}^n$ as a set of equivalence classes in the unit sphere $S^{2n+1} \subseteq \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. The resulting quotient map

$$\pi:\ S^{2n+1} \to \mathbb{C}\mathrm{P}^n$$

is again smooth, because it can be written as a composition of two smooth maps

$$\pi = q \circ \iota$$

where $\iota:\ S^{2n+1} \to \mathbb{R}^{2n+2}\backslash\{0\} = \mathbb{C}^{n+1}\backslash\{0\}$ is the inclusion map.

For any $p \in \mathbb{C}\mathrm{P}^n$, the corresponding fiber $\pi^{-1}(p) \subseteq S^{2n+1}$ is diffeomorphic to a circle S^1 (which we may regard as complex numbers of absolute value 1). Indeed, given any point $(z^0, \ldots, z^n) \in \pi^{-1}(p)$ in the fiber, the other points are obtained as $(\lambda z^0, \ldots, \lambda z^n)$ where $|\lambda| = 1$.

In other words, we can think of

$$S^{2n+1} = \bigcup_{p \in \mathbb{C}\mathrm{P}^n} \pi^{-1}(p)$$

as a union of circles, parametrized by the points of $\mathbb{C}\mathrm{P}^n$. This is an example of what differential geometers call a *fiber bundle* or *fibration*. We won't give a formal definition here, but let us try to 'visualize' the fibration for the important case $n = 1$.

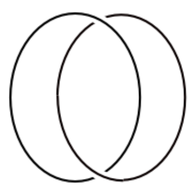
Identifying $\mathbb{C}\mathrm{P}^1 \cong S^2$ as above, the map π becomes a smooth map

$$\pi:\ S^3 \to S^2$$

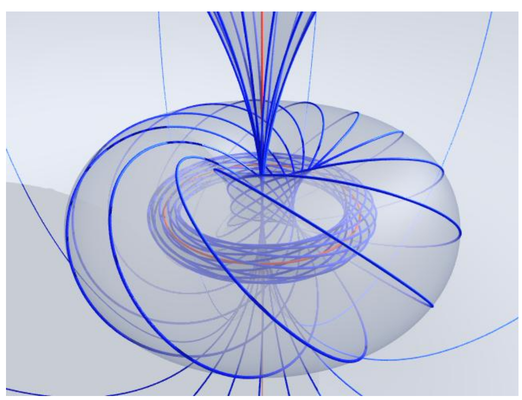
with fibers diffeomorphic to S^1. This map appears in many contexts; it is called the *Hopf fibration* (after *Heinz Hopf* (1894-1971)).

Let $S \in S^3$ be the ‘south pole’, and $N \in S^3$ the ‘north pole’. We have that $S^3 - \{S\} \cong \mathbb{R}^3$ by stereographic projection. The set $\pi^{-1}(\pi(S)) - \{S\}$ projects to a straight line (think of it as a circle with ‘infinite radius’). The fiber $\pi^{-1}(N)$ is a circle that goes around the straight line. If $Z \subseteq S^2$ is a circle at a given ‘latitude’, then $\pi^{-1}(Z)$ is is a 2-torus. For Z close to N this 2-torus is very thin, while for Z approaching the south pole S the radius goes to infinity. Each such 2-torus is itself a union of circles $\pi^{-1}(p),\ p \in Z$. Those circles are neither the usual ‘vertical’ or ‘horizontal’ circles of a 2-torus in $\mathbb{R}^3$, but instead are ‘tilted’. In fact, each such circle is a ‘perfect geometric circle’ obtained as the intersection of its 2-torus with a carefully positioned affine 2-plane.

Moreover, any two of the circles $\pi^{-1}(p)$ are *linked*:



The full picture looks as follows:[1]



[1] http://perso-math.univ-mlv.fr/users/kloeckner.benoit/images.html

A calculation shows that over the charts U_+, U_- (from stereographic projection), the Hopf fibration is just a product. That is, one has

$$\pi^{-1}(U_+) \cong U_+ \times S^1, \quad \pi^{-1}(U_-) \cong U_- \times S^1.$$

In particular, the pre-image of the closed upper hemisphere is a *solid 2-torus* $D^2 \times S^1$ (with $D^2 = \{z \in \mathbb{C} | \, |z| \leq 1\}$ the unit disk), geometrically depicted as a 2-torus in $\mathbb{R}^3$ together with its interior.[2] We hence see that the S^3 may be obtained by gluing two solid 2-tori along their boundaries $S^1 \times S^1$.

3.4 Submanifolds

Let M be a manifold of dimension m. We will define a k-dimensional submanifold $S \subseteq M$ to be a subset that looks locally like $\mathbb{R}^k \subseteq \mathbb{R}^m$ (which we take to be the coordinate subspace defined by $x^{k+1} = \cdots = x^m = 0$.

Definition 3.4. A subset $S \subseteq M$ is called a *submanifold* of dimension $k \leq m$, if for all $p \in S$ there exists a coordinate chart (U, φ) around p such that

$$\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k.$$

Charts (U, φ) of M with this property are called *submanifold charts* for S.

Remark 3.5. 1. A chart (U, φ) such that $U \cap S = \emptyset$ and $\varphi(U) \cap \mathbb{R}^k = \emptyset$ is considered a submanifold chart.
2. We stress that the existence of submanifold charts is only required for points p that lie in S. For example, the half-open line $S = (0, \infty)$ is a submanifold of $\mathbb{R}$. There does not exist a submanifold chart containing 0, but this is not a problem since $0 \notin S$.

Strictly speaking, a submanifold chart for S is not a chart for S, but is a chart for M which is adapted to S. On the other hand, submanifold charts *restrict* to charts for S, and this may be used to construct an atlas for S:

Proposition 3.3. *Suppose S is a submanifold of M. Then S is a k-dimensional manifold in its own right, with atlas consisting of all charts $(U \cap S, \varphi|_{U \cap S})$ such that (U, φ) is a submanifold chart.*

Proof. Let (U, φ) and (V, ψ) be two submanifold charts for S. We have to show that the charts $(U \cap S, \varphi|_{U \cap S})$ and $(V \cap S, \psi|_{V \cap S})$ are compatible. The map

$$\psi|_{V \cap S} \circ \varphi|_{U \cap S}^{-1} : \ \varphi(U \cap V) \cap \mathbb{R}^k \to \psi(U \cap V) \cap \mathbb{R}^k$$

[2] A solid torus is an example of a ”manifold with boundary”, a concept we haven’t properly discussed yet.

is smooth, because it is the restriction of $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ to the coordinate subspace $\mathbb{R}^k$. Likewise its inverse map is smooth. The Hausdorff condition follows because any two distinct points $p, q \in S$, one can take disjoint submanifold charts around p, q. (Just take any submanifold charts, and intersect with the domains of disjoint charts around p, q.)

The proof that S admits a *countable* atlas is unfortunately a bit technical[3]. We use the following

Fact: Every open subset of $\mathbb{R}^m$ is a union of *rational* ε-balls $B_\varepsilon(x)$, $\varepsilon > 0$. Here, 'rational' means that both the center of the ball and its radius are rational: $x \in \mathbb{Q}^n$, $\varepsilon \in \mathbb{Q}$.

(We leave this as an exercise.) Our goal is to construct a countable collection of submanifold charts covering S. (The atlas for S itself is then obtained by restriction.) Start with any countable atlas $(U_\alpha, \varphi_\alpha)$ for M. Given $p \in S \cap U_\alpha$, we can choose a submanifold chart (V, ψ) containing p. Using the above fact, we can choose a rational ε-ball with

$$\varphi(p) \in B_\varepsilon(x) \subseteq \varphi_\alpha(U_\alpha \cap V).$$

This shows that the subsets of the form $\varphi_\alpha^{-1}(B_\varepsilon(x))$, with $B_\varepsilon(x) \subseteq \varphi_\alpha(U_\alpha)$ a rational ε-ball such that $\varphi_\alpha^{-1}(B_\varepsilon(x))$ is contained in *some* submanifold chart, cover all of S. Take these to be the domains of a charts (V_β, ψ_β), where V_β is one of the $\varphi_\alpha^{-1}(B_\varepsilon(x))$, and ψ_β is the restriction of the coordinate maps of a submanifold chart containing $\varphi_\alpha^{-1}(B_\varepsilon(x))$. Then $\{(V_\beta, \psi_\beta)\}$ is a countable collection of submanifold charts covering S. (Recall that a countable union of countable sets is again countable.) $\square$

Example 3.9 (Open subsets). The m-dimensional submanifolds of an m-dimensional manifold are exactly the open subsets.

Example 3.10 (Spheres). Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x||^2 = 1\}$. Write $x = (x^0, \ldots, x^n)$, and regard

$$S^k \subseteq S^n$$

for $k < n$ as the subset where the last $n - k$ coordinates are zero. These are submanifolds: The charts $(U_\pm, \varphi_\pm)$ for S^n given by stereographic projection

$$\varphi_\pm(x^0, \ldots, x^n) = \frac{1}{1 \pm x^0}(x^1, \ldots, x^n)$$

are submanifold charts. In fact, the charts $U_i^\pm$, given by the condition that $\pm x^i > 0$, with $\varphi_i^\pm$ the projection to the remaining coordinates, are submanifold charts as well.

Example 3.11 (Projective spaces). For $k < n$, regard

$$\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$$

as the subset of all $(x^0 : \ldots : x^n)$ for which $x^{k+1} = \ldots = x^n = 0$. These are submanifolds, with the standard charts (U_i, φ_i) for $\mathbb{R}\mathrm{P}^n$ as submanifold charts. (Note that the

[3] You are welcome to ignore the following proof.

charts $U_{k+1}, \ldots, U_n$ don't meet $\mathbb{R}\mathrm{P}^k$, but this does not cause a problem.) In fact, the resulting charts for $\mathbb{R}\mathrm{P}^k$ obtained by restricting these submanifold charts, are just the standard charts of $\mathbb{R}\mathrm{P}^k$. Similarly,

$$\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$$

are submanifolds, and for $n < n'$ we have $\mathrm{Gr}(k,n) \subseteq \mathrm{Gr}(k,n')$ as a submanifold.

Proposition 3.4. *Let $F: M \to N$ be a smooth map between manifolds of dimensions m and n. Then*

$$\mathrm{graph}(F) = \{(F(p),p) |\ p \in M\} \subseteq N \times M$$

is a submanifold of $N \times M$, of dimension equal to the dimension of M.

Proof. Given $p \in M$, choose charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, and let $W = V \times U$. We claim that (W, κ) with

$$\kappa(q,p) = (\varphi(p),\ \psi(q) - \psi(F(p))) \tag{3.8}$$

is a submanifold chart for $\mathrm{graph}(F) \subseteq N \times M$. Note that this is indeed a chart of $N \times M$, because it is obtained from the product chart $(V \times U, \psi \times \varphi)$ by composition with the diffeomorphism $\psi(V) \times \varphi(U) \to \varphi(U) \times \psi(V),\ (v,u) \mapsto (u,v)$, followed by the diffeomorphism

$$\varphi(U) \times \psi(V) \to \kappa(W),\ \ (u,v) \mapsto (u, v - \widetilde{F}(u)). \tag{3.9}$$

where $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$. (The map (3.9) is smooth and injective, and its Jacobian has determinant one, hence is invertible everywhere.) Furthermore, the second component in (3.8) vanishes if and only if $F(p) = q$. That is,

$$\kappa(W \cap \mathrm{graph}(F)) = \kappa(W) \cap \mathbb{R}^m$$

as required. $\square$

This result has the following consequence: If a subset of a manifold, $S \subseteq M$, can be *locally* described as the graph of a smooth map, then S is a submanifold. In more detail, suppose that S can be covered by open sets U, such that for each U there is a diffeomorphism $U \to P \times Q$ taking $S \cap U$ to the graph of a smooth map $Q \to P$, then S is a submanifold.

Example 3.12. The 2-torus $S = f^{-1}(0) \subseteq \mathbb{R}^3$, where

$$f(x,y,z) = (\sqrt{x^2+y^2} - R)^2 + z^2 - r^2$$

is a submanifold of $\mathbb{R}^3$, since it can locally be expressed as the graph of a function of x,y, or of y,z, or of x,z. For example, on the subset where $z > 0$, it is the graph of the smooth function on the annulus $\{(x,y) |\ (R-r)^2 < x^2+y^2 < (R+r)^2\}$, given as

$$H(x,y) = \sqrt{r^2 - (\sqrt{x^2+y^2} - R)^2}.$$

This function is obtained by solving the equation $f(x,y,z)$ for z. Similarly, on each of the four components of the subset where $x^2+y^2 \neq R^2$ and $x \neq 0$ (respectively, $y \neq 0$), one can solve the equation $f(x,y,z)=0$ uniquely for x (respectively, for y), expressing S as the graph of a smooth function of y and z (respectively, of x and z).

Exercise: Work out the formula for the $F(y,z)$ on the subset where $x^2+y^2<R^2$ and $x>0$.

Example 3.13. More generally, suppose $S \subseteq \mathbb{R}^3$ is given as a level set $S=f^{-1}(0)$ for a smooth map $f \in C^\infty(\mathbb{R}^3)$. (Actually, we only need f to be defined and smooth on an open neighborhood of S.) Let $p \in S$, and suppose

$$\frac{\partial f}{\partial x}\Big|_p \neq 0.$$

By the *implicit function theorem* from multivariable calculus, there is an open neighborhood $U \subseteq \mathbb{R}^3$ of p on which the equation $f(x,y,z)=0$ can be uniquely solved for x. That is,

$$S \cap U = \{(x,y,z) \in U \,|\, x = F(y,z)\}$$

for a smooth function F, defined on a suitable open subset of $\mathbb{R}^2$. This shows that S is a submanifold near p, and in fact we may use y,z as coordinates near p. Similar arguments apply for $\frac{\partial f}{\partial y}|_p \neq 0$ or $\frac{\partial f}{\partial z}|_p \neq 0$. Hence, if the gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is non-vanishing at all points $p \in S = f^{-1}(0)$, then S is a 2-dimensional submanifold. Of course, there is nothing special about 2-dimensional submanifolds of $\mathbb{R}^3$, and below, we will put this discussion in a more general framework.

As we saw, submanifolds S of manifolds M are themselves manifolds. They come with an *inclusion map*

$$i:\ S \to M,\ p \mapsto p,$$

taking any point of S to the same point but viewed as a point of M. Unsurprisingly, we have:

Proposition 3.5. *The inclusion map $i:\ S \to M$ is smooth.*

Proof. Given $p \in S \subseteq M$, let (U,φ) be a submanifold chart around $p \in M$, and $(U \cap S, \varphi|_{U\cap S})$ the corresponding chart around $p \in S$. The composition

$$\varphi \circ i \circ (\varphi|_{U\cap S})^{-1}:\ \varphi(U \cap S) \to \varphi(U)$$

is simply the inclusion map from $\varphi(U) \cap \mathbb{R}^k$ to $\varphi(U)$, which is obviously smooth. □

This shows in particular that if $F \in C^\infty(M,N)$ is a smooth map, then its restriction $F|_S : S \to N$ is again smooth. Indeed, $F|_S = F \circ i$ is a composition of smooth maps. This is useful in practice, because in such cases there is no need to verify smoothness in local coordinates of S! For example, the map $S^2 \to \mathbb{R}$, $(x,y,z) \mapsto z$ is smooth since it is the restriction of a smooth map $\mathbb{R}^3 \to \mathbb{R}$ to the submanifold S^2. A related result, which we leave as an exercise, is the following:

Exercise. Let $S \subseteq M$ be a submanifold, with inclusion map i, and let $F : Q \to S$ be a map from another manifold Q. Then F is smooth if and only if $i \circ F$ is smooth. (In other words, if and only if F is smooth as a map into M.)

For the following proposition, recall that a subset U of a manifold is open if and only if for all $p \in U$, and any coordinate chart (V, ψ) around p, the subset $\psi(U \cap V) \subseteq \mathbb{R}^m$ is open. (This does not depend on the choice of chart.)

Proposition 3.6. *Suppose S is a submanifold of M. Then the open subsets of S for its manifold structure are exactly those of the form $U \cap S$, where U is an open subset of M.*

In other words, the topology of S as a manifold coincides with the 'subspace topology' as a subset of the manifold M.

Proof. We have to show:

$$U' \subseteq S \quad \text{is open} \quad \Leftrightarrow \quad U' = U \cap S \quad \text{where} \quad U \subseteq M \quad \text{is open}.$$

"$\Rightarrow$". Suppose $U \subseteq M$ is open, and let $U' = U \cap S$. For any submanifold chart (V, ψ), with corresponding chart $(V \cap S, \psi|_{V \cap S})$ for S, we have that

$$\psi((V \cap S) \cap U') = \psi(V \cap S \cap U) = \psi(U) \cap \psi(V) \cap \mathbb{R}^k$$

is the intersection of the open set $\psi(U) \cap \psi(V) \subseteq \mathbb{R}^n$ with the subspace $\mathbb{R}^k$, hence is open in $\mathbb{R}^k$. Since submanifold charts cover all of S, this shows that U' is open.

"$\Leftarrow$" Suppose $U' \subseteq S$ is open in S. Define

$$U = \bigcup_V \psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k}) \subseteq M,$$

where the union is over any collection of submanifold charts (V, ψ) that cover all of S. Since U' is open in S, so is $U' \cap V \equiv U' \cap (V \cap S)$. Hence $\psi(U' \cap V) = \psi(U' \cap (V \cap S))$ is open in $\mathbb{R}^k$, and its cartesian product with $\mathbb{R}^{m-k}$ is open in $\mathbb{R}^m$. The pre-image $\psi^{-1}(\psi(U' \cap V) \times \mathbb{R}^{m-k})$ is thus open in V, hence also in M, and the union over all such sets is open in M. Since $U \cap S = U'$ (see exercise below) we are done. $\square$

Exercise: Fill in the last detail of this proof: Check that $U \cap S = U'$.

Remark 3.6. As a consequence, if a manifold M can be realized realized as a submanifold $M \subseteq \mathbb{R}^n$, then M is compact with respect to its manifold topology if and only if it is compact as a subset of $\mathbb{R}^n$, if and only if it is a closed and bounded subset of $\mathbb{R}^n$. This can be used to give quick proofs of the facts that the real or complex projective spaces, as well as the real or complex Grassmannians, are all compact.

Remark 3.7. Sometimes, the result can be used to show that certain subsets are *not* submanifolds. Consider for example the subset

$$S = \{(x,y) \in \mathbb{R}^2 | \ xy = 0\} \subseteq \mathbb{R}^2$$

given as the union of the coordinate axes. If S were a 1-dimensional submanifold, then there would exist an open neighborhood U' of $p = (0,0)$ in S which is diffeomorphic to an open interval. But for any open subset $U \subseteq \mathbb{R}^2$ containing p, the intersection $U' = U \cap S$ cannot possibly be an open interval, since $(U \cap S) \backslash \{p\}$ has at least four components, while removing a point from an open interval gives only two components.

3.5 Smooth maps of maximal rank

Let $F \in C^\infty(M,N)$ be a smooth map. Then the fibers (level sets)

$$F^{-1}(q) = \{x \in M | \ F(x) = q\}$$

for $q \in N$ need not be submanifolds, in general. Similarly, the image $F(M) \subseteq N$ need not be a submanifold – even if we allow self-intersections. (More precisely, there may be points p such that the image $F(U) \subseteq N$ of any open neighborhood U of p is never a submanifold.) Here are some counter-examples:

1. The fibers $f^{-1}(c)$ of the map $f(x,y) = xy$ are hyperbolas for $c \neq 0$, but $f^{-1}(0)$ is the union of coordinate axes. What makes this possible is that the gradient of f is zero at the origin.
2. As we mentioned earlier, the image of the smooth map

$$\gamma : \ \mathbb{R} \to \mathbb{R}^2, \ \ \gamma(t) = (t^2, t^3)$$

does not look smooth near $(0,0)$ (and replacing $\mathbb{R}$ by an open interval around 0 does not help). [4] What makes this is possible is that the velocity $\dot{\gamma}(t)$ vanishes for $t = 0$: the curve described by γ 'comes to a halt' at $t = 0$, and then turns around.

In both cases, the problems arise at points where the map does not have maximal rank. After reviewing the notion of rank of a map from multivariable calculus, we will generalize to manifolds.

[4] It is not a submanifold, although we haven't proved it (yet).

3.5.1 The rank of a smooth map

The following discussion will involve some notions from multivariable calculus. Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open subsets, and $F \in C^\infty(U,V)$ a smooth map.

Definition 3.5. The *derivative* of F at $p \in U$ is the linear map

$$D_pF : \mathbb{R}^m \to \mathbb{R}^n,\;\; v \mapsto \frac{d}{dt}\Big|_{t=0} F(p+tv).$$

The *rank of F at p* is the rank of this linear map:

$$\mathrm{rank}_p(F) = \mathrm{rank}(D_pF).$$

(Recall that the rank of a linear map is the dimension of its range.) Equivalently, D_pF is the $n \times m$ matrix of partial derivatives $(D_pF)^i_j = \frac{\partial F^i}{\partial x^j}\Big|_p$:

$$D_pF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}\big|_p & \frac{\partial F^1}{\partial x^2}\big|_p & \cdots & \frac{\partial F^1}{\partial x^m}\big|_p \\ \frac{\partial F^2}{\partial x^1}\big|_p & \frac{\partial F^2}{\partial x^2}\big|_p & \cdots & \frac{\partial F^2}{\partial x^m}\big|_p \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F^n}{\partial x^1}\big|_p & \frac{\partial F^n}{\partial x^2}\big|_p & \cdots & \frac{\partial F^n}{\partial x^m}\big|_p \end{pmatrix}$$

and the rank of F at p is the rank of this matrix (i.e., the number of linearly independent rows, or equivalently the number of linearly independent columns). Note $\mathrm{rank}_p(F) \le \min(m,n)$. By the chain rule for differentiation, the derivative of a composition of two smooth maps satisfies

$$D_p(F' \circ F) = D_{F(p)}(F') \circ D_p(F). \tag{3.10}$$

In particular, if F' is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_p(F)$, and if F is a diffeomorphism then $\mathrm{rank}_p(F' \circ F) = \mathrm{rank}_{F(p)}(F')$.

Definition 3.6. Let $F \in C^\infty(M,N)$ be a smooth map between manifolds, and $p \in M$. The *rank* of F at $p \in M$ is defined as

$$\mathrm{rank}_p(F) = \mathrm{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1})$$

for any two coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$.

By (3.10), this is well-defined: if we use different charts (U',φ') and (V',ψ'), then the rank of

$$\psi' \circ F \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$$

at $\varphi'(p)$ equals that of $\psi \circ F \circ \varphi^{-1}$ at $\varphi(p)$, since the two maps are related by diffeomorphisms.

The following discussion will focus on maps of *maximal rank*. We have that

$$\mathrm{rank}_p(F) \leq \min(\dim M,\ \dim N)$$

for all $p \in M$; the map F is said to have *maximal rank* at p if $\mathrm{rank}_p(F) = \min(\dim M,\ \dim N)$. A point $p \in M$ is called a *critical point* for F if $\mathrm{rank}_p(F) < \min(\dim M,\ \dim N)$.

3.5.2 Local diffeomorphisms

In this section we will consider the case $\dim M = \dim N$. Our 'workhorse theorem' from multivariable calculus is going to be the following fact.

Theorem 3.1 (Inverse Function Theorem for $\mathbb{R}^m$). *Let $F \in C^\infty(U,V)$ be a smooth map between open subsets of $\mathbb{R}^m$, and suppose that the derivative D_pF at $p \in U$ is invertible. Then there exists an open neighborhood $U_1 \subseteq U$ of p such that F restricts to a diffeomorphism $U_1 \to F(U_1)$.*

Remark 3.8. The theorem tells us that for a smooth bijection, a sufficient condition for smoothness of the inverse map is that the differential (i.e., the *first* derivative) is invertible everywhere. It is good to see, in just one dimensions, how this is possible. Given an invertible smooth function $y = f(x)$, with inverse $x = g(y)$, and using $\frac{\mathrm{d}}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}x}$, we have

$$\begin{aligned} g'(y) &= \frac{1}{f'(x)}, \\ g''(y) &= = \frac{-f''(x)}{f'(x)^3}, \\ g'''(y) &= = \frac{-f'''(x)}{f'(x)^4} + 3\frac{f''(x)^2}{f'(x)^5}, \end{aligned}$$

and so on; only powers of $f'(x)$ appear in the denominator.

Theorem 3.2 (Inverse function theorem for manifolds). *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of the same dimension $m = n$. If $p \in M$ is such that* $\mathrm{rank}_p(F) = m$*, then there exists an open neighborhood $U \subseteq M$ of p such that F restricts to a diffeomorphism $U \to F(U)$.*

Proof. Choose charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$. The map

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \ \widetilde{U} := \varphi(U) \to \widetilde{V} := \psi(V)$$

has rank m at $\varphi(p)$. Hence, by the inverse function theorem for $\mathbb{R}^m$, after replacing $\widetilde{U}$ with a smaller open neighborhood of $\varphi(p)$ (equivalently, replacing U with a smaller open neighborhood of p) the map $\widetilde{F}$ becomes a diffeomorphism from $\widetilde{U}$ onto $\widetilde{F}(\widetilde{U}) = \psi(F(U))$. It then follows that

$$F = \psi^{-1} \circ \widetilde{F} \circ \varphi : \ U \to V$$

is a diffeomorphism $U \to F(U)$. $\square$

A smooth map $F \in C^\infty(M,N)$ is called a *local diffeomorphism* if $\dim M = \dim N$, and F has maximal rank everywhere. By the theorem, this is equivalent to the condition that every point p has an open neighborhood U such that F restricts to a diffeomorphism $U \to F(U)$. It depends on the map in question which of these two conditions is easier to verify.

Example 3.14. The quotient map $\pi : \ S^n \to \mathbb{R}\mathrm{P}^n$ is a local diffeomorphism. Indeed, one can see (using suitable coordinates) that π restricts to diffeomorphisms from each $U_j^\pm = \{x \in S^n | \ \pm x^j > 0\}$ to the standard chart U_j.

Example 3.15. The map $\mathbb{R} \to S^1, \ t \mapsto (\cos(2\pi t), \ \sin(2\pi t))$ is a local diffeomorphism. (Exercise.)

Example 3.16. Let M be a manifold with a countable open cover $\{U_\alpha\}$, and let

$$Q = \bigsqcup_\alpha U_\alpha$$

be the *disjoint* union. Then the map $\pi : \ Q \to M$, given on $U_\alpha \subseteq Q$ by the inclusion into M, is a local diffeomorphism. Since π is surjective, it determines an equivalence relation on Q, with π as the quotient map and $M = Q/\sim$.

We leave it as an exercise to show that if the U_α's are the domains of coordinate charts, then Q is diffeomorphic to an open subset of $\mathbb{R}^m$. This then shows that any manifold is realized as a quotient of an open subset of $\mathbb{R}^m$, in such a way that the quotient map is a local diffeomorphism.

3.5.3 Level sets, submersions

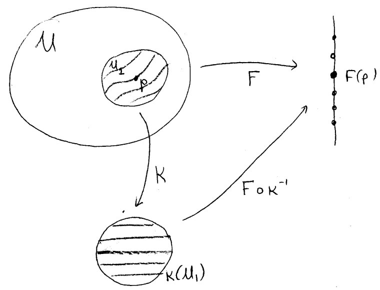
The inverse function theorem is closely related to the *implicit function theorem*, and one may be obtained as a consequence of the other. (We have chosen to take the inverse function theorem as our starting point.)

Proposition 3.7. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is surjective. Then there exists an open neighborhood $U_1 \subseteq U$ of p and a diffeomorphism $\kappa : U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$ such that*

$$(F \circ \kappa^{-1})(u^1, \ldots, u^m) = (u^{m-n+1}, \ldots, u^m)$$

for all $u = (u^1, \ldots, u^m) \in \kappa(U_1)$.

Thus, in suitable coordinates F is given by a projection onto the last n coordinates.



Although it belongs to multivariable calculus, let us recall how to get this result from the inverse function theorem.

Proof. The idea is to extend F to a map between open subsets of $\mathbb{R}^m$, and then apply the inverse function theorem.

By assumption, the derivative D_pF has rank equal to n. Hence it has n linearly independent columns. By re-indexing the coordinates of $\mathbb{R}^m$ (this permutation is itself a change of coordinates) we may assume that these are the last n columns. That is, writing

$$D_pF = \Big(C, D\Big)$$

where C is the $n \times (m-n)$-matrix formed by the first $m-n$ columns and D the $n \times n$-matrix formed by the last n columns, the square matrix D is invertible. Write elements $x \in \mathbb{R}^m$ in the form $x = (x', x'')$ where x' are the first $m-n$ coordinates and x'' the last n coordinates. Let

$$G\colon\ U \to \mathbb{R}^m,\ x = (x', x'') \mapsto (x', F(x)).$$

Then the derivative D_pG has block form

$$D_pG = \begin{pmatrix} I_{m-n} & 0 \\ C & D \end{pmatrix},$$

(where I_{m-n} is the square $(m-n) \times (m-n)$ matrix), and is hence is invertible. Hence, by the inverse function theorem there exists a smaller open neighborhood U_1 of p such that G restricts to a diffeomorphism $\kappa\colon\ U_1 \to \kappa(U_1) \subseteq \mathbb{R}^m$. We have,

$$G \circ \kappa^{-1}(u', u'') = (u', u'')$$

for all $(u', u'') \in \kappa(U_1)$. Since F is just G followed by projection to the x'' component, we conclude

$$F \circ \kappa^{-1}(u', u'') = u''.$$

$\square$

Again, this result has a version for manifolds:

Theorem 3.3. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m \geq n$, and suppose $p \in M$ is such that $\mathrm{rank}_p(F) = n$. Then there exist coordinate charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$, such that*

$$(\psi \circ F \circ \varphi^{-1})(u',u'') = u''$$

for all $u = (u',u'') \in \varphi(U)$. In particular, for all $q \in V$ the intersection

$$F^{-1}(q) \cap U$$

is a submanifold of dimension $m-n$.

Proof. Start with coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$. Apply Proposition 3.7 to the map $\widetilde{F} = \psi \circ F \circ \varphi^{-1}: \ \varphi(U) \to \psi(V)$, to define a smaller neighborhood $\varphi(U_1) \subseteq \varphi(U)$ and change of coordinates κ so that $\widetilde{F} \circ \kappa^{-1}(u',u'') = u''$. After renaming $(U_1, \kappa \circ \varphi|_{U_1})$ as (U,φ) we have the desired charts for F. The last part of the Theorem follows since (U,φ) becomes a submanifold chart for $F^{-1}(q) \cap U$ (after shifting φ by $\psi(q) \in \mathbb{R}^n$). $\square$

Definition 3.7. Let $F \in C^\infty(M,N)$. A point $q \in N$ is called a *regular value* of $F \in C^\infty(M,N)$ if for all $x \in F^{-1}(q)$, one has $\mathrm{rank}_x(F) = \dim N$. It is called a *singular value* if it is not a regular value.

Note that regular values are only possible if $\dim N \leq \dim M$. Note also that all points of N that are not in the image of the map F are considered regular values. We may restate Theorem 3.3 as follows:

Theorem 3.4 (Regular Value Theorem). *For any regular value $q \in N$ of a smooth map $F \in C^\infty(M,N)$, the level set $S = F^{-1}(q)$ is a submanifold of dimension*

$$\dim S = \dim M - \dim N.$$

Example 3.17. The n-sphere S^n may be defined as the level set $F^{-1}(1)$ of the function $F \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ given by

$$F(x^0, \ldots, x^n) = (x^0)^2 + \ldots + (x^n)^2.$$

The derivative of F is the $1 \times (n+1)$-matrix of partial derivatives, that is, the gradient ∇F:

$$D_p F = (2x^0, \ldots, 2x^n).$$

For $x \neq 0$ this has maximal rank. Note that any nonzero real number q is a regular value since $0 \notin F^{-1}(q)$. Hence all the level sets $F^{-1}(q)$ for $q \neq 0$ are submanifolds.

Example 3.18. Let $0 < r < R$. Then

$$F(x,y,z) = (\sqrt{x^2+y^2} - R)^2 + z^2$$

has r^2 as a regular value, with corresponding level set the 2-torus.

Example 3.19. The *orthogonal group* $\mathrm{O}(n)$ is the group of matrices $A \in \mathrm{Mat}_{\mathbb{R}}(n)$ satisfying $A^\top = A^{-1}$. We claim that $\mathrm{O}(n)$ is a submanifold of $\mathrm{Mat}_{\mathbb{R}}(n)$. To see this, consider the map

$$F:\ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Sym}_{\mathbb{R}}(n),\ \ A \mapsto A^\top A,$$

where $\mathrm{Sym}_{\mathbb{R}}(n) \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$ denotes the subspace of symmetric matrices. We want to show that the identity matrix I is a regular value of F. We compute the differential $D_AF:\ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Sym}_{\mathbb{R}}(n)$ using the definition[5]

$$\begin{aligned}(D_AF)(X) &= \frac{d}{dt}\Big|_{t=0} F(A+tX)\\ &= \frac{d}{dt}\Big|_{t=0} ((A^\top + tX^\top)(A+tX))\\ &= A^\top X + X^\top A.\end{aligned}$$

To see that this is surjective, for $A \in F^{-1}(I)$, we need to show that for any $Y \in \mathrm{Sym}_{\mathbb{R}}(n)$ there exists a solution of

$$A^\top X + X^\top A = Y.$$

Using $A^\top A = F(A) = I$ we see that $X = \frac{1}{2}AY$ is a solution. We conclude that I is a regular value, and hence that $\mathrm{O}(n) = F^{-1}(I)$ is a submanifold. Its dimension is

$$\dim \mathrm{O}(n) = \dim \mathrm{Mat}_{\mathbb{R}}(n) - \dim \mathrm{Sym}_{\mathbb{R}}(n) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1).$$

Note that it was important here to regard F as a map to $\mathrm{Sym}_{\mathbb{R}}(n)$; for F viewed as a map to $\mathrm{Mat}_{\mathbb{R}}(n)$ the identity would *not* be a regular value.

Definition 3.8. A smooth map $F \in C^\infty(M,N)$ is a *submersion* if $\mathrm{rank}_p(F) = \dim N$ for all $p \in M$.

Thus, for a submersion *all* level sets $F^{-1}(q)$ are submanifolds.

Example 3.20. Local diffeomorphisms are submersions; here the level sets $F^{-1}(q)$ are discrete points, i.e. 0-dimensional manifolds.

Example 3.21. Recall that $\mathbb{C}\mathrm{P}^n$ can be regarded as a quotient of S^{2n+1}. Using charts, one can check that the quotient map $\pi:\ S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a submersion. Hence its fibers $\pi^{-1}(q)$ are 1-dimensional submanifolds. Indeed, as discussed before these fibers are circles. As a special case, the Hopf fibration $S^3 \to S^2$ is a submersion. As a special case, the Hopf fibration $S^3 \to S^2$ is a submersion.

Remark 3.9. (For those who are familiar with quaternions.) Let $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$ be the quaternionic numbers. The unit quaternions are a 3-sphere S^3. Generalizing the

[5] Note that it would have been confusing to work with the description of D_AF as a matrix of partial derivatives.

definition of $\mathbb{R}\mathrm{P}^n$ and $C\mathrm{P}^n$, there are also quaternionic projective spaces, $\mathbb{H}\mathrm{P}^n$. These are quotients of the unit sphere inside $\mathbb{H}^{n+1}$, hence one obtains submersions

$$S^{4n+3} \to \mathbb{H}\mathrm{P}^n;$$

the fibers of this submersion are diffeomorphic to S^3. For $n = 1$, one can show that $\mathbb{H}\mathrm{P}^1 = S^4$, hence one obtains a submersion

$$\pi:\ S^7 \to S^4$$

with fibers diffeomorphic to S^3.

3.5.4 Example: The Steiner surface

In this section, we will give more lengthy examples, investigating the smoothness of level sets. [6]

Example 3.22 (Steiner's surface). Let $S \subseteq \mathbb{R}^3$ be the solution set of

$$y^2z^2 + x^2z^2 + x^2y^2 = xyz.$$

in $\mathbb{R}^3$. Is this a smooth surface in $\mathbb{R}^3$? (We use *surface* as another term for *2-dimensional manifold*; by a *surface in M* we mean a 2-dimensional submanifold.) Actually, we can easily see that it's *not*. If we take one of x, y, z equal to 0, then the equation holds if and only if one of the other two coordinates is 0. Hence, the intersection of S with the set where $xyz = 0$ (the union of the coordinate hyperplanes) is the union of the three coordinate axes.

Hence, let us rephrase the question: Letting $U \subseteq \mathbb{R}^3$ be the subset where $xyz \neq 0$, is $S \cap U$ is surface? To investigate the problem, consider the function

$$f(x,y,z) = y^2z^2 + x^2z^2 + x^2y^2 - xyz.$$

The differential (which in this case is the same as the gradient) is the 1×3-matrix

$$D_{(x,y,z)}f = \begin{pmatrix} 2x(y^2+z^2) - yz & 2y(z^2+x^2) - zx & 2z(x^2+y^2) - xy \end{pmatrix}$$

This vanishes if and only if all three entries are zero. Vanishing of the first entry gives, after dividing by $2xy^2z^2$, the condition

$$\frac{1}{z^2} + \frac{1}{y^2} = \frac{1}{2xyz};$$

we get similar conditions after cyclic permutation of x, y, z. Thus we have

[6] We won't cover this example in class, for lack of time, but you're encouraged to read it.

$$\frac{1}{z^2}+\frac{1}{y^2}=\frac{1}{x^2}+\frac{1}{z^2}=\frac{1}{y^2}+\frac{1}{x^2}=\frac{1}{2xyz},$$

with a unique solution $x=y=z=\frac{1}{4}$. Thus, $D_{(x,y,z)}f$ has maximal rank (i.e., it is nonzero) except at this point. But this point doesn't lie on S. We conclude that $S\cap U$ is a submanifold. How does it look like? It turns out that there is a nice answer. First, let's divide the equation for $S\cap U$ by xyz. The equation takes on the form

$$xyz(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2})=1. \tag{3.11}$$

The solution set of (3.11) is contaned in the set of all (x,y,z) such that $xyz>0$. On this subset, we introduce new variables

$$\alpha=\frac{\sqrt{xyz}}{x},\ \ \beta=\frac{\sqrt{xyz}}{y},\ \ \gamma=\frac{\sqrt{xyz}}{z};$$

the old variables x,y,z are recovered as

$$x=\beta\gamma,\ \ y=\alpha\gamma,\ \ z=\alpha\beta.$$

In terms of α,β,γ, Equation (3.11) becomes the equation $\alpha^2+\beta^2+\gamma^2=1$. Actually, it is even better to consider the corresponding points

$$(\alpha:\beta:\gamma)=(\frac{1}{x}:\ \frac{1}{y}:\ \frac{1}{z})\in\mathbb{R}\mathrm{P}^2,$$

because we could take either square root of xyz (changing the sign of all α,β,γ doesn't affect x,y,z). We conclude that the map $U\to\mathbb{R}\mathrm{P}^2,\ (x,y,z)\mapsto(\frac{1}{x}:\ \frac{1}{y}:\ \frac{1}{z})$ restricts to a diffeomorphism from $S\cap U$ onto

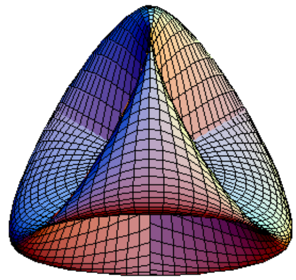
$$\mathbb{R}\mathrm{P}^2\backslash\{(\alpha:\beta:\gamma)|\ \alpha\beta\gamma=0\}.$$

The image of the map

$$\mathbb{R}\mathrm{P}^2\to\mathbb{R}^3,\ \ (\alpha:\beta:\gamma)\mapsto\frac{1}{|\alpha|^2+|\beta|^2+|\gamma|^2}(\beta\gamma,\alpha\beta,\alpha\gamma).$$

is called *Steiner's surface*, even though it is not a submanifold (not even an *immersed* submanifold). Here is a picture: [7]

[7] Source: `http://upload.wikimedia.org/wikipedia/commons/7/7a/RomanSurfaceFrontalView.PNG`



Note that the subset of $\mathbb{R}\mathrm{P}^2$ defined by $\alpha\beta\gamma = 0$ is a union of three $\mathbb{R}\mathrm{P}^1 \cong S^1$, each of which maps into a coordinate axis (but not the entire coordinate axis). For example, the circle defined by $\alpha = 0$ maps to the set of all $(0,0,z)$ with $-\frac{1}{2} \le z \le \frac{1}{2}$. In any case, S is the Steiner surface together with the three coordinate axes. See `http://www.math.rutgers.edu/courses/535/535-f02/pictures/romancon.jpg` for a very nice picture.

Example 3.23. Let $S \subseteq \mathbb{R}^4$ be the solution set of

$$y^2x^2 + x^2z^2 + x^2y^2 = xyz, \quad y^2x^2 + 2x^2z^2 + 3x^2y^2 = xyzw.$$

Again, this cannot quite be a surface because it contains the coordinate axes for x,y,z. Closer investigation shows that S is the union of the three coordinate axes, together with the image of an injective map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4, \ (\alpha : \beta : \gamma) \mapsto \frac{1}{\alpha^2+\beta^2+\gamma^2}(\beta\gamma, \alpha\beta, \alpha\gamma, \alpha^2 + 2\beta^2 + 3\gamma^2).$$

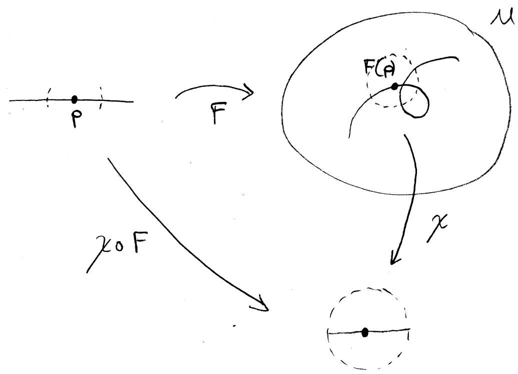
It turns out (see Section 4.2.4 below) that the latter is a submanifold, which realizes $\mathbb{R}\mathrm{P}^2$ as a surface in $\mathbb{R}^4$.

3.5.5 Immersions

We next consider maps $F: M \to N$ of maximal rank between manifolds of dimensions $m \le n$. Once again, such a map can be put into a 'normal form': By choosing suitable coordinates it becomes linear.

Proposition 3.8. *Suppose $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, and suppose $p \in U$ is such that the derivative D_pF is injective. Then there exist smaller neighborhoods $U_1 \subseteq U$ of p and $V_1 \subseteq V$ of $F(p)$, with $F(U_1) \subseteq V_1$, and a diffeomorphism $\chi: V_1 \to \chi(V_1)$, such that*

$$(\chi \circ F)(u) = (u,0) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$



Proof. Since D_pF is injective, it has m linearly independent rows. By re-indexing the rows (which amounts to a change of coordinates on V), we may assume that these are the first m rows.

That is, writing

$$D_pF = \begin{pmatrix} A \\ C \end{pmatrix}$$

where A is the $m \times m$-matrix formed by the first m rows and C is the $(n-m) \times m$-matrix formed by the last $n-m$ rows, the square matrix A is invertible. Consider the map

$$H: \ U \times \mathbb{R}^{n-m} \to \mathbb{R}^n, \ (x,y) \mapsto F(x) + (0,y)$$

Then

$$D_{(p,0)}H = \begin{pmatrix} A & 0 \\ C & I_{n-m} \end{pmatrix}$$

is invertible. Hence, by the inverse function theorem for $\mathbb{R}^n$, H is a diffeomorphism from some neighborhood of $(p,0)$ in $U \times \mathbb{R}^{n-m}$ onto some neighborhood V_1 of $H(p,0) = F(p)$, which we may take to be contained in V. Let

$$\chi: \ V_1 \to \chi(V_1) \subseteq U \times \mathbb{R}^{n-m}$$

be the inverse; thus

$$(\chi \circ H)(x,y) = (x,y)$$

for all $(x,y) \in \chi(V_1)$. Replace U with the smaller open neighborhood

$$U_1 = F^{-1}(V_1) \cap U$$

of p. Then $F(U_1) \subseteq V_1$, and

$$(\chi \circ F)(u) = (\chi \circ H)(u,0) = (u,0)$$

for all $u \in U_1$. $\square$

The manifolds version reads as follows:

Theorem 3.5. *Let $F \in C^\infty(M,N)$ be a smooth map between manifolds of dimensions $m \le n$, and $p \in M$ a point with $\mathrm{rank}_p(F) = m$. Then there are coordinate charts (U,φ) around p and (V,ψ) around $F(p)$ such that $F(U) \subseteq V$ and*

$$(\psi \circ F \circ \varphi^{-1})(u) = (u,0).$$

In particular, $F(U) \subseteq N$ is a submanifold of dimension m.

Proof. Once again, this is proved by introducing charts around p, $F(p)$ to reduce to a map between open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, and then use the multivariable version of the result to obtain a change of coordinates, putting the map into normal form. $\square$

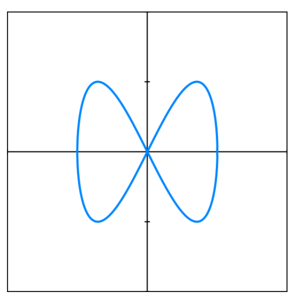
Definition 3.9. A smooth map $F\colon M \to N$ is an *immersion* if $\mathrm{rank}_p(F) = \dim M$ for all $p \in M$.

Example 3.24. Let $J \subseteq \mathbb{R}$ be an open interval. A smooth map $\gamma\colon J \to M$ is also called a *smooth curve*. We see that the image of γ is an immersed submanifold, provided that $\mathrm{rank}_p(\gamma) = 1$ for all $p \in M$. In local coordinates (U,φ), this means that $\frac{d}{dt}(\varphi \circ \gamma)(t) \neq 0$ for all t with $\gamma(t) \in U$. For example, the curve $\gamma(t) = (t^2, t^3)$ fails to have this property at $t = 0$.

Example 3.25 (Figure eight). The map

$$\gamma\colon\ \mathbb{R} \to \mathbb{R}^2,\ \ t \mapsto \big(\sin(t), \sin(2t)\big)$$

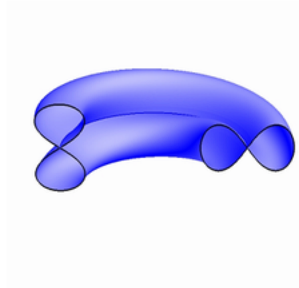
is an immersion; the image is a figure eight.



(Indeed, for all $t \in \mathbb{R}$ we have $D_t\gamma \equiv \dot{\gamma}(t) \neq 0$.)

Example 3.26 (Immersion of the Klein bottle). The Klein bottle admits a ‘figure eight’ immersion into $\mathbb{R}^3$, obtained by taking the figure eight in the $x-z$-plane, moving in the x-direction by $R > 1$, and then rotating about the z-axis while at the same time rotating the figure eight, so that after a full turn $\varphi \mapsto \varphi + 2\pi$ the figure eight has performed a half turn. [8]

[8] Picture source: `http://en.wikipedia.org/wiki/Klein_bottle`



We can regard this procedure as a composition of the following maps:

$$\begin{aligned}
F_1 :\ & (t,\varphi) \mapsto (\sin(t), \sin(2t), \varphi) = (u,v,\varphi),\\
F_2 :\ & (u,v,\varphi) \mapsto \Big(u\cos(\frac{\varphi}{2}) + v\sin(\frac{\varphi}{2}),\ v\cos(\frac{\varphi}{2}) - u\sin(\frac{\varphi}{2}), \varphi\Big) = (a,b,\varphi)\\
F_3 :\ & \big(a,b,\varphi\big) \mapsto ((a+R)\cos\varphi,\ (a+R)\sin\varphi,\ b\big) = (x,y,z).
\end{aligned}$$

Here F_1 is the figure eight in the $u-v$-plane (with φ just a bystander). F_2 rotates the $u-v$-plane as it moves in the direction of φ, by an angle of $\varphi/2$; thus $\varphi = 2\pi$ corresponds to a half-turn. The map F_3 takes this family of rotating u,v-planes, and wraps it around the circle in the $x-y$-plane of radius R, with φ now playing the role of the angular coordinate.

The resulting map $F = F_3 \circ F_2 \circ F_1 :\ \mathbb{R}^2 \to \mathbb{R}^3$ is given by $F(t,\varphi) = (x,y,z)$, where with

$$\begin{aligned}
x &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\cos\varphi,\\
y &= \Big(R + \cos(\frac{\varphi}{2})\sin(t) + \sin(\frac{\varphi}{2})\sin(2t)\Big)\sin\varphi,\\
z &= \cos(\frac{\varphi}{2})\sin(2t) - \sin(\frac{\varphi}{2})\sin(t)
\end{aligned}$$

is an immersion. To verify that this is an immersion, it would be cumbersome to work out the Jacobian matrix directly. It is much easier to use that F is obtained as a composition $F = F_3 \circ F_2 \circ F_1$ of the three maps considered above, where F_1 is an immersion, F_2 is a diffeomorphism, and F_3 is a local diffeomorphism from the open subset where $|a| < R$ onto its image.

Since the right hand side of the equation for F does not change under the transformations

$$(t,\varphi) \mapsto (t + 2\pi, \varphi), \quad (t,\varphi) \mapsto (-t, \varphi + 2\pi),$$

this descends to an immersion of the Klein bottle. It is straightforward to check that this immersion of the Klein bottle is injective, except over the 'central circle' corresponding to $t = 0$, where it is 2-to-1.

Note that under the above construction, any point of the figure eight creates a circle after two'full turns', $\varphi \mapsto \varphi + 4\pi$. The complement of the circle generated by the point $t = \pi/2$ consists of two subsets of the Klein bottle, generated by the parts of the figure eight defined by $-\pi/2 < t < \pi/2$, and by $\pi/2 < t < 3\pi/2$. Each of these is a 'curled-up' immersion of an open Möbius strip. (Remember, it is possible to remove a circle from the Klein bottle to create two Möbius strips!) The point $t = 0$ also creates a circle; its complement is the subset of the Klein bottle generated by $0 < t < 2\pi$. (Remember, it is possible to remove a circle from a Klein bottle to create one Möbius strip.) We can also remove one copy of the figure eight itself; then the 'rotation' no longer matters and the complement is an open cylinder. (Remember, it is possible to remove a circle from a Klein bottle to create a cylinder.)

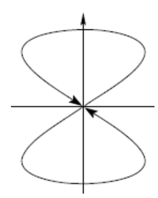
Example 3.27. Let M be a manifold, and $S \subseteq M$ a k-dimensional submanifold. Then the inclusion map $\iota : S \to M, \; x \mapsto x$ is an immersion. Indeed, if (V, ψ) is a submanifold chart for S, with $p \in U = V \cap S, \; \varphi = \psi|_{V \cap S}$ then

$$(\psi \circ F \circ \varphi^{-1})(u) = (u, 0),$$

which shows that

$$\mathrm{rank}_p(F) = \mathrm{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1}) = k.$$

By an *embedding*, we will mean an immersion given as the inclusion map for a submanifold. Not every injective immersion is an embedding; the following picture gives a counter-example:

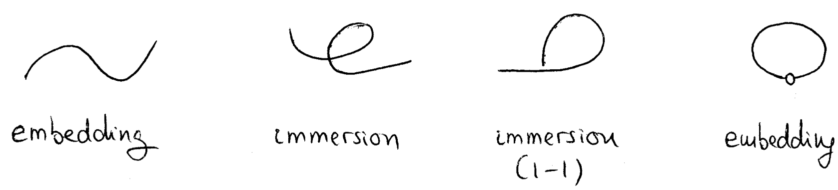


In practice, showing that an injective smooth map is an immersion tends to be easier than proving that its image is a submanifold. Fortunately, for compact manifolds we have the following fact:

Theorem 3.6. *If M is a* compact *manifold, then every injective immersion $F : M \to N$ is an embedding as a submanifold $S = F(M)$.*

Proof. Let $p \in M$ be given. By Theorem 3.5, we can find charts (U, φ) around p and (V, ψ) around $F(p)$, with $F(U) \subseteq V$, such that $\widetilde{F} = \psi \circ F \circ \varphi^{-1}$ is in normal form: i.e., $\widetilde{F}(u) = (u, 0)$. We would like to take (V, ψ) as a submanifold chart for $S = F(M)$, but this may not work yet since $F(M) \cap V = S \cap V$ may be strictly larger than $F(U) \cap V$. Note however that $A := M \backslash U$ is compact, hence its image $F(A)$ is compact, and therefore closed (here we are using that N is Haudorff). Since F is injective, we have that $p \notin F(A)$. Replace V with the smaller open neighborhood $V_1 = V \backslash (V \cap F(A))$. Then $(V_1, \psi|_{V_1})$ is the desired submanifold chart.

Remark 3.10. Some authors refer to injective immersions $\iota : S \to M$ as 'submanifolds' (thus, a submanifold is taken to be a map rather than a subset). To clarify, 'our' submanifolds are sometimes called 'embedded submanifolds' or 'regular submanifolds'.



Example 3.28. Let A,B,C be distinct real numbers. We will leave it as a homework problem to verify that the map

$$F:\ \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4,\ \ (\alpha:\beta:\gamma) \mapsto (\beta\gamma, \alpha\gamma, \alpha\beta,\ A\alpha^2 + B\beta^2 + C\gamma^2),$$

where we use representatives (α,β,γ) such that $\alpha^2+\beta^2+\gamma^2=1$, is an injective immersion. Hence, by Theorem 3.6, it is an embedding of $\mathbb{R}\mathrm{P}^2$ as a submanifold of $\mathbb{R}^4$.

To summarize the outcome from the last few sections: If $F \in C^\infty(M,N)$ has maximal rank near $p \in M$, then one can always choose local coordinates around p and around $F(p)$ such that the coordinate expression of F becomes a linear map of maximal rank. (This simple statement contains the inverse and implicit function theorems from multivariable calculus are special cases.)

Remark 3.11. This generalizes further to maps of *constant rank*. In fact, if $\mathrm{rank}_p(F)$ is independent of p on some open subset U, then for all $p \in U$ one can choose coordinates in which F becomes linear.

3.6 Appendix: Algebras

An *algebra* (over the field $\mathbb{R}$ of real numbers) is a vector space $\mathscr{A}$, together with a *multiplication* (product) $\mathscr{A} \times \mathscr{A} \to \mathscr{A},\ (a,b) \mapsto ab$ such that

1. The multiplication is associative: That is, for all $a,b,c \in \mathscr{A}$

$$(ab)c = a(bc).$$

2. The multiplication map is linear in both arguments: That is,

$$(\lambda_1 a_1 + \lambda_2 a_2)b = \lambda_1(a_1 b) + \lambda_2(a_2 b),$$

$$a(\mu_1 b_1 + \mu_2 b_2) = \mu_1(ab_1) + \mu_2(ab_2),$$

for all $a, a_1, a_2, b, b_1, b_2 \in \mathscr{A}$ and all scalars $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$.

The algebra is called *commutative* if $ab = ba$ for all $a, b \in \mathscr{A}$. A *unital algebra* is an algebra $\mathscr{A}$ with a distinguished element $1_{\mathscr{A}} \in \mathscr{A}$ (called the unit), with

$$1_{\mathscr{A}} a = a = a 1_{\mathscr{A}}$$

for all $a \in \mathscr{A}$.

Remark 3.12. One can also consider non-associative product operations on vector spaces, most importantly one has the class of *Lie algebras*. If there is risk of confusion with these or other concepts, we may refer to *associative algebras*.

For example, the space $\mathbb{C}$ of complex numbers (regarded as a real vector space $\mathbb{R}^2$) is a unital, commutative algebra. A more sophisticated example is the algebra $\mathbb{H} \cong \mathbb{R}^4$ of quaternions, which is a unital non-commutative algebra. (Recall that elements of $\mathbb{H}$ are expressions $x + iu + jv + kw$, where i, j, k have products $i^2 = j^2 = k^2 = -1, \;\; ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik$.) For any n, the space $\mathrm{Mat}_{\mathbb{R}}(n)$ of $n \times n$ matrices, with product the matrix multiplication, is a non-commutative unital algebra. One can also consider matrices with coefficients in $\mathbb{C}$, or in fact with coefficients in any given algebra. For any set X, the space of functions $f: \; X \to \mathbb{R}$ is a unital commutative algebra, where the product is given by pointwise multiplication. Given a topological space X, one has the algebra $C(X)$ of *continuous* $\mathbb{R}$-valued functions. A *homomorphism of algebras* $\Phi: \; \mathscr{A} \to \mathscr{A}'$ is a linear map preserving products: $\Phi(ab) = \Phi(a)\Phi(b)$. (For a homomorphism of unital algebras, one asks in addition that $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{A}'}$.) It is called an *isomorphism of algebras* if Φ is invertible. For the special case $\mathscr{A}' = \mathscr{A}$, these are also called *algebra automorphisms* of $\mathscr{A}$. Note that the algebra automorphisms form a group under composition.

Example 3.29. Consider $\mathbb{R}^2$ as an algebra, with product coming from the identification $\mathbb{R}^2 = \mathbb{C}$. The complex conjugation $z \mapsto \bar{z}$ defines an automorphism $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ of this algebra.

Example 3.30. The algebra $\mathbb{H}$ of quaternions has an automorphism given by cyclic permutation of the three imaginary units. That is, $\Phi(x + iu + jv + kw) = x + ju + kv + iw$

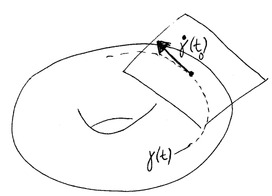
Example 3.31. Let $\mathscr{A} = \mathrm{Mat}_{\mathbb{R}}(n)$ the algebra of $n \times n$-matrices. If $U \in \mathscr{A}$ is invertible, then $X \mapsto \Phi(X) = UXU^{-1}$ is an algebra automorphism.

Example 3.32. Suppose $\mathscr{A}$ is a unital algebra. Let $\mathscr{A}^{\times}$ be the set of invertible elements, that is, elements $u \in \mathscr{A}$ for which there exists $v \in \mathscr{A}$ with $uv = vu = 1_{\mathscr{A}}$. Given u, such v is necessarily unique (write $v = u^{-1}$), and the map $\mathscr{A} \to \mathscr{A}, \; a \mapsto uau^{-1}$ is an algebra automorphism. Such automorphisms are called 'inner'.

Chapter 4
The tangent bundle

4.1 Tangent spaces

For embedded submanifolds $M \subseteq \mathbb{R}^n$, the tangent space T_pM at $p \in M$ can be defined as the set of all *velocity vectors* $v = \dot{\gamma}(0)$, where $\gamma\colon\ J \to M$ is a smooth curve with $\gamma(0) = p$; here $J \subseteq \mathbb{R}$ is an open interval around 0.



It turns out (not entirely obvious!) that T_pM becomes a vector subspace of $\mathbb{R}^n$. (Warning: In pictures we tend to draw the tangent space as an *affine* subspace, where the origin has been moved to p.)

Example 4.1. Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of x such that $||x||^2 = 1$. A curve $\gamma(t)$ lies in S^n if and only if $||\gamma(t)|| = 1$. Taking the derivative of the equation $\gamma(t) \cdot \gamma(t) = 1$ at $t = 0$, we obtain (after dividing by 2, and using $\gamma(0) = p$)

$$p \cdot \dot{\gamma}(0) = 0.$$

That is, T_pM consists of vectors $v \in \mathbb{R}^{n+1}$ that are orthogonal to $p \in \mathbb{R}^3 \backslash \{0\}$. It is not hard to see that every such vector v is of the form $\dot{\gamma}(0)$,[1] hence that

$$T_pS^n = (\mathbb{R}p)^\perp,$$

the hyperplane orthogonal to the line through p.

[1] Given v, take $\gamma(t) = (p + tv)/||p + tv||$.

To extend this idea to general manifolds, note that the vector $v = \dot{\gamma}(0)$ defines a "directional derivative" $C^\infty(M) \to \mathbb{R}$:

$$v : f \mapsto \tfrac{d}{dt}|_{t=0} f(\gamma(t)).$$

For a general manifold, we will define T_pM as a set of directional derivatives.

Definition 4.1 (Tangent spaces – first definition). Let M be a manifold, $p \in M$. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form

$$v(f) = \tfrac{d}{dt}|_{t=0} f(\gamma(t))$$

for some smooth curve $\gamma \in C^\infty(J, M)$ with $\gamma(0) = p$.

The elements $v \in T_pM$ are called the *tangent vectors* to M at p.

The following local coordinate description makes it clear that T_pM is a linear subspace of the vector space $L(C^\infty(M), \mathbb{R})$ of linear maps $C^\infty(M) \to \mathbb{R}$, of dimension equal to the dimension of M.

Theorem 4.1. *Let (U, φ) be a coordinate chart around p. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in T_pM if and only if it has the form,*

$$v(f) = \sum_{i=1}^m a^i \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)}$$

for some $a = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Proof. Given a linear map v of this form, let $\tilde{\gamma} : \mathbb{R} \to \varphi(U)$ be a curve with $\tilde{\gamma}(t) = \varphi(p) + ta$ for $|t|$ sufficiently small. Let $\gamma = \varphi^{-1} \circ \tilde{\gamma}$. Then

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\varphi(p) + ta) \\ &= \sum_{i=1}^m a^i \; \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)}, \end{aligned}$$

by the chain rule. Conversely, given any curve γ with $\gamma(0) = p$, let $\tilde{\gamma} = \varphi \circ \gamma$ be the corresponding curve in $\varphi(U)$ (defined for small $|t|$). Then $\widetilde{\gamma}(0) = \varphi(p)$, and

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) &= \frac{d}{dt}\Big|_{t=0} (f \circ \varphi^{-1})(\tilde{\gamma}(t)) \\ &= \sum_{i=1}^m a^i \; \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}|_{u=\gamma(p)}, \end{aligned}$$

where $a = \frac{d\tilde{\gamma}}{dt}\Big|_{t=0}$. ☐

We can use this result as an alternative definition of the tangent space, namely:

Definition 4.2 (Tangent spaces – second definition). Let (U, φ) be a chart around p. The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form

$$v(f) = \sum_{i=1}^{m} a^i \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{u=\varphi(p)} \tag{4.1}$$

for some $a = (a^1, \ldots, a^m) \in \mathbb{R}^m$.

Remark 4.1. From this version of the definition, it is immediate that T_pM is an m-dimensional vector space. It is not immediately obvious from this second definition that T_pM is independent of the choice of coordinate chart, but this follows from the equivalence with the first definition. Alternatively, one may check directly that the subspace of $L(C^\infty(M), \mathbb{R})$ characterized by (4.1) does not depend on the chart, by studying the effect of a change of coordinates.

According to (4.1), any choice of coordinate chart (U, φ) around p defines a vector space isomorphism $T_pM \cong \mathbb{R}^m$, taking v to $a = (a^1, \ldots, a^m)$. In particular, we see that if $U \subseteq \mathbb{R}^m$ is an open subset, and $p \in U$, then T_pU is the subspace of the space of linear maps $C^\infty(M) \to \mathbb{R}$ spanned by the partial derivatives at p. That is, T_pU has a basis

$$\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p$$

identifying $T_pU \equiv \mathbb{R}^m$. Given

$$v = \sum a^i \frac{\partial}{\partial x^i}|_p$$

the coefficients a^i are obtained by applying v to the coordinate functions $x^1, \ldots, x^m : U \to \mathbb{R}$, that is, $a^i = v(x^i)$.

We now describe yet another approach to tangent spaces which again characterizes "directional derivatives" in a coordinate-free way, but without reference to curves γ. Note first that every tangent vector satisfies the *product rule*, also called the *Leibniz rule*:

Lemma 4.1. *Let $v \in T_pM$ be a tangent vector at $p \in M$. Then*

$$v(fg) = f(p)\, v(g) + v(f)\, g(p) \tag{4.2}$$

for all $f, g \in C^\infty(M)$.

Proof. Letting v be represented by a curve γ, this follows from

$$\frac{d}{dt}\Big|_{t=0}\Big(f\big(\gamma(t)\big)g\big(\gamma(t)\big)\Big) = f(p)\Big(\frac{d}{dt}\Big|_{t=0} g\big(\gamma(t)\big)\Big) + \Big(\frac{d}{dt}\Big|_{t=0} f\big(\gamma(t)\big)\Big)g(p).$$

□

Alternatively, in local coordinates it is just the product rule for partial derivatives. It turns out that the product rule completely characterizes tangent vectors:

Theorem 4.2. *A linear map $v : C^\infty(M) \to \mathbb{R}$ defines an element of T_pM if and only if it satisfies the product rule (4.2).*

The proof of this result will require the following fact from multivariable calculus:

Lemma 4.2 (Hadamard Lemma). *Let $U = B_R(0) \subseteq \mathbb{R}^m$ be an open ball of radius $R > 0$ and $h \in C^\infty(U)$ a smooth function. Then there exist smooth functions $h_i \in C^\infty(U)$ with*

$$h(u) = h(0) + \sum_{i=1}^{m} u^i h_i(u)$$

for all $u \in U$. Here $h_i(0) = \frac{\partial h}{\partial u^i}(0)$.

Proof. Let h_i be the functions defined for $u = (u^1, \dots, u^m) \in U$ by

$$h_i(u) = \begin{cases} \frac{1}{u^i}\Big(h(u^1,\dots,u^i,0,\dots,0) - h(u^1,\dots,u^{i-1},0,0,\dots,0)\Big) & \text{if} \quad u^i \neq 0 \\ \frac{\partial h}{\partial u^i}(u^1,\dots,u^{i-1},0,0,\dots,0) & \text{if} \quad u^i = 0 \end{cases}$$

Using Taylor's formula with remainder, one sees that these functions are smooth. [2] If all $u^i \neq 0$, then the sum $\sum_{i=1}^m u^i h_i(u)$ is a telescoping sum, equal to $h(u) - h(0)$. By continuity, this result extends to all u. Finally, evaluating the derivative

$$\frac{\partial h}{\partial u^i} = h_i(u) + \sum_k u^k \frac{\partial h_k}{\partial u^i}$$

at $u = 0$, we see that $\frac{\partial h}{\partial u^i}\big|_{u=0} = h_i(0)$. □

Proof (Theorem 4.2). Let $v : C^\infty(M) \to \mathbb{R}$ be a linear map satisfying the product rule (4.2).

Step 1: *v vanishes on constants.*

By the product rule, applied to the constant function $1 = 1 \cdot 1$, we have $v(1) = 0$. Thus v vanishes on constants.

Step 2: *If $f_1 = f_2$ on some open neighborhood U of p, then $v(f_1) = v(f_2)$.*

Equivalently, letting $f = f_1 - f_2$, we show that $v(f) = 0$ if $f = 0$ on U. Choose a 'bump function' $\chi \in C^\infty(M)$ with $\chi(p) = 1$, with $\chi|_{M\setminus U} = 0$. Then $f\chi = 0$. The product rule tells us that

$$0 = v(f\chi) = v(f)\chi(p) + v(\chi)f(p) = v(f).$$

Step 3: *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

---

[2] It is a well-known fact from calculus (proved e.g. by using Taylor's theorem with remainder) that if f is a smooth function of a real variable x, then the function g, defined as $g(x) = x^{-1}(f(x) - f(0))$ for $x \neq 0$ and $g(0) = f'(0)$, is smooth.

This is immediate from the product rule.

Step 4: *Let (U, φ) be a chart around p, with image $\widetilde{U} = \varphi(U)$. Then there is unique linear map $\widetilde{v}: C^\infty(\widetilde{U}) \to \mathbb{R}$ such that $\widetilde{v}(\widetilde{f}) = v(f)$ whenever $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$.*

Given $\widetilde{f}$, we can always find a function f such that $\widetilde{f}$ agrees with $f \circ \varphi^{-1}$ on some neighborhood of $\widetilde{p}$. Given another such function f', it follows from Step 2 that $v(f) = v(f')$.

Step 5: *In a chart (U, φ) around p, the map $v: C^\infty(M) \to \mathbb{R}$ is of the form (4.1).*

Since the condition (4.1) does not depend on the choice of chart around p, we may assume that $\widetilde{p} = \varphi(p) = 0$, and that $\widetilde{U}$ is an open ball of some radius $R > 0$ around 0. Define $\widetilde{v}$ as in Step 4. Since v satisfies the product rule on $C^\infty(M)$, the map $\widetilde{v}$ satisfies the product rule on $C^\infty(\widetilde{U})$. Given $f \in C^\infty(M)$, consider the Taylor expansion of the coordinate expression $\tilde{f} = f \circ \varphi^{-1}$ near $u = 0$:

$$\tilde{f}(u) = \widetilde{f}(0) + \sum_i u^i \frac{\partial \tilde{f}}{\partial u^i}\Big|_{u=0} + \tilde{r}(u)$$

The remainder term $\tilde{r}$ is a smooth function that vanishes at $u = 0$ together with its first derivatives. By Lemma 4.2, it can be written in the form $\tilde{r}(u) = \sum_i u^i \tilde{r}_i(u)$ where $\tilde{r}_i$ are smooth functions that vanish at 0. Let us now apply $\widetilde{v}$ to the formula for $\widetilde{f}$. Since $\widetilde{v}$ vanishes on products of functions vanishing at 0 (by Step 3), we have that $\widetilde{v}(\widetilde{r}) = 0$. Since it also vanishes on constants (by Step 1), we obtain

$$v(f) = \widetilde{v}(\widetilde{f}) = \sum_i a^i \frac{\partial \tilde{f}}{\partial u^i}\Big|_{u=0},$$

where we put $a^i = \widetilde{v}(u^i)$.

To summarize, we have the following alternative definition of tangent spaces:

Definition 4.3 (Tangent spaces – third definition). The tangent space T_pM is the space of linear maps $C^\infty(M) \to \mathbb{R}$ satisfying the product rule,

$$v(fg) = f(p)v(g) + v(f)\,g(p)$$

for all $f, g \in C^\infty(M)$.

At first sight, this characterization may seem a bit less intuitive then the definition as directional derivatives along curves. But it has the advantage of being less redundant – a tangent vector may be represented by many curves. Also, as in the coordinate definition it is immediate that T_pM is a linear subspace of the vector space $L(C^\infty(M), \mathbb{R})$. One may still want to use local charts, however, to prove that this vector subspace has dimension equal to the dimension of M.

The following remark gives yet another characterization of the tangent space. Please read it only if you like it abstract – *otherwise skip this*!

Remark 4.2 (A fourth definition). There is a fourth definition of T_pM, as follows. For any $p \in M$, let $C_p^\infty(M)$ denotes the subspace of functions vanishing at p, and let $C_p^\infty(M)^2$ consist of finite sums $\sum_i f_i g_i$ where $f_i, g_i \in C_p^\infty(M)$. We have a direct sum decomposition

$$C^\infty(M) = \mathbb{R} \oplus C_p^\infty(M),$$

where $\mathbb{R}$ is regarded as the constant functions. Since any tangent vector $v\colon C^\infty(M) \to \mathbb{R}$ vanishes on constants, v is effectively a map $v : C_p^\infty(M) \to \mathbb{R}$. By the product rule, v vanishes on the subspace $C_p^\infty(M)^2 \subseteq C_p^\infty(M)$. Thus v descends to a linear map $C_p^\infty(M)/C_p^\infty(M)^2 \to \mathbb{R}$, i.e. an element of the dual space $(C_p^\infty(M)/C_p^\infty(M)^2)^*$. The map

$$T_pM \to (C_p^\infty(M)/C_p^\infty(M)^2)^*$$

just defined is an *isomorphism*, and can therefore be used as a definition of T_pM. This may appear very fancy on first sight, but really just says that a tangent vector is a linear functional on $C^\infty(M)$ that vanishes on constants and depends only on the first order Taylor expansion of the function at p. Furthermore, this viewpoint lends itself to generalizations which are relevant to algebraic geometry and non-commutative geometry: The 'vanishing ideals' $C_p^\infty(M)$ are the maximal ideals in the algebra of smooth functions, with $C_p^\infty(M)^2$ their second power (in the sense of products of ideals). Thus, for any maximal ideal $\mathscr{I}$ in a commutative algebra $\mathscr{A}$ one may regard $(\mathscr{I}/\mathscr{I}^2)^*$ as a 'tangent space'.

After this lengthy discussion of tangent spaces, observe that the velocity vectors of curves are naturally elements of the tangent space. Indeed, let $J \subseteq \mathbb{R}$ be an open interval, and $\gamma \in C^\infty(J, M)$ a smooth curve. Then for any $t_0 \in J$, the tangent (or *velocity*) vector

$$\dot\gamma(t_0) \in T_{\gamma(t_0)}M.$$

at time t_0 is given in terms of its action on functions by

$$(\dot\gamma(t_0))(f) = \frac{d}{dt}\Big|_{t=t_0} f(\gamma(t))$$

We will also use the notation $\frac{d\gamma}{dt}(t_0)$ or $\frac{d\gamma}{dt}|_{t_0}$ to denote the velocity vector.

4.2 Tangent map

4.2.1 Definition of the tangent map, basic properties

For smooth maps $F \in C^\infty(U,V)$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ of Euclidean spaces, and any given $p \in U$, we considered the derivative to be the linear map

$$D_pF\colon\ \mathbb{R}^m \to \mathbb{R}^n,\ a \mapsto \frac{d}{dt}\Big|_{t=0} F(p+ta).$$

The following definition generalizes the derivative to smooth maps between manifolds.

Definition 4.4. Let M,N be manifolds and $F \in C^\infty(M,N)$. For any $p \in M$, we define the *tangent map* to be the linear map

$$T_pF:\ T_pM \to T_{F(p)}N$$

given by

$$\big(T_pF(v)\big)(g) = v(g \circ F)$$

for $v \in T_pM$ and $g \in C^\infty(N)$.

We leave it as an exercise to check that the right hand side does indeed define a tangent vector:

Exercise: Show that for all $v \in T_pM$, the map $g \mapsto v(g \circ F)$ satisfies the product rule at $q = F(p)$, hence defines an element of T_qN.

Proposition 4.1. *If $v \in T_pM$ is represented by a curve $\gamma:\ J \to M$, then $(T_pF)(v)$ is represented by the curve $F \circ \gamma$.*

Proof. For $g \in C^\infty(N)$,

$$T_pF(v)(g) = v(g \circ F) = \frac{d}{dt}\Big|_{t=0}(g \circ F)(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} g\big((F \circ \gamma)(t)\big).$$

This shows that $T_pF(v)$ is represented by $F \circ \gamma : \mathbb{R} \to N$.

Remark 4.3 (Pull-backs, push-forwards). For smooth maps $F \in C^\infty(M,N)$, one can consider various 'pull-backs' of objects on N to objects on M, and 'push-forwards' of objects on M to objects on N. Pull-backs are generally denoted by F^*, push-forwards by F_*. For example, functions on N pull back

$$g \in C^\infty(N) \quad \rightsquigarrow \quad F^*g = g \circ F \in C^\infty(M).$$

Curves push on M forward:

$$\gamma:\ J \to M \quad \rightsquigarrow \quad F_*\gamma = F \circ \gamma:\ J \to N.$$

Tangent vectors to M also push forward,

$$v \in T_pM \quad \rightsquigarrow \quad F_*(v) = (T_pF)(v).$$

The definition of the tangent map can be phrased in these terms as $(F_*v)(g) = v(F^*g)$. Note also that if v is represented by the curve γ, then F_*v is represented by the curve $F_*\gamma$.

Proposition 4.2 (Chain rule). *Let M,N,Q be manifolds. Under composition of maps $F \in C^\infty(M,N)$ and $F' \in C^\infty(N,Q)$,*

$$T_p(F' \circ F) = T_{F(p)}F' \circ T_pF.$$

Proof. Let $v \in T_pM$ be represented by a curve γ. Then both $T_p(F' \circ F)(v)$ and $T_{F(p)}F'(T_pF(v))$ are represented by the curve $F' \circ (F \circ \gamma) = (F' \circ F) \circ \gamma$. □

Exercise: a) Show that the tangent map of the identity map $\mathrm{id}_M :\ M \to M$ at $p \in M$ is the identity map on the tangent space:

$$T_p\,\mathrm{id}_M = \mathrm{id}_{T_pM}.$$

b) Show that if $F \in C^\infty(M,N)$ is a diffeomorphism, then T_pF is a linear isomorphism, with inverse

$$(T_pF)^{-1} = (T_{F(p)}F^{-1}).$$

4.2.2 Coordinate description of the tangent map

To get a better understanding of the tangent map, let us first consider the spacial case that $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For $p \in U$, the tangent space T_pU is canonically identified with $\mathbb{R}^m$, using the basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU$$

of the tangent space. Similarly, $T_{F(p)}V \cong \mathbb{R}^n$, using the basis given by partial derivatives $\frac{\partial}{\partial y^j}|_{F(p)}$. Using this identifications, the tangent map becomes a linear map $T_pF :\ \mathbb{R}^m \to \mathbb{R}^n$, i.e. it is given by an $n \times m$-matrix. This matrix is exactly the Jacobian:

Proposition 4.3. *Let $F \in C^\infty(U,V)$ is a smooth map between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. For all $p \in M$, the tangent map T_pF is just the derivative (i.e., Jacobian matrix) D_pF of F at p.*

Proof. For $g \in C^\infty(V)$, we calculate

$$\begin{aligned}\Big((T_pF)\big(\frac{\partial}{\partial x^i}\Big|_p\big)\Big)(g) &= \frac{\partial}{\partial x^i}\Big|_p (g \circ F) \\ &= \sum_{j=1}^n \frac{\partial g}{\partial y^j}\Big|_{F(p)} \frac{\partial F^j}{\partial x^i}\Big|_p \\ &= \Big(\sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}\Big)(g).\end{aligned}$$

This shows

$$(T_pF)\big(\frac{\partial}{\partial x^i}\Big|_p\big) = \sum_{j=1}^n \frac{\partial F^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

Hence, in terms of the given bases of T_pU and $T_{F(p)}V$, the matrix of the linear map T_pF has entries $\frac{\partial F^j}{\partial x^i}\Big|_p$.

Remark 4.4. For $F \in C^\infty(U,V)$, it is common to write $y = F(x)$, and accordingly write $(\frac{\partial y^j}{\partial x^i})_{i,j}$ for the Jacobian. In these terms, the derivative reads as

$$T_pF\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_j \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

This formula is often used for explicit calculations.

For a general smooth map $F \in C^\infty(M,N)$, we obtain a similar description once we pick coordinate charts. Given $p \in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U) \subseteq V$. Let $\widetilde{U} = \varphi(U)$, $\widetilde{V} = \psi(V)$, and put

$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \widetilde{U} \to \widetilde{V}.$$

Since the coordinate map $\varphi : U \to \mathbb{R}^m$ is a diffeomorphism onto $\widetilde{U}$, It gives an isomorphism

$$T_p\varphi : T_pU \to T_{\varphi(p)}\widetilde{U} = \mathbb{R}^m.$$

Similarly, $T_{F(p)}\psi$ gives an isomorphism of $T_{F(p)}V$ with $\mathbb{R}^n$. Note also that since $U \subseteq M$ is open, we have that $T_pU = T_pM$. We obtain,

$$T_{\varphi(p)}\widetilde{F} = T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1}.$$

which may be depicted in a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "D_{\varphi(p)}\widetilde{F}"] & \mathbb{R}^n \\
T_pM = T_pU \arrow[u, "T_p\varphi", "\cong"'] \arrow[r, "T_pF"'] & T_{F(p)}V = T_{F(p)}N \arrow[u, "\cong", "T_{F(p)}\psi"']
\end{tikzcd}$$

Now that we have recognized T_pF *as the derivative expressed in a coordinate-free way*, we may liberate some of our earlier definitions from coordinates:

Definition 4.5. Let $F \in C^\infty(M,N)$.

- The *rank of* F *at* $p \in M$, denoted $\mathrm{rank}_p(F)$, is the rank of the linear map T_pF.
- F has *maximal rank at* p if $\mathrm{rank}_p(F) = \min(\dim M, \dim N)$.
- F is a *submersion* if T_pF is surjective for all $p \in M$,
- F is an *immersion* if T_pF is injective for all $p \in M$,
- F is a *local diffeomorphism* if T_pF is an isomorphism for all $p \in M$.
- $p \in M$ is a *critical point* of F is T_pF does not have maximal rank at p.
- $q \in N$ is a *regular value* of F if T_pF is surjective for all $p \in F^{-1}(q)$ (in particular, if $q \notin F(M)$).

- $q \in N$ is a *singular value* if it is not a regular value.

Exercise: Using this new definitions, show that the compositions of two submersions is again a submersion, and that the composition of two immersions is an immersion.

4.2.3 Tangent spaces of submanifolds

Suppose $S \subseteq M$ is a submanifold, and $p \in S$. Then the tangent space T_pS is canonically identified as a subspace of T_pM. Indeed, since the inclusion $i: \ S \hookrightarrow M$ is an immersion, the tangent map is an injective linear map,

$$T_pi: \ T_pS \to T_pM,$$

and we identify T_pS with the subspace given as the image of this map. (Hopefully, the identifications are not getting too confusing: S gets identified with $i(S) \subseteq M$, hence also $p \in S$ with its image $i(p)$ in M, and T_pS gets identified with $(T_pi)(T_pS) \subseteq T_pM$.) As a special case, we see that whenever M is realized as a submanifold of $\mathbb{R}^n$, then its tangent spaces T_pM may be viewed as subspaces of $T_p\mathbb{R}^n = \mathbb{R}^n$.

Proposition 4.4. *Let $F \in C^\infty(M,N)$ be a smooth map, having $q \in N$ as a regular value, and let $S = F^{-1}(q)$. For all $p \in S$,*

$$T_pS = \ker(T_pF),$$

as subspaces of T_pM.

Proof. Let $m = \dim M, \ n = \dim N$. Since T_pF is surjective, its kernel has dimension $m-n$. By the normal form for submersions, this is also the dimension of S, hence of T_pS. It is therefore enough to show that $T_pS \subseteq \ker(T_pF)$. Letting $i: \ S \to M$ be he inclusion, we have to show that

$$T_pF \circ T_pi = T_p(F \circ i)$$

is the zero map. But $F \circ i$ is a *constant map*, taking all points of S to the constant value $q \in N$. The tangent map to a constant map is just zero. (See below.) Hence $T_p(F \circ i) = 0$. $\square$

Exercise: Suppose that $F \in C^\infty(M,N)$ is a *constant map*, that is, $F(M) = \{q\}$ for some element $q \in N$. Show that $T_pF = 0$ for all $p \in M$. (Hint: Use the definition of T_pF, and observe that for $g \in C^\infty(N)$ the pull-back $F^*g = g \circ F$ is a constant function.)

As a special case, we can describe the tangent spaces to level sets:

Corollary 4.1. *Suppose $V \subseteq \mathbb{R}^n$ is open, and $q \in \mathbb{R}^k$ is a regular value of $F \in C^\infty(M, \mathbb{R}^k)$, defining an embedded submanifold $M = F^{-1}(q)$. For all $p \in M$, the tangent space $T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$ is given as*

$$T_pM = \ker(T_pF) \equiv \ker(D_pF).$$

Example 4.2. Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be the map $F(x) = x \cdot x = (x^0)^2 + \ldots + (x^n)^2$. Then, for all $p \in F^{-1}(1) = S^n$,

$$(D_pF)(a) = \frac{d}{dt}\Big|_{t=0} F(p+ta) = \frac{d}{dt}\Big|_{t=0} (p+ta)\cdot(p+ta) = 2p\cdot a,$$

hence

$$T_pS^n = \{a \in \mathbb{R}^{n+1} |\ a \cdot p = 0\} = \operatorname{span}(p)^\perp.$$

As another typical application, suppose that $S \subseteq M$ is a submanifold, and $f \in C^\infty(S)$ is a smooth function given as the restriction $f = h|_S$ of a smooth function $h \in C^\infty(M)$. Consider the problem of finding the critical points $p \in S$ of f, that is,

$$\operatorname{Crit}(f) = \{p \in S |\ T_pf = 0\}.$$

Letting $i : S \to M$ be the inclusion, we have $f = h|_S = h \circ i$, hence $T_pf = T_ph \circ T_pi$. It follows that $T_pf = 0$ if and only if T_ph vanishes on the range of T_pi, that is on T_pS:

$$\operatorname{Crit}(f) = \{p \in S |\ T_pS \subseteq \ker(T_ph)\}.$$

If $M = \mathbb{R}^m$, then T_ph is just the Jacobian D_ph, whose kernel is sometimes rather easy to compute – in any case this approach tends to be much faster than a calculation in charts. Here is a concrete example:

Example 4.3. **Problem.** Find the critical points of

$$f : S^2 \to \mathbb{R}, \quad f(x,y,z) = xy.$$

Solution. Following the strategy outlined above, we write $f = h \circ i$ with $h(x,y,z) = xy$. For $p = (x,y,z)$ we have

$$T_ph = D_ph = (y \ \ x \ \ 0),$$

as a linear map $\mathbb{R}^3 \to \mathbb{R}$. There are two cases: **Case 1.** $D_ph = 0$, i.e. $x = y = 0$. This means $p = (0,0,\pm 1)$. In this case $\ker(T_ph) = \mathbb{R}^3$, which of course contains T_pS. Thus both

$$(0,0,\pm 1) \tag{4.3}$$

are critical points of f.

Case 2. $D_ph \neq 0$. Then

$$\ker(T_p h) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \right\}.$$

This contains T_pS if and only if it is equal to T_pS. To check whether the two basis vectors are in T_pS, we just have to check their dot products with $p = (x,y,z)$. This gives the conditions $z = 0$ and $x^2 - y^2 = 0$, which together with $x^2 + y^2 + z^2 = 1$ leads to the four critical points

$$(\pm\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ 0). \tag{4.4}$$

In summary, the function F has six critical points, corresponding to two sign choices in (4.3) and four sign choices in (4.4). □

As another application of the same idea, you should try to prove:

Exercise: Let $S \subseteq \mathbb{R}^3$ be a surface. Show that $p \in S$ is a critical point of the function $f \in C^\infty(S)$ given by $f(x,y,z) = z$, if and only if T_pS is the x-y-plane.

Example 4.4. **Problem.** Show that the equations

$$x^2 + y = 0,\ \ x^2 + y^2 + z^3 + w^4 + y = 1$$

define a two dimensional submanifold S of $\mathbb{R}^4$, and find the equation of the tangent space at the point $(x_0, y_0, z_0, w_0) = (-1,-1,-1,-1)$.

Solution. Let $F \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)$ be the function

$$F(x,y,z,w) = (x^2 + y,\ x^2 + y^2 + z^3 + w^4 + y).$$

The Jacobian matrix is

$$\begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 3z^2 & 4w^3 \end{pmatrix}.$$

Since the first row is non-zero, this has rank 2 unless the second row is a scalar multiple of the first row. This is the case if either $x \neq 0$ and $y = z = w$, or $x = 0$ and $z = w = 0$. In particular, $x = 0$ or $y = 0$. But if such a point (x,y,z,w) also satisfies the first equation for S, that is $x^2 + y = 0$, we see that x, y must *both* be zero. This only leaves the point $(0,0,0,0)$, which however does not solve the second equation $x^2 + y^2 + z^3 + w^4 + y = 1$. This shows that S is a submanifold of dimension $4 - 2 = 2$. At $(-1,-1,-1,-1)$ the Jacobian matrix becomes

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ -2 & -1 & 3 & -4 \end{pmatrix};$$

so the equation of the tangent space $T_pS = \ker(D_pF)$ reads as

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ -2 & -1 & 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is,

$$-2x+y=0, \quad -2x-y+3z-4w=0.$$

Example 4.5. We had discussed various *matrix Lie groups G* as examples of manifolds. By definition, these are submanifolds $G \subseteq \mathrm{Mat}_{\mathbb{R}}(n)$, consisting of invertible matrices with the properties

$$A,B \in G \Rightarrow AB \in G, \quad A \in G \Rightarrow A^{-1} \in G.$$

The tangent space to the identity (group unit) for such matrix Lie groups G turns out to be important; it is commonly denoted by lower case fracture letters:

$$\mathfrak{g} = T_I G.$$

Some concrete examples:

1. The matrix Lie group

$$\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) \neq 0\}$$

of *all* invertible matrices is an open subset of $\mathrm{Mat}_{\mathbb{R}}(n)$, hence

$$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}_{\mathbb{R}}(n)$$

is the entire space of matrices.

2. For the group $\mathrm{O}(n)$, consisting of matrices with $F(A) := A^\top A = I$, we has computed $T_A F(X) = X^\top A + AX^\top$. For $A = I$, the kernel of this map is

$$\mathfrak{o}(n) = \{X \in \mathrm{Mat}_{\mathbb{R}}(n) |\ X^\top = -X\}.$$

3. For the group $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \det(A) = 1\}$, given as the level set $F^{-1}(1)$ of the function $\det: \ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathbb{R}$, we calculate

$$D_A F(X) = \frac{d}{dt}\Big|_{t=0} F(A+tX) = \frac{d}{dt}\Big|_{t=0} \det(A+tX) = \frac{d}{dt}\Big|_{t=0} \det(I+tA^{-1}X) = \mathrm{tr}(A^{-1}X),$$

where $\mathrm{tr}: \ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathbb{R}$ is the trace (sum of diagonal entries). (See exercise below.) Hence

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathrm{Mat}_{\mathbb{R}}(n) |\ \mathrm{tr}(X) = 0\}.$$

Exercise: Show that for every $X \in \mathrm{Mat}_{\mathbb{R}}(n)$,

$$\frac{d}{dt}\Big|_{t=0} \det(I+tX) = \mathrm{tr}(X).$$

(Hint: Use that every matrix is conjugate (i.e., similar) to an upper triangular matrix, and that both determinant and trace are unchanged under conjugation (i.e., similarity transformation).)

4.2.4 Example: Steiner's surface revisited

As we discussed in Section 3.5.4, Steiner's 'Roman surface' is the image of the map

$$\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3,\;\; (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}(yz,\; xz,\; xy).$$

(We changed notation from α, β, γ to x, y, z.) At what points $p \in \mathbb{R}\mathrm{P}^2$ does this map have maximal rank (so that the map is an immersion on an open neighborhood of p?). To investigate this question, one can express the map in local charts, and compute the resulting Jacobian matrix. However, while this approach is perfectly fine, the resulting expressions will become rather complicated. A simpler approach is to consider the composition with the local diffeomorphism $\pi :\; S^2 \to \mathbb{R}\mathrm{P}^2$, given as

$$S^2 \to \mathbb{R}^3,\;\; (x,y,z) \mapsto (yz,\; xz,\; xy).$$

In turn, this map is the restriction $F|_{S^2}$ of the map

$$F :\; \mathbb{R}^3 \to \mathbb{R}^3,\;\; (x,y,z) \mapsto (yz,\; xz,\; xy).$$

We have $T_p(F|_{S^2}) = T_pF|_{T_pS^2}$, hence $\ker(T_p(F|_{S^2})) = \ker(T_pF) \cap T_pS^2$. But $T_pF = D_pF$ for $p = (x,y,z)$ is given by the Jacobian matrix

$$D_pF = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}.$$

This has determinant $\det(D_pF) = 2xyz$, hence its kernel is zero unless $x = 0$ or $y = 0$ or $z = 0$. If $x = 0$, thus $p = (0,y,z)$, the matrix simplifies to

$$D_pF = \begin{pmatrix} 0 & z & y \\ z & 0 & 0 \\ y & 0 & 0 \end{pmatrix},$$

which (unless both y and z are zero as well) has a 1-dimensional kernel spanned by column vectors of the form $(0,-y,z)^\top$. Such a vector is tangent to S^2 if and only if its dot product with $p = (0,y,z)$ is zero, that is, $y^2 = z^2$. Since $p \in S^2$ this means $p = (0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$. Similarly if $y = 0$, or if $z = 0$. We have thus shown: The map $F|_{S^2}$ has maximal rank at all points of S^2, except at the following twelve points:

$$(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}),\ (\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}),\ (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0).$$

The kernel of $T_p(F|_{S^2})$ at $(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$ is the 1-dimensional space spanned by $(0, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$ (sign change in the last entry), and similarly for the points where $y = 0$ or $z = 0$. We conclude that the map $\mathbb{R}\mathrm{P}^2 \to \mathbb{R}^3$ defining Steiner's surface has exactly six points where it fails to be an immersion, and we have computed the kernel of the tangent map at those points.

4.3 The tangent bundle

Proposition 4.5. *For any manifold M of dimension m, the* tangent bundle

$$TM = \bigsqcup_{p \in M} T_pM$$

(disjoint union of vector spaces) is a manifold of dimension $2m$. The map

$$\pi :\ TM \to M$$

taking $v \in T_pM$ to the base point p, is a smooth submersion, with fibers the tangent spaces.

Proof. The idea is simple: Take charts for M, and use the tangent map to get charts for TM. For any open subset U of M, we have

$$TU = \bigsqcup_{p \in U} T_pM = \pi^{-1}(U).$$

(Note $T_pU = T_pM$.) Every chart (U, φ) for M, with $\varphi :\ U \to \mathbb{R}^m$, gives vector space isomorphisms

$$T_p\varphi :\ T_pM \to T_{\varphi(p)}\mathbb{R}^m = \mathbb{R}^m$$

for all $p \in U$. The collection of all maps $T_p\varphi$ for $p \in U$ gives a bijection,

$$T\varphi :\ TU \to \varphi(U) \times \mathbb{R}^m,\ v \mapsto (\varphi(p), (T_p\varphi)(v))$$

for $v \in T_pU \subseteq TU$. The image of these bijections are the open subsets subsets

$$(T\varphi)(TU) = \varphi(U) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m},$$

hence they define charts. We take the collection of all such charts as an atlas for TM:

$$\begin{tikzcd} TU \arrow[r, "T\varphi"'] \arrow[d, "\pi"'] & \varphi(U)\times\mathbb{R}^m \arrow[d, "(u,v)\mapsto u"] \\ U \arrow[r, "\varphi"'] & \varphi(U) \end{tikzcd}$$

We need to check that the transition maps are smooth. If (V,ψ) is another coordinate chart with $U\cap V\neq\emptyset$, the transition map for $TU\cap TV = T(U\cap V) = \pi^{-1}(U\cap V)$ is given by,

$$T\psi\circ(T\varphi)^{-1}:\ \varphi(U\cap V)\times\mathbb{R}^m\to\psi(U\cap V)\times\mathbb{R}^m. \tag{4.5}$$

But $T_p\psi\circ(T_p\varphi)^{-1} = T_{\varphi(p)}(\psi\circ\varphi^{-1})$ is just the derivative (Jacobian matrix) for the change of coordinates $\psi\circ\varphi^{-1}$; hence (4.5) is given by

$$(x,a)\mapsto\Big((\psi\circ\varphi^{-1})(x),\ D_x(\psi\circ\varphi^{-1})(a)\Big)$$

Since the Jacobian matrix depends smoothly on x, this is a smooth map. This shows that any atlas $\mathscr{A}=\{(U_\alpha,\varphi_\alpha)\}$ for M defines an atlas $\{(TU_\alpha,T\varphi_\alpha)\}$ for TM. Taking $\mathscr{A}$ to be countable the atlas for TM is countable. The Hausdorff property is easily checked as well. $\square$

Proposition 4.6. *For any smooth map $F\in C^\infty(M,N)$, the map*

$$TF:\ TM\to TN$$

given on T_pM as the tangent maps $T_pF:\ T_pM\to T_{F(p)}N$, is a smooth map.

Proof. Given $p\in M$, choose charts (U,φ) around p and (V,ψ) around $F(p)$, with $F(U)\subseteq V$. Then $(TU,T\varphi)$ and $(TV,T\psi)$ are charts for TM and TN, respectively, with $TF(TU)\subseteq TV$. Let $\widetilde{F}=\psi\circ F\circ\varphi^{-1}:\ \varphi(U)\to\psi(V)$. The map

$$T\widetilde{F}=T\psi\circ TF\circ(T\varphi)^{-1}:\ \varphi(U)\times\mathbb{R}^m\to\psi(V)\times\mathbb{R}^n$$

is given by

$$(x,a)\mapsto\Big((\widetilde{F})(x),\ D_x(\widetilde{F})(a)\Big).$$

It is smooth, by smooth dependence of the differential $D_x\widetilde{F}$ on the base point. Consequently, TF is smooth, $\square$

Chapter 5
Vector fields

5.1 Vector fields as derivations

A vector field on a manifold may be regarded as a family of tangent vectors $X_p \in T_pM$ for $p \in M$, depending smoothly on the base points $p \in M$. One way of making precise what is meant by 'depending smoothly' is the following.

Definition 5.1 (Vector fields – first definition). A collection of tangent vectors $X_p,\ p \in M$ defines a vector field $X \in \mathfrak{X} \in M$ if and only if for all functions $f \in C^\infty(M)$ the function $p \mapsto X_p(f)$ is smooth. The space of all vector fields on M is denoted $\mathfrak{X}(M)$.

We hence obtain a linear map $X:\ C^\infty(M) \to C^\infty(M)$ such that

$$X(f)|_p = X_p(f). \tag{5.1}$$

Since each X_p satisfy the product rule (at p), it follows that X itself satisfies a product rule. We can use this as an alternative definition:

Definition 5.2 (Vector fields – second definition). A vector field on M is a linear map

$$X:\ C^\infty(M) \to C^\infty(M)$$

satisfying the product rule,

$$X(fg) = X(f)g + fX(g) \tag{5.2}$$

for $f, g \in C^\infty(M)$.

Remark 5.1. The condition (5.2) says that X is a *derivation* of the algebra $C^\infty(M)$ of smooth functions. More generally, a derivation of an algebra A is a linear map $D:\ A \to A$ such that

$$D(a_1a_2) = D(a_1)\, a_2 + a_1 D(a_2).$$

(Appendix 5.8 reviews some facts about derivations.)

We can also express the smoothness of the tangent vectors X_p in terms of coordinate charts (U,φ). Recall that for any $p \in U$, and all $f \in C^\infty(M)$, the tangent vector X_p is expressed as

$$X_p(f) = \sum_{i=1}^m a^i \frac{\partial}{\partial u^i}\Big|_{u=\varphi(p)} (f \circ \varphi^{-1}).$$

The vector $a = (a^1,\dots,a^m) \in \mathbb{R}^m$ represents X_p in the chart; i.e., $(T_p\varphi)(X_p) = a$ under the identification $T_{\varphi(p)}\varphi(U) = \mathbb{R}^m$. As p varies in U, the vector a becomes a function of $p \in U$, or equivalently of $u = \varphi(p)$.

Proposition 5.1. *The collection of tangent vectors $X_p,\ p \in M$ define a vector field if and only if for all charts (U,φ), the functions $a^i:\ \varphi(U) \to \mathbb{R}$ defined by*

$$X_{\varphi^{-1}(u)}(f) = \sum_{i=1}^m a^i(u) \frac{\partial}{\partial u^i}(f \circ \varphi^{-1}),$$

are smooth.

Proof. If the a^i are smooth functions, then for every $f \in C^\infty(M)$ the function $X(f) \circ \varphi^{-1}:\ \varphi(U) \to \mathbb{R}$ is smooth, and hence $X(f)|_U$ is smooth. Since this is true for all charts, it follows that $X(f)$ is smooth. Conversely, if X is a vector field, and $p \in M$ some point in a coordinate chart (U,φ), and $i \in \{1,\dots,m\}$ a given index, choose $f \in C^\infty(M)$ such that $f(\varphi^{-1}(u)) = u^i$. Then $X(f) \circ \varphi^{-1} = a^i(u)$, which shows that the a^i are smooth.

Exercise: In the proof, we used that for any coordinate chart (U,φ) around p, one can choose $f \in C^\infty(M)$ such that $f \circ \varphi^{-1}:\ \varphi(U) \to \mathbb{R}$ coincides with u^j near $\varphi(p)$. Write out the details in the construction of such a function f, using a choice of 'bump function'.

In particular, we see that vector fields on open subsets $U \subseteq \mathbb{R}^m$ are of the form

$$X = \sum_i a^i \frac{\partial}{\partial x^i}$$

where $a^i \in C^\infty(U)$. Under a diffeomorphism $F:\ U \to V,\ x \mapsto y = F(x)$, the coordinate vector fields transform with the Jacobian

$$TF\Big(\frac{\partial}{\partial x^i}\Big) = \sum_j \frac{\partial F^j}{\partial x^i}\Big|_{x=F^{-1}(y)} \frac{\partial}{\partial y^j}$$

Informally, this 'change of coordinates' is often written

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

Here one thinks of the x^i and y^j as coordinates on the same set, and doesn't worry about writing coordinate maps, and one uses the (somewhat sloppy, but convenient) notation $y = y(x)$ instead of $y = F(x)$.)

Example 5.1. **Problem.** Express the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ in polar coordinates, given by

$$x = r\cos\theta,\ y = r\sin\theta$$

(valid for $r > 0$ and $-\pi < \theta < \pi$).
Solution. We have

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

and similarly

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

The matrix of coefficients is of course the Jacobian. Inverting this matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}$$

(in other words, solving the equations for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$) we obtain

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta},$$
$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}.$$

5.2 Vector fields as sections of the tangent bundle

The 'best' way of describing the smoothness of $p \mapsto X_p$ is that it is literally a smooth map into the tangent bundle.

Definition 5.3 (Vector fields – third definition). A vector field on M is a smooth map $X \in C^\infty(M, TM)$ such that $\pi \circ X$ is the identity.

It is common practice to use the same symbol X both as a linear map from smooth functions to smooth functions, or as a map into the tangent bundle. Thus

$$X :\ M \to TM, \quad X :\ C^\infty(M) \to C^\infty(M)$$

coexist. But if it gets too confusing, one uses a symbol

$$L_X : C^\infty(M) \to C^\infty(M)$$

for the interpretation as a derivation; here the L stands for 'Lie derivative' (named after Sophus Lie). Both viewpoints are useful and important, and both have their advantages and disadvantages. For instance, from (A) it is immediate that vector fields on M restrict to open subsets $U \subseteq M$; this map

$$\mathfrak{X}(M) \to \mathfrak{X}(U), \;\; X \mapsto X|_U$$

may seem a little awkward from viewpoint (B) since $C^\infty(U)$ is not a subspace of $C^\infty(M)$. (There is a restriction map $C^\infty(M) \to C^\infty(U)$, but no natural map in the other direction.) On the other hand, (B) gives the *Lie bracket operation* discussed below, which seems unexpected from viewpoint (A).

5.3 Lie brackets

Let M be a manifold. Given vector fields $X,Y : C^\infty(M) \to C^\infty(M)$, the composition $X \circ Y$ is not a vector field: For example, if $X = Y = \frac{\partial}{\partial x}$ as vector fields on $\mathbb{R}$, then $X \circ Y = \frac{\partial^2}{\partial x^2}$ is a second order derivative, which is not a vector field (it does not satisfy the Leibnitz rule). However, the commutator turns out to be a vector field:

Theorem 5.1. *For any two vector fields $X,Y \in \mathfrak{X}(M)$ (regarded as derivations), the commutator*

$$[X,Y] := X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

is again a vector field.

Proof. To check that $[X,Y]$ is a vector field, we verify the derivation property, by direct calculation. We have that

$$\begin{aligned}(X \circ Y)(f_1 f_2) &= X\big(Y(f_1) f_2 + f_1 Y(f_2)\big)\\ &= X(Y(f_1))\, f_2 + f_1\, X(Y(f_2)) + X(f_1)Y(f_2) + Y(f_1)X(f_2);\end{aligned}$$

subtracting a similar expression with 1 and 2 interchanged, some terms cancel, and we obtain

$$\begin{aligned}[X,Y](f_1 f_2) &= X(Y(f_1))\, f_2 + f_1\, X(Y(f_2)) - X(Y(f_2))\, f_1 + f_2\, X(Y(f_1))\\ &= [X,Y](f_1)\, f_2 + f_1\, [X,Y](f_2)\end{aligned}$$

as required.

Remark 5.2. A similar calculation applies to derivations of algebras in general: The commutator of two derivations is again a vector field.

Definition 5.4. The vector field

$$[X,Y] := X \circ Y - Y \circ X$$

is called the *Lie bracket* of $X, Y \in \mathfrak{X}(M)$.

It is instructive to see how this works in local coordinates. For open subsets $U \subseteq \mathbb{R}^m$, if

$$X = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^m b^i \frac{\partial}{\partial x^i},$$

with coefficient functions $a^i, b^i \in C^\infty(U)$, the composition $X \circ Y$ is a second order differential operators on functions $f \in C^\infty(U)$:

$$X \circ Y = \sum_{i=1}^m \sum_{j=1}^m a^j \frac{\partial b^i}{\partial x^j} \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{j=1}^m a^i b^j \frac{\partial^2}{\partial x^i \partial x^j}$$

Subtracting a similar expression for $Y \circ X$, the terms involving second derivatives cancel, and we obtain

$$[X,Y] = \sum_{i=1}^m \sum_{j=1}^m \Big(a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j} \Big) \frac{\partial}{\partial x^i}.$$

(This calculation applies to general manifolds, by taking local coordinates.) The significance of the Lie bracket will become clear later. At this stage, let us give some examples.

Example 5.2. Consider the following two vector fields on $\mathbb{R}^2$,

$$X = \frac{\partial}{\partial x}, \quad Y = (1+x^2)\frac{\partial}{\partial y}.$$

We have

$$X \circ Y = (1+x^2)\frac{\partial^2}{\partial x \partial y} + 2x\frac{\partial}{\partial y}, \qquad Y \circ X = (1+x^2)\frac{\partial^2}{\partial y \partial x}.$$

Both a second order differential operators. Taking the difference, the second order derivatives cancel, due to the equality of mixed partials. We obtain

$$[X,Y] = 2x\frac{\partial}{\partial y}.$$

Note that the vector fields X, Y are linearly independent everywhere. Is it possible to introduce coordinates $(u,v) = \varphi(x,y)$, such that in the new coordinates, these vector fields are the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$? The answer is no: the coordinate vector fields have zero Lie bracket

$$[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}] = 0,$$

(they '*commute*'), but $[X,Y] \neq 0$.

Example 5.3. Consider the following two vector fields on $\mathbb{R}^2$, on the open subset where $xy > 0$,

$$X = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad Y = 2\sqrt{xy}\frac{\partial}{\partial x}$$

Indicating second order derivatives by dots we have

$$X \circ Y = \Big(\frac{x}{y}\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}}\Big)\frac{\partial}{\partial x} + \ldots = 2\sqrt{\frac{x}{y}}\frac{\partial}{\partial x} + \ldots$$

$$Y \circ X = 2\sqrt{\frac{x}{y}}\frac{\partial}{\partial x} + \ldots.$$

Thus, $[X,Y] = X \circ Y - Y \circ X = 0$. Can one introduce coordinates u, v in which these vector fields become the coordinate vector fields? This time, the answer is yes: Define a change of coordinates u, v by putting $x = uv^2, \;\; y = u$. Then

$$\begin{aligned}\frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = v^2\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = \frac{x}{y}\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = X,\\ \frac{\partial}{\partial v} &= \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = 2uv\frac{\partial}{\partial x} = 2\sqrt{xy}\frac{\partial}{\partial x} = Y,\end{aligned}$$

Note: When calculating Lie brackets $X \circ Y - Y \circ X$ of vector fields X, Y in local coordinates, it is not necessary to work out the second order derivatives – we know in advance that these are going to cancel out! This is why we indicated second order derivatives by "…" in the calculation above.

Example 5.4. Consider the same problem for the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \;\; Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

This time, we may verify that $[X,Y] = 0$. Introduce polar coordinates,

$$x = r\cos\theta, \;\; y = r\sin\theta.$$

(this is a well-defined coordinate chart for $r > 0$ and $-\pi < \theta < \pi$). We have [1]

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{1}{r}Y$$

and

[1] In the following, we are using somewhat sloppy notation. Given $(\theta, r) = \varphi(x,y)$, we should more properly write $\varphi_* X, \varphi_* Y$ for the vector fields in the new coordinates.

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = X$$

Hence $X = \frac{\partial}{\partial\theta}$, $Y = r\frac{\partial}{\partial r}$. To get this into the desired form, we make another change of coordinates $\rho = f(r)$ in such a way that Y becomes $\frac{\partial}{\partial\rho}$. Since

$$\frac{\partial}{\partial r} = \frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho} = f'(r)\frac{\partial}{\partial\rho}$$

we want $f'(r) = \frac{1}{r}$, thus $f(r) = \ln(r)$. So, $r = e^\rho$. Hence, the desired change of coordinates is

$$x = e^\rho\cos\theta, \;\; y = e^\rho\sin\theta.$$

Let $S \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is called *tangent to S* if for all $p \in S$, the tangent vector X_p lies in $T_pS \subseteq T_pM$. (Thus X restricts to a vector field $X|_S \in \mathfrak{X}(S)$.)

Example 5.5. The three vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

on $\mathbb{R}^3$ are tangent to the 2-sphere S^2: For example, under the identification $T_p\mathbb{R}^3 = \mathbb{R}^3$, for $p = (x,y,z)$, each of

$$X_p = (0,-z,y), \;\; Y_p = (z,0,-x), \;\; Z_p = (-y,x,0)$$

have zero dot product with p. The bracket of any two of X,Y,Z is again tangent; in fact we have

$$[X,Y] = Z, \;\; [Y,Z] = X, \;\; [Z,X] = Y.$$

More generally, we have:

Proposition 5.2. *If two vector fields $X,Y \in \mathfrak{X}(M)$ are tangent to a submanifold $S \subseteq M$, then their Lie bracket is again tangent to S.*

Proposition 5.2 can be proved by using the coordinate expressions of X,Y in submanifold charts. But we will postpone the proof for now since there is a much shorter, coordinate-independent proof, see the next section.

Example 5.6. Consider the vector fields on $\mathbb{R}^3$,

$$X = \frac{\partial}{\partial x}, \;\; Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

We have

$$[X,Y] = \frac{\partial}{\partial z};$$

hence $X_p,\ Y_p,\ Z_p$ are a basis of $\mathbb{R}^3$ for all $p \in \mathbb{R}^3$. In particular, there cannot exist a surface $S \subseteq \mathbb{R}^3$ such that both X and Y are tangent to S.

5.4 Related vector fields

Definition 5.5. Let $F \in C^\infty(M,N)$ be a smooth map. Vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called F-related, written as

$$X \sim_F Y,$$

if

$$T_pF(X_p) = Y_{F(p)}$$

for all $p \in M$.

Example 5.7. If F is a diffeomorphism, then $X \sim_F Y$ if and only if $Y = F_* X$. In particular, if $N = M$, then an equation $X \sim_F X$ means that X is *invariant* under F.

Example 5.8. Let $S \subseteq M$ be an embedded submanifold and $i:\ S \to M$ the inclusion. Let $X \in \mathfrak{X}(S)$ and $Y \in \mathfrak{X}(M)$. Then

$$X \sim_i Y$$

if and only if Y is tangent to S, with X as its restriction. In particular,

$$0 \sim_i Y$$

if and only if Y vanishes along the submanifold S.

Example 5.9. If $F:\ M \to N$ is a submersion, and $X \in \mathfrak{X}(M)$, then $X \sim_F 0$ if and only if X is tangent to the fibers of F.

Example 5.10. Let $\pi:\ S^n \to \mathbb{R}\mathrm{P}^n$ be the quotient map. Then $X \sim_\pi Y$ if and only if the vector field X is invariant under the transformation $F:\ S^n \to S^n,\ x \mapsto -x$ (that is, $TF \circ X = X \circ F$, and with Y the induced vector field on the quotient.

The F-relation of vector fields also has a simple interpretation in terms of the 'differential operator' picture.

Proposition 5.3. *One has $X \sim_F Y$ if and only if for all $g \in C^\infty(N)$,*

$$X(g \circ F) = Y(g) \circ F.$$

In terms of the pull-back notation, with $F^*g = g \circ F$ for $g \in C^\infty(N)$, this means $X \circ F^* = F^* \circ Y$:

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"'] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"'] \arrow[r, "Y"'] & C^\infty(N) \arrow[u, "F^*"'] \end{tikzcd}$$

Proof. The condition $X(g \circ F) = Y(g) \circ F$ says that

$$(T_pF(X_p))(g) = Y_{F(p)}(g)$$

for all $p \in M$. □

The key fact concerning related vector fields is the following.

Theorem 5.2. *Let $F \in C^\infty(M,N)$ For vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(M)$, we have*

$$X_1 \sim_F Y_1, \quad X_2 \sim_F Y_2 \Rightarrow \quad [X_1, X_2] \sim_F [Y_1, Y_2].$$

Proof. Using the differential operator picture, we have that

$$\begin{aligned} [X_1, X_2](g \circ F) &= X_1(X_2(g \circ F)) - X_2(X_1(g \circ F)) \\ &= X_1(Y_2(g) \circ F) - X_2(Y_1(g) \circ F) \\ &= Y_1(Y_2(g)) \circ F - Y_2(Y_1(g)) \circ F \\ &= [Y_1, Y_2](g) \circ F. \end{aligned}$$

□

Example 5.11. If two vector fields Y_1, Y_2 are tangent to a submanifold $S \subseteq M$ then their Lie bracket $[Y_1, Y_2]$ is again tangent to S, and the Lie bracket of their restriction is the restriction of the Lie brackets. Indeed, letting X_i be the restrictions, we have

$$X_1 \sim_i Y_1, \; X_2 \sim_i Y_2 \;\Rightarrow\; [X_1, X_2] \sim_i [Y_1, Y_2].$$

Similarly, if Y_1 is tangent to S and Y_2 vanishes along S, then the Lie bracket vanishes along S. This follows from the above by putting $X_2 = 0$, since $[X_1, 0] = 0$.

Exercise. Explore the consequences of Proposition 5.3 for the other examples of related vector fields given above.

Exercise. Show that in the description of vector fields as sections of the tangent bundle, two vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$ are F-related if and only if the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "TF"] & TN \\ M \arrow[u, "X"] \arrow[r, "F"'] & N \arrow[u, "Y"'] \end{tikzcd}$$

5.5 Flows of vector fields

For any curve $\gamma\colon\ J \to M$, with $J \subseteq \mathbb{R}$ an open interval, and any $t \in J$, the velocity vector

$$\dot{\gamma}(t) \equiv \frac{d\gamma}{dt} \in T_{\gamma(t)}M$$

is defined as the tangent vector, given in terms of its action on functions as

$$(\dot{\gamma}(t))(f) = \frac{d}{dt} f(\gamma(t)).$$

(The dot signifies a t-derivative.) The curve representing this tangent vector for a given t, in the sense of our earlier definition, is the shifted curve $\tau \mapsto \gamma(t+\tau)$. Equivalently, one may think of the velocity vector as the image of $\frac{\partial}{\partial t}|_t \in T_t J \cong \mathbb{R}$ under the tangent map $T_t\gamma$:

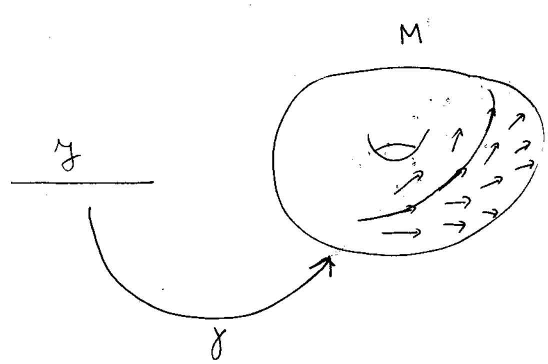
$$\dot{\gamma}(t) = (T_t\gamma)(\frac{\partial}{\partial t}|_t).$$

Definition 5.6. Suppose $X \in \mathfrak{X}(M)$ is a vector field on a manifold M. A smooth curve $\gamma \in C^\infty(J,M)$, where $J \subseteq \mathbb{R}$ is an open interval, is called a *solution curve* to X if

$$\dot{\gamma}(t) = X_{\gamma(t)} \tag{5.3}$$

for all $t \in J$.

Geometrically, Equation (5.3) means that at any given time t, the value of X at $\gamma(t)$ agrees with the velocity vector to γ at t.



Equivalently, in terms of related vector fields,

$$\frac{\partial}{\partial t} \sim_\gamma X.$$

Consider first the case that $M = U \subseteq \mathbb{R}^m$. Here curves $\gamma(t)$ are of the form

$$\gamma(t) = x(t) = (x^1(t), \ldots, x^m(t)),$$

hence

$$\dot{\gamma}(t)(f) = \frac{d}{dt} f(x(t)) = \sum_{i=1}^{m} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial f}{\partial x^i}(x(t)).$$

That is

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial}{\partial x^i}\Big|_{x(t)}.$$

On the other hand, the vector field has the form $X = \sum_{i=1}^{m} a^i(x) \frac{\partial}{\partial x^i}$. Hence (5.3) becomes the system of first order ordinary differential equations,

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = a^i(x(t)), \quad i = 1, \ldots, m. \tag{5.4}$$

Example 5.12. The solution curves of the coordinate vector field $\frac{\partial}{\partial x^j}$ are of the form

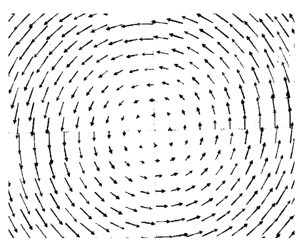
$$x^i(t) = x_0^i, \ \ i \neq j, \quad \ x^j(t) = x_0^j + t.$$

More generally, if $a = (a^1, \ldots, a^m)$ is a constant function of x (so that $X = \sum a^i \frac{\partial}{\partial x^i}$ is the *constant vector field*, the solution curves are affine lines,

$$x(t) = x_0 + ta.$$

Example 5.13. Consider the vector field on $\mathbb{R}^2$,

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$



The corresponding differential equation is $\dot{x} = -y, \ \ \dot{y} = x$. Its solutions are $\gamma(t) = (x(t), y(t))$, where

$$x(t) = x_0 \cos(t) - y_0 \sin(t), \ \ y(t) = y_0 \cos(t) + x_0 \sin(t),$$

for any given $(x_0, y_0) \in \mathbb{R}^2$.

Example 5.14. Consider the following vector field on $\mathbb{R}^m$,

$$X = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}.$$

The corresponding differential equation is

$$\dot{x^i} = x^i(t),$$

with solution $x^i(t) = e^t\, x_0^i$, for $i = 1, \ldots, m$. That is,

$$x(t) = e^t\, x_0.$$

One of the main results from the theory of ODE's says that for any given initial condition $x(0) = x_0$, a solution to the system (5.4) exists and is (essentially) unique:

Theorem 5.3 (Existence and uniqueness theorem for ODE's). *Let $U \subseteq \mathbb{R}^m$ be an open subset, and $a \in C^\infty(U, \mathbb{R}^m)$. For any given $x_0 \in U$, there is an open interval $J_{x_0} \subseteq \mathbb{R}$ around 0, and a solution $x:\ J_{x_0} \to U$ of the ODE*

$$\frac{dx^i}{dt} = a^i(x(t)), \;\; i = 1, \ldots, m$$

with initial condition $x(0) = x_0$, and which is maximal in the sense that any other solution to this initial value problem is obtained by restriction to some subinterval of J_{x_0}.

Thus, J_{x_0} is the maximal open interval on which the solution is defined. The solution depends smoothly on initial conditions, in the following sense. For any given x_0, let $\Phi(t, x_0)$ be the solution $x(t)$ of the initial value problem with initial condition x_0, that is,

$$\Phi(0, x_0) = x_0, \;\; \frac{d}{dt}\Phi(t, x_0) = a(\Phi(t, x_0)).$$

Theorem 5.4 (Dependence on initial conditions for ODE's). *For $a \in C^\infty(U, \mathbb{R}^m)$ as above, the set*

$$\mathscr{J} = \{(t, x) \in \mathbb{R} \times U \,|\, t \in J_x\}.$$

is an open neighborhood of $\{0\} \times U$ in $\mathbb{R} \times U$, and the map

$$\Phi:\ \mathscr{J} \to U,\ (t, x) \mapsto \Phi(t, x)$$

is smooth.

In general, the interval J_{x_0} may be strictly smaller than $\mathbb{R}$, because a solution might escape to infinity in finite time.

Examples 5.15. 1. Consider the ODE

$$\dot{x} = 1$$

on $U = (0, 1) \subseteq \mathbb{R}$. Thus $a(x) = 1$. The solution curves with initial condition $x_0 \in U$ are $x(t) = x_0 + t$, defined for $-x_0 < t < 1 - x_0$. Thus $J_{x_0} = (-x_0, 1 - x_0)$, and

$$\mathscr{J} = \{(t,x)|x \in (0,1), t+x \in (0,1)\}, \ \ \Phi(t,x) = t+x.$$

2. Conside the ODE
$$\dot{x} = x^2$$
on $U = \mathbb{R}$. Here the solution curves escape to infinity in finite time. The initial value problem has solutions
$$x(t) = \frac{x_0}{1-tx_0},$$
with domain of definition
$$J_{x_0} = \{t \in \mathbb{R}|\ tx_0 < 1\}.$$
The set $\mathscr{J} = \{(t,x)|tx < 1\}$ is the region between the two branches of the hyperbola $tx = 1$, and $\Phi(t,x) = \frac{x}{1-tx}$.
3. A similar example, which we leave as an exercise, is
$$\dot{x} = 1 + x^2.$$

For a general vector field $X \in \mathfrak{X}(M)$ on manifolds, Equation (5.3) becomes (5.4) after introduction of local coordinates. In detail: Let (U,φ) be a coordinate chart. In the chart, X becomes the vector field

$$\varphi_*(X) = \sum_{j=1}^m a^j(u)\frac{\partial}{\partial u^j}$$

and $\varphi(\gamma(t)) = u(t)$ with
$$\dot{u}^i = a^i(u(t)).$$

If $a = (a^1,\ldots,a^m): \ \varphi(U) \to \mathbb{R}^m$ corresponds to X in a local chart (U,φ), then any solution curve $x: \ J \to \varphi(U)$ for a defines a solution curve $\gamma(t) = \varphi^{-1}(x(t))$ for X. The existence and uniqueness theorem for ODE's extends to manifolds, as follows:

Theorem 5.5 (Solutions of vector fields on manifolds). *Let $X \in \mathfrak{X}(M)$ be a vector field on a manifold M. For any given $p \in M$, there is an open interval $\mathscr{J}_p \subseteq \mathbb{R}$ around 0, and a solution $\gamma: \ \mathscr{J}_p \to M$ of the initial value problem*

$$\dot{\gamma}(t) = X_{\gamma(t)}, \ \ \gamma(0) = p, \tag{5.5}$$

which is maximal in the sense that any other solution of the initial value problem is obtained by restriction to a subinterval. The set

$$\mathscr{J} = \{(t,p) \in \mathbb{R}\times M|\ t \in \mathscr{J}_p\}$$

is an open neighborhood of $\{0\}\times M$, and the map

$$\Phi: \ \mathscr{J} \to M, \ (t,p) \mapsto \Phi(t,p)$$

such that $\gamma(t) = \Phi(t,p)$ solves the initial value problem (5.5), is smooth.

Proof. Existence and uniqueness of solutions for small times t follows from the existence and uniqueness theorem for ODE's, by considering the vector field in local charts. To prove uniqueness even for large times t, let $\gamma\colon J \to M$ be a maximal solution of (5.5) (i.e., a solution that cannot be extended to a larger open interval), and let $\gamma_1 : J_1 \to M$ be another solution of the same initial value problem, but with $\gamma_1(t) \neq \gamma(t)$ for some $t \in J,\ t > 0$. (There is a similar discussion if the solution is different for some $t < 0$). Then we can define

$$b = \inf\{t \in J \mid t > 0,\ \gamma_1(t) \neq \gamma(t)\}.$$

By the uniqueness for small t, we have $b > 0$. We will get a contradiction in both of the following cases:

Case 1: $\gamma_1(b) = \gamma(b) =: q$. Then both $\lambda_1(s) = \gamma_1(b+s)$ and $\lambda(s) = \gamma(b+s)$ are solutions to the initial value problem

$$\lambda(0) = q, \quad \dot{\lambda}(s) = X_{\lambda(s)};$$

hence they have to agree for small $|s|$, and consequently $\gamma_1(t), \gamma(t)$ have to agree for t close to b. This contradicts the definition of b.

Case 2: $\gamma_1(b) \neq \gamma(b)$. Using the Hausdorff property of M, we can choose disjoint open neighborhoods U of $\gamma(b)$ and U_1 of $\gamma(b_1)$. For $t = b - \varepsilon$ with $\varepsilon > 0$ sufficiently small, $\gamma(t) \in U$ while $\gamma_1(t) \in U_1$. But this is impossible since $\gamma(t) = \gamma_1(t)$ for $0 \leq t < b$.

The result for ODE's about the smooth dependence on initial conditions shows, by taking local coordinate charts, that $\mathscr{J}$ contains an open neighborhood of $\{0\} \times M$, on which Φ is given by a smooth map. The fact that $\mathscr{J}$ itself is open, and the map Φ is smooth everywhere, follows by the 'flow property' to be discussed below. (We will omit the details of this part of the proof.) □

Note that the uniqueness part uses the Hausdorff property in the definition of manifolds. Indeed, the uniqueness part may fail for non-Hausdorff manifolds.

Example 5.16. A counter-example is the non-Hausdorff manifold

$$Y = (\mathbb{R} \times \{1\}) \cup (\mathbb{R} \times \{-1\}) / \sim,$$

where $\sim$ glues two copies of the real line along the strictly negative real axis. Let $U_\pm$ denote the charts obtained as images of $\mathbb{R} \times \{\pm 1\}$. Let X be the vector field on Y, given by $\frac{\partial}{\partial x}$ in both charts. It is well-defined, since the transition map is just the identity map. Then $\gamma_+(t) = \pi(t,1)$ and $\gamma_-(t) = \pi(t,-1)$ are both solution curves, and they agree for negative t but not for positive t.

Given a vector field X, the map $\Phi : \mathscr{J} \to M$ is called the *flow* of X. For any given p, the curve $\gamma(t) = \Phi(t,p)$ is a solution curve. But one can also fix t and consider the time-t flow,

$$\Phi_t(p) \equiv \Phi(t,p).$$

It is a smooth map $\Phi_t : U_t \to M$, defined on the open subset

$$U_t = \{p \in M | \ (t,p) \in \mathscr{J}\}.$$

Note that $\Phi_0 = \mathrm{id}_M$.

Intuitively, $\Phi_t(p)$ is obtained from the initial point $p \in M$ by flowing for time t along the vector field X. One expects that first flowing for time t, and then flowing for time s, should be the same as flowing for time $t+s$. Indeed one has the following *flow property*.

Theorem 5.6 (Flow property). *Let $X \in \mathfrak{X}(M)$, with flow $\Phi : \mathscr{J} \to M$. Let $(t_2, p) \in \mathscr{J}$, and $t_1 \in \mathbb{R}$. Then*

$$(t_1, \Phi_{t_2}(p)) \in \mathscr{J} \Leftrightarrow (t_1+t_2, p) \in \mathscr{J},$$

and one has

$$\Phi_{t_1}(\Phi_{t_2}(p)) = \Phi_{t_1+t_2}(p).$$

Proof. Given $t_2 \in J_p$, we consider both sides as functions of $t_1 = t$. Write $q = \Phi_{t_2}(p)$. We claim that both

$$t \mapsto \Phi_t(\Phi_{t_2}(p)), \quad t \mapsto \Phi_{t+t_2}(p)$$

are maximal solution curves of X, for the same initial condition q. This is clear for the first curve, and follows for the second curve by the calculation, for $f \in C^\infty(M)$,

$$\frac{d}{dt} f(\Phi_{t+t_2}(p)) = \frac{d}{ds}\Big|_{s=t+t_2} \Phi_s(p) = X_{\Phi_s(p)}(f)\Big|_{s=t+t_2} = X_{\Phi_{t+t_2}(p)}(f).$$

Hence, the two curves must coincide. The domain of definition of $t \mapsto \Phi_{t+t_2}(p)$ is the interval $\mathscr{J}_p$, shifted by t_2. Hence, $t_1 \in J_{\Phi(t_2,p)}$ if and only if $t_1 + t_2 \in J_p$. $\square$

We see in particular that for any t, the map $\Phi_t : U_t \to M$ is a diffeomorphism onto its image $\Phi_t(U_t) = U_{-t}$, with inverse Φ_{-t}.

Example 5.17. Let us illustrate the flow property for various vector fields on $\mathbb{R}$. The flow property is evident for $\frac{\partial}{\partial x}$ with flow $\Phi_t(x) = x+t$, as well as for $x\frac{\partial}{\partial x}$, with flow $\Phi_t(x) = e^t x$. The vector field $x^2\frac{\partial}{\partial x}$ has flow $\Phi_t(x) = x/(1-tx)$, defined for $1-tx < 1$. We can explicitly verify the flow property:

$$\Phi_{t_1}(\Phi_{t_2}(x)) = \frac{\Phi_{t_2}(x)}{1-t_1\Phi_{t_2}(x)} = \frac{\frac{x}{1-t_2x}}{1-t_1\frac{x}{1-t_2x}} = \frac{x}{1-(t_1+t_2)x} = \Phi_{t_1+t_2}(x).$$

Let X be a vector field, and $\mathscr{J} = \mathscr{J}^X$ be the domain of definition for the flow $\Phi = \Phi^X$.

Definition 5.7. A vector field $X \in \mathfrak{X}(M)$ is called complete if $\mathscr{J}^X = R \times M$.

Thus X is complete if and only if all solution curves exist for all time.

Example 5.18. The vector field $x\frac{\partial}{\partial x}$ on $M = \mathbb{R}$ is complete, but $x^2\frac{\partial}{\partial x}$ is incomplete.

A vector field may fail to be complete if a solution curve escapes to infinity in finite time. This suggests that a vector fields X that vanishes outside a compact set must be complete, because the solution curves are 'trapped' and cannot escape to infinity.

Proposition 5.4. *If $X \in \mathfrak{X}(M)$ is a vector field that has compact support, in the sense that $X|_{M-A} = 0$ for some compact subset A, then X is complete. In particular, every vector field on a compact manifold is complete.*

Proof. By the uniqueness theorem for solution curves γ, and since X vanishes outside A, if $\gamma(t_0) \in M - A$ for some t_0, then $\gamma(t) = \gamma(t_0)$ for all t. Hence, if a solution curve $\gamma : \; J \to M$ has $\gamma(0) \in A$, then $\gamma(t) \in A$ for all t. Let $U_\varepsilon \subseteq M$ be the set of all p such that the solution curve γ with initial condition $\gamma(0) = p$ exists for $|t| < \varepsilon$ (that is, $(-\varepsilon, \varepsilon) \subseteq J_p$). By smooth dependence on initial conditions, U_ε is open. The collection of all U_ε with $\varepsilon > 0$ covers A, since every solution curve exists for sufficiently small time. Since A is compact, there exists a finite subcover $U_{\varepsilon_1}, \ldots, U_{\varepsilon_k}$. Let ε be the smallest of $\varepsilon_1, \ldots, \varepsilon_k$. Then $U_{\varepsilon_i} \subseteq U_\varepsilon$, for all i, and hence $A \subseteq U_\varepsilon$. Hence, for any $p \in A$ we have $(-\varepsilon, \varepsilon) \subseteq J_p$, that is any solution curve $\gamma(t)$ starting in A exists for times $|t| < \varepsilon$. But $\gamma(-\varepsilon/2), \gamma(\varepsilon/2) \in A$, hence the solution curve starting at those points again exist for times $< \varepsilon$. This shows $(-3\varepsilon/2, 3\varepsilon/2) \subseteq J_p$. Continuing in this way, we find that $(-\varepsilon - N\varepsilon/2, \varepsilon + N\varepsilon/2) \subseteq J_p$ for all N, thus $J_p = \mathbb{R}$ for all $p \in A$. For points $p \in M - A$, it is clear anyhow that $J_p = \mathbb{R}$, since the solution curves are constant. $\square$

Theorem 5.7. *If X is a complete vector field, the flow Φ_t defines a 1-parameter group of diffeomorphisms. That is, each Φ_t is a diffeomorphism and*

$$\Phi_0 = \mathrm{id}_M, \quad \Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}.$$

Conversely, if Φ_t is a 1-parameter group of diffeomorphisms such that the map $(t, p) \mapsto \Phi_t(p)$ is smooth, the equation

$$X_p(f) = \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

defines a complete vector field X on M, with flow Φ_t.

Proof. It remains to show the second statement. Given Φ_t, the linear map

$$C^\infty(M) \to C^\infty(M), \quad f \mapsto \frac{d}{dt}\Big|_{t=0} f(\Phi_t(p))$$

satisfies the product rule, hence it is a vector field X. Given $p \in M$ the curve $\gamma(t) = \Phi_t(p)$ is an integral curve of X since

$$\frac{d}{dt}\Phi_t(p) = \frac{d}{ds}\Big|_{s=0} \Phi_{t+s}(p) = \frac{d}{ds}\Big|_{s=0} \Phi_s(\Phi_t(p)) = X_{\Phi_t(p)}.$$

□

Remark 5.3. In terms of pull-backs, the relation between the vector field and its flow reads as

$$\frac{d}{dt}\Phi_t^*(f) = \Phi_t^* \frac{d}{ds}\Big|_{s=0}\Phi_s^*(f) = \Phi_t^* X(f).$$

This identity

$$\frac{d}{dt}\Phi_t^* = \Phi_t^* \circ X$$

as linear maps $C^\infty(M) \to C^\infty(M)$ may be viewed as the definition of the flow.

Example 5.19. Given $A \in \mathrm{Mat}_{\mathbb{R}}(m)$ let

$$\Phi_t:\ \mathbb{R}^m \to \mathbb{R}^m,\ \ x \mapsto e^{tA}x = \Big(\sum_{j=0}^{\infty} \frac{t^j}{j!}A^j\Big)x$$

(using the exponential map of matrices). Since $e^{(t_1+t_2)A} = e^{t_1 A}e^{t_2 A}$, and since $(t,x) \mapsto e^{tA}x$ is a smooth map, Φ_t defines a flow. What is the corresponding vector field X? For any function $f \in C^\infty(\mathbb{R}^m)$ we calculate,

$$\begin{aligned} X(f)(x) &= \frac{d}{dt}\Big|_{t=0} f(e^{tA}x) \\ &= \sum_j \frac{\partial f}{\partial x^j}(Ax)^j \\ &= \sum_{ij} A_i{}^j x^i \frac{\partial f}{\partial x^j} \end{aligned}$$

showing that

$$X = \sum_{ij} A_i{}^j x^i \frac{\partial}{\partial x^j}.$$

2

As a special case, taking A to be the identity matrix, we recover the Euler vector field $X = \sum_i x^i \frac{\partial}{\partial x^i}$, and its flow $\Phi_t(x) = e^t x$.

Example 5.20. Let X be a complete vector field, with flow Φ_t. For each $t \in \mathbb{R}$, the tangent map $T\Phi_t:\ TM \to TM$ has the flow property,

$$T\Phi_{t_1} \circ T\Phi_{t_2} = T(\Phi_{t_1} \circ \Phi_{t_2}) = T(\Phi_{t_1+t_2}),$$

---

[2] Here we wrote the matrix entries for the i-th row and j-th column as $A_i{}^j$ rather than A_{ij}. That is, one standard basis vectors $e_i \in \mathbb{R}^m$ (written as column vectors), we have $A(e_i) = \sum_j A_i{}^j e_j$, hence for $x = \sum x^i e_i$ we get

$$Ax = \sum_{ij} A_i{}^j x^i e_j$$

from which we read off $(Ax)^j = \sum_i A_i{}^j x^i$.

and the map $\mathbb{R}\times TM \to TM, (t,v)\mapsto \Phi_t(v)$ is smooth (since it is just the restriction of the map $T\Phi : T(\mathbb{R}\times M)\to TM$ to the submanifold $\mathbb{R}\times TM$). Hence, $T\Phi_t$ is a flow on TM, and therefore corresponds to a complete vector field $\widehat{X}\in\mathfrak{X}(TM)$. This is called the *tangent lift* of X.

Proposition 5.5. *Let $F\in C^\infty(M,N)$, and $X\in\mathfrak{X}(M)$, $Y\in\mathfrak{X}(N)$ complete vector fields, with flows Φ_t^X, Φ_t^Y.*

$$X\sim_F Y \quad\Leftrightarrow\quad F\circ\Phi_t^X=\Phi_t^Y\circ F \quad \textit{for all } t.$$

[3]

In short, vector fields are F-related if and only if their flows are F-related.

Proof. Suppose $F\circ\Phi_t^X=\Phi_t^Y\circ F$ for all t. For $g\in C^\infty(N)$, and $p\in M$, taking a t-derivative of

$$g(F(\Phi_t^X(p)))=g(\Phi_t^Y(F(p)))$$

at $t=0$ on both sides, we get

$$\big(T_pF(X_p)\big)(g)=Y_{F(p)}(g)$$

i.e. $T_pF(X_p)=Y_{F(p)}$. Hence $X\sim_F Y$. Conversely, suppose $X\sim_F Y$. As we had seen, if $\gamma:\ J\to M$ is a solution curve for X, with initial condition $\gamma(0)=p$ then $F\circ\gamma:\ J\to M$ is a solution curve for Y, with initial condition $F(p)$. That is, $F(\Phi_t^X(p))=\Phi_t^Y(F(p))$, or $F\circ\Phi_t^X=\Phi_t^Y\circ F$. $\square$

5.6 Geometric interpretation of the Lie bracket

For any smooth map $F\in C^\infty(M,N)$ we defined the pull-back

$$F^*:\ C^\infty(N)\to C^\infty(M),\quad g\mapsto g\circ F.$$

If F is a diffeomorphism, then we can also pull back vector fields:

$$F^*:\ \mathfrak{X}(N)\to\mathfrak{X}(M),\ Y\mapsto F^*Y,$$

by the condition $(F^*Y)(F^*g)=F^*(Y(g))$ for all functions g. That is, $F^*Y\sim_F Y$, or in more detail

$$(F^*Y)_p=(T_pF)^{-1}Y_{F(p)}.$$

By Theorem 5.2, we have $F^*[X,Y]=[F^*X,F^*Y]$.

Any complete vector field $X\in\mathfrak{X}(M)$ with flow Φ_t gives rise to a families of pull-back maps

[3] This generalizes to possibly incomplete vector fields: The vector fields are related if and only if $F\circ\Phi=\Phi\circ(\mathrm{id}_{\mathbb{R}}\times F)$. But for simplicity, we only consider the complete case.

$$\Phi_t^*:\ C^\infty(M)\to C^\infty(M),\quad \Phi_t^*:\ \mathfrak{X}(M)\to\mathfrak{X}(M).$$

The Lie derivative of a function f with respect to X is the function

$$L_X(f)=\frac{d}{dt}\Big|_{t=0}\Phi_t^* f;$$

thus $L_X(f)=X(f)$. The Lie derivative measures how f changes in the direction of X. Similarly, for a vector field Y one defines the *Lie derivative* $L_X(Y)$ by

$$L_X(Y)=\frac{d}{dt}\Big|_{t=0}\Phi_t^* Y\in\mathfrak{X}(M).$$

The definition of Lie derivative also works for incomplete vector fields, since the definition only involves derivatives at $t=0$. The Lie derivative measures how Y changes in the direction of X. Note that

$$(\Phi_t^*Y)_p=(T_p\Phi_t^{-1})\,Y_{\Phi_t(p)};$$

that is, we use the inverse to the tangent map of the flow of X to move $Y_{\Phi_t(p)}$ to p. If Y were invariant under the flow of X, this would agree with Y_p; hence $(\Phi_t^*Y)_p-Y_p$ measures how Y fails to be Φ_t-invariant. L_XY is the infinitesimal version of this. As we will see below, the infinitesimal version actually implies the global version.

Theorem 5.8. *For any $X,Y\in\mathfrak{X}(M)$, the Lie derivative L_XY is just the Lie bracket:*

$$L_X(Y)=[X,Y].$$

Proof. Let $\Phi_t=\Phi_t^X$ be the flow of X. For all $f\in C^\infty(M)$ we obtain, by taking the t-derivative at $t=0$ of both sides of

$$\Phi_t^*(Y(f))=(\Phi_t^*Y)(\Phi_t^*f),$$

that

$$X(Y(f))=\Big(\frac{d}{dt}\Big|_{t=0}\Phi_t^*Y\Big)(f)+Y\Big(\frac{d}{dt}\Big|_{t=0}\Phi_t^*f\Big)=(L_XY)(f)+Y(X(f)).$$

That is, $L_XY=X\circ Y-Y\circ X=[X,Y]$. □

Thus, the Lie bracket $[X,Y]$ measures 'infinitesimally' how the vector field Y changes along the flow of X. Note that in particular, L_XY is skew-symmetric in X and Y – this is not obvious from the definition.

One can also interpret the Lie bracket as measuring how the flows of X and Y fail to commute.

Theorem 5.9. *Let X,Y be complete vector fields, with flows Φ_t,Ψ_s. Then*

$$\begin{aligned}[X,Y]=0 &\Leftrightarrow \Phi_t^*Y=Y \textit{ for all } t\\ &\Leftrightarrow \Psi_s^*X=X \textit{ for all } s\end{aligned}$$

$$\Leftrightarrow \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \textit{ for all } s,t.$$

Proof. The calculation

$$\frac{d}{dt}(\Phi_t)^* Y = (\Phi_t)^* L_X Y = (\Phi_t)^* [X,Y]$$

shows that $\Phi_t^* Y$ is independent of t if and only if $[X,Y] = 0$. Since $[Y,X] = -[X,Y]$, interchanging the roles of X,Y this is also equivalent to $\Psi_s^* X$ being independent of s. The property $\Phi_t^* Y = Y$ means that Y is Φ_t-related to itself, hence it takes the flow of Ψ_s to itself, that is

$$\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t.$$

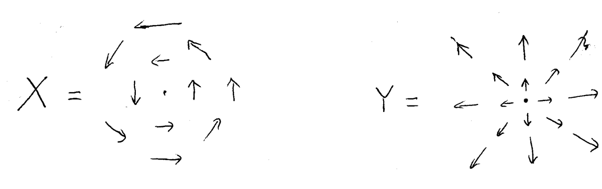
Conversely, if this equation holds then $\Phi_t^*(\Psi_s^* f) = \Psi_s^*(\Phi_t^* f)$ for all $f \in C^\infty(M)$. Differentiating with respect to s at $s = 0$, we obtain

$$\Phi_t^*(Y(f)) = Y(\Phi_t^* f).$$

Hence $\Phi_t^*(Y) = Y$. Differentiating with respect to t at $t = 0$, we get that $[X,Y] = 0$. $\square$

Example 5.21. If $X = \frac{\partial}{\partial y}$ as a vector field on $\mathbb{R}^2$, then $[X,Y] = 0$ if and only if Y is invariant under translation in the y-direction.

Example 5.22. The vector fields $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ commute. This is verified by direct calculation but can also be 'seen' in the following picture



The flow of X is rotations around the origin, but Y is invariant under rotations. Likewise, the flow of Y is by dilations away from the origin,, but X is invariant under dilations.

Aside from being skew-symmetric $[X,Y] = -[Y,X]$, the Lie bracket of vector fields satisfies the important *Jacobi identity*.

Proposition 5.6. *The Lie bracket of vector fields satisfies the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Proof. This may be proved 'by hand', expanding the definition of the Lie bracket $[X,Y] = X \circ Y - Y \circ X$. Each summand gives rise to 4 terms, hence there are altogether 12 terms. Each of the $3! = 6$ orderings of X,Y,Z appears twice, with opposite signs. For example, the term $Y \circ Z \circ X$ appears with coefficient -1 in $[X,[Y,Z]]$, with coefficient $+1$ in $[Y,[Z,X]]$, with coefficient 0 in $[Z,[X,Y]]$. □

The identity may be equivalently stated as

$$[L_X, L_Y]Z = L_{[X,Y]}Z,$$

or also as a 'derivation property'

$$L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z].$$

This last form gives an 'explanation' of the Jacobi identity, as the derivative at $t = 0$ of the identity

$$\Phi_t^*[Y,Z] = [\Phi_t^* Y, \Phi_t^* Z],$$

where Φ_t is the flow of X.

5.7 Frobenius theorem

We saw that for any vector field $X \in \mathfrak{X}(M)$, there are *solution curves* through any given point $p \in M$. The image of this curve is an (immersed) submanifold to which X is everywhere tangent. One might similarly 'integral surfaces' for pairs of vector fields, and 'integral submanifolds' for collections of vector fields.

Suppose $X_1, \ldots, X_r$ are vector fields on the manifold M, such that the tangent vectors $X_1|_p, \ldots, X_r|_p \in T_pM$ are linearly independent for all $p \in M$. A r-dimensional submanifold $S \subseteq M$ is called an *integral submanifold* if the vector fields $X_1, \ldots, X_r$ are all tangent to S.

Suppose that there exists an integral submanifold S through any given point $p \in M$. Then each Lie bracket $[X_i, X_j]|_p \in T_pS$, and hence is a linear combination of $X_1|_p, \ldots, X_r|_p$. It follows that

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^k X_k \tag{5.6}$$

for certain (smooth) functions c_{ij}^k.

A bit more generally, consider a sub-bundle $E \subseteq TM$ of rank r. Such a subbundle is called *involutive* if the Lie bracket of any two sections of E is again a section of E. For vector fields X_i as above, the pointwise spans

$$E_p = \operatorname{span}\{X_1|_p, \ldots, X_r|_p\}$$

define a subbundle with this property. Indeed, given $X = \sum_{i=1}^m a^i X_i$ and $Y = \sum_{i=1}^m b^i X_i$ with functions a^i, b^i, the condition (5.6) guarantees that E is involutive. Given any rank r subbundle $E \subseteq TM$ (not necessarily involutive), a submanifold $S \subseteq M$ is called an *integral submanifold* if $E_p = T_pS$ for all $p \in S$. The following result is due to F. G. Frobenius. [4]

Theorem 5.10 (Frobenius theorem). *Let $E \subseteq TM$ be a subbundle of rank r. The following are equivalent:*

1. *There exists an integral submanifold through every $p \in M$.*
2. *E is involutive.*

In fact, if E is involutive, then it is possible to find a coordinate chart (U,φ) near any given p, in such a way that the subbundle $(T\varphi)(E|_U) \subseteq T\varphi(U)$ is spanned by the first $r \le m$ coordinate vector fields

$$\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^r}.$$

Proof. The statement is local; hence, by choosing coordinates we may assume M is an open subset $U \subseteq \mathbb{R}^m$, with $p = (0,0)$. In particular, the tangent spaces are all identified with $\mathbb{R}^m$. By re-indexing the coordinates, we may assume that $E_p \cap (0 \oplus \mathbb{R}^{m-r}) = 0$. It is convenient to denote the first r coordinates by $x^1, \dots, x^r$ and the remaining coordinates by $y^1, \dots, y^{m-r}$. Thus E_p projects isomorphically onto the coordinate subspace spanned by $x^1, \dots, x^r$. Taking U smaller if necessary, we may assume that this remains true for all points in U. Then E is spanned by vector fields of the form

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m-r} a_i^j(x,y) \frac{\partial}{\partial y^j}.$$

We claim that the X_i commute. Indeed, since $[X_i, X_j]$ takes values in E, it is of the form $\sum_k c_{ij}^k X_k$ for some functions c_{ij}^k. By comparing the coefficients in front of $\frac{\partial}{\partial x^k}$, we see that $c_{ij}^k = 0$. (Indeed, $[X_i, X_j]$ is a linear combination of vector fields in the y-direction.) Thus

$$[X_i, X_j] = 0.$$

Since the X_i commute, also their flows Φ_{i,t_i} commute:

$$\Phi_{i,t_i} \circ \Phi_{j,t_j} = \Phi_{j,t_j} \circ \Phi_{i,t_i}.$$

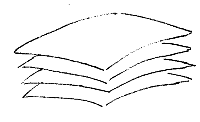
[4] http://upload.wikimedia.org/wikipedia/en/c/c9/Ferdinand_Georg_Frobenius.jpg

Note also that $\Phi_{i,t_i}(x^1,\ldots,x^r,*) = (x^1,\ldots,x^i+t_i,\ldots,x^r,*)$. (This follows because X_i is related to $\frac{\partial}{\partial x^i}$ under projection $(x,y)\mapsto x$.) We define a change of coordinates by the equation

$$(x,y) = \kappa(u,v) := \Phi_{1,u^1}\circ\cdots\circ\Phi_{r,u^r}(0,v).$$

(The Jacobian for this change of variables is invertible at $(0,0)$. Indeed, note that $\kappa(0,v)=(0,v)$, while $\kappa(u,0)=(u,*)$ where $*$ indicates some function. This implies that the Jacobian matrix at $(0,0)$ is upper triangular with 1's along the diagonal, hence that its determinant is 1.) In these new coordinated the flow of the X_i is simply addition of t^i in the i-th entry. This means that $X_i = \frac{\partial}{\partial u^i}$. Each subspace consisting of elements (u,v) with $v=$ const is an integral submanifold. [MORE DETAILS NEEDED] ☐

Thus, for any involutive subbundle $E\subseteq TM$, then any $p\in M$ has an open neighborhood U with a nice decomposition into r-dimensional submanifolds.



One calls such a decomposition (or sometimes the involutive subbundle E itself) a (local) *foliation*.

Example 5.23. Let $\Phi:\ M\to N$ be a submersion. Then the subbundle $E\subseteq TM$ with fibers

$$E_p = \ker(T_p\Phi)\subseteq T_pM$$

is an involutive subbundle of rank $\dim M - \dim N$. Every fiber $\Phi^{-1}(q)$ is an integral submanifold.

Example 5.24. Consider the vector fields on $\mathbb{R}^3$,

$$X = (y-z)\frac{\partial}{\partial x},\ \ Y = \frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

Away from $y=z$ these are linearly independent. Since $[X,Y]=0$, the Frobenius theorem tells us there are integral submanifolds. Indeed, for any given $C\neq 0$ one has the integral submanifold given by the equation $y-z=C$.

Remark 5.4. A foliation gives a decomposition into submanifolds on a neighborhood of any given point. Globally, the integral submanifolds are often only *immersed* submanifolds, given by immersions $i:\ S\to M$ with $(T_pi)(T_pS)=E_p$ for all $p\in S$. The problem is already present for the foliation defined by a single non-vanishing vector field $X=X_1$: It may happen that a solution curve γ through p gets arbitrarily close to p for large t; hence one cannot get a submanifold chart at p unless one restricts the domain of γ.

5.8 Appendix: Derivations

A vector field on a manifold can be regarded as a derivation of the algebra of smooth functions.

Let us quickly recall the notion of a derivation.

Definition 5.8. A *derivation* of an algebra $\mathscr{A}$ is a linear map $D: \mathscr{A} \to \mathscr{A}$ satisfying the product rule

$$D(a_1 a_2) = D(a_1)a_2 + a_1 D(a_2).$$

Remarks 5.5. 1. If $\dim A < \infty$, a derivation is an *infinitesimal automorphism* of an algebra. Indeed, let $U: \mathbb{R} \to \mathrm{End}(A),\ t \mapsto U_t$ be a smooth curve with $U_0 = I$, such that each U_t is an algebra automorphism. Consider the Taylor expansion,

$$U_t = I + tD + \dots$$

here

$$D = \frac{d}{dt}\Big|_{t=0} U_t$$

is the velocity vector at $t = 0$. By taking the derivative of the condition

$$U_t(a_1 a_2) = U_t(a_1)U_t(a_2)$$

at $t = 0$, we get the derivation property for D. Conversely, if D is a derivation, then

$$U_t = \exp(tD) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n$$

(using the exponential of a matrix) is a well-defined curve of algebra automorphisms. We leave it as an exercise to check the automorphism property; it involves proving the property

$$D^n(a_1 a_2) = \sum_k \binom{n}{k} D^k(a_1)\, D^{n-k}(a_2)$$

for all $a_1, a_2 \in A$.

If A has infinite dimensions, one may still want to think of derivations D as infinitesimal automorphisms, even though the discussion will run into technical problems. (For instance, the exponential map of infinite rank endomorphisms is not well-defined in general.)

2. Any given $x \in A$ defines a derivation

$$D(a) = [x, a] := xa - ax.$$

(Exercise: Verify that this is a derivation.) These are called *inner derivations*. If A is commutative (for example $A = C^\infty(M)$) the inner derivations are all trivial. At the other extreme, for the matrix algebra $A = \mathrm{Mat}_{\mathbb{R}}(n)$, one may show that every derivation is inner.

3. If A is a unital algebra, with unit 1_A, then $D(1_A) = 0$ for all derivations D. (This follows by applying the defining property of derivations to $1_A = 1_A 1_A$.)
4. Given two derivations D_1, D_2 of an algebra A, their commutator

$$[D_1, D_2] = D_1 D_2 - D_2 D_1$$

is again a derivation. Indeed, if $a, b \in A$ then

$$\begin{aligned} D_1 D_2(ab) &= D_1\big(D_2(a)b + aD_2(b)\big) \\ &= (D_1 D_2)(a)b + a(D_1 D_2)(b) + D_1(a)D_2(b) + D_2(a)D_1(b). \end{aligned}$$

Subtracting a similar expression with $1,2$ interchanged, one obtains the derivation property of $[D_1, D_2]$.
5. If the algebra A is commutative, then the space of derivations is a 'left-module over A'. That is, if D is a derivation and $x \in A$ then $a \mapsto (xD)(a) := xD(a)$ is again a derivation:

$$(xD)(ab) = x(D(ab)) = x(D(a)\,b + a(D(b)) = (xD)(a)b + a\,(xD)(b),$$

where we used $xa = ax$.

Chapter 6
Differential forms

6.1 Review: Differential forms on $\mathbb{R}^m$

A differential k-form on an open subset $U \subseteq \mathbb{R}^m$ is an expression of the form

$$\omega = \sum_{i_1 \cdots i_k} \omega_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

where $\omega_{i_1 \ldots i_k} \in C^\infty(U)$ are functions, and the indices are numbers

$$1 \leq i_1 < \cdots < i_k \leq m.$$

Let $\Omega^k(U)$ be the vector space consisting of such expressions, with the pointwise addition. It is convenient to introduce a short hand notation $I = \{i_1, \ldots, i_k\}$ for the index set, and write $\omega = \sum_I \omega_I \mathrm{d}x^I$ with

$$\omega_I = \omega_{i_1 \ldots i_k}, \quad \mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Since a k-form is determined by these functions ω_I, and since there are $\frac{m!}{k!(m-k)!}$ ways of picking k-element subsets from $\{1, \ldots, m\}$, the space $\Omega^k(U)$ can be identified with vector-valued smooth functions,

$$\Omega^k(U) = C^\infty(U,\ \mathbb{R}^{\frac{m!}{k!(m-k)!}}).$$

The $\mathrm{d}x^I$ are just formal expressions; at this stage they don't have any particular meaning. They are used, however, to define an associative product operation

$$\Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$

by the 'rule of computation'

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$$

for all i,j; in particular $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$. In turn, using the product structure we may define the *exterior differential*

$$\mathrm{d}: \ \Omega^k(U) \to \Omega^{k+1}(U), \ \ \mathrm{d}\Big(\sum_I \omega_I \mathrm{d}x^I\Big) = \sum_{i=1}^m \sum_I \frac{\partial \omega_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I. \tag{6.1}$$

The key property of the exterior differential is the following fact:

Proposition 6.1. *The exterior differential satisfies*

$$\mathrm{d} \circ \mathrm{d} = 0,$$

i.e. $\mathrm{d}\mathrm{d}\omega = 0$ *for all* ω.

Proof. By definition,

$$\mathrm{d}\mathrm{d}\omega = \sum_{j=1}^m \sum_{i=1}^m \sum_I \frac{\partial^2 \omega_I}{\partial x^j \partial x^i} \mathrm{d}x^j \wedge \mathrm{d}x^i \wedge \mathrm{d}x^I,$$

which vanishes by equality of mixed partials $\frac{\partial \omega_I}{\partial x^i \partial x^j} = \frac{\partial \omega_I}{\partial x^j \partial x^i}$. (We have $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$, but the coefficients in front of $\mathrm{d}x^i \wedge \mathrm{d}x^j$ and $\mathrm{d}x^j \wedge \mathrm{d}x^i$ are the same.) □

Example 6.1. Consider forms on $\mathbb{R}^3$.

- The differential of a function $f \in \Omega^0(\mathbb{R}^3)$ is a 1-form

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z,$$

  with components the *gradient*

$$\mathrm{grad} f = \nabla f.$$

- A 1-form $\omega \in \Omega^1(\mathbb{R}^3)$ is an expression

$$\omega = f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z$$

  with functions f,g,h. The differential is

$$\mathrm{d}\omega = \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big)\mathrm{d}x \wedge \mathrm{d}y + \Big(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\Big)\mathrm{d}y \wedge \mathrm{d}z + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big)\mathrm{d}z \wedge \mathrm{d}x.$$

  Thinking of the coefficients of ω as the components of a function $F = (f,g,h) : U \to \mathbb{R}^3$, we see that the coefficients of $\mathrm{d}\omega$ give the curl of F,

$$\mathrm{curl}(F) = \nabla \times F.$$

- Finally, any 2-form $\omega \in \Omega^2(\mathbb{R}^3)$ may be written

$$\omega = a\ \mathrm{d}y \wedge \mathrm{d}z + b\ \mathrm{d}z \wedge \mathrm{d}x + c\ \mathrm{d}x \wedge \mathrm{d}y,$$

with $A = (a,b,c) : U \to \mathbb{R}^3$. We obtain

$$\mathrm{d}\omega = (\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z})\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z;$$

the coefficient is the divergence

$$\mathrm{div}(A) = \nabla \cdot A$$

The usual properties

$$\mathrm{curl}(\mathrm{grad}(f)) = 0, \quad \mathrm{div}(\mathrm{curl}(F)) = 0$$

are both special cases of $\mathrm{d} \circ \mathrm{d} = 0$.

The *support* $\mathrm{supp}(\omega) \subseteq U$ of a differential form is the smallest closed subset such that ω vanishes on $U \backslash \mathrm{supp}(\omega)$. Suppose $\omega \in \Omega^m(U)$ is a compactly supported form of the top degree $k = m$. Such a differential form is an expression

$$\omega = f\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f \in C^\infty(U)$ is a compactly supported function. One defines the integral of ω to be the usual Riemann integral:

$$\int_U \omega = \int_{\mathbb{R}^m} f(x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m. \tag{6.2}$$

Note that we can regard ω as a form on all of $\mathbb{R}^m$, due to the compact support condition.

Our aim is now to define differential forms on manifolds, beginning with 1-forms. Even though 1-forms on $U \subseteq \mathbb{R}^m$ are identified with functions $U \to \mathbb{R}^m$, they should not be regarded as vector fields, since their transformation properties under coordinate changes are different. In fact, while vector fields are sections of the tangent bundle, the 1-forms are sections of its dual, the cotangent bundle. We will thus begin with a review of dual spaces in general.

6.2 Dual spaces

For any real vector space E, we denote by $E^* = L(E, \mathbb{R})$ its dual space, consisting of all linear maps $\alpha : E \to \mathbb{R}$. We will assume that E is finite-dimensional. Then the dual space is also finite-dimensional, and $\dim E^* = \dim E$. [1] It is common to write the value of $\alpha \in E^*$ on $v \in E$ as a *pairing*, using the bracket notation:[2]

[1] For possibly infinite-dimensional vector spaces, the dual space E^* is not isomorphic to E, in general.

[2] In physics, one also uses the *Dirac bra-ket* notation $\langle \alpha |\, v \rangle := \alpha(v)$; here $\alpha = \langle \alpha |$ is the 'bra' and $v = | v \rangle$ is the 'ket'.

$$\langle \alpha, v\rangle := \alpha(v).$$

Let $e_1,\ldots,e_r$ be a basis of E. Any element of E^* is determined by its values on these basis vectors. For $i = 1,\ldots,r$, let $e^i \in E^*$ (with *upper* indices) be the linear functional such that

$$\langle e^i,\ e_j\rangle = \delta^i{}_j = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

The elements $e^1,\ldots,e^r$ are a basis of E^*; this is called the *dual basis*. The element $\alpha \in E^*$ is described in terms of the dual bases as

$$\alpha = \sum_{j=1}^{r} \alpha_j\, e^j, \quad \alpha_j = \langle \alpha, e_j\rangle.$$

Similarly, for vectors $v \in E$ we have

$$v = \sum_{i=1}^{r} v^i e_i, \quad v^i = \langle e^i, v\rangle.$$

Notice the placement of indices: In a given summation over $i, j, \ldots$, upper indices are always paired with lower indices.

Remark 6.1. As a special case, for $\mathbb{R}^r$ with its standard basis, we have a canonical identification $(\mathbb{R}^r)^* = \mathbb{R}^r$. For more general E with $\dim E < \infty$, there is no *canonical* isomorphism between E and E^* unless more structure is given.

Given a linear map $R:\ E \to F$ between vector spaces, one defines the *dual map*

$$R^*:\ F^* \to E^*$$

(note the direction), by setting

$$\langle R^*\beta,\ v\rangle = \langle \beta, R(v)\rangle$$

for $\beta \in F^*$ and $v \in E$. This satisfies $(R^*)^* = R$, and under the composition of linear maps,

$$(R_1 \circ R_2)^* = R_2^* \circ R_1^*.$$

In terms of basis $e_1,\ldots,e_r$ of E and $f_1,\ldots,f_s$ of F, and the corresponding dual bases (with upper indices), a linear map $R:\ E \to F$ is given by the matrix with entries

$$R_i{}^j = \langle f^j,\ R(e_i)\rangle,$$

while R^* is described by the *transpose* of this matrix (the roles of i and j are reversed). Namely,[3]

[3] In bra-ket notation, we have $R_i{}^j = \langle f^j\,|R\,|e_i\rangle$, and

$$|Re_i\rangle = R|e_i\rangle = \sum_j |f_j\rangle\langle f^j\,|R\,|e_i\rangle, \qquad \langle R^*(f^j)| = \langle (f^j)|R = \langle f^j\,|R\,|e_i\rangle\langle e^i|$$

$$R(e_i) = \sum_{j=1}^{s} R_i^{\ j} f_j, \qquad R^*(f^j) = \sum_{i=1}^{r} R_i^{\ j}\ f^i.$$

Thus,

$$(R^*)^j_{\ i} = R_i^{\ j}.$$

6.3 Cotangent spaces

Definition 6.1. The dual of the tangent space T_pM of a manifold M is called the *cotangent space* at p, denoted

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called cotangent vectors, or simply *covectors*. Given a smooth map $F \in C^\infty(M,N)$, and any $p \in M$ we have the *cotangent map*

$$T_p^*F = (T_pF)^* :\ T^*_{F(p)}N \to T_p^*M$$

defined as the dual to the tangent map.

Thus, a co(tangent) vector at p is a linear functional on the tangent space, assigning to each tangent vector at p a number. The very definition of the tangent space suggests one such functional: Every function $f \in C^\infty(M)$ defines a linear map, $T_pM \to \mathbb{R},\ \ v \mapsto v(f)$. This linear functional is denoted $(\mathrm{d}f)_p \in T_p^*M$.[4]

Definition 6.2. Let $f \in C^\infty(M)$ and $p \in M$. The covector

$$(\mathrm{d}f)_p \in T_p^*M, \quad \langle (\mathrm{d}f)_p, v\rangle = v(f).$$

is called the *differential of f at p*.

Lemma 6.1. *For $F \in C^\infty(M,N)$ and $g \in C^\infty(N)$,*

$$\mathrm{d}(F^*g)_p = T_p^*F((\mathrm{d}g)_{F(p)}).$$

Proof. Check on tangent vectors $v \in T_pM$,

$$\begin{aligned}\langle T_p^*F((\mathrm{d}g)_{F(p)}),\ v\rangle &= \langle (\mathrm{d}g)_{F(p)}),\ (T_pF)(v)\rangle \\ &= ((T_pF)(v))(g) \\ &= v(F^*g) \\ &= \langle \mathrm{d}(F^*g)_p,\ v\rangle.\end{aligned}$$

□

[4] Note that this is actually the same as the tangent map $T_pf :\ T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$.

Consider an open subset $U \subseteq \mathbb{R}^m$, with coordinates $x^1, \ldots, x^m$. Here $T_pU \cong \mathbb{R}^m$, with basis

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^m}\Big|_p \in T_pU \tag{6.3}$$

The basis of the dual space T_p^*U, dual to the basis (6.3), is given by the differentials of the coordinate functions:

$$(\mathrm{d}x^1)_p, \ \ldots, \ (\mathrm{d}x^m)_p \in T_p^*U.$$

Indeed,

$$\Big\langle (\mathrm{d}x^i)_p, \ \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \frac{\partial}{\partial x^j}\Big|_p (x^i) = \delta^i{}_j$$

as required. For $f \in C^\infty(M)$, the coefficients of $(\mathrm{d}f)_p = \sum_i \langle (\mathrm{d}f)_p, \ e_i \rangle e^i$ are determined as

$$\Big\langle (\mathrm{d}f)_p, \ \frac{\partial}{\partial x^j}\Big|_p \Big\rangle = \frac{\partial}{\partial x^j}\Big|_p (f) = \frac{\partial f}{\partial x^j}\Big|_p.$$

Thus,

$$(\mathrm{d}f)_p = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\Big|_p \ (\mathrm{d}x^i)_p.$$

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, with coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$. For $F \in C^\infty(U, V)$, the tangent map is described by the Jacobian matrix, with entries

$$(D_pF)_i{}^j = \frac{\partial F^j}{\partial x^i}(p)$$

for $i = 1, \ldots, m, \ j = 1, \ldots, n$. We have:

$$(T_pF)\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{j=1}^n (D_pF)_i{}^j \ \frac{\partial}{\partial y^j}\Big|_{F(p)},$$

hence dually

$$(T_pF)^*(\mathrm{d}y^j)_{F(p)} = \sum_{i=1}^m (D_pF)_i{}^j \ (\mathrm{d}x^i)_p. \tag{6.4}$$

Thought of as matrices, *the coefficients of the cotangent map are the transpose of the coefficients of the tangent map*.

6.4 1-forms

Similar to the definition of vector fields, one can define *co-vector fields*, more commonly known as *1-forms*: Collections of covectors $\alpha_p \in T_p^*M$ depending smoothly on the base point. One approach of making precise the smooth dependence on the base point is to endow the *cotangent bundle*

$$T^*M = \bigcup_p T_p^*M.$$

(disjoint union of all cotangent spaces), and require that the map $p \mapsto \alpha_p$ is smooth. The construction of charts on T^*M is similar to that for the tangent bundle: Charts (U, φ) of M give cotangent charts $(T^*U, T^*\varphi^{-1})$ of T^*M, using the fact that $T^*(\varphi(U)) = \varphi(U) \times \mathbb{R}^m$ canonically (since $\varphi(U)$ is an open subset of $\mathbb{R}^m$). Here $T^*\varphi^{-1}: T^*U \to T^*\varphi(U)$ is the union of inverses of all cotangent maps $T_p^*\varphi: T_{\varphi(p)}^*\varphi(U) \to T_p^*U$. A second approach is observe that in local coordinates, 1-forms are given by expressions $\sum_i f_i \mathrm{d}x^i$, and smoothness should mean that the coefficient functions are smooth.

We will use the following (equivalent) approach.

Definition 6.3. A 1-form on M is a linear map

$$\alpha: \mathfrak{X}(M) \to C^\infty(M), \quad X \mapsto \alpha(X) = \langle \alpha, X \rangle,$$

which is $C^\infty(M)$*-linear* in the sense that

$$\alpha(fX) = f\alpha(X)$$

for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$. The space of 1-forms is denoted $\Omega^1(M)$.

Let us verify that a 1-form can be regarded as a collection of covectors:

Lemma 6.2. *Let $\alpha \in \Omega^1(M)$ be a 1-form, and $p \in M$. Then there is a unique covector $\alpha_p \in T_p^*M$ such that*

$$\alpha(X)_p = \alpha_p(X_p)$$

for all $X \in \mathfrak{X}(M)$.

(We indicate the value of the function $\alpha(X)$ at p by a subscript, just like we did for vector fields.)

Proof. We have to show that $\alpha(X)_p$ depends only on the value of X at p. By considering the difference of vector fields having the same value at p, it is enough to show that if $X_p = 0$, then $\alpha(X)_p = 0$. But any vector field vanishing at p can be written as a finite sum $X = \sum_i f_i Y_i$ where $f_i \in C^\infty(M)$ vanish at p. [5] By C^∞-linearity, this implies that

$$\alpha(X) = \alpha(\sum_i f_i Y_i) = \sum_i f_i \alpha(Y_i)$$

vanishes at p. $\square$

The first example of a 1-form is described in the following definition.

Definition 6.4. The *exterior differential* of a function $f \in C^\infty(M)$ is the 1-form

$$\mathrm{d}f \in \Omega^1(M),$$

[5] For example, using local coordinates, we can take the Y_i to correspond to $\frac{\partial}{\partial x^i}$ near p, and the f_i to the coefficient functions.

defined in terms of its pairings with vector fields $X \in \mathfrak{X}(M)$ as $\langle \mathrm{d}f,\, X\rangle = X(f)$.

Clearly, $\mathrm{d}f$ is the 1-form defined by the family of covectors $(\mathrm{d}f)_p$. Note that critical points of f may be described in terms of this 1-form: $p \in M$ is a critical point of f if and only if $(\mathrm{d}f)_p = 0$.

Similar to vector fields, 1-forms can be multiplied by functions; hence one has more general examples of 1-forms as finite sums,

$$\alpha = \sum_i f_i \,\mathrm{d}g_i$$

where $f_i, g_i \in C^\infty(M)$.

Let us examine what the 1-forms are for open subsets $U \subseteq \mathbb{R}^m$. Given $\alpha \in \Omega^1(U)$, we have

$$\alpha = \sum_{i=1}^m \alpha_i \,\mathrm{d}x^i$$

with coefficient functions $\alpha_i = \left\langle \alpha,\, \frac{\partial}{\partial x^i}\right\rangle \in C^\infty(U)$. (Indeed, the right hand side takes on the correct values at any $p \in U$, and is uniquely determined by those values.) General vector fields on U may be written

$$X = \sum_{j=1}^m X^j \frac{\partial}{\partial x^j}$$

(to match the notation for 1-forms, we write the coefficients as X^i rather than a^i, as we did in the past), where the coefficient functions are recovered as $X^j = \langle \mathrm{d}x^j,\, X\rangle$. The pairing of the 1-form α with the vector field X is then

$$\langle \alpha,\, X\rangle = \sum_{i=1}^m \alpha_i X^i.$$

Lemma 6.3. *Let $\alpha :\ p \mapsto \alpha_p \in T_p^*M$ be a collection of covectors. Then α defines a 1-form, with*

$$\alpha(X)_p = \alpha_p(X_p)$$

for $p \in M$, if and only if for all charts (U, φ), the coefficient functions for α in the chart are smooth.

Proof. This is similar to the discussion for vector fields, and is left as an exercise.

6.5 Pull-backs of function and 1-forms

Recall again that for any manifold M, the vector space $C^\infty(M)$ of smooth functions is an algebra, with product the pointwise multiplication. Any smooth map $F :\ M \to M'$ between manifolds defined an algebra homomorphism, called the *pull-back*

$$F^* : C^\infty(M') \to C^\infty(M), \;\; f \mapsto F^*(f) := f \circ F.$$

The fact that this preserves products is the following simple calculation:

$$(F^*(f)F^*(g))(p) = f(F(p))g(F(p)) = (fg)(F(p)) = F^*(fg)(p).$$

Given another smooth map $F' : M' \to M''$ we have

$$(F' \circ F)^* = F^* \circ (F')^*$$

(note the ordering).

Let $F \in C^\infty(M,N)$ be a smooth map. Recall that for vector fields, there is no general 'push-forward' or 'pull-back' operation, unless F is a diffeomorphism. For 1-forms the situation is better. Indeed, for any $p \in M$ one has the dual to the tangent map

$$T_p^* F = (T_p F)^* : T_{F(p)}^* N \to T_p^* M.$$

For a 1-form $\beta \in \Omega^1(N)$, we can therefore define

$$(F^*\beta)_p := (T_p^* F)(\beta_{F(p)}).$$

Lemma 6.4. *The collection of co-vectors $(F^*\beta)_p \in T_p^* M$ depends smoothly on p, defining a 1-form $F^*\beta \in \Omega^1(M)$.*

Proof. By working on local coordinates, we may assume that M is an open subset $U \subseteq \mathbb{R}^m$, and N is an open subset $V \subseteq \mathbb{R}^n$. Write

$$\beta = \sum_{j=1}^n \beta_j(y) \mathrm{d}y^j.$$

By (6.4), the pull-back of β is given by

$$F^*\beta = \sum_{i=1}^m \Big(\sum_{j=1}^n \beta_j(F(x)) \frac{\partial F^j}{\partial x^i} \Big) \mathrm{d}x^i.$$

In particular, the coefficients are smooth. □

The Lemma shows that we have a well-defined pull-back map

$$F^* : \Omega^1(N) \to \Omega^1(M), \;\; \beta \mapsto F^*\beta.$$

Under composition of two maps, $(F_1 \circ F_2)^* = F_2^* \circ F_1^*$. The pull-back of forms is related to the pull-back of functions, $g \mapsto F^* g = g \circ F$:

Proposition 6.2. *For $g \in C^\infty(N)$,*

$$F^*(\mathrm{d}g) = \mathrm{d}(F^* g).$$

Proof. We have to show $(F^*(\mathrm{d}g))_p = (\mathrm{d}(F^*g))_p$ for all $p \in M$. But this is just Lemma 6.1. □

Remark 6.2. Recall once again that while $F \in C^\infty(M,N)$ induces a tangent map $TF \in C^\infty(TM,TN)$, there is no natural push-forward operation for vector fields. By contrast, for cotangent bundles there is no naturally induced map from T^*N to T^*M (or the other way), yet there is a natural pull-back operation for 1-forms!

In the case of vector fields, rather than working with '$F_*(X)$' one has the notion of related vector fields, $X \sim_F Y$. For any related vector fields $X \sim_F Y$, and $\beta \in \Omega^1(N)$, we then have that

$$(F^*\beta)(X) = F^*(\beta(Y)).$$

Indeed, at any given $p \in M$ this just becomes the definition of the pullback map.

6.6 Integration of 1-forms

Given a curve $\gamma\colon J \to M$ in a manifold, and any 1-form $\alpha \in \Omega^1(M)$, we can consider the pull-back $\gamma^*\alpha \in \Omega^1(J)$. By the description of 1-forms on $\mathbb{R}$, this is of the form

$$\gamma^*\alpha = f(t)\mathrm{d}t$$

for a smooth function $f \in C^\infty(J)$.

To discuss integration, it is convenient to work with closed intervals rather than open intervals. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A map $\gamma\colon\ [a,b] \to M$ into a manifold will be called *smooth* if it extends to a smooth map from an open interval containing $[a,b]$. We will call such a map a smooth *path*.

Definition 6.5. Given a smooth path $\gamma\colon\ [a,b] \to M$, we define the integral of a 1-form $\alpha \in \Omega^1(M)$ along γ as

$$\int_\gamma \alpha = \int_a^b \gamma^*\alpha.$$

The fundamental theorem of calculus has the following consequence for manifolds. It is a special case of *Stokes' theorem*.

Proposition 6.3. *Let $\gamma\colon\ [a,b] \to M$ be a smooth path, with $\gamma(a) = p,\ \gamma(b) = q$. For any $f \in C^\infty(M)$, we have*

$$\int_\gamma \mathrm{d}f = f(q) - f(p).$$

In particular, the integral of $\mathrm{d}f$ *depends only on the end points of the path, rather than the path itself.*

Proof. We have

$$\gamma^*\mathrm{d}f = \mathrm{d}\gamma^*f = \mathrm{d}(f\circ\gamma) = \frac{\partial(f\circ\gamma)}{\partial t}\mathrm{d}t.$$

Integrating from a to b, we obtain, by the fundamental theorem of calculus, $f(\gamma(b)) - f(\gamma(a))$. $\square$

A 1-form $\alpha \in \Omega^1(M)$ such that $\alpha = \mathrm{d}f$ for some function $f \in C^\infty(M)$ is called *exact*.

Example 6.2. Consider the 1-form

$$\alpha = y^2 e^x \mathrm{d}x + 2ye^x \mathrm{d}y \in \Omega(\mathbb{R}^2).$$

Problem: Find the integral of α along the path

$$\gamma:\ [0,1] \to M,\ \ t \mapsto (\sin(\pi t/2), t^3).$$

Solution: Observe that the 1-form α is exact:

$$\alpha = \mathrm{d}\big(y^2 e^x\big) = \mathrm{d}f$$

with $f(x,y) = y^2 e^x$. The path has end points $\gamma(0) = (0,0)$ and $\gamma(1) = (1,1)$. Hence,

$$\int_\gamma \alpha = f(\gamma(1)) - f(\gamma(0)) = e.$$

Note that the integral over, say, $\alpha = y^2 e^x \mathrm{d}x$ would be much harder.

Remark 6.3. The proposition gives a necessary condition for exactness: The integral of α along paths should depend only on the end points. This condition is also sufficient, since we can define f on the connected components of M, by fixing a base point p_0 on each such component, and putting $f(p) = \int_\gamma \alpha$ for any path from p_0 to p.

If M is an open subset $U \subseteq \mathbb{R}^m$, so that $\alpha = \sum_i \alpha_i \mathrm{d}x^i$, then $\alpha = \mathrm{d}f$ means that $\alpha_i = \frac{\partial f}{\partial x^i}$. A necessary condition is the equality of partial derivatives,

$$\frac{\partial \alpha_i}{\partial x^j} = \frac{\partial \alpha_j}{\partial x^i},$$

In multivariable calculus one learns that this condition is also sufficient, provided U is simply connected (e.g., convex). Using the exterior differential of forms in $\Omega^1(U)$, this condition becomes $\mathrm{d}\alpha = 0$. To obtain a coordinate-free version of the condition, we need higher order forms.

6.7 2-forms

To get a feeling for higher degree forms, and constructions with higher forms, we first discuss 2-forms.

Definition 6.6. A 2-form on M is a $C^\infty(M)$-bilinear skew-symmetric map

$$\alpha:\ \mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M),\ \ (X,Y)\mapsto\alpha(X,Y).$$

Here skew-symmetry means that $\alpha(X,Y)=-\alpha(Y,X)$ for all vector fields X,Y, while $C^\infty(M)$-bilinearity means

$$\alpha(fX,Y)=f\alpha(X,Y)=\alpha(X,fY)$$

for $f\in C^\infty(M)$, as well as $\alpha(X'+X'',Y)=\alpha(X',Y)+\alpha(X'',Y)$, and similarly in the second argument. (Actually, by skew-symmetry it suffices to require $C^\infty(M)$-linearity in the first argument.) By the same argument as for 1-forms, the value $\alpha(X,Y)_p$ depends only on the values X_p,Y_p. Also, if α is a 2-form then so is $f\alpha$ for any smooth function f.

First examples of 2-forms are obtained from 1-forms: Let $\alpha,\beta\in\Omega^1(M)$. Then we define a *wedge product* $\alpha\wedge\beta\in\Omega^2(M)$, as follows:

$$(\alpha\wedge\beta)(X,Y)=\alpha(X)\beta(Y)-\alpha(Y)\beta(X).$$

This is well-defined, since the right hand side is skew-symmetric and bi-linear in X and Y.

For an open subset $U\subseteq\mathbb{R}^m$, a 2-form $\omega\in\Omega^2(U)$ is uniquely determined by its values on coordinate vector fields. By skew-symmetry the functions

$$\omega_{ij}=\omega\Big(\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^j}\Big)$$

satisfy $\omega_{ij}=-\omega_{ji}$; hence it suffices to know these functions for $i<j$. As a consequence, we see that the most general 2-form on U is

$$\omega=\frac{1}{2}\sum_{i,j=1}^{m}\omega_{ij}\mathrm{d}x^i\wedge dx^j=\sum_{i<j}\omega_{ij}\mathrm{d}x^i\wedge dx^j.$$

6.8 k-forms

We now generalize to forms of arbitrary degree.

6.8.1 Definition

Definition 6.7. Let k be a non-negative integer. A *k-form* on M is a $C^\infty(M)$-multilinear, skew-symmetric map

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k \text{ times}} \to C^\infty(M).$$

The space of k-forms is denoted $\Omega^k(M)$; in particular $\Omega^0(M) = C^\infty(M)$.

Here, *skew-symmetry* means that $\alpha(X_1, \ldots, X_k)$ changes sign under exchange of any two of its elements. For example, $\alpha(X_1, X_2, X_3, \ldots) = -\alpha(X_2, X_1, X_3, \ldots)$. More generally, if $\mathfrak{S}_k$ is the group of permutations of $\{1, \ldots, k\}$, and $\operatorname{sign}(s)$ is the sign of a permutation $s \in \mathfrak{S}_k$ ($+1$ for an even permutation, -1 for an odd permutation) then

$$\alpha(X_{s(1)}, \ldots, X_{s(k)}) = \operatorname{sign}(s)\alpha(X_1, \ldots, X_k).$$

The $C^\infty(M)$*-multilinearity* means $C^\infty(M)$-linearity in each argument, similar to the condition for 2-forms. It implies, in particular, α is *local* in the sense that the value of $\alpha(X_1, \ldots, X_k)$ at any given $p \in M$ depends only on the values $X_1|_p, \ldots, X_k|_p \in T_pM$. One thus obtains a skew-symmetric multilinear form

$$\alpha_p : \ T_pM \times \cdots \times T_pM \to \mathbb{R},$$

for all $p \in M$.

If $\alpha_1, \ldots, \alpha_k$ are 1-forms, then one obtains a k-form $\alpha =: \alpha_1 \wedge \ldots \wedge \alpha_k$ by 'wedge product'.

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(X_1, \ldots, X_k) = \sum_{s \in S_k} \operatorname{sign}(s)\alpha_1(X_{s(1)}) \cdots \alpha_k(X_{s(k)}).$$

(More general wedge products will be discussed below.) Here, the signed summation over the permutation group guarantees that the result is skew-symmetric.

Using C^∞-multilinearity, a k-form on $U \subseteq \mathbb{R}^m$ is uniquely determined by its values on coordinate vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$, i.e. by the functions

$$\alpha_{i_1 \ldots i_k} = \alpha\Big(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\Big).$$

Moreover, by skew-symmetry we only need to consider *ordered* index sets $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, that is, $i_1 < \ldots < i_k$. Using the wedge product notation, we obtain

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \ldots i_k} \mathrm{d}x^{i_1} \wedge \cdots \mathrm{d}x^{i_k}.$$

6.8.2 Wedge product

We next turn to the definition of a *wedge product* of forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$. A permutation $s \in \mathfrak{S}_{k+l}$ is called a k, l *shuffle* if it satisfies

$$s(1) < \ldots < s(k), \quad s(k+1) < \ldots < s(k+l).$$

Definition 6.8. The wedge product of $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$ is the element

$$\alpha \wedge \beta \in \Omega^{k+l}(M)$$

given as

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) = \sum \mathrm{sign}(s) \alpha(X_{s(1)}, \ldots, X_{s(k)})\, \beta(X_{s(k+1)}, \ldots, X_{s(k+l)})$$

where the sum is over all k,l-shuffles.

Example 6.3. For $\alpha, \beta \in \Omega^2(M)$,

$$\begin{aligned}(\alpha \wedge \beta)(X,Y,Z,W) = \; & \alpha(X,Y)\beta(Z,W) - \alpha(X,Z)\beta(Y,W) \\ & + \alpha(X,W)\beta(Y,Z) + \alpha(Y,Z)\beta(X,W) \\ & - \alpha(Y,W)\alpha(X,Z) + \alpha(Z,W)\beta(X,Y).\end{aligned}$$

□

The wedge product is *graded commutative*: If $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

Furthermore, it is associative:

Lemma 6.5. *Given* $\alpha_i \in \Omega_{k_i}(M)$ *we have*

$$(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 = \alpha_1 \wedge (\alpha_2 \wedge \alpha_3)$$

Proof. For both sides, the evaluation on $X_1, \ldots, X_k$ with $k = k_1 + k_2 + k_3$, is a signed sum over all k_1, k_2, k_3-shuffles (it should be clear how this is defined).

So, we may in fact drop the parentheses when writing wedge products.

6.8.3 Exterior differential

Recall that we defined the exterior differential on functions by the formula

$$(df)(X) = X(f). \tag{6.5}$$

we will now extend this definition to all forms.

Theorem 6.1. *There is a unique collection of linear maps* $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$*, extending the map (6.5) for* $k = 0$*, such that* $\mathrm{d}(\mathrm{d}f) = 0$ *and satisfying the graded product rule,*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta \tag{6.6}$$

for $\alpha \in \Omega^k(M)$ *and* $\beta \in \Omega^l(M)$*. This* exterior differential *satisfies* $\mathrm{d} \circ \mathrm{d} = 0$.

Proof. Suppose first that such an exterior differential is given. Then d is local, in the sense that for any open subset $U \subseteq M$ the restriction $(\mathrm{d}\alpha)|_U$ depends only on $\alpha|_U$, or equivalently $(\mathrm{d}\alpha)|_U = 0$ when $\alpha|_U = 0$. Indeed, if this is the case and $p \in U$, we may choose $f \in C^\infty(M) = \Omega^0(M)$ such that f vanishes on $M \backslash U$ and $f|_p = 1$. Then $f\alpha = 0$, hence the product rule (6.6) gives

$$0 = \mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

Evaluating at p we obtain $(\mathrm{d}\alpha)_p = 0$ as claimed. Using locality, we may thus work in local coordinates. If $\alpha \in \Omega^1(M)$ is locally given by

$$\alpha = \sum_{i_1 < \cdots < i_k} \alpha_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

then the product rule together with $\mathrm{dd}x^i = 0$ forces us to define

$$\mathrm{d}\alpha = \sum_{i_1 < \cdots < i_k} \mathrm{d}\alpha_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \sum_{l=1}^{m} \sum_{i_1 < \cdots < i_k} \frac{\partial \alpha_{i_1 \cdots i_k}}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Conversely, we may use this explicit formula (cf. (6.1)) to *define* $\mathrm{d}\alpha|_U$ for a coordinate chart domain U; by uniqueness the local definitions on overlas of coordinate chart domains agree. Proposition 6.1 shows that $(\mathrm{dd}\alpha)|_U = 0$, hence it also holds globally. ☐

Definition 6.9. A k-form $\omega \in \Omega^k(M)$ is called *exact* if $\omega = \mathrm{d}\alpha$ for some $\alpha \in \Omega^{k-1}(M)$. It is called *closed* if $\mathrm{d}\omega = 0$.

Since $\mathrm{d} \circ \mathrm{d} = 0$, the exact k-forms are a subspace of the space of closed k-forms. For the case of 1-forms, we had seen that the integral $\int_\gamma \alpha$ of an exact 1-form $\alpha = \mathrm{d}f$ along a smooth path $\gamma\colon\ [a,b] \to M$ is given by the difference of the values at the end points; a necessary condition for α to be exact is that it is closed. An example of a 1-form that is closed but not exact is

$$\alpha = \frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \backslash \{0\}).$$

Remark 6.4. The quotient space (closed k-forms modulo exact k-forms) is a vector space called the k-th (de Rham) *cohomology*

$$H^k(M) = \frac{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is closed }\}}{\{\alpha \in \Omega^k(M) |\ \alpha \text{ is exact }\}}.$$

It turns out that whenever M is compact (and often also if M is non-compact), $H^k(M)$ is a finite-dimensional vector space. The dimension of this vector space

$$b_k(M) = \dim H^k(M)$$

is called the *k-th Betti number* of M; these numbers are important invariants of M which one can use to distinguish non-diffeomorphic manifolds. For example, if $M = \mathbb{C}\mathrm{P}^n$ one can show that

$$b_k(\mathbb{C}\mathrm{P}^n) = 1 \quad \text{for} \quad k = 0, 2, \ldots, 2n$$

and $b_k(\mathbb{C}\mathrm{P}^n) = 0$ otherwise. For $M = S^N$ the Betti numbers are

$$b_k(S^n) = 1 \quad \text{for} \quad k = 0, n$$

while $b_k(S^n) = 0$ for all other k. Hence $\mathbb{C}\mathrm{P}^n$ cannot be diffeomorphic to S^{2n} unless $n = 1$.

6.9 Lie derivatives and contractions

Given a vector field X, and a k-form $\alpha \in \Omega^k(M)$, we can define a $k-1$-form

$$\iota_X \alpha \in \Omega^{k-1}(M)$$

by *contraction*: Thinking of α as a multi-linear form, one simply puts X into the first slot:

$$(\iota_X \alpha)(X_1, \ldots, X_{k-1}) = \alpha(X, X_1, \ldots, X_{k-1}).$$

Contractions have the following compatibility with the wedge product, similar to that for the exterior differential:

$$\iota_X(\alpha \wedge \beta) = \iota_X \alpha \wedge \beta + (-1)^k \alpha \wedge \iota_X \beta, \tag{6.7}$$

for $\alpha \in \Omega^k(M), \beta \in \Omega^l(M)$, which one verifies by evaluating both sides on vector fields. Another important operator on forms is the *Lie derivative*:

Theorem 6.2. *Given a vector field X, there is a unique collection of linear maps $L_X : \ \Omega^k(M) \to \Omega^k(M)$, such that*

$$L_X(f) = X(f), \ \ L_X(\mathrm{d}f) = \mathrm{d}X(f),$$

and satisfying the product rule,

$$L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta \tag{6.8}$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

Proof. As in the case of the exterior differential, we can use the product rule to show that L_X is local: $(L_X \alpha)|_U$ depends only on $\alpha|_U$ and $X|_U$. Since any differential form is a sum of wedge products of 1-forms, L_X is uniquely determined by its action on functions and differential of functions. This proves uniqueness. For existence, we give the following formula:

$$L_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}.$$

On functions, this gives the correct result since

$$L_X f = \iota_X \mathrm{d} f = X(f),$$

and also on differentials of functions since

$$L_X \mathrm{d} f = \mathrm{d} \iota_X \mathrm{d} f = \mathrm{d} L_X f = 0.$$

$\square$

To summarize, we have introduced three operators

$$\mathrm{d}: \ \Omega^k(M) \to \Omega^{k+1}(M), \ \ L_X: \ \Omega^k(M) \to \Omega^k(M), \ \ \iota_X: \ \Omega^k(M) \to \Omega^{k-1}(M).$$

These have the following compatibilities with the wedge product: For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ one has

$$\begin{aligned}
\mathrm{d}(\alpha \wedge \beta) &= (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta, \\
L_X(\alpha \wedge \beta) &= (L_X \alpha) \wedge \beta + \alpha \wedge L_X \beta, \\
\iota_X(\alpha \wedge \beta) &= (\iota_X \alpha) \wedge \beta + (-1)^k \alpha \wedge \iota_X \beta.
\end{aligned}$$

One says that L_X is an *even derivation* relative to the wedge product, whereas d, ι_X are *odd derivations*. They also satisfy important relations among each other:

$$\begin{aligned}
\mathrm{d} \circ \mathrm{d} &= 0 \\
L_X \circ L_Y - L_Y \circ L_X &= L_{[X,Y]} \\
\iota_X \circ \iota_Y + \iota_Y \circ \iota_X &= 0 \\
\mathrm{d} \circ L_X - L_X \circ \mathrm{d} &= 0 \\
L_X \circ \iota_Y - \iota_Y \circ L_X &= \iota_{[X,Y]} \\
\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X &= L_X.
\end{aligned}$$

Again, the signs are determined by the even/odd parity of these operators; one should think of the left hand side as 'graded' commutators, where a plus sign appears whenever two entries are odd. Writing $[\cdot,\cdot]$ for the graded commutators (with the agreement that the commutator of two odd operators has a sign built in) the identities becomes $[\mathrm{d},\mathrm{d}] = 0$, $[L_X, L_Y] = L_{[X,Y]}$, $[\iota_X, \iota_Y] = 0$, $[\mathrm{d}, L_X] = 0$, $[L_X, \iota_Y] = \iota_{[X,Y]}$ and $[\mathrm{d}, \iota_X] = L_X$.

This collection of identities is referred to as the *Cartan calculus*, after Élie Cartan (1861-1951), and in particular the last identity (which certainly is the most intriguing) is called the *Cartan formula*. Basic contributions to the theory of differential forms were made by his son Henri Cartan (1906-1980), who also wrote a textbook on the subject.

Exercise: Prove these identities. (Note that some have already been established.) Hint: First check that the left hand side satisfies a (graded) product rule with respect to wedge product. It therefore suffices to check that both sides agree on functions f and differentials of functions $\mathrm{d}f$.

As an illustration of the Cartan identities, let us prove the following formula for the exterior differential of a 1-form $\alpha \in \Omega^1(M)$:

$$(\mathrm{d}\alpha)(X,Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).$$

(In the Cartan Calculus, we prefer to wrote $L_X f$ instead of $X(f)$ since expressions such as $X(\alpha(Y))$ would look too confusing.) The calculation goes as follows:

$$\begin{aligned}(\mathrm{d}\alpha)(X,Y) &= \iota_Y \iota_X \mathrm{d}\alpha \\ &= \iota_Y L_X \alpha - \iota_Y \mathrm{d}\iota_X \alpha \\ &= L_X \iota_Y \alpha - \iota_{[X,Y]}\alpha - L_Y \iota_X \alpha + \mathrm{d}\iota_Y \iota_X \alpha \\ &= L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X,Y]).\end{aligned}$$

In the last step we used that $\iota_Y \iota_X \alpha = 0$, because α is a 1-form.

Exercise: Prove a similar formula for the exterior differential of a 2-form, and try to generalize to arbitrary k-forms.

6.9.1 Pull-backs

Similar to the pull-back of functions (0-forms) and 1-forms, we have a pull-back operation for k-forms,

$$F^* : \ \Omega^k(N) \to \Omega^k(M)$$

for any smooth map between manifolds, $F \in C^\infty(M,N)$. Its evaluation at any $p \in M$ is given by

$$(F^*\beta)_p(v_1,\ldots,v_k) = \beta_{F(p)}(T_pF(v_1),\ldots,T_pF(v_k)).$$

The pull-back map satisfies $\mathrm{d}(F^*\beta) = F^*\mathrm{d}\beta$, and for a wedge product of forms,

$$F^*(\beta_1 \wedge \beta_2) = F^*\beta_1 \wedge F^*\beta_2.$$

In local coordinates, if $F : \ U \to V$ is a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, with coordinates x,y, the pull-back just amounts to 'putting $y = F(x)$'.

Example 6.4. If $F : \ \mathbb{R}^3 \to \mathbb{R}^2$ is given by $(u,v) = F(x,y,z) = (y^2 z, \ x)$ then

$$F^*(\mathrm{d}u \wedge \mathrm{d}v) = \mathrm{d}(y^2 z) \wedge \mathrm{d}x = y^2 \mathrm{d}z \wedge \mathrm{d}x + 2yz\mathrm{d}y \wedge \mathrm{d}x.$$

The next example is very important, hence we state it as a proposition. It is the 'key fact' toward the definition of an integral.

Proposition 6.4. *Let $U \subseteq \mathbb{R}^m$ with coordinates x^i, and $V \subseteq \mathbb{R}^n$ with coordinates y^j. Suppose $m = j$, and $F \in C^\infty(U,V)$. Then*

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n) = J\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

where $J(x)$ is the determinant of the Jacobian matrix,

$$J(x) = \det\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j=1}^n.$$

Proof.

$$\begin{aligned} F^*\beta &= \mathrm{d}F^1 \wedge \cdots \wedge \mathrm{d}F^n \\ &= \sum_{i_1 \dots i_n} \frac{\partial F^1}{\partial x^{i_1}} \cdots \frac{\partial F^n}{\partial x^{i_n}} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\ &= \sum_{s \in S_n} \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^{s(1)} \wedge \cdots \wedge \mathrm{d}x^{s(n)} \\ &= \sum_{s \in S_n} \mathrm{sign}(s) \frac{\partial F^1}{\partial x^{s(1)}} \cdots \frac{\partial F^n}{\partial x^{s(n)}} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \\ &= J\ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \end{aligned}$$

Here we noted that the wedge product $\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$ is zero unless $i_1, \dots, i_n$ are a permutation of $1, \dots, n$. □

One may regard this result as giving a new, 'better' definition of the Jacobian determinant.

Remark 6.5. The Lie derivative $L_X \alpha$ of a differential form with respect to a vector field X has an important interpretation in terms of the flow Φ_t of X. Assuming for simplicity that X is complete (so that Φ_t is a globally defined diffeomorphism), one has the formula

$$L_X \alpha = \frac{d}{dt}\Big|_{t=0} \Phi_t^* \alpha.$$

(If X is incomplete, the flow Φ_t is defined only locally, but the definition still works.) To prove this identity, it suffices to check that the right hand side satisfies a product rule with respect to the wedge product of forms, and that it takes on the correct values on functions and on differentials of functions. The formula shows that L_X measures to what extent α is invariant under the flow of X.

6.10 Integration of differential forms

Differential forms of top degree can be integrated over *oriented* manifolds. Let M be an oriented manifold of dimension m, and $\omega \in \Omega^m(M)$. Let $\mathrm{supp}(\omega)$ be the support of ω. [6]

If $\mathrm{supp}(\omega)$ is contained in an oriented coordinate chart (U, φ), then one defines

$$\int_M \omega = \int_{\mathbb{R}^m} f(x) \mathrm{d}x^1 \cdots \mathrm{d}x^m$$

where $f \in C^\infty(\mathbb{R}^m)$ is the function, with $\mathrm{supp}(f) \subseteq \varphi(U)$, determined from

$$(\varphi^{-1})^* \omega = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

This definition does not depend on the choice of oriented chart. Indeed, suppose (V, ψ) is another oriented chart with $\mathrm{supp}(\omega) \subseteq V$, and write

$$(\psi^{-1})^* \omega = g \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m.$$

where we write $y^1, \ldots, y^m$ for the coordinates on V. Letting $F = \psi \circ \varphi^{-1}$ be the change of coordinates $y = F(x)$, Proposition 6.4 says that

$$F^*(\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m) = J(x) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,$$

where $J(x) = \det(DF(x))$ is the determinant of the Jacobian matrix of F at x. Hence, $f(x) = g(F(x))J(x)$, and we obtain

$$\int_{\psi(U)} g(y) \mathrm{d}y^1 \cdots y^m = \int_{\varphi(U)} g(F(x))J(x) \mathrm{d}x^1 \cdots \mathrm{d}x^m = \int_{\varphi(U)} f(x) \mathrm{d}x^1 \cdots \mathrm{d}x^m,$$

as required.

Remark 6.6. Here we used the *change-of-variables formula* from multivariable calculus. It was very important that the charts are oriented, so that $J > 0$ everywhere. Indeed, for general changes of variables, the change-of-variables formula involves $|J|$ rather than J itself.

If ω is not necessarily supported in a single oriented chart, we proceed as follows. Let $(U_i, \varphi_i), \;\; i = 1, \ldots, r$ be a finite collection of oriented charts covering $\mathrm{supp}(\omega)$. Together with $U_0 = M \backslash \mathrm{supp}(\omega)$ this is an open cover of M.

Lemma 6.6. *Given a finite open cover of a manifold there exists a* partition of unity subordinate to the cover, *i.e. functions* $\chi_i \in C^\infty(M)$ *with* $\mathrm{supp}(\chi_i) \subseteq U_i$ *and* $\sum_{i=0}^r \chi_i = 1$.

---

[6] The support of a form is defined similar to the support of a function, or support of a vector field. For any differential form $\alpha \in \Omega^k(M)$, we define the *support* $\mathrm{supp}(\alpha)$ to be the smallest closed subset of M outside of which α is zero. (Equivalently, it is the *closure* of the subset over which α is non-zero.)

Indeed, partitions of unity exists for *any* open cover, not only finite ones. A proof is given in the appendix on 'topology of manifolds'.

Let $\chi_0, \ldots, \chi_r$ be a partition of unity subordinate to this cover. We define

$$\int_M \omega = \sum_{i=1}^{r} \int_M \chi_i \omega$$

where the summands are defined as above, since $\chi_i \omega$ is supported in U_i for $i \geq 1$. (We didn't include the term for $i = 0$, since $\chi_0 \omega = 0$.) We have to check that this is well-defined, independent of the choices. Thus, let (V_j, ψ_j) for $j = 1, \ldots, s$ be another collection of oriented coordinate charts covering $\mathrm{supp}(\omega)$, put $V_0 = M - \mathrm{supp}(\omega)$, and let $\sigma_0, \ldots, \sigma_s$ a corresponding partition of unity subordinate to the cover by the V_i's.

Then the $U_i \cap V_j$ form an open cover, with the collection of $\chi_i \sigma_j$ as a partition of unity. We obtain

$$\sum_{j=1}^{s} \int_M \sigma_j \omega = \sum_{j=1}^{s} \int_M (\sum_{i=1}^{r} \chi_i) \, \sigma_j \omega = \sum_{j=1}^{s} \sum_{i=1}^{r} \int_M \sigma_j \chi_i \omega.$$

This is the same as the corresponding expression for $\sum_{i=1}^{r} \int_M \chi_i \omega$.

6.11 Integration over oriented submanifolds

Let M be a manifold, not necessarily oriented, and S is a k-dimensional oriented submanifold, with inclusion $i: S \to M$. We define the integral over S, of any k-form $\omega \in \Omega^k(M)$ such that $S \cap \mathrm{supp}(\omega)$ is compact, as follows:

$$\int_S \omega = \int_S i^* \omega.$$

Of course, this definition works equally well for *any* smooth map from S into M. For example, the integral of compactly supported 1-forms along arbitrary paths $\gamma : \mathbb{R} \to M$ is defined. Note also that M itself does not have to be oriented, it suffices that S is oriented.

6.12 Stokes' theorem

Let M be an m-dimensional oriented manifold.

Definition 6.10. A *region with (smooth) boundary* in M is a closed subset $D \subseteq M$ with the following property: There exists a smooth function $f \in C^\infty(M, \mathbb{R})$ such that 0 is a regular value of f, and

$$D = \{p \in M | \, f(p) \le 0\}.$$

We do not consider f itself as part of the definition of D, only the existence of f is required. The interior of a region with boundary, given as the largest open subset contained in D, is $\mathrm{int}(D) = \{p \in M | \, f(p) < 0$, and the boundary itself is

$$\partial D = \{p \in M | \, f(p) = 0\},$$

a codimension 1 submanifold (i.e., hypersurface) in M.

Example 6.5. The region with bounday defined by the function $f \in C^\infty(\mathbb{R}^2)$, given by $f(x,y) = x^2 + y^2 - 1$, is the unit disk $D \subseteq \mathbb{R}^2$; its boundary is the unit circle.

Example 6.6. Recall that for $0 < r < R$, zero is a regular value of the function on $\mathbb{R}^3$,

$$f(x,y,z) = z^2 + (\sqrt{x^2+y^2} - R)^2 - r^2.$$

The corresponding region with boundary $D \subseteq \mathbb{R}^3$ is the solid torus, its boundary is the torus.

Recall that we are considering D inside an *oriented* manifold M. The boundary ∂D may be covered by oriented submanifold charts (U, φ), in such a way that ∂D is given in the chart by the condition $x^1 = 0$, and D by the condition $x^1 \le 0$: [7]

$$\varphi(U \cap D) = \varphi(U) \cap \{x \in \mathbb{R}^m | \, x^1 \le 0\}.$$

(Indeed, given an oriented submanifold chart for which D lies on the side where $x_1 \ge 0$, one obtains a region chart by composing with the orientation-preserving coordinate change $(x^1, \ldots, x^m) \mapsto (-x^1, -x^2, x^3 \ldots, x^m)$.) We call oriented submanifold charts of this kind 'region charts'.[8]

Lemma 6.7. *The restriction of the region charts to ∂D form an oriented atlas for ∂D.*

Proof. Let (U, φ) and (V, ψ) be two region charts, defining coordinates $x^1, \ldots, x^m$ and $y^1, \ldots, y^m$, and let $F = \psi \circ \varphi^{-1} : \ \varphi(U \cap V) \to \psi(U \cap V), \ x \mapsto y = F(x)$. It restricts to a map

$$F_1 : \ \{x \in \varphi(U \cap V) | \, x_1 = 0\} \to \{y \in \psi(U \cap V) | \, y_1 = 0\}.$$

Since $y^1 > 0$ if and only if $x^1 > 0$, the change of coordinates satisfies

$$\frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} > 0, \quad \frac{\partial y^1}{\partial x^j}\Big|_{x^1=0} = 0, \quad \text{for } j > 0.$$

[7] Note that while we originally defined submanifold charts in such a way that the last $m-k$ coordinates are zero on S, here we require that the first coordinate be zero. It doesn't matter, since one can simply reorder coordinates, but works better for our description of the 'induced orientation'.

[8] This is not a standard name.

Hence, the Jacobian matrix $DF(x)|_{x^1=0}$ has a positive $(1,1)$ entry, and all other entries in the first row equal to zero. Using expansion of the determinant across the first row, it follows that

$$\det(DF(0,x^2,\ldots,x^m)) = \frac{\partial y^1}{\partial x^1}\Big|_{x^1=0} \det(DF'(x^2,\ldots,x^m)).$$

which shows that $\det(DF') > 0$.

In particular, ∂D is again an oriented manifold. To repeat: If $x^1,\ldots,x^m$ are local coordinates near $p \in \partial D$, compatible with the orientation and such that D lies on the side $x^1 \leq 0$, then $x^2,\ldots,x^m$ are local coordinates on ∂D. This convention of 'induced orientation' is arranged in such a way that the Stokes' theorem holds without extra signs.

For an m-form ω such that $\mathrm{supp}(\omega)\cap D$ is compact, the integral

$$\int_D \omega$$

is defined similar to the case of $D = M$: One covers $D \cap \mathrm{supp}(\omega)$ by finitely many submanifold charts (U_i, φ_i) with respect to ∂D (this includes charts that are entirely in the interior of D), and puts

$$\int_D \omega = \sum \int_{D\cap U_i} \chi_i \omega$$

where the χ_i are supported in U_i and satisfy $\sum_i \chi_i$ over $D \cap \mathrm{supp}(\omega)$. By the same argument as for $D = M$, this definition of the integral is independent of the choice made.

Theorem 6.3 (Stokes' theorem). *Let M be an oriented manifold of dimension m, and $D \subseteq M$ a region with smooth boundary ∂D. Let $\alpha \in \Omega^{m-1}(M)$ be a form of degree $m-1$, such that* $\mathrm{supp}(\alpha)\cap D$ *is compact. Then*

$$\int_D \mathrm{d}\alpha = \int_{\partial D} \alpha.$$

As explained above, the right hand side means $\int_{\partial D} i^*\alpha$, where $i\colon\ \partial D \hookrightarrow M$ is the inclusion map.

Proof. We will see that Stokes' theorem is just a coordinate-free version of the fundamental theorem of calculus. Let (U_i, φ_i) for $i = 1,\ldots,r$ be a finite collection of region charts covering $\mathrm{supp}(\alpha)\cap D$. Let $\chi_1,\ldots,\chi_r \in C^\infty(M)$ be functions with $\chi_i \geq 0$, $\mathrm{supp}(\chi_i) \subseteq U_i$, and such that $\chi_1 + \ldots + \chi_r$ is equal to 1 on $\mathrm{supp}(\alpha)\cap D$. (E.g., we may take $U_1,\ldots,U_r$ together with $U_0 = M\backslash \mathrm{supp}(\omega)$ as an open covering, and take the $\chi_0,\ldots,\chi_r \in C^\infty(M)$ to be a partition of unity subordinate to this cover.) Since

$$\int_D \mathrm{d}\alpha = \sum_{i=1}^r \int_D \mathrm{d}(\chi_i\alpha), \qquad \int_{\partial D} \alpha = \sum_{i=1}^r \int_{\partial D} \chi_i\alpha,$$

it suffices to consider the case that α is supported in a region chart.

Using the corresponding coordinates, it hence suffices to prove Stokes' theorem for the case that $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ is a compactly supported form in $\mathbb{R}^m$, and $D = \{x \in \mathbb{R}^m | x^1 \leq 0\}$. That is, α has the form

$$\alpha = \sum_{i=1}^{m} f_i \, \mathrm{d}x^1 \wedge \cdots \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^m,$$

with compactly supported f_i where the hat means that the corresponding factor is to be omitted. Only the $i = 1$ term contributes to the integral over $\partial D = \mathbb{R}^{m-1}$, and

$$\int_{\mathbb{R}^{m-1}} \alpha = \int f_1(0, x^2, \ldots, x^m) \, \mathrm{d}x^2 \cdots \mathrm{d}x^m.$$

On the other hand,

$$\mathrm{d}\alpha = \Big(\sum_{i=1}^{m} (-1)^{i+1} \, \frac{\partial f_i}{\partial x^i} \Big) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

Let us integrate each summand over the region D given by $x^1 \leq 0$. For $i > 1$, we have

$$\int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_i}{\partial x_i} (x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m = 0$$

where we used Fubini's theorem to carry out the x^i-integration first, and applied the fundamental theorem of calculus to the x^i-integration (keeping the other variables fixed, the integrand is the derivative of a compactly supported function). It remains to consider the case $i = 1$. Here we have, again by applying the fundamental theorem of calculus,

$$\begin{aligned} \int_D \mathrm{d}\alpha &= \int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} (x^1, \ldots, x^m) \mathrm{d}x^1 \cdots \mathrm{d}x^m \\ &= \int_{\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_m(0, x^2, \ldots, x^m) \mathrm{d}x^2 \cdots \mathrm{d}x^m = \int_{\partial D} \alpha \end{aligned}$$

□

As a special case, we have

Corollary 6.1. *Let $\alpha \in \Omega^{m-1}(M)$ be a compactly supported form on the oriented manifold M. Then*

$$\int_M \mathrm{d}\alpha = 0.$$

Note that it does not suffice that $\mathrm{d}\alpha$ has compact support. For example, if $f(t)$ is a function with $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t > 0$, then $\mathrm{d}f$ has compact support, but $\int_{\mathbb{R}} \mathrm{d}f = 1$.

A typical application of Stokes' theorem shows that for a closed form $\omega \in \Omega^k(M)$, the integral of ω over an oriented compact submanifold does not change with smooth deformations of the submanifold.

Theorem 6.4. *Let $\omega \in \Omega^k(M)$ be a closed form on a manifold M, and S a compact, oriented manifold of dimension k. Let $F \in C^\infty(\mathbb{R} \times S, M)$ be a smooth map, thought of as a smooth family of maps*

$$F_t = F(t,\cdot): \ S \to M.$$

Then the integrals

$$\int_S F_t^* \omega$$

do not depend on t.

If F_t is an embedding, then this is the integral of ω over the submanifold $F_t(S) \subseteq M$.

Proof. Let $a < b$, and consider the domain $D = [a,b] \times S \subseteq \mathbb{R} \times S$. The boundary ∂D has two components, both diffeomorphic to S. At $t = b$ the orientation is the given orientation on S, while at $t = a$ we get the opposite orientation. Hence,

$$0 = \int_D F^* \mathrm{d}\omega = \int_D \mathrm{d}F^* \omega = \int_{\partial D} F^* \omega = \int_S F_b^* \omega - \int_S F_a^* \omega.$$

Hence $\int_S F_b^* \omega = \int_S F_a^* \omega$. □

Remark 6.7. Note that if one member of this family of maps, say the map F_1, takes values in a $k-1$-dimensional submanifold (for instance, if F_1 is a constant map), then $F_1^* \omega = 0$. (Indeed, the assumption means that $F_1 = j \circ F_1'$, where j is the inclusion of a $k-1$-submanifold and F_1' takes values in that submanifold. But $j^* \omega = 0$ for degree reasons.) It then follows that $\int_S F_t^* \omega = 0$ for all t.

Given a smooth map $\varphi: \ S \to M$, one refers to a smooth map $F: \ \mathbb{R} \times S \to M$ with $F_0 = \varphi$ as an 'smooth deformation' (or 'isotopy') of φ. We say that φ can be smoothly deformed into φ' if there exists a smooth isotopy F with $\varphi = F_0$ and $\varphi' = F_1$. The theorem shows that if S is oriented, and if there is a closed form $\omega \in \Omega^k(M)$ with

$$\int_S \varphi^* \omega \neq \int_S (\varphi')^* \omega$$

then φ cannot be smoothly deformed into φ'. This observation has many applications; here are some of them. [9]

Example 6.7. Suppose $\varphi: S \to M$ is a smooth map, where S is oriented of dimension k, and $\omega \in \Omega^k(M)$ is closed. If $\int_S \varphi^* \omega \neq 0$, then φ cannot smoothly be 'deformed' into a map taking values in a lower-dimensional submanifold. (In particular it cannot be deformed into a constant map.) Indeed, if φ' takes values in a lower-dimensional submanifold, then $\varphi' = j \circ \varphi_1'$ where j is the inclusion of that submanifold. But then

[9] You may wonder if it is still possible to find a continuous deformation, rather than smooth. It turns out that it doesn't help: Results from differential topology show that two smooth maps can be smoothly deformed into each other if and only if they can be continuously deformed into each other.

$j^*\omega = 0$, hence $(\varphi')^*\omega = 0$. For instance, the inclusion $\varphi:\ S^2 \to M = \mathbb{R}^3\backslash\{0\}$ cannot be smoothly deformed inside M so that φ' would take values in $\mathbb{R}^2\backslash\{0\} \subseteq \mathbb{R}^3\backslash\{0\}$.

Example 6.8 (Winding number). Let $\omega \in \Omega^2(\mathbb{R}^2\backslash\{0\})$ be the 1-form

$$\omega = \frac{1}{x^2+y^2}(x\mathrm{d}y - y\mathrm{d}x)$$

In polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, one has that $\omega = \mathrm{d}\theta$. Using this fact one sees that ω is closed (but not exact, since θ is not a globally defined function on $\mathbb{R}^2\backslash\{0\}$.) Hence, if

$$\gamma:\ S^1 \to \mathbb{R}^2\backslash\{0\}$$

is any smooth map (a 'loop'), then the integral

$$\int_{S^1} \gamma^*\omega$$

does not change under deformations (isotopies) of the loop. In particular, γ cannot be deformed into a constant map, unless the integral is zero. The number

$$w(\gamma) = \frac{1}{2\pi}\int_{S^1} \gamma^*\omega$$

is the *winding number* of γ. (One can show that this is always an integer, and that two loops can be deformed into each other if and only if they have the same winding number.)

Example 6.9 (Linking number). Let $f,g:\ S^1 \to \mathbb{R}^3$ be two smooth maps whose images don't intersect, that is, with $f(z) \neq g(w)$ for all $z,w \in S^1$ (we regard S^1 as the unit circle in $\mathbb{C}$). Define a new map

$$F:\ S^1 \times S^1 \to S^2,\ \ (z,w) \mapsto \frac{f(z)-g(w)}{||f(z)-g(w)||}.$$

On S^2, we have a 2-form ω of total integral 4π. It is the pullback of

$$x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y \in \Omega^2(\mathbb{R}^3)$$

to the 2-sphere. The integral

$$L(f,g) = \frac{1}{4\pi}\int_{S^1\times S^1} F^*\omega$$

is called the *linking number* of f and g. (One can show that this is always an integer.) Note that if it is possible to deform one of the loops, say f, into a constant loop through loops that are always disjoint from g, then the linking number is zero. In his case, we consider f,g as 'unlinked'.

6.13 Volume forms

A top degree differential form $\Gamma \in \Omega^m(M)$ is called a *volume form* if it is non-vanishing everywhere: $\Gamma_p \neq 0$ for all $p \in M$. In a local coordinate chart (U, φ), this means that

$$(\varphi^{-1})^*\Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$$

where $f(x) \neq 0$ for all $x \in \varphi(U)$.

Example 6.10. The Euclidean space $\mathbb{R}^n$ has a *standard volume form* $\Gamma_0 = dx^1 \wedge \cdots \wedge \mathrm{d}x^n$. Suppose $S \subseteq \mathbb{R}^n$ is a submanifold of dimension $n-1$, and X a vector field that is *nowhere tangent* to S. Let $i: \ S \to \mathbb{R}^n$ be the inclusion. Then

$$\Gamma := i^*\big(\iota_X \Gamma_0\big) \in \Omega^{n-1}(S)$$

is a volume form. For instance, if S is given as a level set $f^{-1}(0)$, where 0 is a regular value of f, then the gradient vector field

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

has this property.

Exercise: Verify the claim that $\Gamma := i^*\big(\iota_X \Gamma_0\big)$ is a volume form.

Example 6.11. Let $i: \ S^n \to \mathbb{R}^{n+1}$ be the inclusion of the standard n-sphere. Let $X = \sum_{i=0}^n x^i \frac{\partial}{\partial x^i}$. Then

$$\iota_X(\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n) = \sum_{i=0}^{n} (-1)^i x^i \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n$$

pulls back to a volume form on S^n.

Lemma 6.8. *A volume form $\Gamma \in \Omega^m(M)$ determines an orientation on M, by taking as the oriented charts those charts (U, φ) such that $(\varphi^{-1})^*\Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ with $f > 0$ everywhere on $\Phi(U)$.*

Proof. We have to check that the condition is consistent. Suppose (U, φ) and (V, ψ) are two charts, where $(\varphi^{-1})^*\Gamma = f \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$ and $(\psi^{-1})^*\Gamma = g \, \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^m$ with $f > 0$ and $g > 0$. If $U \cap V$ is non-empty, let $F = \psi \circ \varphi^{-1}: \ \varphi(U) \to \psi(V)$ be the transition function. Then

$$F^*(\psi^{-1})^*\Gamma|_{U \cap V} = (\varphi^{-1})^*\Gamma|_{U \cap V},$$

hence

$$g(F(x)) \, J(x) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m = f(x) \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m.$$

where J is that Jacobian determinant of the transition map $F = \psi \circ \varphi^{-1}$. Hence $f = J\,(g \circ F)$ on $\varphi(U \cap V)$. Since $f > 0$ and $g > 0$, it follows that $J > 0$. Hence the two charts are oriented compatible. $\square$

Theorem 6.5. *A manifold M is orientable if and only if it admits a volume form. In this case, any two volume forms compatible with the orientation differ by an everywhere positive smooth function:*

$$\Gamma' = f\Gamma, \quad f > 0.$$

Proof. As we saw above, any volume form determines an orientation. Conversely, if M is an oriented manifold, there exists a volume form compatible with the orientation: Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas on M. Then each

$$\Gamma_\alpha = \varphi_\alpha^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m) \in \Omega^m(U_\alpha)$$

is a volume form on U_α; on overlaps $U_\alpha \cap U_\beta$ these are related by the Jacobian determinants of the transition functions, which are *strictly positive* functions. Let $\{\chi_\alpha\}$ be a locally finite partition of unity subordinate to the cover $\{U_\alpha\}$, see Appendix A.4. The forms $\chi_\alpha \Gamma_\alpha$ have compact support in U_α, hence they extend by zero to global forms on M (somewhat imprecisely, we use the same notation for this extension). The sum

$$\Gamma = \sum_\alpha \chi_\alpha \Gamma_\alpha \in \Omega^m(M)$$

is a well-defined volume form. Indeed, near any point p at least one of the summands is non-zero; and if other summands in this sum are non-zero, they differ by a positive function.

For a compact manifold M with a given volume form $\Gamma \in \Omega^m(M)$, one can define the volume of M,

$$\mathrm{vol}(M) = \int_M \Gamma.$$

Here the orientation used in the definition of the integral is taken to be the orientation given by Γ. Thus $\mathrm{vol}(M) > 0$.

Note that volume forms are always closed, for degree reasons (since $\Omega^{m+1}(M) = 0$). But on a compact manifold, they cannot be exact:

Theorem 6.6. *Let M be a compact manifold with a volume form $\Gamma \in \Omega^m(M)$. Then Γ cannot be exact.*

Proof. We have $\mathrm{vol}(M) = \int_M \Gamma > 0$. But if Γ were exact, then Stokes' theorem would give $\int_M \Gamma = 0$.

Of course, the compactness of M is essential here: For instance, $\mathrm{d}x$ is an exact volume form on $\mathbb{R}$.

Appendix A
Topology of manifolds

A.1 Topological notions

A *topological space* is a set X together with a collection of subsets $U \subseteq X$ called *open subsets*, with the following properties:

- $\emptyset, X$ are open.
- If U, U' are open then $U \cap U'$ is open.
- For any collection U_i of open subsets, the union $\bigcup_i U_i$ is open.

The collection of open subsets is called the *topology* of X. In the third condition, the index set need not be finite, or even countable.

The space $\mathbb{R}^m$ has a standard topology given by the usual open subsets. Likewise, the open subsets of a manifold M define a topology on M. For any set X, one has the *trivial topology* where the only open subsets are $\emptyset$ and X, and the *discrete topology* where every subset is considered open. An *open neighborhood* of a point p is an open subset containing it. A topological space is called *Hausdorff* of any two distinct points have disjoint open neighborhoods.

Let X be a topological space. Then any subset $A \subseteq X$ has a *subspace topology*, with open sets the collection of all intersections $U \cap A$ such that $U \subseteq X$ is open. Given a surjective map $q: X \to Y$, the space Y inherits a *quotient topology*, whose open sets are all $V \subseteq Y$ such that the pre-image $q^{-1}(V) = \{x \in X | \, q(x) \in V\}$ is open.

A subset A is *closed* if its complement $X \backslash A$ is open. Dual to the statements for open sets, one has

- $\emptyset, X$ are closed.
- If A, A' are closed then $A \cup A'$ is closed.
- For any collection A_i of open subsets, the intersection $\bigcap_i A_i$ is closed.

For any subset A, denote by $\overline{A}$ its *closure*, given as the smallest closed subset containing A.

A.2 Manifolds are second countable

A *basis* for the topology on X is a collection $\mathscr{B} = \{U_\alpha\}$ of open subsets of X such that every U is a union from sets from $\mathscr{B}$.

Example A.1. The collection of *all* open subsets of a topological space X is a basis of the topology.

Example A.2. Let $X = \mathbb{R}^n$. Then the collection of all open balls $B_\varepsilon(x)$, with $\varepsilon > 0$ and $x \in \mathbb{R}^n$, is a basis for the topology on $\mathbb{R}^n$.

A topological space is said to be *second countable* if its topology has a countable basis.

Proposition A.1. $\mathbb{R}^n$ *is second countable.*

Proof. A countable basis is given by the collection of all *rational balls*, by which we mean ε-balls $B_\varepsilon(x)$ such that $x \in \mathbb{Q}^m$ and $\varepsilon \in \mathbb{Q}_{>0}$. To check it is a basis, let $U \subseteq \mathbb{R}^m$ be open, and $p \in U$. Choose $\varepsilon \in \mathbb{Q}_{>0}$ such that $B_{2\varepsilon}(p) \subseteq U$. There exists a rational point $x \in \mathbb{Q}^n$ with $||x - p|| < \varepsilon$. This then satisfies $p \in B_\varepsilon(x) \subseteq U$. Since p was arbitrary, this proves the claim.

The same reasoning shows that for any open subset $U \subseteq \mathbb{R}^m$, the rational ε-balls that are contained in U form a basis of the topology of U.

Proposition A.2. *Manifolds are second countable.*

Proof. Given a manifold M, let $\mathscr{A} = \{(U_\alpha, \varphi_\alpha)\}$ be a countable atlas. Then the set of all $\varphi_\alpha^{-1}(B_\varepsilon(x))$, where $B_\varepsilon(x)$ is a rational ball contained in $\varphi_\alpha(U_\alpha)$, is a countable basis for the topology of M. Indeed, any open subset U is a countable union over all $U \cap U_\alpha$, and each of these intersections is a countable union over all $\varphi_\alpha^{-1}(B_\varepsilon(x))$ such that $B_\varepsilon(x)$ is a rational ε-ball contained in $U \cap U_\alpha$. $\square$

A.3 Manifolds are paracompact

A collection $\{U_\alpha\}$ of open subsets of X is called an *open covering* of $A \subseteq X$ if $A \subseteq \bigcup_\alpha U_\alpha$. Consider the case $A = X$. A *refinement* of an open cover $\{U_\alpha\}$ of X is an open cover $\{V_\beta\}$ of X such that each V_β is contained in some U_α. It is a *subcover* if each V_β's is equal to some U_α.

A topological space X is called *compact* if every open cover of X has a finite subcover. A topological space is called *paracompact* if every open cover $\{U_\alpha\}$ has a locally finite refinement $\{V_\beta\}$: that is, every point has an open neighborhood meeting only finitely many V_β's.

Proposition A.3. *Manifolds are paracompact.*

We will need the following auxiliary result.

Lemma A.1. *For any manifold M, there exists a sequence of open subsets $W_1, W_2, \ldots$ of M such that*

$$\bigcup W_i = M,$$

and such that each W_i has compact closure with $\overline{W_i} \subseteq W_{i+1}$.

Proof. Start with a a countable open cover $O_1, O_2, \ldots$ of M such that each O_i has compact closure $\overline{O}_i$. (We saw in the proof of Proposition A.2 how to construct such a cover, by taking pre-images of ε-balls in coordinate charts.) Replacing O_i with $O_1 \cup \cdots \cup O_i$ we may assume $O_1 \subseteq O_2 \subseteq \cdots$. For each i, the covering of the compact set $\overline{O}_i$ by the collection of all O_j's admits a finite subcover. Since the sequence of O_j's is nested, this just means $\overline{O}_i$ is contained in O_j for j sufficiently large. We can thus define $W_1, W_2, \ldots$ as a subsequence $W_i = O_{j(i)}$, starting with $W_1 = O_1$, and inductively letting $j(i)$ for $i > 1$ be the smallest index $j(i)$ such that $\overline{W}_{i-1} \subseteq O_{j(i)}$.

Proof (Proof or Proposition A.3). Let $\{U_\alpha\}$ be an open cover of M. Let W_i be a sequence of open sets as in the lemma. For every i, the compact subset $\overline{W}_{i+1} \backslash W_i$ is contained in the open set $W_{i+2} \backslash \overline{W}_{i-1}$, hence it is covered by the collection of open sets

$$(W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha. \tag{A.1}$$

By compactness, it is already covered by finitely many of the subsets (A.1). Let $\mathscr{V}^{(i)}$ be this finite collection, and $\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}^{(i)} = \{V_\beta\}$ the resulting countable open cover of M.

Note that by construction, if $V_\beta \in \mathscr{V}^{(i)}$, then $V_\beta \cap W_{i-1} = \emptyset$. That is, a given W_i meets only V_β's from $\mathscr{V}^{(k)}$ with $k \leq i$. Since these are finitely many V_β's, it follows that the cover $\mathscr{V} = \{V_\beta\}$ is locally finite. $\square$

Remark A.1. (Cf. Lang, page 35.) One can strengthen the result a bit, as follows: Given a cover $\{U_\alpha\}$, we can find a refinement to a cover $\{V_\beta\}$ such that each V_β is the domain of a coordinate chart (V_β, ψ_β), with the following extra properties, for some $0 < r < R$:

(i) $\psi_\beta(V_\beta) = B_R(0)$, and
(ii) M is already covered by the smaller subsets $V'_\beta = \psi_\beta^{-1}(B_r(0))$.

To prove this, we change the second half of the proof as follows: For each $p \in \overline{W}_{i+1} \backslash W_i$ choose a coordinate chart (V_p, ψ_p) such that $\psi_p(p) = 0$, $\psi_p(V_p) = B_R(0)$, and $V_p \subseteq (W_{i+2} \backslash \overline{W}_{i-1}) \cap U_\alpha$. Let $V'_p \subseteq V_p$ be the pre-image of $B_r(0)$. The V'_p cover $\overline{W}_{i+1} \backslash W_i$; let $\mathscr{V}^{(i)}$ be a finite subcover and proceed as before. This remark is useful for the construction of partitions of unity.

A.4 Partitions of unity

We will need the following result from multivariable calculus.

Lemma A.2 (Bump functions). *For all $0 < r < R$, there exists a function $f \in C^\infty(\mathbb{R}^m)$, with $f(x) = 0$ for $||x|| \geq R$ and $f(x) = 1$ for $||x|| \leq r$.*

Proof. It suffices to prove the existence of a function $g \in C^\infty(\mathbb{R})$ such that $g(t) = 0$ for $t \geq R$ and $g(t) = 1$ for $t \leq r$: Given such g we may take $f(x) = g(||x||)$. To construct g, recall that the function

$$h(t) = 0 \quad \text{if} \quad t \leq 0, \qquad h(t) = \exp(-1/t) \quad \text{if} \quad x > 0$$

is smooth even at $t = 0$. The function $h(t-r) + h(R-t)$ is strictly positive everywhere, since for $t \geq r$ the first summand is positive and for $t \geq R$ the second summand is positive. Furthermore, it agrees with $h(t-r)$ for $t \geq R$. Hence the function $g \in C^\infty(\mathbb{R})$ given as

$$g(t) = 1 - \frac{h(t-r)}{h(t-r) + h(R-t)}$$

is 1 for $t \leq r$, and 0 for $t \leq R$. $\square$

The support $\text{supp}(f)$ of a function f on M is the smallest closed subset such that f vanishes on $M \backslash \text{supp}(f)$. Equivalently, $p \in M \backslash \text{supp}(f)$ if and only if f vanishes on some open neighborhood of p. In the Lemma above, we can take f to have support in $B_R(0)$ – simply apply the Lemma to $0 < r < R' := \frac{1}{2}(R+r)$.

Definition A.1. A *partition of unity subordinate to an open cover* $\{U_\alpha\}$ of a manifold M is a collection of smooth functions $\chi_\alpha \in C^\infty(M)$, with $0 \leq \chi_\alpha \leq 1$, such that $\text{supp}(\chi_\alpha) \subseteq U_\alpha$, and

$$\sum_\alpha \chi_\alpha = 1.$$

Proposition A.4. *For any open cover $\{U_\alpha\}$ of a manifold, there exists a partition of unity $\{\chi_\alpha\}$ subordinate to that cover. One can take this partition of unity to be locally finite: That is, for any $p \in M$ there is an open neighborhood U meeting the support of only finitely many χ_α's.*

Proof. Let V_β be a locally finite refinement of the cover U_α, given by coordinate charts of the kind described in Remark A.1, and let $V'_\beta \subseteq V_\beta$ be as described there. Since the images of $V'_\beta \subseteq V_\beta$ are $B_r(0) \subseteq B_R(0)$, we can use Lemma A.2 to define a function $f_\beta \in C^\infty(M)$ with $f_\beta(p) = 1$ for $p \in \overline{V'_p}$ and $\text{supp}(f_\beta) \subseteq V_\beta$. Since the V'_β are already a cover, the sum $\sum_\beta f_\beta$ is strictly positive everywhere.

For each index β, pick an index α such that $V_\beta \subseteq U_\alpha$. This defines a map $d : \beta \mapsto d(\beta)$ between the indexing sets. The functions

$$\chi_\alpha = \frac{\sum_{\beta \in d^{-1}(\alpha)} f_\beta}{\sum_\gamma f_\gamma}.$$

give the desired partition of unity. $\square$

An important application of partitions of unity is the following result, a weak version of the *Whitney embedding theorem*.

Theorem A.1. *Let M be a manifold admitting a finite atlas with r charts. Then there is an embedding of M as a submanifold of* $\mathbb{R}^{r(m+1)}$.

Proof. Let $(U_i, \varphi_i),\ i = 1, \ldots, r$ be a finite atlas for M, and $\chi_1, \ldots, \chi_r$ a partition of unity subordinate to the cover by coordinate charts. Then the products $\chi_i \varphi_i :\ U_i \to \mathbb{R}^m$ extend by zero to smooth functions $\psi_i :\ M \to \mathbb{R}^m$. The map

$$F :\ M \to \mathbb{R}^{r(m+1)},\ p \mapsto (\psi_1(p), \ldots, \psi_r(p), \chi_1(p) \ldots, \chi_r(p))$$

is the desired embedding. Indeed, F is injective: if $F(p) = F(q)$, choose i with $\chi_i(p) > 0$. Then $\chi_i(q) = \chi_i(p) > 0$, hence both $p, q \in U_i$, and the condition $\psi_i(p) = \psi_i(q)$ gives $\varphi_i(p) = \varphi_i(q)$, hence $p = q$. Similarly T_pF is injective: For $v \in T_pM$ in the kernel of T_pF, choose i such that $\chi_i(p) > 0$, thus $v \in T_pU_i$. Then v being in the kernel of $T_p\psi_i$ and of $T_p\chi_i$ implies that it is in the kernel of $T_p\varphi_i$, hence $v = 0$ since φ_i is a diffeomorphism. This shows that we get an injective immersion, we leave it as an exercise to verify that the image is a submanifold (e.g., by constructing submanifold charts).

The theorem applies in particular to all compact manifolds. Actually, one can show that *all* manifolds admit a finite atlas; for a proof see e.g. Greub-Halperin-Vanstone, *connections, curvature and cohomology*, volume I. Hence, every manifold can be realized as a submanifold.

Appendix B
Vector bundles

B.1 Tangent bundle

Let M be a manifold of dimension m. The disjoint union over all the tangent spaces

$$TM = \bigcup_{p \in M} T_pM$$

is called the *tangent bundle* of M. It comes with a projection map

$$\pi: \; TM \to M$$

taking a tangent vector $v \in T_pM$ to its base point p, and with an inclusion map

$$i: \; M \to TM$$

taking $p \in M$ to the zero vector in $T_pM \subseteq TM$. We will show that TM is itself a manifold of dimension $2m$, in such a way that π and i are smooth maps.

Example B.1. Suppose M is given as a submanifold $M \subseteq \mathbb{R}^n$. Then each tangent space T_pM is realized as a subspace of $\mathbb{R}^n$, and

$$TM = \{(p,v) \in \mathbb{R}^n \times \mathbb{R}^n | \; p \in M, \;\; v \in T_pM\}.$$

We will show that this subset is a submanifold of dimension $2m$. (The dimension is to be expected: p varies in an m-dimensional manifold, and once p is fixed then v varies in an m-dimensional vector space.) For instance, if $M = S^1$, and using coordinates x, y on the first copy of $\mathbb{R}^2$ and a, b on the second copy, then

$$TS^1 = \{(x,y,r,s) \in \mathbb{R}^4 | \; x^2+y^2 = 1, \;\; xr+ys = 0\};$$

one can check directly that $(1,0)$ is a regular value of the function $\Phi(x,y,r,s) = (x^2+y^2, \; xr+ys)$.

The tangent bundle of a general manifold M is a special case of the more general concept of a *vector bundle*, which we will first define.

B.1.1 Vector bundles

Let E_p be k-dimensional vector spaces indexed by the points $p \in M$ of an m-dimensional manifold M, and

$$E = \bigcup_{p \in M} E_p$$

their disjoint union. We denote by $\pi : E \to M$ the projection and $i : M \to E$ the inclusion of zeroes.

Definition B.1. E is called a *vector bundle of rank r over M* if for each $p \in M$ there are charts (U, φ) around p and $(\widehat{U}, \widehat{\varphi})$ around $i(p)$, with $\widehat{U} = \pi^{-1}(U)$, such that $\widehat{\varphi}$ restricts to vector space isomorphisms

$$E_p = \pi^{-1}(p) \to \{\varphi(p)\} \times \mathbb{R}^r \cong \mathbb{R}^r$$

for all $p \in M$. One calls E the *total space* and M the *base* of the vector bundle. Charts $(\widehat{U}, \widehat{\varphi})$ are called *vector bundle charts.*

The vector bundle charts may be pictured in terms of a diagram,

$$\begin{tikzcd}
E \supseteq \widehat{U} \arrow[r, "\widehat{\varphi}"'] \arrow[d, "\pi"'] & \varphi(U) \times \mathbb{R}^r \arrow[d, "(u,v) \mapsto u"] \\
M \supseteq U \arrow[r, "\varphi"'] & \varphi(U)
\end{tikzcd}$$

The key condition is that $\widehat{\varphi}$ restricts to vector space isomorphisms fiberwise. Thus, just like a manifold M looks locally like an open subset of $\mathbb{R}^m$, a vector bundle looks over M locally like a product of an open subset of $\mathbb{R}^m$ with the vector space $\mathbb{R}^r$.

Proposition B.1. *For any vector bundle E over M, the projection $\pi : E \to M$ is a smooth submersion, while the inclusion $i : M \to E$ is an embedding as a submanifold.*

Proof. In vector bundle charts, the maps π and i are just the obvious projection $\varphi(U) \times \mathbb{R}^r \to \varphi(U)$ and inclusion $\varphi(U) \to \varphi(U) \times \mathbb{R}^r$, which obviously are submersions and embeddings respectively. $\square$

Note that the vector bundle charts $(\widehat{U}, \widehat{\varphi})$ are submanifold charts for $i(M) \subseteq E$. We will identify M with its image $i(M)$; it is called the *zero section* of the vector bundle.

Example B.2 (Trivial bundles). The *trivial vector bundle* over M is the direct product $M \times \mathbb{R}^r$. Charts for M directly give vector bundle charts for $M \times \mathbb{R}^r$.

Example B.3 (The infinite Möbius strip). View $M = S^1$ as a quotient $\mathbb{R}/\sim$ for the equivalence relation $x \sim x+1$. Let $E = (\mathbb{R}\times\mathbb{R})/\sim$ be the quotient under the equivalence relation $(x,y) \sim (x+1,-y)$, with the natural map

$$\pi:\ E \to M,\ [(x,y)] \mapsto [x].$$

Then E is a rank 1 vector bundle (a *line bundle*) over S^1. Its total space is an infinite Möbius strip.

Example B.4 (Vector bundles over the Grassmannians). For any $p \in \mathrm{Gr}(k,n)$, let $E_p \subseteq \mathbb{R}^n$ be the k-dimensional subspace that it represents. Then

$$E = \cup_{p\in \mathrm{Gr}(k,n)} E_p$$

is a vector bundle over $\mathrm{Gr}(k,n)$, called the *tautological vector bundle*. Recall our construction of charts (U_I, φ_I) for the Grassmannian, where U_I is the set of all p such that the projection map $\Pi_I:\ \mathbb{R}^n \to \mathbb{R}^I$ restricts to an isomorphism $E_p \to \mathbb{R}^I$, and

$$\varphi_I : U_I \to L(\mathbb{R}^I, \mathbb{R}^{I'}),$$

takes $p \in U_I$ to the linear map A having E_p as its graph. Let

$$\hat{\varphi}_I : \pi^{-1}(U_I) \to L(\mathbb{R}^I, \mathbb{R}^{I'}) \times \mathbb{R}^I$$

be the map given on the fiber E_p by

$$\hat{\varphi}_I(v) = (\varphi_I(p), \Pi_I(v)).$$

The $\hat{\varphi}_I$ are vector bundle charts for E (once we identify $L(\mathbb{R}^I, \mathbb{R}^{I'}) = \mathbb{R}^{k(n-k)}$ and $\mathbb{R}^I = \mathbb{R}^k$).

There is another natural vector bundle E' over $\mathrm{Gr}(k,n)$, with fiber $E'_p := E_p^\perp$ the orthogonal complement of E_p. In terms of the identification $\mathrm{Gr}(k,n) = \mathrm{Gr}(n-k,n)$, E' is the tautological vector bundle over $\mathrm{Gr}(n-k,n)$.

Remark B.1. Note that we did not worry about the 'Hausdorff property' for the total space. We leave it as an exercise to show that for any (possibly non-Hausdorff) vector bundle $E \to M$, the Hausdorff property for the total space E follows from the Hausdorff property of the base M.

Example B.5. As a special case, we obtain the *tautological line bundle E* and the *hyperplane bundle E'* over $\mathbb{R}\mathrm{P}^n = \mathrm{Gr}(1,n+1)$. The tautological line bundle is non-trivial: i.e., there do not exist global trivializations $E \to \mathbb{R}\mathrm{P}^n \times \mathbb{R}$. In the case $n = 1$ the line bundle over $\mathbb{R}\mathrm{P}^1 \cong S^1$ is the 'infinite Möbius strip' considered above.

Definition B.2. A *vector bundle map* (also called *vector bundle morphism*) from $E \to M$ to $F \to N$ is a smooth map $\widehat{\Phi}:\ E \to F$ of the total spaces, together with a smooth map $\Phi:\ M \to N$ of the base manifolds, such that $\widehat{\Phi}$ restricts to linear maps

$$\widehat{\Phi}:\ E_p \to F_{\Phi(p)}.$$

If $\widehat{\Phi}$ is a diffeomorphisms, then it is called a *vector bundle isomorphism*. An isomorphism $E \to M \times \mathbb{R}^r$ with a trivial vector bundle is called a *trivialization* of M.

Vector bundle maps are pictured by commutative diagrams:

$$\begin{array}{ccc} E & \xrightarrow[\widehat{\Phi}]{} & F \\ \downarrow & & \downarrow \\ M & \xrightarrow[\Phi]{} & N \end{array}$$

Example B.6. Any vector bundle chart $(\widehat{U}, \widehat{\varphi})$ defines a vector bundle map $\widehat{\varphi}$: $E|_U := \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^r$.

Example B.7. The tautological line bundle E over $\mathbb{R}\mathrm{P}^1$ is the simplest example of a vector bundle that does not admit a global trivialization. (For example, one can note that if one removes the zero section from E, then $E - M$ stays connected; on the other hand $(M \times \mathbb{R}) - M$ falls into two components.)

B.1.2 Tangent bundles

We return to the discussion of tangent bundles of manifolds.

Proposition B.2. *For any manifold M of dimension m, the* tangent bundle

$$TM = \cup_{p \in M} T_p M$$

(disjoint union of vector spaces) has the structure of a rank m vector bundle over M. Here $\pi: TM \to M$ takes $v \in T_pM$ to the base point p.

Proof. Recall that any chart (U, φ) for M gives vector space isomorphisms

$$T_p\varphi : T_pM = T_pU \to T_{\varphi(p)}\varphi(U) = \mathbb{R}^m$$

for all $p \in U$. Let $TU = \cup_{p \in U} T_pM = \pi^{-1}(U)$. The collection of maps $T_p\varphi$ gives a bijection,

$$T\varphi : TU \to \varphi(U) \times \mathbb{R}^m.$$

We take the collection of $(\widehat{U}, \widehat{\varphi}) = (TU, T\varphi)$ as vector bundle charts for TM:

$$\begin{array}{ccc} TM \supseteq TU & \xrightarrow[T\varphi]{} & \varphi(U) \times \mathbb{R}^m \\ \pi \downarrow & & \downarrow (u,v) \mapsto u \\ M \supseteq U & \xrightarrow[\varphi]{} & \varphi(U) \end{array}$$

We need to check that the transition maps are smooth. If (V,ψ) is another coordinate chart with $U\cap V\neq\emptyset$, the transition map for $TU\cap TV=T(U\cap V)=\pi^{-1}(U\cap V)$ is given by,

$$T\psi\circ(T\varphi)^{-1}:\ \varphi(U\cap V)\times\mathbb{R}^m\to\psi(U\cap V)\times\mathbb{R}^m.$$

But $T_p\psi\circ(T_p\varphi)^{-1}=T_{\varphi(p)}(\psi\circ\varphi^{-1})$ is just the derivative (Jacobian matrix) for the change of coordinates $\psi\circ\varphi^{-1}$; hence this map is given by

$$\varphi(U\cap V)\times\mathbb{R}^m\to\psi(U\cap V)\times\mathbb{R}^m,\ \ (x,a)\mapsto((\psi\circ\varphi^{-1})(x),\ D_x(\psi\circ\varphi^{-1})(a))$$

Since the Jacobian matrix depends smoothly on x, this is a smooth map. □

Proposition B.3. *For any smooth map $\Phi\in C^\infty(M,N)$, the map*

$$T\Phi:\ TM\to TN$$

given on $T_p\Phi$ as the tangent maps $T_p\Phi:\ T_pM\to T_{\Phi(p)}N$, is a vector bundle map.

Proof. Given $p\in M$, choose charts (U,φ) around p and (V,ψ) around $\Phi(p)$, with $\Phi(U)\subseteq V$. As explained above, these give vector bundle charts $(TU,T\varphi)$ and $(TV,T\psi)$. Let $\widetilde{\Phi}=\psi\circ\Phi\circ\varphi^{-1}:\ \varphi(U)\to\psi(V)$. The map

$$T\widetilde{\Phi}=T\psi\circ T\Phi\circ(T\varphi)^{-1}:\ T\varphi(TU)\to T\psi(TV)$$

is smooth, since by smooth dependence of the differential $D_x\widetilde{\Phi}$ on the base point. Consequently, $T\Phi$ is smooth, □

B.1.3 Some constructions with vector bundles

There are several natural constructions producing new vector bundles out of given vector bundles.

1. If $E_1\to M_1$ and $E_2\to M_2$ are vector bundles of rank r_1,r_2, then the cartesian product $E_1\times E_2$ is a vector bundle over $M_1\times M_2$, of rank r_1+r_2.
2. Let $\pi:\ E\to M$ be a given vector bundle.

   Proposition B.4. *Given a submanifold $S\subseteq M$, the restriction $E|_S:=\pi^{-1}(S)$ is a vector bundle over S, in such a way that the inclusion map $E|_S\to E$ is a vector bundle map (and also an embedding as a submanifold).*

   Proof. Given $p\in S\subseteq M$, let $(\widehat{U},\widehat{\varphi})$ be a vector bundle chart for E, with underlying chart (U,φ) containing p. Let (U',φ') be a submanifold chart for S at p. Replacing U,U' with their intersection, we may assume $U'=U$. Let $\widehat{\varphi}'$ be the composition of $\widehat{\varphi}:\ \pi^{-1}(U)\to\varphi(U)\times\mathbb{R}^r$ with the map

   $$(\varphi'\circ\varphi^{-1})\times\mathrm{id}:\ \varphi(U)\times\mathbb{R}^r\to\varphi'(U)\times\mathbb{R}^r.$$

Then $\widehat{\varphi}'$ takes $\pi^{-1}(S)$ to $(\mathbb{R}^l \cap \varphi'(U)) \times \mathbb{R}^r$; hence it is a vector bundle chart and also a submanifold chart. $\square$

More generally, suppose $\Phi \in C^\infty(M,N)$ is a smooth map between manifolds, and $\pi:\ E \to N$ is a vector bundle. Then the *pull-back bundle*

$$\Phi^*E := \cup_{p\in M} E_{\Phi(p)}$$

is a vector bundle over M. One way to prove this is as follows: Consider the embedding of M as the submanifold of $N \times M$ given as the graph of Φ:

$$M \cong \mathrm{gr}(\Phi) = \{(\Phi(x),x)|\ x \in M\}.$$

The vector bundle $E \to N$ defines a vector bundle over $N \times M$, by cartesian product with the zero bundle $0_M \to M$. We have

$$\Phi^*E \cong (E \times 0_M)|_{\mathrm{graph}(\Phi)}.$$

3. Let E, E' be two vector bundles over M. Then the *direct sum* (also called *Whitney sum*)
$$E \oplus E' := \cup_{p\in M} E_p \oplus E'_p$$
is again a vector bundle over M. One way to see this is to regard $E \oplus E'$ as the pull-back of the cartesian product $E \times E'$ under the diagonal inclusion $M \to M \times M,\ x \mapsto (x,x)$.
4. Suppose $\pi:\ E \to M$ is a vector bundle of rank r, and $E' \subseteq E$ is a *vector subbundle* of rank $r' \le r$. That is, E' is a submanifold of E, and is itself a vector bundle over M, with the map $\pi':\ E' \to M$ given by restriction of π. In particular, each $E'_p = \pi'^{-1}(p)$ a vector subspace of E_p. The *quotient bundle*
$$E/E' := \bigcup_p E_p/E'_p$$
is again a vector bundle over M.
5. For any vector bundle $E \to M$, the *dual bundle*
$$E^* = \cup_{p\in M} E_p^*$$
(where $E_p^* = L(E_p, \mathbb{R})$ is the dual space to E_p) is again a vector bundle.

Example B.8. Given a manifold M with a submanifold S, one calls $TM|_S$ the *tangent bundle of M along S*. It contains TS as a subbundle; the *normal bundle* of S in M is defined as a quotient bundle $\nu_S = TM|_S/TS$ with fibers,

$$(\nu_S)_p = T_pM/T_pS.$$

Example B.9. The dual of the tangent bundle TM is called the *cotangent bundle*, and is denoted T^*M. Given a submanifold $S \subseteq M$, one can consider the set of covectors

$\alpha \in T_p^*M$ for $p \in S$ that annihilate T_pS, that is, $\langle \alpha, v\rangle = 0$ for all $v \in T_pS$. This is a vector bundle called the *conormal bundle* ν_S^* of S. The notation is justified, since it is the dual bundle to ν_S.

Example B.10. The direct sum of the two natural bundles E, E' over the Grassmannian $\mathrm{Gr}(k,n)$ has fibers $E_p \oplus E'_p = \mathbb{R}^n$, hence $E \oplus E'$ is the trivial bundle $\mathrm{Gr}(k,n) \times \mathbb{R}^n$.

Definition B.3. A *smooth section* of a vector bundle $\pi : E \to M$ is a smooth map $\sigma : M \to E$ with the property $\pi \circ \sigma = \mathrm{id}_M$. The space of smooth sections of E is denoted $\Gamma^\infty(M,E)$, or simply $\Gamma^\infty(E)$.

Thus, a section is a family of vectors $\sigma_p \in E_p$ depending smoothly on p.

Examples B.11. 1. Every vector bundle has a distinguished section, the *zero section*

$$p \mapsto \sigma_p = 0,$$

where 0 is the zero vector in the fiber E_p. One usually denotes the zero section itself by 0.

2. For a *trivial bundle* $M \times \mathbb{R}^r$, a section is the same thing as a smooth function from M to R^r:
$$\Gamma^\infty(M, M \times \mathbb{R}^r) = C^\infty(M, \mathbb{R}^r).$$
Indeed, any such function $f : M \to \mathbb{R}^r$ defines a section $\sigma(p) = (p, f(p))$; conversely, any section $\sigma : M \to E = M \times \mathbb{R}^r$ defines a function by composition with the projection $M \times \mathbb{R}^r \to \mathbb{R}^r$.
In particular, if $\kappa : E|_U \to U \times \mathbb{R}^r$ is a local trivialization of a vector bundle E over an open subset U, then a section $\sigma \in \Gamma^\infty(E)$ restricts to a smooth function $\psi \circ \sigma|_U : U \to \mathbb{R}^r$.
3. Let $\pi : E \to M$ be a rank r vector bundle. A *frame* for E over $U \subseteq M$ is a collection of sections $\sigma_1, \ldots, \sigma_r$ of E_U, such that $(\sigma_j)_p$ are linearly independent at each point $p \in U$. Any frame over U defines a local trivialization $\psi : E_U \to U \times \mathbb{R}^r$, given in terms of its inverse map $\psi^{-1}(p,a) = \sum_j a_j (\sigma_j)_p$. Conversely, each local trivialization gives rise to a frame.

The space $\Gamma^\infty(M,E)$ is a vector space under pointwise addition:

$$(\sigma_1 + \sigma_2)_p = (\sigma_1)_p + (\sigma_2)_p.$$

Moreover, it is a module over the algebra $C^\infty(M)$, under multiplication[1]: $(f\sigma)_p = f_p \sigma_p$.

[1] Here and from now on, we will often write f_p or $f|_p$ for the value $f(p)$.

B.2 Dual bundles

More generally, if $E \to M$ is a vector bundle of rank r, then we can define its dual $E^* \to M$ with fibers $E_p^* = (E_p)^*$.

Proposition B.5. *The dual E^* of a vector bundle E is itself a vector bundle.*

Proof. For any open subset $U \subseteq M$, write $E_U = \bigcup_{p \in U} E_p$ for the restriction of E to U, and similarly $E_U^* = \bigcup_{p \in U} E_p$. Recall that a vector bundle chart for E is given by a chart (U, φ) for M together with a chart for E, of the form $(E_U, \widehat{\varphi})$, where

$$\widehat{\varphi} : \ E_U \to \varphi(U) \times \mathbb{R}^r$$

is a fiberwise linear isomorphism. Taking the inverse of the dual of the vector space isomorphism $E_p \to \mathbb{R}^r$, we get maps $E_p^* \to (\mathbb{R}^r)^*$, hence

$$(\widehat{\varphi}^*)^{-1} : \ E_U^* \to \varphi(U) \times (\mathbb{R}^r)^*.$$

Identifying $(\mathbb{R}^r)^* \cong \mathbb{R}^r$, these serve as vector bundle charts $(E_U^*, \ (\widehat{\varphi}^*)^{-1})$ for E^*. □

Given smooth sections $\sigma \in \Gamma^\infty(M, E)$ and $\tau \in \Gamma^\infty(M, E^*)$, one can take the pairing to define a function

$$\langle \tau, \sigma \rangle \in C^\infty(M), \quad \langle \tau, \sigma \rangle(p) = \langle \tau_p, \ \sigma_p \rangle.$$

This pairing is $C^\infty(M)$-linear in both entries: That is,

$$\langle \tau, \ f\sigma \rangle = f \langle \tau, \ \sigma \rangle = \langle f\tau, \ \sigma \rangle$$

for all $\tau \in \Gamma^\infty(M, E^*)$, $\sigma \in \Gamma^\infty(M, E)$, $f \in C^\infty(M)$. We can use pairings with smooth sections of E to characterize the smooth sections of E^*.

Proposition B.6. *1. A family of elements $\tau_p \in E_p^*$ for $p \in M$ defines a smooth section of E^* if and only if for all $\sigma \in \Gamma^\infty(M, E)$, the function*

$$M \to \mathbb{R}, \ \ p \mapsto \langle \tau_p, \sigma_p \rangle$$

is smooth.

2. The space of sections of the dual bundle is identified with the space of $C^\infty(M)$-linear maps

$$\tau : \ \Gamma^\infty(M, E) \to C^\infty(M), \quad \sigma \mapsto \langle \tau, \sigma \rangle.$$

Here $C^\infty(M)$-linear means that $\langle \tau, \ f\sigma \rangle = f \langle \tau, \ \sigma \rangle$ for all functions f.

Proof. The proof uses local bundle charts and bump functions. Details are left as an exercise. [2]

[2] It is similar to the fact, proved earlier, that a collection of tangent vectors $X_p \in T_pM$ defines a smooth vector field if and only if for any $f \in C^\infty(M)$ the map $\mapsto X_p(f)$ is smooth.

Remark B.2. A slightly more precise version of the second part of this proposition is as follows: Regard $\mathscr{E} = \Gamma^\infty(M,E)$ as a module over the algebra $A = C^\infty(M)$ of smooth functions, and likewise for $\mathscr{E}^* = \Gamma^\infty(M,E^*)$. The space $\mathrm{Hom}_A(\mathscr{E},A)$ of $A = C^\infty(M)$-linear maps $\mathscr{E} \to A$ is again an A-module. There is a natural A-module map

$$\mathscr{E}^* \to \mathrm{Hom}_A(\mathscr{E},A)$$

defined by the pairing of sections. This map is an isomorphism.