Category $\mathcal{O}$, Problem Set 4

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Contents

1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$-module defined by letting x act as 0 and h act as $\left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right)$.

Show that the induced $U(\mathfrak{g})$-module structure $M \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

Hence $M \notin \mathcal{O}$.

We first unpack all definitions in terms of tensor products, using the fact that $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(\lambda) & \longrightarrow & M & \longrightarrow & M(\lambda) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccc}
1 \otimes 1 & \xmapsto{\psi} & 1 \otimes \mathbf{u} & \xmapsto{\phi} & 1 \otimes 0 \\
& & & & \\
& & 1 \otimes \mathbf{v} & \longmapsto & 1 \otimes 1
\end{array}$$

where $N = \operatorname{span}_{\mathbb{C}} \{\mathbf{u}, \mathbf{v}\}$.

We make the following claims:

1. The action defined on N lifts to a $U(\mathfrak{g})$-action on M.
2. This is an exact sequence of $U(\mathfrak{g})$-modules.
3. $M \ncong M(\lambda) \oplus M(\lambda)$, showing that this sequence can not split.

We note that by using a PBW basis for $\mathfrak{sl}(2, \mathbb{C})$, M is spanned by infinitely many basis elements of the form $\left\{ y^k \otimes \mathbf{u}, y^j \otimes \mathbf{v} \;\middle|\; j, k \in \mathbb{Z}^{\geq 0} \right\}$. We also note that identifying the image $\psi(M(\lambda)) = \mathbb{C} \cdot \mathbf{u} \cong \mathbb{C}_\lambda$ (as $U(\mathfrak{b})$-modules) we obtain a submodule of N, which lifts to a submodule of the induced module M. Similarly,

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

2.1 Solution

We first note that $M \in \mathcal{O} \implies M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ where each M_λ is a finite-dimensional weight space.

Moreover, $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$ is defined to be a direct sum of duals of weight spaces, which are still finite-dimensional.

So let $M, N \in \mathcal{O}$; we will proceed by showing that both $(M \otimes_{\mathbb{C}} L)^\vee$ and $M^\vee \otimes_{\mathbb{C}}^\vee$ have identical direct sum decompositions.

We first have

$$\begin{aligned}
(M \otimes_{\mathbb{C}} L)^{\vee} &\coloneqq \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} (M \otimes_{\mathbb{C}} L)_{\lambda}^{\vee}, && \text{the } \lambda \text{ weight spaces of } M \otimes_{\mathbb{C}} L \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} \left(\bigoplus_{\alpha+\beta=\lambda} (M_{\alpha} \otimes_{\mathbb{C}} L_{\beta}) \right)^{\vee} && \text{by an exercise on the weight spaces of a tensor product} \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} \left(\bigoplus_{\alpha+\beta=\lambda} (M_{\alpha} \otimes_{\mathbb{C}} L_{\beta})^{\vee} \right) && \text{since the inner term is a finite sum} \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} \left(\bigoplus_{\alpha+\beta=\lambda} \left(M_{\alpha}^{\vee} \otimes_{\mathbb{C}} L_{\beta}^{\vee} \right) \right) && \text{since the weight spaces are finite-dimensional,}
\end{aligned}$$

where we've repeatedly used the fact that $(V \otimes W)^{\vee} \cong V^{\vee} \otimes W^{\vee}$ for finite-dimensional vector spaces, which inductively holds for any finite direct sum of vector spaces.

On the other hand, using the fact that

$$\begin{aligned}
(A \oplus B) \otimes (C \oplus D) &= ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \\
&= (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \\
\implies \left(\bigoplus_{j \in J} A_i \right) \otimes \left(\bigoplus_{k \in K} B_k \right) &= \bigoplus_{j \in J} \bigoplus_{k \in K} (A_j \otimes B_k) \quad \text{by induction} \quad .
\end{aligned}$$

we can write

$$\begin{aligned}
M^{\vee} \otimes_{\mathbb{C}} L^{\vee} &\coloneqq \left(\bigoplus_{\alpha \in \mathfrak{h}^{\vee}} M_{\alpha}^{\vee} \right) \otimes_{\mathbb{C}} \left(\bigoplus_{\beta \in \mathfrak{h}^{\vee}} L_{\beta}^{\vee} \right) \\
&\cong \bigoplus_{\lambda \in \mathfrak{h}^{\vee}} \left(\bigoplus_{\alpha+\beta=\lambda} \left(M_{\alpha}^{\vee} \otimes_{\mathbb{C}} L_{\beta}^{\vee} \right) \right),
\end{aligned}$$

which equals what was obtained above.

This exhibits the isomorphism as $\mathbb{C}$-vector spaces, to see that this is in fact as isomorphism of $U(\mathfrak{g})$-modules we can use the fact that for $M \in \mathcal{O}$, a twisted $\mathfrak{g}$-action was defined as

$$\mathbf{v} \in M, \ f \in M^{\vee}, \ g \in \mathfrak{g} \implies (g \cdot f)(\mathbf{v}) = f(\tau(g) \cdot \mathbf{v})$$

for the transpose map τ. This action can be "linearly extended" over direct products and tensor products by taking the action component-wise, and is thus preserved by all of the isomorphisms appearing above.

Since the final terms $\bigoplus_{\lambda \in \mathfrak{h}} \bigoplus_{\alpha+\beta=\lambda} M_{\alpha}^{\vee} \otimes L_{\beta}^{\vee}$ are identical, they carry the same action, and since they are preserved by the isomorphisms, working backwards shows that the actions on $(M \otimes L)^{\vee}$ and $M^{\vee} \otimes L^{\vee}$ must also agree, yielding the desired isomorphism.

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ.

> For a concrete example, take Φ of type B_2 with a short simple root α and a long simple root β. If $\lambda \coloneqq \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in Φ.

3.1 Solution

We would like to show the following two propositions:

1. $\Phi^+_{[\lambda]} \coloneqq \Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. In general, the associated simple system $\Delta_{[\lambda]} \neq \Phi^+_{[\lambda]} \bigcap \Delta$.

3.1.1 Proof of Proposition 1

We'll use the definition that for an abstract root system Φ, a positive system Φ^+ is defined by picking a hyperplane H not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of Φ^+ is on one side of H, then any subset satisfies this property as well, thus $\Phi_{[\lambda]} \bigcap \Phi^+$ consists only of positive roots and thus forms a positive system.

3.1.2 Proof of Proposition 2

Concretely, we can realize Φ and Δ as subsets of $\mathbb{R}^2$ in the following way:

$$
\begin{aligned}
&\Phi = P_1 \coprod P_2 \coloneqq \{[1,0],[0,1],[-1,0],[0,-1]\} \coprod \{[1,1],[-1,1],[1,-1],[-1,-1]\} \\
&\Delta \coloneqq \{\alpha, \beta\} \coloneqq \{[1,0],[-1,1]\},
\end{aligned}
$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$
\begin{aligned}
\Phi_{[\lambda]} &\coloneqq \left\{ \gamma \in \Phi \;\middle|\; \langle \lambda,\ \gamma^\vee \rangle \in \mathbb{Z} \right\}, \qquad \gamma^\vee \coloneqq \frac{2}{\|\gamma\|^2}\,\gamma, \\
\Delta_{[\lambda]} &\coloneqq \left\{ \gamma \in \Delta \;\middle|\; \langle \lambda,\ \gamma^\vee \rangle \in \mathbb{Z} \right\}.
\end{aligned}
$$

Now choosing $\lambda \coloneqq \dfrac{\alpha}{2} = \left[\dfrac{1}{2}, 0\right]$, we now consider the inner products $\langle \lambda,\ \gamma^\vee \rangle$ for $\gamma \in \Phi$:

Thus

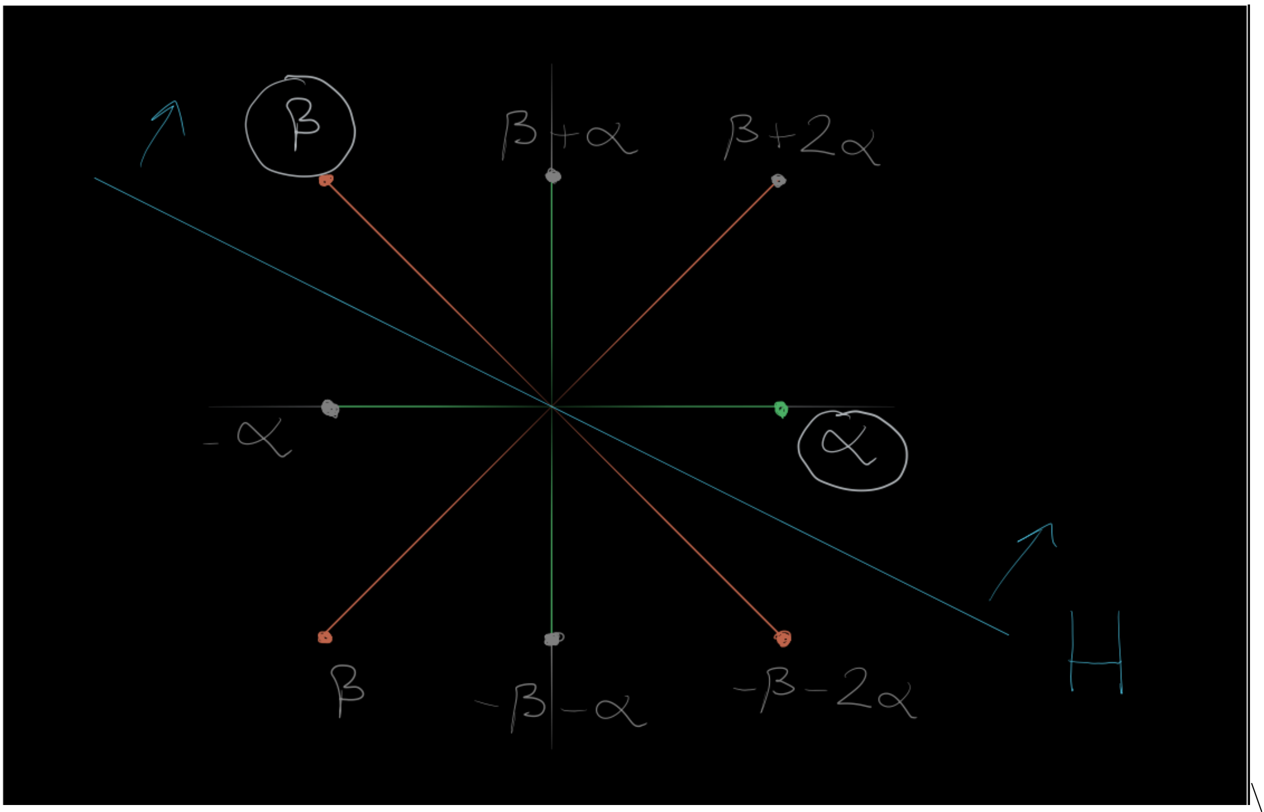
$$
\begin{aligned}
\gamma_1 \in P_1 &\implies \left\langle \left[\frac{1}{2}, 0\right],\ 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right)\langle [1,0],\ \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z} \\
\gamma_2 \in P_2 &\implies \langle \lambda,\ \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right],\ \frac{2}{\left(\sqrt{2}\right)^2}[\pm 1, \pm 1] \right\rangle = \pm\frac{1}{2} \notin \mathbb{Z}
\end{aligned}
$$

where $(\gamma_1)_1$ denotes the first component of γ_1.

We thus find that

$$\begin{aligned} \Phi_{[\lambda]} &= P_1 && \text{the short roots} \\ \Delta_{[\lambda]} = \Phi_{[\lambda]} \bigcap \Delta &= \{\alpha\} && \text{the single short simple root.} \end{aligned}$$

Choosing the following hyperplane H not containing any root, we can choose a positive system:



$$\Phi^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$$

where we can note that $\Phi^+ \bigcap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi^+_{[\lambda]} = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$. But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \neq \{\alpha\} = \{\alpha, \alpha + \beta\} \bigcap \{\alpha, \beta\} = \Phi^+_{[\lambda]} \bigcap \Delta,$$

which is what we wanted to show.

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that $\operatorname{coker} \phi$ has a standard filtration.