HW#4 Qual Problems

Zack Garza

① We will use the fact that if $\psi: \mathbb{Q}(\alpha) \to F$ is any field homomorphism, then it must be the case that $\psi(\alpha) = \beta$, some conjugate of α in $\overline{\mathbb{Q}}$. In particular, we must have $\beta \in \overline{\mathbb{Q}} \cap F$. There are 17 possibilities here: $\beta \in \{ \sqrt[17]{2}\, \zeta_{17}^k \mid 1 \leq k \leq 17 \} := S_\alpha$. Thus

ⓐ $\#\{\sigma: \mathbb{Q}(\alpha) \to \mathbb{C}\} = \# S_\alpha \cap \mathbb{C} = \# S_\alpha = \underline{17}$ since each $\sqrt[17]{2}\, \zeta_{17}^k \in \mathbb{C}$,

ⓑ $\#\{\sigma: \mathbb{Q}(\alpha) \to \mathbb{R}\} = \# S_\alpha \cap \mathbb{R} = \#\{\sqrt[17]{2}\, \zeta_{17}^{17}\} = \underline{1}$ since each other $\sqrt[17]{2}\, \zeta_{17}^k$ has a nonzero imaginary component,

ⓒ $\#\{\sigma: \mathbb{Q}(\alpha) \to \overline{\mathbb{Q}}\} = \# S_\alpha \cap \overline{\mathbb{Q}} = \# S_\alpha = 17$, since $\overline{\mathbb{Q}}$ contains every root of every polynomial in $\mathbb{Q}[x]$.

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② $\underline{\text{Claim}}$: C is algebraically closed. Supposing otherwise, there would be some $f(x) \in C[x]$ s.t. $\underline{f \text{ has no roots in } C}$. Let α be one root; then $f(\alpha) = 0$ in $C(\alpha)$. Write $f(x) = \sum_{j=1}^{n} c_j x^j$; then α is also a root in $F(c_1, c_2, \cdots, c_n, \alpha)$, which is a $\underline{\text{finite}}$ extension of F (since C itself was an algebraic extension). This means that $g := \min(\alpha, F)$ is actually in $F[x]$; but then g splits into linear factors in C, so $\underline{\alpha \in C}$. $\#$

$\boxed{1 \Rightarrow 2:}$ Let $E \geq F$ be an arbitrary algebraic extension. Then consider $S = \left\{ (A, \tau) \;\middle|\; \begin{array}{l} F \leq A \leq E, \\ \tau: A \to C,\ \tau|_F = id_F \end{array} \right\}$

Make S a poset by defining $(A, \tau) \prec (B, \sigma)$ iff $B \geq A$ and $\sigma|_A = \tau$.

For any chain $\cdots \prec (A_i, \tau_i) \prec (A_{i+1}, \tau_{i+1}) \prec \cdots$

define $(A, \tau) := (\bigcup A_i, f)$ where $f: A \to C$ is defined as $x \in A \Rightarrow x \in A_i$ for some $i \Rightarrow f(x) = \tau_i(x)$.

This is an upper bound, so Zorn's lemma applies to yield some maximal elt $(A, \tau: A \to C)$.

$\underline{\text{Claim}: A = E.}$

Otherwise, let $\alpha \in E \setminus A$, and consider $f = \min(\alpha, F)$. Then $\underline{\tau(f) \in C[x] \text{ splits into linear factors}}$, and has some root $\beta \in C$. So consider $E \geq \underline{A(\alpha)} \geq A$; we could then define $\tau': A(\alpha) \to C$ by

$x \mapsto \begin{cases} \beta & \text{if } x = \alpha \\ \tau(x) & \text{else} \end{cases}$ which makes $(A(\alpha), \tau') \succ (A, \tau)$, contradicting maximality.

So $(A, \tau: A \to C) = (E, \tau: E \to C)$, and τ is the desired extension.

$\boxed{2 \Rightarrow 1:}$ Take $E = \overline{F}$ and $\tau : \overline{F} \to C$ be the supplied map, then let $f \in F[x]$ be arbitrary. Then f splits in $\overline{F}[x]$, so $f(x) = \prod (x - \alpha_i) \in \overline{F}[x]$; but then $\tau(f(x)) = \prod (x - \tau(\alpha_i)) \in C[x]$, so f splits in $C[x]$ as well.

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4 (a)

Suppose $\dim_k R = n < \infty$, then any basis of R is a linearly indepent spanning set containing m elts. Then suppose that there is no pair $(m, \{c_0, \cdots, c_{m-1}\})$ s.t. $\alpha^m + \sum_{j=0}^{m-1} c_j \alpha^j = 0$. Then, for example, if R has basis $\{v_1, \cdots, v_n\}$, then $\{v_1, \cdots, v_n, \alpha^1, \cdots, \alpha^m\}$ is a linearly independent spanning set of size $> m$. ⨳

So some pair $(m, \{c_i\})$ must exist.

(b) $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0$ iff $a^n + c_{n-1}a^{n-1} + \cdots + c_1 a = -c_0$

$$\text{iff } \underbrace{a(a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_1)(-c_0^{-1})}_{:= a^{-1}} = 1$$

Exists since $c_0 \neq 0$.

So $a \in R^\times$.

(c) Suppose $a \neq 0$ is not a zero divisor. If $c_0 \neq 0$, we are done by (b), so suppose $c_0 = 0$. Then

$$a^n + c_{n-1}a^{n-1} + \cdots + c_2 a^2 + c_1 a = 0 \text{ iff}$$

$$a(a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_2 a + c_1) = 0 \text{ iff}$$

$$a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_2 a + c_1 = 0$$

Since a is not a zero divisor. Proceeding inductively, there is some smallest j such that $c_j \neq 0$ and

$a^{n-j} + \cdots + c_j = 0$. But case (b) again yields $a \in R^\times$. ∎