Title

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1 Lecture 10

Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of **??**; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$, then the data of a GL_n-torsor split by a τ-cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$\begin{tikzcd} U \times_X U \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\pi_2"] \\ U \arrow[d] \\ X \end{tikzcd}$$

That U trivializes our torsor means that $\pi^* T = \pi^* G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\begin{tikzcd} \pi_1^* \pi^* T \arrow[r, "\sim"] \arrow[d] & \pi_2^* \pi^* T \arrow[d] \\ \pi_1^* \pi^* G \arrow[r, "\sim"] & \pi_2^* \pi^* G \end{tikzcd}$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n-torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\text{ét}}, \mathrm{GL}_n)$ is in bijection with n-dimensional vector bundles on X.

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Exercise 1.0.3 *(?)*: See if Hilbert 90 is true for groups other than GL_n.

1.1 Representability and Local Triviality

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).
You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:

$$\begin{tikzcd} T \times_X T \arrow[r] \arrow[d, shift right] & T \arrow[d] \\ T \arrow[r, "f"] \arrow[u, bend right, shift right, "\exists"'] & X \end{tikzcd}$$

The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that

$$\begin{tikzcd} & T \times_X T \arrow[r] \arrow[d, shift right] & T \arrow[d] \\ & T \arrow[r, "f"] \arrow[u, bend right, shift right, "\exists"'] & X \\ U \arrow[ur, hook, "\text{closed}"] \arrow[urr, bend right, "\exists \text{ét}"'] & & \end{tikzcd}$$

Here "slicing" means finding such a U, and this can be done using the structure theorem for smooth morphisms.

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Example 1.1.5 ***(non-smooth group schemes):***

- α_p, the kernel of Frobenius on $\mathbb{A}^1$ or $\mathbb{G}_a$,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on $\mathbb{G}_m$,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$, etc.

1.1.1 What Hilbert 90 Means

Example 1.1.6 ***(?):*** Let $X = \operatorname{Spec} k, n = 1$, so we're looking at $H^{\cdot}(\operatorname{Spec} k, \mathbb{G}_m)$.

$$\begin{aligned} H^1\left((\operatorname{Spec} k)_{\text{zar}}, \mathbb{G}_m\right) &= 0 \\ &= H^1\left((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m\right) \\ &= H^1(\operatorname{Gal}(k^s/k), \bar{k}^\times). \end{aligned}$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.[1]

Let's generalize this observation.

Example 1.1.7 ***(?):*** Let X be any scheme and $n = 1$, then $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

Example 1.1.8 ***(?):*** Let's compute $H^1(X_{\text{ét}}, \mu_\ell)$ where ℓ is an invertible function on X. We have a SES of étale sheaves, the **Kummer sequence**,

$$1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{z \mapsto z^p} \mathbb{G}_m \to 1.$$

This is exact in the étale topology since adjoining an ℓth power of any function gives an étale cover. We get a LES in cohomology

$$\begin{tikzcd} & & 0 \arrow[dll] \\ H^0(X_{\text{ét}}, \mu_\ell) \arrow[r] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[r, "z \mapsto z^\ell"] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[dll] \\ {\color{red} H^1(X_{\text{ét}}, \mu_\ell)} \arrow[r] & \operatorname{Pic}(X) \arrow[r, "{[\ell]}"] & \operatorname{Pic}(X) \arrow[dll] \\ H^2(X_{\text{ét}}, \mu_\ell) \arrow[r] & \cdots & \end{tikzcd}$$

We know that $H^0(X_{\text{ét}}, \mathbb{G}_m)$ are invertible functions on X, and the red term is what we'd like to compute.

Suppose now $H^0(X, \mathcal{O}_X) = k = \bar{k}$, then $H^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k)$ since it is the kernel of the ℓth power map. We can also compute $H^1(X_{\text{ét}}, \mu_\ell)$, since our diagram reduces to

[1]This is called "90" since Hilbert numbered his theorems in at least one of his books.

$$
\begin{tikzcd}
 & & 0 \arrow[dll, hook] \\
\mu_\ell(k) \arrow[r] & k^\times \arrow[r, "z\mapsto z^\ell"] & k^\times \arrow[dll, two heads, "\delta"'] \\
H^1(X_{\text{ét}},\mu_\ell) \arrow[r] & \mathrm{Pic}(X)[\ell] \arrow[r, "{[\ell]}"] & \mathrm{Pic}(X) \arrow[dll] \\
H^2(X_{\text{ét}},\mu_\ell) \arrow[r] & \cdots &
\end{tikzcd}
$$

where surjectivity of δ follows from the fact that $k = \overline{k}$ and thus every element has an ℓth root, making H^1 the kernel of $[\ell]$.