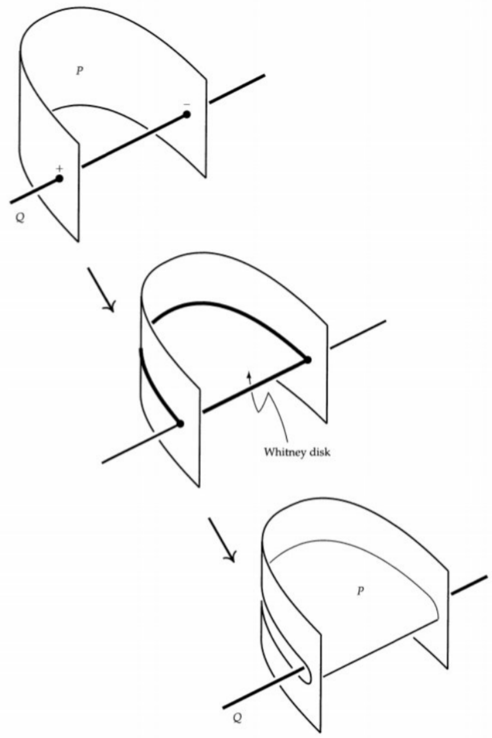

1.26. The Whitney trick

Notes: These are notes live-tex'd from a graduate course in 4-Manifolds taught by Philip Engel at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2021

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1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

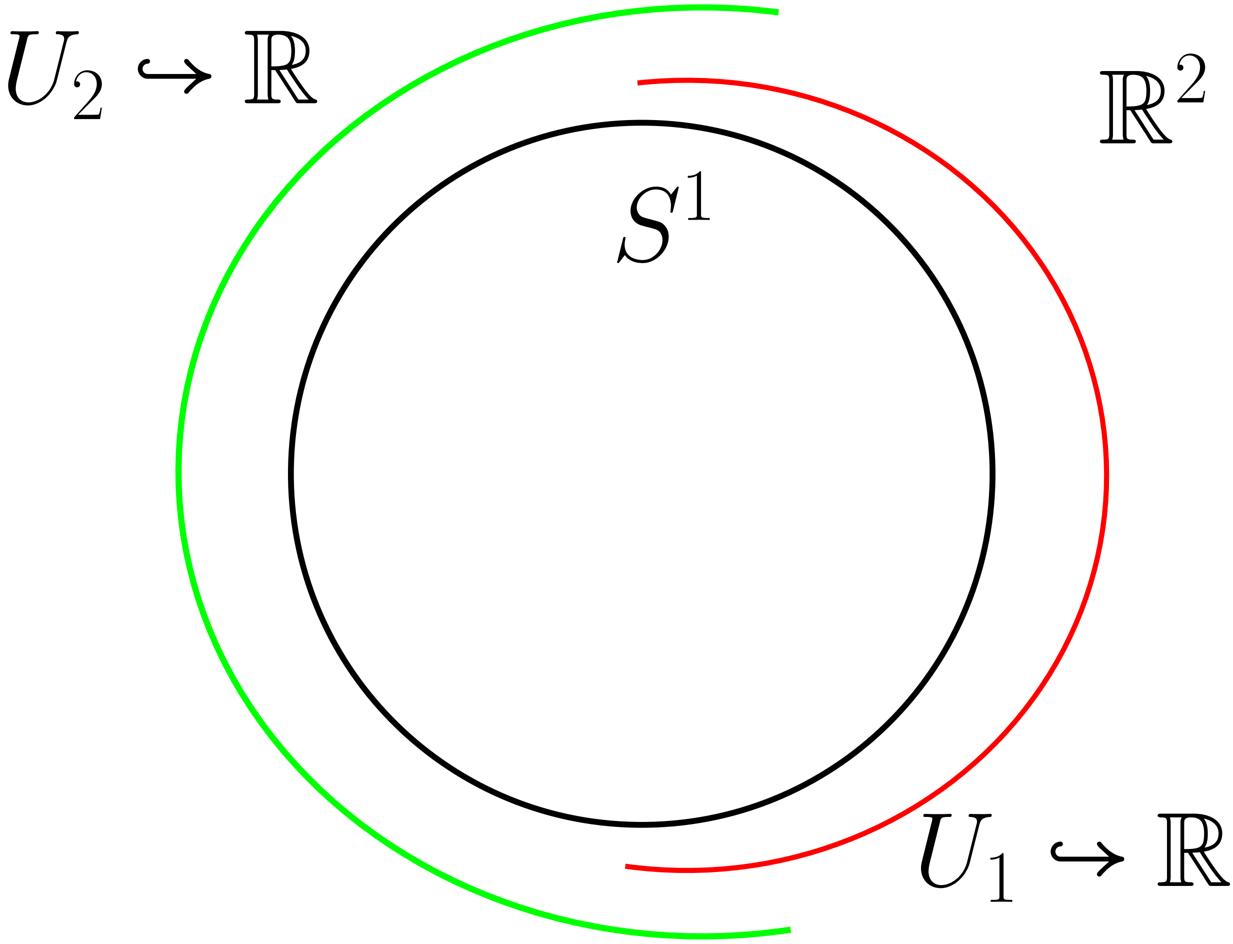
- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [**Dietmar99**]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [**Mandelbaum1980**]
    - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic Rˆ4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [**Calegari**]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [**Rudyak**]
- Akhil Mathew: The Dirac Operator [**Matthew**]
- Tom Weston: An Introduction to Cobordism Theory [**Weston**]

  A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

Definition 1.2.1 (Topological Manifold)
Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to $\mathbb{R}^n$ with a countable topological base, so we have charts $\varphi_u : U \to \mathbb{R}^n$ which are homeomorphisms from open sets covering X.

Example 1.2.2 ***(The circle):*** S^1 is covered by two charts homeomorphic to intervals:



Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} \coloneqq \varphi_V \to \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on $\mathbb{CP}^N$.

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6
Consider $\mathbb{R}^n$ as a topological manifold: are any two smooth structures on $\mathbb{R}^n$ diffeomorphic?

Remark 1.2.7: Fix a copy of $\mathbb{R}$ and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\to \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

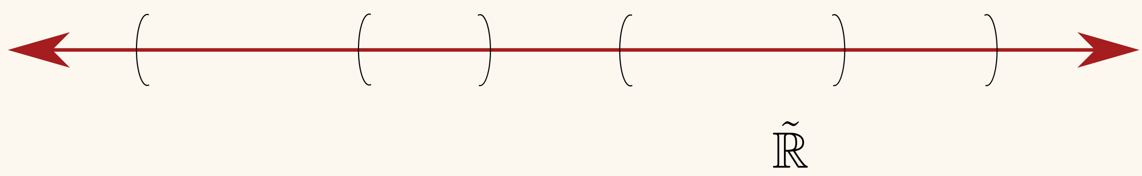
This is also a smooth structure on X, since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

$$\begin{aligned} X &\to X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

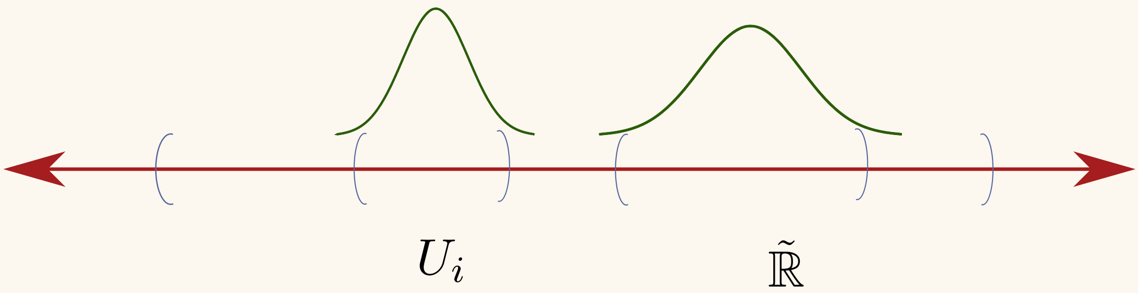
Claim: $\mathbb{R}$ admits a unique smooth structure.

Proof (sketch).
Let $\tilde{\mathbb{R}}$ be some exotic $\mathbb{R}$, i.e. a smooth manifold homeomorphic to $\mathbb{R}$. Cover this by coordinate charts to the standard $\mathbb{R}$:



Fact
There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \to \mathbb{R}$ with $f_i \geq 0$ and $\operatorname{supp} f \subseteq U_i$ such that $\sum f_i = 1$ (*i.e., bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:



Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\to \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan

To understand smooth structures on X, we should try to solve differential equations on X.

■

Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2.

Question 1.2.11

What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.[1]

2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.

3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.

4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $\operatorname{ks}(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure.

[1]Note that this doesn't start at C^0, so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.

Moreover, if $\mathrm{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X;\mathbb{Z}/2\mathbb{Z})$.

5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.

6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, $\mathbb{R}^n$ for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!

7. (Taubes) $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures.

8. A compact oriented smooth surface Σ, the space of complex-analytic structures is a complex orbifold [2] of dimension $3g-2$ where g is the genus of Σ, up to biholomorphism (i.e. *moduli*).

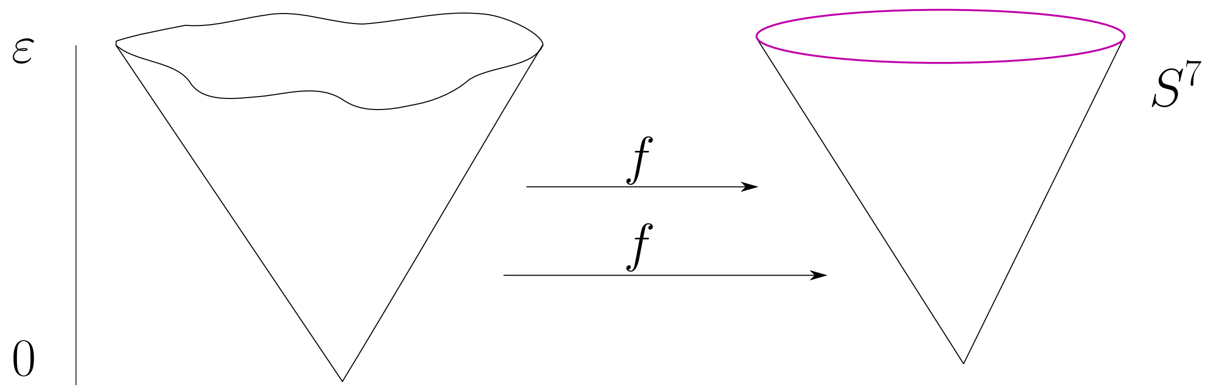
Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group.

2 | Friday, January 15

Remark 2.0.1: Let

$$V \coloneqq \left\{ a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0 \right\} \subseteq \mathbb{C}^5$$
$$S_\varepsilon \coloneqq \left\{ |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 \right\}.$$

Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \cdots, 28$ yields the 28 smooth structures on S^7. Note that V_k is the cone over $V_k \cap S_\varepsilon$.

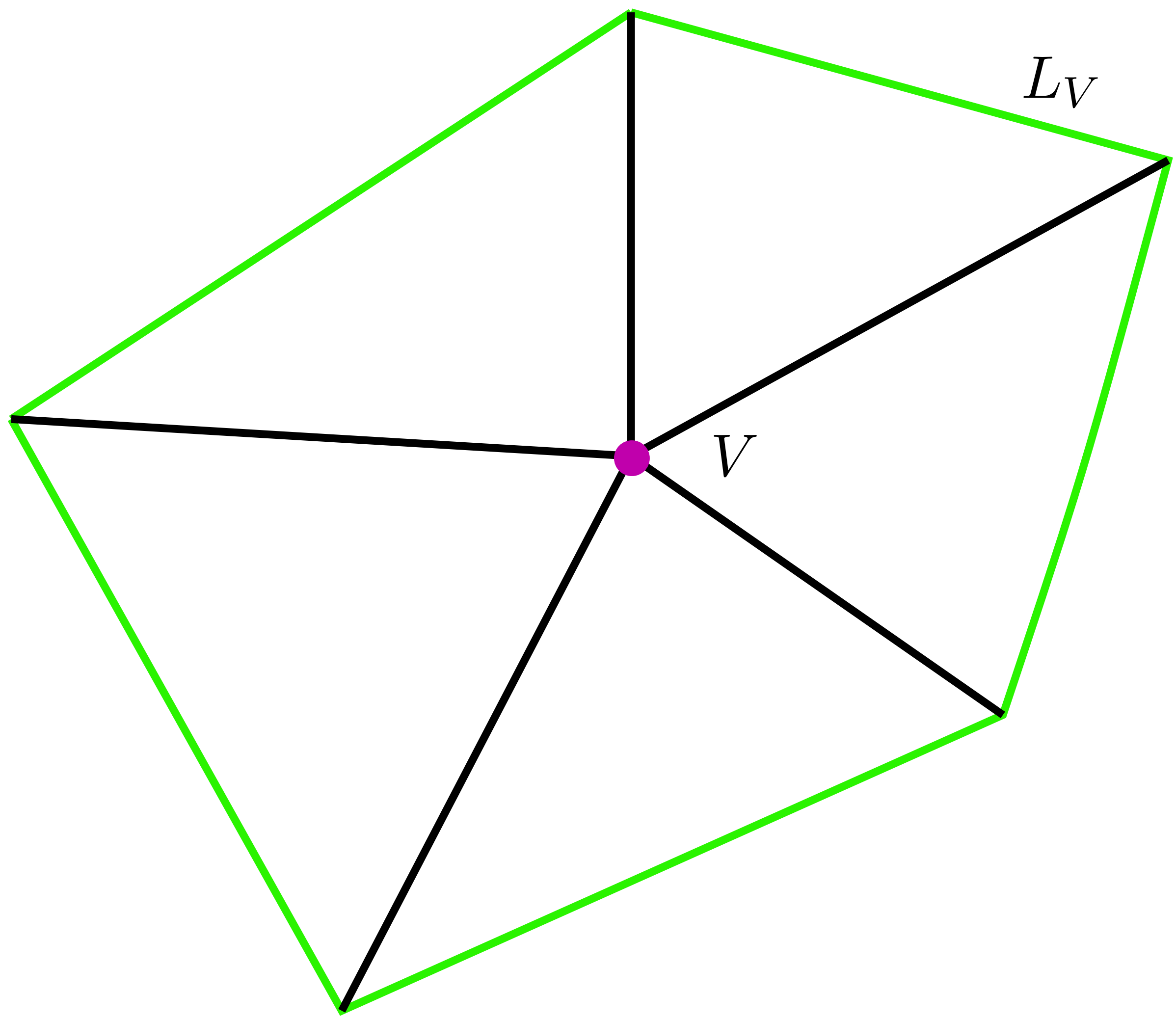


? Admits a smooth structure, and $\overline{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure.

Question 2.0.2
Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

[2] Locally admits a chart to $\mathbb{C}^n/\Gamma$ for Γ a finite group.

Answer 2.0.3
No! Given a simplicial complex, there is a notion of the **combinatorial link** L_V of a vertex V:



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a "PL triangulation" where the links are spheres.

Remark 2.0.4: What's special in dimension 4? Recall the **Kirby-Siebenmann** invariant $\mathrm{ks}(x) \in H^4(X;\mathbb{Z}_2)$ for X a topological manifold where $\mathrm{ks}(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)
Let X be a topological 4-manifold, then

$$\mathrm{ks}(X) \coloneqq \mathrm{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X;\mathbb{Z}_2) = H^4(X \times \mathbb{R};\mathbb{Z}_2) = \mathbb{Z}_2,.$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\mathrm{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} \end{cases}.$$

Remark 2.0.7: $\mathrm{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X;\mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha, \ \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 | Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 ***(Freedman).***
If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\mathrm{ks}(X) = \mathrm{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha, \ \varphi\beta \rangle = \langle \alpha, \ \beta \rangle$.
Conversely, every **unimodular** bilinear form appears as $H^2(X;\mathbb{Z})$ for some X, i.e. the pairing induces a map

$$\begin{aligned} H^2(X;\mathbb{Z}) &\to H^2(X;\mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha, \ \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for "being an element of the orthogonal group", where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X;\mathbb{Z}) \xrightarrow{PD} H_2(X;\mathbb{Z})$ from Poincaré , where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1, \ \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)
The **signature** of a topological 4- manifold is the signature of Q_X, where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X;\mathbb{R})$ and for some a, b

$$(H^2(X;\mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of $+1$s appearing in the matrix and b is the number of -1s. This is $\mathbb{R}^{ab}$ where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a+1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a\times a} \oplus -I_{b\times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 ***(Rokhlin's Theorem).***
Suppose $\langle \alpha, \ \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X;\mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\operatorname{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 ***(Donaldson).***
Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X;\mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

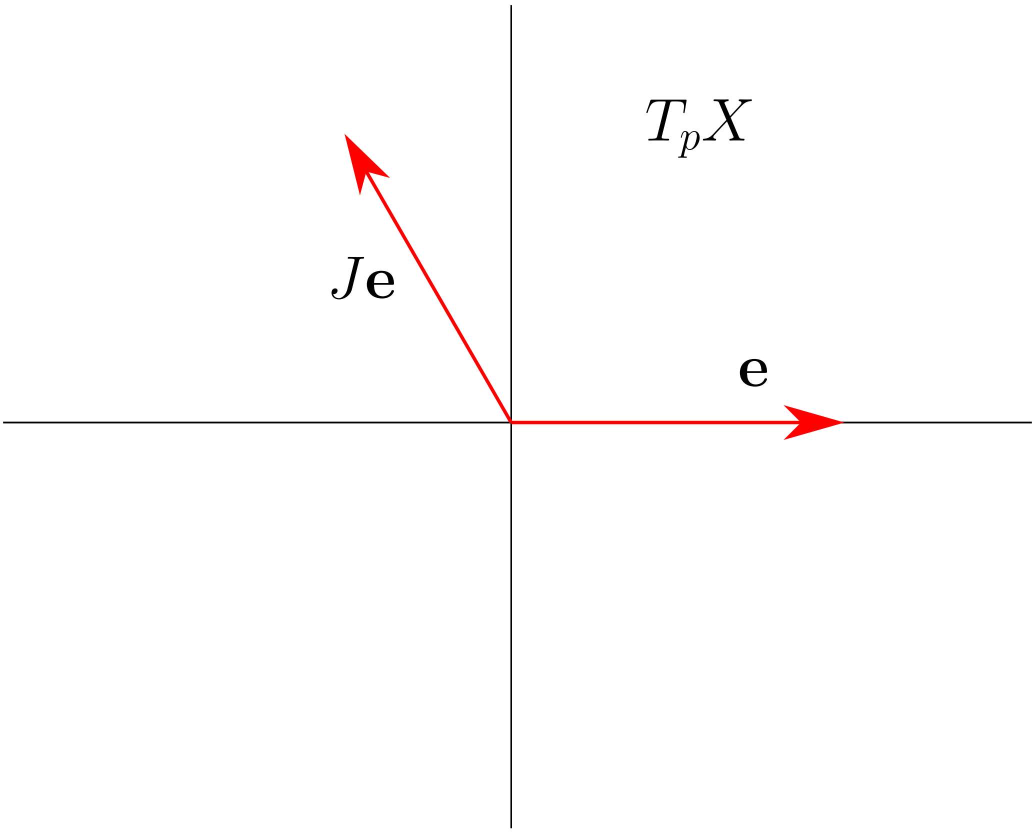
Remark 3.1.1: Last time we showed $\mathbb{R}^1$ had a unique smooth structure, so now we'll do this for $\mathbb{R}^2$. The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)
A **Riemannian metric** $g \in \mathrm{Sym}^2 T^* X$ for X a smooth manifold is a metric on every $T_p X$ given by

$$g_p : T_p X \times T_p X \to \mathbb{R}$$
$$g(v,v) \geq 0, g(v,v) = 0 \iff v = 0.$$

Definition 3.1.3 (Almost complex structure)
An **almost complex structure** is a $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{id}$.

Remark 3.1.4: Let $e \in T_p X$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of $T_p X$.



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

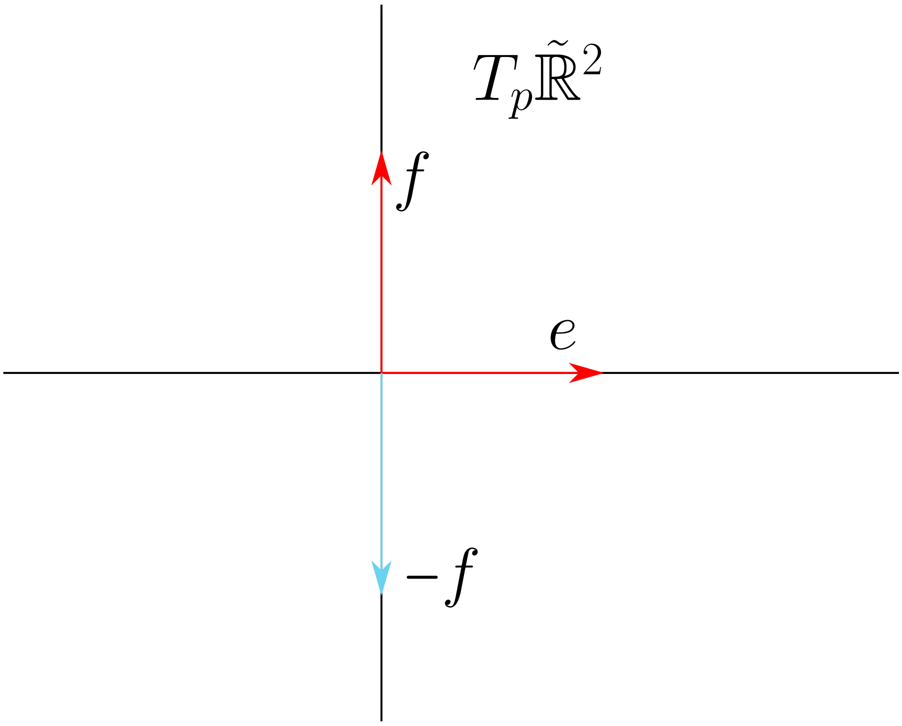
Let $\tilde{\mathbb{R}}^2$ be an exotic $\mathbb{R}^2$.

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g \coloneqq \sum f_I g_i$ with g_i metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e \coloneqq f \in T_p\tilde{\mathbb{R}}^2$ such that

- $g(e,e) = g(f,f)$
- $g(e,f) = 0$.
- $\{e,f\}$ is an oriented basis of $T_p\tilde{\mathbb{R}}^2$

This is because after choosing e, there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5 *(?)*.
Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \to \mathbb{C}$ such that J is multiplication by i.

So $d\varphi(J\cdot e) = i\cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to $\mathbb{D}$ or $\mathbb{C}$, both of which are diffeomorphic to $\mathbb{R}^2$.

See the Newlander-Nirenberg theorem, a result in complex geometry.

4 Lecture 3 (Wednesday, January 20)

Today: some background material on sheaves, bundles, connections.

4.1 Sheaves

Definition 4.1.1 (Presheaves and Sheaves)
Recall that if X is a topological space, a **presheaf** of abelian groups $\mathcal{F}$ is an assignment $U \to \mathcal{F}(U)$ of an abelian group to every open set $U \subseteq X$ together with a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any inclusion $V \subseteq U$ of open sets. This data has to satisfying certain conditions:

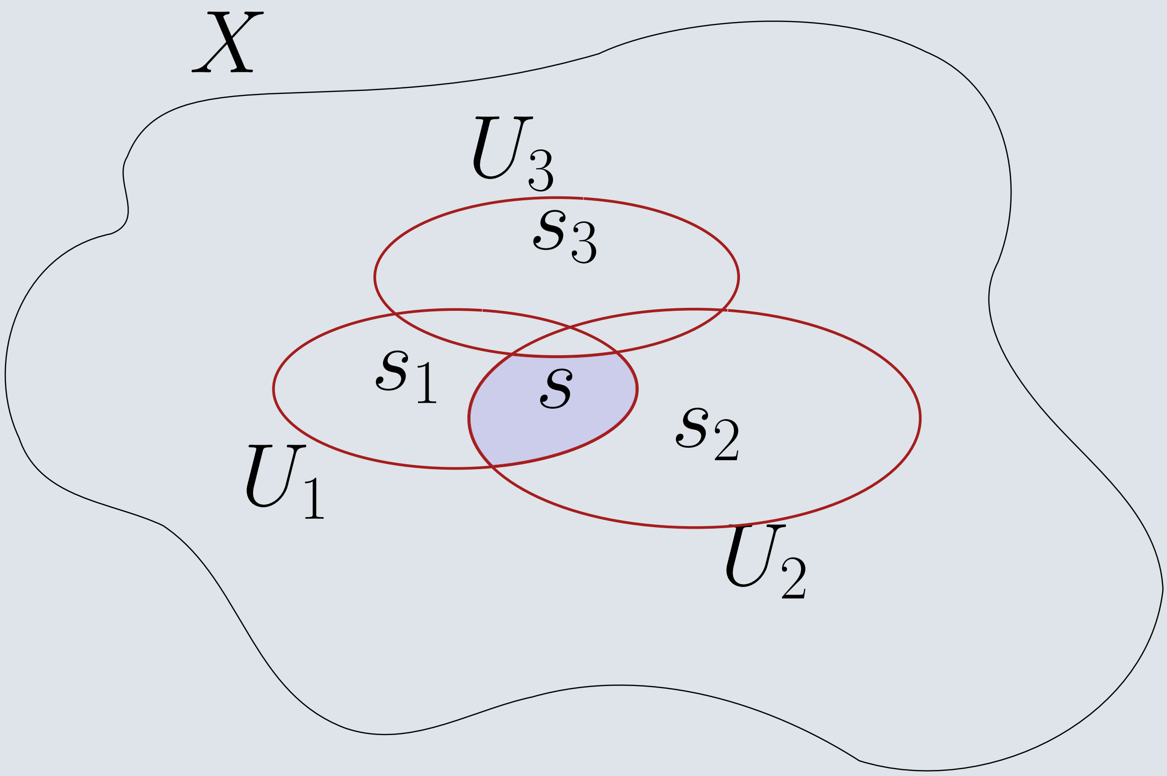
a. $\mathcal{F}(\varnothing) = 0$, the trivial abelian group.

b. $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U) = \mathrm{id}_{\mathcal{F}(U)}$

c. Compatibility if restriction is taken in steps: $U \subseteq V \subseteq W \implies \rho_{VW} \circ \rho_{UV} = \rho_{UW}$.

We say $\mathcal{F}$ is a **sheaf** if additionally:

d. Given $s_i \in \mathcal{F}(U_i)$ such that $\rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)$ implies that there exists a unique $s \in \mathcal{F}(\bigcup_i U_i)$ such that $\rho_{U_i}(s) = s_i$.



Example 4.1.2 *(?):* Let X be a topological manifold, then $\mathcal{F} \coloneqq C^0(\cdot, \mathbb{R})$ the set of continuous functionals form a sheaf. We have a diagram

$$\begin{tikzcd}
U \arrow[r, "\mathcal{F}"] & C^0(U;\mathbb{R}) \arrow[d, hook, dashed, "\text{restrict cts. functions}"] \\
V \arrow[u, hook] \arrow[r, "\mathcal{F}"'] & C^0(V;\mathbb{R})
\end{tikzcd}$$

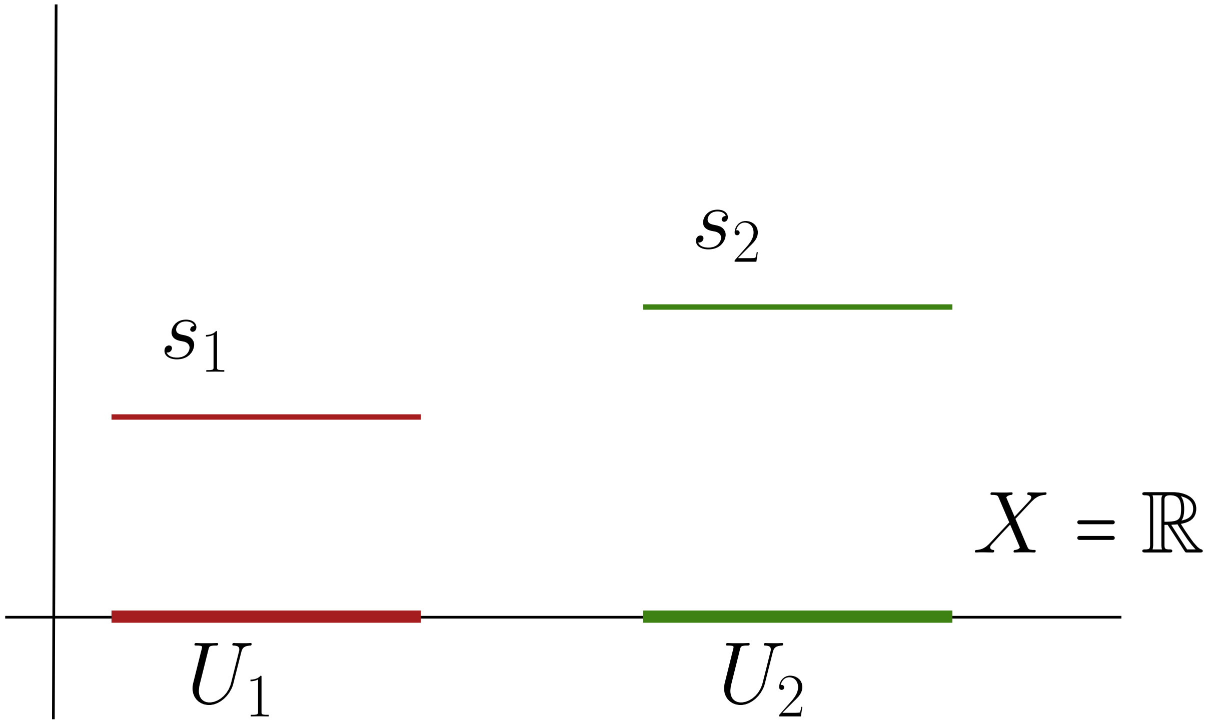
Link to diagram

Property (d) holds because given sections $s_i \in C^0(U_i;\mathbb{R})$ agreeing on overlaps, so $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, there exists a unique $s \in C^0(\bigcup_i U_i;\mathbb{R})$ such that $s|_{U_i} = s_i$ for all i – continuous functions glue.

Remark 4.1.3: Recall that we discussed various structures on manifolds: PL, continuous, smooth, complex-analytic, etc. We can characterize these by their sheaves of functions, which we'll denote $\mathcal{O}$. For example, $\mathcal{O} \coloneqq C^0(\,\cdot\,;\mathbb{R})$ for topological manifolds, and $\mathcal{O} \coloneqq C^\infty(\,\cdot\,;\mathbb{R})$ is the sheaf for smooth manifolds. Note that this also works for PL functions, since pullbacks of PL functions are again PL. For complex manifolds, we set $\mathcal{O}$ to be the sheaf of holomorphic functions.

Example 4.1.4 ***(Locally Constant Sheaves):*** Let $A \in \mathsf{Ab}$ be an abelian group, then $\underline{A}$ is the sheaf defined by setting $\underline{A}(U)$ to be the locally constant functions $U \to A$. E.g. let $X \in \mathsf{Mfd}_{\mathsf{Top}}$ be a topological manifold, then $\underline{\mathbb{R}}(U) = \mathbb{R}$ if U is connected since locally constant $\implies$ globally constant in this case.

⚠ **Warning 4.1.5**
Note that the presheaf of constant functions doesn't satisfy (d)! Take $\mathbb{R}$ and a function with two different values on disjoint intervals:



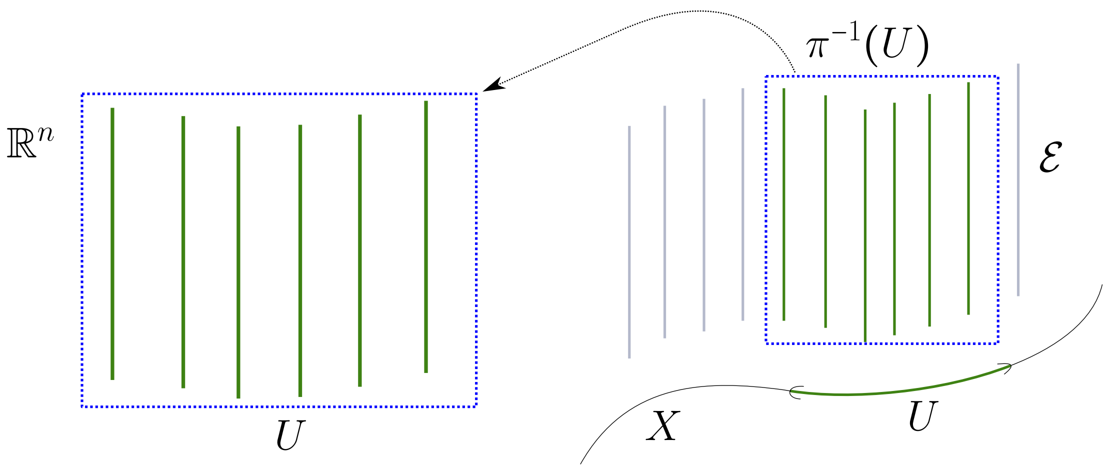
Note that $s_1|_{U_1\cap U_2} = s_2|_{U_1\cap U_2}$ since the intersection is empty, but there is no constant function that restricts to the two different values.

4.2 Bundles

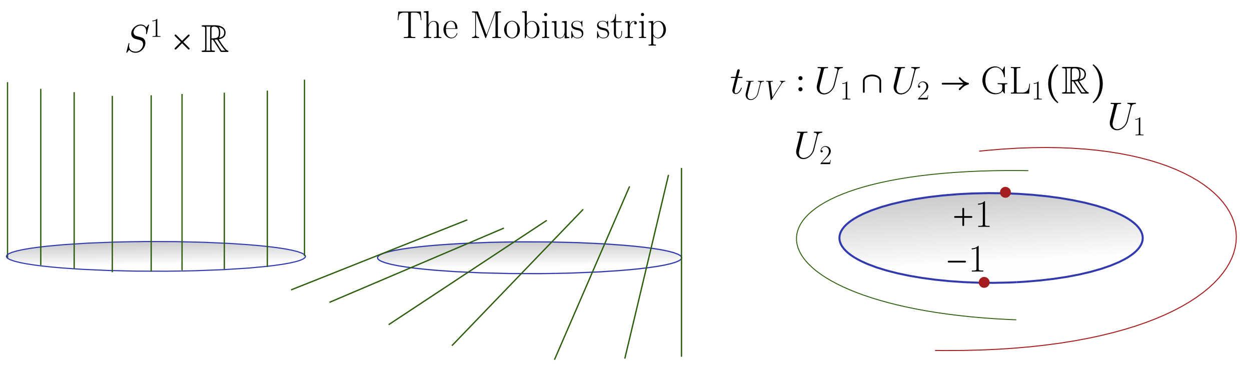
Remark 4.2.1: Let $\pi : \mathcal{E} \to X$ be a **vector bundle**, so we have local trivializations $\pi^{-1}(U) \xrightarrow{h_u} Y^d \times U$ where we take either $Y = \mathbb{R}, \mathbb{C}$, such that $h_v \circ h_u^{-1}$ preserves the fibers of π and acts linearly on each fiber of $Y \times (U \cap V)$. Define

$$t_{UV} : U \cap V \to \mathrm{GL}_d(Y)$$

where we require that t_{UV} is continuous, smooth, complex-analytic, etc depending on the context.



Example 4.2.2 ***(Bundles over*** S^1***):*** There are two $\mathbb{R}^1$ bundles over S^1:



Note that the Mobius bundle is not trivial, but can be locally trivialized.

Remark 4.2.3: We abuse notation: $\mathcal{E}$ is also a sheaf, and we write $\mathcal{E}(U)$ to be the set of sections $s : U \to \mathcal{E}$ where s is continuous, smooth, holomorphic, etc where $\pi \circ s = \mathrm{id}_U$. I.e. a bundle is a sheaf in the sense that its sections *form* a sheaf.

Example 4.2.4 ***(?):*** The trivial line bundle gives the sheaf $\mathcal{O}$: maps $U \xrightarrow{s} U \times Y$ for $Y = \mathbb{R}, \mathbb{C}$ such that $\pi \circ s = \text{id}$ are the same as maps $U \to Y$.

Definition 4.2.5 ($\mathcal{O}$-modules)
An **$\mathcal{O}$-module** is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ has an action of $\mathcal{O}(U)$ compatible with restriction.

Example 4.2.6 ***(?):*** If $\mathcal{E}$ is a vector bundle, then $\mathcal{E}(U)$ has a natural action of $\mathcal{O}(U)$ given by $f \curvearrowright s \coloneqq fs$, i.e. just multiplying functions.

Example 4.2.7 ***(Non-example):*** The locally constant sheaf $\underline{\mathbb{R}}$ is not an $\mathcal{O}$-module: there isn't natural action since the sections of $\mathcal{O}$ are generally non-constant functions, and multiplying a constant function by a non-constant function doesn't generally give back a constant function.

We'd like a notion of maps between sheaves:

Definition 4.2.8 (Morphisms of Sheaves)
A **morphism** of sheaves $\mathcal{F} \to \mathcal{G}$ is a group morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all opens $U \subseteq X$ such that the diagram involving restrictions commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{UV}"] & \mathcal{G}(U) \arrow[d, "\rho_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{F}(V) \end{tikzcd}$$

Example 4.2.9 ***(An $\mathcal{O}$-module that is not a vector bundle.):*** Let $X = \mathbb{R}$ and define the **skyscraper sheaf** at $p \in \mathbb{R}$ as

$$\mathbb{R}_p(U) \coloneqq \begin{cases} \mathbb{R} & p \in U \\ 0 & p \notin U. \end{cases} .$$

The $\mathcal{O}(U)$-module structure is given by

$$\begin{aligned} \mathcal{O}(U) \times \mathcal{O}(U) &\to \mathbb{R}_p(U) \\ (f, s) &\mapsto f(p)s. \end{aligned}$$

This is not a vector bundle since $\mathbb{R}_p(U)$ is not an infinite dimensional vector space, whereas the space of sections of a vector bundle is generally infinite dimensional (?). Alternatively, there are arbitrarily small punctured open neighborhoods of p for which the sheaf makes trivial assignments.

Example 4.2.10 ***(of morphisms):*** Let $X = \mathbb{R} \in \text{Mfd}_{\text{Sm}}$ viewed as a smooth manifold, then multiplication by x induces a morphism of structure sheaves:

$$\begin{aligned} (x\cdot) : \mathcal{O} &\to \mathcal{O} \\ s &\mapsto x \cdot s \end{aligned}$$

for any $x \in \mathcal{O}(U)$, noting that $x \cdot s \in \mathcal{O}(U)$ again.

Exercise 4.2.11 *(?)*
Check that $\ker\varphi$ is naturally a sheaf and $\ker(\varphi)(U) = \ker(\varphi(U)) : \mathcal{F}(U) \to \mathcal{G}(U)$

Here the kernel is trivial, i.e. on any open U we have $(x\cdot) : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)$ is injective. Taking the cokernel $\operatorname{coker}(x\cdot)$ as a presheaf, this assigns to U the quotient presheaf $\mathcal{O}(U)/x\mathcal{O}(U)$, which turns out to be equal to $\mathbb{R}_0$. So $\mathcal{O} \to \mathbb{R}_0$ by restricting to the value at 0, and there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{(x\cdot)} \mathcal{O} \to \mathbb{R}_0 \to 0.$$

This is one reason sheaves are better than vector bundles: the category is closed under taking quotients, whereas quotients of vector bundles may not be vector bundles.

5 Lecture 4 (Friday, January 22)

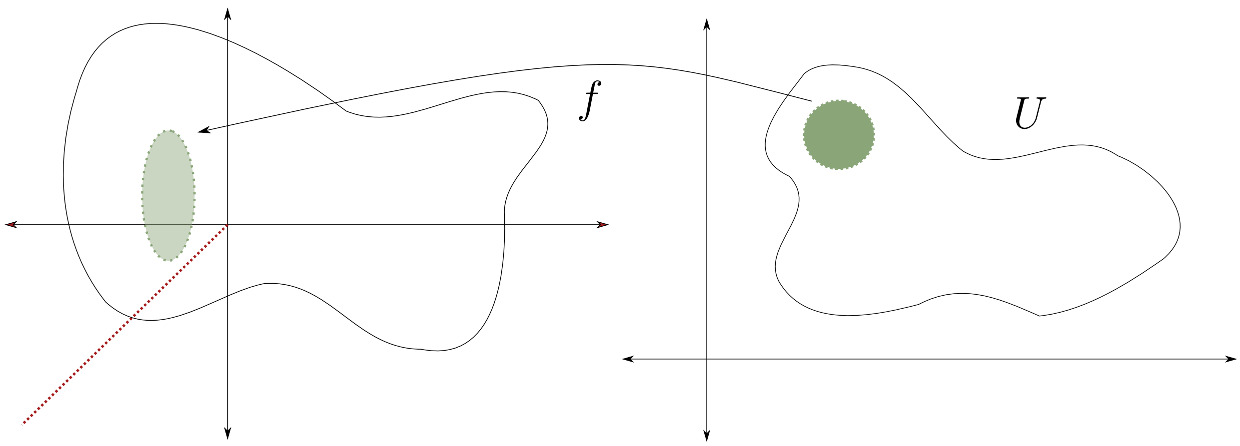
5.1 The Exponential Exact Sequence

Let $X = \mathbb{C}$ and consider $\mathcal{O}$ the sheaf of holomorphic functions and $\mathcal{O}^\times$ the sheaf of *nonvanishing* holomorphic functions. The former is a vector bundle and the latter is a sheaf of abelian groups. There is a map $\exp : \mathcal{O} \to \mathcal{O}^\times$, the **exponential map**, which is the data $\exp(U) : \mathcal{O}(U) \to \mathcal{O}^\times(U)$ on every open U given by $f \mapsto e^f$. There is a kernel sheaf $2\pi i\underline{\mathbb{Z}}$, and we get an exact sequence

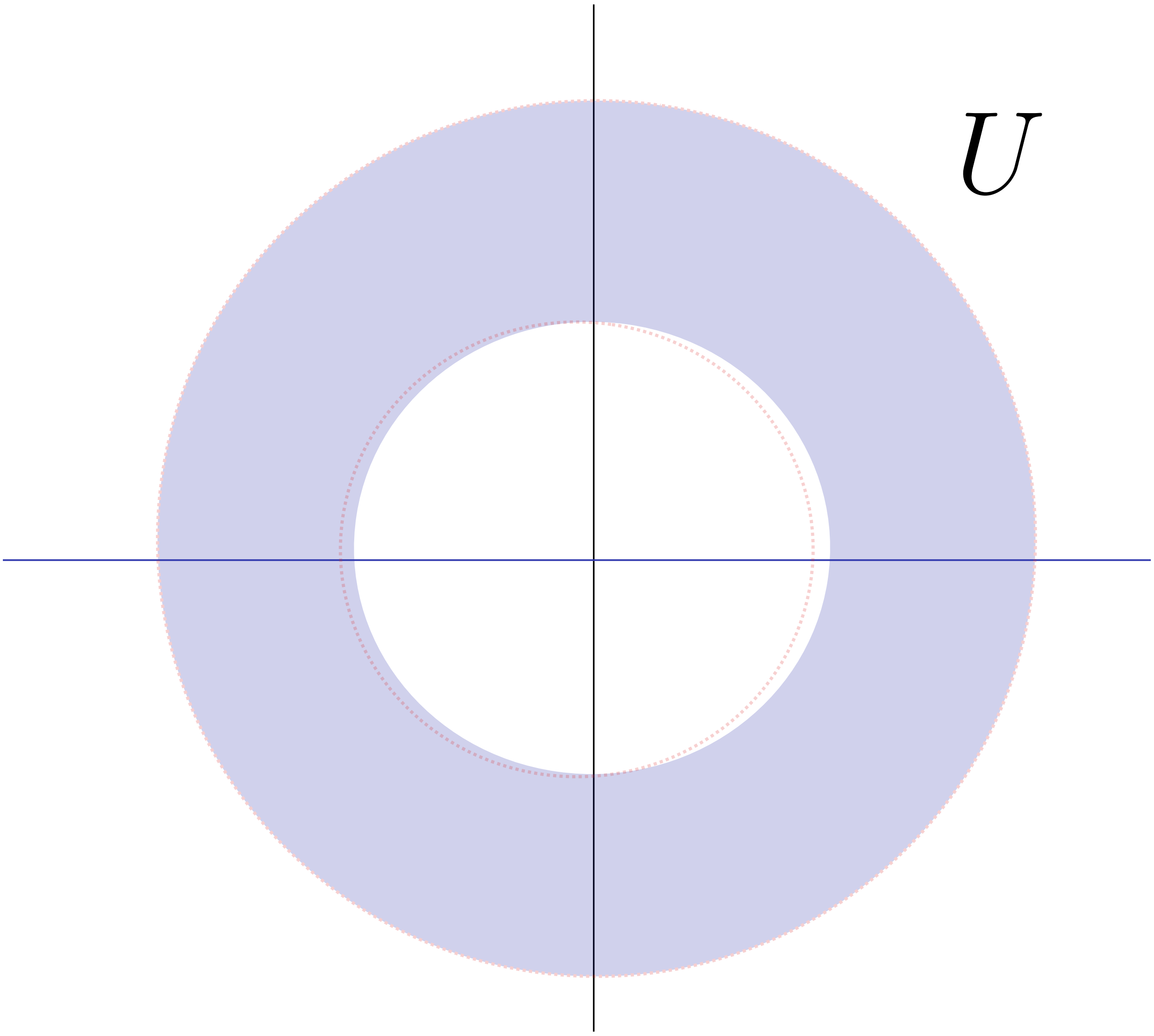
$$0 \to 2\pi i\underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to \operatorname{coker}(\exp) \to 0.$$

Question 5.1.1
What is the cokernel sheaf here?

Let U be a contractible open set, then we can identify $\mathcal{O}^\times(U)/\exp(\mathcal{O}^\times(U)) = 1$.



Any $f \in \mathcal{O}^\times(U)$ has a logarithm, say by taking a branch cut, since $\pi_1(U) = 0 \implies \log f$ has an analytic continuation. Consider the annulus U and the function $z \in \mathcal{O}^\times(U)$, then $z \notin \exp(\mathcal{O}(U))$ – if $z = e^f$ then $f = \log(z)$, but $\log(z)$ has monodromy on U:



Thus on any sufficiently small open set, $\operatorname{coker}(\exp) = 1$. This is only a presheaf: there exists an open cover of the annulus for which $z|_{U_i}$, and so the naive cokernel doesn't define a sheaf. This is because we have a locally trivial section which glues to z, which is nontrivial.

Exercise 5.1.2 (?)
Redefine the cokernel so that it is a sheaf. Hint: look at sheafification, which has the defining property $\mathrm{Hom}_{\mathrm{Presh}}(\mathcal{G}, \mathcal{F}^{\mathrm{Presh}}) = \mathrm{Hom}_{\mathrm{Sh}}(\mathcal{G}, \mathcal{F}^{\mathrm{Sh}})$ for any sheaf $\mathcal{G}$.

Definition 5.1.3 (Global Sections Sheaf)
The **global sections** sheaf of $\mathcal{F}$ on X is given by $H^0(X; \mathcal{F}) = \mathcal{F}(X)$.

Example 5.1.4 ***(?):***

- $C^\infty(X) = H^0(X, C^\infty)$ are the smooth functions on X
- $VF(X) = H^0(X; T)$ are the smooth vector fields on X for T the tangent bundle
- If X is a complex manifold then $\mathcal{O}(X) = H^0(X; \mathcal{O})$ are the globally holomorphic functions on X.
- $H^0(X; \mathbb{Z}) = \underline{\mathbb{Z}}(X)$ are ??

Remark 5.1.5: Given vector bundles V, W, we have constructions $V \oplus W, V \otimes W, V^\vee, \mathrm{Hom}(V, W) = V^\vee \otimes W, \mathrm{Sym}^n V, \Lambda^p V$, and so on. Some of these work directly for sheaves:

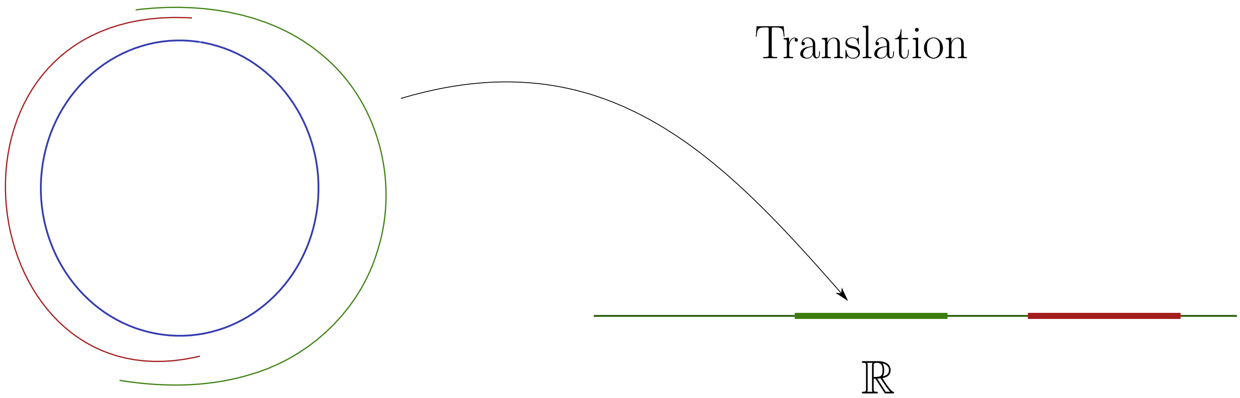
- $\mathcal{F} \oplus \mathcal{G}(U) \coloneqq \mathcal{F}(U) \oplus \mathcal{G}(U)$
- For tensors, duals, and homs $\mathscr{H}\mathrm{om}(V, W)$ we only get presheaves, so we need to sheafify.

⚠ **Warning 5.1.6**

$\mathrm{Hom}(V, W)$ will denote the *global* homomorphisms $\mathscr{H}\mathrm{om}(V, W)(X)$, which is a sheaf.

Example 5.1.7 ***(?):*** Let $X^n \in \mathrm{Mfd}_{\mathrm{sm}}$ and let Ω^p be the sheaf of smooth p-forms, i.e $\Lambda^p T^\vee$, i.e. $\Omega^p(U)$ are the smooth p forms on U, which are locally of the form $\sum f_{i_1, \cdots, i_p}(x_1, \cdots, x_n) dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_p}$ where the $f_{i_1, \cdots, i_p}$ are smooth functions.

Example 5.1.8 ***(Sub-example):*** Take $X = S^1$, writing this as $\mathbb{R}/\mathbb{Z}$, we have $\Omega^1(X) \ni dx$. There are two coordinate charts which differ by a translation on their overlaps, and $dx(x + c) = dx$ for c a constant:



Exercise 5.1.9 ***(?)***

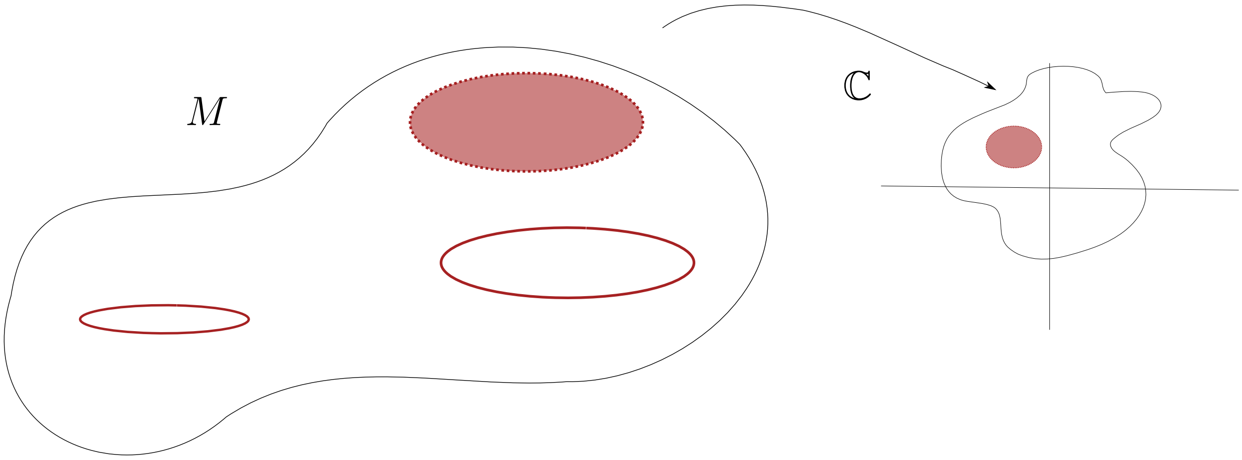
Check that on a torus, dx_i is a well-defined 1-form.

Remark 5.1.10: Note that there is a map $d : \Omega^p \to \Omega^{p+1}$ where $\omega \mapsto d\omega$.

⚠ **Warning 5.1.11**

d is **not** a map of $\mathcal{O}$-modules: $d(f \cdot \omega) = f \cdot \omega + df \wedge \omega$, where the latter is a correction term. In particular, it is not a map of vector bundles, but is a map of sheaves of abelian groups since $d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$, making d a sheaf morphism.

Let $X \in \mathrm{Mfd}_{\mathbb{C}}$, we'll use the fact that TX is complex-linear and thus a $\mathbb{C}$-vector bundle.



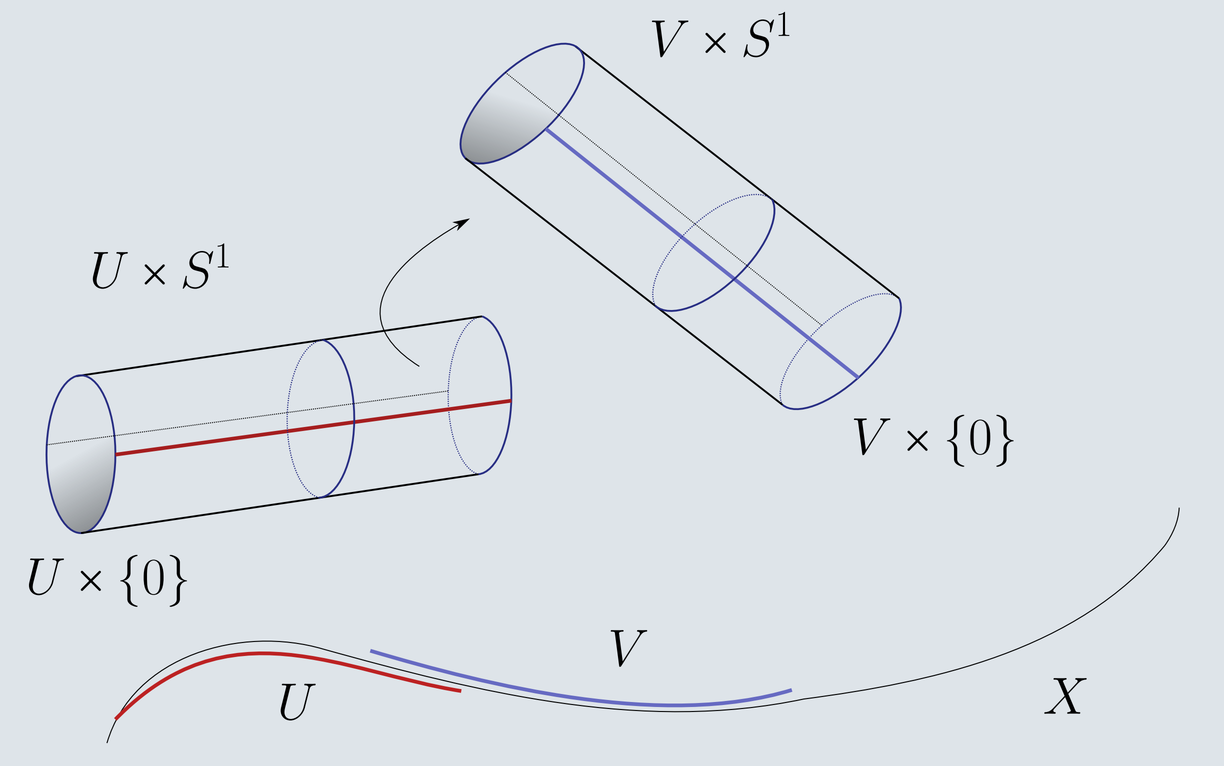
Remark 5.1.12 ***(Subtlety 1):*** Note that Ω^p for complex manifolds is $\Lambda^p T^\vee$, and so if we want to view $X \in \mathrm{Mfd}_{\mathbb{R}}$ we'll write $X_{\mathbb{R}}$. $TX_{\mathbb{R}}$ is then a real vector bundle of rank $2n$.

Remark 5.1.13 ***(Subtlety 2):*** Ω^p will denote *holomorphic* p-forms, i.e. local expressions $\sum f_I(z_1, \cdots, z_n) \Lambda dz_I$. For example, $e^z dz \in \Omega^1(\mathbb{C})$ but $z\bar{z}dz$ is not, where $dz = dx + idy$. We'll use a different notation when we allow the f_I to just be smooth: $A^{p,0}$, the sheaf of $(p,0)$-forms. Then $z\bar{z}dz \in A^{1,0}$.

Remark 5.1.14: Note that $T^\vee X_{\mathbb{R} \otimes \mathbb{C}} = A^{1,0} \oplus A^{0,1}$ since there is a unique decomposition $\omega = fdz + gd\bar{z}$ where f, g are smooth. Then $\Omega^d X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=d} A^{p,q}$. Note that $\Omega^p_{\nwarrow} \neq A^{p,q}$ and these are really quite different: the former are more like holomorphic bundles, and the latter smooth. Moreover $\dim \Omega^p(X) < \infty$, whereas $\Omega^1_{\nwarrow}$ is infinite-dimensional.

6 | Principal G-Bundles and Connections (Monday, January 25)

Definition 6.0.1 (Principal Bundles)
Let G be a (possibly disconnected) Lie group. Then a **principal G-bundle** $\pi : P \to X$ is a space admitting local trivializations $h_u : \pi^{-1}(U) \to G \times U$ such that the transition functions are given by left multiplication by a continuous function $t_{UV} : U \cap V \to G$.



Remark 6.0.2: Setup: we'll consider TX for $X \in \mathrm{Mfd}_{\searrow}$, and let g be a metric on the tangent bundle given by

$$g_p : T_p X^{\otimes 2} \to \mathbb{R},$$

a symmetric bilinear form with $g_p(u, v) \geq 0$ with equality if and only if $v = 0$.

Definition 6.0.3 (The Frame Bundle)
Define $\mathrm{Frame}_p(X) \coloneqq \{\text{bases of } T_p X\}$, and $\mathrm{Frame} X \coloneqq \bigcup_{p \in X} \mathrm{Frame}_p X$.

Remark 6.0.4: More generally, $\mathrm{Frame}\mathcal{E}$ can be defined for any vector bundle $\mathcal{E}$, so $\mathrm{Frame} X \coloneqq \mathrm{Frame} TX$. Note that $\mathrm{Frame} X$ is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle where $n \coloneqq \mathrm{rank}(\mathcal{E})$. This follows from the fact that the transition functions are fiberwise in $\mathrm{GL}_n(\mathbb{R})$, so the transition functions are given by left-multiplication by matrices.

Remark 6.0.5 ***(Important):*** A principal G-bundle admits a G-action where G acts by *right* multiplication:

$$\begin{aligned} P \times G &\to P \\ ((g, x), h) &\mapsto (gh, x). \end{aligned}$$

This is necessary for compatibility on overlaps. **Key point**: the actions of left and right multiplication commute.

Definition 6.0.6 (Orthogonal Frame Bundle)
The **orthogonal frame bundle** of a vector bundle $\mathcal{E}$ equipped with a metric g is defined as $\mathrm{OFrame}_p\mathcal{E} \coloneqq \{\text{orthonormal bases of } \mathcal{E}_p\}$, also written $O_r(\mathbb{R})$ where $r \coloneqq \operatorname{rank}(\mathcal{E})$.

Remark 6.0.7: The fibers $P_x \to \{x\}$ of a principal G-bundle are naturally **torsors** over G, i.e. a set with a free transitive G-action.

Definition 6.0.8 (?)
Let $\mathcal{E} \to X$ be a complex vector bundle. Then a **hermitian metric** is a hermitian form on every fiber, i.e.

$$h_p : \mathcal{E}_p \times \overline{\mathcal{E}_p} \to \mathbb{C}.$$

where $h_p(v, \bar{v}) \geq 0$ with equality if and only if $v = 0$. Here we define $\overline{\mathcal{E}_p}$ as the fiber of the complex vector bundle $\overline{\mathcal{E}}$ whose transition functions are given by the complex conjugates of those from $\mathcal{E}$.

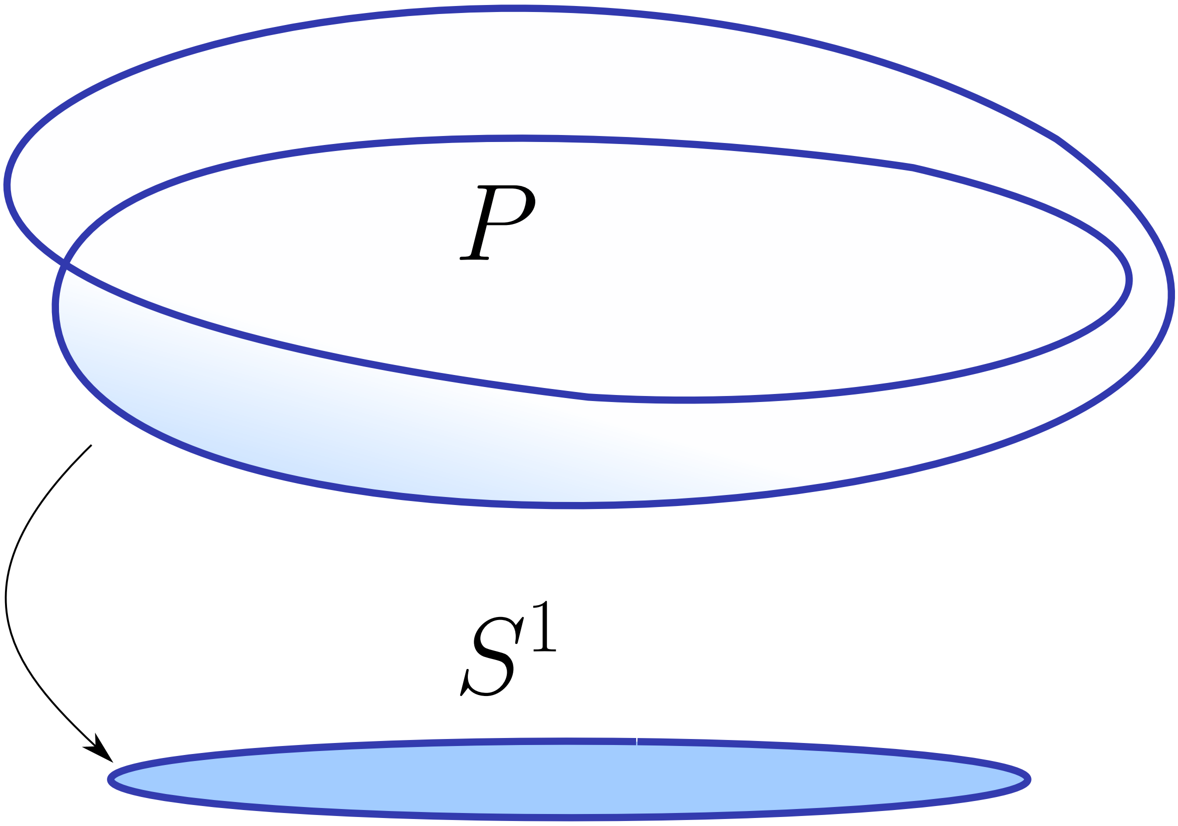
Remark 6.0.9: Note that $\mathcal{E}, \overline{\mathcal{E}}$ are genuinely different as complex bundles. There is a *conjugate-linear* map given by conjugation, i.e. $L(cv) = \bar{c}L(v)$, where the canonical example is

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{C}^n \\ (z_1, \cdots, z_n) &\mapsto (\overline{z_1}, \cdots, \overline{z_n}). \end{aligned}$$

Definition 6.0.10 (Unitary Frame Bundle)
We define the **unitary frame bundle** $\mathrm{UFrame}(\mathcal{E}) \coloneqq \bigcup_p \mathrm{UFrame}(\mathcal{E})_p$, where at each point this is given by the set of orthogonal frames of $\mathcal{E}_p$ given by $(e_1, \cdots, e_n)$ where $h(e_i, \overline{e_j}) = \delta_{ij}$.

Remark 6.0.11: This is a principal G-bundle for $G = U_r(\mathbb{C})$, the invertible matrices $A_{/\mathbb{C}}$ satisfy $A\overline{A}^t = \mathrm{id}$.

Example 6.0.12 ***(of more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$ and $X = S^1$, the Möbius band is a principal G-bundle:



Example 6.0.13 ***(more principal bundles):*** For $G = \mathbb{Z}/2\mathbb{Z}$, for any (possibly non-oriented) manifold X there is an **orientation principal bundle** P which is locally a set of orientations on U, i.e.

$$P \coloneqq \left\{ (x, O) \;\middle|\; x \in X, \, O \text{ is an orientation of } T_p X \right\}.$$

Note that P is an oriented manifold, $P \to X$ is a local isomorphism, and has a canonical orientation. (?) This can also be written as $P = \mathrm{Frame} X / \mathrm{GL}_n^+(\mathbb{R})$, since an orientation can be specified by a choice of n linearly independent vectors where we identify any two sets that differ by a matrix of positive determinant.

Definition 6.0.14 (Associated Bundles)
Let $P \to X$ be a principal G-bundle and let $G \to \mathrm{GL}(V)$ be a continuous representation. The **associated bundle** is defined as

$$P \times_G V = \left\{ (p, v) \;\middle|\; p \in P, \, v \in V \right\} / \sim \qquad\qquad \text{where } (p, v) \sim (pg, g^{-1} v),$$

which is well-defined since there is a right action on the first component and a left action on the second.

Example 6.0.15 ***(?):*** Note that $\mathrm{Frame}(\mathcal{E})$ is a $\mathrm{GL}_r(\mathbb{R})$-bundle and the map $\mathrm{GL}_r(\mathbb{R}) \xrightarrow{\mathrm{id}} \mathrm{GL}(\mathbb{R}^r)$ is a representation. At every fiber, we have $G \times_G V = (p, v)/\sim$ where there is a unique representative of this equivalence class given by (e, pv). So $P \times_G V_p \to \{p\} \cong V_x$.

Exercise 6.0.16 *(?)*
Show that $\mathrm{Frame}(\mathcal{E}) \times_{\mathrm{GL}_r(\mathbb{R})} \mathbb{R}^r \cong \mathcal{E}$. This follows from the fact that the transition functions of $P \times_G V$ are given by left multiplication of $t_{UV} : U \cap V \to G$, and so by the equivalence relation, $\operatorname{im} t_{UV} \in \mathrm{GL}(V)$.

Remark 6.0.17: Suppose that M^3 is an oriented Riemannian 3-manifold. Them $TM \to \mathrm{Frame}(M)$ which is a principal $\mathrm{SO}(3)$-bundle. The universal cover is the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, so can the transition functions be lifted? This shows up for spin structures, and we can get a $\mathbb{C}^2$ bundle out of this.

7 | Wednesday, January 27

7.1 Bundles and Connections

Definition 7.1.1 (Connections)
Let $\mathcal{E} \to X$ be a vector bundle, then a **connection** on $\mathcal{E}$ is a map of sheaves of abelian groups

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1$$

satisfying the *Leibniz rule*:

$$\nabla(fs) = f\nabla s + s \otimes ds$$

for all opens U with $f \in \mathcal{O}(U)$ and $s \in \mathcal{E}(U)$. Note that this works in the category of complex manifolds, in which case ∇ is referred to as a **holomorphic connection**.

Remark 7.1.2: A connection ∇ induces a map

$$\begin{aligned} \tilde{\nabla} : \mathcal{E} \otimes \Omega^p &\to \mathcal{E} \otimes \Omega^{p+1} \\ s \otimes \omega &\mapsto \nabla s \wedge w + s \otimes d\omega. \end{aligned}$$

where $\wedge : \Omega^p \otimes \Omega^1 \to \Omega^{p+1}$. The standard example is

$$\begin{aligned} d : \mathcal{O} &\to \Omega^1 \\ f &\mapsto df. \end{aligned}$$

where the induced map is the usual de Rham differential.

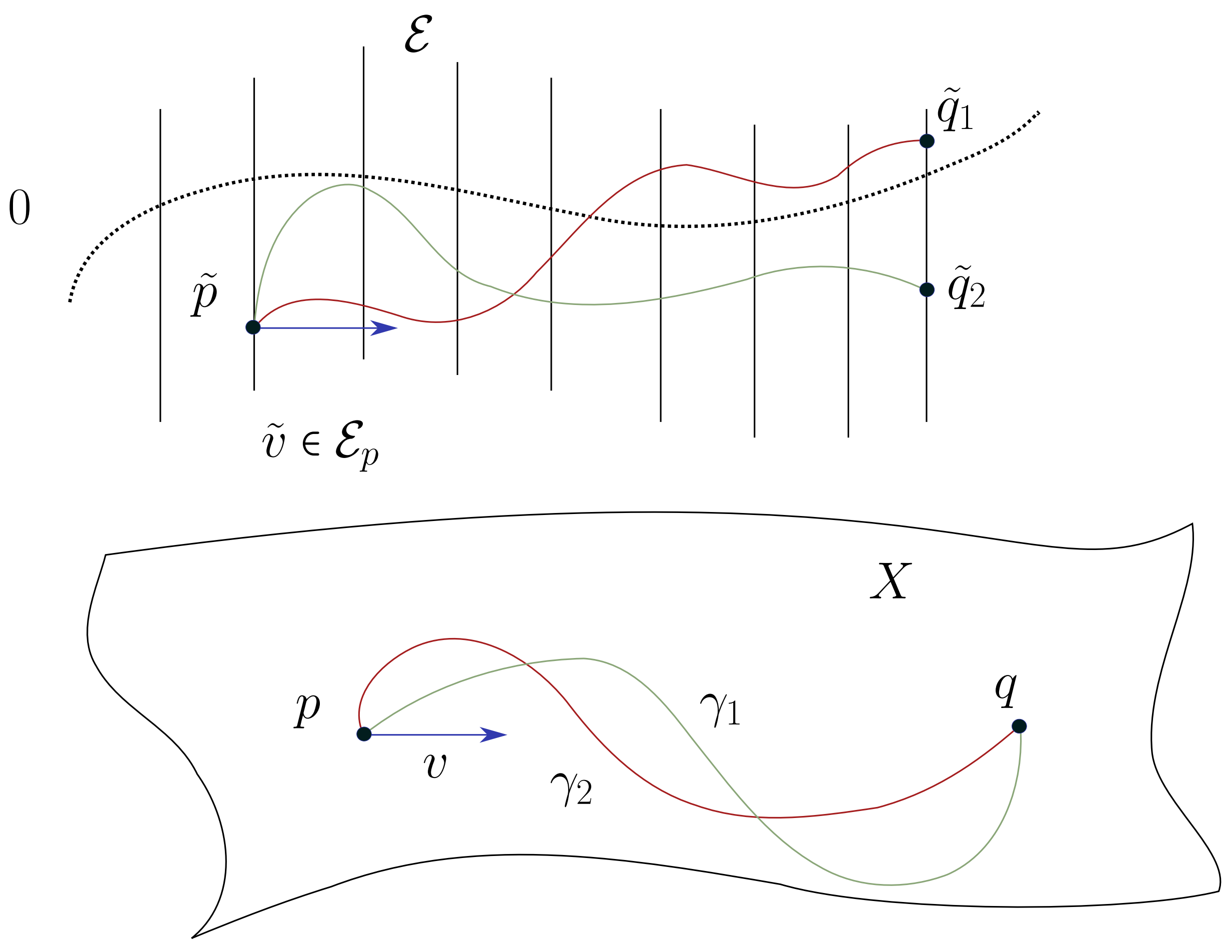
Exercise 7.1.3 (?)

Prove that the *curvature* of ∇, i.e. the map

$$F_\nabla \coloneqq \nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^2$$

is $\mathcal{O}$-linear, so $F_\nabla(fs) = f\nabla \circ \nabla(s)$. Use the fact that $\nabla s \in \mathcal{E} \otimes \Omega^1$ and $\omega \in \Omega^p$ and so $\nabla s \otimes \omega \in \mathcal{E}\Omega^1 \otimes \Omega^p$ and thus reassociating the tensor product yields $\nabla s \wedge \omega \in \mathcal{E} \otimes \Omega^{p+1}$.

Remark 7.1.4: Why is this called a connection?

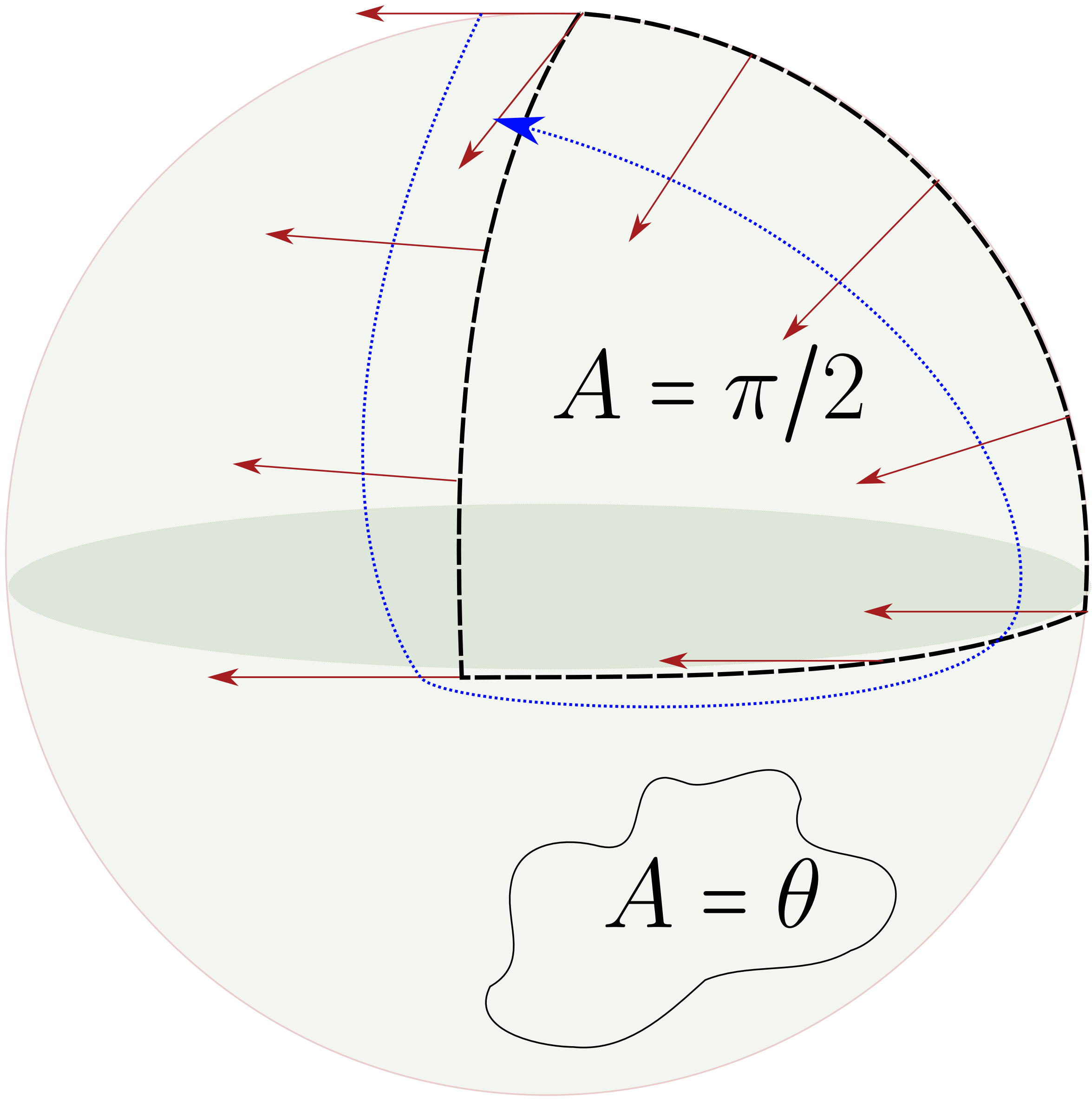


This gives us a way to transport $v \in \mathcal{E}_p$ over a path γ in the base, and ∇ provides a differential equation (a flow equation) to solve that lifts this path. Solving this is referred to as **parallel transport**. This works by pairing $\gamma'(t) \in T_{\gamma(t)}X$ with Ω^1, yielding $\nabla s = (\gamma'(t)) = s(\gamma(t))$ which are sections of γ.

Note that taking a different path yields an endpoint in the same fiber but potentially at a different point, and $F_\nabla = 0$ if and only if the parallel transport from p to q depends only on the homotopy class of γ.

Note: this works for any bundle, so can become confusing in Riemannian geometry when all of the bundles taken are tangent bundles!

Example 7.1.5 ***(A classic example):*** The Levi-Cevita connection ∇^{LC} on TX, which depends on a metric g. Taking $X = S^2$ and g is the round metric, there is nonzero curvature:



In general, every such transport will be rotation by some vector, and the angle is given by the area of the enclosed region.

Definition 7.1.6 (Flat Connection and Flat Sections)
A connection is **flat** if $F_\nabla = 0$. A section $s \in \mathcal{E}(U)$ is **flat** if it is given by

$$L(U) \coloneqq \left\{ s \in \mathcal{E}(U) \;\middle|\; \nabla s = 0 \right\}.$$

Exercise 7.1.7 (?)
Show that if ∇ is flat then L is a *local system*: a sheaf that assigns to any sufficiently small open set a vector space of fixed dimension. An example is the constant sheaf $\underline{\mathbb{C}^d}$. Furthermore $\operatorname{rank}(L) = \operatorname{rank}(\mathcal{E})$.

Remark 7.1.8: Given a local system, we can construct a vector bundle whose transition functions are the same as those of the local system, e.g. for vector bundles this is a fixed matrix, and in general these will be constant transition functions. Equivalently, we can take $L \otimes_{\mathbb{R}} \mathcal{O}$, and $L \otimes 1$ form flat sections of a connection.

7.2 Sheaf Cohomology

Definition 7.2.1 (?)
Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and let $\mathfrak{U} \coloneqq \{U_i\} \rightrightarrows X$ be an open cover of X. Let $U_{i_1, \cdots, i_p} \coloneqq U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}$. Then the **Čech Complex** is defined as

$$C^p_{\mathfrak{U}}(X, \mathcal{F}) \coloneqq \prod_{i_1 < \cdots < i_p} \mathcal{F}(U_{i_1, \cdots, i_p})$$

with a differential

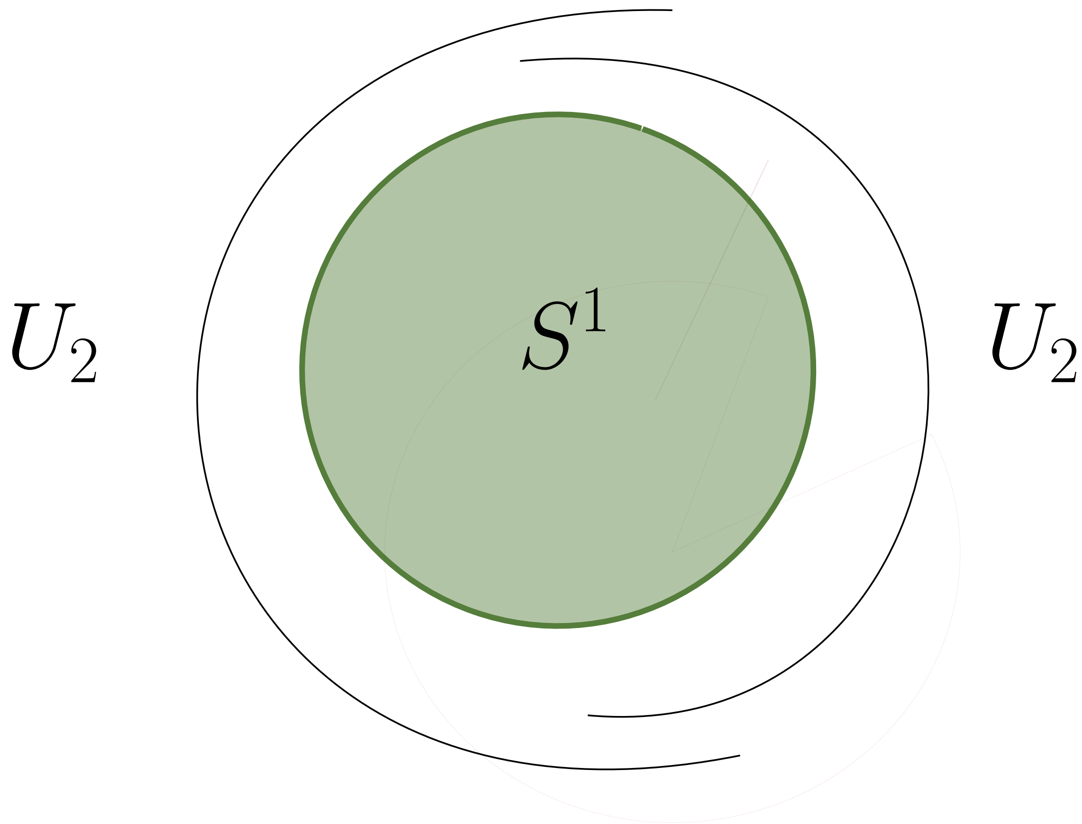
$$\begin{aligned} \partial^p : C^p_{\mathfrak{U}}(X, \mathcal{F}) &\to C^{p+1}_{\mathfrak{U}}(X\mathcal{F}) \\ \sigma &\mapsto (\partial\sigma)_{i_0, \cdots, i_p} \coloneqq \prod_j (-1)^j \, \sigma_{i_0, \cdots, \widehat{i_j}, \cdots, i_p}\Big|_{U_{i_0, \cdots, i_p}} \end{aligned}$$

where we've defined this just on one given term in the product, i.e. a p-fold intersection.

Exercise 7.2.2 (?)
Check that $\partial^2 = 0$.

Remark 7.2.3: The Čech cohomology $H^p_{\mathfrak{U}}(X, \mathcal{F})$ with respect to the cover $\mathfrak{U}$ is defined as $\ker \partial^p / \operatorname{im} \partial^{p-1}$. It is a difficult theorem, but we write $H^p(X, \mathcal{F})$ for the Čech cohomology for any sufficiently refined open cover when X is assumed paracompact.

Example 7.2.4*(?)*: Consider S^1 and the constant sheaf $\underline{\mathbb{Z}}$:



Here we have

$$C^0(S^1, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U_1) \oplus \underline{\mathbb{Z}}(U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}},$$

and

$$C^1(S^1, \mathbb{Z}) = \bigoplus_{\substack{\text{double} \\ \text{intersections}}} \underline{\mathbb{Z}}(U_{ij})\underline{\mathbb{Z}}(U_{12}) = \underline{\mathbb{Z}}(U_1 \cap U_2) = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}.$$

We then get

$$\begin{aligned} C^0(S^1, \underline{\mathbb{Z}}) &\xrightarrow{\partial} C^1(S^1, \underline{\mathbb{Z}}) \\ \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z} \\ (a, b) &\mapsto (a - b, a - b), \end{aligned}$$

Which yields $H^*(S^1, \underline{\mathbb{Z}}) = [\mathbb{Z}, \mathbb{Z}, 0, \cdots]$.

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