Research Notes 1-21

Stuff popping up everywhere:

- Pushforwards
- Derived functors (a little)
- Projective/Injective resolutions

Motto for homology: kernel of what's going out mod image of what's coming in

Easy definition: A *spectral sequence* is the data $\{(E_r, d_r)\}_{r\in\mathbb{Z}}$ where each E_r is an abelian group, $d_r : E_r \to E_r$ is a homomorphism satisfying $d_r^2 = 0$, and $E_{r+1} \cong \frac{\ker d_r}{\operatorname{im} d_r}$.

Another definition: a *homological spectral sequence* is a sequence of $\mathbb{Z}$-bigraded modules $\{E^r_{p,q}\}_{r>0}$ with differentials $d_r : E^r_{p,q} \to E^r_{p-r,q+(r-1)}$ such that $E^{r+1} = H_*(E^r)$.

A *cohomological spectral sequence* is the same, except $d_r : E_r^{p,q} \to E_r^{p+r,q-(r-1)}$

The 'lines' with slope $-\frac{r-1}{r}$ form chain complexes.

Define cycles to be $Z_i := \ker d_i$, boundaries to be $B_i : \operatorname{im} d_i$.

Concrete examples for pages:

$r = 1$: Differential is $d_1 : E^1_{p,q} \to E^1_{p-1,q}$

$r = 2$: Differential is $d_2 : E^2_{p,q} \to E^2_{p-2,q+1}$

Equivalently, $d_2 : H_*(E^1_{p,q}) \to H_*(E^1_{p-1,q})$?

$r = 3$: Differential is $d_3 : E^3_{p,q} \to E^3_{p-3,q+2}$

Should be able to compute the cohomology rings of fiber bundles $E \xrightarrow{f} B$ pretty easily, using the map induced by the cup product $E_r^{i,j} \times E_r^{k,l} \to E_r^{i+k,j+l}$ and the fact that $E_2^{i,j} = H^i(B, H^j(F)) \Rightarrow H^{i+j}(E, \mathbb{Q})$. (For example, try $SO_{n-1} \to SO_n \to S^{n-1}$)

How to put a filtration in the E^1 page: ?

Any complex with a two step filtration $F_1 \subset F_0 = K$ is exactly the long exact arising from $0 \hookrightarrow F^1 \hookrightarrow F_0 \twoheadrightarrow \frac{F_1}{F_0} \twoheadrightarrow 0$.

Next simplest example: a three step filtration $F_2 \subset F_1 \subset F_0 = K$. Write down all of the short exact sequences, and relate $H^*(K)$ to $H^*(\frac{F^i}{F^{i+1}})$.

Index Reference

The E_0 page

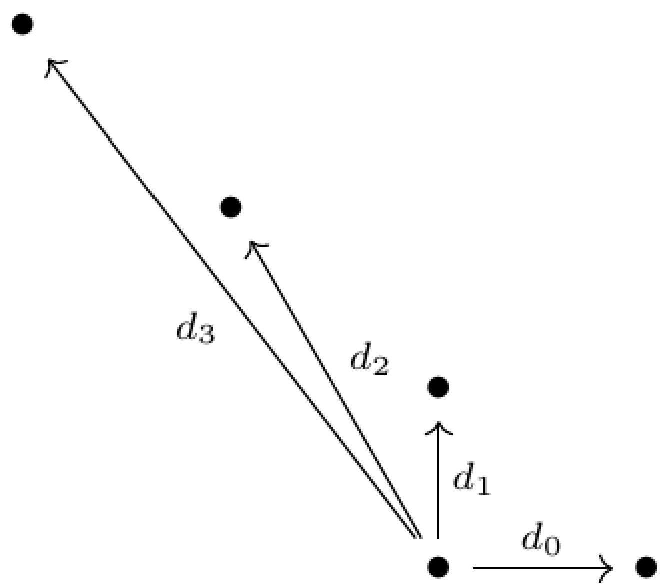
$\xrightarrow{d_0^{-2,2}} E_0^{-1,2} \xrightarrow{d_0^{-1,2}}$	$E_0^{0,2} \xrightarrow{d_0^{0,2}}$	$E_0^{1,2} \xrightarrow{d_0^{1,2}}$	$E_0^{2,2} \xrightarrow{d_0^{2,2}}$	$E_0^{3,2} \xrightarrow{d_0^{3,2}}$	$E_0^{4,2} \xrightarrow{d_0^{4,2}}$	$E_0^{5,2} \xrightarrow{d_0^{5,2}}$
$\xrightarrow{d_0^{-2,1}} E_0^{-1,1} \xrightarrow{d_0^{-1,1}}$	$E_0^{0,1} \xrightarrow{d_0^{0,1}}$	$E_0^{1,1} \xrightarrow{d_0^{1,1}}$	$E_0^{2,1} \xrightarrow{d_0^{2,1}}$	$E_0^{3,1} \xrightarrow{d_0^{3,1}}$	$E_0^{4,1} \xrightarrow{d_0^{4,1}}$	$E_0^{5,1} \xrightarrow{d_0^{5,1}}$
$\xrightarrow{d_0^{-2,0}} E_0^{-1,0} \xrightarrow{d_0^{-1,0}}$	$E_0^{0,0} \xrightarrow{d_0^{0,0}}$	$E_0^{1,0} \xrightarrow{d_0^{1,0}}$	$E_0^{2,0} \xrightarrow{d_0^{2,0}}$	$E_0^{3,0} \xrightarrow{d_0^{3,0}}$	$E_0^{4,0} \xrightarrow{d_0^{4,0}}$	$E_0^{5,0} \xrightarrow{d_0^{5,0}}$
$\xrightarrow{d_1^{-2,2}} E_1^{-1,2} \xrightarrow{d_1^{-1,2}}$	$E_1^{0,2} \xrightarrow{d_1^{0,2}}$	$E_1^{1,2} \xrightarrow{d_1^{1,2}}$	$E_1^{2,2} \xrightarrow{d_1^{2,2}}$	$E_1^{3,2} \xrightarrow{d_1^{3,2}}$	$E_1^{4,2} \xrightarrow{d_1^{4,2}}$	$E_1^{5,2} \xrightarrow{d_1^{5,2}}$

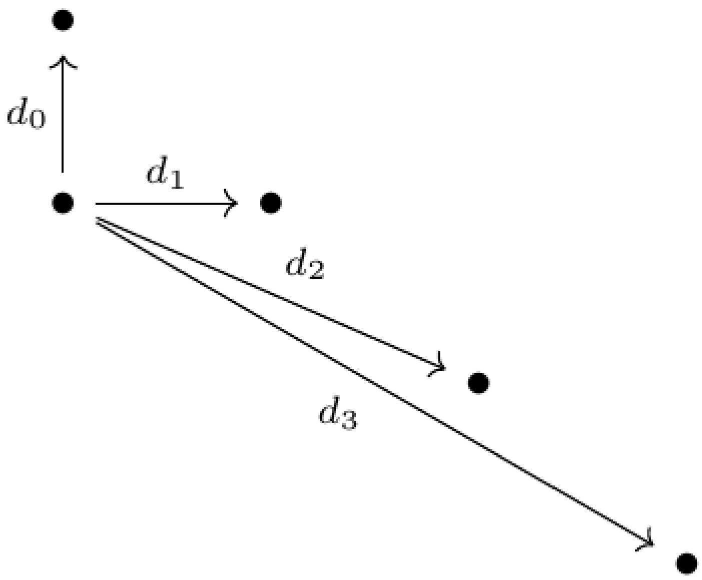
The E_1 page

$\xrightarrow{d_1^{-2,2}} E_1^{-1,2} \xrightarrow{d_1^{-1,2}}$	$E_1^{0,2} \xrightarrow{d_1^{0,2}}$	$E_1^{1,2} \xrightarrow{d_1^{1,2}}$	$E_1^{2,2} \xrightarrow{d_1^{2,2}}$	$E_1^{3,2} \xrightarrow{d_1^{3,2}}$	$E_1^{4,2} \xrightarrow{d_1^{4,2}}$	$E_1^{5,2} \xrightarrow{d_1^{5,2}}$
$\xrightarrow{d_1^{-2,1}} E_1^{-1,1} \xrightarrow{d_1^{-1,1}}$	$E_1^{0,1} \xrightarrow{d_1^{0,1}}$	$E_1^{1,1} \xrightarrow{d_1^{1,1}}$	$E_1^{2,1} \xrightarrow{d_1^{2,1}}$	$E_1^{3,1} \xrightarrow{d_1^{3,1}}$	$E_1^{4,1} \xrightarrow{d_1^{4,1}}$	$E_1^{5,1} \xrightarrow{d_1^{5,1}}$
$\xrightarrow{d_1^{-2,0}} E_1^{-1,0} \xrightarrow{d_1^{-1,0}}$	$E_1^{0,0} \xrightarrow{d_1^{0,0}}$	$E_1^{1,0} \xrightarrow{d_1^{1,0}}$	$E_1^{2,0} \xrightarrow{d_1^{2,0}}$	$E_1^{3,0} \xrightarrow{d_1^{3,0}}$	$E_1^{4,0} \xrightarrow{d_1^{4,0}}$	$E_1^{5,0} \xrightarrow{d_1^{5,0}}$

$\frac{\ker d_0^{-1,2}}{\operatorname{im} d_0^{-2,2}}$	$\frac{\ker d_0^{0,2}}{\operatorname{im} d_0^{-1,2}}$	$\frac{\ker d_0^{1,2}}{\operatorname{im} d_0^{0,2}}$	$\frac{\ker d_0^{2,2}}{\operatorname{im} d_0^{1,2}}$	$\frac{\ker d_0^{3,2}}{\operatorname{im} d_0^{2,2}}$	$\frac{\ker d_0^{4,2}}{\operatorname{im} d_0^{3,2}}$	$\frac{\ker d_0^{5,2}}{\operatorname{im} d_0^{4,2}}$
$\frac{\ker d_0^{-1,1}}{\operatorname{im} d_0^{-2,1}}$	$\frac{\ker d_0^{0,1}}{\operatorname{im} d_0^{-1,1}}$	$\frac{\ker d_0^{1,1}}{\operatorname{im} d_0^{0,1}}$	$\frac{\ker d_0^{2,1}}{\operatorname{im} d_0^{1,1}}$	$\frac{\ker d_0^{3,1}}{\operatorname{im} d_0^{2,1}}$	$\frac{\ker d_0^{4,1}}{\operatorname{im} d_0^{3,1}}$	$\frac{\ker d_0^{5,1}}{\operatorname{im} d_0^{4,1}}$
$\frac{\ker d_0^{-1,0}}{\operatorname{im} d_0^{-2,0}}$	$\frac{\ker d_0^{0,0}}{\operatorname{im} d_0^{-1,0}}$	$\frac{\ker d_0^{1,0}}{\operatorname{im} d_0^{0,0}}$	$\frac{\ker d_0^{2,0}}{\operatorname{im} d_0^{1,0}}$	$\frac{\ker d_0^{3,0}}{\operatorname{im} d_0^{2,0}}$	$\frac{\ker d_0^{4,0}}{\operatorname{im} d_0^{3,0}}$	$\frac{\ker d_0^{5,0}}{\operatorname{im} d_0^{4,0}}$

Differentials





Example: Proving the Snake Lemma without chasing elements

Given the following diagram, with exact rows and commuting squares:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

We want to show that this sequence is exact:

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{im}\alpha \to \operatorname{im}\beta \to \operatorname{im}\gamma \to 0$$

The usual proof involves pushing around elements - all of the maps are "obvious", except for δ.

Example: Proving the 5 lemma

Recall the definition: if $X \xrightarrow{f} Y$ then $\operatorname{coker} f = \frac{Y}{\operatorname{im} f}$

Expand the usual diagram into a double complex by filling in zeros:

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & E & \longleftarrow & D & \longleftarrow & C & \longleftarrow & B & \longleftarrow & A & \longleftarrow & 0 \\
\downarrow & & {\scriptstyle k}\downarrow & & {\scriptstyle -j}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle -g}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow \\
0 & \longleftarrow & E' & \longleftarrow & D' & \longleftarrow & C' & \longleftarrow & B' & \longleftarrow & A' & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

Here we assume that $k, -j, -g, f$ are all isomorphisms. Since this is the E_0 page, we first take homology starting with the vertical arrows as differentials, this yields the E_1 page

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dfrac{\ker h}{\operatorname{im} 0 \mapsto C} \cong \dfrac{\ker h}{0} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dfrac{\ker C' \mapsto 0}{\operatorname{im} h} \cong \dfrac{C'}{\operatorname{im} h} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

Making suitable identifications, this page equals

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \operatorname{Ker} h & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \operatorname{Coker} h & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

The differentials on this page are now all horizontal arrows - but these are all zero maps, so the spectral sequence has collapsed at this page. We now know this page is the complex that the spectral sequence converges to, even if we don't know what $\ker k$ and $\operatorname{coker} h$ are.

We also know that taking the "dual" spectral sequence converges to the same thing. We start with the same E_1 page, but now start with the horizontal arrows as differentials. By exactness of the rows, we have the E_1 page

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

and since the differentials are necessarily at this page, the spectral sequence has collapsed. But this must be equal to what it converged to in the dual setting, so we obtain $\ker h = 0$ and $\operatorname{coker} h = 0$. But $\ker h = 0$ iff h is injective, and $\operatorname{coker} h = 0$ iff h is surjective, so h is an isomorphism.

Recovering the homology

If a spectral sequence collapses, say $E_\infty^{p,q} = E_N^{p,q}$, then $H_n(X)$ is the unique $E_N^{p,q}$ where $p + q = n$. In general, the homology can be read off as the single nonzero element on the diagonal when this happens.