Title

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1.1 General Notes

- Say what you're assuming at the start of the proof.
  - If flipping logic and not using a direct proof (contradiction, contrapositive, etc), then signpost/announce it near the beginning of the proof.
  - Examples: for $P \implies Q$,
    * Direct proof: "Suppose $P \cdots$"
    * Contradiction: "Suppose toward a contradiction P but not $Q \cdots$"
      (Usually show $\neg P$. If you show Q, a direct proof might be simpler.)
    * Contrapositive: "Suppose by contrapositive that $\neg Q$ holds, $\cdots$"
- Put any important equations (i.e. major steps of the proof) on their own lines or in displaymath environments.
- Use some whitespace to separate parts of the proof and increase readability.
- Remember that limits of sequences need not exist, but liminfs/limsups always do

  (just may be $\pm\infty$).
- Try to avoid abbreviating the names of major theorems (example: "AP" can stand for many results, not just the Archimedean property!)
- It's not generally true that $a \leq M \implies |a| \leq M$, e.g. take $a = -1 \leq M = 0$. This only holds for $a \geq 0$.

- A generic set **may not** contain its inf or sup. Example: $\inf\left\{\frac{1}{n}\right\} = 0$ and $0 \notin \left\{\frac{1}{n}\right\}$, or $\sup\left\{1 - \frac{1}{n}\right\} = 1$ with $1 \notin \left\{1 - \frac{1}{n}\right\}$.
- If there exists some element of a set or sequence with a given property, try to say where it comes from and why the property holds for it.
- Similarly, if a property holds for all elements of a set or sequence, try to say why.
- The crux of many proofs are certain inequalities, so try to justify every inequality that appears.
- If you use a theorem, be sure to mention it by its full name.
- Useful counterexamples:
  - $x_n = (-1)^n$
  - Literal lists of numbers: $[0, 1, 0, 2, \cdots]$.

1.2 1.a

Proof ($A \implies B$).

- Suppose $\{a_n\}$ is not bounded above.
- Then any $k \in \mathbb{N}$ is not an upper bound for $\{a_n\}$.
- So choose a subsequence $a_{n_k} > k$, then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k\to\infty} a_{n_k} > \liminf_{k\to\infty} k = \infty.$$

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Proof ($\not A \implies \not B$).

- Suppose $\{a_n\}$ is bounded by M, so $a_n < M < \infty$ for all $n \in \mathbb{N}$.
- Then if $\{a_{n_k}\}$ is a subsequence, we have $a_{n_k} \in \{a_n\}$, so $a_{n_k} < M$ for all $k \in \mathbb{N}$.
- But then

$$a_{n_k} < M \implies \limsup_{k\to\infty} a_{n_k} \leq M,$$

- Now note that if $\lim_{k\to\infty} a_{n_k}$ exists,

$$\lim_{k\to\infty} a_{n_k} < \limsup_{k\to\infty} a_{n_k} \leq M < \infty,$$

so every subsequence is bounded and thus can not converge to ∞.

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1.3 3.a

Proof (Using definition (i)).

- Suppose $x_n \le M$ for all n, we will show that every subsequential limit is also bounded by M.
- Let

$$S := \left\{ x \in \mathbb{R} \;\middle|\; x \text{ is a subsequential limit of } \{x_n\} \right\}$$

be the set of subsequential limits.
  - Note that $\inf S := \liminf_{n\to\infty} x_n$ by definition (i).
- Let $\{x_{n_k}\} \in S$ be an arbitrary convergent subsequence (since we are only concerned about subsequences with well-defined limits).
- Then for every k we have $x_{n_k} \in \{x_n\}$, so

$$|x_{n_k}| \le M.$$

- By order limit laws,

$$|x_{n_k}| \le M \implies \lim_{k\to\infty} |x_{n_k}| \le M,$$

- Since the map $x \mapsto |x|$ is continuous, using the sequential definition of continuity we can pass the limit through the absolute value to obtain

$$\left| \lim_{k\to\infty} x_{n_k} \right| \le M.$$

- Since the subsequence was arbitrary, we find that M is an upper bound for S and so $\sup S \le M$.
- But

$$\inf S \le \sup S \le M \implies \inf S \le M.$$

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Proof (Using definition (ii)).

- Suppose $|x_n| \le M$ for every n, we will directly show that $\left| \lim_{n\to\infty} \inf_{k\ge n} x_n \right| \le M$.
- By order-limit laws, for every fixed n we have

$$|x_n| \le M \iff -M \le x_n \le M \implies -M \le \inf_{k>n} x_k \le M,$$

where we've used the fact that $x_n \ge -M$ for all n implies that $\inf_{k\ge n} x_k \ge -M$.
- Again applying order-limit laws,

$$-M \le \inf_{k\ge n} x_k \le M \implies -M \le \lim_{n\to\infty} \inf_{k\ge n} x_k \le M \iff \left| \lim_{n\to\infty} \inf_{k\ge n} x_{n_k} \right| \le M.$$

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1.4 3.b

Proof (Using definition (i)).

Note that here we define S to be the set of all subsequential limits of $\{x_n\}$ and

$$\liminf_n x_n := \inf S.$$

- Suppose toward a contradiction that $\beta < \liminf_n x_n$ **but** there does not exist any N such that $n \geq N \implies x_n > \beta$.
- Then for all N there exists an $n > N$ with $x_n \leq \beta$, so the set

$$B := \left\{ n \in \mathbb{N} \;\middle|\; x_n \leq \beta \right\}$$

  is countably infinite.
- Then by Bolzano-Weierstrass, since B is bounded it contains a convergent subsequence x_{n_k} which satisfies

$$x_{n_k} \leq \beta \quad \forall k \implies L := \lim_{k \to \infty} x_{n_k} \leq \beta$$

  where we've used order-limit laws.
- We now have $L \in S$, a subsequential limit satisfying $L \leq \beta$ and since $\inf S$ is a lower bound for S,

$$\inf S \leq L \leq \beta.$$

  which contradicts $\beta < \liminf_n x_n$.

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Proof (Using definition (ii)).

Note that here we define

$$\liminf_n x_n := \lim_{n \to \infty} S_n \quad \text{where} \quad S_n := \inf \left\{ x_k \;\middle|\; k \geq n \right\}.$$

- Write $L := \lim_{n \to \infty} S_n$ and suppose $\beta < L$.
- Then we have

$$\forall \varepsilon > 0, \exists N \quad \text{such that} \quad n \neq N \implies |S_n - L| < \varepsilon.$$

- Since $\beta < L \iff L - \beta > 0$, we can set $\varepsilon := L - \beta$ to produce an N such that

$$n \geq N \implies |L - S_n| < L - \beta \iff \beta - L < S_n - L < L - \beta.$$

- Just taking the first part of this composite inequality we have

$$n \geq N \implies \beta - L < S_n - L \iff \beta < S_n := \inf_{k \geq n} x_k \leq x_n,$$

  supplying the N for which $n \geq N \implies \beta < x_n$ as desired.

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Proof (Using definition (ii), alternative).

- Suppose toward a contradiction that $\beta < \liminf_n x_n$ **but** there is no N such that $n \geq N \implies x_n > \beta$.
- Then for all N there exists an n with $x_n \leq \beta$, so if we form the set

$$B_n := \left\{ k \in \mathbb{N} \;\middle|\; k \geq n \text{ and } x_k \leq \beta \right\},$$

then B_n is countably infinite for every n
- But then $B_n \subseteq \left\{ k \in \mathbb{N} \;\middle|\; k \geq n \right\}$ for every n implies that

$$\inf_{k \geq n} x_k \leq \inf_{k \in B_n} x_k \leq \beta \qquad \forall n,$$

since an infimum over a larger set can only get smaller.
- Applying order-limit laws, we then have

$$\inf_{k \geq n} \leq \beta \ \forall n \implies \lim_{n \to \infty} \inf_{k \geq n} x_n \leq \beta,$$

but this contradicts $\liminf_n x_n > \beta$.

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1.5 4.a

Proof .

- Suppose $\{x_n\}$ is bounded and $\limsup |x_n| = 0$.
- Then using the supremum definition, $\lim_{n \to \infty} \sup_{k \geq n} |x_k| = 0$.
- Note that

$$\lim_{n \to \infty} x_n = 0 \iff \forall \varepsilon \quad \exists N \text{ such that } n \geq N \implies |x_n| < \varepsilon.$$

- So let $\varepsilon > 0$ be arbitrary.
- By the definition of the limit appearing in the $\limsup$, there exists an N_0 such that

$$n \geq N_0 \implies \sup_{k \geq n} |x_k| < \varepsilon.$$

- But then taking $N = N_0$ in the first equation yields the result, since

$$n \geq N_0 \implies |x_n| \leq \sup_{k \geq n} |x_k| < \varepsilon.$$

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1.6 4.c

Proof .

- Note that $-1 \leq \sin(x) \leq 1$ and $\sin(x) = \pm 1 \iff x = 2k\pi \pm \frac{\pi}{2}$.
- Since π is irrational, $\sin(n)$ will never be of this form, so $-1 < \sin(n) < 1$.
- Taking floors, we have $-1 \leq \lfloor \sin(n) \rfloor \leq 0$, which in fact means that $\sin(n) \in \{-1, 0\}$ can only take on one of two values.
- The set of subsequential limits is then just $\{-1, 0\}$.
- **Claim**: $\limsup \lfloor \sin(n) \rfloor = 0$.
  - It suffices to show that $\lfloor \sin(n) \rfloor = 0$ infinitely often
  - But note that there is an integer in any interval of the form $[2k\pi, 2k\pi + \pi]$ for $k \in \mathbb{N}$, since it is of length $\pi > 1$.
  - In these intervals, $0 < \sin(n) < 1$, and so $\lfloor \sin(n) \rfloor = 0$, and there infinitely many such intervals.
  - So form a subsequence $x_{n_k} = \lfloor \sin(n_k) \rfloor$ by choosing n_k to be any integer in
- **Claim**: $\liminf \lfloor \sin(n) \rfloor = -1$.
  - By the exact same argument applied to intervals of the form $[3k\pi, 3k\pi + \pi]$ where $-1 < \sin(n) < 0$, we find that $\lfloor \sin(n) \rfloor = -1$ infinitely often.

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