Weil Conjectures

D. Zack Garza

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1 Notes from Daniel's Office Hours

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3. Medium example: $E/\urcorner$ an elliptic curve.
4. Work out a harder example as in Weil

1.1 Definition of Zeta Function

Fix q a prime and $\mathbb{F} := \mathbb{F}_q$ the finite field with q elements, along with its unique degree n extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_p \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\, n \in \mathbb{Z}^{\geq 2}$$

Definition 1.0.1.
Let

$$J = \langle f_1, \cdots, f_N \rangle \trianglelefteq k[x_0, \cdots, x_n]$$

be an ideal, then a *projective algebraic* variety $X \hookrightarrow \mathbb{P}^\infty_{\mathbb{F}}$ can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^\infty_{\mathbb{F}} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_N(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is some fixed $d \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1,\cdots,i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Example: Take $f_1(x) = x \in \mathbb{F}[x]$, consider $V(\langle f_1 \rangle) \subset \mathbb{P}^1_{\mathbb{F}_n}$. This is given by the single point $x = \mathbf{0}$.

Fix $X/\mathbb{F}$ an N-dimensional projective algebraic variety. Note that it then has points in any finite extension L/K.

Definition 1.0.2.
Define the *local zeta function* of X as follows:

$$\begin{aligned} \zeta_X &: \mathbb{C} \longrightarrow \mathbb{C} \\ \zeta_X(t) &= \exp\left(\sum_{n=1}^{\infty} \frac{\alpha_n}{n} t^n\right) \quad \text{where} \quad \alpha_n \coloneqq \#X(\mathbb{F}_n). \end{aligned}$$

Note the following two properties:

$$\begin{aligned} \zeta_X(0) &= 1 \\ t\left(\frac{\partial}{\partial t}\right) \log \zeta_X(t) &= t\left(\frac{\zeta_X'(t)}{\zeta_X(t)}\right) = \sum_{n=1}^{\infty} \alpha_n t^n = \alpha_1 t + \alpha_2 t^2 + \cdots, \end{aligned}$$

which is an *ordinary generating function* for the sequence (α_n).

Todo: why not an OGF.

Remark: Note that for an OGF $F(x) = \sum_{n=0}^{\infty} f_n x^n$, we can extract coefficients in the following way:

$$[x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n F(x)\Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

Example (Point): $X = \{x = 0\} / \mathbb{F}$ a single point over $\mathbb{F}$, then

$$\begin{aligned} \#X(\mathbb{F}) &\coloneqq \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &\coloneqq \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &\coloneqq \alpha_n = 1 \\ &\ddots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned} \log(1+t) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} \\ \implies \log(1-t) &= -\sum_{n=1}^{\infty} \frac{t^n}{n} \\ \implies -\log(1-t) &= \sum_{n=1}^{\infty} \frac{t^n}{n} \\ &= 1 \cdot t + 1 \cdot t^2 + 1 \cdots t^3 + \cdots . \end{aligned}$$

and so

$$\zeta_X(t) = \exp\left(-\log\left(1-t\right)\right) = \frac{1}{1-t}.$$

Example (Affine Line): $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then

$$\begin{aligned} X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n \\ &. \end{aligned}$$

where we just note that we can write $\mathbb{A}^1(\mathbb{F}_n) = \left\{ (x_1) \;\middle|\; x_1 \in \mathbb{F}_n \right\}$.

Example (Projective Line): $X = \mathbb{P}^1/\mathbb{F}$ the projective line over $\mathbb{F}$, then

$$\begin{aligned} X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &. \end{aligned}$$

where we write $\mathbb{P}^1_{\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}} \coprod \{\infty\}$ is the affine line with a point at infinity. We can also count by coordinates:

$$\mathbb{P}^1(\mathbb{F}^n) = \left\{ [x_1, x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}^n \right\} / \sim \qquad = \left\{ [x_1, 1] \;\middle|\; x_1 \in \mathbb{F}^n \right\} \coprod \{[1,0]\}.$$

Example (Affine Space): Take $X = \mathbb{A}^n/\mathbb{F}$, then $\alpha_n = q^m + 1$ for a point at infinity, so

$$X(\mathbb{F}) = .$$

Thus

$$\zeta_X(t) = \frac{1}{(1-q^{-t})(1-q^{1-t})}$$

.

Example (Projective Space): Take $X = \mathbb{P}^n_{\mathbb{F}}$, then $\alpha_n = 1 + q^m + (q^m)^2 + \cdots + (q^m)^n$, so

$$\zeta_X(t) = \left(\frac{1}{1-q^{-t}}\right)\left(\frac{1}{1-q^{1-t}}\right)\left(\frac{1}{1-q^{2-t}}\right)\cdots\left(\frac{1}{q^{n-t}}\right)$$

,

or equivalently, take your favorite curve $\gamma \in \mathbb{C}$ homotopic to $\mathbb{S}^1$.

> Note: this is extremely amenable to numerical approximation if you have a closed form for F or even just a black-box numerical version of F! I.e. easy to throw at a computer.

> Todo: how to manually count points in $\mathbb{P}^n$!

Example: Take $X = \mathrm{Gr}_{\mathbb{F}}(k, n)$, then ????? so

$$\zeta_X(t) = ?.$$

Questions about properties

- $\zeta_{X \coprod Y}(t) =_? \zeta_X(t)\zeta_Y(t)$?
- $\zeta_{X \times Y} = ?$

1.2 Statement of Weil Conjectures

1. (Rationality)

$$\zeta_X(t) = \frac{p_1(t)p_3(t)\cdots p_{2N-1}(t)}{p_0(t)p_2(t)\cdots p_N(t)} \in \mathbb{Z}(t), \quad \text{i.e.} \quad p_i(t) \in \mathbb{Z}[t]$$

$$\begin{aligned} P_0(t) &= 1 - t \\ P_{2n}(t) &= 1 - q^n t \\ P_i(t) &= \prod_j (1 - a_{ij} t), \quad a_{ij} \in \mathbb{C}. \end{aligned}$$

2. (Functional Equation and Poincare Duality)

$$\zeta_X(n-t) = \pm q^{\frac{1}{2}(nE)-Et}\zeta(x,t).$$

3. (Riemann Hypothesis)
4. (Betti Numbers)

1.3 Hard Example: An Elliptic Curve

Take $X = E/\mathbb{F}$, then $\alpha_n = q^n - (a^n + \bar{a}^n - 1)$ where $|a|_{\mathbb{C}} = |\bar{\alpha}|_{\mathbb{C}} = \sqrt{q}$. Then

$$\zeta_X(t) = \frac{(1-aq^{-t})(1-\bar{a}q^{-t})}{(1-q^{-t})(1-q^{1-s})}.$$