Title

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> Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$".

> Course Website: `https://faculty.franklin.uga.edu/brian/math-8030-spring-2020`

1.1 Chapter Zero: Review

> Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase mathfrak characters!

In this course, we'll take $k = \mathbb{C}$.

Recall that a Lie Algebra is a vector space $\mathfrak{g}$ with a bracket $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

- $[xx] = 0$ for all $x \in \mathfrak{g}$
  - Exercise: this implies $[xy] = -[yx]$.
    > Hint: Consider $[x+y, x+y]$. Note that the converse holds iff char $k \neq 2$.

    > Exercise: This implies Lie Algebras never have an identity.
- $[x[yz]] = [[xy]z] + [y[xz]]$ (The Jacobi identity)
  - This says x acts as a derivation.

Definition: $\mathfrak{g}$ is *abelian* iff $[xy] = 0$ for all $x, y \in \mathfrak{g}$.

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
  - A vector subspace that is closed under brackets.
- Homomorphisms
  - I.e. a linear transformation ϕ that commutes with the bracket, i.e. $\phi([xy]) = [\phi(x)\phi(y)]$.
- Ideals

Exercise: Given a vector space (possibly infinite-dimensional) over k, then (exercise) $\mathfrak{gl}(V) := \mathrm{End}_k(V)$ is a Lie algebra when equipped with $[fg] = f \circ g - g \circ f$.

Definition: A *representation* of $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ for some V.

Example: The adjoint representation is $\mathrm{ad}\ : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\mathrm{ad}\ (x)(y) := [xy]$.

Representations give $\mathfrak{g}$ the structure of a module over V, where $x \cdot v := \phi(x)(v)$. All of the usual module axioms hold, where now $[xy] \cdot v = x \cdot (\cdot v) - y \cdot (x \cdot v)$.

Example: The trivial representation $V = k$ where $x \cdot a = 0$.

Definition: V is *irreducible* (or *simple*) iff V as exactly two $\mathfrak{g}$-invariant subspaces, namely $0, V$.

Definition: V is *completely reducible* iff V is a direct sum of simple modules, and *indecomposable* iff V can not be written as $V = M \oplus N$, a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragradient/dual* $V^\vee := \hom_k(V, k)$ where $(x \cdot f) = -f(x \cdot v)$ for $f \in V^\vee, x \in \mathfrak{g}, v \in V$.
- The direct sum $V \oplus W$ where $x \cdot (v, w) = (x \cdot v, x \cdot w)$ and $x \cdot (v + w) = x \cdot v_x \cdot w$.
- The tensor product where $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.
- $\hom_k(V, W)$ where $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$.
  - Note that if we take $W = k$ then the first term vanishes and this recovers the dual.

1.2 Semisimple Lie Algebras

Definition: The derived ideal is given by $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \mathrm{span}_k\left(\left\{[xy] \;\middle|\; x, y \in \mathfrak{g}\right\}\right)$.

This is the analog of the commutator subgroup.

Lemma: $\mathfrak{g}$ is abelian iff $\mathfrak{g}^{(1)} = \{0\}$, and 1-dimensional algebras are always abelian.

This follows because if $[xy] := xy = yx$ then $[xy] = 0 \iff xy = yx$.

Definition: A lie group $\mathfrak{g}$ is *simple* iff the only ideals of $\mathfrak{g}$ are $0, \mathfrak{g}$ and $\mathfrak{g}^{(1)} \neq \{0\}$.

Note that thus rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

Definition: The derived series is defined by $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$, continuing inductively. $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

Lemma: Abelian implies solvable.

Review definition of nilpotent algebras.

Definition: $\mathfrak{g}$ is semisimple (s.s.) iff $\mathfrak{g}$ has no nonzero solvable ideals.

Exercise: Simple implies semisimple.

Some remarks:

1. Semisimple algebras $\mathfrak{g}$ will usually have solvable subalgebras.
2. $\mathfrak{g}$ is semisimple iff $\mathfrak{g}$ has no nonzero abelian ideals.

Definition: The Killing form is given by $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ where $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$, which is a symmetric bilinear form.

Lemma: $\kappa([xy], z) = \kappa(x, [yz])$.

Recall that if $\beta : V^{\otimes 2} \to k$ is any symmetric bilinear form, then its radical is defined by

$$\mathrm{rad}\beta = \left\{ v \in V \;\middle|\; \beta(v, w) = 0 \ \forall w \in V \right\}.$$

Definition: A bilinear form β is nondegenerate iff $\mathrm{rad}\beta = 0$.

Lemma: $\mathrm{rad}\kappa \trianglelefteq \mathfrak{g}$ is an ideal, which follows by the above associative property.

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Example: The standard example of a semisimple lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \;\middle|\; \mathrm{Tr}(x) = 0 \right\}$.

> Note: from now on, $\mathfrak{g}$ will denote a semisimple lie algebra over $\mathbb{C}$.

Theorem (Weyl): Every finite dimensional representation of a semisimple $\mathfrak{g}$ is completely reducible.

> I.e., the category of finite-dimensional representations is relatively uninteresting – there are no extensions, everything is a direct sum, so once you classify the simple algebras (which isn't terribly difficult) then you have complete information.

2 Friday January 10th

Let $\mathfrak{g}$ be a finite dimensional semisimple lie algebra over $\mathbb{C}$.

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the trace zero $n \times n$ matrices.

Let $\mathfrak{h}$ be a maximal toral subalgebra, where $x \in \mathfrak{g}$ is *toral* if x is semisimple, i.e. ad x is semisimple (i.e. diagonalizable).

Example: $\mathfrak{h}$ is the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$.

Fact: $\mathfrak{h}$ is abelian, so ad $\mathfrak{h}$ consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \alpha(h)x \; \forall h \in \mathfrak{h} \right\}$ where $\alpha \in \mathfrak{h}^\vee$ is a linear functional.

Here $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, so $[hx] = 0$ corresponds to zero eigenvalues, and (fact) it turns out that $\mathfrak{h}$ is its own centralizer.

We then obtain a set of roots of $\mathfrak{h}, \mathfrak{g}$ given by $\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \;\middle|\; \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\} \right\}$.

Example: $\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$ for some $i \neq j$, the matrix with a 1 in the i, j position and zero elsewhere.

Fact: The restriction $\kappa|_{\mathfrak{h}}$ is nondegenerate, so we can identify $\mathfrak{h}, \mathfrak{h}^\vee$ via κ (can always do this with vector spaces with a nondegenerate bilinear form), where κ maps to another bilinear form $(\cdot, \cdot)$.

$$\begin{aligned} \mathfrak{h}^\vee \ni \lambda &\iff t_\lambda \in \mathfrak{h} \\ \lambda(h) = \kappa(t_\lambda, h) &\quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu). \end{aligned}$$

2.1 Facts About Φ and Root Spaces

Let $\alpha, \beta \in \Phi$ be roots.

1. ϕ spans $\mathfrak{h}^\vee$ and does not contain zero.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other scalar multiple of α is in Φ.

Aside:

- $\dim \mathfrak{g}_\alpha = 1$.
- If $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ then there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha \coloneqq [x_\alpha, y_\alpha]$ spans a 3-dimensional subalgebra in $\mathfrak{sl}_2$, given by $x_\alpha = [0, 1; 0, 0], y_\alpha = [0, 0; 1, 0], h_\alpha = [1, 0; 0, -1]$.
- Under the correspondence $\mathfrak{h} \iff \mathfrak{h}^\vee$ induced by κ, $h_\alpha \iff \alpha^\vee \coloneqq \dfrac{2\alpha}{(\alpha, \alpha)}$. Thus for all $\lambda \in \mathfrak{h}^\vee$,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}.$$

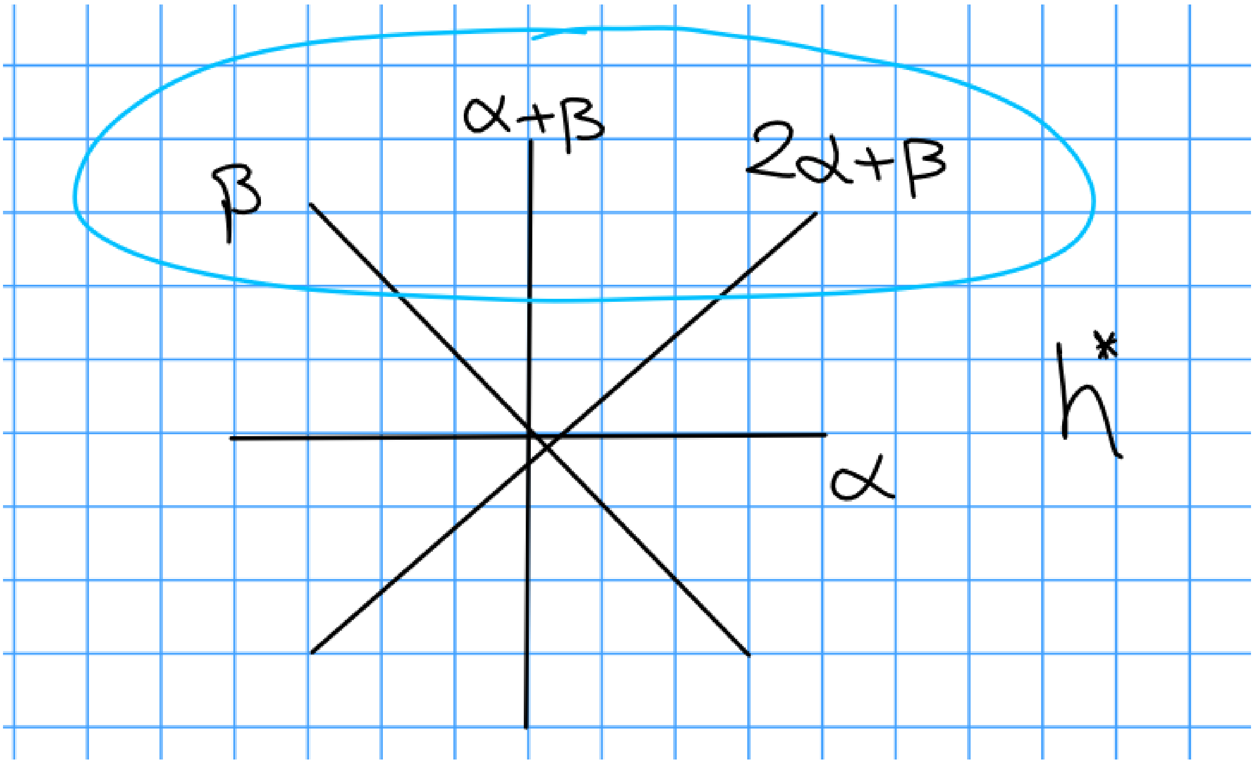
- If $\alpha + \beta \neq 0$, then $\kappa(g_\alpha, g_\beta) = 0$.

3. $(\beta, \alpha^\vee) \in \mathbb{Z}$
4. $S_\alpha(\beta) \coloneqq \beta - (\beta, \alpha^\vee)\alpha \in \Phi$.

If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Example: If $\alpha = E_{ij}, \beta = E_{jk}$ where $k \neq i$, then $[E_{ij}, E_{jk}] = E_{ik}$.

- $\mathfrak{g}$ is generated as an algebra by the root spaces $\mathfrak{g}_\alpha$
- Root strings: If $\beta \neq \pm\alpha$, then the roots of the form $\alpha + k\beta$ for $k \in \mathbb{Z}$ form an unbroken string $\alpha - r\beta, \cdots, \alpha - \beta, \alpha, \alpha + \beta, \cdots, \alpha + \ell\beta$ consisting of at most 4 roots where $r - s = (\alpha, \beta^\vee)$.

Example: The circled roots below form the root string for β:



In general, a subset Φ of a real euclidean space E satisfying conditions (1) through (4) is an *(abstract) root system*.

When Φ comes from a $\mathfrak{g}$, $E := \mathbb{R}\Phi$.

2.1.1 The Root System

There exists a subset $\Delta \subseteq \Phi$ such that

- Δ is a $\mathbb{C}$-basis for $\mathfrak{g}^\vee$
- $\beta \in \Phi$ implies that $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ with either
  - All $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$ or $\beta < 0$.
  - All $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$ or $\beta > 0$.

Δ is called a *simple system*. If $\Delta = \{a_1, \cdots, a_\ell\}$ then $\Phi^+$are the *positive roots*, and $\Phi^+ \ni \beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$, then the *height* of β is defined as $\sum c_\alpha \in \mathbb{Z}_{>0}$.

Note that $\mathbb{Z}\Phi := \Lambda_r$ is a lattice, and is referred to as the *root lattice*, and $\Lambda_r \subset E = \mathbb{R}\Phi$. We also have $\Phi^+ = \left\{ \beta^\vee \mid \beta \in \Phi \right\}$, the *dual root system*, is a root system with simple system $\Delta^\vee$.

Important subalgebras of $\mathfrak{g}$:

- Upper triangular with zero diagonal $\mathfrak{n} = \mathfrak{n}^+ = \sum_{\beta > 0} \mathfrak{g}_\beta$

- Lower triangular with zero diagonal $\mathfrak{n}^- = \sum \beta > 0\mathfrak{g}_{-\beta}$
- Upper triangular, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra
- Lower triangular, $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

There is thus a triangular (Cartan) decomposition, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Fact: If $\beta \in \Phi^+ \setminus \Delta$, and if $\alpha \in \Delta$ such that $(\beta, \alpha^\vee) > 0$, then $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$ has height strictly less than the height of β.

> By root strings, $\beta - \alpha \in \Phi^+$ is positive root of height one less than β, yielding a way to induct on heights (useful technique).

2.1.2 Weyl Groups

For $\alpha \in \Phi$, define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\to \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in E perpendicular to α:

> Note that $S_\alpha^2 = \mathrm{id}$.

Define W as the subgroup of $\mathrm{gl}(E)$ generated by all s_α for $\alpha \in \Phi$, this is the *Weyl group* of $\mathfrak{g}$ or Φ, which is finite and $W = \left\langle s_\alpha \mathrel{}\middle|\mathrel{} \alpha \in \Delta \right\rangle$ is generated by simple reflections.

By (4), W leaves Φ invariant. In fact W is a finite Coxeter group with generators $S = \left\{ s_\alpha \mathrel{}\middle|\mathrel{} \alpha \in \Delta \right\}$ and defining relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ for $\alpha, \beta \in \Delta$ where $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$ and $m(\alpha, \alpha) = 1$.

> Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group.*

3 Monday January 13th

3.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over $\mathbb{C}$. We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group $W = \left\langle s_\beta \mathrel{}\middle|\mathrel{} \beta \in \Phi \right\rangle = \left\langle s_\alpha \mathrel{}\middle|\mathrel{} \alpha \in \Delta \right\rangle$ where $\Delta = \{a_i\}$ are the simple roots. For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) \coloneqq n$ where $\ell(1) \coloneqq 0$.

Facts:

1. $\ell(w)$ is the size of the set $\left\{ \beta \in \Phi^+ \mathrel{}\middle|\mathrel{} w\beta < 0 \right\}$

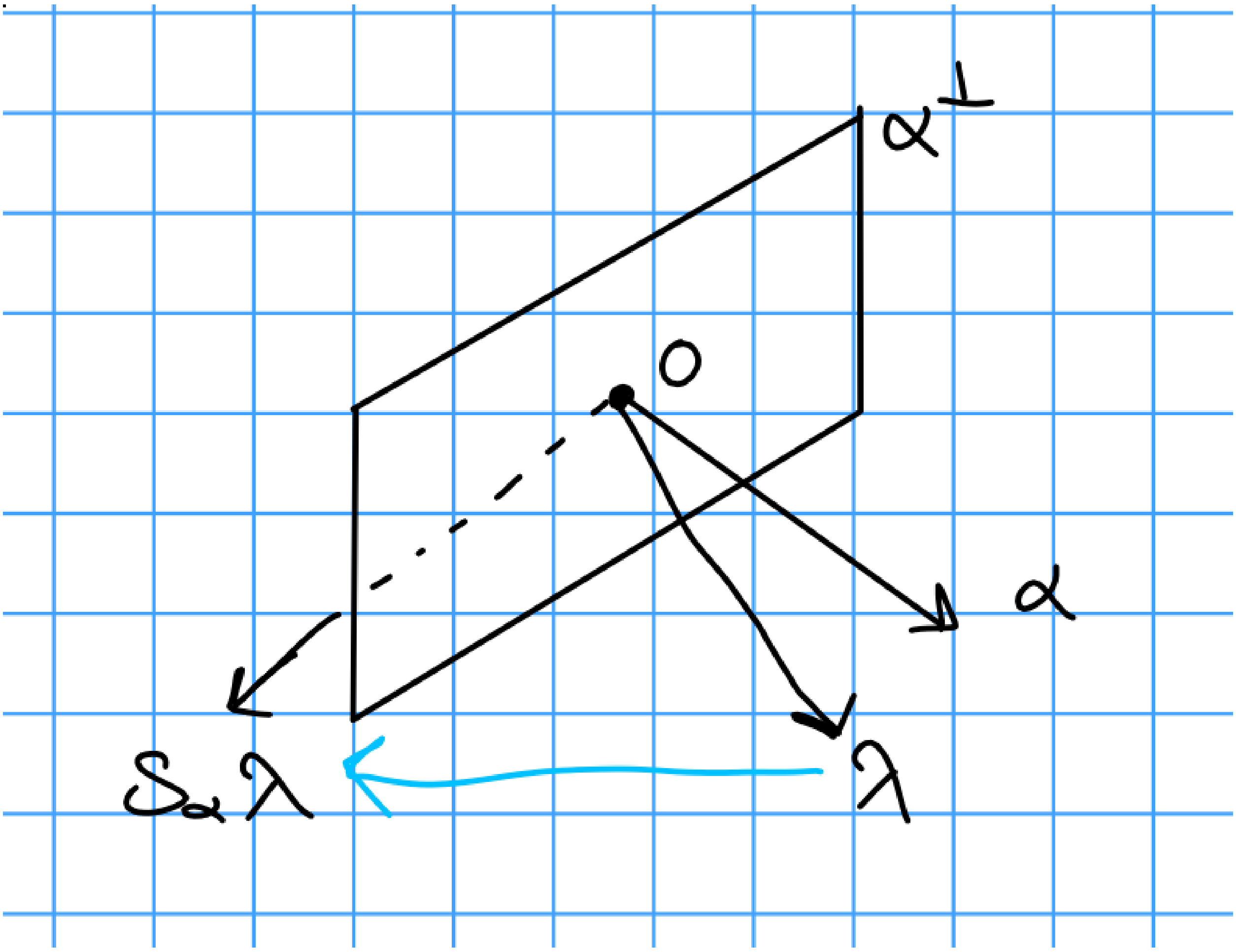


Figure 1: Image

- The above set is equal to $\Phi^+ \bigcap w^{-1}\Phi^{-1}$.
- In particular, for $\beta \in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
- Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.

2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \bigcap w\Phi$ (replacing w^{-1} with w)

3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = \left|\Phi^+\right|$ and $w_0(\Phi^+) = \Phi^-$.

- Also $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$\begin{aligned} \ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0 \end{aligned}.$$

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

3.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$. Then define

$$T \coloneqq \bigcup_{w \in W} wSw^{-1} = \left\{ s_\beta \;\middle|\; \beta \in \Phi^+ \right\}.$$

This is the set of *all* reflections in W through hyperplanes in E.

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w, and refer to this as "the covering relation". So $w' \to w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that $w' = w_0 \to w_1 \to \cdots \to w_n = w$. This is called the **Bruhat-Chevalley order** on W.

Corollary: $w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order*. In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

Recall that lie algebras yield finite crystallographic coxeter groups.

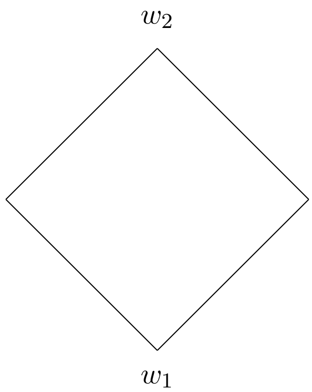
Properties: For (W, S) a coxeter group,

a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \cdots s_n$ for \$w, where a subexpression is any subcollection of s_i in the same order.

Note that this implies that 1 is not only a minimal element in this order, but an infimum.

b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.

c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\left\{ w \in W \;\middle|\; w_1 < w < w_2 \right\}$ is either 0 or 2.



3.3 Properties of Universal Enveloping Algebras

Let $\mathfrak{g}$ be any lie algebra, and $\phi : \mathfrak{g} \to A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[dd, dotted, "\tilde{\phi}"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[dr, "\phi"'] & \\ & A \end{tikzcd}$$

Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_1$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, where $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \left\langle [x, y] - x \otimes y - y \otimes x \;\middle|\; x, y \in \mathfrak{g} \right\rangle.$$

Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\text{ad } x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Big Theorem (Poincaré-Birkhoff-Witt, i.e. PBW): If $\{x_1, \cdots x_n\}$ is a basis for $\mathfrak{g}$, then $\left\{x_1^{t_1}, \cdots, x_n^{t_n} \;\middle|\; t_i \in \mathbb{Z}^+\right\}$ (noting that $x^n = x \otimes x \otimes \cdots x$ and $\mathbb{Z}^+$ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If $\mathfrak{g}$ is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for $\mathfrak{g}$, then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

Notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

3.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the $\mathbb{R}$-span of the roots. We also have the **integral weight lattice**

$$\Lambda = \left\{\lambda \in E \;\middle|\; (\lambda, \alpha^\vee) \in \mathbb{Z}\ \forall \alpha \in \Phi (\text{or } \Phi^+ \text{ or } \Delta)\right\}.$$

There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and $\mathfrak{h}^\vee$. We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+\Delta = \mathbb{Z}^+\Phi^+$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a $\mathbb{Z}$-basis for Λ. Then Λ is a free abelian group of rank ℓ, and $\Lambda^+ = \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.