Lie Algebras

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Contents

1 Monday August 12

> The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

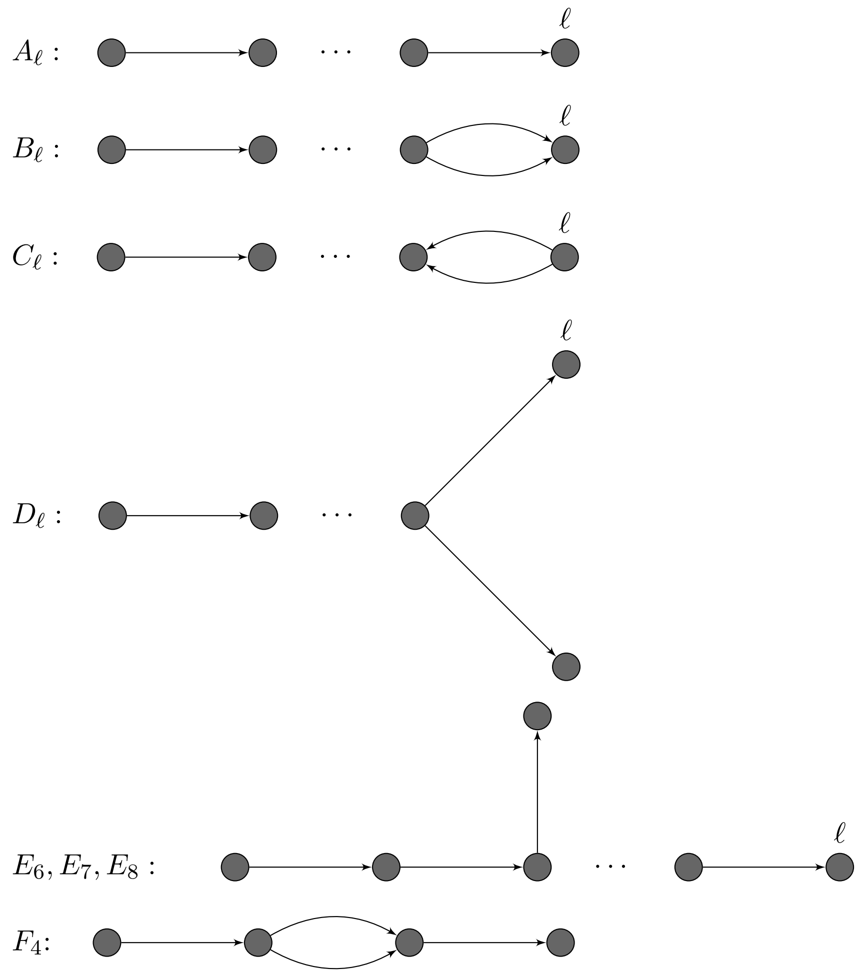
1.1.2 Chapter 3-4

We will describe the following series of correspondences:

$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\oplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:



1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Exercise 1. Show that $[x, y] = -[y, x]$.

Definition 2. Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\mathrm{End}(V) = \left\{ f : V \to V \;\middle|\; V \text{ is linear} \right\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n, V) = \left\{ f \in \mathfrak{gl}(n, V) \;\middle|\; \mathrm{Tr}(f) = 0 \right\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n, V)$.).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

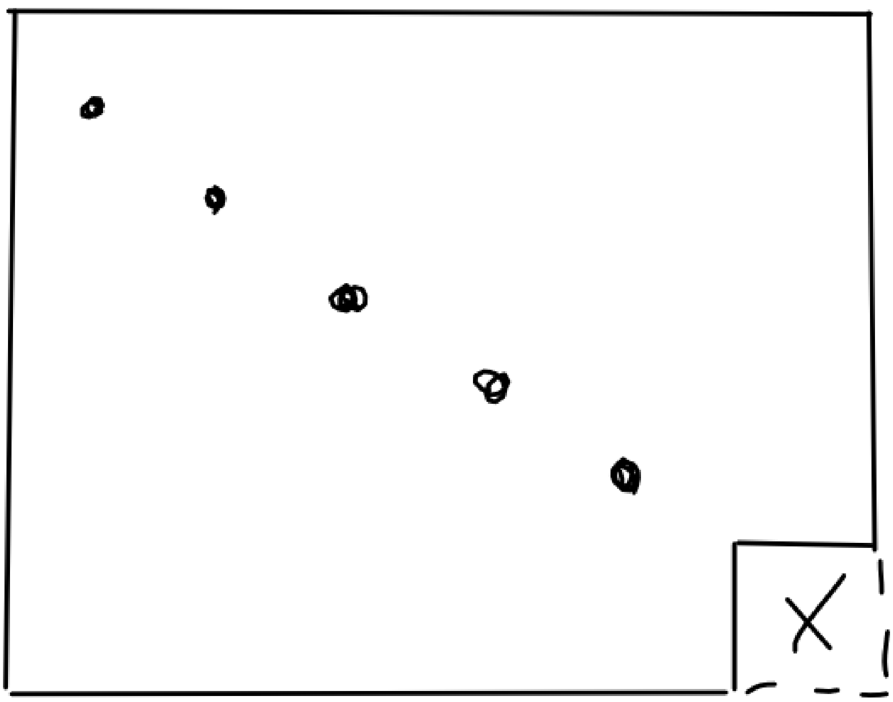
$$\mathfrak{sp}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \;\middle|\; MA - A^T M = 0 \right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right).$$

Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \;\middle|\; MA - A^T M = 0 \right\} \text{ where}$$
$$M = \begin{cases} \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \end{array} \right) & n = 2\ell + 1 \text{ odd}, \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) & \text{else}. \end{cases}$$

Proposition 7. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and zero elsewhere.



- For type A_ℓ, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell+1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type B_ℓ, we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$
\left(\begin{array}{c|cc}
0 & M & N \\
\hline
-N^t & A & C = C^t \\
-M^t & B = B^t & -A^t
\end{array}\right).
$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$
\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}
$$

Theorem 8. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Wednesday August 14

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$
[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.
$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

Exercise: Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition: An F**-algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition: An algebra is **associative** if $x(yz) = (xy)z$.

Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition: Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition: We define $\mathrm{Der}(A) = \left\{\delta \;\middle|\; \delta \text{ is a derivation } \right\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\,\cdot\,,\,\cdot\,]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x, [y, z]] = 0$.

Exercise: Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise: Define the adjoint by $\mathrm{ad}\,_x : \mathfrak{g}\circlearrowleft,\; y \mapsto [x, y]$. Show that $\mathrm{ad}\,_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example: Any F-vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\,\cdot\,,\,\cdot\,] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \dfrac{y}{a}$. Now compute $[x', y'] = [ax + by, \dfrac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\,\cdot\,,\,\cdot\,]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example: Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise: Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\mathrm{ad}\,(e), \mathrm{ad}\,(h), \mathrm{ad}\,(g)$ with respect to this basis.

2.3 Ideals

Definition: A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise: Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \left\{ z \in \mathfrak{g} \,\middle|\, [x, z] = 0 \quad \forall x \in \mathfrak{g} \right\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \left\{ \sum_i [x_i, y_i] \,\middle|\, x_i, y_i \in \mathfrak{g} \right\}$.
  - Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \left\{ x + y \,\middle|\, x \in I, y \in J \right\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \left\{ \sum_i [x_i, y_i] \,\middle|\, x_i \in I, y_i \in J \right\} \trianglelefteq \mathfrak{g}$

Definition: A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem: Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof:

Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \operatorname{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$\begin{aligned} 2e = [h, e] \in I &\implies e \in I, \\ 2f = [h, -f] \in I &\implies f \in I. \end{aligned}$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

■

3 Friday August 16

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x+I, y+I] = [x,y]+I$.

Exercise: Check that this is well-defined, so that if $x+I = x'+I$ and $y+I = y'+I$ then $[x,y]+I = [x',y']+I$.

3.1 Homomorphisms and Representations

Definition: A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x,y] = [\phi(x), \phi(y)]$.

Remark: $\ker \phi \trianglelefteq \mathfrak{g}_1$ and im $\phi \leq \mathfrak{g}_2$ are subalgebras.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \left\{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\right\}$ where $I \mapsto (x \mapsto x+I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I+J)/J \cong I/(I \bigcap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V.

We call V a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned}
\text{ad } &: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \\
&x \mapsto [x, \cdot\,].
\end{aligned}$$

Corollary: Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof:

Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned}
Z(\mathfrak{g}) = \left\{ x \in \mathfrak{g} \;\middle|\; \text{ad }_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \right\} \\
= \ker \text{ad }_x.
\end{aligned}$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad } \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

3.2 Automorphisms

Definition: An automorphism of $\mathfrak{g}$ is an isomorphism $\mathfrak{g}\circlearrowleft$, and we define

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \phi : \mathfrak{g}\circlearrowleft \;\middle|\; \phi \text{ is an isomorphism } \right\}.$$

Proposition: If $\delta \in \mathrm{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) \coloneqq \sum \frac{\delta^n}{n!} \in \mathrm{Aut}(\mathfrak{g}).$$

This is well-defined because δ is nilpotent, and a binomial formula holds:

$$\frac{\delta^{n([x,y])}}{n!} = \sum_{i=0}^{n} [\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!}].$$

and for $n = 1, \delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x, y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\mathrm{ad}_{\ e}) \exp(\mathrm{ad}_{\ -f}) \exp(\mathrm{ad}_{\ e}) \in \mathrm{Aut}\mathfrak{g}.$$

where e, f are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of s is 2 and $W = \{1, s\}$.

4 Monday August 19

4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\cdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ is *solvable* if there is some n for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let $\mathfrak{g}$ be a Lie algebra, then

1. If $\mathfrak{g}$ is solvable, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also solvable.
2. If $I \trianglelefteq \mathfrak{g}$ and both I and $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\mathrm{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

4.2 Nilpotency

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\ &\cdots \\ \mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i]. \end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Proposition: If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}_{\,x} \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof:

This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \cdots [x_n, y]]] = 0$, and so $\mathrm{ad}_{\,x_1} \circ \mathrm{ad}_{\,x_2} \circ \cdots \mathrm{ad}_{\,x_n} = 0$ which implies that $\mathrm{ad}_{\,x}^{\,n} = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If $\mathrm{ad}_{\,x}$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while $\mathrm{ad}_{\,x}$ *is* nilpotent, which contradicts the above theorem.

Proof:

We'll first establish a lemma.

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V. If x is nilpotent as an endomorphism on V for all $x \in V$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma: Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases:

- Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$.
- Case $\dim g = 1$, left as an exercise.

Inductive step: Let A be a maximal proper subalgebra and define $\phi : A \to \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that ϕ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that $\operatorname{im} \phi \leq \mathfrak{gl}(\mathfrak{g}/a)$ is a Lie subalgebra, and $\dim \operatorname{im} \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \operatorname{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\operatorname{im} \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that A is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2 [y, y]$. The last term is zero, the middle two terms are in A, and because A is closed under the bracket, the first term is in A as well.

But then $A \oplus F_y$ is a larger subalgebra than A, which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in $A, A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in A.

Proof to be continued.

5 Wednesday August 21

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of V, constructing $\operatorname{im} \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus Fy$. Now consider

$$W = \left\{ v \in V \;\middle|\; Av = 0 \right\},$$

which is $\mathfrak{g}$-invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But V is killed by any element in A, and both of these terms are in A. In particular, the y appearing in Fy also satisfies $y \in W$. Consider $y|_W \in \operatorname{End}(w)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \operatorname{End}(V)$, which is true exactly because y is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \operatorname{End}(\mathfrak{g}/A)$ is nilpotent.

Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by a, $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since m_ℓ, m_r commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly ad $_a$. But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if ad $_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Proof:

By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to ad $g \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that ad $\mathfrak{g}v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension. Since ad $_x$ is nilpotent, so is ad $_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space. ■

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of F, then the matrices of $\mathfrak{g}$ with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on n, where the base case is easy. For the inductive step, we use the previous theorem to get a v_1 such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\overline{V} = F^n / F v_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \to \mathfrak{gl}(\overline{V})$ where $x \mapsto (\overline{y} \mapsto \overline{y(x)})$.

Then im $\phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \mathrm{End}(F^{n-1})$ is nilpotent, since x was nilpotent on the larger space. But (see notes) then x can be written as a strictly upper-triangular matrix.

5.1 Chapter 2: Semisimple Lie Algebras

We now assume char $F = 0$ and $\overline{F} = F$.

Theorem: If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V, then V contains a common eigenvector for a $x \in \mathfrak{g}$, i.e. a $\lambda : \mathfrak{g} \to F, x \mapsto \lambda(x)$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$.

Proof: We will use induction on the dimension of $\mathfrak{g}$. For the inductive step:

Claim 1: There is an ideal $A \trianglelefteq \mathfrak{g}$ such that $\mathfrak{g} = A \oplus Fy$ for some $y \neq 0$, so A is a subalgebra of a solvable Lie algebra $\mathfrak{g}$ and thus solvable itself. By hypothesis, we can produce a $w \in V \setminus \{0\}$, and thus a functional $\lambda : A \to F$ such that $aw = \lambda(a)w$ for all $a \in A$. So we define

$$V_\lambda = \left\{ v \in V \;\middle|\; av = \lambda(a)v \forall a \in A \right\}$$

where $w \in V_\lambda$.

Claim 2: $y(V_\lambda) \subseteq V_\lambda$, or $y|_{V_\lambda} \in \mathrm{End}(V_\lambda)$.

Thus $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$ is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a $v \in V_\lambda$ and some $\mu \in F$ such that $y(v) = \mu v$. An arbitrary element $x \in \mathfrak{g}$ can be written as $x = a + cy$ for some $a \in A, c \in F$ and it acts by $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c)v \in V_\lambda$.

6 Friday August 23

Chapter 3: Theorems of Lie and Cartan

6.1 4.1: Lie's Theorem

Theorem: Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, where V is finite-dimensional. If $V \neq 0$, then V contains a common eigenvector for all of the endomorphisms in L.

Proof:

Use induction on $\dim L$. The case $\dim L = 0$ is trivial. We'll attempt to mimic the proof of Theorem 3.3. The idea is to

1. Locate an ideal of K of codimension 1,
2. Show by induction that common eigenvectors exist for k,
3. Verify that L stabilizes a space consisting of such eigenvectors,
4. Find in that space an eigenvector for a single $z \in L$ satisfying $L = K + Fz$.

Step (1): Since L is solvable and of positive dimension, then $L \subsetneq [L, L]$. Otherwise, if $L = [L, L]$, then $L^{(1)} = L \implies L^{(n)} = L$, which would contradict L being solvable.

Since $[L, L]$ is abelian, any subspace is automatically an ideal. So take a subspace of codimension one, then its inverse image $K \trianglelefteq L$ is an ideal satisfying $[L, L] \subseteq K$.

Step (2): Use induction to find a common eigenvector $v \in V$ for K. (K is solvable; if $K = 0$ then L is abelian of dimension 1 and any eigenvector for a basis vector of L finishes the proof.)

This means that $x \in K \implies x \curvearrowright v = \lambda(x)v$ for some $\lambda : K \to F$ a linear functional. Fix this λ, and let $W = \left\{ w \in V \;\middle|\; x \curvearrowright w = \lambda(x)w \forall x \in K \right\}$; note that $W \neq 0$.

Step (3): This will involve showing that L leaves W invariant. Assume for the moment that this is done, and proceed to step (4).

Step (4): Write $L = K + Fz$. Since F is algebraically closed, we can find an eigenvector $v_0 \in W$ of z for some eigenvalue of z. Then v_0 is a common eigenvector for L, and λ can be extended to a linear function on L satisfying $x \curvearrowright v_0 = \lambda(x)v_0$ where $x \in L$.

It remains to show that L stabilizes W. Let $w \in W, x \in L$. To test whether or not $x \curvearrowright w \in W$, we take an arbitrary $y \in K$ and examine

$$yx \curvearrowright w = xy \curvearrowright w - [x, y] \curvearrowright w = \lambda(y)x \curvearrowright w - \lambda([x, y])w.$$

Note: the above equality is an important trick.

Thus we need to show that $\lambda([x, y]) = 0$. To this end, fix $w \in W, x \in L$. Let $n > 0$ be the smallest integer for which $w, x \curvearrowright w, \cdots x^n \curvearrowright w$ are all linearly *independent*. Let $W_i = \operatorname{span}(\left\{ w, x \curvearrowright w, \cdots x^{i-1} \curvearrowright w \right\})$ and set $W_0 = 0$. Then $\dim W_n = n$, and $W_{n+i} = W_n$ for all $i \geq 0$. Moreover, x maps W_n into itself. It is easy to check that each $y \in K$ is represented by an upper-triangular matrix with diagonal entries equal to $\lambda(y)$. This follows immediately from the congruence

$$yx^i \curvearrowright w = \lambda(y)x^i \curvearrowright w \mod W_i,$$

which can by proved by induction on i. The case $i = 0$ is trivial. For the inductive step, write

$$yx^i \curvearrowright i = yxx^{i-1} \curvearrowright w = xyx^{i-1} \curvearrowright w = [x,y]x^{i-1} \curvearrowright w$$

By induction,

$$yx^{i-1} = \lambda(y)x^{i-1} \curvearrowright w + w',$$

where $w' \in W_{i-1}$. Since x maps W_{i-1} into W_i by construction, the congruence holds for all i.

According to our description of the action of $y \in K$ on W_n, we have $\mathrm{Tr}_{W_n}(y) = n\lambda(y)$. In particular, this is true for elements k of f of the special form $[x,y]$ where x is as it was above and y is in K.

But both x and y stabilize W_n, so $[x,y]$ acts on W_n as the commutator of two endomorphisms of W_n, and the trace is therefore zero.

We conclude that $n\lambda([x,y]) = 0$. Since $\mathrm{char} F = 0$, this forces $\lambda([x,y]) = 0$ as required. ■

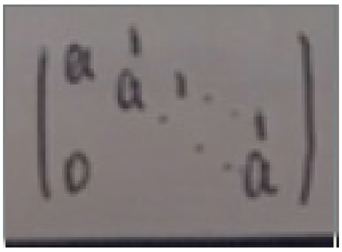
Corollary A (Lie's Theorem): Let $L \leq \mathfrak{gl}(V)$ be a solvable subalgebra where $\dim V = n < \infty$. Then L stabilizes some flag in V, i.e. the matrices of L relative to a suitable basis of V are upper triangular.

Proof: Use the above theorem, along with induction on $\dim V$. This is similar to the proof of corollary 3.3.

6.2 4.2: Jordan-Chevalley Decomposition

Fact 1:

The Jordan Canonical Form of a single endomorphism x over F algebraically closed is an expression of x in matrix form as a sum of blocks:



Fact 2:

Call $x \in \mathrm{End}V$ *semisimple* if the roots of its minimal polynomial over F are all distinct. Equivalently, if F is algebraically closed, then x is semisimple iff x is diagonalizable.

Fact 3:

Two commuting semisimple endomorphisms can be simultaneously diagonalized. Therefore, their sum or difference is again semisimple.

Proposition: Let V be a finite dimensional vector space over F and $x \in \mathrm{End}V$. Then

a. There exist unique $x_s, x_n \in \mathrm{End}V$ satisfying the conditions $x = x_s + x_n$, x_s is semisimple, x_n is nilpotent, and x_s, x_n commute.

b. There exists polynomials $p(t), g(t)$ such that $x_s = p(x)$ and $x_n = g(x)$. In particular, x_s, x_n commute with any endomorphism commuting with x.

c. If $A < B < V$ are subspaces and x maps B into A, then x_s, x_n also map B into A.

The decomposition $x = x_s + x_n$ is called the (additive) **Jordan-Chevalley decomposition** of x, or just the Jordan decomposition. x_s, x_n are respectively called the **semisimple part** and the **nilpotent part** of x.

Example:

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies x_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that $x_s x_n = x_n = x_n x_s$, $x_s = 2x - x^2$, and $x_n = x^2 - x$. We thus have $p(t) = 2t - t^2$ and $q(t) = t^2 - t$.

7 Monday August 26

Definition (Jordan Decomposition):

Let $X \in \mathrm{End}(V)$ for V finite dimensional. Then,

(a) There exists a unique $X_s, X_n \in \mathrm{End}(V)$ such that $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.

(b) There exists a $p(t), q(t) \in t\mathbb{F}[t]$ such that $X_s = p(X), X_n = q(X)$.

(Polynomials with no constant term.)

Proof of (a): Assume $X_s = X_s + X_n = X_s' + X_n'$, so both have bracket zero. Assuming that (b) holds, we have $X_s = p(X)$, and so

$$[X, X_s] = [X_s + X_n', X_s'] = [X_s', X_s'] + [X_s', X_n'] = 0 \implies [p(X), X_S'] = 0 = [X_s, X_s']$$

Using fact (c) from last time, then X_s, X_s' can be diagonalized simultaneously, and so $X_s - X_s'$ is semisimple.

On the other hand, if X_n', X_n are nilpotent, and since these commute, $X_n - X_n'$ is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X_s') + (X_n + X_n')$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let $m(t) = \prod_{i=1}^{r} (t - \lambda_i)^{m_i}$ be the minimal polynomial of X, where each $m_i \geq 1$ and the λ_i are distinct. Then the primary composition of V is given by

$$V = \bigoplus_{i=1}^{r} V_i, \quad V_i = \ker(X - \lambda_i I_V) \neq 0, \quad X(V_i) \subseteq V_i$$

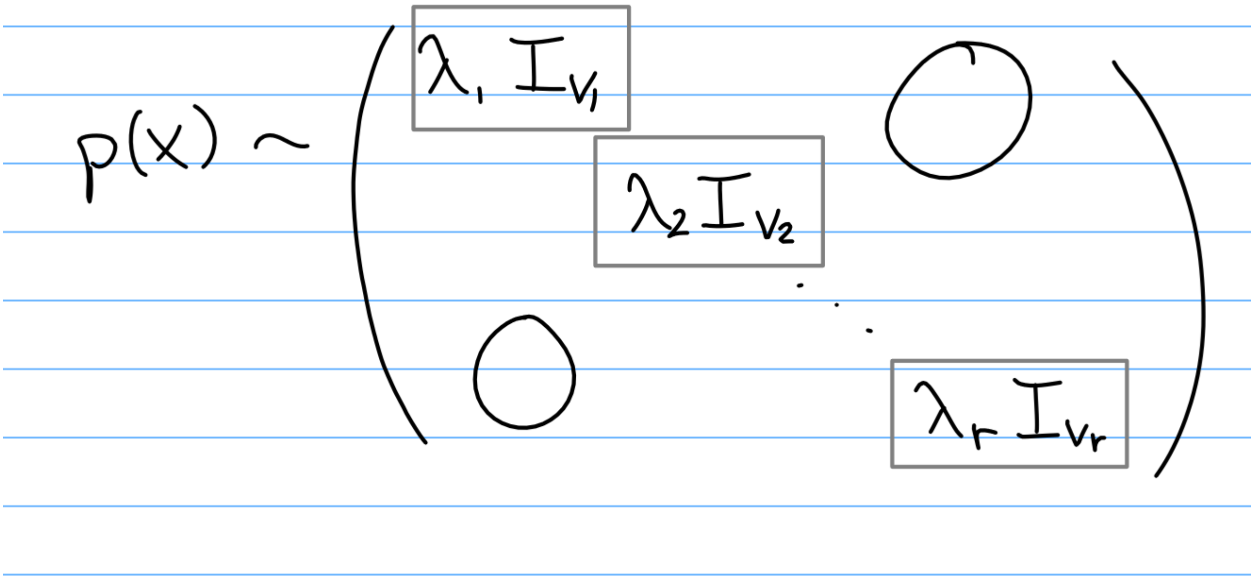


Figure 1: ???

Claim: There exists a polynomial $p \in F[t]$ such that

$$
\begin{aligned}
p &= \lambda \quad \text{mod } (t - \lambda_i)^{m_i} \quad \forall i, \\
p &= 0 \quad \text{mod } t.
\end{aligned}
$$

The existence follows from the Chinese Remainder Theorem.

What is $p(x) \curvearrowright V_i$? This acts by scalar multiplication by λ_i for all i. (Check). Because of the restrictive conditions, $p(x)$ has no constant term.

So $p(X) = X_s$ is the semisimple part we want. Now just set $q(t) = t - p(t)$, then $X_n := q(X) = X - X_s$ is nilpotent.

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have $X = X_s + X_n$, then $\text{ad}_X \in \text{End}(\text{End}(V))$. It can be shown that $(\text{ad}_X)_s + (\text{ad}_X)_n = \text{ad}(X_s) + \text{ad}(X_n)$.

Let e_{ii} be the elementary matrix with a 1 in the i, j position. You can write ad_X as a 4×4 matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)$$

$$
= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array}
\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)
$$

You can check that $(\text{ad }_X)_S = 0, \text{ad }(X_s) = 0$, and $(\text{ad }X)_n$ is the Jordan form given above.

Lemma:

(a) $x \in \text{End}(V) \implies \text{ad }(x)_s = \text{ad }(x_s)$ and $\text{ad }(x)_n = \text{ad }(x_n)$.

(b) If A is a finite dimensional $\mathbb{F}$-algebra, then $\delta \in \text{Der}(A) \implies \delta_s, \delta_n \in \text{Der}(A)$ as well.

Proof of (a):

Check that $\text{ad }(x) = \text{ad }(x_s) + \text{ad }(x_n)$. Then for $y \in \text{End}(V)$, we have

$$
\begin{aligned}
(\text{ad }(x))(y) &= [x, y] \\
&= [x_s + x_n, y] \\
&= [x_s, y] + [x_n, y] \\
&- (\text{ad }(x_s))(y) + (\text{ad }(x_n))(y).
\end{aligned}
$$

Using theorem 3.3, x_n nilpotent $\implies$ $\text{ad }(x_n)$ is also nilpotent. So write $x_s = \sum \lambda_i e_{ii}$ with the eigenvalues on the diagonal. Then $\text{ad }x_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$ for all i, j. But then $\text{ad }x_s$ is given by a matrix with $\lambda_i - \lambda_j$ in the i, j position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

$$(\delta - (\lambda+\mu)I)^n ([x,y])$$

$$= \sum_{i=0}^{n} \binom{n}{i} \left[(\delta - \lambda I)^i (x), (\delta - \mu I)^{n-i} (y) \right]$$

Figure 2: Image

Proof of (b):

Since $\delta \in \mathrm{Der}(A)$, the primary decomposition with respect to δ is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \left\{ a \in A \;\middle|\; (\delta - \lambda I)^k a = 0 \text{ for some } k >> 0 \right\}.$$

So $\delta_s \curvearrowright A_\lambda$ by scalar multiplication (by λ). Then for $\lambda, \mu \in \mathbb{F}$, we have

So $[A_x, A_y] \subseteq A_{\lambda+\mu}$ for all $x, y \in A$. But then

and so $\delta_s \in \mathrm{Der}(A)$, and $\delta_n = \delta - \delta_s \in \mathrm{Der}(A)$ as well.

8 Wednesday August 28

Todo

9 Friday August 30

Review of bilinear forms: let $V = \mathbb{F}^n$.

Definition: A bilinear form $\beta : V^2 \to \mathbb{F}$ can be represented by a matrix B with respect to a basis $\{\mathbf{v}_i\}$ such that

$$\beta\beta(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i) = (a_1 \ a_2 \ \cdots) B (b_1 \ b_2 \ \cdots)$$

- β is *symmetric* iff $\beta(a,b) = \beta(b,a)$.
- β is *symplectic* iff $\beta(a,b) = -\beta(b,a)$.
- β is *isotropic* iff $\beta(a,a) = 0$.

For a subspace $U \leq V$, define

$$U^\perp := \left\{ \mathbf{v} \in V \;\middle|\; \beta(\mathbf{u}, \mathbf{v}) = \mathbf{0} \ \forall u \in U \right\}.$$

$$\delta_s([x,y])$$

$$\| $$

$$(\lambda+\mu)[x,y] = [\lambda x, y] + [x, \mu y]$$

$$\|$$

$$[\delta_s(x), y] + [x, \delta_s(y)]$$

Figure 3: Image

> Note: in general, left/right orthogonality are distinguished, but these will be identical when β is symmetric/symplectic.

The form β is said to be *non-degenerate* iff $V^{\perp} = 0$ iff $\det B \neq 0$.

Assume F is an algebraically closed field, so $\overline{F} = F$, and $\mathrm{char} F \neq 2$, then

- If β is non-degenerate and symmetric, then $B \sim I_n$
- If β is non-degenerate and symplectic, then $B \sim [0, I_{n/2}; I_{n/2}, 0]$.

Remark: $\mathfrak{so}(n, \mathbb{F}) = \left\{ x \in \mathfrak{gl}(n, F) \;\middle|\; \beta(x(u), v) = -\beta(u, x(v)) \right\}$, where B has the matrix $[0, I; I, 0]$ if n is odd, or this matrix with a 1 in the top-left corner if n is odd.

Similarly, $\mathfrak{sp}(2m, \mathbb{F})$ can be described this way with the matrix $[0, -I_m; -I_m, 0]$.

Overview: The killing form is defined as $\kappa : \mathfrak{g}^2 \to \mathbb{F}$ where $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\,_x \circ \mathrm{ad}\,_y)$.

Then we have **Cartan's Criteria**:

- $\mathfrak{g}$ solvable $\iff$ $\kappa(x, y) = 0 \forall x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$.
- $\mathfrak{g}$ semisimple $\iff$ κ is non-degenerate.

Note that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \bigoplus_i I_i$ with each $I_i \trianglelefteq \mathfrak{g}$ and simple.

9.1 Cartan's Criteria

Some facts:

1. κ is symmetric
2. If $\mathfrak{g}$ is finite dimensional, then κ is associative, i.e $\kappa([x, y], z) = \kappa(x, [y, z])$.

Exercise: Show that if $I \trianglelefteq \mathfrak{g}$, then $I^\perp \leq \mathfrak{g}$ is an ideal.

Proof of (2): In section 4.3, it was shown that $\operatorname{tr}([a, b] \circ c) = \operatorname{tr}(a \circ [b, c])$ for all $a, b, c \in \operatorname{End}(V)$ (provided V is finite dimensional).

So [

$$\begin{aligned} \kappa([x, y], z) &= \operatorname{tr}(\operatorname{ad}_{[x,y]} \circ \operatorname{ad}_z) \\ &= \operatorname{tr}([\operatorname{ad}_x, \operatorname{ad}_y] \circ \operatorname{ad}_z) \\ &= \operatorname{tr}(\operatorname{ad}_x \circ [\operatorname{ad}_y, \operatorname{ad}_z]) \\ &= \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_{[y,z]}) \\ &= \operatorname{tr}(x, [y, z]).. \end{aligned}$$

]

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Proof:

$\implies$: We want to show that $\mathfrak{g}^\perp = 0$. Note that $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] \subseteq \mathfrak{g}$, and so for all $x \in [\mathfrak{g}^\perp, \mathfrak{g}^\perp]$ and for any $y \in \mathfrak{g}^\perp$, we have

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0$$

by the const(?) of $\mathfrak{g}^\perp$. This implies $\mathfrak{g}^\perp$ is solvable.

Using fact (2), we have $\mathfrak{g}^\perp \trianglelefteq \mathfrak{g}$ and thus $\mathfrak{g}^\perp \subseteq \operatorname{rad}(\mathfrak{g})$, which is 0 since because $\mathfrak{g}$ is semisimple. So either $\mathfrak{g}^\perp = 0$ or κ is nondegenerate.

Used the fact that the radical was a maximal solvable ideal.

$\impliedby$: We want to show that for all $I \trianglelefteq \mathfrak{g}$ where $[I, I] = 0$, we have $I^\perp \subseteq \mathfrak{g}^\perp$.

For $x \in I, y \in \mathfrak{g}$, we have

$$(\operatorname{ad}_x \circ \operatorname{ad}_y)^2 = \mathfrak{g} \xrightarrow{\operatorname{ad}_y} \mathfrak{g} \xrightarrow{\operatorname{ad}_x} I \xrightarrow{\operatorname{ad}_y} I \xrightarrow{\operatorname{ad}_x} 0$$

And thus $\operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0$ and $I \subseteq \mathfrak{g}^\perp$.

Suppose that $\mathfrak{g}$ is *not* semisimple. Then there exists a solvable ideal $J \neq 0$ such that the last term J^i in the derived series is an ideal $I \trianglelefteq \mathfrak{g}$ such that $[I, I] = 0$, forcing $J^i \subset \mathfrak{g}^\perp = 0$, which is a contradiction.

9.2 Section 5.2

Theorem: If $\mathfrak{g}$ is semisimple, then

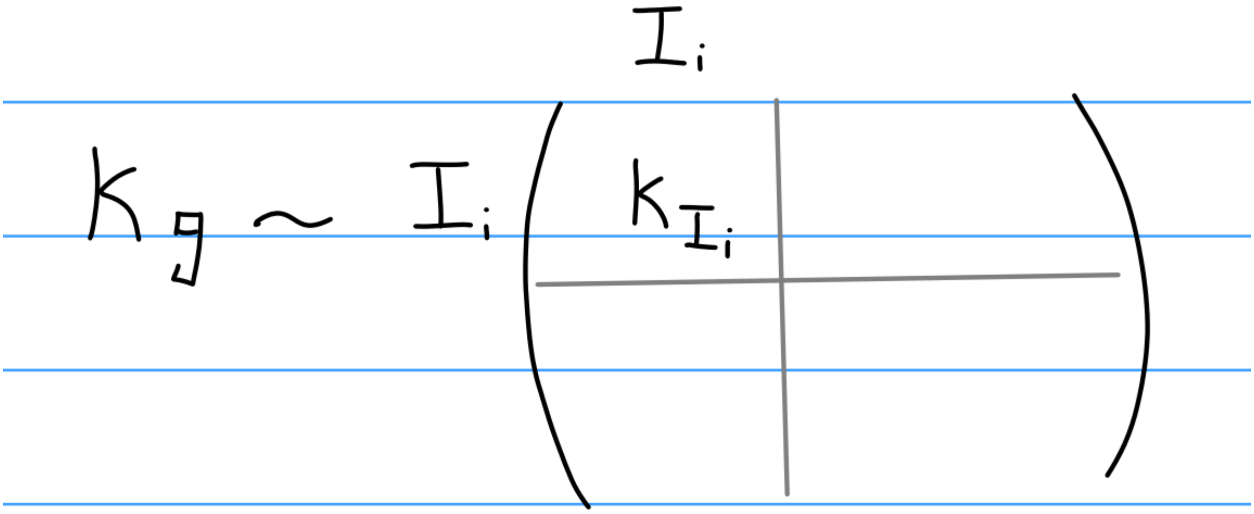


Figure 4: Image

a. There exist ideals $I_i \trianglelefteq \mathfrak{g}$ which are simple Lie algebras satisfying $\mathfrak{g} = \bigoplus I_i$. Note that $[I_i, I_j] \subseteq I_i \bigcap I_j = 0$, since direct summands intersect only trivially.

b. Every simple $I \trianglelefteq \mathfrak{g}$ is one of these I_i.

c. $\kappa_{I_i} = \kappa_{\mathfrak{g}}\big|_{I_i \times I_i}$, so

Remark: $\mathfrak{g}$ is semisimple $\iff$ $\mathfrak{g} = \bigoplus_i I_i$ for some simple Lie algebras I_i.

$\impliedby$: For all $i, S \coloneqq \mathrm{rad}\mathfrak{g}, I_i \trianglelefteq I_i$ is a solvable ideal. This implies that it is 0, since I_i is simple.

By definition, simple Lie algebras are not abelian.

Supposing that $S = I_i$, we would then have $[S.S] \neq 0$ since $[I_i, I_i] \neq 0$ by definition. But $[S, S] \neq S$ because S is solvable, which says that S is not simple (a contradiction).

Note that $[\mathrm{rad}\mathfrak{g}, \mathfrak{g}] \subseteq \bigoplus [\mathrm{rad}\mathfrak{g}, I_i] = 0$, which forces $\mathrm{rad}\mathfrak{g} \subseteq Z(\mathfrak{g})$. Since I_i is simple, $Z(I_i) = 0$ for all i. But $Z(\mathfrak{g}) = \bigoplus Z(I_i) = 0$, and this forces $\mathrm{rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g}) \implies \mathrm{rad}\mathfrak{g} = 0$. So $\mathfrak{g}$ is semisimple.

Next time – starting the representation theory with $\mathfrak{sl}(2, \mathbb{F})$.

10 Monday September 2

Recall the killing form: [

$$\begin{aligned} \kappa : \mathfrak{g}^2 &\to \mathbb{F} \\ (x, y) &\mapsto \mathrm{tr}(\mathrm{ad}_{\ x} \circ \mathrm{ad}_{\ y}). \end{aligned}$$

]

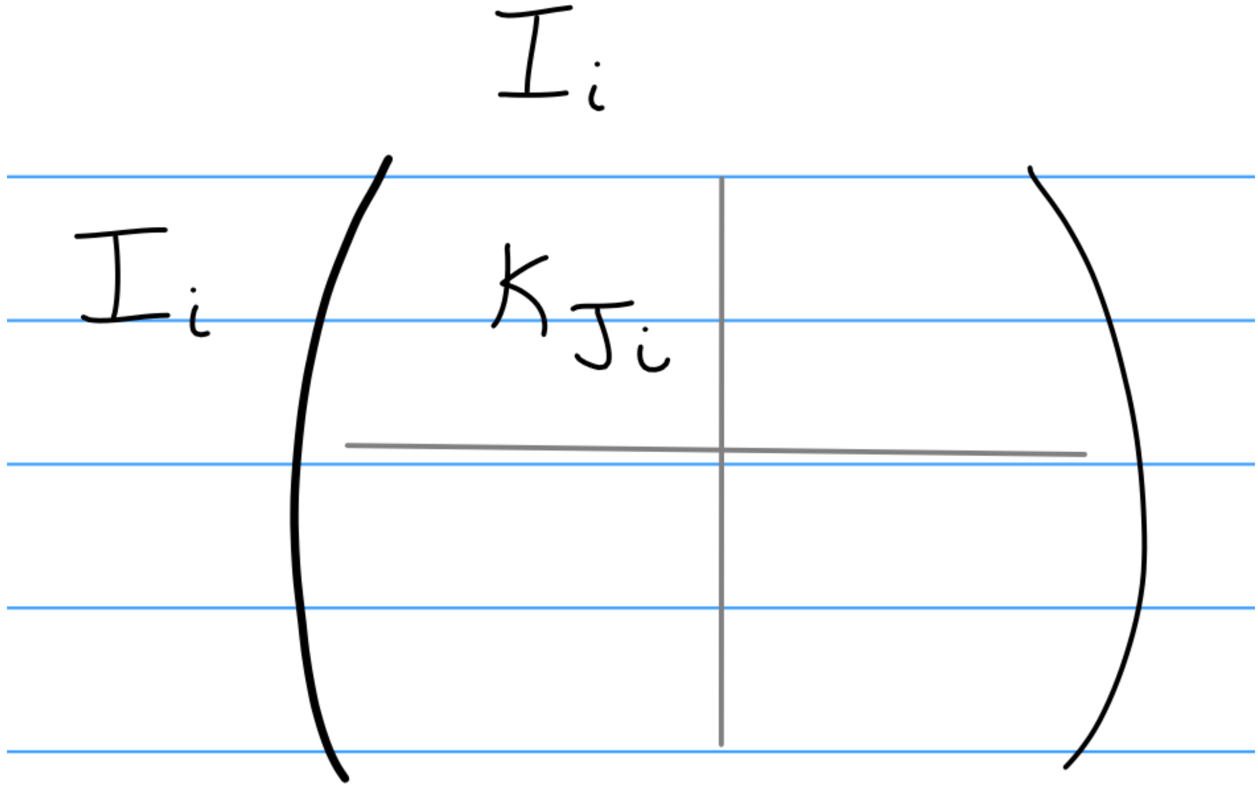


Figure 5: Image

and Cartan's criteria:

1. $\mathfrak{g}$ is solvable $\iff$ $\kappa(x,y) = 0\ \forall x \in [\mathfrak{g},\mathfrak{g}],\ y \in \mathfrak{g}$.
2. $\mathfrak{g}$ is semisimple $\iff$ κ is non-degenerate.

Theorem: If $\mathfrak{g}$ is semisimple, then

a. $\mathfrak{g} = \bigoplus_{i=1}^{n} I_i$ for some $I_i \trianglelefteq \mathfrak{g}$ which are all simple.
b. Every simple ideal $I \trianglelefteq \mathfrak{g}$ is one of the I_i.
c. $\kappa_{I_i} = \kappa_{\mathfrak{g}}\Big|_{I_i \times I_i}$.

Proof of (a): Use induction on $\dim \mathfrak{g}$. If $\mathfrak{g}$ has no nonzero proper ideals, then $\mathfrak{g}$ is simple and we're done.

Otherwise, let I_1 be a minimal nonzero ideal of $\mathfrak{g}$. Then $I_1^\perp \trianglelefteq \mathfrak{g}$ is also an ideal, and thus $I := I_1 \bigcap I_1^\perp \trianglelefteq \mathfrak{g}$ is as well. Then for all $x \in [I, I]$, we must have $\kappa(x,y) = 0$ for any $y \in I \subseteq I_1^\perp$. So I is solvable, and thus $I = 0$. So $\mathfrak{g} = I_1 \oplus I_1^\perp$.

Note that any ideal of $I_1^\perp$ is also an ideal of $\mathfrak{g}$, which implies that $\operatorname{rad}(I_1^\perp) \subseteq \operatorname{rad}(\mathfrak{g})$, which is zero since $\mathfrak{g}$ is semisimple, and thus $I_1^\perp$ is semisimple as well.

$$\operatorname{ad} x \sim \left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right) \qquad \kappa_{\mathfrak{g}}(x,y) = \operatorname{tr}\left(\left(\begin{array}{c|c} A_x & B_x \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right)\right)$$

$$\operatorname{ad} y \sim \left(\begin{array}{c|c} A_y & B_y \\ \hline 0 & 0 \end{array}\right) \qquad = \operatorname{tr}\left(\begin{array}{c|c} A_x A_y & B_x B_y \\ \hline 0 & 0 \end{array}\right)$$

$$= \operatorname{tr}(A_x A_y)$$

$$= \kappa_{I_i}(x,y)$$

Figure 6: Image

By the inductive hypothesis, $I_1^\perp = I_2 \oplus \cdots \oplus I_n$ where each $I_j \trianglelefteq I_i^\perp$ is simple. Then $I_j \trianglelefteq \mathfrak{g} \implies [I_1, I_j] \subset I_1 \bigcap I_j$, since I_1 has no contribution. But this is a subset of $I_1 \bigcap I_1^\perp = 0$. ■

Proof of (b): If $I \trianglelefteq \mathfrak{g}$, then $[I, \mathfrak{g}] \trianglelefteq I$ because $[[I, \mathfrak{g}], I] \subseteq [I, I] \subseteq [I, \mathfrak{g}]$.

Since $\mathfrak{g}$ is semisimple, $0 = \operatorname{rad}(\mathfrak{g}) \supseteq Z(\mathfrak{g})$. So $[I, \mathfrak{g}] \neq 0$, and thus $[I, \mathfrak{g}] = I$ since I is simple. But then $[I, \mathfrak{g}] = \bigoplus [I, I_i]$ is simple as well. So only one direct summand can survive, since otherwise this would produce at least 2 nontrivial ideals, and $[I, \mathfrak{g}] = [I, I_i]$ for some i.

So for all $j \neq i$, we must have $I_j \bigcap I = I_j \bigcap [I, I_i] = 0$, and so $I \subseteq I_i$. But then $I = I_i$ since I_i itself is simple, and we're done.

Proof of (c):

(Without using the simplicity of I_i)

For $x, y \in I_i$, we have

10.1 Inner Derivations

Recall that $\operatorname{ad} \mathfrak{g} \subseteq \operatorname{Der}\mathfrak{g}$, and in fact (lemma) this is an ideal.

Theorem: If $\mathfrak{g}$ is semisimple, then $\operatorname{ad} \mathfrak{g} = \operatorname{Der}\mathfrak{g}$.

Proof of lemma:

For all $\delta \in \operatorname{Der}\mathfrak{g}$ and all $x, y \in \mathfrak{g}$, we have

[

$$\begin{aligned}[][\delta, \mathrm{ad}_{\ x}](y) &= \delta([x,y]) - [x, \delta(y)] \\ &= [\delta(x), y] \\ &= [\mathrm{ad}_{\ \delta(x)}](y),\end{aligned}$$

]

and so $[\delta, \mathrm{ad}\ x] \subseteq \mathrm{ad}\ \mathfrak{g}$. ■

Proof of theorem:

If $\mathfrak{g}$ is semisimple, then $0 = \mathrm{rad}\mathfrak{g} \supseteq Z(\mathfrak{g}) = \ker \mathrm{ad}$. Thus $\mathrm{ad}\ \mathfrak{g} \cong g/\ker \mathrm{ad}\ \cong \mathfrak{g}$ is also semisimple.

This means that $\kappa_{\mathrm{ad}\ \mathfrak{g}}$ is non-degenerate, and thus $\mathrm{ad}\ \mathfrak{g} \bigcap (\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, where $(\mathrm{ad}\ \mathfrak{g})^{\perp} \trianglelefteq \mathrm{Der}(\mathfrak{g})$.

(Note that the non-degeneracy of κ already forces $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$.)

Then $[(\mathrm{ad}\ \mathfrak{g})^{\perp}, \mathrm{ad}\ \mathfrak{g}] = 0$, and so for all $\delta \in (\mathrm{ad}\ \mathfrak{g})^{\perp}$, we have $\delta(x) = [\delta, \mathrm{ad}\ x]$ by the lemma, but we've shown that this is zero.

But then δ must be zero because ad is an isomorphism, and in particular it is injective. This means that $(\mathrm{ad}\ \mathfrak{g})^{\perp} = 0$, and thus $\mathrm{ad}\ \mathfrak{g} = \mathfrak{g}$. ■

We can use this to define an abstract Jordan decomposition by pulling back decompositions on adjoints:

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11.1 4.3: Cartan's Criterion

Lemma: Let $A \subset B$ be two subspaces of $\mathfrak{gl}(V)$ where $\dim V < \infty$. Set $M = \left\{ x \in \mathfrak{gl}(V) \;\middle|\; [x, B] \subset A \right\}$. Suppose that $x \in M$ satisfies $\mathrm{Tr}(xy) = 0$ for all $y \in M$. Then x is nilpotent.

Proof: Let $x = s + n$ (where $s = x_s$ and $n = x_n$) be be the Jordan decomposition of x. Fix a basis $v_1 \cdots v_m$ of V relative to which s has matrix $\mathrm{diag}(a_1 \cdots a_m)$. Let E be the vector subspace of F over the prime field Q spanned by the eigenvalues $a_1 \cdots a_m$. We have to show that $s = 0$, or equivalently that $E = 0$, since E has finite Q-dimension by construction. It will suffice to show that the dual space $E^{\vee}$ is 0, i.e. that any linear functional $f : E \to Q$ is zero.

Given f, let y be the element of $\mathfrak{gl}(V)$ whose matrix is given by $\mathrm{diag}(\mathrm{f(a_1)}, \cdots \mathrm{f(a_m)})$. If $\{e_{ij}\}$ is a basis of $\mathfrak{gl}(V)$, then $\mathrm{ad}_{\ s}(e_{ij}) = (a_i - a_j)e_{ij}$ and $\mathrm{ad}_{\ y}(e_{ij}) = (f(a_i) - f(a_j))e_{ij}$.

Now let $r(t) \in F[t]$ be a polynomial with no constant term, satisfying $r(a_i - a_j) = f(a_i) - f(a_j)$ for all pairs i, j. The existence of such $r(t)$ follows from Lagrange interpolation, and the fact that if $a_i = a_j$ then $0 = r(a_j) - r(a_i) = r(a_i - a_j) = r(0)$, so r has no constant term. Thus there is no ambiguity in the assigned values, since $a_i - a_j = a_j - a_l$ would imply (by linearity of f) that $f(a_i) - f(a_j) = f(a_k) - f(a_l)$. Thus $\mathrm{ad}_{\ y} = r(\mathrm{ad}_{\ s})$.

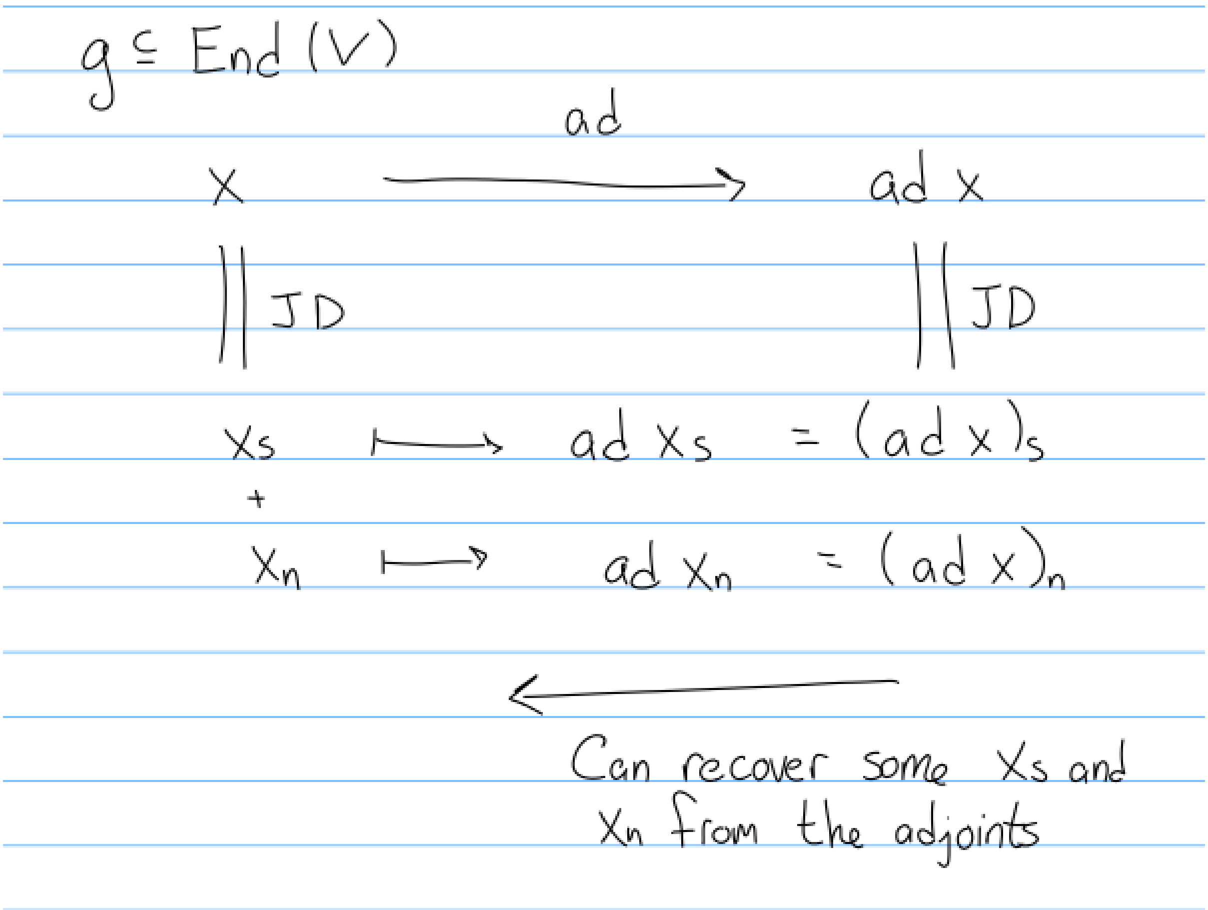

Figure 7: Image

Note that Lagrange Interpolation is a special case of the Chinese Remainder Theorem for polynomials. If all x_is are distinct, then $p_i(x) = x - x_i$ are all pairwise coprime. Then dividing $\dfrac{p(x)}{p_i(x)} = p(x_i)$. So letting $A_1 \cdots A_k$ be constants in k, there is a unique polynomial of degree less than k such that $p(x_i) = A_i$. Thus there is a polynomial $p(x)$ such that $p(x) = A_i \mod p_i(x)$, and $p(x_i) = A_i$.

Now ad $_s$ is the semisimple part of ad $_x$. By lemma A of 4.2, ad $_s$ can be written as a polynomial in ad $_x$ without a constant term. Therefore ad $_y$ is also a polynomial in ad $_x$ without constant term. By hypothesis, ad $_x$ maps B into A, so we have ad $_y(B) \subset A$, and so $y \in M$. Using the hypothesis of the lemma, $\mathrm{Tr}(xy) = 0$, and so $\sum a_i f(a_i) = 0$. The left side is a Q-linear combination of elements of E. Applying f, we obtain $\sum f(a_i)^2 = 0$. But the numbers $f(a_i)$ are rational, so this forces all of them to be zero. Finally, f must be identically 0 because the a_i span E. ■

Note that $\mathrm{Tr}([x,y]z) = \mathrm{Tr}(x[y,z])$. To verify this, write $[x,y]z = xyz - yxz$ and $x[y,z] = xyz - xzy$, then use the fact that $\mathrm{Tr}(y(xz)) = \mathrm{Tr}((xz)y)$.

Theorem (Cartan's Criterion): Let $L \leq \mathfrak{gl}(V)$ be a subalgebra with V finite dimensional. Suppose $\mathrm{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. Then L is solvable.

Proof: It suffices to show that $[L,L]$ is nilpotent, or just that all $x \in [L,L]$ are nilpotent endomorphisms. We apply the above lemma, with V as given, $A = [L,L]$, and $B = L$, so $M = \left\{x \in \mathfrak{gl}(V) \;\middle|\; [x,L] \subset [L,L]\right\}$. We have $L \subset M$. Our hypothesis is that $\mathrm{Tr}(xy) = 0$ for all $x \in [L,L]$ and $y \in L$. To use the lemma to reach the desired conclusion, we need a stronger result: that $\mathrm{Tr}(xy) = 0$ for $x \in [L,L]$ and $y \in M$.

If $[x,y]$ is a generator of $[L,L]$ and $z \in M$, then $\mathrm{Tr}([x,y]z) = \mathrm{Tr}(x[y,z]) = \mathrm{Tr}([y,z]x)$. By definition of M, $[y,z] \in [L,L]$, so the right side is 0 by hypothesis.

Corollary: Let L be a Lie algebraic such that $\mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = 0$ for all $x \in [L,L], y \in L$. Then L is solvable.

Proof: Apply the theorem to the adjoint representation of L. We then get ad L is solvable. Since $\ker \mathrm{ad} = Z(L)$ is also solvable, L itself is solvable.

11.2 Killing Form

11.2.1 Criterion for Semisimplicity

Let L be any lie algebra. If $x, y \in L$, then define $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. Then k is a symmetric bilinear form on L, called the **killing form**.

Theorem: $\mathfrak{g}$ is solvable $\iff \kappa(x,y) = 0$ for all $x \in [\mathfrak{g},\mathfrak{g}], y \in \mathfrak{g}$.

Proof: $\Longleftarrow$: By Cartan's Criterion.

$\Longrightarrow$: Exercise.

Example: The killing form of $\mathfrak{sl}(2,F)$.

We have [

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

]

Then ad $_h = \text{diag}(2, 0, -2)$, and [

$$\text{ad }_x = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\text{ad }_y == \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

]

and thus k has the matrix [

$$\begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

]

where $k_{ij} = \kappa(x_i, x_j)$ where x_i is a basis of L.

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Theorem: If L is semisimple and $x \in L$, there exists a unique x_s, x_n in L such that $x = x_s + x_n$, $[x_n, x_s] = 0$, ad x_s is semisimple, and ad x_n is nilpotent.

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Todo

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Let $S = \exp(\text{ad } e) \circ \exp(\text{ad } -f) \circ \exp(\text{ad } ei)$, which has the following matrix:

Where $\exp(\text{ad } e) = 1 + \text{ad } e + \frac{1}{2}(\text{ad } e)^2$, which would have the form

Theorem: If $\mathfrak{g}$ is semisimple, then any finite dimensional $\mathfrak{g}$-module V is completely reducible, i.e. it splits into a direct sum of simple modules.

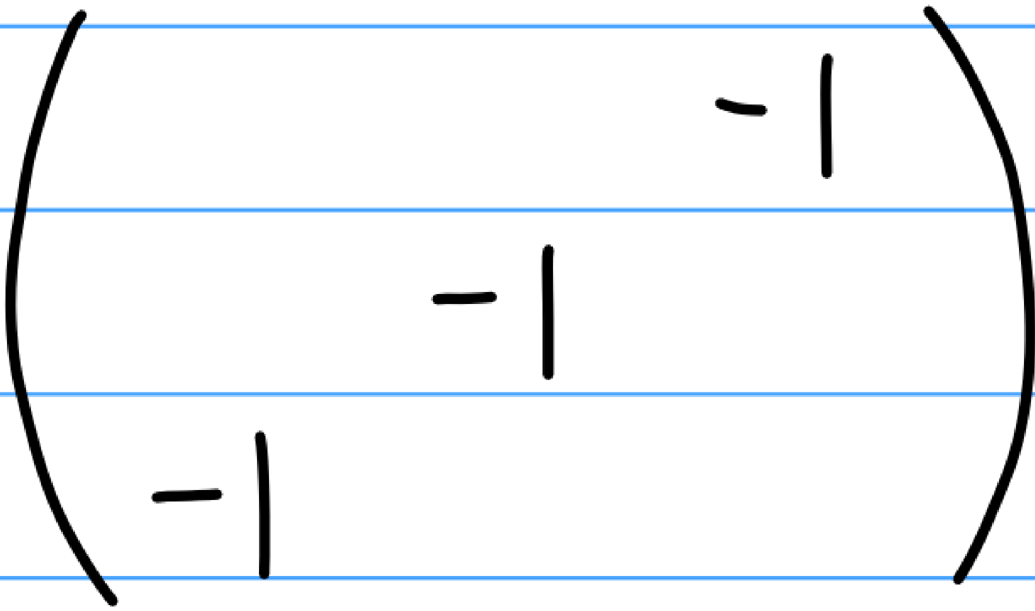

Figure 8: Image

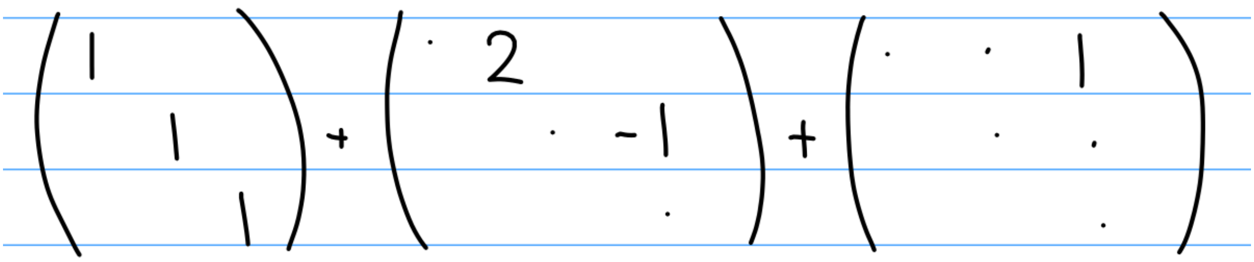

Figure 9: Image

14.1 Proof of Weyl's(?) Theorem

If V itself is simple, then we're done, so suppose it is not.

Assume there exists a nonzero submodule $U \subsetneq V$. It suffices to show that $V = U \oplus U'$ for some U'.

14.1.1 Step 1:

If $\dim V = 2$ and $\dim U = 1$.

Then $U, V/U$ are both trivial modules. So $g \curvearrowright u = 0$ for all $u \in U$. But then $g \curvearrowright (v + U) = U$ for all $v \in V$, since $g \curvearrowright v \in U$.

So for all $x, y \in lieg$ and all $v \in V$, we have $[x, y] \curvearrowright v = x \curvearrowright (y \curvearrowright v) - y \curvearrowright (x \curvearrowright v)$. But both of the terms in parenthesis are in U, and all elements in $\mathfrak{g}$ kill elements in U, so this is zero. So $[\mathfrak{g}, \mathfrak{g}] \curvearrowright V$ trivially.

Exercise: If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

So $\mathfrak{g} \curvearrowright V$ trivially. Thus any U' that is a complementary subspace of U will be a submodule of V.

14.1.2 Step 2:

Suppose U is simple and $\dim U > 1$, so $\dim V/U = 1$.

Let Ω be the Casimir element on U (faithful representation?). Then $\Omega u = cu$ for some $c \in \mathbb{F}$, and so $\Omega(U) \subseteq U$.

Since $\Omega : V \circlearrowleft$ is a homomorphism, $\ker \Omega \subseteq V$ is a $\mathfrak{g}$-submodule. Then $\dim V/U = 1 \implies V/U$ is a trivial module. So $\mathfrak{g} \curvearrowright V/U = 0$, i.e. $\mathfrak{g} \curvearrowright V \subseteq U$.

Then $\Omega(v) = \sum_i x_i \curvearrowright (y_i \curvearrowright v) \in U$ for all $v \in V$. What is the matrix of Ω?

In particular, $\mathrm{Tr}(\Omega \big|_{V/U}) = 0$. So $\mathrm{Tr}(\Omega) = \mathrm{Tr}(\Omega \big|_{U})$. From 6.2, we know that $\mathrm{Tr}(\Omega) \neq 0 \implies c \neq 0$, where c is the scalar appearing above. So $\ker \Omega$ is 1-dimensional, and $\ker \Omega \bigcap U = \{0\}$.

So take $U' = \ker \Omega$.

14.1.3 Step 3:

Suppose U is *not* simple, but $\dim V/U = 1$.

We will induct on the dimension of U. Pick a proper nonzero submodule $\overline{U} \subsetneq U$, so that $\dim U/\overline{U} < \dim U$. Now $V/U \cong (V/\overline{U})/(U/\overline{U})$ by an isomorphism theorem. So $U/\overline{U}$ is a submodule of $V/\overline{U}$ of codimension 1. Applying the inductive hypothesis, we obtain $V/\overline{U} = U/\overline{U} \oplus \overline{V}/\overline{U}$ for some $\overline{V}$ such that $U \subseteq \overline{V} \subseteq V$.

In particular, since $U \subseteq \overline{V}$ has codimension 1, $\dim \overline{U} < \dim U$. So apply the inductive hypothesis again: $\overline{V} = \overline{U} \oplus U'$ for some U', and $V = U \oplus U'$.

$$\begin{array}{cc} & \begin{array}{cc} \quad U \quad & \quad V/U \end{array} \\ \begin{array}{c} U \\ \\ V/U \end{array} & \left(\begin{array}{c|c} cI & * \\ & \\ \hline 0 \cdots\cdots 0 & 0 \end{array}\right) \end{array}$$

Figure 10: Image

14.1.4 Step 4: The general case

Recall that $\hom(V, U)$ is a $\mathfrak{g}$-module where [

$$(g \curvearrowright \phi)(v) = g \curvearrowright \phi(v) - \phi(g \curvearrowright v).$$

]

Define

$$S = \left\{ \phi \in \hom(V, U) \;\middle|\; \phi \Big|_U \in F1_U \right\}.$$

Then $S \leq \hom(V, U)$ as a submodule. Define $T = \left\{ \phi \in S \;\middle|\; \phi \Big|_U = 0 \right\}$. Then $T \leq S$ as a submodule, and $\mathfrak{g}(S) \subseteq T$.

Now each $\phi \in S$is determined ($\mod T$) by the scalar $\phi \Big|_U$. Note that $\dim(S/T) = 1$. By steps 1-3, we know that $S = T \oplus T'$ for some $T' \subseteq S$ of dimension 1. Then $T' = \operatorname{span}_{\mathbb{F}}(f)$ for some nonzero map $f : V \to U$ such that $f(u) = cu$ for some $c \neq 0$.

Then $\mathfrak{g}(T \oplus T') = \mathfrak{g}(S) \subseteq T \implies \mathfrak{g}(T') = 0$. So for all $g \in \mathfrak{g}$, we have $0 = (g \curvearrowright f)(v) = f \curvearrowright f(v) - f(g \curvearrowright v)$. Then $f : V \to U$ is a lie algebra homomorphism, $\ker f = U'$, and thus $V = U \oplus U'$. ■

Some consequences of Weyl's theorem:

14.2 Preservation of Jordan Decomposition

Recall that when $\mathfrak{g} \in \mathfrak{gl}(V)$ is a linear lie algebra, then for $x \in \mathfrak{g}$ we have:

Jordan Decomposition: $x = x_s + x_n$ where $x_s, x_n \in \operatorname{End}(V)$.

Abstract Jordan Decomposition: [

$$\begin{aligned} \mathfrak{g} &\xrightarrow{\text{ad}} \text{ad }(\mathfrak{g}) \\ x &\mapsto \text{ad } x \\ x_s &\leftarrow (\text{ad } x)_s \\ x_n &\leftarrow (\text{ad } x)_n. \end{aligned}$$

] and so $x = x'_s + x'_n$ for some x'. The theorem will be that these recover the usual Jordan decomposition.

Theorem: If $\mathfrak{g} \in \mathfrak{gl}(V)$ is semisimple and V is finite dimensional, then $x_s, x_n \in \mathfrak{g}$, and $x_s = x'_s, x'_n$.

Corollary: If $\mathfrak{g}$ is semisimple and finite dimensional and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation, then if $x = x_s + x_n$ is the abstract Jordan decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the Jordan decomposition in $\mathfrak{gl}(V)$.

Example: If $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is semisimple and finite dimensional, and h is diagonal, then by JD $h = h + 0, \phi(h) = \phi(h) + 0$. Then $h \curvearrowright V$ semisimply, or $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, where $V_\lambda = \left\{ v \in V \;\middle|\; h \curvearrowright v = \lambda v \right\}$ are the eigenspaces.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Figure 11: Image

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Last time: The abstract Jordan Decomposition coincides with the actual Jordan Decomposition. [

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x \mapsto \phi(x) = \phi(x)_s + \phi(x)_n &= \phi(x_n) + \phi(x_s) \\ x_s + x_n &\mapsto \phi(x_s) + \phi(x_n). \end{aligned}$$

]

Therefore $x_s \curvearrowright V$ semisimply. The example we saw last time was $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, with a matrix $h = [1, 0; 0, -1]$ and $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$.

15.1 Finite Dimensional Representations of $\mathfrak{sl}(2, \mathbb{C})$

15.1.1 Weights and Maximal Vectors

Definition: If $V_\lambda \neq 0$, then V_λ is a *weight space* of V and $\lambda \in \mathbb{C}$ is a *weight* of h in V. We then define $W_t(V) = \{\text{weights in } V\}$.

Lemma: If $v \in V_\lambda$ then $e \curvearrowright v \in V_{\lambda+2}$ and $f \curvearrowright v \in V_{\lambda-2}$.

Proof: [

$$\begin{aligned} h \curvearrowright (e \curvearrowright v) &= [h, e] \curvearrowright v + e \curvearrowright (h \curvearrowright v) \\ &= 2e \curvearrowright v + \lambda e \curvearrowright v \\ &= (\lambda + 2)e \curvearrowright v. \end{aligned}$$

]

and [

$$\begin{aligned} h \curvearrowright (f \curvearrowright v) &= [h, f] \curvearrowright v + f \curvearrowright (h \curvearrowright v) \\ &= -2f \curvearrowright v + \lambda f \curvearrowright v \\ &= (\lambda - 2) f \curvearrowright v. \end{aligned}$$

]

So if V is a finite-dimensional $\mathfrak{g}$-module, then there exists a $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. Any nonzero $v \in V_\lambda$ is called a *maximal vector*.

> Note: in category $\mathcal{O}$, these always exist?

Some computations:

- $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ Then $V = \mathbb{C}$ is the trivial module, and $g \curvearrowright V = 0$. So $W_t(V) = \{0\}$, and $V = V_0$.

If $V = \mathbb{C}^2$, then take the natural representation $\operatorname{span}_{\mathbb{C}} \{v_1 = [1, 0], v_2 = [0, 1]\}$. Then $g \curvearrowright V$ by matrix multiplication, and if $h = [1, 0; 0, -1]$ then $h \curvearrowright v_1 = v_1$ and $h \curvearrowright v_2 = -v_2$ by just doing the matrix-vector multiplication. Then $\mathbb{C}([1, 0]) = V_1, \mathbb{C}([0, 1]) = V_{-1}$, so $W_t(V) = \{\pm 1\}$.

Taking $V = \mathbb{C}^3 = \operatorname{ad} \mathfrak{g} = \operatorname{span})_{\mathbb{C}} \{e, f, h\}$, then [

$$\begin{aligned} h \curvearrowright f &= [h, f] = -2f \\ h \curvearrowright h &= [h, h] = 0h \\ h \curvearrowright e &= [h, e] = 2e. \end{aligned}$$

]

So $W_t(V) = \{2, 0, -2\}$ and $V_2 = \mathbb{C}e, V_0 = \mathbb{C}h, V_{-2} = \mathbb{C}f$.

> Note the pattern: some largest value, then jumping by 2 to lower values, ending at negative the largest value. In some sense, the rest of the theory will reduce to the case of $\mathfrak{sl}(2, \mathbb{C})$.

Lemma: Let V be a finite dimensional simple $\mathfrak{sl}(2, \mathbb{C})$-module, and $V_0 \in V_\lambda$ a maximal vector.

Set $V_{-1} = 0, V_i = f^{(i)} \curvearrowright v_0$ (where $f^{(i)} = \dfrac{f^i}{i!}$). Then for all $i \geq 0$, we have

a. $h \curvearrowright v_i = (\lambda - 2i) v_i$
b. $f \curvearrowright v_i = (i + 1) v_{i+1}$
c. $e \curvearrowright v_i = (\lambda - i + 1) v_{i-1}$

Proof of (a): By lemma 7.1, we have $f \curvearrowright v_0 \in V_{\lambda-2}$, and so inductively $f^{(i)} \curvearrowright v_0 \in V_{\lambda-2i}$

Proof of (b): By definition.

Proof of (c): [

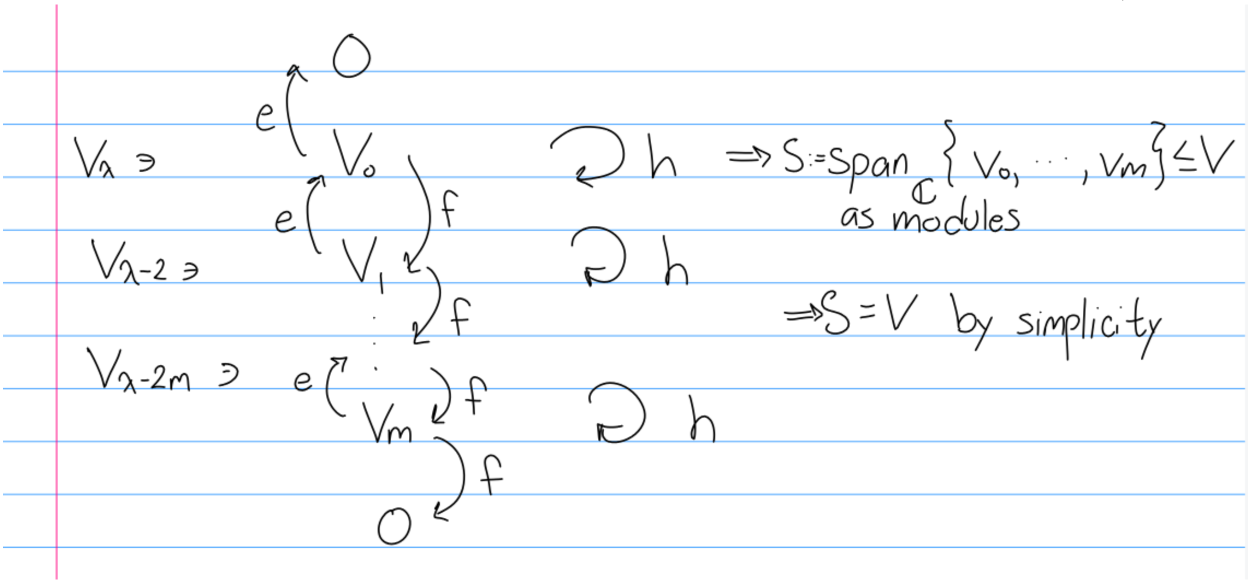
$$\begin{aligned} ie \curvearrowright v_i &= ie \curvearrowright \frac{f^i \curvearrowright v_0}{i!} \\ &= e \curvearrowright (f \curvearrowright v_{i-1}) \\ &= [e, f] \curvearrowright v_{i-1} + f \curvearrowright (e \curvearrowright v_{i-1}) \\ &= h \curvearrowright v_{i-1} + f \curvearrowright ((\lambda - i + 2) v_{i-2}) \text{ind} \\ &= (\lambda - 2i + 2) v_{i-2} + (\lambda + i - 2)(i - 1 v_{i-1}) \\ &= i(\text{RHS}). \end{aligned}$$

]

Theorem: If V is a finite dimensional and simple, then $V \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$ where $L(m) = \operatorname{span}_{\mathbb{C}} \{v_0, v)1, \cdots v_m\}$ where each v_i is of weight $m - 2i$.

Thus $L(m) = L(m)_m \oplus L(m)_{m-2} \oplus \cdots \oplus L(m)_{-m}$ where $\dim L(m)_\mu = 1$ for all μ and $\dim L(m = m + 1.)$

Proof: Pick a maximal vector $v_0 \in V_\lambda$ for any weight λ. Define v_i as usual. Let $m = \min \left\{ i \;\middle|\; V_i \neq 0,\ V_{i+1} = 0 \right\}$



Definition: A module V is a *highest weight module* of weight λ if $V = \mathfrak{g} \curvearrowright v_0$ for some maximal vector $v_0 \in V_\lambda$.

Then λ is referred to as the *highest weight*, and v_0 is the *highest weight vector*.

Corollary: If V is finite-dimensional, then

a. $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$
b. The number of summands $= \dim V_0 + \dim V_1$.

Proof of (a): By Weyl's theorem, we know $V = \oplus W_i$ for some simple W_i. By theorem 7.2, this is equal to $\oplus_{m \in \mathbb{Z}_{\geq 0}} L(m)^{\mu_m}$

Proof of (b): $\dim V_0 = \# \{\text{summands where } m \text{ is even}\}$ $\dim V_1 = \# \{\text{summands where } m \text{ is odd}\}$

Remark: Let

$$V_d = \left\{ f \in \mathbb{C}[x, y] \;\middle|\; f \text{ is homogeneous of total degree } d \right\} = \operatorname{span}_{\mathbb{C}} \left\{ x^d, x^{d-1} y, \cdots, y^d \right\}.$$

Then $\mathfrak{sl}(2, \mathbb{C}) \curvearrowright V_d$ by

$$\begin{aligned}
e &\mapsto x\frac{\partial}{\partial y} \\
f &\mapsto y\frac{\partial}{\partial x} \\
h &\mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.
\end{aligned}$$

Fact: For $L(m), \phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(L(m))$, define

$$s = (\exp \phi(e)) \circ (\exp \phi(-f)) \circ (\exp \phi(e))$$

Then $s(v_i) = -v_{m-i}$.

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Last time: Construction of simple finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ module.

Today: Root space decomposition for semisimple finite-dimensional $\mathfrak{g}$.

16.1 Root Space Decomposition

Let $\mathfrak{g}$ be semisimple and finite dimensional, and let $\mathbb{F} = \mathbb{C}$.

16.1.1 Maximal Toral subalgebra and roots

Definition: A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is *toral* if $\mathfrak{h} \neq 0$ and it consists of only semisimple elements (i.e. $x_n = 0 \forall x \in \mathfrak{h}$)

Lemma:

a. There exists a toral subalgebra of $\mathfrak{g}$, which is a nontrivial maximal toral subalgebra.
b. Any toral subalgebra is abelian.

Proof of (a): Want to show that there exists an $x \in \mathfrak{g}$ such that $x_s \neq 0$, which will imply that $\mathfrak{h} = \mathbb{C}x_s$ is toral.

Suppose $x_s = 0$ for all $x \in \mathfrak{g}$, then $\operatorname{ad} x = \operatorname{ad} x_n$ is nilpotent. By Engel's theorem, this means $\mathfrak{g}$ must be nilpotent. But this contradicts $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (since $\mathfrak{g}$ is semisimple) so the derived series can never reach zero.

Proof of (b): Fix $x \in \mathfrak{h}$, want to show that $[x, h] = 0 \forall h \in \mathfrak{h}$. Then $x = x_s$, and so $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. It suffices to show that $\operatorname{ad} x|_{\mathfrak{h}} = 0$ for all $\mathfrak{h}$.

Suppose that $[x, h] = ah$ for some vector h where $a \neq 0$. Decompose $\mathfrak{h}$ into eigenspaces, so $\mathfrak{h} = \bigoplus_{\lambda} \mathfrak{h}_\lambda$ where $\mathfrak{h}_\lambda = \left\{ y \in \mathfrak{h} \;\middle|\; [h, y] = \lambda y \right\}$. But then $[h, x] \in \mathfrak{h}_0$, since $[h, [h, x]] = [h, -ah] = 0$.

So write $x = \sum_\lambda c_\lambda x_\lambda$, where $c_\lambda \in \mathbb{C}$ and $x_\lambda \in \mathfrak{h}_\lambda$. Then [

$$\begin{aligned}[h, x] &= \sum_\lambda c_\lambda [h, x_\lambda] \\ &= \sum_\lambda c_\lambda \lambda x_\lambda \in \mathfrak{h}_0,\end{aligned}$$

] so $\lambda c \lambda = 0 \forall \lambda \neq 0$, which means $c_\lambda = 0 \forall \lambda \neq 0$, and thus $x \in \mathfrak{h}_0$ and $[h, x] = 0$. But this contradicts $[x, h] = ah$.

Now $\forall x, h \in \mathfrak{h}, g \in \mathfrak{g}$, we have $[h, [x, y]] = [x, [h, y]] + [y, [x, h]] = [x, [h, y]]$. Thus $\text{ad } h \circ \text{ad } x = \text{ad } x \circ \text{ad } h$ as elements of $\text{End}(\mathfrak{g})$.

So $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \left\{x \in \mathfrak{g} \;\middle|\; [h, x] = \alpha(h) x \forall h \in \mathfrak{h}\right\}$.

Note that $\mathfrak{g}_0 = \left\{x \in \mathfrak{g} \;\middle|\; [h, x] = 0 \forall h \in \mathfrak{h}\right\} = C_{\mathfrak{g}}(\mathfrak{h}) \supseteq \mathfrak{h}$, i.e. the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Definition: Fix a toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then a *root* is a nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$. $\mathfrak{g}_\alpha$ is referred to as the *root space.*

We write $\Phi = \{\text{roots}\}$ and $\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right)$.

Example: $\mathfrak{sl}(3, \mathbb{C})$.

TODO: Insert image from phone.

Then $\Phi = \{\alpha : \mathfrak{h} \to \mathbb{C}, h_1 \mapsto \alpha(h_1) \in \{\pm 1, \pm 2\}\}$. So

- $\mathfrak{g}_0 = \mathbb{C} h_1 \oplus \mathbb{C} h_2$
- $\mathfrak{g}_1 = \mathbb{C} f_2 \oplus \mathbb{C} e_3$
- $\mathfrak{g}_2 = \mathbb{C} e_1$
- $\mathfrak{g}_{-1} \mathbb{C} f_3 \oplus \mathbb{C} e_2$
- $\mathfrak{g}_{-2} = \mathbb{C} f_1$.

TODO: Insert second and third image from phone

From these computations, we collect the eigenvalues as ordered pairs. If we choose a larger toral subalgebra, we get a finer decomposition. And if we take a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$ and all $\dim \mathfrak{g}_\alpha = 1$.

Proposition (a): $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{h}^*$.

Proposition (b): If $x \in \mathfrak{g}_\alpha$ and $\alpha \neq 0$ then $\text{ad } x$ is nilpotent.

Proposition (c): If $\alpha, \beta \in \mathfrak{h}^*$ and $\alpha + \beta = 0$, then $\kappa(x, y) = 0 \forall x \in \mathfrak{g}_\alpha, y = \mathfrak{g}_\beta$.

Proof of (a): Easy exercise:

Proof of (b): For all $y \in \mathfrak{g}$, $y \in \mathfrak{g}_\mu$ for some $\mu \in \mathfrak{h}^*$. We have $\mathfrak{g}_u \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+\alpha} \xrightarrow{\text{ad } x} \mathfrak{g}_{\mu+2\alpha} \to \cdots$ by $y \mapsto [x, y] \mapsto \cdots$. Since $\mathfrak{g}$ is finite dimensional, this must terminate, so $(\text{ad } x)^n(y) = 0$ for some n.

Proof of (c): If $\alpha + \beta = 0$, then there exists an $h \in \mathfrak{h}$ such that $\alpha(h) + \beta(h) \neq 0$. Since the killing form is associative, we have

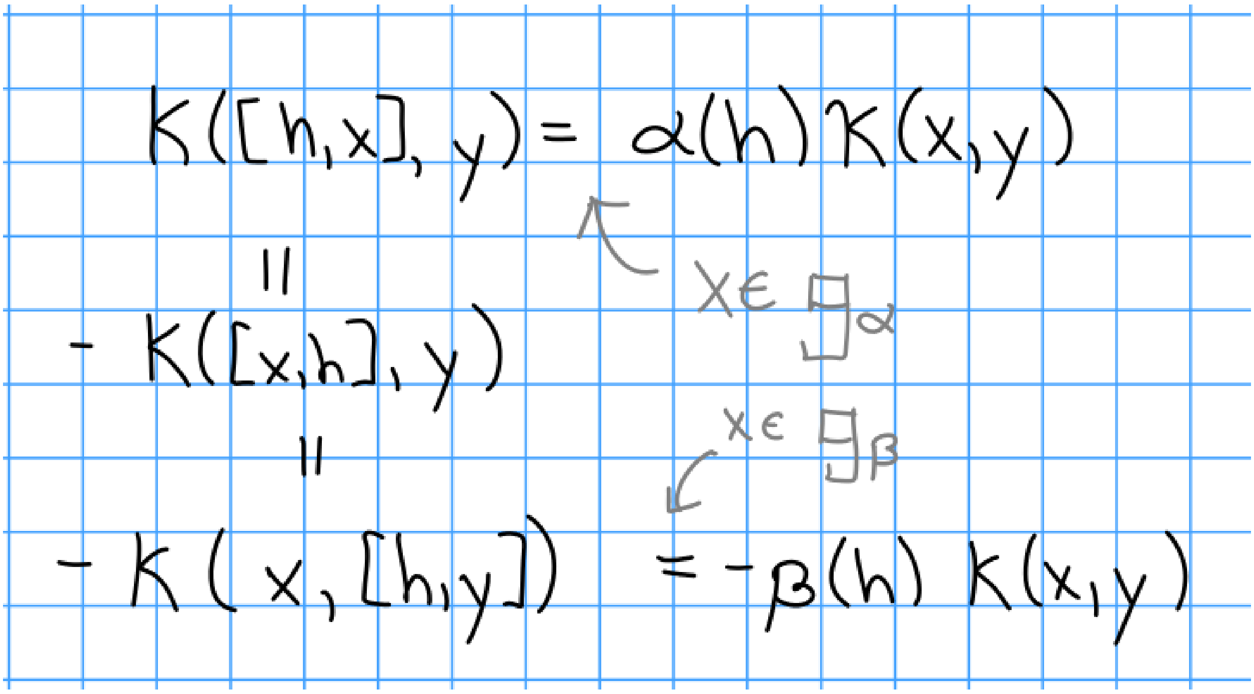

Figure 12: Image

Corollary: $\kappa\Big|_{\mathfrak{g}_0}$ is nondegenerate.

Proof: We want to show $\kappa(h,y) = 0 \forall y \in \mathfrak{g}_0 \implies h = 0$ holds for any choice of $y \in \mathfrak{g}_\alpha$ with $\alpha \neq 0$. By proposition (c), we have $\kappa(h,y) = 0$. Note that we have $\mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha)$. This implies that $\kappa(h,y) = 0 \forall y \in \mathfrak{g}$. But then $h = 0$ because κ is nondegenerate and $\mathfrak{g}$ is semisimple.

17 Monday September 23

Last time: $\mathfrak{h}$ is a *toral* subalgebra if it contains only semisimple elements, and implies that there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [h,x] = \alpha(h)x \;\forall h \in \mathfrak{h} \right\}$ and $\Phi = \left\{ \alpha : \mathfrak{h} \to \mathbb{C} \;\middle|\; \mathfrak{g}_\alpha \neq 0, \alpha \neq 0 \right\}$ and $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$. Take larger $\mathfrak{h}$ yields finer decompositions, and a maximal $\mathfrak{h}$ gives $\dim \mathfrak{g}_\alpha = 1 \forall \alpha \in Phi$.

Corollary: $\kappa|_{\mathfrak{g}_0}$ is nondegenerate.

17.1 The Centralizer of $\mathfrak{h}$

If $x, y \in \mathrm{End}(V)$ where V is finite dimensional, $xy = yx$, and y is nilpotent, then xy is nilpotent and $\mathrm{Tr}(xy) = 0$.

Proposition: If $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal toral subalgebra, then $\mathfrak{h} = \mathfrak{g}_0$.

Proof:

Step 1: If $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$.

If $x \in \mathfrak{g}_0$, then $\operatorname{ad} x(\mathfrak{h}) \subseteq 0$. By proposition 4.2, $\operatorname{ad} x_s(\mathfrak{h}) \subseteq 0, \operatorname{ad} x_n(\mathfrak{h})$, and so $x_s, x_n \in \mathfrak{g}_0$.

Step 2: $\left\{ x_s \,\middle|\, x \in \mathfrak{g}_0 \right\} \subseteq \mathfrak{h}$.

If $x \in \mathfrak{g}_0$, then by step 1 we have $x_s \in \mathfrak{g}_0$ and so $\mathfrak{h} + \mathbb{C}x_s$ is toral, and thus $x_s \in \mathfrak{h}$.

Step 3: $\kappa|_{\mathfrak{h}}$ is non-degenerate.

We want to show that $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g} \implies h = 0$. By the corollary, it suffices to show that $\kappa(h, x) = 0 \ \forall x \in \mathfrak{g}_0$. By step 2, it suffices to check this only for $x \in \mathfrak{g}_0$ such that $x = x_n$.

If $x = x_n$, then $\operatorname{ad} x_n$ is nilpotent and $\operatorname{ad} h$ commutes with $\operatorname{ad} x$ because $[h, x] = 0$ (since $x \in \mathfrak{g}_0$). By the lemma, $\operatorname{Tr}(\operatorname{ad} h \circ \operatorname{ad} x) = 0$, since $\operatorname{ad} h = \kappa(h, x)$.

Step 4: $\mathfrak{g}_0$ is nilpotent.

Pick $x \in \mathfrak{g}_0$. Then by step 2, $x_s \in \mathfrak{h}$, so $\operatorname{ad} x_s : \mathfrak{g}_0 \circlearrowleft$ is a zero map and thus nilpotent.

So $\operatorname{ad} x_n$ is nilpotent, meaning that $\operatorname{ad} x$ is nilpotent. By Engel's theorem, this implies that $\mathfrak{g}_0$ itself is nilpotent.

Step 5: $\mathfrak{g}_0$ is abelian.

Suppose that $I \coloneqq [\mathfrak{g}_0, \mathfrak{g}_0] \neq 0$. We have $I \trianglelefteq \mathfrak{g}_0$, and I is not nilpotent whereas $\mathfrak{g}_0$ is.

By Lemma 3.3, we have $I \bigcap Z(\mathfrak{g}_0) \neq 0$, so pick x in the intersection. Note that $\kappa(\mathfrak{h}, I) = \kappa(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0])$, which by associativity equals $\kappa([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = 0$.

By step 3, we have $\mathfrak{h} \bigcap I = 0$. By step 2, $x \neq x_s$, and thus $x_n \neq 0$. But we also have $x \in Z(\mathfrak{g}_0)$, so $[x, \mathfrak{g}_0] = 0$ and $\operatorname{ad} x(\mathfrak{g}_0) \subseteq 0$. By Proposition 4.2, this holds for x_s, x_n as well, which are both in the center. So for all $y \in \mathfrak{g}_0$, $\operatorname{ad} y$ commutes with $\operatorname{ad} x_n$, which is nilpotent.

By the lemma, this implies that $0 = \operatorname{Tr}(\operatorname{ad} y \circ \operatorname{ad} x_n) = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$. So $x_n = 0$.

Step 6: Suppose $\mathfrak{g}_0 \not\subset \mathfrak{h}$. By step 2, there exists an $x \in \mathfrak{g}_0$ such that $x \notin \mathfrak{h}$, where $x_n \neq 0$. By step 5, $[x_n, y] = 0$ for all $y \in \mathfrak{g}_0$. Then $\operatorname{ad} x$ (which is nilpotent) commutes with $\operatorname{ad} y$. By the lemma, $0 = \kappa(x_n, y)$ for all $y \in \mathfrak{g}_0$, and thus $x_n = 0$. ■

> Main idea: Choose a maximal toral subalgebra to get a nice root space decomposition, and so it coincides with $\mathfrak{g}_0$.

Corollary: $\kappa|_{\mathfrak{g}}$ is nondegenerate.

Thus for all $\alpha \in \mathfrak{h}^*$, there exists a unique $t_\alpha \in \mathfrak{h}$ such that $\alpha = \kappa(t_\alpha, \cdot\,) : \mathfrak{h} \to \mathbb{C}$.

In other words, there is an identification [

$$
\begin{aligned}
\mathfrak{h} &\xrightarrow{1-1} \mathfrak{h}^* \\
h &\mapsto \kappa(h, \cdot\,) \\
t_\alpha &\leftarrow \alpha.
\end{aligned}
$$

]

Definition: A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \left\{ x \in \mathfrak{g} \;\middle|\; [x, h] \subseteq \mathfrak{h} \right\}.$$

Note that $N_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.

Remark: If $\mathfrak{g}$ is semisimple and finite dimensional with $\mathrm{char}(F) = 0$, we will have a correspondence: [

$$\{\text{CSAs of } \mathfrak{g}\} \iff \{\text{maximal toral subalgebras of } \mathfrak{g}\}.$$

]

Maximal toral subalgebras advantages over Cartan subalgebra definition:

- Yields the finest root space decomposition
- $\mathfrak{h}^* = \mathfrak{h}$, Weyl group?
- Existence is easy compared to CSAs

On the other hand, CSA advantages:

- All CSAs are conjugate under G (some group to be defined)
- The dimensions of all CSAs are the same, giving a well-defined notion of dimension ($\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$).

17.2 8.3: Orthogonality Properties

From now on, $\mathfrak{h}$ will be a maximal toral subalgebra.

Proposition: Let $\alpha \in \Phi$. Then

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$
c. $\forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \kappa(x, y) t_\alpha$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} t_\alpha$ (let the nonzero scalar be λ)
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.
f. For any nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha \coloneqq \dfrac{\lambda}{\kappa(t_\alpha, t_\alpha)} t_\alpha$. Moreover, $\langle e_\alpha, f_\alpha, h_\alpha \rangle = \mathfrak{sl}(2, \mathbb{C})$.

18 Wednesday September 25

Today: Properties of the root space when the toral subalgebra is maximal.

Last time: We have $\mathfrak{g} = \mathfrak{g} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$ where $\kappa \Big|_{\mathfrak{h}}$ is nondegenerate. We also have a correspondence [

$$\begin{aligned} \mathfrak{h} &\iff \mathfrak{h}^* h \mapsto \kappa(\mathfrak{h}, \cdot) \\ t_\alpha &\leftarrow \alpha \coloneqq \kappa(t_\alpha, \cdot). \end{aligned}$$

] ## Orthogonality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. Φ spans $\mathfrak{h}^*$
b. $-\alpha \in \Phi$
c. $[x, y] = \kappa(x, y)t_\alpha$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$
d. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}t_\alpha$
e. $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$
f. For each nonzero $e_\alpha \in \mathfrak{g}_\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha := \dfrac{2}{\kappa(t_\alpha, t_\alpha)}t_\alpha$. Moreover, $\langle e_\alpha, t_\alpha, h_\alpha\rangle \cong \mathfrak{sl}(2, \mathbb{C})$.

Proof of (a): We want to show that $h \in \mathfrak{h}$ implies that if $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $h = 0$.

Take $x \in \mathfrak{g}_\alpha$. Then $[h, x] = \alpha(h)x = 0$. So $[\mathfrak{h}, \mathfrak{g}] = 0$ because $\mathfrak{h}$ is abelian. But then $[h, \mathfrak{g}] = 0$, or $h \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

Proof of (b): By Proposition 8.1c, we have $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$.

If $-\alpha \notin \Phi$, then $\mathfrak{g}_{-\alpha} = 0$. So $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$ by the non-degeneracy of κ.

Proof of (c): For all $h \in \mathfrak{h}$, we have [

$$\begin{aligned}
\kappa(h, [x, y]) &= \kappa([h, x], y) \\
&= \kappa(\alpha(h)x, y) \\
&= \kappa(t_\alpha, h)\kappa(x, y) \\
&= \kappa(\kappa(x, y)t_\alpha, h) \\
&= \kappa(h, \kappa(x, y)t_\alpha).
\end{aligned}$$

]

which implies that $\kappa(h, [x, y] - \kappa(x, y)t_\alpha) = 0$, which forces the second argument to be zero by non-degeneracy.

Proof of (d): We will show that (d) implies (c), i.e. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}t_\alpha$.

We want to show $\kappa(x, y)$ is not always zero.

Pick any nonzero $x \in \mathfrak{g}_\alpha$. Then $\kappa(x, \mathfrak{g}_\beta) = 0$ for all $\beta \neq -\alpha$. If $\kappa(x, \mathfrak{g}_{-\alpha}) = 0$, then $\kappa(x, \mathfrak{g}) = 0$. By non-degeneracy, this forces $x = 0$.

Proof of (e): We will skip this for now, and revisit with methods from later sections that make this proof simpler.

Proof of (f): Let $e_\alpha \neq 0$ in $\mathfrak{g}_\alpha$. Then there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, y) \neq 0$. Set $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, f_\alpha) = \dfrac{2}{\kappa(t_\alpha, t_\alpha)}$.

By (c), we have

[

$$\begin{aligned}
[e_\alpha, f_\alpha] &= \kappa(e_\alpha, t_\alpha)t_\alpha \\
&= \frac{2}{\kappa(t_\alpha, t_\alpha)}t_\alpha \\
&= h_\alpha.
\end{aligned}$$

]

and [

$$\begin{aligned}[h_\alpha, e_\alpha] &= \frac{2}{\kappa(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] \\ &= \frac{2}{\kappa(t_\alpha, t_\alpha)}\alpha(t_\alpha)e_\alpha \\ &= 2e_\alpha.\end{aligned}$$

]

and similarly $[h_\alpha, f_\alpha] = -2f_\alpha$.

Definition:

Let $\mathfrak{sl}(2,\alpha) = \langle e_\alpha, f_\alpha, h_\alpha\rangle$ as in (f). A priori, this depends on a choice of $e_\alpha \neq 0$. We will show that this only depends on α.

18.1 Orthogonality/Integrality Properties

Proposition: Let $\alpha \in \Phi$. Then:

a. $\dim \mathfrak{g}_\alpha = 1$. (Note that in general, $\dim \mathfrak{g}_0 = \dim \mathfrak{h} \geq 1$)
b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
c. If $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
d. If $\beta \neq -\alpha \in \Phi$, then let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $\beta + q\alpha$ are both in Φ. Then $\beta + i\alpha \in \Phi$ for every $-p \leq i \leq q$, and [

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = p - q \in \mathbb{Z}.$$

]

Proof of (a) and (b):

Let $M = \mathfrak{h} \oplus \left(\bigoplus_{c\neq 0} \mathfrak{g}_{c\alpha}\right) \leq \mathfrak{g}$ as a subspace. By a routine check, M is an $\mathfrak{sl}(2,\alpha)$ submodule of $\mathfrak{g}$. Recall that $M = \bigoplus_{\lambda\in\mathbb{Z}}$ as a direct sum of vector spaces. Applying Weyl's theorem, we also have $M = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} L(m)^{\oplus\mu_m}$ as a direct sum of (irreducible?) modules.

For $\mathfrak{h}$, if we have $[h_\alpha, h] = 0$ for all $h \in \mathfrak{h}$, then $h \in M_0$. For $\mathfrak{g}_{c\alpha}$, $[h_\alpha, x] = c\alpha(h_\alpha)x$ for all $x \in \mathfrak{g}_{c\alpha}$. But this equals $2cx$. So this implies that $\mathfrak{g}_{c\alpha} \subseteq M_{2c}$.

Thus $2c \in \mathbb{Z}$, and thus $c \in \frac{1}{2}\mathbb{Z}$, and $M_0 = \mathfrak{h}$.

We then have $\dim M_0 = \sum_{m\in 2\mathbb{Z}} \mu_m$. So write $h = \mathbb{C}t_\alpha \oplus \ker\alpha$ as vector spaces. Consider the action $\mathfrak{sl}(2,\alpha) \curvearrowright \ker\alpha$, which is trivial since $h \in \ker\alpha$. We $[h_\alpha, h] = 0$, $[e_\alpha, h] = -\alpha(h)e_\alpha = 0$ since $h \in \ker\alpha$, and similarly $[f_\alpha, h] = 0$.

Thus $\ker\alpha = L(0)^{\oplus \dim\mathfrak{h}-1}$. Moreover, $\mathfrak{sl}(2,\alpha) = L(2) = \operatorname{span}(e_\alpha, t_\alpha, f_\alpha)^T$. But this forces the case that there is no other summand of the form $L(k)$ for k even in M.

Then $\mathfrak{g}_{2\alpha} \subseteq M_4$, which must be zero. So $2\alpha \notin \Phi$, so 2α is not a root. ("Twice a root is never a root")

So $\dfrac{1}{2}\alpha \notin \Phi$, otherwise we could apply this argument to conclude that α is not a root and reach a contradiction. Thus $M_1 = 0$, since $c \neq \dfrac{1}{2}$ implies that there is not summand of the form $L(k)$ for k odd in M. But this forces $M = \mathfrak{h} \oplus \mathfrak{sl}(2, \alpha)$.

> Motto: reduce the complexity by using the $\mathfrak{sl}(2)$ module structure and its representation theory!

19 Friday September 27

Last time, we saw $\Phi \subseteq \mathfrak{h}^* = \{\alpha : \mathfrak{h} \to \mathbb{C}\}$.

Suppose $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a maximal toral subalgebra and take $F = \mathbb{C}$.

We have the following propositions:

a. $\dim \mathfrak{g}_\alpha = 1 \ \forall \alpha \in \Phi$

b. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$, and $2\alpha \notin \Phi$ where $c\alpha : \mathfrak{h} \to \mathbb{C}$, $h \mapsto c \curvearrowright \alpha(h)$. Moreover, $M = \mathfrak{h} \oplus \left(\bigoplus_{c \neq 0} \mathfrak{g}_{c\alpha} \right)$

c. If $\alpha, \beta \in \Phi$ and $\beta \neq -\alpha$ Let $p, q \in \mathbb{Z}$ be the largest such that $\beta - p\alpha$ and $b + q\alpha$ are in Φ. Moreover, $\beta(h_\alpha) = \kappa(t_\beta, t_\alpha) = p - q \in \mathbb{Z}$.

Proof of (c):

Set $M = \sum_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$, which is an $\mathfrak{sl}(2, \alpha)$ module. By (a), we have $\dim \mathfrak{g}_{\beta+i\alpha} = 1 \iff \beta + i\alpha \in \Phi$. But for all $x \in \mathfrak{g}_{\beta+i\alpha}$, we have $[h, x] = (\beta + i\alpha)(h)x$ for all $h \in \mathfrak{h}$. But then $[h_\alpha, x] = (\beta(h_\alpha) + i\alpha(h_\alpha))x = (\beta(h_\alpha) + 2i)x$

Then $\mathfrak{g}_{\beta+i\alpha} \subseteq M_{\beta(h_\alpha)+2i}$, so $\beta(h_\alpha) \in \mathbb{Z}$.

Moreover, $\mathrm{Wt}(M) = 2\mathbb{Z}$ or $2\mathbb{Z} + 1$, and in particular $\dim M_0 + \dim M_1 = 1$.

Thus M is irreducible, and $M \cong L(m)$ for some $m \in \mathbb{Z}_{\geq 0}$. Moreover, $\mathrm{Wt}(M) = \{m, m-2, \cdots - m\}$, and$simg\mathfrak{g}_{\beta+i\alpha} = 1$ for all $i \in [-p, q]$. Thus $\beta + i\alpha \in \Phi$.

Proof of 8.3(e): $\alpha(t_\alpha) \neq 0$. The claim is that for all $\beta \in Phi$, there exists an $r \in \mathbb{Q}$ such that $\beta(h) = r\alpha(h)$ for all $h \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

There are two cases: if $\beta = -\alpha$, then we're done by the previous argument.

Otherwise, $\beta \neq -\alpha$. Take $M = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}$.

Then, [

$$\begin{aligned}
\mathrm{Tr}_M(\mathrm{ad}\ h) = \sum_i \mathrm{Tr}_M((\mathrm{ad}\ e_i \circ \mathrm{ad}\ f_i) - (\mathrm{ad}\ f_i \circ \mathrm{ad}\ e_i)) = \sum_i \mathrm{Tr}_{\mathfrak{g}_{\beta+i\alpha}}(\mathrm{ad}\ h) \\
= \sum (\beta + i\alpha)(h) \dim \mathfrak{g}_{\beta+i\alpha} \\
= \sum_i \dim \mathfrak{g}_{\beta+i\alpha}\beta(h) + \sum_i i \dim(\mathfrak{g}_{\beta+i\alpha}) \\
\implies \beta(h) = \frac{-\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}{\sum_i \dim \mathfrak{g}_{\beta+i\alpha}}\alpha(h).
\end{aligned}$$

]

Now consider the killing form $\kappa(t_\beta, t_\alpha) = \beta(t_\alpha) = r\alpha(t_\alpha)$, where the last equality is what we are claiming.

Suppose that $\alpha(t_\alpha) = 0$. Then $\kappa(t_\beta, t_\alpha) = 0$ for all $\beta \in Phi$. By the non-degeneracy of κ, we have $t_\alpha = 0$ and thus $\alpha = 0$.

19.1 Summary

We have $\mathfrak{g}$ semisimple, finite dimensional, and $\mathfrak{h}$ a maximal toral subalgebra (i.e. the Cartan subalgebra). This implies that κ is nondegenerate, and we have a correspondence [

$$\begin{aligned}
\mathfrak{h} \iff \mathfrak{h}^\vee \\
h \mapsto \kappa(h, \cdot) \\
t_\alpha \leftarrow \alpha.
\end{aligned}$$

]

This gives a symmetric bilinear form $(\cdot, \cdot) : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$.

For $\alpha \in Phi$, define its *coroot* $\alpha^\vee = \dfrac{2}{(\alpha, \alpha)}\alpha$.

Note that $(\cdot)^\vee$ is not linear: note that [

$$(2\alpha)^\vee = \frac{2}{(2\alpha, 2\alpha)}2\alpha = \frac{\alpha}{(\alpha, \alpha)} = \frac{\alpha^\vee}{2}.$$

]

Assume that $\Phi = \{\alpha_i\}$. Define $E_{\mathbb{Q}} = \bigoplus_{i=1}^{\ell} \mathbb{Q}_{\alpha_i}$, and $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$.

Lemma: If $\alpha, \beta \in \Phi$, then

a. $(\beta, \alpha) \in \mathbb{Q}$,
b. $(\cdot, \cdot)$ on $E_{\mathbb{Q}}$ is positive definite, i.e. $x \neq 0 \implies (x, x) > 0$.

An immediate consequence of (b) is that $(\cdot, wait)$ on E is an inner product.

Proof: For all $\lambda, \mu \in \mathfrak{h}^\vee$, we have [

$$(\lambda, \mu) = \kappa(t_\lambda, t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\ t_\lambda \circ \mathrm{ad}\ t_\mu) = \mathrm{Tr}_{\mathfrak{g}}(...) + \sum_{\alpha \in \Phi} \mathrm{Tr}_{\mathfrak{g}_\alpha}(...) = 0 + \sum_{\alpha \in \Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{[} \alpha \in \Phi] = \kappa(t_\alpha, t_\lambda)\kappa($$

]

So pick $\lambda = \mu = \alpha \in \Phi$. Then $(\alpha, \alpha) = \sum_{\beta \in \Phi} (\beta, \alpha)^2$.

Then [

$$\frac{1}{(\alpha, \alpha)} = \sum_{\beta \in \Phi} \left(\frac{(\beta, \alpha^\vee)}{2} \right)^2 .$$

]

where $(\beta, \alpha^\vee) = \cdots = \beta(h_\alpha) \in \mathbb{Z}$.

This means that $(\alpha, \alpha) \in \mathbb{Q}_{>0}$.

Summary of properties proved:

Let $\alpha, \beta \in \Phi$. Then

1. $0 \notin \Phi$ and Φ spans E
2. $\mathbb{C}\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\beta - (\beta, \alpha^\vee)\alpha \in \Phi$
4. $(\beta, \alpha^\vee) \in \mathbb{Z}$

Thus the assignment $(\mathfrak{g}, \mathfrak{h}) \mapsto (\Phi, E)$ defines a **root system**. This only works when $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is maximal toral.

Proof of (3):

We computed $(\beta, \alpha^\vee) = p - q$. Then $-p \leq -(\beta, \alpha^\vee) = q - p \leq q$. So this must be something on the root stream.

20 Monday September 30

Last time: Let $\mathfrak{g}$ be finite dimensional and $\mathfrak{h}$ a maximal toral subalgebra.

Then (Φ, E) is a *root system*, and we obtain a bilinear product [

$$\begin{aligned} \langle \cdot, \ \cdot \rangle : E \times E \to \mathbb{R} \\ (\alpha, \beta) \mapsto \kappa(t_\alpha, t_\beta). \end{aligned}$$

] Examples: $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ where

Todo: Insert clip image h1, h2

[

$$\begin{aligned} \alpha_1 : \mathfrak{h} \to \mathbb{C} \\ h_1 \mapsto 2h_2 \mapsto -1. \end{aligned}$$

]

[

$$\alpha_2 : \mathfrak{h} \to \mathbb{C}$$
$$h_1 \mapsto -1 h_2 \mapsto 2.$$

]

To find t_{α_i}, we need to look at $\kappa \Big|_{\mathfrak{h}}$.

Todo: Insert phone image

Since we only need the trace, this suffice, and we find

[

$$\left.\begin{array}{c|cc} h_1 & h_2 & \\ h_1 & 12 & -6 \\ h_2 & -6 & 12 \end{array}\right] .$$

]

We then get $t_{\alpha_1} = \dfrac{h_1}{6}$ and $t_{\alpha_2} = \dfrac{h_2}{6}$. Moreover [

$$\langle \alpha_1,\ \alpha_1 \rangle = \kappa(t_{\alpha_1}, t_{\alpha_1}) = \frac{1}{3} \in \mathbb{Q}$$
$$\langle \alpha_1,\ \alpha_1 \rangle = \frac{1}{3}$$
$$\langle \alpha_1,\ \alpha_2 \rangle = -\frac{1}{6}$$
$$\langle \alpha_1,\ \alpha_2^\vee \rangle = \frac{2\langle \alpha_1,\ \alpha_2 \rangle}{\langle \alpha_2,\ \alpha_2 \rangle} = -1 \in \mathbb{Z} \langle \alpha_i,\ \alpha_i^\vee \rangle = \frac{2\langle \alpha_i,\ \alpha_i \rangle}{\langle \alpha_i,\ \alpha_i \rangle} = 2 \in \mathbb{Z}.$$

]

This leads to a nice fact: the matrix $\left\langle \alpha_i,\ \alpha_j^\vee \right\rangle$ has $\mathbb{Z}$ entries, and this is called the *Cartan matrix*.

20.1 Ch.3: Root Systems

20.1.1 Axiomatics: Reflections

Fix a Euclidean space E.

Definition: A *hyperplane* in E is a subspace of codimension 1. A *reflection* in E is an element $s \in \mathfrak{gl}(E)$ such that

$$\left\{ E^s := \left\{ x \in E \;\middle|\; sx = s \right\} \text{ is a hyperplane } H \text{ and } s(x) = -x \quad \forall x \in E \;\middle|\; (x, H) = 0 \right\}$$

For nonzero $\alpha \in E$, its reflection is [

$$S_\alpha : E \to E$$
$$\beta \mapsto \beta - \langle \beta,\ \alpha^\vee \rangle \alpha.$$

]

with respect to $H_\alpha = \left\{ x \in E \;\middle|\; \langle x,\ \alpha\rangle = 0 \right\}$, where $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha,\ \alpha\rangle}$.

Lemma: Let $\Phi \subseteq E$ be finite such that $S_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

Suppose that $S \in \mathfrak{gl}(E)$ satisfies

1. $S(\Phi) = \Phi$,
2. $S(h) = h$ for all $h \in H$, and
3. $S(\alpha) = -\alpha$ for some $\alpha \in \Phi$,

then $S = S_\alpha$, i.e. this uniquely characterizes S

Proof:

Let $\tau = S \circ S_\alpha$. Then $\tau(\Phi) = \Phi$ and $\tau(\alpha) = \alpha$. This $\tau \curvearrowright \mathbb{R}\alpha$ by 1, and similarly $\tau \curvearrowright E/\mathbb{R}\alpha$ by 1 by picking a representative in H. Moreover, all eigenvalues of τ are 1. So the minimal polynomial of τ divides $(t-1)^{\dim E}$.

We want to show that $\tau \;\middle|\; (t-1)^N$ for some large N, which forces $\tau \;\middle|\; \gcd((t-1)^{\dim E}, t^N - 1) = 1$. For any $\beta \in \Phi$ and $k > |\Phi|$, not all vectors $\beta, \tau(\beta), \cdots \tau^k(\beta)$. So $\beta = \tau^{k_\beta}(\beta)$ for some k_β depending on β (noting that τ is invertible.)

Multiplying all of these k_βs together, we can get some k_Φ that is larger than $|\Phi|$, and so $\beta = \tau^{k_\Phi}$ for *all* $\beta \in \Phi$. But then $\tau^{k_\Phi} = 1$ in $\mathfrak{gl}(E)$.

20.1.2 Root Systems

Definition: A subset Φ of E a Euclidean space is called a *root system* iff

1. $|\Phi| < \infty, 0 \notin \Phi$, and $E = \bigoplus_{\alpha \in \Phi} \mathbb{R}\alpha$
2. $\alpha \in \phi \implies \mathbb{C}\alpha \bigcap \Phi = \{\pm \alpha\}$
3. $\alpha \in \Phi \implies S_\alpha(\Phi) = \Phi$
4. $\alpha, \beta \in \Phi \implies \langle \beta,\ \alpha^\vee \rangle \in \mathbb{Z}$.

Definition: The *rank* of a root system is the dimension on E.

Definition: The *Weyl Group* of Φ is defined as

$$W = \left\langle S_\alpha \;\middle|\; \alpha \in \Phi \right\rangle \subseteq \mathfrak{gl}(E)$$

Note that $W \hookrightarrow \Sigma_{|\Phi|}$, a permutation group of size $|\Phi|$.

Lemma: If $g \in \mathfrak{gl}(E)$ and $g(\Phi) = \Phi$, then for all $\alpha, \beta \in \Phi$, we have [

$$\begin{aligned} g s_\alpha g^{-1} &= s_{g(\alpha)}, \\ \langle \beta,\ \alpha^\vee \rangle &= \langle g(\beta),\ g(\alpha)^\vee \rangle, \\ \langle \beta,\ \alpha^\vee \rangle &= \langle w(\beta),\ w(\alpha)^\vee \rangle \quad \forall w \in W. \end{aligned}$$

]

Proof: Check 1-3 in Lemma 9.1.

Proof of 1: We have [

$$gs_\alpha g^{-1}(g(\beta)) = gs_\alpha(\beta) \in g(\Phi) = \Phi \quad \forall \beta \in \Phi,$$
.

]

Proof of 2: We have [

$$\left\{ g(\beta) \;\middle|\; \beta \in \Phi \right\} = \Phi \implies gs_\alpha g^{-1}(\Phi) = \Phi \quad \forall h \in gH_\alpha$$
.

] and so $gs_\alpha g^{-1}(h) = gg^{-1}(h) = h$, so h is a fixed point of this map.

Proof of 3: We have $gs_\alpha g^{-1}(g(\alpha) = gs_\alpha(\alpha) = -g(\alpha)$, and so $gs_\alpha g^{-1} = s_{g(\alpha)})$ by Lemma 9.1.

Finally, we have [

$$\begin{aligned} gs_\alpha g^{-1}(g(\beta)) = g(s_\alpha(\beta)) = g(\beta - \langle \beta,\ \alpha^\vee \rangle \alpha) = g(\beta) - \langle \beta,\ \alpha^\vee \rangle g(\alpha) \\ = \\ s_{g(\alpha)} = g(\beta) - \langle g(\beta),\ g(\alpha)^\vee \rangle g(\alpha). \end{aligned}$$

]

21 Wednesday October 2

Recall from last time:

1. $|\Phi| < \infty$ and Φ spans E, where $0 \notin \Phi$
2. If $\alpha \in \Phi$, then $C\alpha \bigcap \Phi = \{\pm\alpha\}$
3. $\alpha \in \Phi$, then $S_\alpha(\Phi) = \Phi$.
4. If $\alpha, \beta \in \Phi$, then $\langle \beta,\ \alpha^\vee \rangle \in \mathbb{Z}$ where $(E, \langle \cdot,\ \cdot \rangle)$is Euclidean and [

$$\begin{aligned} S_\alpha : E \to E \\ \beta \mapsto \beta - \langle \beta,\ \alpha^\vee \rangle \alpha, \quad \alpha^\vee = \frac{2}{\langle \alpha,\ \alpha \rangle} \alpha. \end{aligned}$$

]

Examples:

In Rank 1:

1. Prop 2 implies $\Phi = \{\pm\alpha\}$
2. Prop 1 implies $E = \mathbb{R}\alpha$
3. Prop 3: $S_\alpha(\alpha) = -\alpha$
4. Prop 4 implies $\langle \pm\alpha,\ \pm\alpha \rangle = \pm \dfrac{2\langle \alpha,\ \alpha \rangle}{\langle \alpha,\ \alpha \rangle} = \pm 2$

Rank 1 Diagram: Todo: Insert phone image

In Rank 2: Todo: Insert phone image

Exercise:

- Show that $\mathrm{ord}(S_\alpha, S_\beta) = 2, 3, 4, 6$ for types $A_1 \times A_1, B_2, G_2$.
- Show that $W(A_2) \cong \mathbb{Z}_3$ and $W(B_2) \cong D_8$.

21.1 Pairs of Roots

Lemma: Let $\alpha, \beta \in \Phi$ where $\beta \neq \pm\alpha$, then

1. $\langle \alpha,\ \beta^\vee \rangle \langle \beta,\ \alpha^\vee \rangle \in \{0, 1, 2, 3\}$ Moreover, assuming $|\beta| \geq |\alpha|$, we have the following table Todo: Insert table
2. If $\langle \alpha,\ \beta \rangle > 0$, then $\alpha - \beta \in \Phi$. Similarly, if $\langle \alpha,\ \beta \rangle > 0$, then $\alpha + \beta \in \Phi$.
3. Any root string is unbroken and has length greater than 4.

Proof of (1):

By the Law of Cosines, we can write $x \coloneqq \langle \beta,\ \alpha^\vee \rangle \langle \alpha,\ \beta^\vee \rangle = 4\cos^2(\theta) \in \mathbb{Z}$. This restricts the possibilities to $x \leq 4$. But $x = 4 \iff \alpha = c\beta$, i.e. $\theta = 0$, but we are assuming that $\alpha \neq \pm\beta$, so this can not happen.

Proof of (2):

Since $\langle \alpha,\ \beta \rangle > 0$ and $|\beta| \geq |\alpha|$, then $\langle \alpha,\ \beta^\vee \rangle = 1$. But then $S_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee,\ \beta \rangle \in \Phi$ by Prop 3. So this is equal to $\alpha - \beta$.

A similar argument works for $|\beta| \leq |\alpha|$.

Proof of (3): Let p, q be the largest integers such that $b - p\alpha, b + q\alpha \in \Phi$ respectively. Suppose that the root stream between these two is broken somewhere, say $\beta + s\alpha \in \Phi$ and $\beta + (s+1)\alpha \notin \Phi$ by counting up from $\beta - p\alpha$. Similarly, there is some t counting down from $b + q\alpha$ then $\beta + t\alpha \in \Phi$ but $\beta + (t-1)\alpha \notin \Phi$. In particular, $s < t$. From (2), we have $\langle \alpha,\ \beta + s\alpha \rangle \geq 0$, $\langle \alpha,\ \beta + t\alpha \rangle \leq 0$.

We have

$$\langle \alpha,\ \beta \rangle + t\langle \alpha,\ \alpha \rangle = \langle \alpha,\ \beta + t\alpha \rangle \leq 0 \leq \langle \alpha,\ beta + s\alpha \rangle = \langle \alpha,\ \beta \rangle + s\langle \alpha,\ \alpha \rangle$$

where we know that $\langle \alpha,\ \alpha \rangle > 0$.

Since $S_\alpha(\Phi) = \Phi$ and these $S_\alpha(\beta + i\alpha) = \beta - \mathbb{Z}\alpha$, we find that reflections permute the root string. We then find that $p = \langle \beta,\ \alpha^\vee \rangle + q$, and so $\langle \beta,\ \alpha^\vee \rangle = p - q \in [-3, 3]$.

21.2 Chapter 10: Simple Roots and Weyl Groups

Definition: A *base* of a root system Φ is a subset $\Pi \subseteq \Phi$ such that

1. Π is a basis for the underlying vector space E, and
2. Each $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Pi} \kappa_\alpha^\beta \alpha$ where all of the coefficients κ_α^β all have the same sign.

The roots in Π are called *simple*. A root β is *positive* (resp. *negative*) if the $\kappa_\alpha^\beta \geq 0$ for all $\beta in \Phi^+$ (resp ≤ 0 in Φ^-). The *height* of a β is the sum of the coefficients. Π defines a partial order on E where $\mu \leq \lambda \iff \lambda - \mu \in \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$.

Note that this is defined on the roots themselves, and can then be extended to all of E.

Todo: Insert phone image

22 Monday October 7

Last time:

Lemma 10.2

a. ?
b. $\alpha \in \Pi \implies S_\alpha \curvearrowright \Phi^+ \setminus \{\alpha\}$ by permutation
c. $\alpha_i \in \Pi$ and $S_{\alpha_1}, \cdots, S_{\alpha_{j-1}}(\alpha_j) \in \Phi^-$ then $S_{\alpha_1} \cdots S_{\alpha_j} = S_{\alpha_1} \cdots S_{alpha_{t-1}} \cdots S_{\alpha_{j-1}}$ for some t, where the former has j terms and the latter has $j-2$ terms.

Proof of (a): ?

Proof of (b):

Suppose towards a contradiction that $w(\alpha_j) \in \Phi^+$. Then consider $WS(\alpha_j) = -W(\alpha_j) \in \Phi^-$.

By Lemma 10.2(c), we have $W = S_{\alpha_1} \cdots S_{alpha_{t-1}} S_{\alpha_{t+1}} \cdots S_{\alpha_{j-1}} S_{\alpha_j}$, where this is $j-1$ terms. So $w = S_\alpha \cdots S_{\alpha_j}$ is not reduced.

22.1 Weyl Groups

Recall that the *chambers* are given by the connected component of $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

Theorem: Fix Π of Φ. Then

a. $W \curvearrowright \{\text{chambers}\}$ transitively
b. $W \curvearrowright \{\text{bases}\}$ transitively
c. $\forall \alpha \Phi,\ \exists w \in W \mathrel{\Big|} w(\alpha) \in Pi$
d. $W := \left\{ S_\alpha \mathrel{\Big|} \alpha \in \Phi \right\} = \left\langle S_\alpha \mathrel{\Big|} \alpha \in \Pi \right\rangle := W_0$
e. $W \curvearrowright \{\text{bases}\}$ simply transitively, i.e. $w(\Pi) = \Pi \implies w = e$.

I.e. we can describe the Weyl group using only simple reflections

Proof: We will prove (a) – (c)$ for $W_0#.

Proof of (a): Recall the fundamental chamber, $C(\Pi) = \left\{ x \in E \mathrel{\Big|} (x, a) > 0\ \forall \alpha \in \Pi \right\}$. We want to show that any chamber C is equal to $wC(\Pi)$.

Pick $\gamma \in C$ and $g \in W_0$ such that $(g(\gamma), \rho) = \max \left\{ (w(\gamma), \rho) \mathrel{\Big|} w \in W_0 \right\}$, which exists because W_0 is a finite group.

For all $\alpha \in \Pi, S_\alpha g \in W_0$ and so by maximality we have [

$$\begin{aligned}(g(\gamma),\rho) &\geq (s_\alpha g(\gamma)\rho) \\ &= (g(\gamma), S_\alpha(\rho)) \\ &= (g(\gamma), \rho - \alpha) \\ &= (g(\gamma),\rho) - (g(\gamma),\alpha).\end{aligned}$$

]

and so $(g(\gamma), 0) \geq 0$, because this can never be an equality since $\gamma \in C$. Thus $g(\gamma) \in C(\Pi)$.

Proof of (b):

This holds because there a correspondence between $\{C(\Pi)\} \iff \{\text{bases}\Pi\}$.

Proof of (c):

It suffices to show that $\alpha \in \Phi$ lies in some base $\Pi' = W(\Pi)$. Note that $\beta \neq \alpha \implies H_\beta \neq H_\alpha$, and so we can pick a $\gamma \in H_\alpha \bigcap H_\beta^c$ for every $\beta \in \Phi \setminus \pm\alpha$. Since $\langle\gamma,\ \alpha\rangle = 0$ but $\langle\gamma,\ \beta\rangle \neq 0$ for all $\beta \neq \pm\alpha$, we can choose some $\varepsilon > 0$ such that $|\langle\gamma',\ \beta\rangle| > \varepsilon$ for every $\beta \neq \pm\alpha$. Then $\gamma' \in C(\Pi')$ and thus $\alpha \in \Pi'$.

Proof of (d):

By definition, $W_0 \subseteq W$, so we need to show the reverse containment. For all $\alpha \in Phi$, we want to show $S_\alpha \in W_0$. By (c), there exists a $w \in W_0$ such that $w(\alpha) \coloneqq \beta \in \Pi$, Then $S_\beta = S_{w(\alpha)} = ws_\alpha w^{-1}$. So $S_\alpha = w^{-1}S_\beta w$, where each term is in W_0, so the whole thing is in W_0 as well.

Proof of (e):

Suppose $W(\Pi) = \Pi$. Let $W = S_{\alpha_1} \cdots S_{\alpha_\ell}$ be a reduced expression, which exists by (d). By corollary 10.2b, we have $W(\alpha_\ell \in \Phi^-)$. But this forces $w = e$. ■

Remarks:

By (d), there is a well-defined notion of *length* for $w \in W$. We will now show that $\ell(w) = n(w) \coloneqq \#N_w \coloneqq \#\left\{\alpha \in \Phi^+ \,\middle|\, W(\alpha) \in \Phi^-\right\}$, i.e. the number of roots that get sent to a negative root.

23 Wednesday October 9

Last time:

We have the Weyl group $W \coloneqq \left\{S_\alpha \,\middle|\, \alpha \in \Phi\right\} = \left\{S_\alpha \,\middle|\, S_\alpha \in \Pi\right\}$. If $W \ni w = \prod i = 1^\ell W_{\alpha_i}$ is a product of simple reflections, then W is said to be *reduced* if ℓ is the smallest among all such products. Call $\ell(w)$ the length of W and let $n(W) = \#N_W$. By Corollary 10.2b, $N_W = \left\{\alpha \in \Phi^+ \,\middle|\, W(\alpha) \in \Phi^-\right\}$, and if $W = \prod S_{\alpha_i}$ is reduced, then $w(\alpha_j) \in \Phi^-$.

Lemma: $\ell(w) = n(w)$.

Proof: Done in class, but see Humphrey's.

23.1 Classification

23.1.1 Cartan Matrix

Fix a base $\Pi \subset \Phi$ of rank ℓ.

Definition: Fix an order $(\alpha_1, \alpha_2, \cdots \alpha_\ell$ of Π. Then the *Cartan matrix* is given by $A_{ij} = \left\langle \alpha_i,\ \alpha_j^\vee \right\rangle \in \mathrm{Mat}(\ell \times \ell, \mathbb{Z})$.

Examples:

Facts:

a. A depends on the chosen ordering of Π.
b. A is independent of the choice of Π.
c. A is invertible.
d. A uniquely determines the root system (up to isomorphism). I.e., if $A(\Phi) = A(\Phi')$ then there is an isomorphism $E \xrightarrow{\phi} E$ on the underlying Euclidean space such that $phi(\Phi) = \Phi'$ and $\langle \alpha,\ \beta^\vee \rangle = \langle \phi(\alpha),\ \phi(\beta)^\vee \rangle$ for all $\alpha, \beta \in \Phi$.

23.1.2 Dynkin Diagrams

Recall from Lemma 9.4 that $a_{ij}a_{ji} \in \{0, 1, 2, 3\}$.

Definition: Given a Cartan matrix A, its Coxeter diagram is an undirected multigraph $\Gamma = (I, E)$ where I is a vertex set and the edge set is given by edges between vertices corresponding to i, j (where $i \neq j$) with weight $a_{ij}a_{ji}$.

Examples:

Note that these diagrams don't encode which roots are longer, so we can decorate these diagrams with arrows to indicate this and obtain a partially-directed multigraph.

Definition: A *Dynkin diagram* is the partially-directed multigraph obtained from the Coxeter diagram by adding arrows on the double or triple edges between i, j precisely when $|a_i| > |a_j|$. (Note that this also occurs when $|a|_{ij} < |a|_{ji}$)

Definition: A non-empty root system is *irreducible* if $\Phi \neq \Phi_1 \oplus \Phi_2$ for some nonempty root system Φ_2 where $\alpha \in \Phi_1, \beta \in \Phi_2 \implies \langle \alpha,\ \beta \rangle = 0$.

For example: $\Phi(A_1 \times A_1)$ can be written as $\Phi(A_1) \oplus \phi(A_1)$ since the off-diagonal entries were zero, so it is reducible.

Facts:

a. Φ is irreducible iff the Dynkin diagram is connected
b. Φ can be uniquely written as the union of irreducible root systems (where the multiplicity of each system appearing is well-defined)

Thus to classify root systems, it suffices to classify connected Dynkin diagrams.

Examples of Dynkin diagrams:

24 Friday October 11

Recall from last time the Dynkin diagrams. If Φ is irreducible, then its diagram is one of the following:

Definition: A subset $A = \{v_1, \cdots, v_n\} \subseteq E$ is *admissible* iff

1. A is linearly independent.
2. $\langle v_i,\ v_i\rangle = 1$ for all i, and $\langle v_i,\ v_j\rangle \leq 0$ if $i \neq j$.
3. $s_{ij} = 4\langle v_i,\ v_j\rangle^2 \in \{0,1,2,3\}$ if $i \neq j$.

Define a graph $\Gamma_A = (V_A, E_A)$ where $V_A = A$ and $E_A = \left\{ s_{ij} \;\middle|\; i \neq j \right\}$. If $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subseteq \Phi$ is a base, then $A := \left\{ v_i = \dfrac{\alpha_i}{\sqrt{\langle \alpha_i,\ \alpha_i\rangle}} \right\}$.

Lemma:

a. If A is admissible, then $\#\left\{ (v_i, v_j) \in E_A \;\middle|\; 4\langle v_i,\ v_j\rangle^2 \neq 0 \right\} \leq |A| - 1$, and Γ_A contains no graph cycles.
b. $\deg V_i \leq 3$ for all i.
c. If Γ_A contains a path $p_1 \to \cdots \to p_t$, then $A' := \{p\} \bigcup A \setminus \{p_1, \cdots, p_t\}$ where $p := \sum p_i$. Moreover, $\Gamma_{A'}$ is obtained from Γ_A by contracting this path onto p.

Proof of theorem:

Assume the lemma holds. Let Γ be the Coxeter diagram of Φ; then Φ is connected.

Case 1: Γ has a triple edge. But then both vertices on this edge have degree 3, so this is the maximal number of edges between them. But since Γ must be connected, this is everything.

Case 2: Γ has no triple edges but some double edge. We will first show that Γ has only one double edge.

Suppose otherwise; then Γ has at least two double edges occurring. Without loss of generality (e.g. by taking a subgraph), these are connected by a path of single edges. By the lemma, we can contract this path to get an admissible subset. But then there is a vertex of degree 4, contracting $\deg V_i \leq 3$ for all i.

Now we'll show that Γ has no branching point, i.e. a vertex of degree exactly 3. If this occurs, then a double edge is connected to such a vertex by a path. Contracting this path yields a vertex of degree 4, again a contradiction.

By these two statements, Γ has the general form: [

$$\Gamma = v_1 \to \circ \to \cdots \to v_p \to\to w_q \to \circ \to \cdots \to w_1.$$

]

Let $v = \sum i v_i$ and $w = \sum i w_i$, then $\langle v,\ v\rangle = \dfrac{1}{2}p(p+1)$, and $\langle w,\ w\rangle = \dfrac{1}{2}q(q+1)$. Note that $\langle v_i,\ w_j\rangle = -1/\sqrt{2}$ if $i = p$ and $j = q$, and 0 otherwise.

Thus $\langle v,\ w\rangle = \cdots = \dfrac{1}{2}p^2q^2$. By Cauchy-Schwarz, this is strictly less than $\langle v,\ v\rangle\langle w,\ w\rangle = \dfrac{1}{4}p(p+1)q(q+1)$. We then obtain $(p-1)(q-1) < 2$. Supposing wlog that $p \geq q$, we have either $p = q = 2$, in which case we get $\circ \to \circ \to\to \circ \to \circ$. Otherwise $q = 1$, and we get $\circ \to \cdots \to \circ \to\to \circ$.

Case 3: Γ has only single edges. We want to show Γ has only one branching point, i.e. a vertex of degree 3. If it has 2, we can contract the intermediate path to get a vertex of degree 4. So we have the following situation:

Define $x = \sum ix_i, y = \sum iy_i, w = \sum iw_i$, and $\hat{w}, \hat{x}, \hat{y}$ to be their normalization. Then $B = \{b_i\} := \{\hat{w}, \hat{y}, \hat{y}, z\}$ is orthonormal and linearly independent, so we can apply Gram-Schmidt. This yields a $z' \neq 0$ such that

$$z = \sum \left\langle z,\ \hat{b}_i \right\rangle \hat{b}_i$$

In particular, $\langle z,\ z' \rangle z' \neq 0z'$, otherwise z is a linear combination of the x_i, y_i, w_i. Thus $\langle z,\ \hat{w} \rangle^2 + \langle z,\ \hat{x} \rangle^2 + \langle z,\ \hat{y} \rangle^2 > 1$. We can compute $\langle z,\ \hat{w} \rangle = \dfrac{-q/2}{\sqrt{\frac{1}{2}q(q+1)}}$, and so $\langle z,\ \hat{w} \rangle^2 = \dfrac{q}{2(q+1)}$.

From this, we can obtain $\dfrac{1}{q+1} + \dfrac{1}{r+1} + \dfrac{1}{p+1} > 1$. We can assume $p \geq q \geq r \geq 1$, since these correspond to the lengths of paths in the above image. This allows us to do some case-by-case analysis.

Using this, we find $\dfrac{3}{r+1} > 1$, and so $r = 1$ must hold. Similarly, $\dfrac{2}{q+1} > \dfrac{1}{2}$, which forces $q =\in \{1, 2\}$.

Supposing $r = q = 1$, then we get type D_ℓ because p can be anything. Supposing otherwise that $r = 1, q = 2, p \in \{2, 3, 4\}$, we get type E.

25 Monday October 14

Last time:

Theorem: If Φ is irreducible, then the Dynkin diagram is given by $A - G$.

Definition: A subset $A = \{v_1, \cdots, v_n\}$ is *admissible* if

1. A is a linearly independent set,
2. $\langle v_i,\ v_i \rangle = 1$ for all i, and $\langle v_i,\ v_j \rangle \leq 0$ if $n \neq 0$.
3. $4\langle v_i,\ v_j \rangle^2 \in \{0, 1, 2, 3\}$ if $i \neq j$.

Thus the graph $\Gamma_A = (V_A, E_A)$ is given by $V_A = A$ and $E_A = \left\{ v_i \xrightarrow{4\langle v_i,\ v_j \rangle^2} v_j \;\middle|\; i \neq j \right\}$.

Lemma:

a. If A is admissible, then the number of edges such that $4\langle v_i,\ v_j \rangle \neq 0$ is at most $|A| - 1$.
b. For every i, we have $\deg v_i \leq 3$.
c. If Γ_A contains a straight path of length t, then the graph Γ' obtained by contracting this path is also admissible.

Let p be the point obtained by contracting such a path.

Proof of (a): If $\{p_1, \cdots, p_t\}$ are linearly independent, then $p \neq 0$. Thus by positive-definiteness, we have $0 <_{pd} \langle p,\ p \rangle =_{\#2} t + \sum_{i<j} 2\langle p_i,\ p_j \rangle$. Then $t > \sum_{i<j} (-2)\langle p_i,\ p_j \rangle = \sum_{i<j} \sqrt{4\langle p_i,\ p_j \rangle^2}$, where the

quantity in the square root is the number of edges, which is thus greater than or equal to the number of pairs connected.

Proof of (b): Fix i. Let $u_1 \cdots u_k$ be the vertices in A that are connected to v_i by a single edge. Then by (a), we have $\langle u_i,\ u_j\rangle = 0$ for all $i \neq j$.

Then the set $\{u_1, \cdots, u_k\}$ is an orthonormal basis for their span. Applying Gram-Schmidt, we can write each $v_i = \sum_{j=0}^{k} \langle v_i,\ u_j\rangle u_j$, where we pick u_0 such that the new set $\{u_0\} \bigcup \{u_1, \cdots, u_k\}$. Then $\langle v_i,\ u_0\rangle \neq 0$ for all i; otherwise we would have $\{u_1, \cdots, u_k, v_i\}$ would be linearly dependent, since $v_i = \sum c_i u_i$ from above, which contradicts our initial axiom/assumption. Then $1 = \langle v_i,\ v_i\rangle$ by A2, which equals $\sum_{j=0}^{k} \langle v_i,\ u_j\rangle^2 = \langle v_i,\ u_0\rangle^2 + \sum_{j=1}^{k} \langle v_i,\ u_j\rangle^2$, where the first term is strictly positive.

But then $1 > \sum_{j=1}^{k} \langle v_i,\ v_j\rangle^2 \geq \frac{k}{4}$ by A3, which then forces $k = \deg v_i \leq 3$.

Proof of (c): The conditions of A1 are satisfied. For A2, we have

$$\langle p_i,\ p_j\rangle = \begin{cases} -\frac{1}{2} & |i-j| = 1 \\ 0 & |i-j| > 1 \\ 1 & i = j. \end{cases}$$

We then have $\langle p,\ p\rangle = t + 2\sum i < j\langle p_i,\ p_j\rangle = t + 2\sum_{i=1}^{t-1} \langle p_i,\ p_{i+1}\rangle = 1$. Thus $\langle p,\ v_i\rangle = \sum_{j=1}^{t} \langle p_j,\ v_i\rangle \leq 0$.

For A3, fix $v_i \in A'$. Then v_i is connected (by a single edge) to at most one point p_j, otherwise there would be a cycle. Thus

$$\langle v_i,\ p\rangle = \begin{cases} \langle v_i,\ p_j\rangle & \text{if } v_i \text{ is connected to } p_j \\ 0 & \text{else.} \end{cases}$$

We thus have $4\langle v_i,\ p\rangle^2 = 4\langle v_i,\ p_j\rangle \in \{0, 1, 2, 3\}\, 1 v_i \sim p_j$.

25.1 Construction of Root Systems and Automorphisms

We'll start with the construction of types $A - G$.

Theorem: For Dynkin diagrams of type $A - G$, there exists an irreducible root system having the given diagram.

Proof: By explicit construction. Fix an orthonormal basis $\{\varepsilon_i\}$.

Type A_ℓ: Let

$$\Phi = \left\{ \varepsilon_i - \varepsilon_j \;\middle|\; 1 \leq i \neq j \leq \ell + 1 \right\}$$

Then $|\Phi| = \ell^2 + \ell$, and $\Pi = \left\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \;\middle|\; 1 \leq i \leq \ell\right\}$. We then find that $\dim \mathfrak{g} = \ell^2 + 2\ell$.

Note that we don't know anything about $\mathfrak{g}$ yet, but already know its dimension.

Example: A_2. We have $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_3 - \varepsilon_2\}$. Then $A = (a_{ij})$ with $a_{ij} = \left\langle a_i,\ a_j^\vee \right\rangle$, and $\alpha_1^\vee = \dfrac{2\alpha_1}{\langle \alpha_1,\ \alpha_1 \rangle} = \dfrac{2(\varepsilon_1 - \varepsilon_2)}{\langle \varepsilon_1 - \varepsilon_2,\ \varepsilon_1 - \varepsilon_2 \rangle} = \varepsilon_1 - \varepsilon_2 = \alpha_1$. Doing the computations, it turns out that $\langle \alpha_1,\ \alpha_2^\vee \rangle = -1$, $\langle \alpha_2,\ \alpha_1^\vee \rangle = -1$, and $\langle \alpha_i,\ \alpha_i^\vee \rangle = 2$.

Thus $A = [2, -1; -1, 2]$, which has Dynkin diagram given by:

Type B_ℓ: Recall that these have one "short root":

Then $\Phi = \left\{\pm\varepsilon_j, \pm\varepsilon_j \;\middle|\; 1 \leq i \neq j \leq \ell\right\} \bigcup \left\{\pm\varepsilon_i \;\middle|\; 1 \leq i \leq \ell\right\}$, and we have $\Pi = \left\{\alpha_i = \varepsilon_i - \varepsilon_{i-1} \;\middle|\; 1 \leq i \leq \ell - 1\right\} \bigcup$

After carrying out the computation, we have the following Cartan matrix:

And $\dim \mathfrak{g} = 2\ell^2 + \ell$, since $|\Phi| = 2\ell(\ell - 1) + 2\ell = 2\ell^2$.

Type D_ℓ:

We obtain $\Phi = \left\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \;\middle|\; 1 \leq i \leq \ell - 1\right\} \bigcup \{\alpha_\ell \coloneqq \varepsilon_{\ell-1} + \varepsilon_\ell\}$. We then find $\langle \alpha_{\ell-1},\ \alpha_\ell^\vee \rangle = 0$ and $\langle \alpha_{\ell-2},\ \alpha_\ell^\vee \rangle = -1$.

Type E_ℓ: We have $\Pi(E_\ell) = \Pi(D_{\ell-1}) \bigcup \left\{\alpha_\ell \coloneqq -\dfrac{1}{2}\displaystyle\sum_{i=1}^{8} \varepsilon_i\right\}$.

This yields $|\Phi| = 72, 126, 240$ and $\dim \mathfrak{g} = 78, 133, 248$, corresponding to $\ell = 6, 7, 8$.

More results on exceptional Lie Algebras:

26 Wednesday October 16 (TODO)

Todo

27 Friday October 18 (TODO)

Todo

28 Monday October 21

28.1 Chapter 5: Existence Theorem

28.1.1 Universal Enveloping Algebra (UAE)

Some applications/motivations for UAEs:

1. Groups G are to group algebras $F[G]$ as Lie algebras $\mathfrak{g}$ are to UAE $U(\mathfrak{g})$. Any $\mathfrak{g}$-module then becomes a module over a ring, so the general theory applies.

2. PBW theorem: this yields a concrete F-basis of $U(\mathfrak{g})$. There is a triangular decomposition $U(\mathfrak{g}) = U(h) \otimes U(f) \otimes U(n)$. This allows constructing the Vermo module (and hence irreducible modules) for $\mathfrak{g}$, allowing for a description of BGG Category $\mathcal{O}$.
3. Harish-Chandra theorem: $Z)U(\mathfrak{g})) = S(\mathfrak{g})^W$. This characterizes central characters $\chi : Z(U(\mathfrak{g})) \to F$, which further allows describing the blocks of $\mathcal{O}$, i.e. when two irreducible modules have non-trivial extensions.
4. $U(\mathfrak{g})$ deforms to a quantum group $U_q(\mathfrak{g})$.

28.1.2 Tensor and Symmetric Algebras

Definition: For V a f.d. vector space, the *tensor algebra* over V is given by $T(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)$ where $T^n(V) = \otimes_{i=1}^n V$ with an associative multiplication $T^a \times T^b \to T^{a+b}$ given by $(\otimes_{i=1}^a v_i, \otimes_{i=1}^b w_i) \mapsto \otimes_{i=1}^a v_i \otimes \otimes_{i=1}^b w_i$.

The tensor algebra satisfies a universal property: given any F-linear map $\phi : V \to A$. (See phone image)

Definition: The symmetric algebra on V is given by $S(V) = T(V)/I$ where $I = \langle x \otimes y - y \otimes x \rangle \trianglelefteq T(\mathfrak{g})$.

Some facts:

a. There is a natural grading $S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V)$ where $S^0(V) = F, S^1(V) = V, S^n(V) = T^n(V)/(I \bigcap T^n V)$,
b. If $\{x_i\}^n$ is a basis of V, then $S(V) \cong F[x_1, \cdots, x_n]$.

28.1.3 Construction of UEA

Definition: For $\mathfrak{g}$ a lie algebra, define $U(\mathfrak{g}) = T(\mathfrak{g})/J$ where $J = \langle x \otimes y - y \otimes x - [x, y] \rangle \trianglelefteq T(\mathfrak{g})$.

Thus we have the following type of equation that holds in $U(\mathfrak{g})$:

[

$$v_1 \otimes \cdots \otimes v_a \otimes (x \otimes y) \otimes w_1 \otimes \cdots \otimes w_b =$$
$$v_1 \otimes \cdots \otimes v_a \otimes (y \otimes x) \otimes w_1 \otimes \cdots \otimes w_b + v_1 \otimes \cdots \otimes v_a \otimes ([x, y]) \otimes w_1 \otimes \cdots \otimes w_b.$$

]

Proposition: $U(\mathfrak{g})$ has a universal property: given a lie algebra hom $\theta : \mathfrak{g} \to \mathcal{A}$ where $\mathcal{A}$ is any unital associative F-algebra with a lie bracket, there exists a unique $\psi : U(\mathfrak{g}) \to \mathcal{A}$ making the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\theta"'] & U(\mathfrak{g}) \arrow[d, dotted, "\exists\psi"] \\ & \mathcal{A} \end{tikzcd}$$

where $\iota : \mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ is given by $x \mapsto x + J$.

The upshot: There is a 1 to 1 correspondence [

$$\left\{\begin{matrix}\text{Lie algebra}\\ \text{representations}\\ \mathfrak{g}\to\mathfrak{gl}(V)\end{matrix}\right\} \to \left\{\begin{matrix}\text{Algebras from}\\ U(\mathfrak{g})\to\mathrm{End}(V)\end{matrix}\right\}$$

$$\theta \mapsto \psi$$

$$\theta = \psi\circ\iota \mapsfrom \psi$$

]

Proof (existence):

$\theta : \mathfrak{g} \to \mathcal{A}$ extends to an algebra homomorphism $\tilde{\theta} : T(\mathfrak{g}) \to \mathcal{A}$ given by $\otimes_{i=1}^{n} x_i \mapsto \prod \theta(x_i)$. Note that $\tilde{\theta}(x\otimes y - y\otimes x - [x,y]) = \theta(x)\theta(y) - \theta(y)\theta(x) - \theta([x,y]) = 0$, and thus $J \trianglelefteq \ker\tilde{\theta}$ and $\phi : T(\mathfrak{g})/J \to \mathcal{A}$ is well-defined.

Uniqueness: Suppose that $\psi' : U(\mathfrak{g}) \to \mathcal{A}$ is another hom ψ' such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\theta"'] & U(\mathfrak{g}) \arrow[d, bend right, "\psi'"'] \arrow[d, bend left, "\psi"] \\
& \mathcal{A}
\end{tikzcd}$$

Since $T(\mathfrak{g})$ is generated by $T^1(\mathfrak{g})$, $U(\mathfrak{g})$ is generated by $\iota(\mathfrak{g}) \in U(\mathfrak{g})$. Thus for all $x \in \mathfrak{g}$, $\psi\circ\iota(x) = \theta(x) = \psi'\circ\iota(x)$ by the commuting of each triangle. We then have $\psi = \psi'$ on $\iota(\mathfrak{g})$, and hence on $U(\mathfrak{g})$.

28.1.4 PBW Theorem

PBW: Poincaré-Birkhoff-Witt

Theorem: If $\mathfrak{g}$ has a basis $\{x_i\}_{i\in I}$ where $\leq$ is a total order on I, then let $y_i \coloneqq \iota(x_i) \in U(\mathfrak{g})$. Then $U(\mathfrak{g})$ has an F-basis called a *PBW basis* which is given by [

$$\left\{ y_{i_1}^{r_1}\cdots y_{i_n}^{r_n} \;\middle|\; n\in\mathbb{N}, r_i\in\mathbb{N}, i_1\leq\cdots\leq i_n \right\}.$$

]

We refer to each term appearing as a *PBW monomial*.

Examples:

Type A, $\mathfrak{g} = \mathfrak{sl}(2,F) = \langle f,h,e\rangle$. Pick an order $x_1 = f, x_2 = h, x_3 = e$, so $f < h < e$.

Then $U(\mathfrak{g})$ has a basis [

$$B = \{1\}\bigcup\left\{f^{r^1}\right\}\left\{f^{r^1}h^{r_2}\right\}\bigcup\{f^{r_1}h^{r_2}e^{r_3}\}\bigcup\{h^{r_1}\}\bigcup\{f^{r_1}e^{r_2}\}\bigcup\{e^{r_1}\}\bigcup\{h^{r_1}e^{r_2}\}.$$

]

i.e. $B = \left\{ f^a h^b e^c \;\middle|\; a,b,c\in\mathbb{N}\right\}$.

If you pick a different order, say $f < e < h$, then we obtain $B = \left\{ f^a e^b h^c \;\middle|\; a,b,c\in\mathbb{N}\right\}$.

29 Wednesday October 23

Recall from last time:

For $\mathfrak{g}$ a lie algebra, we define $T(\mathfrak{g})$ the tensor algebra, and the universal enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$ where $x \otimes y - y \otimes x \sim [x, y]$.

We also described the *PBW Theorem*, which provides a basis for $U(\mathfrak{g})$.

Proof of PBW Theorem:

We have $T(\mathfrak{g}) = \operatorname{span}\left\{ x_{j_1} \otimes \cdots \otimes x_{j_k} \;\middle|\; j_1, \cdots, j_k \in I \right\}$, where we note that there are not required to be ordered. Thus $U(\mathfrak{g}) = \operatorname{span}\left\{ y_{j_1} \otimes \cdots \otimes y_{j_k} \;\middle|\; j_1, \cdots, j_k \in I \right\}$, where which are again not required to be ordered. We would thus like to express every term here as some linear combination of monomials in the y_{i_j} with increasing indices. We proceed by inducting on k, the number of tensor factors occurring. The base case is clear.

For $k > 1$, supposing that the element is *not* a PBW monomial, then there is some inversion in the indices $(j_1, \cdots, j_k)$, i.e. there is at least one i such that $j_{i+1} < j_i$. Now for any two indices $a, b \in I$, we have

$$\iota(x_b \otimes x_a) = \iota(x_a \otimes x_b + [x_b, x_a]) \implies y_b y_a = y_a y_b + \iota([x_b, x_a])$$

Since $[x_b, x_a] = \sum_t F x_t$ and $\iota[x_b, x_a] = \sum_t F y_t$.

But then $y_{j_1} \cdots y_{j_k} = y_{i_1} y_{i_2} \cdots y_{j_k} +$ lower degree terms where $i_1 \leq i_2 \cdots i_k$ is a non-decreasing rearrangement of the j_i. By the inductive hypothesis, the lower degree terms are spanned by PBW monomials, so we're done.

Proof of linear independence:

Claim: let $\mathbf{x} \coloneqq x_{j_1} \otimes \cdots \otimes x_{j_n}$ for an arbitrary indexing sequence, and $\mathbf{x}_{(k)}$ be this tensor with the j_k and j_{k+1} terms swapped, and $\mathbf{x}_{[k]}$ be this tensor with $x_{j_k}, x_{j_{k+1}}$ replaced by their bracket.

Then there exists a linear map [

$$\begin{aligned} f : T(\mathfrak{g}) &\to R \coloneqq F[\{z_i\}_{i \in I}] \\ f(x_{i_1} \otimes \cdots \otimes x_{i_n}) &= z_{i_1} \cdots z_{i_n} \\ f(\mathbf{x} - \mathbf{x}_{(k)}) &= f(\mathbf{x}_{[k]}). \end{aligned}$$

]

By collecting terms, we can write

$$\mathbf{x} - \mathbf{x}_{(k)} - \mathbf{x}_{[k]} = x_{j_1} \otimes \cdots \otimes x_{j_{k-1}} \otimes \left((x_{j_k} \otimes x_{j_{k+1}}) - (x_{j_{k+1}} \otimes x_{j_k}) - [x_{j_k}, x_{j_{k+}}] \right) \otimes \cdots$$

So we can take J to be the ideal generated by all elements of this form, and we find that $J \subset \ker f$, and thus f descends to a map $\overline{f}$ on $U(\mathfrak{g})$. We then know that if $\overline{f}$ applied to any PBW monomial is $z_{i_1}^{r_1} \cdots z_{i_n}^{r_n}$, which are linearly independent in R, then any PBW monomial will be linearly independent in $U(\mathfrak{g})$.

Proof of claim:

For each $\mathbf{x}$, define an *index*

$$\lambda(\mathbf{x}) = \#\left\{(a,b) \in \{1,\cdots,n\}^2 \;\middle|\; a<b, j_a<j_b\right\}.$$

Then

$$\left\{\mathbf{x} \;\middle|\; \lambda(\mathbf{x})=0\right\} = \left\{x_{i_1}\otimes\cdots\otimes x_{i_n} \;\middle|\; i_1\leq\cdots\leq i_n\right\}.$$

So set $T^{n,k} = \left\{\mathbf{x}\in T^n(\mathfrak{g}) \;\middle|\; \lambda(\mathbf{x})\leq k\right\}$; we then have a filtration $T^{n,0}\hookrightarrow T^{n,1}\hookrightarrow\cdots\hookrightarrow T^n(\mathfrak{g})$.

Step 1: We'll construct f by induction on n.

For $n>0$, set $f(\mathbf{x}) = z_{j_1}\cdots z_{j_n}$ if $\lambda(\mathbf{x})=0$. We now induct on the index k at a fixed power $n>0$. The base case is clear.

For $k>0$, there exists an inversion $(\ell,\ell+1)$, i.e. some indices $i_\ell > ß_{\ell+1}$. Set $f(\mathbf{x}) = f(\mathbf{x}_{(\ell)}) - f(\mathbf{x}_{[\ell]})$, where the LHS is in $T^{n,k}$ and the RHS terms are in $T^{n,k-1}$ and $T^{n-1}(\mathfrak{g}$ respectively.

Step 2: We'll check that f is well-defined.

In the above definition, note that $f(\mathbf{x})$ can be defined using different inversions of the indices, we'd like to show that these yield the same map.

Let $(\ell,\ell+1)$ and $(\ell',\ell'+1)$ be two distinct inversions. Then set

[

$$\begin{aligned} a &= x_{j_\ell}\\ b &= x_{j_{\ell+1}}\\ c &= x_{j'_\ell}\\ d &= x_{j_{\ell'+1}}\\ &. \end{aligned}$$

]

Then we have several cases:

Case 1: $\ell+1<\ell'$.

Then [

$$\begin{aligned} f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\cdots b\otimes a\cdots c\otimes d\cdots)\\ +f(\cdots\otimes[a,b]\otimes\cdots c\otimes d\cdots)\\ &= f(\cdots b\otimes a\cdots d\otimes c\cdots) + f(\cdots b\otimes a\cdots[c,d]\cdots) + f(\cdots\otimes[a,b]\otimes\cdots d\otimes c\cdots\\ &= f(\mathbf{x}_{(\ell']}) + f(\mathbf{x}_{[\ell']}). \end{aligned}$$

]

Case 2: $\ell+1=\ell'$

Then [

$$
\begin{aligned}
f(\mathbf{x}_{(\ell)}) + f(\mathbf{x}_{[\ell]}) &= f(\cdots b \otimes a \otimes x) + f(\cdots [a,b] \otimes c) \\
&= f(b \otimes c \otimes a) + f(c \otimes [a,b]) + f(b \otimes [a,c]) + f([[a,b],c]) \\
&= f(c \otimes b \otimes a) + f(c \otimes [a,b]) + f(b \otimes [a,c]) + f([[a,b],c]) + f(b \otimes [a,c]) + f(a \otimes [b,c]) + f([[b,
\\
&= f(\mathbf{x}_{(\ell')}) + f(\mathbf{x}_{[\ell']}).
\end{aligned}
$$

]

where the last equality is found by expanding the expression backwards.

30 Friday October 25

Theorem (PBW): The universal enveloping algebra $U(\mathfrak{g})$ has a basis consisting of the PBW monomials. If we fix a basis $\left\{x_i \;\middle|\; i \in I\right\}$ of $\mathfrak{g}$ with a total order, then $\left\{y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \;\middle|\; n \in \mathbb{N} > 0,\ i_j \in I,\ r_i \geq 1\right\}$.

We will construct a map [

$$
\begin{aligned}
\iota : \mathfrak{g} &\to U(\mathfrak{g}) \\
x_i &\mapsto x_i + J \coloneqq y_i,
\end{aligned}
$$

]

where we can recall that $U(\mathfrak{g}) \coloneqq T(\mathfrak{g})/J$ where J was an ideal of specific relations.

Corollary:

a. The map ι is injective.
b. The map ι has no *zero divisors*.

> We will use property (b) to study properties of Verma modules

Proof of (a): If $\sum c_i x_i \in \ker(\iota)$, then [

$$
\begin{aligned}
0 = \iota(\textstyle\sum c_i x_i) &= \textstyle\sum c_i y_i \\
&\implies c_i = 0 \quad \forall i \text{ since } \{y_i\} \subsetneq \{ \text{ PBW monomials } \} \\
&\implies \ker(\iota) = 0.
\end{aligned}
$$

] Proof of (b): An arbitrary element in $U(\mathfrak{g})$ is of the form [

$$
\begin{aligned}
a = \textstyle\sum c_{\mathbf{i},\mathbf{r}}^{a} y_{i_1}^{r_1} \cdots y_{i_n}^{r_n} \text{ for some } c \in F \\
\coloneqq f_a(\mathbf{y}) + \text{ terms with smaller total degree }.
\end{aligned}
$$

]

where f is defined by picking out only those terms of highest total degree, e.g. $f(2y_1 + y_1y_2y_3 + y_2^2) = y_1y_2y_3$, which is of total degree 3.

We want to show that $a \neq 0$ and $b \neq 0$ then $ab \neq 0$, i.e. $(f_a(\mathbf{y}) + \cdots)(f_b(\mathbf{y}) + \cdots) \neq 0$.

Recall that $y_a y_b = y_b y_a + \sum_{a,b \in I}$ degree 1 monomials . Thus $f_a(\mathbf{y})(f_b(\mathbf{y})) \coloneqq f_a f_b(\mathbf{y}) + \sum$ terms of smaller total d

Here we define $f_a(\mathbf{y})f_b(\mathbf{y})$ by e.g. if $b = y_2$, then $f_b(\mathbf{y}) = y_2$, and $f_a(\mathbf{y})f_b(\mathbf{y}) = y_1y_2y_3y_2 = y_1y_2^2y_3 + y_1y_2[y_3, y_2]$. Note that the leading term is of total degree 4, and the remaining term is a sum of lower degree terms.

30.1 Free Lie Algebra

Let $X := \left\{ x_i \;\middle|\; i \in I \right\}$ be a set. Define the *free associative algebra* $\mathcal{F}(X)$ as $\left\{ \sum_k c_{\mathbf{i}} X_{\mathbf{i}} \;\middle|\; \mathbf{i} = (i_1, \cdots, i_k) \in I^k,\ c_{\mathbf{i}} \in \right.$

Then the associated *free lie algebra* $\mathcal{FL}(x) = \bigcap_{\mathfrak{g}} \mathfrak{g}$ where $X \subseteq \mathfrak{g} \subseteq \mathcal{F}(X)$ is a containment of lie algebras.

Let $\iota : X \hookrightarrow \mathcal{FL}(X)$.

Proposition:

a. $\mathcal{FL}(X)$ satisfies a universal property – for any map $\theta : X \to \mathfrak{g}$ a lie algebra, there exists a unique ψ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "\theta"'] & \mathcal{FL}(X) \arrow[d, dotted, "\exists!\psi"] \\
& \mathfrak{g}
\end{tikzcd}$$

b. $U(\mathcal{FL}(X)) = \mathcal{F}(X)$.

Upshot: we can define a Lie algebra $\mathfrak{g}$ using generators and relations, and define $\mathfrak{g} := \mathcal{FL}(X)/(R)$ for some set of relations R.

30.2 Generators and Relations

Recall that we have a correspondence [

$$\left\{ \mathfrak{g} \;\middle|\; \mathfrak{g} \text{ is a semisimple Lie Algebra } \right\}$$

$$\begin{aligned}
&\iff \{\Phi,\ \text{root systems }\} \\
&\iff \{\text{Dynkin diagrams (Cartan Matrices)}\} \\
(\mathfrak{g}, \mathfrak{h}) \to \Phi, \quad \{a_i\} \subseteq \{a\} := \Pi \subseteq \Phi \qquad &\mapsto A_{i,j} = \left\langle \alpha_i,\ \alpha_j^\vee \right\rangle \\
\mathfrak{g}(A) < -_?\Phi \qquad &< -A.
\end{aligned}$$

] We had an explicit construction to go from Dynkin diagrams to root systems, and an existence theorem of Serre's will take root systems Φ and produce semisimple Lie algebras from them. The question will be whether or not there is a one-to-one correspondence here, and that's what we'll spend the rest of the semester showing.

30.3 Cartan/Serre Relations

Recall from (8.3): For all $\alpha \in \Phi$, we have $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, then there exists a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha := \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$, where $t_\alpha := \alpha = \kappa(t_\alpha, \cdot)$.

Fix $\Pi = \left\{\alpha_i \;\middle|\; i \in I\right\}$, and write $h_i := h_{\alpha_i},\ e_i = e_{\alpha_i}$ for each i. Then $\alpha_i(h_j) = a_{ij}$. Now fix $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha}$ such that $[e_i, f_i] = h_i$ for every $i \in I$.

Proposition: $\mathfrak{g}$ is generated by $\left\{e_i, f_i, h_i \;\middle|\; i \in I\right\}$.

We have the Cartan relations for each $i, j \in I$: [

$$\begin{aligned} [h_i, h_j] &= 0, \qquad [e_i, f_j] = \delta_{ij} h_i \\ [h_i, e_j] &a_{ji} e_j \qquad [h_i, f_j] = -a_{ji} f_j. \end{aligned}$$

]

as well as Serre relations for each $i \neq j$: [

$$(\text{ad } e_i)^{1-a_{ji}}(e_j) = 0 \quad (adf_i)^{1-a_{ji}}(f_j) = 0.$$

]

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \langle e_1 := e, f_1 := f, h_1 := h \rangle$ satisfies $[h, e] = 2e$ and $[h, f] = -2f$, and since there are no higher order relation, there are no Serre relations. So we get $A = (2)$ as a matrix.

Example: $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ is of type C_2, and is generated by $\langle e_1, e_2, f_1, f_2, h_1, h_2 \rangle$ satisfying

- $[h_1, h_2] = 0$
- $[h_1, e_1] = 2e_1$
- $[h_1, e_2] = -2e_2$
- $\cdots$

Then e.g. we have $(\text{ad } e_1)^{1-a_{ij}}(e_2) = (\text{ad } e_1)^3(e_2) = 0$.

31 Monday October 28

31.1 Algebra Generated by a Cartan Matrix

Last time: The claim was that for a Cartan matrix A, there is a lie algebra $\mathfrak{g}(A)$ that is semisimple with CSA $\mathfrak{h}$ and a root system Φ that defines that Cartan matrix A.

The algebra $\mathfrak{g}$ is generated by $\left\{e_i, f_i, h_i \;\middle|\; i \in I = \{1, 2, \cdots \ell\}\right\}$, with relations

[

$$\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ji} e_j \\ [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, f_j] &= -a_{ji} f_j, \end{aligned}$$

]

along with the Serre relations (which only appear in higher degrees): [

$$\begin{aligned} s_{ij}^+ &:= \text{ad } (e_i)^{1-a_{ji}}(e_j) = 0 \quad \text{if } i \neq j \\ s_{ij}^- &:= \text{ad } (f_i)^{1-a_{ji}}(f_j) = 0 \quad \text{if } i \neq j \end{aligned}$$

.

]

Proof:

1. Show that $\{e_i, f_i, h_i\}$ generates $\mathfrak{g}$.

The subalgebra $\mathfrak{h}$ is spanned by $\left\{t_{\alpha_i} \;\middle|\; i \in I\right\}$ and hence spanned by $\left\{h_i \;\middle|\; i \in I\right\}$. So it suffices to show that $\mathfrak{g}_\alpha \subseteq \langle e_i \rangle$ for all $\alpha - in\Phi^+$.

Write $\alpha = \alpha_i + \beta$ for each $i \in I, \beta \in \Phi^+$. Then $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta] = \mathfrak{g}_\alpha = \mathbb{C}e_\alpha$, so $e_\alpha = [e_i, e_\beta]$ for some nonzero $e_\beta \in \mathfrak{g}_\beta$.

By repeating this argument, we find that $e_\alpha = [[\cdots[e_{i_1}, e_{i_2}], e_{i_3}]\cdots], \cdots e_{i_k}]$.

2. Verify the relations

We need to check that $s_{ij}^+ = 0$. The α_i root string through α_j is given by

$$\alpha_j + p\alpha_i \to \cdots \to \alpha_j + q\alpha_i$$

where $p \neq 0$ because $\alpha_j - \alpha_i \notin \Phi$ for any i, so the smallest root must be $\alpha_j \in \Phi$. By prop 8.4d, this means that $-q = \alpha_j(h_i) = \alpha_{ji}$.

Thus ad $(e_i)^{1-\alpha_{ji}}(e_j) =$ ad $(e_i)^{1+q} \in \mathfrak{g}_{\alpha_j + (q+1)\alpha_i} = \{0\}$.

31.2 The Lie Algebra $\tilde{\mathfrak{g}}(A)$

Fix a Cartan matrix $A = (a_ij)_{i,j\in I}$where $I = \{1, \cdots, \ell\}$. Let $\tilde{J} \trianglelefteq \mathcal{FL}(\left\{e_i, f_i, h_i \;\middle|\; i \in I\right\})$ generated by

- $[h_i, h_j]$,
- $[h_i, e_j] - a_{ji}e_j$,
- $[e_i, f_j] - \delta_{ij}h_i$
- $[h_i, f_j] + a_{ji}f_j$.

Then let J be the same ideal with the additional relations s^+, s^-, and set

- $\tilde{\mathfrak{g}}(A) = \mathcal{FL}(\{e_i, f_i, h_i\})/\tilde{J}$,
- $\mathfrak{g}(A) = \mathcal{FL}(\{e_i, f_i, h_i\})/J$.

Proposition:

a. Let $V = \mathcal{F}(\{f_1, \cdots, f_\ell\})$. Then $\pi : \tilde{\mathfrak{g}} \to \mathfrak{gl}(V)$ is a *representation* with

- $f_j : f_{i_1} \cdots f_{i_r} \mapsto f_j f_{i_1}$
- $h_j : f_{i_1} \cdots f_{i_r} \mapsto (\alpha_{ji_1} + \cdots) f_{i_1} \cdots f_{i_r}$
- $e_j : f_{i_1} \cdots f_{i_r} \mapsto (\sum \delta \sum a)(\alpha_{ji_1} + \cdots) f_{i_1} \cdots f_{i_r}$

b. $\{h_1, \cdots h_\ell\}$ is linearly independent set in $\tilde{\mathfrak{g}}$.

For (a), it suffices to check $[\pi(h_i), \pi(h_j)] = 0$, $[\pi(h_i), \pi(e_j)] = a_{ji}\pi(e_j)$, etc. For (b), it suffices to show that $\left\{\pi(h_i) \;\middle|\; i \in I\right\}$ is linearly independent.

Suppose $\sum c_i \pi(h_i) = 0$ in $\mathfrak{gl}(V)$. Then, [

$$
\begin{aligned}
0 = \left(\sum_c c_i \pi(h_i)\right)(f_j) = -\left(\sum_i c_i \alpha_{ji}\right) f_j \\
\implies \sum c_i \alpha_{ji} = 0 \quad \forall\, j \\
\implies c_i = 0 \quad \forall\, i, .
\end{aligned}
$$

] since A is invertible.

Thus $\tilde{\mathfrak{h}} := \operatorname{span}_{\mathbb{C}} \{h_i\}$ is a lie subalgebra of $\tilde{\mathfrak{g}}$..

Theorem:

a. $\tilde{\mathfrak{g}} = \bigoplus_{u \in \tilde{\mathfrak{h}}^*} \tilde{\mathfrak{g}}_\mu$ as vector spaces, where

$$
\tilde{\mathfrak{g}}_\mu := \left\{ x \in \tilde{\mathfrak{g}} \;\middle|\; [h, x] = \mu(h)x \quad \forall h \in \tilde{\mathfrak{h}} \right\}.
$$

b. $\tilde{\mathfrak{g}} = \tilde{n}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{n}$ as vector spaces, where $\tilde{n}^- := \langle f_i \rangle$ and $\tilde{n} := \langle e_i \rangle$.

Proof of (a):

It's easy to check that $[\tilde{\mathfrak{g}}_\lambda, \tilde{\mathfrak{g}}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ for all $\lambda, \mu \in \tilde{\mathfrak{h}}^*$. Define $\alpha_i \in \tilde{\mathfrak{h}}^*$ by $h_j \mapsto a_{ij}$. Then

- $e_i \in \tilde{\mathfrak{g}}_{\alpha_i}, f_i \in \tilde{\mathfrak{g}}_{-\alpha_i}, h_i \in \tilde{\mathfrak{g}}_0$ for all i.
- Any $x \in \tilde{\mathfrak{g}}$ lies in $\tilde{\mathfrak{g}}_\mu$ for *some* μ.
- $\tilde{\mathfrak{g}} = \sum_\mu \tilde{\mathfrak{g}}_\mu$.

We just need to show that the last sum is in fact a direct sum.

Suppose that $\exists x \neq 0$ such that $x \in \tilde{\mathfrak{g}}_\mu, x = \sum_\nu x_\nu$ where $x_\nu \in \tilde{\mathfrak{g}}_\nu - \{0\}$ and ν runs over a finite set of weights that are not equal to μ.

Then $[h, x] = \mu(h)x$, and so $(\text{ad } h - \mu(h))(x) = 0$. On the other hand, $\prod_\nu (\text{ad } h - \nu(h))(x_\nu) = 0$. So pick some $h \in \tilde{\mathfrak{h}}$ such that $\mu(h) \neq \nu(h)$ for all ν. Then the polynomials $t - \mu(h), \prod_\nu (t - \nu(h))$ are coprime, and so there exist a, b such that

$$
a(t - \mu(h) + b \prod_\nu (t - \nu(h)) = 1,
$$

Then evaluating at $t = \text{ad } h$, we get [

$$
x = 1(x) = a(\text{ad } h)(\text{ad } h - \mu(h))(x) + b(\text{ad } h)(\prod_\nu \text{ad } h - \nu(h))(x) = 0,
$$

]

and so $\tilde{\mathfrak{g}} = \oplus_\nu \tilde{\mathfrak{g}}_\mu$.

32 Wednesday October 30

Last time:

[

$$W \curvearrowright \mathfrak{h}^*, \lambda \mapsto w(\lambda), W \curvearrowright \mathfrak{h}, \mathfrak{h} \mapsto w \cdot \mathfrak{h}$$

] such that $\lambda(w \cdot h) = (w^{-1}\lambda)(h) \forall \lambda \in \mathfrak{h}^*$.

We then get compatible squares:

<!--& & & &

Proposition:

a. $\Theta_i := \exp(\text{ad } e_i) \circ \exp(\text{ad } (-f_i)) \circ \exp(\text{ad } e_i)$,
b. $\Theta_i(\mathfrak{h}) = \mathfrak{h}$, so it fixes Cartan subalgebra.
c. $\Theta_i \Big|_{\mathfrak{h}} = s_i$ where s_i is the Weyl group action

Proof of (a):

We want to show that $\exp(\text{ad } e_i)$ is well-defined as an automorphism of $\mathfrak{g}$. It suffices to check that ad e_i is *locally nilpotent*, i.e. for all $x \in \mathfrak{g}$, there exists some $n_x > 0$ such that ad $(e_i)^n = 0$. We will also need to check that $\exp \text{ad } e_i$ is a derivation.

To see the local nilpotency, we can check [

$$(\text{ad } e_i)^n([x, y]) = \sum_{t=0}^{n} \binom{n}{t} \left[(\text{ad } e_i)^t, (\text{ad } e_i)^{n-t} \right]$$

]

for all $x, y \in \mathfrak{g}$.

If x, y are locally nilpotent, then $[x, y]$ is as well.

It thus suffices to check that ad e_i acts on generators in a nilpotent way.

A direct computation shows ad $e_i = [e_i, e_i] = 0$, and $(\text{ad } e_i)^{1-a_{ji}}(e_j) = 0$ by the Serre relations.

We also find that ad $e_i(h_j) = [e_i, h_j] = -[h_j, e_i] = -a_{ij}e_i$, and applying it again yields $(\text{ad } e_i)^2(h_j) = -a_{ij}[e_i, e_i] = 0$.

We have ad $e_i(h_j) = 0$, and applying ad $e_i h_i$ multiple times yields $h_i, [e_i, h_i], 0$, so ad ${}^3 e_i(h_i) = 0$.

Proof of (b):

By a direct computation, we have $\Theta_i(h_j) = h_j - a_{ij}h_i \in \mathfrak{h}$. (See CJ's notes for full computation.)

Proof of (c):

Consider computing $s_i \cdot h_j$. This is the unique element satisfying $\lambda(s_i \cdot h_j) = (s_i^{-1}\lambda)(h_j)$, but we can compute [

$$(s_i^{-1}\lambda)(h_j) = h_j - a_{ij}h_i = \Theta_i(h_j).$$

]

Theorem (Serre): Fix $\Phi \supseteq \Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ and $I = \{1, \cdots, \ell\}$. Define A by $a_{ij} = (\alpha_j, \alpha_i^\vee)$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the algebra generated by these elements.

Then

a. $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n$ as vector spaces, where $n^- \cong \tilde{n}^-/s^-$, $\mathfrak{h} \cong \tilde{\mathfrak{h}}$, and $n \cong \tilde{n}/s^+$.
b. $\mathfrak{g} = \oplus_{\mu \in \mathfrak{h}^*} \mathfrak{g}_\mu$ as vector spaces, where $\mathfrak{g}_\mu = \left\{x \in \mathfrak{g} \;\middle|\; [h, x] = \mu(h)x \forall h \in \mathfrak{h}\right\}$
c. $\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_\mu$ if $\lambda \in W_\mu$,
d. $\dim \mathfrak{g} = \ell + |\Phi|$,
e. $\mathfrak{g}$ is semisimple,
f. $\mathfrak{h}$ is a Cartan subalgebra with root system Φ.

Proofs:

a. Follows from Theorem 18.2b and Lemma b.
b. Similar to Theorem 18.2a.

Proof of (c):

We may assume that $\lambda = s_i\mu$. Pick $x \in \mathfrak{g}_\lambda$. Then for all $h \in \mathfrak{h}$, we have

[

$$\begin{aligned}
[\Theta_i(h), \Theta_i(x)] &= \Theta_i(h, x) \\
&= \lambda(h)\Theta_i(x) \\
&= \lambda(\Theta_i^{-1}(h))\Theta_i(x) \\
&= \lambda(s_i^{-1} \cdot h)\Theta_i(x) \\
&= (s_i\lambda)\Theta_i(x),
\end{aligned}$$

]

so $\Theta_i(x) \in \mathfrak{g}_{s_i\lambda}$, and thus $\Theta_i(\mathfrak{g}_\lambda) \subseteq \mathfrak{g}_{s_i\lambda}$.

Replacing Θ_i with Θ_i^{-1} and λ by $s_i\lambda$, we find $\Theta_i^{-1}(\mathfrak{g}_{s_i\lambda}) \subseteq \mathfrak{g}_{s_is_i\lambda} = \mathfrak{g}_\lambda$, and so $\mathfrak{g}_\lambda \cong \mathfrak{g}_{s_i\lambda}$, i.e. $\mathfrak{g}_{s_i\lambda} \subseteq \Theta_i(\mathfrak{g})$.

Proof of (d):

By Corollary 18.2b, we have [

$$\dim \mathfrak{g}_{k\alpha_{ii}} = \begin{cases} 1, & k = \pm 1 \\ 0, & k \notin \{0, \pm 1\} \\ \ell, & k = 0 \end{cases}.$$

]

Thus $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{h}}$.

Since s_{ij}^+ is of height $1 + a_{ji} \geq 2$, we have $\dim \mathfrak{g}_{\alpha_i} = \dim \tilde{\mathfrak{g}}_{\alpha_i} = 1$ for all $i \in I$. Thus for any $\alpha \in \Phi$, we have $\alpha = w\alpha_i$ for some element of the Weyl group $w \in W$.

By parts (a) and (c), we have $\dim \mathfrak{g}_\alpha = 1$, so $\dim \mathfrak{g}_{k\alpha}$ satisfies the same cases as $\dim \mathfrak{g}_{k\alpha_{ii}}$ above.

It remains to show that there are no other root spaces, i.e. $\mathfrak{g}_\mu = 0$ if $\mu \notin \mathbb{Z}\alpha$ for all $\alpha \in \Phi$.

We can show this by considering reflections about hyperplanes again, i.e. that $\alpha \in \Phi \implies H_\mu \neq H_\alpha$.

If this is the case, it implies that there exists an $h \in \mathfrak{h}$ such that $h \in H_\mu \setminus H_\alpha$ for all $\alpha \in \Phi$. But then $\mu(h) = 0$ when $h \notin H_\alpha$ for all $\alpha \in \Phi$, so pick $w \in W$ such that $w^{-1}a_i(h) \in C(\Pi)$, the fundamental chamber. Thus $w^{-1}\alpha_i(h) > 0$ for all i, and is equal to $\alpha_i(w \cdot h)$, and

$$0 = \mu(h) = \kappa(t_\mu, h) = \cdots = (w\mu)(w \cdot h)$$

Writing $w_\mu = \displaystyle\sum_{i=1}^{\ell} m_i \alpha_i$, we have $0 = \displaystyle\sum_{i=1}^{\ell} m_i \alpha_i(w \cdot h)$, we find that note all m_i have the same sign, which is a contradiction. ■

33 Wednesday November 6th

Last time: We considered the finite dimensional representation theory of $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$. We showed Weyl's complete reducibility theorem: any finite dimensional $\mathfrak{g}$ module is semisimple and $\mathfrak{g} = \bigoplus \mathfrak{s}_i$, a sum of simple modules.

Therefore, it suffices to understand the *characters* for simple modules, i.e. what are the dimensions of the weight spaces?

We can answer this question for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$: we have $L(\lambda) = \operatorname{span}_{\mathbb{C}} \{v_i\}_{i=1}^{\lambda}$ where

$$\dim L(\lambda)_\mu = \begin{cases} 1 & \mu \in \{\lambda, \lambda - 2, \cdots, -\lambda\} \\ 0 & otherwise \end{cases}$$

For an arbitrary $\mathfrak{g}$, what is $L(\lambda)$? We'll describe this using Weyl's character theorem, the Verma module (which is an infinite-dimensional highest weight module), and the PBW theorem of the universal enveloping algebra.

In general, the representation of $\mathfrak{g}$ is complicated, so we restrict ourselves to a subcategory BGG category $\mathcal{O}$, which contains the simple and Verma modules. Here, the irreducible character problem is solved if we know that the *multiplicity* of simple modules in any Verma module. The multiplicity is the number of simple modules occurring in a filtration, and the Kazhdan–Lusztig conjecture says that this multiplicity should be the evaluation of a certain KL polynomial at 1. This was first proved using perverse sheaves and D-modules in the 1980s (geometric), and then with purely algebraic proof is due to Williamson around 2013. This was obtained using something called the Soergel bimodule. This is all over $\mathbb{C}$, and there are some generalizations that work for characteristic p. It was thought that the original polynomial would work here, but it turns out that there is another one called the $p - KL$ polynomial.

> These come from the KLR algebra, where there is a change of basis that induces a change of basis on the Hecke algebra, where the KL polynomial takes that standard basis to the KL basis.

Recall that M is a weight module if $M = \oplus_\lambda M_\lambda$, where $M_\lambda := \left\{ m \in M \;\middle|\; h.m = \lambda(h)m \;\forall h \in \mathfrak{g} \right\}$.

Non-example: Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \langle e, h, f \rangle$ and $M = U(\mathfrak{g})/I$ where $I = U(\mathfrak{g})(1 - e) \trianglelefteq U(\mathfrak{g})$ is a left ideal. Then M has basis $\left\{ f^a h^b + I \;\middle|\; a, b \in \mathbb{Z}_{\geq 0} \right\}$. The claim is that $h + I$ is not in any weight space.

If so, we would have $h \curvearrowright (h+I) = h^2 + I$, which is not a multiple of $h+I$, i.e. it's not in $\mathbb{C}(h+I)$. So it is not a weight module.

33.1 Section 20.2: Highest Weight Modules

Definition: A *maximal vector* $v^+ \in M$ is a nonzero vector such that $\eta v^+ = 0$, i.e. $\mathfrak{g}_\alpha v^+$ for all $\alpha \in \Phi^+$.

Definition: A $\mathfrak{g}$-Module M is a *highest weight module* of weight λ if $M = U(\mathfrak{g})v^+$ for some maximal vector v^+.

Example: Consider $\mathfrak{sl}(2, \mathbb{C})$ and $L(\lambda)$, we have the following situation:

Then a similar picture holds for $M(\lambda)$, thus v_0 is a maximal vector, and $L(\lambda), M(\lambda)$ are weight modules.

Theorem: Let M by a highest weight module of weight λ with maximal vector v^+. Fix an ordering $\Phi^+ = \{\beta_1, \beta_2, \cdots, \beta_m\}$ where $m = |\Phi|^+$. Pick a nonzero $e_i \in \mathfrak{g}_{\beta_i}$, then there exists a nonzero $f_i \in \mathfrak{g}_{-\beta_i}$ such that $[e_i, f_i] = h_i$ (a Cartan element) for all i.

a. We can write a basis for the highest weight module, $M = \text{span}_\mathbb{C} \left\{ \prod_{i=1}^m f_i^{r_i} v^+ \;\middle|\; r_I \in \mathbb{Z}_{\geq 0} \right\}$,
b. $\text{Wt}(M) \subseteq \left\{ \mu \in \mathfrak{h}^* \;\middle|\; \mu \leq \lambda \right\}$.
c. $\dim M_\lambda = 1$ and $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$.
d. Submodules of M are weight modules.
e. If M has a *unique* submodule, then M has a unique simple quotient and M is indecomposable.
f. Every non-zero homomorphic image of M is a highest weight module of weight λ.

Proof of (a): $M = U(\mathfrak{g})v^+$, which is in $\sum \mathbb{C} f \ldots h \ldots e \ldots v^+$ where $e \ldots v^+ = 0$, which is in $\sum \mathbb{C} f \ldots v^+$.

34 Wednesday

Todo

35 Friday

Todo

36 Monday November 18th (TODO)

Todo

37 Wednesday November 20

Last time:

$$\mathbb{Z}\Lambda \iff \left\{\mathfrak{h}^* \to \mathbb{Z}_{\geq 0} \,\middle|\, \sim\right\}$$
$$e(\mu) \mapsto e_\mu$$
$$e(\lambda)e(\mu) = e(\lambda+\mu) \mapsto f \star g(\lambda) = \sum_{a+b=\lambda} f(a)g(b)$$

and $\mathrm{ch} L(\lambda) = \sum_{\mu\in\Lambda} \dim L(\lambda)_\mu e(\mu)$.

We have the Kostant function $p(\lambda) = \#\left\{(k_\alpha)_\alpha \,\middle|\, -\lambda = \sum_{\alpha\in\Phi^+} k_\alpha \alpha\right\}$ and the Weyl function $q = e_\rho \star \prod_{\alpha\in\Phi^+}(1-e_{-\alpha}) = \prod_{\alpha\in\Phi^+}(e_{\alpha/2} - e_{-\alpha/2})$.

Lemma: $p \star e_\lambda = \mathrm{ch} M(\lambda)$, so $q \star \mathrm{ch} M(\lambda) = e_{\lambda+\rho}$ and $q \star p = e_\rho$.

37.1 Weyl's Character Formula (24.2-3)

Definition: The *dot action* of W is given by $w \cdot \lambda = w(\lambda+\rho) - \rho$, i.e. a reflection for hyperplanes passing through $-\rho$.

E.g. for type $A2$, where $W(0) = 0$, we have:

And for the dot action, we have

where $W \cdot 0 = 0$ and $s(\alpha_1) = -\alpha_1$.

Theorem (Harish-Chandra): If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$ for $\mu \leq \lambda$.

Proof: Postponed.

ch are characters, $L(\lambda)$ is a Verma module.

Remark: if we sum over $\mu \leq \lambda$, we obtain

$$\begin{aligned}
\mathrm{ch} M(\lambda) &= \sum_{\mu\in W\cdot\lambda} a_{\lambda\mu} \mathrm{ch} L(\mu) \\
\mathrm{ch} L(\lambda) &= \sum_{\mu\in W\cdot\lambda} b_{\lambda\mu} \mathrm{ch} M(\mu) \\
&= \sum_{W\cdot\lambda\in\Lambda} c_{\lambda W} \mathrm{ch} M(w\cdot\lambda).
\end{aligned}$$

Theorem (Weyl's Character Formula): If $\lambda \in \Lambda^+$, then

$$\mathrm{ch} L(\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)} e(w\cdot\lambda)}{\sum_{w\in W}(-1)^{\ell(w)} e(w\cdot 0)}$$

Proof:

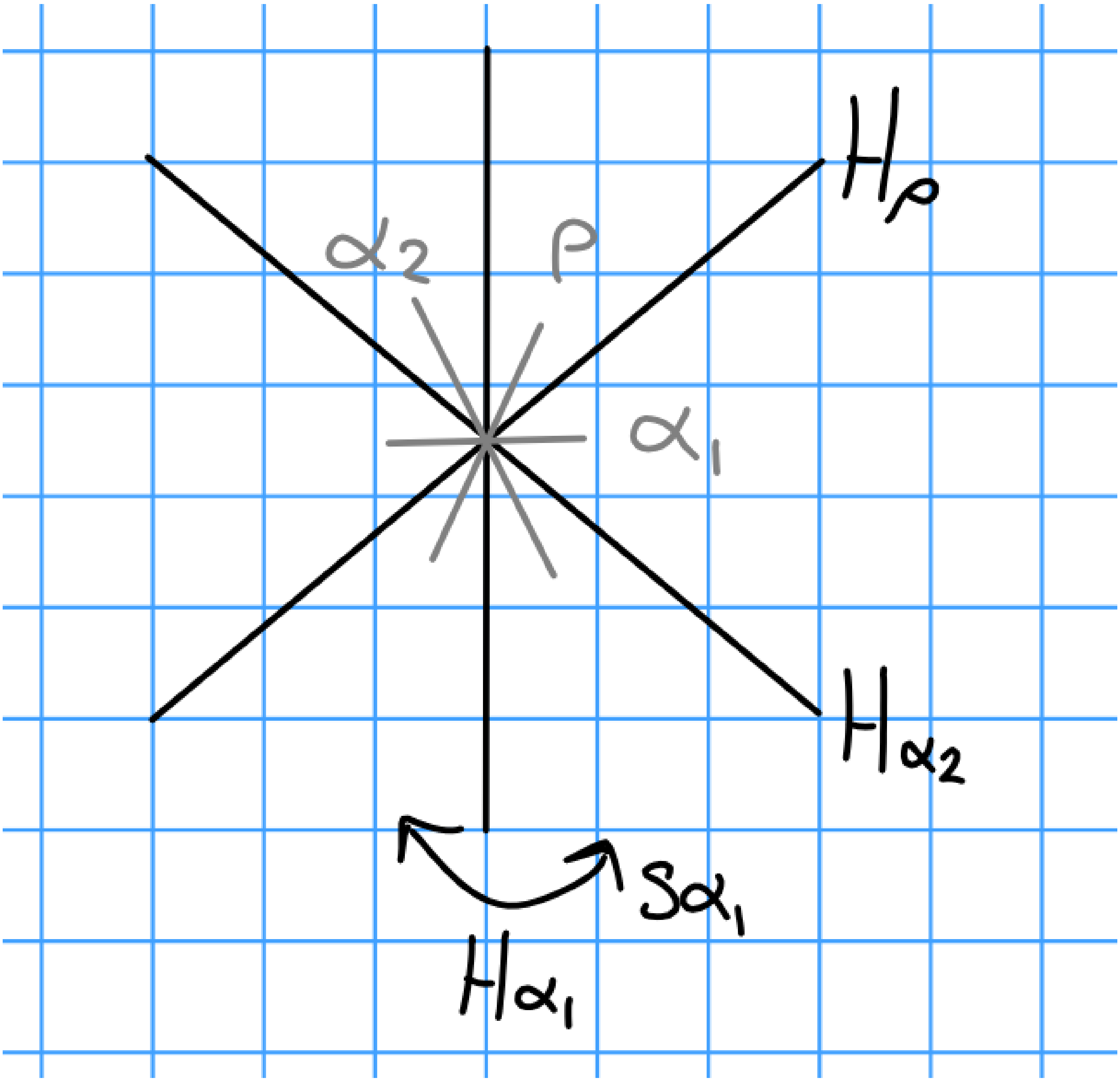

Figure 13: Type A2

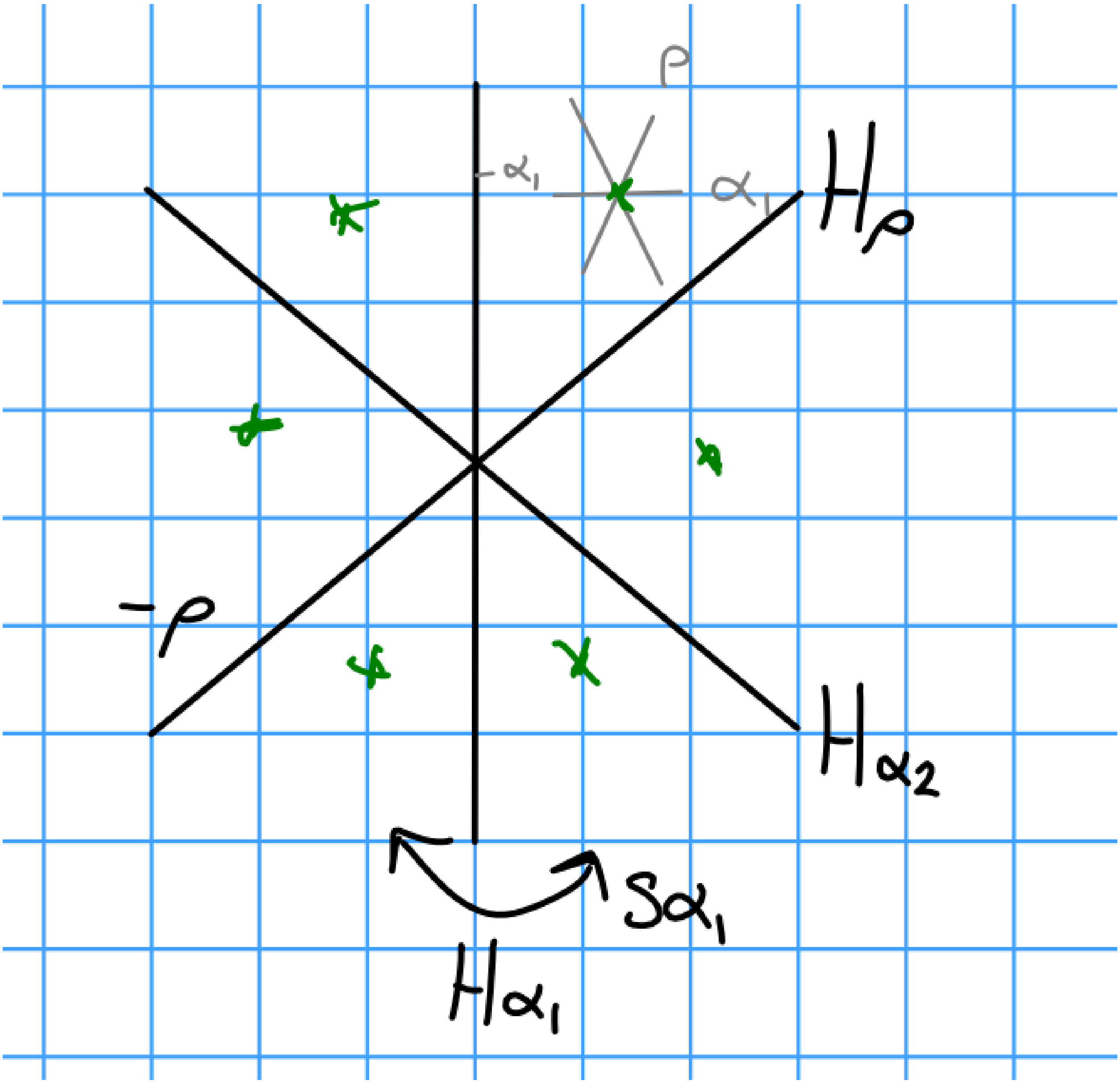

Figure 14: Image

We have $\mathrm{ch}L(\lambda) = \sum_w c_{\lambda w} \mathrm{ch}M(w \cdot \lambda)$, and so by the lemma,

$$q * \mathrm{ch}L(\lambda) = \sum c_{\lambda w} q \star \mathrm{ch}M(W(\lambda+\rho)-\rho) = \sum_w c_{\lambda w} e_{W(\lambda+p)}$$

Thus for all $\alpha \in \Phi^+$, we have

$$s_\alpha(q \star \mathrm{ch}L(\lambda)) = \sum_w c_{\lambda, s_\alpha w} e_{w(\lambda+\rho)}$$

On the other hand, by part (c) of the lemma, we have

$$(s_\alpha \star q) \star \mathrm{ch}L(\lambda) = -q \star \mathrm{ch}L(\lambda) = \sum_w -c_{\lambda,w} e_{w(\lambda+\rho)}$$

which implies that $c_{\lambda, s_\alpha w} = -c_{\lambda, w}$ by comparing term-by-term, and thus $c_{\lambda,w} = (-1)^{\ell(w)}$ because $c_{\lambda e} = 1$.

In particular, $q = q \star e(0) = q \star \mathrm{ch}L(0) = \sum_{w \in W} (-1)^{\ell(w)} e_{w(\rho)}$, and thus

$$\begin{aligned}
\mathrm{ch}L(\lambda) &= \frac{\sum_w (-1)^{\ell(w)} e_{w(\lambda+p)}}{\sum_w (-1)^{\ell(w)} e_{w(p)}} \\
&= \frac{\sum_w (-1)^{\ell(w)} e(w \cdot \lambda)}{\sum_w (-1)^{\ell(w)} e(w \cdot 0)}.
\end{aligned}$$

■

Example: For type $A1$, we have $W = \Sigma_2 = \{\mathbf{1}, s\}$. Take $\lambda = 3$ under

$$\begin{aligned}
\Lambda &\equiv \mathbb{Z} \\
\alpha_1 &\to 2 \\
w_1 = \rho &\to 1,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\mathrm{ch}L(3) &= \frac{e(\mathbf{1} \cdot 3) - e(s \cdot 3)}{e(\mathbf{1} \cdot 0) - e(s \cdot 0)} \\
&= \frac{e(3) - e(-5)}{e(0) - e(-2)} \\
&= e(3) + e(1) + e(-1) + e(-3) \quad \text{by long division.}
\end{aligned}$$

Corollary (Kostant's Dimension Formula):

If $\mu \leq \lambda \in \Lambda^+$, then

$$\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} P(w \cdot \lambda - \mu).$$

Proof: $p \star e_\mu(w \cdot \lambda) = \sum_{a+b=w\cdot\lambda} p(a)e_\mu(h) = p(w \cdot \lambda - \mu)$, since this is the only term that survives.

Then $p(w \cdot \lambda - \mu)$ is the coefficient for $e(\mu)$ in $\mathrm{ch} M(w \cdot \lambda) = \dim M(\lambda)_\mu$. Thus $\dim L(\lambda)_\mu = \sum_{w \in W} (-1)^{\ell(w)} \dim M(w \cdot \lambda)_\mu$.

Corollary (Weyl's Dimension Formula):

If $\lambda \in \Lambda^+$, then

$$\dim L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha^\vee)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha^\vee)}$$

Proof (sketch):

Define an operator $\partial = \prod_{\alpha \in \Phi^+} \partial_a$, where $\partial_a : e(\mu) \mapsto (u, \alpha^\vee) e(\mu)$. Then ∂ is well-defined since $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ for all α, β, and (exercise) ∂ is a derivation.

Define an evaluation homomorphism $\nu : \sum_\mu c_\mu e(\mu) \mapsto \prod_\mu c_\mu$. Note that $\nu(\mathrm{ch} L(\lambda)) = \dim L(\lambda)$, and $\nu(q) = 0$ because $\nu(e_{\alpha_i - 1}) = 0$.

Claim:

$$\nu(\partial(q \star \mathrm{ch} L(\mu - \rho))) = |w| \prod_{\alpha \in \Phi^+} (\mu, \alpha^\vee)$$

This is relatively straightforward once you know that you have a derivation and a homomorphism.

With this claim, we have

$$\nu(\partial(q \star \mathrm{ch} L(\lambda))) = \nu(\partial q)\nu(\mathrm{ch} L(\lambda)) + \nu(q)\nu(\partial \mathrm{ch} L(\lambda))$$

where we can identify a number of terms, and then taking ratios yields Weyl's dimension formula.

38 Friday November 22

Remark: For $\mathfrak{g}$ semisimple, studying $\mathrm{Rep}(\mathfrak{g})$ is too hard. So we study category $\mathcal{O}$, which contains simple modules $L(\lambda)$ for $\lambda \in \mathfrak{g}^*$.

Case 1, $\lambda \in \Lambda^+$: In the finite-dimensional setting, we use Weyl's character formula.

Case 2, $\lambda \notin \Lambda^+$: If suffices to consider $\lambda \in \Lambda$, then we apply Soergel's translation functor $\mathbb{V}$. Then Ł(λ) for $\mathfrak{g}$ corresponds to $L(\lambda^\sharp)$ for $\mathfrak{g}^\sharp$ such that $\lambda^\sharp \in \Lambda(\mathfrak{g}^\sharp)$.

For $\lambda \in \Lambda$, it suffices to consider $\lambda \in W \cdot 0$ using Jantzen's translation functor.

Then $\mathrm{ch}L(w \cdot 0) = \sum_{x \leq W} (-1)^{\ell(w)-\ell(x)} P_{w_0 w, w_0 x}(1) \mathrm{ch} M(x \cdot 0)$.

The $x \leq w$ index indicates the Bruhat order on W, and P is the Kazhdan-Lusztig polynomial and w_0 is the longest element in W.

Example: Type A_2, the $W = \Sigma_3, w_0 = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} = |3\ 2\ 1|$.

Last time: If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$.

38.1 Central Characters (Ch. 23)

38.1.1 Action of the Center (23.2)

Let $Z \coloneqq Z(U(\mathfrak{g}))$ be the center of the universal enveloping algebra. Then there is a Casimir element $\Omega \in Z$, and $\Omega \curvearrowright L(\lambda)$ by scalar multiplication.

Definition/Proposition: For $\lambda \in \mathfrak{h}^*$, its *central character* is $\chi_\lambda : Z \to \mathbb{C}$ such that $z \cdot m = \chi_\lambda(z) m$ for all $z \in Z, m \in M$, where M is a highest weight module with highest weight λ and v^+ is a highest weight vector.

Proof: For all $h \in \mathfrak{h}$, we have $h.(z.v^+) = z.(h.v^+)$ since $z \in Z$, but this equals $\lambda(h) z.v^+$. Then $z.v^+ \in M_\lambda = \mathbb{C} v^+$, so $z.v^+ = \chi_\lambda(z) v^+$ for some $\chi_\lambda(z) \in \mathbb{C}$.

An arbitrary element in $M = U(\mathfrak{g}).v^+$ is $m = x.v^+$ for $x \in U(\mathfrak{g})$. Then

[

$$\begin{aligned} z.m &= z.(x.v^+) \\ &= x.(z.v^+) \\ &= \chi_\lambda(z) x.v^+ \\ &= \chi_\lambda(z) m. \end{aligned}$$

] ∎

Remark: We also have $Z \curvearrowright M(\lambda)$ and any submodule or composition factor by the same scalar.

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\Omega = h^2 + 2h + fe \in Z$.

Take $\lambda \in \Lambda^+ \equiv \mathbb{Z}_{\geq 0}$ and $v^+ \in M(\lambda)_\lambda$. Then $\Omega.v^+ = (h^2 + 2h + fe) v^+ = (\lambda^2 + 2\lambda) v^+$, which means that $\chi_\lambda(\Omega) = \lambda(\lambda + 2) \in \mathbb{C}$.

38.1.2 Harish-Chandra Theorem

Definition: The *Harish-Chandra* homomorphism is the algebra homomorphism [

$$\begin{aligned} \xi : Z &\to U(\mathfrak{h}) \\ \mathbf{f^a h^b e^c} &\mapsto \begin{cases} \mathbf{h^b} & if \mathbf{a} = \mathbf{0} = \mathbf{c} \\ 0 & else \end{cases}. \end{aligned}$$

]

Example: $\xi(\Omega) = h^2 + 2h$.

Lemma: $\chi_\lambda(z) = \lambda(\xi(z))$ implies that $\Omega \curvearrowright L(\lambda), M(\lambda)$ by $(\lambda+\rho, \lambda)$.

Proof: If $z = \mathbf{f}^{\mathbf{a}}\mathbf{h}^{\mathbf{b}}\mathbf{e}^{\mathbf{c}}$ with $\mathbf{c} \neq \mathbf{0}$, the $z.v^+ = 0$ which implies that both sides are zero. If $\mathbf{c} = \mathbf{0}$, then $\mathbf{a} = \mathbf{0}$. Otherwise $z \in U(\mathfrak{g})_\beta$ for some $\beta \neq 0$, so there exists an $h \in \mathfrak{h}$ such that $[h, z] = \beta(h)z \neq 0$, while $[h, z] = 0$ and $z \in Z$.

Thus $\chi_\lambda(z) = \lambda(\mathbf{h}^{\mathbf{b}}) = \lambda(\xi(z))$ if $z = \mathbf{h}^{\mathbf{b}}$. ■

Recall that $\Omega = \sum_{j=1}^{\ell} h_j h_j' + \sum_{i=1}^{m}(e_i t_i + f_i e_i)$ for $h_i = [e_i, f_i] = e_i f_i - f_i e_i$.

Then if we have a basis $\{h_i, e_i, f_i\}$, we can produce a dual basis $\{h_i', e_i', f_i'\}$ with respect to the killing form. Thus $\Omega = \sum_j h_j h_j' + \sum_{i=1}^{m} h_i + 2 f_i e_i$ and $\xi(\Omega) = \sum_j h_j h_j' + \sum i = 1^m h_i$.

Now by writing $t_\lambda = \sum_i a_i h_i = \sum_i b_i h_j'$, where $\kappa(t_\lambda, h_j) = b_j, \kappa(t_\lambda, h_j')$, and $\kappa(t_\lambda, t_\lambda) = \sum a_i b_i$, we can write [

$$
\begin{aligned}
\lambda(\xi(\Omega)) = \sum_j \lambda(h_j)\lambda(h_j') + \sum_i \lambda(h_i) &= \sum_j \kappa(t_\lambda, h_j)\kappa(t_\lambda, h]_j) + \sum_i \lambda(h_i) \\
&= \kappa(t_\lambda, t_\lambda) + \sum_i \lambda(h_i) \\
&= \kappa(t_\lambda, t_\lambda) + \sum_{i=1}^{m}(\lambda, \alpha_i) \\
&= \kappa(t_\lambda, t_\lambda) + (\lambda, \sum_{\alpha\in\Phi^+} \alpha) \\
&= \kappa(t_\lambda, t_\lambda) + (\lambda, 2\rho \\
&= (\lambda, \lambda) + (\lambda, 2\rho) \\
&= (\lambda + 2\rho, \lambda).
\end{aligned}
$$

] ■

Example: $\lambda(\xi(\Omega)) = \lambda(h^2 + h) = \lambda^2 + 2\lambda = (\lambda + 2, \lambda)$ under $\mathfrak{h}^* \equiv \mathbb{Z}$ where $\alpha \mapsto 2, \rho \mapsto 1$.

Definition: The *twisted Harish-Chandra homomorphism* is the algebra homomorphism $\psi = \zeta \circ \xi : Z \to S(\mathfrak{h})$ where [

$$
\begin{aligned}
\zeta : U(\mathfrak{h}) \cong S(\mathfrak{h}) &\to S(\mathfrak{h}) \\
\rho(h_1, \cdots, h_\ell) &\mapsto \rho(h_1 - 1, \cdots, h_\ell - 1).
\end{aligned}
$$

]

Example: $\xi(\Omega) = h^2 + 2h$, so $\psi(\Omega) = \zeta(h^2 + 2h) = (h-1)^2 + 2(h-1) = h^2 - 1$.

Theorem (Harish-Chandra): For all $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\lambda = \chi_\mu \iff \mu = W \cdot \lambda$.

Corollary: If $L(\mu)$ is a composition factor of $M(\lambda)$, then $Z \curvearrowright M(\lambda)$ by the same scalar $\chi_\lambda(z) = \chi_\mu(z)$ for all z.

Then $\chi_\lambda = \chi_\mu \implies \mu = W \cdot \lambda$.

Remark: Assuming this theorem, this completes the proof of the Weyl Character Formula.

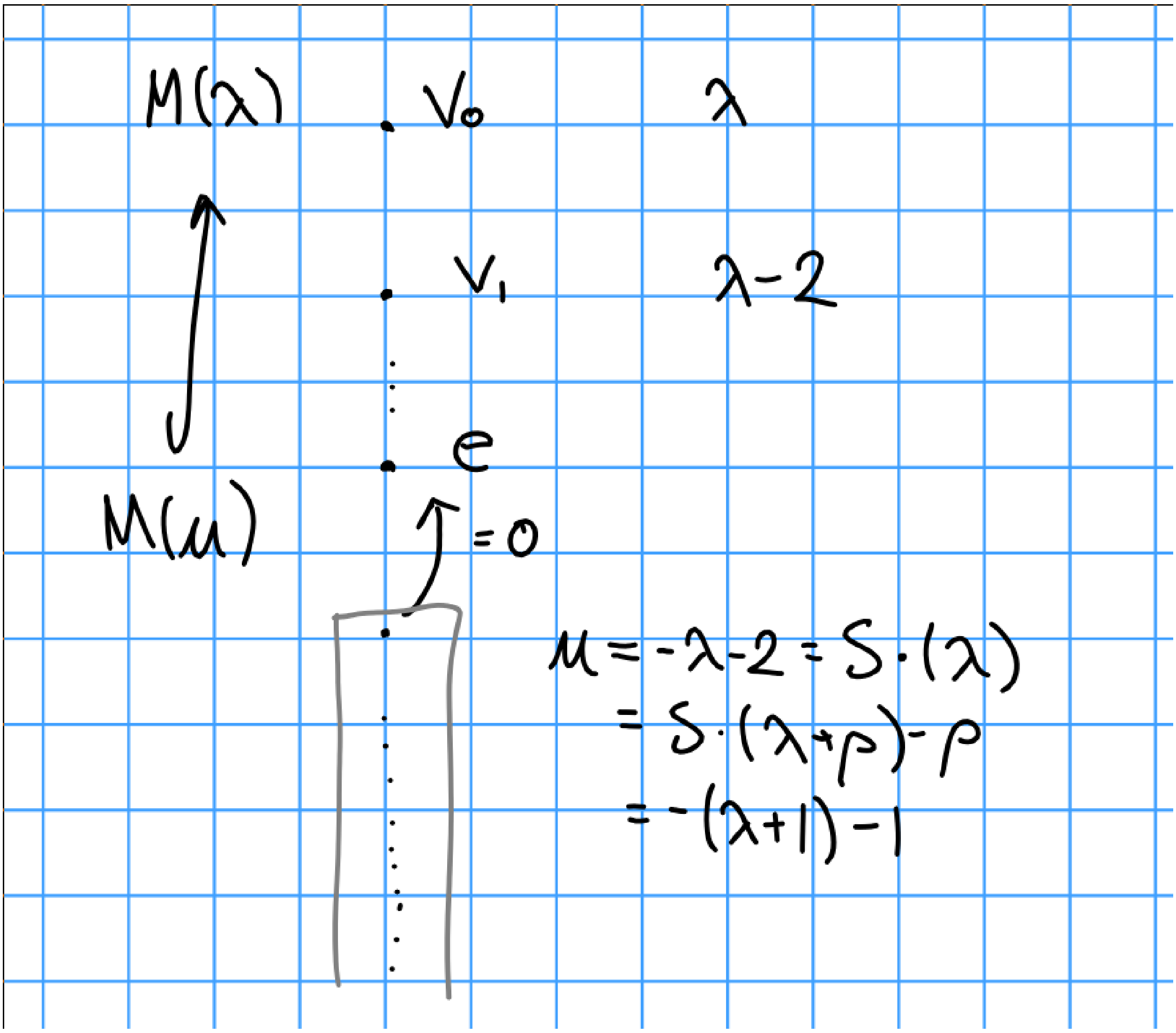

Figure 15: Image

39 Monday November 25

Today: The Conjugacy theorem

December 2nd: Kac-Moody Algebras (i.e. infinite-dimensional lie algebras)

December 4th: Summary of semisimple lie algebras over $\mathbb{C}$

Last time: We had the following goal: If $L(\mu)$ is a composition factor of $M(\lambda)$, then $\mu \in W \cdot \lambda$. We then get a central character $\chi : Z(U(\mathfrak{g})) \to \mathbb{C}$, e.g. $\chi_\lambda : Z \to \mathbb{C}$.

Any $z \in Z$ acts on the highest weight module M of highest weight λ by $\chi_\lambda(z)$.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\lambda \in \mathbb{Z}_{\geq 0}$.

We now want to prove this using the twisted Harish-Chandra homomorphism, where here we have

$$\begin{aligned}\psi : Z &\to U(\mathfrak{h}) = S(\mathfrak{g}) \\ z &\mapsto \iota \circ \xi(z)\end{aligned}$$

where

$$\begin{aligned}\iota : S(\mathfrak{h}) &\to S(\mathfrak{h}) \\ \rho(h_1, \cdots, h_\ell) &\mapsto \rho(h_1 - 1, \cdots, h_\ell - 1).\end{aligned}$$

For example, $h^2 + 2h \mapsto (h-1)^2 + 2(h-1)$.

Theorem (Harish-Chandra): For all $\lambda, \mu \in \mathfrak{h}^*$, we have $\chi_\mu = \chi_\lambda \iff \mu \in W \cdot \lambda$.

Proof (sketch):

1. Assuming *Chevalley's restriction theorem*, we have

$$P(\mathfrak{g})^G \cong P(\mathfrak{h})^W$$

where $G := \left\langle \exp(\text{ad } x) \,\middle|\, x \in \mathfrak{g} \right\rangle \subseteq \text{Aut}(\mathfrak{g})$.

Then the map $Z \to S(\mathfrak{h})^W$ given by $z \mapsto \psi(z)$ is an isomorphism.

Note: the RHS denote a subset invariant under the Weyl group action.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $W = \{e, s\}$ where $s(h) = -h$. Then $\Omega = h^2 + 2h + fe$, and $\psi(\Omega) = (h-1)^2 + 2(h-1) = h^2 - 1 \in S(\mathfrak{h})^W$ because $S \cdot (h^2 - 1) = (-h)^2 - 1 = h^2 - 1$.

2.

$\Longleftarrow$ It suffices to prove the case $\lambda, \mu \in \Lambda$ since Λ is dense in $\mathfrak{h}^*$ in the Zariski topology.

We can check that

$$\begin{aligned}&\chi_\lambda = \chi_{W \cdot \lambda} \\ \iff &\lambda(\xi(z)) = (W \cdot \lambda)(\xi(z)) \forall z \in Z \\ \iff &(\lambda + \rho)(\psi(z)) = ((\lambda + \rho))(\psi(z)) \forall z \in Z \\ &= (\lambda + \rho)(W^{-1} \cdot \psi(z)) \\ &= (\lambda + \rho)(\psi(z)) \quad \text{by (1)}.\end{aligned}$$

$\Longrightarrow$ Suppose that $\chi_\lambda = \chi_\mu$ but $\mu \notin W \cdot \lambda$.

Construct $g \in S(\mathfrak{h})^W$ such that $g(W \cdot \lambda) = 1$ and $g(W \cdot \mu) = 0$.

By 1, there exists a $z = \psi^{-1}(g) \in Z$ such that

$$\chi_\lambda(z) = (\lambda + \rho)(g) = g(\lambda) + g(\rho) \neq g(\mu) + g(\rho) = \cdots = \chi_\mu(z).$$

■

39.1 Cartan Subalgebra (Chapter 15)

Recall that the Cartan subalgebra (CSA) is equal to the maximal toral subalgebra, which is nilpotent and self-normalizing. Then rank $\mathfrak{g} = \dim \mathfrak{h}$ is well-defined by any CSA, since they are all conjugate under G.

39.1.1 Engel Subalgebras

Definition: The *Engel subalgebra* $x \in \mathfrak{g}$ of $\mathfrak{g}$ is the generalized eigenspace of ad x with eigenvalue 0. We can then define $\mathfrak{g}_{0,x} \coloneqq \left\{ y \in \mathfrak{g} \,\middle|\, (\text{ad } x)^n(y) = 0 \text{ for } n \gg 0 \right\}$.

An element $x \in \mathfrak{g}$ is *regular* if $\dim \mathfrak{g}_{0,x}$ is minimal.

Some facts:

a. $\mathfrak{g}_{0,x} = N_{\mathfrak{g}}(\mathfrak{g}_{0,x})$
b. If x is regular then $\mathfrak{g}_{0,x}$ is nilpotent
c. Combining (a) + (b), if x is regular then $\mathfrak{g}_{0,x}$ is a Cartan subalgebra.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, then

- $\mathfrak{g}_{0,e} = \mathfrak{g}$ since ad e kills everything eventually.
- $\mathfrak{g}_{0,h} = \mathbb{C}h$, a 1-dimensional algebra spanned by h, and h is regular, and $\mathbb{C}h = \mathfrak{h}$ is a CSA, which is nilpotent.

39.1.2 CSAs

Theorem: Let $\mathfrak{h} \leq \mathfrak{g}$, then

a. $\mathfrak{h}$ is a CSA $\iff$ $\mathfrak{h} = \mathfrak{g}_{0,x}$ for some regular $x \in \mathfrak{g}$
b. If $\mathfrak{g}$ is semisimple and char $\mathbb{F} \neq 0$, then $\mathfrak{h}$ is maximal toral $\iff$ $\mathfrak{h}$ is a CSA.

Proof (sketch):

Proof of (a):

$\impliedby$: Easy to check.

$\implies$: Suppose $\mathfrak{h}$ is nilpotent and $\mathfrak{h} \subseteq \mathfrak{g}_{0,x}$ for all $x \in \mathfrak{h}$. Then suppose that $\mathfrak{h} \not\subseteq \mathfrak{g}_{0,x}$ for all $x \in \mathfrak{h}$.

So pick $z \in \mathfrak{h}$ such that $\dim \mathfrak{g}_{0,z} \leq \dim \mathfrak{g}_{0,x}$ for all $x \in \mathfrak{h}$. Then $\mathfrak{g}_{0,z} \subseteq \mathfrak{g}_{0,x}$ for all $x \in \mathfrak{h}$. This implies that $x \in \mathfrak{h} \curvearrowright \mathfrak{g}0, z/\mathfrak{h}$ is a nilpotent action (where this quotient is nonzero). By Theorem 3.3, there exists a $y + \mathfrak{h} \neq \mathfrak{g}$ such that ad $\mathfrak{h}(y + \mathfrak{h}) = \mathfrak{h}$, or there exists a $y \notin \mathfrak{h}$ such that $[\mathfrak{h}, y] \subseteq \mathfrak{h}$ with $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. ■

Proof of (b):

$\implies$: $\mathfrak{h}$ is abelian and thus nilpotent, so $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{g}_\alpha$ with $[\mathfrak{h}, \mathfrak{h}] = \{0\} \subseteq \mathfrak{h}$, and $[\mathfrak{h}, \mathfrak{g}_\alpha] \subseteq \mathfrak{g}_\alpha$ Thus $N_{\mathfrak{g}}(\mathfrak{h})$.

$\impliedby$: Next time.

40 Monday December 02

Last time: Something about Engel.

Sketch of proof of (b):

$\implies$: If $\mathfrak{h}$ is abelian, then it is nilpotent, so $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha} \mathfrak{g}_\alpha$ and $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

$\impliedby$: (a) implies that $\mathfrak{h} = \mathfrak{g}_{0,x}$ for some x, write $x = x_s + x_n$ using Jordan decomposition, then $\mathfrak{g}_{0,x} \subseteq \sum \binom{n}{i} (\text{ad } x_j)^i (\text{ad } x_n)^{n-i}$. From this, you can deduce that

$$\begin{aligned}
\mathfrak{h} &= \mathfrak{g}_{0,x_s} \text{by regularity of } x \\
&= C_{\mathfrak{g}}(x_s) \text{because ad } x_s \text{ is diagonalizable} \\
&\supseteq \mathfrak{h} \text{for some maximal toral} \\
&= CSA \text{from the forward implication} \\
&= \mathfrak{g}_{0,x'} \text{from (a) for some regular } x'.
\end{aligned}$$

Thus equality holds by regularity and $\mathfrak{h} = CSA$.

40.1 Conjugacy Theorems [Carter '05]

Now we show that any two CSAs are conjugate under

$$G = \left\langle \exp \text{ad } x \;\middle|\; \text{ad } x \text{ is nilpotent } \right\rangle \trianglelefteq \text{Aut}(G).$$

Thus $\text{rank}\, \mathfrak{g} \coloneqq \dim CSA$ is well-defined.

For a CSA $\mathfrak{h}$, define $f = f(\mathfrak{h})$ by

$$f(x) = \left(\exp(\text{ad } x_1) \circ \cdots \circ \exp(\text{ad } x_m)\right)(x_0)$$

where

$$\begin{aligned}
\mathfrak{g} &\xrightarrow{\sim} \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \cdots \mathfrak{g}_{\alpha_m} \\
x* &\mapsto (x_0, x_1, \cdots, x_m).
\end{aligned}$$

Some facts:

a. $p(x) \neq 0 \iff \mathfrak{h} = \mathfrak{g}_{0,x}$.
b. For nonzero polynomials $p : \mathfrak{g} \to \mathbb{C}$, there exists a nonzero polynomial $q : \mathfrak{g} \to \mathbb{C}$ such that $f(V_p) \supset V_q$ where

$$V_p = \left\{ x \in \mathfrak{g} \;\middle|\; p(x) \neq 0 \right\}, \quad V_q = \cdots$$

Theorem: Any 2 CSAs are conjugate under G.

Proof Define $f = f(\mathfrak{h}), p = p(\mathfrak{h}), q = q(\mathfrak{h})$ for the CSA $\mathfrak{h}$, and similarly $f' = f(\mathfrak{h}')$, etc.

Since q, q' are nonzero, $V_q \bigcap V_{q'} \neq \emptyset$, or $\exists z \neq 0 \in V_q \bigcap V_{q'} \subset f(V_p) \bigcap f'(V_p)$. We then get can $x \in \mathfrak{g}, x' \in \mathfrak{g}$ such that $z = f(x) = f'(x')$ with $p(x) \neq 0, p'(x') \neq 0 \iff \mathfrak{h} = \mathfrak{g}_{0,x}, \mathfrak{h}' = \mathfrak{g}_{0,x'}$.

Then there exists some $\theta \in G$ such that $\theta(x_0) = x_0'$. For all $h \in \mathfrak{h}$, we have $(\operatorname{ad} x_0)^{n(h)}(h) = 0$ and $(\operatorname{ad} x_0')^{n(h)}(h) = 0 \implies \theta(h) \in \mathfrak{h}'$. Thus $\theta(h) \subseteq \mathfrak{h}'$, and by symmetry $\theta(h) \supseteq \mathfrak{h}'$.

Note: this concludes the content of Humphrey's book.

40.2 Affine Lie Algebras

Recall from section 18 that we had a 1-to-1 correspondence

$$\{\text{Cartan matrices } A\} \iff \{\text{semisimple Lie algebras } \mathfrak{g}(A)\}.$$

Definition: A matrix $A = (a_{ij})$ is a *generalized Cartan matrix* if $a_{ii} = 2, i \neq j \implies a_{ij} \in \mathbb{Z}_{\leq 0}$, and $a_{ij} = 0 \iff = a_{ji} = 0$.

Definition: A generalized Cartan matrix A is of *finite type* if there exists a vector $\mathbf{v} > \mathbf{0}$ (coordinate-wise) such that $A\mathbf{v} > \mathbf{0}$. It is of *affine* type if $\exists \mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. It is of *indefinite* type if $\exists \mathbf{v}$ such that $A\mathbf{v} < \mathbf{0}$.

Examples:

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \text{ finite type, take } \mathbf{v} = [5,3]^t \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \text{ affine type, take } \mathbf{v} = [0,0]^t \quad \begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix} \text{ indefinite t}$$

Theorem: If A is indecomposable, i.e. $A \neq A_1 \oplus A_2$, then A has exactly one of these three types.

Facts: Let A be an indecomposable generalized Cartan matrix. Then

a. A has finite type $\iff$ A is a Cartan matrix
b. A has affine type $\iff$ $\det(A) = 0$

Every connected proper subgraph of Dynkin(A) is a Dynkin diagram of finite type. This allows us to classify all affine generalized Cartan matrices.

Affine Coxeter diagrams:

Image

A Comparison:

Finite	Affine
Killing form	Standard invariant form using data from A
Weyl group (finite)	Affine Weyl group (infinite)
Roots Φ	Real roots and imaginary roots

Finite	Affine
Verma modules $M(\lambda) \twoheadrightarrow L(\lambda)$	Similar
Weyl character formula for finite dimensional irreducible modules	Kac character formula for integrable modules
Kazhdan–Lusztig theory	Similar

41 Wednesday December 04

41.1 Summary of Lie Algebras

- Overview
  - Definition of Lie Algebra, abelian, nilpotent, solvable, (semi)simple, reductive = semisimple and abelian.
  - Killing form $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\ x \circ \mathrm{ad}\ y)$
    - Solvable iff $\kappa(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$.
    - Semisimple iff κ is non-degenerate
  - Interested in Kac-Moody algebras
    - Finite = semisimple Lie algebra, finite dimensional
    - Affine = infinite dimensional
    - Indefinite = hard!
- Structure theory for semisimple Lie Algebras
  - Semisimple = direct sum of simples
  - Semisimples are in 1-to-1 correspondence with Dynkin diagrams for $A_\ell \to D_\ell$ (classical) or E_{6-8}, F_4, G_2 (exceptional), which are also in 1-to-1 correspondence with Cartan matrices A
  - Presentations of $\mathfrak{g}(A) = \langle e_i, f_i, h_i \rangle$ mod Cartan relations and Serre relations using a_{ij}
  - $\mathfrak{sl}(2) = \langle e, f, h \rangle / \sim$ where $[h, e] = 2e, [e, f] = h, [h, f] = -2f$
  - Cartan subalgebras $\mathfrak{h} :=$ nilpotent + self-normalizing $\iff$ maximal toral subalgebra
    - By the conjugacy theorem, $\mathrm{rank}\, \mathfrak{g} := \dim \mathfrak{h}$ is well-defined
    - By the abstract Jordan decomposition yields a root-space decomposition $\mathfrak{g} = \mathfrak{h} + \sum\limits_{\alpha \in \Phi} \mathfrak{g}_\alpha$
  - If Π is a fixed set of simple roots, then there exists a triangular decomposition $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n$ where $n^- = f's, \mathfrak{h} = h's, n = e's$
  - Semisimple $\iff$ κ non-degenerate $\iff$ $\mathfrak{h}^* \cong \mathfrak{h}$ by the map $\alpha \mapsto t\alpha$, where $\alpha = \kappa(t_\alpha, \cdot)$
  - $(\alpha, \beta) := \kappa(t_\alpha, t_\beta)$, coroots $\beta^\vee = 2\beta/(\beta, \beta)$
  - $(\alpha, \beta^\vee) = \kappa(t_\alpha, h_\beta) = \alpha(h_\beta)$ yields an inner product
  - Generates reflections $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}*$ where $\lambda \mapsto \lambda - (\lambda, \alpha^\vee)\alpha$
  - Yields the Weyl group $W = \left\langle s_\alpha \;\middle|\; \alpha \in \Pi \right\rangle$
    - Every $w \in W$ has a reduced expression $w = \prod\limits_i S_{\alpha_i}$
    - $\ell(w) =$ length of w $= \# \left\{ \alpha \in \Phi^+ \;\middle|\; w\alpha \in \Phi^- \right\}$
  - Universal enveloping algebra has a PBW basis
  - $Z(U(\mathfrak{g})) \cong \mathcal{S}(\mathfrak{h})^W$
  - Yields central characters $x_\lambda = x_\mu \iff \lambda \in W \cdot \mu$ where $w \cdot \mu = ?$
- $Z(U(\mathfrak{g})) \ni \Omega = \sum x_i x_i'$ where $\kappa(x_i, x_j') = \delta_{ij}$ the Casimir element

- This acts on simple modules by a scalar, where $\Omega \curvearrowright M(\lambda)$ by $(\lambda + p, \lambda + p) - (p, p) = (\lambda + 2p, \lambda)$

41.2 Representation Theory of Semisimple Lie Algebras

- Simple = irreducible modules, but simple $\neq$ indecomposable modules
- Composition series, completely reducible = direct sum of irreducibles
- Construct new modules by $V \otimes W, V^\vee, \hom(V, W) = V^\vee \otimes W$
- Theorem (Weyl): If $\mathfrak{g}$ is semisimple, then any finite-dimensional module is completely reducible
- Integral weights $\Lambda = \sum_i \mathbb{Z} w_i$, where w_i is a fundamental weight such that $(w_i, \alpha_j^\vee) = \delta_{ij}$
- The dominant integral weights are given by $\Lambda^+ = \sum_i \mathbb{Z}_{\geq 0} w_i$
  - For $\mathfrak{g} = \mathfrak{sl}(2)$, we have
    - $\mathfrak{g}^* \cong \mathbb{C}$
    - $\lambda \mapsto \mathbb{Z}$
    - $\alpha_1 \mapsto 2$
    - $\rho = w_j \mapsto 1$
    - Verma $M(\lambda) = \text{span}(v_0, v_1, \cdots)$ corresponding to weights $\lambda, \lambda - 2, \cdots - \lambda$.
    - Irreducible $L(\lambda) = \text{span}(\overline{v}_0, \overline{v}_1, \cdots)$
    - Formal characters char $M(\lambda) = e(\lambda) + e(\lambda - 2) + \cdots \sim e(\lambda)(1 + e(-2) + e(-2)^2 + \cdots) \sim \dfrac{e(\lambda)}{1 - e(-2)}$ as a formal power series
    - Similarly, char $L(\lambda) = e(\lambda) + e(\lambda - 2) + \cdots$
- If $\mathfrak{g}$ is semisimple, then there is a weight module, highest weight module, and maximal vectors
- Verma modules $M(\lambda) = \text{End}_{\mathfrak{h}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$
- Yields $\lfloor$ the Borel subalgebra given by $\lfloor = n \oplus \mathfrak{h}$, has basis $\left\{ \mathbf{f}^{\mathbf{b}} v^+ \right\}$
- Irreducible modules $L(\lambda) = M(\lambda)/N(\lambda)$ where $N(\lambda)$ is the sum of proper submodules of $M(\lambda)$.
- char $M(\lambda) = e(\lambda) / \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))$
- Theorem (Weyl): If $\lambda \in \Lambda^+$, then there is a formula for char $L(\lambda)$.
- If $\lambda \notin \Lambda^+$, then char $L(\lambda)$ can be deduced using composition multiplicity $[M(\lambda) : L(\mu)]$.
  - These are obtained from the Kazhdan-Lusztig polynomials
  - Extended to category $\mathcal{O}$

42 Some Possible Generalizations

- Swap $\mathbb{C}$ with $\mathbb{R}$ of $\overline{\mathbb{F}}_p$
- Finite leads to affine or indefinite
- Lie Algebras lead to Algebraic groups/ Lie groups
- Can also consider Lie super-algebras
- Quantisation leads to quantum groups