Title

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Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of **??**; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\mathrm{zar}}, X_{\mathrm{ét}}, X_{\mathrm{fppf}}$, then the data of a GL_n-torsor split by a τ-cover $U \to X$ is the same as descent data for a vector bundle relative to $U_{/X}$.

This descent data comes from the following:

$$
\begin{array}{c}
U \times_X U \\
\pi_1 \downarrow\downarrow \pi_2 \\
U \\
\downarrow \\
X
\end{array}
$$

That U trivializes our torsor means that $\pi^* T = \pi^* G$ as a G-torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$
\begin{array}{ccc}
\pi_1^* \pi^* T & \overset{\sim}{\longrightarrow} & \pi_2^* \pi^* T \\
\downarrow & & \downarrow \\
\pi_1^* \pi^* G & \overset{\sim}{\longrightarrow} & \pi_2^* \pi^* G
\end{array}
$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T, and the bottom horizontal map is an automorphism of a G-torsor and thus is a section to G. I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n-torsor on the fppf site, it's also trivial on the other two sites, yieldings the desired maps back and forth. Thus $H^1(X_{\mathrm{ét}}, \mathrm{GL}_n)$ is in bijection with n-dimensional vector bundles on X. ■

Exercise 1.0.3(?): See if Hilbert 90 is true for groups other than GL_n.

1.1 Representability and Local Triviality

Question 1.1.1: Suppose G is an affine flat X-group scheme. Are all G-torsors representable by a X-scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G-torsor flat locally and use fppf descent.

Question 1.1.3: Given a G-torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

> *Proof (Sketch).*
> You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \to X$, we can base change it to itself:
>
> $$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow\!\!\uparrow\!\exists & & \downarrow \\ T & \xrightarrow{\ f\ } & X \end{array}$$
>
> The torsor $T \times_X T \to T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G-action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G. We can thus find some U such that
>
> $$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow\!\!\uparrow\!\exists & & \downarrow \\ T & \xrightarrow{\ f\ } & X \end{array}$$
>
> U $\xrightarrow{\text{closed}}$ T, $U \xrightarrow{\exists\text{ét}} X$
>
> Here "slicing" means finding such a U, and this can be done using the structure theorem for smooth morphisms. ∎

Example 1.1.5 *(non-smooth group schemes):*
- α_p, the kernel of Frobenius on $\mathbb{A}^1$ or $\mathbb{G}_a$,
- μ_p in characteristic p, representing pth roots of unity, the kernel of Frobenius on $\mathbb{G}_m$,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$, etc.

1.1.1 What Hilbert 90 Means

Example 1.1.6 *(?):* Let $X = \operatorname{Spec} k, n = 1$, so we're looking at $H^{\cdot}(\operatorname{Spec} k, \mathbb{G}_m)$.

$$H^1\left((\operatorname{Spec} k)_{\text{zar}}, \mathbb{G}_m\right) = 0$$
$$= H^1\left((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m\right)$$
$$= H^1(\operatorname{Gal}(k^s/k), \bar{k}^{\times}).$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.[1]

Let's generalize this observation.

Example 1.1.7 *(?):* Let X be any scheme and $n = 1$, then $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

Example 1.1.8 *(?):* Let's compute $H^1(X_{\text{ét}}, \mu_\ell)$ where ℓ is an invertible function on X. We have a SES of étale sheaves, the **Kummer sequence**,

$$1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{z \mapsto z^p} \mathbb{G}_m \to 1.$$

This is exact in the étale topology since adjoining an ℓth power of any function gives an étale cover. We get a LES in cohomology

We know that $H^0(X_{\text{ét}}, \mathbb{G}_m)$ are invertible functions on X, and the red term is what we'd like to compute.

Suppose now $H^0(X, \mathcal{O}_X) = k = \bar{k}$, then $H^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k)$ since it is the kernel of the ℓth power map. We can also compute $H^1(X_{\text{ét}}, \mu_\ell)$, since our diagram reduces to

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[1]This is called "90" since Hilbert numbered his theorems in at least one of his books.

$$\mu_\ell(k) \xleftarrow{\qquad\qquad\quad} k^\times \xrightarrow{z \mapsto z^\ell} k^\times \xrightarrow{\quad} 0$$

$$H^1(X_{\text{ét}}, \mu_\ell) \xrightarrow{\delta\qquad} \text{Pic}(X)[\ell] \xrightarrow{[\ell]} \text{Pic}(X)$$

$$H^2(X_{\text{ét}}, \mu_\ell) \xrightarrow{\qquad\qquad} \cdots$$

where surjectivity of δ follows from the fact that $k = \bar{k}$ and thus every element has an ℓth root, making H^1 the kernel of $[\ell]$.

Example 1.1.9*(?):* Let $X_{/k}$ with $k = \bar{k}$ with ℓ invertible in k, then (claim) $\underline{\mathbb{Z}/\ell\mathbb{Z}} \cong \mu_\ell$ given by sending a generator to some choice of a primitive ℓth root of unity. To be explicit, we have a representation $\underline{\mathbb{Z}/\ell\mathbb{Z}} = \hom(\,\cdot\,, \text{Spec}\, k[t]/t(t-1)\cdots(t-\ell+1))$ and $\mu_\ell = \text{Spec}\, k[t]/t^\ell - 1$. These are both disjoint unions of points, and hence schemes of dimension zero since ℓ is invertible in the base and the Chinese Remainder Theorem, so one can write down the isomorphism explicitly between the schemes and hence the functors they represent.

Corollary 1.1.10*(?).*
If $\mu_\ell \subseteq k$, then

$$H^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell\mathbb{Z}}) = H^i(X_{\text{ét}}, \mu_\ell).$$

Since the isomorphism depends on the choice of a primitive root, this will not be Galois equivariant, which will come up when we talk about Galois actions on étale cohomology. This already happens for H^0, since $G \curvearrowright \mathbb{Z}/\ell\mathbb{Z}$ trivially but not on μ_ℓ.

1.1.2 Geometric Interpretations

Let X be an affine scheme, we now know $H^1(X_{\text{ét}}, \mathbb{F}_p) = \text{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_x)$, the Artin-Schreier map, and these are $\mathbb{F}_p$-torsors. We also know $H^1(X_{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z})$ in terms of the LES if $k = \bar{k}$ and $\text{ch}(k) = p$, and this is a $\mathbb{Z}/\ell\mathbb{Z}$-torsor. Being torsors here geometrically means they're covering spaces with those groups as Galois groups.

Question 1.1.11: How does one write down these torsors/covering spaces?

Example 1.1.12*(?):* Given

$$[Y] \in H^1(X_{\text{ét}}, \mathbb{F}_p) = \text{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$$

where we write $[Y]$ to denote thinking of the torsor as some geometric object, how to we write down the covering space? Using Artin-Schreier, we can write $Y = \{y^p - y = a\}$ for some $a \in \mathcal{O}_X$, an **Artin-Schreier covering**.

If $\ell \neq \mathrm{ch}(k)$ and $[Z] \in H^1(X_{\text{ét}}, \mu_\ell)$ and assume $\mathrm{Pic}(X) = 0$. Then we can write

$$H^1(X_{\text{ét}}, \mu_\ell) = \mathrm{coker}(\mathcal{O}_X \xrightarrow{x \mapsto x^\ell} \mathcal{O}_X{}^\times)$$

In this case, $Z = \{z^\ell = f\}$ where $f \in \mathcal{O}_X{}^\times$ is an element representing the class in cohomology, and $\mu_\ell \curvearrowright Z$ by multiplication by z.

Remark 1.1.13: The process of explicitly writing down covers is called **explicit geometric class field theory**, which gives a recipe for writing down abelian covers of covers. In general, for $\mathrm{Pic}(X) \neq 0$, the Picard group plays a crucial role.

1.2 Computing the Cohomology of Curves

This is one of Daniel's favorite topics in the entire course!

Theorem 1.2.1*(?)*.
Let $X_{/k}$ be a smooth curve over $k = \bar{k}$, then

$$H^i(X_{\text{ét}}, \mathbb{G}_m) = \begin{cases} \mathcal{O}_X(X)^\times & i = 0 \\ \mathrm{Pic}(X) & i = 1 \\ 0 & \text{else,} \end{cases}$$

noting that $\mathcal{O}_X(X)^\times$ are the global sections of $\mathbb{G}_m$, i.e. invertible functions on X.

The first two cases we've done, $i > 1$ is the hard case.

Corollary 1.2.2*(?)*.
For X a smooth proper connected curve of genus g, $k = \bar{k}$, and $\ell \neq \mathrm{ch}(k)$ is prime,

$$H^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & i = 0 \\ \mathrm{Pic}(X)[\ell^n] = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g} & i = 1 \\ \mathbb{Z}/\ell^n\mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}.$$

Proof (of corollary).
We'll use some theory of abelian varieties: $\mathrm{Pic}^0(X) = \mathrm{Jac}(X)$, and we have a SES

$$0 \to \mathrm{Jac}(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where we identify the Néron-Severi group as $\mathbb{Z}$.[a] We'll use that $\mathrm{Jac}(X)$ is a g-dimensional abelian variety, and so $\mathrm{Jac}(X)[\ell^n] \cong_{\mathrm{Grp}} (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$.
The Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

yields a LES where we identify $\mu_{\ell^n} \cong \mathbb{Z}/\ell^n\mathbb{Z}$:

$$H^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \longrightarrow \text{Pic}(X) \xrightarrow{[\ell]} \text{Pic}(X)$$

with arrows descending from 0 and descending to

$$H^2(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) \longrightarrow 0 \longrightarrow 0$$

So we're just computing the kernel and cokernel of $[\ell]$.

Computing H^1: We'll need one more fact: $\text{Jac}(X)(\bar{k})$ is a divisible group. We can identify

$$H^1(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}}) = \text{Pic}(X)[\ell^n] = \text{Jac}(X) = (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}.$$

where the 2nd equality uses the fact that $\text{Pic}(X)$ is an extension of $\mathbb{Z}$ by an abelian variety and $\mathbb{Z}$ has no torsion, and the last equality is general theory of abelian varieties.

Computing H^2: Since $\text{Jac}(X)$ is divisible, we can identify

$$\text{coker}(\text{Pic}(X) \xrightarrow{[\ell^n]} \text{Pic}(X)) \cong \text{coker}(\mathbb{Z} \xrightarrow{[\ell^n]} \mathbb{Z}) = \mathbb{Z}/\ell^n\mathbb{Z}.$$

The vanishing of higher cohomology follows from the vanishing for $\mathbb{G}_m$. So assuming the theorem and the theory of abelian varieties proves this corollary. ∎

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[a]See Hartshorne Ch. 4, or anything that discusses cohomology of curves.

Exercise 1.2.3(?): Check this using the snake lemma after applying multiplication by ℓ to the SES.

Remark 1.2.4: X is a scheme over $\bar{k}$, and if it started over some subfield L then $\text{Gal}(L_{/k}) \curvearrowright X$ and thus the corresponding functors. These isomorphisms will not be Galois equivariant, and the $\mathbb{Z}/\ell^n\mathbb{Z}$ showing up in degree 2 cohomology will admit a Galois action via the cyclotomic character.

1.2.1 Proof of Theorem

Goal: we want to show that $H^{>1}(X_{\text{ét}}, \mathbb{G}_m) = 0$ for X a smooth curve over $k = \bar{k}$. Three ingredients:

1. The Leray spectral sequence,

2. The divisor exact sequence,

3. Brauer groups.

1.3 Pushforwards and the Leray Spectral Sequence

Suppose $X \xrightarrow{f} Y$ is a morphism of schemes, then we get a functor $f_* \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$: given $\mathcal{F} \in Sh(X_{\text{ét}})$, we have $f_* \mathcal{F}(U \to Y) := \mathcal{F}(U \times_Y X)$. This is left-exact and thus has right-derived functors $R^{\cdot} f_* : \mathrm{Sh}^{\mathrm{Ab}}(X_{\text{ét}}) \to \mathrm{Sh}^{\mathrm{Ab}}(Y_{\text{ét}})$.
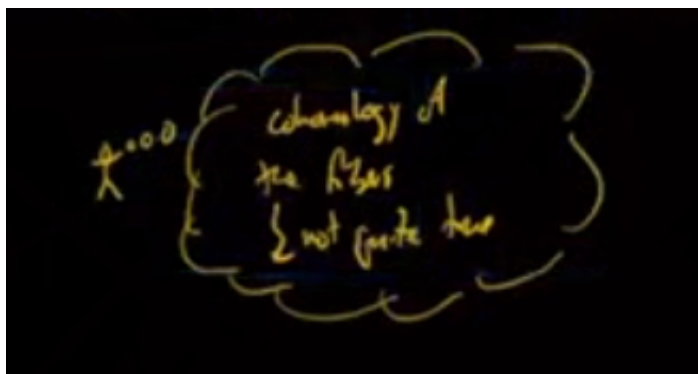
How to think about this:

Figure 1: Cohomology of the fibers: but not quite!

This is not quite true, and the obstruction is called **the base change property**, which we'll see later in the course. What's true in general is that $R^i f_* \mathcal{F}$ is the sheafification of the presheaf $V \to H^i(f^{-1}(V), \mathcal{F})$, which is not quite the cohomology of the fibers since sheafification is somewhat brutal.

> **Proposition 1.3.1** *(Derived pushforwards for finite morphisms).*
> If f is a finite morphism (e.g. a closed immersion) then $R^{>0} f_* = 0$.

Exercise 1.3.2 *(Proof, must-do!):* Prove this. The claim is that f_* is right-exact, which in this case shows it is exact. Check on stalks. Compute that the stalk of $f_* \mathcal{F}$ at $\bar{y} \in Y$ is given by

$$f_* \mathcal{F}_{\bar{y}} = \bigoplus_{\bar{x} \in f^{-1}(\bar{y})} \mathcal{F}_{\bar{x}}$$

for f a finite morphism (not necessarily unramified).

> **Proposition 1.3.3** *(technical).*
> f_* preserves injectives.

Exercise 1.3.4 *(proof):* Prove this! You can do this by showing the following fact from category theory: this is true for any functor with an exact left adjoint, which here is f^* and is exact since filtered colimits and sheafification are both exact, or alternatively you can check on stalks, since the stalks of f^{-1} are the stalks of the original functor.

Corollary 1.3.5*(The Leray Spectral Sequence).*

Suppose $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are morphisms of schemes, then there is a spectral sequence

$$R^i g_* R^j f_* \mathcal{F} \Rightarrow R^{i+j}(g \circ f)_* \mathcal{F}.$$

As a special case, for $Z = \operatorname{Spec} k$ with $k = \bar{k}$, then g_*, f_* are taking global sections so we get

$$H^i(Y, R^j f_* \mathcal{F}) \Rightarrow H^{i+j}(X, \mathcal{F}).$$

Proof (sketch).
There is a general statement (see Tohoku) for the spectral sequence of a composition of two functors between abelian functors where the first preserves injectives, you get such a spectral sequence. ∎