Title

D. Zack Garza

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- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \hom_{\text{Top}}(X, Y) := \left\{ f : X \to Y \mid f \text{ is continuous} \right\}.$$

  - Topologize with the *compact-open* topology: $U \in \hom_T(X, X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
    * If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \overset{n \to \infty}{\to} 0 \quad \forall K \subseteq X \text{ compact.}$$

      In words: $f_n \to f$ uniformly on every compact set.
  - If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with $d(f, g) = \sup_{x \in X} (f(x), g(x))$.
  - Useful in analysis: when is a family of functions $\mathcal{F} = \{f_\alpha\} \subset \hom_{\text{Top}}(X, Y)$ compact? Essentially answered by Arzela-Ascoli

> **Theorem 1.1** *(Ascoli).*
> If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \hom_{\text{Top}}(X, Y)$ has compact closure $\iff$ $\mathcal{F}$ is equicontinuous and $F_x := \left\{ f(x) \mid f \in \mathcal{F} \right\} \subset Y$ has compact closure.

> **Corollary 1.2** *(Arzela).*
> If $\{f_n\} \subset \hom_{\text{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{f_n(x)\}$ is bounded for every X, it contains a uniformly convergent subsequence.

  - Useful in Number Theory / Rep Theory / Fourier Series: can take $
- So define $\text{Map}(X, Y) = \hom_{\text{Top}}(X, Y)$ equipped with the compact-open topology.

- Can immediately consider a lot of interesting spaces by considering $\mathrm{Map}(\,\cdot\,,Y)$:

$$X = I := [0,1] \rightsquigarrow \quad \mathcal{P}Y := \{f : I \to Y\} = Y^I$$
$$X = S^1 \rightsquigarrow \quad \mathcal{L}Y := \left\{f : S^1 \to Y\right\} = Y^{S^1}.$$

Note: take basepoints to obtain the base path space PY, the based loop space ΩY.

- Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences).
- Adjoint property: there is a homeomorphism

$$\mathrm{Map}(X \times Z, Y) \leftrightarrow_{\cong} \mathrm{Map}(Z, \mathrm{Map}(X,Y))$$
$$H : X \times Z \to Y \iff \tilde{H} : Z \to \mathrm{Map}(X,Y)$$
$$(x,z) \mapsto H(x,z) \iff z \mapsto H(\,\cdot\,,z).$$

Categorically, $\hom(X,\,\cdot\,) \leftrightarrow (X \times \,\cdot\,)$ form an adjoint pair in Top.
- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0\mathrm{Map}(X,Y) = \{[f], \text{homotopy classes of maps } f : X \to Y\},$$

i.e. two maps f,g are homotopic $\iff$ they are connected by a path in $\mathrm{Map}(X,Y)$.
  * Proof:

$$\mathcal{P}\mathrm{Map}(X,Y) = \mathrm{Map}(I, \mathrm{Map}(X,Y)) \cong \mathrm{Map}(Y \times I, X),$$

and just check that $\gamma(0) = f \iff H(x,0) = f$ and $\gamma(1) = g \iff H(x,1) = g$.
  * Note that we can interpret the RHS as the space of paths
- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I,\,\cdot\,)$: define

$$\mathrm{Map}^1_I(X,Y) := \mathrm{Map}(I, \mathrm{Map}(X,Y)) = \mathcal{P}\mathrm{Map}(X,Y)$$

and then

$$\begin{aligned}
\mathrm{Map}^2_I(X,Y) &:= \mathrm{Map}(I, \mathrm{Map}^1_I(X,Y)) \\
&\cong \mathrm{Map}(I, \mathrm{Map}(I, \mathrm{Map}(X,Y))) \quad = \mathcal{P}(\mathcal{P}(X,Y)) \\
&\cong \mathrm{Map}(I, \mathrm{Map}(Y \times I, X)) \\
&:= \mathcal{P}\mathrm{Map}(Y \times I, X).
\end{aligned}$$

Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!
- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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