Title

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1 Wednesday, August 26

1.1 Review

- G a reductive algebraic group over k
- $T = \prod_{i=1}^{n} \mathbb{G}_m$ a maximal split torus
- $\mathfrak{g} = \mathrm{Lie}(G)$
- There's an induced root space decomposition $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$
- When G is simple, Φ is an *irreducible* root system
  - There is a classification of these by Dynkin diagrams

Example 1.1.
A_n corresponds to $\mathfrak{sl}(n+1, k)$ (mnemonic: A_1 corresponds to $\mathfrak{sl}(2)$)

- We have representations $\rho : G \longrightarrow \mathrm{GL}(V)$, i.e. V is a G-module
- For $T \subseteq G$, we have a weight space decomposition: $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ where $X(T) = \hom(T, \mathbb{G}_m)$.

> Note that $X(T) \cong \mathbb{Z}^n$, the number of copies of $\mathbb{G}_m$ in T.

1.2 Root Systems and Weights

Example 1.2.
Let $\Phi = A_2$, then we have the following root system:

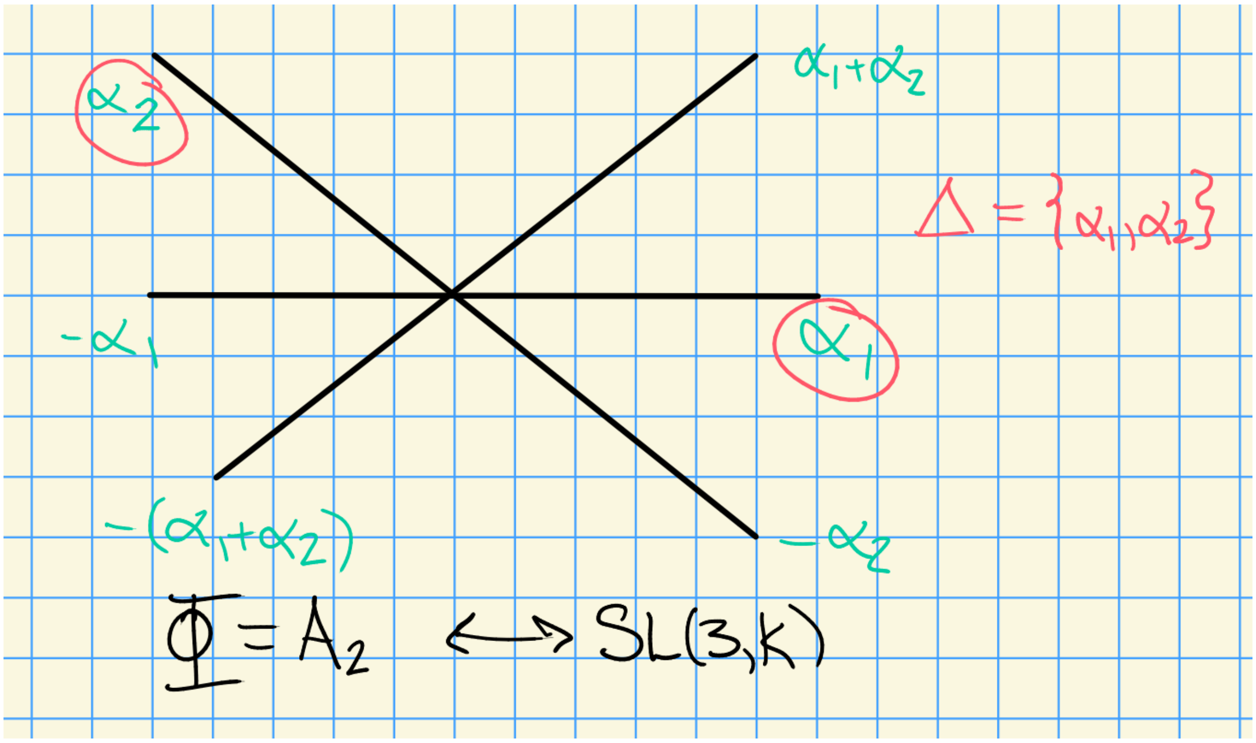

Figure 1: Image

In general, we'll have $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ a basis of *simple roots*.

Remark 1.
Every root $\alpha \in I$ can be expressed as either positive integer linear combination (or negative) of simple roots.

For any $\alpha \in \Phi$, let s_α be the reflection across H_α, the hyperplane orthogonal to α. Then define the *Weyl group* $W = \left\{ s_\alpha \;\middle|\; \alpha \in \Phi \right\}$.

Example 1.3.
Here the Weyl group is S_3:

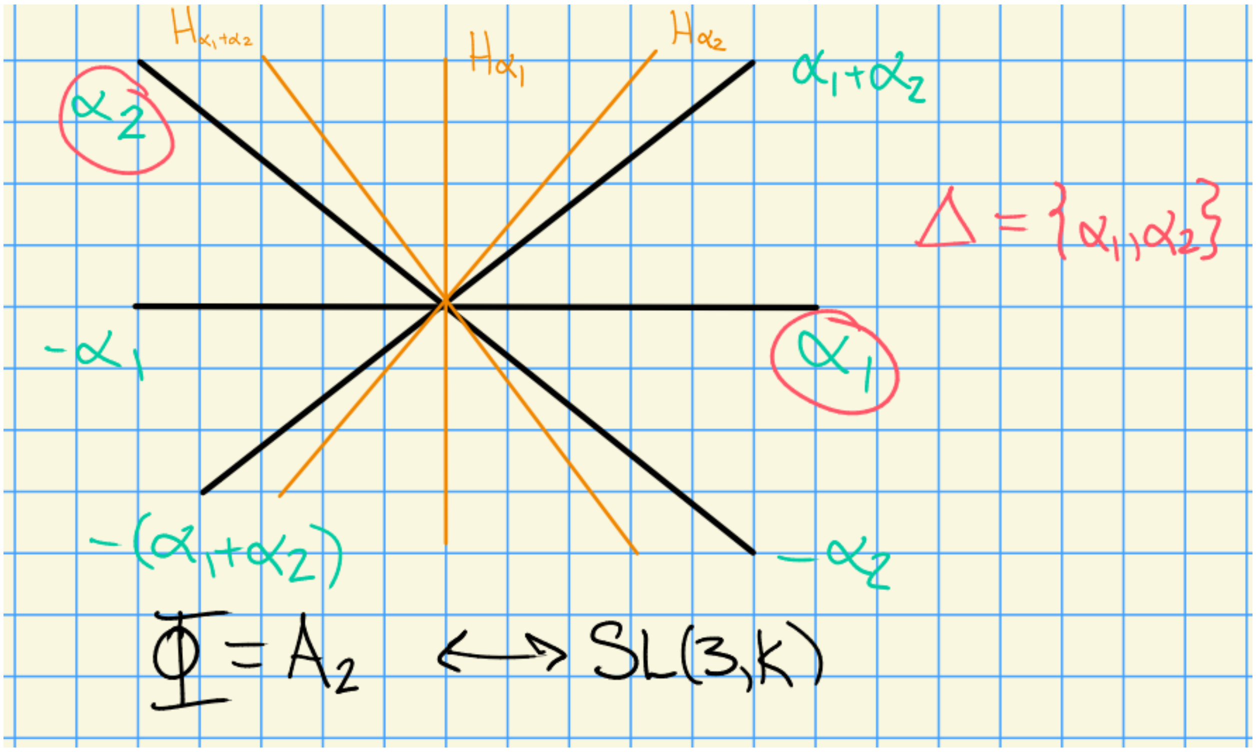

Figure 2: Image

Remark 2.
W acts transitively on bases.

Remark 3.
$X(T) \subseteq \mathbb{Z}\Phi$, recalling that $X(T) = \hom(T, \mathbb{G}_m) = \mathbb{Z}^n$ for some n. Denote $\mathbb{Z}\Phi$ the *root lattice* and $X(T)$ the *weight lattice*.

Example 1.4.
Let $G = \mathfrak{sl}(2, \mathbb{C})$ then $X(T) = \mathbb{Z}\omega$ where $\omega = 1$, $\mathbb{Z}\Phi = \mathbb{Z}\{\alpha\}$ Then there is one weight α, and the root lattice $\mathbb{Z}\Phi$ is just $2\mathbb{Z}$. However, the weight lattice is $\mathbb{Z}\omega = \mathbb{Z}$, and these are not equal in general.

Remark 4.
There is partial ordering on $X(T)$ given by $\lambda \geq \mu \iff \lambda - \mu = \sum_{\alpha\in\Delta} n_\alpha \alpha$ where $n_\alpha \geq 0$. (We say λ *dominates* μ.)

Definition 1.0.1 (Fundamental Dominant Weights)**.**
We extend scalars for the weight lattice to obtain $E \coloneqq X(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, a Euclidean space with an inner product $\langle \cdot, \ \cdot \rangle$.
For $\alpha \in \Phi$, define its *coroot* $\alpha^\vee \coloneqq \dfrac{2\alpha}{\langle \alpha, \ \alpha \rangle}$. Define the *simple coroots* as $\Delta^\vee \coloneqq \{\alpha_i^\vee\}_{i=1}^n$, which has a dual basis $\Omega \coloneqq \{\omega_i\}_{i=1}^n$ the *fundamental weights*. These satisfy $\left\langle \omega_i, \ \alpha_j^\vee \right\rangle = \delta_{ij}$.

What is the notation for fundamental weights? Definitely not Ω usually!

Important because we can index irreducible representations by fundamental weights.

A weight $\lambda \in X(T)$ is *dominant* iff $\lambda \in \mathbb{Z}^{\geq 0}\Omega$, i.e. $\lambda = \sum n_i \omega_i$ with $n_i \in \mathbb{Z}^{\geq 0}$.

If G is simply connected, then $X(T) = \bigoplus \mathbb{Z}\omega_i$.

See Jantzen for definition of simply-connected, $\mathrm{SL}(n+1)$ is simply connected but its adjoint $PGL(n+1)$ is not simply connected.

1.3 Complex Semisimple Lie Algebras

When doing representation theory, we look at the Verma modules $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda \twoheadrightarrow L(\lambda)$.

Theorem 1.1 *(?)*.
$L(\lambda)$ as a finite-dimensional $U(\mathfrak{g})$-module $\iff$ λ is dominant, i.e. $\lambda \in X(T)_+$.

Thus the representations are indexed by lattice points in a particular region:

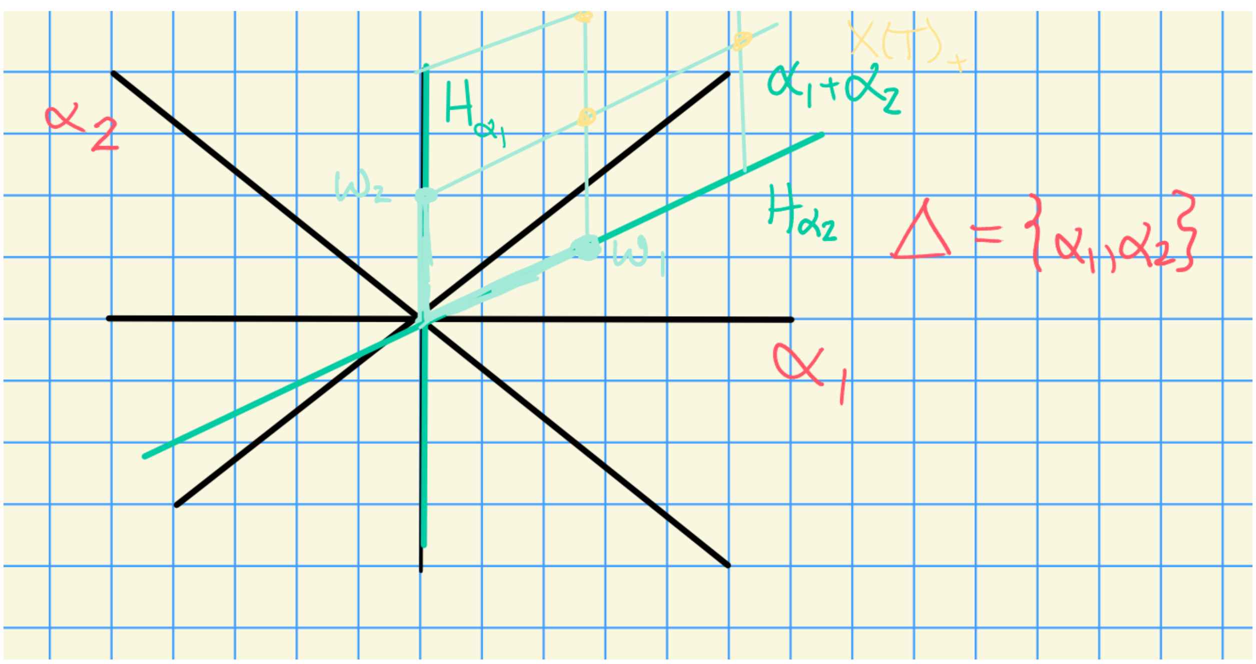

Figure 3: Image

Question 1:

Suppose G is a simple (simply connected) algebraic group. How do you parameterize *irreducible* representations?

For $\rho : G$
$to \mathrm{GL}(V)$, V is a *simple module* (an *irreducible representation*) iff the only proper G-submodules of V are trivial.

Answer 1: They are also parameterized by $X(T)_+$. We'll show this using the induction functor $\mathrm{Ind}_B^G \lambda = H^0(G/B, \mathcal{L}(\lambda))$ (sheaf cohomology of the flag variety with coefficients in some line bundle).

> We'll define what B is later, essentially upper-triangular matrices.

Question 2: What are the dimensions of the irreducible representations for G?

Answer 2: Over $k = \mathbb{C}$ using Weyl's dimension formula.

For $k = \overline{\mathbb{F}_p}$: conjectured to be known for $p \geq h$ (the *Coxeter number*), but by Williamson (2013) there are counterexamples. Current work being done!