

# Linearization Continued

## Section 8.4 Follow-Up

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Linearization  
Continued

D. Zack Garza

## Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

# Review

# Definitions

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \mapsto \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:  
*For every  $w \in C^\infty(S^2, W)$  there exists a symplectic trivialization of the fiber bundle  $w^*TW$ , i.e.  $\langle c_1(TW), \pi_2(W) \rangle = 0$  where  $c_1$  denotes the first Chern class of the bundle  $TW$ .*
- We use this to trivialize the pullback  $\tilde{u}^*TW$  to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

# The Frame

Linearization  
Continued

D. Zack Garza

## Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

- We used the chosen frame  $\{Z_i\}$  to define a chart centered at  $u$  of  $\mathcal{P}^{1,p}(x, y)$  given by

$$\iota : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow \mathcal{P}^{1,p}(x, y)$$

$$Y = (y_1, \dots, y_{2n}) \longmapsto \exp_u \left( \sum y_i Z_i \right).$$

- We regard  $Y(s, t)$  as a tangent vector to  $W$  in some Euclidean embedding.

# Charts

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

- We seek to compute the composite map in charts:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & \mathcal{F}_u & & & & \\
 & \nearrow & & \searrow & & & \\
 W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) & \xrightarrow{\iota} & \mathcal{P}^{1,p}(x, y) & \xrightarrow{\mathcal{F}} & L^p(\mathbb{R} \times S^1; TW) & \longrightarrow & L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\
 & & \mathcal{F} & & & & \\
 & & \nearrow & & \searrow & & \\
 & & & & & & 
 \end{array} \\
 \\
 u & \xrightarrow{\mathcal{F}} & \frac{\partial u}{\partial s} + J(u) \left( \frac{\partial u}{\partial t} - X_t(u) \right) \\
 \\
 (y_1, \dots, y_{2n}) & \longrightarrow & \exp_u \left( \sum y_i Z_i \right)
 \end{array}$$

# Add a Tangent

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} - J(u) X_t(u)$$

$$\mathcal{F}(u + Y) = \frac{\partial(u+Y)}{\partial s} + J(u+Y) \frac{\partial(u+Y)}{\partial t} - J(u+Y) X_t(u+Y)$$

Extract the part that is linear in  $Y$  and collect terms:

$$\begin{aligned} (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + (dJ)_u(Y) \frac{\partial u}{\partial t} + J(u) \frac{\partial Y}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \\ &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \end{aligned}$$

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

## 8.4.1: Leibniz Rule

# Leibniz Rule

Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}(d\mathcal{F})_u(Y) &= \left( \frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\ &\quad + \left( (dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) (dX_t)_u(Y) \right) \\ &= \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) \\ &\quad + \sum_{i=1}^{2n} y_i \left( \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u) (dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).\end{aligned}$$

Use the fact that this is  $O_1 + O_0$  in  $Y$ .



# Order 1

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

Study  $O_1$  first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left( \frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \dots, y_{2n}).$$

where  $J_0$  is the standard complex structure on  $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial}Y + SY$$

where  $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^n))$  is a linear operator of order 0.

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

## Order 0 Part is Symmetric in the Limit

## 8.4.4: Order 0 Part is Symmetric in the Limit

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

### Theorem (8.4.4, CR + Symmetric in the Limit)

*If  $u$  solves Floer's equation, then*

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

*where*

- 1**  $S$  is linear
- 2**  $S$  tends to a symmetric operator as  $s \rightarrow \pm\infty$ , and
- 3** We have the limiting behavior

$$\frac{\partial S}{\partial s}(s, t) \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{uniformly in } t$$

# Proof

Collect terms in the order zero part:

$$\begin{aligned} O_0 = S(y_1, \dots, y_{2n}) &= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\ &\quad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right] \\ &= \sum_{i=1}^{2n} y_i \left[ \frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i) X_t \right) \right. \\ &\quad \left. + J(u) \frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right]. \end{aligned}$$

- Claim: the terms in blue and orange vanish in the limit  $s \rightarrow \pm\infty$ , so it suffices to prove that the red term limits to a symmetric operator.

# Proof: Blue Term Vanishes

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

$$(dJ)_u(Z_i) \left( \frac{\partial u}{\partial t} - (Z_i)X_t \right) \longrightarrow 0$$

The term in blue vanishes: since  $u$  is a solution and

$$\frac{\partial u}{\partial s} \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left( \frac{\partial u}{\partial t} - X_t(u) \right) \xrightarrow{s \rightarrow \pm\infty} 0$$

*This seems to be the full argument for the blue term.*

# Proof: Orange Term Vanishes (1 and 3)

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

$$\frac{\partial Z_i}{\partial s} \xrightarrow{s \rightarrow \pm\infty} 0$$

Follows since the frame  $Z_i$  was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \partial t} \quad \curvearrowright \quad Z_i \xrightarrow{s \rightarrow \pm\infty} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \xrightarrow{s \rightarrow \pm\infty} 0.$$

*This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in  $t$ ?*

# Proof: Symmetry

Write the remaining red term as

$$A := A(y_1, \dots, y_{2n}) = \sum y_i \left( J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u(Z_i) \right).$$

Extract the  $j$ th component:

$$A_j = \sum y_i \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u(Z_i), Z_j \right\rangle.$$

We'll show that

$$\begin{aligned} \lim_{s \rightarrow \pm\infty} & \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u(Z_i), Z_j \right\rangle \\ & - \left\langle J(u) \frac{\partial Z_j}{\partial t} - J(u) (dX_t)_u(Z_j), Z_i \right\rangle = 0. \end{aligned}$$

# Proof: Symmetry

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

Use the fact that the frame  $\{Z_i\}$  is unitary:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \langle J(u)Z_i, Z_j \rangle \\ &= \left\langle (dJ)_u \left( \frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle + \left\langle J(u)Z_i, \frac{\partial Z_j}{\partial t} \right\rangle \\ &= \left\langle (dJ)_u \left( \frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle - \left\langle Z_i, J(u) \frac{\partial Z_j}{\partial t} \right\rangle. \end{aligned}$$



# Proof: Symmetry

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

Therefore it suffices to show

$$\begin{aligned} & - \left\langle J(u) (dX_t)_u (Z_i), \quad Z_j \right\rangle \\ & + \left\langle J(u) (dX_t)_u (Z_j), \quad Z_i \right\rangle \\ & - \left\langle (dJ)_u \left( \frac{\partial u}{\partial t} \right) Z_i, \quad Z_j \right\rangle \end{aligned}$$

$$\xrightarrow{s \rightarrow \pm\infty} 0.$$

# Proof: Symmetry

Using the fact that

$$\left( \frac{\partial u}{\partial t} - X_t(u) \right) \xrightarrow{s \rightarrow \pm \infty} 0$$

we can equivalently show

$$\begin{aligned} & - \langle J(u) (dX_t)_u (Z_i), \quad Z_j \rangle \\ & + \langle J(u) (dX_t)_u (Z_j), \quad Z_i \rangle \\ & - \langle (dJ)_u (X_t) Z_i, \quad Z_j \rangle \end{aligned}$$

$$\xrightarrow{s \rightarrow \pm \infty} 0$$

*For a fixed  $(s, t)$ , this expression only depends on  $Z_i$  at the point  $u(s, t)$ .*

## Lemma That Concludes the Proof

# Lemma

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

**Lemma:** For  $p \in W$ ,  $\{Z_i\}$  a unitary basis of  $T_p W$ ,

$$\begin{aligned} & - \langle J(p)(dX_t)_p(Z_i), Z_j \rangle \\ & + \langle J(p)(dX_t)_p(Z_j), Z_i \rangle \\ & - \langle (dJ)_p(X_t)Z_i, Z_j \rangle \\ & = 0. \end{aligned}$$

**Claim:** This lemma immediately concludes the previous proof?

# Proof of Lemma

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

Extend  $\{Z_i\}$  to a chart containing  $p$  and use the Leibniz rule to rewrite

$$- \langle J(p)(dX_t)_p(Z_i), Z_j \rangle + \langle J(p)(dX_t)_p(Z_j), Z_i \rangle - \langle (dJ)_p(X_t) Z_i, Z_j \rangle = 0$$

as

$$- \langle J(dX_t)(Z_i), Z_j \rangle + \langle J(dX_t)(Z_j), Z_i \rangle + \langle J(dZ_i)(X_t), Z_j \rangle - \langle d(JZ_i)(X_t), Z_j \rangle$$

$$= \langle J[X_t, Z_i], Z_j \rangle + \langle J(dX_t)(Z_j), Z_i \rangle - \langle d(JZ_i)(X_t), Z_j \rangle.$$

where we'll rewrite the red terms.

# Proof of Lemma

Now use

$$X_t \langle JZ_i, Z_j \rangle = 0 \implies \langle d(JZ_i)(X_t), Z_j \rangle + \langle JZ_i, (dZ_j)(X_t) \rangle = 0.$$

We now rewrite the RHS from before:

$$\begin{aligned} \langle J[X_t, Z_i], Z_j \rangle + \langle J(dX_t)(Z_j), Z_i \rangle + \langle JZ_i, (dZ_j)(X_t) \rangle \\ = \langle J[X_t, Z_i], Z_j \rangle + \langle J(dX_t)(Z_j) - J(dZ_j)(X_t), Z_i \rangle \\ = \langle J[X_t, Z_i], Z_j \rangle - \langle J[X_t, Z_j], Z_i \rangle \\ = \omega([X_t, Z_i], Z_j) - \omega([X_t, Z_j], Z_i). \end{aligned}$$

The symmetry follows from  $\omega$  being closed and

$$\begin{aligned} 0 &= d\omega(X_t, Z_i, Z_j) \\ &= X_t \cdot \omega(Z_i, Z_j) - Z_i \cdot \omega(X_t, Z_j) + Z_j \cdot \omega(X_t, Z_i) \\ &\quad - \omega([X_t, Z_i], Z_j) + \omega([X_t, Z_j], Z_i) - \omega([Z_i, Z_j], X_t) \\ &= -X_t \cdot \langle Z_j, JZ_i \rangle + Z_i \cdot (dH_t)(Z_j) - Z_j \cdot (dH_t)(Z_i) \\ &\quad - (dH_t)([Z_i, Z_j]) - \omega([X_t, Z_i], Z_j) + \omega([X_t, Z_j], Z_i) \\ &= d(dH_t)(Z_i, Z_j) - \omega([X_t, Z_i], Z_j) + \omega([X_t, Z_j], Z_i) \\ &= -\omega([X_t, Z_i], Z_j) + \omega([X_t, Z_j], Z_i). \end{aligned}$$



Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

## 8.4.6: Linearization of Hamilton's Equation

# Linearization of Hamilton's Equation

## Recall

$$(d\mathcal{F})_u = \bar{\partial}Y + SY = (\bar{\partial} + S)Y$$

Now think of  $S$  as a map  $Y \mapsto S \cdot Y$ , so  $S \in C^\infty(\mathbb{R} \times S^1; \text{End}(\mathbb{R}^{2n}))$  and define the symmetric operators

$$S^\pm := \lim_{s \rightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

## Theorem

*The equation*

$$\partial_t Y = J_0 S^\pm Y$$

*is a linearization of Hamilton's equation*

$$\frac{\partial z}{\partial t} = X_t(z) \quad \text{at} \quad \begin{cases} x = \lim_{s \rightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \rightarrow \infty} u & \text{for } S^+ \end{cases} \quad \text{respectively.}$$



# Proof

We first linearize Hamilton's equation at  $x$ :

$$\frac{\partial z}{\partial t} = X_t(z) \xrightarrow{\text{linearized}} \frac{\partial Y}{\partial t} = (dX_t)_x Y.$$

So write  $Y = \sum y_i Z_i$  to obtain

$$\begin{aligned} \sum_i \frac{\partial y_i}{\partial t} Z_i &= \sum_i y_i \left( -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i) \right) \\ &= \sum_i \sum_j y_i \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i), Z_j \right\rangle Z_j \\ &= \sum_i \sum_j y_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle Z_i \end{aligned}$$

$$\implies \frac{\partial y_i}{\partial t} = \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle y_j.$$

# Proof

Thus we can rewrite the linearized equation as

$$\frac{\partial Y}{\partial t} = (dX_t)_x Y = B^- \cdot Y, \quad b_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x(Z_j), Z_i \right\rangle.$$

Recall

$$A := A(y_1, \dots, y_{2n}) = \sum y_i \left( J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u(Z_i) \right).$$

Now take  $s \rightarrow -\infty$  and look at the order zero part of  $(d\mathcal{F})_u$ . By the proof of 8.4.4, we have

$$\begin{aligned} A\left(\sum y_i Z_i\right) &= \sum_i \left( J(x) \frac{\partial Z_i}{\partial t} - J(x) (dX_t)_x(Z_i) \right) \\ &= \sum_i \sum_j y_i \left\langle J \frac{\partial Z_i}{\partial t} - J(dX_t)(Z_i), Z_j \right\rangle Z_j \\ &= \sum_i \sum_j y_j \left\langle J \frac{\partial Z_j}{\partial t} - J(dX_t)(Z_j), Z_i \right\rangle Z_i \\ &= \sum_i \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), JZ_i \right\rangle y_j Z_i. \end{aligned}$$

# Proof

Linearization  
Continued

D. Zack Garza

Review

8.4.1: Leibniz  
Rule

Order 0 Part is  
Symmetric in the  
Limit

Lemma That  
Concludes the  
Proof

8.4.6:  
Linearization of  
Hamilton's  
Equation

Deduce that

$$S = (s_{ij}), s_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x(Z_j), J(x)Z_i \right\rangle$$

Deduce

$$JZ_i = \begin{cases} Z_{i+n} & i \leq n \\ -Z_{i-n} & i \geq n+1 \end{cases} \implies s_{ij} = \begin{cases} b_{i+n,j} & i \leq n \\ -b_{i-n,j} & i \geq n+1 \end{cases}$$

$$\iff S^- = -J_0 B^-.$$

