Title

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Contents

1 | Friday, September 18

1.1 Frobenius Kernels

Let char $(k)p > 0$ and let G be an algebraic group scheme. We have a Frobenius map $F : G \to G$ given by $F((x_{ij})) = (x_{ij}^p)$, which we can iterate to get F^r for $r \in \mathbb{N}$. Setting $G_r = \ker F^r$ the rth Frobenius kernel, we get a normal series of group schemes

$$G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G.$$

There is an associated chain of finite dimensional Hopf algebras

$$\operatorname{Dist}(G_1) \leq \operatorname{Dist}(G_2) \leq \cdots \leq \operatorname{Dist}(G).$$

Then $k[G]^\vee = \operatorname{Dist}(G_r)$, and we get an equivalence of representations for G_r to representations for $\operatorname{Dist}(G_r)$.

A special case will be when G is a reductive algebraic group scheme. We'll start by finding a basis for $\operatorname{Dist}(G_r)$.

Recall the PBW theorem: we have a basis for $\mathfrak{g}$ given by

$$\begin{aligned}
\left\{ x_\alpha \mathrel{\Big|} \alpha \in \Phi^+ \right\} & \text{ Positive root vectors} \\
\left\{ h_i \mathrel{\Big|} i = 1, \cdots, n \right\} & \text{ A basis for } t \\
\left\{ x_\alpha \mathrel{\Big|} \alpha \in \Phi^- \right\} & \text{ Negative root vectors}
\end{aligned}$$

.

We can then obtain a basis for $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \left\langle \prod_{\alpha\in\Phi^+} x_\alpha^{n(\alpha)} \prod_{i=1}^{n} h_i^{k_i} \prod_{\alpha\in\Phi^+} x_{-\alpha}^{m(\alpha)} \right\rangle.$$

We can similarly obtain a basis for the distribution algebra

$$\mathrm{Dist}(G) = \left\langle \prod_{\alpha\in\Phi^+} \frac{x_\alpha^{n(\alpha)}}{n!} \prod_{i=1}^{n} \binom{h_i}{k_i} \prod_{\alpha\in\Phi^+} \frac{x_{-\alpha}^{n(\alpha)}}{n!} \right\rangle,$$

and we can similar get $\mathrm{Dist}(G_r)$ by restricting to $0 \leq n(\alpha), k_i, m(\alpha) \leq p^r - 1$. Above the k_i are allowed to be any integers. This yields a triangular decomposition

$$\mathrm{Dist}(G_r) = \mathrm{Dist}(U_r^+)\mathrm{Dist}(T_r)\mathrm{Dist}(U_r^-),$$

where we'll denote the first two terms $\mathrm{Dist}(B_r^+)$ and the last two as $\mathrm{Dist}(B_r)$.

1.2 Induced and Coinduced Modules

Goal: Classify simple G_r-modules. We know the classification of simple G-modules, so we'll follow similar reasoning. We started by realizing $L(\lambda) \hookrightarrow \mathrm{Ind}_B^G \lambda$ as a submodule (the socle) of some "universal" module.

Let M be a B_r-module, we can then define

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r},$$

where we're now taking the B_r-invariants. We get a decomposition as vector spaces,

$$k[G_r] = k[U_r^+] \otimes_k k[B_r]$$

and thus an isomorphism

$$\mathrm{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes M$$

since $k[B_r] \otimes M \cong \mathrm{Ind}_{B_r}^{B_r} M \cong M$.