

# Discussion Notes

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## 1 Discussion 1

If  $X$  is an  $F_\sigma$  set, then

$$X = \bigcup_{i=1}^{\infty} F_i \quad \text{with each } F_i \text{ closed.}$$

If  $X$  is a  $G_\delta$  set, then

$$X = \bigcap_{i=1}^{\infty} G_i \quad \text{with each } G_i \text{ open.}$$

A set  $A$  is *nowhere dense* iff  $(\overline{A})^\circ = \emptyset$  iff for any interval  $I$ , there exists a subinterval  $S$  such that  $S \cap A = \emptyset$ . This is a set that is not dense in any nonempty open set. If the closure of a subset of  $\mathbb{R}$  contains no open intervals, it will be nowhere dense.

A set  $A$  is *meager* or *first category* if it can be written as

$$A = \bigcup_{i \in \mathbb{N}} A_i \quad \text{with each } A_i \text{ nowhere dense}$$

A set  $A$  is *null* if for any  $\varepsilon$ , there exists a cover of  $A$  by countably many intervals of total length less than  $\varepsilon$ , i.e. there exists  $\{I_k\}_{k \in \mathbb{N}}$  such that  $A \subseteq \bigcup_{k \in \mathbb{N}} I_k$  and  $\sum_{k \in \mathbb{N}} \mu(I_k) < \varepsilon$ . If  $A$  is null, we say  $\mu(A) = 0$ .

Some facts:

- If  $f_n \rightarrow f$  and each  $f_n$  is continuous, then  $D_f$  is meager.
- If  $f \in \mathcal{R}(a, b)$  and  $f$  is bounded, then  $D_f$  is null.
- If  $f$  is monotone, then  $D_f$  is countable.
- If  $f$  is monotone and differentiable on  $(a, b)$ , then  $D_f$  is null.

We define the *oscillation of  $f$*  as

$$\omega_f(x) := \lim_{\delta \rightarrow 0^+} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|$$

## 1.1 Uniform Convergence

We say that  $f_n \rightarrow f$  *converges uniformly on  $A$*  if  $\|f_n - f\|_\infty = \sup_{x \in A} |f_n(x) - f(x)| \rightarrow 0$ . (Note that this defines a sequence of *numbers* in  $\mathbb{R}$ .)

This means that one can find an  $n$  large enough that for every  $x \in A$ , we have  $|f_n(x) - f(x)| \leq \varepsilon$  for any  $\varepsilon$ .

- Showing uniform convergence: find some  $M_n$ , independent of  $x$ , such that  $|f_n(x) - f(x)| \leq M_n$  where  $M_n \rightarrow 0$ .
- Negating: Fix  $\varepsilon$ , let  $n$  be arbitrary, and find a bad  $x$  (which can depend on  $n$ ) such that  $|f_n(x) - f(x)| \geq \varepsilon$ .

Example:  $\frac{1}{1+nx} \rightarrow 0$  pointwise on  $(0, \infty)$ , which can be seen by fixing  $x$  and taking  $n \rightarrow \infty$ . To see the convergence is not uniform, choose  $x = \frac{1}{n}$  and  $\varepsilon = \frac{1}{2}$ . Then

$$\sup_{x>0} \left| \frac{1}{1+nx} - 0 \right| \geq \frac{1}{2} \not\rightarrow 0.$$

Here, the problem is at small scales – note that the convergence *is* uniform on  $[a, \infty)$  for any  $a > 0$ . To see this, note that

$$x > a \implies \frac{1}{x} < \frac{1}{a} \implies \left| \frac{1}{1+nx} \right| \leq \left| \frac{1}{nx} \right| \leq \frac{1}{na} \rightarrow 0$$

since  $a$  is fixed.

## 1.2 Uniformly Cauchy

Let  $C^0([a, b], \|\cdot\|_\infty)$  be the metric space of continuous functions of  $[a, b]$ , endowed with the metric

$$d(f, g) = \|f - g\|_\infty = \sup_{x \in [a, b]} |f(x) - g(x)|$$

This is a complete metric space, and

$$f_n \xrightarrow{U} f \iff \forall \varepsilon \exists N \ni m \geq n \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon \forall x \in X$$

$\implies$  : Use the triangle inequality.

$\impliedby$  : Use Bolzano-Weierstrass.