Complex Analysis

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Contents

1 Friday January 10

Recall that $\mathbb{C}$ is a field, where $z = x + iy \implies \overline{z} = x - iy$, and if $z \neq 0$ then $z^{-1} = \overline{z}/|z|^2$.

Lemma (Triangle Inequality: $|z + w| \leq |z| + |w|$

Proof:

$$(|z| + |w|)^2 - |z + w|^2 = 2(|z\overline{w}| - \Re z\overline{w}) \geq 0.$$

Lemma (Reverse Triangle Inequality): $||z| - |w|| \leq |z - w|$.

Proof:

$$|z| = |z - w + w| \leq |z - w| + |w| \implies |w| - |z| \leq |z - w| = |w - z|.$$

Claim: $(\mathbb{C}, |\cdot|)$ is a normed space.

Definition: $\lim z_n = z \iff |z_n - z| \longrightarrow 0 \in \mathbb{R}$.

Definition: A disc is defined as $D_r(z_0) := \left\{ z \in \mathbb{C} \;\middle|\; |z - z_0| < r \right\}$, and a subset is open iff it contains a disc. By convention, D_r denotes a disc about $z_0 = 0$.

Definition: $\displaystyle\sum_k z_k$ converges iff $S_N := \displaystyle\sum_{|k|<N} z_k$ converges.

Note that $z_n \longrightarrow z$ and $z_n = x_n + iy_n$, and

$$|z_n - z| = \sqrt{(x_n - x)^2 - (y_n - y)^2} < \varepsilon \implies |x - x_n|, |y - y_n| < \varepsilon.$$

Since $\mathbb{R}$ is complete iff every Cauchy sequence converges iff every bounded monotone sequence has a limit.

> Note: This is useful precisely when you don't know the limiting term.

Note that $\displaystyle\sum_k z_k$ thus converges if $\left|\displaystyle\sum_{k=m}^{n} z_k\right| < \varepsilon$ for m, n large enough, so sums converges iff they have small tails.

Definition: $S_N = \displaystyle\sum^{N} z_k$ converges absolutely iff $\tilde{S} := \displaystyle\sum^{N} |z_k|$ converges.

Note that the partial sums $\displaystyle\sum^{N} |z_k|$ are monotone, so $\tilde{S}_N$ converges iff the partial sums are bounded above.

Definition: A sum of the form $\sum_{k=0}^{\infty} a_k z_k$ is a power series.

Examples:

$$\sum x^k = \frac{1}{1-x}$$
$$\sum (-x^2)^k = \frac{1}{1+x^2}.$$

Note that both of these have a radius of convergence equal to 1, since the first has a pole at $x = 1$ and the second as a pole at $x = i$.

2 Monday January 13th

Recall that $\sum z_k$ converges iff $s_n = \sum_{k=1}^{n} z_k$ converges.

Lemma: Absolute convergence implies convergence.

The most interesting series: $f(z) = \sum a_k z^k$, i.e. power series.

Divergence lemma: If $\sum z_k$ converges, then $\lim z_k = 0$.

Corollary: If $\sum z_k$ converges, $\{z_k\}$ is uniformly bounded by a constant $C > 0$, i.e. $|z_k| < C$ for all k.

Proposition: If $\sum a_k z_k$ converges at some point z_0, then it converges for all $|z| < |z|_0$.

The inequality is necessarily strict. For example, $\sum \frac{z^{n-1}}{n}$ converges at $z = -1$ (alternating harmonic series) but not at $z = 1$ (harmonic series).

Proof: Suppose $\sum a_k z_1^k$ converges. The terms are uniformly bounded, so $\left| a_k z_1^k \right| \leq C$ for all k. Then we have

$$|a_k| \leq C/|z_1|^k$$

, so if $|z| < |z_1|$ we have

$$\left| a_k z^k \right| \leq |z|^k \frac{C}{|z_1|^k} = C(|z|/|z_1|)^k.$$

So if $|z| < |z_1|$, the parenthesized quantity is less than 1, and the original series is bounded by a geometric series. Letting $r = |z|/|z_1|$, we have

$$\sum \left| a_k z^k \right| \leq \sum c r^k = \frac{c}{1-r},$$

and so we have absolute convergence.

■

Exercise (future problem set): Show that $\sum \frac{1}{k} z^{k-1}$ converges for all $|z| = 1$ except for $z = 1$. (Use summation by parts.)

Definition The radius of convergence is the real number R such that $f(z) = \sum a_k z^k$ converges precisely for $|z| < R$ and diverges for $|z| > R$. We denote a disc of radius R centered at zero by D_R.

If $R = \infty$, then f is said to be *entire*.

Proposition: Suppose that $\sum a_k z^k$ converges for all $|z| < R$. Then $f(z) = \sum a_k z^k$ is continuous on D_R, i.e. using the sequential definition of continuity, $\lim_{z \longrightarrow z_0} f(z) = f(z_0)$ for all $z_0 \in D_R$.

Recall that $S_n(z) \longrightarrow S(z)$ uniformly on Ω iff $\forall \varepsilon > 0$, there exists a $M \in \mathbb{N}$ such that $n > M \implies |S_n(z) - S(z)| < \varepsilon$ for all $z \in \Omega$

Note that arbitrary limits of continuous functions may not be continuous. Counterexample: $f_n(x) = x^n$ on $[0, 1]$; then $f_n \longrightarrow \delta(1)$. Note that it uniformly converges on $[0, 1 - \varepsilon]$ for any $\varepsilon > 0$.

Exercise: Show that the uniform limit of continuous functions is continuous.

Hint: Use the triangle inequality.

Proof of proposition: Write $f(z) = \sum_{k=0}^{N} a_k z^k + \sum_{N+1}^{\infty} a_k z^k := S_N(z) + R_N(z)$. Note that if $|z| < R$, then there exists a T such that $|z| < T < R$ where $f(z)$ converges uniformly on D_T.

Check!

We need to show that $|R_N(z)|$ is uniformly small for $|z| < s < T$. Note that $\sum a_k z^k$ converges on D_T, so we can find a C such that $\left|a_k z^k\right| \leq C$ for all k. Then $|a_k| \leq C/T^k$ for all k, and so

$$\begin{aligned} \left|\sum_{k=N+1}^{\infty} a_k z^k\right| &\leq \sum_{k=N+1}^{\infty} |a_k||z|^k \\ &\leq \sum_{k=N+1}^{\infty} (c/T^k) s^k \\ &= c \sum |s/T|^k \\ &= c \frac{r^{N+!}}{1-r} \qquad = C\varepsilon_n \longrightarrow 0, \end{aligned}$$

which follows because $0 < r = s/T < 1$.

So $S_N(z) \longrightarrow f(z)$ uniformly on $|z| < s$ and $S_N(z)$ are all continuous, so $f(z)$ is continuous.

There are two ways to compute the radius of convergence:

- Root test: $\lim_k |a_k|^{1/k} = L \implies R = \frac{1}{L}$.
- Ratio test: $\lim_k |a_{k+1}/a_k| = L \implies R = \frac{1}{L}$.

As long as these series converge, we can compute derivatives and integrals term-by-term, and they have the same radius of convergence.

3 Wednesday January 15th

See references: Taylor's Complex Analysis, Stein, Barry Simon (5 volume set), Hormander (technically a PDEs book, but mostly analysis)

Good Paper: Hormander 1955

We'll mostly be working from Simon Vol. 2A, most problems from from Stein's Complex.

3.1 Topology and Algebra of $\mathbb{C}$

To do analysis, we'll need the following notions:

1. Continuity of a complex-valued function $f : \Omega \longrightarrow \Omega$
2. Complex-differentiability: For $\Omega \subset \mathbb{C}$ open and $z_0 \in \Omega$, there exists $\varepsilon > 0$ such that $D_\varepsilon = \left\{ z \;\middle|\; |z - z_0| < \varepsilon \right\} \subset \Omega$, and f is **holomorphic** (complex-differentiable) at z_0 iff

$$\lim_{h \longrightarrow 0} \frac{1}{h}(f(z_0 + h) - f(z_0))$$

exists; if so we denote it by $f'(z_0)$.

Example: $f(z) = z$ is holomorphic, since $f(z+h) - f(z) = z + h - z = h$, so $f'(z_0) = \dfrac{h}{h} = 1$ for all z_0.

Example: Given $f(z) = \overline{z}$, we have $f(z+h) - f(z) = \overline{h}$, so the ratio is $\dfrac{\overline{h}}{h}$ and the limit doesn't exist. Note that if $h \in \mathbb{R}$, then $\overline{h} = h$ and the ratio is identically 1, while if h is purely imaginary, then $\overline{h} = -h$ and the limit is identically -1.

We say f is holomorphic on an open set Ω iff it is holomorphic at every point, and is holomorphic on a closed set C iff there exists an open $\Omega \supset C$ such that f is holomorphic on Ω.

If f is holomorphic, writing $h = h_1 + ih_2$, then the following two limits exist and are equal:

$$\begin{aligned}
\lim_{h_1 \longrightarrow 0} \frac{f(x_0 + iy_0 + h_1) - f(x_0 + iy_0)}{h_1} &= \frac{\partial f}{\partial x}(x_0, y_0) \\
\lim_{h_2 \longrightarrow 0} \frac{f(x_0 + iy_0 + ih_2) - f(x_0 + iy_0)}{ih_2} &= \frac{1}{i}\frac{\partial f}{\partial y}(x_0, y_0) \\
\implies \frac{\partial f}{\partial x} &= \frac{1}{i}\frac{\partial f}{\partial y}.
\end{aligned}$$

So if we write $f(z) = u(x,y) + iv(x,y)$, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Big|_{(x_0,y_0)},$$

and equating real and imaginary parts yields the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$
$$\iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The usual rules of derivatives apply:

1. $(\sum f)' = \sum f'$

   Proof: Direct.

2. $(\prod f)' =$ product rule

   Proof: Consider $(f(z+h)g(z+h) - f(z)g(z))/h$ and use continuity of g at z.

3. Quotient rule

   Proof: Nice trick, write $q = \frac{f}{g}$ so $qg = f$, then $f' = q'g + qg'$and $q' = \frac{f'}{g} - \frac{fg'}{g^2}$.

4. Chain rule

   Proof: Use the fact that if $f'(g(z)) = a$, then

   $$f(z+h) - f(z) = ah + r(z,h), \quad |r(z,h)| = o(|h|) \longrightarrow 0.$$

   Write $b = g'(z)$, then

   $$f(g(z+h)) = f(g(z) + bh + r_1) = f(g(z)) + f'(g(z))bh + r_2$$

   by considering error terms, and so

   $$\frac{1}{h}(f(g(z+h)) - f(g(z))) \longrightarrow f'(g(z))g'(z)$$

   .

4 Friday January 17th

Reference: See Lang's Complex Analysis, there are plenty of solution manuals.

Let $f;\Omega \longrightarrow \mathbb{C}$ be a complex-valued function. Recall that f is *complex differentiable* iff the usual ratio/limit exists. Note that $h = x + iy$ and $h \longrightarrow 0 \iff x, y \longrightarrow 0$.

We can write $f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$. This follows from Cauchy-Riemann since $u_x = v_y$ and $u_y = -v_x$.

Definition: We want to define $\partial, \overline{\partial}$ operators. We have the identities

$$x = \frac{z + \overline{z}}{z} \quad y = \frac{z - \overline{z}}{iz}.$$

We can then write

$$dz = dx + idy$$
$$d\overline{z} = dx - idy.$$

We define the dual operators by $\left\langle \frac{\partial}{\partial z},\ dz \right\rangle = 1$ and similarly $\left\langle \frac{\partial}{\partial \overline{z}},\ d\overline{z} \right\rangle = 1$. By the chain rule, we can write

$$\begin{aligned} f_z &= f_x x_z + f_y y_z \\ &= \frac{1}{2} f_x + f_y \frac{1}{2i} \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f, \end{aligned}$$

and similarly $f_{\overline{z}} = f_x x_{\overline{z}} + f_y z_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\right) f$.

We thus find $\partial_x = \partial_z + \partial_{\overline{z}}$ and $\partial_y = i(\partial_z - \partial_{\overline{z}})$, and define

$$\begin{aligned} \partial f &= \frac{\partial f}{\partial z} dz \\ \overline{\partial} f &= \frac{\partial f}{\partial \overline{z}} d\overline{z} \\ df = f_z dz &+ f_{\overline{z}} d\overline{z}. \end{aligned}$$

Proposition: f is holomorphic iff $f_{\overline{z}} = 0$.

This means that f depends on z alone and not $\overline{z}$.

Proof: $\overline{\partial} f = 0$ iff $\frac{1}{2}(f_x + if_y) = 0$, so $(u_x - v_y) + i(v_x + u_y) = 0$. ■

Application to PDEs: We can write $u_{xx} = v_{xy}, u_{yy} = v_{yx}$ and so $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. Thus $\Delta f = 0$, and f satisfies Laplace's equation and is said to be *harmonic*.

Corollary: If f is analytic, then u, v are both harmonic functions.

Theorem (Chain Rule): Let $w = f(z)$ and $g(w) = g(f(z))$. Then

$$h_z = g_w f_z + g_{\overline{w}} \overline{f}_z$$
$$h_{\overline{z}} = g_w f_{\overline{z}} + g_{\overline{w}} \overline{f}_{\overline{z}}.$$

If f, g are holomorphic, $f_{\overline{z}} = g_{\overline{w}} = 0$, so $h_{\overline{z}} = 0$ and h is holomorphic and $h_z = g_w f_z$.

Example: Given a power series $f = \sum a_n (z - z_0)^n$. Then

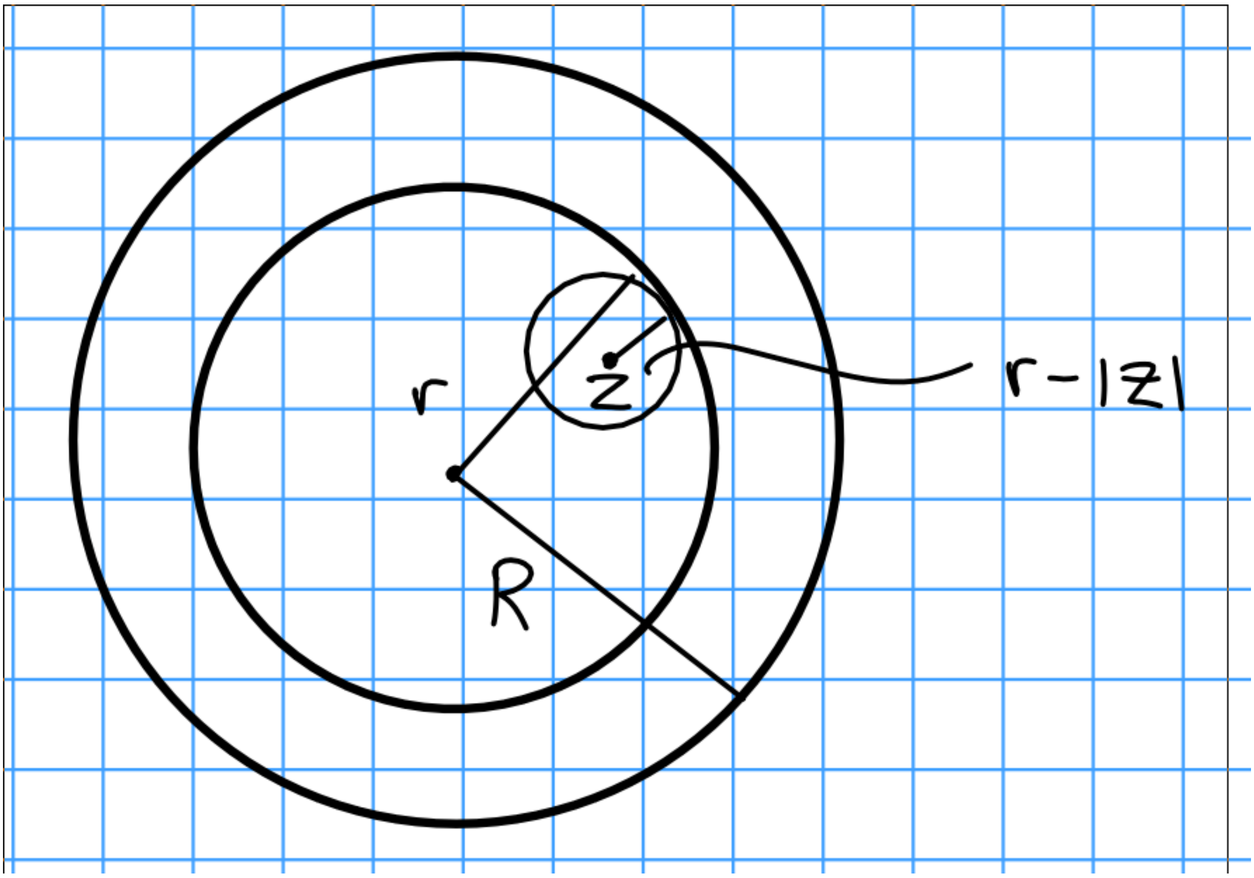


Figure 1: Image

1. There exists a radius of convergence R such that f converges precisely on $D_R(z_0)$.
2. f is continuous on $D_R(z_0)^\circ$.
3. By the root test, $R = (\limsup |a_n|^{1/n})^{-1} = \liminf |a_n/a_{n+1}| = (\limsup |a_{k+1}/a_k|)^{-1}$.

Recall the ratio test: $\sum a_k$ converges absolutely iff $\limsup |a_{k+1}/a_k| < 1$

Theorem: If $f(z) = \sum\limits_{n=0} a_n z^n$ is holomorphic on $|z| < R$ for $R > 0$ then $f'(z) = \sum\limits_{n=1} a_n n z^{n-1}$.

Exercise: Show $\lim\limits_n n^{\frac{1}{n}} = 1$. Also tricky: show $\lim \sin(n)$ doesn't exist, and $\sin(n)$ is dense in $[-1, 1]$.

Proof: Consider $\limsup |a_n n|^{\frac{1}{n}}$.

Remark: An analytic function is holomorphic in its domain of convergence, so analytic implies holomorphic. The converse requires Cauchy's integral formula.

Note: look for 13 equivalent statements, Springer GTM Lipman.

Proof: Given $|z| < R$, fix $r > 0$ such that $|z| < r < R$. Suppose that $|w - z| < r - |z|$, so $|w| < r$.

We want to show

$$|S| = \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1} a_n n z^{n-1} \right| \longrightarrow 0 \quad \text{as } w \longrightarrow z.$$

Idea: write everything in terms of power series. Use the fact that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots)$, and so $\left|(w^k - z^k)/(w-z)\right| \leq kr^{k-1}$.

$$\begin{aligned}
S &= \sum_{n=1} a_n \left(\frac{w^n - z^n}{w - z} - nz^{n-1} \right) \\
&= \sum a_n \Big(w^{n-1} + w^{n-2}z + \cdots + z^{n-1} + nz^{n-1} \Big) \\
&= \sum a_n \Big((w^{n-1} - z^{n-1}) + (w^{n-2} - z^{n-2})z + \cdots + (w-z)z^{n-2} \Big) \quad = \sum a_n (w-z) \Big(\cdots + z^{n-2} \Big) \\
&\leq \sum_{n=2} |a_n| \frac{1}{2} n(n-1) r^{n-2} |z - w|.
\end{aligned}$$

■

Next time: trying to prove holomorphic functions are analytic.

5 Wednesday January 22nd

Note: multiple complex variables, see Hormander or Steven Krantz

Recall from last time that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $z_0 \neq 0$ has radius of convergence $R = (\limsup |a_n|^{1/n})^{-1} > 0$, then f' exists and is obtained by differentiating term-by-term. We have f analytic implies f holomorphic (and smooth), we want to show the converse. For this, we need integration.

Definition: A parameterized curve is a function $z(t)$ which maps a closed interval $[a, b] \subset \mathbb{R}$ to $\mathbb{C}$.

Definition: The curve is said to be smooth iff z' exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for any t. At the boundary $\{a, b\}$, we define the derivative by taking one-sided limits.

Definition: A curve is said to be piecewise smooth iff $z(t)$ is continuous on $[a, b]$ and there are $a < a_1 < \cdots < a_n = b$ with z smooth on each $[a_k, a_{k+1}]$.

Note: may fail to have tangent lines at a_i.

Definition: Two parameterizations $z : [a, b] \longrightarrow \mathbb{C}, \tilde{z} : [c, d] \longrightarrow \mathbb{C}$ are equivalent iff there exists a C^1 bijection $s : [c, d] \longrightarrow [a, b]$ where $s \mapsto t(s)$ such that $s' > 0$ and $\tilde{z}(s) = z(s(t))$.

Note that $s' > 0$ preserves orientation and $s' < 0$ reverses orientation.

Definition:

$$\gamma : [a, b] \longrightarrow \mathbb{C} \implies \gamma^- := [a, b] to \mathbb{C}, \ t \mapsto \gamma(a + b - t).$$

Definition: A curve is closed iff $z(a) = z(b)$, and is simple iff $z(t) \neq z_{t_1}$ for $t \neq t_1$.

Definition: For $C_r(z_0) := \left\{ z \;\middle|\; |z - z_0| = r \right\}$, the positive orientation is given by $z(t) = z_0 + re^{2\pi it}$ for $t \in [0, 1]$.

Definition: The integral of f over γ is defined as

$$\int_\gamma f\ dz = \int_a^b f(z(t))z'(t)\ dt.$$

Note: This doesn't depend on parameterization, since if $t = t(s)$, then a change of variables yields

$$\int_\gamma f\ dz - \int_c^d f(z(t(s)))\ z'(t(s))\ t'(s)\ ds = \int_c^d f(\tilde{z}(s))\ \tilde{z}'(s)\ ds.$$

Definition: The length of γ is defined as $|\gamma| = \displaystyle\int |z'(t)|\ dt$.

Proposition:

1. We can extend this definition to piecewise smooth curves by
$$\int_\gamma f\ dz = \sum \int_{a_k}^{a_{k+1}} f\ dz$$
2. This integral is linear and $\displaystyle\int_\gamma f = -\int_{\gamma^-} f$.
3. We have an inequality
$$\left| \int_\gamma f \right| \leq \max_{a \leq t \leq b} |f(z(t))||\gamma|.$$

Definition: A function F is a primitive for f on Ω iff F is holomorphic on Ω and $F'(z) = f(z)$ on Ω.

Recall that in $\mathbb{R}$, we have $F(x) \displaystyle\int_a^x f(t)\ dt$ as an antiderivative with $F'(x) = f(x)$, and $\displaystyle\int f = F(b) - F(a)$.

Theorem: If f is continuous, has a primitive F in Ω, and γ is a curve beginning at w_0 and ending at w_1, then $\displaystyle\int_\gamma f = F(w_1) - F(w_0)$.

Proof: Use definitions, write $z(t)$ where $z(a) = w_1, z(b) = w_2$. Then

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(z(t))z'(t)\ dt \\
&= \int_a^b F'(z(t))z'(t)\ dt \\
&= \int_a^b F_t\ dt \\
&= F(z(b)) - F(z(a)) \quad \text{by FTC} \\
&= F(w_1) - F(w_2).
\end{aligned}$$

Note that if γ is piecewise smooth, the sum of the integrals telescopes to yield the same conclusion.

Corollary: If f is continuous and γ is a closed curve in Ω, and f has a primitive in Ω, then $\oint f = 0$.

6 Friday January 24th

Corollary: If γ is a closed curve on Ω an open set and f is continuous with a primitive in Ω (i.e. an F holomorphic in Ω with $F' = f$) then $\int_\gamma f\ dz = 0$.

Proof (easy):

$$\int_\gamma f\ dz = \int_\gamma F' = F'(z)z(t)\ dt = F(z(b)) - F(z(a)) = 0.$$

Corollary: If f is holomorphic with $f' = 0$ on Ω, then f is constant.

Proof (easy): Pick $w_0 \in \Omega$; we want to fix $w_0 \in \Omega$ and show $f(w) = f(w_0)$ for all $w \in \Omega$.

Take any path $\gamma : w_0 \longrightarrow w$, then

$$0 = \int_\gamma f' = f(w) - f(w_0).$$

Example: Let $f(z) = e^{-z^2}$, this is holomorphic. Write $f(z) = \sum(-1)^n z^{2n}/n!$, so $\int f = \sum(-1)^n z^{2n+1}/(n!(2n+1))$. Since f is entire, $\int f$ is entire, and $(\int f)' = f$ so this function has a primitive. Thus $\int_\gamma f(z) = 0$ for *any* closed curve. So take γ a rectangle with vertices $\pm a, \pm a + ib$.

So

$$\int_\gamma f = \int_{-a}^a e^{-x^2}\ dx + \int e^{-(a+iy)^2} i\ dy - \int_{-a}^a e^{-(x+ib)^2}\ dx - \int_0^b e^{-(a+iy)^2} i dy = 0.$$

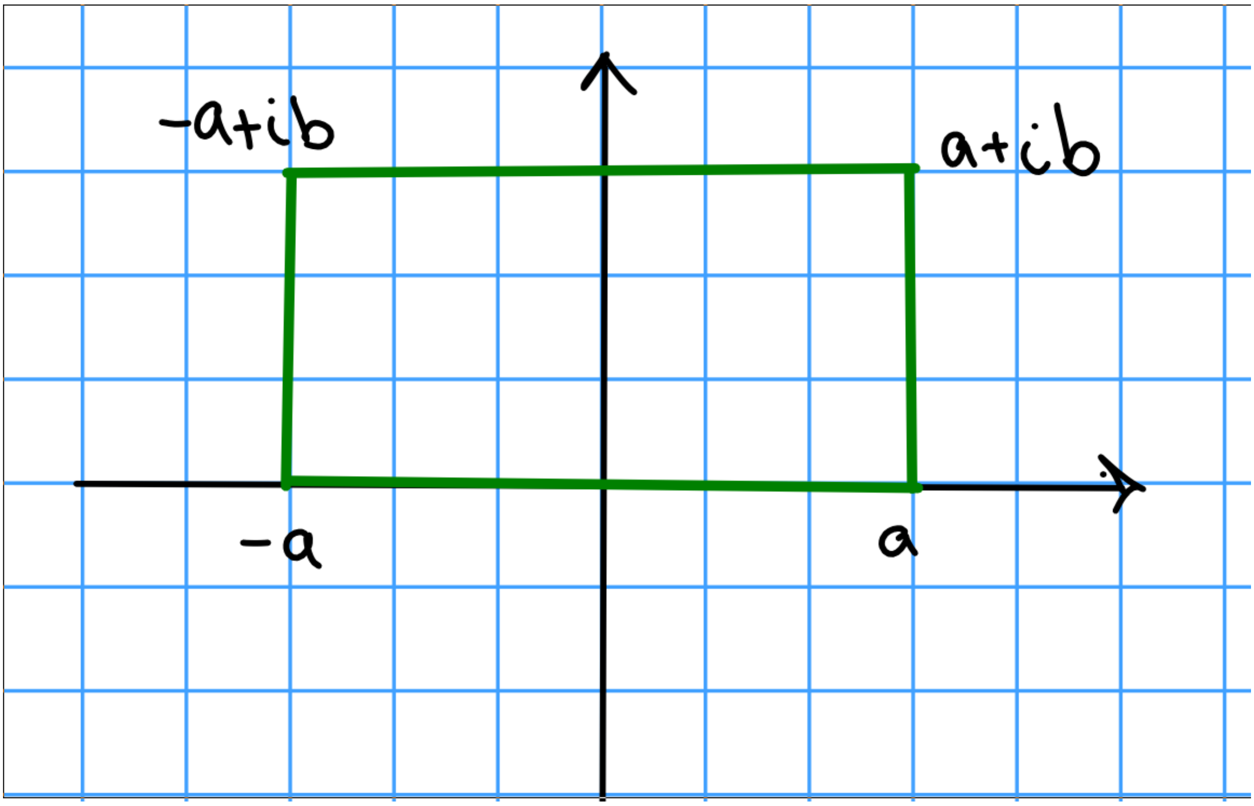


Figure 2: Image

We can do some estimates,

$$\begin{aligned} e^{-(a+iy)^2} &= e^{-(a^2+2iay-y^2)} \\ &= e^{-a^2+y^2} e^{2iay} \\ &\leq e^{-a^2+y^2} \\ &\leq e^{-a^2+b^2}, \end{aligned}$$

$$\left| \int_0^b e^{-(a+ib)^2} i \ dy \right| \leq e^{-a^2+b^2} \cdot b$$

$$\begin{aligned} \int_{-a}^{a} e^{-(x^2+2ibx)-b^2} &= e^{b^2} \int_{-a}^{a} e^{-x^2} (\cos(2bx) - i \sin(2bx)) \\ &\overset{\text{odd fn}}{=} e^{b^2} \int_{-a}^{a} e^{-x^2} \cos(2bx) \ dx. \end{aligned}$$

Now take $a \longrightarrow \infty$ to obtain

$$\int_{\mathbb{R}} e^{-x^2} \ dx = e^{b^2} \int_{\mathbb{R}} e^{-x^2} \cos(2bx) \ dx.$$

We can compute

$$\int_{\mathbb{R}} e^{-x^2} = \left[\left(\int_{\mathbb{R}} e^{-x^2} \right)^2 \right]^{1/2} = \left(\int_0^{2\pi} \int_0^{\infty} e^{r^2} r \ dr \ d\theta \right) = \sqrt{\pi}.$$

and then conclude

$$\int_{\mathbb{R}} e^{-x^2} \cos(2bx) = \sqrt{\pi} e^{-b^2}.$$

Make a change of variables $2b = 2\pi\xi$, so $b = \pi\xi$, then

$$\int_{\mathbb{R}} e^{-x^2} \cos(2\pi\xi x) \ dx = \sqrt{\pi} e^{-\pi^2\xi^2}.$$

Thus $\mathcal{F}(e^{-x^2}) = \sqrt{\pi} e^{-\pi^2\xi^2}$, allowing computation of the Fourier transform. Note that this can be used to prove the Fourier inversion formula.

Exercise: Show that this is an approximate identity and prove the Fourier inversion formula.

Exercise: Show $\mathcal{F}(e^{-ax^2}) = \sqrt{\pi/a}e^{-\pi^2/a\cdot\xi^2}$, and thus taking $a = \pi$ makes $e^{\pi x^2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue 1.

Theorem: If f has a primitive on Ω then $F(z)$ is holomorphic and $\int_\gamma f = 0$. If f is holomorphic, then $\int_\gamma f = 0$.

Theorem (Green's): Take $\Omega \in \mathbb{R}^2$ bounded with $\partial\Omega$ piecewise smooth. If $f, g \in C^1\overline{\Omega}$, then

$$\int_{\partial\Omega} f\ dx + g\ dy = \iint_\Omega (g_x - f_y)\ dA.$$

Proof: Not given here!

Proof of Theorem: Write $\gamma = \partial\Gamma$, and noting that $f_z = f_x = \frac{1}{i} f_y$ implies that $\frac{\partial f}{\partial \overline{z}}$, so

$$\begin{aligned}
\int_\gamma f\ dz &= \int_\gamma f(z)\ (dx + idy) \\
&= \int f(z)\ dx + if(z)\ dy \\
&= \iint_\Gamma (if_x - f_y)\ dA \\
&= i \iint_\Gamma \left(f_x - \frac{1}{i} f_y \right)\ dA \\
&= i \iint 0\ dA = 0.
\end{aligned}$$

Next class: We'll prove that this integral over any triangle is zero by a limiting process.

7 Monday January 27th

Fix a connected domain Ω which is bounded with a piecewise C^1 boundary.

Theorem (Green's): Given $f, g \in C^1\overline{\Omega}$, we can take a vector field $F = \langle f, g \rangle$ and have

$$\begin{aligned}
\int_{\partial\Omega} f\ dx + g\ dy &= \iint_\Omega \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\ dA \\
\int_{\partial\Omega} -f\ dx + g\ dy &= \iint_\Omega \left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \right)\ dA \\
\int_{\partial\Omega} f\ dy - g\ dy &= \iint_\Omega \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right)\ dA \\
\int_{\partial\Omega} F \cdot \mathbf{n}\ ds &= \iint_\Omega \nabla \cdot F\ dA \\
\int_{\partial\Omega} \mathrm{curl}(F)\ ds &= \iint_\Omega \mathrm{div}(F)\ dA,
\end{aligned}$$

where we take $\mathbf{n}$ to be orthogonal to $\partial\Omega$. The quantities appearing on the RHS are referred to as the flux.

For $f(z) \in C^1(\Omega)$ holomorphic, we can then write

$$\begin{aligned}\int_{\partial\Omega} f\ dz &= \int_{\partial\Omega} f\ (dx + idy)\\ &= \int_{\partial\Omega} f\ dx + if\ dy\\ &= \iint_{\Omega} (if_x - f_y)\ dA\\ &= 0,\end{aligned}$$

which follows since f holomorphic, we can write $f'(z) = f_x = \frac{1}{i} f_y$, so $if_x = f_y$ and thus $\frac{\partial f}{\partial \overline{z}} = 0$.

See Taylor's Introduction to Complex Analysis

Theorem (Cauchy's Integral Formula): If $f \in C^1(\overline{\Omega})$ and f is holomorphic, then for any $z \in \Omega$

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{d(\xi)}{\xi - z}\ d\xi.$$

Proof: Since $z \in \Omega$ an open set, we can find some $r > 0$ such that $D_r(z) \subset \Omega$. Then $\frac{f(\xi)}{\xi - z}$ is holomorphic on $\Omega \setminus D_r(z)$. Let $C_r = \partial D_r(z)$.

Claim 1: $\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\ d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z}\ d\xi$.

Proof: Use the parameterization of C_r given by $\xi = z + re^{i\theta}$. Then

$$\begin{aligned}\frac{1}{2\pi i} \int_{C_r} \frac{f(\xi)}{\xi - z}\ d\xi &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}}\ ird\theta\\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\ d\theta\\ &\xrightarrow{r \to 0} \frac{1}{2\pi} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z}.\end{aligned}$$

where we use the fact that $f(z + re^{i\theta}) = f(z) + f'(z)re^{i\theta} + o(r) \longrightarrow f(z)$.

Letting $F(\xi) = f(\xi)/(\xi - z)$, this is holomorphic on $\Omega \setminus D_r(z)$. Let $\Omega_r = \partial\Omega \bigcup (-C_r)$. Take the following path integral:

Then

$$0 = \int_{\partial\Omega_r} F(\xi)\ d\xi = \int_{\partial\Omega} F(\xi)\ d\xi - \int_{C_r} F(\xi)\ d\xi,$$

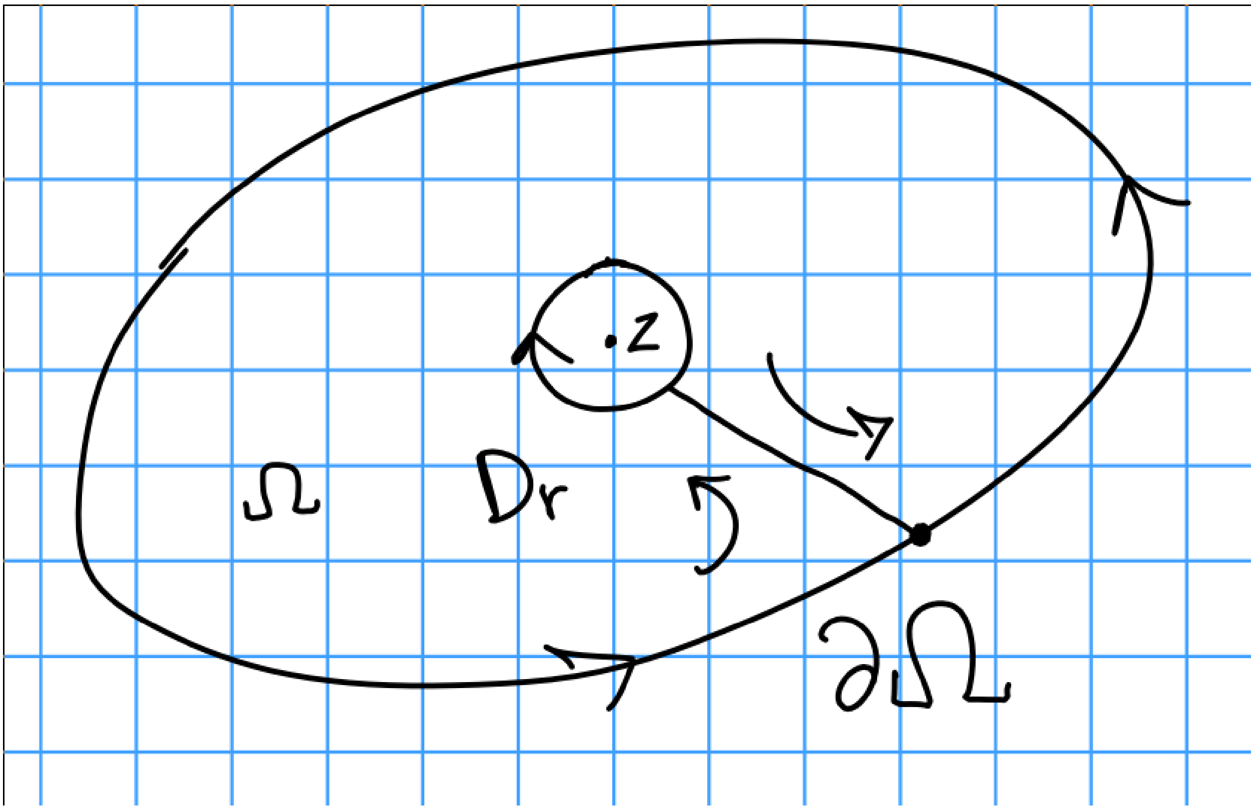


Figure 3: Image

which forces these integrals to be equal.

■

If we can differentiate through the integral, we can obtain

$$\frac{\partial}{\partial z} f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{(\xi - z)^2}\ d\xi.$$

and thus inductively

$$(D_z)^n f(z) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)\ d\xi}{(\xi - z)^{n+1}}.$$

To prove rigorously, need to write

$$\Delta_h f(z) = \frac{1}{h}(f(z+h) - f(z))$$
$$= \frac{1}{2\pi i h} \int_{\partial\Omega} f(\xi)\left(\frac{1}{\xi - (z+h)} - \frac{1}{\xi - z}\right)\ d\xi = \frac{1}{2\pi i h} \int_{\partial\Omega} f(\xi)\left(\frac{1}{(\xi - z - h)(\xi - z)}\right)\ d\xi,$$

and show the integrand converges uniformly, where

$$\frac{1}{(\xi - z - h)(\xi - z)} \xrightarrow{u} \frac{1}{(\xi - z)^2}.$$

Continuing inductively yields the integral formula.

Corollary: If f is holomorphic, then $f \in C^1(\Omega)$ implies that $f \in C^\infty(\Omega)$.

Theorem: If f is holomorphic in Ω, then f is equal to its Taylor series (i.e. $f(z_0$ is analytic.)

Fix $z_0 \in \Omega$ and let $r = |z - z_0|$.

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{\xi - z_0 - (z - z_0)} \\ &= \frac{1}{\xi - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\xi - z_0}\right)} \\ &= \frac{1}{\xi - z_0} \sum_n \left(\frac{z - z_0}{\xi - z_0}\right)^n \quad \text{for } |z - z_0| < |\xi - z_0|. \end{aligned}$$

Note that $\sum z^n$ converges uniformly for any $|z| < \delta < 1$.

Thus

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\xi\in\partial\Omega} f(\xi)\sum \frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}\,d\xi \\
&= \sum\left(\frac{1}{2\pi i}\int \frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi\right)(z-z_0)^n \\
&= \sum \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.
\end{aligned}$$

■

Thus f is holomorphic iff f is analytic.

Counterexample to keep in mind:

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

In the case of $\mathbb{R}$, smooth and analytic are very different categories of functions.

Open question: does a PDE involving analytic functions always have solutions? Or does this hold for smooth functions instead?

8 Wednesday January 29th

8.1 Cauchy's Integral Formula

Theorem 8.1 ***(Cauchy's Integral Formula).***
Let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic, so $f \in C^1(\overline{\Omega})$. Then for any $z \in \Omega$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\,d\xi.$$

In general,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{(\xi-z)^{n+1}}\,d\xi.$$

This implies that f is analytic, i.e.

$$f(z) = \sum a_n(z-z_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Thus f is holomorphic iff f is analytic,

and

$$\int_{\partial\Omega} f = 0 \implies \int_{\partial\Omega_\gamma} \frac{f(\xi)}{\xi - z}\ d\xi = 0.$$

where $\Omega_r = \Omega \setminus D_r(z)$, and $\partial\Omega_r = \partial\Omega \bigcup(-\partial D_r)$.

We can thus shrink integrals:

$$\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\ d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z}\ d\xi.$$

Proposition 8.2 (Homotopy Invariance)**.**
Let $f \in C^1(\Omega)$ be holomorphic on Ω. Let $\gamma_s(t)$ be a family of smooth curves in Ω; then $\displaystyle\int_{\gamma_s} f$ is independent of s.

Proof.
Write

$$\gamma_s(t) = \gamma(s,t) : [a,b] \times [0,1] \longrightarrow \Omega.$$

We have $\gamma_s(0) = \gamma_s(1)$ so $\dfrac{\partial\gamma}{\partial s}(s,0) = \dfrac{\partial\gamma}{\partial s}(s,1)$. Then

$$\begin{aligned}
\frac{\partial\gamma}{\partial s} &= \int_0^1 \left(f'(r(s,t))\frac{\partial r}{\partial s}\frac{\partial r}{\partial t} + f(r(s,t))\frac{\partial^2\gamma}{\partial s\partial t}\right)\ dt \\
&= \int_0^1 \left(f'(r(s,t))\frac{\partial r}{\partial s}\frac{\partial r}{\partial t} + f(r(s,t))\frac{\partial^2\gamma}{\partial \mathbf{t}\partial \mathbf{s}}\right)\ dt \\
&= \int_0^1 \frac{\partial}{\partial t}(f(\gamma(s,t))\gamma_s) \\
&= f(\gamma(s,1))\gamma_s(s,1) - f(\gamma(s,0))\gamma_s(s,0) \\
&= 0.
\end{aligned}$$

where we can just take the paths $\gamma(s,t) = z_0 \in \Omega$ for all s,t.

■

Proposition 8.3 (Pointwise Limit of Locally Uniform is Locally Uniform)**.**
Let $\Omega \subset \mathbb{C}$ be open and $f_v : \Omega \longrightarrow \mathbb{C}$. Suppose that each f_v is holomorphic, $f_v \longrightarrow f$ pointwise, and *locally uniform*, i.e. $f_v \longrightarrow f$ uniformly on every compact $K \subset \Omega$. Then f is holomorphic in Ω and f is locally uniform.

Proof.
Given a compact set $K \subset \Omega$, pick an O with smooth boundary such that $K \subset O \subset \overline{O} \subset \Omega$. We have

$$\begin{aligned} f_v(z) &= \frac{1}{2\pi i} \int_{\partial O} \frac{f_v(\xi)}{\xi - z} \, d\xi \\ f_v^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\partial O} \frac{f_v(\xi)}{(\xi - z)^{n+1}} \, d\xi \end{aligned}$$

.

Then on ∂O, we have uniform convergence

$$\frac{f_v(\xi)}{(\xi - z)^{n+1}} \xrightarrow{u} \frac{f(\xi)}{(\xi - z)^{n+1}}.$$

By moving the limits inside, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial O} \frac{f(\xi)}{\xi - z} \, d\xi \\ f^{(n)}(z) &= \frac{n!}{2\pi i} \int_{\partial O} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi \end{aligned}$$

.

■

Theorem 8.4 ***(Cauchy's Inequality).***
Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C_R = \partial D_R$. Using the integral formula, defining $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R \, d\theta = \frac{n! \|f\|_{C_R}}{R^n}.$$

Corollary 8.5 (Liouville's Theorem)**.**
If f is entire and bounded, then f is constant.

Proof.
For all $z_0 \in \mathbb{C}$, there exists an M such that $|f(z)| \leq M$. Then $|f'(z_0)| \leq \dfrac{M}{R}$ for any $R > 0$. Taking $R \longrightarrow \infty$ yields $f'(z_0) = 0$, so f is constant.

■

Corollary 8.6 (Weak Fundamental Theorem of Algebra)**.**
Every non-constant polynomial $p(z) = a_0 + a_1 z + \cdots a_n z^n$ has a root in $\mathbb{C}$.

Remark: A general proof technique is when proving something for $f(z)$, consider $\dfrac{1}{f(z)}$ and $f(\dfrac{1}{z})$.

Proof.
Suppose p is nonconstant and does not have a root, $\dfrac{1}{p}$ is entire. Assume that $a_n \neq 0$, then

$$\frac{p(z)}{z^n} = a_n \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) := a_n + y$$

We can note that $\lim\limits_{z \longrightarrow \infty} \dfrac{a_{n-k}}{z^k} \longrightarrow 0$, so there exists an $R > 0$ such that

$$\left| \frac{p(z)}{z^n} \right| \geq \frac{1}{2}|a_n| \quad \text{for } |z| > R$$
$$\implies |p(z)| \geq \frac{1}{2}|a_n||z|^n \geq \frac{1}{2}|a_n|R^n.$$

Since $p(z)$ is continuous and has no root in the disc $|z| \leq R$, $|p(z)|$ is bounded from below in this disc. Since $p(z)$ is continuous on a compact set, it attains a minimum, and so $|p(z)| \geq \min\limits_{|z| \leq R} |p(z)| = c_2 \neq 0$. Then $|p(z)| \geq A = \min(C_2, \dfrac{1}{2}|a_n|R^n)$, so $\dfrac{1}{p}$ is bounded. Then f is constant, a contradiction.

■

9 Friday January 31st

9.1 Fundamental Theorem of Algebra

Recall that if f is holomorphic, we have Cauchy's integral formula.

Corollary 9.1 (Weak Fundamental Theorem of Algebra)**.**
If $P(z)$ is a polynomial in $\mathbb{C}$ then P has a root in $\mathbb{C}$.

Proof.
See previous notes.

■

Corollary 9.2 (Fundamental Theorem of Algebra)**.**
Every polynomial of degree n has precisely n roots in $\mathbb{C}$.

Proof.
By induction on the degree of P. From the first corollary, P has a root w_1, so write $z = z - w_1 + w_1$. Then

$$\begin{aligned}
p(z) &= p(z - w_1 + w_1) \\
&= \sum_k^n a_k (z - w_1 + w_1)^k \\
&= \sum_k^n a_k \sum_j^k \binom{k}{j} w_1 k - j (z - w_1)^j \\
&= \sum_k^n \sum_j^k a_k \binom{k}{j} w_1^{k-j} (z - w_1)^j \\
&= \sum_j^n \left(\sum_{k \geq j} a_k \binom{k}{j} \right) (z - w_1)^j \\
&= b_0 + b_1(z - w_1) + \cdots + b_n (z - w_1)^n.
\end{aligned}$$

Since $P(w_1) = 0$, we must have $b_0 = 0$, and thus this equals

$$\begin{aligned}
b_1(z - w_1) + \cdots + b_n(z - w_1)^n &= (z - w_1)\Big(b_1 + \cdots + b_n(z - w_1)^{n-1}\Big) \\
&\coloneqq (z - w_1)\phi(z),
\end{aligned}$$

where $\phi(z)$ is degree $n - 1$, which has $n - 1$ roots by induction.

■

Definition 9.1.
For a sequence $\{z_n\}$, TFAE
1. z is a limit point.
2. There exists a subsequence $\{z_{n_k}\}$ converging to z.
3. For every $\varepsilon > 0$, there are infinitely many z_i in $D_\varepsilon(z)$.

Theorem 9.3.
Suppose f is holomorphic on a bounded connected region Ω and f vanishes on a sequence of distinct points with a limit point in Ω.

Proof.
WLOG by restricting to a subsequence, suppose that $\{w_k\} \in \Omega$ with $f(w_i) = 0$ for all i and z_0 is a limit point of $\{w_i\}$. Let $U = \left\{ z \in \Omega \;\middle|\; f(z) = 0 \right\}$. Then
1. U is nonempty since $f(w_k) = f(z_0) = 0$.
2. Since holomorphic functions are continuous, if $w_k \longrightarrow z$ then $z \in U$, so U is closed.
3. (To prove) U is open.

Since U is closed and open, $U = \Omega$.
We will first show that $f(z) \equiv 0$ in a disk containing z_0. Choose a disc D containing z_0 and

contained in Ω. Since f is holomorphic on D, we can write

$$f(z) = \sum a_n n(z - z_0)^n.$$

Since $f(z_0) = 0$, we have $a_0 = 0$.
Suppose $f \not\equiv 0$. Then there exists a smallest $n \in \mathbb{Z}^+$ such that $a_n \neq 0$, so $f(z) = a_n(z-z_0)^n + \cdots$.
Since $a_n \neq 0$, we can factor this as $a_n(z - z_0)^n(1 + g(z - z_0))$ where

$$g(z - z_0) = \sum_{k=n+1}^{\infty} \frac{a_k}{a_n}(z - z_0)^{k-n}.$$

Note that g is holomorphic, and $g(z_0 - z_0) = 0$.
Choose some w_k such that $f(w_k) = 0$ and $|g(w_k - z_0)| \leq \frac{1}{2}$ by continuity of g. Then

$$|1 + g(w_k - z_0)| > 1 - \frac{1}{2} = \frac{1}{2}.$$

So

$$|f(w_k)| = |a_n(w_k - z_0)^n(1 + g(w_k - z_0))| > |a_n||w_k - z_0|^n \frac{1}{2} > 0,$$

a contradiction. So U is open, closed, and nonempty, so $U = \Omega$.

■

Corollary 9.4.
Suppose f, g are holomorphic in a region Ω with $f(z_k) = g(z_k)$ where $\{z_k\}$ has a limit point. Then $f(z) \equiv g(z)$.

Theorem 9.5 ***(Mean Value).***
Let z_0 be a point in Ω and C_γ the boundary of $D_r(z_0)$. Then

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i} \int_{C_\gamma} f(z)/(z - z_0)dz \\
&= \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{i\theta})/re^{i\theta} rie^{i\theta} \, d\theta \quad \text{by } z = z_0 + re^{i\theta} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \\
&= \frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, r d\theta \\
&= \frac{1}{|C_\gamma|} \int_0^{2\pi} f(z) \, ds,
\end{aligned}$$

which is the average value of f on the circle.

Note that there is another formula that averages over the disc (see book for derivation?)

$$f(z_0) = \frac{1}{D_s(z_0)} \int_{P_s} \int_{D_s} f(z)\ dA.$$

These imply the maximum modulus principle, since the average can not be the max or min unless f is constant. Note that $|f(z)|$ is continuous!

Next time: maximum modulus principle.

10 Monday February 3rd

10.1 Mean Value Theorem

Theorem 10.1 ***(Mean Value for Holomorphic functions).***

$$f(z_0) = \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z)\ dA$$

Proof of MVT?.
Let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic where Ω is open and connected. Then by Cauchy's integral formula, we have $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\ d\theta$ for any $z_0 \in \Omega$.
We can consider $D_r(z_0)$, in which case we have for all $0 < s < r$, ■

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + se^{i\theta})\ d\theta \\
\implies s \cdot f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} s \cdot f(z_0 + se^{i\theta})\ d\theta \\
\implies \cdot f(z_0) \int_0^r s\ ds &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^r f(z_0 + se^{i\theta}) \cdot s\ ds\ d\theta \\
\implies \frac{1}{2} r^2 f(z_0) &= \frac{1}{2\pi} \iint_{D_r(z_0)} f(z)\ dA \\
\implies f(z_0) &= \frac{1}{\pi r^2} \iint_{D_r(z_0)} f(z)\ dA \\
\implies f(z_0) &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\ d\theta.
\end{aligned}$$

Proposition 10.2 (Maximum in Interior Implies Constant)**.**
Let f be holomorphic on Ω be open and connected, and suppose that there is a $z_0 \in \Omega$ such that

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)|,$$

i.e. z_0 is a maximal point of f. Then f is constant on Ω.

If Ω is additionally **bounded**, then f is continuous on $\overline{\Omega}$, then

$$\sup_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \overline{\Omega}} |f(z)|.$$

Proof.
Since $|f|$ is continuous and $\overline{\Omega}$ is compact, $|f|$ attains a maximum at some point in $\overline{\Omega}$. We want to show that if $|f(z_0)| = \sup_{z \in \Omega} |f(z)|$, then f is constant.
Assume that there exists a $z_0 \in \Omega$ such that $f(z) = f(z_0)$. Let $O = \left\{ \xi \in \Omega \;\middle|\; f(\xi) = f(z_0) \right\}$.

Claim. 1. O is not empty, since $z_0 \in O$.
2. O is closed, since if $\xi_n \longrightarrow \xi$ then $f(\xi_n) = f(z_0)$ implies $f(\xi) = f(z_0)$ since f is continuous.
3. (**Claim**) O is open.

Suppose $\xi_0 \in O$, then there exists a disc $D_\rho(\xi_0) \subset \Omega$ such that

$$f(\xi_0) = \frac{1}{\pi \rho^2} \int_{D_\rho(\xi_0)} f(z) dA.$$

Then (claim) $|f(\xi_0)| \geq |f(z)|$ for all $z \in D_\rho(\xi_0)$, which forces $f(z) = f(\xi_0)$ for all $z \in D_\rho(\xi_0)$. ∎

Proof of the claim):.
Suppose that $\sup_{a \in \Omega} |f(z)| = |f(\xi_0)|$ and write $f(\xi_0) = Be^{i\alpha}$ for $B > 0$ and $\alpha \in \mathbb{R}$. Then define $g(z) = f(z)e^{-i\alpha}$; then $g(\xi_0) = B$ is real, and thus

$$0 = g(\xi_0) - B = \frac{1}{\pi \rho^2} \iint_{D_\rho(\xi_0)} \Re(g(z) - B)\ dA.$$

Note that $\Re(g(z) - B) \leq 0$ implies that $\Re(g(z) - B) \equiv 0$ on $D_\rho(z_0)$, so we can write $g(z) = B + iI(z)$ for some real-valued function I.
But then $|g(z)|^2 = B^2 + I(z)^2 = B^2$ by the previous statement, and so $I(z) = 0$, forcing $g(z) = B$ and thus $f(z) = Be^{i\alpha}$. This shows that O is open, and thus $O = \Omega$.

∎

Proposition 10.3 (Stein 2.1, Biholomorphisms of the Open Disc are Contractions)**.**
Suppose f is holomorphic on $D_1(0)$ and $|f(z)| \leq 1$ for all $|z| < 1$ with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $|z| < 1$.

Moreover, there is a point $z_0 \in D_1(0)$ such that $|f(z_0)| = |z_0|$ iff $f(z) = c(z)$ for some $c \in S^1$.

Proof.
Define

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}.$$

Then g is holomorphic on $D_1(0)$ and $|g(z)| \leq \dfrac{1}{\rho}$ for all $|z| < \rho < 1$. Now apply the maximum principle: since this is true for all $\rho < 1$, consider the limit $\rho \longrightarrow 1^-$.
Then $|g(z)| \leq 1$, so $\left|\dfrac{f(z)}{z}\right| \leq 1$ and $|f(z)| \leq |z|$. If $|f(z_0)| = |z_0|$ for any point, then $|g(z_0)| = 1$ implies $g(z_0) = c$ and $c \in S^1$.
Thus $f(z) = cz$ for some $c \in S^1$.

∎

Corollary 10.4 (Characterization of Biholomorphisms of the Disc)**.**
Recall that

$$\Phi_a(z) := \frac{z-a}{1-az}.$$

If $f : D_1(0) \longrightarrow D_1(0)$ is a biholomorphism, then

$$f(z) = c\Phi_a(z) = e^{i\theta}\Phi_a(z)$$

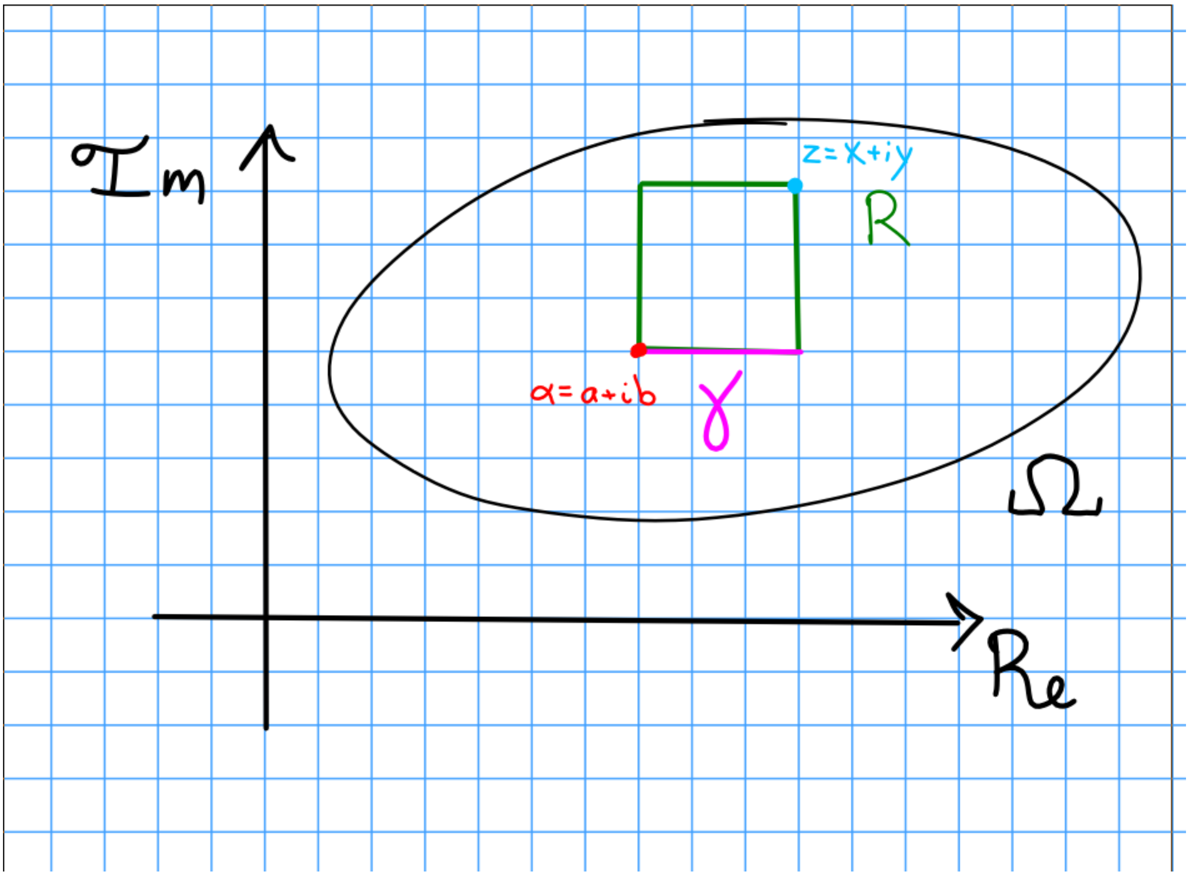
So every such function is a rotated form of Φ_a.

Let Ω be a connected open domain and $f : \Omega \longrightarrow \mathbb{C}$ holomorphic with $f \in C^1$. Then

$$\int_\gamma f(z)\ dz = 0$$

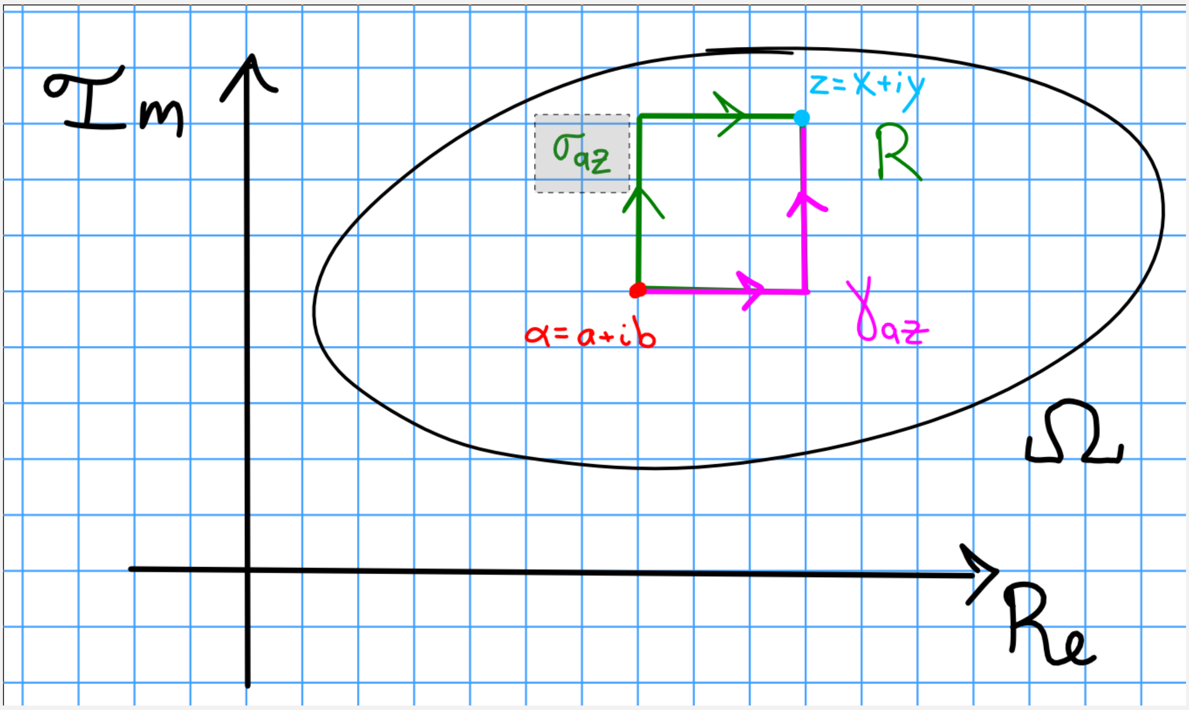
for every closed curve $\gamma \subset \Omega$, which implies that $f^{(k)}(z)$ exists for all $k \in \mathbb{N}$ and f is smooth/holomorphic.

Theorem 10.5 ***(Morera, Partial Converse to Cauchy's Integral Theorem):).***
Suppose $g : \Omega \longrightarrow \mathbb{C}$ is continuous and $\displaystyle\int_\gamma g(z)\ dz = 0$ whenever $\gamma = \partial R$ for some rectangle $R \subset \Omega$ with sides parallel to the axes:



Then $g(z)$ is holomorphic in Ω.

Proof.
Fix a point $\alpha = a + ib$ and given $z = x + iy$, construct a rectangle R containing z. Then by assumption, $\int_{\partial R} g(z)\, dz = 0$. Let γ_{az} be the path given by traversing the bottom edge of R, and σ_{az} by the top path.



Let

$$
\begin{aligned}
f(z) &= \int_{\gamma_{az}} g(z)\ dz \\
&= \int_a^x g(s+ib)\ ds + i\int_b^y g(x+it)\ dt.
\end{aligned}
$$

Since

$$\int_{\partial R} g(z)\ dz = 0 = \int_{\gamma_{az}} \cdots - \int_{\sigma_{az}} \cdots ,$$

we have

$$
\begin{aligned}
f(z) &= \int_{\sigma_{az}} g(z)\ dz \\
&= i\int_b^y g(a+it)\ dt + \int_x^a g(s+iy)\ ds.
\end{aligned}
$$

Exercise: Apply $\dfrac{\partial}{\partial y}$ to the first identity and $\dfrac{\partial}{\partial x}$ to the second.

This yields

$$\frac{\partial f}{\partial x} = g(z) \quad \text{and} \quad \frac{\partial f}{\partial y} = ig(z) = i\frac{\partial f}{\partial x}$$

by applying the FTC, which are precisely the Cauchy-Riemann equations for f. So f is holomorphic, and thus $f(z) = g(z)$.

■

11 Wednesday February 5th

11.1 Cauchy/Morera Theorems

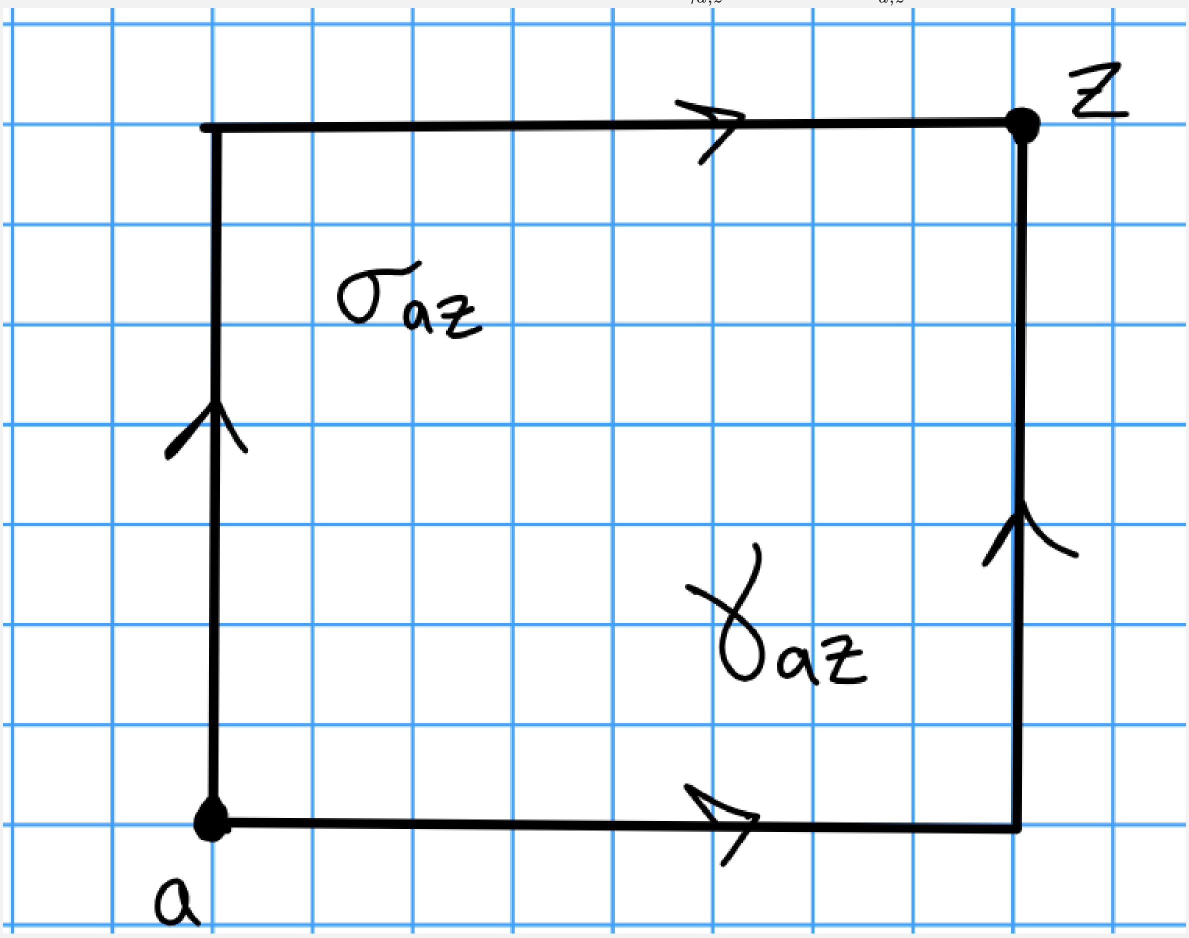
Recall last time: We have Cauchy's theorem, which says that if $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic then

$$\int_\gamma f \; dz = 0.$$

We have a partial converse:

Theorem 11.1 ***(Morera).***
If $g : \Omega \longrightarrow \mathbb{C}$ is continuous and $\int_R g \; dz = 0$ for every rectangle $R \subset \Omega$ with sides parallel to the axes, then g is holomorphic.

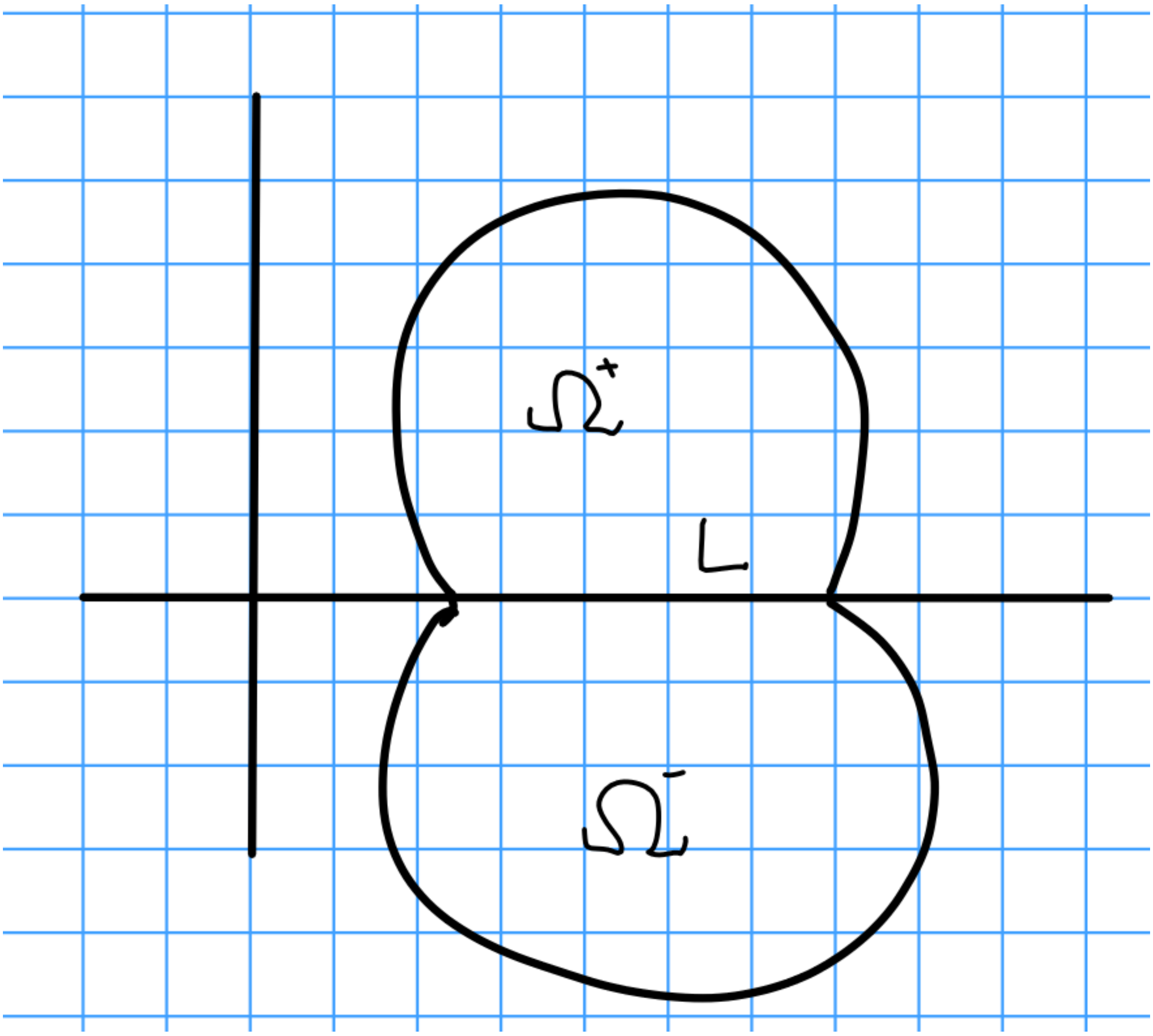
Proof Morera.
Fix a point $a \in \Omega$, then for any $z \in \Omega$ define $f(z) = \int_{\gamma_{a,z}} g(\xi) d\xi = \int_{\sigma_{a,z}} g(\xi) d\xi$.



Then $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = g(z)$, making g holomorphic. ∎

11.2 Schwarz Reflection

Theorem 11.2 ***(Schwarz Reflection, Extending Holomorphic Functions Across Reflected Regions).***
Let $\Omega = \Omega^{+} \bigcup L \bigcup \Omega^{-}$ be a region of the following form:

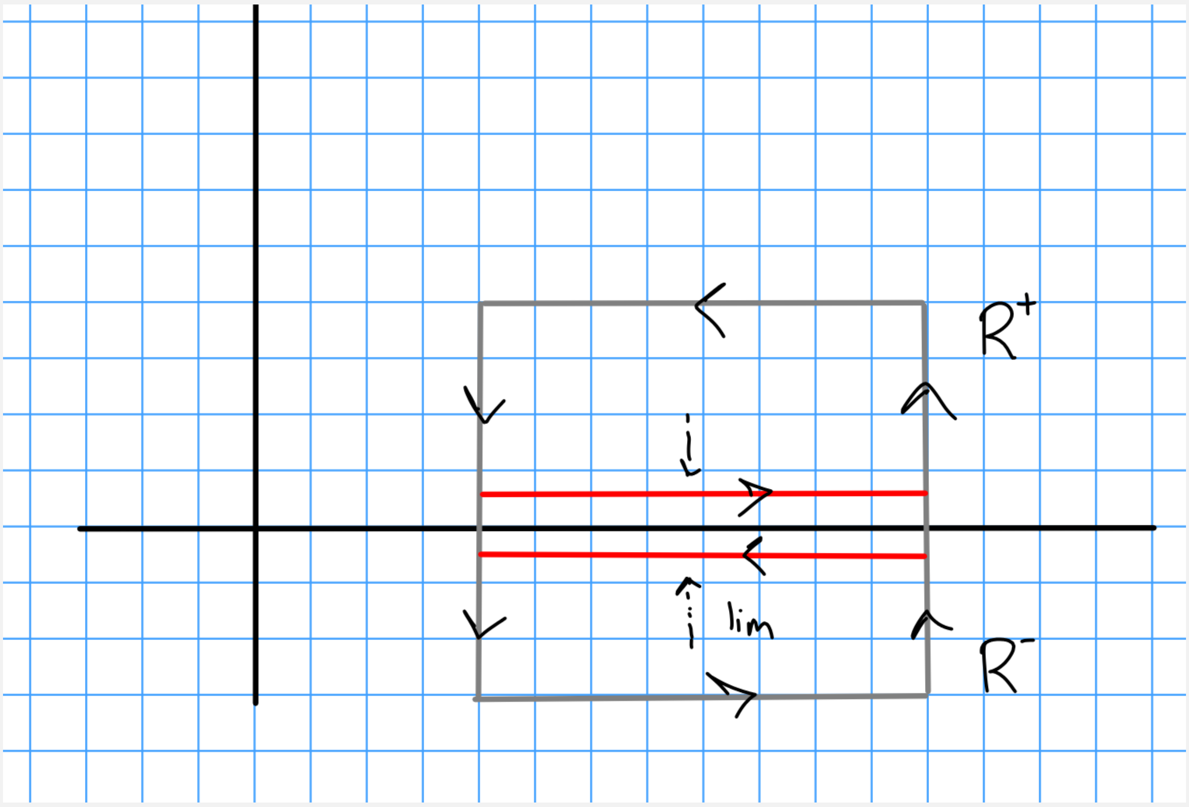


I.e., $L = \left\{ z \in \Omega \;\middle|\; \text{im } z = 0 \right\}$, $\Omega^{\pm} = \{\pm \text{im } z > 0\}$ where Ω is symmetric about the real axis, i.e. $z \in \Omega \implies \overline{z} \in \Omega$.
Assume that $f : \Omega^{+} \bigcup L \longrightarrow \mathbb{C}$ is continuous and holomorphic in Ω^{+} and real-valued on L. Define

$$g(z) = \begin{cases} f(z) & z \in \Omega^{+} \bigcup L \\ \overline{f(\overline{z})} & z \in \Omega^{-} \end{cases}.$$

Then $g(z)$ is defined and holomorphic on Ω.

Proof Schwarz Reflection.
Since g is C^1 in Ω^-, check that g satisfies the Cauchy-Riemann equations on Ω^- and thus holomorphic there. To see that g is holomorphic on all of Ω, we'll show the integral over every rectangle is zero.
It's clear that if $R \subset \Omega^{\pm}$, $\int_R g = 0$ since g is holomorphic there, so it suffices to check rectangles intersecting the real axis. Write $R = R^+ \bigcup R^-$:



We then have $R^+ = \lim_{\varepsilon \longrightarrow 0} R_\varepsilon$ and $R^- = \lim_{\varepsilon \longrightarrow 0} R_{-\varepsilon}$, and $\int_{R_{\pm\varepsilon}} g = 0$ for all $\varepsilon > 0$. By continuity of f on L, we have $\lim \int_{R_\varepsilon} g(z)\, dz = 0$.

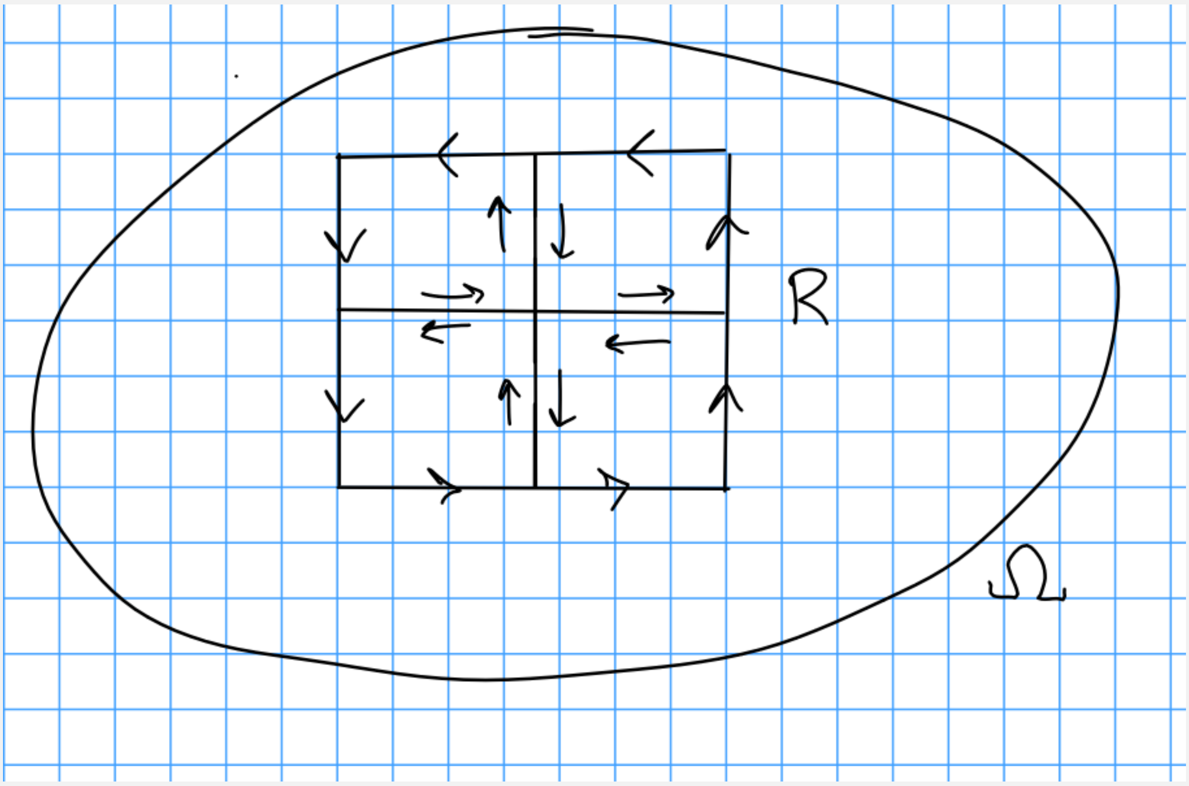
■

11.3 Goursat's Theorem

Theorem 11.3 ***(Goursat, implies smooth).***
If $f : \Omega \longrightarrow \mathbb{C}$ is complex differentiable at each point of Ω, then f is holomorphic. I.e.,

$$f \in C^1(\Omega) \implies f \in C^\infty(\Omega).$$

Proof Goursat.
We have $\int_R f\, dz = 0$ for all rectangles R. Write $I = \int_R f\, dz$. Break R into 4 sub-rectangles:



Then rewriting the integral and applying the triangle inequality yields

$$I = \int_R f = \sum_{j=1}^{4} \int_{R_j} f = \sum_{j=1}^{4} I_j \implies |I| \leq \sum_j |I_j|.$$

So for at least one j, we have $|I_j| \geq \frac{1}{4}|I|$; wlog call it R_1. By continuing to subdivide, we can write

$$|I| \leq 4|I_k| = 4\left|\int_{R_1} f\right| \leq 4\left(4\left|\int_{R_2} f\right|\right) \cdots \leq 4^k\left|\int_{R_k} f\right|.$$

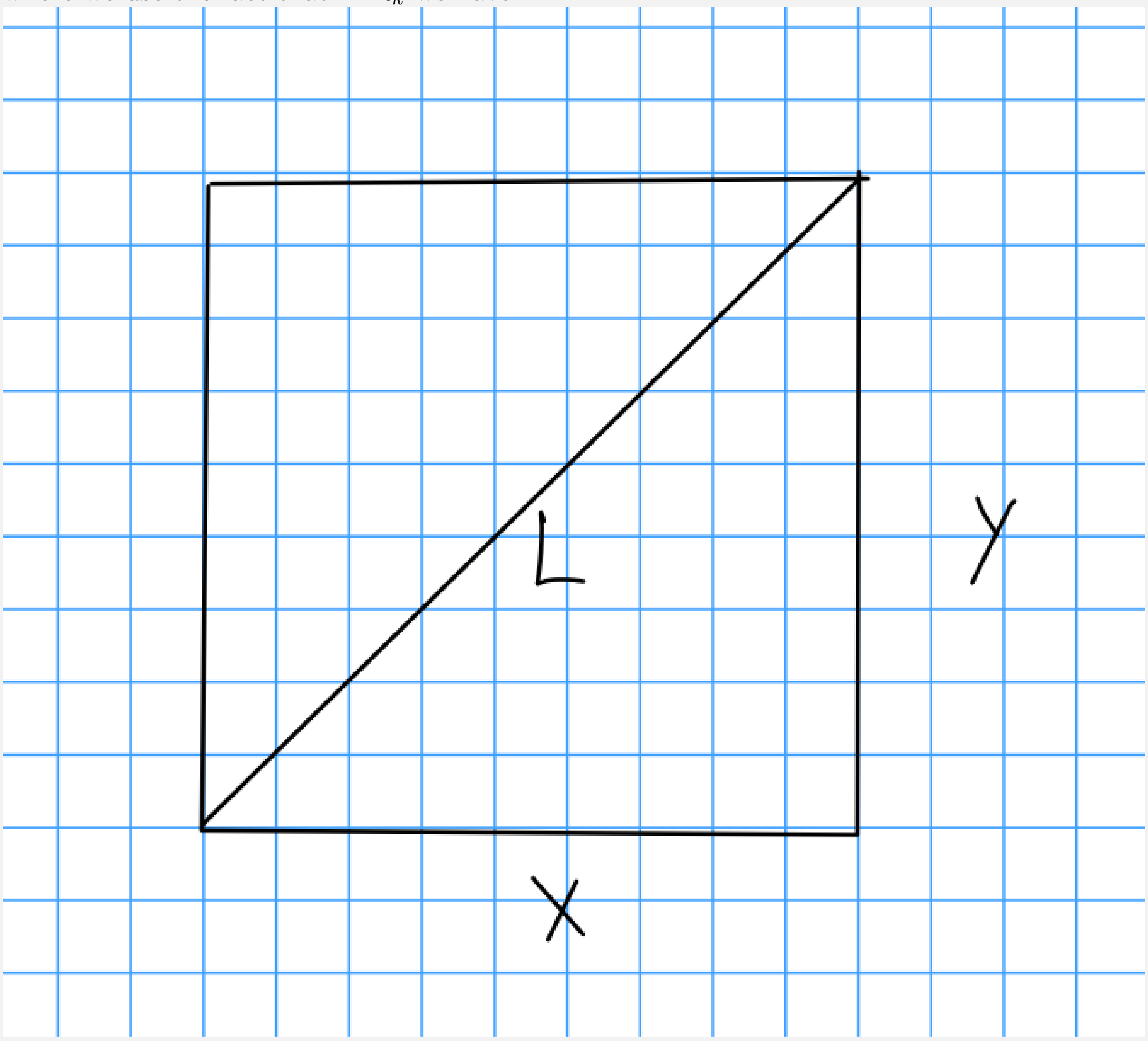
This is a sequence of nested compact intervals, so there is some $z_0 \in \bigcap R_k$.
Write $f(z) = f(z_0) + f'(z_0)(z - z_0) + \delta(z, z_0)$, and since

$$\lim_{z \longrightarrow z_0} \frac{|\delta(z, z_0)|}{z - z_0} = 0,$$

we have $\delta(z, z_0) = o(z - z_0)$. Then $|I| \leq 4^k \frac{1}{2^k}|R|$. We then try to estimate the integral using the fact that $|\delta(z, z_0)| \leq \delta_k |z - z_0|$ for some constant $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

$$\begin{aligned}\int_{R_k} fi &= \int f(z_0) + f'(z_0)(z - z_0) + \delta(z, z_0) \\ &= \int_{R_k} \delta(z, z_0) \quad \text{since the first two terms are holomorphic} \\ &\leq \frac{1}{2^k}|R|\delta_k \frac{C}{2^k}|R| \\ &= c/4^k |R|^2 \delta_k \\ &\overset{k\longrightarrow\infty}{\longrightarrow} 0,\end{aligned}$$

where we use the fact that in R_k we have



$$R_k = 2(x+y) \implies R^2/4 = x^2+y^2+x+y \leq_{CS} x^2+y^2+x^2+y^2 = 2(x^2+y^2)$$
$$\implies x^2+y^2 \leq R^2/8 \implies L = \sqrt{x^2+y^2} \leq R^8/2\sqrt{2}$$
$$\implies |z-z_0| \leq \sqrt{x^2+y^2} \leq R_k/2\sqrt{2} \text{ and } R_k = \frac{1}{2^k}|R|.$$

Note that triangles implies rectangles, but think about how to use triangles to prove it for rectangles (note that sides should be parallel to axes!)

■

12 Friday February 7th

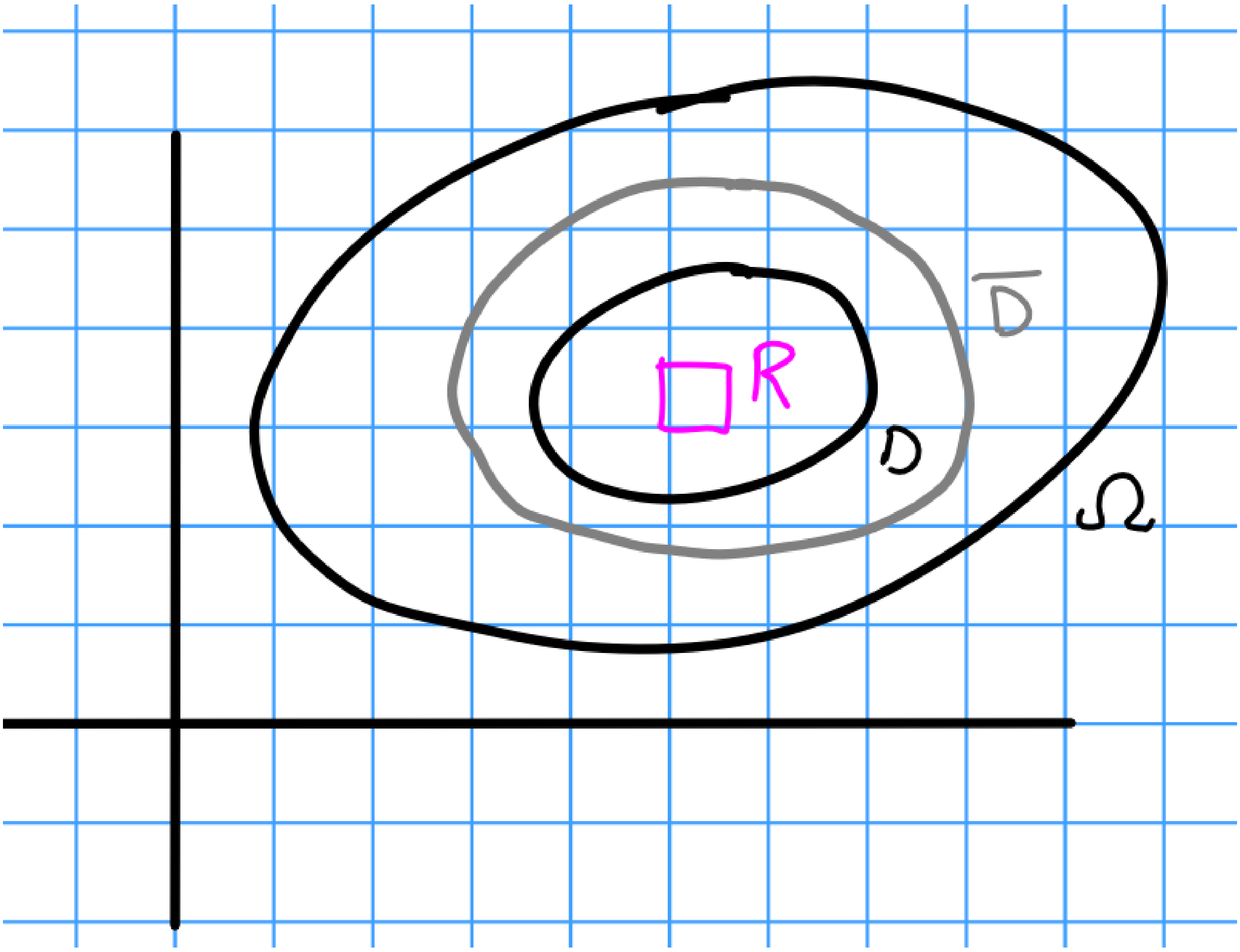
Theorem 12.1 ***(The Uniform Limit of Holomorphic Functions is Holomorphic).*** Suppose $\{f_n\} \longrightarrow f$ is a sequence of holomorphic functions converging uniformly on any compact subset $K \subset \Omega$. Then f is holomorphic.

Proof.
Let D be any disc such that $\overline{D} \subset \Omega$. For any rectangle $R \subset D$, we have

$$\int_R f_n \, dz = 0.$$

Since $f_n \longrightarrow f$ uniformly, $\int_R f \, dz = 0$ and thus f is holomorphic in D.

■



Theorem 12.2 ***(Uniform Convergence of Derivatives).***
Under the same hypotheses, $f_n' \longrightarrow f$ uniformly on any compact subset $K \subset \Omega$.

Proof.
See Stein. ∎

Corollary 12.3 (When Functions Defined by Integrals are Holomorphic)**.**
Suppose $F(z,s) : \Omega \times [a,b] \longrightarrow \mathbb{C}$ and

1. $F(z,s)$ is holomorphic in z for each fixed $s \in [a,b]$.
2. $F(z,s)$ is continuous in $\Omega \times [a,b]$.

Then $f(z) = \displaystyle\int_a^b F(z,s)\ ds$ is holomorphic on Ω.

Proof.
Define $f_n(z) = \left(\sum_{k=1}^{n} F(z, s_k)\right)\frac{b-a}{n}$ where each $s_k = a + \frac{b-a}{n}k \in [a,b]$. Need to show $f_n(z)$ converges uniformly on any compact $K \subset \Omega$, i.e. it's uniformly Cauchy. Fix K compact, then by a theorem in topology $K \times [a,b]$ is again compact.
Using the fact that F is continuous on a compact set and thus uniformly continuous, fix $\varepsilon > 0$ and find $\delta > 0$ such that $\max_{z \in K} |F(z,s) - F(z,t)| < \varepsilon$ for all $s,t \in [a,b]$ with $|t - s| < \delta$.
Thus if $\frac{b-a}{n} < \delta$ and $z \in K$, we have an estimate

$$\begin{aligned}
|f_n(z) - f(z)| &= \left|\sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} F(z, s_k) - F(z,s)\ ds\right| \\
&= \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} |F(z,s_k) - F(z,s)|\ ds \\
&\leq \varepsilon(b-a).
\end{aligned}$$

Thus $f_n \xrightarrow{u} f$. ∎

Remark: this is useful for showing

$$\Gamma(z) = \int_0^\infty e^{-s} s^{z-1}\ ds$$

is holomorphic for $\Re z > 0$.

Question: can every function be uniformly approximated by polynomials?

Answer: in general, no. Take $f(z) = \frac{1}{z}$, which is holomorphic on $\mathbb{C} \setminus 0$, but $\int_\gamma P_N(z) = 0$ for any polynomial (since)hey are entire) for any loop γ around 0, but $\int_\gamma \frac{1}{z} = 2\pi i$.

Theorem 12.4 ***(5.2, Uniform Approximation by Polynomials).***
If f_n is a sequence of holomorphic functions converging uniformly on any compact subset K of Ω then f is holomorphic in Ω and if $f(z) = \sum a_n(z - z_0)^n$ then $P_N(z) = \sum^{N} a_n(z-z_0)^n$.

Theorem 12.5 ***(5.7, Uniform Approximation by Rational Functions).***
Any holomorphic function in a neighborhood of a compact set K can be approximated by a *rational* function with singularities only in K^c. If K^c is connected, it can be approximated by a *polynomial.*

Lemma 12.6 (5.8, ???)**.**
Suppose f is holomorphic in an open set Ω with $K \subset \Omega$ compact. Then there exist finitely many segments $\{\gamma_i\}_{i=1}^N$ in $\Omega \setminus K$ such that for all $z \in K$, ???.

Proof of Lemma, Idea.
Divide region into squares, take γ_i to be line segments such that they enclose K.

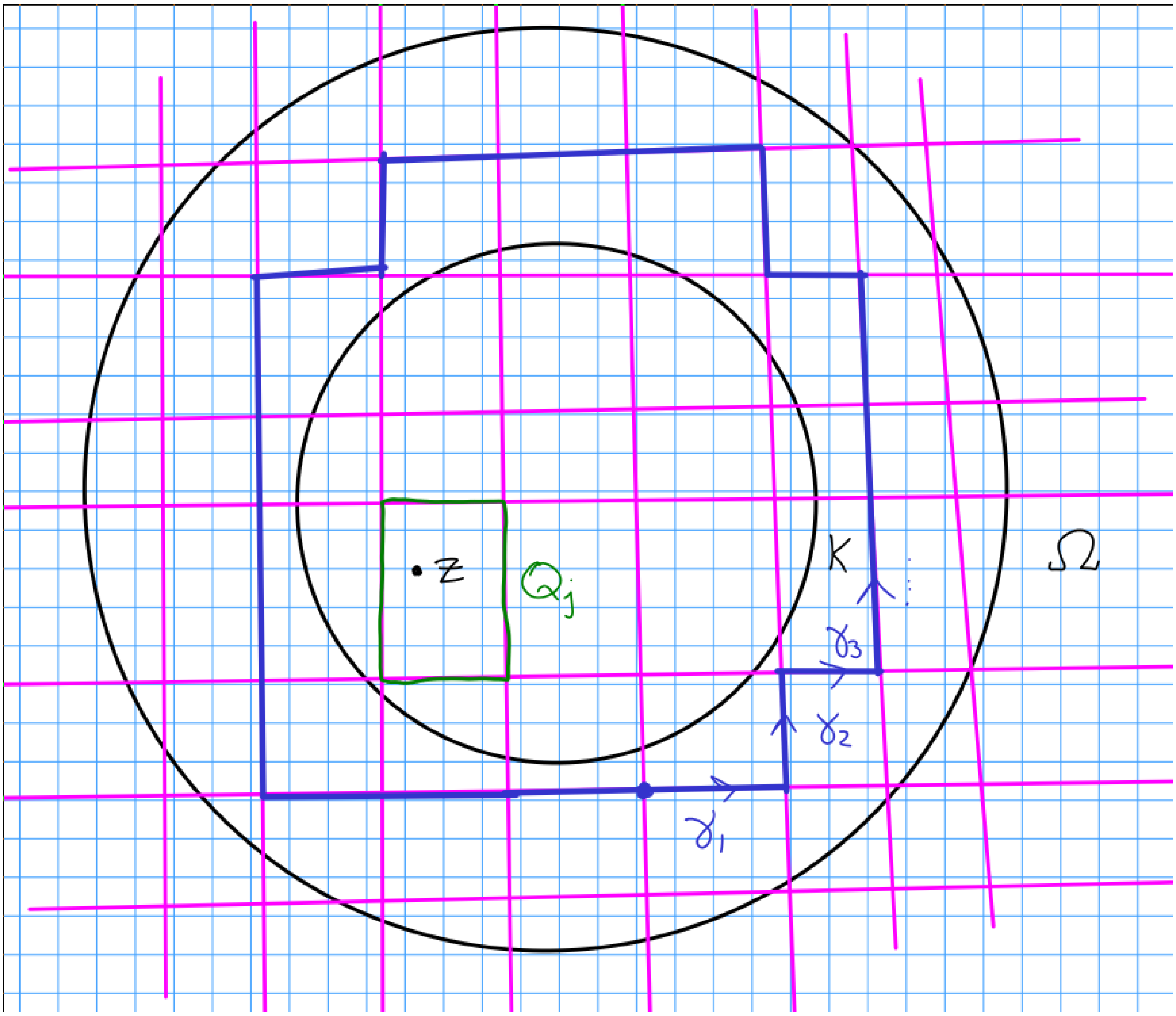
$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{\omega_n} \frac{f(\xi)}{z-\xi}\, d\xi \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{z-\xi}\, d\xi. \end{aligned}$$

where we can rewrite

$$\int_{\gamma_n} \cdots = \int_0^1 \frac{f(\gamma_n(t))}{\gamma_n(t)-z_0} \gamma_n'(t)\, dt = \int_0^1 F(z,s)\, ds$$

The idea is that we can then write $\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1-\frac{z}{\xi}} = \xi^{-1} \sum_k \left(\frac{z}{\xi}\right)^k$, which allows uniform approximation by polynomials.

∎



13 Wednesday February 12th

13.1 Singularities

Let $f(z)$ be holomorphic on Ω, then we have Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi.$$

Example: Note that $f(z) = \dfrac{1}{z}$ is holomorphic on $\mathbb{C} \setminus 0$.

Let Ω be an open set containing a disk D and $\Omega \setminus p$ be a punctured domain.

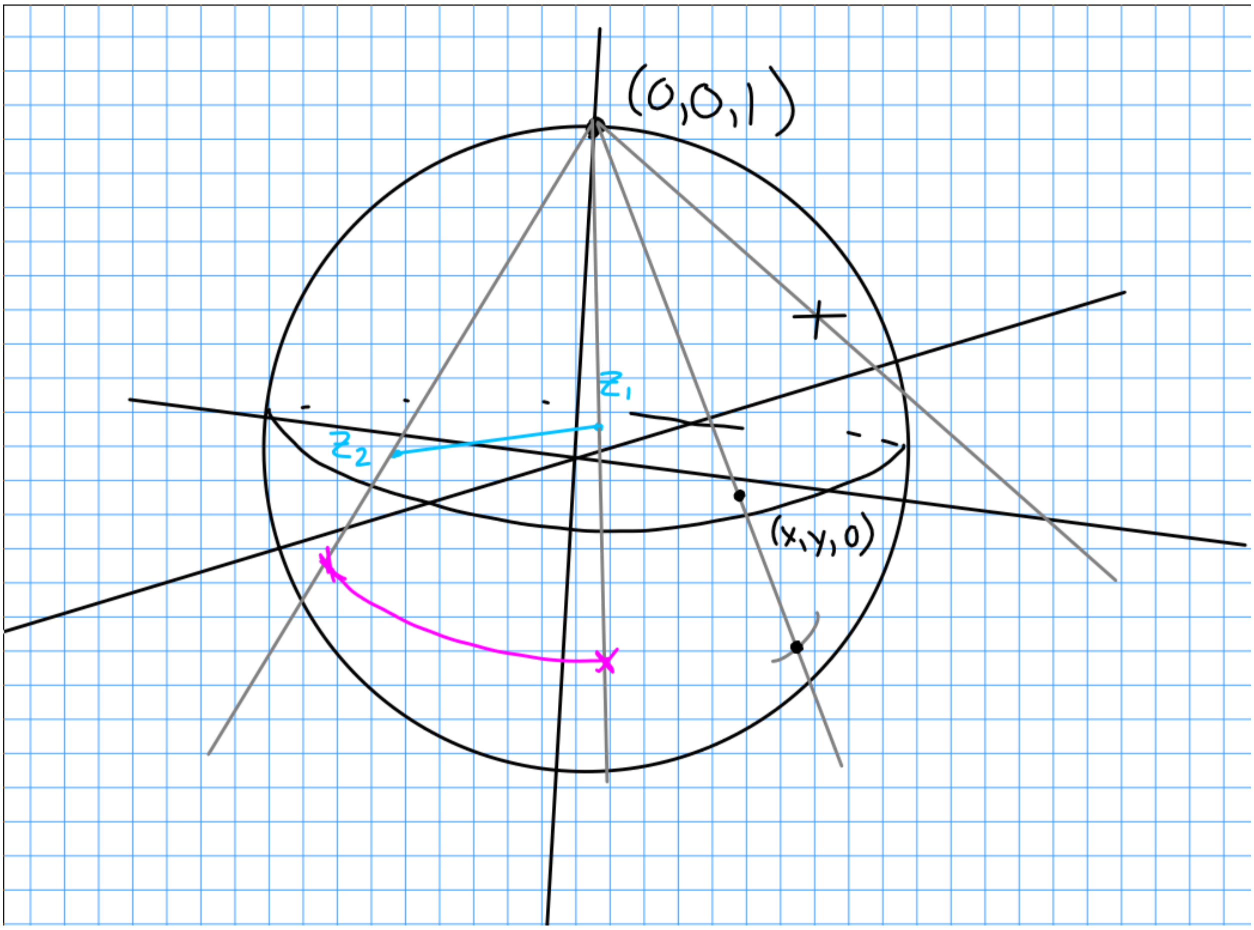
> **Definition 13.1.**
> We say f has an *isolated singularity* at p iff f is defined and holomorphic on some deleted neighborhood of p.
> Classification of singularities:
> 1. **Removable**: $|f(z)|$ is bounded on some $D_r(p) \setminus p$.
>   *Example*: $f(z) = \sin(z)/z$.
> 2. **Poles**: $\lim\limits_{z \longrightarrow p} |f(z)| = \infty$.
>   *Example*: $f_n(z) = \dfrac{1}{z^n}$ at $p = 0$
> 3. **Essential**: neither 1 nor 2.
>   *Example*: $f(z) = e^{\frac{1}{z}}$ at $z = 0$.

Note that for singularities at ∞, we can just make the change of variables $z \mapsto \dfrac{1}{z}$. Defining $F(z) = f(\dfrac{1}{z})$, the singularities at 0 of f correspond to singularities at infinity for F.

13.2 Spherical Projection

We can solve for a spherical projection map $S^2 \longrightarrow \mathbb{C}$. Let $(0, 0, 1)$ be the North pole of the sphere; then to map to $(x, y, 0)$ on the plane we can take the parameterization $\ell : (tx, ty, 1 - t)$. This yields

$$t \mapsto \left(\frac{2\Re(z)}{1 + |z|^2}, \frac{2\Im(z)}{1 + |z|^2}, 1 - \frac{2}{1 + |z|^2}\right).$$



From this we can induce a spherical metric:

$$\phi(z_1, z_2) = \frac{z|z_1 - z_2|}{\sqrt{|z|_1^2 + 1}\sqrt{|z|_2^2 + 1}}.$$

Proposition 13.1 (Continuous Extension Over Removable Singularities)**.**
Let p be a removable singularity of f. Then

1. $\lim_{z \longrightarrow p} f(z)$ exists.
2. The function

$$\tilde{f}(x) = \begin{cases} f(z) & z \neq p \\ \lim_{z \longrightarrow p} f(z) & z = p \end{cases}.$$

   is holomorphic on $D_r(p)$.

Example 13.1.
Consider

$$\frac{\sin(z)}{z} \overset{z \longrightarrow 0}{\longrightarrow} 1.$$

Proof of Proposition.
Take $p = 0$ and consider $g(z) = z^2 f(z)$. We can verify directly that g satisfies the Cauchy-Riemann equations on $D_r(0)$. Then g is holomorphic on $D_r(0)$ and vanishes to order 2 at $z = 0$, and

$$f(z) = \frac{g(z)}{z^2}$$

is holomorphic on $D_r(0)$.
If $f(z)$ has a pole at z_0, then $\lim_{z \longrightarrow z_0} |f(z)| \longrightarrow \infty$ by definition, iff $\lim_{z \longrightarrow z_0} \frac{1}{|f(z)|} = 0$ and thus the reciprocal has a zero at $z = z + 0$. If z_0 is a zero of a nontrivial holomorphic function f, then z_0 is isolated, i.e. there exists a punctured disc $D_r(z_0) \setminus z_0$ on which f is nonzero. ∎

Theorem 13.2 ***(???).***
If f is holomorphic in a connected domain Ω with a zero z_0, then there exists a non-vanishing holomorphic function $g(z)$ and some $n \in \mathbb{N}$ such that

$$f(z) = (z - z_0)^n g(z)$$

.

Proof.
Since f is holomorphic, expand its power series $f(z) = \sum a_k(z - z_0)^k$. Since $f(z_0) = 0$, we have $a_0 = 0$. Choose the smallest n such that $a_n \neq 0$, so

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\ &= (z - z_0)^n(a_n + \cdots) \\ &:= (z - z_0)^n g(z). \end{aligned}$$

Then $g(z_0) \neq 0$, so by continuity there exists an r such that $|g(z)| \geq |a_n|/2$. ∎

Definition 13.2.
A function f defined on a deleted neighborhood of z_0 has a pole at z_0 if the function $F = \frac{1}{f}$ with $F(z_0) := 0$ is holomorphic in a full neighborhood of z_0.

14 Friday February 14th

If f is holomorphic in Ω with $f(z_0) = 0$ then there exists a disc on which $f(z) = \sum a_n(z - z_0)^n$ where $a_0 = f(z_0) = 0$. There is then a minimal k such that $f(z) = (z - z_0)^k g(z)$ where $g(z_0) \neq 0$; this k is the *order* of the zero a_0.

Definition 14.1.
A function defined in a deleted neighborhood of z_0 has a *pole* at z_0 iff $F = \dfrac{1}{f}$ with $F(z_0) := 0$ is holomorphic in a full neighborhood of z_0.

Theorem 14.1.
If f has a pole at z_0, then there exists a holomorphic function h and a unique k such that $f(z) = (z - z_0)^{-k} h(z)$.

Proof.
Write

$$\frac{1}{f} = (z - z_0)^k g(z)$$

with $g(z_0) \neq 0$. Then there is an r such that $|g(z)| \geq \frac{1}{2}|g(z_0)|$ in a disc about z_0. Then

$$f(z) = \frac{1}{(z - z_0)g(z)} := (z - z_0)^{-k} h(z)$$

where $h = 1/g$.
We can then write

$$f(z) = \left(\sum_{i=0}^{k-1} b_k (z - z_0)^{-k} \right) + b_k + \sum_{i=1}^{\infty} b_{k+i} (z - z_0)^i$$

for some fixed k, where $\sum b_i (z - z_0)^i$ is the power series expansion of h. Write this as $P(z) + G(z)$ where $G(z) = \sum_{i=0}^{\infty} b_{i+k} (z - z_0)^i$. Denote P *the principal part* of f at the pole $z = z_0$.
Note that

$$\int_{D_r(z_0)} f = \int_{D_r(z_0)} P(z) = 2\pi i \; a_{-1}.$$

The coefficient a_{-1} is referred to as the *residue* of f at $z = z_0$. ■

Interesting open problems: dynamical systems on $\mathbb{C}^2$.

14.1 Residues

Note that

$$\int \frac{1}{(z-z_0)^{\ell}} = \begin{cases} 2\pi i & \ell = 1 \\ 0 & \text{else} \end{cases}$$

.

How to compute residues:

$$a_{-1} = \frac{1}{2\pi i} \int_{D_r(z_0)} f.$$

Theorem :

$$\mathrm{Res}_{z=z_0} f = \lim_{z \longrightarrow z_0} \frac{1}{(k-1)!} \left(\frac{\partial}{\partial z} \right)^{k-1} (z-z_0)^k f(z).$$

Proof.
Expand in power series, direct check. ■

A useful special case: if z_0 is a pole of order 1, then

$$\mathrm{Res}_{z=z_0} f = \lim_{z \longrightarrow z_0} (z-z_0) f(z).$$

A useful formula:

$$\frac{1}{2}\pi i \int_{\Gamma(z_0)} f = \mathrm{Res}_{z=z_0} f.$$

Theorem 14.2.
Suppose that f is holomorphic in an open set containing a toy contour γ and its interior except for finitely many poles $\{z_i\}$. Then

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum \mathrm{Res}_{z=z_i} f(z).$$

Proof.
Omitted to cover some material needed for homework. ■

Note that if f has a pole of order k, we can expand it in *Laurent series* as

$$\sum_{n=-k}^{1} a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^k.$$

How to determine the radius of convergence of a Laurent series: break

$$\sum_{-\infty}^{\infty} a_n z_n = \sum_{n\in\mathbb{N}} c_n z^n + \sum_{n\in\mathbb{N}} d_n z^{-n}.$$

Applying the root test,

$$\begin{aligned}
&\limsup_n |c_n(z-a)|^{1/n} < 1 \\
\iff &\limsup_n |c_n|^{1/n} |z-z_0|^n < 1 \\
&\iff |z-a| \leq \frac{1}{\limsup_n |c_n|^{1/n}} := \rho_1.
\end{aligned}$$

Similarly, we need

$$\rho_2 := \limsup_n |d_n|^{1/n} < |z-a|.$$

If $\rho_1 > \rho_2$, this will converge on an annulus.

15 Monday February 17th

See Hans Lewy 1957 Annals, Folland and Stein 1973. Does a linear system of PDEs with analytic functions have an analytic solution? What about just C^∞?

15.1 Laurent Series

We can write a formal series

$$\begin{aligned}
f(z) &= \sum_{n\in\mathbb{Z}} a_n(z-a)^n \\
&= \sum_{n\geq 0} a_n(z-z_0)^n + \sum n \leq -1 a_n(z-z_0)^n \\
&:= A(z) + B(z).
\end{aligned}$$

Part A converges for

$$|z-a| < R_1 = \left(\limsup |x_n|^{1/n}\right)^{-1}.$$

Part B converges for

$$|z-a| > R_2 = \limsup |c_{-n}|^{1/n}.$$

If $R_1 < R_2$, this does not converge. Note that if $R_1 > R_2$, then f converges and defines a holomorphic function on the annulus $R_2 < |z-a| < R_1$. Moreover, f converges uniformly on any compact subset of this annulus, so it can be differentiated term-by-term, and the derivative has the same region of convergence.

Note that if f equals its Laurent expansion, then

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(xi)}{(\xi - a)^{n+1}}\ dz$$

where γ is contained in the annulus of convergence, and $c_{n\le -1}0$.

We also have

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma f(z)/(z-a)^m\ dz &= \sum_{n\in\mathbb{Z}} \frac{c_n}{2\pi i}\int_\gamma \frac{1}{(z-a)^{m-n}} \\ &= c_{m-1},\end{aligned}$$

since

$$\int \frac{1}{(z-a)^k} = \begin{cases} 2\pi i & k=1 \\ 0 & \text{else}\end{cases},$$

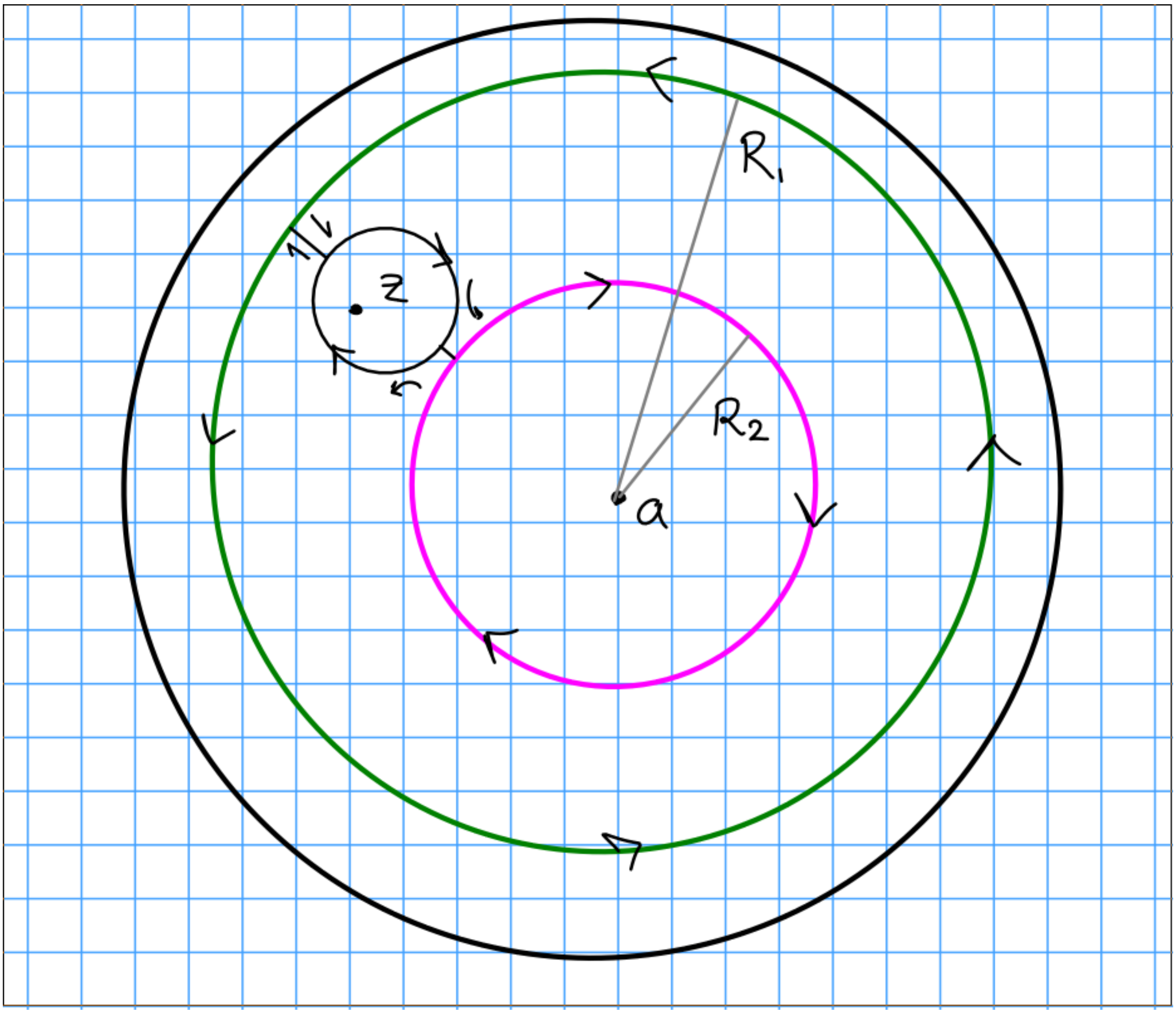
so we have the following formula for the coefficients:

$$c_m = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{m+1}}. \tag{1}$$

So we can start with a series and get a holomorphic function on some region.

We can also start with a holomorphic function and get a Laurent series. Suppose f is holomorphic on an annulus $R_2 < |z| < R_1$. We can then write

$$f(z) = \frac{1}{2\pi i}\int_{|w-a|=R_1} \frac{f(w)}{w-z}\ dw - \int_{|w-z|=R_2} \frac{f(w)}{w-z}\ dw$$



Since $|z-a|/|w-a|<1$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|w-a|=R_1}\frac{f(w)}{w-z}\,dz &= \frac{1}{2\pi i}\int_{|w-a|=R_1}\frac{f(w)}{(w-a)-(z-a)}\,dz \\
&= \frac{1}{2\pi i}\int_{|w-a|=R_1}\frac{f(w)}{(w-a)}\sum_{n\in\mathbb{N}}\frac{(z-a)^n}{(w-a)^n}\,dz \\
&= \sum_{n\in\mathbb{N}}(z-a)^n\frac{1}{2\pi i}\int_{|w-a|=R_1}\frac{f(w)}{(w-a)^{n+1}}\,dw \\
&= \sum_{n\in\mathbb{N}}c_n(z-a)^n.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{|w-a|=R_2}\frac{f(w)}{w-z}\,dw &= -\frac{1}{2\pi i}\int_{|w-a|=R_2}\frac{f(w)}{(w-a)-(z-a)}\,dw \\
&= -\frac{1}{2\pi i}\frac{1}{z-a}\int_{|w-a|=R_2}\frac{f(w)}{\frac{w-a}{z-a}-1}\,dw \\
&= \frac{1}{2\pi i}\frac{1}{z-a}\int_{|w-a|=R_2}\frac{f(w)}{1-\frac{w-a}{z-a}}\,dw \\
&= \frac{1}{2\pi i}\frac{1}{z-a}\int_{|w-a|=R_2}f(w)\sum_{n\in\mathbb{N}}\frac{(w-a)^n}{(z-a)^n}\,dw \\
&= \sum_{n\in\mathbb{N}}\frac{1}{2\pi i}\frac{1}{(z-a)^{n+1}}\int_{|w-a|=R_2}f(w)(w-a)^n\,dw \\
&= \sum_{n=-\infty}^{-1}c_n(z-a)^n.
\end{aligned}
$$

This yields a formula

$$
c_m = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{m+1}}\,dz.
$$

In practice, we don't use this formula for extracting coefficients.

Example 15.1.
Let $f(z) = \dfrac{1}{z(z-1)}$. This has four Laurent series.

Let $C(a, R_1, R_2)$ be the annulus centered at a. Then at $C(0,0,1) = \mathbb{D}\setminus\{0\}$, we have

$$
f(z) = \frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\sum_{k\in\mathbb{N}}z^k.
$$

In $C(1,1,0) = \mathbb{D}(1,1)\setminus\{1\}$, we have

$$
\begin{aligned}
f(z) &= \frac{1}{z-1}\frac{1}{z} \\
&= \frac{1}{z-1}\frac{1}{1+(z-1)} \\
&= \frac{1}{z-1}\sum_{k\in\mathbb{N}}(-1)^k(z-1)^k.
\end{aligned}
$$

In $C(0,1,\infty)$, we can write

$$\begin{aligned} f(z) &= \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} \\ &= \frac{1}{z^2}\sum_{k\in\mathbb{N}}\frac{1}{z^k}. \end{aligned}$$

And in $C(1,1,\infty)$ we have

$$f(z) = \frac{1}{z-1}\frac{1}{z-1+1}.$$

16 Appendix

$$\begin{aligned} dz &= dx + i\ dy \\ d\overline{z} &= dx - i\ dy \\ f_z &= f_x = i^{-1}f_y \end{aligned}$$

$$\int_0^{2\pi} e^{i\ell x}dx = \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}.$$

- Holomorphic: once complex differentiable in neighborhoods of every point.
- Analytic: equal to its Taylor series expansion

Cauchy Inequality: Given $z_0 \in \Omega$, pick the largest disc $D_R(z_0) \subset \Omega$ and let $C_R = \partial D_R$. Using the integral formula, defining $\|f\|_{C_R} = \max_{|z-z_0|=R} |f(z)|$

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\int_0^{2\pi} \frac{\|f\|_{C_R}}{R^{n+1}} R\ d\theta = \frac{n!\|f\|_{C_R}}{R^n}.$$

Collection of facts used on problem sets

Standard forms of conic sections:

- Circle: $x^2 + y^2 = r^2$
- Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$
- Hyperbola: $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$
  - Rectangular Hyperbola: $xy = \frac{c^2}{2}$.
- Parabola: $-4ax + y^2 = 0$.

Mnemonic: Write $f(x,y) = Ax^2 + Bxy + Cy^2 + \cdots$, then consider the discriminant $\Delta = B^2 - 4AC$:

- $\Delta < 0 \iff$ ellipse
  - $\Delta < 0$ and $A = C, B = 0 \iff$ circle
- $\Delta = 0 \iff$ parabola
- $\Delta > 0 \iff$ hyperbola

Completing the square:

$$\begin{aligned} x^2 - bx &= (x-s)^2 - s^2 \quad \text{where} s = \frac{b}{2} \\ x^2 + bx &= (x+s)^2 - s^2 \quad \text{where} s = \frac{b}{2}. \end{aligned}$$

Useful Properties

- $\Re(z) = \frac{1}{2}(z + \overline{z})$ and $\Im(z) = \frac{1}{2i}(z - \overline{z})$.
- $z\overline{z} = |z|^2$
- $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
- $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Useful Series

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{n(n+1)}{2} \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} \end{aligned}$$

Cauchy-Riemann Equations

$$\begin{aligned} u_x = v_y \quad &\text{and} \quad u_y = -v_x \\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad &\text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \end{aligned}.$$

16.1 Useful Techniques

Showing a function is constant: Write $f = u + iv$ and use Cauchy-Riemann to show $u_x, u_y = 0$, etc.

Deriving Polar Cauchy-Riemann: See walkthrough here. Take derivative along two paths, along a ray with constant angle θ_0 and along a circular arc of constant radius r_0. Then equate real and imaginary parts. See problem set 1.

Computing Arguments: $\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w)$.

The sum of the interior angles of an n-gon is $(n-2)\pi$, where each angle is $\frac{n-2}{n}\pi$.

16.2 Residues

If p is a simple pole, $\mathrm{Res}(p, f) = \lim_{z \longrightarrow p} (z-p)f(z)$. Example: Let $f(z) = \dfrac{1}{1+z^2}$, then $\mathrm{Res}(i, f) = \dfrac{1}{2i}$.

Green's Theorem: Todo

$\dfrac{\partial}{\partial z} \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_{j+1} z^j$.