Math 174 HW 4

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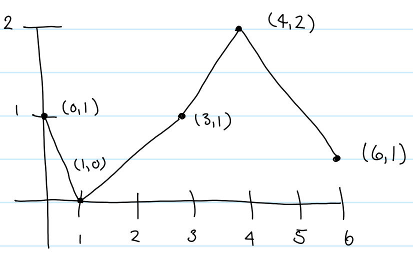
1)



b) $m = \frac{\Delta y}{\Delta x} = \frac{1-2}{6-4} = \frac{-1}{2}$,

so using $(x_1, y_1) = (6, 1)$, we have

$y - 1 = \frac{-1}{2}(x - 6)$

$\longrightarrow \boxed{y = \frac{-1}{2}x + 4.}$

c) Quadratic 1: $x \in [0,3]$

$\rightarrow (0,1), (1,0), (3,1)$ on parabola

Let $f(x) = ax^2 + bx + c$

- $f(0) = 1 \rightarrow \underline{c = 1}$
- $f(1) = 0 \rightarrow \underline{a + b + 1 = 0}$
- $f(3) = 1 \rightarrow 9a + 3b + 1 = 1$

$\rightarrow 9a + 3b = 0$

$\rightarrow \underline{b = -3a}$

$a + b = -1 \rightarrow a + (-3a) = -1 \rightarrow -2a = -1 \rightarrow \underline{a = \frac{1}{2}}, \ \underline{b = -3a = \frac{-3}{2}}$

$\rightarrow \boxed{f_1(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1}$ on $[0,3]$

On $[3,6]$: $(3,1), (4,2), (6,1)$

$f_2(3) = 1 \rightarrow 9a + 3b + c = 1$

$f_2(4) = 2 \rightarrow 16a + 4b + c = 2$

$f_2(6) = 1 \rightarrow 36a + 6b + c = 1$

$$\longrightarrow \begin{bmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 6 & 1 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

$$A\bar{c} = \bar{d}$$

$$\to \bar{c} = A^{-1}\bar{d}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ -\frac{10}{3} & \frac{9}{2} & -\frac{7}{6} \\ 8 & -9 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \underbrace{\left(-\frac{1}{2}, \frac{9}{2}, -8\right)^T}.$$

So $\boxed{f_2(x) = -\frac{1}{2}x^2 + \frac{9}{2}x - 8}$

- $x = 2 \to f(2) = f_1(2) = \frac{1}{2}\cdot 4 - \frac{3}{2}\cdot 2 + 1$
  $= 2 - 3 + 1$
  $= 0$

$\boxed{\to f(2) = 0}$

- $x = 5 \to f(5) = f_2(5) = -\frac{1}{2}\cdot 25 + \frac{9}{2}5 + 8$
  $= \frac{1}{2}(-25 + 45 + 8)$
  $= 14$

$\boxed{\to f(5) = 14}$

2) If $x = x_i$ for any i, then $f(x) = p(x)$ and the LHS is zero, and $x - x_i = 0$ so the RHS is zero as well. Just let $\xi(x) = 1$. (arbitrary)

Otherwise, $x \neq x_i$ for any i, so define

$$g(t) = (f(t) - p(t)) - (f(x) - p(x)) \prod_{j=1}^{n} \frac{(t - x_j)}{(x - x_j)}$$

Note $g(x_i) = \underbrace{(f(x_i) - p(x_i))}_{=0} - \underbrace{(f(x) - p(x)) \prod_{j=1}^{n} \frac{(x_i - x_j)}{(x - x_j)}}_{=0\text{, since } i=j \text{ for some } j}$

So $g(x_i) = 0 \ \forall i$. In particular, $g(x_1) = g(x_n) = 0$.

By generalized Rolle, $\exists$ a constant $c \in (a, b)$ such that

$(n+1)$ ... $\partial^{n+1}/$ ...

By generalized Rolle, $\exists$ a constant $c \in (a,b)$ such that

$$g^{(n+1)}(c) = \partial^{n+1}/\partial t^{n+1}\, g(t)\big|_c = 0$$

But
$$\partial^{n+1}/\partial t^{n+1}\, g(t) = \left(f^{(n+1)}(t) - p^{(n+1)}(t)\right) - (f(x)-p(x)) \cdot \frac{\partial^{n+1}}{\partial t^{n+1}}\left[t^{n+1}\left(\prod_{j=1}^{n}\frac{1}{(x-x_j)}\right) + O(t^n)\right)$$
$$= f^{(n+1)}(t) - (f(x)-p(x))\,(n+1)!\prod_{j=1}^{n}\frac{1}{(x-x_j)}$$

and so
$$g^{(n+1)}(c) = f^{(n+1)}(c) - (f(x)-p(x))\prod_{j=1}^{n}\frac{(n+1)!}{(x-x_j)}$$

$$\rightarrow f(x)-p(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\prod_{j=1}^{n}(x-x_j)$$

So taking $\xi = c$ yields the desired constant. ∎

3)

i	x_i	$f(x_i)$	$f'(x_i)$
0	-1	0	1
1	1	2	-1
2	0	1	2

i	z_i	$f(z_i)$	D^1	D^2	D^3
0	-1	0	1	$\frac{1-1}{1+1} = 0$	$\frac{1-0}{1+1} = \frac{1}{2}$
1	-1	0	$\frac{2-0}{1+1} = 1$	$\frac{1+1}{1+1} = 1$	
2	1	2	-1		
3	1	2			

$$\rightarrow H_3(x) = f[z_0] + \sum_{k=1}^{3} f[z_0 \ldots z_k]\prod_{i=0}^{k-1}(x-z_i)$$

$$= 0 + 1\left[(x-z_0)\right] + 0\left[(x-z_0)(x-z_1)\right] + \tfrac{1}{2}\left[(x-z_0)(x-z_1)(x-z_2)\right]$$

$$= (x+1) + \tfrac{1}{2}(x+1)^2(x-1)$$

$$\rightarrow \boxed{H_3(x) = (x+1) + \tfrac{1}{2}(x+1)^2(x-1)}$$

$H_4(x) = H_3(x) + f[z_0\, z_1\, z_2\, z_3\, z_4] \prod_{i=1}^{3} (x - z_i) + f[z_i \cdots z_5] \prod_{i=1}^{4} (x - z_i)$

i	z_i	$f(z_i)$	D^1	D^2	D^3	D^4	D^5
0	-1	0	1	0	$\frac{1}{2}$	$\frac{-3-\frac{1}{2}}{0+1} = -\frac{7}{2}$	$\frac{2+7/2}{0+1} = 9/2$
1	-1	0	1	1	$\frac{-2-1}{0+1} = -3$	$\frac{-1+3}{0+1} = 2$	
2	1	2	-1	$\frac{1+1}{0-1} = -2$	$\frac{-1+2}{0-1} = -1$		
3	1	2	$\frac{1-2}{0-1} = 1$	$\frac{2-1}{0-1} = -1$			
4	0	1	2				
5	0	1					

$\rightarrow H_4(x) = H_3(x) + \left(\frac{-7}{2}\right)\left[(x+1)^2(x-1)^2\right] + \left(\frac{9}{2}\right)\left[(x+1)^2(x-1)^2(x-0)\right]$

$$\boxed{= (x+1) + \frac{1}{2}(x+1)^2(x-1) - \frac{7}{2}(x+1)^2(x-1)^2 + \frac{9}{2}(x+1)^2(x-1)^2 x.}$$

· For $(x_0, y_0, x_0'), (x_1, y_1, x_1') = (-1, 0, 1), (1, 2, -1)$

$p_1(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1)$

Then,

$a = y_0 \qquad = 0$

$b = x_0' \qquad = 1$

$c = \left(y_1 - y_0 - x_0'(x_1 - x_0)\right) \cdot (x_1 - x_0)^{-2} \qquad = (2 - 0 - 1(1+1)) \cdot (1+1)^{-2} = \frac{2-2}{4} = 0$

$d = (x_1 - x_0)^{-2}\left(x_0' + x_1' - 2\left(\frac{y_1 - y_0}{x_1 - x_0}\right)\right) \qquad = \frac{1}{4}\left(1 - 1 - 2\left(\frac{2-0}{1+1}\right)\right) = \frac{-1}{2}$

$\longrightarrow f_1(x) = 0 + 1(x - x_0) + 0 + \left(\frac{-1}{2}\right)(x - x_0)^2(x - x_1)$

$$\boxed{\longrightarrow f_1(x) = (x+1) - \frac{1}{2}(x+1)^2(x-1)}$$

$f_2(x)$ for $(x_1, f(x_1), f'(x_1)), (x_2, f(x_2), f'(x_2)) = (1, 2, -1), (0, 1, 2)$

$a = y_1 = f(x_1) = 2$

$b = x_1' = f'(x_1) = -1$

$c = \left(f(x_2) - f(x_1) - f'(x_1)(x_2 - x_1)\right)(x_2 - x_1)^{-2}$

$= (1 - 2 + 1(0-1))(0-1)^{-2}$

$= -2$

$d = \left(f'(x_1) + f'(x_2) - 2\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)\right)(x_2 - x_1)^{-2}$

$= (1 - 2 + 1(0-1))(0-1)$

$= -2$

$$d = \left(f'(x_1) + f'(x_2) - 2\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\right)\right)(x_2-x_1)^{-2}$$

$= \left(-1 + 2 - 2\left(\frac{1-2}{0-1}\right)\right)$

$= -1$

$\rightarrow f_2(x) = 2 - 1(x-x_1) - 2(x-x_1)^2 - 1(x-x_1)^2(x-x_2)$

$= 2 - (x-1) - 2(x-1)^2 - (x-1)^2 x$

Therefore $f(x) = \begin{cases} (x+1) - \frac{1}{2}(x+1)^2(x-1), & x \in [x_0, x_1] \\ 2 - (x-1) - 2(x-1)^2 - x(x-1)^2, & x \in [x_1, x_2]. \end{cases}$

4) Conditions

- $f(x_i)$ specified: 4
- $f'(x_i)$ specified: 3
- $f''(x_i)$ specified: 2
- $f'''(x_i)$ specified: 1

Total: 10

$\rightarrow$ Degree $d = 10 + 1 = \underline{11}$.

5) a) There are n intervals $[t_i, t_{i+1})$. A degree K polynomial has $K+1$ degrees of freedom.

This yields $\boxed{n(k+1)}$ degrees of freedom.

b) This imposes $\underline{1}$ equation at each x_i

$f_{i-1}(x_i) = f_i(x_i)$ for $i \in \underbrace{\{1, \dots, n-1\}}_{n-2 \text{ indices}}$, due to excluded endpoints

This yields $\underline{n-2}$ conditions

c) This imposes

$f_{i-1}(x_i) = f_i(x_i)$

$f'_{i-1}(x_i) = f'_i(x_i)$ for $i \in \{1 \cdots n-1\}$

$\vdots$

$f_{i-1}(x_i) = f_i(x_i)$

$\vdots$

$f^{(k)}_{i-1}(x_i) = f^{(k)}_i(x_i)$

Which is $(n-2)(k+1) = n-2+k(n-2)$ total conditions

thus $\underline{k(n-2)}$ new conditions

d) Total dof: $n(k+1) = 6n$

Total eqns: $(n-2)(k+1) = 6(n-2) = 6n-12$

$\rightarrow$ Remaining dof: $n(k+1)-(n-2)(k+1) = 2(k+1) = \boxed{12.}$

6) • $S''(2)=0 \rightarrow 2c+6d(x-1)\big|_2 = 0$

$\rightarrow 2c+6d=0$

$\boxed{\rightarrow c+3d=0}$

• $S''(0)=0$: Already true $\left(\frac{\partial^2}{\partial x^2}\Big|_0 1+2x-x^3 = \frac{\partial}{\partial x}\Big|_0 2-3x^2 = -6x\big|_0 = 0.\right)$

• $S_1(1)=S_2(1)$: Already true

$LHS\big|_1 \quad 1+2x-x^3\big|_1 = 1+2-1 = 2$

$RHS\big|_1 = 2+0+0+0 = 2. \checkmark$

• $S_1'(1)=S_2'(1)$:

$\frac{\partial}{\partial x}\Big|_1 LHS = 2-3x^2\big|_1 = 2-3 = -1$

$\frac{\partial}{\partial x}\Big|_1 RHS = b+2c(x-1)+3d(x-1)^2\big|_1 = b$

$\rightarrow \boxed{b=-1.}$

So far, only two equations in 3 unknowns - need one more equation for uniqueness.

So enforce $S_1''(1)=S_2''(1)$, then

$\frac{\partial^2}{\partial x^2}\Big|_1 1+2x-x^3 = \frac{\partial}{\partial x}\Big|_1 2-3x^2 = -6x\big|_1 = -6$

$\frac{\partial^2}{\partial x^2}\Big|_1 2+b(x-1)+\ldots = \frac{\partial}{\partial x}\Big|_1 b+2c(x-1)+3d(x-1)^2$

$$= 2c + 6d(x-1)\Big|_1$$

$$= 2c$$

$$\longrightarrow -6 = 2c$$

$$\longrightarrow \underline{c = -3}$$

So $c = -3$, $b = -1$, $c + 3d = 0 \to 3d = -c \to d = \frac{-1}{3}c = \frac{-1}{3}(-3) = 1$

$$\boxed{\text{and } S_2(x) = 2 + (-1)(x-1) + (-3)(x-1)^2 + (1)(x-1)^3.}$$

7) • $\underline{S'(0) = -1}$: $b + 2cx + 3dx^2\Big|_0 = -1$

$$\to \boxed{b = -1}$$

• $\underline{S_1(1) = S_2(1)}$: $\boxed{a + b + c + d = 1}$

• $\underline{S_1'(1) = S_2'(1)}$: $\boxed{b + 2c + 3d = 1}$

• $\underline{S'(2) = 6}$: $1 + 2(x-1) + 3(x-1)^2\Big|_2 = 6$

$\longrightarrow 1 + 2 + 3 = 6$, already true.

• $\underline{S_1''(1) = S_2''(1)}$: $2c + 6d = 2 + 6(x-1)\Big|_1 = 2$

$$\longrightarrow \boxed{c + 3d = 1}$$

$$\longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A\bar{c} = \bar{b} \to \bar{c} = A^{-1}\bar{b}$$

$$\to (a\ b\ c\ d)^T = A^{-1}\bar{b}$$
$$= (1, -1, 1, 0)$$

$$\boxed{\to S(x) = \begin{cases} 1 - x + x^2, & x \in [0,1] \\ 1 + (x-1) + (x-1)^2 + (x-1)^3, & x \in [1,2]. \end{cases}}$$

8)

a)

approxSinCos.m

```
function y = approxSinCos(xs, ys, z)
    index = 1;
    for i = 1:size(xs, 2)
        if xs(i) >= z
            index = i;
            break;
        end
    end
    x0 = xs(index - 1);
    x1 = xs(index);
    y0 = ys(index-1);
    y1 = ys(index);
    m = (y1 - y0) / (x1 - x0);
    f = @(x) m*(x-x0) + y0;
    y = f(z);
end
```

b)

COMMAND WINDOW

```
    0.7174    0.7833    0.8415
```

b)

COMMAND WINDOW

```
    0.7174    0.7833    0.8415


>> xs = -1:0.1:1;
>> ys = sin(xs);
>> z = sqrt(2)/2;
>> approxSinCos(xs, ys, z)


ans =


    0.6494


>>
```