# Category $\mathcal{O}$, Problem Set 4

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## 1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let $N$ be a 2-dimensional $U(\mathfrak{b})$-module defined by letting $x$ act as 0 and $h$ act as $\left(\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right)$.

Show that the induced $U(\mathfrak{g})$-module structure $M \coloneqq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

Hence $M \notin \mathcal{O}$.

## 2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

## 3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \bigcap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to $\Delta$.

For a concrete example, take $\Phi$ of type $B_2$ with a short simple root $\alpha$ and a long simple root $\beta$. If $\lambda \coloneqq \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in $\Phi$.

# 4 Humphreys 3.7

## 4.1 a

If a module $M$ has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

## 4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where $M$ has a standard filtration fails to imply that $\operatorname{coker} \phi$ has a standard filtration.