Title

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Recall that the Zariski topology is defined on an affine variety $X = V(J)$ with $J \trianglelefteq k[x_1, \cdots, x_n]$ by describing the closed sets.

Proposition 1.1 *(?).*
X is irreducible if its coordinate ring $A(X)$ is a domain.

Proposition 1.2 *(?).*
There is a 1-to-1 correspondence

$$\left\{ \substack{\text{Irreducible subvarieties} \\ \text{of } X} \right\} \iff \left\{ \substack{\text{Prime ideals} \\ \text{in } A(X)} \right\}.$$

Proof .
Suppose $Y \subset X$ is an affine subvariety. Then

$$A(X)/I_X(Y) = A(Y).$$

By NSS, there is a bijection between subvarieties of X and radical ideals of $A(X)$ where $Y \mapsto I_X(Y)$. A quotient is a domain iff quotienting by a prime ideal, so $A(Y)$ is a domain iff $I_X(Y)$ is prime.

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Recall that $\mathfrak{p} \trianglelefteq R$ is prime when $fg \in \mathfrak{p} \iff f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Thus $\bar{f}\bar{g} = 0$ in $R/\mathfrak{p}$ implies $\bar{f} = 0$ or $\bar{g} = 0$ in $R/\mathfrak{p}$, i.e. $R/\mathfrak{p}$ is a domain.

Finally note that prime ideals are radical (easy proof).

Example 1.1.
Consider $\mathbb{A}^2/\mathbb{C}$ and some subvarieties C_i:

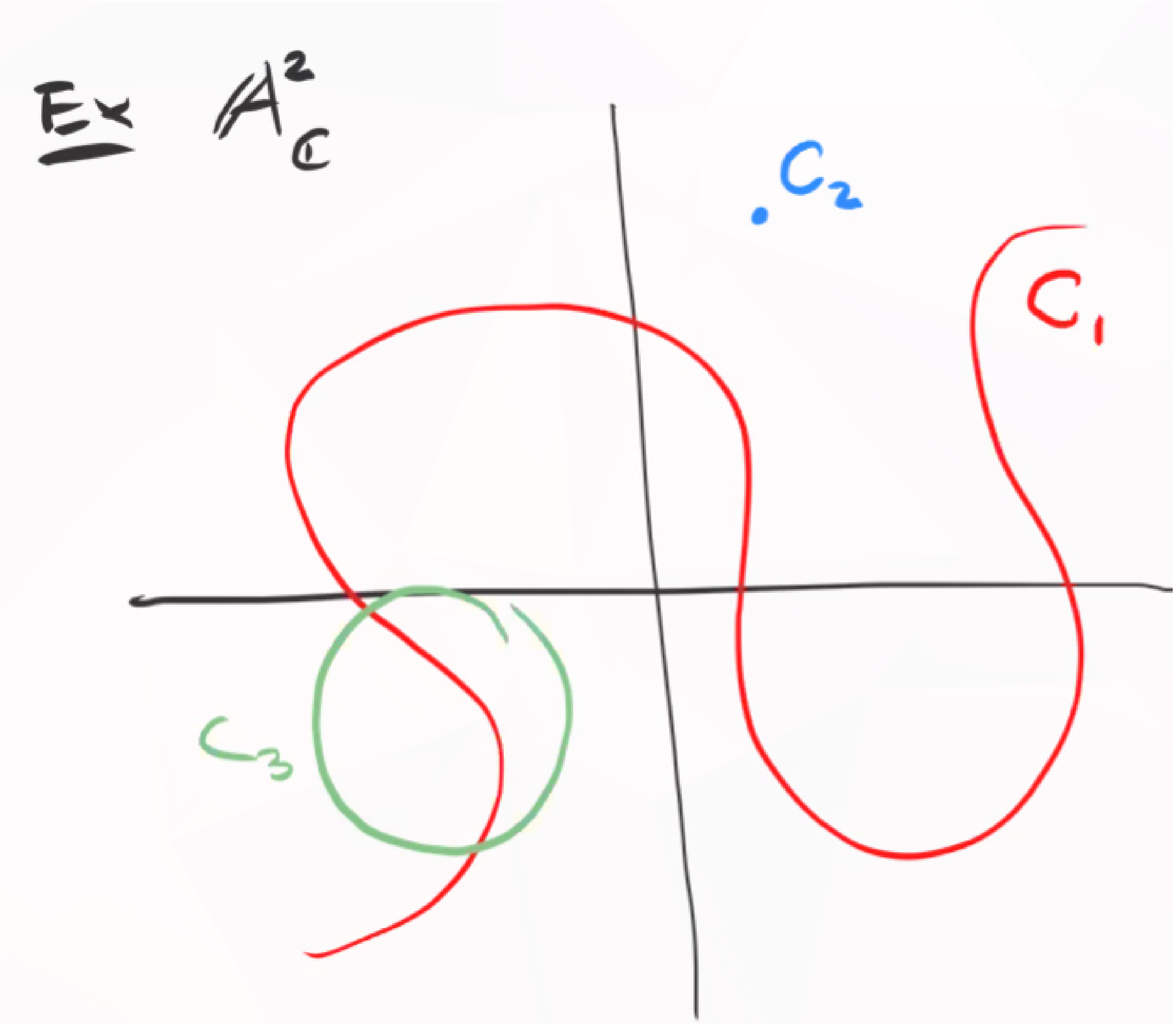

Figure 1: Subvarieties

Then irreducible subvarieties correspond to prime ideals in $\mathbb{C}[x, y]$. Here C_1, C_3 correspond to $V(f), V(g)$ for f, g irreducible polynomials, whereas C_2 corresponds to a maximal ideal, i.e. $V(x_1 - a_1, x_2 - a_2)$.

Note that $I(C_1 \cup C_2 \cup C_3)$ is not a prime ideal, since the variety is reducible as the union of 3 closed subsets.

Example 1.2.
A finite set is irreducible iff it contains only one point.

Example 1.3.

Any irreducible topological space is connected, since irreducible requires a union but connectedness requires a *disjoint* union.

Example 1.4.
A^n/k is irreducible: by prop 2.8, its irreducible iff the coordinate ring is a domain. However $A(\mathbb{A}^n) = k[x_1, \cdots, x_n]$, which is a domain.

Example 1.5.
$V(x_1x_2)$ is not irreducible, since it's equal to $V(x_1) \cup V(x_2)$.

Definition 1.2.1 (Noetherian Space)**.**
A *Noetherian* topological space X is a space with no infinite strictly decreasing sequence of closed subsets.

Proposition 1.3*(?).*
An affine variety X with the zariski topology is a noetherian space.

Proof .
Let $X_0 \supsetneq X_1 \supsetneq \cdots$ be a decreasing sequence of closed subspaces. Then $I(X_0) \subsetneq I(X_1) \subsetneq$. Note that these containments are strict, otherwise we could use $V(I(X_1)) = X_1$ to get an equality in the original chain.
Recall that a ring R is Noetherian iff every ascending chain of ideals terminates. Thus it suffices to show that $A(X)$ is Noetherian.
We have $A(X) = k[x_1, \cdots, x_n]/I(X)$, and if this had an infinite chain $I_1 \subsetneq I_2 \subsetneq \cdots$ lifts to a chain in $k[x_1, \cdots, x_n]$, which is Noetherian. A useful fact: R noetherian implies that $R[x]$ is noetherian, and fields are always noetherian.

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Remark 1.
Any subspace $A \subset X$ of a noetherian space is noetherian. To see why, suppose we have a chain of closed sets in the subspace topology,

$$A \cap X_0 \supsetneq A \cap X_1 \supsetneq \cdots.$$

Then $X_0 \supsetneq X_1 \supsetneq \cdots$ is a strictly decreasing chain of closed sets in X. Why strictly decreasing: $\cap^n X_i = \cap^{n+1} X_i \implies A \cap^n X_i = A \cap^{n+1} X_i$, a contradiction.

Proposition 1.4*(Important).*
Every noetherian space X is a finite union of irreducible closed subsets, i.e. $X = \displaystyle\bigcup_{i=1}^{k} X_i$. If we further assume $X_i \not\subset X_j$ for all i, j, then the X_i are unique up to permutation.

Proof .

If X is irreducible, then $X = X$ and this holds.

Otherwise, write $X = X_1 \cup X_2$ with X_i proper closed subsets. If X_1 and X_1' are irreducible, we're done, so otherwise suppose wlog X_1' is not irreducible.

Then we can express $X = X_1 \cup (X_2 \cup X_2')$ with $X_2, X_2' \subset X_1'$ closed and proper.

Thus we can obtain a tree whose leaves are proper closed subsets:

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