Linearization Continued

Section 8.4 Follow-Up

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April 2020

Review

- The Floer equation is given by
$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$
- We fixed a solution and lifted it to a sphere:
$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$
- We use the assumption:
For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.
- We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame
$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

Review

- We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

1. *S is linear*
2. *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s, t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \textit{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$O_0 = S(y_1, \cdots, y_{2n}) = \sum^{2n} y_i \left[\frac{\partial Z_i}{} + J(u)\frac{\partial Z_i}{} + (dJ)_u(Z_i)\frac{\partial u}{} \right.$$

Lemma

Lemma: For $p \in W$, $\{Z_i\}$ a unitary basis of $T_p W$,

$$\begin{aligned}
&- \left\langle J(p) \left(dX_t\right)_p \left(Z_i\right), \quad Z_j \right\rangle \\
&+ \left\langle J(p) \left(dX_t\right)_p \left(Z_j\right), \quad Z_i \right\rangle \\
&- \left\langle (dJ)_p \left(X_t\right) Z_i, \qquad Z_j \right\rangle \\
&= 0.
\end{aligned}$$

Claim: This lemma immediately concludes the previous proof?

Proof of Lemma

Extend $\{Z_i\}$ to a chart containing p and use the Leibniz rule to rewrite

$$-\left\langle J(p)\left(dX_t\right)_p\left(Z_i\right), Z_j\right\rangle + \left\langle J(p)\left(dX_t\right)_p\left(Z_j\right), Z_i\right\rangle - \langle (dJ)_p\left(X_t\right) Z_i, Z_j\rangle = 0$$

as

$$-\left\langle J\left(dX_t\right)\left(Z_i\right), Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle + \left\langle J\left(dZ_i\right)\left(X_t\right), Z_j\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle$$

$$= \left\langle J\left[X_t, Z_i\right], Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle.$$

where we'll rewrite the red terms.

Proof of Lemma

Now use

$$X_t\langle JZ_i,\ Z_j\rangle = 0 \implies \langle d\left(JZ_i\right)\left(X_t\right), Z_j\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle = 0.$$

We now rewrite the RHS from before:

$$\langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right), Z_i\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle$$

$$\begin{aligned}
&= \langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right) - J\left(dZ_j\right)\left(X_t\right), Z_i\rangle \\
&= \langle J\left[X_t, Z_i\right], Z_j\rangle - \langle J\left[X_t, Z_j\right], Z_i\rangle \\
&= \omega\left(\left[X_t, Z_i\right], Z_j\right) - \omega\left(\left[X_t, Z_j\right], Z_i\right).
\end{aligned}$$

The symmetry follows from ω being closed and

$$\begin{aligned}
0 &= d\omega\left(X_t, Z_i, Z_j\right) \\
&= X_t \cdot \omega\left(Z_i, Z_j\right) - Z_i \cdot \omega\left(X_t, Z_j\right) + Z_j \cdot \omega\left(X_t, Z_i\right) \\
&\quad - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) - \omega\left(\left[Z_i, Z_j\right], X_t\right) \\
&= -X_t \cdot \langle Z_i, JZ_j\rangle + Z_i \cdot \left(dH_t\right)\left(Z_j\right) - Z_j \cdot \left(dH_t\right)\left(Z_i\right) \\
&\quad - \left(dH_t\right)\left(\left[Z_i, Z_j\right]\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\
&= d\left(dH_t\right)\left(Z_i, Z_j\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\
&= -\omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right). \qquad \blacksquare
\end{aligned}$$

Linearization of Hamilton's Equation

Recall

$$(d\mathcal{F})_u = \bar{\partial} Y + SY = \left(\bar{\partial} + S\right) Y$$

Now think of S as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

$$\partial_t Y = J_0 S^\pm Y$$

is a linearization of Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad \text{at} \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \longrightarrow \infty} u & \text{for } S^+ \end{cases} \quad \text{respectively.}$$

Proof

We first linearize Hamilton's equation at x:

$$\frac{\partial z}{\partial t} = X_t(z) \quad \overset{\text{linearized}}{\Longrightarrow} \quad \frac{\partial Y}{\partial t} = (dX_t)_x Y.$$

So write $Y = \sum y_i Z_i$ to obtain

$$\begin{aligned}
\sum_i \frac{\partial y_i}{\partial t} Z_i &= \sum_i y_i \left(-\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i) \right) \\
&= \sum_i \sum_j y_i \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i), Z_j \right\rangle Z_j \\
&= \sum_i \sum_j y_j \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle Z_i
\end{aligned}$$

$$\implies \frac{\partial y_i}{\partial t} = \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle y_j.$$

Proof

Thus we can rewrite the linearized equation as

$$\frac{\partial Y}{\partial t} = (dX_t)_x Y = B^- \cdot Y, \quad b_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x (Z_j), Z_i \right\rangle.$$

Recall

$$A \coloneqq A(y_1, \dots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i) \right).$$

Now take $s \longrightarrow -\infty$ and look at the order zero part of $(d\mathcal{F})_u$:

$$\begin{aligned} A\left(\sum y_i Z_i\right) &= \sum_i \left(J(x) \frac{\partial Z_i}{\partial t} - J(x) (dX_t)_x (Z_i) \right) \\ &= \sum_i \sum_j y_i \left\langle J \frac{\partial Z_i}{\partial t} - J (dX_t)(Z_i), Z_j \right\rangle Z_j \\ &= \sum_i \sum_j y_j \left\langle J \frac{\partial Z_j}{\partial t} - J (dX_t)(Z_j), Z_i \right\rangle Z_i \\ &= \sum_i \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), JZ_i \right\rangle_{y_j} Z_i. \end{aligned}$$