Teichmuller Theory in Outer Space (Mladen Bestvina)

F_n : Free group of rank n

$Out(F_n) = Aut(F_n)/Inn(F_n)$

$\leftrightsquigarrow MCG(\Sigma)$ a hyperbolic surface

$T(\Sigma) :=$ Teichmuller space

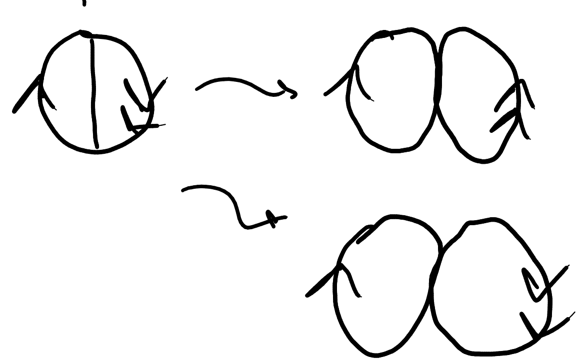
$CV_n \leftrightsquigarrow T(\Sigma)$ where

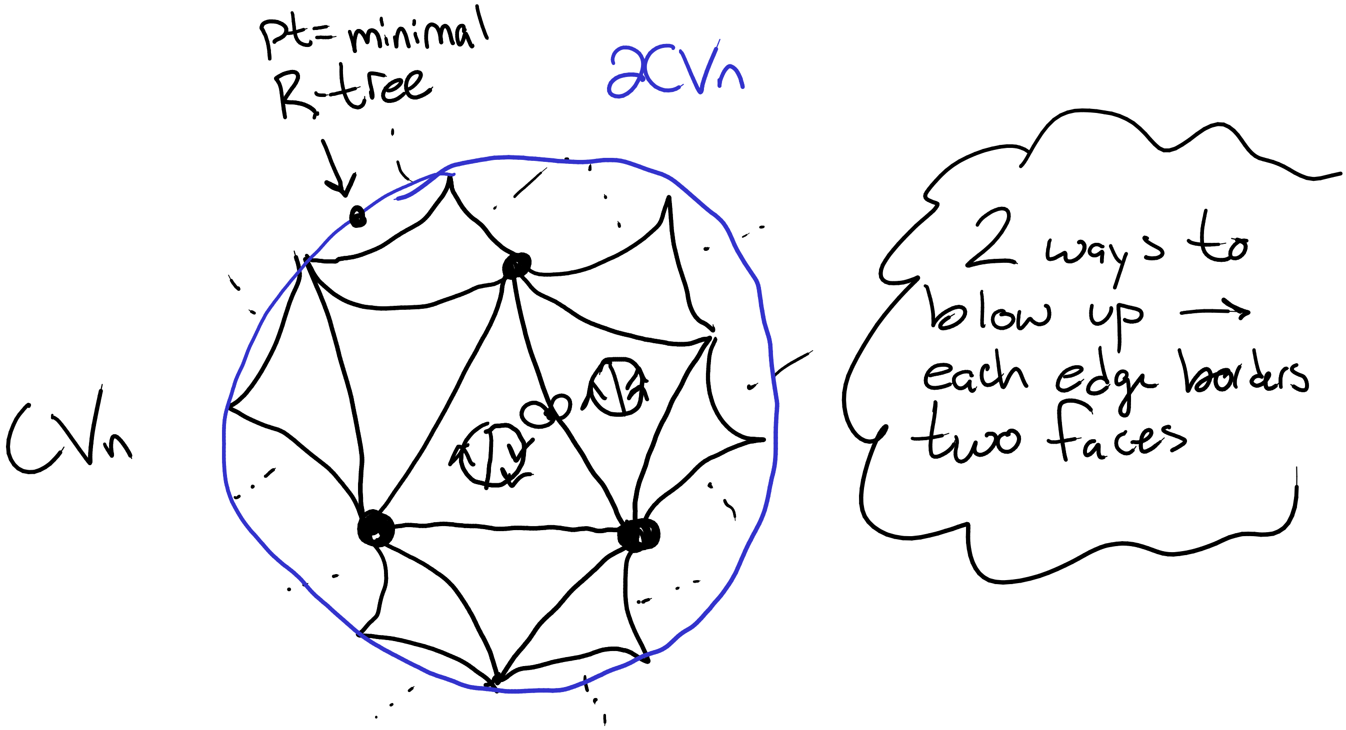
$CV_n = \{(g, \Gamma) \mid \Gamma$ is a metric graph, $vol(\Gamma) = 1$, $g . R_n \simeq \Gamma\}$

<u>Note</u>: g is a marking that we can homotope around.

$CV_n :=$ "Outer space"

$n = 2$, collapse 2 ways





<u>Thm</u> CV_n is contractible

There is a metric-preserving action given by changing the marking.

Now to compactify

$\partial CV_n \longleftrightarrow$ PML projectivized measured lamination

Can think of

$CV_n = \{$Free simplicial minimal F_n-trees$\}$ / $\underbrace{\text{Homotopy}}_{\text{normalization}}$

$\hookrightarrow \{$Minimal F_n - R tree$\}$

where $\mathbb{R}$ trees: any two points have a unique arc isometric to an interval.

Thm $\overline{CV_n}$ (the closure) is compact

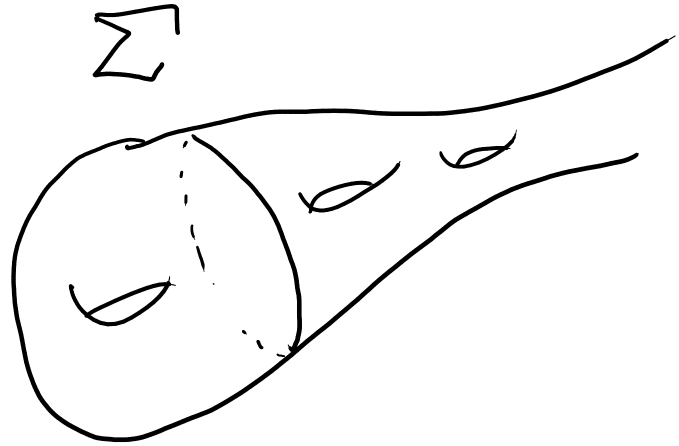
$\{$Minimal F_n - $\mathbb{R}$ tree$\}$

$\rightsquigarrow \ell : \{$Conj classes in $F_n\} \to \mathbb{R}_n$

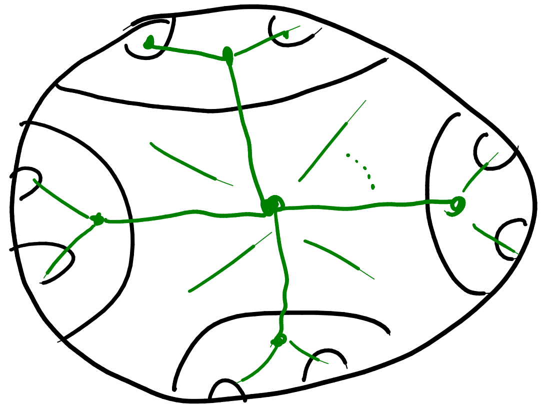
$\ell(\gamma)$ = tr. length of γ on T

So $CV_n \hookrightarrow \mathbb{P}^\infty$

Ex: Σ



Take universal cover = $\mathbb{H}$



Take dual tree: pt for each complementary component, edge across each sep. line

Can equip Σ' with a measure and lift triangles to yield a foliation (measurable arcs)

Not every $T \in \partial CV_n$ comes from such a construction on some Σ'. Why?

1) $\exists T$ dual to measured laminations on a finite 2-complex

See Rips machine, Rauzy induction.

2) $\exists T$ not dual to <u>any</u> such complex

This happens generically.

Lipschitz metric on CV_n $\leftrightsquigarrow$ Teichmuller distance, Thurston dist, WP

Given $(g, \Gamma), (g', \Gamma')$

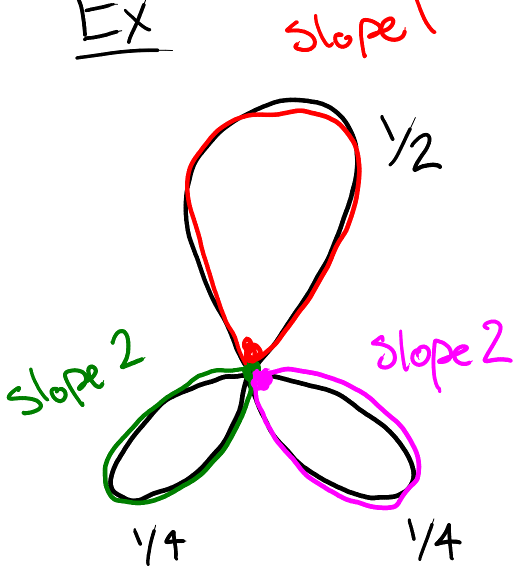
$f : \Gamma \to \Gamma'$ st.

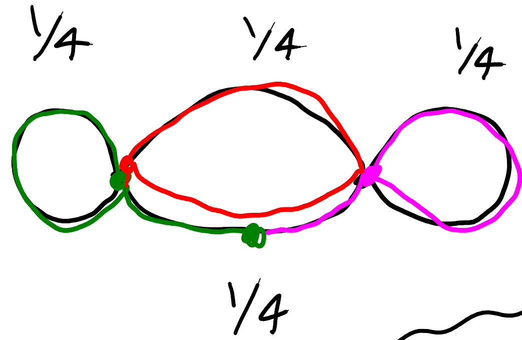
$$\begin{tikzcd} \Gamma \arrow[rr, "f"] & & \Gamma' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] \arrow[u, phantom, "\simeq"] & \end{tikzcd}$$

$$\sigma(\Gamma, \Gamma') := \min_{f} \underbrace{\mathrm{Lip}(f)}_{\text{Lipschitz const}}$$

$$\to d(\Gamma, \Gamma') = \log(\sigma(\Gamma, \Gamma')).$$

Ex



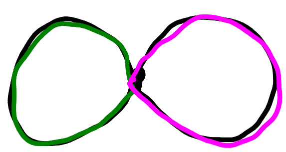


max flow - min cut principle, can take sups/infs for equality

Define $\Delta \subseteq \Gamma$ the tension graph

$= \bigcup \{\text{edges where slope}(f) \text{ is max}\}$

Δ:

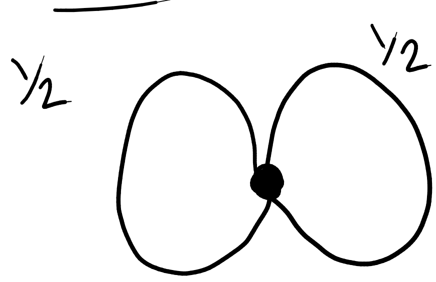


naturally carries a train track structure

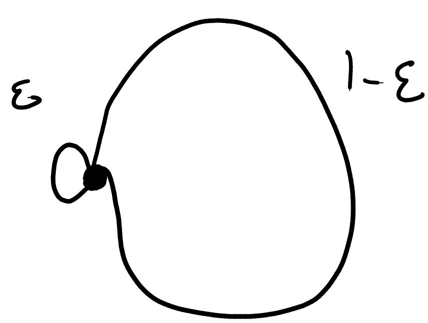
For loops α, set $\alpha \sim \alpha'$ iff $DF(\alpha) = DF(\alpha')$

Note: not symmetric

Ex:



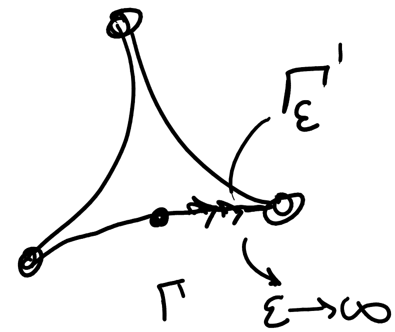
Γ



Γ'_ε

$$d(\Gamma_\varepsilon', \Gamma) \approx \frac{1}{2\varepsilon} \to \infty$$

$$d(\Gamma, \Gamma_\varepsilon') \leq \log 2$$



Yields geodesics

Let Γ be a finite simple graph, with integer edge lengths. Define a group

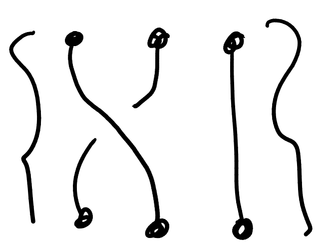
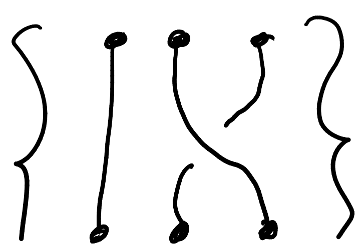
$$A_\Gamma = \langle V(\Gamma) \mid \{ (ab)^{m_{ab}} = (ba)^{m_{ba}} : (a,b) \in E(\Gamma) \} \rangle$$

Don't know much about it generally; center, homology, etc = ?.

$$C_\Gamma = \langle V(\Gamma) \mid R(A_\Gamma) \cup \{ v_i^2 = e \} \rangle$$

the Coxeter group

Ex $B_3 = \langle a, b \mid aba = bab \rangle$

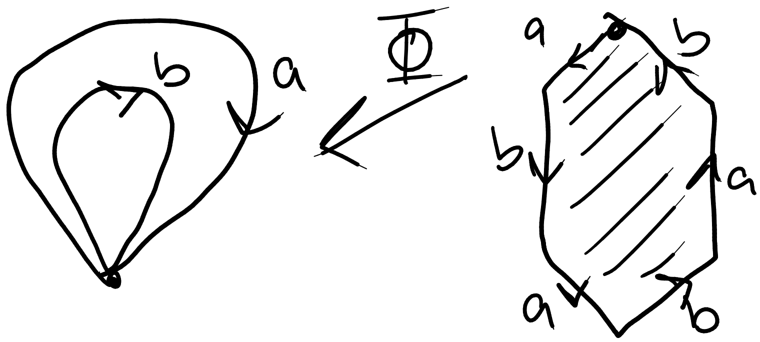
$a =$  , $b =$ 

Note $B_3 \hookrightarrow S_3$

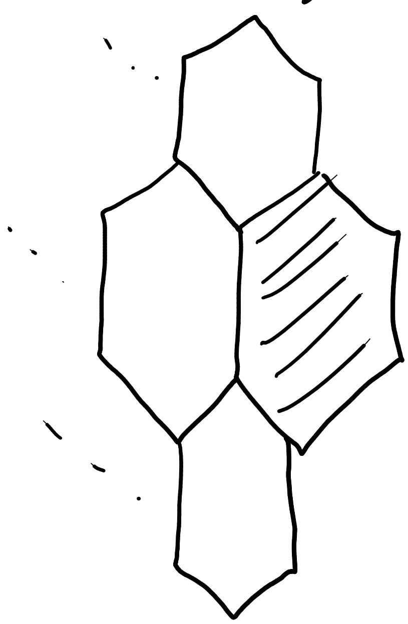
Conjecture: Artin groups are non-positively curved

(Known in special cases)

Can construct a CW complex X with $\pi_1(X) = B_3$; want to put a metric on $\tilde{X}$.



$\tilde{X}$ yields a tiling by "hexagons" but really rectangles after identification

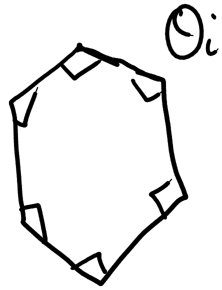


So declare right angles & straight lines... naive method yields a conflict. How to resolve? Restrict to planar regions.

Appel-Schup:

$\longrightarrow \tilde{X}_P$

Define area = # of two cells; Fix boundary and take min. Can produce a non-positively curved metric on D



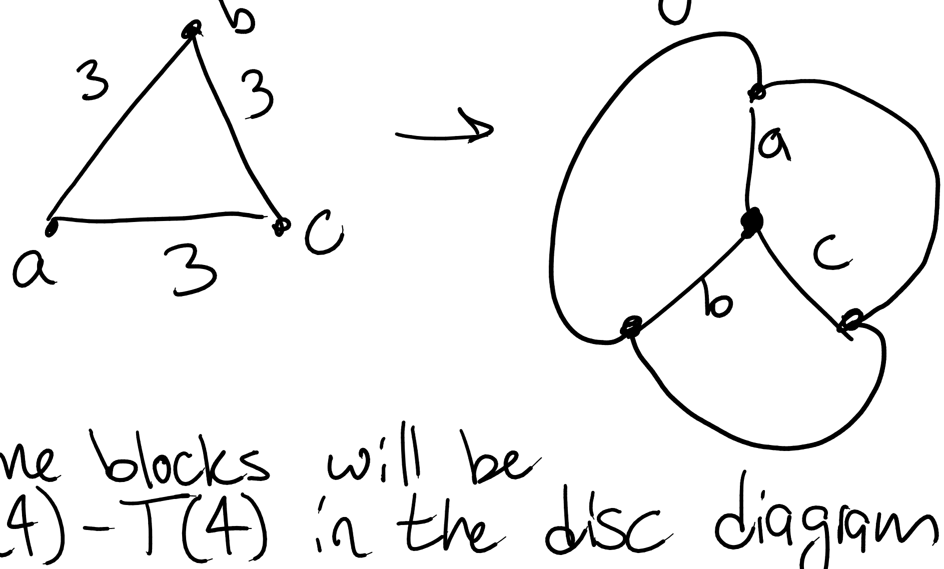
sides ≥ 4

$\Rightarrow \sum \theta_i \geq 2\pi$

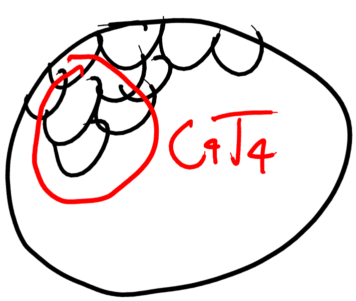
Thm (Pride 1986) If Γ does not have triangles, then

$$\widetilde{X}_\Gamma = C(4)-T(4)$$

What if there are triangles?

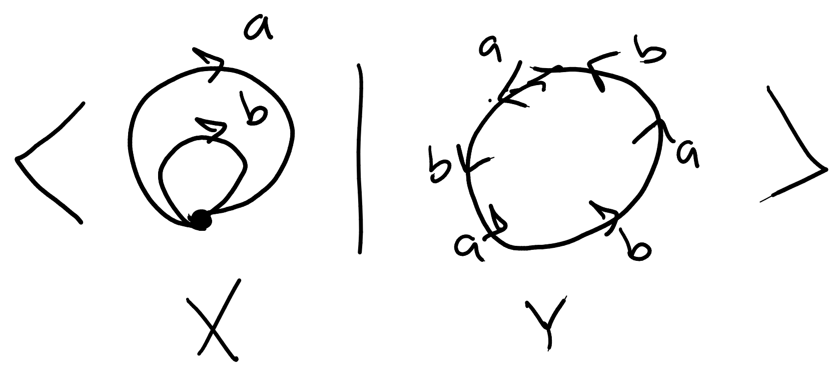


Some blocks will be $C(4)-T(4)$ in the disc diagram



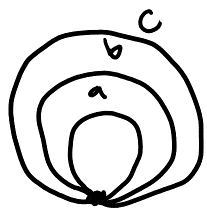
$\rightarrow \widetilde{X}_P$

Cone them off, put a metric on what remains



Yields an immersion $f: Y \to X$

Take $X \sqcup_f CY$, gives $\pi_1 = B_3$



$\langle \; \cdot \; \mid Y_1, Y_2, Y_3 \rangle$

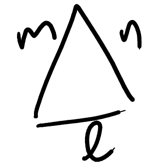
$Y_1 = \widetilde{X}_{ab}^{(1)}$, $Y_2 = \widetilde{X}_{bc}^{(1)}$, $Y_3 = \widetilde{X}_{ca}^{(1)}$

$$\boxed{Z = X \sqcup_{f_i} Y_i}$$

Thm (Appel-Schupp)
If A_Γ is of large type ($m_{ij} \geq 3$) then Z is $C(6)$.

Another class of Artin groups: "2 dim"

Thm TFAE

- For every 

$\subseteq P$, $\frac{1}{m}+\frac{1}{n}+\frac{1}{\ell} \leq 1$
- Cohomological dim $(P) \leq 2$
- $\nexists\, \mathbb{Z}^3 \leq A_P$

A a similar coning procedure, modified Deligne complex

Start with $P = \{g_1\{e\}, g_2 A_{v(p)}, g_3 A_e : e \in P\}$
a poset, take geom. realization

Simplex in $|P|$ $\leftrightsquigarrow$ Chain in P
$c_1 < c_2 < c_3$

Thm: $|P|$ admits a CAT(0) metric.

Allows coning off NPC blocks, left with something ???
?

Question: Is A_P non-positively curved?

The Cohomology of the mapping class group (Andrew Putman)

Let Σ_g be a closed genus g surface

$\mathrm{Mod}_g = \mathrm{Diff}^+(\Sigma_g)/\text{isotopy}$

Topic: $H^n(\mathrm{Mod}_g) = H^n(B\mathrm{Mod}_g)$?

Thm (Milnor): For any g there is a principal G-bundle $EG \to BG$ s.t.

$$\{\text{p. } G\text{-bundles}/X\} \longleftrightarrow [X, BG]$$

$$f^*(EG) \longleftarrow f$$

Ex $BGl_n\mathbb{R} = Gr_n(\mathbb{R}^\infty)$

Ex G discrete: G-bundles = regular G-covers

$$\pi_n(G) = [S^n, BG]$$

$$= G\text{-covers of } S^n$$

$$= \begin{cases} G, & n=1 \\ 0, & \text{else} \end{cases} \quad (S^{n>1} \text{ simply connected})$$

So $BG \simeq K(G,1)$.

Recognizing BG: unique bundle (up to hty) $EG \to BG$ with EG contractible

Thm (Earle-Eells): For $g \geq 2$, $\mathrm{Diff}_0(\Sigma_g) \simeq *$

So Mod_g-Bundles $= \mathrm{Diff}^+(\Sigma_g\text{-bundles})$

$$= \left\{ \begin{array}{c} \Sigma_g \to E \\ \downarrow \\ B \end{array} \quad \text{oriented} \right\}$$

surface bundles

Constructing $B\mathrm{Diff}^+(\Sigma_g)$

By Whitney, the space $\mathrm{Emb}(\Sigma_g, \mathbb{R}^\infty) \simeq *$

$\mathrm{Diff}^+(\Sigma_g) \curvearrowright \mathrm{Emb}(\Sigma_g, \mathbb{R}^\infty)$ free + proper

Define $\mathrm{Gr}(\Sigma_g, \mathbb{R}^\infty) = \mathrm{Emb}(\sim)/\mathrm{Diff}^+(\sim)$

So

$$\begin{array}{ccc} \mathrm{Diff}^+(\Sigma_g) \hookrightarrow & \mathrm{Emb}(\Sigma_g, \mathbb{R}^\infty) & \therefore\!= B\mathrm{Diff}^+(\Sigma_g) \\ & \downarrow & \\ & \mathrm{Gr}(\Sigma_g, \mathbb{R}^\infty) & \end{array}$$

Not much cohom. is known

MMM-classes

$$\kappa_i \in H^{2i}(\mathrm{Mod}_g) = H^{2i}(Gr(\Sigma_g, \mathbb{R}^\infty))$$

Make the tautological

$$\tau = \{(S,p) \in Gr(\Sigma_g, \mathbb{R}^\infty) \times \mathbb{R}^\infty \mid p \in S\}$$

yields

$$\begin{array}{ccc} \Sigma_g & \hookrightarrow & \tau \\ & & \downarrow \\ & & Gr \end{array}$$

Can take vertical tangent bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \hookrightarrow & V \\ & & \downarrow \\ & & \tau \end{array} \quad \text{where } V_{(S,p)} = T_pS$$

and its Euler class $e^i(V) \in H^{2i}(\tau)$

There is an Umkehr map since the fibers are surfs.

$$\pi_!: H^n(\tau) \longrightarrow H^{n-2}(Gr(\Sigma_g, \mathbb{R}^\infty))$$

"integrate along fibers"

So define $\kappa_i := \pi_!(e^{i+1}(V)) \in H^{2i}(\sim)$

$\underline{\text{Ex}}$

$$\begin{tikzcd} \Sigma_g \arrow[r] & M^4 \arrow[d] \\ & \Sigma_h \end{tikzcd}$$

Classified by $[\Sigma_h, BDiff\Sigma_g]$

pull back κ_1 to get $f^*(\kappa)$

$\langle f^*(\kappa), [\Sigma_h]\rangle = 3 \cdot \text{signature}(M_4)$

As always, look at <u>Hirzebirch</u> <u>signature</u> <u>formula</u>

$TM = VM \oplus \pi^* T\Sigma_h$

$\to Sig(M^4) = \langle p_1 VM, [M]\rangle + \boxed{\langle p_1 \pi^* \cdots, \cdots \rangle}^{\to 0}$ pontrayagin class

$= \langle e^2 VM, [M]\rangle$

$\cdots = f^*(\kappa)[\Sigma_h]$

so $\kappa_1 \neq 0$ iff $\exists \Sigma_g$ bundle/surface with nonzero signature (existence of some 4-mfd)

$\underline{\text{Thm}}$: $\mathbb{Q}[\kappa_i] \longrightarrow H^*(Mod_g)$ is an iso in deg $\leq 2/3\, g$ (stable cohomology)

Q: Unstable H^*?

Harer **Thm**: $H^*(\mathrm{Mod}_g) = 0$ for deg $> 4g-5$

$(\mathrm{VCd}(\mathrm{Mod}_g) = 4g-5)$

Harer-Zagier **Thm**: $\chi(\mathrm{Mod}_g)$ is huge (exponential or super in g) and often < 0

$\therefore \exists$ huge odd-dim classes

Cor: $\exists$ a huge amount of unstable cohom

What is known?

Thm (Church-Farb-P, Morita-Sakasai-Suzuki)

$H^{4g-5} = 0$ (nonzero with some complicated local coef system)

Thm (Chan-Galatius-Payne) $H^{4g-6} \neq 0$, but not enough to account for largeness.

Open question: What else is there?

(Dark Matter problem)