Title

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1.1 General Notes

- If flipping logic and not using a direct proof (contradiction, contrapositive, etc), then sign-post/announce it near the beginning of the proof.
- Say what you're assuming at the start of the proof.
- Put any important equations (i.e. major steps of the proof) on their own lines or in displaymath environments.
- Use some whitespace to separate parts of the proof and increase readability.
- Remember that limits of sequences need not exist, but liminfs/limsups always do

  (just may be $\pm\infty$).
- Try to avoid abbreviating the names of major theorems (example: "AP" can stand for many results, not just the Archimedean property!)
- It's not generally true that $a \leq M \implies |a| \leq M$, e.g. take $a = -1$. This only holds $a \geq 0$.
- A generic set may not contain its inf or sup. Example: $\inf \left\{ \frac{1}{n} \right\} = 0$ and $0 \notin \left\{ \frac{1}{n} \right\}$, or $\sup \left\{ 1 - \frac{1}{n} \right\} = 1$ with $1 \notin \left\{ 1 - \frac{1}{n} \right\}$.
- If there exists some element of a set or sequence with a given property, try to say where it comes from and why the property holds for it.
- Similarly, if a property holds for all elements of a set or sequence, try to say why.

1.2 1.a

Proof ($A \implies B$).

- Suppose $\{a_n\}$ is not bounded above.
- Then any $k \in \mathbb{N}$ is not an upper bound for $\{a_n\}$.
- So choose a subsequence $a_{n_k} > k$, then by order-limit laws,

$$a_{n_k} > k \implies \liminf_{k\to\infty} a_{n_k} > \liminf_{k\to\infty} k = \infty.$$

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Proof ($\not A \implies \not B$).

- Suppose $\{a_n\}$ is bounded by M, so $a_n < M < \infty$ for all $n \in \mathbb{N}$.
- Then if $\{a_{n_k}\}$ is a subsequence, we have $a_{n_k} \in \{a_n\}$, so $a_{n_k} < M$ for all $k \in \mathbb{N}$.
- But then

$$a_{n_k} < M \implies \limsup_{k\to\infty} a_{n_k} \leq M,$$

- Now note that if $\lim_{k\to\infty} a_{n_k}$ exists,

$$\lim_{k\to\infty} a_{n_k} < \limsup_{k\to\infty} a_{n_k} \leq M < \infty,$$

  so every subsequence is bounded and thus can not converge to ∞.

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1.3 3.a

Proof (Using definition (i)).

- Suppose $x_n \leq M$ for all n, we will show that every subsequential limit is also bounded by M.
- Let

$$S := \left\{ x \in \mathbb{R} \;\middle|\; x \text{ is a subsequential limit of } \{x_n\} \right\}$$

  be the set of subsequential limits.
  - Note that $\inf S := \liminf_{n\to\infty} x_n$ by definition (i).
- Let $\{x_{n_k}\} \in S$ be an arbitrary convergent subsequence (since we are only concerned about subsequences with well-defined limits).
- Then for every k we have $x_{n_k} \in \{x_n\}$, so

$$|x_{n_k}| \leq M.$$

- By order limit laws,

$$|x_{n_k}| \leq M \implies \lim_{k\to\infty} |x_{n_k}| \leq M,$$

- Since the map $x \mapsto |x|$ is continuous, using the sequential definition of continuity we can pass the limit through the absolute value to obtain

$$\left| \lim_{k\to\infty} x_{n_k} \right| \leq M.$$

- Since the subsequence was arbitrary, we find that M is an upper bound for S and so $\sup S \leq M$.
- But

$$\inf S \leq \sup S \leq M \implies \inf S \leq M.$$

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Proof (Using definition (ii)).

- Suppose $|x_n| \leq M$ for every n, we will directly show that $\left| \lim_{n\to\infty} \inf_{k\geq n} x_n \right| \leq M$.
- By order-limit laws, for every fixed n we have

$$|x_n| \leq M \iff -M \leq x_n \leq M \implies -M \leq \inf_{k>n} x_k \leq M,$$

  where we've used the fact that $x_n \geq -M$ for all n implies that $\inf_{k\geq n} x_k \geq -M$.
- Again applying order-limit laws,

$$-M \leq \inf_{k\geq n} x_k \leq M \implies -M \leq \lim_{n\to\infty} \inf_{k\geq n} x_k \leq M \iff \left| \lim_{n\to\infty} \inf_{k\geq n} x_{n_k} \right| \leq M.$$

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1.4 3.b

Proof .

- Suppose $\beta < \liminf_n x_n$, where by definition (i) we define $\liminf_n x_n = \inf S$ where S is the set of subsequential limits of $\{x_n\}$.
- Then let $M := \inf S$, so we have $\beta < M$ by assumption, and recall

$$M = \inf S \implies \begin{cases} M \leq x & \forall x \in S \\ M < M' \implies & \exists x \in S \text{ such that } M \leq x \leq M'. \end{cases}$$

- To the contrapositive, suppose $M \leq \beta$; then β is of the form M' above and there thus exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} |x_{n_k} - \beta| = 0 \iff \forall \varepsilon, \exists K \text{ such that } k \geq K \implies |x_{n_k} - \beta| < \varepsilon.$$

- Since $M \leq \beta$ are constants, choose $\varepsilon < \beta - M$

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