

# Weil Conjectures

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## 1 Notes from Daniel's Office Hours

0. Definition of Zeta functions
1. Statement of the conjectures
2. Easy examples:  $\mathbb{P}^n_{\mathbb{F}_q}, \text{Gr}_\gamma(k, n) = \text{GL}(n, \mathbb{F}_q)/P$  the stabilizer of an  $\gamma$ -point in  $\mathbb{C}^n, \mathbb{F}_{p^n}$ .
3. Medium example:  $E/\mathbb{F}_q$  an elliptic curve.
4. Work out a harder example as in Weil

### References

- [http://www-personal.umich.edu/~mmustata/zeta\\_book.pdf](http://www-personal.umich.edu/~mmustata/zeta_book.pdf)
- <https://youtu.be/wEz7fCvK6sM?t=293>
- Explanation of exponential appearing
- <https://arxiv.org/pdf/1807.10812.pdf>
- [http://www.math.canterbury.ac.nz/~j.boomer/expos/weil\\_conjectures.pdf](http://www.math.canterbury.ac.nz/~j.boomer/expos/weil_conjectures.pdf)
- Weil's Paper

### 1.1 Definition of Zeta Function

Fix  $q$  a prime and  $\mathbb{F} := \mathbb{F}_q$  the finite field with  $q$  elements, along with its unique degree  $n$  extensions

$$\mathbb{F}_n := \mathbb{F}_{q^n} = \left\{ x \in \overline{\mathbb{F}_q} \mid x^{q^n} - x = 0 \right\} \quad \forall n \in \mathbb{Z}^{\geq 2}$$

#### Definition 1.0.1.

Let

$$J = \langle f_1, \dots, f_M \rangle \trianglelefteq k[x_0, \dots, x_n]$$

be an ideal, then a *projective algebraic* variety  $X \subset \mathbb{P}_{\mathbb{F}}^N$  can be given by

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}}^{\infty} \mid f_1(\mathbf{x}) = \dots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where an ideal generated by *homogeneous* polynomials in  $n + 1$  variables, i.e. there is some fixed  $d \in \mathbb{Z}^{\geq 1}$  such that

$$f(\mathbf{x}) = \sum_{\substack{\mathbf{I}=(i_1, \dots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathbf{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}).$$

For the experts: we can take a reduced (possibly reducible) scheme of finite type over a field  $\mathbb{F}_p$ . We will be thinking of  $K$ -valued points for  $K/\mathbb{F}_p$  algebraic extensions. From the audience: what condition do we need to put on such a scheme to guarantee an embedding into  $\mathbb{P}^{\infty}$ ?

Examples:

- Dimension 1: Curves
- Dimension 2: Surfaces
- Codimension 1: Hypersurfaces

Fix  $X/\mathbb{F} \subset \mathbb{P}$  an  $N$ -dimensional projective algebraic variety, and say it's cut out by the equations  $f_1, \dots, f_M \in \mathbb{F}[x_0, \dots, x_n]$ . Note that it then has points in any finite extension  $L/K$ .

**Definition 1.0.2.**

Define the *local zeta function* of  $X$  the following formal power series:

$$\zeta_X(z) = \exp \left( \sum_{n=1}^{\infty} \alpha_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad \alpha_n := \#X(\mathbb{F}_n).$$

Concretely, for  $X \subset \mathbb{P}^M$  a variety cut out by  $\{f_i\} \subset \mathbb{F}[x_0, \dots, x_M]$  we are measuring the sizes of the sets

$$\alpha_n := \# \left\{ \mathbf{x} \in \mathbb{P}_{\mathbb{F}_{q^n}}^M \mid f_i(\mathbf{x}) = \mathbf{0} \, \forall i \right\}.$$

Note the following two properties:

$$\begin{aligned} \zeta_X(0) &= 1 \\ z \left( \frac{\partial}{\partial z} \right) \log \zeta_X(z) &= t \left( \frac{\zeta'_X(z)}{\zeta_X(z)} \right) = \sum_{n=1}^{\infty} \alpha_n z^n = \alpha_1 z + \alpha_2 z^2 + \dots, \end{aligned}$$

which is an *ordinary generating function* for the sequence  $(\alpha_n)$ .

Todo: why not an OGF.

Remark: Note that for an OGF  $F(x) = \sum_{n=0}^{\infty} f_n x^n$ , we can extract coefficients in the following way:

$$f_n := [x^n]F(x) = [x^n]T_{F,0}(x) = \frac{1}{n!} \left( \frac{\partial}{\partial x} \right)^n F(x) \Big|_{x=0}.$$

Using the Residue theorem, we can also extract in the following way:

$$[x^n]F(x) = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}}.$$

Note: this is extremely amenable to numerical approximation if you have a closed form for  $F$  or even just a black-box numerical version of  $F$ ! I.e. easy to throw at a computer.

### 1.1.1 Simple but Useful Example: A Point

Take  $X = \{x = 0\} / \mathbb{F}$  a single point over  $\mathbb{F}$ , then

$$\begin{aligned} \#X(\mathbb{F}) &:= \alpha_1 = 1 \\ \#X(\mathbb{F}_2) &:= \alpha_2 = 1 \\ &\vdots \\ \#X(\mathbb{F}_n) &:= \alpha_n = 1 \\ &\vdots \end{aligned}$$

Recall that by integrating a geometric series we can derive

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n &&= 1 + z + z^2 + \dots \\ \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n &&= \sum_{n=0}^{\infty} \int z^n = \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots \\ \implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n}. \end{aligned}$$

and so

$$\begin{aligned} \zeta_{\{\text{pt}\}}(z) &= \exp \left( 1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \dots \right) \\ &= \exp(-\log(1-z)) \\ &= \frac{1}{1-z}. \end{aligned}$$

## 1.2 Statement of Weil Conjectures

(Weil 1949)

Let  $X$  be a smooth projective variety of dimension  $N$  over  $\mathbb{F}_q$  for  $q$  a prime and let  $\zeta_X(z)$  be its zeta function.

### 1. (Rationality)

$\zeta_X(z)$  is a rational function:

$$\zeta_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$

$$P_{2N}(z) = 1 - q^N z$$

$$P_j(z) = \prod_{k=1}^{\beta_j} (1 - a_{j,k} z) \quad \text{for some} \quad a_{j,k} \in \mathbb{C}$$

where we've factored each  $P_i$  using its reciprocal roots  $a_{ij}$ .

### 2. (Functional Equation and Poincare Duality)

Let  $\chi(X)$  be the Euler characteristic of  $X$ , i.e. the self-intersection number of the diagonal embedding  $\Delta \hookrightarrow X \times X$ ; then  $\zeta_X(z)$  satisfies the following *functional equation*:

$$\zeta_X\left(\frac{1}{q^N z}\right) = \pm \left(q^{\frac{N}{2}} z\right)^{\chi(X)} \zeta_X(z).$$

Equivalently,

$$\zeta_X(N - z) = \pm q^{\chi(X) \cdot (\frac{n}{2} - 2)} \zeta_X(z).$$

Equivalently, there is an involutive map on the (reciprocal) roots

$$\begin{aligned} z &\longleftrightarrow \frac{q^N}{z} \\ \alpha_{i,j} &\longleftrightarrow \alpha_{2N-i,j} \end{aligned}$$

which sends roots of  $p_i$  to roots of  $p_{2N-i}$ .

### 3. (Riemann Hypothesis)

The  $a_{j,k}$  are *algebraic* integers (roots of some monic  $p \in \mathbb{Z}[x]$ ) which satisfy

$$|a_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}} \quad \forall 1 \leq j \leq 2N - 1, \forall k.$$

### 4. (Betti Numbers) If $X$ is a “good reduction mod $q$ ” of a nonsingular projective variety $\tilde{X}$ in characteristic zero, then the $\beta_i = \deg p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$ .

Moral: the Diophantine properties of a variety's zeta function are governed by its (algebraic) topology. Conversely, the analytic properties encode a lot of geometric/topological/algebraic information.

Historical note

- Desire for a “cohomology theory of varieties” drove 25 years of progress in AG

Remarks:

- Resolved for varieties over  $\mathbb{F}_q$
- On  $L_X$ :
  - Conjectured for smooth varieties over  $\mathbb{Q}$  (rationality  $\sim$  analytically continues to a meromorphic function, some functional equation), little is known.
  - Resolved for elliptic curves (Taylor-Wiles c/o the Taniyama-Shimura conjecture), implies  $L_X$  is an  $L$  function coming from a modular form.

### 1.2.1 Aside: Why call it a Zeta function?

Knowing the zeta function of a point, we can now make a precise analogy.

Suppose we have an algebraic variety cut out by equations:

$$\mathbb{A}_{\mathbb{Z}}^n \supseteq X = V(\langle f_1, \dots, f_d \rangle) \quad \text{where} \quad f_i \in \mathbb{Z}[x_0, \dots, x_{n-1}].$$

Then for every prime  $q$ , we can reduce the equations mod  $p$  and consider

$$\mathbb{A}_{\mathbb{F}_q}^n \supseteq X_q := V(\langle f_1 \bmod q, \dots, f_d \bmod q \rangle) \quad \text{where} \quad f_i \bmod q \in \mathbb{F}_q[x_0, \dots, x_{n-1}]$$

Then define the *Hasse-Weil* zeta function:

$$L_X(s) = \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}).$$

Take  $X = \text{Spec } \mathbb{Q}$  and  $X_p = \text{Spec } \mathbb{F}_p$ , which is a single point since  $\mathbb{F}_p$  is a field. The previous example shows that

$$\zeta_{X_p}(z) = \frac{1}{1-z},$$

We then find that

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left( \frac{1}{1-p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

which is the Euler product expansion of the classical Riemann Zeta function.

Moreover, it is a theorem (difficult, not proved here!) that for any variety  $X/\mathbb{F}_p$ , we have

$$\zeta_X(t) = \prod_{x \in X_{\text{cl}}} \left( \frac{1}{1 - t^{\deg(x)}} \right) \xrightarrow{t=p^{-s}} \zeta_X(s) = \prod_{x \in X_{\text{cl}}} \left( \frac{1}{1 - (p^{\deg(x)})^{-s}} \right),$$

which we can think of as attaching a “weight” to each closed point,  $|x| := p^{\deg(x)}$ , and the usual Riemann Zeta corresponds to assigning a weight of 1 to each point.

Note that this immediately implies that  $\zeta_X(t) \in \mathbb{Z}[[t]]$  is a *rational* function.

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

After modifying  $\zeta$  to make it symmetric about  $\Re(s) = \frac{1}{2}$  and eliminate the trivial zeros at  $-2\mathbb{Z}$  to obtain  $\widehat{\zeta}(s)$ , there are three relevant properties

- “Rationality”:  $\widehat{\zeta}(s)$  has a meromorphic continuation to  $\mathbb{C}$  with simple poles at  $s = 0, 1$ .
- “Functional equation”:  $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
- “Riemann Hypothesis”: The only zeros of  $\widehat{\zeta}$  have  $\Re(s) = \frac{1}{2}$ .

### 1.2.2 More Examples

**Example (Affine Line):**  $X = \mathbb{A}^1/\mathbb{F}$  the affine line over  $\mathbb{F}$ , then Note that we can write

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1] \mid x_1 \in \mathbb{F}_n \right\}$$

as the set of one-component vectors with entries in  $\mathbb{F}_n$ , so

$$\begin{aligned} X(\mathbb{F}) &= q \\ X(\mathbb{F}_2) &= q^2 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp \left( \sum_{n=1}^{\infty} \frac{q^n}{n} z^n \right) = \frac{1}{1 - qz}.$$

**Example (Affine Space):** Set  $X = \mathbb{A}^m/\mathbb{F}$ , affine  $m$ -space over  $\mathbb{F}$ , so we can just repeat with now  $m$  coordinates

$$\mathbb{A}^1(\mathbb{F}_n) = \left\{ \mathbf{x} = [x_1, \dots, x_m] \mid x_i \in \mathbb{F}_n \right\}$$

Counting yields

$$\begin{aligned} X(\mathbb{F}) &= q^m \\ X(\mathbb{F}_2) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_n) &= (q^n)^m. \end{aligned}$$

Thus

$$\zeta_X(z) = \exp\left(\sum_{n=1}^{\infty} \frac{q^{nm}}{n} z^n\right) = \frac{1}{1 - q^m z}.$$

**Example (Projective Line):**  $X = \mathbb{P}^1/\mathbb{F}$  the projective line over  $\mathbb{F}$ , then we can write use some geometry to write

$$\mathbb{P}_{\mathbb{F}}^1 = \mathbb{A}_{\mathbb{F}}^1 \coprod \{\infty\}$$

as the affine line with a point added at infinity.

We can then count by enumerating coordinates:

$$\begin{aligned} \mathbb{P}^1(\mathbb{F}_n) &= \left\{ [x_1, x_2] \mid x_1, x_2 \neq 0 \in \mathbb{F}_n \right\} / \sim \\ &= \left\{ [x_1, 1] \mid x_1 \in \mathbb{F}_n \right\} \coprod \{[1, 0]\}. \end{aligned}$$

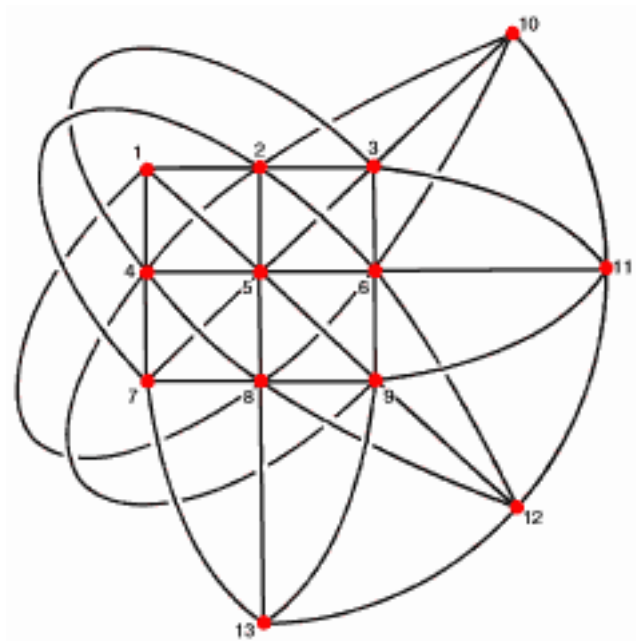
Thus

$$\begin{aligned} X(\mathbb{F}) &= q + 1 \\ X(\mathbb{F}_2) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_n) &= q^n + 1 \\ &\cdot \end{aligned}$$

Thus

$$\zeta_X(z) = \frac{1}{(1 - z)(1 - qz)}.$$

**Example (Projective Space):** Take  $X = \mathbb{P}_{\mathbb{F}}^n$ ,



Example image of  $\mathbb{P}_{\mathbb{GF}(3)}^2$ :

Note that we can identify  $X = \text{Gr}_{\mathbb{F}}(1, n)$  as the space of lines in  $\mathbb{A}_{\mathbb{F}}^n$ .

**Proposition 1.1.**

The number of  $k$ -dimensional subspaces of  $\mathbb{A}_{\mathbb{F}}^m$  is the  $q$ -binomial coefficient:

$$\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

*Proof .*

To choose a  $k$ -dimensional subspace,

- Choose a nonzero vector  $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$  in

$$q^m - 1$$

ways.

- Identify  $\#\text{span}\{\mathbf{v}_1\} = \#\{\lambda \mathbf{v}_1 \mid \lambda \in \mathbb{F}\} = \#\mathbb{F} = q$ .

- Choose a nonzero vector  $\mathbf{v}_2$  *not* in the span of  $\mathbf{v}_1$  in

$$q^m - q$$

ways.

- Identify  $\#\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \#\{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mid \lambda_i \in \mathbb{F}\} = q \cdot q = q^2$ .

- Choose a nonzero vector  $\mathbf{v}_3$  not in the span of  $\mathbf{v}_1, \mathbf{v}_2$  in

$$q^m - q^2$$

ways.



- ... until  $\mathbf{v}_k$  is chosen in

$$(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1})$$

ways.

- This yields a  $k$ -tuple of linearly independent vectors spanning a  $k$ -dimensional subspace  $V_k$
- This overcounts because many linearly independent sets span  $V_k$ , we need to divide out by the number of choose a basis inside of  $V_k$ .
- By the same argument, this is given by

$$(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$$

Thus

$$\begin{aligned} \# \text{subspaces} &= \frac{(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})} \\ &= \frac{q^m - 1}{q^k - 1} \cdot \left(\frac{q}{q}\right) \frac{q^{m-1} - 1}{q^{k-1} - 1} \cdot \left(\frac{q^2}{q^2}\right) \frac{q^{m-2} - 1}{q^{k-2} - 1} \cdots \left(\frac{q^{k-1}}{q^{k-1}}\right) \frac{q^{m-(k-1)} - 1}{q^{k-(k-1)-1}}. \end{aligned}$$

■

We obtain a nice simplification for the number of lines corresponding to setting  $k = 1$ :

$$\begin{bmatrix} m \\ 1 \end{bmatrix}_q = \frac{q^m - 1}{q - 1} = q^{m-1} + q^{m-2} + \cdots + q + 1 = \sum_{j=0}^{m-1} q^j.$$

Thus

$$\begin{aligned} X(\mathbb{F}) &= \sum_{j=0}^{m-1} q^j \\ X(\mathbb{F}_2) &= \sum_{j=0}^{m-1} (q^2)^j \\ &\vdots \\ X(\mathbb{F}_n) &= \sum_{j=0}^{m-1} (q^n)^j. \end{aligned}$$

So

$$\zeta_X(z) = \left(\frac{1}{1-z}\right) \left(\frac{1}{1-qz}\right) \left(\frac{1}{1-q^2z}\right) \cdots \left(\frac{1}{1-q^mz}\right),$$

Note that geometry can help us here: we have a “cell decomposition”  $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$ , and so inductively

$$\mathbb{P}^n = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^n,$$

### 1.3 Hard Example: An Elliptic Curve

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and it's straightforward to prove that

$$\zeta_{X \coprod Y}(z) = \zeta_X(z) \cdot \zeta_Y(z)$$

and recalling that  $\zeta_{\mathbb{A}^j}(z) = \frac{1}{1 - q^j z}$  we have

$$\zeta_{\mathbb{P}^m}(z) = \prod_{j=0}^m \zeta_{\mathbb{A}^j}(z) = \prod_{j=0}^m \frac{1}{1 - q^j z}.$$

Example: Take  $X = \text{Gr}_{\mathbb{F}}(k, n)$ , then ????? so

$$\zeta_X(t) = ?.$$

### 1.3 Hard Example: An Elliptic Curve

The Weyl conjectures take on a particularly nice form for curves. Let  $X/\mathbb{F}$  be a smooth projective curve of genus  $g$ , then

1. (Rationality)

$$\zeta_X(z) = \frac{p(z)}{(1-z)(1-qz)}$$

2. (Functional Equation)

$$\zeta_X\left(\frac{1}{qz}\right) = q^{1-g} z^{2-2g} \zeta_X(z)$$

3. (Riemann Hypothesis)

$$p(t) = \prod_{i=1}^{2g} (q - a_i z) \quad \text{where} \quad |a_i| = \frac{1}{\sqrt{q}}$$

Take  $X = E/\mathbb{F}$ .

Consider the curve  $E$  defined by the following equation:

$$E : y^2 + y = x^3 - x^2$$

This is a cubic, whose graph is presented in Figure 1.

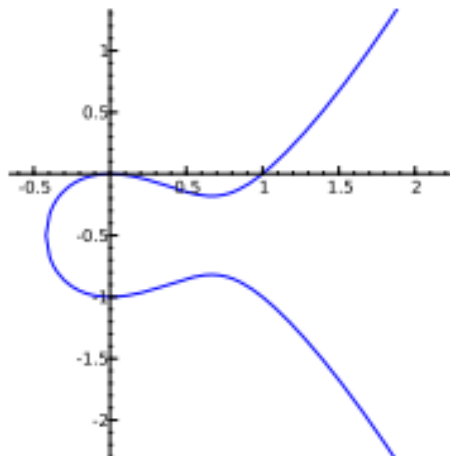


Figure 1: Implicit plot of  $E$

Then

$$\zeta_X(t) = \frac{(1 - aq^{-t})(1 - \bar{a}q^{-t})}{(1 - q^{-t})(1 - q^{1-t})}.$$

The betti numbers are  $[1, 2, 1, 0, \dots]$ .

The number of points are

$$X(\mathbb{F}_n) = (q^n + 1) - (\alpha^n + \bar{\alpha}^n) \quad \text{where} \quad |\alpha| = |\bar{\alpha}| = \sqrt{q}$$

Rough outline of proof:

- ??

The (complex?) dimension of  $X$  is  $N = 1$ , The WC say we should be able to write this as

$$\frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1 - \alpha_{1,1}z)(1 - \alpha_{1,2}z)}{(1-z)(1-qz)}.$$

Since we know the number of points, we can compute

$$\begin{aligned}
 \zeta_X(z) &= \exp \sum_{n=1}^{\infty} \#X(\mathbb{F}_n) \frac{z^n}{n} \\
 &= \exp \sum_{n=1}^{\infty} (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
 &= \exp \left( \sum_{n=1}^{\infty} q^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left( \sum_{n=1}^{\infty} -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
 &= \exp(-\log(1 - qz)) \exp(-\log(1 - z)) \exp(\log(1 - \alpha z)) \exp(\log(1 - \bar{\alpha} z)) \\
 &= \frac{(1 - \alpha z)(1 - \bar{\alpha} z)}{(1 - z)(1 - qz)} \in \mathbb{Q}(z),
 \end{aligned}$$

which is indeed a rational function.

Originally conjectured for curves by Artin Proved by Weil in 1949, proposed generalization to projective varieties Proof had work contributed by Dwork (rationality using p-adic analysis), Artin, Grothendieck (etale cohomology), with completion by Deligne in 1970s (RH)