Title

D. Zack Garza

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- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \hom_{\mathrm{Top}}(X, Y) := \left\{ f : X \to Y \;\middle|\; f \text{ is continuous} \right\}.$$

  - Topologize with the *compact-open* topology: $U \in \hom_T(X, X)$ open iff for every $f \in U$, $f(K)$ is open for every compact $K \subseteq X$.
    - If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \;\middle|\; x \in K \right\} \overset{n \to \infty}{\to} 0 \quad \forall K \subseteq X \text{ compact}.$$

      In words: $f_n \to f$ uniformly on every compact set.
  - If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with $d(f, g) = \sup_{x \in X} (f(x), g(x))$.
- So define $\mathrm{Map}(X, Y) = \hom_{\mathrm{Top}}(X, Y)$ equipped with the compact-open topology.
  - Can immediately consider a lot of interesting spaces by considering $\mathrm{Map}(\,\cdot\,, Y)$:

$$\begin{aligned} X = I := [0,1] &\rightsquigarrow \quad \mathcal{P}Y := \{ f : I \to Y \} = Y^I \\ X = S^1 &\rightsquigarrow \quad \mathcal{L}Y := \left\{ f : S^1 \to Y \right\} = Y^{S^1}. \end{aligned}$$

Note: take basepoints to obtain the base path space PY, the based loop space ΩY.

  - Importance in homotopy theory: the path space fibration $\Omega(Y) \hookrightarrow P(Y) \xrightarrow{\gamma \mapsto \gamma(1)} Y$ (plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences).
  - Adjoint property: there is a homeomorphism

$$\begin{aligned} \mathrm{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \mathrm{Map}(Z, \mathrm{Map}(X, Y)) \\ H : X \times Z \to Y &\iff \tilde{H} : Z \to \mathrm{Map}(X, Y) \\ (x, z) \mapsto H(x, z) &\iff z \mapsto H(\,\cdot\,, z). \end{aligned}$$

Categorically, $\hom(X, \cdot) \leftrightarrow (X \times \cdot)$ form an adjoint pair in Top.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$\pi_0 \mathrm{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},$$

  i.e. two maps f, g are homotopic $\iff$ they are connected by a path in $\mathrm{Map}(X, Y)$.
  * Proof:

$$\mathcal{P}\mathrm{Map}(X, Y) = \mathrm{Map}(I, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(Y \times I, X),$$

    and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.
  * Note that we can interpret the RHS as the space of paths
- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I, \cdot)$: define

$$\mathrm{Map}_I^1(X, Y) := \mathrm{Map}(I, \mathrm{Map}(X, Y)) = \mathcal{P}\mathrm{Map}(X, Y)$$

  and then

$$\begin{aligned} \mathrm{Map}_I^2(X, Y) &:= \mathrm{Map}(I, \mathrm{Map}_I^1(X, Y)) \\ &\cong \mathrm{Map}(I, \mathrm{Map}(I, \mathrm{Map}(X, Y))) \quad = \mathcal{P}(\mathcal{P}(X, Y)) \\ &\cong \mathrm{Map}(I, \mathrm{Map}(Y \times I, X)) \\ &:= \mathcal{P}\mathrm{Map}(Y \times I, X). \end{aligned}$$

  Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!
- Since these are homeomorphisms, everything is invertible, so equip with function composition to form a group.
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