Title

D. Zack Garza

November 26, 2019

Contents

1 Tuesday November 26th

Question: Let $f \in L^1([a,b])$ and $F(x) = \int_a^x f(y)\, dy$ – is F differentiable a.e. and $F' = f$?

If f is continuous, then absolutely yes.

Otherwise, we are considering

$$\frac{f(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)\, dy \to_? f(x)$$

so the more general question is

$$\lim_{m(I)\to 0, x\in I} \frac{1}{m(I)}\int_I f(y)\, dy =_? f(x) \text{ } a.e.$$

Note that if f is continuous, since $[a,b]$ is compact, we have uniform continuity and $\frac{1}{m(I)}\int_I (f(y) - f(x))\, dy < \frac{1}{m(I)}\int_I \varepsilon$.

1.1 Lebesgue Differentiation Theorem

Theorem: If $f \in L^1(\mathbb{R}^n)$ then

$$\lim_{m(B)\to 0, x\in B} \int \frac{1}{m(B)}\int_B f(y)\, dy = f(x) \text{ } a.e.$$

> Note: although it's not obvious at first glance, this really is a theorem about differentiation.

Corollary (Lebesgue Density Theorem): For any measurable set $E \subseteq \mathbb{R}^n$, we have

$$\lim_{r\to 0} \frac{m(E \bigcap B_r(x))}{m(B_r(x))} = 1 \text{ } a.e.$$

Proof: Let $f = \chi_E$ in the theorem.

Proof of theorem: We want to show

$$Df(x) := \limsup_{m(B)\to 0, x\in B} \left| \frac{1}{m(B)} \int_B (f(y) - f(x))\, dy \right| \to 0$$

Note that we can replace the lim sup with $\lim_{\varepsilon\to 0} \sup_{0\leq m(B)\leq\varepsilon, x\in B}$, which is well defined as it is a decreasing sequence of numbers bounded below by zero.

Next we'll introduce that *Hardy-Littlewood Maximal Function*, given by

$$Mf(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy$$

> Exercise: show that this is a measurable function. (Hint: it's easy to show that the appropriate preimage is open.)

Theorem (Hardy-Littlewood Maximal Function Theorem): Let $f \in L^1(\mathbb{R}^n)$, then

$$m(x \in \mathbb{R}^n \ni Mf(x) > \alpha) \leq \frac{3^n}{\alpha} \|f\|_1.$$

> Idea: if you look at all balls intersecting a given ball of radius α, the worst case is that the other ball doesn't intersect and is of the same radius. But then you can draw a ball of radius 3α and cover every such intersecting ball.

> Exercise: As a corollary, $Mf(x) < \infty$ *a.e.*

This is called a *weak type* estimate, compared to a strong type $\|Mf\|_1 \leq C\|f\|_1$. Note that by Chebyshev, a strong estimate would imply the weak one because

$$m(\{x \ni mf(x) > \alpha\}) \leq \frac{1}{\alpha}\|Mf\|_1 \not\leq \frac{C}{\alpha}\|f\|_1,$$

which is an inequality that doesn't hold (hence the theorem) because there is an L^1 function for which Mf is *not* L^1.

Proof of differentiation theorem: The goal is to show $Df(x) = 0$ a.e.

We will show that $m(\{x \ni Df(x) > \alpha\}) = 0$ for all $\alpha > 0$.

Some facts:

1. If g is continuous, then $Dg(x) = 0$ a.e. by uniform convergence.
2. $D(f_1 + f_2)(x) \leq Df_1(x) + Df_2(x)$ by applying the triangle inequality and distributing the lim sup.
3. $Df(x) \leq Mf(x) + |f(x)|$

Fix an α and fix an ε. Choose a continuous g such that $\|f - g\|_1 < \varepsilon$. Writing $f = f - g + g$, we have

$$\begin{aligned} Df(x) &\leq D(f-g)(x) + Dg(x) \\ &= D(f-g)(x) + 0 \\ &\leq M(f-g)(x) + |(f-g)(x)|., \end{aligned}$$

Then $Df(x) \geq \alpha \implies M(f-g)(x) \geq \frac{\alpha}{2}$ or $|(f-g)(x)| \geq \frac{\alpha}{2}$. So we have $\{x \ni Df(x) > \alpha\} \subseteq \{x \ni M(f-g)(x) > \frac{\alpha}{2}\} \bigcup \{x \ni |f(x) - g(x)| > \frac{\alpha}{2}\}$. Applying measures turns this into an inequality.

But then applying the maximal function theorem, we have

$$
\begin{aligned}
m(\{x \ni Df(x) > \alpha\}) &\leq \frac{3^n}{\alpha/2} \|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \\
&\leq \varepsilon(\frac{2(3^n+1)}{\alpha}).
\end{aligned}
$$

□

> Note that somehow proving the maximal function theorem here really paved the way, and allows some generalization. Here we computed an average over a solid ball, but there is a notion of surface measure, so we can consider averaging over the surface of spheres, which can include more exotic objects like spheres in $\mathbb{Z}^d$.

Proof of HL Maximal Function Theorem: Let $E_\alpha = \{x \ni Mf(x) > \alpha\}$. If $x \in E_\alpha$, then it follows that there is a B_x such that $\frac{1}{m(B_x)} \int_{B_x} |f(y)|\ dy > \alpha \iff m(B_x) < \frac{1}{\alpha} \int_{B_x} |f(y)|\ dy$.

Note that if E_α were compact, there would only be finitely many such balls, so let $K \subseteq E_\alpha$ be a compact subset. We will be done if we can show that $m(K) < \frac{3^n}{\alpha} \|f\|_1$, since we can always find a compact K such that $m(E_\alpha \setminus K)$ is small.