

Problem Set 1

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1 Problem 5

1.1 Part 1

Let $A \in \text{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v, Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \text{Mat}(n, n)$ be positive semi-definite, so

$$\langle v, Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

We'd like to show

$$\langle v, (A + B)v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

which follows directly from

$$\begin{aligned} \langle v, (A + B)v \rangle &= \langle v, Av \rangle + \langle v, Bv \rangle \\ &> \langle v, Av \rangle + 0 \\ &\geq 0 + 0 \\ &= 0. \end{aligned}$$

1.2 Part 2

Let M be a smooth manifold with a maximal smooth atlas \mathcal{A} , and choose a covering of M by charts $\mathcal{C} = \{(U_i, \phi_i) \mid i \in I\} \subseteq \mathcal{A}$ such that $M \subseteq \bigcup_{i \in I} U_i$.

Then choose a partition of unity $\{f_i\}_{i \in I}$ subordinate to \mathcal{C} . In each copy of $\phi_i(U_i) \cong \mathbb{R}^n$, choose the metric given by the identity matrix, $g_{jk}^i := \delta_{jk}$

2 Problem 6

2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M ,

$$X : M \rightarrow TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

given by solving the initial value problems

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(p) \Big|_{s=t} &= X(\phi_t(p)), \\ \phi_0(p) &= p \end{aligned}$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \text{id}_M$, and $\phi_1 \in \text{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f : M \rightarrow M$, we have

$$\Lambda(f) = \sum_k \text{Tr} \left(f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is the induced map on homology, and

$\Lambda(f) \neq 0 \iff f$ has at least one fixed point.

In particular, we have

$$\begin{aligned}\Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M),\end{aligned}$$

the Euler characteristic of M .

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0 , thus

$\left. \frac{\partial}{\partial t} \phi_t(p_0) \right|_{t=1}$ so

$$\begin{aligned}\phi_t(p) &= p \\ \implies \frac{\partial}{\partial t} \phi_t(p) &= \frac{\partial}{\partial t} p = 0 && \text{by differentiating wrt } t \\ \implies \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} &= 0 \Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\ \implies X(\phi_1(p_0)) &:= \left. \frac{\partial}{\partial t} \phi_t(p) \right|_{t=1} = 0 && \text{by definition of } \phi_1\end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X . So X has at least one zero, as desired. \square

2.2 Part 2

The trivial bundle

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^2 \times \mathbb{R}^2 \\ & & \downarrow \scriptstyle s \\ & & S^2 \end{array}$$

has a nowhere vanishing section, namely

$$\begin{aligned}s : S^2 &\rightarrow S^2 \times \mathbb{R}^2 \\ \mathbf{x} &\rightarrow (\mathbf{x}, [1, 1])\end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1, 1]$ to every point.

However, as part 1 shows, the bundle

$$\begin{array}{ccc}
 \mathbb{R}^2 & \longrightarrow & TS^2 \\
 & & \downarrow s \\
 & & S^2
 \end{array}$$

can *not* have a nowhere vanishing section. \square