

# Category O

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Sunday 3<sup>rd</sup> May, 2020

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## List of Definitions

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## List of Theorems

### 1 Definitions

- Indecomposable: doesn't decompose as  $A \oplus B$ . Weaker than irreducible.
- Irreducible: simple, i.e. no nontrivial proper submodules. Implies indecomposable.
- Completely reducible: Direct sum of irreducibles.
- Solvable: Derived series terminates.
- Borel: maximal solvable subalgebra.
- Radical: Largest solvable ideal.
- Semisimple: Direct sum of simple modules.
  - Acts in a diagonalizable way.
- Reductive: Radical equals center.
- Artinian: Descending chain condition on submodules.
- Antidominant weight:  $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ , equivalently  $M(\lambda) = L(\lambda)$ .
- Dominant weight:  $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$ .
- Regular weight:  $\lambda$  is regular iff the isotropy/stabilizer group  $\text{Stab}_W(\lambda) := \{w \in W \mid w\lambda = \lambda\} = 1$ , equivalently  $|W\lambda| = |W|$  so  $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$  for all  $\alpha \in \Phi$ .

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- Singular weight: Not regular.
  - Linked:  $\mu \sim \lambda \iff \mu \in W \cdot \lambda$ , the orbit of  $\lambda$  under  $W$ , a.k.a. the linkage class of  $\lambda$ .
  - Socle: Direct sum of all simple submodules.
  - Radical: Intersection of all maximal submodules, smallest submodule such that quotient is semisimple.
  - Head:  $M/\text{rad}(M)$ .

## 2 List of Notation

- $M(\lambda)$ : Verma Modules
- $L(\lambda)$ : Unique simple *quotient* of  $M(\lambda)$ .
- $N(\lambda)$  the maximal *submodule* of  $M(\lambda)$
- The root system

$$\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \mid [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$$

containing roots  $\alpha$

- Abstractly: spans a Euclidean space,  $\lambda\alpha \in \phi \implies \lambda = \pm 1$ , and closed under reflections about orthogonal hyperplanes.
- $\Phi^+$  the corresponding positive system (choose a hyperplane not containing any root),  $\Phi := \Phi^+ \amalg \Phi^-$ .
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$$s_\alpha(\cdot) := (\cdot) - 2\langle \cdot, \alpha \rangle \frac{\alpha}{\|\alpha\|^2}$$

the corresponding reflection about the hyperplane  $H_\alpha$

- $\mathfrak{g}_\alpha := \left\{ x \in \mathfrak{g} \mid [hx] = \alpha(h)x \ \forall h \in \mathfrak{h} \right\}$  the corresponding root space
- The triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{-\alpha} := \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

- $\Delta$  the corresponding simple system of size  $\ell$ , i.e  $\alpha = \sum_{\delta_k \in \Delta} c_\delta \delta_k$  with  $c_\delta \in \mathbb{Z}^{\geq 0}$ .
- $\Lambda = \left\{ \lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi \right\}$  the integral weight lattice
- $\Lambda^+ = \mathbb{Z}^+ \Omega$  the dominant integral weights
  - $\Omega := \{\bar{\omega}_1, \dots, \bar{\omega}_\ell\}$  the fundamental weights
- $[A : B]$  the composition factor multiplicity of  $B$  in a composition series for  $A$ .
- $(A : B)$  the composition factor multiplicity of  $B$  in a *standard filtration* for  $A$ .
- $\phi_{[\lambda]}$  the integral root system of  $\lambda$

- $\Delta_{[\lambda]}$  the corresponding simple system
- $W_{[\lambda]}$  the integral Weyl group of  $\lambda$
- $\mu \uparrow \lambda$ : strong linkage of weights
- $\mathcal{O}_{\chi_\lambda}$ : the block corresponding to  $\lambda$ .
- $\text{char } M := \sum_{\lambda \in \mathfrak{h}^\vee} (\dim M_\lambda) e^\lambda$  the formal character.

### 3 Useful Facts

- $\lambda$  dominant integral  $\implies w\lambda \leq \lambda$  for all  $W$ .
- The dot action is given by  $w \cdot \lambda = w(\lambda + \rho) - \rho$ .
- For any filtration  $0 \hookrightarrow M^n \hookrightarrow M^{n-1} \hookrightarrow \dots \hookrightarrow M^1 \hookrightarrow M^0 = M$ , we have

$$\text{char } M = \sum_{i=1}^n \text{char } (M^i / M^{i-1}),$$

i.e. the character of  $M$  is the sum of the characters of its composition factors (with multiplicity).

### 4 SL2 Theory

#### Definition 4.0.1.

The group and the algebra:

$$\begin{aligned} \mathfrak{sl}(n, \mathbb{C}) &= \left\{ M \in \text{GL}(n, \mathbb{C}) \mid \det(M) = 1 \right\} \\ \mathfrak{sl}(n, \mathbb{C}) &= \left\{ M \in \text{GL}(n, \mathbb{C}) \mid \text{Tr}(M) = 0 \right\}. \end{aligned}$$

- The usual representation on  $\mathbb{C}^2$ :  $h$  has eigenvalues  $\pm 1$ , yields  $L(1)$ .
- The adjoint representation on  $\mathbb{C}^3$ :  $\text{ad } h = \text{diag}(2, 0, -2)$  with eigenvalues  $0, \pm 2$ , yields  $L(2)$ .

Generated by

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with relations

$$\begin{aligned} [hx] &= 2x \\ [hy] &= -2y \\ [xy] &= h \\ &\cdot \end{aligned}$$

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Some identifications:

$$\begin{aligned}
\Phi &= A_1 \\
\dim \mathfrak{h} &= 1 \\
\Lambda &\cong \mathbb{Z} \\
\Lambda_r &\cong \mathbb{Z}/2\mathbb{Z} \\
W &= \{1, s_0\} \quad \lambda - 2i \xleftrightarrow{s_0} -(\lambda - 2i) \\
\chi_\lambda = \chi_\mu &\iff \mu = \lambda, -\lambda - 2 \quad (\text{linked}) \\
\Pi(M(\lambda)) &= \{\lambda, \lambda - 2, \dots\}.
\end{aligned}$$

For  $\lambda$  dominant integral

$$\begin{aligned}
N(\lambda) &\cong L(-\lambda - 2) \\
\dim L(\lambda) &= \lambda + 1 \\
\Pi(L(\lambda)) &= \{\lambda, \lambda - 2, \dots, -\lambda\} \\
\dim (L(\lambda))_\mu &= 1 \quad \forall \mu = \lambda - 2i.
\end{aligned}$$

- Simple modules are parameterized by dominant integral weights:

$$M(\lambda) \text{ is simple} \iff \lambda \notin \mathbb{Z}^{\geq 0} = \Lambda^+ \iff \dim L(\lambda) = \infty$$

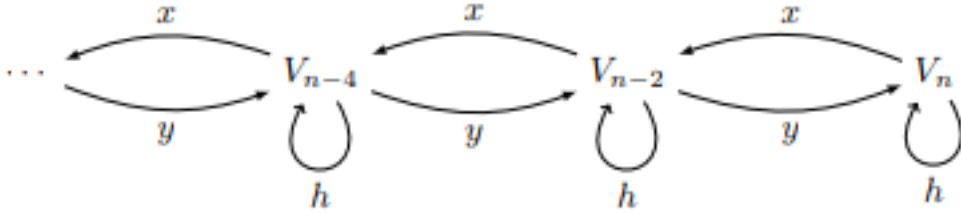


FIGURE 2.2. The action of  $x$  and  $y$  on the eigenspaces of an irreducible  $\mathfrak{sl}_2$ -module.

Finite-dimensional irreducible representations (i.e. simple modules) of  $\mathfrak{sl}(2, \mathbb{C})$  are in bijection with dominant integral weights  $n \in \Lambda$ , i.e.  $n \in \mathbb{Z}^{\geq 0}$ , are denoted  $M(n)$ , and each admits a basis  $\{v_i \mid 0 \leq i \leq n\}$  where

$$\begin{aligned}
h \cdot v_i &= (n - 2i)v_i \\
x \cdot v_i &= (n - i + 1)v_{i-1} \\
y \cdot v_i &= (i + 1)v_{i+1},
\end{aligned}$$

setting  $v_{-1} = v_{n+1} = 0$  and letting  $v_0$  be the unique vector in  $L(n)$  annihilated by  $x$ .

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- $\text{rad } M(\lambda) = N(\lambda)$
  - $\text{hd } M(\lambda) = L(\lambda)$ .
  - $M(\lambda)$  for  $\lambda > 0$  not integral is simple, however  $-\lambda - 2 \notin W \cdot \lambda$ .
  - $\lambda \geq 0 \implies \text{char } L(\lambda) = \text{char } M(\lambda) - \text{char } M(s_\alpha \cdot \lambda)$  where  $s_\alpha \cdot \lambda = -\lambda - 2$ .
  - For  $\lambda \geq 0$ ,  $\dim L(\lambda) = \lambda + 1$  and so

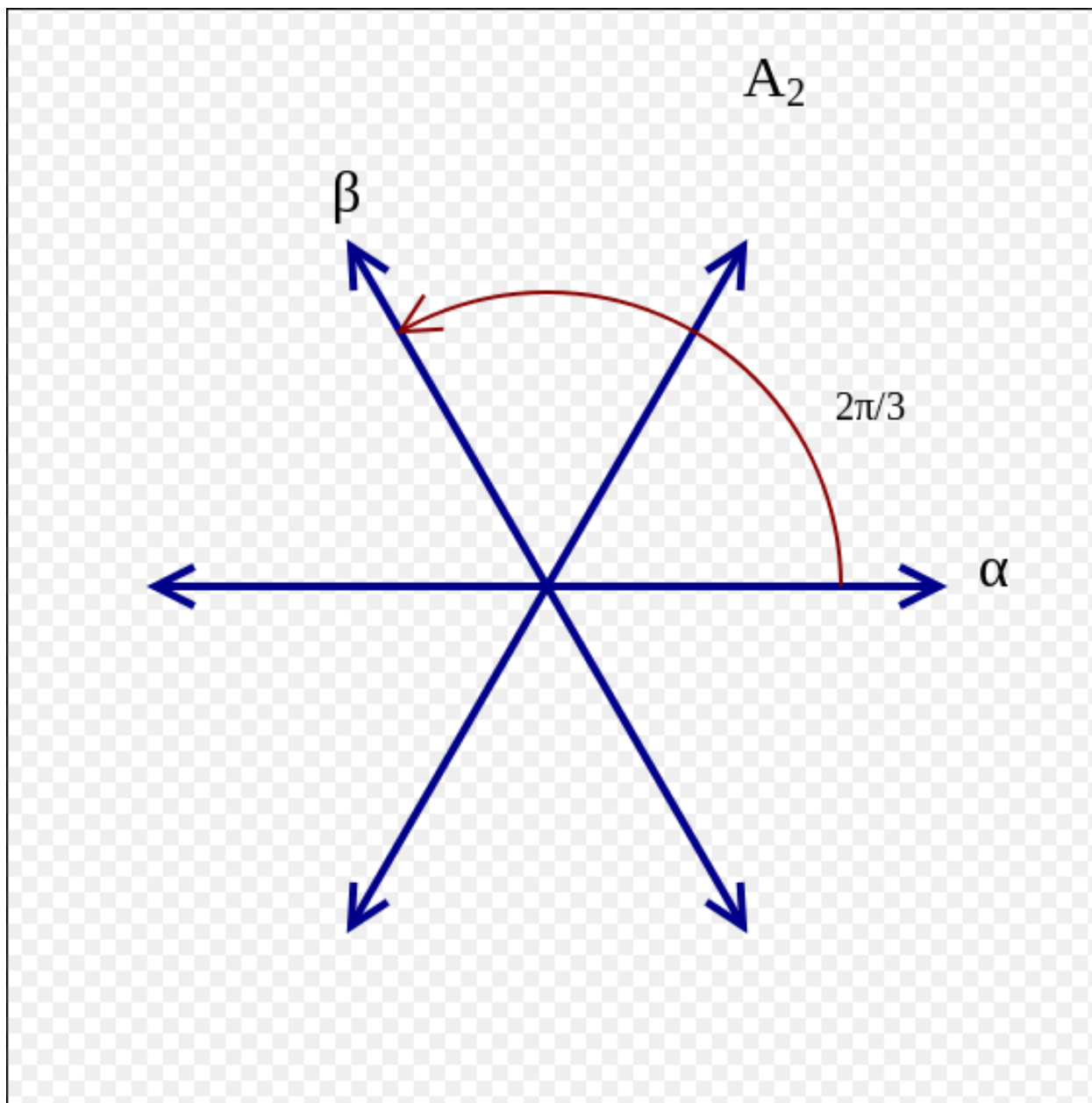
$$\text{char } L(\lambda) = e^\lambda + e^{\lambda-2} + \dots + e^{-\lambda} = \frac{e^{\lambda+1} - e^{\lambda-1}}{e^1 - e^{-1}}.$$

- For  $\lambda \neq \rho \in \mathbb{Z}$ , the composition factors of  $M(\lambda)$  are  $M(\lambda), L(-\lambda - 2)$ .
- There is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(\lambda) & \longrightarrow & M(\lambda) & \longrightarrow & L(\lambda) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & L(-\lambda - 2) & \longrightarrow & M(\lambda) & \longrightarrow & L(\lambda) & \longrightarrow & 0 \end{array}$$

## 5 SL3

$\mathfrak{sl}(3, \mathbb{C})$  has root system  $A_2$ :



$$\Delta = \{\alpha, \beta\}$$

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, w_0\}.$$

For  $\lambda$  regular, integral, and antidominant:

- $M(\lambda) = L(\lambda)$
- No  $M(w \cdot \lambda)$  is simple, all have  $L(\lambda) = M(\lambda)$  as unique simple submodule.
- $[M(w \cdot \lambda) : L(\lambda)] = [M(w \cdot \lambda) : L(w \cdot \lambda)] = 1$  for all  $w$ .
- $\text{char } L(s_\alpha \cdot \lambda) = \text{char } M(s_\alpha \cdot \lambda) - \text{char } M(\lambda)$ .
- $\text{char } M(s_\alpha \cdot \lambda) = \text{char } L(s_\alpha \cdot \lambda) + \text{char } L(\lambda)$ .

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- The Jantzen filtration when  $w \in \{s_{\alpha\beta}, s_{\beta\alpha}, w_0\}$  is given by

$$M(w \cdot \lambda)^0 = M(w \cdot \lambda)$$

$$M(w \cdot \lambda)^1 = ?$$

$$M(w \cdot \lambda)^2 = L(\lambda)$$

$$M(w \cdot \lambda)^{\geq 3} = 0.$$