Topology Problems Q2

Mayer Vietoris (Sheet 7)

1. Compute the homology of:
    1. $\mathbb{RP}^2 = M \bigcup_\partial D^2$
    2. $T^2 = S^1 \times S^1 = (S^1 \times I) \bigcup_f (S^1 \times I)$
       where $(x, 0) \sim (x, 1) \sim (\bar{x}, 0) \in \mathbb{C}$
    3. $S^1 \bigcup_f B^2$ attached along ∂B^2 using $z \mapsto z^n$
2. Show $\tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$
    1. Show $\Sigma S^n \cong S^{n+1}$
3. For $f : S^n \circlearrowleft$, show $\deg f = \deg \Sigma f$
    1. Conclude $\pi_n(S^n) = \mathbb{Z}$
4. Let $\{A_i\}^n \in \mathbf{Ab}$ be finitely generated, show $\exists X \mid H_i(X) \cong A_i$ for $i \leq n$ and 0 otherwise.
5. Suppose $X = \bigcup_i^n A_i$ such that for any $1 \leq k \leq n, \ \bigcap_i^k A_i$ is either empty or contractible, show $i \geq n-1 \implies \tilde{H}_i(X) = 0$ and that this bound is sharp.
6. Compute $H_*(X \times S^n)$ in terms of $H_*(X)$
    1. Compute $H_*(T^n)$
7. Let $M = (S^1 \times B^2) \bigcup_{\mathrm{id}_\partial} (S^1 \times B^2)$ and compute $H_*(M; \mathbb{Z})$
8. Let $X = S^n \times I$ with its ends glued together by a map $S^n \circlearrowleft$ of degree d, calculate $H_*(X)$.
9. Compute $H_*(X)$ for $X = S^3 - N$, with N a knotted solid torus and $\partial N = T$ its boundary torus
10. Let CA be the cone on A, show that $\tilde{H}_*(X \bigcup CA) \cong \tilde{H}_*(X, A)$.
11. Show that the Mayer-Vietoris sequence is natural, i.e. If $X \xrightarrow{f} Y$ where $f(A) \subset C$ and $f(B) \subset D$, then this commutes:

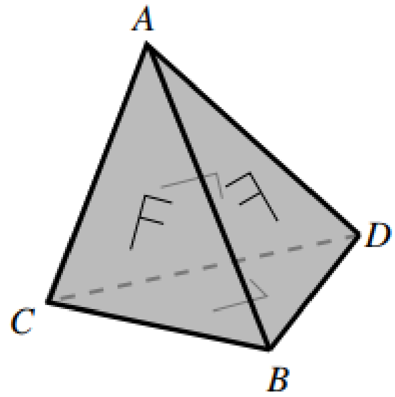
$$\begin{array}{ccccccc} H_n(X) & \longrightarrow & H_n(A \bigcap B) & \longrightarrow & H_n(A) \oplus H_n(B) & \longrightarrow & H_{n-1}(X) \\ \downarrow{f_*} & & \downarrow{f_*} & & \downarrow{f_*} & & \downarrow{f_*} \\ H_n(Y) & \longrightarrow & H_n(C \bigcap D) & \longrightarrow & H_n(C) \oplus H_n(D) & \longrightarrow & H_{n-1}(Y) \end{array}$$

Cellular Homology (Sheet 8)

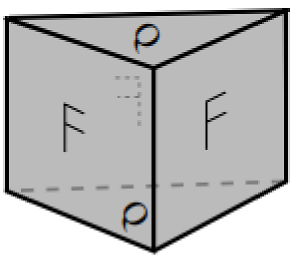
Compute the homology of these spaces

1. $S_m \vee S_n$
    1. $S^m \times S^n$
2. A hexagon with the identifications $a + b + c - a - b - c$

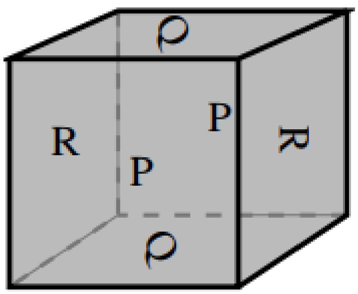
3. Orientable surface of genus g
    1. $g = 2$ is given by $a + b - a - b + c + d - c - d$
4. Nonorientable surface of genus g Obtain by removing g discs from S^2 and attaching g mobius strips
5. $S_1 \vee S_1$ with two discs attached via $(ab)^3$ and $(ab)^6$
6. This identification space:



7. This identification space:



8. This identification space:



9. Describe a CW complex structure for the lens space $L(p, 1)$ and compute π_1, H_* for it.

Degree

1. Let $p(x) = \sum_i^n a_i x^i$, view $p : \mathbb{C} \bigcup \infty \circlearrowleft$ and determine its topological degree
2. Let $p(z) = \frac{\prod_i^n z-a_i}{\prod_j^m z-b_j}$ with all a_i, b_j distinct. What is its topological degree?
3. Show that if $f : S^m \to S^n$ and $\exists U \subset S^m$ such that $f|_U \cong f(U)$, then $m = n$ and f is surjective.

Universal Coefficient Theorem (Sheet 10)

1. Identify the following groups up to isomorphism
    1. $\mathbb{Z}_m \otimes \mathbb{Z}_n$
    2. $\mathbb{Z}_{60}^4 \otimes (\mathbb{Z}_{24}^3 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{120})$
    3. $\mathbb{Z}_n \otimes \mathbb{Q}$
    4. $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Q}/\mathbb{Z})$
2. Compute:
    1. $\mathbf{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$
    2. $\mathbf{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$
3. Compute the following directly from chain complexes and check using UCT:
    1. $H_*(\mathbb{RP}^n; \mathbb{Z}_2)$
    2. $H_*(\mathbb{RP}^n, \mathbb{Z}_3)$
    3. $H^*(\mathbb{RP}^n, \mathbb{Z}_6)$
4. For any space X, show that $H^1(X)$ is free abelian
5. Show that $H_*(X;\mathbb{Q}) = H_*(X;\mathbb{Z}) \otimes \mathbb{Q}$ $H^*(X;\mathbb{Z}) = \mathbf{Hom}(H_*(X;\mathbb{Z}), \mathbb{Q})$
6. Construct a space X such that $H_*(X;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_{12}, \mathbb{Z} \oplus \mathbb{Z}_4, 0 \cdots)$ Compute $H^*(X;\mathbb{Z})$
7. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}_2)$
8. Compute $H_*(\Sigma\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$
9. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z})$
10. Let G be a topological group. Show that $H_*(G)$ is an algebra. Show that $G \curvearrowright H_*(G)$, which factors through the homomorphism $G \to \pi_0(G)$ yielding a trivial action if G is path-connected.

Homological Algebra (Sheet 11)

1. Show that $\ker A \to A \otimes \mathbb{Q}$ given by $a \mapsto a \otimes 1$ is the torsion subgroup of A.
2. Show that $A \hookrightarrow B \implies A \otimes \mathbb{Q} \hookrightarrow B \otimes \mathbb{Q}$
3. Find a free resolution of $\mathbb{Q}$ as a $\mathbb{Z}$-module.
4. Compute $\mathbf{Tor}(\mathbb{Q}, A)$
    1. Compute $\mathbf{Tor}(\mathbb{Q}/\mathbb{Z}, A)$
5.
6. Let $R = \mathbb{Z}[x, y]$, and $M = R/(x-y), N = R/(x, y)$. Construct free resolutions of M, N to compute:
    - $\mathbf{Ext}_R^*(M, M)$
    - $\mathbf{Ext}_R^*(M, N)$
    - $\mathbf{Ext}_R^*(N, M)$
    - $\mathbf{Ext}_R^*(N, N)$
7. Let Λ_* be the exterior algebra generated by the symbols $\{dx_i\}^n$ over a field k. Show that letting $d = \cdot \vee dx_1$ yields a chain complex $0 \to \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^n \to 0$ with trivial homology. Compute what happens when dx_1 is replaced with an arbitrary non-zero element in Λ^1.
8. Define M as the group ring $R = \mathbb{Z}[\mathbb{Z}_2]$ with the action $(\cdot) \times -1$. Construct a free resolution of M and compute $\mathbf{Tor}_R^*(M, M)$.

9. Show $\mathbf{Tor}_R^*(\cdot,\cdot)$ is symmetric in the following way: Given M, N, take free resolutions, view $M_* \to M$ as a chain map and tensor with N_* to get a chain map$\psi : M_* \otimes_R N_* \to M \otimes_R N_*$. Show that ψ is a quasi-isomorphism using the exact sequence $0 \to (Z_n, 0) \to (N_n, 0) \to (B_{n-1}, 0) \to 0$, then switch the roles of M, N.
10. Prove that for a SES $0 \to A \to B \to C$, the group $\mathbf{Ext}(C, A)$ classifies extensions of C by A up to isomorphism.

Cohomology Ring (Sheet 12)

1.