Linearization Continued
Section 8.4 Follow-Up

D. Zack Garza

April 2020

Review

– The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

– We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

– We use the assumption:
For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW), \pi_2(W) \rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

– We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

– We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow \mathcal{P}^{1,p}(x, y)$$

$$Y = (y_1, \ldots, y_{2n}) \longmapsto \exp_u\left(\sum y_i Z_i\right).$$

– We regard $Y(s, t)$ as a tangent vector to W in some Euclidean embedding.

Linearization
Continued

D. Zack Garza

– We seek to compute the composite map in charts:

$$
\begin{array}{ccccc}
& & \mathcal{F}_u & & \\
W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) & \xrightarrow{\ \iota\ } & \mathcal{P}^{1,p}(x,y) \xrightarrow{\ \mathcal{F}\ } L^p\left(\mathbb{R} \times S^1; TW\right) & \longrightarrow & L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right) \\
& & & & \\
& & u \xrightarrow{\qquad\qquad \mathcal{F} \qquad\qquad} & & \dfrac{\partial u}{\partial s} + J(u)\left(\dfrac{\partial u}{\partial t} - X_t(u)\right) \\
& & & & \\
(y_1, \ldots, y_{2n}) & \longrightarrow & \exp_u\left(\sum y_i Z_i\right) & &
\end{array}
$$

Add a Tangent

$$\mathcal{F}(u) = \qquad \frac{\partial u}{\partial s} \qquad +J(u)\frac{\partial u}{\partial t} \qquad -J(u)X_t(u)$$

$$\mathcal{F}(u+Y) = \quad \frac{\partial(u+Y)}{\partial s} \quad +J(u+Y)\frac{\partial(u+Y)}{\partial t} \quad -J(u+Y)X_t(u+Y)$$

Extract the part that is linear in Y and collect terms:

$$(d\mathcal{F})_u(Y)$$

$$= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\,(dX_t)_u(Y)$$

$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right)$$

$$\qquad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\,(dX_t)_u(Y) \right)$$

.

Review

Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$
\begin{aligned}
(d\mathcal{F})_u(Y) &= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t} \right) \\
&\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)(dX_t)_u(Y) \right) \\
&= \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u)Z_i \right) \\
&\quad + \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right. \\
&\qquad\qquad\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right).
\end{aligned}
$$

Use the fact that this is $O_1 + O_0$ in Y.

Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathsf{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s, t)$$

where

1. *S is linear*
2. *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s, t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$O_0 = S(y_1, \cdots, y_{2n}) = \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right.$$

$$\left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right]$$

$$= \sum_{i=1}^{2n} y_i \left[\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t \right) \right.$$

$$\left. + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right].$$

– Claim: the terms in blue and orange vanish in the limit
$s \longrightarrow \pm\infty$, so it suffices to prove that the red term limits to a
symmetric operator.

$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since u is a solution and

$$\frac{\partial u}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

This seems to be the full argument for the blue term.

Proof: Orange Term Vanishes (1 and 3)

Linearization
Continued

D. Zack Garza

$$\frac{\partial Z_i}{\partial s} \overset{s \to \pm\infty}{\longrightarrow} 0$$

Follows since the frame Z_i was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s\,\partial t} \quad \curvearrowright Z_i \overset{s \to \pm\infty}{\longrightarrow} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \overset{s \to \pm\infty}{\longrightarrow} 0.$$

This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in t?

Write the remaining red term as

$$A := A(y_1, \ldots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i) \right).$$

Extract the jth component:

$$A_j = \sum y_i \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i), \quad Z_j \right\rangle.$$

We'll show that

$$\lim_{s \to \pm\infty} \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i), Z_j \right\rangle$$
$$- \left\langle J(u) \frac{\partial Z_j}{\partial t} - J(u) (dX_t)_u Z_j, \quad Z_i \right\rangle = 0.$$

Proof: Symmetry

Use the fact that the frame $\{Z_i\}$ is unitary:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t} \langle J(u) Z_i, Z_j \rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle + \left\langle J(u) Z_i, \frac{\partial Z_j}{\partial t} \right\rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t} \right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle - \left\langle Z_i, J(u) \frac{\partial Z_j}{\partial t} \right\rangle.
\end{aligned}
$$

Proof: Symmetry

Therefore it suffices to show

$$- \left\langle J(u) \left(dX_t \right)_u (Z_i), \quad Z_j \right\rangle$$
$$+ \left\langle J(u) \left(dX_t \right)_u (Z_j), \quad Z_i \right\rangle$$
$$- \left\langle (dJ)_u \left(\frac{\partial u}{\partial t} \right) Z_i, \quad Z_j \right\rangle$$

$$\xrightarrow{s \longrightarrow \pm \infty} 0.$$

Proof: Symmetry

Using the fact that

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

we can equivalently show

$$
\begin{aligned}
&- \left\langle J(u)\left(dX_t\right)_u (Z_i), \quad Z_j \right\rangle \\
&+ \left\langle J(u)\left(dX_t\right)_u (Z_j), \quad Z_i \right\rangle \\
&- \left\langle (dJ)_u \left(X_t\right) Z_i, \quad\quad Z_j \right\rangle
\end{aligned}
$$

$$\overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

For a fixed (s, t), this expression only depends on Z_i at the point $u(s, t)$.

Lemma

Lemma: For $p \in W$, $\{Z_i\}$ a unitary basis of $T_p W$,

$$
\begin{aligned}
&- \left\langle J(p) \left(dX_t\right)_p (Z_i), \quad Z_j \right\rangle \\
&+ \left\langle J(p) \left(dX_t\right)_p (Z_j), \quad Z_i \right\rangle \\
&- \left\langle (dJ)_p \left(X_t\right) Z_i, \quad Z_j \right\rangle \\
&= 0.
\end{aligned}
$$

Claim: This lemma immediately concludes the previous proof?

Proof of Lemma

Extend $\{Z_i\}$ to a chart containing p and use the Leibniz rule to write

$$
\begin{aligned}
&- \langle J(dX_t)(Z_i), \quad Z_j \rangle \\
&+ \langle J(dX_t)(Z_j), \quad Z_i \rangle \\
&+ \langle J(dZ_i)(X_t), \quad Z_j \rangle \\
&- \langle d(JZ_i)(X_t), \quad Z_j \rangle \\
&= \langle J[X_t, Z_i], \quad Z_j \rangle \\
&+ \langle J(dX_t)(Z_j), Z_i \rangle \\
&- \langle d(JZ_i)(X_t), Z_j \rangle.
\end{aligned}
$$