Linearization Continued

Section 8.4 Follow-Up

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Review

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:
  For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

- We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

Review

- We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ Y = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

- We regard $Y(s, t)$ as a tangent vector to W in some Euclidean embedding.

Review

- We seek to compute the composite map in charts:

$$\begin{tikzcd}
W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \arrow[r, "\iota"] \arrow[rrr, bend left, dashed, "\mathcal{F}_u"] & \mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, bend left, dotted, "\bar{\mathcal{F}}"] & L^p\left(\mathbb{R}\times S^1; TW\right) \arrow[r] & L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right) \\
 & u \arrow[rr, "\bar{\mathcal{F}}"] & & \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right) \\
(y_1,\dots,y_{2n}) \arrow[r] & \exp_u\left(\sum y_i Z_i\right) & &
\end{tikzcd}$$

Add a Tangent

$$\begin{aligned}
\mathcal{F}(u) = \quad & \tfrac{\partial u}{\partial s} & +J(u)\tfrac{\partial u}{\partial t} & \quad -J(u)X_t(u) \\
\mathcal{F}(u+Y) = \quad & \tfrac{\partial (u+Y)}{\partial s} & +J(u+Y)\tfrac{\partial (u+Y)}{\partial t} & \quad -J(u+Y)X_t(u+Y)
\end{aligned}$$

Extract the part that is linear in Y and collect terms:

$$\begin{aligned}
&(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\qquad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)
\end{aligned}$$

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Review

- Recall the Leibniz rule

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}
(d\mathcal{F})_u(Y) &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\
&\quad + \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) \left(dX_t\right)_u (Y) \right) \\
&= \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) \\
&\quad + \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} \right. \\
&\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).
\end{aligned}$$