# Lie Algebras

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## Contents

# 1 Lecture 1

> The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also
> a useful appendix which has been uploaded to the ELC system online.
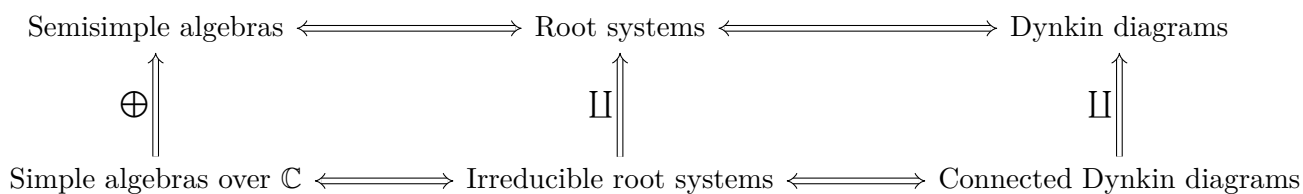
## 1.1 Overview

Here is a short overview of the topics we expect to cover:

### 1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
    - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

### 1.1.2 Chapter 3-4

We will describe the following series of correspondences:

$$
\begin{array}{ccccc}
\text{Semisimple algebras} & \Longleftrightarrow & \text{Root systems} & \Longleftrightarrow & \text{Dynkin diagrams} \\
\Big\Updownarrow{\oplus} & & \Big\Updownarrow{\amalg} & & \Big\Updownarrow{\amalg} \\
\text{Simple algebras over } \mathbb{C} & \Longleftrightarrow & \text{Irreducible root systems} & \Longleftrightarrow & \text{Connected Dynkin diagrams}
\end{array}
$$

## 1.2 Classification

The classical Lie algebras can be essentially classified by certain classes of diagrams:

$E_6, E_7, E_8:$



$F_4:$
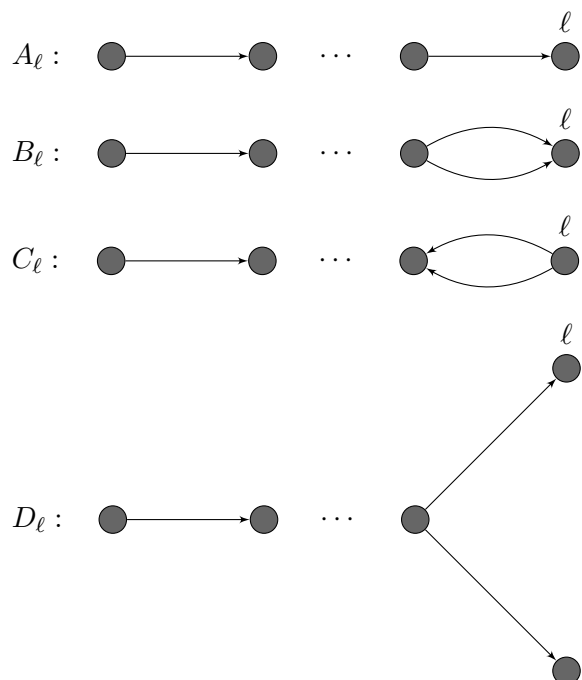
## 1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

### 1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
    - This will be computed for the specific Lie algebras seen earlier

We will also see the type $A_\ell$ algebra used for the first time; however, it differs from the other types in several important/significant ways.

### 1.3.2 Chapter 7

Skip!

### 1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

## 1.4 Content

Fix $F$ a field of characteristic zero – note that prime characteristic is closer to a research topic.

**Definition 1.** A **Lie Algebra** $\mathfrak{g}$ over $F$ is an $F$-vector space with an operation denoted the Lie bracket,

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$(x, y) \mapsto [x, y].$$

satisfying the following properties:

- $[\cdot, \cdot]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

**Exercise 1.** Show that $[x, y] = -[y, x]$.

**Definition 2.** Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

## 1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\operatorname{End}(V) = \{f : V \to V \ni V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

**Exercise 2.** Verify that $V$ is a Lie algebra.

**Definition 3.** Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \ni \operatorname{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group* $\operatorname{SL}(n, V)$.).

**Definition 4.** A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

**Definition 5.** The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$
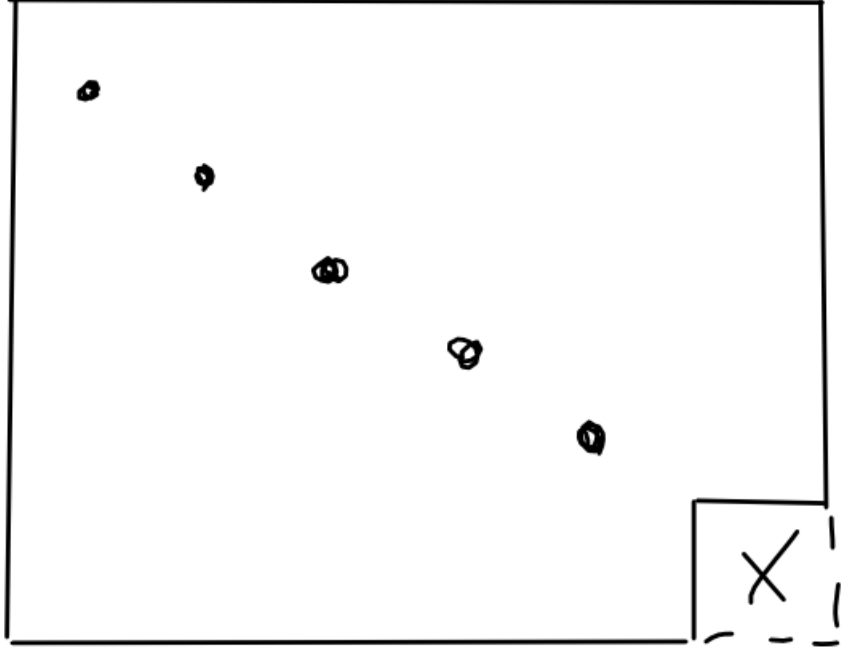
**Definition 6.** The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where}$$

$$M = \begin{cases} \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & 0 & I_n \\ & -I_n & 0 \end{array}\right) & n = 2\ell + 1 \text{ odd,} \\[3em] \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) & \text{else.} \end{cases}$$

**Proposition 1.** The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is $n^2$, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the $i, j$ position and



  zero elsewhere.

- For type $A_\ell$, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$.

- For type $C_\ell$, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type $D_\ell$ we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type $B_\ell$, we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left( \begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

**Exercise 3.** Use the relation $MA = A^{t}M$ to reduce restrictions on the blocks.



**Theorem 1.** These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

## 2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

**Exercise 4.** Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

### 2.1 Lie Algebras of Derivations

**Definition 7.** An $F$-**algebra** $A$ is an $F$-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

**Definition 8.** An algebra is **associative** if $x(yz) = (xy)z$.

**Definition 9.** Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of $A$, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

**Definition 10.** We define $\mathrm{Der}(A) = \{\delta \ni \delta$ is a derivation $\}$.

Any Lie algebra $\mathfrak{g}$ is an $F$-algebra, since $[\,\cdot\,,\,\cdot\,]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x, [y, z]] = 0$.

**Exercise 5.** Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

**Exercise 6** (Turn in). Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x, y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

## 2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

**Example 1.** Any $F$-vector space can be made into a Lie algebra by setting $[x, y] = 0$; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\,\cdot\,,\,\cdot\,] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x, y]$. Some cases:
  - $[x, y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x, y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x', y'] = [ax + by, \frac{y}{a}] = [x, y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x', y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

| $[\,\cdot\,,\,\cdot\,]$ | $x'$ | $y'$ |
|---|---|---|
| $x'$ | $0$ | $x'$ |
| $y'$ | $-x'$ | $0$ |

**Example 2.** Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

**Exercise 7.** Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\mathrm{ad}(e), \mathrm{ad}(h), \mathrm{ad}(g)$ with respect to this basis.

## 2.3 Ideals

**Definition 11.** A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

**Exercise 8.** Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \cap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

**Definition 12.** A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

**Theorem 2.** Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

*Proof.* Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \text{ad}_h$, so $h$ is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal $I$; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I,$$
$$2f = [h, -f] \in I \implies f \in I.$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

$\square$

> Note that there is a homework coming due next Monday, about 4 questions.

# 3 Lecture 3

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x + I, y + I] = [x, y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x, y] + I = [x', y'] + I$.

## 3.1 Homomorphisms and Representations

**Definition 13.** A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x, y] = [\phi(x), \phi()]$.

**Remark.** $\ker \phi \trianglelefteq \mathfrak{g}_1$ and $\operatorname{im} \phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \operatorname{im} \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I + J)/J \cong I/(I \cap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space $V$.

We call $V$ a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$
$$x \mapsto [x, \cdot].$$

**Corollary 1.** Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$Z(\mathfrak{g}) = \left\{ x \in \mathfrak{g} \ni \operatorname{ad}_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \right\}$$
$$= \ker \operatorname{ad}_x.$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \operatorname{im} \operatorname{ad} \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

## 3.2 Automorphisms

Definition: An automorphism of $\mathfrak{g}$ is an isomorphism $\mathfrak{g} \circlearrowleft$, and we define

$$\operatorname{Aut}(\mathfrak{g}) = \{ \phi : \mathfrak{g} \circlearrowleft \ni \phi \text{ is an isomorphism } \}.$$

Proposition: If $\delta \in \text{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) := \sum \frac{\delta^n}{n!} \in \text{Aut}(\mathfrak{g}).$$

This is well-defined because $\delta$ is nilpotent, and a binomial formula holds:

$$\frac{\delta^{n([x,y])}}{n!} = \sum_{i=0}^{n} [\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!}].$$

and for $n = 1, \delta([x,y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp \delta)(x), (\exp \delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x,y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\text{ad}_e) \exp(\text{ad}_{-f}) \exp(\text{ad}_e) \in \text{Aut}\mathfrak{g}.$$

where $e, f$ are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of $s$ is 2 and $W = \{1, s\}$.

## 4 Lecture 4

### 4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for $\mathfrak{g}$ is given by

$$\mathfrak{g}^{(0)} = \mathfrak{g}$$
$$\mathfrak{g}^{(1)} = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}]$$
$$\dots$$
$$\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}].$$

The Lie algebra $\mathfrak{g}$ is *solvable* if there is some $n$ for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let $\mathfrak{g}$ be a Lie algebra, then

1. If $\mathfrak{g}$ is solvable, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also solvable.
2. If $I \trianglelefteq \mathfrak{g}$ and both $I$ and $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\mathrm{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.


## 4.2 Nilpotency

$$\mathfrak{g}^0 = \mathfrak{g}$$
$$\mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0]$$
$$\cdots$$
$$\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i].$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Claim: If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof: This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \cdots [x_n, y]]] = 0$, and so $\mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} \circ \cdots \mathrm{ad}_{x_n} = 0$ which implies that $\mathrm{ad}_x^n = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If $\mathrm{ad}_x$ is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case $x$ being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an $x$ which is *not* nilpotent while $\mathrm{ad}_x$ *is* nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space $V$. If $x$ is nilpotent as an endomorphism on $V$ for all $x \in V$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases: - Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$. - Case $\dim g = 1$, left as an exercise.

Inductive step: Let $A$ be a maximal proper subalgebra and define $\phi : A \to \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that $\phi$ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that $\operatorname{im} \phi \leq \mathfrak{gl}(\mathfrak{g}/a)$ is a Lie subalgebra, and $\dim \operatorname{im} \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \operatorname{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\operatorname{im} \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that $A$ is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2[y, y]$. The last term is zero, the middle two terms are in $A$, and because $A$ is closed under the bracket, the first term is in $A$ as well.

But then $A \oplus F_y$ is a larger subalgebra than $A$, which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in $A$, $A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in $A$.

Proof to be continued on Friday!

# 5 Lecture 5

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of $V$, constructing $\operatorname{im} \phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus Fy$. Now consider

$$ W = \{v \in V \ni Av = 0\}, $$

which is $\mathfrak{g}$-invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But $V$ is killed by any element in $A$, and both of these terms are in $A$. In particular, the $y$ appearing in $Fy$ also satisfies $y \in W$. Consider $y|_W \in \operatorname{End}(w)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \operatorname{End}(V)$, which is true exactly because $y$ is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \operatorname{End}(\mathfrak{g}/A)$ is nilpotent. Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by $a$, $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since $m_\ell, m_r$ commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly $\operatorname{ad}_a$. But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to $\mathrm{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that $\mathrm{ad}\mathfrak{g}v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension. Since $\mathrm{ad}_x$ is nilpotent, so is $\mathrm{ad}_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space.   $\square$

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of $F$, then the matrices of $\mathfrak{g}$ with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on $n$, where the base case is easy. For the inductive step, we use the previous theorem to get a $v_1$ such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\overline{V} = F^n/Fv_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \to \mathfrak{gl}(\overline{V})$ where $x \mapsto (\overline{y} \mapsto \overline{y(x)})$.

Then $\mathrm{im}\,\phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \mathrm{End}(F^{n-1})$ is nilpotent, since $x$ was nilpotent on the larger space. But (see notes) then $x$ can be written as a strictly upper-triangular matrix.

## 5.1 Chapter 2: Semisimple Lie Algebras

We now assume $\mathrm{char}F = 0$ and $\overline{F} = F$.