Problem Set 10

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1 Problem 1

Let ϕ be an n-form. If suffices to show these statements for $n = 2$.

$\implies$: Suppose ϕ is alternating, then $\phi(b, b) = 0$ for all $b \in B$.

Letting $a, b \in B$ be arbitrary, we then have

$$\begin{aligned}
0 &= \phi(a + b, a + b) \\
&= \phi(a, a + b) + \phi(b, a + b) \\
&= \phi(a, a) + \phi(a, b) + \phi(b, a) + \phi(b, b) \\
&= \phi(a, b) + \phi(b, a) \\
&\implies \phi(a, b) = -\phi(b, a),
\end{aligned}$$

which shows that ϕ is skew-symmetric.

$\Longleftarrow$ Suppose ϕ is skew-symmetric, so $\phi(a, b) = -\phi(b, a)$ for all $a, b \in B$. Then $\phi(b, b) = -\phi(b, b)$ by transposing the terms, which says that $\phi(b, b) = 0$ for all $b \in B$ and thus ϕ is alternating.

2 Problem 2

Let $f(x) = \det(P + xQ) \in R[x]$, then f is a polynomial in x which is not identically zero.

To see that $f \not\equiv 0$, we can use that fact that P is invertible to evaluate $f(0) = \det(P) \neq 0$.

We can now note that f has finite degree, and thus finitely many zeroes in R.

3 Problem 3

Letting $k[x] \curvearrowright_{\phi} E$ to yield a $k[x]$-module structure on E and take an invariant factor decomposition,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_t, \quad E_i = \frac{k[x]}{(q_i)}, \quad q_1 \,\Big|\, q_2 \,\Big|\, \cdots \,\Big|\, q_t$$

where $E_i = k[x]/(q_i)$. Then $q_t = q$, the minimal polynomial of E.

In particular, E_t is a ϕ-invariant subspace of E, and if $\deg q_t = m$, then E_t is in fact an m-dimensional cyclic module with basis $\left\{\mathbf{v}, \phi(\mathbf{v}), \phi^2(\mathbf{v}), \cdots, \phi^{m-1}(\mathbf{v})\right\}$ for some $\mathbf{v} \in E_t$.

But since $E_t \leq E$ is a subspace, we have

$$m = \deg q(x) = \deg q_t(x) = \dim E_t \leq \dim E.$$

4 Problem 4

$\Longrightarrow$: Suppose $A \sim D$ where D is diagonal. Then $JCF(A) = JCF(D) = D$, which means that every Jordan block of A has size exactly 1.

Since the elementary divisors of A are precisely the minimal polynomials of the Jordan blocks of A, and the minimal polynomial of any 1×1 matrix $[a_{ij}]$ is given by the linear polynomial $x - a_{ij}$, every elementary divisor of A must be linear.

$\Longleftarrow$: Suppose all of the elementary divisors of A are linear. Every elementary divisor is the minimal polynomial of a Jordan block of A, and so if we write $JCF(A) = \bigoplus M_i$, then the minimal polynomial of each M_i is linear.

Supposing that M_i has minimal polynomial $p_i(x) = x - c$ for some scalar c, we have

$$p_i(M_i) = 0 \implies M_i - cI_n = 0 \implies M_i = cI_n,$$

which shows that M_i is a diagonal matrix with only c on its diagonal.

But if every Jordan block of A is diagonal, then $JCF(A) = D$ is diagonal and $A \sim D$.

5 Problem 5

5.1 Part 1

We'll use the fact that the minimal polynomial q is the invariant factor of highest degree, and so every other invariant factor must divide q.

Moreover, $RCF(A) = C_1 \oplus C_2 \oplus \cdots \oplus C_k$ where each C_i is the companion matrix of the ith invariant factor if we write $V \cong \bigoplus_{i=1}^{k} k[x]/(a_i)$. So it suffices to determine all of the possible distinct combinations of invariant factors.

We can restrict this list by noting that the characteristic polynomial satisfies $\chi_A(x) = \prod a_i$, and in particular, $\deg \chi_A(x) = 6$. Noting that $\deg q(x) = 3$, the degrees of the remaining invariant factors must sum to 3.

So the possibilities are:

$$\begin{aligned}
R_1 : a_1 &= (x-2), & a_2 &= (x-2)^2, & a_3 &= q(x), & & \\
R_2 : a_1 &= (x-2), & a_2 &= (x-2)(x+3), & a_3 &= q(x), & & \\
R_3 : a_1 &= (x+3), & a_2 &= (x-2)(x+3), & a_3 &= q(x), & & \\
R_4 : a_1 &= (x-2), & a_2 &= (x-2), & a_3 &= (x-2) & a_4 &= q(x), \\
R_5 : a_1 &= (x+3), & a_2 &= (x+3), & a_3 &= (x+3) & a_4 &= q(x).
\end{aligned}$$

> This exhausts all possibilities, because the degrees of a_i must be a weakly increasing integer partitions of 3, namely $(1,2)$ or $(1,1,1)$. A $(1,2)$ partition can only yield a quadratic factor for a_2, and since $a_2 \mid a_3$ there are only two choices. If a repeated factor is chosen like $(x-2)^2$, then $a_1 \mid a_2$ forces $a_1 = x - 2$,yielding R_1. Otherwise, we can pick either distinct factor of a_2 as a choice for a_1, yielding R_2, R_3. Any $(1,1,1)$ partition can only be a repeated linear factor, since we must have $a_1 \mid a_2 \mid a_3$, and there are only two choices. This yields R_4, R_5.

Noting that

$$\begin{aligned}
(x-2)^2 &= x^2 - 4x + 4 \\
(x-2)(x+3) &= x^2 + x - 6 \\
q(x) &= x^3 - x^2 - 8x + 12,
\end{aligned}$$

these choices correspond to the matrices

$$R_1 = \left[\begin{array}{c|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right], \quad R_2 = \left[\begin{array}{c|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right], R_3 = \left[\begin{array}{c|cc|ccc} 3 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_4 = \left[\begin{array}{c|c|c|ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad R_5 = \left[\begin{array}{c|c|c|ccc} -3 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -3 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Note: these are perhaps transposed from Hungerford's notation.

Since none of the associated polynomials were irreducible over $\mathbb{Q}$, $RCF(A)$ takes these forms over $\mathbb{C}$ as well.

To obtain the possible Jordan Canonical forms, we'll instead need to consider elementary divisors. These can be obtained from the invariant factors above, yielding the possibilities:

$$\begin{aligned} R_1 :& (x-2),\ (x-2),\ (x-2)^2\ (x+3) \\ R_2 :& (x-2),\ (x-2),\ (x-2)^2,\ (x+3),\ (x+3) \\ R_3 :& (x-2),\ (x-2)^2,\ (x+3),\ (x+3),\ (x+3) \\ R_4 :& (x-2),\ (x-2),\ (x-2),\ (x-2)^2,\ (x+3) \\ R_5 :& (x+3),\ (x+3),\ (x+3),\ (x+3),\ (x-2)^2 \end{aligned}$$

For the sake of notation, write J_λ^k for a $k \times k$ Jordan block with λ on the diagonal and 0_k for the $k \times k$ zero matrix. We then have

$$R_1 : 0_2 \oplus J_2^1 \oplus J_2^1 \oplus J_2^2 \oplus J_3^1$$

$$R_2 : J_2^1 \oplus J_2^1 \oplus J_2^2 \oplus J_3^1 \oplus J_3^1$$

$$R_3 : J_2^1 \oplus J_2^2 \oplus J_3^1 \oplus J_3^1 \oplus J_3^1$$

$$R_4 : J_2^1 \oplus J_2^1 \oplus J_2^1 \oplus J_2^2 \oplus J_3$$

$$R_5 : J_2^2 \oplus J_3^1 \oplus J_3^1 \oplus J_3^1 \oplus J_3^1$$

.

5.2 Part 2

We'll first exhibit the possibilities over $\mathbb{C}$, then show what subset can be obtained over $\mathbb{Q}$.

Over $\mathbb{C}$, we have $x^2 + 1 = (x - i)(x + i)$. By the same argument used in Part 1, we know that $q(x)$ is the largest invariant factor, and since $\deg q = 3$, the degrees of the remaining factors must sum to 4 (since the degree χ_A will be 7, and it's the product of these factors).

We also know that the degrees must forma weakly decreasing partition of 4, which are

- $(1, 1, 1, 1)$
  - This can only be $a_1 = a_2 = a_3 = a_4$, a repeated linear factor, so there are 3 possibilities
- $(1, 1, 2)$
  - This must satisfy $a_1 = a_2$, so there are 3 possibilities for $a_1 = a_2$ and 2 for a_3, for 6 total.
- $(2, 2)$
  - This also must satisfy $a_1 = a_2$, so there are $\binom{3}{2}/2 = 3$ possibilities

The possibilities are thus

$$
\begin{array}{rrrrr}
R_1 : a_1 = (x-i) & a_2 = (x-i) & a_3 = (x-i) & a_4 = (x-i) & a_5 = q(x) \\
R_2 : a_1 = (x+i) & a_2 = (x+i) & a_3 = (x+i) & a_4 = (x+i) & a_5 = q(x) \\
R_3 : a_1 = (x-7) & a_2 = (x-7) & a_3 = (x-7) & a_4 = (x-7) & a_5 = q(x) \\
\\
R_4 : a_1 = (x+i) & a_2 = (x+i) & a_3 = (x+i)(x-i) & a_4 = q(x) & \\
R_5 : a_1 = (x+i) & a_2 = (x+i) & a_3 = (x+i)(x-7) & a_4 = q(x) & \\
R_6 : a_1 = (x-i) & a_2 = (x-i) & a_3 = (x-i)(x+i) & a_4 = q(x) & \\
R_7 : a_1 = (x-i) & a_2 = (x-i) & a_3 = (x-i)(x-7) & a_4 = q(x) & \\
R_8 : a_1 = (x-7) & a_2 = (x-7) & a_3 = (x-7)(x+i) & a_4 = q(x) & \\
R_9 : a_1 = (x-7) & a_2 = (x-7) & a_3 = (x-7)(x-i) & a_4 = q(x) & \\
\\
R_{10} : a_1 = (x+i)(x-i) & a_2 = (x+i)(x-i) & a_3 = q(x) & & \\
R_{11} : a_1 = (x+i)(x-7) & a_2 = (x+i)(x-7) & a_3 = q(x) & & \\
R_{12} : a_1 = (x-i)(x-7) & a_2 = (x-i)(x-7) & a_3 = q(x) & &
\end{array}
.
$$

The corresponding Rational Canonical Forms for each R_j can be obtained by writing the companion matrix for the blocks a_i and taking their direct sum.

It is then easy to see that if A is taken over $\mathbb{Q}$ instead, only form R_3 is possible (since x^2+1 does not split over $\mathbb{Q}$).

Let nJ_λ^k denote $J_\lambda^k \oplus J_\lambda^k \oplus \cdots \oplus J_\lambda^k$, where n copies appear in the direct sum corresponding to n Jordan blocks. We can immediately obtain the corresponding Jordan forms:

$$
\begin{aligned}
R_1 &: 5J_i^1 \oplus J_{-i}^1 \oplus J_7^1 \\
R_2 &: 5J_{-i}^1 \oplus J_i^1 \oplus J_7^1 \\
R_3 &: 5J_7^1 \oplus J_i^1 \oplus J_{-i}^1 \\
\\
R_4 &: 4J_{-i}^1 \oplus 2J_i^1 \oplus J_7^1 \\
R_5 &: 4J_{-i}^1 \oplus J_i^1 \oplus 2J_7^1 \\
R_6 &: 4J_i^1 \oplus 2J_{-i}^1 \oplus J_7^1 \\
R_7 &: 4J_i^1 \oplus J_{-i}^1 \oplus 2J_7^1 \\
R_8 &: 2J_{-i}^1 \oplus J_i^1 \oplus 2J_7^1 \\
R_9 &: 2J_i^1 \oplus J_{-i}^1 \oplus 4J_7^1 \\
\\
R_{10} &: 3J_i^1 \oplus 3J_{-i}^1 \oplus J_7^1 \\
R_{11} &: J_i^1 \oplus 3J_{-i}^1 \oplus 3J_7^1 \\
R_{12} &: 3J_i^1 \oplus J_{-i}^1 \oplus 3J_7^1.
\end{aligned}
$$

6 Problem 6

Let $\phi \in \operatorname{End}(V)$, then following a different proof than what is suggested in Hungerford, define an action

$$\begin{aligned}
k[x] &\curvearrowright V \\
p(x) &\curvearrowright \mathbf{v} = p(\phi)(\mathbf{v}),
\end{aligned}$$

which induces an invariant factor decomposition

$$V \cong \bigoplus_{i=1}^{n} \frac{k[x]}{(f_i)}, \quad f_i \in k[x], \quad f_1 \mathrel{\Big|} f_2 \mathrel{\Big|} \cdots \mathrel{\Big|} f_n.$$

Then $f_n(x)$ is the minimal polynomial of ϕ, and the characteristic polynomial is given by $p_\phi(x) = \prod_{i=1}^{n} f_i(x)$. In particular, $f_n(x) \mathrel{\Big|} p_\phi(x)$ and $f_n(\phi) = 0$ by definition, so $p_\phi(\phi) = 0$ as well. $\square$

7 Problem 7

7.1 Part 1

Suppose $\phi\psi = \psi\phi$ and both ϕ, ψ have bases of eigenvectors.

Letting λ_i denote the eigenvalues of ϕ, write

$$V = \bigoplus_i V_{\lambda_i}.$$

Now let $\mathbf{v}$ be an eigenvector corresponding to λ_i. We have $\phi(\mathbf{v}) = \lambda_i \mathbf{v}$, and

$$\phi\psi(\mathbf{v}) = \psi\phi(\mathbf{v}) = \psi(\lambda_i \mathbf{v}) = \lambda_i \psi(\mathbf{v}),$$

which demonstrates that $\psi(\mathbf{v})$ is also an eigenvector for ϕ, and moreover $\psi(V_{\lambda_i}) \subseteq V_{\lambda_i}$, so it only sends λ_i eigenvectors to other λ_i eigenvectors.

Now consider $\psi|_{V_{\lambda_i}}$, the restriction of ψ this eigenspace. Since ψ had an eigenbasis on V, this restricts to an eigenbasis $\mathcal{B}_i = \{\mathbf{w}_i\}$ of V_{λ_i}. But then every element of $\mathbf{w}_i$ is an eigenvector of ψ by definition, and we also have $\mathbf{w}_i \in V_{\lambda_i}$, so the $\mathbf{w}_i$ are **also** eigenvectors for ϕ.

Doing this for every i, we obtain $\mathcal{B} = \coprod_i \mathcal{B}_i$ where $\operatorname{span}(\mathcal{B}) = E$, which yields a simultaneous eigenbasis of E for both ψ and ϕ.

7.2 Part 2

Writing $\mathcal{B} = \left\{ \mathbf{v}_i \mathrel{\Big|} 1 \leq i \leq n \right\}$, this means we can form an invertible matrix $P = [\mathbf{v}_1^t, \cdots, \mathbf{v}_n^t]$. Then if A is the matrix of ϕ in the standard basis and B is the matrix of ψ, we have

$$PAP^{-1} = D_1 \quad \text{and} \quad PBP^{-1} = D_2$$

where D_1, D_2 are diagonal. In other words, P simultaneously diagonalizes both A and B.