Title

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1.1 Review

Let $k = \bar{k}$, we're setting up correspondences

Ring Theory	Geometry/Topology of Affine Varieties
Polynomial functions	Affine space
$k[x_1, \cdots, x_n]$	$\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$
Maximal ideals $\langle x_1 - a_1, \cdots, x_n - a_n \rangle$	Points $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$
Radical ideals $I \trianglelefteq k[x_1, \cdots, x_n]$	Affine varieties $X \subset \mathbb{A}^n/k$, vanishing locii of polynomials

$$I \mapsto V(I) := \left\{ a \;\middle|\; f(a) = 0 \, \forall f \in I \right\}$$

$$I(X) := \left\{ f \;\middle|\; f|_X = 0 \right\} \hookleftarrow X$$

Radical ideals containing $I(X)$, i.e. ideals in $A(X)$	closed subsets of X, i.e. affine subvarieties
$A(X)$ is a domain	X irreducible
$A(X)$ is not a direct sum	X connected
Prime ideals in $A(X)$	Irreducible closed subsets of X
Krull dimension n (longest chain of prime ideals)	$\dim X = n$, (longest chain of irreducible closed subsets).

> Recall that we defined the coordinate ring $A(X) := k[x_1, \cdots, x_n]/I(X)$, which contained no nilpotents.

We had some results about dimension

1. $\dim X < \infty$ and $\dim \mathbb{A}^n = n$.
2. $\dim Y + \operatorname{codim}_X Y = \dim X$ when $Y \subset X$ is irreducible.
3. Only over $\bar{k} = k$, $\operatorname{codim}_X V(f) = 1$.

Example 1.1.
Take $V(x^2 + y^2) \subset \mathbb{A}^2/\mathbb{R}$

> **Definition 1.0.1** (?)**.**
> An affine variety Y of
> - $\dim Y = 1$ is a **curve**,
> - $\dim Y = 2$ is a **surface**,
> - $\operatorname{codim}_X Y = 1$ is a **hypersurface in** X

Question: Is every hypersurface the vanishing locus of a *single* polynomials $f \in A(X)$?

Answer: This is true iff $A(X)$ is a UFD.

> **Definition 1.0.2** (Codimension in a Ring)**.**
> $\operatorname{codim}_R \mathfrak{p}$ is the length of the longest chain
>
> $$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = \mathfrak{p}.$$

Recall that f is irreducible if $f = f_1 f_2 \implies f_i \in R^\times$ for one i, and f is prime iff $\langle f \rangle$ is a prime ideal, or equivalently $f \mid ab \implies f \mid a$ or $f \mid b$.

Note that prime implies irreducible, since f divides itself.

> **Proposition 1.1*(?)*.**
> Let R be a Noetherian domain, then TFAE
>   a. All prime ideals of codimension 1 are principal.
>   b. R is a UFD.

> *Proof .*
> $a \implies b$:
> Let f be a nonzero non-unit, we'll show it admits a prime factorization. If f is not irreducible, then $f = f_1 f_1'$, both non-units. If f_1' is not irreducible, we can repeat this, to get a chain
>
> $$\langle f \rangle \subsetneq \langle f_1' \rangle \subsetneq \langle f_2' \rangle \subsetneq \cdots,$$
>
> which must terminate.
> This yields a factorization $f = \prod f_i$ with f_i irreducible. To show that R is a UFD, it thus suffices to show that the f_i are prime. Choose a minimal prime ideal containing f. We'll use Krull's Principal Ideal Theorem: if you have a minimal prime ideal $\mathfrak{p}$ containing f, its codimension $\operatorname{codim}_R \mathfrak{p}$ is one. By assumption, this implies that $\mathfrak{p} = \langle g \rangle$ is principal. But $g \mid f$ with f irreducible, so f, g differ by a unit, forcing $\mathfrak{p} = \langle f \rangle$. So $\langle f \rangle$ is a prime ideal.
>
> $b \implies a$:
> Let $\mathfrak{p}$ be a prime ideal of codimension 1. If $\mathfrak{p} = \langle 0 \rangle$, it is principal, so assume not. Then there exists some nonzero non-unit $f \in \mathfrak{p}$, which by assumption has a prime factorization since R is assumed a UFD. So $f = \prod f_i$.
> Since $\mathfrak{p}$ is a prime ideal and $f \in \mathfrak{p}$, some $f_i \in \mathfrak{p}$. Then $\langle f_i \rangle \subset \mathfrak{p}$ and $\mathfrak{p}$ minimal implies $\langle f_i \rangle = \mathfrak{p}$,

so $\mathfrak{p}$ is principal.

∎

> **Corollary 1.2** *(?)*.
> Every hypersurface $Y \subset X$ is cut out by a single polynomial, so $Y = V(f)$, iff $A(X)$ is a UFD.

Example 1.2.
Apply this to $R = A(X)$, we find that there is a bijection

codim1 prime ideals $\iff$ codim1 closed irreducible subsets $Y \subset X$, i.e. hypersurfaces.

Taking $A(X) = \mathbb{C}[x, y, z]/\left\langle x^2 + y^2 - z^2 \right\rangle$.