Problem Set 7

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1 Problem 1

1.1 Part a

We want to show that $\ell^2(\mathbb{N})$ is complete, so let $\{x_n\} \subseteq \ell^2(\mathbb{N})$ be a Cauchy sequence, so $\|x^j - x^k\|_{\ell^2} \to 0$. We want to produce some $\mathbf{x} := \lim_{n \to \infty} x^n$ such that $x \in \ell^2$.

To this end, for each fixed index i, define

$$\mathbf{x}_i := \lim_{n \to \infty} x_i^n.$$

This is well-defined since $\|x^j - x^k\|_{\ell^2} = \sum_i \left| x_i^j - x_i^k \right|^2 \to 0$, and since this is a sum of positive real numbers that approaches zero, each term must approach zero. But then for a fixed i, the sequence $\left| x_i^j - x_i^k \right|^2$ is a Cauchy sequence of real numbers which necessarily converges by completeness of $\mathbb{R}$.

We also have $\|\mathbf{x} - x^j\|_{\ell^2} \to 0$ since

$$\|\mathbf{x} - x^j\|_{\ell^2} = \|\lim_{k \to \infty} x^k - x^j\|_{\ell^2} = \lim_{k \to \infty} \|x^k - x^j\|_{\ell^2} \to 0$$

where the limit can be passed through the norm because the map $t \mapsto \|t\|_{\ell^2}$ is continuous. So $x^j \to \mathbf{x}$ in ℓ^2 as well.

It remains to show that $\mathbf{x} \in \ell^2(\mathbb{N})$, i.e. that $\sum_i |\mathbf{x}_i|^2 < \infty$. To this end, we write

$$
\begin{aligned}
\|\mathbf{x}\|_{\ell^2} &= \|\mathbf{x} - x^j + x^j\|_{\ell^2} \\
&\leq \|\mathbf{x} - x^j\|_{\ell^2} + \|x^j\|_{\ell^2} \\
&\to M < \infty,
\end{aligned}
$$

where $\lim_j \|\mathbf{x} - x^j\|_{\ell^2} = 0$ by the previous argument, and the second term is bounded because $x^j \in \ell^2 \iff \|x^j\|_{\ell^2} := M < \infty$. $\quad \square$

1.2 Part b

Let H be a Hilbert space with inner product $\langle \cdot, \ \cdot \rangle$ and induced norm $\| \cdot \|$.

Lemma: For any complex number z, we have

$$\Im(z) = \Re(-iz),$$

and as a corollary, since the inner product on H takes values in $\mathbb{C}$, we have

$$\Re(\langle x, \ iy \rangle) = \Re(-i\langle x, \ y \rangle) = \Im(\langle x, \ y \rangle).$$

We can compute the following:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x, \ y \rangle\right)$$

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x, \ y \rangle\right)$$

$$
\begin{aligned}
\|x + iy\|^2 &= \|x\|^2 + \|y\|^2 + 2\,\Re\left(\langle x, \ iy \rangle\right) \\
&= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle)
\end{aligned}
$$

$$
\begin{aligned}
\|x - iy\|^2 &= \|x\|^2 + \|y\|^2 - 2\,\Re\left(\langle x, \ iy \rangle\right) \\
&= \|x\|^2 + \|y\|^2 + \Im(\langle x, \ y \rangle)
\end{aligned}
$$

and summing these all

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\| = 4\,\Re\left(\langle x,\ y\rangle\right) + 4i\,\Im\left(\langle x,\ y\rangle\right)$$
$$= 4\langle x,\ y\rangle.$$

To conclude that a linear map U is an isometry iff U is unitary, if we assume U is unitary then we can write

$$\|x\|^2 := \langle x,\ x\rangle = \langle Ux,\ Ux\rangle := \|Ux\|^2.$$

Assuming now that U is an isometry, by the polarization identity we can write

$$\begin{aligned}
\langle Ux,\ Uy\rangle &= \frac{1}{4}\left(\|Ux+Uy\|^2 + \|Ux-Uy\|^2 + i\|Ux+Uy\|^2 - i\|Ux+Uy\|^2\right) \\
&= \frac{1}{4}\left(\|U(x+y)\|^2 + \|U(x-y)\|^2 + i\|U(x+y)\|^2 - i\|U(x+y)\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 + \|x-y\|^2 + i\|x+y\|^2 - i\|x+y\|^2\right) \\
&= \langle x,\ y\rangle.
\end{aligned}$$

$\square$

2 Problem 2

Lemma: The map $\langle\,\cdot,\ \cdot\,\rangle : H \times H \to \mathbb{R}$ is continuous.

Proof:

Let $x_n \to x$ and $y_n \to y$, then

$$\begin{aligned}
|\langle x_n,\ y_n\rangle - \langle x,\ y\rangle| &= |\langle x_n,\ y_n\rangle - \langle x,\ y_n\rangle + \langle x,\ y_n\rangle - \langle x,\ y\rangle| \\
&= |\langle x_n - x,\ y_n\rangle + \langle x,\ y_n - y\rangle| \\
&\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\
&\to 0 \cdot M + C \cdot 0 < \infty,
\end{aligned}$$

where $\|y_n\| \to M$ since $y_n \to y$ implies that $\|y_n\|$ is bounded.

2.1 Part a:

We want to show that sequences in $E^\perp$ converge to elements of $E^\perp$. Using the lemma, letting $\{e_n\}$ be a sequence in $E^\perp$, so $y \in E \implies \langle e_n,\ y\rangle = 0$. Since H is complete, $e_n \to e \in H$; we can show that $e \in E^\perp$ by letting $y \in E$ be arbitrary and computing

$$\langle e, \ y \rangle = \left\langle \lim_n e_n, \ y \right\rangle = \lim_n \langle e_n, \ y \rangle = \lim_n 0 = 0,$$

so $e \in E^{\perp}$.

2.2 Part b:

Let $S := \operatorname{span}_H(E)$; then the smallest closed subspace containing E is $\overline{S}$, the closure of S. We will proceed by showing that $E^{\perp\perp} = \overline{S}$.

$\overline{S} \subseteq E^{\perp\perp}$:

Let $\{x_n\}$ be a sequence in S, so $x_n \to x \in \overline{S}$.

First, each x_n is in $E^{\perp\perp}$, since if we write $x_n = \sum a_i e_i$ where $e_i \in E$, we have

$$y \in E^{\perp} \implies \langle x_n, \ y \rangle = \left\langle \sum_i a_i e_i, \ y \right\rangle = \sum_i a_i \langle e_i, \ y \rangle = 0 \implies x_n \in (E^{\perp})^{\perp}.$$

It remains to show that $x \in E^{\perp\perp}$, which follows from

$$y \in E^{\perp} \implies \langle x, \ y \rangle = \left\langle \lim_n x_n, \ y \right\rangle = \lim_n \langle x_n, \ y \rangle = 0 \implies x \in (E^{\perp})^{\perp},$$

where we've used continuity of the inner product.

$E^{\perp\perp} \subseteq \overline{S}$:

For notation convenience, we'll just write S for $\overline{S}$. Let $x \in E^{\perp\perp}$. Noting that S is closed, we can define P, the operator projecting elements onto S, and write

$$x = Px + (x - Px) \in S \oplus S^{\perp}$$

But since $\langle x, \ x - Px \rangle = 0$ because $x - Px \in E^{\perp}$ and $x \in (E^{\perp})^{\perp}$, we can rewrite the first term in this inner product to obtain

$$0 = \langle x, \ x - Px \rangle = \langle Px + (x - Px), \ x - Px \rangle = \langle Px, \ x - Px \rangle + \langle x - Px, \ x - Px \rangle,$$

where we can note that the first term is zero because $Px \in S$ and $x - Px \in S^{\perp}$, and the second term is $\|x - Px\|^2$.

But this says $\|x - Px\|^2 = 0$, so $x - Px = 0$ and thus $x = Px \in S$, which is what we wanted to show.

3 Problem 3

3.1 Part a

We compute

$$\|e_0\|^2 = \int_0^1 1^2 \, dx = 1$$

$$\|e_1\|^2 = \int_0^1 3(2x-1)^2 = \frac{1}{2}(2x-1)^2 \Big|_0^1 = 1$$

$$\langle e_0, \, e_1 \rangle = \int_0^1 \sqrt{3}(2x-1) \, dx = \frac{\sqrt{3}}{4}(2x-1) \Big|_0^1 = 0.$$

which verifies that this is an orthonormal system.

3.2 Part b

We first note that this system spans the degree 1 polynomials in $L^2([0,1])$, since we have

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{3} & \sqrt{3} \end{bmatrix} [1, x]^t = [e_0, e_1]$$

which exhibits a matrix that changes basis from $\{1, x\}$ to $\{e_0, e_1\}$ which is invertible, so both sets span the same subspace.

Thus the closest degree 1 polynomial f to x^3 is given by the projection onto this subspace, and since $\{e_i\}$ is orthonormal this is given by

$$\begin{aligned}
f(x) &= \sum_i \langle x^3, \, e_i \rangle e_i \\
&= \langle x^3, \, 1 \rangle 1 + \langle x^3, \, \sqrt{3}(2x-1) \rangle \sqrt{3}(2x-1) \\
&= \int_0^1 x^2 \, dx + \sqrt{3}(2x-1) \int_0^1 \sqrt{3}x^2(2x-1) \, dx \\
&= \frac{1}{3} + \sqrt{3}(2x-1)\frac{\sqrt{3}}{6} \\
&= x - \frac{1}{6}.
\end{aligned}$$

We can also compute

$$\|f - g\|_2^2 = \int_0^1 (x^2 - x + \frac{1}{6})^2 \ dx$$
$$= \frac{1}{180}$$
$$\implies \|f - g\|_2 = \frac{1}{\sqrt{180}}.$$

4 Problem 4

4.1 Part a

4.1.1 i

We can first note that $\langle 1/\sqrt{2}, \ \cos(2\pi n x)\rangle = \langle 1/\sqrt{2}, \ \sin(2\pi m x)\rangle = 0$ for any n or m, since this involves integrating either sine or cosine over an integer multiple of its period.

Letting $m, n \in \mathbb{Z}$, we can then compute

$$\langle \cos(2\pi n x), \ \sin(2\pi m x)\rangle = \int_0^1 \cos(2\pi n x)\sin(2\pi m x) \ dx$$
$$= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \sin(2\pi(n-m)x) \ dx$$
$$= \frac{1}{2}\int_0^1 \sin(2\pi(n+m)x) - \frac{1}{2}\int_0^1 \sin(2\pi(n-m)x) \ dx$$
$$= 0,$$

which again follows from integration of sine over a multiple of its period (where we use the fact that $m + n, m - n \in \mathbb{Z}$).

Similarly,

$$\langle \cos(2\pi n x), \ \cos(2\pi m x)\rangle = \int_0^1 \cos(2\pi n x)\cos(2\pi m x) \ dx$$
$$= \frac{1}{2}\int_0^1 \cos(2\pi(m+n)x) + \cos(2\pi(m-n)x) \ dx$$
$$= \begin{cases} \frac{1}{2}\int_0^1 \cos(4\pi n x) + 1 \ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.$$

$$\langle \sin(2\pi n x), \ \sin(2\pi m x)\rangle = \int_0^1 \sin(2\pi n x)\sin(2\pi m x) \ dx$$
$$= \frac{1}{2}\int_0^1 \cos(2\pi(m-n)x) + \cos(2\pi(m+n)x) \ dx$$
$$= \begin{cases} \frac{1}{2}\int_0^1 1 + \cos(4\pi n x) \ dx = 1 & m = n \\ 0 & m \neq n \end{cases}.$$

Thus each pairwise combination of elements are orthonormal, making the entire set orthonormal.

4.1.2 ii

We have

$$
\begin{aligned}
\left\langle e^{2\pi kx},\ e^{-2\pi i\ell x}\right\rangle &= \int_0^1 e^{2\pi ikx}\overline{e^{2\pi i\ell x}}\ dx \\
&= \int_0^1 e^{2\pi ikx}e^{-2\pi i\ell x}\ dx \\
&= \int_0^1 e^{2\pi i(k-\ell)x}\ dx \\
&\left(= \int_0^1 1\ dx = 1 \quad \text{if } k=\ell, \text{ otherwise:}\right) \\
&= \frac{e^{2\pi i(k-\ell)x}}{2\pi i(k-\ell)}\Big|_0^1 \\
&= \frac{e^{2\pi i(k-\ell)}-1}{2\pi i(k-\ell)} \\
&= 0,
\end{aligned}
$$

since $e^{2\pi ik}=1$ for every $k \in Z$, and $k-\ell \in \mathbb{Z}$. Thus this set is orthonormal.

4.2 Part b

4.2.1 i

By the Weierstrass approximation theorem for functions on a bounded interval, we can find a polynomials $P_n(x)$ such that $\|f - P_n\|_\infty \to 0$, i.e. the P_n uniformly approximate f on $[0,1]$.

Letting $\varepsilon > 0$, we can thus choose a P such that $\|f - P\|_\infty < \varepsilon$, which necessarily implies that $\|f - P\|_{L^1} < \varepsilon$ since we have

$$
\int_0^1 |f(x) - P(x)|\ dx \le \int_0^1 \varepsilon\ dx = \varepsilon.
$$

Thus we can write

$$
f(x) = P(x) + (f(x) - P(x))
$$

where $h(x) := f(x) - P(x)$ satisfies $\|h\|_{L^1} < \varepsilon$. It only remains to show that $P \in L^2([0,1])$, but this follows from the fact that any polynomial on a compact interval is uniformly bounded, say $|P(x)| \le M < \infty$ for all $x \in [0,1]$, and thus

$$
\|P\|_{L^2}^2 = \int_0^1 |P(x)|^2\ dx \le \int_0^1 M^2\ dx = M^2 < \infty.
$$

It follows that we can let $g = P$ and $h = f - P$ to obtain the desired result.

4.2.2 ii

By part (i), the claim is that it suffices to show this is true for $f \in L^2$. In this case, we can identify

$$\int_0^1 f(x) \cos(2\pi kx) \; dx := \Re(\hat{f}(k))$$

$$\int_0^1 f(x) \sin(2\pi kx) \; dx := \Im(\hat{f}(k)),$$

the real and imaginary parts of the kth Fourier coefficient of f respectively.

By Bessel's inequality, we know that $\left\{\hat{f}(k)\right\}_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$, and so $\sum_k \left|\hat{f}(k)\right| < \infty$.

But this is a convergent sequence of real numbers, which necessarily implies that $\left|\hat{f}(k)\right| \to 0$. In particular, this also means that its real and imaginary parts tend to zero, which is exactly what we wanted to show.

If we instead have $f \in L^1$, write $f = g + h$ where $g \in L^2$ and $\|h\|_{L^1} \to 0$. Then

$$\left|\int_0^1 f(x) \cos(2\pi kx) \; dx\right| = \left|\int_0^1 (g(x) + h(x)) \cos(2\pi kx) \; dx\right|$$

$$\leq \left|\int_0^1 g(x) \cos(2\pi kx) \; dx\right| + \left|\int_0^1 h(x) \cos(2\pi kx) \; dx\right|$$

$$\leq \left|\int_0^1 g(x) \cos(2\pi kx) \; dx\right| + \int_0^1 |h(x)||\cos(2\pi kx)| \; dx$$

$$= |\hat{g}(k)| + \varepsilon$$

$$\to 0,$$

5 Problem 5

5.1 Part 1

We use the following algorithm: given $\{v\}_i$, we set

- $e_1 = v_1$, and then normalize to obtain $\hat{e}_1 = e_1/\|e_1\|$
- $e_i = v_i - \sum_{k \leq i-1} \langle v_i, \; \hat{e}_i \rangle \hat{e}_i$

The result set $\{\hat{e}_i\}$ is the orthonormalized basis.

We set $e_1 = 1$, and check that $\|e_1\|^2 = 2$, and thus set $\hat{e}_1 = \frac{1}{\sqrt{2}}$.

We then set

$$e_2 = x - \langle x, \ \hat{e}_1 \rangle \hat{e}_1$$
$$= x - \langle x, \ 1 \rangle 1$$
$$= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x \ dx$$
$$= x - \int \text{odd function}$$
$$= x,$$

and so $e_2 = x$. We can then check that

$$\|e_2\| = \left(\int_{-1}^{1} x^2 \ dx \right)^{1/2} = \sqrt{\frac{2}{3}},$$

and so we set $\hat{e}_2 = \sqrt{\frac{3}{2}} x$.

We continue to compute

$$e_3 = x^2 - \langle x^2, \ \hat{e}_1 \rangle \hat{e}_1 - \langle x^2, \ \hat{e}_2 \rangle \hat{e}_2$$
$$= x^2 - \frac{1}{2} \int_{-1}^{1} x^2 \ dx - \frac{3}{2} x \int_{-1}^{1} x^3 \ dx$$
$$= x^2 - \left(\frac{1}{6} x^3 \right)|_{-1}^{1} + \frac{3}{2} x \int_{-1}^{1} \text{odd function}$$
$$= x^2 - \frac{1}{3}.$$

We can then check that $\|e_3\|^2 = \frac{8}{45}$, so we set

$$\hat{e}_3 = \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3})$$
$$= \frac{1}{2} \sqrt{\frac{45}{2}} \frac{1}{3} (3x^2 - 1)$$
$$= \frac{1}{3} \sqrt{\frac{45}{2}} \left(\frac{3x^2 - 1}{2} \right).$$

In summary, this yields

$$\hat{e}_1 = \frac{1}{\sqrt{2}}$$
$$\hat{e}_2 = x$$
$$\hat{e}_3 = \frac{1}{3} \sqrt{\frac{45}{2}} \left(\frac{3x^2 - 1}{2} \right),$$

which are scalar multiples of the first three Legendre polynomials.

5.2 Part b

Let $p(x) = a + bx + cx^2$, we are then looking for p such that $\|x^3 - p(x)\|_2^2$ is minimized. Noting that

$$p(x) \in \text{span}\left\{1, x, x^2\right\} = \text{span}\left\{P_0(x), P_1(x), P_2(x)\right\},$$

we can conclude that $p(x)$ will be the projection of x^3 onto this subspace of $L^2([0,1])$. Thus $p(x) = \sum_{i=0}^{2} \langle x^3, \ \hat{e}_i \rangle \hat{e}_i$.

Proceeding to compute the terms in this expansion, we can note that $\langle x^3, \ f \rangle$ for any f that is even will result in integrating an odd function over a symmetric interval, yielding zero. So only one term doesn't vanish:

$$\left\langle x^3, \ x \right\rangle x = x \int_{-1}^{1} \int_{-1}^{1} x^4 \ dx = \frac{2}{5}x$$

.

And thus $p(x) = \frac{2}{5}x$ is the minimizer.

5.3 Part c