Fundamentals

Algebra

- Properties of logs:
  - $\ln(\prod) = \sum \ln$ but $\prod \ln \neq \ln \sum$
  - $\log_b x = \frac{\ln x}{\ln b}$

Be careful! $\frac{\ln x}{\ln y} \neq \ln \frac{x}{y} = \ln x - \ln y$

- Completing the square:
  - $p(x) = ax^2 + bx + c \implies p(x) = a(x + \frac{b}{2a})^2 + -\frac{1}{2}\left(\frac{b^2-4ac}{2a}\right)$
- Pascal's Triangle:

n	**Sequence**
3	$1, 2, 1$
4	$1, 3, 3, 1$
5	$1, 4, 6, 4, 1$
6	$1, 5, 10, 10, 5, 1$
7	$1, 6, 15, 20, 15, 16, 1$
8	$1, 7, 21, 35, 35, 21, 7, 1$

Obtain new entries by adding in $\mathcal{T}$ pattern (e.g. 7 = 1+6, 12 = 6 + 15, etc).
Note that $\binom{n}{i}$ is given by the entry in the n-th row, i- column.

Table of Small Factorials

n	$n!$
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

$\pi \approx 3.1415926535$
$e \approx 2.71828$
$\sqrt{2} \approx 1.4142135$

Primes Under 100:

$$\begin{array}{l} 2, 3, 5, 7, 11, 13, 17, 19, \\ 23, 29, 31, 37, 41, 43, 47, \\ 53, 59, 61, 67, \\ 71, 73, 79, 83, 89, 97, 101 \end{array}$$

Checking Divisibility by Small Primes

p	$p \mid n \iff$
2	$p \mod 10 = 2, 4$
3	$\sum \text{digits} \mid 3$
5	$p \mod 5 = 0, 5$
7	
11	
13	
17	
23	
27	

Geometry

- Generic Conic Sections

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$\frac{(x - x_0)^2}{w_0} \pm \frac{(y - y_0)^2}{h_0} = c$$

- Circles:

$$Ax^2 + By^2 + C = 0 \qquad (x - x_0)^2 + (y - y_0)^2 = r^2$$

  - Defining trait: locus of points at a constant distance from the **center**
  - **Center** at (x_0, y_0)
- Parabolas:

$$Ax^2 + Bx + Cy + D = 0 \qquad y = ax^2$$

  - Defining Trait:
    - Locus of points equidistant from the **focus** (a point) and the **directrix** (a line)
    - #todo add image
  - **Focus** at $(0, \frac{1}{4a})$
  - **Directrix** at the line $y = -\frac{1}{4a}$
    - For an arbitrary quadratic: complete the square to write in the form $y = a(x - w_0)^2 + h_0$, and translate points of interest by by $(x + w_0, y + h_0)$
- Ellipses:

$$\frac{x^2}{w^2} + \frac{y^2}{h^2} = 1$$

- Defining trait:
  - The locus of points where the *sum* of distances to two **focii** are constant.
- **Center** at $(0,0)$ (can translate easily)
- **Vertices** at $(\pm w, 0)$ and $(0, \pm h)$
- **Focii** at $F_1 = (\sqrt{w^2 - h^2}, 0), F_2 = (-\sqrt{w^2 - h^2}, 0)$
- Another useful shortcut form:

- Hyperbolas:

$$\frac{x^2}{w^2} - \frac{y^2}{h^2} = 1$$

  - Defining trait:
    - Locus of points where the *difference* between the distances to two **focii** are constant.
  - **Vertices** at $(0, \pm h)$ and $(\pm w, 0)$
  - **Focii** at $F_1 = (\sqrt{w^2 + h^2}, 0), F_2 = (-\sqrt{w^2 + h^2}, 0)$
- Summary of Traits:
  - One point p:
    - Distance to p is constant: circle
  - Two points a, b:
    - Distance to a equal to distance to b equals a constant: a line bisecting the midpoint of the line connecting them
    - Difference of distances constant: ellipse
    - Sum of differences constant: hyperbola
  - Point p and a line l:
    - Distance to p equals distance to l equals a constant: parabola
- Areas of certain figures:

Shape	Area / Volume
Circle	πr^2
Annulus	$\pi(r_0 - r_1)^2$
Cylinder	$2\pi r h$
Ellipse	$\frac{1}{2} wh$
Trapezoid:	$\frac{a+b}{2} h$
Any Triangle:	$\frac{1}{2} bh$
Parallelograms:	bh
Cones:	$?$

- Polar coordinates: $(x, y) \mapsto (\sqrt{x^2 + y^2}, \tan^{-1}(\frac{y}{x}))$
- Spherical Coordinates: $[\rho, \phi, \theta]$ where

$$x^2 + y^2 + z^1 = \rho^2$$
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$

Trigonometry

- Trig Values
  - Useful note: $\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$

	sin	cos	tan
0	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$	0
$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{1}$
$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$	∞

- Identities
  - $\sin^2 x + \cos^2 x = 1$ (from Pythagorean theorem)
    - Divide through by $\cos^2 x$ or $\sin^2 x$ to obtain:
      - $\tan^2 + 1 = \sec^2$
      - $1 + \cot^2 x = \csc^2 x$
    - Just listing what conclusions you can pull out of these permutations:

$$\begin{aligned} \sin^2 x &= 1 - \cos^2 x \\ \cos^2 x &= 1 - \sin^2 x \\ \tan^2 x &= \sec^2 - 1 \\ \csc^2 x &= 1 + \cot^2 x \\ \sec^2 x &= 1 + \tan^2 x \\ \cot^2 x &= \csc^2 - 1 \end{aligned}$$

$$\begin{aligned} 1 + \cos^2 x &= \text{nothing!} \\ 1 + \sin^2 x &= \text{nothing!} \\ 1 - \tan^2 x &= \text{nothing!} \\ 1 - \cot^2 x &= \text{nothing!} \\ \tan^2 x - 1 &= \text{nothing!} \\ \cot^2 x - 1 &= \text{nothing!} \end{aligned}$$

  - A derivation with multiple payoffs:

$$\begin{aligned} \cos(a+b) + i\sin(a+b) &= e^{i(a-b)} \\ &= (\cos a + i \sin a)(\cos b - i \sin b) \\ &= (\cos a \cos b + \sin a \sin b) + i(\sin a \cos b + \cos a \sin b) \end{aligned}$$

    - $\sin(a+b) = \sin a \cos b + \cos a \sin b$
    - $\cos(a+b) = \cos a \cos b - \sin a \sin b$
    - $a = b \implies \sin(2a) = 2 \sin a \cos a$
    - $a = b \implies \cos(2a) = \cos^2 a - \sin^2 a$
  - $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
  - $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
  - $\sin 2x = 2 \sin x \cos x$
  - Less useful: $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

    - $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\theta_A$$

  - Totally symmetric under any swap of two symbols
  - Derivation: pick the vertex corresponding to A, label the vectors to the other two vertices $\mathbf{x}, \mathbf{y}$, then

$$\begin{aligned} |\mathbf{x}-\mathbf{y}|^2 &= \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}|\cos\theta \end{aligned}$$

Polynomials

- Vieta's Formulas: Write $p(x) = \sum a_k x^k = \prod (x_k - r_k)$ and expand the product to obtain

$$p(x) = a_n x^n - (\sum_k r_k) x^{n-1} + (\sum_{i<j} r_i r_j) x^{n-1} + \cdots = \sum_{k=1}^{n} (-1)^k \sigma_{n-k}(\{r_i\}_{i=1}^n) x^k$$

where σ_i is the i-th elementary symmetric sum.

  - Example :

$$p(x) = x^2 + bx + c = x^2 - (r_1 + r_2)x + (r_1 r_2)$$

  - Example:

$$\begin{gathered} p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \\ = a_3 x^3 - a_3 (r_1 + r_2 + r_3) x^2 + a_3 (r_1 r_2 + r_1 r_3 + r_2 r_3) x - a_3 (r_1 r_2 r_3) \\ \implies -\frac{a_2}{a_3} = r_1 + r_2 + r_3 \\ \implies \frac{a_1}{a_3} = r_1 r_2 + r_1 r_3 + r_2 r_3 \\ \implies \frac{a_0}{a_3} = r_1 r_2 r_3 \end{gathered}$$

  - Quick conclusions:
    - Sum of roots of a monic polynomial is the **negative** coefficient of x^{n-1}
    - Product of roots of a monic polynomial is the constant coefficient.
- Common enough to memorize:

$$
\begin{aligned}
(a+b)^2 &= a^2+b^2+2ab \\
(a-b)^2 &= a^2+b^2-2ab \\
a^2+b^2 &= (a+b)^2+2ab \\
a^2-b^2 &= (a+b)(a-b) \\
(a+b)^3 &= a^3+b^3+3(a^2b+ab^2) \\
(a-b)^3 &= a^3-b^3+3(-a^2b+ab^2) \\
a^3+b^3 &= (a+b)(a^2+b^2-ab) \\
a^3-b^3 &= (a-b)(a^2+b^2+ab) \\
(\sqrt{a}+\sqrt{b})^2 &= a+b+2\sqrt{ab} \\
(\sqrt{a}-\sqrt{b})^2 &= a+b-2\sqrt{ab} \\
(a+\sqrt{b})(a-\sqrt{b}) &= a^2-b \\
(a+i\sqrt{b})(a-i\sqrt{b}) &= a^2+b \\
(a+b)(a-b) &= a^2+b^2
\end{aligned}
$$

- Polynomial long division
- Rational roots theorem
- Synthetic Division: #todo

General Techniques:

- Gather everything and interpret as a polynomial in some variable
    - Example: given $\sinh(x) = \frac{1}{2}e^x - e^{-x}$, find $\sinh^{-1} x$
        - Let $u = e^x$, and note that $u > 0$ and $x = \ln u$.

$$
\begin{aligned}
x = \frac{1}{2}u + u^{-1} &\implies \\
u - u^{-1} = 2x &\implies \\
u - u^{-1} - 2x = 0 &\implies \\
\mathbf{u}^2 - 2x\mathbf{u} - 1 = 0 &\implies \\
u = \frac{1}{2}(2x \pm \sqrt{4x^2+4})
\end{aligned}
$$

so $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

- Example: you don't always need the quadratic formula. Given $\cosh x = e^x + e^{-x}$, find $\tanh^{-1}(x)$.

$$
\begin{aligned}
x = \frac{u-u^{-1}}{u+u^{-1}} &\implies \\
x(u+u^{-1}) - (u-u^{-1}) = 0 &\implies \\
x(u^2+1) - (u^2-1) = 0 &\implies \\
\mathbf{u}^2(x-1) + (x+1) = 0 &\implies \\
u^2 = \frac{1+x}{1-x}
\end{aligned}
$$

- If you see x^2+y^2, try adding $2xy$ to reduce to $(x+y)^2$
- Finding the minimal polynomial of a number $a+b$: #todo

Single Variable Calculus

Big Theorems / Tools:

- The Fundamental Theorem of Calculus:

$$\frac{\partial}{\partial x} \int_a^x f(t)dt = f(x)$$
$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} g(t)dt = g(b(x))b'(x) - g(a(x))a'(x)$$

- The generalized Fundamental Theorem of Calculus

$$\begin{aligned}\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt &= f(x,\cdot)\frac{\partial}{\partial x}(\cdot)\Big|_{a(x)}^{b(x)} \\ &= f(x,b(x))\, b'(x) - f(x,a(x))\, a'(x)\end{aligned}$$

  - Recover FTC by taking $a(x) = c, b(x) = x, f(x,t) = f(t)$.
    - Note that if $f(x,t) = f(t)$ alone, then $\frac{\partial}{\partial x} f(t) = 0$ and the second integral vanishes
- Extreme Value Theorem
- Involving the Derivative:
  - Mean Value Theorem:

$$f \in C^0(I) \implies \exists p \in I : f(b) - f(a) = f'(p)(b-a)$$

    - Useful variant for integrals and average value:

$$f \in C^0(I) \implies \exists p \in I : \int_a^b f(x)\, dx = f(p)(b-a)$$

  - Rolle's Theorem
- L'Hopital's Rule: If
  - $f(x), g(x)$ differentiable on $I - \{\text{pt}\}$
  - $\lim_{x\to pt} f(x) = \lim_{x\to\{\text{pt}\}} g(x) \in \{0, \pm\infty\}$
  - $\forall x \in I, g'(x) \neq 0$
  - $\lim_{x\to\{\text{pt}\}} \frac{f'(x)}{g'(x)}$ exists

$$\implies \lim_{x\to\{\text{pt}\}} \frac{f(x)}{g(x)} = \lim_{x\to\{\text{pt}\}} \frac{f'(x)}{g'(x)}$$

- Taylor Expansions:

$$\begin{aligned}T(a,x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ &\quad + \frac{1}{6}f'''(a)(x-a)^3 + \frac{1}{24}f^{(4)}(a)(x-a)^4 + \cdots\end{aligned}$$

  Bounded error: $|f(x) - T_k(a,x)| < \left|\frac{1}{(k+1)!}f^{(k+1)}(a)\right|$ where $T_k(a,x)$ is the Taylor series truncated up to and including the x^k term.

Differential

Limits

- Tools for finding $\lim_{x\to a} f(x)$, in order of difficulty:
  - Plug in: equal to $f(a)$ if continuous
  - L'Hopital's Rule (only for indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$)
    - For $\lim f(x)^{g(x)} = 1^\infty, \infty^0, 0^0$, let $L = \lim f^g \implies \ln L = \lim g \ln f$
  - Algebraic rules
  - Squeeze theorem
  - Expand in Taylor series at a
  - Monotonic + bounded
- One-sided limits: $\lim_{x\to a^-} f(x) = \lim_{\varepsilon\to 0} f(a-\varepsilon)$
- Limits at zero or infinity:

$$\lim_{x\to\infty} f(x) = \lim_{\frac{1}{x}\to 0} f(\frac{1}{x}) \text{ and } \lim_{x\to 0} f(x) = \lim_{x\to\infty} f(1/x)$$

  - Also useful:

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \deg p < \deg q \\ \infty & \deg p > \deg q \\ \frac{p_n}{q_n} & \deg p = \deg q \end{cases}$$

- Be careful: limits may not exist!!
  - Example : $\lim_{x\to 0} \frac{1}{x} \neq 0$
- Asymptotes:
  - Vertical asymptotes: at values $x = p$ where $\lim_{x\to p} = \pm\infty$
  - Horizontal asymptotes: given by points $y = L$ where $L \lim_{x\to\pm\infty} f(x) < \infty$
  - Oblique asymptotes: for rational functions, divide - terms without denominators yield equation of asymptote (i.e. look at the asymptotic order or “limiting behavior”).
    - Concretely: $f(x) = \frac{p(x)}{q(x)} = r(x) + \frac{s(x)}{t(x)} \sim r(x)$
- Limit of a recurrence: $x_n = f(x_{n-1}, x_{n-2}, \cdots)$
  - If the limit exists, it is a solution to $x = f(x)$

Derivatives

- Chain rule: $\frac{\partial}{\partial x}(f\circ g)(x) = f'(g(x))g'(x)$
- Product rule: $\frac{\partial}{\partial x} f(x)g(x) = f'g + g'f$
  - Note for all rules: always prime the first thing!
- Quotient rule: $\frac{\partial}{\partial x}\frac{f(x)}{g(x)} = \frac{f'g - g'f}{g^2}$
- Implicit differentiation: $(f(x))' = f'(x)\ dx, (f(y))' = f'(y)\ dy$
  - Often able to solve for $\frac{\partial y}{\partial x}$ this way.
- Obtaining derivatives of inverse functions: if $y = f^{-1}(x)$ then write $f(y) = x$ and implicitly differentiate.
- Approximating change: $\Delta y \approx f'(x)\Delta x$

Related Rates

General series of steps: want to know some unknown rate y_t

- Lay out known relation that involves y

- Take derivative implicitly (say w.r.t t) to obtain a relation between y_t and other stuff.
- Isolate $y_t = \text{ known stuff}$
- Example: ladder sliding down wall
  - Setup: l, x_t and $x(t)$ are known for a given t, want y_t.
  - $x(t)^2 + y(t)^2 = l^2 \implies 2xx_t + 2yy_t = 2ll_t = 0$ (noting that l is constant)
  - So $y_t = -\frac{x(t)}{y(t)} x_t$
  - $x(t)$ is known, so obtain $y(t) = \sqrt{l^2 - x(t)^2}$ and solve.

Integral

- Average values:

$$f_{\text{avg}}(x) = \frac{1}{b-a} \int_a^b f(t)dt$$

  - Proof: apply MVT to $F(x)$.
- Area Between Curves
  - Area in polar coordinates:

$$A = \int_{r_1}^{r_2} \frac{1}{2} r^2(\theta)\ d\theta$$

- Solids of Revolution
  - Disks: $A = \int \pi r(t)^2\ dt$
  - Cylinders: $A = \int 2\pi r(t) h(t)\ dt$

- # Arc lengths

$$\begin{aligned} ds = \sqrt{dx^2 + dy^2} \qquad L &= \int\ ds \\ &= \int_{x_0}^{x_1} \sqrt{1 + \frac{\partial y}{\partial x}}\ dx \\ &= \int_{y_0}^{y_1} \sqrt{\frac{\partial x}{\partial y} + 1}\ dy \end{aligned}$$

  - $SA = \int 2\pi r(x)\ ds$

Big List of Integration Techniques

Given $f(x)$, we want to find an antiderivative $F(x) = \int f$ satisfying $\frac{\partial}{\partial x} F(x) = f(x)$

- Guess and check: look for a function that differentiates to f.
- u- substitution
- Integration by Parts:
  - The standard form:

$$\int u dv = uv - \int v du$$

  - A more general form for repeated applications: let $v^{-1} = \int v$, $v^{-2} = \int \int v$, etc.

$$
\begin{aligned}
\int_a^b uv &= uv^{-1}\Big|_a^b - \int_a^b u^1 v^{-1} \\
&= uv^{-1} - u^1 v^{-2}\Big|_a^b + \int_a^b u^2 v^{-2} \\
&= uv^{-1} - u^1 v^{-2} + u^2 v^{-3}\Big|_a^b - \int_a^b u^3 v^{-3} \\
&\vdots \\
\implies \int_a^b uv &= (-1)^n \int_a^b u^n v^{-n} + \sum_{k=1}^{n} (-1)^k u^{k-1} v^{-k}\Big|_a^b
\end{aligned}
$$

  - Generally useful when one term's nth derivative is a constant.
  - Shoelace method:
  - Note: you can choose u or v equal to 1! Useful if you know the derivative of the integrand.
- Differentiating under the integral

$$
\begin{aligned}
\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)dt - \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt &= f(x,\cdot)\frac{\partial}{\partial x}(\cdot)\Big|_{a(x)}^{b(x)} \\
&= f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x)
\end{aligned}
$$

  - Proof: let $F(x)$ be an antiderivative and compute $F'(x)$ using the chain rule.
  - #todo for constants, this should allow differentiating under the integral when f, f_x are "jointly continuous"

Derivatives	**Integrals**	**Signs**	**Result**
u	v	NA	NA
u'	$\int v$	$+$	$u\int v$
u''	$\int\int v$	$-$	$-u'\int\int v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fill out until one column is zero (alternate signs). Get the result column by multiplying diagonally, then sum down the column.

- Trigonometric Substitution

$$
\begin{aligned}
\sqrt{a^2 - x^2} &\quad\Rightarrow\quad & x = a\sin(\theta) &\quad & dx = a\cos(\theta)\, d\theta \\
\sqrt{a^2 + x^2} &\quad\Rightarrow\quad & x = a\tan(\theta) &\quad & dx = a\sec^2(\theta)\, d\theta \\
\sqrt{x^2 - a^2} &\quad\Rightarrow\quad & x = a\sec(\theta) &\quad & dx = a\sec(\theta)\tan(\theta)\, d\theta
\end{aligned}
$$

- Partial Fractions
- Completing the Square #todo
- Trig Formulas
  - Double angle formulas:

$$
\begin{aligned}
\sin^2(x) &= \frac{1}{2}(1 - 2\cos x) \\
&= \\
&= \\
&= \\
&=
\end{aligned}
$$

- Products of trig functions
  - Setup: $\int \sin^a(x) \cos^b(x) \ dx$
    - Both a, b even: $\sin(x) \cos(x) = \frac{1}{2} \sin(x)$
    - a odd: $\sin^2 = 1 - \cos^2, \ u = \cos(x)$
    - b odd: $\cos^2 = 1 - \sin^2, \ u = \sin(x)$
  - Setup: $\int \tan^a(x) \sec^b(x) \ dx$
    - a odd: $\tan^2 = \sec^2 - 1, \ u = \sec(x)$
    - b even: $\sec^2 = \tan^2 - 1, u = \tan(x)$

Big Derivative / Integral Table

$\frac{\partial f}{\partial x} \Leftarrow$	f	$\Rightarrow \int f dx$
$\frac{1}{2\sqrt{x}}$	$\sqrt{x}$	$\frac{2}{3}x^{\frac{3}{2}}$
nx^{n-1}	$x^n, n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$-nx^{-(n+1)}$	$\frac{1}{x^n}, n \neq 1$	$-\frac{1}{n-1}x^{-(n-1)}$
$\frac{1}{x}$	$\ln(x)$	$x\ln(x) - x$
$a^x \ln(a)$	a^x	$\frac{a^x}{\ln a}$
$\cos(x)$	$\sin(x)$	$-\cos(x)$
$-\csc(x)\cot(x)$	$\csc(x)$	$\ln\|\csc(x) - \cot(x)\|$
$-\sin(x)$	$\cos(x)$	$\sin(x)$
$\sec(x)\tan(x)$	$\sec(x)$	$\ln\|\sec(x) + \tan(x)\|$
$\sec^2(x)$	$\tan(x)$	$\ln\left\|\frac{1}{\cos x}\right\|$
$-\csc^2(x)$	$\cot(x)$	$\ln\|\sin x\|$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$	$x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$	$x\sin^{-1}x + \sqrt{1-x^2}$
$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1}(x)$	$x\cos^{-1}x - \sqrt{1-x^2}$
$\frac{1}{\sqrt{x^2+a}}$	$\ln\left\|x + \sqrt{x^2+a}\right\|$	.
$-2\sin x\cos x$	$\cos^2(x)$	$\frac{1}{2}(x + \sin x\cos x)$
$2\sin x\cos x$	$\sin^2(x)$	$\frac{1}{2}(x - \sin x\cos x)$
$2\csc^2(x)\cot(x)$	$\csc^2(x)$	$-\cot(x)$
$2\sec^2(x)\tan(x)$	$\sec^2(x)$	$\tan(x)$
?	?	?
?	?	?
?	?	?
?	?	?
?	?	?
?	?	?
?	?	?
$(ax+1)e^{ax}$	xe^{ax}	$\frac{1}{a^2}(ax-1)e^{ax}$
?	$e^{ax}\sin(bx)$	$\frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx)$
?	$e^{ax}\cos(bx)$	$\frac{1}{a^2+b^2}e^{ax}(a\sin bx + b\cos bx)$
?	?	?

Other small but useful facts:

$$\int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \cos\theta \, d\theta = 0$$

Optimization

- Critical points: boundary points and wherever $f'(x) = 0$
- Second derivative test:
  - $f''(p) > 0 \implies p$ is a min
  - $f''(p) < 0 \implies p$ is a max

Multivariable Calculus

Notation

$\phi : \mathbb{R}^n \to \mathbb{R}, \quad \phi(x_1, x_2, \cdots) = \cdots$

$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \ \mathbf{F}(x_1, x_2, \cdots) = [\mathbf{F}_1(x_1, x_2, \cdots), \mathbf{F}_2(x_1, x_2, \cdots), \cdots, \mathbf{F}_n(x_1, x_2, \cdots)]$

$\vec{v} = [v_1, v_2, \cdots]$

Partial Derivatives

- Chain Rule: Write out tree of dependent variables:

$$\begin{tikzcd}
 & u \arrow[r] \arrow[dd] \arrow[ddr] & x \\
z \arrow[ur] \arrow[urr] \arrow[dr] \arrow[drr] & & \\
 & v \arrow[r] \arrow[uur] & y
\end{tikzcd}$$

Then sum each possible path, e.g.

$$\begin{aligned}
\left(\frac{\partial z}{\partial x}\right)_y = & \left(\frac{\partial z}{\partial x}\right)_{u,y,v} \\
& + \left(\frac{\partial z}{\partial v}\right)_{x,y,u} \left(\frac{\partial v}{\partial x}\right)_y \\
& + \left(\frac{\partial z}{\partial u}\right)_{x,y,v} \left(\frac{\partial u}{\partial x}\right)_{v,y} \\
& + \left(\frac{\partial z}{\partial u}\right)_{x,y,v} \left(\frac{\partial u}{\partial v}\right)_{x,y} \left(\frac{\partial v}{\partial x}\right)_y
\end{aligned}$$

Where the subscripts denote which variables are held constant.

Approximation and Optimization

- Linear Approximation:
  - $z = f(x, y)$: use Tangent plane formulation to obtain

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- Optimization
  - Critical points of $f(\vec{x})$ given by points $\vec{p}_0$ such that $\nabla f \mid_{\vec{p}_0} = 0$

- Second derivative test: compute $H_f(p_0) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} (\vec{p}_0)$.
- By cases:
  - $H(\mathbf{p}_0) = 0$: No conclusion
  - $H(\mathbf{p}_0) < 0$: Saddle point
  - $H(\mathbf{p}_0) > 0$:
    - $f_{xx}(\mathbf{p}_0) > 0 \implies$ local min
    - $f_{xx}(\mathbf{p}_0) < 0 \implies$ local max
- Mnemonic: make matrix with ∇f as the columns, and then differentiate variables left to right.

- Constrained by domain:
  - Extrema occur on boundaries, so parametrize each boundary to obtain a function in one less variable and apply standard optimization techniques to yield critical points. Test all critical points to find extrema.
- Constrained by an equation:
  - If possible, use constraint to just reduce equation to one dimension and optimze like single-variable case. Otherwise,
  - **Lagrange Multipliers**. The setup:

$$\begin{gathered} \text{Optimize } f(\mathbf{x}) \\ \text{subject to } g(\mathbf{x}) = c \\ \implies \nabla f = \lambda \nabla g \end{gathered}$$

1. Use this formula to obtain a system of equations in the components of x and the parameter λ.
2. Use λ to obtain a relation involving only components of $\mathbf{x}$.
3. Substitute relations **back into constraint** to obtain a collection of critical points.
4. Evaluate f at critical points to find max/min.

Geometry in $\mathbb{R}^3$

Plane Geometry

- Useful to know: rotation matrices

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \implies \mathbf{R}_{\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies \mathbf{R}_{\frac{\pi}{2}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

  - Example use: given $\mathbf{v}, \mathbf{R}_{\frac{\pi}{2}} \mathbf{v} \perp \mathbf{v}$, so useful to obtain normals or other perpendicular vectors in the plane.
- Useful trick: given $\mathbf{v} = [a, b, c]$, one perpendicular vector is $\mathbf{v}^\perp = [c, c, -(a+b)]$ as long as $\mathbf{v} \neq [-1, -1, 0]$ - in this case, choose $\mathbf{v}^\perp = [-(b+c), a, a]$.
- Slope of a line in $\mathbb{R}^2$:

$$\mathbf{v} = [x, y] \in \mathbb{R}^2 \implies m = \frac{y}{x}$$

- Normal to a line in $\mathbb{R}^2$:

$$m^\perp = \frac{-1}{m} \implies \mathbf{v}^\perp = [-y, x]$$

Important Equations
$$

$$

Lines

$$Ax + By + C = 0 \qquad\qquad \mathbf{x} = \mathbf{p} + t\mathbf{v}$$
$$\mathbf{x} \in L \iff \langle \mathbf{x}, \mathbf{n} \rangle = 0?$$

- Determined by a point $\mathbf{p}$ and a vector $\mathbf{v}$ on the line.
  - Also determined by two points $\mathbf{p}_0, \mathbf{p}_1$ by taking $\mathbf{v} = \mathbf{p}_1 - \mathbf{p}_0$
- Symmetric Equation (sometimes useful)
  - Obtained by isolating t in each component and setting results equal:

$$(x, y, z) \in L \iff \frac{x - p_x}{v_x} = \frac{y - p_y}{v_y} = \frac{z - p_z}{v_z}$$

(Note that the denominators are just the coefficients of t in the parametric equation.)

Planes

$$Ax + By + Cz + D = 0 \qquad ax + by + cz = d \qquad \mathbf{x}(t, s) = \mathbf{p} + t\mathbf{v}_1 + s\mathbf{v}_2$$
$$\mathbf{x} \in P \iff \langle \mathbf{n}, \mathbf{x} - \mathbf{p}_0 \rangle = 0$$

- Determined by a point $\mathbf{p}_0$ and a normal vector $\mathbf{n}$
  - Also determined by two points $\mathbf{p}_0, \mathbf{p}_1$ using $\mathbf{n} = \mathbf{p}_0 \times \mathbf{p}_1$
- **Normal vector to a plane**
  - Can read normal off of equation: $\mathbf{n} = [a, b, c]$
- Other Facts

$$d = \langle \mathbf{n}, \mathbf{p}_0 \rangle = n_1 p_1 + n_2 p_2 + n_3 p_3$$

  - Useful trick: once you compute $\mathbf{n}$, you can compute $d = \langle \mathbf{n}, \mathbf{p} \rangle$ for *any* point in the plane (don't necessarily need to use the one you started with, so pick any point that's convenient to calculate)

Surfaces

$$S = \{(x, y, z) \mid f(x, y, z) = 0\} \qquad\qquad z = f(x, y)$$

- **Tangent plane to a surface**:
  - Need a point $\mathbf{p}$ and a normal $\mathbf{n}$. By cases:
  - $f(x, y, z) = 0$
    - ∇f is a normal vector.
    - Write the tangent plane equation $\langle \mathbf{n}, \mathbf{x} - \mathbf{p}_0 \rangle$, done.
  - $z = g(x, y)$:
    - Let $f(x, y, z) = g(x, y) - z$, then $\mathbf{p} \in S \iff \mathbf{p}$ is in a level set of f.
    - ∇f is normal to level sets (and thus the surface), so compute $\nabla f = [g_x, g_y, -1]$
    - Proceed as in previous case
- **Surfaces of revolution**:
  - Given $f(x_1, x_2) = 0$, can be revolved around either the x_1 or x_2 axis.
    - $f(x, y)$ around the x axis yields $f(x, \pm\sqrt{y^2 + z^2}) = 0$
    - $f(x, y)$ around the y axis yields $f(\pm\sqrt{x^2 + z^2}, y) = 0$
    - Remaining cases proceed similarly - leave the axis variable alone, replace other variable with square root involving missing axis.

- Equations of lines tangent to an intersection of surfaces $f(x, y, z) = g(x, y, z)$:
  - Find two normal vectors and take their cross product, e.g. $n = \nabla f \times \nabla g$, then

$$L = \{\mathbf{x} \mid \mathbf{x} = \mathbf{p} + t\mathbf{n}\}$$

- Level curves:
  - Given a surface $f(x, y, z) = 0$, the level curves are obtained by looking at e.g. $f(x, y, c) = 0$.

Curves

$$\mathbf{r}(t) = [x(t), y(t), z(t)]$$

- **Tangent line to a curve**
  - Use the fact that $\mathbf{r}'(t)$ is a tangent vector to $\mathbf{r}(t)$

$$\mathbf{T}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t)$$

- **Normal line to a curve**
  - Use the fact that $\mathbf{r}''(t) \perp \mathbf{r}'(t)$

$$\mathbf{N}(t) = \mathbf{r}(t_0) + t\mathbf{r}''(t)$$

  - Special case: Planar Curves and Lines: $y = f(x)$,
    - Let $g(x, y) = f(x) - y$, then

$$\nabla g = [f_x(x), -1] \implies m = -\frac{1}{f_x(x)}$$

Tangent Lines / Planes

- Key insight: just need a point and a normal vector, and the gradient is normal to level sets.
  The Tangent Plane Equation: for any locus $f(\mathbf{x}) = 0$, we have

$$\mathbf{x} \in T_f(\mathbf{p}_0) \implies \langle \nabla f(\mathbf{p}_0), \mathbf{x} - \mathbf{p}_0 \rangle = 0$$

Normal Lines

Key insight: the gradient is normal.

To find a normal line, you just need a single point $\mathbf{p}$ and a normal vector $\mathbf{n}$; then

$$L = \{\mathbf{x} \mid \mathbf{x} = \mathbf{p} + t\mathbf{v}\}$$

Minimal Distances

Fix a point $\mathbf{p}$. Key idea: find a subspace and project onto it.

Key equations: projection and orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{b}) = \langle \mathbf{b}, \mathbf{a} \rangle \hat{\mathbf{a}} \qquad \mathrm{proj}_{\mathbf{a}}^{\perp}(\mathbf{b}) = \mathbf{b} - \mathrm{proj}_{\mathbf{a}}(\mathbf{b})$$

- **Point to plane**:
  - Given a plane $S = \{\mathbf{x} \in \mathbb{R}^3 \mid n_0 x + n_1 y + n_2 z = d\}$, project onto $S^\perp$ using

$$d = \|\text{proj}_{\mathbf{n}}(\mathbf{p})\|$$

  - Given just two vectors $\mathbf{u}, \mathbf{v}$: manufacture a normal vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ and continue as above.
- **Point to line**:
  - Given a line $L : \mathbf{x}(t) = t\mathbf{v}$ for some fixed $\mathbf{v}$, use

$$d = \|\text{proj}_{\mathbf{v}}^{\perp}(\mathbf{p})\|$$

  - Given a line $L : \mathbf{x}(t) = \mathbf{w}_0 + t\mathbf{w}$, let $\mathbf{v} = \mathbf{x}(1) - \mathbf{x}(0)$ and proceed as above.
- **Line to line**:
  - Given $\mathbf{r}_1(t) = \mathbf{p}_1 + t\mathbf{v}_2$ and $\mathbf{r}_2(t) = \mathbf{p}_2 + t\mathbf{v}_2$:
    - Let $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$, which is normal to both lines.
    - Then project the vector between any two points onto this normal:

$$d = \|\text{proj}_{\mathbf{n}}(\mathbf{p}_2 - \mathbf{p}_1)\|$$

Vector Calculus

Notation

R is a region, S is a surface, V is a solid.

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial S} [\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3] \cdot [dx, dy, dz] = \oint_{\partial S} \mathbf{F}_1 dx + \mathbf{F}_2 dy + \mathbf{F}_3 dz$$

Big Theorems:

- Green's Theorem:

$$\oint_{\partial R} (L\,dx + M\,dy) = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dxdy$$

- Divergence Theorem:

$$\iint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

- Stokes' Theorem:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

  - Equals zero if S is a closed surface $(\partial S = \emptyset)$
- Computing Areas with Green's Theorem:

$$A(R) = \oint_{\partial R} x\,dy = -\oint_{\partial R} y\,dx = \frac{1}{2} \oint_{\partial R} -y\,dx + x\,dy$$

- $\nabla \times (\nabla \phi) = 0$
- $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

Definitions

$$\mathbf{a}\cdot\mathbf{b} = \langle \mathbf{x},\mathbf{y}\rangle = \sum_{i=1}^n x_i y_i = x_1y_1 + x_2y_2 + \cdots \qquad \text{inner/dot product}$$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x_1^2 + x_2^2 + \cdots} \qquad \text{norm}$$

$$\mathbf{a}\times\mathbf{b} = \hat{\mathbf{n}}\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta_{\mathbf{a},\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{cross product}$$

$$\nabla := \sum_{i=1}^n \frac{\partial}{\partial x_i}\mathbf{e}_i = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right] \qquad \text{del operator}$$

$$\nabla\phi := \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}\,\mathbf{e}_i = [\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \cdots, \frac{\partial\phi}{\partial x_n}] \qquad \text{gradient}$$

$$D_{\mathbf{u}}(\phi) = \nabla\phi\cdot\hat{\mathbf{u}} \qquad \text{directional derivative}$$

$$\nabla\cdot\mathbf{F} := \sum_{i=1}^n \frac{\partial\mathbf{F}_i}{\partial x_i} = \frac{\partial\mathbf{F}_1}{\partial x_1} + \frac{\partial\mathbf{F}_2}{\partial x_2} + \cdots + \frac{\partial\mathbf{F}_n}{\partial x_n} \qquad \text{divergence}$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \end{vmatrix} = [\mathbf{F}_{3y} - \mathbf{F}_{2z}, \mathbf{F}_{1z} - \mathbf{F}_{3x}, \mathbf{F}_{2x} - \mathbf{F}_{1y}] \qquad \text{curl}$$

- Note that the directional derivative uses a normalized direction vector!
- Function Types

$$\begin{aligned} \nabla : (\mathbb{R}^n \to \mathbb{R}) &\to (\mathbb{R}^n \to \mathbb{R}^n) \\ \phi &\mapsto \nabla\phi := \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}\,\mathbf{e}_i \end{aligned}$$

$$\begin{aligned} \operatorname{div}(\mathbf{F}) : (\mathbb{R}^n \to \mathbb{R}^n) &\to (\mathbb{R}^n \to \mathbb{R}) \\ \mathbf{F} &\mapsto \nabla\cdot\mathbf{F} := \sum_{i=1}^n \frac{\partial\mathbf{F}_i}{\partial x_i} \end{aligned}$$

$$\begin{aligned} \operatorname{curl}(\mathbf{F}) : (\mathbb{R}^3 \to \mathbb{R}^3) &\to (\mathbb{R}^3 \to \mathbb{R}^3) \\ \mathbf{F} &\mapsto \nabla\times\mathbf{F} \end{aligned}$$

- Some terminology:

$$\begin{aligned} &\text{Scalar Field} && \phi : X \to \mathbb{R} \\ &\text{Vector Field} && \mathbf{F} : X \to \mathbb{R}^n \\ &\text{Gradient Field} && \mathbf{F} : X \to \mathbb{R}^n \mid \exists\phi : X \to \mathbb{R} \mid \nabla\phi = F \end{aligned}$$

- The Gradient: lifts scalar fields on $\mathbb{R}^n$ to vector fields on $\mathbb{R}^n$
- Divergence: drops vector fields on $\mathbb{R}^n$ to scalar fields on $\mathbb{R}^n$
- Curl: takes vector fields on $\mathbb{R}^3$ to vector fields on $\mathbb{R}^3$
- Spherical Coordinates:

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \end{aligned}$$

Computations

- **Line Integrals Of Curves**

- Parametrize the path C as $\{\mathbf{r}(t) : t \in [a,b]\}$, then

$$\int_C f\ ds := \int_a^b (f \circ \mathbf{r})(t)\ \|\mathbf{r}'(t)\|\ dt$$
$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x_t^2 + y_t^2 + z_t^2}\ dt$$

- **Line Integrals of Vector Fields**
  - If exact:

$$\frac{\partial}{\partial y}\mathbf{F}_1 = \frac{\partial}{\partial x}\mathbf{F}_2 \implies \int \mathbf{F}_1\ dx + \mathbf{F}_2\ dy = \phi(\mathbf{p}_1) - \phi(\mathbf{p}_0)$$

    The function ϕ can be found using the same method from ODEs.
  - Parametrize the path C as $\{\mathbf{r}(t) : t \in [a,b]\}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_a^b (\mathbf{F} \circ \mathbf{r})(t) \cdot \mathbf{r}'(t)\ dt$$
$$= \int_a^b [\mathbf{F}_1(x(t), y(t), \cdots), \mathbf{F}_2(x(t), y(t), \cdots)] \cdot [x_t, y_t, \cdots]\ dt$$
$$\int_a^b \mathbf{F}_1(x(t), y(t) \cdots)x_t + \mathbf{F}_2(x(t), y(t), \cdots)y_t + \cdots\ dt$$

  - Equivalently written:

$$\int_a^b \mathbf{F}_1\ dx + \mathbf{F}_2\ dy + \cdots := \int_C \mathbf{F} \cdot d\mathbf{r}$$

    in which case $[dx, dy, \cdots] := [x_t, y_t, \cdots] = \mathbf{r}'(t)$.
  - *Remember to substitute dx back into the integrand!!*
- **Flux Integrals**:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS$$

- **Computing Areas with Green's Theorem**
  - Given R and $f(x,y) = 0$
    - Compute

$$\oint_{\partial R} x\ dy = -\oint_{\partial R} y\ dx$$
$$= \frac{1}{2}\oint_{\partial R} -y\ dx + x\ dy = \frac{1}{2}\iint_R 1 - (-1)\ dA = \iint_R 1\ dA$$

  - Steps:
    - Parametrize C

Other Results

- $\nabla \cdot \mathbf{F} = 0 \not\Longrightarrow \exists G : \mathbf{F} = \nabla \times G$
  - Counterexample

$$\mathbf{F}(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}[x,y,z]\,, \quad S = S^2 \subset \mathbb{R}^3$$
$$\implies \nabla \mathbf{F} = 0 \text{ but } \iint_{S^2} \mathbf{F}\cdot d\mathbf{S} = 4\pi \neq 0$$

Where by Stokes' theorem,

$$\mathbf{F} = \nabla\times\mathbf{G} \implies \iint_{S^2}\mathbf{F} = \iint_{S^2}\nabla\times\mathbf{G} \overset{\text{Stokes}}{=} \oint_{\partial S^2}\mathbf{G}\,d\mathbf{r} = 0$$

since $\partial S^2 = \emptyset$.

- Sufficient condition: $\mathbf{F}$ is everywhere C^1

$$\exists\mathbf{G}:\ \mathbf{F} = \nabla\times\mathbf{G} \iff \forall \text{ closed } S, \iint_S \mathbf{F}\cdot d\mathbf{S} = 0$$

- Recovering Green's Theorem from Stokes' Theorem:
  - Let $\mathbf{F} = [L, M, 0]$, then $\nabla\times\mathbf{F} = [0, 0, \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}]$

Ordinary Differential Equations

Techniques Overview

$$\begin{aligned} p(y)y' = q(x) && \text{separable} \\ y' + p(x)y = q(x) && \text{integrating factor} \\ y' = f(x,y), f(tx,ty) = f(x,y) && y = xV(x) \text{ COV reduces to separable} \\ y' + p(x)y = q(x)y^n && \text{Bernoulli, divide by } y^n \text{ and COV } u = y^{1-n} \\ M(x,y)dx + N(x,y)dy = 0 && M_y = N_x : \phi(x,y) = c(\phi_x = M, \phi_y = N) \\ P(D)y = f(x,y) && x^k e^{rx} \text{ for each root} \end{aligned}$$

Where e^{zx} yields $e^{ax}\cos bx, e^{ax}\sin bx$

Ordinary Differential Equations

- Separable equations:

$$p(y)\frac{dy}{dx} - q(x) = 0 \implies \int p(y)dy = \int q(x)dx + C$$

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

- Population growth:

$$\frac{dP}{dt} = kP \implies \qquad P = P_0 e^{kt}$$

- Logistic growth:
  - General form: $\frac{dP}{dt} = (B(t) - D(t))P(t)$
  - Assume birth rate is constant $B(t) = B_0$ and death rate is proportional to instantaneous population $D(t) = D_0 P(t)$. Then let $r = B_0, C = B_0/D_0$ be the *carrying capacity*:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P \implies \qquad P(t) = \frac{P_0}{\frac{P_0}{C} + e^{-rt}(1 - \frac{P_0}{C})}$$

- First order linear:

$$\frac{dy}{dx} + p(x)y = q(x) \implies I(x) = e^{\int p(x)dx}, \qquad y(x) = \frac{1}{I(x)}\left(\int q(x)I(x)dx + C\right)$$

- Exact:
  - $M(x, y)dx + N(x, y)dy = 0 \text{ is exact } \iff \exists\phi : \frac{\partial\phi}{\partial x} = M(x, y),\ \frac{\partial\phi}{\partial y} = N(x, y)$ $\iff \frac{\partial M}{\partial y} = \frac{\partial N}{x}$
  - General solution:

$$\phi(x, y) = \int^x M(s, y)ds + \int^y N(x, t)dt - \int^y \frac{\partial}{\partial t}\left(\int^x M(s, t)ds\right)dt$$

  (where $\int^x f(t)dt$ means take the antiderivative of f and consider it a function of x)
- Cauchy Euler: #todo
- Bernoulli: $todo

Linear Homogeneous

General form:

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + cy' + cy = 0$$
$$p(D)y = \prod (D - r_i)^{m_i} y = 0$$

where p is a polynomial in the differential operator D with roots r_i:

- Real roots: contribute m_i solutions of the form

$$e^{rx}, xe^{rx}, \cdots, x^{m_i - 1}e^{rx}$$

- Complex conjugate roots: for $r = a + bi$, contribute $2m_i$ solutions of the form

$$e^{(a \pm bi)x}, xe^{(a \pm bi)x},\ \cdots,\ x^{m_i - 1}e^{(a \pm bi)x}$$
$$= e^{ax}\cos(bx), e^{ax}\sin(bx),\ xe^{ax}\cos(bx), xe^{ax}\sin(bx),\ \cdots,$$

Example: by cases, second order equation of the form

$$ay'' + by' + cy = 0$$

- Two distinct roots: $c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- One real root: $c_1 e^{rx} + c_2 xe^{rx}$

- Complex conjugates $\alpha \pm i\beta$: $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Linear Inhomogeneous

General form:

$$y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + cy' + cy = F(x)$$
$$p(D)y = \prod (D - r_i)^{m_i} y = 0$$

Then solutions are of the form $y_c + y_p$, where y_c is the solution to the associated homogeneous system and y_p is a particular solution.

Methods of obtaining particular solutions

Undetermined Coefficients

- Find an operator $p(D)$ the annihilates $F(x)$ (so $q(D)F = 0$)
- Find solution of $q(D)p(D) = 0$, subtract of known solutions from homogeneous part to obtain the form of the trial solution $A_0 f(x)$, where A_0 is the undetermined coefficient
- Substitute trial solution into original equation to determine A_0

Useful Annihilators:

$$\begin{aligned}
&F(x) = p(x): && D^{\deg(p)+1} \\
&F(x) = p(x)e^{ax}: && (D-a)^{\deg(p)+1} \\
&F(x) = \cos(ax) + \sin(ax): && D^2 + a^2 \\
&F(x) = e^{ax}(a_0 \cos(bx) + b_0 \sin(bx)): && (D-z)(D-\overline{z}) = D^2 - 2aD + a^2 + b^2 \\
&F(x) = p(x)e^{ax}\cos(bx) + p(x)e^{ax}\cos(bx): && ((D-z)(D-\overline{z}))^{\max(\deg(p),\deg(q))+1}
\end{aligned}$$

Variation of Parameters

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#todo
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Reduction of Order

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#todo
```

Systems of Differential Equations

General form:

$$\frac{\partial \overrightarrow{x(t)}}{\partial t} = A\overrightarrow{x(t)} + \overrightarrow{b(t)} \text{ or } \mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$$

General solution to homogeneous equation:

$$c_1 \overrightarrow{x_1(t)} + c_2 \overrightarrow{x_2(t)} + \cdots + c_n \overrightarrow{x_n(t)} = X(t)\vec{c}$$

If A is a matrix of constants: $\overrightarrow{x(t)} = e^{\lambda_i t}\ \vec{v}_i$ is a solution for each eigenvalue/eigenvector pair (λ_i, v_i)

- If A is defective: #todo generalized eigenvectors…?

Nonhomogeneous Equation: particular solutions given by $\vec{x}_p(t) = X(t) \int^t X^{-1}(s)\overrightarrow{b(s)}\, ds$

Laplace Transforms

Definitions:

$$H_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$
$$\delta(t) : \int_{\mathbb{R}} \delta(t-a) f(t)\, dt = f(a), \quad \int_{\mathbb{R}} \delta(t-a)\, dt = 1$$
$$(f * g)(t) = \int_0^t f(t-s) g(s)\, ds$$

Useful property: for $a \le b$, $H_a(t) - H_b(t) = \mathbb{1}\,[a, b]$.

$$\begin{aligned}
t^n, n \in \mathbb{N} \quad &\Longleftrightarrow \quad & n! \frac{1}{s^{n+1}}, \quad s > 0 \\
t^{-\frac{1}{2}} \quad &\Longleftrightarrow \quad & \sqrt{\pi}\, s^{-\frac{1}{2}} \quad s > 0 \\
e^{at} \quad &\Longleftrightarrow \quad & \frac{1}{s-a}, \quad s > a \\
\cos(bt) \quad &\Longleftrightarrow \quad & \frac{s}{s^2 + b^2}, \quad s > 0 \\
\sin(bt) \quad &\Longleftrightarrow \quad & \frac{b}{s^2 + b^2}, \quad s > 0 \\
\delta(t-a) \quad &\Longleftrightarrow \quad & e^{-as} \\
H_a(t) \quad &\Longleftrightarrow \quad & s^{-1} e^{-as} \\
e^{at} f(t) \quad &\Longleftrightarrow \quad & F(s-a) \\
H_a(t) f(t-a) \quad &\Longleftrightarrow \quad & e^{-as} F(s) \\
f'(t) \quad &\Longleftrightarrow \quad & sL(f) - f(0) \\
f''(t) \quad &\Longleftrightarrow \quad & s^2 L(f) - sf(0) - f'(0) \\
f^{(n)}(t) \quad &\Longleftrightarrow \quad & s^n L(f) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0) \\
f(t) g(t) \quad &\Longleftrightarrow \quad & F(s) * G(s)
\end{aligned}$$

- For f periodic with period T, $L(f) = \frac{1}{1+e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

Linear Algebra

Assume everywhere that A is an $m \times n$ matrix that represents a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$

General Notes

- Rank: number of nonzero rows in RREF
- $\mathrm{Trace}(A) = \sum_{i=1}^m A_{i,i}$
- Elementary row operations / matrices:
  - Permute rows
  - Multiple a row by a scalar

    - Add any row to another
- $A(m \times n),\ B(n \times p),\ AB = C \implies c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \langle \mathbf{a_i^T}, \mathbf{b_j} \rangle$
    - i.e., the c_{ij} entry is just dotting row i of A with column j of B.

Systems of Linear Equations

Notation: $A\vec{x} = \vec{b}$ a linear system, $r = \operatorname{rank}(A)$ and $r' = \operatorname{rank}(A \mid \vec{b})$ an augmented matrix.

- Consistent: A system of linear equations is *consistent* when it has at least one solution.
- Inconsistent: A system of linear equations is *inconsistent* when it has no solutions.
- Tall matrices: more equations than unknowns
- Wide matrices: more unknowns than equations
- Three possibilities for a system of linear equations:
    - No solutions
    - One unique solution
    - Infinitely many solutions
- Possibilities:
    - $r < r'$: the system is inconsistent.
    - $r = r'$: the system is consistent, and
        - $r' = n \implies$ there is a unique solution (square, tall)
        - $r' < n \implies$ there are infinitely many solutions (wide)
- Homogeneous systems are **always** consistent.

The Determinant

- Properties of the Determinant $A : m \times n$
    - $\det(AB) = \det(A)\det(B)$
    - Permute two Rows: $\det A' = -\det A$
    - Factor a scalar t out of one row: $\det A' = t \det A$
        - $\det(tA) = t^m \det(A)$
    - Add one row to another: $\det(A') = \det(A)$
    - $\det(L) = \det(U) = \prod_{i=1}^{n} a_{ii}$ for upper or lower triangular matrices.
    - $\det(A^{-1}) = \frac{1}{\det(A)}$
    - $\det A^k = k \det A$
    - $\det A^T = \det A$
    - $\det(\mathbf{a}_1 + \mathbf{a}_2, \cdots) = \det(\mathbf{a}_1, \cdots) + \det(\mathbf{a}_2, \cdots)$
    - If any row of A is all zeros, $\det(A) = 0$.
    - Take $A = \begin{pmatrix} \vec{a} \rightarrow \\ \vec{b} \rightarrow \\ \vdots \end{pmatrix}$, then in $\mathbb{R}^3$, $\det(A)$ is the volume of the parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$.

The Spaces of a Matrix / Linear Map

- Finding bases for various spaces of A:
    - Rowspace: reduce to RREF, and take nonzero rows of RREF $(\subseteq \mathbb{R}^n)$
    - Colspace: reduce to RREF, and take columns with pivots from original A $(\subseteq \mathbb{R}^m)$
    - Nullspace: reduce to RREF, zero rows are free variables, convert back to equations and pull free variables out as scalar multipliers.

Eigenvalues and Eigenvectors

- Defining equation: $\lambda \in E(A) \iff \forall x \in \mathbb{R}^m, A\vec{x} = \lambda\vec{x}$
- Finding: solve $A - I\lambda_i = 0$ for each i.
- $\lambda \in E(A) \implies \lambda^2 \in E(A^2)$ (with the same eigenvector).
- Eigenvectors corresponding to distinct eigenvalues are **always** linearly independent
- A has n distinct eigenvalues $\implies A$ has n linearly independent eigenvectors.
- Similar matrices have identical eigenvalues and multiplicities.
- A matrix A is diagonalizable $\iff A$ has n linearly independent eigenvectors.
- Useful Facts
  - $\prod \lambda_i = \det A$
  - $\sum \lambda_i = \mathrm{Tr}\ A$

Misc

- $|\mathrm{rowspace}(A)| = |\mathrm{colspace}(A)|$
- Proof of Cauchy-Schwarz: See Goode page 346.
- Distance from a point p to a line $\vec{a} + t\vec{b}$: let $\vec{w} = \vec{p} - \vec{a}$, then: $\|w - P(w, v)\|$
  - distance from line to point
- Computing change of basis matrices: #todo
- Two step vector subspace test:
  - Ensure it contains the zero vector
  - Ensure it's closed under scalar multiplication and vector addition
- Any set of two vectors $\{\vec{v}, \vec{w}\}$ is linearly dependent $\iff \exists\lambda :\ \vec{v} = \lambda\vec{w}$.
- A set of functions $\{f_i\}$ is linearly independent on $I \iff \exists x_0 \in I : W(x_0) \neq 0$ (where W is the Wronskian)
  - NOTE: $W \equiv 0$ on $I \not\Longrightarrow \{f_i\}$ is linearly dependent!
  - Counterexample: $\{x, x + x^2, 2x - x^2\}$ where $W \equiv 0$ but $x + x^2 = 3(x) + (2x - x^2)$.
  - Sufficient condition: each f_i is the solution to a linear homogeneous ODE $L(y) = 0$.
- Every square matrix is similar to a matrix in jordan canonical form.
- Projection onto column space of A: given by $P(\vec{x}) = A(A^t A)^{-1} A^T \vec{x}$
- Normal equations: $\vec{x}$ is a least squares solution to $A\vec{x} = \vec{b} \iff A^T A\vec{x} = A^T\vec{b}$

Gram-Schmidt Process

Extending $\{\mathbf{x}_i\}$ to an orthonormal basis

$$
\begin{gathered}
\mathbf{v}_1 = \mathbf{x_1} \\
\mathbf{v}_2 = \mathbf{x}_2 - P(\mathbf{x}_2, \mathbf{v}_1) \\
\mathbf{v}_3 = \mathbf{x_3} - P(\mathbf{x}_3, \mathbf{v_1}) - P(\mathbf{x}_3, \mathbf{v}_2) \\
\cdots \\
\mathbf{v}_i = \mathbf{x_i} - \sum_{k=1}^{i-1} P(\mathbf{x}_i, \mathbf{v}_k) = \mathbf{x}_i - \sum_{k=1}^{i-1} \frac{\langle \mathbf{x}_i, \mathbf{v}_k \rangle}{\|\mathbf{v}_k\|^2} \mathbf{v}_k
\end{gathered}
$$

Inverting a Matrix

Equivalent formulas for A^{-1}:

- Adjoints: $A^{-1} = \frac{\mathrm{adjugate(A)}}{\det(A)}$
- Gauss Jordan: $[A \mid I] \sim [I \mid A^{-1}]$

- Cramer's Rule: $A\vec{x} = \vec{b} \implies x_k = \frac{\det(B_k)}{\det(A)}$ where B_k is A where the k-th column is replaced by $\vec{b}$

Big List of Equivalent Properties

Let A be an $m \times n$ matrix. TFAE:

- A is invertible and has a unique inverse A^{-1}
- A^T is invertible
- $\det(A) \neq 0$
- The linear system $A\bar{x} = \bar{b}$ has a unique solution for every $b \in \mathbb{R}^m$
- The homogeneous system $A\bar{x} = 0$ has only the trivial solution $\bar{x} = 0$
- $\operatorname{rank}(A) = m$ (i.e. A is full rank)
- $\operatorname{nullity}(A) := \dim \operatorname{nullspace}(A) = 0$
- $A = \prod_{i=1}^k E_i$ for some finite k, where each E_i is an elementary matrix.
- A is row-equivalent to the identity matrix I_n
- A has exactly n pivots
- The columns of A are a basis for $\mathbb{R}^n$
    - i.e. $\operatorname{colspace}(A) = \mathbb{R}^n$
- The rows of A are a basis for $\mathbb{R}^m$
    - i.e. $\operatorname{rowspace}(A) = \mathbb{R}^m$
- $(\operatorname{colspace} A)^\perp = (\operatorname{rowspace} A)^\perp = \{\vec{0}\}$
- Zero is not an eigenvalue of A.
- A has n linearly independent eigenvectors
- The rows of A are coplanar.

As a consequence, all of the following negations are equivalent:

- A is not invertible/singular
- At least one row of A is a linear combination of the others
- The $RREF$ of A has a row of all zeros.

Reformulated in terms of linear maps T, TFAE:

- $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ exists
- $\operatorname{im}(T) = \mathbb{R}^n$
- $\ker(T) = 0$
- T is injective
- T is surjective
- T is an isomorphism
- The system $A\bar{x} = 0$ has infinitely many solutions

Complex Analysis

- Properties of modulus:
    - $z = a + ib \implies |z| = \sqrt{a^2 + b^2}$
    - $|z|^2 = z\bar{z} = a^2 + b^2$
    - $\frac{z\bar{z}}{|z|^2} = \frac{(a+ib)(a-ib)}{a^2+b^2} = 1$
- $\frac{1}{a+ib} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{a-ib}{a^2+b^2}$
- $e^{zx} = e^{(a+ib)x} = e^{ax}(\cos(bx) + i\sin(bx))$

- Complex exponential: $x^z := e^{z \ln x}$
- n-th roots: $e^{\frac{ki}{2\pi n}}$
- For $z = a + bi$, $(x - z)(x - \overline{z}) = x^2 - 2\mathcal{R}e(z)x + (a^2 + b^2)$

Real Analysis

Summary for GRE exam:

- limits,
- continuity,
- boundedness,
- compactness,
- definitions of topological spaces,
- Lipschitz continuity
- sequences and series of functions.

Notation used throughout: $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, K is a compact set, and "integrable" or $L_R(K)$ denotes "Riemann integrable on K".

Big Theorems / Formulas

- **Generalized Mean Value Theorem**

$$f, g \text{ differentiable on } [a, b] \implies \exists c \in [a, b] : [f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$$

  - Recover MVT: #todo
- **Bolzano-Weierstrass**: every bounded sequence has a convergent subsequence.
- **Heine-Borel**: in $\mathbb{R}^n$, X is compact $\iff$ X is closed and bounded.

Big Examples

- A function continuous and discontinuous at infinitely many points:

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \end{cases}$$

  - Then f is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{R} - \mathbb{Q}$. Proof
  - Fix ε, let $x_0 \in \mathbb{R} - \mathbb{Q}$, choose $n : \frac{1}{n} < \varepsilon$ using Archimedean property.
    - Define $S = \{x \in \mathbb{Q} : x \in (0, 1), x = \frac{m}{n'}, n' < n\}$
    - Then $|S| \leq 1 + 2 + \cdots (n - 1)$, so choose $\delta = \min_{s \in S} |s - x_0|$
    - Then

$$x \in N_\delta(x_0) \implies f(x) < \frac{1}{n} < \varepsilon \qquad \blacksquare$$

    - #todo, revisit and spell out more
  - Let $x_0 = \frac{p}{q} \in \mathbb{Q}$ and $\{x_n\} = \{x - \frac{1}{n\sqrt{2}}\}$. Then

$$x_n \uparrow x_0 \text{ but } f(x_n) = 0 \to 0 \neq \frac{1}{q} = f(x_0) \qquad \blacksquare$$

Motivation: Commuting Limit Operations

- Suppose $f_n \to f$ (pointwise, not necessarily uniformly)
- Let $F(x) = \int f(t)$ be an antiderivative of f
- Let $f'(x) = \frac{\partial f}{\partial x}(x)$ be the derivative of f.

Then consider the following possible ways to commute various limiting operations:

Does taking the derivative of the integral of a function always return the original function?

$$[\frac{\partial}{\partial x}, \int dx]: \qquad \frac{\partial}{\partial x}\int f(x,t)dt =_? \int \frac{\partial}{\partial x} f(x,t)dt$$

Answer: Sort of (but possibly not).

Counterexample:

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases} \implies \int f \approx |x|,$$

which is not differentiable. (This is remedied by the so-called "weak derivative")

Sufficient Condition: If f is continuous, then both are always equal to $f(x)$ by the FTC.

Is the derivative of a continuous function always continuous?

$$[\frac{\partial}{\partial x}, \lim_{x_i \to x}]: \qquad \lim_{x_i \to x} f'(x_n) =_? f'(\lim_{x_i \to x} x)$$

Answer: No.

Counterexample:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \implies f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which is discontinuous at zero.

Sufficient Condition: There doesn't seem to be a general one (which is perhaps why we study C^k functions).

Is the limit of a sequence of differentiable functions differentiable **and** the derivative of the limit?

$$[\frac{\partial}{\partial x}, \lim_{f_n \to f}]: \qquad \lim_{f_n \to f} \frac{\partial}{\partial x} f_n(x) =_? \frac{\partial}{\partial x} \lim_{f_n \to f} f_n(x)$$

Answer: *Super* no – even the uniform limit of differentiable functions need not be differentiable!

Counterexample: $f_n(x) = \frac{\sin(nx)}{\sqrt{n}} \rightrightarrows f = 0$ but $f_n' \nrightarrow f' = 0$

Sufficient Condition: $f_n \rightrightarrows f$ and $f_n \in C^1$.

Is the limit of a sequence of integrable functions integrable **and** the integral of the limit?

$$[\int dx, \lim_{f_n \to f}](f): \qquad \lim_{f_n \to f} \int f_n(x)dx =_? \int \lim_{f_n \to f} f_n(x)dx$$

Answer: No.

Counterexample: Order $\mathbb{Q} \bigcap [0,1]$ as $\{q_i\}_{i \in \mathbb{N}}$, then take

$$f_n(x) = \sum_{i=1}^{n} \mathbb{1}\left[q_n\right] \to \mathbb{1}\left[\mathbb{Q} \bigcap [0,1]\right]$$

where each f_n integrates to zero (only finitely many discontinuities) but f is not Riemann-integrable.

Sufficient Condition: $

- $f_n \rightrightarrows f$, or
- f integrable and $\exists M: \forall n, |f_n| < M$ (f_n uniformly bounded)

Is the integral of a continuous function also continuous?

$$[\int dx, \lim_{x_i \to x}]: \qquad \lim_{x_i \to x} F(x_i) =_? F(\lim_{x_i \to x} x_i)$$

Answer: Yes.

Proof: $|f(x)| < M$ on I, so given c pick a sequence $x \to c$. Then

$$|f(x)| < M \implies \left|\int_c^x f(t)dt\right| < \int_c^x M dt \implies |F(x) - F(c)| < M(b-a) \to 0$$

Is the limit of a sequence of continuous functions also continuous?

$$[\lim_{x_i \to x}, \lim_{f_n \to f}]: \qquad \lim_{f_n \to f} \lim_{x_i \to x} f(x_i) =_? \lim_{x_i \to x} \lim_{f_n \to f} f_n(x_i)$$

Answer: No.

Counterexample: $f_n(x) = x^n \to \delta(1)$

Sufficient Condition: $f_n \rightrightarrows f$

Does a sum of differentiable functions necessarily converge to a differentiable function?

$$\left[\frac{\partial}{\partial x}, \sum_{f_n}\right]: \qquad \frac{\partial}{\partial x}\sum_{k=1}^{\infty} f_k =_? \sum_{k=1}^{\infty} \frac{\partial}{\partial x} f_k$$

Answer: No.

Counterexample: $f_n(x) = \frac{\sin(nx)}{\sqrt{n}} \rightrightarrows 0 := f$, but $f_n' = \sqrt{n}\cos(nx) \nrightarrow 0 = f'$ (at, say, $x = 0$)

Sufficient Condition: When $f_n \in C^1, \exists x_0 : f_n(x_0) \to f(x_0)$, and $\sum \|f_n'\|_\infty < \infty$ (continuously differentiable, converges at a point, and the derivatives absolutely converge)

Continuity

$$f \text{ cts} \iff \lim_{x \to p} f(x) = f(p)$$

Example of a discontinuous function: $\sin(\frac{1}{x})$ at $x = 0$.

Uniiform continuity #todo

Differentiability

$$f'(p) := \frac{\partial f}{\partial x}(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

- For multivariable functions: existence and continuity of $\frac{\partial \mathbf{f}}{\partial x_i} \forall i \implies \mathbf{f}$ differentiable
  - Necessity of continuity: example of a continuous functions with all partial and directional derivatives that is not differentiable:

$$f(x, y) = \begin{cases} \frac{y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & \text{else} \end{cases}$$

Properties, strongest to weakest

$$C^\infty \subsetneq C^k \subsetneq \text{ differentiable } \subsetneq C^0 \subset L_R(K)$$

- Example showing $f \in C^0 \not\Longrightarrow f$ is differentiable **and** f not differentiable $\not\Longrightarrow f \notin C^0$.
  - Take $f(x) = |x|$ at $x = 0$.
- Example showing that f differentiable $\not\Longrightarrow f \in C^1$:
  - Take

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} \implies f'(x) = \begin{cases} -\cos(\frac{1}{x}) + 2x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

but $\lim_{x \to 0} f'(x)$ does not exist and thus f' is not continuous at zero.

Proof that f differentiable $\implies f \in C^0$:

$$f(x) - f(p) = \frac{f(x) - f(p)}{x - p}(x - p) \overset{\text{hypothesis}}{=} f'(p)(x - p) \overset{x \to p}{\rightrightarrows} 0$$

Giant Table of Relations

Bold are assumed hypothesis, regular text is the strongest conclusion you can reach, strikeout denotes implications that aren't necessarily true.

f'	f	$\therefore f$	F
~~exists~~	**continuous**	K-integrable	exists
~~continuous~~	**differentiable**	continuous	exists
~~exists~~	**integrable**	~~continuous~~	differentiable

Explanation of items in table:

- K-integrable: compactly integrable.
- f integrable $\implies$ F differentiable $\implies$ $F \in C^0$
  - By definition and FTC, and differentiability $\implies$ continuity
- f differentiable and K compact $\implies$ f integrable on K.
  - In general, f differentiable $\not\implies$ f integrable. Necessity of compactness:

$$f(x) = e^x \in C^\infty(\mathbb{R}) \text{ but } \int_{\mathbb{R}} e^x dx \to \infty$$

- f integrable $\not\implies$ f differentiable
  - An integrable function that is not differentiable: $f(x) = |x|$ on $\mathbb{R}$
- f differentiable $\implies$ f continuous a.e.

Integrability

- Sufficient criteria for integrability:
  - f continuous, montone, bounded, finitely many discontinuities, or
  - Uniformly continuous, or
  - Finitely many discontinuities
- f integrable $\iff$ bounded and continuous a.e.
  - Prime example of a non-integrable function: $f = \mathbb{1}[\mathbb{Q}]$
- FTC for the Riemann Integral.
  - If F is a differentiable function on the interval $[a, b]$, and F' is bounded and continuous a.e., then $F' \in L_R([a, b])$ and

$$\forall x \in [a, b] : \int_a^x F'(t)\, dt = F(x) - F(a)$$

  - Suppose f bounded and continuous a.e. on $[a, b]$, and define $F(x) := \int_a^x f(t)\, dt$. Then F is absolutely continuous on $[a, b]$, and for $p \in [a, b]$,

$$f \in C^0(p) \implies F \text{ differentiable at } p,\ F'(p) = f(p), \text{ and } F' \overset{\text{a.e}}{=} f.$$

List of Free Conclusions:

- f integrable on U:
  - f is bounded
  - f is continuous a.e. (finitely many discontinuities)
  - F is continuous
  - F is differentiable
- f continuous on U:
  - f is integrable on compact subsets of U
  - F exists
- f differentiable at a point p:
  - f is continuous

- f is differentiable in U
    - f is continuous a.e.
- Defining the Riemann integral: #todo

Pointwise convergence

$$f_n \to f = \lim_{n\to\infty} f_n$$

Summary:

$$\lim_{f_n\to f}\lim_{x_i\to x} f_n(x_i) \neq \lim_{x_i\to x}\lim_{f_n\to f} f_n(x_i)$$

$$\lim_{f_n\to f}\int_I f_n \neq \int_I \lim_{f_n\to f} f_n$$

- Pointwise convergence is strictly weaker than uniform convergence.
    - Example of a function that converges pointwise but not uniformly: $f_n(x) = x^n$ on $[0,1]$
    - Proof: towards a contradiction let $\varepsilon = \frac{1}{2}$. Then let $n = N(\frac{1}{2})$ and $x = \left(\frac{3}{4}\right)^{\frac{1}{n}}$: then $f(x) = 0$ but $|f_n(x) - f(x)| = x^n = \frac{3}{4} > \frac{1}{2}$.

$\blacksquare$

- f_n continuous $\not\Longrightarrow$ f is continuous
    - i.e. "the pointwise limit of continuous functions is not necessarily continuous"
    - Take

$$f_n(x) = x^n, \quad f_n(x) \to \mathbb{1}[x=1]$$

- f_n differentiable $\not\Longrightarrow$ f_n' converges
    - Take

$$f_n(x) = \frac{1}{n}\sin(n^2x) \to 0, \quad f_n' = n\cos(n^2x) \text{ does not converge}$$

- f_n integrable $\not\Longrightarrow$ $\lim_{f_n\to f}\int_I f_n \neq \int_I \lim_{f_n\to f} f_n$
    - May fail to converge to same value, take

$$f_n(x) = \frac{2n^2x}{(1+n^2x^2)^2} \to 0, \quad \int_0^1 f_n = 1 - \frac{1}{n^2+1} \to 1$$

    -

Uniform Convergence

$$f_n \rightrightarrows f = \lim_{n\to\infty} f_n \text{ and } \sum_{n=1}^{\infty} f_n \rightrightarrows S$$

Summary:

$$\lim_{x_i\to x}\lim_{f_n\to f} f_n(x_i) = \lim_{f_n\to f}\lim_{x_i\to x} f_n(x_i) = \lim_{f_n\to f} f_n(\lim_{x_i\to x} x_i)$$

$$\lim_{f_n\to f}\int_I f_n = \int_I \lim_{f_n\to f} f_n$$

$$\sum_{n=1}^{\infty}\int_I f_n = \int_I \sum_{n=1}^{\infty} f_n$$

"The uniform limit of a(n) x function is x", for $x \in$ {continuous, bounded}

- Equivalent to convergence in the uniform metric on the metric space of bounded functions on X:

$$f_n \rightrightarrows f \iff \sup_{x\in X}|f_n(x) - f(x)| \to 0$$

  - $(B(X,Y), \|\|_\infty)$ is a metric space and $f_n \rightrightarrows f \iff \|f_n - f\|_\infty \to 0$
    (where $B(X,Y)$ are bounded functions from X to Y and $\|f\|_\infty = \sup_{x\in I}\{f(x)\}$
- $f_n \rightrightarrows f \implies f_n \to f$ pointwise
- f_n continuous $\implies f$ continuous
  - i.e. “the uniform limit of continuous functions is continuous”
- $f_n \in C^1$, $\exists x_0 : f_n(x_0) \to f(x_0)$, and $f_n' \rightrightarrows g \implies f$ differentiable and $f' = g$ (i.e. $f_n' \to f'$)
  - Necessity of C^1 – look at failures of f_n' to be continuous:
    - Take $f_n(x) = \sqrt{\frac{1}{n^2} + x^2} \rightrightarrows |x|$, not differentiable
    - Take $f_n(x) = n^{-\frac{1}{2}}\sin(nx) \rightrightarrows 0$ but $f_n' \nrightarrow f' = 0$ and $f' \neq g$
- f_n integrable $\implies f$ integrable and $\int f_n \to \int f$
- f_n bounded $\implies f$ bounded
- $f_n \rightrightarrows f_n \not\Longrightarrow f_n'$ converges
  - Says nothing about it general
- $f_n' \rightrightarrows f' \not\Longrightarrow f_n \rightrightarrows f$
  - Unless f converges at one or more points.

Sequences and Metric Spaces

- Big Theorems:
  - **Bolzano-Weierstrass**: every bounded sequence has a convergent subsequence.
  - **Heine-Borel**: in $\mathbb{R}^n, X$ is compact $\iff X$ is closed and bounded.
    - Necessity of $\mathbb{R}^n$: $X = (\mathbb{Z}, d(x,y) = 1)$ is closed, complete, bounded, but not compact since $\{1, 2, \cdots\}$ has no convergent subsequence
    - Converse holds iff bounded is replaced with totally bounded
  - X compact $\iff X$ sequentially compact
- $\{x_i\} \to p \implies$ every subsequence also converges to p.
- $\{x_i\} \to p \implies \{x_i\}$ is Cauchy
  - Converse holds in complete metric spaces. Example of a Cauchy sequence that doesn’t converge: $x_i = \pi$ truncated to i decimal places in $\mathbb{Q} \subset \mathbb{R}$.
- X complete and $X \subset Y \implies X$ closed in Y
  - Necessity of completeness: $\mathbb{Q} \subset \mathbb{Q}$ is closed but $\mathbb{Q} \subset \mathbb{R}$ is not.
- X compact $\implies X$ complete and bounded.
  - Holds for any metric space, converse generally does not
- X compact and $Y \subset X \implies Y$ compact $\iff Y$ closed.

Series

- Define $s_n(x) = \sum_{k=1}^n f_k(x)$ and $S(x) = \lim_{n\to\infty} s_n(x)$, which can converge pointwise or uniformly.

Sequences and Series of Functions

Notation: $\sum_{k\in\mathbb{N}} f_k$ is a "series"

- $\limsup |f_k(x)| \neq 0 \implies$ not convergent- $\limsup |f_k(x)| \neq 0 \implies$ not convergent

Topology

todo

Number Theory

- Totient Function

$$\phi(p) = p - 1$$
$$\phi(p^k) = p^{k-1}(p-1)$$
$$n = pq, (p,q) = 1 \implies \phi(n) = \phi(p)\phi(q)$$

  - With these properties, the following formulas can be derived:

$$\begin{aligned}\phi(n) = \phi(\prod_i p_i^{k_i}) &= \prod_i p_i^{k_i-1}(p_i - 1) \\ &= n\left(\frac{\prod_i (p_i-1)}{\prod_i p_i}\right) \\ &= n\prod_i (1 - \frac{1}{p_i})\end{aligned}$$

- Fermat's Little Theorem

$$x^n - x = 0 \mod n$$
$$x^{p-1} - 1 = 0 \mod p$$

- The Euclidean Algorithm
- The Jacobi symbol

Abstract Algebra

To Sort

- Fermat's Little Theorem
- The Euclidean Algorithm
- Burnside's Lemma
- The Sylow Theorems
- Galois Theory
- http://mathroughguides.wikidot.com/glossary:abstract-algebra

Big List of Notation

$$
\begin{array}{rrrr}
C(x) = & \{g \in G : gxg^{-1} = x\} & \subseteq G & \text{Centralizer} \\
C_G(x) = & \{gxg^{-1} : g \in G\} & \subseteq G & \text{Conjugacy Class} \\
G_x = & \{g.x : x \in X\} & \subseteq X & \text{Orbit} \\
x_0 = & \{g \in G : g.x = x\} & \subseteq G & \text{Stabilizer} \\
Z(G) = & \{x \in G : \forall g \in G,\ gxg^{-1} = x\} & \subseteq G & \text{Center} \\
\mathrm{Inn}(G) = & \{\phi_g(x) = gxg^{-1}\} & \subseteq \mathrm{Aut}(G) & \text{Inner Aut.} \\
\mathrm{Out}(G) = & \mathrm{Aut}(G)/\mathrm{Inn}(G) & \hookrightarrow \mathrm{Aut}(G) & \text{Outer Aut.} \\
N(H) = & \{g \in G : gHg^{-1} = H\} & \subseteq G & \text{Normalizer}
\end{array}
$$

Group Theory

Notation: $H < G$ a subgroup, $N < G$ a normal subgroup, concatenation is a generic group operation.

- $\mathbb{Z}_n$ the unique cyclic group of order n
- $\mathbf{Q}$ the quaternion group
- $G^n = G \times G \times \cdots G$
- $Z(G)$ the center of G
- $o(G)$ the order of a group
- S_n the symmetric group
- A_n the alternating group
- D_n the dihedral group of order $2n$
- Group Axioms
  - Closure: $a, b \in G \implies ab \in G$
  - Identity: $\exists e \in G \mid a \in G \implies ae = ea = a$
  - Associativity: $a, b, c \in G \implies (ab)c = a(bc)$
  - Invertibility: $a \in G \implies \exists b \in G \mid ab = ba = e$
- Definitions:
  - Order
    - Of a group: $o(G) = |G|$, the cardinality of G
    - Of an element: $o(g) = \min\{n \in \mathbb{N} : g^n = e\}$
  - Index
  - Center: the elements that commute with everything
  - Centralizer: all elements that commute with a given element/subgroup.
  - Group Action: a function $f : X \times G \to G$ satisfying
    - $x \in X, g_1, g_2 \in G \implies g_1 . (g_2 . x) = (g_1 g_2) . x$
    - $x \in X \implies e . x = x$
  - Orbits partition any set
  - Transitive Action
  - Conjugacy Class: $C \subset G$ is a conjugacy class $\iff$
    - $x \in C, g \in G \implies gxg^{-1} \in C$
    - $x, y \in C \implies \exists g \in G : gxg^{-1} = y$
    - i.e. subsets that are closed under G acting on itself by conjugation and on which the action is transitive
    - i.e. orbits under the conjugation action
    - The order of any conjugacy class divides the order of G
  - p-group: Any group of order p^n.
  - Simple Group: no nontrivial normal subgroups
  - Normal Series: $0 \trianglelefteq H_0 \trianglelefteq H_1 \cdots \trianglelefteq G$
  - Composition Series: The successive quotients of the normal series
  - Solvable: G is solvable $\iff$ G has an abelian composition series.
- One step subgroup test:

$$a, b \in H \implies ab^{-1} \in H$$

- Useful isomorphism invariants:
  - Order profile of elements: n_1 elements of order p_1, n_2 elements of order p_2, etc
    - Useful to look at elements of order 2!
  - Order profile of subgroups
  - $Z(A) \cong Z(B)$
  - Number of generators (generators are sent to generators)
  - Number and size of conjugacy classes
  - Number of Sylow-p subgroups.
  - Commutativity
  - “Being cyclic”
  - Automorphism Groups
  - Solvability
  - Nilpotency

Big Theorems

- Isomorphism Theorems

$$\begin{aligned}
\phi : G \to G' &\implies \qquad \frac{G}{\ker \phi} \cong \phi(G) \\
H \trianglelefteq G,\ K < G &\implies \qquad \frac{K}{H \bigcap K} \cong \frac{HK}{H} \\
H, K \trianglelefteq G,\ K < H &\implies \qquad \frac{G/K}{H/K} \cong \frac{G}{H}
\end{aligned}$$

- Lagrange’s Theorem: $H < G \implies o(H) \mid o(G)$
  - Converse is false: $o(A_4) = 12$ but has no order 6 subgroup.
- The GZ Theorem: $G/Z(G)$ cyclic $\implies G \in \mathbf{Ab}$
- Orbit Stabilizer Theorem: $G/x_0 \cong Gx$
- The Class Equation
  - Let $G \curvearrowright X$ and $\mathcal{O}_i \subseteq X$ be the nontrivial orbits, then

$$|X| = |X_0| + \sum_{[x_i] \in X/G} |Gx|$$

  - The right hand side is the number of fixed points, plus a sum over all of the orbits of size greater than 1, where any representative within the orbit is chosen and we look at the index of its stabilizer in G.
  - Let $G \curvearrowright G$ and for each nontrivial conjugacy class C_G choose a representative $[x_i] = C_G = C_G(x_i)$ to obtain

$$|G| = |Z(G)| + \sum_{[x_i] = C_G(x_i)} [G : [x_i]]$$

- Useful facts:
  - $H < G \in \mathbf{Ab} \implies H \trianglelefteq G$
    - Converse doesn’t hold, even if all subgroups are normal. Counterexample: $\mathbf{Q}$
  - $G/Z(G) \cong \mathrm{Inn}(G)$
  - $H, K < G$ with $H \cong K \not\Longrightarrow G/H \cong G/K$
    - Counterexample: $G = \mathbb{Z}_4 \times \mathbb{Z}_2, H =< (0,1) >, K =< (2,0) >$. Then $G/H \cong \mathbb{Z}_4 \not\cong \mathbb{Z}_2^2 \cong G/K$

- $G \in \mathbf{Ab} \implies$ for each p dividing $o(G)$, there is an element of order p
  - Any surjective homomorphism $\phi : A \twoheadrightarrow B$ where $o(A) = o(B)$ is an isomorphism
- Sylow Subgroups:
  - Todo
- Big List of Interesting Groups
  - $\mathbb{Z}_4, \mathbb{Z}_2^2$
  - D_4
  - $Q = \langle a, b | a^4 = 1, a^2 = b^2, ab = ba^3 \rangle$ the quaternion group
  - S^3, the smallest nonabelian group
- Chinese Remainder Theorem:

$$\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q \iff (p, q) = 1$$

- Fundamental Theorem of Finitely Generated Abelian Groups:
  - $G = \mathbb{Z}^n \oplus \bigoplus \mathbb{Z}_{q_i}$
- Finding all of the unique groups of a given order: #todo

Cyclic Groups

- Generated by ?
- For each d dividing $o(G)$, there exists a subgroup H of order d.
  - If $G =< a >$, then take $H =< a^{\frac{n}{d}} >$

The Symmetric Group

- Generated by:
  - Transpositions
  - #todo
- Cycle types: characterized by the number of elements in the cycle.
  - Two elements are in the same conjugacy class $\iff$ they have the same cycle type.
- Inversions: given $\tau = (p_1 \cdots p_n)$, a pair p_i, p_j is *inverted* iff $i < j$ but $p_j < p_i$
- Can count inversions $N(\tau)$
  - Equal to minimum number of transpositions to obtain non-decreasing permutation
- Sign of a permutation: $\sigma(\tau) = (-1)^{N(\tau)}$
- Parity of permutations $\cong (\mathbb{Z}, +)$
  - even $\circ$ even = even
  - odd $\circ$ odd = even
  - even $\circ$ odd = odd

Ring Theory

- Ring Axioms: #todo
  - Examples:
  - Non-Examples:
- Definition of an Ideal
- Definitions of types of rings:
  - Field
  - Unique Factorization Domain (UFD)
  - Principal Ideal Domain (PID)
  - Euclidean Domain:
  - Integral Domain
  - Division Ring

$$\text{field} \implies \text{Euclidean Domain} \implies \text{PID} \implies \text{UFD} \implies \text{integral domain}$$

Counterexamples to inclusions are strict:
- An ED that is not a field:
- A PID that is not an ED: $\mathbb{Q}[\sqrt{19}]$
- A UFD that is not a PID:
- An integral domain that is not a UFD:

- Integral Domains
- Unique Factorization Domains
- Prime Elements
- Prime Ideals
- Field Extensions
- The Chinese Remainder Theorem for Rings
- Polynomial Rings
  - Irreducible Polynomials
    - Over $\mathbb{Z}_2 : x, x+1, x^2+x+1, x^3+x+1, x^3+x^2+1$
    - Eisenstein's Criterion
- Gauss' Lemma
- When is $\mathbb{Q}[\sqrt{d}]$ a field? #todo

Combinatorics

- Choosing: $\binom{n}{k}$

Probability

Summary for GRE:

- Calculating Mean, standard deviation, and variance from PDF,
- Bernoulli trials.

Numerical Analysis

- Euler's Method:
  - To solve $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$, choose a step size ε, and let $x_{n+1} = x_0 + n\varepsilon$. Then

$$y_{n+1} = y_n + \varepsilon f(x_n, y_n)$$

- Decompositions of Matrices:
  - LU
  - Cholesky
  - Singular Value

Appendix

Neat Tricks

- Commuting differentials and integrals:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)dt$$

  - Need $f, \frac{df}{dx}$ to be continuous in both variables. Also need $a(x), b(x) \in C_1$.
  - If a, b are constant, boundary terms vanish.
  - Recover the fundamental theorem with $a(x) = a, b(x) = b, f(x,t) = f(t)$.

Useful Series and Sequences

Notation: $\uparrow, \downarrow$: monotonically converges from below/above.

- Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

- Cauchy Product:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} b_i x^n\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_n b_n\right) x^k$$

- Differentiation:

$$\frac{\partial}{\partial x}\sum_{k=i}^{\infty} a_k x^k = \sum_{k=i+1}^{\infty} k\, a_k x^{k-1}$$

- Common Series

$$
\begin{aligned}
\sum_{k=0}^{N} x^k &= \frac{1-x^{N+1}}{1-x} \\
\sum_{k=1}^{\infty} x^k &= \frac{1}{1-x} && \text{for } |x|<1 \\
\sum_{k=1}^{\infty} kx^{k-1} &= \frac{1}{(1-x)^2} && \text{for } |x|<1 \\
\sum_{k=2}^{\infty} k(k-1)x^{k-2} &= \frac{2}{(1-x)^3} && \text{for } |x|<1 \\
\sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} &= \frac{6}{(1-x)^4} && \text{for } |x|<1 \\
\sum_{k=1}^{\infty} \binom{n}{k} x^k y^{n-k} &= (x+y)^n \\
\sum_{k=1}^{\infty} \frac{x^k}{k} &= -\log(1-x) \\
\sum_{k=0}^{\infty} \frac{x^k}{k!} &= e^x \\
\sum_{n=0}^{\infty} \frac{(-1)^k}{(2n+1)!} x^{2k+1} \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} &= \sin(x) \\
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2n)!} x^{2k} \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} &= \cos(x) \\
\sum_{k=0}^{\infty} \frac{(-1)^k}{2n+1} x^{2k+1} \quad = x - \frac{x^3}{3} + \frac{x^5}{5} &= \arctan(x) \\
\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2n+1} \quad = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots &= \sinh(x) \\
\sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \quad = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots &= \cosh(x) \\
\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} &= \operatorname{arctanh} x \\
\sum_{k=1}^{\infty} \frac{1}{k} &= \infty \\
\sum_{k=1}^{\infty} (-1)^k \frac{1}{k} &= \ln(2) \\
\sum_{k=1}^{N} \frac{1}{k} &\approx \ln(N) + \gamma + \frac{1}{2N} \\
\sum_{k=1}^{\infty} \frac{1}{k^2} &= \frac{\pi^2}{6}
\end{aligned}
$$

Rational Roots Theorem

Partial Fraction Decomposition

Given $R(x) = \frac{p(x)}{q(x)}$, factor $q(x)$ into $\prod q_i(x)$.

- Linear factors of the form $q_i(x) = (ax+b)^n$ contribute

$$r_i(x) = \sum_{k=1}^{n} \frac{A_k}{(ax+b)^k} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots$$

- Irreducible quadratics of the form $q_i(x) = (ax^2+bx+c)^n$ contribute

$$r_i(x) = \sum_{k=1}^{n} \frac{A_k x + B_k}{(ax^2+bx+c)^k} = \frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \cdots$$

  - Note: ax^2+bx+c is irreducible $\iff b^2 < 4ac$
- Write $R(x) = \frac{p(x)}{\prod q_i(x)} = \sum r_i(x)$, then solve for the unknown coefficients A_k, B_k.
  - IMPORTANT SHORTCUT: don't try to solve the resulting linear system: for each $q_i(x)$, multiply through by that factor and evaluate at its root to zero out many terms!
  - For linear terms $q_i(x) = (ax+b)^n$, define $P(x) = (ax+b)^n R(x)$; then

$$A_k = \frac{1}{(n-k)!} P^{(n-k)}(a), \quad k = 1, 2, \cdots n$$
$$\implies A_n = P(a),\ A_{n-1} = P'(a),\ \cdots,\ A_1 = \frac{1}{(n-1)!} P^{(n-1)}(A)$$

  - Note: #todo check, might need to evaluate at $-b/a$ instead, extend to quadratics.