

Title

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Recall the definition of a presheaf: a sheaf of rings on a space is a contravariant functor from its category of open sets to ring, such that

1. $F(\emptyset) = 0$
2. The restriction from U to itself is the identity,
3. Restrictions compose.

Examples:

- Smooth functions on \mathbb{R}^n
- Holomorphic functions on \mathbb{C}

Recall the definition of sheaf: a presheaf satisfying *unique* gluing: given $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ implies that there exists a unique $f \in \mathcal{F}(\cup U_i)$ such that $f|_{U_i} = f_i$.

Question: Are the constant functions on \mathbb{R} a presheaf and/or a sheaf?

Answer: This is a presheaf but not a sheaf. Set $\mathcal{F}(U) = \{f : U \rightarrow \mathbb{R} \mid f(x) = c\} \cong \mathbb{R}$ with $\mathcal{F}(\emptyset) = 0$.

Can check that restrictions of constant functions are constant, the composition of restrictions is the overall restriction, and restriction from U to itself gives the function back.

Given constant functions $f_i \in \mathcal{F}(U_i)$, does there exist a unique constant function $\mathcal{F}(\cup U_i)$ restricting to them? No: take $f_1 = 1$ on $(0, 1)$ and $f_2 = 2$ on $(2, 3)$. Can check that they both restrict to the zero function on the intersection, since these sets are disjoint.

How can we make this into a sheaf? One way: weaken the topology. Another way: define another presheaf \mathcal{G} on \mathbb{R} given by *locally* constant function, i.e. $\{f : U \rightarrow \mathbb{R} \mid \forall p \in U, \exists U_p \ni p, f|_{U_p} \text{ is constant}\}$. Reminiscent of definition of regular functions in terms of local properties.

Example 1.1.

Let $X = \{p, q\}$ be a two-point space with the discrete topology, i.e. every subset is open. Then

define a sheaf by

$$\begin{aligned}
\emptyset &\mapsto 0 \\
\{p\} &\mapsto R \\
\{q\} &\mapsto S \\
\implies \{p, q\} &\mapsto R \times S,
\end{aligned}$$

where the sheaf condition forces the assignment of the whole space to be the product. Note that the first 3 assignments are automatically compatible, which means that we need a unique $f \in \mathcal{F}(X)$ restricting to R and S . In other words, $\mathcal{F}(X)$ needs to be unique and have maps to R, S , but this is exactly the universal property of the product.

Example 1.2.

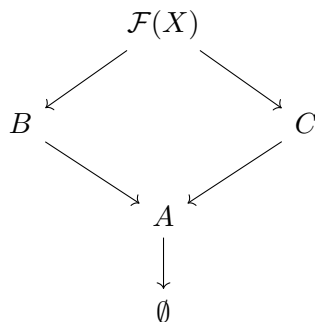
Consider the presheaf on X given by $\mathcal{F}(X) = R \times S \times T$. Taking $T = \mathbb{Z}/2\mathbb{Z}$, we can force uniqueness to fail: by projecting to R, S , there are two elements in the fiber, namely $(r, s, 0) \mapsto r, s$ and $(r, s, 1) \mapsto r, s$.

Example 1.3.

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Can check that it's closed under finite intersections and arbitrary unions, so this forms a topology. Now make the assignments

$$\begin{aligned}
\{a\} &\mapsto A \\
\{b\} &\mapsto B \\
\{a, b\} &\mapsto C \\
X &\mapsto ?.
\end{aligned}$$

We have a situation like this:



Unique gluing says that given $r \in B, s \in C$ such that $\varphi_B(r) = \varphi_C(s)$, there should exist a unique $t \in \mathcal{F}(X)$ such that $t|_{\{a,b\}} = r$ and $t|_{\{a,c\}} = s$. This recovers exactly the fiber product.

$$B \times_A C := \left\{ (r, s) \in B \times C \mid \varphi_B(r) = \varphi_C(s) \in A \right\}.$$

Example 1.4.

Let X be an affine variety with the Zariski topology and let $\mathcal{F} := \mathcal{O}_X$ be the sheaf of regular functions:

$$\mathcal{O}_X(U) := \left\{ f : U \rightarrow k \mid \forall p \in U, \exists U_p \ni p, f|_{U_p} = \frac{g_p}{h_p} \right\}.$$

Is this a presheaf? We can check that there are restriction maps:

$$\begin{aligned} \mathcal{O}_X(U) &\rightarrow \mathcal{O}_X(V) \\ \{f : U \rightarrow k\} &\mapsto \{f|_V(x) := f(x) \text{ for } x \in V\}. \end{aligned}$$

This makes sense because if $V \subset U$, any $x \in V$ is in the domain of f . Given that f is locally a fraction, say $\rho = g_p/h_p$ on $U_p \ni p$, is $\varphi|_V$ locally a fraction? Yes: for all $p \in V \subset U$, $\varphi = g_p/h_p$ on U_p and this remains true on $U_p \cap V$.

To check that \mathcal{O}_X is a sheaf, given a set of regular functions $\{\varphi_i : U_i \rightarrow k\}$ agreeing on intersections, define

$$\begin{aligned} \varphi : \cup U_i &\rightarrow k \\ \varphi(x) &:= \varphi_i(x) \text{ if } x \in U_i. \end{aligned}$$

This is well-defined, since if $x \in U_i \cap U_j$, $\varphi_i(x) = \varphi_j(x)$ since both restrict to the same function on $U_i \cap U_j$ by assumption.

Why is φ locally a fraction? We need to check that for all $p \in U := \cup U_i$ there exists a $U_p \ni p$ with $\varphi|_{U_p} = g_p/h_p$. But any $p \in \cup U_i$ implies $p \in U_i$ for some i . Then there exists an open set $U_{i,p} \ni p$ in U_i such that $\varphi|_{U_{i,p}} = g_p/h_p$ by definition of a regular function. So take $U_p = U_{i,p}$ and use the fact that $\varphi|_{U_i} = \varphi_i$ along with compatibility of restriction.

Remark 1.

General observation: any presheaf of functions is a sheaf when the functions are defined by a local property, i.e any property that can be checked at p by considering an open set $U_p \ni p$.

As in the examples of smooth or holomorphic functions, these were local properties. E.g. checking that a function is smooth involves checking on an open set around each point. On the other hand, being a constant function is not a local property.

Definition 1.0.1 (Restriction of a (Pre)sheaf).

Given a sheaf \mathcal{F} on X and an open set $U \subset X$, we can define a sheaf $\mathcal{F}|_U$ on U (with the subspace topology) by defining $\mathcal{F}|_U(V) := \mathcal{F}(V)$ for $U \subseteq V$.

Definition 1.0.2 (Stalks).

Let \mathcal{F} be a sheaf on X and $p \in X$ a point. The *stalk* of \mathcal{F} at p , denoted \mathcal{F}_p , is defined by

$$\mathcal{F}_p := \{ \}.$$