Title

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Contents

1 | Thursday, December 03

We showed last time that if X is an affine variety, then $T_pX = V\left(f_1 \mid f \in I(X)\right)$ for $p = \mathbf{0} \in \mathbb{A}^n$, and we showed this is naturally isomorphic to $\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)$. Then there was a claim that generalizing this definition to an arbitrary variety X involved taking $\mathfrak{n}_p \leq \mathcal{O}_{X,p}$, a maximal ideal in this local ring of germs of regular functions, given by $\left\{(U, \varphi) \mid p \in U,\ \varphi \in \mathcal{O}_X(U)\right\}$. In this case, $T_p = \left(\mathfrak{n}_p/\mathfrak{n}_p^2\right)$. To prove this, it suffices to show that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathfrak{n}_p/\mathfrak{n}_p^2$. Note that for any affine open $U_i \ni p$, we have $\mathcal{O}_{X,p} = \mathcal{O}_{U_i,p}$.

When X is affine, we have $\mathcal{O}_{X,p} = A(X)_{\mathfrak{m}_p} := \left\{f/g \mid f \in A(X), g \notin \mathfrak{m}_p\right\}/\sim$. Note that this localization makes sense, since the complement of a maximal ideal is multiplicatively closed since it is prime. The equivalence relation was $f/g = f'/g'$ if there exists an $s \notin \mathfrak{m}_p$ such that $s(fg' - f'g) = 0$. We want to show that $\mathfrak{m}_p/\mathfrak{m}_p^2 = \mathfrak{m}_p A(X)_{\mathfrak{m}_p}/\mathfrak{m}_p A(X)_{\mathfrak{m}_p}^2$, i.e. this doesn't change when we localize. In other words, we want to show that $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong S^{-1}\mathfrak{m}_[/(S^{-1}\mathfrak{m}_p)^2$.

Let $f \in S$ so $f(p) \neq 0$. Then $\overline{f} \in A(X)/\mathfrak{m}_p \cong K$ is a nonzero element in a field and thus invertible. Thus $c := 1/\overline{f}$ is an element of $K^\times$, and for all $g \in \mathfrak{m}_p$ we have $g/f \cong cg$ in $\mathfrak{m}_p/\mathfrak{m}_p^2$. So multiplying by elements of S is invertible in $\mathfrak{m}_p/\mathfrak{m}_p^2$. Thus $S^{-1}\left(\mathfrak{m}_p/\mathfrak{m}_p^2\right) \cong \mathfrak{m}_p/\mathfrak{m}_p^2$, where the LHS is isomorphic to $S^{-1}\mathfrak{m}_p/\left(S^{-1}\mathfrak{m}_p^2\right)$.

> **Definition 1.0.1** (Smooth/Regular Varieties)
> A connected variety X is **smooth** (or **regular**) if $\dim T_pX = \dim X$ for all $p \in X$. More generally, an arbitrary (potentially disconnected) variety is smooth if every connected component is smooth.

Example 1.0.2 *(?):* $\mathbb{A}^n$ is smooth since $T_p\mathbb{A}^n = k^n$ for all points p, which has dimension n.

Example 1.0.3 *(?):* $\mathbb{A}^n \coprod \mathbb{A}^{n-1}$ is also smooth since each connected component is smooth.

> **Definition 1.0.4** (Singular Varieties)
> A variety that is not smooth is **singular** at p if $\dim T_pX \neq \dim X$.

Fact 1.0.5: $\dim T_pX \geq \dim X$ for X equidimensional, i.e. every component has the same dimension. This rules out counterexamples like the following in $\mathbb{A}^3$:

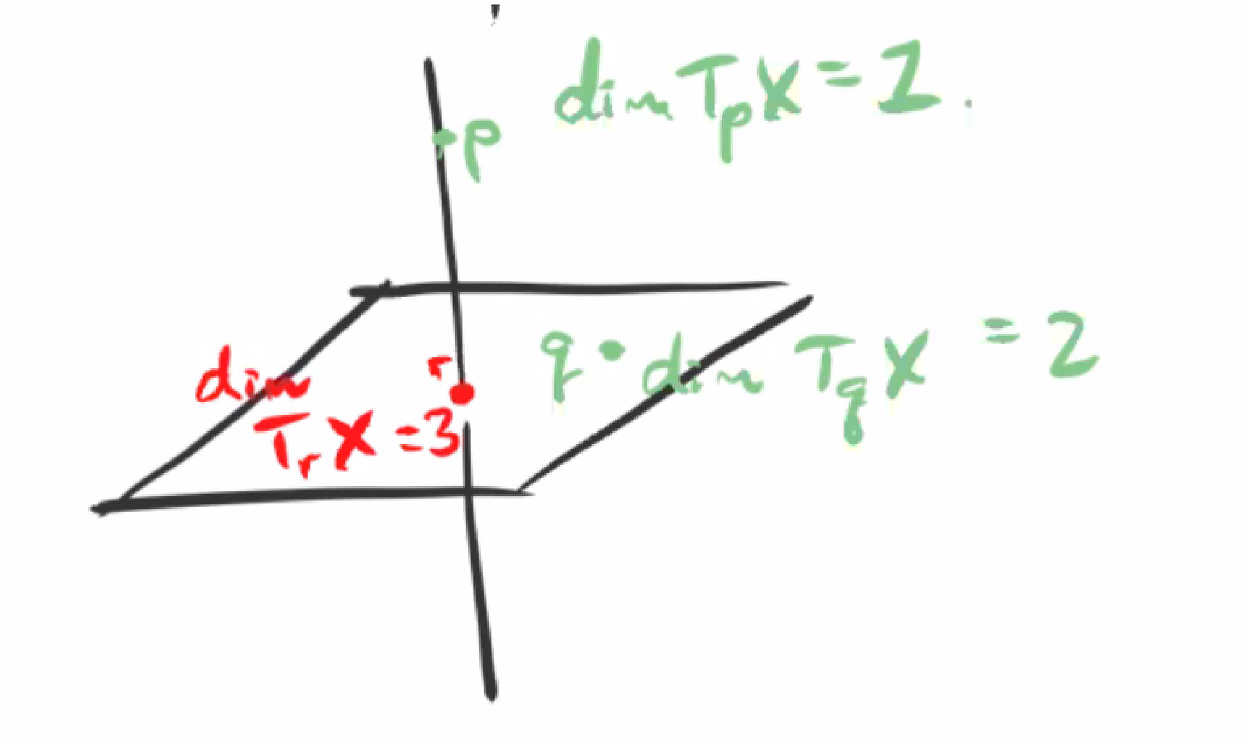

Figure 1: Union of Plane and Axis

Example 1.0.6 ***(?):*** Consider $X := V(y^2 - x^3) \subset \mathbb{A}^2$:

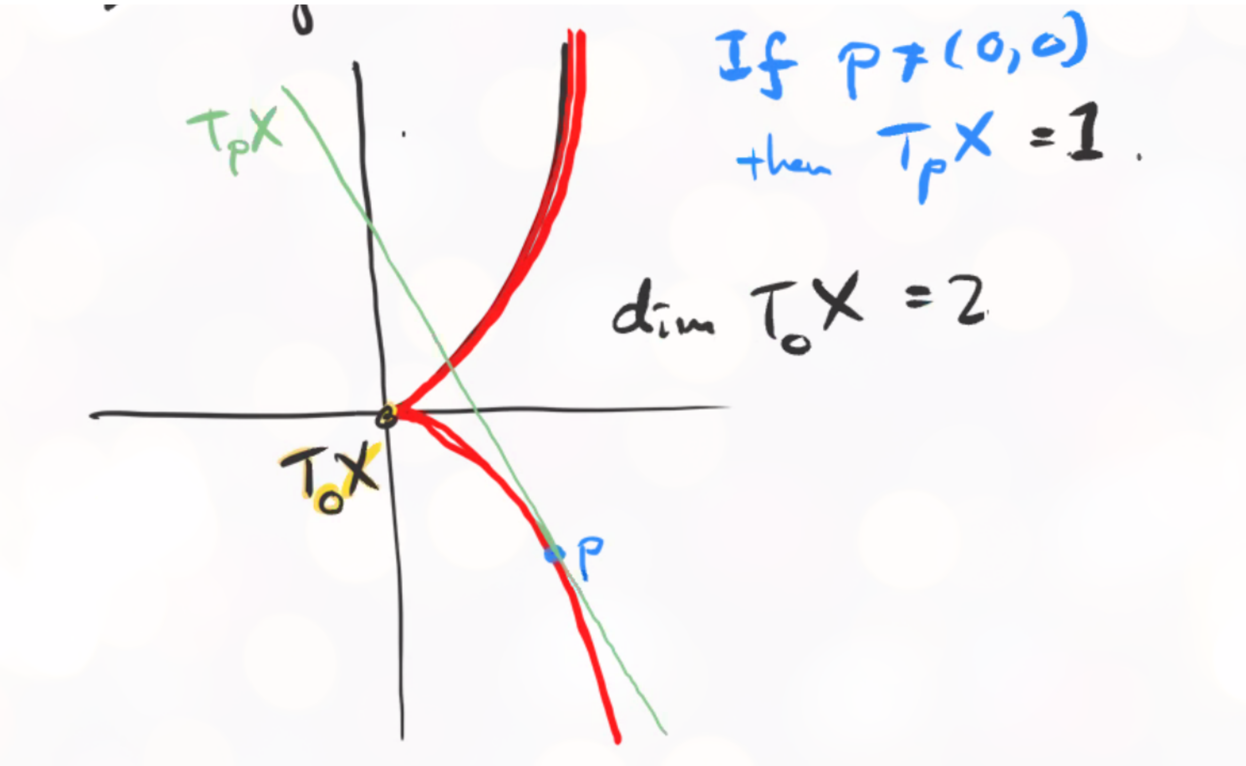

Figure 2: Image

Note that $\dim T_0 X = 2$ is easy to see since it's equal to $V\left(f_1 \;\middle|\; f \in \left\langle y^2 - x^3 \right\rangle\right) = V(0) = k^2$. Thus $p \neq 0$ are smooth points and $p = 0$ is the unique singular point. So X is not smooth, but $X \setminus \{0\}$ is.

Definition 1.0.7 (Regular Ring)
A local ring R over a field k is **regular** iff $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$, the length of the longest chain of prime ideals. Note that we'll add the additional assumption that $R/\mathfrak{m} \cong k$.

Remark 1.0.8: A variety X is thus smooth iff $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim_p X = \dim \mathcal{O}_{X,p}$.

Theorem 1.0.9 ***(A hard theorem from commutative algebra (Auslander-Buchsbanm, 1940s)).***
A regular local ring is a UFD.

Corollary 1.0.10 ***(?).***
Each connected component of a smooth variety is irreducible.

Proof (?).
If a connected component$X$ is not irreducible, then there exists a point $p \in X$ such that $\mathcal{O}_{X,p}$

is not a domain, and thus a nonzero pair $f, g \in \mathcal{O}_{X,p}$ such that $fg = 0$. These exist by simply taking an indicator function on each component. So 0 doesn't have a unique factorization. So $\mathcal{O}_{X,p}$ is not regular, and thus $\dim T_pX > \dim_p X$, which is a contradiction. ■

Remark 1.0.11: How can we check if a variety X is smooth then? Just checking dimensions from the definitions is difficult in general.

Proposition 1.0.12 ***(Jacobi Criterion).***
Let $p \in X$ an affine variety embedded in $\mathbb{A}^n$, and suppose $I(X) = \langle f_1, \cdots, f_r \rangle$. Then X is smooth at $p \iff$ the matrix $\left(\frac{\partial f}{\partial x_j} \right) \Big|_p$ has rank $n - \dim X$.

Example 1.0.13 ***(?):*** Is $V(x^2 - y^2 - 1) \subset \mathbb{A}^2$ smooth? We have $I(X) = \langle f_1 \rangle := \left\langle x^2 - y^2 - 1 \right\rangle$, so let $p \in X$. Then consider the matrix

$$\left[J := \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right] = \left[2x \quad -2y \right].$$

We want to show that at any $p \in X$, we have $\mathrm{rank}(J) = 1$. This is true for $p \neq (0,0)$, but this is not a point in X.

Example 1.0.14 ***(?):*** Consider $X := V(y^2 - x^3 + x^2) \subset \mathbb{A}^2$:

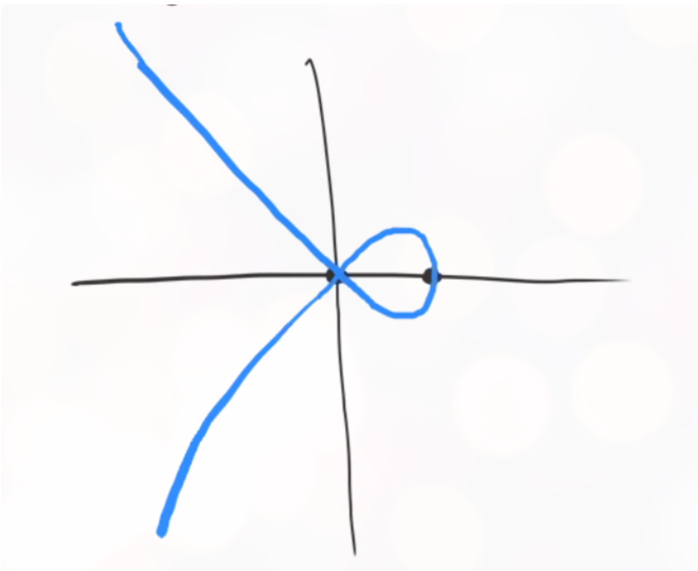

Figure 3: Image

Then $I(X) = \left\langle y^2 - x^3 + x^2 \right\rangle = \langle f \rangle$, and

$$J = \left[2y \quad -3x^2 + 2x \right].$$

Then $\mathrm{rank}(J) = 0$ at $p = (0, 0)$, which is a point in X, so X is not smooth.

Example 1.0.15 *(?)*: Consider $X := V(x^2 + y^2, 1 + z^3) \subset \mathbb{A}^3$, then $I(X) = \left\langle x^2 + y^2, 1 + z^3 \right\rangle \langle f, g \rangle$ which is clearly a radical ideal.

We then have

$$J = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{bmatrix} = \begin{bmatrix} 2x & 2y & 0 \\ 0 & 0 & 3z^2 \end{bmatrix},$$

and thus

$$\mathrm{rank}(J) = \begin{cases} 0 & x = y = z = 0 \\ 1 & x = y = 0 \text{ xor } z = 0 \\ 2 & \text{else.} \end{cases}$$

We can check that $\dim X = 1$ and $\mathrm{codim}_{\mathbb{A}^3} X = 3 - 1 = 2$, so a point $(x, y, z) \in X$ is smooth iff $\mathrm{rank}(J) = 2$. The singular locus is where $x = y = 0$ and $z = \zeta_6$ is any generator of the 6th roots of unity, i.e. $\zeta_6, \zeta_6^3, \zeta_6^5$, along with the point 0. Note that $z = 0$ is not a point on X, since $1 + z^3 \neq 0$ in this case.

Thus the singular locus is $V(x^2 + y^2) = V((x + iy)(x - iy)) \cap V(1 + z^3)$, which results in 3 singular points after intersecting:

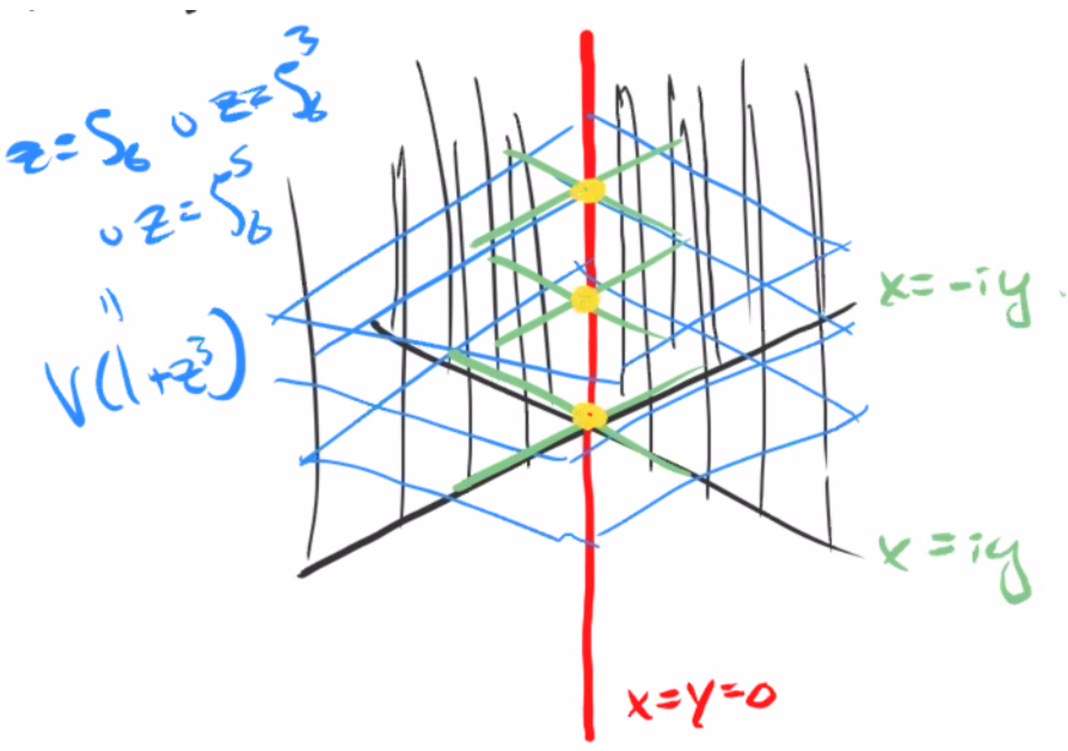

Figure 4: Image

Note that it doesn't matter that $V(1+z^3)$ was intersected here, as long as it's anything that intersects the z-axis nontrivially we will still get something singular.