Problem Set 5

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1 Problem 1

We first make the following definitions:

$$
\begin{aligned}
S &:= \sum_{j=1}^\infty \sum_{k=1}^\infty a_{jk} = \sup \left\{ \sum_{(j,k)\in B} a_{jk} \ni B \subset \mathbb{N}^2,\ |B| < \infty \right\} \\
T &:= \sum_{k=1}^\infty \sum_{j=1}^\infty a_{kj} = \sup \left\{ \sum_{(j,k)\in C} a_{kj} \ni C \subset \mathbb{N}^2,\ |B| < \infty \right\}.
\end{aligned}
$$

We will show that $S = T$ by showing that $S \leq T$ and $T \leq S$.

Let $B \subset \mathbb{N}^2$ be finite, so $B \subseteq [0, I] \times [0, J] \subset \mathbb{N}^2$.

Now letting $R > \max(I, J)$, we can define $C = [0, R]^2$, which satisfies $B \subseteq C \subset \mathbb{N}^2$ and $|C| < \infty$.

Moreover, since $a_{jk} \geq 0$ for all pairs (j, k), we have the following inequality:

$$
\sum_{(j,k)\in B} a_{jk} < \sum_{(k,j)\in C} a_{jk} \leq \sum_{(k,j)\in C} a_{jk} \leq T,
$$

since T is a supremum over *all* such sets C, and the terms of any finite sum can be rearranged.

But since this holds for every B, we this inequality also holds for the supremum of the smaller term by order-limit laws, and so

$$
S := \sup_B \sum_{(k,j)\in B} a_{jk} \leq T.
$$

An identical argument shows that $T \leq S$, yielding the desired equality. $\square$

1.1 Problem 2

We want to show the following equality:

$$\int_0^1 g(x)\ dx = \int_0^1 f(x)\ dx.$$

To that end, we can rewrite this using the integral definition of $g(x)$:

$$\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx = \int_0^1 f(x)\ dx$$

Note that if we can switch the order of integration, we would have

$$\begin{aligned}
\int_0^1 \int_x^1 \frac{f(t)}{t}\ dt\ dx &=_? \int_0^1 \int_0^t \frac{f(t)}{t}\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t} \int_0^t\ dx\ dt \\
&= \int_0^1 \frac{f(t)}{t}(t-0)\ dt \\
&= \int_0^1 f(t)\ dt,
\end{aligned}$$

which is what we wanted to show, and so we are simply left with the task of showing that this is switch of integrals is justified.

To this end, define

$$\begin{aligned}
F : \mathbb{R}^2 &\to \mathbb{R} \\
(x,t) &\mapsto \frac{\chi_A(x,t)\hat{f}(x,t)}{t}.
\end{aligned}$$

where $A = \{(x,t) \subset \mathbb{R}^2 \ni 0 \leq x \leq t \leq 1\}$ and $\hat{f}(x,t) \coloneqq f(t)$ is the cylinder on f.

This defines a measurable function on $\mathbb{R}^2$, since characteristic functions are measurable, the cylinder over a measurable function is measurable, and products/quotients of measurable functions are measurable.

In particular, $|F|$ is measurable and non-negative, and so we can apply Tonelli to $|F|$. This allows us to write

$$\begin{aligned}
\int_{\mathbb{R}^2} |F| &= \int_0^1 \int_0^t \left|\frac{f(t)}{t}\right|\ dx\ dt \\
&= \int_0^1 \int_0^t \frac{|f(t)|}{t}\ dx\ dt \quad \text{since } t > 0 \\
&= \int_0^1 \frac{|f(t)|}{t} \int_0^t\ dx\ dt \\
&= \int_0^1 |f(t)| < \infty,
\end{aligned}$$

where the last inequality holds because f was assumed to be measurable. So F is measurable on the product space $\mathbb{R}^2$, and we can thus apply Tonelli to F to justify the initial switch. $\square$

2 Problem 3

Let $A = \{0 \leq x \leq y\} \subset \mathbb{R}^2$, and define

$$F(x, y) = \frac{\chi_A(x, y) x^{1/3}}{(1 + xy)^{3/2}}.$$

Then $F \in L^1(\mathbb{R}^2) \iff f \in L^1(\mathbb{R}^2)$, and if this is true then we would have

$$\int_{\mathbb{R}^2} F = \int_0^\infty \int_y^\infty f(x, y)\ dx\ dy$$

and if an interchange of integrals is justified, this yields

$$\begin{aligned}
\int_{\mathbb{R}^2} F &=_? \int_0^\infty \int_x^\infty \frac{x^{1/3}}{(1 + xy)^{3/2}}\ dy\ dx \\
&= 2 \int_{\mathbb{R}} \frac{1}{x^{2/3}\sqrt{1 + x^2}}\ dx \\
&= 2 \int_0^1 \frac{1}{x^{2/3}\sqrt{1 + x^2}}\ dx + 2 \int_1^\infty \frac{1}{x^{2/3}\sqrt{1 + x^2}}\ dx \\
&\leq \int_0^1 x^{-2/3}\ dx + \int_0^\infty x^{-5/3} \\
&= 2(3) + 2\left(\frac{3}{2}\right) < \infty,
\end{aligned}$$

where the first term in the split integral is bounded by using the fact that $\sqrt{1 + x^2} \geq \sqrt{x^2} = x$, and the second term from $x > 1 \implies x > 0 \implies \sqrt{1 + x^2} \geq \sqrt{1}$.

Since this implies that F is measurable, it remains to show that this interchange of integrals is justified.

Toward that end, ...