# Title

D. Zack Garza

#### Contents

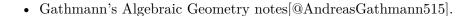
## **Contents**

Prologu		3
0.1	References	3
0.2	Notation	3
0.3	Summary of Important Concepts	4
0.4	Useful Examples	5
	0.4.1 Varieties	5
	0.4.2 Presheaves / Sheaves	5
0.5	Useful Algebra Facts	5
0.6	The Algebra-Geometry Dictionary	6

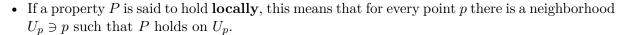
Contents 2

## **Prologue**





#### 0.2 Notation



$$k[\mathbf{x}] \coloneqq k[x_1, \cdots, x_n] \qquad \text{The polynomial ring in $n$ indeterminates} \\ k(\mathbf{x}) \coloneqq k(x_1, \cdots, x_n) \qquad \text{The rational function field} \\ k(\mathbf{x}) \coloneqq \left\{ f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}), \; \middle| \; p, q, \in k[x_1, \cdots, x_n] \right\} \\ V(J), V_a(J) \qquad \text{The variety associated to an ideal } J \trianglelefteq k[x_1, \cdots, x_n] \\ V(J) \coloneqq \left\{ \mathbf{x} \in \mathbb{A}^n \; \middle| \; f(\mathbf{x}) = 0, \, \forall f \in J \right\} \\ I(S), I_a(S) \qquad \text{The ideal associated to a subset } S \subseteq \mathbb{A}^n_k \\ I(S) \coloneqq \left\{ f \in k[x_1, \cdots, x_n] \; \middle| \; f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in X \right\} \\ A(X) \coloneqq k[x_1, \cdots, x_n]/I(X) \qquad \text{The coordinate ring of a variety} \\ \mathcal{O}_X \qquad \text{The structure sheaf } \left\{ f : U \to k \; \middle| \; f \in k(\mathbf{x}) \text{ locally} \right\} \\ D(f) \qquad \qquad \text{A distinguished open set} \\ D(f) \coloneqq V(f)^c = \left\{ x \in \mathbb{A}^n \; \middle| \; f(x) \neq 0 \right\} \\ \Delta_X \qquad \qquad \text{The diagonal } \left\{ (x, x) \; \middle| \; x \in X \right\} \subseteq X \times X \\ \mathcal{U} \rightrightarrows X \qquad \qquad \text{An open cover.}$$

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \le k[x_1, \dots, x_n]$	V(J) the zero locus
I(S) the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
I + J	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$\dot{I}\cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
I(V):I(W)	$\overline{V\setminus W}$

0.1 References 3

Algebra	Geometry
Prime ideals $\mathfrak{p} \in \operatorname{Spec}(k[x_1, \cdots, x_n])$	Irreducible subsets

## 0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions  $V(\cdot)$  and  $I(\cdot)$ ?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?
  - Irreducibility?
  - Connectedness?
  - Dimension?
- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of  $\mathcal{O}_X$ , the sheaf of regular functions?
  - How does one compute  $\mathcal{O}_X$  for X = D(f) a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

### 0.4 Useful Examples



- $V(xy-1) \subseteq \mathbb{A}^2$  a hyperbola
- V(x) a coordinate axis
- V(x-p) a point.

#### 0.4.2 Presheaves / Sheaves

- $C^{\infty}(\cdot,\mathbb{R})$ , a sheaf of smooth functions
- $C^0(\cdot,\mathbb{R})$ , a sheaf of continuous functions
- $\mathcal{O}_X(\cdot)$ , the sheaf of regular functions on X
- $\underline{\mathbb{R}}(\cdot)$ , the constant sheaf associated to  $\mathbb{R}$  (locally constant real-valued functions)
- $\operatorname{Hol}(\cdot,\mathbb{C})$ , a sheaf of holomorphic functions
- $K_p$  the skyscraper sheaf:

$$K_p(U) := \begin{cases} k & p \in U \\ 0 & \text{else.} \end{cases}$$

## 0.5 Useful Algebra Facts

#### Fact 0.5.1:

- $\mathfrak{p} \leq R$  is prime  $\iff R/\mathfrak{p}$  is a domain.
- $\mathfrak{p} \leq R$  is maximal  $\iff R/\mathfrak{p}$  is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and  $\langle f \rangle \leq R$  is generated by an irreducible element f, then  $\langle f \rangle$  is maximal

#### Proposition 0.5.2 (Finitely generated polynomial rings are Noetherian).

A polynomial ring  $k[x_1, \dots, x_n]$  on finitely many generators is Noetherian. In particular, every ideal  $I \subseteq k[x_1, \dots, x_n]$  has a finite set of generators and can be written as  $I = \langle f_1, \dots, f_m \rangle$ .

Proof(?).

A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.5.5),  $k[x_1, \dots, x_n]$  is thus Noetherian.

0.4 Useful Examples 5

Proposition 0.5.3 (Properties and Definitions of Ideal Operations).

$$\begin{split} I+J &\coloneqq \left\{f+g \ \middle| \ f \in I, \ g \in J\right\} \\ IJ &\coloneqq \left\{\sum_{i=1}^N f_i g_i \ \middle| \ f_i \in I, \ g_i \in J, N \in \mathbb{N}\right\} \\ I+J &= \langle 1 \rangle \implies I \cap J = IJ \end{split} \qquad \text{(coprime or comaximal)} \ \langle a \rangle + \langle b \rangle = \langle a,b \rangle \ . \end{split}$$

#### Theorem 0.5.4 (Noether Normalization).

Any finitely-generated field extension  $k_1 \hookrightarrow k_2$  is a finite extension of a purely transcendental extension, i.e. there exist  $t_1, \dots, t_\ell$  such that  $k_2$  is finite over  $k_1(t_1, \dots, t_\ell)$ .

### Theorem 0.5.5 (Hilbert's Basis Theorem).

If R is a Noetherian ring, then R[x] is again Noetherian.

## 0.6 The Algebra-Geometry Dictionary

 $\sim$ 

Let  $k = \bar{k}$ , we're setting up correspondences

Ring Theory Geometry/Topology of Affine Varieties
Polynomial functions Affine space

$$k[x_1, \cdots, x_n]$$
  $\mathbb{A}^n/k := \{[a_1, \cdots, a_n] \in k^n\}$ 

Maximal ideals 
$$\langle x_1 - a_1, \cdots, x_n - a_n \rangle$$
 Points  $[a_1, \cdots, a_n] \in \mathbb{A}^n/k$ 

Radical ideals 
$$I \leq k[x_1, \dots, x_n]$$
 Affine varieties  $X \subset \mathbb{A}^n/k$ , vanishing locii of polynomia

$$I\mapsto V(I)\coloneqq \Big\{a\ \Big|\ f(a)=0 \forall f\in I\Big\}$$
 
$$I(X)\coloneqq \Big\{f\ \Big|\ f|_X=0\Big\} \leftrightarrow X$$

Radical ideals containing 
$$I(X)$$
, i.e. ideals in  $A(X)$  closed subsets of X, i.e. affine subvarieties

$$A(X)$$
 is a domain  $X$  irreducible

$$A(X)$$
 is not a direct sum  $X$  connected  
Prime ideals in  $A(X)$  Irreducible closed subsets of  $X$ 

Krull dimension 
$$n$$
 (longest chain of prime ideals) dim  $X = n$ , (longest chain of irreducible closed subsets