

# Problem Set 1

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## Contents

Source: Section 1 of Gathmann

### Exercise 0.1 (1.19).

Prove that every affine variety  $X \subset \mathbb{A}^n/k$  consisting of only finitely many points can be written as the zero locus of  $n$  polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in  $X$  have different  $x_1$ -coordinates.

### Exercise 0.2 (1.21).

Determine  $\sqrt{I}$  for

$$I := \langle x_1^3 - x_2^6, x_1x_2 - x_2^3 \rangle \subseteq \mathbb{C}[x_1, x_2].$$

#### Solution:

For notational purposes, let  $\mathcal{I}, \mathcal{V}$  denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider  $\mathcal{V}(I)$ , the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 &= 0 \\ xy - y^3 &= 0. \end{cases}$$

In the second equation, we have  $(x - y^2)y = 0$ , and since  $k[x, y]$  is an integral domain

1. If  $y = 0$ , then  $x^3 = 0 \implies x = 0$ , and thus the  $(0, 0) \in \mathcal{V}(I)$ , i.e. the origin of  $\mathbb{A}^2/\mathbb{C}$  is contained in this vanishing locus.
2. Otherwise,

### Exercise 0.3 (1.22).

Let  $X \subset \mathbb{A}^3/k$  be the union of the three coordinate axes. Compute generators for the ideal  $I(X)$  and show that it can not be generated by fewer than 3 elements.

**Exercise 0.4** (1.23: Relative Nullstellensatz).

Let  $Y \subset \mathbb{A}^n/k$  be an affine variety and define  $A(Y)$  by the quotient

$$\pi : k[x_1, \dots, x_n] \longrightarrow A(Y) := k[x_1, \dots, x_n]/I(Y).$$

- a. Show that  $V_Y(J) = V(\pi^{-1}(J))$  for every  $J \trianglelefteq A(Y)$ .
- b. Show that  $\pi^{-1}(I_Y(X)) = I(X)$  for every affine subvariety  $X \subseteq Y$ .
- c. Using the fact that  $I(V(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq k[x_1, \dots, x_n]$ , deduce that  $I_Y(V_Y(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq A(Y)$ .

Conclude that there is an inclusion-reversing bijection

$$\left\{ \begin{array}{c} \text{Affine subvarieties} \\ \text{of } Y \end{array} \right\} \iff \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{in } A(Y) \end{array} \right\}.$$

**Exercise 0.5** (Extra).

Let  $J \trianglelefteq k[x_1, \dots, x_n]$  be an ideal, and find a counterexample to  $I(V(J)) = \sqrt{J}$  when  $k$  is not algebraically closed.