4-Manifolds

Lectures by Philip Engel. University of Georgia, Spring 2020

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

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1.1 Background

From Phil's email:

There are very few references in the notes, and I'll try to update them to include more as we go. Personally, I found the following online references particularly useful:

- Dietmar Salamon: Spin Geometry and Seiberg-Witten Invariants [**Dietmar99**]
- Richard Mandelbaum: Four-dimensional Topology: An Introduction [**Mandelbaum1980**]
  - This book has a nice introduction to surgery aspects of four-manifolds, but as a warning: It was published right before Freedman's famous theorem. For instance, the existence of an exotic R^4 was not known. This actually makes it quite useful, as a summary of what was known before, and provides the historical context in which Freedman's theorem was proven.
- Danny Calegari: Notes on 4-Manifolds [**Calegari**]
- Yuli Rudyak: Piecewise Linear Structures on Topological Manifolds [**Rudyak**]
- Akhil Mathew: The Dirac Operator [**Matthew**]
- Tom Weston: An Introduction to Cobordism Theory [**Weston**]

  A wide variety of lecture notes on the Atiyah-Singer index theorem, which are available online.

1.2 Introduction

Definition 1.2.1 (Topological Manifold)
Recall that a **topological manifold** (or C^0 manifold) X is a Hausdorff topological space *locally homeomorphic* to $\mathbb{R}^n$ with a countable topological base, so we have charts $\varphi_u : U \to \mathbb{R}^n$ which are homeomorphisms from open sets covering X.

Example 1.2.2 ***(The circle):*** S^1 is covered by two charts homeomorphic to intervals:

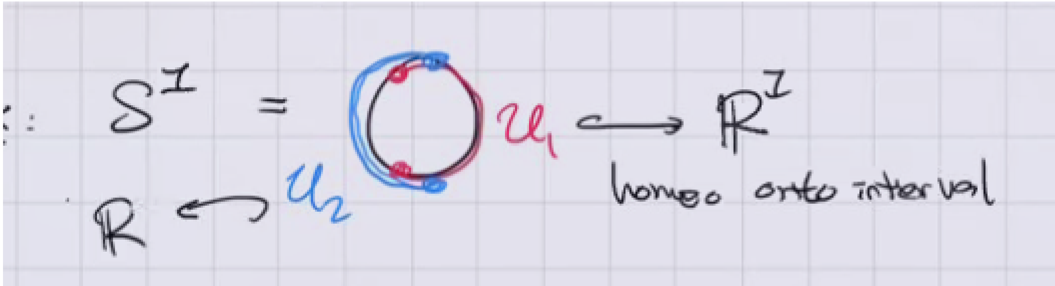

Figure 1: image_2021-01-13-14-02-19

Remark 1.2.3: Maps that are merely continuous are poorly behaved, so we may want to impose extra structure. This can be done by imposing restrictions on the transition functions, defined as

$$t_{uv} \coloneqq \varphi_V \to \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V).$$

Definition 1.2.4 (Restricted Structures on Manifolds)

- We say X is a **PL manifold** if and only if t_{UV} are piecewise-linear. Note that an invertible PL map has a PL inverse.
- We say X is a C^k **manifold** if they are k times continuously differentiable, and **smooth** if infinitely differentiable.
- We say X is **real-analytic** if they are locally given by convergent power series.
- We say X is **complex-analytic** if under the identification $\mathbb{R}^n \cong \mathbb{C}^{n/2}$ if they are holomorphic, i.e. the differential of t_{UV} is complex linear.
- We say X is a **projective variety** if it is the vanishing locus of homogeneous polynomials on $\mathbb{CP}^N$.

Remark 1.2.5: Is this a strictly increasing hierarchy? It's not clear e.g. that every C^k manifold is PL.

Question 1.2.6
Consider $\mathbb{R}^n$ as a topological manifold: are any two smooth structures on $\mathbb{R}^n$ diffeomorphic?

Remark 1.2.7: Fix a copy of $\mathbb{R}$ and form a single chart $\mathbb{R} \xrightarrow{\text{id}} \mathbb{R}$. There is only a single transition function, the identity, which is smooth. But consider

$$\begin{aligned} X &\to \mathbb{R} \\ t &\mapsto t^3. \end{aligned}$$

This is also a smooth structure on X, since the transition function is the identity. This yields a different smooth structure, since these two charts don't like in the same maximal atlas. Otherwise there would be a transition function of the form $t_{VU} : t \mapsto t^{1/3}$, which is not smooth at zero. However, the map

$$\begin{aligned} X &\to X \\ t &\mapsto t^3. \end{aligned}$$

defines a diffeomorphism between the two smooth structures.

Claim: $\mathbb{R}$ admits a unique smooth structure.

Proof (sketch).
Let $\tilde{\mathbb{R}}$ be some exotic $\mathbb{R}$, i.e. a smooth manifold homeomorphic to $\mathbb{R}$. Cover this by coordinate charts to the standard $\mathbb{R}$:

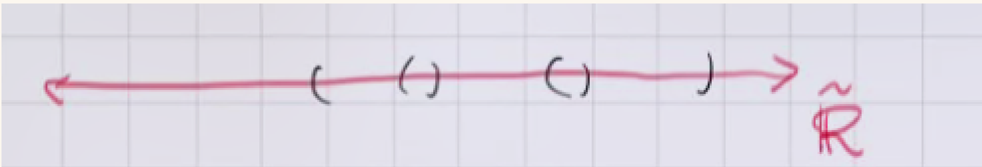

Figure 2: image_2021-01-13-14-22-18

Fact
There exists a cover which is *locally finite* and supports a *partition of unity*: a collection of smooth functions $f_i : U_i \to \mathbb{R}$ with $f_i \geq 0$ and $\operatorname{supp} f \subseteq U_i$ such that $\sum f_i = 1$ (*i.e., bump functions*). It is also a purely topological fact that $\tilde{\mathbb{R}}$ is orientable.

So we have bump functions:

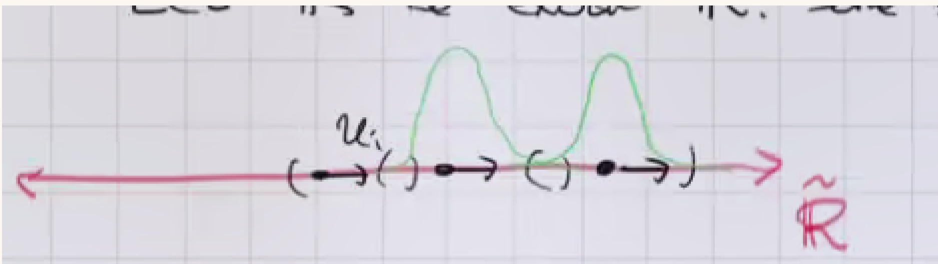

Figure 3: image_2021-01-13-14-25-30

Take a smooth vector field V_i on U_i everywhere aligning with the orientation. Then $\sum f_i V_i$ is a smooth nowhere vector field on X that is nowhere zero in the direction of the orientation. Taking the associated flow

$$\begin{aligned} \mathbb{R} &\to \tilde{\mathbb{R}} \\ t &\mapsto \varphi(t). \end{aligned}$$

such that $\varphi'(t) = V(\varphi(t))$. Then φ is a smooth map that defines a diffeomorphism. This follows from the fact that the vector field is everywhere positive.

Slogan
To understand smooth structures on X, we should try to solve differential equations on X. ■

Remark 1.2.10: Note that here we used the existence of a global frame, i.e. a trivialization of the tangent bundle, so this doesn't quite work for e.g. S^2.

Question 1.2.11
What is the difference between all of the above structures? Are there obstructions to admitting any particular one?

Answer 1.2.12

1. (Munkres) Every C^1 structure gives a unique C^k and C^∞ structure.[1]
2. (Grauert) Every C^∞ structure gives a unique real-analytic structure.
3. Every PL manifold admits a smooth structure in $\dim X \leq 7$, and it's unique in $\dim X \leq 6$, and above these dimensions there exists PL manifolds with no smooth structure.
4. (Kirby–Siebenmann) Let X be a topological manifold of $\dim X \geq 5$, then there exists a cohomology class $\mathrm{ks}(X) \in H^4(X; \mathbb{Z}/2\mathbb{Z})$ which is 0 if and only if X admits a PL structure. Moreover, if $\mathrm{ks}(X) = 0$, then (up to concordance) the set of PL structures is given by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.
5. (Moise) Every topological manifold in $\dim X \leq 3$ admits a unique smooth structure.
6. (Smale et al.): In $\dim X \geq 5$, the number of smooth structures on a topological manifold X is finite. In particular, $\mathbb{R}^n$ for $n \neq 4$ has a unique smooth structure. So dimension 4 is interesting!
7. (Taubes) $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures.
8. A compact oriented smooth surface Σ, the space of complex-analytic structures is a complex orbifold [2] of dimension $3g - 2$ where g is the genus of Σ, up to biholomorphism (i.e. *moduli*).

Remark 1.2.13: Kervaire-Milnor: S^7 admits 28 smooth structures, which form a group.

[1] Note that this doesn't start at C^0, so topological manifolds are genuinely different! There exist topological manifolds with no smooth structure.
[2] Locally admits a chart to $\mathbb{C}^n/\Gamma$ for Γ a finite group.

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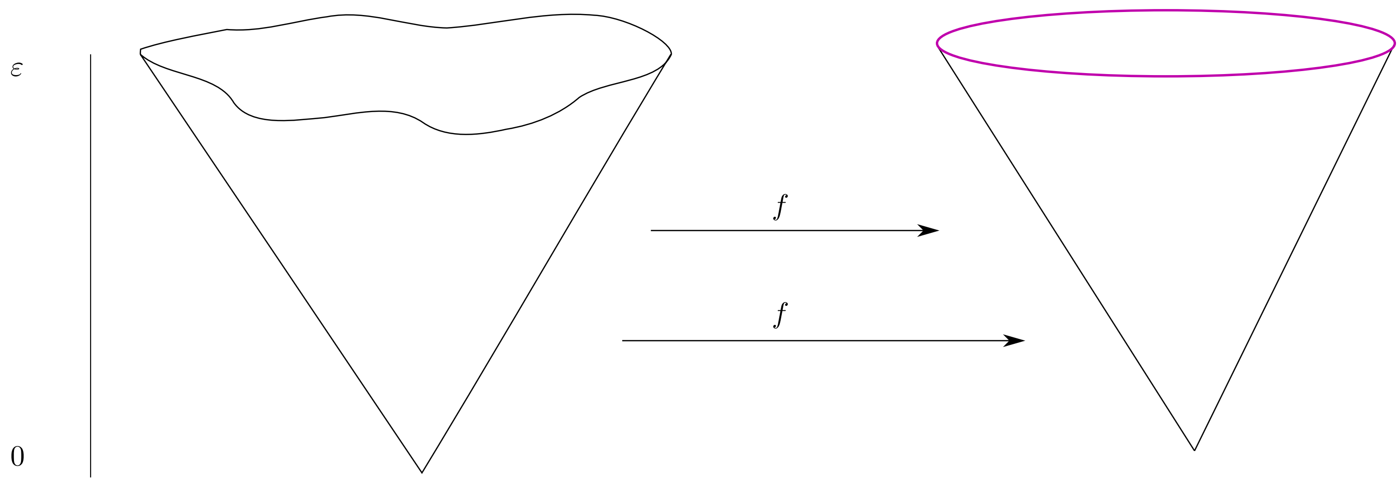
Remark 2.0.1: Let

$$V \coloneqq \left\{a^2 + b^2 + c^2 + d^3 + e^{6k-1} = 0\right\} \subseteq \mathbb{C}^5$$
$$S_\varepsilon \coloneqq \left\{|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2\right\}.$$

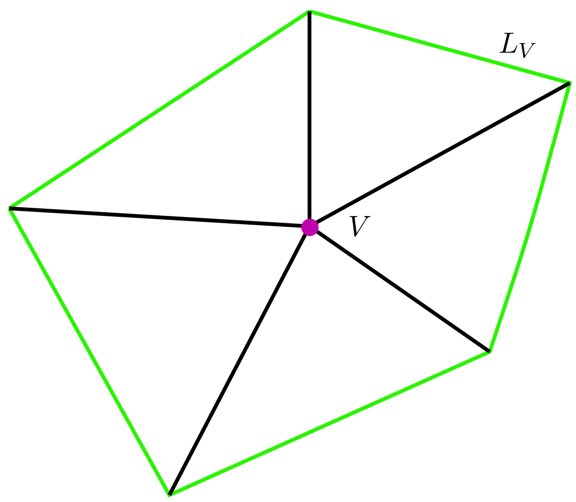
Then $V_k \cap S_\varepsilon \cong S^7$ is a homeomorphism, and taking $k = 1, 2, \cdots, 28$ yields the 28 smooth structures on S^7. Note that V_k is the cone over $V_k \cap S_\varepsilon$.



? Admits a smooth structure, and $\overline{V}_k \subseteq \mathbb{CP}^5$ admits no smooth structure.

Question 2.0.2
Is every triangulable manifold PL, i.e. homeomorphic to a simplicial complex?

Answer 2.0.3
No! Given a simplicial complex, there is a notion of the **combinatorial link** of a vertex.



It turns out that there exist simplicial manifolds such that the link is not homeomorphic to a sphere, whereas every PL manifold admits a "PL triangulation" where the links are spheres.

Remark 2.0.4: What's special in dimension 4? Recall the **Kirby-Siebenmann** invariant $\mathrm{ks}(x) \in H^4(X; \mathbb{Z}_2)$ for X a topological manifold where $\mathrm{ks}(X) = 0 \iff X$ admits a PL structure, with the caveat that $\dim X \geq 5$. We can use this to cook up an invariant of 4-manifolds.

Definition 2.0.5 (Kirby-Siebenmann Invariant of a 4-manifold)
Let X be a topological 4-manifold, then

$$\mathrm{ks}(X) \coloneqq \mathrm{ks}(X \times \mathbb{R}).$$

Remark 2.0.6: Recall that in $\dim X \geq 7$, every PL manifold admits a smooth structure, and we can note that

$$H^4(X; \mathbb{Z}_2) = H^4(X \times \mathbb{R}; \mathbb{Z}_2) = \mathbb{Z}_2, .$$

since every oriented 4-manifold admits a fundamental class. Thus

$$\mathrm{ks}(X) = \begin{cases} 0 & X \times \mathbb{R} \text{ admits a PL and smooth structure} \\ 1 & X \times \mathbb{R} \text{ admits no PL or smooth structures} \end{cases}.$$

Remark 2.0.7: $\mathrm{ks}(X) \neq 0$ implies that X has no smooth structure, since $X \times \mathbb{R}$ doesn't. Note that it was not known if this invariant was nonzero for a while!

Remark 2.0.8: Note that $H^2(X; \mathbb{Z})$ admits a symmetric bilinear form Q_X defined by

$$\langle \alpha, \ \beta \rangle \mapsto \int_X \alpha \wedge \beta = \alpha \smile \beta([X]) \in \mathbb{Z}.$$

where $[X]$ is the fundamental class.

3 Main Theorems for the Course

Proving the following theorems is the main goal of this course.

Theorem 3.0.1 ***(Freedman).***
If X, Y are compact oriented topological 4-manifolds, then $X \cong Y$ are homeomorphic if and only if $\mathrm{ks}(X) = \mathrm{ks}(Y)$ and $Q_X \cong Q_Y$ are isometric, i.e. there exists an isometry

$$\varphi : H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z}).$$

that preserves the two bilinear forms in the sense that $\langle \varphi\alpha,\ \varphi\beta \rangle = \langle \alpha,\ \beta \rangle$.
Conversely, every **unimodular** bilinear form appears as $H^2(X;\mathbb{Z})$ for some X, i.e. the pairing induces a map

$$\begin{aligned} H^2(X;\mathbb{Z}) &\to H^2(X;\mathbb{Z})^\vee \\ \alpha &\mapsto \langle \alpha,\ \cdot \rangle. \end{aligned}$$

which is an isomorphism. This is essentially a classification of simply-connected 4-manifolds.

Remark 3.0.2: Note that preservation of a bilinear form is a stand-in for "being an element of the orthogonal group", where we only have a lattice instead of a full vector space.

Remark 3.0.3: There is a map $H^2(X;\mathbb{Z}) \xrightarrow{PD} H_2(X;\mathbb{Z})$ from Poincaré , where we can think of elements in the latter as closed surfaces $[\Sigma]$, and

$$\langle \Sigma_1,\ \Sigma_2 \rangle = \text{signed number of intersections points of } \Sigma_1 \pitchfork \Sigma_2.$$

Note that Freedman's theorem is only about homeomorphism, and is not true smoothly. This gives a way to show that two 4-manifolds are homeomorphic, but this is hard to prove! So we'll black-box this, and focus on ways to show that two *smooth* 4-manifolds are *not* diffeomorphic, since we want homeomorphic but non-diffeomorphic manifolds.

Definition 3.0.4 (Signature)
The **signature** of a topological 4- manifold is the signature of Q_X, where we note that Q_X is a symmetric nondegenerate bilinear form on $H^2(X;\mathbb{R})$ and for some a, b

$$(H^2(X;\mathbb{R}), Q_x) \xrightarrow{\text{isometric}} \mathbb{R}^{a,b}.$$

where a is the number of $+1$s appearing in the matrix and b is the number of -1s. This is $\mathbb{R}^{ab}$ where $e_i^2 = 1, i = 1 \cdots a$ and $e_i^2 = -1, i = a+1, \cdots b$, and is thus equipped with a specific bilinear form corresponding to the Gram matrix of this basis.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = I_{a \times a} \oplus -I_{b \times b}.$$

Then the signature is $a - b$, the dimension of the positive-definite space minus the dimension of the negative-definite space.

Theorem 3.0.5 ***(Rokhlin's Theorem).***
Suppose $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ and $\alpha \in H^2(X;\mathbb{Z})$ and X a simply connected **smooth** 4-manifold. Then 16 divides $\operatorname{sig}(X)$.

Remark 3.0.6: Note that Freedman's theorem implies that there exists topological 4-manifolds with no smooth structure.

Theorem 3.0.7 ***(Donaldson).***
Let X be a smooth simply-connected 4-manifold. If $a = 0$ or $b = 0$, then Q_X is diagonalizable and there exists an orthonormal basis of $H^2(X;\mathbb{Z})$.

Remark 3.0.8: This comes from Gram-Schmidt, and restricts what types of intersection forms can occur.

3.1 Warm Up: $\mathbb{R}^2$ Has a Unique Smooth Structure

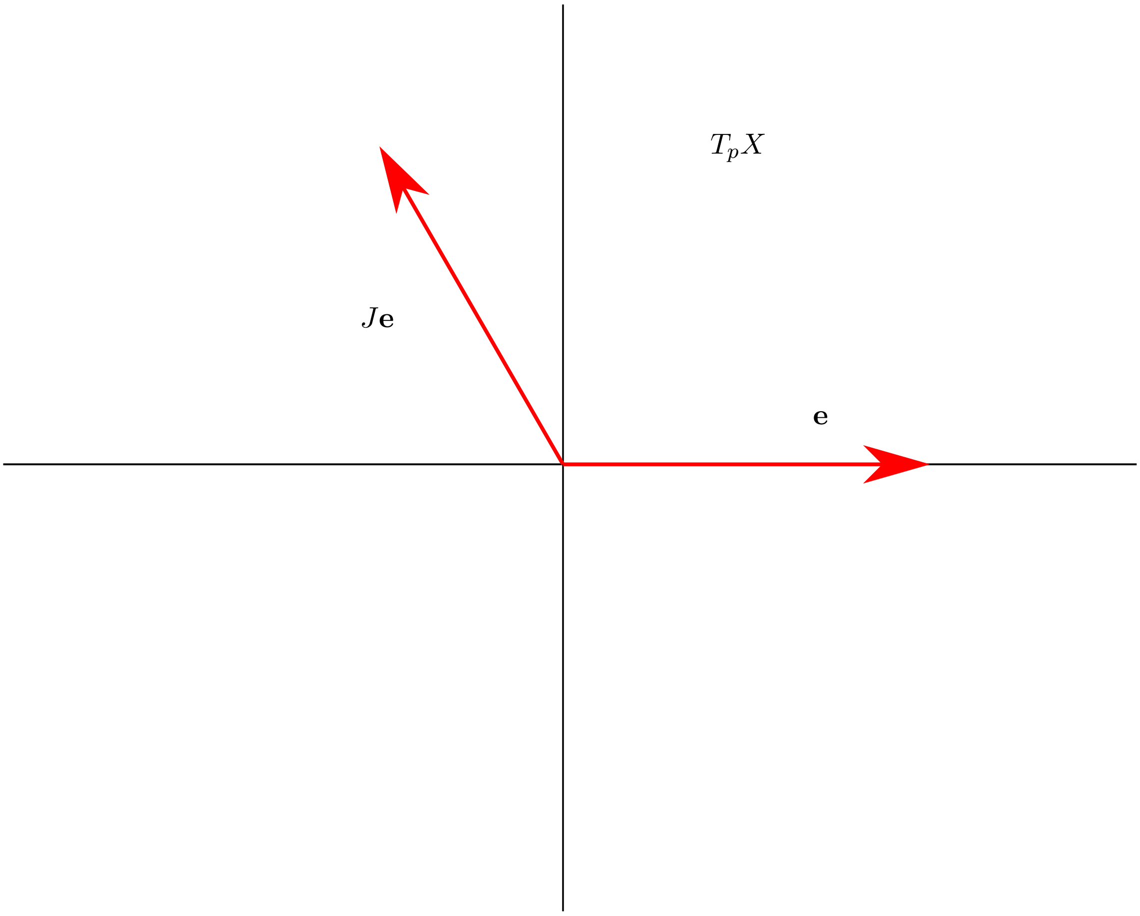
Remark 3.1.1: Last time we showed $\mathbb{R}^1$ had a unique smooth structure, so now we'll do this for $\mathbb{R}^2$. The strategy of solving a differential equation, we'll now sketch the proof.

Definition 3.1.2 (Riemannian Metrics)
A **Riemannian metric** $g \in \operatorname{Sym}^2 T^* X$ for X a smooth manifold is a metric on every $T_p X$ given by

$$\begin{gathered} g_p : T_p X \times T_p X \to \mathbb{R} \\ g(v,v) \geq 0, g(v,v) = 0 \iff v = 0. \end{gathered}$$

Definition 3.1.3 (Almost complex structure)
An **almost complex structure** is a $J \in \operatorname{End}(TX)$ such that $J^2 = -\operatorname{id}$.

Remark 3.1.4: Let $e \in T_p X$ and $e \neq 0$, then if X is a surface then $\{e, Je\}$ is a basis of $T_p X$.



This is a basis because if Je and e are parallel, then ??? In particular, J_p is determined by a point in $\mathbb{R}^2 \setminus \{\text{the } x\text{-axis}\}$

3.1.1 Sketch of Proof

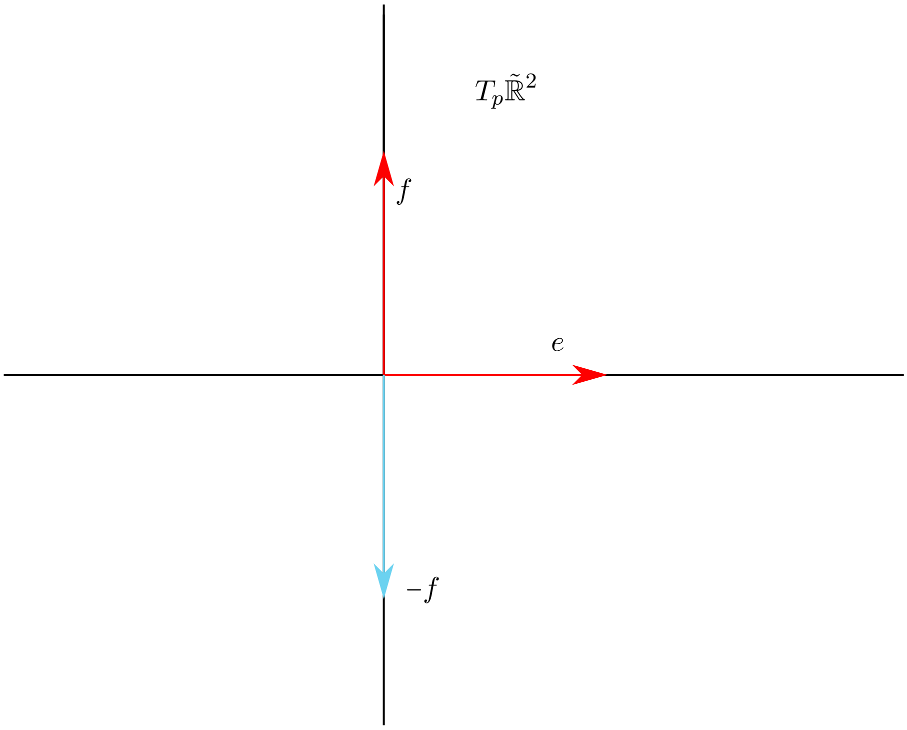
Let $\tilde{\mathbb{R}}^2$ be an exotic $\mathbb{R}^2$.

Step 1 Choose a metric on $\tilde{\mathbb{R}}^2$ $g \coloneqq \sum f_I g_i$ with g_i metrics on coordinate charts U_i and f_i a partition of unity.

Step 2 Find an almost complex structure on $\tilde{\mathbb{R}}^2$. Choosing an orientation of $\tilde{\mathbb{R}}^2$, g defines a unique almost complex structure $J_p e \coloneqq f \in T_p\tilde{\mathbb{R}}^2$ such that

- $g(e,e) = g(f,f)$
- $g(e,f) = 0.$
- $\{e,f\}$ is an oriented basis of $T_p\tilde{\mathbb{R}}^2$

This is because after choosing e, there are two orthogonal vectors, but only one choice yields an *oriented* basis.



Step 3 We then apply a theorem:

Theorem 3.1.5 ***(?).***
Any almost complex structure on a surface comes from a complex structure, in the sense that there exist charts $\varphi_i : U_i \to \mathbb{C}$ such that J is multiplication by i.

So $d\varphi(J \cdot e) = i \cdot d\varphi_i(e)$, and $(\tilde{\mathbb{R}}^2, J)$ is a complex manifold. Since it's simply connected, the Riemann Mapping Theorem shows that it's biholomorphic to $\mathbb{D}$ or $\mathbb{C}$, both of which are diffeomorphic to $\mathbb{R}^2$.

See the Newlander-Nirenberg theorem, a result in complex geometry.