Graduate Student Topology Seminar Notes

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Contents

Topic: Weinstein Surgery and More

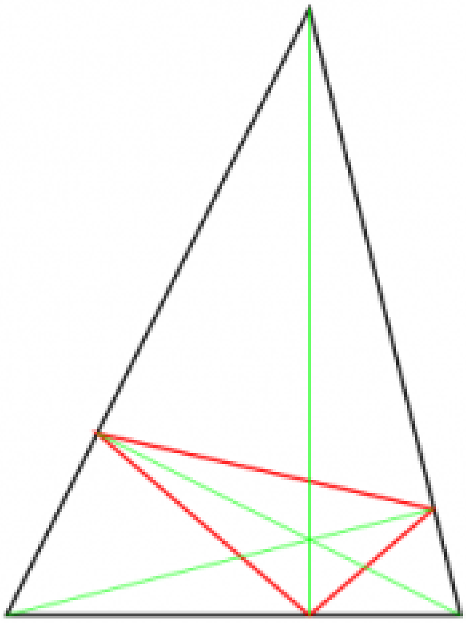
References:

- Mike Usher's Notes
- From Stein to Weinstein and Back (Cieliebak and Eliashberg)

1 Why Care About Contact/Symplectic Geometry

*Open Problem:** Does every triangular billiards admit a periodic orbit?

Answer (1775): Yes for acute triangles, there is at least one periodic orbit:



For arbitrary triangles: unknown!

Historically, the study of periodic orbits motivated the definition of *contact structures*.

Definition (Hyperplane Field): A *hyperplane* field ξ is a codimension 1 sub-bundle $\mathbb{R}^{n-1} \to \xi \to M$ of the tangent bundle $\mathbb{R}^n \to TM \to M$.

> See examples.

Definition (Contact Manifold) A smooth manifold with a hyperplane field (M^{2n+1}, ξ) is *contact* iff $\xi = \ker \alpha$ for some $\alpha \in \Omega^1(M)$ where $\alpha \wedge (d\alpha)^n$ is a top/volume form in $\Omega^{2n+1}(M)$

> Note that $\lambda \wedge (d\lambda)^n = 0$ defines a foliation?

Definition (Reeb Vector Field): There is a canonical vector field on every contact manifold: the Reeb vector field X. This satisfies $\lambda(X) = 1$ and $\iota_x d\lambda = 0$.

> *Remark:* Contact manifolds are cylinder-like boundaries of symplectic manifolds; namely if M is contact then we can pick any C^1 increasing function $f : \mathbb{R} \to \mathbb{R}^+$ (e.g. $f(t) = e^t$) and obtain an exact symplectic form $\omega = d(f\lambda)$ on $M_C \coloneqq M \times \mathbb{R}$.

> Any such f induces a Hamiltonian vector field on M_C, and the Reeb vector field is the restriction to $M \times \{0\}$.

2 Why Care About Contact Structures

Let M be a symplectic manifold and $H : M \to \mathbb{R}$ a Hamiltonian.

For regular values $r \in \mathbb{R}$ of the Hamiltonian, $H^{-1}(r) \subset M$ is a submanifold $Y \subset M$ with a smooth vector field X_H called a *regular level set*.

Question: Does X_H have a closed orbit on every regular level set? What conditions do you need to guarantee the existence of a closed orbit?

Turns out not to depend on H, and only on the hypersurface Y. The existence of a closed orbit is equivalent to the existence of a closed embedded curve γ that is everywhere tangent to $\ker(\omega|_Y)$.

Question: When is such a curve guaranteed to exist?

Theorem (Weinstein, 1972): If Y is convex.

Theorem (Rabinowitz): If Y is "star-shaped" (exists a point p that can "see" all points via straight lines).

Theorem (1987): Every contact-type hypersurface in the symplectic manifold $(\mathbb{R}^{2n}, \omega)$ contains a periodic orbit.

Conjecture (Weinstein, 1978): Let (M, ξ) be a closed (compact) contact manifold with a Reeb vector field X and $H^1(M; \mathbb{R}) = 0$. Then X admits a periodic orbit.

Theorem (Weinstein, Dimension 3, Overtwisted. 1993): Let (M, λ, ξ) be a closed contact 3-manifold where λ is overtwisted. Then the **Reeb vector field** X admits a periodic orbit.

3 Definitions

Definition (Hamiltonian): A smooth function $H : M \to \mathbb{R}$ will be referred to as an energy functional or a *Hamiltonian.* If we have $H : M \times I \to \mathbb{R}$, we'll refer to this as a *time-dependent Hamiltonian*, i.e. the time slices $H_t : M \to \mathbb{R}$ given by $H_t(p) = H(p, t)$ are Hamiltonians.

> *Remark:* If (M, ω) is a symplectic manifold, each $H_t : M \to I$ induces a unique vector field X_{H_t} characterized by the property $\iota_{X_{H_t}} \omega = -dH_t$ where ι is the interior product.

Definition (Symplectic Manifold): Recall that M^{2n} is a **symplectic manifold** iff W is smooth of even dimension and admits a

- closed: $d\omega = 0$
- nondegenerate $\omega_p : T_pM \times T_pM \to \mathbb{R}$; $\omega_p(\mathbf{v}, \mathbf{w}) = 0 \ \forall \mathbf{w} \implies \mathbf{v} = 0$.
- skew-symmetric: $\omega_p(\mathbf{v}, \mathbf{w}) = \omega_p(\mathbf{w}, \mathbf{v})$.
- bilinear: Lifts to a map $T_pM \otimes T_PM \to \mathbb{R}$
- 2-form

$$\omega \in \Omega^2(M) = \Gamma^\infty(\bigwedge^2 T^\vee M).$$

> *Motivation*: There is Hamiltonian H in the background, we want this to induce a vector field V_H and thus a flow.

> Motivation for definitions: we want an "antisymmetric inner product".
>
> - *Closed:* The lie derivative of ω along V_H is 0, i.e. $\mathcal{L}_{V_H}(\omega) = 0$.
> - *Nondegenerate:* Implies that for every dH there exists a vector field V_H such that $dH = \omega(V_H, \cdot)$.
> - *Skew-symmetry:* H should be constant along flow lines, i.e. $dH(V_H) = \omega(V_H, V_H) = 0$
> - *Bilinear:* Send any form $\langle \cdot, \ \cdot \rangle : V \times V \to k$ to the linear map $f : V \to V^\vee$ where $f(v) = \langle v, \ \cdot \rangle$. If the pairing is nondegenerate, $\ker f = 0$, and we get an identification $V \cong V^\vee$.

Important Remark ω being nondegenerate yields $TM \cong T^\vee M$, which can be combined with ι to obtain an isomorphism $\mathfrak{X}(M) \cong \Omega^1(M)$. So we can freely trade 1-forms for vector fields. Very useful!
MOST IMPORTANTLY: for any smooth functional $f: M \to \mathbb{R}$, we can associate to it a vector field X_f.

Definition (Hamiltonian vector field): Given a smooth functional $H : (M, \omega) \to \mathbb{R}$, the associated *Hamiltonian vector field* is the unique field X_H satisfying $\omega(X_H, \cdot\,) = dH$.

Remark: Conservation of energy Since ω is alternating,

$$X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0.$$

Proposition: $(M, \omega \in \Omega^2(M))$ is symplectic iff $\omega^n \neq 0$ everywhere (c.f. Mike Hutchings).

Corollary: Every symplectic manifold is orientable (since it has a nonvanishing volume form).

Important Remark: Symplectic structures on smooth manifolds give us a way to generate *flows* on a manifold (by defining a Hamiltonian or a symplectic vector field).

Definition (Exact Symplectic Manifold): W is an *exact* symplectic manifold iff there exists a 1-form $\lambda \in \Omega^1(W)$ such that $d\lambda \in \Omega^2(W)$ is non-degenerate.

Remark: If (W, λ) is exact symplectic then $(W, d\lambda)$ is symplectic. λ is sometimes referred to as a *Liouville form.*

Important Remark: If (W, λ) is exact and $H : \mathbb{R} \times M \to \mathbb{R}$ is smooth, then the *Hamiltonian flow* $\phi_H^t : M \to M$ is defined for all time and is an *exact symplectomorphism.*

Theorem: There are no closed (compact and boundaryless) exact symplectic manifolds.

Proof:

$$\begin{aligned}
\int_{\partial M} \lambda \wedge \omega^{n-1} &= \int_M d(\lambda \wedge \omega^{n-1}) \\
&= \int_M d\lambda \wedge \omega^{n-1} + (-1)^{|\lambda|} \lambda \wedge d\omega^{n-1} \\
&= \int_M \omega \wedge \omega^{n-1} + (-1)^{|\lambda|} \lambda \wedge 0 \\
&= \int_M \omega^n \\
&= \mathrm{Vol}_{\mathrm{Sp}}(M) \\
&> 0,
\end{aligned}$$

so $\partial M \neq 0$, and thus M can not be closed.

$\blacksquare$

Definition (Liouville Vector Field): Let (W, λ) be exact symplectic. The Liouville vector field on $(W, \omega = d\lambda)$ is the (unique) vector field X such that $\iota_x \omega = \lambda$.

Remark: X induces a flow $\psi^{X,t}$, and for any compact embedded surface $\Sigma_g \hookrightarrow W$ we have

$$\begin{aligned} \psi^{X,t*} d\lambda = e^t d\lambda \implies \\ \mathrm{Area}_{d\lambda}\left(\psi^{X,t}(S)\right) &:= \int_{\psi^{X,t}(S)} d\lambda \\ &= \int_S (\psi^{X,t})^* d\lambda \\ &= e^t \, \mathrm{Area}_{d\lambda}(S) \end{aligned}$$

This says that the flow lines of X "dilate" the areas of surfaces at an exponential rate, or that X is an "infinitesimal generator" of a canonical dilation..

Remark: This is useful because even if W isn't compact, we can obtain W as the "limit" of compact submanifolds where we inflate along this flow.

Theorem: A Liouville vector field X satisfies $\mathcal{L}_X \omega = \omega$, where $\mathcal{L}_X$ is the Lie Derivative.

Proof:

$$\mathcal{L}_X \omega = [d, \iota_X]\omega = \iota_X(d\omega) + d(\iota_x \omega) = \iota_x(d\omega) + d\lambda = \iota_X(0) + d\lambda = d\lambda = \omega.$$

Use the fact that ω is closed, so $d\omega = 0$.

■

Definition (Contact Type): For (W, λ) an exact symplectic manifold, a codimension 1 submanifold $Y \subset W$ is of *restricted contact-type* iff X is transverse to Y, i.e. for every $p \in Y$, we have $X(p) \notin T_p(Y)$.

We say Y is of *contact type* iff there is a neighborhood $U \supset Y$ and a one-form λ with $d\lambda = \omega|_U$ making (U, λ) of restricted contact type.

Remark: (U, λ) is of restricted contact type iff $\lambda|_U$ is a contact form.

Definition (Hypersurface of contact type): For (X, ω) a symplectic manifold, a hypersurface $\Sigma \hookrightarrow X$ is of *contact-type* iff there is a contact form λ such that $d\lambda = \omega|_Y$.

Definition (Liouville Domain): (W, λ) is a Liouville domain iff W is a *compact* exact symplectic manifold with boundary such that the Liouville vector field X points outwards on ∂W transversally.

Remark: This condition implies that ∂W is a contact manifold with contact form $\alpha = \lambda|_{\partial W}$.

Definition (Isotropic): Let Λ be the image of an embedded sphere $S^k \to W$. Then Λ is *isotropic* iff $\lambda|_\Lambda = 0$.

Definition (Weinstein Surgery): Let (W, λ) be a Liouville domain (although we won't need compactness).

Recall: (W, λ) is a $2n$-dimensional exact symplectic manifold with contact-type boundary ∂W such that the Liouville vector field X points outwards along ∂W.

Weinstein surgery takes

- (W, λ) a $2n$-dimensional manifold
- That is exact and symplectic

- With contact-type boundary ∂W
- Where the Liouville vector field points outward along ∂W

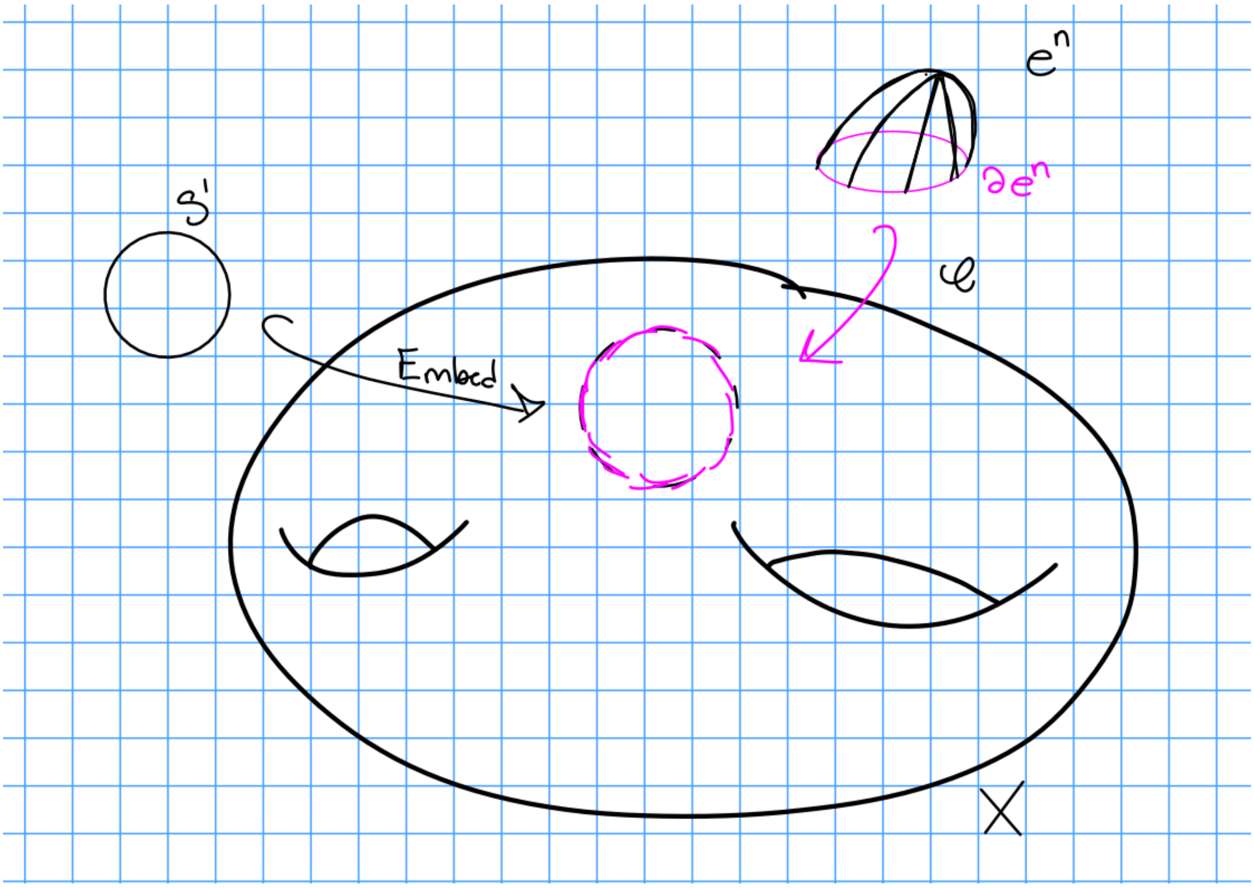
and produces a distinct manifold $(W(\Lambda), \lambda')$ with the above properties which is obtained by surgery along Λ an isotropic embedded sphere. Thus $W(\Lambda)$ is obtained from attaching a k-handle to W along Λ.

3.1 Aside on Surgery

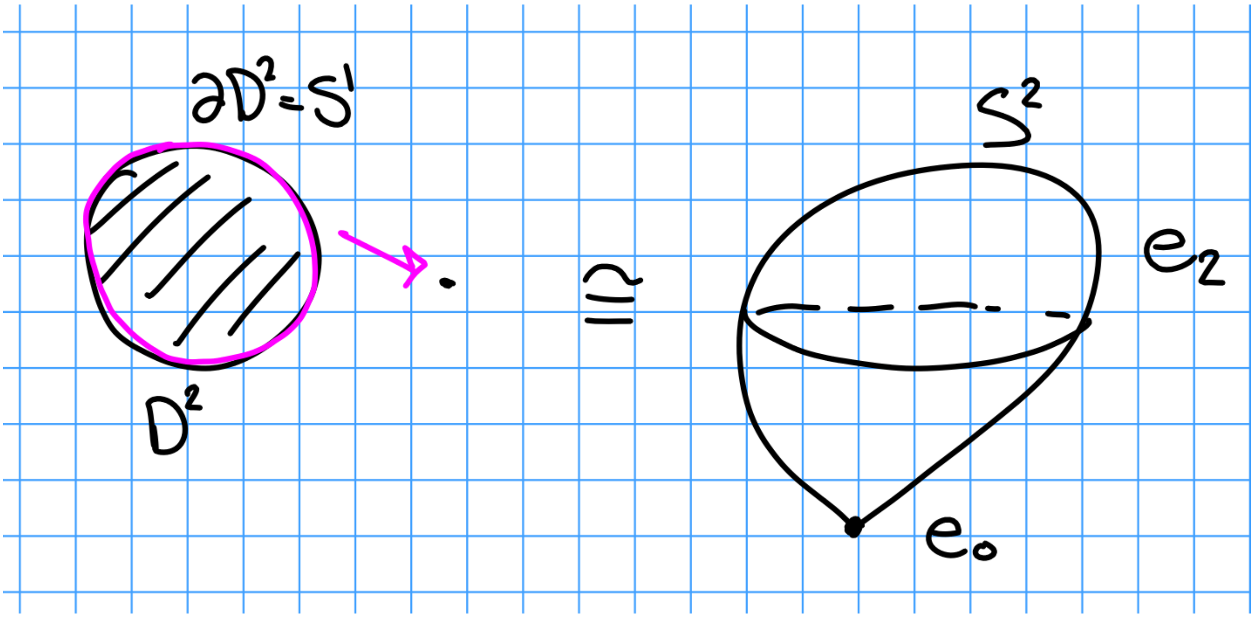
3.1.1 Analogy: CW Cell Attachment

Given X, we can form $\tilde{X} = X^n \coprod_\phi e^n$ where $e^n \cong D^n$ is an n-cell and $\phi : S^{n-1} \to X$ is the characteristic/attaching map.

> Remark: Why S^{n-1}? Really it's $\partial e^n = \partial D^n = S^{n-1}$.



Problem: This doesn't "see" the smooth structure, and CW complexes can have singular points (e.g. $S^2 = e^0 \coprod e^2$.



Solution: Use *tubular neighborhoods.*

Definition (Surgery): Given a manifold M^n where $n = p + q$, then p-surgery on M, denoted $\mathcal{S}(M)$, result of cutting out $S^p \times D^q$ and gluing back in $D^{p+1} \times S^{q-1}$.

Let $\Gamma_{p,q} = S^p \times D^q$, call this our "surgery cell". As in the CW case, we want to attach this cell via an embedding of its boundary into M.

We can compute

$$\partial(S^p \times D^q) = S^p \times S^{q-1} = \partial(\mathbf{D^{p+1}} \times \mathbf{S^{q-1}})$$

then the above says

$$\partial\Gamma_{p,q} = S^p \times S^q = \partial\Gamma_{p+1,q-1}$$

So fix any embedding

$$\phi : \Gamma_{p,q} \to M$$

Note that this restricts to some map (abusing notation)

$$\phi : \partial\Gamma_{p,q} \to M$$

So by the above observation, we can trade this in for a map

$$\phi : \partial\Gamma_{p+1,q-1} \to M.$$

And so we can use this as an attaching map:

$$\mathcal{S}_p(M) := M \setminus \phi(\Gamma_{p,q})^\circ \coprod\nolimits_\phi \Gamma_{p+1,q-1}.$$

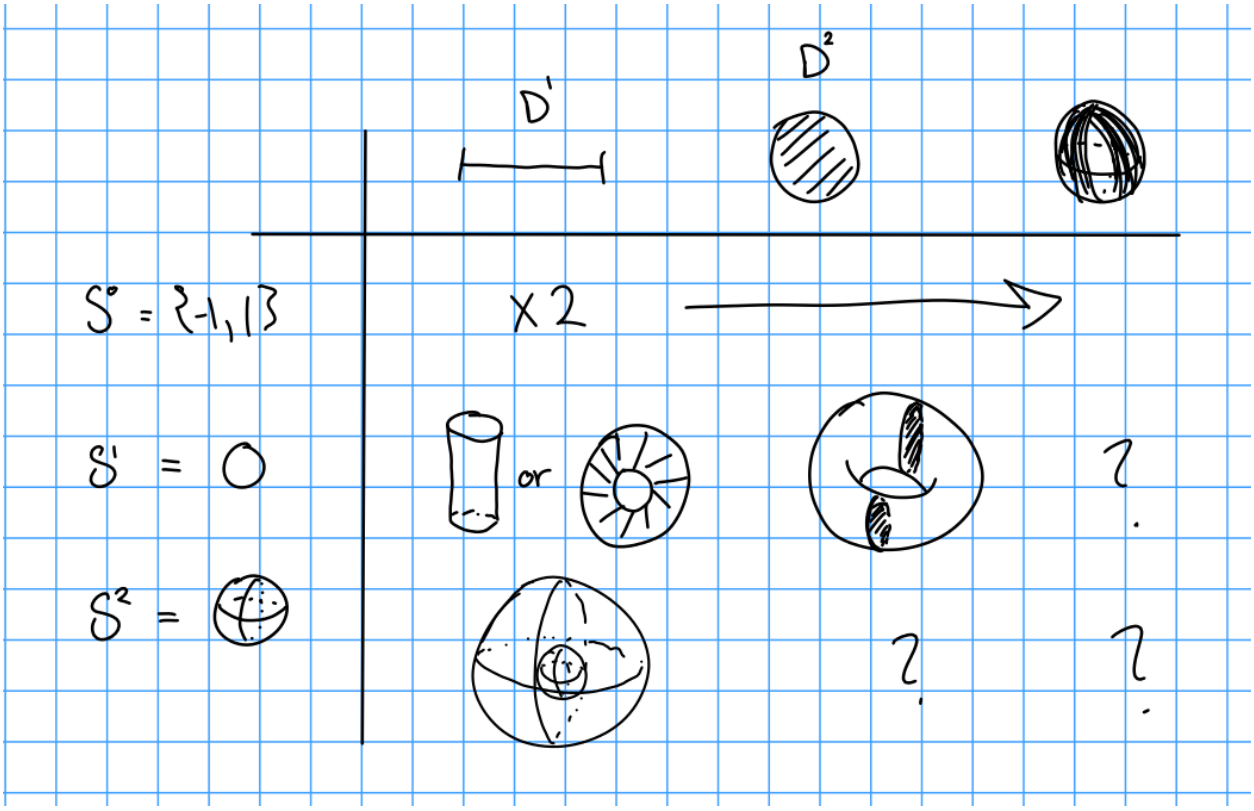
Definition (Handle Attachment) Given a manifold $(M^n, \partial M^n)$ with boundary, attaching a p-handle to M, denoted $H_p(M)$, is given by p-surgery on ∂M, i.e.

$$H_p(M)^\circ = M$$
$$\partial H_p(M) = \mathcal{S}_k(\partial M).$$

Remark: we need conditions on the embedding of the normal bundle for this to work.

Examples of Handles : $S^1 \times D^2 \cong \overline{T}$, a solid torus.

A useful table:



See examples of surgery

Definition (Weinstein Surgery): We now add a k-handle along $\Lambda \subset \partial W$ to yield some W', where $\partial W'$ is obtained by surgery in ∂W along Λ.

4 Why Care About Weinstein Surgery

Theorem: Every compact contact 3-manifold arises as a combination of (2 different variants of) Weinstein surgeries on S^3.

Compare to theorem: Every compact 3-manifold arises as surgery on a link.

Theorem: Weinstein surgery on a *loose Legendrian* knot yields an overtwisted contact structure.

Compare: Every compact manifold is surgery on a link and admits a contact structure.

However, not every compact 3-manifold M admits a *fillable* contact structure (M, ξ) (roughly: admits a symplectic manifold (X, ω) with $\partial X = M$ and some compatibility between ξ, ω) – need framing to be realizable as a *Legendrian* framing.

Theorem: Weinstein surgery along a loose Legendrian sphere yields an overtwisted contact manifold.