# Title

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## Contents



# 1 Tuesday, September 08

Review: we discussed irreducible components. Recall that the *Zariski topology* on an affine variety $X$ has affine subvarieties as closed sets, and a *noetherian space* has no infinitely decreasing chains of closed subspaces.

We showed that any noetherian space has a decomposition into irreducible components $X = \cup X_i$ with $X_i$ closed, irreducible, and unique such that no two are subsets of each other. Applying this to affine varieties, a descending chain of subspaces $X_0 \supsetneq X_1 \cdots$ in $X$ corresponds to an increasing chain of ideals $I(X_0) \subsetneq I(X_1) \cdots$ in $A(X)$. Since $k[x_1, \cdots, x_n]$ is a noetherian ring, this chain terminates, so affine varieties are noetherian.

## 1.1 Dimension

**Definition 1.0.1** (Dimensions)**.**
Let $X$ be a topological space.

1. The *dimension* $\dim X \in \mathbb{N} \cup \{\infty\}$ is either $\infty$ or the length $n$ of the longest chain of **irreducible** closed subsets $\emptyset \neq Y_0 \subsetneq \cdots \subsetneq Y_n \subset X$ where $Y_n$ need not be equal to $X$.
2. The *codimension* of $Y$ in $X$, $\operatorname{codim}_X(Y)$, for an irreducible subset $Y \subseteq X$ is the length of the longest chain $Y \subset Y_0 \subsetneq Y_1 \cdots \subset X$.

**Example 1.1.**
Consider $\mathbb{A}^1/k$, what are the closed subsets? The finite sets, the empty set, and the entire space.

What are the irreducible closed subsets? Every point is a closed subset, so sets with more than one point are reducible. So the only irreducible closed subsets are $\{a\}, \mathbb{A}^1/k$, since an affine variety is irreducible iff its coordinate ring is a domain and $A(\mathbb{A}^1/k) = k[x]$. We can check

$$\emptyset \subsetneq Y_0 = \{a\} \subsetneq Y_1 = \mathbb{A}^1/k,$$

which is of length 1, so $\dim(\mathbb{A}^1/k) = 1$.

Note that we count the number of nontrivial strict subset containments in this chain.

**Example 1.2.**
Consider $V(x_1x_2) \subset \mathbb{A}^2/k$, the union of the $x_i$ axes. Then the closed subsets are $V(x_1), V(x_2)$, along with finite sets and their unions. What is the longest chain of irreducible closed subsets?

Note that $k[x_1, x_2]/\langle x_1 \rangle \cong k[x_2]$ is a domain, so $V(x_i)$ are irreducible. So we can have a chain

$$\emptyset \subsetneq \{a\} \subsetneq V(x_1) \subset X,$$

where $a$ is any point on the $x_2$-axis, so $\dim(X) = 1$.

The only closed sets containing $V(x_1)$ are $V(x_1) \cup S$ for $S$ some finite set, which can not be irreducible.

**Remark 1.**
You may be tempted to think that if $X$ is noetherian then the dimension is finite. However, finite dimension requires a bounded length on descending/ascending chains, whereas noetherian only requires "termination", which may not happen in a bounded number of steps. So this is **false**!

**Example 1.3.**
Take $X = \mathbb{N}$ and define a topology by setting closed subsets be the sets $\{0, \cdots, n\}$ as $n$ ranges over $\mathbb{N}$, along with $\mathbb{N}$ itself. Is $X$ noetherian? Check descending chains of closed sets:

$$\mathbb{N} \supsetneq \{0, \cdots, N\} \supsetneq \{0, \cdots, N-1\} \cdots,$$

which has length at most $N$, so it terminates and $X$ is noetherian.

But note that all of these closed subsets $X_N := \{0, \cdots, N\}$ are irreducible. Why? If $X_n = X_i \cup X_j$ then one of $i, j$ is equal to $N$, i.e $X_i, X_j = X_N$.

So for every $N$, there exists a chain of irreducible closed subsets of length $N$, implying that $\dim(\mathbb{N}) = \infty$.

**Remark 2.**
Let $X$ be an affine variety. There is a correspondence

$$\left\{ \substack{\text{Chains of irreducible closed subsets} \\ Y_0 \subsetneq \cdots \subsetneq Y_n \text{ in } X} \right\} \left\{ \substack{\text{Chains of prime ideals} \\ P_0 \supsetneq \cdots \supsetneq P_n \text{ in } A(X)} \right\}.$$

Why? We have a correspondence between closed subsets and radical ideals. If we specialize to irreducible, we saw that these correspond to radical ideals $I \subset A(X)$ such that $A(Y) := A(X)/I$ is a domain, which precisely correspond to prime ideal in $A(X)$.

We thus make the following definition:

**Definition 1.0.2** (Krull Dimension)**.**
The *krull dimension* of a ring $R$ is the length $n$ of the longest chain of prime ideals

$$P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n.$$

**Remark 3.**
This uses the key fact from commutative algebra: a finitely generated $k$-algebra $M$ satisfies

1. $M$ has finite $k$-dimension
2. If $M$ is a domain, every maximal chain has the same length.

**Remark 4.**
From scheme theory: for any ring $R$, there is an associated topological space $\operatorname{Spec} R$ given by the set of prime ideals in $R$, where the closed sets are given by

$$V(I) = \left\{ \text{Prime ideals } \mathfrak{p} \trianglelefteq R \;\middle|\; I \subseteq \mathfrak{p} \right\}.$$

If $R$ is a noetherian ring, then $\operatorname{Spec}(R)$ is a noetherian space.

**Example 1.4.**
Using the fact above, let's compute $\dim \mathbb{A}^n/k$. We can take the following chain of prime ideals in $k[x_1, \cdots, x_n]$:

$$0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \cdots \subsetneq \langle x_1, \cdots, x_n \rangle.$$

By applying $V(\,\cdot\,)$ we obtain

$$\mathbb{A}^n/k \supsetneq \mathbb{A}^{n-1}/k \cdots \supsetneq \mathbb{A}^0/k = \{0\} \supsetneq \emptyset,$$

where we know each is irreducible and closed, and it's easy to check that these are maximal:

If there were an ideal $\langle x_1, x_2 \rangle \subset P \subset \langle x_1, x_2, x_3 \rangle$, then take $P \cap k[x_1, x_2, x_3]/\langle x_1, x_2 \rangle$ which would yield a polynomial ring in $k[x_1]$. But we know the only irreducible sets in $\mathbb{A}^1/k$ are a point and the entire space.

So this is a chain of maximal length, implying $\dim \mathbb{A}^n/k = n$.