Analysis Qual Solutions

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1 Fall 2019

1.1 1

Cesaro mean/summation. Break series apart into pieces that can be handled separately.

1.2 a

Prove a stronger result:

$$a_n \to A \implies \frac{1}{N} \sum^N a_k \to A.$$

Idea: once N is large enough, $a_k \approx A$, and all smaller terms will die off as $N \to \infty$.
See this MSE answer.

Suppose $S_k \to S$. Choose ℓ large enough such that

$$k \geq \ell \implies |S_k - S| < \varepsilon.$$

With ℓ fixed, choose N large enough such that

$$k \leq \ell \implies \frac{|S_k - S|}{N} < \varepsilon.$$

Then

$$\begin{aligned}
\left| \left(\frac{1}{N} \sum_{k=1}^N S_k \right) - S \right| &= \frac{1}{N} \left| \sum_{k=1}^N (S_k - S) \right| \\
&\leq \frac{1}{N} \sum_{k=1}^N |S_k - S| \\
&= \sum_{k=1}^{\ell} \frac{|S_k - S|}{N} + \sum_{k=\ell+1}^{N} \frac{|S_k - S|}{N} \\
&\to 0.
\end{aligned}$$

1.3 b

Define

$$\Gamma_n \coloneqq \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

Then $\Gamma_1 = \sum_k \frac{a_k}{k}$ and each Γ_n is a tail of this series, so by assumption $\Gamma_n \to 0$.

Then

$$\begin{aligned}
\frac{1}{n} \sum_{k=1}^n a_k &= \frac{1}{n} (\Gamma_0 + \Gamma_1 + \cdots + \Gamma_n - \mathbf{\Gamma_{n+1}}) \\
&\to 0.
\end{aligned}$$

This comes from consider the following summation:

$$
\begin{array}{l|lllllll}
\Gamma_1 : & a_1 & +\frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_2 : & & \frac{a_2}{2} & +\frac{a_3}{3} & +\cdots & & & \\
\Gamma_3 : & & & \frac{a_3}{3} & +\cdots & & & \\
\hline
\sum_{i=1}^{n} \Gamma_i : & a_1 & +a_2 & +a_3 & +\cdots & a_n & +\frac{a_{n+1}}{n+1} & +\cdots
\end{array}
$$

■

1.4 2

DCT, and bounding in the right place. Don't evaluate the actual integral!

Use the fact that $\int_0^1 \cos(tx)\ dt = \sin(x)/x$, then

$$
\begin{aligned}
\left|\frac{\partial^n}{\partial x} \sin(x)/x\right| &= \left|\frac{\partial^n}{\partial x} \int_0^1 \cos(tx)\ dt\right| \\
&=_? \left|\int_0^1 \frac{\partial^n}{\partial x} \cos(tx)\ dt\right| \\
&= \left|\int_0^1 -t^n \sin(tx)\ dt\right| \quad \text{for } n \text{ odd} \\
&\leq \int_0^1 |t^n \sin(tx)|\ dt \\
&\leq \int_0^1 t^n\ dt \\
&= \frac{1}{n+1} \\
&< \frac{1}{n}.
\end{aligned}
$$

Where the DCT is justified by noting that $f(t) = \cos(tx)$ is dominated by $g(t) = 1$ on $[0, 1]$, which integrates to 1.

■

1.5 3

Borel-Cantelli.
Use the following observation: for a sequence of sets X_n,

$$\limsup_n X_n = \left\{x \,\middle|\, x \in X_n \text{ for infinitely many } n\right\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} X_n$$
$$\liminf_n X_n = \left\{x \,\middle|\, x \in X_n \text{ for all but finitely many } n\right\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} X_n.$$

And recall

$$\prod_n e^{x_n} = e^{\sum_n x_n} \quad \text{and} \quad \sum_n \log(x_n) = \log\left(\prod_n x_n\right).$$

1.5.1 a

The Borel σ-algebra is closed under countable unions/intersections/complements, and $B = \limsup_n B_n$ is an intersection of unions of measurable sets.

1.5.2 b

We'll use the fact that tails of convergent sums go to zero, so $\sum_{n\geq M} \mu(B_n) \xrightarrow{M\to\infty} 0$, and $B_M := \bigcap_{m=1}^{M} \bigcup_{n\geq m} B_n \searrow B$.

$$\begin{aligned}
\mu(B_M) &= \mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} B_n\right) \\
&\leq \mu\left(\bigcup_{n\geq m} B_n\right) \quad \text{for all } m \in \mathbb{N} \\
&\to 0,
\end{aligned}$$

and the result follows by continuity of measure.

1.5.3 c

To show $\mu(B) = 1$, we'll show $\mu(B^c) = 0$.

Let $B_k = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{K} B_n$. Then

$$
\begin{aligned}
\mu(B_K^c) &= \mu\left(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{K} B_n^c\right) \\
&\le \sum_{m=1}^{\infty}\mu\left(\bigcap_{n=m}^{K} B_n^c\right) \qquad \text{by subadditivity} \\
&= \sum_{m=1}^{\infty}\prod_{n=m}^{K} 1-\mu(B_n) \\
&\le \sum_{m=1}^{\infty}\prod_{n=m}^{K} e^{-\mu(B_n^c)} \quad \text{by hint} \\
&= \sum_{m=1}^{\infty} e^{-\sum_{n=m}^{K}\mu(B_n^c)} \\
&\to 0
\end{aligned}
$$

since $\sum_{n=m}^{K}\mu(B_n^c)\to\infty$, and we can apply continuity of measure since $B_K^c \xrightarrow{K\to\infty} B^c$.

■

1.6 4

Bessel's Inequality, surjectivity of Riesz map, and Parseval's Identity.
Trick – remember to write out finite sum S_N, and consider $\|x - S_N\|$.

1.6.1 a

Claim:

$$
\begin{aligned}
0 \le \left\| x - \sum_{n=1}^{N}\langle x, u_n\rangle\, u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{N}|\langle x, u_n\rangle|^2 \\
&\implies \sum_{n=1}^{\infty}|\langle x, u_n\rangle|^2 \le \|x\|^2.
\end{aligned}
$$

Proof: Let $S_N = \sum_{n=1}^{N}\langle x,\ u_n\rangle u_n$. Then

$$
\begin{aligned}
0 &\le \|x - S_N\|^2 \\
&= \langle x - S_n,\ x - S_N\rangle \\
&= \|x\|^2 - \sum_{n=1}^{N}|\langle x,\ u_n\rangle|^2 \\
&\xrightarrow{N\to\infty} \|x\|^2 - \sum_{n=1}^{N}|\langle x,\ u_n\rangle|^2.
\end{aligned}
$$

1.6.2 b

1. Fix $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
2. Define

$$x := \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{k=1}^{N} a_k u_k$$

3. $\{S_N\}$ Cauchy (by 1) and H complete $\implies x \in H$.
4. $$\langle x,\ u_n\rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n\rangle = a_n \quad \forall n \in \mathbb{N}$$

   since the u_k are all orthogonal.
5. $$\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$$

   by Pythagoras since the u_k are normal.

Bonus: We didn't use completeness here, so the Fourier series may not actually converge to x. If $\{u_n\}$ is **complete** (so $x = 0 \iff \langle x,\ u_n\rangle = 0\ \forall n$) then the Fourier series *does* converge to x and $\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 = \|x\|^2$ for all $x \in H$.

■

1.7 5

Continuity in L^1 (recall that DCT won't work! Notes 19.4, prove it for a dense subset first). Lebesgue differentiation in 1-dimensional case. See HW 5.6.

1.8 a

Choose $g \in C_c^0$ such that $\|f - g\|_1 \to 0$.

By translation invariance, $\|\tau_h f - \tau_h g\|_1 \to 0$.

Write

$$\begin{aligned}\|\tau f - f\|_1 &= \|\tau_h f - g + g - \tau_h g + \tau_h g - f\|_1 \\ &\leq \|\tau_h f - \tau_h g\| + \|g - f\| + \|\tau_h g - g\| \\ &\to \|\tau_h g - g\|,\end{aligned}$$

so it suffices to show that $\|\tau_h g - g\| \to 0$ for $g \in C_c^0$.

Fix $\varepsilon > 0$. Enlarge the support of g to K such that

$$|h| \le 1 \text{ and } x \in K^c \implies |g(x-h) - g(x)| = 0.$$

By uniform continuity of g, pick $\delta \le 1$ small enough such that

$$x \in K,\ |h| \le \delta \implies |g(x-h) - g(x)| < \varepsilon,$$

then

$$\int_K |g(x-h) - g(x)| \le \int_K \varepsilon = \varepsilon \cdot m(K) \to 0.$$

1.9 b

We have

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x)|\ dx &= \int_{\mathbb{R}} \left| \frac{1}{2h} \int_{x-h}^{x+h} f(y)\ dy \right| dx \\
&\le \frac{1}{2h} \int_{\mathbb{R}} \int_{x-h}^{x+h} |f(y)|\ dy\ dx \\
&=_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{y-h}^{y+h} |f(y)|\ \mathbf{dx\ dy} \\
&= \int_{\mathbb{R}} |f(y)|\ dy \\
&= \|f\|_1.
\end{aligned}$$

and (rough sketch)

$$\begin{aligned}
\int_{\mathbb{R}} |A_h(f)(x) - f(x)|\ dx &= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - f(x) \right| dx \\
&= \int_{\mathbb{R}} \left| \left(\frac{1}{2h} \int_{B(h,x)} f(y)\ dy \right) - \frac{1}{2h} \int_{B(h,x)} f(x)\ dy \right| dx \\
&\le_{FT} \frac{1}{2h} \int_{\mathbb{R}} \int_{B(h,x)} |f(y-x) - f(x)|\ \mathbf{dx\ dy} \\
&\le \frac{1}{2h} \int_{\mathbb{R}} \|\tau_x f - f\|_1\ dy \\
&\to 0 \quad \text{by (a).}
\end{aligned}$$

$\blacksquare$

2 Spring 2019

2.1 1

2.1.1 a

Let $\{f_k\}$ be a Cauchy sequence in $C(I)$. For each fixed $x \in [0,1]$, the sequence of real numbers $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$, which is complete, since

$$x_0 \in I \implies |f_k(x_0) - f_j(x_0)| \leq \sup_{x \in I} |f_k(x) - f_j(x)| = \|f_k - f_j\|_\infty \to 0,$$

so we can define $f(x) := \lim_k f_k(x)$.

We also have

$$\|f_k - f\|_\infty = \left\| f_k - \lim_{j \to \infty} f_j \right\|_\infty = \lim_{j \to \infty} \|f_k - f_j\|_\infty \to 0.$$

Finally, f is the uniform limit of continuous functions and thus continuous.

■

2.1.2 b

It suffices to produce a Cauchy sequence that does not converge to a continuous function. Take

$$f_k(x) = \begin{cases} (x + \frac{1}{2})^k & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases} \xrightarrow{k \to \infty} f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}) \\ 1 & x \in [\frac{1}{2}, 1] \end{cases},$$

which is Cauchy, but there is no $g \in L^1$ that is continuous such that $\|f - g\|_1 = 0$.

2.2 2

2.2.1 a

Lemma 1: $\mu(\coprod_{k=1}^\infty E_k) = \lim_{N \to \infty} \sum_{k=1}^N \mu(E_k)$.

Lemma 2: $A = A \setminus B \coprod A \bigcap B$.

Let $A_k = F_k \setminus F_{k+1}$, so the A_k are disjoint, and let $A = \coprod_k A_k$.

Let $F = \bigcap_k F_k$. Then $F_1 = F \coprod A$ by lemma 2, so

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(A) \\
&= \mu(F) + \lim_{N\to\infty} \sum_k^N \mu(A_k) \quad \text{by Lemma 1} \\
&= \mu(F) + \lim_{N\to\infty} \sum_k^N \mu(F_k) - \mu(F_{k+1}) \\
&= \mu(F) + \lim_{N\to\infty} (\mu(F_1) - \mu(F_N)) \quad \text{(Telescoping)} \\
&= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_N),
\end{aligned}$$

and since the measure is finite, $\mu(F_1) < \infty$ and can be subtracted, yielding

$$\begin{aligned}
\mu(F_1) &= \mu(F) + \mu(F_1) - \lim_{N\to\infty} \mu(F_N) \\
\implies \mu(F) &= \lim_{N\to\infty} \mu(F_N).
\end{aligned}$$

2.2.2 b

Suppose toward a contradiction that there is some $\varepsilon > 0$ for which no such δ exists.

This means that we can take any sequence $\delta_n \to 0$ and produce sets A_n such $m(A) < \delta_n$ but $\mu(A) > \varepsilon$.

So choose the sequence $\delta_n = \dfrac{1}{2^n}$ and define A_n accordingly, and let

$$A = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Since

$$\mu\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \sum_{k=n}^{\infty} \mu(A_k) \approx \frac{1}{2^n} \to 0,$$

by part (a) we have $m(A) = 0$. Now by assumption, we should thus have $\mu(A) = 0$ as well.

However, again by part (a), we have

$$\mu(A) = \lim_n \mu\left(\bigcup_{k=n}^{\infty} A_k\right) \geq \lim_n \mu(A_n) = \lim_n \varepsilon = \varepsilon > 0.$$

2.3 3

Since $f_k \to f$ almost everywhere, we have $\liminf_k f_k(x) = f(x)$ and since $|f|^2 \in L^+$ we can apply Fatou:

$$
\begin{aligned}
\|f\|_2^2 &= \int |f(x)|^2 \\
&= \int \liminf_k |f_k(x)|^2 \\
&\underset{\text{Fatou}}{\leq} \liminf_k \int |f_k(x)|^2 \\
&= M^2,
\end{aligned}
$$

so $\|f\| \leq M < \infty$ and $f \in L^2$.

Let $I = [0,1]$. Applying Egorov's theorem to produce sets F_ε such that $f_k \overset{u}{\to} f$ on F_ε and taking $F = \bigcap F_\varepsilon$, we have

$$
\int_I f_k = \int_{F_\varepsilon} f_k + \int_{F_\varepsilon^c} f_k \quad \overset{\varepsilon \to 0}{\to} \quad \int_F f_k + 0 \quad \overset{k \to \infty}{\to} \quad \int_F f,
$$

using that fact that uniform converges allows commuting limits and integrals.

2.4 4

2.4.1 a

$\implies$:

Idea: $\mathcal{A} = \{f(x) - t \geq 0\} \bigcap \{t \geq 0\}$.

Define $F(x,t) = f(x)$, $G(x,t) = t$, and $H(x,y) = F(x,t) - G(x,t)$, which are all measurable functions.

Then $\mathcal{A} = \{H \geq 0\} \bigcap \{G \geq 0\}$ which is an intersection of measurable sets.

$\impliedby$:

By F.T., for almost every $x \in \mathbb{R}^n$, the x-slices are measurable, so

$$
\mathcal{A}_x := \left\{ t \in \mathbb{R} \;\middle|\; (x,t) \in \mathcal{A} \right\} = [0, f(x)] \implies m(\mathcal{A}_x) = f(x)
$$

But $x \mapsto m(\mathcal{A}_x)$ is a measurable function, and is exactly to $x \mapsto f(x)$, so f is measurable.

2.4.2 b

We first note

$$
\begin{aligned}
\mathcal{A} &= \left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; 0 \leq t \leq f(x) \right\} \\
\mathcal{A}_t &= \left\{ x \in \mathbb{R}^n \;\middle|\; t \leq f(x) \right\}.
\end{aligned}
$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\ dx &= \int_{\mathbb{R}^n} \int_0^{f(x)} 1\ dt\ dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \chi_{\mathcal{A}}\ dt\ dx \\
&\overset{F.T.}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\ dx\ dt \\
&= \int_0^{\infty} \int_{\mathbb{R}^n} \chi_{\mathcal{A}}\ dx\ dt \\
&= \int_0^{\infty} m(\mathcal{A}_t)\ dt,
\end{aligned}$$

where we just note that $\int\int \chi_{\mathcal{A}} = m(\mathcal{A})$, and by F.T., all of these integrals are equal.

2.5 5

2.5.1 a

By Holder's inequality with $p = q = 2$, we have

$$\|f\|_1 = \|f \cdot 1\|_1 \leq \|f\|_2 \|1\|_2 = \|f\|_2 m(X)^{\frac{1}{2}} = \|f\|_2,$$

since $X = [0, 1] \implies m(X) = 1$.

So $L^2(X) \subseteq L^1(X)$, and since simple functions are dense in both spaces, L^2 is dense in L^1.

2.5.2 b

Step 1 Let $\Lambda \in L^1(X)^\vee$; we'll show that in fact $\Lambda \in L^2(X)^\vee$, and by Riesz Representation for L^2 there will be a $g \in L^2$ such that $\Lambda(f) = \langle f,\ g\rangle$.

Lemma: $m(X) < \infty \implies L^p(X) \subset L^2(X)$.

Proof: Write Holder's inequality as $\|fg\|_1 \leq \|f\|_a \|g\|_b$ where $\dfrac{1}{a} + \dfrac{1}{b} = 1$, then

$$\|f\|_p^p = \||f|^p\|_1 \leq \||f|^p\|_a\ \|1\|_b.$$

Now take $a = \dfrac{2}{p}$ and this reduces to

$$\begin{aligned}
&\|f\|_p^p \leq \|f\|_2^p\ m(X)^{\frac{1}{b}} \\
\implies &\|f\|_p \leq \|f\|_2 \cdot O(m(X)) < \infty.
\end{aligned}$$

Let $f \in L^2$ be arbitrary – by the lemma, $\|f\|_1 \leq C\|f\|_2$ for some constant $C = O(m(X))$.

Since $\|\Lambda\|_{1^\vee} := \sup\limits_{\|f\|_1 = 1} |\Lambda(f)|$, given an arbitrary $f \in L^1$, we can define $\hat{f} = f/\|f\|_1$, so $\left\|\hat{f}\right\|_1 = 1$, and obtain

$$\left|\Lambda(\hat{f})\right| \leq \|\Lambda\|_{1^\vee},$$

since $\|\Lambda\|_{1^\vee}$ is the *least* such bound over all $f \in L^1$, and thus

$$\begin{aligned} \frac{|\Lambda(f)|}{\|f\|_1} &= \left|\Lambda(\hat{f})\right| \leq \|\Lambda\|_{1^\vee} \\ \implies |\Lambda(f)| &\leq \|\Lambda\|_{1^\vee} \cdot \|f\|_1 \\ &\leq \|\Lambda\|_{1^\vee} \cdot C\|f\|_2, \end{aligned}$$

which is finite by assumption. So $\Lambda \in (L^2)^\vee$ since it is bounded and thus continuous.

By Riesz Representation for L^2, there is a $g \in L^2$ such that for all $f \in L^2$, $\Lambda(f) = \langle f,\ g\rangle$

Step 2 By Holder, we already have

$$\begin{aligned} \|\Lambda\|_{1^\vee} &= \sup_{\|f\|_1 = 1} |\Lambda(f)| \\ &= \sup_{\|f\|_1 = 1} \left|\int_X fg\right| \\ &\leq \sup_{\|f\|_1 = 1} \|fg\|_1 \\ &\leq \sup_{\|f\|_1 = 1} \|f\|_1 \|g\|_\infty \\ &= \|g\|_\infty, \end{aligned}$$

so it just remains to show that $\|g\|_\infty \leq \|\Lambda\|_{1^\vee}$.

Suppose otherwise, so $\|g\|_\infty > \|\Lambda\|_{1^\vee}$.

Then there exists some $E \subseteq X$ with $m(E) > 0$ such that $x \in E \implies |g(x)| > \|\Lambda\|_{1^\vee}$.

Define

$$h = \frac{1}{m(E)} \frac{\overline{g}}{|g|} \chi_E.$$

$$
\begin{aligned}
\Lambda(h) &= \int_X hg \\
&= \int_X \frac{1}{m(E)} \frac{g\overline{g}}{|g|} \chi_E \\
&= \frac{1}{m(E)} \int_E |g| \\
&\geq \frac{1}{m(E)} \|g\|_\infty m(E) \\
&= \|g\|_\infty \\
&> \|\Lambda\|_{1^\vee},
\end{aligned}
$$

a contradiction.

$\blacksquare$