Problem Set 5

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> **Proposition 1.1.**
> Suppose $\lambda + \rho \in \Lambda^+$. Then $M(w \cdot \lambda) \subset M(\lambda)$ for all $w \in W$. Thus all $[M(\lambda) : L(w \cdot \lambda)] > 0$.
>
> More precisely, if $w = s_n \cdots s_1$ is a reduced expression for w in terms of simple reflections corresponding to roots α_i, then there is a sequence of embeddings:
>
> $$M(w \cdot \lambda) = M(\lambda_n) \subset M(\lambda_{n-1}) \subset \cdots \subset M(\lambda_0) = M(\lambda)$$
>
> Here
>
> $$\lambda_0 := \lambda, \lambda_k := s_k \cdot \lambda_{k-1} = (s_k \ldots s_1) \cdot \lambda \implies \lambda_n = s_n \cdot \lambda_{n-1} = w \cdot \lambda$$
> $$w \cdot \lambda = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0 = \lambda \text{with} \quad \langle \lambda_k + \rho, \alpha_{k+1}^\vee \rangle \in \mathbb{Z}^+ \text{ for } k = 0, \cdots, n-1.$$

Assume $\lambda + \rho \in \Lambda^+$.

a. Prove that the unique simple submodule of $M(\lambda)$ is isomorphic to $M(w_\diamond \cdot \lambda)$, where $w_\diamond$ is the longest element of W.

b. In case $\lambda \in \Lambda^+$, show that the inclusions obtained in the above proposition are all proper.

2 4.6

> **Theorem 2.1** *(Verma).*
> Let $\lambda \in \mathfrak{h}^\vee$. Given $\alpha > 0$, suppose $\mu := s_\alpha \cdot \lambda \leq \lambda$. Then there exists an embedding $M(\mu) \subset M(\lambda)$.

Work through the steps of Verma's Theorem in the special case discussed in the previous problem

2.1 Solution

Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and identify its root system A_2 with $\Delta = \{\alpha, \beta\}$ and $\Phi^+ = \{\alpha, \beta, \gamma := \alpha + \beta\}$ We can also identify the Weyl group as $W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\gamma\}$ where there is a reduced expression $s_\gamma = w_0 = s_\alpha s_\beta s_\alpha$.

We can begin by letting $\lambda \in \Lambda$ be an arbitrary integral weight and let $\mu \neq \lambda$ be an arbitrary weight linked to λ, where WLOG apply some Weyl group element to μ to place it in the dominant chamber and assume

$$\mu := s_\alpha \cdot \lambda < \lambda$$

(where the inequality is strict).

2.1.1 Part 1

Since μ is assumed integral, we can find some $w \in W$ such that

$$\mu' := w^{-1} \cdot \mu \in \Lambda^+ - \rho.$$

Claim: $w = s_\alpha s_\beta$, so $w^{-1} = s_\beta s_\alpha$ and thus

$$\mu' = s_\beta s_\alpha \cdot \mu$$

As in Proposition 4.3, we then write

$$\begin{aligned}
\mu_0 &= \mu' \\
\mu_1 &= s_\beta \cdot \mu' \\
\mu_2 &= s_\alpha s_\beta \cdot \mu' = w \cdot \mu' = \mu
\end{aligned}$$

which satisfies

$$\mu = \mu_2 \leq \mu_1 \leq \mu_0 = \mu'$$
$$\mu = s_\alpha s_\beta \cdot \mu' \leq s_\beta \mu' \leq \mu'.$$

which (by the proposition) gives a sequence of embeddings

$$M(\mu) = M(\mu_2) \hookrightarrow M(\mu_1) \hookrightarrow M(\mu_0) = M(\mu')$$
$$\text{i.e.}$$
$$M(\mu) = M(s_\alpha s_\beta \cdot \mu') \hookrightarrow M(s_\beta \cdot \mu') \hookrightarrow M(\mu').$$

2.1.2 Step 2

We now define

$$\lambda' := w^{-1}\lambda = s_\beta s_\alpha \cdot \lambda$$

and the parallel list of weights

$$\lambda_0 = \lambda'$$
$$\lambda_1 = s_\beta \cdot \lambda'$$
$$\lambda_2 = s_\alpha s_\beta \cdot \lambda' := \lambda.$$

We can similarly use the fact that $\lambda \neq \mu \implies \mu_k \neq \lambda_k$ for any k.

2.1.3 Step 3

We now define

3 4.11

In the case of $\mathfrak{sl}(3, \mathbb{C})$, what can be said at this point about Verma modules with a singular integral highest weight?

> Aside from the trivial case $-\rho$, a typical linkage class has 3 elements. For example, if λ lies in the α hyperplane and is antidominant, the linked weights are λ, $s_\beta \cdot \lambda$, $s_\alpha s_\beta \cdot \lambda$.