

Title

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1.1 Frobenius Kernels

Let $\text{char}(k)p > 0$ and let G be an algebraic group scheme. We have a Frobenius map $F : G \rightarrow G$ given by $F((x_{ij})) = (x_{ij}^p)$, which we can iterate to get F^r for $r \in \mathbb{N}$. Setting $G_r = \ker F^r$ the r th Frobenius kernel, we get a normal series of group schemes

$$G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G.$$

There is an associated chain of finite dimensional Hopf algebras

$$\text{Dist}(G_1) \leq \text{Dist}(G_2) \leq \cdots \leq \text{Dist}(G).$$

Then $k[G]^\vee = \text{Dist}(G_r)$, and we get an equivalence of representations for G_r to representations for $\text{Dist}(G_r)$.

A special case will be when G is a reductive algebraic group scheme. We'll start by finding a basis for $\text{Dist}(G_r)$.

Recall the PBW theorem: we have a basis for \mathfrak{g} given by

$$\begin{aligned} & \{x_\alpha \mid \alpha \in \Phi^+\} \text{ Positive root vectors} \\ & \{h_i \mid i = 1, \dots, n\} \text{ A basis for } \mathfrak{t} \\ & \{x_\alpha \mid \alpha \in \Phi^-\} \text{ Negative root vectors} \end{aligned}$$

We can then obtain a basis for $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \left\langle \prod_{\alpha \in \Phi^+} x_{\alpha}^{n(\alpha)} \prod_{i=1}^n h_i^{k_i} \prod_{\alpha \in \Phi^+} x_{-\alpha}^{m(\alpha)} \right\rangle.$$

We can similarly obtain a basis for the distribution algebra

$$\text{Dist}(G) = \left\langle \prod_{\alpha \in \Phi^+} \frac{x_{\alpha}^{n(\alpha)}}{n!} \prod_{i=1}^n \binom{h_i}{k_i} \prod_{\alpha \in \Phi^+} \frac{x_{-\alpha}^{n(\alpha)}}{n!} \right\rangle,$$

and we can similar get $\text{Dist}(G_r)$ by restricting to $0 \leq n(\alpha), k_i, m(\alpha) \leq p^r - 1$. Above the k_i are allowed to be any integers. This yields a triangular decomposition

$$\text{Dist}(G_r) = \text{Dist}(U_r^+) \text{Dist}(T_r) \text{Dist}(U_r^-),$$

where we'll denote the first two terms $\text{Dist}(B_r^+)$ and the last two as $\text{Dist}(B_r)$.

1.2 Induced and Coinduced Modules

Goal: Classify simple G_r -modules. We know the classification of simple G -modules, so we'll follow similar reasoning. We started by realizing $L(\lambda) \hookrightarrow \text{Ind}_B^G \lambda$ as a submodule (the socle) of some “universal” module.

Let M be a B_r -module, we can then define

$$\text{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r},$$

where we're now taking the B_r -invariants. We get a decomposition as vector spaces,

$$k[G_r] = k[U_r^+] \otimes_k k[B_r]$$

and thus an isomorphism

$$\text{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes M$$

since $k[B_r] \otimes M \cong \text{Ind}_{B_r}^{B_r} M \cong M$.

We then define

$$\text{Coind}_{B_r}^{G_r} = \text{Dist}(G_r) \otimes_{\text{Dist}(B_r)} M,$$

which is an analog of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} M$.

We have $\text{Dist}(U_r^+) \otimes \text{Dist}(B_r) \cong \text{Dist}(G_r)$, so

$$\text{Coind}_{B_r}^{G_r} = \text{Dist}(G_r) \otimes_{\text{Dist}(B_r)} M \cong \text{Dist}(U_r^+) \otimes_k \text{Dist}(B_r) \otimes_{\text{Dist}(B_r)} M \cong \text{Dist}(U_r^+) \otimes_k M,$$

which we'll define as the **coinduced module**.

We can compute the dimension:

$$\dim \text{Ind}_{B_r}^{G_r} M = \dim \text{Coind}_{B_r}^{G_r} M = (\dim M) p^{r|\Phi^+|}.$$

Open: don't know how to compute composition factors.

Proposition 1.1 (?).

1.

$$\mathrm{Coind}_{B_r}^{G_r} M \equiv \mathrm{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho,$$

where the last term is a one-dimensional B_r -module and ρ is the *Weyl weight*.

2.

$$\mathrm{Coind}_{B_r^+}^{G_r} M \cong \mathrm{Ind}_{B_r^+}^{G_r} M \otimes -2(p^r - 1)\rho$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^n w_i.$$

Proof (Sketch for (1)).

Since the tensor product satisfies a universal property, we have a map

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \mathrm{Dist}(G_r) \otimes_{\mathrm{Dist}(B_r)} M \\ & \searrow B_r & \swarrow \exists \psi \\ & N = M \mathrm{Ind}_{B_r}^{G_r} \otimes 2(p^r - 1)\rho & \end{array}$$

1. We need to find a B_r morphism $f : M \rightarrow N$.

2. We need to show that f generates N as a G_r -module.

Note that if (1) and (2) hold, then ψ is surjective, but since $\dim \mathrm{Coind}_{B_r}^{G_r} M = \dim N$ this forces ψ to be an isomorphism.

We can write

$$\begin{aligned} \mathrm{Ind}_{B_r}^{G_r} M \otimes 2(p^r - 1)\rho &= (k[G_r] \otimes M \otimes 2(p^r - 1)\rho)^{B_r} \\ &\cong \mathrm{hom}_{B_r}(\mathrm{Dist}(G_r), M \otimes 2(p^r - 1)\rho). \end{aligned}$$

Let $g_m(x) := m \otimes 2(p^r - 1)\rho$ for any $x = \prod_{\alpha \in \Phi^+} \frac{x_\alpha^{p^r - 1}}{(p^r - 1)!}$, and $g_m(x) = 0$ for any other x . ■