

8.8 Part 2, Computing the Index of L

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Monday 25th May, 2020

Contents

0.1	Main Results	2
0.2	8.8.5:	2

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s, t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \text{Mat}(2n; \mathbb{R})$$

$$S(s, t) \xrightarrow{s \rightarrow \pm\infty} S^\pm(t).$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
 - (not the same kernel/cokernel)
- Compute $\text{Ind } L_1$: explicitly describe $\ker L_1, \text{coker } L_1$.
- Replace in two steps:
 - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s .
 - * Replace $S(s, t)$ by a matrix

$$\tilde{S}(s, t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

- * Idea: approximate by cylinders at infinity.
- * Use invariance of index under small perturbations.
- $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
 - * Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\text{Ind}(L) = \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$\begin{aligned} L_1 : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

where $S : \mathbb{R} \longrightarrow \text{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\text{Ind}(R^\pm(t))$; then

$$\text{Ind}(L) = \text{Ind}(L_1) = \text{Ind}(R^-(t)) - \text{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing L_1 to R^\pm . Let $k^\pm := \text{Ind}(R^\pm)$; then $\text{Ind}(L_1) = k^- - k^+$.
- Lemma 8.8.4: $\text{Ind}(L_0) = \text{Ind}(L)$.
- Han's Talk:
 - Prop 8.8.3, using Lemma 8.8.5
- Me
 - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\text{Ind}(L_1) = K^- - k^+$.

Statement: let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^2) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^2) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\begin{aligned} \dim \text{Ker } F &= 2\# \left\{ \ell \in \mathbf{Z} \mid a^- < 2\pi\ell < a^+ \right\} \\ \dim \text{Ker } F^* &= 2\# \left\{ \ell \in \mathbf{Z} \mid a^+ < 2\pi\ell < a^- \right\}. \end{aligned}$$