Moduli Spaces

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Contents

List of Definitions

List of Theorems

1 Preface

These are notes live-tex'd from a course in Moduli Spaces taught by Ben Bakker at the University of Georgia in Spring 2020. Any errors or inaccuracies are almost certainly my own.

D. Zack Garza, March 18, 2020

2 Thursday January 9th

Some references:

- Course Notes
- Hilbert schemes/functors of points: Notes by Stromme
  - Slightly more detailed: Nitsure, ...Hilbert schemes, Fundamentals of Algebraic Geometry
  - Mumford, Curves on Surfaces
- Harris-Harrison, Moduli of Curves (chatty and less rigorous)

2.1 Representability

Last time: Fix an S-scheme, i.e. a scheme over S.

Then there is a map

$$\begin{aligned} \mathrm{Sch}/S &\longrightarrow \mathrm{Fun}(\mathrm{Sch}/S^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{\mathrm{Sch}/S}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\longrightarrow h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form} \end{aligned}$$

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd}.$$

Theorem 2.1 *(Yoneda)*.

$$\hom_{\mathrm{Fun}}(h_x, F) = F(x).$$

Corollary 2.2.

$$\hom_{\mathrm{Sch}/S}(x, y) \cong \hom_{\mathrm{Fun}}(h_x, h_y).$$

Definition 2.2.1 (Moduli Functor)**.**
A **moduli functor** is a map

$$\begin{aligned} F : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback".} \end{aligned}$$

Definition 2.2.2 (Moduli Space)**.**
A **moduli space** for that "something" appearing above is an $M \in \mathrm{Obj}(\mathrm{Sch}/S)$ such that $F \cong h_M$.

Now fix $S = \mathrm{Spec}\ (k)$.

h_m is the functor of points over M.

Remark (1) $h_m(\mathrm{Spec}\ (k)) = M(\mathrm{Spec}\ (k)) \cong$ "families over Spec k" $= F(\mathrm{Spec}\ k)$.

Remark (2) $h_M(M) \cong F(M)$ are families over M, and $\mathrm{id}_M \in \mathrm{Mor}_{\mathrm{Sch}/S}(M, M) = \xi_{Univ}$ is the universal family.

Every family is uniquely the pullback of ξ_{Univ}. This makes it much like a classifying space.

For $T \in \mathrm{Sch}/S$,

$$h_M \xrightarrow{\cong} F$$
$$f \in h_M(T) \xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\mathrm{Univ}}).$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

Remark (3) If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\begin{array}{ccc} \xi_M & & \xi_{M'} \\ & & \\ M & \xrightarrow{\quad f \quad} & M' \\ & & \\ & & \\ M' & \xrightarrow{\quad g \quad} & M \\ & & \\ \xi_{M'} & & \xi_M \end{array}$$

which shows that f, g must be mutually inverse by using universal properties.

Example 2.1.
A length 2 subscheme of $\mathbb{A}^1_k$ (??) then

$$F(S) = \left\{ V(x^2 + bx + c) \right\} \subset \mathbb{A}'_5$$

where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback.

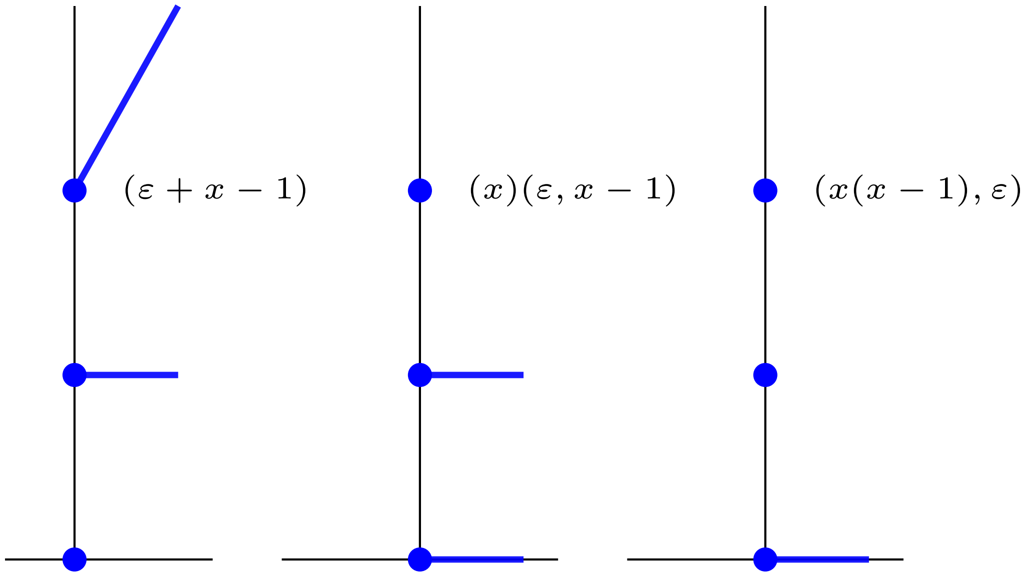
Then F is representable by $\mathbb{A}^2_k(b,c)$ and the universal object is given by

$$V(x^2+bx+c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b,c)$$

where $b,c \in k[b,c]$.

Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider Spec $k[\varepsilon]$.



F	✓	x	x
F'	✓	x	x
F''	✓	✓	✓

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, hook, "i"] & \operatorname{Spec} k[\varepsilon] & & = F'(\operatorname{Spec} k) \arrow[dr] & \\
F(\operatorname{Spec} k[\varepsilon]) \arrow[r, "F(i)"] & F(\operatorname{Spec} k) \arrow[urr] & & & = F''(\operatorname{Spec} k) \\
T_pF^{','} \arrow[u, hook, "\subset"] & P = V(x(x-1)) \arrow[u, hook, "\in"] & & &
\end{tikzcd}$$

We think of $T_pF^{','}$ as the tangent space at p.

If F is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\text{Spec } k[\varepsilon]) \arrow[r] & M(\text{Spec } k) \\
T_pM \arrow[u, hook, "\subset"] \arrow[r] & p \arrow[u, hook, "\subset"]
\end{tikzcd}$$

$$\begin{tikzcd}
& \text{Spec } k \arrow[dl, hook'] \arrow[dr, "?"] & \\
\text{Spec } k[\varepsilon] \arrow[rr] & & \text{Spec } \mathcal{O}_{M,p} \subset M
\end{tikzcd}$$

$$\begin{tikzcd}
& k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
\mathfrak{m}_p \arrow[u, hook] & (\varepsilon) \arrow[u, hook] \\
\mathfrak{m}_p^2 \arrow[u, hook] & 0 \arrow[u, hook]
\end{tikzcd}$$

Moreover, $T_pM = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned}
\lambda : k[\varepsilon] &\longrightarrow k[\varepsilon] \\
\varepsilon &\mapsto \lambda\varepsilon
\end{aligned}$$

$$\lambda^* : \text{Spec }(k[\varepsilon]) \longrightarrow \text{Spec }(k[\varepsilon])$$

$$\begin{array}{ccc}
\lambda : M(\text{Spec }(k[\varepsilon])) & \longrightarrow & M(\text{Spec }(k[\varepsilon])) \\
\cup & & \cup \\
T_pM & \longrightarrow & T_pM.
\end{array}$$

Conclusion: If F is representable, for each $p \in F(\text{Spec } k)$ there exists a unique point of T_pF that are invariant under scaling.

1. If $F, F', G \in \text{Fun}((\text{Sch}/S)^{\text{op}}, \text{Set})$, there exists a fiber product

$$\begin{tikzcd}
F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\
F \arrow[r] & G
\end{tikzcd}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

2. This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \longrightarrow T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

3. If F, F', G are representable, then so is the fiber product $F \times_G F'$.
4. For any functor

$$F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}/T) &\longrightarrow \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

5. F is representable by M/S implies that F_T is representable by $M_T = M \times_S T/T$.

2.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

Proposition 2.3.
$\mathbb{P}^n_{\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ F(S) = \mathcal{O}_s^{n+1} &\longrightarrow L \longrightarrow 0/\sim. \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $F(f)$ is given by pullbacks.

Remark $\mathbb{P}^n_S$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ T &\mapsto F_S(T) = \left\{ \mathcal{O}_T^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim. \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

Proof (of Proposition).
Note: $\mathcal{O}^{n+1} \longrightarrow L \longrightarrow 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section.
So

$$F_i(S) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point.
There is a natural transformation $F_i \longrightarrow F$ by forgetting the latter condition, and is in fact a subfunctor.

$F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

Claim: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{tikzcd}
F_i \arrow[r] & F \\
F_{ij} \arrow[u] \arrow[r] & F_j \arrow[u]
\end{tikzcd}$$

and each $F_{ij} \longrightarrow F_i$ is an open embedding (on the level of their representing schemes).

Example .
For $n = 1$, we can glue along open subschemes

$$\begin{tikzcd}
& F_0 \\
F_{01} \arrow[ur] \arrow[dr] & \\
& F_1
\end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd}
& & & F_0 \arrow[ddr] & \\
& & F_{01} \arrow[ur] \arrow[dr] & & \\
F_{012} \arrow[r] \arrow[urr] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\
& & F_{12} \arrow[ur] \arrow[dr] & & \\
& & & F_2 \arrow[uur] &
\end{tikzcd}$$

This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS).
Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S.

RHS to LHS comes from gluing.

■

Proof (of Claim).

$$F_i(S) = \left\{ \mathcal{O}_S^{n+1} \longrightarrow L \cong \mathcal{O}_s \longrightarrow 0, s_i \neq 0 \right\},$$

but there are no conditions on the sections other than s_i.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$.
We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. Spec $k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.
What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0, s_i \neq 0 \text{ for } i \in I \right\} = \bigtimes_{i \in I \; F} F_i,$$

and $F_I \longrightarrow F_J$ when $I \supset J$.

■

3 Tuesday January 14th

Last time: Representability of functors, and specifically projective space $\mathbb{P}^n_{\mathbb{Z}}$ constructed via a functor of points, i.e.

$$\begin{aligned} h_{\mathbb{P}^n_{\mathbb{Z}}} : \mathbb{P}^n_{\mathbb{Z}}\mathrm{Sch}^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ s &\mapsto \mathbb{P}^n_{\mathbb{Z}}(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}. \end{aligned}$$

for L a line bundle, up to isomorphisms of diagrams:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, Rightarrow] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism.

The point was that for $F_i \subset \mathbb{P}^n_{\mathbb{Z}}$ where

$$F_i(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \;\middle|\; s_i \text{ is invertible} \right\}$$

are representable and can be glued together, and projective space represents this functor.

Remark Because projective space represents this functor, there is a universal object:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) & \end{tikzcd}$$

and other functors are pullbacks of the universal one. (Moduli Space)

Exercise Show that $\mathbb{P}^n_{\mathbb{Z}}$ is proper over Spec $\mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

Definition 3.0.1 (Equalizer)**.**
For a category C, we say a diagram $X \longrightarrow Y \rightrightarrows Z$ is an *equalizer* iff it is universal with respect to the property:

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, shift left] \arrow[r, shift right] & Z \\ & S \arrow[ul, dashed, "\exists!"] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Note that X is the universal object here.

Example 3.1.
For sets, $X = \left\{y \;\middle|\; f(y) = g(y)\right\}$ for $Y \xrightarrow{f,g} Z$.

Definition 3.0.2 (Coequalizer)**.**
A **coequalizer** is the dual notion,

$$\begin{tikzcd} & S & \\ Z \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[u] \arrow[r] & X \arrow[ul, dashed, "\exists!"'] \end{tikzcd}$$

Example 3.2.
Take $C = \mathrm{Sch}/S$, X/S a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$
\begin{tikzcd}
X_\alpha \arrow[r] & X \\
X_{\alpha\beta} \arrow[u] \arrow[r] & X_\beta \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_\beta \arrow[r] & X \\
X_{\beta\alpha} \arrow[u] \arrow[r] & X_\alpha \arrow[u]
\end{tikzcd}
$$

These are canonically isomorphic.

In Sch$/S$, we have

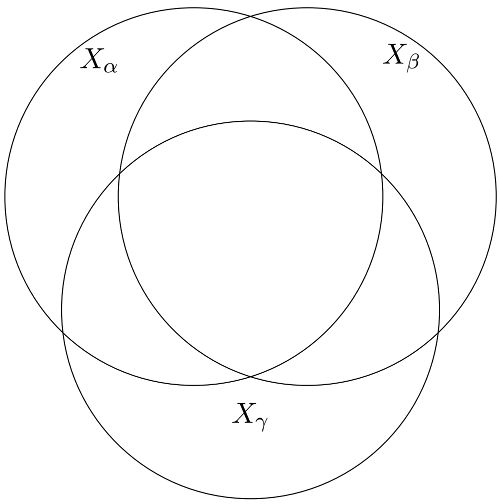
$$
\begin{tikzcd}
\coprod_{\alpha\beta} X_{\alpha\beta} \arrow[r, shift left, "f_{\alpha\beta}"] \arrow[r, shift right, "g_{\alpha\beta}"'] & \coprod_\alpha X_\alpha \arrow[r] & X
\end{tikzcd}
$$

where

$$
\begin{aligned}
f_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\alpha \\
g_{\alpha\beta} &: X_{\alpha\beta} \longrightarrow X_\beta;
\end{aligned}
$$

this is a coequalizer.

Conversely, we can glue schemes. Given $X_\alpha \longrightarrow X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the **cocycle condition**:

1. $$\varphi_{\alpha\beta}^{-1}\Big(X_{\beta\alpha} \bigcap X_{\beta\gamma}\Big) = X_{\alpha\beta} \bigcap X_{\alpha\gamma},$$

   noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

$$\begin{tikzcd}
X_{\alpha\beta}\bigcap X_{\alpha\gamma} \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\alpha\beta}"'] & & X_{\gamma\alpha}\bigcap X_{\gamma\beta} \\
& X_{\beta\alpha}\bigcap X_{\beta\gamma} \arrow[ur, "\varphi_{\beta\gamma}"'] &
\end{tikzcd}$$

Then there exists a scheme X/S such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \longrightarrow X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 3.0.3 (Zariski Sheaf)**.**
A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a *Zariski sheaf* iff for any scheme T/S and any open cover T_α, the following is an equalizer:

$$F(T) \longrightarrow \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 3.3.
Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \longrightarrow \coprod_{\alpha} T_\alpha \longrightarrow T,$$

since giving a local map to X that agrees on intersections if enough to specify a map from $T \longrightarrow X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 3.0.4 (Subfunctors, Open/Closed Functors)**.**
Suppose we have a morphism $F' \longrightarrow F$ in the category $\mathrm{Fun}(\mathrm{Sch}/S, \mathrm{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all T/S.
- ι is **open/closed/locally closed** iff for any scheme T/S and any section $\xi \in F(T)$ over T, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U.

I.e. we can test if pullbacks are contained in a subfunctors by checking factorization.

Note This is the same as asking if the subfunctor F', which maps to F (noting a section is the same as a map to the functor of points), and since $T \longrightarrow F$ and $F' \longrightarrow F$, we can form the fiber product $F' \times_F T$:

$$\begin{tikzcd} F' \arrow[r] & F \\ F' \times_F T \arrow[u] \arrow[r, "g"] & T \arrow[u, "\xi"'] \end{tikzcd}$$

and $F' \times_F T \cong U$.

Note: this is almost tautological!

Thus $F' \longrightarrow F$ is open/closed/locally closed iff $F' \times_F T$ is representable and g is open/closed/locally closed.

I.e. base change is representable, and (?).

Exercise (Tautologous)

1. If $F' \longrightarrow F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Mantra: Treat functors as spaces. We have a definition of open, so now we'll define coverings.

Definition 3.0.5 (Open Covers)**.**
A collection of open subfunctors $F_\alpha \subset F$ is an **open cover** iff for any T/S and any section $\xi \in F(T)$, i.e. $\xi : T \longrightarrow F$, the T_α in the following diagram are an open cover of T:

$$\begin{tikzcd} F_\alpha \arrow[r] & F \\ T_\alpha \arrow[u] \arrow[r] & T \arrow[u, "\xi"'] \end{tikzcd}$$

Example 3.4.
Given

$$F(s) = \left\{ \mathcal{O}_s^{n+1} \longrightarrow L \longrightarrow 0 \right\}$$

and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \longrightarrow F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

Proposition 3.1.
A Zariski sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ with a representable open cover is representable.

Proof .
Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$

representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \longrightarrow F_\alpha$ have the correct gluing maps.

This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \longrightarrow x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x; we'll show that x represents F.
First produce a map $x \longrightarrow F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \longrightarrow \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

Lemma 3.2.
If $E \longrightarrow F$ is a map of functors and E, F are Zariski sheaves, where there are open covers $E_\alpha \longrightarrow E, F_\alpha \longrightarrow F$ with commutative diagrams

$$\begin{tikzcd} E \arrow[r] & F \\ E_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

(i.e. these are isomorphisms locally) then the map is an isomorphism.
With the following diagram, we're done by the lemma:

$$\begin{tikzcd} X \arrow[r] & F \\ X_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

Example 3.5.
For S and E a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \longrightarrow L \longrightarrow 0\} \,/ \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E.

Then the restriction $F_i \longrightarrow \mathbb{P}E$ were $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i, $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i-1}$ by the proposition. (Note that this is clearly a sheaf.)

Example 3.6.
For E locally free over S of rank n, take $r < n$ and consider the functor $\mathrm{Gr}(k, E)(T) = \{f^*E \longrightarrow Q \longrightarrow 0\} \,/ \sim$ (a Grassmannian) where Q is locally free of rank k.

Exercise

a. Show that this is representable

b. For the Plucker embedding

$$\mathrm{Gr}(k, E) \longrightarrow \mathbb{P} \wedge^k E,$$

a section over T is given by $f^*E \longrightarrow Q \longrightarrow 0$ corresponding to

$$\wedge^k f^*E \longrightarrow \wedge^k Q \longrightarrow 0,$$

noting that the left-most term is $f^* \wedge^k E$.

Show that this is a closed subfunctor. (That it's a functor is clear, that it's closed is not.)

Take $S = \operatorname{Spec} k$, then E is a k-vector space V, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n.

Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \longrightarrow Q \longrightarrow 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The Hilbert scheme functor gives e.g. for $\mathbb{P}^n$ of all flat families of subschemes.

4 Thursday January 16th

4.1 Subfunctors

A functor $F' \subset F : (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is **open** iff for all $T \xrightarrow{\xi} F$ where $T = h_T$ and $\xi \in F(T)$.

We can take fiber products:

$$\begin{tikzcd}
F' \arrow[r] & F \\
\begin{matrix} F' \times_F T \\ \text{Representable} \end{matrix} \arrow[u] \arrow[r, "\text{Open}"] & T \arrow[u]
\end{tikzcd}$$

So we can think of "inclusion in F" as being an *open condition*: for all T/S and $\xi \in F(T)$, there exists an open $U \subset T$ such that for all covers $f : T' \longrightarrow T$, we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff f factors through U.

Suppose $U \subset T$ in Sch/T, we then have

$$h_{U/T}(T') = \begin{cases} \emptyset & T' \longrightarrow T \text{ doesn't factor} \\ \{\mathrm{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is $h_{U/T}(T') = \hom_T(T', U)$.

By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{ (a, b) \in F'(T) \times T(T) \;\middle|\; \xi(b) = \iota(a) \text{ in } F(T) \right\},$$

where $F' \xrightarrow{\iota} F$ and $T \xrightarrow{\xi} F$.

So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of F/T' as sections of F over T/T' (?).

$$\begin{tikzcd}
F' \arrow[r, "\iota"] & F & \\
F' \times_F T \arrow[u] \arrow[r] & T \arrow[u, "\xi"] & \\
 & & T' \arrow[ul, "f"] \arrow[uul, bend right, "f\circ\xi"']
\end{tikzcd}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U/S}(T'),$$

and so for $U \subset T$ in Sch/S we have $h_{U/S} \subset h_{T/S}$ is the functor of maps that factor through U. We just identify $h_{U/S}(T') = \hom_S(T', U)$ and $h_{T/S}(T') = \hom_S(T', T)$.

Example 4.1.
$\mathbb{G}_m, \mathbb{G}_a$. $\mathbb{G}_a$ represents giving a global function, $\mathbb{G}_m$ represents giving an invertible function.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{G}_a \\
T' \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & T \arrow[u, "f \in \mathcal{O}_T(T)"']
\end{tikzcd}$$

where $T' = \{f \neq 0\}$ and $\mathcal{O}_T(T)$ are global functions.

4.2 Actual Geometry: Hilbert Schemes

The best moduli space!

Want to parameterize families of subschemes over a fixed object. Fix k a field, X/k a scheme; we'll parameterize subschemes of X.

Definition 4.0.1 (Hilbert Functor)**.**
The hilbert functor is given by

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$

which sends T to closed subschemes $Z \subset X \times_S T \longrightarrow T$ which are flat over T.

Here flatness replaces the Cartier condition.

Definition 4.0.2 (Flatness)**.**
For $X \xrightarrow{f} Y$ and $\mathbb{F}$ a coherent sheaf on X, f is flat over Y iff for all $x \in X$ the stalk F_x is a flat $\mathcal{O}_{y,f(x)}$-module.

Note that f is flat if $\mathcal{O}_x$ is.

Flatness corresponds to varying continuously.

Warning: Unless otherwise stated, assume schemes are Noetherian.

Note that everything works out if we only path with finite covers.

Remark If X/k is projective, so $X \subset \mathbb{P}_k^n$, we have line bundles $\mathcal{O}_x(1) = \mathcal{O}(1)$. For any sheaf F over X, there is a hilbert polynomial $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$. (i.e. we twist by $\mathcal{O}(1)$ n times.)

The cohomology of F isn't changed by the pushforward into $\mathbb{P}_n$ since it's a closed embedding, i.e.

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

Fact (First) For $n \gg 0$, $\dim_k H^0 = \dim M_n$, the nth graded piece of M, which is a graded module over the homogeneous coordinate ring whose $i_* F = \tilde{M}$.

In general, for L ample of X and F coherent on X, we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

Proposition 4.1.
For $f : X \longrightarrow S$ projective, i.e. there is a factorization:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^n \times S \ni \mathcal{O}(1) \arrow[d] \\ & S \end{tikzcd}$$

If S is reduced, irreducible, locally Noetherian, then f is flat $\iff$ $P_{\mathcal{O}_{x_s}}$ is constant for all

$s \in S$.

To be more precise, look the base change to X_1, and the pullback of the fiber? $\mathcal{O}\Big|_{x_i}$?

Note: not using the word "integral" here! S is flat $\iff$ the hilbert polynomial over the fibers are constant.

Example 4.2.
The zero-dimensional subschemes $Z \in \mathbb{P}^n_k$, then P_Z is the length of Z, i.e. $\dim_k(\mathcal{O}_Z)$, and

$$P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z).$$

For two closed points in $\mathbb{P}^2$, $P_Z = 2$.

Consider the affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$, which is given by

$$\text{Spec } k[x,y]/(y,x^2) \cong k[x]/(x^2)$$

and $P_Z = 2$. I.e. in flat families, it has to record how the tangent directions come together.

Example 4.3.
Consider the flat family $xy = 1$ (flat because it's an open embedding) over $k[x]$, here we have points running off to infinity.

Proposition 4.2 ***(Modified Characterization of Flatness for Sheaves).***
A sheaf F is flat iff P_{F_S} is constant.

4.2.1 Proof

Assume $S = \text{Spec } A$ for A a local Noetherian domain.

Lemma 4.3.
For F a coherent sheaf on X/A is flat, we can take the cohomology via global sections $H^0(X; F(n))$. This is an A-module, and is a free A-module for $n \gg 0$.

Proof (of Lemma).
Assumed X was projective, so just take $X = \mathbb{P}^n_A$ and let F be the pushforward. There is a correspondence sending F to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module $\sum_{n \gg 0} H^0(\Pi^m_A; F(n))$ This is equal to $\oplus_{n \gg 0} H^0(\mathbb{P}^m_A; F(n))$ and taking the associated sheaf ($Y \mapsto \tilde{Y}$, as per Hartshorne's notation) which is free, and thus F is free.

See tilde construction in Hartshorne, essentially amounts to localizing free tings.

Conversely, take an affine cover U_i of X. We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume $H^i(\mathbb{P}^m; F(n)) = 0$ for $n \gg 0$, and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^m; F(n)) \longrightarrow \mathcal{C}^0(\underline{U}; F(n)) \longrightarrow \cdots \longrightarrow C^m(\underline{U}; F(n)) \longrightarrow 0.$$

Assuming F is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right.
Finally, local Noetherian + finitely generated flat implies free.

■

By the lemma, we want to show $H^0(\mathbb{P}^m; F(n))$ is free for $n \gg 0$ iff the hilbert polynomials on the fibers P_{F_S} are all constant.

Claim 1 (1)**.**
It suffices to show that for each point $s \in \operatorname{Spec} A$, we have

$$H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$$

for $k(S)$ the residue field, for $n \gg 0$.

Note that P_{F_s} measures the rank of the LHS.

$\Longrightarrow:$ The dimension of RHS is constant, whereas the LHS equals $P_{F_S}(n)$.

$\Longleftarrow:$ If the dimension of the RHS is constant, so the LHS is free.

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank.

For M a finitely generated module over A, find $0 \longrightarrow A^n \longrightarrow M \longrightarrow Q$ is surjective after tensoring with $\operatorname{Frac}(A)$, and tensoring with $k(S)$ for a closed point, if $\dim A^n = \dim M$ then $Q = 0$.

Proof (of Claim 1).
By localizing, we can assume s is a closed point. Since A is Noetherian, its ideal is f.g. and we have

$$A^m \longrightarrow A \longrightarrow k(S) \longrightarrow 0.$$

We can tensor with F (viewed as restricting to fiber) to obtain

$$F(n)^m \longrightarrow F(n) \longrightarrow F_S(n) \longrightarrow 0.$$

Because F is flat, this is still exact.

We can take $H^*(x, \cdot)$, and for $n \gg 0$ only H^0 survives. This is the same as tensoring with $H^0(x, F(n))$.

■

Definition 4.3.1 (Hilbert Polynomial Subfunctor)**.**
Given a polynomial $P \in \mathbb{Z}[n]$ for X/S projective, we define a subfunctor by picking only those with Hilbert polynomial p fiberwise as $\mathrm{Hilb}^P_{X/S} \subset \mathrm{Hilb}_{X/S}$. This is given by $Z \subset X \times_S T$ with $P_Z = P$.

Theorem 4.4 ***(Grothendieck).***
If S is Noetherian and X/S projective, then $\mathrm{Hilb}^P_{X/S}$ is representable by a projective S-scheme.

See cycle spaces in analytic geometry.

5 Thursday January 23

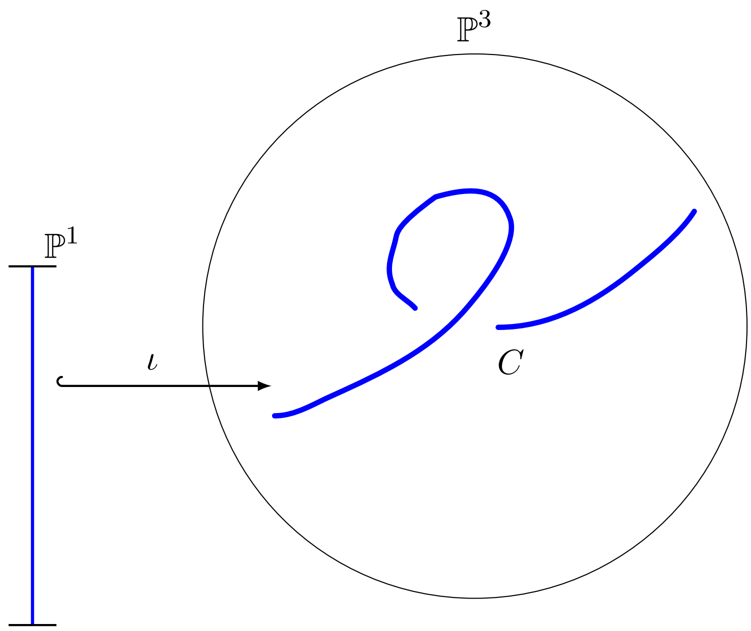
Some facts about the Hilbert polynomial:

1. For a subscheme $Z \subset \mathbb{P}^n_k$ with $\deg P_z = \dim Z = n$, then
$$p_z(t) = \deg z t^n/(n!) + O(t^{n-1}).$$

2. We have $p_z(t) = \chi(\mathcal{O}_z(t))$, consider the sequence
$$0 \longrightarrow I_z(t) \longrightarrow \mathcal{O}^{(t)}_{\mathbb{P}^n} \longrightarrow \mathcal{O}^{(t)}_z \longrightarrow 0,$$
then $\chi(I_z(t)) = \dim H^0(\mathbb{P}^n, J_z(t))$ for $t \gg 0$, and $p_z(0)$ is the Euler characteristic of $\mathcal{O}_Z$.

Serre vanishing, Riemann-Roch, ideal sheaf.

Example 5.1 (Good to keep in mind)**.**
The twisted cubic:



Then

$$p_C(t) = (\deg C)t + \chi(\mathcal{O}_{\mathbb{P}^1}) = 3t + 1.$$

5.0.1 Hypersurfaces

Recall that length 2 subschemes of $\mathbb{P}^1$ are the same as specifying quadratics that cut them out, each such $Z \subset \mathbb{P}^1$ satisfies $Z = V(f)$ where $\deg f = d$ and f is homogeneous. So we'll be looking at $\mathbb{P}H^0(\mathbb{P}^n_k, \mathcal{O}(d))^\vee$, and the guess would be that this is $\mathrm{Hilb}_{\mathbb{P}^n_k}$

Resolve the structure sheaf

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

so we can twist to obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t) \longrightarrow \mathcal{O}_D(t) \longrightarrow 0.$$

Then

$$\chi(\mathcal{O}_D(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(t-d)),$$

which is

$$\binom{n+t}{n} - \binom{n+t-d}{n} = \frac{dt^{n-1}}{(n-1)!} + O(t^{n-2}).$$

Lemma 5.1.
Anything with the Hilbert polynomial of a degree d hypersurface is in fact a degree d hypersurface.

We want to write a morphism of functors

$$\mathrm{Hilb}^{P_{n,d}}_{\mathbb{P}^n_k} \longrightarrow \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee.$$

which sends flat families to families of equations cutting them out.

Want

$$Z \subset \mathbb{P}^n \times S \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee \longrightarrow L \longrightarrow 0.$$

This happens iff

$$0 \longrightarrow L^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$$

with torsion-free quotient.

Note that we use $L^\vee$ instead of $\mathcal{O}_s$ because of scaling.

We have

$$
\begin{aligned}
&0 \longrightarrow I_z \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S} \longrightarrow \mathcal{O}_z \longrightarrow 0 \\
&0 \longrightarrow I_z(d) \longrightarrow \mathcal{O}_{\mathbb{P}^n \times S}(d) \longrightarrow \mathcal{O}_z(d) \longrightarrow 0 \quad \text{by twisting.}
\end{aligned}
$$

We then consider $\pi_s : \mathbb{P}^n \times S \longrightarrow S$, and apply the pushforward to the above sequence noting that it is not right-exact.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \pi_{s*} I_z(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_{\mathbb{P}^n \times S}(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_z(d) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L^\vee = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) \arrow[r] & \text{locally free} \arrow[r] & 0 &
\end{tikzcd}
$$

Note: above diagram may be off horizontally? Todo: check.

This equality follows from flatness, cohomology, and base change. In particular, we need the following facts.

The scheme-theoretic fibers, given by $H^0(\mathbb{P}^n, I_z(d))$ and $H^0(\mathbb{P}^n, \mathcal{O}_z(d))$, are all the same dimension.

Using

1. Cohomology and base change, i.e. for $X \xrightarrow{f} Y$ a map of Noetherian schemes (or just finite-type) and F a sheaf on X which is flat over Y, there is a natural map (not usually an isomorphism)

$$R^i f_* f \otimes k(y) \longrightarrow H^i(x_y, F|_{x_y}),$$

but is an isomorphism if $\dim H^i(x_y, F|_{x_y})$ is constant, in which case $R^i f_* f$ is locally free.

2. If $Z \subset \mathbb{P}^n_k$ is a degree d hypersurface, then independently we know

$$\dim H^0(\mathbb{P}^n, I_z(d)) = 1 \text{ and } \dim H^0(\mathbb{P}^n, \mathcal{O}_z(d)) = \binom{d+n}{n} - 1.$$

To get a map going backwards, we take the universal degree 2 polynomial and form

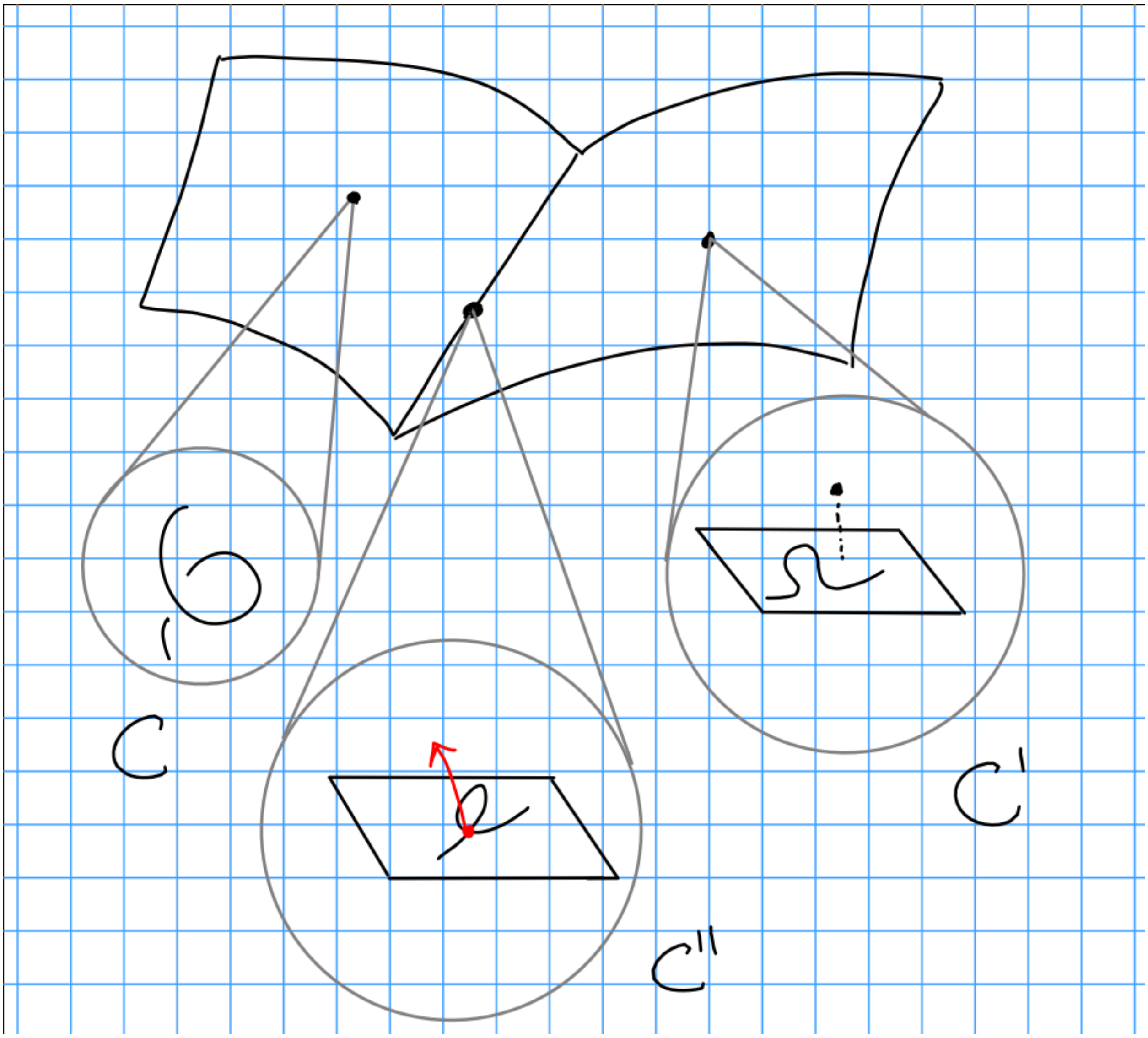
$$V(a_{00}x_0^2 + a_{11}x_1^2 + a_{12}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2) \subset \mathbb{P}^2 \times \mathbb{P}^5.$$

5.0.2 Example: Twisted Cubics

Consider a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ obtained by taking a basis of a homogeneous cubic polynomial. The canonical example is

$$(x, y) \longrightarrow (x^3, x^2y, xy^2, y^3).$$

Then $P_C(t) = 3t+1$, and $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_k}$ has a component with generic point a twisted cubic, and another component with points a curve disjoint union a point, and the overlap are nodal curves with a "fat" 3-dimensional point:



Then $P_{C'} = 1 + \tilde{P}$, the hilbert polynomial of just the base without the disjoint point, so this equals $1 + P_{2,3} = 1 + (3t+0) = 3t+1$. For $P_{C''}$, we take the sequence

$$0 \longrightarrow k \longrightarrow \mathcal{O}_{C''} \longrightarrow \mathcal{O}_{C''\text{reduced}} \longrightarrow 0,$$

so

$$P_{C''} = 1 + P_{C''\text{red}} = 3t+1.$$

Note: flat families have to have the same constant Hilbert polynomial.

Note that we can get paths in this space from $C \longrightarrow C''$ and $C' \longrightarrow C''$ by collapsing a twisted cubic onto a plane, and sending a disjoint point crashing into the node on a nodal cubic.

We're mapping $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, and there is a natural action of $\mathbb{P}\mathrm{GL}(4) \curvearrowright \mathbb{P}^3$, so we get a map

$$\mathbb{P}\mathrm{GL}(4) \times \mathbb{P}^3 \longrightarrow \mathbb{P}^3.$$

Let $c \in \mathbb{P}^3$ and let $\mathcal{C}$ be the preimage. This induces (?) a map

$$\mathbb{P}\mathrm{GL}(4) \longrightarrow \mathrm{Hilb}_{\mathbb{P}^3}^{3t+1}$$

where the fiber over $[C]$ in the latter is $\mathbb{P}\mathrm{GL}(2) = \mathrm{Aut}(\mathbb{P}^1)$. By dimension counting, we find that the dimension of the twisted cubic component is $15 - 3 = 12$.

The 15 in the other component comes from 3-dim choices of plane, 3-dim choices of a disjoint point, and

$$\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee \cong \mathbb{P}^9,$$

yielding 15 dimensions.

To show that these are actually different components, we use Zariski tangent spaces. Let T_1 be the tangent space of the twisted cubic component, then

$$\dim T_1 \mathrm{Hilb}_{\mathbb{P}^3_k}^{3t+1} = 12,$$

and similarly the dimension of the tangent space over the C' component is 15.

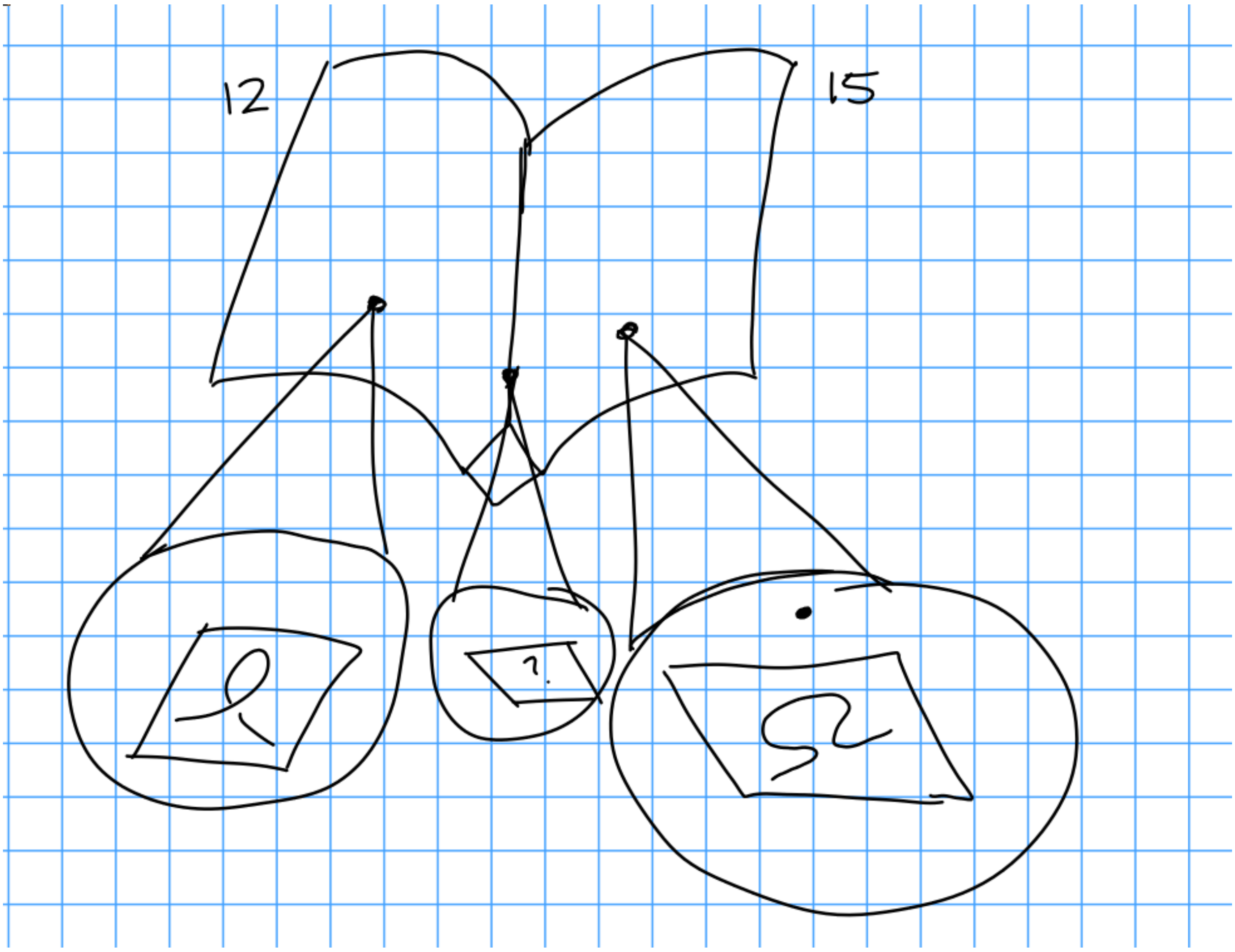
Fact (from Algebra) Let A be Noetherian and local, then the dimension of the Zariski tangent space, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$, the Krull dimension. If this is an equality, then A is regular.

Thus dimensions of the tangent spaces give an upper bound.

Proposition: If X/k is projective and P is a Hilbert polynomial, then $[Z] \in \mathrm{Hilb}_{X/k}^P$, i.e. a closed subscheme of X with hilbert polynomial p (note there's an ample bundle floating around) then the tangent space is $\hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

6 Tuesday January 28th

Last time: Twisted cubics, given by $\mathrm{Hilb}_{\mathbb{P}^3_k}^{3t+1}$.



We got lower (?) bounds on the dimension by constructing families, but want an exact dimension.

Key: Let $Z \subset X$ be a closed k-dimensional subspace.

Proposition 6.1.
For $[z] \in \mathrm{Hilb}^P_{X/k}(k)$, we have an identification of the Zariski tangent space $T_[z]\mathrm{Hilb}^P_{X/k} = \hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

Say $F : (\mathrm{Sch}/K)^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is a function and let $x \in F(k)$. There is an inclusion $i : \operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$. Then there is an induced map $F(\operatorname{Spec} k[\varepsilon]) \xrightarrow{i^*} F(\operatorname{Spec} k)$ where $T_x F := (i^*)^{-1}(x) \mapsto x$. So if F is represented by a scheme H/k, then $T_x h_J = T_x H = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ over k.

Will need a criterion for flatness later, esp. for Artinian thickenings.

Lemma 6.2.
Assume A' is a Noetherian ring and $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$ with $J^2 = 0$. Assume we have $X'/\operatorname{Spec} A'$, and a coherent sheaf F' on X', where X' is Noetherian.
Then F' is flat over A' iff

$$\begin{tikzcd}
F & F' \\
\operatorname{Spec} A' \times_{\operatorname{Spec} A} X' =: X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\
\operatorname{Spec} A' \arrow[r] & \operatorname{Spec} A'
\end{tikzcd}$$

1. F is flat
2. $0 \longrightarrow F \otimes_A J \longrightarrow F'$ is exact.

Take the first exact sequence and tensor with F' (which is right-exact), then $J \otimes_{A'} F' = J \otimes_A$ canonically. This follows because $J = J \otimes_{A'} A$, and there is an isomorphism $J \otimes_{A'} A' \longrightarrow J \otimes_{A'} A$. And $F = F' \otimes_{A'} A$ is a pullback of F'. If flat, then tensoring is exact.

6.0.1 Proof of Lemma

Both conditions are necessary since pullbacks of flats are flat (1), and (2) gives the flatness condition.)

Recall that for a module over a Noetherian ring, M/A, M is flat over A iff $\operatorname{Tor}_1^A(M, A/p) = 0$ for all prime p. Reason: Tor commutes with direct limits, so M is flat iff $\operatorname{Tor}_1^A(M, N) = 0$ for all finitely generated N. Since A is Noetherian, N has a finite filtration $N^{\cdot}$ where $N_i/N_{i+1} \cong A/p_i$. Use the fact that every ideal is contained in a prime ideal.

Take $x \in N$, this yields a map $A \longrightarrow N$ which factors through A/I. If we make such a filtration on A/I, then we can quotient N by $\operatorname{im} f$ where $f : A/I \longrightarrow N$. Continuing inductively, the resulting filtration must stabilize. So we can assume $N = A/I$.

Then I is contained in a maximal.

Exercise Finish proof. See Aatiyah Macdonald.

So it's enough to show that $\operatorname{Tor}_1^{A'}(F', A'/p') = 0$ for all primes $p' \subset A'$.

Observation Since J is nilpotent, $J \subset p'$.

Let $p = p'/J$, this is a prime ideal.

We have an exact diagram by taking quotients:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] & p' \arrow[r] \arrow[d] & p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & A'/p' \arrow[d] & A/p \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

So we can tensor with F' everywhere, and get a map from kernels to cokernels using the snake lemma:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \mathrm{Tor}(A, F) = 0 \arrow[d] & \\
 & 0 \arrow[r, "\text{snake}"] \arrow[d] & \mathrm{Tor}_1^{A_1}(A'/p', F') \arrow[d] & \mathrm{Tor}_1^{A_1}(A/p, F') \arrow[d] & \\
0 \arrow[r] & J \otimes_{A'} F' \arrow[r, hook, "\text{by commuting square}"] & p' \otimes_{A'} F' \arrow[r] \arrow[d] & p \otimes_{A'} F' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \otimes_{A'} F' \arrow[r, hook, "\text{by (2)}"'] & A' \otimes_{A'} F' \arrow[r] \arrow[d] & A \otimes_{A'} F' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r, "\text{snake}"] & A'/p' \otimes_{A'} F' \arrow[r, "="] \arrow[d] & A/p \otimes_{A'} F' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Then by (1), we have $\mathrm{Tor}_1^{A'}(A'/p', F') = \mathrm{Tor}_1^{A'}(A/p, F') = 0$.

■

We will just need this for $A' = k[\varepsilon]$ and $A = k$.

Proposition 6.3.
$T_z \mathrm{Hilb}_{X/k} = \hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

Proof .
Again we have $T_z \mathrm{Hilb}_{X/k} \subset \mathrm{Hilb}_{X/k}(k[\varepsilon])$, and is given by $\left\{ Z' \subset X \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k[\varepsilon] \mathrm{flat}/k[\varepsilon] \;\middle|\; Z' \times_{\mathrm{Spec}\ k[\varepsilon]} \mathrm{Spec}\ k = Z \right\}$.
We have an exact diagram

$$0 \longrightarrow I_{Z'} \longrightarrow \mathcal{O}_{X[\varepsilon]} \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[d] & & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
k \arrow[d] & {} \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_x \arrow[r] \arrow[d] & \mathcal{O}_z \arrow[r] \arrow[d] & {} \\
k[\varepsilon] \arrow[d] & {} \arrow[r] & I_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_{x[\varepsilon]} \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & {} \\
k \arrow[d] & {} \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_x \arrow[r] \arrow[d] \arrow[u, dotted, bend right] & \mathcal{O}_Z \arrow[r] \arrow[d] & {} \\
0 & & {} & {} & {} &
\end{tikzcd}$$

Note the existence of a splitting above.
Given $\phi \in \hom_{\mathcal{O}_x}(I_Z, \mathcal{O}_Z)$. We have

$$I_{Z'} = \left\{ f + \varepsilon g \;\middle|\; f \in I_Z,\ \phi(f) = g \mod I_Z,\ \phi(f) \in \mathcal{O}_Z,\ g \mod I_Z \in \mathcal{O}_x / I_Z = \mathcal{O}_Z \right\}.$$

It's easy to see that $Z' \subset x'$, and

1. $Z' \times k = Z$
2. It's flat over $k[\varepsilon]$, looking at $0 \longrightarrow k \otimes I_{Z'} \longrightarrow I_{Z'}$.

For the converse, take $f \in I_Z$ and lift to $f' = f + \varepsilon g \in I_{Z'}$, then $g \in \mathcal{O}_x$ is well-defined wrt I_Z. Then $g \in \hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

■

The main point: these hom sets are extremely computable.

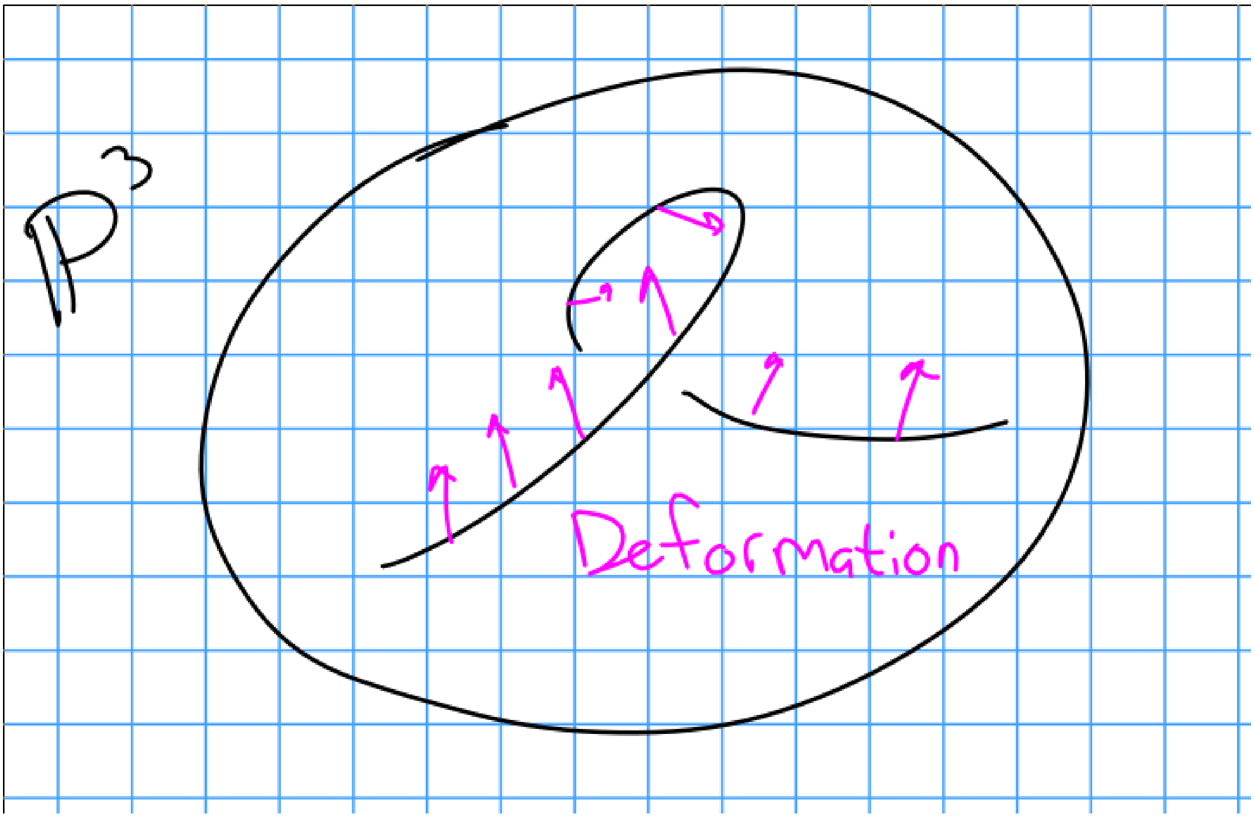
Example: Let Z be a twisted cubic in $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3/k}(k)$.

Observation

$$\hom_{\mathcal{O}_x}(I_Z, \mathcal{O}_Z) = \hom_{\mathcal{O}_X}(I_Z/I_Z^2, \mathcal{O}_Z) = \hom_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)$$

If I_Z/I_Z^2 is locally free, these are global sections of the dual, i.e. $H^0((I_Z/I_Z^2)^\vee)$.

In this case, $Z \hookrightarrow X$ is regularly embedded, and thus $(I_Z/I_Z^2)^\vee$ should be regarded as the normal bundle. Sections of the normal bundle match up with directions to take first-order deformations:



For $i : C \hookrightarrow \mathbb{P}^3$, there is an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow i^*\Omega_{\mathbb{P}^3} \longrightarrow \Omega_\varepsilon \longrightarrow 0,$$

taking duals, which induces

$$0 \longrightarrow T_C \longrightarrow i^*T_{\mathbb{P}^3} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow 0.$$

How do we compute $T_{\mathbb{P}^3}$? Fit into the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow i^*\mathcal{O}(1)^4 \longrightarrow i^*T_{\mathbb{P}^3} \longrightarrow 0,$$

which we can restrict to C.

We have $i^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$, so

$$0 \longrightarrow H^0\mathcal{O}_c \longrightarrow H^*(\mathcal{O}(3)^4) \longrightarrow H^0(i^*T_{\mathbb{P}^3}) \longrightarrow 0$$

which looks like $k \longrightarrow k^{16} \longrightarrow k^{15}$. This yields

$$0 \longrightarrow H^0(T_c) \longrightarrow H^0(i^*T_{\mathbb{P}^3}) \longrightarrow H^0(N_{C/\mathbb{P}^3}) \longrightarrow H^1T_c,$$

which reduces to $0 \longrightarrow k^3 \longrightarrow k^{15} \longrightarrow k^{12} \longrightarrow 0$.

Example $\mathrm{Hilb}^{P_?}_{\mathbb{P}^n)k} \cong \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee$ which has dimension $\binom{n+1}{n} - 1$.

Pick Z a k point in this Hilbert scheme, then $T_Z H = \hom(I_Z, \mathcal{O}_Z)$. Since $I_Z \cong \mathcal{O}_{\mathbb{P}}(-d)$ which fits into

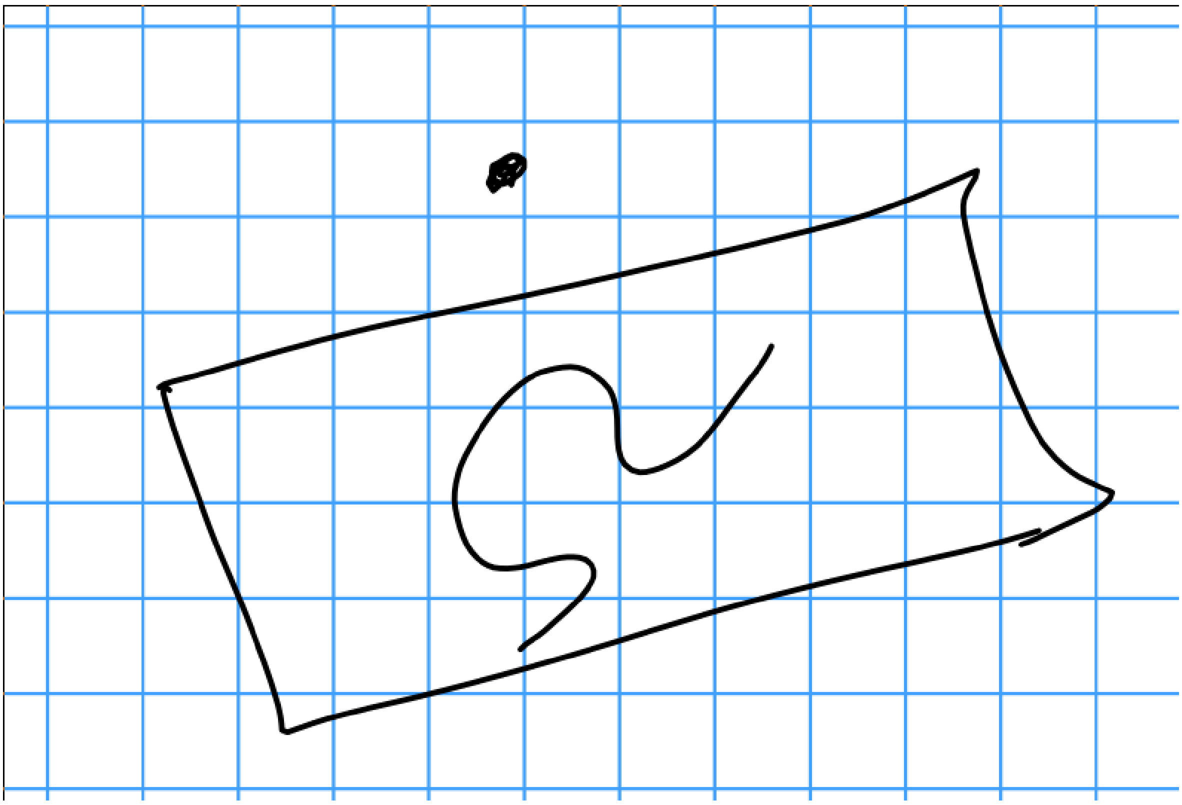
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We can identify

$$\hom(I_Z, \mathcal{O}_Z) = H^0((I_Z/I_Z^2)^\vee) = H^0(\mathcal{O}_Z(d)).$$

$$\begin{array}{ccccc}
0 & \mathcal{O}_{\mathbb{P}^n} & \mathcal{O}_{\mathbb{P}^n}(d) & \mathcal{O}_Z(d) & 0 \\
 & & & & \\
0 & H^0(\cdot) & H^0(\cdot) & H^0(\cdot) & 0 \\
 & & & & \\
 & k & k^{\binom{n+d}{n}} & k^{\binom{n+d}{n}-1} &
\end{array}$$

Example 6.1.
The tangent space of the following cubic:



We can identify $\hom_{\mathcal{O}_k}(I_Z, \mathcal{O}_Z) = H^0((I_Z/I_Z^2)^\vee) = 3 + H^0((I_{Z_0}/I_{Z_0}^2)^\vee)$, where the latter equals $H^0(\mathcal{O}_1\Big|_{z_0} \oplus \mathcal{O}(\zeta)\Big|_{z_0})$ yielding

$$3 + 9 = 12.$$

7 Thursday January 30th

Recall how we constructed the hilbert scheme of hypersurfaces

$$\mathrm{Hilb}_{\mathbb{P}^n_k}^{P_{m,d}} = \mathbb{P}H^0(\mathbb{P}^n; \mathcal{O}(d))^\vee$$

A section $\mathrm{Hilb}_{\mathbb{P}^n_k}^{P}(s)$ corresponds to $z \in \mathbb{P}^n_s$. We can look at the exact sequence

$$0 \longrightarrow I_Z(m) \longrightarrow \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{\text{restrict}} \mathcal{O}_z(m) \longrightarrow 0.$$

as $\mathbb{P}^n_s \xrightarrow{\pi_s} S$, so we can pushforward along π, which is left-exact, so

$$0 \longrightarrow \pi_{s*}I_Z(m) \longrightarrow \pi_{s*}\mathcal{O}_{\mathbb{P}^n_S} = \mathcal{O}_S \otimes H^0(\mathbb{P}^n; \mathcal{O}(m)) \longrightarrow \mathcal{O}_z(m) \longrightarrow R^1\pi_{s*}I_Z(m) \longrightarrow \cdots.$$

Idea: $Z \subset \mathbb{P}^n_k$ will be determined (in families!) by the space of degree d polynomials vanishing on Z (?), i.e. $H^0(\mathbb{P}^n, I_z(m)) \subset H^0(\mathbb{P}^n, \mathcal{O}(m))$ for m very large. This would give a map of functors

$$\mathrm{Hilb}_{\mathbb{P}^n_k}^{P} \longrightarrow \mathrm{Gr}(N, H^0(\mathbb{P}^n, \mathcal{O}(m))).$$

If this is a closed subfunctor, a closed subfunctor of a representable functor is representable and we're done.

Note: We need to get an m uniform in Z, and more concretely:

1. First need to make sense of what it means for Z to be determined by $H^0(\mathbb{P}^n, I_Z(m))$ for m only depending on P.
2. This works point by point, but we need to do this in families. I.e. we'll use the previous exact sequence, and want the R^1 to vanish.

Slogan: We need uniform vanishing statements. There is a convenient way to package the vanishing requirements needed here. From now on, take $k = \overline{k}$ and $\mathbb{P}^n = \mathbb{P}^n_k$.

Definition 7.0.1 (m-Regularity of Coherent Sheaves)**.**
A coherent sheaf F on $\mathbb{P}^n$ is m-regular if $H^i(\mathbb{P}^n; F(m-i)) = 0$ for all $i > 0$.

Example 7.1.
Consider $\mathcal{O}_{\mathbb{P}^n}$, this is 0-regular. Line bundles on $\mathbb{P}_n$ only have 0 and top cohomology. Just need to check that $H^n(\mathbb{P}^n; \mathcal{O}(-n)) = 0$, but by Serre duality this is $H^0(\mathbb{P}^n; \mathcal{O}(n) \otimes \omega_{\mathbb{P}^n})^\vee$ which is $H^0(\mathbb{P}^n; \mathcal{O}(-1))^\vee = 0$.

Proposition 7.1.
Assume F is m-regular. Then

1. There is a natural multiplication map from linear forms on $\mathbb{P}^n$, $H^0(\mathbb{P}^n; \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n; F(k)) \longrightarrow H^0(\mathbb{P}^n; F(k+1))$ which is surjective for $k \geq n$.

   Think of this as a graded module, this tells you the lowest number of small grade pieces needed to determine the entire thing.

2. F is m'-regular for $m' \geq m$.
3. $F(k)$ is globally generated for $k \geq m$, i.e. the restriction

$$H^0(\mathbb{P}^n; F(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow F(k) \longrightarrow 0$$

is exact (i.e. surjective).

Example 7.2.
$\mathcal{O}$ is m-regular for $m \geq 0$ implies $\mathcal{O}(k)$ is $-k$-regular and is also m-regular for$m \geq -k$.

7.0.1 Proof of 2 and 3

Induction on dimension of n in $\mathbb{P}^n$. Coherent sheaves on $\mathbb{P}^0$ are vector spaces, so no higher cohomology.

Step 1:

Take a generic hyperplane $H \subset \mathbb{P}^n$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_H \longrightarrow 0$$

where $\mathcal{O}_H$ is the structure sheaf. Tensoring with H remains exact, so we get

$$0 \longrightarrow F(-1) \longrightarrow F \longrightarrow F_H \longrightarrow 0.$$

Why? $\mathbb{A}^n \subset \mathbb{P}^n$, let $A = \mathcal{O}_{\mathbb{P}^n}(\mathbb{A}^n)$ be the polynomial ring over $\mathbb{A}^n$. Then the restriction of the first sequence to $\mathbb{A}^n$ yields

$$0 \longrightarrow A \xrightarrow{f} A \longrightarrow A/f \longrightarrow 0,$$

and thus we want $F \xrightarrow{f} F \longrightarrow F/fF \longrightarrow 0$ which results after restricting the second sequence to $\mathbb{A}^n$.

Thus we just want f to not be a zero divisor. If we take f not vanishing on any associated point of F, then this will be exact. Associated points: generic points arising by supports of sections of F. F is coherent, so it has finitely many associated points.

If H does not contain any of the associated points of F, then the second sequence is indeed exact.

Step 2:

Twist up by k to obtain

$$0 \longrightarrow F(k-1) \longrightarrow F(k) \longrightarrow F_H(k) \longrightarrow 0.$$

Look at the LES in cohomology to get

$$H^i(F(m-i)) \longrightarrow H^i(F_H(m-i)) \longrightarrow H^{i+1}(F(m-(i+1))).$$

So F_H is m-regular. By induction, this proves statements 1 and 2 for all F_H.

So take $k = m+1-i$ and consider $H^i(F(m-i)) \longrightarrow H^i(F(m+1-i)) \longrightarrow H^i(F_H(m+1-i))$. We know 2 is satisfied, so the RHS is zero, and we know the LHS is zero, so the middle term is zero. Thus F itself is $m+1$ regular, and by inducting on m we get statement 2.

By multiplication maps, we get a commutative diagram:

$$\begin{tikzcd}
 & H^0(\mathcal{O}(1)) \otimes H^0(F(k)) \arrow[rr] \arrow[d, "\beta"] \arrow[drr] & & H^0(\mathcal{O}(1)) \otimes H^0(F_H(k)) \arrow[d] \\
H^0(F(k)) \arrow[ur, "H\otimes \mathrm{id}"] \arrow[r, "H"] & H^0(F(k+1)) \arrow[rr, dashed, "\alpha"] & & H^0(F_H(k+1))
\end{tikzcd}$$

We'd like to show the diagonal map is surjective.

Observations

1. The top map is a surjection, since $H^0(F(k)) \longrightarrow H^0(F_H(k)) \longrightarrow H^1(F(k-1)) = 0$ for $k \geq m$ by (2).
2. The right-hand map is surjective for $k \geq m$.
3. $\ker(\alpha) \subset \text{im}\,(\beta)$ by a small diagram chase, so β is surjective.

This shows (1) and (2) completely.

Proof (of (3)).
We know $F(k)$ is globally generated for $k \gg 0$. Thus for all $k \geq m$, $F(k)$ is globally generated by (1). ∎

Theorem 7.2.
Let $P \in \mathbb{Q}[t]$ be a Hilbert polynomial. There exists an m_0 only depending on P such that for all subschemes $Z \subset \mathbb{P}^n_k$ with hilbert polynomial $P_Z = P$, the ideal sheaf I_z is m_0-regular.

7.0.2 Proof of Theorem

Induct on n. For $n = 0$, again clear because higher cohomology vanishes and there are no nontrivial subschemes.

For a fixed Z, pick H in $\mathbb{P}^n$ (and setting $I := I_z$ for notation) such that $0 \longrightarrow I(-1) \longrightarrow I \longrightarrow I_H \longrightarrow 0$ is exact. Note that the hilbert polynomial $P_{I_H}(t) = P_I(t) - P_I(t-1)$ and $P_I = P_{\mathcal{O}_{\mathbb{P}^n}} - P_Z$. By induction, there exists some m_1 depending only on P such that I_H is m_1-regular. We get $H^{i-1}(I_H(k)) \longrightarrow H^i(I(k-1)) \longrightarrow H^i(I(k)) \longrightarrow H^i(I_H(k))$, and for $k \geq m_1 - i$ the LHS and RHS vanish so we get an isomorphism in the middle. By Serre vanishing, for $k \gg 0$ we have $H^i(I(k)) = 0$ and thus $H^i(I(k)) = 0$ for $k \geq m_i - i$. This works for all $i > 1$, we have $H^i(I(m_i - i)) = 0$.

We now need to find $m_0 \geq m_1$ such that $H^1(I(m_0 - 1)) = 0$ (trickiest part of the proof).

Lemma 7.3.
The sequence $\left(\dim H^1(I(k))\right)_{k\geq m_i-1}$ is *strictly* decreasing.

Note: $h^1 = \dim H^1$.

Given the lemma, it's enough to take $m_0 \geq m_1 + h^1(I(m_1-1))$. Consider the LES we have a surjection $H^0(\mathcal{O}_Z(m_1-1)) \longrightarrow H^1(I(m_1-1)) \longrightarrow 0$. So the dimension of the LHS is equal to $P_Z(m_1-1)$, using the fact that terms vanish and make the Euler characteristic equal to P_Z. Thus we can take $m_0 = m_1 + P(m_1-1)$.

Proof (of Lemma).
Considering the LES

$$H^0(I(k+1)) \xrightarrow{\alpha_{k+1}} H^0(I_H(k+1)) \longrightarrow H^1(I(k)) \longrightarrow H^1(I(k+1)) \longrightarrow 0,$$

where the last term is zero because I_H is m_1-regular. So the sequence $h^1(I(k))$ is non-increasing. Observation: If it does *not* strictly decrease for some k, then there is an equality on the RHS, which makes α_{k+1} surjective. This means that α_{k+2} is surjective, since $H^0(\mathcal{O}(1)) \otimes H^0(I_H(k+1)) \twoheadrightarrow H^0(I_H(k+2))$. So if one is surjective, everything above it is surjective, but by Serre vanishing we eventually get zeros. So α_{k+i} is surjective for all $i \geq 1$, contradicting Serre vanishing, since the RHS are isomorphisms for all k.

■

Thus for any $Z \subset \mathbb{P}^n_k$ with $P_Z = P$, we uniformly know that I_Z is m_0-regular for some m_0 depending only on P.

Claim 2.
Z is determined by the degree m_0 polynomials vanishing on Z, i.e. $H^0(I_z(m_0))$ as a subspace of all degree m_0 polynomials $H^0(\mathcal{O}(m_0))$ and has fixed dimension. We have $H^i(I_Z(m_0)) = 0$ for all $i > 0$, and in particular $h^0(I_Z(m_0)) = P(m_0)$ is constant.

It is determined by these polynomials because we have a sequence

$$0 \longrightarrow I_Z(m_0) \longrightarrow \mathcal{O}(m_0) \longrightarrow \mathcal{O}_Z(m_0) \longrightarrow 0.$$

We can get a commuting diagram over it

$$0 \longrightarrow H^0(I_Z(m_0)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow H^0(\mathcal{O}(m_0)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \cdots$$

where the middle map down is just evaluation and.the first map down is a surjection. Hence $I_Z(m_0)$, hence $\mathcal{O}_Z$, hence Z is determined by $H^0(I_Z(m_0))$.

■

Next time: we'll show that this is a subfunctor that is locally closed.

8 Thursday February 6th

For $k = \overline{k}$, and C/k a smooth projective curve, then $\mathrm{Hilb}^n_{C/k} = \mathrm{Sym}^n C$. For$X/k$ a smooth projective *surface*, $\mathrm{Hilb}^n_{X/k} \neq \mathrm{Sym}^n X$. There is a map (the Hilbert-Chow map)

$$\begin{aligned} \mathrm{Hilb}^n_{X/k} &\longrightarrow \mathrm{Sym}^n X \\ Z &\mapsto \mathrm{supp}(Z) \\ U = \text{reduced subschemes} &\mapsto U' = \text{ reduced multisets} \\ \mathbb{P}^1 &\mapsto (x, x). \end{aligned}$$

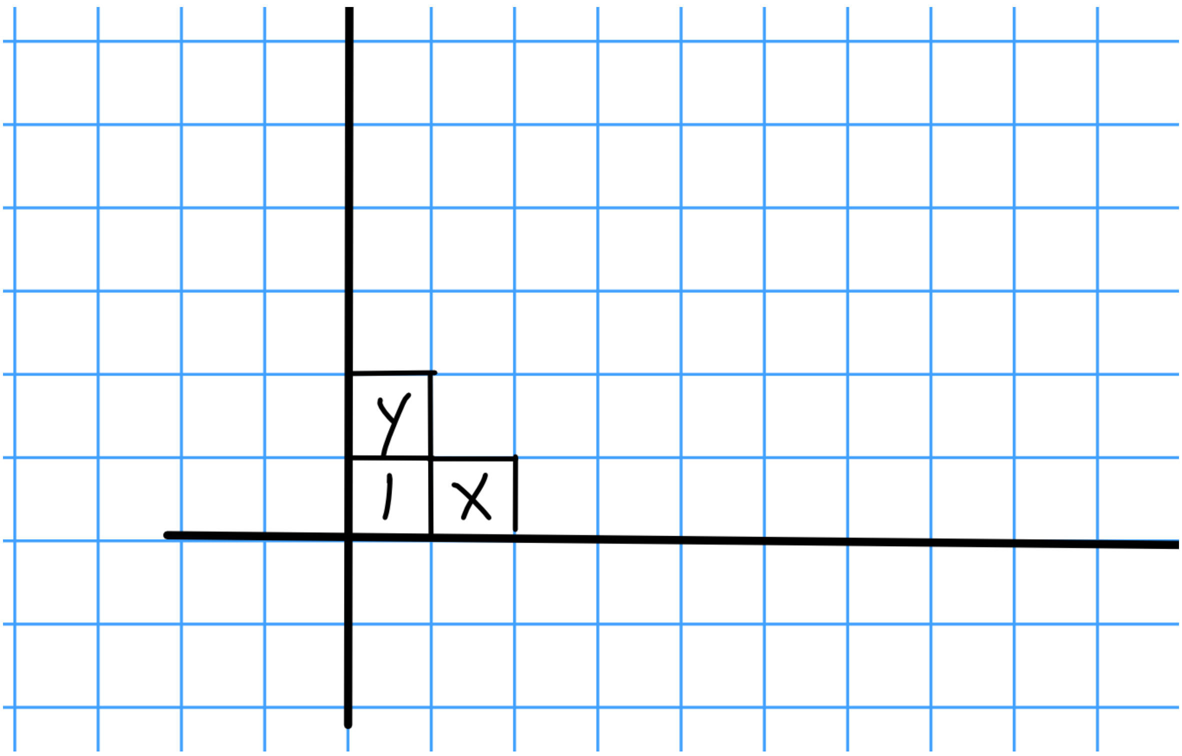
Example: Consider $\mathbb{A}^2 \times \mathbb{A}^2$ under the $\mathbb{Z}/2\mathbb{Z}$ action $((x_1, y_1), (x_2, y_2)) \mapsto ((x_2, y_2), (x_1, y_1))$. Then $(\mathbb{A}^2)^2/\mathbb{Z}/2 = \mathrm{Spec}\ k[x_1, y_1, x_2, y_2]^{\mathbb{Z}/2} = \mathrm{Spec}\ k[x_1x_2, y_1y_2, x_1 + x_2, y_1 + y_2, x_1y_2 + x_2y_1, \cdots]$ with a bunch of symmetric polynomials adjoined.

Example: Take $\mathbb{A}^2$ and consider $\mathrm{Hilb}^3_{\mathbb{P}^2}$. If I is a monomial ideal in $\mathbb{A}^2$, there is a nice picture. We can identify the tangent space

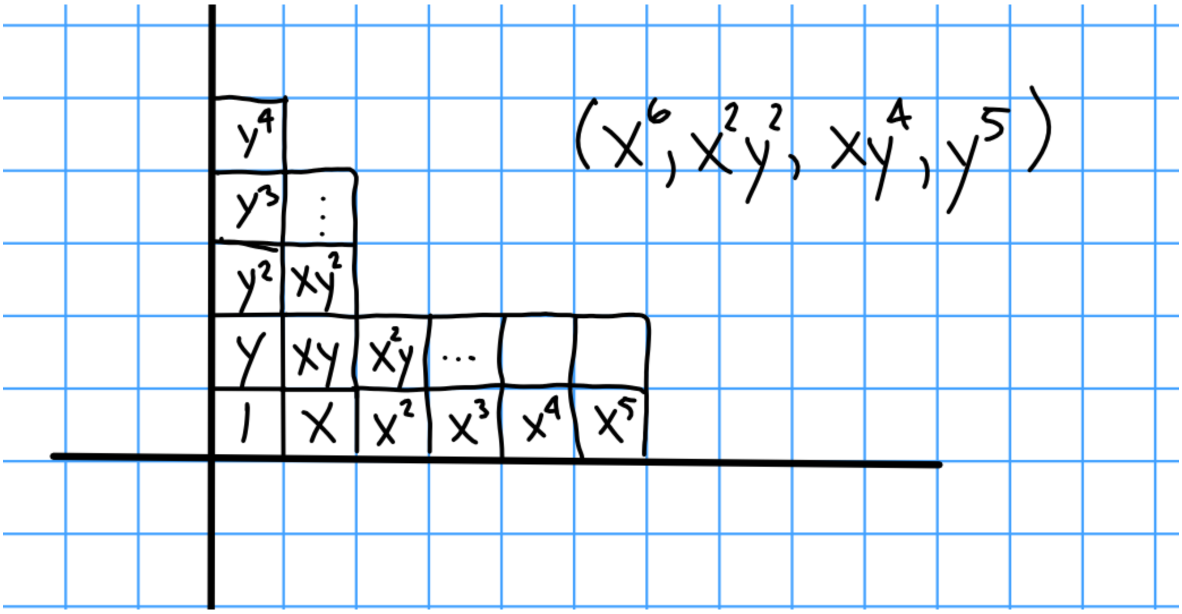
$$T_Z \mathrm{Hilb}^n_{\mathbb{P}^2} = \mathrm{hom}_{\mathcal{O}_{\mathbb{P}^2}}(I_2, \mathcal{O}_Z) = \bigoplus \mathrm{hom}(I_{Z_i}, \mathcal{O}_{Z_i})$$

if $Z = \coprod Z_i$. If I is supported at 0, then we can identify the ideal with the generators it leaves out.

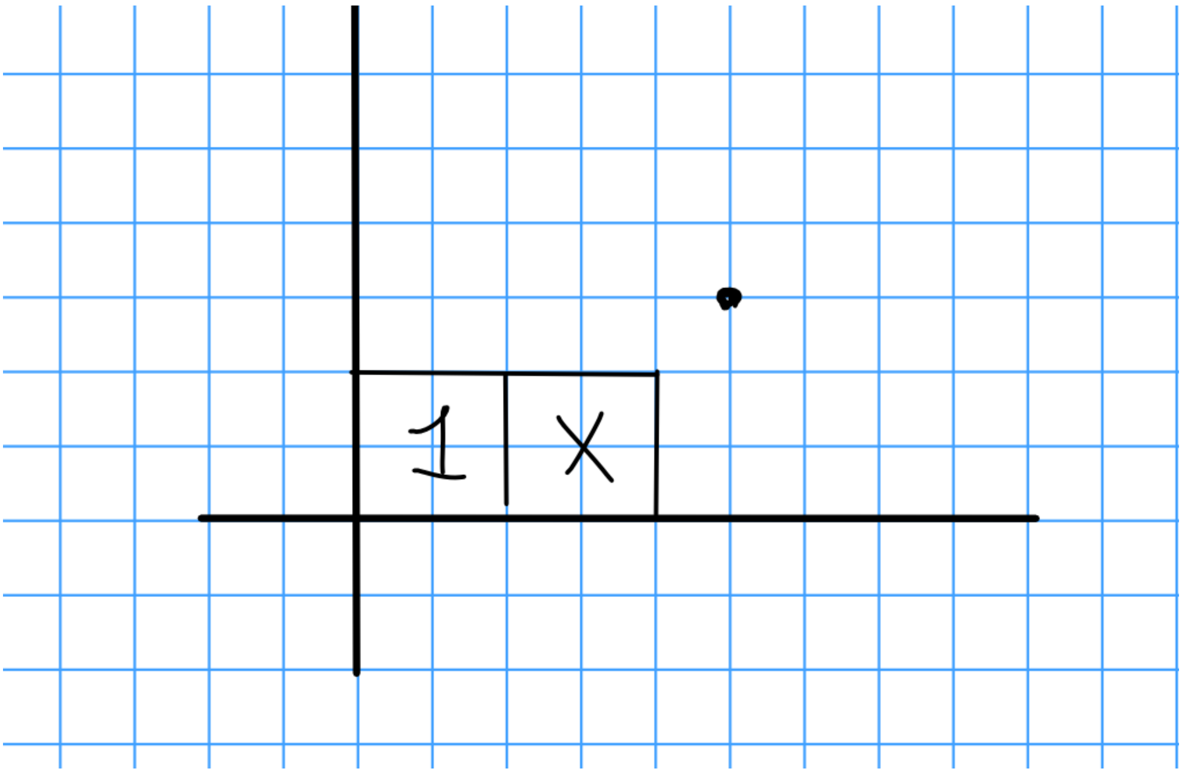
Example 8.1.
$I = (x^2, xy, y^2)$:



Example 8.2.
$I = (x^6, x^2y^2, xy^4, y^5)$:



Example 8.3.
$I = (x^2, y)$. Let $e = x^2$, $f = y$.

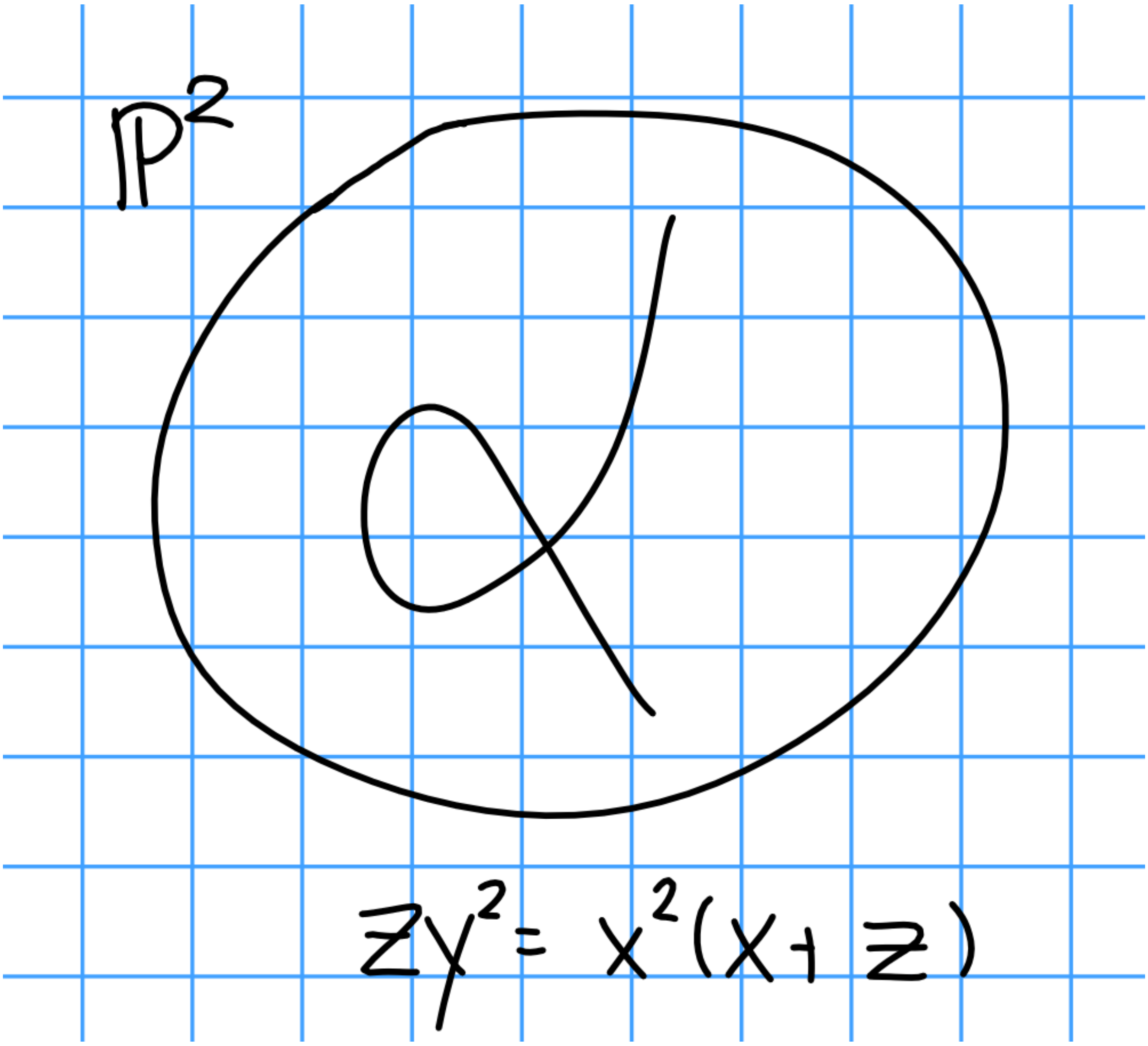


By comparing rows to columns, we obtain a relation $ye = x^2 f$. Write $\mathcal{O} = \{1, x\}$, then note that this relation is trivial in $\mathcal{O}$ since $y = x^2 = 0$.

Thus $\hom(I, \mathcal{O}) = \hom(k^2, k^2)$ is 4-dimensional.

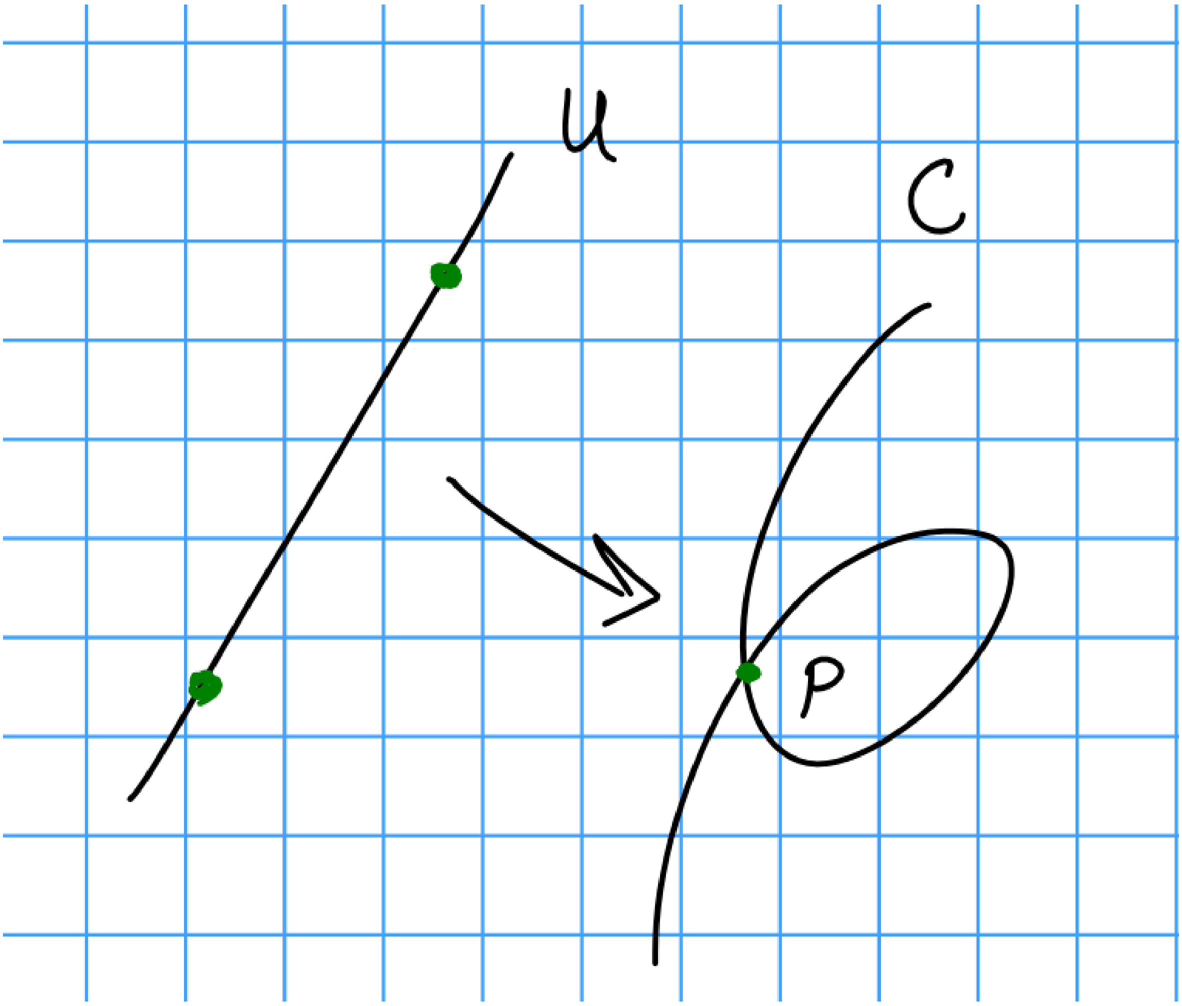
Note that C/k for curves is an important case to know. Take $Z \subset C \times C^n$, then quotient by the symmetric group S^n (need to show this can be done), then $Z/S^n \subset C \times \mathrm{Sym}^n C$ and composing with the functor Hilb represents yields a map $\mathrm{Sym}^n C \longrightarrow \mathrm{Hilb}^n_{C/k}$. This is bijective on points, and a tangent space computation shows it's an isomorphism.

Example 8.4.
Consider the nodal cubic in $\mathbb{P}^2$:



Consider the open subscheme $V \subset \mathrm{Hilb}^2_{C/k}$ of points $z \subset U$ for $U \subset C$ open.

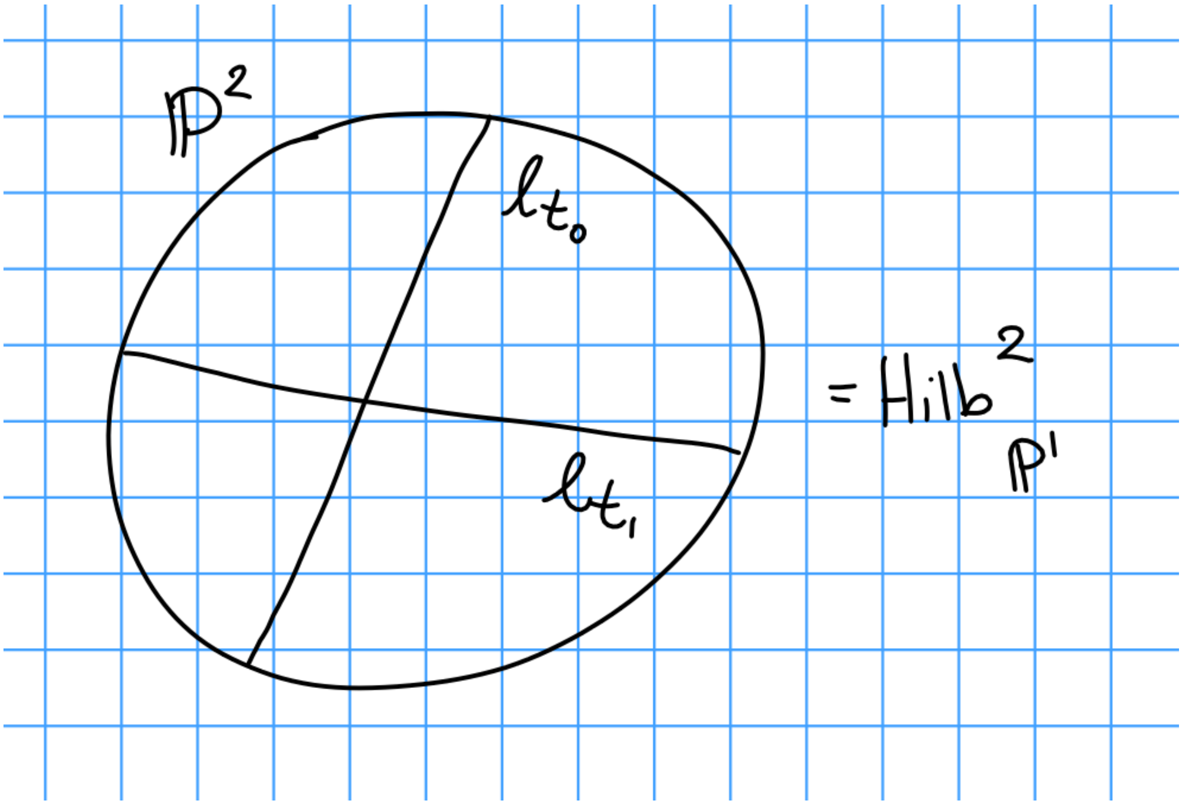
We can normalize:



This yields a map fro $\mathbb{P}^1 \setminus 2$ points. This gives us a stratification, i.e. a locally closed embedding

$$(\text{z supported on U})\coprod(\text{1 point at p})\coprod(\text{both points at p}) \longrightarrow \text{Hilb}^2_{C/k}.$$

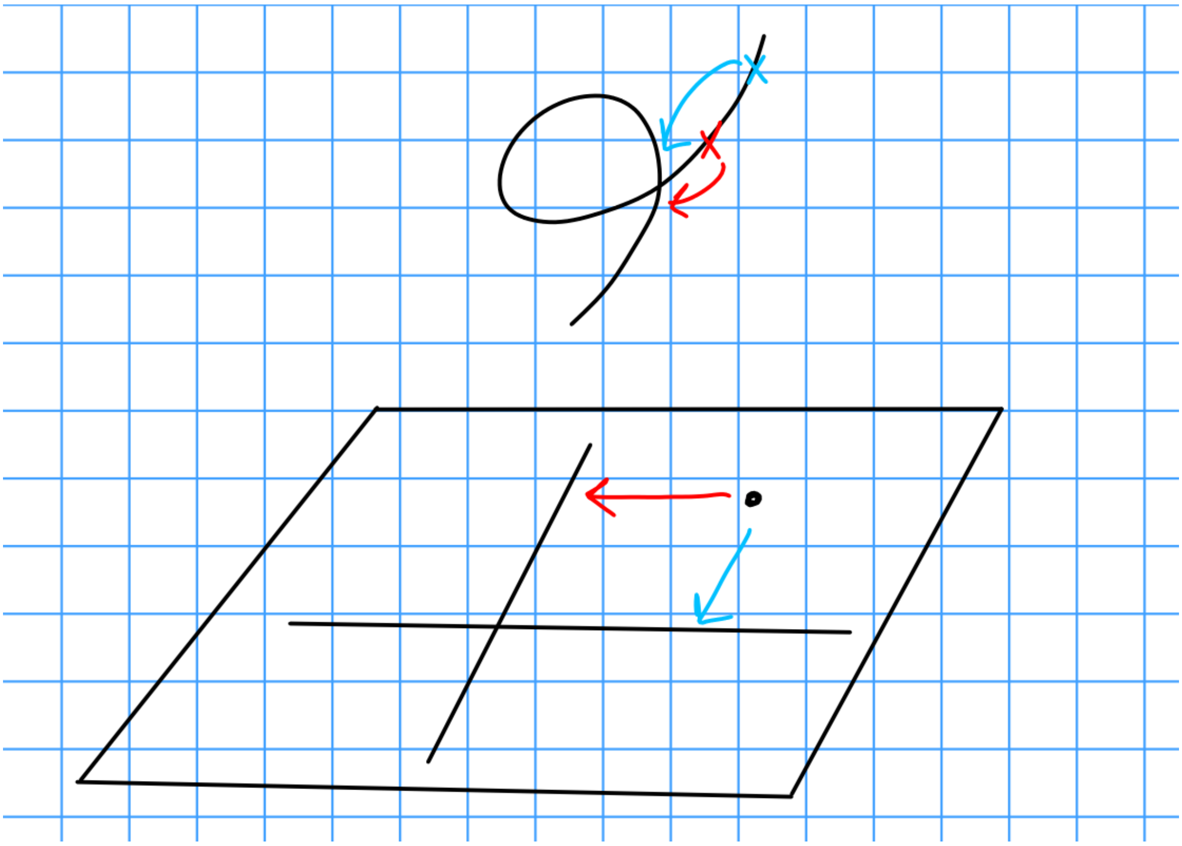
The first locus is given by the complement of two lines:



The third locus is given by arrows at p pointing in any direction, which gives a copy of $\mathbb{P}^1$.

The second is $\mathbb{P}^1$ minus two points.

Above each point is a nodal cubic with two marked points, and moving the base point towards a line correspond to moving one of the points toward the node:



More precisely, we're considering the cover $\mathbb{P}^1 \setminus 2$ points $\longrightarrow C$ and thinking about ways in which two points and approach the missing points. These give specific tangent directions at the node on the cubic, depending on how this approach happens – either both points approach missing point #1, both approach missing point #2, or each approach a separate missing point.

Useful example to think about. Not normal, reduced, but glued in a weird way. Possibly easier to think about: cuspidal cubic.

8.1 Representability

Definition 8.0.1 (m-Regularity)**.**
A coherent sheaf F on $\mathbb{P}^n_k$ for k a field is m-regular iff $H^i(F(m-i)) = 0$ for all $i > 0$.

Proposition 8.1.
For every Hilbert polynomial P, there exists some m_0 depending on P such that any $Z \subset \mathbb{P}^n_k$ with $P_Z = P$ satisfies I_Z is m-regular.

Remark (1) F is m- regular iff $\overline{F} = F \times_{\text{Spec } k} \text{Spec } \overline{k}$ is m-regular.

Remark (2) The m_0 produced does not depend on k.

Lemma 8.2.
For $m_0 = m_0(P)$ and $N = N(P)$, we have an embedding as a subfunctor

$$\mathrm{Hilb}^P_{\mathbb{P}^m_{\mathbb{Z}}} \longrightarrow \mathrm{Gr}(N, H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0))^\vee).$$

For any $Z \subset \mathbb{P}^n_S$ flat over S with $P_{Z_s} = P$ for all $s \in S$ points, we want to send this to $0 \longrightarrow R^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0))^\vee \longrightarrow Q \longrightarrow 0$ or equivalently $0 \longrightarrow Q^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0)) \longrightarrow R \longrightarrow 0$ with R locally free.

So instead of the quotient Q being locally free, we can ask for the sub $Q^\vee$ to be locally free instead, which is a weaker condition.

We thus send Z to $0 \longrightarrow \pi_{s*} I_Z(m_0) \longrightarrow \pi_{s*}\mathcal{O}_{\mathbb{P}^n_s}(m_0) = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0))$ which we obtain by taking the pushforward from this square:

$$\begin{tikzcd}
\mathbb{P}^n_s \arrow[r] \arrow[d, "\pi_s"] & \mathbb{P}^n_Z \arrow[d] \\
S \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

We have a sequence $0 \longrightarrow I_Z(m_0) \longrightarrow \mathcal{O}(m_0) \longrightarrow \mathcal{O}_Z(m_0) \longrightarrow 0$.

Review base-change!

This we get a sequence $0 \longrightarrow \pi_{s*} I_Z(m_0) \longrightarrow \pi_{s*}\mathcal{O}(m_o) \longrightarrow \pi_{s*}\mathcal{O}_Z(m_0) \longrightarrow R^1\pi_{s*} I_Z(m_0) \longrightarrow \cdots$.

Step 1: $R^1\pi_* I_Z(m_0) = 0$.

By base change, it's enough to show that $H^1(Z_s, I_{Z_s}(m_0)) = 0$. This follows by m_0-regularity.

Step 2: $\pi_{s*} I_Z(m_0)$ and $\pi_{s*}\mathcal{O}_Z(m_0)$ are locally free.

For all $i > 0$, we have

- $R^i\pi_{s*} I_Z(m_0) = 0$ by m_0-regularity,
- $R^9\pi_{s*}\mathcal{O}(m_0) = 0$ by base change,
- and thus $R^i\pi_{s*}\mathcal{O}_Z(m_0) = 0$.

Step 3: $\pi_{s*} I_Z(m_0)$ has rank $N = N(P)$.

Again by base change, there is a map $\pi_* I_Z(m_0) \otimes k(s) \longrightarrow H^0(Z_S, I_{Z_s}(m_0))$ which we know is an isomorphism. Because $h^i(I_{Z_S}(m_0)) = 0$ for $i > 0$ by m-regularity and $h^0(I_{Z_S}(m_0)) = P_{\mathcal{O}}(m_0) - P_{\mathcal{O}_{Z_s}}(m_0) = P_{\mathcal{O}}(m_0) - P(m_0)$.

This yields a well-defined functor $\mathrm{Hilb}^P_{\mathbb{P}^n_{\mathbb{Z}}} \longrightarrow \mathrm{Gr}(N, H^0(\mathbb{P}^n, \mathcal{O}(m_0))^\vee)$.

Note that we've just said what happens to objects; strictly speaking we should define what happens for morphisms, but they're always give by pullback.

We want to show injectivity, i.e. that we can recover Z from the data of a number f polynomials vanishing on it, which is the data $0 \longrightarrow \pi_{s*} I_Z(m_0) \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0))$.

Given $0 \longrightarrow Q^\vee \longrightarrow \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0)) = \pi_{s*}\mathcal{O}_{\mathbb{P}^n_S}(m_0)$, we get a daigram

$$\begin{tikzcd}
\pi_s^* Q^\vee \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n_s}(m_0) \\
& I(m_0) \arrow[ur, hook] &
\end{tikzcd}$$

where $Q^\vee = \pi_{s*} I_Z(m_0)$, so we're looking at

$$\begin{tikzcd}
Q^\vee = \pi_{s*}^* \pi_{s*} I_Z(m_0) \arrow[rr] \arrow[dr, tail, "\twoheadrightarrow"] & & \mathcal{O}_{\mathbb{P}^n_s}(m_0) \\
& I(m_0) \arrow[ur, hook] &
\end{tikzcd}$$

The surjectivity here follows from $\mathcal{O}_{Z_s} \otimes H^0(I_{Z_s}(m_0)) \longrightarrow I_{Z_s}(m_0)$ (?).

Given a universal family $G = \mathrm{Gr}(N, H^0(\mathcal{O}(m_0))^\vee)$ and $Q^\vee \subset \mathcal{O}_G \otimes H^0(\mathcal{O}(m_0))^\vee$, we obtain $I_W \subset \mathcal{O}_G$ and $W \subset \mathbb{P}^n_G$.

9 Tuesday February 18th

Theorem: Let X/S be a projective subscheme (i.e. $X \subset \mathbb{P}^n$ for some n). The Hilbert functor of flat families $\mathrm{Hilb}^p_{X/S}$ is representable by a projective S-scheme.

> Note that without a fixed P, this is *locally* of finite type but not finite type. After fixing P, it becomes finite type.

Example: For a curve of genus g, there is a smooth family $\mathcal{C} \xrightarrow{\pi} S$ with S finite-type over $\mathbb{Z}$ where every genus g curve appears as a fiber. I.e., genus g curves form a *bounded family* (here there are only finitely many algebraic parameters to specify a curve).

> How did we construct? Take the third power of the canonical bundle and show it's very ample, so it embeds into some projective space and has a hilbert polynomial.

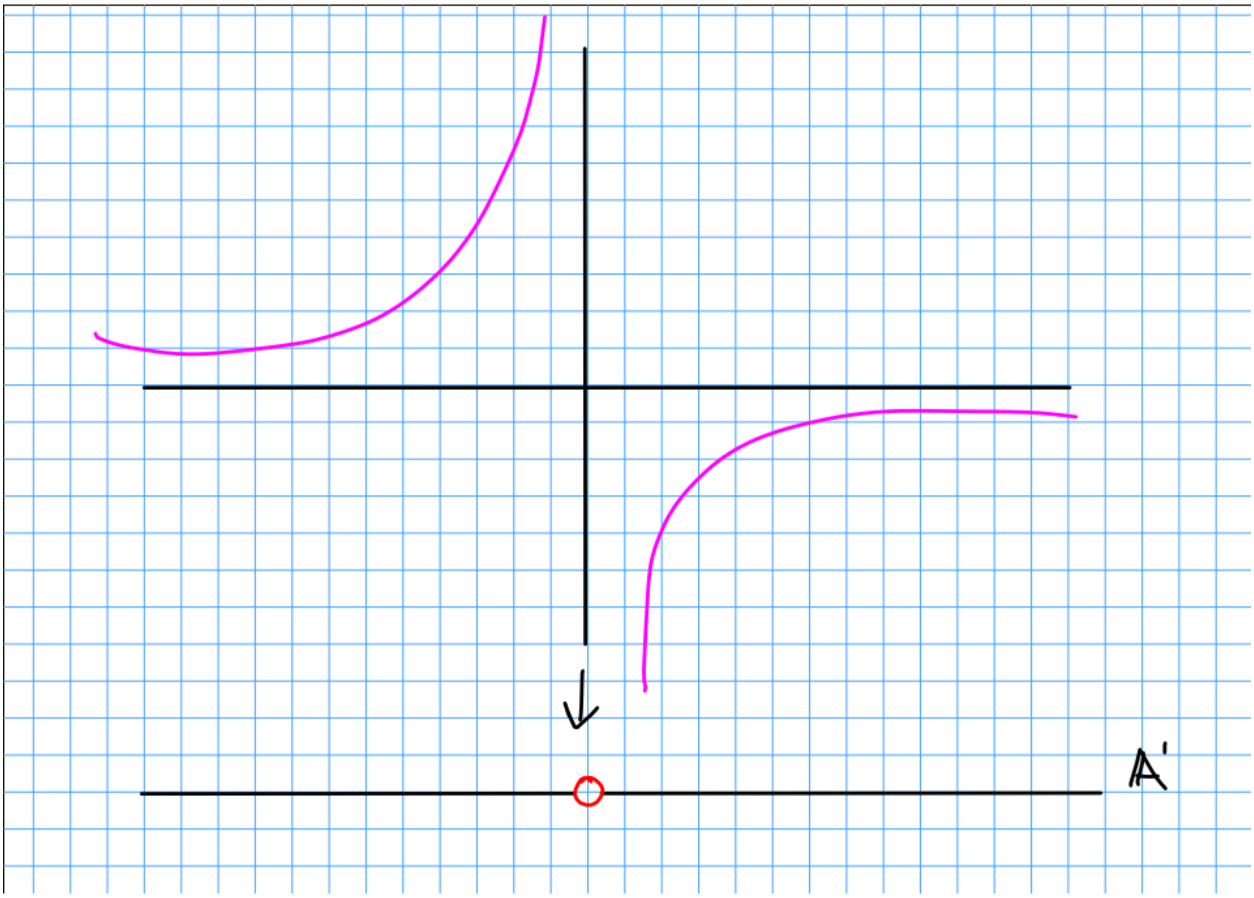
In fact, there is a finite type *moduli stack* $\mathcal{M}_g/\mathbb{Z}$ of genus g curves. There will be a map $S \twoheadrightarrow \mathcal{M}_g$, noting that $\mathcal{C}$ is not a moduli space since it may have redundancy.

> We'll use the fact that a finite-type scheme surjects onto $\mathcal{M}_g$ to show it is finite type.

Remarks: 1. If X/S is proper, we can't talk about the Hilbert polynomial, but the functor $\mathrm{Hilb}_{X/S}$ is still representable by a locally finite-type scheme with connected components which are proper over S.

2. If X/S is *quasiprojective* (so locally closed, i.e. $X \hookrightarrow \mathbb{P}^n_S$), then $\mathrm{Hilb}^P_{X/S}(T) \coloneqq \{z \in X_T$ projective, flat over
   is still representable, but now by a quasiprojective scheme.

Example: Length Z subschemes of $\mathbb{A}^1$: representable by $\mathbb{A}^2$.



Upstairs: parametrizing length 1 subschemes, i.e. points.

3. If $X \subset \mathbb{P}^n_S$ and E is a coherent sheaf on X, then

$$\mathrm{Quot}^P_{E,X/S}(T) = \{j^*E \longrightarrow F \longrightarrow 0, \text{ over } X_T \longrightarrow T, \ F \text{ flat with fiberwise Hilbert polynomial } P\}$$

where $T \xrightarrow{g} S$ is representable by an S-projective scheme.

Example: Take $E = \mathcal{O}_x$, X and S a point, and E is a vector space, then $\mathrm{Quot}^P_{E/S} = \mathrm{Gr}(\mathrm{rank}, E)$.

The hilbert scheme of 2 points on a surface is more complicated than just the symmetric product.

Example:

$$\begin{aligned} \left(\mathbb{A}^2\right)^3 &\longrightarrow \left(\mathbb{A}^2\right)^2 \\ \supseteq \Delta \coloneqq \Delta_{01} \times \Delta 02 &\longrightarrow \left(\mathbb{A}^2\right)^2 \end{aligned}$$

where Δ_{ij} denote the diagonals on the i, j factors. Here all associate points of Δ dominate the image, but it is not flat. Note that if we take the complement of the diagonal in the image, then the restriction $\Delta' \longrightarrow \left(\mathbb{A}^2\right)^2 \setminus D$ is in fact flat.

Mumford example: The Hilbert scheme may have nontrivial scheme structure, i.e. this will be a "nice" hilbert scheme with is generally not reduced. We will find a component H of a $\mathrm{Hilb}^P_{\mathbb{P}^3_C}$ whose generic point corresponds to a smooth irreducible $C \subset \mathbb{P}^3$ which is generically non-reduced.

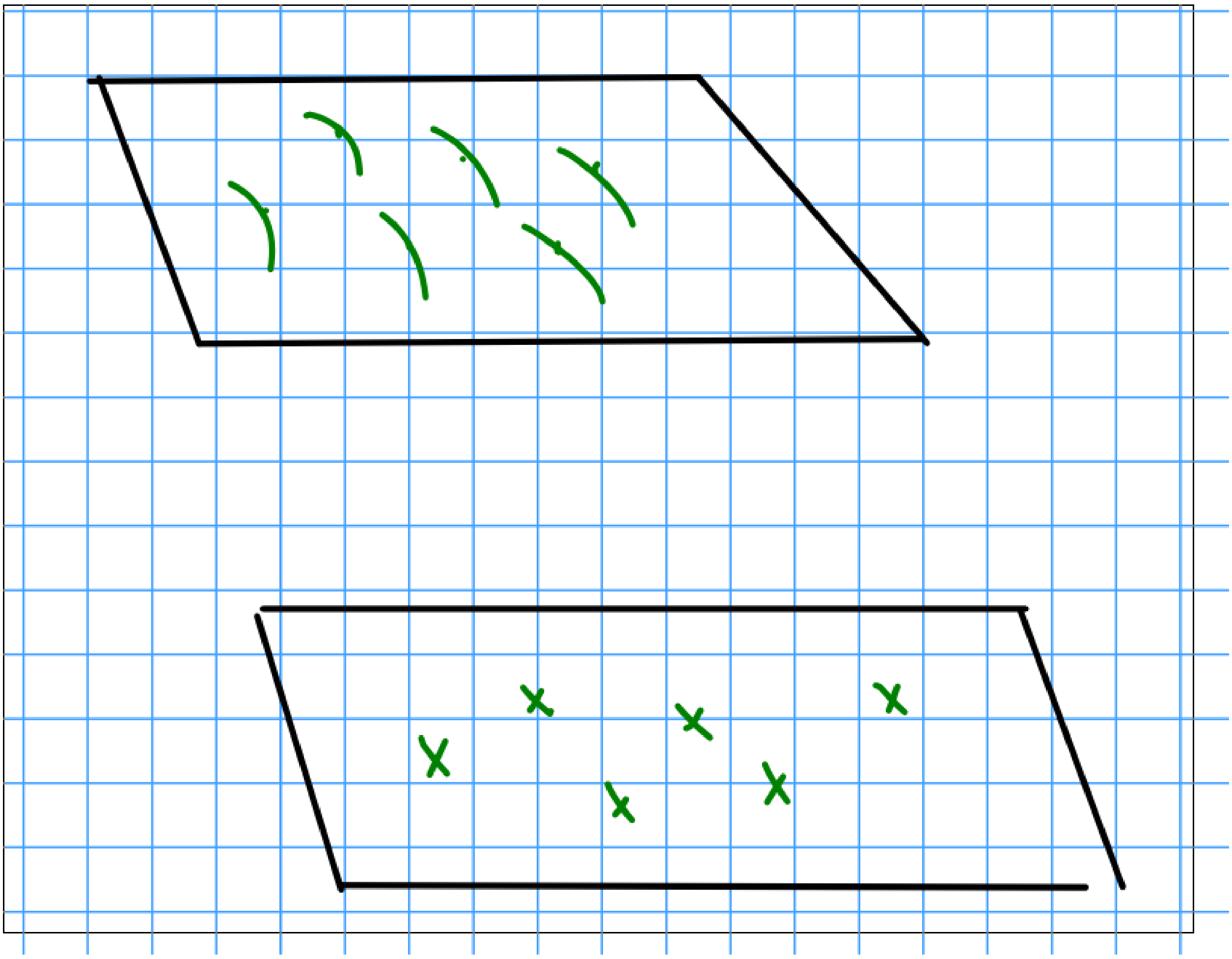
9.1 Cubic Surfaces

See Hartshorne Chapter 5.

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface, then $\mathcal{O}(1)$ on $\mathbb{P}^3$ restricts to a divisor class H of a hyperplane section, i.e. the associated line bundle $\mathcal{O}_x(H) = \mathcal{O}_x(1)$.

Two important facts:

1. X is the blowup of $\mathbb{P}^2$ minus 6 points (replace each point with a curve). There is thus a blowdown map $X \xrightarrow{\pi} \mathbb{P}^2$.



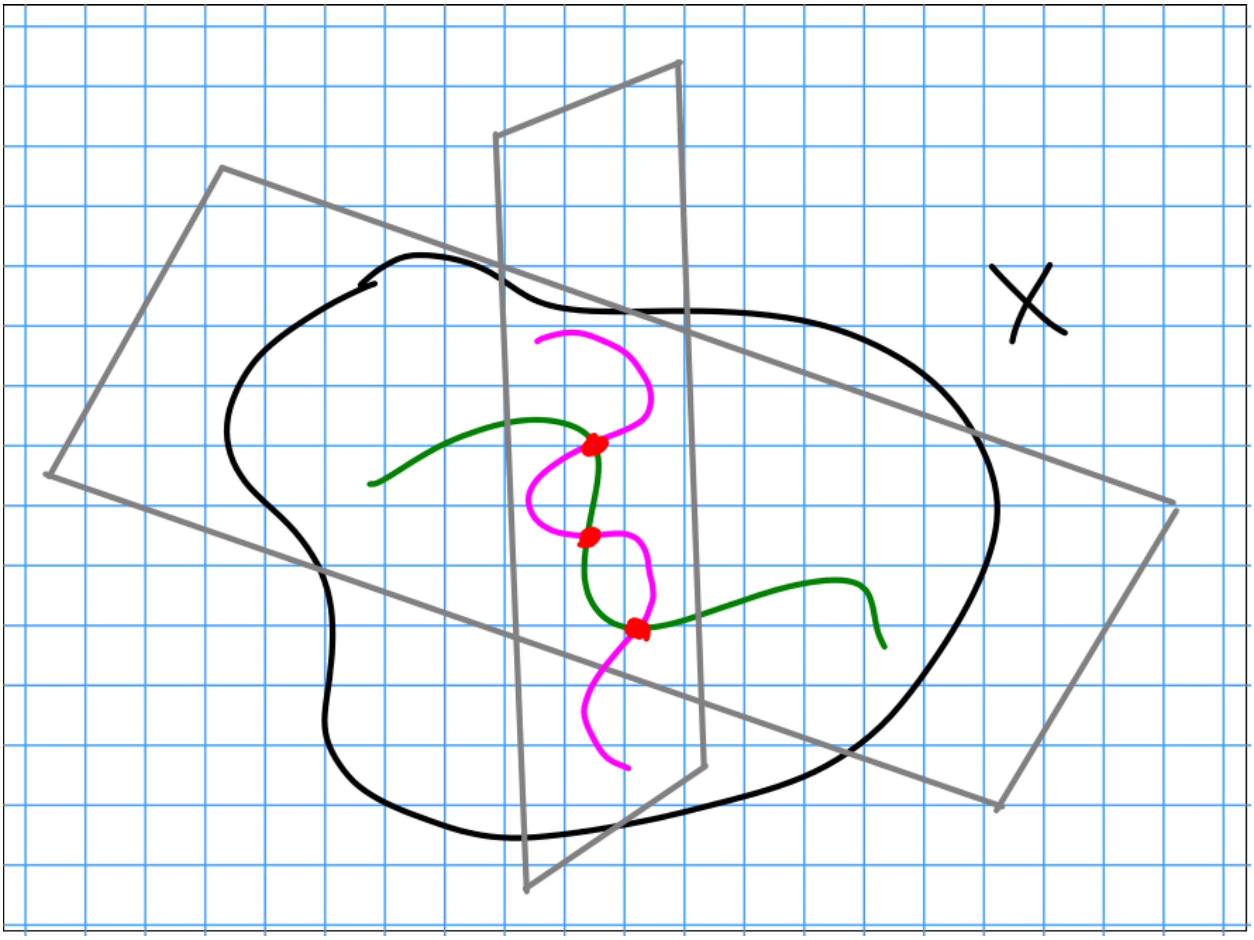
Let $\ell = \pi^*(\text{line})$, then a fact is that $3\ell - E_1 - \cdots - E_6$ (where E_i are the curves about the p_i) is very ample and embeds X into $\mathbb{P}^3$ as a cubic.

2. Every smooth cubic surface X has *precisely* 27 lines. Any 6 pairwise skew lines arise as $E_1, \cdots, E_6$ as in the previous construction.

Take an X and a line $L \subset X$. Consider any C in the linear system $|4H + 2L|$. Fact: $\mathcal{O}(4H + 2L)$ is very ample, so embeds into a big projective space, and thus C is smooth and irreducible by Bertini.

Then the hilbert polynomial of C is of the form $at + b$ where $b = \chi(\mathcal{O}_c)$, the Euler characteristic of the structure sheaf of C, and $a = \deg C$. So we'll compute these.

We have $\deg C = H \cdot C$ (intersection) $= H \cdot (4H + 2L) = 4H^2 + 2H \cdot L = 4 \cdot 3 + 2 = 14$. The intersections here correspond to taking hyperplane sections, intersecting with X to get a curve, and counting intersection points:



In general, for X a surface and $C \subset X$ a smooth curve, then $\omega_C = \omega_X(C) \Big|_C$. Since $X \subset \mathbb{P}^3$, we have $\omega_X = \omega_{\mathbb{P}^3}(X) \Big|_X = \mathcal{O}(-4) \oplus \mathcal{O}(3) \Big|_X = \mathcal{O}_X(-1) = \mathcal{O}_X(-H)$. We also have $\omega_C = \omega_X(C) \Big|_X = (\mathcal{O}_X(-H) \oplus \mathcal{O}_X(4H + 2L)) \Big|_C$, so taking degrees yields $\deg \omega_C = C \cdot (3H + 2L) = (4H + 2L)(3H + 2L) = 12H^2 + 14HL + 4L^2 = 36 + 14 + (-4) = 46$. Since this equals $2g(C) - 2$, we can conclude that the genus is given by $g(C) = 24$.

Thus P is given by $14t + (1 - g) = 14t - 23$.

Good to know: moving a cubic surface moves the lines, you get a monodromy action, and the Weyl group of E_6 acts transitively so lines look the same.

Claim: There is a flat family $Z \subset \mathbb{P}^3_S$ with fiberwise hilbert polynomial P of cures of this form such that the image of the map $S \longrightarrow \mathrm{Hilb}^P_{\mathbb{P}^3}$ has dimension 56.

Proof: We can compute the dimension of the space of smooth cubic surfaces, since these live in $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(3))$, which has dimension $\binom{3+3}{3} - 1 = 19$. Since there are 27 lines, the dimension of the space of such cubics with a choices of a line is also 19. Choose a general C in the linear system $|4H + 2L|$ will add $\dim |4H + 2L| = \dim \mathbb{P}H^0(x, \mathcal{O}_x(C))$. We have an exact sequence

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0 \\ H^0(0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0) \\ . \end{gathered}$$

Since the first H^0 vanishes (?) we get an isomorphism.

By Riemann-Roch, we have $\deg \mathcal{O}_C(C) = C^2 = (4H+2L)^2 = 16H^2 + 16HL + 4L^2 = 64 - 4 = 60$. We can also compute $\chi(\mathcal{O}_C(C)) = 60 - 23 = 37$. We have $h^0(\mathcal{O}_C(C)) - h^1(\mathcal{O}_C(C)) = h^0(\mathcal{O}_C(C)) - h^0(\omega_C(-C))) = 2(23) - 60 < 0$, so there are no sections.

So $\dim |4H + 2L| = 37$. Thus letting S be the space of cubic surfaces X, a line L, and a general $C \in |4H + 2L|$, $\dim S = 56$. We get a map $S \longrightarrow \mathrm{Hilb}^P_{\mathbb{P}^3}$, and we need to check that the fibers are 0-dimensional (so there are no redundancies).

We then just need that every such C lies on a unique cubic. Why does this have to be the case? If $C \subset X, X'$ then $C \subset X \bigcap X'$ is degree 14 curve sitting inside a degree 6 curve, which can't happen.

Thus if H is a component of $\mathrm{Hilb}^P_{\mathbb{P}^3}$ containing the image of S, the $\dim H \geq 56$.

Claim 3: For any C above, we have $\dim T_C H = 57$.

When the subscheme is smooth, we have an identification with sections of the normal bundle $T_C H = H^0(C, N_{C/\mathbb{P}^3})$. There's an exact sequence

$$0 \longrightarrow N_{C/X} = \mathcal{O}_C(C) \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{X/\mathbb{P}^3} \Big|_C = \mathcal{O}_C(x) \Big|_C = \mathcal{O}_C(3H) \Big|_C \longrightarrow 0.$$

Note $\omega_C = \mathcal{O}_C(3H + 2L)$.

As we computed,

$$\begin{gathered} H^0(\mathcal{O}_C(C)) = 37 \\ H^1(\mathcal{O}_C(C)) = 0 \\ . \end{gathered}$$

So we need to understand the right-hand term $H^0(\mathcal{O}_C(3H))$. By Serre duality, this equals $h^1(\omega_C(-3H)) = h^1(\mathcal{O}_C(3L))$. We get an exact sequence

$$0 \longrightarrow \mathcal{O}_X(2L - C) \longrightarrow \mathcal{O}_X(2L) \longrightarrow \mathcal{O}_C(2L) \longrightarrow 0.$$

Taking homology, we have $0 \longrightarrow 0 \longrightarrow 1\ to1\ to0$ since $2L - C = -4H$. Computing degrees yields $h^0(\mathcal{O}_C(3H)) = 20$.

Thus the original exact sequence yields $0 \longrightarrow 37 \longrightarrow ? \longrightarrow 20 \longrightarrow 0$, so $? = 57$. So $\dim N_{C/\mathbb{P}^3} = 57$.

Claim 3: $\dim H = 56$.

Proof: Suppose otherwise. Then we have a family over H^{red} of *smooth* curves, where $f(S) \subset H^{\text{red}}$, where the generic element is not on a cubic or any lower degree surface.

Let C' be a generic fiber. Then C' lies on a pencil of quartics, i.e. 2 linearly independent quartics. Let $I = I_{C'}$ be the ideal of this curve in $\mathbb{P}^3$, there is a SES $0 \longrightarrow I(4) \longrightarrow \mathcal{O}(4) \longrightarrow \mathcal{O}_C(4) \longrightarrow 0$. It can be shown that $\dim H^0(I(4)) \geq 2$.

Fact: A generic quartic in this pencil is *smooth* (can be argued because of low degree and smoothness).

We can compute the dimension of quartics, which is $\binom{4+3}{3} - 1 = 35 - 1 = 34$. The dimension of C's lying on a fixed quartic is 24. But then the dimension of the image in the Hilbert scheme is at most $24 + 34 - 1 = 57$. It can be shown that the picard rank of such a quartic is 1, generated by $\mathcal{O}(1)$, so this is a *strict* inequality, which is a contradiction since $\dim \text{Hilb} = 56$.

> Use the fact that these curves are K3 surfaces? Get the fact about the generator of the picard group from hodge theory.

So we can deform curves a bit, but not construct an algebraic family that escapes a particular cubic.

10 Tuesday February 25th

Let k be a field, X/k projective, then the k-points $\text{Hilb}^P_{X/k}(k)$ corresponds to closed subschemes $Z \subset X$ with hilbert polynomial $P_z = P$. Given a P, we want to understand the local structure of $\text{Hilb}^p_{X/k}$, i.e. diagrams of the form

$$\begin{tikzcd}
 & & \text{Hilb}^P_{X/k} \arrow[d] \\
\text{Spec }(k) \arrow[r] \arrow[urr, "p"] & \text{Spec }(A) \arrow[r] \arrow[ur, dashed, "?"] & \text{Spec }(k) \\
 & A/k \text{ Artinian local} \arrow[u] &
\end{tikzcd}$$

Example 10.1.
For $A = k[\varepsilon]$, the set of extensions is the Zariski tangent space.

Definition 10.0.1 (Category of Artinian Algebras)**.**
Let (Art/k) be the category of local Artinian k-algebras with local residue field k.

Note: these will be the types of algebras appearing in the above diagrams.

Remark This category has fiber coproducts, i.e. there are pushouts:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d, dashed] \\ B \arrow[r, dashed] & A \otimes_C B \end{tikzcd}$$

There are also fibered products,

$$\begin{tikzcd} A \times_C B \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

Here, $A \times_C B := \left\{(a,b) \;\middle|\; f(a) = g(b)\right\} \subset A \times B$.

Example 10.2.
If $A = B = k[\varepsilon]/(\varepsilon^2)$ and $C = k$, then $A \times_C B = k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1, \varepsilon_2)^2$

Note that on the Spec side, these should be viewed as $\mathrm{Spec}\,(A) \coprod_{\mathrm{Spec}\,(C)} \mathrm{Spec}\,(B) = \mathrm{Spec}\,(A \times_C B)$.

Definition 10.0.2 (Deformation Functor (loose definition))**.**
A *deformation functor* is a functor $F : (\mathrm{Art}/k) \longrightarrow \mathrm{Set}$ such that $F(k) = \{\{\mathrm{pt}\}\}$ is a singleton.

Example 10.3.
Let X/k be any scheme and let $x \in X(k)$ be a k-point. We can consider the deformation functor F such that $F(A)$ is the set of extensions f of the following form:

$$\begin{tikzcd} & & X \arrow[d] \\ \mathrm{Spec}\,(k) \arrow[r, hook] \arrow[urr, "x"] & \mathrm{Spec}\,(A) \arrow[r] \arrow[ur, dashed, "f"] & \mathrm{Spec}\,(k) \end{tikzcd}$$

If $A' \longrightarrow A$ is a morphism, then we define $F(A') \longrightarrow F(A)$ is defined because we can precompose to fill in the following diagram

$$\begin{tikzcd}
 & & & X \arrow[d] \\
\text{Spec }(k) \arrow[urrr] \arrow[dr] & & & \text{Spec }(k) \\
 & \text{Spec }(A) \arrow[uurr, "\exists \tilde{f}"] \arrow[r] & \text{Spec }(A') \arrow[uur, dashed, "f"'] \arrow[ur] &
\end{tikzcd}$$

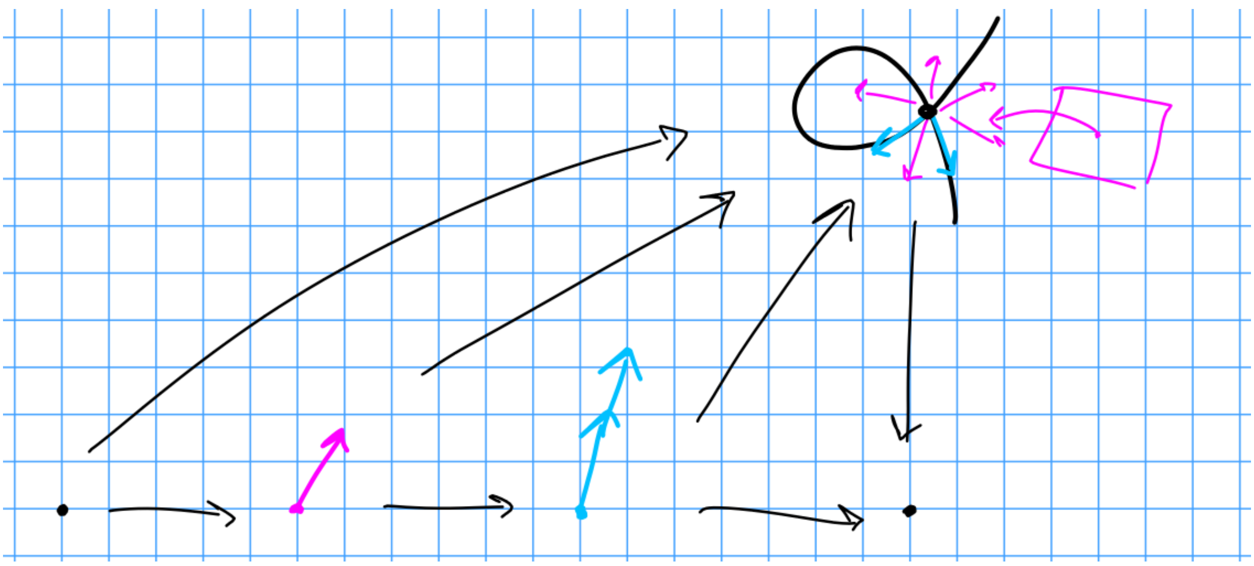
So this is indeed a deformation functor.

Motivating example: the Zariski tangent space on the nodal cubic doesn't "see" the two branches, so we allow "second order" tangent vectors.

We can consider parametrizing the functors above as $F_{X,x}(A)$, which is isomorphic to $F_{\text{Spec }(\mathcal{O}_x)_{X,x}}$ and further isomorphic to $F_{\text{Spec }\widehat{\mathcal{O}_{x}}_{x,X}}$. This is because for Artinian algebras, we have maps $\text{Spec }(\mathcal{O}_{x,X})/\mathfrak{m}^N \longrightarrow \text{Spec }\mathcal{O}_{X,x} \longrightarrow X$.

Remark: $\widehat{\mathcal{O}}_{X,x}$ will be determined by $F_{X,x}$.

Example 10.4.
Consider $y^2 = x^2(x+1)$, and think about solving this over $k[t]/t^n$ with solutions equivalent to $(0,0) \mod t$.



Note that the 'second order' tangent vector comes from Spec $k[t]/t^3$.

We can write $F_{X,x}(A) = \pi^{-1}(x)$ where $\hom_{\mathrm{Sch}/k}(\mathrm{Spec}\ k, X) \xrightarrow{\pi} \hom_{\mathrm{Sch}/k}(\mathrm{Spec}\ k, x) \ni x$. Thus $F_{X,x}(A) = \hom_{\mathrm{Sch}/k}(\mathrm{Spec}\ A, \mathrm{Spec}\ \mathcal{O}_{x,X}) = \hom_{k\text{-alg}}(\widehat{\mathcal{O}}_{X,x}, A)$.

Example 10.5.
Given any local k-algebra R, we can consider

$$\begin{aligned} h_R : (\mathrm{Art}/k) &\longrightarrow \mathrm{Set} \\ A &\mapsto \hom(R, A). \end{aligned}$$

and

$$\begin{aligned} h_{\mathrm{Spec}\ R} : (\mathrm{ArtSch}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\ \mathrm{Spec}\ (A) &\mapsto \hom(\mathrm{Spec}\ A, \mathrm{Spec}\ R). \end{aligned}$$

Definition 10.0.3 (Representable Deformations)**.**
A deformation F is **representable** if it is of the form h_R as above for some $R \in \mathrm{Art}/k$.

Remark There is a Yoneda Lemma for $A \in \mathrm{Art}/k$,

$$\hom_{\mathrm{Fun}}(h_A, F) = F(A).$$

We are thus looking for things that are representable in a larger category, which restrict.

Definition 10.0.4 (Pro-representability)**.**
A deformation functor is *pro-representable* if it is of the form h_R for R a complete local k-algebra (i.e. a limit of Artinian local k-algebras).

We will see that there are simple criteria for a deformation functor to be pro-representable. This will eventually give us the complete local ring, which will give us the scheme representing the functor we want.

Remark It is difficult to understand even $F_{X,x}(A)$ directly, but it's easier to understand small extensions.

Definition 10.0.5 (Small Extensions)**.**
A *small extension* is a SES of Artinian k-algebras of the form $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$ such that J is annihilated by the maximal ideal fo A'.

Lemma 10.1.
Given any quotient $B \longrightarrow A \longrightarrow 0$ of Artinian k-algebras, there is a sequence of small

extensions (quotients):

$$
\begin{tikzcd}
0 & & & \\
B_0 \arrow[u] & B_1 \arrow[ul] & \cdots & B_n = A \arrow[ulll] \\
B \arrow[u] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}
$$

This yields

$$
\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[dddr, Rightarrow] & \operatorname{Spec} B \\
& \operatorname{Spec} B_0 \arrow[u, hook] \\
& \vdots \arrow[u, hook] \\
& \operatorname{Spec} B_n \arrow[u, hook]
\end{tikzcd}
$$

where the Spec B_i are all small.

Remark In most cases, extending deformations over small extensions is easy.

Example 10.6.
Suppose $k = \overline{k}$ and let X/k be connected. We have a picard functor

$$
\begin{aligned}
\operatorname{Pic}_{X/k} : (\mathrm{Sch}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Set} \\
S &\mapsto \operatorname{Pic}(X_S)/\operatorname{Pic}(S).
\end{aligned}
$$

If we take a point $x \in \operatorname{Pic}_{X/k}(k)$, which is equivalent to line bundles on X up to equivalence, we obtain a deformation functor

$$
\begin{aligned}
F := F_{\operatorname{Pic}_{X/k,x}} &\longrightarrow \mathrm{Set} \\
A &\mapsto \pi^{-1}(x)
\end{aligned}
$$

where

$$\begin{aligned}\pi : \text{Pic }_{X/k}(\text{Spec } A) &\longrightarrow \text{Pic }_{X/k}(\text{Spec } k) \\ \pi^{-1}(x) &\mapsto x.\end{aligned}$$

This is given by taking a line bundle on the thickening and restricting to a closed point. Thus the functor is given by sending A to the set of line bundles on X_A which restrict to X_x.

That is, $F(A) \subset \text{Pic }_{X/k}(\text{Spec } A)$ which restrict to x. So just pick the subspace Pic (X_A) (base changing to A) which restrict.

There is a natural identification of Pic $(X_A) = H^1(X_A, \mathcal{O}^*_{X_A})$. If $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$ is a thickening of Artinian k-algebras, there is a restriction map of invertible functions $\mathcal{O}^*_{X_A} \longrightarrow \mathcal{O}^*_{X'_A} \longrightarrow 0$ which is surjective since the map on structure sheaves is surjective and its a nilpotent extension. The kernel is then just $\mathcal{O}_{X_{A'}} \otimes J$.

If this is a small extension, we get a SES $0 \longrightarrow \mathcal{O}_X \otimes J \longrightarrow \mathcal{O}^*_{X_{A'}} \longrightarrow \mathcal{O}^*_{x_A} \longrightarrow 0$.

Taking the LES in cohomology, we obtain $H^1\mathcal{O}_X \otimes J \longrightarrow H^1\mathcal{O}^*_{X_{A'}} \longrightarrow H^1\mathcal{O}^*_{x_A} \longrightarrow H^0\mathcal{O}_X \otimes J$. Thus there is an obstruction class in H^2, and the ambiguity is detected by H^1. H^1 is referred to as the *deformation space*, since it counts the extensions, and H^2 is the *obstruction space*.

11 Thursday February 27th

Big picture idea: We have moduli functors, such as

$$\begin{aligned}F_{S'} : (\text{Sch}/k)^{\text{op}} &\longrightarrow \text{Set} \\ \text{Hilb} : S &\longrightarrow \text{flat subschemes of } X_S \\ \text{Pic } : S &\longrightarrow \text{Pic } (X_S)/\text{Pic } (S) \\ \text{Def} : S &\longrightarrow \text{flat families } /S, \text{ smooth, finite, of genus } g.\end{aligned}$$

Deformation Theory: Choose a point f the scheme representing $F_{S'}$ with $\xi_0 \in F_{gl}(\text{Spec } K)$. Define

$$F_{\text{loc}} : (\text{Artinian local schemes}/K)^{\text{op}} \longrightarrow \text{Set}.$$

$$\begin{tikzcd} \text{Spec } (K) \arrow[r, hook, "i"] & \text{Spec } (A) \arrow[r] & F(i)^{-1}(\xi_0) \arrow[r] & F_{gr}(\text{Spec } K) \arrow[d, "F(i)"] \\ & & & F_{gl}(\text{Spec } K) \end{tikzcd}$$

Deformation functors: Let $F : (\text{Art}/k) \longrightarrow \text{Set}$ where $F(k)$ is a point. Denote $\widehat{\text{Art}}/k$ the set of complete local k-algebras. Since $\text{Art}/k \subset \widehat{\text{Art}}/k$, we can make extensions $\widehat{F}$ by just taking limits:

$$\begin{tikzcd}
 & \mathrm{Art}/k \arrow[r, "F"] & \mathrm{Set} \\
\varprojlim R/\mathfrak{m}_R^n = R \in & \widehat{\mathrm{Art}}/k \arrow[u] \arrow[ur, "\widehat{F}"] &
\end{tikzcd}$$

where we define $\widehat{F}(R) = \varprojlim F(R/\mathfrak{m}_R^n)$.

Question: when is F pro-representable, which happens iff $\widehat{F}$ is representable? In particular, we want $h_R \xrightarrow{\cong} \widehat{F}$ for $R \in \widehat{\mathrm{Art}}/k$, so $h_R = \hom_{\widehat{\mathrm{Art}}/k}(R, \cdot) = \hom_?(\cdot, \mathrm{Spec}\ k)$.

Example 11.1.
Let $F_{gl} = \mathrm{Hilb}^p_{X/k}$, which is represented by H/k. Then $\xi_0 = F_{gl}(k) = H(k) = \left\{ Z \subset X \mid P_z = f \right\}$.
Then F_{loc} is representable by $\widehat{\mathcal{O}}_{H/\xi_0}$.

Definition 11.0.1 (Thickening)**.**
Given an Artinian k-algebra $A \in \mathrm{Art}/k$, a *thickening* is an $A' \in \mathrm{Art}/k$ such that $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$, so $\mathrm{Spec}\ A \hookrightarrow \mathrm{Spec}\ A'$.

Definition 11.0.2 (Small Thickening)**.**
A *small thickening* is a thickening such that $0 = \mathfrak{m}_{A'} J$, so J becomes a module for the residue field, and $\dim_k J = 1$.

Lemma 11.1.
Any thickening of A, say $B \longrightarrow A$, fits into a diagram:

$$\begin{tikzcd}
 & & 0 & & \\
 & J \arrow[r] & A' \arrow[u] \arrow[r] & A \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & I \arrow[u] \arrow[r] & B \arrow[u] \arrow[r] & A \arrow[r] & 0 \\
 & I' \arrow[u] \arrow[r, Rightarrow] & I' \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

> *Proof* .
> We just need $I' \subset I$ with $\mathfrak{m}_S I \subset J' \subset I \iff J\mathfrak{m}_B = 0$. ■

Choose J' to be a preimage of a codimension 1 vector space in $I/\mathfrak{m}_B I$. Thus $J = I/I'$ is 1-dimensional.

Thus any thickening A can be obtained by a sequence of small thickenings. By the lemma, in principle F and thus $\widehat{F}$ are determined by their behavior under small extensions.

11.0.1 Example

Consider Pic , fix X/k, start with a line bundle $L_0 \in \text{Pic }(x)/\text{Pic }(k) = \text{Pic }(x)$ and the deformation functor $F(A)$ being the set of line bundles L on $X_A$with $L|_x \cong L_0$, modulo isomorphism.

Note that this yields a diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[d, hook] & k \arrow[d, "\text{unique closed point}"] \\
X_A \arrow[r] & \text{Spec } A
\end{tikzcd}$$

This is equal to $(I_x)^{-1}(L_0)$, where $\text{Pic }(X_a) \xrightarrow{I_x} \text{Pic }(x)$.

If $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$ is a small thickening, we can identify

$$\begin{tikzcd}
0 \arrow[r] & J \otimes_x \mathcal{O}_x \cong \mathcal{O}_x \arrow[r] & \mathcal{O}_{X_{A'}} \arrow[r] & \mathcal{O}_{X_A} \arrow[r] & 0 & \in \text{AbSheaves} \\
0 \arrow[r] & \mathcal{O}_x \arrow[r, "f \mapsto 1+f"] & \mathcal{O}^*_{X_{A'}} \arrow[r] \arrow[u, hook] & \mathcal{O}^*_{X_A} \arrow[r] & 0 &
\end{tikzcd}$$

This yields a LES

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{O}_x) = k \arrow[r] & H^0(X_{A'}, \mathcal{O}^*_{x_{A'}}) = A'^* \arrow[r] & H^0(X_{A'}, \mathcal{O}^*_{x_A}) = A^* \arrow[r] \arrow[dll] & \therefore 0 \\
\therefore 0 \arrow[r] & H^1(X, \mathcal{O}_x) \arrow[r] & H^1(X_{A'}, \mathcal{O}^*_{x_{A'}}) = \text{Pic }(X_{A'}) \arrow[r, "\text{restriction to } X_A"] & H^1(X_A, \mathcal{O}^*_{x_A}) = \text{Pic }(X_A) \arrow[dll, "\text{obs}"] & \\
& H^2(X, \mathcal{O}_x) & & &
\end{tikzcd}$$

To understand F on small extensions, we're interested in

1. Given $L \in F_{loc}(A)$, i.e. L on X_A restricting to L_0, when does it extend to $L' \in F_{loc}(A')$? I.e., does there exist an L' on $X_{A'}$ restricting to L?

2. Provided such an extension L' exists, how many are there, and what is the structure of the space of extensions?

We have an $L \in \text{Pic}\ (X_A)$, when does it extend? By exactness, L' exists iff $\text{obs}(L) = 0 \in H^2(X, \mathcal{O}_x)$, which answers 1. To answer 2, $(I_x)^{-1}(L)$ is the set of extensions of L, which is a torsor under $H^1(x, \mathcal{O}_x)$. Note that these are fixed k-vector spaces.

Note #3: $H^1(X, \mathcal{O}_x)$ is interpreted as the *tangent space* of the functor F, i.e. $F_{loc}(K[\varepsilon])$.

> Note that if X is projective, line bundles can be unobstructed without the group itself being zero.

For (3), just play with $A = k[\varepsilon]$, which yields $0 \longrightarrow k \xrightarrow{\varepsilon} k[\varepsilon] \longrightarrow k \longrightarrow 0$, then

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{O}_x) \arrow[r] & H^1(X_{k[\varepsilon]}, \mathcal{O}^*_{k[\varepsilon]}) \arrow[r, "I_x"] & H^1(X, \mathcal{O}^*_x) \arrow[l, bend right] \\
& & (I_x)^{-1}(L_0) \in \text{Pic}\ (X_{k[\varepsilon]}) & L_0 \in \text{Pic}\ (x)
\end{tikzcd}$$

i.e., there is a canonical trivial extension $L_0[\varepsilon]$.

Example 11.2.
Let $X \supset Z_0 \in \text{Hilb}_{X/k}(k)$, we computed $T_{Z_0}\text{Hilb}_{X/k} = \hom_{\mathcal{O}_x}(I_{Z_0}, \mathcal{O}_z)$. We took $Z_0 \subset X$ and extended to $Z' \subset X_{k[\varepsilon]}$ by base change.

In this case, $F_{loc}(A)$ was the set of $Z' \subset X_A$ which are flat over A, such that base-changing $Z' \times_{\text{Spec}\ A} \text{Spec}\ k \cong Z$. This was the same as looking at the preimage restricted to the closed point,

$$\begin{aligned} \text{Hilb}_{X/k}(A) &\xrightarrow{i^*} \text{Hilb}_{X/k}(k) \\ (i^*)^{-1}(z_0) &\hookleftarrow z_0. \end{aligned}$$

Recall how we did the thickening: we had $0 \longrightarrow J \longrightarrow A' \longrightarrow A \longrightarrow 0$ with $J^2 = 0$, along with F on X_A which is flat over A with X/k projective, and finally an F' on $X_{A'}$ restricting to F.

The criterion we had was F' was flat over A' iff $0 \longrightarrow J \otimes_{A'} F' \longrightarrow F'$, i.e. this is injective.

Suppose $z \in F_{loc}(A)$ and an extension $z' \in F_{loc}(A')$. By tensoring the two exact sequences here, we get an exact grid:

$$
\begin{tikzcd}
0 \arrow[rr] \arrow[dd] & & I_{Z'} \arrow[r] & \mathcal{O}_{X_{A'}} \arrow[r] & \mathcal{O}_{Z'} \arrow[r] & 0 \\
& & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
J \arrow[d] & 0 \arrow[r] & I_{Z_0} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{Z_0} \arrow[r] \arrow[d] & 0 \\
A' \arrow[d] & 0 \arrow[r] & I_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_{X_{A'}} \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & 0 \\
A \arrow[dd] & 0 \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_{X_A} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 & 0 & \\
0 & & & & &
\end{tikzcd}
$$

The space of extension should be a torsor under $\hom_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0})$, which we want to think of as $\hom_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0})$. Picking a ϕ in this hom space, we want to take an extension $I_{Z'} \xrightarrow{\phi} I_{Z''}$. We'll cover how to make this extension next time.