Lie Algebras

D. Zack Garza

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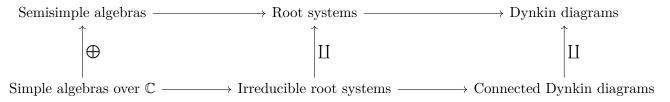
1 Lecture 1

The material for this class will roughly come from Humphrey, Chapters 1 to 5.

Here is a short overview of the topics we expect to cover:

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
 - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

Around chapter 3 or 4, we will describe the following series of correspondences:



2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_{\ell} \iff \mathfrak{sl}(\ell+1,F)$
- $B_{\ell} \iff \mathfrak{so}(2\ell+1,F)$
- $C_{\ell} \iff \mathfrak{sp}(2\ell, F)$
- $D_{\ell} \iff \mathfrak{so}(2\ell, F)$

Exercise 1. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 1. An *F*-algebra *A* is an *F*-vector space endowed with a bilinear map $A^2 \to A$, $(x, y) \mapsto xy$.

Definition 2. An algebra is associative if x(yz) = (xy)z.

Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 3. Any map $\delta: A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 4. We define $Der(A) = \{\delta \ni \delta \text{ is a derivation } \}.$

Any Lie algebra \mathfrak{g} is an F-algebra, since $[\cdot,\cdot]$ is bilinear. Moreover, \mathfrak{g} is associative iff [x,[y,z]]=0.

Exercise 2. Show that $\operatorname{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \Longrightarrow [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 3 (Turn in). Define the adjoint by $ad_x : \mathfrak{g} \circlearrowleft, y \mapsto [x,y]$. Show that $ad_x \in Der(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 1. Any F-vector space can be made into a Lie algebra by setting [x, y] = 0; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\cdot, \cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is [x,y]. Some cases:
 - $-[x,y]=0 \implies \mathfrak{g}$ is abelian.
 - $-[x,y]=ax+by\neq 0$. Assume $a\neq 0$ and set $x'=ax+by, y'=\frac{y}{a}$. Now compute $[x',y']=[ax+by,\frac{y}{a}]=[x,y]=ax+by=x'$. Punchline: $\mathfrak{g}\cong Fx'\oplus Fy',[x',y']=x'$.

We can fill in a table with all of the various combinations of brackets:

Example 2. Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise 4. Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Compute the matrices of ad(e), ad(h), ad(g) with respect to this basis.

2.3 Ideals

Definition 5. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \subseteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using |x,y| = |-y,x|.

Exercise 5. Check that the following are all ideals of \mathfrak{g} :

- $\mathfrak{z}(\mathfrak{g}) = \{ z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g} \}$
- The commutator (or derived) algebra $[\mathfrak{g},\mathfrak{g}] = \{\sum_i [x_i,y_i] \ni x_i,y_i \in \mathfrak{g}\}.$ - Moreover, $[\mathfrak{gl}(n,F),\mathfrak{gl}(n,F)] = \mathfrak{sl}(n,F)$.

Fact: If $I, J \leq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \leq \mathfrak{g}$
- $\begin{array}{l} \bullet \ I \bigcap J \trianglelefteq \mathfrak{g} \\ \bullet \ [I,J] = \{ \sum_i [x_i,y_i] \ni x_i \in I, y_i \in J \} \ \trianglelefteq \mathfrak{g} \end{array}$

Definition 6. A Lie algebra is simple if $[\mathfrak{g},\mathfrak{g}] \neq 0$ (i.e. when \mathfrak{g} is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Theorem 1. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2,F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2,F)$ given by $B=\{e,h,f\}$ where

- [e, f] = h,
- [h, e] = 2e,
- [h, f] = -2f.

So think of $[h, e] = \mathrm{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2,F)$ has a nontrivial ideal I; then pick $x=ae+bh+cf\in I$. Then [e,x]=0-2be+ch, and [e,[e,x]]=0-0+2ce. Again since char $F\neq 2$, then if $c\neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So c = 0.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I,$$

$$2f = [h, -f] \in I \implies f \in I.$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

Note that there is a homework coming due next Monday, about 4 questions.