

# Homotopy Groups of Spheres

## Graduate Student Seminar

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# Introduction

# Outline

Homotopy  
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Spheres

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Introduction

Examples

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

# Definitions

- Definition of homotopy:

$$f, g \in \text{hom}_{\text{Top}}(X, Y) \quad f \sim g \iff \exists F \in \text{hom}_{\text{Top}}(X \times I, Y)$$

such that  $F(0) = f, F(1) = g$ .

- This yields an equivalence relation on morphisms, write *homotopy classes of maps*

$$[X, Y] := \text{hom}_{\text{Top}}(X, Y) / \sim$$

- Definition of homotopy equivalence:

$$X \sim Y \iff \exists \begin{cases} f \in \text{hom}(X, Y) \\ g \in \text{hom}(Y, X) \end{cases} \quad \text{such that } \begin{cases} f \circ g \sim \text{id}_Y \\ g \circ f \sim \text{id}_X \end{cases}$$

- Similarly write

$$[X] = \left\{ Y \in \text{Top} \mid Y \sim X \right\}.$$

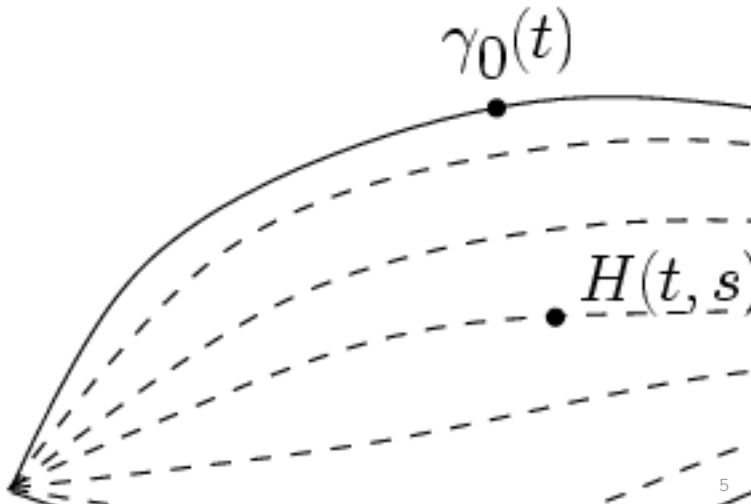
# The Fundamental Group

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Introduction

Examples



# Classification

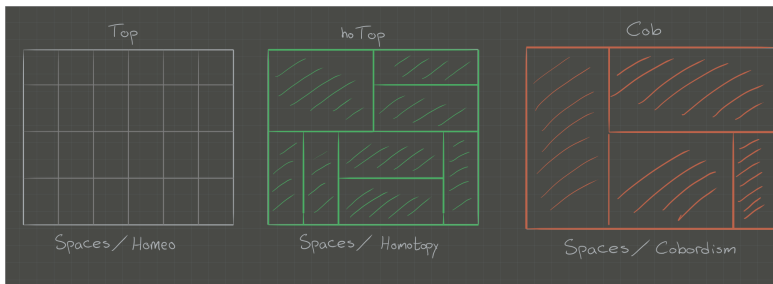
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Groups of  
Spheres

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Introduction

Examples

- Holy grail: understand the topological category completely
  - I.e. have a well-understood geometric model one space of each homeomorphism type



*Also have the derived category  $D\text{Top}$ , its interplay with  $\text{hoTop}$  is the subject of e.g. the Poincare conjecture(s).*

- Any representative from a green box: a *homotopy type*.

# Example: Homotopy Equivalence is Useful

**Proposition:** Let  $B$  be a CW complex; then isomorphism classes of  $\mathbb{R}^1$ -bundles over  $B$  are given by  $H^1(X, \mathbb{Z}/2\mathbb{Z})$ .

- Use the fact that for any fixed group  $G$ , the functor

$$h_G(\cdot) : \text{hoTop}^{\text{op}} \longrightarrow \text{Set}$$

$$X \mapsto G\text{-bundles over } X$$

is representable by a space called  $BG$  (Brown's representability theorem).

- Letting  $I(G, X) = \{G\text{-bundles}/B\} / \sim$ , there is an isomorphism  $I(G, X) \cong [X, BG]$ . In general, identify  $G = \text{Aut}(F)$  the automorphism group of the fibers – for vector bundles of rank  $n$ , take  $G = GL(n, \mathbb{R})$ .

*Note that for a poset of spaces  $(M_i, \hookrightarrow)$ , the space  $M^\infty := \varinjlim M_i$ . This are infinite dimensional “Hilbert manifolds”.*

Proof:

$$I(\mathbb{R}^1, X) = [X, B(GL(1, \mathbb{R}))]$$

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Examples



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# Examples

# Sphere 1

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Examples