Title

D. Zack Garza

Friday 16^{th} October, 2020

Contents

1 | Friday, October 16

1.1 Example: Weyl's Character Formula

Review: suppose the following is invariant under the Weyl group, so $\sum a_\mu e^\mu \in \mathbb{Z}[X(T)]^W$. In this case, we have an equality

$$\sum a_\mu e^\mu = \sum a_\mu \chi(\mu),$$

where $\chi(\mu) = \sum_{i\geq 0} (-1)^i \mathrm{char}\, H^i(\mu)$. We also had a relation

$$\chi(w\cdot \mu) = (-1)^{\ell(w)}\chi(\mu) = \mathrm{sgn}(w)\chi(\mu).$$

Now let $\lambda \in X(T)\otimes \mathbb{Q}$, then we defined

$$A(\lambda) = \sum_{w\in W} \mathrm{sgn}(w) e^{w\lambda} \in \mathbb{Z}[X(T)\otimes \mathbb{Q}].$$

We obtain

1. $w' A(\lambda) = \mathrm{sgn}(w') A(\lambda)$
2. $A(\mu)A(\lambda) = \mathbb{Z}[X(T)\otimes \mathbb{Q}]^W$.

Theorem 1.1.1 ***(Weyl's Character Formula).***

$$\lambda \in X(T) \implies \chi(\lambda) = \frac{A(\lambda+\rho)}{A(\lambda)}.$$

As a special case when $\lambda \in X(T)_+$, all higher sheaf cohomology vanishes and thus

$$\operatorname{char} H^0(\lambda) = \frac{A(\lambda+\rho)}{A(\lambda)}.$$

Proof .
We first perform a *reindexing* step:

$$\begin{aligned}
\sum_{w,w'} \operatorname{sgn}(w \cdot w') e^{w(\lambda+\rho)+w'\rho} &= \sum_{w,w'} \operatorname{sgn}(w^{-1}w') e^{w(\lambda+\rho)+w'\rho} \\
&= \sum_{w,y} \operatorname{sgn}(y) e^{w(\lambda+\rho)+wy\rho} && y = w^{-1}w' \implies w' = wy \\
&= \sum_{w,y} \operatorname{sgn}(y) e^{w(\lambda+\rho+y\rho)}.
\end{aligned}$$

Now let $\lambda \in X(T)$, we then compute

$$\begin{aligned}
A(\lambda+\rho)A(\rho) &= \sum_{w} \operatorname{sgn}(w) e^{w(\lambda+\rho)} + \sum_{w'} \operatorname{sgn}(w') e^{w'(\lambda+\rho)} \\
&= \sum_{w,w'} \operatorname{sgn}(ww') e^{w(\lambda+\rho)+w'\rho} \\
&= \sum_{w,w'} \operatorname{sgn}(w') e^{w(\lambda+\rho+w'\rho)} && \text{from reindexing above, setting } y := w' \\
&= \sum_{w,w'} \operatorname{sgn}(w') \chi(w(\lambda+\rho+w'\rho)) \\
&= \sum_{w,w'} \operatorname{sgn}(w') \chi\Big(w \cdot (\lambda + w'\rho + w^{-1}\rho)\Big) && \text{definition of dot action} \\
&= \sum_{w,w'} \operatorname{sgn}(ww') \chi(\lambda + w'\rho + w\rho) && \text{swapping } w \rightsquigarrow w^{-1}.
\end{aligned}$$

Note that χ can be introduced since $A(\lambda+\rho)A(\rho) \in \mathbb{Z}[X(T) \otimes \mathbb{Q}]^{W\cdot}$.

Not sure, double check.

We can now conclude that

$$A(\rho)^2 = \sum_{w,w'} \operatorname{sgn}(ww') e^{w\rho+w'\rho}.$$

Since this quantity is W-invariant, since it's a square, we can move the χ inside:

$$\begin{aligned}
\chi(\lambda)\Big(\sum a_\mu e^\mu\Big) &= \sum a_\mu \chi(\lambda+\mu) \\
\implies \chi(\lambda) A(\rho)^2 &= \sum_{w,w'} \operatorname{sgn}(ww') \chi(\lambda + w\rho + w'\rho),
\end{aligned}$$

which is exactly what the first calculation resulted in. So we can conclude

$$A(\lambda+\rho)A(\rho) = \chi(\lambda)A(\rho)^2.$$

Note that $A(\rho) \neq 0$ since $w\rho \neq \rho$ unless $w = \mathrm{id}$. Thus we are actually working in $\mathbb{Z}[X(T)+\mathbb{Z}\rho]$, which is an integral domain, and thus we can apply cancellation laws to obtains

$$A(\lambda+\rho) = \chi(\lambda)A(\rho).$$

■

Example 1.1.1.
Let $G = \mathrm{GL}_3(k)$, which has a natural 3-dimensional representation V. Let $\lambda = (1,0,0)$, so $L(1,0,0) = V$. This is a polynomial representation, so by permuting we can obtain

$$\operatorname{char} V = e^{(1,0,0)} + e^{(0,1,0)} + e^{(0,0,1)} = \chi(1,0,0),$$

where the last equality holds since λ is dominant.

We can write $\rho = (2,1,0)$, since the fundamental weights are given by $w_1 = (1,0,0)$ and $w_2 = (1,1,0)$ (since we're in an SL_2 and/or A_2 situation). We then obtain $\lambda + \rho = (3,1,0)$, and since $W = S_3$,

$$A(\lambda+\rho) = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)} = e^{(3,1,0)} - e^{(1,3,0)} + e^{(1,0,3)} - e^{(0,1,3)} + e^{(0,3,1)} - e^{(3,0,1)}.$$

Thus

$$A(\rho) = e^{(2,1,0)} - e^{(1,2,0)} + e^{(1,0,2)} - e^{(0,1,2)} + e^{(0,2,1)} - e^{(2,0,1)}.$$

We can then compute

$$\begin{aligned}\chi(1,0,0)A(\rho) = & e^{(3,1,0)} - e^{(2,2,0)} + e^{(2,0,2)} - e^{(1,1,2)} + e^{(1,2,1)} - e^{(3,0,1)} + \\ & e^{(2,2,0)} - e^{(1,3,0)} + e^{(1,1,2)} - e^{(0,2,2)} + e^{(0,3,1)} - e^{(2,1,1)} + \\ & e^{(2,1,1)} - e^{(1,2,1)} + e^{(1,0,3)} - e^{(0,1,3)} + e^{(0,2,2)} - e^{(2,0,2)}.\end{aligned}$$

After cancellation, you'll find that this expression is equal to $A(\lambda + \rho)$.

1.2 Strong Linkage Principle

We'll consider representations in characteristic zero, so we can take $k = \mathbb{C}$. Let G bet a complex simple group, $\mathfrak{g} = \mathrm{Lie}(G)$, t a maximal torus, X the weights, and X_+ the dominant weights. We have a correspondence

$$\left\{ (g,t) \right\} \iff \left\{ (\Phi, W) \right\}$$

where Φ is an irreducible root system and W is the Weyl group. We'll have a set of simple roots $\Delta \subseteq \Phi^+$. For $\lambda \in X$, we have

$$Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} \lambda \twoheadrightarrow L(\lambda).$$

Then $\lambda \in X_+ \iff L(\lambda)$ is finite dimensional. We have W acting on X via reflections, which we can extend to a dot action

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \qquad \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

We define Category $\mathcal{O}$ which has objects $\mathfrak{g}$-modules with a weight space decomposition which is locally finite wrt $\mathfrak{n}^+$.

1.2.1 Linkage in Category $\mathcal{O}$

Set $Z(\lambda) = \Delta(\lambda)$, then

$$[Z(\lambda) : L(\mu)] \neq 0 \implies \lambda \in W \cdot \mu.$$

The LHS is computed by evaluating certain Kazhdan-Lusztig polynomials at $x = 1$.

Example 1.2.1.
Let $\Phi = A_2$, then

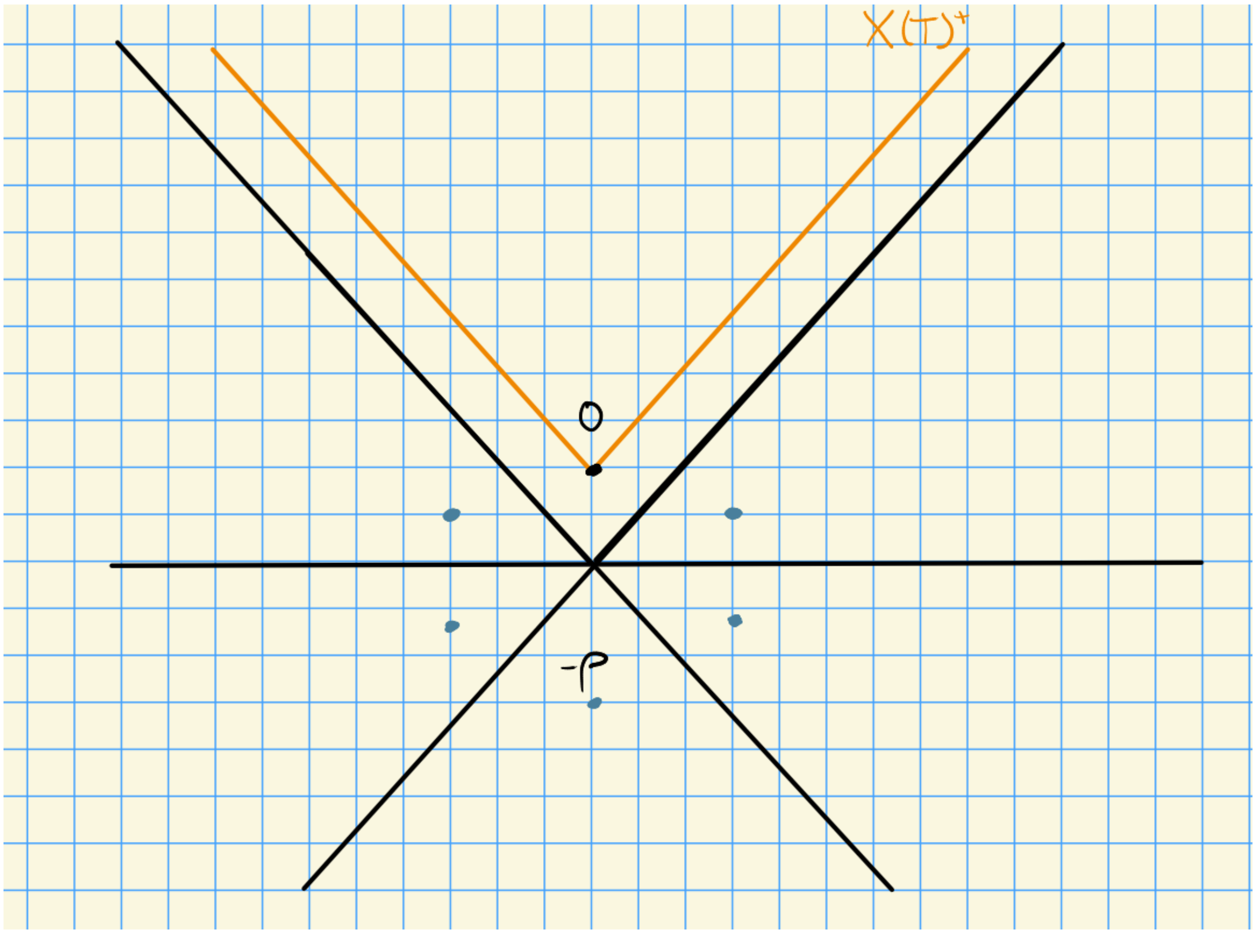

Figure 1: Image

$\mathcal{O}_0$ is the principal block, and the irreducibles correspond to $\left\{ L(w \cdot 0) \;\middle|\; w \in W \right\}$, and the number of irreducibles in given by $|W|$. In this case, there is only 1 finite-dimensional module in any given block of category $\mathcal{O}$.

Example 1.2.2.
For $\Phi = A_1$, we have the following situation:

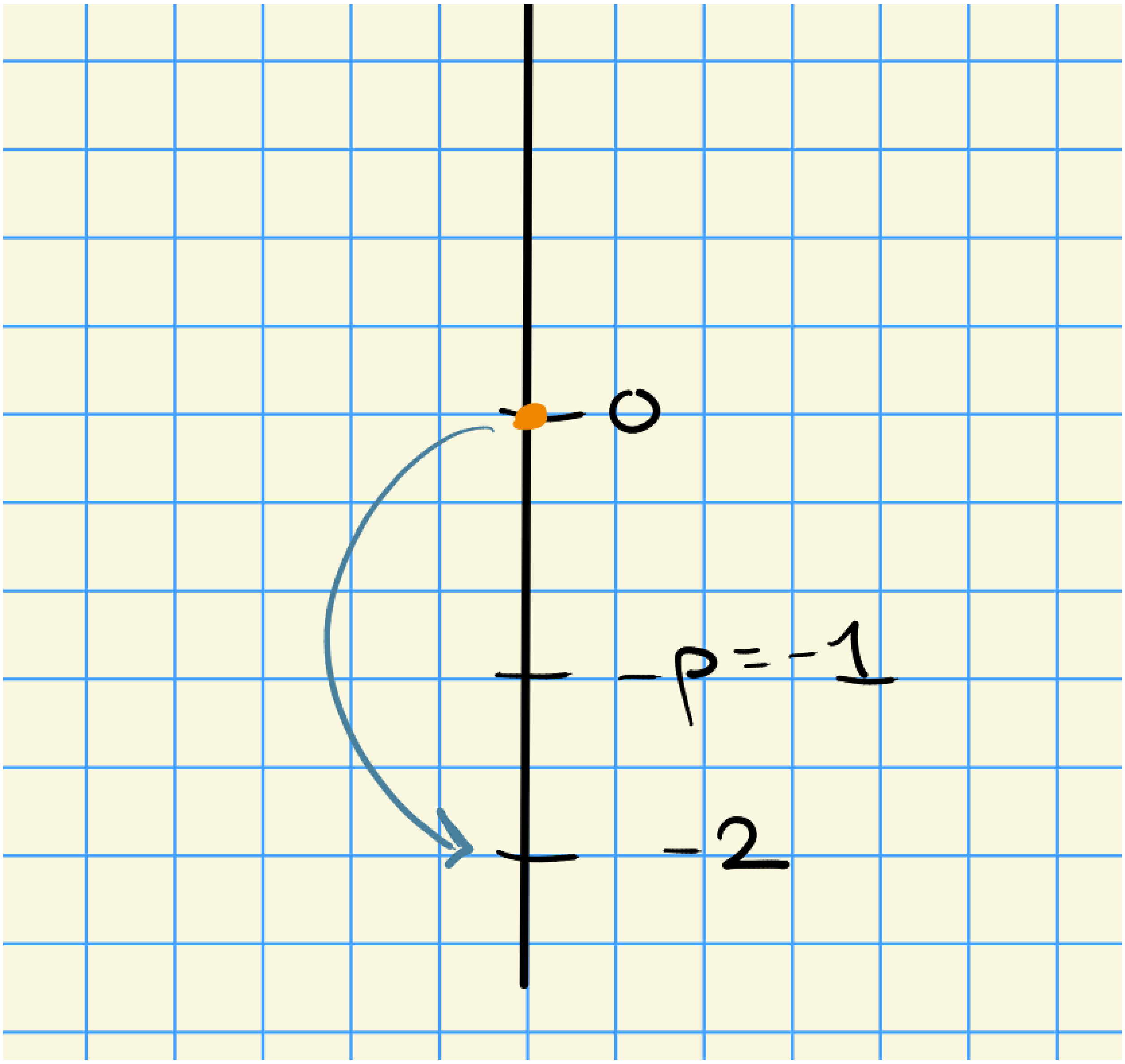

Figure 2: Image

In $\mathcal{O}_0$, there are two irreducible representations given by the Verma modules $L(0), L(-2)$, and we find that

$$Z(0): \begin{bmatrix} L(0) \\ L(-2) \end{bmatrix} \qquad Z(-2): \begin{bmatrix} L(-2) \end{bmatrix}$$

Figure 3: Image

In this case, the projectives are given by

$$P(-2): \begin{bmatrix} L(-2) \\ \vert \\ L(0) \\ \vert \\ L(-2) \end{bmatrix} \qquad P(0): \begin{bmatrix} L(0) \\ \vert \\ L(-2) \end{bmatrix}$$

Figure 4: Image