8.8 Part 2, Computing the Index of L

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Monday 25^{th} May, 2020

Contents

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

  where

$$\begin{aligned} S : \mathbb{R} \times S^1 &\longrightarrow \mathrm{Mat}(2n; \mathbb{R}) \\ S(s,t) &\overset{s \longrightarrow \pm\infty}{\longrightarrow} S^{\pm}(t). \end{aligned}$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
  - *(not the same kernel/cokernel)*
- Compute $\operatorname{Ind} L_1$: explicitly describe $\ker L_1, \operatorname{coker} L_1$.
- Replace in two steps:
  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in s.
    - Replace $S(s,t)$ by a matrix

$$\tilde{S}(s,t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

    - Idea: approximate by cylinders at infinity.
    - Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    - Use invariance of index under homotopy.

0.1 Main Results

- Theorem 8.8.1:

$$\operatorname{Ind}(L) = \mu\left(R^{-}(t)\right) - \mu\left(R^{+}(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing L to L_1 Construct an operator

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

  where $S : \mathbb{R} \longrightarrow \operatorname{Mat}(2n;\mathbb{R})$ is a path of diagonal matrices depending on $\operatorname{Ind}(R^{\pm}(t))$; then

$$\operatorname{Ind}(L) = \operatorname{Ind}(L_1) = \operatorname{Ind}(R^{-}(t)) - \operatorname{Ind}(R^{+}(t)).$$

- Prop 8.8.3: Reducing L_1 to $R^{\pm}$. Let $k^{\pm} := \operatorname{Ind}(R^{\pm})$; then $\operatorname{Ind}(L_1) = k^{-} - k^{+}$.
- Lemma 8.8.4: $\operatorname{Ind}(L_0) = \operatorname{Ind}(L)$.
- Han's Talk:
  - Prop 8.8.3, using Lemma 8.8.5
- Me
  - Proof of 8.8.5

0.2 8.8.5:

Used in the proof of 8.8.3, $\operatorname{Ind}(L_1) = K^{-} - k^{+}$.

Setup:

$$S(s) = \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \qquad \text{with } a_i(s) = \begin{cases} a_i^{-} & \text{if } s \le -s_0 \\ a_i^{+} & \text{if } s \ge s_0 \end{cases}.$$

Statement: let $p > 2$ and define

$$\begin{aligned} F : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2}\right) \\ Y &\mapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y. \end{aligned}$$

This looks like L_1 for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^{\pm} := a_1^{\pm} = a_2^{\pm}$. Then

$$\begin{aligned} \dim\operatorname{Ker} F &= 2\cdot\#\left\{\ell \in \mathbf{Z} | a^{-} < 2\pi\ell < a^{+}\right\} \\ \dim\operatorname{Ker} F^{\star} &= 2\cdot\#\left\{\ell \in \mathbf{Z} | a^{+} < 2\pi\ell < a^{-}\right\}. \end{aligned}$$

2. Suppose $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$, then

$$\dim \operatorname{Ker} F = \#\left\{ i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$
$$\dim \operatorname{Ker} F^\star = \#\left\{ i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

0.3 Proof

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) = a_1(s) = a_2(s)$.
- Start with equation on $\mathbb{R}^2$,

$$Y(s,t) = (Y_1(s,t), Y_2(s,t))$$

- Replace with equation on $\mathbb{C}$:

$$Y(s,y) = Y_1(s,t) + iY_2(s,t)$$

.

- Rewrite the PDE $F(Y) = 0$ as $\bar{\partial} Y + S(s)Y = 0$, i.e.

$$\frac{\partial}{\partial s}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let $Y = B\tilde{Y}$ where $B \in \mathrm{GL}(1,\mathbb{C})$ satisfies $\left(\bar{\partial} + S\right)B = 0$ to obtain $\bar{\partial}\tilde{Y} = 0$.
  - Can choose $B = \left[b(s))0\ 0 \quad b(s)\right]$ where $\dfrac{\partial b}{\partial s} = -a(s)b(s)$.