Title

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1.1 Regular Functions

See chapter 3 in the notes.

Some examples:

- X a manifold or an open set in $\mathbb{R}^n$ has a ring of C^∞ functions.
- $X \subset \mathbb{C}$ has a ring of holomorphic functions.
- $X \subset \mathbb{R}$ has a ring of real analytic functions

These all share a common feature: it suffices to check if a function is a member on an arbitrary open set about a point, i.e. they are *local*.

Definition 1.0.1 (?)**.**
Let X be an affine variety and $U \subseteq X$ open. A **regular function** on U is a function $\varphi : U \to k$ such that φ is "locally a fraction", i.e. a ratio of polynomial functions.
More formally, for all $p \in U$ there exists a U_p with $p \in U_p \subseteq U$ such that $\varphi(x) = g(x)/f(x)$ for all $x \in U_p$ with $f, g \in A(X)$.

Example 1.1.
For X an affine variety and $f \in A(X)$, consider the open set $U \coloneqq V(f)^c$. Then $\dfrac{1}{f}$ is a regular function on U, so for $p \in U$ we can take U_p to be all of U.

Example 1.2.
For $X = \mathbb{A}^1$, take $f = x - 1$. Then $\dfrac{x}{x-1}$ is a regular function on $\mathbb{A}^1 \setminus \{1\}$.

Example 1.3.
Let $X + V(x_1x_4 - x_2x_3)$ and $U \coloneqq X \setminus V(x_2, x_4) = \left\{ [x_1, x_2, x_3, x_4] \;\middle|\; x_1x_4 = x_2x_3, x_2 \neq 0 \text{ or } x_4 \neq 0 \right\}$.
Define

$$\begin{aligned} \varphi : U &\to K \\ [x_1, x_2, x_3, x_4] &\mapsto \begin{cases} \dfrac{x_1}{x_2} & \text{if } x_2 \neq 0 \\ \dfrac{x_3}{x_4} & \text{if } x_4 \neq 0 \end{cases}. \end{aligned}$$

This is well-defined on $\{x_2 \neq 0\} \cap \{x_4 \neq 0\}$, since $\dfrac{x_1}{x_2} = \dfrac{x_3}{x_4}$. Note that this doesn't define an element of k at $[0, 0, 0, 1] \in U$. So this is not globally a fraction.

Notation: we'll let $\mathcal{O}_X(U)$ is the ring of regular function on U.

Proposition 1.1 ***(?).***
Let $U \subset X$ be an affine variety and $\varphi \in \mathcal{O}_X(U)$. Then $V(\varphi) \coloneqq \left\{ x \in U \;\middle|\; \varphi(x) = 0 \right\}$ is closed in the subspace topology on U.

Proof .
For all $a \in U$ there exists $U_a \subset U$ such that $\varphi = g_a/f_a$ on U_a with $f_a, g_a \in A(X)$ with $f_a \neq 0$ on U_a.
Then [
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