Title

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Contents

1 | Lecture 8: Riemann-Roch Spaces (Part 2)

Recall the proposition we ended with last time:

:::{.proposition title="?"}{#prop:deg_bounded_above} There exists a $\delta = \delta(K/k) \in \mathbb{Z}$ such that for all $A \in \operatorname{Div} K$, we have

$$\deg A - \ell(A) \leq \delta.$$

:::

Exercise 1.0.1 ***(?):*** This proposition is enough to show the existence of rational functions whose polar divisor has as its support any finite subset $S \subset \Sigma(K/k)$.

Most of the lecture will be the proof of this statement.

1.1 Proof of Upper Bound

Rewriting [@lemma:divisor_order_to_subspaces] yields

$$A_2, A_2 \in \operatorname{Div} K,\ A_1 \leq A_2 \implies \deg A_1 - \ell(A_1) \leq \deg A_2 - \ell(A_2).$$

1.1.1 Step 1

Choose an $x \in K \setminus k$ and set $B \coloneqq (x)_-$.

Claim: There exists a $C \geq 0$ such that for all $n \geq 0$,

$$\ell(nB + C) \geq (n+1) \deg B.$$

So we give ourselves a certain effective divisor: the divisor of poles of an arbitrary nonconstant element. We can then get a preliminary asymptotic lower bound, not on the same Riemann-Roch space, but on a new one after augmenting the space by some fixed effective divisor C.

Proof (?).
Since $K/k(x)$ has finite degree, let $u_1, \cdots, u_d$ be a basis for K consisting of finitely many rational functions. Note that $d = [K : k(x)]$, and is also equal to $\deg B$ since B was a divisor of poles. Noting that the divisor groups are free commutative groups, so taking any finite number of elements in $\bigoplus \mathbb{Z}$, we can find an element that is less than or equal to all of them. Thus we can choose a $C \geq 0$ such that

$$(u_i) \geq -C \qquad \forall 1 \leq i \leq d.$$

Since the u_i are $k(x)$-linearly independent in K, the functions $\left\{x^i u_j \;\middle|\; 0 \leq i \leq n,\ 1 \leq j \leq d\right\}$ are k-linearly independent, since any k-linear relation would immediately yield a $k(x)$-linear relation among the u_i.
Exercise 1.1.1 ***(?):*** If $f_i \in \mathcal{L}(D_i)$, so the poles of f are no worse than D_i, then the poles of $f_1 f_2$ are bounded by $D_1 + D_2$ and thus $f_1 f_2 \in \mathcal{L}(D_1 + D_2)$.
Now we can note that there are $(n+1)d = \deg B$ many elements here, and moreover, these all lie in $\mathcal{L}(nB + C)$ since each $(u_j) \geq -C$ and $(x) \geq -B$ and $i \leq n$. From this we can conclude

$$\ell(nB + c) \geq (n+1)d = (n+1)\deg B.$$

■

1.1.2 Step 2

We'll now show that throwing in the fixed divisor C can't increase the Riemann-Roch space that much, and in fact

$$\ell(nB + C) \leq \ell(nB) + \deg C,$$

and so we get a bound

$$\begin{aligned}
\ell(nB) &\geq \ell(nB + C) - \deg C \\
&\geq (n+1)\deg B - \deg C \\
&= \deg(nB) + ([K : k(x)] - \deg C) \\
&:= \deg(nB) \pm \gamma,
\end{aligned}$$

which shows that

$$\forall n \geq 0,\ \deg(nB) - \ell(nB) \leq \gamma. \tag{1}$$

A problem here is that γ depends upon everything that we've done so far, and this inequality only holds for multiples of a fixed divisor (an infinite ray emanating from B).

1.1.3 Step 3

Claim: For all $A \in \operatorname{Div} K$, there exist $A_1, D \in \operatorname{Div} K$ and $n \geq 0$ such that $A \leq A_1$, $A_1 \sim D$, and $D \leq nB$. I.e. although it can't literally be true that $A \leq nB$, it will be up to linear equivalence.

To see this, set $A_1 := \max(A, 0)$. Using the bound from eq. 1, for $n \gg 0$ we have

$$\begin{aligned}
\ell(nB - A_1) &\geq \ell(nB) - \deg A_1 \\
&\geq \deg(nB) - \gamma - \deg A_1 \\
&> 0,
\end{aligned}$$

and so there exists a $z \in \mathcal{L}(nB - A_1)^\bullet$, a nontrivial element in the linear system.

Remark 1.1.2: The first inequality is an application of our lemma because A_1 is effective, which was the point of this maneuver. I.e., in order to get from $nB - A_1$ to nB, we added A_1, which can only increase the dimension of the space by at most $\deg A_1$. Finally, in the last inequality, we use the fact that B has positive degree since it's a divisor of poles of a nonconstant rational function, and the remaining terms don't depend on n, so we can make $\deg(nB)$ arbitrarily large.

So now set $D \coloneqq A_1 - (z)$, then $A_1 \sim D$ and since it's in the linear system,

$$(z) \geq -(nB - A_1) = A_1 - nB$$

so $-(z) \leq nB - A_1$ and by adding A_1 to both sides, we obtain

$$0 = A_1 - (z) \leq nB.$$

What have we shown? For any divisor D, we can make it less than nB for some n, up to linear equivalence.

1.1.4 Step 4

Finally, for $A \in \operatorname{Div} K$, choose A_1, D as in the previous step, so $A \leq A_1 \sim D \leq nB$. Then

$$\begin{aligned}
\deg A - \ell(A) &\leq \deg(A_1) - \ell(A) && \text{using } A \leq A_1 \\
&= \deg(D) - \ell(D) && \text{changing within linear equivalence class} \\
&\leq \deg(nB) - \ell(nB) \\
&\leq \gamma.
\end{aligned}$$

■

1.2 Genus

Definition 1.2.1 (Genus (Important!))
The **genus** of K/k is defined as

$$g \coloneqq \max_{A \in \operatorname{Div} K} (\deg(A) - \ell(A) + 1).$$

This exists by the [@prop:deg_bounded_above], since this set is bounded above.

Exercise 1.2.2 ***(?):*** Show that $g \geq 0$ always and

$$g(k(t)/k) = 0.$$

Remark 1.2.3: Note that if the $+1$ is mostly a correction factor to match up with the topological genus of $\mathbb{P}^1_{/\mathbb{C}}$. That the genus is non-negative should come from the lower bound we had from before. It turns out that over $k = \mathbb{C}$, this genus will agree on the nose with the topological genus of the corresponding compact Riemann surface.

Theorem 1.2.4 ***(Riemann's Inequality).***
If K/k is a function field of genus g,

a. For all $A \in \operatorname{Div} K$,

$$\ell(A) \geq \deg(A) + 1 - g.$$

b. There exists a $c = c(K) \in \mathbb{Z}$ such that for all $A \in \operatorname{Div} K$,

$$\deg(A) \geq c \implies \ell(A) = \deg(A) - g + 1.$$

Remark 1.2.5: This says that the dimension of the linear system is very close to the degree of the corresponding divisor, and is only off by a constant factor g. Part (a) is literally just a rearrangement of the definition of the genus. Part (b) says that if you assume A has sufficiently large degree, this upper bound becomes an equality.

Proof (of b).
By the definition of g, since it is a maximum there exists an A_0 such that

$$g = \deg(A_0) - \ell(A_0) + 1.$$

Set $c := \deg(A_0) + g$. Then if $\deg(A) \geq c$, we have

$$\begin{aligned} \ell(A - A_0) &\geq \deg(A - A_0) - g + 1 \\ &\geq c - \deg(A_0) - g + 1 \\ &= 1, \end{aligned}$$

so there exists a $z \in \mathcal{L}(A - A_0)^{\bullet}$ since the dimension is at least 1.
Now set $A' := A + (z)$, and note that $A' \geq A_0$. Thus

$$\begin{aligned} \deg(A) - \ell(A) &= \deg(A') - \ell(A') \\ &\geq \deg(A_0) - \ell(A_0) && \text{by the lemma} \\ &= g - 1. \end{aligned}$$

By maximality of the genus, we have $\deg(A) - \ell(A) \leq g - 1$, which forces equality ■

Next up: how to we make this inequality into an equality? It turns out that there is some different divisor D' and we can subtract off $\ell(D')$, and that will be the Riemann-Roch theorem.