Less Common Topics

- Integrating factors
- Change of Variables
- Inhomogeneous ODEs (need a *particular solution*)
    - Variation of parameters
    - Annihilators
    - Undetermined coefficients
    - Reduction of Order
    - Laplace Transforms
    - Series solutions
- Special ODEs
    - Exact
    - Bernoulli
    - Cauchy-Euler

Topics: Number Theory

Definitions

- The fundamental theorem of arithmetic:

$$n \in \mathbb{Z} \implies n = \prod_{i=1}^{n} p_i^{k_i}, \quad p_i \text{ prime}$$

- Divisibility and modular congruence:

$$x \mid y \iff y = 0 \mod x \iff \exists c \ni y = xc$$

- Useful fact:

$$x = 0 \mod n \iff x = 0 \mod p_i^{k_i} \; \forall i$$

(Follows from the Chinese remainder theorem since all of the $p_i^{k_i}$ are coprime)

Definitions

- GCD, LCM

$$xy = \gcd(x, y)\,\mathrm{lcm}(x, y)$$

$$d \mid x \text{ and } d \mid y \implies d \mid \gcd(x, y)$$

$$\text{and } \gcd(x, y) = d\gcd\left(\frac{x}{d}, \frac{y}{d}\right)$$

  - Also works for $\mathrm{lcm}(x, y)$
  - Computing $\gcd(x, y)$:
    - Take prime factorization of x and y,
    - Take only the distinct primes they have in common,
    - Take the minimum exponent appearing

The Euclidean Algorithm

Computes GCD, can also be used to find modular inverses:

$$
\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\vdots \\
r_k &= q_{k+2} r_{k+1} + \mathbf{r_{k+2}} \\
r_{k+1} &= q_{k+3} r_{k+2} + 0
\end{aligned}
$$

Back-substitute to write $ax + by = \mathbf{r_{k+2}} = \gcd(a, b)$.

(Also works for polynomials!)

Definitions

- Coprime

$$a \text{ is coprime to } b \iff \gcd(a, b) = 1$$

- Euler's Totient Funtion

$$\phi(a) = |\{x \in \mathbb{N} \ni x \leq a \text{ and } \gcd(x, a) = 1\}|$$

  - Computing ϕ:

$$\gcd(a, b) = 1 \implies \phi(ab) = \phi(a)\phi(b)$$

$$\phi(p^k) = p^k - p^{k-1}$$

  - Just take the prime factorization and apply these.