

# Assignment 6: The Fourier Transform

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## Contents

|          |                    |          |
|----------|--------------------|----------|
| <b>1</b> | <b>Problem 1</b>   | <b>1</b> |
| <b>2</b> | <b>Problem 2</b>   | <b>2</b> |
| 2.1      | Part (a) . . . . . | 2        |
| 2.2      | Part (b) . . . . . | 3        |
| 2.2.1    | (i) . . . . .      | 3        |
| 2.2.2    | (ii) . . . . .     | 3        |
| <b>3</b> | <b>Problem 3</b>   | <b>3</b> |
| 3.1      | (a) . . . . .      | 3        |
| 3.1.1    | (i) . . . . .      | 3        |
| 3.1.2    | (ii) . . . . .     | 4        |
| 3.2      | (b) . . . . .      | 4        |
| <b>4</b> | <b>Problem 4</b>   | <b>4</b> |
| 4.1      | (a) . . . . .      | 4        |
| 4.1.1    | (i) . . . . .      | 4        |
| 4.1.2    | (ii) . . . . .     | 5        |
| 4.2      | (b) . . . . .      | 5        |
| <b>5</b> | <b>Problem 5</b>   | <b>6</b> |
| 5.1      | (a) . . . . .      | 6        |
| 5.2      | (b) . . . . .      | 7        |
| 5.2.1    | (i) . . . . .      | 7        |
| 5.2.2    | (ii) . . . . .     | 7        |
| 5.3      | (c) . . . . .      | 7        |
| <b>6</b> | <b>Problem 6</b>   | <b>8</b> |

## 1 Problem 1

Assuming the hint, we have

$$\lim_{|\xi| \rightarrow \infty} \hat{f}(\xi) = \lim_{\xi' \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx$$

But as an immediate consequence, this yields

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} (f(x) - f(x - \xi')) \exp(-2\pi i x \cdot \xi) \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| |\exp(-2\pi i x \cdot \xi)| \, dx \\ &\leq \int_{\mathbb{R}^n} |f(x) - f(x - \xi')| \, dx \\ &\rightarrow 0, \end{aligned}$$

which follows from continuity in  $L^1$  since  $f(x - \xi') \rightarrow f(x)$  as  $\xi' \rightarrow 0$ .

It thus only remains to show that the hint holds, and that  $\xi' \rightarrow 0$  as  $\xi \rightarrow \infty$ .

## 2 Problem 2

### 2.1 Part (a)

Assuming an interchange of integrals is justified, we have

$$\begin{aligned} \widehat{(f * g)}(\xi) &:= \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dy \, dx \\ &= \int \int f(x - y) g(y) \exp(-2\pi i x \cdot \xi) \, dx \, dy \\ &= \int \int f(t) \exp(-2\pi i(x - y) \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dx \, dy \\ &\quad (t = x - y, \, dt = \, dx) \\ &= \int \int f(t) \exp(-2\pi i t \cdot \xi) g(y) \exp(-2\pi i y \cdot \xi) \, dt \, dy \\ &= \int f(t) \exp(-2\pi i t \cdot \xi) \left( \int g(y) \exp(-2\pi i y \cdot \xi) \, dy \right) \, dt \\ &= \int f(t) \exp(-2\pi i t \cdot \xi) \hat{g}(\xi) \, dt \\ &= \hat{g}(\xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\ &= \hat{g}(\xi) \hat{f}(\xi). \end{aligned}$$

It thus remains to show that this swap is justified.

## 2.2 Part (b)

We'll use the following lemma: if  $\hat{f} = \hat{g}$ , then  $f = g$  almost everywhere.

### 2.2.1 (i)

By part 1, we have

$$\widehat{f * g} = \hat{f} \hat{g} = \hat{g} \hat{f} = \widehat{g * f},$$

and so by the lemma,  $f * g = g * f$ .

Similarly, we have

$$(\widehat{f * g}) * h = \widehat{f * g} \hat{h} = \hat{f} \hat{g} \hat{h} = \hat{f} \widehat{g * h} = f * (g * h).$$

### 2.2.2 (ii)

Suppose that there exists some  $I \in L^1$  such that  $f * I = f$ . Then  $\widehat{f * I} = \hat{f}$  by the lemma, so  $\hat{f} \hat{I} = \hat{f}$  by the above result.

But this says that  $\hat{f}(\xi) \hat{I}(\xi) = \hat{f}(\xi)$  almost everywhere, and thus  $\hat{I}(\xi) = 1$  almost everywhere. Then  $\lim_{|\xi| \rightarrow \infty} \hat{I}(\xi) \neq 0$ , which by Problem 1 shows that  $I$  can not be in  $L^1$ , a contradiction.

## 3 Problem 3

### 3.1 (a)

#### 3.1.1 (i)

Let  $g(x) = f(x - y)$ . We then have

$$\begin{aligned} \hat{g}(\xi) &:= \int g(x) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \exp(-2\pi i y \cdot \xi) \, dx \\ &= \exp(-2\pi i y \cdot \xi) \int f(x - y) \exp(-2\pi i(x - y) \cdot \xi) \, dx \\ &\quad (t = x - y, dt = dx) \\ &= \exp(-2\pi i y \cdot \xi) \int f(t) \exp(-2\pi i t \cdot \xi) \, dt \\ &= \exp(-2\pi i y \cdot \xi) \hat{f}(\xi). \end{aligned}$$

### 3.1.2 (ii)

Let  $h(x) = \exp(2\pi i x \cdot y) f(x)$ . We then have

$$\begin{aligned}\hat{h}(\xi) &:= \int \exp(2\pi i x \cdot y) f(x) \exp(-2\pi i x \cdot \xi) \, dx \\ &= \int \exp(2\pi i x \cdot y - 2\pi i x \cdot \xi) f(x) \, dx \\ &= \int f(\xi - y) \exp(-2\pi i x \cdot (\xi - y)) \, dx \\ &= \hat{f}(\xi - y).\end{aligned}$$

### 3.2 (b)

We'll use the fact that if  $\langle \cdot, \cdot \rangle$  is an inner product on a vector space  $V$  and  $A$  is an invertible linear transformation, then for all  $\mathbf{x}, \mathbf{y} \in V$  we have

$$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle$$

where  $A^{-T}$  denotes the transpose of the inverse of  $A$  (or  $(A^{-1})^*$  if  $V$  is complex).

We then have

$$\begin{aligned}\frac{1}{|\det T|} \hat{f}(T^{-T} \xi) &= \frac{1}{|\det T|} \int f(x) \exp(-2\pi i x \cdot T^{-T} \xi) \, dx \\ &\quad x \mapsto Tx, \, dx \mapsto |\det T| \, dx \\ &= \frac{1}{|\det T|} \int f(Tx) \exp(-2\pi i Tx \cdot T^{-T} \xi) |\det T| \, dx \\ &= \int f(Tx) \exp(-2\pi i x \cdot \xi) \, dx \\ &\quad \text{since } Tx \cdot T^{-T} \xi = T^{-1}Tx \cdot \xi = x \cdot \xi \\ &= \widehat{(f \circ T)}(\xi).\end{aligned}$$

## 4 Problem 4

### 4.1 (a)

#### 4.1.1 (i)

Let  $g(x) = xf(x)$ . Then if an interchange of the derivative and the integral is justified, we have

$$\begin{aligned}
\frac{\partial}{\partial \xi} \hat{f}(\xi) &:= \frac{\partial}{\partial \xi} \int f(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= ? \int f(x) \frac{\partial}{\partial \xi} \exp(-2\pi i x \cdot \xi) \, dx \\
&= \int f(x) 2\pi i x \exp(-2\pi i x \cdot \xi) \, dx \\
&= 2\pi i \int x f(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&:= 2\pi i \hat{g}(\xi).
\end{aligned}$$

It thus remains to show that this interchange is justified. TODO

#### 4.1.2 (ii)

We have

$$\begin{aligned}
\hat{h}(\xi) &:= \int \frac{\partial f}{\partial x}(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&= f(x) \exp(-2\pi i x \cdot \xi) \Big|_{x=-\infty}^{x=\infty} - \int f(x) (2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx \\
&\quad \text{(integrating by parts)} \\
&= - \int f(x) (-2\pi i \xi) \exp(-2\pi i x \cdot \xi) \, dx \\
&\quad \text{(since } f(\infty) = f(-\infty) = 0) \\
&= 2\pi i \xi \int f(x) \exp(-2\pi i x \cdot \xi) \, dx \\
&:= 2\pi i \xi \hat{f}(\xi).
\end{aligned}$$

#### 4.2 (b)

Let  $G(x) = \exp(-\pi x^2)$  and  $\partial_\xi$  be the operator that differentiates with respect to  $\xi$ .

Then

$$\partial_\xi \left( \frac{\hat{G}(\xi)}{G(\xi)} \right) = \frac{G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi)}{G(\xi)^2},$$

and the claim is that this is zero. This happens precisely when the numerator is zero, so we'd like to show that

$$G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi) = 0.$$

Using the following facts,

- $\partial_\xi G(\xi) = -2\pi\xi G(\xi)$  by computing directly,
- $\partial_\xi \hat{G}(\xi) = -2\pi\xi \hat{G}(\xi)$ , which follows from the following computation

$$\begin{aligned}
\partial_\xi \hat{G}(\xi) &:= \partial_\xi \int G(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int G(x) \partial_\xi e^{-2\pi i x \cdot \xi} dx \\
&= \int G(x) (-2\pi i x) e^{-2\pi i x \cdot \xi} dx \\
&= \int G(x) (-2\pi i x) e^{-2\pi i x \cdot \xi} dx \\
&= -i \int 2\pi x G(x) e^{-2\pi i x \cdot \xi} dx \\
&= -i \int \partial_x G(x) \exp(-2\pi i x \cdot \xi) dx \\
&:= -i \widehat{\partial_x G(x)}(\xi) \\
&= -i (2\pi i \xi \hat{G}(\xi)) \\
&= -2\pi \xi \hat{G}(\xi),
\end{aligned}$$

we can thus write

$$G(\xi) \partial_\xi \hat{G}(\xi) - \hat{G}(\xi) \partial_\xi G(\xi) = G(\xi) (-2\pi \xi \hat{G}(\xi)) - \hat{G}(\xi) (-2\pi \xi G(\xi)),$$

which is patently zero.

It follows that  $\frac{\hat{G}(\xi)}{G(\xi)} = c_0$  for some constant  $c_0$ , from which it follows that  $\hat{G}(\xi) = c_0 G(\xi)$ .

Using the fact that  $G(0) = 1$  by direct evaluation and  $\hat{G}(0) = \int G(x) dx = 1$ , we can conclude that  $c_0 = 1$  and thus  $\hat{G}(\xi) = G(\xi)$ .

## 5 Problem 5

### 5.1 (a)

By a direct computation. we have

$$\begin{aligned}
\hat{D}(\xi) &:= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-2\pi i x \xi} dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) + i \sin(-2\pi x \xi) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(-2\pi x \xi) dx \\
&\quad (\text{since } \sin \text{ is odd and the domain is symmetric about } 0) \\
&= 2 \int_0^{\frac{1}{2}} \cos(-2\pi x \xi) dx \\
&\quad (\text{since } \cos \text{ is even and the domain is symmetric about } 0) \\
&= 2 \left( \frac{1}{2\pi \xi} \sin(-2\pi x \xi) \Big|_{x=0}^{x=\frac{1}{2}} \right) \\
&= \frac{\sin(\pi \xi)}{\pi \xi}.
\end{aligned}$$

## 5.2 (b)

### 5.2.1 (i)

Since  $F(x) = D(x) * D(x)$ , we have  $\hat{F}(\xi) = (\hat{D}(\xi))^2$  by question 2a, and so  $\hat{F}(\xi) = \left( \frac{\sin(\pi \xi)}{\pi \xi} \right)^2$ .

### 5.2.2 (ii)

Letting  $\mathcal{F}$  denote the Fourier transform operator, we have  $\mathcal{F}^2(h)(\xi) = h(-\xi)$  for any  $h \in L^1$ . In particular, if  $f$  is an even function, then  $f(\xi) = -f(\xi)$  and  $\mathcal{F}^2(f) = f$ .

In this case, letting  $F$  be the box function,  $F$  can be seen to be even from its definition. Since  $f := \mathcal{F}(F)$  by part (i), we have

$$\hat{f} := \mathcal{F}(f) = \mathcal{F}(\mathcal{F}(F)) = \mathcal{F}^2(F) = F,$$

which says that  $\hat{f}(x) = F(x)$ , the original box function.

## 5.3 (c)

By a direct computation of the integral in question, we have

$$\begin{aligned}
I(x) &:= \int e^{-2\pi|\xi|} e^{2\pi i x \xi} d\xi \\
&= \int_{-\infty}^0 e^{-2\pi(-\xi)} e^{-2\pi i x \xi} d\xi + \int_0^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&= \int_0^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} d\xi + \int_0^{\infty} e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&\quad \text{by the change of variables } \xi \mapsto -\xi, d\xi \mapsto -d\xi \text{ and swapping integration bounds} \\
&= \int_0^{\infty} e^{-2\pi\xi} e^{-2\pi i x \xi} + e^{2\pi\xi} e^{2\pi i x \xi} d\xi \\
&= \frac{1}{2\pi} \int_0^{\infty} e^{-u} e^{-i x u} + e^{-u} e^{i x u} du \\
&= \frac{1}{2\pi} \int_0^{\infty} e^{-u(1+ix)} + e^{-u(1-ix)} du \\
&= \frac{1}{2\pi} \left( \left. \frac{-e^{-u(1+ix)}}{1+ix} \right|_{u=0}^{u=\infty} + \left. \frac{-e^{-u(1-ix)}}{1-ix} \right|_{u=0}^{u=\infty} \right) \\
&= \frac{1}{2\pi} \left( \frac{1}{1+ix} + \frac{1}{1-ix} \right) \\
&= \frac{1}{2\pi} \frac{2}{1+x^2} \\
&= \frac{1}{\pi} \frac{1}{1+x^2},
\end{aligned}$$

so  $P(x) = I(x)$ .

Then, by the Fourier inversion formula, we have

$$\begin{aligned}
I(x) = P(x) &= \int \hat{P}(\xi) e^{-2\pi i x \xi} dx \\
\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} &= \int \hat{P}(\xi) e^{-2\pi i x \xi} dx \\
\implies \int e^{-2\pi|\xi|} e^{2\pi i x \xi} - \hat{P}(\xi) e^{-2\pi i x \xi} dx &= 0 \\
\implies \int \left( e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} dx &= 0 \\
\implies \left( e^{-2\pi|\xi|} - \hat{P}(\xi) \right) e^{-2\pi i x \xi} &=_{a.e.} 0 \\
\implies e^{-2\pi|\xi|} &=_{a.e.} \hat{P}(\xi),
\end{aligned}$$

where equality is almost everywhere and follows from the fact that if  $\int f = 0$  then  $f = 0$  almost everywhere.

## 6 Problem 6

We first note that if  $G_t(x) := t^{-n} e^{-\pi|x|^2/t^2}$ , then  $\hat{G}_t(\xi) = e^{-\pi t^2|\xi|^2}$ .



Moreover, if an interchange of integrals is justified, we have have

$$\begin{aligned}
\|f\|_1 &:= \int_{\mathbb{R}^n} \left| \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt \right| dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt dx \\
&\quad \text{since the integrand and thus integral is positive.} \\
&\stackrel{?}{=} \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dx dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left( \int_{\mathbb{R}^n} G_t(x) dx \right) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} (1) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt,
\end{aligned}$$

which we claim is finite, so  $f \in L^1$ .

To see that the norm is finite, we note that

$$t \in [0, 1] \implies e^{-\pi t^2} < 1$$

and if we take  $\varepsilon < \frac{1}{2}$ , we have  $2\varepsilon - 1 < 0$  and thus

$$t \in [1, \infty) \implies t^{2\varepsilon-1} \leq 1.$$

Thus

$$\begin{aligned}
\int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt &= \int_0^1 e^{-\pi t^2} t^{2\varepsilon-1} dt + \int_1^\infty e^{-\pi t^2} t^{2\varepsilon-1} dt \\
&\leq \int_0^1 t^{2\varepsilon-1} dt + \int_1^\infty e^{-\pi t^2} dt \\
&\leq \int_0^1 t^{2\varepsilon-1} dt + \int_0^\infty e^{-\pi t^2} dt \\
&= \frac{1}{2\varepsilon} + \frac{1}{2} < \infty,
\end{aligned}$$

where the first term is obtained by directly evaluating the integral, and the second is derived from the fact that its integral over the real line is 1 and it is an even function.

Justifying the interchange: we note that the integrand  $G_t(x)e^{-\pi t^2}t^{2\varepsilon-1}$  is non-negative, and we've just showed that one of the iterated integrals is absolutely convergent, so Tonelli will apply if the integrand is measurable. But  $G_t(x)$  is a continuous function on  $\mathbb{R}^n$  and the remaining terms are continuous on  $\mathbb{R}$ , so they are all measurable on  $\mathbb{R}^n$  and  $\mathbb{R}$  respectively. But then taking cylinders on everything in sight yields measurable functions, and the product of measurable functions is measurable.

If another interchange of integrals is justified, we can compute

$$\begin{aligned}
\hat{f}(\xi) &:= \int_{\mathbb{R}^n} \left( \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} dt \right) e^{-2\pi i x \cdot \xi} dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} dt dx \\
&= ? \int_0^\infty \int_{\mathbb{R}^n} G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi} dx dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \left( \int_{\mathbb{R}^n} G_t(x) e^{-2\pi i x \cdot \xi} dx \right) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} \hat{G}_t(\xi) dt \\
&= \int_0^\infty e^{-\pi t^2} t^{2\varepsilon-1} e^{-\pi t^2 |\xi|^2} dt \\
&= \int_0^\infty e^{-\pi t^2 (1+|\xi|^2)} t^{2\varepsilon-1} dt \\
&= \int_0^\infty e^{-\pi (t\sqrt{1+|\xi|^2})^2} t^{2\varepsilon-1} dt \\
&\quad s = t\sqrt{1+|\xi|^2}, \quad ds = \sqrt{1+|\xi|^2} dt \\
&= \int_0^\infty e^{-\pi s^2} \left( \frac{s}{\sqrt{1+|\xi|^2}} \right)^{2\varepsilon-1} \frac{1}{\sqrt{1+|\xi|^2}} ds \\
&= (1+|\xi|^2)^{-\frac{2\varepsilon-1}{2}} (1+|\xi|^2)^{-\frac{1}{2}} \int_0^\infty e^{-\pi s^2} s^{2\varepsilon-1} ds \\
&= (1+|\xi|^2)^{-\varepsilon} \int e^{-\pi t^2} t^{2\varepsilon-1} dt \\
&:= F(\xi) \|f\|_1.
\end{aligned}$$

To see that the interchange is justified, note that

$$\int_{\mathbb{R}^n} \int_0^\infty |G_t(x) e^{-\pi t^2} t^{2\varepsilon-1} e^{-2\pi i x \cdot \xi}| dt dx = \int_{\mathbb{R}^n} \int_0^\infty |G_t(x) e^{-\pi t^2} t^{2\varepsilon-1}| dt dx,$$

since  $|e^{2\pi i x \cdot \xi}| = 1$ . The integrand appearing is precisely what we showed was measurable when computed  $\|f\|_1$  above, so Tonelli applies.

Thus  $F(\xi)$  is the Fourier transform of the function  $g(x) := f(x)/\|f\|_1$ .  $\square$