Section 8.6 - 8.8: Setup for Computing the Index

May 27, 2020

Summary/Outline

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

What we have so far:

- Define

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$$
$$S(s,t) \xrightarrow{s \to \pm\infty} S^{\pm}(t).$$

Outline

- Took $R^{\pm} : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$: symplectic paths associated to $S^{\pm}$
- These paths defined $\mu(x), \mu(y)$
- Section 8.7:

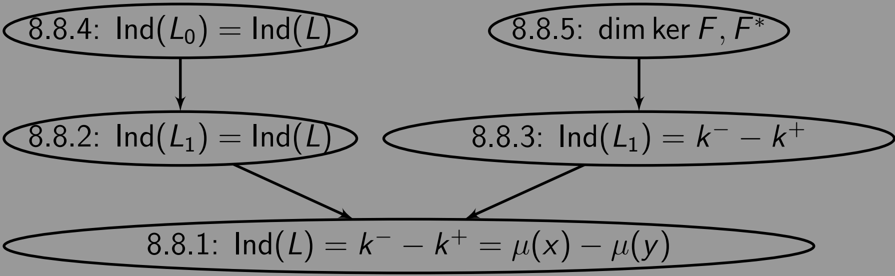
$$R^{\pm} \in \mathcal{S} := \left\{ R(t) \;\middle|\; R(0) = \mathrm{id},\ \det(R(1) - \mathrm{id}) \neq 0 \right\} \implies L \text{ is Fredholm.}$$

- WTS 8.8.1:

$$\mathrm{Ind}(L) \overset{\text{Thm?}}{=} \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y).$$

From Yesterday

- Han proved 8.8.2 and 8.8.4.
  - So we know $\mathrm{Ind}(L) = \mathrm{Ind}(L_1)$
- Today: 8.8.5 and 8.8.3:
  - Computing $\mathrm{Ind}(L_1)$ by computing kernels.



8.8.3: $\mathrm{Ind}(L_1) = k^- - k^+$

Recall

$$\begin{aligned} L : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

$$\begin{aligned} L_1 : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$

$$\begin{aligned} L_1^\star : W^{1,q}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) &\longrightarrow L^q\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right) \\ Z &\longmapsto -\frac{\partial Z}{\partial s} + J_0\frac{\partial Z}{\partial t} + S(s)^t Z \end{aligned}$$

Here $\frac{1}{p} + \frac{1}{q} = 1$ are conjugate exponents.

Reductions

$$L_1^* = -\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S(s)^t.$$

- Since coker $L_1 \cong$ ker L_1^*, it suffices to compute ker L_1^*
- We have

$$J_0^1 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^{n} \mathrm{Mat}(2; \mathbb{R}).$$

- This allows us to reduce to the $n = 1$ case.

Outline

We had a path of diagonal matrices:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) :=_? \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases} .$$

Outline

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8.8.5: $\dim\ker F$, F^*

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