Problem Set 8

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Contents

1 Problem 1

1.1 Part a

Define a map

$$\phi_{\mathrm{ev}} : \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \to A$$
$$(f : \mathbb{Z}_m \to A) \mapsto f(1)$$

Then noting that ϕ_{ev} is a homomorphism, forcing $f(\overline{0}) = 0_A$ (where $\overline{0} : \mathbb{Z}_m \to A$ is the zero map), we must have

$$0 = f(0) = f(m) = mf(1),$$

we must have $mf(1) = 0$ in A. So

$$\operatorname{im} \phi_{\mathrm{ev}} = \{a \in A \mid ma = 0\} \coloneqq A[m].$$

It is also the case that

$$\ker \phi_{\mathrm{ev}} = \{f \in \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \mid f(1) = 0\} = \{\overline{0}\},$$

which follows from the fact that $\mathbb{Z}_m = \langle 1 \mod m \rangle$ and $A = \langle 1_A \rangle$ as $\mathbb{Z}$-modules, so if $f(1 \mod m) = 0_A$ then

$$f(n \mod m) = nf(1 \mod m) = 0$$

and so f is necessarily the zero map. So $ker\phi = \overline{0}$.

We can then apply the first isomorphism theorem,

$$\frac{\hom_{\mathbb{Z}}(\mathbb{Z}_m, A)}{\ker \phi_{\mathrm{ev}}} \cong \mathrm{im}\ \phi_{\mathrm{ev}} \implies \hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m].$$

1.2 Part 2

The claim is that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{(m,n)}$, from which the result immediately follows by part 1.

Define a map

$$\phi : \mathbb{Z} \to \mathbb{Z}_n[m]$$
$$x \mapsto x \mod n$$

Then ϕ is clearly surjective, since it is a quotient map, and

$$
\begin{aligned}
\ker \phi &= \{x \in \mathbb{Z} \ni x \equiv 0 \mod n \ \text{ and } \ mx = 0\} \\
&= \{x \in \mathbb{Z} \ni x \equiv 0 \mod m \ \text{ and } \ x \equiv 0 \mod n\} \\
&= \{x \in \mathbb{Z} \ni x \equiv 0 \mod \gcd(m,n)\} \\
&= \mathbb{Z}_{\gcd(m,n)}.
\end{aligned}
$$

By the first isomorphism theorem, we have

$$\frac{\mathbb{Z}}{\ker \phi} \cong \mathrm{im}\ \phi \implies \mathbb{Z}_{\gcd(m,n)} := \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}} \cong \mathbb{Z}_n[m].$$

1.3 Part 3

We identify

$$\mathbb{Z}^* = \hom(\mathbb{Z}, \mathbb{Z})$$