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University of Georgia

D. Zack Garza

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

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Problem 1.0.1 (Weibel 1.1.2)
Show that a morphism $u : C \to D$ of chain complexes preserves boundaries and cycles respectively, hence inducing a map $H_n(C) \to H_n(D)$ for each n. Prove that $H_n : \mathrm{Ch}(R\text{-}\mathrm{mod}) \to R\text{-}\mathrm{mod}$ is a functor.

Solution:

Claim 1: The chain map u induces the following well-defined maps:

$$Z_n(u) : Z_n(C) \to Z_n(D)$$
$$B_n(u) : B_n(C) \to B_n(D).$$

Proof (of claim (1)).
We'll use the convention that $Z_n \coloneqq \ker d_n$ and $B_n \coloneqq \operatorname{im} d_{n+1}$ where we index chain complexes as $C. = \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}^C} C_n \xrightarrow{d_n^C} C_{n-1} \to \cdots\right)$. Unraveling definitions, we would like to show the existence of maps

$$\begin{aligned} Z_n(u) &: \ker d_n^C \to \ker d_n^D \\ B_n(u) &: \operatorname{im} d_{n+1}^C \to \operatorname{im} d_{n+1}^D. \end{aligned}$$

It suffices to show

a. $x \in \ker d_n^C \implies u_n(x) \in \ker d_n^D$, and
b. $y \in \operatorname{im} d_{n+1}^C \implies u_n(y) \in \operatorname{im} d_{n+1}^D$.

Since u is a morphism of chain complexes, we have a commuting ladder where $u_{n-1} \circ d_n^C = d_n^D \circ u_n$:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n^C"] \arrow[d, "u_n"] & C_{n-1} \arrow[r, "d_{n-1}^C"] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}^D"'] & D_n \arrow[r, "d_n^D"'] & D_{n-1} \arrow[r, "d_{n-1}^D"'] & \cdots
\end{tikzcd}$$

Link to Diagram

To see that (a) holds, we compute

$$\begin{aligned}
x \in \ker d_n^C & & \leq C_n & \\
\iff d_n^C(x) = 0_R & & \in C_{n-1} & \\
\iff (u_{n-1} \circ d_n^C)(x) = 0_R & & \in D_{n-1} & \quad \text{since } u_n \text{ is a ring morphism and sends } 0_r \to 0_R \\
\implies (d_n^D \circ u_n)(x) = 0_R & & \in D_{n-1} & \quad \text{using commutativity} \\
\implies x \in \ker(d_n^D \circ u_n) & & \leq D_{n-1} & \\
\iff u_n(x) \in \ker d_n^D & & \leq D_n. &
\end{aligned}$$

Similarly, for (b) we have

$$\begin{aligned}
y \in \operatorname{im} d_{n+1}^C &\iff \exists x \in C_{n+1} \text{ such that } d_{n+1}^C(x) = y & \\
&\implies u_{n+1}(x) \in D_{n+1} & \\
&\implies (d_{n+1}^D \circ u_{n+1})(x) \in \operatorname{im} d_{n+1}^D \leq D_n & \\
&\implies (u_n \circ d_{n+1}^C)(x) \in \operatorname{im} d_{n+1}^D \leq D_n & \quad \text{using commutativity} \\
&\iff u_n(y) \in \operatorname{im} d_{n+1}^D & \quad \text{using } d_{n+1}^C(x) = y.
\end{aligned}$$

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Now noting that $H_n(C) \coloneqq Z_n(C)/B_n(C)$, since u_n preserves Z_n there is a well-defined restriction of each $u_n : C_n \to D_n$ to $u_n : Z_n(C) \to Z_n(D)$. Composing with the projection $Z_n(D) \to Z_n(D)/B_n(D) \coloneqq H_n(D)$ yields maps $u_n : Z_n(C) \to H_n(D)$.

Problem 1.0.2 (Weibel 1.1.4)
Show that for every $A \in R\text{-mod}$ and $C \in \mathrm{Ch}(R\text{-mod})$ that $D. \coloneqq \mathrm{Hom}_{R\text{-mod}}(A, C.)$ is a chain complex of abelian groups. Taking $A \coloneqq Z_n$, show that $H_n(D.) = 0 \implies H_n(C.) = 0$. Is the converse true?

Problem 1.0.3 (Weibel 1.1.6: Homology of a graph)
Let Γ be a finite graph with vertices $V \coloneqq \{v_1, \cdots, v_V\}$ and edge $E \coloneqq \{e_1, \cdots, e_E\}$. Define the **incidence matrix** of Γ to be the $V \times E$ matrix A where

$$A_{ij} = \begin{cases} 1 & e_j \text{ starts at } v_i, \\ -1 & e_j \text{ ends at } v_i, \\ 0 & \text{else.} \end{cases}$$

Define a chain complex by taking free R-modules:

$$C. = (\cdots \to 0 \to C_1 \coloneqq R[V] \xrightarrow{d} C_0 \coloneqq R[V] \to 0 \to \cdots),$$

where $d : C_1 \to C_0$ is given by A. If Γ is connected, show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and $V - E - 1$ respectively.

Hint: choose a basis $\{v_0, v_1 - v_0, \cdots, v_V - v_0\}$ and use a path from $v_0 \rightsquigarrow v_i$ to produce an element $e \in C_1$ with $d(e) = v_i - v_0$.

Problem 1.0.4 (Weibel 1.1.7: Tetrahedra)
The **tetrahedron** T is a surface with 4 vertices, 6 edges, and 4 faces of dimension 2, and its homology is the homology of the complex

$$C. \coloneqq (\cdots \to 0 \to R^4 \to R^6 \to R^4 \to 0 \to \cdots).$$

Write down the matrices in this complex and computationally verify that

$$H_*(T) = [R, 0, R, 0, \cdots].$$

Problem 1.0.5 (Weibel 1.2.3)
Let $\mathcal{A}$ be the category $\mathrm{Ch}(R\text{-mod})$ and let f be a chain map. Show that the complex $\ker f$ is a (categorical) kernel of f and that $\mathrm{coker}\, f$ is a (categorical) cokernel of f.

Verify exactness in the Snake Lemma in at least two other positions.

Problem 1.0.6 (Weibel 1.4.3)
Show that C is a split exact chain complex if and only if id_C is nullhomotopic.

Problem 1.0.7 (Weibel 1.4.5)
Show that chain homotopy classes of maps form a quotient category K of $\mathrm{Ch}(R\text{-}\mathrm{mod})$ and that the functors H_n factor through the quotient functor $\mathrm{Ch}(R\text{-}\mathrm{mod}) \to K$ using the following steps:

1. Show that chain homotopy equivalence is an equivalence relation on $\left\{ f : C \to D \;\middle|\; f \text{ is a chain map} \right\}$. Define $\mathrm{Hom}_K(C, D)$ to be the equivalence classes of such maps and show that it is an abelian group.
2. Let $f \simeq g : C \to D$ be two chain homotopic maps. If $u : B \to C, v : D \to E$ are chain maps, show that vfu, vgu are chain homotopic. Deduce that K is a category when the objects are defined as chain complexes and the morphisms are defined as in (1).
3. Let $f_0, f_1, g_0, g_1 : C \to D$ all be chain maps such that each pair $f_i \simeq g_i$ are chain homotopic. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that K is an additive category and $\mathrm{Ch}(R\text{-}\mathrm{mod}) \to K$ is an additive functor.
4. Is K an abelian category? Explain.

Try at least two parts.