Problem Set 7

D. Zack Garza

October 26, 2019

Contents

1 Regular Problems														1											
	1.1	Problem 1															1								
		1.1.1	Case	1: <i>p</i>	= q.												 								1
		1.1.2	Case	2: <i>p</i>	> q.												 								2
		1.1.3	Case	3: q	> <i>p</i> .												 								2
2	Qua	l Probl	ems																						2

1 Regular Problems

1.1 Problem 1

Note that if either p = 1 or q = 1, G is a p-group, which is a nontrivial center that is always normal. So assume $p \neq 1$ and $q \neq 1$.

We want to show that G has a non-trivial normal subgroup. Noting that $\#G = p^2q$, we will proceed by showing that either n_p or n_q must be 1.

We immediately note that

$$n_p \equiv 1 \mod p$$

$$n_q \equiv 1 \mod q$$

$$n_p \mid q \qquad \qquad n_q \mid p^2,$$

which forces

$$n_p \in \{1, q\}, \quad n_1 \in \{1, p, p^2\}.$$

If either $n_p = 1$ or $n_q = 1$, we are done, so suppose $n_p \neq 1$ and $n_1 \neq 1$. This forces $n_p = q$, and we proceed by cases:

1.1.1 Case 1: p = q.

Then $\#G = p^3$ and G is a p-group. But every p-group has a non-trivial center $Z(G) \leq G$, and the center is always a normal subgroup.

1.1.2 Case 2: p > q.

Here, since $n_p \mid q$, we must have $n_p < q$. But if $n_p < q < p$ and $n_p = 1 \mod p$, then $n_p = 1$.

1.1.3 Case 3: q > p.

Since $n_p \neq 1$ by assumption, we must have $n_p = q$. Now consider sub-cases for n_q :

- $n_q = p$: If $n_q = p = 1 \mod q$ and p < q, this forces p = 1.
- $n_q = p^2$: We will reach a contradiction by showing that this forces

$$|P := \bigcup_{S_p \in \operatorname{Syl}(p,G)} S_p \setminus \{e\} | + |Q := \bigcup_{S_q \in \operatorname{Syl}(q,G)} S_q \setminus \{e\} | + |\{e\}| > |G|.$$

We have

$$\begin{split} |P| + |Q| + |\{e\}| &= n_p(q-1) + n_q(p^2 - 1) + 1 \\ &= p^2(q-1) + q(p^2 - 1) + 1 \\ &= p^2(q-1) + 1(p^2 - 1) + (q-1)(p^2 - 1) + 1 \qquad \text{(since } q > 1) \\ &= p^2q - p^2 + p^2 - 1 + (q-1)(p^2 - 1) \\ &= p^2q + (q-1)(p^2 - 1) \\ &\geq p^2q + (2-1)(2^2 - 1) \qquad \text{(since } p, q \ge 2) \\ &= p^2q + 3 \\ &> p^2q = |G|, \end{split}$$

which is a contradiction.

2 Qual Problems