Real Analysis Qual Prep Week 2: Measure Theory, Fubini Tonelli

D. Zack Garza

Table of Contents

Contents

1 Study Guide

References:

- Folland's "Real Analysis: Modern Techniques", Ch.1
- Stein and Shakarchi Ch.1, Ch.2

1.1 Convergence Tips/Tricks

- Our favorite tools: **metrics** and **norms**!

  A **metric** on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

  - $\rho(x, y) = 0$ iff $x = y$;
  - $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
  - $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

  –
  - So show things are equal by showing $|x - y| = 0$. Know the triangle inequality by heart!

- **Uniform convergence**:

  Definition 1.2.7 (Uniform Convergence)

  $$(\forall \varepsilon > 0)\,(\exists n_0 = n_0(\varepsilon))\,(\forall x \in S)\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon)\,.$$

  Negated:[a]

  $$(\exists \varepsilon > 0)\,(\forall n_0 = n_0(\varepsilon))\,(\exists x = x(n_0) \in S)\,(\exists n > n_0)\,(|f_n(x) - f(x)| \geq \varepsilon)\,.$$

  [a] Slogan: to negate, find a bad x depending on n_0 that are larger than some ε.

  –
  - Negating: find a bad ε and a single bad point x.
  - Showing a sum converges uniformly: remember that $\sum_{k \geq 1} a_k$ is *defined* to be $\lim_{N \to \infty} \sum_{k \leq N} a_k$.

    So the trick is to define $f_n(x) := \sum_{k \leq n} a_k$ and then apply the usual criteria above.
  - It's sometimes useful to trade the $\forall x$ in the definition with $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$ instead.

- Compare and contrast to **pointwise convergence**, which is strictly weaker:

> **Definition 1.2.8** (Pointwise Convergence)
> A sequence of functions $\{f_j\}$ is said to **converge pointwise** to f if and only if
> $$(\forall \varepsilon > 0)(\forall x \in S)\,(\exists n_0 = n_0(x, \varepsilon))\,(\forall n > n_0)\,(|f_n(x) - f(x)| < \varepsilon)\,.$$

  -
  - The main difference: pointwise can depend on the x and the ε, but uniform needs one ε that works for *all* x simultaneously.
  - Note uniform implies pointwise but not conversely.

- The sup norm: $\|f\|_\infty := \sup\limits_{x \in X} |f_n(x)|$
  - A useful way to force uniform convergence: bound your sequence uniformly by a sequence that goes to zero:

> **Proposition 1.4.1** *(Testing Uniform Convergence: The Sup Norm Test).*
> $f_n \to f$ uniformly iff there exists an M_n such that $\|f_n - f\|_\infty \leq M_n \to 0$.

  -

- **Sups and infs**: sup is the least upper bound, inf is the greatest lower bound.
- The $p-$test:

$$\sum_{n \geq 1} \frac{1}{n^p} < \infty \iff p > 1$$

- Useful fact: convergent sums have **small tails**, i.e.

$$\sum_{n \geq 1} a_n < \infty \implies \lim_{N \to \infty} \sum_{n \geq N} a_n = 0$$

- So try bounding things from above by the *tail* of a sum!
- If you can't bound by a tail: as long as you have control over the coefficients, you can pick them to make the sum to converge "fast enough".
  - Example: for a fixed ε, choose $a_n = 1/2^n$. Note that $\sum\limits_{n \geq 1} 1/2^n = 1$, so choose $a_n := \varepsilon/2^n$:

$$\cdots \leq \sum_{n \geq 1} a_n := \sum_{n \geq 1} \frac{\varepsilon}{2^n} = \varepsilon \to 0$$

- The $\varepsilon/3$ trick:

Theorem 1.4.4 (*Uniform Limit Theorem*).
If $f_n \to f$ pointwise and uniformly with each f_n continuous, then f is continuous. [a]

[a] Slogan: a uniform limit of continuous functions is continuous.

Proof .

- Follows from an $\varepsilon/3$ argument:

$$|F(x) - F(y| \le |F(x) - F_N(x)| + |F_N(x) - F_N(y)| + |F_N(y) - F(y)| \le \varepsilon \to 0.$$

  - The first and last $\varepsilon/3$ come from uniform convergence of $F_N \to F$.
  - The middle $\varepsilon/3$ comes from continuity of each F_N.

- So just need to choose N large enough and δ small enough to make all 3 ε bounds hold.

∎

- **The M-test:**

Weierstrass M-test. Suppose that (f_n) is a sequence of real- or complex-valued functions defined on a set A, and that there is a sequence of non-negative numbers (M_n) satisfying the conditions

- $|f_n(x)| \le M_n$ for all $n \ge 1$ and all $x \in A$, and
- $\sum_{n=1}^{\infty} M_n$ converges.

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely and uniformly on A.

1.2 Measure Theory

- F_σ sets: unions of closed sets (F for *fermi*, French for closed. Sigma for sums, ie unions)
- G_δ sets: intersections of open sets
- σ algebras: closed under complements, countable intersections, countable unions
- Some of the most useful properties of measures:

Let X be a set equipped with a σ-algebra $\mathcal{M}$. A **measure** on $\mathcal{M}$ (or on $(X, \mathcal{M})$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

i. $\mu(\varnothing) = 0$,

ii. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$.

Property (ii) is called **countable additivity**. It implies **finite additivity**:

1.8 Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

a. **(Monotonicity)** *If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$.*

b. **(Subadditivity)** *If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.*

c. **(Continuity from below)** *If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.*

d. **(Continuity from above)** *If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \supset E_2 \supset \cdots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.*

- The proof of continuity of measure contains a very useful trick: replace a sequence of sets $\{E_k\}$ with a sequence of *disjoint* sets that either union or intersect to the same thing.
  - Example: if $A_1 \subseteq A_2 \subseteq \cdots$, set $F_1 = A_1$ and $F_k = A_k \setminus A_{k-1}$ for $k \geq 2$. Then $\bigcup\limits_{k\geq 1} A_k = \coprod\limits_{k\geq 1} F_k$.
- Occasionally you need some properties of **outer measures**:

containing E, the inner area of E is just the area of R minus the outer area of $R \setminus E$.

The abstract generalization of the notion of outer area is as follows. An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

- $\mu^*(\varnothing) = 0$,
- $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$,
- $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.

The most common way to obtain outer measures is to start with a family $\mathcal{E}$ of

1.10 Proposition. *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define*

$$\mu^*(A) = \inf\left\{\sum_1^\infty \mu(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\}.$$

Then μ^* is an outer measure.

- Outer measure for $\mathbb{R}^n$: you consider all collections of cubes that cover your set, sum up their volumes, and take the infimum over all such collections:

Definition 1.2.9 (Outer Measure)
The **outer measure** of a set is given by

$$m_*(E) \coloneqq \inf_{\substack{\{Q_i\} \rightrightarrows E \\ \text{closed cubes}}} \sum |Q_i|.$$

- "Almost everywhere *blah*" : the set where *blah* does not happen has measure zero.
- "Infinitely many/all but finitely many" types of sets, which show up in Borel-Cantelli style problems

and likewise for unions and intersections. In this situation, the notions of **limit superior** and **limit inferior** are sometimes useful:

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n, \qquad \liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

The reader may verify that

$$\begin{aligned} \limsup E_n &= \{x : x \in E_n \text{ for infinitely many } n\}, \\ \liminf E_n &= \{x : x \in E_n \text{ for all but finitely many } n\}. \end{aligned}$$

$$E^c = X \setminus E.$$

In this situation we have **deMorgan's laws:**

$$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c, \qquad \left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c.$$

- Lemmas that sometimes show up on quals:

1.18 Theorem. *If $E \in \mathcal{M}_\mu$, then*

$$\mu(E) = \inf\{\mu(U) : U \supset E \text{ and } U \text{ is open}\}$$
$$= \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

1.19 Theorem. *If $E \subset \mathbb{R}$, the following are equivalent.*

a. $E \in \mathcal{M}_\mu$.

b. $E = V \setminus N_1$ *where* V *is a* G_δ *set and* $\mu(N_1) = 0$.

c. $E = H \cup N_2$ *where* H *is an* F_σ *set and* $\mu(N_2) = 0$.

1.3 Fubini-Tonelli

Quick statement:

Theorem 3.1.10 ***(Tonelli (Non-Negative, Measurable)).***
For $f(x,y)$ **non-negative and measurable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is a **measurable** function
- $F(x) = \int f(x,y)\, dy$ is a **measurable** function,
- For E measurable, the slices $E_x \coloneqq \left\{y \mid (x,y) \in E\right\}$ are measurable.
- $\int f = \int \int F$, i.e. any iterated integral is equal to the original.

Theorem 3.1.11 ***(Fubini (Integrable)).***
For $f(x,y)$ **integrable**, for almost every $x \in \mathbb{R}^n$,

- $f_x(y)$ is an **integrable** function
- $F(x) \coloneqq \int f(x,y)\, dy$ is an **integrable** function,
- For E measurable, the slices $E_x \coloneqq \left\{y \mid (x,y) \in E\right\}$ are measurable.

Explained in Stein and Shakarchi (Fubini, which requires **integrability**) interesting issues arise.

In general, we may write $\mathbb{R}^d$ as a product

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \qquad \text{where } d = d_1 + d_2, \text{ and } d_1, d_2 \geq 1.$$

A point in $\mathbb{R}^d$ then takes the form (x, y), where $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. With such a decomposition of $\mathbb{R}^d$ in mind, the general notion of a slice, formed by fixing one variable, becomes natural. If f is a function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the **slice** of f corresponding to $y \in \mathbb{R}^{d_2}$ is the function f^y of the $x \in \mathbb{R}^{d_1}$ variable, given by

$$f^y(x) = f(x, y).$$

Similarly, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is $f_x(y) = f(x, y)$.

In the case of a set $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ we define its **slices** by

$$E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\} \qquad \text{and} \qquad E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in E\}.$$

See Figure 1 for an illustration.

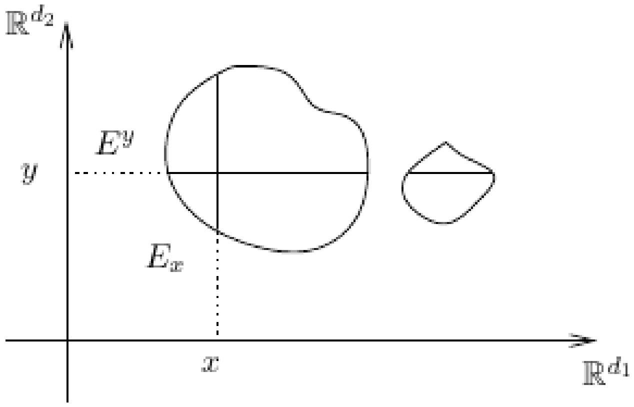

Figure 1. Slices E^y and E_x (for fixed x and y) of a set E

The main theorem is as follows. We recall that by definition all integrable functions are measurable.

Theorem 3.1 *Suppose $f(x, y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

(i) *The slice f^y is integrable on $\mathbb{R}^{d_1}$.*

(ii) *The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\, dx$ is integrable on $\mathbb{R}^{d_2}$.*

Moreover:

(iii) $$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)\, dx \right) dy = \int_{\mathbb{R}^d} f.$$

Clearly, the theorem is symmetric in x and y so that we also may conclude that the slice f_x is integrable on $\mathbb{R}^{d_2}$ for a.e. x. Moreover, $\int_{\mathbb{R}^{d_2}} f_x(y)\, dy$ is integrable, and

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y)\, dy \right) dx = \int_{\mathbb{R}^d} f.$$

In particular, Fubini's theorem states that the integral of f on $\mathbb{R}^d$ can be computed by iterating lower-dimensional integrals, and that the iterations can be taken in any order

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)\, dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y)\, dy \right) dx = \int_{\mathbb{R}^d} f.$$

And Tonelli, which only requires **measurability**:

Theorem 3.2 *Suppose $f(x,y)$ is a non-negative measurable function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for almost every $y \in \mathbb{R}^{d_2}$:*

(i) *The slice f^y is measurable on $\mathbb{R}^{d_1}$.*

(ii) *The function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)\,dx$ is measurable on $\mathbb{R}^{d_2}$.*

Moreover:

(iii) $\displaystyle\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^d} f(x,y)\,dx\,dy$ *in the extended sense.*

In practice, this theorem is often used in conjunction with Fubini's theorem.[3] Indeed, suppose we are given a measurable function f on $\mathbb{R}^d$ and asked to compute $\int_{\mathbb{R}^d} f$. To justify the use of iterated integration, we first apply the present theorem to $|f|$. Using it, we may freely compute (or estimate) the iterated integrals of the non-negative function $|f|$. If these are finite, Theorem 3.2 guarantees that f is integrable, that is, $\int |f| < \infty$. Then the hypothesis in Fubini's theorem is verified, and we may use that theorem in the calculation of the integral of f.

A more precise statement from Folland:

In this section we fix a measure space $(X, \mathcal{M}, \mu)$, and we define

$$L^+ = \text{the space of all measurable functions from } X \text{ to } [0, \infty].$$

2.37 The Fubini-Tonelli Theorem. *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.*

a. (Tonelli) *If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x \, d\nu$ and $h(y) = \int f^y \, d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and*

$$\int f \, d(\mu \times \nu) = \int \left[\int f(x,y) \, d\nu(y) \right] d\mu(x) = \int \left[\int f(x,y) \, d\mu(x) \right] d\nu(y). \tag{2.38}$$

b. (Fubini) *If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x \, d\nu$ and $h(x) = \int f^y \, d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and (2.38) holds.*

Some things that qual questions are commonly based on:

Corollary 3.3 *If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for almost every $y \in \mathbb{R}^{d_2}$ the slice*

$$E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$$

is a measurable subset of $\mathbb{R}^{d_1}$. Moreover $m(E^y)$ is a measurable function of y and

$$m(E) = \int_{\mathbb{R}^{d_2}} m(E^y) \, dy.$$

This is an immediate consequence of the first part of Theorem 3.2 applied to the function χ_E. Clearly a symmetric result holds for the x-slices in $\mathbb{R}^{d_2}$.

We have thus established the basic fact that if E is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for almost every $y \in \mathbb{R}^{d_2}$ the slice E^y is measurable in $\mathbb{R}^{d_1}$ (and also the symmetric statement with the roles of x and y interchanged). One might be tempted to think that the converse assertion holds. To see that this is not the case, note that if we let $\mathcal{N}$ denote a

Corollary 3.8 *Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$, and let*

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \ 0 \le y \le f(x)\}.$$

Then:

(i) *f is measurable on $\mathbb{R}^d$ if and only if $\mathcal{A}$ is measurable in $\mathbb{R}^{d+1}$.*

(ii) *If the conditions in* (i) *hold, then*

$$\int_{\mathbb{R}^d} f(x)\, dx = m(\mathcal{A}).$$

Proof. If f is measurable on $\mathbb{R}^d$, then the previous proposition guarantees that the function

$$F(x, y) = y - f(x)$$

is measurable on $\mathbb{R}^{d+1}$, so we immediately see that $\mathcal{A} = \{y \ge 0\} \cap \{F \le 0\}$ is measurable.

Conversely, suppose that $\mathcal{A}$ is measurable. We note that for each $x \in \mathbb{R}^{d_1}$ the slice $\mathcal{A}_x = \{y \in \mathbb{R} : \ (x, y) \in \mathcal{A}\}$ is a closed segment, namely $\mathcal{A}_x = [0, f(x)]$. Consequently Corollary 3.3 (with the roles of x and y interchanged) yields the measurability of $m(\mathcal{A}_x) = f(x)$. Moreover

$$m(\mathcal{A}) = \int \chi_{\mathcal{A}}(x, y)\, dx\, dy = \int_{\mathbb{R}^{d_1}} m(\mathcal{A}_x)\, dx = \int_{\mathbb{R}^{d_1}} f(x)\, dx,$$

as was to be shown.

2 | Qual Problems

Suggested by Peter Woolfitt!

Spring 2012 #4

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative integrable function.

a. Show that $\sin \circ f$ is integrable.

b. Use Fubini's theorem to show that

$$\int_{[0,\infty)} m(\{x : f(x) \geq y\}) \cos y \, dy = \int_{\mathbb{R}} \sin(f(x))\, dx.$$

Fall 2016 #2

2. Let f and g be real valued measurable functions on $[a, b]$ with $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$. Show that either $f(x) = g(x)$ a.e., or there exists measurable subset E of $[a, b]$ such that $\int_E f(x)\, dx > \int_E g(x)\, dx$.

Fall 2018 #5

Problem 5. *Let $f \geq 0$ be a Lebesgue measurable function on $\mathbb{R}$. Show that*

$$\int_{\mathbb{R}} f = \int_0^\infty m(\{x : f(x) > t\})\, dt.$$

Spring 2019 #4: This is an expanded version of Fall 2018 #5 above.

4. Let f be a non-negative function on $\mathbb{R}^n$ and $\mathcal{A} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq f(x)\}$.

Prove the validity of the following two statements:

(a) f is a Lebesgue measurable function on $\mathbb{R}^n$ $\iff$ $\mathcal{A}$ is a Lebesgue measurable subset of $\mathbb{R}^{n+1}$

(b) If f is a Lebesgue measurable function on $\mathbb{R}^n$, then

$$m(\mathcal{A}) = \int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty m(\{x \in \mathbb{R}^n : f(x) \geq t\})\, dt$$