Lie Algebras

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Contents

1 Lecture 1

> The material for this class will roughly come from Humphrey, Chapters 1 to 5. There is also a useful appendix which has been uploaded to the ELC system online.

1.1 Overview

Here is a short overview of the topics we expect to cover:

1.1.1 Chapter 2

- Ideals, solvability, and nilpotency
- Semisimple Lie algebras
  - These have a particularly nice structure and representation theory
- Determining if a Lie algebra is semisimple using Killing forms
- Weyl's theorem for complete reducibility for finite dimensional representations
- Root space decompositions

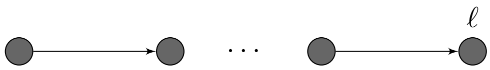
1.1.2 Chapter 3-4

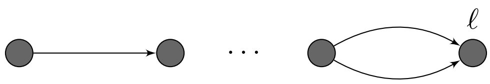
We will describe the following series of correspondences:

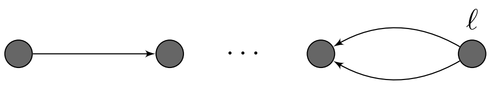
$$\begin{tikzcd}
\text{Semisimple algebras} \arrow[r, Leftrightarrow] & \text{Root systems} \arrow[r, Leftrightarrow] & \text{Dynkin diagrams} \\
\text{Simple algebras over } \mathbb{C} \arrow[u, Rightarrow, "\bigoplus"] \arrow[r, Leftrightarrow] & \text{Irreducible root systems} \arrow[u, Rightarrow, "\coprod"] \arrow[r, Leftrightarrow] & \text{Connected Dynkin diagrams} \arrow[u, Rightarrow, "\coprod"]
\end{tikzcd}$$

1.2 Classification

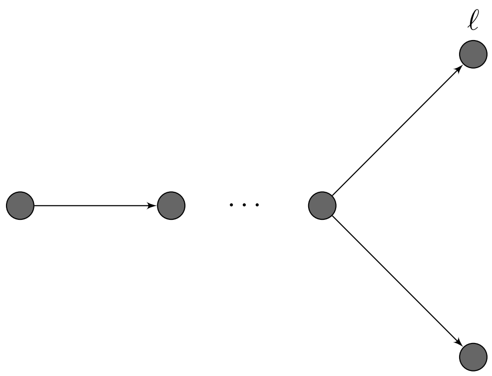
The classical Lie algebras can be essentially classified by certain classes of diagrams:

A_ℓ : 

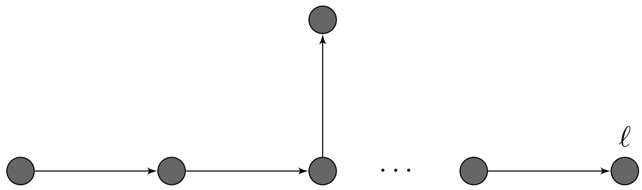
B_ℓ : 

C_ℓ : 

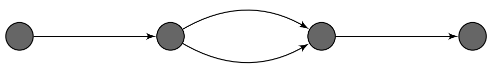
D_ℓ :



E_6, E_7, E_8 :



F_4:



1.3 Chapters 4-5

These cover the following topics:

- Conjugacy classes of Cartan subalgebras
- The PBW theorem for the universal enveloping algebra
- Serre relations

1.3.1 Chapter 6

Some import topics include:

- Weight space decompositions
- Finite dimensional modules
- Character and the Harish-Chandra theorem
- The Weyl character formula
  - This will be computed for the specific Lie algebras seen earlier

We will also see the type A_ℓ algebra used for the first time; however, it differs from the other types in several important/significant ways.

1.3.2 Chapter 7

Skip!

1.3.3 Topics

Time permitting, we may also cover the following extra topics:

- Infinite dimensional Lie algebras [Carter 05]
- BGG Cat-$\mathcal{O}$ [Humphrey 08]

1.4 Content

Fix F a field of characteristic zero – note that prime characteristic is closer to a research topic.

Definition 1. A **Lie Algebra** $\mathfrak{g}$ over F is an F-vector space with an operation denoted the Lie bracket,

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y]. \end{aligned}$$

satisfying the following properties:

- $[\,\cdot\,,\,\cdot\,]$ is bilinear
- $[x, x] = 0$
- The Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Exercise 1. Show that $[x, y] = -[y, x]$.

Definition 2. Two Lie algebras $\mathfrak{g}, \mathfrak{g}'$ are said to be isomorphic if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.5 Linear Lie Algebras

Let $V = \mathbb{F}^n$, and define $\mathrm{End}(V) = \{f : V \to V \ni V \text{ is linear}\}$. We can then define $\mathfrak{gl}(n, V)$ by setting $[x, y] = (x \circ y) - (y \circ x)$.

Exercise 2. Verify that V is a Lie algebra.

Definition 3. Define

$$\mathfrak{sl}(n, V) = \{f \in \mathfrak{gl}(n, V) \ni \mathrm{Tr}(f) = 0\}.$$

(Note the different in definition compared to the lie *group* $\mathrm{SL}(n, V)$.).

Definition 4. A *subalgebra* of a Lie algebra is a vector subspace that is closed under the bracket.

Definition 5. The symplectic algebra

$$\mathfrak{sp}(2\ell, F) = \left\{A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0\right\} \text{ where } M = \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right).$$

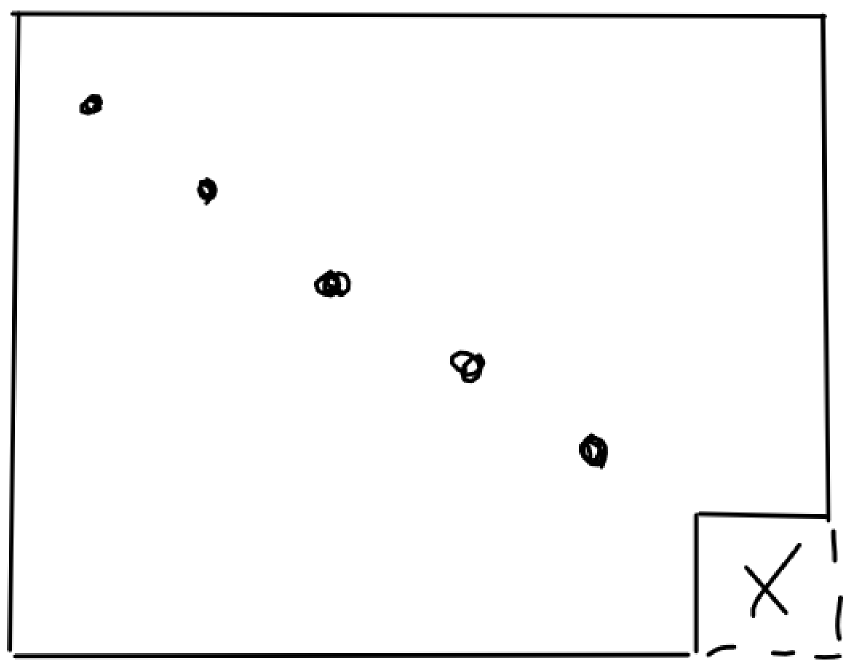
Definition 6. The orthogonal algebra

$$\mathfrak{so}(2\ell, F) = \left\{ A \in \mathfrak{gl}(2\ell, F) \ni MA - A^T M = 0 \right\} \text{ where}$$

$$M = \begin{cases} \left(\begin{array}{c|cc} 1 & 0 & \\ \hline 0 & \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} & \end{array}\right) & n = 2\ell + 1 \text{ odd}, \\ \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) & \text{else.} \end{cases}$$

Proposition 7. The dimensions of these algebras can be computed;

- The dimension of $\mathfrak{gl}(n, \mathbb{F})$ is n^2, and has basis $\{e_{i,j}\}$ the matrices if a 1 in the i, j position and



  zero elsewhere.
- For type A_ℓ, we have $\dim \mathfrak{sl}(n, \mathbb{F}) = (\ell + 1)^2 - 1$.
- For type C_ℓ, we have $||\mathfrak{sp}(n, \mathbb{F}) = \ell^2 + 2\left(\frac{\ell(\ell+1)}{2}\right)$, and so elements here

$$\begin{pmatrix} A & B = B^t \\ C = C^t & A^t \end{pmatrix}.$$

- For type D_ℓ we have

$$||\mathfrak{so}(2\ell, \mathbb{F}) = \dim \left\{ \begin{pmatrix} A & B = -B^t \\ C = -C^t & -A^t \end{pmatrix} \right\},$$

which turns out to be $2\ell^2 - \ell$.

- For type B_ℓ, we have $\dim \mathfrak{so}(2\ell, \mathbb{F}) = 2\ell^2 - \ell + 2\ell = 2\ell^2 + \ell$, with elements of the form

$$\left(\begin{array}{c|cc} 0 & M & N \\ \hline -N^t & A & C = C^t \\ -M^t & B = B^t & -A^t \end{array} \right).$$

Exercise 3. Use the relation $MA = A^{tM}$ to reduce restrictions on the blocks.

$$\begin{tikzcd}
 & & \mathfrak{so}(6) & \\
 & & \mathfrak{so}(5) \arrow[dr] & \\
\mathfrak{sl}(4) \arrow[uurr] & \mathfrak{sl}(2)^2 \arrow[r] & \mathfrak{so}(4) & \mathfrak{sp}(4) \\
 & & \mathfrak{so}(3) & \\
 & \mathfrak{sl}(2) \arrow[ur] & & \mathfrak{sp}(2) \arrow[ul]
\end{tikzcd}$$

Theorem 8. These are *all* of the isomorphisms between any of these types of algebras, in any dimension.

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_\ell \iff \mathfrak{sl}(\ell + 1, F)$
- $B_\ell \iff \mathfrak{so}(2\ell + 1, F)$
- $C_\ell \iff \mathfrak{sp}(2\ell, F)$
- $D_\ell \iff \mathfrak{so}(2\ell, F)$

Exercise 4. Characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition 9. An F-**algebra** A is an F-vector space endowed with a bilinear map $A^2 \to A,\ (x,y) \mapsto xy$.

Definition 10. An algebra is **associative** if $x(yz) = (xy)z$.

Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition 11. Any map $\delta : A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

Definition 12. We define $\mathrm{Der}(A) = \{\delta \ni \delta \text{ is a derivation }\}$.

Any Lie algebra $\mathfrak{g}$ is an F-algebra, since $[\,\cdot\,,\,\cdot\,]$ is bilinear. Moreover, $\mathfrak{g}$ is associative iff $[x,[y,z]] = 0$.

Exercise 5. Show that $\mathrm{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

Exercise 6 (Turn in)**.** Define the adjoint by $\mathrm{ad}_x : \mathfrak{g}\circlearrowleft,\ y \mapsto [x,y]$. Show that $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$.

2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example 13. Any F-vector space can be made into a Lie algebra by setting $[x,y] = 0$; such algebras are referred to as *abelian.*

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x,x] = 0 \implies [\,\cdot\,,\,\cdot\,] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is $[x,y]$. Some cases:
  - $[x,y] = 0 \implies \mathfrak{g}$ is abelian.
  - $[x,y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x',y'] = [ax+by, \frac{y}{a}] = [x,y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x',y'] = x'$.

We can fill in a table with all of the various combinations of brackets:

$[\,\cdot\,,\,\cdot\,]$	x'	y'
x'	0	x'
y'	$-x'$	0

Example 14. Let $V = \mathbb{R}^3$, and define $[a,b] = a \times b$ to be the usual cross product.

Exercise 7. Look at notes for basis elements of $\mathfrak{sl}(2,F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute the matrices of $\mathrm{ad}(e), \mathrm{ad}(h), \mathrm{ad}(g)$ with respect to this basis.

2.3 Ideals

Definition 15. A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \trianglelefteq \mathfrak{g}$, if $x, y \in I \implies [x, y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using $[x, y] = [-y, x]$.

Exercise 8. Check that the following are all ideals of $\mathfrak{g}$:

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g}\}$
- The commutator (or derived) algebra $[\mathfrak{g}, \mathfrak{g}] = \{\sum_i [x_i, y_i] \ni x_i, y_i \in \mathfrak{g}\}$.
  - Moreover, $[\mathfrak{gl}(n, F), \mathfrak{gl}(n, F)] = \mathfrak{sl}(n, F)$.

Fact: If $I, J \trianglelefteq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \trianglelefteq \mathfrak{g}$
- $I \bigcap J \trianglelefteq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \trianglelefteq \mathfrak{g}$

Definition 16. A Lie algebra is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e. when $\mathfrak{g}$ is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 17. Suppose that char $F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof. Recall that we have a basis of $\mathfrak{sl}(2, F)$ given by $B = \{e, h, f\}$ where

- $[e, f] = h$,
- $[h, e] = 2e$,
- $[h, f] = -2f$.

So think of $[h, e] = \mathrm{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since char $F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2, F)$ has a nontrivial ideal I; then pick $x = ae + bh + cf \in I$. Then $[e, x] = 0 - 2be + ch$, and $[e, [e, x]] = 0 - 0 + 2ce$. Again since char $F \neq 2$, then if $c \neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So $c = 0$.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I,$$
$$2f = [h, -f] \in I \implies f \in I.$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

$\square$

Note that there is a homework coming due next Monday, about 4 questions.

3 Lecture 3

Last time, we looked at ideals such as $0, \mathfrak{g}, Z(\mathfrak{g})$, and $[\mathfrak{g}, \mathfrak{g}]$.

Definition: If $I \trianglelefteq \mathfrak{g}$ is an ideal, then the quotient $\mathfrak{g}/I$ also yields a Lie algebra with the bracket given by $[x + I, y + I] = [x, y] + I$.

Exercise: Check that this is well-defined, so that if $x + I = x' + I$ and $y + I = y' + I$ then $[x, y] + I = [x', y'] + I$.

3.1 Homomorphisms and Representations

Definition 18. A linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie homomorphism* if $\phi[x, y] = [\phi(x), \phi()]$.

Remark. $\ker \phi \trianglelefteq \mathfrak{g}_1$ and im $\phi \leq \mathfrak{g}_2$ is a subalgebra.

Fact: There is a canonical way to set up a 1-to-1 correspondence $\{I \trianglelefteq \mathfrak{g}\} \iff \{\hom \phi : \mathfrak{g} \to \mathfrak{g}'\}$ where $I \mapsto (x \mapsto x + I)$ and the inverse is given by $\phi \mapsto \ker \phi$.

Theorem (Isomorphism theorem for Lie algebras):

- If $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then $\mathfrak{g}/\ker \phi \cong \text{im } \phi$
- If $I, J \trianglelefteq \mathfrak{g}$ are ideals and $I \subset J$ then $J/I \trianglelefteq \mathfrak{g}g/I$ and $(\mathfrak{g}/I)/(J/I) \cong \mathfrak{g}/J$.
- If $I, J \trianglelefteq \mathfrak{g}$ then $(I + J)/J \cong I/(I \bigcap J)$.

Definition: A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ into a linear Lie algebra for some vector space V.

We call V a $\mathfrak{g}$-module with action $g \cdot v = \phi(g)(v)$.

Example: The *adjoint representation*:

$$\begin{aligned} \text{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto [x, \cdot\,]. \end{aligned}$$

Corollary 19. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: Since $\mathfrak{g}$ is simple, the center $Z(\mathfrak{g}) = 0$. We can rewrite the center as

$$\begin{aligned} Z(\mathfrak{g}) = \Big\{ x \in \mathfrak{g} \ni \text{ad}_{x(y)} = 0 \quad \forall y \in \mathfrak{g} \Big\} \\ = \ker \text{ad}_x. \end{aligned}$$

Using the first isomorphism theorem, we have $\mathfrak{g}/Z(\mathfrak{g}) \cong \text{im ad} \subseteq \mathfrak{gl}(\mathfrak{g})$. But $\mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}$ here, so we are done.

3.2 Automorphisms

Definition: An automorphism of $\mathfrak{g}$ is an isomorphism $\mathfrak{g}\circlearrowleft$, and we define

$$\mathrm{Aut}(\mathfrak{g}) = \{\phi : \mathfrak{g}\circlearrowleft \ni \phi \text{ is an isomorphism } \}.$$

Proposition: If $\delta \in \mathrm{Der}(\mathfrak{g})$ is nilpotent, then

$$\exp(\delta) \coloneqq \sum \frac{\delta^n}{n!} \in \mathrm{Aut}(\mathfrak{g}).$$

This is well-defined because δ is nilpotent, and a binomial formula holds:

$$\frac{\delta^{n([x,y])}}{n!} = \sum_{i=0}^{n} [\frac{\delta^i(x)}{i!}, \frac{\delta^{n-i}(y)}{(n-i)!}].$$

and for $n = 1, \delta([x,y]) = [x, \delta(y)] + [\delta(x), y]$.

Exercise: Show that

$$[(\exp\delta)(x), (\exp\delta)(y)] = \sum_{n=0}^{k-1} \frac{\delta^n([x,y])}{n!}.$$

Example: Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$ and define

$$s = \exp(\mathrm{ad}_e)\exp(\mathrm{ad}_{-f})\exp(\mathrm{ad}_e) \in \mathrm{Aut}\mathfrak{g}.$$

where e, f are defined as (todo, see written notes).

Then define the Weyl group $W = \langle s \rangle$.

Exercise: Check that $s(e) = -f, s(f) = -e, s(h) = -h$, and so the order of s is 2 and $W = \{1, s\}$.

4 Lecture 4

4.1 Solvability

Idea: Define a semisimple Lie algebra

Definition: The derived series for $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \\ &\cdots \\ \mathfrak{g}^{(i+1)} &= [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]. \end{aligned}$$

The Lie algebra $\mathfrak{g}$ is *solvable* if there is some n for which $\mathfrak{g}^{(n)} = 0$.

Exercise (to turn in): Check that the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{F})$.

Example: Abelian Lie algebras are solvable

Example: Simple Lie algebras are *not* solvable.

Proposition: Let $\mathfrak{g}$ be a Lie algebra, then

1. If $\mathfrak{g}$ is solvable, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also solvable.
2. If $I \trianglelefteq \mathfrak{g}$ and both I and $\mathfrak{g}/I$ are solvable, then so is $\mathfrak{g}$.
3. If $I, J \trianglelefteq \mathfrak{g}$ are solvable, then so is $I + J$.

Corollary (of part 3 above): Any Lie algebra has a unique maximal solvable ideal, which we denote the *radical* $\mathrm{Rad}(\mathfrak{g})$.

Definition: A Lie algebra is semisimple if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Example: Any simple Lie algebra is semisimple.

Example: Using part (2) above, we can deduce that we can construct a semisimple Lie algebra from *any* Lie algebra: for any $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

4.2 Nilpotency

$$\begin{aligned}
\mathfrak{g}^0 &= \mathfrak{g} \\
\mathfrak{g}^1 &= [\mathfrak{g}^0, \mathfrak{g}^0] \\
&\cdots \\
\mathfrak{g}^{i+1} &= [\mathfrak{g}^i, \mathfrak{g}^i].
\end{aligned}$$

Much like the previous case, we have

Example: Abelian Lie algebras are nilpotent.

Example: Nilpotent Lie algebras are solvable.

Example: The *strictly* upper triangular matrices (with zero on the diagonal) are nilpotent.

1. If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are also nilpotent.
2. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.
3. If $\mathfrak{g} \neq 0$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.

Claim: If $\mathfrak{g}$ is nilpotent, then $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for all $x \in \mathfrak{g}$.

Proof: This is because $\mathfrak{g}^n = 0 \iff [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \cdots]]] = 0$, and so for every $x_i, y \in \mathfrak{g}$ we have $[x_1, [x_2, \cdots [x_n, y]]] = 0$, and so $\mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} \circ \cdots \mathrm{ad}_{x_n} = 0$ which implies that $\mathrm{ad}_x^n = 0$ for all $x \in \mathfrak{g}$.

Theorem [Engel]: If ad_x is nilpotent for all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

> Remark: This can be confusing if $\mathfrak{g}$ is a linear algebra, we can consider elements $x \in \mathfrak{g}$ and ask if it is the case x being nilpotent (as an endomorphism) iff $\mathfrak{g}g$ is nilpotent? False, a counterexample is $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$, where there exists an x which is *not* nilpotent while ad_x *is* nilpotent, which contradicts the above theorem.

Proof:

Lemma: Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra for some finite dimensional vector space V. If x is nilpotent as an endomorphism on V for all $x \in V$, then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$, so $x \in \mathfrak{g} \implies x(v) = 0$.

Proof of lemma Use induction on $\dim \mathfrak{g}$, splitting into two separate base cases: - Case $\dim \mathfrak{g} = 0$, then $\mathfrak{g} = \{0\}$. - Case $\dim g = 1$, left as an exercise.

Inductive step: Let A be a maximal proper subalgebra and define $\phi : A \to \mathfrak{gl}(\mathfrak{g}/A)$ where $a \mapsto (x + A \mapsto [a, x] + A)$. We need to check that ϕ is a homomorphism, this just follows from using the Jacobi identity.

We also need to show that im $\phi \leq \mathfrak{gl}(\mathfrak{g}/a)$ is a Lie subalgebra, and $\dim \mathrm{im}\ \phi < \dim \mathfrak{g}$. The claim is that $\phi(a) \in \mathrm{End}(\mathfrak{g}/A)$ is nilpotent for all $a \in A$. By the inductive hypothesis, there is a nonzero coset $y + A \in \mathfrak{g}/A$ such that $(\mathrm{im}\ \phi) \cdot (y + A) = A$. Since $y \notin A$, then $\phi(a)(y + A) = A$ for all $a \in A$, and so $[a, y] \in A$.

We want to show that A is a subalgebra of codimension 1, and $A \oplus F_y \leq \mathfrak{g}$ is a Lie subalgebra. This is because $[a_1 + c_1 y, a_2 + c_2 y] = [a_1, a_2] + c_2[a_1, y] - c_2[a_2, y] + c_1 c_2[y, y]$. The last term is zero, the middle two terms are in A, and because A is closed under the bracket, the first term is in A as well.

But then $A \oplus F_y$ is a larger subalgebra than A, which was maximal, so it must be everything. So $A \oplus F_y = \mathfrak{g}$. So $A \trianglelefteq \mathfrak{g}$ because $[a_1, a_2 + cy]$ is in $A, A \oplus F_y = \mathfrak{g}$ respectively, and this equals $[a_1, a_2] + c[a_1, y]$, where both terms are in A.

Proof to be continued on Friday!

5 Lecture 5

Last time: we had a theorem that said that if $\mathfrak{g} \in \mathfrak{gl}(V)$ and every $x \in \mathfrak{g}$ is nilpotent, then there exists a nonzero $v \in V$ such that $\mathfrak{g}v = 0$.

We proceeded by induction on the dimension of V, constructing im $\phi \subseteq \mathfrak{gl}(\mathfrak{g}/A)$, and showed that $\mathfrak{g} = A \oplus Fy$. Now consider

$$W = \{v \in V \ni Av = 0\},$$

which is $\mathfrak{g}$-invariant, so $\mathfrak{g}(W) \subseteq W$, or for all $a \in A, x \in \mathfrak{g}, v \in W$, we have $a \curvearrowright x(v) = 0$. This is true because $a \curvearrowright x = x \circ a + [a, x] \in \mathfrak{gl}(V)$. But V is killed by any element in A, and both of these terms are in A. In particular, the y appearing in Fy also satisfies $y \in W$. Consider $y|_W \in \mathrm{End}(w)$, and we want to apply the inductive hypothesis to $Fy|_W \subseteq \mathfrak{gl}(V)$.

We need to check that $y|_W \in \mathrm{End}(V)$, which is true exactly because y is nilpotent. So we can construct a nonzero $v \in W \subset V$ such that $y(v) = 0$, and so $\mathfrak{g}v = 0$.

Claim: $\phi(a) \in \mathrm{End}(\mathfrak{g}/A)$ is nilpotent. Each $a \in A \subset \mathfrak{g}$ is nilpotent by assumption. Define the maps for left multiplication by a, $m_\ell : x \mapsto ax$, and the right multiplication $m_r : x \mapsto xa$. These are nilpotent, and since m_ℓ, m_r commute, the difference $m_\ell - m_r$ is nilpotent, and this is exactly ad_a. But then $\phi(a)$ is nilpotent.

Good proof for using all of the definitions!

Now we can see what the consequences of having such a nonzero vector are. This theorem implies Engel's theorem, which says that if $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is nilpotent for every $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

Proof: By induction on dimension. The base case is easy. For the inductive step, the previous theorem applies to $\mathrm{ad}g \subset \mathfrak{gl}(\mathfrak{g})$. So we can produce the nonzero $v \in \mathfrak{g}$ such that $\mathrm{ad}\mathfrak{g}v = 0$. Then $[x, v] = 0$ for all $x \in \mathfrak{g}$, so either $v \in Z(\mathfrak{g})$ or $Z(\mathfrak{g}) \neq 0$. In either case, $\mathfrak{g}/Z(\mathfrak{g})$ has smaller dimension. Since ad_x is nilpotent, so is $\mathrm{ad}_x + Z(\mathfrak{g})$, and so $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. By an earlier proposition, since the quotient is nilpotent, so is the total space. $\square$

Let $\mathfrak{N}(F)$ be the subalgebra of $\mathfrak{gl}(F)$ consisting of strictly upper triangular matrices. We have a corollary: if $\mathfrak{g} \subset \mathfrak{gl}(n, F)$ is a Lie subalgebra such every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of F, then the matrices of $\mathfrak{g}$ with respect to some bases of in $\mathfrak{N}(n, F)$.

The proof is by induction on n, where the base case is easy. For the inductive step, we use the previous theorem to get a v_1 such that $x(v_1) = 0$ for all $x \in \mathfrak{g}$. Let $\overline{V} = F^n/Fv_1 \cong F^{n-1}$, and define $\phi : \mathfrak{g} \to \mathfrak{gl}(\overline{V})$ where $x \mapsto (\overline{y} \mapsto \overline{y(x)})$.

Then $\mathrm{im}\,\phi \leq \mathfrak{gl}(n-1, F)$ as a subalgebra, and every $\phi(x) \in \mathrm{End}(F^{n-1})$ is nilpotent, since x was nilpotent on the larger space. But (see notes) then x can be written as a strictly upper-triangular matrix.

5.1 Chapter 2: Semisimple Lie Algebras

We now assume $\mathrm{char}\, F = 0$ and $\overline{F} = F$.

Theorem: If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional V, then V contains a common eigenvector for a $x \in \mathfrak{g}$, i.e. a $\lambda : \mathfrak{g} \to F, x \mapsto \lambda(x)$ such that $x(v) = \lambda(x)v$ for all $x \in \mathfrak{g}$.

Proof: We will use induction on the dimension of $\mathfrak{g}$. For the inductive step:

Claim 1: There is an ideal $A \trianglelefteq \mathfrak{g}$ such that $\mathfrak{g} = A \oplus Fy$ for some $y \neq 0$, so A is a subalgebra of a solvable Lie algebra $\mathfrak{g}$ and thus solvable itself. By hypothesis, we can produce a $w \in V \setminus \{0\}$, and thus a functional $\lambda : A \to F$ such that $aw = \lambda(a)w$ for all $a \in A$. So we define

$$V_\lambda = \{v \in V \ni av = \lambda(a)v \forall a \in A\}$$

where $w \in V_\lambda$.

Claim 2: $y(V_\lambda) \subseteq V_\lambda$, or $y|_{V_\lambda} \in \mathrm{End}(V_\lambda)$.

Thus $F(y|_{V_\lambda}) \leq \mathfrak{gl}(V_\lambda)$ is a Lie algebra of dimension 1, and thus solvable. By the inductive hypothesis, we can find a $v \in V_\lambda$ and some $\mu \in F$ such that $y(v) = \mu v$. An arbitrary element $x \in \mathfrak{g}$ can be written as $x = a + cy$ for some $a \in A, c \in F$ and it acts by $x(v) = a(v) + cy(v) = \lambda(a)v + c\mu v = (\lambda(a) + c)v \in V_\lambda$.

6 Lecture n+1

Todo

7 Lecture n+2

Definition (Jordan Decomposition)

Let $X \in \mathrm{End}(V)$ for V finite dimensional. Then,

(a) There exists a unique $X_s, X_n \in \mathrm{End}(V)$ such that $X = X_s + X_n$ where X_s is semisimple, X_n is nilpotent, and $[X_s, X_n] = 0$.

(b) There exists a $p(t), q(t) \in t\mathbb{F}[t]$ such that $X_s = p(X), X_n = q(X)$.

(Polynomials with no constant term.)

Proof of (a): Assume $X_s = X_s + X_n = X_s' + X_n'$, so both have bracket zero. Assuming that (b) holds, we have $X_s = p(X)$, and so

$$[X, X_s] = [X_s + X_n', X_s'] = [X_s', X_s'] + [X_s', X_n'] = 0 \implies [p(X), X_S'] = 0 = [X_s, X_s']$$

Using fact (c) from last time, then X_s, X_s' can be diagonalized simultaneously, and so $X_s - X_s'$ is semisimple.

On the other hand, if X_n', X_n are nilpotent, and since these commute, $X_n - X_n'$ is nilpotent. But then this is a Jordan decomposition of the zero map, i.e.

$$0 = X - X = (X_s - X_s') + (X_n + X_n')$$

where the first term is semisimple and the second is nilpotent. Then each term is both semisimple *and* nilpotent, so they must be zero, which is what we wanted to show.

Proof of part (b): Let $m(t) = \prod_{i=1}^r (t - \lambda_i)^{m_i}$ be the minimal polynomial of X, where each $m_i \geq 1$ and the λ_i are distinct. Then the primary composition of V is given by

$$V = \bigoplus_{i=1}^r V_i, \quad V_i = \ker(X - \lambda_i I_V) \neq 0, \quad X(V_i) \subseteq V_i$$

Claim: There exists a polynomail $p \in F[t]$ such that

$$\begin{aligned} p &= \lambda \mod (t - \lambda_i)^{m_i} \quad \forall i, \\ p &= 0 \mod t. \end{aligned}$$

The existence follows from the Chinese Remainder Theorem.

What is $p(x) \curvearrowright V_i$? This acts by scalar multiplication by λ_i for all i. (Check). Because of the restrictive conditions, $p(x)$ has no constant term.

So $p(X) = X_s$ is the semisimple part we want. Now just set $q(t) = t - p(t)$, then $X_n \coloneqq q(X) = X - X_s$ is nilpotent.

Example: The Jordan Decomposition is invariant under taking adjoints.

If we have $X = X_s + X_n$, then $\mathrm{ad}_X \in \mathrm{End}(\mathrm{End}(V))$. It can be shown that $(\mathrm{ad}_X)_s + (\mathrm{ad}_X)_n = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$.

$$p(X) \sim \begin{pmatrix} \boxed{\lambda_1 I_{V_1}} & & & 0 \\ & \boxed{\lambda_2 I_{V_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_r I_{V_r}} \end{pmatrix}$$

Figure 1: ???

Let e_{ii} be the elementary matrix with a 1 in the i, j position. You can write ad_X as a 4×4 matrix (see image).

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$X = X_s + X_n$$

$$
= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array}
\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}
$$

$$
\xrightarrow[\text{JNF}]{}
\left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)
$$

$$= \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{JNF}} \left(\begin{array}{c|cccc} 0 & & & & \\ \hline & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 \end{array}\right)$$

You can check that $(\mathrm{ad}_X)_S = 0, \mathrm{ad}(X_s) = 0$, and $(\mathrm{ad}X)_n$ is the Jordan form given above.

Lemma:

(a) $x \in \mathrm{End}(V) \implies \mathrm{ad}(x)_s = \mathrm{ad}(x_s)$ and $\mathrm{ad}(x)_n = \mathrm{ad}(x_n)$.

(b) If A is a finite dimensional $\mathbb{F}$-algebra, then $\delta \in \mathrm{Der}(A) \implies \delta_s, \delta_n \in \mathrm{Der}(A)$ as well.

Proof of (a):

Check that $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$. Then for $y \in \mathrm{End}(V)$, we have

$$\begin{aligned} (\mathrm{ad}(x))(y) &= [x, y] \\ &= [x_s + x_n, y] \\ &= [x_s, y] + [x_n, y] \\ &- (\mathrm{ad}(x_s))(y) + (\mathrm{ad}(x_n))(y). \end{aligned}$$

Using theorem 3.3, x_n nilpotent $\implies$ $\mathrm{ad}(x_n)$ is also nilpotent. So write $x_s = \sum \lambda_i e_{ii}$ with the eigenvalues on the diagonal. Then $\mathrm{ad}x_s(e_{ij}) = (\lambda_i - \lambda_j)e_{ij}$ for all i, j. But then $\mathrm{ad}x_s$ is given by

$$(\delta-(\lambda+\mu)I)^n([x,y])$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left[(\delta-\lambda I)^i(x),\ (\delta-\mu I)^{n-i}(y)\right]$$

Figure 2: Image

a matrix with $\lambda_i - \lambda_j$ in the i, j position and zeros elsewhere. By the uniqueness of the Jordan decomposition, the statement follows.

Proof of (b):

Since $\delta \in \mathrm{Der}(A)$, the primary decomposition with respect to δ is given by

$$A = \bigoplus_{\lambda \in F} A_\lambda \quad \text{where } A_\lambda = \left\{ a \in A \ni (\delta - \lambda I)^k a = 0 \text{ for some } k >> 0 \right\}.$$

So $\delta_s \curvearrowright A_\lambda$ by scalar multiplication (by λ). Then for $\lambda, \mu \in \mathbb{F}$, we have

So $[A_x, A_y] \subseteq A_{\lambda+\mu}$ for all $x, y \in A$. But then

and so $\delta_s \in \mathrm{Der}(A)$, and $\delta_n = \delta - \delta_s \in \mathrm{Der}(A)$ as well.

8 Lecture n+3

Todo

9 Lecture n+4

Review of bilinear forms: let $V = \mathbb{F}^n$.

Definition: A bilinear form $\beta : V^2 \to \mathbb{F}$ can be represented by a matrix B with respect to a basis $\{\mathbf{v}_i\}$ such that

$$\beta\beta(\sum a_i \mathbf{v}_i, \sum b_i \mathbf{v}_i) = (a_1\ a_2\ \cdots)B(b_1\ b_2\ \cdots)$$

- β is *symmetric* iff $\beta(a, b) = \beta(b, a)$.
- β is *symplectic* iff $\beta(a, b) = -\beta(b, a)$.
- β is *isotropic* iff $\beta(a, a) = 0$.

$$\delta_s([x,y])$$

$$\|$$

$$(\lambda+\mu)[x,y] = [\lambda x, y] + [x, \mu y]$$

$$\|$$

$$[\delta_s(x), y] + [x, \delta_s(y)]$$

Figure 3: Image

For a subspace $U \leq V$, define

$$U^{\perp} := \{\mathbf{v} \in V \ni \beta(\mathbf{u}, \mathbf{v}) = \mathbf{0} \forall u \in U\}.$$

> Note: in general, left/right orthogonality are distinguished, but these will be identical when β is symmetric/symplectic.

The form β is said to be *non-degenerate* iff $V^{\perp} = 0$ iff $\det B \neq 0$.

Assume F is an algebraically closed field, so $\overline{F} = F$, and $\mathrm{char} F \neq 2$, then

- If β is non-degenerate and symmetric, then $B \sim I_n$
- If β is non-degenerate and symplectic, then $B \sim [0, I_{n/2}; I_{n/2}, 0]$.

Remark:

$\mathfrak{so}(n, \mathbb{F}) = \{x \in \mathfrak{gl}(n, F) \ni \beta(x(u), v) = -\beta(u, x(v))\}$, where B has the matrix $[0, I; I, 0]$ if n is odd, or this matrix with a 1 in the top-left corner if n is odd.

Similarly, $\mathfrak{Sp}(2m, \mathbb{F})$ can be described this way with the matrix $[0, -I; -I, 0]$.