Title

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Contents

1 Lecture 3

Last of preliminaries. Upcoming: one-variable function fields and their valuation rings.

1.1 Polynomials Defining Regular Function Fields

Where's the curve: $f(x,y)=0$.

Exercise 1.1.1: Let R_1, R_2 be k-algebras that are also domains with fraction fields K_i. Show $R_1 \otimes_k R_2$ is a domain $\iff$ $K_1 \otimes_k K_2$ is a domain.[1]

Definition 1.1.2 (Geometrically Irreducible)
A polynomial of positive degree $f \in k[t_1, \cdots, t_n]$ is **geometrically irreducible** if $f \in \bar{k}[t_1, \cdots, t_n]$ is irreducible as a polynomial.

If $n=1$ then f is geometrically irreducible $\iff$ it's linear, i.e. of degree 1. Let f be irreducible, then since polynomial rings are UFDs then $\langle f \rangle$ is a prime ideal (irreducibles generate principal ideals) and $k[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. Let K_f be the fraction field.

Exercise 1.1.3 ***(an easy one):***

a. Above for $1 \leq i \leq n$ let x_i be the image of t_i in K_f. Show that $K_f = k(x_1, \cdots, x_n)$.

b. Show that if K/k is generated by $x_1, \cdots, x_n$, then it is the fraction field of $k[t_1, \cdots, t_n]/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ (equivalently, a height 1 ideal).

Proposition 1.1.4 ***(?).***
Suppose that f is geometrically irreducible.

a. The function field K/k is regular.

b. For all ℓ/k, $f \in \ell[t_1, \cdots, t_n]$ is irreducible.

In this case we say f is *absolutely irreducible* as a synonym for geometrically irreducible.

Proof .
By definition of geometric irreducibility, $\bar{k}[t_1, \cdots, t_n]/\langle f \rangle = k[t_1, \cdots, t_n]/\langle f \rangle \otimes_k \bar{k}$ is a domain. The exercise shows that $K_f \otimes_k k$ is a domain, so K_f is regular. It follows that for all ℓ/k, $K_f \otimes_k \ell$ is a domain, so $\ell[t_1, \cdots, t_n]/\langle f \rangle$ is a domain. ■

[1] Hint: use a denominator clearing argument.

Moral: geometrically irreducible polynomials are good sources of regular function fields.

Exercise 1.1.5: Let k be a field, $d \in \mathbb{Z}^+$ such that $4 \nmid d$ and $p(x) \in k[x]$ be positive degree. Factor $p(x) = \prod_{i=1}^{r} (x - a_i)^{\ell_i}$ in $\bar{k}[x]$.

a. Suppose that for some i, $d \nmid \ell_i$. Show that $f(x, y) := y^d - p(x) \in k[x, y]$ is geometrically irreducible. Conclude that $K_f := ff\left(k[x, y]/\left\langle y^d - p(x)\right\rangle\right)$ is a regular one-variable function field over k, and thus elliptic curves yield regular function fields.[2]

b. What happens when $4 \mid d$?

Exercise 1.1.6 ***(Nice, Recommended):*** Assume k is a field, if necessary assuming $\mathrm{ch}(k) \neq 2$.

a. Let $f(x, y) = x^2 - y^2 - 1$ and show K_f is is rational: $K_f = k(z)$.

b. Let $f(x, y) = x^2 + y^2 - 1$. Show that K_f is again rational.

c. Let $k = \mathbb{C}$ and $f(x, y) = x^2 + y^2 + 1$, K_f is rational.

d. Let $k = \mathbb{R}$. For $f(x, y) = x^2 + y^2 + 1$, is K_f rational?[3]

Question 1.1.7: Can we always construct regular function fields using geometrically irreducible polynomials?

Answer 1.1.8: In several variables, no, since not every variety is birational to a hypersurface. In one variable, yes, as the following theorem shows:

Theorem 1.1.9 ***(Regular Function Fields in One Variable are Geometrically Irreducible).***
Let K/k be a one variable function field (finitely generated, transcendence degree one). Then

a. If K/k is separable, then $K = k(x, y)$ for some $x, y \in K$.

b. If K/k is regular (separable + constant subfield is k, so stronger) then $K \cong K_f$ for a geometrically irreducible $f \in k[x, y]$.

Recall separable implies there exists a separating transcendence basis.

Proof (of a).
This means there exists a primitive element $x \in K$ such that $K/k(x)$ is finite and separable. By the Primitive Element Corollary (FT 7.2), there exist a $y \in K$ such that $K = k(x, y)$. ∎

[2]Referred to as *hyperelliptic* or *superelliptic* function fields. Hint: use FT 9.21 or Lang's Algebra.
[3]This is an example of a non-rational genus zero function field.

Proof (of b).
Omitted for now, slightly technical. ■

Importance of last result: a regular function field on one variable corresponds to a nice geometrically irreducible polynomial f.

Remark 1.1.10: Note: the plane curve module may not be smooth, and in fact usually is not possible. I.e. $k[x,y]/\langle f\rangle$ is a one-dimensional noetherian domain, which need not be integrally closed.

Question 1.1.11: Can every one variable function field be 2-generated?

Answer 1.1.12: Yes, as long as the ground field is perfect. In positive characteristic, the suspicion is no: there exists finite inseparable extensions ℓ/k that need arbitrarily many generators. However, what if K/k has constant field k but is not separable? Riemann-Roch may have something to say about this.

Example 1.1.13: Example from earlier lecture:

$$ax^p + b - y^b$$

Moral: look for examples of nice function fields by taking irreducible polynomials in two variables. This will define a one-variable function field. If the polynomial is geometrical reducible, this produces regular function fields.

Next: One variable function fields and their valuations.