

Algebra Notes

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Contents

1	Group Theory	2
1.1	Big List of Notation	2
1.2	Basics	2
1.3	Finitely Generated Abelian Groups	3
1.4	The Symmetric Group	4
1.5	Counting Theorems	5
1.5.1	Examples of Orbit-Stabilizer	6
1.5.2	Sylow Theorems	7
1.5.3	Sylow 1 (Cauchy for Prime Powers)	7
1.5.4	Sylow 2 (Sylows are Conjugate)	7
1.5.5	Sylow 3 (Numerical Constraints)	7
1.6	Products	8
1.7	Isomorphism Theorems	8
1.8	Special Classes of Groups	9
1.9	Series of Groups	11
2	Rings	11
2.1	Definitions	11
2.2	Nontrivial Properties	13
2.3	Ideals	13
2.3.1	Maximal and Prime Ideals	13
2.3.2	Nilradical and Jacobson Radical	13
2.3.3	Zorn's Lemma	14
3	Fields	15
3.1	Finite Fields	15
3.2	Galois Theory	16
3.2.1	Lemmas About Towers	16
3.2.2	Examples	18
3.3	Cyclotomic Polynomials	19
4	Modules	20
4.1	General Modules	20
4.2	Classification of Modules over a PID	20
4.3	Minimal / Characteristic Polynomials	20

4.4	Canonical Forms	21
4.4.1	Rational Canonical Form	21
4.4.2	Jordan Canonical Form	22
4.5	Using Canonical Forms	22
4.6	Diagonalizability	23
4.7	Matrix Counterexamples	24
4.8	Miscellaneous	24

1 Group Theory

1.1 Big List of Notation

$C(x) =$	$\{g \in G \mid gxg^{-1} = x\}$	$\subseteq G$	Centralizer
$C_G(h) =$	$\{ghg^{-1} \mid g \in G\}$	$\subseteq G$	Conjugacy Class
$Gx =$	$\{g.x \mid x \in X\}$	$\subseteq X$	Orbit
$G_x =$	$\{g \in G \mid g.x = x\}$	$\subseteq G$	Stabilizer
$X_g =$	$\{x \in X \mid \forall g \in G, g.x = x\}$	$\subseteq X$	Fixed Points
$Z(G) =$	$\{x \in G \mid \forall g \in G, gxg^{-1} = x\}$	$\subseteq G$	Center
$\text{Inn}(G) =$	$\{\phi_g(x) = gxg^{-1}\}$	$\subseteq \text{Aut}(G)$	Inner Aut.
$\text{Out}(G) =$	$\text{Aut}(G)/\text{Inn}(G)$	$\hookrightarrow \text{Aut}(G)$	Outer Aut.
$N(H) =$	$\{g \in G \mid gHg^{-1} = H\}$	$\subseteq G$	Normalizer

1.2 Basics

Definition (Centralizer):

$$C_G(H) = \{g \in G \mid ghg^{-1} = h \forall h \in H\}$$

Definition (Normalizer):

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

Lemma: $C_G(H) \trianglelefteq N_G(H)$

Lemma: The size of the conjugacy class of H is the index of its centralizer, i.e.

$$|\{gHg^{-1} \mid g \in G\}| = [G : C_G(H)].$$

Proof: Orbit-stabilizer.

Lemma (“The Fundamental Theorem of Cosets”):

$$aH = bH \iff a^{-1}b \in H \text{ or } aH \cap bH = \emptyset$$

Definition: $[x, y] = x^{-1}y^{-1}xy$ is the **commutator**, and $[G, G] := \{[x, y] \mid x, y \in G\}$ is the **commutator subgroup**.

Lemma:

$$[G, G] \leq H \text{ and } H \trianglelefteq G \implies G/H \text{ is abelian.}$$

Lemmas:

- Every subgroup of a cyclic group is itself cyclic.
- Intersections of subgroups are still subgroups
 - Intersections of distinct coprime-order subgroups are trivial
 - Intersections of subgroups of the same prime order are either trivial or equality
- The Quaternion group has only one element of order 2, namely -1 .
 - They also have the presentation

$$\begin{aligned} Q &= \langle x, y, z \mid x^2 = y^2 = z^2 = xyz = -1 \rangle \\ &= \langle x, y \mid x^4 = y^4 = e, x^2 = y^2, yxy^{-1} = x^{-1} \rangle. \end{aligned}$$

- A dihedral group always has a presentation of the form

$$D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle,$$

yielding at least 2 distinct elements of order 2.

1.3 Finitely Generated Abelian Groups

Invariant factor decomposition:

$$G \cong \mathbb{Z}^r \times \prod_{j=1}^m \mathbb{Z}/(n_j) \quad \text{where } n_1 \mid \cdots \mid n_m.$$

Going from invariant divisors to elementary divisors:

- Take prime factorization of each factor
- Split into coprime pieces

Example:

$$\begin{aligned} &\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3 \cdot 5^2 \cdot 7) \\ &\cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2^3) \oplus \mathbb{Z}/(5^2) \oplus \mathbb{Z}/(7) \end{aligned}$$

Going from elementary divisors to invariant factors:

- Bin up by primes occurring (keeping exponents)
- Take highest power from each prime as *last* invariant factor
- Take highest power from all remaining primes as next, etc

Example: Given the invariant factor decomposition

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}, .$$

$p = 2$	$p = 3$	$p = 5$
2, 2, 2	3, 3	5^2

$$\implies n_m = 5^2 \cdot 3 \cdot 2$$

$p = 2$	$p = 3$	$p = 5$
2, 2	3	\emptyset

$$\implies n_{m-1} = 3 \cdot 2$$

$p = 2$	$p = 3$	$p = 5$
2	\emptyset	\emptyset

$$\implies n_{m-2} = 2$$

and thus

$$G \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3 \cdot 2) \oplus \mathbb{Z}/(5^2 \cdot 3 \cdot 2).$$

1.4 The Symmetric Group

Definitions:

- A cycle is **even** \iff product of an *even* number of transpositions.
 - A cycle of even *length* is **odd**
 - A cycle of odd *length* is **even**

Definition The **alternating group** is the subgroup of **even** permutations, i.e. $A_n := \left\{ \sigma \in S_n \mid \text{sign}(\sigma) = 1 \right\}$ where $\text{sign}(\sigma) = (-1)^m$ where m is the number of cycles of even length.

Corollary: Every $\sigma \in A_n$ has an even number of *odd* cycles (i.e. an even number of *even-length* cycles).

Example:

$$A_4 = \{\text{id}, \\ (1, 3)(2, 4), (1, 2)(3, 4), (1, 4)(2, 3), \\ (1, 2, 3), (1, 3, 2), \\ (1, 2, 4), (1, 4, 2), \\ (1, 3, 4), (1, 4, 3), \\ (2, 3, 4), (2, 4, 3)\}.$$

Lemmas:

- The transitive subgroups of S_3 are S_3, A_3
- The transitive subgroups of S_4 are $S_4, A_4, D_4, \mathbb{Z}_2^2, \mathbb{Z}_4$.
- S_4 has two normal subgroups: A_4, \mathbb{Z}_2^2 .
- $S_{n \geq 5}$ has one normal subgroup: A_n .
- $Z(S_n) = 1$ for $n \geq 3$
- $Z(A_n) = 1$ for $n \geq 4$
- $[S_n, S_n] = A_n$
- $[A_4, A_4] \cong \mathbb{Z}_2^2$
- $[A_n, A_n] = A_n$ for $n \geq 5$, so $A_{n \geq 5}$ is nonabelian.
- $A_{n \geq 5}$ is *simple*.

1.5 Counting Theorems

Lagrange's Theorem:

$$H \leq G \implies |H| \mid |G|.$$

Corollary: The order of every element divides the size of G , i.e.

$$g \in G \implies o(g) \mid o(G) \implies g^{|G|} = e.$$

Warning: There does **not** necessarily exist $H \leq G$ with $|H| = n$ for every $n \mid |G|$.
Counterexample: $|A_4| = 12$ but has no subgroup of order 6.

Cauchy's Theorem:

For every prime p dividing $|G|$, there is an element (and thus a subgroup) of order p .

This is a partial converse to Lagrange's theorem, and strengthened by Sylow's theorem.

Notation: For a group G acting on a set X ,

- $G \cdot x = \{g \curvearrowright x \mid g \in G\} \subseteq X$ is the orbit
- $G_x = \{g \in G \mid g \curvearrowright x = x\} \subseteq G$ is the stabilizer
- $X/G \subset \mathcal{P}(X)$ is the set of orbits

- $X^g = \{x \in X \mid g \curvearrowright x = x\} \subseteq X$ are the fixed points

Orbit-Stabilizer:

$$|G \cdot x| = [G : G_x] = |G|/|G_x| \quad \text{if } G \text{ is finite}$$

Mnemonic: $G/G_x \cong G \cdot x$.

1.5.1 Examples of Orbit-Stabilizer

1. Let G act on itself by conjugation.

- $G \cdot x$ is the **conjugacy class** of x
- $G_x = Z(x) := C_G(x) = \{g \mid [g, x] = e\}$, the **centralizer** of x .
- G^g (the fixed points) is the **center** $Z(G)$.

Corollary: The number of conjugates of an element (i.e. the size of its conjugacy class) is the index of its centralizer, $[G : C_G(x)]$.

Corollary: the **Class Equation**:

$$|G| = |Z(G)| + \sum_{\substack{\text{One } x_i \text{ from} \\ \text{each conjugacy} \\ \text{class}}} [G : Z(x_i)]$$

1. Let G act on S , its set of *subgroups*, by conjugation.

- $G \cdot H = \{gHg^{-1}\}$ is the **set of conjugate subgroups** of H
- $G_H = N_G(H)$ is the **normalizer** of H in G
- S^G is the set of **normal subgroups** of G

Corollary: Given $H \leq G$, the number of conjugate subgroups is $[G : N_G(H)]$.

1. For a fixed proper subgroup $H < G$, let G act on its cosets $G/H = \{gH \mid g \in G\}$ by left-multiplication.

- $G \cdot gH = G/H$, i.e. this is a *transitive* action.
- $G_{gH} = gHg^{-1}$ is a *conjugate subgroup* of H
- $(G/H)^G = \emptyset$

Application: If G is simple, $H < G$ proper, and $[G : H] = n$, then there exists an injective map $\phi : G \hookrightarrow S_n$.

Proof: This action induces ϕ ; it is nontrivial since $gH \neq H$ for all g implies $H = G$; $\ker \phi \trianglelefteq G$ and G simple implies $\ker \phi = 1$.

Burnside's Formula:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

1.5.2 Sylow Theorems

Notation: For any p , let $\text{Syl}_p(G)$ be the set of Sylow- p subgroups of G .

Write

- $|G| = p^n m$ where $(m, p) = 1$,
- S_p a Sylow- p subgroup, and
- n_p the number of Sylow- p subgroups.

Definition: A p -group is a group G such that every element is order p^k for some k . If G is a finite p -group, then $|G| = p^j$ for some j .

Lemma: p -groups have nontrivial centers.

Some useful facts:

- Coprime order subgroups are disjoint, or more generally $\mathbb{Z}_p, \mathbb{Z}_q \subset G \implies \mathbb{Z}_p \cap \mathbb{Z}_q = \mathbb{Z}_{(p,q)}$.
- The Chinese Remainder theorem: $(p, q) = 1 \implies \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$

1.5.3 Sylow 1 (Cauchy for Prime Powers)

$\forall p^n$ dividing $|G|$ there exists a subgroup of size p^n .

If $|G| = \prod p_i^{\alpha_i}$, then there exist subgroups of order $p_i^{\beta_i}$ for every i and every $0 \leq \beta_i \leq \alpha_i$. In particular, Sylow p -subgroups always exist.

1.5.4 Sylow 2 (Sylows are Conjugate)

All sylow- p subgroups S_p are conjugate, i.e.

$$S_p^1, S_p^2 \in \text{Syl}_p(G) \implies \exists g \text{ such that } gS_p^1g^{-1} = S_p^2.$$

Corollary: $n_p = 1 \iff S_p \trianglelefteq G$

1.5.5 Sylow 3 (Numerical Constraints)

1. $n_p \mid m$ (in particular, $n_p \leq m$),
2. $n_p \equiv 1 \pmod{p}$,
3. $n_p = [G : N_G(S_p)]$ where N_G is the normalizer.

Corollary: p does not divide n_p .

Lemma: Every p -subgroup of G is contained in a Sylow p -subgroup.

Proof: Let $H \leq G$ be a p -subgroup. If H is not *properly* contained in any other p -subgroup, it is a Sylow p -subgroup by definition. Otherwise, it is contained in some p -subgroup H^1 . Inductively this yields a chain $H \subsetneq H^1 \subsetneq \dots$, and by Zorn's lemma $H := \bigcup_i H^i$ is maximal and thus a Sylow p -subgroup.

Fratini's Argument: If $H \trianglelefteq G$ and $P \in \text{Syl}_p(G)$, then $HN_G(P) = G$ and $[G : H]$ divides $|N_G(P)|$.

1.6 Products

Characterizing direct products: $G \cong H \times K$ when

- $G = HK = \{hk \mid h \in H, k \in K\}$
- $H \cap K = \{e\} \subset G$
- $H, K \trianglelefteq G$

Can relax to only $H \trianglelefteq G$ to get a semidirect product instead

Characterizing semidirect products: $G = N \rtimes_{\psi} H$ when

- $G = NH$
- $N \trianglelefteq G$
- $H \curvearrowright N$ by conjugation via a map

$$\begin{aligned} \psi : H &\rightarrow \text{Aut}(N) \\ h &\mapsto h(\cdot)h^{-1}. \end{aligned}$$

Useful Facts

- If $\sigma \in \text{Aut}(H)$, then $N \rtimes_{\psi} H \cong N \rtimes_{\psi \circ \sigma} H$.
- $\text{Aut}(\mathbb{Z}/(p)^n) \cong \text{GL}(n, \mathbb{F}_p)$
 - If this occurs in a semidirect product, it suffices to consider similarity classes of matrices (i.e. just use canonical forms)
- $\text{Aut}(\mathbb{Z}/(n)) \cong \mathbb{Z}/(n)^{\times} \cong \mathbb{Z}/(\varphi(n))$ where φ is the totient function.
 - $\varphi(p^k) = p^{k-1}(p-1)$
- If G, H have coprime order then $\text{Aut}(G \oplus H) \cong \text{Aut}(G) \oplus \text{Aut}(H)$.

1.7 Isomorphism Theorems

Lemma: If $H, K \leq G$ and $H \leq N_G(K)$ (or $K \trianglelefteq G$) then $HK \leq G$ is a subgroup.

Note that this implies that HK is not always a subgroup.

Diamond Theorem / 2nd Isomorphism Theorem:

If $S \leq G$ and $N \trianglelefteq G$, then

$$\frac{SN}{N} \cong \frac{S}{S \cap N} \quad \text{and} \quad |SN| = \frac{|S||N|}{|S \cap N|}$$



Mnemonic:

Note: for this to make sense, we also have

- $SN \leq G$,
- $S \cap N \leq S$,

Cancellation / 3rd Isomorphism Theorem

If $H, K \trianglelefteq G$ with $H \trianglelefteq K$, then

$$\frac{G/H}{G/K} \cong \frac{G}{K}$$

Note: for this to make sense, we also have $G/K \trianglelefteq G/H$.

The Correspondence Theorem / 4th Isomorphism Theorem: Suppose $N \trianglelefteq G$, then there exists a correspondence:

$$\begin{aligned} \left\{ H < G \mid N \subseteq H \right\} &\iff \left\{ H \mid H < \frac{G}{N} \right\} \\ \left\{ \begin{array}{l} \text{Subgroups of } G \\ \text{containing } N \end{array} \right\} &\iff \left\{ \begin{array}{l} \text{Subgroups of the} \\ \text{quotient } G/N \end{array} \right\}. \end{aligned}$$

In words, subgroups of G containing N correspond to subgroups of the quotient group G/N . This is given by the map $H \mapsto H/N$.

Note: $N \trianglelefteq G$ and $N \subseteq H < G \implies N \trianglelefteq H$.

1.8 Special Classes of Groups

Definition: The “**2 out of 3 property**” is satisfied by a class of groups \mathcal{C} iff whenever $G \in \mathcal{C}$, then $N, G/N \in \mathcal{C}$ for any $N \trianglelefteq G$.

Definition: If $|G| = p^k$, then G is a **p-group**.

Facts about p-groups:

- If $k = 1$ then G is cyclic
- If $k = 2$, then $G \cong \mathbb{Z}/(p)^2$ or $\mathbb{Z}/(p^2)$.
- p-groups have nontrivial centers
 - Proof: Use class equation.
- Every normal subgroup is contained in the center
- Normalizers grow
- Every maximal is normal
- Every maximal has index p
- p-groups are *nilpotent*
- p-groups are *solvable*

Facts about other special order groups:

- $|G| = pq$: Two possibilities. By cases:
 - If p divides $q - 1$, two cases:
 - * $G \cong \mathbb{Z}/(pq)$ or $\mathbb{Z}/(p) \times \mathbb{Z}/(q)$
 - Otherwise, $G \cong \mathbb{Z}/(pq)$
- Proof: Sylow theorems. Note: Such groups are never simple.
- $|G| = p^2q$:
 - Two abelian: $\mathbb{Z}/(p) \times \mathbb{Z}/(q^2)$, or $\mathbb{Z}/(pq) \times \mathbb{Z}/(q)$.
 -

Definition: A group G is **simple** iff $H \trianglelefteq G \implies H = \{e\}, G$, i.e. it has no non-trivial proper subgroups.

Lemma: If G is *not* simple, then for any $N \trianglelefteq G$, it is the case that $G \cong E$ for an extension of the form $N \rightarrow E \rightarrow G/N$. $>$

Definition: A group G is **solvable** iff G has a terminating normal series with abelian factors, i.e.

$$G \rightarrow G^1 \rightarrow \cdots \rightarrow \{e\} \text{ with } G^i/G^{i+1} \text{ abelian for all } i.$$

Lemmas:

- G is solvable iff G has a terminating *derived series*.
- Solvable groups satisfy the 2 out of 3 property
- Abelian \implies solvable
- Every group of order less than 60 is solvable.

Definition: A group G is **nilpotent** iff G has a terminating central series, upper central series, or lower central series.

Moral: the adjoint map is nilpotent.

Lemma: For G a finite group, TFAE:

- G is nilpotent
- Normalizers grow (i.e. $H < N_G(H)$ whenever H is proper)
- Every Sylow- p subgroup is normal
- G is the direct product of its Sylow p -subgroups
- Every maximal subgroup is normal
- G has a terminating *Lower Central Series*
- G has a terminating *Upper Central Series*

Lemmas:

- G nilpotent $\implies G$ solvable
- Nilpotent groups satisfy the 2 out of 3 property.
- G has normal subgroups of order d for *every* d dividing $|G|$
- G nilpotent $\implies Z(G) \neq 0$
- Abelian \implies nilpotent
- p -groups \implies nilpotent

1.9 Series of Groups

Definition: A **normal series** of a group G is a sequence $G \rightarrow G^1 \rightarrow G^2 \rightarrow \dots$ such that $G^{i+1} \trianglelefteq G_i$ for every i .

Definition A **composition series** of a group G is a finite normal series such that G^{i+1} is a *maximal proper* normal subgroup of G^i .

Theorem (Jordan-Hölder): Any two composition series of a group have the same length and isomorphic factors (up to permutation).¹

Definition A **derived series** of a group G is a normal series $G \rightarrow G^1 \rightarrow G^2 \rightarrow \dots$ where $G^{i+1} = [G^i, G^i]$ is the commutator subgroup.

The derived series terminates iff G is *solvable*.

Definition: A **central series** for a group G is a terminating normal series $G \rightarrow G^1 \rightarrow \dots \rightarrow \{e\}$ such that each quotient is **central**, i.e. $[G, G^i] \leq G^{i-1}$ for all i .

Definition: A **lower central series** is a terminating normal series $G \rightarrow G^1 \rightarrow \dots \rightarrow \{e\}$ such that $G^{i+1} = [G^i, G]$

Moral: Iterate the adjoint map $[\cdot, G]$.

G is nilpotent \iff the LCS terminates.

Definition: An **upper central series** is a terminating normal series $G \rightarrow G^1 \rightarrow \dots \rightarrow \{e\}$ such that $G^1 = Z(G)$ and G^{i+1} is defined such that $G^{i+1}/G^i = Z(G^i)$.

Moral: Iterate taking “higher centers”.

2 Rings

2.1 Definitions

Definition: A ring R is **simple** iff every ideal $I \trianglelefteq R$ is either 0 or R .

Definition: An element $r \in R$ is **irreducible** iff $r = ab \implies a$ is a unit or b is a unit.

Definition: An element $r \in R$ is **prime** iff $ab \mid r \implies a \mid r$ or $b \mid r$ whenever a, b are nonzero and not units.

Definition: \mathfrak{p} is a **prime ideal** $\iff ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition: $\text{Spec}(R) = \{\mathfrak{p} \trianglelefteq R \mid \mathfrak{p} \text{ is prime}\}$ is the **spectrum** of R .

Definition: \mathfrak{m} is **maximal** $\iff I \not\trianglelefteq R \implies I \subseteq \mathfrak{m}$.

Definition: $\text{Spec}_{\max}(R) = \{\mathfrak{m} \trianglelefteq R \mid \mathfrak{m} \text{ is maximal}\}$ is the **max-spectrum** of R .

Note: nonstandard notation / definition.

Lemmas (Quotienting):

- R/I is a domain $\iff I$ is prime,
- R/I is a field $\iff I$ is maximal.
- For R a PID, I is prime $\iff I$ is maximal.

Lemma (Characterizations of Rings):

- R a commutative division ring $\implies R$ is a field
- R a finite integral domain $\implies R$ is a field.
- \mathbb{F} a field $\implies \mathbb{F}[x]$ is a Euclidean domain.
- \mathbb{F} a field $\implies \mathbb{F}[x]$ is a PID.
- \mathbb{F} is a field $\iff \mathbb{F}$ is a commutative simple ring.
- R is a UFD $\iff R[x]$ is a UFD.
- R a PID $\implies R[x]$ is a UFD
- R a PID $\implies R$ Noetherian
- $R[x]$ a PID $\implies R$ is a field.

Lemma: Fields \subset Euclidean domains \subset PIDs \subset UFDs \subset Integral Domains \subset Rings

- A Euclidean Domain that is not a field: $\mathbb{F}[x]$ for \mathbb{F} a field
– *Proof:* Use previous lemma, and x is not invertible
- A PID that is not a Euclidean Domain: $\mathbb{Z}\left[\frac{1 + \sqrt{-19}}{2}\right]$.
– *Proof:* complicated.
- A UFD that is not a PID: $\mathbb{F}[x, y]$.
– *Proof:* $\langle x, y \rangle$ is not principal
- An integral domain that is not a UFD: $\mathbb{Z}[\sqrt{-5}]$
– *Proof:* $(2 + \sqrt{-5})(2 - \sqrt{-5}) = 9 = 3 \cdot 3$, where all factors are irreducible (check norm).
- A ring that is not an integral domain: $\mathbb{Z}/(4)$
– *Proof:* $2 \pmod{4}$ is a zero divisor.

Lemma: In R a UFD, an element $r \in R$ is prime $\iff r$ is irreducible.

Note: For R an integral domain, prime \implies irreducible, but generally not the converse.

Example of a prime that is not irreducible: $x^2 \pmod{(x^2 + x)} \in \mathbb{Q}[x]/(x^2 + x)$. Check that x is prime directly, but $x = x \cdot x$ and x is not a unit.

Example of an irreducible that is not prime: $3 \in \mathbb{Z}[\sqrt{-5}]$. Check norm to see irreducibility, but $3 \mid 9 = (2 + \sqrt{-5})(2 - \sqrt{-5})$ and doesn't divide either factor.

Lemma: If R is a PID, then every element in R has a unique prime factorization.

Definition: A nonzero unital ring R is **semisimple** iff $R \cong \bigoplus_{i=1}^n M_i$ with each M_i a simple module.

Theorem (Artin-Wedderburn): If R is a nonzero, unital, *semisimple* ring then $R \cong \bigoplus_{i=1}^m \text{Mat}(n_i, D_i)$, a finite sum of matrix rings over division rings.

Corollary: If M is a simple ring over R a division ring, the M is isomorphic to a matrix ring.

2.2 Nontrivial Properties

Lemma: Every $a \in R$ for a finite ring is either a unit or a zero divisor.

Proof: Let $a \in R$ and define $\phi(x) = ax$. If ϕ is injective, then it is surjective, so $1 = ax$ for some $x \implies x^{-1} = a$. Otherwise, $ax_1 = ax_2$ with $x_1 \neq x_2 \implies a(x_1 - x_2) = 0$ and $x_1 - x_2 \neq 0$, so a is a zero divisor.

2.3 Ideals

2.3.1 Maximal and Prime Ideals

Lemma: Maximal \implies prime, but generally not the converse.

Counterexample: $(0) \in \mathbb{Z}$ is prime since \mathbb{Z} is a domain, but not maximal since it is properly contained in any other ideal.

Proof: Suppose \mathfrak{m} is maximal, $ab \in \mathfrak{m}$, and $b \notin \mathfrak{m}$. Then there is a containment of ideals $\mathfrak{m} \subsetneq \mathfrak{m} + (b) \implies \mathfrak{m} + (b) = R$.
So

$$1 = m + rb \implies a = am + r(ab),$$

but $am \in \mathfrak{m}$ and $ab \in \mathfrak{m} \implies a \in \mathfrak{m}$. ■

Lemma: If x is not a unit, then x is contained in some maximal ideal \mathfrak{m} .

Proof: Zorn's lemma.

Lemma: R/\mathfrak{m} is a field $\iff \mathfrak{m}$ is maximal.

Lemma: R/\mathfrak{p} is an integral domain $\iff \mathfrak{p}$ is prime.

2.3.2 Nilradical and Jacobson Radical

Definition: $\mathfrak{N} := \left\{ x \in R \mid x^n = 0 \text{ for some } n \right\}$ is the **nilradical** of R .

Lemma: The nilradical is the intersection of all **prime** ideals, i.e.

$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p}$$

Proof:

$$\mathfrak{N} \subseteq \bigcap \mathfrak{p}: x \in \mathfrak{N} \implies x^n = 0 \in \mathfrak{p} \implies x \in \mathfrak{p} \text{ or } x^{n-1} \in \mathfrak{p}.$$

$\mathfrak{N}^c \subseteq \bigcup \mathfrak{p}^c$: Define $S = \{I \trianglelefteq R \mid a^n \notin I \text{ for any } n\}$. Then apply Zorn's lemma to get a maximal ideal \mathfrak{m} , and maximal \implies prime.

Lemma: $R/\mathfrak{N}(R)$ has no nonzero nilpotent elements.

Proof:

$$\begin{aligned} a + \mathfrak{N}(R) \text{ nilpotent} &\implies (a + \mathfrak{N}(R))^n := a^n + \mathfrak{N}(R) = \mathfrak{N}(R) \\ &\implies a^n \in \mathfrak{N}(R) \\ &\implies \exists \ell \text{ such that } (a^n)^\ell = 0 \\ &\implies a \in \mathfrak{N}(R). \end{aligned}$$

Definition: The **Jacobson radical** is the intersection of all **maximal** ideals, i.e.

$$J(R) = \bigcap_{\mathfrak{m} \in \text{Spec}_{\max}} \mathfrak{m}$$

Lemma: $\mathfrak{N}(R) \subseteq J(R)$.

Proof: Maximal \implies prime, and so if x is in every prime ideal, it is necessarily in every maximal ideal as well.

2.3.3 Zorn's Lemma

Lemma: A field has no nontrivial proper ideals.

Lemma: If $I \trianglelefteq R$ is a proper ideal $\iff I$ contains no units.

$$\text{Proof: } r \in R^\times \bigcap I \implies r^{-1}r \in I \implies 1 \in I \implies x \cdot 1 \in I \quad \forall x \in R.$$

Lemma: If $I_1 \subseteq I_2 \subseteq \dots$ are ideals then $\bigcup_j I_j$ is an ideal.

Example Application of Zorn's Lemma: Every proper ideal is contained in a maximal ideal.

Proof: Let $0 < I < R$ be a proper ideal, and consider the set

$$S = \{J \mid I \subseteq J < R\}.$$

Note $I \in S$, so S is nonempty. The claim is that S contains a maximal element M .

S is a poset, ordered by set inclusion, so if we can show that every chain has an upper bound, we can apply Zorn's lemma to produce M .

Let $C \subseteq S$ be a chain in S , so $C = \{C_1 \subseteq C_2 \subseteq \dots\}$ and define $\hat{C} = \bigcup_i C_i$.

\hat{C} is an upper bound for C :

This follows because every $C_i \subseteq \hat{C}$.

\hat{C} is in S :

Use the fact that $I \subseteq C_i < R$ for every C_i and since no C_i contains a unit, \hat{C} doesn't contain a unit, and is thus proper. ■

3 Fields

Let k denote a field.

Lemmas:

- The characteristic of \mathbb{F} is either 0 or p a prime.
- All fields are simple rings
- Any homomorphism of fields is either 0 or injective
- If L/k is algebraic, then $\min(\alpha, L)$ divides $\min(\alpha, k)$.

Lemma: Every finite extension is algebraic.

Eisenstein's Criterion: If $f(x) = \sum_{i=0}^n \alpha_i x^i \in \mathbb{Q}[x]$ and $\exists p$ such that

- p divides every coefficient *except* a_n and
- p^2 does not divide a_0 ,

then f is irreducible.

Definition: For R a UFD, a polynomial $p \in R[x]$ is **primitive** iff the greatest common divisors of its coefficients is a unit.

Gauss' Lemma: Let R be a UFD and F its field of fractions. Then a primitive $p \in R[x]$ is irreducible in $R[x] \iff p$ is irreducible in $F[x]$.

Corollary: A primitive polynomial $p \in \mathbb{Q}[x]$ is irreducible iff p is irreducible in $\mathbb{Z}[x]$.

3.1 Finite Fields

Definition: The prime subfield of a field F is the subfield generated by 1.

Lemma (Characterization of Prime Subfields): The prime subfield of any field is isomorphic to either \mathbb{Q} or \mathbb{F}_p for some p .

Lemma ("Freshman's Dream"): If $\text{char } k = p$ then $(a + b)^p = a^p + b^p$ and $(ab)^p = a^p b^p$.

Theorem (Construction of Finite Fields): $\mathbb{GF}(p^n) \cong \frac{\mathbb{F}_p[x]}{(f)}$ where $f \in \mathbb{F}_p[x]$ is any irreducible of degree n , and $\mathbb{GF}(p^n) \cong \mathbb{F}[\alpha] \cong \text{span}_{\mathbb{F}} \{1, \alpha, \dots, \alpha^{n-1}\}$ for any root α of f .

Lemma (Prime Subfields of Finite Fields): Every finite field F is isomorphic to a unique field of the form $\mathbb{GF}(p^n)$ and if $\text{char } F = p$, it has prime subfield \mathbb{F}_p .

Lemma (Containment of Finite Fields): $\mathbb{GF}(p^\ell) \leq \mathbb{GF}(p^k) \iff \ell$ divides k .

Lemma (Identification of Finite Fields as Splitting Fields): $\mathbb{GF}(p^n)$ is the splitting field of $\rho(x) = x^{p^n} - x$, and the elements are exactly the roots of ρ .

Every element is a root by Cauchy's theorem, and the p^n roots are distinct since its derivative is identically -1 .

Lemma (Splits Product of Irreducibles): Let $\rho_n := x^{p^n} - x$. Then $f(x) \mid \rho_n(x) \iff \deg f \mid n$ and f is irreducible.

Corollary: $x^{p^n} - x = \prod f_i(x)$ over all irreducible monic $f_i \in \mathbb{F}_p[x]$ of degree d dividing n .

Proof:

\Leftarrow : Suppose f is irreducible of degree d . Then $f \mid x^{p^d} - x$ (consider $F[x]/\langle f \rangle$) and $x^{p^d} - x \mid x^{p^n} - x \iff d \mid n$.
 \Rightarrow :

- $\alpha \in \mathbb{GF}(p^n) \iff \alpha^{p^n} - \alpha = 0$, so every element is a root of ϕ_n and $\deg \min(\alpha, \mathbb{F}_p) \mid n$ since $\mathbb{F}_p(\alpha)$ is an intermediate extension.
- So if f is an irreducible factor of ϕ_n , f is the minimal polynomial of some root α of ϕ_n , so $\deg f \mid n$.
 $\phi_n'(x) = p^n x^{p^n-1} \neq 0$, so ϕ_n has distinct roots and thus no repeated factors. So ϕ_n is the product of all such irreducible f .

Lemma: No finite field is algebraically closed.

3.2 Galois Theory

Definition: A field extension L/k is **algebraic** iff every $\alpha \in L$ is the root of some polynomial $f \in k[x]$.

Definition: Let L/k be a finite extension. Then TFAE:

- L/k is **normal**.
- Every irreducible $f \in k[x]$ that has one root in L has *all* of its roots in L
 – i.e. every polynomial splits into linear factors
- Every embedding $\sigma : L \hookrightarrow \bar{k}$ that is a lift of the identity on k satisfies $\sigma(L) = L$.
- If L is separable: L is the splitting field of some irreducible $f \in k[x]$.

Definition: Let L/k be a field extension, $\alpha \in L$ be arbitrary, and $f(x) := \min(\alpha, k)$. TFAE:

- L/k is **separable**
- f has no repeated factors/roots
- $\gcd(f, f') = 1$, i.e. f is coprime to its derivative
- $f' \neq 0$

Lemma: If $\text{char } k = 0$ or k is finite, then every *algebraic* extension L/k is separable.

Definition: $\text{Aut}(L/k) = \left\{ \sigma : L \rightarrow L \mid \sigma|_k = \text{id}_k \right\}$.

Lemma: If L/k is algebraic, then $\text{Aut}(L/k)$ permutes the roots of irreducible polynomials.

Lemma: $|\text{Aut}(L/k)| \leq [L : k]$ with equality precisely when L/k is normal.

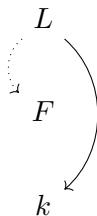
Definition: If L/k is Galois, we define $\text{Gal}(L/k) := \text{Aut}(L/k)$.

3.2.1 Lemmas About Towers

Let $L/F/k$ be a finite tower of field extensions

- Multiplicativity: $[L : k] = [L : F][F : k]$

- L/k normal/algebraic/Galois $\implies L/F$ normal/algebraic/Galois.
 - *Proof (normal)*: $\min(\alpha, F) \mid \min(\alpha, k)$, so if the latter splits in L then so does the former.
 - *Corollary*: $\alpha \in L$ algebraic over $k \implies \alpha$ algebraic over F .
 - *Corollary*: E_1/k normal and E_2/k normal $\implies E_1E_2/k$ normal and $E_1 \cap E_2/k$ normal.



- F/k algebraic and L/F algebraic $\implies L/k$ algebraic.
- If L/k is algebraic, then F/k separable and L/F separable $\iff L/k$ separable



- F/k Galois and L/F Galois $\implies F/k$ Galois **only if** $\text{Gal}(L/F) \trianglelefteq \text{Gal}(L/k)$
 - $\implies \text{Gal}(F/k) \cong \frac{\text{Gal}(L/k)}{\text{Gal}(L/F)}$



Common Counterexamples:

- $\mathbb{Q}(\zeta_3, 2^{1/3})$ is normal but $\mathbb{Q}(2^{1/3})$ is not since the irreducible polynomial $x^3 - 2$ has only one root in it.

Definition (Characterizations of Galois Extensions): Let L/k be a finite field extension. TFAE:

- L/k is **Galois**
- L/k is finite, normal, and separable.
- L/k is the splitting field of a separable polynomial
- $|\text{Aut}(L/k)| = [L : k]$
- The fixed field of $\text{Aut}(L/k)$ is exactly k .

Fundamental Theorem of Galois Theory: Let L/k be a Galois extension, then there is a

correspondence:

$$\begin{aligned} \{\text{Subgroups } H \leq \text{Gal}(L/k)\} &\iff \left\{ \begin{array}{l} \text{Fields } F \text{ such} \\ \text{that } L/F/k \end{array} \right\} \\ H &\rightarrow \{E^H := \text{The fixed field of } H\} \\ \left\{ \text{Gal}(L/F) := \left\{ \sigma \in \text{Gal}(L/k) \mid \sigma(F) = F \right\} \right\} &\leftarrow F. \end{aligned}$$

- This is contravariant with respect to subgroups/subfields.
- $[F : k] = [G : H]$, so degrees of extensions over the base field correspond to indices of subgroups.
- $[K : F] = |H|$
- L/F is Galois and $\text{Gal}(K/F) = H$
- F/k is Galois $\iff H$ is normal, and $\text{Gal}(F/k) = \text{Gal}(L/k)/H$.
- The compositum $F_1 F_2$ corresponds to $H_1 \cap H_2$.
- The subfield $F_1 \cap F_2$ corresponds to $H_1 H_2$.

3.2.2 Examples

1. $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}/(n)^\times$ and is generated by maps of the form $\zeta_n \mapsto \zeta_n^j$ where $(j, n) = 1$.
I.e., the following map is an isomorphism:

$$\begin{aligned} \mathbb{Z}/(n)^\times &\rightarrow \text{Gal}(\mathbb{Q}(\zeta_n), \mathbb{Q}) \\ r \pmod n &\mapsto (\phi_r : \zeta_n \mapsto \zeta_n^r). \end{aligned}$$

2. $\text{Gal}(\mathbb{GF}(p^n)/\mathbb{GF}(p)) \cong \mathbb{Z}/(n)$, a cyclic group generated by powers of the Frobenius automorphism:

$$\begin{aligned} \varphi_p : \mathbb{GF}(p^n) &\rightarrow \mathbb{GF}(p^n) \\ x &\mapsto x^p. \end{aligned}$$

Lemma: Every quadratic extension is Galois.

Lemma: If K is the splitting field of an irreducible polynomial of degree n , then $\text{Gal}(K/\mathbb{Q}) \leq S_n$ is a transitive subgroup.

Corollary: n divides the order $|\text{Gal}(K/\mathbb{Q})|$.

Definition: TFAE

- k is a **perfect** field.
- Every irreducible polynomial $p \in k[x]$ is separable
- Every finite extension F/k is separable.
- If $\text{char } k > 0$, the Frobenius is an automorphism of k .

Theorem:

- If $\text{char } k = 0$ or k is finite, then k is perfect.
- $k = \mathbb{Q}, \mathbb{F}_p$ are perfect, and any finite normal extension is Galois.
- Every splitting field of a polynomial over a perfect field is Galois.

Lemma (Composite Extensions): If F/k is finite and Galois and L/k is arbitrary, then FL/L is Galois and

$$\text{Gal}(FL/L) = \text{Gal}(F/F \cap L) \subset \text{Gal}(F/k).$$

3.3 Cyclotomic Polynomials

Definition: Let $\zeta_n = e^{2\pi i/n}$, then

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (k,n)=1}}^n (x - \zeta_n^k),$$

which is a product over primitive roots of unity.

Lemma: $\deg \Phi_n(x) = \phi(n)$ for ϕ the totient function.

Computing Φ_n :

1.

$$\Phi_n(z) = \prod_{d|n, d>0} (z^d - 1)^{\mu(\frac{n}{d})}$$

where

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \end{cases}$$

2.

$$x^n - 1 = \prod_{d|n} \Phi_d(x) \implies \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d < n}} \Phi_d(x)},$$

so just use polynomial long division.

Lemma:

$$\begin{aligned} \Phi_p(x) &= x^{p-1} + x^{p-2} + \cdots + x + 1 \\ \Phi_{2p}(x) &= x^{p-1} - x^{p-2} + \cdots - x + 1. \end{aligned}$$

Lemma:

$$k \mid n \implies \Phi_{nk}(x) = \Phi_n(x^k)$$

Definition: An extension F/k is **simple** if $F = k[\alpha]$ for a single element α .

Theorem (Primitive Element): Every finite separable extension is simple.

Corollary: $\mathbb{GF}(p^n)$ is a simple extension over \mathbb{F}_p .

4 Modules

4.1 General Modules

Definition: A module is **simple** iff it has no nontrivial proper submodules.

Definition: A **free** module is a module with a basis (i.e. a spanning, linearly independent set).

Example: $\mathbb{Z}/(6)$ is a \mathbb{Z} -module that is *not* free.

Definition: A module M is **projective** iff M is a direct summand of a free module $F = M \oplus \cdots$.

Free implies projective, but not the converse.

Definition: A sequence of homomorphisms $0 \xrightarrow{d_1} A \xrightarrow{d_2} B \xrightarrow{d_3} C \rightarrow 0$ is *exact* iff $\text{im } d_i = \ker d_{i+1}$.

Lemma: If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence, then

- C free \implies the sequence splits
- C projective \implies the sequence splits
- A injective \implies the sequence splits

Moreover, if this sequence splits, then $B \cong A \oplus C$.

4.2 Classification of Modules over a PID

Let M be a finitely generated modules over a PID R . Then there is an invariant factor decomposition

$$M \cong F \bigoplus R/(r_i) \quad \text{where } r_1 \mid r_2 \mid \cdots,$$

and similarly an elementary divisor decomposition.

4.3 Minimal / Characteristic Polynomials

Fix some notation:

$\min_A(x)$: The minimal polynomial of A

$\chi_A(x)$: The characteristic polynomial of A .

Definition: The minimal polynomial is the unique polynomial $\min_A(x)$ of minimal degree such that $\min_A(A) = 0$.

Definition: The **characteristic polynomial** of A is given by

$$\chi_A(x) = \det(A - xI) = \det(SNF(A - xI)).$$

Useful lemma: If A is upper triangular, then $\det(A) = \prod_i a_{ii}$

Theorem (Cayley-Hamilton): The minimal polynomial divides the characteristic polynomial, and in particular $\chi_A(A) = 0$.

Lemma: Writing

$$\begin{aligned}\min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i}\end{aligned}$$

- $a_i \leq b_i$
- The roots both polynomials are precisely the eigenvalues of A .

Proof: By Cayley-Hamilton, \min_A divides χ_A . Every λ_i is a root of μ_M :

Let $(\mathbf{v}_i, \lambda_i)$ be a nontrivial eigenpair. Then by linearity,

$$\min_A(\lambda_i)\mathbf{v}_i = \min_A(A)\mathbf{v}_i = \mathbf{0},$$

which forces $\min_A(\lambda_i) = 0$.

Definition: Two matrices A, B are **similar** (i.e. $A = PBP^{-1}$) $\iff A, B$ have the same Jordan Canonical Form (JCF).

Definition: Two matrices A, B are **equivalent** (i.e. $A = PBQ$) \iff

- They have the same rank,
- They have the same invariant factors, *and*
- They have the same (JCF)

Finding the minimal polynomial:

Let $m(x)$ denote the minimal polynomial A .

1. Find the characteristic polynomial $\chi(x)$; this annihilates A by Cayley-Hamilton. Then $m(x) \mid \chi(x)$, so just test the finitely many products of irreducible factors.
2. Pick any \mathbf{v} and compute $T\mathbf{v}, T^2\mathbf{v}, \dots, T^k\mathbf{v}$ until a linear dependence is introduced. Write this as $p(T) = 0$; then $\min_A(x) \mid p(x)$.

Definition: Given a monic $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + x^n$, the **companion matrix** of p is given by

$$C_p := \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}.$$

4.4 Canonical Forms

4.4.1 Rational Canonical Form

Corresponds to the **Invariant Factor Decomposition** of T .

Lemma: $RCF(A)$ is a block matrix where each block is the companion matrix of an invariant factor of A .

Derivation:

- Let $k[x] \curvearrowright V$ using T , take invariant factors a_i ,
- Note that $T \curvearrowright V$ by multiplication by x
- Write $\bar{x} = \pi(x)$ where $F[x] \xrightarrow{\pi} F[x]/(a_i)$; then $\text{span}\{\bar{x}\} = F[x]/(a_i)$.
- Write $a_i(x) = \sum b_i x^i$, note that $V \rightarrow F[x]$ pushes $T \curvearrowright V$ to $T \curvearrowright k[x]$ by multiplication by \bar{x}
- WRT the basis \bar{x} , T then acts via the companion matrix on this summand.
- Each invariant factor corresponds to a block of the RCF.

4.4.2 Jordan Canonical Form

Corresponds to the **Elementary Divisor Decomposition** of T .

Lemma: The elementary divisors of A are the minimal polynomials of the Jordan blocks.

Lemma: Writing

$$\begin{aligned}\min_A(x) &= \prod (x - \lambda_i)^{a_i} \\ \chi_A(x) &= \prod (x - \lambda_i)^{b_i}\end{aligned}$$

- $a_i \leq b_i$
- a_i tells you the size of the **largest** Jordan block associated to λ_i ,
- b_i is the **sum of sizes** of all Jordan blocks associated to λ_i
- $\dim E_{\lambda_i}$ is the **number of Jordan blocks** associated to λ_i

4.5 Using Canonical Forms

Lemma: The characteristic polynomial is the *product of the invariant factors*, i.e.

$$\chi_A(x) = \prod_{j=1}^n f_j(x).$$

Lemma: The minimal polynomial of A is the *invariant factor of highest degree*, i.e.

$$\min_A(x) = f_n(x).$$

Lemma: For a linear operator on a vector space of nonzero finite dimension, TFAE:

- The minimal polynomial is equal to the characteristic polynomial.
- The list of invariant factors has length one.
- The Rational Canonical Form has a single block.

- The operator has a matrix similar to a companion matrix.
- There exists a *cyclic vector* \mathbf{v} such that $\text{span}_k \{T^j \mathbf{v} \mid j = 1, 2, \dots\} = V$.
- T has $\dim V$ distinct eigenvalues

4.6 Diagonalizability

Notation: A^* denotes the conjugate transpose of A .

Lemma: Let V be a vector space over k an algebraically closed and $A \in \text{End}(V)$. Then if $W \subseteq V$ is an invariant subspace, so $A(W) \subseteq W$, the A has an eigenvector in W .

Theorem (The Spectral Theorem):

1. Hermitian matrices (i.e. $A^* = A$) are diagonalizable over \mathbb{C} .
2. Symmetric matrices (i.e. $A^t = A$) are diagonalizable over \mathbb{R} .

Proof: Suppose A is Hermitian. Since V itself is an invariant subspace, A has an eigenvector $\mathbf{v}_1 \in V$. Let $W_1 = \text{span}_k \{\mathbf{v}_1\}^\perp$. Then for any $\mathbf{w}_1 \in W_1$,

$$\langle \mathbf{v}_1, A\mathbf{w}_1 \rangle = \langle A\mathbf{v}_1, \mathbf{w}_1 \rangle = \lambda \langle \mathbf{v}_1, \mathbf{w}_1 \rangle = 0,$$

so $A(W_1) \subseteq W_1$ is an invariant subspace, etc.

Suppose now that A is symmetric. Then there is an eigenvector of norm 1, $\mathbf{v} \in V$.

$$\lambda = \lambda \langle \mathbf{v}, \mathbf{v} \rangle = \langle A\mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{v}, A\mathbf{v} \rangle = \bar{\lambda} \implies \lambda \in \mathbb{R}.$$

Lemma: $\{A_i\}$ pairwise commute \iff they are all simultaneously diagonalizable.

Proof: By induction on number of operators

- A_n is diagonalizable, so $V = \bigoplus E_i$ a sum of eigenspaces
- Restrict all $n - 1$ operators A to E_n .
- The commute in V so they commute in E_n
- **(Lemma)** They were diagonalizable in V , so they're diagonalizable in E_n
- So they're simultaneously diagonalizable by I.H.
- But these eigenvectors for the A_i are all in E_n , so they're eigenvectors for A_n too.
- Can do this for each eigenspace. ■

Full details here

Theorem (Characterizations of Diagonalizability)

M is diagonalizable over $\mathbb{F} \iff \min_M(x, \mathbb{F})$ splits into distinct linear factors over \mathbb{F} , or equivalently iff all of the roots of \min_M lie in \mathbb{F} .

Proof: \implies : If \min_A factors into linear factors, so does each invariant factor, so every elementary divisor is linear and $JCF(A)$ is diagonal.

\impliedby : If A is diagonalizable, every elementary divisor is linear, so every invariant factor factors into linear pieces. But the minimal polynomial is just the largest invariant factor.

4.7 Matrix Counterexamples

1. A matrix that is:

- Not diagonalizable over \mathbb{R} but diagonalizable over \mathbb{C}
- No eigenvalues in \mathbb{R} but distinct eigenvalues over \mathbb{C}
- $\min_M(x) = \chi_M(x) = x^2 + 1$

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \sim \left[\begin{array}{c|c} -1\sqrt{-1} & 0 \\ \hline 0 & 1\sqrt{-1} \end{array} \right].$$

2.

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Not diagonalizable over \mathbb{C}
 - Eigenvalues $[1, 1]$ (repeated, multiplicity 2)
 - $\min_M(x) = \chi_M(x) = x^2 - 2x + 1$
3. Non-similar matrices with the same characteristic polynomial

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

4. A full-rank matrix that is not diagonalizable:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Matrix roots of unity:

$$\sqrt{I_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\sqrt{-I_2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

4.8 Miscellaneous

Lemma: $I \trianglelefteq R$ is a free R -module iff I is a principal ideal.

Proof: \implies :

Suppose I is free as an R -module, and let $B = \{\mathbf{m}_j\}_{j \in J} \subseteq I$ be a basis so we can write $M = \langle B \rangle$.

Suppose that $|B| \geq 2$, so we can pick at least 2 basis elements $\mathbf{m}_1 \neq \mathbf{m}_2$, and consider

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1,$$

which is also an element of M .

Since R is an integral domain, R is commutative, and so

$$\mathbf{c} = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_2\mathbf{m}_1 = \mathbf{m}_1\mathbf{m}_2 - \mathbf{m}_1\mathbf{m}_2 = \mathbf{0}_M$$

However, this exhibits a linear dependence between \mathbf{m}_1 and \mathbf{m}_2 , namely that there exist $\alpha_1, \alpha_2 \neq 0_R$ such that $\alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 = \mathbf{0}_M$; this follows because $M \subset R$ means that we can take $\alpha_1 = -m_2, \alpha_2 = m_1$. This contradicts the assumption that B was a basis, so we must have $|B| = 1$ and so $B = \{\mathbf{m}\}$ for some $\mathbf{m} \in I$. But then $M = \langle B \rangle = \langle \mathbf{m} \rangle$ is generated by a single element, so M is principal.

\impliedby :

Suppose $M \trianglelefteq R$ is principal, so $M = \langle \mathbf{m} \rangle$ for some $\mathbf{m} \neq \mathbf{0}_M \in M \subset R$.

Then $x \in M \implies x = \alpha\mathbf{m}$ for some element $\alpha \in R$ and we just need to show that $\alpha\mathbf{m} = \mathbf{0}_M \implies \alpha = 0_R$ in order for $\{\mathbf{m}\}$ to be a basis for M , making M a free R -module.

But since $M \subset R$, we have $\alpha, m \in R$ and $\mathbf{0}_M = 0_R$, and since R is an integral domain, we have $\alpha m = 0_R \implies \alpha = 0_R$ or $m = 0_R$.

Since $m \neq 0_R$, this forces $\alpha = 0_R$, which allows $\{\mathbf{m}\}$ to be a linearly independent set and thus a basis for M as an R -module. ■