

Math 200A Homework Question Compendium

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Contents

| | | |
|---|-------|---|
| 1 | One | 1 |
| 2 | Two | 2 |
| 3 | Three | 3 |
| 4 | Four | 4 |
| 5 | Five | 5 |
| 6 | Six | 6 |
| 7 | Seven | 7 |
| 8 | Eight | 7 |

1 One

1. Given: $\forall x \in G, x^2 = e$

Show: $G \in \mathbf{Ab}$

2. Given: $|G| < \infty, |G| \equiv 0 \pmod{2}$

Show: $\exists g \in G \ni o(g) = 2$

3. Given: $G \in \mathbf{Ab}$

Show: $T(G) \leq G$ (where $T(G) = \{g \in G : |g| < \infty\}$)

4. Show: Every finite group is finitely generated.

- Show: \mathbb{Z} is finitely generated
- Show: $H \leq (\mathbb{Q}, +) \implies H$ is cyclic
- Show: \mathbb{Q} is not finitely generated

5. Show: \mathbb{Q}/\mathbb{Z} has, for each coset, exactly one representative in $[0, 1) \cap \mathbb{Q}$

- Show: Every element of \mathbb{Q}/\mathbb{Z} has finite order.

- Show: There are elements in \mathbb{Q}/\mathbb{Z} of arbitrarily large order.
 - Show: $\mathbb{Q}/\mathbb{Z} = T(\mathbb{R}/\mathbb{Z})$
 - Show: $\mathbb{Q}/\mathbb{Z} \cong \mathbb{C}^x$
6. Given: $G/Z(G)$ is cyclic
Show: G is abelian
7. Given: $H \trianglelefteq G, K \trianglelefteq G, H \cap K = e$
Show: $\forall h \in H, \forall k \in K, hk = kh$
8. Given: $|G| < \infty, H \leq G, N \trianglelefteq G, (|H|, [G : N]) = 1$
Show: $H \leq N$
9. Given: $|G| < \infty, N \trianglelefteq G, (|N|, [G : N]) = 1$
Show: N is the unique subgroup of order $|N|$

2 Two

1. Given: For every triplet in G , two elements commute
Show: G is abelian
2. Given: $H_1, H_2, H_3 \leq G, G = H_1 \cup H_2$
Show: $G = H_1 \vee G = H_2$
3. Given: $G = H_1 \cup H_2 \cup H_3, G$ finite
Show: $G = H_i \vee \forall i, [G : H_i] = 2$
4. Show: TFAE; $\text{clos}(H)$ satisfies:
- It is the smallest normal subgroup of G containing H .
 - It is the subgroup generated by all conjugates of H .
 - $\text{clos}(H) = \bigcap_{H \leq N \trianglelefteq G} N$
 - If $\phi : G \rightarrow X$ and $\phi(H) = e$, then ϕ factors through $G/\text{clos}(H)$
5. Given: $H, K \trianglelefteq HK \leq G$
Show:

$$\frac{HK}{H \cap K} \cong \frac{HK}{H} \times \frac{HK}{K}$$

6. Given: $H \leq G, N \trianglelefteq G, H \in \text{Hall}(G)$
Show:

$$H \cap N \in \text{Hall}(N) \text{ and } \frac{HN}{N} \in \text{Hall}\left(\frac{G}{N}\right)$$

7. Given: $|G| = n, G$ cyclic, $\sigma_i : G \rightarrow G \ni x \mapsto x^i$

- Show $\sigma_i \in \text{End}(G)$
- Show $\sigma_i \in \text{Aut}(G)$ iff $(i, n) = 1$
- $\sigma_i = \sigma_j$ iff $i = j \pmod n$
- $\tau \in \text{Aut}(G) \implies \exists i \ni \tau = \sigma_i$
- $\sigma_i \circ \sigma_j = \sigma_{ij}$
- The map

$$\begin{aligned} \psi : \mathbb{Z}_n^\times &\rightarrow \text{Aut}(G) \\ i &\mapsto \sigma_i \end{aligned}$$

is an isomorphism.

8. Given: G is cyclic

Show: $\text{Aut}(G)$ is abelian of order $\phi(n)$

9. Show: $D_\infty \cong \langle a, b \mid b^2 = e, ba = a^{-1}b \rangle$
10. Show: $Q_8 \cong \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1} \rangle$
11. Show: $\langle x, y \mid xy^2 = y^3, yx^2 = x^3y \rangle = \langle e \rangle$

3 Three

1. Given: $G \curvearrowright X$ transitively, $H \trianglelefteq G$

- Show: $H \curvearrowright X$, but possibly not transitively
- Show: G acts transitively on $\{\mathcal{O}_h : h \in H\}$
- Show: $\forall i, j, |\mathcal{O}_{h_i}| = |\mathcal{O}_{h_j}|$
- Given: $x \in \mathcal{O}_h$ Show: $|\mathcal{O}_h| = |H : H \cap G_x|$
- Show: $|\{\mathcal{O}_h\}_{h \in H}| = [G : HG_x]$

2. Given: \mathcal{K} a conjugacy class in S_n , $\{\mathcal{O}_s : s \in S_n\}$ orbits of an A_n -action on S_n

Show: $\mathcal{K} = \mathcal{O}_s \vee \mathcal{K} = \mathcal{O}_{s_i} \cup \mathcal{O}_{s_j}$

Show: Case 2 occurs iff $\{k_i\}$, the cycle lengths in disjoint cycle form, are odd and distinct

3. i: $|G| < \infty, H < G$

- Show: $\{gHg^{-1} : g \in G\} = [G : N_G(H)]$
- Show:

$$G \neq \bigcup_{g \in G} gHg^{-1}$$

4. Prove Cauchy's Theorem. Given $p \mid o(G) < \infty$

$$X = \left\{ (a_i)_{i=1}^p \in G^p \ni \prod_{i=1}^p a_i = e \right\}$$

- Show: $(a_1 a_2 \cdots a_p) = e \implies (a_2 a_3 \cdots a_p a_1) = e$
 - Show: $(\mathbb{Z}_p, +) \curvearrowright X$ and $\bar{1} \curvearrowright (a_1 a_2 \cdots a_p) = (a_2 a_3 \cdots a_p a_1)$
 - Show: $|X| = |G|^{p-1}$
 - Show: $\{\mathcal{O}_x : |\mathcal{O}_x| = 1\} > 1$ and $\exists a \in G \ni a^p = e$
5. Given: $G \curvearrowright X$, $|G| < \infty$, $1 < |X| < \infty$
- Show: $\exists g \in G$ such that $\forall x \in X, g \curvearrowright x \neq x$
 - Show: This holds if $|G| = \infty$, but not if $|X| = \infty$ as well.
6. Given: $H \leq G$. Show: $\text{core}(H)$ is
- The largest $N \trianglelefteq G, N \subseteq H$
 - Generated by all normal subgroups contained in H
 - Given by $\bigcap_{g \in G} gHg^{-1}$
 - The kernel of $G \curvearrowright \frac{G}{H} \ni x \curvearrowright gH = (xg)H$
7. Given: $[H : G] = n < \infty$
- Show: $[\text{core}(H) : G]$ divides $n!$
 - Show: G simple $\implies o(G) \mid n!$ and $|G| < \infty$
8. Given: A_n is simple for $n \geq 5$
- Show: $\nexists H \in A_n \ni [H : A_n] < n$
- Show: $\exists H [H : A_n] = n$
9. Given: r beads of n colors
- Show: How many distinct circular bracelets can be made.

4 Four

- Given: $H \text{ char } G$
Show: $H \trianglelefteq G$
- Given: $H \text{ char } K \trianglelefteq G$
Show: $H \trianglelefteq G$
- Given: $K = \langle k \rangle \trianglelefteq G$
Show: $H \leq K \implies H \trianglelefteq G$
- Show $H \trianglelefteq K \trianglelefteq G \not\implies H \trianglelefteq G$
- Given: $P \leq H \leq K \leq G < \infty, P \in \text{Syl}_p(G)$
Show: $P, H \trianglelefteq K \implies P \trianglelefteq K$
- Show: $N_G(N_G(P)) = N_G(P)$
- Given: $\sigma \in \text{Aut}(G)$
Show: $\sigma \text{Inn}(G) \sigma^{-1} = \text{Inn}(G)$ iff $\forall g \in G, g^{-1} \sigma(g) \in \mathbb{Z}(G)$

8. Show: $\text{Inn}(G) \text{ char } \text{Aut}(G)$
9. Given: $H \subseteq G, P \in \text{Syl}_p(G)$
 Show: $\exists g \in G \ni gPg^{-1} \in \text{Syl}_p(H)$
 Given: $H \trianglelefteq G$
 Show: $P \cap H \in \text{Syl}_p(H)$
 Given: $P \trianglelefteq G$
 Show: $P \cap H \in \text{Syl}_p(H)$ and $|\text{Syl}_p(H)| = 1$
10. Given: $|G| = pqr, p < q < r$
 Show: $\exists P_i \in \text{Syl}_i(G) \trianglelefteq G$
11. Given: $|G| = 595$
 Show: All sylow subgroups are normal
12. Given: $|G| = p(p+1)$
 Show: $\exists N \trianglelefteq G$ where $|N| = p$ or $p+1$

5 Five

1. Given: $G = H \rtimes_{\psi} K$

$$\begin{aligned}\psi : K &\rightarrow \text{Aut}(H) \\ k &\mapsto \psi(k)\end{aligned}$$

$$\theta \in \text{Aut}(H) \quad \rho : K \rightarrow K$$

$$\begin{aligned}\phi_{\theta} : \text{Aut}(H) &\rightarrow \text{Aut}(H) \\ \rho &\mapsto \theta \circ \rho \circ \theta^{-1}\end{aligned}$$

$$\begin{aligned}\psi_2 : K &\rightarrow \text{Aut}(H) \\ k &\mapsto (\phi_{\theta} \circ \psi)(k)\end{aligned}$$

$$\begin{aligned}\psi_3 : K &\rightarrow \text{Aut}(H) \\ k &\mapsto (\psi \circ \rho)(k)\end{aligned}$$

$$\text{Show: } H \rtimes_{\psi} K \cong H \rtimes_{\psi_2} K \cong H \rtimes_{\psi_3} K$$

2. Classify groups of order $pq, p < q, p \mid q-1$
3. Classify groups of order 20.
4. Classify groups of order 75.
5. Show: $|G| < 60 \implies G$ is not simple.

6. Show: $|G| < 60 \implies G$ is solvable
7. Given: $|G| < \infty$, $H \leq G$ maximal $\implies [G : H] = p$, a prime.
 - Show: $|G|$ is solvable
 - Given: $P \in \text{Syl}_p(G)$ and $\exists H \ni N_G(P) \leq H \leq G$
 - * Show: $[G : H] = 1 \pmod p$
 - Given: $p \mid o(G)$, the largest such prime
 - * Show: $\exists P \leq G \in \text{Syl}_p(G)$,
8. $|G| < \infty$
 - Given: G is characteristically simple
 - Show: $\exists H$ (simple) $\ni G \cong H^n$.
 - Show: Whether or not the converse holds
 - Given: $N \leq G$ minimal
 - Show: N is characteristically simple, $N \cong H^n$

6 Six

1. Given: G is nilpotent

Show: $H \leq G \implies H, G/H$ are nilpotent
2. Show: $G/Z(G)$ is nilpotent $\implies G$ is nilpotent
3. Given: $|G| < \infty$

Show: $|G|$ is nilpotent iff $a, b \in G, (a, b) = 1 \implies ab = ba$
4. Show: D_{2n} is nilpotent iff $n = 2^i$
5. Given: $|G| < \infty$
 - Show $\Phi(G)$ char G
 - Show $\Phi(G)$ is nilpotent
 - Given: $|P| = p^e$
 - Show: $P/\Phi(P)$ is an elementary abelian p-group
 - Show: $N \leq P, P/N$ is elementary abelian $\implies \Phi(P) \subseteq N$
6. Given: R a commutative ring, $x, y \in R$ nilpotent
 - Show: $x + y$ is nilpotent
 - Show: $\{x \in R : x \text{ is nilpotent}\} \leq R$
 - Given: $u \in R^\times, x \in R$ nilpotent
 - Show: $u + x \in R^\times$
 - Show: An counterexample to 1 when R is noncommutative.
7. Given: R a commutative ring, $R[[x]]$ its formal power series
 - Show that

$$\sum_{i=0}^{\infty} a_i x^i \in R[[x]]^\times \iff a_0 \in R^\times$$

- Show: R a domain $\implies R[[x]]$ a domain
- Given: R a field
 - Show that

$$I = \{r \in R[[x]] : r_0 = 0\}$$

is a maximal ideal of $R[[x]]$.

- Show: I is the unique maximal ideal
8. Given: R a commutative ring, G a finite group, RG a group ring.
- Given: $\mathcal{K} = \{k_1, k_2, \dots, k_m\}$ a conjugacy class in G
 - Show:

$$K = \sum_{i=1}^m k_i \in RG \implies K \in Z(RG)$$

- Given: $\mathcal{K}_1 \dots \mathcal{K}_r$ distinct conjugacy classes in G , $K_i = \sum_j k_j \ni k_j \in \mathcal{K}_i$
 - Show that

$$Z(RG) = \left\{ \sum a_l K_l : \forall 1 \leq l \leq r, a_l \in R \right\}$$

(All R -linear combinations of the \mathcal{K}_i)

9. Given: R a ring, $M_n(R)$ its matrix ring
- Given: $I \trianglelefteq R$ (two-sided)
 - Show: $M_n(I) \trianglelefteq M_n(R)$
 - Show:

$$\frac{M_n(R)}{M_n(I)} \cong M_n\left(\frac{R}{I}\right)$$

- Show: $\forall I_M \trianglelefteq M_n(R)$, I is of the form $M_n(I)$ for some $I \trianglelefteq R$
 - Show: R a division ring $\implies M_n(R)$ is a simple ring.

7 Seven

8 Eight