Title

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Contents

Prologue

0.1 References

- Gathmann's Algebraic Geometry notes[@AndreasGathmann515].

0.2 Notation

$k[x_1, \cdots, x_n]$	The polynomial ring in n indeterminates
$k(x_1, \cdots, x_n)$	The rational function field $\{f(\mathbf{x}) = p(\mathbf{x})/q(\mathbf{x}),\ p, q, \in k[x_1, \cdots, x_n]\}$
$V(J), V_a(J)$	The variety associated to an ideal $J \trianglelefteq k[x_1, \cdots, x_n]$
$I(S), I_a(S)$	The ideal associated to a subset $S \subseteq \mathbb{A}_k^n$
$A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$	The coordinate ring of a variety
$\mathcal{O}_X$	The structure sheaf $\left\{ f : U \to k \;\middle|\; \right\}.$

Lots of notation to fill in.

Algebra	Geometry
Radical ideals $J = \sqrt{J} \trianglelefteq k[x_1, \cdots, x_n]$	$V(J)$ the zero locus
$I(S)$ the ideal of a set	$S \subseteq \mathbb{A}^n$ a subset
$I + J$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$V \cap W$
$I \cap J, IJ$	$V(I) \cup V(J)$
$I(V) \cap I(W), \sqrt{I(V)I(W)}$	$V \cup W$
$I(V) : I(W)$	$\overline{V \setminus W}$

0.3 Summary of Important Concepts

- What is an affine variety?
- What is the coordinate ring of an affine variety?
- What are the constructions $V(\,\cdot\,)$ and $I(\,\cdot\,)$?
- What is the Nullstellensatz?
- What are the definitions and some examples of:
  - The Zariski topology?

  - Irreducibility?
  - Connectedness?
  - Dimension?

- What is the definition of a presheaf?
  - What are some examples and counterexamples?
- What is the definition of sheaf?
  - What are some examples?
  - What are some presheaves that are not sheaves?
- What is the definition of $\mathcal{O}_X$, the sheaf of regular functions?
  - How does one compute $\mathcal{O}_X$ for $X = D(f)$ a distinguished open?
- What is a morphism between two affine varieties?
- What is the definition of separatedness?
  - What are some examples of spaces that are and are not separated?
- What is a projective space?
- What is a projective variety?
- What is the projective coordinate ring?
- How does one take the closure of an affine variety X in projective space?
- What is completeness?
  - What are some examples and counterexamples of complete spaces?

0.4 Useful Algebra Facts

Fact 0.4.1:

- $\mathfrak{p} \trianglelefteq R$ is prime $\iff$ $R/\mathfrak{p}$ is a domain.
- $\mathfrak{p} \trianglelefteq R$ is maximal $\iff$ $R/\mathfrak{p}$ is a field.
- Maximal ideals are prime.
- Prime ideals are radical.
- If R is a PID and $\langle f \rangle \trianglelefteq R$ is generated by an irreducible element f, then $\langle f \rangle$ is maximal

Proposition 0.4.2 ***(Finitely generated polynomial rings are Noetherian).***
A polynomial ring $k[x_1, \cdots, x_n]$ on finitely many generators is Noetherian. In particular, every ideal $I \trianglelefteq k[x_1, \cdots, x_n]$ has a finite set of generators and can be written as $I = \langle f_1, \cdots, f_m \rangle$.

Proof (?).
A field k is both Artinian and Noetherian, since it has only two ideals and thus any chain of ideals necessarily terminates. By Hilbert's basis theorem (Theorem 0.4.5), $k[x_1, \cdots, x_n]$ is thus Noetherian. ■

Proposition 0.4.3 ***(Properties and Definitions of Ideal Operations).***

$$I + J := \left\{ f + g \;\middle|\; f \in I,\, g \in J \right\}$$

$$IJ := \left\{ \sum_{i=1}^{N} f_i g_i \;\middle|\; f_i \in I,\, g_i \in J, N \in \mathbb{N} \right\}$$

$$I + J = \langle 1 \rangle \implies I \cap J = IJ \qquad \text{(coprime or comaximal)}\ \langle a \rangle + \langle b \rangle = \langle a, b \rangle\,.$$

Theorem 0.4.4 ***(Noether Normalization).***
Any finitely-generated field extension $k_1 \hookrightarrow k_2$ is a finite extension of a purely transcendental extension, i.e. there exist $t_1, \cdots, t_\ell$ such that k_2 is finite over $k_1(t_1, \cdots, t_\ell)$.

Theorem 0.4.5 ***(Hilbert's Basis Theorem).***
If R is a Noetherian ring, then $R[x]$ is again Noetherian.

0.5 The Algebra-Geometry Dictionary

Let $k = \bar{k}$, we're setting up correspondences

$$
\begin{aligned}
\text{Ring Theory} & \qquad \text{Geometry/Topology of Affine Varieties} \\
\text{Polynomial functions} & \qquad \text{Affine space} \\
k[x_1, \cdots, x_n] & \qquad \mathbb{A}^n/k \coloneqq \{[a_1, \cdots, a_n] \in k^n\} \\
\text{Maximal ideals } \langle x_1 - a_1, \cdots, x_n - a_n \rangle & \qquad \text{Points } [a_1, \cdots, a_n] \in \mathbb{A}^n/k \\
\text{Radical ideals } I \trianglelefteq k[x_1, \cdots, x_n] & \qquad \text{Affine varieties } X \subset \mathbb{A}^n/k, \text{ vanishing locii of polynomi} \\
& I \mapsto V(I) \coloneqq \left\{ a \;\middle|\; f(a) = 0 \forall f \in I \right\} \\
I(X) \coloneqq \left\{ f \;\middle|\; f|_X = 0 \right\} \hookleftarrow X & \\
\text{Radical ideals containing } I(X) \text{, i.e. ideals in } A(X) & \qquad \text{closed subsets of } X \text{, i.e. affine subvarieties} \\
A(X) \text{ is a domain} & \qquad X \text{ irreducible} \\
A(X) \text{ is not a direct sum} & \qquad X \text{ connected} \\
\text{Prime ideals in } A(X) & \qquad \text{Irreducible closed subsets of } X \\
\text{Krull dimension } n \text{ (longest chain of prime ideals)} & \qquad \dim X = n, \text{ (longest chain of irreducible closed subsets}
\end{aligned}
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