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Recall that a sheaf of rings on a topological space X is a ring $\mathcal{F}(U)$ for all open sets $U \subset X$ satisfying four properties:

1. The empty set is mapped to zero.
2. The morphism $\mathcal{F}(U) \rightarrow \mathcal{F}(U)$ is the identity.
3. Given $W \subset V \subset U$ we have
4. Gluing: given sections $s_i \in \mathcal{F}(U_i)$ which agree on overlaps (restrict to the same function on $U_i \cap U_j$), there is a unique $s \in \mathcal{F}(\cup U_i)$.

Example 1.1.

If X is an affine variety with the zariski topology, \mathcal{O}_X is a sheaf of regular functions, where we recall $\mathcal{O}_X(U)$ are the functions $\varphi : U \rightarrow k$ that are locally a fraction.

Recall that the *stalk* of a sheaf \mathcal{F} at a point $p \in X$, is defined as

$$\mathcal{F}_p := \left\{ (U, \varphi) \mid p \in U \text{ open}, \varphi \in \mathcal{F}(U) \right\} / \sim.$$

where $(U, \varphi) \sim (U', \varphi')$ if there exists a $p \in W \subset U \cap U'$ such φ, φ' restricted to W are equal.

Recall that a *local ring* is a ring with a unique maximal ideal \mathfrak{m} . Given a prime ideal $\mathfrak{p} \in R$, so $ab \in \mathfrak{p} \implies a, b \in \mathfrak{p}$, the complement $R \setminus \mathfrak{p}$ is closed under multiplication. So we can localize to obtain $R_{\mathfrak{p}} = \left\{ a/s \mid s \in R \setminus \mathfrak{p}, a \in R \right\} / \sim$ where $a'/s' \sim a/s$ iff there exists a $t \in R \setminus \mathfrak{p}$ such that $t(a's - as') = 0$.

⚠ Warning: Note that R_f is localizing at the powers of f , whereas $R_{\mathfrak{p}}$ is localizing at the *complement* of \mathfrak{p} .

Since maximal ideals are prime, we can localize any ring R at a maximal ideal $R_{\mathfrak{m}}$, and this will be a local ring. Why? The ideals in $R_{\mathfrak{m}}$ biject with ideals in R contained in \mathfrak{m} . Thus all ideals in $R_{\mathfrak{m}}$ are contained in the maximal ideal generated by \mathfrak{m} , i.e. $\mathfrak{m}R_{\mathfrak{m}}$.

Lemma 1.1 (?).

Let X be an affine variety. The stalk of the sheaf of regular functions $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$ is isomorphic to the localization $A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p := I(\{p\})$.

Proof .

We can write

$$A(X)_{\mathfrak{m}_p} := \left\{ \frac{g}{f} \mid g \in A(X), f \in A(X) \setminus \mathfrak{m}_p \right\} / \sim$$

$$\text{where } g_1/f_1 \sim g_2/f_2 \iff \exists h(p) \neq 0 \text{ where } 0 = h(f_2g_1 - f_1g_2).$$

where the f are regular functions on X such that $f(p) \neq 0$.

We can also write

$$\mathcal{O}_{X,p} := \left\{ (U, \varphi) \mid p \in U, \varphi \in \mathcal{O}_X(U) \right\} / \sim$$

$$\text{where } (U, \varphi) \sim (U', \varphi') \iff \exists p \in W \subset U \cap U' \text{ s.t. } \varphi|_W = \varphi'|_W.$$

So we can define a map

$$\begin{aligned} \Phi : A(X)_{\mathfrak{m}_p} &\rightarrow \mathcal{O}_{X,p} \\ \frac{g}{f} &\mapsto \left(D_f, \frac{g}{f} \right). \end{aligned}$$

Step 1: There are equivalence relations on both sides, so we need to check that things are well-defined.

We have

$$\begin{aligned} g/f \sim g'/f' &\iff \exists g \text{ such that } h(p) \neq 0, h(gf' - g'f) = 0 \in A(X) \\ &\iff \text{the functions } \frac{g}{f}, \frac{g'}{f'} \text{ agree on } W := D(f) \cap D(f') \cap D(h) \\ &\implies (D_f, g/f) \sim (D_{f'}, g'/f'), \end{aligned}$$

since there exists a $W \subset D_f \cap D_{f'}$ such that $g/f, g'/f'$ are equal.

Step 2: Surjectivity, since this is clearly a ring map with pointwise operations.

Any germ can be represented by (U, φ) with $\varphi \in \mathcal{O}_X(U)$. Since the sets D_f form a base for the topology, there exists a $D_f \subset U$ containing p . By definition, $(U, \varphi) = (D_f, \varphi|_{D_f})$ in $\mathcal{O}_{X,p}$. Using the proposition that $\mathcal{O}_X(D(f)) = A(X)_f$, this implies that $\varphi|_{D_f} = g/f^n$ for some n and $f(p) \neq 0$, so (U, φ) is in the image of Φ .

Step 3: Injectivity. We want to show that $g/f \mapsto 0$ implies that $g/f = 0 \in A(X)_{\mathfrak{m}_p}$.

Suppose that $(D_f, g/f) = 0 \in \mathcal{O}_{X,p}$ and $(U, \varphi) = 0 \in \mathcal{O}_{X,p}$, then there exists an open $W \subset D_f$ containing p such that after passing to some distinguished open $D_h \ni p$ such that $\varphi = 0$ on D_h . Wlog we can assume $\varphi = 0$ on U , since we could shrink U (staying in the same equivalence class) to make this true otherwise. Then $\varphi = g/f$ on D_h , using that $\mathcal{O}_X(D_f) = A(X)_f$, so $g/f = 0$ here. So there exists a k such that $f^k(g \cdot 1 - 0 \cdot f) = 0$ in $A(X)$, so $f^k g = 0 \in A(X)_{\mathfrak{m}_p}$. ■

Conclusion:

$$\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}.$$

Example 1.2.

Let $X = \{p, q\}$ with the discrete topology with the sheaf \mathcal{F} given by $p \mapsto R, q \mapsto S, X \mapsto R \times S$.

Then $\mathcal{F}_p = R$, since if U is open and $p \in U$ then either $U = \{p\}$ or $U = X$. We can check that for (r, s) a section of \mathcal{F} , we have an equivalence of germs $(X, (r, s)) \sim (\{p\}, r)$ since $\{p\} \subset X \cap \{p\}$. Here X plays the role of U , $\{p\}$ of U' , and