Topology Qual Problems

D. Zack Garza

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Contents

1 Problems

1.1 Homotopy

1. Show that any non-surjective map $f : X \to S^n$ is homotopic to the constant map.
2. Let $f, g \to S^n$ be such that $\forall x \in X, f(x) \neq -g(x)$. Show that $f \simeq g$.
3. Let $\alpha : S^n \to S^n,\ \alpha(p) = -p$ be the antipodal map on S^n. Show that n odd $\implies f \simeq \mathrm{id}$.
4. Show that X is homotopy-equivalent to a point $\iff \mathrm{id}_X \simeq g$ for some constant map g.
5. Show that $S^1 \times I \simeq M$, the Mobius strip.
6. Show that $\mathbb{R}^3 - S^1 \simeq S^1 \vee S^2$.
7. Classify the letters of the alphabet up to homeomorphism, and up to homotopy.
8. **REVISIT** Let $f, g : S^1 \to X,\ P = X \cup_f B^2 \cong X \coprod B^2 / \sim$, where $x \sim f(x)$, $Q = X \cup_g B^2$. Show that $f \simeq g \implies P \simeq Q$.

1.2 Fundamental Group

1. Show that $x, y \in X$ path & simply-connected $\implies$ all paths from x to y are homotopic rel $\{0, 1\}$.
2. Show that for X path connected, $\pi_1(X) = \nVdash \iff \forall$cts. $f : S^1 \to X$ f, extends to a continuous map $F : B^2 \to X$.
3. Show $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.
4. Show $\pi_1(S^n) = 1$ for $n \geq 2$.
5. Show that $S^2 - \{p_0, p_1\} \simeq S^1$.
6. Show that $S^3 - \{p_0, p_1\} \simeq S^2$
7. Show that $S^2 \not\cong S^3$.
8. For each of the following $f : S^1 \to S^1$, identify the corresponding $f_* : \mathbb{Z} \to \mathbb{Z}$:
   1. $z \mapsto z^n$
   2. $\bar{x} \mapsto -\bar{x}$
   3. $e^{i\theta} \mapsto e^{2\pi i \sin\theta}$
9. Determine the winding number of the following map: $f : S^1 \to \mathbb{C} - \{0\}, z \mapsto 8z^4 + 4z^3 + 2z^2 + z^{-1}$
10. Identify $\pi_1(M, [(1, \frac{1}{2})])$, and identify the class of ∂M.
11. Let $X = S^1 \times S^1$ and γ a loop based at x_0. What is the induced map $\gamma_\sharp$?

1.3 Group Actions

1. Show that octagon pasting is homeomorphic to the $T = \mathbb{R}^2/\mathbb{Z}^2$.
2. Let x_0 be the image of 0, show that there is an order 6 homeomorphism $f : T \to T$ fixing x_0. Find a representation of f_* as a matrix, and find its determinant.
3. Show that $\pi_1(K)$, the Klein bottle, is given by pairs (m, n) where $(m, n) \star (p, q) = (m + (-1)^n p, n + q)$
   1. Show this is torsion-free
   2. Show that T is a double cover of K.
4. For each of these actions of $\mathbb{Z}_2$ on S^n, compute $\pi_1(S^n/\mathbb{Z}_2)$
   1. $S^1, z \mapsto -z$

2. $S^2, (x, y, z) \mapsto (-x, -y, z)$
3. $S^3, (z, w) \mapsto (-z, -w)$

1.4 Applications

1. Let $i : \mathbb{RP}^2 \to \mathbb{RP}^3$, induced by $S^2 \hookrightarrow S^3$ as the equator. Show that $i \not\simeq$ const.
2. Show that there is no map $f : S^2 \to S^1$ that commutes with the antipodal map.
3. Prove that for any $f : S^2 \to \mathbb{R}^2$, there exists $x \in S^2$ such that $f(x) = f(-x)$.
4. Prove the Ham Sandwich theorem.
5. Show that K can not be a topological group.

1.5 Van Kampen's Theorem

1. Compute a presentation of $\pi_1(T)$ and prove it is isomorphic to $\mathbb{Z}_2$.
2. (Images)
3. Show that $T - D^1 := X \simeq S^1 \vee S^1$.
    1. Show there does not exist a retraction $r : X \to \partial X$.
4. Images
5. IMages
6. Images
7. Calculate a presentation of $\pi_1(S^3 - K)$
8. Show that all 3 presentations of $\pi_1(K)$ are isomorphic
    1. Square with sides glued
    2. Two mobius strips glues along boundary
    3. Multiplication rule
9. Given a group $G =< A : R >$, show how to construct a CW-complex X such that $\pi_1(X) = G$.
10. Write down the fundamental group of the following spaces:
11. $\mathbb{R}^2 - \{0, 1\}$
12. $\mathbb{R}^2 - I$
13. The symbol $\oplus \in \mathbb{R}^2$
14. $S^2 - \{p_i\}_{i=1}^4$
15. $T - \{p_0\}$
16. $S^2/\mathbb{Z}_2$ via the antipodal map
17. $S^2/\mathbb{Z}_3$ via a $2\pi/3$ rotation about the z-axis.
18. $S_2 \cup \{(0, 0, z) \mid -1 \leq z \leq 1\}$
19. $\mathbb{R}^3 - \{(x, y, 0) \mid x^2 + y^2 = 1\}$
20. $\mathbb{R}^2 - H$, the Hopf link
21. Prove that the homophony group is trivial.

1.6 Mayer Vietoris (Sheet 7)

1. Compute the homology of:
    1. $\mathbb{RP}^2 = M \bigcup_\partial D^2$
    2. $T^2 = S^1 \times S^1 = (S^1 \times I) \bigcup_f (S^1 \times I)$ where $(x, 0) \sim (x, 1) \sim (\bar{x}, 0) \in \mathbb{C}$
    3. $S^1 \bigcup_f B^2$ attached along ∂B^2 using $z \mapsto z^n$
2. Show $\tilde{H}_i(\Sigma X) \cong \tilde{H}_{i-1}(X)$

1. Show $\Sigma S^n \cong S^{n+1}$

3. For $f : S^n \circlearrowleft$, show $\deg f = \deg \Sigma f$

   1. Conclude $\pi_n(S^n) = \mathbb{Z}$

4. Let $\{A_i\}^n \in \mathbf{Ab}$ be finitely generated, show $\exists X \mid H_i(X) \cong A_i$ for $i \leq n$ and 0 otherwise.

5. Suppose $X = \bigcup_i^n A_i$ such that for any $1 \leq k \leq n$, $\bigcap_i^k A_i$ is either empty or contractible, show $i \geq n-1 \implies \tilde{H}_i(X) = 0$ and that this bound is sharp.

6. Compute $H_*(X \times S^n)$ in terms of $H_*(X)$

   1. Compute $H_*(T^n)$

7. Let $M = (S^1 \times B^2) \bigcup_{\mathrm{id}_\partial} (S^1 \times B^2)$ and compute $H_*(M; \mathbb{Z})$

8. Let $X = S^n \times I$ with its ends glued together by a map $S^n \circlearrowleft$ of degree d, calculate $H_*(X)$.

9. Compute $H_*(X)$ for $X = S^3 - N$, with N a knotted solid torus and $\partial N = T$ its boundary torus

10. Let CA be the cone on A, show that $\tilde{H}_*(X \bigcup CA) \cong \tilde{H}_*(X, A)$.

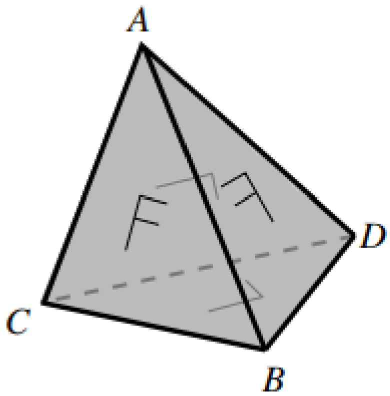
11. Show that the Mayer-Vietoris sequence is natural, i.e. If $X \xrightarrow{f} Y$ where $f(A) \subset C$ and $f(B) \subset D$, then this commutes:

$$\begin{tikzcd}
H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(A \bigcap B) \arrow[r] \arrow[d, "f_*"] & H_n(A) \oplus H_n(B) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\
H_n(Y) \arrow[r] & H_n(C \bigcap D) \arrow[r] & H_n(C) \oplus H_n(D) \arrow[r] & H_{n-1}(Y)
\end{tikzcd}$$

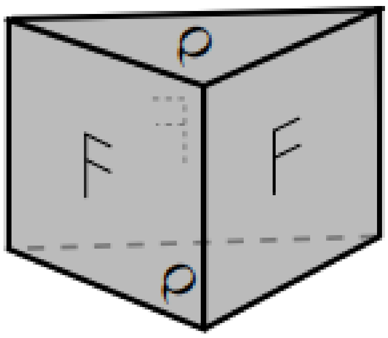
1.7 Cellular Homology (Sheet 8)

Compute the homology of these spaces

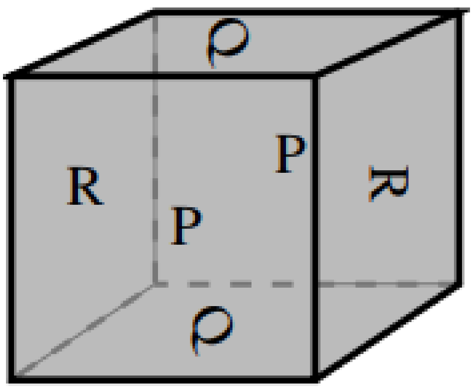
1. $S_m \vee S_n$
2. $S^m \times S^n$
3. A hexagon with the identifications $a + b + c - a - b - c$
4. Orientable surface of genus g
   1. $g = 2$ is given by $a + b - a - b + c + d - c - d$
5. Nonorientable surface of genus g Obtain by removing g discs from S^2 and attaching g mobius strips
6. $S_1 \vee S_1$ with two discs attached via $(ab)^3$ and $(ab)^6$



7. This identification space:



8. This identification space:



9. This identification space:
10. Describe a CW complex structure for the lens space $L(p,1)$ and compute π_1, H_* for it.

1.8 Degree

1. Let $p(x) = \sum_i^n a_i x^i$, view $p : \mathbb{C} \bigcup \infty \circlearrowleft$ and determine its topological degree
2. Let $p(z) = \frac{\prod_i^n z - a_i}{\prod_j^m z - b_j}$ with all a_i, b_j distinct. What is its topological degree?
3. Show that if $f : S^m \to S^n$ and $\exists U \subset S^m$ such that $f|_U \cong f(U)$, then $m = n$ and f is surjective.

1.9 Universal Coefficient Theorem (Sheet 10)

1. Identify the following groups up to isomorphism
   1. $\mathbb{Z}_m \otimes \mathbb{Z}_n$
   2. $\mathbb{Z}_{60}^4 \otimes (\mathbb{Z}_{24}^3 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{120})$
   3. $\mathbb{Z}_n \otimes \mathbb{Q}$
   4. $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Q}/\mathbb{Z})$
2. Compute:
   1. $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$
   2. $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$
3. Compute the following directly from chain complexes and check using UCT:
   1. $H_*(\mathbb{RP}^n; \mathbb{Z}_2)$
   2. $H_*(\mathbb{RP}^n, \mathbb{Z}_3)$
   3. $H^*(\mathbb{RP}^n, \mathbb{Z}_6)$
4. For any space X, show that $H^1(X)$ is free abelian
5. Show that $H_*(X;\mathbb{Q}) = H_*(X;\mathbb{Z}) \otimes \mathbb{Q}$ $H^*(X;\mathbb{Z}) = \hom(H_*(X;\mathbb{Z}), \mathbb{Q})$
6. Construct a space X such that $H_*(X;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_{12}, \mathbb{Z} \oplus \mathbb{Z}_4, 0 \cdots)$ Compute $H^*(X;\mathbb{Z})$
7. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z}_2)$
8. Compute $H_*(\Sigma\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Z})$
9. Compute $H_*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z})$
10. Let G be a topological group. Show that $H_*(G)$ is an algebra. Show that $G \curvearrowright H_*(G)$, which factors through the homomorphism $G \to \pi_0(G)$ yielding a trivial action if G is path-connected.

1.10 Homological Algebra (Sheet 11)

1. Show that $\ker A \to A \otimes \mathbb{Q}$ given by $a \mapsto a \otimes 1$ is the torsion subgroup of A.

2. Show that $A \hookrightarrow B \implies A \otimes \mathbb{Q} \hookrightarrow B \otimes \mathbb{Q}$

3. Find a free resolution of $\mathbb{Q}$ as a $\mathbb{Z}$-module.

4. Compute $\mathrm{Tor}(\mathbb{Q}, A)$

   1. Compute $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$

5.

6. Let $R = \mathbb{Z}[x, y]$, and $M = R/(x - y), N = R/(x, y)$. Construct free resolutions of M, N to compute:

   - $\mathrm{Ext}_R^*(M, M)$
   - $\mathrm{Ext}_R^*(M, N)$
   - $\mathrm{Ext}_R^*(N, M)$
   - $\mathrm{Ext}_R^*(N, N)$

7. Let Λ_* be the exterior algebra generated by the symbols $\{dx_i\}^n$ over a field k. Show that letting $d = \cdot \vee dx_1$ yields a chain complex $0 \to \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^n \to 0$ with trivial homology. Compute what happens when dx_1 is replaced with an arbitrary non-zero element in Λ^1.

8. Define M as the group ring $R = \mathbb{Z}[\mathbb{Z}_2]$ with the action $(\cdot) \times -1$. Construct a free resolution of M and compute $\mathrm{Tor}_R^*(M, M)$.

9. Show $\operatorname{Tor}_R^*(\cdot,\cdot)$ is symmetric in the following way: Given M, N, take free resolutions, view $M_* \to M$ as a chain map and tensor with N_* to get a chain map$\psi : M_* \otimes_R N_* \to M \otimes_R N_*$. Show that ψ is a quasi-isomorphism using the exact sequence $0 \to (Z_n, 0) \to (N_n, 0) \to (B_{n-1}, 0) \to 0$, then switch the roles of M, N.

10. Prove that for a SES $0 \to A \to B \to C$, the group $\operatorname{Ext}(C, A)$ classifies extensions of C by A up to isomorphism.

1.11 Cohomology Ring (Sheet 12)

Todo

2 Topology Problems: Solutions

2.1 Homotopy

1. **Main Idea**: A linear homotopy projected onto the sphere works.

Let $f : X \to S^n \subset \mathbb{R}^{n+1}$ be an arbitrary map that fails to be surjective. Then, by definition, there is at least one point $s_0 \in S^n - f(X)$.

Then, $\forall x \in X$, since $f(x) \neq s_0$, there is a unique geodesic C connecting $f(x)$ and s_0. So a variant of the straight line homotopy will work, by interpolating between $f(x)$ and s_0 along C.

So let $H : X \times I \to S^n$ be defined by $H(x,t) = P(ts_0 + (1-t)f(x))$, where $P : \mathbb{R}^{n+1} \to S^n$ is given by $P(x) = x/\|x\|$. This is well defined, since the denominator is zero iff $f(x) = s_0$, which by assumption is not the case. This is a homotopy, since $H(x,0) = P(f(x)) = f(x)$ (since P fixes S^n) and $H(x,1) = P(s_0) = s_0$ (since $s_0 \in S^n$).

2. **Main Idea**: Exact same idea as 1, just a more complicated check.

Take $H(x,t) = P(tf(x) + (1-t)g(x))$. This is well defined; the only case to check is when the denominator is zero. But $\|x\| = 0$ iff $x = 0$, which would imply $tf(x) + (1-t)g(x) = 0$ and so $tf(x) = -(1-t)g(x)$.

Taking norms and observing that since $f, g \in S^n \implies \|f\| = \|g\| = 1$, this forces $t = 1-t$ and thus $t = 1/2$. But this would force $(1/2)f(x) = (-1/2)g(x)$ and thus $f(x) = -g(x)$, which we assumed was not the case.

3. **Main Idea**: Linear homotopy fails continuity without the condition from (2), so use complex embedding to avoid the origin at $t = 1/2$.

Suppose n is odd and define $f : S^n \to S^n$ to be the antipodal map. Since $n+1$ is even, we have $n + 1 = 2m$ for some $m \in \mathbb{N}$, so identify $S^n = S^{2m-1} \subset \mathbb{R}^{2m} \cong \mathbb{C}^m$

Then $z \in S^n$ can be written as a vector $z \in \mathbb{C}^m$ such that $\|z\| = 1$.

Then define $P : \mathbb{C}^m \to \mathbb{C}^m$ by $P(z) = z/|z|$, the projection onto the complex unit sphere, and define $H : \mathbb{C}^m \times I \to \mathbb{C}^m$ by $H(z,t) = P(e^{i\pi t}z)$.

This is a homotopy, since $H(z,0) = P(z) = z$ (since $\|z\| = 1$), so this is the identity map. We also have $H(z,1) = P(-z) = -z$, the antipodal map.

This is well-defined, since $e^{i\pi t} > 0$ and $z \neq 0$, so the linear homotopy in ambient $\mathbb{C}^m$ avoids the origin and thus the denominator when taking the projection is never zero.

4. $\Leftarrow$: **Main Idea**: Projection and inclusion are homotopy inverses. One composition is equality, the other is just equality *up to homotopy*, but that's all we need!

Suppose id_X is nullhomotopic.

Then there exists some constant map $g : X \to \{x_0\}$ for some $x_0 \in X$ where $g(x) = x_0$ and $g \simeq \mathrm{id}_X$.

This means there is some homotopy $F : X \times I \to X$ such that $F(x, 0) = \mathrm{id}_X(x) = x$ and $F(x, 1) = g(x) = x_0$ for all $x \in X$.

So let $p : X \to \{x_0\}$ be the projection map sending every point to x_0, and $\iota : \{x_0\} \to X$ be the inclusion. We will show that the two compositions are homotopy inverses, from which it follows that $X \simeq \{x_0\}$. This means that X is homotopy-equivalent to a point, and thus by definition contractible.

Then $(p \circ \iota) : \{x_0\} \to \{x_0\}$ is given by $p(\iota(x_0)) = p(x_0) = x_0$, so this is the identity on the target space $\{x_0\}$.

Similarly, $(\iota \circ p) : X \to X$ is given by $\iota(p(x)) = \iota(x_0) = x_0$, so this is the constant map on X mapping every point from X to x_0. But then this map is exactly g, and by assumption this is homotopic to the identity on X

But then we have $p \circ \iota \simeq \mathrm{id}_{\{x_0\}}$ and $\iota \circ p \simeq \mathrm{id}_X$, so they are homotopy inverses.

$\Rightarrow$: **Main Idea**: One of the homotopy inverses *is* just a constant map.

Suppose $X \simeq \{x_0\}$, then there exist a pair of homotopy inverses

$f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ such that $f \circ g \simeq \mathrm{id}_{\{x_0\}}$ and $g \circ f \simeq \mathrm{id}_X$.

Since $\{x_0\}$ is a single point space, f is necessarily a constant map (i.e. $f(x) = x_0$ for every $x \in X$.) But then $(g \circ f)(x) = g(x_0) = y_0$ for some constant $y_0 \in X$, so $g \circ f$ is a constant map. By assumption, $g \circ f \simeq \mathrm{id}_X$, so the identity is homotopic to a constant map.

5. **Main Idea**: Deformation retract M onto its center circle; two spaces that deformation retract onto a common space are themselves homotopy equivalent.

Claim: $S^1 \times I \simeq S^1 \times \{*\}$ This is because I is contractible, so $I \simeq \{*\}$. (Maybe needs further proof)

Claim: $M \simeq S^1 \times \{*\}$.

If both of these claims hold, then we will have $M \simeq S^1 \times I$ as two spaces that deformation retract onto a common space. Identifying $M = I \times I/\sim$ where $(x, 0) \sim (1 - x, 1)$, fix $x = 1/2$.

Then consider the subspace $U = \{(1/2, y) \mid y \in [0, 1]\} \subset M$. Claim: $U \cong \{*\} \times S^1$ for some point $*$.

U can be written $\{1/2\} \times (I/\sim)$, and since $(1/2, 0) \sim (1/2, 1)$, we have $I/\sim = I/\partial I \cong S^1$, so $U \cong \{1/2\} \times S^1$ as desired (taking $* = \frac{1}{2}$).

However, we can define a homotopy from M onto U, in the form of a deformation retract.

Let $F : M \times I \to M$ be defined by $F((x, y), t) = F_t(x, y) = ((1 - t)x + \frac{1}{2}t, y)$. Then $F((x, y), 0) = (x, y) = \mathrm{id}_M$, and $F((x, y), 1) = (\frac{1}{2}, y) \subseteq U$. Moreover, if $(x, y) \in U$, then $(x, y) = (\frac{1}{2}, y)$ and

$F((x,y),t) = ((1-t)\frac{1}{2} + \frac{1}{2}t, y) = (\frac{1}{2} - t\frac{1}{2} + \frac{1}{2}t, y) = (\frac{1}{2}, y) = (x,y)$, so $F = \mathrm{id}_U$. This makes F a deformation retract from M onto U, and so $M \simeq U$.

But then, summarizing our results, we have $S^1 \times I \simeq S^1 \times \{*\} \cong S^1 \times \left\{\frac{1}{2}\right\} = U \simeq M$, and so $S^1 \times I \simeq M$ as desired.

6. **Main Idea**: Using a funky deformation retract. See Hatcher, PDF page 55, Example 1.23. Add picture!!

Deformation retract
$R^3 - S^1$ onto $S^2 - U$, where U is a diameter inside S^2 also passing through the middle of S^1 in the interior. This can be done by moving points outside of S^2 towards the surface, and points inside S^2 just move away from the S^1 inside (either towards U or towards the surface of S^2, so they don't hit S^1).

Then take a geodesic between the endpoints of the diameter on S^2, pick any point p on the geodesic, and move both diameter points towards it. This yields $S^2 \vee S^1$ at the point p.

7. **Main Idea**: Nothing to it. Homotopy:
8. $A \simeq \Delta \simeq S^1$
9. $a \simeq d \simeq o \simeq S^1$
10. $B \simeq 8 \simeq S^1 \vee S^1$
11. $b \simeq o \simeq S^1$
12. $C \simeq *$
13. $c \simeq l \simeq *$
14. $D \simeq S^1$
15. $d \simeq o \simeq S^1$
16. $E \simeq *$
17. $e \simeq d \simeq S^1$
18. $F \simeq *$
19. $f \simeq *$
20. $G \simeq *$
21. $g \simeq 8 \simeq S^1 \vee S^1$
22. $H \simeq *$
23. $h \simeq l \simeq *$
24. $I \simeq *$
25. $i \simeq \{*_1, *_2\}$
26. $J \simeq *$
27. $j \simeq i \simeq \{*_1, *_2\}$

28. $K \simeq *$
    1. $k \simeq K \simeq *$
29. $L \simeq *$
    1. $l \simeq *$
30. $M \simeq *$
    1. $m \simeq *$
31. $N \simeq *$
    1. $n \simeq *$
32. $O \simeq S^1$
    1. $o \simeq S^1$
33. $P \simeq D \simeq S^1$
    1. $p \simeq P \simeq S^1$
34. $Q \simeq O \simeq S^1$
    1. $q \simeq p \simeq o \simeq S^1$
35. $R \simeq D \simeq S^1$.
    1. $r \simeq l \simeq S^1$
36. $S \simeq *$
    1. $s \simeq S \simeq *$
37. $T \simeq *$
    1. $t \simeq l \simeq *$
38. $U \simeq *$
    1. $u \simeq U \simeq *$
39. $V \simeq *$
    1. $v \simeq V \simeq *$
40. $W \simeq *$
    1. $w \simeq W \simeq *$
41. $X \simeq *$
    1. $x \simeq X \simeq *$
42. $Y \simeq *$
    1. $y \simeq v \simeq *$
43. $Z \simeq *$
    1. $z \simeq Z \simeq *$

This results in a partition of the alphabet into the following homotopy types:

- $\{A, D, O, P, Q, R, S^1\} \bigcup \{a, b, d, e, g, o, p, q\}$
- $\{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z, *\} \bigcup \{c, f, h, k, l, m, n, r, s, t, u, v, w, x, y, z\}$
- $\{B, S^1 \vee S^1\}$
- $\{i, j, \{*, *\}\}$

Homeomorphisms: ignore ligatures!!

1. $\{A, R\}$ Can remove a point to obtain two components homeomorphic to $\{I, F\}$ respectively.
2. $\{D, O, S^1\}$ These all have no single point that can be removed to disconnect the space.
3. $\{B, S^1 \vee S^1\}$ Remove point at junction
4. $\{C, G, I, J, L, M, N, S, U, V, W, Z, [0, 1]\}$ These all have a point that can be removed to yield **two** components, but no points that yield **three**. (Intuitively, all can be obtained by twisting a straight wire.)
5. $\left\{E, F, T, Y, \bigvee_{i=1}^{3}[0, 1]\right\}$ These all have a point that can be removed to yield 3 connected components homeomorphic to I. This is the "pasting" point in the vee.
6. $\left\{H, K, \bigvee_{i=1}^{5}[0, 1]\right\}$ Can remove **two** points to disconnect each into **five** components.
7. $\{P, Q, S^1 \vee [0, 1]\}$ Both contain a nontrivial loop.
8. $\left\{X, \bigvee_{i=1}^{4}[0, 1]\right\}$ Can remove **one** point to separate into **four** components.
9. **Main Idea**: Show that both spaces are a deformation retract of the same space. (See Hatcher, Proposition 0.18, p. 25)

Suppose we have the following maps

$$f : S^1 \to X$$
$$g : S^1 \to X$$

where $f \simeq g$. Then there exists a homotopy

$$H : S^1 \times I \to X$$

such that $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$.

Then define

$$P := X \coprod_f B^2$$
$$Q := X \coprod_g B^2$$

We want to that P and Q are homotopy-equivalent. In order to do so, we will construct a larger space which deformation retracts onto both P and Q, which is a homotopy equivalence.

With H in hand, we can define the space $R = X \coprod_H B^2 \times I$, where we recognize $S^1 = \partial B^2$. In particular, S^1 is a subspace of B^2.

Claim: Both P and Q are subspaces of R. Since $H(z, 0) = f(z)$. So considering $X \coprod_H B^2 \times \{0\} \cong X \coprod_f B^2 = P$. A similar argument holds at the point $1 \in I$. (*Not a strong argument*)

But note that $B^2 \times I$ is a solid cylinder, and so can be deformation retracted onto the outer shell plus one of the "lids". Formally, this would be given by $S^1 \times I \cup B^2 \times \{p\}$ for some $p \in [0, 1]$.

Claim: choosing $p = 0$ induces a deformation retract of R onto P, and choosing $p = 1$ induces a deformation retract of R onto Q.

Proof: ?

2.2 Fundamental Group

1. **Main idea**: just algebraic manipulations using the π_1 functor and unravelling definitions.

Let X be path connected and simply connected, and let $x, y \in X$ be two arbitrary points. Then consider two paths, $\gamma : I \to X, \gamma(0) = x, \gamma(1) = y$ $\alpha : I \to X, \alpha(0) = x, \alpha(1) = y$.

We would like to show $\gamma \simeq \alpha$. Since X is simply connected, we know that $\pi_1(X) = 0$. This means that for any $a, b \in \pi_1(X), a = b = e$, the identity element in this group.

So we construct two loops: one as $\gamma\bar{\alpha}$, the other as $\alpha\bar{\gamma}$. Apply the π_1 functor yields $[\gamma\bar{\alpha}] = e = [c_x] = [\alpha\bar{\gamma}]$, where $[c_x]$ is the equivalence class of the constant path at x, and equivalently the identity element in $\pi_1(X)$. Lemma: If $f \simeq g$, then $f \circ h \simeq g \circ h$ for any h.

But this says $\gamma\bar{\alpha} \simeq c_x$ and $\alpha\bar{\gamma} \simeq c_x$. But $\gamma \simeq c_x \circ \gamma \simeq (\alpha\bar{\gamma}) \circ \gamma \simeq \alpha \circ (\bar{\gamma} \circ \gamma) \simeq \alpha$, which is what we desired.

2. **Main Idea** Homotopies on maps $S^1 \to X$ are cylinders, find a way to continuously map a cylinder onto a disk given the existence of such a homotopy. Let X be path connected, $\pi_1(X) = 0$, and let $f : S^1 \to X$ be arbitrary. Then $f(S^1) \subseteq X$ is a path in X, and since $\pi_1(X) = 0$, this path is homotopic to a point x_0. So f is homotopic to the constant map $c_{x_0} : S^1 \to X, z \mapsto x_0$.

So let $H : S^1 \times I \to X$ be this homotopy. We know that $H(z, 0) = f(z)$ and $H(z, 1) = c_{x_0}(z) = x_0$.

Claim: Consider quotient $\frac{S^1 \times I}{S^1 \times \{1\}}$ with the projection map $p : S^1 \times I \to S^1 \times \{1\}$. Then H factors through the quotient uniquely (why?), and there exists a unique $\hat{H}$ making this diagram commute:

This follow from the universal property of the quotient in **Top**, where it is sufficient that H is constant on $S^1 \times \{1\}$ - but this is exactly what was deduced above.

However, the quotient object constructed is homeomorphic to D^2, as per the following diagram

Here, we just recognize that $S^1 \times I$ is a cylinder, and quotienting at the $t = 1$ point in I simply collapses the top portion of the cylinder to a point, forming a cone. We then take the flattening map to just project every point on the cone directly downwards onto the base circle, yielding D^2.

(Note: I guess this map can be constructed as $\Phi : S^1 \times I \to D^2$ where $\Phi(z, t) = z(1-t)$. Since $t = 1$ on $S^1 \times \{1\}$, $\Phi(z, 1) = 0$ and this is exactly the kernel of Φ. Continuous as product of continuous functions, need to check injective/surjective and show inverse is continuous.)

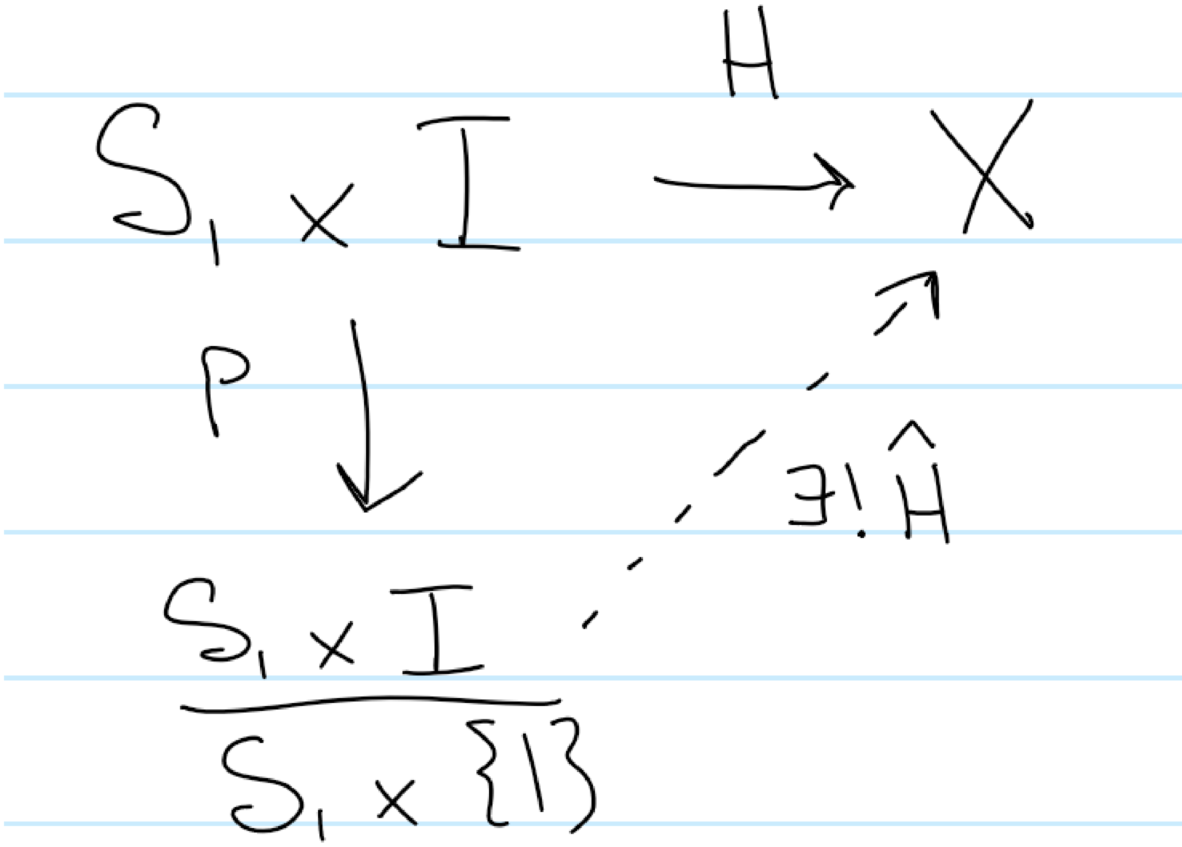


Figure 1: universal1

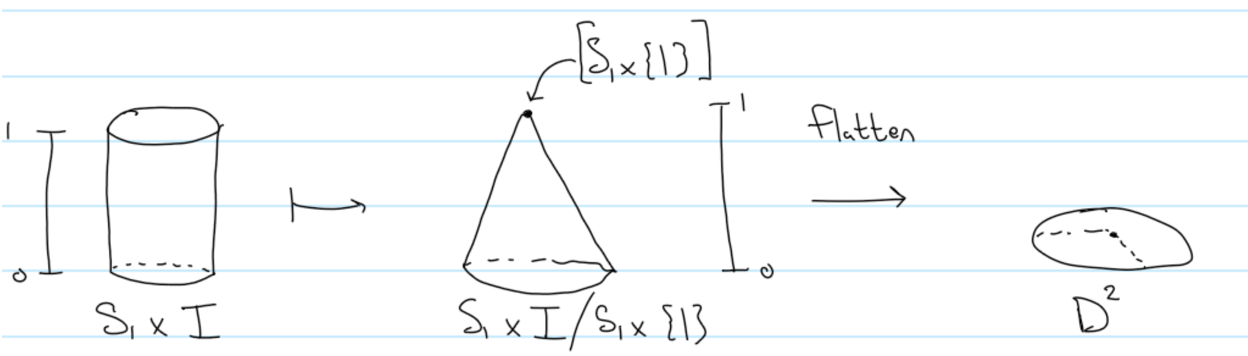


Figure 2: 2017-11-24 14_59_29-Untitled page - OneNote

Need to check injective/surjective, show that kernel is $S^1 \times 1$, then use first isomorphism theorem.)

But then $\hat{H}$ is exactly a continuous map from $D^2 \to X$, as desired.

3. $\Rightarrow$ Let $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$ be an arbitrary loop in $X \times Y$. Then α is equivalently a map $S^1 \to X \times Y$. Considering S^1 to be a subset of $\mathbb{R}^2$, we can parameterize α as $\alpha(z) = \alpha(x+iy) = (\alpha_x(x), \alpha_y(y))$ in components. In particular, since α is continuous, so are α_x, α_y. Moreover, since $\alpha(0) = \alpha(0 + i0) = (x_0, y_0)$, we have $\alpha_x(0) = x_0, \alpha_y(0) = y_0$. (Note: alternatively, given the product, we have projections p_X, p_y, so we can define the map $\alpha \mapsto (p_X \circ \alpha, p_Y \circ \alpha)$)

But then $\alpha_x : S^1 \to X$ and $\alpha_y : S^1 \to Y$ are loops entirely in X, Y at the respective base points, and so we can define the map $F : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by $[\alpha] = [(\alpha_x, \alpha_y)] \mapsto ([\alpha_x], [\alpha_y])$

This is injective, since $([a], [b]) = ([c], [d])$ on the RHS means that $[a] = [c], [b] = [d]$ in the fundamental groups, and thus $a \simeq c, b \simeq d$ in the spaces. We want to show that $[(a, b)] = [(c, d)]$, which would follow if $\alpha(x+iy) = (a(x), b(y)) \simeq \beta(x+iy) = (c(x), d(y))$ in $X \ \times Y$. ...?

This is surjective, because if $([a], [b])$ are elements in the right-hand side, then $a(0) = a(1) = x_0$ and $b(0) = b(1) = y_0$, so we can consider $(a, b) : I \to X \times Y$ where $(a, b)(z) = (a, b)(x+iy) = (a(x), b(y))$. This is then a loop in $X \times Y$, since $(a, b)(0) = (a(0), b(0)) = (0, 0) = (x_0, y_0)$ and similarly $(a, b)(1) = (a(1), b(1)) = (x_0, y_0)$. So this is actually a map $(a, b) : S^1 \to X \times Y$, or in other words, a loop in $X \times Y$ based at (x_0, y_0), which lifts to an element of the fundamental group on the LHS.

Maps in both directions are continuous, since a vector function is continuous iff its component functions are continuous.

This is well-defined, due to the fact that if $a \simeq b$, then $p_X \circ a \simeq p_X \circ b$, and $F = (f_x, f_y)$ is a homotopy iff its components functions are homotopies.

4. Let $A = S^n - \{n_p = \text{North Pole}\}, B = S^n - \{s_p = \text{South Pole}\}$. Then $A \bigcup B = S^n$ and $A \bigcap B = S^n - \{n_p, s_p\}$. Since A, B are open and path connected, we can apply van Kampen's theorem to obtain $\pi_1(X) = \pi_1(A) * \pi_1(B)$ amalgamated over $\pi_1(A \cap B)$. But $A \cong \mathbb{R}^n \cong B$ via stereographic projection, and since $\mathbb{R}^n$ is contractible, $\pi_1(\mathbb{R}^n) = 0 = \pi_1(A) = \pi_1(B)$. So $\pi_1(X) = 0 * 0 = 0$ as desired.

This follow because we can compute $A \cap B \cong \mathbb{R}^n - \{\text{pt}\} \cong S^n - 1$, and so $\pi_1(A \bigcap B) = \pi_1(S^n) \times \pi_1(\mathbb{R}^1) = 0 \times 0 = 0$, and so has the presentation $\pi_1(A \cap B) = \langle w \mid w^1 = e \rangle$. We can then look at the inclusions $i : A \cap B \to A$ $j : A \cap B \to B$ and the induced homomorphisms $I : \pi_1(A \cap B) \to \pi_1(A)$ $J : \pi_1(A \cap B) \to \pi_1(B)$. But since both sides in both maps are trivial, these are constant maps between identities. We can then present the group $0 = \pi_1(A) = \langle a \mid a^1 = e \rangle$ and since $I(w)J(w)^{-1} = ee^{-1} = e$, we have $\pi_1(B) = \langle b \mid b^1 = e \rangle$, so $\pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B) = \langle a, b \mid a^1 = b^1 = e \rangle$.

(See https://en.wikipedia.org/wiki/Seifert%E2%80%93van_Kampen_theorem for presentation of amalgamated product)

5. WLOG, assume p_0, p_1 are the north and south poles of S^2. We can then form a deformation retract of X onto the equator of S^2, which is equal to S^1. To do so, just move every point x along the unique great circle connecting x, p_0, p_1, and proceed at linear speed towards the equator. This is well defined at every point on S^2 *except* the poles, which are not included in X, and the equator is fixed at every instant. So this forms a deformation retract. Alternatively,

use the fact that $\mathbb{R}^n - \{\text{pt}\} \cong S^{n-1} \times \mathbb{R}$ via polar coordinates, and $S^n - \{\text{pt}\} \cong \mathbb{R}^n$ by stereographic projection. So $S^2 - \{p_0, p_1\} \cong \mathbb{R}^2 - \{p_1\} \cong S^1 \times \mathbb{R}$. But since $\mathbb{R}$ is contractible, the last one is homotopic to $S^1 \times \{0\} \cong S^1$. **Alternatively**: use the lemma, then $k = 2$ and so $S^2 - \{p_1, p_2\} \simeq \bigvee_{i=1}^1 S^1 = S^1$.

6. Lemma: $S^n - \{p_i\}_{i=1}^k = \bigvee_{k-1} S^{n-1}$, i.e. S^n minus k points is equal to $k-1$ copies of of S^{n-1}. Proof: $S^n - \{p_1\} \cong \mathbb{R}^n$ by stereographic projection, so $S^n - \{p_1, p_2 \cdots p_k\} \cong \mathbb{R}^n - \{p_2, \cdots p_k\}$. WLOG, suppose none of these points are zero (otherwise, take a translation away from zero. This is affine and continuous.) Then fix 0 as the base point, and form $k-1$ loops α_i, where the ith loop encircles p_i. Then $\mathbb{R}^n$ deformation retracts onto $\cup_{i=1}^{k-1} \alpha_i$, which is homeomorphic to $\bigvee_{i=1}^{k-1} S^1$.

7. Theorem: $\pi_1(\bigvee_{i=1}^k S^1) \cong *_{i=1}^n \mathbb{Z}$, the free product of n copies of $\mathbb{Z}$. Proof: By induction, using Van-Kampen's theorem. Base case: Take $i = 1$, then $\pi_1(S^1) = \mathbb{Z}$ as proved in Hatcher. Inductive step: Suppose this holds for all $k < n$, then we have $X = \bigvee^n S^1 = \left(\bigvee^{n-1} S^1\right) \vee S^1$. Let p be the point of common intersection, then let $U = \bigvee^{n-1} S^1$ $V = S^1 \bigcup \{p\}$

Then $U \bigcup V = X$, $U \bigcap V = \{p\}$, both U, V are path-connected. Since we have $\pi_1(\{\text{pt}\}) = 0$, the amalgamated free product reduces to the usual free product. By the IH, we have $\pi_1(U) = *^{n-1}\mathbb{Z}$, so

$$\pi_1(X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V) =_{\text{IH}} (*^{n-1}\mathbb{Z}) * \pi_1(V) = (*^{n-1}\mathbb{Z}) * \mathbb{Z} = *^n\mathbb{Z}.$$

Definition: Let $F_n := *^n\mathbb{Z}$ be the free abelian group on n generators. Lemma: If $n \neq m, F_n \ncong F_m$. Proof: If $F^n \cong F^m$, then $\mathbb{Z}^n \cong \mathbb{Z}^m$. But then tensor both sides with $\mathbb{Z}_2$ over $\mathbb{Z}$, yielding $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong Z^m \otimes_{\mathbb{Z}} \mathbb{Z}_2$. But the LHS is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, while the RHS is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$. *(Why?)* These are both finite groups - there are 2 elements in $\mathbb{Z}/2\mathbb{Z}$, so the first has 2^n elements and the latter has 2^m elements. But if $2^n = 2^m$, then $n = m$. The lemma follows from the contrapositive.

Now we have all we need - let $X = S^2 - \{p_1, p_2\}$ and $Y = S^3 - \{q_1, q_2\}$. Then by the previous problems, $X \simeq S^1$ and $Y \simeq S^2$, so if $S^2 \cong S^3$ then $X \simeq Y$ and $S^1 \simeq S^2$. But $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(S^2) = 0$, so $S^1 \not\simeq S^2$, a contradiction.

8. Here we go:

9. Let $\alpha(t) = e^{2\pi i t}$ where $t \in [0, 1]$, be a loop in S^1 parameterized by t, which goes around S^1 exactly once. Then under the map $f : z \mapsto z^n$, we obtain $f(\alpha(t)) = e^{2\pi n i t}$ where $t \in [0, 1]$. This resulting loop then goes around S^1 n times, so the induced homomorphism on $\pi_1(S^1) = \mathbb{Z}$ is the map $f^* : \mathbb{Z} \to \mathbb{Z}$ given by $f^*(a) = na$.

10. Define α as above, and define $f : S^1 \to S^1$ to be the antipodal map, so $f(z) = -z$ for $z \in S^1 \subset \mathbb{C}$. We then left α to the fundamental group, and define $f_*([\alpha]) = [f \circ \alpha]$. Computing, we have $(f \circ \alpha)(t) = f(\alpha(t)) = -e^{2\pi i t}$. Where $\alpha(0) = \alpha(1) = 1 + 0i$, we have $(f \circ \alpha)(0) = (f \circ \alpha)(1) = -1 + 0i$. But note that α was a counter-clockwise loop in S^1, and the image of α is also a counter-clockwise loop. So this maps the generator $[\alpha] \in \pi_1(S^1, 1)$ to the generator $[\alpha'] \in \pi_1(S^1, -1)$. But since S^1 is path-connected, the fundamental groups at these two base points are isomorphic. Alternatively: the antipodal map on S^1 is homotopic to the identity map (since $n = 1$ is odd), so $[f \circ \alpha] = [f][\alpha] = [\text{id}][\alpha] = [\alpha]$, so the induced homomorphism on $\pi_1(S^1)$ is the identity map.

11. Let $\alpha(t) = e^{it}$ where $t \in [0, 2\pi]$ be a counter-clockwise loop in S^1; then $[\alpha]$ generates the

fundamental group. Then $f^*([\alpha]) = [(f \circ \alpha)(t)] = [e^{it} \mapsto e^{2\pi i \sin t}]$. Then just consider how sin behaves in each quadrant. In quadrant 1, as t ranges from $0, \pi/2$ then $\sin t$ ranges from 0 to 1, so α is exactly traced out. In quadrant two, $\bar{\alpha}$ is traced out, since $\sin t$ decreases from 1 to 0. This happens again in the bottom quadrants, so we have $f^*([\alpha]) = [\alpha\bar{\alpha}\alpha\bar{\alpha}] = [\alpha][\alpha]^{-1}[\alpha][\alpha]^{-1} = [\mathrm{id}]$. But the identity element in $\mathbb{Z}$ is 0, so the induced homomorphism on $\mathbb{Z}$ is $f^*(a) = 0$, the homomorphism sending everything to 0.

12. From complex analysis, $W(f(\alpha(t))) = Z_f - P_f = 4 - 1 = 3$. No idea how to approach with induced maps on the fundamental group of S^1 or $\mathbb{C} - \{0\}$.

13. Let M be the mobius strip, identified as $I \times I/(t, 0) \sim (1-t, 1)$, and let $x_0 = [(1, \frac{1}{2})] = [(0, \frac{1}{2})]$. Let X be the line $(t, \frac{1}{2})$ for $t \in I$; by the identification of the endpoints this is actually a copy of $I/\partial I \cong S^1$ inside of M representing the middle circle of the strip. But then M deformation retracts onto S^1 by just moving every point in $I \times I$ horizontally towards this line, so $M \simeq S^1$ and $\pi_1(M) \cong \mathbb{Z}$, generated by the loop described which we'll call α.

To see what the boundary curve is, label the corners a, b with the suitable identification. Then take a path from a to b on the right-hand boundary of the square. By sliding this through $I \times I$, this is homotopic α. But similarly, the path from b to a on the LHS of the square is also homotopic to α, so the loop $a \to b \to a \simeq \alpha^2$, so if $[\alpha] = 1 \in \pi_1(M)$, then $[a \to b \to a] = 2$.

11. First note that $\pi_1(S^1 \times S^1) \cong F^2$, the free group on two generators, say $[\alpha], [\beta]$ corresponding to the two nontrivial loops on the torus - say α is the longitudinal loop, and β is the meridian. Then if γ is a loop on a torus, then you can just count how many times it winds longitudinally and around the meridian, say m and n times respectively. Then γ can be homotoped into m copies of α and n copies of β based at x_0. So the induced map is $f_\sharp : F^2 \to F^2$ given by $\alpha \mapsto \alpha^m, \beta \mapsto \beta^n$. Since $F^2 \cong Z \times Z$, we equivalently have $[\alpha] = (1, 0), [\beta] = (0, 1)$, and then $f_\sharp : Z^2 \to Z^2$ is given by $(1, 0) \mapsto (m, 0)$ and $(0, 1) \mapsto (0, n)$.

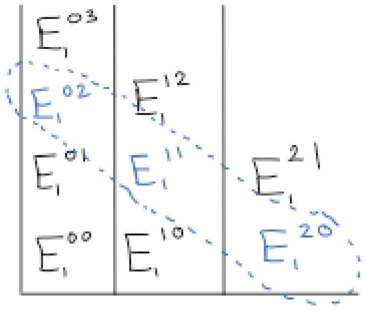
3 Group Actions

1.

4 Covering Spaces

1. Any covering map $p : S^1 \times S^1 \to \mathbb{RP}^2$ would induce an injection on fundamental groups, but $\pi_1(T) = \mathbb{Z}^2$ and $\pi_1(\mathbb{Z}_2)$ - but there are no homomorphisms between these groups. Why? One of them has an element of order 2, the other does not.

2. Theorem: if $M_g \twoheadrightarrow M_h$ is an $n-$sheeted covering space, then $g = n(h - 1) + 1$.



3. Draw CW square for T and cut down the center to see two copies of K.
4. Let $p : \tilde{G} \twoheadrightarrow G$ be such a covering, $a, b \in \tilde{G}$, we then want to show that $p(a)p(b) = p(a \star b)$ for some group operation $\star$ which we need to construct.

Pick a basepoint $x \in G$ and any point $\tilde{x} \in p^{-1}(x)$. Since $\tilde{G}$ is path connected, pick two paths α, β from $\tilde{x}$ to a, b respectively.

Now define a path $f : I \to G$ by $f(t) = (p \circ \alpha)(t) \cdot (p \circ \beta)(t)$, that is, evaluating f, g at a given time in $\tilde{G}$, projecting the results down into G, and multiplying them there. By uniqueness of path lifting, this yields a lift $\tilde{f} : I \to \tilde{G}$

Then define $a \star b = \tilde{f}(1)$, the endpoint of $\tilde{f}$ in $\tilde{G}$. Then by construction,

$p(a \star b) = p(\tilde{f}(1)) = f(1) = (p \circ \alpha)(1) \cdot (p \circ \beta)(1) = p(a)p(b)$. (Need to show this is continuous, and doesn't depend on α, β?)

5. Since $T^n = \prod_n S^1$, we have $\pi_1(T^n) = \prod_n \pi_1(S^1) = \mathbb{Z}^n$. We can also construct a cover $p : \mathbb{R}^n \to T^n$ by just taking $\mathbb{R} \twoheadrightarrow S^1$ the usual cover in each coordinate, yielding the covering space $\tilde{X} = \mathbb{R}^n$ over $X = T^n$.

By Hatcher (prop 4.1), the induced maps $p_*^i : \pi_i(\tilde{X}) \to \pi_i(X)$ is an isomorphism for $i \geq 2$. But $\pi_i(\mathbb{R}^n) = 0$ for $i \neq 0$, so by this isomorphism $\pi_i(T^n) = i \geq 2$.

6. General construction: construct a tree T by picking a basepoint in G and adding a vertex for every non-backtracking walk in G.

In this case, it's the infinite 3-valent graph (also called the infinite $k-$regular tree)

This is the universal cover, because T is connected and acyclic (i.e. a tree). This means that $\pi_1(T) = 0$, so T is simply connected. Since universal covers are simply connected and unique up to isomorphism, this is it.

7. Generators of the subgroups:
8. $\langle ab^{-1}, aba^{-2}, a^3b^{-1}a^{-2}, a^3 \rangle$
9. $\langle b, aba^{-1}, a^2ba^{-2}, a^3 \rangle$
10. $\langle b^2, ba, a^3, aba^{-1} \rangle$
11. $\langle b \rangle$
12. $\langle ba, b^{-1}a \rangle$

Relevant covers:

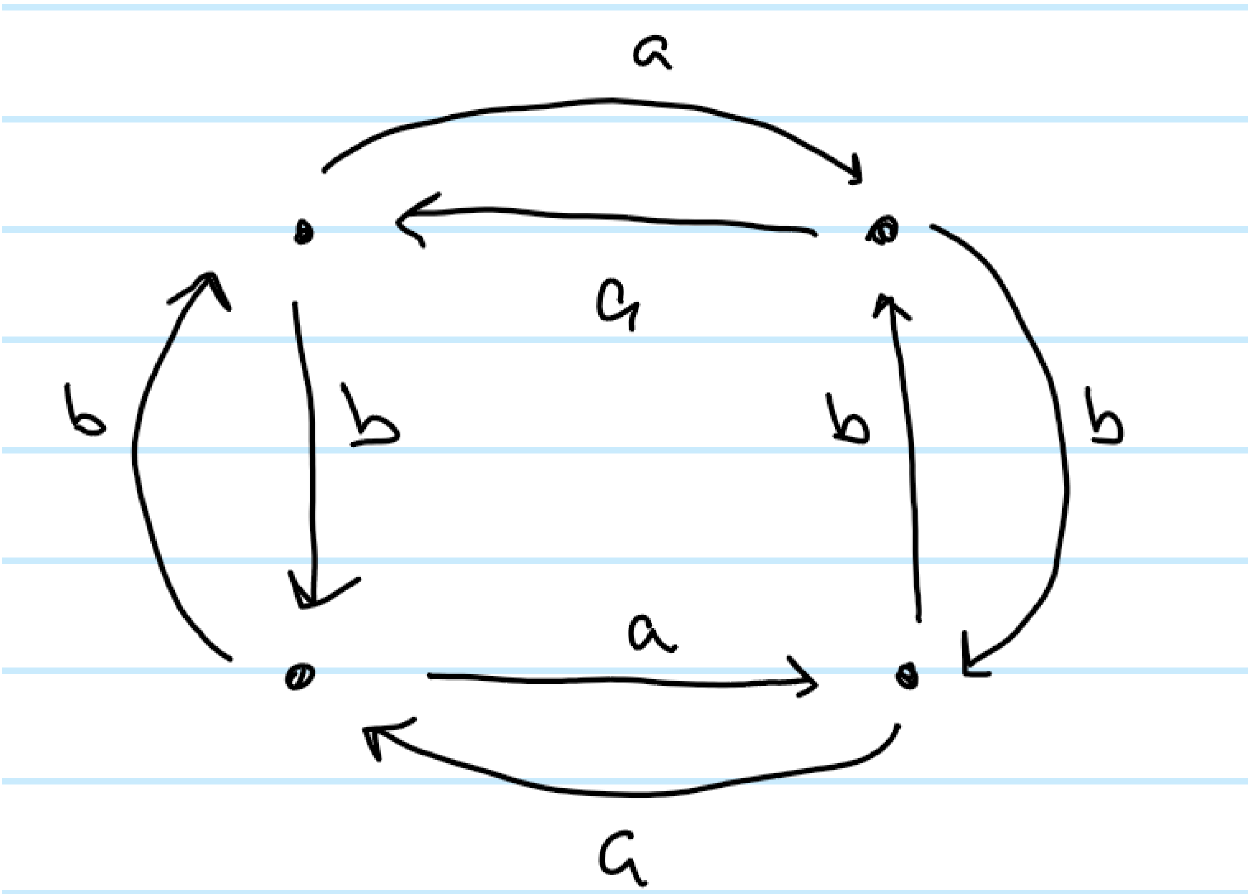


Figure 3: 1512964258737

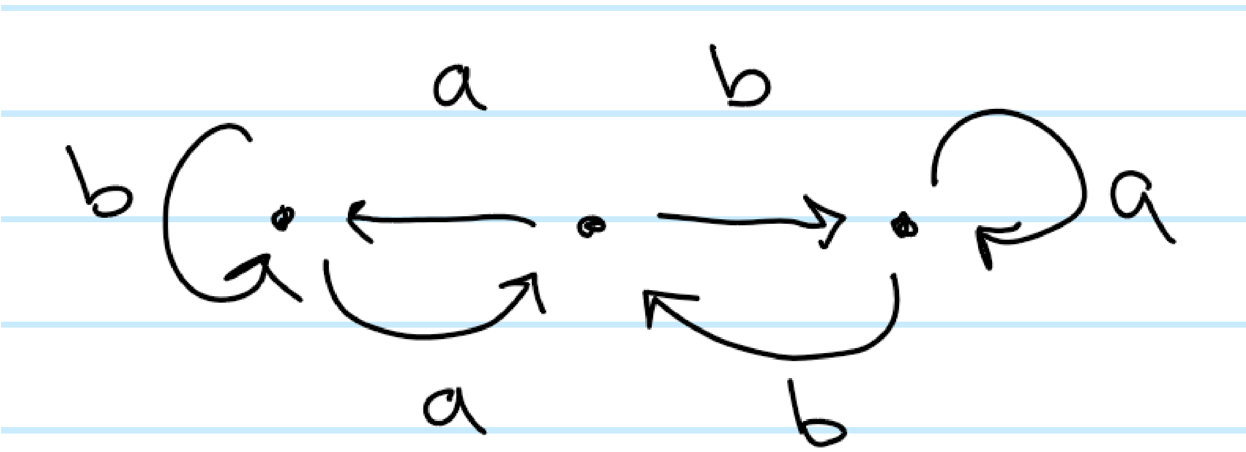


Figure 4: 1512964650272

1.
2.
3.
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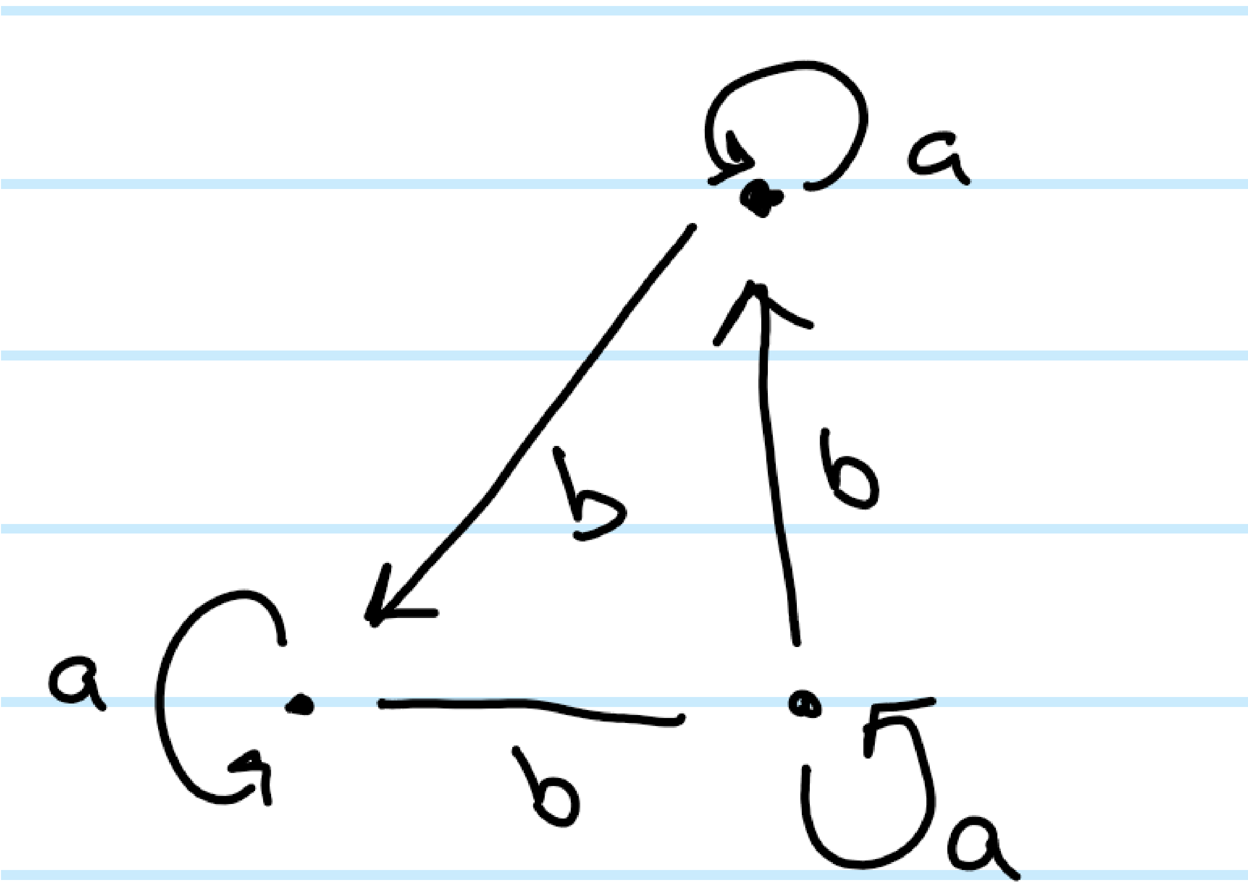


Figure 5: 1512965253808

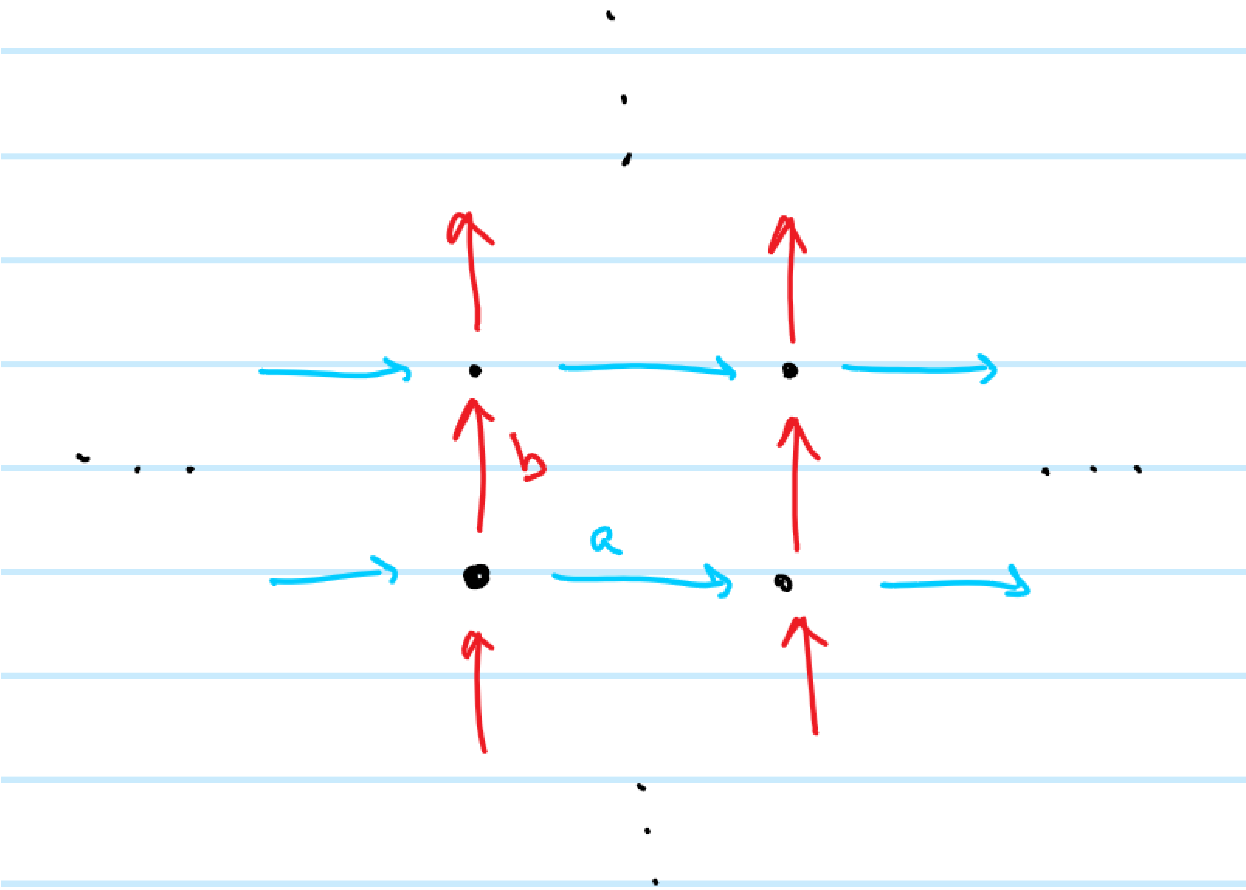
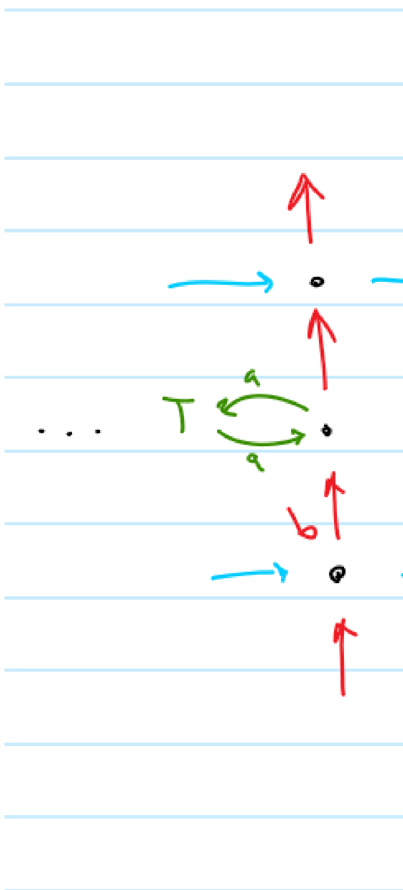
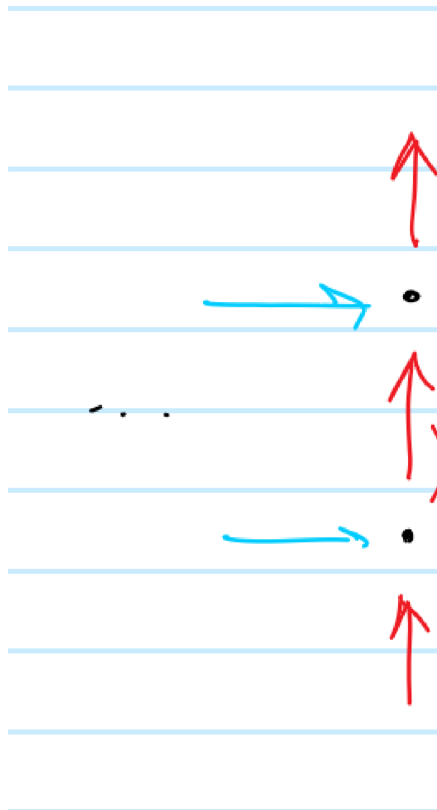


Figure 6: 1512965792844



5. Let T be a copy of the Cayley Tree on two on the two generators a, b, then:

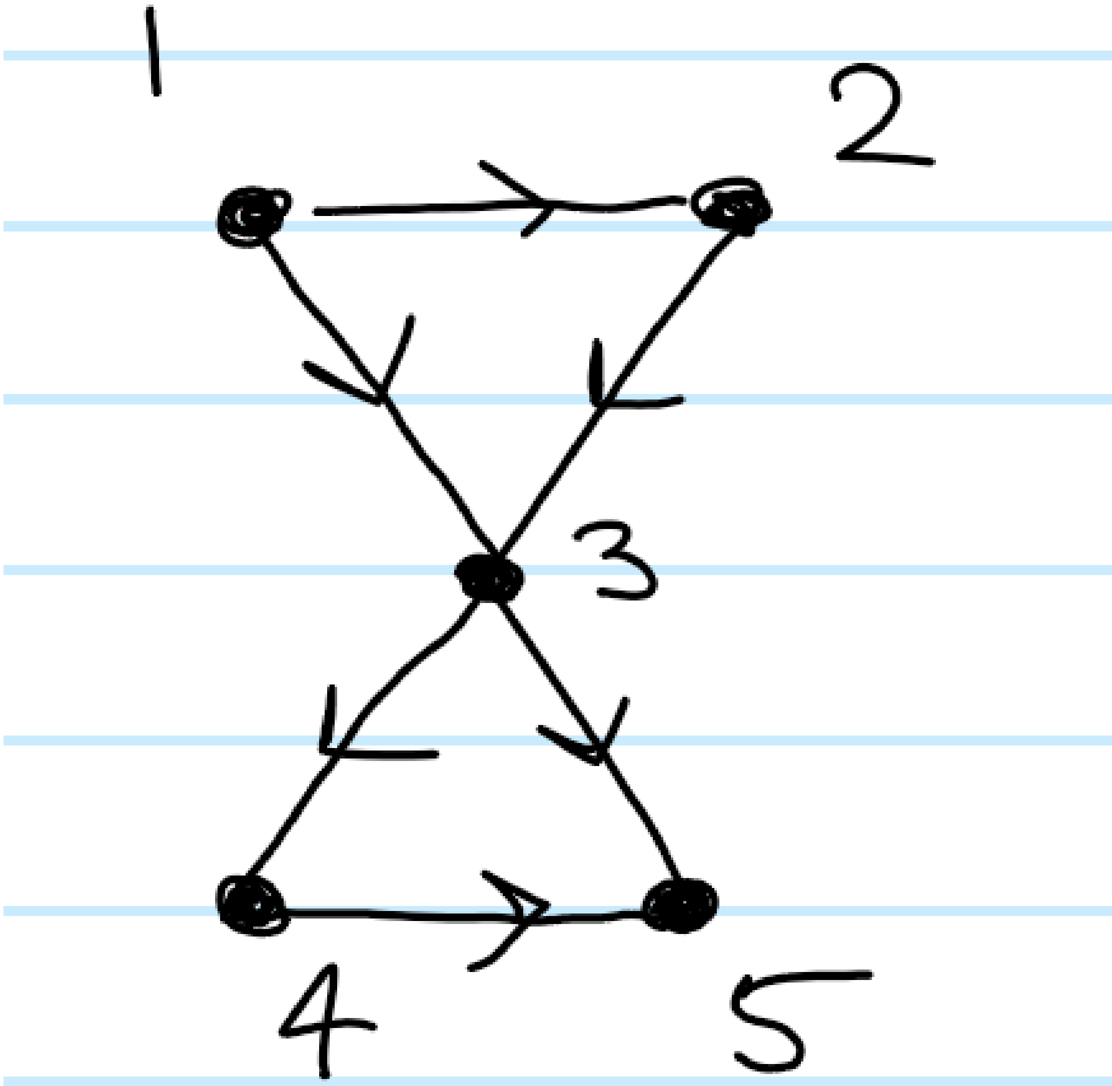
6. This is just the Cayley graph over $\mathbb{Z}\times\mathbb{Z}$, or essentially the integer lattice:



7. It's helpful to note that $\langle(1,0),(0,p)\rangle \subset \langle(1,0),(0,1)\rangle \cong \mathbb{Z}\times\mathbb{Z} \subset \mathbb{R}\times\mathbb{R}$ is an index p subgroup.

4.1 Simplicial Homology

1. Todo



2. Figure 8
   Here we have: $C_3 = \emptyset$ $C_1 = [12], [23], [13], [34], [35], [45] \cong \mathbb{Z}^6$ $C_0 = [1], [2], [3], [4], [5] \cong \mathbb{Z}^5$

So we have $C_2 \to C_1 \to C_0 \cong 0 \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^5 \xrightarrow{\partial_0} 0$

Computing boundary operators, we have

$\partial_1([12]) = [2] - [1]$ $\partial_1([23]) = [3] - [2]$ $\partial_1([13]) = [3] - [1]$ $\partial_1([34]) = [4] - [3]$ $\partial_1([35]) = [5] - [3]$ $\partial_1([45]) = [5] - [4]$

$\partial_0 = 0$

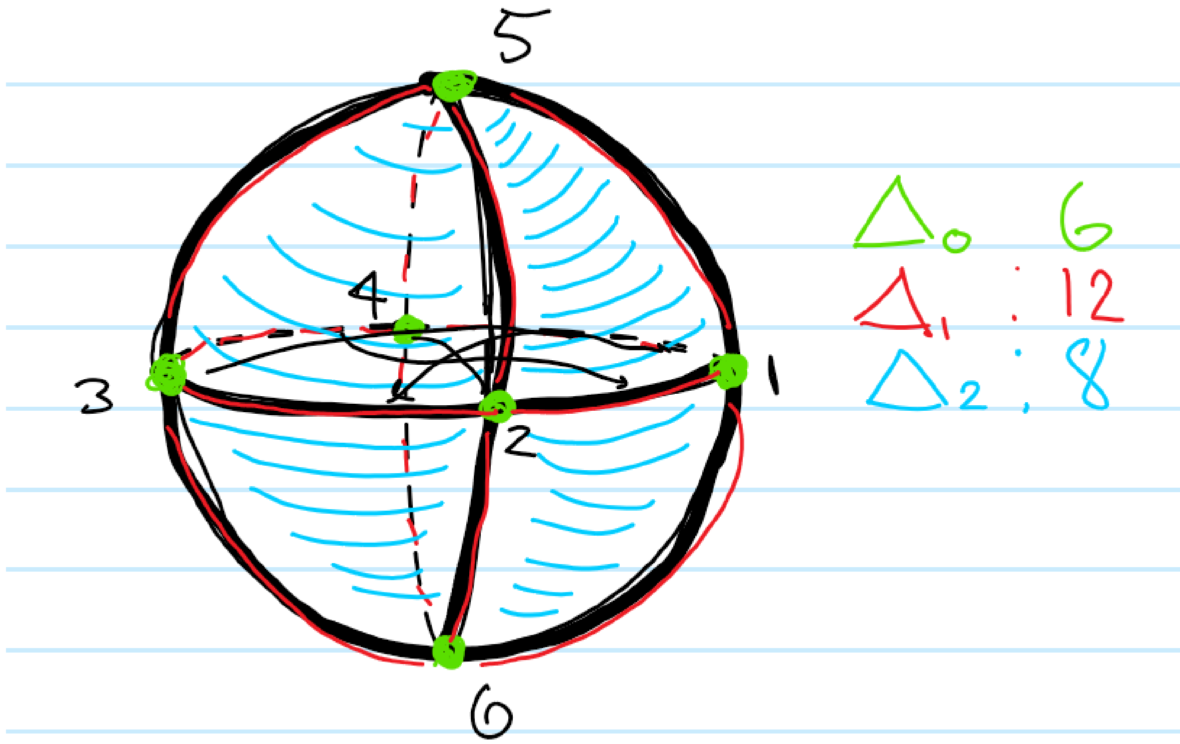
And so $H_0 = \ker \partial_0 / \mathrm{im}\ \partial_1 = \frac{C_0}{<\partial_1([ij])>}$, but from the above calculation we have $[5] = [4] = [3] = [2] = [1]$ in the quotient, so there is just one generator and $H_0 \cong \mathbb{Z}$.

Note that ∂_2 is an injection from 0 into C_1, since there are no 2-simplices. Moreover, one can generate two 1-cycles, so we have $H_1 = \frac{\ker \partial_1}{\mathrm{im}\ \partial_2} = \frac{<[23]-[31]+[12],\ [45]-[35]+[34]>}{0} \cong \mathbb{Z}^2$.

One way to see that these are the generators is to pretend there are two 2-simplices, $[123], [345]$

and compute ∂_2 of both of them. Since $\partial_1\partial_2 = 0$, anything in the image of ∂_2 would have to go to zero anyways, and would thus be in the kernel of ∂_1. Since it's not actually the boundary of any 2-chain, it doesn't become trivial in homology.

So we have $H_2 \to H_1 \to H_0 = 0 \to \mathbb{Z}^2 \to \mathbb{Z}$.



2. S^2

   So we have $C_0 = 1, 2, 3, 4, 5, 6$ $C_1 = 12, 14, 15, 16, 23, 25, 26, 34, 35, 36, 45, 46$ $C_2 = 126, 236, 346, 146, 125, 23$

$C_3 = \emptyset$

And $0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \cong 0 \xrightarrow{\partial_3} \mathbb{Z}^8 \xrightarrow{\partial_2} \mathbb{Z}^{12} \xrightarrow{\partial_1} \mathbb{Z}^6 \xrightarrow{\partial_0} 0$ We have $\partial_1([ij]) = j - i$ and $\partial_2([ijk]) = jk - ik + ij$.

We know in advance we should have $\prod H_n = (\cdots, 0, \mathbb{Z}, 0, \mathbb{Z})$.

For $H_0 = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{C_0}{\langle \{j-i | i<j\} \rangle}$. In the quotient, we see $1 = 6 = 3 = 2 = 5 = 4$ by just taking the indicated walk on the graph, so there is one generator in the quotient and $H_0 \cong \mathbb{Z}$.

For $H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2}$, we just note that there are 6 2-cycles, so each are in the kernel of ∂_1, but each of them comes from a 2-cell, so is in the image of ∂_2. So both groups in question are $\mathbb{Z}^8$, and the quotient is zero. For $H_3 = \frac{\ker \partial_2}{\operatorname{im} \partial_3}$, since $\operatorname{im} \partial_3 = 0$, we can just look at $\partial_3([123456]) = 23456 - 13456 + 12456 - 12356 + 12346 - 12345$. This is an element (and the only one) that goes to zero under ∂_2, it generates $\ker \partial_2$. So there is one generator, and $H_3 = \mathbb{Z}$.

3. $\mathbb{RP}^2$

4. $S^2 \bigcup_f D^2$, where f attaches to the equator

5. $T \bigcup_f D^2$, where f attaches inside the torus

5 Mayer Vietoris Problems

6 $\mathbb{RP}^2$

We start with a few known facts. Let $A = M$, the Mobius strip, and $B = D^2$, the solid disk.

- $\mathbb{RP}^2 = M \coprod_\partial D^2$
- $H_*(M) = H_*(S^1)$, by a deformation retract of M onto its center circle.
- $H_*(D^2) = \mathbb{Z}\delta_0$
- $H_*(S^1) = \mathbb{Z}(\delta_0 + \delta_1)$
- $M \bigcap D^2 = \partial M = S^1$

From Mayer-Vietoris, we have

$$\begin{tikzcd}
 & & \cdots 0 \arrow[dll, "\delta_3"] \\
H_2\partial M \arrow[r, "(i^*,-j^*)_2"] & H_2M \oplus H_2D^2 \arrow[r, "(l^*-r^*)_2"] & H_2\mathbb{RP}^2 \arrow[dll, "\delta_2"] \\
H_1\partial M \arrow[r, "(i^*,-j^*)_1"] & H_1M \oplus H_1D^2 \arrow[r, "(l^*-r^*)_1"] & H_1\mathbb{RP}^2 \arrow[dll, "\delta_1"] \\
H_0\partial M \arrow[r, "(i^*,-j^*)_0"] & H_0M \oplus H_0D^2 \arrow[r, "(l^*-r^*)_0"] & H_0\mathbb{RP}^2 \arrow[dll, "\delta_0"] \\
0 & &
\end{tikzcd}$$

and plugging in what is known yields

$$\begin{tikzcd}
 & & 0 \arrow[dll, "\delta_3"] \\
0 \arrow[r, "(i^2,-j^2)"] & 0 \oplus 0 \arrow[r, "l^2-r^2"] & H_2\mathbb{RP}^2 \arrow[dll, "\delta_2"] \\
\mathbb{Z} \arrow[r, "(i^1,-j^1)"] & \mathbb{Z} \oplus 0 \arrow[r, "l^1-r^1"] & H_1\mathbb{RP}^2 \arrow[dll, "\delta_1"] \\
\mathbb{Z} \arrow[r, "(i^0,-j^0)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "l^0-r^0"] & H_0\mathbb{RP}^2 \arrow[dll, "\delta_0"] \\
0 & &
\end{tikzcd}$$

where $i : S^1 \to M$ and $j : S^1 \to D^2$.

We can then identify all of the induced maps:

- $i^2 : H_2\partial M \to H_2M \implies i^2 : 0 \to 0 \implies i^2 = 0$
- $i^1 : H_1\partial M \to H_1M$, i.e. $i^1 : \mathbb{Z} \to \mathbb{Z}$ where $1 \mapsto 2$
  - Since M deformation retracts onto its center circle, $H_1M \cong H_1S_M$ where S_M is the center circle (homotopies induce isomorphisms on homology). But $H_1\partial M$ is generated by a cycle of edges with includes into ∂M, which retracts onto a cycle that double covers S_M, so this map acts by doubling the generator.
- $i^0 : H_0\partial M \to H_0M$, i.e. $i^0 : \mathbb{Z} \to \mathbb{Z}$
- $j^2 : H_2\partial M \to H_2D^2 \implies j^2 : 0 \to 0 \implies j^2 = 0$
- $j^1 : H_1\partial M \to H_1D^2 \implies j^1 : \mathbb{Z} \to 0 \implies j^1 = 0$
- $j^0 : H_0\partial M \to H_0D^2 \implies j_0 : \mathbb{Z} \to \mathbb{Z}$

So we can that the only nontrivial maps are j^0, i^0, i^1.

6.1 Claim: $H_2(\mathbb{RP}^2) = 0$:

We consider the portion of the sequence

$$\cdots 0 \to H_2\mathbb{RP}^2 \xrightarrow{\delta_2} H_1\partial M \xrightarrow{(i^1,-j^1)} H_1M \oplus H_1D^2 \cdots$$
$$\cdots 0 \to H_2\mathbb{RP}^2 \xrightarrow{\delta_2} \mathbb{Z} \xrightarrow{(i^1,-j^1)} \mathbb{Z} \oplus 0 \cdots$$

We will show that $\ker \delta_2 = \text{im } \delta_2 = 0$. By the first isomorphism theorem, we would then have $\frac{H_2\mathbb{RP}^2}{\ker \delta_2} \cong \text{im } \delta_2$ yielding $\frac{H_2\mathbb{RP}^2}{0} = H_2\mathbb{RP}^2 \cong 0$.

- *Claim:* $\ker \delta_2 = 0$

  This follows because it is on the left tail of an exact sequence, where $\ker \delta_2 = \text{im } 0 = 0$.

- *Claim: im* $\delta_2 = 0$

$$(i^1, -j^1) : H_1 \partial M \to H_1 M \oplus H_1 D^2$$

is injective; explicitly, it is the map

$$\begin{aligned} M_2 : \mathbb{Z} &\to \mathbb{Z} \oplus 0 \\ 1 &\mapsto (2, 0) \end{aligned}$$

From above, know that $-j^1$ is a zero map, and that i^1 doubles each generator. By this explicit construction, it is injective since 0 maps to 0.

But then $\ker(i^1, -j^1) = \text{im } \delta_2 = 0$ by exactness.

So now we have:

$$\begin{tikzcd}
& & 0 \arrow[dll, "0"'] \\
0 \arrow[r, "0\times 0"] & 0 \oplus 0 \arrow[r, "0"] & 0 \arrow[dll, "0"'] \\
\mathbb{Z} \arrow[r, "x\mapsto(2x{,}0)"] & \mathbb{Z} \oplus 0 \arrow[r, "l^1 - r^1"] & H_1\mathbb{RP}^2 \arrow[dll, "\delta_1"'] \\
\mathbb{Z} \arrow[r, "(i^0{,}-j^0)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "l^0 - r^0"] & H_0\mathbb{RP}^2 \arrow[dll, "\delta_0"'] \\
0 & &
\end{tikzcd}$$

6.2 Claim: $H_1(\mathbb{RP}^2) = \mathbb{Z}_2$

Here we are examining this portion of the sequence:

$$\begin{aligned} \cdots \mathbb{Z} \xrightarrow{x\mapsto(2x,0)} H_1 M \oplus H_1 D^2 \xrightarrow{l^1 - r^1} H_1\mathbb{RP}^1 \xrightarrow{\delta_1} H_0 \partial M \xrightarrow{(i^0,-j^0)} H_0 M \oplus H_0 D^2 \cdots \\ \cdots \mathbb{Z} \xrightarrow{x\mapsto(2x,0)} \mathbb{Z} \oplus 0 \xrightarrow{l^1 - r^1} H_1\mathbb{RP}^1 \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{(i^0,-j^0)} \mathbb{Z} \oplus \mathbb{Z} \cdots \end{aligned}$$

In general, we have the first isomorphism theorem: given any map f we have $\frac{\text{dom} f}{\ker f} \cong \text{im } f$. Here we will take $f = l^1 - r^1$ and identify the necessary components to apply this theorem.

- Claim: im $l^1 - r^1 = H_1\mathbb{RP}^2$.

  - We use the fact that the maps (i^*, j^*) are all injections, so in particular $0 = \ker(i^0, j^0) = \text{im } \delta_1$ by exactness. Consequently $\ker \delta_1 = H_1 \mathbb{RP}^1 = \text{im } l^1 - r^1$ by exactness.
- What is $\ker(l^1 - r^1)$?
  - By exactness, $\ker(l^1 - r^1) = \text{im } (x \mapsto (2x, 0)) = 2\mathbb{Z} \oplus 0$

By the first isomorphism theorem, we have $\text{im } (l^1 - r^1) \cong \frac{\text{dom}(l^1 - r^1)}{\ker(l^1 - r^1)} = \frac{\mathbb{Z} \oplus 0}{2\mathbb{Z} \oplus 0} \cong \mathbb{Z}_2$.

Note that $l^1 - r^1$ is a nontrivial homomorphism from $2\mathbb{Z} \cong \mathbb{Z}$ to $\mathbb{Z}_2$, of which there is only one: the natural quotient map $x \mapsto x \mod 2$.

There is also no nontrivial homomorphism from $\mathbb{Z}_2 \to \mathbb{Z}$, so $\delta_1 = 0$.

We now have:

$$
\begin{tikzcd}
 & & 0 \arrow[dll, "0"'] \\
0 \arrow[r, "0 \times 0"] & 0 \oplus 0 \arrow[r, "0"] & 0 \arrow[dll, "0"'] \\
\mathbb{Z} \arrow[r, "x \mapsto (2x,0)"] & \mathbb{Z} \oplus 0 \arrow[r, "x \mapsto x \mod 2"] & \mathbb{Z}_2 \arrow[dll, "0"'] \\
\mathbb{Z} \arrow[r, "(i^0, -j^0)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "l^0 - r^0"] & H_0 \mathbb{RP}^2 \arrow[dll, "\delta_0"'] \\
0 & &
\end{tikzcd}
$$

6.3 Claim: $H_0(\mathbb{RP}^2) = \mathbb{Z}$

Here we examine

$$
H_1 \mathbb{RP}^2 \xrightarrow{\delta_1} H_0 \partial M \xrightarrow{(i^0, j^0)} H_0 M \oplus H_0 D^2 \xrightarrow{l^0 - r^0} H_0 \mathbb{RP}^2 \xrightarrow{\delta_0} 0
$$
$$
\mathbb{Z}_2 \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{(i^0, j^0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{l^0 + r^0} H_0 \mathbb{RP}^2 \xrightarrow{\delta_0} 0
$$

Since there is no nontrivial homomorphism from $\mathbb{Z}_2 \to \mathbb{Z}$, we have $\delta_1 = 0$.

We also have $\delta_0 = 0$ and $\ker \delta_0 = H_0 \mathbb{RP}^2 = \text{im } l^0 + r^0$ making $l^0 + r^0$ surjective, so by the first isomorphism theorem we have $H_0 \mathbb{RP}^2 \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\ker l^0 + r^0} = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\text{im } (i^0, j^0)}$

By a similar argument used earlier, the double covering of the boundary circle ∂M over S^1 yields the map $(i^0, j^0) : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ given by $x \mapsto (2x, 2x)$ with

6.4 Summary

With all of this information, we finally have

$$\begin{tikzcd}
 & & 0 \arrow[dll, "0"'] \\
0 \arrow[r, "0\mapsto(0{,}0)"] & 0 \oplus 0 \arrow[r, "(0{,}0)\mapsto 0"] & 0 \arrow[dll, "0"'] \\
\mathbb{Z} \arrow[r, "x\mapsto(2x{,}0)"] & 2\mathbb{Z} \oplus 0 \arrow[r, "(x{,}0)\mapsto x \mod 2"] & \mathbb{Z}_2 \arrow[dll, "0"'] \\
\mathbb{Z} \arrow[r, "x\mapsto(2x{,}x)"] & 2\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "(x{,}y)\mapsto x-y"] & \mathbb{Z} \arrow[dll, "0"'] \\
0 & &
\end{tikzcd}$$

And so we find $H_*(\mathbb{RP}^2) = \mathbb{Z}\delta_0 + \mathbb{Z}_2\delta_1$

7 Cellular Homology

8 Degree

9 UCT

10 Homological Algebra