Chapter 9

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Contents

Important Ideas:

- Compactness of $\mathcal{L}(x, y)$.
- $\partial^2 = 0$.
- Using broken trajectories to compactify
- Gluing

1 | Background, Notation, Setup

Goals

- Construct Floer homology and prove the Arnold Conjecture

  ("Symplectic Morse Inequalities?"):

$$\# \left\{ \text{1-Periodic trajectories of } X_H \right\} \geq \sum_{k \in \mathbb{Z}} HM_k(W; \mathbb{Z}/2\mathbb{Z}).$$

> Here $HM_*(W)$ is the Morse homology.

Strategy:

1. Define the action functional $\mathcal{A}_H$.
2. Construct the chain complex (graded vector space) CF_*.
3. Define X_H, which will be used to define ∂ later.
4. Count trajectories.
5. Show finite-energy trajectories connect critical points of $\mathcal{A}_H$.
6. Show compactness property for space of trajectories of finite energy.
7. Define ∂ (uses a compactness property in 9.1c)
8. Show space of trajectories is a manifold (plus genericity, "Smale property")
9. Show that $\partial^2 = 0$.
10. Show that HF_* doesn't depend on $\mathcal{A}_H$ or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_*.

Ingredients:

- (W, ω) with $\omega \in \Omega^2(W)$ is a symplectic manifold with an almost complex structure J.
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in $T_x M$:

$$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$

  - Zeros of vector field X_H correspond to critical points of H:

$$X_H(x) = 0 \iff (dH)_x = 0.$$

  - Take the associated flow $\psi^t : W \to W$, assumed 1-periodic so $\psi^1(x) = x$: critical points of H are periodic trajectories.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}{}_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y \end{aligned}$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \operatorname{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}W$ is the *free loop space* on W, i.e. space of contractible loops on W, i.e. $C^\infty(S^1; W)$ with the C^∞ topology
  - Elements $x \in \mathcal{L}W$ can be viewed as maps $S^1 \to W$.
  - Can extend to maps from a closed disc, $u : \overline{\mathbb{D}}^2 \to M$.

- Loops in $\mathcal{L}W$ can be viewed as maps $S^2 \to W$, since they're maps $I \times S^1 \to W$ with the boundaries pinched:

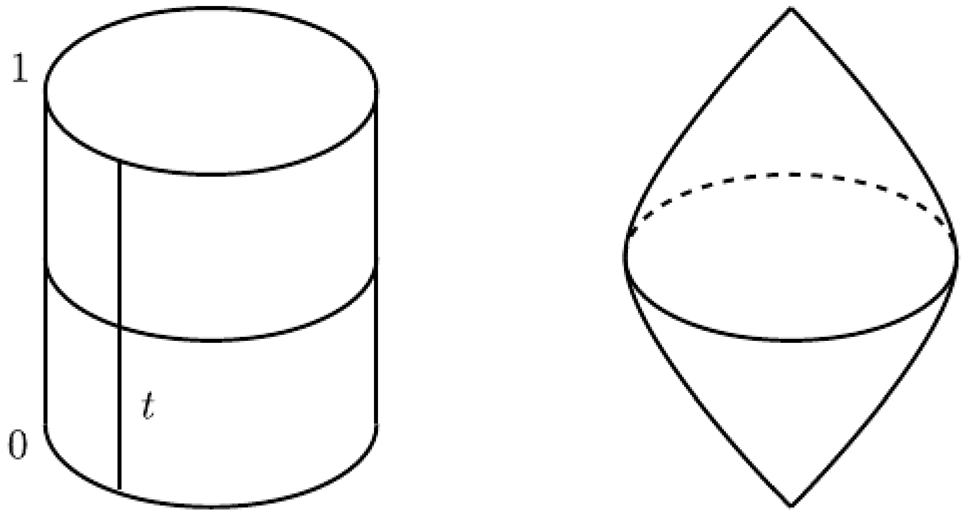

Figure 1: Loops in $\mathcal{L}W$

- The action functional is given by

$$\begin{aligned} \mathcal{A}_H &: \mathcal{L}W \to \mathbb{R} \\ & x \mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \displaystyle\int_0^1 (H_t\ dt - p\ dq)$.
  - Correspondence between trajectories of the gradient of $\mathcal{A}_H$ and solutions to Floer equations.

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of $\mathcal{A}_H$.
- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, W)$,

$$\int_{S^2} u^*\omega = 0 \quad \text{or equivalently} \quad \langle \omega,\ \pi_2 W \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM, equivalently

$$\langle c_1 TW,\ \pi_2 W \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- Maslov index: used the fact that

- $\mathrm{Sp}(2n, \mathbb{R})$ retracts onto $U(n)$: use a polar decomposition $S = PQ$ as a PSD times orthogonal, then homotope P to I.
- $\pi_1 U_n = \mathbb{Z}$: use $U(n, \mathbb{C}) \simeq SU(n, \mathbb{C}) \times S^1$ by the determinant, and $\pi_1 SU(n, \mathbb{C}) = 0$.
- Thus every path in $\gamma : I \to \mathrm{Sp}(2n, \mathbb{R})$ can be assigned an integer by getting a map $\tilde{\gamma} : I \to S^1$ and taking (approximately) its winding number.

- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y.
  - After perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
  - How we did it:
    * Describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$

      (Banach manifold modeled on the Sobolev spaces $W^{1,p}$),
    * Apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality.
    * Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

2 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

$$\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$$

$$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x,y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting x to y:

$$\mathcal{L}(x, y) \coloneqq \mathcal{M}(x, y)/\mathbb{R}.$$

  - Use the quotient topology, define sequentially:

$$\tilde{u}_n \overset{n\to\infty}{\longrightarrow} \tilde{u} \quad \iff \quad \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot) \overset{n\to\infty}{\longrightarrow} u(s, \cdot).$$

  - When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

$$n(x, y) \coloneqq \#\mathcal{L}(x, y)$$

    is well-defined.

- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k.
  - Finitely many since they are nondegeneracy implies they are isolated.

Remark 1.
Some notation:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathcal{M}(x,z) \arrow[d, "\pi"] \\ & \mathcal{L}(x,z) \end{tikzcd}$$

Hats will generally denote maps induced on quotient.

- Defined a differential

$$\begin{aligned} \partial : C_k(H) &\to C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x,y)y \end{aligned}$$

$$\begin{aligned} n(x,y) &\coloneqq \#\left\{\text{Trajectories of grad } \mathcal{A}_H \text{ connecting } x,y\right\} \mod 2 \\ &= \#\mathcal{L}(x,y) \mod 2. \end{aligned}$$

- Examined ∂^2:

$$\begin{aligned} \partial^2 : C_k(H) &\to C_{k-2}(H) \\ x &\mapsto \partial(\partial(x)) \\ &= \partial\left(\sum_{\mu(y)=\mu(x)-1} n(x,y)y\right) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x,y)\partial(y) \\ &= \sum_{\mu(y)=\mu(x)-1} n(x,y)\left(\sum_{\mu(z)=\mu(y)-1} n(y,z)z\right) \\ &= \sum_{\mu(y)=\mu(x)-1} \sum_{\mu(z)=\mu(y)-1} n(x,y)n(y,z)\, z \\ &= \sum_{\mu(z)=\mu(y)-1} \left(\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z)\right) z \qquad \text{(finite sums, swap order)}, \end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x,z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\overline{\mathcal{L}}(x,y)$.
- We'll then have

$$\overline{\mathcal{L}}(x,z) = \mathcal{L}(x,z) \cup \partial\overline{\mathcal{L}}(x,z), \qquad \partial\overline{L}(x,z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x,y) \times \mathcal{L}(y,z),$$

  which "space-ifies" the equation we want.
- We'll show $\partial\overline{\mathcal{L}}(x,z)$ is a 1-manifold, which must have an even number of points, and thus

$$\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = \#\Big(\partial\overline{\mathcal{L}}(x,z)\Big) \equiv 0 \mod 2.$$

3 | Three Important Theorems

3.1 First Theorem: Convergence to Broken Trajectories

- Recall: *broken trajectories* are unions of intermediate trajectories connecting intermediate critical points.
- Shown last time: a sequence of trajectories can converge to a broken trajectory, i.e. there are broken trajectories in the closure of $\mathcal{L}(x, z)$.
- This theorem describes their behavior:

Theorem 3.1 ***(9.1.7: Convergence to Broken Trajectories).***
Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, z)$, then there exist

- A subsequence $\left\{u_{n_j}\right\}$
- Critical points $\{x_0, x_1, \cdots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = z$
- Sequences $\left\{s_n^1\right\}, \left\{s_n^2\right\}, \cdots, \left\{s_n^\ell\right\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_{n_j} \cdot s_n^k \overset{n\to\infty}{\longrightarrow} u^k.$$

- Upshots:
  - Every sequence upstairs has a subsequence which (after reparameterizing) converges
  - This descends to actual convergence after quotienting by $\mathbb{R}$?
  - Yields uniqueness of limits in $\mathcal{L}(x, z)$, thus a separated topology
  - Sequentially compact $\iff$ compact since $\mathcal{L}(x, z)$ is a metric space?

Corollary 3.2 ***(Compactness).***
$\overline{\mathcal{L}}(x, z)$ is compact.

3.2 Second Theorem: Compactness of $\overline{\mathcal{L}}(x,z)$

Definition 3.2.1 (Regular Pair)**.**
For an almost complex structure J and a Hamiltonian H, the pair (H,J) is **regular** if the Floer map $\mathcal{F}$ is transverse to the zero section in the following vector bundle:

$$\begin{tikzcd}
E_u \coloneqq \{\text{Vector fields tangent to } M \text{ along } u\} \arrow[r] & C^\infty(\mathbb{R}\times S^1; TM) \arrow[d] \\
& C^\infty(\mathbb{R}\times S^1; M) \arrow[u, bend left, dotted, "\mathcal{F}"] \arrow[u, bend right, dotted, "\mathbf{0}"']
\end{tikzcd}$$

Most of chapter 9 is spent proving this theorem:

Theorem 3.3 ***(9.2.1).***
Let (H,J) be a regular pair with H nondegenerate and x,z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\overline{\mathcal{L}}(x,z)$ is a compact 1-manifold with boundary with

$$\partial\overline{\mathcal{L}}(x,z) = \bigcup_{y\in\mathcal{I}(x,z)} \mathcal{L}(x,y)\times\mathcal{L}(y,z) \qquad \text{where} \quad \mathcal{I}(x,z) = \left\{ y \;\middle|\; \mu(x) < \mu(y) < \mu(z) \right\}.$$

Note: possibly a typo in the book? Has x,y on the LHS.

Corollary 3.4.

$$\partial^2 = 0.$$

3.3 Third Theorem: Gluing

- We already know that $\overline{\mathcal{L}}(x,z)$ is compact and $\mathcal{L}(x,z)$ is a 1-manifold, so we look at neighborhoods of boundary points.

Theorem 3.5 ***(9.2.3: Gluing).***
Let x, y, z be three critical points of $\mathcal{A}_H$ with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x,y) \times \mathcal{M}(y,z) \quad \rightsquigarrow \quad (\widehat{u}, \widehat{v}) \in \mathcal{L}(x,y) \times \mathcal{L}(y,z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

$$\psi : [\rho_0, \infty) \to \mathcal{M}(x,z)$$

such that $\widehat{\psi}$, the induced map on the quotient

$$\begin{tikzcd} {[\rho_0, \infty)} \arrow[r, "\psi"] \arrow[dr, dotted, "\widehat{\psi}"'] & \mathcal{M}(x,z) \arrow[d, "\pi"] \\ & \mathcal{L}(x,z) \end{tikzcd}$$

is an embedding that satisfies

$$\widehat{\psi}(\rho) \overset{\rho \to \infty}{\longrightarrow} (\widehat{u}, \widehat{v}) \in \overline{\mathcal{L}}(x,z).$$

2. (“Uniqueness”) For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x,z)$,

$$\ell_n \overset{n \to \infty}{\longrightarrow} (\widehat{u}, \widehat{v}) \quad \Longrightarrow \quad \ell_n \in \operatorname{im}(\widehat{\psi}) \text{ for } n \gg 0.$$

4 | Gluing Theorem

Broken into three steps:

1. **Pre-gluing**:

- Get a function w_ρ which interpolates between u and v in the parameter ρ.
    - Not exactly a solution itself, just an “approximation”.

2. **Newton’s Method**:

- Apply the Newton-Picard method to w_p to construct a true solution

$$\begin{aligned}\psi : [-\rho, \infty) &\to \mathcal{M}(x, z) \\ \rho &\mapsto \exp_{w_p}(\gamma(p))\end{aligned}$$

$$\text{for some } \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x, z)$$

  where $\mathcal{P} =?$.

3. **Project and Verify Properties**:

- Check that the projection $\widehat{\psi} = \pi \circ \psi$ satisfies the conditions from the theorem.

5 | 9.3: Pre-gluing

- Choose (once and for all) a bump function β on $B_\varepsilon(0)^c \subset \mathbb{R} \to [0, 1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^\pm(s)$:

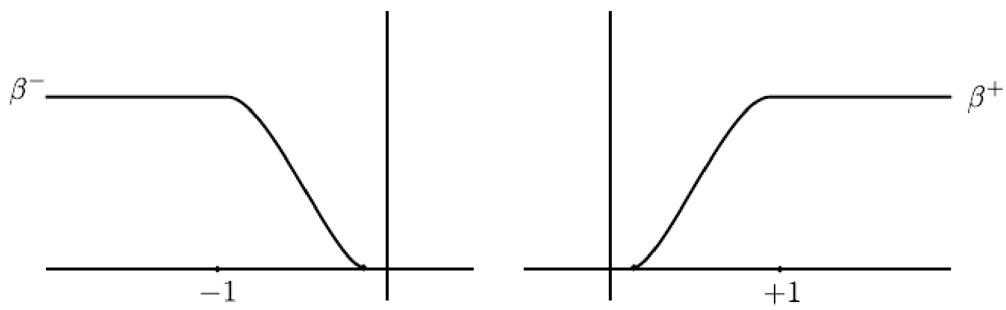

Figure 2: Bump away from zero

- Define an interpolation w_ρ from u to v in the following way: let
  - $\exp[\,\cdot\,] \coloneqq \exp_{y(t)}(\,\cdot\,)$ and
  - $\ln(\,\cdot\,) \coloneqq \exp^{-1}_{y(t)}(\,\cdot\,)$,

  then

$$w_\rho : x \to z$$

$$w_\rho(s, t) \coloneqq \begin{cases} u(s + \rho, t) & s \in (-\infty, -1] \\ \exp\left[\beta^-(s)\ln(u(s + \rho, t)) + \beta^+(s)\ln(u(s - \rho, t))\right] & s \in [-1, 1] \\ u(s - \rho, t) & s \in [1, \infty) \end{cases}.$$

- Why does this make sense?

$$|s| \le 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \;\middle|\; \sup_{t \in S^1} \|Y(t)\| \le r_0 \right\} \subseteq \operatorname{im} \exp_{y(t)}(\,\cdot\,).,$$

so we can apply $\exp_{y(t)}^{-1}(\,\cdot\,)$.

- Can make $|s| \le 1$ for large ρ, since

$$u(s,t) \xrightarrow{s \to \infty} y(t)$$
$$v(s,t) \xrightarrow{s \to -\infty} y(t).$$

  - So pick a ρ_0 such that this holds for $\rho > \rho_0$.
  - Might have to increase ρ_0 later in the proof, so $\rho > \rho_0$ just means $\rho \gg 0$.

- Some properties:
  - $w_\rho \in C^\infty(x,z)$ and is differentiable in ρ.
  - $s \in [-\varepsilon, \varepsilon] \implies w_\rho(s,t) = y(t)$.

$$w_\rho(s - \rho, t) \xrightarrow{\rho \to \infty} u(s,t) \quad \text{in} \quad C^\infty_{\text{loc}}$$

$$w_\rho(s,t) \xrightarrow{\rho \to \infty} y(t) \quad \text{in} \quad C^\infty_{\text{loc}}.$$

- Now carry out the linearized version on tangent vectors
  - Let $Y \in T_u \mathcal{P}(x,y)$
  - Let $Z \in T_v \mathcal{P}(x,y)$
  - Replace w_ρ with the interpolation

$$Y \#_\rho Z \in T_{w_\rho} \mathcal{P}(x,y) = W^{1,p}(w_\rho^* TW).$$

defined by

$$(Y \#_\rho Z)(s,t) = \begin{cases} Y(s+\rho,t) & s \in (-\infty,-1] \\ \exp\left[\beta^-(s) T_{u(s+\rho,t)} \ln(Y(s+\rho,t)) \;+ \beta^+(s) T_{u(s-\rho,t)} \ln(Z(s-\rho,t))\right] & s \in [-1,1] \\ Z(s-\rho,t) & s \in [1,\infty) \end{cases}.$$

6 | 9.4: Construction of ψ.

- Have constructed $w_\rho \in C^\infty_\searrow(x,z) C^\infty(x,z)$ for every $\rho \ge \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x,z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \text{grad } Hx$$

in the weak sense.

- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Trivialization
- Defining $\mathcal{F}_\rho$.

$$
\begin{aligned}
W^{1,p}\left(\mathbf{R}\times S^1;\mathbf{R}^{2n}\right) &\xrightarrow{\mathcal{F}_\rho} L^p\left(\mathbf{R}\times S^1;\mathbf{R}^{2n}\right)\\
(y_1,\dots,y_{2n}) &\longmapsto \left[\left(\frac{\partial}{\partial s}+J\frac{\partial}{\partial t}+\operatorname{grad} H_t\right)\left(\exp_{w_\rho}\sum y_i Z_i^\rho\right)\right]_{Z_i}
\end{aligned}
$$

where $\mathcal{F}_\rho \coloneqq \mathcal{F}\circ\exp_{w_\rho}$ written in the bases Z_i. sd - Newton-Picard method, general idea

- Original method and variant: find the limit of a sequence

$$
x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\qquad x_{n+1}=x_n-\frac{f(x_n)}{f'(x_0)}.
$$

- Allows finding zeros of f given an approximate zero x_0.
- Linearize $\mathcal{F}_\rho$.
- Applying Newton-Picard
  - Decompose $W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})$ as $\ker(L_{\rho_n})\oplus W_{\rho_n}^\perp$.
  - Apply Newton-Picard in $W_{\rho_n}^\perp$
- Will obtain for every $\rho\geq\rho_0$ an element $\gamma(\rho)\in W_\rho^\perp$ with

$$
\mathcal{F}_\rho(\gamma(\rho))=0.
$$

  - Apply the implicit function theorem to show differentiability of γ in ρ.
  - Use a trivialization Z_i^ρ to get a vector field along w_ρ (also called $\gamma(\rho)$)
    * This is exactly an element of $T_{w_\rho}\mathcal{P}(x,z)$
  - Exponentiate to get an element of $\mathcal{M}(x,z)$:

$$
\psi(\rho)\coloneqq\exp_{w_\rho}(\gamma(\rho)).
$$

  - Project onto $\mathcal{L}(x,z)$ to get $\widehat{\psi}$.
  - Prove $\widehat{\psi}$ is a proper injective immersion and thus an embedding.
  - Show that the broken trajectory $(\widehat{u},\widehat{v})$ is the endpoint of an embedded interval in $\overline{\mathcal{L}}(x,z)$.
    * Then show that any other sequence converging to $(\widehat{u},\widehat{v})$ must approach via this interval, otherwise could have cuspidal points:

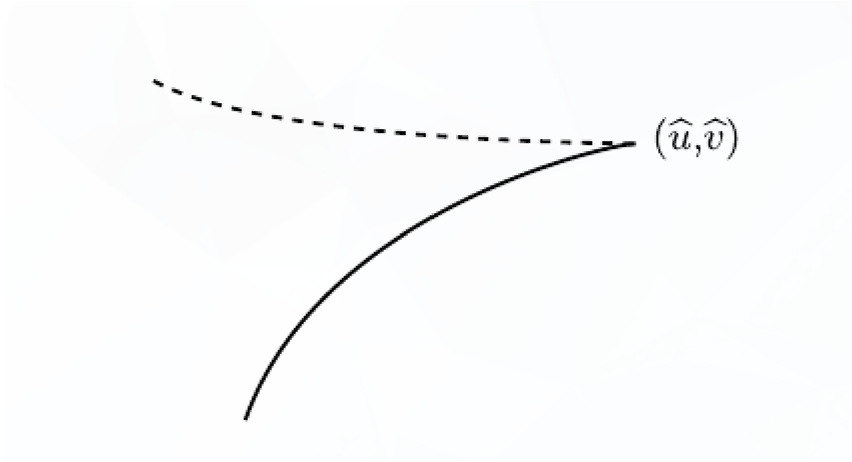

Figure 3: Cuspidal Point on Boundary