Title

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1 General

Definition (Tangent Bundle): $TM = \coprod_{p \in M} T_p M$, which fits into the vector bundle $\mathbb{R}^n \to TM \to M$ so $T_p M \cong \mathbb{R}^n$.

$T_p M = \operatorname{span}_{\mathbb{R}} \{\partial x_i\}$

Definition (Cotangent Bundle): Since $T_p M$ is a vector space, we can consider its dual $T_p^\vee M$, and similarly the cotangent bundle $\mathbb{R}^n \to T^\vee M \to M$.

$T_p^\vee M = \operatorname{span}_{\mathbb{R}} \{dx_i\}$.

Definition (Derivative of a Map): Recall that for any smooth function $H : M \to N$, the *derivative* of H at $p \in M$ is defined by $dH_p : T_p M \to T_p N$ which we define using the derivation definition of tangent vectors: given a derivation $v \in T_p M : C^\infty(M) \to \mathbb{R}$, we send it to the derivation $w_v \in T_q M = C^\infty(M) \to \mathbb{R}$ where w_v actson on $f \in C^\infty(M)$ by precomposition, i.e. $w_v \curvearrowright f = v(f \circ H)$.

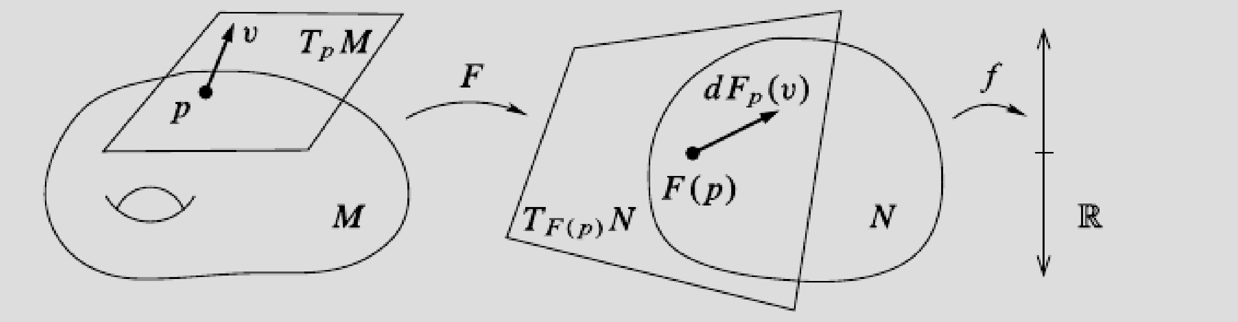


Fig. 3.3 The differential

Definition: Fields and Forms A section of TM is a vector field, and a section of $T^\vee M$ is a 1-form.

More generally, differential k-forms are in $\Omega^k(M) := \Gamma(\Lambda^k T^\vee M)$, i.e. sections of exterior powers of the cotangent bundle.

Definition (Closed and Exact Forms): Let $d_p : \Omega^p(M) \to \Omega^{p+1}(M)$ be the exterior derivative. Then a form ω is *closed* (or is a *cocycle*) iff $\omega \in \ker d_p$, and *exact* (or a *coboundary*) iff $\omega \in \operatorname{im} d_{p-1}$.

Note that closed forms are exact, since $d^2 = 0$, i.e. ω closed implies $\omega = d\lambda$ implies $d\omega = d^2\lambda = 0$ implies ω is exact.

If $\alpha, \beta \in \Omega^p(M)$ with $\alpha - \beta$ exact, they are said to be *cohomologous*.

Definition (Vector Field): A *vector field* X on M is a section of the tangent bundle $TM \xrightarrow{\pi} M$. Recall that these form an algebra $\mathfrak{X}(M)$ under the Lie bracket.

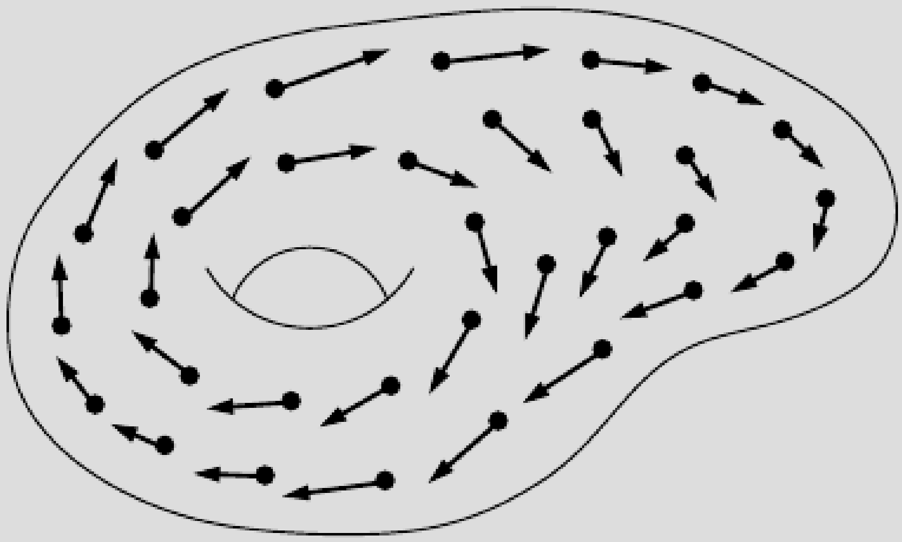


Fig. 8.1 A vector field

Note that vector fields can be *time-dependent* as a section of $T(M \times I) \to M \times I$.

Definition (Regular Value): Let $H : M \to \mathbb{R}$ be a smooth function, then $c \in \mathbb{R}$ is a *regular value* iff for every $p \in H^{-1}(c)$, the induced map $H^* : T_pM \to T_P\mathbb{R}$ is surjective.

Definition (Closed Orbit): An *closed orbit* of a vector field X on M is an element in the loop space $\gamma \in \Omega M$ (equivalently a map $\gamma : S^1 \to M$) satisfying the ODE $\dfrac{\partial \gamma}{\partial t}(t) = X(\gamma(t))$.

In words: the ODE says that the tangent vector at every point along the loop γ should precisely be the tangent vector that the vector field X prescribes at that point.

Note: Every fixed point of X is trivially a closed orbit.

Definition (Flow): A *flow* is a group homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$ given by $t \mapsto \phi_t$. Its integral curves are given by $\gamma_p : \mathbb{R} \to M$ where $t \mapsto \phi_t(p)$.

Remark: Note that $X(p) \in T_pM$ is a tangent vector at each point, so we can ask that $\dfrac{\partial \phi_t}{\partial t}(p) = X(\phi_t(p))$, i.e. that the tangent vectors to a flow are given by a vector field. This works locally, and can be extended globally if X is compactly supported.

Definition (Interior Product): Let M be a manifold and X a vector field. The interior product is a map

$$\begin{aligned} \iota_X : \Omega^{p+1}(M) &\to \Omega^p(M) \\ \omega &\mapsto \iota_X \omega : \Lambda^p TM \to \mathbb{R} \\ &\quad (X_1, \cdots, X_p) \to \omega(\mathbf{X}, X_1, \cdots, X_p). \end{aligned}$$

Note that this *contracts* a vector field with a differential form, coming from a natural pairing on (i, j) tensors $V^{\otimes i} \otimes (V^\vee)^{\otimes j}$.

Definition (Lie Derivative):

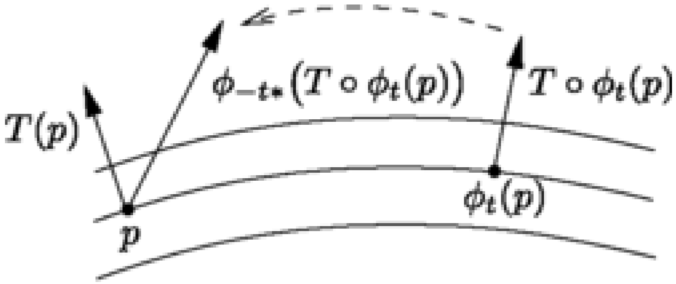
General definition: For an arbitrary tensor field T (a section of some tensor bundle $V \to TM^{\otimes n} \to M$, example: Riemann curvature tensor, or any differential form) and a vector field X (a section of the tangent bundle $W \to TM \to M$), we can define a "derivative" of T along X. Namely,

$$(\mathcal{L}_X T)_p = \left[\frac{\partial}{\partial t}\Big((\phi_{-t})_* T_{\phi_t(p)}\Big)\right]_{t=0}$$

where

- ϕ_t is the 1-parameter group of diffeomorphisms induced by the flow induced by X,
- $(\,\cdot\,)_*$ is the pushforward

> This measures how a tensor field changes as we flow it along a vector field.



Specialized definition: If $\omega \in \Omega^{p+1}(M)$ is a differential form, we define

$$\mathcal{L}_x\omega = [d, \iota_x]\omega = d(\iota_x\omega) - \iota_x(d\omega)$$

where d is the exterior product.

> This is a consequence of "Cartan's Magic Formula", not the actual definition!

2 Symplectic

Definition (Symplectic Vector Field): A vector field X is symplectic iff $\mathcal{L}_X(\omega) = 0$.

> Remark: Then the flow ϕ_X preserves the symplectic structure.

Definition (Hamiltonian Vector Field): If X is a vector field and $\iota_X\omega$ is both closed and exact, then X is a *Hamiltonian vector field.*

3 Contact

Definition (Overtwisted Contact Structure): (M, ξ) is *overtwisted* iff there exists an embedded disc $D^n \xrightarrow{i} M$ such that $T(\partial D^n)_p \subset \xi_p$ pointwise for all $p \in \partial D^n$ and TD^n_p is transverse to ξ for every $p \in (D^n)^\circ$.

4 Handles

Definition (Normal Bundle): Let $i : S \hookrightarrow M$ be an embedding, and let $N_M(S)$ denote the *normal bundle* of S in M, which fits into an exact sequence

$$0 \to TS \to i^*TM \to N_M(S) \to 0,$$

where $i * TM$ is the pullback:

$$\begin{tikzcd} i^*TS \arrow[r, dashed] \arrow[d, dashed] & TM \arrow[d] \\ S \arrow[r, "i"] & M \end{tikzcd}$$

so we can identify $N_M(S) \cong TM|_{i(S)}/TS$.

Remark: We can "symplectify" this definition by requiring that the pullback of ω is constant rank.

Definition (Tubular Neighborhood): For $S \hookrightarrow M$ an embedded submanifold, a *tubular neighborhood* of S is an embedding of the total space of a vector bundle $E \to S$ along with a smooth map $J : E \to M$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[d, "\pi"] \arrow[dr, "J"] & \\ S \arrow[u, dashed, bend left, "0_E"] \arrow[r, "i"] & M \end{tikzcd}$$

where 0_E is the zero section.