Group Theory: Classification

· Semidirect Products

Notation: $G \cong N \rtimes_\psi H$ where

$$\psi: H \to \mathrm{Inn}(N) \trianglelefteq \mathrm{Aut}_{Grp}(H)$$

$$h \mapsto \left\{ \begin{array}{l} \psi_h: H \to H \\ g \mapsto g^h := hgh^{-1} \end{array} \right\}$$

$$\boxed{\text{"} h \mapsto h(\cdot)h^{-1} \text{"}}$$

Thm (Recognizing semidirect prods)

- $N \trianglelefteq G$, $H \leq G$ (Note: $N, H \trianglelefteq G \Rightarrow G \cong N \times H$)
- $G = NH$ ($g \in G \Rightarrow \exists n, h$ s.t. $g = nh$) (Need one normal for $HK \leq G$)
- $H \underset{\psi}{\curvearrowright} N$ by conjugation

$\Rightarrow G \cong N \rtimes_\psi H$ $\psi: H \to \mathrm{Aut}(N)$, $h \mapsto \psi_h$

Where $(n_1, h_2) \cdot (n_1, h_2) = (n_1 \cdot \psi_{h_1}(n_2), h_1 \cdot h_2)$

Group law on N

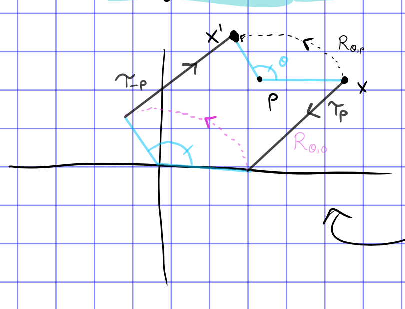
Group law on H

Group law on $N \times H$

eg: $(n_1 (h_1 n_2 h_1^{-1}), h_1 h_2)$

Mnemonic: $N \trianglelefteq G$ necessary for conjugation to be an aut.

Motivating the weird defn:



$$R_{\theta,p} = \tau_p R_{\theta,0} \tau_{-p} = \tau_p R_{\theta,0} \tau_p^{-1}$$

Thm: $E(\mathbb{R}^2)$ is generated by
1) Rotations about 0
2) Reflections through lines (through 0)
3) Translations

$$E(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes O_2(\mathbb{R})$$

Translate   rotate, reflect (linear)

[Euclidean gp = isometries
$|x - y| = |f(x) - f(y)|$
$\leq \mathrm{Aff}(V) = V \rtimes_\psi GL(V)$
$f \curvearrowright v := f(v)$]

Enough to just specify a pair $(\vec{v}, M) \in \overbrace{\mathbb{R}^2 \times O_2(\mathbb{R})}^{\text{As a set}}$

rot: $\begin{bmatrix} a & -b \\ +b & a \end{bmatrix}$, ref: $\begin{bmatrix} +c & d \\ d & -c \end{bmatrix}$

But can't be the direct product:

$G = N \times H \Rightarrow [N, H] = 1$

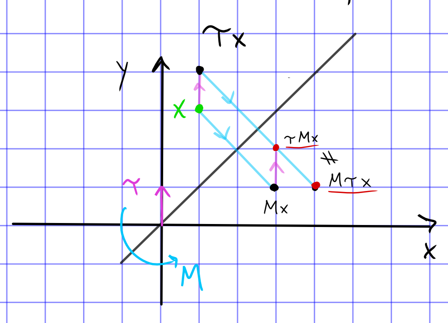
$$[n, h] := nhn^{-1}h^{-1} = (nhn^{-1})h^{-1} = h_0 h^{-1} \in H \text{ since } H \trianglelefteq G$$
$$= n(hn^{-1}h^{-1}) = n n_0 \in N \text{ since } N \trianglelefteq G$$

$\Rightarrow [n, h] \in N \cap H = \{1_G\}$. ◻

But $\tau M \neq M \tau$, take $\left\{ \begin{array}{l} M = \text{reflect about } y = x \\ \tau = \text{translate by } (1,0) \end{array} \right.$

$(0,1) \mapsto (1,0)$
$(1,0) \mapsto (0,1)$

$$M = \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 0 & 1 \\ e_2 & 1 & 0 \end{array}$$



or just compute:

$$M[x,y]^t + \tau = [y, x] + \tau$$
$$= [y, x+1]$$

$$M(\tau + [x,y]) = M[x, y+1]^t$$
$$= [y+1, x]$$

not equal!

So how do they compose?

$$f(\vec{x}) = A\vec{x} + \vec{v}$$
$$g(\vec{x}) = B\vec{x} + \vec{\omega}$$

$$(f \circ g)(\vec{x}) = A(B\vec{x} + \vec{v}) + \vec{\omega}$$
$$= \underbrace{AB}_{M}\vec{x} + \underbrace{A\vec{v} + \vec{\omega}}_{\tau}$$

$$\Rightarrow (\vec{v}, A) \circ (\vec{\omega}, B) = (\underbrace{A\vec{v} + \vec{\omega}}_{\tau}, \underbrace{AB}_{M})$$

Can find a representation:

$$\underset{1\times 2,\ 2\times 2}{(\vec{v}, A)} \to \underset{3\times 3}{\left[\begin{array}{c|c} A & \vec{v} \\ \hline 0 & 1 \end{array}\right]} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}} = \underset{3\times 1}{\left[\begin{array}{c} A\vec{x} + \vec{v} \\ \hline 1 \end{array}\right]}$$

Check: $\begin{bmatrix} A & \vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B & \vec{\omega} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} AB & A\vec{\omega} + \vec{v} \\ 0 & 1 \end{bmatrix}$ (works!)

$\Rightarrow G \cong \mathbb{R}^2 \rtimes_\psi O_2(\mathbb{R})$ $\psi: O_2(\mathbb{R}) \to \mathrm{Aut}_{Grp}(\mathbb{R}^2)$, $A \mapsto \left\{ \begin{array}{l} \psi_A: \mathbb{R}^2 \to \mathbb{R}^2 \\ \vec{x} \mapsto A\vec{x} \end{array} \right\}$

Really $O(\mathbb{R}^2) \to GL(\mathbb{R}^2)$ the "natural" rep.

Where $(\vec{v}, A) \circ (\vec{\omega}, B) = (v + \psi_A(\omega), AB)$

$$:= (v + A\omega, AB)$$

Slogan: Group law on first component "twisted" by a representation ψ, G is a "twisted product" of N and H.

Really:

$0 \to N \xrightarrow{i} G \xrightarrow{\pi} H \to 0$ is a SES in Grp

$\Rightarrow G$ is an extension of H by N (note $N \cong \ker(\pi) \trianglelefteq G$)

Thm | Split by $s: H \to G \Rightarrow H \curvearrowright N$, ie N is an H-module.

Pf |
- Identify $N \cong i(N) \trianglelefteq G$
- Define $\gamma: G \to \mathrm{Aut}\, G$, $g \mapsto \left\{ \begin{array}{l} \gamma_g: G \to G \\ x \mapsto gxg^{-1} \end{array} \right\}$
- Descends to $\mathrm{Aut}(N)$ $\tilde{\gamma}: G \to \mathrm{Aut}(N)$, $g \mapsto \left\{ \begin{array}{l} \gamma_g: N \to N \\ n \mapsto gng^{-1} \end{array} \right\}$

  (Well-def since $N \trianglelefteq G \Rightarrow gng^{-1} \in N$ $\forall n \in N$

- Use splitting to pull back to K $\tilde{\gamma}^K: H \to \mathrm{Aut}(N)$, $h \mapsto \left\{ \begin{array}{l} \gamma_{s(h)}: N \to N \\ n \mapsto s(h) \cdot n \cdot s(h)^{-1} \end{array} \right\}$

Interesting aside:

Thm There is a correspondence

$$\left\{ \begin{array}{c} \text{SESs} \\ 0 \to N \to G \to H \to 0 \end{array} \right\} \Big/ \sim \;\rightleftharpoons\; H^2(H; N)$$

(Group cohomology: $H^n(H; N) := \mathrm{Ext}^n_{\mathbb{Z}[H]}(\mathbb{Z}, N)$)

= Derived functors of $F(\cdot) = H$-invariants $(\cdot)$ on H-mod
Fixed pts of H-action.

$$\left\{ \begin{array}{c} \text{Split SESs} \\ 0 \to N \to G \underset{\curvearrowleft s}{\to} H \to 0 \end{array} \right\} \Big/ \sim \;\rightleftharpoons\; \{(N, H) \in H\text{-mod} \times Grp\}$$

$$\| $$

$$\{ N \rtimes_\psi H \mid \psi: H \to \mathrm{Aut}(N) \}$$

Back to classifying groups

Ex

1) $A_n \trianglelefteq S_n$ index 2

⇒ $1 \to A_n \to S_n \to \mathbb{Z}/2 \to 1$

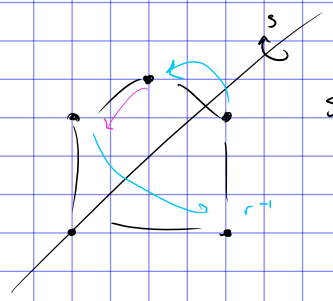
⇒ $S_n \cong A_n \rtimes \mathbb{Z}_2$

2) $D_n = \langle r, s \mid r^n, s^2, srs^{-1} = r^{-1} \rangle$

$\langle r \rangle$ index 2

⇒ $1 \to \langle r \rangle \to D_n \to \langle s \rangle \to 1$

⇒ $D_n \cong \mathbb{Z}/n \rtimes \mathbb{Z}/2$



$r^5 = s^2$

$srs^{-1} = r^{-1}$

D_5

Ex

$\mathbb{Z}/5 \rtimes_\psi \mathbb{Z}/3 \cong \mathbb{Z}/5 \times \mathbb{Z}/3 \quad \forall \psi$ since

$\mathrm{Aut}\,\mathbb{Z}/n \cong (\mathbb{Z}/n)^\times = \mathbb{Z}/\phi(n)$, $\phi(3) = 3^1 - 3^0 = 2$] $\phi(p^k) = p^k - p^{k-1}$

⇒ $\psi \in \mathrm{Hom}_{Grp}(\mathbb{Z}/2, \mathbb{Z}/5) = \{0\}$

General fact: $\mathrm{Hom}_{Grp}(\mathbb{Z}/n, \mathbb{Z}/m) = \{0\}$ when m, n coprime

Why? $f \in$ ↑ ⇒ $o(1_n) = n \in \mathbb{Z}/n$, $o(f(1_n)) \mid o(1_n)$

⇒ $o(f(1_n)) \mid n$

By Lagrange, $o(f(1_n)) \mid |\mathbb{Z}/m| = m$

⇒ $o(f(1_n)) \mid \gcd(m,n) = 1$ when coprime.

Fact: $\forall \Gamma \in \mathrm{Aut}(H)$, $N \rtimes_\psi H \cong N \rtimes_{\psi \circ \Gamma} H$

Fact: $\psi(h) = \mathbb{1}_N \ \forall h \Rightarrow N \rtimes_\psi H \cong N \times H$

Really: $\mathrm{Hom}_{Grp}(H, \mathrm{Aut}(N)) \in (\mathrm{Aut}(H) \times \mathrm{Aut}(N))$-mod (compose)

ψ_1, ψ_2 in the same orbit ⇒ $N \rtimes_{\psi_1} H \cong N \rtimes_{\psi_2} H$.

Warning: not conversely! Diff orbits can yield iso groups.

Fact: $\mathrm{Aut}((\mathbb{Z}/n)^\ell) \cong GL_\ell(\mathbb{Z}/n)$

$\mathrm{Hom}_{Grp}(\underbrace{\mathbb{Z}/m}_{H}, \underbrace{GL_\ell(\mathbb{Z}/n)}_{\mathrm{Aut}(N)}) \cong \{ M \in GL_\ell(\mathbb{Z}/n) \mid M^m = 1 \}$

Conjugation is an Aut on GL_ℓ, so just identify conjugacy classes (e.g. by invariant factors)

$\mathrm{Aut}(\mathbb{Z}/p^e) \cong (\mathbb{Z}/p^e)^\times \cong \mathbb{Z}/\phi(p^e)$

Fact: $G = N \rtimes_\psi H$, $N \cong \langle \mathrm{Gens}(N) \mid \mathrm{Relns}(N) \rangle$
$H \cong \langle \mathrm{Gens}(H) \mid \mathrm{Relns}(H) \rangle$

$$\Rightarrow G = \left\langle \mathrm{Gens}(N) \cup \mathrm{Gens}(H) \;\middle|\; \mathrm{Relns}(N) \cup \mathrm{Relns}(H) \cup \left\{ hnh^{-1} = \psi_h(n) \ \begin{array}{l} \forall n \in \mathrm{Gens}(N) \\ \forall h \in \mathrm{Gens}(H) \end{array} \right\} \right\rangle$$

eg $N = \langle a \mid a^n \rangle$, $H = \langle b \mid b^m \rangle$ ⇒ $N \rtimes_\psi H = \langle a, b \mid a^n, b^m, bab^{-1} = \psi_b(a) \rangle$

Note $N \times H = \langle a, b \mid a^n, b^m, bab^{-1} = a \rangle$ $\quad N, H \trianglelefteq G \Rightarrow [N, H] = 1$

Hungerford 2.6.9

Classify all groups of order 18

Soln

- $18 = 2 \cdot 3^2$
  - $n_2 \mid 3^2 \Rightarrow n_2 \in \{1, 3, 3^2\}$, $n_2 \equiv 1 \bmod 2$] All possible ¨
  - $n_3 \mid 2 \Rightarrow n_3 \in \{1, \cancel{2}\}$, $n_3 \equiv 1 \bmod 3$] $\Rightarrow n_3 = 1$
- $n_3 = 1 \Rightarrow$ Unique normal $Syl_3(G) \geq S_3 \trianglelefteq G$
- Recognizing semidirect products:
  1) $S_3 \trianglelefteq G$
  2) $G = S_2 S_3$?
     - $\hookrightarrow |G| = 2^2 \cdot 3, \quad |S_2| = 2, \quad |S_3| = 3^2$
     - $\hookrightarrow S_2 \cap S_3 = \{1_G\}$ since all elts have order dividing $\gcd(2, 3^2) = 1$
     - $\Rightarrow |S_2 S_3| = \dfrac{|S_2| \cdot |S_3|}{|S_2 \cap S_3|} = \dfrac{2^2 \cdot 3}{1} = |G|$
     - $S_2 S_3 \leq G \Rightarrow \underline{S_2 S_3 = G}$ (subgroup of same size can only be the entire gp.)
  3) $S_2 \curvearrowright S_3$? Sure
     $\psi : S_2 \to Aut(G)$, $s \mapsto (g \mapsto sgs^{-1})$ $\xrightarrow[\text{since } gS_3g = S_3]{\text{restricts to } Aut(S_2)}$ $\psi|_{S_2} : S_2 \to Aut(S_3)$, $s \mapsto (g \mapsto sgs^{-1})$

$\Rightarrow G \cong S_3 \rtimes_\psi S_2, \quad \psi : S_2 \to Aut(S_3)$

- $|S_2| = 2 \Rightarrow S_2 \cong \mathbb{Z}/2$
- $|S_3| = 3^2$, need to classify gps. of order 9

Thm. Every $|G| = p^2$ is abelian

- $|G| = p^2$
- Consider $Z(G)$

$G/Z(G) = \{x^i Z(G)\} = \langle x Z(G) \rangle$
$g \to x^\ell Z(G) \to g = x^\ell z_1 \to gh = hg$
$h \to x^m Z(G) \to h = x^m z_2 \to x^\ell z_1 x^n z_2$

a) $|Z(G)| = p^2 \Rightarrow$ Abelian, use classif. of AbGrp.

b) $|Z(G)| = p \Rightarrow |G/Z(G)| = p$
$\Rightarrow$ cyclic quotient $\Rightarrow$ abelian

c) $|Z(G)| = 1 \Rightarrow$ Not possible

Thm: p-groups have nontrivial center.

Class eqn

Herstein p.73

Pf:
- Let $X = \{h \in G \mid o(h) = p\} \leq G$, $|G| = p^k$
- $G \curvearrowright X \quad g \curvearrowright x := gxg^{-1}$
- $Gx = \{gxg^{-1} \mid g \in G\} = \bigcup_{g \in G} \{g \curvearrowright x\}$
  $= Conj(x)$. Conj class $G \curvearrowright X$
- $G_x = \{g \in G \mid gxg^{-1} = x\} = \{g \in G \mid g \curvearrowright x = x\}$
  $= Z(x)$. Centralizer
- Orbit-Stabilizer: Mnemonic $Gx \cong G/G_x$
  $\Rightarrow |Gx| = [G : G_x]$
  $|Conj_G(x)| = [G : Z(x)]$
  $\Rightarrow Z(x) \leq G \Rightarrow |Z(x)| = p^\ell, \ell \leq k$
  $\Rightarrow [G : Z(x)] = p^k / p^\ell = p^{k-\ell} := p^n \quad (n \geq 0)$
- $|X| \equiv 1 \bmod p$ (claim)
  But $X = \bigsqcup_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} Gx_i$
  $\Rightarrow |X| = \sum_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} |Gx_i|$
  $= \sum_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} |Conj_G(x_i)|$
  $= \sum_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} [G : Z(x_i)]$
  $= \sum_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} p^{n_i}$
  $\Rightarrow |X| = \sum_{\substack{\text{one } x_i \text{ in} \\ \text{each orbit}}} p^{n_i}$ (take mod p)
  1 mod p $\Rightarrow$ not all $n_i \geq 1$
  $\Rightarrow$ some $n_i = 0$
  $\Rightarrow [G : Z(x_i)] = p^0 = 1$
  $\Rightarrow Z(x_i) = G$ for some x_i
  $\Rightarrow x_i \in Z(G)$.

$|G| = |Z(G)| + \underbrace{\sum_{\substack{\text{one } x_i \\ \text{each class}}} |Conj(x_i)|}_{p}$; $\sum [G : C_G(x_i)]$, $\sum_{n_i \geq 1} p^{n_i}$

!! p can't divide both!

$\therefore S_3$ is abelian, so $S_3 \in \{\mathbb{Z}/3^2, (\mathbb{Z}/3)^{\oplus 2}\}$
$\hookrightarrow \langle z \mid z^9 \rangle$; $\hookrightarrow \langle b, c \mid b^3, c^3, cbc^{-1} = b \rangle$

Recall $G \cong S_3 \rtimes_\psi \mathbb{Z}/2 = \langle y, Gens(S_3) \mid y^2, \{yhy^{-1} = \psi_y(h) \ \forall h \in S^3\} \rangle$

Case 1: $S_3 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 = \langle b, c \rangle$; $b = (1,0)$, $c = (0,1)$

- Check $Aut((\mathbb{Z}/3)^{\oplus 2}) \cong (GL_2(\mathbb{Z}/3), \cdot)$
- Look at $\boxed{Hom_{Grp}(\mathbb{Z}/m, Aut(S_3))}$ $\{= Hom_{\mathbb{Z}\text{-mod}}\}$
  $\hookrightarrow$ Send $[1]_2 \mapsto M$; $M^2 = id$; y
  $\boxed{\Rightarrow M^2 - 1 = 0} \Rightarrow p(x) := x^2 - 1$; $p(M) = 0$.
  $\hookrightarrow x^2 - 1 = (x+1)(x-1) \Rightarrow min_M(x) \in \left\{ \begin{matrix} x^2 - 1 \\ x + 1 \\ x - 1 \end{matrix} \right\} := q(x)$

Pre/post-compose by Auts $\Rightarrow$ only need similarity classes! (JCF)

Subcase (a): If $q(x) = x - 1$, $M \simeq 1$, $\psi_y(\vec{v}) = M\vec{v}$,

$\Rightarrow G \cong \left\langle y, b, c \,\middle|\, \begin{matrix} y^2, b^3, c^3, \quad bcb^{-1} = c \\ yby^{-1} = \psi_y(b) \\ ycy^{-1} = \psi_y(c) \end{matrix} \right\rangle$ new relations

$b = (0,1)$, $c = (1,0)$, $M = 1$
$\left\{ \begin{array}{l} \psi_y(b) = Mb = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{1} \cdot \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b \\ \psi_y(c) = Mc = c. \end{array} \right.$

$\Rightarrow G \cong \left\langle y, b, c \,\middle|\, \begin{matrix} y^2, b^3, c^3, \quad bcb^{-1} = c \\ yby^{-1} = b \\ ycy^{-1} = c \end{matrix} \right\rangle$

Conjugation relations $\Rightarrow \langle b, c \rangle \trianglelefteq G$, $\langle b \rangle, \langle c \rangle \trianglelefteq \langle b, c \rangle$

Recognizing direct products $\Rightarrow G \cong \langle a \rangle \times \langle b \rangle \times \langle c \rangle$

$\boxed{\Rightarrow G \cong \mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/3}$

Subcase (b): $q(x) = x + 1 \Rightarrow M \simeq -1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Check $\psi_y(b) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -b \mapsto b^{-1} = b^2$; $b \mapsto b^{-1}$

since $b^3 = e$.

$\psi_y(c) = -c \mapsto c^{-1} = c^2$ since $c^3 = e$

$\Rightarrow G \cong \left\langle y, b, c \,\middle|\, \begin{matrix} y^2, b^3, c^3, \quad bcb^{-1} = c, \\ yby^{-1} = b^2 = b^{-1} \\ ycy^{-1} = c^2 = c^{-1} \end{matrix} \right\rangle$ = Who knows, some group!

Subcase (c): $q(x) = x^2 - 1$

Min poly = char poly $\Rightarrow JCF(M) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\psi_y(b) = Mb = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = b \mapsto b$

$\psi_y(c) = Mc = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -c \mapsto c^{-1} = c^2$

Recognizing direct prods

$\Rightarrow G \cong \left\langle y, b, c \,\middle|\, \begin{matrix} y^2, b^3, c^3, \quad bcb^{-1} = c \\ yby^{-1} = b \\ ycy^{-1} = c^2 \end{matrix} \right\rangle \cong \langle b \rangle \times \left\langle y, c \,\middle|\, \begin{matrix} y^2, c^3 \\ ycy^{-1} = c^2 \end{matrix} \right\rangle$

$\cong \mathbb{Z}/3 \times D_3$

Case 2: $S_3 \cong \mathbb{Z}/3^2 = \langle z \mid z^9 \rangle$; $\phi(3^2) = 3^2 - 3^1 = 9 - 3 = 6$

- Check $Aut(\mathbb{G}_a(\mathbb{Z}/9)) \cong \mathbb{G}_m(\mathbb{Z}/\phi(9)) \cong (\mathbb{Z}/6, \cdot)$ mult
- $o([1]_2) = 2$ in $\mathbb{Z}/2 \Rightarrow o(\psi([1]_2)) \mid 2$
  $\Rightarrow o(\psi([1]_2)) \in \{1, 2\}$.

Subcase a: $o(\psi([1]_2)) = 1 \Rightarrow \psi([1]_2) \cong [1]_6$
$\Rightarrow [1]_2 \mapsto 1 \in Aut(\mathbb{Z}/9)$
$\Rightarrow$ Direct product!

$G \cong \mathbb{Z}/2 \times \mathbb{Z}/3^2$

Subcase b: $o(\psi([1]_2)) = 2$

- Elts of order 2 in $\mathbb{G}_m(\mathbb{Z}/6) = \{[-1]_6\}$; $\equiv 5 \bmod 6$
- Pulls back to $f : \mathbb{Z}/9 \to \mathbb{Z}/9 \in Aut(\mathbb{Z}/9)$, $[1]_9 \mapsto [-1]_9$; $\mathbb{G}_a$
- In mult. notation, $\psi_y(z) = z^{-1}$

$G \cong \mathbb{Z}/2 \rtimes_\psi \mathbb{Z}/3^2$

$\Rightarrow G \cong \left\langle y, z \,\middle|\, \begin{matrix} y^2, z^9, \\ yzy^{-1} = z^{-1} \end{matrix} \right\rangle \cong D_9$

$\rightsquigarrow$ 5 groups

- $\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/3$ $\in Ab$
- $\mathbb{Z}/2 \rtimes (\mathbb{Z}/3 \times \mathbb{Z}/3)$
- $\mathbb{Z}/3 \times D_3$
- $\mathbb{Z}/2 \times (\mathbb{Z}/3^2)$ $\in Ab$
- D_9

Case 1: $M^2 = I \rightsquigarrow \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

Case 2

Classify all abelian groups of order

$p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

$\hookrightarrow p(n_1) \cdot p(n_2) \cdots p(n_k)$

$G \cong \prod_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$ p_i not nec. distinct

Elem divisors: ↗

Invt factors: $\prod_{i=1}^{n} \mathbb{Z}/m_i$

$m_1 \mid m_2 \mid \cdots \mid m_n$

$\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^n} \cong \mathbb{Z}_{pq^n}$

Order $4225 = 65^2 = 5^2 \cdot 13^2$

$p(2) \cdot p(2) = 2 \cdot 2 = 4$

CRT

$(2, 2)$	$\to$ $\mathbb{Z}/5^2 \times \mathbb{Z}/13^2$	$= \mathbb{Z}/4225$
$(1{+}1, 2)$	$\mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/13^2$	$= \mathbb{Z}/845 \times \mathbb{Z}/5$
$(2, 1{+}1)$	$\mathbb{Z}/5^2 \times \mathbb{Z}/13 \times \mathbb{Z}/13$	$= \mathbb{Z}/325 \times \mathbb{Z}/13$
$(1{+}1, 1{+}1)$	$(\mathbb{Z}/5)^2 \times (\mathbb{Z}/13)^2$	$= \mathbb{Z}/65 \times \mathbb{Z}/65$

ED	IF
$5^2, 13^2$	$(5^2 \cdot 13^2)$
$5, 5, 13^2$	$(5)(5 \cdot 13^2)$
$5^2, 13, 13$	$(13)(5^2 \cdot 13)$
$5, 5, 13, 13$	$(5 \cdot 13)(5 \cdot 13)$

$R/\langle x^2-1 \rangle \to \dfrac{R}{\langle x-1 \rangle} \oplus \dfrac{R}{\langle x+1 \rangle}$

5, 5, 13

n^{th}	$n-1$
5	5
13	

	n^{th}	$n-1$	$n-2$	$n-3$
2	2^5,	2^2,	2,	2
3	3^{389},	3		
17	17^2,	17,	17	

Classification gps