Topology Qual Prep Week 1: Point-Set

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  - topologies,
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2 Warmups

- State the axioms of a topology.
- What does it mean for a set to be open? Closed?
- State the definition of the product topology, the subspace topology, and the quotient topology.
- What does it mean for a family of sets to form a **basis** for a topology?
- What is an interior point? An isolated point? A limit point?
- What is the **closure** of a subspace $E \subseteq X$?
- What does it mean for a topological space to be **compact**?
- What does it mean for $E \subseteq X$ to be a **dense** subspace?
- Come up with 6 different topologies on $\mathbb{R}^d$.
- What is a **separable** space?

- What is a **nowhere dense** subspace?

3 Exercises

- Prove Cantor's intersection theorem.
- Determine if the following subsets of $\mathbb{R}$ are opened, closed, both, or neither:
  - $\mathbb{Q}$
  - $\mathbb{Z}$
  - $\{1\}$
  - $\left\{p \in \mathbb{Z}^{\geq 0} \,\middle|\, p \text{ is prime}\right\}$
  - $\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{Z}^{\geq 0}\right\}$
  - $\left\{\frac{1}{n} \,\middle|\, n \in \mathbb{Z}^{\geq 0}\right\} \cup \{0\}$
- Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ for any $n \geq 2$.
- Is it true that the closure of a product is the product of the closures?
  - Is it true that the interior of a product is the product of the interiors?
- Find a space that is connected but not locally connected. Can there be a space that is locally connected but not connected?
- Show that for X an arbitrary topological space, the one-point compactification $\widehat{X}$ (with its corresponding topology) is compact.
- Prove that path-connected implies connected
  - Show that the topologist's sine curve is connected but not path-connected.
- Is every product (finite or infinite) of Hausdorff spaces Hausdorff?
- Is $\mathbb{R}$ homeomorphic to $[0, \infty)$?
- Show that X is connected iff the only subsets of X which are both closed and open are $\emptyset, X$.
- Show that a closed subset A of a compact space X is compact. Does this hold when A is instead an open subset?
- Show that if $f : X \to Y$ is continuous and X is compact then the image $f(X) \subseteq Y$ is compact.
- Show that every compact metric space is complete.
- Show that a compact subset of a Hausdorff space is closed. Does the converse hold?
  - What property on a space guarantees that compact sets are closed
  - What property on a space guarantees that closed sets are compact?
- Show that a continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism.

Do three problems from:

1. (May 2016) Given any topological space Z and subset $D \subseteq Z$, let $Cl_Z(D)$ denote the closure of D in Z. Show that if X and Y are topological spaces and $A \subseteq X$, $B \subseteq Y$, then $Cl_{X\times Y}(A \times B) = Cl_X(A) \times Cl_Y(B)$.

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2. (May 2016) Let X be a connected space and $A, B \subseteq X$ be closed subsets of X with $X = A \cup B$ and $A \cap B$ a connected subset of X. Show that both A and B are connected.

- $T_4 \Rightarrow$

- Prove the following implications of separation axioms, and show that they are strict:
- Show that every compact metrizable space has a countable basis.

4 Qual Questions

Problem 1.1.4 (Fall 2010, 8)
Show that for any two topological spaces X and Y , $X \times Y$ is compact if and only if both X and Y are compact.

Tube lemma:

Solution:

Problem 1.1.9 (?)
If X is a topological space and $S \subset X$, define in terms of open subsets of X what it means for S **not** to be connected.
Show that if S is not connected there are nonempty subsets $A, B \subset X$ such that

$$A \cup B = S \quad \text{and} \quad A \cap \overline{B} = \overline{A} \cap B = \emptyset$$

Here $\overline{A}$ and $\overline{B}$ denote closure with respect to the topology on the ambient space X.

Problem 1.3.3 (?)
Let

$$X = \left\{ (x, y) \in \mathbb{R}^2 | x > 0, y \geq 0, \text{ and } \frac{y}{x} \text{ is rational} \right\}$$

and equip X with the subspace topology induced by the usual topology on $\mathbb{R}^2$.
Prove or disprove that X is connected.

Problem 1.4.3 (Spring 2009, 31)

a. Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

b. Give an example that shows that the "Hausdorff" hypothesis in part (a) is necessary.

Problem 1.4.4 (?)
Let X be a topological space and let

$$\Delta = \left\{ (x, y) \in X \times X \,\middle|\, x = y \right\}.$$

Show that X is a Hausdorff space if and only if Δ is closed in $X \times X$.