5a) Let $X = (C(I), \|\cdot\|_\infty)$ where $I = [0,1]$,

$C(I) = \{ f : I \to \mathbb{R} \mid f \text{ is continuous} \}$, and

$$d(f,g) = \|f-g\|_\infty = \sup_{x \in I} |f(x) - g(x)| .$$

<u>Claim</u>: X is a metric space.

1) $d(f,g) = 0 \Rightarrow f = g$

If $\sup\limits_{x \in I} |f(x) - g(x)| = 0$ then $|f(x) - g(x)| = 0 \ \forall x \in \mathbb{R}$,

so $f(x) = g(x) \ \forall x \in \mathbb{R}$ and $f = g$.

2) $d(f,g) = d(g,f)$

We have
$$\begin{aligned} d(f,g) &= \sup_{x \in I} |f(x) - g(x)| \\ &\sup_{x \in I} |g(x) - f(x)| \\ &= d(g,f). \end{aligned}$$

3) $d(f,h) \leq d(f,g) + d(g,h)$

We have
$$\begin{aligned} d(f,g) &= \sup_{x \in I} |f(x) - g(x)| \\ &= \sup_{x \in I} |f(x) - h(x) + h(x) - g(x)| \end{aligned}$$

$$\leq \sup_{x\in I}\left(|f(x)-h(x)|+|h(x)-g(x)|\right) \quad \text{Δ-ineq in $\mathbb{R}$}$$

$$= \sup_{x\in I}|f(x)-h(x)| + \sup_{x\in I}|h(x)-g(x)|$$

$$= d(f,h)+d(h,g).$$

So X is a metric space. $\square$

<u>Claim</u>: X is complete.

Let $\{f_i\}$ be a Cauchy sequence in X, we will show that it converges in X. Since $\{f_i\}$ is Cauchy in X, we have

$$\forall \varepsilon>0,\ \exists N_0 \mid n\geq m\geq N_0 \Rightarrow \|f_n-f_m\|_\infty<\varepsilon$$

First we will define a candidate limit function f, then show $f\in X$.

1) Define $f:=\lim\limits_{n\to\infty} f_n$ by $f(x)=\lim\limits_{n\to\infty} f_n(x)$.

This is well-defined; let $S_x=\{f_i(x)\}\subseteq\mathbb{R}$ for a fixed x, and we claim S_x is Cauchy in $\underline{\mathbb{R}}$, which is complete. This follows because if $\{f_i\}$ is Cauchy in X, then

$$|f_n(x)-f_m(x)|\leq \sup_{x\in I}|f_n(x)-f_m(x)| = \|f_n-f_m\|_\infty \to 0.$$

2) $f \in X$, for which it suffices to show f is continuous.

Let $\varepsilon > 0$, and since $\{f_i\}$ is Cauchy, choose N_0 large s.t.

$$n \geq N_0 \Rightarrow \|f_n - f\|_\infty < \frac{\varepsilon}{3}.$$

Now fix $n \geq N_0$; since <u>f_n is continuous,</u> choose δ such that

$$|x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

Then

$$\begin{aligned}
|x-y| < \delta \Rightarrow |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq \sup_{x \in I} |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + \sup_{y \in I} |f_n(y) - f(y)| \\
&= \|f - f_n\|_\infty + |f_n(x) - f_n(y)| + \|f_n - f\|_\infty \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

So f is continuous, $f = \lim f_n \in X$, and X is complete. ∎

5b) Let $B = \{f \in X \mid \|f\|_\infty \leq 1\}$

Claim: B is closed.

Let f be a limit point of B, so there is some sequence $f_n \to f$ in X with each $f_n \in B$ so $\|f_n\|_\infty \leq 1$ $\forall n$.

Let $\varepsilon > 0$, and since $f_n \to f$ in X, choose N_0 such that

$$n \geq N_0 \Rightarrow \|f_n - f\| < \varepsilon$$

Then,

$$\begin{aligned}\|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &< \varepsilon + 1,\end{aligned}$$

and taking $\varepsilon \to 0$ yields $\|f\|_\infty \leq 1$. ∎

Claim: B is bounded

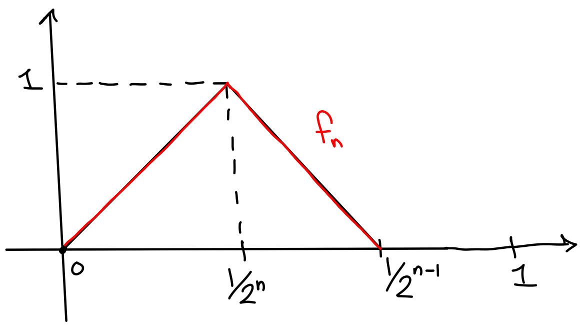
A subset $B \subseteq X$ is bounded iff there is some $x \in X$ and some $r > 0$ in $\mathbb{R}$ where $B \subset N(r, x) = \{y \in X \mid d(y,x) < r\}$.

Choose $x = 0$, $r = 2$, then $f \in B \Rightarrow d(f, 0) = \|f - 0\|_\infty = 1 < 2$, so $f \in N(2,0)$.

<u>Claim</u>: B is not compact.

Since B is a metric space, B is compact iff B is sequentially compact.

Define f_n as the triangle:



Then $f_n \xrightarrow{\mathbb{R}} f$ where $f(x) = \begin{cases} 1, & x=0 \\ 0, & x\in(0,1] \end{cases}$,

pointwise in $\mathbb{R}$

and so $\forall n$, $\|f_n - f\|_\infty = 1$, attained at $x=0$. So $\lim_{n\to\infty} \|f_n - f\|_\infty \neq 0$,

and $\{f_n\}$ does not converge in X, nor can any subsequence. $\blacksquare$

<u>Claim</u>: B is not totally bounded.

If it were, $\forall \varepsilon$ there would exist a finite collection

$\{g_i\}_{i=1}^N \subseteq B$ such that $B \subseteq \bigcup_{i=1}^N N(\varepsilon, g_i)$ where

$N(\varepsilon, g_i) = \{h \in B \mid \|h - g_i\| < \varepsilon\}$.

Note that if $h_1, h_2 \in N(\varepsilon, g_i)$ then $\|h_1 - h_2\| \leq \|h_1 - g\| + \|g - h_2\| < 2\varepsilon$.

So choose $\varepsilon = \frac{1}{2}$, and consider the collection $\{f_n\}_{n=1}^{\infty}$.

Since $\|f_n - f_m\| = 1$, each $N(\varepsilon, g_i)$ can contain at <u>most</u> one f_n, since $f_n, f_m \in N(\varepsilon, g_i)$ for $n \neq m$ would imply $\|f_n - f_m\|_\infty < 2\varepsilon = 2(\frac{1}{2}) = 1$. But there are finitely many $N(\varepsilon, g_i)$ and infinitely many f_n, so if this is a cover of B, so $N(\varepsilon, g_i)$ must contain at least 2 f_n's. $\#$

(6a) <u>Claim</u>: If $\sum g_n \xrightarrow{U} G$, then $g_n \xrightarrow{U} 0$.

Let $G_N = \sum_{n=1}^{N} g_n$ and $G = \lim_{N\to\infty} G_N$.

Suppose $G_N \xrightarrow{u} G$, then choose N large enough so that

$$\forall x \in X,\ n \geq N \Rightarrow |G_n(x) - G(x)| < \frac{\varepsilon}{2}$$

Then letting $n > n-1 > N$, we have

$$\begin{aligned}
|g_n(x)| &= \left|\sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{n-1} g_i(x)\right| \\
&= \left|\left(\sum_{i=1}^{n} g_i(x) - G(x)\right) - \left(\sum_{i=1}^{n-1} g_i - G(x)\right)\right| \\
&\leq \left|\sum_{i=1}^{n} g_i(x) - G(x)\right| + \left|\sum_{i=1}^{n-1} g_i - G(x)\right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

So $\forall x \in X$, $|g_n(x)| < \varepsilon \Rightarrow g_n \xrightarrow{u} 0$. $\square$

Now let $g_n = 1/1+n^2x$, we'll show g_n does <u>not</u> converge to 0 uniformly.

Note $g_n \xrightarrow{u} g$ iff $\forall \varepsilon, \exists N_0 \mid \forall x, n \geq N_0 \Rightarrow |g_n(x) - g(x)| < \varepsilon$,

so let $\varepsilon < \frac{1}{2}$, N_0 be arbitrary, and choose $x_0 < 1/N_0^2$. Then,

$$|g_{N_0}(x_0)| = \frac{1}{|1+N_0^2 x|} = \frac{1}{|1+N_0^2(1/N_0^2)|} = \frac{1}{2} > \varepsilon.$$

<u>Claim</u>: g is continuous on $(0,\infty)$.

Let $x \in (0,\infty)$ be arbitrary, and choose $a < x$. We will show g converges uniformly on $[a,\infty)$, and since each g_n is continuous on $[a,\infty)$ as well, g will be the uniform limit of continuous functions and thus continuous itself.

We can use the M-test. Since $x > a$,

$$\left|1/1+n^2x\right| \leq \left|1/n^2x\right| \leq \left|1/n^2a\right| = \frac{1}{a}\left|\frac{1}{n^2}\right|,$$

where $$\sum_{n=1}^{\infty} \frac{1}{a}\frac{1}{n^2} = \frac{1}{a}\sum \frac{1}{n^2} < \infty,$$

So g converges uniformly on $[a,\infty)$.

(6b) <u>Claim:</u> g is differentiable on $(0,\infty)$.

If $g'(x)$ exists, we have

$$g'(x) = \lim_{a \to x} (x-a)^{-1}(g(x)-g(a))$$

$$= \lim_{a \to x} (x-a)^{-1} \sum_{n=1}^{\infty} \frac{-n^2 (x-a)}{(1+n^2x)(1+n^2a)}$$

$$= \lim_{a \to x} \sum_{n=1}^{\infty} \frac{-n^2}{(1+n^2x)(1+n^2a)}$$

$$= \sum (-n^2)/(1+n^2x)^2 ,$$

which exists because it converges uniformly on $[a,\infty)$, as

$$\left|\frac{-n^2}{(1+n^2x)^2}\right| \leq \left|\frac{n^2}{(n^2x)^2}\right| = \left|\frac{1}{n^2x^2}\right| \leq \left|\frac{1}{a^2n^2}\right| := M_n$$

where $\sum M_n = \sum \frac{1}{a^2n^2} = \frac{1}{a^2}\sum \frac{1}{n^2} < \infty$.

So g is <u>continuously differentiable</u> on $(0,\infty)$. ■

(7a) <u>Claim</u>: $h_n \xrightarrow{u} 0$ on $[0, \infty)$

Note that $h_n'(x) = \dfrac{1-nx}{(1+x)^n}$ $\Rightarrow h_n' = 0$ iff $x = 1/n$ and

$$h_n''(x) = \frac{1+x+nx}{nx^2(1+x)^{n-1}} \quad \text{and} \quad h_n''(\tfrac{1}{n}) < 0,$$

so $x = \frac{1}{n}$ is a global maximum and thus

$$\forall x, \ |h_n(x)| \leq |h_n(\tfrac{1}{n})| = \left| \frac{1/n}{(1+\frac{1}{n})^n} \right| = \frac{1}{n(1+\frac{1}{n})^n} \leq \frac{1}{2n} \quad \text{for } n>1$$

so $\sup\limits_{x\in[0,\infty)} |h_n(x)| = |h_n(\frac{1}{n})| = O(\frac{1}{n}) \to 0$, thus $\|h_n\|_\infty \to 0$

and $h_n \to 0$ uniformly.

(7b) Let $h(x) = \sum\limits_{n=1}^{\infty} h_n(x) = \sum\limits_{n=1}^{\infty} x/(1+x)^{n+1}$

i) Demonstrably, $h(0) = 0$, and for a fixed x we have

$$h(x) = \sum_{n=1}^{\infty} x/(1+x)^{n+1} = (x/1+x) \sum_{n=1}^{\infty} (1/1+x)^n$$

$$= \frac{x}{1+x}\left(\frac{1}{1-(1/1+x)}\right) \quad \text{since } x>0 \Rightarrow (1/1+x) < 1$$

$$= 1. \quad \square$$

ii) It can not converge uniformly on $[0,\infty)$, otherwise h would be the uniform limit of continuous functions, but h is discontinuous.

7c) Let $a>0$ and $X=[a,\infty)$.

Claim: $\sum h_n \xrightarrow{u} h$ on X.

$$(1+x)^{n+1} = \sum_{j=1}^{n} \binom{n}{j} x^j \geq \underbrace{1+nx+n^2x^2}_{x>a>0, \text{ so positive terms.}}$$

Since $x>a$, we have

$$|h_n(x)| = \left|\frac{x}{(1+x)^{n+1}}\right| \leq \left|\frac{x}{1+nx+n^2x^2}\right| \leq \left|\frac{a}{1+na+n^2a^2}\right| \leq \left|\frac{a}{n^2a^2}\right| = \left|\frac{1}{n^2a}\right|$$

So let $M_n = 1/an^2$, then $\sum M_n < \infty \Rightarrow \sum h_n \xrightarrow{u} h$ by the M test. ■