

# Problem Set 6

D. Zack Garza

Monday 4<sup>th</sup> May, 2020

## Contents

<b>1</b>	<b>Humphreys 5.3</b>	<b>1</b>
<b>2</b>	<b>Humphreys 7.2</b>	<b>1</b>
2.1	Solution . . . . .	1
<b>3</b>	<b>Exercise p.108</b>	<b>3</b>

## 1 Humphreys 5.3

Let  $\lambda$  be regular, antidominant, and integral, and suppose  $M(\lambda)^n \neq 0$  but  $M(\lambda)^{n+1} = 0$ . In the Jantzen filtration of  $M(w \cdot \lambda)$ , show that  $n = \ell_\lambda(w)$  where  $\ell_\lambda$  is the length function of the system  $(W_{[\lambda]}, \Delta_{[\lambda]})$ . Thus there are  $\ell(w) + 1$  nonzero layers in this filtration.

Use 0.3(2) to describe  $\Phi_{w \cdot \lambda}^+$ .

## 2 Humphreys 7.2

Let  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$  and show that  $T_\lambda^\mu$  need not take Verma modules to Verma modules.

For example, let  $\lambda = 1$  and  $\mu = -3$ .

### 2.1 Solution

Let  $\lambda = 1$  and  $\mu = -3$ , noting that both are integral,  $\mu$  is antidominant, and  $\mu, \lambda$  are *compatible* as in the definition in 7.1. We can then consider  $\nu := \mu - \lambda = -3 - 1 = -4$ , and to compute the  $\bar{\nu}$  that appears in the definition of  $T_\lambda^\mu$ , we consider the (usual)  $W$ -orbit of  $\nu$ . In  $\mathfrak{sl}(2, \mathbb{C})$ , we identify  $\Lambda = \mathbb{Z}$ ,  $W = \{\text{id}, s_\alpha\}$ , and  $s_\alpha \lambda = -\lambda$  as reflection about 0. Thus the orbit is given by  $W\nu = \{-4, 4\}$ , which contains the unique dominant weight  $\bar{\nu} = 4$ . We thus have

$$T_1^{-3}(\cdot) = \text{pr}_{-3}(L(4) \otimes \text{pr}_1(\cdot)).$$

We use the fact that we always have an exact sequence of the form

$$0 \longrightarrow N(\lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

where in  $\mathfrak{sl}(2, \mathbb{C})$  we can identify  $N(\lambda) = L(-\lambda - 2)$ , thus we have

$$0 \longrightarrow L(-\lambda - 2) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0.$$

Here we can identify

$$\begin{aligned} L(-\lambda - 2) &= L(-1 - 2) \\ &= L(-3) \\ &= L(\mu) \\ &= M(\mu) \quad \text{since } \mu = -3 \text{ is integral and antidominant,} \end{aligned}$$

thus we can rewrite the exact sequence as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(\mu) & \longrightarrow & M(\lambda) & \longrightarrow & L(\lambda) & \longrightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 0 & \longrightarrow & M(-3) & \longrightarrow & M(1) & \longrightarrow & L(1) & \longrightarrow & 0 \end{array}$$

We know that the translation functor is exact, so applying  $T_\lambda^\mu$  yields the following short exact sequence:

$$0 \longrightarrow T_1^{-3}M(-3) \longrightarrow T_1^{-3}M(1) \longrightarrow T_1^{-3}L(1) \longrightarrow 0$$

Since not *both*  $\lambda, \mu$  are antidominant, we can not apply Theorem 7.6 to compute these, so we instead turn to the definition. We claim that

$$\begin{aligned} T_1^{-3}L(1) &= \text{pr}_{-3}(L(4) \otimes \text{pr}_1(L(1))) \\ &= \text{pr}_{-3}(L(4) \otimes L(1)) \\ &= 0. \end{aligned}$$

This follows from the fact that any module in  $\mathcal{O}_{\chi_{-3}}$  has a composition series for which *all* of the composition factors have highest weight in  $\$,$  but  $L(4) \otimes L(1) \cong L(3) \oplus L(5)$  has only composition factors with highest weight 3 or 5.

This forces an isomorphism  $T_1^{-3}M(-3) \xrightarrow{\sim} T_1^{-3}M(1)$ , so it suffices to show that either of these is not a Verma module.

and considering the weights of the parenthesized term.

This follows by considering

$$\begin{aligned} T_1^{-3}M(-3) &= \text{pr}_{-3}(L(4) \otimes \text{pr}_1 M(-3)) \\ &= \text{pr}_{-3}(L(4) \otimes M(-3)). \end{aligned}$$

We'll use the fact that

$$\begin{aligned} \Pi(M(-3)) &= \{-3, -5, \dots\} \\ \Pi(L(4)) &= \{-4, -2, 0, 2, 4\}, \end{aligned}$$

---

and by Theorem 3.6, the parenthesized term has a finite filtration with quotients  $Q(\mu) \in \left\{ M(\lambda + \mu) \mid \mu \in \Pi(L(4)) \right\}$ .

Identifying  $\Pi(L(4)) = \{-4, -2, 0, 2, 4\}$  we thus see that there are quotients of the form

$$Q(\mu) \in \{\dots, M(-3+2), M(-3+4), \dots\} = \{\dots, M(-1), M(3), \dots\}$$

and since  $W_{[\lambda]} = \{\lambda, -\lambda - 2\} = \{1, -3\}$ , the projection onto  $\mathcal{O}_{\chi_{-3}}$  has a composition series with these factors. But then the resulting projection must have at least *two distinct* simple quotients, whereas every Verma module has a unique simple quotient, so the projection is not a Verma module. ■

### 3 Exercise p.108

- a. Work out the Jantzen filtration sections for  $M(w_0 \cdot \lambda)$ . List carefully any additional assumptions or facts needed to deduce  $M(w_0 \cdot \lambda)^i$  uniquely.
- b. Continue #4.11 for the case of singular  $\lambda$ , e.g.  $(\lambda + \rho, \hat{\alpha}) = 1$ . If you didn't deduce the structure of all  $M(w \cdot \lambda)$  there, can you complete it now?
- c. Work out the non-integral case. (There are several different cases to consider.)