Mapping Class Groups

D. Zack Garza

Wednesday 30$^{\text{th}}$ September, 2020

Contents

1 | Setup

- All manifolds:

  - Connected
  - Oriented
  - 2nd countable (countable basis)
  - Hausdorff (separate with disjoint neighborhoods, uniqueness of limits)
  - With boundary (possibly empty)

- Weakly Hausdorff: every continuous image of a compact Hausdorff space into it is closed.

- Compactly generated: sets are closed iff their intersection with every compact subspace is closed.

- Curves: simple, closed, oriented

- For X, Y topological spaces, consider

$$Y^X = C(X, Y) = \hom_{\mathrm{Top}}(X, Y) := \left\{ f : X \to Y \,\middle|\, f \text{ is continuous} \right\}.$$

1.1 The Compact-Open Topology

- General idea: *cartesian closed* categories, require *exponential objects* or *internal homs*: i.e. for every hom set, there is some object in the category that represents it

  - Slogan: we'd like homs to be spaces.

- Can make this work if we assume WHCG: weakly Hausdorff and compactly generated.

- Topologize with the *compact-open* topology $\mathcal{O}_{\mathrm{CO}}$:

$$U \in \mathcal{O}_{\mathrm{CO}} \iff \forall f \in U, \quad f(K) \subset Y \text{ is open for every compact } K \subseteq X.$$

1.1.1 Mapping Spaces

- So define

$$\mathrm{Map}(X, Y) := (\hom_{\mathrm{Top}}(X, Y), \mathcal{O}_{\mathrm{CO}}) \qquad \text{where } \mathcal{O}_{\mathrm{CO}} \text{ is the compact-open topology.}$$

- Can immediately define interesting derived spaces:

  - $\mathrm{Homeo}(X, Y)$ the subspace of homeomorphisms
  - $\mathrm{Imm}(X, Y)$, the subspace of immersions (injective map on tangent spaces)
  - $\mathrm{Emb}(X, Y)$, the subspace of embeddings (immersion + diffeomorphic onto image)
  - $C^k(X, Y)$, the subspace of $k\times$ differentiable maps
  - $C^\infty(X, Y)$ the subspace of smooth maps
  - $\mathrm{Diffeo}(X, Y)$ the subspace of diffeomorphisms
  - $C^\omega(X, Y)$ the subspace of analytic maps
  - $\mathrm{Isom}(X, Y)$ the subspace of isometric maps (for Riemannian metrics)
  - $[X, Y]$ homotopy classes of maps

1.2 Aside on Analysis

- If $Y = (Y, d)$ is a metric space, this is the topology of "uniform convergence on compact sets": for $f_n \to f$ in this topology iff

$$\|f_n - f\|_{\infty, K} := \sup \left\{ d(f_n(x), f(x)) \mid x \in K \right\} \overset{n \to \infty}{\Rightarrow} 0 \quad \forall K \subseteq X \text{ compact.}$$

  - In words: $f_n \to f$ uniformly on every compact set.

- If X itself is compact and Y is a metric space, $C(X, Y)$ can be promoted to a metric space with

$$d(f, g) = \sup_{x \in X} (f(x), g(x)).$$

1.2.1 Application in Analysis

- Useful in analysis: when does a family of functions

$$\mathcal{F} = \{ f_\alpha \} \subset \hom_{\mathrm{Top}}(X, Y)$$

  form a compact subset of $\mathrm{Map}(X, Y)$?

- Essentially answered by:

> **Theorem 1.1** *(Ascoli).*
> If X is locally compact Hausdorff and (Y, d) is a metric space, a family $\mathcal{F} \subset \hom_{\mathrm{Top}}(X, Y)$ has compact closure $\iff$ $\mathcal{F}$ is equicontinuous and $F_x := \left\{ f(x) \mid f \in \mathcal{F} \right\} \subset Y$ has compact closure.

> **Corollary 1.2** *(Arzela).*
> If $\{ f_n \} \subset \hom_{\mathrm{Top}}(X, Y)$ is an equicontinuous sequence and $F_x := \{ f_n(x) \}$ is bounded for every X, it contains a uniformly convergent subsequence.

1.3 Aside on Number Theory

- Useful in Number Theory / Rep Theory / Fourier Series:
  - Can take G to be a locally compact abelian topological group and define its Pontryagin dual

$$\widehat{G} := \hom_{\mathrm{TopGrp}}(G, S^1)$$

  where we consider $S^1 \subset \mathbb{C}$.
- Can integrate with respect to the Haar measure μ, define L^p spaces, and for $f \in L^p(G)$ define a Fourier transform $\widehat{f} \in L^p(\widehat{G})$.

$$\widehat{f}(\chi) := \int_G f(x) \overline{\chi(x)} d\mu(x).$$

2 | Path Spaces

- Can immediately consider some interesting spaces via the functor $\text{Map}(\,\cdot\,, Y)$:

$$
\begin{aligned}
X = \{\text{pt}\} &\rightsquigarrow \quad \text{Map}(\{\text{pt}\}, Y) \cong Y \\
X = I &\rightsquigarrow \quad \mathcal{P}Y := \{f : I \to Y\} = Y^I \\
X = S^1 &\rightsquigarrow \quad \mathcal{L}Y := \left\{f : S^1 \to Y\right\} = Y^{S^1}.
\end{aligned}
$$

- Adjoint property: there is a homeomorphism

$$
\begin{aligned}
\text{Map}(X \times Z, Y) &\leftrightarrow_{\cong} \text{Map}(Z, Y^X) \\
H : X \times Z \to Y &\iff \tilde{H} : Z \to \text{Map}(X, Y) \\
(x, z) \mapsto H(x, z) &\iff z \mapsto H(\,\cdot\,, z).
\end{aligned}
$$

    - Categorically, $\hom(X, \,\cdot\,) \leftrightarrow (X \times \,\cdot\,)$ form an adjoint pair in Top.

    - A form of this adjunction holds in any cartesian closed category (terminal objects, products, and exponentials)

2.1 Homotopy and Isotopy in Terms of Path Spaces

- Can take basepoints to obtain the base path space PY, the based loop space ΩY.

- Importance in homotopy theory: the path space fibration

$$
\Omega Y \hookrightarrow PY \xrightarrow{\gamma \mapsto \gamma(1)} Y
$$

    - Plays a role in "homotopy replacement", allows you to assume everything is a fibration and use homotopy long exact sequences.

- Fun fact: with some mild point-set conditions (Locally compact and Hausdorff),

$$
\pi_0 \text{Map}(X, Y) = \{[f], \text{ homotopy classes of maps } f : X \to Y\},
$$

    i.e. two maps f, g are homotopic $\iff$ they are connected by a path in $\text{Map}(X, Y)$.

> Picture!

2.1.1 Proof

$$
\mathcal{P}\text{Map}(X, Y) = \text{Map}(I, Y^X) \cong \text{Map}(X \times I, Y),
$$

and just check that $\gamma(0) = f \iff H(x, 0) = f$ and $\gamma(1) = g \iff H(x, 1) = g$.

- Interpretation: the RHS contains homotopies for maps $X \to Y$, the LHS are paths in the space of maps.

2.2 Iterated Path Spaces

- Now we can bootstrap up to play fun recursive games by applying the pathspace *endofunctor* $\mathrm{Map}(I, \cdot)$:

$$\mathcal{P}\mathrm{Map}(X,Y) := \mathrm{Map}(I, Y^X)$$

$$\mathcal{P}^2\mathrm{Map}(X,Y) := \mathcal{P}\mathrm{Map}(I, Y^X) = \mathrm{Map}(I, (Y^X)^I) = \mathrm{Map}(I, Y^{XI})$$

$$\vdots$$

$$\mathcal{P}^n\mathrm{Map}(X,Y) := \mathcal{P}^{n-1}\mathrm{Map}(I, Y^{XI}) = \mathrm{Map}(X, Y^{XI^n}).$$

- Can interpret

$$\mathcal{P}^2\mathrm{Map}(X,Y) = \mathcal{P}\mathrm{Map}(X \times I, Y).$$

as the space of paths between homotopies.

- Interpretation: we can consider paths in the *space* of paths, and paths between homotopies, and homotopies between homotopies, ad infinitum!

This in fact defines a *monad* on spaces: an endofunctor that behaves like a monoid.

Picture, link to infinity categories.

3 | Defining the Mapping Class Group

3.1 Isotopy

- Define a homotopy between $f, g : X \to Y$ as a map $F : X \times I \to Y$ restricting to f, g on the ends.
    - Equivalently: a *path*, an element of $\mathrm{Map}(I, C(X, Y))$.
- Isotopy: require the partially-applied function $F_t : X \to Y$ to be homeomorphisms for every t.
    - Equivalently: a path in the subspace of homeomorphisms, an element of $\mathrm{Map}(I, \mathrm{Homeo}(X, Y))$

Picture: picture of homotopy, paths in $\mathrm{Map}(X, Y)$, subspace of homeomorphisms.

3.2 Self-Homeomorphisms

- In any category, the automorphisms form a group.

  - In a general category $\mathcal{C}$, we can always define the group $\mathrm{Aut}_{\mathcal{C}}(X)$.

    * If the group has a topology, we can consider $\pi_0\mathrm{Aut}_{\mathcal{C}}(X)$, the set of path components.
    * Since groups have identities, we can consider $\mathrm{Aut}_{\mathcal{C}}^0(X)$, the path component containing the identity.

  - So we make a general definition, the *extended mapping class group*:

  $$\mathrm{MCG}_{\mathcal{C}}^{\pm}(X) \coloneqq \mathrm{Aut}_{\mathcal{C}}(X)/\mathrm{Aut}_{\mathcal{C}}^0(X).$$

  - Here the $\pm$ indicates that we take both orientation preserving and non-preserving automorphisms.

  - Has an index 2 subgroup of orientation-preserving automorphisms, $\mathrm{MCG}^+(X)$.

  - Can define $\mathrm{MCG}_{\partial}(X)$ as those that restrict to the identity on ∂X.

Picture: quotienting out by identity component

3.3 Definitions in Several Categories

- Now restrict attention to

$$\mathrm{Homeo}(X) \coloneqq \mathrm{Aut}_{\mathrm{Top}}(X) = \left\{ f \in \mathrm{Map}(X,X) \;\middle|\; f \text{ is an isomorphism} \right\}$$

$$\text{equipped with } \mathcal{O}_{\mathrm{CO}}.$$

  - Taking $\mathrm{MCG}^{\pm}_{\mathrm{Top}}(X)$ yields *homeomorphism up to homotopy*
- Similarly, we can do all of this in the smooth category:

$$\mathrm{Diffeo}(X) \coloneqq \mathrm{Aut}_{C^{\infty}}(X).$$

  - Taking $\mathrm{MCG}_{C^{\infty}}(X)$ yields *diffeomorphism up to isotopy*
- Similarly, we can do this for the homotopy category of spaces:

$$\mathrm{ho}(X) \coloneqq \left\{ [f] \in [X,Y] \right\}.$$

  - Taking $\mathrm{MCG}(X)$ here yields *homotopy classes of self-homotopy equivalences.*
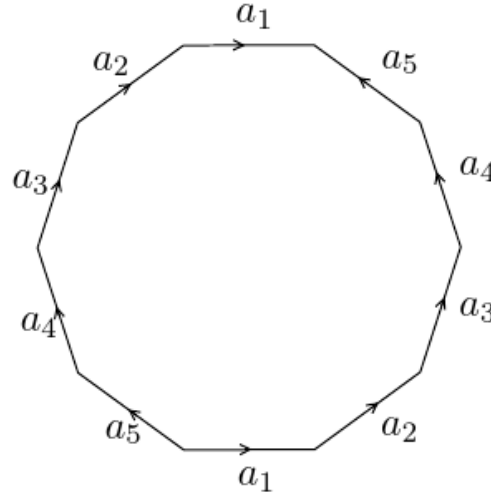
Image

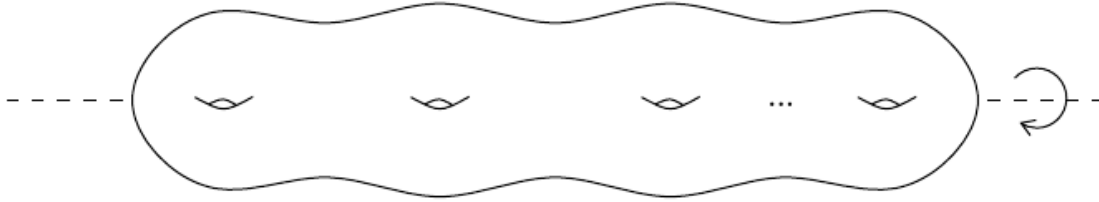Figure 2.2 Rotation by $2\pi/10$ gives an order 10 element of $\mathrm{Mod}(S_2)$.

Figure 2.3 The rotation by π about the indicated axis is a hyperelliptic involution.

Figure 1: Image

3.4 Relation to Moduli Spaces

- For topological manifolds: Isotopy classes of homeomorphisms

  - In the compact-open topology, two maps are isotopic iff they are in the same component of $\pi \operatorname{Aut}(X)$.

- For surfaces: For Σ a genus g surface, $\operatorname{MCG}(S)$ acts on the Teichmuller space $T(S)$, yielding a SES

$$0 \to \operatorname{MCG}(\Sigma) \to T(\Sigma) \to \mathcal{M}_g \to 0$$

  where the last term is the moduli space of Riemann surfaces homeomorphic to X.

- $T(S)$ is the moduli space of complex structures on S, up to the action of homeomorphisms that are isotopic to the identity:

  - Points are isomorphism classes of marked Riemann surfaces
  - Equivalently the space of hyperbolic metrics

- Used in the Neilsen-Thurston Classification: for a compact orientable surface, a self-homeomorphism is isotopic to one which is any of:

  - Periodic,
  - Reducible (preserves some simple closed curves), or
  - Pseudo-Anosov (has directions of expansion/contraction)

Picture: $\mathcal{M}_g$.

4 | Examples of MCG

4.1 The Plane: Straight Lines

- $\mathrm{MCG}_{\mathrm{Top}}(\mathbb{R}^2) = 1$: for any $f : \mathbb{R}^2 \to \mathbb{R}^2$, take the straight-line homotopy:

$$F : \mathbb{R}^2 \times I \to \mathbb{R}^2$$
$$F(x, t) = t f(x) + (1 - t)x.$$

Picture: parameterize line between x and $f(x)$ and flow along it over time.

4.2 The Closed Disc: The Alexander Trick

- $\mathrm{MCG}_{\mathrm{Top}}(\overline{\mathbb{D}}^2) = 1$: for any $f : \overline{\mathbb{D}}^2 \to \overline{\mathbb{D}}^2$ such that $f|_{\partial \overline{\mathbb{D}}^2} = \mathrm{id}$, take

$$F : \overline{\mathbb{D}}^2 \times I \to \overline{\mathbb{D}}^2$$

$$F(x, t) := \begin{cases} tf\left(\dfrac{x}{t}\right) & \|x\| \in [0, t) \\ x & \|x\| \in [1 - t, 1] \end{cases}.$$

  - This is an isotopy from f to the identity.

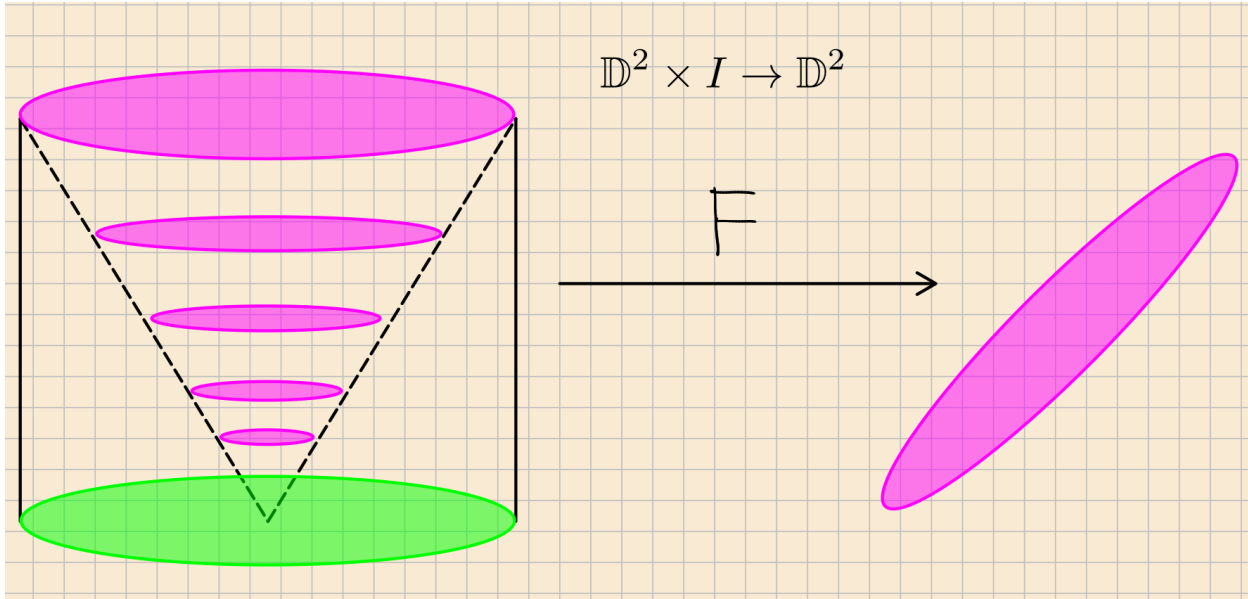 - Interpretation: "cone off" your homeomorphism over time:

Figure 2: Image

  - Note that this won't work in the smooth category: singularity at origin

4.3 Overview of Big Results

- The word problem in $\mathrm{MCG}(\Sigma_g)$ is solvable

- Any finite group is $\mathrm{MCG}(X)$ for some compact hyperbolic 3-manifold X.

- For $g \geq 3$, the center of $\mathrm{MCG}(\Sigma_g)$ is trivial and $H_1(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = 1$

  - Why care: same as abelianization of the group.

> **Theorem 4.1 (Dehn-Neilsen-Baer).**
> Let Σ_g be compact and oriented with $\chi(\Sigma_g) < 0$. Then
>
> $$\mathrm{MCG}_{\partial}^+(\Sigma_g) \cong \mathrm{Out}_{\partial}(\pi_1(\Sigma_g)) \cong_{\mathrm{Grp}} \pi_0 \mathrm{ho}_{\partial}(\Sigma_g).$$

- For $g \geq 4$, $H_2(\mathrm{MCG}(\Sigma_g); \mathbb{Z}) = \mathbb{Z}$

  - Why care: used to understand surface bundles

$$\Sigma_g \longrightarrow E$$
$$\downarrow$$
$$B$$

  - Find the classifying space $B\mathrm{Diffeo}$

  - Understand its homotopy type, since the homotopy LES yields

$$[S^n, B\mathrm{Diffeo}(\Sigma_g)] \cong [S^{n-1}, \mathrm{Diffeo}(\Sigma_g)]$$

  - Theorem (Earle-Ells): For $g \geq 2$, $\mathrm{Diffeo}_0(\Sigma_g)$ is contractible. As a consequence, $\mathrm{Diffeo}(\Sigma_g) \twoheadrightarrow \mathrm{Mod}(\Sigma_g)$ is a homotopy equivalence, and there is a correspondence:

$$\left\{ \begin{matrix} \text{Oriented } \Sigma_g \text{ bundles} \\ \text{over } B \end{matrix} \right\} \Big/ {\scriptstyle\text{Bundle isomorphism}} \iff \left\{ \begin{matrix} \text{Monodromy Representations} \\ \rho: \pi_1(B) \to \mathrm{MCG}(\Sigma_g) \end{matrix} \right\} \Big/ {\scriptstyle\text{Conjugacy}}.$$

---

5 | Dehn Twists

- $\mathrm{MCG}(\Sigma_g)$ is generated by finitely many **Dehn twists**, and always has a finite presentation

Claim: Let $A := \left\{ z \in \mathbb{C} \ \middle| \ 1 \leq |z| \leq 2 \right\}$, then $\mathrm{MCG}(A) \cong \mathbb{Z}$, generated by the map

$$
\begin{aligned}
\tau_0 : \mathbb{C} &\to \mathbb{C} \\
z &\mapsto \exp\left(2\pi i |z|\right) z.
\end{aligned}
$$

6 | MCG of the Torus

6.1 Setup

Definition 6.0.1 (Special Linear Group)**.**

$$\mathrm{SL}(n, \Bbbk) = \Big\{ M \in \mathrm{GL}(n, \Bbbk) \ \Big| \ \det M = 1 \Big\} = \ker \det_{\mathbb{G}_m}.$$

Definition 6.0.2 (Symplectic Group)**.**

$$\mathrm{Sp}(2n, \Bbbk) = \Big\{ M \in \mathrm{GL}(2n, \Bbbk) \ \Big| \ M^t \Omega M = \Omega \Big\} \leq \mathrm{SL}(2n, \Bbbk)$$

where Ω is a nondegenerate skew-symmetric bilinear form on $\Bbbk$.
Example:

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Definition 6.0.3 (Algebraic Intersection)**.**
A bilinear antisymmetric form on middle homology:

$$\widehat{\imath} : H_1(\Sigma_g; \mathbb{Z}) \otimes H_1(\Sigma_g; \mathbb{Z}) \to \mathbb{Z}.$$

> Note that this is a symplectic pairing.

- There is a natural action of $\mathrm{MCG}(\Sigma)$ on $H_1(\Sigma; \mathbb{Z})$, i.e. a *homology representation* of $\mathrm{MCG}(\Sigma)$:

$$\rho : \mathrm{MCG}(\Sigma) \to \mathrm{Aut}_{\mathrm{Grp}}(H_1(\Sigma; \mathbb{Z}))$$
$$f \mapsto f_*.$$

- For a surface of finite genus $g \geq 1$, elements in $\mathrm{im}\rho$ preserve the *algebraic intersection form*

- Thus there is an interesting surjective representation:

$$0 \to \mathrm{Tor}(\Sigma_g) \hookrightarrow \mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}(2g; \mathbb{Z}) \to 0.$$

- Kernel is the *Torelli group*, interesting because the symplectic group is well understood, so questions about MCG reduce to questions about Tor.

Theorem 6.1 *(Mapping Class Group of the Torus).*
The homology representation of the torus induces an isomorphism

$$\sigma : \mathrm{MCG}(\Sigma_2) \xrightarrow{\cong} \mathrm{SL}(2, \mathbb{Z})$$

6.2 Proof

- For f any automorphism, the induced map $f_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is a group automorphism, so we can consider the group morphism

$$\tilde{\sigma} : (\mathrm{Homeo}(X, X), \circ) \to (\mathrm{GL}(2, \mathbb{Z}), \circ)$$
$$f \mapsto f_*.$$

- This will descend to the quotient $\mathrm{MCG}(X)$ iff

$$\mathrm{Homeo}^0(X, X) \subseteq \ker \tilde{\sigma} = \tilde{\sigma}^{-1}(\mathrm{id})$$

  - This is true here, since any map in the identity component is homotopic to the identity, and homotopic maps induce the equal maps on homology.

- So we have a (now injective) map

$$\tilde{\sigma} : \mathrm{MCG}(X) \to \mathrm{GL}(2, \mathbb{Z})$$
$$f \mapsto f_*.$$

Claim: $\mathrm{im}(\tilde{\sigma}) \subseteq \mathrm{SL}(2, \mathbb{Z})$.

Proof .

- Algebraic intersection numbers in Σ_2 correspond to determinants
- $f \in \mathrm{Homeo}^+(X)$ preserve algebraic intersection numbers.
- See section 1.2 of Farb and Margalit   ∎

- We can thus freely restrict the codomain to define the map

$$\sigma : \mathrm{MCG}(X) \to \mathrm{SL}(2, \mathbb{Z})$$
$$f \mapsto f_*.$$

Claim: σ is surjective.

- $\mathbb{R}^2$ is the universal cover of Σ_2, with deck transformation group $\mathbb{Z}^2$.

- Any $A \in \mathrm{SL}(2, \mathbb{Z})$ extends to $\tilde{A} \in \mathrm{GL}(2, \mathbb{R})$, a linear self-homeomorphism of the plane that is orientation-preserving.

Claim: $\tilde{A}$ is equivariant wrt $\mathbb{Z}^2$

Proof .

$$\mathrm{SL}(2, \mathbb{Z}) = \left\langle S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

Note that

$$S^2 = 1, \qquad T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

So if $\mathbf{x} = [x_1, x_2] \in \mathbb{Z} \oplus \mathbb{Z}$ and $\tilde{A} \in \mathrm{SL}(2, \mathbb{Z})$, we have $\tilde{A}\mathbf{x} \in \mathbb{Z} \oplus \mathbb{Z}$, i.e. A preserves any integer lattice

$$\Lambda = \left\{ p\mathbf{v}_1 + q\mathbf{v}_2 \ \middle| \ p, q \in \mathbb{Z} \right\}.$$

∎

- So $\tilde{A}$ descends to a well-defined map $\psi_{\tilde{A}}$ on $\Sigma_2 := \mathbb{R}^2/\mathbb{Z}^2$, which is still a linear self-homeomorphism

- There is a correspondence

$$\left\{ \begin{smallmatrix} \text{Primitive curves in} \\ \pi_1(\Sigma_2) \cong \mathbb{Z}^2 \end{smallmatrix} \right\} \iff \left\{ \text{Primitive vectors in } \mathbb{Z}^2 \right\} \iff \left\{ \begin{smallmatrix} \text{Oriented simple closed} \\ \text{curves in } \Sigma_2 \end{smallmatrix} \right\} \Big/ {}_{\text{homotopy}},$$

  where an element x is *primitive* iff it is not a multiple of another element.

- Thus $\sigma([\psi_{\tilde{A}}]) = \tilde{A}$, and we have surjectivity.

Claim: σ is injective.

- Useful fact: $\Sigma_2 \simeq K(\mathbb{Z}^2, 1)$.

Proposition 6.2 *(Hatcher 1B.9).*
Let X be a connected CW complex and Y a $K(G, 1)$. Then there is a map

$$\hom_{\mathrm{Grp}}(\pi_1(X; x_0), \pi_1(Y; y_0)) \to \hom_{\mathrm{Top}}((X; x_0), (Y; y_0)),$$

i.e. every homomorphism of fundamental groups is induced by a continuous pointed map. Moreover, the map is unique up to homotopies fixing x_0.

- Thus there is a correspondence

$$\left\{ \begin{smallmatrix} \text{Homotopy classes of} \\ \text{maps } \Sigma_2 \circlearrowleft \end{smallmatrix} \right\} \iff \left\{ \text{Group morphisms } \mathbb{Z}^2 \circlearrowleft \right\}.$$

- Claim: any element $f \in \mathrm{MCG}(\Sigma_2)$ has a representative φ which fixes any given basepoint

- So if $f \in \ker \sigma$, then $f \simeq \varphi \simeq \mathrm{id}$ are homotopic, so $\ker \sigma = 1$.