Algebra Problems

UGA

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1 Problem Set One

1.1 Exercises

Problem 1.1 (Hungerford 1.6.3)**.**
If $\sigma = (i_1 i_2 \cdots i_r) \in S_n$ and $\tau \in S_n$, then show that $\tau\sigma\tau^{-1} = (\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

Problem 1.2 (Hungerford 1.6.4)**.**
Show that $S_n \cong \langle (12), (123\cdots n)\rangle$ and also that $S_n \cong \langle (12), (23\cdots n)\rangle$

Problem 1.3 (Hungerford 2.2.1)**.**
Let G be a finite abelian group that is not cyclic. Show that G contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p.

Problem 1.4 (Hungerford 2.2.12.b)**.**
Determine (up to isomorphism) all abelian groups of order 64; do the same for order 96.

Problem 1.5 (Hungerford 2.4.1)**.**
Let G be a group and $A \trianglelefteq G$ be a normal abelian subgroup. Show that G/A acts on A by conjugation and construct a homomorphism $\varphi : G/A \to \mathrm{Aut}(A)$.

Problem 1.6 (Hungerford 2.4.9)**.**
Let $Z(G)$ be the center of G. Show that if $G/Z(G)$ is cyclic, then G is abelian.

Note that Hungerford uses the notation $C(G)$ for the center.

Problem 1.7 (Hungerford 2.5.6)**.**
Let G be a finite group and $H \trianglelefteq G$ a normal subgroup of order p^k. Show that H is contained in every Sylow p-subgroup of G.

Problem 1.8 (Hungerford 2.5.9)**.**
Let $|G| = p^n q$ for some primes $p > q$. Show that G contains a unique normal subgroup of index q.

1.2 Qual Problems

Problem 1.9.
Let G be a finite group and p a prime number. Let X_p be the set of Sylow-p subgroups of G and n_p be the cardinality of X_p. Let $\mathrm{Sym}(X)$ be the permutation group on the set X_p.

1. Construct a homomorphism $\rho : G \to \mathrm{Sym}(X_p)$ with image a transitive subgroup (i.e. with a single orbit).
2. Deduce that if G is simple then the order of G divides $n_p!$.
3. Show that for any $1 \leq a \leq 4$ and any prime power p^k, no group of order ap^k is simple.

Problem 1.10.
Let G be a finite group and let $N \trianglelefteq G$, and let p be a prime number and Q a subgroup of G such that $N \subset Q$ and Q/N is a Sylow $p-$subgroup of G/N.

1. Prove that Q contains a Sylow $p-$subgroup of G.
2. Prove that every Sylow $p-$subgroup of G/N is the image of a Sylow $p-$subgroup of G.

Problem 1.11.
Let G be a finite group and $H < G$ a subgroup. Let n_H be the number of subgroups of G that are conjugate to H. Show that n_H divides the order of G.

Problem 1.12.
Let $G = S_5$, the symmetric group on 5 elements. Identify all conjugacy classes of elements in G, provide a representative from each class, and prove that this list is complete.

2 Problem Set Two

2.1 Exercises

Problem 2.1 (Hungerford 2.1.9)**.**
Let G be a finitely generated abelian group in which no element (except 0) has finite order. Show that G is a free abelian group.

Problem 2.2 (Hungerford 2.1.10)**.**

1. Show that the additive group of rationals $\mathbb{Q}$ is not finitely generated.
2. Show that $\mathbb{Q}$ is not free.
3. Conclude that Exercise 9 is false if the hypothesis "finitely generated" is omitted.

Problem 2.3 (Hungerford 2.5.8)**.**
Show that if every Sylow $p-$subgroup of a finite group G is normal for every prime p, then G is the direct product of its Sylow subgroups.

Problem 2.4 (Hungerford 2.6.4)**.**
What is the center of the quaternion group Q_8? Show that $Q_8/Z(Q_8)$ is abelian.

Problem 2.5 (Hungerford 2.6.9)**.**
Classify up to isomorphism all groups of order 18. Do the same for orders 20 and 30.

Problem 2.6 (Hungerford 1.9.1)**.**
Show that every non-identity element in a free group F has infinite order.

Problem 2.7 (Hungerford 1.9.3)**.**
Let F be a free group and for a fixed integer n, let H_n be the subgroup generated by the set $\{x^n \mid x \in F\}$. Show that $H_n \trianglelefteq F$.

2.2 Qual Problems

Problem 2.8.
List all groups of order 14 up to isomorphism.

Problem 2.9.
Let G be a group of order p^3 for some prime p. Show that either G is abelian, or $|Z(G)| = p$.

Problem 2.10.
Let p, q be distinct primes, and let k denote the smallest positive integer such that p divides $q^k - 1$. Show that no group of order pq^k is simple.

Problem 2.11.
Show that S_4 is a solvable, nonabelian group.

3 Problem Set Three

3.1 Exercises

Problem 3.1 (Hungerford 2.7.10)**.**
Show that S_n is solvable for $n \leq 4$ but S_3 and S_4 are not nilpotent.

Problem 3.2 (Hungerford 2.8.3)**.**
Show that if N is a simple normal subgroup of a group G and G/N has a composition series, then G has a composition series.

Problem 3.3 (Hungerford 2.8.9)**.**
Show that any group of order p^2q (for primes p, q) is solvable.

Problem 3.4 (Hungerford 5.1.1)**.**
Let F/K be a field extension. Show that

1. $[F : K] = 1$ iff $F = K$.
2. If $[F : K]$ is prime, then there are no intermediate fields between F and K.
3. If $u \in F$ has degree n over K, then n divides $[F : K]$.

Problem 3.5 (Hungerford 5.1.8)**.**
Show that if $u \in F$ is algebraic of odd degree over K, then so is u^2, and moreover $K(u) = K(u^2)$.

Problem 3.6 (Hungerford 5.1.14)**.**
1. If $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, compute $[F : \mathbb{Q}]$ and find a basis of $F/\mathbb{Q}$.
2. Do the same for $\mathbb{Q}(i, \sqrt{3}, \zeta_3)$ where ζ_3 is a complex third root of 1.

Problem 3.7 (Hungerford 5.1.16)**.**
Show that in $\mathbb{C}$, the fields $\mathbb{Q}(i) \cong \mathbb{Q}(\sqrt{2})$ as vector spaces, but not as fields.

3.2 Qual Problems

Problem 3.8.
Let R and S be commutative rings with multiplicative identity.

1. Prove that when R is a field, every non-zero ring homomorphism $\phi : R \to S$ is injective.
2. Does (a) still hold if we only assume that R is a domain? If so, prove it, and if not provide a counterexample.

Problem 3.9.
Determine for which integers the ring $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of fields. Carefully prove your answer.

Problem 3.10.
Suppose that R is a commutative ring. Show that an element $r \in R$ is not invertible iff it is contained in a maximal ideal.

Problem 3.11.

1. Give the definition that a group G must satisfy the be solvable.
2. Show that every group G of order 36 is solvable.

   Hint: You may assume that S^4 is solvable.

4 Problem Set Four

4.1 Exercises

Problem 4.1 (Hungerford 5.3.7)**.**
If F is algebraically closed and E is the set of all elements in F that are algebraic over a field K, then E is an algebraic closure of K.

Problem 4.2 (Hungerford 5.3.8)**.**
Show that no finite field is algebraically closed.

Hint: if $K = \{a_i\}_{i=0}^n$, consider

$$f(x) = a_1 + \prod_{i=0}^{n}(x - a_i) \in K[x]$$

where $a_1 \neq 0$.

Problem 4.3 (Hungerford 5.5.2)**.**
Show that if $p \in \mathbb{Z}$ is prime, then $a^p = a$ for all $a \in \mathbb{Z}_p$, or equivalently $c^p \equiv c \mod p$ for all $c \in \mathbb{Z}$.

Problem 4.4 (Hungerford 5.5.3)**.**
Show that if $|K| = p^n$, then every element of K has a unique pth root in K.

Problem 4.5 (Hungerford 5.5.10)**.**
Show that every element in a finite field can be written as the sum of two squares.

Problem 4.6 (Hungerford 5.6.1)**.**
Let F/K be a field extension. Let $\text{char}K = p \neq 0$ and let $n \geq 1$ be an integer such that $(p, n) = 1$. If $v \in F$ and $nv \in K$, then $v \in K$.

Problem 4.7 (Hungerford 5.6.8)**.**
If $\text{char}K = p \neq 0$ and $[F : K]$ is finite and not divisible by p, then F is separable over K.

4.2 Qual Problems

Problem 4.8.
Suppose that α is a root in $\mathbb{C}$ of $P(x) = x^{17} - 2$. How many field homomorphisms are there from $\mathbb{Q}(\alpha)$ to:

1. $\mathbb{C}$,
2. $\mathbb{R}$,
3. $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$?

Problem 4.9.
Let C/F be an algebraic field extension. Prove that the following are equivalent:

1. Every non-constant polynomial $f \in F[x]$ factors into linear factors over $C[x]$.
2. For every (not necessarily finite) algebraic extension E/F, there is a ring homomorphism $\alpha : E \to C$ such that $\alpha\,|_F$ is the identity on F.

   Hint: use Zorn's Lemma.

Problem 4.10.
Let R be a commutative ring containing a field k, and suppose that $\dim_k R < \infty$. Let $\alpha \in R$.

1. Show that there exist $n \in \mathbb{N}$ and $\{c_0, c_1, \cdots c_{n-1}\} \subseteq k$ such that
$$a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = 0.$$
2. Suppose that (a) holds and show that if $c_0 \neq 0$ then a is a unit in R.
3. Suppose that (a) holds and show that if a is not a zero divisor in R, then a is invertible.

5 Problem Set Five

5.1 Exercises

Problem 5.1 (Hungerford 5.3.5)**.**
Show that if $f \in K[x]$ has degree n and F is a splitting field of f over K, the $[F : K]$ divides $n!$.

Problem 5.2 (Hungerford 5.3.12)**.**
Let E be an intermediate field extension in $K \leq E \leq F$.

1. Show that if $u \in F$ is separable over over K, then u is separable over E.
2. Show that if F is separable over K, then F is separable over E and E is separable over K.

Problem 5.3 (Hungerford 5.3.13)**.**
Show that if $[F : K] < \infty$, then the following conditions are equivalent:

1. F is Galois over K
2. F is separable over K and F is a splitting field of some polynomial $f \in K[x]$.
3. F is a splitting field over K of some polynomial $f \in K[x]$ whose irreducible factors are separable.

Problem 5.4 (Hungerford 5.4.1)**.**
Suppose that $f \in K[x]$ splits in F as

$$f = \prod_{i=1}^{k}(x - u_i)^{n_i}$$

with the u_i distinct and each $n_i \geq 1$. Let

$$g(x) = \prod_{i=1}^{k}(x - u_i) = \sum_{i=1}^{k} v_i x^i$$

and let $E = K(\{v_i\}_{i=1}^{k})$. Then show that the following hold:

1. F is a splitting field of g over E.
2. F is Galois over E.
3. $\mathrm{Aut}_E(F) = \mathrm{Aut}_K(F)$.

Problem 5.5 (Hungerford 5.4.10 a/g/h)**.**
Determine the Galois groups of the following polynomials over the corresponding fields:

1. $x^4 - 5$ over $\mathbb{Q}, \mathbb{Q}(\sqrt{5}), \mathbb{Q}(i\sqrt{5})$.
2. $x^3 - 2$ over $\mathbb{Q}$.
3. $(x^3 - 2)(x^2 - 5)$ over $\mathbb{Q}$.

Problem 5.6 (Hungerford 5.6.11)**.**
If $f \in K[x]$ is irreducible of degree $m > 0$ and $\text{char}(K)$ does not divide m, then f is separable.

5.2 Qual Problems

Problem 5.7.
Let E/F be a Galois field extension, and let K/F be an intermediate field of E/F. Show that K is normal over F iff $\mathrm{Gal}(E/K) \trianglelefteq \mathrm{Gal}(E/F)$.

Problem 5.8.
Let $F \subset L$ be fields such that L/F is a Galois field extension with Galois group equal to $D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1,\ \sigma\tau = \tau\sigma^3 \rangle$. Show that there are fields $F \subset E \subset K \subset L$ such that E/F and K/E are Galois field extensions, but K/F is not Galois.

Problem 5.9.
Let $f(x) = x^3 - 7$.

1. Let K be the splitting field for f over $\mathbb{Q}$. Describe the Galois group of $K/\mathbb{Q}$ and the intermediate fields between $\mathbb{Q}$ and K. Which intermediate fields are not Galois over $\mathbb{Q}$?
2. Let L be the splitting field for f over $\mathbb{R}$. What is the Galois group $L/\mathbb{R}$?
3. Let M be the splitting field for f over $\mathbb{F}_{13}$, the field with 13 elements. What is the Galois group of $M/\mathbb{F}_{13}$?

6 Problem Set Six

6.1 Exercises

Problem 6.1 (Hungerford 5.4.11)**.**
Determine all subgroups of the Galois group and all intermedate fields of the splitting (over $\mathbb{Q}$) of the polynomial $(x^3 - 2)(x^2 - 3) \in \mathbb{Q}[x]$.

Problem 6.2 (Hungerford 5.4.12)**.**
Let K be a subfield of $\mathbb{R}$ and let $f \in K[x]$ be an irreducible quartic. If f has exactly 2 real roots, the Galois group of f is either S_4 or D_4.

Problem 6.3 (Hungerford 5.8.3)**.**
Let ϕ be the Euler function.

1. $\phi(n)$ is even for $n > 2$.
2. find all $n > 0$ such that $\phi(n) = 2$.

Problem 6.4 (Hungerford 5.8.9)**.**
If $n > 2$ and ζ is a primitive n-th root of unity over $\mathbb{Q}$, then $[\mathbb{Q}(\zeta + \zeta^{-1}) : \mathbb{Q}] = \phi(n)/2$.

Problem 6.5 (Hungerford 5.9.1)**.**
If F is a radical extension field of K and E is an intermediate field, then F is a radical extension of E.

Problem 6.6 (Hungerford 5.9.3)**.**
Let K be a field, $f \in K[x]$ an irreducible polynomial of degree $n \geq 5$ and F a splitting field of f over K. Assume that $Aut_k(F) \simeq S_n$. Let u be a root of f in F. Then,

1. $K(u)$ is not Galois over K; $[K(u) : K] = n$ and $Aut_K(K(u)) = 1$ (and hence solvable).
2. Every normal closure over K that contains u also contains an isomorphic copy of F.
3. There is no radical extension field E of K such that $K \subset K(u) \subset E$.

6.2 Qual Problems

Problem 6.7. 1. Let K be a field. State the main theorem of Galois theory for a finite field extension L/K

2. Let $\zeta_{43} := e^{2\pi i/43}$. Describe the group of all field automorphisms $\sigma : \mathbb{Q}(\zeta_{43}) \to \mathbb{Q}(\zeta_{43})$.

3. How many proper subfields are there in the field $\mathbb{Q}(\zeta_{43})$?

Problem 6.8.
Let F be a field and let $f(x) \in F[x]$.

1. Define what is a splitting field of $f(x)$ over F.

2. Let F be a finite field with q elements. Let E/F be a finite extension of degree $n > 0$. Exhibit an explicit polynomial $g(x) \in F[x]$ such that E/F is a splitting of $g(x)$ over F. Fully justify your answer.

3. Show that the extension E/F in (2) is a Galois extension.

Problem 6.9.
Let $K \subset L \subset M$ be a tower of finite degree field extensions. In each of the following parts, either prove the assertion or give a counterexample (with justification).

1. If M/K is Galois, then L/K is Galois

2. If M/K is Galois, then M/L is Galois.

7 Problem Set Seven

7.1 Exercises

Problem 7.1 (Hungerford 4.1.3)**.**
Let I be a left ideal of a ring R, and let A be an $R-$module.

1. Show that if S is a nonempty subset of A, then
$$IS := \left\{ \sum_{i=1}^{n} r_i a_i \mid n \in \mathbb{N}^*; r_i \in I; a_i \in S \right\}$$
is a submodule of A.

   Note that if $S = \{a\}$, then $IS = Ia = \{ra \mid r \in I\}$.

2. If I is a two-sided ideal, then A/IA is an R/I module with the action of R/I given by
$$(r + I)(a + IA) = ra + IA.$$

Problem 7.2 (Hungerford 4.1.5)**.**
If R has an identity, then a nonzero unitary $R-$module is **simple** if its only submodules are 0 and A.

1. Show that every simple $R-$module is cyclic.
2. If A is simple, every $R-$module endomorphism is either the zero map or an isomorphism.

Problem 7.3 (Hungerford 4.1.7)**.**
1. Show that if A, B are R-modules, then the set $\mathrm{Hom}_R(A, B)$ is all R-module homomorphisms $A \to B$ is an abelian group with $f + g$ given on $a \in A$ by
$$(f + g)(a) := f(a) + g(a) \in B.$$
Also show that the identity element is the zero map.

2. Show that $\mathrm{Hom}_R(A, A)$ is a ring with identity, where multiplication is given by composition of functions.

   Note that $\mathrm{Hom}_R(A, A)$ *is called the* ***endomorphism ring*** *of A.*

3. Show that A is a left $\mathrm{Hom}_R(A, A)$-module with an action defined by
$$a \in A, f \in \mathrm{Hom}_R(A, A) \implies f \curvearrowright a := f(a).$$

Problem 7.4 (Hungerford 4.1.12)**.**
Let the following be a commutative diagram of R-modules and R-module homomorphisms with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

Prove the following:

1. If α_1 is an epimorphisms and α_2, α_4 are monomorphisms then α_3 is a monomorphism.
2. If α_5 is a monomorphism and α_2, α_4 are epimorphisms then α_3 is an epimorphism.

Problem 7.5 (Hungerford 4.2.4)**.**
Let R be a principal ideal domain, A a unitary left R-module, and $p \in R$ a prime (and thus irreducible) element. Define

$$pA \coloneqq \{pa \mid a \in A\}$$
$$A[p] \coloneqq \{a \in A \mid pa = 0\}.$$

Show the following:

1. $R/(p)$ is a field.
2. pA and $A[p]$ are submodules of A.
3. A/pA is a vector space over $R/(p)$, with
$$(r + (p))(a + pA) = ra + pA.$$
4. $A[p]$ is a vector space over $R/(p)$ with
$$(r + (p))a = ra.$$

Problem 7.6 (Hungerford 4.2.8)**.**
If V is a finite dimensional vector space and

$$V^m \coloneqq V \oplus V \oplus \cdots \oplus V \quad (m \text{ summands}),$$

then for each $m \geq 1$, V^m is finite dimensional and $\dim V^m = m(\dim V)$.

Problem 7.7 (Hungerford 4.2.9)**.**
If F_1, F_2 are free modules of a ring with the invariant dimension proerty, then

$$\text{rank}(F_1 \oplus F_2) = \text{rank}F_1 + \text{rank}F_2.$$

7.2 Qual Problems

Problem 7.8.
Let F be a field and let $f(x) \in F[x]$.

1. State the definition of a splitting field of $f(x)$ over F.
2. Let F be a finite field with q elements. Let E/F be a finite extension of degree $n > 0$. Exhibit an explicit polynomial $g(x) \in F[x]$ such that E/F is a splitting field of g over F. Fully justify your answer.
3. Show that the extension in (b) is a Galois extension.

Problem 7.9.
Let R be a commutative ring and let M be an R-module. Recall that for $\mu \in M$, the *annihilator* of μ is the set

$$\mathrm{Ann}(\mu) = \{r \in R \mid r\mu = 0\}.$$

Suppose that I is an ideal in R which is maximal with respect to the property there exists a nonzero element $\mu \in M$ such that $= \mathrm{Ann}(\mu)$.

Prove that I is a *prime* ideal in R.

Problem 7.10.
Suppose that R is a principal ideal domain and $I \trianglelefteq R$ is an ideal. If $a \in I$ is an irreducible element, show that $I = Ra$.

8 Problem Set Eight

8.1 Exercises

Problem 8.1 (Hungerford 4.4.1)**.**
Show the following:

1. For any abelian group A and any positive integer m,
$$\operatorname{Hom}(\mathbb{Z}_m, A) \cong A[m] := \{a \in A \mid ma = 0\}.$$
2. $\operatorname{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}$.
3. As a $\mathbb{Z}-$module, $\mathbb{Z}_m^* = 0$.
4. For each $k \geq 1$, $\mathbb{Z}_m$ is a $\mathbb{Z}_{mk}-$module, and as a $\mathbb{Z}_{mk}$ module, $\mathbb{Z}_m^* \cong \mathbb{Z}_m$.

Problem 8.2 (Hungerford 4.4.3)**.**
Let $\pi : \mathbb{Z} \to \mathbb{Z}_2$ be the canonical epimorphism. Show that the induced map $\overline{\pi} : \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the zero map. Conclude that $\overline{\pi}$ is not an epimorphism.

Problem 8.3 (Hungerford 4.4.5)**.**
Let R be a unital ring, show that there is a ring homomorphism $\operatorname{Hom}_R(R, R) \to R^{op}$ where Hom_R denotes left $R-$module homomorphisms. Conclude that if R is commutative, then there is a ring isomorphism $\operatorname{Hom}_R(R, R) \cong R$.

Problem 8.4 (Hungerford 4.4.9)**.**
Show that for any homomorphism $f : A \to B$ of left $R-$modules the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\theta_A"] \arrow[d, "f"] & A^{**} \arrow[d, "f^*"] \\ B \arrow[r, "\theta_B"] & B^{**} \end{tikzcd}$$

where θ_A, θ_B are as in Theorem 4.12 and f^* is the map induced on $A^{**} := \operatorname{Hom}_R(\operatorname{Hom}(A, R), R)$ by the map
$$\overline{f} : \operatorname{Hom}(B, R) \to \operatorname{Hom}_R(A, R).$$

Problem 8.5 (Hungerford 4.6.2)**.**
Show that every free module over a unital integral domain is torsion-free. Show that the converse is false.

Problem 8.6 (Hungerford 4.6.3)**.**
Let A be a cyclic $R-$module of order $r \in R$.

1. Show that if s is relatively prime to r, then $sA = A$ and $A[s] = 0$.
2. If s divides r, so $sk = r$, then $sA \cong R/(k)$ and $A[s] \cong R/(s)$.

Problem 8.7 (Hungerford 4.6.6)**.**
Let A, B be cyclic modules over R of nonzero orders r, s respectively, where r is *not* relatively prime to s. Show that the invariant factors of $A \oplus B$ are $\gcd(r, s)$ and $\operatorname{lcm}(r, s)$.

8.2 Qual Problems

Problem 8.8.
Let R be a PID. Let $n > 0$ and $A \in M_n(R)$ be a square $n \times n$ matrix with coefficients in R.

Consider the R-module $M := R^n/\mathrm{im}(A)$.

1. Give a necessary and sufficient condition for M to be a torsion module (i.e. every nonzero element is torsion). Justify your answer.
2. Let F be a field and now let $R := F[x]$. Give an example of an integer $n > 0$ and an $n \times n$ square matrix $A \in M_n(R)$ such that $M := R^n/\mathrm{im}(A)$ is isomorphic as an $R-$module to $R \times F$.

Problem 8.9.
1. State the structure theorem for finitely generated modules over a PID.
2. Find the decomposition of the $\mathbb{Z}-$module M generated by w, x, y, z satisfying the relations

$$\begin{aligned} 3w + 12y + 3x + 6z &= 0 \\ 6y &= 0 \\ -3w - 3x + 6y &= 0. \end{aligned}$$

Problem 8.10.
Let R be a commutative ring and M an $R-$module.

1. Define what a torsion element of M is .
2. Given an example of a ring R and a cyclic $R-$module M such that M is infinite and M contains a nontrivial torsion element m. Justify why m is torsion.
3. Show that if R is a domain, then the subset of elements of M that are torsion is an $R-$submodule of M. Clearly show where the hypothesis that R is a domain is used.

9 Problem Set Nine

9.1 Exercises

Problem 9.1 (Hungerford 7.1.3)**.**

1. Show that the center of the ring $M_n(R)$ consists of matrices of the form rI_n where r is in the center of R.

   Hint: Every such matrix must commute with ϵ_{ij}, the matrix with 1_R in the i, j position and zeros elsewhere.

2. Show that $Z(M_n(R)) \cong Z(R)$.

Problem 9.2 (Hungerford 7.1.5)**.**

1. Show that if A, B are (skew)-symmetric then $A + B$ is (skew)-symmetric.
2. Let R be commutative. Show that if A, B are symmetric, then AB is symmetric $\iff AB = BA$. Also show that for any matrix $B \in M_n(R)$, both BB^t and $B + B^t$ are always symmetric, and $B - B^t$ is always skew-symmetric.

Problem 9.3 (Hungerford 7.1.7)**.**
Show that similarity is an equivalence relation on $M_n(R)$, and *equivalence* is an equivalence relation on $M_{m\times n}(R)$.

Problem 9.4 (Hungerford 7.2.2)**.**
Show that an $n \times m$ matrix A over a division ring D has an $m \times n$ left inverse B (so $BA = I_m$) $\iff$ rank$A = m$. Similarly, show A has a right $m \times n$ inverse $\iff$ rank$A = n$.

Problem 9.5 (Hungerford 7.2.4)**.**

1. Show that a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n$$

   has a simultaneous solution $\iff$ the corresponding matrix equation

$AX = B$ has a solution, where $A = (a_{ij}), X = [x_1, \cdots, x_m]^t$, and $B = [b_1, \cdots, b_n]^t$.

2. If A_1, B_1 are matrices obtained from A, B respectively by performing the same sequence of elementary **row** operations, then X is a solution of $AX = B \iff X$ is a solution of $A_1X = B_1$.

3. Let C be the $n \times (m+1)$ matrix given by

$$C = \begin{pmatrix} a_{11} & \cdots & a_{1m} & b_1 \\ . & & & \\ a_{n1} & \cdots & a_{nm} & b_n \end{pmatrix}.$$

   Then $AX = B$ has a solution $\iff$ rank$A$ = rank$C$ and the solution is unique $\iff$ rank$(A) = m$.

   Hint: use part 2.

4. If $B = 0$, so the system $AX = B$ is homogeneous, then it has a nontrivial solution $\iff$ $A < m$ and in particular $n < m$.

Problem 9.6 (Hungerford 7.2.5)**.**
Let R be a PID. For each positive integer r and sequence of nonzero ideals $I_1 \supset I_2 \supset \cdots \supset I_r$, choose a sequence $d_i \in R$ such that $(d_i) = I_i$ and $d_i \mid d_{i+1}$.

For a given pair of positive integers n, m, let S be the set of all $n \times m$ matrices of the form $\begin{pmatrix} L_r & 0 \\ 0 & 0 \end{pmatrix}$ where $r = 1, 2, \cdots, \min(m, n)$ and L_r is a diagonal $r \times r$ matrix with main diagonal d_i.

Show that S is a set of canonical forms under equivalence for the set of all $n \times m$ matrices over R.

9.2 Qual Problems

Problem 9.7.
Let R be a commutative ring.

1. Say what it means for R to be a unique factorization domain (UFD).
2. Say what it means for R to be a principal ideal domain (PID)
3. Give an example of a UFD that is not a PID. Prove that it is not a PID.

Problem 9.8.
Let A be an $n \times n$ matrix over a field F such that A is diagonalizable. Prove that the following are equivalent:

1. There is a vector $v \in F^n$ such that $v, Av, \cdots A^{n-1}v$ is a basis for F^n.
2. The eigenvalues of A are distinct.

Problem 9.9.
Let $x, y \in \mathbb{C}$ and consider the matrix

$$M = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix}$$

1. Show that $[0, 1, 0]^t$ is an eigenvector of M.
2. Compute the rank of M as a function of x and y.
3. Find all values of x and y for which M is diagonalizable.

10 Problem Set Ten

10.1 Exercises

Problem 10.1 (Hungerford 7.3.1)**.**
Let B be an R-module. Show that if $r+r \neq 0$ for all $r \neq 0 \in R$, then an n-linear form $B^n \to R$ is alternating $\iff$ it is skew-symmetric.

Problem 10.2 (Hungerford 7.3.5)**.**
If R is a field and $A, B \in M_n(R)$ are invertible then the matrix $A + rB$ is invertible for all but a finite number of $r \in R$.

Problem 10.3 (Hungerford 7.4.4)**.**
Show that if q is the minimal polynomial of a linear transformation $\phi : E \to E$ with $\dim_k E = n$ then $\deg q \leq n$.

Problem 10.4 (Hungerford 7.4.8)**.**
Show that $A \in M_n(K)$ is similar to a diagonal matrix $\iff$ the elementary divisors of A are all linear.

Problem 10.5 (Hungerford 7.4.10)**.**
Find all possible rational canonical forms for a matrix $A \in M_n(\mathbb{Q})$ such that

1. A is 6×6 with minimal polynomial $q(x) = (x-2)^2(x+3)$.
2. A is 7×7 with $q(x) = (x^2+1)(x-7)$.

Also find all such forms when $A \in M_n(\mathbb{C})$ instead, and find all possible Jordan Canonical Forms over $\mathbb{C}$.

Problem 10.6 (Hungerford 7.5.2)**.**
Show that if ϕ is an endomorphism of a free k-module E of finite rank, then $p_\phi(\phi) = 0$.

Hint:
If A is the matrix of ϕ and $B = xI_n - A$ then $B^a B = BI_n = p_\phi I_n$ in $M_n(k[x])$. If E is a $k[x]$-module with structure induced by ϕ, and ψ is the $k[x]$-module endomorphism $E \to E$ with matrix given by B, then

$$\psi(u) = xu - \phi(u) = \phi(u) - \phi(u) = 0$$

for all $u \in E$.

Problem 10.7 (Hungerford 7.5.7)**.**

1. Let ϕ, ψ be endomorphisms of a finite-dimensional vector space E such that $\phi\psi = \psi\phi$. Show that if E has a basis of eigenvectors of ψ, then it has a basis of eigenvectors for both ψ and ϕ simultaneously.
2. Interpret the previous part as a statement about matrices similar to a diagonal matrix.

10.2 Qual Problems

Problem 10.8.
Let $M \in M_5(R)$ be a 5×5 square matrix with real coefficients defining a linear map $L : \mathbb{R}^5 \to \mathbb{R}^5$. Assume that when considered as an element of $M_5(\mathbb{C})$, then the scalars $0, 1+i, 1+2i$ are eigenvalues of M.

1. Show that the associated linear map L is neither injective nor surjective.
2. Compute the characteristic polynomial and minimal polynomial of M.
3. How many fixed points can L have?

   (That is, how many solutions are there to the equation $L(v) = v$ with $v \in \mathbb{R}^5$?)

Problem 10.9.
Let n be a positive integer and let B denote the $n \times n$ matrix over $\mathbb{C}$ such that every entry is 1. Find the Jordan normal form of B.

Problem 10.10.
Suppose that V is a 6-dimensional vector space and that T is a linear transformation on V such that $T^6 = 0$ and $T^5 \neq 0$.

1. Find a matrix for T in Jordan Canonical form.
2. Show that if S, T are linear transformations on a 6-dimensional vector space V which both satisfy $T^6 = S^6 = 0$ and $T^5, S^5 \neq 0$, then there exists a linear transformation A from V to itself such that $ATA^{-1} = S$.