

Title

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1 | Lecture 10

Remark 1.0.1: What we've been calling a *torsor* (a sheaf with a group action plus conditions) is called by some sources a **pseudotorsor** (e.g. the Stacks Project), and what we've been calling a *locally trivial torsor* is referred to as a *torsor* instead.

Recall that statement of ??; we'll now continue with the proof:

Proof (of Hilbert 90).

Observation 1.0.2: Let $\tau = X_{\text{zar}}, X_{\text{ét}}, X_{\text{fppf}}$, then the data of a GL_n -torsor split by a τ -cover $U \rightarrow X$ is the same as descent data for a vector bundle relative to U/X .

This descent data comes from the following:

$$\begin{array}{c} U \times_X U \\ \pi_1 \downarrow \quad \downarrow \pi_2 \\ U \\ \downarrow \\ X \end{array}$$

That U trivializes our torsor means that $\pi^*T = \pi^*G$ as a G -torsor, where G acts on itself by left-multiplication. We have two different ways of pulling back, and identifications with G in both, yielding

$$\begin{array}{ccc} \pi_1^* \pi^* T & \xrightarrow{\sim} & \pi_2^* \pi^* T \\ \downarrow & & \downarrow \\ \pi_1^* \pi^* G & \xrightarrow{\sim} & \pi_2^* \pi^* G \end{array}$$

Both of the bottom objects are isomorphic to $G|_{U \times U}$.

Claim: The top horizontal map is descent data for T , and the bottom horizontal map is an automorphism of a G -torsor and thus is a section to G . I.e. a section to GL_n is an invertible matrix on double intersections (satisfying the cocycle condition) and a cover, which is precisely descent data for a vector bundle.

Using fppf descent, proved previously, we know that descent data for vector bundles is effective. So if we have a locally trivial GL_n -torsor on the fppf site, it's also trivial on the other two sites, yielding the desired maps back and forth. Thus $H^1(X_{\text{ét}}, \text{GL}_n)$ is in bijection with n -dimensional vector bundles on X . ■

Exercise 1.0.3(?): See if Hilbert 90 is true for groups other than GL_n .

1.1 Representability and Local Triviality

Question 1.1.1: Suppose G is an affine flat X -group scheme. Are all G -torsors representable by a X -scheme?

Answer 1.1.2: Yes, by the same proof as last time, try working out the details. Idea: you can trivialize a G -torsor flat locally and use fppf descent.

Question 1.1.3: Given a G -torsor T that is fppf locally trivial, is it étale locally trivial?

Answer 1.1.4: In general no, but yes if G is smooth.

Proof (Sketch).

You can take an fppf local trivialization, trivialize by p itself, then slice to get an étale trivialization. Given a torsor $T \rightarrow X$, we can base change it to itself:

$$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow \wr \exists & & \downarrow \\ T & \xrightarrow{f} & X \end{array}$$

The torsor $T \times_X T \rightarrow T$ is trivial since there exists the indicated section given by the diagonal map. Another way to see this is that $T \times T \cong T \times G$ by the G -action map, which is equivalent to triviality here. Here f is smooth map since G itself was smooth and the fibers of T are isomorphic to the fibers of G . We can thus find some U such that

$$\begin{array}{ccc} T \times_X T & \longrightarrow & T \\ \downarrow \wr \exists & & \downarrow \\ T & \xrightarrow{f} & X \\ \uparrow \text{closed} & & \uparrow \\ U & \xrightarrow{\exists \text{ét}} & X \end{array}$$

Here “slicing” means finding such a U , and this can be done using the structure theorem for smooth morphisms. ■

Example 1.1.5 (non-smooth group schemes):

- α_p , the kernel of Frobenius on \mathbb{A}^1 or \mathbb{G}_a ,
- μ_p in characteristic p , representing p th roots of unity, the kernel of Frobenius on \mathbb{G}_m ,
- The kernel of Frobenius on any positive dimensional affine group scheme.
- $\mu_p \times \mathrm{GL}_n$, etc.

1.1.1 What Hilbert 90 Means

Example 1.1.6(?): Let $X = \operatorname{Spec} k, n = 1$, so we're looking at $H^*(\operatorname{Spec} k, \mathbb{G}_m)$.

$$\begin{aligned} H^1((\operatorname{Spec} k)_{\text{zar}}, \mathbb{G}_m) &= 0 \\ &= H^1((\operatorname{Spec} k)_{\text{ét}}, \mathbb{G}_m) \\ &= H^1(\operatorname{Gal}(k^s/k), \bar{k}^\times). \end{aligned}$$

The first comes from the fact that we're looking at line bundles of spec of a field, i.e. a point, which are all trivial. The last line comes from our previous discussion of the isomorphism between étale cohomology of fields and Galois cohomology. Etymology: the fact that this cohomology is zero is usually what's called **Hilbert 90**.¹

Let's generalize this observation.

Example 1.1.7(?): Let X be any scheme and $n = 1$, then $H^1(X_{\text{ét}}, \mathbb{G}_m) = \operatorname{Pic}(X)$.

Example 1.1.8(?): Let's compute $H^1(X_{\text{ét}}, \mu_\ell)$ where ℓ is an invertible function on X . We have a SES of étale sheaves, the **Kummer sequence**,

$$1 \rightarrow \mu_\ell \rightarrow \mathbb{G}_m \xrightarrow{z \mapsto z^\ell} \mathbb{G}_m \rightarrow 1.$$

This is exact in the étale topology since adjoining an ℓ th power of any function gives an étale cover. We get a LES in cohomology

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & \swarrow & & & \\ H^0(X_{\text{ét}}, \mu_\ell) & \longrightarrow & H^0(X_{\text{ét}}, \mathbb{G}_m) & \xrightarrow{z \mapsto z^\ell} & H^0(X_{\text{ét}}, \mathbb{G}_m) & & \\ & & \swarrow & & & & \\ H^1(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \operatorname{Pic}(X) & \xrightarrow{[\ell]} & \operatorname{Pic}(X) & & \\ & & \swarrow & & & & \\ H^2(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \dots & & & & \end{array}$$

We know that $H^0(X_{\text{ét}}, \mathbb{G}_m)$ are invertible functions on X , and the red term is what we'd like to compute.

Suppose now $H^0(X, \mathcal{O}_X) = k = \bar{k}$, then $H^0(X_{\text{ét}}, \mu_\ell) = \mu_\ell(k)$ since it is the kernel of the ℓ th power map. We can also compute $H^1(X_{\text{ét}}, \mu_\ell)$, since our diagram reduces to

¹This is called "90" since Hilbert numbered his theorems in at least one of his books.

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & \swarrow & \\
 \mu_\ell(k) & \xleftarrow{\quad} & k^\times & \xrightarrow{z \mapsto z^\ell} & k^\times \\
 & \nwarrow & \searrow & & \\
 H^1(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \text{Pic}(X) & \xrightarrow{[\ell]} & \text{Pic}(X) \\
 & \nwarrow & \searrow & & \\
 H^2(X_{\text{ét}}, \mu_\ell) & \longrightarrow & \dots & &
 \end{array}$$

where surjectivity of δ follows from the fact that $k = \bar{k}$ and thus every element has an ℓ th root. 