

# Title

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## 1 | Monday, November 09

### 1.1 Strong Linkage

We have two categories:

- $G_r T$ , with a notion of *strong linkage*, and
- $G_r$ , which instead only has *linkage*.

We'll restate a few theorems.

#### **Theorem 1.1.1(?)**.

Let  $\lambda, \mu \in X(T)$ .

1. If  $[\widehat{Z}_r(\lambda) : \widehat{L}_r(\mu)]_{G_r T} \neq 0$ , then  $\mu \uparrow \lambda$  are strongly linked.
2. If  $[Z_r(\lambda) : L_r(\mu)]_{G_r} \neq 0$ , then  $\mu \in W_p \cdot \lambda + p^r X(T)$ .

*Example 1.1.1(?):* In the case of  $\Phi = A_2$ , we'll consider the two different categories.

We have the following picture for  $\widehat{Z}$ :

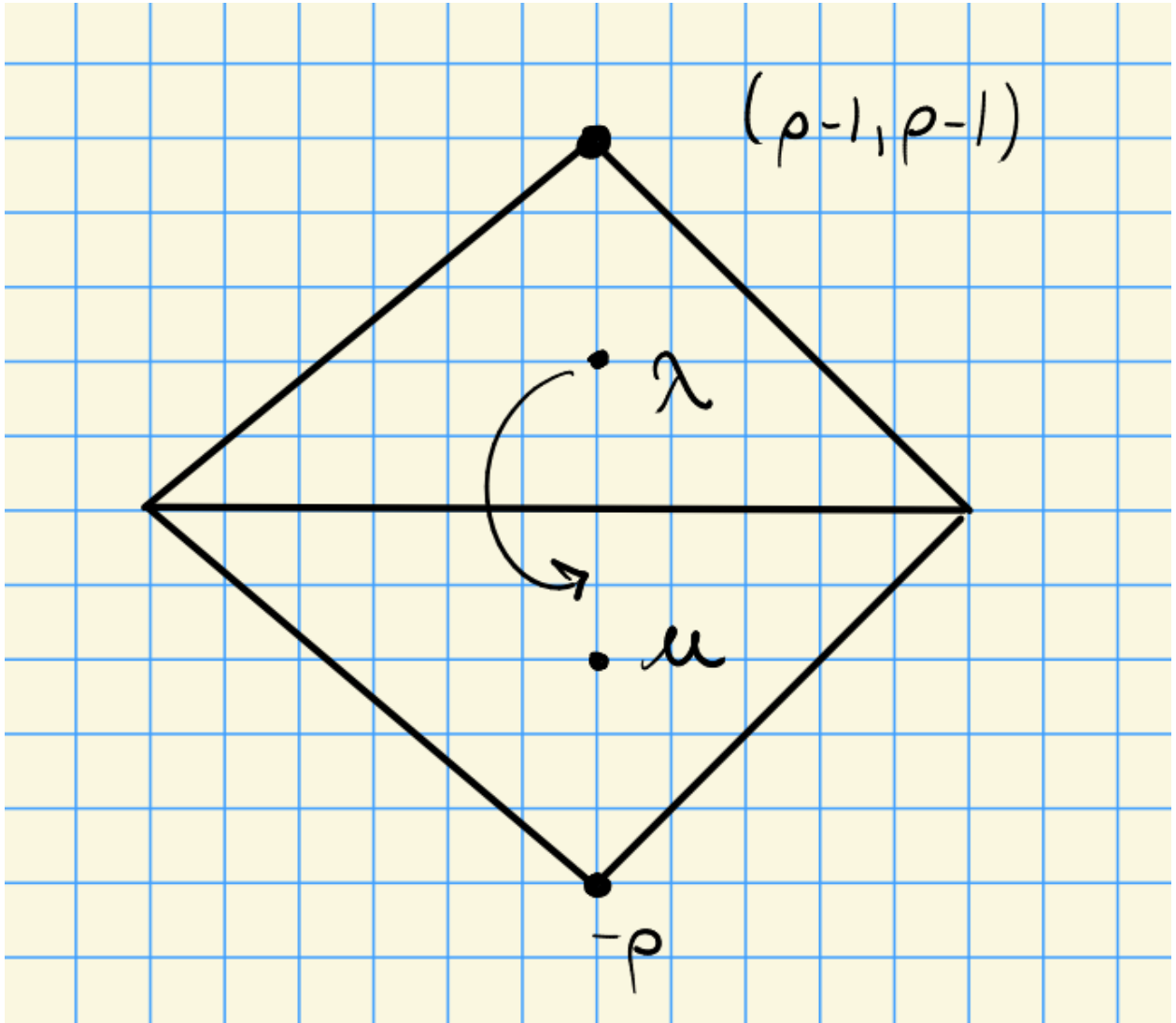


Figure 1: Image

Considering  $X_1(T)$  and  $[\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] \neq 0$ , and  $\hat{Z}_1(\lambda)$  has 6 composition factors as  $G_1T$ -modules.

On the other hand, for  $Z$ , we have the following:

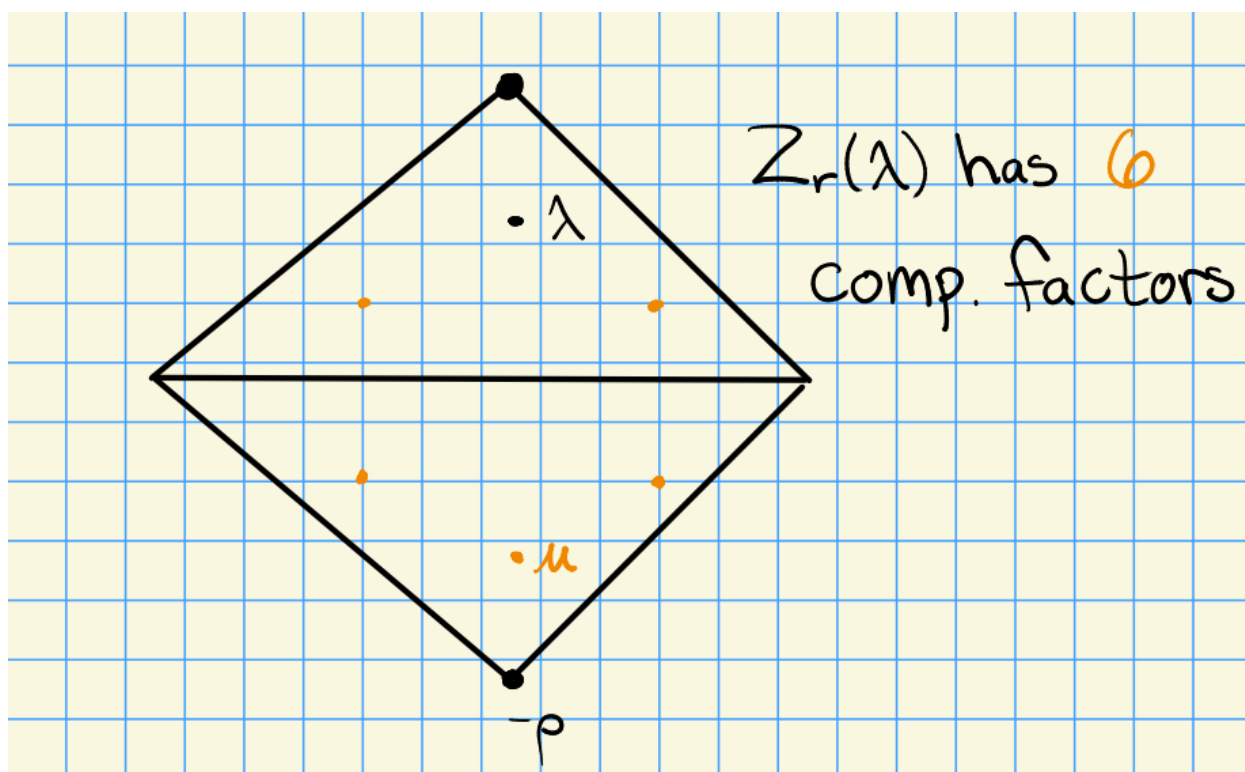


Figure 2: Image

This again has 6 composition factors, obtained by ??

What's the main difference?

## 1.2 Extensions

Let  $\lambda, \mu \in X(T)$ . We can use the Chevalley anti-automorphism (essentially the transpose) to obtain a form of duality for extensions:

$$\mathrm{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) = \mathrm{Ext}_{G_r}^j(\widehat{L}_r(\mu), \widehat{L}_r(\lambda)) \quad \text{for } j \geq 0.$$

We have a form of a weight space decomposition

$$\mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu)) = \bigoplus_{\gamma \in X(T)} \mathrm{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu))_{\gamma}$$

where we are taking the fixed points under the torus  $T$  action on the first factor (for which  $T_r$  acts

trivially). We can write this as

$$\begin{aligned}
 \cdots &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r}^j(L_r(\lambda), L_r(\mu) \otimes \gamma) \\
 &= \bigoplus_{\gamma \in X(T)} \text{Ext}_{G_r T}^j(L_r(\lambda), L_r(\mu) \otimes p^r v) \\
 &= \bigoplus_{v \in X(T)} \text{Ext}_{G_r T}^j(\widehat{L}_r(\lambda), \widehat{L}_r(\mu + p^r v)).
 \end{aligned}$$

So if we know extensions in the  $G_r$  category, we know them in the  $G_r T$  category.

There is an isomorphism

$$\text{Ext}_{G_r T}^1(\widehat{L}_r(\lambda), \widehat{L}_r(\mu)) \cong \text{Hom}_{G_r T}(\text{rad}_{G_r T} \widehat{Z}_r(\lambda), \widehat{L}_r(\mu)).$$

Finally, for  $\lambda, \mu \in X(T)$ , if the above  $\text{Ext}^1$  vanishes, then  $\lambda \in W_p \cdot \mu$  (i.e.  $\lambda$  and  $\mu$  are linked).

### 1.3 The Steinberg Modules