Homework 7

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1 Problem 1

1.1 Part 1

In order for IS to be a submodule of A, we need to show the following implication:

$$x \in IS,\ a \in A \implies xa, ax \in IS.$$

Suppose $x \in IS$. Then by definition, $x = \sum_{i=1}^n r_i a_i$ for some $r_i \in R, a_i \in A$.

But then

$$\begin{aligned} xa &= \left(\sum_{i=1}^n r_i a_i \right) a \\ &= \sum_{i=1}^n r_i a_i a \\ &\coloneqq \sum_{i=1}^n r_i a_i', \end{aligned}$$

where $a_i' \coloneqq a_i a$ for each i, which is still an element of A since A itself is a module and thus closed under multiplication.

But this expresses xa as an element of IS. Similarly, we have

$$
\begin{aligned}
ax &= a\left(\sum_{i=1}^{n} r_i a_i\right)\\
&= \sum_{i=1}^{n} a r_i a_i a\\
&:= \sum_{i=1}^{n} r_i a a_i,\\
&:= \sum_{i=1}^{n} r_i a_i',
\end{aligned}
$$

and so $ax \in IS$ as well.

1.2 Part 2

Letting $R/I \curvearrowright A/IA$ be the action given by $r + I \curvearrowright +IA := ra + IA$, we need to show the following:

- $r.(x + y) = r.x + r.y$,
- $(r + r').x = r.x + r'.x$,
- $(rs).x = r.(s.x)$, and
- $1.x = x$.

Letting $\oplus$ denote the addition defined on cosets, we have

$$
\begin{aligned}
r \curvearrowright (x + IA \oplus y + IA) &:= r \curvearrowright x + y + IA\\
&:= r(x + y) + IA\\
&= rx + ry + IA\\
&:= rx + IA \oplus ry + IA\\
&:= (r \curvearrowright x + IA) \oplus (r \curvearrowright y + IA).
\end{aligned}
$$

$$
\begin{aligned}
(r + s) \curvearrowright x + IA &:= (r + s)x + IA\\
&:= rx + sx + IA\\
&:= rx + IA \oplus sx + IA\\
&:= (rs \curvearrowright IA) \oplus (sx \curvearrowright IA).
\end{aligned}
$$