Homological Algebra Problem Sets

Problem Set 3

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Problem 1.0.1 (Prove Corollary 2.3.2)
For R a PID, show that an R-module A is divisible if and only if A is injective.

Recall that a module is divisible *if and only if for every $r \neq 0 \in R$ and every $a \in A$, we have $a = br$ for some $b \in A$.*

Solution:
Note: we'll assume R is commutative, and since R is a domain, it has no nonzero zero divisors and thus all elements $r \in R$ are left-cancelable.
$\implies$: Suppose A is divisible, we then want to show every R-module morphism of the following form lifts, where we regard the ideal J and the ring R as R-modules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Since R is a PID, we have $J = jR$ for some $j \in R$, so it suffices to produce lifts of the following form:

$$\begin{tikzcd}
0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f"] & R \arrow[dl, dashed, "\exists \tilde{f}"] \\
& A &
\end{tikzcd}$$

Link to Diagram

Consider $f(j) \in A$. Since A is divisible, we have $A = jA$, so we can write $f(j) = j\mathbf{a}'$ for some $\mathbf{a}' \in A$. Using R-linearity and the fact that j is left-cancelable, we have

$$jf(1_R) = f(j) = j\mathbf{a}' \implies f(1_R) = \mathbf{a}'.$$

Thus we can set

$$\begin{aligned}
\tilde{f} &: R \to A \\
1_R &\mapsto \mathbf{a}',
\end{aligned}$$

and extending R-linearly yields a well-defined R-module morphism. Moreover, the diagram commutes by construction, since $\iota(1_R) = 1_R$.

$\Longleftarrow$: Suppose $A \in R$-**Mod** is injective, where by Baer's criterion we equivalently have a lift of the following form for every $J \trianglelefteq R$:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, hook] \arrow[d] & R \arrow[dl, dashed] \\ & A & \end{tikzcd}$$

Link to Diagram

Let $j \in R$ be a nonzero element that is not a zero-divisor, we then want to show that $A = jA$, i.e. that for every $\mathbf{a} \in A$, there is a $\mathbf{a}' \in A$ such that $\mathbf{a} = j\mathbf{a}'$. Fixing $\mathbf{a} \in A$, define a map $f_a : J \to A$ in the following way: for $x \in J$, use the fact that $\langle j \rangle \coloneqq jR$ to first write $x = jr$ for some $r \in R$, and then set $f_a(x) = f_a(jr) \coloneqq r\mathbf{a}$. To summarize, we have

$$\begin{aligned} f_a : J = jR &\to R \\ x = jr &\mapsto r\mathbf{a}. \end{aligned}$$

By injectivity, we can take the inclusion $jR \hookrightarrow R$ and get a lift:

$$\begin{tikzcd} 0 \arrow[r] & jR \arrow[r, hook, "\iota"] \arrow[d, "f_a"] & R \arrow[dl, dashed, "\exists \tilde{f}_a"] \\ & A & \end{tikzcd}$$

Link to Diagram

We can now use the fact that

$$\begin{aligned} r\mathbf{a} &= f_a(jr) \\ &= \tilde{f}_a(\iota(jr)) \\ &= \tilde{f}_a(jr) \\ &= jr\tilde{f}_a(1_R) && \text{using } R\text{-linearity and } j, r \in R \\ &= rj\tilde{f}_a(1_R) && \text{since } R \text{ is commutative} \\ \implies \mathbf{a} &= j\tilde{f}_a(1_R) \in jA, \end{aligned}$$

where in the last step we have canceled an r on the left. So in the definition of divisibility, we can take

$$\mathbf{a}' \coloneqq \tilde{f}_a(1_R),$$

and letting $\mathbf{a}$ range over all elements of A yields the desired result.

Problem 1.0.2 (Calculating Ext Groups)
Calculate $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q)$ for distinct primes p, q.

The following are several claims that are later used in the actual solution:

Claim 1: For any $m \in \mathbb{Z}$,

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n.$$

Proof (?).
Note that there is an injection

$$1 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),$$

which follows from the fact that there is a SES

$$1 \to \mathbb{Z} \xrightarrow{x \mapsto nx} \mathbb{Z} \xrightarrow{\pi_n} \mathbb{Z}/n \to 1$$

where π_m is the canonical quotient morphism, and applying the left-exact contravariant functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ yields the first exact sequence above. We use this to identify the former as a submodule of the latter, and note that for any $\mathbb{Z}$-module morphism $\mathbb{Z} \xrightarrow{f} \mathbb{Q}/\mathbb{Z}$,

1. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module with generator 1, f is entirely determined by $f(1)$, and
2. f descends to a map $\tilde{f} : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$ if and only if $f(n) \in \mathbb{Z}$, i.e. $f(n) = [0]$ is in the equivalence class of zero in the quotient, and so

$$[1] = [0] = f(n) = nf(1).$$

Using this injection, we can identify the submodule $\operatorname{Hom}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$ as all of those morphism $\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ which descend to make the following diagram commute.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "f"] \arrow[d, "\pi_m"'] & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/n \arrow[ur, dashed, "\exists \tilde{f}"'] &
\end{tikzcd}$$

Link to Diagram

To characterize these, it suffices to determine all of the possible images $f(1)$. Moreover, we can restrict our attention to coset representatives in the interval $[0,1) \cap \mathbb{Q} \subseteq \mathbb{R}$, where we want to find all $q := f(1) \in [0,1)$ such that $nq = 1$. A complete list of n such representatives is given by

$$q \in \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\right\}.$$

Setting $f_i(1) := \left[\frac{i}{n}\right]$ (where we take the equivalence class mod $\mathbb{Z}$) yields n distinct morphisms $f_i : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ that descend to $\tilde{f}_i : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$. We can define a map

$$\begin{aligned}\Psi : \mathbb{Z} &\to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}) \\ i &\mapsto f_i,\end{aligned}$$

and using the fact that if $i = i' \pmod{n}$, write $i' = i + kn$ for some $k \in \mathbb{Z}$, then

$$f_{i'}(1) = f_{i+kn}(1) = \left[\frac{i+kn}{n}\right] = \left[\frac{i}{n} + k\right] = \left[\frac{i}{n}\right] = f_i(1),$$

since $k \in \mathbb{Z}$, so by the first isomorphism theorem Ψ descends to an isomorphism

$$\tilde{\Psi} : \mathbb{Z}/n \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z}).$$

■

Claim 2: $\mathbb{Q}/\mathbb{Z}$ is an injective object in $\mathbb{Z}$-modules.

Proof (?).
By the previous exercise, it suffices to show that $\mathbb{Q}/\mathbb{Z}$ is divisible. More generally, if any group G is divisible and $N \trianglelefteq G$ is a normal subgroup, then G/N will be divisible. This follows from the fact that if $\bar{a}, \bar{b} \in G/N$ and $n \in \mathbb{Z}$, we can write $\bar{a} = a + N$ and $\bar{b} = b + N$ for some coset representatives, use divisibility to write $a = nb$, and then compute

$$\bar{a} = a + N = (nb) + N := n(b + N) = n\bar{b}.$$

That $\mathbb{Q}$ is divisible is a straightforward check: let $n \in \mathbb{Z}$ and $a \in \mathbb{Q}$, we then want a $b \in \mathbb{Q}$ such that $a = nb$, and $b := \frac{a}{n} \in \mathbb{Q}$ works. Since $\mathbb{Q}$ is an abelian group, $\mathbb{Z}$ is automatically normal, and the result follows.

■

Claim:

$$\frac{\mathbb{Z}/n}{m(\mathbb{Z}/n)} \cong \mathbb{Z}/d \qquad\qquad d := \gcd(\mathbb{Z}/m, \mathbb{Z}/n).$$

Proof (?).
Using

$$M \otimes_R \frac{A}{I} \cong \frac{M}{IM} \in R\text{-}\mathbf{Mod},$$

and taking

- $M := \mathbb{Z}/m$,
- $A := \mathbb{Z}$,
- $I := n\mathbb{Z}$,

we have

$$\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \frac{\mathbb{Z}/m}{n\,(\mathbb{Z}/m)} \qquad\qquad \in \mathbb{Z}\text{-}\mathbf{Mod}.$$

We can now use the map

$$\begin{aligned} \varphi : \mathbb{Z} &\to \mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n \\ x &\mapsto x(1 \otimes 1) \end{aligned}$$

and compute

$$\begin{aligned}
\ker\varphi &= \left\{x \in \mathbb{Z} \mathrel{\Big|} x(1\otimes 1) = 0\right\} \\
&= \left\{x \in \mathbb{Z} \mathrel{\Big|} n \mathrel{\Big|} x \text{ or } m \mathrel{\Big|} x\right\} \\
&= \langle n, m\rangle \\
&= \langle \gcd(n,m)\rangle && \text{by Bezout's theorem} \\
&:= \langle d\rangle\,.
\end{aligned}$$

Now apply the first isomorphism theorem yields the result.

■

Solution:
We'll follow the procedure outlined in Weibel:

- Define the contravariant functor $F(\,\cdot\,) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p,\,\cdot\,)$, then noting that it is left-exact, it has right-derived functors.
- Find an injective resolution I of $\mathbb{Z}/q$.
- Write $F(I)$ as a new (not necessarily exact) chain complex.
- Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) := R^iF(\mathbb{Z}/q) := H^i(F(\mathbb{Z}/q))$.

We can first take the following injective resolution:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/q \arrow[r, "d^{-1}"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^0"] & \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
& {[1]_q} \arrow[r] & {\left[\frac{1}{q}\right]} & & \\
& & {[x]} \arrow[r] & {[qx]} &
\end{tikzcd}$$

Link to Diagram

This is a chain complex by construction, since $d^2([1]_q) = \left[q\left(\frac{1}{q}\right)\right] = [1] = [0]$. We now delete the augmentation and apply $F(\,\cdot\,)$:

$$
\begin{tikzcd}
1 \arrow[r] & I^0 := \mathbb{Q}/\mathbb{Z} \arrow[rr, "d^0"] & & I^1 := \mathbb{Q}/\mathbb{Z} \arrow[r, "d^1"] & 1 \\
& & {} \arrow[d, Rightarrow, "F(\,\cdot\,)"] & & \\
1 \arrow[r] \arrow[d, equal] & F(I^0) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[rr, "\partial^0 := F(d^0)"] & & F(I^1) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Q}/\mathbb{Z}) \arrow[r, "\partial^1 := F(d^1)"] & 1 \arrow[d, equal] \\
1 \arrow[r] & \mathbb{Z}/p \arrow[u, "\Psi \cong"] \arrow[rr, "\tilde{\partial}^0"] & & \mathbb{Z}/p \arrow[u, "\Psi \cong"] \arrow[r, "\tilde{\partial}^1"] & 1
\end{tikzcd}
$$

Link to Diagram

Here we immediately simplify by applying the isomorphism from the earlier claim. Noting that $d^0(x) := qx$ was multiplication by q, we have $\partial^0(f) = d^0 \circ f$ is post-composition by the multiplication by q map, and $\tilde{\partial}^0$ similarly becomes multiplication by q.
We now take homology:

$$
\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}/q) := R^0 F(\mathbb{Z}/p) := \frac{\ker \partial^1}{\operatorname{im} \partial^0} = \frac{\mathbb{Z}/p}{q\,(\mathbb{Z}/p)} \cong \mathbb{Z}/d\mathbb{Z} \cong 1,
$$

where $d := \gcd(p, q) = 1$ if p, q are coprime.

Problem 1.0.3 (Weibel 2.3.2)
For $A \in \mathbf{Ab}$, define $I(A) := \bigoplus_{f \in \mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$, and let $e_A : A \to I(A)$. Show that e_A is injective.
Hint: if $a \in A$, find a map $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend this to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$.

Problem 1.0.4 (Weibel 2.4.2)
If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$
U(L_i F) \cong L_i(UF).
$$

Problem 1.0.5 (Weibel 2.4.3)
If $0 \to M \to P \to A \to 0$ is exact with P projective or F-acyclic, show that

$$
L_i F(A) \cong L_{i-1} F M \qquad\qquad i \geq 2.
$$

Show that $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_i F(A) = H_i(F(P))$.

Problem 1.0.6 (Weibel 2.5.2)
Show that the following are equivalent:

a. A is a projective R-module.

b. $\operatorname{Hom}_R(\,\cdot\,, A)$ is an exact functor.

c. $\operatorname{Ext}_R^{i\neq 0}(A, B) = 0$ and for all B, i.e. A is $\operatorname{Hom}_R(\,\cdot\,, B)$-acyclic for all B.

d. $\operatorname{Ext}_R^1(A, B)$ vanishes for all B.

Problem 1.0.7 (Weibel 2.6.4)
Show that colim is left adjoint to Δ, and conclude that colim is right-exact when when $\mathcal{A}$ is abelian and colim exists. Show that the pushout, i.e. $\bullet \leftarrow \bullet \rightarrow \bullet$, is not an exact functor on **Ab**.