CRAG

The Weil Conjectures

D. Zack Garza

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A Quick Note

- A big thanks to Daniel Litt for organizing this reading seminar, recommending papers, helping with questions!!
- Goals for this talk:
  - Understand the Weil conjectures,
  - Understand *why* the relevant objects should be interesting,
  - See elementary but concrete examples,
  - Count all of the things!

Generating Functions

Varieties

Fix q a prime and $\mathbb{F}_q$ the (unique) finite field with q elements, along with its (unique) degree n extensions

$$\mathbb{F}_{q^n} = \left\{ x \in \bar{\mathbb{F}}_q \;\middle|\; x^{q^n} - x = 0 \right\} \quad \forall\ n \in \mathbb{Z}^{\geq 1}$$

Definition (Projective Algebraic Varieties)

Let $J = \langle f_1, \cdots, f_M \rangle \trianglelefteq k[x_0, \cdots, x_n]$ be an ideal, then a *projective algebraic variety* $X \subset \mathbb{P}^n_{\mathbb{F}}$ can be described as

$$X = V(J) = \left\{ \mathbf{x} \in \mathbb{P}^n_{\mathbb{F}_q} \;\middle|\; f_1(\mathbf{x}) = \cdots = f_M(\mathbf{x}) = \mathbf{0} \right\}$$

where J is generated by *homogeneous* polynomials in $n+1$ variables, i.e. there is a fixed $d = \deg f_i \in \mathbb{Z}^{\geq 1}$ such that

$$f(\mathbf{x}) = \sum_{\substack{\mathrm{I} = (i_1, \cdots, i_n) \\ \sum_j i_j = d}} \alpha_{\mathrm{I}} \cdot x_0^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad f(\lambda \cdot \mathbf{x}) = \lambda^d f(\mathbf{x}), \lambda \in \mathbb{F}^\times.$$

Point Counts

- For a fixed variety X, we can consider its $\mathbb{F}_q$-points $X(\mathbb{F}_q)$.
  - Note that $\#X(\mathbb{F}_q) < \infty$ is an integer
- For any $L/\mathbb{F}_q$, we can also consider $X(L)$
  - For $[L : \mathbb{F}_q]$ finite, $\#X(L) < \infty$ and are integers for every such n.
  - In particular, we can consider the finite counts $\#X(\mathbb{F}_{q^n})$ for any $n \geq 2$.
- So we can consider the sequence

$$[N_1, N_2, \cdots, N_n, \cdots] := [\#X(\mathbb{F}_q),\ \#X(\mathbb{F}_{q^2}), \cdots,\ \#X(\mathbb{F}_{q^n}), \cdots].$$

- Idea: associate some generating function (a formal power series) encoding the sequence, e.g.

$$F(z) = \sum_{n=1}^{\infty} N_n z^n = N_1 z + N_2 z^2 + \cdots.$$

Why Generating Functions?

For *ordinary* generating functions, the coefficients are related to the real-analytic properties of F:

$$[z^n] \cdot F(z) = [z^n] \cdot T_{F,z=0}(z) = \frac{1}{n!} \left(\frac{\partial}{\partial z} \right)^n F(z) \bigg|_{z=0} = N_n$$

and also to the complex analytic properties:

$$[z^n] \cdot F(z) := \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{F(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_{\mathbb{S}^1} \frac{N_n}{z} \, dz = N_n.$$

Using the Residue theorem. The latter form is very amenable to computer calculation.

Why Generating Functions?

- An OGF is an infinite series, which we can interpret as an analytic function $\mathbb{C} \longrightarrow \mathbb{C}$
- In nice situations, we can hope for a closed-form representation.
- A useful example: by integrating a geometric series we can derive

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \qquad\qquad (= 1 + z + z^2 + \cdots) \\
\implies \int \frac{1}{1-z} &= \int \sum_{n=0}^{\infty} z^n \\
&= \sum_{n=0}^{\infty} \int z^n \quad for |z| < 1 \quad \text{by uniform convergence} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \\
\implies -\log(1-z) &= \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad\qquad \left(= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).
\end{aligned}$$

Exponential

- For completeness, also recall that

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

- We can regard $\exp, \log$ as elements in the ring of formal power series $\mathbb{Q}[[z]]$.
- In particular, for any $p(z) \in z \cdot \mathbb{Q}[[z]]$ we can consider $\exp(p(z)), \log(1 + p(z))$
- Since $\mathbb{Q} \hookrightarrow \mathbb{C}$, we can consider these as a complex-analytic functions, ask where they converge, etc.

Zeta Functions

Definition: Local Zeta Function

Problem: count points of a (smooth?) projective variety $X/\mathbb{F}$ in all (finite) degree n extensions of $\mathbb{F}$.

Definition (Local Zeta Function)

The *local zeta function* of an algebraic variety X is the following formal power series:

$$Z_X(z) = \exp\left(\sum_{n=1}^{\infty} N_n \frac{z^n}{n} \right) \in \mathbb{Q}[[z]] \quad \text{where} \quad N_n := \#X(\mathbb{F}_n).$$

Note that

$$\begin{aligned} z\left(\frac{\partial}{\partial z}\right) \log Z_X(z) &= z \frac{\partial}{\partial z}\left(N_1 z + N_2 \frac{z^2}{2} + N_3 \frac{z^3}{3} + \cdots \right) \\ &= z\left(N_1 + N_2 z + N_3 z^2 + \cdots\right) \qquad \text{(unif. conv.)} \\ &= N_1 z + N_2 z^2 + \cdots = \sum_{n=1}^{\infty} N_n z^n, \end{aligned}$$

which is an *ordinary* generating function for the sequence (N_n).

Examples

Example: A Point

Take $X = \{\mathrm{pt}\} = V(\{f(x) = x - c\})/\mathbb{F}$ for c a fixed element of $\mathbb{F}$. This yields a single point over $\mathbb{F}$, then

$$\begin{aligned}
\#X(\mathbb{F}_q) &:= N_1 = 1 \\
\#X(\mathbb{F}_{q^2}) &:= N_2 = 1 \\
&\vdots \\
\#X(\mathbb{F}_{q^n}) &:= N_n = 1
\end{aligned}$$

and so

$$\begin{aligned}
Z_{\{\mathrm{pt}\}}(z) &= \exp\left(1 \cdot z + 1 \cdot \frac{z^2}{2} + 1 \cdot \frac{z^3}{3} + \cdots\right) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) \\
&= \exp\left(-\log\left(1 - z\right)\right) \\
&= \frac{1}{1 - z}.
\end{aligned}$$

Notice: Z admits a closed form **and** *is a rational function.*

Example: The Affine Line

Take $X = \mathbb{A}^1/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^1(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1] \;\middle|\; x_1 \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned} X(\mathbb{F}_q) &= q \\ X(\mathbb{F}_{q^2}) &= q^2 \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^n. \end{aligned}$$

Then

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} \right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{(qz)^n}{n} \right) \\ &= \exp(-\log(1-qz)) \\ &= \frac{1}{1-qz}. \end{aligned}$$

Example: Affine m-space

Take $X = \mathbb{A}^m/\mathbb{F}$ the affine line over $\mathbb{F}$, then We can write

$$\mathbb{A}^m(\mathbb{F}_{q^n}) = \left\{ \mathbf{x} = [x_1, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\}$$

as the set of one-component vectors with entries in $\mathbb{F}_n$, so

$$\begin{aligned} X(\mathbb{F}_q) &= q^m \\ X(\mathbb{F}_{q^2}) &= (q^2)^m \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^{nm}. \end{aligned}$$

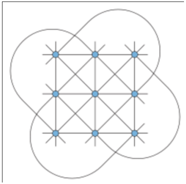


Figure: $\mathbb{A}^2/\mathbb{F}_3$ $(q = 3, m = 2, n = 1)$

Then

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} q^{nm} \frac{z^n}{n} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{(q^m z)^n}{n} \right) \\ &= \exp(-\log(1 - q^m z)) \\ &= \frac{1}{1 - q^m z}. \end{aligned}$$

Example: Projective Line

Take $X = \mathbb{P}^1/\mathbb{F}$, we can still count by enumerating coordinates:

$$\mathbb{P}^1(\mathbb{F}_{q^n}) = \left\{ [x_1 : x_2] \;\middle|\; x_1, x_2 \neq 0 \in \mathbb{F}_{q^n} \right\} / \sim \quad = \left\{ [x_1 : 1] \;\middle|\; x_1 \in \mathbb{F}_{q^n} \right\} \coprod \{[1:0]\}.$$

Thus

$$\begin{aligned} X(\mathbb{F}_q) &= q + 1 \\ X(\mathbb{F}_{q^2}) &= q^2 + 1 \\ &\vdots \\ X(\mathbb{F}_{q^n}) &= q^n + 1. \end{aligned}$$

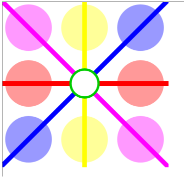


Figure: $\mathbb{P}^1/\mathbb{F}_3$ $(q = 3, n = 1)$

Thus

$$\begin{aligned} Z_X(z) &= \exp\left(\sum_{n=1}^{\infty} (q^n + 1) \frac{z^n}{n} \right) \\ &= \exp\left(\sum_{n=1}^{\infty} q^n \frac{z^n}{n} + \sum_{n=1}^{\infty} 1 \cdot \frac{z^n}{n} \right) \\ &= \frac{1}{(1 - qz)(1 - z)}. \end{aligned}$$

The Weil Conjectures

Weil 1

(Weil 1949)

Let X be a smooth projective variety of dimension N over $\mathbb{F}_q$ for q a prime, let $Z_X(z)$ be its zeta function, and define $\zeta_X(s) = Z_X(q^{-s})$.

1. (Rationality)
   $Z_X(z)$ is a rational function:

$$Z_X(z) = \frac{p_1(z) \cdot p_3(z) \cdots p_{2N-1}(z)}{p_0(z) \cdot p_2(z) \cdots p_{2N}(z)} \in \mathbb{Q}(z), \quad \text{i.e.} \quad p_i(z) \in \mathbb{Z}[z]$$

$$P_0(z) = 1 - z$$

$$P_{2N}(z) = 1 - q^N z$$

$$P_j(z) = \prod_{j=1}^{\beta_i} (1 - a_{j,k} z) \quad \text{for some reciprocal roots} \quad a_{j,k} \in \mathbb{C}$$

where we've factored each P_i using its reciprocal roots a_{ij}.
In particular, this implies the existence of a meromorphic continuation of the associated function $\zeta_X(s)$, which a priori only converges for $\Re(s) \gg 0$. This also implies that for n large enough, N_n satisfies a linear recurrence relation.

Weil 2

2. (Functional Equation and Poincare Duality)
   Let $\chi(X)$ be the Euler characteristic of X, i.e. the self-intersection number of the diagonal embedding $\Delta \hookrightarrow X \times X$; then $Z_X(z)$ satisfies the following *functional equation*:

$$Z_X\left(\frac{1}{q^N z}\right) = \pm\left(q^{\frac{N}{2}} z\right)^{\chi(X)} \ Z_X(z).$$

Equivalently,

$$\zeta_X(N-s) = \pm\left(q^{\frac{N}{2}-s}\right)^{\chi(X)} \zeta_X(s).$$

Note that when $N = 1$, e.g. for a curve, this relates $\zeta_X(s)$ to $\zeta_X(1-s)$.

Equivalently, there is an involutive map on the (reciprocal) roots

$$\begin{aligned} z &\iff \frac{q^N}{z} \\ \alpha_{j,k} &\iff \alpha_{2N-j,k} \end{aligned}$$

which sends interchanges the coefficients in p_j and p_{2N-j}.

Weil 3

3. (Riemann Hypothesis)
   The reciprocal roots $\alpha_{j,k}$ are *algebraic* integers (roots of some monic $p \in \mathbb{Z}[x]$) which satisfy

$$|\alpha_{j,k}|_{\mathbb{C}} = q^{\frac{j}{2}}, \qquad 1 \leq j \leq 2N-1, \ \forall k.$$

4. (Betti Numbers)
   If X is a "good reduction mod q" of a nonsingular projective variety $\tilde{X}$ in characteristic zero, then the $\beta_i = \mathbf{deg}\, p_i(z)$ are the Betti numbers of the topological space $\tilde{X}(\mathbb{C})$.

Moral:

- The Diophantine properties of a variety's zeta function are governed by its (algebraic) topology.
- Conversely, the analytic properties of encode a lot of geometric/topological/algebraic information.

Why is (3) called the "Riemann Hypothesis"?

Recall the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

After modifying ζ to make it symmetric about $\Re(s) = \frac{1}{2}$ and eliminate the trivial zeros to obtain $\widehat{\zeta}(s)$, there are three relevant properties

1. "Rationality": $\widehat{\zeta}(s)$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$.
2. "Functional equation": $\widehat{\zeta}(1-s) = \widehat{\zeta}(s)$
3. "Riemann Hypothesis": The only zeros of $\widehat{\zeta}$ have $\Re(s) = \frac{1}{2}$.

Why is (3) called the "Riemann Hypothesis"?

Suppose it holds for some X. Use the facts:

a. $|\exp(z)| = \exp(\Re(z))$ and
b. $a^z := \exp(z \operatorname{Log}(a))$,

and to replace the polynomials P_j with

$$L_j(s) := P_j(q^{-s}) = \prod_{k=1}^{\beta_j} \left(1 - \alpha_{j,k} q^{-s}\right).$$

Now consider the roots of $L_j(s)$: we have

$$L_j(s_0) = 0 \iff (1 - \alpha_{j,k} q^{-s}) = 0 \quad \text{for some} \quad k$$

$$\iff q^{-s_0} = \frac{1}{\alpha_{j,k}}$$

$$\iff \left| q^{-s_0} \right| = \left| \frac{1}{\alpha_{j,k}} \right| \qquad \overset{\text{by assumption}}{=} \quad q^{-\frac{j}{2}}$$

$$\iff q^{-\frac{j}{2}} \overset{(a)}{=} \exp\left(-\frac{j}{2} \cdot \mathrm{Log}(q)\right) = \left|\exp\left(-s_0 \cdot \mathrm{Log}(q)\right)\right|$$

$$\overset{(b)}{=} \left|\exp\left(-(\Re(s_0) + i \cdot \Im(s_0)) \cdot \mathrm{Log}(q)\right)\right|$$

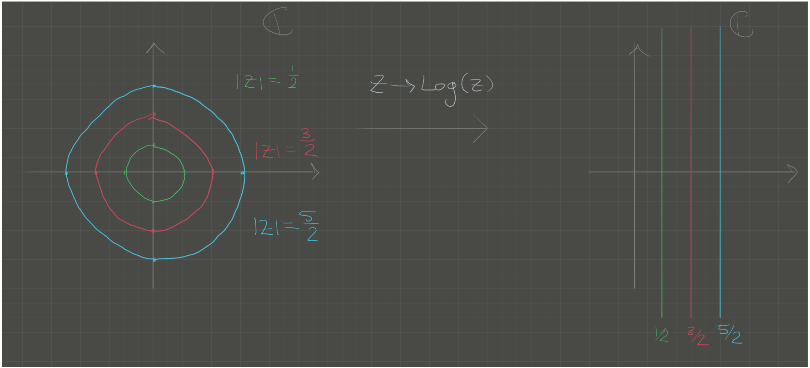
$$\overset{(a)}{=} \exp\left(-(\Re(s_0)) \cdot \mathrm{Log}(q)\right)$$

$$\iff -\frac{j}{2} \cdot \mathrm{Log}(q) = -\Re(s_0) \cdot \mathrm{Log}(q) \quad \text{by injectivity}$$

$$\iff \Re(s_0) = \frac{j}{2}.$$

Analogy with Riemann Hypothesis

Roughly speaking, we can apply Log (a conformal map) to send the $\alpha_{j,k}$ to zeros of the L_j, this says that the zeros all must lie on the "critical lines" $\frac{j}{2}$.



In particular, the zeros of L_1 have real part $\frac{1}{2}$ (analogy: classical Riemann hypothesis).

Precise Relation

- Difficult to find in the literature!
- Idea: make a similar definition for schemes, then take $X = \mathrm{Spec}\, \mathbb{Z}$.
- Define the "reductions mod p" X_p for closed points p.
- Define the *local* zeta functions $\zeta_{X_p}(s) = Z_{X_p}(p^{-s})$.
- (Potentially incorrect) Evaluate to find $Z_{X_p}(z) = \frac{1}{1-z}$.
- Take a product over all closed points to define

$$\begin{aligned} L_X(s) &= \prod_{p \text{ prime}} \zeta_{X_p}(p^{-s}) \\ &= \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}} \right) \\ &= \zeta(s), \end{aligned}$$

which is the Euler product expansion of the classical Riemann Zeta function.

If anyone knows a reference for this, let me know!

Weil for Curves

Weil for Curves

The Weil conjectures take on a particularly nice form for curves. Let $X/\mathbb{F}_q$ be a smooth projective curve of genus g, then

1. (Rationality)
$$Z_X(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)}$$
2. (Functional Equation)
$$Z_X\left(\frac{1}{qz}\right) = (z\sqrt{q})^{2-2g} Z_X(z)$$
3. (Riemann Hypothesis)
$$p_1(z) = \prod_{i=1}^{\beta_1} (1 - a_i z) \quad \text{where} \quad |a_i| = \sqrt{q}$$
4. (Betti Numbers) $\mathcal{P}_{\Sigma_g}(x) = 1 + 2g \cdot x + x^2 \implies \mathbf{deg}\, p_1 = \beta_1 = 2g$.

$\mathcal{P}$ here is the Poincaré polynomial, the generating function for the Betti numbers. Σ_g is the surface (real 2-dimensional smooth manifold) of genus g.

The Projective Line

Recall $Z_{\mathbb{P}^1/\mathbb{F}_q}(z) = \frac{1}{(1-z)(1-qz)}$.

1. Rationality: Clear!
2. Functional Equation: $g = 0 \implies 2g - 2 = 2$

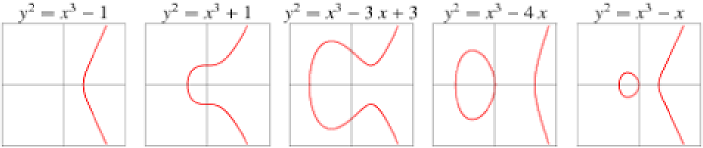
$$Z_{\mathbb{P}^1}\left(\frac{1}{qz}\right) = \frac{1}{(1-\frac{1}{qz})(1-\frac{q}{qz})} = \frac{qz^2}{(1-z)(1-qz)} = \frac{(\sqrt{q}z)^2}{(1-z)(1-qz)}.$$

3. Riemann Hypothesis: Nothing to check (no $p_1(z)$)
4. Betti numbers: Use the fact that $\mathcal{P}_{\mathbb{CP}^1} = 1 + 0 \cdot x + x^2$, and indeed $\deg p_0 = \deg p_2 = 1, \;\; \deg p_1 = 0$.

   Note that even Betti numbers show up as degrees in the denominator, odd in the numerator. Allows us to immediately guess the zeta function for $\mathbb{P}^n/\mathbb{F}_q$ by knowing $H^*\mathbb{CP}^\infty$!

Elliptic Curves

Figure: Some Elliptic Curves



Consider $E/\mathbb{F}_q$.

- (Nontrivial!) The number of points is given by

$$N_n := E(\mathbb{F}_{q^n}) = (q^n + 1) - (\alpha^n + \overline{\alpha}^n) \quad \text{where} \quad |\alpha| = |\overline{\alpha}| = \sqrt{q}$$

  - Proof: Involves trace (or eigenvalues?) of Frobenius, (could use references)
- $\dim_{\mathbb{C}} E/\mathbb{C} = N = 1$ and $g = 1$.

The Weil Conjectures say we should be able to write

$$Z_E(z) = \frac{p_1(z)}{p_0(z)p_2(z)} = \frac{p_1(z)}{(1-z)(1-qz)} = \frac{(1-a_1 z)(1-a_2 z)}{(1-z)(1-qz)}.$$

1. Rationality: using the point count, we can compute

$$
\begin{aligned}
Z_E(z) &= \exp \sum_{n=1}^\infty \#E(\mathbb{F}_{q^n}) \frac{z^n}{n} \\
&= \exp \sum_{n=1}^\infty (q^n + 1 - (\alpha^n + \bar{\alpha}^n)) \frac{z^n}{n} \\
&= \exp \left(\sum_{n=1}^\infty q^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^\infty 1 \cdot \frac{z^n}{n} \right) \\
&\qquad \exp \left(\sum_{n=1}^\infty -\alpha^n \cdot \frac{z^n}{n} \right) \exp \left(\sum_{n=1}^\infty -\bar{\alpha}^n \cdot \frac{z^n}{n} \right) \\
&= \exp\left(-\log\left(1 - qz\right)\right) \cdot \exp\left(-\log\left(1 - z\right)\right) \\
&\qquad \exp\left(\log\left(1 - \alpha z\right)\right) \cdot \exp\left(\log\left(1 - \bar{\alpha} z\right)\right) \\
&= \frac{(1-\alpha z)(1-\bar{\alpha} z)}{(1-z)(1-qz)} \in \mathbb{Q}(z),
\end{aligned}
$$

which is a rational function of the expected form (Weil 1).

Note that the "expected" point counts show up in the denominator, along with the even Betti numbers, while the "correction" factor appears in the denominator and odd Betti numbers.

Elliptic Curves: Weil 2 and 3

2. Functional Equation: we use the equivalent formulation of "Poincaré duality":

$$\frac{(1-\alpha z)(1-\bar{\alpha} z)}{(1-z)(1-qz)} = \frac{p_1(z)}{p_0(z)p_2(z)} \implies \begin{cases} z & \iff \frac{q}{z} \\ \alpha_{j,k} & \iff \alpha_{2-j,k} \end{cases}$$

This amounts to checking that the coefficients of p_0, p_2 are interchanged, and that the two coefficients of p_1 are interchanged:

$$\mathrm{Coefs}(p_0) = \{1\} \xrightarrow{z \longrightarrow \frac{q}{z}} \left\{ \frac{1}{q} \right\} = \mathrm{Coefs}(p_2)$$

$$\mathrm{Coefs}(p_1) = \{\alpha, \bar{\alpha}\} \xrightarrow{z \longrightarrow \frac{q}{z}} \left\{ \frac{q}{\alpha}, \frac{q}{\bar{\alpha}} \right\} = \{\bar{\alpha}, \alpha\} \quad \text{using} \quad \alpha\bar{\alpha} = q.$$

3. RH: Assumed as part of the point count ($|\alpha| = q^{\frac{1}{2}}$)
4. Betti Numbers: $\mathcal{P}_{\Sigma_1}(x) = 1 + 2x + x^2$, and indeed $\deg p_0 = \deg p_2 = 1, \ \deg p_1 = 2$.

History

- 1801, Gauss: Point count and RH showed for specific elliptic curves
- 1924, Artin: Conjectured for algebraic curves ,
- 1934, Hasse: proved for elliptic curves.
- 1949, Weil: Proved for smooth projective curves over finite fields, proposed generalization to projective varieties
- 1960, Dwork: Rationality via p-adic analysis
- 1965, Grothendieck et al.: Rationality, functional equation, Betti numbers using étale cohomology
  - Trace of Frobenius on ℓ-adic cohomology
  - Expected proof via *the standard conjectures*. Wide open!
- 1974, Deligne: Riemann Hypothesis using étale cohomology, circumvented the standard conjectures
- Recent: Hasse-Weil conjecture for arbitrary algebraic varieties over number fields
  - Similar requirements on L-functions: functional equation, meromorphic continuation
  - 2001: Full modularity theorem proved, extending Wiles, implies Hasse-Weil for elliptic curves
  - Inroad to Langlands: show every L function coming from an algebraic variety also comes from an automorphic representation.

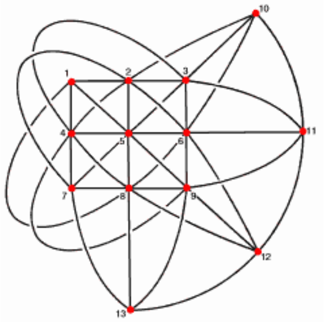
Weil for Projective m-space

Take $X = \mathbb{P}^m/\mathbb{F}$ We can write

$$\mathbb{P}^m(\mathbb{F}_{q^n}) = \mathbb{A}^{m+1}(\mathbb{F}_{q^n}) \setminus \{\mathbf{0}\} / \sim \quad = \left\{ \mathbf{x} = [x_0, \cdots, x_m] \;\middle|\; x_i \in \mathbb{F}_{q^n} \right\} / \sim$$

But how many points are actually in this space?

Figure: Points and Lines in $\mathbb{P}^2/\mathbb{F}_3$



A nontrivial combinatorial problem!

To illustrate, this can be done combinatorially: identify $\mathbb{P}_{\mathbb{F}}^m = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}_{\mathbb{F}}^{m+1}$.

Theorem

The number of k-dimensional subspaces of $\mathbb{A}_{\mathbb{F}_q}^N$ is the q-analog of the binomial coefficient:

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q^N - 1)(q^{N-1} - 1)\cdots(q^{N-(k-1)} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}.$$

Remark: Note $\lim_{q \longrightarrow 1} \begin{bmatrix} N \\ k \end{bmatrix}_q = \binom{N}{k}$, *the usual binomial coefficient.*

Proof: To choose a k-dimensional subspace,

- Choose a nonzero vector $\mathbf{v}_1 \in \mathbb{A}_{\mathbb{F}}^n$ in $q^N - 1$ ways.
  - For next step, note that $\#\mathrm{span}\left\{v_1\right\} = \#\left\{\lambda v_1 \mid \lambda \in \mathbb{F}_q\right\} = \#\mathbb{F}_q = q$.
- Choose a nonzero vector $\mathbf{v}_2$ *not* in the span of $\mathbf{v}_1$ in $q^N - q$ ways.
  - Now note $\#\mathrm{span}\left\{v_1, v_2\right\} = \#\left\{\lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_i \in \mathbb{F}\right\} = q \cdot q = q^2$.

Proof continued

- Choose a nonzero vector $\mathbf{v}_3$ *not* in the span of $\mathbf{v}_1, \mathbf{v}_2$ in $q^N - q^2$ ways.
- $\cdots$ until $\mathbf{v}_k$ is chosen in

$$(q^N - 1)(q^N - q)\cdots(q^N - q^{k-1}) \quad \text{ways} \quad .$$

  - This yields a k-tuple of linearly independent vectors spanning a k-dimensional subspace V_k
- This overcounts because many linearly independent sets span V_k, we need to divide out by the number of ways to choose a basis inside of V_k.
- By the same argument, this is given by

$$(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$$

Thus

$$\begin{aligned}
\#\text{subspaces} &= \frac{(q^N-1)(q^N-q)(q^N-q^2)\cdots(q^N-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})} \\
&= \frac{q^N-1}{q^k-1}\cdot\left(\frac{q}{q}\right)\frac{q^{N-1}-1}{q^{k-1}-1}\cdot\left(\frac{q^2}{q^2}\right)\frac{q^{N-2}-1}{q^{k-2}-1}\cdots\left(\frac{q^{k-1}}{q^{k-1}}\right)\frac{q^{N-(k-1)}-1}{q^{k-(k-1)-1}} \\
&= \frac{(q^N-1)(q^{N-1}-1)\cdots(q^{N-(k-1)}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.
\end{aligned}$$

$\blacksquare$

Note that we've actually computed the number of points in any Grassmannian.

Identify $\mathbb{P}^m_{\mathbb{F}} = \mathrm{Gr}_{\mathbb{F}}(1, m+1)$ as the space of lines in $\mathbb{A}^{m+1}_{\mathbb{F}}$.

We obtain a simplification (importantly, a *sum formula*) when setting $k = 1$:

$$\begin{bmatrix} m+1 \\ 1 \end{bmatrix}_q = \frac{q^{m+1}-1}{q-1} = q^m + q^{m-1} + \cdots + q + 1 = \sum_{j=0}^{m} q^j.$$

Thus

$$\begin{aligned}
X(\mathbb{F}_q) &= \sum_{j=0}^{m} q^j \\
X(\mathbb{F}_{q^2}) &= \sum_{j=0}^{m} (q^2)^j \\
&\vdots \\
X(\mathbb{F}_{q^n}) &= \sum_{j=0}^{m} (q^n)^j.
\end{aligned}$$

Computing the Zeta Function

So

$$\begin{aligned}
Z_X(z) &= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}(q^n)^j \frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{m}\frac{\left(q^j z\right)^n}{n}\right) \\
&= \exp\left(\sum_{j=0}^{m}\sum_{n=1}^{\infty}\frac{\left(q^j z\right)^n}{n}\right) \\
&= \exp\left(\sum_{j=0}^{m-1} -\log(1-q^j z)\right) \\
&= \prod_{j=0}^{m}\left(1-q^j z\right)^{-1} \\
&= \left(\frac{1}{1-z}\right)\left(\frac{1}{1-qz}\right)\left(\frac{1}{1-q^2 z}\right)\cdots\left(\frac{1}{1-q^m z}\right),
\end{aligned}$$

Miraculously, still a rational function! Consequence of sum formula, works in general.

Checking the Weil Conjectures

$$Z_X(z) = \prod_{j=0}^{m} \left(\frac{1}{1 - q^j z} \right).$$

1. Rationality: Clear!
2. Functional Equation: Less clear, but true:

$$\begin{aligned} Z_X\left(\frac{1}{q^m z}\right) &= \frac{1}{(1 - 1/q^m t)\,(1 - q/q^m t) \cdots (1 - q^m/q^m z)} \\ &= \frac{q^m z \cdot q^{m-1} z \ldots qz \cdot z}{(1-z)(1-qz)\ldots(1-q^m z)} \\ &= q^{\frac{m(m+1)}{2}} z^{m+1} \cdot Z_X(z) \\ &= \left(q^{\frac{m}{2}} z\right)^{\chi(X)} \cdot Z_X(z) \end{aligned}$$

.

Checking

$$Z_X(z) = \prod_{j=0}^{m} \left(\frac{1}{1 - q^j z} \right).$$

3. Riemann Hypothesis: Reduces to the statement $\{\alpha_i\} = \left\{ \frac{q^m}{\alpha_j} \right\}$.
   - Clear since $\alpha_j = q^j$ and every α_i is a lower power of q.
4. Betti Numbers: Use the fact that $\mathcal{P}_{\mathbb{CP}^m}(x) = 1 + x^2 + x^4 + \cdots + x^{2m}$
   - Only even dimensions, and correspondingly no numerator.

An Easier Proof: "Paving by Affines"

Quick recap:

$$Z_{\{\mathrm{pt}\}} = \frac{1}{1-z} \qquad Z_{\mathbb{P}^1}(z) = \frac{1}{1-qz} \qquad Z_{\mathbb{A}^1}(z) = \frac{1}{(1-z)(1-qz)}.$$

Note that $\mathbb{P}^1 = \mathbb{A}^1 \coprod \{\infty\}$ and correspondingly $Z_{\mathbb{P}^1}(z) = Z_{\mathbb{A}^1}(z) \cdot Z_{\{\mathrm{pt}\}}(z)$.
This works in general:

Lemma (Excision)

If $Y/\mathbb{F}_q \subset X/\mathbb{F}_q$ is a closed subvariety, for $U = X \setminus Y$,
$Z_X(z) = Z_Y(z) \cdot Z_U(z)$.

Proof: Let $N_n = \#Y(\mathbb{F}_{q^n})$ and $M_n = \#U(\mathbb{F}_{q^n})$, then

$$\begin{aligned}
\zeta_X(z) &= \exp\left(\sum_{n=1}^{\infty} (N_n + M_n) \frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n} + \sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} N_n \cdot \frac{z^n}{n}\right) \cdot \exp\left(\sum_{n=1}^{\infty} M_n \cdot \frac{z^n}{n}\right) = \zeta_Y(z) \cdot \zeta_U(z).
\end{aligned}$$

An Easier Proof: "Paving"

Note that geometry can help us here: we have a decomposition $\mathbb{P}^n = \mathbb{P}^{n-1} \coprod \mathbb{A}^n$, and thus inductively a stratification

$$\mathbb{P}^m = \coprod_{j=0}^{m} \mathbb{A}^j = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m.$$

Recalling that

$$Z_{X \coprod Y}(z) = Z_X(z) \cdot Z_Y(z)$$

and $Z_{\mathbb{A}^j}(z) = \frac{1}{1-q^j z}$, we have

$$Z_{\mathbb{P}^m}(z) = \prod_{j=0}^{m} Z_{\mathbb{A}^j}(z) = \prod_{j=0}^{m} \frac{1}{1-q^j z}.$$

Grassmannians

Consider now $X = \mathrm{Gr}(k, m)/\mathbb{F}$ – by the previous computation, we know

$$X(\mathbb{F}_{q^n}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} := \frac{(q^{nm}-1)(q^{nm-1}-1)\cdots(q^{nm-n(k-1)}-1)}{(q^{nk}-1)(q^{n(k-1)}-1)\cdots(q^n-1)}$$

but the corresponding Zeta function is much more complicated than the previous examples:

$$Z_X(z) = \exp\left(\sum_{n=1}^\infty \begin{bmatrix} m \\ k \end{bmatrix}_{q^n} \frac{z^n}{n}\right) = \cdots ?.$$

Since $\dim_{\mathbb{C}} \mathrm{Gr}_{\mathbb{C}}(k, m) = 2k(m-k)$, by Weil we should expect

$$Z_X(z) = \prod_{j=0}^{2k(m-k)} \frac{p_{2(j+1)}(z)}{p_{2j}(z)}$$

with $\deg p_j = \beta_j$.

Grassmannian

It turns out that (proof omitted) one can show

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \sum_{j=0}^{k(m-k)} \lambda_{m,k}(j) q^j \implies Z_X(z) = \prod_{j=0}^{k(m-k)} \left(\frac{1}{1 - q^j x} \right)^{\lambda_{m,k}(j)}$$

where $\lambda_{m,k}$ is the number of integer partitions of of $[i]$ into at most $m-k$ parts, each of size at most k.

- One proof idea: use combinatorial identities to write q-analog $\begin{bmatrix} m \\ k \end{bmatrix}_q$ as a *sum*
- Second proof idea: "pave by affines" (need a reference!)

This lets us conclude that the Poincare polynomial of the complex Grassmannian is given by

$$\mathcal{P}_{\mathrm{Gr}_{\mathbb{C}}(m,k)}(x) = \sum_{n=1}^{k(m-k)} \lambda_{m,k}(n) x^{2n},$$

In particular, the $H^* \mathrm{Gr}_{\mathbb{C}}(m,k)$ vanishes in odd degree.

Weil's Proof

Proof of rationality of $Z_X(T)$ for X a diagonal hypersurface.

- Set q to be a prime power and consider $X/\mathbb{F}_q$ defined by

$$X = V(a_0 x_0^n + \cdots + a_r x_r^n) \subset \mathbb{F}_q^{r+1}.$$

- We want to compute $N = \#X$.
- Set $d_i = \mathbf{gcd}(n_i, q-1)$.
- Define the character

$$\psi_q : \mathbb{F}_q \longrightarrow \mathbb{C}^\times$$
$$a \mapsto \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}\right).$$

  - By Artin's theorem for linear independence of characters, $\psi_q \not\equiv 1$ and every additive character of $\mathbb{F}_q$ is of the form $a \mapsto \psi_q(ca)$ for some $c \in \mathbb{F}_q$.
- Shorthand notation: say $a \sim 0 \iff a \equiv 0 \mod 1$.

A Diagonal Hypersurface

- Fix an injective multiplicative map

$$\phi : \bar{\mathbb{F}}_q^\times \longrightarrow \mathbb{C}^\times.$$

- Define

$$\begin{aligned} \chi_{\alpha,n} : \mathbb{F}_{q^n}^\times &\longrightarrow \mathbb{C}^\times \\ x &\mapsto \phi(x)^{\alpha(q^n-1)} \end{aligned}$$

$$\text{for} \quad \alpha \in \mathbb{Q}/\mathbb{Z}, n \in \mathbb{Z}, \quad \alpha(q^n - 1) \equiv 0 \mod 1.$$

  - Extend this to $\mathbb{F}_{q^n}$ by

$$\begin{cases} 1 & \alpha \equiv 0 \mod 1 \\ 0 & \text{else} \end{cases}.$$

  - Set $\chi_\alpha = \chi_{\alpha,1}$.

- Proposition:

$$\alpha(q-1) \equiv 0 \mod 1 \implies \chi_{\alpha,n}(x) = \chi_\alpha(\mathrm{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))$$

- Proposition:

$$d := \gcd(n, q-1), u \in \mathbb{F}_q \implies \#\left\{x \in \mathbb{F}_q \;\middle|\; x^n = u\right\} = \sum_{d\alpha \sim 0} \chi_\alpha(u)$$

- This implies

$$N = \sum_{\substack{\alpha=[\alpha_0,\cdots,\alpha_r] \\ d_i\alpha_i\sim 0}} \sum_{\substack{u=[u_0,\cdots,u_r] \\ \sum a_i u_i=0}} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j)$$

$$= q^r + \sum_{\substack{\alpha,\ \alpha_i\in(0,1) \\ d_i\alpha_i\sim 0}} \left(\prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) \sum_{\sum u_i=0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) \right).$$

since the inner sum is zero if some *but not all* of the $\alpha_i \sim 0$.

- Evaluate the innermost sum by restricting to $u_0 \neq 0$ and setting $u_i = u_0 v_i$ and $v_0 := 1$:

$$\sum_{\sum u_i=0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) = \sum_{u_0\neq 0} \chi_{\sum \alpha_i}(u_0) \sum_{\sum v_i=0} \prod_{j=0}^{r} \chi_{\alpha_j}(v_j)$$

$$= \begin{cases} (q-1)\sum_{\sum v_i=0} \prod_{j=0}^{r} \chi_{\alpha_j}(v_j) & \text{if } \sum \alpha_i \sim 0 \\ 0 & \text{else} \end{cases}.$$

A Diagonal Hypersurface

- Define the *Jacobi sum* for α where $\sum \alpha_i \sim 0$:

$$J(\alpha) := \left(\frac{1}{q-1}\right) \sum_{\Sigma\ u_i = 0} \prod_{j=0}^{r} \chi_{\alpha_j}(u_j) = \sum_{\Sigma\ v_i=0} \prod_{j=1}^{r} \chi_{\alpha_j}(v_j)$$

- Express N in terms of Jacobi sums as

$$N = q^r + (q-1) \sum_{\substack{\Sigma \alpha_i \sim 0 \\ d_i \alpha_i \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) J(\alpha).$$

- Evaluate $J(\alpha)$ using Gauss sums: for $\chi : \mathbb{F}_q \longrightarrow \mathbb{C}$ a multiplicative character, define

$$G(\chi) := \sum_{x \in \mathbb{F}_q} \chi(x) \psi_q(x).$$

- Proposition: for any $\chi \neq \chi_0$,
  - $|G(\chi)| = q^{\frac{1}{2}}$
  - $G(\chi)G(\bar{\chi}) = q\chi(-1)$
  - $G(\chi_0) = 0$

$$\chi(t) = \frac{G(\chi)}{q} \sum_{x \in \mathbb{F}_q} \bar{\chi}(x) \psi_q(tx).$$

A Diagonal Hypersurface

- Proposition: if

$$\sum \alpha_i \sim 0 \implies J(\alpha) = \frac{1}{q} \prod_{k=1}^{r} G(\chi_{\alpha_k}) \quad \text{and} \quad |J(\alpha)| = q^{\frac{r-1}{2}}.$$

- We thus obtain

$$N = q^r + \left(\frac{q-1}{q} \right) \sum_{\substack{\sum \alpha_j \sim 0 \\ d_j \alpha_j \sim 0 \\ \alpha \in (0,1)}} \prod_{j=0}^{r} \chi_{\alpha_j}(a_j^{-1}) G(\chi_{\alpha_j}).$$

- We now ask for number of points in $\mathbb{F}_{q^\nu}$ and consider a point count

$$\overline{N}_\nu = \# \left\{ [x_0 : \cdots : x_r] \in \mathbb{P}^r_{\mathbb{F}_q^\nu} \;\middle|\; \sum_{i=0}^{r} a_i x_i^n = 0 \right\}.$$

- Theorem (Davenport, Hasse)

$$(q-1)\alpha \sim 0 \implies -G(\chi_{\alpha,\nu}) = (-G(\chi_\alpha))^\nu.$$

A Diagonal Hypersurface

- We have a relation $(q^\nu - 1)\overline{N}_\nu = N_\nu$.
- This lets us write

$$\overline{N}_\nu = \sum_{j=0}^{r-1} q^{j\nu} + \sum_{\substack{\sum \alpha_i \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \prod_{j=0}^{r} \overline{\chi}_{\alpha_{j,\nu}}(a_i) J_\nu(\alpha).$$

- Set

$$\mu(\alpha) = \min \left\{ \mu \;\middle|\; (q^\mu - 1)\alpha \sim 0 \right\}$$

$$C(\alpha) = (-1)^{r+1} \prod_{j=1}^{r} \overline{\chi}_{\alpha_0, \mu(\alpha)}(a_j) \cdot J_{\mu(\alpha)}(\alpha).$$

- Plugging into the zeta function Z yields

$$\exp\left(\sum_{\nu=1}^{\infty} \overline{N}_\nu \frac{T^\nu}{\nu} \right) = \prod_{j=0}^{r-1} \left(\frac{1}{1 - q^j T} \right) \prod_{\substack{\sum \alpha_i \sim 0 \\ \gcd(n, q^\nu - 1)\alpha_i \sim 0 \\ \alpha_i \in (0,1)}} \left(1 - C(\alpha) T^{\mu(\alpha)} \right)^{\frac{(-1)^r}{\mu(\alpha)}},$$

which is evidently a rational function!