

Category \mathcal{O} , Problem Set 4

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1 Humphreys 3.1

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and identify $\lambda \in \mathfrak{h}^\vee$ with a scalar. Let N be a 2-dimensional $U(\mathfrak{b})$ -module defined by letting x act as 0 and h act as $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Show that the induced $U(\mathfrak{g})$ -module structure $M := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N$ fits into an exact sequence which fails to split:

$$0 \longrightarrow M(\lambda) \longrightarrow M \longrightarrow M(\lambda) \longrightarrow 0$$

1.1 Solution

Reference 1 Reference 2

Hence $M \notin \mathcal{O}$.

We first unpack all definitions in terms of tensor products, using the fact that $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\lambda$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(\lambda) & \longrightarrow & M & \longrightarrow & M(\lambda) & \longrightarrow & 0 \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
0 & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccc}
1 \otimes 1 & \xrightarrow{\psi} & 1 \otimes \mathbf{u} & \xrightarrow{\phi} & 1 \otimes 0 \\
& & & & \\
& & 1 \otimes \mathbf{v} & \xrightarrow{\quad} & 1 \otimes 1
\end{array}$$

where $N = \text{span}_{\mathbb{C}} \{\mathbf{u}, \mathbf{v}\}$.

We make the following claims:

1. The $U(\mathfrak{b})$ action defined on N lifts to a $U(\mathfrak{g})$ -action on M .
2. This is an exact sequence of $U(\mathfrak{g})$ -modules.
3. $M \not\cong M(\lambda) \oplus M(\lambda)$, showing that this sequence can not split.

Claim 1: We choose the basis

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and note that in the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, we have

$$\begin{aligned}
\mathfrak{n}^- &= \mathbb{C} \cdot x \\
\mathfrak{h} &= \mathbb{C} \cdot h \\
\mathfrak{n}^+ &= \mathbb{C} \cdot y
\end{aligned}$$

Since the action is defined over $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ and x acts by zero, we obtain a \mathfrak{g} -action on N which thus extends uniquely to a $U(\mathfrak{g})$ -action.

Claim 2: We first note that since the submodule $\mathbb{C} \cdot \mathbf{u} < M$ is closed under the action of h (since h acts by $u \mapsto \lambda u$) and is equal to the image of ψ , we can identify $\mathbb{C} \cdot \mathbf{u} \cong \mathbb{C}_\lambda$ as $U(\mathfrak{b})$ -modules and identify $M(\lambda)$ as a submodule of N . Since submodules of N lift to submodules of $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N$, the map ψ is an injection. Moreover, the map ϕ is a surjection, since the generator $1 \otimes 1$ of $M(\lambda)$ is precisely the image of one of the generators of M .

To see that the sequence is exact in the middle, we note that by choosing a PBW basis of $\mathfrak{sl}(2, \mathbb{C})$ and a basis $\{\mathbf{u}, \mathbf{v}\}$ for N , we can obtain a basis of M of the form $\{y^j \otimes \mathbf{u}, y^k \otimes \mathbf{v} \mid j, k \in \mathbb{Z}^{\geq 0}\}$. This allows us to identify the lift of the submodule $\mathbb{C} \cdot \mathbf{u}$ to the span of $\{y^k \otimes \mathbf{u}\}$ in M . Then $\text{im } \psi \in \ker \phi$ by construction, since

$$\phi(y^k \otimes \mathbf{u}) = \phi(y^k(1 \otimes \mathbf{u})) = y^k \phi(1 \otimes \mathbf{u}) = y^k(1 \otimes u) = 0$$

we first note that $\text{im } \psi = \mathbb{C} \cdot \mathbf{u} \subset \ker \phi$ by construction, since we explicitly map the aforementioned submodule $\mathbb{C} \cdot \mathbf{u}$ to 0. To see that $\ker \phi \subset \mathbb{C} \cdot \mathbf{u}$, But then we can identify $\mathbb{C} \cdot \mathbf{u} = \{y^k(1 \otimes \mathbf{u})\}$ since ϕ is specified on $1 \otimes \mathbf{u}$, we see that any other basis element maps as

$$\phi(y^k \otimes \mathbf{v}) = \phi(y^k(1 \otimes \mathbf{v})) = y^k \phi(1 \otimes \mathbf{v}) = y^k(1 \otimes 1) = y^k \otimes 1 \neq 0,$$

2 Humphreys 3.2

Show that for $M \in \mathcal{O}$ and $\dim L < \infty$,

$$(M \otimes L)^\vee \cong M^\vee \otimes L^\vee$$

Reference for Dual of Sum

2.1 Solution

We first note that $M \in \mathcal{O} \implies M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$ where each M_λ is a finite-dimensional weight space.

Moreover, $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda^\vee$ is defined to be a direct sum of duals of weight spaces, which are still finite-dimensional.

So let $M, N \in \mathcal{O}$; we will proceed by showing that both $(M \otimes_{\mathbb{C}} L)^\vee$ and $M^\vee \otimes_{\mathbb{C}} L^\vee$ have identical direct sum decompositions.

We first have

$$\begin{aligned} (M \otimes_{\mathbb{C}} L)^\vee &:= \bigoplus_{\lambda \in \mathfrak{h}^\vee} (M \otimes_{\mathbb{C}} L)_\lambda^\vee, && \text{the } \lambda \text{ weight spaces of } M \otimes_{\mathbb{C}} L \\ &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta) \right)^\vee && \text{by an exercise on the weight spaces of a tensor product} \\ &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha \otimes_{\mathbb{C}} L_\beta)^\vee \right) && \text{since the inner term is a finite sum} \\ &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee) \right) && \text{since the weight spaces are finite-dimensional,} \end{aligned}$$

where we've repeatedly used the fact that $(V \otimes W)^\vee \cong V^\vee \otimes W^\vee$ for finite-dimensional vector spaces, which inductively holds for any finite direct sum of vector spaces.

On the other hand, using the fact that

$$\begin{aligned} (A \oplus B) \otimes (C \oplus D) &= ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \\ &= (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D) \\ \implies \left(\bigoplus_{j \in J} A_j \right) \otimes \left(\bigoplus_{k \in K} B_k \right) &= \bigoplus_{j \in J} \bigoplus_{k \in K} (A_j \otimes B_k) \quad \text{by induction} \quad . \end{aligned}$$

we can write

$$\begin{aligned} M^\vee \otimes_{\mathbb{C}} L^\vee &:= \left(\bigoplus_{\alpha \in \mathfrak{h}^\vee} M_\alpha^\vee \right) \otimes_{\mathbb{C}} \left(\bigoplus_{\beta \in \mathfrak{h}^\vee} L_\beta^\vee \right) \\ &\cong \bigoplus_{\lambda \in \mathfrak{h}^\vee} \left(\bigoplus_{\alpha+\beta=\lambda} (M_\alpha^\vee \otimes_{\mathbb{C}} L_\beta^\vee) \right), \end{aligned}$$

which equals what was obtained above.

This exhibits the isomorphism as \mathbb{C} -vector spaces, to see that this is in fact an isomorphism of $U(\mathfrak{g})$ -modules we can use the fact that for $M \in \mathcal{O}$, a twisted \mathfrak{g} -action was defined as

$$\mathbf{v} \in M, f \in M^\vee, g \in \mathfrak{g} \implies (g \cdot f)(\mathbf{v}) = f(\tau(g) \cdot \mathbf{v})$$

for the transpose map τ . This action can be “linearly extended” over direct products and tensor products by taking the action component-wise, and is thus preserved by all of the isomorphisms appearing above.

Since the final terms $\bigoplus_{\lambda \in \mathfrak{h}} \bigoplus_{\alpha+\beta=\lambda} M_\alpha^\vee \otimes L_\beta^\vee$ are identical, they carry the same action, and since they are preserved by the isomorphisms, working backwards shows that the actions on $(M \otimes L)^\vee$ and $M^\vee \otimes L^\vee$ must also agree, yielding the desired isomorphism.

3 Humphreys 3.4

Show that $\Phi_{[\lambda]} \cap \Phi^+$ is a positive system in the root system $\Phi_{[\lambda]}$, but the corresponding simple system $\Delta_{[\lambda]}$ may be unrelated to Δ .

For a concrete example, take Φ of type B_2 with a short simple root α and a long simple root β . If $\lambda := \alpha/2$, check that $\Phi_{[\lambda]}$ contains just the four short roots in Φ .

3.1 Solution

We would like to show the following two propositions:

1. $\Phi_{[\lambda]}^+ := \Phi_{[\lambda]} \cap \Phi^+$ is a positive system in $\Phi_{[\lambda]}$,
2. In general, the associated simple system $\Delta_{[\lambda]} \neq \Phi_{[\lambda]}^+ \cap \Delta$.

3.1.1 Proof of Proposition 1

We’ll use the definition that for an abstract root system Φ , a positive system Φ^+ is defined by picking a hyperplane H not containing any roots and taking all roots on one side of this hyperplane.

However, if every element of Φ^+ is on one side of H , then any subset satisfies this property as well, thus $\Phi_{[\lambda]} \cap \Phi^+$ consists only of positive roots and thus forms a positive system.

3.1.2 Proof of Proposition 2

Concretely, we can realize Φ and Δ as subsets of \mathbb{R}^2 in the following way:

$$\begin{aligned}\Phi &= P_1 \amalg P_2 := \{[1, 0], [0, 1], [-1, 0], [0, -1]\} \amalg \{[1, 1], [-1, 1], [1, -1], [-1, -1]\} \\ \Delta &:= \{\alpha, \beta\} := \{[1, 0], [-1, 1]\},\end{aligned}$$

where we note that P_1 consists of short roots (of norm 1) and P_2 of long roots (of norm $\sqrt{2}$) and we've chosen a simple system consisting of one short root and one long root.

Now by definition,

$$\begin{aligned}\Phi_{[\lambda]} &:= \left\{ \gamma \in \Phi \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}, & \gamma^\vee &:= \frac{2}{\|\gamma\|^2} \gamma, \\ \Delta_{[\lambda]} &:= \left\{ \gamma \in \Delta \mid \langle \lambda, \gamma^\vee \rangle \in \mathbb{Z} \right\}.\end{aligned}$$

Now choosing $\lambda := \frac{\alpha}{2} = \left[\frac{1}{2}, 0\right]$, we now consider the inner products $\langle \lambda, \gamma^\vee \rangle$ for $\gamma \in \Phi$:

Thus

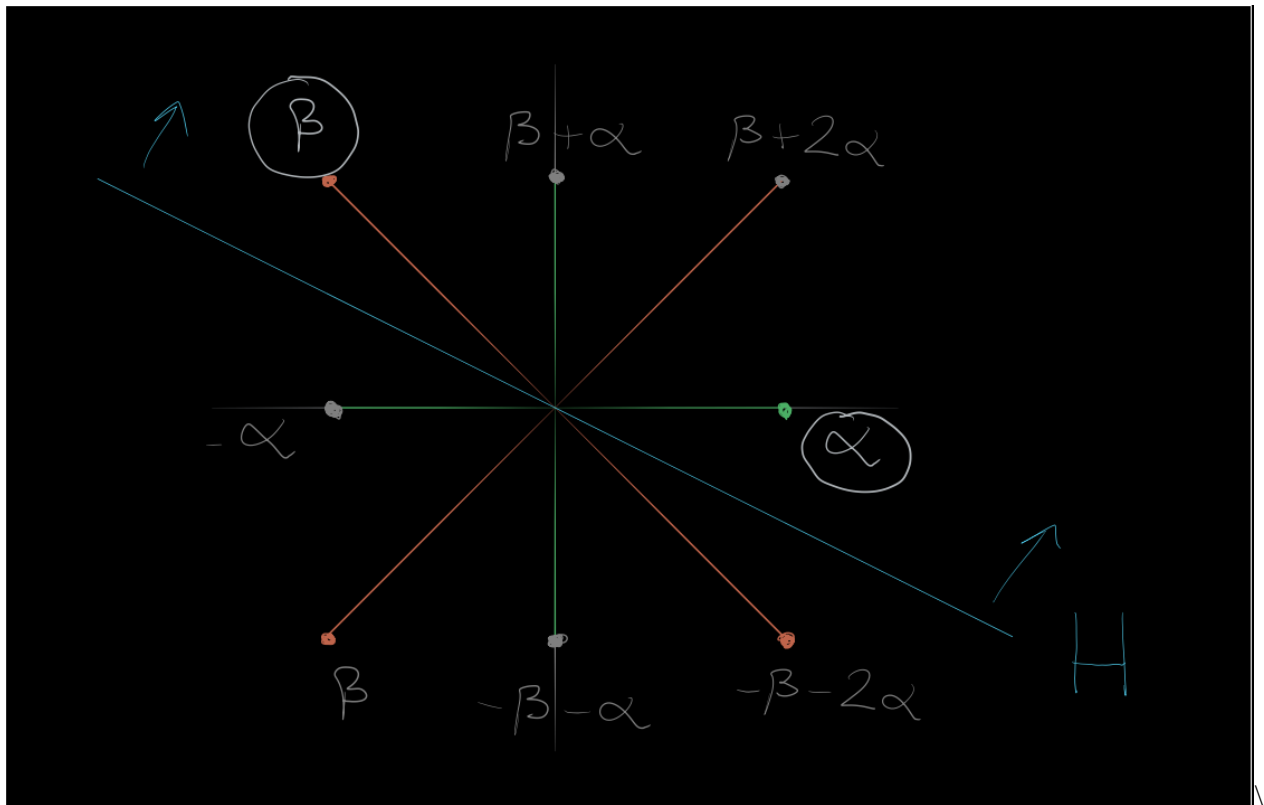
$$\begin{aligned}\gamma_1 \in P_1 &\implies \left\langle \left[\frac{1}{2}, 0\right], 2\gamma_1 \right\rangle = 2\left(\frac{1}{2}\right) \langle [1, 0], \gamma_1 \rangle = (\gamma_1)_1 \in \{0, \pm 1\} \in \mathbb{Z} \\ \gamma_2 \in P_2 &\implies \langle \lambda, \gamma_2^\vee \rangle = \left\langle \left[\frac{1}{2}, 0\right], \frac{2}{(\sqrt{2})^2} [\pm 1, \pm 1] \right\rangle = \pm \frac{1}{2} \notin \mathbb{Z}\end{aligned}$$

where $(\gamma_1)_1$ denotes the first component of γ_1 .

We thus find that

$$\begin{aligned}\Phi_{[\lambda]} &= P_1 && \text{the short roots} \\ \Delta_{[\lambda]} &= \Phi_{[\lambda]} \cap \Delta = \{\alpha\} && \text{the single short simple root.}\end{aligned}$$

Choosing the following hyperplane H not containing any root, we can choose a positive system:



$$\Phi^+ = \{\beta, \beta + \alpha, \beta + 2\alpha, \alpha\}$$

where we can note that $\Phi^+ \cap \Delta = \Delta$, since we've placed both simple roots on the positive side of this hyperplane by construction.

But by taking roots on the positive side of this plane, we have

$$\Phi_{[\lambda]} = \{\alpha, -\alpha, \alpha + \beta, -\alpha - \beta\} \implies \Phi_{[\lambda]}^+ = \{\alpha, \alpha + \beta\}$$

where we can now note that a simple system in *this* root system must still have rank 2, so we can take $\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\}$. But now we can note

$$\Delta_{[\lambda]} = \{\alpha, \alpha + \beta\} \neq \{\alpha\} = \{\alpha, \alpha + \beta\} \cap \{\alpha, \beta\} = \Phi_{[\lambda]}^+ \cap \Delta,$$

which is what we wanted to show.

4 Humphreys 3.7

4.1 a

If a module M has a standard filtration and there exists an epimorphism $\phi : M \longrightarrow M(\lambda)$, prove that $\ker \phi$ admits a standard filtration.

4.2 b

Show by example that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ that the existence of a monomorphism $\phi : M(\lambda) \longrightarrow M$ where M has a standard filtration fails to imply that $\text{coker } \phi$ has a standard filtration.