Lie Algebras

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1 Lecture 1

todo

2 Lecture 2

Recall from last time that a Lie Algebra is a vector space with a bilinear bracket, which importantly satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = \mathbf{0}.$$

Also recall the examples from last time:

- $A_{\ell} \iff \mathfrak{sl}(\ell+1,F)$
- $B_{\ell} \iff \mathfrak{so}(2\ell+1,F)$
- $C_{\ell} \iff \mathfrak{sp}(2\ell, F)$
- $D_{\ell} \iff \mathfrak{so}(2\ell, F)$

Exercise: characterize these matrix subalgebras in terms of basis elements, and compute their dimensions.

2.1 Lie Algebras of Derivations

Definition: An F-algebra A is an F-vector space endowed with a bilinear map $A^2 \to A$, $(x,y) \mapsto xy$.

Definition: An algebra is **associative** if x(yz) = (xy)z.

Modern interest: simple Lie algebras, which have a good representation theory. Take a look a Erdmann-Wildon (Springer) for an introductory look at 3-dimensional algebras.

Definition: Any map $\delta: A^2 \to A$ that satisfies the Leibniz rule is called a **derivation** of A, where the rule is given by $\delta(xy) = \delta(x)y + x\delta(y)$.

We define $Der(A) = \{ \delta \ni \delta \text{ is a derivation } \}.$

Any Lie algebra \mathfrak{g} is an F-algebra, since $[\cdot,\cdot]$ is bilinear. Moreover, \mathfrak{g} is associative iff [x,[y,z]]=0.

Claim: $\operatorname{Der}\mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$ is a Lie subalgebra. One needs to check that $\delta_1, \delta_2 \in \mathfrak{g} \implies [\delta_1, \delta_2] \in \mathfrak{g}$.

2.1.1 Exercise (Turn in)

Define the adjoint by $\operatorname{ad}_x : \mathfrak{g} \circlearrowleft, y \mapsto [x,y]$. Show that $\operatorname{ad}_x \in \operatorname{Der}(\mathfrak{g})$.

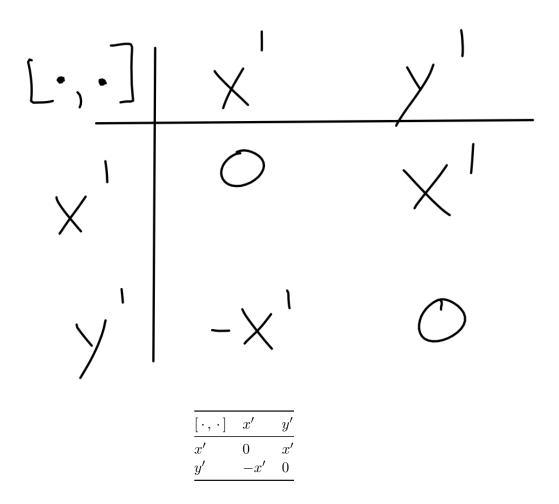
2.2 Abstract Lie Algebras

Fact: Every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e. a subalgebra of $\mathfrak{gl}(V)$. Each isomorphism type can be specified by certain *structure constants* for the Lie bracket.

Example: Any F-vector space can be made into a Lie algebra by setting [x, y] = 0; such algebras are referred to as *abelian*.

Attempting to classify Lie algebras of dimension at most 2.

- 1 dimensional: We can write $\mathfrak{g} = Fx$, and so $[x, x] = 0 \implies [\cdot, \cdot] = 0$. So every bracket must be zero, and thus every Lie algebra is abelian.
- 2 dimensional: Write $\mathfrak{g} = Fx \oplus Fy$, the only nontrivial bracket here is [x,y]. Some cases:
 - $-[x,y]=0 \implies \mathfrak{g}$ is abelian.
 - $-[x,y] = ax + by \neq 0$. Assume $a \neq 0$ and set $x' = ax + by, y' = \frac{y}{a}$. Now compute $[x',y'] = [ax + by, \frac{y}{a}] = [x,y] = ax + by = x'$. Punchline: $\mathfrak{g} \cong Fx' \oplus Fy', [x',y'] = x'$.



$$\begin{array}{c|ccc} [\cdot,\cdot] & x' & y' \\ \hline x' & 0 & x' \\ y' & -x' & 0 \end{array}$$

Example: Let $V = \mathbb{R}^3$, and define $[a, b] = a \times b$ to be the usual cross product.

Exercise: Look at notes for basis elements of $\mathfrak{sl}(2, F)$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Compute the matrices of ade, adh, adg with respect to this basis.

2.3 Ideals

Definition: A subspace $I \subseteq \mathfrak{g}$ is called an **ideal**, and we write $I \unlhd \mathfrak{g}$, if $x,y \in I \implies [x,y] \in I$.

Note that there is no need to distinguish right, left, or two-sided ideals. This can be shown using [x,y]=[-y,x].

Exercise: Check that the following are all ideals of \mathfrak{g} :

- $\{0\}, \mathfrak{g}$.
- $\mathfrak{z}(\mathfrak{g}) = \{ z \in \mathfrak{g} \ni [x, z] = 0 \quad \forall x \in \mathfrak{g} \}$
- The commutator (or derived) algebra $[\mathfrak{g},\mathfrak{g}] = \{\sum_i [x_i,y_i] \ni x_i,y_i \in \mathfrak{g}\}.$ - Moreover, $[\mathfrak{gl}(n,F),\mathfrak{gl}(n,F)] = \mathfrak{sl}(n,F).$

Fact: If $I, J \leq \mathfrak{g}$, then

- $I + J = \{x + y \ni x \in I, y \in J\} \leq \mathfrak{g}$
- $I \cap J \leq \mathfrak{g}$
- $[I, J] = \{\sum_i [x_i, y_i] \ni x_i \in I, y_i \in J\} \leq \mathfrak{g}$

Definition: A Lie algebra is **simple** if $[\mathfrak{g},\mathfrak{g}] \neq 0$ (i.e. when \mathfrak{g} is not abelian) and has no non-trivial ideals. Note that this implies that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Theorem: Suppose that $char F \neq 2$, then $\mathfrak{sl}(2, F)$ is not simple.

Proof: Recall that we have a basis of $\mathfrak{sl}(2,F)$ given by $B=\{e,h,f\}$ where

- [e, f] = h,
- [h, e] = 2e, [h, f] = -2f.

So think of $[h, e] = \mathrm{ad}_h$, so h is an eigenvector of this map with eigenvalues $\{0, \pm 2\}$. Since $\mathrm{char} F \neq 2$, these are all distinct. Suppose $\mathfrak{sl}(2,F)$ has a nontrivial ideal I; then pick $x=ae+bh+cf\in I$. Then [e,x]=0-2be+ch, and [e,[e,x]]=0-0+2ce. Again since $charF\neq 2$, then if $c\neq 0$ then $e \in I$. Now you can show that $h \in I$ and $f \in I$, but then $I = \mathfrak{sl}(2, F)$, a contradiction. So c = 0.

Then $x = bh \neq 0$, so $h \in I$, and we can compute

$$2e = [h, e] \in I \implies e \in I, 2f = [h, -f] \in I \implies f \in I,$$

which implies that $I = \mathfrak{sl}(2, F)$ and thus it is simple.

Note that there is a homework coming due next Monday, about 4 questions.