Title

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1.1 Review and Proposition

From last time: Steinberg's tensor product.

Let G be a reductive algebraic group scheme over k with char $(k) > 0$. We have a Frobenius $F : G \to G$, we iterate to obtain F^r and examine the Frobenius kernels $G_r := \ker F^r$.

If we have a representation $\rho : G \to \mathrm{GL}(M)$, we can "twist" by F^r to obtain $\rho^{(r)} : G \to \mathrm{GL}(M^{(r)})$. We have

Here $M^{(r)}$ has the same underlying vector space as M, but a new module structure coming from $\rho^{(r)}$. Note that G_r acts trivially on $M^{(r)}$.

- $\left\{ L(\lambda) \;\middle|\; \lambda \in X(T)_+ \right\}$ are the simple G-modules,
- $\left\{ L_r(\lambda) \;\middle|\; \lambda \in X_r(T)_+ \right\}$ are the simple G_r-modules,

Note that $L(\lambda) \downarrow_{G_r}$ is semisimple, equal to $L_r(\lambda)$ for $\lambda \in X_r(T)$.

1960's, Curtis and Steinberg.

Proposition 1.1 ***(?).***
Let $\lambda \in X_r(T)$ and $\mu \in X(T)_+$. Then

$$L(\lambda + p^r \mu) \cong L(\lambda) \otimes L(\mu)^{(r)}.$$

Recall that socle formula: letting M be a G-module, we have an isomorphism of G-modules:

$$\mathrm{Soc}_{G_r} \cong \bigoplus_{\lambda \in X_r(T)} L(\lambda) \otimes \hom_{G_r}(L(\lambda), M).$$

1.2 Proof

Let $M = L(\lambda + p^r\mu)$. Then from the socle formula, only one summand is nonzero, and thus $\hom_{G_r}(L(\lambda), M)$ must be simple. Then there exists a $\tilde{\lambda} \in X_r(T)$ and a $\tilde{\mu} \in X(T)_+$ such that

$$M = L(\tilde{\lambda}) \otimes L(\tilde{\mu})^{(r)}.$$

We now compare highest weights:

$$\lambda + p^r\mu = \tilde{\lambda} + p^r\tilde{\mu} \implies \lambda = \tilde{\lambda} \quad \text{and} \quad \mu = \tilde{\mu}.$$

Theorem 1.2 ***(Steinberg).***
Let $\lambda \in X(T)_+$, with a p-adic expansion

$$\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_m p^m.$$

where $\lambda_j \in X_1(T)$ for all j. Then

$$L(\lambda) = L(\lambda_0) \otimes \bigotimes_{j=1}^{m} L(\lambda_j)^{(j)}.$$

Corollary 1.3 ***(?).***
In order to know $\dim L(\lambda)$ for $\lambda \in X(T)_+$, it is enough to know $\dim L_1(\mu)$ for $\mu \in X_1(T)$.