Title

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1 References

- [**bakker__8330**], [**hartshorne__2010**]
- Hilbert schemes/functors of points: [**stromme**], [**hartshorne__def**].
  - Slightly more detailed: [**fantechi__2005**]
- Curves on surfaces: [**mumford__1985**]
- Moduli of Curves: [**harris__morrison__1998**] (chatty and less rigorous)

2 Schemes vs Representable Functors (Thursday January 9th)

Last time: fix an S-scheme, i.e. a scheme over S. Then there is a map

$$\begin{aligned} \mathrm{Sch}_{/S} &\to \mathrm{Fun}(\mathrm{Sch}_{/S}^{\mathrm{op}}, \mathrm{Set}) \\ x &\mapsto h_x(T) = \hom_{\mathrm{Sch}_{/S}}(T, x). \end{aligned}$$

where $T' \xrightarrow{f} T$ is given by

$$\begin{aligned} h_x(f) : h_x(T) &\to h_x(T') \\ (T \mapsto x) &\mapsto \text{triangles of the form.} \end{aligned}$$

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & X \\ & T \arrow[ur] & \end{tikzcd}$$

2.1 Representability

Theorem 2.1.1 *(?).*

$$\hom_{\text{Fun}}(h_x, F) = F(x).$$

Corollary 2.1.2 *(?).*

$$\hom_{\text{Sch}_{/S}}(x, y) \cong \hom_{\text{Fun}}(h_x, h_y).$$

Definition 2.1.3 (Moduli Functor)
A **moduli functor** is a map

$$\begin{aligned} F : (\text{Sch}_{/S})^{\text{op}} &\to \text{Set} \\ F(x) &= \text{"Families of something over } x\text{"} \\ F(f) &= \text{"Pullback"}. \end{aligned}$$

Definition 2.1.4 (Moduli Space)
A **moduli space** for that "something" appearing above is an $M \in \text{Obj}(\text{Sch}_{/S})$ such that $F \cong h_M$.

Remark 2.1.5: Now fix $S = \text{Spec}(k)$, and write h_m for the functor of points over M. Then

$$h_m(\text{Spec}(k)) = M(\text{Spec}(k)) \cong \text{families over } \text{Spec}\, k = F(\text{Spec}\, k).$$

Remark 2.1.6: $h_M(M) \cong F(M)$ are families over M, and $\text{id}_M \in \text{Mor}_{\text{Sch}_{/S}}(M, M) = \xi_{Univ}$ is the universal family.

Every family is uniquely the pullback of ξ_{Univ}. This makes it much like a classifying space. For $T \in \text{Sch}_{/S}$,

$$\begin{aligned} h_M &\xrightarrow{\cong} F \\ f \in h_M(T) &\xrightarrow{\cong} F(T) \ni \xi = F(f)(\xi_{\text{Univ}}). \end{aligned}$$

where $T \xrightarrow{f} M$ and $f = h_M(f)(\mathrm{id}_M)$.

Remark 2.1.7: If M and M' both represent F then $M \cong M'$ up to unique isomorphism.

$$\begin{tikzcd}
\xi_M & \xi_{M'} \\
M \arrow[r, "f"] & M'
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[r, "g"] & M \\
\xi_{M'} & \xi_M
\end{tikzcd}$$

which shows that f, g must be mutually inverse by using universal properties.

Example 2.1.8*(?)*: A length 2 subscheme of $\mathbb{A}^1_k$ (??) then

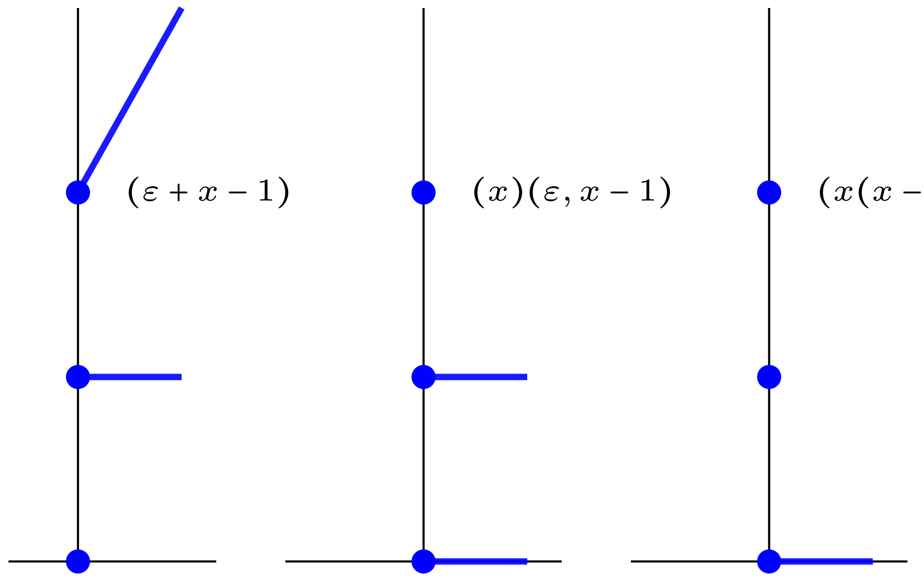
$$F(S) = \left\{V(x^2 + bx + c)\right\} \subset \mathbb{A}^5$$

where $b, c \in \mathcal{O}_s(s)$, which is functorially bijective with $\{b, c \in \mathcal{O}_s(s)\}$ and $F(f)$ is pullback. Then F is representable by $\mathbb{A}^2_k(b, c)$ and the universal object is given by

$$V(x^2 + bx + c) \subset \mathbb{A}^1(?) \times \mathbb{A}^2(b, c)$$

where $b, c \in k[b, c]$. Moreover, $F'(S)$ is the set of effective Cartier divisors in $\mathbb{A}'_5$ which are length 2 for every geometric fiber. $F''(S)$ is the set of subschemes of $\mathbb{A}'_5$ which are length 2 on all geometric fibers. In both cases, $F(f)$ is always given by pullback.

Problem: F'' is not a good moduli functor, as it is not representable. Consider $\operatorname{Spec} k[\varepsilon]$, for which we have the following situation:



F	✓	x	x
F'	✓	x	x
F''	✓	✓	✓

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, hook, "i"] & \operatorname{Spec} k[\varepsilon] & = F'(\operatorname{Spec} k) \arrow[dr] & \\
F(\operatorname{Spec} k[\varepsilon]) \arrow[r, "F(i)"] & F(\operatorname{Spec} k) \arrow[ur] & & = F''(\operatorname{Spec} k) \\
T_pF^{','} \arrow[u, hook, "\subset"] & P = V(x(x-1)) \arrow[u, hook, "\in"] & &
\end{tikzcd}$$

We think of $T_pF^{','}$ as the tangent space at p. If F is representable, then it is actually the Zariski tangent space.

$$\begin{tikzcd}
M(\operatorname{Spec} k[\varepsilon]) \arrow[r] & M(\operatorname{Spec} k) \\
T_pM \arrow[u, hook, "\subset"] \arrow[r] & p \arrow[u, hook, "\subset"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \operatorname{Spec} k \arrow[dl, hook'] \arrow[dr, "?"] & \\
\operatorname{Spec} k[\varepsilon] \arrow[rr] & & \operatorname{Spec} \mathcal{O}_{M,p} \subset M
\end{tikzcd}$$

$$\begin{tikzcd}
 & k \\
\mathcal{O}_{M,p} \arrow[ur] \arrow[r] & k[\varepsilon] \arrow[u] \\
\mathfrak{m}_p \arrow[u, hook] & (\varepsilon) \arrow[u, hook] \\
\mathfrak{m}_p^2 \arrow[u, hook] & 0 \arrow[u, hook]
\end{tikzcd}$$

Moreover, $T_pM = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, and in particular this is a k-vector space. To see the scaling structure, take $\lambda \in k$.

$$\begin{aligned} \lambda : k[\varepsilon] &\to k[\varepsilon] \\ \varepsilon &\mapsto \lambda\varepsilon \end{aligned}$$

$$\lambda^* : \operatorname{Spec}(k[\varepsilon]) \to \operatorname{Spec}(k[\varepsilon])$$

$$\lambda : M(\operatorname{Spec}(k[\varepsilon])) \to M(\operatorname{Spec}(k[\varepsilon])).$$

$$\begin{tikzcd}
M(\operatorname{Spec}(k[\varepsilon])) \arrow[r, "\lambda"] & M(\operatorname{Spec}(k[\varepsilon])) \\
T_pM \arrow[u, "\subseteq"] \arrow[r] & T_pM \arrow[u, "\subseteq"]
\end{tikzcd}$$

Conclusion: If F is representable, for each $p \in F(\operatorname{Spec} k)$ there exists a unique point of T_pF that are invariant under scaling.

Remark 2.1.9: If $F, F', G \in \operatorname{Fun}((\mathrm{Sch}_{/S})^{\mathrm{op}}, \mathrm{Set})$, there exists a fiber product

$$\begin{tikzcd} F \times_G F' \arrow[r, dotted] \arrow[d, dotted] & F' \arrow[d] \\ F \arrow[r] & G \end{tikzcd}$$

where

$$(F \times_G F')(T) = F(T) \times_{G(T)} F'(T).$$

Remark 2.1.10: This works with the functor of points over a fiber product of schemes $X \times_T Y$ for $X, Y \to T$, where

$$h_{X \times_T Y} = h_X \times_{h_t} h_Y.$$

Remark 2.1.11: If F, F', G are representable, then so is the fiber product $F \times_G F'$.

Remark 2.1.12: For any functor

$$F : (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set},$$

for any $T \xrightarrow{f} S$ there is an induced functor

$$\begin{aligned} F_T : (\mathrm{Sch}_{/T}) &\to \mathrm{Set} \\ x &\mapsto F(x). \end{aligned}$$

Remark 2.1.13: F is representable by $M_{/S}$ implies that F_T is representable by $M_T = M \times_S T/T$.

2.2 Projective Space

Consider $\mathbb{P}^n_{\mathbb{Z}}$, i.e. "rank 1 quotient of an $n+1$ dimensional free module".

Proposition 2.2.1 *(?).*
$\mathbb{P}^n_{/\mathbb{Z}}$ represents the following functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}} &\to \mathrm{Set} \\ S &\mapsto \left\{ \mathcal{O}_S^{n+1} \to L \to 0 \right\} / \sim . \end{aligned}$$

where $\sim$ identifies diagrams of the following form:

$$\begin{tikzcd} \mathcal{O}_s^{n+1} \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ \mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and $F(f)$ is given by pullbacks.

Remark 2.2.2: $\mathbb{P}^n_{/S}$ represents the following functor:

$$\begin{aligned} F_S : (\mathrm{Sch}_{/S})^{\mathrm{op}} &\to \mathrm{Set} \\ T \mapsto F_S(T) &= \left\{\mathcal{O}_T^{n+1} \to L \to 0\right\} / \sim . \end{aligned}$$

This gives us a cleaner way of gluing affine data into a scheme.

2.2.1 Proof of Proposition

Remark 2.2.3: Note that $\mathcal{O}^{n+1} \to L \to 0$ is the same as giving $n+1$ sections $s_1, \cdots s_n$ of L, where surjectivity ensures that they are not the zero section. So

$$F_i(S) = \left\{\mathcal{O}_s^{n+1} \to L \to 0\right\} / \sim,$$

with the additional condition that $s_i \neq 0$ at any point. There is a natural transformation $F_i \to F$ by forgetting the latter condition, and is in fact a subfunctor. [1]

Claim: It is enough to show that each F_i and each F_{ij} are representable, since we have natural transformations:

$$\begin{tikzcd} F_i \arrow[r] & F \\ F_{ij} \arrow[r] \arrow[u] & F_j \arrow[u] \end{tikzcd}$$

and each $F_{ij} \to F_i$ is an open embedding on the level of their representing schemes.

Example 2.2.4*(?)*: For $n = 1$, we can glue along open subschemes

[1] $F \leq G$ is a subfunctor iff $F(s) \hookrightarrow G(s)$.

$$\begin{tikzcd}
 & F_0 \\
F_{01} \arrow[ur] \arrow[dr] & \\
 & F_1
\end{tikzcd}$$

For $n = 2$, we get overlaps of the following form:

$$\begin{tikzcd}
 & & & F_0 \arrow[ddr] & \\
 & & F_{01} \arrow[ur] \arrow[dr] & & \\
F_{012} \arrow[r] \arrow[urr] \arrow[drr] & F_{02} \arrow[ur] \arrow[dr] \arrow[uurr, dotted, bend left] \arrow[ddrr, bend right] & & F_1 \arrow[r] & F \\
 & & F_{12} \arrow[ur] \arrow[dr] & & \\
 & & & F_2 \arrow[uur] &
\end{tikzcd}$$

This claim implies that we can glue together F_i to get a scheme M. We want to show that M represents F. $F(s)$ (LHS) is equivalent to an open cover U_i of S and sections of $F_i(U_i)$ satisfying the gluing (RHS). Going from LHS to RHS isn't difficult, since for $\mathcal{O}_s^{n+1} \to L \to 0$, U_i is the locus where $s_i \neq 0$ and by surjectivity, this gives a cover of S. The RHS to LHS comes from gluing.

Proof (of claim).
We have

$$F_i(S) = \left\{\mathcal{O}_S^{n+1} \to L \cong \mathcal{O}_s \to 0, s_i \neq 0\right\},$$

but there are no conditions on the sections other than s_i.
So specifying $F_i(S)$ is equivalent to specifying $n-1$ functions $f_1 \cdots \widehat{f_i} \cdots f_n \in \mathcal{O}_S(s)$ with $f_k \neq 0$. We know this is representable by $\mathbb{A}^n$. We also know F_{ij} is obviously the same set of sequences, where now $s_j \neq 0$ as well, so we need to specify $f_0 \cdots \widehat{f_i} \cdots f_j \cdots f_n$ with $f_j \neq 0$. This is representable by $\mathbb{A}^{n-1} \times \mathbb{G}_m$, i.e. $\operatorname{Spec} k[x_1, \cdots, \widehat{x}_i, \cdots, x_n, x_j^{-1}]$. Moreover, $F_{ij} \hookrightarrow F_i$ is open.
What is the compatibility we are using to glue? For any subset $I \subset \{0, \cdots, n\}$, we can define

$$F_I = \left\{\mathcal{O}_s^{n+1} \to L \to 0, s_i \neq 0 \text{ for } i \in I\right\} = \prod i \in I_F F_i,$$

and $F_I \to F_J$ when $I \supset J$.

■

3 Functors as Spaces (Tuesday January 14th)

Last time: representability of functors, and specifically projective space $\mathbb{P}^n_{/\mathbb{Z}}$ constructed via a functor of points, i.e.

$$
\begin{aligned}
h_{\mathbb{P}^n_{/\mathbb{Z}}} &: \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set} \\
&\quad s \mapsto \mathbb{P}^n_{/\mathbb{Z}}(s) = \left\{ \mathcal{O}_s^{n+1} \to L \to 0 \right\}.
\end{aligned}
$$

for L a line bundle, up to isomorphisms of diagrams:

$$
\begin{tikzcd}
\mathcal{O}_s^{n+1} \arrow[r] \arrow[d, Rightarrow, no head] & L \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathcal{O}_s^{n+1} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}
$$

That is, line bundles with $n+1$ sections that globally generate it, up to isomorphism. The point was that for $F_i \subset \mathbb{P}^n_{/\mathbb{Z}}$ where

$$
F_i(s) = \left\{ \mathcal{O}_s^{n+1} \to L \to 0 \;\middle|\; s_i \text{ is invertible} \right\}
$$

are representable and can be glued together, and projective space represents this functor.

Remark 3.0.1: Because projective space represents this functor, there is a universal object:

$$
\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}^{n+1} \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\
 & \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1) &
\end{tikzcd}
$$

and other functors are pullbacks of the universal one. (Moduli Space)

Exercise 3.0.2 (?)
Show that $\mathbb{P}^n_{/\mathbb{Z}}$ is proper over $\operatorname{Spec} \mathbb{Z}$. Use the evaluative criterion, i.e. there is a unique lift

$$
\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}
$$

3.1 Generalizing Open Covers

Definition 3.1.1 (Equalizer)
For a category C, we say a diagram $X \to Y \rightrightarrows Z$ is an *equalizer* iff it is universal with respect to the following property:

$$
\begin{tikzcd}
X \arrow[r] & Y \arrow[r, shift left] \arrow[r, shift right] & Z \\
& S \arrow[ul, dashed, "\exists!"] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

where X is the universal object.

Example 3.1.2*(?):* For sets, $X = \left\{ y \mid f(y) = g(y) \right\}$ for $Y \xrightarrow{f,g} Z$.

Definition 3.1.3 (?)
A **coequalizer** is the dual notion,

$$
\begin{tikzcd}
& S & \\
Z \arrow[ur] \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[u] \arrow[r] & X \arrow[ul, dashed, "\exists!"']
\end{tikzcd}
$$

Example 3.1.4*(?):* Take $C = \mathrm{Sch}_{/S}$, $X_{/S}$ a scheme, and $X_\alpha \subset X$ an open cover. We can take two fiber products, $X_{\alpha\beta}, X_{\beta,\alpha}$:

$$
\begin{tikzcd}
X_\alpha \arrow[r] & X \\
X_{\alpha\beta} \arrow[u] \arrow[r] & X_\beta \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_\beta \arrow[r] & X \\
X_{\beta\alpha} \arrow[u] \arrow[r] & X_\alpha \arrow[u]
\end{tikzcd}
$$

These are canonically isomorphic.

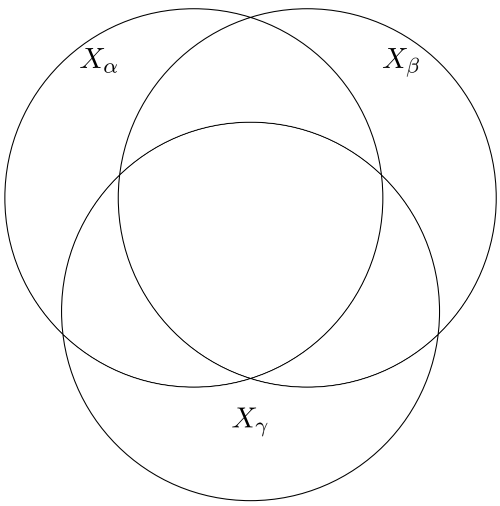
In $\mathrm{Sch}_{/S}$, we have

$$\begin{tikzcd} \coprod_{\alpha\beta} X_{\alpha\beta} \arrow[r, shift left, "f_{\alpha\beta}"] \arrow[r, shift right, "g_{\alpha\beta}"'] & \coprod_{\alpha} X_{\alpha} \arrow[r] & X \end{tikzcd}$$

where

$$\begin{aligned} f_{\alpha\beta} &: X_{\alpha\beta} \to X_\alpha \\ g_{\alpha\beta} &: X_{\alpha\beta} \to X_\beta; \end{aligned}$$

form a coequalizer. Conversely, we can glue schemes. Given $X_\alpha \to X_{\alpha\beta}$ (schemes over open subschemes), we need to check triple intersections:



Then $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ must satisfy the **cocycle condition**:

Definition 3.1.5 (Cocycle Condition)
Maps $\varphi_{\alpha\beta} : X_{\alpha\beta} \xrightarrow{\cong} X_{\beta\alpha}$ satisfy the **cocycle condition** iff

1.
$$\varphi_{\alpha\beta}^{-1}\left(X_{\beta\alpha} \cap X_{\beta\gamma}\right) = X_{\alpha\beta} \cap X_{\alpha\gamma},$$
noting that the intersection is exactly the fiber product $X_{\beta\alpha} \times_{X_\beta} X_{\beta\gamma}$.

2. The following diagram commutes:

$$\begin{tikzcd}
X_{\alpha\beta} \cap X_{\alpha\gamma} \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\alpha\beta}"'] & & X_{\gamma\alpha} \cap X_{\gamma\beta} \\
& X_{\beta\alpha} \cap X_{\beta\gamma} \arrow[ur, "\varphi_{\beta\gamma}"'] &
\end{tikzcd}$$

Then there exists a scheme $X_{/S}$ such that $\coprod_{\alpha\beta} X_{\alpha\beta} \rightrightarrows \coprod X_\alpha \to X$ is a coequalizer; this is the gluing.

Subfunctors satisfy a patching property because morphisms to schemes are locally determined. Thus representable functors (e.g. functors of points) have to be (Zariski) sheaves.

Definition 3.1.6 (Zariski Sheaf)
A functor $F : (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set}$ is a **Zariski sheaf** iff for any scheme $T_{/S}$ and any open cover T_α, the following is an equalizer:

$$F(T) \to \prod F(T_\alpha) \rightrightarrows \prod_{\alpha\beta} F(T_{\alpha\beta})$$

where the maps are given by restrictions.

Example 3.1.7*(?)*: Any representable functor is a Zariski sheaf precisely because the gluing is a coequalizer. Thus if you take the cover

$$\coprod_{\alpha\beta} T_{\alpha\beta} \to \coprod_\alpha T_\alpha \to T,$$

since giving a local map to X that agrees on intersections if enough to specify a map from $T \to X$.

Thus any functor represented by a scheme automatically satisfies the sheaf axioms.

Definition 3.1.8 (Subfunctors and Open/Closed Functors)
Suppose we have a morphism $F' \to F$ in the category $\mathrm{Fun}(\mathrm{Sch}_{/S}, \mathrm{Set})$.

- This is a **subfunctor** if $\iota(T)$ is injective for all $T_{/S}$.
- ι is **open/closed/locally closed** iff for any scheme $T_{/S}$ and any section $\xi \in F(T)$ over T, then there is an open/closed/locally closed set $U \subset T$ such that for all maps of schemes $T' \xrightarrow{f} T$, we can take the pullback $f^*\xi$ and $f^*\xi \in F'(T')$ iff f factors through U.

Remark 3.1.9: This says that we can test if pullbacks are contained in a subfunctors by checking factorization. This is the same as asking if the subfunctor F', which maps to F (noting a section is the same as a map to the functor of points), and since $T \to F$ and $F' \to F$, we can form the fiber product $F' \times_F T$:

$$\begin{tikzcd} F' \arrow[r] & F \\ F' \times_F T \arrow[u] \arrow[r, "g"] & T \arrow[u, "\xi"'] \end{tikzcd}$$

and $F' \times_F T \cong U$. Note: this is almost tautological! Thus $F' \to F$ is open/closed/locally closed iff $F' \times_F T$ is representable and g is open/closed/locally closed. I.e. base change is representable.

Exercise 3.1.10 (?)

1. If $F' \to F$ is open/closed/locally closed and F is representable, then F' is representable as an open/closed/locally closed subscheme
2. If F is representable, then open/etc subschemes yield open/etc subfunctors

Slogan 3.1.11
Treat functors as spaces.

We have a definition of open, so now we'll define coverings.

Definition 3.1.12 (Open Covers)
A collection of open subfunctors $F_\alpha \subset F$ is an **open cover** iff for any $T_{/S}$ and any section $\xi \in F(T)$, i.e. $\xi : T \to F$, the T_α in the following diagram are an open cover of T:

$$\begin{tikzcd} F_\alpha \arrow[r] & F \\ T_\alpha \arrow[u] \arrow[r] & T \arrow[u, "\xi"'] \end{tikzcd}$$

Example 3.1.13*(?):* Given

$$F(s) = \left\{ \mathcal{O}_s^{n+1} \to L \to 0 \right\}$$

and $F_i(s)$ given by those where $s_i \neq 0$ everywhere, the $F_i \to F$ are an open cover. Because the sections generate everything, taking the T_i yields an open cover.

3.2 Results About Zariski Sheaves

Proposition 3.2.1 *(?).*
A Zariski sheaf $F : (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set}$ with a representable open cover is representable.

Proof (?).
Let $F_\alpha \subset F$ be an open cover, say each F_α is representable by x_α. Form the fiber product $F_{\alpha\beta} = F_\alpha \times_F F_\beta$. Then x_β yields a section (plus some openness condition?), so $F_{\alpha\beta} = x_{\alpha\beta}$ representable. Because $F_\alpha \subset F$, the $F_{\alpha\beta} \to F_\alpha$ have the correct gluing maps.
This follows from Yoneda (schemes embed into functors), and we get maps $x_{\alpha\beta} \to x_\alpha$ satisfying the gluing conditions. Call the gluing scheme x; we'll show that x represents F. First produce a map $x \to F$ from the sheaf axioms. We have a map $\xi \in \prod_\alpha F(x_\alpha)$, and because we can pullback, we get a unique element $\xi \in F(X)$ coming from the diagram

$$F(x) \to \prod F(x_\alpha) \rightrightarrows \prod_{\alpha\beta} F(x_{\alpha\beta}).$$

■

Lemma 3.2.2 *(?).*
If $E \to F$ is a map of functors and E, F are Zariski sheaves, where there are open covers $E_\alpha \to E, F_\alpha \to F$ with commutative diagrams

$$\begin{tikzcd} E \arrow[r] & F \\ E_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

(i.e. these are isomorphisms locally), then the map is an isomorphism.

With the following diagram, we're done by the lemma:

$$\begin{tikzcd} X \arrow[r] & F \\ X_\alpha \arrow[u] \arrow[r, "\cong"] & F_\alpha \arrow[u] \end{tikzcd}$$

Example 3.2.3 *(?):* For S and E a locally free coherent $\mathcal{O}_s$ module,

$$\mathbb{P}E(T) = \{f^*E \to L \to 0\} / \sim$$

is a generalization of projectivization, then S admits a cover U_i trivializing E. Then the restriction $F_i \to \mathbb{P}E$ were $F_i(T)$ is the above set if f factors through U_i and empty otherwise. On U_i, $E \cong \mathcal{O}_{U_i}^{n_i}$, so F_i is representable by $\mathbb{P}_{U_i}^{n_i - 1}$ by the proposition. Note that this is clearly a sheaf.

Example 3.2.4 ***(?):*** For E locally free over S of rank n, take $r < n$ and consider the functor

$$\mathrm{Gr}(k, E)(T) = \{f^* E \to Q \to 0\} / \sim$$

(a Grassmannian) where Q is locally free of rank k.

Exercise 3.2.5 (?)

1. Show that this is representable
2. For the Plucker embedding

$$\mathrm{Gr}(k, E) \to \mathbb{P} \wedge^k E,$$

a section over T is given by $f^* E \to Q \to 0$ corresponding to

$$\wedge^k f^* E \to \wedge^k Q \to 0,$$

noting that the left-most term is $f^* \wedge^k E$. Show that this is a closed subfunctor.

That it's a functor is clear, that it's closed is not.

Take $S = \operatorname{Spec} k$, then E is a k-vector space V, then sections of the Grassmannian are quotients of $V \otimes \mathcal{O}$ that are free of rank n. Take the subfunctor $G_w \subset \mathrm{Gr}(k, V)$ where

$$G_w(T) = \{\mathcal{O}_T \otimes V \to Q \to 0\} \text{ with } Q \cong \mathcal{O}_t \otimes W \subset \mathcal{O}_t \otimes V.$$

If we have a splitting $V = W \oplus U$, then $G_W = \mathbb{A}(\hom(U, W))$. If you show it's closed, it follows that it's proper by the exercise at the beginning.

Thursday: Define the Hilbert functor, show it's representable. The of all flat families of subschemes.

4 | Thursday January 16th

4.1 Subfunctors

Definition 4.1.1 (Open Functors)
A functor $F' \subset F : (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathrm{Set}$ is **open** iff for all $T \xrightarrow{\xi} F$ where $T = h_T$ and $\xi \in F(T)$.

We can take fiber products:

$$\begin{tikzcd}
F' \arrow[r] & F \\
\substack{F' \times_F T \\ \text{Representable}} \arrow[u] \arrow[r, "\text{Open}"] & T \arrow[u]
\end{tikzcd}$$

So we can think of "inclusion in F" as being an *open condition*: for all $T_{/S}$ and $\xi \in F(T)$, there exists an open $U \subset T$ such that for all covers $f : T' \to T$, we have

$$F(f)(\xi) = f^*(\xi) \in F'(T')$$

iff f factors through U.

Suppose $U \subset T$ in Sch$/T$, we then have

$$h_{U/T}(T') = \begin{cases} \varnothing & T' \to T \text{ doesn't factor} \\ \{\text{pt}\} & \text{otherwise} \end{cases}.$$

which follows because the literal statement is $h_{U/T}(T') = \hom_T(T', U)$. By the definition of the fiber product,

$$(F' \times_F T)(T') = \left\{(a,b) \in F'(T) \times T(T) \;\middle|\; \xi(b) = \iota(a) \text{ in } F(T)\right\},$$

where $F' \xrightarrow{\iota} F$ and $T \xrightarrow{\xi} F$. So note that the RHS diagram here is exactly given by pullbacks, since we identify sections of F/T' as sections of F over T/T' (?).

$$\begin{tikzcd}
F' \arrow[r, "\iota"] & F & \\
F' \times_F T \arrow[u] \arrow[r] & T \arrow[u, "\xi"] & \\
& & T' \arrow[ul, "f"] \arrow[uul, bend right, "f\circ\xi"']
\end{tikzcd}$$

We can thus identify

$$(F' \times_F T)(T') = h_{U_{/S}}(T'),$$

and so for $U \subset T$ in $\text{Sch}_{/S}$ we have $h_{U_{/S}} \subset h_{T_{/S}}$ is the functor of maps that factor through U. We just identify $h_{U_{/S}}(T') = \hom_S(T', U)$ and $h_{T_{/S}}(T') = \hom_S(T', T)$.

Example 4.1.2 ***(?):*** $\mathbb{G}_m, \mathbb{G}_a$. The scheme/functor $\mathbb{G}_a$ represents giving a global function, $\mathbb{G}_m$ represents giving an invertible function.

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] & \mathbb{G}_a \\
T' \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner", very near start] & T \arrow[u, "f \in \mathcal{O}_T(T)"']
\end{tikzcd}$$

where $T' = \{f \neq 0\}$ and $\mathcal{O}_T(T)$ are global functions.

4.2 Actual Geometry: Hilbert Schemes

The best moduli space!

⚠ **Warning 4.2.1**
Unless otherwise stated, assume all schemes are Noetherian.

We want to parameterize families of subschemes over a fixed object. Fix k a field, $X_{/k}$ a scheme; we'll parameterize subschemes of X.

Definition 4.2.2 (The Hilbert Functor)
The **Hilbert functor** is given by

$$\text{Hilb}_{X_{/S}} : (\text{Sch}_{/S})^{op} \to \text{Set}$$

which sends T to closed subschemes $Z \subset X \times_S T \to T$ which are flat over T.

Here **flatness** will replace the Cartier condition:

Definition 4.2.3 (Flatness)
For $X \xrightarrow{f} Y$ and $\mathbb{F}$ a coherent sheaf on X, f is **flat** over Y iff for all $x \in X$ the stalk F_x is a flat $\mathcal{O}_{y,f(x)}$-module.

Remark 4.2.4: Note that f is flat if $\mathcal{O}_x$ is. Flatness corresponds to varying continuously. Note that everything works out if we only play with finite covers.

Remark 4.2.5: If $X_{/k}$ is projective, so $X \subset \mathbb{P}^n_k$, we have line bundles $\mathcal{O}_x(1) = \mathcal{O}(1)$. For any sheaf F over X, there is a Hilbert polynomial $P_F(n) = \chi(F(n)) \in \mathbb{Z}[n]$, i.e. we twist by $\mathcal{O}(1)$ n times. The cohomology of F isn't changed by the pushforward into $\mathbb{P}_n$ since it's a closed embedding, and so

$$\chi(X, F) = \chi(\mathbb{P}^n, i_* F) = \sum (-1)^i \dim_k H^i(\mathbb{P}^n, i_* F(n)).$$

Fact 4.2.6
For $n \gg 0$, $\dim_k H^0 = \dim M_n$, the nth graded piece of M, which is a graded module over the homogeneous coordinate ring whose $i_* F = \tilde{M}$.

In general, for L ample of X and F coherent on X, we can define a **Hilbert polynomial**,

$$P_F(n) = \chi(F \otimes L^n).$$

This is an invariant of a polarized projective variety, and in particular subschemes. Over irreducible bases, flatness corresponds to this invariant being constant.

Proposition 4.2.7 ***(?).***
For $f : X \to S$ projective, i.e. there is a factorization:

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & \mathbb{P}^n \times S \ni \mathcal{O}(1) \arrow[dl] \\ & S & \end{tikzcd}$$

If S is reduced, irreducible, locally Noetherian, then f is flat $\iff$ $P_{\mathcal{O}_{x_s}}$ is constant for all $s \in S$.

Remark 4.2.8: To be more precise, look at the base change to X_1, and the pullback of the fiber? $\mathcal{O}\Big|_{x_i}$? Note that we're not using the word "integral" here! S is flat $\iff$ the Hilbert polynomial over the fibers are constant.

Example 4.2.9 ***(?):*** The zero-dimensional subschemes $Z \in \mathbb{P}^n_k$, then P_Z is the length of Z, i.e. $\dim_k(\mathcal{O}_Z)$, and

$$P_Z(n) = \chi(\mathcal{O}_Z \otimes \mathcal{O}(n)) = \chi(\mathcal{O}_Z) = \dim_k H^0(Z; \mathcal{O}_Z) = \dim_k \mathcal{O}_Z(Z).$$

For two closed points in $\mathbb{P}^2$, $P_Z = 2$. Consider the affine chart $\mathbb{A}^2 \subset \mathbb{P}^2$, which is given by

$$\operatorname{Spec} k[x,y]/(y, x^2) \cong k[x]/(x^2)$$

and $P_Z = 2$. I.e. in flat families, it has to record how the tangent directions come together.

Example 4.2.10 ***(?):*** Consider the flat family $xy = 1$ (flat because it's an open embedding) over $k[x]$, here we have points running off to infinity.

Proposition 4.2.11 ***(Modified Characterization of Flatness for Sheaves).***
A sheaf F is flat iff P_{F_S} is constant.

4.3 Proof That Flat Sheaves Have Constant Hilbert Polynomials

Assume $S = \operatorname{Spec} A$ for A a local Noetherian domain.

Lemma 4.3.1 *(?).*
For F a coherent sheaf on $X_{/A}$ is flat, we can take the cohomology via global sections $H^0(X; F(n))$. This is an A-module, and is a free A-module for $n \gg 0$.

Proof (of lemma).
Assumed X was projective, so just take $X = \mathbb{P}^n_A$ and let F be the pushforward. There is a correspondence sending F to its ring of homogeneous sections constructed by taking the sheaf associated to the graded module

$$\sum_{n \gg 0} H^0(\mathbb{P}^m_A; F(n)) = \bigoplus_{n \gg 0} H^0(\mathbb{P}^m_A; F(n))$$

and taking the associated sheaf ($Y \mapsto \tilde{Y}$, as per Hartshorne's notation) which is free, and thus F is free. [a]

Conversely, take an affine cover U_i of X. We can compute the cohomology using Čech cohomology, i.e. taking the Čech resolution. We can also assume $H^i(\mathbb{P}^m; F(n)) = 0$ for $n \gg 0$, and the Čech complex vanishes in high enough degree. But then there is an exact sequence

$$0 \to H^0(\mathbb{P}^m; F(n)) \to \mathcal{C}^0(\underline{U}; F(n)) \to \cdots \to C^m(\underline{U}; F(n)) \to 0.$$

Assuming F is flat, and using the fact that flatness is a 2 out of 3 property, the images of these maps are all flat by induction from the right. Finally, local Noetherian and finitely generated flat implies free.

■

[a] See tilde construction in Hartshorne, essentially amounts to localizing free tings.

By the lemma, we want to show $H^0(\mathbb{P}^m; F(n))$ is free for $n \gg 0$ iff the Hilbert polynomials on the fibers P_{F_S} are all constant.

Claim 1: It suffices to show that for each point $s \in \operatorname{Spec} A$, we have

$$H^0(X_s; F_S(n)) = H^0(X; F(n)) \otimes k(S)$$

for $k(S)$ the residue field, for $n \gg 0$.

Claim 2: P_{F_S} measures the rank of the LHS.

Proof (of claim 2).
$\Longrightarrow$: The dimension of RHS is constant, whereas the LHS equals $P_{F_S}(n)$.

$\Longleftarrow$: If the dimension of the RHS is constant, so the LHS is free. ■

For a f.g. module over a local ring, testing if localization at closed point and generic point have the same rank. For M a finitely generated module over A, we find that

$$0 \to A^n \to M \to Q$$

is surjective after tensoring with $\mathrm{Frac}(A)$, and tensoring with $k(S)$ for a closed point, if $\dim A^n = \dim M$ then $Q = 0$.

Proof (of claim 1).
By localizing, we can assume s is a closed point. Since A is Noetherian, its ideal is f.g. and we have

$$A^m \to A \to k(S) \to 0.$$

We can tensor with F (viewed as restricting to fiber) to obtain

$$F(n)^m \to F(n) \to F_S(n) \to 0.$$

Because F is flat, this is still exact. We can take $H^*(x, \cdot)$, and for $n \gg 0$ only H^0 survives. This is the same as tensoring with $H^0(x, F(n))$. ■

Definition 4.3.2 (Hilbert Polynomial Subfunctor)
Given a polynomial $P \in \mathbb{Z}[n]$ for $X_{/S}$ projective, we define a subfunctor by picking only those with Hilbert polynomial p fiberwise as $\mathrm{Hilb}_{X_{/S}}^P \subset \mathrm{Hilb}_{X_{/S}}$. This is given by $Z \subset X \times_S T$ with $P_Z = P$.

Theorem 4.3.3 ***(Grothendieck).***
If S is Noetherian and $X_{/S}$ projective, then $\mathrm{Hilb}_{X_{/S}}^P$ is representable by a projective S-scheme.

See ***cycle spaces*** *in analytic geometry.*

5 | Hilbert Polynomials (Thursday January 23)

Some facts about the Hilbert polynomial:

1. For a subscheme $Z \subset \mathbb{P}_k^n$ with $\deg P_z = \dim Z = n$, then

$$p_z(t) = \deg z \frac{t^n}{n!} + O(t^{n-1}).$$

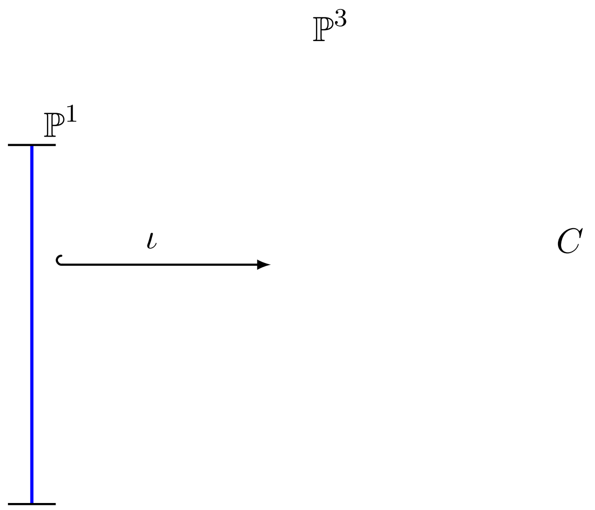
2. We have $p_z(t) = \chi(\mathcal{O}_z(t))$, consider the sequence

$$0 \to I_z(t) \to \mathcal{O}_{\mathbb{P}^n}^{(t)} \to \mathcal{O}_z^{(t)} \to 0,$$

then $\chi(I_z(t)) = \dim H^0(\mathbb{P}^n, J_z(t))$ for $t \gg 0$, and $p_z(0)$ is the Euler characteristic of $\mathcal{O}_Z$.

Remark 5.0.1: Keywords to look up here: Serre vanishing, Riemann-Roch, ideal sheaf.

Example 5.0.2 ***(The twisted cubic):***



Then

$$p_C(t) = (\deg C)t + \chi(\mathcal{O}_{\mathbb{P}^1}) = 3t + 1.$$

5.0.1 Hypersurfaces

Recall that length 2 subschemes of $\mathbb{P}^1$ are the same as specifying quadratics that cut them out, each such $Z \subset \mathbb{P}^1$ satisfies $Z = V(f)$ where $\deg f = d$ and f is homogeneous. So we'll be looking at $\mathbb{P}H^0(\mathbb{P}^n_k, \mathcal{O}(d))^\vee$, and the guess would be that this is $\mathrm{Hilb}_{\mathbb{P}^n_k}$ Resolve the structure sheaf

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_D(t) \to 0.$$

so we can twist to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^n}(t-d) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_D(t) \to 0.$$

Then

$$\chi(\mathcal{O}_D(t)) = \chi(\mathcal{O}_{\mathbb{P}^n}(t)) - \chi(\mathcal{O}_{\mathbb{P}^n}(t-d)),$$

which is

$$\binom{n+t}{n} - \binom{n+t-d}{n} = \frac{dt^{n-1}}{(n-1)!} + O(t^{n-2}).$$

Lemma 5.0.3 *(?).*
Anything with the Hilbert polynomial of a degree d hypersurface is in fact a degree d hypersurface.

We want to write a morphism of functors

$$\mathrm{Hilb}_{\mathbb{P}^n_k}^{P_{n,d}} \to \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee.$$

which sends flat families to families of equations cutting them out. Want

$$Z \subset \mathbb{P}^n \times S \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee \to L \to 0.$$

This happens iff

$$0 \to L^\vee \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$$

with torsion-free quotient. Note that we use $L^\vee$ instead of $\mathcal{O}_s$ because of scaling. We have

$$\begin{aligned} 0 \to I_z \to \mathcal{O}_{\mathbb{P}^n \times S} \to \mathcal{O}_z \to 0 & \\ 0 \to I_z(d) \to \mathcal{O}_{\mathbb{P}^n \times S}(d) \to \mathcal{O}_z(d) \to 0 & \quad \text{by twisting.} \end{aligned}$$

We then consider $\pi_s : \mathbb{P}^n \times S \to S$, and apply the pushforward to the above sequence. Notie that it is not right-exact:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \pi_{s*} I_z(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_{\mathbb{P}^n \times S}(d) \arrow[r] \arrow[d, equal] & \pi_{s*}\mathcal{O}_z(d) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) L^\vee = \arrow[r] & \text{locally free} \arrow[r] & 0 & \end{tikzcd}$$

Note: above diagram may be off horizontally?

This equality follows from flatness, cohomology, and base change. In particular, we need the following:

Fact 5.0.4
The scheme-theoretic fibers, given by $H^0(\mathbb{P}^n, I_z(d))$ and $H^0(\mathbb{P}^n, \mathcal{O}_z(d))$, are all the same dimension.

Using

1. Cohomology and base change, i.e. for $X \xrightarrow{f} Y$ a map of Noetherian schemes (or just finite-type) and F a sheaf on X which is flat over Y, there is a natural map (not usually an isomorphism)

$$R^i f_* f \otimes k(y) \to H^i(x_y, F|_{x_y}),$$

but is an isomorphism if $\dim H^i(x_y, F|_{x_y})$ is constant, in which case $R^i f_* f$ is locally free.

2. If $Z \subset \mathbb{P}^n_k$ is a degree d hypersurface, then independently we know

$$\dim H^0(\mathbb{P}^n, I_z(d)) = 1 \text{ and } \dim H^0(\mathbb{P}^n, \mathcal{O}_z(d)) = \binom{d+n}{n} - 1.$$

To get a map going backwards, we take the universal degree 2 polynomial and form

$$V(a_{00}x_0^2 + a_{11}x_1^2 + a_{12}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2) \subset \mathbb{P}^2 \times \mathbb{P}^5.$$

5.0.2 Example: Twisted Cubics

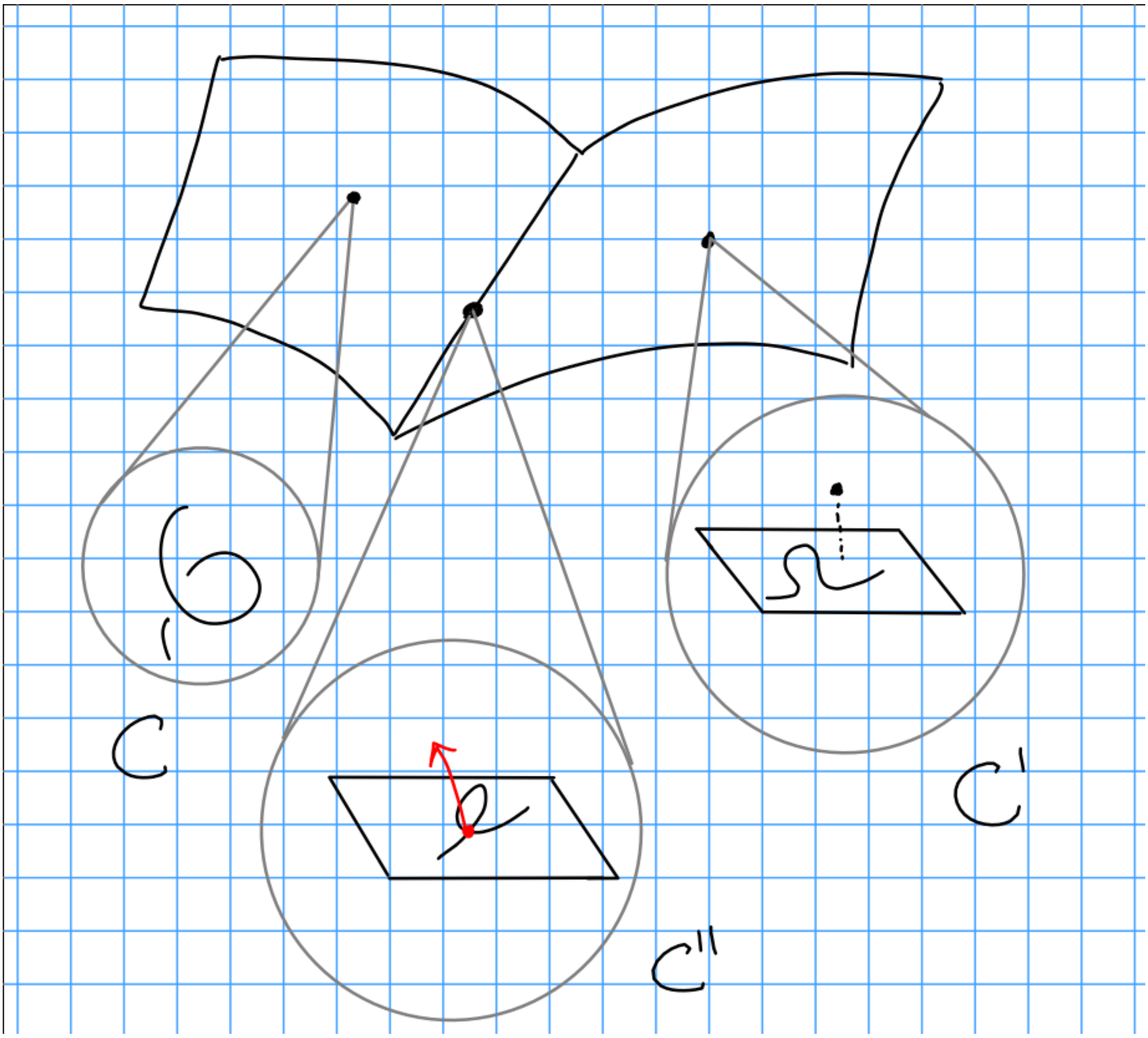
Consider a map $\mathbb{P}^1 \to \mathbb{P}^3$ obtained by taking a basis of a homogeneous cubic polynomial. The canonical example is

$$(x, y) \to (x^3, x^2y, xy^2, y^3).$$

Then

$$P_C(t) = 3t + 1$$

and $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_k}$ has a component with generic point a twisted cubic, and another component with points a curve disjoint union a point, and the overlap are nodal curves with a "fat" 3-dimensional point:



Then $P_{C'} = 1 + \tilde{P}$, the Hilbert polynomial of just the base without the disjoint point, so this equals $1 + P_{2,3} = 1 + (3t + 0) = 3t + 1$. For $P_{C''}$, we take the sequence

$$0 \to k \to \mathcal{O}_{C''} \to \mathcal{O}_{C''\text{reduced}} \to 0,$$

so

$$P_{C''} = 1 + P_{C''\text{red}} = 3t + 1.$$

Remark 5.0.5: Note that flat families *must* have the same (constant) Hilbert polynomial.

Note that we can get paths in this space from $C \to C''$ and $C' \to C''$ by collapsing a twisted cubic onto a plane, and sending a disjoint point crashing into the node on a nodal cubic. We're mapping $\mathbb{P}^1 \to \mathbb{P}^3$, and there is a natural action of $\mathrm{PGL}(4) \curvearrowright \mathbb{P}^3$, so we get a map

$$\mathrm{PGL}(4) \times \mathbb{P}^3 \to \mathbb{P}^3.$$

Let $c \in \mathbb{P}^3$ and let $\mathcal{C}$ be the preimage. This induces (?) a map

$$\mathrm{PGL}(4) \to \mathrm{Hilb}_{\mathbb{P}^3}^{3t+1}$$

where the fiber over $[C]$ in the latter is $\mathrm{PGL}(2) = \mathrm{Aut}(\mathbb{P}^1)$. By dimension counting, we find that the dimension of the twisted cubic component is $15 - 3 = 12$. The 15 in the other component comes from 3-dim choices of plane, 3-dim choices of a disjoint point, and

$$\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))^\vee \cong \mathbb{P}^9,$$

yielding 15 dimensions. To show that these are actually different components, we use Zariski tangent spaces. Let T_1 be the tangent space of the twisted cubic component, then

$$\dim T_1 \,\mathrm{Hilb}_{\mathbb{P}^3_k}^{3t+1} = 12,$$

and similarly the dimension of the tangent space over the C' component is 15.

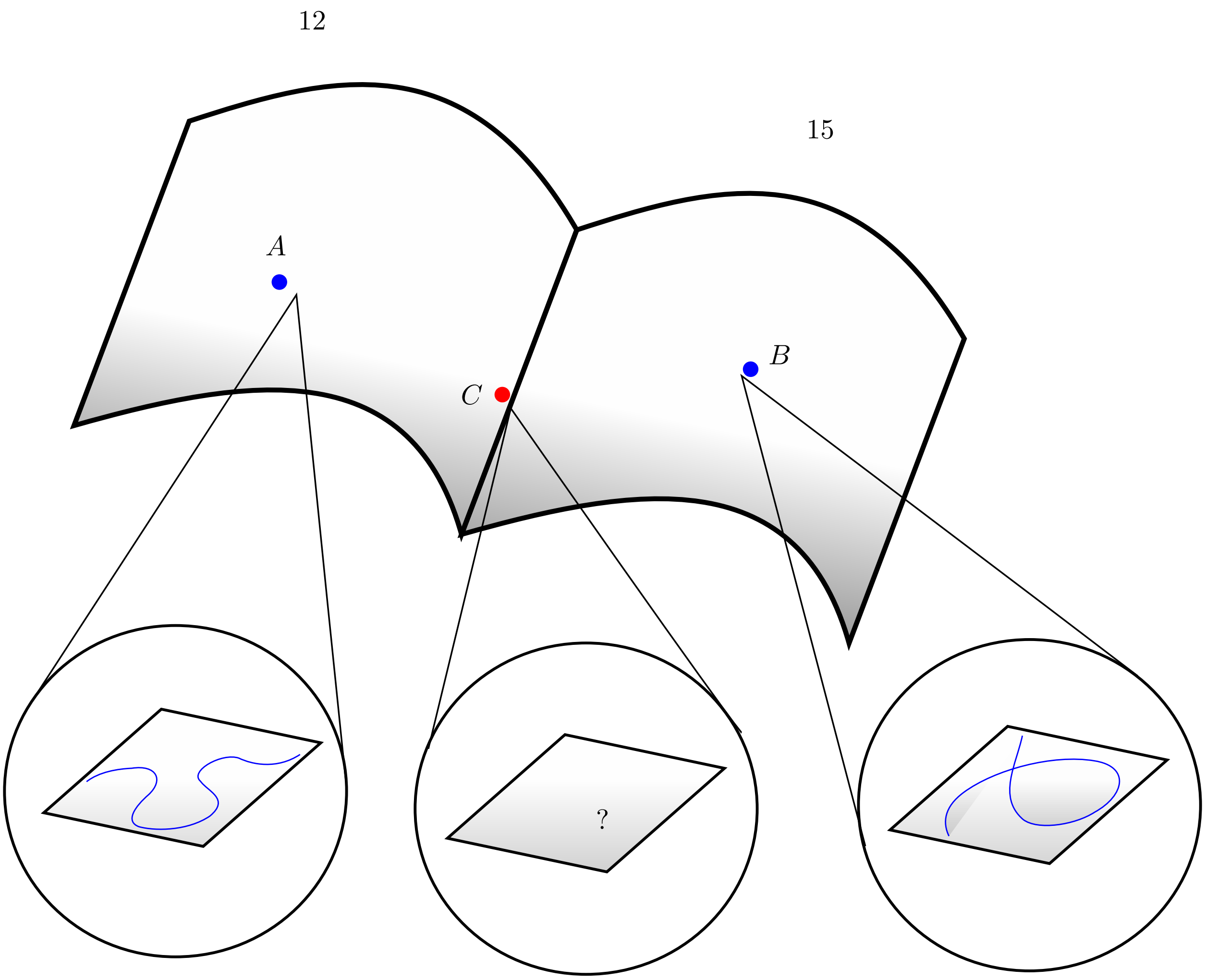
Fact 5.0.6
Let A be Noetherian and local, then the dimension of the Zariski tangent space, $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$, the Krull dimension. If this is an equality, then A is regular.

Slogan 5.0.7
Dimensions of tangent spaces give an upper bound.

Proposition 5.0.8 ***(?).***
If $X_{/k}$ is projective and P is a Hilbert polynomial, then $[Z] \in \mathrm{Hilb}_{X_{/k}}^P$, i.e. a closed subscheme of X with Hilbert polynomial p (note there's an ample bundle floating around) then the tangent space is $\hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

6 Hilbert Schemes of Hypersurfaces (Tuesday January 28th)

Last time: Twisted cubics, given by $\mathrm{Hilb}_{\mathbb{P}^3_k}^{3t+1}$.



Components of the Scheme of Cubic Curves.

We got lower (?) bounds on the dimension by constructing families, but want an exact dimension. The following will be a key fact:

Proposition 6.0.1 ***(?).***
Let $Z \subset X$ be a closed k-dimensional subspace. For $[z] \in \mathrm{Hilb}^P_{X_{/k}}(k)$, we have an identification of the Zariski tangent space

$$T_{[z]} \mathrm{Hilb}^P_{X_{/k}} = \hom_{\mathcal{O}_X}(I_z, \mathcal{O}_Z)$$

Say

$$F : (\mathrm{Sch}_{/k})^{\mathrm{op}} \to \mathrm{Set}$$

is a functor and let $x \in F(k)$. There is an inclusion $i : \operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$ and an induced map

$$\begin{gathered} F(\operatorname{Spec} k[\varepsilon]) \xrightarrow{i^*} F(\operatorname{Spec} k) \\ T_x F \coloneqq (i^*)^{-1}(x) \mapsto x \end{gathered}$$

So if F is represented by a scheme $H_{/k}$, then

$$T_x h_J = T_x H = (\mathfrak{m}_x / \mathfrak{m}_x^2)^\vee \text{ over } k$$

Will need a criterion for flatness later, esp. for Artinian thickenings.

Lemma 6.0.2 ***(?).***
Assume A' is a Noetherian ring and $0 \to J \to A' \to A \to 0$ with $J^2 = 0$. Assume we have $X'_{/\operatorname{Spec} A'}$, and a coherent sheaf F' on X', where X' is Noetherian. Then F' is flat over A' iff

1. F is flat
2. $0 \to F \otimes_A J \to F'$ is exact.

$$\begin{tikzcd} F & F' \\ X \coloneqq \operatorname{Spec} A' \times_{\operatorname{Spec} A} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}$$

6.0.1 Sketch Proof of Lemma

Take the first exact sequence and tensor with F' (which is right-exact), then $J \otimes_{A'} F' = J \otimes_A$ canonically. This follows because $J = J \otimes_{A'} A$, and there is an isomorphism $J \otimes_{A'} A' \to J \otimes_{A'} A$. And $F = F' \otimes_{A'} A$ is a pullback of F'. If flat, then tensoring is exact. Note that both conditions in the lemma are necessary since pullbacks of flats are flat by (1), and (2) gives the flatness condition.

Definition 6.0.3 (Flat Modules)
Recall that for a module over a Noetherian ring, M/A, M is **flat** over A iff

$$\operatorname{Tor}_1^A(M, A/p) = 0 \qquad \text{for all primes } p.$$

Remark 6.0.4: Reason: Tor commutes with direct limits, so M is flat iff

$$\operatorname{Tor}_1^A(M, N) = 0 \qquad \text{for all finitely generated } N.$$

Since A is Noetherian, N has a finite filtration $N^\cdot$ where $N_i/N_{i+1} \cong A/p_i$. Use the fact that every ideal is contained in a prime ideal. Take $x \in N$, this yields a map $A \to N$ which factors through

A/I. If we make such a filtration on A/I, then we can quotient N by $\operatorname{im} f$ where $f : A/I \to N$. Continuing inductively, the resulting filtration must stabilize. So we can assume $N = A/I$. Then I is contained in a maximal.

Exercise 6.0.5 (?)
Finish proof. See Aatiyah Macdonald.

6.0.2 Proof of Proposition

Proof (of proposition, given lemma).
So it's enough to show that $\operatorname{Tor}_1^{A'}(F', A'/p') = 0$ for all primes $p' \subset A'$.
Observation
Since J is nilpotent, $J \subset p'$.

6.1 Consequences of Proof

Let $p = p'/J$, this is a prime ideal. We have an exact diagram by taking quotients:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r] & p' \arrow[r] \arrow[d] & p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & A'/p' \arrow[d] & A/p \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

So we can tensor with F' everywhere, and get a map from kernels to cokernels using the snake lemma:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \operatorname{Tor}(A, F) = 0 \arrow[d] & \\
 & 0 \arrow[r, "\text{snake}"] \arrow[d] & \operatorname{Tor}_1^{A_1}(A'/p', F') \arrow[d] & \operatorname{Tor}_1^{A_1}(A/p, F') \arrow[d] & \\
0 \arrow[r] & J \otimes_{A'} F' \arrow[r, hook, "\text{by commuting square}"] & p' \otimes_{A'} F' \arrow[r] \arrow[d] & p \otimes_{A'} F' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \otimes_{A'} F' \arrow[r, hook, "\text{by (2)}"'] & A' \otimes_{A'} F' \arrow[r] \arrow[d] & A \otimes_{A'} F' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r, "\text{snake}"] & A'/p' \otimes_{A'} F' \arrow[r, "="] \arrow[d] & A/p \otimes_{A'} F' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Then by (1), we have

$$\operatorname{Tor}_1^{A'}(A'/p', F') = \operatorname{Tor}_1^{A'}(A/p, F') = 0.$$

■

We will just need this for $A' = k[\varepsilon]$ and $A = k$.

Proposition 6.1.1 *(?).*

$$T_z \operatorname{Hilb}_{X_{/k}} = \hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z).$$

Proof (?).
Again we have $T_z \operatorname{Hilb}_{X_{/k}} \subset \operatorname{Hilb}_{X_{/k}}(k[\varepsilon])$, and is given by

$$\left\{ Z' \subset X \times_{\operatorname{Spec} k} \operatorname{Spec} k[\varepsilon] \;\middle|\; Z' \text{ is flat}_{/k[\varepsilon]},\ Z' \times_{\operatorname{Spec} k[\varepsilon]} \operatorname{Spec} k = Z \right\}.$$

We have an exact diagram:

$$
\begin{tikzcd}
0 \arrow[r] & I_{Z'} \arrow[r] & \mathcal{O}_{X[\varepsilon]} \arrow[r] & \mathcal{O}_{Z'} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[d] & & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
k \arrow[d] & {} \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_x \arrow[r] \arrow[d] & \mathcal{O}_z \arrow[r] \arrow[d] & {} \\
k[\varepsilon] \arrow[d] & {} \arrow[r] & I_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_{x[\varepsilon]} \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & {} \\
k \arrow[d] & {} \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_x \arrow[r] \arrow[d] \arrow[u, dotted, bend right] & \mathcal{O}_Z \arrow[r] \arrow[d] & {} \\
0 & & {} & {} & {} &
\end{tikzcd}
$$

Note the existence of a splitting above. Given $\varphi \in \hom_{\mathcal{O}_x}(I_Z, \mathcal{O}_Z)$. We have

$$
I_{Z'} = \left\{ f + \varepsilon g \;\middle|\; \begin{array}{l} f, g \\ \in I_Z, \\ \varphi(f) \\ = g \pmod{I}_Z, \\ \varphi(f) \\ \in \mathcal{O}_Z, \\ g \pmod{I}_Z \\ \in \mathcal{O}_x / I_Z = \mathcal{O}_Z \end{array} \right\}.
$$

It's easy to see that $Z' \subset x'$, and

1. $Z' \times k = Z$
2. It's flat over $k[\varepsilon]$, looking at $0 \to k \otimes I_{Z'} \to I_{Z'}$.

For the converse, take $f \in I_Z$ and lift to $f' = f + \varepsilon g \in I_{Z'}$, then $g \in \mathcal{O}_x$ is well-defined wrt I_Z. Then $g \in \hom_{\mathcal{O}_x}(I_z, \mathcal{O}_z)$.

■

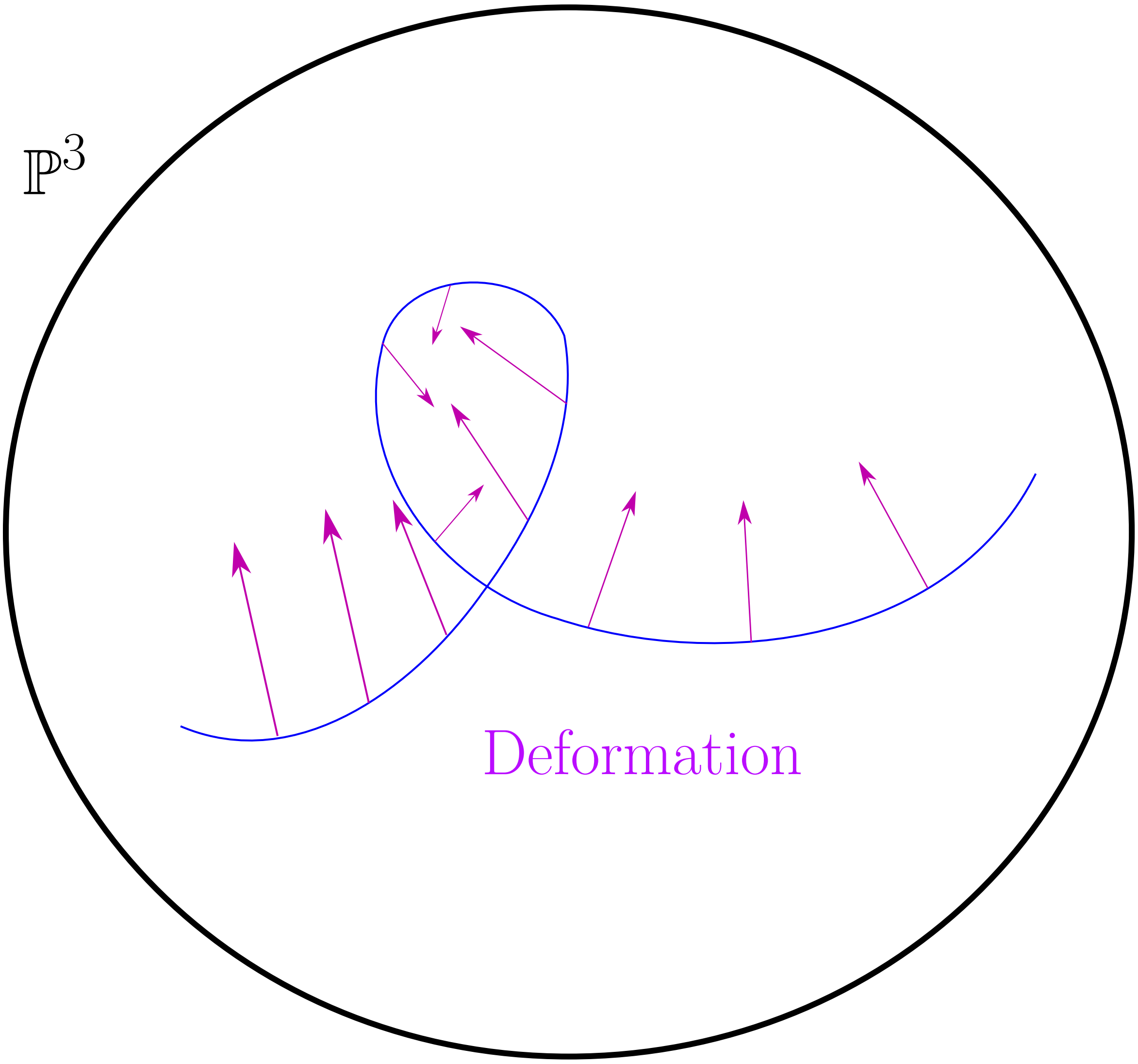
The main point here is that these hom sets are extremely computable.

Example 6.1.2 ***(?):*** Let Z be a twisted cubic in $\mathrm{Hilb}^{3t+1}_{\mathbb{P}^3_{/k}}(k)$.

Observation 6.1.3

$$
\hom_{\mathcal{O}_x}(I_Z, \mathcal{O}_Z) = \hom_{\mathcal{O}_X}(I_Z/I_Z^2, \mathcal{O}_Z) = \hom_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)
$$

If I_Z/I_Z^2 is locally free, these are global sections of the dual, i.e. $H^0((I_Z/I_Z^2)^\vee)$. In this case, $Z \hookrightarrow X$ is regularly embedded, and thus $(I_Z/I_Z^2)^\vee$ should be regarded as the normal bundle. Sections of the normal bundle match up with directions to take first-order deformations:



For $i : C \hookrightarrow \mathbb{P}^3$, there is an exact sequence

$$0 \to I/I^2 \to i^*\Omega_{\mathbb{P}^3} \to \Omega_\varepsilon \to 0$$
$$\Downarrow \quad \text{taking duals}$$
$$0 \to T_C \to i^*T_{\mathbb{P}^3} \to N_{C_{/\mathbb{P}^3}} \to 0,$$

How do we compute $T_{\mathbb{P}^3}$? Fit into the exact sequence

$$0 \to \mathcal{O} \to i^*\mathcal{O}(1)^4 \to i^*T_{\mathbb{P}^3} \to 0,$$

which we can restrict to C.

We have $i^* \mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$, so

$$\begin{gathered} 0 \to H^0 \mathcal{O}_c \to H^*(\mathcal{O}(3)^4) \to H^0(i^* T_{\mathbb{P}^3}) \to 0 \\ \Downarrow \\ 0 \to k \to k^{16} \to k^{15} \to 0. \end{gathered}$$

This yields

$$\begin{gathered} 0 \to H^0(T_c) \to H^0(i^* T_{\mathbb{P}^3}) \to H^0(N_{C_{/\mathbb{P}^3}}) \to H^1 T_c \\ \Downarrow \\ 0 \to k^3 \to k^{15} \to k^{12} \to 0 \end{gathered}$$

Example 6.1.4*(?):* $\mathrm{Hilb}_{\mathbb{P}^n_k}^{P_?} \cong \mathbb{P} H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee$ which has dimension $\binom{n+1}{n} - 1$. Pick Z a k point in this Hilbert scheme, then $T_Z H = \hom(I_Z, \mathcal{O}_Z)$. Since $I_Z \cong \mathcal{O}_{\mathbb{P}}(-d)$ which fits into

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Z \to 0.$$

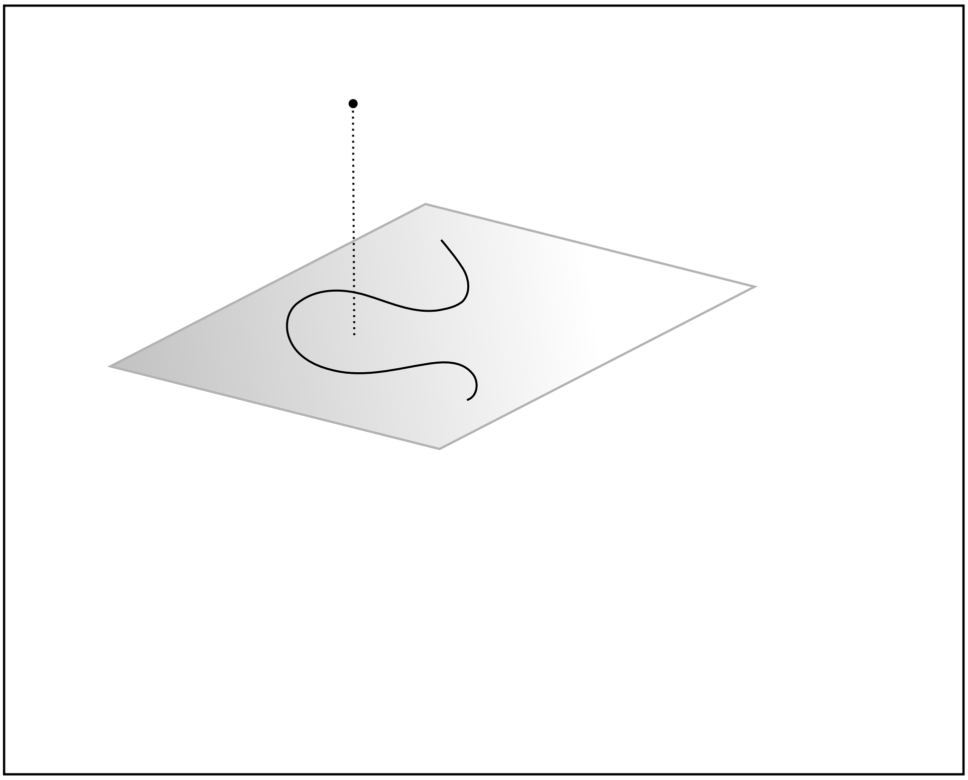
We can identify

$$\hom(I_Z, \mathcal{O}_Z) = H^0((I_Z/I_Z^2)^\vee) = H^0(\mathcal{O}_Z(d)).$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_Z(d) \longrightarrow 0$$

$$\begin{array}{lccccccccc} & 0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & H^0(\mathcal{O}_Z(d)) & \longrightarrow & 0 \\ \text{dim:} & & & k & & k^{\binom{n+d}{n}} & & k^{\binom{n+d}{n}-1} & & \end{array}$$

Example 6.1.5*(?):* The tangent space of the following cubic:



We can identify

$$\hom_{\mathcal{O}_k}(I_Z, \mathcal{O}_Z) = H^0((I_Z/I_Z^2)^\vee) = 3 + H^0((I_{Z_0}/I_{Z_0}^2)^\vee),$$

where the latter equals $H^0\left(\mathcal{O}_1\Big|_{z_0} \oplus \mathcal{O}(\zeta)\Big|_{z_0}\right)$ yielding

$$3 + 9 = 12.$$

7 Uniform Vanishing Statements (Thursday January 30th)

Recall how we constructed the Hilbert scheme of hypersurfaces

$$\mathrm{Hilb}^{P_{m,d}}_{\mathbb{P}^n_k} = \mathbb{P}H^0(\mathbb{P}^n; \mathcal{O}(d))^\vee$$

A section $\mathrm{Hilb}^{P}_{\mathbb{P}^n_k}(s)$ corresponds to $z \in \mathbb{P}^n_s$. We can look at the exact sequence

$$0 \to I_Z(m) \to \mathcal{O}_{\mathbb{P}^n_S} \xrightarrow{\text{restrict}} \mathcal{O}_z(m) \to 0.$$

as $\mathbb{P}^n_s \xrightarrow{\pi_s} S$, so we can pushforward along π, which is left-exact, so

$$0 \to \pi_{s*} I_Z(m) \to \pi_{s*} \mathcal{O}_{\mathbb{P}^n_S} = \mathcal{O}_S \otimes H^0(\mathbb{P}^n; \mathcal{O}(m)) \to \mathcal{O}_z(m) \to R^1 \pi_{s*} I_Z(m) \to \cdots.$$

Idea: $Z \subset \mathbb{P}^n_k$ will be determined (in families!) by the space of degree d polynomials vanishing on Z (?), i.e.

$$H^0(\mathbb{P}^n, I_z(m)) \subset H^0(\mathbb{P}^n, \mathcal{O}(m))$$

for m very large. This would give a map of functors

$$\mathrm{Hilb}^P_{\mathbb{P}^n_k} \to \mathrm{Gr}(N, H^0(\mathbb{P}^n, \mathcal{O}(m))).$$

If this is a closed subfunctor, a closed subfunctor of a representable functor is representable and we're done .

Remark 7.0.1: We need to get an m uniform in Z, and more concretely:

1. First need to make sense of what it means for Z to be determined by $H^0(\mathbb{P}^n, I_Z(m))$ for m only depending on P.

2. This works point by point, but we need to do this in families. I.e. we'll use the previous exact sequence, and want the R^1 to vanish.

Slogan 7.0.2
We need *uniform* vanishing statements. There is a convenient way to package the vanishing requirements needed here. From now on, take $k = \bar{k}$ and $\mathbb{P}^n = \mathbb{P}^n_k$.

7.1 m-Regularity

Definition 7.1.1 (m-Regularity of Coherent Sheaves)
A coherent sheaf F on $\mathbb{P}^n$ is m**-regular** if $H^i(\mathbb{P}^n; F(m-i)) = 0$ for all $i > 0$.

Example 7.1.2 ***(?):*** Consider $\mathcal{O}_{\mathbb{P}^n}$, this is 0-regular. Line bundles on $\mathbb{P}_n$ only have 0 and top cohomology. Just need to check that $H^n(\mathbb{P}^n; \mathcal{O}(-n)) = 0$, but by Serre duality this is

$$H^0(\mathbb{P}^n; \mathcal{O}(n) \otimes \omega_{\mathbb{P}^n})^\vee = H^0(\mathbb{P}^n; \mathcal{O}(-1))^\vee = 0.$$

Proposition 7.1.3 ***(?).***
Assume F is m-regular. Then

1. There is a natural multiplication map from linear forms on $\mathbb{P}^n$,

$$H^0(\mathbb{P}^n; \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n; F(k)) \to H^0(\mathbb{P}^n; F(k+1)),$$

which is surjective for $k \geq n$.[a]

2. F is m'-regular for $m' \geq m$.

3. $F(k)$ is globally generated for $k \geq m$, i.e. the restriction

$$H^0(\mathbb{P}^n; F(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(k) \to 0$$

is exact (i.e. surjective).

[a] Think of this as a graded module, this tells you the lowest number of small grade pieces needed to determine the entire thing.

Example 7.1.4 *(?):* $\mathcal{O}$ is m-regular for $m \geq 0$ implies $\mathcal{O}(k)$ is $-k$-regular and is also m-regular for$m \geq -k$.

7.1.1 Proof of 2 and 3

Induction on dimension of n in $\mathbb{P}^n$. Coherent sheaves on $\mathbb{P}^0$ are vector spaces, so no higher cohomology.

Proof (Step 1).
Take a generic hyperplane $H \subset \mathbb{P}^n$, there is an exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0.$$

where $\mathcal{O}_H$ is the structure sheaf. Tensoring with H remains exact, so we get

$$0 \to F(-1) \to F \to F_H \to 0.$$

Why? $\mathbb{A}^n \subset \mathbb{P}^n$, let $A = \mathcal{O}_{\mathbb{P}^n}(\mathbb{A}^n)$ be the polynomial ring over $\mathbb{A}^n$. Then the restriction of the first sequence to $\mathbb{A}^n$ yields

$$0 \to A \xrightarrow{f} A \to A/f \to 0,$$

and thus we want

$$F \xrightarrow{f} F \to F/fF \to 0$$

which results after restricting the second sequence to $\mathbb{A}^n$. Thus we just want f to not be a zero divisor. If we take f not vanishing on any associated point of F, then this will be exact. Associated points: generic points arising by supports of sections of F. F is coherent, so it has finitely many associated points. If H does not contain any of the associated points of F, then the second sequence is indeed exact.

■

Proof (Step 2).
Twist up by k to obtain

$$0 \to F(k-1) \to F(k) \to F_H(k) \to 0.$$

Look at the LES in cohomology to get

$$H^i(F(m-i)) \to H^i(F_H(m-i)) \to H^{i+1}(F(m-(i+1))).$$

So F_H is m-regular. By induction, this proves statements 1 and 2 for all F_H. So take $k = m+1-i$ and consider

$$H^i(F(m-i)) \to H^i(F(m+1-i)) \to H^i(F_H(m+1-i)).$$

We know 2 is satisfied, so the RHS is zero, and we know the LHS is zero, so the middle term is zero. Thus F itself is $m+1$ regular, and by inducting on m we get statement 2.

■

By multiplication maps, we get a commutative diagram:

$$\begin{tikzcd}
 & H^0(\mathcal{O}(1)) \otimes H^0(F(k)) \arrow[rr] \arrow[d, "\beta"] \arrow[drr] & & H^0(\mathcal{O}(1)) \otimes H^0(F_H(k)) \arrow[d] \\
H^0(F(k)) \arrow[ur, "H\otimes\mathrm{id}"] \arrow[r, "H"] & H^0(F(k+1)) \arrow[rr, dashed, "\alpha"] & & H^0(F_H(k+1))
\end{tikzcd}$$

We'd like to show the diagonal map is surjective.

Observation 7.1.5

1. The top map is a surjection, since

$$H^0(F(k)) \to H^0(F_H(k)) \to H^1(F(k-1)) = 0$$

   for $k \geq m$ by (2).

2. The right-hand map is surjective for $k \geq m$.

3. $\ker(\alpha) \subset \operatorname{im}(\beta)$ by a small diagram chase, so β is surjective.

This shows (1) and (2) completely.

Proof (of 3).
We know $F(k)$ is globally generated for $k \gg 0$. Thus for all $k \geq m$, $F(k)$ is globally generated by (1).

■

Theorem 7.1.6 *(?)*.
Let $P \in \mathbb{Q}[t]$ be a Hilbert polynomial. There exists an m_0 only depending on P such that for all subschemes $Z \subset \mathbb{P}^n_k$ with Hilbert polynomial $P_Z = P$, the ideal sheaf I_z is m_0-regular.

7.1.2 Proof of Theorem

Induct on n. For $n = 0$, again clear because higher cohomology vanishes and there are no nontrivial subschemes. For a fixed Z, pick H in $\mathbb{P}^n$ (and setting $I \coloneqq I_z$ for notation) such that

$$0 \to I(-1) \to I \to I_H \to 0.$$

is exact. Note that the Hilbert polynomial $P_{I_H}(t) = P_I(t) - P_I(t-1)$ and $P_I = P_{\mathcal{O}_{\mathbb{P}^n}} - P_Z$. By induction, there exists some m_1 depending only on P such that I_H is m_1-regular. We get

$$H^{i-1}(I_H(k)) \to H^i(I(k-1)) \to H^i(I(k)) \to H^i(I_H(k)),$$

and for $k \geq m_1 - i$ the LHS and RHS vanish so we get an isomorphism in the middle. By Serre vanishing, for $k \gg 0$ we have $H^i(I(k)) = 0$ and thus $H^i(I(k)) = 0$ for $k \geq m_i - i$. This works for all $i > 1$, we have $H^i(I(m_i - i)) = 0$. We now need to find $m_0 \geq m_1$ such that $H^1(I(m_0 - 1)) = 0$ (trickiest part of the proof).

Lemma 7.1.7 *(?)*.
The sequence $\left(\dim H^1(I(k))\right)_{k \geq m_i - 1}$ is *strictly* decreasing.[a]

[a] Note: $h^1 = \dim H^1$.

Remark 7.1.8: Given the lemma, it's enough to take $m_0 \geq m_1 + h^1(I(m_1 - 1))$. Consider the LES we have a surjection

$$H^0(\mathcal{O}_Z(m_1 - 1)) \to H^1(I(m_1 - 1)) \to 0.$$

So the dimension of the LHS is equal to $P_Z(m_1 - 1)$, using the fact that terms vanish and make the Euler characteristic equal to P_Z. Thus we can take $m_0 = m_1 + P(m_1 - 1)$.

Proof (of Lemma).
Considering the LES

$$H^0(I(k+1)) \xrightarrow{\alpha_{k+1}} H^0(I_H(k+1)) \to H^1(I(k)) \to H^1(I(k+1)) \to 0,$$

where the last term is zero because I_H is m_1-regular. So the sequence $h^1(I(k))$ is non-increasing.
Observation
If it does *not* strictly decrease for some k, then there is an equality on the RHS, which makes α_{k+1} surjective. This means that α_{k+2} is surjective, since

$$H^0(\mathcal{O}(1)) \otimes H^0(I_H(k+1)) \twoheadrightarrow H^0(I_H(k+2)).$$

So if one is surjective, everything above it is surjective, but by Serre vanishing we eventually get zeros. So α_{k+i} is surjective for all $i \geq 1$, contradicting Serre vanishing, since the RHS are isomorphisms for all k.

■

Thus for any $Z \subset \mathbb{P}^n_k$ with $P_Z = P$, we uniformly know that I_Z is m_0-regular for some m_0 depending only on P.

Claim: Z is determined by the degree m_0 polynomials vanishing on Z, i.e. $H^0(I_z(m_0))$ as a subspace of all degree m_0 polynomials $H^0(\mathcal{O}(m_0))$ and has fixed dimension. We have $H^i(I_Z(m_0)) = 0$ for all $i > 0$, and in particular $h^0(I_Z(m_0)) = P(m_0)$ is constant.

It is determined by these polynomials because we have a sequence

$$0 \to I_Z(m_0) \to \mathcal{O}(m_0) \to \mathcal{O}_Z(m_0) \to 0.$$

We can get a commuting diagram over it

$$0 \to H^0(I_Z(m_0)) \otimes \mathcal{O}_{\mathbb{P}^n} \to H^0(\mathcal{O}(m_0)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \cdots$$

where the middle map down is just evaluation and.the first map down is a surjection. Hence $I_Z(m_0)$, hence $\mathcal{O}_Z$, hence Z is determined by $H^0(I_Z(m_0))$.

Next time: we'll show that this is a subfunctor that is locally clos

8 | Thursday February 6th

Review base-change!

For $k = \bar{k}$, and $C_{/k}$ a smooth projective curve, then $\mathrm{Hilb}^n_{C_{/k}} = \mathrm{Sym}^n C$.

Definition 8.0.1 (The Hilbert-Chow Map)
For $X_{/k}$ a smooth projective *surface*, $\mathrm{Hilb}^n_{X_{/k}} \neq \mathrm{Sym}^n X$, there is a map (the Hilbert-Chow map)

$$\begin{aligned} \mathrm{Hilb}^n_{X_{/k}} &\to \mathrm{Sym}^n X \\ Z &\mapsto \mathrm{supp}(Z) \\ U = \text{reduced subschemes} &\mapsto U' = \text{ reduced multisets} \\ \mathbb{P}^1 &\mapsto (x, x). \end{aligned}$$

Example 8.0.2 *(?):* Consider $\mathbb{A}^2 \times \mathbb{A}^2$ under the $\mathbb{Z}/2\mathbb{Z}$ action

$$((x_1, y_1), (x_2, y_2)) \mapsto ((x_2, y_2), (x_1, y_1)).$$

Then

$$\begin{aligned}(\mathbb{A}^2)^2/\mathbb{Z}/2\mathbb{Z} &= \operatorname{Spec} k[x_1, y_1, x_2, y_2]^{\mathbb{Z}/2\mathbb{Z}} \\ &= \operatorname{Spec} k[x_1x_2, y_1y_2, x_1 + x_2, y_1 + y_2, x_1y_2 + x_2y_1, \cdots]\end{aligned}$$

with a bunch of symmetric polynomials adjoined.

Example 8.0.3 *(?):* Take $\mathbb{A}^2$ and consider $\operatorname{Hilb}^3_{\mathbb{P}^2}$. If I is a monomial ideal in $\mathbb{A}^2$, there is a nice picture. We can identify the tangent space

$$T_Z \operatorname{Hilb}^n_{\mathbb{P}^2} = \hom_{\mathcal{O}_{\mathbb{P}^2}}(I_2, \mathcal{O}_Z) = \bigoplus \hom(I_{Z_i}, \mathcal{O}_{Z_i}).$$

if $Z = \coprod Z_i$. If I is supported at 0, then we can identify the ideal with the generators it leaves out.

Example 8.0.4 *(?):* $I = (x^2, xy, y^2)$:

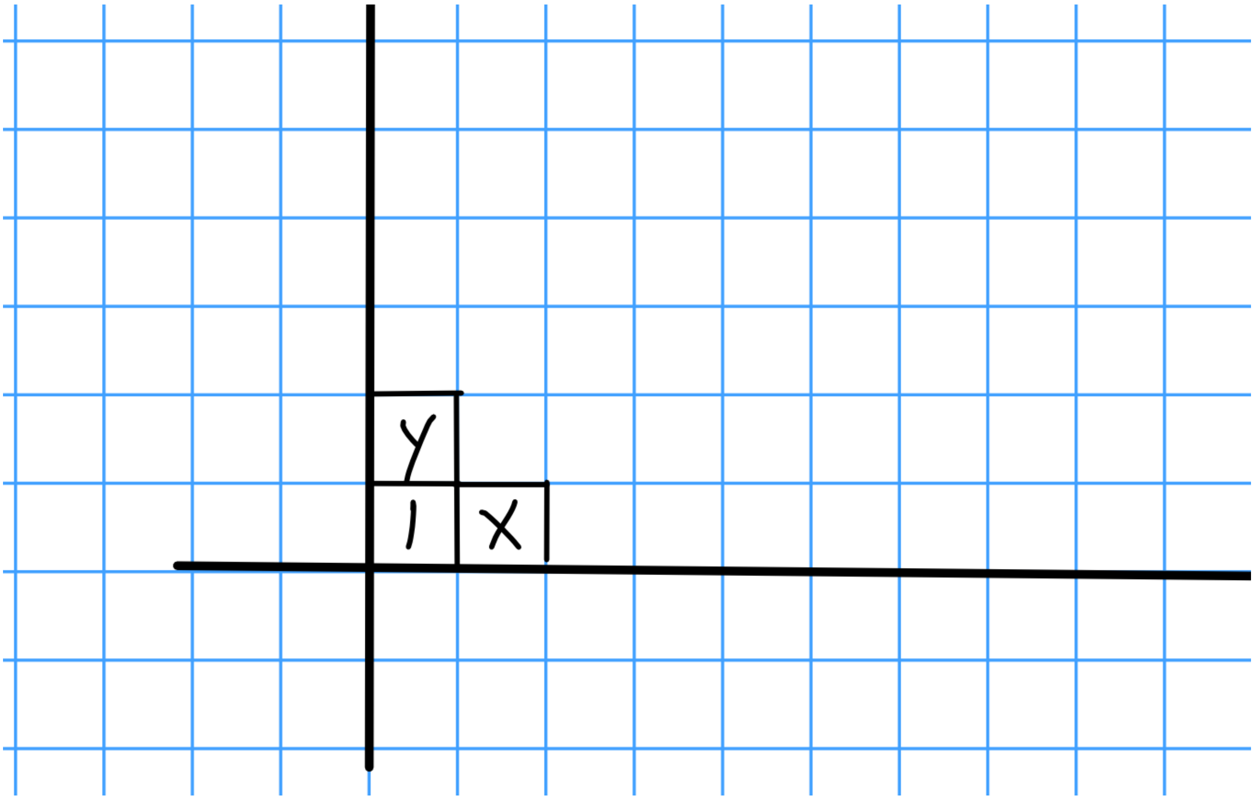


Figure 1: Image

Example 8.0.5 *(?):* $I = (x^6, x^2y^2, xy^4, y^5)$:

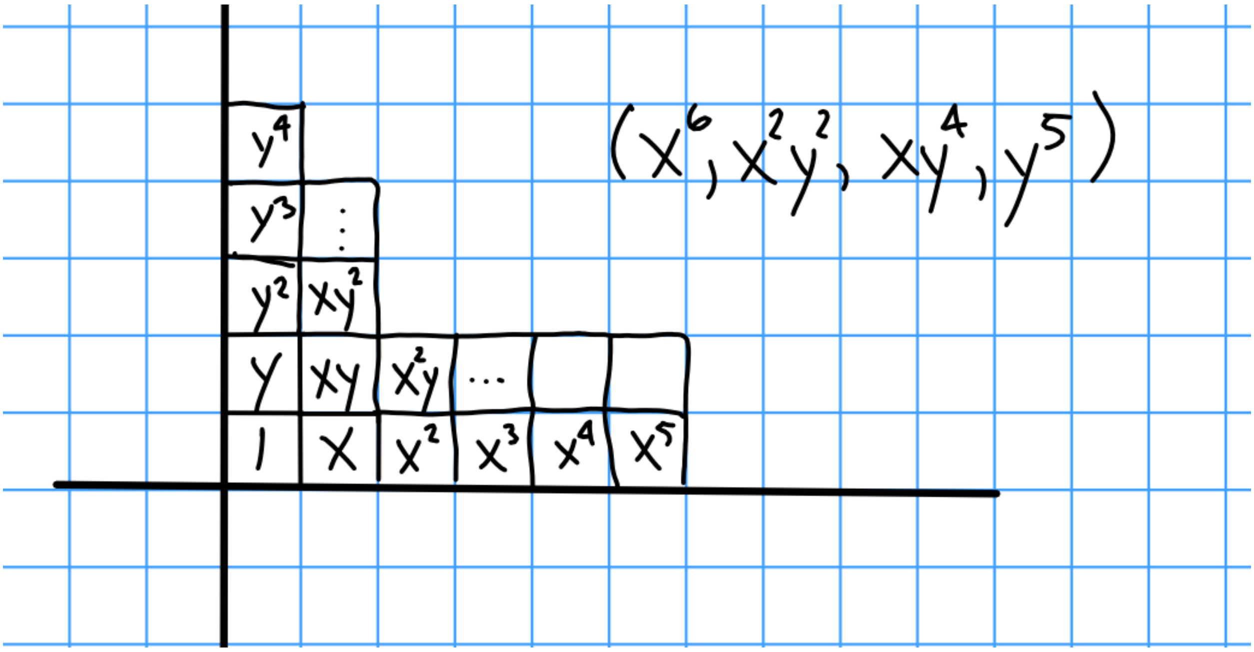


Figure 2: Image

Example 8.0.6 ***(?):*** $I = (x^2, y)$. Let $e = x^2, f = y$.

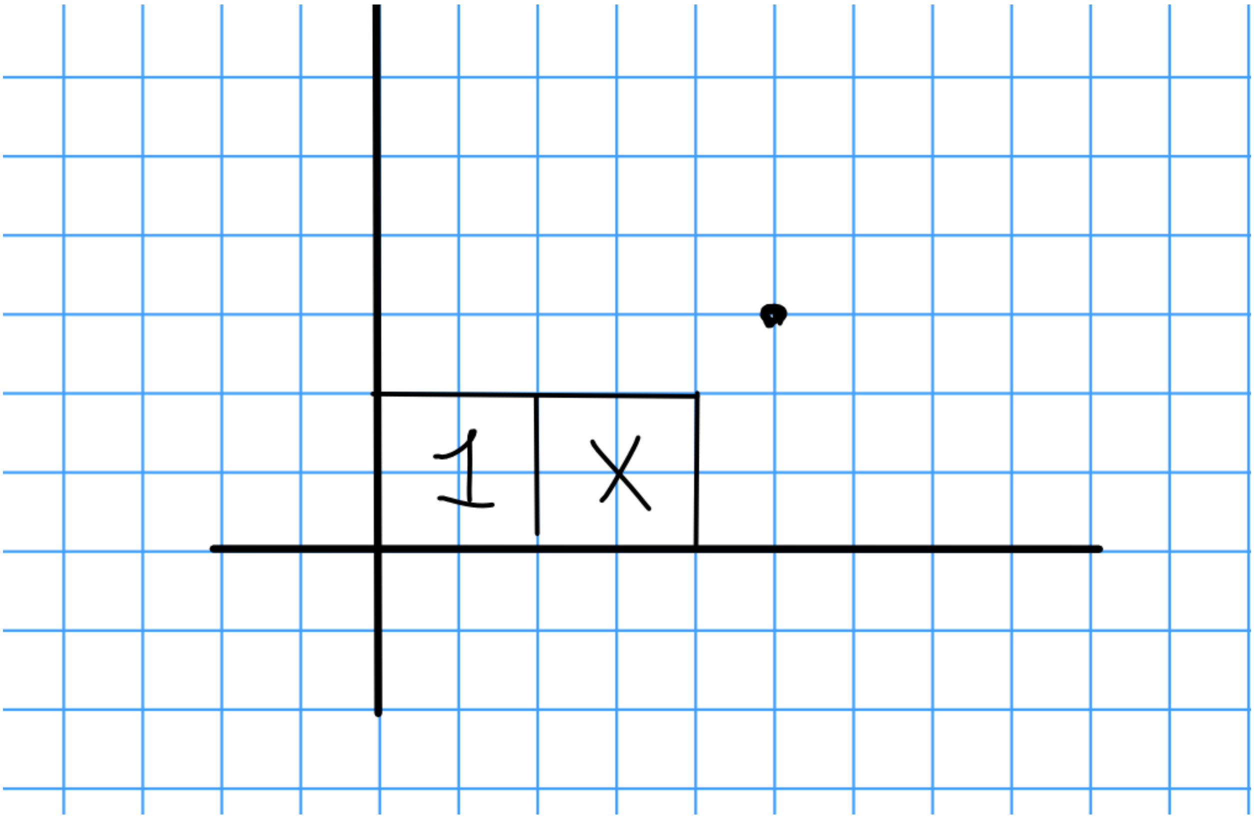
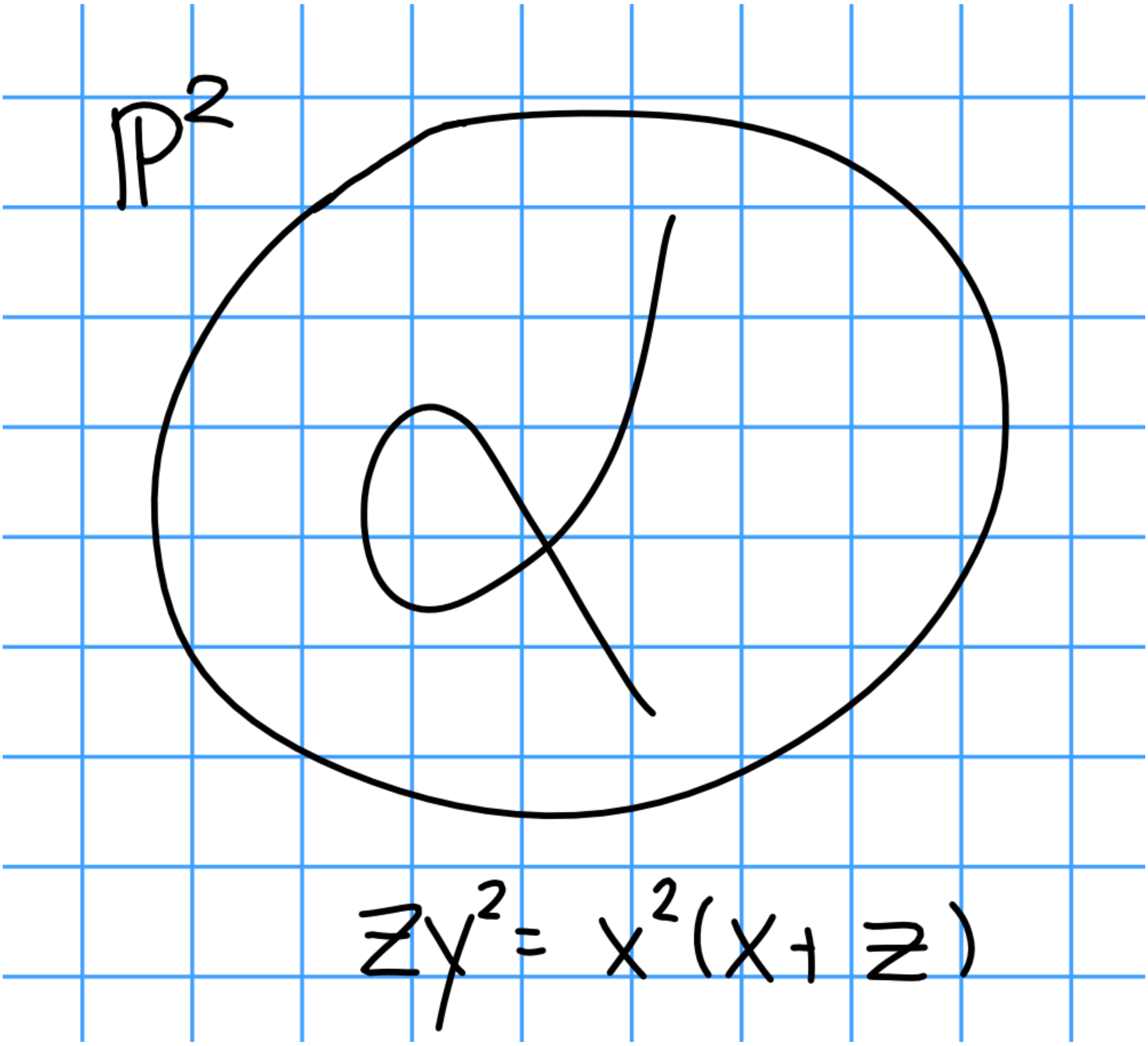


Figure 3: Image

By comparing rows to columns, we obtain a relation $ye = x^2 f$. Write $\mathcal{O} = \{1, x\}$, then note that this relation is trivial in $\mathcal{O}$ since $y = x^2 = 0$. Thus $\hom(I, \mathcal{O}) = \hom(k^2, k^2)$ is 4-dimensional.

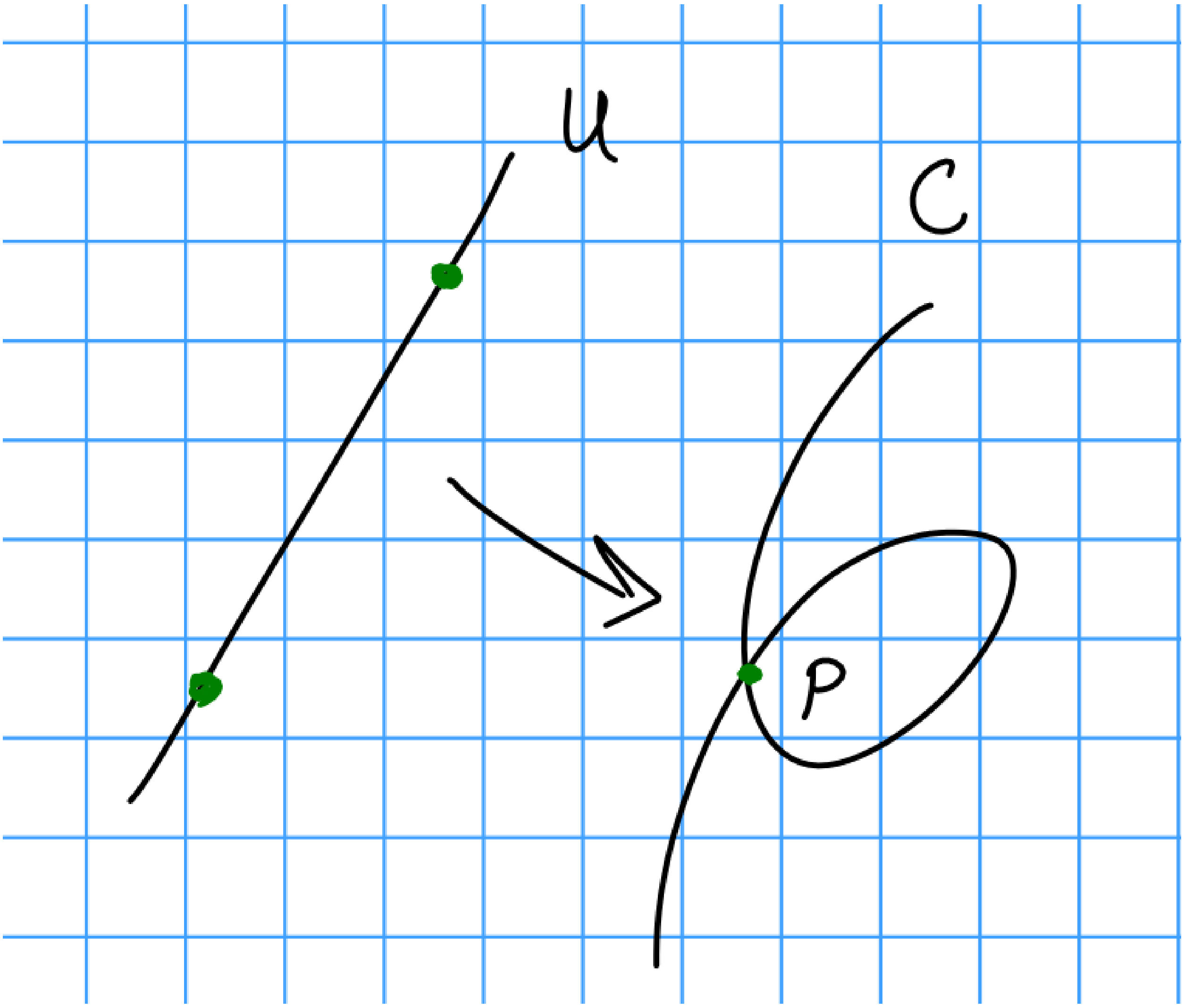
Remark 8.0.7: Note that $C_{/k}$ for curves is an important case to know. Take $Z \subset C \times C^n$, then quotient by the symmetric group S^n (need to show this can be done), then $Z/S^n \subset C \times \operatorname{Sym}^n C$ and composing with the functor Hilb represents yields a map $\operatorname{Sym}^n C \to \operatorname{Hilb}^n_{C_{/k}}$. This is bijective on points, and a tangent space computation shows it's an isomorphism.

Example 8.0.8*(?)*: Consider the nodal cubic in $\mathbb{P}^2$:



The nodal cubic $zy^2 = x^2(x+z)$.

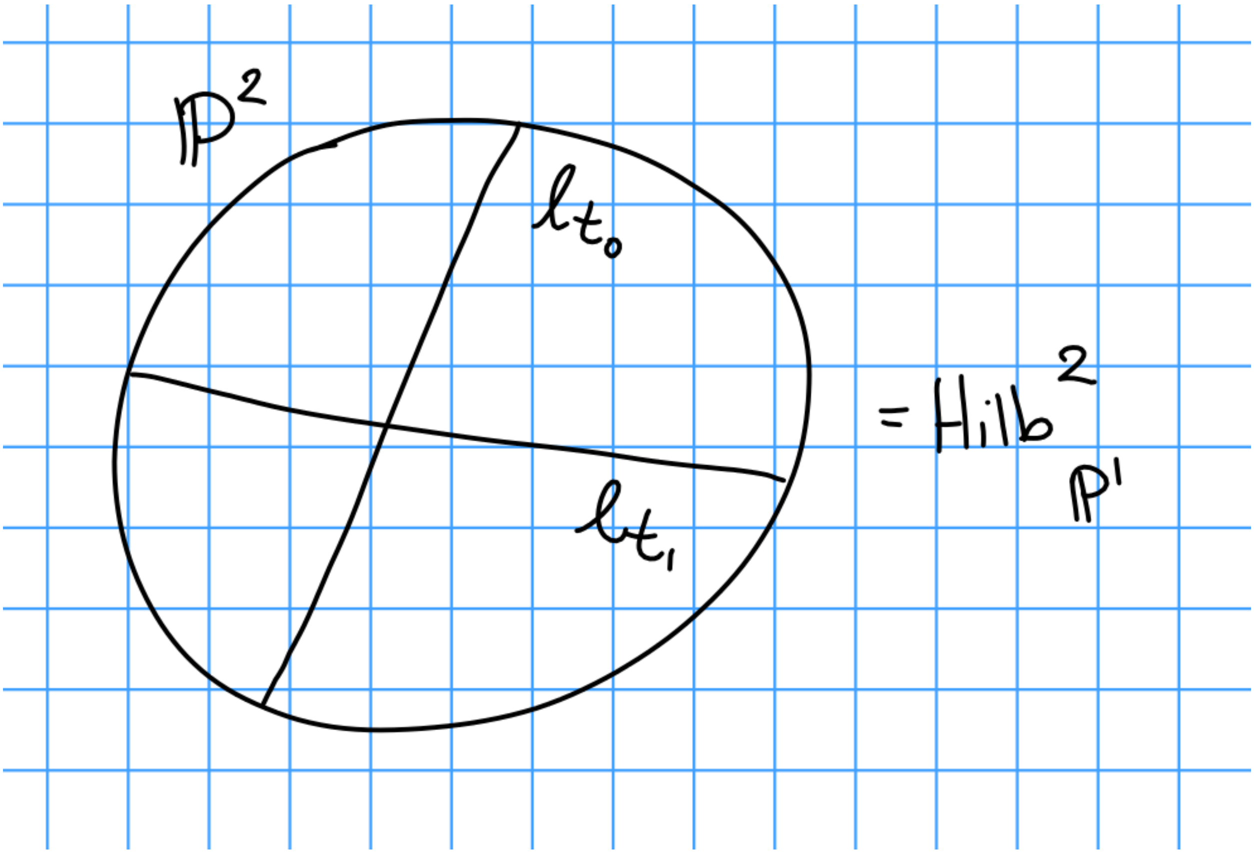
Consider the open subscheme $V \subset \mathrm{Hilb}^2_{C/k}$ of points $z \subset U$ for $U \subset C$ open. We can normalize:



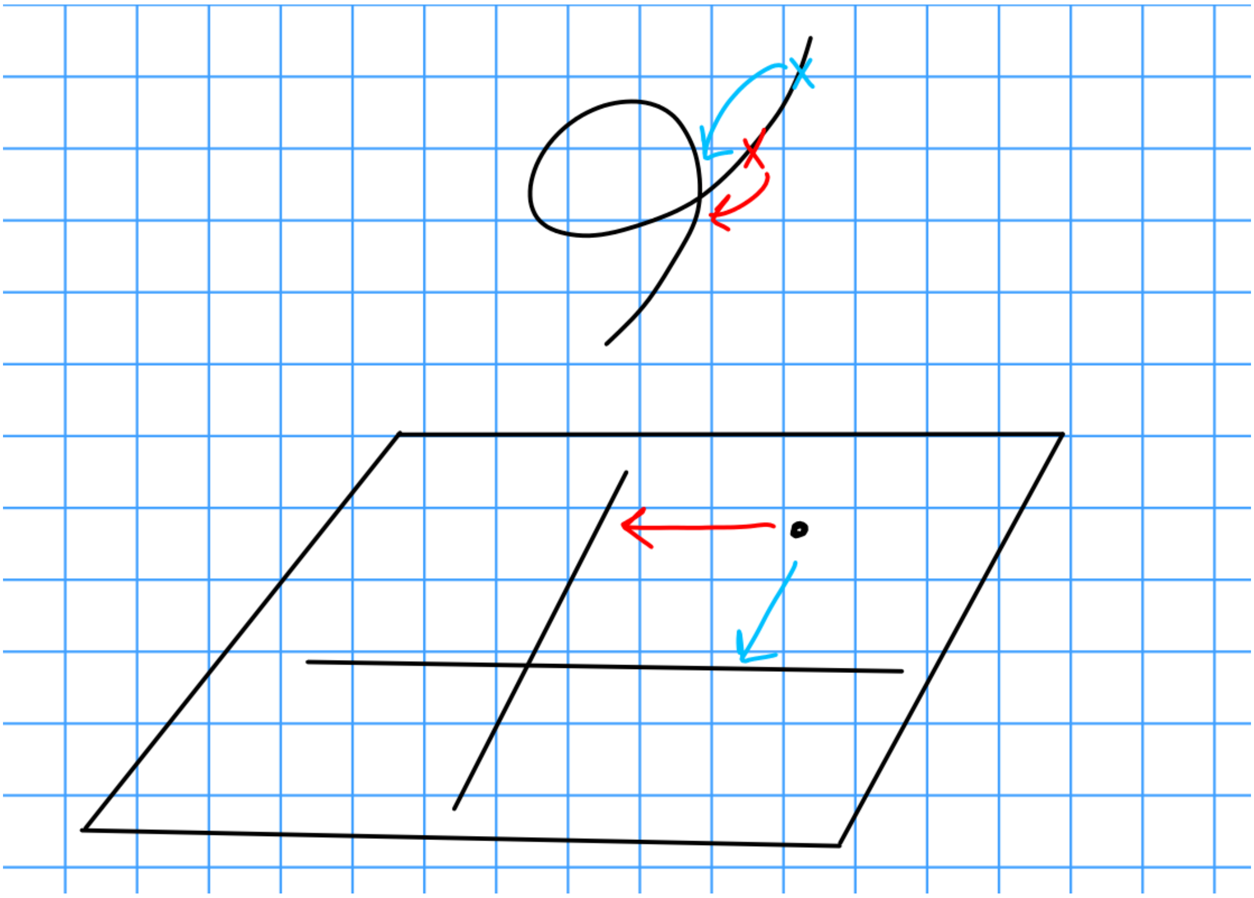
This yields a map fro $\mathbb{P}^1 \setminus 2$ points. This gives us a stratification, i.e. a locally closed embedding

$$(\text{z supported on U})\coprod(\text{1 point at p})\coprod(\text{both points at p}) \to \mathrm{Hilb}^2_{C/k}.$$

The first locus is given by the complement of two lines:



The third locus is given by arrows at p pointing in any direction, which gives a copy of $\mathbb{P}^1$. The second is $\mathbb{P}^1$ minus two points. Above each point is a nodal cubic with two marked points, and moving the base point towards a line correspond to moving one of the points toward the node:



More precisely, we're considering the cover $\mathbb{P}^1 \setminus 2$ points $\to C$ and thinking about ways in which two points and approach the missing points. These give specific tangent directions at the node on the cubic, depending on how this approach happens – either both points approach missing point #1, both approach missing point #2, or each approach a separate missing point.

Remark 8.0.9: Useful example to think about. Not normal, reduced, but glued in a weird way. Possibly easier to think about: cuspidal cubic.

8.1 Representability

Recall the following definition:

Definition 8.1.1 (m-Regularity)
A coherent sheaf F on $\mathbb{P}^n_k$ for k a field is m-regular iff $H^i(F(m-i)) = 0$ for all $i > 0$.

Proposition 8.1.2 *(?)*.
For every Hilbert polynomial P, there exists some m_0 depending on P such that any $Z \subset \mathbb{P}^n_k$ with $P_Z = P$ satisfies I_Z is m-regular.

Remark 8.1.3 *(1):* F is m- regular iff $\overline{F} = F \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is m-regular.

Remark 8.1.4 *(2):* The m_0 produced does not depend on k.

Lemma 8.1.5 *(?).*
For $m_0 = m_0(P)$ and $N = N(P)$, we have an embedding as a subfunctor

$$\mathrm{Hilb}^P_{\mathbb{P}^m_{\mathbb{Z}}} \to \mathrm{Gr}(N, H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0))^\vee).$$

For any $Z \subset \mathbb{P}^n_S$ flat over S with $P_{Z_s} = P$ for all $s \in S$ points, we want to send this to

$$0 \to R^\vee \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0))^\vee \to Q \to 0$$

or equivalently

$$0 \to Q^\vee \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(m_0)) \to R \to 0$$

with R locally free.

So instead of the quotient Q being locally free, we can ask for the sub $Q^\vee$ to be locally free instead, which is a weaker condition.

We thus send Z to

$$0 \to \pi_{s*} I_Z(m_0) \to \pi_{s*} \mathcal{O}_{\mathbb{P}^n_s}(m_0) = \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0))$$

which we obtain by taking the pushforward from this square:

$$\begin{tikzcd}
\mathbb{P}^n_s \arrow[r] \arrow[d, "\pi_s"] & \mathbb{P}^n_Z \arrow[d] \\
S \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

We have a sequence $0 \to I_Z(m_0) \to \mathcal{O}(m_0) \to \mathcal{O}_Z(m_0) \to 0$. Thus we get a sequence

$$0 \to \pi_{s*} I_Z(m_0) \to \pi_{s*} \mathcal{O}(m_o) \to \pi_{s*} \mathcal{O}_Z(m_0) \to R^1 \pi_{s*} I_Z(m_0) \to \cdots.$$

8.1.1 Step 1

$$R^1 \pi_* I_Z(m_0) = 0.$$

By base change, it's enough to show that $H^1(Z_s, I_{Z_s}(m_0)) = 0$. This follows by m_0-regularity.

8.1.2 Step 2

$\pi_{s*} I_Z(m_0)$ and $\pi_{s*}\mathcal{O}_Z(m_0)$ are locally free. For all $i > 0$, we have

- $R^i \pi_{s*} I_Z(m_0) = 0$ by m_0-regularity,
- $R^i \pi_{s*}\mathcal{O}(m_0) = 0$ by base change,
- and thus $R^i \pi_{s*}\mathcal{O}_Z(m_0) = 0$.

8.1.3 Step 3

$\pi_{s*} I_Z(m_0)$ has rank $N = N(P)$.

Again by base change, there is a map $\pi_* I_Z(m_0) \otimes k(s) \to H^0(Z_S, I_{Z_s}(m_0))$ which we know is an isomorphism. Because $h^i(I_{Z_S}(m_0)) = 0$ for $i > 0$ by m-regularity and

$$h^0(I_{Z_S}(m_0)) = P_{\mathcal{O}}(m_0) - P_{\mathcal{O}_{Z_s}}(m_0) = P_{\mathcal{O}}(m_0) - P(m_0).$$

This yields a well-defined functor

$$\mathrm{Hilb}^P_{\mathbb{P}^n_{\mathbb{Z}}} \to \mathrm{Gr}(N, H^0(\mathbb{P}^n, \mathcal{O}(m_0))^\vee).$$

Remark 8.1.6: Note that we've just said what happens to objects; strictly speaking we should define what happens for morphisms, but they're always give by pullback.

We want to show injectivity, i.e. that we can recover Z from the data of a number f polynomials vanishing on it, which is the data $0 \to \pi_{s*} I_Z(m_0) \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0))$.

Given

$$0 \to Q^\vee \to \mathcal{O}_s \otimes H^0(\mathbb{P}^n, \mathcal{O}(m_0)) = \pi_{s*}\mathcal{O}_{\mathbb{P}^n_S}(m_0)$$

we get a diagram

$$\begin{tikzcd}
\pi_s^* Q^\vee \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n_s}(m_0) \\
& I(m_0) \arrow[ur, hook] &
\end{tikzcd}$$

where $Q^\vee = \pi_{s*} I_Z(m_0)$, so we're looking at

$$\begin{tikzcd}
Q^\vee = \pi_{s*}^*\pi_{s*} I_Z(m_0) \arrow[rr] \arrow[dr, two heads] & & \mathcal{O}_{\mathbb{P}^n_s}(m_0) \\
& I(m_0) \arrow[ur, hook] &
\end{tikzcd}$$

The surjectivity here follows from $\mathcal{O}_{Z_s} \otimes H^0(I_{Z_s}(m_0)) \to I_{Z_s}(m_0)$ (?). Given a universal family $G = \mathrm{Gr}(N, H^0(\mathcal{O}(m_0))^\vee)$ and $Q^\vee \subset \mathcal{O}_G \otimes H^0(\mathcal{O}(m_0))^\vee$, we obtain $I_W \subset \mathcal{O}_G$ and $W \subset \mathbb{P}^n_G$.

9 | Tuesday February 18th

Theorem 9.0.1*(?).*
Let X/S be a projective subscheme (i.e. $X \subset \mathbb{P}^n$ for some n). The Hilbert functor of flat families $\mathrm{Hilb}^p_{X/S}$ is representable by a projective S-scheme.

Remark 9.0.2: Note that without a fixed P, this is *locally* of finite type but not finite type. After fixing P, it becomes finite type.

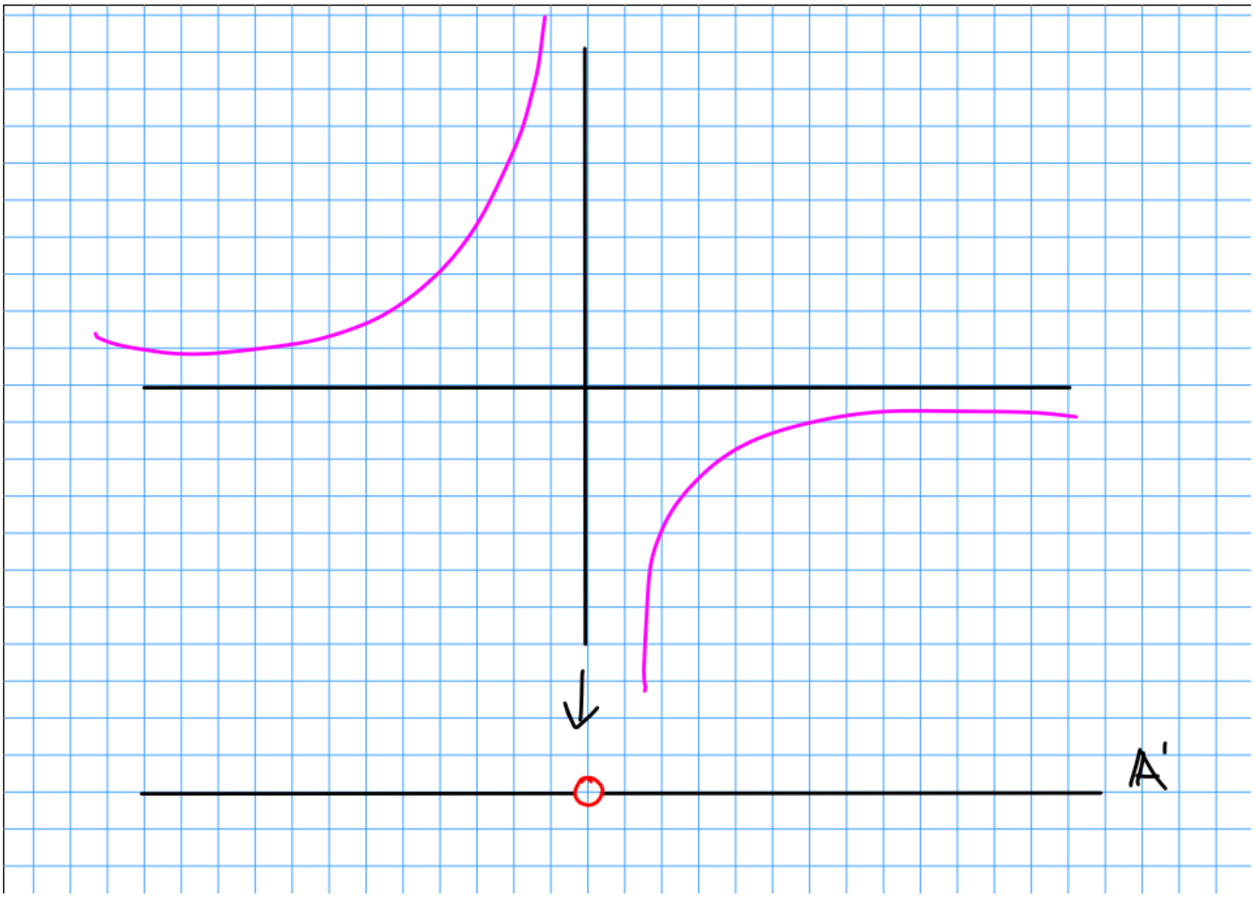
Example 9.0.3*(?):* For a curve of genus g, there is a smooth family $\mathcal{C} \xrightarrow{\pi} S$ with S finite-type over $\mathbb{Z}$ where every genus g curve appears as a fiber. I.e., genus g curves form a *bounded family* (here there are only finitely many algebraic parameters to specify a curve). How did we construct? Take the third power of the canonical bundle and show it's very ample, so it embeds into some projective space and has a Hilbert polynomial.

In fact, there is a finite type *moduli stack* $\mathcal{M}_g/\mathbb{Z}$ of genus g curves. There will be a map $S \twoheadrightarrow \mathcal{M}_g$, noting that $\mathcal{C}$ is not a moduli space since it may have redundancy. We'll use the fact that a finite-type scheme surjects onto $\mathcal{M}_g$ to show it is finite type.

Remark 9.0.4: If X/S is proper, we can't talk about the Hilbert polynomial, but the functor $\mathrm{Hilb}_{X/S}$ is still representable by a locally finite-type scheme with connected components which are proper over S.

Remark 9.0.5: If X/S is *quasiprojective* (so locally closed, i.e. $X \hookrightarrow \mathbb{P}^n_S$), then $\mathrm{Hilb}^P_{X/S}(T) \coloneqq \{z \in X_T$ projective, flat over S with fiberwise Hilbert polynomial P $\}$ is still representable, but now by a quasiprojective scheme.

Example 9.0.6*(?):* Length Z subschemes of $\mathbb{A}^1$: representable by $\mathbb{A}^2$.



Upstairs: parametrizing length 1 subschemes, i.e. points.

Remark 9.0.7: If $X \subset \mathbb{P}^n_S$ and E is a coherent sheaf on X, then

$$\mathrm{Quot}^P_{E,X/S}(T) = \{j^*E \to F \to 0, \text{ over } X_T \to T, \ F \text{ flat with fiberwise Hilbert polynomial } P\}$$

where $T \xrightarrow{g} S$ is representable by an S-projective scheme.

Example 9.0.8 *(?):* Take $E = \mathcal{O}_x$, X and S a point, and E is a vector space, then $\mathrm{Quot}^P_{E/S} = \mathrm{Gr}(\mathrm{rank}, E)$.

⚠ **Warning 9.0.9**
The Hilbert scheme of 2 points on a surface is more complicated than just the symmetric product.

Example 9.0.10 *(?):*

$$\begin{aligned} (\mathbb{A}^2)^3 &\to (\mathbb{A}^2)^2 \\ \supseteq \Delta \coloneqq \Delta_{01} \times \Delta_{02} &\to (\mathbb{A}^2)^2 \end{aligned}$$

where Δ_{ij} denote the diagonals on the i, j factors. Here all associate points of Δ dominate the image, but it is not flat. Note that if we take the complement of the diagonal in the image, then the restriction $\Delta' \to \left(\mathbb{A}^2\right)^2 \smallsetminus D$ is in fact flat.

Example 9.0.11 ***(Mumford):*** The Hilbert scheme may have nontrivial scheme structure, i.e. this will be a "nice" Hilbert scheme with is generally not reduced. We will find a component H of a $\mathrm{Hilb}^{P}_{\mathbb{P}^3_C}$ whose generic point corresponds to a smooth irreducible $C \subset \mathbb{P}^3$ which is generically non-reduced.

9.1 Cubic Surfaces

See Hartshorne Chapter 5.

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface, then $\mathcal{O}(1)$ on $\mathbb{P}^3$ restricts to a divisor class H of a hyperplane section, i.e. the associated line bundle $\mathcal{O}_x(H) = \mathcal{O}_x(1)$.

Fact 9.1.1 (Important fact 1)
X is the blowup of $\mathbb{P}^2$ minus 6 points (replace each point with a curve). There is thus a blowdown map $X \xrightarrow{\pi} \mathbb{P}^2$.

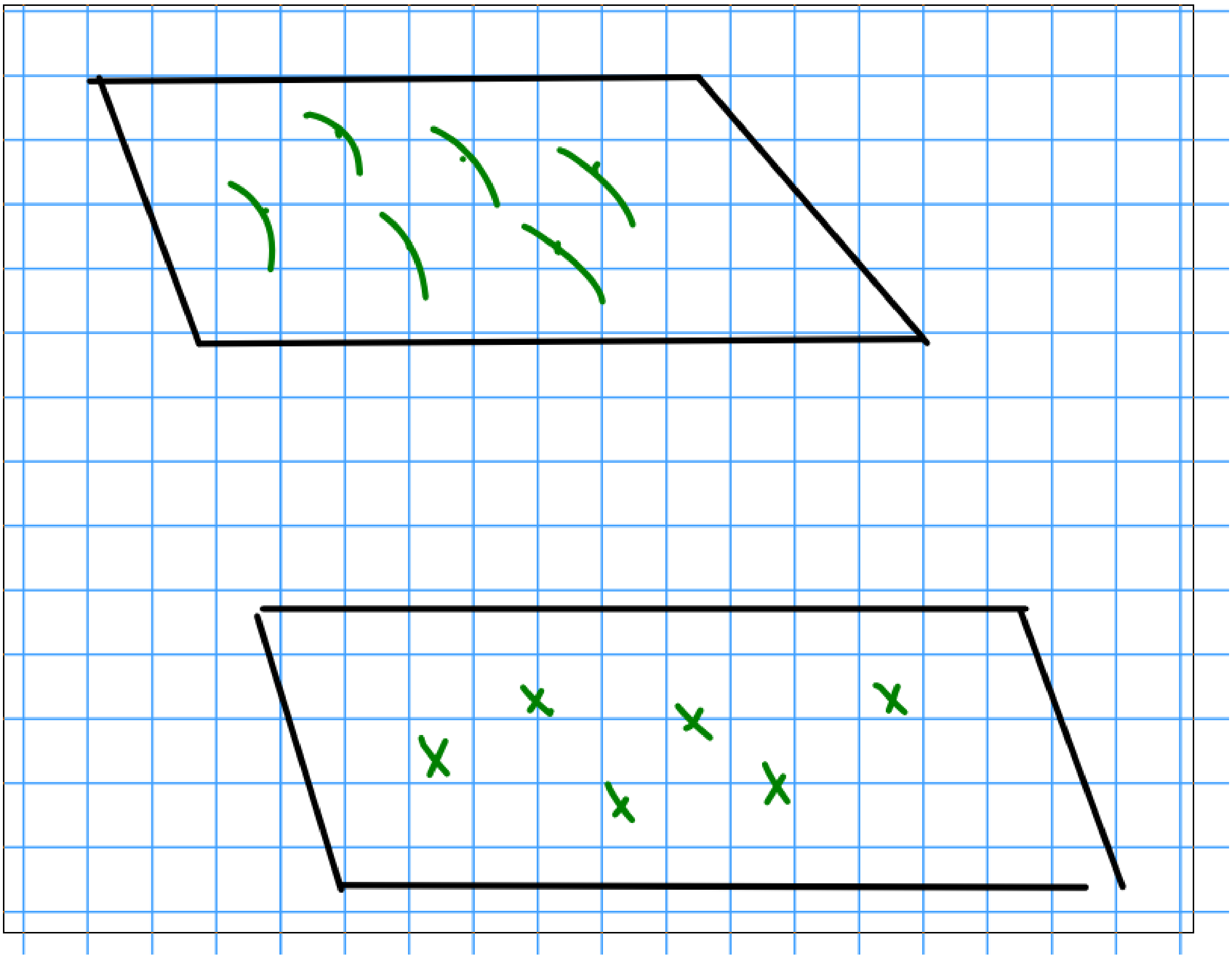
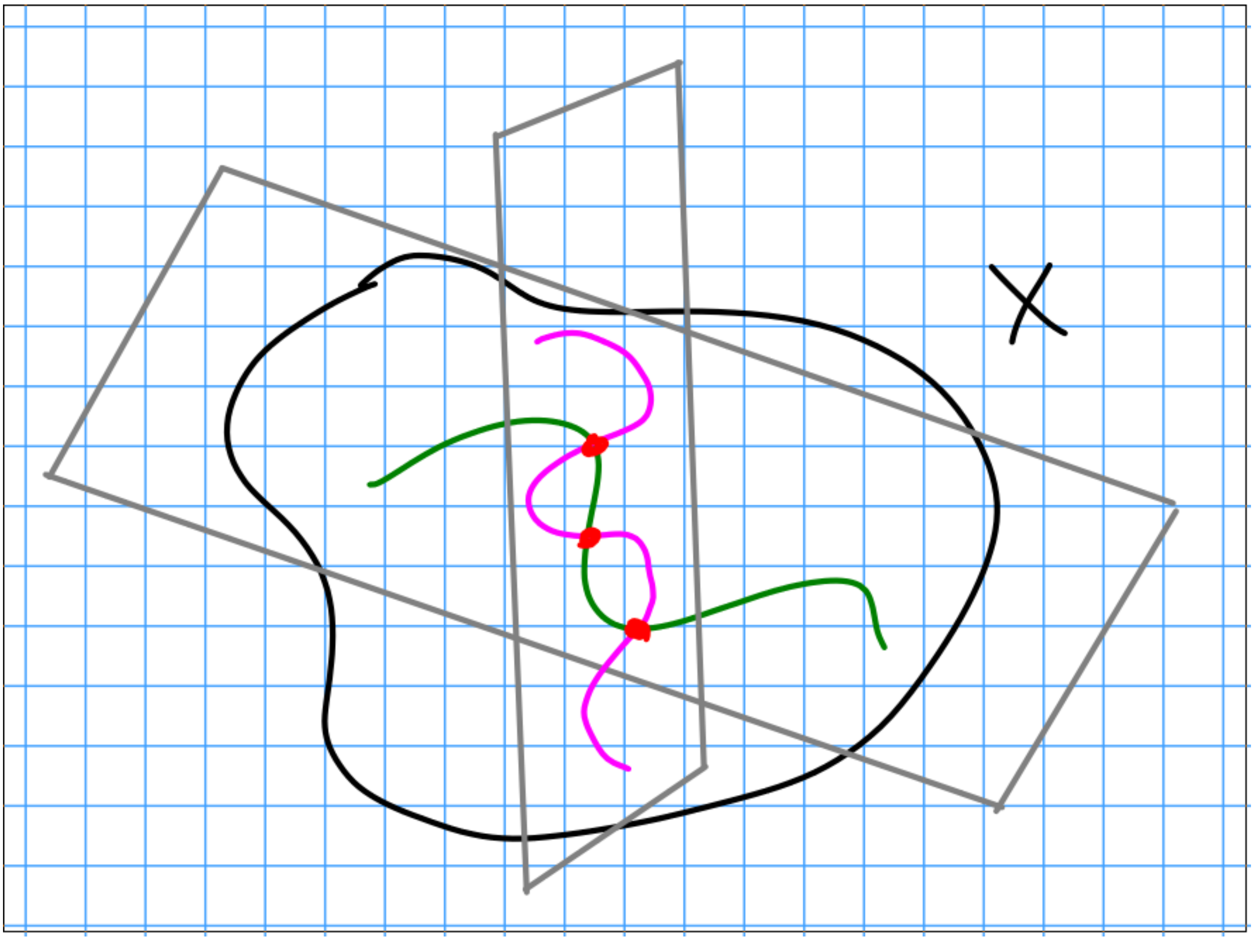


Figure 4: Image

Let $\ell = \pi^*(\text{line})$, then a fact is that $3\ell - E_1 - \cdots - E_6$ (where E_i are the curves about the p_i) is very ample and embeds X into $\mathbb{P}^3$ as a cubic.

Fact 9.1.2 (Important fact 2)
Every smooth cubic surface X has *precisely* 27 lines. Any 6 pairwise skew lines arise as $E_1, \cdots, E_6$ as in the previous construction.

Take an X and a line $L \subset X$. Consider any C in the linear system $|4H + 2L|$. Fact: $\mathcal{O}(4H + 2L)$ is very ample, so embeds into a big projective space, and thus C is smooth and irreducible by Bertini. Then the Hilbert polynomial of C is of the form $at + b$ where $b = \chi(\mathcal{O}_c)$, the Euler characteristic of the structure sheaf of C, and $a = \deg C$. So we'll compute these. We have $\deg C = H \cdot C$ (intersection) $= H \cdot (4H + 2L) = 4H^2 + 2H \cdot L = 4 \cdot 3 + 2 = 14$. The intersections here correspond to taking hyperplane sections, intersecting with X to get a curve, and counting intersection points:



In general, for X a surface and $C \subset X$ a smooth curve, then $\omega_C = \omega_X(C)\Big|_C$. Since $X \subset \mathbb{P}^3$, we have

$$\begin{aligned}
\omega_X &= \omega_{\mathbb{P}^3}(X)\Big|_X \\
&= \mathcal{O}(-4) \oplus \mathcal{O}(3)\Big|_X \\
&= \mathcal{O}_X(-1) \\
&= \mathcal{O}_X(-H).
\end{aligned}$$

We also have

$$\begin{aligned}\omega_C &= \omega_X(C)\Big|_X \\ &= (\mathcal{O}_X(-H) \oplus \mathcal{O}_X(4H+2L))|_C \\ &\Downarrow \qquad \text{taking degrees} \\ \deg \omega_C &= C \cdot (3H+2L) \\ &= (4H+2L)(3H+2L) \\ &= 12H^2 + 14HL + 4L^2 \\ &= 36 + 14 + (-4) \\ &= 46.\end{aligned}$$

Since this equals $2g(C) - 2$, we can conclude that the genus is given by $g(C) = 24$. Thus P is given by $14t + (1-g) = 14t - 23$.

Remark 9.1.3: Good to know: moving a cubic surface moves the lines, you get a monodromy action, and the Weyl group of E_6 acts transitively so lines look the same.

Claim 1: There is a flat family $Z \subset \mathbb{P}^3_S$ with fiberwise Hilbert polynomial P of cures of this form such that the image of the map $S \to \mathrm{Hilb}^P_{\mathbb{P}^3}$ has dimension 56.

Proof (of claim).
We can compute the dimension of the space of smooth cubic surfaces, since these live in $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(3))$, which has dimension $\binom{3+3}{3} - 1 = 19$. Since there are 27 lines, the dimension of the space of such cubics with a choices of a line is also 19. Choose a general C in the linear system $|4H+2L|$ will add $\dim|4H+2L| = \dim \mathbb{P}H^0(x, \mathcal{O}_x(C))$. We have an exact sequence

$$\begin{gathered}0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0 \\ H^0\left(0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0\right) \\ .\end{gathered}$$

Since the first H^0 vanishes (?) we get an isomorphism. By Riemann-Roch, we have

$$\deg \mathcal{O}_C(C) = C^2 = (4H+2L)^2 = 16H^2 + 16HL + 4L^2 = 64 - 4 = 60.$$

We can also compute $\chi(\mathcal{O}_C(C)) = 60 - 23 = 37$. We have

$$h^0(\mathcal{O}_C(C)) - h^1(\mathcal{O}_C(C)) = h^0(\mathcal{O}_C(C)) - h^0(\omega_C(-C))) = 2(23) - 60 < 0,$$

so there are no sections.
So $\dim|4H+2L| = 37$. Thus letting S be the space of cubic surfaces X, a line L, and a general $C \in |4H+2L|$, $\dim S = 56$. We get a map $S \to \mathrm{Hilb}^P_{\mathbb{P}^3}$, and we need to check that the fibers are 0-dimensional (so there are no redundancies). We then just need that every such C lies on a unique cubic. Why does this have to be the case? If $C \subset X, X'$ then $C \subset X \cap X'$ is degree 14 curve sitting inside a degree 6 curve, which can't happen. Thus if H is a component of $\mathrm{Hilb}^P_{\mathbb{P}^3}$ containing the image of S, the $\dim H \geq 56$.

■

Claim 2: For any C above, we have $\dim T_C H = 57$.

When the subscheme is smooth, we have an identification with sections of the normal bundle $T_C H = H^0(C, N_{C/\mathbb{P}^3})$. There's an exact sequence

$$0 \to N_{C/X} = \mathcal{O}_C(C) \to N_{C/\mathbb{P}^3} \to N_{X/\mathbb{P}^3}\Big|_C = \mathcal{O}_C(x)\Big|_C = \mathcal{O}_C(3H)\Big|_C \to 0.$$

Note $\omega_C = \mathcal{O}_C(3H + 2L)$.

As we computed,

$$\begin{aligned} H^0(\mathcal{O}_C(C)) &= 37 \\ H^1(\mathcal{O}_C(C)) &= 0. \end{aligned}$$

So we need to understand the right-hand term $H^0(\mathcal{O}_C(3H))$. By Serre duality, this equals $h^1(\omega_C(-3H)) = h^1(\mathcal{O}_C(3L))$. We get an exact sequence

$$0 \to \mathcal{O}_X(2L - C) \to \mathcal{O}_X(2L) \to \mathcal{O}_C(2L) \to 0.$$

Taking homology, we have $0 \to 0 \to 1 \to 1 \to 0$ since $2L - C = -4H$. Computing degrees yields $h^0(\mathcal{O}_C(3H)) = 20$. Thus the original exact sequence yields

$$0 \to 37 \to ? \to 20 \to 0,$$

so $? = 57$ and thus $\dim N_{C/\mathbb{P}^3} = 57$.

Claim 3:

$$\dim H = 56.$$

9.1.1 Proof That the Dimension is 56

Suppose otherwise. Then we have a family over H^{red} of *smooth* curves, where $f(S) \subset H^{\mathrm{red}}$, where the generic element is not on a cubic or any lower degree surface. Let C' be a generic fiber. Then C' lies on a pencil of quartics, i.e. 2 linearly independent quartics. Let $I = I_{C'}$ be the ideal of this curve in $\mathbb{P}^3$, there is a SES

$$0 \to I(4) \to \mathcal{O}(4) \to \mathcal{O}_C(4) \to 0.$$

It can be shown that $\dim H^0(I(4)) \geq 2$.

Fact 9.1.4
A generic quartic in this pencil is *smooth* (can be argued because of low degree and smoothness).

We can compute the dimension of quartics, which is $\binom{4+3}{3} - 1 = 35 - 1 = 34$. The dimension of C's lying on a fixed quartic is 24. But then the dimension of the image in the Hilbert scheme is at most $24 + 34 - 1 = 57$. It can be shown that the picard rank of such a quartic is 1, generated by $\mathcal{O}(1)$, so this is a *strict* inequality, which is a contradiction since $\dim \mathrm{Hilb} = 56$. This proves the theorem.

Remark 9.1.5: Use the fact that these curves are $K3$ surfaces? Get the fact about the generator of the Picard group from Hodge theory. So we can deform curves a bit, but not construct an algebraic family that escapes a particular cubic.

10 | Tuesday February 25th

Let k be a field, $X_{/k}$ projective, then the k-points $\mathrm{Hilb}^P_{X_{/k}}(k)$ corresponds to closed subschemes $Z \subset X$ with hilbert polynomial $P_z = P$. Given a P, we want to understand the local structure of $\mathrm{Hilb}^p_{X_{/k}}$, i.e. diagrams of the form

$$\begin{tikzcd}
& & \mathrm{Hilb}^P_{X_{/k}} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] \arrow[urr, "p"] & \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Spec}(k) \\
& A_{/k} \text{ Artinian local} \arrow[u] &
\end{tikzcd}$$

Example 10.0.1 ***(?):*** For $A = k[\varepsilon]$, the set of extensions is the Zariski tangent space.

> **Definition 10.0.2** (Category of Artinian Algebras)
> Let $(\mathrm{Art}_{/k})$ be the category of local Artinian k-algebras with local residue field k.

Note that these will be the types of algebras appearing in the above diagrams.

Remark 10.0.3: This category has fiber coproducts, i.e. there are pushouts:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d, dashed] \\
B \arrow[r, dashed] & A \otimes_C B
\end{tikzcd}$$

There are also fibered products,

$$\begin{tikzcd}
A \times_C B \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d] \\
A \arrow[r] & C
\end{tikzcd}$$

Here, $A \times_C B \coloneqq \left\{ (a,b) \;\middle|\; f(a) = g(b) \right\} \subset A \times B$.

Example 10.0.4*(?)*: If $A = B = k[\varepsilon]/(\varepsilon^2)$ and $C = k$, then $A \times_C B = k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1, \varepsilon_2)^2$

Note that on the Spec side, these should be viewed as

$$\operatorname{Spec}(A) \coprod_{\operatorname{Spec}(C)} \operatorname{Spec}(B) = \operatorname{Spec}(A \times_C B).$$

Definition 10.0.5 (Deformation Functor (Preliminary Definition))
A *deformation functor* is a functor $F : (\mathrm{Art}_{/k}) \to \mathrm{Set}$ such that $F(k) = \{\mathrm{pt}\}$ is a singleton.

Example 10.0.6*(?)*: Let $X_{/k}$ be any scheme and let $x \in X(k)$ be a k-point. We can consider the deformation functor F such that $F(A)$ is the set of extensions f of the following form:

$$\begin{tikzcd}
 & & X \arrow[d] \\
\operatorname{Spec}(k) \arrow[r, hook] \arrow[urr, "x"] & \operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed, "f"] & \operatorname{Spec}(k)
\end{tikzcd}$$

If $A' \to A$ is a morphism, then we define $F(A') \to F(A)$ is defined because we can precompose to fill in the following diagram

$$\begin{tikzcd}
 & & & X \arrow[d] \\
\mathrm{Spec}(k) \arrow[urrr] \arrow[dr] & & & \mathrm{Spec}(k) \\
 & \mathrm{Spec}(A) \arrow[r] \arrow[uurr, "\exists \tilde{f}"] & \mathrm{Spec}(A') \arrow[ur] \arrow[uur, dashed, "f"'] &
\end{tikzcd}$$

So this is indeed a deformation functor.

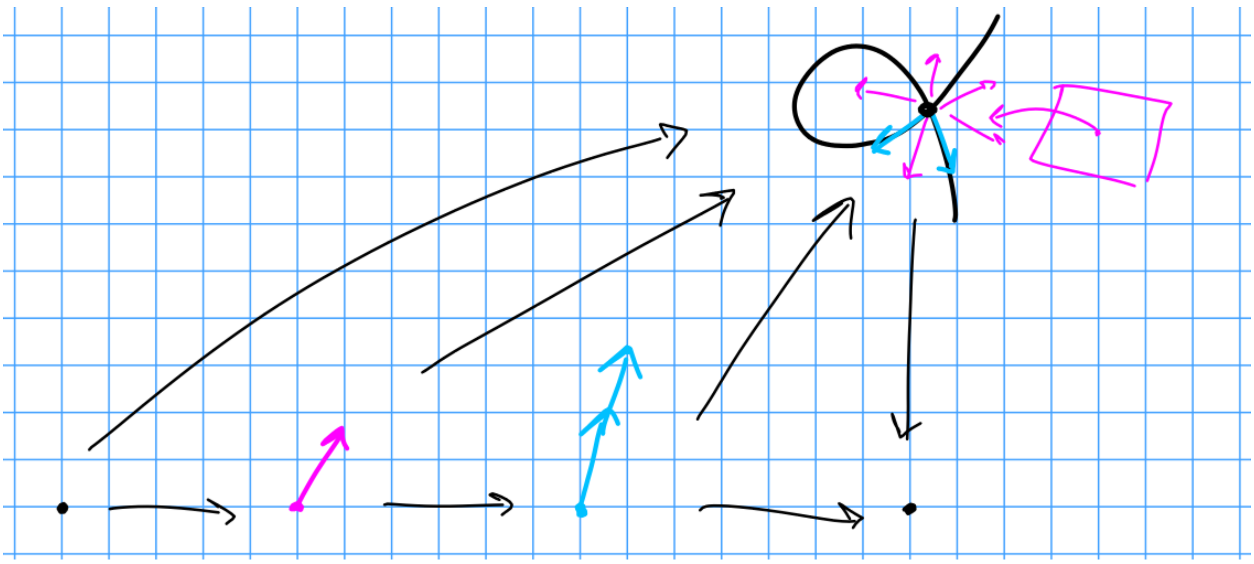
Example 10.0.7 ***(a motivating example):*** The Zariski tangent space on the nodal cubic doesn't "see" the two branches, so we allow "second order" tangent vectors.

We can consider parametrizing the functors above as $F_{X,x}(A)$, which is isomorphic to $F_{\mathrm{Spec}(\mathcal{O}_x)_{X,x}}$ and further isomorphic to $F_{\mathrm{Spec}\,\widehat{\mathcal{O}_{x}}_{x,X}}$. This is because for Artinian algebras, we have maps

$$\mathrm{Spec}(\mathcal{O}_{x,X})/\mathfrak{m}^N \to \mathrm{Spec}\,\mathcal{O}_{X,x} \to X.$$

Remark 10.0.8: $\widehat{\mathcal{O}}_{X,x}$ will be determined by $F_{X,x}$.

Example 10.0.9 ***(?):*** Consider $y^2 = x^2(x+1)$, and think about solving this over $k[t]/t^n$ with solutions equivalent to $(0,0) \pmod{t}$.



Note that the 'second order' tangent vector comes from $\mathrm{Spec}\,k[t]/t^3$.

We can write $F_{X,x}(A) = \pi^{-1}(x)$ where

$$\hom_{\mathrm{Sch}_{/k}}(\operatorname{Spec} k, X) \xrightarrow{\pi} \hom_{\mathrm{Sch}_{/k}}(\operatorname{Spec} k, x) \ni x.$$

Thus

$$F_{X,x}(A) = \hom_{\mathrm{Sch}_{/k}}(\operatorname{Spec} A, \operatorname{Spec} \mathcal{O}_{x,X}) = \hom_{k\text{-Alg}}(\widehat{\mathcal{O}}_{X,x}, A).$$

Example 10.0.10 ***(?):*** Given any local k-algebra R, we can consider

$$\begin{aligned} h_R : (\mathrm{Art}_{/k}) &\to \mathrm{Set} \\ A &\mapsto \hom(R, A). \end{aligned}$$

and

$$\begin{aligned} h_{\operatorname{Spec} R} : (\mathrm{Art}\,\mathrm{Sch}_{/k})^{\mathrm{op}} &\to \mathrm{Set} \\ \operatorname{Spec}(A) &\mapsto \hom(\operatorname{Spec} A, \operatorname{Spec} R). \end{aligned}$$

Definition 10.0.11 (Representable Deformation)
A deformation F is **representable** if it is of the form h_R as above for some $R \in \mathrm{Art}_{/k}$.

Remark 10.0.12: There is a Yoneda Lemma for $A \in \mathrm{Art}_{/k}$,

$$\hom_{\mathrm{Fun}}(h_A, F) = F(A).$$

We are thus looking for things that are representable in a larger category, which restrict.

Definition 10.0.13 (Pro-Representability)
A deformation functor is *pro-representable* if it is of the form h_R for R a complete local k-algebra (i.e. a limit of Artinian local k-algebras).

Remark 10.0.14: We will see that there are simple criteria for a deformation functor to be pro-representable. This will eventually give us the complete local ring, which will give us the scheme representing the functor we want.

Remark 10.0.15: It is difficult to understand even $F_{X,x}(A)$ directly, but it's easier to understand small extensions.

Definition 10.0.16 (Small Extensions)
: A *small extension* is a SES of Artinian k-algebras of the form [0 ' $\to 0$
.] such that J is annihilated by the maximal ideal fo A'.

Lemma 10.0.17*(?)*.
Given any quotient $B \to A \to 0$ of Artinian k-algebras, there is a sequence of small extensions (quotients):

$$\begin{tikzcd}
0 & & & \\
B_0 \arrow[u] & B_1 \arrow[ul] & \cdots & B_n = A \arrow[ulll] \\
B \arrow[u] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

This yields

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, hook] \arrow[dddr, Rightarrow] & \operatorname{Spec} B \\
& \operatorname{Spec} B_0 \arrow[u, hook] \\
& \vdots \arrow[u, hook] \\
& \operatorname{Spec} B_n \arrow[u, hook]
\end{tikzcd}$$

where the $\operatorname{Spec} B_i$ are all small.

Remark 10.0.18: In most cases, extending deformations over small extensions is easy.

10.1 First Example of Deformation and Obstruction Spaces

Suppose $k = \overline{k}$ and let $X_{/k}$ be connected. We have a picard functor

$$\begin{aligned} \mathrm{Pic}_{X_{/k}} : (\mathrm{Sch}_{/k})^{\mathrm{op}} &\to \mathrm{Set} \\ S &\mapsto \mathrm{Pic}(X_S)/\mathrm{Pic}(S). \end{aligned}$$

If we take a point $x \in \mathrm{Pic}_{X_{/k}}(k)$, which is equivalent to line bundles on X up to equivalence, we obtain a deformation functor

$$\begin{aligned} F \coloneqq F_{\mathrm{Pic}_{X_{/k}}, x} &\to \mathrm{Set} \\ A &\mapsto \pi^{-1}(x) \end{aligned}$$

where

$$\begin{aligned} \pi : \mathrm{Pic}_{X_{/k}}(\operatorname{Spec} A) &\to \mathrm{Pic}_{X_{/k}}(\operatorname{Spec} k) \\ \pi^{-1}(x) &\mapsto x. \end{aligned}$$

This is given by taking a line bundle on the thickening and restricting to a closed point. Thus the functor is given by sending A to the set of line bundles on X_A which restrict to X_x. That is, $F(A) \subset \mathrm{Pic}_{X_{/k}}(\operatorname{Spec} A)$ which restrict to x. So just pick the subspace $\mathrm{Pic}(X_A)$ (base changing to A) which restrict. There is a natural identification of $\mathrm{Pic}(X_A) = H^1(X_A, \mathcal{O}_{X_A}^*)$. If [0’ → 0

.] is a thickening of Artinian k-algebras, there is a restriction map of invertible functions [O{X_A}^* → $\mathcal{O}${X_A'}^* → 0

.] which is surjective since the map on structure sheaves is surjective and its a nilpotent extension. The kernel is then just $\mathcal{O}_{X_{A'}} \otimes J$. If this is a small extension, we get a SES

$$0 \to \mathcal{O}_X \otimes J \to \mathcal{O}_{X_{A'}}^* \to \mathcal{O}_{x_A}^* \to 0.$$

Taking the LES in cohomology, we obtain

$$H^1\mathcal{O}_X \otimes J \to H^1\mathcal{O}_{X_{A'}}^* \to H^1\mathcal{O}_{x_A}^* \to H^0\mathcal{O}_X \otimes J.$$

*Thus there is an obstruction class in H^2, and the ambiguity is detected by H^1. Thus H^1 is referred to as the **deformation space**, since it counts the extensions, and H^2 is the **obstruction space**.*

11 Deformation Theory (Thursday February 27th)

Big picture idea: We have moduli functors, such as

$$
\begin{aligned}
F_{S'} &: (\mathrm{Sch}_{/k})^{\mathrm{op}} \to \mathrm{Set} \\
\mathrm{Hilb} &: S \to \textit{flat subschemes of } X_S \\
\mathrm{Pic} &: S \to \mathrm{Pic}(X_S)/\mathrm{Pic}(S) \\
\mathrm{Def} &: S \to \textit{flat families } /S, \textit{ smooth, finite, of genus } g.
\end{aligned}
$$

Definition 11.0.1 (Deformation Theory)
Choose a point f the scheme representing $F_{S'}$ with $\xi_0 \in F_{gl}(\operatorname{Spec} K)$. Define

$$F_{\mathrm{loc}} : (\text{Artinian local schemes}/K)^{\mathrm{op}} \to \mathrm{Set}.$$

$$
\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, hook, "i"] & \mathrm{Spec}(A) \arrow[r] & F(i)^{-1}(\xi_0) \arrow[r] & F_{gr}(\operatorname{Spec} K) \arrow[d, "F(i)"] \\
 & & & F_{gl}(\operatorname{Spec} K)
\end{tikzcd}
$$

Definition 11.0.2 (Deformation Functors)
Let $F : (\mathrm{Art}_{/k}) \to \mathrm{Set}$ where $F(k)$ is a point. Denote $\widehat{\mathrm{Art}}_{/k}$ the set of complete local k-algebras. Since $\mathrm{Art}_{/k} \subset \widehat{\mathrm{Art}}/k$, we can make extensions $\widehat{F}$ by just taking limits:

$$
\begin{tikzcd}
 & \mathrm{Art}_{/k} \arrow[r, "F"] & \mathrm{Set} \\
\varprojlim R/\mathfrak{m}_R^n = R \in & \widehat{\mathrm{Art}}_{/k} \arrow[u] \arrow[ur, "\widehat{F}"] &
\end{tikzcd}
$$

where we define

$$\widehat{F}(R) := \varprojlim F(R/\mathfrak{m}_R^n).$$

Question 11.0.3
When is F pro-representable, which happens iff $\widehat{F}$ is representable? In particular, we want $h_R \xrightarrow{\cong} \widehat{F}$ for $R \in \widehat{\mathrm{Art}}_{/k}$, so

$$h_R = \hom_{\widehat{\mathrm{Art}}_{/k}}(R, \cdot) = \hom_{?}(\cdot, \operatorname{Spec} k).$$

Example 11.0.4 ***(?):*** Let $F_{\mathrm{gl}} = \mathrm{Hilb}^p_{X_{/k}}$, which is represented by $H_{/k}$. Then .

$$\xi_0 = F_{\mathrm{gl}}(k) = H(k) = \left\{ Z \subset X \;\middle|\; P_z = f \right\}.$$

Then F_{loc} is representable by $\widehat{\mathcal{O}}_{H/\xi_0}$.

Definition 11.0.5 (Thickening)
Given an Artinian k-algebra $A \in \mathrm{Art}_{/k}$, a *thickening* is an $A' \in \mathrm{Art}_{/k}$ such that $0 \to J \to A' \to A \to 0$, so $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$.

Definition 11.0.6 (Small Thickening)
A **small thickening** is a thickening such that $0 = \mathfrak{m}_{A'} J$, so J becomes a module for the residue field, and $\dim_k J = 1$.

Lemma 11.0.7 *(?).*
Any thickening of A, say $B \to A$, fits into a diagram:

$$\begin{tikzcd}
 & & 0 & & \\
 & J \arrow[r] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & A \arrow[r] & 0 \\
 & I' \arrow[r, Rightarrow] \arrow[u] & I' \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Proof (of lemma).
We just need $I' \subset I$ with $\mathfrak{m}_S I \subset J' \subset I \iff J\mathfrak{m}_B = 0$. Choose J' to be a preimage of a codimension 1 vector space in $I/\mathfrak{m}_B I$. Thus $J = I/I'$ is 1-dimensional. ■

Thus any thickening A can be obtained by a sequence of small thickenings. By the lemma, in principle F and thus $\widehat{F}$ are determined by their behavior under small extensions.

11.0.1 Example

Consider Pic, *fix* $X_{/k}$, *start with a line bundle* $L_0 \in \mathrm{Pic}(x)/\mathrm{Pic}(k) = \mathrm{Pic}(x)$ *and the deformation functor* $F(A)$ *being the set of line bundles* L *on* X_A*with* $L|_x \cong L_0$, *modulo isomorphism. Note that this yields a diagram*

$$\begin{tikzcd}
x \arrow[r] \arrow[d, hook] & k \arrow[d, "\text{unique closed point}"] \\
X_A \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

This is equal to $(I_x)^{-1}(L_0)$, *where* $\operatorname{Pic}(X_a) \xrightarrow{I_x} \operatorname{Pic}(x)$. *If*

$$0 \to J \to A' \to A \to 0.$$

is a small thickening, we can identify

$$\begin{tikzcd}
0 \arrow[r] & J \otimes_x \mathcal{O}_x \cong \mathcal{O}_x \arrow[r] & \mathcal{O}_{X_{A'}} \arrow[r] & \mathcal{O}_{X_A} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_x \arrow[r, "f \mapsto 1+f"] & \mathcal{O}^*_{X_{A'}} \arrow[r] \arrow[u, hook] & \mathcal{O}^*_{X_A} \arrow[r] & 0
\end{tikzcd} \in \textit{AbSheaves}$$

This yields a LES

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{O}_x) = k \arrow[r] & H^0(X_{A'}, \mathcal{O}^*_{x_{A'}}) = A'^* \arrow[r] & H^0(X_{A'}, \mathcal{O}^*_{x_A}) = A^* \arrow[r] \arrow[dll] & \therefore 0 \\
\therefore 0 \arrow[r] & H^1(X, \mathcal{O}_x) \arrow[r] & H^1(X_{A'}, \mathcal{O}^*_{x_{A'}}) = \operatorname{Pic}(X_{A'}) \arrow[r, "\text{restriction to } X_A"] & H^1(X_A, \mathcal{O}^*_{x_A}) = \operatorname{Pic}(X_A) \arrow[dll, "obs"] & \\
& H^2(X, \mathcal{O}_x) \arrow[r] & \cdots & &
\end{tikzcd}$$

Remark 11.0.8: To understand F on small extensions, we're interested in

1. Given $L \in F_{\mathrm{loc}}(A)$, i.e. L on X_A restricting to L_0, when does it extend to $L' \in F_{\mathrm{loc}}(A')$? I.e., does there exist an L' on $X_{A'}$ restricting to L?

2. Provided such an extension L' exists, how many are there, and what is the structure of the space of extensions?

Question 11.0.9
We have an $L \in \operatorname{Pic}(X_A)$, when does it extend?

By exactness, L' exists iff $obs(L) = 0 \in H^2(X, \mathcal{O}_x)$, which answers 1. To answer 2, $(I_x)^{-1}(L)$ is the set of extensions of L, which is a torsor under $H^1(x, \mathcal{O}_x)$. Note that these are fixed k-vector spaces.

Remark 11.0.10: $H^1(X, \mathcal{O}_x)$ is interpreted as the **tangent space** of the functor F, i.e. $F_{\text{loc}}(K[\varepsilon])$. Note that if X is projective, line bundles can be unobstructed without the group itself being zero.

For (3), just play with $A = k[\varepsilon]$, which yields $0 \to k \xrightarrow{\varepsilon} k[\varepsilon] \to k \to 0$, then

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, \mathcal{O}_x) \arrow[r] & H^1(X_{k[\varepsilon]}, \mathcal{O}^*_{k[\varepsilon]}) \arrow[r, "I_x"] & H^1(X, \mathcal{O}^*_x) \arrow[l, bend right] \\
 & & (I_x)^{-1}(L_0) \in \operatorname{Pic}(X_{k[\varepsilon]}) & L_0 \in \operatorname{Pic}(x)
\end{tikzcd}$$

i.e., there is a canonical trivial extension $L_0[\varepsilon]$.

Example 11.0.11 ***(?):*** Let $X \supset Z_0 \in \operatorname{Hilb}_{X_{/k}}(k)$, we computed

$$T_{Z_0} \operatorname{Hilb}_{X_{/k}} = \hom_{\mathcal{O}_x}(I_{Z_0}, \mathcal{O}_z).$$

We took $Z_0 \subset X$ and extended to $Z' \subset X_{k[\varepsilon]}$ by base change. In this case, $F_{\text{loc}}(A)$ was the set of $Z' \subset X_A$ which are flat over A, such that base-changing $Z' \times_{\operatorname{Spec} A} \operatorname{Spec} k \cong Z$. This was the same as looking at the preimage restricted to the closed point,

$$\begin{aligned}
\operatorname{Hilb}_{X_{/k}}(A) &\xrightarrow{i^*} \operatorname{Hilb}_{X_{/k}}(k) \\
(i^*)^{-1}(z_0) &\leftarrow\!\shortmid z_0.
\end{aligned}$$

Recall how we did the thickening: we had $0 \to J \to A' \to A \to 0$ with $J^2 = 0$, along with F on X_A which is flat over A with $X_{/k}$ projective, and finally an F' on $X_{A'}$ restricting to F. The criterion we had was F' was flat over A' iff $0 \to J \otimes_{A'} F' \to F'$, i.e. this is injective. Suppose $z \in F_{\text{loc}}(A)$ and an extension $z' \in F_{\text{loc}}(A')$. By tensoring the two exact sequences here, we get an exact grid:

$$
\begin{tikzcd}
0 \arrow[rr] \arrow[dd] & & I_{Z'} \arrow[r] & \mathcal{O}_{X_{A'}} \arrow[r] & \mathcal{O}_{Z'} \arrow[rr] & & 0 \\
& & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
J \arrow[d] & 0 \arrow[r] & I_{Z_0} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{Z_0} \arrow[r] \arrow[d] & 0 & \\
A' \arrow[d] & 0 \arrow[r] & I_{Z'} \arrow[r] \arrow[d] & \mathcal{O}_{X_{A'}} \arrow[r] \arrow[d] & \mathcal{O}_{Z'} \arrow[r] \arrow[d] & 0 & \\
A \arrow[dd] & 0 \arrow[r] & I_Z \arrow[r] \arrow[d] & \mathcal{O}_{X_A} \arrow[r] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 & \\
& & 0 & 0 & 0 & & \\
0 & & & & & &
\end{tikzcd}
$$

The space of extension should be a torsor under $\hom_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0})$, which we want to think of as $\hom_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0})$. Picking a φ in this hom space, we want to take an extension $I_{Z'} \xrightarrow{\varphi} I_{Z''}$.

We'll cover how to make this extension next time.

12 | Tuesday March 31st

See notes on Ben's website. We'll review where we were.

12.1 Deformation Theory

We want to represent certain moduli functors by schemes. If we know a functor is representable, it's easier to understand the deformation theory of it and still retain a lot of geometric information. The representability of deformation is much easier to show. We're considering functors $F : \mathrm{Art}_{/k} \to \mathrm{Set}$.

Example 12.1.1 ***(?):*** The Hilbert functor

$$
\begin{aligned}
\mathrm{Hilb}_{X_{/k}} (\mathrm{Sch}_{/k})^{\mathrm{op}} &\to \mathrm{Set} \\
S &\mapsto \{Z \subset X \times S \text{ flat over } S\}.
\end{aligned}
$$

This yields

$$F : \mathrm{Art}_{/k} \to \mathrm{Set}$$
$$???.$$

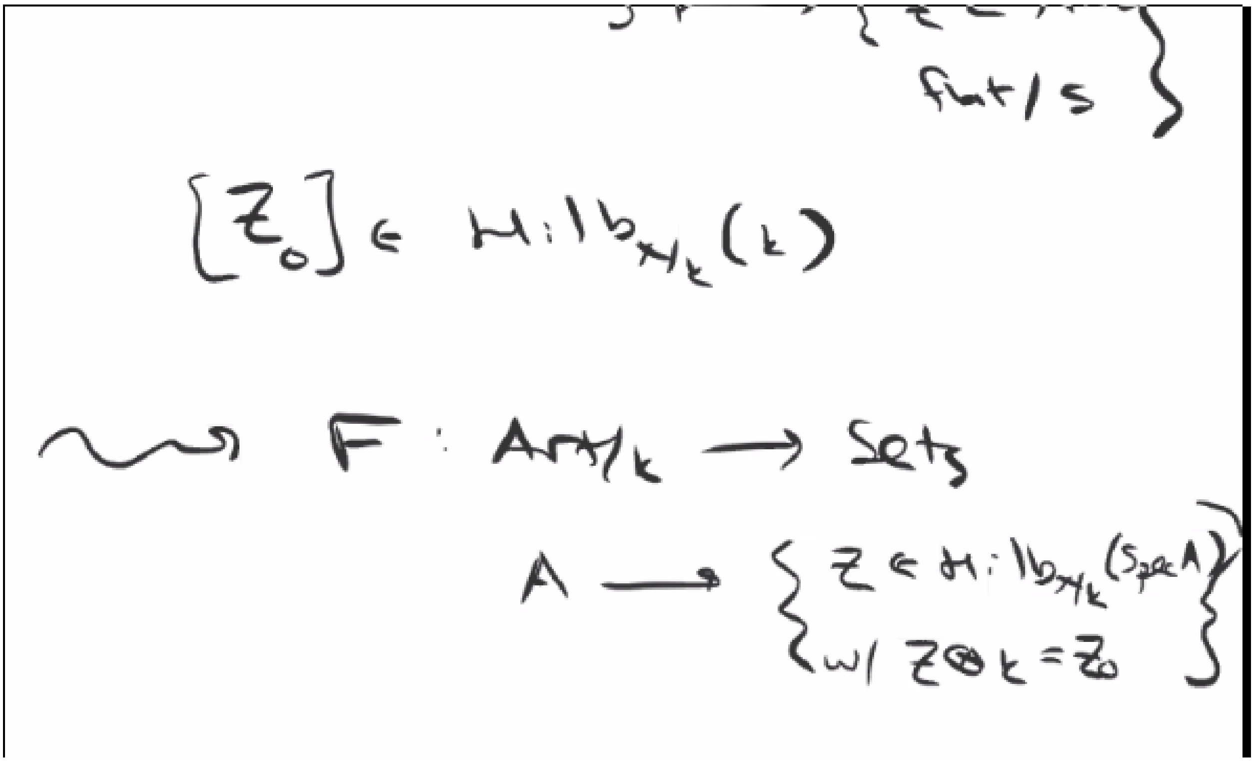


Figure 5: Image

Recall that we're interested in pro-representability, where $\widehat{F}(R) = \varprojlim F(R\mu_R^n)$ *is given by a lift of the form*

$$\begin{tikzcd} \mathrm{Art}_{/k} \arrow[r, "F"] & \mathrm{Set} \\ \widehat{\mathrm{Art}_{/k}} \arrow[u, hook] \arrow[ur, "\widehat{F}"'] & \end{tikzcd}$$

Question 12.1.2
Is $\widehat{F}$ representable, i.e. is F pro-representable?

Example 12.1.3*(?):* The F in the previous example is pro-representable by $\widehat{F} = \hom(\mathcal{O}_{\mathrm{Hilb},z_0}, \cdot)$.

Definition 12.1.4 (Pro-Representable Hull)
F has a *pro-representable hull* iff there is a formally smooth map $h_R \to F$.

Question 12.1.5
Does F have a pro-representable hull?

Recall that a map of functors on artinian k-algebras is ***formally smooth*** *if it can be lifted through nilpotent thickenings. That is, for $F, G : \mathrm{Art}_{/k} \to \mathrm{Set}$, $F \to G$ is* formally smooth *if for any thickening $A' \twoheadrightarrow A$, we have*

$$\begin{tikzcd}
 & & F \arrow[d] \\
h_A \arrow[r] \arrow[urr] \arrow[d, equal] & h_{A'} \arrow[r] \arrow[ur, dotted] \arrow[d, equal] & G \arrow[d, equal] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \arrow[r] & G
\end{tikzcd}$$

We proved for R, A finite type over k, $\operatorname{Spec} R \to \operatorname{Spec} A$ smooth is formally smooth. Given a complete local k-algebra R and a section $\xi \in \widehat{F}(R)$, we make the following definitions:

Definition 12.1.6 (Versal, Miniversal, Universal)
The pair (R, ξ) is

- *Versal* for F iff $h_R \xrightarrow{\xi} F$ is formally smooth.[a]
- *Miniversal* for F iff versal and an isomorphism on Zariski tangent spaces.
- *Universal* for F if $h_R \xrightarrow{\cong} F$ is an isomorphism, i.e. h_R pro-represents F.
  - Pullback by a unique map

[a] Not a unique map, but still a pullback

Remark 12.1.7: Note that **versal** means that any formal section (s, η) where $\eta \in \widehat{F}(s)$ comes from pullback, i.e there exists a map

$$R \to S$$
$$\widehat{F}(R) \to \widehat{F}(s)$$
$$\xi \mapsto \eta.$$

Miniversal means adds that the derivative is uniquely determined, and universal means that $R \to S$ is unique.

Definition 12.1.8 (Obstruction Theory)
An **obstruction theory** for F is the data of $\mathrm{def}(F), \mathrm{obs}(F)$ which are finite-dimensional k-vector spaces, along with a functorial assignment of the following form:

$$
\begin{aligned}
(A' \twoheadrightarrow A) \quad &\text{a small thickening } \mapsto \\
\mathrm{def}(F) \circlearrowleft F(A') \to F(A) &\xrightarrow{\mathrm{obs}} \mathrm{obs}(F)
\end{aligned}
$$

that is exact[a] and if $A = k$, it is exact on the left (so the action was faithful on nonempty fibers).

[a] Recall that right-exactness was a transitive action.

Example 12.1.9 ***(?):*** We have

$$
\begin{aligned}
\mathrm{Pic}_{X_{/k}} : (\mathrm{Sch}_{/k})^{\mathrm{op}} &\to \mathrm{Set} \\
S &\mapsto \mathrm{Pic}(X \times X)/\mathrm{Pic}(S).
\end{aligned}
$$

This yields

$$
\begin{aligned}
F : \mathrm{Art}_{/k} &\to \mathrm{Set} \\
A \mapsto L \in \mathrm{Pic}(X_A), \ L \otimes k &\cong L_0
\end{aligned}
$$

where $X_{/k}$ is proper and irreducible. Then F has an obstruction theory with $\mathrm{def}(F) = H^1(\mathcal{O}_x)$ and $\mathrm{obs}(F) = H^2(\mathcal{O}_x)$. The key was to look at the LES of

$$
0 \to \mathcal{O}_x \to \mathcal{O}_{X_{A'}}^* \to \mathcal{O}_{X_A}^* \to 0.
$$

for $0 \to k \to A' \to A \to 0$ small.

Remark 12.1.10 ***(Summary):*** In both cases, the obstruction theory is exact on the left for any small thickening. We will prove the following:

- F has an obstruction $\iff$ it has a pro-representable hull, i.e. a versal family
- F has an obstruction theory which is always exact at the left $\iff$ it has a universal family.

12.2 Schlessinger's Criterion

Let $F : \mathrm{Art}_{/k} \to \mathrm{Set}$ *be a deformation functor (and it only makes sense to talk about deformation functors when* $F(k) = \{\mathrm{pt}\}$*). This theorem will tell us when a miniversal and a universal family exists.*

Theorem 12.2.1 *(Schlessinger).*
F has a miniversal family iff

1. Gluing along common subspaces: ror any small $A' \to A$ and $A'' \to A$ any other thickening, the map
$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$
is surjective.

2. Unique gluing: if $(A' \to A) = (k[\varepsilon] \to k)$, then the above map is bijective.

3. $t_F = F(k[\varepsilon])$ is a finite dimensional k-vector space, i.e.
$$F(k[\varepsilon] \times_k k[\varepsilon]) \xrightarrow{\cong} F(k[\varepsilon]) \times F(k[\varepsilon]).$$

4. For $A' \to A$ small,
$$\begin{tikzcd} F(A') \arrow[r, "f"] & F(A) \\ t_f \circlearrowright f^{-1}(\eta) \arrow[u, hook, "\subseteq"] & \eta \arrow[u, "\in"] \end{tikzcd}$$

where the action is simply transitive.
F has a miniversal family iff (1)-(3) hold, and universal iff all 4 hold.

Exercise 12.2.2 (?)
Show that the existence of an obstruction theory which is exact on the left implies (1)-(4).

The following diagram commutes:

$$\begin{tikzcd} \mathrm{def} \circlearrowright F(A' \times_A A'') \ni \eta \arrow[r] \arrow[d] & F(A'') \ni \xi'' \arrow[r, "\mathrm{obs}"] \arrow[d] & \mathrm{obs} \\ \mathrm{def} \circlearrowright F(A') \ni \eta' m \xi' \arrow[r] & F(A') \ni \xi \arrow[r, "\mathrm{obs}"] & \mathrm{obs} \end{tikzcd}$$

So we have a map $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \ni (\xi', \xi'')$. Using transitivity of the def *action, we can get $\xi' = \eta' + \theta$ and thus $\eta + \theta$ is the lift.*

12.3 Abstract Deformation Theory

Example 12.3.1 *(?):* We start with $(X_0)_{/k}$ and define the functor F sending A to X/A flat families over A with $X_0 \hookrightarrow^i X$ such that $i \otimes k$ is an isomorphism. The punchline is that F has an obstruction theory if X_0 is smooth with

- $\operatorname{def}(F) = H^1(T_{X_0})$
- $\operatorname{obs}(F) = H^2(T_{X_0})$

Remark 12.3.2:

1. If X is a deformation of X_0 over A and we have a small extension $k \to A' \to A$ with X' over A' a lift of X. Then there is an exact sequence
$$0 \to \operatorname{Der}_R(\mathcal{O}_{X_0}) \to \operatorname{Aut}_{A'}(X') \to \operatorname{Aut}_A(X).$$

2. If $(X_0)_{/k}$ is smooth and *affine*, then any deformation X over A (a flat family restricting to X_0) is trivial, i.e. $X \cong X_0 \times_k \operatorname{Spec}(A)$.

$$\begin{tikzcd}
 & & X_0 \times \operatorname{Spec}(A) \arrow[d] \\
X_0 \arrow[r, hook] & X \arrow[r] \arrow[ur, dotted, "f"] & \operatorname{Spec}(A)
\end{tikzcd}$$

Thus $X_0 \hookrightarrow X$ has a section $X \to X_0$, and the claim is that this forces X to be trivial.

We have

$$\begin{tikzcd}
0 \arrow[r] & J \otimes \mathcal{O}_X \arrow[r] & \mathcal{O}_x \arrow[r] & \mathcal{O}_{X_0} \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

yielding

$$\begin{aligned}
0 \to K \to \mathcal{O}_{X_0} \otimes A \to \mathcal{O}_X \to 0 \\
(\cdot \otimes k) \\
1 \to k \otimes k = 0 \to \mathcal{O}_{X_0} \xrightarrow{\cong} \mathcal{O}_{X_0} \to 0.
\end{aligned}$$

Remark 12.3.3: Why does this involve cohomology of the tangent bundle? For X_0 smooth, $\operatorname{Der}_k(\mathcal{O}_{X_0}) = \mathcal{H}(T_{X_0})$, but the LHS is equal to $\hom(\Omega_{(X_0)_{/k}}, \mathcal{O}_{X_0}) = H^0(T_{X_0})$.

Upcoming: proof of Schlessinger so we can use it!

13 | Thursday April 2nd

13.1 Abstract Deformations

Let X_0 be smooth and consider the deformation functor

$$\begin{aligned} F : \mathrm{Art}_{/k} &\to \mathrm{Set} \\ A &\mapsto (X_{/A}, \iota) \end{aligned}$$

where X is flat (and thus smooth) and i is a closed embedding $i : X_0 \hookrightarrow X$ with $i \otimes k$ an isomorphism.

Then F has an obstruction theory with

- $\mathrm{def}(F) = H^1(X_0, T_0)$ *of the tangent bundle*
- $\mathrm{obs}(F) = H^2(X_0, T_0)$.

Additionally assume X_0 is smooth and projective, which will force the above cohomology groups to be finite-dimensional over k.

Remark 13.1.1 ***(Key points):***

- All deformations of smooth affine schemes are trivial
- Automorphisms of a deformation X/A which are the identity on X_0 are $\mathrm{id} + \delta$ for δ a derivation in $\mathrm{Der}_k(\mathcal{O}_{X_0}) = \hom_{\mathcal{O}_{X_0}}(\Omega_{(X_0)_{/k}}, \mathcal{O}_{X_0})$.

See screenshot.

Suppose we have a small thickening $k \to \mathbb{A}^1 \to \mathbb{A}$ and $X/\mathbb{A}$ with an affine cover X_α of X. This comes with gluing information $\varphi_{\alpha\beta} : X_{\alpha\beta} \to X_{\beta\alpha} = X_\alpha \cap X_\beta$. These maps satisfy a cocycle condition:

$$\begin{tikzcd} X_{\alpha\beta} \cap X_{\alpha\gamma} \arrow[rr] \arrow[dr] & & X_{\gamma\alpha} \cap X_{\gamma\beta} \arrow[dl] \\ & X_{\beta\alpha} \cap X_{\beta\gamma} & \end{tikzcd}$$

Question 13.1.2
Can we extend this to $X'/\mathbb{A}$?

We have $X_\alpha \cong X_\alpha^{\mathrm{red}} \times \mathbb{A}$? Choose $\varphi'_{\alpha\beta}$ such that

$$\begin{tikzcd} X'_{\alpha\beta} \arrow[r, "\varphi'_{\alpha\beta}"] & X'_{\beta\alpha} = X^{\mathrm{red}}_{\beta\alpha} \times \mathbb{A} \\ X_{\alpha\beta} \arrow[u, hook] \arrow[r, "\varphi_{\alpha\beta}"] & X_{\beta\alpha} \arrow[u, hook] \end{tikzcd}$$

We need $\varphi'_{\alpha\beta}$ to satisfy the cocycle condition in order to glue. We want the following map to be the identity: $(\varphi'_{\alpha\gamma})^{-1}\varphi'_{\beta\gamma}\varphi'_{\alpha\beta}$. This is an automorphism of $X'_{\alpha\beta} \cap X'_{\alpha\beta}$ and is thus the identity in $\mathrm{Aut}(X_{\alpha\beta} \cap X_{\alpha\gamma})$*. So it makes sense to talk about*

$$\delta_{\alpha\beta\gamma} \coloneqq (\varphi'_{\alpha\gamma})^{-1}\varphi'_{\beta\gamma}\varphi'_{\alpha\beta} - \mathrm{id} \in M^0(T_{X^{\mathrm{red}}_{\alpha\beta\gamma}}).$$

Exercise 13.1.3 (?)
In parts,

1. $\delta_{\alpha\beta\gamma}$ is a 2-cocycle for T_{X_0}, so it has trivial boundary in terms of Cech cocycles. Thus $[\delta_{\alpha\beta\gamma}] \in H^2(T_{X_0})$.
2. The class $[\delta_{\alpha\beta\gamma}]$ is independent of choice of $\varphi'_{\alpha\beta}$, i.e. $\varphi'_{\alpha\beta} - \varphi''_{\alpha\beta} \in H^0((T_X)_{\alpha\beta})$ gives a coboundary η and thus $\delta = \delta' + \eta$. This yields $\mathrm{obs}(X) \in H^2(T_{X_0})$.
3. $\mathrm{obs}(X) = 0 \iff X$ lifts to some X' (i.e. a lift exists)

Remark 13.1.4: For the sufficiency, we have $\delta_{\alpha\beta\gamma} = \partial\eta_{\alpha\beta} \in H^0(T_{X_{\alpha\beta}})$. Let $\varphi''_{\alpha\beta} = \varphi'_{\alpha\beta} - \eta_{\alpha\beta}$, the claim is that $\varphi''_{\alpha\beta}$ satisfies the gluing condition. This covers the obstruction, so now we need to show that the set of lifts is a torsor for the action of the deformation space $\mathrm{def}(F) = H^1(T_{X_0})$. From an X', we obtain $X'_{\alpha\beta} \xrightarrow{\varphi'_{\alpha\beta}} X'_{\beta\alpha}$ where the LHS is isomorphic to $(X'_{\alpha\beta})^{\mathrm{red}} \times \mathbb{A}^r$? Given $\eta_{\alpha\beta} \in H^0(T_{X_{\alpha\beta}})$, then $\varphi'_{\alpha\beta} + \eta_{\alpha\beta} = \varphi''_{\alpha\beta}$ is another such identification.

Exercise 13.1.5 (?)
In parts

1. $\partial\eta_{\alpha\beta} = 0$.
2. Given an X' and 1-coboundary η, we get a new lift $X'' = X' + \eta$. If $[\eta] = [\eta'] \in H^1(T_{X_0})$, then $X' + \eta \cong X' + \eta'$.

By construction, $(X'+\eta)_\alpha \cong (X'+\eta')_\alpha$, but these may not patch together. However, if $[\eta] = [\eta']$ then this isomorphism can be modified by by ε defined by $\eta - \eta' = \partial\varepsilon$, and it patches.

Remark 13.1.6: This kind of patching is ubiquitous – essentially patching together local obstructions to get a global one. In general, there is a local-to-global spectral sequence that computes the obstruction space

13.2 Proving Schlessinger

13.2.1 The Schlessinger Axioms

H1 *For any two small thickenings*

$$A' \to A$$
$$A'' \to A$$

we have a natural map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

and we require that this map is surjective. So deformations agreeing on the sub glue together.

H2 *When $(A' \to A) = (k[\varepsilon] \to k)$ is the trivial extension, the map in H1 is an isomorphism.*

Doing things to first order is especially simple.

H3 *The tangent space of F is given by $t_F = F(k[\varepsilon])$, and we require that $\dim_k t_F < \infty$, which makes sense due to H2.*

H4 *If we have two equal small thickenings $(A' \to A) = (A'' \to A)$, then the map in H1 is an isomorphism.*

H4' *For $A' \to A$ small,*

$$t_F \circlearrowleft F(A') \to F(A)$$

is exact in the middle and left.

Remark 13.2.1: Note that the existence of this action uses H2.

⚠ **Warning 13.2.2**
For (R, ξ) a complete local ring and $\xi \in \widehat{F}(R)$ a formal family, this is a hull $\iff$ miniversal, i.e. for $h_R \xrightarrow{\xi} F$, this is smooth an isomorphism on tangent spaces.

Theorem 13.2.3 ***(1, Schlessinger).***

a. F has a miniversal family (R, ξ) with $\dim t_R < \infty$, noting that $t_R = \mathfrak{m}_R/\mathfrak{m}_R^2$, iff H1-H3 hold.
b. F has a universal family (R, ξ) with $\dim t_R < \infty$ iff h1-H4 hold.

Theorem 13.2.4 *(2).*

a. F having an obstruction theory implies H1-H3.
b. F having a strong obstruction theory (exact on the left) is equivalent to H1-H4.

Some preliminary observations:

Exercise 13.2.5 (Easy, fun, diagram chase)
If F has an obstruction theory, then H1-H3 hold.

Exercise 13.2.6 (?)
An obstruction theory being exact on the left implies H4.

13.2.2 Example

Exercise 13.2.7 (?)
For R a complete local k-algebra with t_R finite dimensional has a strong obstruction theory.

Can always find a surjection from a power series ring:

$$S \coloneqq k[[t_R^\vee]] \twoheadrightarrow R$$

which yields an obstruction theory

- def = t_R
- obs = $I/\mathfrak{m}_S I$

i.e., if F is pro-representable, then it has a strong obstruction theory. Suppose that (R,ξ) is versal for F, this implies H1. We get $F(A' \times_A A'') \twoheadrightarrow F(A') \times_{F(A)} F(A'')$ For versal, if we have $h_R \xrightarrow{\xi} F$ smooth, we have

$$\begin{tikzcd}
 & & & h_r \arrow[d] \\
h_k \arrow[r] \arrow[urrr] & h_A \arrow[r] \arrow[urr, dotted] \arrow[rr, bend right, "\eta"'] & h_{A'} \arrow[r] \arrow[ur, dotted] & F
\end{tikzcd}$$

and we can find a lift from $h_{A''}$ as well, so we get a diagram

$$\begin{tikzcd}
 & & F \\
h_{A''} \arrow[r] & h_R \arrow[ur] & \\
h_A \arrow[r] \arrow[u] & h_{A'} \arrow[u] &
\end{tikzcd}$$

and thus

$$\begin{tikzcd} A'' \arrow[r] & R \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

So we get the left $\tilde{\eta}$ of (η', η'') we want from

$$\begin{tikzcd} h_{A' \times_A A''} \arrow[r, "f"] \arrow[rr, bend right, "\tilde{\eta}"'] & h_R \arrow[r] & F \end{tikzcd}$$

If (R, ξ) is miniversal, then H2 holds. We want to show that the map

$$F(A'' \times_K k[\varepsilon]) \xrightarrow{\sim} ??$$

is a bijection.

Suppose we have two maps

$$\begin{tikzcd} & & h_R \\ h_{A''} \arrow[r] \arrow[urr, bend left] & h_{A'' \times k[\varepsilon]} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[r, shift left] \arrow[r, shift right] & F \\ & h_{k[\varepsilon]} \arrow[ur, bend right] & \end{tikzcd}$$

Then the two lifts are in fact equal, and

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & A'' \times k[\varepsilon] \arrow[r] \arrow[d] & k[\varepsilon] \\ & A'' & \end{tikzcd}$$

If (R, ξ) is miniversal with t_R finite dimensional, then H3 holds immediately. If (R, ξ) is universal, then H4 holds.

Question 13.2.8
Why are H4 and H4' connected?

Answer 13.2.9
Let $A' \to A$ be small, then

$$A' \times_A A' = A' \times_k k[\varepsilon](x, y) \qquad \mapsto ??.$$

Using H2, we can identify $F(A; \times_A A') \cong t_F \times F(A')$. We can thus define an action

$$(\theta, \xi) \mapsto (\theta + \xi, \xi).$$

If this is an isomorphism, then this action is simply transitive. The map $\theta \mapsto \theta + \xi$ gives an isomorphism on the fiber of $F(A') \to F(A)$.

Next time we'll show the interesting part of the sufficiency proof.

14 | Tuesday April 7th

(Missing first few minutes.)

Take I_{q+1} to be the minimal I such that $\mathfrak{m}_q I_q \subset I \subset I_1$ and ξ_q lifts to S/I.

Claim: Such a minimal I exists, i.e. if I, I' satisfy the two conditions then $I \cap I'$ does as well. So I, I' are determined by their images v, v' in the vector space $I_q \otimes k$.

So enlarge either v or v' such that $v + v' = I_q \otimes k$ but $v \cap v'$ is the same. We can thus assume that $I + I' = I_q$, and so

$$S/I \cap I' = S/I \times_{S/I_q} S/I'$$

which by H1 yields a map

$$F(S/I \cap I') \to F(S/I) \times_{F(S/I_q)} F(S/I')$$

So $I \cap I'$ satisfies both conditions and thus a minimal I_{q+1} exists. Let ξ_{q+1} be a lift of ξ_q over S/I_{q+1} (noting that there may be many lifts).

14.1 Showing Miniversality

Claim: Define $R = \varinjlim R_q$ and $\xi = \varinjlim \xi_q$, the claim is that (R, ξ) is miniversal.

We already have $h_R \xrightarrow{\xi} F$ and thus $t_R \xrightarrow{\cong} t_F$ is fulfilled. We need to show formal smoothness, i.e. for $A' \to A$ a small thickening, suppose we have a lift

$$\begin{tikzcd}
 & & h_R \arrow[d, "\xi"] \\
h_a \arrow[r] \arrow[urr, "n"] \arrow[rr, bend right] & h_{A'} \arrow[r] & F
\end{tikzcd}$$

If we have a u' such that commutativity in square 1 holds (?) then we can form a lift u' satisfying commutativity in both squares 1 and 2. We can restrict sections to get a map $F(A') \to F(A)$ and using representability obtain $h_R(A') \to h_R(A)$. Combining H1 and H2, we know t_F acts transitively on fibers, yielding

$$\begin{tikzcd}
t_R \circlearrowleft \arrow[d, "\cong"] & u' \in h_R(A') \arrow[r] \arrow[d] & u \in h_R(A) \arrow[d] \\
t_F \circlearrowleft & \eta' \in F(A') \arrow[r] & \eta \in F(A)
\end{tikzcd}$$

Then $u' \mapsto u$ is equivalent to (1), and $u' \mapsto \eta'$ is equivalent to (2). Let η_0 be the image of u' and define $\eta' = \eta_0 + \theta, \theta \in t_F$ then $u' = u' + \theta, \theta \in t_R$. So we can modify the lift to make these agree. Thus it suffices to show

$$\begin{tikzcd}
A' \arrow[r] & A & R_q \arrow[l] \\
S \arrow[u, "v"] \arrow[r] & {} \arrow[u, "u"] \arrow[ur] \arrow[ul, dotted, "\exists_? u'"] &
\end{tikzcd}$$

We get a diagram of the form

$$\begin{tikzcd}
S \arrow[r, "w"] \arrow[d] & A' \times_A R_1 \arrow[r] \arrow[d, "{\pi_2, small}"] & A' \arrow[d, "small"] \\
R \arrow[r] & R_q \arrow[r] & A
\end{tikzcd}$$

Observation 14.1.1

- $S \to R_q$ is surjective.

- $\mathrm{im}(w) \subset A' \times_A R_1$ is a subring, so either
    - $\mathrm{im}(w) \xrightarrow{\cong} R_q$ if it doesn't meet the kernel, or
    - $\mathrm{im}(w) = A' \times_A R_q$

In case (a), this yields a section of the middle map and we'd get a map $R_q \to A'$ and thus the original map we were after $R \to A$.

So assume w is surjective and consider

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & S \arrow[r] & A' \times_A R_q \arrow[r] \arrow[d, "small"] & 0 \\
& & & R_q &
\end{tikzcd}$$

and we have $\mathfrak{m}_S I_1 \subset I \subset I_q$ where the second containment is because I a quotient of R_q factors through S/I and the first is because S/I is a small thickening of R_q. But ξ_q lifts of S/I, and we have

$$\xi \in F(S/I) \twoheadrightarrow \xi = \xi' \times \xi_q?.$$

Therefore $I_{q+1} \subset I$ and we have a factorization

$$\begin{tikzcd}
S \arrow[rr] \arrow[dr, dotted] & & S/I \\
& R_{q+1} \arrow[ur, dotted] &
\end{tikzcd}$$

Recall that we had

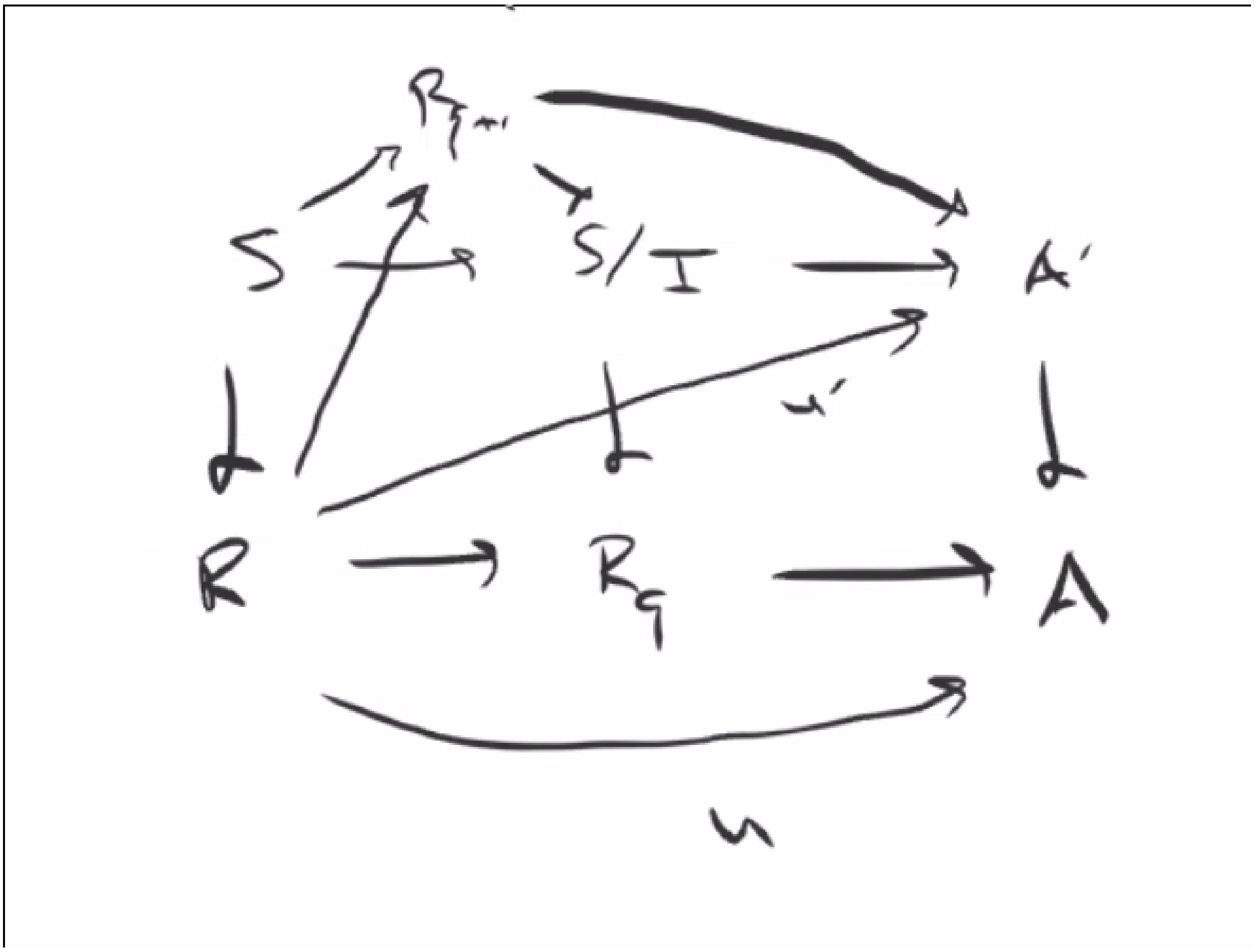


Image to diagram

where the diagonal map u' gives us the desired lift, and thus

$$\begin{tikzcd} R \arrow[r] \arrow[rr, bend left] & R_{q+1} \arrow[r] & A' \end{tikzcd}$$

exists. This concludes showing miniversality.

14.2 Part of Proof

To finish, we want to show that H4 implies that the map on sections $h_R \xrightarrow{\xi} F$ is bijective.

where the map ξ is "formal etale", which will necessarily imply that it's a bijection over all artinian rings. So we just need to show formal étaleness. We have a diagram

<!–> Content: Hartshorne's Deformation Theory, section in FGA is in less generality but has many good examples. See "Fundamental Algebraic Geometry". See also representability of the Picard scheme.–>

15 | Thursday April 9th

Let $F : \mathrm{Art}_{/k} \to \mathrm{Set}$ *be a deformation functor with an obstruction theory. Then H1-H3 imply the existence of a miniversal family, and gives us some control on the hull* $h_R \to F$*, namely*

$$\dim \operatorname{def}(F) \geq \dim R \geq \dim \operatorname{def}(F) - \dim \operatorname{obs}(F).$$

In particular, if $\operatorname{obs}(F) = 0$*, then* $R \cong k[[\operatorname{def}(F)^\vee]] = k[[t_F^\vee]]$.

Example 15.0.1 ***(?):*** Let $M = \mathrm{Hilb}_{\mathbb{P}^n_{/k}}^{dt+(1-g)}$ where $k = \bar{k}$, and suppose $[Z] \in M$ is a smooth point.

Then

$$\operatorname{def} = \hom_{\mathcal{O}_x\text{-}\mathrm{mod}}(I_Z, \mathcal{O}_Z) = \hom_Z(I_Z/I_Z^2, \mathcal{O}_Z) = H^0(N_{Z/X}).$$

the normal bundle $N_{Z/X} = (I/I^2)^\vee$ of the regular embedding, and obs $= H^1(N_{Z/X})$.

Claim: If $H^1(\mathcal{O}_Z(1)) = 0$ (e.g. if $d > 2g - 2$) then M is smooth.

Proof (of claim).
The tangent bundle of $\mathbb{P}^n$ sits in the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

And the normal bundles satisfies

$$0 \to T_Z \to T_{\mathbb{P}^n}\Big|_Z \to N_{Z/\mathbb{P}^n} \to 0$$

$$\Downarrow \text{ is the dual of}$$

$$0 \to I/I^2 \to \Omega\Big|_Z \to \Omega \to 0.$$

There is another SES:

$$?????.$$

Taking the LES in cohomology yields

$$H^1(\mathcal{O}_Z(1)^{n+1}) = 0 \to H^1(N_{Z/\mathbb{P}^n}) = 0 \to 0$$

and thus M is smooth at $[Z]$. We can compute the dimension using Riemann-Roch:

$$\begin{aligned}\dim_{[Z]} M &= \dim H^0(N_{Z/\mathbb{P}^n}) \\ &= \chi(N_{Z/\mathbb{P}^n}) \\ &= \deg N + \operatorname{rank} N(1-g) \\ &= \deg T_{\mathbb{P}^n}\Big|_Z - \deg T_Z + (n-1)(1-g) \\ &= d(n+1) + (2-2g) + (n-1)(1-g).\end{aligned}$$

■

Remark 15.0.2: This is one of the key outputs of obstruction theory: being able to compute these dimensions.

Example 15.0.3 ***(?):*** Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface and let $H = \mathrm{Hilb}_{X/k}^{\text{lines}=t+1} \subset \mathrm{Hilb}_{\mathbb{P}^5/k}^{t+1} = \mathrm{Gr}(1, \mathbb{P}^5)$, the usual Grassmannian.

Claim: Let $[\ell] \in H$, then the claim is that H is smooth at $[\ell]$ of dimension 4.

Proof (of claim).
We have

- def $= H^0(N_{\ell/X})$
- obs $= H^1(N_{\ell/X})$

We have an exact sequence

$$0 \to N_{\ell/X} \to N_{\ell/\mathbb{P}} \to N_{X/\mathbb{P}}\Big|_\ell \to 0$$
.

There are surjections from $\mathcal{O}_\ell(1)^6$ onto the last two terms.

Claim Subclaim: For $N = N_{\ell/\mathbb{P}}$ or $N_{X/\mathbb{P}}\Big|_\ell$, we have $H^1(N) = 0$ and $\mathcal{O}(1)^6 \twoheadrightarrow N$ is surjective on global sections.

Proof (of subclaim).
Because ℓ is a line, $\mathcal{O}_\ell(1) = \mathcal{O}(1)$ and $H^1(\mathcal{O}_\ell(1)) = 0$ and the previous proof applies, so $H^1(N) = 0$.

■

We thus have a diagram:

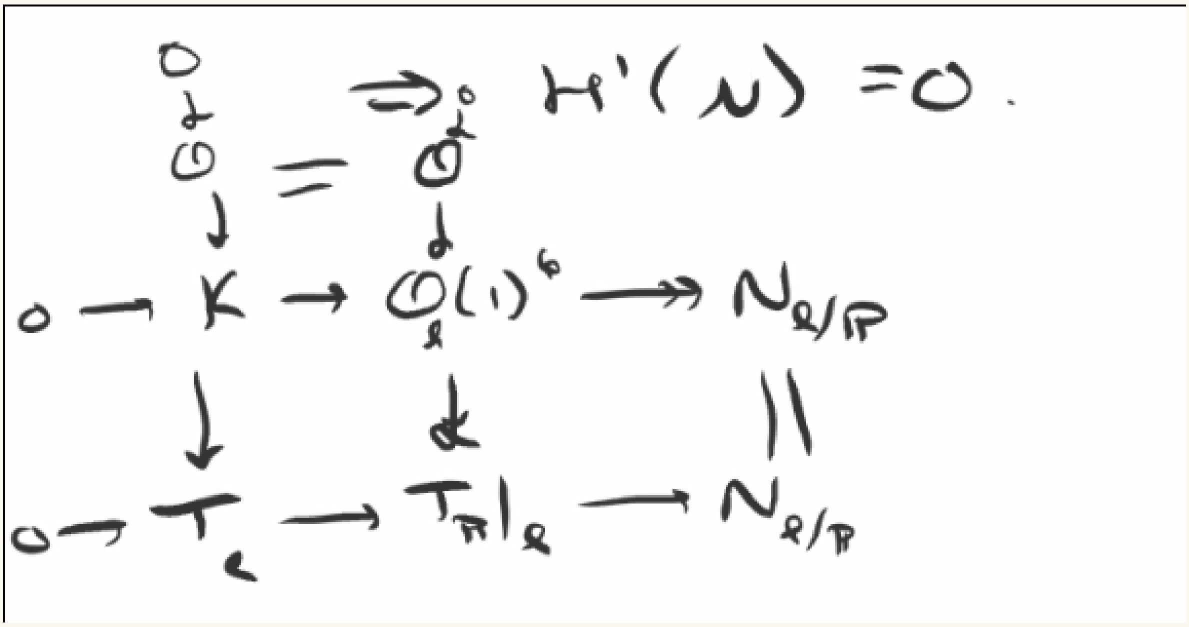
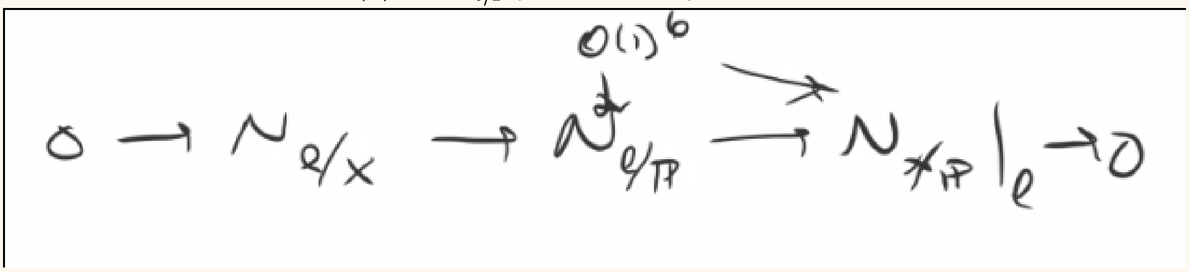
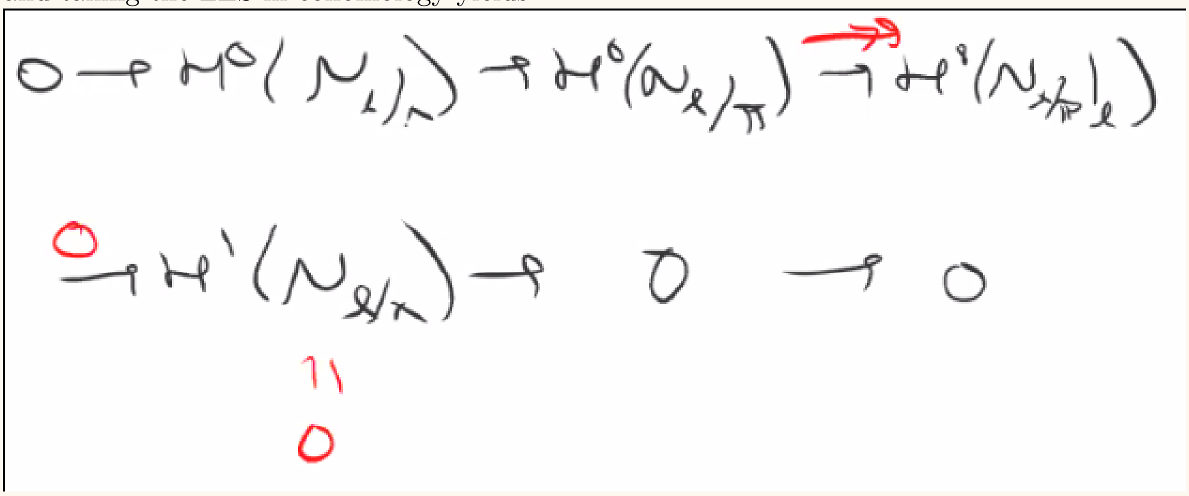


Figure 6: Image

In particular, $T_\ell = \mathcal{O}(2)$, and the LES for $0 \to \mathcal{O} \to K \to T_\ell$ shows $H^1(K) = 0$. Looking at the horizontal SES $0 \to K \to \mathcal{O}_\ell(1)^6 \twoheadrightarrow N_{\ell/\mathbb{P}}$ yields the surjection claim. We have



and taking the LES in cohomology yields



Therefore H is smooth at ℓ and

$$\begin{aligned}\dim_\ell H &= \chi(N_{\ell/X}) \\ &= \deg T_X - \deg T_\ell + 3 \\ &= \deg T_{\mathbb{P}} - \deg N_{X/\mathbb{P}} - \deg T_\ell + 3 \\ &= 6 - 3 - 2 + 3 = 4.\end{aligned}$$

■

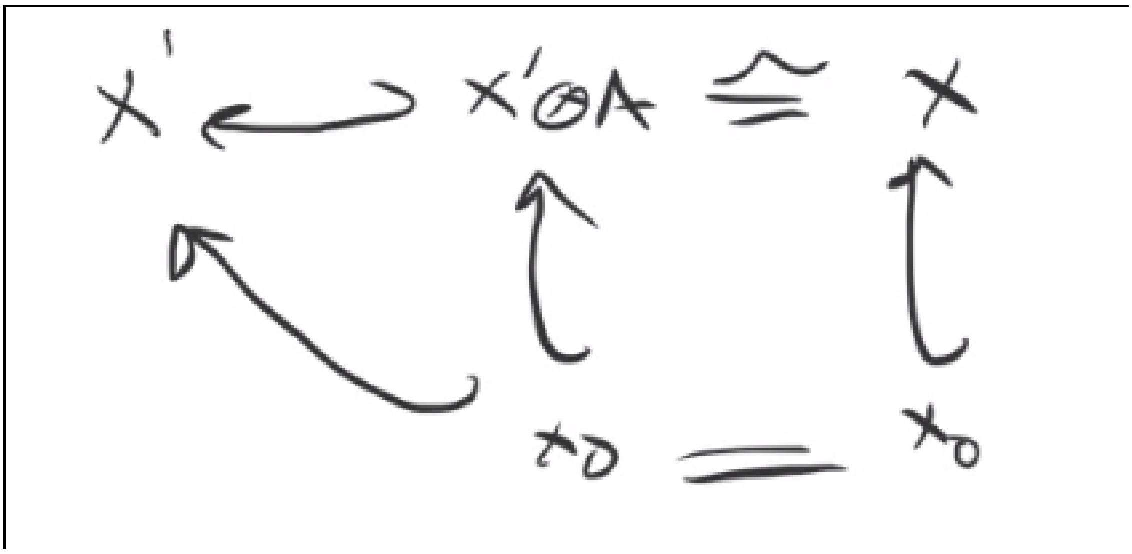
Remark 15.0.4: It turns out that the Hilbert scheme of lines on a cubic has some geometry: the Hilbert scheme of two points on a K3 surface.

15.1 Abstract Deformations Revisited

Take X_0/k some scheme and consider the deformation functor $F(A)$ taking A to X/A flat with an embedding $\iota : X_0 \hookrightarrow X$ with $\iota \otimes k$ an isomorphism. Start with H1, the gluing axiom (regarding small thickenings $A' \to A$ and a thickening $A'' \to A$). Suppose

$$X_0 \hookrightarrow X' \in F(A') \to F(A).$$

which restricts to $X_0 \hookrightarrow X$. Then in $F(A)$, we have $X_0 \hookrightarrow X' \otimes_{A'} A$, and we obtain a commutative diagram where $X' \otimes A \hookrightarrow X'$ is a closed immersion:



The restriction $X' \to X$ means that there exists a diagram

$$\begin{tikzcd} X' & & X \arrow[ll, dotted, "\exists"'] \\ & X \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Note that this is not necessarily unique. We have

$$H^1 \quad F(A' \times_A A'') \longrightarrow F(A) \times_{F(A)} F(A'')$$

$$\begin{array}{ccc} X' & X & X'' \\ \uparrow & \uparrow & \uparrow \\ x_0 & x_0 & x_0 \end{array}$$

This means that we can find embeddings such that

$$\begin{tikzcd} Z = \mathcal{O}_{X'} \times_{\mathcal{O}_X} \mathcal{O}_{X''} \arrow[r] \arrow[d] & \mathcal{O}_{X''} \arrow[d] \\ \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

And thus if we have

$$\begin{tikzcd}
Z & & X' \arrow[ll, hook'] \\
X'' \arrow[u, hook] \arrow[urr, phantom, "\circlearrowleft"] & & X \arrow[ll, hook'] \arrow[u, hook] \\
& & & X_0 \arrow[ul, hook']
\end{tikzcd}$$

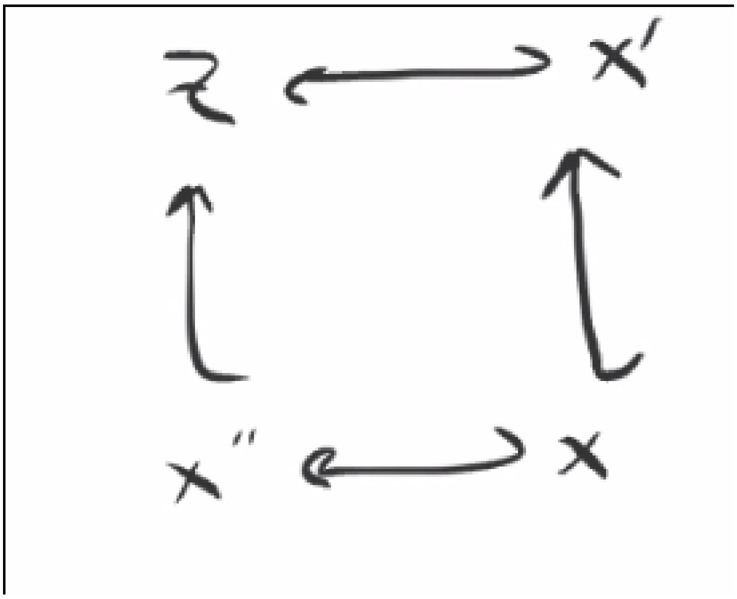
*then $X_0 \hookrightarrow Z$ is **a** required lift (again not unique).*

Question 15.1.1
When is such a lift unique?

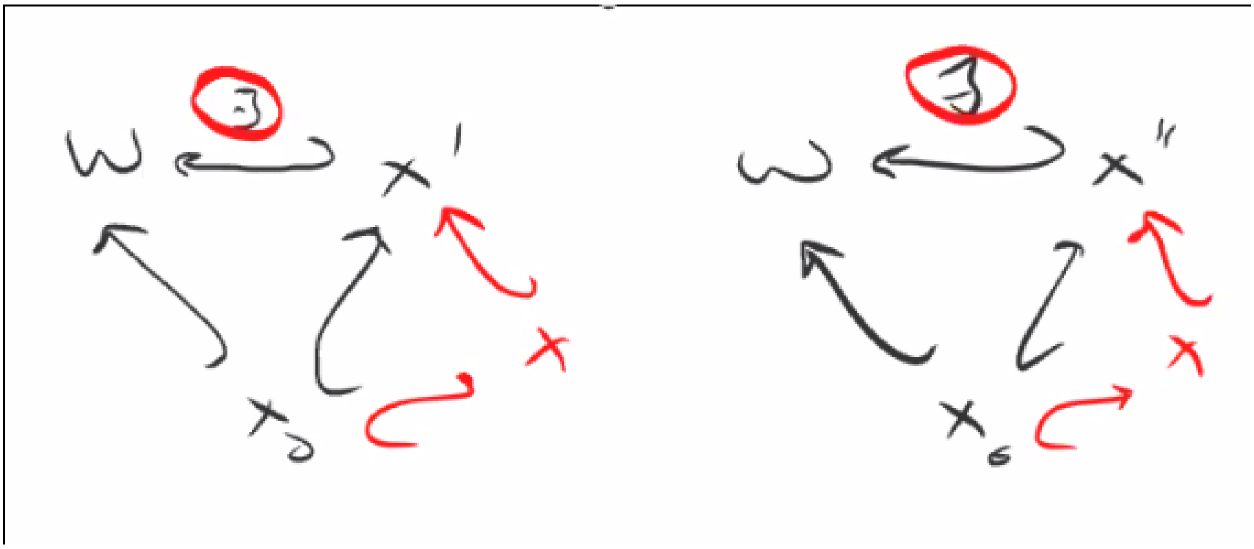
Suppose $X_0 \hookrightarrow W$ is another lift, then it restricts to both X, X' and we can fill in the following diagrams:

$$\begin{tikzcd}
W & & X' \arrow[ll, hook', "\exists"'] \\
& X_0 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
W & & X'' \arrow[ll, hook', "\exists"'] \\
& X_0 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

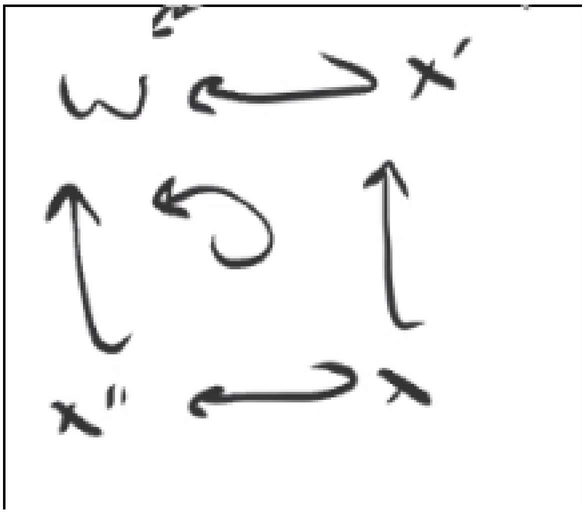
Using the universal property of Z, which is the coproduct of this diagram:



However, there may be no such way to fill in the following diagram:



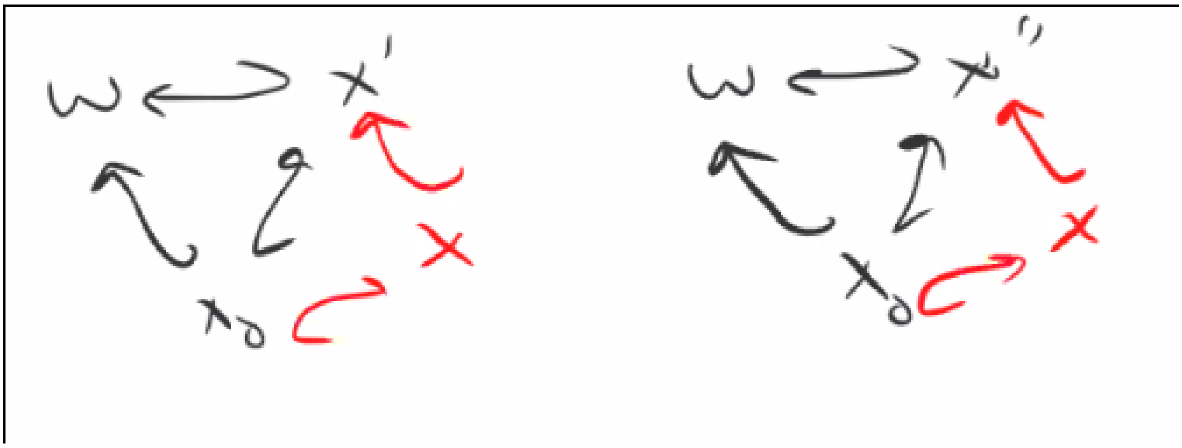
But if there exists a map making this diagram commute:



Then there is a map $Z \to W$ which is flat after tensoring with k, which is thus an isomorphism.[2]

Remark 15.1.2: Thus the lift is unique if

- $X = X_0$, then the following diagrams commute by taking the identity and the embedding you have. Note that in particular, this implies H2.



- Generally, these diagrams can be completed (and thus the gluing maps are bijective) if the map

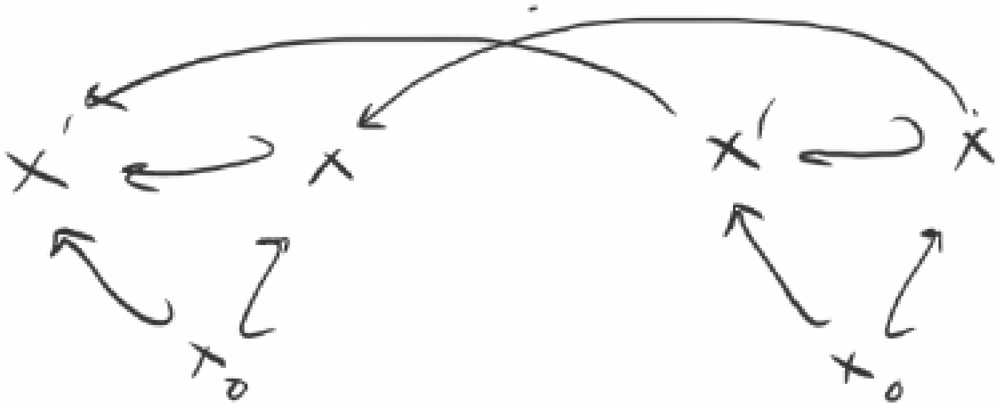
$$\mathrm{Aut}(X_0 \hookrightarrow X') \to \mathrm{Aut}(X_0 \hookrightarrow X).$$

  of automorphisms of X' commuting with $X_0 \hookrightarrow X$ is surjective.

So in this situation, there is only one *way to fill in this diagram up to isomorphism:*

[2]Recall that by Nakayama, a nonzero module tensor k can not be zero.

If we had two ways of filling it in, we obtain bridging maps:



Lemma 15.1.3 *(?).*
If $H^0(X_0, T_{X_0}) = 0$ (where the tangent bundle always makes sense as the dual of the sheaf of Kahler differentials) which we can identify as derivations $D_{\mathcal{O}_k}(\mathcal{O}_{X_0}, \mathcal{O}_{X_0})$, then the gluing map is bijective.

Proof (?).
The claim is that $\mathrm{Aut}(X_0 \hookrightarrow X) = 1$ are always trivial. This would imply that all random choices lead to triangles that commute. Proceeding by induction, for the base case $\mathrm{Aut}(X_0 \hookrightarrow X_0) = 1$ trivially. Assume $X_0 \hookrightarrow X_i$ lifts $X_0 \hookrightarrow X$, then there's an exact sequence

$$0 \to \mathrm{Der}_k(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}) \to \mathrm{Aut}(X_0 \hookrightarrow X_0') \to \mathrm{Aut}(X_0 \hookrightarrow X).$$

■

Thus F always satisfies H1 and H2, and $H^0(T_{X_0}) = 0$ (so no "infinitesimal automorphism") implies H4. Recall that the dimension of deformations of F over $k[\varepsilon]$ is finite, i.e. $\dim t_F < \infty$ *This is* *where some assumptions are needed.*

If $X_{/K}$ is either

- *Projective, or*
- *Affine with isolated singularities,*

this is enough to imply H3. Thus by Schlessinger, under these conditions F has a miniversal family. Moreover, if $H^0(T_{X_0}) = 0$ then F is pro-representable.

Example 15.1.4*(?)*: If X_0 is a smooth projective genus $g \geq 2$ curve, then

- Obstruction theory gives the existence of a miniversal family
- We have obs $= H^2(T_{X_0}) = 0$, and thus the base of the miniversal family is smooth of dimension $\operatorname{def}(F) \dim H^1(T_{X_0})$,
- $H^0(T_{X_0}) = 0$ and $\deg T_{X_0} = 2 - 2g < 0$, which implies that the miniversal family is universal.

We can conclude

$$\dim H^1(T_{X_0}) = -\chi(T_{X_0}) = -\deg T_{X_0} + g - 1 = 3(g-1).$$

Remark 15.1.5: Note that the global deformation functor is not representable by a scheme, and instead requires a stack. However, the same fact shows smoothness in that setting.

15.2 Hypersurface Singularities

Consider $X(f) \subset \mathbb{A}^n$, and for simplicity, $(f = 0) \subset \mathbb{A}^2$, and let

- $S = \mathbb{C}[x, y]$.
- $B = \mathbb{C}[x, y]/(f)$

Question 15.2.1
What are the deformations over $A := k[\varepsilon]$?

This means we have a ring B' flat over k and tensors to an isomorphism, so tensoring $k \to A \to k$ yields

Thus any such B' is the quotient of $S[\varepsilon]$ by an ideal. We have $f' = f + \varepsilon g$.

Question 15.2.2
When do two f's give the same B'?

We have $\varepsilon f' = \varepsilon f$, *so* $\varepsilon f \in (f')$ *and we can modify* g *by any* cf *where* $c \in S$, *where only the equivalence class* $g \in S/(f)$ *matters. Now consider* $\operatorname{Aut}(B \hookrightarrow B')$, *i.e. maps of the form*

$$\begin{aligned} x &\mapsto x + \varepsilon a \\ y &\mapsto y + cb \end{aligned}$$

for $a, b \in S$.

Under this map,

$$f_0' = f + \varepsilon g \mapsto f(x + \varepsilon a, y + \varepsilon b) + \varepsilon g(x, y)$$

$$\Downarrow \quad \textit{implies}$$

$$f(x, y) = \varepsilon a \frac{\partial}{\partial x} f + \varepsilon b \frac{\partial}{\partial y} f + \varepsilon g(x, y),$$

so in fact only the class of $g \in S/(f, \partial_x f, \partial_y f)$. *This is the ideal of the singular locus, and will be Artinian (and thus finite-dimensional) if the singularities are isolated, which implies H3.*

We can in fact exhibit the miniversal family explicitly by taking $g_i \in S$, *yielding a basis of the above quotient. The hull will be given by setting* $R = \mathbb{C}[[t_1, \cdots, t_m]]$ *and taking the locus* $V(f + \sum t_i g_i) \subset \mathbb{A}^2_R$.

Example 15.2.3 ***(simple):*** For $f = xy$, then the ideal is $I = (xy, y, x) = (x, y)$ and C/I is 1-dimensional, so the miniversal family is given by $V(xy + t) \subset \mathbb{C}[[t_1]][x, y]$. The greater generality is needed because there are deformation functors with a hull but no universal families.

16 | Tuesday April 14th

Recall that we are looking at $(X_0)_{/k}$ *and* $F : \mathrm{Art}_{/k} \to \mathrm{Set}$ *where* A *is sent to* $X_{/A}$ *flat with* $i : X_0 \hookrightarrow X$ *where* $i \otimes k$ *is an isomorphism. The second condition is equivalent to a cartesian diagram*

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} A \end{tikzcd}$$

We showed we always have H1 and H2, and H3 if X_0/k *is projective or* X_0 *is affine with isolated singularities. In this situation we have a miniversal family. This occurs iff for* $A' \to A$ *a small thickening and* $(X_0 \hookrightarrow X) \in F(A)$, *we have a surjection*

$$\operatorname{Aut}_{A'}(X_0 \hookrightarrow X') \twoheadrightarrow \operatorname{Aut}_A(X_0 \hookrightarrow X).$$

where the RHS are automorphisms of $X_{/A}$, i.e. those which commute with the identity on A and X_0. We had a naive functor F_n where we don't include the inclusion $X_0 \hookrightarrow X$. When F has a hull then the naive functor has a versal family, since there is a forgetful map that is formally smooth. If it's the case that for all $A' \to A$ small and $F_n \to F_n(A)$ we have $\mathrm{Aut}_{A'}(X') \twoheadrightarrow \mathrm{Aut}_A(X)$, then $F = F_n$ and both are pro-representable. The forgetful map is smooth because given $X_{/A}$ in $F_n(A)$, we have some inclusion $X_0 \hookrightarrow X$, so one gives surjectivity. Using the surjectivity on automorphisms, we get

$$\begin{tikzcd}
X_0 \arrow[rr, hook] \arrow[dr, hook] & & X \arrow[dl, dotted] \\
& X &
\end{tikzcd}$$

Deformation theory is better at answering when the following diagrams exist:

$$\begin{tikzcd}
X \arrow[r, hook, dotted, "\exists?"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d, dotted, "\exists?"] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} A'
\end{tikzcd}$$

i.e., the existence of an extension of X to A'. This is different than understanding diagrams of the following type, where we're considering isomorphism classes of the squares, and deformation theory helps understand the blue one:

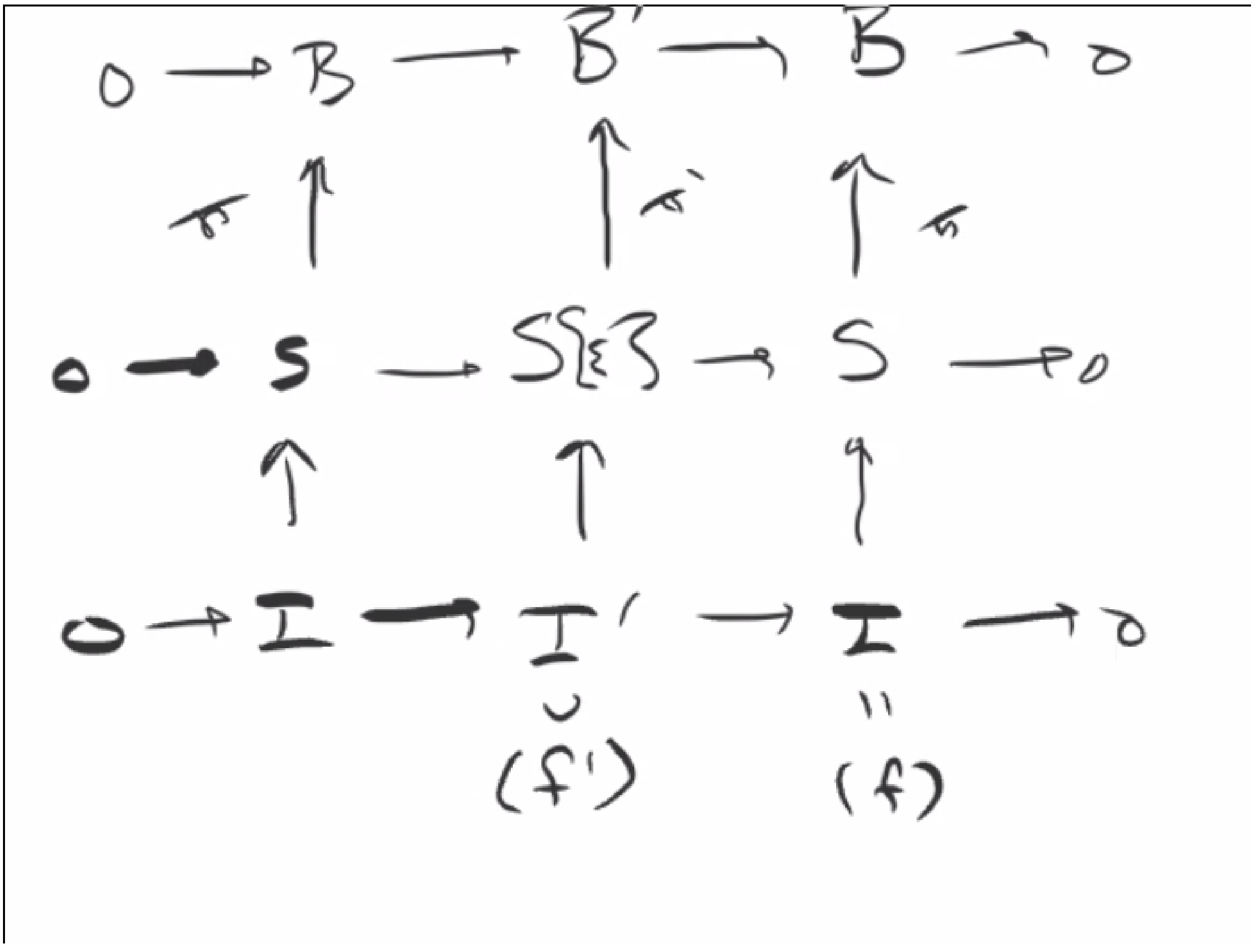
$$\begin{tikzcd}
& F(A') \arrow[r] & F(A) \\
X_0 \arrow[r, hook] \arrow[d] & X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A'
\end{tikzcd}$$

Example 16.0.1 ***(Hypersurface Singularities):*** Take $S = k[x, y]$ and $B = S/(f)$, then deformations of $\operatorname{Spec} B$ to ? Given $k \to k[\varepsilon] \to k$ we can tensor[3] to obtain

[3]For flat maps, tensoring up to an isomorphism implies isomorphism.

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r] \arrow[u, "\pi"] & S[\varepsilon] \arrow[r] \arrow[u, "\pi'"] & S \arrow[r] \arrow[u, "\pi"] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[u] & I' \arrow[r] \arrow[u] & I \arrow[r] \arrow[u] & 0 \\
& & \langle f' \rangle \arrow[u, phantom, sloped, "\subseteq"] & \langle f \rangle \arrow[u, phantom, sloped, "\subseteq"] &
\end{tikzcd}
$$

Link to diagram.



We want to understand $F(k[\varepsilon])$. We know $f' = f + \varepsilon g$ for some $g \in S$.

Observation 16.0.2

1. $g \in B$ and $f'' = f + \varepsilon(g + cf)$ generates the same ideal.
2. We're free to reparameterize, i.e. $x \mapsto x + \varepsilon a$ and $y \mapsto y + \varepsilon b$ and thus

$$g \mapsto g + af_x + bf_y$$

, i.e. the partial derivatives.

Thus isomorphism classes of B' in deformations $B' \to B$ only depend on the isomorphism classes $g \in B/(f_x, f_y)B$. When the singularities are isolated, this quotient is finite-dimensional as a k-vector space.

Example 16.0.3*(?):* $F(k[\varepsilon]) = B/(f_x, f_y)B$. Thus H3 holds and there is a miniversal family $h_R \to F$. We can describe it explicitly: take $g_i \in S$, yielding a k-basis in $S/(f, f_x, f_y)$. Then

$$V(f + \sum t_i g_i) \subset \operatorname{Spec} k[[t_1, \cdots, t_n]][x, y].$$

Set $R = k[[t_1, \cdots, t_n]]$, then this lands in $\mathbb{A}^2_R$.

Example 16.0.4*(?):* The nodal curve $y^2 = x^3$, take .

$$S/(y^2 - x^3, 2y, -3x^2) = S/(y, x^2).$$

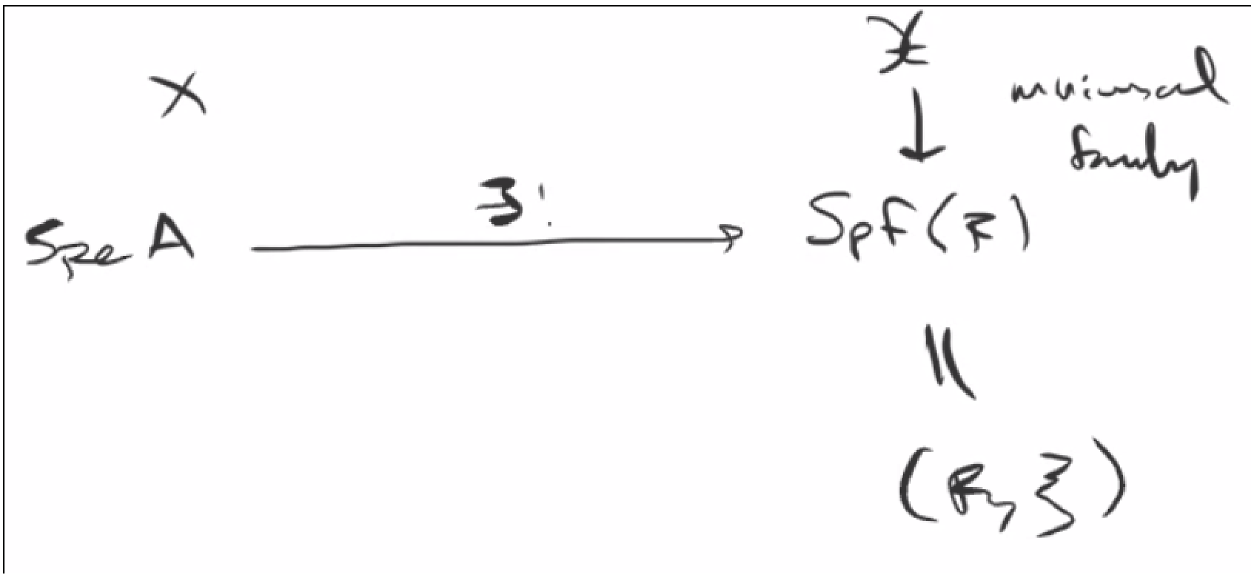
So take $g_1 = 1, g_2 = x$, then the miniversal family is .

$$V(y^2 - x^3 + t + t_2 x) \subset \mathbb{A}^2_{k[[t_1, t_2]]}.$$

This gives all ways of smoothing the node.

Remark 16.0.5: Note that none of these are pro-representable.

Given X and A, we obtain a miniversal family over the formal spectrum $\operatorname{Spf}(R) = (R, \xi)$ *and a unique map:*



We can take two deformations over $A = k[\xi]/S^n$:

- $X_1 = V(x+y)$??
- $X_2 = V(x+uy)$??

As deformations over A, $X_1 \cong X_2$ where we send ,

$$\begin{aligned} s &\mapsto s, \\ y &\mapsto y, \\ x &\mapsto ux. \end{aligned}$$

since

$$(xy + us) = (uxy + us) = (u(xy+s)) = (xy+s).$$

But we have two different classifying maps, which do commute up to an automorphism of A, but are not equal. Since they pullback to different elements (?), F can not be pro-representable.

$$\mathrm{Aut}_{k[\xi]/S^n}(X_0 \hookrightarrow X_1) \longrightarrow \mathrm{Aut}_{k[s]}(X_0 \hookrightarrow X_1)$$

u

$x \longmapsto ux$

$y \longmapsto y$

$s \longmapsto s$

has no lift

because we would

have to send

$s \longmapsto us$

$\implies$ <u>not</u> an A-automorphism.

So reparameterization in A yield different objects in $F(A)$. In other words, $\mathcal{X} \to \mathrm{Spf}(R)$ has automorphisms inducing reparameterizations of R. This indicates why we need maps restricting to the identity.

16.1 The Cotangent Complex

For $X \xrightarrow{f} Y$, we have $L_{X/Y} \in D\mathrm{QCoh}(X)$, the derived category of quasicoherent sheaves on X. This answers the extension question: For any square-zero thickening $Y \hookrightarrow Y'$ (a closed immersion) with ideal I yields an $\mathcal{O}_Y$-module.

1. *An extension exists iff $0 = \mathrm{obs} \in \mathrm{Ext}^2(L_{X/Y}, f^*I)$*
2. *If so, the set of ways to do so is a torsor over this ext group.*
3. *The automorphisms of the completion are given by $\hom(L_{X/Y}, f^*I)$.*

Special cases: $X \to Y$ smooth yields $L_{X/Y} = \Omega_{X/Y}[0]$ concentrated in degree zero. Example: $Y = \mathrm{Spec}\, k$ and $Y' = \mathrm{Spec}\, k[\varepsilon]$ yields

$$\mathrm{obs} \in \mathrm{Ext}^2_x(\Omega_{X/Y}, \mathcal{O}_x) = H^2(T_{X_{/k}}).$$

For $X \hookrightarrow Y$ is a regular embedding (closed immersion and locally a regular sequence) $L_{X/Y} = \left(I/I^2\right)[1]$, the conormal bundle.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$

exact triangle:

$$f^* L_{Y/Z} \longrightarrow L_{X/Z} \longrightarrow L_{X/Y} \longrightarrow f^* L_{Y/Z}[1]$$

Example 16.1.1 *(?):* For Y smooth, $X \hookrightarrow Y$ a regular embedding, $L_{X_{/k}} = \Omega_{X_{/k}}$ with obs/def = $\mathrm{Ext}^{2/1}(\Omega_x, \mathcal{O})$ and the infinitesimal automorphisms are the homs.

Example 16.1.2*(?):* For $Y = \operatorname{Spec} k[x,y] = \mathbb{A}^2$ and $X = \operatorname{Spec} B = V(f) \subset \mathbb{A}^2$ we get

$$0 \to I/I^2 \to \Omega_{X_{/k}} \otimes B \to \Omega_? X_{/k} \to 0$$

$$\Downarrow \quad \text{equals}$$

$$0 \to B \xrightarrow{1 \mapsto (f_x, f_y)} B^2 \to \Omega_{B_{/k}} = L_{X_{/k}} \to 0.$$

Taking $\hom(\,\cdot\,, B)$ yields

$$\begin{tikzcd}
0 \arrow[r] & \hom(\Omega, B) \arrow[r] & B^2 \arrow[dll, "(f_x,f_y)^t"'] \\
\mathrm{Ext}^1(\Omega, B) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
\mathrm{Ext}^2(\Omega, B) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So ,

$$\begin{aligned}
\text{obs} &= 0 \\
\text{def} &= B/(f_x, f_y)B \\
\text{Aut} &\neq 0.
\end{aligned}$$

and

Remark 16.1.3: We have the following obstruction theories:

- For abstract deformations, we have
$$X_{0/k} \text{ smooth} \implies \operatorname{Aut}/\text{def}/\text{obs} = H^{0/1/2}(T_{X_0}).$$

- For embedded deformations, Y_0/k smooth, $X_0 \hookrightarrow Y_0$ regular, we have
$$\operatorname{Aut}/\text{def}/\text{obs} = 0, H^{0/1}(N_{X_0/Y_0}).$$

As an exercise, interpret this in terms of L_{X_0/Y_0}.

- For maps $X_0 \xrightarrow{f_0} Y_0$, i.e. maps
$$X_0 \times k[\varepsilon] \xrightarrow{f} Y_0 \times k[\varepsilon].$$
we consider the graph $\Gamma(f_0) \subset X_0 \times Y_0$.

$$\Gamma(f_0) \subset X_0 \times Y_0$$

$$\begin{tikzcd}
 & & f^* T_{Y_0} \arrow[d, phantom, "\cong" rotate=90] & \\
0 \arrow[r] & T_\Gamma \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & T_{X_0} \oplus f_0^* T_{Y_0} \arrow[r] & N_\Gamma \arrow[r] & 0 \\
 & T_{X_0} & & &
\end{tikzcd}$$

Since all of these structures are special cases of the cotangent complex, they place nicely together in the following sense: Given $X \hookrightarrow_i Y$ we have

$$0 \to T_X \to i^* T_Y \to N_{X/Y} \to 0.$$

Yielding a LES

$$\begin{aligned}
0 &\to H^0(T_X) \to H^0(i^* T_Y) \to H^0(N_{X/Y}) \\
&\to H^1(T_X) \to H^1(i^* T_Y) \to H^1(N_{X/Y}) \\
&\to H^2(T_X).
\end{aligned}$$

$$0 \to T_{X_0} \to i_0^* T_{Y_0} \to N_{X_0/Y_0} \to 0$$

$$0 \to H^0(T_{X_0}) \to \overset{\text{def map.}}{\underline{H^0(i^* T_Y)}} \to \overset{\text{Embed. def.}}{\underline{H^0(N_{X/Y})}}$$

$$\to \overset{\text{Ab. def}}{H^1(T_X)} \xrightarrow{\text{obs}} \underset{\text{Obs(map)}}{\underline{H^1(i^* T_Y)}} \to H^1(N_{X/Y})$$

$$\to H^2(T_X)$$

Exercise 16.1.4 (?)
Consider $X \subset \mathbb{P}^3$ a smooth quartic, and show that $\operatorname{def}(X) \cong k^{20}$ but $\operatorname{def}_{\text{embedded}} \cong k^{19}$. This is a quartic K3 surface for which deformations don't lift (non-algebraic, don't sit inside any $\mathbb{P}^n$).

Next time: Obstruction theory of sheaves, T1 lifting as a way to

17 Characterization of Smoothness (Thursday April 16th)

Recap from last time: the cotangent complex answers an extension problem. Given $X \xrightarrow{f} Y$ and $Y \hookrightarrow Y'$ a square zero thickening. When can the pullback diagram be filled in?

$$\begin{tikzcd}
X \arrow[r, dotted] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, dotted] \\
Y \arrow[r] & Y'
\end{tikzcd}$$

- *The existence is governed by* $\mathrm{obs} \in \mathrm{Ext}^2(L_{X/Y}, f^*I)$
- *The number of extensions by* $\mathrm{Ext}^1(L_{X/Y}, f^*I)$
- *The automorphisms by* $\mathrm{Ext}^0(L_{X/Y}, f^*I)$

Suppose we're considering $k[\varepsilon] \to k$*, where* $L_{X_{/k}} = \Omega_{X_{/k}}$*, and* $H^*(T_{X_{/k}})$ *houses the obstruction theory. For an embedded deformation* $X \hookrightarrow Y$*, we have*

$$\begin{tikzcd} X \arrow[r, dotted] & X' \arrow[d, dotted] \\ Y \arrow[r] & Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[\varepsilon] \end{tikzcd}$$

then $L_{X/Y} = I/I^2[1] = N^\vee_{X/Y}[1]$ *and*

$$\mathrm{obs} \in \mathrm{Ext}^2(N^\vee[1], \mathcal{O}) = \mathrm{Ext}^1(N^\vee, \mathcal{O}) = H^1(N).$$

and similarly $\mathrm{def} = H^0(N)$ *and* $\mathrm{Aut} = 0$*. For* $X \xrightarrow{f} Y$*, we can think of this as an embedded deformation of* $\Gamma \subset X \times Y$*, in which case* $N^\vee = F^*\Omega_{Y_{/k}}$*. Then* $\mathrm{obs}, \mathrm{def} \in H^{1,0}(f^*T_{X_{/k}})$ *respectively and* $\mathrm{Aut} = 0$*. There is an exact triangle*

$$f^*L_{Y_{/k}} \to L_{X_{/k}} \to L_{X/Y} \to f^*L_{Y_{/k}}[1].$$

17.1 T1 Lifting

This will give a criterion for a pro-representable functor to be smooth. We've seen a condition on F *with obstruction theory for the hull to be smooth, namely* $\mathrm{obs}(F) = 0$*. However, often* $F = h_R$ *will have* R *smooth with a natural obstruction theory for which* $\mathrm{obs}(F) \neq 0$*.*

Example 17.1.1 *(?):* For $X_{/k}$ smooth projective, the picard functor $\mathrm{Pic}_{X_{/k}}$ is smooth because we know it's an abelian variety. We also know that the natural obstruction space is $\mathrm{obs} = H^2(\mathcal{O}_X)$, which may be nonzero. We could also have abstract deformations given by $H^2(T_X)$

Given $A \in \mathrm{Art}_{/k}$ and M a finite length A-module, we can form the ring $A \oplus M$ where M is square zero and $A \curvearrowright M$ by the module structure. This yields

$$0 \to M \to A \oplus M \to A \to 0$$

The explicit ring structure is given by $(x, y) \cdot (x, y') = (xx', x'y + xy')$.

Proposition 17.1.2 ***(Characterization of Smoothness).***
Assume $\mathrm{ch}\, k = 0$ and F is a pro-representable deformation functor, so $F = \hom(R, \cdot)$ where R is a complete local k-algebra with $\dim t_R < \infty$.

Then R is smooth[a] over k $\iff$ for all $A \in \mathrm{Art}_{/k}$ and all $M, M' \in A$-mod finite dimensional with $M \twoheadrightarrow M'$, we have

$$F(A \oplus M) \twoheadrightarrow F(A \oplus M').$$

[a] I.e. $R \cong k[[t_R^\vee]]$.

17.1.1 Proof of Proposition

Observation 17.1.3
First observe that $\ker(F(A \oplus M) \to F(A)) = \ker(\hom(R, A \oplus M) \to \hom(R, A))$, note that if we have two morphisms

$$\begin{tikzcd} R \arrow[r] & R \arrow[r, shift left, "g \oplus g"] \arrow[r, shift right, "f \oplus g'"'] & A \oplus M \end{tikzcd}$$

denoting these maps h, h' we have

1. $g - g' \in \mathrm{Der}_k(R, M)$, since

$$\begin{aligned}(h - h')(x, y) &= h(x)h(y) - h'(x)h'(y) \\ &= (f(x)f(y), f(x)g(y) + f(y)g(x)) - (f(x)f(y), f(x)g'(y) + f(y)g'(x)) \\ &= f(x)(g - g')(y) + f(y)(g - g')(x).\end{aligned}$$

2. Given $g : R \to A \oplus M$ and $\theta \in \mathrm{Der}_k(R, M)$, then $g + \theta : R \to A \oplus M$.

We conclude that the fibers are naturally torsors for $\mathrm{Der}_k(R, M)$ *if nonempty. It is in fact a canonically trivial torsor, since there is a distinguished element in each fiber. Thus to show the following, it is enough to show surjection on fibers and trivial extensions go to trivial ones, then* $\mathrm{Der}_k(R, M) \to \mathrm{Der}_k(R, M')$ *with* $0 \mapsto 0$*.*

$$\begin{tikzcd} F(A \oplus M) \arrow[rr] \arrow[dr] & & F(A \oplus M') \arrow[dl] \\ & F(A) & \end{tikzcd}$$

The criterion for F being surjective is equivalent to

$$\mathrm{Der}_k(R, M) \twoheadrightarrow \mathrm{Der}_k(R, M')$$

$$\Downarrow \qquad \textit{identified as}$$

$$\hom_R(\Omega_{R_{/k}}, M) \twoheadrightarrow \hom(\Omega_{R'_{/k}}, M').$$

Warning 17.1.4
$\Omega_{R_{/k}}$ is complicated. An example is

$$\Omega_{k[[x]]/k} \otimes k((x)) = \Omega_{k((x))/k}.$$

which is an infinite dimensional $k((x))$ vector space.

Here we only need to consider the completions $\hom_R(\widehat{\Omega}_{R_{/k}}, M) \twoheadrightarrow \hom(\widehat{\Omega}_{R'_{/k}}, M') = k[[x]]\ dx$.

Fact 17.1.5
In characteristic zero, $R?k$ is smooth iff $\widehat{\Omega}_{R_{/k}}$ is free.

Thus the surjectivity condition is equivalent to checking that $\hom(\widehat{\Omega}_{R_{/k}}, \cdot)$ *is right-exact on finite length modules. This happens iff* $\widehat{\Omega}$ *are projective iff they are free.*

Fact 17.1.6 (from algebra)
Uses an algebra fact: for a complete finitely-generated module M over a complete ring, then M is free if M projective with respect to sequences of finite-length modules. Over a local ring, finitely-generated and projective implies free.

Remark 17.1.7: This is powerful – allows showing deformations of Calabi-Yaus are unobstructed!

Definition 17.1.8 (Calabi-Yau)
A smooth projective $X_{/k}$ is **Calabi-Yau** iff

$$\omega_x \cong \mathcal{O}_x,$$

i.e. the canonical bundle is trivial.

Proposition 17.1.9 *(?).*
$X_{/k}$ CY with $H^0(T_X) = 0$ (implying that the deformation functor F of X is pro-representable, say by R, and has no infinitesimal automorphisms) has unobstructed deformations, i.e. R is smooth of dimension $H^1(T_X)$.

Note that $H^2(T_X) \neq 0$ in general, so this is a finer criterion.

Example 17.1.10 ***(?):*** Take $X \subset \mathbb{P}^4$ a smooth quintic threefold.

- By adjunction, this is Calabi-Yau since
$$\omega_x = \omega_{\mathbb{P}^4}(5)\Big|_X = \mathcal{O}_x.$$

- By Lefschetz,
$$H^i_{\text{sing}}(\mathbb{P}^4, \mathbb{C}) \xrightarrow{\cong} H^i_{\text{sing}}(X, \mathbb{C}) \qquad \text{except in middle dimension}$$
$$\Downarrow \quad \text{implies}$$
$$H^{3,1} = H^{1,3} = 0.$$

- By Serre duality,
$$H^0(T_x) = 0 \cong H^4(\Omega_x \otimes \omega_x)$$
$$\Downarrow \quad \text{implies}$$
$$H^3(\Omega_x) = H^{3,1} = 0.$$

Exercise 17.1.11 (?)
There are nontrivial embedded deformations that yield the same abstract deformations, write them down for the quintic threefold.

Claim: The abstract moduli space here is given by $\mathrm{PGL}(5) \diagdown \mathrm{Hilb}$ where Hilb is smooth.

17.1.2 Proof that obstructions to deformations of Calabi-Yaus are unobstructed

We need to show that for any $M \twoheadrightarrow M'$ *that*
$$F(A \oplus M) \twoheadrightarrow F(A \oplus M').$$
The fibers of the LHS are extensions from A *to* $A \oplus M$*, and the RHS are extensions of* X/A*? By dualizing, we need to show* $H^1(T_{X/A} \otimes M) \twoheadrightarrow H^1(T_{X/A} \otimes M')$ *since the LHS is* $\mathrm{Ext}^1(\Omega_{X/A}, M)$*. We want the bottom map here to be surjective:*

$$\begin{tikzcd} X \arrow[d] & X' \arrow[d] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A \oplus M \end{tikzcd}$$

Fact 17.1.12 (Important)
For X/A a deformation of a CY, $H^*(T_{X/A})$ is free. This will finish the proof, since the map is given by $H^1(T_{X/A}) \otimes M \twoheadrightarrow H^1(T_{X/A}) \otimes M'$ by exactness. This uses the fact that there's a spectral sequence

$$\mathrm{Tor}_q(H^p(T_{X/A}), M) \implies H^{p+q}(T_{X/A} \otimes M)$$

which follows from base change and uses the fact that $T_{X/A}$ is flat.

We'll be looking at $\mathrm{Tor}_1(H^0(T_{X/A}), M)$ *which is zero by freeness. Hodge theory is now used: by Deligne-Illusie, for* $X \xrightarrow{f} S$ *smooth projective, taking pushforwards* $R^p f_* \Omega^q_{X/S}$ *are free (coming from degeneration of Hodge to de Rham) and commutes with base change.*

Remark 17.1.13: This implies that $\omega_{X/A} = \mathcal{O}_X$ is trivial. Using Deligne-Illusie, since ω is trivial on the special fiber, $H^0(\omega_{X/A}) = A$ is free of rank 1. We thus have a section $\mathcal{O}_X \to \omega_{X/A}$ which is an isomorphism by flatness, since it's an isomorphism on the special fiber.

Remark 17.1.14: By Serre duality, $H^1(T_{X/A}) = H^{n-1}(\Omega_{X/A} \otimes \omega_{X/A})^\vee = H^{n-1}(\Omega_{X/A})^\vee$, which is free by Deligne-Illusie. This also holds for $H^0(T_{X/A}) = H^n(\Omega_{X/A})^\vee$ is free.

Thus deformations of Calabi-Yaus are unobstructed.

17.1.3 Remarks

Remark 17.1.15: In fact we need much less. Take $A_n = k[t]/t^n$, then consider

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A_n \arrow[r] \arrow[d, equal] & A_n[\varepsilon] \arrow[r] \arrow[d, equal] & A_n \arrow[d, equal] \\ 0 \arrow[r] & A_n \arrow[r] & A_n \oplus \varepsilon A_n \arrow[r] & A_n \end{tikzcd}$$

For a deformation X/A_n, let $T^1(X/A_n) = \ker(F(A_n[\varepsilon]) \to F(A_n))$, the fiber above X/A_n. Then Kuramata shows that one only needs to show surjectivity for these kinds of extensions, which is quite a bit less.

In the T1 lifting theorem, the condition is equivalent to the following: For any deformation X/A_{n+1}*, there is a map*

$$T^1(X/A_{n+1}) \to T^1(X \otimes A_n/A_n).$$

and surjectivity is equivalent to the lifting condition. In the CY situation, the extension group $T^1(X/A_{n+1}) = H^1(T_{X/A_{n+1}})$ *and the RHS is* $H^1(T_{X\otimes A_n/A_n})$*. So the slogan for the T1 lifting property is the following:*

Slogan 17.1.16
If the deformation space is free and commutes with base change, then deformations are unobstructed.

Commuting with base change means the RHS is $H^1(T_{X/A_n}) \otimes A_n$*, so we just need to show it's free?*

18 | Monday April 27th

18.1 Principle of Galois Cohomology

Let $\ell_{/k}$ *a galois extension and* $X_{/k}$ *some "object" for which it makes sense to associate another object over* ℓ*. We'll prove that there's a correspondence*

$$\left\{ \begin{matrix} \ell_{/k}, \textit{ twisted forms} \\ Y \textit{ of } X_{/k} \end{matrix} \right\} \rightleftharpoons H^1(\ell_{/k}, \operatorname{Aut}(X_{/\ell})).$$

Recall that $\operatorname{PGL}(n, \ell) \coloneqq \operatorname{GL}(n, \ell)/\ell^\times$*.*

Example 18.1.1 ***(?):*** Let $X = \mathbb{P}^{n-1}/k$, then $H^1(\ell_{/k}, \operatorname{PGL}(n, \ell)$ parameterizes twisted forms of $\mathbb{P}^{n-1}$, e.g. for $n = 2$ twisted forms of $\mathbb{P}^1$ and plane curves.

Example 18.1.2 ***(?):*** Take $X = M_n(k)$ the algebra of $n \times n$ matrices. Then by a theorem (Skolern-Noether) $\operatorname{Aut}(M_n(k)) = \operatorname{PGL}(n, k)$. Thus $H^1(\ell_{/k}, \operatorname{PGL}(n, k))$ also parameterizes twisted forms of $M_n(k)$ in the category of unital (not necessarily commutative) k-algebras. These are exactly central simple algebras $A_{/k}$ where $\dim_k A = n^2$ with center $Z(A) = k$ with no nontrivial two-sided ideals. By taking $\ell = k^s$, we get a correspondence

$$\left\{ \text{CSAs} A_{/k} \text{ of degree } n \right\} \rightleftharpoons \left\{ \text{ Severi-Brauer varieties of dimension n-1} \right\}.$$

Taking $n = 2$ we obtain

$$\left\{ \text{Quaternion algebras } A_{/k} \right\} \rightleftharpoons \left\{ \text{Genus 0 curves } \ell_{/k} \right\}.$$

18.2 The Weil Descent Criterion

Fix $\ell_{/k}$ *finite Galois with* $g \coloneqq \operatorname{Aut}(\ell_{/k})$*.*

1. $X_{/k} \to X_{/\ell}$ *with a* g*-action.*

2. *What additional data on an ℓ-variety $Y_{/\ell}$ do we need in order to "descend the base" from ℓ to k?*

For $\sigma \in g$, write ℓ^σ to denote ℓ given the structure of an ℓ-algebra via $\sigma : \ell \to \ell^\sigma$. If $X_{/\ell}$ is a variety, so is $X^\sigma_{/\ell}$?

$$\begin{tikzcd}
X^\sigma \arrow[r] \arrow[d] \arrow[dr, dotted] & X \arrow[d] \\
\operatorname{Spec} \ell^\sigma \arrow[r, "f"] & \operatorname{Spec} \ell
\end{tikzcd}$$

where f is the map induced on Spec *by σ. We can also think of these on defining equations:*

$$\begin{aligned}
X &= \operatorname{Spec} \ell[t_1, \cdots, t_n] / \langle p_1, \cdots, p^n \rangle \\
X^\sigma &= \operatorname{Spec} \ell[t_1, \cdots, t_n] / \langle \sigma_{p_1}, \cdots, \sigma p^n \rangle
\end{aligned}$$

.

For $X_{/k}, X_{/\ell}$, we canonically identify X with X^σ by the map $f_\sigma : X \xrightarrow{\cong} X^\sigma$, a canonical isomorphism of ℓ-varieties. We thus have

$$\begin{tikzcd}
X \arrow[r, "f_\sigma"] \arrow[rr, bend left, "f_{\sigma\tau}"] & X^\sigma \arrow[r, "f_\sigma"] & X^{\sigma\tau}
\end{tikzcd}$$

under a "cocycle condition" $f_{\sigma\tau} = {}^\sigma f_\tau \circ f_\sigma$.

Theorem 18.2.1 ***(Weil).***
Given $Y_{/\ell}$ quasi-projective and $\forall \sigma \in g$ we have descent datum $f_\sigma : Y \xrightarrow{\cong} Y^\sigma$ satisfying the above cocycle condition, and there exists a unique $X_{/k}$ such that $X_{/\ell} \xrightarrow{\cong} Y_{/\ell}$ and the descent data coincide.

18.2.1 An Application

Let $X_{/k}$ be a quasiprojective variety and $Y_{/k}$ and $\ell_{/k}$ twisted forms. Then $a_0 \in Z'(\ell_{/k}, \operatorname{Aut} X)$. Conversely, we have the following:

Definition 18.2.2 (Twisted Descent Data)
Let a_0 be such a cocycle and $\{s_\sigma : X \to X^\sigma\}$ be descent datum attached to X. Define twisted descent datum $g_\sigma := f_\sigma \circ a_\sigma$ from

$$X/\ell \xrightarrow{a_\sigma} X_{/\ell} \xrightarrow{f_\sigma} X^\sigma/\ell.$$

Exercise 18.2.3 (?)
Check that g_σ satisfies the cocycle condition, so by Weil uniquely determines a (k-model) $Y_{/k}$ of $X_{/\ell}$.

Example 18.2.4*(?):* Let $G_{/k}$ be a smooth algebraic group and $X_{/k}$ a torsor under G. Then $\mathrm{Aut}(G) \supset \mathrm{Aut}_{G\text{-torsor}}(G) = G$, since in general the translations will only be a subgroup of the full group of automorphisms. Then

$$H^1(\ell_{/k}, G) \to H^1(\ell_{/k}, \mathrm{Aut}\, G)$$

defines a twisted form X of G. How do you descend the torsor structure? This is possible, but not covered in Bjoern's book! This requires expressing the descent data more functorially – see the book on Neron models.

18.3 The Cohomology Theory

18.3.1 Motivation

Let $G_{/k}$ be a smooth connected commutative algebraic group where Γk does not divide n, so the map $[n] : G \to G$ is an isogeny. Then

$$0 \to G[n](k^s) \to G(k^s) \xrightarrow{[n]} G(k^s) \to 0$$

is a SES of $g = \mathrm{Aut}(k^s_{/k})$-modules.

Claim: Taking the associated cohomology sequence yields the Kummer sequence:

$$0 \to G(k)/nG(k) \to H^1(k, G[n]) \to H^1(k, G)[n] \to 0$$

where the RHS is the **Weil–Châtelet** group and the LHS is the **Mordell-Weil** group.

For g a profinite group, a commutative discrete g-group is by definition a g-module. These form an abelian category with enough injectives, so we can take right-derived functors of left-exact functors. We will consider the functor

$$A \mapsto A^g \coloneqq \left\{ x \in A \;\middle|\; \sigma x = x \;\forall \sigma \in g \right\},$$

then define $H^i(g, A)$ to be the ith right-derived functor of $A \mapsto A^\sigma$. This is abstractly defined by taking an injective resolution, applying the functor, then taking cohomology. A concrete description is given by $C^n(g, A) = \mathrm{Map}(g^n, A)$ with

$$\begin{aligned}
d : C^n(g, A) &\to C^{n+1}(g, A) \\
(df)(\sigma_1, \cdots, \sigma_{n+1} &\coloneqq \sigma_1 f(\sigma_2, \cdots, \sigma_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1) f(\sigma_1, \cdots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \cdots, \sigma_{n+1}) \\
&\quad + (-1)^{n+1} f(\sigma_1, \cdots, \sigma_n).
\end{aligned}$$

Then $d^2 = 0$, H^n *is kernels mod images, and this agrees with* H^1 *as defined before with* $H^0 = A^g$. *We'll see that that*

$$H^i(g, A) = \varinjlim_U G^i(g/U, A^U).$$

If g *is finite,* A *is a* g*-module* $\iff$ A *is a* $\mathbb{Z}[g]$*-module, and thus*

$$A^g = \hom_{\mathbb{Z}[g]\text{-}mod}(\mathbb{Z}, A).$$

where $\mathbb{Z}$ *is equipped with a trivial* g*-action. We can thus think of*

$$H^i(g, A) = \operatorname{Ext}^i_{\mathbb{Z}[g]}(\mathbb{Z}, A).$$