Full Notes

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Reference: Humphrey's "Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$".

Course Website: `https://faculty.franklin.uga.edu/brian/math-8030-spring-2020`

1.1 Chapter Zero: Review

> Material can be found in Humphreys 1972. Assignment zero: practice writing lowercase mathfrak characters!

In this course, we'll take $k = \mathbb{C}$.

Recall that a Lie Algebra is a vector space $\mathfrak{g}$ with a bracket $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

- $[xx] = 0$ for all $x \in \mathfrak{g}$
  - Exercise: this implies $[xy] = -[yx]$.

    > Hint: Consider $[x+y, x+y]$. Note that the converse holds iff char $k \neq 2$.

    > Exercise: This implies Lie Algebras never have an identity.
- $[x[yz]] = [[xy]z] + [y[xz]]$ (The Jacobi identity)
  - This says x acts as a derivation.

Definition: $\mathfrak{g}$ is *abelian* iff $[xy] = 0$ for all $x, y \in \mathfrak{g}$.

There are also the usual notions (define for rings/algebras) of:

- Subalgebras,
  - A vector subspace that is closed under brackets.
- Homomorphisms
  - I.e. a linear transformation ϕ that commutes with the bracket, i.e. $\phi([xy]) = [\phi(x)\phi(y)]$.
- Ideals

Exercise: Given a vector space (possibly infinite-dimensional) over k, then (exercise) $\mathfrak{gl}(V) := \mathrm{End}_k(V)$ is a Lie algebra when equipped with $[fg] = f \circ g - g \circ f$.

Definition: A *representation* of $\mathfrak{g}$ is a homomorphism $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ for some V.

Example: The adjoint representation is $\mathrm{ad}\ : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\mathrm{ad}\ (x)(y) := [xy]$.

Representations give $\mathfrak{g}$ the structure of a module over V, where $x \cdot v := \phi(x)(v)$. All of the usual module axioms hold, where now $[xy] \cdot v = x \cdot (\cdot v) - y \cdot (x \cdot v)$.

Example: The trivial representation $V = k$ where $x \cdot a = 0$.

Definition: V is *irreducible* (or *simple*) iff V as exactly two $\mathfrak{g}$-invariant subspaces, namely $0, V$.

Definition: V is *completely reducible* iff V is a direct sum of simple modules, and *indecomposable* iff V can not be written as $V = M \oplus N$, a direct sum of proper submodules.

There are several constructions for creating new modules from old ones:

- The *contragradient/dual* $V^\vee := \hom_k(V, k)$ where $(x \cdot f) = -f(x \cdot v)$ for $f \in V^\vee, x \in \mathfrak{g}, v \in V$.
- The direct sum $V \oplus W$ where $x \cdot (v, w) = (x \cdot v, x \cdot w)$ and $x \cdot (v + w) = x \cdot v_x \cdot w$.
- The tensor product where $x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w$.
- $\hom_k(V, W)$ where $(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$.
  - Note that if we take $W = k$ then the first term vanishes and this recovers the dual.

1.2 Semisimple Lie Algebras

Definition: The derived ideal is given by $\mathfrak{g}^{(1)} := [\mathfrak{g}\mathfrak{g}] := \mathrm{span}_k\left(\left\{ [xy] \;\middle|\; x, y \in \mathfrak{g} \right\}\right)$.

> This is the analog of the commutator subgroup.

Lemma: $\mathfrak{g}$ is abelian iff $\mathfrak{g}^{(1)} = \{0\}$, and 1-dimensional algebras are always abelian.

> This follows because if $[xy] \coloneqq xy = yx$ then $[xy] = 0 \iff xy = yx$.

Definition: A lie group $\mathfrak{g}$ is *simple* iff the only ideals of $\mathfrak{g}$ are $0, \mathfrak{g}$ and $\mathfrak{g}^{(1)} \neq \{0\}$.

> Note that thus rules out the zero modules, abelian lie algebras, and particularly 1-dimensional lie algebras.

Definition: The derived series is defined by $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}\mathfrak{g}^{(1)}]$, continuing inductively. $\mathfrak{g}$ is said to be solvable if $\mathfrak{g}^{(n)} = 0$ for some n.

Lemma: Abelian implies solvable.

> Review definition of nilpotent algebras.

Definition: $\mathfrak{g}$ is semisimple (s.s.) iff $\mathfrak{g}$ has no nonzero solvable ideals.

Exercise: Simple implies semisimple.

Some remarks:

1. Semisimple algebras $\mathfrak{g}$ will usually have solvable subalgebras.
2. $\mathfrak{g}$ is semisimple iff $\mathfrak{g}$ has no nonzero abelian ideals.

Definition: The Killing form is given by $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ where $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$, which is a symmetric bilinear form.

Lemma: $\kappa([xy], z) = \kappa(x, [yz])$.

Recall that if $\beta : V^{\otimes 2} \to k$ is any symmetric bilinear form, then its radical is defined by

$$\mathrm{rad}\beta = \left\{ v \in V \;\middle|\; \beta(v, w) = 0 \ \forall w \in V \right\}.$$

Definition: A bilinear form β is nondegenerate iff $\mathrm{rad}\beta = 0$.

Lemma: $\mathrm{rad}\kappa \trianglelefteq \mathfrak{g}$ is an ideal, which follows by the above associative property.

Theorem: $\mathfrak{g}$ is semisimple iff κ is nondegenerate.

Example: The standard example of a semisimple lie algebra is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \coloneqq \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \;\middle|\; \mathrm{Tr}(x) = 0 \right\}$.

> Note: from now on, $\mathfrak{g}$ will denote a semisimple lie algebra over $\mathbb{C}$.

Theorem (Weyl): Every finite dimensional representation of a semisimple $\mathfrak{g}$ is completely reducible.

> I.e., the category of finite-dimensional representations is relatively uninteresting – there are no extensions, everything is a direct sum, so once you classify the simple algebras (which isn't terribly difficult) then you have complete information.

2 Friday January 10th

Let $\mathfrak{g}$ be a finite dimensional semisimple lie algebra over $\mathbb{C}$.

Recall that this means it has no proper solvable ideals.

A more useful characterization is that the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a *non-degenerate* symmetric (associative) bilinear form.

The running example we'll use is $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the trace zero $n \times n$ matrices.

Let $\mathfrak{h}$ be a maximal toral subalgebra, where $x \in \mathfrak{g}$ is *toral* if x is semisimple, i.e. ad x is semisimple (i.e. diagonalizable).

Example: $\mathfrak{h}$ is the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$.

Fact: $\mathfrak{h}$ is abelian, so ad $\mathfrak{h}$ consists of commuting semisimple elements, which (theorem from linear algebra) can be simultaneously diagonalized.

This leads to the root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

where $\mathfrak{g}_\alpha = \left\{ x \in \mathfrak{g} \;\middle|\; [hx] = \alpha(h) x \; \forall h \in \mathfrak{h} \right\}$ where $\alpha \in \mathfrak{h}^\vee$ is a linear functional.

Here $\mathfrak{h} = C_\mathfrak{g}(\mathfrak{h})$, so $[hx] = 0$ corresponds to zero eigenvalues, and (fact) it turns out that $\mathfrak{h}$ is its own centralizer.

We then obtain a set of roots of $\mathfrak{h}, \mathfrak{g}$ given by $\Phi = \left\{ \alpha \in \mathfrak{h}^\vee \;\middle|\; \alpha \neq 0, \mathfrak{g}_\alpha \neq \{0\} \right\}$.

Example: $\mathfrak{g}_\alpha = \mathbb{C} E_{ij}$ for some $i \neq j$, the matrix with a 1 in the i, j position and zero elsewhere.

Fact: The restriction $\kappa|_\mathfrak{h}$ is nondegenerate, so we can identify $\mathfrak{h}, \mathfrak{h}^\vee$ via κ (can always do this with vector spaces with a nondegenerate bilinear form), where κ maps to another bilinear form $(\,\cdot\,,\,\cdot\,)$.

$$\mathfrak{h}^\vee \ni \lambda \iff t_\lambda \in \mathfrak{h}$$
$$\lambda(h) = \kappa(t_\lambda, h) \quad \text{where } (\lambda, \mu) = \kappa(t_\lambda, t_\mu).$$

2.1 Facts About Φ and Root Spaces

Let $\alpha, \beta \in \Phi$ be roots.

1. Φ spans $\mathfrak{h}^\vee$ and does not contain zero.
2. If $\alpha \in \Phi$ then $-\alpha \in \Phi$, but no other scalar multiple of α is in Φ.

Aside:

- $\dim \mathfrak{g}_\alpha = 1$.
- If $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ then there exists a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha := [x_\alpha, y_\alpha]$ spans a 3-dimensional subalgebra in $\mathfrak{sl}_2$, given by $x_\alpha = [0,1;0,0], y_\alpha = [0,0;1,0], h_\alpha = [1,0;0,-1]$.
- Under the correspondence $\mathfrak{h} \iff \mathfrak{h}^\vee$ induced by κ, $h_\alpha \iff \alpha^\vee := \dfrac{2\alpha}{(\alpha,\alpha)}$. Thus for all $\lambda \in \mathfrak{h}^\vee$,

$$\lambda(h_\alpha) = (\lambda, \alpha^\vee) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}.$$

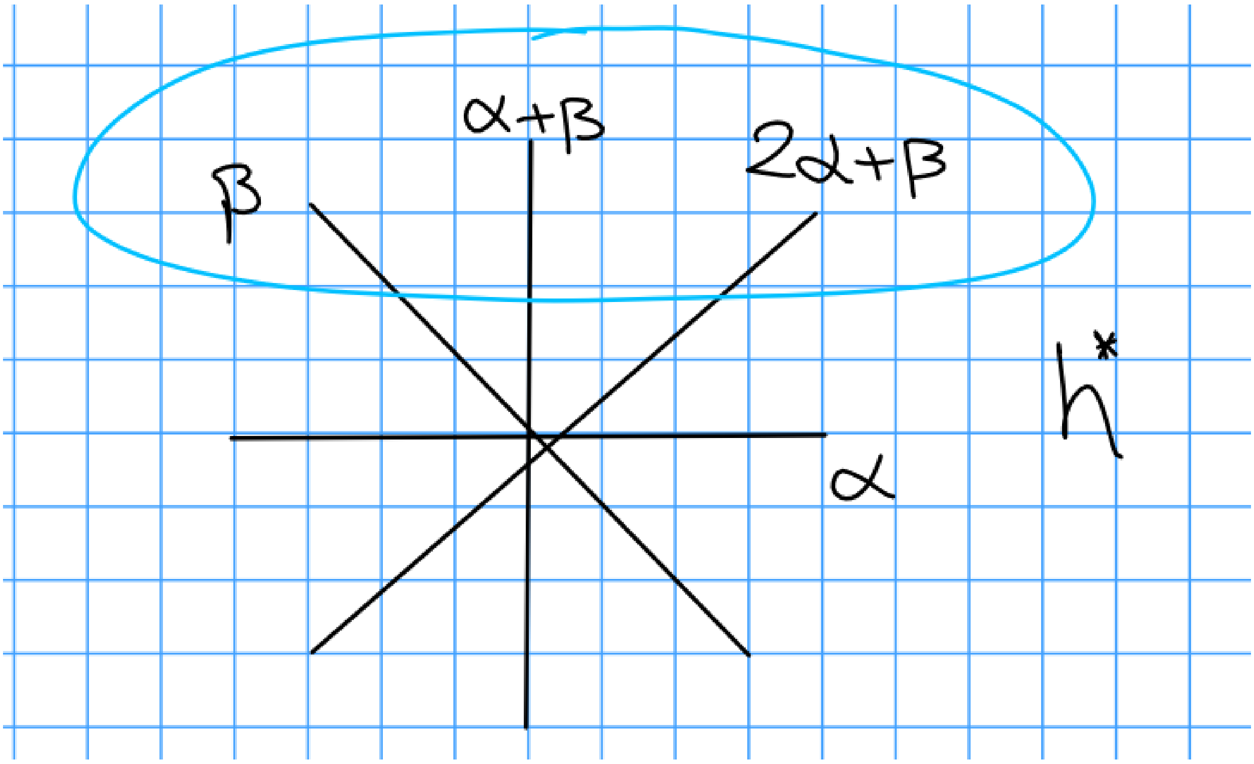
- If $\alpha + \beta \neq 0$, then $\kappa(g_\alpha, g_\beta) = 0$.

3. $(\beta, \alpha^\vee) \in \mathbb{Z}$
4. $S_\alpha(\beta) := \beta - (\beta, \alpha^\vee)\alpha \in \Phi$.

If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. Example: If $\alpha = E_{ij}, \beta = E_{jk}$ where $k \neq i$, then $[E_{ij}, E_{jk}] = E_{ik}$.

- $\mathfrak{g}$ is generated as an algebra by the root spaces $\mathfrak{g}_\alpha$
- Root strings: If $\beta \neq \pm\alpha$, then the roots of the form $\alpha + k\beta$ for $k \in \mathbb{Z}$ form an unbroken string $\alpha - r\beta, \cdots, \alpha - \beta, \alpha, \alpha + \beta, \cdots, \alpha + \ell\beta$ consisting of at most 4 roots where $r - s = (\alpha, \beta^\vee)$.

Example: The circled roots below form the root string for β:



In general, a subset Φ of a real euclidean space E satisfying conditions (1) through (4) is an *(abstract) root system.*

When Φ comes from a $\mathfrak{g}$, $E := \mathbb{R}\Phi$.

2.1.1 The Root System

There exists a subset $\Delta \subseteq \Phi$ such that

- Δ is a $\mathbb{C}$-basis for $\mathfrak{g}^\vee$
- $\beta \in \Phi$ implies that $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$ with either
  - All $c_\alpha \in \mathbb{Z}_{\geq 0} \iff \beta \in \Phi^+$ or $\beta < 0$.
  - All $c_\alpha \in \mathbb{Z}_{\leq 0} \iff \beta \in \Phi^-$ or $\beta > 0$.

Δ is called a *simple system.* If $\Delta = \{a_1, \cdots, a_\ell\}$ then $\Phi^+$are the *positive roots*, and $\Phi^+ \ni \beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$, then the *height* of β is defined as $\sum c_\alpha \in \mathbb{Z}_{>0}$.

Note that $\mathbb{Z}\Phi := \Lambda_r$ is a lattice, and is referred to as the *root lattice*, and $\Lambda_r \subset E = \mathbb{R}\Phi$. We also have $\Phi^+ = \left\{\beta^\vee \mathrel{}\middle|\mathrel{} \beta \in \Phi\right\}$, the *dual root system*, is a root system with simple system $\Delta^\vee$.

Important subalgebras of $\mathfrak{g}$:

- Upper triangular with zero diagonal $\mathfrak{n} = \mathfrak{n}^+ = \sum_{\beta>0} \mathfrak{g}_\beta$
- Lower triangular with zero diagonal $\mathfrak{n}^- = \sum_{\beta>0} \mathfrak{g}_{-\beta}$
- Upper triangular, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra
- Lower triangular, $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

There is thus a triangular (Cartan) decomposition, $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

Fact: If $\beta \in \Phi^+ \setminus \Delta$, and if $\alpha \in \Delta$ such that $(\beta, \alpha^\vee) > 0$, then $\beta - (\beta, \alpha^\vee)\alpha \in \Phi^+$ has height strictly less than the height of β.

By root strings, $\beta - \alpha \in \Phi^+$ is positive root of height one less than β, yielding a way to induct on heights (useful technique).

2.1.2 Weyl Groups

For $\alpha \in \Phi$, define

$$\begin{aligned} S_\alpha : \mathfrak{h}^\vee &\to \mathfrak{h}^\vee \\ \lambda &\mapsto \lambda - (\lambda, \alpha^\vee)\alpha. \end{aligned}$$

This is reflection in the hyperplane in E perpendicular to α:

Note that $S_\alpha^2 = \mathrm{id}$.

Define W as the subgroup of $\mathrm{gl}(E)$ generated by all s_α for $\alpha \in \Phi$, this is the *Weyl group* of $\mathfrak{g}$ or Φ, which is finite and $W = \left\langle s_\alpha \mathrel{}\middle|\mathrel{} \alpha \in \Delta\right\rangle$ is generated by simple reflections.

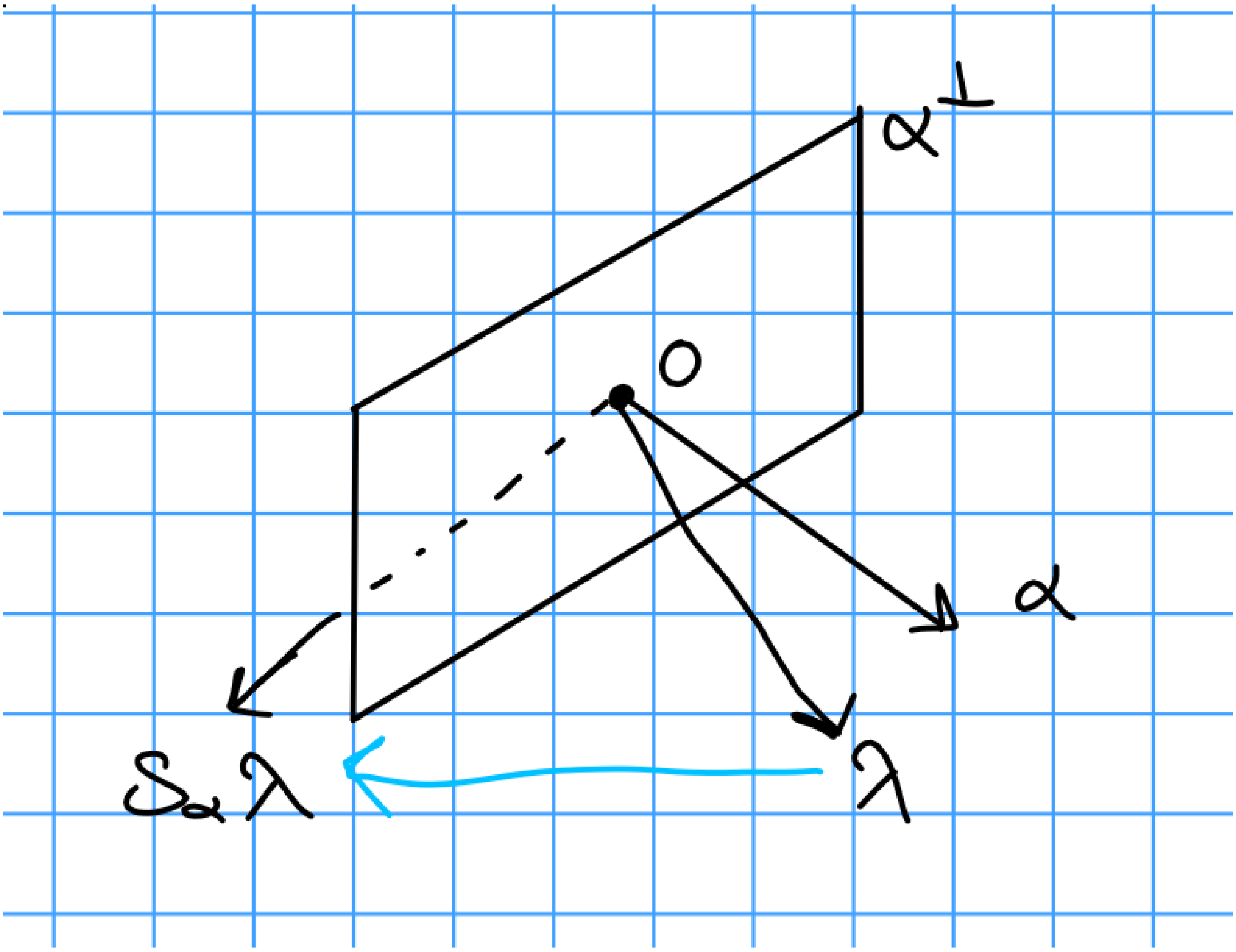


Figure 1: Image

By (4), W leaves Φ invariant. In fact W is a finite Coxeter group with generators $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$ and defining relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ for $\alpha, \beta \in \Delta$ where $m(\alpha, \beta) \in \{2, 3, 4, 6\}$ when $\alpha \neq \beta$ and $m(\alpha, \alpha) = 1$.

Note that if this finiteness on numerical conditions are met, then this is referred to as a *Crystallographic group.*

3 Monday January 13th

3.1 Lengths

Recall that we have a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta$ for finite dimensional semisimple lie algebras over $\mathbb{C}$. We have $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$, for $\lambda \in \mathfrak{h}^\vee$ and the Weyl group $W = \left\langle s_\beta \;\middle|\; \beta \in \Phi \right\rangle = \left\langle s_\alpha \;\middle|\; \alpha \in \Delta \right\rangle$ where $\Delta = \{a_i\}$ are the simple roots. For $w \in W$, we can take the reduced expression for w by writing $w = s_1 \cdots s_n$ with s_i simple and n minimal. The length is uniquely determined, but not the expression. So we define $\ell(w) \coloneqq n$ where $\ell(1) \coloneqq 0$.

Facts:

1. $\ell(w)$ is the size of the set $\left\{ \beta \in \Phi^+ \;\middle|\; w\beta < 0 \right\}$

- The above set is equal to $\Phi^+ \bigcap w^{-1}\Phi^{-1}$.
- In particular, for $\beta \in \Phi^+$, β is simple (i.e. $\beta \ni \Delta$ iff $\ell(s_\beta) = 1$).
- Note: α is the only root that s_α sends to a negative root, so $s_\alpha(\beta) > 0$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$.

2. $\ell(w) = \ell(w^{-1})$ for all $w \in W$, so $\ell(w)$ is also the size of $\Phi \bigcap w\Phi$ (replacing w^{-1} with w)

3. There exists a unique $w_0 \in W$ with $\ell(w_0)$ maximal such that $\ell(w_0) = \left|\Phi^+\right|$ and $w_0(\Phi^+) = \Phi^-$.

- Also $\ell(w_0 w) = \ell(w_0) - \ell(w)$

Note that the product of reduced expressions is not usually reduced, so the length isn't additive.

4. For $\alpha \in \Phi^+$, $w \in W$, we have either

$$\begin{aligned} \ell(ws_\alpha) > \ell(w) &\iff w(\alpha) > 0 \\ \ell(ws_\alpha) < \ell(w) &\iff w(\alpha) < 0 \end{aligned}.$$

Taking inverses yields $\ell(s_\alpha w) > \ell(w) \iff w^{-1}\alpha > 0$.

3.2 Bruhat Order

Let S be the set of simple reflections, i.e. $S = \left\{ s_\alpha \;\middle|\; \alpha \in \Delta \right\}$. Then define

$$T \coloneqq \bigcup_{w \in W} wSw^{-1} = \left\{ s_\beta \;\middle|\; \beta \in \Phi^+ \right\}.$$

This is the set of *all* reflections in W through hyperplanes in E.

We'll write $w' \xrightarrow{t} w$ means $w = tw'$ and $\ell(w') < \ell(w)$. Note that in the literature, it's also often assumed that that $\ell(w') = \ell(w) - 1$. In this case, we say w' covers w, and refer to this as "the covering relation". So $w' \to w$ means that $w' \xrightarrow{t} w$ for some $t \in T$. We extend this to a partial order: $w' < w$ means that there exists a w such that $w' = w_0 \to w_1 \to \cdots \to w_n = w$. This is called the **Bruhat-Chevalley order** on W.

Corollary: $w' < w \implies \ell(w') < \ell(w)$, so $1 \in W$ is the unique minimal element in W under this order.

> It turns out that if we set $w = w't$ instead, this results in the same partial order.

If you restrict T to simple reflections, this yields the *weak Bruhat order.* In this case, the left and right versions differ, yielding the *left/right weak Bruhat orders* respectively. (Note that this is because conjugating a simple reflection may not yield a simple reflection again.)

> Recall that lie algebras yield finite crystallographic coxeter groups.

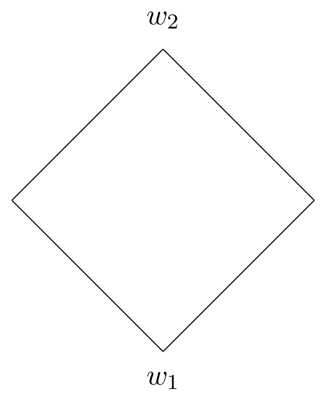
Properties: For (W, S) a coxeter group,

a. $w' \leq w$ iff w' occurs as a subexpression/subword of every reduced expression $s_1 \cdots s_n$ for $w, where a subexpression is any subcollection of s_i in the same order.

> Note that this implies that 1 is not only a minimal element in this order, but an infimum.

b. Adjacent elements w', w (i.e. $w' < w$ and there does not exist a w'' such that $w' < w'' < w$) in the Bruhat order differ in length by 1.

c. If $w' < w$ and $s \in S$, then $w's \leq w$ or $w's \leq ws$ (or both). i.e., if $\ell(w_1) = 2 = \ell(w_2)$, then the size of $\left\{ w \in W \;\middle|\; w_1 < w < w_2 \right\}$ is either 0 or 2.



3.3 Properties of Universal Enveloping Algebras

Let $\mathfrak{g}$ be any lie algebra, and $\phi : \mathfrak{g} \to A$ be any map into an associative algebra. Then there exists an object $U(\mathfrak{g})$ and a map i such that the following diagram commutes:

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[dd, dotted, "\tilde{\phi}"] \\ \mathfrak{g} \arrow[ur, "i"] \arrow[dr, "\phi"'] & \\ & A \end{tikzcd}$$

Note that $\tilde{\phi}$ is a map in the category of associative algebras.

Moreover any lie algebra homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_1$ induces a morphism of associative algebras $U(\mathfrak{g}_1) \to U(\mathfrak{g}_2)$, where $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

$U(\mathfrak{g})$ can be constructed as

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \left\langle [x,y] - x \otimes y - y \otimes x \;\middle|\; x, y \in \mathfrak{g} \right\rangle .$$

Note that this ideal is not necessarily homogeneous.

Properties:

- Usually noncommutative
- Left and right Noetherian
- No zero divisors
- $\mathfrak{g} \curvearrowright U(\mathfrak{g})$ by the extension of the adjoint action, $(\mathrm{ad}\ x)(u) = xu - ux$ for $x \in \mathfrak{g}, u \in U(\mathfrak{g})$.

Big Theorem (Poincaré-Birkhoff-Witt, i.e. PBW): If $\{x_1, \cdots x_n\}$ is a basis for $\mathfrak{g}$, then $\left\{x_1^{t_1}, \cdots, x_n^{t_n} \;\middle|\; t_i \in \mathbb{Z}^+\right\}$ (noting that $x^n = x \otimes x \otimes \cdots x$ and $\mathbb{Z}^+$ includes 0) is a basis for $U(\mathfrak{g})$.

Corollary: $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective, so we can think of $\mathfrak{g} \subseteq U(\mathfrak{g})$.

If $\mathfrak{g}$ is semisimple, then it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and choose a compatible basis for $\mathfrak{g}$, then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$.

If $\phi : \mathfrak{g} \to \mathrm{gl}(V)$ is any lie algebra representation, it induces an algebra representation $U(\mathfrak{g})$ of $U(\mathfrak{g})$ on V and vice-versa. It satisfies $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [xy] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$. Note that this lets us go back and forth between lie algebra representations and associative algebra representations, i.e. the theory of modules over rings.

Notation: $\mathfrak{Z}(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

3.4 Integral Weights

We have a Euclidean space $E = \mathbb{R}\Phi^+$, the $\mathbb{R}$-span of the roots. We also have the **integral weight lattice**

$$\Lambda = \left\{\lambda \in E \;\middle|\; (\lambda, \alpha^\vee) \in \mathbb{Z}\ \forall \alpha \in \Phi(\text{or } \Phi^+ \text{ or } \Delta)\right\}.$$

There is a sublattice $\Lambda_r \subseteq \Lambda$, which is an additive subgroup of finite index.

There is a partial order of Λ on E and $\mathfrak{h}^\vee$. We write $\mu \leq \lambda \iff \lambda - \mu \in \mathbb{Z}^+\Delta = \mathbb{Z}^+\Phi^+$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, define a dual basis $(w_i, \alpha_j^\vee) = \delta_{ij}$. The fundamental weights are given by a $\mathbb{Z}$-basis for Λ. Then Λ is a free abelian group of rank ℓ, and $\Lambda^+ = \mathbb{Z}^+ w_1 + \cdots + \mathbb{Z}^+ w_\ell$ are the **dominant integral weights**.

Note that in Jantzen's book, X is used for Λ and X^+ correspondingly.

4 Wednesday January 15th

4.1 Review

The Weyl vector is given by $\rho = \overline{\omega}_1 + \cdots + \overline{\omega}_\ell = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta \in \Lambda^+$.

- If $\alpha \in \Delta$ then $(\rho, \alpha^\vee) = 1$
- $s_\alpha(\rho) = \rho - \alpha$.

Let $\lambda \in \Lambda^+$; a few facts:

1. The size of $\left\{\mu \in \Lambda^+ \;\middle|\; \mu \leq \lambda\right\}$ (with the partial order from last time) is finite.
2. $w\lambda < \lambda$ for all $w \in W$.

The Weyl chamber (for a fixed root, $E =$ Euclidean space) is $C = \left\{\lambda \in E \;\middle|\; (\lambda, \alpha) > 0 \;\forall \alpha \in \Delta\right\}$ (Note that the hyperplane splits E into connected components, we mark this component as distinguished.)

- A connected component of the union of hyperplanes is orthogonal to roots
- They're in bijection with Δ
- They're permuted simply transitively by W.

And $\overline{C}$ denotes the fundamental domain.

4.2 Weight Representations

For $\lambda \in \mathfrak{h}^\vee$, we let $M_\lambda = \left\{v \in M \;\middle|\; h \cdot v = \lambda(h)v \;\forall h \in \mathfrak{h}\right\}$ denote a *weight space* of M if $M_\lambda \neq 0$. In this case, λ is a *weight* of M. The dimension of M_λ is the *multiplicity* of λ in M, and we define the set of weights as $\Pi(M) = \left\{\lambda \in \mathfrak{h}^\vee \;\middle|\; M_\lambda \neq 0\right\}$.

Example if $M = \mathfrak{g}$ under the adjoint action, then $\Pi(M) = \Phi \bigcup \{0\}$.

Remark: The weight vectors for distinct weights are linearly independent. Thus there is a $\mathfrak{g}$-submodule given by $\sum_\lambda M_\lambda$, which is in fact a direct sum.

Note: It may not be the case that $\sum_\lambda M_\lambda = M$, and can in fact be zero, although this is an odd situation. See Humphreys #1, #20.2, p. 110.

In our case, we'll have a *weight module* $M = \bigoplus_\lambda M_\lambda$, so $\mathfrak{h} \curvearrowright M$ semisimply.

4.3 Finite-rimensional Modules

Recall Weyl's complete reducibility theorem, which implies that any finite dimensional $\mathfrak{g}$-module is a weight module. In fact, $\mathfrak{n}, \mathfrak{n}^- \curvearrowright M$ nilpotently.

Some facts:

- $\Pi(M) \subset \Lambda$ is a subset of the integral lattice.
- $\Pi(M)$ is W-invariant.
- $\dim M_\lambda = \dim M_{W\lambda}$ for any $\lambda \in \Pi(M), w \in W$.

4.4 Simple Finite Dimensional $\mathfrak{sl}(2, \mathbb{C})$-modules

Fix the standard basis $\{x, h, y\}$ of $\mathfrak{sl}(2, \mathbb{C})$ with $[hx] = 2x, [hy] = -2y, [xy] = h$. Since $\dim \mathfrak{h} = 1$, there is a bijection $\mathfrak{h}^\vee \leftrightarrow \mathbb{C}$, $\Lambda \leftrightarrow \mathbb{Z}$, and $\Lambda_r \leftrightarrow 2\mathbb{Z}$ with $\alpha \to 2$ and $\rho \to 1$.

There is a correspondence between weights and simple modules:

$$\begin{aligned} \left\{\text{Isomorphism classes of simple modules}\right\} &\iff \Lambda^+ = \{0, 1, 2, 3, \cdots\} \\ L(\lambda) &\iff \lambda. \end{aligned}$$

Moreover, $L(\lambda)$ has a 1-dimensional weight spaces with weights $\lambda, \lambda - 2, \cdots, -\lambda$ and thus $\dim L(\lambda) = \lambda + 1$.

Examples:

- $L(0) = \mathbb{C}$, the trivial representation,
- $L(1) = \mathbb{C}^2$, the natural representation where $\mathfrak{sl}(2, \mathbb{C})$ acts by matrix multiplication,
- $L(2) = \mathfrak{g}$, the adjoint representation.

Choose a basis $\{v_1, \cdots, v_\lambda\}$ for $L(\lambda)$ so that $\mathbb{C}v_0 = M_\lambda$, $\mathbb{C}v_1 = M_{\lambda-2}$, $\cdots \mathbb{C}v_\lambda M_{-\lambda}$. Then

- $h \cdot v_i = (\lambda - 2i)v_i$
- $x \cdot v_i = (\lambda - i + 1)v_{i-1}$, where $v_{-1} := 0$
- $y \cdot v_i = (i + 1)v_{i+1}$ where $v_{\lambda+1} := 0$.

We then say $L(\lambda)$ is a highest weight module, since it is generated by a highest weight vector λ. Then $W = \{1, s_\alpha\}$, where s_α is reflection through 0 by the identification $\alpha = 2$.

5 Chapter 1: Category $\mathcal{O}$ Basics

The category of $U(\mathfrak{g})$-modules is too big. Motivated by work of Verma in 60s, started by Bernstein-Gelfand-Gelfand in the 1970s. Used to solve the Kazhdan-Lusztig conjecture.

5.1 Axioms and Consequences

Definition: $\mathcal{O}$ is the full subcategory of $U(\mathfrak{g})$ modules consisting of M such that

1. M is finitely generated as a $U(\mathfrak{g})$-module.
2. M is $\mathfrak{h}$-semisimple, i.e. M is a weight module $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$.
3. M is locally n-finite, i.e. the dimension of $U(\mathfrak{n})v < \infty$ for all $v \in M$.

Example: If $\dim M < \infty$, then M is $\mathfrak{h}$-semisimple, and axioms 1, 3 are obvious.

Lemma: Let $M \in \mathcal{O}$, then

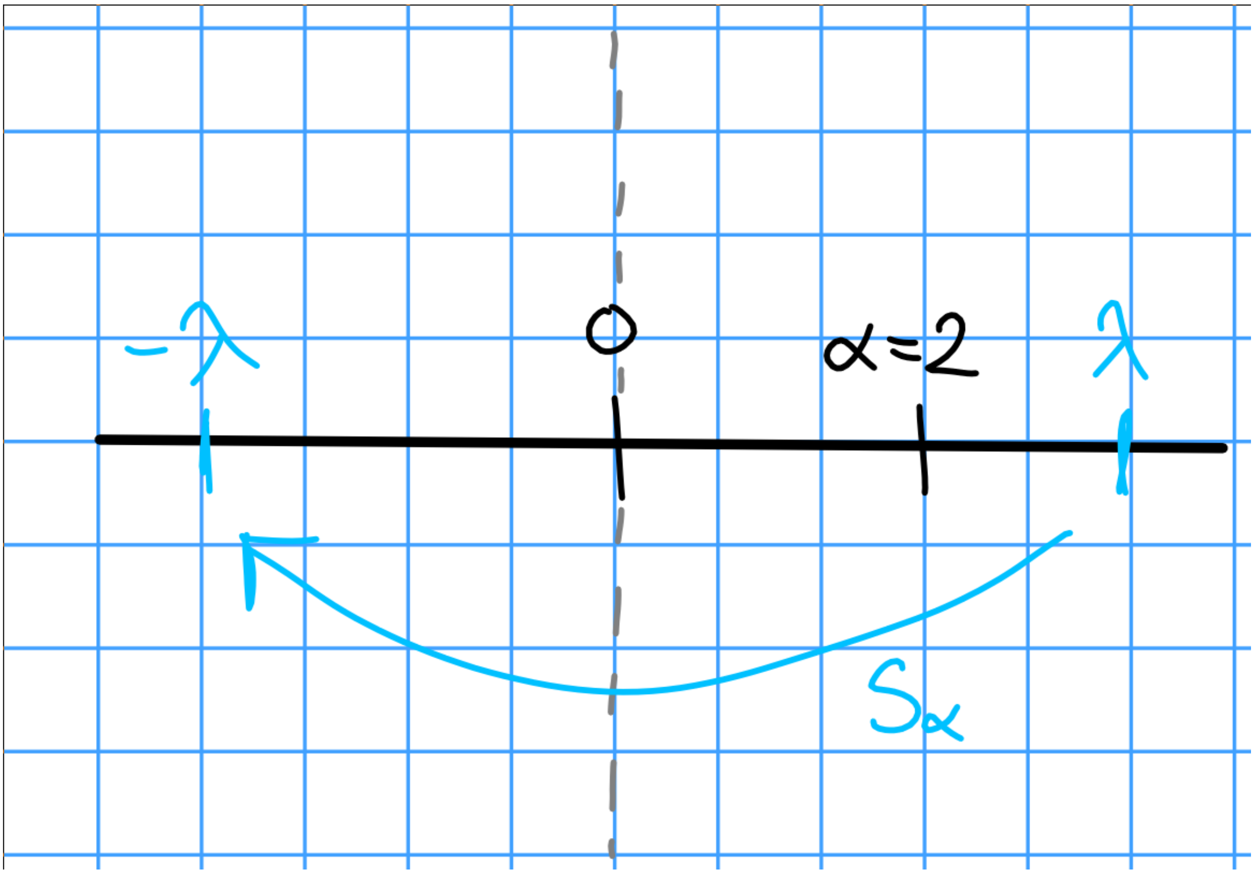


Figure 2: Image

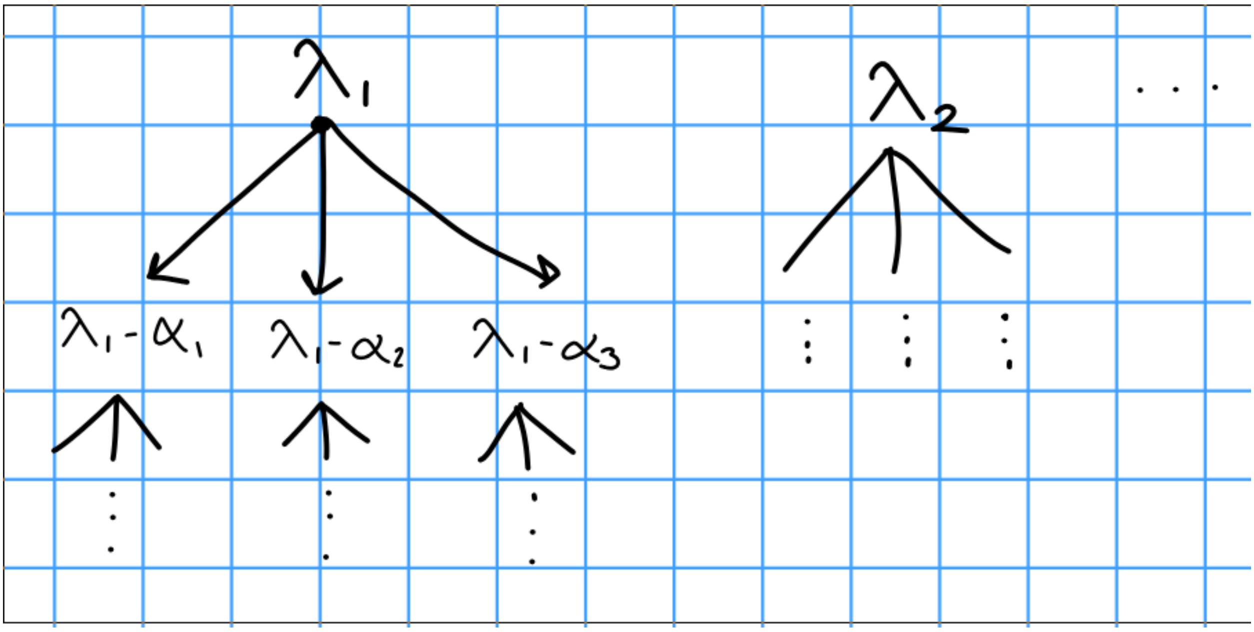


Figure 3: Image

4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$.
5. There exist $\lambda_1, \cdots \lambda_r \in \mathfrak{h}^\vee$ such that $\Pi(M) \subset \bigcup_{i=1}^{\lambda} (\lambda_i - \mathbb{Z}^+\Phi^+)$.

Proof: By axiom 2, every $v \in M$ is a finite sum of weight vectors in M. We can thus assume that our finite generating set consists of weight vectors. We can then reduce to the case where M is generated by a single weight vector v. So consider $U(\mathfrak{g}) \cdot v$. By the PBW theorem, there is a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})$.

By axiom 3, $U(\mathfrak{n}) \cdot v$ is finite dimensional, so there are finitely many weights of finite multiplicity in the image. Then $U(\mathfrak{h})$ acts by scalar multiplication, and $U(\mathfrak{n}^-)$ produces the "cones" that result in the tree structure:

A weight of the form $\mu = \lambda_i - \sum n_i \alpha_i$ can arise from $y_{n_1}^{n_1} \cdots$.

6 Friday January 17th

Let M

1. be finitely generated,
2. semisimple $M = \oplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$,
3. locally finite
4. $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^\vee$,
5. satisfy the forest condition for weights.

Theorem:

a. $\mathcal{O}$ is Noetherian (ascending chain condition on submodules, i.e. no infinite filtrations by submodules)

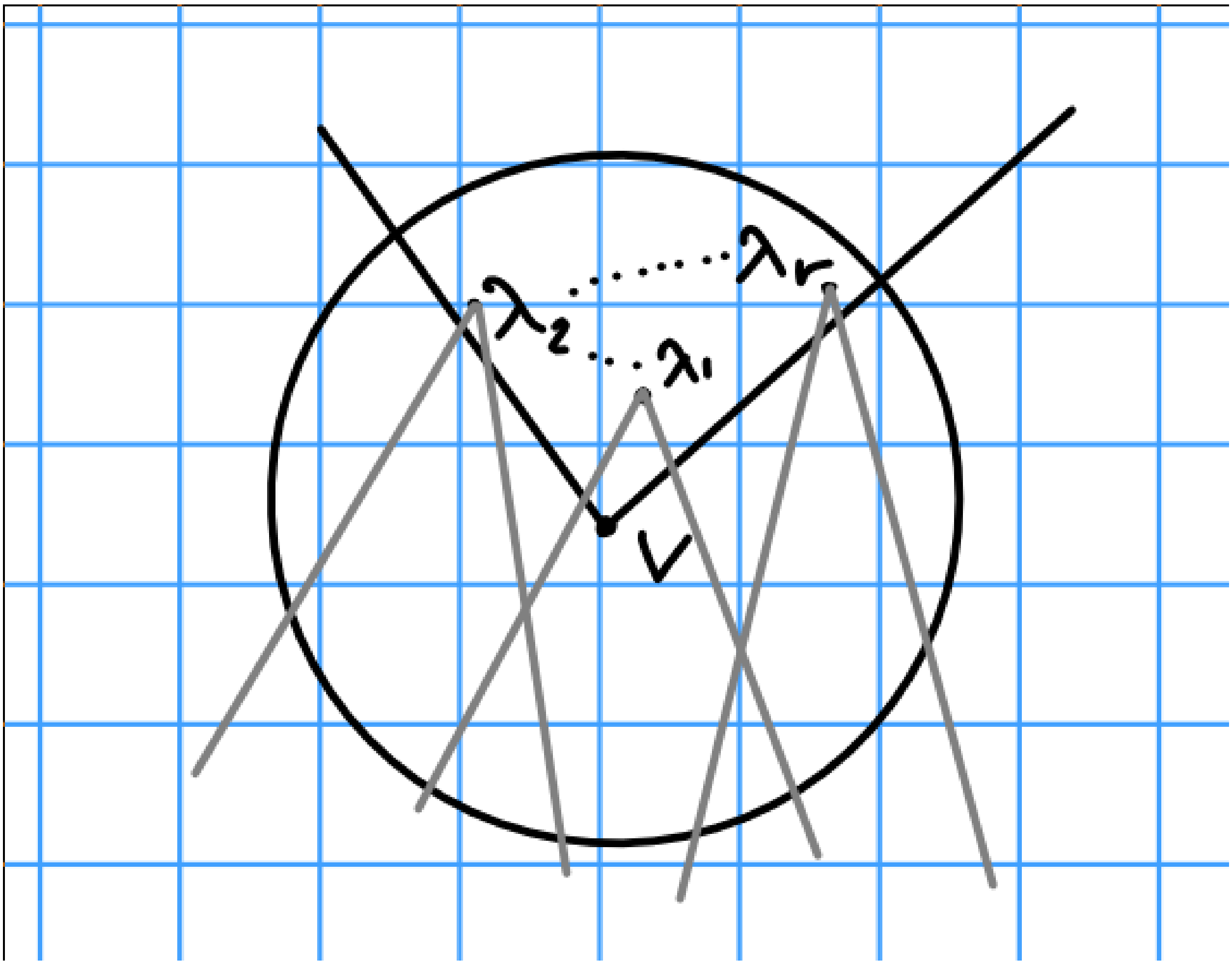


Figure 4: Image

b. $\mathcal{O}$ is closed under quotients, submodules, finite direct sums
c. $\mathcal{O}$ is abelian (similar to a category of modules)
d. If $M \in \mathcal{O}$, $\dim L < \infty$, then $L \otimes M \in \mathcal{O}$ and the endofunctor $M \mapsto L \otimes M$ is exact
e. If $M \in \mathcal{O}$ is locally $Z(\mathfrak{g})$-finite (recall: this is the center of $U(\mathfrak{g})$), i.e. $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ for all $v \in M$.
f. $M \in \mathcal{O}$ is finitely generated module. (?)

Proofs of a and b: See BA II, page 103.

Proof of c: Implied by (b), BA II Page 330.

Proof of d: Can check that $L \otimes M$ satisfies 2 and 3 above. Need to check first condition. Take a basis $\{v_i\}$ for L and $\{w_j\}$ a finite set of generators for M. The claim is that $B = \{v_i \otimes w_j\}$ generates $L \otimes M$. Let N be the submodule generated by B.

For any $v \in V$, $v \otimes w_j \in N$. For arbitrary $x \in \mathfrak{g}$, compute $x \cdot (v \otimes w_j) = (x \cdot v) \otimes w_j + x \otimes (v \cdot w_j)$. Since the LHS is in N and the first term on the RHS is in N, the entire RHS is in N. By iterating, we find that $v \otimes (u \cdot w_j) \in N$ for all PBW monomials u. So $L \otimes M \in \mathcal{O}$.

Proof of e: Since $v \in M$is a sum of weight vectors, wlog we can assume $v \in M_\lambda$ is a weight vector (where $\lambda \in \mathfrak{h}^\vee$). For any central element $z \in Z(\mathfrak{g})$, we can compute $h \cdot (z \cdot v) = z \cdot (h \cdot v) = z \cdot \lambda(h) v = \lambda(h) z \cdot v$. Thus $z \cdot v \in M_\lambda$. By (4), we know that $\dim M_\lambda < \infty$, so $\dim \operatorname{span} Z(\mathfrak{g}) \cdot v < \infty$ as well.

Proof of f: By 5, M is generated by a finite dimensional $U(\mathfrak{b})$ submodule N. Since we have a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)U(\mathfrak{b})$, there is a basis of weight vectors for N that generates M as a $U(\mathfrak{n}^-)$ module.

6.1 Highest Weight Modules

Definition: A maximal vector $v^+ \in M \in \mathcal{O}$ is a nonzero vector such that $\mathfrak{n} \cdot v^+ = 0$.

Note: By 2 and 3, every nonzero $M \in \mathcal{O}$ has a maximal vector.

Definition: A highest weight module M of highest weight λ is a module generated by a maximal vector of weight λ, i.e. $M = U(\mathfrak{g})v^+ = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})v^+ = U(\mathfrak{n}^-)v^+$.

Theorem: Let $M = U(\mathfrak{n}^-)v^+$ be a highest weight module, where $v^+ \in M_\lambda$. Fix $\Phi^+ = \{\beta_1, \cdots, \beta_n\}$ with root vectors $y_i \in \mathfrak{g}_{\beta_i}$.

a. M is the $\mathbb{C}$-span of PBW monomials $\left\langle y_1^{t_1} \cdots y_m^{t_m} \right\rangle$ of weight $\lambda - \sum t_i \beta_i$. Thus M is a module.
b. All weights μ of M satisfy $\mu \le \lambda$
c. $\dim M_\mu < \infty$ for all $\mu \in T(M)$, and $\dim M_\lambda = 1$. In particular, property (3) holds and $M \in \mathcal{O}$.
d. Every nonzero quotient of M is a highest-weight module of highest weight λ.
e. Every submodule of M is a weight module, and any submodule generated by a maximal vector with $\mu < \lambda$ is proper. If M is semisimple, then the set of maximal weight vectors equals $\mathbb{C}^\times v^+$.
f. M has a unique maximal submodule N and a unique simple quotient L, thus M is indecomposable.
g. All simple highest weight modules of highest weight λ are isomorphic. For such M, $\dim \operatorname{End}(M) = 1$. (Category $\mathcal{O}$ version of Schur's Lemma, generalizes to infinite dimensional case)

Proofs of a to e: Either obvious or follows from previous results. First few imply M is in $\mathcal{O}$, and we know the latter hold for such modules.

Proof of f: N is a sum of submodules, so $N = \sum M_i$, proper submodules of M. So take $L = M/N$. To see indecomposability, there exists a better proof in section 1.3.

Proof of g: Let $M_1 = U(\mathfrak{n}^-)v_1^+$ and M_2 be define similarly, where the $v_i \in (M_i)_\lambda$ have the same weight. Then $M_0 \coloneqq M_1 \oplus M_2$ implies that $v^+ \coloneqq (v_1^+, v_2^+)$ is a maximal vector for M_0. So $N \coloneqq U(\mathfrak{n}^-)v^+$ is a highest weight module of highest weight λ.

We have the following diagram:

$$\begin{tikzcd}
 & N \arrow[d, hook] \arrow[ddl, tail] \arrow[ddr, tail] & \\
 & M_1 \oplus M_2 \arrow[dl] \arrow[dr] & \\
M_1 & & M_2
\end{tikzcd}$$

and since e.g. $N \to M_1$ is not the zero map, it is a surjection.

By (f), N is a unique simple quotient, so this forces $M_1 \cong M_2$. Since M is simple, any nonzero $\mathfrak{g}$-endomorphism ϕ must be an isomorphism, and so we take $v^+ \mapsto cv^+$ for some $c \neq 0$. Note that since ϕ is also a $\mathfrak{h}$-morphism, we have $\dim M_\lambda = 1$.

Since v^+ generates M and $\phi(u \cdot v^+) = u\phi(v^+) = cu \cdot v^+$, thus ϕ is multiplication by a constant.

7 Wednesday January 22nd

> Note: Try problems 1.1 and 1.3*.

Recall: In category $\mathcal{O}$, we have finite dimensional, semisimple modules over $\mathbb{C}$ with triangular decompositions.

If M is any $U(\mathfrak{g})$ module than a $v^+ \in M_\lambda$ a weight vector (so $\lambda \in \mathfrak{h}^\vee$) is primitive iff $\mathfrak{n} \cdot v^+ = 0$. Note: it doesn't have to be of maximal weight. M is a highest weight module of highest weight λ iff it's generated over $U(\mathfrak{g})$ as an associative algebra by a maximal vector v^+ of weight λ. Then $M = U(\mathfrak{g}) \cdot v^+$.

See structure of highest weight modules, and irreducibility.

Corollary: If $0 \neq M \in \mathcal{O}$, then M has a finite filtration with quotients highest weight modules, i.e. $M_0 \subset M_1 \subset \cdots \subset M_n = M$ with M_i/M_{i-1} highest weight modules. Note that the quotients are not necessarily simple, so this isn't a composition series, although we'll show such a series exists later.

Theorem: Let V be the $\mathfrak{n}$ submodule of M generated by a finite set of weight vectors which generate M as a $U(\mathfrak{g})$ module. I.e. take the finite set of weight vectors and act on them by $U(\mathfrak{n})$. Then $\dim_{\mathbb{C}} V < \infty$ since M is locally $\mathfrak{n}$-finite.

Proof: Induction on $n = \dim V$. If $n = 1$, M itself is a highest weight module.

> Note that $\mathfrak{n}$ increases weights.

For $n > 1$, choose a weight vector $v_1 \in V$ of weight λ which is maximal among all weights of V. Set $M_1 := U(\mathfrak{g})v_1$; this is a highest weight submodule of M of highest weight λ. ($\mathfrak{n}$ has to kill v_1, otherwise it increases weight and v_1 wouldn't be maximal.)

Let $\overline{M} = M/M_1 \in \mathcal{O}$, this is generated by the image of $\overline{V}$ of V and thus $\dim \overline{V} < \dim V$. By the IH, $\overline{M}$ has the desired filtration, say $0 \subset \overline{M}_2 \subset \overline{M}_{n-1} \subset \overline{M}_n = \overline{M}$. Let $\pi : M \to M/M_1$, then just take the preimages $\pi^{-1}(\overline{M}_i)$ to be the filtration on M.

> Note: by isomorphism theorems, the quotients in the series for M are isomorphic to the quotients for $\overline{M}$.

7.1 Verma and Simple Modules

Constructing *universal* highest weight modules using "algebraic induction". Start with a nice subalgebra of $\mathfrak{g}$ then "induce" via $\otimes$ to a module for $\mathfrak{g}$.

Recall $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, here $\mathfrak{h} \oplus \mathfrak{n}$ is the Borel subalgebra $\mathfrak{b}$, and $\mathfrak{n}$ corresponds to a fixed choice of positive roots in Φ^+ with basis Δ. Then $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{b})$. Given any $\lambda \in \mathfrak{h}^\vee$, let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{h}$-module (i.e. 1-dimensional $\mathbb{C}$-vector space)on which $\mathfrak{h}$ acts by λ.

Let $\{1\}$ be the basis for $\mathbb{C}$, so $h \cdot 1 = \lambda(h)1$ for all $h \in \mathfrak{h}$. Then there is a map $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$, so make C_λ a $\mathfrak{b}$-module via this map. This "inflate" C_λ into a 1-dimensional $\mathfrak{b}$-module.

Note that $\mathfrak{h}$ is solvable, and by Lie's Theorem, every finite dimensional irreducible $\mathfrak{b}$-module is of the form $\mathbb{C}_\lambda$ for some $\lambda \in \mathfrak{h}^\vee$.

Definition: $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda$ is the *Verma module of highest weight* λ. This process is called algebraic/tensor induction. This is a $U(\mathfrak{g})$ module via left multiplication, i.e. acting on the first tensor factor.

Since $U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} U(\mathfrak{h})$, we have $M(\lambda) \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$, but at what level of structure?

- As a vector space (clear)
- As a $\mathfrak{n}^-$-module via left multiplication
- As a $\mathfrak{h}^-$-module via the $\otimes$ action.

In particular, $M(\lambda)$ is a *free* $U(\mathfrak{n}^-)$-module of rank 1.

> Note: this always happens when tensoring with a vector space?

Consider $v^+ := 1 \otimes 1 \in M(\lambda)$ (note that $U(\mathfrak{n}^-)$ is not homogeneous, so not graded, but does have a filtration). Then v^+ is nonzero, and freely generates $M(\lambda)$ as a $U(\mathfrak{n}^-)$-module. Moreover $\mathfrak{n} \cdot v^+ = 0$ since for $x \in \mathfrak{g}_\beta$ for $\beta \in \Phi^+$, we have

$$\begin{aligned} x(1 \otimes 1) &= x \otimes 1 \\ &= 1 \otimes x \cdot 1 \quad \text{since } x \in \mathfrak{b} \\ &= 1 \otimes 0 \implies x \in \mathfrak{n} \\ &= 0, \end{aligned}$$

and for $h \in \mathfrak{h}$,

$$\begin{aligned}h(1 \otimes 1) &= h1 \otimes 1 \\ &= 1 \otimes h1 \\ &= 1 \otimes \lambda(h)1 \\ &= \lambda(h)v^+.\end{aligned}$$

So $M(\lambda)$ is a highest weight module of highest weight λ, and thus $M(\lambda) \in \mathcal{O}$.

Observation: Any weight $\lambda \in \mathfrak{h}^\vee$ is the highest weight of some $M \in \mathcal{O}$. Let $\Pi(M)$ denote the set of weights of a module, then $\Pi(M(\lambda)) = \lambda - \mathbb{Z}^+\Phi^+$.

By PBW, we can obtain a basis for $M(\lambda)$ as $\left\{ y_1^{t_1} \cdots y_m^{t_m} v^+ \;\middle|\; t_i \in \mathbb{Z}^+ \right\}$. Taking a fixed ordering $\{\beta_1, \cdots, \beta_m\} = \Phi^+$, then $0 \neq y_i \in \mathfrak{g}_{-\beta_i}$. Then every weight of this form is a weight of some $M(\lambda)$, and every weight of $M(\lambda)$ is of this form: $\lambda - \sum t_i\beta_i$.

Remark: The functor $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\,\cdot\,) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \cdot$ from the category of finite-dimensional $\mathfrak{g}$-semisimple $\mathfrak{b}$-modules to $\mathcal{O}$ is an exact functor, since it is naturally isomorphic to $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \cdot$ (which is clearly exact?)

Alternate construction of $M(\lambda)$: Let I by a left ideal of $U(\mathfrak{g})$ which annihilates v^+, so $I = \left\langle \mathfrak{n}, h - \lambda(h) \cdot 1 \;\middle|\; h \in \mathfrak{h} \right\rangle$. Since v^+ generates $M(\lambda)$ as a $U(\mathfrak{g})$-module, then (by a standard ring theory result) $M(\lambda) = U(\mathfrak{g})/I$, since I is the annihilator of $M(\lambda)$.

Theorem (Universal property of $M(\lambda)$): Let M be any highest weight module of highest weight λ generated by v. Then $I \cdot v = 0$, so I is the annihilator of v and thus M is a quotient of $M(\lambda)$. Thus $M(\lambda)$ is universal in the sense that every other highest weight module arises as a quotient of $M(\lambda)$.

By theorem 1.2, $M(\lambda)$ has a unique maximal submodule $N(\lambda)$ (nonstandard notation) and a unique simple quotient $L(\lambda)$ (standard notation).

Theorem: Every simple module in $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$ and is determined uniquely up to isomorphism by its highest weight. Moreover, there is an analog of Schur's lemma: $\dim \hom_{\mathcal{O}}(L(\mu), L(\lambda)) = \delta_{\mu\lambda}$, i.e. it's 1 iff $\lambda = \mu$ and 0 otherwise.

> Note: up to isomorphism, we've found all of the simple modules in $\mathcal{O}$, and most are finite-dimensional.

Proof: Next class.