Section 8.6: The Solutions of the Floer Equation are "Somewhere Injective".

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Contents

0.1 Outline

Two Goals:

1. Prove critical points are discrete and regular points are open/dense.
2. Prove the continuation principle that was used in Proposition 8.1.4

0.2 Outline of Statements

$$
\begin{tikzcd}
8.1.5 \arrow[r, Rightarrow] \arrow[dr, dash] & 8.1.4 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & 8.1.3 \arrow[dl, dash] \\
& 8.1.1 \arrow[r, Rightarrow] & 8.1.2
\end{tikzcd}
$$

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- 8.1.4: $\Gamma : W^{1,p} \times C^\infty_\varepsilon \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x, y, J)$ is a Banach manifold
- 8.1.1: For $h \in \mathcal{H}_{\text{reg}}$, $H_0 + h$ is nondegenerate and $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(H_0 + h, J)$.
- 8.1.2: For $h \in \mathcal{H}_{\text{reg}}$ and all contractible orbits x, y of H_0, $\mathcal{M}(x, y, H_0 + h)$ is a manifold of dimension $\mu(x) - \mu(y)$.

0.3 Notation

- The Floer equation and its linearization:

$$\begin{aligned}\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \text{grad}\,_u(H) = 0\\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S\cdot Y\\ &\qquad Y \in u^*TW,\ S \in C^\infty(\mathbb{R}\times S^1; \text{End}(\mathbb{R}^{2n})).\end{aligned}$$

- $z = s + it$
- X is a vector field (time-dependent and periodic) on $\mathbb{R}^{2n}$, J an almost complex structure
  - X, J are smooth
- $u \in C^\infty(\mathbb{R}\times S^1; W)$ is a solution to the equation

$$\frac{\partial u}{\partial s} + J(t,u)\Big(\frac{\partial u}{\partial t} - X(t,u)\Big) = 0$$

- $C(u)$ the set of critical points and $R(u)$ the set of regular points of u:

$$\begin{aligned}(s_0,t_0) \in C(u) \subseteq \mathbb{R}\times S^1 &\iff \frac{\partial u}{\partial s}(s_0,t_0) = 0\\ (s_0,t_0) \in R(u) \subset \mathbb{R}\times S^1 &\iff (s_0,t_0) \notin C(u)\ \&\ s \neq s_0 \implies u(s_0,t_0) \neq u(s,t_0).\end{aligned}$$

0.4 Goal 1: Discrete Critical Points and Dense Regular Points

Goal 1: prove the following theorem

Theorem 0.1 ***(8.5.4).***

1. $C(u)$ is discrete and
2. $R(u) \hookrightarrow \mathbb{R}\times S^1$ is open and dense.

Outline of the proof:

- Prove 8.6.1 (direct, short) which transforms the equation

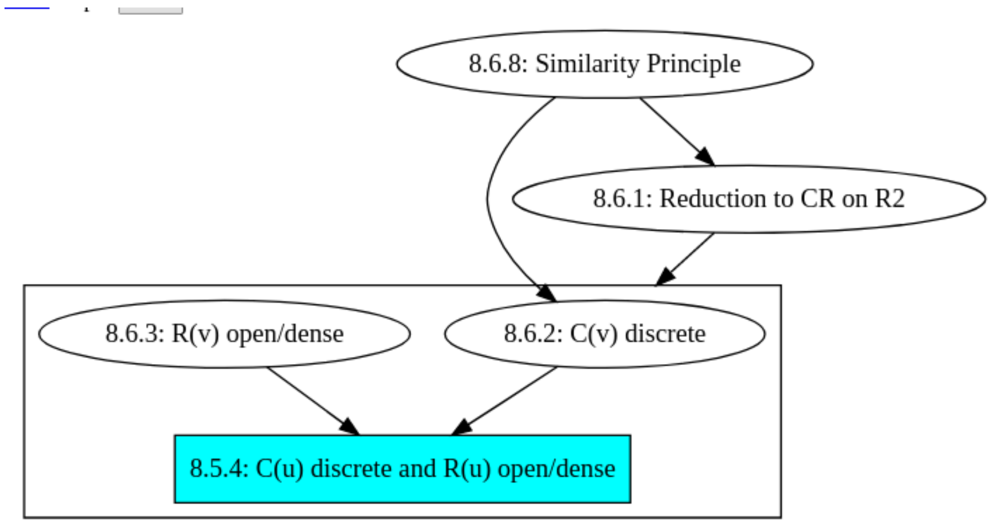
$$\frac{\partial u}{\partial s} + J(t,u)\Big(\frac{\partial u}{\partial t} - X(t,u)\Big) = 0 \quad \text{where} \quad u \in C^\infty(\mathbb{R}\times S^1; W)$$

  to a Cauchy-Riemann equation on $\mathbb{R}^2$:

$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} = 0 \quad \text{where} \quad v \in C^\infty(\mathbb{R}^2; W)$$

- Produces a map v which reduces Theorem 8.5.4 to two statements
  - 8.6.2: $C(v)$ (and thus $C(u)$) is discrete
    - Proved later using similarity principle.
  - 8.6.3 (Injectivity): If v is a smooth periodic solution of CR with $\dfrac{\partial v}{\partial s} \neq 0$ then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.
- Prove 8.6.3 (Injectivity)

  - Show open.
  - Show dense
- Prove 8.6.8 (similarity principle)
- Use similarity principle to prove 8.6.1, yields theorem.



0.5 8.6.1: Transform to Cauchy-Riemann

Proposition 0.2 ***(8.6.1, Transform to CR-equation on R2).***
If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0.$$

Then there exists
- An almost complex structure J_1
- A diffeomorphism φ on W ?
- A map $v \in C^\infty(\mathbb{R}^2; W)$

satisfying

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$

where
1. $v(s,t+1) = \varphi(v(s,t))$
2. $C(u) = C(v)$, i.e. u, v have the same critical points
3. $R(u) = R(v)$.

Proof

- Since $W \times S^1$ is compact, the flow ψ_t of X_t is defined on all of W
  - We thus have a map $\psi_t : W \longrightarrow W$ such that
    * $\frac{\partial}{\partial t}\psi_t = X_t \circ \psi_t$

    * $\psi_0 = \mathrm{id}$

- Define the (important!) map

$$v(s,t) := \left(\psi_t^{-1} \circ u\right)(s,t)$$

- We can then compute

$$\begin{aligned}\frac{\partial u}{\partial s} &= (d\psi_t)\left(\frac{\partial v}{\partial s}\right)\\ \frac{\partial u}{\partial t} &= (d\psi_t)\left(\frac{\partial v}{\partial t}\right) + X_t(u)\end{aligned}$$

.

  - Attempt at explanation: rearrange, use chain rule, and known derivative of ψ_t:

$$\begin{aligned}u(s,t) = (\psi_t \circ v)(s,t) \implies \frac{\partial u}{\partial s} &= \frac{\partial \psi_t}{\partial s}(v(s,t)) \cdot \frac{\partial v}{\partial s}(s,t)\\ \implies \frac{\partial u}{\partial t} &= \frac{\partial \psi_t}{\partial t}(v(s,t)) \cdot \frac{\partial v}{\partial t}(s,t)\\ &= (X_t \circ \psi_t)(v(s,t)) \cdot \frac{\partial v}{\partial t}(s,t)\\ &= (X_t \circ \psi_t \circ v)(s,t) \cdot \frac{\partial v}{\partial t}(s,t).\end{aligned}$$

- Continuing computations,

$$\begin{aligned}0 &= \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_t(u)\right) && \text{since } u \text{ is a solution}\\ &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} - JX_t(u) && \text{expanding terms}\\ &= \left((d\psi_t)\left(\frac{\partial v}{\partial s}\right)\right) + J\left((d\psi_t)\left(\frac{\partial v}{\partial t}\right) + X_t(u)\right) - JX_t(u) && \text{by substitution}\\ &= (d\psi_t)\left(\frac{\partial v}{\partial s}\right) + J(u)\,(d\psi_t)\left(\frac{\partial v}{\partial t}\right) && \text{cancelling}\\ &= (d\psi_t)\left(\frac{\partial v}{\partial s} + (d\psi_t)^{-1}\,J(u)\,(d\psi_t)\left(\frac{\partial v}{\partial t}\right)\right) && \text{collecting terms}\\ &:= (d\psi_t)\left(\frac{\partial v}{\partial s} + \psi_t^* J(v)\right) && \text{by definition.}\end{aligned}$$

- Conclude that v is a solution of

$$\frac{\partial v}{\partial s} + \psi_t^\star J(v)\frac{\partial v}{\partial t} = 0.$$

- Set $\varphi := \psi_1$ and $J_1(v) := \psi_1^* J(v)$ to obtain

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$

  of which v is still a solution

- We can check directly that

$$\begin{aligned} v(s,t+1) &:= (\psi_t^{-1} \circ u)(s,t+1) \\ {}_? &= \left(\psi_1 \circ \psi_t^{-1} \circ u\right)(s,t) \\ &= \psi_1(v(s,t)) \\ &:= \varphi(v(s,t)), \end{aligned}$$

  which verifies property 1.

Note: just a guess from me!

- Verifying that $C(u) = C(v)$: not spelled out. Property of flow?

Lemma 8.6.2: The set of critical points of v above is discrete. Precisely: There exists a constant $\delta > 0$ such that $(dv)_z \neq 0$ for any $0 << |z| < \delta$.

Proof: Postponed to p.264.

Definition: Multiple points

Proposition 8.6.3: Injectivity result. Let v be a smooth 1-periodic (in t) solution of the CR equation, i.e. $v(s,t+1) = \phi(v(s,t))$ for some smooth ϕ ? and $\dfrac{\partial v}{\partial s} \neq 0$. Then $R(v) \hookrightarrow \mathbb{R}^2$ is open and dense.

0.6 Regular Points Are Open and Dense

Proof (BIG):

- Show $R(v)$ is open (easy)
- Show $R(v)$ is dense (delicate)

Long proof.

Lemma 8.6.4: For every $r > 0$ there exists a $\delta > 0$ such that

$$|t - t_0|,\ |s - s_0| < \delta \implies \exists s' \in B_r(s_j) \text{ s.t. } v(s,t) = v(s',t).$$

Proof: short.

Lemma 8.6.5: Let v_1, v_2 be two solutions of the CR-equation with $X_t \equiv 0$ on $B_\varepsilon(0)$, $v_1(0,0) = v_2(0,0)$ such that $(dv_1)_0, (dv_2)_0 \neq 0$. Also suppose

$$\forall \varepsilon\ \exists \delta \text{ s.t.}$$

$$\forall (s,t) \in B_\delta(0),\ \exists s' \in \mathbb{R} \begin{cases} (s',t) \in B_\varepsilon(0) \\ v_1(s,t) = v_2(s',t) \end{cases}.$$

Then

$$\forall z \in B_\varepsilon(0), \quad v_1(s,t) = v_2(s,t).$$

Take perturbed CR equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Fix $S \in C^\infty(\mathbb{R}^2; \mathrm{End}(\mathbb{R}^{2n}))$

Lemma (Similarity Principle, used to prove continuation principle and 8.6.8): Let $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$ be a solution to the perturbed CR equation and let $p > 2$. Then there exists $0 < \delta < \varepsilon$ and a map $A \in W^{1,p}(B_\delta, \mathrm{GL}(\mathbb{R}^{2n}))$ and a holomorphic map $\sigma : B_\delta \longrightarrow \mathbb{C}^n$ such that

$$\forall (s,t) \in B_\delta \quad Y(s,t) = A(s,t)\ \sigma(s+it) \quad \text{and} \quad J_0 A)s,t) = A(s,t) J_0.$$

Use continuation principle to finish proofs of many old theorems/lemmas.

Theorem (8.6.11, Essential property of $\bar\partial$) For every $p > 1$, the following operator is surjective and Fredholm:

$$\bar\partial : W^{1,p}\left(S^2; \mathbb{C}^n\right) \longrightarrow L^p(\Lambda^{0,1} T^* S^2 \otimes \mathbb{C}^n).$$

Lead up to the proof of 8.1.5 in Section 8.7

1 Goal 2: Continuation Principle

Goal 2: prove a continuation principle:

Proposition 1.1 ***(8.6.6, Continuation Principle).***
On an open $U \subset \mathbb{R}^2$, let Y be a solution to the perturbed CR equation

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n}$ and $S \in C^\infty(\mathbb{R}^2, \mathrm{End}(\mathbb{R}^{2n}))$.
Say that f has an *infinite-order zero* at z_0 iff

$$\forall k \geq 0, \qquad \sup_{|z - z_0| \leq t} \frac{|f(z)|}{r^k} \overset{r \longrightarrow 0}{\longrightarrow} 0.$$

For f smooth, equivalently $f^{(k)}(z_0) = 0$ for all k.

Then the set

$$C := \left\{ (s,t) \in U \;\middle|\; Y \text{ has an infinite order zero at } (s,t) \right\}$$

is clopen. In particular, if U is connected and $Y = 0$ on some nonempty $V \subset U$, then $Y \equiv 0$.

Proposition 1.2 ***(8.1.4,).***

Define

$$\mathcal{Z}(x,y,J) := \left\{ (u, H_0 + h) \,|\, h \in \mathcal{C}^\infty_\varepsilon(H_0) \text{ and } u \in \mathcal{M}(x,y,J,H) \right\}.$$

If $(u, H_0 + h) \in \mathcal{Z}(x,y)$ then the following map admits a continuous right-inverse and is surjective:

$$\begin{aligned}\Gamma : W^{1,p}\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right)\times \mathcal{C}^{\infty}_{\varepsilon}\left(H_0\right) &\longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^{2n}\right)\\ (Y,h) &\longmapsto \left(d\mathcal{F}^{H_0+h}\right)_u(Y)+\operatorname{grad}_u h\end{aligned}$$

where $\mathcal{F}^{H_0+h}$ is the Floer operator corresponding to H_+h.

Used to show (via the implicit function theorem) that $\mathcal{Z}(x,y,J)$ is a Banach manifold when $x\neq y$.