

Problem Set 1

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1.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M ,

$$X : M \rightarrow TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \rightarrow M$$

and thus a one-parameter family

$$\phi_t : M \rightarrow M \in \text{Diff}(M, M).$$

In particular, $\phi_0 = \text{id}_M$, and $\phi_1 \in \text{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\rightarrow M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1 . For an arbitrary map $f : M \rightarrow M$, we have

$$\Lambda(f) = \sum_k \text{Tr} \left(f_* \Big|_{H_k(X; \mathbb{Q})} \right).$$

where $f_* : H_*(X; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is the induced map on homology, and $\Lambda_f = 0$ iff f has a fixed point.

In particular, we have

$$\begin{aligned}\Lambda(\text{id}_M) &= \sum_k \text{Tr}(\text{id}_{H_k(X; \mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M),\end{aligned}$$

the Euler characteristic of M .

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and ht