

Title

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1 Week 1

[Link to Notes](#)

1.1 Exercise 1.3H: Right Exactness of Tensoring

Show that the following endofunctor

$$\begin{aligned} F : R\text{-mod} &\longrightarrow R\text{-mod} \\ X &\mapsto X \otimes_R N \\ (X \xrightarrow{f} Y) &\mapsto (X \otimes_R N \xrightarrow{f \otimes \text{id}_N} Y \otimes_R N) \end{aligned}$$

is exact.

Solution:

Note: to make sense of the functor, we may need to show that there is an isomorphism

$$\text{hom}_{R\text{-mod}}(X, Y) \otimes_R \text{hom}_{R\text{-mod}}(A, B) \longrightarrow \text{hom}_{R\text{-mod}}(X \otimes_R A, Y \otimes_R B).$$

This is what makes taking $f : X \longrightarrow Y$ and $g : A \longrightarrow B$ and forming $f \otimes g : X \otimes A \longrightarrow Y \otimes B$ well-defined?

Let $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence, so

- $\text{im } f = \ker g$ by exactness at B
- $\text{im } g = C$ by exactness at C .

Applying the above F yields

$$A \otimes_R N \xrightarrow{f \otimes \text{id}_N} B \otimes_R N \xrightarrow{g \otimes \text{id}_N} C \otimes_R N \longrightarrow 0.$$

We thus need to show

1. Exactness as $C \otimes_R N$: $\text{im}(g \otimes \text{id}_N) = C \otimes_R N$, i.e. this is surjective.
2. Exactness at $B \otimes_R N$: $\text{im}(f \otimes \text{id}_N) = \ker(g \otimes \text{id}_N)$.

We'll use the fact that every element in a tensor product is a finite sum of elementary tensors.

- Claim: $\text{im}(g \otimes \text{id}_N) \subseteq C \otimes_R N$.
 - Let $b \otimes n \in B \otimes_R N$ be an elementary tensor
 - Then $(g \otimes \text{id}_N)(b \otimes n) := g(b) \otimes \text{id}_N(n) = g(b) \otimes n$
 - Since $\text{im}(g) = C$, there exists a $c \in C$ such that $g(b) = c$, so $g(b) \otimes n = c \otimes n \in C \otimes_R N$
 - Extend by linearity:

$$(g \otimes \text{id}_N) \left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i \right) = \sum_{i=1}^m (g \otimes \text{id}_N)(r_i \cdot b_i \otimes n_i) := \sum_{i=1}^m g(r_i \cdot b_i) \otimes \text{id}_N(n_i) \stackrel{H}{=} \sum_{i=1}^m r_i \cdot c_i \otimes n_i \in C \otimes_R N$$

where we've used bilinearity for the first equality, and the equality marked with H uses above the proof for elementary tensors, and noted that we can pull ring scalars $r_i \in R$ through R -mod morphisms. - Claim: $C \otimes_R N \subseteq \text{im}(g \otimes \text{id}_N)$. - Let $c \otimes n \in C \otimes_R N$ be an elementary tensor. - Then $c \in C = \text{im}(g)$ implies $c = g(b)$ for some $b \in B$. - So $c \otimes n = g(b) \otimes n = (g \otimes \text{id}_N)(b \otimes n) \in \text{im}(g \otimes \text{id}_N)$. - Extend by linearity:

$$\sum_{i=1}^m r_i \cdot c_i \otimes n_i \stackrel{H}{=} \sum_{i=1}^m g(r_i \cdot b_i) \otimes n_i = \sum_{i=1}^m (g \otimes \text{id}_N)(r_i \cdot b_i \otimes n_i) = (g \otimes \text{id}_N) \left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i \right).$$

This proves (1).

- Claim: $\text{im}(f \otimes \text{id}_N) \subseteq \ker(g \otimes \text{id}_N)$.
 - Let $b \otimes n \in \text{im}(f \otimes \text{id}_N)$, we want to show $(g \otimes \text{id}_N)(b \otimes n) = 0 \in C \otimes_R N$.
 - Then $b \otimes n = f(a) \otimes n$ for some $a \in A$.
 - By exactness of the original sequence, $\text{im } f \subseteq \ker g$, so $g(f(a)) = 0 \in C$
 - Then

$$(g \otimes \text{id}_N)(b \otimes n) = (g \otimes \text{id}_N)(f(a) \otimes n) := g(f(a)) \otimes n = 0 \otimes n = 0 \in C \otimes_R N$$

where we've used the fact that $0 \otimes x = 0$ in any tensor product.

– Extend by linearity.

- Claim (**nontrivial part**): $\ker(g \otimes \text{id}_N) \subseteq \text{im}(f \otimes \text{id}_N)$.

Note: the problem is that

$$x \in \ker(g \otimes \text{id}_N) \implies x = \sum_{i=1}^m r_i \cdot b_i \otimes n_i \implies (g \otimes \text{id}_N) \left(\sum_{i=1}^m r_i \cdot b_i \otimes n_i \right) = \sum_{i=1}^m r_i \cdot g(b_i) \otimes n_i = 0 \in C \otimes_R N$$

but this does not imply that $g(b_i) = 0 \in C$ for all i , which is what you would need to use $\text{im } f = \ker g$ to write $g(b_i) = 0 \implies \exists a_i, f(a_i) = b_i$ and pull everything back to $A \otimes_R N$.

- Strategy: use the first claim and the first isomorphism theorem to obtain this situation:

$$\begin{array}{ccccc}
 & & \alpha & & \\
 & \nearrow & & \searrow & \\
 \frac{B \otimes_R N}{\text{im}(f \otimes_R \text{id}_N)} & \xrightarrow{i} & \frac{B \otimes_R N}{\ker(g \otimes_R \text{id}_N)} & \xrightarrow{\cong} & \text{im}(g \otimes_R \text{id}_N) = C \otimes_R N
 \end{array}$$

- The first injection i will exist because $\text{im}(g \otimes_R \text{id}_N) \subseteq \ker(g \otimes_R \text{id}_N)$ by the first claim.
- The middle isomorphism is the first isomorphism theorem.
- The RHS equality follows from surjectivity of $g \otimes_R \text{id}_N$
- We then apply a strengthened version of the 1st isomorphism theorem for modules:

Hungerford Ch.4 Thm 1.7: If $f : A \rightarrow B$ is a R -module morphism and $C \leq \ker f$ then there is a unique map $\tilde{f} : A/C \rightarrow B$ which is an isomorphism iff f is an epimorphism and $C = \ker f$.

Following Hungerford Ch.4 Prop. 5.4, p.210.

- Since $\text{im}(g \otimes_R \text{id}_N) \subseteq \ker(g \otimes_R \text{id}_N)$, by the theorem the map α exists and satisfies the same formula, i.e. $\alpha = \tilde{g} \otimes \text{id}_N$ where the tilde denotes the induced map on quotients, so $\alpha([b \otimes n]) = g(b) \otimes n$.

- * We will show it is an isomorphism, which forces $\text{im}(g \otimes_R \text{id}_N) \cong \ker(g \otimes_R \text{id}_N)$ by the above theorem.

- Constructing the inverse map: define

$$\begin{aligned}
 \tilde{\alpha}^{-1} : C \times N &\rightarrow \frac{B \otimes_R N}{\text{im}(g \otimes_R \text{id}_N)} \\
 (c, n) &\mapsto (b \otimes n) \mod \text{im}(g \otimes_R \text{id}_N) \quad \text{where } b \in g^{-1}(c),
 \end{aligned}$$

which we will show well-defined (i.e. independent of choice of b) and R -linear, lifting to a map α^{-1} out of the tensor product by the universal property which is a two-sided inverse for α .

- Well-defined:

- * $g^{-1}(b)$ exists because g is surjective.
- * If $b \neq b'$ and $g(b') = 0$, then $0 = g(b) - g(b') = g(b - b')$ so $b - b' \in \ker g$.
- * By the original exactness, $b - b' \in \text{im } f$ so $b - b' = f(a)$ for some $a \in A$.
- * Then $f(a) \otimes n \in \text{im}(f \otimes \text{id})$ implies $f(a) \otimes n \equiv 0 \mod \text{im}(f \otimes \text{id})$.
- * Then noting that $b - b' = f(a) \implies b = f(a) + b'$, working mod $\text{im}(g \otimes_R \text{id}_N)$ we have

$$b \otimes n \equiv (f(a) + b') \otimes n \equiv (f(a) \otimes n) + (b' \otimes n) \equiv b' \otimes n.$$

- R -linear:

- * ?

- Two-sided identity:

- * $(\alpha \circ \alpha^{-1})(c \otimes n) = \alpha(b \otimes n) = g(b) \otimes n = c \otimes n$, so $\alpha \circ \alpha^{-1} = \text{id}$.
- * $(\alpha^{-1} \circ \alpha)([b \otimes n]) = \alpha^{-1}(g(b) \otimes n) = [b' \otimes n]$ where $b' \in g^{-1}(g(b))$ implies $b' = b$, so $\alpha^{-1} \circ \alpha = \text{id}$.

2 More Exercises

2.1 1.3.K

Note: I think this is an exercise about base change.

Part a: For M an A -module and $\varphi : A \rightarrow B$ a morphism of rings, give $B \otimes_A M$ the structure of a B -module and show that it describes a functor $B\text{-Mod} \rightarrow A\text{-Mod}$.

Solution

- $B \otimes_A M$ makes sense: B is a (B, A) -bimodule with the usual multiplication on the left and the right action

$$\begin{aligned} A &\rightarrow \text{End}(B) \\ a &\mapsto (b \mapsto b \cdot \varphi(a)). \end{aligned}$$

- $B \otimes_A M$ is a left B -module via the following action:

$$\begin{aligned} B &\rightarrow \text{End}(B \otimes_A M) \\ b_0 &\mapsto (b \otimes m \mapsto b_0 b \otimes m). \end{aligned}$$

- This describes a functor:

$$\begin{aligned} F : A\text{-Mod} &\rightarrow B\text{-Mod} \\ X &\mapsto B \otimes_A X \\ (X \xrightarrow{f} Y) &\mapsto (B \otimes_A X \xrightarrow{\text{id}_B \otimes f} B \otimes_A Y). \end{aligned}$$

– Need to check:

- * Preserves identity morphism, i.e. $X \in A\text{-Mod}$ implies $F(\text{id}_X) = \text{id}_{F(X)}$ in $B\text{-Mod}$.
- * Preserves composition: $F(f \circ g) = F(f) \circ F(g)$.
- Preserving identity morphisms:
 - By construction $X \xrightarrow{\text{id}_X} X$ maps to $B \otimes_A X \xrightarrow{\text{id}_B \otimes \text{id}_X} B \otimes_A X$, can argue that this is the identity map for B -modules.
- Preserving composition:

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (B \otimes_A X \xrightarrow{\text{id}_B \otimes f} B \otimes_A Y \xrightarrow{\text{id}_B \otimes g} B \otimes_A Z) = (B \otimes_A X \xrightarrow{\text{id}_B \otimes (g \circ f)} B \otimes_A Z).$$

Note: not sure if there's anything to show here.

Part b: If $\psi : A \rightarrow C$ is another ring morphism, show that $B \otimes_A C$ has a ring structure.

Solution:

- Note $B \otimes_A C$ makes sense, since C is a left A -module via $a \mapsto (c \mapsto \psi(a)c)$.

- Need to define $(B \otimes_A C, P, M)$ such that it's an abelian group under P (plus), a monoid under M (multiplication), and left/right distributivity.
- Start by defining on cartesian products:

$$P : (B \otimes_A C)^{\times 2} \longrightarrow B \otimes_A C$$

$$P((b_1 \otimes c_1), (b_2 \otimes c_2)) = (b_1 +_B b_2) \otimes (c_1 +_C c_2),$$

$$M : (B \otimes_A C)^{\times 2} \longrightarrow B \otimes_A C$$

$$M((b_1 \otimes c_1), (b_2 \otimes c_2)) = (b_1 \cdot_B b_2) \otimes (c_1 \cdot_C c_2).$$

- Check A -bilinearity:

$$\begin{aligned} P(a \cdot (b_1 \otimes c_1), (b_2 \otimes c_2)) &:= (a \cdot (b_1 + b_2)) \otimes (c_1 + c_2) \\ &= ((b_1 + b_2)) \otimes a \cdot (c_1 + c_2) \quad \text{since } C \text{ is a left } A\text{-module} \\ &:= P((b_1 \otimes c_1), a \cdot (b_2 \otimes c_2)). \end{aligned}$$

$$\begin{aligned} M(a \cdot (b_1 \otimes c_1), (b_2 \otimes c_2)) &:= (a \cdot (b_1 \cdot b_2)) \otimes (c_1 \cdot c_2) \\ &= (b_1 \cdot b_2) \otimes (a \cdot (c_1 \cdot c_2)) \quad \text{since } C \text{ is a left } A\text{-module} \\ &:= M((b_1 \otimes c_1), a \cdot (b_2 \otimes c_2)). \end{aligned}$$

- So these lift to maps out of $(B \otimes_A C)^{\otimes 2}$.
- P forms an abelian group: clear because $+_B, +_C$ do, and commuting is just done within each factor.
- M forms a monoid: clear for some reason.
- Checking distributivity, claim: it suffices to check on elementary tensors and extend by linearity?

$$\begin{aligned} (b_0 \otimes c_0) \cdot ((b_1 \otimes c_1) + (b_2 \otimes c_2)) &= (b_0 \otimes c_0) \cdot ((b_1 + b_2) \otimes (c_1 + c_2)) \\ &= (b_0(b_1 + b_2)) \otimes (c_0(c_1 + c_2)) \\ &= (b_0 b_1 + b_0 b_2) \otimes (c_0 c_1 + c_0 c_2) \\ &= \dots \end{aligned}$$

2.2 1.3.L

If $S \subseteq A$ is multiplicative and $M \in A\text{-Mod}$, describe a natural isomorphism

$$\eta : (S^{-1}A) \otimes_A M \longrightarrow (S^{-1}M)$$

as both $S^{-1}A$ -modules and A -modules.

Solution

- Recall the definition

$$S^{-1}A := \left\{ \frac{a}{s} \mid a \in A, s \in S \right\} / \sim$$

$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \iff \exists s \in S \text{ such that } s(s_2 a_1 - s_1 a_2) = 0_A.$$

- Similarly $S^{-1}M = \left\{ \frac{m}{s} \right\} / \sim$.

The universal property: in $A\text{-Mod}$, $M \longrightarrow S^{-1}M$ is initial among all morphisms $\alpha : M \longrightarrow N$ such that $\alpha(S) \subseteq N^\times$:

$$\begin{array}{ccc} & & S^{-1}M \\ & \nearrow^{S^{-1}} & \vdots \exists! \tilde{\alpha} \\ M & \xrightarrow{\alpha} & N \end{array}$$

Strategy: define a map $M \longrightarrow S^{-1}A \otimes_A M$ such that S is invertible in the image to obtain a map? Show they satisfy the same universal property?

- Since $M \in A\text{-Mod}$, we have an action $a \cdot m$, so define

$$\eta : (S^{-1}A) \times M \longrightarrow (S^{-1}M)$$

$$\left(\frac{a}{s}, m \right) \mapsto \frac{a \cdot m}{s}.$$

- The tensor product $S^{-1}A \otimes_A M$ makes sense.
 - $S^{-1}A$ is a right A -module by $a_0 \mapsto \left(\frac{a}{s} \mapsto \frac{a_0 a}{s} \right)$.
 - $S^{-1}M$ is a left A -module by $a_0 \mapsto (m \mapsto a_0 \cdot m)$ where the action comes from the A -module structure of M .
- The map makes sense as an A -module morphism
 - $S^{-1}A \otimes_A M$ is a left A -module by $a_0 \mapsto \left(\frac{a}{s} \otimes m \mapsto \frac{a_0 a}{s} \otimes m \right)$
 - $S^{-1}M$ is a left A -module by $a_0 \mapsto \left(\frac{m}{s} \mapsto \frac{a_0 \cdot m}{s} \right)$ using the A -module structure on M .
- The map makes sense as an $S^{-1}A$ -module morphism
 - $S^{-1}A \otimes_A M$ is a left $S^{-1}A$ -module by $\frac{a_0}{s_0} \mapsto \left(\frac{a}{s} \otimes m \mapsto \frac{a_0 a}{s_0 s} \otimes m \right)$
 - $S^{-1}M$ is a left $S^{-1}A$ -module by $\frac{a_0}{s_0} \mapsto \left(\frac{m}{s} \mapsto \frac{a_0 \cdot m}{s_0 s} \right)$ by the A -module structure on M .
- Well-defined: ?

- A -bilinear: let $r \in A$, then

$$\begin{aligned}
 \eta\left(r \cdot \frac{a}{s}, m\right) &:= \eta\left(\frac{r \cdot a}{s}, m\right) \\
 &:= \frac{\psi(r \cdot a)(m)}{s} \\
 &= \frac{r \cdot \psi(a)(m)}{s} \quad \text{since } \psi \text{ is a ring morphism} \\
 &= \frac{\psi(a)(r \cdot m)}{s} \quad \text{since } \psi(a) \text{ is a ring morphism} \\
 &:= \eta\left(\frac{a}{s}, r \cdot m\right).
 \end{aligned}$$

So this lifts to a map out of the tensor product.

- $S^{-1}A$ -bilinear?

2.3 1.3.P

Show that the fiber product over the terminal object is the cartesian product.

Solution:

- Recall definition: T is terminal iff every object X admits a morphism $X \rightarrow T$.
- Strategy: use both universal products to produce an isomorphism
- Let $\mathbf{p}_X, \mathbf{p}_Y$ be the cartesian product projections, and $\mathbf{p}_X^T, \mathbf{p}_Y^T$ be the fiber product projections
- Let T_X, T_Y be the maps $X \rightarrow T, Y \rightarrow T$.
- Since $X \times Y$ is an object in this category, it admits one unique map to T

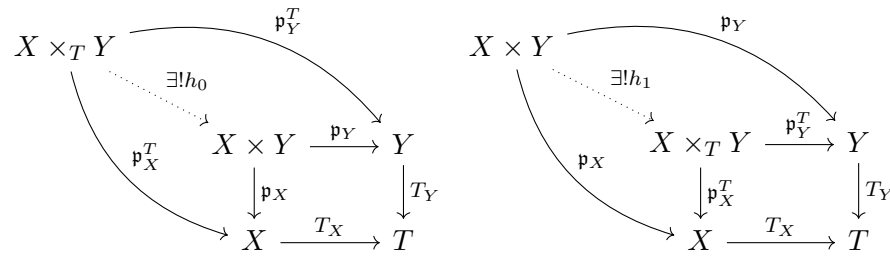
$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\mathbf{p}_Y} & Y \\
 \mathbf{p}_X \downarrow & \searrow T_{X \times Y} & \downarrow T_Y \\
 X & \xrightarrow{T_X} & T
 \end{array}$$

- But now $T_Y \circ \mathbf{p}_Y : X \times Y \rightarrow T$ is another such map, so it must equal $T_{X \times Y}$.
- Similarly $T_X \circ \mathbf{p}_X$ is equal to $T_{X \times Y}$.
- Thus $T_Y \circ \mathbf{p}_Y = T_X \circ \mathbf{p}_X$, which is part of the universal property for $X \times_T Y$.
- By the universal property of $X \times Y$, for every W admitting maps to X, Y we get the following h_0 :

$$\begin{array}{ccccc}
 W & & & & \\
 & \searrow \exists! h_0 & & \searrow & \\
 & X \times Y & \xrightarrow{\mathbf{p}_Y} & Y & \\
 & \downarrow \mathbf{p}_X & & \downarrow T_Y & \\
 & X & \xrightarrow{T_X} & T &
 \end{array}$$

Note that T doesn't matter in this particular diagram.

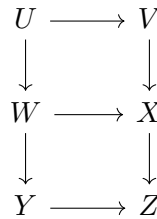
- This gives us the LHS diagram, the RHS comes from the universal property of $X \times Y$:



- By commutativity, $h_0 \circ h_1 = \text{id}_{X \times Y}$ and vice-versa?

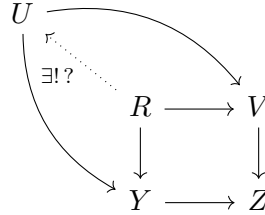
2.4 1.3.Q

Show that if the two squares in this diagram are cartesian, then the outer square is also cartesian:

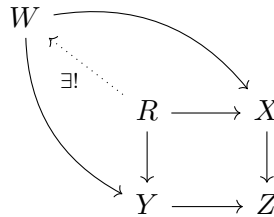


Solution:

- Need to show that given two maps $R \rightarrow V$ and $R \rightarrow Y$ such that $(V \rightarrow Z) \circ (U \rightarrow V) = (Y \rightarrow Z) \circ (R \rightarrow Y)$, then there is a unique map $R \rightarrow U$ giving a commuting diagram:



- Applying the bottom square:
 - Need to produce maps $R \rightarrow X$ and $R \rightarrow Y$
 - We're given a map $R \rightarrow Y$ by assumption.
 - We can build a map $R \rightarrow X$ by taking $(V \rightarrow X) \circ (R \rightarrow V)$.
 - We then get a map $R \rightarrow W$:



- Applying the top square:
 - We have a map $R \rightarrow V$ by assumption
 - We have a map $R \rightarrow W$ from step 1
 - We have maps $V \rightarrow X$ and $W \rightarrow X$ from the top square
 - We thus obtain

