Title

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Last time: $V(I) = \left\{ x \in \mathbb{A}^n \;\middle|\; f(x) = 0 \, \forall x \in I \right\}$ and $I(X) = \left\{ f \in k[x_1, \cdots, x_n] \;\middle|\; f(x) = 0 \, \forall x \in X \right\}$. We proved the Hilbert Nullstellensatz $I(V(J)) = \sqrt{J}$, defined the coordinate ring of an affine variety X as $A(X) \coloneqq k[x_1, \cdots, x_n]/I(X)$, the ring of "regular" (polynomial) functions on X.

Recall that a *topology* on X can be defined as a collection of "closed" subsets of X that are closed under arbitrary intersections and finite unions. A subset $Y \subset X$ inherits a subspace topology with closed sets of the form $Z \cap Y$ for $Z \subset X$ closed.

Definition 1.0.1 (Zariski Topology)**.**
Let X be an affine variety. The closed sets are affine subvarieties $Y \subset X$.

We have $\emptyset, X$ closed, since

1. $V_X(1) = \emptyset$,
2. $V_X(0) = X$

Closure under finite unions: Let $V_X(I), V_X(J)$ be closed in X with $I, J \subset A(X)$ ideals. Then $V_X(IJ) = V_X(I) \cup V_X(J)$.

Closure under intersections: We have $\bigcap_{i \in \sigma} V_X(J) = V_X\left(\sum_{i \in \sigma} J_i \right)$.

Remark 1.
There are few closed sets, so this is a "weak" topology.

Example 1.1.
Compare the classical topology on $\mathbb{A}^1/\mathbb{C}$ to the Zariski topology.

Consider the set $A \coloneqq \left\{ x \in \mathbb{A}^1/\mathbb{C} \;\middle|\; \|x\| \leq 1 \right\}$, which is closed in the classical topology.

But A is not closed in the Zariski topology, since the closed subsets are finite sets or the whole space.

Here the topology is in fact the cofinite topology.

Example 1.2.
Let $f : \mathbb{A}^1/k \longrightarrow \mathbb{A}^1/k$ be any injective map. Then f is necessarily continuous wrt the Zariski topology.

Thus the notion of continuity is too weak in this situation.

Example 1.3.
Consider $X \times Y$ a product of affine varieties. Then there is a product topology where open sets are of the form $\displaystyle\bigcup_{i=1}^{n} U_i \times V_i$ with U_i, V_i open in X, Y respectively.

This is the wrong topology! On $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, the diagonal $\Delta \coloneqq V(x-y)$ is closed in the Zariski topology on $\mathbb{A}^2$ but not in the product topology.

Example 1.4.
Consider $\mathbb{A}^2/\mathbb{C}$, so the closed sets are curves and points. Observation: $V(x_1x_2) \subset \mathbb{A}^2/\mathbb{C}$ decomposed into the union of the coordinate axes $X_1 \coloneqq V(x_1)$ and $X_2 \coloneqq V(x_2)$. The Zariski topology can detect these decompositions.

Definition 1.0.2 (Irreducibility and Connectedness)**.**
Let X be a topological space.

a. X is *reducible* iff there exist nonempty proper closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise, X is said to be *irreducible*.
b. X is *disconnected* if there exist $X_1, X_2 \subset X$ such that $X = X_1 \coprod X_2$. Otherwise, X is said to be *connected*.

Example 1.5.
$V(x_1x_2)$ is reducible but connected.

Remark 2.
$\mathbb{A}^1/\mathbb{C}$ is *not* irreducible, since we can write $\mathbb{A}^1/\mathbb{C} = \{\|x\| \leq 1\} \cup \{\|x\| \geq 1\}$.

Proposition 1.1 ***(?).***
Let X be a disconnected affine variety with $X = X_1 \coprod X_2$. Then $A(X) \cong A(X_1) \times A(X_2)$.

Proof .
We have $X_1 \cup X_2 = X$, so $I(X_1) \cap I(X_2) = I(X) = (0)$ in the coordinate ring $A(X)$ (recalling that it is a quotient by $I(X)$.)
Since $X_1 \cap X_1 \emptyset$, we have

$$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)} = I(\emptyset) = \langle 1 \rangle .$$

Thus $I(X_1) + I(X_2) = \langle 1 \rangle$, and by the Chinese Remainder Theorem, the following map is an isomorphism:

$$A(X) \longrightarrow A(X)/I(X_1) \times A(X)/I(X_2).$$

But the codomain is precisely $A(X_1) \times A(X_2)$.

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Proposition 1.2*(?)*.
An affine variety X is irreducible $\iff$ $A(X)$ is an integral domain.

Proof .
$\implies$: By contrapositive, suppose $f_1, f_2 \in A(X)$ are nonzero with $f_1 f_2 = 0$. Let $X_i = V(f_i)$, then $X = V(0) = V(f_1 f_2) = X_1 \cup X_2$ which are closed and proper since $f_i \neq 0$.

$\impliedby$: Suppose X is reducible with $X = X_1 \cup X_2$ with X_i proper and closed. Define $J_i \coloneqq I(X_i)$, and note $J_i \neq 0$ because $V(J_i) = V(I(X_i)) = X_i$ by part (a) of the Nullstellensatz.
So there exists a nonzero $f_i \in J_i = I(X_i)$, so f_i vanishes on X_i. But then $V(f_1) \cup V(f_2) \supset X_1 \cup X_2 = X$, so $X = V(f_1 f_2)$ and $f_1 f_2 \in I(X) = \langle 0 \rangle$ and $f_1 f_2 = 0$. So $A(X)$ is not a domain.

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Example 1.6.
Let $X = \{p_1, \cdots, p_d\}$ be a finite set in $\mathbb{A}^n$. The Zariski topology on X is the discrete topology, and $X = \coprod \{p_i\}$. So

$$A(X) = A(\coprod \{p_i\}) = \prod_{i=1}^{d} A(\{p_i\}) = \prod_{i=1}^{d} k[x_1, \cdots, x_n] / \langle x_j - a_j(p_i) \rangle_{j=1}^{d} .$$

Example 1.7.
Set $V(x_1 x_2) = X$, then $A(X) = k[x_1, x_2]/\langle x_1 x_2 \rangle$. This not being a domain (since $x_1 x_2 = 0$) corresponds to $X = V(x_1) \cup V(x_2)$ not being irreducible.

Example 1.8.
$\mathbb{A}^2/k$ is irreducible since $k[x_1, \cdots x_n]$ is a domain.

Example 1.9.

Let X_1 be the xy plane and X_2 be the line parallel to the y-axis through $[0, 0, 1]$, and let $X = X_1 \coprod X_2$. Then $X_1 = V(z)$ and $X_2 = V(x, z - 1)$, and $I(X) = \langle z \rangle \cdots \langle x, z - 1 \rangle = \left\langle xz, z^2 - z \right\rangle$.

Then the coordinate ring is given by $A(X) = \mathbb{C}[x, y, z] / \left\langle xz, z^2 - z \right\rangle = \mathbb{C}[x, y, z] / \langle z \rangle \oplus \mathbb{C}[x, y, z] / \langle x, z - 1 \rangle$.