Qual Problems

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Contents

1 Problem 1

1.1 Part 1

Definition: An element $r \in R$ is *irreducible* if whenever $r = st$, then either s or t is a unit.

Definition: Two elements $r, s \in R$ are *associates* if $r = \ell s$ for some unit ℓ.

A ring R is a *unique factorization domain* iff for every $r \in R$, there exists a set $\left\{ p_i \mathrel{\Big|} 1 \leq i \leq n \right\}$ such that $r = u \prod_{i=1}^n p_i$ where u is a unit and each p_i is irreducible.

Moreover, this factorization is unique in the sense that if $r = w \prod_{i=1}^n q_i$ for some w a unit and q_i irreducible elements, then each q_i is an associate of some p_i.

1.2 Part 2

A ring R is a *principal ideal domain* iff whenever $I \trianglelefteq R$ is an ideal of R, there is a single element $r_i \in R$ such that $I = (r_i)$.

1.3 Part 3

An example of a UFD that is not a PID is given by $R = k[x, y]$ for k a field.

That R is a UFD follows from the fact that if k is a field, then k has no prime elements since every non-zero element is a unit. So the factorization condition holds vacuously for k, and k is a UFD. But then we can use the following result:

Theorem: If R is a UFD, then $R[x]$ is a UFD.

Since k is a UFD, the theorem implies that $k[x]$ is a UFD, from which it follows that $k[x][y] = k[x,y]$ is also a UFD.

To see that R is not a PID, consider the ideal $I = (x,y)$, and suppose $I = (g)$ for some single $g \in k[x,y]$.

Note that $I \neq R$, since I contains no degree zero polynomials. Moreover, since $(x) \subset I = (g)$ (and similarly for y), we have $g \,\Big|\, x$ and $g \,\Big|\, y$, which forces $\deg g = 0$.

So in fact $g \in k$ and thus g is invertible, but then $(g) = g^{-1}(g) = (1) = k$, so this forces $I = k \trianglelefteq k[x,y]$. However, $x \notin k$ (nor y), which is a contradiction.

2 Problem 2

Lemma: Let A be a linear operator, $m_A(x)$ its minimal polynomial, and $p_A(x)$ its characteristic polynomial. Then A has distinct eigenvalues $\iff m_A(x) = p_A(x)$.

Proof: We always have $m_A(x) \,\Big|\, p_A(x)$, since $p_A(A) = 0$ by Cayley-Hamilton and $m_A(x)$ is the *minimal* polynomial for which this holds. We can also note that $p_A(\lambda_i) = 0$ for all eigenvalues λ_i by construction. Moreover, if every eigenvalue is distinct, then we have $p_A(x) = \prod_i (x - \lambda_i)$ in $\overline{k}[x]$.

Given an invariant factor decomposition of A with factors $f_1 \,\Big|\, f_2 \,\Big|\, \cdots \,\Big|\, f_n := m_A(x)$, it is the case that if λ_i is an eigenvalue of A, then $f_j(\lambda_i) = 0$ for some f_j.

But then $m_A(\lambda_i) = 0$ for all λ_i, since each f_j divides m_A. So every eigenvalue is a root of $m_A(x)$, and thus $\prod_i (x - \lambda_i) \,\Big|\, m_A(x)$.

But then we have both $m_A(x) \,\Big|\, p_A(x)$ and $p_A(x) \,\Big|\, m_A(x)$, so these polynomials are equal.

$\square$

Lemma If A is a linear operator and we write

$$V = \frac{k[x]}{(f_1)} \oplus \frac{k[x]}{(f_2)} \oplus \cdots \oplus \frac{k[x]}{(f_n)}$$

as an invariant factor decomposition of V as a $k[x]$ module (defined by $p(x) \curvearrowright \mathbf{v} := p(A)(\mathbf{v})$), with $f_1 \,\Big|\, f_2 \,\Big|\, \cdots$, then we always have

- $m_A(x) = f_n(x)$, i.e. the minimal polynomial is the invariant factor of largest degree,
- $p_A(x) = \prod_{i=j}^n f_j(x)$, i.e. the characteristic polynomial is the product of all of the invariant factors.

$\square$

$\Longleftarrow$: Suppose A has distinct eigenvalues, then by Lemma 1 we have

$$f_n(x) = \prod_{j=1}^{n} f_j(x),$$

which can only happen if $f_1(x) = f_2(x) = \cdots = f_{n-1}(x) = 1$, in which case there is only one nontrivial invariant factor and we have

$$V \cong \frac{k[x]}{(f_n)}, \quad \mathrm{Ann}(V) = (f_n)$$

which exhibits V as a cyclic $k[x]$-module and thus we have $V = k[x]\mathbf{v}$ for some $\mathbf{v} \in V$.

We can now note that if $\deg f_n = \dim V = m$, we have

$$k[x]/(f_n) = \mathrm{span}_{k[x]} \left\{ 1, x, \cdots, x^{m-1} \right\} \iff V \cong k[x]\mathbf{v} = \mathrm{span}_{k[x]} \left\{ \mathbf{1v}, x\mathbf{v}, \cdots x^{m-1}\mathbf{v} \right\},$$

But then noting that the $k[x] \curvearrowright V$ by $w \mapsto xw$, so $k[T] \curvearrowright V$ by $w \mapsto Tw$.

$\Longrightarrow$:

Suppose

$$V = \mathrm{span}_k \left\{ \mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \cdots A^{n-1}\mathbf{v} \right\} := \mathrm{span}_k \mathcal{B}$$

where $\dim_k V = n$.

Then $A^n\mathbf{v}$ is necessarily a linear combination of these basis elements, and in particular, there are coefficients c_i (not all zero) such that

$$A^n\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i \mathbf{v}.$$

This means that with respect to the basis $\mathcal{B}$, A has the following matrix representation:

$$[A]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \dots & 0 & c_0 \\ 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & c_{n-1} \end{bmatrix},$$

which we compute by considering the images of all basis elements under A. Letting $\mathcal{B} = \left\{ \mathbf{w}_i := A^i\mathbf{v} \;\middle|\; 0 \leq i \leq n -\right.$
we have

$$\begin{aligned}
\mathbf{w}_0 &:= \mathbf{v} \mapsto A\mathbf{v} := \mathbf{w}_1 \\
\mathbf{w}_1 &:= A\mathbf{v} \mapsto A^2\mathbf{v} := \mathbf{w}_2 \\
\mathbf{w}_2 &:= A^2\mathbf{v} \mapsto A^3\mathbf{v} := \mathbf{w}_3 \\
&\vdots \qquad \vdots \\
\mathbf{w}_{n-2} &:= A^{n-2}\mathbf{v} \mapsto A^{n-1}\mathbf{v} := \mathbf{w}_{n-1} \\
\mathbf{w}_{n-1} &:= A^{n-1}\mathbf{v} \mapsto A^{n}\mathbf{v} = \sum_{i=0}^{n-1} c_i A^i \mathbf{v}_i := \sum_{i=0}^{n-1} c_i \mathbf{w}_i.
\end{aligned}$$

3 Problem 3

3.1 Part 1

Let $\mathbf{v} = [0, 1, 0]^t$, We compute

$$M\mathbf{v} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(0) + 0(1) + x(0) \\ 0(0) + 1(1) + 0(0) \\ y(0) + 0(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which shows that $\mathbf{v}$ is an eigenvector of M with eigenvalue $\lambda = 1$.

3.2 Part 2

Noting that the rank is the dimension of the column space, we find that

- $\text{rank}(M) \geq 1$, since it is not the zero matrix,
- $\text{rank}(M) \geq 2$, since neither $[1, 0, y]^t$ or $[x, 0, 1]^t$ can be in the span of $[0, 1, 0]^t$, and
- $\text{rank}(M) = 3 \iff \det(M) \neq 0$.

So we compute

$$\det_M(x, y) = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ y & 0 & 1 \end{vmatrix} = 1(1 - 0) - 0(1 - xy) + x(-y) = 1 - xy,$$

and so $\det_M(x, y) = 0 \iff xy = 1$. Thus

$$\text{rank}(M) = \begin{cases} 3 & xy = 1 \\ 2 & \text{else.} \end{cases}$$

3.3 Part 3

Since M is diagonalizable $\iff$ M is full rank, which in this case means $\operatorname{rank}(M) = 3$, we have

$$S = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; M \text{ is diagonalizable } \right\} = \left\{ \left(x, \frac{1}{x} \right) \;\middle|\; x \in \mathbb{R} \setminus \{0\} \right\} \subset \mathbb{R}^2.$$