

Title

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Contents

1 Thursday, August 27	3
1.1 Consequence of the Nullstellensatz	3

1 | Thursday, August 27

1.1 Consequence of the Nullstellensatz

Recall Hilbert's Nullstellensatz:

- a. For any affine variety, $V(I(X)) = X$.
- b. For any ideal $J \subseteq k[x_1, \dots, x_n]$, $I(V(J)) = \sqrt{J}$.

So there's an order-reversing bijection

$$\{\text{Radical ideals } k[x_1, \dots, x_n]\} \rightarrow V(\cdot)I(\cdot) \{\text{Affine varieties in } \mathbb{A}^n\}.$$

In proving $I(V(J)) \subseteq \sqrt{J}$, we had an important lemma (Noether Normalization): the maximal ideals of $k[x_1, \dots, x_n]$ are of the form $\langle x - a_1, \dots, x - a_n \rangle$.

Corollary 1.1.1(?).

If $V(I)$ is empty, then $I = \langle 1 \rangle$.

Remark 1.1.2: This is because no common vanishing locus \implies trivial ideal, so there's a linear combination that equals 1.

Slogan 1.1.3: The only ideals that vanish nowhere are trivial.

Proof.

By contrapositive, suppose $I \neq \langle 1 \rangle$. By Zorn's Lemma, there exists a maximal ideal \mathfrak{m} such that $I \subset \mathfrak{m}$. By the order-reversing property of $V(\cdot)$, $V(\mathfrak{m}) \subseteq V(I)$. By the classification of maximal ideals, $\mathfrak{m} = \langle x - a_1, \dots, x - a_n \rangle$, so $V(\mathfrak{m}) = \{a_1, \dots, a_n\}$ is nonempty. ■

We now return to the remaining hard part of the proof of the Nullstellensatz:

Proof ($I(V(J)) \subseteq \sqrt{J}$ (hard part)).

Let $f \in V(I(J))$, we want to show $f \in \sqrt{J}$. Consider the ideal

$$\tilde{J} := J + \langle ft - 1 \rangle \subseteq k[x_1, \dots, x_n, t]$$

Observation 1.1.4: $f = 0$ on all of $V(J)$ by the definition of $I(V(J))$. However, if $f = 0$, then $ft - 1 \neq 0$, so

$$V(\tilde{J}) = V(J) \cap V(ft - 1) = \emptyset$$

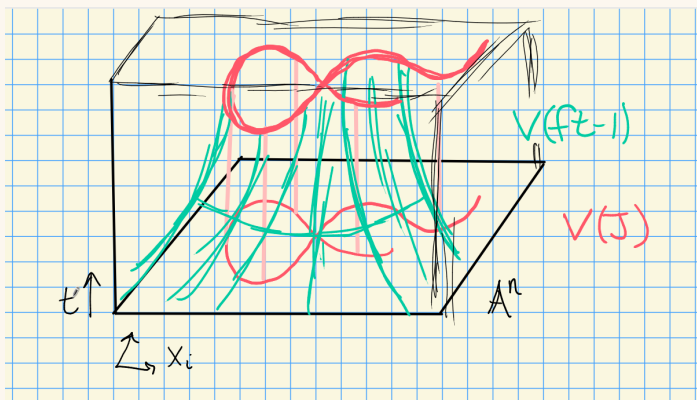


Figure 1: Effect, a hyperbolic tube around $V(J)$, so both can't vanish

Applying the corollary $\tilde{J} = (1)$, so

$$1 = \langle ft - 1 \rangle g_0(x_1, \dots, x_n, t) + \sum f_i g_i(x_1, \dots, x_n, t)$$

with $f_i \in J$. Let t^N be the largest power of t in any g_i . Thus for some polynomials G_i , we have

$$f^N := (ft - 1)G_0(x_1, \dots, x_n, ft) + \sum f_i G_i(x_1, \dots, x_n, ft)$$

noting that f does not depend on t . Now take $k[x_1, \dots, x_n, t]/\langle ft - 1 \rangle$, so $ft = 1$ in this ring. This kills the first term above, yielding

$$f^N = \sum f_i G_i(x_1, \dots, x_n, 1) \in k[x_1, \dots, x_n, t]/\langle ft - 1 \rangle.$$

Observation 1.1.5: There is an inclusion

$$k[x_1, \dots, x_n] \hookrightarrow k[x_1, \dots, x_n, t]/\langle ft - 1 \rangle.$$

Exercise 1.1.6: Why is this true?

Since this is injective, this identity also holds in $k[x_1, \dots, x_n]$. But $f_i \in J$, so $f \in \sqrt{I}$. ■

Example 1.1.7: Consider $k[x]$. If $J \subset k[x]$ is an ideal, it is principal, so $J = \langle f \rangle$. We can factor $f(x) = \prod_{i=1}^k (x - a_i)^{n_i}$ and $V(f) = \{a_1, \dots, a_k\}$. Then

$$I(V(f)) = \langle (x - a_1)(x - a_2) \cdots (x - a_k) \rangle = \sqrt{J} \subsetneq J,$$

so this loses information.

Example 1.1.8: Let $J = \langle x - a_1, \dots, x - a_n \rangle$, then $I(V(J)) = \sqrt{J} = J$ with J maximal. Thus there is a correspondence

$$\{\text{Points of } \mathbb{A}^n\} \iff \{\text{Maximal ideals of } k[x_1, \dots, x_n]\}.$$

Theorem 1.1.9 (Properties of I).

- a. $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$.
- b. $I(X_1) \cap I(X_2) = \sqrt{I(X_1) + I(X_2)}$.

Proof.

We proved (a) on the variety side.

For (b), by the Nullstellensatz, $X_i = V(I(X_i))$, so

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= IV(I(X_1) + I(X_2)) \\ &= \sqrt{I(X_1) + I(X_2)}. \end{aligned}$$

■

Example 1.1.10: Example of property (b):

Take $X_1 = V(y - x^2)$ and $X_2 = V(y)$, a parabola and the x -axis.

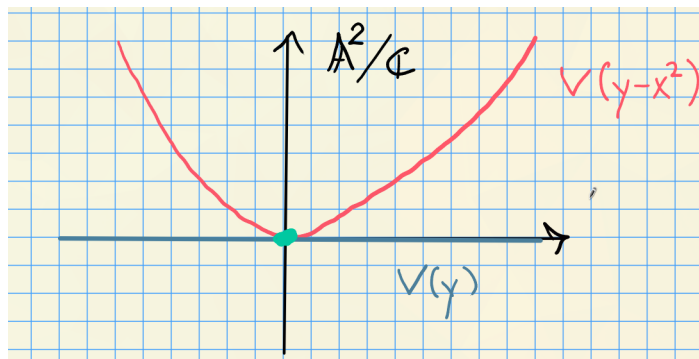


Figure 2: Intersecting $V(y - x^2)$ and $V(y)$

Then $X_1 \cap X_2 = \{(0,0)\}$, and $I(X_1) + I(X_2) = \langle y - x^2, y \rangle = \langle x^2, y \rangle$, but

$$I(X_1 \cap X_2) = \langle x, y \rangle = \sqrt{\langle x^2, y \rangle}$$

Proposition 1.1.11 (?).

If $f, g \in k[x_1, \dots, x_n]$, and suppose $f(x) = g(x)$ for all $x \in \mathbb{A}^n$. Then $f = g$.

Proof.

Since $f - g$ vanishes everywhere, $f - g \in I(\mathbb{A}^n) = I(V(0)) = \sqrt{0} = 0$.

■

More generally suppose $f(x) = g(x)$ for all $x \in X$, where X is some affine variety. Then by definition, $f - g \in I(X)$, so a “natural” space of functions on X is $k[x_1, \dots, x_n]/I(X)$.

Definition 1.1.12 (Coordinate Ring)

For an affine variety X , the *coordinate ring* of X is

$$A(X) := k[x_1, \dots, x_n]/I(X).$$

Elements $f \in A(X)$ are called *polynomial* or *regular* functions on X .

Observation 1.1.13: The constructions $V(\cdot), I(\cdot)$ work just as well for $A(X)$ and X .

Given any $S \subset A(Y)$ for Y an affine variety,

$$V(S) = V_Y(S) := \{x \in Y \mid f(x) = 0 \forall f \in S\}.$$

Given $X \subset Y$ a subset,

$$I(X) = I_Y(X) := \{f \in A(Y) \mid f(x) = 0 \forall x \in X\} \subseteq A(Y).$$

Example 1.1.14: For $X \subset Y \subset \mathbb{A}^n$, we have $I(X) \supset I(Y) \supset I(\mathbb{A}^n)$, so we have maps

$$A(\mathbb{A}^n) \xrightarrow{\cdot/I(Y)} A(Y) \xrightarrow{\cdot/I(X)} A(X)$$

$\nearrow \cdot/I(X)$

Theorem 1.1.15(?).

Let $X \subset Y$ be an affine subvariety, then

- a. $A(X) = A(Y)/I_Y(X)$
- b. There is a correspondence

$$\begin{aligned} \{\text{Affine subvarieties of } Y\} &\iff \{\text{Radical ideals in } A(Y)\} \\ X &\mapsto I_Y(X) \\ V_Y(J) &\leftarrow J. \end{aligned}$$

Proof.

Properties are inherited from the case of \mathbb{A}^n , see exercise in Gathmann. ■

Example 1.1.16: Let $Y = V(y - x^2) \subset \mathbb{A}^2/\mathbb{C}$ and $X = \{(1, 1)\} = V(x - 1, y - 1) \subset \mathbb{A}^2/\mathbb{C}$.

Then there is an inclusion $\langle y - x^2 \rangle \subset \langle x - 1, y - 1 \rangle$ (e.g. by Taylor expanding about the point $(1, 1)$), and there is a map

$$\begin{array}{ccccc}
 A(\mathbb{A}^n) & \longrightarrow & A(Y) & \longrightarrow & A(X) \\
 \parallel & & \parallel & & \parallel \\
 k[x, y] & \longrightarrow & k[x, y]/\langle y - x^2 \rangle & \longrightarrow & k[x, y]/\langle x - 1, y - 1 \rangle
 \end{array}$$

