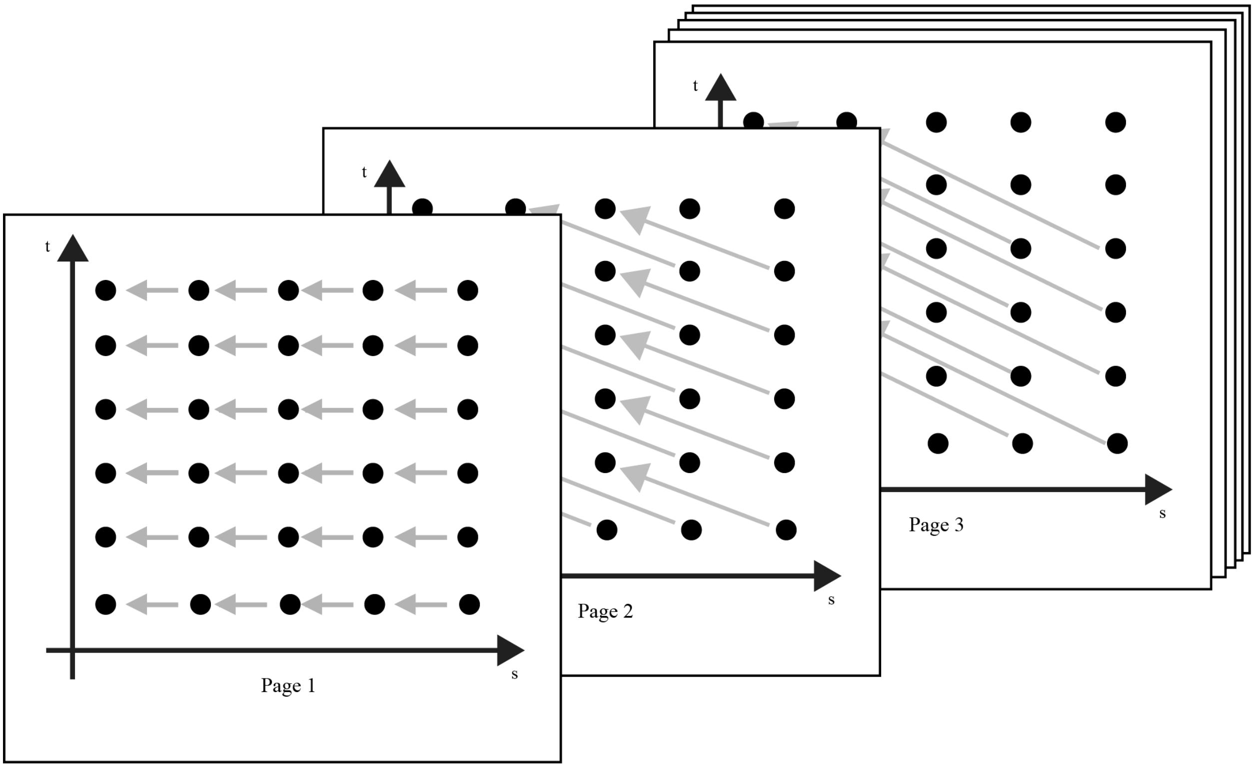

Notes: These are notes live-tex'd from a graduate course in Homological Algebra taught by Brian Boe at the University of Georgia in Spring 2021. As such, any errors or inaccuracies are almost certainly my own.

Homological Algebra

Lectures by Brian Boe. University of Georgia, Spring 2021

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1 Wednesday, January 13

Reference:

- The course text is Weibel [1].
- See the many corrections/errata: http://www.math.rutgers.edu/~weibel/Hbook-corrections.html
- Sections we'll cover:
  - 1.1-1.5,
  - 2.2-2.7,
  - 3.4,
  - 3.6,
  - 6.1,
  - 5.1-5.2,
  - 5.4-5.8,
  - 6.8,
  - 6.7,
  - 6.3,
  - 7.1-7.5,
  - 7.7-7.8,
  - Appendix A (when needed)
- Course Website: https://uga.view.usg.edu/d2l/le/content/2218619/viewContent/33763436/View

1.1 Overview

Definition 1.1.1 (Exact complexes)
A **complex** is given by

$$\cdots \xrightarrow{d_{i-1}} M_{i-1} \xrightarrow{d_i} M_i \xrightarrow{d_{i+1}} M_{i+1} \to \cdots.$$

where $M_i \in R$-mod and $d_i \circ d_{i-1} = 0$, which happens if and only if $\operatorname{im} d_{i-1} \subseteq \ker d_i$. If $\operatorname{im} d_{i-1} = \ker d_i$, this complex is **exact**.

Example 1.1.2 ***(?):*** We can apply a functor such as $\otimes_R N$ to get a new complex

$$\cdots \xrightarrow{d_{i-1}\otimes 1_N} M_{i-1}\otimes_R N \xrightarrow{d_i \otimes 1} M_i \otimes N \to M_{i+1} \xrightarrow{d_{i+1}\otimes 1} \cdots.$$

Example 1.1.3 ***(?):*** Applying $\mathrm{Hom}(N, \cdot)$ similarly yields

$$\mathrm{Hom}_R(N, M_i) \xrightarrow{d_{i-1}^*} \mathrm{Hom}_R(N, M_{i+1}),$$

where $d_i^* = d_i \circ (\cdot)$ is given by composition.

Example 1.1.4 ***(?):*** Applying $\mathrm{Hom}(\cdot, N)$ yields

$$\mathrm{Hom}_R(M_i, N) \xrightarrow{d_i^*} \mathrm{Hom}_R(M_{i+1}, N)$$

where $d_i^* = (\cdot) \circ d_i$.

Remark 1.1.5: Note that we can also take complexes with arrows in the other direction. For F a functor, we can rewrite these examples as

$$d_i^* \circ d_{i-1}^* = F(d_i) \circ F(d_{i-1}) = F(d_i \circ d_{i-1}) = F(0) = 0,$$

provided F is nice enough and sends zero to zero. This follows from the fact that functors preserve composition. Even if the original complex is exact, the new one may not be, so we can define the following:

Definition 1.1.6 (Cohomology)

$$H^i(M^*) = \ker d_i^* / \operatorname{im} d_{i-1}^*.$$

Remark 1.1.7: These will lead to i**th derived functors**, and category theory will be useful here. See appendix in Weibel. For a category $\mathcal{C}$ we'll define

- $\mathrm{Obj}(\mathcal{C})$ as the objects
- $\mathrm{Hom}_{\mathcal{C}}(A, B)$ a set of morphisms between them, where a more modern notation might be $\mathrm{Mor}(A, B)$.
- Morphisms compose: $A \xrightarrow{f} B \xrightarrow{g} C$ means that $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$
- Associativity
- Identity morphisms

See the appendix for diagrams defining zero objects and the zero map, which we'll need to make sense of exactness. We'll also needs notions of kernels and images, or potentially cokernels instead of images since they're closely related.

Remark 1.1.8: In the examples, we had $\ker d_i \subseteq M_i$, but this need not be true since the objects in the category may not be sets. Such an example is the category of complexes of R-modules: $\mathrm{Cx}(R\text{-mod})$. In this setting, kernels will be subcomplexes but not subsets.

Definition 1.1.9 (Functors)
Recall that **functors** are "functions" between categories $F : \mathcal{C} \to \mathcal{D}$ such that

- Objects are sent to objects,
- Morphisms are sent to morphisms, so $A \xrightarrow{f} B \rightsquigarrow F(A) \xrightarrow{F(f)} F(B)$,
- F respects composition and identities

Example 1.1.10 ***(Hom):*** $\mathrm{Hom}_R(N, \cdot) : R\text{-mod} \to \mathrm{Ab}$, noting that the hom set may not have an R-module structure.

Remark 1.1.11: Taking cohomology yields the ith derived functors of F, for example $\mathrm{Ext}^i, \mathrm{Tor}_i$. Recall that functors can be *covariant* or contravariant. See section 1 for formulating simplicial and singular homology (from topology) in this language.

1.2 Chapter 1: Chain Complexes

1.2.1 Complexes of R-modules

Definition 1.2.1 (Exactness)
Let R be a ring with 1 and define R-mod to be the category of *right* R-modules. $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if and only if $\ker g = \operatorname{im} f$, and in particular $g \circ f = 0$.

Definition 1.2.2 (Chain Complex)
A **chain complex** is

$$C_\cdot \coloneqq (C_\cdot, d_\cdot) \coloneqq \left(\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \right)$$

for $n \in \mathbb{Z}$ such that $d_n \circ d_{n+1} = 0$. We drop the n from the notation and write $d^2 \coloneqq d \circ d = 0$.

Definition 1.2.3 (Cycles and boundaries)

- $Z_n = Z_n(C_\cdot) = \ker d_n$ are referred to as n**-cycles**.
- $B_n = B_n(C_\cdot) = \operatorname{im} d_{n+1}$ are the n**-boundaries**.

Definition 1.2.4 (Homology of a chain complex)
Note that if $d^2 = 0$ then $B_n \leq Z_n \leq C_n$. In this case, it makes sense to define the quotient module $H^n(C_\cdot) \coloneqq Z_n/B_n$, the n**th homology** of $C_\cdot$.

Definition 1.2.5 (Maps of chain complexes)
A map $u : C_\cdot \to D_\cdot$ of chain complexes is a sequence of maps $u_n : C_n \to D_n$ such that all of the following squares commute:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & C_n \arrow[r] \arrow[d, "u_n"] & C_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Remark 1.2.6: We can thus define a category $\mathrm{Ch}(R\text{-mod})$ where

- The objects are chain complexes,
- The morphisms are chain maps.

Exercise 1.2.7 (Weibel 1.1.2)
A chain complex map $u : C. \to D.$ restricts to

$$u_n : Z_n(C.) \to Z_n(D.)$$
$$u_n : B_n(D.) \to B_n(D.)$$

and thus induces a well-defined map $u_{n,*} : H_n(C.) \to H_n(D.)$.

Remark 1.2.8: Each H_n thus becomes a functor $\mathrm{Ch}(R\text{-mod}) \to R\text{-mod}$ where $H_n(u) \coloneqq u_{*.n}$.

2 | Friday, January 15

2.1 Review

See assignment posted on ELC, due Wed Jan 27

Remark 2.1.1: Recall that a chain complex is $C.$ where $d^2 = 0$, and a map of chain complex is a ladder of commuting squares

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "u_{n-1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n+1} \arrow[r] \arrow[d, "u_{n+1}"] & \cdots \\
\cdots \arrow[r] & D_{n-1} \arrow[r, "d_{n-1}"] & D_n \arrow[r, "d_n"'] & D_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to diagram Recall that $u_n : Z_n(C) \to Z_n(D)$ and $u_n : B_n(C) \to B_n(D)$ preserves these submodules, so there are induced maps $u_{\cdot,n} : H_n(D) \to H_n(D)$ where $H_n(C) \coloneqq Z_n(C)/B_n n{-}1(C)$. Moreover, taking $H_n(\cdot)$ is a functor from $\mathrm{Ch}(R\text{-mod}) \to R\text{-mod}$ for any fixed n and on objects $C \mapsto H_n(C)$ and chain maps $u_n \to H_n(u) \coloneqq u_{*,n}$. Note the lower indices denote maps going down in degree.

2.2 Cohomology

Definition 2.2.1 (Quasi-isomorphism)
A chain map $u : C \to D$ is a **quasi-isomorphism** if and only if the induced map $u_{*,n} : H^n(C) \to H^n(D)$ is an isomorphism of R-modules.

Remark 2.2.2: Note that the usual notion of an isomorphism in the categorical sense might be too strong here.

Definition 2.2.3 (Cohomology)
A **cochain complex** is a complex of the form

$$\cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \cdots$$

where $d^n \circ d^{n-1} = 0$. We similarly write $Z^n(C) \coloneqq \ker d^n$ and $B^n(C) \coloneqq \operatorname{im} d^{n-1}$ and write the R-module $H^n(C) \coloneqq Z^n/B^n$ for the nth **cohomology** of C.

Remark 2.2.4: There is a way to go back and forth bw chain complexes and cochain complexes: set $C_n \coloneqq C^{-n}$ and $d_n \coloneqq d^{-n}$. This yields

$$C^{-n} \xrightarrow{d^{-n}} C^{-n+1} \iff C_n \xrightarrow{d^n} C_{n-1},$$

and the notions of $d^2 = 0$ coincide.

Definition 2.2.5 (Bounded complexes)
A cochain complex C is **bounded** if and only if there exists an $a \le b \in \mathbb{Z}$ such that $C_n \neq 0 \iff a \le n \le b$. Similarly C^m is bounded above if there is just a b, and **bounded below** for just an a. All of the same definitions are made for cochain complexes.

Remark 2.2.6: See the book for classical applications:

- 1.1.3: Simplicial homology
- 1.1.5: Singular homology

2.3 Operations on Chain Complexes

Remark 2.3.1: Write Ch for $\mathrm{Ch}(R\text{-mod})$, then if $f, g : C \to D$ are chain maps then $f + g : C \to D$ can be defined as $(f+g)(x) = f(x) + g(x)$, since D has an addition coming from its R-module structure. Thus the hom sets $\mathrm{Hom}_{\mathrm{Ch}}(C, D)$ becomes an abelian group. There is a distinguished **zero object**[1] 0, defined as the chain complex with all zero objects and all zero maps. Note that we also have a zero map given by the composition $(C \to 0) \circ (0 \to D)$.

Definition 2.3.2 (Products and Coproducts)
If $\{A_\alpha\}$ is a family of complexes, we can form two new complexes:

- The **product** $\left(\prod_\alpha A_\alpha\right)_n \coloneqq \prod_\alpha A_{\alpha,n}$ with the differential

$$\left(\prod d_\alpha\right)_n : \prod A_{\alpha,n} \xrightarrow{d_{\alpha,n}} \prod A_{\alpha,n-1}.$$

- The **coproduct** $\left(\coprod_\alpha A_\alpha\right)_n \coloneqq \bigoplus_\alpha A_{\alpha,n}$, i.e. there are only finitely many nonzero entries, with exactly the same definition as above for the differential.

Remark 2.3.3: Note that if the index set is finite, these notions coincide. By convention, finite direct products are written as direct sums.

These structures make Ch into an **additive category**. See appendix for definition: the homs are abelian groups where composition distributes over addition, existence of a zero object, and existence of finite products. Note that here we have arbitrary products.

Definition 2.3.4 (Subcomplexes)
We say B is a **subcomplex** of C if and only if

- $B_n \leq C_n \in R\text{-mod}$ for all n,
- The differentials of B_n are the restrictions of the differentials of C_n.

Remark 2.3.5: This can be alternatively stated as saying the inclusion $i : B \to C$ given by $i_n : B_n \to C_n$ is a morphism of chain complexes. Recall that some squares need to commute, and this forces the condition on restrictions.

Definition 2.3.6 (Quotient Complexes)
When $B \leq C$, we can form the quotient complex C/B where

$$C_n/B_n \xrightarrow{d_n} C_{n-1}/B_{n-1}.$$

Moreover there is a natural projection $\pi : C \to C/B$ which is a chain map.

[1] See appendix A 1.6 for initial and terminal objects. Note that $\varnothing$ is an initial but non-terminal object in Set, whereas zero objects are both.

Remark 2.3.7: Suppose $f : B \to C$ is a chain map, then there exist induced maps on the levelwise kernels and cokernels, so we can form the **kernel** and **cokernel** complex:

$$\begin{tikzcd}
\cdots \arrow[r] & \ker f_n \arrow[r, dashed, "\exists d_n"] \arrow[d, "i_n"] & \ker f_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n \arrow[r, "d_n"] \arrow[d, "\pi_n"] & C_{n-1} \arrow[r] \arrow[d, "\pi_{n-1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{coker} f_n \arrow[r, dashed, "\exists d_n"] & \operatorname{coker} f_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Link to Diagram

Here $\ker f \leq B$ is a subcomplex, and $\operatorname{coker} f$ is a quotient complex of C. The chain map $i : \ker f \to B$ is a categorical kernel of f in Ch, and π is similarly a cokernel. See appendix A 1.6. These constructions make Ch into an **abelian category**: roughly an additive category where every morphism has a kernel and a cokernel.

3 | 1.2 (Wednesday, January 20)

3.1 Taking Chain Complexes of Chain Complexes

See phone pic for missed first 10m

3.1.1 Double Complexes

Remark 3.1.1: Consider a double complex:

$$\begin{tikzcd}
 & & C_{p,\cdot} & \\
 & C_{p-1,q+1} & C_{p,q+1} \arrow[d] & C_{p-1,q+1} \\
C_{\cdot,q} & C_{p-1,q} & C_{p,q} \arrow[l, "d^h_{p,q}"'] \arrow[d, "d^v_{p,q}"'] & C_{p+1,q} \\
 & C_{p-1,q+1} & C_{p-1,q+1} & C_{p-1,q+1}
\end{tikzcd}$$

Link to Diagram

All of the individual rows and columns are chain complexes, where $(d^h)^2 = 0$ and $(d^v)^2 = 0$, and the square anticommute: $d^v d^h + d^h d^v - 0$, so $d^v d^h = -d^h d^v$. This is almost a chain complex of chain complexes, i.e. an element of $\mathrm{Ch}(\mathrm{Ch}\, R\text{-mod})$). It's useful here to consider lines parallel to the line $y = x$.

> **Definition 3.1.2** (Bounded Complexes)
> A double complex $C_{\cdot,\cdot}$ is **bounded** if and only if there are only finitely many nonzero terms along each constant diagonal $p + q = n$.

Example 3.1.3 ***(?):*** A *first quadrant* double complex $\{C_{p,q}\}_{p,q\geq 0}$ is bounded: note that this can still have infinitely many terms, but each diagonal is finite because each will hit a coordinate axis.

Remark 3.1.4 ***(The sign trick):*** The squares anticommute, since the d^v are not chain maps between the horizontal chain complexes. This can be fixed by changing every one out of four signs, defining

$$\begin{aligned} f_{*,q} &: C_{*,q} \to C_{*,q-1} \\ f_{p,q} \coloneqq (-1)^p d^v_{p,q} &: C_{p,q} \to C_{p,q-1}. \end{aligned}$$

This yields a new double complex where the signs of each column alternate:

$$\begin{tikzcd}
C_{0,q} \arrow[d, "d^v"] & C_{1,q} \arrow[l, "d^h"] \arrow[d, "-d^v"] & C_{2,q} \arrow[l, "d^h"] \arrow[d, "d^v"] \\
C_{0,q-1} & C_{1,q-1} \arrow[l, "d^h"] & C_{2,q-1} \arrow[l, "d^h"]
\end{tikzcd}$$

Now the squares commute and $f_{\cdot,q}$ are chain maps, so this object is an element of $\mathrm{Ch}(\mathrm{Ch}\, R\text{-mod})$.

3.1.2 Total Complexes

Recall that products and coproducts of R-modules coincide when the indexing set is finite.

Definition 3.1.5 (Total Complexes)
Given a double complex $C_{\cdot,\cdot}$, there are two ordinary chain complexes associated to it referred to as **total complexes**:

$$(\mathrm{Tot}^{\Pi} C)_n \coloneqq \prod_{p+q=n} C_{p,q} (\mathrm{Tot}^{\oplus} C)_n \coloneqq \bigoplus_{p+q=n} C_{p,q}.$$

Writing $\mathrm{Tot}(C)$ usually refers to the former. The differentials are given by

$$d_{p,q} = d^h + d^v : C_{p,q} \to C_{p-1,q} \oplus C_{p,q-1},$$

where $C_{p,q} \subseteq \mathrm{Tot}^{\oplus}(C)_n$ and $C_{p-1,q} \oplus C_{p,q-1} \subseteq \mathrm{Tot}^{\oplus}(C)_{n-1}$. Then you extend this to a differential on the entire diagonal by defining $d = \bigoplus d_{p,q}$.

Exercise 3.1.6 (?)
Check that $d^2 = 0$, using $d^v d^h + d^h d^v = 0$.

Remark 3.1.7: Some notes:

- $\mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\Pi}(C)$ when C is bounded.
- The total complexes need not exist if C is unbounded: one needs infinite direct products and infinite coproducts to exist in $\mathcal{C}$. A category admitting these is called **complete** or **cocomplete**.[2]

3.1.3 More Operations

[2] Recall that abelian categories are additive and only require *finite* products/coproducts. A counterexample: categories of *finite* abelian groups, where e.g. you can't take infinite sums and stay within the category.

Definition 3.1.8 (Truncation below)
Fix $n \in \mathbb{Z}$, and define the n**th truncation** $\tau_{\geq n}(C)$ by

$$\tau_{\geq n}(C) = \begin{cases} 0 & i < n \\ Z_n & i = n \\ C_i & i > n. \end{cases}.$$

Pictorially:

$$\cdots \longleftarrow 0 \xleftarrow{d_n} Z_n \xleftarrow{d_{n+1}} C_{n+1} \xleftarrow{d_{n+2}} C_{n+2} \longleftarrow \cdots$$

Link to diagram

This is sometimes call the **good truncation of** C **below** n.

Remark 3.1.9: Note that

$$H_i(\tau_{\geq n} C) = \begin{cases} 0 & i < n \\ H_i(C) & i \geq n. \end{cases}.$$

Definition 3.1.10 (Truncation above)
We define the quotient complex

$$\tau_{<n} C \coloneqq C/\tau_{\geq n} C.$$

which is C_i below n, C_n/Z_n at n. Thus is has homology

$$\begin{cases} H_i(C) & i < n. \\ 0 & i \geq n \end{cases}.$$

Definition 3.1.11 (Translation)
If C is a chain complex and $p \in \mathbb{Z}$, define a new complex $C[p]$ by

$$C[p]_n \coloneqq C_{n+p}.$$

$$\begin{tikzcd}
\text{Degrees} & -p & & 0 & & p \\
C & C_{-p} & \cdots & C_0 \arrow[dll, dashed] & \cdots & C_p \arrow[dll, dashed] \\
C[p] & C_0 & \cdots & C_p & \cdots & C_{2p}
\end{tikzcd}$$

Link to Diagram

Similarly, if C is a *cochain* complex, we set $C[p]^n \coloneqq C^{n-p}$:

$$\begin{tikzcd}
\text{Degrees} & -p & & 0 & & p \\
C & C^{-p} \arrow[r] \arrow[drr, dashed] & \cdots \arrow[r] & C^0 \arrow[r] \arrow[drr, dashed] & \cdots \arrow[r] & C^p \\
C[p] & C^0 \arrow[r] & \cdots \arrow[r] & C^{-p} \arrow[r] & \cdots \arrow[r] & C^0
\end{tikzcd}$$

Link to Diagram

Mnemonic: Shift p positions in the same direction as the arrows.

In both cases, the differentials are given by the shifted differential $d[p] \coloneqq (-1)^p d$. Note that these are not alternating: p is the fixed translation, so this is a constant that changes the signs of all differentials. Thus $H_n(C[p]) = H_{n+p}(C)$ and $H^n(C[p]) = H^{n-p}$.

Exercise 3.1.12
Check that if $C^n \coloneqq C_{-n}$, then $C[p]^n = C[p]_{-n}$.

Remark 3.1.13: We can make translation into a functor $[p] : \mathrm{Ch} \to \mathrm{Ch}$: given $f : C \to D$, define $f[p] : C[p] \to D[p]$ by $f[p]_n \coloneqq f_{n+p}$, and a similar definition for cochain complexes changing p to $-p$.

4 | Lecture 4 (Friday, January 22)

4.1 Long Exact Sequences

Remark 4.1.1: Some terminology: in an abelian category $\mathcal{A}$ an example of an **exact complex** in $\mathrm{Ch}(\mathcal{A})$ is

$$\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \cdots.$$

where *exactness* means ker = im at each position, i.e. $\ker f = 0, \operatorname{im} f = \ker g, \operatorname{im} g = C$. We say f is monic and g epic.

As a special case, if $0 \to A \to 0$ is exact then A must be zero, since the image of the incoming map must be 0. This also happens when every other term is zero. If $0 \to A \xrightarrow{f} B \to 0$, then $A \cong B$ since f is both injective and surjective (say for R-modules).

Theorem 4.1.2 ***(Long Exact Sequences).***
Suppose $0 \to A \to B \to C \to 0$ is a SES in $\mathrm{Ch}(\mathcal{A})$ (note: this is a sequence of *complexes*), then there are natural maps

$$\delta : H_n(C) \to H_{n-1}(A)$$

called **connecting morphisms** which decrease degree such that the following sequence is exact:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
 & H_n(A) \arrow[r, "f_* = H_n(f)"] & H_n(B) \arrow[r, "g_* = H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
 & H_{n-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Link to Diagram

This is referred to as the **long exact sequence in homology**. Similarly, replacing chain complexes by cochain complexes yields a similar connecting morphism that increases degree.

Note on notation: some books use ∂ for homology and δ for cohomology.

The proof that this sequence exists is a consequence of the *snake lemma*.

Lemma 4.1.3 ***(The Snake Lemma).***
The sequence highlighted in red in the following diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] & \ker(\alpha) \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & \ker(\gamma) \arrow[d] \arrow[ddddll, dotted, "\exists\delta"] & & \\
 & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \\
 & 0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 & \\
 & & \mathrm{coker}(\alpha) \arrow[r] & \mathrm{coker}(\beta) \arrow[r] & \mathrm{coker}(\gamma) \arrow[r] & \mathrm{coker}(g') \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proof (of the Snake Lemma: Existence).

- Start with $c \in \ker(\gamma) \leq C$, so $\gamma(c) = 0 \in C'$
- **Choose** $b \in B$ by surjectivity
  - We'll show it's independent of this choice.
- Then $b' \in B'$ goes to $0 \in C'$, so $b' \in \ker(B' \to C')$
- By exactness, $b' \in \ker(B' \to C') = \operatorname{im}(A' \to B')$, and now produce a unique $a' \in A'$ by injectivity
- Take the image $[a'] \in \operatorname{coker} \alpha$
- Define $\partial(c) \coloneqq [a']$.

■

Proof (of the Snake Lemma: Uniqueness).

- We chose b, suppose we chose a different $\tilde{b}$.
- Then $\tilde{b} - b \mapsto c - c = 0$, so the difference is in $\ker g = \operatorname{im} f$.
- Produce an $\tilde{a} \in A$ such that $\tilde{a} \mapsto \tilde{b} - b$
- Then $\bar{a} \coloneqq \alpha(\tilde{a})$, so apply f'.
- Define $\beta(\tilde{b}) = \tilde{b}' \in B$.
- Commutativity of the LHS square forces $\tilde{a}' \mapsto \tilde{b}' - b'$.
- Then $\bar{a} + a' \mapsto \tilde{b}' - b' + b' = \tilde{b}'$.
- So $\tilde{a}' + a'$ is the desired pullback of $\tilde{b}'$
- Then take $[\tilde{a}'] \in \operatorname{coker} \alpha$; are $a', \tilde{a}'$ in the same equivalence class?
- Use that fact that $\tilde{a} = a' + \bar{a}$, where $\bar{a} \in \operatorname{im} \alpha$, so $[\tilde{a}] = [a' + \bar{a}] = [a'] \in \operatorname{coker} \alpha \coloneqq A' / \operatorname{im} \alpha$.

■

A few changes in the middle, redo!

Proof (of the Snake Lemma: Exactness).

- Let's show $g : \ker \beta \to \ker \gamma$.
  - Let $b \in \ker \beta$, then consider $\gamma(g(\beta)) = g'(\beta(b)) = g'(0) = 0$ and so $g(b) \in \ker \gamma$.
- Now we'll show $\operatorname{im}(g|_{\ker \beta}) \subseteq \ker \delta$
  - Let $b \in \ker \beta, c = g(b)$, then how is $\delta(c)$ defined?
  - Use this b, then apply β to get $b' = \beta(b) = 0$ since $b \in \ker \beta$.
  - So the unique thing mapping to it a' is zero, and thus $[a'] = 0 = \delta(c)$.
- $\ker \delta \subseteq \operatorname{im}(g|_{\ker \beta})$
  - Let $c \in \ker \delta$, then $\delta(c) = 0 = [a'] \in \operatorname{coker} \alpha$ which implies that $a' \in \operatorname{im} \alpha$.
  - Write $a' = \alpha(a)$, then $\beta(b) = b' = f'(a') = f'(\alpha(a))$ by going one way around the LHS square, and is equal to $\beta(f(a))$ going the other way.
  - So $\tilde{b} \coloneqq b - f(a) \in \ker \beta$, since $\beta(b) = \beta(f(a))$ implies their difference is zero.

– Then $g(\tilde{b}) = g(b) - g(f(a)) = g(b) = c$, which puts $c \in g(\ker \beta)$ as desired.

■

Exercise 4.1.4 (?)
Show exactness at the remaining places – the most interesting place is at $\operatorname{coker} \alpha$. Also check that all of these maps make sense.

Remark 4.1.5: We assumed that $\mathcal{A} = R$-mod here, so we could chase elements, but this happens to also be true in any abelian category $\mathcal{A}$ but by a different proof. The idea is to embed $\mathcal{A} \to R$-mod for some ring R, do the construction there, and pull the results back – but this doesn't quite work! $\mathcal{A}$ can be too big. Instead, do this for the smallest subcategory $\mathcal{A}_0$ containing all of the modules and maps involved in the snake lemma. Then $\mathcal{A}_0$ is small enough to embed into R-mod by the **Freyd-Mitchell Embedding Theorem**.

5 | Lecture 5 (Monday, January 25)

5.1 LES Associated to a SES

Theorem 5.1.1 *(?).*
For every SES of chain complexes, there is a long exact sequence in homology.

Proof (?).
Suppose we have a SES of chain complexes

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

which means that for every n there is a SES of R-modules. Recall the diagram for the snake lemma, involving kernels across the top and cokernels across the bottom. Applying the snake lemma, by hypothesis $\operatorname{coker} g = 0$ and $\ker f = 0$. There is a SES

$$A_n/dA_{n+1} \to B_n/dB_{n+1} \to C_n/dC_{n+1} \to 0$$

Using the fact that $B_n \subseteq Z_n$, we can use the 1st and 2nd isomorphism theorems to produce

$$
\begin{tikzcd}
 & H_n(A) \arrow[r, "f_*"] \arrow[d] & H_n(B) \arrow[r, "g_*"] \arrow[d] & H_n(C) \arrow[d] \arrow[dddll, dotted, bend left, "\exists\delta"] & \\
 & A_n/dA_{n+1} \arrow[r, "f"] \arrow[d, "d_n"] & B/dB_{n+1} \arrow[r, "g"] \arrow[d, "d_n"] & C/dC_{n+1} \arrow[r] \arrow[d, "d_n"] & {} \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r] \arrow[d] & Z_{n-1}(B) \arrow[r] \arrow[d] & Z_{n-1}(C) \arrow[d] & \\
 & \operatorname{coker} d_n = Z_{n-1}(A)/dA_n = H_{n-1}(A) \arrow[r, "f_*"'] & H_{n-1}(B) \arrow[r, "g_*"'] & H_{n-1}(C) &
\end{tikzcd}
$$

Link to diagram

This yields an exact sequence relating H_n to H_{n-1}, and these can all be spliced together.

- $\ker(A_n/dA_{n-1} \to Z_{n-1}(A) = Z_n(A)/dA_{n+1} \coloneqq H_n(A)$ using the 2nd isomorphism theorem

■

Remark 5.1.2: Note that d is *natural*, which means the following: there is a category $\mathcal{S}$ whose objects are SESs of chain complexes and whose maps are chain maps:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

There is another full subcategory $\mathcal{L}$ of Ch whose objects are LESs of objects in the original abelian category, i.e. exact chain complexes. The claim is that the LES construction in the theorem defines a functor $\mathcal{S} \to \mathcal{L}$. We've seen how this maps objects, so what is the map on morphisms? Given a morphism as in the above diagram, there is an induced morphism:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "H_n(u_A)"] & H_n(B) \arrow[r] \arrow[d, "H_n(u_B)"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "H_n(u_C)"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_{n-1}(u_A)"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r, "\partial"] & H_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}
$$

The first two squares commute, and *naturality* means that the third square commutes as well.

> **Exercise 5.1.3** (?)
> Check the details!

Remark 5.1.4: It is sometimes useful to explicitly know how to compute snake lemma boundary elements. See the book for a recipe for computing $\partial(\xi)$:

- Lift ξ to a cycle $c \in Z_n(C) \subseteq C_n$.
- Pull c back to a preimage $b \in B_n$ by surjectivity.
- Apply the differential to get $d(b) \in Z_{n-1}(B)$, using that images are contained in kernels.
- Since this is in kernel of the outgoing map, it's in the kernel of the incoming map and thus there exists an $a \in Z_{n-1}(A)$ such that $f(a) = db$
- So set $\delta(\xi) \coloneqq [a] \in H_{n-1}(A)$.

Remark 5.1.5: Why is naturality useful? Suppose $H_n(B) = 0$, you get isomorphisms, and this allows inductive arguments up the LES. The LES in homology is sometimes abbreviated as an **exact triangle**:

$$\begin{tikzcd}
& H_*(A) \arrow[dr, "f"] & \\
H_*(C) \arrow[ur, squiggly, "\partial"] & & H_*(B) \arrow[ll, "g"]
\end{tikzcd}$$

Here $\partial : H_*(C) \to H_*(A)[1]$ shifts degrees. Note that this motivates the idea of **triangulated categories**, which is important in modern research. See Weibel Ch.10, and exercise 1.4.5 for how to construct these as quotients of Ch.

5.2 1.4: Chain Homotopies

Remark 5.2.1: Assume for now that we're in the situation of R-modules where R is a field, i.e. vector spaces. The main fact/advantage here that is not generally true for R-modules: every subspace has a complement. Since $B_n \subseteq Z_n \subseteq C_n$, we can write $C_n = Z_n \oplus B_n'$ for every n, and $Z_n = B_n \oplus H_n$. This notation is suggestive, since $H_n \cong Z_n/B_n$ as a quotient of vector spaces. Substituting, we get $C_n = B_n \oplus H_n \oplus B_n'$. Consider the projection $C_n \to B_n$ by projecting onto the first factor. Identifying $B_n \coloneqq \operatorname{im}(C_{n+1} \to C_n) \cong C_{n+1}/Z_{n+1}$ by the 1st isomorphism theorem in the reverse direction. But this image is equal to B_{n+1}', and we can embed this in C_{n+1}, so define $s_n : C_n \to C_{n+1}$ as the composition

$$s_n \coloneqq (C_n \xrightarrow{\text{Proj}} B_n = \operatorname{im}(C_{n+1} \to C_n) \xrightarrow{d_{n+1}^{-1}} C_{n+1}/Z_{n+1} \xrightarrow{\cong} B_{n+1}' \hookrightarrow C_{n+1}.$$

Claim 1: $d_{n+1} s_n d_{n+1} = d_{n+1}$ are equal as maps.

Proof (?).

- Check on the first factor $B'_{n+1} \subseteq C_{n+1}$ directly to get $s_n d_{n+1}(x) = d_{n+1}(x)$ for $x \in B'_{n+1}$, and then applying d_{n+1} to both sides is the desired equality.
- On the second factor Z_{n+1}, both sides give zero since this is exactly the kernel.

■

Claim 2: $d_{n+1}s_n + s_{n-1}d_n = \mathrm{id}_{C_n}$ if and only if $H_n = 0$, i.e. the complex C is exact at C_n. This map is the sum of taking the two triangle paths in this diagram:

$$\begin{tikzcd}
 & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d_{n+1}"] & C_n &
\end{tikzcd}$$

Proof (?).
We again check this on both factors:

- Using the first claim, $s_n = 0$ on B'_n and thus $s_{n-1}d_n = \mathrm{id}_{B'_n}$.
- On H_n, $s_n = 0$ and $d_n = 0$, and so the LHS is $0 = \mathrm{id}_{H_n}$ *if and only if* $H_n = 0$.
- On B_n, and tracing through the definition of s_n yields $d_{n+1}s_n(x) = x$ and this yields id_{B_n}.

■

Next time: summary of decompositions, start general section on chain homotopies.

6 | Wednesday, January 27

See phone pic for missed first 10m.

6.1 1.4: Chain Homotopies

Definition 6.1.1 (Split Exact)
A complex is called **split** if there are maps $s_n : C_n \to C_{n+1}$ such that $d = dsd$. In this case, the maps s_n are referred to as the **splitting maps**, and if C is additionally acyclic, we say C is **split exact**.

Remark 6.1.2: Note that when C is split exact, we have

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
C_{n+1} \arrow[r, "d"] & C_n &
\end{tikzcd}$$

Link to Diagram

Example 6.1.3 ***(Not all complexes split):*** Take

$$C = \left(0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{d} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0\right).$$

Then $\operatorname{im} d = \{0, 2\} = \ker d$, but this does not split since $\mathbb{Z}/2\mathbb{Z}^2 \ncong \mathbb{Z}/4\mathbb{Z}$: one has an element of order 4 in the underlying additive group. Equivalently, there is no complement to the image. What might be familiar from algebra is $ds = \mathrm{id}$, but the more general notion is $dsd = d$.

Example 6.1.4 ***(?):*** The following complex is not split exact for the same reason:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z} \to \cdots.$$

Question 6.1.5
Given $f, g : C \to D$, when do we get equality $f_* = g_* : H_*(C) \to H_*(D)$?

Definition 6.1.6 (Homotopy Terminology for Chains)
A chain map $f : C \to D$ is **nullhomotopic** if and only if there exist maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$:

$$\begin{tikzcd}
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n &
\end{tikzcd}$$

Link to Diagram

The map s is called a **chain contraction**. Two maps are **chain homotopic** (or initially: f is chain homotopic to g, since we don't yet know if this relation is symmetric) if and only if $f - g$ is nullhomotopic, i.e. $f - g = ds + sd$. The map s is called a **chain homotopy** from f to g. A map f is a **chain homotopy equivalence** if both fg and gf are chain homotopic to the identities on C and D respectively.

Lemma 6.1.7 *(?).*
If map $f : C \to D$ is nullhomotopic then $f_* : H_*(C) \to H_*(D)$ is the zero map. Thus if f, g are chain homotopic, then they induce equal maps.

Proof (?).
An element in the quotient $H_n(C)$ is represented by an n-cycle $x \in Z_n(C)$. By a previous exercise, $f(x)$ is a well-defined element of $H_n(D)$, and using that $d(x) = 0$ we have

$$f(x) = (ds + sd)(x) = d(s(x)),$$

and so $f[x] = [f(x)] = [0]$.

$$\begin{tikzcd}
 & x & d(x) = 0 \\
 & C_n \arrow[r, "d"] \arrow[dl, "s_n"'] \arrow[d, "\mathrm{id}"] & C_{n-1} \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r, "d"] & D_n & \\
 & d(s(x)) &
\end{tikzcd}$$

Link to Diagram

Now applying the first part to $f - g$ to get the second part. ∎

See Weibel for topological motivations.

6.2 1.5 Mapping Cones

Remark 6.2.1: Note that we'll skip *mapping cylinders*, since they don't come up until the section on triangulated categories. The goal is to see how any two maps between homologies can be fit into a LES. This helps reduce questions about *quasi-isomorphisms* to questions about split exact complexes.

Definition 6.2.2 (Mapping Cones)
Suppose we have a chain map $f : B \to C$, then there is a chain complex $\operatorname{cone}(f)$, the **mapping cone of** f, defined by

$$\operatorname{cone}(f)_n = B_{n-1} \oplus C_n.$$

The maps are given by the following:

$$\begin{tikzcd} B_{n-1} \arrow[r, "-d^B"] \arrow[ddr, "-f"] & B_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "d^C"] & C_{n-1} \end{tikzcd}$$

Link to Diagram

We can write this down: $d(b, c) = (-d(b), -f(b) + d(c))$, or as a matrix

$$\begin{bmatrix} -d^b & 0 \\ -f & d^C \end{bmatrix}.$$

Exercise 6.2.3 (?)
Check that the differential on $\operatorname{cone}(f)$ squares to zero.

Exercise 6.2.4 (Weibel 1.5.1)
When $f = \mathrm{id} : C \to C$, we write $\operatorname{cone}(C)$ instead of $\operatorname{cone}(\mathrm{id})$. Show that $\operatorname{cone}(C)$ is split exact, with splitting map $s(b, c) = (-c, 0)$ for $b \in C_{n-1}, c \in C_n$.

Proposition 6.2.5 *(?).*
Suppose $f : B \to C$ is a chain map, then the induced maps $f_* : H(B) \to H(C)$ fit into a LES. There is a SES of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & B[-1] & \longrightarrow & 0 \\ & & c & \longrightarrow & (0,c) & & & & \\ & & & & (b,c) & \longrightarrow & -b & & \end{array}$$

Link to Diagram

Exercise 6.2.6 *(?)*
Check that these are chain maps, i.e. they commute with the respective differentials d.

The corresponding LES is given by the following:

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}\,\mathrm{cone}(f) \arrow[r, "\delta_*"] & H_{n+1}(B[-1]) = H_n(B) \arrow[dll, "\partial"] \\ H_n(C) \arrow[r] & H_n\,\mathrm{cone}(f) \arrow[r] & H_n(B[-1]) = H_{n-1}(B) \arrow[dll] \\ \cdots & & \end{tikzcd}$$

Link to Diagram

Lemma 6.2.7 *(?)*.
The map $\partial = f_*$

Proof (?).
Letting $b \in B_n$ is an n-cycle.

1. Lift b to anything via δ, say $(-b, 0)$.
2. Apply the differential d to get $(db, fb) = (0, fb)$ since b was a cycle.
3. Pull back to C_n by the map $C \to \mathrm{cone}(f)$ to get fb.
4. Then the connecting morphism is given by $\partial[b] = [fb]$. But by definition of f_*, we have $[fb] = f_*[b]$.

■

7 Friday, January 29

7.1 Mapping Cones

Remark 7.1.1: Given $f : B \to C$ we defined $\operatorname{cone}(f)_n \coloneqq B_{n-1} \oplus C_n$, which fits into a SES

$$0 \to C \to \operatorname{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

and thus yields a LES in cohomology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(\operatorname{cone}(f)) \arrow[r] & H_n(B) \arrow[dll, "\delta = f_*"] \\
H_n(C) \arrow[r] & H_n(\operatorname{cone}(f)) \arrow[r] & H_{n-1}(B) \arrow[dll, "\delta"] \\
\cdots & &
\end{tikzcd}$$

Link to Diagram

Corollary 7.1.2 *(?).*
$f : B \to C$ is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is exact.

Proof (?).
In the LES, all of the maps f_* are isomorphisms, which forces $H_n(\operatorname{cone}(f)) = 0$ for all n. ∎

Remark 7.1.3: So we can convert statements about quasi-isomorphisms of complexes into exactness of a single complex.

We'll skip the rest, e.g. mapping cylinders which aren't used until the section on triangulated categories. We'll also skip the section on δ-functors, which is a slightly abstract language.

7.2 Ch. 2: Derived Functors

Remark 7.2.1: Setup: fix $M \in R$-mod, where R is a ring with unit. Note that by an upcoming exercise, $\mathrm{Hom}_R(M, \cdot) : \text{mod-}R \to \mathrm{Ab}$ is a *left-exact* functor, but not in general right-exact: given a SES $0 \to A \to B \to C \to 0 \in \mathrm{Ch}(\text{mod-}R)$, there is an exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_* = f \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_* = g \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \end{tikzcd}$$

Link to Diagram

However, this is not generally surjective: not every $M \to C$ is given by composition with a morphism $M \to B$ (*lifting*). To create a LES here, one could use the cokernel construction, but we'd like to do this functorially by defining a sequence functors F^n that extend this on on the right to form a LES:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_* = f \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_* = g \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \arrow[dll, out=0, in=180] \\ & F^1(A) \arrow[r] & F^1(B) \arrow[r] & F^1(C) \arrow[dll, out=0, in=180] \\ & F^2(A) \arrow[r] & \cdots & \end{tikzcd}$$

Link to Diagram

It turns out such functors exist and are denoted $F^n(\,\cdot\,) \coloneqq \mathrm{Ext}_R^n(M, \cdot)$:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(M, A) \arrow[r, "f_* = f \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \arrow[r, "g_* = g \circ (\,\cdot\,)"] & \mathrm{Hom}_R(M, A) \arrow[dll, out=0, in=180] \\ & \mathrm{Ext}_R^1(A) \arrow[r] & \mathrm{Ext}_R^1(B) \arrow[r] & \mathrm{Ext}_R^1(C) \arrow[dll, out=0, in=180] \\ & \mathrm{Ext}_R^2(A) \arrow[r] & \cdots & \end{tikzcd}$$

Link to Diagram

By convention, we set $\mathrm{Ext}^0_R(\cdot) \coloneqq \mathrm{Hom}_R(M, \cdot)$. This is an example of a general construction: **right-derived functors** of $\mathrm{Hom}_R(M, \cdot)$. More generally, if $\mathcal{A}$ is an abelian category (with a certain additional property) and $F : \mathcal{A} \to \mathcal{B}$ is a left-exact functor (where $\mathcal{B}$ is another abelian category) then we can define right-derived functors $R^n F : \mathcal{A} \to \mathcal{B}$. These send SESs in $\mathcal{A}$ to LESs in $\mathcal{B}$:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & FA \arrow[r] & FB \arrow[r] & FC \arrow[dll, out=0, in=180] & \\
& R^1FA \arrow[r] & R^1FB \arrow[r] & R^1FC \arrow[dll, out=0, in=180] & \\
& \cdots & & &
\end{tikzcd}
$$

Link to Diagram

Similarly, if F is *right-exact* instead, there are left-derived functors $L^n F$ which form a LES ending with 0 at the right:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
& & & \cdots \arrow[dll, out=0, in=180] & \\
& LFA \arrow[r] & LFB \arrow[r] & LFC \arrow[dll, out=0, in=180] & \\
& FA \arrow[r] & FB \arrow[r] & FC \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

7.3 2.2: Projective Resolutions

Definition 7.3.1 (Projective Modules)
Let $\mathcal{A} = R$-mod, then $P \in R$-mod satisfies the following universal property:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Link to Diagram

Remark 7.3.2: Free modules are projective. Let $F = R^X$ be the free module on the set X. Then consider $\gamma(x) \in C$, by surjectivity these can be pulled back to some elements in B:

$$\begin{tikzcd} & X \arrow[dl, dotted, "\exists\tilde{\beta}"'] \arrow[d, hook, "\iota_X"] & \\ & F \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \\ \exists b \in g^{-1}(\gamma(x)) \coloneqq \beta(x) & \gamma(x) & \end{tikzcd}$$

Link to Diagram

This follows from the universal property of free modules:

$$\begin{tikzcd} & \exists F(X) \arrow[d, dashed, "\exists ! f' \in \mathrm{Hom}_R(F(X), X)"] & \\ X \arrow[ur, "\exists g \in \mathrm{Hom}_{\mathrm{Set}}(X, F(X))"] \arrow[r, "f \in \mathrm{Hom}_{\mathrm{Set}}(X, M)"] & M & \in R\text{-mod} \end{tikzcd}$$

Link to Diagram

Proposition 7.3.3 *(?)*.
An R-module is projective if and only if it is a direct summand of a free module.

Exercise 7.3.4 (?)
Prove the $\Longleftarrow$ direction!

Proof (?).
$\Longrightarrow$: Assume P is projective, and let $F(P)$ be the free R-module on the underlying set of P. We can start with this diagram:

$$
\begin{tikzcd}
 & F(P) \arrow[d, dashed] \\
P \arrow[ur] \arrow[r, "\mathrm{id}_P"] & P
\end{tikzcd}
$$

Link to Diagram

And rearranging, we get

$$
\begin{tikzcd}
 & & & P \arrow[dl, "\exists \iota"'] \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & \ker \pi \arrow[r, hook] & F(P) \arrow[r, two heads, "\pi"] & P \arrow[r] \arrow[l, dashed, bend left, "\iota"] & 0
\end{tikzcd}
$$

Link to Diagram

Since $\pi \circ \iota$, the SES splits and this $F(P) \cong P \oplus \ker \pi$, making P a direct summand of a free module.

■

Example 7.3.5 *(?)*: Not every projective module is free. Let $R = R_1 \times R_2$ a direct product of unital rings. Then $P \coloneqq R_1 \times \{0\}$ and $P' \coloneqq \{0\} \times R_2$ are R-modules that are submodules of R. They're projective since R is free over itself as an R-module, and their direct sum is R. However they can not be free, since e.g. P has a nonzero annihilator: taking $(0,1) \in R$, we have $(0,1) \cdot P = \{(0,0)\} = 0_R$. No free module has a nonzero annihilator, since ix $0 \neq r \in R$ then $rR \neq 0$ since $r1_R \in rR$, which implies that $r\left(\bigoplus R\right) \neq 0$.

Example 7.3.6 *(?)*: Taking $R = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ admits projective R-modules which are not free.

Example 7.3.7 *(?)*: Let F be a field, define the ring $R \coloneqq \mathrm{Mat}(n \times n, F)$ with $n \geq 2$, and set $V = F^n$ thought of as column vectors. This is left R-module, and decomposes as $R = \bigoplus_{i=1}^{n} V$ corresponding to the columns of R, using that $AB = [Ab_1, \cdots, Ab_n]$. Then V is a projective R-module as a direct summand of a free module, but it is not free. We have vector spaces, so we can consider dimensions: $\dim_F R = n^2$ and $\dim_F V = n$, so V can't be a free R-module since this would force $\dim_F V = kn^2$ for some k.

Example 7.3.8 ***(?):*** How many projective modules are there in a given category? Let $\mathcal{C} \coloneqq \mathrm{Ab}^{\mathrm{fin}}$ be the category of *finite* abelian groups, where we take the full subcategory of the category of all abelian groups. This is an abelian category, although it is not closed under *infinite* direct sums or products, which has no projective objects.

Proof (?).
Over a PID, every submodule of a free module is free, and so we have free $\iff$ projective in this case. So equivalently, we can show there are no free $\mathbb{Z}$-modules, which is true because $\mathbb{Z}$ is infinite, and any such module would have to contain a copy of $\mathbb{Z}$.

■

Remark 7.3.9: The definition of projective objects extends to any abelian category, not just R-modules.

8 | Monday, February 01

Recall the universal of projective modules.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Definition 8.0.1 (Enough Projective)
If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ has **enough projectives** if and only if for all $a \in \mathcal{A}$ there exists a projective object $P \in \mathcal{A}$ and a surjective morphism $P \twoheadrightarrow A$.

Example 8.0.2 ***(?):*** mod-R has enough projectives: for all $A \in$ mod-R, one can take $F(A) \twoheadrightarrow A$.

Example 8.0.3 ***(?):*** The category of finite abelian groups does *not* have enough projectives.

Why?

Lemma 8.0.4 ***(?).***
P is projective if and only if $\mathrm{Hom}_{\mathcal{A}}(P, \cdot\,)$ is an exact functor.

Exercise 8.0.5 (?)
Prove this!

Definition 8.0.6 ((Key))
Let $M \in \text{mod-}R$, then a **projective resolution** of M is an exact complex

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_1 \overset{\varepsilon}{\twoheadrightarrow} M \to 0.$$

We write $P. \overset{\epsilon}{\twoheadrightarrow} M$.

Lemma 8.0.7 *((Key)).*
Every object $M \in \text{mod-}R$ has a projective resolution. This is true in any abelian category with enough projectives.

Proof (?).

- Since there are enough projectives, choose $P_0 \xrightarrow{\epsilon_0} M \to 0$.
- To extend this, set $M_0 \coloneqq \ker \epsilon_0$, then find a projective cover $P_1 \xrightarrow{\epsilon_1} M_0$
- Use that $d_1 \coloneqq \iota_0 \circ \epsilon_1$ and $\operatorname{im} d_1 = M_0 = \ker \epsilon_0$
- Then $d_2 \coloneqq \iota_1 \circ \epsilon_2$ with $\operatorname{im} d_2 = M_1$, and $\ker d_1 = \ker \epsilon_1 = M_1$.
- Continuing in this fashion makes the complex exact at every stage.

$$\begin{tikzcd}
 & & 0 \arrow[dr, hook] & & 0 & & & & \\
 & & & M_1 \arrow[ur, two heads] \arrow[dr, hook] & & & & & \\
\cdots \arrow[rr, dashed] & & P_2 \arrow[ur, two heads] \arrow[rr, dashed, "\exists d_2"] & & P_1 \arrow[rr, dashed, "\exists d_1"] \arrow[dr, two heads, "\varepsilon_0"] & & P_0 \arrow[r, two heads, "\varepsilon_0"] & M & 0 \\
 & M_2 \arrow[ur, hook] & & & & M_0 \arrow[ur, "\iota_0"] \arrow[dr] & & & \\
0 \arrow[ur, hook] & & & & 0 \arrow[ur, hook] & & 0 & &
\end{tikzcd}$$

Link to Diagram

∎

8.1 Comparison Theorem

Theorem 8.1.1 *(Comparison Theorem).*
Suppose $P. \xrightarrow{\epsilon} M$ is a projective resolution of an object in $\mathcal{A}$ and $(M \xrightarrow{f} N \in \operatorname{Mor}(\mathcal{A})$ and $Q. \xrightarrow{\eta} N$ a resolution of N. Then there exists a chain map $P \xrightarrow{f} Q$ lifting f which is unique up

to chain homotopy:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2^P"] \arrow[d, dashed, "\exists f_2"] & P_1 \arrow[r, "d_1^P"] \arrow[d, dashed, "\exists f_1"] & P_0 \arrow[r, "\varepsilon = d_0^P"] \arrow[d, dashed, "\exists f_0"] & M \arrow[r] \arrow[d, "f_{-1} := f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2^Q"'] & Q_1 \arrow[r, "d_1^Q"'] & Q_0 \arrow[r, "\eta = d_0^Q"'] & N \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Remark 8.1.2: The proof will only use that $P \xrightarrow{\epsilon} M$ is a chain complex of projective objects, i.e. $d^2 = 0$, and that $\epsilon \circ d_1^p = 0$. To make the notation more consistent, we'll write $Z_{-1}(P) \coloneqq M$ and $Z_{-1}(Q) \coloneqq N$. Toward an induction, suppose that the f_i have been constructed for $i \leq n$, so $f_{i-1} \circ d = d \circ f_i$.

Proof (Existence).
A fact about chain maps is that they induce maps on the kernels of the outgoing maps, so there is a map $f_n' : Z_n(P) \to Z_n(Q)$. We get a diagram where the top row is not necessarily exact:

$$
\begin{tikzcd}
P_{n+1} \arrow[r, "d"] \arrow[d, dashed, "\exists f_{n+1}"] & Z_n(P) \arrow[d, "f_{n'}"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0
\end{tikzcd}
$$

Link to Diagram

Using the definition of projective, since P_{n+1} is projective, the map $f_{n+1} : P_{n+1} \to Q_{n+1}$ exists where $d \circ f_{n+1} = f_n' \circ d = f_n \circ d$, since $f_n = f_n'$ on $\operatorname{im} d \subseteq Z_n(P)$. This yields commutativity of the above square.

■

Proof (Uniqueness).
Suppose $g : P \to Q$ is another lift of f', the consider $h \coloneqq f - g$. This is a chain map $P \to Q$ lifting of $f' - f' = 0$. We'll construct a chain contraction $\{s_n :; P_n \to Q_{n+1}\}$ by induction on n: We have the following diagram:

$$
\begin{tikzcd}
 & P_0 \arrow[r, "\varepsilon"] \arrow[d, "h_0 := f_0 - f_0'"'] & M \arrow[d, "f - f' = 0"] \\
Q_1 \arrow[r, "d"'] & Q_0 \arrow[r, "\eta"'] & N
\end{tikzcd}
$$

Link to Diagram

Setting $P_{-1} \coloneqq 0$ and $s_{-1} : P_{-1} \to Q_0$ to be the zero map, we have $\eta \circ h_0 = \varepsilon(f' - f') = 0$. Using projectivity of P_0, there exists an s_0 as shown below which satisfies $h_0 = d \circ s_0 = ds_0 + s_{-1}d$ where $s_{-1}d = 0$:

$$\begin{tikzcd}
 & P_0 \arrow[r, "d_0=0"] \arrow[dl, dashed, "\exists s_1"'] \arrow[d, "h_0"] & P_{-1} = 0 \arrow[dl, "s_{-1}=0"] \\
Q_1 \arrow[r, two heads] & d(Q_1) \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Proceeding inductively, assume we have maps $s_i : P_i \to Q_{i+1}$ such that $h_{n-1} = ds_{n-1} + s_{n-2}d$, or equivalently $ds_{n-1} = h_{n-1} - s_{n-2}d$. We want to construct s_n in the following diagram:

$$\begin{tikzcd}
 & P_n \arrow[r, "d"] \arrow[dl, dashed, "\exists s_n"'] \arrow[d, "h_n"] & P_{n-1} \arrow[r, "d"] \arrow[dl, "s_{n-1}"] \arrow[d, "h_{n-1}"] & P_{n-2} \arrow[dl, "s_{n-2}"] \\
Q_{n+1} \arrow[r, "d"] & Q_n \arrow[r, "d"] & Q_{n-1} &
\end{tikzcd}$$

Link to Diagram

So consider $h_n - s_{n-1}d : P_n \to Q_n$, which we want to equal $d(s_n)$. We want exactness, so we need better control of the image! We have $d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d$. But this is equal to $dh_n - h_{n-1}d = 0$ since h is a chain map. Thus we get $h_n - s_{n-1}d : P_n \to Z_n(Q)$, and thus using projectivity one last time, we obtain the following:

$$\begin{tikzcd}
 & P_n \arrow[dl, dashed, "\exists s_n"'] \arrow[d, "h_n - s_{n-1}d"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r, "d"] & 0
\end{tikzcd}$$

Link to Diagram

Since P_n is projective, there exists an $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$.

■

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Remark 9.0.1: All rings have 1 in this course!

9.1 Horseshoe Lemma

Proposition 9.1.1 ***(Horseshoe Lemma).***
Suppose we have a diagram like the following, where the columns are exact and the rows are projective resolutions:

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\varepsilon'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
 & & & & A \arrow[d, "\pi_A"] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}
$$

Link to Diagram

Note that if the vertical sequence were split, one could sum together to two resolutions to get a resolution of the middle. This still works: there is a projective resolution of P of A given by

$$P_n \coloneqq P_n' \oplus P_n''$$

which lifts the vertical column in the above diagram to an exact sequence of complexes

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0,$$

where $\iota_n : P_n' \hookrightarrow P_n$ is the natural inclusion and $\pi_i : P_n \twoheadrightarrow P_n''$ the natural projection.

9.1.1 Proof of the Horseshoe Lemma

We can construct this inductively:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \\
\ker(\varepsilon) \arrow[r] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \\
\ker(\varepsilon'') \arrow[r] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Link to Diagram

- P_0'' projective and π_A surjective implies ε'' lifts to $\eta'' : P_0'' \to A$
- Composing yields $\eta' \coloneqq \iota_A \circ \eta' : P_0' \to A$
- Get $\varepsilon \coloneqq \eta' \oplus \eta'' : P_0 \coloneqq P_0' \oplus P_0'' \to A$.

Flipping the diagram, we can apply the snake lemma to the two columns:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[uurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"] & A'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 &
\end{tikzcd}
$$

Link to Diagram

We can now conclude that

- $\operatorname{coker} \varepsilon = 0$
- $\partial = 0$ since it lands on the zero moduli

So append a zero onto the far left column:

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker(\varepsilon') \arrow[r] \arrow[d] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "\iota"] \arrow[dr, "\eta'"] & A' \arrow[r] \arrow[d, "\iota_A"] & 0 \arrow[d] \\
\ker(\varepsilon) \arrow[r] \arrow[d] & P_0 \arrow[r, "\eta"] \arrow[d, "\pi"] & A \arrow[r] \arrow[d, "\pi_A"] & \operatorname{coker}(\varepsilon) \arrow[d] \\
\ker(\varepsilon'') \arrow[r] \arrow[d] \arrow[uurrr, squiggly, bend left, "\partial"] & P_0'' \arrow[r, "\varepsilon''"] \arrow[d] \arrow[ur, dashed, "\eta''"] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

Link to Diagram

Thus the LHS column is a SES, and we have the first step of a resolution. Proceeding inductively, at the next step we have

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_1' \arrow[r, "d_1'"] & \ker(\varepsilon') \arrow[r] \arrow[d] & 0 \\
 & & \ker(\varepsilon) \arrow[d] & \\
\cdots \arrow[r] & P_1'' \arrow[r, "d_1''"] & \ker(\varepsilon'') \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}
$$

Link to Diagram

However, this is precisely the situation that appeared before, so the same procedure works.

Exercise 9.1.2 (?)
Check that the middle complex is exact! Follows by construction.

9.2 Injective Resolutions

Definition 9.2.1 (Injective Objects)
Let $\mathcal{A}$ be an abelian category, then $I \in \mathcal{A}$ is **injective** if and only if it satisfies the following universal property: A is projective if and only if for every monic $\alpha : A \to I$, any map $f : A \to B$ lifts to a map $B \to I$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \\
& I \arrow[ur, dashed, "\exists\beta"'] &
\end{tikzcd}$$

Link to Diagram

We say $\mathcal{A}$ **has enough injectives** if and only if for all A, there exists $A \hookrightarrow I$ where I is injective.

Slogan 9.2.2
Maps on subobjects extend.

Proposition 9.2.3 ***(Products of Injectives are Injective).***
If $\{I_\alpha\}$ is a family of injectives and $I \coloneqq \prod_\alpha I_\alpha \in A$, then I is again injective.

Proof (?).
Use the universal property of direct products. ■

9.3 Baer's Criterion

Proposition 9.3.1 *(Baer's Criterion).*
An object $E \in R$-mod is injective if and only if for every right ideal $J \trianglelefteq R$, every map $J \to E$ extends to a map $R \to E$. Note that J is a right R-submodule.

Proof (?).
$\implies$: This is essentially by definition. Instead of taking arbitrary submodules, we're just taking R itself and *its* submodules:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\
& E &
\end{tikzcd}$$

Link to Diagram

$\impliedby$: Suppose we have the following:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \\
& E &
\end{tikzcd}$$

Link to Diagram

Let $\mathcal{E} \coloneqq \left\{ \alpha' : A' \to E \;\middle|\; A \leq A' \leq B \right\}$, i.e. all of the intermediate extensions:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & A' \arrow[r] & B \\
& E & &
\end{tikzcd}$$

Link to Diagram

Add a partial order to $\mathcal{E}$ where $\alpha' \leq \alpha''$ if and only if α'' extends α'. Applying Zorn's lemma (and abusing notation slightly), we can produce a maximal $\alpha' : A' \to E$. The claim is that $A' = B$. Supposing not, then A' is a proper submodule, so choose a $b \in B \setminus A'$. Then define the set $J \coloneqq \left\{ r \in R \;\middle|\; br \in A' \right\}$, this is a right ideal of R since A' was a right R-module. Now applying the assumption of Baer's condition on E, we can produce a map $f : R \to E$:C

$$
\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, "b\cdot"'] & R \arrow[ddl, dashed, "\exists f"] \\
 & A' \arrow[d, "\alpha'"'] & \\
 & E &
\end{tikzcd}
$$

Link to Diagram

Now let $A'' \coloneqq A' + bR \leq B$, and provisionally define

$$
\begin{aligned}
\alpha'' : A'' &\to E \\
a + br &\mapsto \alpha'(a) + f(r).
\end{aligned}
$$

Remark 9.3.2: Is this well-defined? Consider overlapping terms, it's enough to consider elements of the form $br \in A'$. In this case, $r \in J$ by definition, and so $\alpha'(br) = f(r)$ by commutativity in the previous diagram, which shows that the two maps agree on anything in the intersection.

Note that α'' now extends α', but $A' \subsetneq A''$ since $b \in A'' \setminus A'$. But then A'' strictly contains A', contradicting its maximality from Zorn's lemma.

■

Remark 9.3.3: Big question: what *are* injective modules really? These are pretty nonintuitive objects.

10 Appendix: Extra Definitions

Definition 10.0.1 (Acyclic)
A chain complex C is **acyclic** if and only if $H_*(C) = 0$.

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Bibliography

[1] Charles A. Weibel. *An introduction to homological algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.