Title

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1 | Wednesday, October 28

1.1 Review of Last Time

Suppose we have two weights in the same facet, i.e. they're in the same stabilizer under the action of the affine Weyl group:

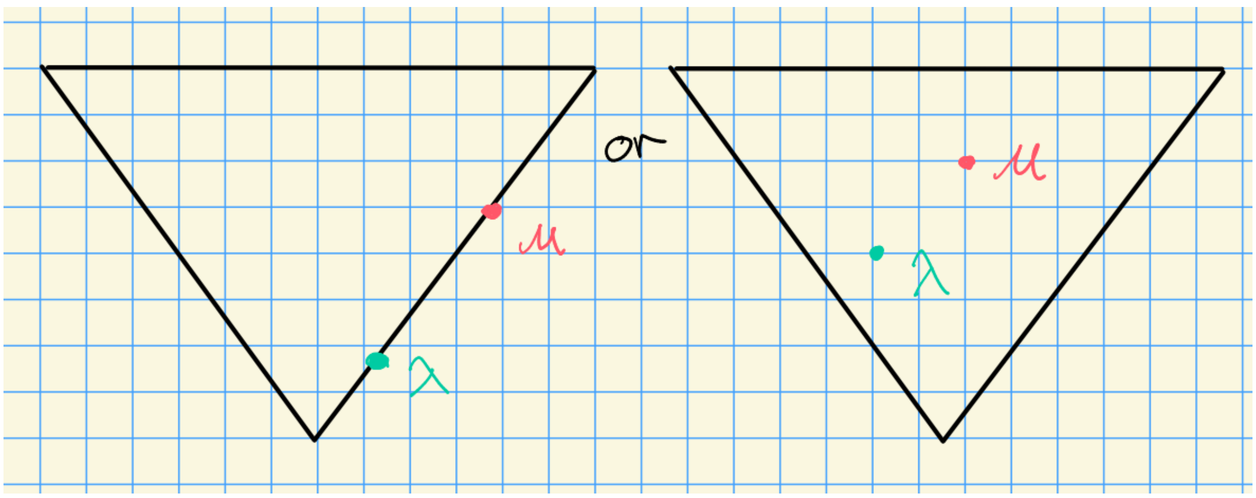

Figure 1: Weights in the same facet

We had a theorem: if λ, μ are in the same facet, then $\mathcal{B}_\lambda \cong \mathcal{B}_\mu$ is an equivalence of categories, where the map is via the translation functors.

1.2 Description of $T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big)$

We can write

$$\begin{aligned}
T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big) &= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes \mathrm{pr}_\lambda\Big(H^i(w\cdot\lambda)\Big)\Big) \\
&= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes H^i(w\cdot\lambda)\Big) \\
&= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes R^i\,\mathrm{Ind}_B^G\, w\cdot\lambda\Big) \\
&= \mathrm{pr}_\mu\Big(R^i\,\mathrm{Ind}_B^G\,(L(\nu_1)\otimes w\cdot\lambda)\Big).
\end{aligned}$$

Take a composition series by B-modules of $L(\nu_1)\otimes w\cdot\lambda$, say

$$0 = M_0 \subseteq M_1 \cdots \subseteq M_r = L(\nu_1)\otimes w\cdot\lambda.$$

where $M_j/M_{j-1} \cong \lambda + j + w\cdot\lambda$ and $\lambda_j < \lambda_{j'} \implies j < j'$, i.e. we can order them in a decreasing way.

Consider the SES

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow M_j/M_{j-1} \longrightarrow 0$$

where applying $\mathrm{pr}_\mu(\,\cdot\,)$ induces the LES

$$\cdots \longrightarrow \mathrm{pr}_\mu M_{j-1} \longrightarrow \mathrm{pr}_\mu M_j \longrightarrow \mathrm{pr}_\mu(M_j/M_{j-1}) \longrightarrow \cdots$$

We know that

$$\mathrm{pr}_\mu H^i(\lambda_j + w\cdot\lambda) = \begin{cases} H^i(\lambda_j + w\cdot\lambda) & \lambda + j + w\cdot\lambda \in W_p\cdot\mu \\ 0 & \text{else} \end{cases},$$

i.e. this projection is the identity for weights linked to μ and zero otherwise. We also have

$$\mathrm{pr}_\mu H^i(M_r) = T_\lambda^\mu H^i(w\cdot\lambda).$$

Theorem 1.2.1 ***(?).***
Let $\lambda,\mu\in\overline{C}_{\mathbb{Z}}$ and F be a facet with $\lambda\in F$. If $\mu\in\overline{F}$, then we have

$$T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big) = H^i(w\cdot\mu) \qquad \forall w\in W_p.$$

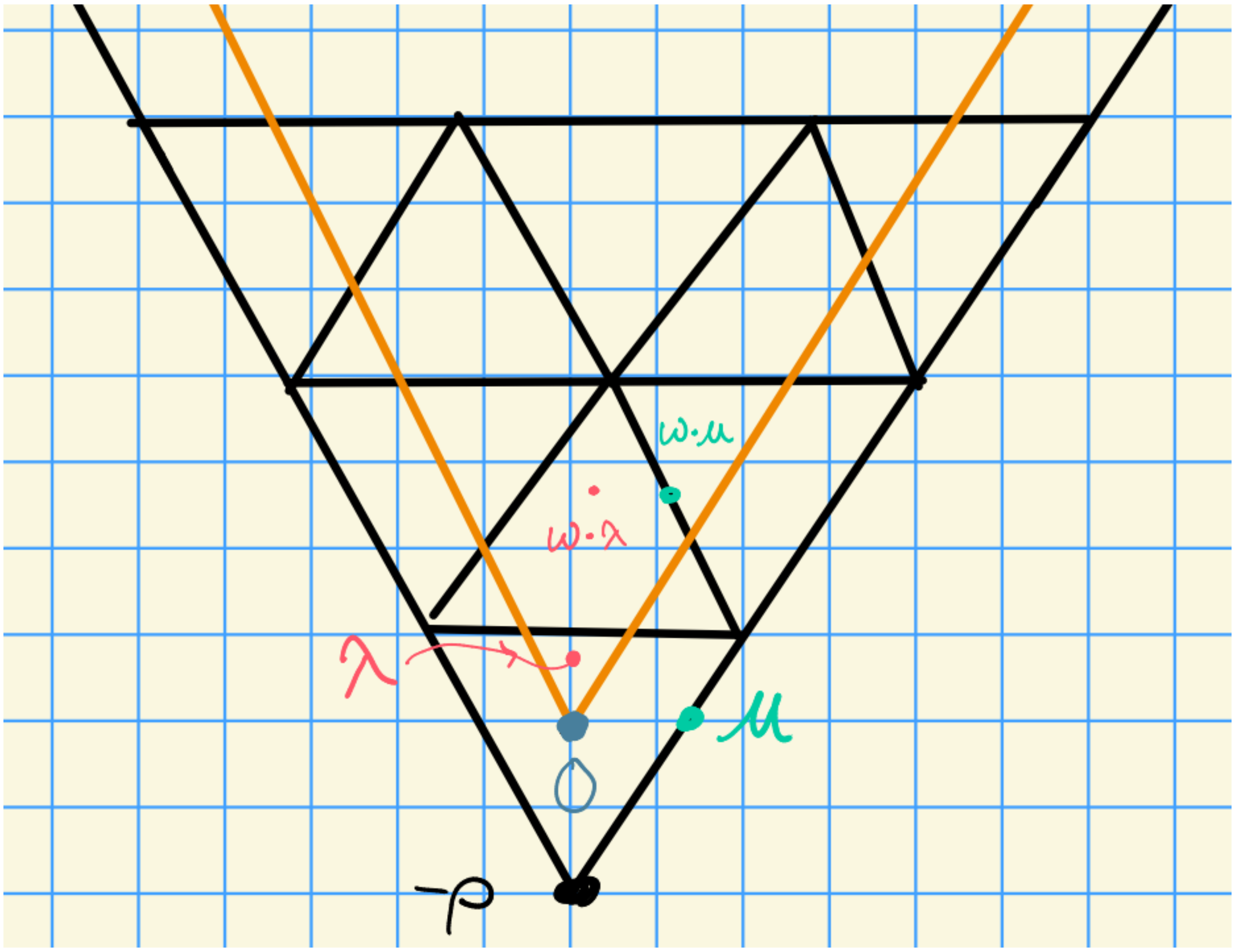

Figure 2: Image

Example 1.2.1 (?)**.**

Here consider $H_0(\lambda) \xrightarrow{T_\lambda^\mu} H_0(\mu) = 0$, since μ is outside of the dominant region (in orange.) We also have $H^0(w\cdot\lambda) \to H^0(w\cdot\mu) \neq 0$, since this falls *into* the dominant region.

Proof (?).
Let $\lambda \in F$ and $\mu \in \overline{F}$. Then $\mathrm{Stab}_{W_p}(\lambda) \subseteq \mathrm{Stab}_{W_p}(\mu)$. By a previous technical lemma, we had a formula for computing $\operatorname{ch} T_\lambda^\mu V$, which involved considering

$$w_1 \in \frac{\mathrm{Stab}_{W_p}(\lambda)}{\mathrm{Stab}_{W_p}(\lambda) \cap \mathrm{Stab}_{W_p}(\mu)}.$$

In this case, we get $w_1 = \mathrm{id}$, since the top and bottom are equal.
By that lemma, there exists a unique ℓ such that $w\cdot\lambda + \lambda_\ell \in W_p\cdot\mu$, where λ_ℓ is a weight of $L(\nu_1)$. From the LES, we have

$$\cdots \longrightarrow \mathrm{pr}_\mu M_{j-1} \longrightarrow \mathrm{pr}_\mu M_j \longrightarrow \mathrm{pr}_\mu(M_j/M_{j-1}) = \lambda_j + w\cdot\lambda \longrightarrow \cdots$$

where the last term will only be nonzero in restricted cases. We can thus conclude that

$$\mathrm{pr}_\mu(H^i(M_j)) = \begin{cases} 0 & j < \ell \\ H^i(w\cdot\mu) & j \geq \ell. \end{cases}$$

Setting $j = r$, we have

$$T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big) = \mathrm{pr}_\mu H^j(M_r) = H^i(w\cdot\mu).$$

Proposition 1.2.1 ***(?).***

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