Homotopy Groups of Spheres

Graduate Student Seminar

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April 2020

Introduction

Outline

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- "Measuring stick" for current tools, similar to special values of L-functions
- Serre's computation

Intuition

Homotopies of paths:

Spheres

Setup

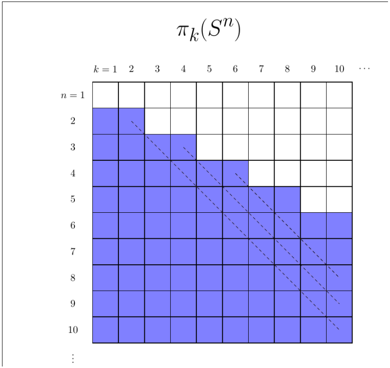
- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate: $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:

$$\pi_k(S^n)$$

	$k=1$	2	3	4	5	6	7	8	9	10	$\cdots$
$n=1$											
2											
3											
4											
5											
6											
7											
8											
9											
10											
$\vdots$											

n = k: Stabilization

- Theorem (1937, Freudenthal): For $k \gg 0$, $[\Sigma^k X, \Sigma^k Y] \cong [\Sigma^{k+1} X, \Sigma^{k+1} Y]$.
- Use the fact that $\Sigma S^k \cong S^{k+1}$, then in some *stable range* $\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1}$



n = k: Stabilization

We can thus suspend something we already know:

$$\pi_k(S^n)$$