Chapter 9

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1 | Background, Notation, Setup

Goals

Theorem 1.1 ***(Arnold Conjecture (Symplectic Morse Inequalities?)).***
Let (W, ω) be a compact symplectic manifold and

$$H : W \to \mathbb{R}$$

a time-dependent Hamiltonian with nondegenerate 1-periodic solutions. Then

$$\#\left\{\text{1-Periodic trajectories of } X_H\right\} \geq \sum_{k \in \mathbb{Z}} \dim_? HM_k(W;\, \mathbb{Z}/2\mathbb{Z}).$$

Here $HM_*(W)$ is the Morse homology, and *nondegenerate* means the differential of the flow at time 1 has no fixed vectors.

Important Ideas for This Chapter:

Theorem 1.2 ***(Use Broken Trajectories to Compactify).***
$\mathcal{L}(x, y)$ is compact, where the compactification is given by adding in

$$\partial \mathcal{L}(x, y) = \{\text{"Broken Trajectories"}\}$$

Theorem 1.3 ***(Gluing Yields a Chain Complex).***

$$\partial^2 = 0$$

Strategy:

In the background, have a Hamiltonian $H : W \to \mathbb{R}$. Basic idea: cook up a gradient flow.

1. Define the action functional $\mathcal{A}_H$

On an infinite-dimensional space, critical points are periodic solutions of H

2. Construct the chain complex (graded vector space) CF_*.

Uses analog of the *index* of a critical point.

3. Define the vector field X_H using $-\text{grad } \mathcal{A}_H$.

This will be used to define ∂ later.

4. Count the trajectories of X_H
5. Show finite-energy trajectories connect critical points of $\mathcal{A}_H$.
6. Show *Gromov Compactness* for space of trajectories of finite energy
7. Define ∂

Uses another compactness property

8. Show space of trajectories is a manifold, plus analog of "Smale property"
9. **Show that** $\partial^2 = 0$ using a gluing property
10. Show that HF_* doesn't depend on $\mathcal{A}_H$ or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_*.

Ingredients:

- (W, ω, J) with $\omega \in \Omega^2(W)$ is a symplectic manifold
  - With $J : T_pW \to T_pW$ an almost complex structure, so $J^2 = -\mathrm{id}$.
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian
  - X_H the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in T_xM:

$$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$

  - Zeros of vector field X_H correspond to critical points of H:

$$X_H(x) = 0 \iff (dH)_x = 0.$$

  - Take the associated flow, assumed 1-periodic:

$$\psi^t \in C^\infty(W, W) \qquad \psi^1 = \mathrm{id},$$

  - Critical points of H are periodic trajectories.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \mathrm{grad}\,_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y \end{aligned}$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}W$ is the *free loop space* on W, i.e. space of contractible loops on W, i.e. $C^\infty(S^1; W)$ with the C^∞ topology
  - Elements $x \in \mathcal{L}W$ can be viewed as maps $S^1 \to W$.
  - Can extend to maps from a closed disc, $u : \overline{\mathbb{D}}^2 \to M$.
  - Loops in $\mathcal{L}W$ can be viewed as maps $S^2 \to W$, since they're maps $I \times S^1 \to W$ with the boundaries pinched:

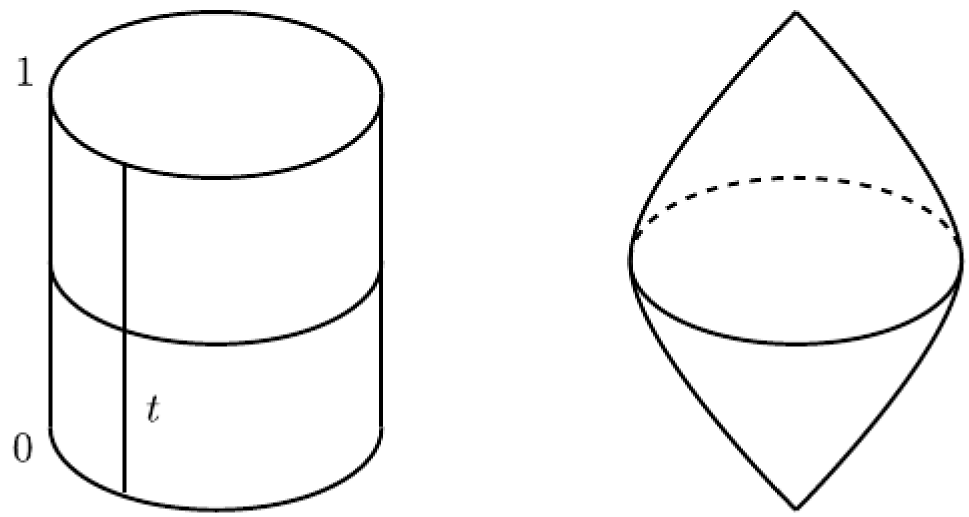

Figure 1: Loops in $\mathcal{L}W$

- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\to \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq)$.
  - A correspondence

$$\left\{ \begin{matrix} \text{Solutions to the} \\ \text{Floer equation} \end{matrix} \right\} \iff \left\{ \begin{matrix} \text{Trajectories} \\ \text{of grad } \mathcal{A}_H \end{matrix} \right\}.$$

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of $\mathcal{A}_H$.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, W)$,

$$\int_{S^2} u^*\omega = 0 \quad \text{or equivalently} \quad \langle \omega,\ \pi_2 W \rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM, equivalently

$$\langle c_1 TW,\ \pi_2 W \rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- Maslov index: used the fact that
  - Every path in $\gamma : I \to \mathrm{Sp}(2n, \mathbb{R})$ can be assigned an integer coming from a map $\tilde{\gamma} : I \to S^1$ and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y.
  - After perturbing H to get transversality, get a manifold of dimension $\mu(x) - \mu(y)$.
    - Dimension:

$$\dim \mathcal{M}.$$

  - How we did it:
    - Describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$

      (Banach manifold modeled on the Sobolev spaces $W^{1,p}$),
    - Apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality.
    - Followed from showing $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

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2 | Reminder of Goals

- Construct Floer homology to prove

Theorem 2.1 ***(Symplectic Morse Inequalities).***

$$\#\{\text{1-Periodic trajectories of } X_H\} \geq \sum_{k\in\mathbb{Z}} HM_k(W;\, \mathbb{Z}/2\mathbb{Z}).$$

Important Ideas for This Chapter:

Theorem 2.2 ***(Using Broken Trajectories to Compactify).***
$\mathcal{L}(x, y)$ is compact,

$$\partial\mathcal{L}(x, y) = \{\text{"Broken Trajectories"}\}$$

:::{.theorem title="Using Gluing to Get a Chain Complex"}

$$\partial^2 = 0$$

3 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

  $\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$

  $$\mathcal{M} = \{\text{All contractible finite-energy solutions to the Floer equation}\} = \bigcup_{x,y} \mathcal{M}(x, y).$$

- Defined the moduli space of (unparameterized) **trajectories** connecting x to y:

  $$\mathcal{L}(x, y) \coloneqq \mathcal{M}(x, y)/\mathbb{R}.$$

  - Use the quotient topology, define sequentially:

    $$\tilde{u}_n \overset{n\to\infty}{\longrightarrow} \tilde{u} \quad\iff\quad \exists\,\{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s,\,\cdot\,) \overset{n\to\infty}{\longrightarrow} u(s,\,\cdot\,).$$

- When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

$$n(x,y) \coloneqq \#\mathcal{L}(x,y)$$

is well-defined.

- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[S]$ where S is the set of periodic orbits of X_H of Maslov index k.
  - Finitely many since they are nondegeneracy implies they are isolated.

Remark 1.
Some notation:

$$\begin{tikzcd} \mathbb{R} \arrow[r] & \mathcal{M}(x,z) \arrow[d, "\pi"] \\ & \mathcal{L}(x,z) \end{tikzcd}$$

Hats will generally denote maps induced on quotient.

- Defined a differential

$$\begin{aligned} \partial : C_k(H) &\to C_{k-1}(H) \\ x &\mapsto \sum_{\mu(y)=k-1} n(x,y)y \end{aligned}$$

$$\begin{aligned} n(x,y) &\coloneqq \#\left\{\text{Trajectories of grad } \mathcal{A}_H \text{ connecting } x,y\right\} \mod 2 \\ &= \#\mathcal{L}(x,y) \mod 2. \end{aligned}$$

- Examined ∂^2:

$$
\begin{aligned}
\partial^2 : C_k(H) &\to C_{k-2}(H) \\
x &\mapsto \partial(\partial(x)) \\
&= \partial\left(\sum_{\mu(y)=\mu(x)-1} n(x,y)y\right) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x,y)\partial(y) \\
&= \sum_{\mu(y)=\mu(x)-1} n(x,y)\left(\sum_{\mu(z)=\mu(y)-1} n(y,z)z\right) \\
&= \sum_{\mu(y)=\mu(x)-1} \ \sum_{\mu(z)=\mu(y)-1} n(x,y)n(y,z)\, z \\
&= \sum_{\mu(z)=\mu(y)-1} \left(\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z)\right) z \qquad \text{(finite sums, swap order)},
\end{aligned}
$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x, z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\overline{\mathcal{L}}(x, y)$.
- We'll then have

$$\overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial\overline{\mathcal{L}}(x, z), \qquad \partial\overline{L}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z),$$

  which "space-ifies" the equation we want.
- We'll show $\partial\overline{\mathcal{L}}(x, z)$ is a 1-manifold, which must have an even number of points, and thus

$$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = \#\big(\partial\overline{\mathcal{L}}(x, z)\big) \equiv 0 \mod 2.$$

Image here of relations between spaces!

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4 Three Important Theorems

4.1 First Theorem: Convergence to Broken Trajectories

- Recall: *broken trajectories* are unions of intermediate trajectories connecting intermediate critical points.
- Shown last time: a sequence of trajectories can converge to a broken trajectory, i.e. there are broken trajectories in the closure of $\mathcal{L}(x, z)$.
- This theorem describes their behavior:

Theorem 4.1 ***(9.1.7: Convergence to Broken Trajectories).***
Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, z)$, then there exist

- A subsequence $\left\{u_{n_j}\right\}$
- Critical points $\{x_0, x_1, \cdots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = z$
- Sequences $\left\{s_n^1\right\}, \left\{s_n^2\right\}, \cdots, \left\{s_n^\ell\right\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_{n_j} \cdot s_n^k \xrightarrow{n\to\infty} u^k.$$

- Upshots:
  - Every sequence upstairs has a subsequence which (after reparameterizing) converges
  - This descends to actual convergence after quotienting by $\mathbb{R}$?
  - Yields uniqueness of limits in $\mathcal{L}(x, z)$, thus a separated topology
  - Sequentially compact $\iff$ compact since $\mathcal{L}(x, z)$ is a metric space?

Corollary 4.2 ***(Compactness).***
$\overline{\mathcal{L}}(x, z)$ is compact.

4.2 Second Theorem: Compactness of $\overline{\mathcal{L}}(x, z)$

Definition 4.2.1 (Regular Pair)**.**
For an almost complex structure J and a Hamiltonian H, the pair (H, J) is **regular** if the Floer map $\mathcal{F}$ is transverse to the zero section in the following vector bundle:

$$\begin{tikzcd} E_u \coloneqq \{\text{Vector fields tangent to } M \text{ along } u\} \arrow[r] & C^\infty(\mathbb{R} \times S^1; TM) \arrow[d] \\ & C^\infty(\mathbb{R} \times S^1; M) \arrow[u, bend left, dotted, "\mathcal{F}"] \arrow[u, bend right, dotted, "\mathbf{0}"'] \end{tikzcd}$$

Most of chapter 9 is spent proving this theorem:

Theorem 4.3 ***(9.2.1).***
Let (H, J) be a regular pair with H nondegenerate and x, z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\overline{\mathcal{L}}(x, z)$ is a compact 1-manifold with boundary with

$$\partial \overline{\mathcal{L}}(x, z) = \bigcup_{y \in \mathcal{I}(x,z)} \mathcal{L}(x, y) \times \mathcal{L}(y, z)$$

$$\text{where} \qquad \mathcal{I}(x, z) = \left\{ y \;\middle|\; \mu(x) < \mu(y) < \mu(z) \right\}.$$

Note: possibly a typo in the book? Has x, y on the LHS.

Corollary 4.4.

$$\partial^2 = 0.$$

4.3 Third Theorem: Gluing

Theorem 4.5 ***(9.2.3: Gluing).***
Let x, y, z be three critical points of $\mathcal{A}_H$ with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

   $$\psi : [\rho_0, \infty) \to \mathcal{M}(x, z)$$

   such that $\widehat{\psi}$, the induced map on the quotient

   $$\begin{tikzcd} {[\rho_0, \infty)} \arrow[r, "\psi"] \arrow[dr, dotted, "\widehat{\psi}"'] & \mathcal{M}(x, z) \arrow[d, "\pi"] \\ & \mathcal{L}(x, z) \end{tikzcd}$$

   is an embedding that satisfies

   $$\widehat{\psi}(\rho) \xrightarrow{\rho \to \infty} (\hat{u}, \hat{v}) \in \overline{\mathcal{L}}(x, z).$$

2. ("Uniqueness") For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x, z)$,

   $$\ell_n \xrightarrow{n \to \infty} (\hat{u}, \hat{v}) \quad \Longrightarrow \quad \ell_n \in \operatorname{im}(\widehat{\psi}) \text{ for } n \gg 0.$$

- We already know that $\overline{\mathcal{L}}(x, z)$ is compact and $\mathcal{L}(x, z)$ is a 1-manifold, so we look at neighborhoods of boundary points.
- Why unique: will show that the broken trajectory $(\hat{u}, \hat{v})$ is the endpoint of an embedded interval in $\overline{\mathcal{L}}(x, z)$.
  - Then show that any other sequence converging to $(\hat{u}, \hat{v})$ must approach via this interval, otherwise could have cuspidal points:

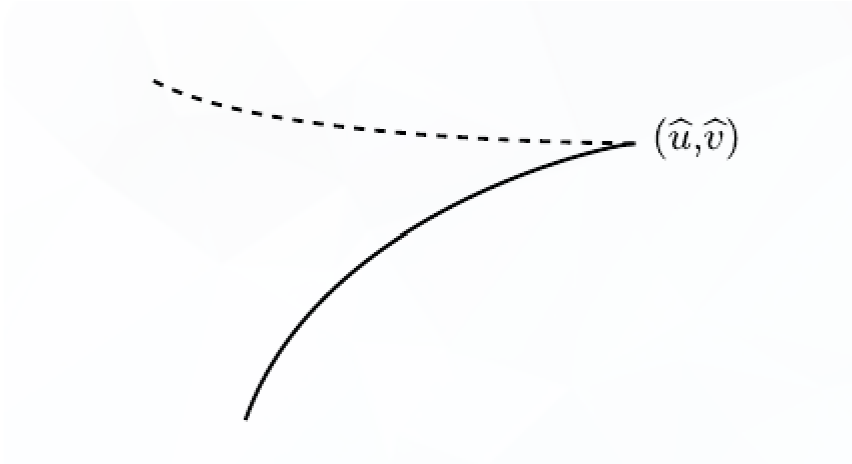

Figure 2: Cuspidal Point on Boundary

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5 | Gluing Theorem

Broken into three steps:

1. **Pre-gluing**:

- Get a function w_ρ which interpolates between u and v in the parameter ρ.
  - Not exactly a solution itself, just an "approximation".

2. **Newton's Method**:

- Apply the Newton-Picard method to w_p to construct a true solution

$$\begin{aligned}\psi : [-\rho, \infty) &\to \mathcal{M}(x, z) \\ \rho &\mapsto \exp_{w_p}(\gamma(p))\end{aligned}$$

$$\text{for some } \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x, z)$$

- General idea: guess, intersect x-axis with tangent of graph, use that as a new guess and iterate.
- GIF of Newton's Method

3. **Project and Verify Properties**:

- Check that the projection $\widehat{\psi} = \pi \circ \psi$ satisfies the conditions from the theorem.

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6 | 9.3: Pre-gluing, Construction of w_ρ

- Choose (once and for all) a bump function β on $B_\varepsilon(0)^c \subset \mathbb{R} \to [0,1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^\pm(s)$:

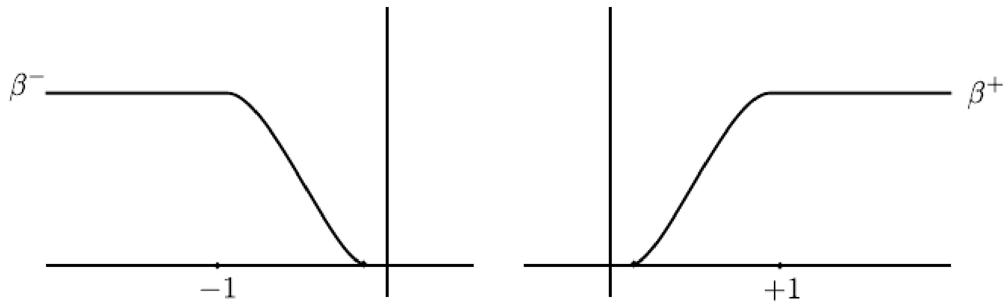

Figure 3: Bump away from zero

- Define an interpolation w_ρ from u to v in the following way: let
  - $\exp[\,\cdot\,] \coloneqq \exp_{y(t)}(\,\cdot\,)$ and
  - $\ln(\,\cdot\,) \coloneqq \exp_{y(t)}^{-1}(\,\cdot\,)$,

  then

$$w_\rho : x \to z$$

$$w_\rho(s,t) \coloneqq \begin{cases} u(s+\rho,t) & s \in (-\infty,-1] \\ \\ \exp\left[\beta^-(s)\ln(u(s+\rho,t)) + \beta^+(s)\ln(u(s-\rho,t))\right] & s \in [-1,1] \\ \\ u(s-\rho,t) & s \in [1,\infty) \end{cases}.$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \;\middle|\; \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\} \subseteq \operatorname{im}\exp_{y(t)}(\,\cdot\,).,$$

so we can apply $\exp_{y(t)}^{-1}(\,\cdot\,)$.

- Can make $|s| \leq 1$ for large ρ, since

$$u(s,t) \overset{s\to\infty}{\longrightarrow} y(t)$$
$$v(s,t) \overset{s\to-\infty}{\longrightarrow} y(t).$$

  - So pick a ρ_0 such that this holds for $\rho > \rho_0$.
  - Might have to increase ρ_0 later in the proof, so $\rho > \rho_0$ just means $\rho \gg 0$.

- Some properties:
  - $w_\rho \in C^\infty(x,z)$ and is differentiable in ρ.
  - $s \in [-\varepsilon, \varepsilon] \implies w_\rho(s,t) = y(t)$.

$$w_\rho(s-\rho,t) \overset{\rho\to\infty}{\longrightarrow} u(s,t) \quad \text{in} \quad C^\infty_{\text{loc}}$$

$$w_\rho(s,t) \overset{\rho\to\infty}{\longrightarrow} y(t) \quad \text{in} \quad C^\infty_{\text{loc}}.$$

- Now carry out the linearized version on tangent vectors
  - Let $Y \in T_u\mathcal{P}(x,y)$
  - Let $Z \in T_v\mathcal{P}(x,y)$
  - Replace w_ρ with the interpolation

$$Y\#_\rho Z \in T_{w_\rho}\mathcal{P}(x,y) = W^{1,p}(w_\rho^* TW).$$

  defined by

$$(Y\#_\rho Z)(s,t) = \begin{cases} Y(s+\rho,t) & s \in (-\infty,-1] \\ \exp_T\left[\beta^-(s)\ln_T(Y(s+\rho,t)) + \beta^+(s)\ln_T(Z(s-\rho,t))\right] & s \in [-1,1] \\ Z(s-\rho,t) & s \in [1,\infty) \end{cases},$$

  where the subscript T indicates taking tangents of the exponential maps at appropriate points.

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7 | 9.4: Construction of ψ.

7.1 Summary

- $\mathcal{F}_\rho$ will be $F \circ \exp w_\rho$ expanded in the bases Z_i coming from a trivialization of TW.
- $L_\rho = (d\mathcal{F}_\rho)_0$ will be the linearization of the Floer operator at zero.
- Newton-Picard method, general idea:
  - Allows finding zeros of f given an approximate zero x_0, using the extra information of the 1st derivative f'.
  - Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

  - Second variant more useful: only need to know the derivative at one point.

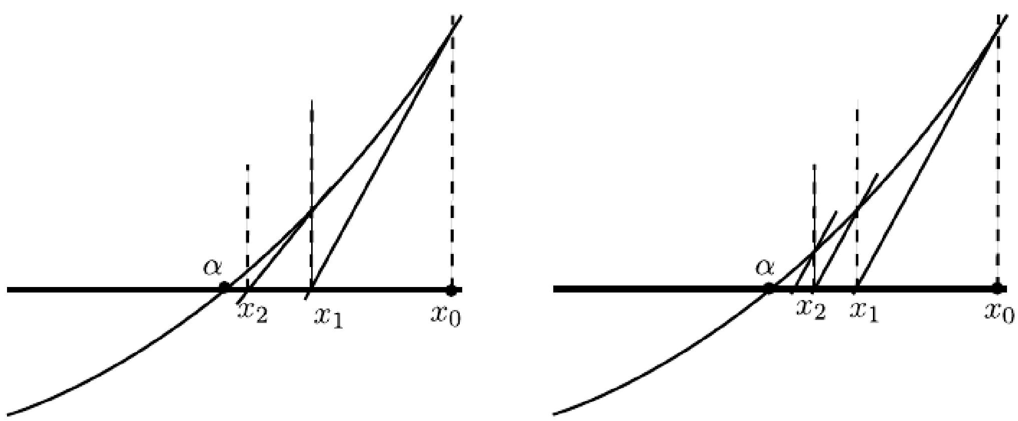

Fig. 9.6

Figure 4: Newton Method Variants

- Adapting Newton-Picard to operators:
  - $\dfrac{1}{f'(x_0)}$ will correspond to L_ρ^{-1}, but it won't be invertible on entire space.
  - Decompose
  $$T_{w_\rho}\mathcal{P}(x,z) = W^{1,p}(w_\rho^* TW) = W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) = \ker(L_\rho)\oplus W_\rho^\perp,$$
  where L_ρ will have a right inverse on $W_\rho^\perp \subseteq T_{w_\rho}\mathcal{P}(x,z)$.
  - Find that $x_0 = 0$ is "approximately" a solution and run Newton-Picard in $W_\rho^\perp$
  - Where does $W_\rho^\perp$ come from? Essentially the kernel of some linear functional given by an integral:
  $$W_\rho^\perp := \left\{ Y \in W^{1,p} \;\middle|\; \int_{\mathbb{R}\times S^1} \langle Y,\ \cdots\rangle\, ds\, dt = 0,\ \text{ plus conditions}\right\}.$$
- Will obtain for every $\rho \geq \rho_0$ an element $\gamma(\rho)\in W_\rho^\perp$ with
$$\mathcal{F}_\rho(\gamma(\rho)) = 0.$$
  - Where does γ come from? Intersection-theoretic interpretation on page 320:

$$
\begin{aligned}
\left(\exp_{w_\rho}\right)^{-1}\mathcal{M}(x,z) \cap W_\rho^\perp \subseteq T_{w_\rho}\mathcal{P}(x,z) && \rightsquigarrow \gamma \\
\mathcal{M}(x,z) \cap \left\{\exp_{w_\rho} W_\rho^\perp \;\middle|\; \rho \geq \rho_0\right\} \subseteq \mathcal{P}(x,z) && \rightsquigarrow \psi(\rho),
\end{aligned}
$$

which we get by exponentiating.

- This gives a codimension 1 subspace in $\mathcal{M}(x,z)$, which we take to be $\psi(\rho)$:

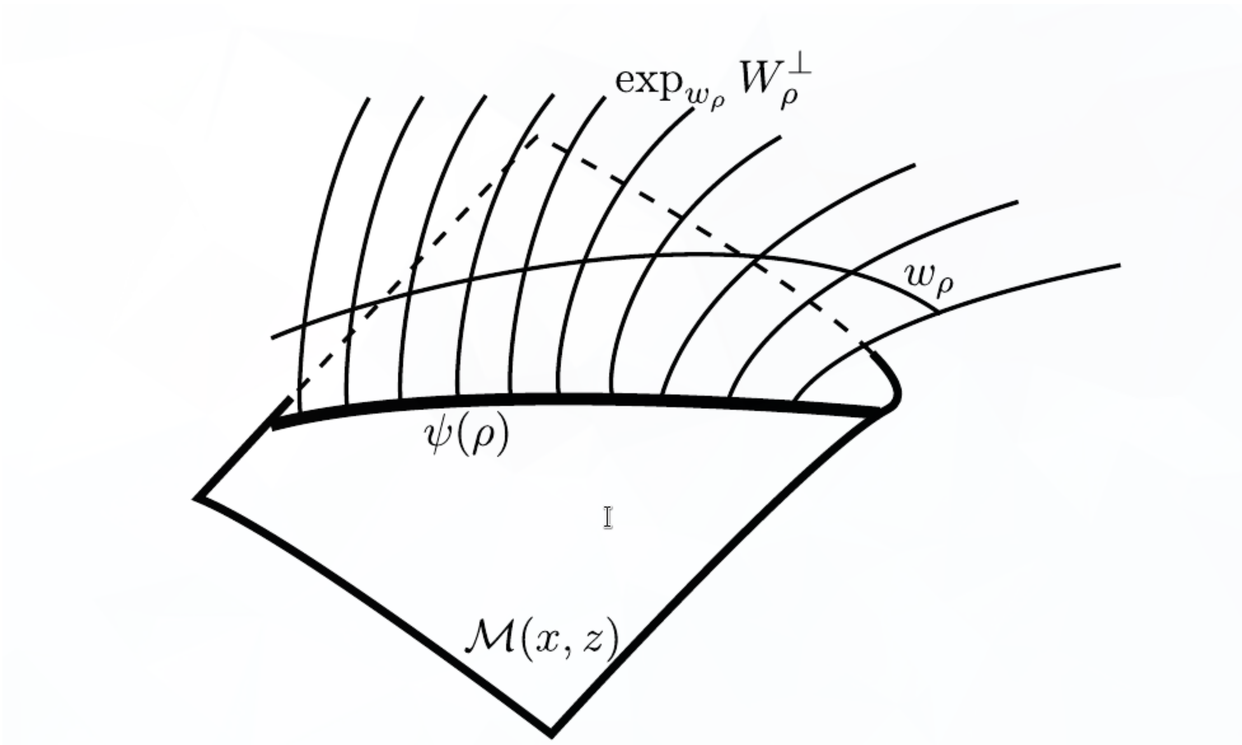

Figure 5: Intersection interpretation

7.1 Summary

Schematic picture here for $\gamma, \psi(\rho)$.

- Apply the implicit function theorem to show differentiability of γ in ρ.
- Use a trivialization Z_i^ρ of TW to get a vector field along w_ρ (also called $\gamma(\rho)$)
  - This is exactly an element of $T_{w_\rho}\mathcal{P}(x,z)$
- Exponentiate to get an element of $\mathcal{M}(x,z)$:

$$\psi(\rho) := \exp_{w_\rho}(\gamma(\rho)).$$

- Project onto $\mathcal{L}(x,z)$ to get $\widehat{\psi}$.
- Prove $\widehat{\psi}$ is a proper injective immersion and thus an embedding.
- Show that the broken trajectory $(\hat{u},\hat{v})$ is the endpoint of an embedded interval in $\overline{\mathcal{L}}(x,z)$.
  - Then show that any other sequence converging to $(\hat{u},\hat{v})$ must approach via this interval, otherwise could have cuspidal points:

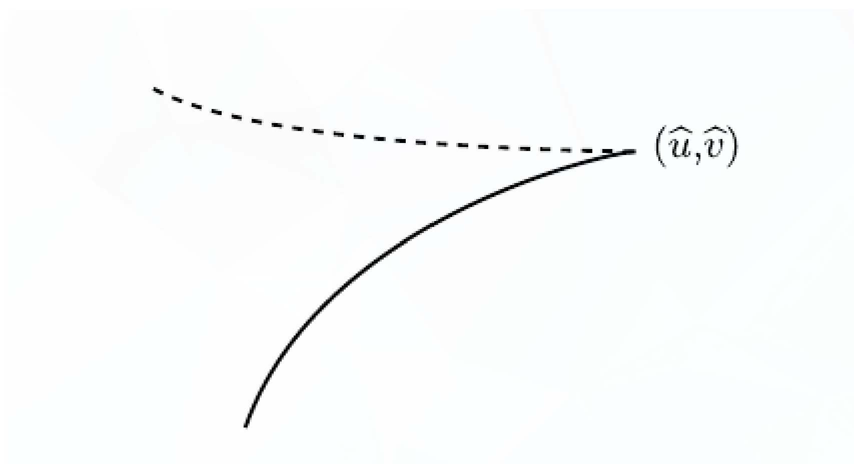

Figure 6: Cuspidal Point on Boundary

Probably not worth going farther than this!

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7.2 Details

- Have constructed $w_\rho \in C^\infty_\searrow(x, z)$ for every $\rho \geq \rho_0$, since there is exponential decay.
- Yields $\psi_\rho \in \mathcal{M}(x, z)$ a true solution (to be defined).
- Need to check that $\mathcal{F}(\psi_\rho) = 0$ where

$$\mathcal{F} = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad } Hx$$

  in the weak sense.
- ψ_ρ already continuous, and by elliptic regularity, makes it a strong solution.
- Defining $\mathcal{F}_\rho$.

$$\begin{aligned} W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\xrightarrow{\mathcal{F}_\rho} L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ (y_1, \ldots, y_{2n}) &\longmapsto \left[\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \text{grad}\, H_t\right)\left(\exp_{w_\rho} \sum y_i Z_i^\rho\right)\right]_{Z_i} \end{aligned}$$

where $\mathcal{F}_\rho \coloneqq \mathcal{F} \circ \exp_{w_\rho}$ written in the bases Z_i. sd

- Linearize $\mathcal{F}_\rho$.