Problem Set 1

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1.1 Part 1

Let $A \in \mathrm{Mat}(n, n)$ be a positive definite $n \times n$ matrix, so

$$\langle v,\ Av \rangle > 0 \quad \forall v \in \mathbb{R}^n,$$

and $B \in \mathrm{Math}(n, n)$ be non-negative definite, so

$$\langle v,\ Bv \rangle \geq 0 \quad \forall v \in \mathbb{R}^n,$$

1.2 Part 2

2 Problem 6

2.1 Part 1

Let $M = S^2$ as a smooth manifold, and consider a vector field on M,

$$X : M \to TM$$

We want to show that there is a point $p \in M$ such that $X(p) = 0$.

Every vector field on a compact manifold without boundary is complete, and since S^2 is compact with $\partial S^2 = \emptyset$, X is necessarily a complete vector field.

Thus every integral curve of X exists for all time, yielding a well-defined flow

$$\phi : M \times \mathbb{R} \to M$$

given by solving the initial value problems

$$\frac{\partial}{\partial s}\phi_s(p)\Big|_{s=t} = X(\phi_t(p)),$$
$$\phi_0(p) = p$$

at every point $p \in M$.

This yields a one-parameter family

$$\phi_t : M \to M \in \mathrm{Diff}(M, M).$$

In particular, $\phi_0 = \mathrm{id}_M$, and $\phi_1 \in \mathrm{Diff}(M, M)$. Moreover ϕ_0 is homotopic to ϕ_1 via the homotopy

$$\begin{aligned} H : M \times I &\to M \\ (p, t) &\mapsto \phi_t(p). \end{aligned}$$

We can now apply the Lefschetz fixed-point theorem to ϕ_0 and ϕ_1. For an arbitrary map $f : M \to M$, we have

$$\Lambda(f) = \sum_k \mathrm{Tr}\left(f_*\Big|_{H_k(X;\mathbb{Q})}\right).$$

where $f_* : H_*(X; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the induced map on homology, and

$$\Lambda(f) \neq 0 \iff f \text{ has at least one fixed point.}$$

In particular, we have

$$\begin{aligned} \Lambda(\mathrm{id}_M) &= \sum_k \mathrm{Tr}(\mathrm{id}_{H_k(X;\mathbb{Q})}) \\ &= \sum_k \dim H_k(X; \mathbb{Q}) \\ &= \chi(M), \end{aligned}$$

the Euler characteristic of M.

Since homotopic maps induce equal maps on homology, we also have $\Lambda(\phi_1) = \chi(M)$.

Since

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we have $\chi(S^2) = 2 \neq 0$, and thus ϕ_1 has a fixed point p_0, thus

$\frac{\partial}{\partial t}\phi_t(p_0)\Big|_{t=1}$ so

$$\begin{aligned}
\phi_t(p) =& p \\
\implies \frac{\partial}{\partial t}\phi_t(p) =& \frac{\partial}{\partial t}p = 0 && \text{by differentiating wrt } t \\
\implies \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& 0\Big|_{t=0} = 0 && \text{by evaluating at } t = 0 \\
\implies X(\phi_1(p_0)) \coloneqq \frac{\partial}{\partial t}\phi_t(p)\Big|_{t=1} =& 0 && \text{by definition of } \phi_1
\end{aligned}$$

so $X(\phi_1(p_0)) = 0$, which shows that p_0 is a zero of X. So X has at least one zero, as desired. $\square$

2.2 Part 2

The trivial bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & S^2 \times \mathbb{R}^2 \arrow[d] \\
& S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

has a nowhere vanishing section, namely

$$\begin{aligned}
s : S^2 &\to S^2 \times \mathbb{R}^2 \\
\mathbf{x} &\to (\mathbf{x}, [1, 1])
\end{aligned}$$

which is the identity on the S^2 component and assigns the constant vector $[1, 1]$ to every point. However, as part 1 shows, the bundle

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] & TS^2 \arrow[d] \\
& S^2 \arrow[u, bend right, dotted, "s"']
\end{tikzcd}$$

can *not* have a nowhere vanishing section. $\square$