Title

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1 Lecture 12

1.1 Brauer Groups

Goal: for C a curve over $k = \bar{k}$, we've computed

$$H^i(C, \mathbb{G}_m) = \begin{cases} \mathcal{O}_C^\times(C) & i = 0 \\ \mathrm{Pic}(C) & i = 1 \\ 0 & i > 1 \end{cases}.$$

Currently $i > 1$ is a mystery, so today we'll look at $i = 2$. Recall that we've reduced this to the Galois cohomology of the function field $H^i(k(C), \mathbb{G}_m)$ and of the strict Henselization [1] $H^i(K_{\bar{x}}, \mathbb{G}_m)$.

Today we'll try to understand the Galois cohomology of a field with coefficient in $\bar{k}^\times$, or $\mathbb{G}_m$ thought of as a sheaf on the étale site. We'll discuss $i = 2$, and a general principle in group cohomology is that if one understands $i = 1, 2$ then one can often understand all degrees.

In general, H^1 has a geometric interpretation: torsors. H^2 is much harder: they classify more general objects called **gerbes**. A miracle is that $H^2(\mathbb{G}_m)$ has real meaning, and is very closely related to real physical objects (certain torsors). Recall that we defined the *cohomological Brauer group of X* (**??**) as

$$\mathrm{Br}^{\mathrm{coh}} \coloneqq \mathrm{Br}'(X) \coloneqq H^i(X_{\text{ét}}, \mathbb{G}_m)_{\mathrm{tors}}.$$

We also started defining the Brauer group by considering

$$\bigcup_n \{\text{étale locally trivial } \mathrm{PGL}_n\text{-torsors}\} \xrightarrow{\delta} H^2(X_{\text{ét}}, \mathbb{G}_m),$$

and defining $\mathrm{Br}(X) \coloneqq \operatorname{im} f$ as a set, which is a reasonably concrete geometric object. This map came from a LES in cohomology, coming from a SES of sheaves, not all of which were abelian. The definition of δ was the boundary map of

$$\bigcup_n H^1(X_{\text{ét}}, \mathrm{PGL}_n) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mathbb{G}_m) \tag{1}$$

arising from the SES of sheaves of groups on $X_{\text{ét}}$,

$$1 \to \mathbb{G}_m \to \mathrm{GL}_m \to \mathrm{PGL}_n \to 1.$$

We argued last time that this was exact in the Zariski topology since the RHS map was a $\mathbb{G}_m$-torsor and thus Zariski locally trivial. What does δ mean? [2]

[1] The stalk of the structure sheaf, $\mathcal{O}_{C,x}$.

[2] Best reference: Giraud, "Cohomologie non Abelienne".

Remark 1.1.1: Making the LES here is a little subtle. You get a long exact sequence of *sets* here which terminates at the H^2 we're interested in, although one usually doesn't get a map of the form $H^1(C) \to H^2(B)$ for a SES $A \to B \to C$, you need that A is abelian (or in the center).

We'll now try to make δ explicit in terms of Čech cohomology, which is the only way we have to make sense of the LHS set in equation (1). We defined it to be the set of étale locally trivial PGL_n-torsors, which has a description in terms of $\check{H}^1$: the boundary map doesn't usually make sense for a nonabelian group, but it does in very low degrees such as $i = 1$. So we need to implement the snake lemma. Start with $[T] \in H^i(X_{\text{ét}}, \mathrm{PGL}_n)$ where T is a PGL_n- torsor split by $U \to X$. On $U \times_X U$, descent data is given by a section $\Gamma(U \times_X U, \mathrm{PGL}_n)$ as a sheaf. This is because descent data is an isomorphism on this double intersection and an automorphism of PGL_n is the same as a section to PGL_n. This descent data satisfies the cocycle condition. How do we apply the boundary map to an element in the Čech complex? After refining U we can lift this descent data to a section $s \in \Gamma(U \times_X U, GL_n)$. Note that $H^2(\mathbb{G}_m)$ is represented by a section to $\mathbb{G}_m$ of $U \times_X U \times_X U$. We started with something satisfying the cocycle condition but lifted in an arbitrary way, so it may not still satisfy the cocycle condition. We can write

$$\pi_{23}^* \pi_{12}^* s \left(\pi_{13} s\right)^{-1} \in \Gamma\left(U \times_X U \times_X U, \mathbb{G}_m\right).$$

A priori it's just a section to GL_n but we know that this goes to 1 in PGL_n, meaning it must comes from $\mathbb{G}_m$. The LHS is a 2-cocycle representing an element of $H^2(X_{\text{ét}}, \mathbb{G}_m)$.

Exercise 1.1.2*(?):* Check that d of this element is zero.

Slogan 1.1.3: $\delta([T])$ is the obstruction to lifting a PGL_n-torsor T to a GL_n-torsor. If this class vanishes, a lift exists.

Remark 1.1.4: This is what you might expect: the image of something coming from a boundary map is the obstruction to coming from the previous map. This class is called the **Brauer class of** T.

We've just mapped from a set to a group, so we don't know that the image is a group yet, and we don't yet know that the image is in $\mathrm{Br}^{\mathrm{coh}}$ since we don't know if the image is torsion.

1.1.1 Geometric Interpretations of PGL_n-Torsors Brauer Classes

Suppose $T \in \mathrm{Sh}^{\mathrm{Set}}(X_{\text{ét}})$ and $G = \underline{\mathrm{Aut}}(T)$ as a sheaf, whose sections are given on an open U by pulling back T to U and compute its automorphisms, where for example T is a scheme. There is a natural bijection

$$\left\{ \begin{array}{c} \text{Locally split} \\ G\text{-torsors} \end{array} \right\} \rightleftharpoons \left\{ \text{forms of } T \right\}.$$

An example of this has come up before: namely that GL_n-torsors corresponded to vector bundles.