Exams 2 Review

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Contents

1 Exam 2 (Practice)

Link to PDF File

Proving uniform continuity: show

$$\|f - \tau_h f\|_1 \xrightarrow{h \to 0} 0$$

Notation: C_0 is the set of functions that vanish at infinity.

1.1 1: Fubini-Tonelli

Theorem (Fubini):

Suppose

- $f : \mathbb{R}^{n_1 + n_2} \to \mathbb{R}$ is measurable on its domain
- f is non-negative

Then for almost every $x \in \mathbb{R}^{n_1}$,

1. Every slice

$$\begin{aligned} f_x : \mathbb{R}^{n_2} &\to \mathbb{R} \\ y &\mapsto f(x, y) \end{aligned}$$

is measurable on $\mathbb{R}^{n_2}$.

2. The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\to \mathbb{R} \\ x &\mapsto \int_{\mathbb{R}^{n_2}} f_x(y)\ dy \end{aligned}$$

is measurable on $\mathbb{R}^{n_1}$

3. The function

$$G(y) = \int_{\mathbb{R}^n} F(x)\ dx$$

is measurable and

$$G(y) = \int_{\mathbb{R}^n} f = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f(x, y)\ dy\ dx$$

for any iterated version of this integral.

Corollary (Measurable Slices):

Let E be a measurable subset of $\mathbb{R}^n$. Then

- For almost every $x \in \mathbb{R}^{n_1}$, the slice $E_x \coloneqq \left\{ y \in \mathbb{R}^{n_2} \mathrel{\Big|} (x, y) \in E \right\}$ is measurable in $\mathbb{R}^{n_2}$.
- The function

$$\begin{aligned} F : \mathbb{R}^{n_1} &\to \mathbb{R} \\ x &\mapsto m(E_x) = \int_{\mathbb{R}^{n_2}} \chi_{E_x}\ dy \end{aligned}$$

is measurable and

$$m(E) = \int_{\mathbb{R}^{n_1}} m(E_x)\ dx = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \chi_{E_x}\ dy\ dx$$

$\implies$:

- Let f be measurable on $\mathbb{R}^n$.
- Then the cylinders $F(x,y) = f(x)$ and $G(x,y) = f(y)$ are both measurable on $\mathbb{R}^{n+1}$.
- Write $\mathcal{A} = \{G \leq F\} \bigcap \{G \geq 0\}$; both are measurable.

$\impliedby$:

- Let A be measurable in $\mathbb{R}^{n+1}$.
- Define $A_x = \left\{ y \in \mathbb{R} \;\middle|\; (x,y) \in \mathcal{A} \right\}$, then $m(A_x) = f(x)$.
- By the corollary, A_x is measurable set, $x \mapsto A_x$ is a measurable function, and $m(A) = \displaystyle\int f(x)\ dx$.
- Then explicitly, $f(x) = \chi_A$, which makes f a measurable function.

1.1.1 b

- Define $A_y = \left\{ x \in \mathbb{R}^n \;\middle|\; (x,y) \in A \right\}$, and notice that $A_y = \left\{ x \in \mathbb{R}^n \;\middle|\; 0 \leq y \leq f(x) \right\}$.
- By the corollary, A_y is measurable and

$$m(\mathcal{A}) = \int m\left(\mathcal{A}_y\right) dy = \int_0^y m\left(\{x \in \mathbb{R}^n \ni f(x) \geq y\}\right) dy$$

1.2 2: Convolutions and the Fourier Transform

1.2.1 a

Definition (Convolution):

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\ dy.$$

Facts:

- $f, g \in L^1 \implies f * g \in L^1$
- $f \in L^1, g \leq M \implies f * g \leq M'$ and is uniformly continuous.
- $f, g \in L^1, f \leq M, g \leq M' \implies f * g \xrightarrow{x \to \infty} 0.2$
- $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- $f \in L^1, g'$ exists, $\dfrac{\partial g}{\partial x_i}$ all bounded $\implies \dfrac{\partial}{\partial x_i}(f * g) = f * \dfrac{\partial g}{\partial x_i}$
- $f, g \in C_c^\infty \implies f * g \in C^\infty$ and $f * g \xrightarrow{x \to \infty} 0$.

1.2.2 b

Definition (Approximation to the Identity):

$$\phi(x) = e^{-\pi x^2}$$
$$\phi_t(x) = t^{-n} \phi(\frac{x}{t}).$$

Facts:

- $\int \phi = \int \phi_t = 1$
- f bounded and uniformly continuous $\implies f * \phi_t \rightrightarrows f$

Theorem (Norm Convergence of Approximate Identities):

$$\|f * \phi_t - f\|_1 \xrightarrow{t \to 0} 0.$$

Proof:

$$\begin{aligned}
\|f - f * \phi_t\|_1 &= \int f(x) - \int f(x-y)\phi_t(y)\ dydx \\
&= \int f(x) \int \phi_t(y)\ dy - \int f(x-y)\phi_t(y)\ dydx \\
&= \int \int \phi_t(y)[f(x) - f(x-y)]\ dydx \\
&=_{FT} \int \int \phi_t(y)[f(x) - f(x-y)]\ dxdy \\
&= \int \phi_t(y) \int f(x) - f(x-y)\ dxdy \\
&= \int \phi_t(y) \|f - \tau_y f\|_1 dy \\
&= \int_{y<\delta} \phi_t(y)\|f - \tau_y f\|_1 dy + \int_{y \geq \delta} \phi_t(y)\|f - \tau_y f\|_1 dy \\
&\leq \int_{y<\delta} \phi_t(y)\varepsilon + \int_{y \geq \delta} \phi_t(y) \left(\|f\|_1 + \|\tau_y f\|_1\right) dy \quad \text{by continuity in } L^1 \\
&\leq \varepsilon + 2\|f\|_1 \int_{y \geq \delta} \phi_t(y) dy \\
&\leq \varepsilon + 2\|f\|_1 \varepsilon \quad \text{since } \phi_t \text{ has small tails} \\
&\to 0 \quad \square.
\end{aligned}$$

Theorem (Convolutions Vanish at Infinity)

$$f, g \in L^1 \text{ and bounded } \implies \lim_{|x| \to \infty} (f * g)(x) = 0.$$

Proof:

- Choose $M \geq f, g$.
- By small tails, choose N such that $\int_{B_N^c} |f|, \int_{B_n^c} |g| < \varepsilon$
- Note

$$|f * g| \leq \int |f(x-y)|\ |g(y)|\ dy \coloneqq I$$

- Use $|x| \leq |x-y| + |y|$, take $|x| \geq 2N$ so either

–

$$|x-y| \geq N \implies I \leq \int_{\{x-y \geq N\}} |f(x-y)|M \; dy \leq \varepsilon M \to 0$$

–

$$|y| \geq N \implies I \leq \int_{\{y \geq N\}} M|g(y)| \; dy \leq M\varepsilon \to 0$$

□

1.2.3 c

Definition (The Fourier Transform):

$$\hat{f}(\xi) = \int f(x)e^{-2\pi i x \cdot \xi} \; dx.$$

Facts:

- *Riemann-Lebesgue lemma:* $\hat{f}$ vanishes at infinity
- $f \in L^1 \implies \hat{f}$ is bounded and uniformly continuous
- $f, \hat{f} \in L^1 \implies f$ is bounded, uniformly continuous, and vanishes at infinity
- $f \in L^1$ and $\hat{f} = 0$ almost everywhere $\implies f = 0$ almost everywhere.

Theorem (Fourier Inversion):

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(x)e^{2\pi i x \cdot \xi} d\xi.$$

Proof: Idea: Fubini-Tonelli doesn't work directly, so introduce a convergence factor, take limits, and use uniqueness of limits.

Use the following facts:

- $f, g \in L^1 \implies \displaystyle\int \hat{f}g = \int f\hat{g}.$
- $g(x) := e^{-\pi|t|^2} \implies \hat{g}(\xi) = g(\xi)$
- $g_t(x) = t^{-n}g(x/t) = t^{-n}e^{-\pi|x|^2/t^2}$
- $\hat{g}_t(x) = g(tx) = e^{-\pi t^2|x|^2}$
- $\phi(\xi) := e^{2\pi i x \cdot \xi} \; \hat{g}_t(\xi)$
- $\hat{\phi}(\xi) = \mathcal{F}(\hat{g}_t(\xi - x)) = g_t(x - \xi)$
- $\lim\limits_{t \to 0} \phi(\xi) = e^{2\pi i x \cdot \xi}$

Take the modified integral:

$$\begin{aligned}
I_t(x) &= \int \hat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \int \hat{f}(\xi)\phi(\xi) \\
&= \int f(\xi)\hat{\phi}(\xi) \\
&= \int f(\xi)\widehat{\hat{g}}(\xi - x) \\
&= \int f(\xi)g_t(x-\xi)\ d\xi \\
&= \int f(y-x)g_t(y)\ dy \quad (\xi = y - x) \\
&= (f * g_t) \\
&\to f \text{ in } L^1 \text{ as } t \to 0,
\end{aligned}$$

but we also have

$$\begin{aligned}
\lim_{t\to 0} I_t(x) &= \lim_{t\to 0}\int \hat{f}(\xi)\ e^{2\pi i x\cdot\xi}\ e^{-\pi t^2|\xi|^2} \\
&= \lim_{t\to 0}\int \hat{f}(\xi)\phi(\xi) \\
&=_{DCT} \int \hat{f}(\xi)\lim_{t\to 0}\phi(\xi) \\
&= \int \hat{f}(\xi)\ e^{2\pi i x\cdot\xi}
\end{aligned}$$

.

So

$$I_t(x) \to \int \hat{f}(\xi)\ e^{2\pi i x\cdot\xi} \ \text{ pointwise and } \ \|I_t(x) - f(x)\|_1 \to 0.$$

So there is a subsequence I_{t_n} such that $I_{t_n}(x) \to f(x)$ almost everywhere, so $f(x) = \int \hat{f}(\xi)\ e^{2\pi i x\cdot\xi}$ almost everywhere by uniqueness of limits. $\square$

1.3 3: Hilbert Spaces

1.3.1 a

Theorem (Bessel's Inequality):

$$\left\| x - \sum_{n=1}^{N} \langle x, u_n\rangle\, u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, u_n\rangle|^2$$

and thus

$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \le \|x\|^2$$

Proof: Let $S_N = \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n$

$$
\begin{aligned}
\|x - S_N\|^2 &= \langle x - S_n,\ x - S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\langle x,\ S_N \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re\left\langle x,\ \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \langle x,\ \langle x,\ u_n \rangle u_n \rangle \\
&= \|x\|^2 + \|S_N\|^2 - 2\Re \sum_{n=1}^{N} \overline{\langle x,\ u_n \rangle} \langle x,\ u_n \rangle \\
&= \|x\|^2 + \left\| \sum_{n=1}^{N} \langle x,\ u_n \rangle u_n \right\|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 + \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 - 2 \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
&= \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

And by continuity of the norm and inner product, we have

$$
\begin{aligned}
\lim_{N\to\infty} \|x - S_N\|^2 &= \lim_{N\to\infty} \|x\|^2 - \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \lim_{N\to\infty} S_N \right\|^2 &= \|x\|^2 - \lim_{N\to\infty} \sum_{n=1}^{N} |\langle x,\ u_n \rangle|^2 \\
\implies \left\| x - \sum_{n=1}^{\infty} \langle x,\ u_n \rangle u_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2.
\end{aligned}
$$

Then noting that $0 \leq \|x - S_N\|^2$, we have

$$
\begin{aligned}
0 &\leq \|x\|^2 - \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \\
\implies \sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 &\leq \|x\|^2 \quad \square.
\end{aligned}
$$

1.3.2 b

- Take $\{a_n\} \in \ell^2$, then note that $\sum |a_n|^2 < \infty \implies$ the tails vanish.
- Define $x = \lim_{N\to\infty} S_N$ where $S_N = \sum_{k=1}^{N} a_k u_k$
- $\{S_N\}$ is Cauchy and H is complete, so $x \in H$.

- By construction, $\langle x,\ u_n\rangle = \left\langle \sum_k a_k u_k,\ u_n \right\rangle = \sum_k a_k \langle u_k,\ u_n\rangle = a_n$ since the u_k are all orthogonal.
- $\|x\|^2 = \left\| \sum_k a_k u_k \right\|^2 = \sum_k \|a_k u_k\|^2 = \sum_k |a_k|^2$ by Pythagoras since the u_k are normal.

1.3.3 c

Let x and u_n be arbitrary. Then

$$\begin{aligned}
\left\langle x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle &= \langle x,\ u_n\rangle - \left\langle \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k,\ u_n \right\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle \langle x,\ u_k\rangle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle \langle u_k,\ u_n\rangle \\
&= \langle x,\ u_n\rangle - \langle x,\ u_n\rangle = 0 \\
\implies x - \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k = 0 &\quad \text{by completeness.}
\end{aligned}$$

So

$$x = \sum_{k=1}^{\infty} \langle x,\ u_k\rangle u_k \implies \|x\|^2 = \sum_{k=1}^{\infty} |\langle x,\ u_k\rangle|^2. \quad \square$$

1.4 4: Lp Spaces

p-test for integrals:

$$\begin{aligned}
\int_0^1 x^{-p} < \infty &\iff p < 1 \\
\int_1^{\infty} x^{-p} < \infty &\iff p > 1.
\end{aligned}$$

Yields a general technique: break integrals apart at $x = 1$.

Inclusions and relationships:

$$\begin{aligned}
m(X) < \infty \implies & L^{\infty} \subset L^2 \subset L^1 \\
& \ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z}) \subset \ell^{\infty}(\mathbb{Z}).
\end{aligned}$$

1.4.1 a

Theorem (Holder's Inequality):

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proof:

It suffices to show this when $\|f\|_p = \|g\|_q = 1$, since

$$\|fg\|_1 \leq \|f\|_p \|f\|_q \iff \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq 1.$$

Using $AB \leq \frac{1}{p}A^p + \frac{1}{q}B^q$, we have

$$\int |f||g| \leq \int \frac{|f|^p}{p} \frac{|g|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \quad \square.$$

Theorem (Minkowski's Inequality):

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof:

We first note

$$|f+g|^p = |f+g||f+g|^{p-1} \leq (|f| + |g|)\, |f+g|^{p-1}.$$

Note that if p, q are conjugate exponents then

$$\begin{aligned} \frac{1}{q} &= 1 - \frac{1}{p} = \frac{p-1}{p} \\ q &= \frac{p}{p-1}. \end{aligned}$$

Then taking integrals yields

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)\,|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&= \left\|f(f+g)^{p-1}\right\|_1 + \left\|g(f+g)^{p-1}\right\|_1 \\
&\leq \|f\|_p \left\|(f+g)^{p-1})\right\|_q + \|g\|_p \left\|(f+g)^{p-1})\right\|_q \\
&= \left(\|f\|_p + \|g\|_p\right) \left\|(f+g)^{p-1})\right\|_q \\
&= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^{p}\right)^{1-\frac{1}{p}} \\
&= \left(\|f\|_p + \|g\|_p\right) \frac{\int |f+g|^p}{\left(\int |f+g|^p\right)^{\frac{1}{p}}} \\
&= \left(\|f\|_p + \|g\|_p\right) \frac{\|f+g\|_p^p}{\|f+g\|_p}
\end{aligned}$$

and canceling common terms yields

$$\begin{aligned}
1 &\leq \left(\|f\|_p + \|g\|_p\right) \frac{1}{\|f+g\|_p} \\
&\implies \|f+g\|_p \leq \|f\|_p + \|g\|_p \quad \square.
\end{aligned}$$

1.4.2 c

Definition (Infinity Norm):

$$\begin{aligned}
L^\infty(X) = \{f : X \to \mathbb{C} \ni \|f\|_\infty < \infty\} \\
\text{where} \\
\|f\|_\infty = \inf_{\alpha \geq 0} \{\alpha \ni m\{|f| \geq \alpha\} = 0\}.
\end{aligned}$$

Theorem:

$$m(X) < \infty \implies \lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Proof: Let $M = \|f\|_\infty$. For any $L < M$, let $S = \{|f| \geq L\}$. Then $m(S) > 0$ and

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\geq \left(\int_S |f|^p\right)^{\frac{1}{p}} \\
&\geq L\ m(S)^{\frac{1}{p}} \xrightarrow{p\to\infty} L \\
&\implies \liminf_p \|f\|_p \geq M.
\end{aligned}$$

We also have

$$\begin{aligned}
\|f\|_p &= \left(\int_X |f|^p\right)^{\frac{1}{p}} \\
&\leq \left(\int_X M^p\right)^{\frac{1}{p}} \\
&= M\ m(X)^{\frac{1}{p}} \xrightarrow{p\to\infty} M \\
&\implies \limsup_p \|f\|_p \leq M \quad \square.
\end{aligned}$$

Note: this doesn't help with this problem at all.

Solution:

$$\begin{aligned}
\int f^p &= \int_{x\leq 1} f^p + \int_{x=1} f^p + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + \int_{x=1} 1 + \int_{x\geq 1} f^p \\
&= \int_{x\leq 1} f^p + m(\{f=1\}) + \int_{x\geq 1} f^p \\
&\xrightarrow{p\to\infty} 0 + m(\{f=1\}) + \begin{cases} 0 & m(\{x\geq 1\}) = 0 \\ \infty & m(\{x\geq 1\}) > 0. \end{cases}
\end{aligned}$$

1.5 5: Dual Spaces

Definition: A map $L : X \to \mathbb{C}$ is a linear functional iff

$$L(\alpha\mathbf{x} + \mathbf{y}) = \alpha L(\mathbf{x}) + L(\mathbf{y}).$$

Theorem (Riesz Representation for Hilbert Spaces): If Λ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$\forall x \in H, \quad \Lambda(x) = \langle x,\ y\rangle.$$

Proof:

- Define $M \coloneqq \ker \Lambda$.
- Then M is a closed subspace and so $H = M \oplus M^\perp$
- There is some $z \in M^\perp$ such that $\|z\| = 1$.
- Set $u \coloneqq \Lambda(x)z - \Lambda(z)x$
- Check

$$\Lambda(u) = \Lambda(\Lambda(x)z - \Lambda(z)x) = \Lambda(x)\Lambda(z) - \Lambda(z)\Lambda(x) = 0 \implies u \in M$$

- Compute

$$\begin{aligned}
0 &= \langle u,\ z\rangle \\
&= \langle \Lambda(x)z - \Lambda(z)x,\ z\rangle \\
&= \langle \Lambda(x)z,\ z\rangle - \langle \Lambda(z)x,\ z\rangle \\
&= \Lambda(x)\langle z,\ z\rangle - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x)\|z\|^2 - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \Lambda(z)\langle x,\ z\rangle \\
&= \Lambda(x) - \left\langle x,\ \overline{\Lambda(z)}z\right\rangle,
\end{aligned}$$

- Choose $y \coloneqq \overline{\Lambda(z)}z$.
- Check uniqueness:

$$\begin{aligned}
\langle x,\ y\rangle &= \langle x,\ y'\rangle \quad \forall x \\
\implies \langle x,\ y - y'\rangle &= 0 \quad \forall x \\
\implies \langle y - y',\ y - y'\rangle &= 0 \\
\implies \|y - y'\| &= 0 \\
\implies y - y' &= \mathbf{0} \implies y = y'.
\end{aligned}$$

1.5.1 b

Theorem (Continuous iff Bounded): Let $L : X \to \mathbb{C}$ be a linear functional, then the following are equivalent:

1. L is continuous
2. L is continuous at zero
3. L is bounded, i.e. $\exists c \geq 0 \ni |L(x)| \leq c\|x\|$ for all $x \in H$

$2 \implies 3$: Choose $\delta < 1$ such that

$$\|x\| \leq \delta \implies |L(x)| < 1.$$

Then

$$\begin{aligned}|L(x)| &= \left| L\left(\frac{\|x\|}{\delta}\frac{\delta}{\|x\|}x\right)\right| \\ &= \frac{\|x\|}{\delta}\left| L\left(\delta\frac{x}{\|x\|}\right)\right| \\ &\leq \frac{\|x\|}{\delta}1,\end{aligned}$$

so we can take $c = \dfrac{1}{\delta}$. $\square$

$3 \implies 1$:

We have $|L(x-y)| \leq c\|x-y\|$, so given $\varepsilon \geq 0$ simply choose $\delta = \dfrac{\varepsilon}{c}$.

1.5.2 c

Definition (Dual Space):

$$X^\vee := \{L : X \to \mathbb{C} \ni L \text{ is continuous } \}$$

Definition (Operator Norm):

$$\|L\|_{X^\vee} := \sup_{\substack{x\in X \\ \|x\|=1}} |L(x)|$$

Theorem: (Operator Norm is a Norm)

Proof: The only nontrivial property is the triangle inequality, but

$$\|L_1 + L_2\| = \sup |L_1(x) + L_2(x)| \leq \sup L_1(x) + \sup L_2(x) = \|L_1\| + \|L_2\|.$$

Theorem (Completeness in Operator Norm):

$X^\vee$ equipped with the operator norm is a Banach space.

Proof:

- Let $\{L_n\}$ be Cauchy in $X^\vee$.
- Then for all $x \in C$, $\{L_n(x)\} \subset \mathbb{C}$ is Cauchy and converges to something denoted $L(x)$.
- Need to show L is continuous and $\|L_n - L\| \to 0$.
- Since $\{L_n\}$ is Cauchy in $X^\vee$, choose N large enough so that

$$n, m \geq N \implies \|L_n - L_m\| < \varepsilon \implies |L_m(x) - L_n(x)| < \varepsilon \quad \forall x \ni \|x\| = 1.$$

- Take $n \to \infty$ to obtain

$$\begin{aligned} m \geq N &\implies |L_m(x) - L(x)| < \varepsilon \quad \forall x \ni \|x\| = 1 \\ &\implies \|L_m - L\| < \varepsilon \to 0.\end{aligned}$$

- Continuity:

$$\begin{aligned}
|L(x)| &= |L(x) - L_n(x) + L_n(x)| \\
&\leq |L(x) - L_n(x)| + |L_n(x)| \\
&\leq \varepsilon \|x\| + c\|x\| \\
&= (\varepsilon + c)\|x\| \quad \square.
\end{aligned}$$

2 Exam 2 (2018)

Link to PDF File

3 Exam 2 (2014)

Link to PDF File

4 Qual: Fall 2019

4.1 1

See phone photo?

4.2 2

DCT?

4.3 3

"Follow your nose."

4.4 4

See Problem Set 8.

Bessel's Inequality: For any orthonormal set in a Hilbert space (not necessarily a basis), we have

$$\sum_{n=1}^{\infty} |\langle x,\ u_n \rangle|^2 \leq \|x\|^2$$

Proof:

$$0 \leq \left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k \right\|^2$$

Corollary (Parseval's Identity): If span $\{u_n\}$ is dense in $\mathcal{H}$, so it is a basis, then this is an equality.

Riesz-Fischer: Let $U = \{u_n\}_{n=1}^{\infty}$ be an orthonormal set (not necessarily a basis), then

1. There is an isometric surjection

$$\begin{aligned}\mathcal{H} &\to \ell^2(\mathbb{N}) \\ \mathbf{x} &\mapsto \{\langle \mathbf{x},\ \mathbf{u}_n\rangle\}_{n=1}^{\infty}\end{aligned}$$

i.e. if $\{a_n\} \in \ell^2(\mathbb{N})$, so $\sum |a_n|^2 < \infty$, then there exists a $\mathbf{x} \in \mathcal{H}$ such that

$$a_n = \langle \mathbf{x},\ \mathbf{u}_n\rangle \quad \forall n.$$

2. $\mathbf{x}$ can be chosen such that

$$\|\mathbf{x}\|^2 = \sum |a_n|^2$$

> Note: the choice of $\mathbf{x}$ is unique $\iff$ $\{u_n\}$ is **complete**, i.e. $\langle \mathbf{x},\ \mathbf{u}_n\rangle = 0$ for all n implies $\mathbf{x} = \mathbf{0}$.

Proof:

- Given $\{a_n\}$, define $S_N = \sum^{N} a_n \mathbf{u}_n$.
- S_N is Cauchy in $\mathcal{H}$ and so $S_N \to \mathbf{x}$ for some $\mathbf{x} \in \mathcal{H}$.
- $\langle x,\ u_n\rangle = \langle x - S_N,\ u_n\rangle + \langle S_N,\ u_n\rangle \to a_n$
- By construction, $\|x - S_N\|^2 = \|x\|^2 - \sum^{N} |a_n|^2 \to 0$, so $\|x\|^2 = \sum^{\infty} |a_n|^2$.

4.5 5

See Problem Set 5.

Heine-Cantor theorem: Every continuous function on a compact set is uniformly continuous.

Uniform continuity:

$$\begin{aligned} &\forall \varepsilon \quad \exists \delta(\varepsilon) \,\Big|\quad \forall x, y, \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\ \iff &\forall \varepsilon \quad \exists \delta(\varepsilon) \,\Big|\quad \forall x, y, \quad |y| < \delta \implies |f(x - y) - f(y)| < \varepsilon \end{aligned}$$

.

Fubini-Tonelli interchange of integrals, where the change of bounds becomes very important.

Continuity in L^1:

$$\lim_{y \to 0} \|\tau_y f - f\|_1 = 0.$$