

# Problem Sets

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## 1 1.1

### 1.1 a

If  $M \in \mathcal{O}$  and  $[\lambda] = \lambda + \Lambda_r$  is any coset of  $\mathfrak{h}^\vee / \Lambda_r$ , let  $M^{[\lambda]}$  be the sum of weight spaces  $M_\mu$  for which  $\mu \in [\lambda]$ . Prove that  $M^{[\lambda]}$  is a  $U(\mathfrak{g})$ -submodule of  $M$  and that  $M$  is the direct sum of finitely many such submodules.

### 1.2 b

Deduce that the weights of an indecomposable module  $M \in \mathcal{O}$  lie in a single coset of  $\mathfrak{h}^\vee / \Lambda_r$ .

## 2 1.3\*

Show that  $M(\lambda)$  has the following property: for any  $M \in \mathcal{O}$ ,

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) = \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M) \cong \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M),$$

where  $\mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}}$  is the restriction functor.

Hint: use the universal mapping property of tensor products.

### 3 Relevant information (?):

#### 3.1 1

- $\mathfrak{h} \leq \mathfrak{g}$  is the Cartan subalgebra.
  - In finite-dimensional setting: maximal toral
  - Nilpotent subalgebra, i.e. LCS terminates, i.e.  $\text{ad}_h = [h, \cdot]$  is a nilpotent operator so  $\text{ad}_h^n = 0$  for some  $n$ .
- $\Phi \subset \mathfrak{h}^\vee$  is the root system of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ .
  - Each  $\alpha \in \Phi$  is a root
  - Each root  $\alpha$  has a corresponding root space  $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [hx] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ .
- $\Lambda_r := \text{span}_{\mathbb{Z}} \{\alpha \in \Phi\} \subset \mathbb{C}^n$  is the root lattice.
- $M_\mu := \{v \in M \mid h \cdot v = \mu(h)v \ \forall h \in \mathfrak{h}\}$  is the weight space for  $\mu$ .
- 

$M \in \mathcal{O} \implies$

- $M$  is finitely generated as a  $U(\mathfrak{g})$ -module.
- $M$  is a weight module, so  $M = \bigoplus_{\lambda \in \mathfrak{h}^\vee} M_\lambda$
- For every  $v \in M$ ,  $U(\mathfrak{n}) \cdot v$  is finite-dimensional

#### 3.2 2

$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$  where  $\mathfrak{b} \leq \mathfrak{g}$  is a fixed Borel subalgebra corresponding to a choice of positive roots, and  $\mathbb{C}_\lambda$  is the 1-dimensional  $\mathfrak{b}$ -module defined for any  $\lambda \in \mathfrak{h}^\vee$  by the fact that  $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{h}$  and thus  $\mathfrak{n} \curvearrowright \mathfrak{h}$  can be taken to be a trivial action. The induction functor is given by  $\text{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})}(\cdot)$ .

The restriction functor is given by  $\text{Res}_{\mathfrak{b}}^{\mathfrak{g}}(\cdot) = ?$

Frobenius Reciprocity for groups looks like

$$\begin{aligned} \text{hom}_{k[G]}(k[G] \otimes_{k[H]} V, W) &\rightarrow \text{hom}_{k[H]}(V, W) \\ \lambda &\mapsto 1 \otimes (\cdot) = (v \mapsto \lambda(1 \otimes v)) \\ (g \otimes v \mapsto g \cdot f(v)) &\leftarrow f. \end{aligned}$$