

Homotopy Groups of Spheres

Graduate Student Seminar

D. Zack Garza

April 2020

Introduction

Homotopy
Groups of
Spheres

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Outline

- Homotopy as a means of classification somewhere between homeomorphism and cobordism
- Comparison to homology
- Higher homotopy groups of spheres exist
- Homotopy groups of spheres govern gluing of CW complexes
- CW complexes fully capture that homotopy category of spaces
- There are concrete topological constructions of many important algebraic operations at the level of spaces (quotients, tensor products)
- Relation to framed cobordism?
- “Measuring stick” for current tools, similar to special values of L-functions
- Serre’s computation

Intuition

Homotopies of paths:

Spheres

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Setup

- Defining $\pi_k(X) = [S^k, X]$, the simplest objects to investigate:
 $X = S^n$
- Can consider the bigraded group $\pi_S := [S^k, S^n]$:

| | | $\pi_k(S^n)$ | | | | | | | | | | |
|-------|--|--------------|---|---|---|---|---|---|---|---|----|-----|
| | | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
| $n=1$ | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | |
| 9 | | | | | | | | | | | | |
| 10 | | | | | | | | | | | | |
| : | | | | | | | | | | | | |

$n = k$: Stabilization

- Theorem (1937, Freudenthal): For $k \gg 0$,
 $[\Sigma^k X, \Sigma^k Y] \cong [\Sigma^{k+1} X, \Sigma^{k+1} Y]$.
- Use the fact that $\Sigma S^k \cong S^{k+1}$, then in some *stable range*
 $\pi_{n+k} S^n \cong \pi_{n+k+1} S^{n+1}$