Title

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References: `https://www.daniellitt.com/tale-cohomology`

Prerequisites:

- Homological Algebra
  - Abelian Categories
  - Derived Functors
  - Spectral Sequences (just exposure!)
- Sheaf theory and sheaf cohomology
- Schemes (Hartshorne II and III)

Outline/Goals:

- Basics of etale cohomology
  - Etale morphism
  - Grothendieck topologies
  - The etale topology
  - Etale cohomology and the basis theorems
  - Etale cohomology of curves
  - Comparison theorems to singular cohomology
  - Focused on the case where coefficients are a constructible sheaf.
- Prove the Weil Conjectures (more than one proof)
  - Proving the Riemann Hypothesis for varieties over finite fields

  One of the greatest pieces of 20th century mathematics!
- Topics
  - Weil 2 (Strengthening of RH, used in practice)
  - Formality of algebraic varieties (topological features unique to varieties)
  - Other things (monodromy, refer to Katz' AWS notes)

What is Etale Cohomology? Suppose $X/\mathbb{C}$ is a quasiprojective variety: a finite type separated integral $\mathbb{C}$-scheme.

If you take the complex points, it naturally has the structure of a complex analytic space $X(\mathbb{C})^{\mathrm{an}}$: you can give it the Euclidean topology, which is much finer than the Zariski topology.

For a nice topological space, we can associate the singular cohomology $H^i(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Z})$, which satisfies several nice properties:

- Finitely generated $\mathbb{Z}$-modules
- Extra Hodge structure when tensored up to $\mathbb{C}$ (same as $\mathbb{C}$ coefficients)
- Cycle classes (i.e. associate to a subvariety a class in cohomology)

Goal of etale cohomology: do something similar for much more general "nice" schemes. Note that some of these properties are special to complex varieties

E.g. finitely generated: not true for a random topological space

We'll associate X a "nice scheme" $\rightsquigarrow H^i(X_{\mathrm{et}}, \mathbb{Z}/\ell^n\mathbb{Z})$. Take the inverse limit over all n to obtain the ℓ-adic cohomology $H^i(X_{\mathrm{et}}, \mathbb{Z}_\ell)$. You can tensor with $\mathbb{Q}$ to get something with $\mathbb{Q}_\ell$ coefficients. And as in singular cohomology, you can a "twisted coefficient system".

What are nice schemes:

- $X = \operatorname{Spec} \mathcal{O}_k$, the ring of integers over a number field.
- X a variety over an algebraically closed field
  - Typical, most analogous to taking a variety over $\mathbb{C}$.
- X a variety over a non-algebraically closed field

Some comparisons between the last two cases:

- For $\mathbb{C}$- variety, H^i_{sing} will vanish above $i = 2d$.
- Over a finite field, H^i will vanish for $i > 2d + 1$ but generally not vanish for $i = 2d + 1$.

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