

Title

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1.1 Frobenius Kernels

Let $\text{char}(k)p > 0$ and let G be an algebraic group scheme. We have a Frobenius map $F : G \rightarrow G$ given by $F((x_{ij})) = (x_{ij}^p)$, which we can iterate to get F^r for $r \in \mathbb{N}$. Setting $G_r = \ker F^r$ the r th Frobenius kernel, we get a normal series of group schemes

$$G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G.$$

There is an associated chain of finite dimensional Hopf algebras

$$\text{Dist}(G_1) \leq \text{Dist}(G_2) \leq \cdots \leq \text{Dist}(G).$$

Then $k[G]^\vee = \text{Dist}(G_r)$, and we get an equivalence of representations for G_r to representations for $\text{Dist}(G_r)$.

A special case will be when G is a reductive algebraic group scheme. We'll start by finding a basis for $\text{Dist}(G_r)$.

Recall the PBW theorem: we have a basis for \mathfrak{g} given by

$$\begin{aligned} & \{x_\alpha \mid \alpha \in \Phi^+\} \text{ Positive root vectors} \\ & \{h_i \mid i = 1, \dots, n\} \text{ A basis for } \mathfrak{t} \\ & \{x_\alpha \mid \alpha \in \Phi^-\} \text{ Negative root vectors} \end{aligned}$$

We can then obtain a basis for $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \left\langle \prod_{\alpha \in \Phi^+} x_{\alpha}^{n(\alpha)} \prod_{i=1}^n h_i^{k_i} \prod_{\alpha \in \Phi^+} x_{-\alpha}^{m(\alpha)} \right\rangle.$$

We can similarly obtain a basis for the distribution algebra

$$\text{Dist}(G) = \left\langle \prod_{\alpha \in \Phi^+} \frac{x_{\alpha}^{n(\alpha)}}{n!} \prod_{i=1}^n \binom{h_i}{k_i} \prod_{\alpha \in \Phi^+} \frac{x_{-\alpha}^{m(\alpha)}}{n!} \right\rangle,$$

and we can similar get $\text{Dist}(G_r)$ by restricting to $0 \leq n(\alpha), k_i, m(\alpha) \leq p^r - 1$. Above the k_i are allowed to be any integers. This yields a triangular decomposition

$$\text{Dist}(G_r) = \text{Dist}(U_r^+) \text{Dist}(T_r) \text{Dist}(U_r^-),$$

where we'll denote the first two terms $\text{Dist}(B_r^+)$ and the last two as $\text{Dist}(B_r)$.

1.2 Induced and Coinduced Modules

Goal: Classify simple G_r -modules. We know the classification of simple G -modules, so we'll follow similar reasoning. We started by realizing $L(\lambda) \hookrightarrow \text{Ind}_B^G \lambda$ as a submodule (the socle) of some “universal” module.

Let M be a B_r -module, we can then define

$$\text{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r},$$

where we're now taking the B_r -invariants. We get a decomposition as vector spaces,

$$k[G_r] = k[U_r^+] \otimes_k k[B_r]$$

and thus an isomorphism

$$\text{Ind}_{B_r}^{G_r} M = (k[G_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes (k[B_r] \otimes M)^{B_r} \cong k[U_r^+] \otimes M$$

since $k[B_r] \otimes M \cong \text{Ind}_{B_r}^{B_r} M \cong M$.