

# Problem Set 1

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## Contents

Source: Section 1 of Gathmann

**Exercise 0.1** (Gathmann 1.19).

Prove that every affine variety  $X \subset \mathbb{A}^n/k$  consisting of only finitely many points can be written as the zero locus of  $n$  polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in  $X$  have different  $x_1$ -coordinates.

**Solution:**

Let  $X = \{p_1, \dots, p_d\} = \{p_j\}_{j=1}^d$ , where each  $p_j$  can be written in coordinates  $p_j := [p_j^1, p_j^2, \dots, p_j^n]$ .

**Claim:** Without loss of generality, we can assume all of the first components  $\{p_j^1\}_{j=1}^d$  are distinct.

Todo: by some change of basis?

We will use the following fact

**Theorem 0.1 (Lagrange).**

Given a set of  $d$  points  $\{(x_i, y_i)\}_{i=1}^d$  with all  $x_i$  distinct, there exists a unique polynomial of degree  $d$  in  $f \in k[x]$  such that  $f(x_i) = y_i$  for every  $i$ .

This can be explicitly given by

$$\tilde{f}(x) = \sum_{i=1}^d y_i \left( \prod_{\substack{0 \leq m \leq d \\ m \neq i}} \left( \frac{x - x_m}{x_i - x_m} \right) \right).$$

Equivalently, there is a polynomial  $f$  defined by  $f(x_i) = \tilde{f}(x_i) - y_i$  of degree  $d$  whose roots are precisely the  $x_i$ .

Using this theorem, we define a system of  $n$  polynomials in the following way:

- Define  $f_1 \in k[x_1] \subseteq k[x_1, \dots, x_n]$  by

$$f_1(x) = \prod_{i=1}^d (x - p_i^1).$$

Then the roots of  $f_1$  are precisely the first components of the points  $p$ .

- Define  $f_2 \in k[x_1, x_2] \subseteq k[x_1, \dots, x_n]$  by considering the ordered pairs

$$\{(x_1, x_2) = (p_j^1, p_j^2)\},$$

then taking the unique Lagrange interpolating polynomial satisfying  $f_2(p_j^1) = p_j^2$  for all  $1 \leq j \leq d$ . Then set  $f_2 := \tilde{f}(x_1) - x_2$ .

- Define  $f_3 \in k[x_1, x_3] \subseteq k[x_1, \dots, x_n]$  by considering the ordered pairs

$$\{(x_1, x_3) = (p_j^1, p_j^3)\},$$

then taking the unique Lagrange interpolating polynomial  $\tilde{f}$  satisfying  $\tilde{f}_2(p_j^1) = p_j^3$  for all  $1 \leq j \leq d$ . Then set  $f_3 := \tilde{f}(x_1) - x_3$ .

• ...

Continuing in this way up to  $f_n \in k[x_1, x_n]$  yields a system of  $n$  polynomials.

**Claim:**  $V(f_1, \dots, f_n) = X$ .

$X \subset V(f_1, \dots, f_n)$ : This follows by construction; letting  $p_j \in X$  be arbitrary, we find that

$$f_1(p_j) = \prod_{i=1}^d (p_j^1 - p_i^1) = (p_j^1 - p_j^1) \prod_{\substack{i \leq d \\ i \neq j}} (p_j^1 - p_i^1) = 0.$$

**Exercise 0.2** (Gathmann 1.21).

Determine  $\sqrt{I}$  for

$$I := \langle x_1^3 - x_2^6, x_1 x_2 - x_2^3 \rangle \subseteq \mathbb{C}[x_1, x_2].$$

**Solution:**

For notational purposes, let  $\mathcal{I}, \mathcal{V}$  denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider  $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$ , the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 &= 0 \\ xy - y^3 &= 0. \end{cases}$$

In the second equation, we have  $(x - y^2)y = 0$ , and since  $\mathbb{C}[x, y]$  is an integral domain, one term must be zero.

1. If  $y = 0$ , then  $x^3 = 0 \implies x = 0$ , and thus  $(0, 0) \in \mathcal{V}(I)$  ,i.e. the origin is contained in this vanishing locus.

2. Otherwise, if  $x - y^2 = 0$ , then  $x = y^2$ , with no further conditions coming from the first equation. So

$$P := \{(t^2, t) \mid t \in \mathbb{C}\} \subset \mathcal{V}(I).$$

The corresponding ideal

Since the origin is in the latter set, this simplifies to  $P = \mathcal{V}(I)$ , and so taking the ideal generated by  $P$  yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \langle y - x^2 \rangle \in \mathbb{C}[x, y]$$

and thus  $\sqrt{I} = \langle y - x^2 \rangle$ .

**Exercise 0.3** (Gathmann 1.22).

Let  $X \subset \mathbb{A}^3/k$  be the union of the three coordinate axes. Compute generators for the ideal  $I(X)$  and show that it can not be generated by fewer than 3 elements.

**Solution:**

**Claim:**

$$I(X) = \langle x_1x_2, x_1x_3, x_2x_3 \rangle.$$

**Exercise 0.4** (Gathmann 1.23: Relative Nullstellensatz).

Let  $Y \subset \mathbb{A}^n/k$  be an affine variety and define  $A(Y)$  by the quotient

$$\pi : k[x_1, \dots, x_n] \longrightarrow A(Y) := k[x_1, \dots, x_n]/I(Y).$$

- Show that  $V_Y(J) = V(\pi^{-1}(J))$  for every  $J \trianglelefteq A(Y)$ .
- Show that  $\pi^{-1}(I_Y(X)) = I(X)$  for every affine subvariety  $X \subseteq Y$ .
- Using the fact that  $I(V(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq k[x_1, \dots, x_n]$ , deduce that  $I_Y(V_Y(J)) \subset \sqrt{J}$  for every  $J \trianglelefteq A(Y)$ .

Conclude that there is an inclusion-reversing bijection

$$\left\{ \begin{array}{c} \text{Affine subvarieties} \\ \text{of } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Radical ideals} \\ \text{in } A(Y) \end{array} \right\}.$$

**Exercise 0.5** (Extra).

Let  $J \trianglelefteq k[x_1, \dots, x_n]$  be an ideal, and find a counterexample to  $I(V(J)) = \sqrt{J}$  when  $k$  is not algebraically closed.