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1.1 Review of Last Time

Suppose we have two weights in the same facet, i.e. they're in the same stabilizer under the action of the affine Weyl group:

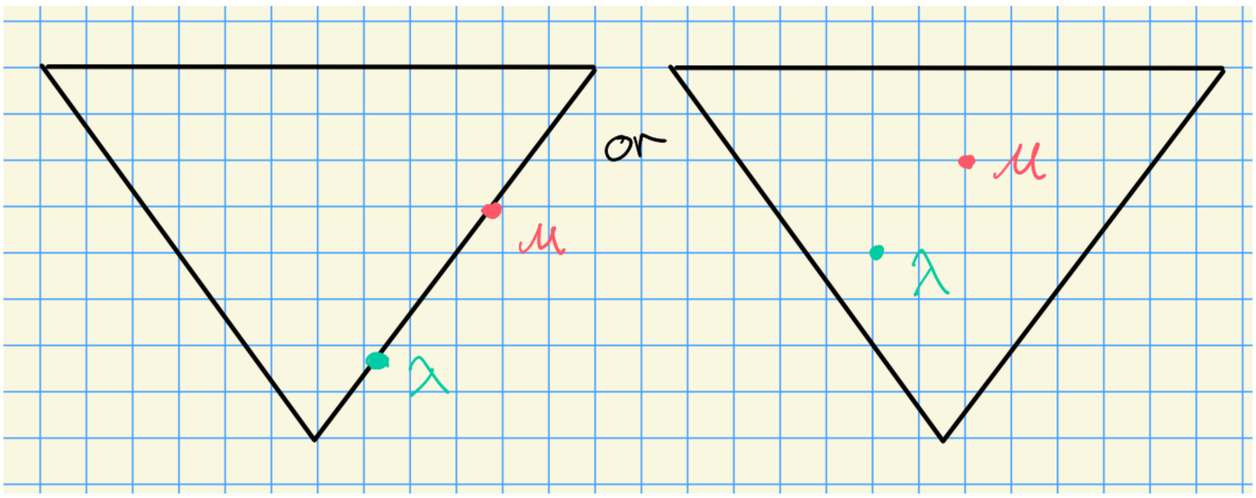

Figure 1: Weights in the same facet

We had a theorem: if λ, μ are in the same facet, then $\mathcal{B}_\lambda \cong \mathcal{B}_\mu$ is an equivalence of categories, where the map is via the translation functors.

1.2 Description of $T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big)$

We can write

$$\begin{aligned}
T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big) &= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes \mathrm{pr}_\lambda\Big(H^i(w\cdot\lambda)\Big)\Big) \\
&= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes H^i(w\cdot\lambda)\Big) \\
&= \mathrm{pr}_\mu\Big(L(\nu_1)\otimes R^i\,\mathrm{Ind}_B^G\, w\cdot\lambda\Big) \\
&= \mathrm{pr}_\mu\Big(R^i\,\mathrm{Ind}_B^G\,(L(\nu_1)\otimes w\cdot\lambda)\Big).
\end{aligned}$$

Take a composition series by B-modules of $L(\nu_1)\otimes w\cdot\lambda$, say

$$0=M_0\subseteq M_1\cdots\subseteq M_r=L(\nu_1)\otimes w\cdot\lambda.$$

where $M_j/M_{j-1}\cong\lambda+j+w\cdot\lambda$ and $\lambda_j<\lambda_{j'}\implies j<j'$, i.e. we can order them in a decreasing way.

Consider the SES

$$0\longrightarrow M_{j-1}\longrightarrow M_j\longrightarrow M_j/M_{j-1}\longrightarrow 0$$

where applying $\mathrm{pr}_\mu(\,\cdot\,)$ induces the LES

$$\cdots\longrightarrow \mathrm{pr}_\mu M_{j-1}\longrightarrow \mathrm{pr}_\mu M_j\longrightarrow \mathrm{pr}_\mu(M_j/M_{j-1})\longrightarrow\cdots$$

We know that

$$\mathrm{pr}_\mu H^i(\lambda_j+w\cdot\lambda)=\begin{cases}H^i(\lambda_j+w\cdot\lambda) & \lambda+j+w\cdot\lambda\in W_p\cdot\mu\\ 0 & \text{else}\end{cases},$$

i.e. this projection is the identity for weights linked to μ and zero otherwise. We also have

$$\mathrm{pr}_\mu H^i(M_r)=T_\lambda^\mu H^i(w\cdot\lambda).$$

Theorem 1.2.1 ***(?).***
Let $\lambda,\mu\in\overline{C}_{\mathbb{Z}}$ and F be a facet with $\lambda\in F$. If $\mu\in\overline{F}$, then we have

$$T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big)=H^i(w\cdot\mu)\qquad\forall w\in W_p.$$

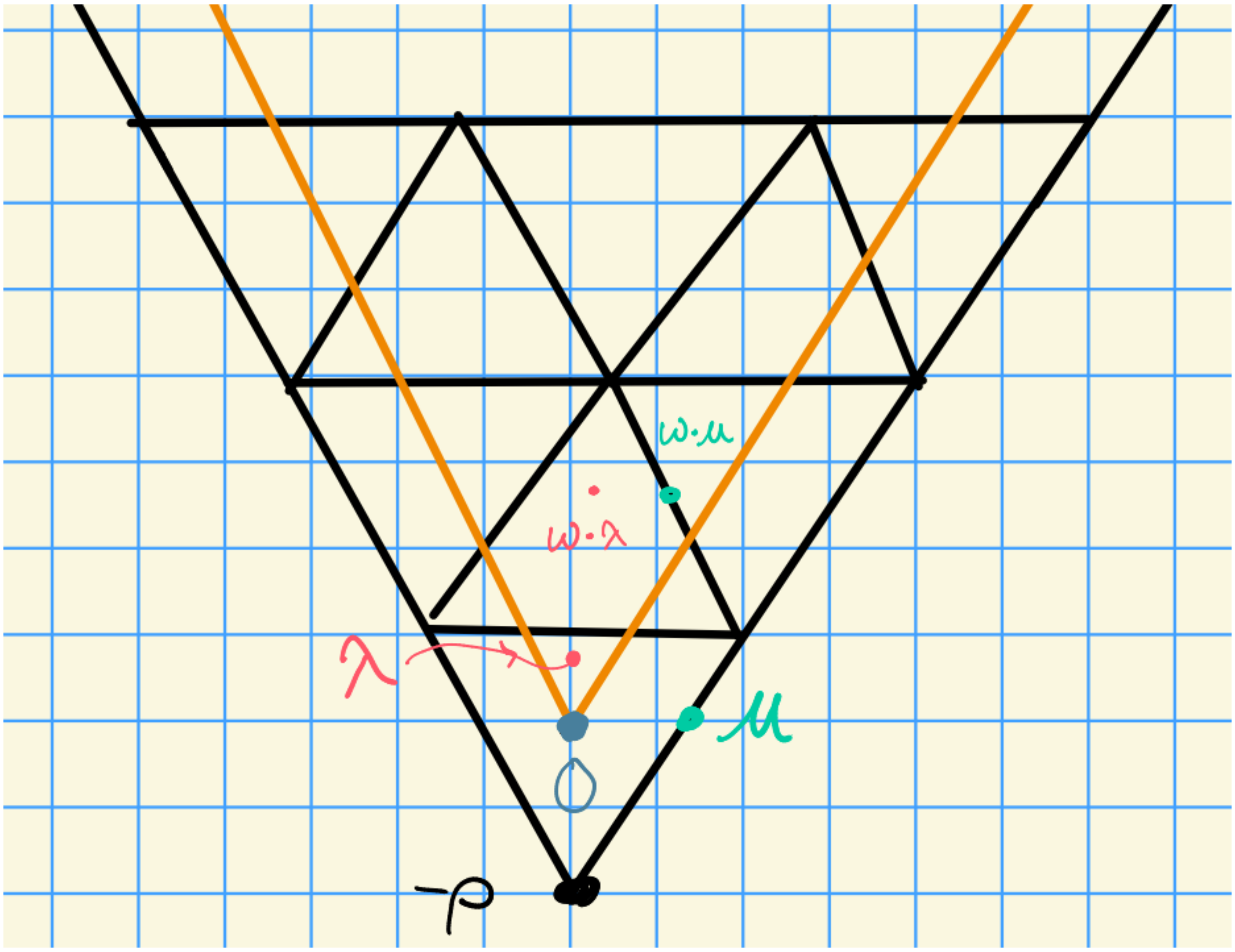

Figure 2: Image

Example 1.2.1 (?).

Here consider $H_0(\lambda) \xrightarrow{T_\lambda^\mu} H_0(\mu) = 0$, since μ is outside of the dominant region (in orange.) We also have $H^0(w\cdot\lambda) \to H^0(w\cdot\mu) \neq 0$, since this falls *into* the dominant region.

Proof (?).
Let $\lambda \in F$ and $\mu \in \overline{F}$. Then $\mathrm{Stab}_{W_p}(\lambda) \subseteq \mathrm{Stab}_{W_p}(\mu)$. By a previous technical lemma, we had a formula for computing $\operatorname{ch} T_\lambda^\mu V$, which involved considering

$$w_1 \in \frac{\mathrm{Stab}_{W_p}(\lambda)}{\mathrm{Stab}_{W_p}(\lambda) \cap \mathrm{Stab}_{W_p}(\mu)}.$$

In this case, we get $w_1 = \mathrm{id}$, since the top and bottom are equal.
By that lemma, there exists a unique ℓ such that $w\cdot\lambda + \lambda_\ell \in W_p\cdot\mu$, where λ_ℓ is a weight of $L(\nu_1)$. From the LES, we have

$$\cdots \longrightarrow \mathrm{pr}_\mu M_{j-1} \longrightarrow \mathrm{pr}_\mu M_j \longrightarrow \mathrm{pr}_\mu(M_j/M_{j-1}) = \lambda_j + w\cdot\lambda \longrightarrow \cdots$$

where the last term will only be nonzero in restricted cases. We can thus conclude that

$$\mathrm{pr}_\mu(H^i(M_j)) = \begin{cases} 0 & j < \ell \\ H^i(w\cdot\mu) & j \geq \ell. \end{cases}$$

Setting $j = r$, we have

$$T_\lambda^\mu\Big(H^i(w\cdot\lambda)\Big) = \mathrm{pr}_\mu H^j(M_r) = H^i(w\cdot\mu).$$

■

Suppose $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\mu \in C_{\mathbb{Z}}$. What happens when you translate λ (blue) off of a wall? $T_\lambda^\mu\Big(H^0(w\cdot\lambda)\Big)$ has a filtration with factors $H^0(w_1\cdot\mu)$ and $H^0(w_2\cdot\mu)$ (shown in green).

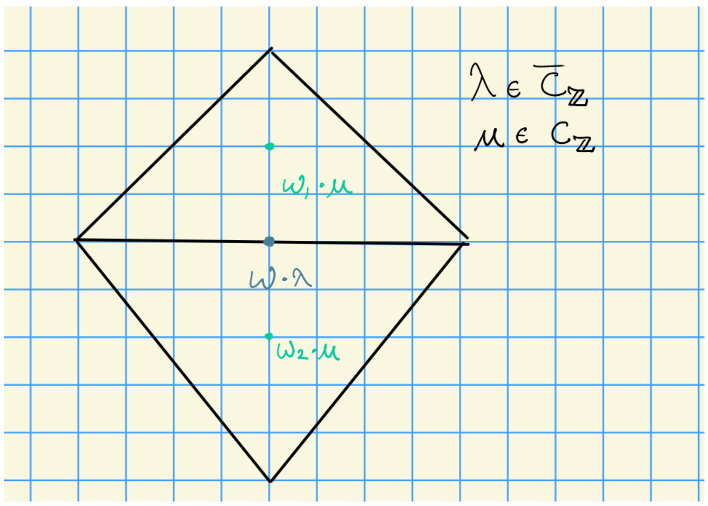

Figure 3: Filtration coming from translating off of a wall

If $w\lambda$ is a vertex with $\mu \in C_{\mathbb{Z}}$, then $T_\lambda^\mu(H^0(w\cdot\lambda))$ has six factors:

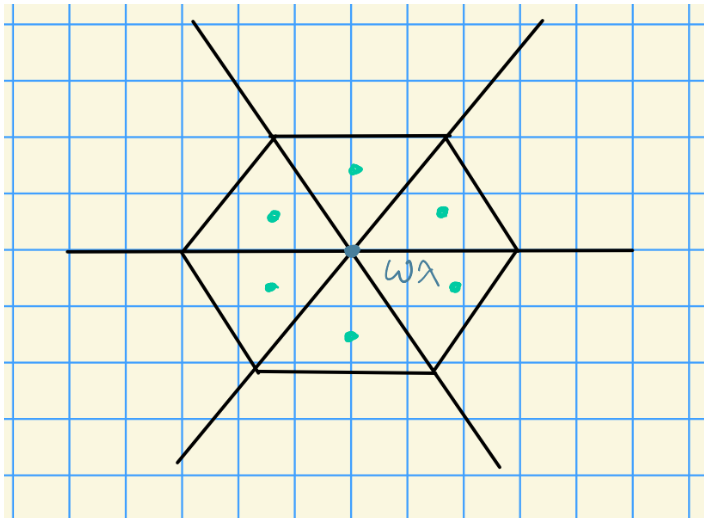

Figure 4: Weight where the translation has six factors

Proposition 1.2.1 ***(?).***
Suppose $\lambda \in \overline{C}_{\mathbb{Z}}$ and $\mu \in C_{\mathbb{Z}}$, and let $w \in W_p$ where $w \cdot \lambda \in X(T)_+$. Then $T_\lambda^\mu(H^0(w\cdot\lambda))$ has a filtration such that all of the composition factors are of the form $H^0(ww_1\cdot\mu)$ where $w_1 \in \mathrm{Stab}_{W_p}(\lambda)$ and each of the factors occurs at most once.

Recall that $\widehat{F}$ denotes the *upper closure.* :::{.proposition title="?"} Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$ be in the bottom alcove, where $\mu \in \overline{F}_1$ but $\lambda \in F_1$. Let F be the facet containing $w\cdot\lambda$, then

$$T_\lambda^\mu(L(w\cdot\lambda)) = \begin{cases} L(w\cdot\mu) & w\cdot\mu \in \widehat{F} \\ 0 & \text{else.} \end{cases}$$

:::

Example 1.2.2 (?)**.**
In this situation, we have $T_\lambda^\mu(L(\lambda)) = 0$:

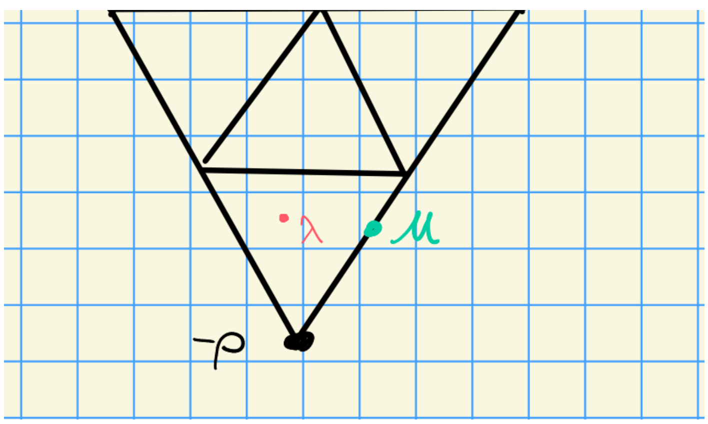

Figure 5: Image

If instead $\mu \in \widehat{C}_{\mathbb{Z}}$, we have $T_\lambda^\mu(L(\lambda)) = L(\mu)$:

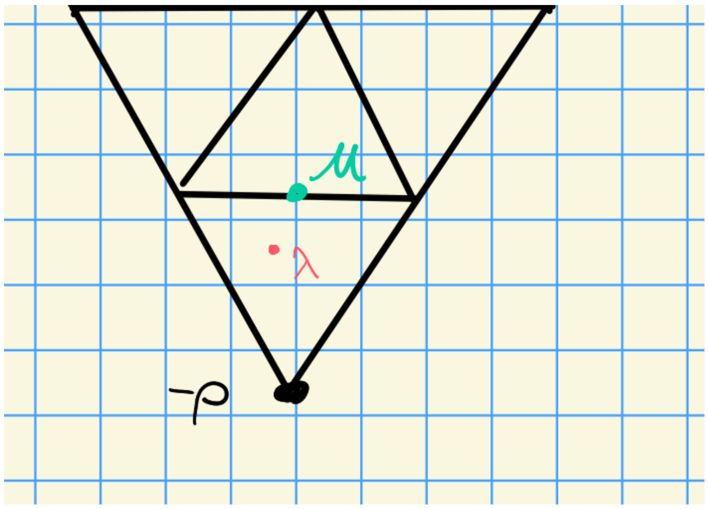

Figure 6: Image

Big Question: What happens to $L(w\cdot\lambda)$ when translating away from a wall?

Definition 1.2.1 (Walls)**.**
A facet F is a **wall** $\iff$ $\Big|\Phi_0^+(F)\Big| = 1$. In this case, there exists a unique α such that $\langle \lambda+\rho,\ \alpha^\vee\rangle = n_\alpha p$.

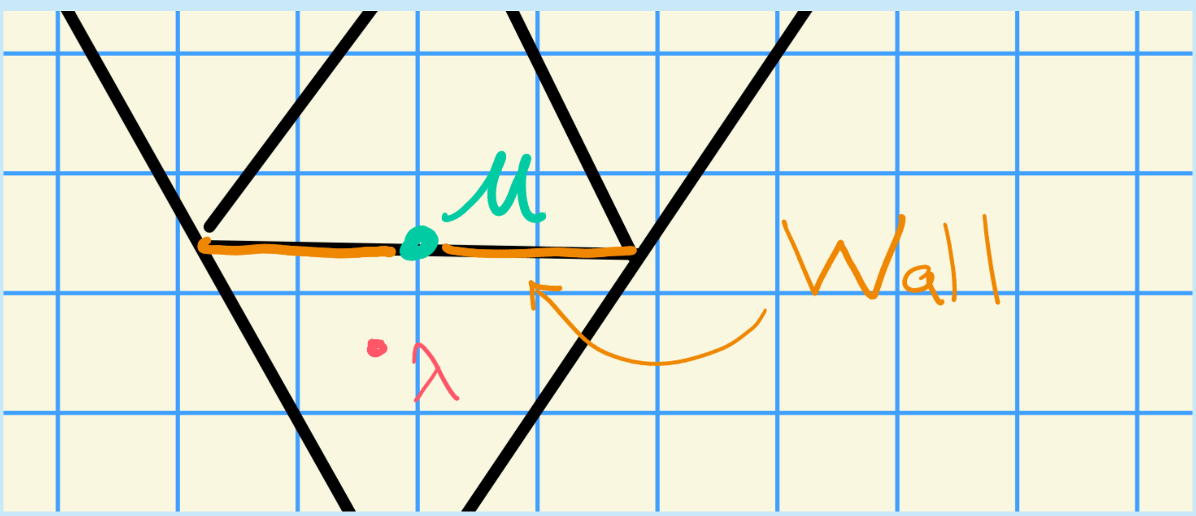

Figure 7: Example of a wall

Remark 1.2.1.
Note that $s_F = s_{\beta,n_p}$ where $n_p = \langle \lambda+\rho,\ \beta^\vee\rangle$ acts on the wall as the identity and reflects across it:

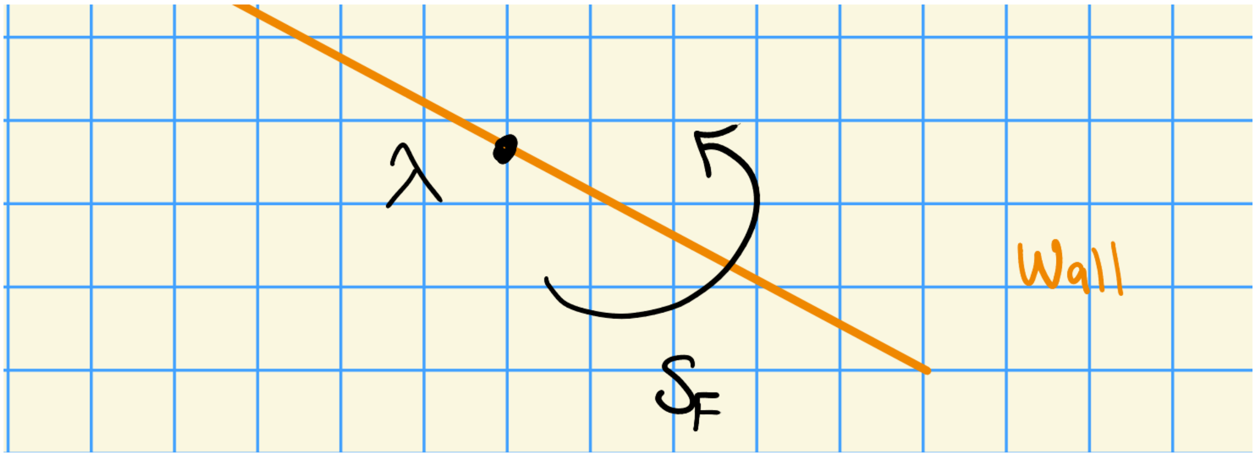

Figure 8: Image

Here $\mathrm{Stab}_{W_p}(\lambda) = \{1, s_F\}$.

Proposition 1.2.2*(?)*.
Consider the following situation:

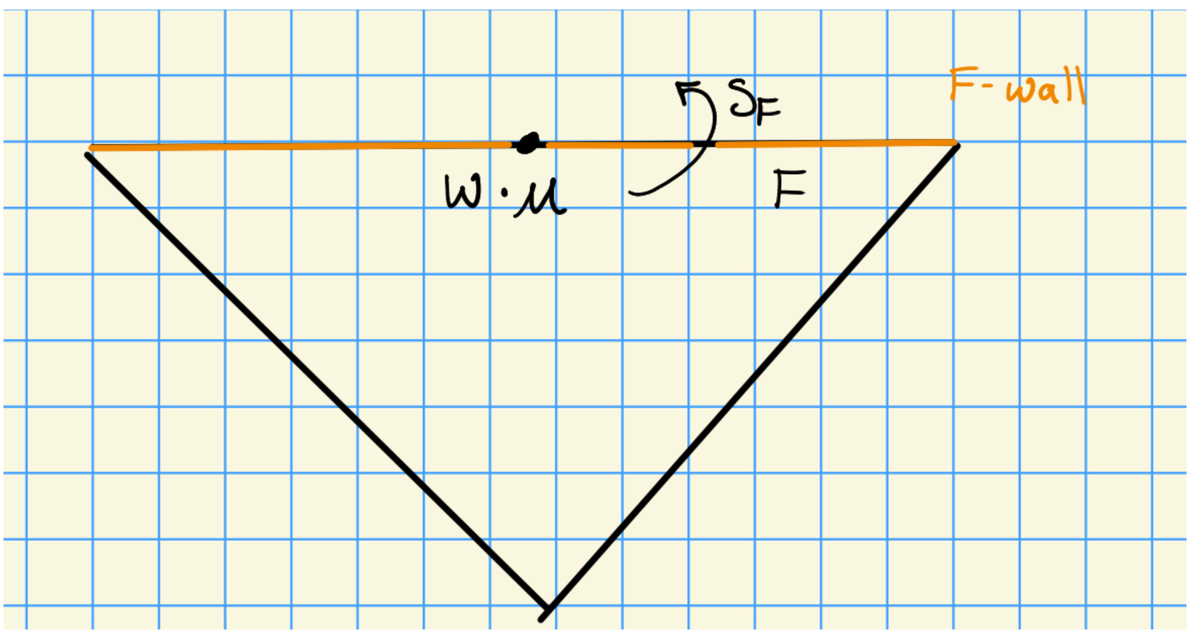

Figure 9: Image

1. $[T_\mu^\lambda(L(w\cdot\mu)) : L(w\cdot\lambda)] = 2$, appearing once in the socle and once in the head.
2. $L(w\cdot\lambda) = \mathrm{Soc}\ _G T_\mu^\lambda(L(w\cdot\mu)) = T_\mu^\lambda(L(w\cdot\mu))/\mathrm{rad} T_\mu^\lambda(L(w\cdot\mu))$.

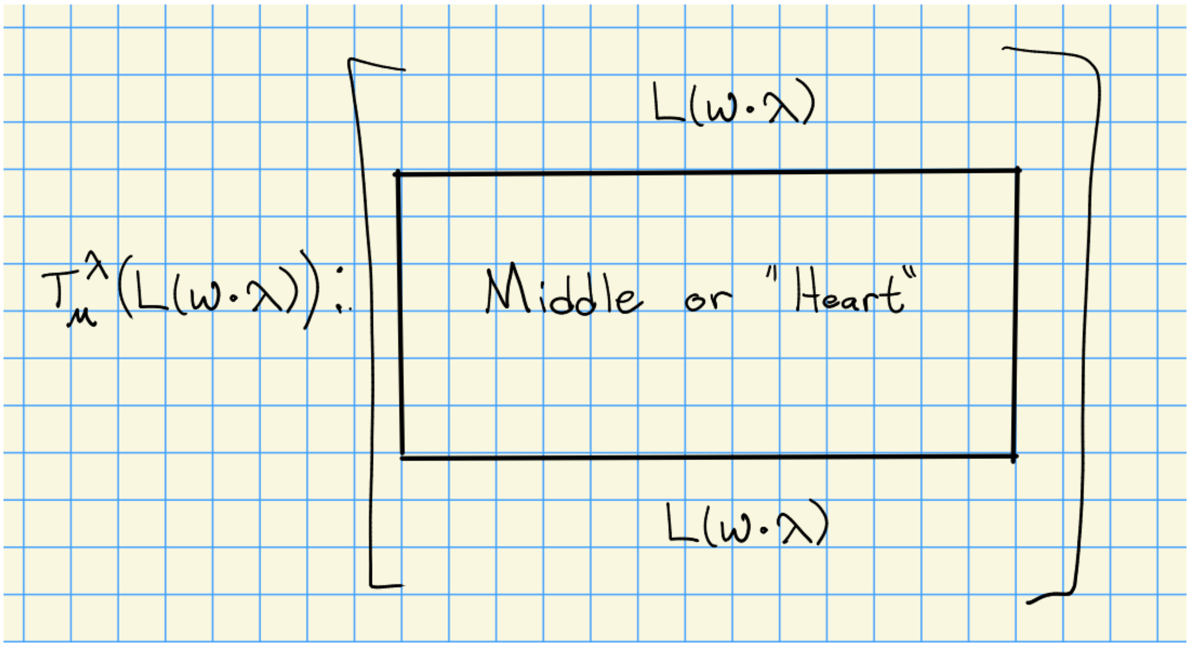

Figure 10: Heart of the module

Big Problems:

1. When is the heart semisimple?
2. Determine the composition factors in the heart?

Given these, you could compute dimensions of irreducible representations.