MTH 674 Differential Geometry of manifolds

Midterm Sample Problems

Problem I. Homework assignments I and II.

Problem II. Define in detail

a) A topological manifold M.
b) A differentiable manifold M.
c) Transition functions.
d) An atlas of coordinate charts.
e) Real and complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.
f) Grassmannians $\mathbf{Gr}(m, k)$.
g) The product manifold $M_1 \times M_2$ of two differentiable manifolds M_1, M_2.
h) Coordinate representation of a mapping $F\colon M_1 \to M_2$ from M_1 into M_2, where M_1, M_2 are topological manifolds.
i) A smooth mapping $F\colon M_1 \to M_2$ from a differentiable manifold M_1 into a differentiable manifold M_2.
j) A diffeomorphism $F\colon M_1 \to M_2$ between two manifolds M_1 and M_2.
k) A bump function.
l) Partition of unity subordinate to a cover.
m) A derivation at a point $p \in M$, where M is a differentiable manifold.
n) The tangent space $T_p(M)$ to a manifold M at point p.
o) The pushforward $F_*(\mathbf{v})$ of a tangent vector $\mathbf{v} \in T_pM$ under a mapping $F\colon M \to N$.
p) Coordinate vectors $\partial/\partial x^i$, where $(\mathcal{U}, \mathbf{x} = (x^1, \dots, x^m))$ is a coordinate chart on a differentiable manifold.
q) A smooth vector field X on a differentiable manifold M.
r) The rank of a mapping $F\colon M_1 \to M_2$.
s) The Lie bracket $[X, Y]$ of two smooth vector fields on M.
t) F-related vector fields on M_1, M_2, where $F\colon M_1 \to M_2$ is a smooth mapping.
u) Quotient topology
v) Immersion, submersion
w) Immersed, embedded submanifold.
x) Slice coordinates
y) Lie group and the Lie algebra of a Lie group.
z) The Einstein summation convention.

Problem III.

a) Let $M_1 = \mathbb{R}$ with the coordinate map $\phi(x) = x$ and let $M_2 = \mathbb{R}$ with the coordinate map $\psi(x) = x^{1/3}$. Show that M_1 and M_2 are not equal as manifolds but that they are diffeomorphic as manifolds.

b) The figure eight is the image of the mapping

$$F\colon (-\pi,\pi) \to \mathbb{R}^2, \qquad F(t) = (2\cos(t-\frac{\pi}{2}), \sin 2(t-\frac{\pi}{2})).$$

Show that F is an injective immersion but not an embedding.

Problem IV.

a) Let M_1, M_2 be smooth manifolds. Show that $M_1 \times M_2$ is diffeomorphic with $M_2 \times M_1$ in the standard product manifold structure.

b) Let $F\colon \mathbb{R}\mathrm{P}^2 \to \mathbf{Gr}(3,2)$ map a line in $\mathbb{R}^3$ to the plane perpendicular to it. Show that F is a diffeomorphism.

Problem V. Let M be a differentiable manifold.

a) State the n-submanifold property for a subset $N \subset M$.

b) Let $F\colon M \to P$ be a smooth mapping with constant rank k. Starting from the rank theorem, show that for any $p \in P$ the level set $N_p = \{q \in M \,|\, f(q) = p\}$ satisfies the n submanifold property (provided it is not empty).

c) Show that the 3-dimensional unit sphere

$$S^3 = \{(x,y,z,u) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^1 + u^2 = 1\}$$

is an embedded submanifold of $\mathbb{R}^4$.

d) Show that the subgroup $SL(n) \subset Gl(n)$ of matrices with unit determinant is an embedded submanifold of $Gl(n)$.

e) Show that the group $SO(n) \subset Gl(n)$ of orthogonal matrices with unit determinant is an embedded submanifold of $Gl(n)$.

f) Let M, N be differentiable manifolds and let $b \in N$. Show that the mapping $i_b\colon M \to M \times N$, $i_b(p) = (p,b)$ is an embedding.

Problem VI. Let D be a derivation at $p \in M$.

a) Suppose that $f\colon M \to \mathbb{R}$ is zero in some neighborhood of the point p. Show that $Df = 0$ (that is, that D is a local operator).

b) Let $M = \mathbb{R}^n$ with the coordinates x^1,$\ldots x^n$, and let D be a derivation at $0 \in \mathbb{R}^n$. Show that D can be expressed as a linear combination of the coordinate differentials $\partial/\partial x^i$, $i = 1,2,\ldots,n$. You may assume without proof that given a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f(0,\cdots,0) = 0$ then there are smooth functions $g_i\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(x^1,\ldots,x^n) = \sum_{i=1}^n x^i g_i(x^1,\ldots,x^n)$.

Problem VII.

a) Let $\pi_{\mathrm{P}}\colon S^2 \to \mathbb{R}\mathrm{P}^2$, $\pi_{\mathrm{P}}(x,y,z) = [x,y,z]$, be the natural projection, where S^2 is the unit sphere and $\mathbb{R}\mathrm{P}^2$ the projective space. Show that open sets in $\mathbb{R}\mathrm{P}^2$ are precisely the images of open sets in S^2 under the mapping π_{P}.

b) Show that the projection $\pi_{\mathrm{P}}\colon S^2 \to \mathbb{R}\mathrm{P}^2$ is smooth and everywhere of rank 2.

c) Let $F\colon \mathbb{R}^2 \to \mathbb{R}\mathrm{P}^2$ be the smooth map $F(x,y) = [x,y,1]$, and let $X = x\partial_x - y\partial_y$ be a smooth vector field on $\mathbb{R}^2$. Prove that there is a smooth vector field on $\mathbb{R}\mathrm{P}^2$

that is F-related to X, and find its coordinate expressions in the standard charts for $\mathbb{R}\mathrm{P}^2$.

Solution. (4c) Let $(\mathcal{U}_x, \psi_x)$, $(\mathcal{U}_y, \psi_y)$, $(\mathcal{U}_z, \psi_z)$ be the standard charts on $\mathbb{R}\mathrm{P}^2$. Note that $F(\mathbb{R}^2) \subset \mathcal{U}_z$ and that $\psi_z \circ F(x, y) = (x, y)$. Hence we can define a vector field Y on $\mathcal{U}_z$ that is F-related to X by $Y = (\psi_z^{-1})_*(X)$. We need show that Y can be extended to a smooth vector field on $\mathbb{R}\mathrm{P}^2$.

The coordinate expression for Y in ψ_z coordinates is simply $X = x\partial_x - y\partial_y$. We will use coordinates (u^1, u^2) and (v^1, v^2) on the images $(= \mathbb{R}^2)$ of ψ_x and ψ_y, respectively. The u^1-component of Y in ψ_x coordinates is given by

$$Y(\psi_x^1) = X(\psi_x^1 \circ \psi_z^{-1}) = X(\frac{y}{x}) = -2\frac{y}{x} = -2u^1.$$

Also,

$$Y(\psi_x^2) = X(\frac{1}{x}) = -\frac{1}{x} = -u^2.$$

Thus the coordinate expression for Y in ψ_x coordinates is

$$Y = -2u^1\partial_{u^1} - u^2\partial_{u^2}.$$

Note that the above expression defines a smooth vector field on all of $\mathcal{U}_x$.

Similarly, the coordinate expression of Y in ψ_y coordinates is given by

$$Y = 2v^1\partial_{v^1} + v^2\partial_{v^2}.$$

It follows that Y extends to a smooth vector field on all of $\mathbb{R}\mathrm{P}^2$. ■

Problem VIII.

a) Let $X = x^2y\partial_y - z\partial_z$, $Y = xy\partial_x + y\partial_y - z^2\partial_z$ be vector fields on $\mathbb{R}^3$. Compute $[X, Y]$.

b) Let $(\mathcal{U}, \phi)$, $(\mathcal{U}, \psi)$ be two coordinate systems on a 2-dimensional manifold M with the same domain $\mathcal{U} \subset M$, and suppose that the transition function is given by $(u^1, u^2) = \psi \circ \phi^{-1}(x^1, x^2) = ((x^1)^2 - (x^2)^2, 2x^1x^2)$. A vector field X on M has the coordinate expression $X = x^2\partial_{x^1}$ in the chart $(\mathcal{U}, \phi)$. Find its coordinate expression in the chart $(\mathcal{U}, \psi)$.

c) Find the expression for the vector field $X = y\partial_x$ in polar coordinates.

Problem IX. Let $M \subset \mathbb{R}^3$ be the paraboloid determined by the equation $z = x^2 + y^2$ with the subspace topology.

a) Let $\varphi\colon M \to \mathbb{R}^2$ be given by $\varphi(x, y, z) = (x, y)$. Show that φ provides a global coordinate map for M.

b) Let $F\colon M \to S^2$ be the map

$$F(x, y, z) = (1 + 4x^2 + 4y^2)^{-1/2}(-2x, -2y, 1).$$

Is F smooth?

c) Let $\mathbf{v} = \partial/\partial\varphi^1{}_{|(x,y,z)=(0,1,1)}$. Compute a coordinate expression for $F_*(\mathbf{v})$, where F is as in part b.

d) Let $f\colon M \to \mathbb{R}$, $f(x,y,z) = xyz^2$. Find the coordinate derivative

$$\frac{\partial}{\partial \varphi^1}_{|(1,-1,2)} f$$

at the indicated point.

Problem X. The equations $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$ define spherical coordinates ρ, θ, ϕ on $\mathbb{R}^3$.

a) Let $\mathbf{V} = \partial/\partial\phi$, $\mathbf{W} = \partial/\partial\rho$ be the coordinate partial derivative vector fields. Express $\mathbf{V}$, $\mathbf{W}$ in Cartesian coordinates x, y, z.

b) Let $\pi_{\mathrm{P}}\colon \mathbb{R}^3 \smallsetminus \{(0,0,0)\} \to \mathbb{R}\mathrm{P}^2$ be the projection. Find an expression for $\pi_{\mathrm{P}*}(\mathbf{V})$ in suitable coordinates for $\mathbb{R}\mathrm{P}^2$.

Problem XI. The complex projective space $\mathbb{C}\mathrm{P}^1$ is defined as the set of all 1 dimensional complex subspaces of $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$, or alternatively, the orbit space of the action of $\mathbb{C}^* = \mathbb{C}\smallsetminus\{0\}$ on $\mathbb{C}^{2^*} = \mathbb{C}^2 \smallsetminus \{(0,0)\}$ by multiplication. Write $\pi_{\mathrm{P}}\colon \mathbb{C}^{2^*} \to \mathbb{C}\mathrm{P}^1$ for the projection. Endow $\mathbb{C}\mathrm{P}^1$ with the usual quotient topology so that a set $U \subset \mathbb{C}\mathrm{P}^1$ is open if and only if the union of all lines constituting U (with the origin removed!) is open in $\mathbb{C}^{2^*}$.

a) Explain why the quotient topology on $\mathbb{C}\mathrm{P}^1$ is second countable and Hausdorff, and show that the projection map π_{P} is open.

b) Define an atlas of coordinate charts on $\mathbb{C}\mathrm{P}^1$ in analogy with the standard coordinate charts for $\mathbb{R}\mathrm{P}^1$ constructed in class, and show that these are homeomorphisms. Also identify the image of each coordinate map.

c) Find the transition functions (*a.k.a.* the change of coordinate maps) for the atlas for $\mathbb{C}\mathrm{P}^1$ constructed in part b.

d) Compute the rank of the projection π_{P} at every point in $\mathbb{C}^{2^*}$.

e) Use part c to show that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to the unit sphere S^2.

Problem XII. Let $\mathbf{Gr}(n,2)$ denote the Grassmannian space of 2–dimensional (vector) subspaces in $\mathbb{R}^n$.

a) Let $\mathcal{P} \in \mathbf{Gr}(n,2)$. Then any two bases $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathcal{P}$ are related by

$$\mathbf{w}_i = \sum_{j=1,2} a_i^j \mathbf{v}_j, \qquad i = 1,2, \tag{1}$$

where $(a_i^j) \in GL(2)$ is an invertible 2×2 matrix. Conclude that $\mathbf{Gr}(n,2)$ can be identified with the equivalence classes of linearly independent pairs of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\{\mathbf{v}_1, \mathbf{v}_2\} \sim \{\mathbf{w}_1, \mathbf{w}_2\}$ provided that equation (1) holds for some $(a_i^j) \in GL(2)$.

b) Let $GL(2)$ act on the space $\mathcal{T}$ of $2\times n$ matrices of rank 2 by left multiplication. Use part a to show that the orbit space $\mathcal{T}/GL(2)$ of the action can be identified with $\mathbf{Gr}(n,2)$.

- **c)** Equip $\mathcal{T}$ with the subspace topology as an open subset of $\mathbb{R}^{2n}$, and equip $\mathbf{Gr}(n,2)$ with the usual quotient topology. Conclude that the projection $\pi_{Gr}\colon \mathcal{T} \to \mathbf{Gr}(n,2)$ is an open mapping and that $\mathbf{Gr}(n,2)$ is Hausdorff.
- **d)** Let $\mathcal{T}(i,j) \subset \mathcal{T}$ denote the set of rank 2 matrices whose minor consisting of the ith and jth columns is invertible. Show that the action of $GL(2)$ on $\mathcal{T}$ preserves the sets $\mathcal{T}(i,j)$ and that every $GL(2)$ orbit in $\mathcal{T}(i,j)$ contains a unique matrix whose i,j-minor is the identity matrix.
- **e)** Conclude that every plane $\mathcal{P} \in \mathcal{T}(i,j)$ can be identified with a unique $(n-2)\times 2$ matrix. (For example, with $n=4$ and $i=1$, $j=3$, the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ would correspond to the plane admitting the basis $\mathbf{v}_1 = (1,a,0,b)$, $\mathbf{v}_2 = (0,c,1,d)$.)
- **f)** Show that each $\mathbf{G}_{ij} = \pi_{Gr}(\mathcal{T}(i,j)) \subset \mathbf{Gr}(n,2)$ is open, and use part e to define coordinate maps φ_{ij} in each $\mathbf{G}_{ij}$. In particular, show that each $\varphi_{ij}\colon \mathbf{G}_{ij} \to \mathbb{R}^{2n-4}$ is a homeomorphism.
- **g)** Finally compute a representative sample of transition functions to conclude that $\mathbf{Gr}(n,2)$ forms a differentiable manifold.

Problem XIII. Let $\mathbb{C}\mathrm{P}^n$ and $\mathbf{Gr}(n,2)$ denote the complex projective space and Grassmann manifold constructed in problems XI and XII.

- **a)** Let S^3 be the unit sphere in $\mathbb{C}^2$ identified with $\mathbb{R}^4$, and let the Hopf map $\pi_H : S^3 \to S^2$ be the restriction of the projection π_{P} to S^3. Describe the inverse image ${\pi_H}^{-1}(p)$ of a point $p \in S^2$.
- **b)** Show that the projection $\pi_{Gr}\colon \mathcal{T} \to \mathbf{Gr}(n,2)$ is differentiable.
- **c)** Show that each $\mathbf{Gr}(n,2)$ is compact.

Problem XIV.

- **a)** Let φ_3 denote the standard coordinates on $\mathcal{U}_3 = \{[x,y,z] \in \mathbb{R}\mathrm{P}^2 \mid z \neq 0\}$ given by $(u^1,u^2) = \varphi_3([x,y,z]) = (x/z, y/z)$. The coordinate differential $\mathbf{v} = \partial/\partial u^1$ corresponds to the directional derivative under some curve $\boldsymbol{\alpha}$ in $\mathbb{R}\mathrm{P}^2$. Geometrically, a curve represents a 1-parameter family of lines in $\mathbb{R}^3$. Describe this family for a curve $\boldsymbol{\alpha}$ of your choice.
- **b)** Let φ_{34}, $\begin{pmatrix} u_1^1 & u_1^2 \\ u_2^1 & u_2^2 \end{pmatrix} = \varphi_{34}(\mathcal{P})$, denote the coordinates for $\mathbf{Gr}(4,2)$ as constructed in problem XII. The coordinate differential $\mathbf{v} = \partial/\partial u_2^2$ corresponds to the directional derivative along some curve $\boldsymbol{\alpha}$ in $\mathbf{Gr}(4,2)$, that is, 1-parameter family of planes in $\mathbb{R}^4$. Describe this family for a curve $\boldsymbol{\alpha}$ of your choice.

Problem XV.

- **a)** Let $L_A\colon Gl(m) \to Gl(m)$, $A \in Gl(m)$, be the left translation $L_A(X) = AX$. Describe $L_{A*}\colon T_{\mathbb{I}}Gl(m) \to T_A Gl(m)$.
- **b)** Let $\Psi\colon Gl(m) \to Gl(m)$ be the mapping $\Psi(X) = X^T X$. Describe $\Psi_*\colon T_{\mathbb{I}}Gl(m) \to T_{\mathbb{I}}Gl(m)$.

- **c)** Let $\iota\colon Gl(m) \to Gl(m)$ denote the inverse $\iota(X) = X^{-1}$. Describe $\iota_*\colon T_{\mathbb{I}}Gl(m) \to T_{\mathbb{I}}Gl(m)$.
- **d)** Let $m\colon Gl(m) \times Gl(m) \to Gl(m)$, $m(A, B) = AB$ denote the multiplication map. Show that the differential $m_*\colon T_{\mathbb{I}}Gl(m) \oplus T_{\mathbb{I}}Gl(m) \to T_{\mathbb{I}}Gl(m)$ is given by $m_*(V, W) = V + W$.

Problem XVI.

- **a)** Show that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$, where X, $Y \in \mathcal{X}(M)$.
- **b)** Let $F\colon M \to N$ be smooth, where M, N are differentiable manifolds. Suppose that vector fields $X, Y \in \mathcal{X}(M)$ on M are F-related to vector fields $\widehat{X}$, $\widehat{Y} \in \mathcal{X}(N)$. Show that the bracket $[X, Y]$ is F-related to $[\widehat{X}, \widehat{Y}]$.
- **c)** Let M, N be smooth manifolds and let $F\colon M \to N$, $G\colon N \to M$ be smooth maps satisfying $G \circ F = \mathrm{id}$. Given $Y \in \mathcal{X}(N)$, define X_p by $X_p = G_*(Y_{F(p)})$. Show that the assignment $p \to X_p$ defines a smooth vector field on M.
- **d)** Let F and G be as in part c. Show that $F(M) \subset N$ is an embedded submanifold.

Problem XVII.

- **a)** Let $F\colon M \to N$ be smooth, where M, N are differentiable manifolds, and suppose that $F_*\colon T_pM \to T_{F(p)}N$ is an isomorphism for all $p \in M$. Prove that F is an open mapping. Show, in addition, that F must be surjective if M is compact and N is connected.
- **b)** Suppose that $F\colon M \to N$ is of constant rank and surjective. Prove that F is a smooth submersion.

Problem XVIII.

- **a)** Let $F\colon \mathbb{R}^m \to \mathbb{R}\mathrm{P}^m$ be defined by $F(x^1, x^2, \ldots, x^m) = [x^1, x^2, \ldots, x^m, 1]$. Show that F is a smooth map onto a dense open subset of $\mathbb{R}\mathrm{P}^m$.
- **b)** Define similarly $G\colon \mathbb{C}^m \to \mathbb{C}\mathrm{P}^m$ by $G(z^1, z^2, \ldots, z^m) = [z^1, z^2, \ldots, z^m, 1]$. Show that G is a diffeomorphism onto a dense, open set of $\mathbb{C}\mathrm{P}^m$.
- **c)** Let $\widehat{p}(z)$ be a complex polynomial in one variable and let $G\colon \mathbb{C}^1 \to \mathcal{U} \subset \mathbb{C}\mathrm{P}^1$ be as in part b, where $\mathcal{U}$ denotes the image of G. Define $p\colon \mathcal{U} \to \mathcal{U}$ by the condition that $p \circ G = G \circ \widehat{p}$. Show that p can be extended to a smooth map on the entire $\mathbb{C}\mathrm{P}^1$.

Problem XIX. Let $\mathcal{U} \subset \mathbb{R}^2$ be an open set. A *proper coordinate patch* is a one-to-one immersion $\mathbf{x}\colon \mathcal{U} \to \mathbb{R}^3$ such that the inverse function $\mathbf{x}^{-1}\colon \mathbf{x}(\mathcal{U}) \to \mathcal{U}$ is continuous in the subspace topology of $\mathbf{x}(\mathcal{U})$. A *surface* is a subset $S \subset \mathbb{R}^3$ equipped with the subspace topology such that for each point $p \in S$, there is a proper coordinate patch whose image contains a neighborhood of p in S. Show that a surface is a differentiable manifold.

Problem XX.

- **a)** Show that the set of all lines in $\mathbb{R}^2$ can be identified with an open subset of $\mathbb{R}\mathrm{P}^2$.

b) Show that the set of all planes in $\mathbb{R}^3$ can be identified with an open subset of $\mathbb{RP}^3$.

c) Show that the space of all *oriented* lines in $\mathbb{R}^3$ can be identified with TS^2.